Some New Exact Results for the q-State Potts Model on Ladder Graphs

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ABSTRACT

We present exact calculations of the partition function for the q-state Potts model for general q, temperature and magnetic field on strips of the square lattices of width $L_y = 2$ and arbitrary length $L_x = m$ with periodic longitudinal boundary conditions. A new representation of the transfer matrix for the q-state Potts model is introduced which can be used to calculate the determinant of the transfer matrix for an arbitrary $m \times m$ lattice with periodic boundary conditions.

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I. Introduction

The two dimensional q-state Potts models [1,2] for various q have been of interest as examples of different universality classes for phase transitions and, for q=3,4 as models for the adsorption of gases on certain substrates [3,4,5]. For q ≥ 3 the free energy has never been calculated in closed form for arbitrary temperature. It is thus of continuing value to obtain further information about the two dimensional potts model. Some exact results have been established for the model: from a duality relation, the critical point has been identified [1]. The free energy and latent heat [6,7,8], and magnetization [9] have been calculated exactly by Baxter at this critical point, establishing that the model has a continuous, second order transition for q ≤ 4 and a first order transition for q ≥ 5. Baxter has also shown that although the q = 3 model has no phase with antiferromagnetic long-range order at any finite temperature there is an antiferromagnetic critical point at T = 0 [9]. The values of the critical exponents (for the range of q where the transition is continuous) have been determined [10,11,12]. Further insight into the critical behaviour was gained using the methods of conformal field theory [13]. In this paper, a new representation for transfer matrix of the q-state Potts model is introduced. The determinant of the transfer matrices can be calculated, which corresponds to the results suggested in [14] for square lattices. There is a duality relation for the largest eigenvalues of Potts model on square lattices with periodic boundary conditions which determines a size-independent value for the reduced internal energy at the critical point. In the last section, an exact solution is obtained for the q-state Potts model on ladder graphs and in a magnetic field. The paper is organized as follows: In section II, a new representation for the transfer matrix of the q-state Potts model is given. This representation requires operator-like numbers. Determinant of the transfer matrices for m × m lattices and exact solution for the largest eigenvalue for q-state Potts model on a 2 × m square lattice with periodic boundary conditions is calculated. In section III, exact solution for the q-state Potts model in a magnetic field and on a ladder graph is given.

II. New representation for the transfer matrices

The q-state Potts model has served as a valuable model in the study of phase transition and critical phenomena. On a lattice, or more generally on a graph G, at temperature T this model is defined by the partition function:

\[ Z(G, q, a) = \sum_{\{\sigma_n\}} e^{-\beta H} \]  

(1)

with the Hamiltonian (the critical temperature of this model is half of the standard definition of the Potts model)

\[ H = -J \sum_{<i,j>} (2 \delta_{\sigma_i \sigma_j} - 1) \]  

(2)

where \( \sigma_i = 1, \cdots, q \) are the spin variables on each vertex \( i \in G \), \( \beta = (k_B T)^{-1} \); and \(<i,j>\) denotes pairs of adjacent vertices. We use the notation:

\[ k = \beta J ; a = e^k \]  

(3)
Consider an $m \times m$ square lattice with periodic boundary conditions. For the Ising model, partition function can be written as product of transfer matrices [15,16]

$$Z = Tr P^m$$

$$P = V_2 V'_1$$

where

$$V_2 = \prod_{\alpha=1}^{m} e^{k Z_\alpha Z_{\alpha+1}}$$

$$= \exp ( k \sum_{\alpha=1}^{m} Z_\alpha Z_{\alpha+1} )$$

$$V'_1 = [2 \sinh 2k]^{m \over m} \exp ( \tilde{k} \sum_{\alpha=1}^{m} X_\alpha )$$

$$X_\alpha = 1 \otimes \ldots \otimes 1 \otimes X \otimes 1 \otimes \ldots \otimes 1$$

$$Z_\alpha = 1 \otimes \ldots \otimes 1 \otimes Z \otimes 1 \otimes \ldots \otimes 1$$

where $X$ and $Z$ are $\alpha$th factors and given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\tilde{k}$ is given by the following duality relation $(q=2)$:

$$e^{-2\tilde{k}} = \frac{e^k - e^{-k}}{e^k + (q-1)e^{-k}}$$

Feynman once said that "every theoretical physicist who is any good knows six or seven different theoretical representations for the same physics" [17]. There are several representations for the transfer matrix of the q-state Potts model [18,19,20] but we introduce a new one which helps us to obtain the determinant of the matrices easily. In this new formulation, the matrices $V'_1$ and $X$ have to be replaced by:

$$V'_1 = [2 \sinh k (e^k + (q-1) e^{-k})]^{m \over m} \exp ( \tilde{k} \sum_{\alpha=1}^{m} X_\alpha )$$

$$= [2 \sinh k (e^k + (q-1) e^{-k})]^{m \over m} V_1$$

$$X = \frac{2}{q} \left( \sigma + \left(1 - \frac{q}{2}\right) I \right)$$
where $\sigma$ is a $q \times q$ matrix with zero diagonal elements and unit elements on all other entries and $I$ is a $q \times q$ unit matrix.

\[
X = \begin{pmatrix}
\frac{2}{q} - 1 & \frac{2}{q} & \ldots & \frac{2}{q} \\
\frac{2}{q} & \frac{2}{q} - 1 & \ldots & \frac{2}{q} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{2}{q} & \ldots & \frac{2}{q} - 1
\end{pmatrix}
\] (17)

$Z$ also has to be replaced by a diagonal $q \times q$ matrix and its elements satisfy the following product by definition:

\[
Z = \begin{pmatrix}
a_1 & 0 & \ldots & 0 \\
0 & a_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_q
\end{pmatrix}
\] (18)

\[a_i a_j = 2 \delta_{ij} - 1\] (19)

Expect $q = 2$ case, one cannot represent $a_i$ elements by real or complex numbers because of the defined product (19); however, elements of $V_2$ and the transfer matrix are real numbers (in calculation of $V_2$ only product of two $a_i$ elements appear). The determinant of the transfer matrix can be calculated easily.

\[
det V_1 = \exp \left[ \text{Tr} \left( \kbar \sum_{\alpha=1}^{m} X_{\alpha} \right) \right] = \exp \left[ m \kbar q^{m-1} (2 - q) \right] \] (20)

\[
det V_2 = \exp \left[ \text{Tr} \left( k \sum_{\alpha=1}^{m} Z_{\alpha} Z_{\alpha+1} \right) \right]
= \exp \left[ m k q^{m-2} \sum_{i,j} a_i a_j \right]
= \exp \left[ m k q^{m-1} (2 - q) \right] \] (21)

and

\[
det(V_2 V_1) = \exp[-m q^{m-1} (q - 2)(k + \kbar)] \] (22)

$\det(V_2 V_1)$ has a maximum at the self dual point $k = \kbar$ and finally determinant of the transfer matrix is given by,

\[
det P = [2 (e^k + (q - 1)e^{-k}) \sinh k \frac{m^2}{2} \exp[-m q^{m-1} (q - 2)(k + \kbar)]] \] (23)
For a $2 \times m$ square lattice one can calculate the largest eigenvalue of the transfer matrix using different methods [21, 22]. For the Hamiltonian (2) the largest eigenvalue for the transfer matrix is given by

$$\lambda_{\text{max}}(k, \tilde{k}) = [2 \left( e^k + (q - 1)e^{-k} \right) \sinh k ] \exp[\gamma]$$

where $\gamma$ is given by

$$\cosh \gamma = \cosh[2k] \cosh[2\tilde{k}] - \left( \frac{q-2}{q} \right) \sinh[2k] \sinh[2\tilde{k}]$$

The above solution for $q=2$ is equal to the exact solution for a $2 \times m$ square lattice obtained by Onsager [23]. For an arbitrary $m \times m$ lattice, the largest eigenvalue for $q$ state Potts model has a general form which is the result of duality, for duality in Potts model see[18,24].

$$\lambda_{\text{max}}(k, \tilde{k}) = [2 \left( e^k + (q - 1)e^{-k} \right) \sinh k ] \frac{m}{2} f(k, \tilde{k})$$

where $f(k, \tilde{k})$ is a symmetric function of $k$ and $\tilde{k}$ and is not known ($q \geq 3$) for an arbitrary $m \times m$ lattice.

Using the same method, one can prove the formula for the determinant of the transfer matrix which has been conjectured in reference [14]. The Hamiltonian is defined by

$$H = -J \sum_{<i,j>} \delta_{\sigma_i \sigma_j}$$

and the matrices $X$ and $V'_1$ and the duality relation are given by

$$e^{-\tilde{k}} = \frac{e^k - 1}{e^k + (q - 1)}$$

$$V'_1 = [e^k - 1]^m \exp(\tilde{k} \sum_{\alpha=1}^{m} X_{\alpha})$$

$$= [e^k - 1]^m V_1 ;$$

$$X = \frac{2}{q} \left( \sigma + \left( 1 - \frac{q}{2} \right) I \right)$$

and $V_2$ is given by the same formula as before. The definition of the product of the elements of the $Z$ matrix is different and is defined by

$$a_i a_j = \delta_{ij}$$

After straightforward calculation the determinant of the transfer matrix in this case is given by

$$\det P = (e^k - 1)^m q^m (e^k e^{\tilde{k}})^m q^{m-1}$$
II. Potts model on ladder graphs and in a magnetic field

Exact solution of the two dimensional Ising model in a magnetic field is not known; however, on a one dimensional lattice $1 \times m$ the partition function can be calculated (e.g. see [25]). The simplest generalization is to solve this problem for ladder graphs $(2 \times m)$ with longitudinal periodic boundary condition. There could be several applications for this solution. On a ladder graph the transfer matrix for the $q$ state Potts model is of order $2^q$. For $q = 2$, the transfer matrix is of order 4 and the characteristic equation can be obtained. It is possible to do the same thing for 3, 4, 5, and 6 state Potts model and then generalize it to an arbitrary $q$. Consider the following Hamiltonian for the $q$ state Potts model on a ladder graph with longitudinal periodic boundary conditions.

$$H = \sum_{<i,j>} (-J \delta_{\sigma_i,\sigma_j} - h \delta_{\sigma_i,1})$$ (37)

where $h$ is the magnetic field. The partition function is given by

$$Z(G, q, x, y) = \sum_{\{\sigma_n\}} e^{-\beta H}$$ (38)

where

$$x = e^{\beta J}, \ y = e^{\beta h}$$ (39)

Our result for the partition function of the $q$ state Potts model on a $2 \times m$ lattice with longitudinal periodic boundary condition is given by,

$$Z = \sum_{j=1}^{4} \lambda_{4,j}^m + (q-2) \sum_{j=1}^{3} (\lambda_{3,j}^m + \lambda_{2,j}^m) + (q^2 - 5q + 5) \lambda_{1,1}^m + \lambda_{1,1}'^m$$ (40)

where $\lambda_{i,j}$ is the jth root of an equation of order i. $\lambda_{1,1}'$ is a root for the following first order equation,

$$\lambda + (q - 1) y + (2 - q) xy - x^2 y = 0$$ (41)

$\lambda_{1,1}$ is a root for the following equation,

$$(\lambda - x^2 + 2x - 1)^{q^2 - 5q + 5} = 0$$ (42)

The terms $\lambda_{2,j}$ for $j = 1, 2$ are the roots of the second order equation,

$$[\lambda^2 + c_{21} \lambda + c_{22}]q^{-2} = 0$$ (43)

with

$$c_{21} = (q - 2) + (3 - q)x - x^2 + xy - x^2y$$ (44)

$$c_{22} = (1 - q)y + (3q - 4)xy - (3q - 6)x^2y + (q - 4)x^3y + x^4y$$ (45)
The terms $\lambda_{3,j}$ for $j = 1, 2, 3$ are roots of the cubic equation,

$$\left(\lambda^3 + \lambda^2 c_{31} + \lambda c_{32} + c_{33}\right)^{q-2} = 0 \quad (46)$$

with

$$c_{31} = q - 4 + (6 - q)x - x^2 - x^3 + xy - x^2y \quad (47)$$

$$c_{32} = (2 - q)x + (3q - 7)x^2 - (3q - 9)x^3 + (q - 5)x^4 + x^5 +$$

$$(3 - q)y + (17 - 3q)x^2y + (q - 9)x^3y + x^5y + (3q - 12)xy \quad (48)$$

$$c_{33} = -xy(x - 1)^5(x + q - 1) \quad (49)$$

The terms $\lambda_{4,j}$ for $j = 1, 2, 3, 4$ are roots of an algebraic equation of degree four,

$$\lambda^4 + c_{41}\lambda^3 + c_{42}\lambda^2 + c_{43}\lambda + c_{44} = 0 \quad (50)$$

with

$$c_{41} = -x^3y^2 - x^3 - x^2 - (q^2 - 5q + 7) - (3q - 8)x - (x^2 + (q - 2)x + (q - 1))y \quad (51)$$

$$c_{42} = (x - 1)(q + x - 1)x^3 y^3 + (x - 1)(x^4 + 2x^3 + (q - 3)x^2 + (q - 1)x + (x - 1)y^2 +$$

$$(q - 5)(q - 4)(q - 3)(x - 1)y + (x - 1)(x^4 + qx^3 + (6q - 13)x^2 + (4q^2 - 18q + 21)x +$$

$$(6q^2 - 35q + 51))y + (q + x - 2)^2(x - 1)^2x \quad (52)$$

$$c_{43} = -(x(x - 1)^3((q + x - 1)x^3 + 3x^2 + (3q - 5)x + (q - 2)(q - 1))y^3 +$$

$$(q + x - 1)(x^2 + (q - 2)x + (q - 1))xy^2 + (q + x - 2)^2(q + x - 1)y) \quad (53)$$

$$c_{44} = (x - 1)^5(q + x - 1)^3x^2y^3 \quad (54)$$

This solution is interesting as one may use it to generalize the results of [26,27] and to identify exact location of the partition function zeros, namely Yang-Lee[28], Fisher(complex temperature)[29] and Potts (complex q) zeros for the Potts model on the ladder graphs. Another application could be a generalization of the arguments given in [30,31] from one dimensional lattices to the ladder graphs.

V. Conclusion

In this work a new representation for the transfer matrix of the q state Potts model on an square lattice is introduced. Exact result for the partition function of the q state Potts model for general q, temperature and magnetic field on strips of the square lattices $2 \times m$ with periodic boundary conditions are given. It will be interesting to generalize some exact results for the one dimensional lattices [27,30,31] to the ladder graphs.

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References

[1] R. B. Potts, Proc. Camb. Phil. Soc. 48, 106 (1952).
[2] F. Y. Wu, Rev. Mod. Phys. 54 (1982) 235.
[3] S. Alexander, Phys. Lett. A54, 353 (1975).
[4] A. N. Berker, S. Oslund, and F. Putnam, Phys. Rev. B17, 3650 (1978).
[5] E. Domany et al, Phys. Rev. B18, 2209 (1978).
[6] R. J. Baxter, J. Phys. C 6 L445 (1973).
[7] R. J. Baxter et al, Proc. Roy. Soc. London, Ser. A 358, 535 (1978).
[8] R. J. Baxter, J. Stat. Phys. 28, 1 (1982).
[9] R. J. Baxter, Proc. Roy. Soc. London, Ser. A 383, 43 (1982).
[10] M. P. M. den Nijs, J. Phys A 12, 1825 (1979); Phys. Rev. B27, 1674.
[11] J. L. Black and V. J. Emery, Phys. Rev. B23, 429 (1981).
[12] B. Nienhuis, J. Appl. Phys. 15, 199 (1982).
[13] V. S. Dotsenko, Nucl. Phys. B235, 54 (1984), and refs therein.
[14] S. Chang and R. Schrok, Physica A 296, 234-288 (2001). cond-mat/0011503.
(e.g. Eq. 3.4.39)
[15] K. Huang, Statistical Physics, 2nd edition, (Wiley). Chapter 15.
[16] B. Bergersen and M. Plischke, Equilibrium Statistical Physics, 2nd edition,
(World Scientific). Chapter 5.
[17] R. Feynman, The character of Physical Law (MIT Press, 1965), p. 168.
[18] L. Mittag and M. J. Stephen. J. Math. Phys. Vol. 12, No. 3 (1971).
[19] R. J. Baxter, Exactly Solved Models in St. Phys. (Academic Press, 1982).
[20] P. Martin, Potts Models and related Problems in St. Phys. (World Scientific).
[21] R. Shrock, Physica A 283, 388-446 (2000); cond-mat/0001389.
[22] M. A. Yurishchev, cond-mat/0111401.
[23] L. Onsager, Phys. Rev. 65, 117-49 (1944).
[24] F. Y. Wu and Y. K. Wang, J. Math. Phys. 17, 439 (1976).
[25] F. Y. Wu, cond-mat/9805301.
[26] Z. Glumac and K. Uzelac, J. Phys A: Math. Gen. 27 7709-17 (1994).
[27] R.G. Ghulghazaryan and N.S. Ananikian. cond-mat/0204424.
[28] C. N. Yang and T. D. Lee, Phys. Rev. 87 404-410, (1952).
[29] M. E. Fisher, Lectures in Theoretical Physics, Vol 7 C, ed W. E. Brittin
(Boulder, Co: University of Colorado Press) pp 1.
[30] P.B. Dolan and D. Johanston, 1D Potts, Yang-Lee Edges and Chaos; cond-mat/0010372.
[31] B. P. Dolan, D. A. Johanston, R. Kenna, The Information Geometry of the
One Dimensional Potts Model; cond-mat/0207180.