Proposal for optically realizing quantum game

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We present a proposal for optically implementing the quantum game of the two-player quantum prisoner’s dilemma involving nonmaximally entangled states by using beam splitters, phase shifters, cross-Kerr medium, photon detector and the single-photon representation of quantum bits.
PACS number(s): 03.67.-a, 03.67.Lx, 42.50.Dv

I. INTRODUCTION

Recently, much attention has been paid to the topic of quantum games \[1,2,3,4,5,6,7,8\], which is forming a new area of study within quantum information and applications of quantum theory. In addition to their own intrinsic interest, quantum games open a new window for exploring the fascinating world of quantum information and quantum mechanics. Actually, various problems in quantum information and computation, such as quantum cryptography, quantum cloning, quantum algorithms, can be regarded as quantum games \[6\]. The quantum prisoner’s dilemma \[1\] is a famous two player quantum game which is quantization of the so-called Prisoner’s Dilemma. Eiser and coworkers \[1\] showed that with proper quantum strategies, the paradox in the classical two-player Prisoner’s Dilemma can be solved under the maximal entangled states, and discussed the generalized quantum game of the quantum prisoner’s dilemma where the players share a nonmaximally entangled states. Then Du and coworkers \[2\] experimentally realized the quantum game on a nuclear magnetic resonance quantum computer. Quantum games with multi-players \[3,6,7,8\] have also been studied to some extent. It has been found that through properly choosing quantum strategies, the players may gain more payoffs in the quantum case than in classical case, and quantum games can exhibit certain forms of nonclassical equilibria since quantum entanglement is introduced.

As well known, optical realization is one of the most effective methods for quantum information processing. The important experimental implementation of quantum cryptography \[1,2\], quantum teleportation \[10\], quantum dense coding \[11,12\] and quantum computation \[13\], is completed in quantum optical systems which generally consist of beam splitters, phase shifters, mirrors, Kerr medium, and so on. In this paper we add quantum games to the list. The purpose of the present paper is to present an optical realization of the quantum prisoner’s dilemma which involves nonmaximally entangled states. This paper is organized as follows. In Sec. II, we briefly review the quantum game of the quantum prisoner’s dilemma with nonmaximally entangled states. In Sec. III and IV, we present optical preparation of the initial state and optical realization of the strategic operations in the quantum game. Finally, we summarize our results in Sec. VI.

II. QUANTUM GAME WITH NONMAXIMALLY ENTANGLED STATES

Before going into the optical realization, let us briefly review the quantum game of the two-player quantum prisoner’s dilemma with nonmaximally entangled states \[2\]. There are two players, and the players have two possible strategies: cooperate (\(C\)) and defect (\(D\)). The payoff table for the players is shown in Table I. Classically the dominant strategy for both players is to defect (the Nash equilibrium) since no player can improve his/her payoff by unilaterally changing his/her own strategy, even though the Pareto optimal is for both players to cooperate. This is the dilemma. In the quantum version involving nonmaximally entangled states, (see Fig. 1), the game is modelled by two qubits, one for each player, with the basis states denoted by \(|C\rangle\) and \(|D\rangle\). The physical model of the quantum game consists of three ingredients: a entangling source of two qubits, a set of physical instruments that enables the players to manipulate his or her own qubits in a strategic manner, and a physical measurement device which determines the players’ payoff from the entstate of the two qubits. Starting with the product state \(|CC\rangle\) of the two players one acts on the state with the entangling gate \(\hat{J}\) to obtain an entangled state given by

\[|in\rangle = \cos \left(\frac{\gamma}{2}\right) |C\rangle |C\rangle + i \sin \left(\frac{\gamma}{2}\right) |D\rangle |D\rangle, \tag{1}\]

which acts as the initial state of the quantum game. Here \(\gamma\) measures the entanglement of the initial state, it changes from 0 (no entanglement) to \(\pi/2\) (maximal entanglement).

The two players now act with local unitary operators \(\hat{U}_A\) and \(\hat{U}_B\) on their qubits. Finally, the disentangling gate \(\hat{J}^\dagger\) is carried out and the system is measured in the computational basis, giving rise to one of the four outcomes \(|CC\rangle, |CD\rangle, |DC\rangle, \) and \(|DD\rangle\). If one allows quantum strategies of the form

\[\hat{U}(\theta, \phi) = \begin{pmatrix} e^{i\phi} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & e^{-i\phi} \cos \frac{\theta}{2} \end{pmatrix}, \tag{2}\]
where \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq \frac{\pi}{2} \), the two classical strategies \( C \) and \( D \) is correspond to \( U(0,0) \) and \( U(\pi,0) \) given by

\[
\hat{C} = \hat{U}(0,0), \quad \hat{D} = \hat{U}(\pi,0).
\]

It is worthwhile to mention that Eq. \((2)\) is a restriction of the possible unitary strategies. It is a subset of the possible strategies. This subset is being chosen because of the ease of physical implementation.

| Alice: C | Bob: C | Bob: D |
|---------|--------|--------|
| (3,3)   | (0,5)  | (1,1)  |

Du and coworkers\(^2\) showed that the game exhibits an intriguing structure as a function of the amount of entanglement with two thresholds \( \gamma_1 = \arcsin \sqrt{1/5} \) and \( \gamma_2 = \arcsin \sqrt{2/3} \) which separate a classical region, an intermediate region, and a fully quantum region when \( \gamma \) changes from 0 (no entanglement) to \( \pi/2 \) (maximal entanglement). \( 0 \leq \gamma \leq \gamma_1 \) is the classical region in which the quantum game behaves classically, i.e., the only Nash equilibrium is \( \hat{D} \otimes \hat{D} \) and the payoffs for both players are one, which is the same as in the classical game. The intermediate region is given by \( \gamma_1 \leq \gamma \leq \gamma_2 \). In this region the quantum game can not resolve the dilemma. The region of \( \gamma > \gamma_2 \) is the fully quantum region in which a unique and new Nash equilibrium is only a Nash equilibrium in the space of strategies given by Eq. \((2)\), which has the property of being Pareto optimal. In the next two sections, we will give rise to optical realization of the players’ strategic operations and the \( J \)-gate operation, respectively.

A typical two-mode mixer that preserves the total number of photons in the mode pair, is a lossless beam splitter with transparency \( T \). Denoting the input annihilation operators as \( \hat{a} \) and \( \hat{b} \), and the output’s as \( \hat{a}' \) and \( \hat{b}' \). The beam splitter transform the input \( \hat{a} \) and \( \hat{b} \) into the output \( \hat{a}' \) and \( \hat{b}' \) as following

\[
\begin{pmatrix}
\hat{a}' \\
\hat{b}'
\end{pmatrix} = \begin{pmatrix}
\cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\
-i \sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix} \begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix},
\]

where the transparency \( T = \cos^2(\theta/2) \). This is one ways to relate the input state to the output state. A beam splitter also can be described by a unitary operator with the form

\[
\hat{B}(\theta) = \exp \left[ -i\theta (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \right].
\]

If impinging on a beam splitter with one photon in one mode and zero photon in the other, the output lights are entangled through the following expressions

\[
\hat{B}(\theta)|1\rangle_a|0\rangle_b = \cos \frac{\theta}{2} |1\rangle_a|0\rangle_b - i \sin \frac{\theta}{2} |0\rangle_a|1\rangle_b,
\]

\[
\hat{B}(\theta)|0\rangle_a|1\rangle_b = \cos \frac{\theta}{2} |0\rangle_a|1\rangle_b - i \sin \frac{\theta}{2} |1\rangle_a|0\rangle_b.
\]

In the above single-photon representation of quantum bits, let \( |C\rangle = |1\rangle|0\rangle, |D\rangle = |0\rangle|1\rangle \), under the beam-splitter transformation we have

\[
\begin{align*}
\hat{B}(\theta)|C\rangle &= \cos \frac{\theta}{2} |C\rangle - i \sin \frac{\theta}{2} |D\rangle, \\
\hat{B}(\theta)|D\rangle &= \cos \frac{\theta}{2} |D\rangle - i \sin \frac{\theta}{2} |C\rangle,
\end{align*}
\]

where if we set \( \theta = \pi/2 \), it is a 50:50 beam splitter with transparency \( T = 1/2 \).

The second device we should introduce is a phase shifter in mode \( a \), which can be described by the unitary operator

\[
\hat{P}(\phi) = \exp \left[ i\phi \hat{a}^\dagger \hat{a} \right].
\]

Putting a phase shifter in mode \( a \) and mode \( b \), respectively, with the phase shifter in mode \( b \) conjugate to the

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**TABLE I:** Payoff matrix for the prisoner’s dilemma. The first entry in the parenthesis denotes the payoff of Alice and the second number is Bob’s payoff.

| Alice: C | Bob: C | Bob: D |
|---------|--------|--------|
| (3,3)   | (0,5)  | (1,1)  |

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**III. REALIZATION OF QUANTUM STRATEGIC OPERATIONS**

In this section we propose a method to realize the player’s strategy of the quantum prisoner’s dilemma with nonmaximally entangled state only by 50:50 lossless beam splitters, phase shifters and mirrors, which are easy to operate for players.

![FIG. 1: The setup of a two-player quantum game.](image)

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**FIG. 2:** A beam splitter with two inputs \( a, b \), and two outputs \( a', b' \).
respectively, can be realized by the following form
\[
\hat{P}(\phi, -\phi) = e^{i\phi}\hat{P}(\phi)\hat{P}_b(-\phi),
\]
which correspond to the rotation around the z axes by \(\phi\). The rotations around the y and x axes by \(\theta\) and \(\eta\), respectively, can be realized by the following form
\[
\hat{U}_y\left(\frac{\theta}{2}\right) = \hat{B}\left(-\frac{\pi}{2}\right)\hat{P}\left(-\frac{\theta}{2}\frac{\pi}{2}\right)\hat{B}\left(\frac{\pi}{2}\right),
\]
\[
\hat{U}_x\left(\frac{\eta}{2}\right) = \hat{P}\left(-\frac{\pi}{4}\frac{\pi}{4}\right)\hat{U}_y\left(-\frac{\eta}{2}\right)\hat{P}\left(\frac{\pi}{4}\frac{\pi}{4}\right),
\]
where the rotations around the y and x axes are defined by \(\frac{\pi}{4}\).

Then the unitary operator of the player’s strategy shown in Eq. (11), can be realized using the following identity:
\[
\hat{U}(\theta, \phi) = \hat{P}(0, -\phi)\hat{U}_y\left(-\frac{\theta}{2}\right)\hat{P}(\phi, 0),
\]
and the experiment setup is the combination of the beam splitters, phase shifters and mirrors shown in Fig. 3.

IV. REALIZATION OF THE \(\hat{J}\)-GATE OPERATION

The \(\hat{J}\) gate is the most important operation in the quantum game since it introduces quantum entanglement. In this section we show that the \(\hat{J}\) gate can be realized in terms of a sequence of beam splitters, cross Kerr medium, and phase shifters. Unitary transformations corresponding to beam splitters and phase shifters are given by Eqs. 8 and 9, respectively, while a unitary transformation which characterizes a cross-Kerr medium is given by
\[
\hat{K}(\chi) = \exp\left[-i\chi\hat{a}^\dagger\hat{a}\hat{b}\hat{b}^\dagger\right],
\]
where the nonlinear Kerr coefficient \(\chi\) is proportional to the third order susceptibility \(\chi^{(3)}\) of the medium and the interaction time within the medium. Here we assume the self-modulation terms \(\hat{a}^2\hat{a}^\dagger\hat{b}^2\) and \(\hat{b}^\dagger\hat{b}\hat{b}\) can be ignored by an appropriate choice of resonance \(\chi^{(2)}\).

We use the setup in Fig. 4 which consists of four 50:50 beam splitters, four phase shifters, one cross-Kerr medium and one mirrors. All of the BSs are 50:50 beam splitters. The cross-Kerr medium couples the lights in mode \(a\) and \(d\). The lights of Four modes first pass through BS1 and BS2, and two of them go through the cross-Kerr medium, then they change their phases by four phase shifters, whose value is connected with the nonlinear Kerr coefficient, at last they propagate through beam splitters, BS3 and BS4.

Then the unitary transformation to realize the \(\hat{J}\)-gate is indicated in Fig. 4, which consists of four 50:50 beam splitters, four phase shifters, one cross-Kerr medium and one mirrors. All of the BSs are 50:50 beam splitters. The cross-Kerr medium couples the lights in mode \(a\) and \(d\). The lights of Four modes first pass through BS1 and BS2, and two of them go through the cross-Kerr medium, then they change their phases by four phase shifters, whose value is connected with the nonlinear Kerr coefficient.

The \(\hat{J}\) gate can be expressed as the product of a sequence of beam-splitter transformations, the cross-Kerr transformation and phase-shifter transformations, which is given by
\[
\hat{J} = \hat{B}_4\left(-\frac{\pi}{2}\right)\hat{B}_3\left(-\frac{\pi}{2}\right)\hat{P}(\theta_2, -\theta_2)\hat{P}(\theta_1, -\theta_1)\times\hat{K}(\gamma)\hat{B}_2\left(\frac{\pi}{2}\right)\hat{B}_1\left(\frac{\pi}{2}\right),
\]
which can be further reduced to the following simple form
\[
\hat{J} = \exp\left[-i\frac{\gamma}{4}\left(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger\right)\left(\hat{c}^\dagger\hat{d} - \hat{c}\hat{d}^\dagger\right)\right].
\]
If we assume that the input state of the setup indicated in Fig. 4 is \(\langle CC\rangle\), then its out state is given by
\[
|\psi_0\rangle = \exp\left[-i\frac{\gamma}{4}\left(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger\right)\left(\hat{c}^\dagger\hat{d} - \hat{c}\hat{d}^\dagger\right)\right]|CC\rangle.
\]
which can be explicitly written as
\[ |\psi_0\rangle = \cos \frac{\gamma}{2} |C\rangle |C\rangle + i \sin \frac{\gamma}{2} |D\rangle |D\rangle. \] (19)

This just is the entangled state as the initial state of the quantum game \(|\in\rangle\) given by \[1\].

It is easy to verify the following commutators
\[ [\hat{J}, \hat{D} \otimes \hat{D}] = 0, \quad [\hat{J}, \hat{D} \otimes \hat{D}] = 0, \]
\[ [\hat{J}, \hat{C} \otimes \hat{D}] = 0, \] (20)

which are the subsidiary conditions which the entangling operation \(\hat{J}\) must be satisfied in order to guarantee that the ordinary prisoner’s dilemma is faithfully represented. Hence, the unitary transformation \[17\] faithfully realize the \(\hat{J}\)-gate operation to generate the initial entangled state \[11\].

V. CONCLUSION

In this paper we have presented a optical scheme to realize the quantum prisoner’s dilemma by using beam splitters, phase shifters, cross-Kerr medium and mirrors with single-photon sources. It should also be possible to use this method to realize multi-player quantum game. We believe that the practical realization of such quantum games should provide further insight into the studies of quantum networks.

Acknowledgments

This work was supported in part the National Fundamental Research Program (2001CB309310), the National Natural Science Foundation of China under Grant Nos. 90203018 and 10075018, the State Education Ministry of China and the Educational Committee of Hunan Province.

[1] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. 83 (1999) 3077.
[2] J. Du, H. Li, X. Xu, M. Shi, J. Wu, X. Zhou, and R. Han, Phys. Rev. Lett. 88 (2002) 137902.
[3] J. Du, H. Li, X. Xu, X. Zhou, and R. Han, Phys. Lett. A 302 (2002) 229.
[4] D.A. Meyer, Phys. Rev. Lett. 82 (1999) 1052.
[5] A. P. Flitney et al., Fluctuation and Noise Lett. 4 (2002) R175.
[6] S.C. Benjamin, P.M. Hayden, Phys. Rev. A 64 (2001) 030301(R).
[7] Y.M. Ma, G.L. Long, F.G. Deng, F. Li, S.X. Zhang, Phys. Lett. A 301 (2002) 117.
[8] A. Iqbal, A.H. Toor, Phys. Lett. A 293 (2002) 103.
[9] T. Jennewein, C. Simon, G. Wehls, J. Weinfurter and A Zeilinger, Phys. Rev. Lett 84 (2000) 4729.
[10] J.W. Pan, S. Gasparoni, M. Aspelmeyer, T. Jennewein and A. Zeilinger, Nature 421 (2003) 721.
[11] K. Mattle, H. Weinfurter, P.G. Kwiat, and A. Zeilinger, Phys. Rev. Lett. 76 (1997) 4656.
[12] X. Li, Q. Pan, J. Jing, J. Zhang, C. Xie, and K. Peng, Phys. Rev. Lett. 88 (2002) 047904.
[13] E. Knill, R. Laflamme, and G.J. Milburn, Nature 409 (2001) 46.
[14] R.A. Campos, B.E.A. Saleh, M.C. Teich, Phys. Rev. A 40 (1989) 1371.
[15] C.C. Gerry and R.A. Campos, Phys. Rev. A 64 (2001) 063814.
[16] N. Imoto, H. A. Haus, and Y. Yamamoto, Phys. Rev. A 32 (1985) 3387.
[17] C.C. Gerry, Phys. Rev. A 61 (2000) 043811.