One-Loop Amplitudes in
Supersymmetric QCD from MHV Vertices

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Abstract

The Cachazo-Svrcek-Witten (CSW) rule for efficiently calculating gauge theory amplitudes is extended to $\mathcal{N} = 1$ supersymmetric QCD (SQCD), incorporating massless quarks, in a manner that preserves the manifest supersymmetry. Using this extended CSW rule, we obtain compact expressions for all the one-loop MHV amplitudes in SQCD including one or two external quark-antiquark (chiral-antichiral multiplets) pairs. The collinear singularities of the five-point amplitudes are investigated to confirm the consistency of the results.
§1. Introduction

A new type of weak-weak duality proposed by Witten gives a correspondence between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and topological strings on supertwistor space.\textsuperscript{1)} The perturbative amplitudes of the $\mathcal{N} = 4$ SYM theory can be interpreted as D-instanton contributions in a topological string.

From the perturbative SYM viewpoint, Cachazo, Svrcek and Witten showed that the essence of this duality can be extracted as a rule\textsuperscript{2)} which provides an alternative method to efficiently compute the perturbative YM amplitudes. The building blocks of the CSW rule are tree-level maximally helicity violating (MHV) amplitudes and the scalar propagator, $1/P^2$. It also includes a definite prescription for using the (on-shell) MHV amplitudes as (off-shell) vertices. In comparison with the usual Feynman rule, the CSW rule has many practical advantages that simplify the computation of amplitudes. The CSW rule was immediately extended to incorporate (extended) supersymmetry\textsuperscript{3)–5)} and applied to obtain several gluon tree amplitudes.\textsuperscript{2), 6)} It was also applied to calculate the one-loop gluon MHV amplitudes of the $\mathcal{N} = 4$\textsuperscript{7)} and $\mathcal{N} = 1$\textsuperscript{8), 9)} SYM theory, which reproduce previously obtained results.\textsuperscript{10), 11)}

Another extension of the CSW rule is obtained by introducing massless quarks in the fundamental representation.\textsuperscript{12), 13)\*} In this case, there are two types of new MHV amplitudes including one or two external quark-antiquark pairs, which must be added as extra vertices. Several tree-level amplitudes with external fermions have been computed by using this extended CSW rule and shown to be completely consistent with the results obtained using the usual Feynman rules.\textsuperscript{12), 13)}

The purpose of this paper is to combine these two extensions. By introducing a scalar partner in the fundamental representation (squark), we derive an extended CSW rule for the $\mathcal{N} = 1$ supersymmetric QCD (SQCD). We use a superfield formulation and adopt the momentum representation of the supertwistor space by introducing some fermionic variables.\textsuperscript{7)} Because this formulation involves only the physical helicity states without auxiliary fields, it provides a drastically simple alternative method efficiently compute the perturbative SQCD amplitudes, in a manner that preserves the manifest supersymmetry. As an application of this extended CSW rule, we calculate all the one-loop MHV amplitudes with arbitrary numbers of legs including one or two external quark-antiquark pairs. The supersymmetry acts non-trivially, because the superpartners contribute through the internal loop. The new rule, using super-vertices, drastically reduces the number of Feynman-like diagrams, called MHV diagrams, which we must sum up. Only three (eight) MHV diagrams are needed for

\textsuperscript{*} See also Refs. 14) and 15), in which some tree-level MHV amplitudes involving quark-antiquark pairs in supersymmetric quiver gauge theories were studied.
the amplitudes, including one (two) external quark-antiquark pairs. However, because we have no string theory interpretation, there is no a priori reason that this extended CSW rule should lead to correct results in general. For this reason, as a non-trivial check, we confirm that the five-point amplitudes, as examples, have the correct collinear singularities.

This paper is organized as follows. In §2 we rearrange the known tree-level MHV amplitudes in SQCD in a manifestly supersymmetric form by introducing three fermionic variables, $\eta$, $\chi$ and $\rho$. These supersymmetric MHV amplitudes are used as the MHV vertices in our extended CSW rule, which is given at the beginning of §3. Using this rule, we compute all the one-loop MHV amplitudes in $\mathcal{N} = 1$ SQCD. In order to make this paper self-contained, in §§3.1 we first present the gluon one-loop MHV amplitudes, which have already been obtained.\(^8\),\(^9\) Compact expressions of the one-loop MHV amplitudes with one or two external quark-antiquark pairs are given in §§3.2 and 3.3 respectively. As a non-trivial check of the results, we investigate the collinear singularities of five-point amplitudes in §4. Section 5 is devoted to a summary and discussion. Finally, five appendices are provided. In Appendix A our spinor conventions are summarized. Some useful formulas are also given there. Definitions and properties of the functions appearing in the one-loop amplitudes are given in Appendix B. The formulas needed to evaluate the phase space integral are presented in Appendix C. Tree-splitting amplitudes and four-point amplitudes, which are needed to study the collinear behavior of five-point amplitudes, are summarized in Appendices D and E respectively.

§2. Tree-level MHV amplitudes in $\mathcal{N} = 1$ SQCD

In this section we present the tree-level MHV amplitudes of the $\mathcal{N} = 1$ SQCD in a manifestly supersymmetric way by introducing fermionic variables. We employ the $U(N_c)$ gauge group for simplicity.

Let us first consider partial amplitudes of $n$ gluons with fixed color ordering multiplied by the color factor

$$C^{(0)} = \text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n}).$$

(2.1)

Here $a_k$ ($k = 1, 2, \cdots, n$) is the color (adjoint) index of the $k$-th gluon. The $U(N_c)$ generators $T^a$ are $N_c \times N_c$ hermitian matrices normalized so as to satisfy $\text{Tr}(T^aT^b) = \delta^{ab}/2$. The MHV helicity structure of the amplitudes is such that two of the external gluons have negative helicities and the other $n - 2$ have positive helicities. We refer to the part other than the color factor and the delta function yielding total momentum conservation, $i(2\pi)^4\delta^4(\sum_i p_i)$, as the "amplitude" in this paper.

When a particle is massless, and hence $p^2 = 0$, the momentum vector $p_\mu$ in four dimen-
ions can be conveniently rewritten in terms of the commutative spinor \( \lambda_a \) [and its complex conjugate, \( \bar{\lambda}_{\dot{a}} \)] as

\[
p_{a\dot{a}} = (\sigma_{\mu})_{a\dot{a}}p^\mu = \lambda_a \bar{\lambda}_{\dot{a}}.
\]

The helicity amplitudes are functions of these commutative spinors and the helicities of the external particles, \( \lambda_{ia} \), \( \bar{\lambda}_{i\dot{a}} \) and \( h_i \). The tree-level MHV amplitudes of \( n \) gluons can thus be written in the holomorphic form

\[
\mathcal{A}_n^{JM} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle},
\]

where the \( i \)-th and \( j \)-th gluons are assumed to have negative helicity, and the shorthand notation \( \langle ij \rangle = \langle \lambda_i \lambda_j \rangle = \lambda_{ia} \lambda_{j\dot{a}} \) is used. We use the convention that the momenta of all the external particles are outgoing.

In order to incorporate supersymmetry, we also have to introduce two anticommuting variables, \( \eta \) and \( \chi \), for \( \mathcal{N} = 1 \) supersymmetry.\(^1\) Using these variables, the helicity operator can be written

\[
h = 1 - \frac{1}{2} \eta \frac{\partial}{\partial \eta} - \frac{3}{2} \chi \frac{\partial}{\partial \chi}.
\]

This is equivalent to the assumption that the vector superfield \( V^a \) in (the momentum representation of) the supertwistor space is a function of the variables \( \lambda, \bar{\lambda}, \eta \) and \( \chi \) and can be expanded as

\[
V^a(\lambda, \bar{\lambda}, \eta, \chi) = g^{a(+)}(\lambda, \bar{\lambda}) + \eta A^{a(+)}(\lambda, \bar{\lambda}) + \chi A^{a(-)}(\lambda, \bar{\lambda}) + \eta \chi g^{a(-)}(\lambda, \bar{\lambda}),
\]

where the component fields \( g^{a(+)}, A^{a(+)}, A^{a(-)} \) and \( g^{a(-)} \) has helicity 1, 1/2, -1/2 and -1, respectively. It should be remarked that the fermionic variable \( \eta \) is directly related to the supersymmetry, but \( \chi \) is not. It has the role of combining the positive helicity multiplet \( (g^{a(+)}, A^{a(+)}) \) and negative helicity multiplet \( (g^{a(-)}, A^{a(-)}) \) into one superfield. Thus the tree-level MHV amplitudes of \( n \) external \( \mathcal{N} = 1 \) vector multiplets can be written as

\[
\mathcal{A}_n^{(0)} = - \sum_{k<l} (-1)^{\epsilon_k + \epsilon_l} \eta_k \eta_l \sum_{i<j} (-1)^{\epsilon_i + \epsilon_j} \chi_i \chi_j \frac{\langle kl \rangle \langle ij \rangle^3}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle},
\]

where \( \epsilon_i = 0 \ (1) \) if the momentum of the \( i \)-th particle is outgoing (incoming).\(^{**}\) We use these tree-level MHV amplitudes as the vertices depicted in Fig. I

\(^{**}\) In order to use them as vertices, we must also consider the incoming momenta, because the legs of a vertex do not have to be external.
The amplitude (2.6) can be expanded in terms of the anticommuting variables $\eta_i$ and $\chi_i$, and each coefficient corresponds to the amplitude of external states whose helicities can be read off by using the helicity operator (2.4). For example, the coefficient of $\eta_i \chi_i \eta_j \chi_j$, with fixed $i$ and $j$, is the $n$-gluon MHV amplitude (2.3).

Here we rewrite the amplitude (2.6) as a product of three parts for later use:

$$A_n^{(0)} = \Delta(\eta) \Delta^{(0)}(\chi) \prod_{i=1}^{n} \frac{1}{\langle i(i+1) \rangle}, \quad (n + 1 \equiv 1) \quad (2.7a)$$

$$\Delta(\eta) = \sum_{k<l} (-1)^{\epsilon_k+\epsilon_l} \langle kl \rangle \eta_k \eta_l = \delta^2 \left( \sum_k (-1)^{\epsilon_k} (\lambda_k^a \eta_k) \right), \quad (2.7b)$$

$$\Delta^{(0)}(\chi) = -\sum_{i<j} (-1)^{\epsilon_i+\epsilon_j} \langle ij \rangle^3 \chi_i \chi_j. \quad (2.7c)$$

It should be noted that the first factor, $\Delta(\eta)$, can be written in the form of a delta function. The second factor, $\Delta^{(0)}(\chi)$, however, does not have such an interpretation. This is probably related to the unusual role of the $\chi$ variable mentioned above and makes it difficult to interpret it as amplitudes of some string theory.

In order to incorporate quark chiral multiplets, we need to introduce another fermionic variable, $\rho$. The quark chiral superfield $Q_i$ is a fermionic superfield and a function of the variables $\lambda$, $\bar{\lambda}$, $\eta$ and $\rho$. It can be expanded as

$$Q_i(\lambda, \bar{\lambda}, \eta, \rho) = q_i^{(+)}(\lambda, \bar{\lambda}) + \eta \phi_i^{(+)}(\lambda, \bar{\lambda}) + \rho \phi_i^{(-)}(\lambda, \bar{\lambda}) + \eta \rho q_i^{(-)}(\lambda, \bar{\lambda}). \quad (2.8)$$

The component fields $q_i^{(+)}$, $\phi_i^{(+)}$, $\phi_i^{(-)}$ and $q_i^{(-)}$ have the helicities $1/2$, $0$, $0$ and $-1/2$, respectively, and form a chiral multiplet of four-dimensional $\mathcal{N} = 1$ supersymmetry. The helicity operator for the chiral superfield is given by

$$h_c = \frac{1}{2} - \frac{1}{2} \eta \frac{\partial}{\partial \eta} - \frac{1}{2} \rho \frac{\partial}{\partial \rho}. \quad (2.9)$$
The new fermionic variable $\rho$ combines positive-helicity and negative-helicity multiplets, $(q_i^+, \phi_i^+)$ and $(\phi_i^-, q_i^-)$, into the single superfield (2.8). The antiquark antichiral superfield $\bar{Q}^i$ has properties similar to those of the quark superfield, except for the color quantum number. The (anti)quark superfield $Q^i$ ($\bar{Q}^i$) is in the (anti)fundamental representation of the gauge group $U(N_c)$.

When quark multiplets are incorporated, there are two types of additional MHV amplitudes, which we call two-quark and four-quark MHV amplitudes, including one or two external quark-antiquark pairs ordered as depicted in Fig. 2. We note that the fermion line, denoted by the solid line, is directed from the antiquark $\bar{q}$ to the quark $q$, and along this line, the helicity is always conserved. The tree-level two-quark MHV amplitudes $A_n^{(2)}$, depicted in Fig. 2 (a), can be written in terms of the fermionic variables $\eta$ and $\rho$ as

$$A_n^{(2)} = - \sum_{k<l=1}^n (-1)^{\epsilon_k+\epsilon_l} \sum_{\alpha=1}^{2n} \sum_{j=3}^{n} (-1)^{\epsilon_\alpha+\epsilon_j} \rho_\alpha \chi_j \frac{\langle kl \rangle \langle 1j \rangle \langle 2j \rangle \langle \alpha j \rangle}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \equiv \sum_{k<l} \sum_{\alpha=1}^{2n} \sum_{j=3}^{n} A_n^{(2)}(k, l, \alpha, j),$$

where $\epsilon_\alpha = 0 \ (1)$ if $\alpha = q \ (\bar{q})$, and we also denote $\bar{q}$ by 1 and $q$ by 2. The color factor,

$$C^{(2)} = (T^{a_3} \cdots T^{a_n})_{i_1}^{i_2},$$

is omitted. This can also be written as a product:

$$A_n^{(2)} = \Delta(\eta) \Delta^{(2)}(\rho, \chi) \prod_{i=1}^{n} \frac{1}{i(i+1)}, \quad (2.12a)$$

$$\Delta^{(2)}(\rho, \chi) = - \sum_{\alpha=1}^{2n} \sum_{j=3}^{n} (-1)^{\epsilon_\alpha+\epsilon_j} \langle 1j \rangle \langle 2j \rangle \langle \alpha j \rangle \rho_\alpha \chi_j. \quad (2.12b)$$

The tree-level four-quark MHV amplitudes $A_n^{(4)}(j)$ have the form

$$A_n^{(4)}(j) = - \sum_{k<l=1}^n (-1)^{\epsilon_k+\epsilon_l} \sum_{\alpha=1}^{2n} \sum_{\beta=1}^{j+1} (-1)^{\epsilon_\alpha+\epsilon_\beta} \rho_\alpha \rho_\beta \frac{\langle kl \rangle \langle 1(j+1) \rangle \langle 2j \rangle \langle \alpha \beta \rangle}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \equiv \sum_{k<l} \sum_{\alpha=1}^{2n} \sum_{\beta=1}^{j+1} A_n^{(4)}(k, l, \alpha, \beta), \quad (2.13)$$

where the color factor

$$C^{(4)} = (T^{a_3} \cdots T^{a_{j-1}})_{i_1}^{i_2} (T^{a_{j+2}} \cdots T^{a_n})_{i_{2q}}^{i_{q1}}, \quad (2.14)$$

*) This should not be confused with the direction of the momentum.
Fig. 2. The MHV vertices including (a) one external quark-antiquark pair (b) two external quark-antiquark pairs.

is omitted and $\bar{q}_1$, $q_1$, $\bar{q}_2$ and $q_2$ are also denoted by 1, 2, $j$ and $j + 1$, respectively, for simplicity. The amplitudes are distinguished by $j = 3, 4, \cdots, n - 1$, which determine the manner in which $n - 4$ vector multiplets are divided into two sets, as depicted in Fig. 2(b).

We can write the amplitude (2.13) in the product form

$$A_n^{(4)}(j) = \Delta(\eta)\Delta^{(4)}(\rho) \prod_{i=1}^{n} \frac{1}{(i(i+1))},$$  

(2.15a)

$$\Delta^{(4)}(\rho) = -\sum_{\alpha=1}^{2} \sum_{\beta=j}^{j+1} (-1)^{\epsilon_\alpha + \epsilon_\beta} \langle 1(j+1) \rangle \langle 2j \rangle \langle \alpha \beta \rangle \rho_\alpha \rho_\beta.$$  

(2.15b)

§3. Extended CSW rule and one-loop MHV amplitudes in $\mathcal{N} = 1$ SQCD

Another important piece of the CSW rule is a prescription for extending the external momenta of the tree-level MHV amplitudes into off-shell momenta to take the amplitudes as the vertices and to connect them by propagators. The method appropriate for the one-loop calculation is to construct an off-shell momentum $L_\mu$ from the on-shell one $l_\mu$, $l^2 = 0$, and a real variable $z$ by using an arbitrary but fixed null vector $\hat{\eta}_\mu$, $\hat{\eta}^2 = 0$ as

$$L_\mu = l_\mu + z\hat{\eta}_\mu.$$  

(3.1)

Including this prescription, the extended CSW rule for $\mathcal{N} = 1$ SQCD is summarized as follows.

1. Draw all the MHV diagrams with given external legs that have the appropriate topology fixed by the color and helicity configurations.

2. Assign the following building blocks for each diagram.

   (a) Vertex: the tree-level MHV amplitude $\langle 21 \rangle$, $\langle 21 \rangle$, or $\langle 21 \rangle$ with a delta function for total momentum conservation, represented by $i(2\pi)^4 \delta^4(\sum_i p_i)$. 


(b) Propagator: $1/L^2$ using the off-shell momentum extended by the prescription (3.1).

Two momenta and color indices connected by a propagator must be chosen so as to be conserved.

3. Assign the integral $\int \frac{d^4L}{(2\pi)^4} d\eta d\psi$ to each propagator, where we have $d\psi = d\chi (d\rho)$ for the propagator of a vector (chiral) multiplet. An extra minus sign is also assigned to the chiral multiplet loop. This comes from the fact that the quark superfield $Q$, given in (2.8), is fermionic.

We calculate all the one-loop MHV amplitudes as an application of this extended CSW rule for $\mathcal{N} = 1$ SQCD.

Before considering individual amplitudes, let us first outline the general structure of the one-loop MHV amplitudes obtained from the CSW rule. The MHV diagrams producing one-loop MHV amplitudes are depicted in Figs. 3-5. All these diagrams have the same topology, with two vertices connected by two propagators. Each partial amplitude for a fixed diagram is schematically given by

$$A_n = \int \frac{d^4L_1}{(2\pi)^4} \frac{d^4L_2}{(2\pi)^4} (2\pi)^4 i\delta^4(L_2 - L_1 + P_L) d\eta_1 d\psi_1 d\eta_2 d\psi_2 \frac{1}{L_1^2} A(L) \frac{1}{L_2^2} A(R),$$

(3.2)

where $P_L$ is the sum of external momenta attached to the left vertex. The factor yielding total momentum conservation, $i(2\pi)^4 \delta^4(\sum_i p_i)$, and the one-loop color factor $N_c C^{(0,2,4)}$ are omitted, as mentioned above. The amplitude is obtained by summing up the contributions from all the possible MHV diagrams. We use the convention that all the external momenta are outgoing and loop momentum flows in the clockwise direction. The factor $A(L)(A(R))$ is an appropriate MHV vertex corresponding to the left (right) vertex in the MHV diagram.

Using the variables $(l, z)$ in Eq. (3.1), the loop-integration measure can be rewritten as

$$\frac{d^4L_1}{L_1^2} \frac{d^4L_2}{L_2^2} \delta^4(L_2 - L_1 + P_L) = \frac{dz_1 dz_2}{z_1 z_2} dLIPS(l_2, -l_1; P_{L;z}),$$

(3.3)

where the Lorentz-invariant phase-space integration measure $dLIPS(l_2, -l_1; P_{L;z})$ is defined by

$$dLIPS(l_2, -l_1; P_{L;z}) = d^4l_1 \delta^{(+)}(l_1^2) d^4l_2 \delta^{(+)}(l_2^2) \delta^4(l_2 - l_1 + P_{L;z}),$$

(3.4)

with $\delta^{(+)}(l^2) = \theta(l^0) \delta(l^2)$, $z = z_1 - z_2$ and $P_{L;z} = P_L - z \hat{n}$.

Because the loop integral in (3.2) generally has both ultraviolet and infrared divergences, we use the supersymmetric regularization, i.e. the four-dimensional helicity scheme. That is, after carrying out all the spinor calculations in four dimensions, the loop integral is
evaluated in $4 - 2\epsilon$ dimensions. After some manipulations, the general form of the one-loop MHV amplitudes \[ A_n = \int d\mathcal{M} d\eta_1 d\eta_2 d\psi_1 d\psi_2 \mathcal{A}(L)\mathcal{A}(R), \] can be written as

\[ d\mathcal{M} = \frac{\mu^2}{(2\pi)^{3-2\epsilon}} \frac{dz}{z} d\text{LIPS}(l_2, -l_1; P_{L;z}). \] (3.6)

The renormalization scale $\mu$ is introduced, and the phase-space integral must be evaluated in $4 - 2\epsilon$ dimensions. Moreover, the $\eta$ integration can be easily carried out independently of the specific diagrams, since it can be written in the form of the conservation, as mentioned above:

\[ \int d\eta_1 d\eta_2 \Delta(\eta_L)\Delta(\eta_R) = \langle l_1 l_2 \rangle \sum_{i<j=1}^{n} \langle ij \rangle \eta_i \eta_j = \langle l_1 l_2 \rangle \Delta(\eta). \] (3.7)

The remaining parts depend on the individual diagrams and are computed in the following subsections.

### 3.1. All-gluon MHV amplitudes

In order to specifically explain how the one-loop amplitudes are obtained from the CSW rule, we first consider the one-loop MHV amplitudes whose external particles are all gluon vector multiplets. The superfield formulation naturally reproduces supersymmetrically decomposed amplitude (3.10). The MHV diagrams contributing to such amplitudes are depicted in Fig. 3. The first diagram, Fig. 3 (a), represents the SYM contribution obtained in previous works. The second one, Fig. 3 (b), obtained by using the new vertex $A_n^{(2)}$, gives a new contribution coming from the chiral-multiplet loop.

Let us first compute $A_n^{(a)}$, the contribution of the diagram (a). The loop $\chi$-integral can
be easily evaluated, after a calculation using Schouten’s identity (A.7), as
\[ \int d \chi_i d \chi_j A^{(0)}(\chi_L) A^{(0)}(\chi_R) = \langle l_1 l_2 \rangle^3 \sum_{i<j} \langle ij \rangle^3 \chi_i \chi_j \]
\[ + 3 \langle l_1 l_2 \rangle \sum_{i=m_1}^{m_2} \sum_{j=m_2+1}^{m_1-1} \langle ij \rangle \langle il_1 \rangle \langle il_2 \rangle \langle jl_1 \rangle \langle jl_2 \rangle \chi_i \chi_j. \]

(3.8)

Then we obtain
\[ A_n^{(a)} = \sum_{k<l} \sum_{i<j} A_n^{(0)}(k, l, i, j) \int d M \frac{\langle m_1 (m_1 - 1) \rangle \langle l_2 l_1 \rangle \langle m_2 (m_2 + 1) \rangle \langle l_1 l_2 \rangle}{\langle (m_1 - 1) l_1 \rangle \langle l_1 m_1 \rangle \langle m_2 l_2 \rangle \langle l_2 (m_2 + 1) \rangle} \]
\[ - 3 \sum_{k<l} \sum_{i=m_1}^{m_2} \sum_{j=m_2+1}^{m_1-1} A_n^{(0)}(k, l, i, j) \int d M \frac{\langle m_1 (m_1 - 1) \rangle \langle m_2 (m_2 + 1) \rangle \langle il_1 \rangle \langle il_2 \rangle \langle jl_1 \rangle \langle jl_2 \rangle}{\langle ij \rangle^2 \langle (m_1 - 1) l_1 \rangle \langle l_1 m_1 \rangle \langle m_2 l_2 \rangle \langle l_2 (m_2 + 1) \rangle}. \]

(3.9)

Each term here coincides with the contribution of the loop of the $\mathcal{N} = 4$ vector and the $\mathcal{N} = 1$ adjoint chiral multiplet obtained in Refs. 7) and 8), 9), respectively. We have
\[ A_n^{(a)} = A_n^{N=1 \text{vector}} = A_n^{N=4} - 3 A_n^{N=1 \text{chiral}}, \]

(3.10)

where the $\mathcal{N} = 4$ SYM contribution $A_n^{N=4}$ is given by 7)
\[ A_n^{N=4} = \sum_{k<l} \sum_{i<j} c_r A_n^{(0)}(k, l, i, j) \sum_{m=1}^{n \left[\frac{3}{2}\right]-1} \sum_{r=1}^{\left(1 - \frac{1}{2} \delta_{2, r_{-}} \right)} F(s, t, P^2, Q^2), \]

(3.11)

with
\[ c_r = \frac{1}{(4 \pi)^2 \epsilon} \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2 \epsilon)}. \]

(3.12)

The summation is taken over all possible MHV diagrams of the type depicted in Fig. 3(a), which are obtained by dividing the $n$ external legs into two vertices, fixing the ordering. The factor of 1/2 is needed for the diagrams whose two vertices have the same number, $n/2$, of legs because these are counted twice in the summation. 7) The scalar box function $F$ and its finite part $B$ are defined by
\[ F(s, t, P^2, Q^2) = - \frac{1}{\epsilon^2} \left[ \left( \frac{\mu^2}{-s} \right)^\epsilon + \left( \frac{\mu^2}{-t} \right)^\epsilon - \left( \frac{\mu^2}{-P^2} \right)^\epsilon - \left( \frac{\mu^2}{-Q^2} \right)^\epsilon \right] + B(s, t, P^2, Q^2), \]

(3.13)

\[ B(s, t, P^2, Q^2) = \text{Li}_2 \left( 1 - \frac{P^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{P^2}{t} \right) + \text{Li}_2 \left( 1 - \frac{Q^2}{s} \right) 

+ \text{Li}_2 \left( 1 - \frac{Q^2}{t} \right) - \text{Li}_2 \left( 1 - \frac{P^2 Q^2}{st} \right) + \frac{1}{2} \log^2 \left( \frac{s}{t} \right). \]

(3.14)
where $Li_2(x) = -\int_0^x \frac{\log(1-y)}{y} dy$ is the dilogarithm function.\(^{20}\) The scalar invariants in the arguments are defined by $s = (p + P)^2$ and $t = (P + q)^2$, with $p = p_m$, $q = p_{m+r+1}$, $P = p_{m+1} + \cdots + p_{m+r}$ and $Q = p_{m+r+2} + \cdots + p_{m-1}$, which satisfy $p + q + P + Q = 0$, due to the conservation of total momentum.

The contribution of the $\mathcal{N} = 1$ chiral multiplet, $\mathcal{A}_n^{N=1\text{chiral}}$, has the form\(^8,9\)

$$\mathcal{A}_n^{N=1\text{chiral}} = \sum_{k<l} \sum_{i<j} c_{i,j,k,l} A_n^{(0)}(k, l, i, j) \left[ \sum_{m=j+1}^{i-1} \sum_{a=i+1}^{j-1} b_{ia}^{ij} B(s, t, P^2, Q^2) \right. $$

$$ + \left. \frac{1}{1 - 2\epsilon} \left( \sum_{m=i+1}^{j-1} \sum_{a=j}^{i-1} c_{ia}^{ij} T_\epsilon(p, P^2, Q^2) + (i \leftrightarrow j) \right) \right], \quad (3.15)$$

with coefficients

$$b_{ia}^{ij} = \frac{\text{tr}(ijma)\text{tr}(ijam)}{s_{ij}^2 s_{ma}^2}, \quad (3.16a)$$

$$c_{ia}^{ij} = \frac{1}{2} \left[ \frac{\text{tr}(ijam)}{s_{ij} s_{ma}} - \frac{\text{tr}(ij(a+1)m)}{s_{ij} s_{ma+1}} \right] \frac{\text{tr}(ijmP_{m+1a}) - \text{tr}(ijmP_{m+1a})}{s_{ij}}. \quad (3.16b)$$

Here we introduce the notation $s_{ij} = (p_i + p_j)^2$, $P_{m+1a} = p_{m+1} + p_{m+2} + \cdots + p_a$ and $\text{tr}(ijkl) = \text{tr}(\gamma_i \gamma_j \gamma_k \gamma_l)$, with the trace convention defined in Appendix A. The $\epsilon$-dependent triangle function $T_\epsilon$ is defined by\(^8\)

$$T_\epsilon(p, P, Q) = \frac{(\mu^2)^\epsilon (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}}{Q^2 - P^2}, \quad (3.17)$$

where the renormalization point $\mu^2$ dependence is included. This extended definition also contains the bubble function, which has a divergent contribution, as the degenerate case,\(^8\) as summarized in Appendix B. As in the case of the $\mathcal{N} = 4$ part, the summation is taken over all possible MHV diagrams constrained now such that the $i$-th and $j$-th external legs, corresponding to the negative helicity states, are attached to different vertices.\(^8,9\)

The contribution of the diagram (b) can be similarly calculated by using the the MHV vertex $\mathcal{A}^{(2)}$. As the loop $\rho$-integral gives

$$\int d\rho_1 d\rho_2 \Delta^{(2)}(\rho_L, \chi_L) \Delta^{(2)}(\rho_R, \chi_R) = \langle l_1 l_2 \rangle \sum_{i=m_1}^{m_2} \sum_{j=m_2+1}^{m_1-1} \langle ij \rangle \langle il_1 \rangle \langle il_2 \rangle \langle jl_1 \rangle \langle jl_2 \rangle \chi_i \chi_j, \quad (3.18)$$

the result becomes $\mathcal{A}_n^{(b)} = \mathcal{A}_n^{N=1\text{chiral}}$, taking into account an extra minus sign. If we have $N_f$ chiral multiplets (flavors), the total one-loop all-gluon MHV amplitude is given by

$$\mathcal{A}_n = \mathcal{A}_n^{N=4} - 3 \mathcal{A}_n^{N=1\text{chiral}} + \frac{N_f}{N_c} \mathcal{A}_n^{N=1\text{chiral}}. \quad (3.19)$$

The factor of $1/N_c$ comes from the color factor.
3.2. Two-quark MHV amplitudes

The one-loop two-quark MHV amplitudes can be computed from the MHV diagrams depicted in Fig. 4.

Fig. 4. One-loop MHV diagrams for two-quark MHV amplitudes.

Let us begin by evaluating the diagram (a). The loop χ-integral gives

\[
\int d\chi_1 d\chi_2 \Delta^{(2)}(L) \Delta^{(0)}(R) = \langle l_1 l_2 \rangle^3 \sum_{\alpha=1}^{n} \sum_{j=3}^{n} \langle 1j \rangle \langle 2j \rangle \langle \alpha j \rangle \rho_\alpha \chi_j \\
+ 3 \langle l_1 l_2 \rangle \sum_{\alpha=1}^{2} \sum_{j=m_2+1}^{m_1-1} \langle \hat{\alpha} j \rangle \langle \alpha l_1 \rangle \langle j l_1 \rangle \langle j l_2 \rangle \rho_\alpha \chi_j \\
+ \langle l_1 l_2 \rangle \sum_{\alpha=1}^{2} \sum_{j=m_2+1}^{m_1-1} \langle \alpha \hat{\alpha} \rangle \left( \langle \alpha l_1 \rangle \langle j l_2 \rangle + \langle \alpha l_2 \rangle \langle j l_1 \rangle \right) \langle j l_1 \rangle \langle j l_2 \rangle \rho_\alpha \chi_j,
\]

(3.20)

where \( \hat{\alpha} = 2 \) (1) for \( \alpha = 1 \) (2). The first two terms give, after summing up all the diagrams, the same amplitudes \( A_n^{N=4} \) and \( A_n^{N=1, \text{chiral}} \), as in the previous case. Actually, these amplitudes can be decomposed into primitive amplitudes as\(^{21}\)

\[
A_n = A_n^{N=4} - 3 A_n^{N=1, \text{chiral}} - A_n^R - \frac{N_f}{N_c} A_f^R,
\]

(3.21)

where the last term is the contribution of the chiral-multiplet loop coming from the diagram (c). Therefore, here we compute the other contributions \( A_n^R \). The contribution to \( A_n^R \) comes from the third term in Eq. (3.20), and yields

\[
A_n^{R(a)} = \sum_{k<l=1}^{n} \sum_{\alpha=1}^{2} \sum_{j=3}^{n} A_n^{(2)}(k, l, \alpha, j) \sum_{m_1=j+1}^{n+1} \sum_{m_2=2}^{j-1} \int dM f_{(m_1, m_2)}^{(\alpha, j)(a)}(l_1, l_2),
\]

(3.22)

where \( n+1 \equiv 1 \) and

\[
f_{(m_1, m_2)}^{(\alpha, j)(a)}(l_1, l_2) = \frac{\langle \alpha \hat{\alpha} \rangle \langle (m_1 - 1)m_1 \rangle \langle m_2(m_2 + 1) \rangle \langle (\alpha l_1) \langle j l_2 \rangle + \langle \alpha l_2 \rangle \langle j l_1 \rangle \rangle \langle j l_1 \rangle \langle j l_2 \rangle}{\langle 1j \rangle \langle 2j \rangle \langle \alpha j \rangle \langle (m_1 - 1)l_1 \rangle \langle l_1 m_1 \rangle \langle m_2 l_2 \rangle \langle l_2(m_2 + 1) \rangle}.
\]

(3.23)
The integrand $I^{(a,j)(a)}_{(m_1,m_2)}(l_1,l_2)$ can be written as

$$I^{(a,j)(a)}_{(m_1,m_2)}(l_1,l_2) = R^{(a,j)}(m_1 - 1, m_2) - R^{(a,j)}(m_1 - 1, m_2 + 1)$$

$$- R^{(a,j)}(m_1, m_2) + R^{(a,j)}(m_1, m_2 + 1), \quad (3.24)$$

where

$$R^{(a,j)}(m_1, m_2) = \frac{1}{4(m_1 \cdot l_1)(m_2 \cdot l_2)} \left( \frac{\text{tr}(j\alpha m_1 l_1)\text{tr}(j\beta m_2 l_2)}{4(\alpha \cdot j)(\alpha \cdot \beta)} + \frac{\text{tr}(j\alpha m_1 l_1)\text{tr}(j\beta m_2 l_2)}{4(\alpha \cdot j)(\alpha \cdot \beta)} \right)$$

$$- 2 \frac{\text{tr}(j\alpha m_1 l_1)\text{tr}(j\beta m_2 l_2)}{4(\alpha \cdot j)^2}$$

$$\equiv \frac{r^{(a,j)}_{(m_1,m_2)}(l_1,l_2)}{4(m_1 \cdot l_1)(m_2 \cdot l_2)}. \quad (3.25)$$

We use the same notation for (holomorphic) spinors and the corresponding four-vectors, e.g. $\langle ij \rangle = \langle \lambda_i \lambda_j \rangle$ and $(i \cdot j) = p_{i\mu}p_{j}^{\mu}$, with $p_{i\alpha} = \lambda_{i\alpha} \bar{\lambda}_{i\dot{\alpha}}$.

Next, we carry out the loop integration. Using the formula (C.6), the phase-space integral can be evaluated, except in the case $(m_1, m_2) = (1, 2)$, which is considered later, as

$$\int d\mathcal{M} R^{(a,j)}(m_1, m_2) = c_F \left( \frac{\mu^2}{\pi \epsilon \csc(\pi \epsilon)} \right) \int \frac{dz}{z} (P_{L;i}^2)^{-\epsilon} \left( - \frac{r^{(a,j)}_{(m_1,m_2)}(m_2,m_1)}{4(m_1 \cdot m_2)^2} \log (1 - a P_{L;i}^2) \right)$$

$$+ \frac{1}{1 - 2\epsilon} \left[ \frac{r^{(a,j)}_{(m_1,m_2)}(P_{L;i}, m_1)}{4(m_1 \cdot m_2)(m_1 \cdot P_{L;i})} + \frac{r^{(a,j)}_{(m_1,m_2)}(m_2, P_{L;i})}{4(m_1 \cdot m_2)(m_2 \cdot P_{L;i})} \right]. \quad (3.26)$$

where we use the $\hat{\eta} = p_{m_1}$ or $p_{m_2}$ gauge so that

$$a = \frac{(m_1 \cdot m_2)}{(m_1 \cdot m_2)P_{L;i}^2 - 2(m_1 \cdot P_{L;i})(m_2 \cdot P_{L;i})}. \quad (3.27)$$

It should be noted that in the center-of-mass frame, in which we have $P_{L;i} = P_{L;i}(1,0)$, the $P_{L;i}$ dependence in [ ] in the integrand cancels out.\(^{17}\) The dispersion integral therefore reduces to the two types of integrations evaluated in Refs. 7)-8),17):

$$\int \frac{dz}{z} (P_{L;i}^2)^{-\epsilon} \log (1 - a P_{L;i}^2) = (\pi \epsilon \csc(\pi \epsilon)) \text{Li}_2 \left( 1 - a P_{L;i}^2 \right), \quad (3.28a)$$

$$\int \frac{dz}{z} (P_{L;i}^2)^{-\epsilon} = (\pi \epsilon \csc(\pi \epsilon)) \frac{(-P_{L;i}^2)^{-\epsilon}}{\epsilon}. \quad (3.28b)$$

Consequently, the dispersion integral in (3.26) is calculated as

$$\int d\mathcal{M} R^{(a,j)}(m_1, m_2) = c_F \left( - \frac{r^{(a,j)}_{(m_1,m_2)}(m_2,m_1)}{4(m_1 \cdot m_2)^2} \text{Li}_2 \left( 1 - a P_{L;i}^2 \right) \right).$$
\[
\frac{1}{1 - 2\epsilon} \left[ \frac{r_{(\alpha,j)}^{(\alpha,j)}(P_L, m_1)}{4(m_1 \cdot m_2)(m_1 \cdot P_L)} + \frac{r_{(m_1, m_2)}^{(\alpha,j)}(m_2, P_L)}{4(m_1 \cdot m_2)(m_2 \cdot P_L)} \right] \frac{1}{\epsilon} \left( \frac{\mu^2}{-P_L^2} \right)^\epsilon.
\]

(3.29)

We must evaluate the integral separately for the case \((m_1, m_2) = (1, 2)\), because in that case, \(a\) defined by Eq. (3.27) becomes infinity. This appears only for the case \(P_L = P_{12}\) and can be computed by using the formula (C.13) as 

\[
\int d\mathcal{M}|_{P_L=\mathcal{P}_{12}} R^{(\alpha,j)}(1, 2) = \frac{c_f}{\epsilon^2(1 - 2\epsilon)} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon.
\]

(3.30)

By adding up the four (two) terms in (3.24) and appropriately shifting the summation indices \(m_1\) and \(m_2\), we generically get the box (triangle) function defined by (3.14) (3.17) due to the formula (B.1). However, some terms at the boundary of the summation region do not have partners.

The missing pieces of the primitive amplitude \(A_n^R\) are supplied by the diagram (b). The loop fermion integration \(\int d\chi_I d\rho_{l_2} \Delta^{(2)}(L) \Delta^{(2)}(R)\) in this case can be computed similarly, and the contribution to the primitive amplitudes \(A_n^R\) is given by

\[
\langle l_1 l_2 \rangle \sum_{j=3}^{m_2} \left( \langle 2j \rangle \langle l_1 l_2 \rangle (\langle j l_1 \rangle \langle j l_2 \rangle + \langle j l_2 \rangle \langle j l_1 \rangle) \rho_1 \chi_j - \langle 2j \rangle \langle l_1 l_2 \rangle \langle j l_2 \rangle \rho_2 \chi_j \right)
\]

\[+ \langle l_1 l_2 \rangle \sum_{j=m_2+1}^{n} \left( \langle 2j \rangle \langle l_1 l_2 \rangle \langle l_1 l_2 \rangle \langle j \rangle \rho_1 \chi_j - \langle 2j \rangle \langle l_1 l_2 \rangle \langle j l_2 \rangle \rho_1 \chi_j + \langle 2j \rangle \langle l_1 l_2 \rangle \langle j l_2 \rangle \rho_2 \chi_j \right).\]

(3.31)

The amplitudes \(A_n^{R(b)}\) of the diagram (b) then become

\[
A_n^{R(b)} = \sum_{k<l=1}^{n} \sum_{\alpha=1}^{2} \sum_{j=3}^{n} A_n^{(2)}(k, l, \alpha, j) \sum_{m=2}^{j-1} \int d\mathcal{M}_{(R,m)}^{(\alpha,j)}(l_1, l_2)
\]

\[+ \sum_{k<l=1}^{n} \sum_{\alpha=1}^{2} \sum_{j=3}^{n} A_n^{(2)}(k, l, \alpha, j) \sum_{m=j}^{n} \int d\mathcal{M}_{(L,m)}^{(\alpha,j)}(l_1, l_2),
\]

(3.32)

where

\[
I_{(J,m)}^{(\alpha,j)} = \begin{cases} 
\frac{12 \langle m(m+1) \rangle (\langle \alpha l_1 \rangle \langle j l_2 \rangle) (\langle j l_1 \rangle \langle \alpha l_2 \rangle) \langle j l_2 \rangle}{\langle \alpha j \rangle \langle l_1 l_2 \rangle \langle j l_2 \rangle}, & \text{for } (J, \alpha) = (L, 2), (R, 1), \\
\frac{12 \langle m(m+1) \rangle (\langle \alpha l_1 \rangle \langle m l_2 \rangle) (\langle \alpha l_2 \rangle \langle l_2(m+1) \rangle)}{\langle \alpha j \rangle \langle l_1 l_2 \rangle \langle j l_2 \rangle}, & \text{for } (J, \alpha) = (L, 1), (R, 2), \\
\end{cases}
\]

(3.33)

\(^{(*)}\) This is effectively obtained by replacing the corresponding (divergent) dilogarithm function \(\text{Li}_2\) in (3.29) with the factor \(-\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon\).
The integrand $I_{(j,m)}^{(a,j)b}$ can be rewritten as

$$I_{(j,m)}^{(a,j)b} = R^{(a,j)}_j(m) - R^{(a,j)}_j(m+1), \quad (3.34)$$

with

$$R^{(a,j)}_j(m) = \begin{cases} 
\gamma(\alpha) \left[ \frac{1}{4(\bar{\alpha} \cdot l_1)(m \cdot l_2)} \left( \frac{\text{tr}(j\bar{\alpha}l_1)\text{tr}(j\alpha l_2)}{4(\alpha \cdot j)(\bar{\alpha} \cdot j)} - 2\frac{\text{tr}(j\bar{\alpha}l_1)\text{tr}(j\alpha l_2)}{4(\alpha \cdot j)^2} \right) 
\right. 
+ \frac{1}{2(m \cdot l_2)} \left( \frac{\text{tr}(j\alpha l_2)}{2(\alpha \cdot j)} - \frac{\text{tr}(j\bar{\alpha} l_2)}{2(\bar{\alpha} \cdot j)} \right), 
\left. \right. 
& \text{for } (J, \alpha) = (L, 1), (R, 2), \\
\gamma(\alpha) \left[ -\frac{1}{4(\alpha \cdot l_1)(m \cdot l_2)} \left( \frac{\text{tr}(j\alpha l_1)\text{tr}(j\alpha l_2)}{4(\alpha \cdot j)(\bar{\alpha} \cdot j)} \right) 
\right. 
+ \frac{1}{2(m \cdot l_2)} \left( \frac{\text{tr}(j\alpha l_2)}{2(\alpha \cdot j)} - \frac{\text{tr}(j\bar{\alpha} l_2)}{2(\bar{\alpha} \cdot j)} \right), 
\right. 
& \text{for } (J, \alpha) = (L, 2), (R, 1), 
\end{cases} \quad (3.35)$$

where $\gamma$ is a sign factor defined by

$$\gamma(\alpha) = \begin{cases} 
-1 & \text{for } \alpha = 1, \\
+1 & \text{for } \alpha = 2. 
\end{cases} \quad (3.36)$$

We can decompose $R^{(a,j)}_j(m)$ as

$$R^{(a,j)}_j(m) = \begin{cases} 
\gamma(\alpha) \left( R^{(a,j)}_j(\bar{\alpha}, m) + R^{(a,j)}_j(m) \right), & \text{for } (J, \alpha) = (L, 1), (R, 2), \\
\gamma(\alpha) \left( -R^{(a,j)}_j(\alpha, m) + R^{(a,j)}_j(m) \right), & \text{for } (J, \alpha) = (L, 2), (R, 1), 
\end{cases} \quad (3.37)$$

where

$$\tilde{R}^{(a,j)}_m(m) = \frac{1}{2(m \cdot l_2)} \left( \frac{\text{tr}(j\alpha l_2)}{2(\alpha \cdot j)} - \frac{\text{tr}(j\bar{\alpha} l_2)}{2(\bar{\alpha} \cdot j)} \right)$$

$$R^{(a,j)}_m(m) = \frac{\tilde{R}^{(a,j)}_m(l_2)}{2(m \cdot l_2)}. \quad (3.38)$$

The loop integral of the first term is given in [K28]. The second term can be similarly computed as

$$\int d\mathcal{M} \tilde{R}^{(a,j)}_m(m) = \frac{c_T (\mu^2)^\epsilon}{(\pi \epsilon \csc(\pi \epsilon))(1-2\epsilon)} \int \frac{dz}{z} \left( P_{L;z}^2 \right)^{-\epsilon} \tilde{r}^{(a,j)}_m(P_{L;z})$$

$$= \frac{c_T \tilde{r}^{(a,j)}_m(P_L)}{1-2\epsilon} \frac{1}{2(m \cdot P_L)} \frac{1}{\epsilon} \left( \frac{\mu^2}{-P_L^2} \right)^\epsilon, \quad (3.39)$$
using formulas (C.1b) and (3.28b). We replace $P_{L;z}$ with $P_L$, except for the $(P_{L;z}^2)^{-\epsilon}$ factor, in the dispersion integral, for the same reason as above.

By summing up the contributions from the two diagrams (a) and (b), the primitive amplitudes $A_n^R$ of the two-quark MHV amplitudes are, after some calculation, obtained as

$$A_n^R = \sum_{k<l=1}^{n} \sum_{a=1}^{2} \sum_{j=3}^{n} c_r A_n^{(2)}(k, l, \alpha, j) \left[ -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon + \sum_{m=j+1}^{n+1} \sum_{a=2}^{j-1} b_{ma}^{\alpha j} B(m, a) \right] + \frac{1}{1-2\epsilon} \left( \sum_{m=2}^{n} \sum_{a=j}^{n+1} \sum_{j=1}^{j-1} \right) c_{ma}^{\alpha j} T_\epsilon(m, a) + \frac{1}{1-2\epsilon} \sum_{m \in \Gamma(\alpha)} \bar{c}_{m}^{\alpha j} T_\epsilon(m, 1),$$

(3.40)

with

$$b_{ma}^{\alpha j} = \frac{\text{tr}(jama)\text{tr}(j\alpha am)}{s_{ja}s_{ja}^2ma} + \frac{\text{tr}(jama)\text{tr}(j\alpha am)}{s_{ja}s_{ja}^2ma} - 2\frac{\text{tr}(jama)\text{tr}(j\alpha am)}{s_{ja}^2ma},$$

(3.41a)

$$c_{ma}^{\alpha j} = \left( \frac{\text{tr}(jama) - \text{tr}(j\alpha(a+1)m)}{s_{ja}s_{ma} + 1} \right) \frac{\text{tr}(j\alpha m P_{m+1a})}{s_{ja}} + \left( \frac{\text{tr}(jama) - \text{tr}(j\alpha(a+1)m)}{s_{ja}s_{ma} + 1} \right) \frac{s_{ja}}{s_{ja}},$$

(3.41b)

$$\bar{c}_{m}^{\alpha j} = -2\gamma(\alpha) \frac{\text{tr}(j \alpha m)\text{tr}(j \alpha m P_{m+11})}{s_{\alpha j}^2},$$

(3.41c)

where the summation region in the last term is defined by

$$\Gamma(\alpha) = \begin{cases} 
\{ j+1, \ldots, n \}, & \text{for } \alpha = 1, \\
\{ 3, \ldots, j - 1 \}, & \text{for } \alpha = 2,
\end{cases}$$

(3.42)

and we have introduced the notation

$$B(m, a) \equiv B(P_{ma-1}^2, P_{ma+1}^2, P_{m+1a-1}^2, P_{m+1a-1}^2),$$

(3.43a)

$$T_\epsilon(m, a) \equiv T_\epsilon(p_m, P_{a+1m-1}, P_{m+1a}).$$

(3.43b)

The quark-loop diagram (c) leads the primitive amplitude $A_n^f$ in the decomposition (3.21). After carrying out the loop fermion integration, we have

$$A_n^f = -\sum_{k<l=1}^{n} \sum_{a=1}^{2} \sum_{j=3}^{n} A_n^{(2)}(k, l, \alpha, j) \sum_{m_1=j+1}^{n+1} \sum_{m_2=2}^{j-1} \int dM_{(m_1, m_2)}^{(j)}(l_1, l_2),$$

(3.44)

where

$$I_{(m_1, m_2)}^{(j)}(l_1, l_2) = \frac{\langle (m_1 - 1)m_1 | m_2(m_2 + 1) \rangle \langle 1l_1 | 2l_2 \rangle \langle jl_1 | jl_2 \rangle}{\langle 1j | 2j \rangle \langle (m_1 - 1)l_1 | m_2l_2 \rangle \langle l_2(m_2 + 1) \rangle}. $$

(3.45)
This can also be written as

$$I_{(m_1,m_2)}^{(j)}(l_1,l_2) = R_f^{(j)}(m_1 - 1, m_2) - R_f^{(j)}(m_1 - 1, m_2 + 1) - R_f^{(j)}(m_1, m_2) + R_f^{(j)}(m_1, m_2 + 1),$$

(3.46)

where

$$R_f^{(j)}(m_1, m_2) = -\frac{\text{tr}(j_1 m_1 k) \text{tr}(j_2 m_2 l)}{16(1 \cdot j)(2 \cdot j)(m_1 \cdot l_1)(m_2 \cdot l_2)}.$$  

(3.47)

The loop integral can be easily carried out similarly to those in the previous cases, and we obtain

$$\mathcal{A}_n^f = \sum_{k<l=1}^{2} \sum_{a=1}^{n} \sum_{j=3}^{m_1} c_f A_n^{(2)}(k, l, \alpha, j) \left[ \sum_{m=j+1}^{n} \sum_{a=3}^{n} b_{ma} B(m, a) \right] + \frac{1}{1 - 2\epsilon} \left( \sum_{m=3}^{n} \sum_{a=j}^{n} + \sum_{m=j+1}^{n} \sum_{a=2}^{n} \right) c_{ma}^j T_\epsilon(m, a),$$

(3.48)

where the coefficient functions are defined by

$$b_{ma}^j = -\frac{\text{tr}(j_1 ma) \text{tr}(j_2 am)}{s_{1j} s_{ma} s_{2j}}.$$  

(3.49a)

$$c_{ma}^j = \frac{1}{2} \left[ \left( \frac{\text{tr}(1jam)}{s_{1j} s_{ma}} - \frac{\text{tr}(1j(a+1)m)}{s_{1j} s_{ma+1}} \right) \frac{\text{tr}(j_2 mp_{m+1a})}{s_{2j}} \right. \right.$$

$$+ \left. \left( \frac{\text{tr}(2jam)}{s_{2j} s_{ma}} - \frac{\text{tr}(2j(a+1)m)}{s_{2j} s_{ma+1}} \right) \frac{\text{tr}(j_1 mp_{m+1a})}{s_{1j}} \right].$$  

(3.49b)

### 3.3. Four-quark MHV amplitudes

We can similarly calculate the one-loop four-quark MHV amplitudes from the MHV diagrams depicted Fig. 5. The seven diagrams (a)-(g) give contributions to the first three terms of the same decomposition as in the previous case, (3.21):

$$\mathcal{A}_n = \mathcal{A}_n^{N=4} - 3\mathcal{A}_n^{N=1\text{chir}} - \mathcal{A}_n^R - \frac{N_f}{N_c} \mathcal{A}_n^f.$$  

We present a compact expression of the unknown piece $\mathcal{A}_n^R$ in the following.

There are two types of amplitudes characterized by the helicity configurations $(\alpha, \beta) = (1, j), (2, j + 1)$ and $(1, j + 1), (2, j)$. The two quarks have the same helicity in the former configuration and different helicities in the latter. After the loop fermion integration, the contribution to $\mathcal{A}_n^{R(\alpha)}$ from the diagram (a) becomes

$$\mathcal{A}_n^{R(\alpha)} = \sum_{k<l=1}^{n} \sum_{\alpha=1}^{2} \sum_{\beta=j}^{j} \mathcal{A}_n^{(k, l, \alpha, \beta)} \sum_{m_1=j+2}^{j+1} \sum_{m_2=2}^{j-1} \int dM_{j}^{(\alpha, \beta)(a)}(l_1, l_2),$$

(3.50)
where we have

\[
J^{(\alpha,\beta)(a)}_{(m_1,m_2)}(l_1,l_2) = \begin{cases} 
- \frac{\langle \alpha \bar{\alpha} \rangle \langle \beta \bar{\beta} \rangle \langle \alpha l_1 \rangle \langle \alpha l_2 \rangle \langle \beta l_1 \rangle \langle \beta l_2 \rangle}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle^2} \\
\frac{\langle \alpha \bar{\alpha} \rangle \langle \beta \bar{\beta} \rangle \langle \alpha l_1 \rangle \langle \alpha l_2 \rangle \langle \beta l_1 \rangle \langle \beta l_2 \rangle}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle^2} \\
- \frac{\langle \beta \bar{\beta} \rangle \langle \alpha l_1 \rangle \langle \alpha l_2 \rangle (\langle \alpha l_1 \rangle \langle \beta l_2 \rangle + \langle \beta l_1 \rangle \langle \alpha l_2 \rangle)}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle^2} \\
\frac{\langle \beta \bar{\beta} \rangle \langle \alpha l_1 \rangle \langle \alpha l_2 \rangle (\langle \alpha l_1 \rangle \langle \beta l_2 \rangle + \langle \beta l_1 \rangle \langle \alpha l_2 \rangle)}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle^2} \\
D(m_1,m_2), \\
\end{cases}
\]

for \((\alpha,\beta) = (1,j)\) or \((2,j+1)\),

\[
-2 \frac{\langle \alpha l_1 \rangle \langle \alpha l_2 \rangle \langle \beta l_1 \rangle \langle \beta l_2 \rangle}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle^2} \\
+ \frac{\langle \alpha l_1 \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \beta l_1 \rangle \langle \beta l_2 \rangle}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle} \\
+ \frac{\langle \alpha l_1 \rangle \langle \alpha l_2 \rangle \langle \beta l_1 \rangle \langle \beta l_2 \rangle}{\langle \alpha \beta \rangle \langle \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta \rangle} \\
D(m_1,m_2),
\]

for \((\alpha,\beta) = (1,j+1)\) or \((2,j)\).

(3.51)
Here we define the common factor $D$ as

$$D(m_1, m_2) = \frac{\langle (m_1 - 1)m_1 \rangle \langle m_2 (m_2 + 1) \rangle}{\langle (m_1 - 1)l_1 \rangle \langle l_1 m_1 \rangle \langle m_2 l_2 \rangle \langle l_2 (m_2 + 1) \rangle}. \tag{3.52}$$

The integrand $J_{(m_1, m_2)}(l_1, l_2)$ can be decomposed as

$$J_{(m_1, m_2)}(l_1, l_2) = S^{(\alpha, \beta)}(m_1 - 1, m_2) - S^{(\alpha, \beta)}(m_1 - 1, m_2 + 1)$$

$$- S^{(\alpha, \beta)}(m_1, m_2) + S^{(\alpha, \beta)}(m_1, m_2 + 1), \tag{3.53}$$

with

$$S^{(\alpha, \beta)}(m_1, m_2) = \begin{cases} 
\frac{1}{4} (\Omega - 4) \left( S(\alpha, \beta, \alpha, \beta) + S(\beta, \alpha, \beta, \alpha) \right) \\
+ S(\alpha, \beta, \alpha, \tilde{\beta}) + S(\beta, \alpha, \tilde{\beta}, \alpha), \\
\text{for } (\alpha, \beta) = (1, j) \text{ or } (2, j + 1), \\
- \frac{1}{2} S(\alpha, \beta, \alpha, \beta) + S(\beta, \alpha, \beta, \alpha) \\
+ \frac{1}{2} \Omega \left( S(\tilde{\alpha}, \alpha, \tilde{\alpha}, \beta) + S(\tilde{\beta}, \beta, \tilde{\beta}, \alpha) \right) \\
+ \frac{1}{2} \left( S(\alpha, \beta, \alpha, \tilde{\beta}) + S(\beta, \alpha, \tilde{\beta}, \alpha) \right), \\
\text{for } (\alpha, \beta) = (1, j + 1) \text{ or } (2, j),
\end{cases} \tag{3.54}$$

where

$$\Omega = \frac{\text{tr}(1(1 + 1)j^2)}{4(1 \cdot (j + 1))(2 \cdot j)}, \tag{3.55}$$

$$S(a, b, c, d) = \frac{\text{tr}(abm_1 l_1) \text{tr}(cdm_2 l_2) + \text{tr}(cdm_1 l_1) \text{tr}(abm_2 l_2)}{16(a \cdot b)(c \cdot d)(m_1 \cdot l_1)(m_2 \cdot l_2)}. \tag{3.56}$$

The loop integral can be carried out similarly to those in the previous cases. We can sum the generic terms of (3.53) in the summation (3.50) by renaming the indices, and we thereby obtain some combinations of the box and triangle functions. There are, however, some missing terms, for which the contributions of the other diagrams, (b)-(g), compensate.

The four diagrams (b)-(e) have similar structures, and for this reason it is sufficient to evaluate just two of them (b) and (c). The primitive amplitudes $A^{R(b,c)}_n$ after the fermion integration can be written as

$$A^{R(b,c)}_n = \sum_{k < l=1}^n \sum_{\alpha=1}^{2} \sum_{\beta=1}^{j+1} A^{(4)}(k, l, \alpha, \beta) \sum_{m=2}^{j-1} \int dM J^{(\alpha, \beta)(b,c)}_m(l_1, l_2), \tag{3.57}$$
where

\[ J_m^{(\alpha,\beta)(b)}(l_1, l_2) = \begin{cases} 
- \frac{\langle l_1 l_2 \rangle \langle j l_2 \rangle}{\langle 2 j \rangle} (2l_1) D(2, m), & \text{for } \alpha = 1, \\
- \frac{\langle 2l_1 \rangle \langle \beta l_2 \rangle + \langle \beta l_1 \rangle \langle 2l_2 \rangle}{\langle 2 \beta \rangle^2} + \frac{\langle j \beta \rangle \langle \beta l_1 \rangle \langle 2l_2 \rangle^2}{\langle 2j \rangle \langle 2 \beta \rangle^2} (2l_1) D(2, m), & \text{for } \alpha = 2,
\end{cases} \]

and

\[ J_m^{(\alpha,\beta)(c)}(l_1, l_2) = \begin{cases} 
\left( \frac{\langle j l_1 \rangle \langle \alpha l_2 \rangle + \langle \alpha l_1 \rangle \langle j l_2 \rangle}{\langle j \alpha \rangle^2} - \frac{\langle 2 \alpha \rangle \langle \alpha l_1 \rangle \langle j l_2 \rangle^2}{\langle j 2 \rangle \langle j \alpha \rangle^2} \right) (j l_1) D(j + 1, m), & \text{for } \beta = j, \\
\frac{\langle l_1 l_2 \rangle \langle 2l_2 \rangle}{\langle j 2 \rangle} (j l_1) D(j + 1, m), & \text{for } \beta = j + 1.
\end{cases} \]

The integrand \( J_m^{(\alpha,\beta)(b,c)}(l_1, l_2) \) can be decomposed as

\[ J_m^{(\alpha,\beta)(b,c)}(l_1, l_2) = S_{b,c}^{(\alpha,\beta)}(m) - S_{b,c}^{(\alpha,\beta)}(m + 1), \]

with

\[ S_{b}^{(\alpha,\beta)}(m) = \begin{cases} 
- \frac{\text{tr}(2j 1l_1) \text{tr}(j 1ml_2)}{16(1 \cdot j)(2 \cdot j)(1 \cdot l_1)(m \cdot l_2)} + \frac{\text{tr}(j 2ml_2)}{4(2 \cdot j)(m \cdot l_2)}, & \text{for } \alpha = 1, \\
-3 \frac{\text{tr}(\beta 21l_1) \text{tr}(\beta 2ml_2)}{16(2 \cdot \beta)^2(1 \cdot l_1)(m \cdot l_2)} + \frac{\text{tr}(\beta 2l_1) \text{tr}(j 2ml_2)}{16(2 \cdot j)(2 \cdot \beta)(1 \cdot l_1)(m \cdot l_2)} \\
- \frac{\text{tr}(2\beta 1l_1) \text{tr}(\beta 1ml_2)}{16(1 \cdot \beta)(2 \cdot \beta)(1 \cdot l_1)(m \cdot l_2)} + \frac{\text{tr}(\beta 2ml_2)}{4(2 \cdot \beta)(m \cdot l_2)}, & \text{for } \alpha = 2,
\end{cases} \]

and

\[ S_{c}^{(\alpha,\beta)}(m) = \begin{cases} 
3 \frac{\text{tr}(\alpha j(\alpha + 1)l_1) \text{tr}(\alpha jml_2)}{16(\alpha \cdot j)^2(\alpha + 1 \cdot l_1)(m \cdot l_2)} - \frac{\text{tr}(\alpha j(\alpha + 1)l_1) \text{tr}(2jml_2)}{16(\alpha \cdot j)(\alpha + 1 \cdot l_1)(m \cdot l_2)} \\
+ \frac{\text{tr}(j 2l_1) \text{tr}(2j ml_2)}{16(2 \cdot j)(2 \cdot j + 1)(\alpha + 1 \cdot l_1)(m \cdot l_2)} - \frac{\text{tr}(2jml_2)}{4(2 \cdot j)(m \cdot l_2)}, & \text{for } \beta = j, \\
\frac{\text{tr}(j 2l_1) \text{tr}(2j + 1ml_2)}{16(2 \cdot j)(2 \cdot j + 1)(j + 1 \cdot l_1)(m \cdot l_2)} - \frac{\text{tr}(2jml_2)}{4(2 \cdot j)(m \cdot l_2)}, & \text{for } \beta = j + 1.
\end{cases} \]

The loop integral can be easily evaluated similarly to those in the previous cases.
The diagrams (d) and (e) can be obtained by rotating the diagrams (b) and (c) upside down, thus exchanging indices $(1, 2) \leftrightarrow (j, j + 1), l_1 \leftrightarrow l_2$ and $m \rightarrow m - 1$. The index $m$ is then summed over the region $j + 2 \leq m \leq n + 1$.

The diagram (f) gives a contribution in the case $(\alpha, \beta) = (1, j), (2, j + 1)$ that is the same order in $N_c$ as the first five diagrams, which fill in the final pieces of $A_n^R$:

\[
A_n^{R(f)} = \sum_{k<l=1}^n \sum_{\alpha=1}^{j+1} \sum_{\beta=1}^{j+1} A_n^{(4)}(k, l, \alpha, \beta) \int dM J_m^{(\alpha, \beta)}(f)(l_1, l_2),
\]

for $(\alpha, \beta) = (1, j), (2, j + 1), \quad \text{(3.63)}$

with

\[
J_m^{(\alpha, \beta)}(l_1, l_2) = -\frac{\langle \alpha \bar{\alpha} \rangle \langle \beta \bar{\beta} \rangle (\langle \alpha \bar{l}_1 \rangle \langle \beta \bar{l}_2 \rangle + 2 \langle \beta \bar{l}_1 \rangle \langle \alpha \bar{l}_2 \rangle)}{\langle \alpha \beta \rangle^2 \langle \bar{\alpha} \bar{l}_1 \rangle \langle \bar{\beta} \bar{l}_2 \rangle} \frac{\text{tr}(\bar{\beta} \alpha \bar{l}_1)\text{tr}(\alpha \beta \bar{l}_2)}{16(\alpha \cdot \beta)^2(\bar{\alpha} \cdot l_1)(\bar{\beta} \cdot l_2)} + \frac{\text{tr}(\bar{\beta} \alpha \bar{l}_1)\text{tr}(\beta \alpha \bar{l}_2)}{16(\alpha \cdot \beta)(\beta \cdot \bar{l}_1)(\alpha \cdot l_2)} - \frac{\text{tr}(\beta \alpha \bar{l}_1)}{4(\alpha \cdot \beta)(\bar{\alpha} \cdot l_1)}. \quad \text{(3.64)}
\]

We need not consider the diagram (g) since it only contributes to the primitive amplitude $A_n^{N=4}$, not to $A_n^R$.

Summing up all the contributions, $A_n^R$ for the four-quark MHV amplitudes are given by

\[
A_n^R = \sum_{k<l=1}^n \sum_{\alpha=1}^{j+1} \sum_{\beta=1}^{j+1} c_{\Gamma} A_n^{(4)}(k, l, \alpha, \beta) \left[ -\frac{1}{e^2} \left( -\frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{1}{e^2} \left( -\frac{s_{jj+1}^2}{-s_{12}} \right)^{\epsilon} \right] + \frac{1}{1-2\epsilon} \sum_{m=j=1}^{n+1} \sum_{m=j+1}^{j} b_{\alpha \beta} B(m, a) + \frac{1}{1-2\epsilon} \left( \sum_{m=j+1}^{n+1} \sum_{m=j+1}^{j} \sum_{m=2}^{n} \sum_{m=2}^{j} \epsilon_{\alpha \beta} T_\epsilon(m, a) \right) \]

\[
+ \frac{1}{1-2\epsilon} \sum_{m \in \Gamma_1(\alpha, \beta)} \left( \bar{c}_{m}^{(1)\alpha \beta} T_\epsilon(m, 1) + d_{m}^{(1)\alpha \beta} K_\epsilon(m, 1) \right) + \frac{1}{1-2\epsilon} \sum_{m \in \Gamma_j(\alpha, \beta)} \left( \bar{c}_{m}^{(j)\alpha \beta} T_\epsilon(m, j) + d_{m}^{(j)\alpha \beta} K_\epsilon(m, j) \right), \quad \text{(3.65)}
\]

where the summation regions $\Gamma_1$ and $\Gamma_j$ are defined by

\[
(\Gamma_1(\alpha, \beta), \Gamma_j(\alpha, \beta)) = \begin{cases} 
(\{j+1, \cdots, n\}, \{2, \cdots, j-1\}), & \text{for } (\alpha, \beta) = (1, j), \\
(\{j+2, \cdots, n\}, \{j+2, \cdots, n\}), & \text{for } (\alpha, \beta) = (1, j+1), \\
(\{3, \cdots, j-1\}, \{3, \cdots, j-1\}), & \text{for } (\alpha, \beta) = (2, j), \\
(\{3, \cdots, j\}, \{j+2, \cdots, n+1\}), & \text{for } (\alpha, \beta) = (2, j+1). 
\end{cases}
\quad \text{(3.66)}
\]
We next introduce a combination of the bubble function $K$ as

$$K_{\epsilon}(p, P, Q) = \frac{(\mu^2)^{\epsilon}}{\epsilon} \left( (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon} \right),$$  \hfill (3.67)

and use the same abbreviated notation as in (3.43b). For the case $(\alpha, \beta) = (1, j)$, $(2, j + 1)$, the coefficient functions are given by

$$b_{\alpha\beta}^{a\beta} = \frac{\text{tr}(\alpha \tilde{\beta} ma)\text{tr}(\tilde{\alpha} \beta am)}{2s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{ma}^2} - 3\frac{\text{tr}(\alpha \beta ma)\text{tr}(\alpha \beta am)}{2s_{\alpha \beta} s_{ma}^2}$$

$$+ \frac{\text{tr}(\alpha \beta ma)}{2s_{\alpha \beta} s_{ma}} \left( \frac{\text{tr}(\alpha \tilde{\beta} am)}{s_{\alpha \beta} s_{ma}} + \frac{\text{tr}(\tilde{\alpha} \beta am)}{s_{\tilde{\alpha} \beta} s_{ma}} \right) + (m \leftrightarrow a),$$  \hfill (3.68a)

$$c_{\alpha \beta}^{a\beta} = \mathcal{C}(a) - \mathcal{C}(a + 1),$$  \hfill (3.68b)

$$C(A) = -\frac{\text{tr}(\alpha \tilde{\beta} \alpha \beta Am)\text{tr}(\alpha \beta mP_{m+1a}) + \text{tr}(\tilde{\beta} \alpha \tilde{\beta} \alpha Am)\text{tr}(\beta \alpha mP_{m+1a})}{2s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{\alpha \beta} s_{mA}}$$

$$- 3\frac{\text{tr}(\alpha \beta Am)\text{tr}(\alpha \beta mP_{m+1a}) + \text{tr}(\beta \alpha Am)\text{tr}(\beta \alpha mP_{m+1a})}{2s_{\alpha \beta} s_{mA}}$$

$$+ \frac{\text{tr}(\alpha \beta Am)\text{tr}(\alpha \beta mP_{m+1a}) + \text{tr}(\beta \alpha Am)\text{tr}(\beta \alpha mP_{m+1a})}{s_{\alpha \beta} s_{\alpha \beta} s_{mA}}$$

$$+ \frac{\text{tr}(\beta \alpha Am)\text{tr}(\beta \alpha mP_{m+1a}) + \text{tr}(\tilde{\beta} \alpha Am)\text{tr}(\tilde{\beta} \alpha mP_{m+1a})}{s_{\alpha \beta} s_{\alpha \beta} s_{mA}},$$  \hfill (3.68c)

$$\tilde{c}_{m}^{(1)\alpha \beta} = \frac{\gamma(\alpha)}{2} \left( \frac{\text{tr}(\alpha \beta m\hat{\alpha})}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{m\hat{\alpha}}} - \frac{\text{tr}(\alpha \tilde{\beta} m\hat{\alpha})}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{m\hat{\alpha}}} \right) \frac{\text{tr}(\alpha \beta mP_{m+1}) - \text{tr}(\beta \alpha mP_{m+1})}{s_{\alpha \beta}},$$  \hfill (3.68d)

$$\tilde{c}_{m}^{(j)\alpha \beta} = \frac{\gamma(\alpha)}{2} \left( \frac{\text{tr}(\beta \alpha m\hat{\beta})}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{m\hat{\beta}}} - \frac{\text{tr}(\alpha \tilde{\beta} m\hat{\beta})}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{m\hat{\beta}}} \right) \frac{\text{tr}(\beta \alpha mP_{m+1}) - \text{tr}(\alpha \beta mP_{m+1})}{s_{\alpha \beta}},$$  \hfill (3.68e)

$$d_{m}^{(1)\alpha \beta} = \frac{\gamma(\alpha)}{2} \frac{\text{tr}(\alpha \beta m\hat{\alpha})}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{m\hat{\alpha}}},$$  \hfill (3.68f)

$$d_{m}^{(j)\alpha \beta} = \frac{\gamma(\alpha)}{2} \frac{\text{tr}(\beta \alpha m\hat{\beta})}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{m\hat{\beta}}},$$  \hfill (3.68g)

and for the case $(\alpha, \beta) = (1, j + 1)$, $(2, j)$ by

$$b_{\alpha\beta}^{a\beta} = \frac{\text{tr}(\alpha \beta ma)\text{tr}(\tilde{\alpha} \beta am)}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{ma}^2} + \frac{\text{tr}(\tilde{\alpha} \beta ma)\text{tr}(\alpha \beta am)}{s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{ma}^2} - 2\frac{\text{tr}(\alpha \beta ma)\text{tr}(\alpha \beta am)}{s_{\alpha \beta} s_{ma}^2} s_{\alpha \beta} s_{ma}^2$$  \hfill (3.69a)

$$c_{\alpha \beta}^{a\beta} = \mathcal{C}(a) - \mathcal{C}(a + 1),$$  \hfill (3.69b)

$$C(A) = \frac{\text{tr}(\tilde{\alpha} \beta \alpha Am)\text{tr}(\tilde{\alpha} \beta mP_{m+1a}) + \text{tr}(\tilde{\alpha} \beta Am)\text{tr}(\tilde{\alpha} \beta mP_{m+1a})}{2s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{\alpha \beta} s_{mA}}$$

$$+ \frac{\text{tr}(\tilde{\beta} \alpha Am)\text{tr}(\tilde{\beta} \alpha mP_{m+1a}) + \text{tr}(\tilde{\beta} \alpha Am)\text{tr}(\tilde{\beta} \alpha mP_{m+1a})}{2s_{\alpha \beta} s_{\tilde{\alpha} \beta} s_{\alpha \beta} s_{mA}}.$$
or, in a form more useful for later use, as

\[
S_h(m_1, m_2) = \frac{1}{16(m_1 \cdot l_1)(m_2 \cdot l_2)} \times \\
\left( \Omega \frac{\text{tr}(j(j + 1)m_1 l_1)\text{tr}(j2m_2 l_2)}{4(2 \cdot j)(j \cdot j + 1)} - \tilde{\Omega} \frac{\text{tr}(2(j + 1)m_1 l_1)\text{tr}(2jm_2 l_2)}{4(2 \cdot j)(2 \cdot j + 1)} \right)
\]
\[
\begin{align*}
+ \Omega \frac{\text{tr}(21m_1l_1)\text{tr}(2jm_2l_2)}{4(1 \cdot 2)(2 \cdot j)} - \Omega \frac{\text{tr}(j1m_1l_1)\text{tr}(j2m_2l_2)}{4(1 \cdot j)(2 \cdot j)} \\
+ \Omega \frac{\text{tr}((j + 1)1m_1l_1)\text{tr}((j + 1)jm_2l_2)}{4(1 \cdot j + 1)(j \cdot j + 1)} - \Omega \frac{\text{tr}(1(j + 1)m_1l_1)\text{tr}(1jm_2l_2)}{4(1 \cdot j)(1 \cdot j + 1)} \\
+ \Omega \frac{\text{tr}(1(j + 1)m_1l_1)\text{tr}(12m_2l_2)}{4(1 \cdot 2)(1 \cdot j + 1)} - \Omega \frac{\text{tr}((j + 1)1m_1l_1)\text{tr}((j + 1)2m_2l_2)}{4(1 \cdot j + 1)(2 \cdot j + 1)}
\end{align*}
\]

where

\[
\hat{\Omega} = \frac{\text{tr}(1(j + 1)2j)}{4(1 \cdot j + 1)(2 \cdot j)}.
\]

After the loop integration, we obtain the final result:

\[
\mathcal{A}_n^f = \sum_{k<l=1}^n \sum_{a=1}^2 \sum_{j=j}^{j+1} c_{r} \mathcal{A}_n^{(4)}(k, l, a, \beta) \\
\times \left[ \sum_{m=j+2}^n \sum_{a=3}^{j+1} b_{ma} B(m, a) + \frac{1}{1 - 2\epsilon} \left( \sum_{m=3}^{j-1} \sum_{a=j+1}^n + \sum_{m=j+2}^n \sum_{a=2}^{j-1} \right) c_{ma} T_{\epsilon}(m, a) \\
+ \frac{1}{1 - 2\epsilon} \left( \sum_{m=2}^{j-1} d_{m}^{(n)} K_{\epsilon}(m, n) + \sum_{m=3}^{j} d_{m}^{(j+1)} K_{\epsilon}(m, j + 1) \\
+ \sum_{m=j+2}^{n+1} d_{m}^{(2)} K_{\epsilon}(m, 2) + \sum_{m=j+1}^n d_{m}^{(j-1)} K_{\epsilon}(m, j - 1) \right) \right],
\]

where

\[
\begin{align*}
b_{ma} &= \frac{\text{tr}(1(j + 1)ma)\text{tr}(j2am) - sm_{ma}\text{tr}(2j1(j + 1)ma)}{s_{1j+1}s_{2j}s_{ma}^2}, \\
c_{ma} &= -\frac{1}{4} \left[ \left( \frac{\text{tr}(1(j + 1)am)}{s_{1j+1}s_{ma}} - \frac{\text{tr}(1(j + 1)(a + 1)m)}{s_{1j+1}s_{ma+1}} \right) \frac{\text{tr}(2jmP_{m+1a}) - \text{tr}(j2mP_{m+1a})}{s_{2j}} \\
&\quad \quad + \left( \frac{\text{tr}(2jam)}{s_{2j}s_{ma}} - \frac{\text{tr}(2j(a + 1)m)}{s_{2j}s_{ma+1}} \right) \frac{\text{tr}(1(j + 1)mP_{m+1a}) - \text{tr}((j + 1)1mP_{m+1a})}{s_{1j+1}} \right],
\end{align*}
\]

\[
\begin{align*}
d_{m}^{(n)} &= -\frac{\text{tr}(j21m)}{4s_{2j}s_{1m}}, \\
d_{m}^{(j+1)} &= \frac{\text{tr}(2j(j + 1)m)}{4s_{2j}s_{mj+1}}, \\
d_{m}^{(2)} &= \frac{\text{tr}((j + 1)12m)}{4s_{1j+1}s_{2m}}, \\
d_{m}^{(j-1)} &= -\frac{\text{tr}(1(j + 1)jm)}{s_{1j+1}s_{mj}}.
\end{align*}
\]
§4. Five-point amplitudes and collinear singularities

In the previous section, we extended the CSW rule to $\mathcal{N} = 1$ SQCD, incorporating massless chiral multiplets, and calculated all the one-loop MHV amplitudes including external quarks. There is no a priori reason, however, that this extended CSW rule should generally give the correct result, since it does not have any string theoretic interpretation. For this reason, in this section, we explicitly consider the five-point amplitudes and confirm that they exhibit the correct collinear behavior.

4.1. Collinear behavior of the amplitudes

Let us begin by explaining the general collinear behavior of the gauge theory (MHV) amplitudes with external quark chiral multiplets, as well as gluon vector multiplets. First, consider the $n$-point tree-level amplitudes with a fixed color ordering and an arbitrary helicity configuration. We can easily see that collinear singularities arise only when neighboring legs, $a$ and $b$, become collinear and have the form

$$A_{n}^{\text{tree}} \xrightarrow{a||b} \sum_{h = \pm} \text{Split}_{-h}^{\text{tree}}(A^{h_a}, B^{h_b}) A_{n-1}^{\text{tree}}((a + b)^h \cdots),$$

where $A$ and $B$ denote the species of the particles and the corresponding $h_a$ and $h_b$ represent the signs of their helicities. The species of the intermediate state is determined by the two collinear particles $A$ and $B$. The same symbols $a$ and $b$ appearing in the arguments of the amplitudes denote the momenta of the corresponding particles. The non-vanishing splitting amplitudes diverge as $1/\langle ab \rangle \sim 1/\sqrt{s_{ab}}$ in the collinear limit, $s_{ab} \to 0$. In this limit, we have $p_a = zP$ and $p_b = (1-z)P$, where $P$ and $h$ are the momentum and helicity of the intermediate state. The tree-level-splitting amplitudes $\text{Split}_{-h}^{\text{tree}}(A^{h_a}, B^{h_b})$ can be found in Ref. 10) and references therein. It should be noted that these splitting amplitudes $\text{Split}_{-h}^{\text{tree}}(A^{h_a}, B^{h_b})$ are universal. They depend only on two collinear legs but not on the specific amplitude. We summarize the tree-level splitting amplitudes needed to investigate the MHV amplitudes in Appendix D.

The collinear limits of the one-loop amplitudes have the form

$$A_{n}^{\text{loop}} \xrightarrow{a||b} \sum_{\lambda = \pm} \left( \text{Split}_{-h}^{\text{tree}}(A^{h_a}, B^{h_b}) A_{n-1}^{\text{loop}}((a + b)^h \cdots) \right. \left. + \text{Split}_{-h}^{\text{loop}}(A^{h_a}, B^{h_b}) A_{n-1}^{\text{tree}}((a + b)^h \cdots) \right).$$

Because the one-loop MHV amplitudes of the $\mathcal{N} = 1$ SQCD, obtained in the previous section, are proportional to the corresponding tree-level MHV amplitudes, the loop-splitting
amplitudes must be proportional to the tree-splitting amplitudes,

\[
\text{Split}_{-h}(A^{h_a}, B^{h_b}) = c_r N_c \times \text{Split}_{-h}(A^{h_a}, B^{h_b}) \times r_S(-h, A^{h_a}, B^{h_b}),
\]

(4.3)

where the extra \(N_c\) comes from the one-loop color factor. Like the amplitudes, the proportionality function \(r_S(-h, A^{h_a}, B^{h_b})\) can be decomposed into the primitive parts \(r^S_{x}(-h, A^{h_a}, B^{h_b})\) \((x = (N = 4), (N = 1\text{chiral}), R, f)\).

4.2. Five-point amplitudes

As an explicit example of our general results, (3.40), (3.48), (3.65) and (3.76), here we give the five-point amplitudes and investigate their collinear behavior to check their consistency.

The two-quark primitive amplitudes \(A^R_5\) have the form

\[
A^R_5 = \sum_{k<l}^{5} \sum_{\alpha=1}^{5} \sum_{j=3}^{5} c_r A_5^{(2)}(k, l, \alpha, j) A^R_5(\alpha, j),
\]

(4.4)

where the quantities \(A^R_5(\alpha, j)\) are given as follows, corresponding to each helicity configuration:

\[
A^R_5(1, 3) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - 4 + \left( \frac{2 \text{tr}(3124)\text{tr}(3142)}{s_{13}^2} - \frac{s_{24}\text{tr}(3124)}{s_{13}} \right) L_{s1} \left( \frac{-s_{94}}{-s_{51}}, \frac{s_{14}}{-s_{51}} \right) \left( \frac{-s_{23}}{-s_{51}} \right) \frac{s_{14}}{s_{34}},
\]

(4.5a)

\[
A^R_5(2, 3) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - 4 - \text{tr}(3124) L_{s0} \left( \frac{-s_{93}}{-s_{51}}, \frac{-s_{34}}{-s_{51}} \right) + \left( \frac{2 \text{tr}(3125)\text{tr}(3152)}{s_{13}^2} - \frac{s_{25}\text{tr}(3125)}{s_{13}} \right) L_{s1} \left( \frac{-s_{25}}{-s_{34}}, \frac{-s_{12}}{-s_{34}} \right) \left( \frac{-s_{34}}{-s_{34}} \right) \frac{s_{14}}{s_{34}},
\]

(4.5b)

\[
A^R_5(1, 4) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - 4 + \left( \frac{2 \text{tr}(4125)\text{tr}(4152)}{s_{13}^2} - \frac{s_{25}\text{tr}(4125)}{s_{13}} \right) L_{s1} \left( \frac{-s_{34}}{-s_{51}}, \frac{-s_{12}}{-s_{51}} \right) \left( \frac{-s_{12}}{-s_{34}} \right) \frac{s_{14}}{s_{34}}.
\]
The definitions of the functions $L_0$, $L_{S0}$ and $L_{S1}$ appearing here are summarized in Appendix B.

The general collinear behavior explained in the previous subsection requires these func-
tions $A_5^R(\alpha, j)$ to behave as

$$A_5^R(\alpha, j) \xrightarrow{a|b} A_4^R + r_5^R(-h, a^{b_{\alpha}}, b^{b_{\beta}}),$$

(4.6)

where the configuration of $A_4^R$ is not given explicitly but fixed by the two collinear legs.\(^*\) It is straightforward to verify that the above five-point amplitudes have the correct collinear singularities for all the neighboring pairs of external legs. We obtain the proportionality functions

$$r_S^R(-, g^+, g^+) = 0,$$

(4.7a)

$$r_S^R(+, g^\pm, g^\mp) = 0,$$

(4.7b)

$$r_S^R(\mp, g^+, q^\mp) = F(1 - z, s),$$

(4.7c)

$$r_S^R(\mp, q^\mp, g^+) = F(z, s),$$

(4.7d)

$$r_S^R(\mp, q^\mp, q^\mp) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s} \right) - 2 \left( \frac{\mu^2}{-s} \right) - 4,$$

(4.7e)

where

$$F(z, s) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{z(-s)} \right) + \frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s} \right) - \text{Li}_2(1 - z).$$

(4.8)

The two-quark primitive amplitudes $A_5^f$ are similarly given by

$$A_5^f = \sum_{k<l=1}^5 \sum_{\alpha=1}^2 \sum_{j=3}^5 c_k A_5^{(2)}(k, l, \alpha, j) A_5^f(j),$$

(4.9)

where

$$A_5^f(3) = \text{tr}(3154) \frac{L_0}{s_{13}} \frac{(-s_{4\alpha})}{s_{34}},$$

(4.10a)

$$A_5^f(4) = \text{tr}(4153) \text{tr}(4235) L s_1 \frac{(-s_{3\alpha})}{s_{14} s_{24}} \frac{(-s_{45})}{s_{12}},$$

(4.10b)

$$A_5^f(5) = \text{tr}(5234) L_0 \frac{(-s_{4\alpha})}{s_{25}} \frac{(-s_{45})}{s_{45}}.$$}

(4.10c)

These amplitudes have the expected collinear singularities, and we obtain

$$r_S^f(-, a^+, b^+) = 0,$$

(4.11a)

$$r_S^f(+, a^\pm, b^\mp) = 0,$$

(4.11b)

$$r_S^f(+, q^\pm, q^\mp) = \frac{1}{\epsilon} \left( \frac{\mu^2}{-s} \right) + 2,$$

(4.11c)

\(^*\) The four-point amplitudes needed here are summarized in Appendix E.
which coincide with the results given in Refs. 21) and 22).

The four-quark primitive amplitudes

$$A^R_5 = \sum_{k<l=1}^{5} \sum_{\alpha=1}^{2} \sum_{\beta=3}^{4} c_{R} A^{(4)}_5(k, l, \alpha, \beta) A^R_5(\alpha, \beta)$$  \hspace{1cm} (4.12)

can be computed from the general formula (3263) and are given by the explicit forms of the function $A^R_5(\alpha, \beta)$:

$$A^R_5(1, 3) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right) - \frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{34}} \right) - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right) - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{34}} \right) - 8$$

$$+ \left( \frac{3}{s_{13}} \frac{\text{tr}(1342)\text{tr}(1324)}{s_{24}^2 \text{tr}(1342)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{21}^2}$$

$$+ \left( \frac{3}{s_{13}} \frac{\text{tr}(1352)\text{tr}(1325)}{s_{24}^2 \text{tr}(1352)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{23}^2}$$

$$- \left( s_{12} + 3 \frac{\text{tr}(1342)}{s_{13}} \right) L_{0} \left( -s_{12}, -s_{12}, -s_{12}, -s_{34} \right)$$  \hspace{1cm} (4.13a)

$$A^R_5(2, 4) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right) - \frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{34}} \right) - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right) - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{34}} \right) - 8$$

$$+ \left( \frac{3}{s_{24}} \frac{\text{tr}(2413)\text{tr}(2431)}{s_{24}^2 \text{tr}(2413)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{24}^2}$$

$$+ \left( \frac{3}{s_{24}} \frac{\text{tr}(2453)\text{tr}(2435)}{s_{24}^2 \text{tr}(2453)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{45}^2}$$

$$- \left( s_{34} + 3 \frac{\text{tr}(4213)}{s_{24}} \right) L_{0} \left( -s_{12}, -s_{12}, -s_{12}, -s_{34} \right)$$  \hspace{1cm} (4.13b)

$$A^R_5(1, 4) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right) - \frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{34}} \right) - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right) - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{34}} \right) - 8$$

$$+ \left( \frac{2}{s_{14}} \frac{\text{tr}(1452)\text{tr}(1425)}{s_{14}^2 \text{tr}(1452)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{34}^2}$$

$$+ \left( \frac{2}{s_{14}} \frac{\text{tr}(1453)\text{tr}(1435)}{s_{14}^2 \text{tr}(1453)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{12}^2}$$

$$+ \left( \frac{2}{s_{14}} \frac{\text{tr}(1425)\text{tr}(1435)}{s_{14}^2 \text{tr}(1425)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{12}^2}$$

$$+ \left( \frac{2}{s_{14}} \frac{\text{tr}(1453)\text{tr}(1435)}{s_{14}^2 \text{tr}(1453)} \right) \frac{Ls_{1}(-s_{34}, -s_{34})}{s_{12}^2}$$
The collinear singularities of these amplitudes can be confirmed by using the functions appearing in (4.7).

Finally, the four-quark primitive amplitudes \( A_{5}^{R} \) are similarly given by

\[
A_{5}^{R}(2,3) = -\frac{1}{\epsilon^{2}} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{1}{\epsilon^{2}} \left( \frac{\mu^2}{-s_{34}} \right)^{\epsilon} - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^{\epsilon} - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{34}} \right)^{\epsilon} - 8
\]

\[
-\left( \text{tr}(152) \right)_{s_{14}} - \left( \text{tr}(4153) \right)_{s_{14}} \left( \text{tr}(2345) + \text{tr}(3215) \right)_{s_{23}} \left( \text{tr}(453) \right)_{s_{34}} \left( \text{tr}(1452) + \text{tr}(4153) \right)_{s_{14}} \left( \text{tr}(1452) + \text{tr}(4153) \right)_{s_{34}}
\]

The collinear behavior can be checked by using (4.11).

\section{Summary and discussion}

In this paper, we have extended the CSW rule to the \( \mathcal{N} = 1 \) SQCD, incorporating massless quark chiral multiplets. The MHV vertices have been arranged into a manifestly supersymmetric form by introducing the fermionic variables, \( \eta \) and \( \chi \) for the vector multiplet and \( \eta \) and \( \rho \) for the chiral multiplet. This formulation involves only the physical helicity states without auxiliary fields, and therefore provides a simple alternative method efficiently compute the perturbative SQCD amplitudes, in a manner that preserves the manifest supersymmetry. As an application of this extended CSW rule, we have calculated all the one-loop MHV amplitudes with arbitrary numbers of external legs including one or two external quark-antiquark pairs. We have confirmed that the five-point amplitudes have the correct collinear singularities as a non-trivial check of the results. We have given explicit forms of (some of) the one-loop splitting amplitudes for the \( \mathcal{N} = 1 \) SQCD.

As a confirmation of our general expressions, the investigations of the collinear singularities should be extended to arbitrary amplitudes, for which the splitting amplitudes obtained in this paper should be useful. As another confirmation, it is important to extend recently
obtained a direct proof which shows that the one-loop gluon amplitudes coincide with the field theoretical results in $\mathcal{N} = 1$ SYM.\cite{18} We hope to report on progress on these remaining problems in the near future.

One of the important applications of the CSW rule is computing real perturbative QCD amplitudes.\cite{17,23} The results obtained in this paper are not directly related to QCD but can be used to study several models beyond the Standard Model, such as the minimal supersymmetric Standard Model or supersymmetric grand unified theories. It is expected that the results will be useful for analyzing perturbative properties of such models.

It would also be interesting to formulate a twistor-string interpretation of the extended CSW rule. Such a twistor string theory, which may be related to $\mathcal{N} = 1$ supersymmetric gauge theory, is suggested in Ref. 1) as a string on some weighted supertwistor space. The fermionic variables $\eta$ and $\chi$ can be interpreted as momentum variables of the fermionic coordinates in this supertwistor space. However, there are some difficulties with this naive expectation, as discussed in Ref. 1). We hope that our study provides some insight that may help to this problem.

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Appendix A

Spinor Conventions and Useful Formulas

In this paper we use the two-component notation defined by

$$\gamma_\mu = \begin{pmatrix} 0 & (\sigma_\mu)_{ab} \\ (\bar{\sigma}_\mu)^{\dot{a}\dot{b}} & 0 \end{pmatrix},$$

(A-1)

with

$$(\sigma_\mu)_{ab} = (1_2, \sigma_i), \quad (\bar{\sigma}_\mu)^{\dot{a}\dot{b}} = (1_2, -\sigma_i),$$

(A-2)

where $1_2$ denotes a $2 \times 2$ unit matrix and the quantities $\sigma_i$ are the Pauli matrices. The antisymmetric spinors are defined by $\epsilon^{01} = \epsilon_{01} = \epsilon^{\dot{0}\dot{1}} = \epsilon_{\dot{0}\dot{1}} = 1$. We next introduce the
shorthand notation $\langle ij \rangle$ and $[i, j]$ as

$$\langle \lambda_i \lambda_j \rangle = \lambda_i^a \lambda_j^a = \epsilon^{ab} \lambda_i^a \lambda_j^b = \epsilon_{ab} \lambda_i^a \lambda_j^b; \quad (A.3a)$$

$$[\lambda_i, \lambda_j] = \bar{\lambda}_i^\alpha \lambda_j^\alpha = \epsilon^{\alpha\beta} \bar{\lambda}_i^\alpha \lambda_j^\beta = \epsilon_{\alpha\beta} \bar{\lambda}_i^\alpha \lambda_j^\beta. \quad (A.3b)$$

Then, with the trace convention

$$\text{tr}(ij \cdots k) = (\sigma^\mu \sigma^\nu \cdots \sigma^p)_a^\alpha p_{i\mu} p_{j\nu} \cdots p_{kp}; \quad (A.4)$$

we can factorize traces contracted with massless momenta as, for example,

$$\text{tr}(ij) = - \langle ij \rangle [j\bar{i}] = 2(i \cdot j), \quad (A.5a)$$

$$\text{tr}(ijkl) = \langle kj \rangle [j\bar{i}] [i\bar{k}], \quad (A.5b)$$

$$\text{tr}(ijklmn) = - \langle ml \rangle [i\bar{k}] \langle kj \rangle [j\bar{i}] \langle in \rangle [\bar{n}\bar{m}]. \quad (A.5c)$$

We can also show that the following quantities are holomorphic:

$$\frac{\text{tr}(ijkl)}{4(i \cdot j)(k \cdot l)} = \frac{\langle kj \rangle \langle il \rangle}{\langle ij \rangle \langle kl \rangle}, \quad (A.6a)$$

$$\frac{\text{tr}(ijklmn)}{8(i \cdot j)(k \cdot l)(m \cdot n)} = \frac{\langle ml \rangle \langle kj \rangle \langle in \rangle}{\langle kl \rangle \langle ij \rangle \langle mn \rangle}. \quad (A.6b)$$

The most frequently used identity is Schouten’s identity, given by

$$\langle ij \rangle \langle kl \rangle + \langle jk \rangle \langle il \rangle + \langle ki \rangle \langle jl \rangle = 0, \quad (A.7)$$

and the identity\footnote{The special case with $j = k$ for this identity was studied in Ref. 8.)}

$$2(m_1 \cdot m_2)\text{tr}(ijm_1 \cdot P)\text{tr}(ikm_2 \cdot P) - 2(m_1 \cdot P)\text{tr}(ijm_1 \cdot m_2)\text{tr}(ikm_2 \cdot P)$$

$$- 2(m_2 \cdot P)\text{tr}(ijm_1 \cdot P)\text{tr}(ikm_2 \cdot m_1) + P^2\text{tr}(ijm_1 \cdot m_2)\text{tr}(ikm_2 \cdot m_1) = 0, \quad (A.8)$$

where $p_i, p_j$ and $p_k$ are massless momenta, while $P$ is not necessarily so.

Now, we present further useful identities used in this paper, in which $p_i, p_j \cdots$ are null vectors, but $P$ and $Q$ are not necessarily so:

$$\text{tr}(ijkl) = \text{tr}(jilk) = \text{tr}(klij), \quad (A.9a)$$

$$\text{tr}(i_1 \cdots i_{2n+1} \sigma^\mu)\text{tr}(j_1 \cdots j_{2n+1} \sigma^a) = 2\text{tr}(i_1 \cdots i_{2n+1} j_{2n+1} \cdots j_1). \quad (A.9b)$$

$$\text{tr}(ijkl)\text{tr}(ilmn) = 2(i \cdot l)\text{tr}(ijklmn), \quad (A.9c)$$

$$\text{tr}(ijkl)\text{tr}(kjil) = s_{ij} s_{jk} s_{kl} s_{li}. \quad (A.9d)$$

$$\text{tr}(i_1 \cdots i_{2n+1} P)\text{tr}(j_1 \cdots j_{2n+1} Q) = \text{tr}(i_1 \cdots i_{2n+1} Q)\text{tr}(j_1 \cdots j_{2n+1} P). \quad (A.9e)$$
Appendix B

The Box, the Triangle and the Bubble Functions

Here we summarize some properties of the box, the triangle and the bubble functions.
The box function $B(s,t,P^2,Q^2)$ defined by Eq. (3.14) can also be written as

$$B(s,t,P^2,Q^2) = \text{Li}_2(1-aP^2) + \text{Li}_2(1-aQ^2) - \text{Li}_2(1-as) - \text{Li}_2(1-at), \quad (B.1)$$

which is frequently used in §3. The $\epsilon$-dependent triangle function defined by Eq. (3.17) has three different $\epsilon \to 0$ limits:

$$T_\epsilon(p,P,Q) \sim \begin{cases} 
\log \left( \frac{Q^2}{P^2} \right), & \text{for } P^2 \neq 0, \ Q^2 \neq 0, \\
-\frac{1}{\epsilon} \frac{1}{P^2} \left( \frac{\mu^2}{-P^2} \right)^\epsilon, & \text{for } P^2 \neq 0, \ Q^2 = 0, \\
-\frac{1}{\epsilon} \frac{1}{Q^2} \left( \frac{\mu^2}{-Q^2} \right)^\epsilon, & \text{for } P^2 = 0, \ Q^2 \neq 0.
\end{cases} \quad (B.2)$$

Here, total momentum conservation, $p + P + Q = 0$, is assumed.

The combination of the bubble function defined by Eq. (3.67) also has three different $\epsilon \to 0$ limits:

$$K_\epsilon(p,P,Q) \sim \begin{cases} 
\log \left( \frac{Q^2}{P^2} \right), & \text{for } P^2 \neq 0, \ Q^2 \neq 0, \\
\frac{1}{\epsilon} \left( \frac{\mu^2}{-P^2} \right)^\epsilon, & \text{for } P^2 \neq 0, \ Q^2 = 0, \\
-\frac{1}{\epsilon} \left( \frac{\mu^2}{-Q^2} \right)^\epsilon, & \text{for } P^2 = 0, \ Q^2 \neq 0.
\end{cases} \quad (B.3)$$

The functions appearing in the general results can be written in terms of these functions in the five-point amplitudes as follows. For the box function $B(s,t,P^2,Q^2)$, a kinematical restriction leads to $P^2 = 0$ or $Q^2 = 0$ for the five-point amplitudes. Here we consider the
case \( Q^2 = 0 \), since two cases are symmetric:

\[
B(s, t, P^2, 0) = \text{Li}_2 \left( 1 - \frac{P^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{P^2}{t} \right) + \frac{\pi^2}{6} + \frac{1}{2} \log^2 \left( \frac{s}{t} \right)
\]

\[
= - \text{Li}_2 \left( 1 - \frac{s}{P^2} \right) - \text{Li}_2 \left( 1 - \frac{t}{P^2} \right) - \log \left( \frac{s}{P^2} \right) \log \left( \frac{t}{P^2} \right) + \frac{\pi^2}{6}
\]

\[
= - \text{Li}_2 \left( \frac{-s}{P^2}, \frac{-t}{P^2} \right)
\]

(B.5)

For triangle functions with \( P^2 \neq 0, Q^2 \neq 0 \), we have

\[
T_\epsilon(p, P, Q) = - \frac{\text{Li}_0 \left( \frac{-Q^2}{P^2} \right)}{P^2} = - \frac{\text{Li}_0 \left( \frac{-P^2}{Q^2} \right)}{Q^2}.
\]

(B.6)

We have used the equality \( \text{Li}_2(1) = \frac{\pi^2}{6} \) and two formulas for the dilogarithm function to study the collinear behavior of the five-point amplitudes:

\[
\text{Li}_2(1 - r) + \text{Li}_2(1 - r^{-1}) = -\frac{1}{2} \log^2 r, \quad \text{(B.7a)}
\]

\[
\text{Li}_2(r) + \text{Li}_2(1 - r) = \frac{\pi^2}{6} - \log r \log(1 - r). \quad \text{(B.7b)}
\]

Appendix C

Phase Space Integrals

The basic formulas used to evaluate the Lorentz-invariant phase-space integrals, Passarino-Veltman reduction,\(^\text{24}\) are given in Refs. 7), 8) and 17). Here we summarize some of the results needed for our calculation.

The first formulas are given by

\[
\int d\text{LIPS}(l_2, -l_1; P) \frac{l_1^\mu}{(m_1 \cdot l_1)} = -m_1^\mu \frac{2\pi \lambda}{2\epsilon(1 - 2\epsilon)(m_1 \cdot P)} + P_\mu \frac{2\pi \lambda}{1 - 2\epsilon} \frac{1}{(m_1 \cdot P)}, \quad \text{(C.1a)}
\]

\[
\int d\text{LIPS}(l_2, -l_1; P) \frac{l_2^\mu}{(m_2 \cdot l_2)} = -m_2^\mu \frac{2\pi \lambda}{2\epsilon(1 - 2\epsilon)(m_2 \cdot P)} + P_\mu \frac{2\pi \lambda}{1 - 2\epsilon} \frac{1}{(m_2 \cdot P)}, \quad \text{(C.1b)}
\]

where

\[
\lambda = \frac{\pi^{\frac{1}{2} - \epsilon} \left( \frac{P^2}{4} \right)^{-\epsilon}}{4 \Gamma \left( \frac{1}{2} - \epsilon \right)}.
\]

(C.2)
Introducing the conventional constant

\[ c_r = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \]  

(C.3)

we obtain the useful relation

\[ \frac{\lambda}{(2\pi)^D} = \frac{c_r}{4\pi^2[\pi\epsilon \csc(\pi\epsilon)]} (P^2)^{-\epsilon}. \]  

(C.4)

The second and the most important formulas are obtained by calculating

\[ \mathcal{I}^{\mu\nu} := \int d\text{LIPS}(l_2, -l_1; P) \frac{l_1^\mu l_2^\nu}{(m_1 \cdot l_1)(m_2 \cdot l_2)}, \]  

(C.5)

which can be expanded as

\[ \mathcal{I}^{\mu\nu} = \eta^{\mu\nu} \mathcal{I}_0 + m_1^\mu m_1^\nu \mathcal{I}_1 + m_2^\mu m_2^\nu \mathcal{I}_2 + P^\mu P^\nu \mathcal{I}_3 + m_1^\mu m_2^\nu \mathcal{I}_4 \\
+ m_2^\mu m_1^\nu \mathcal{I}_5 + m_1^\mu P^\nu \mathcal{I}_6 + P^\mu m_1^\nu \mathcal{I}_7 + m_2^\mu P^\nu \mathcal{I}_8 + P^\mu m_2^\nu \mathcal{I}_9. \]  

(C.6)

The coefficients \( \mathcal{I}_i \) \( (i = 0, 1, \cdots, 9) \) are given by

\[ \mathcal{I}_0 = \frac{1}{(m_1 \cdot m_2)N} \left( -2 \left[ (m_1 \cdot m_2)P^2 - (m_1 \cdot P)(m_2 \cdot P) \right] \tilde{T}^{(0,0)} + (m_1 \cdot P)N\tilde{T}^{(1,0)} \right. \]

\[ - (m_2 \cdot P)N\tilde{T}^{(0,1)} - \frac{1}{4}N^2\tilde{T}^{(1,1)} - (m_1 \cdot P)^2\tilde{T}^{(-1,1)} - (m_2 \cdot P)^2\tilde{T}^{(-1,1)} \right), \]  

(C.7a)

\[ \mathcal{I}_1 = \frac{1}{(m_1 \cdot m_2)^2N^2} \left( 4(m_2 \cdot P)^2 \left[ (m_1 \cdot m_2)P^2 - (m_1 \cdot P)(m_2 \cdot P) \right] \tilde{T}^{(0,0)} \right. \]

\[ + (m_2 \cdot P)N^2\tilde{T}^{(1,0)} + 2(m_2 \cdot P)^3N\tilde{T}^{(0,1)} + \frac{1}{2}(m_2 \cdot P)^2N^2\tilde{T}^{(1,1)} \]

\[ + \left[ (m_1 \cdot m_2)P^2N + 2(m_1 \cdot P)^2(m_2 \cdot P)^2 \right] \tilde{T}^{(-1,1)} + 2(m_2 \cdot P)^4\tilde{T}^{(-1,1)} \right), \]  

(C.7b)

\[ \mathcal{I}_2 = \frac{1}{(m_1 \cdot m_2)^2N^2} \left( 4(m_1 \cdot P)^2 \left[ (m_1 \cdot m_2)P^2 - (m_1 \cdot P)(m_2 \cdot P) \right] \tilde{T}^{(0,0)} \right. \]

\[ - 2(m_1 \cdot P)^3N\tilde{T}^{(1,0)} + \frac{1}{2}(m_1 \cdot P)^2N^2\tilde{T}^{(1,1)} - (m_1 \cdot P)N^2\tilde{T}^{(0,1)} \]

\[ + 2(m_1 \cdot P)^4\tilde{T}^{(-1,1)} + \left[ (m_1 \cdot m_2)P^2N + 2(m_1 \cdot P)^2(m_2 \cdot P)^2 \right] \tilde{T}^{(-1,1)} \right), \]  

(C.7c)

\[ \mathcal{I}_3 = \frac{1}{N^2} \left( 2(m_1 \cdot m_2)P^2\tilde{T}^{(0,0)} - (m_1 \cdot P)N\tilde{T}^{(1,0)} \right). \]
\[ I_4 = \frac{1}{(m_1 \cdot m_2)^2 N^2} \left( [3(m_1 \cdot m_2)P^2N + 4(m_1 \cdot P)^2(m_2 \cdot P)^2] \bar{T}^{(0,0)} + (m_2 \cdot P)N\bar{T}^{(0,1)} + 2(m_1 \cdot P)^2\bar{T}^{(1,-1)} + 2(m_2 \cdot P)^2\bar{T}^{(-1,1)} \right), \] (C.7d)

\[ I_5 = \frac{1}{(m_1 \cdot m_2)^2 N^2} \left( [3(m_1 \cdot m_2)P^2N + 4(m_1 \cdot P)^2(m_2 \cdot P)^2] \bar{T}^{(0,0)} - (m_1 \cdot P) \left[ \frac{3}{2} (m_1 \cdot m_2)P^2 - 2(m_1 \cdot P)(m_2 \cdot P) \right] N\bar{T}^{(1,0)} + \frac{1}{4} N^2\bar{T}^{(1,1)} \right), \] (C.7e)

\[ I_6 = \frac{1}{(m_1 \cdot m_2)^2 N^2} \left( - (m_2 \cdot P) \left[ 3(m_1 \cdot m_2)P^2 - 2(m_1 \cdot P)(m_2 \cdot P) \right] \bar{T}^{(0,0)} \right) \] \[ - \frac{1}{2} N^2\bar{T}^{(1,0)} - 2(m_2 \cdot P)^2N\bar{T}^{(0,1)} - \frac{1}{2} (m_2 \cdot P)N^2\bar{T}^{(1,1)} \] \[ - (m_1 \cdot P)(m_1 \cdot m_2)P^2\bar{T}^{(1,-1)} - 2(m_2 \cdot P)^3\bar{T}^{(-1,1)} \right), \] (C.7f)

\[ I_7 = \frac{1}{(m_1 \cdot m_2)^2 N^2} \left( - (m_2 \cdot P) \left[ 3(m_1 \cdot m_2)P^2 - 2(m_1 \cdot P)(m_2 \cdot P) \right] \bar{T}^{(0,0)} \right) \] \[ + \frac{1}{2} (m_1 \cdot m_2)P^2N\bar{T}^{(1,0)} - (m_2 \cdot P)^2N\bar{T}^{(0,1)} \] \[ - (m_1 \cdot P)(m_1 \cdot m_2)P^2\bar{T}^{(1,-1)} - 2(m_2 \cdot P)^3\bar{T}^{(-1,1)} \right), \] (C.7g)

\[ I_8 = \frac{1}{(m_1 \cdot m_2)^2 N^2} \left( - (m_1 \cdot P) \left[ 3(m_1 \cdot m_2)P^2 - 2(m_1 \cdot P)(m_2 \cdot P) \right] \bar{T}^{(0,0)} \right) \]
\[ + (m_1 \cdot P)^2 N \tilde{T}^{(1,0)} - \frac{1}{2} (m_1 \cdot m_2) P^2 N \tilde{T}^{(0,1)} \]
\[ - 2(m_1 \cdot P)^3 \tilde{T}^{(1,-1)} - (m_2 \cdot P)(m_1 \cdot m_2) P^2 \tilde{T}^{(-1,1)} \] 
\[ \mathcal{I}_0 = \frac{1}{(m_1 \cdot m_2) N^2} \left( - (m_1 \cdot P) \left[ 3(m_1 \cdot m_2) P^2 - 2(m_1 \cdot P)(m_2 \cdot P) \right] \tilde{T}^{(0,0)} + 2(m_1 \cdot P)^2 N \tilde{T}^{(1,0)} + \frac{1}{2} N^2 \tilde{T}^{(0,1)} - \frac{1}{2} (m_1 \cdot P) N^2 \tilde{T}^{(1,1)} - 2(m_1 \cdot P)^3 \tilde{T}^{(1,-1)} - (m_2 \cdot P)(m_1 \cdot m_2) P^2 \tilde{T}^{(-1,1)} \right) , \] 
\[ \text{where} \]
\[ N = (m_1 \cdot m_2) P^2 - 2(m_1 \cdot P)(m_2 \cdot P), \] 
\[ \text{and } \tilde{T}^{(a,b)} \text{ are fundamental integrals defined by} \]
\[ \tilde{T}^{(a,b)} = \int dLIPS(l_2, -l_1; P_{L,z}) (m_1 \cdot l_1)^a (m_2 \cdot l_2)^b. \] 

We only need \( \tilde{T}^{(a,b)} \) for \( a, b = 0, \pm 1 \), which can be computed as
\[ \tilde{T}^{(0,0)} = \frac{2\pi \lambda}{1 - 2\epsilon} ; \] 
\[ \tilde{T}^{(1,0)} = -\frac{1}{\epsilon} \frac{2\pi \lambda}{(m_1 \cdot P)} ; \] 
\[ \tilde{T}^{(0,1)} = \frac{1}{\epsilon} \frac{2\pi \lambda}{(m_2 \cdot P)} ; \] 
\[ \tilde{T}^{(1,1)} = -\frac{8\pi \lambda}{N} \left( \frac{1}{\epsilon} + \log \left( 1 - \frac{(m_1 \cdot m_2) P^2}{N} \right) \right) ; \] 
\[ \tilde{T}^{(1,-1)} = -\frac{2\pi \lambda}{(m_1 \cdot P)^2} \left( \frac{N}{2\epsilon} + \frac{1}{1 - 2\epsilon} \left[ (m_1 \cdot m_2) P^2 - (m_1 \cdot P)(m_2 \cdot P) \right] \right) ; \] 
\[ \tilde{T}^{(-1,1)} = -\frac{2\pi \lambda}{(m_2 \cdot P)^2} \left( \frac{N}{2\epsilon} + \frac{1}{1 - 2\epsilon} \left[ (m_1 \cdot m_2) P^2 - (m_1 \cdot P)(m_2 \cdot P) \right] \right) . \]

It should be noted that we set \( \eta^\mu = 4 - D + 2\epsilon \), which is valid for dimensional reduction regularization, which is adopted in this paper, at the one-loop level. The additional contribution of \( 2\epsilon \) comes form the so-called \( \epsilon \)-scalar. Using these explicit forms of the fundamental integrals, we obtain
\[ \mathcal{I}_0 = \frac{2\pi \lambda}{(m_1 \cdot m_2)} \log \left( 1 - a P^2 \right) ; \]
\[ \mathcal{I}_3 = -\frac{2\pi \lambda}{1 - 2\epsilon N} \cdot \frac{2}{N} . \]
\[ I_7 = \frac{2\pi \lambda}{1 - 2\epsilon (m_1 \cdot P)} N, \]  
\[ I_8 = \frac{2\pi \lambda}{1 - 2\epsilon (m_2 \cdot P)} P^2, \]
\[ I_5 = \frac{P^2}{2(m_1 \cdot m_2)} I_3 = -\frac{2\pi \lambda}{(m_1 \cdot m_2)^2} \log (1 - a P^2), \]
\[ I_7 + (m_2 \cdot P) I_3 = \frac{2\pi \lambda}{1 - 2\epsilon (m_1 \cdot m_2)(m_1 \cdot P)}, \]
\[ I_8 + (m_1 \cdot P) I_3 = \frac{2\pi \lambda}{1 - 2\epsilon (m_1 \cdot m_2)(m_2 \cdot P)}, \]

where
\[ a = \frac{(m_1 \cdot m_2)}{N}. \]

The one-loop phase space integrals can be calculated using these relations.

For the special case \( P_L = m_1 + m_2 \) in the gauge \( \hat{\eta} = m_1 \) or \( \hat{\eta} = m_2 \), the integral \( I^{\mu\nu} \) can be expanded as
\[ I^{\mu\nu} = \eta^{\mu\nu} \tilde{I}_0 + m_1^\mu m_1^\nu \tilde{I}_1 + m_2^\mu m_2^\nu \tilde{I}_2 + m_1^\mu m_3^\nu \tilde{I}_3 + m_2^\mu m_4^\nu \tilde{I}_4, \]

since \( P \) is not independent. The fundamental integrals \( \tilde{I}^{(a,b)} \) can be obtained simply by setting \( P = m_1 + m_2 \) in Eq.\,(C.10), except for \( \tilde{I}^{(1,1)} \), which diverges, because \( N = 0 \). Fortunately, however, we need only \( \tilde{I}_0 \) and \( \tilde{I}_4 \), which can be obtained without using \( \tilde{I}^{(1,1)} \) and evaluated as
\[ \tilde{I}_0 = -\frac{2\pi \lambda}{(m_1 \cdot m_2)} \left( \frac{1}{\epsilon} + \frac{1}{1 - 2\epsilon} \right) = (m_1 \cdot m_2) \tilde{I}_1 = (m_1 \cdot m_2) \tilde{I}_2, \]
\[ \tilde{I}_4 = \frac{2\pi \lambda}{(m_1 \cdot m_2)^2} \frac{1}{\epsilon(1 - 2\epsilon)}. \]

**Appendix D**

--- *Tree-Splitting Amplitudes* ---

Here we present explicit forms of tree-splitting amplitudes, which are needed to study the collinear behavior of the one-loop MHV amplitudes. They can be obtained from the results given in Ref. 10) by using the supersymmetry.

The tree-level splitting amplitudes considered here have the form
\[ \text{Split}^{\text{tree}}_{-h}(A^{h_a}, B^{h_b}) = \frac{f_{-h}(A^{h_a}, B^{h_b})}{\sqrt{z(1 - z)(ab)}}. \]
The functions $f$ for the case in which two collinear particles $A$ and $B$ are in the vector multiplets can be obtained from the tree-level amplitudes (2.6) as

\begin{align}
  f_-(g^{(+)} , g^{(+)} ) &= 1, \tag{D-2a} \\
  f_+(g^{(+)} , g^{(-)} ) &= (1 - z)^2, \tag{D-2b} \\
  f_+(g^{(-)} , g^{(+)} ) &= z^2, \tag{D-2c} \\
  f_-(g^{(+)} , A^{(+)}) &= (1 - z)^{1/2}, \tag{D-2d} \\
  f_-(A^{(+)} , g^{(+)} ) &= z^{1/2}, \tag{D-2e} \\
  f_+(g^{(+)} , A^{(-)}) &= (1 - z)^{3/2}, \tag{D-2f} \\
  f_+(A^{(-)} , g^{(+)} ) &= z^{3/2}, \tag{D-2g} \\
  f_+(A^{(+)} , A^{(-)}) &= z^{1/2}(1 - z)^{3/2}, \tag{D-2h} \\
  f_+(A^{(-)} , A^{(+)} ) &= - z^{3/2}(1 - z)^{1/2}. \tag{D-2i} \\
\end{align}

For the other cases, the functions $f$ can be obtained from the tree-level amplitudes (2.10). The functions $f$ for the case in which the particles in the vector and antichiral multiplets become collinear are given by

\begin{align}
  f_-(g^{(+)} , \bar{g}^{(+)}) &= (1 - z)^{1/2}, \tag{D-3a} \\
  f_+(g^{(+)} , \bar{g}^{(-)}) &= (1 - z)^{3/2}, \tag{D-3b} \\
  f_-(g^{(+)} , \bar{\phi}^{(+)}) &= 1 - z, \tag{D-3c} \\
  f_+(g^{(+)} , \bar{\phi}^{(-)}) &= 1 - z, \tag{D-3d} \\
  f_-(A^{(+)} , \bar{q}^{(+)}) &= z^{1/2}(1 - z)^{1/2}, \tag{D-3e} \\
  f_+(A^{(+)} , \bar{\phi}^{(-)}) &= z^{1/2}(1 - z). \tag{D-3f} \\
\end{align}

The functions $f$ for the two collinear particles in the chiral and vector multiplets are

\begin{align}
  f_-(q^{(+)} , g^{(+)} ) &= z^{1/2}, \tag{D-4a} \\
  f_+(q^{(-)} , g^{(+)} ) &= z^{3/2}, \tag{D-4b} \\
  f_-(\phi^{(+)} , g^{(+)} ) &= z, \tag{D-4c} \\
  f_+(\phi^{(-)} , g^{(+)} ) &= z, \tag{D-4d} \\
  f_-(q^{(+)} , A^{(+)}) &= z^{1/2}(1 - z)^{1/2}, \tag{D-4e} \\
  f_+(\phi^{(-)} , A^{(+)}) &= - z(1 - z)^{1/2}. \tag{D-4f} \\
\end{align}

The collinear behavior of the antichiral and chiral multiplets is characterized by

\begin{align}
  f_+(\bar{q}^{(+)} , q^{(-)}) &= z^{1/2}(1 - z)^{3/2}, \tag{D-5a} \\
\end{align}
\[ f_+ (\bar{q}^-, q^+) = -z^{3/2} (1 - z)^{1/2}, \quad (D.5b) \]
\[ f_+ (\bar{q}^+, \phi^-) = z^{1/2} (1 - z), \quad (D.5c) \]
\[ f_+ (\bar{\phi}^-, q^+) = -z (1 - z)^{1/2}, \quad (D.5d) \]
\[ f_+ (\bar{\phi}^+, \phi^-) = z (1 - z), \quad (D.5e) \]
\[ f_+ (\bar{\phi}^-, \phi^+) = z (1 - z). \quad (D.5f) \]

We can confirm their universality by using the amplitudes (2.13).

**Appendix E**

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**Four-Point Amplitudes**

Here we present the four-point amplitudes needed to study the collinear behavior of the five-point amplitudes. First, the four-gluon primitive amplitudes \( A_{4_{\text{chiral}}} \) are given by

\[
A_{4_{\text{chiral}}} = -\sum_{k<l} \sum_{i<j} c_{\Gamma A} A_{4}^{(0)}(k, l, i, j) A_{4}^{(1)}(i, j), \quad (E.1)
\]

where \( A_{4}^{(1)}(i, j) \) is given by

\[
A_{4}^{(1)}(1, 2) = -\frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{23}} \right)^\epsilon - 2, \quad (E.2a)
\]
\[
A_{4}^{(1)}(2, 3) = -\frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 2, \quad (E.2b)
\]
\[
A_{4}^{(1)}(3, 4) = -\frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{23}} \right)^\epsilon - 2, \quad (E.2c)
\]
\[
A_{4}^{(1)}(4, 1) = -\frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 2, \quad (E.2d)
\]
\[
A_{4}^{(1)}(1, 3) = -\frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 2
\]
\[
- \frac{s_{12} s_{23}}{2 s_{13}^2} \left( \log^2 \left( \frac{s_{12}}{s_{23}} \right) + \pi^2 \right) + \frac{s_{12}}{s_{13}} \log \left( \frac{-s_{12}}{-s_{23}} \right), \quad (E.2e)
\]
\[
A_{4}^{(1)}(2, 4) = -\frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{23}} \right)^\epsilon - 2
\]
\[
- \frac{s_{12} s_{23}}{2 s_{13}^2} \left( \log^2 \left( \frac{s_{12}}{s_{23}} \right) + \pi^2 \right) - \frac{s_{12}}{s_{13}} \log \left( \frac{-s_{12}}{-s_{23}} \right). \quad (E.2f)
\]

The primitive two-quark primitive amplitudes \( A_{4}^{R} \) have the form

\[
A_{4}^{R} = \sum_{k<l=1}^{4} \sum_{\alpha=1}^{2} \sum_{j=3}^{4} c_{\Gamma A} A_{4}^{(2)}(k, l, \alpha, j) A_{4}^{(1)}(\alpha, j), \quad (E.3)
\]

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where

\[ A_4^R(1, 3) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 4 \]
\[ + \frac{s_{12}(s_{12} - s_{23})}{2s_{13}^2} \left( \log^2 \left( \frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + 2 \frac{s_{12}}{s_{13}} \log \left( \frac{-s_{12}}{-s_{23}} \right), \]  

(E.4a)

\[ A_4^R(2, 3) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 4 \]
\[ - \frac{s_{12}}{2s_{14}} \left( \log^2 \left( \frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right), \]  

(E.4b)

\[ A_4^R(1, 4) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 4 \]
\[ - \frac{s_{12}}{2s_{14}} \left( \log^2 \left( \frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right), \]  

(E.4c)

\[ A_4^R(2, 4) = -\frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{2}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 4 \]
\[ + \frac{s_{12}(s_{12} - s_{23})}{2s_{13}^2} \left( \log^2 \left( \frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + 2 \frac{s_{12}}{s_{13}} \log \left( \frac{-s_{12}}{-s_{23}} \right). \]  

(E.4d)

The two-quark primitive amplitude \( A_4^f \) is similarly obtained as

\[ A_4^f = \sum_{k<l=1}^4 \sum_{\alpha=1}^2 \sum_{j=3}^4 c_{\Gamma} A_4^{(2)}(k, l, \alpha, j) A_4^f(j), \]  

(E.5)

but we have

\[ A_4^f(i, j) = 0. \]  

(E.6)

The four-quark primitive amplitudes \( A_4^R \) are give by

\[ A_4^R = \sum_{k<l=1}^4 \sum_{\alpha=1}^2 \sum_{\beta=3}^4 c_{\Gamma} A_4^{(4)}(k, l, \alpha, \beta) A_4^R(\alpha, \beta), \]  

(E.7)

with

\[ A_4^R(1, 3) = -\frac{2}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{4}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 8 \]
\[ + \frac{s_{12}(s_{12} - s_{23})}{2s_{13}^2} \left( \log^2 \left( \frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + 3 \frac{s_{12}}{s_{13}} \log \left( \frac{-s_{12}}{-s_{23}} \right); \]  

(E.8a)

\[ A_4^R(2, 4) = -\frac{2}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{4}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 8 \]
\[ + \frac{s_{12}(s_{12} - s_{23})}{2s_{13}^2} \left( \log^2 \left( \frac{-s_{12}}{-s_{23}} \right) + \pi^2 \right) + 3 \frac{s_{12}}{s_{13}} \log \left( \frac{-s_{12}}{-s_{23}} \right). \]  

(E.8b)
\[ A_R^R(1, 3) = -\frac{2}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{4}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 8, \]  
(E.8c)

\[ A_R^R(2, 3) = -\frac{2}{\epsilon^2} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - \frac{4}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon - 8. \]  
(E.8d)

The remaining primitive amplitude \( A_4^f \) is

\[ A_4^f = \sum_{k<l=1}^2 \sum_{\alpha=1}^4 \sum_{\beta=3}^4 c_f A_4^{(4)}(k, l, \alpha, \beta) A_4^f, \]  
(E.9)

with

\[ A_4^f = \frac{1}{\epsilon} \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon + 2. \]  
(E.10)

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