Experimental and theoretical studies of oscillations of stratified fluid

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Abstract. This article examines the problem of small movements of liquids and the motion of a solid with a circular cylindrical cavity that is completely filled three of incompressible ideal fluids. Assuming irrotational fluid motion is formulated boundary value problem and solutions are obtained for potentials of the displacements of the particles of liquids. Using the Lagrange equations of the 2nd kind the equations of motion of a hydromechanical system. Illustrates the difference between motion of a rigid body having a cavity with fluids from the case considered by N. E. Zhukovsky. Examines the problem of natural vibrations of liquids in a stationary tank and the motion of a rigid body fluids. Given formula, determine the natural frequencies and mode shapes. The article presents the results of an experimental study of motion of rigid body and liquids.

Keywords: rigid body, liquid, natural frequencies, mode shapes, incompressible ideal fluids.

1. Introduction.
Currently, the problem of the motion of a rigid body with a cavity filled with fluid is a classical problem in mechanics. The first such task was considered N. E. Zhukovsky and published in [1]. Recent advances in the aerospace industry, cryogenic industry and gas industry require research of a more general problem of the motion of bodies containing a complex liquid medium. One of such problems is the problem of the motion of a rigid body with a cavity filled with layered liquids. Fluctuations of liquids partially filling the cavity of the stationary and movable rigid body devoted to a fairly large number of works, see for example [2]-[15].

2. Statement of the problem
Let a circular cylinder of radius \( r_0 \), completely filled with three ideal incompressible fluid having a density \( \rho_0, \rho_1, \rho_2 \), (\( \rho_0 \neq \rho_1 \neq \rho_2 \)) (see figure 1). The cylinder is mounted on a movable platform with a total mass M, capable of moving on a smooth horizontal base. Assuming that the fluid in the cylinder, making a small displacement will define the main characteristics of the movement of liquids to a circular cylinder.
To compile the equations of motion of the considered mechanical system will use the equations of motion the Lagrange 2\textsuperscript{nd} kind. The kinetic energy of the system can be written in the form

\[ T = \frac{1}{2} M u^2 + \frac{1}{2} \sum_{i=0}^2 \rho_i \int \int \int (\nabla \Phi_i)^2 \, dV_i, \quad (1) \]

where \( V_i, \Phi_i \) \((i = 0, 1, 2)\) - the volumes and capacities of the absolute velocities of the liquids. At low fluid motion potentials velocities \( \Phi_i \) can be expressed the potentials of displacement of the particles

\[ \Phi_i = \frac{\partial \chi_i(x, y, z, t)}{\partial t}, \quad (i = 0, 1, 2). \quad (2) \]

The potentials of displacement \( \chi_i \) are solutions of boundary value problems that satisfy the Laplace equation and the boundary conditions:

\[ \Delta \chi_i = 0, (i = 0, 1, 2), \quad (3) \]

(a) the conditions of no motion on wetted surfaces

\[ \frac{\partial \chi_i}{\partial V_i} \bigg|_{S_i} = u \cdot \sin(\eta), \quad \frac{\partial \chi_0}{\partial V_0} \bigg|_{x_2 = 0} = 0, \quad \frac{\partial \chi_2}{\partial V_2} \bigg|_{x_2 = 0} = 0, \quad (4) \]

(b) kinematic conditions on the surfaces of the sections

\[ \frac{\partial \chi_0}{\partial V_0} = \frac{\partial \chi_1}{\partial V_1}; \text{ at } \Gamma_1 \quad \text{and} \quad \frac{\partial \chi_1}{\partial V_1} = \frac{\partial \chi_2}{\partial V_2}; \text{ at } \Gamma_2, \quad (5) \]

(c) dynamic conditions on the surfaces of the sections

\[ \left( \rho_0 \cdot \frac{\partial^2 \chi_0}{\partial t^2} - \rho_1 \cdot \frac{\partial^2 \chi_1}{\partial t^2} \right) = \left( \rho_1 - \rho_0 \right) \cdot g \cdot \frac{\partial \chi_0}{\partial V_0}; \text{ at } \Gamma_1, \quad (6) \]

\[ \left( \rho_1 \cdot \frac{\partial^2 \chi_1}{\partial t^2} - \rho_2 \cdot \frac{\partial^2 \chi_2}{\partial t^2} \right) = \left( \rho_2 - \rho_1 \right) \cdot g \cdot \frac{\partial \chi_1}{\partial V_1}; \text{ at } \Gamma_2, \quad (7) \]

The solution of the boundary value problems (2)-(7) the displacement potentials can be written as,
\[ \chi_0 = 2r_0 \sin \eta \sum_{n=1}^{\infty} \frac{J_1(k_n \xi)}{J_1(x_n)} \left( u(t) - thk_n \sigma_n \right), \tag{8} \]

\[ \chi_1 = 2r_0 \sin \eta \sum_{n=1}^{\infty} \frac{J_1(k_n \xi)}{J_1(x_n)} \left( u(t) + thk_n \sigma_n \right), \tag{9} \]

\[ \chi_2 = 2r_0 \sin \eta \sum_{n=1}^{\infty} \frac{J_1(k_n \xi)}{J_1(x_n)} \left( u(t) + thk_n \sigma_n \right). \tag{10} \]

The potential energy for the considered case is given by,

\[ \Pi_\xi = \frac{1}{2} g(\rho - \rho_0) \sum_{n=1}^{\infty} \left( \frac{\partial \chi_1}{\partial \xi} \right)^2 r \sin \eta dr + \frac{1}{2} g(\rho - \rho_1) \sum_{n=1}^{\infty} \left( \frac{\partial \chi_2}{\partial \xi} \right)^2 r \sin \eta dr. \tag{11} \]

Putting the solution of (8) to (10) for potentials of displacements in formulas (1), (11), we obtain final expressions for the kinetic and potential energy,

\[ T_\xi = \frac{1}{2} M^* \ddot{u}_c + \frac{1}{2} \sum_{n=1}^{\infty} \left( m_i^{*} - m_0 \right) \ddot{u}_c \sigma_i + \frac{1}{2} \sum_{n=1}^{\infty} \left( m_2^{*} - m_1 \right) \ddot{u}_c \sigma_2, \tag{12} \]

\[ + \frac{1}{2} \sum_{n=1}^{\infty} \left( m_i^{*} + m_0 \right) \omega_n^2 \sigma_i - \frac{1}{2} \sum_{n=1}^{\infty} \left( m_2^{*} + m_1 \right) \omega_n^2 \sigma_2. \]

\[ \Pi_\xi = \frac{1}{2} \sum_{n=1}^{\infty} (m_i^{*} - m_0) \omega_n^2 \sigma_i - \frac{1}{2} \sum_{n=1}^{\infty} (m_2^{*} - m_1) \omega_n^2 \sigma_2. \tag{13} \]

In formulas (12), (13) adopted notation: \( \sigma_i, \sigma_2 \) - generalized coordinates of wave motion of surfaces of sections of the fluids.

\( m_0 = \pi r_0^{2} \rho_0 h_0, m_1 = \pi r_0^{2} \rho_1 h_0, m_2 = \pi r_0^{2} \rho_2 h_2 \) - the mass of each liquid.

\( m_i^{*} = \rho V, m_0^{*} = \rho V, m_2^{*} = \rho V \) - mass of oscillating fluids.

\[ V_n = \frac{2 \pi r_0^3}{\xi_n (\xi_n^2 - 1)} \text{thk}_n \text{thk}_n, \quad \overline{f}_{0a} = \text{thk}_n \text{thk}_n, \quad \overline{f}_{1a} = \text{thk}_n \text{thk}_n, \quad \omega_n = gk \text{thk}_n. \]

\( M^* = m_0 + m_1 + m_2 + M \) - the total mass of the whole hydro-mechanical system.

3. Derivation of equations of motion

Used the Lagrange equations of the 2nd kind, we get the system of equations of motion in the form

\[ M^* \dddot{u}_c + \sum_{n=1}^{\infty} \left( m_i^{*} - m_0 \right) \dddot{\sigma}_i + \sum_{n=1}^{\infty} \left( m_2^{*} - m_1 \right) \dddot{\sigma}_2 = 0, \tag{14} \]

\[ \dddot{\sigma}_i + \frac{(m_i^{*} - m_0)}{m_i^{*} + m_0} \omega_i^2 \sigma_i - \frac{m_i^{*} \sigma_i}{m_i^{*} + m_0} \dddot{u}_c + \frac{(m_2^{*} - m_0)}{m_i^{*} + m_0} \dddot{u}_c = 0, \quad (n = 1, 2, 3, \ldots) \tag{15} \]

\[ \dddot{\sigma}_2 + \frac{(m_2^{*} - m_1)}{m_i^{*} + m_0} \omega_2^2 \sigma_2 - \frac{m_i^{*} \sigma_i}{m_i^{*} + m_0} \dddot{u}_c + \frac{(m_2^{*} - m_0)}{m_i^{*} + m_0} \dddot{u}_c = 0, \quad (n = 1, 2, 3, \ldots) \tag{16} \]
For the convenience of further research, we introduce new notations: 

\[ m^*_{1n} = (m'_{1n} - m''_{1n}), \quad m^*_{2n} = (m'_{2n} - m''_{2n}), \quad \alpha_{1n} = \frac{(m'_{1n} + m''_{1n})}{\alpha_{1n}}, \quad \alpha_{2n} = \frac{(m'_{2n} + m''_{2n})}{\alpha_{2n}}, \quad a_{12n} = m'_{1n} / \text{ch} \alpha_{1n}, \]

\[ a_{21n} = m'_{2n} / \text{ch} \alpha_{2n}, \quad b_{1n} = \frac{m^*_{1n}}{\alpha_{1n}}, \quad b_{2n} = \frac{m^*_{2n}}{\alpha_{2n}}, \quad \beta_{1n} = b_{1n} \omega_n^2, \quad \beta_{2n} = b_{2n} \omega_n^2. \]

Let the tank completely filled with liquid, and mounted on the movable plate can only move progressively in the horizontal plane. We assume that the plate acts in the horizontal plane of the external force \( F_u = F_0 \sin pt \), the line of action which pass through the plane of symmetry of the tank. Considering only the first tone of the vibrations of the fluid \( (n = 1) \), we define the amplitude of the external force \( F_0 \), if the movable plate moves with acceleration \( u_c = a_0 \sin pt \).

The equations of motion of a plate with liquid in \( (n = 1) \) will be form

\[ M^* u_{\text{c}c} + m^* \sigma_1 + m^* \sigma_2 = F_0 \sin pt, \quad \sigma_1 + \beta_{1n} \sigma_1 - a_{12} \sigma_2 + b_{1n} u_{\text{c}c} = 0, \quad \sigma_2 + \beta_{2n} \sigma_2 - a_{21} \sigma_1 + b_{2n} u_{\text{c}c} = 0. \]

When considering forced oscillations of the liquid, we assume that the generalized coordinates \( \sigma_1(t), \quad \sigma_2(t) \) change the law \( \sigma_1 = \sigma_1^0 \sin pt, \quad \sigma_2 = \sigma_2^0 \sin pt \) and \( u_c = u_c^0 \sin pt \), and rewrite equations (17)-(19),

\[ -p^2 (M^* u_{\text{c}c} + m^* \sigma_1^0 + m^* \sigma_2^0) = F_0, \]

\[ (-p^2 + \beta_{1n}^0) \sigma_1^0 - a_{12} p^2 \sigma_2^0 = b_1 p^2 u_{c0}^0, \]

\[ -a_{21} p^2 \sigma_2^0 + (-p^2 + \beta_{2n}^0) \sigma_2^0 = b_2 p^2 u_{c0}^0, \]

From equations (21) and (22) define values \( \sigma_1^0 \) and \( \sigma_2^0 \),

\[ \sigma_1^0 = \sigma_1^0 - a_{12} \sigma_2^0 = b_1 \left( -p^2 + \beta_{1n}^0 \right) + b_2 \sigma_2^0 = \frac{b_1 \left( -p^2 + \beta_{1n}^0 \right) + b_2 \sigma_2^0}{\Omega_1^2 - p^2}, \quad \sigma_2^0 = \frac{b_2 \left( -p^2 + \beta_{2n}^0 \right) + b_1 \sigma_1^0}{\Omega_2^2 - p^2}, \]

where \( \Omega_1 \) and \( \Omega_2 \) - natural frequencies of the three liquids in a stationary tank,

\[ \Omega_{1,2}^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \]

\[ a = 1 - a_{21} a_{21}, \quad b = -\left( \beta_{1n}^0 + \beta_{2n}^0 \right), \quad c = \left( \beta_{1n}^0 \cdot \beta_{2n}^0 \right). \]

Substitute the obtained expression (23) into equation (20). The result of the equation of motion of a rigid body transformed to the

\[ F_0 = a_0 M_{np}, \]

where \( M_{np} \) - mass converted rigid body,

\[ M_{np} = M^* \left[ 1 + m^* \left[ b_1 \left( -p^2 + \beta_{1n}^0 \right) + b_2 p^2 a_{12} \right] \frac{p^2}{\Omega_1^2 - p^2} \right] + m^* \left[ b_2 \left( -p^2 + \beta_{2n}^0 \right) + b_1 p^2 a_{21} \right] \frac{p^2}{\Omega_2^2 - p^2}. \]

One of the results of studies of the motion of a rigid body with a cavity completely filled with a homogeneous liquid drop is that mass converted rigid body is equal to the sum of the masses of rigid
body and solidified liquid [1]. In this case, as shown by the formula (27) in the motion of a rigid body with a layered liquid mass converted rigid body depends on the frequency of the oscillations.

Graph of the ratio \( M_{np} / M^* \) shown in figure 2. The graph was the following notation,

\[
\begin{align*}
  f(\bar{p}) &= M_{np} / M^*, \\
  \bar{p}^2 &= \frac{p^2}{\Omega_1^2}, \\
  \bar{p}^2 &= \frac{\Omega_2^2}{\Omega_1^2}, \\
  \bar{p}^2 &= \frac{\beta_1^2}{\Omega_1^2}, \\
  \bar{p}^2 &= \frac{\beta_2^2}{\Omega_1^2}, \\
  \Delta_2 &= (\rho_2 - \rho_1), \\
  \Delta_1 &= (\rho_1 - \rho_0).
\end{align*}
\]

(28) (29)

Figure 2. The dependence of the dimensionless mass converted rigid body \( f(\bar{p}) = M_{np} / M^* \) from the dimensionless frequency of oscillation, \( (\Omega_1^{(1)}, \Omega_2^{(1)}, \Omega_2^{(2)}, \Omega_2^{(2)}) \) - the first and second main frequency for the cases \( \Delta_1 = \Delta_2 = 0.1 \) and \( \Delta_1 = \Delta_2 = 0.2 \).

4. Experimental study of fluctuations of the stratified liquid

Subjects liquids selected were water (\( \rho_2 = 1000 \text{ kg} / \text{m}^3 \)), sunflower oil density (\( \rho_1 = 920 \text{ kg} / \text{m}^3 \)) and formic alcohol (\( \rho_0 = 830 \text{ kg} / \text{m}^3 \)), with coefficients of surface tension of water - sunflower oil (\( \sigma_1 = 0.013 \text{ N} / \text{m} \)), and sunflower oil-formic alcohol (\( \sigma_2 = 0.02 \text{ N} / \text{m} \)), the value of the kinematic viscosity of water (\( \nu_1 = 1,006.10^{-6} \text{ m}^2 / \text{s} \)), sunflower oil (\( \nu_2 = 1,54.10^{-6} \text{ m}^2 / \text{s} \)) and formic alcohol (\( \nu_0 = 1,54.10^{-6} \text{ m}^2 / \text{s} \)).

4.1. Description of the experimental setup

Experimental setup figure 3 consisted of the base -1, the movable plate -2, -3 transparent tank, electromechanical exciter of vibrations -4 and measuring complex. The base was a stationary plate of the textolite, which was fixed electromechanical exciter and metal guides ensure smooth motion without tank vibration of the movable plate.

Control and measurement system consisted of a strain test station VI6-6 T -5, pressure sensors and displacement -6 and -7, USB digital oscilloscope Hantek 6022BE -8 and -9 laptop and power supply HY5003-2 -10, which were placed on the laboratory table, close to the fixed base. Power supply HY5003-2 is designed to power the strain test station and motor. Control and data display oscilloscope
Hantek 6022BE implemented personal computer/laptop/netbook/tablet running Windows OS via USB interface.

General view of the experimental setup is shown in figure 3.

![Figure 3. General view of the experimental setup: 1-base, 2-movable plate, 3-cylindrical container, 4-electromechanical exciter, 5-strain station VI6-6 T, 6-pressure transducer, 7-the displacement transducer, 8-USB digital oscilloscope Hantek 6022BE, the 9-ASUS laptop 10-power HY5003-2.](image)

4.2. The kinematic diagram of the setup
Electromechanical oscillation exciter consisted of a DC motor and a crank mechanism that reciprocates the movable plate.
As the vibration measuring instrument used inductive apparatus VI6-6 T-compact six-channel, with solid-state power supply. Installation VI6-6 T is designed to measure vibrations, accelerations, pressures and displacements. On the basis of the instrument lies principle amplitude modulation frequency inductive sensors on differential-transformer scheme.

4.3. Experimental determination of the main vibration characteristics of layered fluid
To find the oscillation frequencies of a layered fluid was used the method of free oscillations. To create the initial conditions of the free oscillations of the fluids moving platform with a tank was set vibrations through the reciprocating motion close to a resonant frequency of oscillations of the fluids. After some time joint fluctuations of the tank with liquids platform instantly stopped, and the liquid made free oscillations in a stationary tank. In the experiment on a laptop recorded the forced and free oscillations of fluids and the movement of the tank itself.
In the experiment it is possible to allocate following stages:
(1) before the experiment was determined the theoretical value of the oscillation frequency of liquids in a stationary tank by the formulas (24), (25),
(2) carried out calibration of displacement sensors and pressure calibration of pressure sensors was carried out by additions of liquid (water) at a certain depth,

(3) the test cylinder was freed from some part of the water and then filled with layers of liquids of different thickness for the experiment,

(4) the experiment was repeated again with a different average depth of the liquid,

(5) the results of the records of the experiments the laptop was determined the natural frequencies of liquids.

The values of the frequencies defined by the formulas (24), (25) and obtained from the experiment shown in figure 4.

![Figure 4](image1.png)

**Figure 4.** Dependence to the first main frequency (a) and the second main frequency (b) on the average thickness of the liquid defined by the formula (24), (25), and defined in the experiment ($\bar{h}_0 = 0.6, \bar{h}_2 = 2.1 - \bar{h}_1$)

In figures 5 and 6 shows typical graphs of the processes recorded during the experiments

![Figure 5](image2.png)

**Figure 5.** Example of a record of the oscillations of the moving platform and forced vibration of fluids
Conclusion. In work the equations of motion of a rigid body performs translational motion and having a hollow cylindrical shape, is completely filled with three ideal homogeneous immiscible liquids. The resulting equations of motion, formulas for the added masses and the results of numerical calculations. Given the results of the pilot study. The results can be used in the design of transport systems with cryogenic liquids or liquefied gases.

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