Vacuum densities and zero-point energy for fields obeying Robin conditions on cylindrical surfaces

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Abstract. The Casimir effect for general Robin conditions on the surface of a cylinder in \( D \)-spacetime dimensions is studied for massive scalar field with general curvature coupling. The energy distribution and vacuum stress are investigated. We separate volumic and superficial energy contributions, for both interior and exterior space regions. The possibility that some special conditions may be energetically singled out is indicated.

1 Introduction

The whole scientific production related to the Casimir effect has already a quite impressive volume (for reviews see e.g. refs.[1]). This effect describes the energy variation undergone by a quantum system in which the field modes are constrained by specific conditions on a given boundary. The most often studied types of boundary and conditions are those associated to well known problems, e.g., plates, spheres, and vanishing conditions, perfectly conducting conditions, etc.

Our aim in the present paper is to analyze a combination of boundary geometry and conditions which hasn’t attracted the researcher’s attention up to now. We are referring to general homogeneous —or ‘Robin’— boundary conditions on infinitely long cylindrical surfaces. Such conditions are an extension of the ones imposed on perfectly conducting boundaries and may, in some geometries, be useful for depicting the presence of interfaces between dielectric media. Robin type of boundary conditions also appear in considerations of the vacuum effects for a confined charged scalar field in external fields [2] and in quantum gravity [3, 4]. Unlike ‘purely-Neumann’ boundary conditions, they can be made conformally invariant. The Casimir energy for cylindrical geometry can be important to the flux tube model of confinement [5, 6] and for determining the structure of the vacuum state in interacting field theories [7]. The cylindrical problem with perfectly conducting conditions was first considered in ref.[8] (see also refs.[9], [10]). While the earliest studies have focused on global quantities, such as the total energy and stress on a shell, the local characteristics of the corresponding electromagnetic vacuum are considered in [11] for the interior and exterior regions of a conducting cylindrical shell, and in [12] for the region between two coaxial shells (see also [13]).

In this paper we will study the vacuum expectation values of the energy-momentum tensor for the massive scalar field with general curvature coupling satisfying Robin boundary condition on the surface of a cylindrical shell in arbitrary spatial dimensions. As for other geometries, an important question (see, for instance, [14]) is the relation between the mode sum energy, as a renormalised sum of the zero-point energies for each normal mode, and the volume integral of the renormalised energy density. We will show that for the geometry under consideration these two quantities are different and will interpret this difference as a result of the additional surface energy contribution (similar considerations for the plane and spherical geometries were made in

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The paper is organized as follows. Sec. 2 is devoted to the energy density inside a cylindrical shell, from the expectation values of the energy-momentum tensor. Calculations of this nature involve infinite sums. The summation method applied in this case is based on a variant of the generalized Abel-Plana formula [13] together with an adequate cutoff function. In sec. 3, the integrated energy inside the cylinder is calculated. One finds that, in general, the total vacuum energy differs from the integral of the volumic density. This is explained by the presence of a surface contribution located at the boundary. The part corresponding to the modes propagating in the exterior is considered in sec. 4. Then, it is possible to sum both pieces and obtain a formal expression for the total integrated Casimir energy. In sec. 5 we study the possibilities of performing actual evaluations of this magnitude. Zeta function is proposed as a summation method, and its working is illustrated in the example of $D = 3 + 1$ and zero mass. The ensuing ending remarks are included into sec. 6.

### 2 Casimir energy density and stress inside a cylindrical shell

We will consider a scalar field $\varphi$, with curvature coupling $\xi$, satisfying a generic Robin boundary condition

$$ (A + B n^i \nabla_i) \varphi(x) = 0 $$

on the cylindrical shell with radius $a$. Here $n^i$ is the normal to the boundary surface, $\nabla_i$ is the covariant derivative operator, $A$ and $B$ are constants. The results in the following will depend on the ratio of these coefficients only. However, to keep the transition to the Dirichlet and Neumann cases transparent we will use the form (2.1). The corresponding field equation is in form

$$ \left( \nabla^i \nabla_i + \xi R + m^2 \right) \varphi(x) = 0, $$

where $R$ is the scalar curvature for the background spacetime, $m$ is the mass for the field quanta. By using this equation, in the case of the flat background the corresponding metric energy-momentum tensor (EMT) may be presented in the form

$$ T_{ik} = \nabla_i \varphi \nabla_k \varphi + \left[ (\xi - \frac{1}{4} \xi R + m^2) - \xi \nabla_i \nabla_k \right] \varphi^2. $$

The vacuum expectation values (v.e.v.’s) for these quantities can be derived by evaluating the mode sum

$$ \langle 0 | T_{ik}(x) | 0 \rangle = \sum_{\alpha} T_{ik} \{ \varphi_{\alpha}(x), \varphi_{\alpha}^*(x) \}. $$

Here $\{ \varphi_{\alpha}(x) \}$ is a complete orthonormal set of positive frequency solutions to the field equation with quantum numbers $\alpha$, satisfying the boundary condition (2.1).

In this section we will consider the scalar vacuum inside a cylindrical shell in $D$-dimensional spacetime. In accordance with the problem symmetry we will take cylindrical coordinates $(r, \phi, z_1, \ldots, z_N)$, $N = D - 3$. The corresponding eigenfunctions to (2.2) have the form

$$ \varphi_{\alpha}(x) = \beta_{\alpha} J_n(\gamma r) \exp \left( i n \phi + i k r^\parallel - i \omega t \right), \quad \omega = \sqrt{\gamma^2 + k^2 + m^2}, $$

where $r^\parallel = (z_1, \ldots, z_N)$ and the coefficients $\beta_{\alpha}$ are determined from the orthonormality condition. The eigenvalues for the quantum number $\gamma$ are quantized by the boundary condition (2.1) on the cylinder surface $r = a$. From this condition it follows that the possible values of $\gamma$ are solutions to the equation

$$ J_n(\gamma a) = AJ_n(\gamma a) + B \gamma J'_n(\gamma a) = 0. $$
Here and below we use the notation
\[ \tilde{f}(z) \equiv Af(z) + (B/a)z f'(z) \] (2.8)
for a given function \( f(z) \). It is well known that for real \( A \) and \( B \) the all zeros for \( \tilde{f}(z) \) are simple and real. Let \( z = \lambda_{n,l}, l = 1, 2, \ldots \) be the corresponding positive zeros, arranged in ascending order: \( \lambda_{n,l} < \lambda_{n,l+1} \). Now the possible values for \( \gamma \) can be expressed as \( \gamma = \lambda_{n,l}/a \).

From the orthonormality condition
\[ \int dV \varphi_{\alpha}(x) \varphi_{\alpha}^*(x) = \frac{1}{2\omega} \delta_{nn'} \delta_{ll'} \delta(k - k'), \] (2.9)
where the integration goes over the region inside a cylinder, one finds for the coefficients \( \beta_{\alpha} \)
\[ \beta_{\alpha}^2 = \frac{\gamma T_n(\gamma a)}{\omega a (2\pi)^{N+1}}, \] (2.10)
and, following [13], we have introduced the notation
\[ T_n(z) = \frac{z}{(z^2 - n^2) J_n^2(z) + z^2 J_n^2(z)}. \] (2.11)

Using eigenfunctions (2.3) for the v.e.v. of the two-field product (positive frequency Wightmann function) we have
\[ \langle 0|\varphi(x)\varphi(x')|0 \rangle = \sum_{\alpha} \varphi_{\alpha}(x)\varphi_{\alpha}^*(x') = \frac{1}{(2\pi)^N a^2} \sum_{\alpha} \frac{\lambda_{n,l} T_n(\lambda_{n,l})}{\sqrt{\lambda_{n,l}^2 + k^2 + m^2}} \times \]
\[ \times J_n(\lambda_{n,l} r/a) J_n(\lambda_{n,l} r'/a) \exp \left[ in(\phi - \phi') + ik(\mathbf{r} - \mathbf{r}') - i\omega(t - t') \right], \]
where
\[ \sum_{\alpha} = \int d^N k \sum_{n=-\infty}^{+\infty} \sum_{l=1}^{\infty}. \] (2.13)

Substituting this expression into (2.4) for the v.e.v. of the EMT one finds
\[ \langle 0|T^k_{\alpha} |0 \rangle = \text{diag}(\epsilon, -p_1, -p_2, -p_3, \ldots, -p_{D-1}). \] (2.14)

Here \( \epsilon \) is the vacuum energy density, \( p_1, p_2, p_3 = p_4 = \cdots = p_{D-1} \) are effective pressures in the radial, azimuthal and longitudinal directions, respectively (vacuum stresses). These quantities are determined by the relations
\[ q(r) = \frac{1}{(2\pi)^{N+1} a^2} \sum_{\alpha} \frac{\lambda_{n,l}^3 T_n(\lambda_{n,l})}{\sqrt{\lambda_{n,l}^2 + k^2 a^2 + m^2 a^2}} f_n^q[\lambda_{n,l}] = \text{diag}(\epsilon, p_1, \ldots, p_{D-1}), \] (2.15)
where, for a given function \( f(z) \), we have introduced the notations
\[ f_n^{(e)}[f(z)] = \left[ 1 + \frac{k^2 + m^2}{z^2} r^2 \right] f^2(z) - \left( 2\xi - \frac{1}{2} \right) f^2(z) + \left( \frac{n^2}{z^2} - 1 \right) f^2(z), \] (2.16)
\[ f_n^{(p_1)}[f(z)] = \frac{1}{2} \left[ f^2(z) - \left( \frac{n^2}{z^2} - 1 \right) f^2(z) \right] + \frac{2\xi}{z} f(z) f'(z), \] (2.17)
\[ f_n^{(p_2)}[f(z)] = \left( 2\xi - \frac{1}{2} \right) \left[ f^2(z) + \left( \frac{n^2}{z^2} - 1 \right) f^2(z) \right] - \frac{2\xi}{z} f(z) f'(z) + \frac{n^2}{z^2} f^2(z), \] (2.18)
\[ f_n^{(p_3)}[f(z)] = \left( 2\xi - \frac{1}{2} \right) \left[ f^2(z) + \left( \frac{n^2}{z^2} - 1 \right) f^2(z) \right] + \frac{k^2 r^2}{N z^2} f^2(z), \] (2.19)
Integrating over $k$ by using the formula

$$
\int \frac{k^s d^Nk}{\sqrt{k^2 + c^2}} = e^{N+s-1} \pi^{(N-1)/2} \frac{\Gamma(N/2)}{\Gamma(N/2)} \Gamma\left(\frac{N+s-1}{2}\right) \Gamma\left(\frac{N+s}{2}\right)
$$

we can see that

$$
p_i = -\varepsilon, \quad i = 3, \ldots, D - 1.
$$

The v.e.v.'s (2.15) are divergent. To make them finite we introduce the cutoff function $\psi_\mu(\gamma)$, which decreases with increasing $\gamma$ and satisfies the condition $\psi_\mu \to 1$, $\mu \to 0$. To extract the divergent parts we will apply to the corresponding sums over $l$ the summation formula [13]

$$
\sum_{l=1}^{\infty} T_n(\lambda_n l) f(\lambda_n l) = \frac{1}{2} \int_0^{\infty} \frac{f(x)}{\sqrt{x^2 + c^2}} dx - \frac{1}{2\pi} \int_0^{c} \frac{K_n(x)}{I_n(x)} \frac{e^{-n\pi i} f(ix) + e^{n\pi i} f(-ix)}{\sqrt{c^2 - x^2}} +
$$

$$
+ \frac{i}{2\pi} \int_c^{\infty} \frac{K_n(x)}{I_n(x)} \frac{e^{-n\pi i} f(ix) - e^{n\pi i} f(-ix)}{\sqrt{x^2 - c^2}}.
$$

This formula is valid for functions $f(z)$ satisfying the conditions

$$
|f(z)| < \epsilon(x)e^{c|y|}, \quad z = x + iy, \quad c < 2,
$$

$$
f(z) = o(z^{2|n|-1}), \quad z \to 0,
$$

where $\epsilon(x) \to 0$ for $x \to \infty$.

To evaluate the mode sum over $l$ in (2.15) in (2.22) as a function $f(z)$ we choose

$$
f(z) = z^3 f_n(q)[J_n(zr/a)]\psi_\mu(z/a).
$$

We will assume a class of cutoff functions for which (2.23) satisfies conditions (2.23) and (2.24) uniformly with respect to cutoff parameter $\mu$ (this is the case, for instance, for $e^{-\mu^2}$, $\mu > 0$).

Using the relation $f_n(q)[J_n(-izr/a)] = e^{2n\pi i} f_n(q)[J_n(izr/a)]$ we see that the subintegrand of the first integral on the right hand side of formula (2.22) is proportional to $\psi_\mu(iz/a) - \psi_\mu(-iz/a)$. Consequently, after removing the cutoff ($\psi_\mu \to 1$) the contribution of the first integral will be zero. Hence, omitting this integral for the v.e.v. we obtain

$$
q = \frac{1}{(2\pi)^{N+1}a^3} \int d^Nk \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2} \int_0^{\infty} dz \frac{z^3 f_n(q)[J_n(zr/a)]\psi_\mu(z/a)}{\sqrt{z^2 + k^2a^2 + m^2a^2}} +
$$

$$
+ e^{-n\pi i} \frac{1}{\pi} \int_{a\sqrt{k^2 + m^2}}^{\infty} \frac{K_n(z)}{I_n(z)} \frac{z^3 f_n(q)[J_n(izr/a)]\chi_\mu(z/a)}{\sqrt{z^2 - k^2a^2 - m^2a^2}} \right\},
$$

where $\chi_\mu(y) = [\psi_\mu(iy) + \psi_\mu(-iy)]/2$. The second integral on the right of this formula vanishes in the limit $a \to \infty$, whereas the first one does not depend on $a$. It follows from here that the latter corresponds to the spacetime without boundaries. This can be also seen directly by explicit summation over $n$ using the formula

$$
\sum_{n=-\infty}^{+\infty} J_{n+1}^2(z) = 1.
$$

For instance, in the case of the energy density it follows from here that

$$
\sum_{n=-\infty}^{+\infty} f_n(q)[J_n(zr)] = 1 + \frac{k^2 + m^2}{z^2}.
$$
Using this relation for the term corresponding to the first integral on the right of (2.26) and removing the cutoff one has

\[
e^{(0)} = \frac{1}{2(2\pi)^{N+1}} \int d^N k \sum_{n=-\infty}^{+\infty} \int_0^\infty dz \frac{z^3 f_n(z)}{z^2 + k^2 + m^2} j_n(z r/a) = (2.28)
\]

\[
e^{(0)} = \frac{1}{2(2\pi)^{N+1}} \int d^N k \int_0^\infty dz z \sqrt{z^2 + k^2 + m^2} = \frac{1}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sqrt{k^2 + m^2}.
\]

Hence summation formula (2.22) allows us to extract the contribution of the unbounded space without specifying the cutoff function. In the remaining part, the integration over \(k\) can be explicitly done using the formula

\[
\int d^N k \int_{|k^2 + m^2|}^\infty dz \frac{k^s g(z)dz}{\sqrt{z^2 - k^2 - m^2}} = \frac{\pi^{N/2}}{\Gamma(N/2)} \frac{B\left(\frac{N + s}{2}, 1\right)}{2} \int_0^\infty dz \left(z^2 - m^2\right)^{(N+s-1)/2} g(z),
\]

where \(B(x, y)\) is the Euler beta function (this formula can be proved by integrating over the angular part of \(d^N k\) and introducing polar coordinates in the plane \((x = \sqrt{z^2 - k^2 - m^2}, k)\)). Removing the cutoff for this part one obtains

\[
q_{SUB} = q - q^{(0)} = \frac{2^{2-D} \pi^{-D/2}}{a^D \Gamma(D/2 - 1)} \sum_{n=-\infty}^{+\infty} \int_0^\infty dz z^3 \left(z^2 - a^2 m^2\right)^{D/2 - 2} \frac{K_n(z)}{I_n(z)} F_n^{(q)}[I_n(zr/a)],
\]

where \(I_n(z)\) and \(K_n(z)\) are the modified Bessel functions, and

\[
F_n^{(e)}[f(z)] = \frac{1}{D-2} \left(1 - \frac{m^2}{z^2} r^2\right) f^2(z) + \left(2\xi - \frac{1}{2}\right) \left[f'^2(z) + \left(\frac{n^2}{z^2} + 1\right) f^2(z)\right]
\]

\[
F_n^{(pv)}[f(z)] = \frac{1}{2} \left[\left(\frac{n^2}{z^2} + 1\right) f^2(z) - f'^2(z) + \frac{2K}{z} f(z)f'(z)\right]
\]

\[
F_n^{(p)}[f(z)] = -\left(2\xi - \frac{1}{2}\right) \left[f'^2(z) + \left(\frac{n^2}{z^2} + 1\right) f^2(z)\right] + \frac{2K}{z} f(z)f'(z) - \frac{n^2}{z^2} f^2(z)
\]

\[
F_n^{(p)}[f(z)] = -F_n^{(e)}[f(z)], \quad i = 3, \ldots, D - 1.
\]

In (2.30) \(q^{(0)}\) are the components of the v.e.v. \(\langle 0_M | T_k^I | 0_M \rangle\) over the vacuum \(|0_M\rangle\) corresponding to the unbounded \(D\)-dimensional Minkowskian spacetime. It can be seen that v.e.v.’s (2.30) satisfy the continuity equation \(\nabla_i T_k^I = 0\), which, for the cylindrical geometry, takes the form

\[
\frac{dp_i}{dr} + \frac{1}{r} (p_1 - p_2) = 0.
\]

For \(D = 3\) formulae (2.30) coincide with the corresponding results in [16] for the spherical case.

Formule (2.30) can be obtained by another equivalent way, applying a certain second-order differential operator on the regularized Wightmann function and taking the coincidence limit. The calculation of the Wightmann function is of interest for its own sake, as this function determines the response of a particle detector moving through the vacuum under consideration [17]. For this reason here we derive formula for the regularized Wightmann function. To do this we apply summation formula (2.22) to the sum over \(l\) in (2.12). It can be shown that the first integral on the right of (2.23) will give the corresponding Wightmann function for \(D\)-dimensional Minkowskian spacetime, and the second integral vanishes. Integrating over \(k\) with the help of standard integrals, in the term corresponding to the third integral one obtains

\[
\langle 0|\varphi(x)\varphi(x')|0\rangle = \langle 0_M|\varphi(x)\varphi(x')|0_M\rangle - \sum_{n=-\infty}^{+\infty} \frac{e^{in(\varphi - \varphi')}}{(2\pi)^{D/2}} \int_0^\infty dz z(z^2 - m^2)^{D/4 - 1} \times (2.36)
\]
\[ I_n(zr)I_n(zr') \frac{\bar{K}_n(z)}{I_n(z)} \frac{J_D/2 - 2}{[(\mathbf{r}-\mathbf{r}')^2 - (t-t')^2]^{D/4-1}} \]

It can be easily seen that for the case of a circle \((D = 3)\) this formula coincides with the corresponding expression for the spherical case derived in [14].

Let's consider vacuum densities (2.30) on the cylinder axis, \(r = 0\). Now the only contributions come from the summands with \(n = 0, \pm 1\) and one has

\[
q(r = 0) = \frac{2^{2-D} \pi^{-D/2}}{a^D \Gamma(D/2 - 1)} \int_{\infty}^{\infty} \frac{dz}{z^3} \left( z^2 - a^2m^2 \right)^{D/2-2} \left\{ u_{q0} \frac{\bar{K}_0(z)}{I_0(z)} + u_{q1} \frac{\bar{K}_1(z)}{I_1(z)} \right\},
\]

where

\[
\begin{align*}
u_{z0} &= \frac{1 - m^2a^2z^2}{D - 2} + 2\xi - \frac{1}{2}, & u_{z1} &= 2\xi - \frac{1}{2} \\
u_{p10} &= u_{p20} = -\xi + \frac{1}{2}, & u_{p11} &= u_{p21} = -\xi.
\end{align*}
\]

The results of the corresponding numerical evaluation for the Dirichlet and Neumann minimally and conformally coupled scalars in \(D = 4\) are presented in Fig.1.

![Figure 1](Figure 1: The Casimir energy density, \(a^D \varepsilon\), (a, c) and vacuum radial pressure, \(a^D p_1\), (b, d) on the cylinder axis for minimally (left) and conformally (right) coupled Dirichlet (a, b) and Neumann (c, d) scalars in \(D = 4\) as functions of \(ma\).)

In the large mass limit, \(ma \gg 1\), using the asymptotic formulae for the modified Bessel function for large values of argument in the leading order from (2.37) one obtains

\[
\varepsilon(r = 0) \sim -2p_1(r = 0) \sim \frac{4\xi - 1}{2^{D-1}\pi D/2-1a^D}(am)^{D/2+1}e^{-2am}(2\delta B_0 - 1), \quad ma \gg 1.
\]

The v.e.v.’s \((2.30)\) diverge on the cylinder surface (for a given \(n\) the \(z\)-integrals diverge as \((a - r)^{1-D}, r \to a\)). The corresponding asymptotic behaviour can be found using the uniform
asymptotic expansions for the modified Bessel functions, and the leading terms have the form

$$\varepsilon \sim - p_2 \sim \frac{(D - 1)(\xi - \xi_0)\Gamma(D/2)}{2^{D-1}\pi^{D/2}(a - r)^D}(2\delta_{B0} - 1)$$

and

$$p_1 \sim - \frac{(\xi - \xi_0)\Gamma(D/2)}{2^{D-1}\pi^{D/2}(a - r)^D}(2\delta_{B0} - 1).$$

Note that these terms are independent of the mass and Robin coefficients. Taking the limit \(a, r \to \infty, a - r = \text{const}\) one obtains the leading terms for the asymptotic behaviour near the single plate considered in ref.[15]. The surface divergences in the renormalized expectation values for the vacuum EMT are well known in quantum field theory. In the case of \(D = 3\) massless fields the corresponding asymptotic series near arbitrary smooth boundary are presented in refs.[14, 18].

### 3 Total Casimir energy inside a cylinder

The unregularized integrated Casimir energy inside a cylindrical shell may be obtained by integrating \((2.15)\) for \(q = \varepsilon\). Using standard formulae for the integrals involving the Bessel functions this energy can be presented in the form

$$E^{(\text{vol})}_{in} = E_{in} - \left(2\xi - \frac{1}{2}\right)\left\langle \frac{L}{2\pi} \right\rangle^N \frac{1}{a} \sum_{\alpha} \frac{\lambda^2_{n,l} T_n(\lambda_{n,l})}{\sqrt{\lambda^2_{n,l} + k^2 a^2 + m^2 a^2}} J_n(\lambda_{n,l}) J'_n(\lambda_{n,l}),$$

where

$$E_{in} = \frac{1}{2} \left\langle \frac{L}{2\pi} \right\rangle^N \frac{1}{a} \sum_{\alpha} \frac{\lambda^2_{n,l} T_n(\lambda_{n,l})}{\sqrt{\lambda^2_{n,l} + k^2 a^2 + m^2}}$$

is the total Casimir energy inside a cylinder, evaluated as a sum of the zero-point energies for each normal mode of frequency. As we see, for the general Robin case, \((3.1)\) differs from the total vacuum energy. The reason for this difference is the existence of an additional surface energy contribution to the volume energy, located on the boundary \(r = a - \text{const}\):

$$E^{(\text{surf})}_{in} = \left(2\xi - \frac{1}{2}\right)\left\langle \frac{L}{2\pi} \right\rangle^N \frac{1}{a} \sum_{\alpha} \frac{\lambda^2_{n,l} T_n(\lambda_{n,l})}{\sqrt{\lambda^2_{n,l} + k^2 a^2 + m^2 a^2}} J_n(\lambda_{n,l}) J'_n(\lambda_{n,l}).$$

To see this, note that there is a surface energy density contribution to the vacuum energy density \([18]\):

$$T_{00}^{(\text{surf})} = - \left(2\xi - \frac{1}{2}\right) \delta(r - a + 0) \varphi n^i \partial_i \varphi. \quad (3.4)$$

The corresponding v.e.v. can be evaluated using eigenfunctions \((2.5)\):

$$\langle 0 | T_{00}^{(\text{surf})} | 0 \rangle = \delta(r - a + 0) \frac{2\xi - 1/2}{a^2 (2\pi)^N} \sum_{\alpha} \frac{\lambda^2_{n,l} T_n(\lambda_{n,l})}{\sqrt{\lambda^2_{n,l} + k^2 a^2 + m^2 a^2}} J_n(\lambda_{n,l} r/a) J'_n(\lambda_{n,l} r/a). \quad (3.5)$$

The integration of this energy density leads to the expression \((3.3)\) for the total interior surface energy. In \((3.7)\), applying to the sum over \(l\) formula \((2.22)\), in analogy to \((2.30)\), the subtracted surface energy density can be presented in the form

$$\langle T_{00}^{(\text{surf})} \rangle_{\text{SUB}} = - \frac{(2\xi - 1/2) \delta(r - a + 0)}{2^{D-2} \pi^{D/2} \Gamma(D/2 - 1) a^{D-1}} \sum_{n=\infty}^{\infty} \int_{ma}^{\infty} \! \! dz \, z^2 \left( m^2 a^2 \right)^{D/2 - 2} \times \frac{\tilde{K}_n(z)}{I_n(z)} \frac{J_n(z r/a)}{I_n(z r/a)} \left( \frac{z}{r} \right). \quad (3.6)$$
This leads to the following subtracted surface energy

\[
\langle E_{\text{in}}^{(\text{surf})} \rangle_{\text{SUB}} = -\frac{(4\xi - 1) L^{D-3}a^{2-D}}{2^{D-2} \pi^{D/2} 1! (D/2 - 1)} \sum_{n=-\infty}^{+\infty} \int_{ma}^{\infty} dz \, z^2 \left(z^2 - m^2 a^2\right)^{D/2-2} \frac{\tilde{K}_n(z)}{I_n(z)} I_n(z) I'_n(z). 
\]  

We can find the subtracted total volume energy inside a cylinder integrating the energy density (2.30):

\[
\langle E_{\text{in}}^{(\text{vol})} \rangle_{\text{SUB}} = \frac{L^{D-3}a^{2-D}}{2^{D-2} \pi^{D/2} 1! (D/2 - 1)} \sum_{n=-\infty}^{+\infty} \int_{ma}^{\infty} dz \, z \left(z^2 - m^2 a^2\right)^{D/2-2} \frac{\tilde{K}_n(z)}{I_n(z)} F_v[I_n(z)], 
\]

where, for a given function \( f(z) \), we have introduced the notation

\[
F_v[f(z)] = (4\xi - 1) z f(z) f'(z) - \frac{z^2 - m^2 a^2}{D-2} \left[f^2(z) - \left(1 + \frac{n^2}{z^2}\right) f^2(z)\right].
\]

The total interior vacuum energy can be obtained as a sum of the surface and volume parts:

\[
\langle E_{\text{in}} \rangle_{\text{SUB}} = -\frac{L^{D-3}a^{2-D}}{2^{D-2} \pi^{D/2} 1! (D/2 - 1)} \sum_{n=-\infty}^{+\infty} \int_{ma}^{\infty} dz \, z \left(z^2 - m^2 a^2\right)^{D/2-1} \times
\]

\[\times \frac{\tilde{K}_n(z)}{I_n(z)} \left[I_n^2(z) - \left(1 + \frac{n^2}{z^2}\right) I_n^2(z)\right].\]

As we see, in the total vacuum energy the dependencies on the curvature coupling \( \xi \) are cancelled out and we obtain \( \xi \)-independent vacuum energy. We could have expected this result, as eigenfrequencies are independent of this parameter. The vacuum force acting from inside per unit surface of the cylinder can be found using the expression (2.30) for the vacuum radial pressure:

\[
F_{\text{in}} = p_1 \big|_{r=a-0} = \frac{2^{2-D} \pi^{D/2}}{a^2 1! (D/2 - 1)} \sum_{n=-\infty}^{+\infty} \int_{ma}^{\infty} dz \, z^3 \left(z^2 - m^2 a^2\right)^{D/2-2} \frac{\tilde{K}_n(z)}{I_n(z)} F_v(p_1)[I_n(z)],
\]

with notation (2.32). For the massless scalar field and in the case of conformal coupling, \( \xi = \xi_c \), the vacuum force and volume energy are related by

\[
\langle E_{\text{in}}^{(\text{vol})} \rangle_{\text{SUB}} = \frac{2\pi L^{D-3}}{D-2} a^2 F_{\text{in}}.
\]

This relation can be also obtained from the continuity equation (2.33) if we take into account the tracelessness condition for the corresponding energy-momentum tensor.

### 4 Casimir densities outside a cylindrical shell

Firstly, let’s consider v.e.v.’s for the energy-momentum tensor in the region between two coaxial cylindrical layers with radii \( a \) and \( b \), \( a < b \). The corresponding boundary conditions are in form

\[
\left(A + B \frac{\partial}{\partial r}\right) \varphi(r) = 0, \quad r = a, b.
\]

The corresponding eigenfunctions can be obtained from (2.5) with the replacement

\[
J_n(\gamma r) \rightarrow g_n(\gamma a, \gamma r) = J_n(\gamma r)Y_n(\gamma a) - Y_n(\gamma r)J_n(\gamma a),
\]
where $Y_n(z)$ is the Neumann function. The functions chosen in this way satisfy the boundary condition on the inner cylinder. From the boundary condition on the shell $r = b$ one obtains that the possible values of $\gamma$ are solutions to the equation
\[ C_n^{ab}(\gamma, \gamma a) \equiv \bar{J}_n(\gamma a)\bar{Y}_n(\gamma b) - \bar{Y}_n(\gamma a)\bar{J}_n(\gamma b) = 0. \quad (4.3) \]

The corresponding roots will be denoted by $\gamma a = \sigma_n l$, $l = 1, 2, \ldots$.

The normalization coefficient is determined from condition (2.9), where now the integration goes over the region between the cylindrical shells, and is equal to
\[ \beta_n^2 = \frac{\pi^2 T_n^{ab}(\gamma a)}{4\omega a (2\pi)^{N+1}}. \quad (4.4) \]

Here we use the notation
\[ T_n^{ab}(z) = z \left\{ \frac{\bar{J}_n^2(z)}{J_n^2(\eta z)} \left[ A^2 + B^2(\eta^2 z^2 - n^2) \right] - A^2 - B^2(z^2 - n^2) \right\}^{-1}, \quad \eta = \frac{b}{a}. \quad (4.5) \]

Using the corresponding eigenfunctions it can be seen that the v.e.v.’s for the energy-momentum tensor have the form (2.14), where the components are given by formulæ
\[ q(r) = \frac{\pi^2}{4(2\pi)^{N+1}a^3} \sum_{\alpha} \frac{\sigma_n^3 T_n^{ab}(\sigma_n, l)}{\sqrt{\sigma_n^2 + k^2 a^2 + m^2 a^2}} f_n^{(q)}[g_n(\sigma_n, l, \sigma_n l r/a)], \quad q = \varepsilon, p, \quad (4.6) \]

with the functions $f_n^{(q)}[f(z)]$ defined as (2.16)-(2.19).

To evaluate the sum over $l$ we will apply the summation formula (13)
\[ \frac{\pi^2}{2} \sum_{l=1}^{\infty} h(\sigma_n, k) T_n^{ab}(\sigma_n, k) = \int_0^\infty \frac{h(x)dx}{J_n^2(x) + Y_n^2(x)} - \frac{\pi}{4} \int_0^\infty \frac{K_n(\eta x)}{K_n(x)I_n(\eta x) - K_n(\eta x)I_n(x)} dx, \quad (4.7) \]

where we have assumed that all zeros for the function $C_n^{ab}(\eta, z)$ are real. In case of existence of purely imaginary zeros we have to include additional residue terms on the left of formula (4.7).

Now, by an evaluation similar to (2.26), one can see that
\[ q = \frac{1}{(2\pi)^{D-2}a^3} \int d^N k \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2} \int_0^\infty dz \frac{z^3 f_n^{(q)}[g_n(z, zr/a)]}{J_n^2(z) + Y_n^2(z) \sqrt{z^2 + k^2 a^2 + m^2 a^2}} + \frac{\pi}{4} \int_{a\sqrt{k^2 + m^2}}^{+\infty} \frac{z^3 f_n^{(q)}[g_n(iz, izr/a)]}{\sqrt{z^2 - k^2 a^2 - m^2 a^2} K_n(x)I_n(\eta x) - K_n(\eta x)I_n(x)} dz \right\}. \quad (4.8) \]

To obtain v.e.v.’s for the exterior of a single cylindrical shell let’s consider the limit $b \to \infty$. It can be easily seen that in this limit the second integral on the right of formula (4.8) tends to zero, whereas the first one is independent from $b$. It follows from here that the expressions
\[ q = \frac{1}{(2\pi)^{D-2}a^3} \int d^N k \sum_{n=-\infty}^{+\infty} \int_0^\infty dz \frac{z^3 f_n^{(q)}[g_n(z, zr/a)]}{J_n^2(z) + Y_n^2(z) \sqrt{z^2 + k^2 a^2 + m^2 a^2}}, \quad (4.9) \]

are v.e.v.’s for the EMT components for the exterior region of a single shell. To regularize this quantities we have to subtract the parts corresponding to the unbounded space. As we saw, the latter can be presented in the form (2.28). Using the identity
\[ \frac{f_n^{(q)}[g_n(z, zr/a)]}{J_n^2(z) + Y_n^2(z)} - f_n^{(q)}[\bar{J}_n(zr/a)] = -\frac{1}{2} \sum_{\sigma=1}^{2} \frac{\bar{J}_n(z)}{\bar{H}_n^{(\sigma)}(z)} f_n^{(q)}[\bar{H}_n^{(\sigma)}(zr/a)] \quad (4.10) \]
with $H_n^{(\sigma)}(z)$, $\sigma = 1, 2$ being the Hankel functions, one obtains

$$
q = -\frac{1}{4(2\pi)^{D-2}a^3} \int d^N k \sum_{n=-\infty}^{+\infty} \sum_{\sigma=1}^{2} \int_0^{+\infty} dz \frac{\tilde{f}_n(z) z^3 f_n^{(\sigma)}(zr/a)}{H_n^{(\sigma)}(z) \sqrt{z^2 + k^2 a^2 + m^2 a^2}}.
$$

(4.11)

Assuming that the function $\tilde{H}_n^{(1)}(z)$, $(\tilde{H}_n^{(2)}(z))$ has no zeros for $0 < \arg z \leq \pi/2$ ($-\pi/2 < \arg z < 0$) we can rotate the integration contour for $z$ by angle $\pi/2$ for $\sigma = 1$ and by angle $-\pi/2$ for $\sigma = 2$. The integrals over $(0, ia\sqrt{k^2 + m^2})$ and $(0, -ia\sqrt{k^2 + m^2})$ cancel out. Introducing the Bessel modified functions and integrating over $k$ with the help of formula (2.24) for the subtracted v.e.v. one obtains

$$
q_{SUB} = \frac{2^{2-D}\pi^{-D/2}}{a^D \Gamma(D/2-1)} \sum_{n=-\infty}^{+\infty} \int_{ma}^{+\infty} dz \, z^3 \left(z^2 - a^2 m^2\right)^{D/2-2} \frac{\tilde{I}_n(z)}{K_n(z)} F_n^{(q)} \left[ K_{n}(zr/a) \right],
$$

(4.12)

where we use notations (2.31)-(2.34). As we see, these quantities can be obtained from the ones for interior region by the replacements $I \rightarrow K$, $K \rightarrow I$. As for the interior region the v.e.v. (4.12) diverge at cylinder surface. The leading terms of these divergences are determined by the same formulae (4.10) with replacement $a - r \to r - a$.

Let’s consider the asymptotic behaviour of v.e.v (4.12) at large distances, $r \to \infty$, for the massless case. Introducing a new integration variable $y = zr/a$, and expanding the subintegral in terms of $r/a$, we can see that the leading term of the asymptotic expansion comes from the summand with $n = 0$:

$$
q_{SUB} = \frac{2^{2-D}\pi^{-D/2}}{a^D \Gamma(D/2-1) r^D \ln r/a} \int_0^{+\infty} dy y^{D-1} F_0^{(q)} \left[ K_0(y) \right], \quad r \to \infty.
$$

(4.13)

The integral in this expression can be evaluated using the formula for the integrals containing the square of the McDonald function (see, for instance, [19]). This leads to the following result

$$
\varepsilon \sim (D - 2)p_1 \sim \frac{D - 2}{D - 1} p_2 \sim \frac{\pi}{\Gamma^3(D/2) \Gamma(D - 1) \Gamma(D/2 - 1)} \frac{\xi - \xi_c}{r^D \ln r/a}, \quad r \to \infty.
$$

(4.14)

For a conformally coupled scalar this leading term is zero and $\varepsilon \sim 1/r^{D+2}$.

Integrating the vacuum energy density over the region outside of a cylindrical shell we obtain the corresponding volume energy:

$$
\langle \bar{E}_{ext}^{(vol)} \rangle_{SUB} = \frac{-L^{D-3} a^{2-D}}{2 D^2 - 2D^2 - 1 \Gamma(D/2 - 1)} \sum_{n = -\infty}^{+\infty} \int_{ma}^{+\infty} dz \, z^{D/2 - 2} \frac{\tilde{I}_n(z)}{K_n(z)} F_v[K_n(z)],
$$

(4.15)

where the functional $F_v[f(z)]$ is defined as (3.9). The outside surface energy can be derived using the surface energy density given by the formula (3.4) with the replacement $a - 0 \to a + 0$ in the argument of the $\delta$-function. This gives

$$
\langle \bar{E}_{ext}^{(surf)} \rangle_{SUB} = \frac{4 \xi - 1}{2 D^2 - 2D^2 - 1 \Gamma(D/2 - 1)} \sum_{n = -\infty}^{+\infty} \int_{ma}^{+\infty} dz \, z^2 \left(z^2 - a^2 m^2\right)^{D/2 - 2} \frac{\tilde{I}_n(z)}{K_n(z)} K_n^2(z).
$$

(4.16)

Now taking the sum of (4.15) and (4.16) for the total exterior vacuum energy one obtains

$$
\langle \bar{E}_{ext} \rangle_{SUB} = \frac{L^{D-3} a^{2-D}}{2 D^2 - 1 \Gamma(D/2 - 1)} \sum_{n = -\infty}^{+\infty} \int_{ma}^{+\infty} dz \, z^{D/2 - 2} \times
$$

$$
\times \frac{\tilde{I}_n(z)}{K_n(z)} \left[ K_n^2(z) - \left(1 + \frac{n^2}{z^2}\right) K_n^2(z) \right].
$$

(4.17)
As we could have expected, the dependencies on the curvature coupling are cancelled. The expression for the radial projection of the vacuum force acting per unit surface of the cylinder from the outside directly follows from (4.14) with \(q = p_1\):

\[
F_{\text{ext}} = -p_1 \bigg|_{r=a+0} = \frac{-2^{2-D} \pi^{D-2}}{a \Gamma(D/2 - 1)} \sum_{n=\infty}^{\infty} \int_{n\pi}^{\infty} dz \left( z^2 - m^2 a^2 \right)^{D/2-2} \frac{I_n(z)}{K_n(z)} F_n^{[p_1]} [K_n(z)],
\]

where the function \(F_n^{[p_1]}\) is defined as (2.32). Now we turn to the case of a cylindrical shell with zero thickness, assuming that the coefficients in the Robin boundary conditions are the same for the interior and exterior regions. The total vacuum energies and force per unit surface can be obtained summing the corresponding quantities for these regions:

\[
\langle E \rangle_{\text{SUB}} = \langle E_{\text{in}} \rangle_{\text{SUB}} + \langle E_{\text{ext}} \rangle_{\text{SUB}}, \quad F = F_{\text{in}} + F_{\text{ext}}.
\]

Using the expressions for the interior and exterior quantities we have

\[
\langle E^{(\text{surf})} \rangle_{\text{SUB}} = \frac{(4\xi - 1) L^{D-3} a^{2-D}}{2^{D-2} \pi^{D-2} \Gamma(D/2 - 1)} \sum_{n=\infty}^{\infty} \int_{n\pi}^{\infty} dz z \left( z^2 - m^2 a^2 \right)^{D/2-2} \times \left[ 1 - \frac{(\tilde{I}_n(z) \tilde{K}_n(z))'}{z I_n'(z) K_n'(z)} \right],
\]

for the surface part of the vacuum energy,

\[
\langle E \rangle_{\text{SUB}} = \frac{-L^{D-3} a^{2-D}}{2^{D-2} \pi^{D-2} \Gamma(D/2)} \sum_{n=\infty}^{\infty} \int_{n\pi}^{\infty} dz \left( z^2 - m^2 a^2 \right)^{D/2-1} \times \left[ 2\beta + (z^2 + n^2 - \beta^2) \frac{(\tilde{I}_n(z) \tilde{K}_n(z))'}{z I_n'(z) K_n'(z)} \right],
\]

for the total vacuum energy, and

\[
F = \frac{-2^{1-D} \pi^{D-2}}{a \Gamma(D/2 - 1)} \sum_{n=\infty}^{\infty} \int_{n\pi}^{\infty} dz z \left( z^2 - m^2 a^2 \right)^{D/2-2} \times \left[ 2\beta - 4\xi + (z^2 + n^2 - \beta^2 + 4\xi\beta) \frac{(\tilde{I}_n(z) \tilde{K}_n(z))'}{z I_n'(z) K_n'(z)} \right],
\]

for the total vacuum force acting per unit surface of the shell. In these formulae we have introduced the notation

\[
\tilde{f}(z) = z^\beta f(z), \quad \beta = A/B
\]

for a given function \(f(z)\). The cases of Dirichlet and Neumann boundary conditions are obtained taking the limits \(\beta = \infty\) and \(\beta = 0\). For the surface energies from (4.20) one obtains

\[
\langle E^{(\text{surf})} \rangle^{(D)}_{\text{SUB}} = -\langle E^{(\text{surf})} \rangle^{(N)}_{\text{SUB}} = \frac{(1 - 4\xi) L^{D-3} a^{2-D}}{2^{D-2} \pi^{D-2} \Gamma(D/2 - 1)} \sum_{n=\infty}^{\infty} \int_{n\pi}^{\infty} dz z \left( z^2 - m^2 a^2 \right)^{D/2-2}.
\]

For the analytical continuation of the expression on the right we note that

\[
\sum_{n=\infty}^{\infty} \int_{n\pi}^{\infty} dz z \left( z^2 - m^2 a^2 \right)^{D/2-2} = \frac{1}{2} B(D/2 - 1, 1 - D/2)(1 + 2\zeta_R(0)) = 0,
\]
where $\zeta_R$ denotes the Riemann zeta function ($\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$). As a result, we conclude that the 

**surface energy is zero for Dirichlet and Neumann scalars.** Due to the relation (4.25) we can omit $n$-independent terms in the subintegrands of (4.20), (4.22) without changing the values of the sums. We can use this to improve the convergence properties of the corresponding integrals.

Let’s present the total Casimir energy (4.21) in another equivalent form. For this, note that the functions $f(z) = \tilde{I}_n(z)$, $\tilde{K}_n(z)$ satisfy the equation

$$z^2 f''(z) + (1 - 2\beta)zf(z) - (z^2 + n^2 - \beta^2)f(z) = 0. \quad (4.26)$$

Using this, it can be easily seen that

$$2\beta + (z^2 + n^2 - \beta^2)\frac{(\tilde{I}_n\tilde{K}_n)'}{\tilde{I}_n\tilde{K}_n} = z\frac{(\tilde{I}_n\tilde{K}_n)'}{\tilde{I}_n\tilde{K}_n} + 2(1 - \beta) = z \tilde{I}_n\tilde{K}_n' = ((\tilde{I}_n\tilde{K}_n)'/\tilde{I}_n\tilde{K}_n). \quad (4.27)$$

Thus, we obtain an equivalent representation for the total Casimir energy:

$$\langle E \rangle_{\text{SUB}} = \frac{-L^{D-3}a^{2-D}}{2^{D-1}\pi^{D/2-1}\Gamma(D/2)} \sum_{n=-\infty}^{\infty} \int_{ma}^{\infty} dz (z^2 - m^2a^2)^{D/2-1} \frac{d}{dz} \ln (\tilde{I}_n(z)\tilde{K}_n(z)). \quad (4.28)$$

This expression is divergent in the given form. A method to extract finite results will be explained below using the case of the $D = 4$ massless scalar field as an example. The corresponding results for arbitrary dimension and massive case will be presented elsewhere.

5 Integrated Casimir energy per unit-length by mode sum

Taking advantage of the usual prescription for evaluating the eigenfrequency sum (see. e.g. ref. [4]), the integrated Casimir energy per unit-length of an infinitely long cylinder is expressed as

$$\varepsilon_c = \frac{1}{2} \zeta_{\text{cyl}}(\sigma) \bigg|_{\sigma \to -1}, \quad (5.1)$$

where $\zeta_{\text{cyl}}(\sigma)$ is the zeta function for the cylinder eigenfrequencies under the chosen boundary conditions. Note that, in order to simplify the procedure, we have not included the arbitrary mass scale typically used in these problems (in fact, it will not be necessary because the result will be finite). This zeta function can be related to the one for a circle, with the same boundary conditions and radius $a$, which we shall call $\zeta_{\text{circ}}(s)$, and corresponds to the restriction of the initial problem to a plane perpendicular to the cylinder axis. Because of the cylindrical symmetry, the eigenvalues of the wave operator—including mass—have the form $(\gamma^2 + k^2 + m^2)^{1/2}$, where the $\gamma$’s are the eigenfrequencies of the restricted problem and $k \in \mathbb{R}$. Then, in a spacetime of $D = N + 2 + 1$ dimensions,

$$\zeta_{\text{cyl}}(\sigma) = \int \frac{d^Nk}{(2\pi)^N} \sum_\gamma (\gamma^2 + k^2 + m^2)^{-\sigma/2} = \frac{1}{(4\pi)^{N/2}} \frac{\Gamma\left(\frac{\sigma-N}{2}\right)}{\Gamma\left(-\frac{N}{2}\right)} \zeta_{\text{circ}}(\sigma - N) \quad (5.2)$$

with

$$\zeta_{\text{circ}}(s) \equiv \sum_\gamma (\gamma^2 + m^2)^{-s/2} = \sum_n d_n \sum_l (\gamma_{n,l}^2 + m^2)^{-s/2}, \quad \gamma_{n,l} = \frac{\lambda_{n,l}}{a}. \quad (5.3)$$

In this section we are considering just one cylindrical surface, and these $\gamma$’s denote now the zeros of $\tilde{J}_n(\gamma a)\tilde{H}_n^{(1)}(\gamma a)$ (the first factor coming from the interior propagation and the second from the exterior propagation). The $n$ index corresponds to angular momentum, $d_n$ indicates
each degeneracy, and the \( l \) index labels the different frequencies for a given \( n \). Expanding (5.3) around \( \sigma = -1 \), one distinguishes two possibilities depending on \( N \):

1. Even \( N \) (\( N = 2p \))

\[
\zeta_{\text{cyl}}(\sigma) = \frac{1}{2^{2p+1}\pi^{p+1/2}} \Gamma \left( -p - \frac{1}{2} \right) \zeta_{\text{circ}}(-2p - 1). \tag{5.4}
\]

2. Odd \( N \) (\( N = 2p + 1 \))

\[
\zeta_{\text{cyl}}(\sigma) = \frac{(-1)^p}{2^{2p+1}(p+1)!} \left( \frac{\zeta_{\text{circ}}(-2p - 2)}{\sigma + 1} - \frac{1}{2} (\psi(-1/2) - \psi(p + 2)) \zeta_{\text{circ}}(-2p - 2) + \zeta_{\text{circ}}'(-2p - 2) + \mathcal{O}(\sigma + 1) \right). \tag{5.5}
\]

Later on we will focus on \( D = 3 + 1 \) (i.e., \( p = 0 \) case). Then, in situations where \( \zeta_{\text{circ}}(-2) = 0 \), (5.5) reduces to

\[
\zeta_{\text{cyl}}(s) = \frac{1}{2\pi} \zeta_{\text{circ}}'(2). \tag{5.6}
\]

Thus, one has to obtain the expansion of \( \zeta_{\text{circ}}(s) \) up to linear terms in \( (\sigma + 1) = (s + 2) \) (in general, \( \sigma + 1 = s + D - 2 \)). By the Cauchy theorem, we may express the circle zeta function \( \zeta_{\text{circ}}(s) \) as the following contour integrations

\[
\zeta_{\text{circ}}(s) = a^s \sum_n d_n \frac{1}{2\pi i} \int_C dz \left( z^2 + a^2m^2 \right)^{-s/2} \frac{d}{dz} \ln \left( \bar{J}_n(z) \bar{H}_n^{(1)}(z) \right)
= a^s \sum_n d_n \frac{s-2}{2\pi i} \int_C dz \left( z^2 + a^2m^2 \right)^{-(s+2)/2} z \ln \left( \bar{J}_n(z) \bar{H}_n^{(1)}(z) \right), \tag{5.7}
\]

where \( C \) is a circuit in the complex \( z \)-plane enclosing all the zeros of \( \bar{J}_n(z) \bar{H}_n^{(1)}(z) \) (it is understood that in this function we have now set \( a = 1 \)). In this way, we are already including the contributions associated to internal and external propagation. Formulas (5.7) have the disadvantage of being valid for an \( s \)-domain in which \( \text{Re} \, s \) is positive, while we need to reach \( s = -(D - 2) \). Just in order to study the formal relations between these functions, we take the first form in (5.7) and consider a specific circuit. Let \( C \) be the contour made of a vertical line along the imaginary axis avoiding — and leaving outside— the cut of the \((\zeta^2 + a^2m^2)^{-s/2}\) function (placed between \(-ian\) and \(+ian\)), and a semicircle of infinite radius to its right. After realizing which parts yield no contribution, one finds

\[
\zeta_{\text{circ}}(s) = a^s \sum_n d_n \frac{1}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_{am}^\infty dx \left( x^2 - a^2m^2 \right)^{-s/2} \frac{d}{dx} \ln \left( \bar{I}_n(x) \bar{K}_n(x) \right). \tag{5.8}
\]

Now, using (5.2), one may formally write the integrated Casimir energy per lateral unit-length as

\[
\frac{E_C}{L^{D-3}} = \frac{1}{2} \zeta_{\text{cyl}}(-1) = \frac{a^{2-D}}{2^{D-1}\pi^{(D-2)/2}} \sum_n d_n \int_{am}^\infty dx \left( x^2 - a^2m^2 \right)^{-s/2} \frac{d}{dx} \ln \left( \bar{I}_n(x) \bar{K}_n(x) \right). \tag{5.9}
\]

Nevertheless, we have to stress that it is not valid to directly set \( s = -(D - 2) \) in order to obtain \( E_C/L^{D-3} \). Analytic continuation to the left of the present \( s \)-domain is required first. In the example below, we explain a method for performing this type of process. Note, by the way, that if one ’naively’ sets \( s = -(D - 2) \) in (5.9), expression (4.28) is recovered.
5.1 $m = 0, D = 3 + 1$ case

In order to obtain the necessary analytic continuation to $s = -2$, we will follow a different path, which has been explained e.g. in ref. [21]. Its application will be illustrated for the case $m = 0$, $D = 3 + 1$ (therefore $d_0 = 1$ and $d_n = 2$ for $n \neq 0$). The mode sum is decomposed into sets of partial waves with a defined angular momentum $n$, which, for convenience, will be separately considered. Thus, we employ the notation

$$
\zeta_{\text{circ}}(s) = \sum_n d_n \zeta_n(s) = \sum_{n=-\infty}^{\infty} \zeta_n(s),
$$

(5.10)

where, taking the second form in (5.7),

$$
\zeta_n(s) = \sum_{l=1}^{\infty} \gamma_{n,l}^{-s} = \frac{s}{2 \pi i} \int_C dz \; z^{-s-1} \ln \left( J_n(z) H_n^{(1)}(z) \right).
$$

(5.11)

A first step in the extension of the $s$-domain to the left is the study of the asymptotic behaviour of the integrand for $|z| \to \infty$, given by $J_n(z) H_n^{(1)}(z) \sim -iz/2 \equiv f_{\text{as}}(z)$. In view of this, we subtract from and add to the integrand a term $z^{-s} \ln f_{\text{as}}(z)$. This move leaves us with

$$
\zeta_n(s) = \frac{s}{2 \pi i} \left[ \int_C dz \; z^{-s} \ln \left( \frac{J_n(z) H_n^{(1)}(z)}{f_{\text{as}}(z)} \right) + \int_C dz \; z^{-s} \ln f_{\text{as}}(z) \right].
$$

(5.12)

Taking a contour $C$ as described (now, since $m = 0$, the cut line has reduced to the point $z = 0$) we must conclude that the second integral vanishes, because its integrand has no singularity in the interior. As for the first integral, there are nonvanishing contributions only from the vertical parts of $C$. Parameterizing them adequately, we may write

$$
\zeta_n(s) = a^s \sin \left( \frac{\pi s}{2} \right) \int_0^{\infty} dx \; x^{-s} \ln L_n(x),
$$

$$
L_n(x) = -\frac{2}{x} \left[ I_n(x) I_n(x) \right].
$$

(5.13)

This formula is slightly better than the initial one, as its domain of validity has been shifted to the left: actually, it holds for $-1 < \text{Re}s < 0$. However, it is not good enough, as we have not arrived at $s = -2$ yet. To this end, we shall have to perform even more subtractions. Before going on, we decompose $\zeta_{\text{circ}}(s)$ into the contribution from nonzero angular momenta ($n \neq 0$) and the contribution from zero angular momentum ($n = 0$):

$$
\zeta_{\text{circ}}(s) = \zeta_{\text{circ}}^{n \neq 0}(s) + \zeta_{\text{circ}}^{n = 0}(s).
$$

(5.14)

When $n \neq 0$ we may perform a rescaling $x \to nx$. Afterwards, summing the internal and external contributions, we find

$$
\zeta_n(s) = a^s n^{-s} \frac{s}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_0^{\infty} dx \; x^{-s-1} \ln L_n(nx).
$$

(5.15)

We shall actually assume $B = 1$ and take $A$ as the only variable. This can be done whenever $B \neq 0$. Therefore, the only case which cannot be described in this way is $B = 0$, $A \neq 0$, i.e., purely Dirichlet conditions, which have already been considered in ref. [9] (formula (34) in that paper). Studying the uniform asymptotic expansions of the involved Bessel functions (for $B \neq 0$) one realizes that

$$
L_n(nx) \sim \frac{(1 + x^2)^{1/2}}{x} \left( B^2 + \text{next-to-leading terms} \right), \text{ for large } nx.
$$

(5.16)
This behaviour motivates (for $B = 1$) the subtraction from and addition to the integrand in eq. (5.13) of a term $x^{-s-1} \ln \left( \frac{(1 + x^2)^{1/2}}{x} \right)$. Using then the result

$$\int_0^\infty dx \, x^{-s-1} \ln \left( \frac{(1 + x^2)^{1/2}}{x} \right) = \frac{\pi}{2s \sin \left( \frac{\pi s}{2} \right)},$$

(5.17)

one obtains

$$\zeta_n(s) = a^s \left\{ n^{-s} \frac{s}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_0^\infty dx \, x^{-s-1} \ln \left| \frac{x}{(1 + x^2)^{1/2}} \right| L_n(nx) + \frac{n^{-s}}{2} \right\}. \tag{5.18}$$

Hence,

$$\zeta^\neq_{\text{circ}}(s) = 2 \sum_{n=1}^{\infty} \zeta_n(s) = a^s \left\{ 2 \frac{s}{\pi} \sin \left( \frac{\pi s}{2} \right) \sum_{n=1}^{\infty} n^{-s} \int_0^\infty dx \, x^{-s-1} \ln \left| \frac{x}{(1 + x^2)^{1/2}} \right| L_n(nx) \right\} + \zeta_R(s), \tag{5.19}$$

where

$$S_n(N, s) = \int_0^\infty dx \, x^{-s} \left[ \ln \left| \frac{x}{(1 + x^2)^{1/2}} \right| L_n(nx) - 2 \sum_{j=1}^{N/2} \frac{\mathcal{U}_2(t(x))}{n^{2j}} \right]. \tag{5.20}$$

In formula (5.20), the objects called $\mathcal{U}_2(t)$ are polynomials in $t$, where

$$t(x) = \frac{1}{(1 + x^2)^{1/2}}, \tag{5.21}$$

which are constructed step by step when obtaining the asymptotic approximation

$$\ln \left| \frac{x}{(1 + x^2)^{1/2}} \right| L_n(nx) \sim 2 \sum_{j=1}^{N/2} \frac{\mathcal{U}_2(t(x))}{n^{2j}}, \tag{5.22}$$

for large $nx$ —in fact, this is like a further elaboration of (5.16) —. Then, the $\mathcal{U}_{2j,k}$’s are the present coefficients of these polynomials, in other words,

$$\mathcal{U}_{2j}(t) = \sum_k \mathcal{U}_{2j,k} t^k, \tag{5.23}$$

where the $k$-index range corresponds to the existing coefficients. The whole process for obtaining this type of approximation has already been described in detail in refs. [20, 21] (see also [22]). By comparison, one may reason that these $\mathcal{U}_{2j}(t)$’s have to be the same as the even-$j$ $U_j^{*, R}(t)$’s in formula (3.37) of ref. [21], after the replacement $\alpha \rightarrow A$. The $N$ variable denotes the order of
the subtractions performed in order to remove negative-s poles from the initial integral. If $N$ is large enough—and in our case this means just $N \geq 2$—the $S_n(N, s)$ integral is finite at $s = -2$. When $N$ is further increased, the value of the numerical integration $S_n(N, s)$ gets smaller and smaller while the algebraic $n$-sum (easier to evaluate) grows accordingly. In practice, we take $N = 8$ or 10.

The $n = 0$ contribution has to be dealt with separately, as a step in the process for $n \neq 0$ involved rescaling by $n$, and cannot be applied now. The same philosophy as in refs. 21, 9 will now be adopted. We will just perform the necessary subtraction to isolate the $s = -2$ divergence from the integral, leaving the rest to numerical evaluation. The result is expressed in the form

$$\zeta_{\text{circ}}^{n=0}(s) = a^2 \frac{s}{\pi} \sin \left( \frac{\pi s}{2} \right) \left[ R_0(s) - \left( A^2 - A + 3/8 \right) \frac{1}{2} B \left( \frac{s + 2}{2}, -\frac{s}{2} \right) \right],$$

(5.25)

where the integral

$$R_0(s) = \int_0^{\infty} dx \: x^{-s} \left[ \ln |L_0(x)| + \left( A^2 - A + 3/8 \right) t^2(x) \right]$$

(5.26)

is, by construction, finite at $s = -2$. Therefore, $R_0(-2)$ can be calculated numerically for given values of $A$ ($B = 1$).

Thus, the circle zeta function is the sum of (5.24) and (5.25). At $s = -2$, (5.24) has a nonzero finite value coming from the pole of $B \left( \frac{s + 2}{2}, -\frac{s}{2} \right)$ multiplied by the zero of $\frac{\pi}{2} \sin \left( \frac{\pi s}{2} \right)$ at $s = -2$. This nonzero value depends also on the involved $U_{2j, 2}$ coefficient(s) and on $\zeta_R(s + 2j)$. Now, making a similar reasoning for (5.25), we realize that the $s = -2$ value of this expression is just $-\frac{1}{a^2} (A^2 - A + 3/8)$. In fact, algebraic calculation shows that the finite value of (5.24) at $s = -2$ amounts to the same quantity with opposite sign. Therefore, $\zeta_{\text{circ}}(-2) = 0$, and one can safely apply formula (5.6). In these circumstances, the matter reduces to finding $\zeta_{\text{circ}}'(-2)$ by expanding the sum of (5.24) and (5.25) around $s = -2$. Once this has been done, the integrated Casimir energy per unit-length is simply

$$\varepsilon_c = \frac{1}{4\pi} \zeta_{\text{circ}}'(-2).$$

(5.27)

Results for a given $A$-range around $A = 0$ (purely Neumann) are shown in Fig. 3 below, where we have plotted the numerical values of $\varepsilon_c \cdot a^2$ as a function of $A$.

6 Final remarks

In the present paper, the vacuum—or zero-point—energy of scalar fields obeying Robin boundary conditions on cylindrical surfaces has been studied. Local quantities have been evaluated starting from vacuum expectation values of energy momentum tensors and applying generalized Abel-Plana summation methods supplemented with a suitable cutoff function. Integrated energies seem to be more easily found from the eigenfrequency sum which determines the Casimir effect. In that case, zeta-function regularization has proven to be an adequate tool. However, both approaches constitute different aspects of one underlying idea. In fact, we have shown that the two methods lead to identical formal expressions —formulas (4.28) and (5.9) at $s = -(D-2)$.

We have obtained an expression for the integrated energy inside the cylinder. In general, the total vacuum energy does not coincide with the integral of the volumic density, because there is a purely superficial contribution, located at the boundary itself, which has to be added to the volumic part. Moreover, the contributions associated to the modes propagating in the exterior of the cylinder has also been considered. In both cases, there is a part coming from the surface (‘viewed’ from inside, in the first, and from outside, in the second). The total Casimir
Figure 2: Total integrated Casimir energy per unit-length, multiplied by $a^2$, for $B = 1$ as a function of the $A$ coefficient, in a region around $A = 0$. Note the presence of a minimum near $A = -0.010$. The value for $A = 0$ (purely Neumann b.c.) is of $-0.014176...$ in agreement with formula (44) of ref. [9].
energy arises as the sum of all these contributions, and may also be obtained by eigenfrequency summation. As an example, we have chosen the massless case in $D = 3 + 1$ dimensions for application of the zeta function method. The result for a given parameter range is shown in fig. [2], which displays the presence of a local minimum near (but not coinciding with) the point where Robin conditions become purely Neumann conditions. This indicates that, in a theoretical space of possible boundary conditions, some linear combinations are energetically privileged.

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