Universal aspects of non-equilibrium currents in a quantum dot

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We study the electric current in the non-equilibrium Kondo model at zero magnetic field, using real-time perturbation theory in the Schwinger-Keldysh formulation. We show that the perturbative coefficients to all orders have a finite limit at large switch-on time ($t_0 \to -\infty$), and we give a prescription for general operators to give finite coefficients in this limit. We explain how this is related to the fact that the leads play the role of thermal baths and allow relaxation to occur and the steady state to form. This proves perturbatively that a steady state is reached in the Schwinger-Keldysh formulation, and specifies which operators correspond to quantities that have a well-defined value in the steady state. Then, we show that the steady state can be described by a special type of density matrix (related to Hershfield’s conjecture for the particular example of the non-equilibrium Kondo model.) In the second part of the paper we perform a renormalization-group analysis of the perturbative series. We give a general argument that strongly suggests that the perturbative series of any average in the steady state satisfies the equilibrium Callan-Symanzik equations, and show in detail how it works to one-loop order for the electric current operator inside any average. We finally compute to two loops order the average of the electric current in the steady state, and perform a renormalization-group improvement. From this, we give a universal prescription, valid in the perturbative regime, for comparing the effect of the electric current to that of the temperature on the “Kondo cloud”.

I. INTRODUCTION AND DISCUSSION

The description of an out-of-equilibrium strongly correlated system is a long standing problem. Even in the simplest case where the system is in a steady state and its properties no longer change with time, the usual formalism of quantum statistical mechanics is inadequate. Theoretical understanding of such systems became all the more pressing with the recent spectacular progress in nanotechnology, which has made it possible to study the Kondo impurity, one of the best understood strongly correlated systems, in out-of-equilibrium conditions.

The Kondo impurity was realized experimentally as a quantum dot, a tiny island of electron liquid, attached via two tunnel junctions to leads (baths or reservoirs of electrons) held at different electric (or chemical) potentials. This set-up allows an electric current to flow across the dot, and measurements of the current were carried out as a function of the potential difference $V$, the temperature $T$ and the magnetic field $B$ [1].

When the dot carries a net spin in the Coulomb blockade regime, it can be modeled by a Kondo Hamiltonian with two channels $\alpha = 1, 2$, corresponding to the two leads, to which the spin of the dot, $\vec{S}$, couples [2]. The resonant tunneling through the dot (elastic co-tunneling) allows the electrons from each bath to jump on the dot and back to the same bath, leading to the formation of Kondo resonance around the Fermi level $\mu_\alpha$ in each lead. Further, electrons from one bath can jump on the dot and onto the other bath, giving “off-diagonal coupling” of the two channels to each other. With the matrix of couplings $J_{\alpha,\alpha'}$ and at zero magnetic field, the Hamiltonian is

$$H = \sum_\alpha \sum_{\vec{k}, a} (\epsilon_{\vec{k}} - \mu_\alpha) c_{\alpha, \vec{k}, a}^\dagger c_{\alpha, \vec{k}, a} + \sum_{\alpha, \alpha'} \sum_{\vec{k}, \vec{k}', a, a'} J_{\alpha, \alpha'} c_{\alpha, \vec{k}, a}^\dagger \vec{\sigma}_{a, a'} c_{\alpha', \vec{k}', a'} \cdot \vec{S}. \quad (1)$$

Here $a$ denotes the spin index $a = \pm 1/2$, and $\vec{S}$ is in the spin-1/2 representation.

The process corresponding to off-diagonal coupling induces a current when the baths are held at nonzero potential difference, $V = \mu_2 - \mu_1$. The development of the Kondo resonance as temperature is lowered enables
FIG. 1: (Color online) (A) The system. (B) The system unfolded.

the system to overcome the Coulomb blockade, producing a significant increase of the conductance. The unitarity limit is reached as $T \to 0$, $V \to 0$.

As we are interested in the universal properties of the system, we shall consider the model in the range $T, V \ll D_\alpha$, where $D_\alpha = D$ are the bandwidths of the leads, each lead being considered a very large conductor (the bandwidths can be assumed to be the same for both channels). We are allowed therefore to carry out the standard steps (linearizing around the Fermi level, keeping only the $s$-wave component in the expansion of $c_{\alpha, \vec{k}, a}$ in spherical modes), to obtain a representation of each lead as a free electron gas on the half line consisting of left and right movers $\psi_{\alpha, L}(x), \psi_{\alpha, R}(x), x \leq 0$, interacting with the impurity localized at $x = 0$. It will be convenient for us to “unfold” the baths, making left and right movers on the half line into right movers on the full line (defining $\psi_{\alpha, R}(x) = \psi_{\alpha, L}(-x), x \geq 0$). See Figure 1.

The field-theoretic Hamiltonian is then:

$$H = -i \sum_\alpha \int_{-\infty}^{\infty} dx \psi_{\alpha}^\dagger(x) \partial_x \psi_{\alpha}(x) + \frac{V}{2} \int_{-\infty}^{\infty} dx \left( \psi_2^\dagger \psi_2 - \psi_1^\dagger \psi_1 \right) + \sum_{\alpha, \alpha' = 1, 2} J_{\alpha, \alpha'} \psi_{\alpha}^\dagger(0) \bar{\sigma} \psi_{\alpha'}(0) \cdot \vec{S}$$

(2)

where we work in units such that $v_F = 1$. We also denote,

$$H_z = \frac{1}{2} \int_{-\infty}^{\infty} dx \left( \psi_2^\dagger \psi_2 - \psi_1^\dagger \psi_1 \right)$$

and

$$H_I = \sum_{\alpha, \alpha' = 1, 2} J_{\alpha, \alpha'} \psi_{\alpha}^\dagger(0) \bar{\sigma} \psi_{\alpha'}(0) \cdot \vec{S}.$$ 

The coupling of the baths to the quantum dot is parametrized by the Hermitian matrix

$$J_{\alpha, \alpha'} = \pi \left( \begin{array}{cc} \lambda_d & \lambda \\ \lambda & \lambda_d \end{array} \right)_{\alpha, \alpha'}$$

where $\lambda_d$ and $\lambda$ are real (the factor $\pi$ is introduced for later convenience). It is possible to diagonalize the matrix of coupling $J_{\alpha, \alpha'}$ by a change of basis in the channel space. In the simple situation of a single level quantum dot, described by the Anderson model, coupled to identical leads one naturally finds the relation $\lambda_d = \lambda$. This makes the matrix of coupling constants degenerate, one of the eigenvalues being zero. After diagonalizing it, the terms representing the interaction with $\vec{S}$ become that of a decoupled free fermion and a one-channel Kondo model. More generally, for $\lambda_d \neq \lambda$, diagonalizing the matrix of couplings gives the interaction term of the usual 2-channel Kondo model. Out of equilibrium, for $V \neq 0$, diagonalizing the matrix of couplings does not lead to
a simplification of the problem, since the out-of-equilibrium term $VH_z$ induces extra coupling between the new fermion fields: it is not invariant under change of basis. We will not perform this diagonalization here, in order to keep the term $VH_z$ simple. Also, we will consider the general case $\lambda_d \neq \lambda$.

In this paper, we are interested in studying the electric current as function of voltage and temperature in the steady state of this model. As in the usual Schwinger-Keldysh formulation of a non-equilibrium steady state \[12, 13\], we will imagine coupling the dot to the leads at some time $t_0$ in the past when the system is in a thermal equilibrium state, then allowing the system to evolve till $t = 0$ when the current is evaluated. One expects that after a transitory regime, as $t_0 \to -\infty$, the system will relax into a steady state with a constant current flowing through the dot.

Many questions arise concerning this formulation. Probably the most obvious one is: Is the model sufficient to describe the establishment of a steady state current? Or does one have to contemplate additional relaxation mechanisms (certainly present in actual experiments) to absorb the continuous flow of energy of the electrons moving from the higher Fermi-level lead to the lower one? In the framework of real-time perturbation theory, a related (but not equivalent) question that one can answer is whether or not the “infrared” limit $t_0 \to -\infty$ exists for the integrals representing the perturbative coefficients in an expansion in $\lambda$ and $\lambda_d$. We develop the real-time Keldysh perturbation theory (in some ways similar to \[3\] and \[14\]), and use it to establish the convergence of every term of the perturbative series as the switch-on time $t_0$ is sent to minus infinity.

This result is highly non-trivial. Real-time perturbation theory often gives divergences as the switch-on time is sent to minus infinity, unless a good relaxation phenomenon is included in the model. In equilibrium, this issue can be easily overcome: an infrared-divergent real-time perturbation theory for a system in equilibrium only means that the particular model we are considering does not have the proper relaxation mechanism. But with an additional interaction, however small, representing a proper relaxation mechanism, the system will have infrared convergent real-time perturbation theory. Then, for equilibrium models, this can be equivalently described by the always-infrared-convergent “imaginary time” perturbation theory (where the integrals in imaginary time are on a finite interval) coming from the description of the model using its equilibrium density matrix. There, the additional interaction representing relaxation can be sent to zero from the beginning.

Out of equilibrium, however, there is no a priori steady-state density matrix description of steady quantities. Thus, if the real-time perturbation theory is infrared divergent, there is no simple way to describe steady state physics. This is very natural: in contrast to the equilibrium case, the relaxation mechanism is used not only to reach the steady state, but also to form it, since we need a continuous absorption of energy. Hence, such infrared divergences are far more pathological. If the limit of large negative switch-on time exists order by order, this superficially seems a good indication that the steady state is reached and that the model indeed describes the steady state (although, strictly speaking, one would still have to analyse the possible non-perturbative contributions if the perturbative series is asymptotic). But if no such limit exists, certainly more questions arise: does the model reach a steady state non-perturbatively (that is, divergences are an artifact of perturbation theory), or are other interactions necessary? These questions of course appeared in the literature before (see for instance \[12, 13\]). An interesting example is the case studied in \[12\], where it was shown that when the model we are discussing is put into a magnetic field, the limit $t_0 \to -\infty$ and $\lambda, \lambda_d \to 0$ do not commute, leading to divergencies in the perturbation theory as $t_0 \to -\infty$. Assuming that a steady state exists in the model, the correct result as $t_0 \to -\infty$ should then be non-perturbative, and it was partially evaluated under this assumption (the “zeroth order” was evaluated), without external thermal reservoir coupled to the dot.

But the convergence of the perturbative series or the technical “way around” its divergencies as just described above do not guarantee that the results are describing the correct physical steady state. Indeed, if the model is believed to have a steady state, perturbatively or not, two questions should still be answered: Is there an element in the model playing the role of a good thermal reservoir to sustain the correct steady state in the model? And if not, is there a guarantee that a coupling to an external reservoir would not have an important effect?

These last two questions have more bearing than it may seem. In the usual Schwinger-Keldysh formulation, one does not assume any exchange with a thermal bath while the system is evolved: one starts with a thermal equilibrium state, then turns on the coupling to the dot and lets the system evolve without thermal bath. This is certainly not the true physical situation; in fact, understanding how a thermal bath affects the evolution of a quantum system was amongst the main points of the study of Caldeira and Leggett \[4\]. We carry out
the real-time perturbation theory both in equilibrium (that is, in the case \( V = 0 \); note that the real-time formalism includes a transitory non-equilibrium region where the system relaxes to equilibrium) and out of equilibrium (\( V \neq 0 \)), and show that it is the same phenomenon that makes the perturbative series infrared convergent in equilibrium and out of equilibrium. This phenomenon is a factorization at large time separation of the correlation functions involved in the perturbative coefficients; interpreting the integrals over time defining these coefficients in terms of physical processes, this signals a decoherence in time induced by the leads and suggests that the leads are good thermal baths. The convergent expression in equilibrium is indeed the right equilibrium density matrix, confirming that the leads themselves play the role of thermal baths for the dot degrees of freedom. The convergent expression out of equilibrium should then be “the right” steady state. Note that the factorization signaling decoherence in time occurs because the Hamiltonian for the leads is conformally invariant: indeed, in conformal field theory, a large time separation is a large distance separation, which, by locality, gives rise to factorization. Physically, this occurs because the separation between the energy levels of the baths is much smaller than all other scales in the problem and essentially energy-independent (although, as will become clear from our investigation, these two conditions may not be sufficient). Note that these ideas are not entirely new: in [3] the results of Caldeira and Leggett for constructing a thermal bath coupled to a quantum mechanical system were re-interpreted as coupling a bulk conformal field theory in a disk to degrees of freedom on the boundary. Our results generalize this to the behavior of the impurity in the Kondo model. We are currently investigating how this can be further generalized (for example, what the general properties of the conformal field theory should be).

Our proof also allows us to describe the steady-state physics in terms of a “steady state density matrix,” as conjectured by Hershfield [6]. The essential difference between the usual density matrix and the steady-state density matrix can be seen as a non-locality in the latter which captures the build-up of the steady state.

Note that a proof of convergence to all order was developed in [3] for a non-equilibrium free boson model with boundary interactions, but the arguments there were very model-dependent (and do not apply to the non-equilibrium Kondo model) and quite different from ours. In particular, our arguments have a much deeper physical meaning and scope, as explained above.

Let us stress here that it is quite important to know that no external bath is required for reaching a non-equilibrium steady state in an impurity model. Indeed, this means that we can use the real-time formalism without addition of a coupling to an external bath in order to 1) study the perturbative scaling properties of the model as we did in this paper for the Kondo model (see below), 2) construct more or less explicitly, without strong assumption, the steady state as an eigenstate of the Hamiltonian, for instance using the the formalism of the “steady-state density matrix”; this eventually can give access to the infrared behavior of the model and to the integrability properties of the steady state (work in progress). The real-time formalism might not be the easiest way of trying to obtain this understanding, but it is probably the clearest, as it is the most closely related to the actual experimental situation.

Other questions that need to be addressed in the study of this out-of-equilibrium steady state are: How will the Kondo effect, the quenching of the impurity spin as the temperature is lowered below the Kondo scale \( T_K \), evolve in the presence of a current? Will new scales make their appearance? Which quantities are universal? To what extent the powerful ideas of the Renormalization Group (RG) apply there? Many interesting attempts were carried out, mainly perturbatively, to understand the flow of couplings as the cut-off (band width) \( D \) is reduced [7, 8, 9, 10, 11, 12, 13]. We shall pursue a different track and study a question related to the universality features of the model, namely: does a limit \( D \rightarrow \infty \) exist? In this limit all results are universal. We shall establish that such a limit exists by running the RG equations “backwards”, referring to them in the field theoretic context, in the usual way, as the Callan-Symanzik equations. We shall deduce an out-of-equilibrium \( \beta \)-function carrying out the calculation directly in the steady state and will show that it is the same as the equilibrium \( \beta \)-function. This may not be too surprising since the singularity structure of the system usually does not depend on the state in which they are evaluated, so that the ground state and the highly excited steady state produce the same singularities. The finite parts of course are different. We show that only one scale arises, the Kondo temperature, \( T_K \), and the current can be written in a universal form as a function of the ratios \( T/V, T/K \), with \( C \) an additional dimensionless parameter characterizing the asymmetry between \( \lambda \) and \( \lambda' \) (in other words, specifying the RG trajectory). We carry out the computation of the current to two-loop order and verify these statements explicitly. We then use the RG arguments to re-sum the leading logarithms. Our results are valid in the regime where both the bias voltage and the temperature are smaller than the band width, and where the bias voltage or the temperature is larger than the Kondo scale: \( V, T \ll D \), and \( T_K \ll V \) or \( T_K \ll T \). In
particular, we verify that there are no divergencies at $T/V \to 0$ in the perturbative results. This means that to two-loop order, the voltage plays the role of a good infrared cutoff. From the RG analysis, we give a universal prescription, valid in this regime, for comparing the effect of the electric current to the effect of the temperature on the destruction of the Kondo cloud.

We also examined the effect of a local magnetic field on the dot but, as expected, were unable to show that the perturbation series converges in this case. We will come back to a discussion of this case, in relation with the results of [12], in the last section of this paper.

II. FORMULATION OF THE PROBLEM AND GENERAL CONSIDERATIONS

The Schwinger-Keldysh formulation [15, 16].

First formulation.

We shall be interested in the electric current that passes from lead 2 to lead 1 across the quantum dot under the action of the potential difference $V$. It can be calculated by evaluating the average of the current operator $J$ with respect to a density matrix that has evolved over sufficiently long time from the initial non-interacting density matrix

$$\rho_0 = e^{-\beta H_0}$$

under the action of the full evolution operator

$$S(v)(t_1, t_2) = e^{-i(t_1-t_2)H} = e^{-i(t_1-t_2)\left(H_0 + VH_z + H_I\right)} .$$

(4)

The operational meaning of this formulation is the following. The non-interacting leads are initially, say at time $t_0$, brought to thermal and chemical equilibrium at zero potential difference exchanging energy and particles with a common external reservoir at fixed temperature and chemical potential. The energy levels of lead 1 and lead 2 are filled up to the same energy (with thermal and particle fluctuations).

Just after time $t_0$, they are separated from the external reservoir, then a potential difference $V$ is applied and the interaction is turned on. The application of the potential $V$ just after time $t_0$, as usual, causes a raising of the energy levels of lead 2 with respect to those of lead 1. For the clarity of the discussion below, it is worth being more precise here. One should imagine both leads having a continuum of available states from the bottom of their bandwidths with increasing energies (the energies grow in a continuous way for infinite leads, of course, so one should think about densities of states). At time $t_0$, the available states of the leads are filled up to equal energies. Then, just after time $t_0$, when the potential is applied, one shifts the energies of all states of lead 2 by, say, $V/2$ (towards higher energies), and the energies of those of lead 1 by $-V/2$, without changing the occupations of the states. Hence, the levels of lead 2 are now filled up to a higher energy than those of lead 1. Since the reservoir is disconnected and the interaction is turned on, there is a current. The steady state current is obtained after an infinite time, which we will take to be time 0 (that is, we will take $t_0 \to -\infty$).

In the equation (4), the raising of the energy levels and the turning on of the interaction strength seem instantaneous and simultaneous. But one can multiply both terms $VH_z$ and $H_I$ by a factor that smoothly increases from 0 at time $t_0 = -\infty$ to 1 at time 0, for instance the factor $e^{-\mu t}$, in order to implement a simultaneous adiabatic increase of both the potential and the interaction strength. Sending $\mu \to 0$ (the adiabatic increase occurring far in the past) gives the steady state. This is really what is understood in this formulation.

It is not obvious, a priori, that this formulation represents adequately the usual experimental situation, where the leads and the quantum dot are always connected to a common thermal reservoir (but not a reservoir of electrons), even while the steady state is being reached. However, it is natural to think that the leads can themselves play the role of thermal reservoirs. As discussed in the introduction, this is indeed the case, and will be made more precise below.
The average of an operator $O$ in the steady state is then given by

$$
\langle O \rangle_{ss} = \lim_{t_0 \to -\infty} \frac{\text{Tr} \left[ S^{(V)}(0, t_0) \ e^{-\beta H_0} \ S^{(V)}(t_0, 0) \ O \right]}{\text{Tr} \left[ e^{-\beta H_0} \right]}.
$$

(5)

The operators act in the Hilbert space for $H_0$ (which is a tensor product of the two-channel free massless fermion Hilbert space and of the impurity space) obtained by imposing asymptotically vanishing conditions for the fermion fields (correlation functions of fermion fields vanish at infinite distance from each other and from the dot). To be more accurate, we could start by taking the fermion fields on a line segment of length $L$ containing the dot, with some free boundary conditions; for instance, $\psi_n(-L) = \psi_n(L)$ (this corresponds to the usual free boundary conditions when fermions are folded back on the half line), then send $L$ to infinity. The steady state would be obtained in the region

$$
L^{-1} \ll |t_0|^{-1} \ll V, T
$$

(6)

where the energy scale of switch-on, $|t_0|^{-1}$, suffices to smear out the energy level spacing $L^{-1}$.

**Second formulation.**

Another formulation can be given. The initial idea of this second formulation is that the current can be created not only by a shift of the energies of the states of leads 1 and 2 (coming from the application of an electric potential), but also by putting additional electrons in lead 2 and taking away electrons from lead 1. With the interaction on and the reservoir disconnected, a current is created. Again, after an infinite time, the steady state should be reached. The current is then given by

$$
\langle J \rangle_{ss} = \lim_{t_0 \to -\infty} \frac{\text{Tr} \left[ \tilde{S}(0, t_0) \ e^{-\beta (H_0-VH)} \ \tilde{S}(t_0, 0) \ J \right]}{\text{Tr} \left[ e^{-\beta (H_0-VH)} \right]}
$$

(8)

(for more general operators $O$, see (32)).

If the size of the bandwidth can be sent to infinity (when evaluating quantum averages of operators that give finite results in this limit), then the operational description above for the second formulation is equivalent to that of the first formulation, since then only the Fermi energies of leads 1 and 2 matter. In particular, raising the energy levels or filling states with electrons are exactly the same operation in this case.

However, there is another difference between both formulations. In the second formulation, we can now think about putting a factor $e^{-\mu t}$ for adiabatically increasing the interaction strength, but there is no such possibility...
for adiabatically increasing the potential (one would have to add the term \(-(1 - e^{-\mu t})VH_z\) in the evolution Hamiltonian). In other words, for practical calculations, this second formulation naturally implies that the energy levels of lead 2 are raised (or the states are filled), with respect to those of lead 1, by an amount \(V\) instantaneously, and that the interaction is then turned on adiabatically. This is to be contrasted with the first formulation, where both the potential difference and the interaction strength were understood as being simultaneously increased adiabatically.

We will show below that both formulations are equivalent.

**Symmetry currents.** The Hamiltonian \(H_0\) is conformally invariant and has a large algebra of symmetries associated with it. It is a WZW “current algebra” of the symmetry currents (not to be confused with the physical current \(J\)) and it will be convenient to carry out many of the calculations in terms of symmetry currents. Introduce the following operators,

\[
J_z = \pi : (\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2) : \\
J_x = i \pi (\psi_2^\dagger \sigma \psi_1 - \psi_1^\dagger \sigma \psi_2) \\
J_y = \pi (\psi_2^\dagger \sigma_1 \psi_1 + \psi_1^\dagger \sigma_2 \psi_2) \\
J_d = \pi : (\psi_2^\dagger \sigma_2 \psi_2 + \psi_1^\dagger \sigma_1 \psi_1) : .
\]

They form the following subalgebra of the \(su(4)_1\) current algebra:

\[
\begin{align*}
[J_x^0(x), J_d(y)] &= 2i\pi (\epsilon_{ijk} J_d^i(x) \delta(x - y) - \delta_{ij} \delta'(x - y)) \\
[J_x^0(x), J_z(y)] &= 2i\pi (\epsilon_{ijk} J_z^i(x) \delta(x - y) - \delta_{ij} \delta'(x - y)) \\
[J_y^0(x), J_d(y)] &= 2i\pi (\epsilon_{ijk} J_d^i(x) \delta(x - y) - \delta_{ij} \delta'(x - y)) \\
[J_x(x), J_z(y)] &= -2i\pi \delta'(x - y) \\
[J_y(x), J_z(y)] &= 2i\pi \epsilon_{ijk} J_z^i(x) \delta(x - y) \\
[J_x^0(x), J_y^0(y)] &= 2i\pi \epsilon_{ijk} J_y^k(x) \delta(x - y) \\
[J_d(x), J_z(y)] &= 0 \\
[J_d(x), J_y^0(y)] &= 2i\pi \delta_{ij} J_z(x) \delta(x - y) \\
[J_y^0(x), J_z(y)] &= 2i\pi J_y^0(x) \delta(x - y) \\
[J_z(x), J_z(y)] &= 2i\pi J_y^0(x) \delta(x - y).
\end{align*}
\]

In terms of these currents the full Hamiltonian \(H = H_0 + VH_z + H_1\) can be expressed as follows:

\[
\begin{align*}
H_0 &= \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \left( J_d(x) \cdot \dot{J}_d(x) : + : J_x(x) \cdot \dot{J}_x(x) : + : J_y(x) \cdot \dot{J}_y(x) : + : J_z(x) \cdot \dot{J}_z(x) : \right), \\
H_z &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx J_z(x) \\
H_1 &= \lambda_d \dot{J}_d(0) \cdot \vec{S} + \lambda \dot{J}_y(0) \cdot \vec{S}.
\end{align*}
\]

The operator \(H_z\) is the total (normalized) isospin \(z\)-component.

**The electric current.** We now turn to discuss in more detail the current across the quantum dot (we set the electric charge \(e = 1\)), and express it also in terms of the symmetry currents. The electric current \(\mathcal{J}\) is given by the operator measuring the difference between the fermion density on, say, the second channel just before hitting the impurity and the fermion density on the same channel just after hitting it:

\[
\mathcal{J} = \lim_{\epsilon \to 0^+} \left( \psi_2^\dagger \psi_2(x = -\epsilon) - \psi_2^\dagger \psi_2(x = \epsilon) \right) = \frac{1}{2\pi} \lim_{\epsilon \to 0^+} (J_z(\epsilon) - J_z(-\epsilon)).
\]

(we use here the “unfolded set-up”). Equivalently one can express the electric current as the rate of decrease of the charge on lead-2 (or increase on lead-1),

\[
\mathcal{J} = -\frac{d}{dt} N_2 = \frac{d}{dt} N_1 = -i[H, H_z] = \lambda \dot{J}_y(0) \cdot \vec{S}.
\]
It is easy to see that the two definitions coincide. We rewrite the first definition using “impurity conditions” - operator relation inherited from boundary conditions. Boundary conditions, in general, are part of the equations of motion and lead to operator relations valid on the full Hilbert space of a boundary quantum field theory. They are often derived from the action of the model in the same way as one derives the equations of motion. From our view point, after “unfolding” the Kondo model, we have a model with an impurity instead of a boundary. As with boundaries, impurities give rise to “impurity conditions” which are part of the equations of motion and are operator relations valid on the full Hilbert space. In operator language (which is more convenient for our purposes), the impurity condition associated to a local operator \(O(x)\) can be written

\[
\lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{-\epsilon} dx + \int_{\epsilon}^{\infty} dx \right) [H, O(x)] = \int_{-\infty}^{\infty} [H, O(x)].
\]

Consider the impurity condition associated to the operator \(J_z(x)\) with the model with Hamiltonian \(H\) (2).

\[
\lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{-\epsilon} dx + \int_{\epsilon}^{\infty} dx \right) [H, J_z(x)] = \int_{-\infty}^{\infty} [H, J_z(x)].
\]

(14)

On the left-hand side, only the free part \(H_0\) of the Hamiltonian is involved, because the operator \(J_z(x)\) is never at the site \(x = 0\) (and \(H_z\) commutes with \(J_z(x)\)). Using the fact that with the free Hamiltonian \(H_0\), \(J_z(x)\) is a right-moving operator \([H_0, J_z(x)] = i \frac{d}{dt} J_z(x)\), and using the asymptotic conditions \(J_z(\infty) = J_z(-\infty)\), we have

\[
\lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{-\epsilon} dx + \int_{\epsilon}^{\infty} dx \right) [H, J_z(x)] = \lim_{\epsilon \to 0^+} i(J_z(-\epsilon) - J_z(\epsilon)).
\]

On the other hand, on the right-hand side of (14), since the integration is on the full interval, the free part of the Hamiltonian does not contribute. Only the impurity term, at \(x = 0\), contributes, and it gives

\[
\int_{-\infty}^{\infty} [H, J_z(x)] = 2i\pi\lambda J_z(0) \cdot \vec{S}.
\]

as expected.

Having now discussed the system and the various operators describing it we turn to discuss in more detail the nature of non-equilibrium in the system.

III. EQUILIBRIUM VS. NON-EQUILIBRIUM

Our model, a quantum impurity coupled to leads at different chemical potentials, describes a non-equilibrium situation – a current is flowing from one lead to another. What is the the precise meaning of this statement?

In this section we show in what sense an out-of-equilibrium model differs from an equilibrium model. We begin by showing how the Keldysh formulation leads, when the system is in equilibrium, to the usual equilibrium density matrix description. We prove, in other words, the following:

\[
\frac{\bar{S}(0, -\infty)e^{-\beta H_0} \bar{S}(-\infty, 0)}{\text{Tr}[e^{-\beta H_0}]} = \frac{e^{-\beta H_0} \mathcal{P} \exp \left( i \int_{-i\beta}^{0} dt H_1^{(0)}(t) \right)}{\text{Tr}[e^{-\beta H_0} \mathcal{P} \exp \left( i \int_{-i\beta}^{0} dt H_1^{(0)}(t) \right)]} = \frac{e^{-\beta H \mid_V = 0}}{\text{Tr}[e^{-\beta H \mid_V = 0}]}
\]

(15)

as an equation to hold when evaluated inside traces with insertion of any number of local operators at fixed positions.

A local operator is, by definition, an operator depending on the position \(x\) (in the sense that its commutator with the momentum operator is a derivative with respect to \(x\)), such that its commutator with the Hamiltonian density at position \(y\) is zero for \(x \neq y\). Note that local charges, for instance conserved charges of the Hamiltonian, are integrals of local operators, and are not local operators themselves. Hence, the limit (15) does not hold with insertion of local charges. This makes physical sense, since conserved charges are not expected to relax to
their equilibrium values. Technically, one must remember that the density matrix is an operator with infinitely many matrix elements, hence any limit applied to it cannot be expected to converge to an object having the same properties (or to converge at all) independently from which subset of matrix elements we are looking at.

This derivation is important for what follows, so we present it in some detail. Recall that \( S(t_1, t_2) \), Eq. (7), is the evolution operator at zero voltage. We now establish some useful identities. In the interaction picture with respect to \( H_0 \), we have,

\[
S(0, t_0)e^{-iH_0t_0}e^{-\beta H_0} = \mathcal{P} \exp \left( i \int_0^{t_0} dt H_1^{(0)}(t) \right) e^{-\beta H_0} = e^{-\beta H_0} \mathcal{P} \exp \left( i \int_{-i\beta}^{t_0-i\beta} dt H_1^{(0)}(t) \right)
\]

(16)

where in the interaction picture

\[
H_1^{(0)}(t) = e^{iH_0t}e^{-iH_0t} = \lambda_d \vec{J}_d(t) \cdot \vec{S} = \lambda_y \vec{J}_y(t) \cdot \vec{S}.
\]

In (15) and in the last two expressions of (16), the symbol \( \mathcal{P} \) indicates path-ordering in time: the operators are positioned from left to right with their time argument going from the lower integral limit to the upper integral limit. In the first occurrence in (16), integrals are ordered from 0 on the left to \( t_0 \) on the right. In the second, the integration contour is from \(-i\beta\) on the left to \( t_0 - i\beta \) on the right. On the other hand, we have,

\[
e^{-iH_0t_0}S(t_0, 0) = \mathcal{P} \exp \left(i \int_{t_0}^{0} dt H_1^{(0)}(t) \right).
\]

(17)

The Keldysh evolution is then (multiplying (17) with (16) and dividing by the trace of this product),

\[
\frac{S(0, t_0)e^{-\beta H_0}S(t_0, 0)}{\text{Tr} (e^{-\beta H_0})} = \frac{e^{-\beta H_0} \mathcal{P} \exp \left(i \int_{-i\beta}^{t_0-i\beta} dt H_1^{(0)}(t) \right) \mathcal{P} \exp \left(i \int_{t_0}^{0} dt H_1^{(0)}(t) \right)}{\text{Tr} \left( e^{-\beta H_0} \mathcal{P} \exp \left(i \int_{-i\beta}^{t_0-i\beta} dt H_1^{(0)}(t) \right) \mathcal{P} \exp \left(i \int_{t_0}^{0} dt H_1^{(0)}(t) \right) \right)}
\]

(18)

The last equality is valid perturbatively if the integration from \( t_0 - i\beta \) to \( t_0 \) is negligible at every order in perturbation theory.

Note that the last equality involves taking \(|t_0|\) much greater than \( \beta \). At zero temperature, when \( \beta \to \infty \), this condition cannot hold, and since the correlation functions then may have algebraic decay with power \(-1\) at large distances, our proofs below (at equilibrium and in the steady state) do not apply. Nevertheless, as will be seen, our two-loop perturbative results for the non-equilibrium current have finite zero-temperature limit; this will be discussed further in the last section.

To show the last equality in (18) we evaluate the expectation value of a local operator (or product of any local operators at fixed positions) \( \mathcal{O} \), inserted at the right-hand side of the first equation of (18). Denoting by

\[
\langle \langle \cdot \cdot \cdot \rangle \rangle_0 = \frac{\text{Tr} (e^{-\beta H_0} \cdot \cdot \cdot )}{\text{Tr} (e^{-\beta H_0})}
\]

(19)

the averaging in the free theory at temperature \( \beta^{-1} \), we consider,

\[
\left\langle \left\langle \mathcal{P} \exp \left(i \int_{-i\beta}^{t_0-i\beta} dt H_1^{(0)}(t) \right) \mathcal{P} \exp \left(i \int_{t_0}^{0} dt H_1^{(0)}(t) \right) \mathcal{O} \right\rangle \right\rangle_0.
\]

(20)

All correlation functions involved are correlation functions where the \( H_1(t) \)’s are \textit{connected} to \( \mathcal{O} \). Connected correlation functions are defined, in the usual way, by subtracting from correlation function appropriate products of expectation values. In Appendix A we recall their precise definition and main properties. Only connected
correlation functions occur, because in (20) we divide by the correlation function of the operator where all the $H_I(t)$’s are involved.

In Appendix B we show that correlation functions of the type

$$\langle\langle H_I^{(0)}(t + t_1)H_I^{(0)}(t + t_2)\cdots H_I^{(0)}(t + t_n)\mathcal{O}\rangle\rangle_0$$

factorize, as $t \to \pm \infty$, into

$$\langle\langle H_I^{(0)}(t_1)H_I^{(0)}(t_2)\cdots H_I^{(0)}(t_n)\rangle\rangle_0 \langle\langle \mathcal{O}\rangle\rangle_0,$$

with sub-leading asymptotic contributions vanishing exponentially for finite $\beta$.

The last step of (18) follows from the factorization property. Due to this property, connected correlation functions of the type

$$\langle\langle H_I^{(0)}(t_1)H_I^{(0)}(t_2)\cdots H_I^{(0)}(t_n)\mathcal{O}\rangle\rangle_{0,\text{connected}}$$

vanish exponentially whenever any subset of consecutive time variables $\{t_i, t_{i+1}, \ldots, t_j\}$ (corresponding to a subset of time-ordered operators $H_I(t)$’s) goes to negative infinity simultaneously. This implies that order by order in perturbation theory of (20), all integrands vanish exponentially in any large-time region, in particular in the segment $t_0 - i\beta$, which then factorizes and cancels between numerator and denominator. Hence, in the limit where $t_0 \to -\infty$ the last step of (18) is exact order by order in perturbation theory, and we have (15), as claimed. See Figure 2.

We wish to note that our argument in Appendix B relied on the fact that $H_I(t)$ is a local, right-moving operator, and that it couples to the external degree of freedom $\mathcal{S}$ (the impurity) in an $SU(2)$-invariant way. For electronic degrees of freedom large time means large distance, and at large distances, correlation functions of local fields factorize. Combined with $SU(2)$ invariance, this implies the factorization of correlation functions of $H_I(t)$’s at large $|t|$.

The implications of the well-defined limit $t_0/\beta \to -\infty$, and in particular of the factorization at large time separation of the correlation functions involved in the perturbative coefficients, are important. It was not necessary to invoke any external relaxation mechanism: the factorization signals a decoherence in time and suggests that $H_0$ represents a good thermal bath, and this bath by itself provides such a mechanism. As in Caldeira-Leggett models, $H_0$ can be seen as an infinity of free oscillators with an appropriate frequency distribution in order to represent a thermal bath. The loss of time-reversal symmetry associated with this relaxation mechanism occurs when taking the limit $|t_0|/\beta \to \infty$.

The same derivation can be carried out for more general unitary conformal field theories perturbed by an interaction $H_I$ at one point, or defined on a finite region of space. Inferring from our derivation, the interaction can be due to an external degree of freedom coupled to any linear combination of fields that factorize into right- and left-movers, and the coupling has to be invariant with respect to a symmetry group acting on the full configuration space.

**The steady state current.** The derivation fails when out of equilibrium, $V \neq 0$. The step that becomes incorrect, if we start with expression (5) for the steady-state average, is the shifting of the integration limits $t_0 - i\beta \to t_0$. Indeed, the correlation functions involving $V H_I^{(0)}(t) = V H_z$ are not suppressed at large negative times since $H_z$ is a conserved charge of the Hamiltonian $H_0$. Moreover, due to quantum fluctuations of the charge $H_z$, made possible by the interaction $H_I$ (that is, $[H_I, H_z] \neq 0$), connected correlation functions involving $H_z$ are not zero. These two conditions are at the origin of the appearance of a non-equilibrium situation. In physical terms, the first condition is that the bath represented by $H_0$ does not provide a relaxation mechanism for reaching Boltzmann’s distribution of states associated to the energy $H_0 + V H_z + H_I$; the second condition is that nevertheless, $H_z$ is subject to quantum fluctuations and evolves with time. In this case then the Keldysh formulation does not reduce to an equilibrium description.

Our analysis, however, has not yet established that a steady state occurs. We shall present below a full proof to this effect. To motivate the proof we begin with a physical argument, based on the first formulation described around Eqs. (3) and (5), by considering the respective ground states of $H_0$ and $H$ (instead of the associated
FIG. 2: (Color online) Deriving the equilibrium formulation starting from real time (Keldysh) formulation. (A) The Keldysh integration contour on the complex time plane; the directions of the arrows indicate the ordering of operators from left to right. (B) Adding the segment \((t_0, t_0 + \beta)\) is allowed by the factorization property (this step is not allowed when the system is out of equilibrium). (C) The equilibrium contour.

thermal density matrices) and showing that they are “far” enough and that the evolution of \(H_z\) is “slow” enough in the limit \(L \to \infty\) so that a steady state is established. Under other circumstances, we might expect some oscillating behavior.

That the ground state of \(H\) is far enough, and the evolution of \(H_z\) is slow enough, can be made more precise in the following way. To begin with, consider the ground state \(|0\rangle\) of \(H_0\) and the ground state \(|V\rangle\) of the Hamiltonian \(H_0 + VH_z\). Later we shall consider the effect of the couplings \(\lambda\) and \(\lambda_d\). The ground state \(|V\rangle\) can be obtained in the following way. Consider the operator

\[
U_V = e^{i \int_{-\infty}^{\infty} dx \, x J_x(x)}
\]  

(21)
This is a unitary operator, and its effect on $U_{-V} e^{-\beta H_0} U_V = e^{-i \int dx J_z(x)} e^{-\beta H_0} e^{i \int dx J_z(x)} = e^{-\beta H_0} e^{-i \int dx J_z(x-i\beta)} e^{i \int dx J_z(x)} = e^{-\beta H_0} e^{-\beta VH_z} e^{i \int dx J_z(x)} = e^{-\beta H_0} e^{-\beta VH_z} e^{i \int dx dx_z (x_1-i\beta) dx_z [J_z(x_1), J_z(x_2)]}.

The last exponential factor is a real number, scaling with the system size $L$ (it could be absorbed into the definition of $U_{-V}$, at the price of losing its unitarity). Hence, the operator $U_{-V}$ takes $H_0$ to $H_0 + VH_z$, up to an additive number, and the ground state of $H_0 + VH_z$ can be obtained by

$$|V\rangle = U_{-V} |0\rangle .$$

Computing the expectation values of $H_z$ (tracing over the two-dimensional impurity space) in these ground states we have, as $L \to \infty$,

$$\langle 0| H_z |0\rangle = 0 , \quad \langle V| H_z |V\rangle \sim -VL .$$

In particular, $U_{-V}$ has the effect, in the infinite-$L$ limit, of changing the asymptotic conditions to $J_z(\pm \infty) = -\pi V/2$. We discussed the ground state of $H_0 + VH_z$. However, the actual ground state of $H$ yields corrections to the expectation values that are of higher order in the couplings with a finite limit as $L \to \infty$, and our conclusion therefore apply to the full Hamiltonian.

While the expectation values are infinitely distant in an infinite system, the rate of change of $\langle H_z \rangle$ as the interaction is switched on is finite since the operator $H_I$ giving rise to the current is local. This will be seen explicitly in the perturbative calculations of the current $\mathcal{J}$ below (recall that $\mathcal{J} = -i[H, H_z]$).

Hence, as $L \to \infty$, it would take more and more time to get from $|0\rangle$ to $|V\rangle$. Here we assume that the average of $H_z$ would decrease monotonically. This is expected for $L$ large enough and elapsed time large enough, though not infinite. More precisely in the region $[a, b]$, we expect the expectation value of $H_z$ to decrease steadily; this is the steady state. In other words, we expect a steady state to occur because $H_z$ scales with the length of the system, whereas its variation does not. For a finite $L$, it does not decrease monotonically at all times, and we might eventually see an oscillating behavior of period characterized by $L$.

We proceed now to the main result of this section: we show that to all orders in perturbation theory the limit of very large negative times, $t_0/\beta \to -\infty$ in (24), is well defined for any local operator, $\mathcal{O}$, supported at a point or on a finite interval. This shows that there is indeed a steady state: the current operator $\mathcal{J}$ acquires a well-defined expectation value.

Using the interaction picture with respect to $H_0 + VH_z$, we can write the steady-state average of any operator $\mathcal{O}$ as

$$\langle \mathcal{O} \rangle_{ss} = \lim_{t_0/\beta \to -\infty} \frac{1}{\text{Tr}[e^{-\beta H_0}]} \text{Tr} \left[ \mathcal{P} \exp \left( i \int_{t_0}^0 dt H_I^{(V)}(t) \right) e^{-\beta H_0} \mathcal{P} \exp \left( i \int_{t_0}^0 dt H_I^{(V)}(t) \right) \mathcal{O} \right]$$

with

$$H_I^{(V)}(t) = e^{i(H_0 + VH_z)t} H_I e^{-i(H_0 + VH_z)t} = e^{iV H_z t} H_I^{(0)}(t) e^{-iV H_z t} .$$

The operator $H_I^{(V)}(t)$ can be expressed in terms of “deformed” current-algebra operators. Consider

$$J_d^{(V)}(x) = e^{-iV H_z x} J_d(x) e^{iV H_z x}$$
$$J_z^{(V)}(x) = e^{-iV H_z x} J_z(x) e^{iV H_z x}$$
$$J_y^{(V)}(x) = e^{-iV H_z x} J_y(x) e^{iV H_z x} .$$
It is a simple matter to use the commutation relations (10) in order to obtain

\[ J_d^{(V)}(x) = \tilde{J}_d(x) \]
\[ J_x^{(V)}(x) = \cos(Vx)\tilde{J}_x(x) + \sin(Vx)\tilde{J}_y(x) \]
\[ J_y^{(V)}(x) = -\sin(Vx)\tilde{J}_x(x) + \cos(Vx)\tilde{J}_y(x) \].

(27)

Using these operators, we have

\[ H_I^{(V)}(t) = \lambda_d\tilde{J}_d^{(V)}(-t) \cdot \vec{S} + \lambda\tilde{J}_y^{(V)}(-t) \cdot \vec{S} \].

(28)

The proof that the limit \( t_0/\beta \to -\infty \) exists in (3) proceeds from arguments similar to those in the previous subsection. Let the operator \( O \) be supported on a finite interval. Moving the operator \( e^{-\beta H_0} \) to the left inside the trace in (25), we have

\[ \langle O \rangle_{ss} = \lim_{t_0/\beta \to -\infty} \frac{\text{Tr} \left[ e^{-\beta H_0} \mathcal{P} \exp \left( i \int_{-i\beta}^{t_0-i\beta} dt H_I^{(V)}(t) \right) \mathcal{P} \exp \left( i \int_{t_0}^{0} dt H_I^{(V)}(t) \right) O \right]}{\text{Tr} \left[ e^{-\beta H_0} \mathcal{P} \exp \left( i \int_{-i\beta}^{t_0-i\beta} dt H_I^{(V)}(t) \right) \mathcal{P} \exp \left( i \int_{t_0}^{0} dt H_I^{(V)}(t) \right) \right]} \]

(29)

where

\[ H_I^{(V)}(t) = e^{\beta VH_x}H_I^{(V)}(t)e^{\beta VH_x} \]
\[ = \lambda_d\tilde{J}_d(-t) \cdot \vec{S} + \lambda\sin(V(t + i\beta))\tilde{J}_x(-t) \cdot \vec{S} + \lambda\cos(V(t + i\beta))\tilde{J}_y(-t) \cdot \vec{S} \].

The exact form of \( \tilde{H}_I \) is actually not important; note only that it is a linear combination of local operators evolved in interaction-picture time. Again using the fact that only connected correlation functions (where \( H_I^{(V)}(t) \) and \( \tilde{H}_I^{(V)}(t) \) are connected to \( O \)) occur order by order in perturbation theory, and the fact that correlation functions involving \( H_I^{(V)}(t) \) and \( \tilde{H}_I^{(V)}(t) \) factorize at large times \( t \), one can see that all integrals are convergent in the limit \( t_0/\beta \to -\infty \) order by order in perturbation theory.

Physically, this means that the bath represented by \( H_0 \) provides the same mechanism for the steady state to occur as the mechanism it provides for the system to reach equilibrium in the case \( V = 0 \).

We now cast the expression for the steady state averages in another suggestive form and derive the alternative formulation, expressed in (3), with the steady state obtained by coupling the dot (i.e. turning on the couplings \( \lambda, \lambda_d \)) to leads initially equilibrated at temperature \( T \) and at potential difference \(-V\).

Observe that the operators with superscript \( (V) \) form the same current algebra, Eq. (10), as those without superscript since they are obtained by the unitary transformation \( U_V \) (21):

\[ J_d^{(V)}(x) = U_V\tilde{J}_d(x)U_V \]
\[ J_x^{(V)}(x) = U_V\tilde{J}_x(x)U_V \]
\[ J_y^{(V)}(x) = U_V\tilde{J}_y(x)U_V \]
\[ J_z^{(V)}(x) = U_V\tilde{J}_z(x)U_V = J_z(x) + \frac{V}{2} \].

(30)

Hence, the steady-state average of an operator \( O \) can be written

\[ \langle O \rangle_{ss} = \lim_{t_0/\beta \to -\infty} \frac{\text{Tr} \left[ \mathcal{P} \exp \left( i \int_{0}^{t_0} dt H_I^{(0)}(t) \right) U_V e^{-\beta H_0} U_V \mathcal{P} \exp \left( i \int_{0}^{t_0} dt H_I^{(0)}(t) \right) U_V O U_V \right]}{\text{Tr} \left[ e^{-\beta H_0} \right]} \]

(31)

where we recall that \( H_I^{(0)}(t) \) is the operator evolved with \( H_0 \) only. Recalling the transformation of \( H_0 \) under \( U_V \) (22), we find

\[ \langle O \rangle_{ss} = \lim_{t_0/\beta \to -\infty} \frac{\text{Tr} \left[ \tilde{S}(0, t_0) e^{-\beta(H_0-VH_x)} \tilde{S}(t_0, 0) O^{(-V)} \right]}{\text{Tr} \left[ e^{-\beta(H_0-VH_x)} \right]} \]

(32)
where

$$O^{(-V)} = U_V O U_{-V}.$$  

(33)

In (32), it was necessary to include the factor $e^{\beta V H_z}$ inside the trace in the denominator of the right-hand side. For a system on a finite interval, the inclusion of this factor has the effect of cancelling the constant term that appears in (32). Then, the limit of infinite interval is well defined.

Since for the current operator (12) we have $J^{(-V)} = J$, this shows the equivalence, for the steady state current, between the formulation (32) and the formulation (33). In general, we will denote the steady state average in the latter formulation by

$$\langle O \rangle_{ss'} = \lim_{t_0/\beta \to -\infty} \lim_{t_0'/\beta \to -\infty} \frac{\text{Tr} \left[ S(0, t_0') e^{-\beta (H_0 - V H_z)} S(t_0, 0) O \right]}{\text{Tr} \left[ e^{-\beta (H_0 - V H_z)} S(t_0, 0) \right]},$$

(34)

That is,

$$\langle O \rangle_{ss'} = \langle U_{-V} O U_V \rangle_{ss}.$$  

(35)

Below, we carry out some formal manipulations which are justified only if we can establish a more stringent convergence property as that used above. We need to establish that the following expression:

$$\langle O \rangle_{ss'} = \lim_{t_0/\beta \to -\infty} \lim_{t_0'/\beta \to -\infty} \frac{\text{Tr} \left[ S(0, t_0') e^{-\beta (H_0 - V H_z)} S(t_0, 0) O^{(-V)} \right]}{\text{Tr} \left[ e^{-\beta (H_0 - V H_z)} S(t_0, 0) \right]},$$

(36)

with the limits on $t_0$ and on $t_0'$ taken independently, will yield a result independent of the order the limits were taken. Note that we have included factors $S(t_0, 0)$ and $S(0, t_0')$ in the denominator. They assure convergence and cancel by cyclicity of the trace if the limit exists. To prove convergence in (36), we consider

$$\langle O \rangle_{ss} = \lim_{t_0/\beta \to -\infty} \lim_{t_0'/\beta \to -\infty} \frac{\text{Tr} \left[ P \exp \left( i \int_{t_0'}^{t_0} dt H_I^{(V)}(t) \right) e^{-\beta H_0} P \exp \left( i \int_{t_0}^{0} dt H_I^{(V)}(t) \right) O \right]}{\text{Tr} \left[ P \exp \left( i \int_{t_0'}^{t_0} dt H_I^{(V)}(t) \right) e^{-\beta H_0} P \exp \left( i \int_{t_0}^{0} dt H_I^{(V)}(t) \right) \right]}.$$  

(37)

Indeed, the same arguments we used to establish the connectedness and factorization allow us to take, for instance, first the limit with $|t_0'|$ large, then the limit with $|t_0|$ large, or vice versa. The result is unique. Using the operator $U_V$ in a manner similar to the one above, it is a simple matter to obtain (36) from (37) and the result then is the same as the one obtained from the formulation (34).

Alternative description of the Steady State. What replaces the density matrix $e^{-\beta H}$ description of equilibrium? We could translate the proof establishing equilibrium when $(V = 0)$ to the proof establishing steady state when $(V \neq 0)$ by means of the current algebra of symmetries. By similar means we shall show that a new operator will play for the system in its steady state a similar role to the one played by the density matrix in equilibrium. Such a steady-state density matrix can be obtained from simple manipulations, now that we have established the convergence of the integrals.

Consider the formulation (36) of the steady-state problem, with $O$ in (36) an operator supported on a finite interval in the theory $H_0$. Bringing $e^{-\beta H_0}$ completely to the left, the right-hand side of (36) can be written as follows:

$$\lim_{t_0/\beta \to -\infty} \lim_{t_0'/\beta \to -\infty} \frac{\text{Tr} \left[ e^{-\beta H_0} P \exp \left( i \int_{t_0'}^{t_0} dt H_I^{(0)}(t) \right) e^{\beta V H_z} P \exp \left( i \int_{t_0}^{0} dt H_I^{(0)}(t) \right) O^{(-V)} \right]}{\text{Tr} \left[ e^{-\beta H_0} P \exp \left( i \int_{t_0'}^{t_0} dt H_I^{(0)}(t) \right) e^{\beta V H_z} P \exp \left( i \int_{t_0}^{0} dt H_I^{(0)}(t) \right) \right]}.$$  

Since we showed that the limits can be taken independently, we can shift $t_0' - \beta$ to $t_0'$ both in the numerator and in the denominator, without shifting $t_0$, with vanishing error in the limit. We can then take $t_0' = t_0$ and keep only one limit symbol. Inserting

$$1 = P \exp \left( i \int_{t_0}^{0} dt H_I^{(0)}(t) \right) P \exp \left( i \int_{0}^{t_0} dt H_I^{(0)}(t) \right)$$
just before the operator $e^{\beta VH_z}$ we have,

$$
\lim_{t_0/\beta \to -\infty} \frac{\text{Tr} \left[ e^{-\beta H|V=0} \mathcal{P} \exp \left( i \int_{t_0}^{t_0} dt \, H_i^{(0)}(t) \right) e^{\beta VH_z} \mathcal{P} \exp \left( i \int_{t_0}^{0} dt \, H_i^{(0)}(t) \right) \mathcal{O}(-V) \right]}{\text{Tr} \left[ e^{-\beta H|V=0} \mathcal{P} \exp \left( i \int_{t_0}^{t_0} dt \, H_i^{(0)}(t) \right) e^{\beta VH_z} \mathcal{P} \exp \left( i \int_{t_0}^{0} dt \, H_i^{(0)}(t) \right) \right]} .
$$

Defining the operator:

$$
Y = \lim_{t_0/\beta \to -\infty} \bar{S}(0,t_0)H_z\bar{S}(t_0,0)
$$

allows us to write the steady-state average of a local operator as

$$
\langle \mathcal{O} \rangle_{ss} = \frac{\text{Tr} \left[ e^{-\beta H|V=0} e^{\beta VY} \mathcal{O} \right]}{\text{Tr} \left[ e^{-\beta H|V=0} e^{\beta VY} \right]} .
$$

Note that the limit (38) cannot be expected to exist as an operator (recall that we are dealing with operators with infinitely many matrix elements), but only when inserted into appropriate traces (or only when appropriate matrix elements are considered). More precisely, we have only proven that (38) is a well-defined operator when it is evaluated in expressions like (39), and that the result is the steady-state average of the local operators inserted. This is a statement solely about a small part of the matrix elements of the operator (38). The meaning of Eq. (39) is that one must first evaluate the traces and their ratio with the expression (38) at finite $t_0/\beta$, then take the limit indicated in (38) on the result. The properties of $Y$ as an operator acting in a Hilbert space will be discussed elsewhere.

Observe, however, that in all situations where the operator $Y$ is well defined (that is, when we consider the appropriate matrix elements), then it is a conserved charge. Indeed, when it is well defined, then the limit $t_0/\beta \to -\infty$ of $\bar{S}(0,t_0)H_zS(t_0,0)$ (or of any function of this operator) must exist. Since $S(t_1, t_2) = S(t_1 + dt, t_2 + dt)$, we have $\frac{d}{dt} \lim_{t_0/\beta \to -\infty} S(t, t_0)H_zS(t_0, t) = 0$, hence $[H|V=0, Y] = 0$. Then we can finally write

$$
\langle \mathcal{O} \rangle_{ss} = \frac{\text{Tr} \left[ e^{-\beta (H|V=0 - VY)} \mathcal{O} \right]}{\text{Tr} \left[ e^{-\beta (H|V=0 - VY)} \right]} .
$$

That is, averages in the steady-state can be obtained by tracing with an appropriate density matrix.20

What difference is there between the equilibrium and the steady state? Consider a quantum mechanical system described by a Hamiltonian $H$. Put the system at equilibrium with a bath where there can be exchange of heat and of any quantity $Q$ that is conserved by the dynamics $H$. The average of observables is then described by the density matrix $e^{-\beta (H + \mu Q)}$ where $\mu$ is the chemical potential associated to $Q$: the energy brought to the system by increasing $Q$ by one unit. In expression (40), the average of a local operator $\mathcal{O}$ in the steady state is a trace with a density matrix of exactly the same form. The main difference is that the operator $Y$ is a non-local conserved charge. A local conserved charge can be written as an integral over space of a local operator of the theory $H|V=0$ plus a local operator at the impurity site, with possible non-trivial impurity-space components. The operator $Y$ (38) cannot be written in that way. To see this, we can write it as follows:

$$
Y = H_z + \int_{-\infty}^{0} dt \, \mathcal{J}(t)
$$

where

$$
\mathcal{J}(t) = \bar{S}(0,t)J\bar{S}(t,0)
$$

is the time-evolved current $\mathcal{J}$ with respect to the theory $H|V=0$. Then it is simple to write it as an integral of a charge density:

$$
Y = \int_{-\infty}^{\infty} dx \, j_{tot}(x,0)
$$

with

$$
j_{tot}(x,t) = \frac{1}{\pi} J_z(x,t) + \delta(x) \int_{-\infty}^{t} dt' \mathcal{J}(t') .
$$
The charge density has a local bulk part, but the term at the impurity is not a local field of the theory $H_{V=0}$: it is the time integral of the current, and the current is not the time derivative of a local field.

The non-locality of $Y$ is the main difference between the description of a steady state and of an equilibrium state. In the formulation (10), only a restricted set of operators $O$ have well-defined average: those that have stationary expectation values. All local operators are of this type, but, for instance, it is simple to see that the operator $H_z$ does not have a well-defined steady-state value.

Note also that the operator $Y$ gives in principle a description of the asymptotic state that one can use in order to describe the steady state: quantities in the steady state can be evaluated as averages in an appropriate asymptotic state. Further analysis in this direction will be presented in our future works.

We wish to remark that sometime ago Hershfield has considered steady-state flow and has argued that under some assumptions concerning the relaxation of correlation function an expression (40) would govern the steady-state current. He gave then implicit equations to determine $Y$. It appears to us that our explicit expressions for the operator $Y$ satisfies his implicit equations, and should hence correspond to the same operator (although we have not thoroughly ascertained the confluence of the two approaches). We must stress, however, that no assumptions were made in our derivation.

IV. RG-IMPROVED REAL-TIME PERTURBATION THEORY

The perturbative expansion. We now turn to real-time perturbation theory for the current (8). We take the formulation where the system is initially brought to equilibrium with a non-zero bias voltage, then disconnected from the external bath before the voltage is turned off and the interaction is turned on. It will be convenient to consider adiabatically turning on the interaction in the infinite past: we introduce a large-time exponential cutoff, $e^{\eta t} H_I$, with $\eta$ a positive scale with dimension of energy, and take the limit $t_0/\beta \to -\infty$ in (8). The quantity $\eta \beta$ will be sent to 0 at the end of the calculations. This means, physically, that the two leads are slowly brought towards the dot after the voltage has been turned off. Our proof that there are no divergencies as $t_0/\beta \to -\infty$ in the previous section shows that there are no divergencies as $\eta \beta \to 0$. The current can then be written

$$\langle J \rangle_{ss} = \lim_{\eta \beta \to 0^+} \sum_{k=0}^\infty \int_{-\infty}^0 dt_1 e^{\eta t_1} \int_{t_1}^0 dt_2 e^{\eta t_2} \cdots \int_{t_{k-1}}^0 dt_k e^{\eta t_k} \langle [H_I(t_1), [H_I(t_2), \cdots, [H_I(t_k), J] \cdots]] \rangle_V \nonumber$$

with

$$S_\eta(t_1, t_2) = \mathcal{P} \exp \int_{t_1}^{t_2} i (H_0 + e^{\eta t} H_I) \, dt .$$

More precisely,

$$\langle J \rangle_{ss} = \lim_{\eta \beta \to 0^+} \sum_{k=0}^\infty \int_{-\infty}^0 dt_1 e^{\eta t_1} \int_{t_1}^0 dt_2 e^{\eta t_2} \cdots \int_{t_{k-1}}^0 dt_k e^{\eta t_k} \langle [H_I(t_1), [H_I(t_2), \cdots, [H_I(t_k), J] \cdots]] \rangle_V \nonumber$$

where

$$\langle \langle \cdots \rangle \rangle_V = \frac{\Tr \left[ e^{-\beta(H_0-VH_z)} \cdots \right]}{\Tr \left[ e^{-\beta(H_0-VH_z)} \right]} .$$

The integrals in this expansion are plagued with ultraviolet divergencies which we have to regularize. This can be done in several ways. For our purposes, it will be convenient to modify the operators $\vec{J}_d(x)$, $\vec{J}_x(x)$, $\vec{J}_y(x)$ in order to render their correlation functions regular at coinciding points. More precisely, we choose the regularization scheme where all operators (in the Hamiltonian and in correlation functions) at the impurity site are regularized, whereas all operators away from it are unaffected. Since the interaction is only at the impurity site, this is
and the choice is a matter of convenience. We will choose a Gaussian regularization, 

\[ \Lambda \]

The parameter \( \Lambda \) plays the role of an effective bandwidth (we do not denote it in order to preserve hermiticity of the regularized operators. The function \( \Lambda \) can be chosen in many ways, and the choice is a matter of convenience. We will choose a Gaussian regularization,

\[ R_\Lambda(p) = e^{-p^2/(2\Lambda^2)}. \] (50)

The parameter \( \Lambda \) plays the role of an effective band width (we do not denote it \( D \) since it is certainly not exactly the band width). The universal part of the limit \( \Lambda \gg V, T \) is the same as that of the limit of large band width.

The time integrals in (47) can now be traded to momentum integrals:

\[ \langle J \rangle_{ss} = \sum_{k=0}^{\infty} (-1)^k \int \frac{dp_1 R_\Lambda(p_1)}{p_1 + i\eta} \int \frac{dp_2 R_\Lambda(p_2)}{p_1 + p_2 + 2i\eta} \cdots \int \frac{dp_k R_\Lambda(p_k)}{p_1 + p_2 + \cdots + p_k + k\eta} \int ds R_\Lambda(s) \left( \langle [\hat{H}_1(p_1), [\hat{H}_1(p_2), \cdots, [\hat{H}_1(p_k), J_s(s)] \cdots] \rangle \right)_V \] (51)

where

\[ \hat{H}_1(p) = \lambda_d \tilde{J}_d(p) \cdot \tilde{S} + 2 \tilde{J}_b(p) \cdot \tilde{S} \] (52)

and

\[ \tilde{J}(p) = J_s(p) \cdot \tilde{S}. \] (53)

The parameter \( \eta \) can be set to zero with the requirement that the momentum integrals be taken on a line parallel to the real axis in the \( p \)-plane with a slight positive imaginary part:

\[ \langle J \rangle_{ss} = \sum_{k=0}^{\infty} (-1)^k \int \frac{dp_1 R_\Lambda(p_1)}{p_1} \int \frac{dp_2 R_\Lambda(p_2)}{p_1 + p_2} \cdots \int \frac{dp_k R_\Lambda(p_k)}{p_1 + p_2 + \cdots + p_k} \int dq R_\Lambda(q) \left( \langle [\hat{H}_1(p_1), [\hat{H}_1(p_2), \cdots, [\hat{H}_1(p_k), \tilde{J}(q)] \cdots] \rangle \right)_V. \] (54)
Note that this can be done since $R_{\Lambda}(p)$ does not have a singularity at $p = 0$ or anywhere on the real $p$-line. No singularity at $p_1 + \cdots + p_j = 0$ (for $j = 1, \ldots, k$) can occur in the averages $\langle \langle \cdots \rangle \rangle_V$ in the expression above, since we know that there are no divergencies as $\eta \beta \to 0$.

In fact, for explicit calculations, it will be more convenient to integrate over the real line in momentum space, and to use the formalism of principal value integrals. For this purpose, recall that

$$\int_+ dp \frac{f(p)}{(p + q)^n} = \int dp \left( \frac{(-1)^n}{(n-1)!} i\pi \delta^{(n-1)}(p + q) + \frac{1}{(p + q)^n} \right) f(p)$$

where $P$ says that we must take the principal value of the integral:

$$\int dp \left( \frac{1}{(p + q)^n} \right) f(p) = \text{finite part of} \left( \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) dp \frac{f(p)}{(p + q)^n} \text{ in power expansion as } \epsilon \to 0 .$$

**Explicit one-loop calculations.** The trace over the impurity space of the operators

$$[\tilde{H}_1(p_1), [\tilde{H}_1(p_2), \cdots, [\tilde{H}_1(p_k), \tilde{J}(q)] \cdots]]$$

can be obtained by using the following general formula, valid for any vector operators $\tilde{A}$ and $\tilde{B}$ that commute with $\tilde{S}$:

$$[\tilde{A} \cdot \tilde{S}, \tilde{B} \cdot \tilde{S}] = i \frac{1}{2} \{ A', B' \} \epsilon_{ijk} S^k + \frac{1}{4} [A', B']$$

where $\{ \cdots \}$ is the anti-commutator. Using the commutation relations [29] for evaluating the commutators, one is left then only with multiple anti-commutators of the mode operators. To one loop we will need the following commutators:

$$C_1 = [\tilde{J}_y(q) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S}]$$
$$C_2 = [\tilde{J}_d(p) \cdot \tilde{S}, [\tilde{J}_y(q) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S}]]$$
$$C_3 = [\tilde{J}_y(p) \cdot \tilde{S}, [\tilde{J}_d(q) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S}]] .$$

Under trace over the impurity space, they give

$$\text{Tr}_\text{impurity}(C_1) = -3iJ_x(q + k)$$
$$\text{Tr}_\text{impurity}(C_2) = 2\{J_y^x(q), J_x^z(p + k)\} - 2\{J_y^z(p + q), J_x^z(k)\}$$
$$\text{Tr}_\text{impurity}(C_3) = -2\{J_y^z(p + q), J_x^z(k)\} .$$

The traces $\langle \langle \cdots \rangle \rangle_V$ over the bulk-CFT Hilbert space of these anti-commutators can be calculated without the need to construct the Hilbert space for $H_0 - VH_x$, but only by using the following exchange relations:

$$\tilde{J}_d(p) e^{-\beta(H_0 - VH_x)} = e^{-\beta(H_0 - VH_x)} \tilde{J}_d(p) e^{-\beta p}$$
$$\tilde{J}_+ (p) e^{-\beta(H_0 - VH_x)} = e^{-\beta(H_0 - VH_x)} \tilde{J}_+(p) e^{-\beta(p + V)}$$
$$\tilde{J}_- (p) e^{-\beta(H_0 - VH_x)} = e^{-\beta(H_0 - VH_x)} \tilde{J}_-(p) e^{-\beta(p - V)}$$

where we use the following linear combinations:

$$\tilde{J}_+ (p) = \frac{1}{2} \left( \tilde{J}_x(p) + i\tilde{J}_y(p) \right) , \quad \tilde{J}_- (p) = \frac{1}{2} \left( \tilde{J}_x(p) - i\tilde{J}_y(p) \right) .$$

The exchange relations imply for any operator $O$

$$\langle \langle O\tilde{J}_d(p) \rangle \rangle_V = e^{-\beta p} \langle \langle (\tilde{J}_d(p) O) \rangle \rangle_V$$
$$\langle \langle O\tilde{J}_+ (p) \rangle \rangle_V = e^{-\beta(p + V)} \langle \langle (\tilde{J}_+ (p) O) \rangle \rangle_V$$
$$\langle \langle O\tilde{J}_- (p) \rangle \rangle_V = e^{-\beta(p - V)} \langle \langle (\tilde{J}_- (p) O) \rangle \rangle_V .$$
Using the commutator formulas (49), using these relations and using the traces of single operators, all traces can be calculated. Traces of single operators can be calculated from the fact that \( \langle \langle \hat{J}_{d,x,y}(p) \rangle \rangle_V = \langle \langle J_z \rangle \rangle_V = 0 \), and from \( \langle \langle O \rangle \rangle_V = \langle \langle U_{-V} \mathcal{O} U_V \rangle \rangle_V \). In terms of modes, the operator \( U_{-V} \) has the representation

\[
U_{-V} = e^{2VJ_z(0)}
\]

where we formally define \( J_z(p) = dJ_z(p)/dp \). The traces of single operators are then given by

\[
\langle \langle \hat{J}_{d,x,y}(p) \rangle \rangle_V = 0, \quad \langle \langle J_z(p) \rangle \rangle_V = V \delta(p).
\]

The trace of the only type of anti-commutator appearing in (59) is then given by

\[
\langle \langle \{J_y(p), J_z(q)\} \rangle \rangle_V = -\frac{1}{8} F(p) \delta(p + q) \delta_{ij}
\]

where

\[
F(p) = (p + V) \frac{1 + e^{-\beta(p+V)}}{1 - e^{-\beta(p+V)}} - (p - V) \frac{1 + e^{-\beta(p-V)}}{1 - e^{-\beta(p-V)}}.
\]

From (54) and using (55), the integrals to be calculated at one loop are

\[
I_1 = - \int dq R_\Lambda(q) \left(-i \pi \delta(q) + \frac{P_1}{q} \right) \int dk R_\Lambda(k) \langle \langle C_1 \rangle \rangle_V
\]

\[
I_{2,3} = \int dp R_\Lambda(p) \left(-i \pi \delta(p) + \frac{P_1}{p} \right) \int dq R_\Lambda(q) \left(-i \pi \delta(p + q) + \frac{P_1}{p + q} \right) \times
\]

\[
\int dk R_\Lambda(k) \langle \langle C_{2,3} \rangle \rangle_V.
\]

The current will then be given by

\[
\langle \langle J \rangle \rangle_{ss} = \lambda^2 \left(I_1 + \lambda_d I_2 + I_3 + O(\lambda_d^2, \lambda^2) \right)
\]

Since all integrals are real, it is clear that in the expressions (67), only terms with an odd number of delta functions in the momentum variables give non-zero contributions. It is a simple matter then to obtain

\[
I_1 = \frac{3\pi V}{2}
\]

\[
I_2 = I_3 = \frac{3\pi V}{4} \int dp \frac{P_1}{p} f(p) R_\Lambda(Vp)^2
\]

where \( p \) is now a dimensionless momentum variable, and

\[
f(p) = (p + 1) \frac{1 + e^{-w(p+1)}}{1 - e^{-w(p+1)}} - (p - 1) \frac{1 + e^{-w(p-1)}}{1 - e^{-w(p-1)}}
\]

and

\[
w = \beta V.
\]

Using the symmetry \( f(-p) = -f(p) \), the integral \( I_2 \) can be calculated in the following way, keeping only the divergent and finite parts as \( \Lambda \to \infty\):

\[
I_2 = I_3 = \frac{3\pi V}{2} \int_0^\infty \frac{dp}{p} f(p) e^{-p^2 V^2/\Lambda^2}
\]

\[
\sim \frac{3\pi V}{2} \left[ \int_0^\infty \frac{dp}{p} (f(p) - 2 + 2e^{-p^2}) - 2 \int_0^\infty \frac{dp}{p} (e^{-p^2} - 1) e^{-p^2 V^2/\Lambda^2} \right]
\]

\[
\sim 3\pi V \left[ P(w) + \ln \left( \frac{\Lambda}{V} \right) \right]
\]
where the symbol $\sim$ means that equality is valid only for the finite and divergent parts as $\Lambda \to \infty$ and where
\[
P(w) = \frac{1}{2} \int_0^\infty \frac{dp}{p} \left( f(p) - 2 + 2e^{-p^2} \right) = \int_0^\infty \frac{dp}{p} \left( \frac{p+1}{e^{w(p+1)} - 1} - \frac{p-1}{e^{w(p-1)} - 1} + e^{-p^2} \right). \tag{74}
\]
This is a well-defined function for all $w$ with positive real part, and is easy to evaluate numerically.

It will be useful to have the asymptotic behavior of the function $P(w)$, at large and small $w$. This is evaluated in Appendix D. The asymptotic expansion at large $w$ is given by
\[
P(w) - 1 + \frac{\gamma}{2} \sim -\frac{\pi^2}{3} w^{-2} \quad \text{as} \quad w \to \infty , \tag{75}
\]
whereas the expansion at small $w$ is
\[
P(w) = \ln(w) + \frac{\sqrt{\pi}}{2} + \gamma - \ln(2\pi) + O(w) \quad \text{as} \quad w \to 0 . \tag{76}
\]

**Summarizing**, the zero- and one-loop divergent and finite contributions to the current are (with $T = 1/\beta$)
\[
\langle J \rangle_{ss} = 3\pi V \lambda^2 \left[ \frac{1}{2} + 2\lambda_d \left( P \left( \frac{V}{T} \right) + \ln \left( \frac{\Lambda}{V} \right) \right) \right] \tag{77}
\]
with $P(w)$ given by the integrals in (74), $P(w) = \int_0^\infty \frac{dp}{p} \left( \frac{p+1}{e^{w(p+1)} - 1} - \frac{p-1}{e^{w(p-1)} - 1} + e^{-p^2} \right)$. Note that only the combination $P(V/T) + \ln(\Lambda/V)$ appears at one loop. This combination has the limits
\[
P \left( \frac{V}{T} \right) + \ln \left( \frac{\Lambda}{V} \right) \sim - \left( \ln \left( \frac{V}{\Lambda} \right) - 1 + \frac{\gamma}{2} \right) \quad \text{as} \quad T \ll V \ll \Lambda
\]
\[
\sim - \left( \ln \left( \frac{T}{\Lambda} \right) - \frac{\sqrt{\pi}}{2} - \gamma + \ln(2\pi) \right) \quad \text{as} \quad V \ll T \ll \Lambda . \tag{78}
\]
The one-loop calculation was also performed in [17] (although the analysis did not go in as much detail as ours), and it can be verified that their results agree with ours.

**Results for the two-loop calculations.** Two-loop integrals come from the commutators (here $p, q, r$ and $k$ are all momentum variables):
\[
C_4 = \{ \tilde{J}_d(p) \cdot \tilde{S}, \{ \tilde{J}_d(q) \cdot \tilde{S}, \{ \tilde{J}_d(r) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S} \} \}
\]
\[
C_5 = \{ \tilde{J}_d(p) \cdot \tilde{S}, \{ \tilde{J}_d(q) \cdot \tilde{S}, \{ \tilde{J}_d(r) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S} \} \}
\]
\[
C_6 = \{ \tilde{J}_d(p) \cdot \tilde{S}, \{ \tilde{J}_d(q) \cdot \tilde{S}, \{ \tilde{J}_d(r) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S} \} \}
\]
\[
C_7 = \{ \tilde{J}_d(p) \cdot \tilde{S}, \{ \tilde{J}_d(q) \cdot \tilde{S}, \{ \tilde{J}_d(r) \cdot \tilde{S}, \tilde{J}_x(k) \cdot \tilde{S} \} \}
\]
(79)
appearing inside the traces in the integrands of (54). Calculating the traces and using (55), we can find the corresponding set of two-loop integrals, $I_4, I_5, I_6$ and $I_7$ (written in Appendix E). These integrals enter the current as
\[
\langle J \rangle_{ss} = \lambda^2 (I_4 + \lambda_d (I_2 + I_3) + \lambda_d^2 (I_4 + I_5 + I_6) + \lambda^2 I_7 + O(\lambda_d^2, \lambda^2 \lambda_d)) . \tag{80}
\]

Explicit calculation of these integrals lead to the following divergent parts as $\Lambda \to \infty$:
\[
I_4 + I_5 + I_6 \sim 3\pi V \left[ 10P(w) + 5 \ln \left( \frac{\Lambda}{V} \right) - 1 \right] \ln \left( \frac{\Lambda}{V} \right) + \text{finite}
\]
\[
I_7 \sim 3\pi V \left[ 2P(w) + \ln \left( \frac{\Lambda}{V} \right) - 1 \right] \ln \left( \frac{\Lambda}{V} \right) + \text{finite} . \tag{81}
\]
The finite contributions are much more complicated, and are reported in Appendix F. Let us only notice that the limit $T/V \to 0$ of these contributions is finite, as in the one-loop results. This indicates that the voltage
$V$ plays the role of a good infrared cutoff for the perturbative calculation of the current to two loops: the temperature may be set to zero without divergences to this order.

**Summarizing**, the zero-, one- and two-loop finite and divergent contributions to the current are,

$$
\langle J \rangle_{ss} = 3\pi V \lambda^2 \left\{ \frac{1}{2} + 2\lambda_d \left[ P(w) + \ln \left( \frac{\Lambda}{V} \right) \right] + 
+ \lambda^2 \left[ 10P(w) + 5 \ln \left( \frac{\Lambda}{V} \right) - 1 \right] \ln \left( \frac{\Lambda}{V} \right) + \lambda^2 \left[ 2P(w) + \ln \left( \frac{\Lambda}{V} \right) - 1 \right] \ln \left( \frac{\Lambda}{V} \right) \right\} + 
+ \lambda^2 \lambda_d^2 [I_4 + I_5 + I_6]_{\text{finite}} + \lambda^6 [I_7]_{\text{finite}}
$$

(82)

where $[I_4 + I_5 + I_6]_{\text{finite}}$ and $[I_7]_{\text{finite}}$ are given in Appendix [F].

**The Renormalization Group equation (The Callan-Symanzik equation).** In systems at equilibrium, Wilson’s renormalization group ideas allow us to understand how physical quantities can have universal forms (independent of the precise form of the interactions at the microscopic level) when all physical energy scales (temperature, voltage, etc.) are much lower than the microscopic scales (band width, inverse lattice spacing, etc.). Out of equilibrium, it is not obvious that Wilson’s renormalization group ideas still apply.

A different, but equivalent, way to look at universality is to study the limit where the cut-off $\Lambda$ is increased and sent to infinity, while the couplings dependence on the cut-off is again governed by an RG equation, valid at very large cut-off. If such a limit exists then all quantities tend to their universal form.

We will argue that the steady-state average of the current operator (or of any operator having a well-defined average in the steady state) satisfies the Callan-Symanzik equation with the same beta function and anomalous dimension as they occur in any average evaluated at equilibrium. More precisely, we will argue that

$$
\left( \Lambda \frac{\partial}{\partial \Lambda} \right)_{\lambda, \lambda_d} + \beta_\lambda (\lambda, \lambda_d) \frac{\partial}{\partial \lambda} + \beta_{\lambda_d} (\lambda, \lambda_d) \frac{\partial}{\partial \lambda_d} \right) \langle J \rangle_{ss} \xrightarrow{\Lambda \to \infty} 0
$$

(83)

where $\beta_\lambda$ and $\beta_{\lambda_d}$ are the beta functions of the anisotropic two-channel Kondo model. Note that the anomalous dimension term does not occur: the current operator $J$ has zero anomalous dimension (this is natural from a physical perspective, as the current is a physical object which should not change with a change of scales; we will verify this explicitly to one loop, and indirectly to two loops, below).

We should note that the Callan-Symanzik equations with one-loop beta functions and zero anomalous dimension was written in [17] for the steady-state average of the current, from physical arguments [22]. Here we present a quantum field theoretic argument that applies to all orders (and all matrix elements), and in the next section we explicitly verify this argument and calculate the beta functions and anomalous dimension in a universal fashion (so that it automatically applies to the steady state) to one-loop order.

The Callan-Symanzik equation embodies Wilson’s renormalization group ideas: it tells us how a change of cutoff $\Lambda$ (for instance, the band width) can be compensated by a change of few relevant coupling constant, as long as all physical energy scales are much lower than $\Lambda$. Solving the Callan-Symanzik equations (this is done below for the current in the steady state) allows us to describe the low energy behavior of the steady-state current in terms of the ratios $V/T_K$ and $T/T_K$, where $T_K$ is an integration constant, as well as of one extra parameter (invariant under the RG flow) characterizing the asymmetry between the couplings $\lambda_d$ and $\lambda$; we will denote this parameter by $C$. The integration constant $T_K$ and the extra parameter $C$ characterize the quantum field theory; when they are fixed, all averages can be evaluated unambiguously. These parameters are not universal: different microscopic theories have low-energy behaviors described by different values for them. Up to these non-universal quantities, the quantum field theory description is universal, independent of the precise choice of the cutoff procedure (precise structure of the band, for instance) and of irrelevant couplings (interactions that give vanishing contributions at low energies). The integration constant $T_K$ is the Kondo temperature: the temperature above which the “Kondo cloud” gets destroyed by the thermal energy. It is related to the microscopic values of the couplings $\lambda$ and $\lambda_d$ (the values when $\Lambda$ is chosen to be of the order of the real band width), and it decreases if the couplings are decreased. At zero couplings, the Kondo temperature is zero and all scales of the low-energy physics disappear: this is a quantum critical point. The quantum field theory with finite ratios $V/T_K$ and $T/T_K$ describes the situation $T_K \ll \Lambda$: the couplings are sent to zero at the same time.
as the voltage and temperature are made much smaller than \( \Lambda \). This is the scaling limit, describing the region around the quantum critical point \( \lambda = \lambda_d = 0 \). In particular, one finds that the perturbative expansion is valid in the region \( T_K \ll V \) or \( T_K \ll T \).

We will see that in the steady state, it will be more convenient to introduce a scale \( \Delta \) characterizing both the effects of the thermal energy and of the electric potential driving the current on the Kondo cloud. We will then be able to compare these effects, using a comparison parameter that is exact in one-loop perturbation theory.

It is natural that the Callan-Symanzik equation is still valid in the steady state, since the steady state can be understood as an appropriate asymptotic state, characterized by a scale \( \Delta \), and since averages of operators in asymptotic states satisfy the Callan-Symanzik equation. In particular, \( V \) should flow trivially with the renormalization group. However, it is instructive to see explicitly how this works in real-time perturbation theory.

We will argue that Eq. (54) holds simply from the fact that the Callan-Symanzik equation is satisfied for any average in equilibrium. The main observation is the following. Consider the Hilbert space \( \mathcal{H}_{-V} \) associated to the Hamiltonian \( H_0 - V H_z \). It is not formed of vectors that are in the Hilbert space \( \mathcal{H}_{0} \) associated to the Hamiltonian \( H_0 \). In particular, its ground state \(| - V \rangle \) can be formally defined as

\[
\text{U}_{V}|0\rangle,
\]

where \(|0\rangle\) is the ground state of \( H_0 \). This definition indeed makes sense for any finite length of the system, and in fact allows to calculate matrix elements of any operators also at infinite length, but it does not give a vector in \( \mathcal{H}_{0} \) at infinite length. Nevertheless, the mode operators associated to current algebra operators still have a well-defined action on \( \mathcal{H}_{-V} \). In order to see this, it is convenient to construct the Hilbert space \( \mathcal{H}_{-V} \) by defining a vacuum state \(|0\rangle \) satisfying \( \tilde{J}_{+,-,d}(p)|0\rangle = 0 \) and \( J_z(p)|0\rangle = 0 \) for all \( p \geq 0 \), and by constructing other states of the Hilbert space by acting with \( \tilde{J}_{+,-,d}(p) \) and \( J_z(p) \) at \( p < 0 \). Then, it is a simple matter to see, using (63), that

\[
\begin{align*}
\tilde{J}_{+}(p)|-V\rangle &= 0 \quad \text{if and only if} \quad p \geq -V \\
\tilde{J}_{-}(p)|-V\rangle &= 0 \quad \text{if and only if} \quad p \geq V \\
\tilde{J}_{d}(p)|-V\rangle &= 0 \quad \text{if and only if} \quad p \geq 0 \\
J_z(p)|-V\rangle &= 0 \quad \text{if and only if} \quad p > 0.
\end{align*}
\]

The main observation is that any normal ordering operation valid on \( \mathcal{H}_0 \) is still a good normal ordering operation on \( \mathcal{H}_{-V} \). Indeed, a state of the form

\[
\tilde{J}^{i_1}_{+}(p_{i_1}) \cdots \tilde{J}^{i_k}_{+}(p_{i_k}) J^{i_1}_{+}(p_{i_1}') \cdots J^{i_k}_{+}(p_{i_k}') J^{i_1}_{d}(p_{i_1}'') \cdots J^{i_k}_{d}(p_{i_k}'') J_z(p_{i_1}''') \cdots J_z(p_{i_k}''') \langle -V | - V
\]

gives zero whenever

\[
p_1 + \ldots + p_a + p_{a+1} + \cdots + p_b + p_{b+1} + \ldots + p_c + p_{c+1} + \ldots + p_d + (a-b)V > 0.
\]

Since \((a-b)V\) is a much smaller than \( \Lambda \) for any finite \( a \) and \( b \), a normal ordering of the type

\[
:\tilde{J}(p_1) J(p_2) \cdots J(p_k) := J(p_{i_1}) J(p_{i_2}) \cdots J(p_{i_k}) \quad \text{with} \quad p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_k},
\]

where the indices \( i_m \)'s are all different and drawn in appropriate fashion from the set \( \{1,2,\ldots,k\} \) as to make the set of inequalities for the \( p_{i_m} \)'s valid, is still a good normal ordering on \( \mathcal{H}_{-V} \).

This observation is enough in order to see that the Callan-Symanzik equation is still valid in the steady state. Indeed, the Callan-Symanzik equation is really an equation for operators, rather than just for particular averages. Recalling the regularized real-time perturbation theory (53), consider the operator

\[
\tilde{J}_A = \sum_{k=0}^{\infty} (-1)^k \int dp_1 R_A(p_1) \left( -i \pi \delta(p_1) + \frac{1}{p_1} \right) \int dp_2 R_A(p_2) \left( -i \pi \delta(p_1 + p_2) + \frac{1}{p_1 + p_2} \right) \times \int dp_k R_A(p_k) \left( -i \pi \delta(p_1 + p_2 + \cdots + p_k) + \frac{1}{p_1 + p_2 + \cdots + p_k} \right) \times \\
\int ds R_A(s) \left[ \bar{H}_1(p_1), [H_1(p_2), \cdots, [H_1(p_k), \tilde{J}(s)]] \right].
\]
It gives the interacting current operator with respect to the free Hilbert space \( \mathcal{H}_0 \) (at equilibrium) or \( \mathcal{H}_{-\mathcal{V}} \) (in the steady state). The validity of the Callan-Symanzik equations at equilibrium means that all matrix elements of \( \tilde{\mathcal{J}}_\Lambda \) in the Hilbert space \( \mathcal{H}_0 \) satisfy the Callan-Symanzik equations. This can be written:

\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta_\Lambda(\lambda, \lambda_d) \frac{\partial}{\partial \lambda} + \beta_{\lambda_d}(\lambda, \lambda_d) \frac{\partial}{\partial \lambda_d} \right) \tilde{\mathcal{J}}_\Lambda |_{\Lambda=\infty} = 0 \quad \text{on} \quad \mathcal{H}_0 .
\]

Indeed, such matrix elements correspond, in perturbation theory, to matrix elements of the current operator \( \mathcal{J} \) in the basis of eigenstates of the full Hamiltonian \( H_{|\mathcal{V}=0} \) (all of which obey the same Callan-Symanzik equation). We give an argument for the validity of \( (87) \) in Appendix C. In \( (87) \), the limit \( \Lambda \to \infty \) should be performed after a matrix element has been calculated, holding the momenta associated to this matrix element fixed. We now extract the divergent and finite part of the operator \( \mathcal{J} \) as \( \Lambda \to \infty \):

\[
\tilde{\mathcal{J}}_\Lambda \sim \sum_{j=0}^{\infty} \frac{1}{j!} \ln^j(\Lambda) : \mathcal{O}_j : \quad (88)
\]

where \( \mathcal{O}_j \)'s are operators built out of the mode operators for the current algebra along with possible impurity-space operators (the normal ordering is defined in \( (85) \)). This is always possible to do by rewriting \( \tilde{\mathcal{J}}_\Lambda \) in terms of normal-ordered operators and evaluating the coefficients as \( \Lambda \to \infty \). The Callan-Symanzik equation \( (87) \) then states,

\[
: \mathcal{O}_{j+1} : + \left( \beta_\Lambda(\lambda, \lambda_d) \frac{\partial}{\partial \lambda} + \beta_{\lambda_d}(\lambda, \lambda_d) \frac{\partial}{\partial \lambda_d} \right) : \mathcal{O}_j : = 0 \quad (j = 0, 1, \ldots, \infty) .
\]

Since the normal ordering operation \( (85) \) is also valid on the Hilbert space \( \mathcal{H}_{-\mathcal{V}} \), it is clear that the expression \( (88) \) also gives the divergent and finite part of the current operator on \( \mathcal{H}_{-\mathcal{V}} \), so that the recursion relation \( (89) \) among operators : \( \mathcal{O}_j : \) implies the Callan-Symanzik equation \( (83) \) also holds for the steady-state average of the current. Of course, the same is true for any operator that has a well-defined average in the steady state.

All this will be explicitly verified for the current operator to one loop in the following sub-section.

**Density-matrix-independent calculation of the beta functions and of the anomalous dimension of the current to one loop.** Let us write \( \tilde{\mathcal{J}}_\Lambda \) using time variables instead of momentum variables:

\[
\tilde{\mathcal{J}}_\Lambda = \sum_{k=0}^{\infty} \int_{-\infty}^{t_0} dt_1 \int_{t_1}^{t_0} dt_2 \cdots \int_{t_{k-1}}^{t_0} dt_k [(H^{(0)}_I)_\Lambda(t_1), [(H^{(0)}_I)_\Lambda(t_2), \cdots, [(H^{(0)}_I)_\Lambda(t_k), (\mathcal{J}^{(0)})_\Lambda(0)] \cdots] .
\]

(90)

Here

\[
(H^{(0)}_I)_\Lambda(t) = \left( \lambda(\tilde{\mathcal{J}}_\Lambda)_\Lambda(-t) + \lambda_d(\tilde{\mathcal{J}}_\Lambda)_\Lambda(-t) \right) \cdot \vec{S} \quad (91)
\]

and

\[
(\mathcal{J}^{(0)})_\Lambda(t) = \lambda(\tilde{\mathcal{J}}_\Lambda)_\Lambda(-t) \cdot \vec{S} .
\]

(92)

In the regularization scheme that we consider, characterized by the regulator \( R_\Lambda(p) \) \( (50) \), it is a simple matter to observe that

\[
\Lambda \frac{\partial}{\partial \Lambda} (H^{(0)}_I)_\Lambda(t) = -\frac{1}{\Lambda^2} (H^{(0)}_I)_\Lambda''(t) , \quad \Lambda \frac{\partial}{\partial \Lambda} (\mathcal{J}^{(0)})_\Lambda(t) = -\frac{1}{\Lambda^2} (\mathcal{J}^{(0)})_\Lambda''(t)
\]

(93)

where primes mean time derivatives. Consider the first few terms of \( (90) \):

\[
\tilde{\mathcal{J}}_\Lambda = (\mathcal{J}^{(0)})_\Lambda + \int_{-\infty}^{0} dt [i(H^{(0)}_I)_\Lambda(t), (\mathcal{J}^{(0)})_\Lambda(0)] + \int_{-\infty}^{0} dt_1 \int_{t_1}^{0} dt_2 [i(H^{(0)}_I)_\Lambda(t_1), [i(H^{(0)}_I)_\Lambda(t_2), (\mathcal{J}^{(0)})_\Lambda(0)] + \cdots
\]

(94)
Using integration by parts, we find

\[-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{J} = \frac{1}{\Lambda^2} ((\mathcal{J}^{(0)})'_\Lambda(0) + \ldots + \left[ i(H_I^{(0)})_\Lambda(0), (\mathcal{J}^{(0)})_\Lambda(0) \right] + \int_{-\infty}^0 dt \left[ i(H_I^{(0)})_\Lambda(t), (\mathcal{J}^{(0)})'_\Lambda(0) \right] + \int_{-\infty}^0 dt \left[ [i(H_I^{(0)})_\Lambda(t), i(H_I^{(0)})_\Lambda(t)], (\mathcal{J}^{(0)})_\Lambda(0) \right] + \int_{-\infty}^0 dt \left[ [H_I^{(0)}_\Lambda(t), [H_I^{(0)}_\Lambda(0), (\mathcal{J}^{(0)})_\Lambda(0)] \right] + \int_{-\infty}^0 dt_1 \int_{t_1}^0 dt_2 \left[ i(H_I^{(0)})_\Lambda(t_1), [i(H_I^{(0)})_\Lambda(t_2), (\mathcal{J}^{(0)})'_\Lambda(0) \right] + \ldots \right].\]

(95)

We want to evaluate all this at \(\Lambda \to \infty\) in order to find the beta function using the equation (87). Inside the parenthesis, we need only keep the contributions of order \(\Lambda^2 \ln^i(\Lambda)\) for non-negative integers \(i\). The very first operator of course does not contribute, but all others do. These leading contributions have to be compared with the leading contributions of

\[
\left( \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{\lambda_d} \frac{\partial}{\partial \lambda_d} \right) \mathcal{J} = \left( \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{\lambda_d} \frac{\partial}{\partial \lambda_d} \right) \left[ (\mathcal{J}^{(0)})_\Lambda(0) + \int_{-\infty}^0 dt \left[ i(H_I^{(0)})_\Lambda(t), (\mathcal{J}^{(0)})_\Lambda(0) \right] + \ldots \right]
\]

(96)

where we wrote only the terms contributing to the one-loop order. The beta functions appearing there can be obtained by requiring that when the derivatives with respect to the couplings are applied to the operator \(i(H_I^{(0)})_\Lambda(t)\), they give the operator \(i(H_I^{(0)})_\Lambda(t), i(H_I^{(0)})_\Lambda(t)\) (appearing on the third line of (95)) in the limit \(\Lambda \to \infty\). The main contribution in this limit of this operator can be obtained from

\[
\frac{1}{\Lambda^2} \left[ i(H_I^{(0)})_\Lambda(t), i(H_I^{(0)})_\Lambda(t) \right] = -i \left( \lambda_2^2 + \lambda^2 \right) \mathcal{J}_\Lambda(-t) + 2\lambda \lambda_d \mathcal{J}_\Lambda(-t) \right] \cdot \tilde{S} + \frac{1}{\Lambda^2} : \mathcal{O} :
\]

where \(\mathcal{O}\) contains products of current algebra operators at the same point \(x = -t\); the explicit form of \(\mathcal{O}\) is not important here. Equating the leading behavior of this operator as \(\Lambda \to \infty\) with that of

\[
\left( \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{\lambda_d} \frac{\partial}{\partial \lambda_d} \right) i(H_I^{(0)})_\Lambda(t)
\]

gives

\[
\beta_{\lambda_d} = -\left( \lambda_2^2 + \lambda^2 \right) + O(\lambda_2^3, \lambda_4^2 \lambda_2), \quad \beta_\lambda = -2\lambda \lambda_d + O(\lambda \lambda_d^2, \lambda^3).\]

(98)

Note that we had to take the limit \(\Lambda \to \infty\) of the operator \([i(H_I^{(0)})_\Lambda(t), i(H_I^{(0)})_\Lambda(t)]\) rather than of the integral where it is involved on the third line of (95), since the beta function does not depend on the particular average that we are calculating. The sub-leading operators in (97) may give contributions to this integral, but these are two-loop contributions to the anomalous dimension of the operator \(\mathcal{J}\) (of course, since this operator should have zero anomalous dimensions, all such contributions should cancel out).

The total one-loop contributions to the anomalous dimension of \(\mathcal{J}\) can be verified to be zero by checking that when the derivatives with respect to the couplings in (100) are applied to the operator \((\mathcal{J}^{(0)})_\Lambda(0)\) (the first term inside the parenthesis), they give the two terms appearing on the second line of (95) in the limit \(\Lambda \to \infty\). In this limit, the second term on the second line of (95) can be written

\[
\int_{-\infty}^0 dt \left[ i(H_I^{(0)})_\Lambda(t), \mathcal{J} \right] \sim \int_{-\infty}^0 dt \left[ i(H_I^{(0)})'_\Lambda(t), \mathcal{J} \right] = [i(H_I^{(0)})'_\Lambda(0), \mathcal{J}].
\]

(99)

That is, it is equal to the first term. Together, their leading behavior at \(\Lambda \to \infty\) can be obtained from

\[
\frac{2}{\Lambda^2} \left[ i(H_I^{(0)})'_\Lambda(0), (\mathcal{J}^{(0)})_\Lambda(0) \right] = -2\lambda_d (\mathcal{J}^{(0)})_\Lambda(0) \cdot \tilde{\mathcal{S}}(0) + \frac{1}{\Lambda^2} : \mathcal{O} :
\]

(100)
where $\tilde{O}$ is of the same form as $O$. But since

$$
\left( \beta_{\lambda} \frac{\partial}{\partial \lambda} + \beta_{\lambda_d} \frac{\partial}{\partial \lambda_d} \right) \mathcal{F} = \lambda^{-1} \beta_{\lambda} \mathcal{F} = -2\lambda_d \mathcal{F} + O(\lambda\lambda_d^2, \lambda^3),
$$

we immediately conclude that the anomalous dimension of the current is zero to one loop.

It is a simple matter to verify that the one-loop steady-state current (68) with the values (69) and (73) indeed satisfies the Callan-Symanzik equation

$$
\left. \Lambda \frac{\partial}{\partial \Lambda} \right|_{\lambda, \lambda_d} - 2\lambda\lambda_d \frac{\partial}{\partial \lambda} - (\lambda_d^2 + \lambda^2) \frac{\partial}{\partial \lambda_d} \langle \mathcal{F} \rangle \xrightarrow{\Lambda \to \infty} 0.
$$

(101)

Note that our derivation of the Callan-Symanzik equation to one-loop did not use the particular initial density matrix $e^{-\beta(H_0 - VH_\lambda)}$; only the fact that normal-ordered operators are finite when averaged with this density matrix.

**Two-loop beta functions.** The two-loop result (80) with two-loop divergent parts (81) and one-loop finite and divergent parts (69) and (73) is

$$
\langle \mathcal{F} \rangle_{ss} \sim 3\pi V \lambda^2 \left( \frac{1}{2} + 2\lambda_d \left( P(w) + \ln \left( \frac{\Lambda}{V} \right) \right) + \lambda_d^2 \left[ 10P(w) + 5 \ln \left( \frac{\Lambda}{V} \right) - 1 \right] \ln \left( \frac{\Lambda}{V} \right) + \lambda^2 \left[ 2P(w) + \ln \left( \frac{\Lambda}{V} \right) - 1 \right] \ln \left( \frac{\Lambda}{V} \right) \right).
$$

(102)

This satisfies perturbatively the Callan-Symanzik equation with beta functions

$$
\beta_{\lambda_d} = -\lambda_d^2 - \lambda^2 + O(\lambda_d^3, \lambda^2 \lambda_d)
$$

$$
\beta_{\lambda} = -2\lambda\lambda_d + \lambda\lambda_d^2 + \lambda^3 + O(\lambda\lambda_d^3, \lambda^3 \lambda_d).
$$

(103)

Note that this calculation does not give the third order coefficient of $\beta_{\lambda_d}$. However, we know the universal beta functions of the one-channel Kondo model $-(g^2 - \frac{1}{2}g^3)$ and that of the symmetric two-channel Kondo model $-(g^2 - g^3)$. Taking $V = 0$ (which does not affect the beta function), these two cases are obtained respectively at $\lambda = 0$, and at $\lambda_d = \lambda$ upon diagonalization of the matrix of couplings $J_{\alpha, \alpha'}$. These two facts essentially fix the two-loop beta function to be of the form

$$
\beta_{\lambda_d} = -(\lambda_d^2 + \lambda^2 - \lambda^3 - \lambda^2 \lambda_d + \ldots)
$$

$$
\beta_{\lambda} = -(2\lambda\lambda_d - a\lambda\lambda_d^2 - (2 - a)\lambda^3 + \ldots)
$$

(104)

where $a$ is a non-universal number. Our results fix $a = 1$, which gives the standard beta functions

$$
\beta_{\lambda_d} = -(\lambda_d^2 + \lambda^2 - \lambda^3 - \lambda^2 \lambda_d + O(\lambda_d^3, \lambda^2 \lambda_d^2, \lambda^4))
$$

$$
\beta_{\lambda} = -(2\lambda\lambda_d - \lambda\lambda_d^2 - \lambda^3 + O(\lambda\lambda_d^3, \lambda^3 \lambda_d)).
$$

(105)

It is convenient now to change variables to $\lambda_\pm = \lambda_d \pm \lambda$ so that (to the same order)

$$
\beta_\pm = -\lambda_\pm^2 + \frac{1}{2} \lambda_\pm (\lambda_\pm^2 + \lambda^2).
$$

(106)

The RG invariant anisotropy parameter can be expressed as:

$$
C = \frac{1}{2} \left( \frac{1}{\lambda_+} \left( 1 - \frac{1}{2} \lambda_+ \right) - \frac{1}{\lambda_+} \left( 1 - \frac{1}{2} \lambda_- \right) \right).
$$

(107)

**The scaling limit of the current.** We now evaluate the current as function of the voltage and of the temperature. For the expression of the current, our analysis will make use solely of the one-loop results (77).
Expression (74) is the perturbative expansion in the limit $T, V \ll \Lambda$ of the model. The couplings $\lambda_d, \lambda$ take their “microscopic” values: the actual values for the physical process represented by the interaction terms, and the cut-off $\Lambda$ is of the scale of the band width. Of course, since we neglected terms vanishing as $\Lambda \to \infty$, this is not an exact expression for finite $\Lambda$. However, it allows us to have an exact expression in the scaling limit.

As explained earlier, we expect universality for $T, V \ll \Lambda$. It is convenient therefore to consider the limit where $\Lambda$ is sent to infinity with the couplings becoming cut-off dependent: they are modified so as to keep the physics unchanged when the cut-off is increased. Denoting for the moment $\Lambda = \Lambda_0$ the physical value of the cut-off (of the order of the band width) and $\lambda = \lambda_0, \lambda_d = \lambda_d^0$ the values of the coupling there, the running couplings $\lambda'(\Lambda), \lambda_d'(\Lambda)$, are governed by the RG equations,

$$
\Lambda \frac{d\lambda_d'}{d\Lambda} = (\lambda_d')^2 + (\lambda')^2 - (\lambda_d^0)^3 - (\lambda')^2 \lambda_d^0
$$
$$
\Lambda \frac{d\lambda'}{d\Lambda} = 2\lambda' \lambda_d' - \lambda' (\lambda_d^0)^2 - (\lambda')^3
$$

(108)

with initial conditions fixed by the microscopic values of the couplings: $\lambda_d'(\Lambda = \Lambda_0) = \lambda_d^0, \lambda'(\Lambda = \Lambda_0) = \lambda_0$.

The solution of the RG flow can then be described by RG invariants - quantities that describe the full trajectory. Such quantities are $C$ introduced earlier, and a scale $T_K$ to be discussed below. Thus any physical quantity $F$ will depend on the cut-off and coupling via these invariants $F = F(T/T_K, V/T_K, C)$. One may reformulate this scaling procedure as follows. Introduce the scale $M$, the physical scale on which the system is examined,

$$
V = M \sin(\alpha) \quad , \quad T = M \cos(\alpha)
$$

for some angle $\alpha$ in the $V - T$ plane so that the previous result for the current is written as,

$$
\langle J \rangle_{ss} = \frac{3\pi}{2} V \lambda^2 \left[ 1 + 4\lambda_d \left( P(\tan(\alpha)) + \ln \left( \frac{\Lambda}{M} \csc(\alpha) \right) \right) \right] + \ldots
$$

Following the previous considerations, the current can be written in terms of couplings that depend on the physical scale $M$, satisfying the equations with respect to $M$,

$$
M \frac{d\lambda_d'}{dM} = - (\lambda_d')^2 - (\lambda')^2 + (\lambda_d^0)^3 + (\lambda')^2 \lambda_d^0
$$
$$
M \frac{d\lambda'}{dM} = - 2\lambda' \lambda_d' + \lambda' (\lambda_d^0)^2 + (\lambda')^3
$$

(109)

with initial conditions $\lambda_d'(M = \Lambda) = \lambda_d, \lambda'(M = \Lambda) = \lambda$ ($\Lambda$ being of the order of the band width), as follows:

$$
\langle J \rangle_{ss} = \frac{3\pi}{2} V (\lambda')^2 \left[ 1 + 4\lambda_d Q(\alpha) + \ldots \right]
$$

(110)

where

$$
Q(\alpha) = P(\tan(\alpha)) + \ln(\csc(\alpha))
$$

(111)

Again, the solution to the RG flow (109) should be described solely as a function of $M/T_K$ and of the RG invariant $C$ (107), instead of $\Lambda$ and the initial conditions $\lambda_d$ and $\lambda$. With such a description, we can trivially take the scaling limit: $T, V, T_K \ll \Lambda$ with fixed ratios $T : V : T_K$ and fixed $C$, since $\Lambda$ does not appear anymore. In this limit, the quantum field theory gives exact results, and the system is in its universal regime. In order to have unambiguous results, we need to define $T_K$. Once $T_K$ is defined, one need only solve the RG flow (109) (a numerical solution is easy to obtain with good precision, for instance), and one obtains the scaling limit in its perturbative region. The actual values of $\lambda_d, \lambda$ and of the band width $\Lambda$ should be such that the theory is near to the scaling limit if we want the system to be meaningfully described by quantum field theory.

The perturbative region of the scaling limit is $T_K \ll \sqrt{V^2 + T^2}$, for any value of $C$. Hence, in order to define $T_K$ in perturbation theory, we need to look at the expansion as $M/T_K \to \infty$ of the solution to the RG flow (109). This expansion, and the value of $T_K$, have different forms depending on $C$. We will consider here two cases:
taking $C = \infty$ then $M/T_K \to \infty$; and keeping $C$ finite then taking $M/T_K \to \infty$. The first case corresponds to
\[ \lambda_d = \lambda \]. In fact, it is true to all orders that if $\lambda_d = \lambda$, then $\lambda'_d = \lambda'$ for all scales $M$. In this case, the equilibrium Hamiltonian $H|_{V=0}$ is in fact a one-channel Kondo model plus a decoupled free massless fermion. The case $C$ finite cor-responds naturally to $\lambda \sim C\lambda_d^2$ in the scaling limit (as will become clear below).

In the case where $\lambda_d = \lambda$, the RG equations lead to the following large-$(M/T_K)$ expansion of the running couplings:

\[ \lambda'_d = \lambda' \left( 1 + \frac{1}{2} \ln \frac{M}{T_K} \right) \left( \frac{\ln \left( \frac{M}{T_K} \right)}{4 \ln^2 \left( \frac{M}{T_K} \right)} + \frac{a}{2 \ln \left( \frac{M}{T_K} \right)} + \frac{\ln^2 \left( \frac{M}{T_K} \right)}{8 \ln^3 \left( \frac{M}{T_K} \right)} + \frac{(4a - 1) \ln \left( \frac{M}{T_K} \right)}{8 \ln^3 \left( \frac{M}{T_K} \right)} + O \left( \frac{1}{\ln^3 \left( \frac{M}{T_K} \right)} \right) \right). \]  

(112)

The coefficient $a$ can be changed by a change of the scale (the integration constant) $T_K$; making the replacement $T_K \rightarrow xT_K$ is equivalent to doing $a \rightarrow a + \ln(x)$. Note also that a change of scale $T_K \rightarrow xT_K$ corresponds simply to a perturbative change of the running coupling constants that keeps the beta functions invariant. Fixing the value of $a$ is making a choice of definition for $T_K$, which is one more condition necessary to totally specify the renormalization procedure. Of course, different definitions of $T_K$ reproduce the same scaling limit. For arbitrary $a$, in terms of the coupling $\lambda_d = \lambda$ and of the scale of the band width $\Lambda$, $T_K$ has the form

\[ T_K(a) = \Lambda \sqrt{2 \lambda e^{-a - \frac{1}{4}} (1 + O(\lambda))} \]  

(\lambda_d = \lambda).

A standard definition for the Kondo temperature is to make the term in $1/\ln^2(M/T_K)$ vanish ($a = 0$), giving:

\[ T_K = \Lambda \sqrt{2 \lambda e^{-\frac{1}{4}} (1 + O(\lambda))} \]  

(\lambda_d = \lambda).  

(113)

In the case where the RG invariant $C$ is finite, the large-$(M/T_K)$ expansion has the form

\[ \lambda'_d = \frac{1}{\ln \left( \frac{M}{T_K} \right)} + \frac{a}{\ln \left( \frac{M}{T_K} \right)} + \frac{1}{\ln \left( \frac{M}{T_K} \right)} + \frac{3}{\ln \left( \frac{M}{T_K} \right)} + \frac{6a + 1}{\ln \left( \frac{M}{T_K} \right)} + \ldots \]  

(114)

where the dots (\ldots) mean $O(\ln^{-3}(M/T_K))$ for $\lambda'_d$ and $O(\ln^{-4}(M/T_K))$ for $\lambda'$. Note that in general, $C$ will not appear only as a normalization of $\lambda$; this is an artifact of the limited perturbative order which we consider. Also, note that $C$ must be positive for $\lambda'$ to be positive. Again, a variation of $T_K$ has the effect of changing $\lambda$: making the replacement $T_K \rightarrow xT_K$ is equivalent to doing $a \rightarrow a + \ln(x)$. For arbitrary $a$, we can use the first or the second equation of (113) in order to determine $T_K$ in terms of the couplings $\lambda_d$, $\lambda$ and of the scale of the band width $\Lambda$. This gives two equivalent forms:

\[ T_K(a) = \Lambda \lambda_d e^{-a - \frac{1}{2\lambda_d}} (1 + O(\lambda_d)) = \Lambda \sqrt{\frac{\lambda}{C}} e^{-a - \frac{1}{2}} - \sqrt{\lambda} (1 + O(\sqrt{\lambda})) \]  

(\lambda_d \neq \lambda).

Recall that from (107), we indeed have that $\lambda \propto \lambda_d^2$ in the scaling limit, so that the second equality above is correct. A more standard way of writing the Kondo temperature can be obtained by considering the linear combination $\lambda'_d + \lambda'$. This gives, again for the same object $T_K(a)$,

\[ T_K(a) = \Lambda (\lambda_d + \lambda) e^{-(a + C) - \frac{1}{\lambda_d + \lambda}} (1 + O(\lambda_d)) \]  

(\lambda_d \neq \lambda).

A standard definition is $a = -C$, giving

\[ T_K = \Lambda (\lambda_d + \lambda) e^{-\frac{1}{\lambda_d + \lambda}} (1 + O(\lambda_d)) \]  

(\lambda_d \neq \lambda).  

(115)

Four comments are now in order.

First, note that taking $C \rightarrow \infty$ in (114) does not give (112). This is expected, since the former corresponds to taking first $M/T_K \rightarrow \infty$ then $C \rightarrow \infty$ in the solution to the RG equations, whereas the latter corresponds to
taking first $C \to \infty$ then $M/T_K \to \infty$, and these two limits do not commute. In particular, one would have obtained still different expressions taking simultaneously $C \to \infty$ and $M/T_K \to \infty$, with a prescribed ratio between them, for instance (it is a simple matter to evaluate in this case, or in any other case, the expansion as $M/T_K \to \infty$ of the solutions to the RG equations). The expressions $\text{(114)}$ and $\text{(112)}$ should not be regarded as describing all solutions to the RG equation; they are rather particular limits of the solutions, as described above, and are used here solely for the purpose of defining the Kondo temperature purely using perturbation theory (the perturbative regime is, again, the regime $M \gg T_K$, for any $C$ taken to infinity or not, simultaneously or not with $M/T_K$). The expansion $\text{(113)}$ offers a way of defining $T_K$ in the regime with $C$ finite, whereas the expansion $\text{(112)}$ offers a way of defining it in the regime with $C = \infty$. Other definitions would have been possible, but we only need these two definitions of the Kondo temperature here.

Second, it is important to recall that the cases $C = \infty$ and $C < \infty$ do exhaust all possible scaling regimes, even though the expressions $\text{(112)}$ and $\text{(113)}$ do not exhaust all possible behavior of the running couplings as $M/T_K \to \infty$. However, there are many ways of reaching any given scaling regime. In particular, the relations $\lambda = \lambda_d$ and $\lambda \sim C \lambda_d^2$ (naturally associated to, respectively, the scale definitions $\text{(113)}$ and $\text{(115)}$) do not exhaust all possible ways the same scaling limit can be reached. In order to understand what this means, first recall that the full RG-improved perturbation theory reproduces the divergent and finite part of the full bare perturbation theory, so that we can talk about the RG trajectory for the bare couplings before taking the scaling limit. Then, from this viewpoint, one can take the scaling limit by fixing a trajectory $C$, and by sending $\Lambda \to \infty$ while keeping $\lambda_d$ and $\lambda$ on the trajectory at scale $\Lambda$. For finite $C$, this gives $\lambda \sim C \lambda_d^2$, whereas for $C = \infty$, this gives $\lambda = \lambda_d$. But one could also take the scaling limit by changing the value of $C$ (changing the shape of the trajectory) while $\Lambda \to \infty$, always keeping $\lambda_d$ and $\lambda$ on the trajectory defined by $C$. For instance, this is what happens if one takes $\lambda = q \lambda_d$ for some fixed $q \neq 1$ when sending $\lambda_d$, $\lambda \to 0$; then one must simultaneously take $C \to \infty$ in order to keep $\lambda$ and $\lambda_d$ on the RG trajectory. The resulting value of $T_K$ in terms of such couplings is different from $\text{(113)}$ and $\text{(115)}$. But in this example, when the scaling limit is reached we have $C = \infty$, so that the quantum field theory is the same as the one obtained by taking $\lambda = \lambda_d$ and sending them to $0$; in particular, in the scaling limit, we still have $\lambda' = \lambda_d'$, and we can still define a scale $T_K$ using $\text{(112)}$. Since we are only interested in the scaling regimes, we do not need to look at all possible ways a given regime can be reached.

Third, it is important to note that the leading behavior of the current at large $M/T_K$ is very different if $C = \infty$ or if $C < \infty$ (again, we only look at the two cases mentioned above). In the first case, it is given by

$$\langle J \rangle_{ss} \sim \frac{3\pi}{8} \frac{M \sin(\alpha)}{\ln^2(M/T_K)} = \frac{3\pi}{8} \frac{V}{\ln^2(\sqrt{V^2 + T^2/T_K})} \quad (\lambda_d = \lambda). \quad (116)$$

On the other hand, in the second case it is given by

$$\langle J \rangle_{ss} \sim \frac{3\pi C^2}{2} \frac{M \sin(\alpha)}{\ln^4(M/T_K)} = \frac{3\pi C^2}{2} \frac{V}{\ln^4(\sqrt{V^2 + T^2/T_K})} \quad (\lambda_d \neq \lambda). \quad (117)$$

Note that since the RG invariant $C$ appears as a coefficient, the leading behavior in this case is non-universal; this is a property of the regime $C$ finite.

Finally, the parameter $\alpha$ can be seen as parametrizing a family of choices of infrared cutoffs for our theory. For instance, at $\alpha = 0$, the temperature is the infrared cutoff, whereas at $\alpha = \pi/2$, the voltage solely plays the role of an infrared cutoff. It is important to note that the voltage is a good infrared cutoff for the average of the current operator to two-loop order. Indeed, the limit $T/V \to 0$ of our one-loop and two-loop bare perturbative results is finite; equivalently, the limit $\alpha \to \pi/2$ of the renormalized perturbative results is finite. We should remark however that this need not be the case when other quantities are considered, see [7].

**Universal ratios.** From the viewpoint of the interpretation of the measurement of non-equilibrium quantities, the Kondo temperature, as defined for instance in $\text{(113)}$, is not very convenient, since the temperature $T$ and the bias voltage $V$ have a different influence on the Kondo screening cloud, and we would like a definition that embodies this difference. Hence, it seems appropriate to define a continuum of scales, $M_K(\alpha)$, depending on the angle $\alpha$ on the $V - T$ plane. Consider first the case $\lambda_d = \lambda$. A possible definition of $M_K(\alpha)$ is the requirement that the average current $J$ in the steady state does not have a term of the form $M/\ln^3(M/M_K(\alpha))$ in its expansion at large $M/M_K(\alpha)$:

$$\langle J \rangle = \frac{3\pi}{8} \frac{M \sin(\alpha)}{\ln^2(M/M_K(\alpha))} \left(1 + \frac{\ln \ln(M/M_K(\alpha))}{\ln(M/M_K(\alpha))} + O \left( \frac{\ln^2 \ln(M/M_K(\alpha))}{\ln^3(M/M_K(\alpha))} \right) \right). \quad (118)$$
This gives:

\[ M_K(\alpha) = T_K e^{Q(\alpha)} \]  

(119)

where \( T_K \) is defined in (113).

As particular cases, we can now define the “decoherence” Kondo scale, \( M_K(\pi/2) \), as the voltage at \( T = 0 \) at which the Kondo cloud is destroyed by the electrons passing through, and the “thermal” Kondo scale \( M_K(0) \). We find that their ratio, which is universal and does not receive corrections from the two-loop (or higher-loop) contributions to the beta function (nor from higher order contributions to the current), is

\[ R_{\text{current}} = \frac{M_K(\pi/2)}{M_K(0)} = 2\pi e^{1 - \frac{3\gamma}{2} - \frac{\sqrt{\pi}}{2}} = 2.96188723... \]  

(120)

In the case \( \lambda_d \neq \lambda \) (that is, for \( C < \infty \)), a similar definition of \( M_K(\alpha) \) can be made: requiring that the current does not have a term of the form \( M/\ln^4(M/M_K(\alpha)) \). Constructing again the ratio \( M_K(\pi/2)/M_K(0) \) gives exactly the same number: this ratio is indeed universal, independent of both \( T_K \) and \( C \). It is important that the current possess an infrared-convergent (convergence at large switch-on times) perturbative expansion to one loop for these universal ratios to have a meaning. Further, that they have the usual one-loop logarithmic accuracy near to the scaling limit is a consequence of the infrared convergence of the two-loop perturbation theory.

The scale \( M_K(\alpha) \) defined above is characteristic of the current; other physical quantities would give different functions \( M_K(\alpha) \), and different ratios. For instance, the same analysis can be applied on the differential conductance

\[ G = \frac{d}{dV} \langle J \rangle_{ss} . \]

From the perturbative calculations, we have

\[ G = \frac{3\pi}{2} \lambda^2 \left[ 1 + 4\lambda_d \left( P \left( \frac{V}{T} \right) + \ln \left( \frac{\Lambda}{V} \right) + \frac{V}{T} P' \left( \frac{V}{T} \right) - 1 \right) + ... \right] . \]  

(121)

Again, in terms of running couplings, we have

\[ G = \frac{3\pi}{2} (\lambda')^2 \left[ 1 + 4\lambda'_d \tilde{Q}(\alpha) + ... \right] \]  

(122)

where

\[ \tilde{Q}(\alpha) = P(\tan(\alpha)) + \ln(\csc(\alpha)) + \tan(\alpha) P'(\tan(\alpha)) - 1 . \]  

(123)

Taking again \( \lambda_d = \lambda \), we can repeat the calculations above and define similarly the scale \( \tilde{M}_K(\alpha) \) associated to the conductance. We find

\[ \tilde{M}_K(\alpha) = T_K e^{\tilde{Q}(\alpha)} . \]  

(124)

We observe that \( \tilde{Q}(0) = Q(0) \) and that \( \tilde{Q}(\pi/2) = Q(\pi/2) - 1 \). Hence, we have

\[ R_{\text{conductance}} = \frac{\tilde{M}_K(\pi/2)}{\tilde{M}_K(0)} = e^{-1} R_{\text{current}} = 2\pi e^{3\gamma - \frac{\sqrt{\pi}}{2}} = 1.08961742... \]  

(125)

V. PERSPECTIVES

Reaching the steady-state. We showed that the bath of free massless fermions suffices to allow the system to reach equilibrium at temperature \( T \) in the case of zero bias voltage, and to allow it to reach steady state when the bias voltage is non-zero. No other relaxation process has to be assumed; the infinite bath of free
massless fermions plays the role of a thermal bath and is able to absorb the energy necessary for relaxation to occur, as well as for the steady state to exist. This is in close connection with the study of Caldeira and Leggett \[4\]: they constructed a model where an infinite number of oscillators provides an explicit dissipation in order to study, from first principles, the effect of dissipation on quantum tunneling. It turns out \[3\] that their construction is simply related to models of field theory that are conformal in the bulk (in general, with non-conformal boundary conditions), as recall in the Introduction. It would be interesting to see to what extent our proof can be generalized to the study of more general models of quantum kinetics with thermal dissipation, and to see whether similar arguments can be applied to other impurity models out of equilibrium. A principle to follow, as can be extracted from our derivation, is the fact that the orbit of a global symmetry of the model should cover the possible values of the boundary degrees of freedom.

Related to the latter point, another question is: What is the effect of a magnetic field on the steady state in the quantum dot? Our proof of factorization in Appendix \[B\] does not hold anymore when a magnetic field is present, since it uses heavily the invariance of the correlation functions under \(SU(2)\) transformations. Hence, the real-time perturbation series is no longer expected to be convergent as the switch-on time is sent to minus infinity. This is simple to understand physically: at small couplings, both the current through the dot and its interaction with the leads are weak, that is, both the non-equilibrating effect and the thermalization effect are weak. At zero magnetic field, since all states of the isolated dot have the same energy, the thermalization effect is more efficient and still stronger. At non-zero magnetic field, however, as the couplings are sent to zero, we cannot expect that the dot smoothly reaches its thermal equilibrium energy distribution when no other thermal bath is coupled to it. The real-time perturbation theory should describe this situation, but obviously its zeroth order cannot give anything else than the thermal equilibrium value of any quantity under study. Hence, in non-zero magnetic field, we can expect some strong non-analyticity in the couplings and we must find large-time (IR) divergences in the perturbative coefficients. This clearly indicates that the real-time perturbative series does not properly describe the approach to the steady state, neither the steady state itself, of the model in magnetic field without external thermal bath.

Questions remain, as raised in the Introduction: Does the quantum field theory still (non-perturbatively) reach a steady state, or does it show other behaviors at large times, like oscillations (of the dot magnetization, for instance)? If it reaches a steady state, is it a good description of realistic systems, where the dot is coupled to an external thermal bath at all times (without exchange of particles), independently of its coupling to the leads?

The former question was partially answered, in \[12\]: it was assumed on physical grounds that the non-equilibrium Kondo model reaches a steady state, and mainly from this assumption, it was explained how to obtain the zeroth order of perturbation theory for the dot magnetization. Indeed, strong non-analyticity is obtained. In a sense, one should start with a density matrix that already contains a non-thermal distribution of the impurity spin states; this density matrix can be obtained by requiring that the perturbative series be convergent at large times. As explained clearly there, this is equivalent to solving a quantum Boltzmann equation in order to determine the non-thermal dot occupation numbers. This should answer partly, in some sense still perturbatively, the question of describing the steady state of the quantum field theory (the non-equilibrium Kondo model with magnetic field) – although the results really start with the assumption of a steady state, and do not establish its existence. However, this does not address the question as to whether the leads correctly play the role of thermal baths, or whether a coupling to an external thermal bath would have important effects.

In relation to the latter question, our argument suggests that, at least at small couplings (or at temperatures much greater than the Kondo temperature), the model \textit{does not} describe the true steady state of the non-equilibrium Kondo dot in contact with a thermal environment. Indeed, from a physical interpretation of the perturbative series, the leads do not provide a strong enough thermalisation to absorb the energy necessary for the steady state to occur, and it is possible that it cannot be trusted to sustain the correct steady state. That it does not provide the thermalisation for the steady state to occur is certainly in agreement with \[12\]: there it was one of the main points that the large-time divergences are due to the absence of a proper thermal bath, and that one needs to put “by hand” a thermal bath connected to the dot. This was done, essentially, by putting a small imaginary part on the evolution time, which was then set to zero before taking small couplings \(\lambda, \lambda_d\) (that is, it was obtained a steady state where the coupling to the thermal bath is much smaller than the couplings \(\lambda, \lambda_d\)). The most delicate question, however, concerns the fact that the steady state itself may be affected non-trivially by a thermal bath. It is probable (but this should be verified) that other, more realistic, representations of a thermal bath give the same results as \[12\] in the limit of small coupling with the thermal
bath. However, as the couplings $\lambda, \lambda_d$ are sent to zero, due to the thermal effect of the environment, the average dot magnetization, for instance, should smoothly reach its thermal equilibrium value, and this may well be a universal cross-over behavior (since it occurs at small couplings, hence near to a second order phase transition). In order to assess this, it would be important to have a more adequate description of a thermal bath, and the theory of Caldeira and Leggett surely provides the most promising avenue.

**The steady-state density matrix.** Our proof that the real-time perturbation theory describes a steady state was immediately adapted to the proof that the steady state can be described by a density matrix (13), almost as in an equilibrium state. As mentioned, a steady-state density matrix was also introduced in [6] for generic models under the assumption that a relaxation time was present. Our derivation is slightly different, and does not make further assumptions, since we showed that relaxation does occur in the non-equilibrium Kondo model. The main characteristics of the steady-state density matrix, as opposed to equilibrium density matrices, is that it is defined by coupling a non-local conserved charge $Y$ to the voltage $V$, instead of coupling a local conserved charge to an appropriate chemical potential. This non-locality is related to the fact that the operator $Y$ describes the build-up of the steady state, or, in some sense, the asymptotic state that characterizes the steady state. The properties of this asymptotic states and other consequences of this description are still to be explored.

**Renormalized real-time perturbation theory.** We developed the two-loop renormalized real-time perturbation theory. We gave an argument for the validity of the Callan-Symanzik equation in the steady state (with the same beta functions as the equilibrium ones), and verified this to one loop for the current operator inside any correlation function. It is tempting to relate the validity of the Callan-Symanzik equation to the fact that the operator $Y$ appearing in the steady-state density matrix is a conserved charge. In particular, the fact that the voltage does not flow is obvious from such consideration, as it is coupled to a conserved charge. However, since $Y$ is non-local, it is hard to make this connection more precise.

The quantum field theory gives physical quantities in the scaling limit (the universal region) $V, T, T_K \ll D$, where $T_K$ is the Kondo temperature, which is a non-universal quantity related to the microscopic values of the couplings. Our renormalized perturbative results give the current in the region $T_K \ll \sqrt{V^2 + T^2} \ll D$. This includes the part of the universal region where the system is strongly out of equilibrium. In this perturbative region, we defined a continuous family of Kondo scales $M(\alpha)$ depending on the ratio $V/T = \tan(\alpha)$. By comparing the scale at $\alpha = \pi/2$ ($T/V \to 0$) with that at $\alpha = 0$ ($V/T \to 0$), we obtained a universal measure of the effect of the voltage on the Kondo cloud, as compared to the effect of the temperature. We noted that such a universal measure was correct since to two-loop order, the voltage plays the role of a good infrared cutoff for the current, so that no divergencies appear to that order as $T/T_K \to 0$ (with fixed $V/T_K$).

In connection to the latter point, it has sometimes been suggested in the literature that at $T/T_K = 0$, the system should be in a “strong coupling regime” (see for instance [18]), and as such, the perturbation theory should not be valid and should show infrared divergencies (divergencies as $T/T_K \to 0$) in higher-loop calculations. In particular, it is clear that this occurs in the calculation of any thermodynamical quantities, which can indeed be deemed “in a strong coupling regime” (at the IR fixed point) at zero temperature. In [8, 9], for instance, one sees logarithmic divergences as $T/T_K \to 0$ for any fixed $V/T_K$ at the one-loop order of perturbation theory for the spin susceptibility, pointing to the fact that for describing the limit $T/T_K \to 0$ of that thermodynamical quantity, one needs to know about the IR fixed point [23] (the perturbation theory only describes the theory around its UV fixed point). Our point, though, is that this might not be so for all quantities. Our two-loop results suggest that in a sense, the current is really a dynamical quantity, ruled by the scale $\alpha$ (UV fixed point). In connection to the latter point, it has sometimes been suggested in the literature that at $T/T_K = 0$, the system should be in a “strong coupling regime” (see for instance [18]), and as such, the perturbation theory should not be valid and should show infrared divergencies (divergencies as $T/T_K \to 0$) in higher-loop calculations. In particular, it is clear that this occurs in the calculation of any thermodynamical quantities, which can indeed be deemed “in a strong coupling regime” (at the IR fixed point) at zero temperature. In [8, 9], for instance, one sees logarithmic divergences as $T/T_K \to 0$ for any fixed $V/T_K$ at the one-loop order of perturbation theory for the spin susceptibility, pointing to the fact that for describing the limit $T/T_K \to 0$ of that thermodynamical quantity, one needs to know about the IR fixed point [23] (the perturbation theory only describes the theory around its UV fixed point). Our point, though, is that this might not be so for all quantities. Our two-loop results suggest that in a sense, the current is really a dynamical quantity, ruled by the scale $\alpha$ (UV fixed point).
field. It is easy to check for instance that the one-loop, finite-magnetic-field results of [8, 9, 11] satisfy the one-loop Callan-Symanzik equations in the universal regime. Although we did not cover the case with a finite magnetic field (and see the discussion above for the subtleties involved), we expect that our general arguments for the validity of the Callan-Symanzik equations to all order still hold since in the universal regime, the magnetic field is a low-energy scale as compared to the bandwidth. It may be useful to note that the usual perturbative renormalization was modified in [8, 9] in order to correctly incorporate the structure of logarithmic divergencies in the region $V > D$ (where $D$ is the bandwidth) of the one-loop calculation of the current and of the impurity magnetization at finite magnetic field; more precisely, energy-dependent coupling constants were introduced.

We want to stress that this region may be non-universal (that is, results may depend on the precise structure of the band), as is the region $T > D$ or $T_K > D$. Then, naturally, it cannot be universally described by a finite number of coupling constants (or, more precisely, by a finite number of RG invariants). In order to recover “scaling”, one needs to use exact RG or similar methods, and the usual Callan-Symanzik equation does not hold; but this is not in disagreement with our results, which only deal with the universal regime $V, T, T_K \ll D$.

Acknowledgments

BD is grateful to J. Cardy, J. Chalker and F. Essler, as well as all members of the QFT research group at Oxford, for sharing their insights in many occasions, to A. Lamacraft for discussions during his visit, and to N. Shah for useful comments on the manuscript. BD acknowledges support from an EPSRC post-doctoral fellowship (grant GR/S91086/01), and also acknowledges Rutgers University, where this work was initiated. NA is grateful to C. Bolech, P. Mehta, O. Parcollet, A. Rosch and A. Schiller, for numerous illuminating discussions, criticisms and suggestions as well as useful comments on the manuscript. We would like to thank S. Kehrein for various discussions during the early stage of this work.

APPENDIX A: CONNECTED CORRELATION FUNCTIONS

Consider a theory with density matrix $\rho$, and denote, for any operator $O$,

$$\langle\langle O \rangle\rangle = \frac{\text{Tr} [\rho O]}{\text{Tr} [\rho]}.$$  

Consider the following average:

$$\langle O \rangle \equiv \frac{\langle\langle \mathcal{P} \exp \left( i \int_{t_0}^0 dt H_I(t) \right) O \rangle\rangle}{\langle\langle \mathcal{P} \exp \left( i \int_{t_0}^0 dt H_I(t) \right) \rangle\rangle} \quad (1)$$

where $H_I(t)$ can be any operator depending on the (time) parameter $t$, and $O$ is also any operator. The path-ordered exponential above is understood as an expansion in time-ordered integrals of multilinear of $H_I(t)$’s. The connected correlation functions, where the $H_I(t)$’s are “connected” to $O$, can then naturally be defined by saying that

$$\langle O \rangle = \sum_{n=0}^{\infty} i^n \int_{t_0}^0 dt_1 \int_{t_1}^0 dt_2 \cdots \int_{t_{n-1}}^0 dt_n \langle\langle H_I(t_1)H_I(t_2)\cdots H_I(t_n) O \rangle\rangle_{\text{connected}} \quad (2)$$

Equivalently, they can be defined recursively by

$$\langle\langle H_I(t_1)\cdots H_I(t_n) O \rangle\rangle_{\text{connected}} = \langle\langle H_I(t_1)\cdots H_I(t_n) O \rangle\rangle_{\text{connected}}$$

$$+ \sum_{m=1}^{n} \sum_{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{n-m} \subseteq \{1, \ldots, n\}} \langle\langle H_I(t_{\alpha_1})\cdots H_I(t_{\alpha_m}) \rangle\rangle_{\text{connected}} \langle\langle H_I(t_{\beta_1})\cdots H_I(t_{\beta_{n-m}}) O \rangle\rangle_{\text{connected}} \quad (3)$$
This recursive definition does not involve integrations over the time parameters \( t_i \)'s, so that it is in fact a general definition for correlation functions where a set of arbitrary operators \( H_I(t_i) \)'s parametrized by \( i = 1, \ldots, n \) are connected to \( \mathcal{O} \).

The main property of connected correlation functions is the following. Consider a fixed set of \( t_i \)'s for \( i = 1, \ldots, n \) and a fixed subset \( t_{a_1}, \ldots, t_{a_m} \) for \( m < n \). Consider correlation functions of \( H_I(t_i) \)'s with and without insertion of \( \mathcal{O} \). One can observe that if the operators \( H_I(t_{a_i}) \)'s factorize out of all such correlation functions (for instance:

\[
\langle\langle H_I(t_1) \cdots H_I(t_{a_1}) \cdots H_I(t_{a_m}) \cdots H_I(t_n) \mathcal{O} \rangle\rangle = \langle\langle H_I(t_1) \cdots H_I(t_{a_1}) \cdots H_I(t_{a_m}) \cdots H_I(t_n) \rangle\rangle \langle\langle H_I(t_{a_1}) \cdots H_I(t_{a_m}) \rangle\rangle
\]

then all connected correlation functions involving at least one of the \( H_I(t_{a_i}) \)'s are zero. This is easy to show from (3) by induction on the number of operators \( H_I(t_i) \)'s inside connected correlation functions.

Connected correlation functions also occur in more general situations:

\[
\langle\langle \mathcal{P} \exp \left(i \int_{t_1}^{t_2} dt H_I^{(1)}(t) \right) \cdots \mathcal{P} \exp \left(i \int_{t_N}^{t_1} dt H_I^{(N)}(t) \right) \mathcal{O} \rangle\rangle \]

\[
= \langle\langle \mathcal{P} \exp \left(i \int_{t_1}^{t_2} dt H_I^{(1)}(t) \right) \cdots \mathcal{P} \exp \left(i \int_{t_N}^{t_1} dt H_I^{(N)}(t) \right) \mathcal{O} \rangle\rangle_{\text{connected}}
\]

On the right-hand side, the operators \( H_I^{(1)}(t_i) \)'s, \( H_I^{(2)}(t_i) \)'s, \( \ldots, H_I^{(N)}(t_i) \)'s are all connected to \( \mathcal{O} \).

**APPENDIX B: PROOF OF FACTORIZATION**

Consider a product of operators of the type

\[
\tilde{J}_1(x + x_1) \cdot \tilde{S} \tilde{J}_2(x + x_2) \cdot \tilde{S} \cdots \tilde{J}_n(x + x_n) \cdot \tilde{S}
\]

where \( \tilde{J}_{1,2,\ldots} \) can be \( \tilde{J}_{x}, \tilde{J}_x \) or \( \tilde{J}_x \). Recall the notation (19). There, the trace is performed over the Hilbert space of the conformal field theory where the currents \( \tilde{J}_{1,2,\ldots} \) act and over the two-dimensional impurity space associated to \( \tilde{S} \). Insert an operator which is composed of products of local operators acting on the CFT Hilbert space tensored with an arbitrary operator on the impurity space. We will denote it by

\[
\sum_a \mathcal{O}^a S_a
\]

where \( a = 0, 1, 2, 3 \) and \( S_0 \) is the identity \( \mathbf{1} \) on the impurity space. The operators in \( \mathcal{O}^a \) can be at any fixed time (with respect to the theory \( H_0 \)) and position, and can also be integrals of such operators over finite time intervals. We consider the SU(2) invariant quantity:

\[
\langle\langle \tilde{J}_1(x + x_1) \cdot \tilde{S} \tilde{J}_2(x + x_2) \cdot \tilde{S} \cdots \tilde{J}_n(x + x_n) \cdot \tilde{S} \mathcal{O}^a S_a \rangle\rangle_0 = \langle\langle J_1^{(1)}(x + x_1) J_2^{(2)}(x + x_2) \cdots J_n^{(n)}(x + x_n) \mathcal{O}^a \rangle\rangle_0 \langle\langle S_{i_1} S_{i_2} \cdots S_{i_n} S_a \rangle\rangle_0 .
\]

The first factor on the right-hand side factorizes as \( |x| \rightarrow \infty \) because of the locality of the operators and because the correlation function is evaluated in a unitary quantum field theory:

\[
\langle\langle J_1^{(1)}(x + x_1) J_2^{(2)}(x + x_2) \cdots J_n^{(n)}(x + x_n) \mathcal{O}^a \rangle\rangle_0 \sim |x| \rightarrow \infty \langle\langle J_1^{(1)}(x_1) J_2^{(2)}(x_2) \cdots J_n^{(n)}(x_n) \rangle\rangle_0 \langle\langle \mathcal{O}^a \rangle\rangle_0
\]

This field theoretic factorization induces a corresponding factorization in the spin space, as we proceed to show. The expression \( \langle\langle J_1^{(1)}(x_1) J_2^{(2)}(x_2) \cdots J_n^{(n)}(x_n) \rangle\rangle_0 \) is a tensor in the product space of \( n \) copies of the fundamental
representation of the rotation group $O(3)$ and the only way to form a rotational invariant with it is to multiply it by a corresponding product of matrices $S_{i_1}S_{i_2} \cdots S_{i_n}$, i.e. \(\langle (J_{i_1}^1(x_1)J_{i_2}^1(x_2) \cdots J_{i_n}^1(x_n)) | S_{i_1}S_{i_2} \cdots S_{i_n} \rangle \propto 1\). Hence, the product $S_{i_1}S_{i_2} \cdots S_{i_n}$ also factorizes, and we have
\[
\langle (J_{i_1}^1(x_1)J_{i_2}^1(x_2) \cdots J_{i_n}^1(x_n)) \rangle_0 \langle (S_{i_1}S_{i_2} \cdots S_{i_n}) \rangle_0 \langle (S_{i_n}) \rangle_0
\]  
(3)
The factorization property then follows:
\[
\langle (\hat{J}_1(x+x_1) \cdot \hat{S} \cdot \hat{J}_2(x+x_2) \cdot \hat{S} \cdots \hat{J}_n(x+x_n) \cdot \hat{S} \cdot \bar{O}^a S_{i_n}) \rangle_0 \langle |x| \to \infty \rangle (\langle \hat{J}_1(x+x_1) \cdot \hat{S} \cdot \hat{J}_2(x+x_2) \cdot \hat{S} \cdots \hat{J}_n(x+x_n) \cdot \hat{S} \rangle S_{i_n}) \rangle_0.
\]  
(4)
The operator $H_I$ is a linear combination of operators of the type $\vec{J} \cdot \vec{S}$. The operators $\vec{J}$ are right-moving fields. Hence when $H_I$ is evolved in time with $H_0$ for a time $t$, it becomes a linear combination of $\vec{J} \cdot \vec{S}$'s at position $-t$. This shows the factorization of juxtaposed $H_I^{(0)}(t)$’s as $|t| \to \infty$ when they are evaluated inside a trace with operators at fixed time and position on the right. The same proof applies if such operator insertion is put on the left of juxtaposed $H_I^{(0)}(t)$’s. This completes the proof.

**APPENDIX C: CALLAN-SYMANZIK EQUATION FOR MATRIX ELEMENTS**

In this appendix, we will justify equation (87). Consider the finite-temperature average, in the theory described by $H_{|V=0}$, of the regularized current operator $J^\Lambda = \lambda(\hat{J}_x(0) \cdot \hat{S})$ with insertions of creation and annihilation operators of the Hamiltonian: $A_i^\dagger(p)$ and $A_i(p)$. More precisely these are creation and annihilation operators for eigenstates of $H_{|V=0}$ with energy $p$, corresponding to the massless particles “naturally” associated to the local operator $J_i(x)$, where $J_i$ is any of the ten operators $\vec{J}_d$, $\vec{J}_x$, $\vec{J}_y$, $J_z$. Thus consider,

\[
\text{Tr} \left( e^{-\beta H_{|V=0}} A_{i_1}(p_1) \cdots A_{i_m}(p_m) J^\Lambda(0) A^\dagger_{m+1}(p_{m+1}) \cdots A^\dagger_{m+n}(p_{m+n}) \right) / \text{Tr} (e^{-\beta H_{|V=0}}).
\]  
(1)

Since the creation and annihilation operators are eigenoperators of the Hamiltonian, the quantity (11) satisfies the Callan-Symanzik equation [33] (that is, in [33] we can replace the steady-state average of the current by this quantity).

But we showed that the interacting density matrix $e^{-\beta H_{|V=0}}$ can be obtained from the free one $e^{-\beta H_0}$ by evolving it for an infinite time [17]. In much the same way, the interacting creation and annihilation operators can be heuristically written in terms of the mode operators associated to the current algebra operators:

\[
A_i^\dagger(p) = S(0,-\infty)J_i^\dagger(p)S(-\infty,0), \quad A_i(p) = S(0,-\infty)J_i(p)S(-\infty,0)
\]

(where $S(t_1,t_2)$ is defined in (17)). Here, $A_i$ is any of the 10 operators $\vec{A}_d$, $\vec{A}_x$, $\vec{A}_y$ or $A_z$. Indeed, the operators $A_i^\dagger(p)$ and $A_i(p)$ written in this way heuristically satisfy the canonical commutation relations amongst them, and the appropriate commutation relations with Hamiltonian $H_{|V=0}$, written as $S(0,-\infty)H_0S(-\infty,0)$ (they are eigeneoperators of $H_{|V=0}$). Hence, the quantity (1) can be written

\[
\text{Tr} \left( e^{-\beta H_0} J_{i_1}(p_1) \cdots J_{i_m}(p_m) S(-\infty,0) J^\Lambda S(0,-\infty) J^\dagger_{m+1}(p_{m+1}) \cdots J^\dagger_{i_m+n}(p_{m+n}) \right) / \text{Tr} (e^{-\beta H_0}).
\]  
(2)

Taking the zero-temperature limit $\beta \to \infty$, this quantity reproduces all matrix elements of the current (see eqn [33]) $\vec{J}_\Lambda = S(-\infty,0)J^\Lambda S(0,-\infty)$ on the Hilbert space of the free theory $H_0$, which proves (87).

In fact, to be more precise, when going more from (2) to (1), one needs to modify slightly [2]. First, the mode operators $J_i^\dagger(p)$, $J_i(p)$ should be replaced by appropriate wave packets $\vec{J}_i^\dagger(p)$, $\vec{J}_i(p)$, obtained by integrating
the associated currents $J_i(x)$ (times the oscillating exponential $e^{-ipx}$) over space with a kernel vanishing (exponentially, say) as $|x| \gg l$ for some wave-packet extent $l$. Then, the evolution operator $S(t_0, 0)$ should be replaced by $S(t_0, 0) = S(0, t_0')$, and these two evolution operators should be put into the trace in the order in which they appear in the numerator. By using arguments involving the vanishing of connected correlation functions, it should be observed that the limits $|t_0| \gg l, \beta$ and $|t_0'| \gg l, \beta$ exist independently. Bringing both evolution operators around the density matrix $e^{-\beta H_0}$, one should then use the steps involved in proving (14). The result, up to vanishing contributions as $t_0, t_0' \gg l, \beta$, is (10), with operators $A_i^1(p)$ and $A_i(p)$ replaced respectively by

$$
\tilde{A}_i^1(p) = S(0, t_0' + i\beta)\tilde{J}_i(p)S(t_0, 0), \quad \tilde{A}_i(p) = S(0, t_0' + i\beta)\tilde{J}_i(p)S(t_0, 0).
$$

These are the operators that should correspond to asymptotic states, in the limit $|t_0|, |t_0'| \gg l \gg \beta$ with $t_0 = t_0' < 0$. In this limit, one indeed recovers (2).

**APPENDIX D: ASYMPTOTIC BEHAVIOR OF THE FUNCTION $P(w)$**

In this appendix we evaluate the asymptotic behavior of the function $P(w)$ (74). The large $w$ behavior is easy to obtain; the first term inside the parenthesis (on the right-hand side of the second equation in (74)) disappears, and the second term $-(p - 1)/(e^{nw(p - 1)} - 1)$ becomes $(p - 1)\Theta(p < 1)$: it is non-zero only for $p < 1$. Taking these contributions into account as well as the last term $e^{-p^2}$, the result is:

$$
P(\infty) = 1 - \frac{\gamma}{2}.
$$

(1)

In fact, we can also obtain the next term in the large $w$ expansion. The next contributions can be written in the following way, by expanding the integrand:

$$
\int_0^\infty \frac{dp}{p} \sum_{n=1}^\infty \left( (p + 1)e^{-nw(p+1)} + \Theta(p < 1)(p - 1)e^{nw(p-1)} - \Theta(p > 1)(p - 1)e^{nw(1-p)} \right).
$$

Interchanging integration and summation (this is valid because the integration variable $p$ is always kept in a region where the expansion of the integrand is convergent), the integral of every term of the sum can be expressed in terms of the exponential integrals. Every term can then be expanded at large $w$, giving

$$
\sum_{n=1}^\infty \left( -\frac{2}{n^2w^2} - \frac{12}{n^3w^4} - \frac{240}{n^5w^6} - \ldots \right).
$$

Now every term can be re-summed, and we get the large-$w$ asymptotic expansion (75).

The small $w$ behavior is more subtle. The integral can be divided into two parts:

$$
P(w) = \left( \int_0^1 + \int_1^\infty \right) \frac{dp}{p} \left( \frac{p+1}{e^{w(p+1)}-1} - \frac{p-1}{e^{w(p-1)}-1} + e^{-p^2} \right).
$$

(2)

The first integral can be evaluated by expanding the integrand in small $w$. The integrand goes as $-1 + e^{-p^2} + O(w)$ at small $w$, so the first integral is convergent at $w \to 0$, and in fact gives an expansion in Taylor series in $w$. In order to obtain the divergent part in $w$, we need only consider the second integral, and we can forget about the term $e^{-p^2}$. Make the transformation of variable $p \to p/w$:

$$
\int_w^\infty \frac{dp}{p} \left( \frac{p/w + 1}{e^{p+w}-1} - \frac{p/w - 1}{e^{p-w} - 1} \right).
$$

The integrand can then be expanded in $w$: this gives a Taylor series in $w^2$ starting with power $0$. Each term of this Taylor series gives a convergent integral: the asymptotic behavior of each term at $p \to \infty$ is exponentially decreasing. Moreover, each term, except for the very first one, has a behavior like $p^2$ as $p \to 0$, so that at each
order in $w^2$, except at the zeroth order, we can evaluate the integral. We thus obtain a Taylor series in $w^2$. At
the zeroth order, we have

$$\int_w^{\infty} \frac{dp}{p} \left( -2 \frac{(p-1)e^p + 1}{(e^p - 1)^2} \right).$$

We can write it as

$$\int_w^{\infty} \frac{dp}{p} \left( -2 \frac{(p-1)e^p + 1}{(e^p - 1)^2} + \Theta(p < 1) \right) + \int_w^{\infty} \frac{dp}{p} (-\Theta(p < 1)).$$

The first integral is convergent as $w \to 0$: it has again a Taylor expansion in $w$. The second integral is easy to
evaluate, and gives $\ln(w).$ Hence,

$$P(w) \sim \ln(w) \quad \text{as} \quad w \to 0.$$  (3)

In fact, we can gather the previous missing parts to get the constant term:

$$\int_0^1 (1) \, dp + \int_0^{\infty} e^{-p^2} + \int_0^{\infty} \frac{dp}{p} \left( -2 \frac{(p-1)e^p + 1}{(e^p - 1)^2} + \Theta(p < 1) \right).$$

This gives

$$\frac{\sqrt{\pi}}{2} + \gamma - \ln(2\pi) = -0.374434476...$$  (5)

so that we get $\Box$.

**APPENDIX E: INTEGRALS FOR THE TWO-LOOP CALCULATIONS**

The integrals, as they enter in $\Box$, are

$$I_4 = \frac{3iV}{8} \int dpR(p) \left( -i\pi\delta(p) + \frac{1}{p} \right) \int dqR(q) \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) \times$$

$$\int drR(r) \left( -i\pi\delta(p+q+r) + \frac{1}{p+q+r} \right) R(-p-q-r) \times$$

$$\left( -2g(p+q)(-f(p+q+r) + f(r)) + g(q)(-f(p+q+r) + f(p+r) - f(q+r) + f(r)) \right)$$

$$I_5 = -\frac{3iV}{8} \int dpR(p) \left( -i\pi\delta(p) + \frac{1}{p} \right) \int dqR(q) \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) \times$$

$$\int drR(r) \left( -i\pi\delta(p+q+r) + \frac{1}{p+q+r} \right) R(-p-q-r) \times$$

$$\left( -2g(r)(f(p+q) - f(p+q+r)) + g(r)(f(q) - f(q+r)) + g(p+r)(f(q) - f(p+q+r)) \right)$$

$$-\frac{3iV}{4} \int dpR(p)^2 \left( \frac{1}{p} \right) \int dqR(q)^2 \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) \left( -i\pi\delta(q) + \frac{1}{q} \right) pf(q)$$

$$-\frac{3iV}{4} \int dpR(p)^2 \left( -i\pi\delta(p) \right) \int dqR(q)^2 \left( i\pi\delta(q) + \frac{1}{q} \right) pf(q)$$

$$I_6 = \frac{3iV}{8} \int dpR(p) \left( -i\pi\delta(p) + \frac{1}{p} \right) \int dqR(q) \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) \times$$

$$\int drR(r) \left( -i\pi\delta(p+q+r) + \frac{1}{p+q+r} \right) R(-p-q-r) \times$$

$$\left( 2g(r)(f(p+q) - f(p+q+r)) + g(q)(-f(p+q+r) + f(p+r)) + 4 \right)$$

...
\[ I_7 = \frac{-3iV}{8} \int dp R(p)^2 \left( \frac{1}{p} \right) \int dq R(q)^2 \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) \times \\
\int \frac{1}{(p+q)R} \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) R(-p-q-r) \times \\
\left( 2g(p+q)(-f(p+q+r) + f(r)) + g(p+r)(f(q) - f(p+q+r)) - g(q+r)(f(q) + f(r)) + 1 \right) \\
- \frac{3iV}{4} \int dp R(p)^2 \left( \frac{1}{p} \right) \int dq R(q)^2 \left( -i\pi\delta(p+q) + \frac{1}{p+q} \right) \left( -i\pi\delta(q) + \frac{1}{q} \right) pf(q) \\
- \frac{3iV}{4} \int dp R(p)^2 (-i\pi\delta(p)) \int dq R(q)^2 \left( i\pi\delta(q) + \frac{1}{q} \right) pf(q) \tag{1} \]

where

\[ R(p) = R_A(Vp) = e^{-\frac{x^2}{2\Lambda^2}}. \]

\[ g(p) = \frac{1 + e^{-wp}}{1 - e^{-wp}}. \tag{2} \]

Recall that \( w = \beta V \). The function \( f(p) \) is as in (71), and is related to \( g(p) \) by

\[ f(p) = (p+1)g(p+1) - (p-1)g(p-1). \tag{3} \]

**APPENDIX F: FINITE CONTRIBUTIONS OF THE TWO-LOOP RESULTS**

They are given by

\[ [I_4 + I_5 + I_6]_{\text{finite}} = 6\pi V \int_0^\infty dq \int_0^q \frac{dr}{r} \left( g_c(q+r)(f(q) + f(r)) - g_c(q-r)(f(q) - f(r)) \right) \\
- \frac{3\pi V}{2} \int_0^\infty dq \int_0^q \frac{dr}{r} \left[ g_c(q-r)f_c(q-r) - g_c(q+r)f_c(q+r) \right] \\
+ \frac{3\pi V}{2} \int_0^\infty dq \int_0^q \frac{dr}{r} \left[ h_c(q+r) - \frac{q}{q-r}h_c(q-r)e^{-r^2} \right] \\
+ \frac{3\pi V}{2} \int_0^\infty \frac{dqdr}{qr} \left( g_c(r) + e^{-r^2} \right)(2f(q) - f(q+r) - f(q-r)) \\
- \frac{3\pi V}{2} \int_0^\infty \frac{dqdr}{qr} \left( 1 - e^{-r^2} \right)(f_c(q+r) + f_c(q-r)) \\
- 12\pi V \int_0^{\infty} \frac{dr}{r} \left( \frac{\gamma}{2} + \ln r \right) f_c(r) \\
- \frac{3\pi V}{2} \int_0^\infty \frac{dr}{r} \left( \frac{\gamma}{2} + \ln r \right) h_c(r) \\
- \frac{3\pi V}{2} \int_0^\infty \frac{dq}{q} f_c(q) \\
+ \frac{3\pi^3 V}{2} \left( \frac{3}{w}G(w) - 1 \right) \\
+ 3\pi V \left( u(1-i) + u(1+i) - u \left( \frac{1-i}{2} \right) - u \left( \frac{1+i}{2} \right) \right) \\
- \frac{\pi^3 V}{24} \]
and
\[
[I_r]_{\text{finite}} = \frac{3\pi V}{2} \int_{0}^{\infty} \frac{dq}{q} \int_{0}^{q} \frac{dr}{r} \left[ g_c(q-r)(f(q) - f(r)) - g_c(q+r)(f(q) + f(r)) \right]
+ \frac{3\pi V}{2} \int_{0}^{\infty} \frac{dq}{q} \int_{0}^{q} \frac{dr}{r} \left[ g_c(q-r)f_c(q-r) - g_c(q+r)f_c(q+r) \right]
- \frac{3\pi V}{2} \int_{0}^{\infty} \frac{dq}{q} \int_{0}^{q} \frac{dr}{r} \left[ h_c(q+r) - \frac{q}{q-r} h_c(q-r)e^{-r^2} \right]
- 6\pi V \int_{0}^{\infty} \frac{dq}{q} \int_{0}^{q} \frac{dr}{r} (1 - e^{-r^2})(1 - e^{-q^2})f_c(q+r)
+ 3\pi V \int_{0}^{\infty} \frac{dr}{r} \left( \frac{\gamma}{2} + \ln r \right) f_c(r)
+ 3\pi V \int_{0}^{\infty} \frac{dr}{r} \left( \frac{\gamma}{2} + \ln r \right) h_c(r)
- 3\pi V \int_{0}^{\infty} \frac{dq}{q} f_c(q)
+ 6\pi V \left[ u(i-1) + u(-i-1) + u \left( \frac{i-1}{2} \right) + u \left( -\frac{i-1}{2} \right) \right]
+ 3\pi V \ln 2 - \frac{5\pi^3 V}{4}.
\]

Here, we have
\[
g(p) = \frac{1 + e^{-wp}}{1 - e^{-wp}}
\]
\[
f(p) = (p+1)g(p+1) - (p-1)g(p-1)
\]
\[
g_c(p) = g(p) - \text{sign}(p)
\]
\[
f_c(p) = f(p) - 2\text{sign}(p)(1 - e^{-p^2})
\]
\[
h_c(p) = f_c(p) + (1 - e^{-p^2})g_c(p)
\]
\[
G(u) = \frac{\sinh(u) - \frac{u}{\sqrt{2}}}{\sinh^{2}(\frac{u}{\sqrt{2}})}
\]
\[
u(x) = -\text{dilog}(1-x)
\]

where \( w = V/T \).

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One could also introduce the operator $J_c = \pi (\psi^\dagger_2 \psi_2 + \psi^\dagger_1 \psi_1)$. Since the total (sum over the two channels) charge is locally conserved, the current $J_c$ commutes with all current operators introduced above, hence it decouples from the theory and can be identically set to 0.

Observe here that the operator $H|_{V=0} - VY$ need not have a spectrum bounded from below in the Hilbert space of $H|_{V=0}$. As usual, in any explicit evaluation, the traces are regularized using a regularisation that respects cyclicity. The resulting expression is well defined with insertion of local operators only and the limit implied in the definition of the operator $Y$ should be taken simultaneously in the numerator and the denominator.

By definition, $dS_\eta(t_1, t_2)/dt_2 = iS_\eta(t_1, t_2)(H_0 + e^{\eta t_2}H_I)$ and $S_\eta(t, t) = 1$.

The authors also considered the case of an oscillating applied voltage, where they had to introduce an extra scale, a “decoherence time,” in order to recover universality. We do not consider this case here.

The large-$V/T_K$ behavior in $1/V$ found at 0-loop order, then, can only be trusted in the region $V \gg T \gg T_K$. 

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[19] One could also introduce the operator $J_c = \pi (\psi^\dagger_2 \psi_2 + \psi^\dagger_1 \psi_1)$. Since the total (sum over the two channels) charge is locally conserved, the current $J_c$ commutes with all current operators introduced above, hence it decouples from the theory and can be identically set to 0.
[20] Observe here that the operator $H|_{V=0} - VY$ need not have a spectrum bounded from below in the Hilbert space of $H|_{V=0}$. As usual, in any explicit evaluation, the traces are regularized using a regularisation that respects cyclicity. The resulting expression is well defined with insertion of local operators only and the limit implied in the definition of the operator $Y$ should be taken simultaneously in the numerator and the denominator.
[21] By definition, $dS_\eta(t_1, t_2)/dt_2 = iS_\eta(t_1, t_2)(H_0 + e^{\eta t_2}H_I)$ and $S_\eta(t, t) = 1$.
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