Abstract Interpretation with Unfoldings*

Marcelo Sousa¹, César Rodríguez², Vijay D’Silva³ and Daniel Kroening¹

¹ University of Oxford, United Kingdom
² Université Paris 13, Sorbonne Paris Cité, LIPN, CNRS, France
³ Google Inc., San Francisco

Abstract. We present and evaluate a technique for computing path-sensitive interference conditions during abstract interpretation of concurrent programs. In lieu of fixed point computation, we use prime event structures to compactly represent causal dependence and interference between sequences of transformers. Our main contribution is an unfolding algorithm that uses a new notion of independence to avoid redundant transformer application, thread-local fixed points to reduce the size of the unfolding, and a novel cutoff criterion based on subsumption to guarantee termination of the analysis. Our experiments show that the abstract unfolding produces an order of magnitude fewer false alarms than a mature abstract interpreter, while being several orders of magnitude faster than solver-based tools that have the same precision.

1 Introduction

This paper is concerned with the problem of extending an abstract interpreter for sequential programs to analyze concurrent programs. A naïve solution to this problem is a global fixed point analysis involving all threads in the program. A distinct solution is to analyze threads in isolation and exchange invariants on global variables between threads [19,18,4]. Related research, including this paper, seeks analyses that preserve the scalability of the local fixed point approach without losing the precision of a global fixed point.

We design and implement an abstract unfoldings data structure and analysis that combines an abstract domain with the type of unfolding algorithm used to analyze Petri nets. An unfolding is a tree-like structure that uses partial orders to represent concurrent executions and conflict relations to represent interference. A challenge in combining unfoldings with abstract domains is that abstract domains typically provide approximations of states and transitions, not traces, and are not equipped with interference information. Another challenge is that unfolding algorithms are typically applied to explicit-state analysis of systems with deterministic transitions while abstract domains are symbolic and non-deterministic owing to abstraction.

The main idea of this paper is to construct an unfolding of an analyzer, rather than a program. An event is the application of a transformer in an analysis

* Supported by ERC project 280053 (CPROVER) and a Google Fellowship.
context, and concurrent executions are replaced by a partial order on transformer applications. We introduce independence for transformers and use this notion to construct an unfolding of a domain given a program and independence relation. The unfolding of a domain is typically large and we use thread-local fixed point computation to reduce its size without losing interference information.

From a static analysis perspective, our analyser is a path-sensitive abstract interpreter that uses an independence relation to compute a history abstraction (or trace partition) and organizes exploration information in an unfolding. From a dynamic analysis perspective, our approach is a super-optimal POR [23] that uses an abstract domain to collapse branches of the computation tree originating from thread-local control decisions.

**Contribution** We make the following contributions towards reusing an abstract interpreter for sequential code for the analysis of a concurrent program.

1. A new notion of transformer independence for unfolding with domains (Sec. 4).
2. The unfolding of a domain, which provides a sound way to combine transformer application and partial-order reduction (Sec. 5.1).
3. A method to construct the unfolding using thread-local analysis and pruning techniques (Sec. 6.1, Sec. 6).
4. An implementation and empirical evaluation demonstrating the trade-offs compared to an abstract interpreter and solver-based tools (Sec. 7).

We provide the proofs of our formal results in the Appendix.

## 2 Motivating Example and Overview

Consider the program given in Fig. 1 (a), which we wish to prove safe using an interval analysis. Thread 1 (resp. 2) increments \( i \) (resp. \( j \)) in a loop that can non-deterministically stop at any iteration. All variables are initialized to 0 and the program is safe as the `assert` in thread 2 cannot be violated.

When we use a POR approach to prove safety of this program, the exploration algorithm exploits the fact that only the interference between statements that modify the variable \( g \) can lead to distinct final states. This interference is typically
known as *independence* [22,11]. The practical relevance of independence is that one can use it to define a safe fragment, given in Fig. 1 (b), of the computation tree of the program which can be efficiently explored [23,1]. At every iteration of each loop, the conditionals open one more branch in the tree. Thus, each branch contains a different write to the global variable, which is dependent with the writes of the other thread as the order of their application reaches different states. As a result, the exploration tree becomes intractable very fast. It is of course possible to bound the depth of the exploration at the expense of completeness of the analysis.

The thread-modular static analysis that is implemented in AstreeA [19] or Frama-C [26] incorrectly triggers an alarm for this program. These tools statically analyze each thread in isolation assuming that \( g \) equals 0. Both discover that thread 1 (resp. 2) can write \([0,100]\) (resp. \([0,150]\)) to \( g \) when it reads 0 from it. Since each thread can modify the variable read by the other, they repeat the analysis starting from the join of the new interval with the initial interval. In this iteration, they discover that thread 2 can write \([0,250]\) to \( g \) when it reads \([0,150]\) from it. The analysis now incorrectly determines that it needs to re-analyze thread 2, because thread 1 also wrote \([0,250]\) in the previous iteration and that is a larger interval than that read by thread 2. This is the reasoning behind the false alarm. The core problem here is that these methods are path-insensitive across thread context switches and that is insufficient to prove this assertion. The analysis is accounting for a thread context switch that can never happen (the one that flows \([0,250]\) to thread 2 before thread 2 increments \( g \)).

More recent approaches [14,20] can achieve a higher degree of flow-sensitivity but they either require manual annotations to guide the trace partitioning or are restricted to program locations outside of a loop body.

Our key contribution is an unfolding that is flow- and path-sensitive across interfering statements of the threads and path-insensitive inside the non-interfering blocks of statements. Figure 1 (c) shows the unfolding structure that our method explores for this program. The boxes in this structure are called *events* and they represent the action of firing a transformer after a history of firings. The arrows depict *causality* constraints between events, i.e., the *happens-before* relation. Dotted lines depict the immediate *conflict relation*, stating that two events cannot be simultaneously present in the same concurrent execution, known as *configuration*. This structure contains three maximal configurations (executions), which correspond to the three ways in which the statements reading or writing to variable \( g \) can interleave.

Conceptually, we can construct this unfolding using the following idea: start by picking an arbitrary interleaving. Initially we pick the empty one which reaches the initial state of the program. Now we run a sequential abstract interpreter on one thread, say thread 1, from that state and stop on every location that reads or writes a global variable. In this case, the analyzer would stop at the statement \( g += i \) with the invariant that \((g \mapsto [0,0], i \mapsto [0,100])\). This invariant corresponds to the first event of the unfolding (top-left corner). The unfolding contains now a new execution, so we iterate again the same procedure by picking
the execution consisting of the event we just discovered. We run the analyser on thread 2 from the invariant reached by that execution and stop on any global action. That gives rise to the event \( g += j \), and in the next step using the execution composed of the two events we have seen, we discover its causal successor \( a() \). Note however that before visiting that event, we could have added event \( g += j \) corresponding to the invariant of running an analyser starting from the initial state on thread 2. Furthermore we know that because both invariants are related to the same shared variable, these two events must be ordered. We enforce that order with the conflict relation.

Our method mitigates the aforementioned branching explosion of the POR tree because it never unfolds the conflicting branches of a naive exploration. In comparison to thread-modular analysis, it remains precise about the context switches because it uses a history-preserving data structure.

Another novelty of our approach is the observation that certain events are \textit{equivalent} in the sense that the state associated with one is \textit{subsumed} by the second. In our example, one of these events, known as a \textit{cutoff event}, is labelled by \( g += i \) and denoted with a striped pattern. Specifically, the configuration \( \{ g += i, g += j \} \) reaches the same state as \( \{ g += j, g += i \} \). Thus, no causal successor of a cutoff event needs to be explored as any action that we can discover from the cutoff event can be found somewhere else in the structure.

Outline. The following diagram displays the various concepts and transformations presented in the paper:

![Overview diagram](image)

Fig. 2. Overview diagram

Let \( M \) be the program under analysis whose concrete semantics \( C_M \) is abstracted by a domain \( D_M \). The relations \( \diamond \) and \( \diamond_i \) are independence relations with different levels of granularity over the transformers of \( M, C_M, \) or \( D_M \). We denote by \( U_{C_M, \diamond} \) the \textit{unfolding} of either \( C_M \) or \( D_M \) under independence relation \( \diamond \) (defined in Sec. 5.1). Whenever we unfold a domain using a weak independence relation \( (\diamond_2 \text{ on } C_M \text{ and } \diamond_3 \text{ on } D_M) \), we can use cutoffs to prune the unfolding represented by the dashed line between unfoldings. The resulting unfolding (defined in Sec. 6.1) is denoted by the letter \( P \). The main contribution of our work is the \textit{compact unfolding}, \( Q_{D_M, \diamond_3} \), described above.
## 3 Preliminaries

There is no new material in this section, but we recommend the reader to review the definition of an analysis instance, which is not standard.

**Concurrent programs.** We model the semantics of a concurrent, non-deterministic program by a labelled transition system \( M = (\Sigma, \rightarrow, A, s_0) \), where \( \Sigma \) is the set of states, \( A \) is the set of program statements, \( \rightarrow \subseteq \Sigma \times A \times \Sigma \) is the transition relation, and \( s_0 \) is the initial state. The identifier of the thread containing a statement \( a \) is given by a function \( p : A \rightarrow \mathbb{N} \). If \( s \xrightarrow{a} s' \) is a transition, the statement \( a \) is enabled at \( s \), and \( a \) can fire at \( s \) to produce \( s' \). We let \( \text{enabl}(s) \) denote the set of statements enabled at \( s \). As statements may be non-deterministic, firing \( a \) may produce more than one such \( s' \). A sequence \( \sigma : = a_1 \ldots a_n \in A^* \) is a run when there are states \( s_1, \ldots, s_n \) satisfying \( s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} s_n \). For such \( \sigma \) we define \( \text{state}(\sigma) : = s_n \). We let \( \text{runs}(M) \) denote the set of all runs of \( M \), and \( \text{reach}(M) : = \{ \text{state}(\sigma) \in \Sigma : \sigma \in \text{runs}(M) \} \) the set of all reachable states of \( M \).

**Analysis Instances.** A lattice \( (D, \sqsubseteq, \sqcup_D, \sqcap_D) \) is a poset with a binary, least upper bound operator \( \sqcup_D \) called join and a binary, greatest lower bound operator \( \sqcap_D \) called meet. A transformer \( f : D \rightarrow D \) is a monotone function on \( D \). A domain \( (D, \sqsubseteq, F) \) consists of a lattice and a set of transformers. We adopt standard assumptions in the literature that \( D \) has a least element \( \bot \), called bottom, and that transformers are bottom-strict, i.e. \( f(\bot) = \bot \). To simplify presentation, we equip domains with sufficient structure to lift notions from transition systems to domains, and assume that domains represent control and data states.

**Definition 1.** An analysis instance \( D : = (D, \sqsubseteq, F, d_0) \), consists of a domain \( (D, \sqsubseteq, F) \) and an initial element \( d_0 \in D \).

A transformer \( f \) is enabled at an element \( d \) when \( f(d) \neq \bot \), and the result of firing \( f \) at \( d \) is \( f(d) \). The element generated by or reached by a sequence of transformers \( \sigma : = f_1, \ldots, f_m \) is the application \( \text{state}(\sigma) : = (f_m \circ \ldots \circ f_1)(d_0) \) of transformers in \( \sigma \) to \( d_0 \). Let \( \text{reach}(D) \) be the set of reachable elements of \( D \). The sequence \( \sigma \) is a run if \( \text{state}(\sigma) \neq \bot \) and \( \text{runs}(D) \) is the set of all runs of \( D \).
The collecting semantics of a transition system $M$ is the analysis instance $C_M := \langle \mathcal{P}(\Sigma), \subseteq, F, \{s_0\} \rangle$, where $F$ contains a transformer $f_a(S) := \{ s' \in \Sigma : s \in S \land s \xrightarrow{a} s' \}$ for every statement $a$ of the program. The pointwise-lifting of a relation $R \subseteq A \times A$ on statements to transformers in $D$ is $R_D = \{(f_a, f_{a'}) \mid \langle a, a' \rangle \in R\}$. Let $m_0: A \rightarrow F$ be map from statements to transformers: $m_0(a) := f_a$. An analysis instance $D = (D, \subseteq, F, d_0)$ is an abstraction of $(D, \subseteq, F, d_0)$ if there exists a concretization function $\gamma: D \rightarrow D$, which is monotone and satisfies that $d_0 \subseteq \gamma(d_0)$, and that $f \circ \gamma \subseteq \gamma \circ f$, where the order between functions is pointwise.

**Labelled Prime Event Structures.** Event structures are tree-like representations of system behaviour that use partial orders to represent concurrent interaction. Fig. 3 (c) depicts an event structure. The nodes are events and solid arrows, represent causal dependencies: events 4 and 7 must fire before 8 can fire. The dotted line represents conflicts: 4 and 7 are not in conflict and may occur in any order, but 4 and 9 are in conflict and cannot occur in the same execution.

A labelled prime event structure [21] (pes) is a tuple $\mathcal{E} := (E, <, \#, h)$ with a set of events $E$, a causality relation $< \subseteq E \times E$, which is a strict partial order, a conflict relation $\# \subseteq E \times E$ that is symmetric and irreflexive, and a labelling function $h: E \rightarrow X$. The components of $\mathcal{E}$ satisfy (1) the axiom of finite causes, that for all $e \in E$, $\{e' \in E : e' < e\}$ is finite, and (2) the axiom of hereditary conflict, that for all $e, e', e'' \in E$, if $e \neq e'$ and $e' < e''$, then $e \neq e''$.

The history of an event $[e] := \{e' \in E : e' < e\}$ is the least set of events that must fire before $e$ can fire. A configuration of $\mathcal{E}$ is a finite set $C \subseteq E$ that is (i) (causally closed) $[e] \subseteq C$ for all $e \in C$, and (ii) (conflict free) $\neg (e \neq e')$ for all $e, e' \in C$. We let $conf(\mathcal{E})$ denote the set of all configurations of $\mathcal{E}$. For any $e \in E$, the local configuration of $e$ is defined as $[e] := [e] \cup \{e\}$. In Fig. 3 (c), the set $\{1, 2\}$ is a configuration, and in fact it is a local configuration, i.e., $[2] = \{1, 2\}$. The set $\{1, 2, 3\}$ is a $\subseteq$-maximal configuration. The local configuration of event 8 is $\{4, 7, 8\}$. Given a configuration $C$, we define the interleavings of $C$ as $\text{inter}(C) := \{ h(e_1), \ldots, h(e_n) : \forall e_i, e_j \in C, e_i < e_j \implies i < j \}$. An interleaving corresponds to the sequence labelling any topological sorting (sequentialization) of the events in the configuration. We say that $\mathcal{E}$ is finite iff $E$ is finite. Fig. 3 (d) shows the interleavings of configuration $\{1, 2, 3\}$.

Event structures are naturally (partially) ordered by a prefix relation $\preceq$. Given two pes $\mathcal{E} := (E, <, \#, h)$ and $\mathcal{E}' := (E', <', \#' , h')$, we say that $\mathcal{E}$ is a prefix of $\mathcal{E}'$, written $\mathcal{E} \preceq \mathcal{E}'$, when $E \subseteq E'$, $<$ and $\#$ are the projections of $<'$ and $\#' \rightarrow E$, and $E \supseteq \{ e' \in E' : e' < e \land e \in E \}$. Moreover, the set of prefixes of a given pes $\mathcal{E}$ equipped with $\preceq$ is a complete lattice.

### 4 Independence for Transformers

Partial-order reduction tools use a notion called independence to avoid exploring concurrent interleavings that lead to the same state. Our analyzer uses independence between transformers to compactly represent transformer applications that lead to the same result. The contribution of this section is a notion of...
independence for transformers (represented by the lowest horizontal line in Fig. 2) and a demonstration that abstraction may both create and violate independence relationships.

We recall a standard notion of independence for statements [22,11]. Two statements \( a, a' \) of a program \( M \) commute at a state \( s \) iff

- if \( a \in \text{enabl}(s) \) and \( s \xrightarrow{a} s' \), then \( a' \in \text{enabl}(s) \) iff \( a' \in \text{enabl}(s') \); and
- if \( a, a' \in \text{enabl}(s) \), then there is a state \( s' \) such that \( s \xrightarrow{a,a'} s' \) and \( s \xrightarrow{a',a} s' \).

Independence between statements is an underapproximation of commutativity. A relation \( \Diamond \subseteq A \times A \) is an independence for \( M \) if it is symmetric, irreflexive, and satisfies that every \( (a, a') \in \Diamond \) commute at every reachable state of \( M \). In general, \( M \) has multiple independence relations; \( \emptyset \) is always one of them.

Suppose independence for transformers is defined by replacing statements and transitions with transformers and transformer application, respectively. Ex. 1 illustrates that an independence relation on statements cannot be lifted to obtain transformers that are independent under such a notion.

**Example 1.** Consider the collecting semantics \( C_M \) of a program \( M \) with two variables, \( x \) and \( y \), two statements \( a := \text{assume}(x=0) \) and \( a' := \text{assume}(y=0) \), and initial element \( d_0 := \{ (x \mapsto 0, y \mapsto 1), (x \mapsto 1, y \mapsto 0) \} \). Since \( a \) and \( a' \) read different variables, \( R := \{ (a, a'), (a', a) \} \) is an independence relation on \( M \). Now observe that \( \{ (f_a, f_{a'}), (f_{a'}, f_a) \} \) is not an independence relation on \( C_M \), as \( f_a \) and \( f_{a'} \) disable each other. Note, however, that \( f_a(f_{a'}(d_0)) \) and \( f_{a'}(f_a(d_0)) \) are both \( \perp \).

Weak independence, defined below, allows transformers to be considered independent even if they disable each other.

**Definition 2.** Let \( \mathcal{D} := (D, \subseteq, F, d_0) \) be an analysis instance. A relation \( \Diamond \subseteq F \times F \) is a weak independence on transformers if it is symmetric, irreflexive, and satisfies that every \( f \Diamond f' \) implies \( f(f'(d)) = f'(f(d)) \) for every \( d \in \text{reach}(\mathcal{D}) \). Moreover, \( \Diamond \) is an independence if it is a weak independence and satisfies that if \( f(d) \neq \perp \), then \( (f \circ f')(d) \neq \perp \) iff \( f'(f(d)) \neq \perp \), for all \( d \in \text{reach}(\mathcal{D}) \).

Recall that \( R_\mathcal{D} \) is the lifting of a relation on statements to transformers. Observe that the relation \( R \) in Ex. 1, when lifted to transformers, is a weak independence on \( C_M \). The proposition below shows that independence relations on statements generate weak independence on transformers over \( C_M \).

**Proposition 1 (Lifted independence).** If \( \Diamond \) is an independence relation on \( M \), the lifted relation \( \Diamond_{C_M} \) is a weak independence on the collecting semantics \( C_M \).

We now show that independence and abstraction are distinct notions in that transformers that are independent in a concrete domain may not be independent in the abstract, and those that are not independent in the concrete may become independent in the abstract.
Consider an analysis instance \( \mathcal{D} := (\bar{D}, \bar{\Xi}, \bar{F}, \bar{d}_0) \) that is an abstraction of \( \mathcal{D} := (D, \Xi, F, d_0) \) and a weak independence \( \diamond \subseteq F \times F \). The inherited relation \( \diamond \subseteq \bar{F} \times \bar{F} \) contains \((\bar{f}, \bar{f}')\) iff \((f, f')\) is in \( \diamond \).

**Example 2 (Abstraction breaks independence).** Consider a system \( M \) with the initial state \((x \mapsto 0, y \mapsto 0)\), and two threads \( t_1 : x = 2\), \( t_2 : y = 7 \). Let \( \mathcal{I} \) be the domain for interval analysis with elements \((i_x, i_y)\) being intervals for values of \( x \) and \( y \). The initial state is \( d_0 = (x \mapsto [0, 0], y \mapsto [0, 0]) \). Abstract transformers for \( t_1 \) and \( t_2 \) are shown below. These transformers are deliberately imprecise to highlight that sound transformers are not the most precise ones.

\[
\begin{align*}
\mathcal{T} & = \{(x \mapsto [2, 4], y \mapsto [i_x, i_y]), (x \mapsto [i_x, i_y], y \mapsto [2, 4])\} \\
\end{align*}
\]

The relation \( \diamond := \{(t_1, t_2), (t_2, t_1)\} \) is an independence on \( M \), and when lifted to \( \diamond_{\mathcal{C}_M} \) is a weak independence on \( \mathcal{C}_M \) (in fact, \( \diamond_{\mathcal{C}_M} \) is an independence). However, the relation \( \diamond_{\mathcal{I}} \) is not a weak independence because \( f_1 \) and \( f_2 \) do not commute at \( d_0 \), due to the imprecision introduced by abstraction. Consider the statements \( \text{assume}(x \neq 9) \) and \( \text{assume}(x < 10) \) applied to \( (x \mapsto [0, 10]) \) to see that even best transformers may not commute.

On the other hand, even when certain transitions are not independent, their transformers may become independent in an abstract domain.

**Example 3 (Abstraction creates independence).** Consider two threads \( t_1 : x = 2 \) and \( t_2 : x = 3 \), with abstract transformers \( f_1(i_x) = [2, 3] \) and \( f_2(i_x) = [2, 3] \). The transitions \( t_1 \) and \( t_2 \) do not commute, but owing to imprecision, \( R = \{(f_1, f_2), (f_2, f_1)\} \) is a weak independence on \( \mathcal{I} \).

5 Unfolding of an Abstract Domain with Independence

This section shows that unfoldings, which have primarily been used to analyze Petri nets, can be applied to abstract interpretation (represented by vertical lines in Fig. 2). An abstract unfolding is an event structure in which an event is recursively defined as the application of a transformer after a minimal set of interfering events; and a configuration represent equivalent sequences of transformer applications (events). Analogous to an invariant map in abstract interpreters and an abstract reachability tree in software model checkers, our abstract unfolding allows for constructing an over-approximation of the set of firable transitions in a program.

5.1 The Unfolding of a Domain

Our construction generates a PES \( \mathcal{E} := (E, <, \#, h) \). Recall that a configuration is a set of events that is closed with respect to < and that is conflict-free. Events in \( \mathcal{E} \) have the form \( e = (f, C) \), representing that the transformer \( f \) is applied after the transformers in configuration \( C \) are applied. The order in which transformers
must be applied is given by \( \prec \), while \( \# \) encodes transformer applications that cannot belong to the same configuration.

The unfolding \( \mathcal{U}_{\mathcal{D}, \bowtie} \) of an analysis instance \( \mathcal{D} := (D, \leq, F, d_0) \) with respect to a relation \( \bowtie \subseteq F \times F \) is defined inductively below. Recall that a configuration \( C \) generates a set of interleavings \( \text{inter}(C) \), which define the state of the configuration.

\[
\text{state}(C) := \bigcap_{\sigma \in \text{inter}(C)} \text{state}(\sigma)
\]

If \( \bowtie \) is a weak independence relation, all interleavings lead to the same state.

**Definition 3 (Unfolding).** The unfolding \( \mathcal{U}_{\mathcal{D}, \bowtie} \) of \( \mathcal{D} \) under the relation \( \bowtie \) is the structure returned by the following procedure:

1. Start with a PES \( \mathcal{E} := (E, \prec, \#, h) \) equal to \( (\emptyset, \emptyset, \emptyset, \emptyset) \).
2. Add a new event \( e := (f, C) \) to \( E \), where the configuration \( C \in \text{conf}(\mathcal{E}) \) and transformer \( f \) satisfy that \( f \) is enabled at state(\( C \)), and \( f \bowtie h(e) \) holds for every \( \prec \)-maximal event \( e \) in \( C \).
3. Update \( \prec \), \( \# \), and \( h \) as follows:
   - for every \( e' \in C \), set \( e' \prec e \);
   - for every \( e' \in E \setminus C \), if \( e \neq e' \) and \( f \bowtie h(e') \), then set \( e' \# e \);
   - set \( h(e) := f \).
4. Repeat steps 2 and 3 until no new event can be added to \( E \); return \( \mathcal{E} \).

Def. 3 defines the events, the causality, and conflict relations of \( \mathcal{U}_{\mathcal{D}, \bowtie} \) by means of a saturation procedure. Step 1 creates an empty PES. Step 2 defines a new event from a transformer \( f \) that can be applied after configuration \( C \). Step 3 defines \( e \) to be a causal successor of every dependent event in \( C \), and defines \( e \) to be in conflict with dependent events not in \( C \). Since conflicts are inherited in a PES, causal successors of \( e \) will also be in conflict with all \( e' \) satisfying \( e \# e' \). Events from \( E \setminus C \), which are unrelated to \( f \) in \( \bowtie \), will remain concurrent to \( e \).

**Proposition 2.** The structure \( \mathcal{U}_{\mathcal{D}, \diamond} \) generated by Def. 3 is a uniquely defined PES.

If \( \bowtie \) is a weak independence, every configuration of \( \mathcal{U}_{\mathcal{D}, \bowtie} \) represents sequences of transformer applications that produce the same element. If \( C \) is a configuration that is local, meaning it has a unique maximal event, or if \( C \) is generated by an independence, then \( \text{state}(C) \) will not be \( \bot \). Treating transformers as independent if they generate \( \bot \) enables greater reduction during analysis.

**Theorem 1 (Well-formedness of \( \mathcal{U}_{\mathcal{D}, \diamond} \)).** Let \( \diamond \) be a weak independence on \( \mathcal{D} \), let \( C \) be a configuration of \( \mathcal{U}_{\mathcal{D}, \diamond} \) and \( \sigma, \sigma' \) be interleavings of \( C \). Then:

1. \( \text{state}(\sigma) = \text{state}(\sigma') \);
2. \( \text{state}(\sigma) \neq \bot \) when \( \diamond \) is additionally an independence relation;
3. If \( C \) is a local configuration, then also \( \text{state}(\sigma) \neq \bot \).
Thm. 2 shows that the unfolding is adequate for analysis in the sense that every sequence of transformer applications leading to non-$\bot$ elements that could be generated during standard analysis with a domain will be contained in the unfolding. We emphasize that these sequences are only symbolically represented.

**Theorem 2 (Adequacy of $U_{D,\diamond}$).** For every weak independence relation $\diamond$ on $D$, and sequence of transformers $\sigma \in \text{runs}(D)$, there is a unique configuration $C$ of $U_{D,\diamond}$ such that $\sigma \in \text{inter}(C)$.

### 5.2 Abstract Unfoldings

The soundness theorems of abstract interpretation show when a fixed point computed in an abstract domain soundly approximates fixed points in a concrete domain. Our analysis constructs unfoldings instead of fixed points. The soundness of our analysis *does not* follow from fixed point soundness because the abstract unfolding we construct depends on the independence relation used. Though independence may not be preserved under lifting, as shown in Ex. 2, lifted relations can still be used to obtain sound results.

**Example 4.** In Ex. 2, the transformer composition $f_1 \circ f_2$ produces $(x \mapsto [2,4], y \mapsto [6,8])$, while $f_2 \circ f_2$ produces $(x \mapsto [2,4], y \mapsto [7,9])$. If $f_1$ and $f_2$ are considered independent, the state of the configuration $\{f_1, f_2\}$ is $\text{state}(f_1, f_2) \cap \text{state}(f_2, f_1)$, which is the abstract element $(x \mapsto [2,4], y \mapsto [7,7])$ and contains the final state $(x \mapsto 2, y \mapsto 7)$ reached in the concrete.

Thus, with sound abstractions of (weakly) independent, concrete transformers, can be treated as independent without compromising soundness of the analysis. The soundness theorem below asserts a correspondence between sequences of concrete transformer applications and the abstract unfolding. The concrete and abstract objects in Thm. 3 have different type: we are not relating a concrete unfolding with an abstract unfolding, but concrete transformer sequences with abstract configurations. Since $\text{state}(C)$ is defined as a meet of transformer sequences, the proof of Thm. 3 relies on the independence relation and has a different structure from standard proofs of fixed point soundness from transformer soundness.

**Theorem 3 (Soundness of the abstraction).** Let $\overline{D}$ be a sound abstraction of the analysis instance $D$, let $\diamond$ be a weak independence on $D$, and $\overline{\diamond}$ be the lifted relation on $\overline{D}$. For every sequence $\sigma \in \text{runs}(D)$ satisfying $\text{state}(\sigma) \neq \bot$, there is a unique configuration $C$ of $U_{\overline{D},\overline{\diamond}}$ such that $m(\sigma) \in \text{inter}(C)$.

Thm. 3 and Thm. 2 are fundamentally different. Thm. 2 shows that an unfolding parameterized by a weak independence relation is a data structure for representing all sequences of transformer applications that may be generated during analysis within a domain. Thm. 3 shows that every concrete sequence of transformers has a corresponding sequence of abstract transformers. However, the abstract unfolding in Thm. 3 may not represent all transformer applications of
the abstract domain in isolation. Formally, let \( \preceq \) be the order between unfolding prefixes and \( m(U_D, \cdot) \) is a lifting of an unfolding over a concrete domain to an abstract domain, we have \( m(U_D, \cdot) \preceq U_D, \cdot \). In fact, every configuration of \( U_D, \cdot \) will be isomorphic to a configuration in \( U_D, \cdot \).

6 Plugging Thread-Local Analysis

Unfoldings compactly represent concurrent executions using partial orders. However, they are a branching structure and one extension of the unfolding can multiply the number of branches, leading to a blow-up in the number of branches. Static analyses of sequential programs often avoid this explosion (at the expense of precision) by over-approximating (using join or widening) the abstract state at the CFG locations where two or more program paths converge. Adequately lifting this simple idea of merging at CFG locations from sequential to concurrent programs is a highly non-trivial problem [9].

In this section, we present a method that addresses this challenge and can mitigate the blow-up in the size of the unfolding caused by conflicts between events of the same thread. The key idea of our method is to merge abstract states generated by statements that work on local data of one thread, i.e., those whose impact over the memory/environment is invisible to other threads. Intuitively, the key insight is that we can merge certain configurations of the unfolding and still preserve its structural properties with respect to interference. The state of the resulting configuration will be a sound over-approximation of the states of the merged configurations at no loss of precision with respect to conflicts between events of different threads.

Our approach is to analyse \( M \) by constructing the unfolding of an abstract domain \( D \) and a weak independence relation \( \cdot \) using a thread-local procedure that over-approximates the effect of transformers altering local variables.

Assume that \( M \) has \( n \) threads. Let \( F_1, \ldots, F_n \) be the partitioning of the set of transformers \( F \) by the thread to which they belong. For \( f \in F_i \), we let \( p(f) := i \) denote the thread to which \( f \) belongs. We define, per thread, the (local) transformers which can be used to run the merging analysis. A transformer \( f \in F_i \) is local when, for all other threads \( j \neq i \) and all transformers \( f' \in F_j \) we have \( f \preceq f' \). A transformer is global if it is not local. We denote by \( F_{i}^{\text{loc}} \) and \( F_{i}^{\text{glo}} \), respectively, the set of local and global transformers in \( F_i \). In Fig. 1 (a), the global transformers would be those representing the actions to the variable \( g \). The remaining statements correspond to local transformers.

We formalize the thread-local analysis using the function \( \text{tla}: \mathbb{N} \times D \rightarrow D \), which plays the role of an off-the-shelf static analyzer for sequential thread code. A call to \( \text{tla}(i, d) \) will run a static analyzer on thread \( i \), restricted to \( F_{i}^{\text{loc}} \), starting from \( d \), and return its result which we assume is a sound fixed point. Formally, we assume that \( \text{tla}(i, d) \) returns \( d' \in D \), such that for every sequence \( f_1 \ldots f_n \in (F_{i}^{\text{loc}})^* \) we have \((f_n \circ \ldots \circ f_1)(d) \preceq d'\). This condition requires any implementation of \( \text{tla}(i, d) \) to return a sound approximation of the
Algorithm 1: Unfolding using thread-local fixpoint analysis

1. Procedure `unfold(D, ◇, n)`
2. Set $E := (E, <$, #, h) to $\emptyset$, $\emptyset$, $\emptyset$, $\emptyset$
3. forall $i, C \in \mathbb{N} \times \text{conf}(E)$
4. for $f$ enabled on $\text{tla}(i, \text{state}(C))$
5. $e := \text{mkevent}(f, C, ◇)$
6. if `iscutoff(e, E)` continue
7. Add $e$ to $E$
8. Extend $<$, #, and $h$ with $e$

10. Procedure `mkevent(f, C, ◇)`
11. do
12. while $C$ changed
13. return $(f, C)$

Theorem 4 (Soundness of the abstraction). Let $\diamondsuit$ be a weak independence on $\mathcal{D}$ and $\mathcal{P}_\mathcal{D}, \diamondsuit$ the PES computed by a call to `unfold(\mathcal{D}, \diamondsuit, n)` with cutoff checking disabled. Then, for any execution $\sigma \in \text{runs}(\mathcal{D})$ there is a unique configuration $C$ in $\mathcal{P}_\mathcal{D}, \diamondsuit$ such that $\hat{\sigma} \in \text{inter}(C)$.
6.1 Cutoff Events: Pruning the Unfolding

If we remove the conditional statement in line 6 of Alg. 1, the algorithm would only terminate if every run of $\mathcal{D}$ contains finitely many global transformers. This conditional check has two purposes: (1) preventing infinite executions from inserting infinitely many events into $\mathcal{E}$; (2) pruning branches of the unfolding that start with equivalent events. The procedure $\text{iscutoff}$ decides when an event is marked as a cutoff [17]. In such cases, no causal successor of the event will be explored. The implementation of $\text{iscutoff}$ cannot prune “too often”, as we want the computed PES to be a complete representation of behaviours of $\mathcal{D}$ (e.g., if a transformer is fireable, then some event in the PES will be labelled by it).

Formally, given $\mathcal{D}$, a PES $\mathcal{E}$ is $\mathcal{D}$-complete iff for every reachable element $d \in \text{reach}(\mathcal{D})$ there is a configuration $C$ of $\mathcal{E}$ such that $\text{state}(C) \supseteq d$. The key idea behind cutoff events is that, if event $e$ is marked as a cutoff, then for any configuration $C$ that includes $e$ it must be possible to find a configuration $C'$ without cutoff events such that $\text{state}(C) \supseteq \text{state}(C')$. This can be achieved by defining $\text{iscutoff}(e, \mathcal{E})$ to be the predicate: $\exists e' \in \mathcal{E}$ such that $\text{state}([e]) \supseteq \text{state}([e'])$ and $|[e']| < |[e]|$. When such $e'$ exists, including the event $e$ in $\mathcal{E}$ is unnecessary because any configuration $C$ such that $e \in C$ can be replayed in $\mathcal{E}$ by first executing $[e']$ and then (copies of) the events in $C \setminus [e]$.

We now would like to prove that Alg. 1 produces a $\mathcal{D}$-complete prefix when instantiated with the above definition of $\text{iscutoff}$. However, a subtle an unexpected interaction between the operators $\text{tla}$ and $\text{iscutoff}$ makes it possible to prove Thm. 5 only when $\text{tla}$ respects independence. Formally, we require $\text{tla}$ to satisfy the following property: for any $d \in \text{reach}(\mathcal{D})$ and any two global transformers $f \in F_{\text{glo}}^i$ and $f' \in F_{\text{glo}}^j$, if $f \mathcal{D} f'$ then

$$(f' \circ \text{tla}(j) \circ f \circ \text{tla}(i))(d) = (f \circ \text{tla}(i) \circ f' \circ \text{tla}(j))(d)$$

When $\text{tla}$ does not respect independence, it may over-approximate the global state (e.g. via joins and widening) in a way that breaks the independence of otherwise independent global transformers. This triggers the cutoff predicate to incorrectly prune necessary events.

**Theorem 5.** Let $\mathcal{D}$ be a weak independence in $\mathcal{D}$. Assume that $\text{tla}$ respects independence and that $\text{iscutoff}$ uses the procedure defined above. Then the PES $\mathcal{Q}_{\mathcal{D},\mathcal{D}}$ computed by Alg. 1 is $\mathcal{D}$-complete.

Note that Alg. 1 terminates if the lattice order $\supseteq$ is a well partial order (every infinite sequence contains an increasing pair). This includes, for instance, all finite domains. Furthermore, it is also possible to accelerate the termination of Alg. 1 using widenings in $\text{tla}$ to force cutoffs. Finally, notice that while we defined $\text{iscutoff}$ using McMillan’s size order [17], Thm. 5 also holds if $\text{iscutoff}$ is defined using adequate orders [6], known to yield smaller prefixes.
7 Experimental Evaluation

In this section we evaluate our approach based on abstract unfoldings. The goal of our experimental evaluation is to explore the following questions:

– Are abstract unfoldings practical? (I.e., is our approach able to yield efficient algorithms that can be used to prove properties of concurrent programs that require precise interference reasoning?)
– How does abstract unfoldings compare with competing approaches such as thread-modular analysis and symbolic partial order reduction?

Implementation. To address these questions, we have implemented a new program analyser based on abstract unfoldings baptized APoET, which implements an efficient variant of the exploration algorithm described in Alg. 1. The exploration strategy is based on PoET [23], an explicit-state model checker that implements a super-optimal partial order reduction method using unfoldings.

As described in Alg. 1, APoET is an analyser parameterized by a domain and a set of procedures: tla, iscutoff and mkevent. As a proof of concept, we have implemented an interval analysis and a basic parametric segmentation functor for arrays [5], which we instantiate with intervals and concrete integers values (to represent offsets). In this way, we are able to precisely handle arrays of threads and mutexes. APoET supports dynamic thread creation and uses CIL to inline functions calls. The analyser receives as input a concurrent C program that uses the POSIX thread library and parameters to control the widening level and the use of cutoffs. We implemented cutoffs according to the definition in Sec. 6.1 using an hash table that maps control locations to abstract values and the size of the local configuration of events.

APoET is parameterized by a domain functor of actions that is used to define independence and control the tla procedure. We have implemented an instance of the domain of actions for memory accesses and thread synchronisations. Transformers record the segments of the memory, intervals of addresses or sets of addresses, that have been read or written and synchronisation actions related to thread creation, join and mutex lock and unlock operations. This approach is used to compute a conditional independence relation as transformers can perform different actions depending on the state. The conditional independence relation is dynamically computed and is used in the procedure mkevent.

Finally, the tla procedure was implemented with a worklist fixpoint algorithm which uses the widening level given as input. In the interval analysis, we guarantee that tla respects independence using a predicate over the actions that identifies whether a transformer is local or global. This modularity allows us to define two modes of analysis for APoET: 1) consider global transformers those that yield actions related to thread synchronisation (i.e., thread creation/join and mutex lock/unlock) assuming that the program is data-race free and 2) consider an action global if it accesses the heap or is related to thread synchronisation which can be used to detect data races.
Benchmarks. We employ 6 benchmarks adapted from the SVCOMP'17 (yielding 9 rows in Table 1) and 4 parametric programs (yielding 15 rows) written by us: map-reduce DNA sequence analysis, producer-consumer, parallel sorting, and a thread pool. Most SVCOMP benchmarks are unsuitable for this comparison because either they are data-deterministic (and our approach fights data-explosion) or create unboundedly many threads, or use non-integer data types (e.g., structs, unsupported by our prototype). Thus we use new benchmarks exposing data non-determinism and complex synchronization patterns, where the correctness of assertions depend on the history of synchronizations. All new benchmarks are as complex as the most complex ones of the SVCOMP (excluding device drivers).

Each program was annotated with assertions enforcing, among others, properties related to thread synchronization (e.g., after spawning the worker threads, the master analyses results only after all workers finished), or invariants about data (e.g., each thread accesses a non-overlapping segment of the input array).

Tools compared. We compare APoet against the two approaches most closely related to ours: abstract interpreters (represented by the tool AstreeA) and partial-order reductions (PORs) handling data-nondeterminism (tools Impara and cbmc 5.6). AstreeA implements thread-modular abstract interpretation for concurrent programs [19], Impara combines POR with interpolation-based reasoning to cope with data non-determinism [25], and cbmc uses a symbolic encoding based on partial orders [2]. We sought to compare against symbolic execution tools but we could not find any available to download or capable of parsing the benchmarks.

Analysis. Table 1 presents the experimental results. When the program contained non-terminating executions (e.g., spinlocks), we used 5 loop unwindings for cbmc as well as cutoffs in APoet and a widening level of 15. For the family of fmax benchmarks, we were not able to run AstreeA on all instances, so we report approximated execution times and warnings based on the results provided by Antoine Miné on some of the instances. With respect to the size of the abstract unfolding, our experiments show that APoet is able to explore unfoldings up to 33K events and it was able to terminate on all benchmarks with an average execution time of 81 seconds. In comparison with AstreeA, APoet is far more precise: we obtain only 12 warnings (of which 5 are false positives) with APoet compared to 43 (32 false positives) with AstreeA. We observe a similar trend when comparing APoet with the mthread plugin for Frama-c [26] and confirm that the main reason for the source of imprecision in AstreeA is imprecise reasoning of thread interference. In the case of APoet, we obtain warnings in benchmarks that are buggy (Lazy*, Sigma* and Tpoli* family), as expected. Furthermore, APoet reports warnings in the ATGC benchmarks caused by imprecise reasoning of arrays combined with widening and also in the Cond benchmark as it contains non-relational assertions.

APoet is able to outperform Impara and CBMC on all benchmarks. We believe that these experiments demonstrate that effective symbolic reasoning with
Table 1. Experimental results. All experiments with APoet, Impara and cbmc were performed on an Intel Xeon CPU with 2.4 GHz and 4 GB memory with a timeout of 30 minutes; AstreeA was ran on HP ZBook with 2.7 GHz i7 processor and 32 GB memory. Columns are: P: nr. of threads; A: nr. of assertions; t(s): running time (TO - timeout); E: nr. of events in the unfolding; Ecut: nr. of cutoff events; W: nr. of warnings; V: verification result (S - safe; U - unsafe); N: nr. of node states; A* marks programs containing bugs. <2 reads as “less than 2”.

| Benchmark | Name     | P  | A  | t(s) | E  | Ecut | W  | t(s) | V  | t(s) | N  | V  | t(s) |
|-----------|----------|----|----|------|----|------|----|------|----|------|----|----|------|
|           | APoet    |     |    |      |    |      |    |      |    |      |    |    |      |
|           | AstreeA  |     |    |      |    |      |    |      |    |      |    |    |      |
|           | Impara   |     |    |      |    |      |    |      |    |      |    |    |      |
|           | cbmc 5.6 |     |    |      |    |      |    |      |    |      |    |    |      |
| ATGC(2)   |          | 3  | 7  | 0.37 | 47 | 0    | 2  | 1.07 | 2  | TO   | S  | 2.37 |
| ATGC(3)   |          | 4  | 7  | 5.78 | 432| 0    | 1  | 1.69 | 2  | TO   | S  | 6.6 |
| ATGC(4)   |          | 5  | 7  | 132.08 | 7195| 0  | 1  | 2.68 | 2  | TO   | S  | 20.22 |
| COND      |          | 5  | 2  | 0.55 | 982| 0    | 2  | 0.71 | 2  | TO   | S  | 34.39 |
| FMAX(2,3) |          | 2  | 8  | 0.70 | 100| 15   | 0  | 0.31 | 0  | TO   |    |    |
| FMAX(3,3) |          | 2  | 8  | 0.58 | 85 | 11   | 0  | <2  | 2  | TO   |    |    |
| FMAX(5,3) |          | 2  | 8  | 0.56 | 85 | 11   | 0  | 1.50 | 2  | TO   |    |    |
| FMAX(2,4) |          | 2  | 8  | 3.38 | 277| 43   | 0  | <2  | 2  | TO   |    |    |
| FMAX(2,6) |          | 2  | 8  | 45.82 | 1663| 321 | 0  | <2  | 2  | TO   |    |    |
| FMAX(4,6) |          | 2  | 8  | 61.32 | 2230| 207 | 0  | <2  | 2  | TO   |    |    |
| FMAX(2,7) |          | 2  | 8  | 146.19 | 3709| 769 | 0  | 1.87 | 2  | TO   |    |    |
| FMAX(4,7) |          | 2  | 8  | 265.23 | 6966| 671 | 0  | <2  | 2  | TO   |    |    |
| LAZY      |          | 4  | 2  | 0.01 | 72 | 0    | 0  | 0.50 | 2  | TO   | S  | 3.59 |
| LAZY*     |          | 4  | 2  | 0.01 | 72 | 0    | 0  | 0.49 | 2  | TO   | U  | 3.50 |
| MONAB1    |          | 5  | 1  | 0.27 | 982| 0    | 0  | 0.61 | 2  | TO   | S  | 38.51 |
| MONAB2    |          | 5  | 1  | 0.25 | 982| 0    | 0  | 0.58 | 2  | TO   | S  | 37.34 |
| RAND      |          | 5  | 1  | 0.40 | 657| 0    | 0  | 3.32 | 0  | TO   |    |    |
| SIGMA     |          | 5  | 5  | 2.62 | 7126| 0   | 0  | 0.43 | 0  | TO   | S  | 189.09 |
| SIGMA*    |          | 5  | 5  | 2.64 | 7126| 0   | 0  | 0.43 | 1  | TO   | U  | 141.38 |
| STF       |          | 3  | 2  | 0.01 | 69 | 0    | 0  | 0.66 | 2  | S  | 5.93 | 250 | S  | 2.12 |
| TPOOLL(2)*|          | 3  | 11 | 1.23 | 141| 7    | 1  | 1.97 | 2  | U  | 0.64 | 80 | -   |
| TPOOLL(3)*|          | 4  | 11 | 109.22 | 1712| 90  | 2  | 3.77 | 3  | U  | 0.72 | 113 | -  |
| TPOOLL(4)*|          | 5  | 11 | 1111.46 | 33018| 1762 | 2  | 8.06 | 3  | U  | 0.78 | 152 | -  |
| THPOOL    |          | 2  | 24 | 33.47 | 353| 103  | 0  | 1.44 | 5  | S  | TO   |    |    |

Partial orders is challenging as cbmc only terminates on 46% of the benchmarks and Impara only on 17%.

8 Related Work

In this section, we compare our approach with closely related program analysis techniques for (i) concurrent programs with (ii) a bounded number of threads and that (iii) handle data non-determinism.

The thread-modular approach in the style of rely-guarantee reasoning has been extensively studied in the past [19,18,4,16,10,14,20]. In [19], Miné proposes a flow-insensitive thread-modular analysis based on the interleaving semantics which forces the abstraction to cope with interleaving explosion. We address the interleaving explosion using the unfolding as an algorithmic approach to compute a flow and path-sensitive thread interference analysis. A recent approach [20] uses relational domains and trace partitioning to recover precision in thread modular analysis but requires manual annotations to guide the partitioning and does not scale with the number of global variables. The analysis in [8] is not as precise
as our approach (confirmed by experiments with Duet on a simpler version of our benchmarks) as it employs an abstraction for unbounded parallelism. The work in [14] presents a thread modular analysis that uses a lightweight interference analysis to achieve an higher level of flow sensitivity similar to [8]. The interference analysis of [14] uses a constraint system to discard unfeasible pairs of read-write actions which is static and less precise than our approach based on independence. The approach is also flow-insensitive in the presence of loops with global read operations.

The interprocedural analysis for recursive concurrent programs of [13] does not address the interleaving explosion. A related approach that uses unfoldings is the causality-based bitvector dataflow analysis proposed in [9]. There, unfoldings are used as a method to obtain dataflow information while in our approach they are the fundamental datastructure to drive the analysis. Thus we can apply thread-local fixpoint analysis while their unfolding suffers from path explosion due to local branching. Furthermore, we can build unfoldings for general domains even with invalid independence relations while their approach is restricted to the independence encoded in the syntax of a Petri net and bitvector domains.

Compared to dynamic analysis of concurrent programs [1,7,15,12], our approach builds on top of a (super-)optimal partial-order reduction [23] and is able to overcome a high degree of path explosion unrelated to thread interference.

9 Conclusion

We introduced a new algorithm for static analysis of concurrent programs based on the combination of abstract interpretation and unfoldings. Our algorithm explores an abstract unfolding using a new notion of independence to avoid redundant transformer application in an optimal POR strategy, thread-local fixed points to reduce the size of the unfolding, and a novel cutoff criterion based on subsumption to guarantee termination of the analysis.

Our experiments show that APoET generates about 10x fewer false positives than a mature thread modular abstract interpreter and is able to terminate on a large set of benchmarks as opposed to solver-based tools that have the same precision. We observed that the major reasons for the success of APoET are: (1) the use of cutoffs to cope with and prune cyclic explorations caused by spinlocks and (2) tla mitigates path explosion in the threads. Our analyser is able to scale with the number of threads as long as the interference between threads does not increase. As future work, we plan to experimentally evaluate the application of local widenings to force cutoffs to increase the scalability of our approach.

Acknowledgments. The authors would like to thank Antoine Miné for the invaluable help with AstreeA and the anonymous reviewers for their helpful feedback.
References

1. Parosh Abdulla, Stavros Aronis, Bengt Jonsson, and Konstantinos Sagonas. Optimal dynamic partial order reduction. In Principles of Programming Languages (POPL), pages 373–384. ACM, 2014.
2. Jade Alglave, Daniel Kroening, and Michael Tautschnig. Partial orders for efficient bounded model checking of concurrent software. In Computer Aided Verification (CAV), volume 8044 of LNCS, pages 141–157. Springer, 2013.
3. Blai Bonet, Patrik Haslum, Victor Khomenko, Sylvie Thiébaux, and Walter Vogler. Recent advances in unfolding technique. Theoretical Comp. Science, 551:84–101, September 2014.
4. Jean-Loup Carre and Charles Hymans. From Single-thread to Multithreaded: An Efficient Static Analysis Algorithm. arXiv:0910.5833 [cs], October 2009.
5. Patrick Cousot, Radhia Cousot, and Francesco Logozzo. A parametric segmentation functor for fully automatic and scalable array content analysis. In Principles of Programming Languages (POPL), pages 105–118. ACM, 2011.
6. Javier Esparza, Stefan Römer, and Walter Vogler. An improvement of McMillan’s unfolding algorithm. Formal Methods in System Design, 20:285–310, 2002.
7. Azadeh Farzan, Andreas Holzer, Niloofar Razavi, and Helmut Veith. Con2Colic testing. In Foundations of Software Engineering (FSE), pages 37–47. ACM, 2013.
8. Azadeh Farzan and Zachary Kincaid. Verification of parameterized concurrent programs by modular reasoning about data and control. In Principles of Programming Languages (POPL), pages 297–308. ACM, 2012.
9. Azadeh Farzan and P. Madhusudan. Causal dataflow analysis for concurrent programs. In Tools and Algorithms for the Construction and Analysis of Systems (TACAS), volume 4424 of LNCS, pages 102–116. Springer, 2007.
10. Cormac Flanagan and Shaz Qadeer. Thread-modular model checking. In Model Checking Software, volume 2648 of LNCS, pages 213–224. Springer, May 2003.
11. Patrice Godefroid. Partial-Order Methods for the Verification of Concurrent Systems – An Approach to the State-Explosion Problem, volume 1032 of LNCS. Springer, 1996.
12. Henning Günther, Alfons Laarman, Ana Sokolova, and Georg Weissenbacher. Dynamic reductions for model checking concurrent software. In Verification, Model Checking, and Abstract Interpretation (VMCAI), volume 10145 of LNCS, pages 246–265. Springer, 2017.
13. Bertrand Jeannet. Relational interprocedural verification of concurrent programs. Software & Systems Modeling, 12(2):285–306, March 2012.
14. Markus Kusano and Chao Wang. Flow-sensitive composition of thread-modular abstract interpretation. In Foundations of Software Engineering (FSE), pages 799–809. ACM, 2016.
15. Kari Kähkönen, Olli Saarikivi, and Keijo Heljanko. Unfolding based automated testing of multithreaded programs. Automated Software Engineering, 22:1–41, May 2014.
16. Alexander Malkis, Andreas Podelski, and Andrey Rybalchenko. Precise thread-modular verification. In Static Analysis (SAS), volume 4634 of LNCS, pages 218–232. Springer, August 2007.
17. Kenneth L. McMillan. Using unfoldings to avoid the state explosion problem in the verification of asynchronous circuits. In Computer Aided Verification (CAV), volume 663 of LNCS, pages 164–177. Springer, 1993.
18. Antoine Miné. Static analysis of run-time errors in embedded real-time parallel C programs. *Logical Methods in Computer Science*, 8(1), March 2012.

19. Antoine Miné. Relational thread-modular static value analysis by abstract interpretation. In *Verification, Model Checking, and Abstract Interpretation (VMCAI)*, volume 8318 of *LNCS*, pages 39–58. Springer, 2014.

20. Raphaël Monat and Antoine Miné. Precise thread-modular abstract interpretation of concurrent programs using relational interference abstractions. In *Verification, Model Checking, and Abstract Interpretation (VMCAI)*, volume 10145 of *LNCS*, pages 386–404. Springer, 2017.

21. Mogens Nielsen, Gordon Plotkin, and Glynn Winskel. Petri nets, event structures and domains, part I. *Theoretical Computer Science*, 13(1):85–108, 1981.

22. Doron Peled. All from one, one for all: on model checking using representatives. In *Computer Aided Verification (CAV)*, volume 697 of *LNCS*, pages 409–423. Springer, 1993.

23. César Rodríguez, Marcelo Sousa, Subodh Sharma, and Daniel Kroening. Unfolding-based partial order reduction. In *Concurrency Theory (CONCUR)*, volume 42 of *LIPIcs*, pages 456–469. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2015.

24. César Rodríguez, Marcelo Sousa, Subodh Sharma, and Daniel Kroening. Unfolding-based partial order reduction. *CoRR*, abs/1507.00980, 2015.

25. Björn Wachter, Daniel Kroening, and Joël Ouaknine. Verifying multi-threaded software with Impact. In *Formal Methods in Computer-Aided Design (FMCAD)*, pages 210–217, 2013.

26. Boris Yakobowski and Richard Bonichon. Frama-C’s Mthread plug-in. Report, Software Reliability Laboratory, 2012.
A Proofs: Abstract Partial-Order Semantics

A.1 Proofs for Sec. 4: Independence of Transformers

Proposition 1 (Lifted independence). If $\triangledown$ is an independence relation on $M$, the lifted relation $\triangledown c_M$ is a weak independence on the collecting semantics $C_M$.

Proof. Let $f_1, f_2$ be transformers of $C_M$ such that $f_1 \triangledown f_2$. Let $a_1 := m^{-1}_0(f_1)$ and $a_2 := m^{-1}_0(f_2)$ be the corresponding program statements. We know that $a_1 \triangledown a_2$. Let $d := \{s_1, \ldots, s_n\} \in \text{reach}(C_M)$ be an element of $C_M$. By definition of $C_M$ we know that $d$ contains only reachable states of $M$. Furthermore, we know that $a_1$ and $a_2$ commute on all of them. Let $d_1 := f_1 \circ f_2(d)$ and $d_2 := f_2 \circ f_1(d)$ be the abstract elements obtained after executing the abstract transformers in both orders. We need to show that $d_1 = d_2$. W.l.o.g., we show that $d_1 \subseteq d_2$ (the opposite direction holds by symmetry). Let $s' \in d_1$ be an state in $d_1$. Then there is some $s \in d$ such that $s \xrightarrow{a_2;a_1} s'$. Since $a_1$ and $a_2$ are independent under $\triangledown$, then also $s \xrightarrow{a_1;a_2} s'$ (by commutativity $a_1$ is enabled at $s$ because it was at the state reached after executing $a_2$ and both orderings reach $s'$). By definition of $f_1, f_2$ we get that $s' \in d_2$. \qed

A.2 Results in Section 5: Unfolding Semantics

This section contains the proofs of the formal statements made in Sec. 5. All notations fixed in Sec. 5 are assumed here. We will need to make some new definitions.

We recall now the definition of a PES. An $X$-labelled prime event structure [21] (X-LPES, or PES in short) is a tuple $\mathcal{E} := (E, <, \#, h)$ where $< \subseteq E \times E$ is a strict partial order, $\# \subseteq E \times E$ is a symmetric, irreflexive relation, and $h : E \rightarrow X$ is a labelling function satisfying:

- for all $e \in E$, $\{e' \in E : e' < e\}$ is finite, and
- for all $e, e', e'' \in E$, if $e \# e'$ and $e' < e''$, then $e \neq e''$.

Event structures are naturally (partially) ordered by a prefix relation $\preceq$. Given two PESs $\mathcal{E} := (E, <, \#, h)$ and $\mathcal{E}' := (E', <', \#', h')$, we say that $\mathcal{E}$ is a prefix of $\mathcal{E}'$, written $\mathcal{E} \preceq \mathcal{E}'$, when $E \subseteq E'$, $<$ and $\#$ are the projections of $<$ and $\#' \text{ to } E$, and $E \ni \{e' \in E' : e' < e \wedge e \in E\}$. Moreover, the set of prefixes of a given PES $\mathcal{E}$ equipped with $\preceq$ is a complete lattice.

Def. 3 defines $\mathcal{U}_{D, \triangledown}$ using an iterative procedure that constructs, possibly, an infinite object. We call unfolding prefix the structure $\mathcal{E}$ that the algorithm constructs after countably many steps, possibly before reaching fixpoint.

We now prove that the set of unfolding prefixes equipped with relation $\preceq$ forms a complete join-semilattice, where the operator $\text{union}(\cdot)$, defined below, is the join operator. In turn, this implies the existence of a unique $\preceq$-maximal element, that will be found by Def. 3 when it reaches fixpoint.
We first define the operator $\text{union}(\cdot)$. Let

$$P := \{(E_1, <_1, \#_1, h_1), (E_2, <_2, \#_2, h_2), \ldots\}$$

be a countable set of finite unfolding prefixes of $\mathcal{D}$. The $\text{union}$ of all of them is the PES $\text{union}(P) := (E, <, \#, h)$, where

$$E := \bigcup_{1 \leq i} E_i \quad < := \bigcup_{1 \leq i} <_i \quad h := \bigcup_{1 \leq i} h_i,$$

and $\#$ is the $\varepsilon$-minimal relation on $E \times E$ that satisfies (2) and such that $e \not\# e'$ holds for every two events $e, e' \in E$ iff

$$e \notin [e'] \text{ and } e' \notin [e] \text{ and } \neg (h(e) \bowtie h(e')).$$  \hspace{1cm} (3)

Since every element of $P$ is a PES, clearly $\text{union}(P)$ is also a PES. (1) and (2) are trivially satisfied. Notice that all events in $E_1, E_2, E_3, \ldots$ are pairs of the form $(t, H)$, and the union of two or more $E_i$’s will “merge” many equal events.

**Lemma 1.** For every set $P$ of unfolding prefixes, $\text{union}(P)$ is the least-upper bound of $P$ with respect to the order $\sqsubseteq$.

**Proof.** Let $F := \{\mathcal{P}_i\}$ with $i \in \mathbb{N}$ be a countable set of finite or infinite prefixes, where $\mathcal{P}_i := (E_i, <_i, \#_i, h_i)$. In fact, we can assume that all $\mathcal{P}_i$ are finite. Either $\mathcal{P}_i$ is finite or it has been constructed by the third rule in Def. 3. In the second case, it is the $\text{union}(\cdot)$ of a countably infinite set of finite prefixes. In both cases, $\mathcal{P}_i$ accounts for countably many finite prefixes.

Now, for any countable set of prefixes $X := \{\mathcal{Q}_i\}, i \in \mathbb{N}$, it is immediate to show that

$$\text{union}(\mathcal{P}, \text{union}(X)) = \text{union}(\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \ldots).$$

Finally, since the union of countably many countable sets is a countable set, we can assume w.l.o.g. that $F$ is a countable set of finite prefixes (this assumes the Axiom of choice).

Let $\mathcal{P} := \text{union}(F)$ be their union, where $\mathcal{P} := (E, <, \#, h)$. We need to show that

- (upper bound) $\mathcal{P}_i \not\subseteq \mathcal{P}$;
- (least element) for any unfolding prefix $\mathcal{P}'$ such that $\mathcal{P}_j \not\subseteq \mathcal{P}'$ holds for all $1 \leq j$, we have that $\mathcal{P} \not\subseteq \mathcal{P}'$.

We start showing that $\mathcal{P}$ is an upper bound. Let $\mathcal{P}_i \in F$ be an arbitrary unfolding prefix. We show that $\mathcal{P}_i \not\subseteq \mathcal{P}$:

- Trivially $E_i \not\subseteq E$.
- $<_i \not\subseteq \cap (E_i \times E_i)$. Trivial.
\[ \prec_i \geq \cap (E_i \times E_i) \]. Let \( e, e' \in E \) be two events of \( \mathcal{P} \). Assume that \( e < e' \) and that both \( e \) and \( e' \) are in \( E_i \). Since \( e' \in E_i \), there is some \( j \in \mathbb{N} \) such that \( e <_j e' \), and both \( e \) and \( e' \) are in \( E_j \). Assume that \( e' := (t, H) \). Since \( \mathcal{P}_j \) is a finite prefix constructed by Def. 3, then necessarily \( e \in H \). As a result, Def. 3 must have found that \( e' \) was in \( H \) when adding \( e \) to the prefix that eventually became \( \mathcal{P}_i \), and consequently \( e' \prec_i e \).

\[ \#_i \leq \# \cap (E_i \times E_i) \]. Trivial.

\[ \#_i \geq \# \cap (E_i \times E_i) \]. Assume that \( e \neq e' \) and that \( e, e' \in E_i \). We need to prove that \( e \neq e' \). Assume w.l.o.g. that \( e' \) was added to \( \mathcal{P}_i \) by Def. 3 after \( e \). If \( e \) and \( e' \) satisfy (3), then trivially \( e \neq e' \). If not, then assume w.l.o.g. that there exists some \( e'' < e' \) such that \( e \neq e'' \), and such that \( e \) and \( e'' \) satisfy (3). Then \( e \neq e'' \) and, since \( \mathcal{P}_i \) is a LES then we have \( e \neq e' \).

\[ h_i = h \cap (E_i \times E_i) \]. Trivial.

We now focus on proving that \( \mathcal{P} \) is the least element among the upper bounds of \( F \). Let \( \mathcal{P}' := \{ E', <', \#', h' \} \) be an upper bound of all elements of \( F \). We show that \( \mathcal{P} \preceq \mathcal{P}' \).

\[ \text{Since } E \text{ is the union of all } E_i \text{ and all } E_i \text{ are by hypothesis in } E', \text{ then necessarily } E \subseteq E'. \]

\[ \prec \leq \prec' \cap (E \times E) \]. Assume that \( e < e' \). By definition \( e \) and \( e' \) are in \( E \), so we only need to show that \( e <' e' \). We know that there is some \( i \in \mathbb{N} \) such that \( e <_i e' \). We also know that \( \mathcal{P}_i \preceq \mathcal{P}' \), which implies that \( e <' e' \).

\[ \preceq \leq \cap (E \times E) \]. Assume that \( e < e' \) and that \( e, e' \in E \). We know that there is some \( i \in \mathbb{N} \) such that \( e, e' \in E_i \). We also know that \( \mathcal{P}_i \preceq \mathcal{P}' \), which implies that \( \prec_i = <' \cap (E_i \times E_i) \). This means that \( e <_i e' \), and so \( e < e' \).

\[ h = h' \cap (E \times E) \]. Trivial.

\[ \# \leq \#' \cap (E \times E) \]. Assume that \( e \neq e' \). Then \( e \) and \( e' \) are in \( E \). Two things are possible. Either \( e, e' \) satisfy (3) or, w.l.o.g., there exists some \( e'' < e' \) such that \( e \) and \( e'' \) satisfy (3). In the former case, using items above, it is trivial to show that \( \neg(e <' e') \), that \( \neg(e' <' e) \), and that \( h'(e) \bowtie h'(e') \). This means that \( e \neq e' \). In the latter case its the same.

\[ \# \geq \#' \cap (E \times E) \]. Trivial.

\[ \square \]

**Proposition 2.** The structure \( \mathcal{U}_{\mathcal{D}, \Diamond} \) generated by Def. 3 is a uniquely defined pes.

**Proof.** It is trivial to show that \( \mathcal{U}_{\mathcal{D}, \Diamond} \) satisfies (1) and (2). When the procedure in Def. 3 reaches fixpoint it obviously computes the least-upper bound of the set of unfolding prefixes. That is a unique element in the lattice. \[ \square \]

**Theorem 1 (Well-formedness of \( \mathcal{U}_{\mathcal{D}, \Diamond} \)).** Let \( \Diamond \) be a weak independence on \( \mathcal{D} \), let \( C \) be a configuration of \( \mathcal{U}_{\mathcal{D}, \Diamond} \) and \( \sigma, \sigma' \) be interleavings of \( C \). Then:

1. \( \text{state}(\sigma) = \text{state}(\sigma') \);
2. \( \text{state}(\sigma) \neq \bot \) when \( \Diamond \) is additionally an independence relation;
3. If \( C \) is a local configuration, then also \( \text{state}(\sigma) \neq \bot \).
Applying the same argument, this time applied to $\tilde{\sigma}$ argument, we have that

\[ \tilde{\sigma} = \tilde{\sigma}.h(e), f_1, \ldots, f_l \]

and assume that $\tilde{\sigma}$ is the local configuration of event $e := (f, H)$. Since there is only one maximal event in $C$ (event $e$), then necessarily $\sigma$ has the form $\sigma := \tilde{\sigma}.h(e)$, for $\tilde{\sigma} \in \text{inter}(H)$. From item (1) we know that all interleavings of $H$ reach the same dataflow fact, and since $\text{state}(H)$ is defined as the meet of all of them, then necessarily $\text{state}(H) = \text{state}(\tilde{\sigma})$. We also know, from Def. 3 that transformer $f$ is enabled at $\text{state}(H)$. This means that $\text{state}(\sigma) \neq \bot$.

Finally we prove (1). The proof is by induction on the size $|C|$ of the configuration.

**Base case.** $|C| = 0$ and so, $C = \emptyset$. The set of interleavings of $C$ contains zero linearizations and the result trivially holds.

**Inductive Step.** Assume that the result holds for configuration of size $k - 1$ and assume that $|C| = k$. Let $e \in C$ be any $<$-maximal event in $C$, and assume that $\sigma$ and $\sigma'$ have the form

\[ \sigma := \tilde{\sigma}.h(e), f_1, \ldots, f_l \]
\[ \sigma' := \tilde{\sigma}'\cdot h(e), g_1, \ldots, g_m. \]

Recall that the interleavings of a configuration are the topological orderings of events w.r.t. causality. As a result transformer $h(e)$ is independent in $\Diamond$ to the transformers that label all events $f_1, \ldots, f_l$ and $g_1, \ldots, g_m$.

Now consider the dataflow fact $d := \text{state}(\tilde{\sigma})$. If $d = \bot$, then clearly $\tilde{\sigma}, h(e), f_1$ reaches $\bot$ as well. If $d \neq \bot$, then $\tilde{\sigma}$ is a run of $D$ and $d \in \text{reach}(D)$. Now, by construction of $\Diamond$, we have that $\tilde{\sigma}, f_1, h(e)$ is also a run of $D$ and

\[ \text{state}(\tilde{\sigma}, h(e), f_1) = \text{state}(\tilde{\sigma}, f_1, h(e)). \]

Applying the same argument $l - 1$ times more we prove that $\tilde{\sigma}, f_1, \ldots, f_l, h(e)$ is a run of $D$ and that

\[ \text{state}(\sigma) = \text{state}(\tilde{\sigma}, f_1, \ldots, f_l, h(e)). \]

That is, we have “pushed back” the occurrence of transition $h(e)$ in the interleaving without changing the state ($\bot$ or not) reached by the interleaving. Using the same argument, this time applied to $\sigma'$ instead of $\sigma$, we can also show that

\[ \text{state}(\sigma') = \text{state}(\tilde{\sigma}', g_1, \ldots, g_m, h(e)). \]

Now, we remark that both $\tilde{\sigma} f_1, \ldots, f_l$ and $\tilde{\sigma}' g_1, \ldots, g_m$ are interleavings of $C \setminus \{e\}$, a configuration of size $k - 1$. By induction hypothesis both interleavings thus satisfy that

\[ \text{state}(\tilde{\sigma}, f_1, \ldots, f_l) = \text{state}(\tilde{\sigma}', g_1, \ldots, g_m) \]

It then follows that

\[ \text{state}(\sigma) = \text{state}(\tilde{\sigma}, f_1, \ldots, f_l, h(e)) \]
\[ = \text{state}(\tilde{\sigma}', g_1, \ldots, g_m, h(e)) \]
\[ = \text{state}(\sigma') \]
Theorem 2 (Adequacy of $U_{D,\Diamond}$). For every weak independence relation $\Diamond$ on $D$, and sequence of transformers $\sigma \in \text{runs}(D)$, there is a unique configuration $C$ of $U_{D,\Diamond}$ such that $\sigma \in \text{inter}(C)$.

Proof. Assume that $\sigma$ fires at least one transition. The proof is by induction on the length $|\sigma|$ of the run.

Base Case. If $\sigma$ fires one transformer $f$, then $f$ is enabled at $d_0$, the initial dataflow fact of $D$. Then $\{\dagger\}$ is a history for $f$, as necessarily $\text{state}(\{\dagger\})$ enables $f$. This means that $\epsilon := (f,\{\dagger\})$ is an event of $U_{D,\Diamond}$, and clearly $\sigma \in \text{inter}(\{\dagger\}).$ It is easy to see that no other event $\epsilon'$ different than $\epsilon$ but such that $h(\epsilon) = h(\epsilon')$ can exist in $U_{M,\Diamond}$ and satisfy that the history $[\epsilon']$ of $\epsilon'$ equals the singleton $\{\dagger\}$. The representative configuration for $\sigma$ therefore exists and is unique.

Inductive Step. Assume that $\sigma := \sigma'f$. By the induction hypothesis, we assume that there exist a unique configuration $C'$ such that $\sigma' \in \text{inter}(C')$. By Thm. 1, all sequences in $\text{inter}(C')$ reach the same dataflow fact $d$. Furthermore, $\sigma'$ is one of them, and it is also a run in $\text{runs}(D)$. This implies that $d \neq \bot$. It also implies that $f$ is enabled at $d$.

If all $\prec$-maximal events $\epsilon' \in C'$ satisfy that $h(\epsilon') \not\bowtie f$, then $C'$ is a history for transformer $f$, and $U_{D,\Diamond}$ contains an event $\epsilon := (f,C')$. Let $C := C' \cup \{\epsilon\} = [\epsilon]$ be the configuration that contains $C'$ and $\epsilon$. Clearly $\sigma \in \text{inter}(C)$. Below we show that such $C$ is unique.

Alternatively, $C''$ could have one or more maximal events $\epsilon'$ such that $h(\epsilon') \bowtie f$. We now find a history for $f$ inside of $C'$, as follows. Let $C'' := C' \setminus \{\epsilon'\}$, for any such $\epsilon'$, and let $d'' := \text{state}(C'')$. Since $f$ is enabled at $d$ and $f \bowtie h(\epsilon')$, then necessary $f$ is also enabled at $d''$, as otherwise $f \circ h(\epsilon')(d'') = h(\epsilon) \circ f(d'')$ would be $\bot$, and $f$ would not be enabled at $d$. If all $\prec$-maximal events of $C''$ are dependent with $f$, then $C''$ is a history for $f$, and we set $\epsilon := (f,C'')$. If not, we can apply again the argument a finite number of times (as $C'$ is finite) until we find a history $H$ for $f$ inside of $C'$ ($\{\dagger\}$ is always a valid history). We set $\epsilon := (f,H)$ and $C := C' \cup \{\epsilon\}$. As before, clearly $\sigma \in \text{inter}(C)$.

In both cases we found a configuration $C \supset C'$ such that $\sigma \in \text{inter}(C)$. We now argue that such $C$ is unique. By induction hypothesis we know that $C'$ is the only configuration that represents $\sigma'$. If there was another $C''$ that represents $\sigma$ and such that $C' \not\subset C''$, then removing the maximal event that represents $f$ in $\sigma$ would yield a second representative for $\sigma'$. This implies that any such $C''$ must include $C'$. Showing uniqueness now reduces to showing that $\epsilon$ is the only event in $U_{D,\Diamond}$ such that $C' \cup \{\epsilon\}$ represents $\sigma$.

By contradiction, assume that $U_{D,\Diamond}$ contains another event $\epsilon' := (f,H')$ such that $C' \cup \{\epsilon'\}$ represents $\sigma$. Assume that $\epsilon$ has the form $(f,H)$. Since $\epsilon \neq \epsilon'$ we know that $H \neq H'$. By construction we know that $H \subseteq C'$. If $H' \not\subseteq C'$, then $C' \cup \{\epsilon'\}$ would not be a configuration (not causally closed). So also $H' \subseteq C'$. Now, since $H \neq H'$, w.l.o.g. at least one of the maximal events in $H$ is not in $H'$. Furthermore, that event is in $C'$. By Def. 3 this means that $\epsilon'$ is in conflict with that event, and so $C' \cup \{\epsilon'\}$ is not a configuration. This is a contradiction. □
A.3 Results in Sec. 5.2: Abstract Unfoldings

Theorem 3 (Soundness of the abstraction). Let $\mathcal{D}$ be a sound abstraction of the analysis instance $\mathcal{D}$, let $\diamondsuit$ be a weak independence on $\mathcal{D}$, and $\bar{\diamondsuit}$ be the lifted relation on $\bar{\mathcal{D}}$. For every sequence $\sigma \in \text{runs}(\mathcal{D})$ satisfying $\text{state}(\sigma) \neq 1$, there is a unique configuration $\bar{C}$ of $\mathcal{U}_{\mathcal{D},\bar{\diamondsuit}}$ such that $m(\sigma) \in \text{inter}(\bar{C})$.

Proof. For the same reasons as in Thm. 2, the statement of the theorem restricts $\sigma$ to have at least one transformer. The proof is by induction on the length $|\sigma|$ of the run.

Base Case. Run $\sigma$ fires one transformer $f$ which is enabled at $d_0$. Let $\bar{f} := m(f)$ be the associated abstract transformer. Then $\bar{f}$ is enabled at $\bar{d}_0$, and $\{\dagger\}$ is a history for $\bar{f}$. As a result $\bar{e} := (\bar{f}, \{\dagger\})$ is an event of $\mathcal{U}_{\mathcal{D},\bar{\diamondsuit}}$, and clearly $m(\sigma) = \bar{f} \in \text{inter}(\{\dagger, \bar{e}\})$. It is immediate to show that $\{\dagger, \bar{e}\}$ is the only configuration that represents $m(\sigma)$.

Inductive Step. Assume that $\sigma := \sigma' f$. By the induction hypothesis, we assume that there exist a unique configuration $\bar{C}'$ in $\mathcal{U}_{\mathcal{D},\bar{\diamondsuit}}$ such that $m(\sigma') \in \text{inter}(\bar{C}')$.

We fix some notation. Let $\bar{d}' := \text{state}(\bar{C}')$ be the abstract state reached by $\bar{C}'$. Let $d' := \text{state}(\sigma')$ be the concrete state reached by $\sigma'$. Let $\bar{f} := m(f)$ be the abstract counterpart of $f$.

Now we show that $d' \subseteq \gamma(\bar{d}')$. Recall that $\bar{d}'$ is defined as the meet of the state reached by all interleavings of $\bar{C}'$. Therefore, $\bar{d}'$ does not satisfy that $\text{state}(m(\sigma')) \subseteq \bar{d}'$, which would probably be the easiest strategy to prove our goal. We follow a different reasoning. Since $m(\sigma') \in \text{inter}(\bar{C}')$, by we get that $m^{-1}(\text{inter}(\bar{C}'))$ is a set of runs of the concrete domain $\mathcal{D}$. Furthermore, all those runs reach the same concrete dataflow fact $d'$ as $\sigma'$. Then all runs in $\text{inter}(\bar{C}')$ reach abstract dataflow facts that soundly approximate $d'$. What is more, in a Galois connection, the concretization map $\gamma$ preserves abstract meets. Formally, for any two abstract facts $\bar{d}_1, \bar{d}_2$, we have

$$\gamma(\bar{d}_1) \cap \gamma(\bar{d}_2) \subseteq \gamma(\bar{d}_1 \cap \bar{d}_2).$$

We thus can make the following development:

$$\gamma(\bar{d}') = \gamma(\bigcap_{\bar{\sigma} \in \text{inter}(\bar{C}')} \text{state}(\bar{\sigma}))$$

$$\supseteq \bigcap_{\bar{\sigma} \in \text{inter}(\bar{C}')} \gamma(\text{state}(\bar{\sigma}))$$

$$\supseteq \bigcap_{\bar{\sigma} \in \text{inter}(\bar{C}')} d'$$

$$= d'$$

This shows that $d' \subseteq \gamma(\bar{d}')$. It also shows that $\bar{f}$ is enabled at $\bar{d}' = \text{state}(\bar{C}')$, since $\bar{f}$ is a sound approximation of $f$.

If all maximal events $e$ of $\bar{C}'$ are such that $h(e) \bar{\diamondsuit} \bar{f}$, then $\bar{C}'$ is a history for $\bar{f}$, and $\bar{e} := (\bar{f}, \bar{C}')$ is an event of $\mathcal{U}_{\mathcal{D},\bar{\diamondsuit}}$. Let $\bar{C} := \bar{C}' \cup \{\bar{e}\} = [\bar{e}]$ be the configuration that contains $\bar{C}'$ and $\bar{e}$. Clearly $m(\sigma) \in \text{inter}(\bar{C})$. Below we show that such $C$ is unique.
Alternatively, \( \hat{C}' \) could have one or more maximal events independent with \( \hat{f} \). Let \( e' \) be any <-maximal event in \( \hat{C}' \) such that \( h(e') \hat{\Diamond} f \). In the sequel we find a history for \( \hat{f} \) inside of \( C' \). Let \( \hat{C}'' := \hat{C}' \setminus \{ e' \} \), and let \( \hat{d}'' := \text{state}(\hat{C}'') \).

We show that \( \hat{f} \) is enabled at \( \hat{d}'' \). Since all interleavings of \( \hat{C}' \) correspond to runs of \( D \), necessarily all interleavings of \( \hat{C}'' \) are also executions of \( D \); and all of them reach the same dataflow fact, say \( d'' \). Using the same reasoning as above, we can show that \( d'' \in \gamma(d''') \). Now, showing that \( \hat{f} \) is enabled at \( d'' \) reduces to showing that \( f \) is enabled at \( d'' \). This, in turn, is a consequence of the fact that \( m^{-1}(h(e'))(d'') = d' \) and \( f(d') \not\perp \) and the fact that \( m^{-1}(h(\hat{e})) \) and \( f \) are independent (we skip details).

This shows that \( \hat{f} \) is enabled at \( \text{state}(\hat{C}'') \). If all maximal events of \( \hat{C}'' \) are dependent with \( \hat{f} \), then \( \hat{C}'' \) is a history for \( \hat{f} \). If not, we can apply again the argument a finite number of times (as \( \hat{C}'' \) is finite) until we find a history \( H \) for \( \hat{f} \) inside of \( \hat{C}'' \) \( \{ \dagger \} \) is always a valid history). We set \( \hat{e} := (\hat{f}, H) \) and \( \hat{C} := \hat{C}' \cup \{ \hat{e} \} \).

In both cases we found a configuration \( \hat{C} \supseteq \hat{C}' \) such that \( m(\sigma) \in \text{inter}(\hat{C}) \).

Showing that \( \hat{C} \) is unique requires the same reasoning than in Thm. 2, which we skip here.

\[ \square \]

### A.4 Results in Sec. 6: Plugging Thread-Local Analysis

Our goal in this section is proving Thm. 4. We will introduce some new notions necessary to formalize the operation performed by the thread-local analysis. In short, given \( D \) and an implementation of \( \text{tla} \), we will define a new analysis instance \( \hat{D} \), called the collapsing domain, that we use to prove the theorem.

For any global transformer \( f \in \text{F}_g \), we define the collapsing transformer \( \hat{f}: D \rightarrow D \) of \( f \) as \( \hat{f} := f \circ \text{tla}(i) \). The set of collapsing transformers induce a new analysis instance

\[ \hat{D} := (\hat{D}, \hat{\xi}, \hat{F}, \hat{d}_0) \]

that soundly approximates \( D \), where \( \hat{D}, \hat{\xi}, \) and \( \hat{d}_0 \) are the same as in \( D \), and the set of transformers is \( \hat{F} := \{ \hat{f} \in D \rightarrow D : f \text{ is global in } D \} \). Our notion of approximation here is different than the one given in Sec. 3. There we required exactly one abstract transformer per concrete one, while \( D \) has only abstract transformers for the global concrete ones. The notion of approximation here is rather suttering simulation of runs. For any run \( \sigma \in \text{runs}(D) \), recall that \( \hat{\sigma} \) is the subsequence of \( \sigma \) obtained by removing the local transformers. Clearly, \( \hat{\sigma} \in \text{runs}(\hat{D}) \) when \( \sigma \in \text{runs}(D) \).

Let \( \hat{\Diamond} \) be a weak independence on \( D \). One would wish that the collapsing transformers exhibits the same independence than the original ones. That is, if \( f_1 \hat{\Diamond} f_2 \), then \( f_1 \) and \( f_2 \) commute on all reachable facts. Unfortunately, this is not true in general. The abstract interpreter hidden behind \( \text{tla}(\cdot, \cdot) \) might apply widening on local loops, or data-flow joins could introduce imprecision when merging paths, all of which may affect the commutativity of the collapsing transformers. In the sequel, we employ the pointwise lifting of relation \( \Diamond \) to \( \hat{D} \), defined as \( \hat{\Diamond} := \{ (\hat{f}_1, \hat{f}_2) : f_1, f_2 \text{ are global and } f_1 \hat{\Diamond} f_2 \} \). As we said, relation \( \hat{\Diamond} \) is
in general not a weak independence relation in \( \hat{D} \). However, using ideas similar to those of Sec. 5.2 we can proof the following:

**Lemma 2 (Soundness of the abstraction).** For any execution \( \sigma \in \text{runs}(D) \) there is a unique configuration \( C \) of \( \hat{U}_D,\Diamond \) such that \( \hat{\sigma} \in \text{inter}(C) \).

**Proof. (Sketch)** The proof of this result is very similar to that of Thm. 3.

**Base Case.** Run \( \sigma \) fires only local transformers and the length of \( \hat{\sigma} \) is zero. As in Thm. 3 there is a unique representative configuration in \( \hat{U}_D,\Diamond \).

**Inductive Step.** Assume that \( \sigma : \sigma' f \). By the induction hypothesis, we assume that there exist a unique configuration \( \bar{C}' \) in \( \hat{U}_D,\Diamond \) such that \( \hat{\sigma}' \in \text{inter}(C') \).

We distinguish two cases

- Transformer \( f \) is local. Then \( \hat{\sigma} = \hat{\sigma}' \) and \( C' \) is a representative configuration for \( \sigma \).
- Transformer \( f \) is global. Then assume that \( \sigma \) is of the form \( \sigma = \sigma_g \sigma_l f \), where \( \sigma_g \) ends in a global transformer and \( \sigma_l \) contains only local transformers. Observe that \( C' \) is also a representative configuration of \( \sigma_g \).

Clearly, \( \text{state}(C') \supseteq \text{state}(\sigma_g) \). Since \text{tla}(\cdot,\cdot) always overapproximates the execution of any arbitrary sequence of local transformers, it must also overapproximate the execution of \( \sigma_l \) from \( \text{state}(\sigma_g) \). This proves that \( \hat{f} \) is enabled at \( \text{state}(C') \).

If all maximal events in \( C' \) are dependent with \( \hat{f} \) in \( \hat{\Diamond} \), then \( C' \) is a history for \( \hat{f} \) and \( e : (\hat{f}, C') \) an event of \( \hat{U}_D,\Diamond \). If not, using the same reasoning as in Thm. 3 we can find a history \( H \subseteq C' \) for \( \hat{f} \), and define event \( e : (\hat{f}, H) \).

In both cases, by construction \( C : = C' \cup \{ e \} \) is a configuration of \( \hat{U}_D,\Diamond \). Configuration \( C \) is a representative of \( \hat{\sigma} \).

We can now easily prove the main theorem of the section.

**Theorem 4 (Soundness of the abstraction).** Let \( \Diamond \) be a weak independence on \( D \) and \( P_D,\Diamond \) the PES computed by a call to \text{unfold}(D,\Diamond,n) \) with cutoff checking disabled. Then, for any execution \( \sigma \in \text{runs}(D) \) there is a unique configuration \( C \) in \( P_D,\Diamond \) such that \( \hat{\sigma} \in \text{inter}(C) \).

**Proof.** A call to \text{unfold}(D,\Diamond,n) \) with cutoff checking disabled computes the unfolding of \( \hat{D} \), so we have that

\[
P_D,\Diamond = \hat{U}_D,\Diamond.
\]

The theorem holds as a consequence of Lemma 2.
A.5 Formalizing Cutoff Events

In this section, a new cutoff criterion is defined that exploits the lattice order \( \preceq \) for more aggressive pruning than standard cutoffs. Let \( D \) be an analysis instance and \( \Diamond \) a weak independence.

In order to prune the unfolding, we need to refer to the order in which it is constructed. A \textit{strategy} is any strict (partial) order \( \prec \) on the finite configurations of \( \mathcal{U}_D, \Diamond \) satisfying that when \( C \subseteq C' \), then \( C \prec C' \). In other words, strategies refine the natural order in which the domain is unfolded.

Each strategy identifies a set of feasible and cutoff events. Intuitively, feasible events will be those which have no cutoff among the set of causal predecessors:

\textbf{Definition 4 (Cutoffs).} An event \( e \) of \( \mathcal{U}_D, \Diamond \) is \( \prec \)-feasible if all causal predecessors \( e' \in \lfloor e \rfloor \) are not \( \prec \)-cutoff. A \( \prec \)-feasible event is a \( \prec \)-cutoff if there exists some \( \prec \)-feasible event \( e' \) in \( \mathcal{U}_D, \Diamond \), called the corresponding event, such that \( [e'] \prec [e] \) and

\[ \text{state}([e]) \preceq \text{state}([e']). \]  

In other words, \( e \) will be a cutoff iff the fact reached by the branch it represents (local configuration) has already been “seen” when \( D \) is unfolded in the order stated by \( \prec \). Observe that “seen” formally means that another equally or less precise element has been unfolded before.

While the notion of cutoffs has been around for a while in the literature of unfoldings \([17,6,3]\), to the best of our knowledge, Def. 4 is the first to use a subsumption relation to match the corresponding event. The most general previous definition \([3]\) only allowed states to be compared using equivalence relations in (4), while we used the partial order \( \preceq \). The set of \( \prec \)-feasible events defines an unfolding prefix of \( U \):

\textbf{Definition 5 (Feasible prefix).} The \( \prec \)-prefix of \( D \) is the unique unfolding prefix \( P^\prec D, \Diamond \) of \( \mathcal{U}_D, \Diamond \) that contains exactly all \( \prec \)-feasible events which are not \( \prec \)-cutoffs.

The shape and properties of the \( \prec \)-prefix strongly depend on the underlying strategy \( \prec \). One is interested in strategies that identify complete prefixes.

A well-known unfolding strategy is the \textit{size order} \( \prec_s \subseteq E \times E \), defined by Ken McMillan in his seminal paper \([17]\) as \( C \prec_s C' \) iff \( |C| < |C'| \). Adequate strategies \([6,3]\) were discovered later and yield up to exponentially smaller prefixes. In order to keep the presentation concise, we restrict next theorem to the size order (although it also holds for adequate strategies).

\textbf{Theorem 6.} The unfolding prefix \( P^\prec_s D, \Diamond \) is \( D \)-complete.

\textit{Proof. (Sketch)} Let \( d \in \text{reach}(D) \) be a reachable state. Then there is some configuration \( C \) in \( \mathcal{U}_D, \Diamond \) such that \( d = \text{state}(C) \). If \( C \) is free of \( \prec \)-cutoff events, then \( C \) is in \( P^\prec_s D, \Diamond \) and we found the configuration that we searched.
If not, let $e \in C$ be a \textless-cutoff event and $e'$ the corresponding event in $U_{D,\Diamond}$. Since $d \neq 1$, any interleaving of $C$ is a run of $D$. Since $\text{state}([e']) \equiv \text{state}([e])$, any interleaving of $[e]$ can be extended with the transformers that label in any topological sorting of the events in $C \setminus [e]$, and the resulting sequence is a run $\sigma' \in \text{runs}(D)$ that satisfies $d \equiv \text{state}(\sigma')$. Furthermore, since all runs of $D$ are represented as (unique) configurations of $U_{D,\Diamond}$, it is possible to extend configuration $[e]$ into a unique configuration that represents $\sigma'$. Let it be $C'$. We have that $d = \text{state}(C) \equiv \text{state}(C')$, and $C'$ is at least one event smaller than $C$. If $C'$ has no cutoff, then we found the configuration that we were searching. If not, we only need to repeat this argument a finite number of times (since every time we remove at least one event from the configuration) until we find a configuration that reaches a state that covers $d$.

A.6 Pruning without Relaxed Independence

In Sec. 5.2 we unfolded an abstract domain $D$ into an event structure $U_{\bar{D},\Diamond}$ under an independence relation $\Diamond$ that was weak in the concrete domain $D$ but non-weak in the abstract one $\bar{D}$.

It would be natural to extend the cutoff criterion introduced above, which requires a weak independence, to employ the non-weak relation $\Diamond$. Unfortunately, in this case the feasible $\textless$-prefix is not necessarily complete. The proof of Thm. 6 relies on the fact that all runs of $D$ will appear under the form of one configuration in $U_{\bar{D},\Diamond}$. However, the non-weak relation $\Diamond$ fails to guarantee that.

Alternatively, one may try to change the completeness criterion, asking that all facts reachable in the concrete domain $D$ are present in the unfolding of the abstract domain. Unfortunately, proving Thm. 6 with this notion of completeness fails again, for the same reason. The reasoning behind the notion of cutoff events fundamentally relies on the fact that arbitrary executions of $D$ must be present in $U_{\bar{D},\Diamond}$.

Therefore, using cutoff criteria for the abstract unfolding is possible only together with weak independence relations. Fortunately, at least for simple domains such as intervals, computing a weak independence seems to be reasonably inexpensive.

A.7 Results in Sec. 6.1: Cutoff Events

In the following proof we make use of the collapsing domain introduced in App. A.4.

**Theorem 5.** Let $\Diamond$ be a weak independence in $D$. Assume that tla respects independence and that iscutoff uses the procedure defined above. Then the PES $Q_{D,\Diamond}$ computed by Alg. 1 is $D$-complete.

**Proof.** If $\Diamond$ respects independence, then it is straightforward to show that $\Diamond$ is a weak independence in $D$. That means that Thm. 6 is applicable and implies that $Q_{D,\Diamond}$ is $D$-complete, as Alg. 1 computes exactly that prefix. \qed