SCHAUDER ESTIMATES FOR EQUATIONS WITH CONE METRICS, II

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Abstract. This is the continuation of our paper [20], to study the linear theory for equations with conical singularities. We derive interior Schauder estimates for linear elliptic and parabolic equations with a background Kähler metric of conical singularities along a divisor of simple normal crossings. As an application, we prove the short-time existence of the conical Kähler-Ricci flow with conical singularities along a divisor with simple normal crossings.

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1. Introduction

Regularity of solutions of Complex Monge-Ampère equations is a central problem in complex geometry. Complex Monge-Ampère equations with singular and degenerate data can be applied to study compactness and moduli problems of canonical Kähler metrics in Kähler geometry. In [43], Yau has already considered special cases of complex singular Monge-Ampère equations as generalization of his solution to the Calabi conjecture. Conical singularities along complex hypersurfaces of a Kähler manifold are among the mildest singularities in Kähler geometry and it has been extensively studied, especially in the case of Riemann surfaces [41, 28]. The study of such Kähler metrics with conical singularities has many geometric applications, for example, the

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Chern number inequality in various settings [39, 38]. Recently, Donaldson [14] initiated the program of studying analytic and geometric properties of Kähler metrics with conical singularities along a smooth complex hypersurface on a Kähler manifold. This is an essential step to the solution of the Yau-Tian-Donaldson conjecture relating existence of Kähler-Einstein metrics and algebraic K-stability on Fano manifolds [5, 6, 7, 40]. In [14], the Schauder estimate for linear Laplace equations with conical background metric is established using classical potential theory. This is crucial for the openness of the continuity method to find desirable (conical) Kähler-Einstein metric. Donaldson’s Schauder estimate is generalized to the parabolic case [8] with similar classical approach. There is also an alternative approach for the conical Schauder estimates using microlocal analysis [23]. There are also various global and local estimates and regularity derived in the conical setting [1, 15, 9, 11, 12, 18, 15, 24, 29, 31, 44, 45].

The Schauder estimates play an important role in the linear PDE theory. Apart from the classical potential theory, various proofs have been established by different analytic techniques. In fact, the blow-up or perturbation techniques developed in [36, 42] (also see [33, 34, 2, 3]) are much more flexible and sharper than the classical method. The authors combined the perturbation method in [20] and geometric gradient estimates to establish sharp Schauder estimates for Laplace equations and heat equations on $\mathbb{C}^n$ with a background flat Kähler metric of conical singularities along the smooth hyperplane $\{z_1 = 0\}$ and derived explicit and optimal dependence on conical parameters.

In algebraic geometry, one often has to consider pairs $(X, D)$ with $X$ being an algebraic variety of complex dimension $n$ and the boundary divisor $D$ as a complex hypersurface of $X$. After possible log resolution, one can always assume the divisor $D$ is a union of smooth hypersurfaces with simple normal crossings. The suitable category of Kähler metrics associated to $(X, D)$ is the family of Kähler metrics on $X$ with conical singularities along $D$. In order to study canonical Kähler metrics on pairs and related moduli problems, we are obliged to study regularity and asymptotics for complex Monge-Ampère equation with prescribed conical singularities of normal crossings. However, the linear theory is still missing and has been open for a while. The goal of this paper is to extend our result [20] and establish the sharp Schauder estimates for linear equations with background Kähler metric of conical singularities along divisors of simple normal crossings. We can apply and extend many techniques developed in [20], however, new estimates and techniques have to be developed because in case of conical singularities along a single smooth divisor, the difficult estimate in the conical direction can sometimes be bypassed and reduced to estimates in the regular directions, while such treatment does not work in the case of simple normal crossings. One is forced to treat regions near high codimensional singularities directly with new and more delicate estimate beyond the scope of [20].

The standard local models for such conical Kähler metrics can be described as below.

Let $\beta = (\beta_1, \ldots, \beta_p) \in (0, 1]^p$ and $p \leq n$ and $\omega_{\beta}$ (or $g_{\beta}$) be the standard cone metric on $\mathbb{C}^p \times \mathbb{C}^{n-p}$ with cone singularity along $S = \cup_{i=1}^p S_i$, where $S_i = \{z_i = 0\}$, that is,

$$\omega_{\beta} = \sum_{j=1}^p \beta_j^{2} \sqrt{-1} dz_j \wedge d\bar{z}_j + \sum_{j=p+1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j. \quad (1.1)$$

We shall use $s_{2p+1}, \ldots, s_{2n}$ to denote the real coordinates of $\mathbb{C}^{n-p} = \mathbb{R}^{2n-2p}$, such that for $j = p+1, \ldots, n$

$$z_j = s_{2j-1} + \sqrt{-1}s_{2j}.\quad (1.1)$$
In this paper we will study the following conical Laplacian equation with the background metric \( g_\beta \) on \( \mathbb{C}^n \)

\[
\Delta_\beta u = f, \quad \text{in } B_\beta(0,1) \setminus \mathcal{S},
\]

(1.2)

where \( B_\beta(0,1) \) is the unit ball with respect to \( g_\beta \) centered at 0. The Laplacian \( \Delta_\beta \) is defined as

\[
\Delta_\beta u = \sum_{j,k} g_\beta^{jk} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = \sum_{j=1}^p |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} + \sum_{j=p+1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}.
\]

We always assume \( f \in C^0(B_\beta(0,1)) \) and \( u \in C^0(B_\beta(0,1)) \cap C^2(B_\beta(0,1) \setminus \mathcal{S}) \). Throughout this paper, given a continuous function \( f \) we denote

\[
\omega(r) := \omega_f(r) = \sup_{z,w \in B_\beta(0,1), d_\beta(z,w) < r} |f(z) - f(w)|
\]

the oscillation of \( f \) with respect to \( g_\beta \) in the ball \( B_\beta(0,1) \). It is clear that \( \omega(2r) \leq 2\omega(r) \) for any \( r < 1/2 \). We say a continuous function \( f \) is Dini continuous if \( \int_0^1 \frac{\omega(r)}{r} dr < \infty \).

**Definition 1.1.** We will write the (weighted) polar coordinates of \( z_j \) for \( 1 \leq j \leq p \) as

\[
r_j = |z_j|^{\beta_j}, \quad \theta_j = \arg z_j.
\]

We denote \( D' \) to be one of the first order operators \( \{ \frac{\partial}{\partial r_j}, \frac{\partial}{\partial \beta_j r_j}, \frac{\partial}{\partial \theta_j} \} \), and \( N_j \) to be one of the operators \( \{ \frac{\partial}{\partial r_j}, \frac{\partial}{\partial \beta_j r_j}, \omega, \theta_j \} \) which as vector fields are transversal to \( S_j \).

Our first main result is the Hölder estimates of the solution \( u \) to the equation (1.2).

**Theorem 1.1.** Suppose \( \beta \in (1/2,1)^p \) and \( f \in C^0(B_\beta(0,1)) \) is Dini continuous with respect to \( g_\beta \). Let \( u \in C^0(B_\beta(0,1)) \cap C^2(B_\beta(0,1) \setminus \mathcal{S}) \) be the solution to the equation (1.2), then there exists \( C = C(n,\beta) > 0 \) such that for any two points \( p,q \in B_\beta(0,1/2) \setminus \mathcal{S} \),

\[
||D'||^2 u(p) - (D')^2 u(q)|| + \sum_{j=1}^p |z_j|^{2(1-\beta_j)} \left| \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(p) - |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(q) \right| 
\]

\[
\leq C \left( d||u||_{L^\infty(B_\beta(0,1))} + \int_0^d \frac{\omega(r)}{r} dr + d \int_0^1 \frac{\omega(r)}{r^2} dr \right),
\]

for any \( 1 \leq j \leq p \),

\[
|N_jD'u(p) - N_jD'u(q)| \leq C \left( d^{\frac{1}{\beta_j}} ||u||_{L^\infty(B_\beta(0,1))} + \int_0^d \frac{\omega(r)}{r} dr + d^{\frac{1}{\beta_j}} \int_0^1 \frac{\omega(r)}{r^{1/\beta_j}} dr \right),
\]

(1.3)

and for any \( 1 \leq j,k \leq p \) with \( j \neq k \),

\[
|N_jN_ku(p) - N_jN_ku(q)| \leq C \left( d^{\frac{1}{\beta_{\max}}} ||u||_{L^\infty(B_\beta(0,1))} + \int_0^d \frac{\omega(r)}{r} dr + d^{\frac{1}{\beta_{\max}}} \int_0^1 \frac{\omega(r)}{r^{1/\beta_{\max}}} dr \right),
\]

(1.4)

where \( d = d_\beta(p,q) > 0 \) is the \( g_\beta \)-distance of \( p \) and \( q \) and \( \beta_{\max} = \max\{\beta_1, \ldots, \beta_p\} \in (1/2,1) \).

**Remark 1.1.** (1) We remark that the number \( \beta_{\max} \) on the RHS of (1.5) can be replaced by \( \max\{\beta_j, \beta_k\} \).

(2) In Theorem 1.1 and Theorem 1.3 below, we assume \( \beta \in (1/2,1)^p \) just for exposition purposes and clearness of the statement. When some of angles \( \beta_j \) lie in \( (0,1/2) \), the pointwise Hölder estimates in Theorem 1.1 are adjusted as follows: if \( \beta_j \in (0,1/2] \) in (1.4), we replace the RHS by the RHS of (1.3). In (1.5), if both \( \beta_j \) and \( \beta_k \in (0,1/2] \), we also replace the RHS by that of (1.3); if at least one of the \( \beta_j, \beta_k \) is bigger than \( 1/2 \), (1.5) remains unchanged. The inequalities in Theorem 1.3 can be adjusted similarly. The proofs of these estimates are contained in the proof of the case when \( \beta_j \in (1/2,1) \) by using the corresponding estimates in (2.2).
An immediate corollary of Theorem 1.1 is a precise form of Schauder estimates for equation (1.2).

**Corollary 1.1.** Given \( \beta \in (0,1)^p \) and \( f \in C_{\beta}^{0,\alpha}(\overline{B_{\beta}(0,1)}) \) for some \( 0 < \alpha < \min\{1, \frac{1}{p_{\max}} - 1\} \), if \( u \in C^0(B_{\beta}(0,1)) \cap C^2(B_{\beta}(0,1) \setminus \mathcal{S}) \) solves equation (1.2), then \( u \in C_{\beta}^{2,\alpha}(B_{\beta}(0,1)) \). Moreover, for any compact subset \( K \subseteq B_{\beta}(0,1) \), there exists a constant \( C = C(n, \beta, K) > 0 \) such that the following estimate holds (see Definition 2.1 for the notations)

\[
\|u\|_{C_{\beta}^{2,\alpha}(K)} \leq C \left( \|u\|_{C^0(B_{\beta}(0,1))} + \frac{\|f\|_{C_{\beta}^{0,\alpha}(B_{\beta}(0,1))}}{\alpha(\min\{1, \frac{1}{p_{\max}} - 1\} - \alpha)} \right). \tag{1.6}
\]

**Remark 1.2.** A scaling-invariant version of the Schauder estimate (1.6) is that for any \( 0 < r < 1 \), there exists a constant \( C = C(n, \beta, \alpha) > 0 \) such that (see Definition 2.3 for the notations)

\[
\|u\|_{C_{\beta}^{2,\alpha}(B_{\beta}(0,r),r)} \leq C \left( \|u\|_{C^0(B_{\beta}(0,r))} + \|f\|_{C_{\beta}^{0,\alpha}(B_{\beta}(0,r))} \right), \tag{1.7}
\]

which follows from a standard rescaling argument by scaling \( r \) to 1.

Let \( g \) be a \( C_{\beta}^{0,\alpha} \)-conical Kähler metric on \( B_{\beta}(0,1) \) (see Definition 3.1 below). By definition \( g \) is equivalent to \( g_{\beta} \). We consider the equation

\[
\Delta_g u = f \text{ in } B_{\beta}(0,1), \quad u = \varphi \text{ on } \partial B_{\beta}(0,1), \tag{1.8}
\]

for some \( \varphi \in C^0(\partial B_{\beta}(0,1)) \). The following theorem is the generalization of Corollary 1.1 for non-flat background conical Kähler metrics, which is useful for applications of global geometric complex Monge-Ampère equations.

**Theorem 1.2.** For any given \( \beta \in (0,1)^p \), \( f \in C_{\beta}^{0,\alpha}(\overline{B_{\beta}(0,1)}) \) and \( \varphi \in C^0(\partial B_{\beta}(0,1)) \), there is a unique solution \( u \in C_{\beta}^{2,\alpha}(B_{\beta}(0,1)) \cap C^0(\overline{B_{\beta}(0,1)}) \) to the equation (1.8). Moreover, for any compact subset \( K \subseteq B_{\beta}(0,1) \), there exists \( C = C(n, \beta, \alpha, g, K) > 0 \) such that

\[
\|u\|_{C_{\beta}^{2,\alpha}(K)} \leq C \left( \|u\|_{C^0(B_{\beta}(0,1))} + \|f\|_{C_{\beta}^{0,\alpha}(B_{\beta}(0,1))} \right).
\]

Theorem 1.2 can immediately be applied to study complex Monge-Ampère equations with prescribed conical singularities along divisors of simple normal crossings and most of the geometric and analytic results for canonical Kähler metrics with conical singularities along a smooth divisor can be generalized to the case of simple normal crossings.

We now turn to the parabolic Schauder estimates for the solution \( u \in C^0(Q_{\beta}) \cap C^2(Q_{\beta}^\#) \) to the equation

\[
\frac{\partial u}{\partial t} = \Delta_g u + f, \tag{1.9}
\]

for a Dini continuous function \( f \) in \( Q_{\beta} \), where for notation convenience we write \( Q_{\beta} := B_{\beta}(0,1) \times (0,1] \) and \( Q_{\beta}^\# := B_{\beta}(0,1) \setminus \mathcal{S} \times (0,1] \). Our second main theorem is the following pointwise estimate.

**Theorem 1.3.** Suppose \( \beta \in (1/2,1)^p \) and \( u \) is the solution to (1.9). Then there exists a computable constant \( C = C(n, \beta) > 0 \) such that for any \( Q_{\beta} = (p, t_p), Q_{\beta} = (q, t_q) \subseteq B_{\beta}(0,1/2) \setminus \mathcal{S} \times (\tilde{t}, 1) \) (for some \( \tilde{t} \in (0,1) \)) such that

\[
|\langle D' \rangle^2 u(Q_{p}) - \langle D' \rangle^2 u(Q_{q})| + \sum_{j=1}^{p} \left| z_j^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(Q_{p}) - z_j^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(Q_{q}) \right|
\]

\[
+ \left| \frac{\partial u}{\partial t}(Q_{p}) - \frac{\partial u}{\partial t}(Q_{q}) \right| \leq C \left( \frac{d}{\tilde{t}^{3/2}} \|u\|_{L^\infty(B_{\beta}(0,1))} + \tilde{t}^{-1} \int_0^d \frac{\omega(r)}{r} dr + \frac{d}{\tilde{t}^{3/2}} \int_0^d \frac{\omega(r)}{r^2} dr \right),
\]
and for any \(1 \leq j \leq p\)
\[
|N_j D'(Q_p) - N_j D'(Q_q)| \leq C\left(\frac{d_{\text{max}}^{\frac{r}{2}}}{r^{\beta/2}} \|u\|_{L^\infty(B_{\beta}(0,1))} + \int_0^d \frac{\omega(r)}{r} dr + \frac{d_{\text{max}}^{\frac{r}{2}}}{r^{\beta/2}} \int_0^d \frac{\omega(r)}{r^{1/\beta}} dr\right),
\]
and for any \(1 \leq j, k \leq p\) with \(j \neq k\)
\[
|N_j N_k u(Q_p) - N_j N_k u(Q_q)| \leq C\left(\frac{d_{\text{max}}^{\frac{r}{2}}}{r^{\beta/2}} \|u\|_{L^\infty(B_{\beta}(0,1))} + \int_0^d \frac{\omega(r)}{r} dr + \frac{d_{\text{max}}^{\frac{r}{2}}}{r^{\beta/2}} \int_0^d \frac{\omega(r)}{r^{1/\beta}} dr\right),
\]
where \(d = d_{p, \beta}(Q_p, Q_q) > 0\) is the parabolic \(g_{\beta}\)-distance of \(Q_p\) and \(Q_q\), and \(\beta_{\text{max}} = \max\{\beta_1, \ldots, \beta_p\}\), and \(\omega(r)\) is the oscillation of \(f\) in \(Q_\beta\) under the parabolic distance \(d_{p, \beta}\) (c.f. Section 2.1.2).

If \(f \in C^{\alpha, \frac{\beta}{2}}_{\beta}(Q_\beta)\) for some \(\alpha \in (0, \min(\frac{1}{\beta_{\text{max}}} - 1, 1))\), then we have the following precise estimates as the parabolic analogue of Corollary 1.1.

**Corollary 1.2.** Suppose \(\beta \in (0, 1)^p\) and \(u \in C^0(Q_\beta) \cap C^2(Q_{\beta}^{\frac{\beta}{2}})\) satisfies the equation (1.9), then there exists a constant \(C = C(n, \beta) > 0\) such that (see Definition 2.5 for the notations)
\[
\|u\|_{C^{\alpha, \frac{\beta}{2}}_{\beta}(B_{\beta}(0,1/2) \times (1/2, 1))} \leq C\left(\|u\|_{C^0(Q_\beta)} + \frac{\|f\|_{C^{\alpha, \frac{\beta}{2}}_{\beta}(Q_\beta)}}{\alpha(\min\{1/\beta_{\text{max}}, 1\} - \alpha)}\right).
\]

For general non-flat \(C^{\alpha, \frac{\beta}{2}}_{\beta}\)-conical Kähler metrics \(g\), we consider the linear parabolic equation
\[
\frac{\partial u}{\partial t} = \Delta_\beta u + f, \quad \text{in } Q_\beta, \quad u = \varphi \text{ on } \partial_{p, \beta} Q_\beta.
\] (1.10)

We then have the following parabolic Schauder estimates as an analogue of Theorem 1.2.

**Theorem 1.4.** Given \(\beta \in (0, 1)^p\), \(f \in C^{\alpha, \frac{\beta}{2}}_{\beta}(\overline{Q_\beta})\) and \(\varphi \in C^0(\partial_{p, \beta} Q_\beta)\), there exists a unique solution \(u \in C^{2+\alpha, \frac{\beta+2}{2}}_{\beta}(B_{\beta}(0,1) \times (0,1]) \cap C^0(\overline{Q_\beta})\) to the Dirichlet boundary value problem (1.10). For any compact subset \(K \subseteq B_{\beta}(0,1)\) and \(\varepsilon_0 > 0\) there exists \(C = C(n, \beta, \alpha, K, \varepsilon_0, g) > 0\) such that the following interior Schauder estimate holds
\[
\|u\|_{C^{2+\alpha, \frac{\beta+2}{2}}_{\beta}(K \times [0,1])} \leq C\left(\|u\|_{C^0(Q_\beta)} + \|f\|_{C^{\alpha, \frac{\beta}{2}}_{\beta}(Q_\beta)}\right).
\]
Furthermore, if we assume \(u|_{t=0} = u_0 \in C^{2+\alpha}_{\beta}(B_{\beta}(0,1))\), then \(u \in C^{2+\alpha, \frac{\beta+2}{2}}_{\beta}(B_{\beta}(0,1) \times [0,1])\) and there exists a constant \(C = C(n, \beta, \alpha, g, K) > 0\) such that
\[
\|u\|_{C^{2+\alpha, \frac{\beta+2}{2}}_{\beta}(K \times [0,1])} \leq C\left(\|u\|_{C^0(Q_\beta)} + \|f\|_{C^{\alpha, \frac{\beta}{2}}_{\beta}(Q_\beta)} + \|u_0\|_{C^{2+\alpha}_{\beta}(B_{\beta}(0,1))}\right).
\]

As an application of Theorem 1.4, we derive the short-time existence of the conical Kähler-Ricci flow with background metric being conical along divisors with simple normal crossings.

Let \((X, D)\) be a compact Kähler manifold, where \(D = \sum_j D_j\) is a finite union of smooth divisors \(\{D_j\}\) and \(D\) has only simple normal crossings. Let \(\omega_0\) be a \(C^{0, \alpha'}(X)\)-conical Kähler metric with cone angle \(2\pi \beta\) along \(D\) (see Definition 2.7) and \(\hat{\omega}_t\) be a family of conical metrics with bounded norm \(\|\hat{\omega}\|_{C^{\alpha', \frac{\alpha'}{2}}_{\beta}}\) and \(\hat{\omega}_0 = \omega_0\). We consider the complex Monge-Ampère flow
\[
\frac{\partial \varphi}{\partial t} = \log \left(\frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_0^n}\right) + f, \quad \text{and } \varphi|_{t=0} = 0,
\] (1.11)
for some \(f \in C^{\alpha', \frac{\alpha'}{2}}_{\beta}(X \times [0,1])\).
Theorem 1.5. Given \( \alpha \in (0, \alpha') \), there exists \( T = T(n, \hat{\omega}, f, \alpha', \alpha) > 0 \) such that (1.11) admits a unique solution \( \phi \in C^{2+\alpha,2+\alpha} _\beta (X \times [0, T]) \).

An immediate corollary of Theorem 1.5 is the short time existence for the conical Kähler-Ricci flow defined as below
\[
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \sum_j (1 - \beta_j)[D_j], \quad \omega|_{t=0} = \omega_0,
\]
(1.12)
where \( \text{Ric}(\omega) \) is the unique extension of the Ricci curvature of \( \omega \) from \( X \setminus D \) to \( X \) and \( [D_j] \) denotes the current of integration over the component \( D_j \). In addition we assume \( \omega_0 \) is a \( C^0,\alpha' \beta (X, D) \)-conical Kähler metric such that
\[
\omega_0^n = \frac{\Omega}{\prod_j (|s_j|^{h_j})^{1-\beta_j}},
\]
(1.13)
where \( s_j, h_j \) are holomorphic sections and hermitian metrics of the line bundle associated to \( D_j \), respectively, and \( \Omega \) is a smooth volume form.

Corollary 1.3. For any given \( \alpha \in (0, \alpha') \), there exists a constant \( T = T(n, \omega_0, \alpha, \alpha') > 0 \) such that the conical Kähler-Ricci flow (1.12) admits a unique solution \( \omega = \omega_t \), such that \( \omega \in C^{0,\alpha/2} _\beta (X \times [0, T]) \) and for each \( t \in [0, T] \), \( \omega_t \) is still a conical metric with cone angle \( 2\pi \beta \) along \( D \).

Furthermore, \( \omega \) is smooth in \( X \setminus D \times (0, T] \) and the (normalized) Ricci potentials of \( \omega \), \( \log \left( \frac{\omega^n}{\omega_0^n} \right) \) is still in \( C^{2+\alpha,2+\alpha} _\beta (X \times [0, T]) \).

The short time existence of the conical Kähler-Ricci flow with singularities along a smooth divisor is derived in [8] by adapting the elliptic potential techniques of Donaldson [14]. Corollary 1.3 treats the general case of conical singularities with simple normal crossings. There have been many results in the analytic aspects of the conical Ricci flow [8, 9, 15, 16, 24, 32, 43]. In [30], the conical Ricci flow on Riemann surfaces is completely classified with jumping conical structure in the limit. Such phenomena is also expected in higher dimension, but it requires much deeper and delicate technical advances both in analysis and geometry.

2. Preliminaries

We explain the notations and give some preliminary tools which will be used later in this section.

2.1. Notations

To distinguish the elliptic from parabolic norms, we will use the ordinary \( C \) to denote the norms in the elliptic case and the script \( C \) to denote the norms in the parabolic case.

We always assume the Hölder component \( \alpha \) appearing in \( C^{0,\alpha} _\beta \) or \( C^{0,\alpha/2} _\beta \) (or other Hölder norms) to be in \( (0, \min\{\beta^{-1}_{\text{max}} - 1, 1\}) \).

2.1.1. Elliptic case. We will denote \( d_\beta(x, y) \) to be the distance of two points \( x, y \in \mathbb{C}^n \) under the metric \( g_\beta \). \( B_\beta(x, r) \) will be the metric ball under the metric induced by \( g_\beta \) with radius \( r \) and center \( x \). It is well-known that \( (\mathbb{C}^n \setminus S, g_\beta) \) is geodesically convex, i.e. any two points \( x, y \in \mathbb{C}^n \setminus S \) can be joined by a \( g_\beta \)-minimal geodesic \( \gamma \) which is disjoint with \( S \).
Definition 2.1. We define the $g_\beta$-Hölder norm of functions $u \in C^0(B_\beta(0,r))$ for some $\alpha \in (0,1)$ as
\[
\|u\|_{C^{0,\alpha}_\beta(B_\beta(0,r))} := \|u\|_{C^0(B_\beta(0,r))} + [u]_{C^{0,\alpha}_\beta(B_\beta(0,r))},
\]
where the semi-norm is defined as $[u]_{C^{0,\alpha}_\beta(B_\beta(0,r))} := \sup_{x \neq y \in B_\beta(0,r)} \frac{|u(x) - u(y)|}{d_\beta(x,y)^\alpha}$. We denote the subspace of all continuous functions $u$ such that $\|u\|_{C^{0,\alpha}_\beta(B_\beta(0,r))} < \infty$ as $C^{0,\alpha}_\beta(B_\beta(0,r))$.

Definition 2.2. The $C^{2,\alpha}_\beta$ norm of a function $u$ on $B_\beta(0,r) = B_\beta$ is defined as:
\[
\|u\|_{C^{2,\alpha}_\beta(B_\beta)} := \|u\|_{C^0(B_\beta)} + \|\nabla_{g_\beta} u\|_{C^0(B_\beta, g_\beta)} + \sum_{j=1}^p \|N_j D'u\|_{C^{0,\alpha}_\beta(B_\beta)} + \sum_{1 \leq j \neq k \leq p} \|N_j N_k u\|_{C^{0,\alpha}_\beta(B_\beta)} + \sum_{j=1}^p \left( \left\|z_j \right\|^{2(1-\beta_j)} \left| \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} \right| \right)_{C^{0,\alpha}_\beta(B_\beta)}.
\]

For a given set $\Omega \subset B_\beta(0,1)$ we define the following weighted (semi)norms.

Definition 2.3. Suppose $\sigma \in \mathbb{R}$ is a given real number and $u$ is a $C^{2,\alpha}_\beta$-function in $\Omega$. We denote $d_x = d_\beta(x, \partial \Omega)$ for any $x \in \Omega$. We define the weighted (semi)norms
\[
[u]_{C^{\alpha}_\beta(\Omega)}^{(\sigma)} = \sup_{x \neq y \in \Omega} \min(d_x, d_y)^{\sigma+\alpha} \frac{|u(x) - u(y)|}{d_\beta(x,y)^\alpha},
\]
\[
\|u\|_{C^0(\Omega)}^{(\sigma)} = \sup_{x \in \Omega} d_x^\sigma |u(x)|, \quad [u]_{C^1_\beta(\Omega)}^{(\sigma)} = \sup_{x \in \Omega, \mathcal{S}} d_x^{\sigma+1} \left( \sum_j |N_j u|(x) + |D'u|(x) \right),
\]
\[
[u]_{C^{2}_\beta(\Omega)}^{(\sigma)} = \sup_{x \in \Omega, \mathcal{S}} d_x^{\sigma+2} |T u(x)|,
\]
\[
[u]_{C^{\alpha}_\beta(\Omega)}^{(\sigma)} = \sup_{x \neq y \in \Omega, \mathcal{S}} \min(d_x, d_y)^{\sigma+2+\alpha} \frac{|T u(x) - T u(y)|}{d_\beta(x,y)^\alpha},
\]
and
\[
\|u\|_{C^{0,\alpha}_\beta(\Omega)}^{(\sigma)} = \|u\|_{C^0(\Omega)}^{(\sigma)} + [u]_{C^1_\beta(\Omega)}^{(\sigma)} + [u]_{C^2_\beta(\Omega)}^{(\sigma)} + [u]_{C^{2,\alpha}_\beta(\Omega)}^{(\sigma)},
\]
where $T$ denotes the following operators of second order:
\[
\left\{ \left| z_j \right|^{2(1-\beta_j)} \frac{\partial^2}{\partial z_j \partial \overline{z}_j}, N_j N_k (j \neq k), N_j D', (D')^2 \right\}. \tag{2.1}
\]
When $\sigma = 0$, we denote the norms above as $[\cdot]^*$, $\|\cdot\|^*$ for notation simplicity.

2.1.2. Parabolic case. We denote $\mathcal{Q}_\beta = \mathcal{Q}_\beta(0,1) = B_\beta(0,1) \times (0,1]$ to be parabolic cylinder and $\partial \mathcal{Q}_\beta = \overline{(B_\beta(0,1) \times \{0\}) \cup (\partial B_\beta(0,1) \times (0,1])}$ to be the parabolic boundary of the cylinder $\mathcal{Q}_\beta$. We write $\mathcal{S}_\mathcal{P} = \mathcal{S} \times [0,1]$ as the singular set and $\mathcal{Q}^\#_\beta = \mathcal{Q}_\beta \setminus \mathcal{S}_\mathcal{P}$ the complement of $\mathcal{S}_\mathcal{P}$. For any two space-time points $Q_i = (p_i, t_i)$, we define their parabolic distance $d_{\mathcal{P},\beta}(Q_1, Q_2)$ as
\[
d_{\mathcal{P},\beta}(Q_1, Q_2) = \max\{\sqrt{|t_1 - t_2|}, d_\beta(p_1, p_2)\}.
\]
Definition 2.4. We define the $g_\beta$-Hölder norm of functions $u \in C^0(\Omega)$ for some $\alpha \in (0,1)$ as

$$\|u\|_{C_\beta^{\alpha,\alpha/2}(\Omega)} := \|u\|_{C^0(\Omega)} + \|\beta \cdot u\|_{C^{\alpha,\alpha/2}(\Omega)},$$

where the semi-norm is defined to be $[u]_{C^{\alpha,\alpha/2}(\Omega)} := \text{sup}_{\Omega \setminus \Omega_1} |u(Q_1) - u(Q_2)|$. We denote the subspace of all continuous functions $u$ such that $\|u\|_{C^{\alpha,\alpha/2}(\Omega)} < \infty$ as $C^{\alpha,\alpha/2}_\beta(\Omega)$.

Definition 2.5. The $C^{2+\alpha,2+\alpha/2}_\beta$ norm of a function $u$ on $\Omega$ is defined as:

$$\|u\|_{C^{2+\alpha,2+\alpha/2}_\beta}(\Omega) := \|u\|_{C^0(\Omega)} + \|\nabla g_\beta u\|_{C^{\alpha,\alpha/2}(\Omega)} + \|T u\|_{C^{\alpha,\alpha/2}(\Omega)},$$

where $T$ denotes all the second order operators in (2.1) and the first order operator $\partial/\partial n$.

For a given set $\Omega \subset \Omega_\beta$ we define the following weighted (semi)norms.

Definition 2.6. Suppose $\sigma \in \mathbb{R}$ is a real number and $u$ is a $C^{2+\alpha,2+\alpha/2}_\beta$-function in $\Omega$. We denote $d_{P,Q} = d_{P,\beta}(Q, d\Omega)$ for any $Q \in \Omega$. We define the weighted (semi)norms

$$[u]_{C^{\alpha,\alpha/2}_\beta}(\Omega),$$

$$\|u\|_{C^{\alpha,\alpha/2}_\beta}(\Omega) := \sup_{Q \in \Omega} d_{P,Q}^{\sigma+1} |\nabla u(Q)|,$$

and

$$\|u\|_{C^{2+\alpha,2+\alpha/2}_\beta}(\Omega) := \sup_{Q \in \Omega} d_{P,Q}^{\sigma+2} |T u(Q)|.$$

When $\sigma = 0$, we denote the norms above as $\| \cdot \|$ or $\| \cdot \|$ for simplicity.

2.1.3. Compact Kähler manifolds. Let $(X,D)$ be a compact Kähler manifold with a divisor $D = \sum_j D_j$ with simple normal crossings, i.e., on an open coordinates chart $(U,z_j)$ of any $x \in D$, $D \cap U$ is given by $\{z_1 \cdots z_p = 0\}$, and $D_j \cap U = \{z_j = 0\}$ for any component $D_j$ of $D$. We fix a finite cover $\{U_a, z_{a,j}\}$ of $D$.

Definition 2.7. A (singular) Kähler metric $\omega$ is called a conical metric with cone angle $2\pi \beta$ along $D$, if locally on any coordinates chart $U_a$, $\omega$ is equivalent to $\omega_\beta$ under the the coordinates $\{\zeta_{a,j}\}$, where $\omega_\beta$ is the standard cone metric (1.1) with cone angle $2\pi \beta$ along $\{\zeta_{a,j} = 0\}$, and on $X \setminus \cup_a U_a \omega$ is a smooth Kähler metric in the usual sense.

A conical metric $\omega$ is in $C^{0,\alpha}_\beta(X,D)$ if for each $a$, $\omega$ is $C^{0,\alpha}_\beta(U_a)$ and on $X \setminus \cup_a U_a \omega$ is smooth in the usual sense. Similarly we can define the $C^{\alpha/2}_\beta$-conical Kähler metrics on $X \times [0,1]$. 


Definition 2.8. A continuous function \( u \in C^0(X) \) is said to be in \( C^{0,\alpha}_\beta(X,D) \) if locally on each \( U_a \), \( u \) is in \( C^{0,\alpha}_\beta(U_a) \) and on \( X \setminus \cup_a U_a \) it is \( C^{0,\alpha} \)-continuous in the usual sense. We define the \( C^{0,\alpha}_\beta(X,D) \)-norm of \( u \) as
\[
\|u\|_{C^{0,\alpha}_\beta(X,D)} := \|u\|_{C^{0,\alpha}(X \setminus \cup_a U_a, \omega)} + \sum_a \|u\|_{C^{0,\alpha}_\beta(U_a)}.
\]

The \( C^{0,\alpha}_\beta(X,D) \)-norm depends on the choice of finite covers, and another cover yields a different but equivalent norm. The space \( C^{0,\alpha}_\beta(X,D) \) is clearly independent of the choice of finite covers.

2.2. A useful lemma

We will frequently use the following elementary estimates from [20].

Lemma 2.1 (Lemma 2.2 in [20]). Given \( r \in (0,1] \), suppose \( v \in C^0(B_\mathcal{C}(0,r)) \cap C^2(B_\mathcal{C}(0,r) \setminus \{0\}) \) satisfies the equation
\[
|z|^{2(1-\beta_1)} \frac{\partial^2 v}{\partial z \partial \bar{z}} = F, \quad \text{in} \ B_\mathcal{C}(0,r) \setminus \{0\},
\]
for some \( F \in L^\infty(B_\mathcal{C}(0,r)) \), then we have the pointwise estimate that for any \( z \in B_\mathcal{C}(0,9r/10) \setminus \{0\} \)
\[
|\frac{\partial v}{\partial z}(z)| \leq C \frac{\|v\|_{L^\infty}}{r} + C \|F\|_{L^\infty} \left\{ \begin{array}{ll}
|z|^{2\beta_1-1}, & \text{if } \beta_1 \in (1/2,1) \\
|\log(|z|/2r)|, & \text{if } \beta_1 = 1/2 \end{array} \right.
\]  
where the \( L^\infty \)-norms are taken in \( B_\mathcal{C}(0,r) \) and \( C > 0 \) is a uniform constant.

Finally we remark that the idea of the proof of the estimates in Theorems 1.1 and 1.3 is the same for general \( 2 \leq p \leq n \). To explain the argument clearer we prove the theorems assuming \( p = 2 \), i.e. the cone metric of \( \omega_\beta \) is singular along the two components \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \).

3. Elliptic estimates

In this section, we will prove Theorems 1.1 and 1.2, the Schauder estimates for the Laplace equation (1.2). To begin with, we first observe the simple \( C^0_\beta \)-estimate based on maximum principle. Suppose \( u \in C^2(B_\beta(0,1) \setminus \mathcal{S}) \cap C^0(B_\beta(0,1)) \) satisfies the equation
\[
\left\{ \begin{array}{ll}
\Delta_\beta u = 0, & \text{in} \ B_\beta(0,1) \setminus \mathcal{S}, \\
u = \varphi, & \text{on} \ \partial B_\beta(0,1)
\end{array} \right.
\]
for some \( \varphi \in C^0(\partial B_\beta(0,1)) \), then

Lemma 3.1. We have the following maximum principle,
\[
\inf_{\partial B_\beta(0,1)} \varphi \leq \inf_{B_\beta(0,1)} u \leq \sup_{B_\beta(0,1)} u \leq \sup_{\partial B_\beta(0,1)} \varphi.
\]

Proof. Consider the functions \( \bar{u}_\epsilon = u \pm \epsilon (\log |z_1|^2 + \log |z_2|^2) \) for any \( \epsilon > 0 \). By the same proof of Lemma 2.1 in [20], (3.2) is established.

Next step is to show the equation (3.1) is solvable for suitable boundary values.
3.1. Conical harmonic functions

3.1.1. Smooth approximating metrics. Let $\epsilon \in (0, 1)$ be a given small positive number and we define a smooth approximating Kähler metric on $B_\beta(0,1)$

$$g_\epsilon = \beta_1^2 \frac{-1dz_1 \wedge d\bar{z}_1}{(|z_1|^2 + \epsilon)^{1-\beta_1}} + \beta_2^2 \frac{-1dz_2 \wedge d\bar{z}_2}{(|z_2|^2 + \epsilon)^{1-\beta_2}} + \sum_{j=3}^{\infty} \frac{-1dz_j \wedge d\bar{z}_j}{|z_2|^2 + \epsilon}.$$  \hspace{1cm} (3.3)

g_\epsilon are product metrics on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-2}$. It is clear that its Ricci curvature satisfies

$$\text{Ric}(g_\epsilon) = \sqrt{-1} \partial \bar{\partial} \log \left( (|z_1|^2 + \epsilon)^{1-\beta_1} (|z_2|^2 + \epsilon)^{1-\beta_2} \right) \geq 0.$$

Let $u_\epsilon \in C^2(B_\beta(0,1))$ be the solution to the equation with a given $\varphi \in C^0(\partial B_\beta(0,1))$

$$\Delta_{g_\epsilon} u_\epsilon = 0, \text{ in } B_\beta(0,1), \text{ and } u_\epsilon = \varphi, \text{ on } \partial B_\beta(0,1).$$ \hspace{1cm} (3.4)

Note that the metric balls $B_\beta(0,1)$ and $B_{g_\epsilon}(0,1)$ are uniformly close when $\epsilon$ is sufficiently small, so for the following estimates we will work on $B_\beta(0,1)$.

Let $u_\epsilon$ be the harmonic function for $\Delta_{g_\epsilon}$ as in (3.4), which we may assume without of loss of generality is positive by replacing $u_\epsilon$ by $u_\epsilon - \inf u_\epsilon$ if necessary. We will study the Cheng-Yau-type gradient estimate of $u_\epsilon$ and the estimate of $\Delta_{g_\epsilon} u_\epsilon := (|z_1|^2 + \epsilon)^{1-\beta_1} \partial_{\bar{z}_1}^2 u_\epsilon$. Let us recall Cheng-Yau’s gradient estimate first.

In Subsections 3.1.2 - 3.1.5, for notation convenience, we will omit the subscript $\epsilon$ in $g_\epsilon$, $u_\epsilon$, in the proofs of lemmas.

3.1.2. Cheng-Yau gradient estimate revisit. We will also assume $u_\epsilon > 0$, otherwise consider the function $u_\epsilon + \delta$, for some $\delta > 0$ then letting $\delta \to 0$. We fix a ball metric $B_{g_\epsilon}(p,R) \subset B_\beta(0,1)$ centered at some point $p \in B_\beta(0,1)$. Since $\text{Ric}(g_\epsilon)$ \geq 0, the Cheng-Yau gradient estimate holds for $\Delta_{g_\epsilon}$-harmonic functions.

Lemma 3.2 ([10]). Let $u_\epsilon \in C^2(B(p,R))$ be a positive $\Delta_{g_\epsilon}$-harmonic function. There exists a uniform constant $C = C(n) > 0$ such that (the metric balls are taken under the metric $g_\epsilon$)

$$\sup_{x \in B(p,3R/4)} |\nabla u_\epsilon|_{g_\epsilon}(x) \leq C(n)^{\frac{\text{osc}_{B(p,R)} u_\epsilon}{R}}.$$ \hspace{1cm} (3.5)

As we mentioned above, we will omit the $\epsilon$ in the subscript of $u_\epsilon$ and $g_\epsilon$. The proof of the lemma is standard ([10]). For completeness and to motivate the proof of Lemmas 3.3 and 3.4 below, we provide a sketched proof. Define $f = \log u_\epsilon$, and it can be calculated that

$$\Delta f = \frac{\Delta u_\epsilon}{u_\epsilon} - \frac{|\nabla u_\epsilon|^2}{u_\epsilon^2} = -|\nabla f|^2.$$ \hspace{1cm} (3.6)

Then by Bochner formula, we have

$$\Delta |\nabla f|^2 = |\nabla \nabla f|^2 + |\bar{\nabla} \nabla f|^2 + 2 \text{Re}(\nabla f, \bar{\nabla} \Delta f) + \text{Ric}(\nabla f, \bar{\nabla} f) \geq |\nabla \nabla f|^2 - 2 \text{Re}(\nabla f, \bar{\nabla} |\nabla f|^2).$$ \hspace{1cm} (3.7)

Let $\phi : [0,1] \to [0,1]$ be a standard cut-off function such that $\phi_{[0,3/4]} = 1$ and $\phi_{[5/6,1]} = 0$ and between 0,1 otherwise. let $r(x) = d_{g_\epsilon}(p,x)$ be the distance function to $p$ under the metric $g = g_\epsilon$. By abusing notation we also write $\phi(x) = \phi \left( \frac{r(x)}{R} \right)$. It can be calculated by Laplacian comparison and the Bochner formula (3.7) that at the (positive) maximum point $p_{\max}$ of $H := \phi^2 |\nabla f|^2$ that

$$\frac{2}{n} H^2 - \frac{4|\phi'|^2}{R} H^{3/2} - \frac{8(\phi')^2}{R^2} H + \frac{2H}{R^2} \left( (2n-1) \phi \phi' + \phi \phi'' + (\phi')^2 \right) \leq 0,$$
Let \( \varphi \) can be checked from the definitions by the property that define \( \varphi \)\( \epsilon \), \( \Delta \) under the same assumptions as in Lemma 3.2, along the “bad” directions. We want to estimate the upper bound of \( G \).

\[ G \]

\[ 3.1.3. \]

As in the proof of Cheng-Yau gradient estimates, we will work on the function \( n \), \( 1 \), \( \epsilon \), \( \Delta \) such that \( \varphi \) satisfies the estimates

\[ \max_{x \in B(p, R/2)} \frac{\partial^2 f}{\partial z_1 \partial z_1} \frac{\partial^2 f}{\partial z_2 \partial z_2} + \sum_j \frac{\partial^2 f}{\partial z_j \partial z_j} \]

\[ = (\Delta f)^2 + 2 \Re \langle \nabla f, \nabla \Delta f \rangle. \]

Then it follows that

\[ \Delta (-\Delta f) = -\Delta \Delta f = \Delta |\nabla f|^2 \geq (\Delta f)^2 + 2 \Re \langle \nabla f, \nabla \Delta f \rangle. \]

Let \( \varphi : [0, 1] \rightarrow [0, 1] \) be a standard cut-off function such that \( \varphi|_{[0, 1/2]} = 1 \) and \( \varphi|_{[2/3, 1]} = 0 \). We also define \( \varphi(x) = \frac{\varphi(x)}{R} \). Then consider the function \( G := \varphi^2 \cdot (-\Delta f) \). We calculate

\[ \Delta G = \Delta (\varphi^2(-\Delta f)) \]

\[ = \varphi^2 \Delta (-\Delta f) + 2 \Re \langle \nabla \varphi^2, \nabla (-\Delta f) \rangle - (-\Delta f) \Delta \varphi^2 \]

\[ \geq \varphi^2 ((\Delta f)^2 + 2 \Re \langle \nabla f, \nabla \Delta f \rangle + 2 \Re \langle \nabla \varphi^2, \nabla (-\Delta f) \rangle + (-\Delta f) \Delta \varphi^2. \]

We want to estimate the upper bound of \( G \). If the maximum value of \( G = \varphi^2(-\Delta f) \) is negative, we are done. So we assume the maximum of \( G \) on \( B(p, R) \) is positive, which is achieved at some
point \( p_{\max} \in B(p, 2R/3) \). Hence at \( p_{\max} \), we have \((-\Delta_1 f) > 0 \). By Laplacian comparison that \( \Delta r \leq \frac{2n-1}{r} \), we get at \( p_{\max} \),
\[
\Delta \varphi^2 \geq \frac{2}{R^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right).
\]
(3.13)
Thus at \( p_{\max} \), the last term on RHS of (3.12) is
\[
\geq (-\Delta_1 f) \frac{2}{R^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right).
\]
Substituting this into (3.12), it follows that at \( p_{\max} \), \( \Delta G \leq 0 \) and \( \nabla \Delta_1 f = -\varphi \Delta_1 f \nabla \varphi \) and
\[
0 \geq \Delta G \geq \varphi^2 (\Delta_1 f)^2 + 2\varphi^2 \Re(\nabla f, \nabla \Delta_1 f) + 4\varphi \Re(\nabla f, \nabla (-\Delta_1 f))
\]
\[
+ (-\Delta_1 f) \frac{2}{R^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right)
\]
\[
\geq \varphi^2 (\Delta_1 f)^2 - 4\varphi|\Delta_1 f||\nabla f||\nabla \varphi| + 8\Delta_1 f|\nabla \varphi|^2 + (-\Delta_1 f) \frac{2}{R^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right)
\]
\[
gamma \geq \frac{G^2}{\varphi^2} - 4\varphi^{-1}G|\nabla f||\nabla \varphi| - 8\varphi G|\nabla \varphi|^2 + 2G \frac{\nabla \varphi}{R^2\varphi^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right)
\]
\[
\geq \frac{G^2}{\varphi^2} - 4\varphi G|\nabla f| G - 8\varphi G|\nabla \varphi|^2 G + 2\frac{\nabla \varphi}{R^2\varphi^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right).
\]
Therefore at \( p_{\max} \in B(p, 2R/3) \), it holds that
\[
G^2 - 4\frac{|\nabla f| G}{\varphi} - 8\frac{|\nabla \varphi|^2 G}{R^2} + 2\frac{\nabla \varphi}{R^2\varphi^2} \left( (2n-1)\varphi' + \varphi'' + (\varphi')^2 \right) \leq 0,
\]
combining (3.8) and the fact that \( \varphi, \varphi', \varphi'' \) are all uniformly bounded, we can get at \( p_{\max} \)
\[
G^2 \leq C(n) R^{-2} G \quad \Rightarrow \quad G(p_{\max}) \leq \frac{C(n)}{R^2}.
\]
Then for any \( x \in B(p, R/2) \), where \( \varphi = 1 \), we have
\[
-\Delta_1 f(x) = G(x) \leq G(p_{\max}) \leq \frac{C(n)}{R^2}.
\]
Moreover, recall that \( f = \log u \) and \( -\Delta_1 f = -\frac{\Delta u}{u} + |\nabla_1 f|^2 \), therefore it follows that
\[
\sup_{x \in B(p, R/2)} \left( -\frac{\Delta_1 u}{u}(x) \right) \leq \frac{C(n)}{R^2}.
\]
(3.15)
This in particular implies that
\[
\sup_{x \in B(p, R/2)} \left( -\frac{\Delta_1 u}{u}(x) \right) \leq C(n) \frac{\osc_{B(p, R/2)} u}{R^2} \leq C(n) \frac{\osc_{B(p, R)} u}{R^2}.
\]
(3.16)
On the other hand, by considering the function \( \hat{u} = \max_{B(p, R)} u - u \), which is still a positive \( g_e \)-harmonic function \( \Delta_g \hat{u} = \Delta_g \hat{u} = 0 \). Applying (3.15) to the function \( \hat{u} \), we get
\[
\sup_{x \in B(p, R/2)} \left( \frac{\Delta_1 u(x)}{\max_{B(p, R)} u - u(x)} \right) = \sup_{x \in B(p, R/2)} \left( -\frac{\Delta_1 \hat{u}}{\hat{u}}(x) \right) \leq \frac{C(n)}{R^2}
\]
(3.17)
which yields that
\[
\sup_{x \in B(p, R/2)} \Delta_1 u(x) \leq C(n) \frac{\osc_{B(p, R/2)} u}{R^2}.
\]
(3.18)
Combining (3.18) and (3.16), we get
\[
\sup_{x \in B(p, R/2)} |\Delta_1 u|(x) \leq C(n) \frac{\text{Osc}_{B(p, R)} u}{R^2}. \tag{3.19}
\]

### 3.1.4. Mixed derivatives estimates

In this subsection, we will estimate the following mixed derivatives
\[
|\nabla_1 \nabla_2 f|^2 = \frac{\partial^2 f}{\partial z_1 \partial z_2} \frac{\partial^2 f}{\partial \bar{z}_1 \partial \bar{z}_2} g^{11} g^{22}, \quad |\nabla_1 \nabla_2 f|^2 = \frac{\partial^2 f}{\partial z_1 \partial \bar{z}_2} \frac{\partial^2 f}{\partial \bar{z}_1 \partial z_2} g^{11} g^{22},
\]
where as before \( f = \log u \) and \( u \) is a positive harmonic function of \( \Delta g_r \). Here for simplicity, we omit the subscript \( \epsilon \) in \( u_\epsilon, f_\epsilon \) and \( g_\epsilon \). Observing that since \( g_\epsilon = g \) is a product metric with the non-zero components \( g_{kk} \) depending only on \( z_k \), it follows that the curvature tensor
\[
R_{ijkl} = -\frac{\partial^2 g_{ij}}{\partial z_k \partial z_l} + g^{pq} \frac{\partial g_{ij}}{\partial z_p} \frac{\partial g_{pq}}{\partial z_l}
\]
vanishes unless \( i = j = k = l = 1 \) or 2, and also \( R_{iiij} \geq 0 \) for all \( i = 1, \ldots, n \).

We fix some notations: we will write \( f_{12} = \nabla_1 \nabla_2 f \) (in fact this is just the ordinary derivative of \( f \) w.r.t. \( g \), since \( g \) is a product metric), \( |f_{12}|^2 = |\nabla_1 \nabla_2 f|^2 \), etc.

Let us first recall that the equation (3.11) implies
\[
\Delta(-\Delta_1 f - \Delta_2 f) = \sum_{k=1}^{n} \left( g^{11} g^{kk} f_{1k} f_{1k} + g^{12} g^{kk} f_{1k} f_{2k} + g^{22} g^{kk} f_{2k} f_{2k} \right)
- 2 \text{Re}(\nabla f, \nabla (-\Delta_1 f - \Delta_2 f)) + f_1 f_1 g^{11} g^{11} R_{1111} + f_2 f_2 g^{22} g^{22} R_{2222} \tag{3.20}
\geq \sum_{k=1}^{n} \left( |\nabla_1 \nabla_k f|^2 + |\nabla_1 \nabla_k f|^2 + |\nabla_2 \nabla_k f|^2 + |\nabla_2 \nabla_k f|^2 \right)
- 2 \text{Re}(\nabla f, \nabla (-\Delta_1 f - \Delta_2 f)).
\]

Next we calculate \( \Delta|\nabla_1 \nabla_2 f|^2 \). For notation convenience we will write \( f_{12} = f_{12} g^{11} g^{22} \), and hence \( |\nabla_1 \nabla_2 f|^2 = f_{12} f_{12} \). We calculate
\[
\Delta|\nabla_1 \nabla_2 f|^2 = g^{kk} (f_{12} f_{12})_{kk} = g^{kk} (f_{12} f_{12})_{kk} (\text{since } g \text{ is a product metric})
= g^{kk} (f_{12k} f_{12}^{12} + f_{12k} f_{12}^{12} + f_{12k} f_{12}^{12} + f_{12k} f_{12}^{12}). \tag{3.21}
\]

The first term on the RHS of (3.21) is (by Ricci identities and switching the indices)
\[
g^{kk} f_{12} \left( f_{kk12} + g^{mn} f_{m1} R_{km2k} + g^{mn} f_{km} R_{1m2k} \right)
= g^{kk} f_{12} \left( f_{kk12} + g^{mn} f_{m2} R_{km1k} + g^{mn} f_{m1} R_{km1k} + g^{mn} f_{m1} R_{km2k} + g^{mn} f_{km} R_{1m2k} \right) \tag{3.22}
\]
and the last term on the RHS of (3.21) is the conjugate of the first term, hence we get
\[
\Delta|\nabla_1 \nabla_2 f|^2 = 2 \text{Re}(f_{12} (\Delta f)_{12}) + 2 f_{12} f_{12} (g^{11} g^{11} R_{1111} + g^{22} g^{22} R_{2222})
+ g^{kk} f_{12k} f_{12}^{12} + g^{kk} f_{12k} f_{12}^{12}. \tag{3.23}
\]
Recall from (3.6) we have \( \Delta f = -\|
abla f\|^2 \), hence the first term on RHS of (3.23) is
\[
2\text{Re}(f^{12}(\Delta f)_{12}) = 2\text{Re}(f^{12}(-\|
abla f\|^2)_{12}) = -2\text{Re}(f^{12}g^{kk}(f_{k1}f_{k2} + f_{k1}f_{k2} + f_{k2}f_{k1} + f_{k2}f_{k1})) = -2\text{Re}(f^{12}g^{kk}(f_{k1}f_{k2} + f_{k2}f_{k1} + f_{k2}f_{k1} - f_{k2}f_{k1} - f_{k2}f_{k1} - f_{k2}f_{k1} - f_{k2}f_{k1})) = -4\text{Re}(\nabla f, \nabla|\nabla_1 \nabla_2 f|^2) - 2\text{Re}(f^{12}g^{kk}f_{k1}f_{k2} + f^{12}g^{kk}f_{k2}f_{k1})
\]
Combining (3.24) and (3.23), we get
\[
\Delta|\nabla_1 \nabla_2 f|^2 \geq -4\text{Re}(\nabla f, \nabla|\nabla_1 \nabla_2 f|^2) + \sum_k (f_{12k}f^{12k} + f_{12k}f^{12k})
\]
On the other hand we have by Kato’s inequality
\[
\Delta|\nabla_1 \nabla_2 f|^2 = 2|\nabla_1 \nabla_2 f|\Delta|\nabla_1 \nabla_2 f| + 2|\nabla_1 \nabla_2 f|^2 \\
\leq 2|\nabla_1 \nabla_2 f|\Delta|\nabla_1 \nabla_2 f| + \sum_k |\nabla_k \nabla_1 \nabla_2 f|^2 + |\nabla_k \nabla_1 \nabla_2 f|^2
\]
Combining (3.25) and (3.26) it follows that
\[
\Delta|\nabla_1 \nabla_2 f| \geq -2\text{Re}(\nabla f, \nabla|\nabla_1 \nabla_2 f|) - \sum_k (|\nabla_1 \nabla_k f||\nabla_2 \nabla_k f| + |\nabla_2 \nabla_k f||\nabla_1 \nabla_k f|).
\]
Combining (3.20), (3.27) and applying Cauchy-Schwarz inequality, we have
\[
\Delta(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f)) \\
\geq -2\text{Re}(\nabla f, \nabla(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))) \\
+ \sum_n (|\nabla_1 \nabla_k f|^2 + |\nabla_1 \nabla_k f|^2 + |\nabla_2 \nabla k f|^2 + |\nabla_2 \nabla k f|^2).
\]
Combining (3.29) and (3.27) it is (recall under our notation |\nabla_1 \nabla_1 f|^2 = (\Delta_1 f)^2)
\[
\geq |\nabla_1 \nabla_2 f|^2 + |\Delta_1 f|^2 + |\Delta_2 f|^2 \geq \frac{1}{12}(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))^2,
\]
so we get the following equation
\[
\Delta(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f)) \\
\geq -2\text{Re}(\nabla f, \nabla(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))) \\
+ \frac{1}{12}(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f))^2.
\]
Denote \( Q = \eta^2(|\nabla_1 \nabla_2 f| + 2(-\Delta_1 f - \Delta_2 f)) =: \eta^2 Q_1 \), where \( \eta(x) = \tilde{\eta}(r(x)/R) \) and \( \tilde{\eta} \) is a cut-off function such that \( \tilde{\eta}|_{[0,1/3]} = 1 \) and \( \tilde{\eta}|_{[1/2,1]} = 0 \). The following arguments are similar to the previous
two cases. We calculate
\[ \Delta Q = \eta^2 \Delta Q_1 + 2 \text{Re}(\nabla \eta^2, \nabla Q_1) + Q_1 \Delta \eta^2 \]
\[ \geq -2\eta^2 \text{Re}(\nabla f, \overline{\nabla \eta}) + 2 \text{Re}(\nabla \eta^2, \nabla Q_1) + \eta^2 \frac{Q_1^2}{12} + Q_1 \Delta \eta^2. \] (3.30)

Apply maximum principle to \( Q \) and if the max \( Q \leq 0 \), we are done. So we may assume that \( \max Q > 0 \) and is attained at \( p_{\text{max}} \), thus at \( p_{\text{max}} \), \( Q_1 > 0 \), \( \Delta Q \leq 0 \), \( \nabla Q_1 = -2\eta^{-1} Q_1 \nabla \eta \) and
\[ Q_1 \Delta \eta^2 \geq Q_1 \frac{2}{R^2} ((2n-1)\eta \eta' + \eta \eta'' + (\eta')^2). \]

So at \( p_{\text{max}} \) it holds that
\[ 0 \geq \Delta Q \]
\[ \geq 4 \eta Q_1 \text{Re}(\nabla f, \overline{\nabla \eta}) - 8 Q_1 |\nabla \eta|^2 + \eta^2 \frac{Q_1^2}{12} + Q_1 \frac{2}{R^2} ((2n-1)\eta \eta' + \eta \eta'' + (\eta')^2) \]
\[ = \frac{Q_1^2}{12} + 4 \eta \text{Re}(\nabla f, \overline{\nabla \eta}) - 8 Q_1 \eta (\eta')^2 + \frac{2 Q_1}{R^2 \eta^2} ((2n-1)\eta \eta' + \eta \eta'' + (\eta')^2) \]
\[ \geq \frac{1}{\eta} \left( \frac{Q_1^2}{12} - \frac{40}{R^2} Q - \frac{800}{\eta^2 R^2} Q - \frac{100}{\eta^2 R^2} Q \right) \]
where we choose \( \eta \) such that \( |\eta'|, |\eta''| \leq 10 \), for example. Therefore at \( p_{\text{max}} \in B(p, R/2) \) we have
\[ \frac{Q_1^2}{12} - Q\left(\frac{40}{R^2} + \frac{800}{\eta^2 R^2} + \frac{100}{\eta^2 R^2}\right) \leq 0 \Rightarrow Q(p_{\text{max}}) \leq \frac{C(n)}{R^2}, \]

since \( \sup_{B(p, R/2)} |\nabla f| \leq C(n) R^{-1} \) from the previous estimates. Then for any \( x \in B(p, R/3) \) we have
\[ Q_1(x) = \eta^2(x) Q_1(x) = Q(x) \leq Q(p_{\text{max}}) \leq \frac{C(n)}{R^2}. \]

Thus it follows that
\[ |\nabla_1 \nabla_2 f|(x) \leq Q_1(x) + 2(\Delta_1 f(x) + \Delta_2 f(x)) \leq \frac{C(n)}{R^2} + 2(\Delta_1 f(x) + \Delta_2 f(x)). \]

On the other hand from \( |\nabla_1 \nabla_2 f| = |\nabla_1 \nabla_2 u - \nabla_1 u \nabla_2 u| \)
\[ |\nabla_1 \nabla_2 u|(x) \leq |\nabla_1 \nabla_2 u|(x) \leq \frac{|\nabla_1 u(x)|}{u} \frac{|\nabla_2 u(x)|}{u} \]
\[ \leq C(n) \frac{u(x)}{R^2} + 2 \Delta_1 u(x) + 2 \Delta_2 u(x) + u(x) \frac{|\nabla_1 u(x)|}{u} \frac{|\nabla_2 u(x)|}{u} \]
\[ \leq C(n) \frac{\text{osc}_{B(p, R)}}{R^2}. \]

Therefore we obtain that
\[ \sup_{B(p, R/3)} |\nabla_1 \nabla_2 u| \leq C(n) \frac{\text{osc}_{B(p, R)}}{R^2}. \]

(3.33)

By exactly the same argument we can also get similar estimates for \( |\nabla_1 \nabla_2 u| \) and \( |\nabla_1 \nabla_k u| \) for \( k \neq 1 \).

Hence we have proved the following lemma:

**Lemma 3.4.** There exists a constant \( C(n) > 0 \) such that for the solution \( u_\epsilon \) to the equation (3.4)
\[ \sup_{B_{g_k}(0, R/2)} \left( |\nabla_1 \nabla_j u_\epsilon|_{g_k} + |\nabla_1 \nabla_j u_\epsilon|_{g_k} \right) \leq C(n) \frac{\text{osc}_{B_{g_k}(0, R)}}{R^2}, \]
for all \( i, j = 1, 2, \cdots, n \).

### 3.1.5. Convergence of \( u_\varepsilon \)

In this section, we will show the Dirichlet problem (3.1) admits a unique solution for any \( \varphi \in C^0(\partial B_\beta(0,1)) \). We will write \( B_\beta = B_\beta(0,1) \) for notation simplicity in this subsection.

**Proposition 3.1.** The Dirichlet boundary value problem (3.1) admits a unique solution \( u \in C^2(B_\beta \setminus \mathcal{S}) \cap C^0(\overline{B_\beta}) \) for any \( \varphi \in C^0(\partial B_\beta) \). Moreover, \( u \) satisfies the estimates in Lemmas 3.2, 3.3 and 3.4 with \( u_\varepsilon \) replaced by \( u \) and the metric balls replaced by those under the metric \( g_\beta \), which we will refer as “derivatives estimates” throughout this section.

**Proof.** Given the estimates of \( u_\varepsilon \) as in lemmas 3.2, 3.3 and 3.4, we can derive the uniform local \( C^{2,\alpha} \) estimates of \( u_\varepsilon \) on any compact subsets of \( B_\beta(0,1) \setminus \mathcal{S} \).

The \( C^0 \) estimates of \( u_\varepsilon \) follow immediately from the maximum principle (see Lemma 3.1).

Take any compact subsets \( K \subset K' \subset B_\beta(0,1) \). By Lemma 3.3, we have

\[
\sup_{K'} \left( |z_1|^{1-\beta_1} \left| \frac{\partial u_\varepsilon}{\partial z_1} \right| + |z_2|^{1-\beta_2} \left| \frac{\partial u_\varepsilon}{\partial z_2} \right| + \left| \frac{\partial^2 u_\varepsilon}{\partial s_j} \right| \right) \leq C(n) \frac{\|u_\varepsilon\|_{\infty}}{d(K',\partial B_\beta)}, \tag{3.34}
\]

and the third-order estimates

\[
\sup_{K'} \left( |z_1|^{1-\beta_1} \left| \frac{\partial^3 u_\varepsilon}{\partial z_1 \partial s_k \partial s_l} \right| + |z_2|^{1-\beta_2} \left| \frac{\partial^3 u_\varepsilon}{\partial z_2 \partial s_k \partial s_l} \right| + \left| \frac{\partial^3 u_\varepsilon}{\partial s_j \partial s_k \partial s_l} \right| \right) \leq C(n) \frac{\|u_\varepsilon\|_{\infty}}{(K',\partial B_\beta)^3}. \tag{3.36}
\]

Moreover, applying the gradient estimate to the \( \Delta_\beta \)-harmonic function \( \Delta_{\varepsilon,1} u_\varepsilon \), we get

\[
\sup_{K'} \left( |z_1|^{1-\beta_1} \left| \frac{\partial}{\partial z_1} \Delta_{\varepsilon,1} u_\varepsilon \right| + |z_2|^{1-\beta_2} \left| \frac{\partial}{\partial z_2} \Delta_{\varepsilon,1} u_\varepsilon \right| + \left| \frac{\partial^2 u_\varepsilon}{\partial s_j} \right| \right) \leq C(n) \frac{\|u_\varepsilon\|_{\infty}}{d(K',\partial B_\beta)^3}. \tag{3.37}
\]

From (3.34), (3.35) and (3.36), we see that the functions \( u_\varepsilon \) have uniform \( C^3 \) estimates in the “tangential directions” on any compact subset of \( B_\beta(0,1) \). Moreover, for any fixed small constant \( \delta > 0 \), let \( T_\delta(\mathcal{S}) \) be the tubular neighborhood of \( \mathcal{S} \). We consider the equation

\[
\Delta_\varepsilon u_\varepsilon = (|z_1|^2 + \varepsilon)^{1-\beta_1} \frac{\partial^2 u_\varepsilon}{\partial z_1 \partial \bar{z}_1} + (|z_2|^2 + \varepsilon)^{1-\beta_2} \frac{\partial^2 u_\varepsilon}{\partial z_2 \partial \bar{z}_2} + \sum_{j=5}^{2n} \frac{\partial^2 u_\varepsilon}{\partial s_j^2} = 0, \quad \text{on } K' \setminus T_{\delta/2}(\mathcal{S}),
\]

which is strictly elliptic (with ellipticity depending only on \( \delta > 0 \)). Hence by standard elliptic Schauder theory, we also have \( C^{2,\alpha} \)-estimates of \( u_\varepsilon \) in the “transversal directions” (i.e. normal to \( \mathcal{S} \)) and the mixed directions, on the compact subset \( K \setminus T_\delta(\mathcal{S}) \). By taking \( \delta \to 0 \), \( K \to B_\beta \), and a diagonal argument, up to a subsequence \( u_\varepsilon \) converge in \( C^2_{loc}(B_\beta \setminus \mathcal{S}) \) to a function \( u \in C^{2,\alpha}(B_\beta \setminus \mathcal{S}) \). Clearly, \( u \) satisfies the equation \( \Delta_\beta u = 0 \) on \( B_\beta \setminus \mathcal{S} \), and the estimates (3.34), (3.35) and (3.36) hold for \( u \) outside \( \mathcal{S} \), which implies that \( u \) can be continuously extended through \( \mathcal{S} \) and defines a continuous function in \( B_\beta(0,1) \). It remains to check the boundary value of \( u \).

**Claim:** \( u = \varphi \) on \( \partial B_\beta(0,1) \)

It remains to show the limit function \( u \) of \( u_\varepsilon \) satisfies the boundary condition \( u = \varphi \) on \( \partial B_\beta(0,1) \), which will be proved by constructing suitable barriers as we did in [20].

The metric ball \( B_\beta(0,1) \) is given by

\[
B_\beta(0,1) = \{ z \in \mathbb{C}^n | d_\beta(0, z)^2 := |z_1|^{2\beta_1} + |z_2|^{2\beta_2} + \sum_{j=5}^{2n} s_j^2 < 1 \}.
\]
$B_\beta(0,1) \subset B_{C^n}(0,1)$ and their boundaries only intersect at $S_1 \cap S_2$, where $z_1 = z_2 = 0$. Fix any point $q \in \partial B_\beta(0,1)$ and we consider the cases when $q \in S_1 \cap S_2$ or $q \not\in S_1 \cap S_2$.

**Case 1:** $q \in S_1 \cap S_2$, i.e. $z_1(q) = z_2(q) = 0$. Consider the point $q' = -q \in \partial B_\beta(0,1) \cap \partial B_{C^n}(0,1)$, and $q$ is the unique farthest point to $q'$ on $\partial B_\beta(0,1)$ under the Euclidean distance, hence the function $\Psi_q(z) := d_{C^n}(z,q)^2 - 4$ satisfies $\Psi_q(q) = 0$ and $\Psi_q(z) < 0$ for all $z \in \partial B_\beta(0,1) \setminus \{q\}$. By the continuity of $\varphi$ for any $\delta > 0$, there is a small neighborhood $V$ of $q$ such that $\varphi(z) - \delta < \varphi(z) < \varphi(q) + \delta$ for all $z \in \partial B_\beta(0,1) \cap V$ and on $\partial B_\beta(0,1) \setminus V$, $\Psi_q$ is bounded above by a negative constant. Hence we can make $\varphi_q(z) := \varphi(q) - \delta + A\Psi_q(z) < \varphi(z)$ for all $z \in \partial B_\beta(0,1)$ if $A$ is chosen large enough. The function $\varphi_q$ is $\Delta_{g_q}$-subharmonic hence by maximum principle we have $u_\epsilon(z) \geq \varphi_q(z)$ for all $z \in B_\beta(0,1)$. Letting $\epsilon \to 0$ we get $u(z) \geq \varphi_q(z)$. Then taking $z \to q$ it follows that $\lim \inf_{z \to q} u(z) \geq \varphi(q) - \delta$, but $\delta > 0$ is arbitrary so we have $\lim \inf_{z \to q} u(z) \geq \varphi(q)$.

By considering the barrier function $\varphi(q) + \delta - A\Psi_q(z)$ and similar argument it is not hard to see that $\lim \sup_{z \to q} u(z) \leq \varphi(q)$, hence $\lim \inf_{z \to q} u(z) = \varphi(q)$ and $u$ is continuous up to $q \in \partial B_\beta(0,1)$.

**Case 2:** $q \in \partial B_\beta(0,1) \setminus S_1 \cap S_2$. We consider the case when $z_1(q) \neq 0$ and $z_2(q) \neq 0$. The boundary $\partial B_\beta(0,1)$ is smooth near $q$, hence satisfies the exterior sphere condition. We choose an exterior Euclidean ball $B_{C^n}(\tilde{q},r_q)$ which is tangential with $\partial B_\beta(0,1)$ (only) at $q$, i.e. under the Euclidean distance $q$ is the unique closest point to $\tilde{q}$ on $\partial B_\beta(0,1)$. So the function $G(z) = \frac{|z - \tilde{q}|^2}{r_q^2}$ satisfies $G(q) = 0$ and $G(z) < 0$ for all $z \in \partial B_\beta(0,1) \setminus \{q\}$. We calculate

$$
\Delta_{g_q} G = (|z_1|^2 + \epsilon)^{-\beta_1+1} \frac{\partial^2 G}{\partial z_1 \partial \bar{z}_1} + (|z_2|^2 + \epsilon)^{-\beta_2+1} \frac{\partial^2 G}{\partial z_2 \partial \bar{z}_2} + \sum_{k=3}^n \frac{\partial^2 G}{\partial z_k \partial \bar{z}_k} \\
= ((|z_1|^2 + \epsilon)^{-\beta_1+1} - 1) \frac{\partial^2 G}{\partial z_1 \partial \bar{z}_1} + ((|z_2|^2 + \epsilon)^{-\beta_2+1} - 1) \frac{\partial^2 G}{\partial z_2 \partial \bar{z}_2} \\
= \sum_{k=1}^2 (-n+1)(|z_k|^2 + \epsilon)^{-\beta_k+1} - 1 \left( - \frac{n|z_k - \tilde{q}_k|^2}{|z - \tilde{q}|^{2n}} + 1 \right) \\
\geq -C(q,r_q).
$$

The function $\Psi_q(z) = A(d_{\beta}(z,0)^2 - 1) + G(z)$ is $\Delta_{g_q}$-subharmonic for $A > 1$ and $\Psi_q(q) = 0$, $\Psi_q(z) < 0$ for $\forall z \in \partial B_\beta(0,1) \setminus \{q\}$. We are in the same situation as the Case 1, so by the same argument as above, we can show the continuity of $u$ at such boundary point $q$.

In case $z_1(q) \neq 0$ but $z_2(q) = 0$. The boundary $\partial B_\beta(0,1)$ is not smooth at $q$ and we cannot apply the exterior sphere condition to construct the barrier. Instead we will use the geometry of the metric ball $B_\beta(0,1)$. Consider the standard cone metric $g_{\beta_1} = \beta_1^2 \frac{dz_k \otimes d\bar{z}_k}{|z_1|^2 + \epsilon} + \sum_{k=2}^n dz_k \otimes d\bar{z}_k$ with cone singularity only along $S_1 = \{z_1 = 0\}$. The metric ball $B_\beta(0,1)$ is strictly contained in $B_{g_{\beta_1}}(0,1)$, and their boundaries are tangential at the points with vanishing $z_2$-coordinate. Thus $q \in \partial B_\beta(0,1) \cap \partial B_{g_{\beta_1}}(0,1)$ and $\partial B_{g_{\beta_1}}(0,1)$ is smooth at $q$ so there exists an exterior sphere for $\partial B_{g_{\beta_1}}(0,1)$ at $q$. We define similar function $G(z)$ as in the last paragraph, and by the strict inclusion of the metric balls $B_\beta(0,1) \subset B_{g_{\beta_1}}(0,1)$, it follows that $G(q) = 0$ and $G(z) < 0$ for all $z \in \partial B_\beta(0,1) \setminus \{q\}$. The remaining argument is the same as before.

\[ \square \]

**Remark 3.1.** For any constant $c \in \mathbb{R}$, the following Dirichlet boundary value problem

$$
\Delta_{g_q} u = c, \quad \text{in } B_\beta(0,1) \setminus S, \quad \text{and } u = \varphi, \quad \text{on } \partial B_\beta(0,1),
$$

where $\varphi$ is the barrier function we defined above, is uniformly solvable for every $c > 0$. Moreover, the solution $u$ is continuous up to the boundary $\partial B_\beta(0,1)$.
admits a solution $u \in C^2(B_\beta \setminus \mathcal{S}) \cap C^0(\partial B_\beta)$ for any given $\varphi \in C^0(\partial B_\beta)$. This follows from the solution $\tilde{u}$ of (3.1) with the boundary value $\tilde{\varphi} = \varphi - \frac{c}{2(n-2)} \sum_{j=5}^{2n} s_j^2$. Then the function $u = \tilde{u} + \frac{c}{2(n-2)} \sum_{j=5}^{2n} s_j^2$ solves the equation above.

For later application, we prove the existence of solution for a more general RHS of the Laplace equation with the standard background metric. Note that this result is not needed in the proof of Theorem 1.1.

**Proposition 3.2.** For any given $\varphi \in C^0(\partial B_\beta(0,1))$ and $f \in C^0_{\beta}(B_\beta(0,1))$, the Dirichlet boundary value problem

$$
\begin{align*}
\Delta_{g_\beta} v &= f, \text{ in } B_\beta(0,1) \setminus \mathcal{S}, \\
v &= \varphi, \text{ on } \partial B_\beta(0,1)
\end{align*}
$$

(3.38)

admits a unique solution $v \in C^2(B_\beta(0,1) \setminus \mathcal{S}) \cap C^0(\partial B_\beta(0,1))$.

By Theorem 1.1, the solution $v$ to (3.38) belongs to $C^2_{\beta}(B_\beta(0,1)) \cap C^0(\overline{B_\beta(0,1)})$.

**Proof.** The proof is similar to that of Proposition 3.1. As before let $g_\epsilon$ be the approximating metrics (3.3) of $g_\beta$ which are smooth metrics on $B_\beta(0,1)$. By standard elliptic theory we can solve the equations

$$
\begin{align*}
\Delta_{g_\epsilon} v_\epsilon &= f, \text{ in } B_\beta(0,1), \\
v_\epsilon &= \varphi, \text{ on } \partial B_\beta(0,1).
\end{align*}
$$

(3.39)

For any compact subset $K \subseteq B_\beta(0,1)$ and small $\delta > 0$, we have uniform $C^{2,\alpha'}$-bound of $v_\epsilon$ on $K \setminus T_\delta(S)$ for some $\alpha' < \alpha$. Thus $v_\epsilon$ converges in $C^{2,\alpha'}$-norm to a function $v$ on $K \setminus T_\delta(S)$, as $\epsilon \to 0$. By a standard diagonal argument, setting $K \to B_\beta(0,1)$ and $\delta \to 0$, we can achieve that

$$
v_\epsilon \to_{C^{2,\alpha'}_{\text{loc}}(B_\beta(0,1) \setminus \mathcal{S})} v \in C^{2,\alpha'}_{\text{loc}}(B_\beta(0,1) \setminus \mathcal{S}), \text{ as } \epsilon \to 0.
$$

And clearly $v$ satisfies the equation (3.38) in $B_\beta(0,1) \setminus \mathcal{S}$. It only remains to show the boundary value of $v$ coincides with $\varphi$ and $v$ is globally continuous in $B_\beta(0,1)$.

- $v \in C^0(\overline{B_\beta(0,1)})$. It suffices to show $v$ is continuous at any $p \in \mathcal{S} \cap B_\beta(0,1)$. Fix such a point $p$ and take $R_0 > 0$ small so that $B_{C_\omega}(p, 10R_0) \cap \partial B_\beta(0,1) = \emptyset$. We observe that $\frac{1}{2} g_{C_\omega} \leq g_\epsilon \leq g_\beta$, so for any $r \in (0, 1/2)$

$$
B_{g_\beta}(p, r) \subset B_{g_\epsilon}(p, r) \subset B_{C_\omega}(p, 2r),
$$

(3.40)
in particular, the balls $B_{g_\beta}(p, 5R_0)$ are also disjoint with $\partial B_\beta(0,1)$.

Since $\text{Ric}(g_\epsilon) \geq 0 \geq 0$ we have the following Sobolev inequality ([25]): there exists a constant $C = C(n) > 0$ such that for any $h \in C^1_{0}(B_{g_\epsilon}(p, r))$, it holds that

$$
\left( \int_{B_{g_\epsilon}(p, r)} h^{\frac{2n}{n-2}} \omega_\epsilon^n \right)^{\frac{n-1}{n}} \leq C \left( \frac{\int_{B_{g_\epsilon}(p, r)} |\nabla h|^2 h^{\frac{2n}{n-2}} \omega_\epsilon^n}{\text{Vol}_{g_\epsilon}(B_{g_\epsilon}(p, r))} \right)^{1/n} \int_{B_{g_\epsilon}(p, r)} |\nabla h|^2 h^{\frac{2n}{n-2}} \omega_\epsilon^n.
$$

(3.41)

It can be checked by straightforward calculations that $\text{Vol}_{g_\epsilon}(B_{g_\epsilon}(p, 1)) \geq c_0(n) > 0$ for some constant $c_0$ independent of $\epsilon$. Then Bishop volume comparison yields that for any $r \in (0, 1)$,

$$
C_1(n) r^{2n} \geq \text{Vol}_{g_\epsilon}(B_{g_\epsilon}(p, r)) \geq c_1(n) r^{2n}.
$$

Thus the Sobolev inequality (3.41) is reduced to

$$
\left( \int_{B_{g_\epsilon}(p, r)} h^{\frac{2n}{n-2}} \omega_\epsilon^n \right)^{\frac{n-1}{n}} \leq C \int_{B_{g_\epsilon}(p, r)} |\nabla h|^2 h^{\frac{2n}{n-2}} \omega_\epsilon^n, \quad \forall h \in C^1_{0}(B_{g_\epsilon}(p, r)).
$$

(3.42)
With (3.42) at hand, we can apply the same proof of the standard De Giorgi-Nash-Moser theory (see the proof of Corollary 4.18 of [22]) to derive the uniform Hölder continuity of $v_\epsilon$ at $p$, i.e. there exists a constant $C = C(n, \beta, R_0) > 0$ such that

$$\text{osc}_{B_\beta(p,r)} v_\epsilon \leq \text{osc}_{B_\beta_0(p,r)} v_\epsilon \leq C r^{\alpha''}, \quad \forall r \in (0, R_0)$$

for some $\alpha'' = \alpha''(n, \beta, R_0) \in (0, 1)$ where in the first inequality we use the relation (3.40). Letting $\epsilon \to 0$ we see the continuity of $v$ at $p$.

• $v = \varphi$ on $\partial B_\beta(0,1)$. The proof is almost identical to that of Proposition 3.1. For example, the function $\varphi_q(z) = \varphi(q) - \delta + A\Psi_q(z)$ defined in Case 1 in the proof of Proposition 3.1 satisfies $\Delta_g \varphi_q(z) \geq \max_X f$ if $A > 0$ is taken large enough. Then from $\Delta_g(\varphi_q - v_\epsilon) \geq 0$ in $B_\beta$ and $\varphi_q - v_\epsilon \leq 0$ on $\partial B_\beta$, applying maximum principle we get $\varphi_q \leq v_\epsilon$ in $B_\beta(0,1)$. The remaining are the same as in Proposition 3.1. The Case 2 can be dealt with similarly.

\[\square\]

Remark 3.2. Let $H^1_0(\beta_0(0,1), g_\beta)$ be the completion of the space $C^0_1(\beta_0(0,1))$-functions under the norm

$$||\nabla u||_{L^2(g_\beta)} = \left( \int_{B_\beta(0,1)} |\nabla u|^2 g_\beta \omega_\beta^n \right)^{1/2}.$$ 

For any $h \in C^0_0(\beta(0,1))$, letting $\epsilon \to 0$ in (3.42) we get

$$\left( \int_{B_\beta(p,r)} |h|^{2\alpha_\beta} \omega_\beta^n \right)^{\alpha_\beta^{-1}} \leq C \int_{B_\beta(p,r)} |\nabla h|^{2} g_\beta \omega_\beta^n,$$ 

(3.43) for the same constant $C$ in (3.42). That is, Sobolev inequality also holds for the conical metric $\omega_\beta$.

3.2. Tangential and Laplacian estimates

In this section, we will prove the Hölder continuity of $\Delta_k u$ for $k = 1, 2$ and $(D')^2 u$ for the solution $u$ to (1.2). The arguments of [20] can be adopted here. We recall that we assume $\beta_1, \beta_2 \in (\frac{1}{2}, 1)$. We fix some notations first.

For a given point $p \notin S$, we denote $r_p = d_{g_\beta}(p, S)$, the $g_\beta$-distance of $p$ to the singular set $S$. For notation simplicity we will fix $\tau = 1/2$ and an integer $k_p \in \mathbb{Z}_+$ to be the smallest integer such that $\tau^k < r_p$, and $k_{i,p} \in \mathbb{Z}_+$ the smallest integer $k$ such that $\tau^k < d_{\beta}(p, S_i)$, for $i = 1, 2$. So $k_p = \max\{k_{1,p}, k_{2,p}\}$ We denote $p_1 \in S_1$ and $p_2 \in S_2$ the projections of $p$ to $S_1, S_2$, respectively.

For $j = 1, 2$ we will write $\Delta_j u := |z_j|^{2(1-\beta_j)} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}$.

We will consider a family of conical Laplace equations with different choices of $k \in \mathbb{Z}_+^+$.

(i) If $k \geq k_p$, the geodesic balls $B_\beta(p, \tau^k)$ are disjoint with $S$ and have smooth boundaries. $g_\beta$ is smooth on such balls. By standard theory we can solve the Dirichlet problem for $u_k \in C^\infty(B_\beta(p, \tau^k)) \cap C^0(B_\beta(p, \tau^k))$

$$\begin{cases} 
\Delta_\beta u_k = f(p), & \text{in } B_\beta(p, \tau^k) \\
u_k = u, & \text{on } \partial B_\beta(p, \tau^k)
\end{cases}$$

(3.44)

(ii) Without loss of generality we assume $d_{\beta}(p, S_1) \leq d_{\beta}(p, S_2)$, i.e. $k_{1,p} \geq k_{2,p}$. We now solve the Dirichlet problem $u_k \in C^2(B_\beta(p_1, 2\tau^k) \setminus S_1) \cap C^0(B_\beta(p_1, 2\tau^k))$ for $k_{2,p} + 2 \leq k < k_{1,p}$

$$\begin{cases} 
\Delta_\beta u_k = f(p), & \text{in } B_\beta(p_1, 2\tau^k) \\
u_k = u, & \text{on } \partial B_\beta(p_1, 2\tau^k)
\end{cases}$$

(3.45)
By similar argument as in the proof of Proposition 3.1, such \( u_k \) exists.  
(iii) For \( 2 \leq k \leq k_{2,p} + 1 \), let \( u_k \in C^2(B_\beta(p_{1,2}, 2\tau^k) \setminus S) \cap C^0(\overline{B_\beta(p_{1,2}, 2\tau^k)}) \) solve the equation  
\[
\begin{cases}
\Delta_\beta u_k = f(p), & \text{in } B_\beta(p_{1,2}, 2\tau^k) \\
u_k = u, & \text{on } \partial B_\beta(p_{1,2}, 2\tau^k)
\end{cases}
\]
whose existence follows from Remark 3.1. Here \( p_{1,2} = (0; 0; s(p)) \in S_1 \cap S_2 \) is the projection of \( p_1 \) to \( S_2 \).

We remark that we may take \( f(p) = 0 \) by considering \( \tilde{u} = u - \frac{f(p)}{2(n-2)}|s - s(p)|^2 \). If the estimate holds for \( \tilde{u} \), it also holds for \( u \). So from now on we assume \( f(p) = 0 \).

**Lemma 3.5.** Let \( u_k \) be the solutions to the equations (3.44), (3.45) and (3.46). There exists a constant \( C = C(n) > 0 \) such that for all \( k \in \mathbb{Z}_+ \), the following estimates hold  
\[
\|u_k - u\|_{L^\infty(B_\beta(p))} \leq C(n)\tau^{2k}\omega(\tau^k),
\]
where we denote \( \hat{B}_k(p) \) by  
\[
\hat{B}_k(p) := \begin{cases}
B_\beta(p, \tau^k), & \text{if } k \geq k_p \\
B_\beta(p_{1,2}, 2\tau^k), & \text{if } k_{2,p} + 2 \leq k < k_{1,p} \\
B_\beta(p_{1,2}, 2\tau^k), & \text{if } 1 \leq k \leq k_{2,p} + 1,
\end{cases}
\]
in different choices of \( k \in \mathbb{Z}_+ \).

We will also denote \( \lambda \hat{B}_k(p) \) to the concentric ball with \( \hat{B}_k(p) \) but the radius scaled by \( \lambda \in (0, 1) \).

This lemma follows straightforwardly from Lemma 3.1 and the definition of \( \omega(r) \). So we omit the proof. By triangle inequality, we get the following estimates  
\[
\|u_k - u_{k+1}\|_{L^\infty(\frac{1}{2}\hat{B}_k)} \leq C(n)\tau^{2k}\omega(\tau^k),
\]
(3.49)

Since \( u_k - u_{k+1} \) are \( g_\beta \)-harmonic functions on \( \frac{1}{2}\hat{B}_k \), applying the gradient and Laplacian estimates (3.5) and (3.9) for harmonic functions, we get:

**Lemma 3.6.** There exists a constant \( C(n) > 0 \) such that for all \( k \in \mathbb{Z}_+ \) it holds that  
\[
\|D'u_k - D'u_{k+1}\|_{L^\infty(\frac{1}{2}\hat{B}_k)} \leq C(n)\tau^k\omega(\tau^k),
\]
and  
\[
\sup_{\frac{1}{2}\hat{B}_k \setminus S} \left( \sum_{i=1}^2 \left| \sum_{i=1}^2 \Delta_i(u_k - u_{k+1}) \right| + \left| (D')^2 u_k - (D')^2 u_{k+1} \right| \right) \leq C(n)\omega(\tau^k),
\]
where we recall that \( D' \) denotes the first order operators \( \frac{\partial}{\partial s_i} \) for \( i = 5, \ldots, 2n \).

The following lemma can be proved by looking at the Taylor expansion of \( u_k \) at \( p \) for \( k >> 1 \) as in Lemma 2.8 of [20].

**Lemma 3.7.** The following limits hold:  
\[
\lim_{k \to \infty} D'u_k(p) = D'u(p), \quad \lim_{k \to \infty} (D')^2 u_k(p) = (D')^2 u(p), \quad \lim_{k \to \infty} \Delta_i u_k(p) = \Delta_i u(p),
\]
where \( i = 1, 2 \).

Combining Lemmas 3.6 and 3.7, we obtain the following estimates on the 2nd-order (tangential) derivatives.
Proposition 3.3. There exists a constant $C = C(n, \beta) > 0$ such that
\[
\sup_{B_\beta(0,1/2) \setminus \mathcal{S}} (|D^i u| + |\Delta_i u|) \leq C \left( \|u\|_{L^\infty(B_\beta(0,1))} + \int_0^1 \frac{\omega(r)}{r} \, dr + |f(0)| \right).
\] (3.53)

Proof. From triangle inequality we have for any given $z \in B_\beta(0,1/2) \setminus \mathcal{S}$
\[
|\Delta^2 u(z)| \leq \sum_{k=0}^{\infty} \left( |(D^i)^2 u_k(z) - (D^i)^2 u_{k+1}(z)| + |(D^i)^2 u_1(z)| \right)
\]
\[
\leq C(n) \sum_{k=1}^{\infty} \omega(\tau^k) + C(n) \osc_{B_\beta(0,1)} u_0
\]
\[
\leq C(n, \beta) \left( \|u\|_{L^\infty} + \int_0^1 \frac{\omega(r)}{r} \, dr + |f(0)| \right).
\]
The estimates for $\Delta_i u$ can be proved similarly. \hfill \Box

For any other given point $q \in B_\beta(0,1/2) \setminus \mathcal{S}$, we can solve Dirichlet boundary problems as $u_k$ with the metric balls centered at $q$, and we obtain a family of functions $v_k$ such that
\[
\Delta q v_k = f(q), \quad \text{in } \tilde{B}_k(q), \quad v_k = u \text{ on } \partial \tilde{B}_k(q),
\] (3.54)
where $\tilde{B}_k(q)$ are metrics balls centered at $q$ given by
\[
\tilde{B}_k(q) = \tilde{B}_k := \begin{cases} 
B_\beta(q, \tau^k), & \text{if } k \geq k_q, \\
B_\beta(q_{i,j}, 2\tau^k), & \text{if } j_{i,q} + 2 \leq k < k_q, \text{ here } k_{i,q} = \max(k_{1,q}, k_{2,q}) \text{ and } j \neq i \\
B_\beta(q_{i,j}, 2\tau^k), & \text{if } k \leq j_{i,q} + 1.
\end{cases}
\]

Similar estimates as in Lemmas 3.5, 3.6 and 3.6 also hold for $v_k$ within the balls $\tilde{B}_k(q)$.

We are now ready to state the main result in this subsection on the continuity of second order derivatives.

Proposition 3.4. Let $d = d_\beta(p, q)$. There exists a constant $C = C(n) > 0$ such that if $u$ solves the conical Laplace equation (1.2), then the following holds for $i = 1, 2$:
\[
|\Delta_i u(p) - \Delta_i u(q)| + |(D^i)^2 u(p) - (D^i)^2 u(q)| \leq C \left( \|u\|_{L^\infty(B_\beta(0,1))} + \int_0^d \frac{\omega(r)}{r} \, dr + d \int_d^1 \frac{\omega(r)}{r^2} \, dr \right).
\]

Proof. We only prove the estimate for $(D^i)^2 u$, and the one for $\Delta_i u$ can be dealt with in the same way.

We may assume $r_p = \min(r_p, r_q)$. We fix an integer $\ell$ such that $\tau^\ell$ is comparable to $d$, more precisely, we take
\[
\tau^{\ell+4} \leq d < \tau^{\ell+3}, \quad \text{or} \quad \tau^{\ell+1} \leq 8d \leq \tau^{\ell}.
\]
We calculate by triangle inequality
\[
|(D^i)^2 u(p) - (D^i)^2 u(q)| \leq |(D^i)^2 u(p) - (D^i)^2 u_\ell(p)| + |(D^i)^2 u_\ell(p) - (D^i)^2 u_\ell(q)|
\]
\[
+ |(D^i)^2 u_\ell(q) - (D^i)^2 v_\ell(q)| + |(D^i)^2 v_\ell(q) - (D^i)^2 u(q)|
\]
\[=: I_1 + I_2 + I_3 + I_4.
\]
We will estimate $I_1 - I_4$ one by one.
• $I_1$ and $I_4$: by (3.51) and (3.52) we have

$$I_1 = |(D')^2u(p) - (D')^2u_\ell(p)| \leq C(n) \sum_{k=\ell}^{\infty} \omega(\tau^k),$$

and similar estimate holds for $I_4$ as well

$$I_4 = |(D')^2u(q) - (D')^2v_\ell(q)| \leq C(n) \sum_{k=\ell}^{\infty} \omega(\tau^k).$$

• $I_3$: by the choice of $\ell$, it is not hard to see that $\frac{2}{3} \tilde{B}_\ell(q) \subset \tilde{B}_\ell(p)$. In particular $u_\ell$ and $v_\ell$ are both defined on $\frac{2}{3} \tilde{B}_\ell(q)$ and satisfy the equations

$$\Delta_{\beta} u_\ell = f(p), \quad \Delta_{\beta} v_\ell = f(q)$$

respectively on this ball. From (3.47) for $u_\ell$ and similar estimate for $v_\ell$ we get

$$\|u_\ell - v_\ell\|_{L^\infty(\frac{2}{3} \tilde{B}_\ell(q))} \leq C\tau^2 \omega(\tau^\ell).$$

Consider the function

$$U := u_\ell - v_\ell - \frac{f(p) - f(q)}{2(n-2)} |s - s(\tilde{q})|^2$$

where $\tilde{q}$ is the center of the ball $\tilde{B}_\ell(q)$. $U$ is $g_\beta$-harmonic in $\frac{2}{3} \tilde{B}_\ell(q)$ and satisfies the estimate:

$$\|U\|_{L^\infty(\frac{2}{3} \tilde{B}_\ell(q))} \leq C\tau^{2\ell} \omega(\tau^\ell) + C\tau^2 \omega(d) \leq C(n)\tau^2 \omega(\tau^\ell).$$

The derivatives estimates imply that

$$|(D')^2U(q)| \leq C\tau^{-2\ell} \|U\|_{L^\infty(\frac{2}{3} \tilde{B}_\ell(q))} \leq C(n)\omega(\tau^\ell).$$

Hence

$$I_3 = |(D')^2u_\ell(q) - (D')^2v_\ell(q)| \leq C(n)\omega(\tau^\ell).$$

• $I_2$: this is a little more complicated than the previous estimates. We define $h_k = u_{k-1} - u_k$ for $k \leq \ell$. $h_k$ is $g_\beta$-harmonic on $\tilde{B}_k(p)$ and by (3.47) $h_k$ satisfies the $L^\infty$-estimate $\|h_k\|_{L^\infty(\tilde{B}_k(p))} \leq C\tau^k \omega(\tau^k)$ and the derivative estimates $\|(D')^2 h_k\|_{L^\infty(\frac{2}{3} \tilde{B}_k(p))} \leq C \omega(\tau^k)$. On the other hand, the function $(D')^2 h_k$ is also $g_\beta$-harmonic on $\frac{2}{3} \tilde{B}_k(p)$ so the gradient estimate implies that

$$\|\nabla g_\beta(D')^2 h_k\|_{L^\infty(\frac{2}{3} \tilde{B}_k(p) \backslash S)} \leq C\tau^{-k} \omega(\tau^k).$$

Integrating this along the minimal $g_\beta$-geodesic $\gamma$ connecting $p$ and $q$ and noting that $\gamma$ avoids $S$ since $(C^n \backslash S, g_\beta)$ is strictly geodesically convex, we get

$$|(D')^2 h_k(p) - (D')^2 h_k(q)| \leq d \cdot \|\nabla g_\beta(D')^2 h_k\|_{L^\infty(\frac{1}{2} \tilde{B}_k(p) \backslash S)} \leq dC\tau^{-k} \omega(\tau^k).$$

By triangle inequality for each $k \leq \ell$

$$I_2 = |(D')^2u_\ell(p) - (D')^2u_\ell(q)| \leq |(D')^2 u_2(p) - (D')^2 u_2(q)| + dC \sum_{k=2}^{\ell} \tau^{-k} \omega(\tau^k).$$

Observe that $p, q \in \tilde{B}_2(p)$ and the function $(D')^2 u_2$ is $g_\beta$-harmonic on $\tilde{B}_2(p)$. From (3.47) and derivative estimates we have

$$\|(D')^2 u_2\|_{L^\infty(\frac{2}{3} \tilde{B}_2(p))} \leq C\|u_2\|_{L^\infty(\tilde{B}_2(p))} \leq C(\|u\|_{L^\infty} + \omega(\tau^2)).$$
Again by gradient estimate we have
\[ \|\nabla g_\beta(D')^2u_2\|_{L_\infty(\hat{B}_2(p))} \leq C(\|u\|_{L_\infty} + \omega(\tau^2)). \]

Integrating along the minimal geodesic \( \gamma \) we arrive at
\[ \| (D')^2u_2(p) - (D')^2u_2(q) \| \leq dC(\|u\|_{L_\infty} + \omega(\tau^2)). \]

Combining (3.57), we obtain that
\[ I_2 \leq Cd\left(\|u\|_{L_\infty(B_\beta(0,1))} + \sum_{k=2}^\ell \tau^{-k}\omega(\tau^k)\right). \]

Combining this with the estimates for \( I_1, I_2, I_3, I_4 \), we get
\[ \| (D')^2u(p) - (D')^2u(q) \| \leq C\left(d(\|u\|_{L_\infty(B_\beta(0,1))} + \sum_{k=2}^\ell \tau^{-k}\omega(\tau^k)) + \sum_{k=\ell}^\infty \omega(\tau^k)\right). \]

Proposition 3.4 now follows from this and the fact that \( \omega(\tau) \) is monotonically increasing. \( \square \)

### 3.3. Mixed normal-tangential estimates along the directions \( S \)

Throughout this section, we fix two points \( p, q \in B_\beta(0,1/2) \setminus S \) and assume \( r_p \leq r_q \). Recall that we introduce the weighted “polar coordinates” \( (r_i, \theta_i) \) for \( (z_1, z_2) \) as
\[ \rho_i = |z_i|, \quad r_i = \rho_i^{\beta_i}, \quad \theta_i = \arg z_i, \quad i = 1, 2. \]

Under these coordinates it holds that
\[ \Delta_i u = |z_i|^{2(1-\beta_i)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_i} - \frac{1}{r_i} \frac{\partial u}{\partial r_i} + \frac{1}{\beta_i r_i^2} \frac{\partial^2 u}{\partial \theta_i^2}. \] (3.58)

Let \( u_k \) (resp. \( v_k \)) be the solutions to equations (3.44), (3.45) and (3.46) on \( \hat{B}_k(p) \) (resp. \( \hat{B}_k(q) \)). Recall \( u_k - u_{k+1} \) satisfies (3.49) and apply gradient estimates to the \( g_\beta \)-harmonic function \( u_k - u_{k+1} \), we get the bound of \( \|\nabla g_\beta(u_k - u_{k+1})\|_{L_\infty(\hat{B}_k(p))} \) which in particular implies that for \( i = 1 \) or \( 2 \)
\[ \|z_i|^{1-\beta_i}(\frac{\partial u_k}{\partial z_i} - \frac{\partial u_{k+1}}{\partial z_i})\|_{L_\infty(\hat{B}_k(p))} \leq C \tau^k \omega(\tau^k). \] (3.59)

Similarly \( D'u_k - D'u_{k+1} \) is also \( g_\beta \)-harmonic on \( \frac{1}{2}\hat{B}_k(p) \) and apply gradient estimates to this function we get for \( i = 1, 2 \)
\[ \|z_i|^{1-\beta_i}(\frac{\partial D'u_k}{\partial z_i} - \frac{\partial D'u_{k+1}}{\partial z_i})\|_{L_\infty(\frac{1}{2}\hat{B}_k(p))} \leq C \omega(\tau^k). \] (3.60)

The following lemma can be proved by the same way as in Lemma 2.10 of [20] since \( p \notin S \), so we omit the proof.

**Lemma 3.8.** The following limits hold: for \( i = 1 \) or \( 2 \)
\[ \lim_{k \to \infty} \frac{\partial u_k}{\partial r_i}(p) = \frac{\partial u}{\partial r_i}(p), \quad \lim_{k \to \infty} \frac{\partial u_k}{\partial \theta_i}(p) = \frac{\partial u}{\partial \theta_i}(p) \]
and
\[ \lim_{k \to \infty} \frac{\partial D'u_k}{\partial r_i}(p) = \frac{\partial D'u}{\partial r_i}(p), \quad \lim_{k \to \infty} \frac{\partial D'u_k}{\partial \theta_i}(p) = \frac{\partial D'u}{\partial \theta_i}(p). \] (3.61)

Similar formulas also hold for \( v_k \) at the point \( q \).
We are going to estimate the quantities
\[
J := \left| \frac{\partial D' u}{\partial r_i} (p) - \frac{\partial D' u}{\partial r_i} (q) \right|, \quad K := \left| \frac{\partial D' u}{r_i \partial \theta_i} (p) - \frac{\partial D' u}{r_i \partial \theta_i} (q) \right|, \quad i = 1, 2.
\]
Note that $J, K$ correspond to $|N_J D^i u(p) - N_J D^i u(q)|$ in Theorem 1.1. We will estimate the case for $i = 1$ and $J$, since the other cases are completely the same. By triangle inequality we have
\[
J \leq \left| \frac{\partial D' u}{\partial r_i} (p) - \frac{\partial D' u}{\partial r_i} (q) \right| + \left| \frac{\partial D' u}{r_i \partial \theta_i} (p) - \frac{\partial D' u}{r_i \partial \theta_i} (q) \right|
\]
\[
= J_1 + J_2 + J_3 + J_4.
\]

**Lemma 3.9.** There exists a constant $C(n) > 0$ such that $J_1, J_3$ and $J_4$ satisfy
\[
J_1 + J_4 \leq C \sum_{k=\ell}^{\infty} \omega(\tau^k), \quad J_3 \leq C \omega(\tau^\ell).
\]

**Proof.** The estimates for $J_1$ and $J_4$ can be proved similarly as in proving those of $I_1$ and $I_4$ as in Section 3.2, using (3.60) and (3.61). $J_3$ can be estimated similar to that of $I_3$ as in Section 3.2, using (3.60). So we omit the details. \qed

To estimate $J_2$, as in Section 3.2 we denote $h_k := u_{k-1} - u_k$ for $2 \leq k \leq \ell$ which is $g_{\beta}$-harmonic on $\hat{B}_k(p)$ and satisfies the $L^\infty$-estimate $\|h_k\|_{L^\infty(\hat{B}_k(p))} \leq C \tau^{2k} \omega(\tau^k)$ by (3.60). We rewrite (3.56) as
\[
\| (D')^3 h_k \|_{L^\infty\left( \frac{1}{2} \hat{B}_k(p) \right)} \leq C \tau^{-k} \omega(\tau^k).
\]

**Lemma 3.10.** There exists a constant $C = C(n, \beta) > 0$ such that for any $z \in \frac{1}{2} \hat{B}_k(p) \setminus S$, the following pointwise estimate holds for all $k \leq \min(\ell, k_p)$
\[
\left| \frac{\partial D' h_k}{\partial r_1} (z) \right| + \left| \frac{\partial D' h_k}{r_1 \partial \theta_1} (z) \right| \leq C r_1(z) \frac{1}{\tau} (\frac{1}{\tau})^{1 - k} \omega(\tau^k).
\]

**Proof.** We define a function $F$ as
\[
|z_1|^{2(1-\beta_1)} \frac{\partial^2 D' h_k}{\partial z_1 \partial \bar{z}_1} = -|z_2|^{2(1-\beta_2)} \frac{\partial^2 D' h_k}{\partial z_2 \partial \bar{z}_2} - \sum_{j=5}^{2n} \frac{\partial^2 D' h_k}{\partial s_j^2} =: F.
\]

The Laplacian estimates (3.9) and derivative estimates applied to the $g_{\beta}$-harmonic function $D' h_k$ implies that $F$ satisfies
\[
\| F \|_{L^\infty(\frac{1}{2} \hat{B}_k(p))} \leq C(n) \tau^{-k} \omega(\tau^k).
\]

For any $k \leq \min(\ell, k_p)$ and $x \in S_1 \cap \frac{1}{2} \hat{B}_k(p)$, $B_{\beta}(x, \tau^k) \subset \frac{1}{2} \hat{B}_k(p)$. The intersection of $B_{\beta}(x, \tau^k)$ with complex plane $\mathbb{C}$ passing through $x$ and orthogonal to the hyperplane $S_1$ lies in a metric ball of radius $\tau^k$ under the standard cone metric $\hat{g}_{\beta_1}$ on $\mathbb{C}$. We view the equation (3.63) as defined on the ball $\hat{B} := B_\mathbb{C}(x, (\tau^k)^{1/\beta_1}) \subset \mathbb{C}$. The estimate (2.2) applied to the function $D' h_k$ gives rise to
\[
\sup_{B_\mathbb{C}(x, (\tau^k)^{1/\beta_1}/2) \setminus \{x\}} \left| \frac{\partial D' h_k}{\partial z_1} \right| \leq C \left\| D' h_k \right\|_{L^\infty(\hat{B})} \frac{(\tau^k)^{1/\beta_1}}{\tau^k} + C \| F \|_{L^\infty(\hat{B})} (\tau^k)^{2-\frac{1}{\beta_1}}.
\]
Therefore on $B_C(x, (\tau^k)^{1/\beta_1}/2) \setminus \{x\}$ the following holds
\[
\left| \frac{\partial D'h_k}{\partial r_1} \right| + \left| \frac{\partial D'h_k}{r_1 \partial \theta_1} \right| \leq r_1^{\frac{\beta_1}{2}-1} \left| \frac{\partial D'h_k}{\partial z_1} \right| \leq C r_1^{\frac{\beta_1}{2}-1} \tau^{k(1-\frac{1}{\beta_1})} \omega(\tau^k) \tag{3.65}
\]

On the other hand, since $B_C(x, (\tau^k)^{1/\beta_1}/2) = B_{\beta_1}(x, 2^{-\beta_1} \tau^k)$
\[
\frac{1}{4} \hat{B}_k(p) \subset \bigcup_{x \in S_1} \bigcap_{\frac{1}{4} \hat{B}_k} B_C(x, (\tau^k)^{1/\beta_1}/2).
\tag{3.66}
\]
equation (3.65) implies the desired estimate on the balls $\frac{1}{4} \hat{B}_k(p)$.

\[\square\]

**Remark 3.3.** By similar arguments we can get the following estimates as well for any $k \leq \min(\ell, k_p)$ and $z \in \frac{1}{4} \hat{B}_k(p) \setminus S_1$
\[
\left| \frac{\partial (D')^2 h_k}{\partial r_1} (z) \right| + \left| \frac{\partial (D')^2 h_k}{r_1 \partial \theta_1} (z) \right| \leq C r_1(z)^{\frac{1}{\beta_1}-1} \tau^{-\frac{k}{\beta_1}} \omega(\tau^k).
\tag{3.67}
\]

**Lemma 3.11.** There exists a constant $C = C(n, \beta) > 0$ such that for all $k \leq \min(k_p, \ell)$ and $z \in \frac{1}{4} \hat{B}_k(p) \setminus S$ the following pointwise estimates hold
\[
\left| \frac{\partial^2 D'h_k}{\partial r_1^2} (z) \right| + \left| \frac{\partial^2 D'h_k}{r_1 \partial r_1 \partial \theta_1} (z) \right| \leq C r_1(z)^{\frac{1}{\beta_1}-2} \tau^{-k(\frac{1}{\beta_1}-1)} \omega(\tau^k),
\tag{3.68}
\]
\[
\left| \frac{\partial^2 D'h_k}{\partial r_1^2} (z) \right| \leq C r_1(z)^{\frac{1}{\beta_1}-2} \tau^{-k(\frac{1}{\beta_1}-1)} \omega(\tau^k).
\tag{3.69}
\]

**Proof.** Applying the gradient estimate to the $g_\beta$-harmonic function $D'h_k$, we get
\[
\| \frac{\partial D'h_k}{r_1 \partial \theta_1} \|_{L^\infty(\frac{1}{4} \hat{B}_k(p))} \leq \| \nabla g_\beta D'h_k \|_{L^\infty(\frac{1}{4} \hat{B}_k(p))} \leq C \omega(\tau^k).
\]
The function $\partial_{\theta_1} D'h_k$ is also a continuous $g_\beta$-harmonic function so the derivatives estimates implies on $\frac{1}{4} \hat{B}_k(p) \setminus S$
\[
|F_1| \leq \left| \frac{\partial^2 (\partial_{\theta_1} D'h_k)}{\partial z_1 \partial z_1} \right| + \left| \frac{\partial^2 (\partial_{\theta_1} D'h_k)}{\partial s_j^2} \right| \leq C \tau^{-k} \omega(\tau^k),
\]
where $F_1$ is defined below
\[
|z_1|^{2(1-\beta_1)} \frac{\partial^2 (\partial_{\theta_1} D'h_k)}{\partial z_1 \partial z_1} = -|z_2|^{2(1-\beta_2)} \frac{\partial^2 (\partial_{\theta_1} D'h_k)}{\partial z_2 \partial z_2} - \sum_{j=0}^{2n} \frac{\partial^2 (\partial_{\theta_1} D'h_k)}{\partial s_j^2} =: F_1.
\tag{3.70}
\]
We apply similar arguments as in the proof of Lemma 3.10. For any $x \in S_1 \cap \frac{1}{4} \hat{B}_k(p)$, we view the equation (3.70) as defined on the $C$-ball $B_C(x, (\tau^k)^{1/\beta_1})$, and by the estimate (2.2) we have on $B_C(x, (\tau^k)^{1/\beta_1}/2) \setminus \{x\}$
\[
\left| \frac{\partial (\partial_{\theta_1} D'h_k)}{\partial z_1} \right| \leq C \| \partial_{\theta_1} D'h_k \|_{L^\infty(\frac{1}{4} \hat{B}_k)} + C \| F_1 \|_{L^\infty(\frac{1}{4} \hat{B}_k)} (\tau^k)^{2-\frac{1}{\beta_1}}.
\]
Equivalently, this means that on $B_C(x, (\tau^k)^{1/\beta_1}/2) \setminus \{x\}$
\[
\left| \frac{\partial^2 D'h_k}{\partial r_1 \partial \theta_1} \right| \leq r_1^{\frac{1}{\beta_1}-1} \left| \frac{\partial (\partial_{\theta_1} D'h_k)}{\partial z_1} \right| \leq C r_1^{\frac{1}{\beta_1}-1} \tau^{k(1-\frac{1}{\beta_1})} \omega(\tau^k).
\]

Again by the inclusion (3.66), we get (3.68). The estimate (3.69) follows from Lemma 3.10, (3.68), (3.64) and the equation (from (3.63)) below
\[
\frac{\partial^2 D'h_k}{\partial r_1^2} = \frac{1}{r_1} \frac{\partial D'h_k}{\partial r_1} - \frac{1}{\beta_1^2 r_1^2} \frac{\partial^2 D'h_k}{\partial \theta_1^2} + F.
\]

\[\square\]

Lemma 3.12. There exists a constant \(C(n, \beta) > 0\) for \(k \leq \min(k_2, p, \ell)\), the following pointwise estimates hold for any \(z \in \frac{1}{2} \hat{B}_k(p) \setminus \mathcal{S}\)
\[
\left| \frac{\partial}{\partial r_2} \left( \frac{\partial D'h_k}{\partial r_1} \right)(z) \right| + \left| \frac{\partial}{\partial r_2 \partial \theta_2} \left( \frac{\partial D'h_k}{\partial r_1} \right)(z) \right| \leq C(n, \beta) r_1^{\frac{1}{\beta_1} - 1} r_2^{\frac{1}{\beta_2} - 1} \tau^{-k(-1 + \frac{1}{\beta_1} + \frac{1}{\beta_2})} \omega(\tau^k). \tag{3.71}
\]

Proof. By the Laplacian estimate in (3.9) for the harmonic function \(D'h_k\) on \(\frac{1}{2} \hat{B}_k(p)\), we have
\[
\sup_{\frac{1}{2} \hat{B}_k(p)} \left( \left| \Delta_1 D'h_k \right| + \left| \Delta_2 D'h_k \right| \right) \leq C(n, \beta) \tau^{-2k \operatorname{osc}_{\frac{1}{2} \hat{B}_k(p)}(D'h_k)} \leq C(n, \beta) \tau^{-k} \omega(\tau^k). \tag{3.72}
\]
Since \(\Delta_1(D'h_k)\) is also \(g_\beta\)-harmonic, the Laplacian estimates (3.9) imply that
\[
\sup_{\frac{1}{2} \hat{B}_k(p)} \left( \left| \Delta_1 D'h_k \right| + \left| \Delta_2 D'h_k \right| \right) \leq C(n, \beta) \tau^{-2k \operatorname{osc}_{\frac{1}{2} \hat{B}_k(p)}(D'h_k)} D_1 D'h_k \leq C \tau^{-3k} \omega(\tau^k). \tag{3.73}
\]
Now from the equation \(\Delta g(\Delta_1 D'h_k) = 0\)
\[
|z_1|^{2(1 - \beta_1)} \frac{\partial^2}{\partial z_1 \partial z_2} \Delta_1 D'h_k = -\Delta_2 \Delta_1 D'h_k - \sum_j \frac{\partial^2}{\partial z_j^2} \Delta_1 D'h_k =: F_2. \tag{3.74}
\]
From (3.73) and the Laplacian estimates (3.9), we see that \(\sup_{\frac{1}{2} \hat{B}_k(p)} |F_2| \leq C \tau^{-3k} \omega(\tau^k)\). By similar argument by considering \(x \in \frac{1}{3} \hat{B}_k(p) \cap \mathcal{S}_1\), we obtain from (3.74) that on \(\hat{B} : = B_C(x, (\tau^k)^{1/\beta_1}/2) \setminus \{x\}\)
\[
\left| \frac{\partial}{\partial z_1} \Delta_1 D'h_k \right| \leq C \frac{\|\Delta_1 D'h_k\|_{L^\infty(\hat{B})}}{(\tau^k)^{1/\beta_1}} + C\|F_2\|_{L^\infty(\hat{B})}(\tau^k)^{2 - \frac{1}{\beta_1}} \leq C \tau^{-k(1 + \frac{1}{\beta_1})} \omega(\tau^k).
\]
This implies that for any \(z \in \frac{1}{3} \hat{B}_k(p) \setminus \mathcal{S}\)
\[
\left| \frac{\partial}{\partial r_1} \Delta_1 D'h_k(z) \right| + \left| \frac{\partial}{\partial r_1 \partial \theta_1} \Delta_1 D'h_k(z) \right| \leq C r_1^{\frac{1}{\beta_1} - 1} \tau^{-k(1 + \frac{1}{\beta_1})} \omega(\tau^k). \tag{3.75}
\]
Now taking \(\frac{\partial}{\partial r_1}\) on both sides of \(\Delta_2 D'h_k = 0\), we get
\[
|z_2|^{2(1 - \beta_2)} \frac{\partial^2}{\partial z_2 \partial z_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) = -\frac{\partial}{\partial r_1} (\Delta_1 D'h_k) - \sum_j \frac{\partial^2}{\partial z_j^2} \left( \frac{\partial D'h_k}{\partial r_1} \right) =: F_3. \tag{3.76}
\]
From (3.75), for any \(z \in \frac{1}{4} \hat{B}_k \setminus \mathcal{S}, |F_3|(z) \leq C r_1^{\frac{1}{\beta_1} - 1} \tau^{-k(1 + \frac{1}{\beta_1})} \omega(\tau^k)\). By similar argument for any \(y \in \frac{1}{3} \hat{B}_k(p) \cap \mathcal{S}_2\), we apply the estimate (2.2) to \(\frac{\partial D'h_k}{\partial r_1}\), we get on \(A_1 := B_C(y, (\tau^k)^{1/\beta_2}/2) \setminus \{y\}\), the punctured ball in the complex plane \(\mathbb{C}\) of (Euclidean) radius \((\tau^k)^{1/\beta_2}/2\) and orthogonal to \(\mathcal{S}_2\) passing through \(y\),
\[
\left| \frac{\partial}{\partial z_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right| \leq C \frac{\|\frac{\partial D'h_k}{\partial r_1}\|_{L^\infty(A_1)}}{(\tau^k)^{1/\beta_2}} + C\|F_3\|_{L^\infty(A_1)}(\tau^k)^{2 - \frac{1}{\beta_2}} \leq C r_1^{\frac{1}{\beta_1} - 1} \tau^{-k(\frac{1}{\beta_1} + \frac{1}{\beta_2})} \omega(\tau^k).
\]
Varying \( y \in \frac{1}{\tau_2} \hat{B}_k(p) \cap S_2 \) we get for any \( z \in \frac{1}{\tau} \hat{B}_k \setminus S \), that the following pointwise estimate holds

\[
| \frac{\partial}{\partial r_2} \left( \frac{\partial D' h_k}{\partial r_1} \right)(z) | + | \frac{\partial}{\partial r_2 \partial \theta_2} \left( \frac{\partial D' h_k}{\partial r_1} \right)(z) | \leq C r_1^{1-1} \tau^{-k \left( \frac{1}{\tau} - 1 \right)} \omega(\tau^k). \tag{3.77}
\]

**Lemma 3.13.** Let \( d = d_\beta(p, q) \). There exists a constant \( C(n, \beta) \) such that for all \( k \leq \ell \)

\[
\left| \frac{\partial D' h_k}{\partial r_1}(p) - \frac{\partial D' h_k}{\partial r_1}(q) \right| \leq C d^{1-1} \tau^{-k \left( \frac{1}{\tau} - 1 \right)} \omega(\tau^k), \tag{3.78}
\]

and

\[
\left| \frac{\partial D' h_k}{r_1 \partial \theta_1}(p) - \frac{\partial D' h_k}{r_1 \partial \theta_1}(q) \right| \leq C d^{1-1} \tau^{-k \left( \frac{1}{\tau} - 1 \right)} \omega(\tau^k). \tag{3.79}
\]

**Proof.** We will consider the different cases \( r_p = \min(r_p, r_q) \leq 2d \) and \( r_p = \min(r_p, r_q) > 2d \).

\( \blacktriangleright \) \( r_p \leq 2d \). In this case, it is clear by the choice of \( \ell \) that \( r_p \approx \tau^k \leq 2d \leq \tau^{\ell+2} \), so \( k \geq \ell + 2 \).

From our assumption when solving (3.45), \( r_p = d_\beta(p, S_1) \), i.e. \( r_1(p) = r_p \leq 2d \). By triangle inequality we have \( r_1(q) \leq 3d \). We also remark that for \( k \leq \ell \), \( \tau^k \geq \tau^{\ell + 2} \). In particular, the geodesics considered below all lie inside the balls \( \frac{1}{\tau} \hat{B}_k(p) \), and the estimates in Lemma 3.10 - Lemma 3.12 holds for points on these geodesics.

Let \( p = (r_1(p), \theta_1(p); r_2(p), \theta_2(p); s(p)) \) and \( q = (r_1(q), \theta_1(q); r_2(q), \theta_2(q); s(q)) \) be the coordinates of the points \( p, q \). Let \( \gamma : [0, d] \to B_\beta(0, q) \setminus S \) be the unique \( g_\beta \)-geodesic connecting \( p \) and \( q \). We know the curve \( \gamma \) is disjoint with \( S \), and we denote \( \gamma(t) = (r_1(t), \theta_1(t); r_2(t), \theta_2(t); s(t)) \) be the coordinates of \( \gamma(t) \) for \( t \in [0, d] \). By definition we have for \( \forall t \in [0, d] \)

\[
|\gamma'(t)|_{g_\beta}^2 = (r_1'(t))^2 + \beta_1^2 r_1(t)^2 (\theta_1'(t))^2 + (r_2'(t))^2 + \beta_2^2 r_2(t)^2 (\theta_2'(t))^2 + |s'(t)|^2 = 1.
\]

So \( |s(p) - s(q)| \leq d \) and \( |r_i(p) - r_i(q)| \leq d \) for \( i = 1, 2 \). We denote

\[
qu := (r_1(q), \theta_1(q); r_2(p), \theta_2(p); s(p)), \quad p' := (r_1(p), \theta_1(q); r_2(p), \theta_2(p); s(p)) \tag{3.80}
\]
the points with coordinates related to \( p \) and \( q \). Let \( \gamma_1 \) be the \( g_\beta \)-geodesic connecting \( q \) and \( q' \); \( \gamma_2 \) the \( g_\beta \)-geodesic joining \( q' \) to \( p' \); \( \gamma_3 \) the \( g_\beta \)-geodesic joining \( p' \) to \( p \).

By triangle inequality, we have

\[
\left| \frac{\partial D' h_k}{\partial r_1}(p) - \frac{\partial D' h_k}{\partial r_1}(q) \right| \leq \left| \frac{\partial D' h_k}{\partial r_1}(p) - \frac{\partial D' h_k}{\partial r_1}(p') \right| + \left| \frac{\partial D' h_k}{\partial r_1}(p') - \frac{\partial D' h_k}{\partial r_1}(q') \right| + \left| \frac{\partial D' h_k}{\partial r_1}(q') - \frac{\partial D' h_k}{\partial r_1}(q) \right| =: J_1' + J_2' + J_3'.
\]

Integrating along \( \gamma_3 \) on which the points have fixed \( r_1 \)-coordinate \( r_1(p) \), we get by (3.68)

\[
J_1' = \left| \int_{\gamma_3} \frac{\partial}{\partial \theta_1} \left( \frac{\partial D' h_k}{\partial r_1} \right) d\theta_1 \right| \leq C(n, \beta) r_1(p)^{1-1} \tau^{-k \left( \frac{1}{\tau} - 1 \right)} \omega(\tau^k). \tag{3.81}
\]
Integrating along $\gamma_2$ and by (3.69) we get

$$J'_2 = \left| \int_{\gamma_2} \frac{\partial}{\partial r_1} \left( \frac{\partial D'h_k}{\partial r_1} \right) dr_1 \right|$$

$$\leq C(n, \beta) \tau^{-k} (\frac{1}{\beta_1}) \omega(\tau^k) \left| \int_{r_1(p)}^{r_1(q)} t^{\frac{1}{\beta_1} - 2} dt \right|$$

$$= C(n, \beta) \tau^{-k} (\frac{1}{\beta_1}) \omega(\tau^k) \left| r_1(p) \frac{1}{\beta_1} - 1 - r_1(q) \frac{1}{\beta_1} - 1 \right|$$

$$\leq C(n, \beta) \tau^{-k} (\frac{1}{\beta_1}) \omega(\tau^k) \left| r_1(p) - r_1(q) \right| \frac{1}{\beta_1} - 1$$

$$\leq C(n, \beta) \tau^{-k} (\frac{1}{\beta_1}) \omega(\tau^k) d \frac{1}{\beta_1} - 1.$$

To deal with $J'_3$, we need to consider different choices of $k \leq \ell$.

- If $k_{2,p} + 1 \leq k \leq \ell$, the balls $\hat{B}_k(p)$ are centered at $p_1 \in S_1$ (recall $p_1$ is the projection of $p$ to $S_1$, hence $p$ and $p_1$ have the same $(r_2, \theta_2; s)$-coordinates). $\tau^{-k} \leq 8^{-1} d^{-1}$ by the choice of $\ell$. The balls $\hat{B}_k(p)$ are disjoint with $S_2$, so we can introduce the smooth coordinates $w_2 = \frac{\beta_2}{2}$, and under the coordinates $(r_1, \theta_1; w_2, z_3, \ldots, z_n)$, the metric $g_{\beta}$ become the smooth cone metric with conical singularity only along $S_1$ with angle $2\pi \beta_1$. Therefore we can derive the following estimate as in (3.62) that

$$\sup_{\frac{1}{2} \hat{B}_k(p) \setminus S_1} \left| \frac{\partial (D')^2 h_k}{\partial r_1} \right| + \left| \frac{\partial}{\partial r_1} \left( \frac{\partial D'h_k}{\partial w_2} \right) \right| \leq C \tau^{-k} \omega(\tau^k).$$

(3.83)

Since $q$ and $q'$ have the same $(r_1, \theta_1)$-coordinates and $g_{\beta}$ is a product metric, $\gamma_1$ is in fact a straight line segment (under the coordinates $(w_2, z_3, \ldots, z_n)$) in the hyperplane with fixed $(r_1, \theta_1)$-coordinates. Integrating over $\gamma_1$, we get

$$J'_3 \leq \int_{\gamma_1} \left| \frac{\partial}{\partial w_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right| + \sum_j \left| \frac{\partial}{\partial s_j} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right|$$

$$\leq C \tau^{-k} \omega(\tau^k) d_{\beta}(q, q') \leq C \tau^{-k} \omega(\tau^k) d$$

$$\leq C(n, \beta) \tau^{-k} (\frac{1}{\beta_1}) \omega(\tau^k) d \frac{1}{\beta_1} - 1.$$
We will use frequently the inequalities that \( r_1(q) \leq 3d \) and \( \max(r_2(q), r_2(p)) \leq 2\tau^k \) in the estimates below. Integrating along \( \hat{\gamma} \) we get by (3.71)
\[
J''_3 \leq \left| \int_{\hat{\gamma}} \frac{\partial}{\partial \theta_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) d\theta_2 \right| \leq C r_1(q)^{\frac{1}{\beta_1}} \frac{1}{r_2(q)} \frac{1}{\tau^{k(-1+\frac{1}{\beta_1}+\frac{1}{\beta_2})}} \omega(\tau^k) \leq C d^{\frac{1}{\beta_1}} \tau^{-k(\frac{1}{\beta_1}-1)} \omega(\tau^k)
\]
Integrating along \( \hat{\gamma} \) we get again by (3.71)
\[
J''_2 \leq \left| \int_{\hat{\gamma}} \frac{\partial}{\partial r_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) dr_2 \right|
\leq C r_1(q)^{\frac{1}{\beta_1}} \frac{1}{r_2(q)} \frac{1}{\tau^{k(-1+\frac{1}{\beta_1}+\frac{1}{\beta_2})}} \omega(\tau^k) \right| \int_{r_2(q)}^{r_2(p)} t^{\frac{1}{\beta_2}-1} dt
\leq C r_1(q)^{\frac{1}{\beta_1}} \frac{1}{\tau^{k(-1+\frac{1}{\beta_1}+\frac{1}{\beta_2})}} \omega(\tau^k) \max(r_2(q), r_2(p))^\frac{1}{\beta_2}-1 d
\leq C d^{\frac{1}{\beta_1}} \tau^{-k(\frac{1}{\beta_1}-1)} \omega(\tau^k)
\]
Integrating along \( \hat{\gamma}_1 \) we get by (3.67)
\[
J''_1 \leq \left| \int_{\hat{\gamma}_1} \frac{\partial}{\partial s_j} \left( \frac{\partial D'h_k}{\partial r_1} \right) ds \right| \leq C r_1(q)^{\frac{1}{\beta_1}} \frac{1}{\tau^{\frac{k}{\beta_1}}} \frac{1}{\omega(\tau^k)} d \leq C d^{\frac{1}{\beta_1}} \tau^{-k(\frac{1}{\beta_1}-1)} \omega(\tau^k).
\]
Combining the above three inequalities, we get in the case \( k \leq k_{2,p} \) that
\[
J''_3 \leq C d^{\frac{1}{\beta_1}} \tau^{-k(\frac{1}{\beta_1}-1)} \omega(\tau^k).
\]
Combining the estimates on \( J'_1, J'_2, J'_3 \), we finish the proof of (3.78) in the case \( r_p \leq 2d \).

\[\blacksquare\]

\( \tau^{k/p} \approx r_p > 2d \geq \tau^{\ell+2} \). From triangle inequality we get \( d\theta(\gamma(t), S) \geq d \), in particular, the \( r_1 \) and \( r_2 \) coordinates of \( \gamma(t) \) are both bigger than \( d \). In this case \( k \leq \ell \leq k_p \), Lemma 3.10 - Lemma 3.12 hold for the points in \( \gamma \). \( r_1(\gamma(t)) \leq r_1(p) + d \leq 2\tau^k \). We calculate the gradient of \( \frac{\partial D'h_k}{\partial r_1} \) along \( \gamma \)
\[
|\nabla_{g_{\gamma}} \frac{\partial D'h_k}{\partial r_1}|^2 = \left| \frac{\partial}{\partial r_1} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right|^2 + \left| \frac{\partial}{\partial \beta_1 r_1 \partial \theta_1} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right|^2 + \left| \frac{\partial}{\partial r_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right|^2 + \left| \frac{\partial}{\partial s_j} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right|^2.
\]
(1). When \( k_{2,p} + 1 \leq k \leq \ell \) we have by (3.83) that
\[
\sup_{\frac{1}{\beta_1} S_1} \left| \frac{\partial}{\partial r_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right| + \left| \frac{\partial}{\partial \theta_2} \left( \frac{\partial D'h_k}{\partial r_1} \right) \right| \leq C \tau^{-k} \omega(\tau^k). \tag{3.85}
\]
Thus by Lemma 3.11, (3.67) and (3.85) along \( \gamma \) we have
\[
|\nabla_{g_{\gamma}} \frac{\partial D'h_k}{\partial r_1}| \leq C \omega(\tau^k) \left( d^{\frac{1}{\beta_1}-2} \tau^{-k(\frac{1}{\beta_1}-1)} + \tau^{-k} \right)
\]
Integrating along \( \gamma \) we get
\[
\left| \frac{\partial D'h_k}{\partial r_1}(p) - \frac{\partial D'h_k}{\partial r_1}(q) \right| \leq \int_{\gamma} \left| \nabla_{g_{\gamma}} \frac{\partial D'h_k}{\partial r_1} \right| \leq C \omega(\tau^k) \left( d^{\frac{1}{\beta_1}-1} \tau^{-k(\frac{1}{\beta_1}-1)} + d^{-k} \right)
\leq C \omega(\tau^k) d^{\frac{1}{\beta_1}-1} \tau^{-k(\frac{1}{\beta_1}-1)}. \]
(2). When \( k \leq k_{2,p} \), we have \( r_2(\gamma(t)) \leq r_2(p) + d \leq \tau^k + d \leq \frac{a}{\beta} \tau^k \) and similar estimates hold for \( r_1(\gamma(t)) \) too. Then by Lemma 3.11, Lemma 3.12 and (3.67) along \( \gamma \) the following estimate holds
\[
|\nabla g_\alpha \frac{\partial D'h_k}{\partial r_1}(\gamma(t))| \leq C \omega(\tau^k) (d^{\frac{1}{\beta 1} - 2} \tau^{-k(\frac{1}{\beta 1} - 1)} + \tau^{-k})
\]
Integrating along \( \gamma \) we get
\[
\left| \frac{\partial D'h_k}{\partial r_1}(p) - \frac{\partial D'h_k}{\partial r_1}(q) \right| \leq \int_\gamma |\nabla g_\alpha \frac{\partial D'h_k}{\partial r_1}| \leq C \omega(\tau^k) (d^{\frac{1}{\beta 1} - 1} \tau^{-k(\frac{1}{\beta 1} - 1)} + d \tau^{-k})
\leq C \omega(\tau^k) d^{\frac{1}{\beta 1} - 1} \tau^{-k(\frac{1}{\beta 1} - 1)}.
\]

This finishes the proof of the lemma in this case.

\( \blacktriangleright \) \( r_p > 2d \) but \( \ell \geq k_p + 1 \). When \( k \leq k_p \), the estimate (3.78) follows in the same way as the case above. So it suffices to consider the case when \( k_p + 1 \leq k \leq \ell \). In this case the balls \( \hat{B}_k(p) = B_\beta(p, \tau^k) \) and it can be seen by triangle inequality that the geodesic \( \gamma \subset \hat{B}_k(p) \). Since the metric balls \( \hat{B}_k(p) \) are disjoint with \( S \) we can use the smooth coordinates \( w_1 = z_1^{\beta_1} \) and \( w_2 = z_2^{\beta_2} \) as before, and everything becomes smooth under these coordinates in \( \hat{B}_k(p) \).

The estimate (3.79) can be shown by the same argument, so we skip the details.

Iteratively applying (3.78) for \( k \leq \ell \), we get
\[
J_2 = |\frac{\partial D'u_{1k}}{\partial r_1}(p) - \frac{\partial D'u_{1k}}{\partial r_1}(q)| \leq |\frac{\partial D'u_{2k}}{\partial r_1}(p) - \frac{\partial D'u_{2k}}{\partial r_1}(q)| + C d^{\frac{1}{\beta 1} - 1} \sum_{k=3}^\ell \tau^{-k(\frac{1}{\beta 1} - 1)} \omega(\tau^k)
\leq C d^{\frac{1}{\beta 1} - 1} (\|u\|_{C^0} + \sum_{k=2}^\ell \tau^{-k(\frac{1}{\beta 1} - 1)} \omega(\tau^k)),
\]
where the inequality
\[
|\frac{\partial D'u_{2k}}{\partial r_1}(p) - \frac{\partial D'u_{2k}}{\partial r_1}(q)| \leq C d^{\frac{1}{\beta 1} - 1} \|u\|_{C^0}
\]
can be proved by the same argument as in proving (3.78).

Combining the estimates for \( J_1, J_2, J_3, J_4 \) we finish the proof of (1.4).

We remark that in solving (3.45), we assume \( r_1(p) \leq r_2(p) \), we need also to deal with the following case, whose proof is more or less parallel to that of Lemma 3.13, so we just point out the differences and sketch the proof.

**Lemma 3.14.** Let \( d = d_\beta(p, q) > 0 \). There exists a constant \( C(n, \beta) > 0 \) such that for all \( k \leq \ell \),
\[
|\frac{\partial D'h_k}{\partial r_2}(p) - \frac{\partial D'h_k}{\partial r_2}(q)| \leq C d^{\frac{1}{\beta 2} - 1} \tau^{-k(\frac{1}{\beta 2} - 1)} \omega(\tau^k),
\]
(3.86)
\[
|\frac{\partial D'h_k}{\partial r_2}(p) - \frac{\partial D'h_k}{\partial r_2}(q)| \leq C d^{\frac{1}{\beta 2} - 1} \tau^{-k(\frac{1}{\beta 2} - 1)} \omega(\tau^k).
\]
(3.87)

**Proof.** We consider two different cases on whether \( k \leq k_{1,p} \) or \( k_{1,p} + 1 \leq k \leq \ell \).

\( \bullet \) \( k_{2,p} + 1 \leq k \leq \ell \). The balls \( \hat{B}_k(p) \) are disjoint with \( S_2 \), so we can introduce the complex coordinate \( w_2 = z_2^{\beta_2} \) on these balls as before. Let \( t_1, t_2 \) be the real and imaginary parts of \( w_2 \), respectively. The derivatives estimates imply that
\[
\|\partial w_2 D' h_k\|_{L^\infty(\frac{1}{2} \hat{B}_k(p))} \leq C \omega(\tau^k), \quad \|\partial^2_{w_2} D' h_k\|_{L^\infty(\frac{1}{2} \hat{B}_k(p))} \leq C \tau^{-k} \omega(\tau^k),
\]

\]
where $\partial^2_{w_k}$ denotes the full second order derivatives in the $\{t_1, t_2\}$-directions. And
\[
\|\frac{\partial}{\partial r_1} \left( \frac{\partial D'h_k}{\partial w_2} \right) \|_{L^\infty(\frac{1}{2}B_k(p))} + \| \frac{\partial}{\partial r_1 \partial \theta_1} \left( \frac{\partial D'h_k}{\partial w_2} \right) \|_{L^\infty(\frac{1}{2}B_k(p))} \leq C\tau^{-k} \omega(\tau^k).
\]
Since
\[
\frac{\partial}{\partial r_2} = \frac{w_2}{r_2} \frac{\partial}{\partial w_2} + \frac{\tilde{w}_2}{r_2} \frac{\partial}{\partial \tilde{w}_2},
\]
(3.88)
it holds that
\[
\frac{\partial}{\partial w_2} \left( \frac{\partial D'h_k}{\partial r_2} \right) = \frac{1}{r_2^2} \frac{\partial D'h_k}{\partial w_2} \frac{|w_2|^2}{r_2^2} \frac{\partial D'h_k}{\partial w_2} - \frac{\tilde{w}_2}{r_2^2} \frac{\partial D'h_k}{\partial w_2} + \frac{w_2}{r_2^2} \frac{\partial^2 D'h_k}{\partial w_2 \partial r_2},
\]
we have on $\frac{1}{2}B_k(p)$
\[
|\frac{\partial}{\partial w_2} \left( \frac{\partial D'h_k}{\partial r_2} \right) | \leq C \frac{\omega(\tau^k)}{r_2} + C\tau^{-k} \omega(\tau^k).
\]
And
\[
\|\frac{\partial}{\partial r_1} \left( \frac{\partial D'h_k}{\partial r_2} \right) \|_{L^\infty(\frac{1}{2}B_k(p))} + \| \frac{\partial}{\partial r_1 \partial \theta_1} \left( \frac{\partial D'h_k}{\partial r_2} \right) \|_{L^\infty(\frac{1}{2}B_k(p))} \leq C\tau^{-k} \omega(\tau^k).
\]
Therefore
\[
|\nabla _{g_{\theta}} \frac{\partial D'h_k}{\partial r_2} |^2 = \left| \frac{\partial^2 D'h_k}{\partial r_1 \partial r_2} \right|^2 + \left| \frac{\partial^2 D'h_k}{\partial r_1 \partial \theta_1} \right|^2 + \left| \frac{\partial^2 D'h_k}{\partial w_2 \partial r_2} \right|^2 + \sum_j \left| \frac{\partial^2 D'h_k}{\partial s_j \partial r_2} \right|^2 \leq C(\tau^{-k} \omega(\tau^k))^2 + C\frac{1}{r_2^2} \omega(\tau^k)^2.
\]
In this case we know that $r_1(p) \approx \tau^k \geq 2\tau^k \geq \tau^\ell > 8d$, so along $\gamma$
\[
r_2(\gamma(t)) \geq r_2(p) - d \geq r_1(p) - d \geq \frac{7}{4} \tau^k.
\]
Integrating along $\gamma$ we get
\[
\left| \frac{\partial D'h_k}{\partial r_2}(p) - \frac{\partial D'h_k}{\partial r_2}(q) \right| \leq \int _\gamma \left| \nabla _{g_{\theta}} \frac{\partial D'h_k}{\partial r_2} \right| \leq C\tau^{-k} \omega(\tau^k)d \leq C\tau^{-k(\frac{1}{2^2} - 1)} \omega(\tau^k)d^{\frac{1}{2^2} - 1}.
\]
\* $k \leq k_2$. This case is completely the same as in the proof of (3.78), by replacing $r_1$ by $r_2$, $\beta_1$ by $\beta_2$. So we omit the details.
(3.87) can be proved similarly. \qed

3.4. Mixed normal directions

In this section, we will deal with the Hölder continuity of the following four mixed derivatives:
\[
\frac{\partial^2 u}{\partial r_1 \partial r_2}, \quad \frac{\partial^2 u}{\partial r_1 \partial \theta_1}, \quad \frac{\partial^2 u}{\partial r_1 \partial \theta_1}, \quad \frac{\partial^2 u}{\partial r_2 \partial \theta_1 \partial \theta_2}, \quad \frac{\partial^2 u}{\partial r_1 \partial \theta_1 \partial \theta_2}, \quad \frac{\partial^2 u}{\partial r_1 \partial r_2},
\]
(3.89)
which by our previous notation correspond to $N_1 N_2 u$. Since the proof for each of them is more or less the same, we will only prove the Hölder continuity for $\frac{\partial^2 u}{\partial r_1 \partial r_2}$. The following holds at $p$ and $q$ by the same reasoning of Lemma 3.7
\[
\lim _{k \to \infty} \frac{\partial^2 u_k}{\partial r_1 \partial r_2}(p) = \frac{\partial^2 u}{\partial r_1 \partial r_2}(p), \quad \lim _{k \to \infty} \frac{\partial^2 v_k}{\partial r_1 \partial r_2}(q) = \frac{\partial^2 u}{\partial r_1 \partial r_2}(q).
\]
By triangle inequality
\[
\left| \frac{\partial^2 u}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q) \right| \leq \left| \frac{\partial^2 u}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u_k}{\partial r_1 \partial r_2}(p) \right| + \left| \frac{\partial^2 u_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u_k}{\partial r_1 \partial r_2}(q) \right| + \left| \frac{\partial^2 u_k}{\partial r_1 \partial r_2}(q) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q) \right|.
\]
\[ + \left| \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2}(q) - \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2}(q) \right| + \left| \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2}(q) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q) \right| \]

\[ =: L_1 + L_2 + L_3 + L_4. \]

**Lemma 3.15.** We have the following estimate

\[ L_1 + L_4 \leq \sum_{k=\ell}^{\infty} \omega(\tau^k) \]

**Proof.** We consider the different cases that \( k \geq k_p + 1 \) and \( \ell \leq k \leq k_p \).

- **Case 1:** \( k \geq k_p + 1 \). In this case the balls \( \hat{B}_k(p) \) are disjoint with \( S \) and so and we can introduce the smooth coordinates \( w_1 = z_1^{\beta_1} \) and \( w_2 = z_2^{\beta_2} \). Under the coordinates \( \{w_1, w_2, z_3, \ldots, z_n\} \), \( g_\beta \) becomes the standard Euclidean metric \( g_{E^n} \) and the metric balls \( \hat{B}_k(p) \) become the standard Euclidean ball with the same radius and center \( p \). Since the \( g_\beta \)-harmonic functions \( u_k - u_{k+1} \) satisfy (3.49), by standard gradient estimates for Euclidean harmonic functions we get

\[ \sup_{\hat{B}_k(p)} \left| D_{w_1} D_{w_2} (u_k - u_{k-1}) \right| \leq C \omega(\tau^k), \]

where we use \( D_{w_i} \) to denote either \( \frac{\partial}{\partial w_i} \) or \( \frac{\partial}{\partial \theta_i} \) for simplicity. From (3.88) and similar formula for \( \frac{\partial}{\partial r_1} \), we get

\[ \sup_{\hat{B}_k(p)} \left| \frac{\partial^2}{\partial r_1 \partial r_2} (u_k - u_{k-1}) \right| \leq C \omega(\tau^k). \]

- **Case 2:** \( \ell \geq k_2, p + 1 \) and \( \ell \leq k_p \). For all \( \ell \leq k \), the balls \( \hat{B}_k(p) \) are disjoint with \( S \) and center at \( p_1 \). We can still use \( w_2 = z_2^{\beta_2} \) as the smooth coordinate. The cone metric \( g_\beta \) becomes smooth in \( w_2 \)-variable and we can apply the standard gradient estimate to the \( g_\beta \)-harmonic function \( D_{w_2} (u_k - u_{k-1}) \) to get

\[ \sup_{\hat{B}_k(p)} \left| \frac{\partial}{\partial r_1} D_{w_2} (u_k - u_{k-1}) \right| + \left| \frac{\partial}{\partial r_1 \partial \theta_1} D_{w_2} (u_k - u_{k-1}) \right| \leq C \omega(\tau^k). \]

Again by (3.88), we get

\[ \sup_{\hat{B}_k(p)} \left| \frac{\partial^2}{\partial r_1 \partial r_2} (u_k - u_{k-1}) \right| + \left| \frac{\partial^2}{\partial r_1 \partial \theta_1 \partial r_2} (u_k - u_{k-1}) \right| \leq C \omega(\tau^k). \]

- **Case 3:** \( \ell \leq k_2, p \), the case where \( k \geq k_2, p + 1 \) can be dealt with similarly as above. In the case \( \ell \leq k \leq k_2, p \), \( r_2(p) \approx \tau^{k_2, p} \). Now the balls \( \hat{B}_k(p) \) are centered at \( p_{1,2} \in S_1 \cap S_2 \). We can proceed as in the proof of Lemma 3.12 with the harmonic functions \( u_k - u_{k-1} \) replacing \( D'h_k \) as in that lemma to prove that for any \( z \in \frac{1}{4} \hat{B}_k(p) \backslash S \)

\[ \left| \frac{\partial^2}{\partial r_1 \partial r_2} (u_k - u_{k-1}) \right| (z) + \left| \frac{\partial^2}{r_2 \partial \theta_2 \partial r_1} (u_k - u_{k-1}) \right| (z) \]

\[ \leq C(n, \beta) r_1(z)^{\frac{1}{\beta_1} - 1} r_2(z)^{-\frac{1}{\beta_2}} \tau^{-k(-\frac{\beta_1}{\beta_1} + \frac{1}{\beta_2})} \omega(\tau^k). \]

In particular, the estimate in each case holds at \( p \) and from \( r_1(p) \leq r_2(p) \leq \tau^k \), we obtain

\[ \left| \frac{\partial^2}{\partial r_1 \partial r_2} (u_k - u_{k-1}) \right| (p) + \left| \frac{\partial^2}{r_2 \partial \theta_2 \partial r_1} (u_k - u_{k-1}) \right| (p) \leq C \omega(\tau^k). \]
Combining each case above, by (3.90), (3.91) and (3.92), we get for all $k \geq \ell$
\[
\left| \frac{\partial^2 u}{\partial r_1 \partial r_2} (u_k - u_{k-1}) \right|(p) \leq C(n, \beta)\omega(\tau^k).
\]
Therefore by triangle inequality
\[
L_1 \leq \sum_{k=\ell+1}^{\infty} \left| \frac{\partial^2 u}{\partial r_1 \partial r_2} (u_k - u_{k-1}) \right|(p) \leq C(n, \beta) \sum_{k=\ell+1}^{\infty} \omega(\tau^k).
\]
The estimate for $L_4$ can be dealt with similarly by studying the derivatives of $v_k$ at $q$.

**Lemma 3.16.** $L_3$ satisfies
\[
L_3 \leq C(n, \beta)\omega(\tau^\ell).
\]

**Proof.** As in the proof of previous lemma, we need to consider different cases: $\ell \geq k_{1,p} + 1$, $\ell \geq k_{2,p} + 1$ or $\ell \leq k_{2,p}$.

- If $\ell \geq k_{1,p} + 1$, then the ball $\hat{B}_\ell(p) = B_\beta(p, \tau^\ell)$ and the function $U$ defined in (3.55) is $g_\beta$-harmonic in $\frac{1}{2}\hat{B}_\ell(p)$, and $\sup_{\frac{1}{2}\hat{B}_\ell(p)} |U| \leq C\omega(2^\ell \omega(\tau^\ell))$. Since $\frac{1}{2}\hat{B}_\ell(p)$ is disjoint with $S$, $w_1$ and $w_2$ are well-defined on $\frac{1}{2}\hat{B}_\ell(p)$ and thus we have the derivative estimate:
\[
\sup_{\frac{1}{2}\hat{B}_\ell(p)} \left| \frac{\partial^2 U}{\partial r_1 \partial r_2} \right| \leq \sup_{\frac{1}{2}\hat{B}_\ell(p)} \left| D_{w_1} D_{w_2} U \right| \leq C(n, \beta)\omega(\tau^\ell).
\]

In particular, at $q \in \frac{1}{2}\hat{B}_\ell(p)$
\[
L_3 = \left| \frac{\partial^2 u_\ell}{\partial r_1 \partial r_2} (q) - \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2} (q) \right| = \left| \frac{\partial^2 U}{\partial r_1 \partial r_2} (q) \right| \leq C(n, \beta)\omega(\tau^\ell).
\]

- If $k_{1,p} \geq \ell \geq k_{2,p}$, then the ball $\hat{B}_\ell(p) = B_\beta(p_1, 2\tau^\ell)$ and the function $U$ defined in (3.55) is $g_\beta$-harmonic and well-defined in a ball $B_{q_2} := B_\beta(q, \tau^{\ell+1}/10) \subset \frac{1}{2}\hat{B}_\ell(p)$, and $\sup_{\frac{1}{2}\hat{B}_\ell(p)} |U| \leq C\omega(2^\ell \omega(\tau^\ell))$. Since $\frac{1}{2}\hat{B}_\ell(p)$ is disjoint with $S$, $w_2$ is well-defined on $\frac{1}{2}\hat{B}_\ell(p)$ and thus we have the derivatives estimates:
\[
\sup_{\frac{1}{2}\hat{B}_{q_2}} \left| \frac{\partial^2 U}{\partial r_1 \partial r_2} \right| \leq \sup_{\frac{1}{2}\hat{B}_{q_2}} \left| \frac{\partial}{\partial r_1} D_{w_2} U \right| \leq C(n, \beta)\omega(\tau^\ell).
\]

In particular, at $q \in \frac{1}{2}B_{q_2}$, we have
\[
L_3 = \left| \frac{\partial^2 u_\ell}{\partial r_1 \partial r_2} (q) - \frac{\partial^2 v_\ell}{\partial r_1 \partial r_2} (q) \right| = \left| \frac{\partial^2 U}{\partial r_1 \partial r_2} (q) \right| \leq C(n, \beta)\omega(\tau^\ell).
\]

- If $\ell \leq k_{2,p} - 1$, then $r_{2}(p) \approx \tau^{k_{2,p}} \leq \tau^{\ell+1} < 8d$, so $r_2(q) \leq r_2(p) + d \leq \frac{5}{8} \tau^\ell$ and $r_1(q) \leq d + r_1(p) \leq d + r_2(p) \leq \frac{5}{8} \tau^\ell$. Therefore the ball $\hat{B}_\ell(q)$ is centered at $q_1$, or $q_2$ or $q_{1,2} \in S_1 \cap S_2$ with radius $2\tau^\ell$. It follows from this that the function $U$ defined in (3.55) is well-defined on the ball $\frac{1}{8}\hat{B}_\ell(p)$.

By the same strategy as in the proof of Lemma 3.12, with the harmonic function $D'h_k$ in that lemma replaced by $U$ on the metric ball $\frac{1}{8}\hat{B}_\ell(p)$, we can prove that for any $z \in \frac{1}{8}\hat{B}_\ell(p) \setminus S$ the following holds:
\[
\left| \frac{\partial^2 U}{\partial r_1 \partial r_2} (z) \right| \leq C(n, \beta) r_1^{\ell-1} r_2^{-1} \tau^{-\ell(-2+\frac{1}{p_1}+\frac{1}{p_2})} \omega(\tau^\ell).
\]
Applying this inequality at \( q \) we get
\[
L_3 = \left| \frac{\partial^2 (u - v)}{\partial r_1 \partial r_2} (q) \right| \leq C(n, \beta) r_1(q)^{1-\frac{1}{2}} r_2(q)^{1-\frac{1}{2}} \tau^{-\ell - 2 + \frac{1}{m}} \omega(\tau^{\ell}) \leq C\omega(\tau^{\ell}).
\]

In sum, in all cases \( L_3 \leq C(n, \beta) \omega(\tau^{\ell}) \).

\[ \square \]

**Lemma 3.17.** There exists a constant \( C = C(n, \beta) > 0 \) such that for all \( k \leq \ell \) and \( z \in \frac{1}{2} \hat{B}_k(p) \backslash S \)
\[
\left| \frac{\partial}{\partial \theta_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) (z) \right| + \left| \left( \frac{\partial^3 h_k}{\partial r_1 \partial r_2^2 \partial r_2} \right) \right| \leq C \cdot \begin{cases} r_1^{1-\frac{1}{k} - \frac{1}{m}} \tau^{-k \left( 1 - \frac{1}{m} \right)} \omega(\tau^{k}), & \text{if } k \in [k_2, p + 1, \min(\ell, k_p)] \\ r_1^{1-\frac{1}{2} - \frac{1}{m}} \tau^{-k \left( 1 - \frac{1}{2} \right)} \omega(\tau^{k}), & \text{if } k \leq k_2, p. \end{cases}
\]

(3.93)

**Proof.** The proof is parallel to that of Lemma 3.12. The function \( \frac{\partial h_k}{\partial \theta_1} \) is \( g_\beta \)-harmonic on \( \hat{B}_k(p) \), and by the Laplacian estimate (3.9), we have
\[
\sup_{\frac{1}{2} \hat{B}_k(p)} \left( |\Delta_1 \frac{\partial h_k}{\partial \theta_1}| + |\Delta_2 \frac{\partial h_k}{\partial \theta_1}| \right) \leq C(n, \beta) \omega(\tau^{k}).
\]

The function \( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \) is also \( g_\beta \)-harmonic, so the Laplacian estimates (3.9) imply
\[
\sup_{\frac{1}{2} \hat{B}_k(p)} \left( |\Delta_1 \Delta_2 \frac{\partial h_k}{\partial \theta_1}| + |\Delta_2 \Delta_2 \frac{\partial h_k}{\partial \theta_1}| + (D')^2 \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \leq C\tau^{-2k} \left( \text{osc}_{\frac{1}{2} \hat{B}_k(p)} \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \leq C\tau^{-2k} \omega(\tau^{k}).
\]

We consider the equation
\[
|z|^2 (1 - |z|^2) \Delta_2 \frac{\partial h_k}{\partial \theta_1} - \Delta_1 \Delta_2 \frac{\partial h_k}{\partial \theta_1} = -\Delta_1 \Delta_2 \frac{\partial h_k}{\partial \theta_1} - \sum_j \frac{\partial^2}{\partial s_j^2} \Delta_2 \frac{\partial h_k}{\partial \theta_1} =: F_5,
\]
(3.94)

where the function \( F_5 \) satisfies \( \sup_{\frac{1}{2} \hat{B}_k(p)} |F_5| \leq C\tau^{-2k} \omega(\tau^{k}). \)

- In case \( k_2, p + 1 \leq k \leq \min(\ell, k_p) \), we can introduce the smooth coordinate \( w_2 = z_2^2 \) in the ball \( \frac{1}{2} \hat{B}_k(p) \) as before, since the ball is disjoint with \( S_2 \) and under the coordinates \( (r_1, \theta_1; w_2, z_3, \ldots, z_n) \) we can use the usual standard gradient estimate to the \( g_\beta \)-harmonic function \( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \) to obtain that
\[
\sup_{\frac{1}{2} \hat{B}_k(p)} \left| \frac{\partial}{\partial r_2} \left( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \right| + \left| \frac{\partial}{\partial r_2 \partial \theta_2} \left( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \right| \leq C\tau^{-k} \omega(\tau^{k}).
\]
(3.95)

- In case \( k \leq k_2, p \), the ball \( \hat{B}_k(p) \) is centered at \( p_{1,2} \). We apply the usual estimate (2.2) to the function \( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \), the solution to the equation (3.94), on any \( \mathbb{C} \)-ball \( A_2 := B_\mathbb{C}(y, (\tau^{k})^{1/\beta_2}) \) for any \( y \in S_2 \cap \frac{1}{16} \hat{B}_k(p) \) where \( A_2 \) denotes the Euclidean ball in the complex plane orthogonal to \( S_2 \) and passing through \( y \). Then for any \( z \in B_\mathbb{C}(y, (\tau^{k})^{1/\beta_2}/2) \backslash \{y\} \)
\[
\left| \frac{\partial}{\partial z_2} \left( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \right| \leq C \left\| \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right\|_{L^\infty(A_2)} + C \| F_5 \|_{L^\infty(A_2)} (\tau^{k})^{2 - \frac{1}{2}} \leq C\tau^{-k/\beta_2} \omega(\tau^{k}).
\]

This implies that on \( \frac{1}{2} \hat{B}_k(p) \backslash S \)
\[
\left| \frac{\partial}{\partial r_2} \left( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \right| + \left| \frac{\partial}{r_2 \partial \theta_2} \left( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) \right| \leq C r_2^{1-\frac{1}{k}} \tau^{-k/\beta_2} \omega(\tau^{k}).
\]
(3.96)
Taking $\frac{\partial}{\partial r_2}$ on both sides of $\Delta_\beta \frac{\partial h_k}{\partial \theta_1} = 0$, we get
\[
|z_1|^{2(1-\beta_1)} \frac{\partial^2 h_k}{\partial z_1 \partial z_1} \left( \frac{\partial^2 h_k}{\partial r_2 \partial \theta_1} \right) = -\frac{\partial}{\partial r_2} \left( \Delta_2 \frac{\partial h_k}{\partial \theta_1} \right) - \sum_j \frac{\partial^2}{\partial s^2_j} \left( \frac{\partial^2 h_k}{\partial r_2 \partial \theta_1} \right) =: F_6. \tag{3.97}
\]
It is not hard to see from (3.95) and (3.96) and standard derivative estimates that on $\frac{1}{3} \mathcal{B}_k(p) \setminus \mathcal{S}$
(i) in case $k_{2,p} + 1 \leq k \leq \min(\ell, k_p)$, $|F_6| \leq C \tau^{-k} \omega(\tau^k)$;
(ii) in case $k \leq k_{2,p}$, $|F_6| \leq C r_2^{-\frac{k-1}{2}} \tau^{-\frac{\beta}{2}} \omega(\tau^k)$.

Then by applying estimate (2.2) to the function $\frac{\partial^2 h_k}{\partial r_2 \partial \theta_1}$ on any $\mathcal{C}$-ball $A_3 := B_\mathcal{C}_C(x, (\tau^k)^{1/\beta_1})$ for any $x \in \frac{1}{3} \mathcal{B}_k(p) \cap \mathcal{S}_1$ we get that on $B_\mathcal{C}(x, (\tau^k)^{1/\beta_1}/2) \setminus \{x\}$
\[
\left| \frac{\partial}{\partial r_1} \left( \frac{\partial^2 h_k}{\partial r_2 \partial \theta_1} \right) \right| \leq C \left( r_1^{-\frac{k-1}{2}} \tau^{-k(1-\beta_1)} \omega(\tau^k), \text{ if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \right)
\]
\[
+ \left( r_1^{-\frac{\beta}{2}} \tau^{-k(1-\beta_1)} \omega(\tau^k), \text{ if } k \leq k_{2,p} \right).
\]

Therefore this estimate holds on $\frac{1}{3} \mathcal{B}_k(p) \setminus \mathcal{S}$.

\[\square\]

**Lemma 3.18.** For any $k \leq \ell$ and any point $z \in \frac{1}{3} \mathcal{B}_k(p) \setminus \mathcal{S}$ the following estimates hold
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (z) \right| \leq C \left( \begin{array}{ll}
\frac{1}{r_1^{\frac{k-1}{2}}} \tau^{-k(1-\beta_1)} \omega(\tau^k), & \text{if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \\
\frac{1}{r_1^{\frac{k-1}{2}}} \tau^{-k(1-\beta_1)} \omega(\tau^k), & \text{if } k \leq k_{2,p}.
\end{array} \right. \tag{3.98}
\]
\[
\left| \frac{\partial^2 D' h_k}{\partial r_1 \partial r_2} (z) \right| \leq C \left( \begin{array}{ll}
\frac{1}{r_1^{\frac{k-1}{2}}} \tau^{-k(1-\beta_1)} \omega(\tau^k), & \text{if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \\
\frac{1}{r_1^{\frac{k-1}{2}}} \tau^{-k(1-\beta_1)} \omega(\tau^k), & \text{if } k \leq k_{2,p}.
\end{array} \right.
\]

**Proof.** This follows from almost the same argument as in the proof of Lemma 3.17, by studying the harmonic functions $h_k$ and $D' h_k$ instead of $\frac{\partial h_k}{\partial \theta_1}$. $\square$

**Lemma 3.19.** The following estimate holds for any $k \leq \ell$ and any $z \in \frac{1}{3} \mathcal{B}_k(p) \setminus \mathcal{S}$
\[
\left| \frac{\partial}{\partial r_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right| (z) \leq C \omega(\tau^k) \left( \begin{array}{ll}
\tau^{-k} + r_1(z)^{\frac{1}{\beta_1}} \tau^{-k(\frac{1}{\beta_1} - 1)}, & \text{if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \\
r_2(z)^{\frac{1}{\beta_2} - 1} \tau^{-k(\frac{1}{\beta_2} + 1)} + r_1(z)^{\frac{1}{\beta_1}} r_2(z)^{\frac{1}{\beta_2} - 1} \tau^{-k(1-\beta_1)} & \text{if } k \leq k_{2,p}.
\end{array} \right. \tag{3.99}
\]

**Proof.** By the Laplacian estimates (3.9) we have
\[
\sup_{\frac{1}{3} \mathcal{B}_k(p) \setminus \mathcal{S}} |\Delta_1 h_k| + |\Delta_2 h_k| \leq C(n, \beta) \omega(\tau^k).
\]
Applying again the Laplacian estimate (3.9) to the $\frac{\partial h_k}{\partial r_1}$ harmonic function $\Delta_1 h_k$,
\[
\sup_{\frac{1}{3} \mathcal{B}_k(p) \setminus \mathcal{S}} \left( |\Delta_1 \Delta_1 h_k| + |\Delta_2 \Delta_1 h_k| + |(D')^2 \Delta_1 h_k| \right) \leq C(n, \beta) \tau^{-2k} \omega(\tau^k).
We consider the equation
\[ |z_2|^{2-2\beta_2} \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} \Delta_1 h_k = -\Delta_1 \Delta_1 h_k - \sum_j \frac{\partial^2}{\partial s_j^2} \Delta_1 h_k =: F_7. \quad (3.100) \]

From the estimates above, we see \[ \|F_7\|_{L^\infty(\frac{1}{15} \hat{B}_k(p))} \leq C \tau^{-2k} \omega(\tau^k). \]

- When \( k_{2,p} + 1 \leq k \leq \min(\ell, k_p) \), we directly apply gradient estimate to \( \Delta_1 h_k \) to get
  \[ \sup_{\frac{1}{15} \hat{B}_k(p) \setminus S} \left| \frac{\partial}{\partial r_2} \Delta_1 h_k \right| + \left| \frac{\partial}{r_2 \partial \theta_2} \Delta_1 h_k \right| \leq C \tau^{-k} \omega(\tau^k). \quad (3.101) \]

- When \( k \leq k_{2,p} \), the balls \( \hat{B}_k(p) \) are centered at \( p_{1,2} \), and we can apply the usual \( \mathbb{C} \)-ball type estimate to get that for all \( z \in \frac{1}{2} \hat{B}_k(p) \setminus S \)
  \[ \left| \frac{\partial}{\partial r_2} \Delta_1 h_k \right|(z) + \left| \frac{\partial}{r_2 \partial \theta_2} \Delta_1 h_k \right| \leq C r_2(z) \frac{1}{\beta_2} \frac{1}{\tau^k / \tau^{2k}} + C r_2(z) \frac{1}{\beta_2} \frac{1}{\tau^k / \tau^{2k}} \|F_7\|_{L^\infty(\tau^{k(2-\frac{1}{2})})} \leq C r_2(z) \frac{1}{\beta_2} \frac{1}{\tau^k / \tau^{2k}} \omega(\tau^k). \]

Recall the following equation holds
\[ \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) = \frac{\partial}{\partial r_2} \Delta_1 h_k - \frac{1}{r_1} \frac{\partial^2 h_k}{\partial r_1 \partial r_2} - \frac{1}{\beta_1^2} \frac{\partial^2 h_k}{\partial \theta_1^2 \partial \theta_2} \]
from which we derive that for any \( z \in \frac{1}{2} \hat{B}_k(p) \setminus S \)
\[ \left| \frac{\partial}{\partial r_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|(z) \leq C \omega(\tau^k) \left\{ \begin{array}{ll} \tau^{-k} + r_1(z) \frac{1}{\beta_1^2} \frac{1}{\tau^k / \tau^{2k}}, & \text{if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \\ r_2(z) \frac{1}{\beta_2} \frac{1}{\tau^k / \tau^{2k}}, & \text{if } k \leq k_{2,p} \end{array} \right. \]

**Lemma 3.20.** There exists a constant \( C = C(n, \beta, \beta_2) > 0 \) such that for all \( k \leq \ell \) and \( z \in \frac{1}{2} \hat{B}_k(p) \setminus S \)
\[ \left| \frac{\partial}{\partial \theta_2} \left( \frac{\partial^2 h_k}{\partial \theta_1 \partial \theta_2} \right) \right|(z) + \left( \frac{\partial^3 h_k}{\partial \theta_1^2 \partial \theta_2 \partial r_1} \right)(z) \leq C \omega(\tau^k) \left\{ \begin{array}{ll} r_1^{\frac{1}{\beta_1^2}} \tau^{-k(-1+\frac{1}{\beta_1})}, & \text{if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \\ r_2^{\frac{1}{\beta_2} / \tau^k} \tau^{-k(-1+\frac{1}{\beta_2})}, & \text{if } k \leq k_{2,p} \end{array} \right. \quad (3.102) \]

**Proof.** It follows from the Laplacian estimate (3.9) that
\[ \sup_{\frac{1}{15} \hat{B}_k(p) \setminus S} \left( |\Delta_1 \frac{\partial h_k}{\partial \theta_2}| + |\Delta_2 \frac{\partial h_k}{\partial \theta_2}| \right) \leq C(n) \omega(\tau^k). \]
And by (3.9) again we have
\[ \sup_{\frac{1}{15} \hat{B}_k(p) \setminus S} \left( |\Delta_1 \Delta_1 \frac{\partial h_k}{\partial \theta_2}| + |\Delta_2 \Delta_1 \frac{\partial h_k}{\partial \theta_2}| + |(D^2 \Delta_1 \frac{\partial h_k}{\partial \theta_2})| \right) \leq C \tau^{-2k} \omega(\tau^k). \]

We look at the equation
\[ |z_1|^{2(1-\beta_1)} \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} \left( \Delta_1 \frac{\partial h_k}{\partial \theta_2} \right) = -\Delta_2 \Delta_1 \frac{\partial h_k}{\partial \theta_2} - \sum_j \frac{\partial^2}{\partial s_j^2} \left( \Delta_1 \frac{\partial h_k}{\partial \theta_2} \right) =: F_8, \]
and $F_8$ satisfies sup $\frac{1}{2} B_k(p) |F_8| \leq C \tau^{-2k} \omega(\tau^k)$. By the estimate (2.2) as we did before it follows that for any $z \in \frac{1}{2} B_k(p) \setminus S$ (remember here $k \leq \min(\ell, k_0)$)

$$\left| \frac{\partial}{\partial r_1} \Delta_1 \frac{\partial h_k}{\partial \theta_1} (z) \right| + \left| \frac{\partial}{\partial r_1} \Delta_1 \frac{\partial h_k}{\partial \theta_2} (z) \right| \leq C r_1 (z) \frac{1}{\tau^{k/\beta_1}} \left\| \Delta_1 \frac{\partial h_k}{\partial \theta_1} \right\|_{L^\infty} + C r_1 (z) \frac{1}{\tau^{k/\beta_1}} \left\| F_8 \right\|_{L^\infty} \tau^{(2 - \beta_1/\beta_2)}$$

$$\leq C r_1 (z) \frac{1}{\tau^{k/\beta_1}} \tau^{-k/\beta_1} \omega(\tau^k).$$

Taking $\frac{\partial}{\partial r_1}$ on both sides of the equation $\Delta \beta \frac{\partial h_k}{\partial \theta_2} = 0$, we get

$$|z^2|^{2(1 - \beta_2)} \frac{\partial^2 h_k}{\partial z^2 \partial z^2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial \theta_2} \right)(z) = - \frac{\partial}{\partial r_1} \left( \Delta_1 \frac{\partial h_k}{\partial \theta_1} \right) - \sum_j \frac{\partial}{\partial r_1} \left( \frac{\partial^2 h_k}{\partial \theta_j \partial \theta_2} \right) =: F_0.$$ (3.103)

Here $|F_0(z)| \leq C r_1 (z) \frac{1}{\tau^{k/\beta_1}} \tau^{-k/\beta_1} \omega(\tau^k)$ for any $z \in \frac{1}{2} B_k(p) \setminus S$. Therefore we get by the usual C-ball argument that

- If $k \leq k_2, p$, then for any $z \in \frac{1}{2} B_k(p) \setminus S$

$$\left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial \theta_2} \right)(z) \right| + \left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial \theta_2} \right)(z) \right| \leq C r_2 (z) \frac{1}{\tau^{k/\beta_1}} r_1 (z) \frac{1}{\tau^{k/\beta_1}} \omega(\tau^k)$$

- If $k_2, p + 1 \leq k \leq \min(\ell, k_0)$, then

$$\left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial \theta_2} \right)(z) \right| + \left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial \theta_2} \right)(z) \right| \leq C r_1 (z) \frac{1}{\tau^{k/\beta_1}} \tau^{k(1 - \beta_1/\beta_2)} \omega(\tau^k).$$

\[ \square \]

**Lemma 3.21.** The following estimate holds for any $k \leq \ell$ and any $z \in \frac{1}{2} B_k(p) \setminus S$

$$\left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right| \leq C \omega(\tau^k) \begin{cases} r_1 (z) \frac{1}{\tau^{k/\beta_1}} \tau^{-k/\beta_1} + r_1 \frac{1}{\tau^{k/\beta_1}} \tau^{-k(1 + \beta_1/\beta_2)}, & \text{if } k \in [k_2, p + 1, \min(\ell, k_0)] \\
 r_1 (z) \frac{1}{\tau^{k/\beta_1}} \tau^{-k/\beta_1} + r_1 \frac{1}{\tau^{k/\beta_1}} \tau^{-k(1 + \beta_1/\beta_2)}, & \text{if } k \leq k_2, p. \end{cases}$$ (3.104)

**Proof.** We first observe the following equation

$$\frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) = \frac{\partial}{\partial r_1} \Delta_2 h_k - \frac{1}{r_2} \frac{\partial^2 h_k}{\partial r_1 \partial r_2} - \frac{1}{\beta_2^2 r_2^2} \frac{\partial^2}{\partial r_2^2} \left( \frac{\partial h_k}{\partial r_1} \right).$$

It can be shown that for any $z \in \frac{1}{2} B_k(p) \setminus S$ by the C-ball argument that

$$\left| \frac{\partial}{\partial r_1} \Delta_2 h_k (z) \right| \leq C r_1 (z) \frac{1}{\tau^{k/\beta_1}} \tau^{-k/\beta_1} \omega(\tau^k).$$

From Lemma 3.18, we have for any $z \in \frac{1}{2} B_k(p) \setminus S$

$$\left| \frac{1}{r_2} \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (z) \right| \leq C \begin{cases} r_1 \frac{1}{\tau^{k/\beta_1}} \tau^{-k(-1 + \beta_1/\beta_2)} \omega(\tau^k), & \text{if } k \in [k_2, p + 1, \min(\ell, k_0)] \\
 r_1 \frac{1}{\tau^{k/\beta_1}} \tau^{-k(2 + \beta_1/\beta_2)} \omega(\tau^k), & \text{if } k \leq k_2, p. \end{cases}$$

From Lemma 3.20, we have for any $z \in \frac{1}{2} B_k(p) \setminus S$

$$\left| \frac{1}{r_2^2} \frac{\partial^3 h_k}{\partial r_1 \partial \theta_2^2} (z) \right| \leq C \omega(\tau^k) \begin{cases} r_1 \frac{1}{\tau^{k/\beta_1}} \tau^{-k(-1 + \beta_1/\beta_2)}, & \text{if } k \in [k_2, p + 1, \min(\ell, k_0)] \\
 r_1 \frac{1}{\tau^{k/\beta_1}} \tau^{-k(-2 + \beta_1/\beta_2)} \omega(\tau^k), & \text{if } k \leq k_2, p. \end{cases}$$
Therefore for any \( z \in \frac{1}{3} \hat{B}_k(p) \setminus S \) we have
\[
\left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right)(z) \right| \leq C \omega(\tau^k) \cdot \begin{cases} 
 r_1(z) \tau^{-k-\frac{1}{\tau}} - r_1(z) \tau^{-k-\frac{1}{\tau}}, & \text{if } k \in [k_{2,p} + 1, \min(\ell, k_p)] \\
 r_1(z) \tau^{-k-\frac{1}{\tau}} - r_1(z) \tau^{-k-\frac{1}{\tau}}, & \text{if } k \leq k_{2,p}.
\end{cases}
\]

It remains to estimate \( L_2 \). For simplicity, we denote \( h_k := -u_k + u_{k-1} \) as before, where we take \( k \leq \ell \). We will denote \( \beta_{\max} = \max(\beta_1, \beta_2) \).

**Lemma 3.22.** Let \( d = d_\beta(p, q) \). There exists a constant \( C(n, \beta) > 0 \) such that for all \( k \leq \ell \)
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| \leq C \omega(\tau^k) \tau^{-k\left(\frac{1}{r_1} - 1\right)} d_{\max}^{-1} \leq C \omega(\tau^k) \tau^{-k\left(\frac{1}{r_1} - 1\right)} d_{\max}^{-1}.
\]

**Proof.**

**Case 1:** First we assume that \( r_p \leq 2d_\beta \) so \( r_q \leq 3d_\beta \) and \( \ell + 2 \leq k_p \), in particular, the balls \( \hat{B}_k(p) \) are centered at either \( p_1 \in S_1 \) or 0, depending on whether \( k \geq k_{2,p} + 1 \) or \( k \leq k_{2,p} \). As in the proof of Lemma 3.13, let \( \gamma : [0, d] \to B_\beta(0, 1) \setminus S \) be the \( g_\beta \)-geodesic connecting \( p \) and \( q \). The two points \( q' \) and \( p' \) are defined as in (3.80), \( \gamma_1, \gamma_2, \gamma_3 \) the \( g_\beta \)-geodesics as defined in that lemma. We calculate by triangle inequality that
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| \leq \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p') \right| + \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p') - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q') \right| \\
+ \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q') - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| =: L'_1 + L'_2 + L'_3.
\]

Integrating along \( \gamma_3 \) on which the coordinates \((r_1, r_2, \theta_2; z_3, \ldots, z_n)\) are the same as \( p \), we get by (3.93)
\[
L'_1 = \int_{\gamma_3} \frac{\partial}{\partial \theta_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) d\theta_1 \\
\leq C \omega(\tau^k) \begin{cases} 
 r_1(p) \tau^{-k-\frac{1}{\tau}} - r_1(q) \tau^{-k-\frac{1}{\tau}}, & \text{if } k \in [k_{2,p} + 1, \ell] \\
 r_1(p) \tau^{-k-\frac{1}{\tau}} - r_1(q) \tau^{-k-\frac{1}{\tau}}, & \text{if } k \leq k_{2,p}.
\end{cases}
\]

Integrating along \( \gamma_2 \) along which the coordinates \((\theta_1, r_2, \theta_2; z_3, \ldots, z_n)\) are the same as \( p' \) or \( q' \), we get by (3.99) that
\[
L'_2 = \int_{\gamma_2} \frac{\partial}{\partial \theta_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) d\theta_1 \\
\leq C \omega(\tau^k) \begin{cases} 
 r_1(p) \tau^{-k-\frac{1}{\tau}} d_{\max}^{-1} - r_1(q) \tau^{-k-\frac{1}{\tau}} d_{\max}^{-1}, & \text{if } k \in [k_{2,p} + 1, \ell] \\
 r_1(p) \tau^{-k-\frac{1}{\tau}} d_{\max}^{-1} - r_1(q) \tau^{-k-\frac{1}{\tau}} d_{\max}^{-1}, & \text{if } k \leq k_{2,p}.
\end{cases}
\]

To deal with the term \( L'_3 \), we consider different cases of \( k \), either \( \ell \geq k \geq k_{2,p} + 1 \) or \( k \leq k_{2,p} \).
• If \( k_{2,p} + 1 \leq k \leq \ell \), the balls \( \hat{B}_k(p) \) are centered at \( p_1 \in S_1 \). Here \( \tau^{-k} \leq \tau^{-\ell} \leq 8^{-1}d^{-1} \) and \( \tau^k \leq \tau^{k_{2,p}+1} \leq \frac{1}{2}r_2(p) \), so \( r_2(q) \geq -d + r_2(p) \geq \tau^k \). The balls \( \hat{B}_k(p) \) are disjoint with \( S_2 \), we can use the smooth coordinate \( w_2 = z_2^{\beta_2} \) as before. The functions \( D_w D'h_k \) are \( g_\beta \)-harmonic, hence by gradient estimate we have

\[
\sup_{\frac{1}{2}\hat{B}_k(p) \setminus S_1} \left| \nabla g_\beta (D_w D'h_k) \right| \leq C(n) \frac{\|D_w D'h_k\|_{L^\infty(\frac{1}{2}\hat{B}_k(p))}}{\tau^k} \leq C\tau^{-k} \omega(\tau^k).
\]

From (3.88), we get that

\[
\sup_{\frac{1}{2}\hat{B}_k(p) \setminus S_1} \left| \frac{\partial^2}{\partial r_1 \partial r_2} D'h_k \right| \leq C(n) \tau^{-k} \omega(\tau^k). \tag{3.105}
\]

Recall \( r_1(p) = \tau_p \leq 2d \leq \frac{1}{2} \tau^k \), triangle inequality implies that \( r_1(q) \leq 3d \leq \frac{1}{2} \tau^k \). The points in \( \gamma_1 \) have the fixed \( (r_1, \theta_1) \)-coordinates \((r_1(q), \theta_1(q))\), so integrating along \( \gamma_1 \) we get by (3.104) and (3.105) that

\[
L'_3 \leq \int_{\gamma_1} \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (q') \right| + \left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right| + \left| D' \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|
\leq C d \omega(\tau^k) \left( r_1(q)^{\frac{1}{m_1} - 1} \tau^{-k/\beta_1} + r_1(q)^{\frac{1}{m_1} - 1} \min(r_2(p), r_2(q))^{-1} \tau^{-k/(m_1 - 1)} + \tau^{-k} \right)
\leq C \tau^{-k} \omega(\tau^k) d \leq C \tau^{-k/(m_1 - 1)} \omega(\tau^k) d^{1/m_1}.
\]

• If \( k \leq k_{2,p} \), the \( \tau^k \geq \tau^{k_{2,p}} \geq r_2(p) \) and \( \tau^k \geq \tau^\ell \geq 8d \). Thus \( r_2(q) \leq r_2(p) + d \leq \frac{3}{2} \tau^k \). We choose points \( \bar{q}, \bar{q} \) as in (3.84), and let \( \bar{\gamma}_1, \bar{\gamma} \) and \( \bar{\gamma} \) be \( g_\beta \)-geodesics defined as in the proof of Lemma 3.13. Then we have

\[
L'_3 \leq \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (\bar{q}') \right| + \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (\bar{q}) \right| + \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (\bar{q}) \right| =: L''_3 + L''_2 + L''_3.
\]

We will estimate \( L''_3, L''_2 \) and \( L''_3 \) term by term by integrating appropriate functions along the geodesics \( \bar{\gamma}_1, \bar{\gamma} \) and \( \bar{\gamma} \) as follows: The points in \( \bar{\gamma} \) have fixed \( (r_1, \theta_1; r_2, s) \)-coordinates \((r_1(q), \theta_1(q); r_2(q), s(q))\) and by (3.102)

\[
L''_3 = \int_{\bar{\gamma}} \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| d\theta_2 \leq C \omega(\tau^k) r_1(q)^{-1} r_2(q)^{-1} \tau^{-k(-2 + \frac{1}{m_1} + \frac{1}{m_2})}
\leq C \omega(\tau^k) r_1(q)^{-1} \tau^{-k(-1 + \frac{1}{m_1})} \leq C \tau^{-k/(m_1 - 1)} \omega(\tau^k) d^{1/m_1}.
\]

Integrating along \( \bar{\gamma} \) on which the points have constant \( r_1 \)-coordinate \( r_1(q) \), we get by (3.104)

\[
L''_2 = \int_{\bar{\gamma}} \left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| d\theta_2 \leq C \omega(\tau^k) \left( r_1(q)^{-1} - r_2(q) - r_2(p) \right) + r_1(q)^{-1} \tau^{-k(-2 + \frac{1}{m_1} + \frac{1}{m_2})} \left( r_2(q)^{\frac{1}{m_2}} - r_2(p)^{\frac{1}{m_2}} - r_2(q)^{\frac{1}{m_2}} \right)
\leq C \omega(\tau^k) \left( r_1(q)^{-1} - r_2(q) - r_2(p) \right) + r_1(q)^{-1} \tau^{-k(-2 + \frac{1}{m_1} + \frac{1}{m_2})} \left( r_2(q)^{\frac{1}{m_2}} - r_2(p)^{\frac{1}{m_2}} - r_2(q)^{\frac{1}{m_2}} \right)
\leq C \omega(\tau^k) r_1(q)^{-1} \tau^{-k(-1 + \frac{1}{m_1})} \leq C \tau^{-k/(m_1 - 1)} \omega(\tau^k) d^{1/m_1}.
\]
Integrating along \( \tilde{\gamma}_1 \) on which the points have constant \((r_1, \theta_1; r_2, \theta_2)\)-coordinates, we have by (3.71) that
\[
L_1'' \leq \int_{\tilde{\gamma}_1} \left| D' \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right| \leq C r_1(q)^{\frac{1}{\gamma_1} - 1} r_2(p)^{\frac{1}{\gamma_2} - 1} \tau^{-k(1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2})} \leq C d \tau^{-k} \omega(\tau^k) \leq C d \tau^{-k} \omega(\tau^k).
\]

Combining both cases, we conclude that \( L_3' \leq C \tau^{-k} \left( \frac{1}{\gamma_1} - 1 \right) \omega(\tau^k) d \tau^{-1} \). Then by the estimates above for \( L_1' \) and \( L_2' \), we finally get for all \( k \leq \ell \)
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| \leq C \omega(\tau^k) \tau^{-k(\frac{1}{\gamma_1} - 1)} d \tau^{-1} \leq C \omega(\tau^k) \tau^{-k(\frac{1}{\gamma_{\max}} - 1)} d \tau^{-1},
\]
where in the last inequality we use the fact that \( \tau^{-k} d \leq 1/8 < 1 \), when \( k \leq \ell \). Hence we finish the proof of Lemma 3.22 in case \( r_p \leq 2d \).

Now we deal with the remaining cases.

**Case 2:** here we assume \( \min(r_p, r_q) = r_p \geq 2d \) and \( \ell \leq k_p \). In this case \( \tau^{k_p} \approx r_p \geq 2d \geq \tau^{\ell+3} \), so \( \ell + 3 \geq k_p \). It follows by triangle inequality that \( d_{\beta}(\gamma(t), S) \geq d \), where \( \gamma \) is the \( g_\beta \)-geodesic joining \( p \) to \( q \) as before. In particular this implies that \( \min(r_1(\gamma(t)), r_2(\gamma(t))) \geq d \).

In this case \( \ell \leq k_p \), Lemmas 3.17 - 3.21 hold for all \( k \leq \ell \) and \( r_1(p) \approx \tau^{k_p} \leq \tau^\ell \), so \( r_1(\gamma(t)) \leq d + r_1(p) \leq \frac{9}{8} \tau^\ell \leq \frac{9}{8} \tau^k \). We calculate the gradient of \( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \) along the geodesic \( \gamma \) as follows
\[
\left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(\gamma(t)) \right|^2 = \left| \frac{\partial}{\partial r_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|^2 + \left| \frac{1}{\beta_1 r_1 \partial \theta_1} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|^2 + \left| \frac{\partial}{\partial r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|^2
+ \left| \frac{\partial}{\partial \theta_2 r_2} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|^2 + \sum_j \left| \frac{\partial}{\partial s_j} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right|^2.
\]

(1) When \( k_{2,p} + 1 \leq k \leq \ell \) we have along \( \gamma \)
\[
r_2(\gamma(t)) \geq r_2(p) - d \geq \tau^k - d \geq \frac{7}{8} \tau^k.
\]
Then by Lemmas 3.17 - 3.21 we have along \( \gamma \) that
\[
\left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(\gamma(t)) \right| \leq C \omega(\tau^k) \left( \tau^{-k} + d \tau^{-1} \tau^{-k(\frac{1}{\gamma_1} - 1)} \right)
\]
Integrating this inequality along \( \gamma \) we get
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2}(q) \right| \leq \int_{\gamma} \left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| \leq C \omega(\tau^k) \left( d \tau^{-k} + d \tau^{-1} \tau^{-k(\frac{1}{\gamma_1} - 1)} \right) \leq C d \tau^{-1} \tau^{-k(\frac{1}{\gamma_1} - 1)} \omega(\tau^k).
\]

(2) When \( k \leq k_{2,p} \), we have along \( \gamma \)
\[
r_2(\gamma(t)) \leq r_2(p) + d \leq \tau^k + d \leq \frac{9}{8} \tau^k.
\]
Then by Lemmas 3.17 - 3.21 we have along \( \gamma \) that
\[
\left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| \leq C \omega(\tau^k) \left( \tau^{-k} + d \tau^{-1} \tau^{-k(\frac{1}{\gamma_1} - 1)} \right).
\]
Integrating this inequality along $\gamma$ we again get
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (q) \right| \leq \int_\gamma \left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| \leq Cd^\frac{n-1}{n-1} \tau^{-k(\frac{n}{n-1}-1)} \omega(\tau^k).
\]

**Case 3:** here we assume $\min(r_p, r_q) = r_p \geq 2d$, but $\ell \geq k_p + 1$. The case when $k \leq k_p$ can be dealt with by the same argument as in **Case 2**, so we omit it and only consider the cases when $k_p + 1 \leq k \leq \ell$, so here $r_2(p) \geq r_1(p) \geq \tau^k \geq \tau^\ell > 8d$ so $r_1(\gamma(t)) \geq \frac{8}{\beta} \tau^k$ and $r_2(\gamma(t)) \geq \frac{8}{\beta} \tau^k$ for any point $\gamma(t)$ in the geodesic $\gamma$. By triangle inequality it follows that $\gamma \subset \frac{1}{8} \tilde{B}_k(p) = B_\beta(p, \tau^k/3)$.

As before, we can introduce smooth coordinates $w_1 = \tilde{z}_1$ and $w_2 = \tilde{z}_2$, and $g_\beta$ becomes the standard smooth Euclidean metric $g_{eu}$ under these coordinates. Moreover, $h_k$ are the usual Euclidean harmonic function $\Delta_{g_{eu}} h_k = 0$ on $\tilde{B}_k(p)$. By the standard derivative estimates we have
\[
\sup_{\frac{1}{8} B_k(p)} \left( |D^3 w_{1, w_2} h_k| + |D' (D^2 w_{1, w_2}) h_k| \right) \leq C \tau^{-k} \omega(\tau^k).
\]

From the equation
\[
\frac{\partial^2 h_k}{\partial r_1 \partial r_2} = \frac{w_1 w_2}{r_1 r_2} \frac{\partial^2 h_k}{\partial w_1 \partial w_2} + \frac{\bar{w}_1 w_2}{r_1 r_2} \frac{\partial^2 h_k}{\partial \bar{w}_1 \partial \bar{w}_2} + \frac{w_1 \bar{w}_2}{r_1 r_2} \frac{\partial^2 h_k}{\partial w_1 \partial \bar{w}_2} + \frac{\bar{w}_1 w_2}{r_1 r_2} \frac{\partial^2 h_k}{\partial \bar{w}_1 \partial w_2}
\]

we see that for $i = 1, 2$
\[
\sup_{\frac{1}{8} B_k(p)} \left| \frac{\partial}{\partial w_i} \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right| \leq C \frac{\omega(\tau^k)}{r_i} + C \tau^{-k} \omega(\tau^k), \quad \sup_{\frac{1}{8} B_k(p)} \left| D' \left( \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right) \right| \leq C \tau^{-k} \omega(\tau^k).
\]

From this we see that
\[
\sup_\gamma \left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| \leq \sup_\gamma \left( C \tau^{-k} \omega(\tau^k) + C \frac{\omega(\tau^k)}{r_1} + C \frac{\omega(\tau^k)}{r_2} \right) \leq C \tau^{-k} \omega(\tau^k).
\]

Integrating along $\gamma$ we see that
\[
\left| \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (p) - \frac{\partial^2 h_k}{\partial r_1 \partial r_2} (q) \right| \leq \int_\gamma \left| \nabla g_\beta \frac{\partial^2 h_k}{\partial r_1 \partial r_2} \right| \leq Cd^\frac{n-1}{n-1} \tau^{-k(\frac{n}{n-1}-1)} \omega(\tau^k).
\]

Combining the estimates in all three cases, we finish the proof of Lemma 3.22.

By Lemma 3.22 it follows that
\[
L_2 = \left| \left| \frac{\partial^2 u_i}{\partial r_1 \partial r_2} (p) - \frac{\partial^2 u_i}{\partial r_1 \partial r_2} (q) \right| \leq \left| \left| \frac{\partial^2 u_2}{\partial r_1 \partial r_2} (p) - \frac{\partial^2 u_2}{\partial r_1 \partial r_2} (q) \right| \right| + Cd^\frac{n-1}{n-1} \sum_{k=3}^\ell \tau^{-k(\frac{n}{n-1}-1)} \omega(\tau^k).
\]

To finish the proof, it suffices to estimate the first term on the RHS of the above equation. Recall we assume $u_2$ is a $g_\beta$-harmonic function defined on the ball $\tilde{B}_2(p)$, which is centered at $p_{1,2} \in S_1 \cap S_2$ and has radius $2\tau^2$. $u_2$ satisfies the $L^\infty$ estimate by maximum principle: there exists some $C = C(n) > 0$ such that
\[
\|u_2\|_{L^\infty(\tilde{B}_2(p))} \leq C (\|u\|_{L^\infty(B_\beta(0,1))} + \omega(\tau^2)).
\]

Recall that the proofs of the estimates in Lemmas 3.17 - 3.21 in the case when $k \leq k_{2, p}$ work for any $g_\beta$-harmonic functions defined on suitable balls, and we can repeat the arguments there replacing the $L^\infty$-estimates of $h_k$ that $\|h_k\|_{L^\infty} \leq C \tau^{2k} \omega(\tau^k)$, by the $L^\infty$-estimate of $u_2$ as in (3.107), to get
similar estimates as in those lemmas, which we will not repeat here. Given these estimates, we can repeat the proof of Lemma 3.22 to prove the following estimates

\[ \left| \frac{\partial^2 u_2}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u_2}{\partial r_1 \partial r_2}(q) \right| \leq C d^{\beta_{\max}^1 - 1}(\|u\|_{L^\infty(B_{\beta}(0, 1))} + \omega(\tau^2)). \]

This inequality, combined with (3.106) give the final estimate of the term \( L_2 \), that

\[ L_2 \leq C d^{\beta_{\max}^1 - 1}\|u\|_{L^\infty(B_{\beta}(0, 1))} + C d^{\beta_{\max}^2 - 1} \sum_{k=2}^{\ell} \tau^{-k(\frac{1}{\beta_{\max}^1} - 1)} \omega(\tau^k). \] (3.108)

By Lemma 3.15, Lemma 3.16 and the estimate (3.108) for \( L_2 \), we are ready to prove the following estimate (see the equation (1.5))

**Proposition 3.5.** For the given \( p, q \in B_{\beta}(0, 1/2) \setminus \mathcal{S} \), there is a constant \( C = C(n, \beta) > 0 \) such that

\[ \left| \frac{\partial^2 u}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q) \right| \leq C \left( d^{\beta_{\max}^1 - 1}\|u\|_{L^\infty(B_{\beta}(0, 1))} + \int_0^d \frac{\omega(r)}{r} dr + d^{\beta_{\max}^1 - 1} \int_d^{1} \frac{\omega(r)}{r^{1/\beta_{\max}}} dr \right). \]

**Proof.** From Lemma 3.15, Lemma 3.16 and the estimate (3.108) for \( L_2 \), we have

\[ \left| \frac{\partial^2 u}{\partial r_1 \partial r_2}(p) - \frac{\partial^2 u}{\partial r_1 \partial r_2}(q) \right| \leq C \left( d^{\beta_{\max}^1 - 1}\|u\|_{L^\infty(B_{\beta}(0, 1))} + d^{\beta_{\max}^1 - 1} \tau^{-k(\frac{1}{\beta_{\max}^1} - 1)} \omega(\tau^k) \right) \]

where the last inequality follows from the fact that \( \omega(r) \) is monotonically increasing.

Finally, we remark that the estimates for the other operators in (3.89) follows similarly, so we omit the detailed proof and just state that the estimates are the same as the estimates for \( \frac{\partial^2 u}{\partial r_1 \partial r_2} \) as in Proposition 3.5.

### 3.5. Non-flat conical Kähler metrics

In this section, we will consider the Schauder estimates for general conical Kähler metrics on \( B_{\beta}(0, 2) \subset \mathbb{C}^n \) with cone angle \( 2\pi \beta \) along the simple normal crossing hyper-surface \( \mathcal{S} \). Let \( \omega \) be such a metric. By definition, there exists a constant \( C \geq 1 \) such that

\[ C^{-1} \omega_{\beta} \leq \omega \leq C \omega_{\beta}, \quad \text{in } B_{\beta}(0, 2) \setminus \mathcal{S}, \] (3.109)

where \( \omega_{\beta} \) is the standard flat conical metric as before. Since \( \omega \) is closed and \( B_{\beta}(0, 2) \) is simply connected, we can write \( \omega = \sqrt{-1} \partial \bar{\partial} \phi \) for some strictly pluri-subharmonic function \( \phi \). By elliptic regularity, \( \phi \) is Holder continuous under the Euclidean metric on \( B_{\beta}(0, 2) \).

We fix an \( \alpha \in (0, \min \{\frac{1}{\beta_{max}^1} - 1, 1\}) \).

**Definition 3.1.** We say \( \omega = g \) a \( C_{\beta}^{0, \alpha} \) Kähler metric on \( B_{\beta}(0, 2) \) if it satisfies (3.109) and the Kähler potential \( \phi \) of \( \omega \) belongs to \( C_{\beta}^{2, \alpha}(B_{\beta}(0, 2)) \).
Our interest is to study the Laplacian equation
\[ \Delta_g u = f, \quad \text{in } B_{\beta}(0,1), \quad (3.110) \]
where \( f \in C^0_{\beta}(B_{\beta}(0,1)) \) and \( u \in C^2_{\beta} \). We will prove the following scaling-invariant interior Schauder estimates, and the proof follows closely from that of Theorem 6.6 in [19]. So we mainly focus on the differences.

**Proposition 3.6.** There exists a constant \( C = C(n, \beta, \|g\|_{C^0_{\beta}(\cdot)}^*) > 0 \) such that if \( u \in C^2_{\beta}(B_{\beta}(0,1)) \) satisfies the equation (3.110), then
\[ \|u\|_{C^2_{\beta}(B_{\beta}(0,1))} \leq C(\|u\|_{C^0_{\beta}(B_{\beta}(0,1))} + \|f\|_{C^0_{\beta}(B_{\beta}(0,1))}^{(2)}). \quad (3.111) \]

**Proof.** Given any points \( x_0 \neq y_0 \in B_{\beta}(0,1) \), assume \( d_{x_0} = \min(d_{x_0}, d_{y_0}) \) (recall \( d_x = d_{\beta}(x, \partial B_{\beta}(0,1)) \)). Let \( \beta \in (0, 1/4) \) be a small number to be determined later. Denote \( d = \mu d_{x_0} \) and \( B := B_{\beta}(x_0, d) \), and \( B/2 := B_{\beta}(x_0, d/2) \).

- **Case 1.** \( d_{\beta}(x_0, y_0) < d/2 \).

  **Case 1.1.** \( B_{\beta}(x_0, d) \cap \mathcal{S} = \emptyset \). On \( B_{\beta}(x_0, d) \) we can introduce the smooth complex coordinates \( \{w_1 = z_1^{\beta_1}, w_2 = z_2^{\beta_2}, z_3, \ldots, z_n\} \), under which \( g_{\beta} \) becomes the Euclidean one and the components of \( g \) become \( C^\alpha \) in the usual sense. The equation (3.110) has \( C^\alpha \) leading coefficients and we can apply Theorem 6.6 in [19] to conclude that (the following inequality is understood in the new coordinates)
\[ [u]^*_{C^{2,\alpha}(B)} \leq C(\|u\|_{C^0(B)} + \|f\|_{C^0_{\beta}(B)}^{(2)}). \quad (3.112) \]

Recall \( T \) denotes the second order operators appearing in (2.1). Let \( D \) denote the ordinary first order operators in \( \{w_1, w_2, z_3, \ldots, z_n\} \). Then we calculate
\[
|Tu(x_0) - Tu(y_0)| \leq |D^2 u(x_0) - D^2 u(y_0)| + \frac{d_{\beta}(x_0, y_0)}{d} (|D^2 u(x_0)| + |D^2 u(y_0)|).
\]
\[ \leq \frac{4d_{\beta}(x_0, y_0)\alpha}{d^{2+\alpha}} [u]^*_{C^{2,\alpha}(B)} + \frac{4d_{\beta}(x_0, y_0)\alpha}{d^{2+\alpha}} [u]^*_{C^{2,\alpha}(B)} \]
(by interpolation inequality)
\[ \leq \frac{8d_{\beta}(x_0, y_0)\alpha}{d^{2+\alpha}} [u]^*_{C^{2,\alpha}(B)} + C \frac{d_{\beta}(x_0, y_0)\alpha}{d^{2+\alpha}} \|u\|_{C^0(B)}. \]

Then we get
\[ d_{x_0}^{2+\alpha} |Tu(x_0) - Tu(y_0)| \leq C \frac{\mu^{2+\alpha}}{d^{2+\alpha}} \|u\|_{C^0_{\beta}(B)} + \frac{C}{\mu^{2+\alpha}} \|u\|_{C^0(B)}. \quad (3.113) \]

- **Case 1.2.** \( B_{\beta}(x_0, d) \cap \mathcal{S} \neq \emptyset \). Let \( \hat{x}_0 \in \mathcal{S} \) be the nearest point of \( x_0 \) to \( \mathcal{S} \). We consider the balls \( \hat{B} := B_{\beta}(\hat{x}_0, 2d) \) which is contained in \( B_{\beta}(0,1) \) by triangle inequality. As in [14], we introduce a (non-holomorphic) basis of \( T_{1,0}^*(\mathbb{C}^n \setminus \mathcal{S}) \) as
\[ \{ \epsilon_j := dr_j + \sqrt{-1} \beta_j r_j d\theta_j, d\z_j \}_{j=1,2, k=3,\ldots,n}, \]
and the dual basis of \( T_{1,0}(\mathbb{C}^n \setminus \mathcal{S}) \):
\[ \{ \gamma_j := \frac{\partial}{\partial r_j} - \sqrt{-1} \frac{1}{\beta_j r_j} \frac{\partial}{\partial \theta_j} \}_{j=1,2, k=3,\ldots,n}. \]

We can write the \((1,1)\)-form \( \omega \) in the basis \( \{ \epsilon_j \wedge \bar{\epsilon}_k, \epsilon_j \wedge d\z_k, d\z_k \wedge \epsilon_j, d\z_j \wedge d\z_k \} \) as
\[ \omega = g_{\epsilon_j \bar{\epsilon}_k} \epsilon_j \wedge \bar{\epsilon}_k + g_{\epsilon_j \hat{\epsilon}_k} \epsilon_j \wedge d\z_k + g_{\epsilon_j \hat{\epsilon}_k} d\z_k \wedge \epsilon_j + g_{\hat{\epsilon}_j \hat{\epsilon}_k} d\z_j \wedge d\z_k, \quad (3.114) \]
where

\[ g_{\epsilon_j \epsilon_k} = \sqrt{-1} \partial \bar{\partial} \phi(\gamma_j, \gamma_k), \quad g_{\epsilon_j k} = \sqrt{-1} \partial \bar{\partial} \phi(\gamma_j, \partial \bar{\partial} z_k), \quad g_{k \epsilon_j} = \sqrt{-1} \partial \bar{\partial} \phi(\partial \bar{\partial} z_k, \gamma_j), \quad g_{k j} = \frac{\partial^2}{\partial z_k \partial \bar{z}_j} \phi. \]  

(3.115)

We remark that all the second order derivatives of \( \phi \) appearing in (3.115) are linear combination of \(|z_j|^2 - 2 \frac{\partial^2}{\partial z_j \partial \bar{z}_j} N_j N_k (j \neq k), N_j D^\alpha \) and \((D^\alpha)^2\), which are studied in Theorem 1.1. The standard metric \( \omega_\beta \) becomes the identity matrix under the basis above for (1,1)-forms. If \( \omega \in C^{0,\alpha}_\beta \), all the coefficients in the expression of \( \omega \) in (3.114) are \( C^{0,\alpha}_\beta \) continuous and the cross terms \( g_{\epsilon_j \epsilon_k} \) with \( j \neq k \), \( g_{\epsilon_j k} \) tend to zero when approaching the corresponding singular sets \( S_j \) or \( S_k \). Moreover, the limit of \( g_{jk} dz_j \wedge d\bar{z}_k \) as tending to \( S_1 \cap \ldots \cap S_p \) defines a Kähler metric on it. Rescaling or rotating the coordinates if necessary we may assume at \( \hat{x}_0 \in S \), \( g_{\epsilon_j \epsilon_j}(\hat{x}_0) = 1 \), \( g_{jk}(\hat{x}_0) = \delta_{jk} \) and the cross terms vanish at \( \hat{x}_0 \). Let \( \omega_\beta \) be the standard cone metric under these new coordinates near \( \hat{x}_0 \), and we can write the equation (3.110) as

\[ \Delta_g u(z) = \Delta_{\omega_\beta} u(z) + \eta(z), \quad i \partial \bar{\partial} u(z) = f(z), \quad \forall \ z \not\in S \]

for some Hermtian matrix \( \eta(z) = (\eta^j)_{j,k=1}^n, \eta^j_k = g^j_k(z) - g^j_\beta \). It is not hard to see the term \( \eta(z), i \partial \bar{\partial} u \) can be written as

\[ \sum_{j,k=1}^2 (g^{\epsilon_j \epsilon_k}(z) - \delta_{jk}) u_{\epsilon_j \epsilon_k} + 2 \text{Re} \sum_{1 \leq j \leq 2, 3 \leq k \leq n} g^{j \epsilon_k} u_{\epsilon_j \epsilon_k} + \sum_{j,k=3}^n (g^{j \epsilon_k}(z) - \delta_{jk}) u_{j \epsilon_k}, \]

(3.116)

and \( g \) with the upper indices denotes the inverse matrix of \( g \). We consider the equivalent form of the equation (3.110) on \( \tilde{B} \),

\[ \Delta_{\omega_\beta} u = f - \eta, \sqrt{-1} \partial \bar{\partial} u =: \hat{f}, \quad u \in C^0(\tilde{B}) \cap C^2(\tilde{B}\setminus S). \]

Observe that \( x_0, y_0 \in B_\beta(\hat{x}_0, 3d/2) \) we can apply the scaled inequality (1.7) of Theorem 1.1 to conclude that

\[ d^{2+\alpha} \frac{|Tu(x_0) - Tu(y_0)|}{d_\beta(x_0, y_0)} \leq C \left( \|u\|_{C^0(\tilde{B})} + \|\hat{f}\|_{C^{0,\alpha}_\beta(\tilde{B})} \right), \]

thus

\[ d^{2+\alpha} \frac{|Tu(x_0) - Tu(y_0)|}{d_\beta(x_0, y_0)} \leq \frac{C}{\mu^{2+\alpha}} \left( \|u\|_{C^0(\tilde{B})} + \|\hat{f}\|_{C^{0,\alpha}_\beta(\tilde{B})} \right). \]  

(3.117)

• **Case 2.** \( d_\beta(x_0, y_0) \geq d/2 \).

\[ d^{2+\alpha} \frac{|Tu(x_0) - Tu(y_0)|}{d_\beta(x_0, y_0)} \leq 4d^{2+\alpha} \frac{|Tu(x_0)| + |Tu(y_0)|}{d^\alpha} \leq \frac{8}{\mu^\alpha} [u]^*_C C^2(\beta(0,1)). \]  

(3.118)

Combining (3.113), (3.117) and (3.118) we get

\[ d^{2+\alpha} \frac{|Tu(x_0) - Tu(y_0)|}{d_\beta(x_0, y_0)} \leq \frac{8}{\mu^\alpha} [u]^*_C C^2(\beta(0,1)) + \frac{C}{\mu^{2+\alpha}} \left( \|u\|_{C^0(\tilde{B})} + \|\hat{f}\|_{C^{0,\alpha}_\beta(\tilde{B})} \right) \]

(3.119)

\[ + \frac{C}{\mu^{2+\alpha}} \|f\|_{C^{0,\alpha}_\beta(\tilde{B})} \]

By definition it is easy to see that (we denote \( B_\beta = B_\beta(0,1) \))

\[ \|f\|_{C^{0,\alpha}_\beta(\tilde{B})} \leq C \mu^2 \|f\|_{C^0(\tilde{B})} + C \mu^{2+\alpha} \|f\|_{C^{0,\alpha}_\beta(\tilde{B})} \leq \mu^2 \|f\|_{C^{0,\alpha}_\beta(\tilde{B})}. \]
We calculate
\[
\|\hat{f}\|_{C^2_\beta(B)}^{(2)} \leq \|\eta\|_{C^0_\beta(B)}^{(0)} \|Tu\|_{C^2_\beta(B)}^{(2)} + \|f\|_{C^2_\beta(B)}^{(2)}
\]
\[
\leq C_0 [g]_{C^2_\beta(B)}^{(1)} \mu^\alpha \left( \mu^2 [u]_{C^2_\beta(B)}^{(1)} + \mu^2 [u]_{C^2_\beta(B)}^{(2)} \right) + 2 \mu^2 [u]_{C^2_\beta(B)}^{(2)}.
\]
\[
\leq C_0 [g]_{C^2_\beta(B)}^{(1)} \mu^\alpha \left( \mu^2 [u]_{C^2_\beta(B)}^{(1)} + 2 \mu^2 [u]_{C^2_\beta(B)}^{(2)} \right) + 2 \mu^2 [u]_{C^2_\beta(B)}^{(2)}.
\]
\[
\frac{8}{\mu^\alpha [u]_{C^2_\beta(B)}^{(1)}} \leq \mu^\alpha [u]_{C^2_\beta(B)}^{(2)} + C(\mu [u]_{C^0_\beta(B)})
\]
If we choose \( \mu > 0 \) small such that \( \mu^\alpha (2 C_0 [g]_{C^2_\beta(B)}^{(1)} + 1) \leq 1/2 \), then we get from (3.119) and the inequalities above that
\[
d_{x_0}^{2+\alpha} \left| Tu(x_0) - Tu(y_0) \right| \leq \frac{1}{2} [u]_{C^2_\beta(B)}^{(2)} + C(\mu [u]_{C^0_\beta(B)} + \|f\|_{C^2_\beta(B)}^{(2)}).
\]
Taking supremum over \( x_0 \neq y_0 \in B_\beta(0,1) \) we conclude from the inequality above that
\[
[u]_{C^2_\beta(B)}^{(2)} \leq C \left( \|u\|_{C^0_\beta(B)} + \|f\|_{C^2_\beta(B)}^{(2)} \right).
\]
Proposition 3.6 then follows from interpolation inequalities.

\[ \square \]

**Remark 3.4.** It follows easily from the proof of Proposition 3.6 that the estimate (3.111) also holds for metric balls \( B_\beta(p,R) \subset B_\beta(0,1) \) whose center \( p \) may not lie at \( S \).

An immediate corollary to Proposition 3.6 is the following interior Schauder estimate.

**Corollary 3.1.** Suppose \( u \) satisfies the equation (3.110). For any compact subset \( K \subset B_\beta(0,1) \), there exists a constant \( C = C(n, \beta, K, \|g\|_{C^{0,\alpha}(B_\beta(0,1))}) > 0 \) such that
\[
\|u\|_{C^{2,\alpha}(K)} \leq C \left( \|u\|_{C^0(B_\beta(0,1))} + \|f\|_{C^2_\beta(B_\beta(0,1))} \right).
\]

Next we will show that the equation (3.110) admits a unique \( C^{2,\alpha}_\beta \)-solution for any \( f \in C^{0,\alpha}(B_\beta(0,1)) \) and boundary value \( \varphi \in C^0(\partial B_\beta(0,1)) \). We will follow the argument in Section 6.5 in [19]. In the following we will write \( B_\beta = B_\beta(0,1) \) for simplicity.

**Lemma 3.23.** Let \( \sigma \in (0,1) \) be a given number. Suppose \( u \in C^{2,\alpha}_\beta(B_\beta) \) solves (3.110) and \( \|u\|_{C^{0,\alpha}(B_\beta)}^{(-\sigma)} < \infty \) and \( \|f\|_{C^{2,\alpha}_\beta(B_\beta)}^{(2-\sigma)} < \infty \). Then there exists a \( C = C(n, \beta, \sigma, \alpha, g, \sigma) > 0 \) such that
\[
\|u\|_{C^{2,\alpha}_\beta(B_\beta)}^{(-\sigma)} \leq C \left( \|u\|_{C^0(B_\beta)}^{(-\sigma)} + \|f\|_{C^{2,\alpha}_\beta(B_\beta)}^{(2-\sigma)} \right).
\]

**Proof.** Given the estimates in Proposition 3.6, the proof is identical to that of Lemma 6.20 in [19]. So we omit the details. \[ \square \]

**Lemma 3.24.** Let \( u \in C^{2}_\beta(B_\beta) \cap C^0(B_\beta) \) solve the equation \( \Delta_{\beta} u = f \) and \( u \equiv 0 \) on \( \partial B_\beta \). For any \( \sigma \in (0,1) \), there exists a constant \( C = C(n, \beta, \sigma, g) > 0 \) such that
\[
\|u\|_{C^0(B_\beta)}^{(-\sigma)} = \sup_{x \in B_\beta} d^\sigma_x |u(x)| \leq C \sup_{x \in B_\beta} d^{2-\sigma}_x |f(x)| = C \|f\|_{C^0(B_\beta)}^{(2-\sigma)},
\]
where as before \( d_x = d_\beta(x, \partial B_\beta) \).
Proof. Consider the function \( w_1 = (1 - d^2_{\beta})^\sigma \) where \( d_\beta(x) = d_\beta(x, 0) \). We calculate
\[
\Delta_g w_1 = \sigma(1 - d^2_{\beta})^{\sigma - 2}(- (1 - d^2_{\beta})\text{tr}g_\beta - (1 - \sigma)|\nabla d^2_{\beta}|^2) \\
\leq \sigma(1 - d^2_{\beta})^{\sigma - 2}(- C^{-1}(1 - d^2_{\beta}) - 4C^{-1}d^2_{\beta}(1 - \sigma)) \\
\leq -c_0\sigma(1 - d^2_{\beta})^{\sigma - 2}.
\]
Take a large constant \( A > 1 \) such that for \( w = Aw_1 \)
\[
\Delta_g w \leq -(1 - d^2_{\beta})^{\sigma - 2} \leq -|f|/N, \quad \text{in } B_\beta,
\]
where \( N = \sup_{x \in B_\beta} d^2_{x - \sigma}|f(x)| = \sup_{x \in B_\beta} (1 - d^2_{\beta}(x))^{2 - \sigma}|f(x)| \). Hence \( \Delta_g(Nw \pm u) \leq 0 \) and from the definition of \( w \) we also have \( w|_{\partial B_\beta} \equiv 0 \), by maximum principle we obtain that \( |u(x)| \leq Nw \leq CN(1 - d^2_{\beta}(x))^\sigma = CNd^\sigma_{\beta} \), hence the lemma is proved.

**Proposition 3.7.** Given any function \( f \in C^{0,\alpha}_{\beta}(\overline{B_\beta}) \), the Dirichlet problem \( \Delta_g u = f \) in \( B_\beta \) and \( u \equiv 0 \) on \( \partial B_\beta \) admits a unique solution \( u \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta}) \).

**Proof.** The proof of this proposition is almost identical to that of Theorem 6.22 in [19]. For completeness, we provide the detailed argument. Fix a \( \sigma \in (0, 1) \). We define a family of operators \( \Delta_t = t\Delta_g + (1 - t)\Delta_{\beta} \) and it is straightforward to see that \( \Delta_t \) is associated to some cone metric which also satisfies (3.109). We study the Dirichlet problem
\[
\Delta_t u_t = f, \quad \text{in } B_\beta, \quad u_t \equiv 0 \text{ on } \partial B_\beta.
\]
Equation \((*)_t\) admits a unique solution \( u_0 \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta}) \) by Proposition 3.2. By Theorem 5.2 in [19], in order to apply the continuity method to solve \((*)_t\), it suffices to show \( \Delta_t^{-1} \) defines a bounded linear operator between some Banach spaces. More precisely, define
\[
B_1 := \{ u \in C^{2,\alpha}_{\beta}(B_\beta) \mid \|u\|_{C^{2,\alpha}_{\beta}(B_\beta)} < \infty \},
\]
\[
B_2 := \{ f \in C^{0,\alpha}_{\beta}(B_\beta) \mid \|f\|_{C^{0,\alpha}_{\beta}(B_\beta)} < \infty \}.
\]
By definition any \( u \in B_1 \) is continuous on \( \overline{B_\beta} \) and \( u = 0 \) on \( \partial B_\beta \). By Lemmas 3.23 and 3.24, we have
\[
\|u\|_{B_1} = \|u\|_{C^{2,\alpha}_{\beta}(B_\beta)} \leq C\|f\|_{C^{0,\alpha}_{\beta}(B_\beta)} = C\|\Delta_t u\|_{B_2},
\]
for some constant \( C \) independent of \( t \in [0, 1] \). Thus \((*)_t\) admits a solution \( u \in B_1 \).

**Corollary 3.2.** For any given \( \varphi \in C^0(\partial B_\beta) \) and \( f \in C^{0,\alpha}_{\beta}(\overline{B_\beta}) \), the Dirichlet problem
\[
\Delta_g u = f, \quad \text{in } B_\beta, \quad u = \varphi, \quad \text{on } \partial B_\beta,
\]
(3.120) admits a unique solution \( u \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta}) \).

**Proof.** We may extend \( \varphi \) continuously to \( B_\beta \) and assume \( \varphi \in C^0(\overline{B_\beta}) \). Take a sequence of functions \( \varphi_k \in C^{2,\alpha}_{\beta}(\overline{B_\beta}) \cap C^0(\overline{B_\beta}) \) which converges uniformly to \( \varphi \) on \( \overline{B_\beta} \). The Dirichlet problem \( \Delta_g v_k = f - \Delta_g \varphi_k \) in \( B_\beta \) and \( v_k = 0 \) on \( \partial B_\beta \) admits a unique solution \( v_k \in C^{2,\alpha}_{\beta}(B_\beta) \cap C^0(\overline{B_\beta}) \). Thus the function \( u_k := v_k + \varphi_k \in C^{2,\alpha}_{\beta} \) satisfies \( \Delta_g u_k = f \) in \( B_\beta \) and \( u_k = \varphi_k \) on \( \partial B_\beta \). \( u_k \) is uniformly bounded in \( C^0(\overline{B_\beta}) \) by maximum principle. Corollary 3.1 gives uniformly \( C^{2,\alpha}_{\beta}(K) \)-bound on any compact subset \( K \subset B_\beta \). Letting \( k \to \infty \) and \( K \to B_\beta \), by a diagonal argument and up to a subsequence \( u_k \to u \in C^{2,\alpha}_{\beta}(B_\beta) \). On the other hand, from \( \Delta_g(u_k - u_l) = 0 \) we see that \( \{u_k\} \) is
a Cauchy sequence in $C^0(B_{\beta})$ thus $u_k$ converges uniformly to $u$ on $\overline{B_{\beta}}$. Hence $u \in C^0(\overline{B_{\beta}})$ and satisfies the equation (3.120).

Corollary 3.3. Given $f \in C^0_{\beta}(B_{\beta})$, suppose $u$ is a weak solution to the equation $\Delta_g u = f$ in the sense that

$$\int_{B_{\beta}} (\nabla u, \nabla \varphi) \omega^n_g = - \int_{B_{\beta}} f \varphi \omega^n_g, \quad \forall \varphi \in H^1_0(B_{\beta}),$$

then $u \in C^2_{\alpha}(B_{\beta})$.

Proof. We first observe that the Sobolev inequality (3.43) also holds for the metric $g$, since $g$ is equivalent to $g_{\beta}$. The metric space $(\beta \beta, g)$ also has maximal volume growth/decay, so we can apply the same proof of De Giorgi-Nash-Moser theory ([22]) to conclude that $u$ is continuous in $B_{\beta}$. The standard elliptic theory implies that $u \in C^2_{\alpha}(B_{\beta} \setminus S)$. For any $r \in (0, 1)$, by Corollary 3.2, the Dirichlet problem $\Delta_g \tilde{u} = f$ in $B_\beta(0, r)$, $\tilde{u} = u$ on $\partial B_\beta(0, r)$ admits a unique solution $\tilde{u} \in C^2_{\alpha}(B_\beta(0, r)) \cap C^0(B_\beta(0, r))$. Then $\Delta_g (u - \tilde{u}) = 0$ in $B_\beta(0, r)$ and $u - \tilde{u} = 0$ on $\partial B_\beta(0, r)$. By maximum principle we get $u = \tilde{u}$ in $B_\beta(0, r)$, so we conclude $u \in C^2_{\alpha}(B_{\beta}(0, r))$. Since $r \in (0, 1)$ is arbitrary, we get $u \in C^2_{\alpha}(B_{\beta})$.

Corollary 3.4. Let $X$ be a compact Kähler manifold and $D = \sum_j D_j$ be a divisor with simple normal crossings. Let $g$ be a conical Kähler metric with cone angle $2\pi \beta$ along $D$. Suppose $u \in H^1(g)$ is a weak solution to the equation $\Delta_g u = f$ in the sense that

$$\int_X (\nabla u, \nabla \varphi) \omega^n_g = - \int_X f \varphi \omega^n_g, \quad \forall \varphi \in C^1(X)$$

for some $f \in C^0_{\beta}(X)$. Then $u \in C^2_{\beta}(X) \cap C^0(X)$ and there exists a constant $C = C(n, \beta, g, \alpha)$ such that

$$\|u\|_{C^2_{\beta}(X)} \leq C(\|u\|_{C^0(X)} + \|f\|_{C^0_{\beta}(X)}).$$

Proof. We can choose finite covers of $D$, $\{B_a\}$, $\{B_a'\}$ with $B_a' \Subset B_a$ and centers at $D$. By assumption $u$ is a weak solution to the equation $\Delta_g u = f$ in each $B_a$, then by Corollary 3.3 we conclude that $u \in C^2_{\beta}(B_a)$ for each $B_a$. On $X \setminus S$, the metric $g$ is smooth so standard elliptic theory implies that $u \in C^2_{\alpha}(X \setminus S)$. Since $\{B_a\}$ covers $D$, $u \in C^2_{\alpha}(X)$.

We can apply Corollary 3.1 to obtain that for some constant $C > 0$

$$\|u\|_{C^2_{\beta}(B_a')} \leq C(\|u\|_{C^0(B_a)} + \|f\|_{C^0_{\beta}(B_a)}).$$

On $X \setminus \cup_a \{B_a\}'$ the metric $g$ is smooth, the usual Schauder estimates apply. We finish the proof of the Corollary by the definition of $C^2_{\beta}(X)$ (c.f. Definition 2.8).

Remark 3.5. Let $(X, D, g)$ be as in Corollary 3.4. It is easy to see by variational method weak solutions to $\Delta_g u = f$ always exist for any $f \in L^2(X, \omega^n_g)$ satisfying $\int_X f \omega^n_g = 0$. 

□
4. Parabolic estimates

In this section, we will study the heat equation with background metric $\omega_\beta$ and prove the Schauder estimates for such solution $u \in C^0(Q_\beta) \cap C^{2,1}(Q^\#_\beta)$ to the equation

$$\frac{\partial u}{\partial t} = \Delta_{g_\beta} u + f,$$  \hspace{1cm} (4.1)

for a function $f \in C^0(Q_\beta)$ with some better regularity.

4.1. Conical heat equations

In this section, we will show that for any $\varphi \in C^0(\partial P Q_\beta)$, the Dirichlet problem (4.2) admits a unique $C^{2,1}(Q^\#_\beta) \cap C^0(Q_\beta)$-solution in $Q_\beta$. We first observe that a maximum principle argument yields the uniqueness of the solution.

Suppose $u \in C^{2,1}(Q^\#_\beta) \cap C^0(Q_\beta)$ solves the Dirichlet problem

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta_{g_\beta} u, & \text{in } Q_\beta \\
u = \varphi, & \text{on } \partial P Q_\beta,
\end{cases}$$  \hspace{1cm} (4.2)

for some given continuous function $\varphi \in C^0(\partial P Q_\beta)$. It follows from maximum principle as in Lemma 3.1 that

$$\inf_{\partial P Q_\beta} u \leq \inf_{Q_\beta} u \leq \sup_{Q_\beta} u \leq \sup_{\partial P Q_\beta} u.$$  \hspace{1cm} (4.3)

So the $C^{2,1}(Q^\#_\beta) \cap C^0(Q_\beta)$-solution to (4.2) is unique, if exists.

Now we prove the existence of solutions to (4.2). As before, we will use an approximation argument. Let $g_\epsilon$ be the smooth approximation metrics in $B_\beta$ as defined in (3.3). Let $u_\epsilon$ be the $C^{2,1}(Q_\beta) \cap C^0(Q_\beta)$-solution to the equation

$$\frac{\partial u_\epsilon}{\partial t} = \Delta_{g_\epsilon} u_\epsilon, \text{ in } Q_\beta,$$

$$u_\epsilon = \varphi, \text{ on } \partial P Q_\beta,$$  \hspace{1cm} (4.4)

4.1.1. Estimates of $u_\epsilon$. We first recall the Li-Yau gradient estimates ([26, 35]) for positive solutions to the heat equations.

**Lemma 4.1.** Let $(M, g)$ be a complete manifold with $\text{Ric}(g) \geq 0$, and $B(p, R)$ be the geodesic ball with center $p \in M$ and radius $R > 0$. Let $u$ be a positive solution to the heat equation $\partial_t u - \Delta_g u = 0$ on $B(p, R)$, then there exists $C = C(n) > 0$ such that for all $t > 0$,

$$\sup_{B(p,2R/3)} \left( \frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u} \right) \leq \frac{C}{R^2} + \frac{2n}{t},$$

where $u_t = \frac{\partial u}{\partial t}$.

By considering the functions $u_\epsilon - \inf u_\epsilon$ and $\sup u_\epsilon - u_\epsilon$, from the Lemma 4.1, we see that there exists a constant $C = C(n) > 0$ such that for any $R \in (0, 1)$ and $t \in (0, R^2)$

$$\sup_{B_{g_\epsilon}(0,2R/3)} |\nabla u_\epsilon|^2 \leq C \left( \frac{1}{R^2} + \frac{1}{t} \right) (\text{osc}_{R} u_\epsilon)^2,$$  \hspace{1cm} (4.5)

and

$$\sup_{B_{g_\epsilon}(0,2R/3)} |\Delta_{g_\epsilon} u_\epsilon| = \sup_{B_{g_\epsilon}(0,2R/3)} |\frac{\partial u_\epsilon}{\partial t}| \leq C \left( \frac{1}{R^2} + \frac{1}{t} \right) \text{osc}_{R} u_\epsilon,$$  \hspace{1cm} (4.6)
where \( \text{osc}_{R}u_{\epsilon} := \text{osc}_{B_{g_{\epsilon}}(0,R) \times (0,R^{2})}u_{\epsilon} \) is the oscillation of \( u_{\epsilon} \) in the cylinder \( B_{g_{\epsilon}}(0,R) \times (0,R^{2}) \).

Replacing \( u_{\epsilon} \) by \( u_{\epsilon} - \inf u_{\epsilon} \), we may assume \( u_{\epsilon} > 0 \) and define \( f_{\epsilon} = \log u_{\epsilon} \). Then we have

\[
\frac{\partial f_{\epsilon}}{\partial t} = \nabla g \cdot \nabla f_{\epsilon} + |\nabla f_{\epsilon}|^{2}.
\]

Let \( \varphi(x) = \varphi\left(\frac{r(x)}{R}\right) \) where \( \varphi \) is a cut-off function equal to 1 on \([0,3/5]\), 0 on \([2/3,\infty)\), and satisfies the inequalities \(|\varphi''| \leq 10 \) and \((\varphi')^{2} \leq 10\varphi\). \( r(x) \) is the distance function under \( g_{\epsilon} \) to the center 0.

**Lemma 4.2.** There exists a constant \( C = C(n) > 0 \) such that for any small \( \epsilon > 0 \)

\[
\sup_{B_{g_{\epsilon}}(0,3R/5)} |\Delta u_{\epsilon}| \leq C\left(\frac{1}{t} + \frac{1}{R^{2}}\right)\text{osc}_{R}u_{\epsilon}, \quad \forall \ t \in (0,R^{2}),
\]

where we denote \( \Delta u_{\epsilon} := (|z_{i}|^{2} + \epsilon)^{1-\beta} \frac{\partial^{2}u}{\partial z_{i}\partial \bar{z}_{i}} \) for \( i = 1, \ldots, p \).

**Proof.** We only prove the case when \( i = 1 \). We denote \( F := t\varphi(-\Delta f_{\epsilon} - 2\dot{f}_{\epsilon}) \), and we calculate

\[
\left(\frac{\partial}{\partial t} - \Delta_{g_{\epsilon}}\right)(-\Delta f_{\epsilon} - 2\dot{f}_{\epsilon})
\]

\[
= -|\nabla v f_{\epsilon}|^{2} - |\nabla v f_{\epsilon}|^{2} - 2\text{Re}(\nabla f_{\epsilon}, \nabla(-\Delta f_{\epsilon} - 2\dot{f}_{\epsilon})) - R_{11j\bar{k}}f_{\epsilon,j} f_{\epsilon,k}
\]

\[
\leq -(-\Delta f_{\epsilon})^{2} - 2\text{Re}(\nabla f_{\epsilon}, \nabla(-\Delta f_{\epsilon} - 2\dot{f}_{\epsilon})).
\]

\( F \) achieves its maximum at a point \((p_{0}, t_{0})\), where we may assume \( F(p_{0}, t_{0}) > 0 \), otherwise we are done yet. In particular, \( p_{0} \in B_{g_{\epsilon}}(0,2r/3) \) by the definition of \( \varphi \) and \( t_{0} > 0 \). Then at \((p_{0}, t_{0})\), we have

\[
0 \leq \left(\frac{\partial}{\partial t} - \Delta g_{\epsilon}\right) F
\]

\[
= \frac{F}{t_{0}} + t_{0}\varphi\left(\frac{\partial}{\partial t} - \Delta g_{\epsilon}\right)(-\Delta f_{\epsilon} - 2\dot{f}_{\epsilon}) - \frac{F}{\varphi}\Delta g_{\epsilon}\varphi - 2t_{0}\text{Re}(\nabla \varphi, \nabla(\frac{F}{t_{0}\varphi}))
\]

\[
\leq \frac{F}{t_{0}} + t_{0}\varphi\left(\left(-\Delta f_{\epsilon}\right)^{2} - 2\frac{F}{t_{0}\varphi^{2}}\text{Re}(\nabla f_{\epsilon}, \nabla \varphi)\right) + C\frac{F}{R^{2}\varphi}(\varphi' + \varphi'') \]

\[
+ 2\frac{F}{R^{2}\varphi^{2}}(\varphi')^{2}
\]

where we use the Laplacian comparison and the fact that \( \nabla F = 0 \) at \((p_{0}, t_{0})\). The second term on the RHS satisfies (we denote \( \tilde{F} := -\Delta f_{\epsilon} - 2\dot{f}_{\epsilon} \) for notation convenience)

\[
t_{0}\varphi\left(\left(-\Delta f_{\epsilon}\right)^{2} - 2\frac{F}{t_{0}\varphi^{2}}\text{Re}(\nabla f_{\epsilon}, \nabla \varphi)\right) \leq t_{0}\varphi\left(-\tilde{F}^{2} - 4\tilde{F}\dot{f}_{\epsilon} - 4(\dot{f}_{\epsilon})^{2} + \frac{2\tilde{F}||\nabla f_{\epsilon}||\varphi'|}{\varphi}\right)
\]

\[
\leq t_{0}\varphi\left(-\tilde{F}^{2} - 4\tilde{F}\dot{f}_{\epsilon} + 2\tilde{F}||\nabla f_{\epsilon}||^{2} + \frac{\tilde{F}||\varphi'||^{2}}{2R^{2}\varphi^{2}}\right)
\]

(by Lemma 4.1) \( \leq t_{0}\varphi\left(-\tilde{F}^{2} + \tilde{F}||\varphi'||^{2} + C\frac{\tilde{F}}{t_{0}} + C\frac{\tilde{F}}{R^{2}}\right) \)

\[
= -\frac{F^{2}}{t_{0}\varphi} + CF\frac{F}{R^{2}\varphi} + CF\frac{F}{t_{0}} + CF\frac{F}{R^{2}},
\]

inserting this to (4.7), we get for some constant \( C = C(n) > 0 \) at \((p_{0}, t_{0})\)

\[
-F^{2} + C\varphi F + \frac{t_{0}\varphi F}{R^{2}} + C t_{0} \frac{F}{R^{2}} \geq 0,
\]
from which we obtain $F(p_0, t_0) \leq \frac{C t_0}{R^2} + C$, and by the choice of $(p_0, t_0)$, we can see that

$$\sup_{B_{g_0}(0,R/2)} (-\Delta_1 f_\epsilon - 2\dot{f}_\epsilon) \leq C\left(\frac{1}{R^2} + \frac{1}{t}\right), \quad \forall \ t \in (0,R^2),$$

which implies that on $B_{g_0}(0,3R/5) \times (0,R^2)$

$$-\Delta_1 u_\epsilon \leq \dot{u}_\epsilon + C\left(\frac{1}{t} + \frac{1}{R^2}\right)u_\epsilon.$$

Applying (4.8) to the function $\sup u_\epsilon - u_\epsilon$, we obtain that on $B_{g_0}(0,3R/5) \times (0,R^2)$

$$|\Delta_1 u_\epsilon| \leq |\dot{u}_\epsilon| + C\left(\frac{1}{t} + \frac{1}{R^2}\right)\text{osc}_R u_\epsilon \leq C\left(\frac{1}{t} + \frac{1}{R^2}\right)\text{osc}_R u_\epsilon$$

by equation (4.6). Thus we finish the proof of the lemma.

**Lemma 4.3.** There exists a constant $C = C(n) > 0$ such that

$$\sup_{i \neq j} \sup_{B_{g_0}(0,R/2)} (|\nabla_i \nabla_j u_\epsilon| + |\nabla_i \nabla_j u_\epsilon|) \leq C\left(\frac{1}{t} + \frac{1}{R^2}\right)\text{osc}_R u_\epsilon,$$

for all $t \in (0,R^2)$. Recall here $|\nabla_i \nabla_j u_\epsilon|^2 = \nabla_i \nabla_j u_\epsilon \nabla_i \nabla_j u_\epsilon \eta^i \eta^j$ (no summation over $i,j$ is taken).

**Proof.** We will only prove the estimate for $|\nabla_1 \nabla_2 u_\epsilon|$. The others are similar, so we omit the proof.

By similar calculations as in deriving (3.27), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\epsilon}\right)|\nabla_1 \nabla_2 f_\epsilon| \leq 2\text{Re} \langle \nabla f_\epsilon, \nabla |\nabla_1 \nabla_2 f_\epsilon| \rangle + \sum_k \left( |\nabla_1 \nabla_k f_\epsilon| |\nabla_2 \nabla_k f_\epsilon| + |\nabla_2 \nabla_k f_\epsilon| |\nabla_1 \nabla_k f_\epsilon| \right),$$

and similar to (3.20)

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\epsilon}\right)(-\Delta_1 f_\epsilon - \Delta_2 f_\epsilon) \leq 2\text{Re} \langle \nabla f_\epsilon, \nabla (-\Delta_1 f_\epsilon - \Delta_2 f_\epsilon) \rangle - \sum_k \left( |\nabla_1 \nabla_k f_\epsilon|^2 + |\nabla_1 \nabla_k f_\epsilon|^2 + |\nabla_2 \nabla_k f_\epsilon|^2 + |\nabla_2 \nabla_k f_\epsilon|^2 \right).$$

Combining (4.10), (4.9) and Cauchy-Schwarz inequality, we get

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\epsilon}\right)(|\nabla_1 \nabla_2 f_\epsilon| + 2(\Delta_1 f_\epsilon + \Delta_2 f_\epsilon)) \leq 2\text{Re} \langle \nabla f_\epsilon, \nabla (|\nabla_1 \nabla_2 f_\epsilon| + 2(\Delta_1 f_\epsilon + \Delta_2 f_\epsilon)) \rangle - \sum_k \left( |\nabla_1 \nabla_k f_\epsilon|^2 + |\nabla_1 \nabla_k f_\epsilon|^2 + |\nabla_2 \nabla_k f_\epsilon|^2 + |\nabla_2 \nabla_k f_\epsilon|^2 \right)$$

$$\leq 2\text{Re} \langle \nabla f_\epsilon, \nabla (|\nabla_1 \nabla_2 f_\epsilon| + 2(\Delta_1 f_\epsilon + \Delta_2 f_\epsilon)) \rangle - \frac{1}{10} \left( |\nabla_1 \nabla_2 f_\epsilon| + 2(\Delta_1 f_\epsilon + \Delta_2 f_\epsilon) \right)^2.$$

We define a similar cut-off function $\eta$ as $\varphi$ in the proof of Lemma 4.2, such that $\eta = 1$ on $B_{g_\epsilon}(0,R/2)$ and vanishes outside $B_{g_\epsilon}(0,3R/5)$. We denote

$$G = t\eta(|\nabla_1 \nabla_2 f_\epsilon| + 2(-\Delta_1 f_\epsilon - \Delta_2 f_\epsilon) - 2\dot{f}_\epsilon).$$

We can argue similarly as the $F$ in the proof of Lemma 4.2 that at the maximum point $(p_0, t_0)$ of $G$, for which we assume $G(p_0, t_0) > 0$

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_{g_\epsilon}\right)G$$

$$\leq \frac{G}{t_0 \eta} - \frac{G^2}{t_0 \eta^2} + C \frac{G}{R^2 \eta} + C \frac{G}{t_0} + C \frac{G}{R^2} + C \frac{G}{R^2 \eta} \eta' + \eta'' + 2G \frac{\eta'}{R^2 \eta^2}$$
Letting $\epsilon (4.13)$ follows by applying the gradient estimate (4.5) to the $\Delta$
standard parabolic Schauder theory yields uniform
we can assume that
thus by Lemmas 4.1 and 4.2, we conclude that on $B_{\gamma}(0, R/2) \times (0, R^2)$

$$|\nabla_1 \nabla_2 f_\epsilon| + 2(-\Delta_1 f_\epsilon - \Delta_2 f_\epsilon) - 2 f_\epsilon \leq C\left(\frac{1}{R^2} + \frac{1}{t}\right),$$

as desired. \qed

4.1.2. Existence of $u$ to (4.2). We will show the limit function of $u_\epsilon$ as $\epsilon \to 0$ solves (4.2).

**Proposition 4.1.** Given any $R \in (0, 1)$ and any $\varphi \in C^0(\partial \Omega \setminus \Omega, R)$, there exists a unique function $u \in C^{2,1}(\Omega, R, \#) \cap C^0(\Omega, R)$ solving the equation (4.2). Moreover, there exists a constant $C = C(n, \beta) > 0$ such that for any $t \in (0, R^2)$ (we denote $B_{\beta}(r)^\# := B_{\beta}(0, r) \setminus S$)

$$\sup_{B_{\beta}(R/2)^\#} \left( \sum_{j=1}^{p} |\nabla_j u|^{2-2\beta_j} \left| \frac{\partial u}{\partial z_j} \right|^2 + |D' u|^2 \right) \leq C\left(\frac{1}{t} + \frac{1}{R^2}\right)(\text{osc}_{R} u)^2, \quad (4.11)$$

$$\sup_{B_{\beta}(R/2)^\#} \left( \sum_{i \neq j} (|\nabla_i \nabla_j u|_{\Omega_{\beta}} + |\nabla_i \nabla_j u|_{\Omega_{\beta}}) + \left| \frac{\partial u}{\partial t} \right| \right) \leq C\left(\frac{1}{t} + \frac{1}{R^2}\right)\text{osc}_{R} u, \quad (4.12)$$

and

$$\sup_{B_{\beta}(R/2)^\#} \left( \sum_{j=1}^{p} |\nabla_{\beta} \Delta_j u| + |\nabla_{\beta} (D')^2 u| + |\nabla_{\beta} \frac{\partial u}{\partial t}| \right) \leq C\left(\frac{1}{t} + \frac{1}{R^2}\right)^{3/2}\text{osc}_{R} u, \quad (4.13)$$

where by abusing notation we denote $\text{osc}_{R} u := \text{osc}_{B_{\beta}(0,R) \times (0, R^2)} u$.

**Proof.** Let $u_\epsilon$ be the $C^{2,1}$-solution to the equation (4.4). The $C^0$-norm of $u_\epsilon$ follows from maximum principle (4.3).

To prove the higher order estimates, for any fixed compact subset $K \subset B_{\beta}(0, R)$ and $\delta > 0$, standard parabolic Schauder theory yields uniform $C^{4,\alpha,\frac{4+\alpha}{2}}$-estimates of $u_\epsilon$ on $(K \setminus T_{\delta} S) \times (\delta, R^2]$, for any $\alpha \in (0, 1)$. As $\epsilon \to 0$, $u_\epsilon$ converges in $C^{4,\alpha,\frac{4+\alpha}{2}} (K \setminus T_{\delta} S \cap (\delta, R^2))$ to some function $u$ which is also $C^{4,\alpha,\frac{4+\alpha}{2}}$ in $(K \setminus T_{\delta} S) \times (\delta, R^2]$. Let $\delta \to 0$, $K \to B_{\beta}(0, R)$ and use a diagonal argument, then we can assume that

$$u_\epsilon \to C^{4,\alpha,\frac{4+\alpha}{2}}(B_{\beta}(0, R)^\# \times (0, R^2)) \to u, \quad \text{as } \epsilon \to 0.$$

Letting $\epsilon \to 0$, the estimate (4.11) follows from (4.5); (4.12) is a consequence of Lemma 4.3; and (4.13) follows by applying the gradient estimate (4.5) to the $\Delta_{g_\delta}$-harmonic functions $\Delta_j u_\epsilon$ and $(D')^2 u_\epsilon$, and then letting $\epsilon \to 0$.

The gradient estimate (4.11) implies that for any compact $K \subset B_{\beta}(0, R)$

$$\sup_{K \setminus S} \left| \frac{\partial u}{\partial z_j} \right| \leq \frac{C(n, K, \beta)(\text{osc}_{R} u)^2}{t} |z_j|^{\beta_j - 1}, \quad \forall t \in (0, R^2).$$
From this we see that for any \( t \in (0, R^2) \), \( u(\cdot, t) \) can be continuously extended to \( S \) and thus \( u \in C^0(B_\beta(0, R) \times (0, R^2)) \).

It only remains to show \( u = \varphi \) on \( \partial_P \mathcal{Q}_\beta(0, R) \). Fix an arbitrary point \( (q_0, t_0) \in \partial_P (\mathcal{Q}_\beta(0, R)) \).

**Case 1.** \( t_0 = 0 \) and \( q_0 \in \overline{B_\alpha(0, R)} \). We define a barrier function \( \phi_1(z, t) = e^{-d_{C_n}(z, q_0)^2 - \lambda t} - 1 \), where \( \lambda > 0 \) is to be determined. We calculate

\[
\frac{\partial}{\partial t} \left( \phi_1 \right) = -\lambda e^{-d_{C_n}(z, q_0)^2 - \lambda t} - (-\Delta_{q_0} d_{C_n}^2 + |\nabla d_{C_n}(q_0)|^2) e^{-d_{C_n}(z, q_0)^2 - \lambda t} 
\leq (-\lambda + \sum_{j=1}^{p} (|z_j|^2 + \epsilon)^{1-\beta_j} + (n-p)) e^{-d_{C_n}(z, q_0)^2 - \lambda t} < 0,
\]

if \( \lambda \geq 4n \). On the other hand, \( \phi_1(q_0, t_0) = 0 \) and \( \phi_1(z, t) < 0 \) for any \( (z, t) \neq (q_0, t_0) \). For any \( \varepsilon > 0 \), we can find a small neighborhood \( V \cap \partial_P (\mathcal{Q}_\beta(0, R)) \) of \( (q_0, t_0) \) such that on \( V \), \( \varphi(q_0, t_0) + \varepsilon > \varphi(z, t) \geq \varphi(q_0, t_0) - \varepsilon \) since \( \varphi \) is continuous. On \( \partial_P (\mathcal{Q}_\beta(0, R)) \setminus V \), the function \( \phi_1 \) is bounded above by a negative constant. Therefore the function \( \varphi_1 := \varphi(q_0, t_0) - \varepsilon + A \phi_1(z, t) \leq \varphi(z, t) \) for any \( (z, t) \in \partial_P (\mathcal{Q}_\beta(0, R)) \) if \( A >> 1 \). Therefore by maximum principle \( \varphi_1(z, t) \leq u_\varepsilon(z, t) \) for any \( (z, t) \in \mathcal{Q}_\beta(0, R) \). Letting \( \varepsilon \to 0 \), \( \varphi_1(z, t) \leq u(z, t) \). So letting \( (z, t) \to (q_0, t_0) \), \( \varphi(q_0, t_0) - \varepsilon \leq \liminf_{(z, t) \to (q_0, t_0)} u(z, t) \). Setting \( \varepsilon \to 0 \) we conclude that \( \varphi(q_0, t_0) \leq \liminf_{(z, t) \to (q_0, t_0)} u(z, t) \). By considering \( \varphi_1(z, t) = \varphi(q_0, t_0) + \varepsilon - A \phi_1(z, t) \) and similar argument as above we can get \( \varphi(q_0, t_0) \geq \limsup_{(z, t) \to (q_0, t_0)} u(z, t) \). Thus \( u \) coincides with \( \varphi \) at \( (q_0, t_0) \).

**Case 2.** \( t_0 > 0 \) and \( q_0 \in \partial B_\beta(0, R) \cap (S_1 \cap S_2) \). In this case \( z_1(q_0) = z_2(q_0) = 0 \). Denote \( q_0' = -q_0 \in \partial B_\beta(0, R) \) to be the (Euclidean) opposite point of \( q_0 \). Define for some small \( \delta > 0 \)

\[
\phi_2(z, t) = d_{C_n}(z, q_0')^2 - 4R^2 - \delta(t-t_0)^2.
\]

\( \phi_2(q_0, z_0) = 0 \) and \( \phi(z, t) < 0 \) for any \( (z, t) \neq (q_0, t_0) \). We calculate that \( \partial_t \phi_2 - \Delta_{q_0} \phi_2 \leq 0 \). Then by similar argument as in **Case 1** replacing \( \phi_1 \) by \( \phi_2 \), we get \( \lim_{(z, t) \to (q_0, t_0)} u(z, t) = \varphi(q_0, t_0) \).

**Case 3.** \( t_0 > 0 \) and \( q_0 \in \partial B_\beta(0, R) \setminus (S_1 \cap S_2) \). As the **Case 2** in the proof of Proposition 3.1, we define a similar function \( G \) as there. Define \( \delta_3(z, t) = A(d_{\beta}(z, 0)^2 - R^2) + G(z) - \delta(t-t_0)^2 \) for \( A >> 1 \) and small \( \delta > 0 \). Then we can calculate that \( \partial_t \delta_3 \leq \Delta_{q_0} \delta_3 \) and \( \delta_3(q_0, t_0) = 0 \) and \( \delta_3(z, t) < 0 \) for any other \( (z, t) \neq (q_0, t_0) \). Similar argument as in **Case 1** proves that

\[
\lim_{(z, t) \to (q_0, t_0)} u(z, t) = \varphi(q_0, t_0).
\]

Combining all the three cases above, we obtain that \( u \) coincides with \( \varphi \) on \( \partial_P \mathcal{Q}_\beta(0, R) \). Thus the Dirichlet problem (4.2) admits a unique solution \( u \in C^0(\overline{\mathcal{Q}_\beta(0, R)}) \cap C^{2,1}(\mathcal{Q}_\beta(0, R)^\#) \).

---

**Corollary 4.1.** Given any \( f \in C^{\alpha, \beta}(\mathcal{Q}_\beta) \) and \( \varphi \in C^0(\partial_P \mathcal{Q}_\beta) \), there exists a unique solution \( v \in C^{2,1}(\mathcal{Q}_\beta^\#) \cap C^0(\overline{\mathcal{Q}_\beta}) \) to the Dirichlet problem

\[
\frac{\partial v}{\partial t} = \Delta_{q_0} v + f, \quad \text{in } \mathcal{Q}_\beta, \quad \text{and } v = \varphi, \quad \text{on } \partial_P \mathcal{Q}_\beta.
\]

**Proof.** Let \( v_\epsilon \in C^{2+\alpha, \frac{\alpha}{\beta}+\alpha}(\mathcal{Q}_\beta) \cap C^0(\mathcal{Q}_\beta) \) be the unique solution to the equations

\[
\frac{\partial v_\epsilon}{\partial t} = \Delta_{q_0} v_\epsilon + f, \quad \text{in } \mathcal{Q}_\beta, \quad \text{and } v_\epsilon = \varphi, \quad \text{on } \partial_P \mathcal{Q}_\beta.
\]
For any compact subset $K \in B_β(0,1)$ and $δ \in (0,1)$, the standard Schauder estimates for parabolic equations provide uniform $C^{2+α,2α}_σ$-estimates for $v_ε$ on $K \setminus T_δ S \times (δ^2,1)$. Then $v_ε \to v$ for some $v \in C^{2+α,2α}_σ(K \setminus T_δ S \times (δ^2,1))$. Taking $δ \to 0$ and $K \to B_β(0,1)$ and by a diagonal argument, we can take $v_ε$ converges in $C^{2+α,2α}_loc(B_β \setminus S \times (0,1))$ to $v$, and $v$ satisfies the equation $\frac{∂v}{∂t} = Δ_g v + f$ on $B_β \setminus S \times (0,1)$. It only remains to show $v \in C^{0}(Q_δ)$ and $v = ϕ$ on $∂_p Q_δ$. The same proof as in Cases 1, 2, 3 in Proposition 4.1 yields that $v$ must coincide with $ϕ$ on $∂_p Q_δ$, since we can always choose $A > 1$ large enough such that (for example in Case 1) $\frac{∂ϕ}{∂t} - Δ_g ϕ ≤ \inf Q_δ f ≤ \frac{∂v}{∂t} - Δ_g v$. To see the continuity of $v$ in $Q_δ$, because of the Sobolev inequality (3.42) for metric spaces $(B_β, g_ε)$, by the proof of the standard De Giorgi-Nash-Moser theory for parabolic equations, we conclude that for any $p \in S$, $t_0 \in (0,1)$, there exists a small number $R_0 = R_0(p,t_0)$ such that on the cylinder $\hat{Q}_{R_0} := B_β(p,R_0) \times (t_0 - R_0^2, t_0)$, osc $v ≤ C r^α$ for any $r \in (0,R_0)$ and some $α' \in (0,1)$. Therefore osc $v ≤ C r^α$ and $v$ is continuous at $(p,t_0)$, as desired.

The uniqueness of the solution to (4.14) follows from maximum principle.

**Remark 4.1.** Corollary 4.1 is not needed in the proof of Theorem 1.3. So by Theorem 1.3, the solution $u$ to (4.14) is in $C^{2+α,2α}_σ(Q_δ) \cap C^{0}(Q_δ)$.

4.2. Sketched proof of Theorem 1.3

With Proposition 4.1, we can prove the Schauder estimates for the solution $u \in C^{0}(Q_δ) \cap C^{2,1}(Q_δ^{β})$ to the equation (4.1) for a Dini-continuous function $f$, by making use of almost the same arguments as in the proof of Theorem 1.1. So we will not provide the full details, and only point out the main differences. For any given points $Q_p = (p,t_p)$, $Q_q = (q,t_q) \in (B_β(0,1/2) \setminus S) \times (i,1)$. To define the approximating functions $u_k$ as in (3.44), we define $u_k$ in this case as the solution to the heat equation

$$\frac{∂u_k}{∂t} = Δ_g u_k + f(Q_p), \quad \text{in} \quad \hat{B}_k(p) \times (t_p - i \cdot τ^2k, t_p]$$

and $u_k = u$ on $∂_p(\hat{B}_k(p) \times (t_p - i \cdot τ^2k, t_p])$, where $\hat{B}_k(p)$ is defined in (3.48). We can now apply the estimates in Proposition 4.1 to the functions $u_k$ or $u_k - u_{k-1}$, instead of the ones in Lemmas 3.3 and 3.4 as we did in Sections 3.2, 3.3 and 3.4 to prove the Schauder estimates for $u$. Thus we finish the proof of Theorem 1.3.

4.3. Interior Schauder estimate for non-flat conical Kähler metrics

Let $g = \sqrt{-1}g_{jk}(z,t)dz_j ∧ dz_k$ be a $C^{α,2}_{β}$ conical Kähler metric on $Q_δ$ with conical singularity along $S$, that is, for any $t \in [0,1]$, $g(·,t)$ is a $C^{0,α}_{β}$ conical Kähler metric as in Section 3.5 and the coefficients of $g$ in the basis $\{ε_j ∧ ε_k, \cdots\}$, are $\frac{2}{t}$-Hölder continuous in $t \in [0,1]$. Suppose $u \in C^{2+α,2α}_σ(Q_δ)$ satisfies the equation

$$\frac{∂u}{∂t} = Δ_g u + f, \quad \text{in} \quad Q_δ,$$

for some $f \in C^{α,2}_{σ}(Q_δ)$.

**Proposition 4.2.** There exists a constant $C = C(n, β, α, g)$ such that

$$\|u\|_{C^{2+α,2α}_σ(Q_δ)} ≤ C(\|u\|_{C^0(\overline{Q_δ})} + \|f\|_{C^{α,2}_σ(Q_δ)}).$$
Proof. The proof is parallel to that of Proposition 3.6. Given any two points \( P_x = (x, t_x), P_y = (y, t_x) \in Q_B, \) we may assume \( d_{P_x} = \min\{d_{P_x}, d_{P_y}\} > 0 \) where \( d_{P_x} := d_{P,B}(P_x, \partial P \mathcal{Q}_B) \) is the parabolic distance of \( P_x \) to the parabolic boundary \( \partial P \mathcal{Q}_B. \) Let \( \mu \in (0, 1/4) \) be a positive number to be determined later. Denote \( d := \mu d_{P_x}, \mathcal{Q} := B_\beta(x, d) \times (t_x - d^2, t_x) \) the “parabolic ball” centered at \( P_x, \) and \( \frac{1}{2} \mathcal{Q} := B_\beta(x, d/2) \times (t_x - d^2/4, t_x). \)

- **Case 1.** \( d_{P,B}(P_x, P_y) < d/2. \) In this case we always have \( P_y \in \frac{1}{2} \mathcal{Q}. \)

**Case 1.1.** \( B_\beta(x, d) \cap \mathcal{S} = \emptyset. \) As in the proof of Proposition 3.6, we can introduce smooth complex coordinates \( \{w_1, w_2, z_3, \ldots, z_n\} \) on \( B_\beta(x, d), \) under which \( g_\beta \) becomes the standard Euclidean metric and the components of \( g \) are \( \mathcal{C}^{\alpha, \frac{2}{2}} \) in the usual sense on \( \mathcal{Q}. \) The leading coefficients and constant term \( f \) in (4.15) are both \( \mathcal{C}^{\alpha, \frac{2}{2}} \) in the usual sense, so we can apply the standard parabolic Schauder estimates (see Theorem 4.9 in [27]) to get that there exists some constant \( C = C(n, \beta, \alpha, g) \) which is independent of \( \mathcal{Q} \)

\[
[u]_{C^{2+\alpha, \frac{2}{2}}(\mathcal{Q})}^* \leq C \left( \|u\|_{\mathcal{C}^{0}(\mathcal{Q})} + \|f\|_{\mathcal{C}^{0, \frac{2}{2}}(\mathcal{Q})}^{(2)} \right).
\]

(4.16)

Let \( D \) denote the ordinary first order differential operators in the coordinates \( \{w_1, w_2, z_3, \ldots, z_n\}. \) We calculate

\[
|Tu(P_x) - Tu(P_y)| \leq |D^2 u(P_x) - D^2 u(P_y)| + \frac{d_{P,B}(P_x, P_y)}{d} \left( |D^2 u(P_x)| + |D^2 u(P_y)| \right)
\]

\[
\leq \frac{4d_{P,B}(P_x, P_y)^\alpha}{d^{2+\alpha}} [u]_{C^{2+\alpha, \frac{2}{2}}(\mathcal{Q})}^* + \frac{4d_{P,B}(P_x, P_y)^\alpha}{d^{2+\alpha}} [u]_{C^{2+1, \frac{2}{2}}(\mathcal{Q})}^*
\]

\[
\leq \frac{8d_{P,B}(P_x, P_y)^\alpha}{d^{2+\alpha}} [u]_{C^{2+\alpha, \frac{2}{2}}(\mathcal{Q})}^* + C \frac{d_{P,B}(P_x, P_y)^\alpha}{d^{2+\alpha}} \|u\|_{\mathcal{C}^{0}(\mathcal{Q})}.
\]

Recall \( T \) denotes the operators in \( T \) and \( \frac{\partial}{\partial t}, \) then by (4.16) it follows that

\[
d_{P_x}^{2+\alpha} |Tu(P_x) - Tu(P_y)| \leq \frac{C}{\mu^{2+\alpha}} \|f\|_{\mathcal{C}^{0, \frac{2}{2}}(\mathcal{Q})}^{(2)} + \frac{C}{\mu^{2+\alpha}} \|u\|_{\mathcal{C}^{0}(\mathcal{Q})}.
\]

(4.17)

**Case 1.2.** \( B_\beta(x, d) \cap \mathcal{S} \neq \emptyset. \) Let \( \hat{x} \in \mathcal{S} \) be the projection of \( x \) to \( \mathcal{S} \) and \( \hat{P}_x = (\hat{x}, t_x) \) be the corresponding space-time point. Denote \( \hat{\mathcal{Q}} := B_\beta(\hat{x}, 2d) \times (t_x - 4d^2, t_x). \) As the Case 1.2 in the proof of Proposition 3.6, we may choose suitable complex coordinates so that \( g_{\gamma_{\hat{P}_x}}(\hat{P}_x) = \delta_{jk} \) and for \( j, k \geq p + 1 \) \( g_{\hat{P}_x}(\hat{P}_x) = \delta_{jk}, \) and the cross terms in the expansion of \( g \) in (3.14) vanish at \( \hat{P}_x. \) Thus the equation (4.15) can be re-written as

\[
\frac{\partial u}{\partial t} = \Delta_{g_\beta} u + \eta \sqrt{-1} \partial \bar{\partial} u + \gamma =: \Delta_{g_\beta} u + \bar{f}, \quad u \in \mathcal{C}^{0}(\hat{\mathcal{Q}}) \cap \mathcal{C}^{2,1}(\hat{\mathcal{Q}}^\#),
\]

for some \((1, 1)\)-form \( \eta \) as in the proof of Proposition 3.6. From the rescaled version of Theorem 1.3 we conclude that

\[
d_{P_x}^{2+\alpha} |Tu(P_x) - Tu(P_y)| \leq C \left( \|u\|_{\mathcal{C}^{0}(\hat{\mathcal{Q}})} + \|\bar{f}\|_{\mathcal{C}^{0, \frac{2}{2}}(\hat{\mathcal{Q}})}^{(2)} \right),
\]

hence

\[
d_{P_x}^{2+\alpha} |Tu(P_x) - Tu(P_y)| \leq \frac{C}{\mu^{2+\alpha}} \left( \|u\|_{\mathcal{C}^{0}(\mathcal{Q})} + \|\bar{f}\|_{\mathcal{C}^{0, \frac{2}{2}}(\mathcal{Q})}^{(2)} \right).
\]

(4.18)
• Case 2. \(d_{\mathcal{P},\beta}(P_x, P_y) \geq d/2\). Then we calculate (recall \(Q_\beta := B_\beta(0, 1) \times (0, 1)\))

\[
d_{\mathcal{P},\beta}(P_x, P_y) \geq d/2 
\]

\[
d_{\mathcal{P},\beta}^2(P_x, P_y)^\alpha 
\]

Combining (4.17), (4.18) and (4.19), we obtain

\[
d_{\mathcal{P},\beta}^2(P_x, P_y)^\alpha 
\]

Observe that for any \(P \in Q\) or \(P \in \hat{Q}\), \(d_{\mathcal{P},\beta}(P, \partial Q_\beta) \geq (1 - 2\mu)d_P\). Then it follows from definition that

\[
\|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2 \leq C\mu^2\|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2 + C\mu^{2+\alpha}\|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2 \leq C\mu^2\|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2.
\]

We calculate

\[
\|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2 \leq \|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2 + \|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q)}^2
\]

where in the last inequality we use the interpolation inequality, by which we also have

\[
\frac{8}{\mu^\alpha}\|u\|_{c_\beta^{2,1}(Q_\beta)}^2 \leq \mu^\alpha\|u\|_{c_\beta^{2,\alpha, \frac{2\mu}{\beta}}(Q_\beta)}^2 + C\mu\|u\|_{c_\beta^{0}(Q_\beta)}^2.
\]

If \(\mu\) is chosen small such that \(\mu^\alpha(2C_1\|g\|_{c_\beta^{\frac{2\mu}{\beta}}(Q_\beta)}) + 1 < 1/2\), combining the above inequalities we get

\[
d_{\mathcal{P},\beta}^2(P_x, P_y)^\alpha \leq \|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q_\beta)}^2 + C\|u\|_{c_\beta^{0}(Q_\beta)}^2.
\]

Taking supremum over all \(P_x \neq P_y \in Q_\beta\), we obtain that

\[
[u]_{c_\beta^{2,\alpha, \frac{2\mu}{\beta}}(Q_\beta)} \leq C\|u\|_{c_\beta^{0}(Q_\beta)}^2 + \|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q_\beta)}^2.
\]

The proposition is proved by invoking the interpolation inequalities.

\[
\square
\]

**Remark 4.2.** It follows from the proof that the estimates in Proposition 4.2 also hold on \(Q_\beta(p, R) := B_{\mathcal{P},\beta}(p, R) \times (0, R^2) \subset Q_\beta\), i.e. the cylinder whose spatial center \(p\) may not lie in \(S\).

It is easy to derive the following local Schauder estimate for \(c_\beta^{2,\alpha, \frac{2\mu}{\beta}}\)-solutions to (4.15) from Proposition 4.2.

**Corollary 4.2.** Let \(K \subset B_{\mathcal{P},\beta}(0, 1)\) be a compact subset and \(\varepsilon_0 \in (0, 1)\) be a given number. Assumptions as in Proposition 4.2, there exists a constant \(C = C(n, \beta, \alpha, g, K, \varepsilon_0) > 0\) such that

\[
\|u\|_{c_\beta^{2,\alpha, \frac{2\mu}{\beta}}(K \times [\varepsilon_0, 1])} \leq C\|u\|_{c_\beta^{0}(Q_\beta)}^2 + \|f\|_{c_\beta^{\alpha, \frac{2\mu}{\beta}}(Q_\beta)}^2.
\]
With the interior Schauder estimates in Proposition 4.2, we can show the existence of \(C^2_{\beta} Q_{\beta}\)-solutions to the Dirichlet problem:

\[
\frac{\partial u}{\partial t} = \Delta_g u + f, \text{ in } Q_{\beta}, \text{ and } u = \varphi \text{ on } \partial_P Q_{\beta},
\]

(4.20)

for any given \(f \in C^0_{\beta} Q_{\beta}\) and \(\varphi \in C^0(\partial_P Q_{\beta})\). We first show the existence of solutions to (4.20) in case \(\varphi \equiv 0\).

**Lemma 4.4.** Let \(\sigma \in (0, 1)\) be a given number and \(u \in C^2_{\beta} Q_{\beta}\) solves (4.20) with \(\|u\|_{C^0(Q_{\beta})} < \infty\) and \(\|f\|_{C^0(Q_{\beta})} < \infty\). Then there is a constant \(C = C(n, \alpha, \beta, g, \sigma) > 0\) such that

\[
\|u\|_{C^0_{\beta} Q_{\beta}} \leq C(\|u\|_{C^0(Q_{\beta})} + \|f\|_{C^0_{\beta} Q_{\beta}}).
\]

Proof. The lemma follows from definitions of the norms and the estimates in Proposition 4.2. \(\square\)

**Lemma 4.5.** Suppose \(u \in C^2_{\beta} Q_{\beta}\) satisfies \(\frac{\partial u}{\partial t} = \Delta_g u + f\) and \(u \equiv 0\) on \(\partial_P Q_{\beta}\). For any \(\sigma \in (0, 1)\), there exists a constant \(C = C(n, \beta, g, \sigma) > 0\) such that

\[
\|u\|_{C^0(Q_{\beta})} = \sup_{P_x \in Q_{\beta}} d_{P_x} |u(P_x)| \leq C \sup_{P_x \in Q_{\beta}} d_{P_x}^\sigma |f(P_x)| = C\|f\|_{C^0(Q_{\beta})},
\]

where \(d_{P_x} = d_{P_{\beta}}(P_x, \partial_P Q_{\beta})\) is the parabolic distance of \(P_x\) to the parabolic boundary \(\partial_P Q_{\beta}\).

Proof. We denote \(N := \|f\|_{C^0(Q_{\beta})} < \infty\) and \(P_x = (x, t_x)\). Define functions

\[
w_1(P_x) = (1 - d_\beta(x)^2)^\sigma, \quad w_2(P_x) = t_x^{\sigma/2},
\]

where \(d_\beta(x) = d_\beta(x, 0)\) is the \(g_\beta\)-distance between \(x\) and \(0\). Observe that by definition \(d_{P_x} = \min\{1 - d_\beta(x), t_x^{1/2}\}\). By a straightforward calculation there is a constant \(c_0 > 0\) such that

\[(\frac{\partial}{\partial t} - \Delta_g)w_1 \geq c_0(1 - d_\beta(x))^{\sigma - 2}, \quad \text{and} \quad (\frac{\partial}{\partial t} - \Delta_g)w_2 \geq c_0(t_x^{1/2})^{\sigma - 2}.
\]

By maximum principle we get

\[
|u(P_x)| \leq Nc_0^{-1}(w_1(P_x) + w_2(P_x)), \quad \forall P_x \in Q_{\beta}.
\]

(4.21)

We decompose \(Q_{\beta}\) into different regions, \(Q_{\beta} = \Omega_1 \cup \Omega_2\), where

\[
\Omega_1 := \{ P_x \in Q_{\beta} \mid t_x^{1/2} > 1 - d_\beta(x) \},
\]

\[
\Omega_2 := \{ P_x \in Q_{\beta} \mid t_x^{1/2} \leq 1 - d_\beta(x) \}.
\]

(4.21) implies that on the parabolic boundaries \(\partial_P \Omega_1, \partial_P \Omega_2, |u(P_x)| \leq 2Nc_0^{-1}d_{P_x}^\sigma\). On \(\Omega_1\) we have \((\frac{\partial}{\partial t} - \Delta_g)(2Nc_0^{-1}w_1 \pm u) \geq 0\) and \(2Nc_0^{-1}w_1 \pm u \geq 0\) on \(\partial_P \Omega_1\), then maximum principle implies that \(2Nc_0^{-1}w_1 \pm u \geq 0\) in \(\Omega_1\), i.e. \(|u(P_x)| \leq 2Nc_0^{-1}d_{P_x}^\sigma\) in \(\Omega_1\). Similarly we also have \(2Nc_0^{-1}w_2 \pm u \geq 0\) in \(\Omega_2\) and thus \(|u(P_x)| \leq 2Nc_0^{-1}d_{P_x}^\sigma\) in \(\Omega_2\). In conclusion, we get

\[
|u(P_x)| \leq 2c_0^{-1}Nd_{P_x}^\sigma, \quad \forall P_x \in Q_{\beta},
\]

and the lemma is proved. \(\square\)
**Proposition 4.3.** If \( \varphi \equiv 0 \), the equation (4.20) admits a unique solution \( u \in C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta}) \cap C^0(Q_{\beta}) \) for any \( f \in C^0(Q_{\beta}) \).

**Proof.** The uniqueness follows from maximum principle, so it suffices to show the existence. We will use the continuity method. Define a continuous family of linear operators: for \( s \in [0,1] \), let \( L_s := s(\frac{\partial}{\partial t} - \Delta_{g_s}) + (1-s)(\frac{\partial}{\partial t} - \Delta_{g_{\beta}}) \). It can be seen that \( L_s = \frac{\partial}{\partial t} - \Delta_{g_s} \) for some conical Kähler metric \( g_s \) which uniformly equivalent to \( g_{\beta} \) and has uniform \( C^{\alpha, \frac{\alpha}{2}}_{\beta} \)-estimate. So the interior Schauder estimates holds also for \( L_s \). Fix a \( \sigma \in (0,1) \). Define

\[
B_1 := \{ u \in C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta}) \mid \| u \|_{C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta})} < \infty \},
\]

\[
B_2 := \{ f \in C^{\alpha, \frac{\alpha}{2}}_{\beta}(Q_{\beta}) \mid \| f \|_{C^{\alpha, \frac{\alpha}{2}}_{\beta}(Q_{\beta})} < \infty \}.
\]

Observe that any \( u \in B_1 \) is continuous in \( Q_{\beta} \) and vanishes on \( \partial_{\beta} Q_{\beta} \). \( L_s \) defines a continuous family of linear operators from \( B_1 \) to \( B_2 \). By Lemmas 4.4 and 4.5 we have

\[
\| u \|_{B_1} \leq C(\| u \|_{C^{0}(Q_{\beta})} + \| L_s u \|_{B_2}) \leq C\| L_s u \|_{B_2}, \quad \forall s \in [0,1], \quad \forall u \in B_1.
\]

By Corollary 4.1 and Remark 4.1, \( L_0 \) is invertible, thus by Theorem 5.2 in [19], \( L_1 \) is also invertible.

**Corollary 4.3.** For any \( \varphi \in C^0(\partial_{\beta} Q_{\beta}) \) and \( f \in C^{\alpha, \frac{\alpha}{2}}_{\beta}(Q_{\beta}) \), the equation (4.20) admits a unique solution \( u \in C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta}) \cap C^0(Q_{\beta}) \).

**Proof.** The proof is identical to that of Corollary 3.2 by an approximation argument. We may assume \( \varphi \in C^0(Q_{\beta}) \) and choose a sequence of \( \varphi_k \in C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta}) \) which converges uniformly to \( \varphi \) on \( Q_{\beta} \). The equations \( \frac{\partial u}{\partial t} = \Delta_{g} v_k + f - \Delta_{g} \varphi_k \), \( v_k \equiv 0 \) on \( \partial_{\beta} Q_{\beta} \) admits a unique \( C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta} \)-solution by Proposition 4.3. The interior Schauder estimates in Corollary 4.2 imply that \( u_k := v_k + \varphi_k \) converges in \( C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta, \text{loc}} \) to some function \( u \) in \( C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta}) \) which solves the equation (4.20). The \( C^0 \)-convergence \( u_k \to u \) is uniform on \( Q_{\beta} \) by maximum principle so \( u = \varphi \) on \( \partial_{\beta} Q_{\beta} \), as desired.

We recall the definition of weak solutions and refer to Section 7.1 in [17] for the notations.

**Definition 4.1.** We say a function \( u \) on \( Q_{\beta} \) is a weak solution to the equation \( \frac{\partial u}{\partial t} = \Delta_{g} u + f \), if

1. \( u \in L^2(0,1; H^1(B_{\beta})) \) and \( \frac{\partial u}{\partial t} \in L^2(0,1; H^{-1}(B_{\beta})) \);
2. For any \( v \in H^1_0(B_{\beta}) \) and \( t \in (0,1) \)

\[
\int_{B_{\beta}} \frac{\partial u}{\partial t}(x,t) v(x) \omega_{\beta}^n = - \int_{B_{\beta}} (\nabla u(x,t), \nabla v(x))_{g} \omega_{\beta}^n + \int_{B_{\beta}} f(x,t) v(x) \omega_{\beta}^n.
\]

On can use the classical Galerkin approximations to construct weak solution to the equation \( \frac{\partial u}{\partial t} = \Delta_{g} u + f \) (see Section 7.1.2 in [17]). If \( f \) has better regularity, so does the weak solution \( u \).

**Lemma 4.6.** If \( f \in C^{\alpha, \frac{\alpha}{2}}(Q_{\beta}) \), then any weak solution to \( \frac{\partial u}{\partial t} = \Delta_{g} u + f \) belongs to \( C^{2+\alpha, \frac{2+\alpha}{2}}_{\beta}(Q_{\beta}) \).
Proof. Sobolev inequality holds for the metric $g$ so by the proof of the standard De Giorgi-Nash-Moser theory for parabolic equations implies that $u$ is in fact continuous on $Q_\beta$. Since the metric $g$ is smooth on $Q_\beta^\#$, the weak solution $u$ is also a weak solution in $Q_\beta^\#$ with the smooth background metric, so $u \in C^{2+\alpha, \frac{2+\alpha}{\alpha}}_{loc}(Q_\beta^\#)$ in the usual sense by the classical Schauder estimates. Thus it suffices to consider points at $S$. We choose the worst such points $0 \in S$ only, since the case when centers are in other components of $S$ is even simpler. We fix the point $P_0 = (0, t_0) \in Q_\beta$ with $t_0 > 0$. Fix an $r \in (0, \sqrt{t_0})$. By Corollary 4.3 the equation

$$\frac{\partial v}{\partial t} = \Delta v + f, \text{ in } Q_\beta(P_0, r) := B_\beta(0, r) \times (t_0 - r^2, t_0],$$

with boundary value $v = u$ on $\partial_{\nu} Q_\beta(P_0, r)$ admits a unique solution $v \in C^{2+\alpha, \frac{2+\alpha}{\alpha}}_{\beta}(Q_\beta(P_0, r))$. Then by maximum principle $u = v$ in $Q_\beta(P_0, r)$. Thus $u \in C^{2+\alpha, \frac{2+\alpha}{\alpha}}_{\beta}(Q_\beta(P_0, r))$ too. The argument also works at other space-time points in $S_\beta$, we see that $u \in C^{2+\alpha, \frac{2+\alpha}{\alpha}}_{\beta}(Q_\beta)$, as desired.

**Corollary 4.4.** Let $(X, g, D)$ be as in Corollary 3.4, $u_0 \in C^0(X)$ and $f \in C^{0, \frac{\alpha}{2}}(X \times (0, 1])$ be given functions. The weak solution $u$ to the equation

$$\frac{\partial u}{\partial t} = \Delta_g u + f, \text{ in } X \times (0, 1], \quad u|_{t=0} = u_0$$

always exists. Moreover, $u \in C^{2+\alpha, \frac{2+\alpha}{\alpha}}_{\beta}(X \times (0, 1])$ and there exists a constant $C = C(n, g, \beta, \alpha) > 0$ such that

$$\|u\|_{C^{2+\alpha, \frac{2+\alpha}{\alpha}}_{\beta}(X \times (1/2, 1])} \leq C\left(\|u_0\|_{C^0(X)} + \|f\|_{C^{0, \frac{\alpha}{2}}(X \times (0, 1])}\right).$$

**Proof.** The weak equation can be constructed using the Galerkin approximations ([17]). The uniqueness is an easy consequence of maximum principle. The regularity of $u$ follows from the local results in Lemma 4.6. The estimate follows from maximum principle, a covering argument as in Corollary 3.4 and the local estimates in Corollary 4.2.

The **interior** estimate in Corollary 4.4 is not good enough to show the existence of solutions to non-linear partial differential equations since the estimate becomes worse as $t$ approaches $t = 0$. We need some global estimates in the whole time interval $t \in [0, 1]$ if the initial $u_0$ has better regularity.

### 4.4. Schauder estimate near $t = 0$

In this subsection, we will prove a Schauder estimate in the whole time interval for the solutions to the heat equation when the initial value is $0$ or has better regularity. We consider the model case with the background metric $g_\beta$ first, then we generalize the estimate to general non-flat conical Kähler metrics.

#### 4.4.1. The model case

In this subsection, we will assume the background metric is $g_\beta$. Let $u$ be the solution to the equation

$$\frac{\partial u}{\partial t} = \Delta_{g_\beta} u + f, \text{ in } Q_\beta, \quad u|_{t=0} = 0, \quad (4.22)$$

and $u = \varphi \in C^0$ on $\partial B_\beta \times (0, 1]$, where $f \in C^{0, \alpha/2}_{\beta}(Q_\beta^\#)$. In the calculations below, we should have used the smooth approximating solutions $u_\epsilon$, where $\partial_t u_\epsilon = \Delta_{g_\epsilon} u_\epsilon + f$ and $u_\epsilon = u$ on $\partial_{\nu} Q_\beta$. But
by letting $\epsilon \to 0$, the corresponding estimates also hold for $u$. So for simplicity, below we will work directly on $u$.

We fix $0 < \rho < R \leq 1$ and denote $B_R := B_R(0, R)$ and $B_R := B_R \times [0, R^2]$ in this section. Let $u$ be the solution to (4.22). We first have the following Cacciopoli inequalities.

**Lemma 4.7.** There exists a constant $C = C(n) > 0$ such that

$$\sup_{t \in [0, \rho^2]} \int_{B_R} u^2 \omega_B^n + \int_{Q_R} |\nabla u|_{\beta g}^2 \omega_B^n dt \leq C \left( \frac{1}{(R - \rho)^2} \int_{Q_R} u^2 \omega_B^n dt + (R - \rho)^2 \int_{Q_R} f^2 \omega_B^n dt \right),$$

(4.23)

and

$$\sup_{t \in [0, \rho^2]} \int_{B_R} |\nabla u|_{\beta g}^2 \omega_B^n + \int_{Q_R} (|\nabla \nabla u|_{\beta g}^2 + |\nabla \nabla u|_{\beta g}^2) \omega_B^n dt \leq C \left( \frac{1}{(R - \rho)^2} \int_{Q_R} |\nabla u|_{\beta g}^2 \omega_B^n dt + \int_{Q_R} (f - f_R)^2 \omega_B^n dt \right),$$

(4.24)

where $f_R := \frac{1}{|Q_R| \omega_B} \int_{Q_R} f \omega_B^n dt$ is the average of $f$ over the cylinder $Q_R$.

**Proof.** We fix a cut-off function $\eta$ such that $\text{supp } \eta \subset B_R$ and $\eta = 1$ on $B_R$, $|\nabla \eta|_{\beta g} \leq \frac{2}{R - \rho}$. Multiplying both sides of the equation (4.22) by $\eta^2 u$, and applying integration by parts we get

$$\frac{d}{dt} \int_{B_R} \eta^2 u^2 = \int_{B_R} 2\eta^2 u \Delta_{\beta g} u + 2\eta^2 u f$$

$$= \int_{B_R} -2\eta^2 |\nabla u|_{\beta g}^2 - 4\eta \langle \nabla u, \nabla \eta \rangle_{\beta g} + 2\eta^2 u f$$

$$\leq \int_{B_R} -\eta^2 |\nabla u|_{\beta g}^2 + 4\eta^2 |\nabla \eta|_{\beta g}^2 + \frac{\eta^2 u}{(R - \rho)^2} + \eta^2 (R - \rho)^2 f^2;$$

(4.23) follows from this inequality by integrating over $t \in [0, s^2]$ for all $s \leq \rho$. To see (4.24), observe that the Bochner formula yields that

$$\frac{\partial}{\partial t} |\nabla u|^2 \leq \Delta_{\beta g} |\nabla u|^2 - |\nabla \nabla u|_{\beta g}^2 - |\nabla \nabla u|_{\beta g}^2 - 2\langle \nabla u, \nabla f \rangle_{\beta g}.$$
Combining (4.23) and (4.24) we conclude that
\[
\sup_{t \in [0, R^2/4]} \int_{B_{R/2}} |\nabla u|^2 + \int_{Q_{R/2}} |\Delta_{g_{\beta}} u|^2 \leq \frac{C}{R^4} \int_{Q_R} u^2 + CR^{2n+2} \|f\|_{\mathcal{C}^0(Q_R)}^2 + CR^{2n+2+2\alpha} ([f]_{C_{g_{\beta}}^{\infty,n/2}(Q_R)})^2.
\] (4.25)

By a standard Moser iteration argument we get the following sub-mean value inequality.

**Lemma 4.8.** If in addition \( f \equiv 0 \), then there exists a constant \( C = C(n, \beta) > 0 \) such that
\[
\sup_{Q_{\rho}} |u| \leq C \left( \frac{1}{(R - \rho)^{2n+2}} \int_{Q_R} u^2 \omega_{g_{\beta}}^n dt \right)^{1/2}.
\]

**Proof.** For any \( p \geq 1 \), multiplying both sides of the equation by \( \eta^2 u_+^p \) where \( u_+ = \max\{u, 0\} \) and applying IBP, we get
\[
\frac{d}{dt} \int_{B_R} \frac{\eta^2 u_+^{p+1}}{p+1} = \int_{B_R} -p\eta^2 u_+^{p-1} |\nabla u_+|^2 - 2\eta u_+^{p} (\nabla u_+, \nabla \eta).
\]

By Cauchy-Schwarz inequality and integrating over \( t \in [0, R^2] \), we conclude that
\[
\sup_{s \in [0, R^2]} \int_{B_R} \eta^2 u_+^{p+1} \bigg|_{t=s} + \int_{Q_R} |\nabla (\eta u_+^{p+1})|^2 \leq \frac{C}{(R - \rho)^2} \int_{Q_R} u_+^{p+1} \omega_{g_{\beta}}^n dt =: A.
\]

By Sobolev inequality we get
\[
\int_0^{R^2} \int_{B_R} (\eta^2 u_+^{p+1})^{1+\frac{1}{n}} \leq \int_0^{R^2} \left( \int_{B_R} \eta^2 u_+^{p+1} \right)^{1/n} \left( \int_{B_R} (\eta u_+^{p+1})^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}}\]
\[
\leq A^{1/n} C \int_0^{R^2} \int_{B_R} |\nabla (\eta u_+^{p+1})|^2 \leq C A^{(n+1)/n}.
\]

If we denote \( H(p, \rho) = \left( \int_0^{R^2} \int_{B_R} u_+^p \right)^{1/p} \), the inequality above implies that
\[
H((p + 1)\xi, \rho) \leq \frac{C^{1/(p+1)}}{(R - \rho)^{2/(p+1)}} H(p + 1, R),
\]
where \( \xi = \frac{n+1}{n} > 1 \). Denote \( p_k + 1 = 2\xi^k \) and \( \rho_k = \rho + (R - \rho)2^{-k} \), then \( H(p_{k+1} + 1, \rho_{k+1}) \leq H(p_k + 1, \rho_k) \). Iterating this inequality we get
\[
H(\infty, \rho) = \sup_{Q_{\rho}} u_+ \leq \frac{C}{(R - \rho)^{n+1}} \left( \int_{Q_R} u_+^2 \right)^{1/2}.
\]

Similarly we get the same inequality for \( u_- = \max\{-u, 0\} \).

**Corollary 4.5.** If in addition \( f \equiv 0 \), then there is a \( C = C(n, \beta) > 0 \) such that
\[
\int_{Q_{\rho}} u^2 \omega_{g_{\beta}}^n dt \leq C \left( \frac{\rho}{R} \right)^{2+2n} \int_{Q_R} u^2 \omega_{g_{\beta}}^n dt.
\] (4.26)

**Proof.** When \( \rho \in \left[ \frac{R}{2}, R \right] \), the inequality is trivial; when \( \rho \in [0, R/2) \), it follows from Lemma 4.8. □
Lemma 4.9. If in addition \( f \equiv 0 \), then there is a \( C = C(n, \beta) > 0 \) such that for any \( \rho \in (0, R) \)
\[
\int_{Q_\rho} u^2 \omega_\beta^n dt \leq C \left( \frac{\rho}{R} \right)^{2n+4} \int_{Q_R} u^2 \omega_\beta^n dt.
\]

Proof. The inequality is trivial in case \( \rho \in [R/2, R] \) so we assume \( \rho < R/2 \). First we observe that \( \Delta_\beta u \) also satisfies the equation \( \partial_t (\Delta_\beta u) = \Delta_\beta (\Delta_\beta u) \) and \((\Delta_\beta u)_{t=0} \equiv 0\), so (4.26) holds with \( u^2 \) replaced by \((\Delta_\beta u)^2\), i.e.
\[
\int_{Q_\rho} (\Delta_\beta u)^2 \omega_\beta^n dt \leq C \left( \frac{\rho}{R} \right)^{2+2n} \int_{Q_R} (\Delta_\beta u)^2 \omega_\beta^n dt.
\]
Since \( u_{t=0} = 0 \), \( u(x, t) = \int_0^t \partial_s u(x, s) ds \), we calculate
\[
\int_{Q_\rho} u^2 \leq \rho^4 \int_{Q_\rho} |\partial_t u|^2 = \rho^4 \int_{Q_\rho} (\Delta_\beta u)^2
\leq C \rho^4 \left( \frac{\rho}{R} \right)^{2n+2} \int_{Q_{R/2}} (\Delta_\beta u)^2
\]
by (4.25) \( \leq C \left( \frac{\rho}{R} \right)^{2n+6} \int_{Q_R} u^2 \omega_\beta^n dt. \)

\[\square\]

Lemma 4.10. Let \( u \) be a solution to (4.22). There exists a constant \( C = C(n, \beta, \alpha) > 0 \) such that
\[
\frac{1}{\rho^{2n+2+2\alpha}} \int_{Q_\rho} (\Delta_\beta u)^2 \leq \frac{C}{R^{2n+2+2\alpha}} \int_{Q_R} (\Delta_\beta u)^2 \omega_\beta^n dt + C ([f]_{C_{\alpha,\alpha/2}(Q_R)})^2.
\]

Proof. Let \( u = u_1 + u_2 \), where
\[
\partial_t u_1 = \Delta_\beta u_1 + f_R, \text{ in } Q_R, \quad u_1 = u \text{ on } \partial P Q_R,
\]
and
\[
\partial_t u_2 = \Delta_\beta u_2 + f - f_R, \text{ in } Q_R, \quad u_2 = 0 \text{ on } \partial P Q_R.
\]
The function \((\Delta_\beta u_1)\) satisfies the assumptions of Lemma 4.9. Thus
\[
\int_{Q_\rho} (\Delta_\beta u_1)^2 \omega_\beta^n dt \leq C \left( \frac{\rho}{R} \right)^{2n+4} \int_{Q_R} (\Delta_\beta u_1)^2 \omega_\beta^n dt.
\]
Multiplying \( \dot{u}_2 = \partial u_2 / \partial t \) on both sides of the equation for \( u_2 \) and noting that \( \dot{u}_2 = 0 \) on \( \partial B_R \times (0, R^2) \), we get
\[
\int_{B_R} (\dot{u}_2)^2 = \int_{B_R} \dot{u}_2 \Delta_\beta u_2 + \dot{u}_2 (f - f_R) = \int_{B_R} -2 \langle \nabla \dot{u}_2, \nabla u_2 \rangle + \dot{u} (f - f_R)
\leq \int_{B_R} -\frac{\partial}{\partial t} |\nabla u_2|^2 + \frac{1}{2} (\dot{u}_2)^2 + 2(f - f_R)^2.
\]
Integrating over \( t \in [0, R^2] \), we obtain
\[
\int_{Q_R} (\dot{u}_2)^2 \leq -2 \int_{B_R} |\nabla u_2|^2 \bigg|_{t=R^2} + 4 \int_{Q_R} (f - f_R)^2,
\]
therefore
\[
\int_{Q_R} (\Delta_\beta u_2)^2 \leq 2 \int_{Q_R} (\dot{u}_2)^2 + 2 \int_{Q_R} (f - f_R)^2 \leq C R^{2n+2+2\alpha} ([f]_{C_{\alpha,\alpha/2}(Q_R)})^2.
\]
Then for \( \rho < R \) we have
\[
\int_{Q_\rho} (\Delta \beta u)^2 \leq 2 \int_{Q_\rho} (\Delta \beta u_1)^2 + 2 \int_{Q_\rho} (\Delta \beta u_2)^2 \\
\leq C \left( \frac{\rho}{R} \right)^{2n+4} \int_{Q_R} (\Delta \beta u_1)^2 \omega_\beta^n dt + CR^{2n+2+2\alpha} (|f| c_\alpha^{o,\alpha/2}(Q_R))^2.
\]
The estimate is proved by an iteration lemma (see Lemma 3.4 in [22]).

\[\Box\]

**Lemma 4.11.** Suppose \( u \) satisfies the equations (4.22). There exists a constant \( C = C(n, \beta, \alpha) > 0 \) such that for any \( \rho \in (0, R/2) \)
\[
\int_{Q_\rho} (\Delta \beta u - (\Delta \beta u)_\rho)^2 \omega_\beta^n dt \leq CM_R \rho^{2n+2+2\alpha},
\]
where
\[
M_R := \frac{1}{R^{2+2\alpha}} \| u \|_{C^0(Q_R)}^2 + \frac{1}{R^{2\alpha}} \| f \|_{C^0(Q_R)}^2 + (|f| c_\alpha^{o,\alpha/2}(Q_R))^2.
\]
**Proof.** From Lemma 4.10, we get
\[
\int_{Q_\rho} (\Delta \beta u)^2 \leq C \rho^{2+2n+2\alpha} \left( \frac{1}{R^{2n+2+2\alpha}} \int_{Q_{2R/3}} (\Delta \beta u)^2 + (|f| c_\alpha^{o,\alpha/2}(Q_{2R/3}))^2 \right)
\]
by (4.25) \( \leq C \rho^{2+2n+2\alpha} \left( \frac{1}{R^{2n+2+2\alpha}} \int_{Q_R} u^2 + \frac{1}{R^{2\alpha}} \| f \|_{C^0(Q_R)}^2 + (|f| c_\alpha^{o,\alpha/2}(Q_R))^2 \right) \)
\( \leq C \rho^{2+2n+2\alpha} M_R. \)

On the other hand by Hölder inequality
\[
(\Delta \beta u)^2 = \frac{1}{|Q_\rho|^{2+\alpha}} \left( \int_{Q_\rho} (\Delta \beta u) \omega_\beta^n dt \right)^2 \leq \frac{C}{\rho^{2+2n}} \int_{Q_\rho} (\Delta \beta u)^2 \leq CM_R \rho^{2n}.
\]

The lemma is proved by combining the two inequalities above.

\[\Box\]

By Campanato’s lemma (see Theorem 3.1 in [22]), we get

**Corollary 4.6.** There is a constant \( C = C(n, \beta, \alpha) > 0 \) such that for any \( x \in B_\beta(0, 3/4) \) and \( R < 1/10 \)
\[
|\Delta \beta u| c_\beta^{o,\alpha/2}(B_\beta(x,R/2) \times [0,R^2/4]) \\
\leq C \left( \frac{1}{R^{2+\alpha}} \| u \|_{C^0(B_\beta(x,R) \times [0,R^2])} + \frac{1}{R^\alpha} \| f \|_{C^0(B_\beta(x,R) \times [0,R^2])} + (|f| c_\alpha^{o,\alpha/2}(B_\beta(x,R) \times [0,R^2])) \right). \tag{4.27}
\]

**Lemma 4.12.** There exists a constant \( C = C(n, \beta, \alpha) > 0 \) such that for any \( x \in B_\beta(0, 3/4) \) and \( R < 1/10 \)
\[
|T \beta u| c_\beta^{o,\alpha/2}(B_\beta(x,R/2) \times [0,R^2/4]) + |\partial u/\partial t| c_\beta^{o,\alpha/2}(B_\beta(x,R/2) \times [0,R^2/4]) \\
\leq C \left( \frac{1}{R^{2+\alpha}} \| u \|_{C^0(B_\beta(x,R) \times [0,R^2])} + \frac{1}{R^\alpha} \| f \|_{C^0(B_\beta(x,R) \times [0,R^2])} + (|f| c_\alpha^{o,\alpha/2}(B_\beta(x,R) \times [0,R^2])) \right). \tag{4.28}
\]
Proof. It follows from (4.27) and the elliptic Schauder estimates in Theorem 1.1 by adjusting $R$ slightly that for any $t \in [0, R^2/4]$ \[ [Tu(\cdot, t)]_{C^{0, \alpha}}(B_\beta(x, R/2)) \leq C \left( \frac{1}{R^{2+\alpha}} \|u\|_{C^0(B_\beta(x, R) \times [0, R^2])} + \frac{1}{R^\alpha} \|f\|_{C^0(B_\beta(x, R) \times [0, R^2])} + \|f\|_{C^{\alpha, \alpha/2}_\beta(B_\beta(x, R) \times [0, R^2])} \right), \] that is, in the spatial variables the estimate (4.28) holds. It only remains to show the Hölder continuity of $Tu$ in the time-variable. For this, we fix any two times $0 \leq t_1 < t_2 \leq R^2/4$ and denote $r := \sqrt{t_2 - t_1}/2$. For any $x_0 \in B_\beta(x, R/4)$, $B_\beta(x_0, r) \subset B_\beta(x, R/2)$. By (4.27) and the equation for $u$, it is not hard to see that the inequality (4.27) holds when the $\Delta_\beta u$ on LHS is replaced by $\hat{u} = \frac{\partial u}{\partial t}$. In particular \[ \frac{|\hat{u}(y, t) - \hat{u}(y, t_1)|}{|t - t_1|^{\alpha/2}} \leq A_R, \forall y \in B_\beta(x, R/2) \] where $A_R :=$ the constant on the RHS of (4.27). Integrating over $t \in [t_1, t_2]$ we get \[ |u(y, t_2) - u(y, t_1) - \hat{u}(y, t_1)(t_2 - t_1)| \leq CA_R(t_2 - t_1)^{1+\frac{\alpha}{2}}, \] thus for any $y \in B_\beta(x_0, r)$ \[ |u(y, t_2) - u(y, t_1) - \hat{u}(x_0, t_1)(t_2 - t_1)| \leq |u(y, t_2) - u(y, t_1) - \hat{u}(y, t_1)(t_2 - t_1)| + |\hat{u}(x_0, t_1) - \hat{u}(y, t_1)|(t_2 - t_1) \leq CA_R(t_2 - t_1)^{1+\frac{\alpha}{2}} + A_Rr^\alpha(t_2 - t_1). \] Denote $\tilde{u}(y) := u(y, t_2) - u(y, t_1) - \hat{u}(x_0, t_1)(t_2 - t_1)$, which is a function on $B_\beta(x_0, r)$ and $\tilde{f} := \Delta_\beta \tilde{u} = \Delta_\beta u(\cdot, t_2) - \Delta_\beta u(\cdot, t_1)$ satisfies $\|	ilde{f}\|_{C^0(B_\beta(x_0, r))} \leq A_R(t_2 - t_1)^\alpha$ and $[\tilde{f}]_{C^{\alpha, \alpha}_\beta(B_\beta(x_0, r))} \leq A_R$ by (4.27). It follows from the rescaled version of Proposition 3.6 that \[ |T\tilde{u}|_{C^0(B_\beta(x_0, r/2))} \leq C(n, \beta, \alpha) \left( \frac{\|	ilde{u}\|_{C^0(B_\beta(x_0, r))}}{r^2} + \|	ilde{f}\|_{C^0(B_\beta(x_0, r))} + r^\alpha[\tilde{f}]_{C^{\alpha, \alpha}_\beta(B_\beta(x_0, r))} \right) \leq C(t_2 - t_1)^{\alpha/2}A_R. \] Therefore for any $x_0 \in B_\beta(x, R/4)$ \[ \frac{|Tu(x_0, t_2) - Tu(x_0, t_1)|}{|t_2 - t_1|^{\alpha/2}} \leq CA_R. \] It is then elementary to see by triangle inequality that (by adjusting $R$ slightly if necessary) \[ [Tu]_{C^{\alpha, \alpha/2}_\beta(B_\beta(x, R/2) \times [0, R^2/4])} \leq CA_R, \] as desired. The estimate for $\hat{u}$ follows from the equation $\hat{u} = \Delta_\beta u + f$. \hfill $\square$

**Remark 4.3.** By a simple parabolic rescaling of the metric and time, we see from (4.28) that for any $0 < r < R < 1/10$ that \[ [Tu]_{C^{\alpha, \alpha/2}_\beta(Q_r)} \leq C \left( \frac{\|u\|_{C^0(Q_R)}}{(R - r)^{2+\alpha}} + \frac{\|f\|_{C^0(Q_R)}}{(R - r)^\alpha} + [f]_{C^{\alpha, \alpha/2}_\beta(Q_R)} \right), \] (4.29)
4.4.2. the non-flat metric case. In this subsection, we will consider the case when the background metrics are general non-flat $C^{a,α/2}_β$-conical Kähler metrics $g = g(z,t)$. Suppose $u \in C^{2+α,2+α/2}_β(\mathcal{Q}_β)$ satisfies the equation
\[
\frac{\partial u}{\partial t} = \Delta g u + f, \quad \text{in } \mathcal{Q}_β, \quad u|_{t=0} = 0,
\]
and $u \in C^0(\partial_0 \mathcal{Q}_β)$.

**Proposition 4.4.** There exists a constant $C = C(n, β, α, g) > 0$ such that
\[
\|u\|_{C^{α,α/2}_β(\hat{Q}_r)} \leq C \left(\|u\|_{C^0(\hat{Q}_R)} + \|f\|_{C^{α,α/2}_β(\hat{Q}_R)}\right)
\]

Proof. Choose suitable complex coordinates at the origin $x = 0$, we may assume the components of $g$ in the basis $\{ε_j \wedge \bar{ε}_k, \ldots\}$ satisfies $g_{ε_j\bar{ε}_k}(0) = δ_{jk}$ and $g_{jk}(0) = δ_{jk}$ at the origin $0$. As in the proof of Proposition 4.2, we can write the equation (4.30) as
\[
\frac{\partial u}{\partial t} = \Delta g u + η \sqrt{-1} \bar{∂}u + f =: \Delta g u + \hat{f},
\]
where $η$ is given in the proof of Proposition 3.6. By (4.29) we get
\[
[Tu]_{C^{α,α/2}_β(\hat{Q}_r)} \leq C \left(\|u\|_{C^0(\hat{Q}_R)} + \frac{1}{(R-r)^α} \|f\|_{C^0(\hat{Q}_R)} + [\hat{f}]_{C^{α,α/2}_β(\hat{Q}_R)}\right),
\]
where $\hat{Q}_R := B_β(0,R) \times [0,R]^2$. Observe that
\[
\frac{1}{(R-r)^α} \|f\|_{C^0(\hat{Q}_R)} \leq \frac{1}{(R-r)^α} \|f\|_{C^0(\hat{Q}_R)} + \frac{1}{(R-r)^α} η \|u\|_{C^0(\hat{Q}_R)}
\]
\[
\leq \frac{1}{(R-r)^α} \|f\|_{C^0(\hat{Q}_R)} + \frac{\|u\|_{C^{2+α,2+α/2}_β(\hat{Q}_R)}}{(R-r)^α} \|\hat{f}\|_{C^{α,α/2}_β(\hat{Q}_R)} + C(ε)\|u\|_{C^0(\hat{Q}_R)}
\]
and
\[
[\hat{f}]_{C^{α,α/2}_β(\hat{Q}_R)} \leq [f]_{C^{α,α/2}_β(\hat{Q}_R)} + η \|u\|_{C^0(\hat{Q}_R)} [Tu]_{C^{α,α/2}_β(\hat{Q}_R)} + [Tu]_{C^0(\hat{Q}_R)} [\hat{f}]_{C^{α,α/2}_β(\hat{Q}_R)}
\]
\[
\leq [f]_{C^{α,α/2}_β(\hat{Q}_R)} + η \|u\|_{C^0(\hat{Q}_R)} R^{α} [Tu]_{C^{α,α/2}_β(\hat{Q}_R)}
\]
\[
+ \|u\|_{C^{2+α,2+α/2}_β(\hat{Q}_R)} [\hat{f}]_{C^{α,α/2}_β(\hat{Q}_R)} + C(ε)\|u\|_{C^0(\hat{Q}_R)}
\]
By choosing $R_0 = R_0(n, β, α, g) > 0$ small enough and suitable $ε > 0$, for any $0 < r < R < R_0 < 1/10$, the combination of the above inequalities yields that
\[
[Tu]_{C^{α,α/2}_β(\hat{Q}_r)} \leq \frac{1}{2} [Tu]_{C^{α,α/2}_β(\hat{Q}_r)} + C \left(\|u\|_{C^0(\hat{Q}_R)} + \frac{1}{(R-r)^α} \|f\|_{C^0(\hat{Q}_R)} + [f]_{C^{α,α/2}_β(\hat{Q}_R)}\right)
\]
By Lemma 4.13 below (setting $φ(r) = [Tu]_{C^{α,α/2}_β(\hat{Q}_r)}$), we conclude that
\[
[Tu]_{C^{α,α/2}_β(\hat{Q}_r)} \leq C \left(\|u\|_{C^0(\hat{Q}_R)} + \|f\|_{C^{α,α/2}_β(\hat{Q}_R)}\right).
\]
This is the desired estimate when the center of the ball is the worst possible. For the other balls $B_β(x,r)$ with center $x \in B_β(0,1/2)$, we can repeat the above procedures and use the smooth coordinates $w_j = z_j^β$ in case the ball is disjoint with $\mathcal{S}_j$. Finitely many such balls cover $B_β(0,1/2)$ so we get the
\[
[Tu]_{C^{α,α/2}_β(B_β(0,1/2) \times [0,1/100])} \leq C \left(\|u\|_{C^0(\hat{Q}_R)} + \|f\|_{C^{α,α/2}_β(\hat{Q}_R)}\right).
\]
The proposition is proved by combining this inequality, the equation for $u$, interpolation inequalities, and the interior Schauder estimates in Corollary 4.2.

□

**Lemma 4.13** (Lemma 4.3 in [22]). Let $\phi(t) \geq 0$ be bounded in $[0, T]$. Suppose for any $0 < t < s \leq T$ we have

$$\phi(t) \leq \frac{1}{2} \phi(s) + \frac{A}{(s-t)^a} + B$$

for some $a > 0$, $A, B > 0$. Then it holds that for any $0 < t < s \leq T$

$$\phi(t) \leq c(a) \left( \frac{A}{(s-t)^a} + B \right).$$

**Corollary 4.7.** Suppose $u$ satisfies the equation

$$\frac{\partial u}{\partial t} = \Delta_g u + f, \text{ in } Q_\beta, \quad u|_{t=0} = u_0 \in C^{2,\alpha}_\beta(B_\beta(0,1)),$$

then

$$\|u\|_{C^{2+\alpha,2+\alpha/2}_\beta(B_\beta(0,1/2) \times [0,1])} \leq C \left( \|u\|_{C^0(Q_\beta)} + \|f\|_{C^{\alpha,\alpha/2}_\beta(Q_\beta)} + \|u_0\|_{C^{2+\alpha}_\beta(B_\beta(0,1))} \right),$$

for some constant $C = C(n, \beta, \alpha, g) > 0$.

**Proof.** We set $\hat{u} = u - u_0$ and $\hat{f} = f - \Delta_g u_0$. $\hat{u}$ satisfies the conditions in Proposition 4.4, so the corollary follows from Proposition 4.4 applied to $\hat{u}$ and triangle inequalities.

□

**Corollary 4.8.** Let the assumptions be as in Corollary 4.4 except that in addition we assume $u_0 \in C^{2,\alpha}_\beta(X)$. Then the weak solution to $\frac{\partial u}{\partial t} = \Delta_g u + f$ with $u|_{t=0} = u_0$ exists and is in $C^{2+\alpha,2+\alpha/2}_\beta(X \times [0,1])$. Moreover there is a $C = C(n, \beta, \alpha, g) > 0$ such that

$$\|u\|_{C^{2+\alpha,2+\alpha/2}_\beta(X \times [0,1])} \leq C \left( \|f\|_{C^{\alpha,\alpha/2}_\beta(X \times [0,1])} + \|u_0\|_{C^{2+\alpha}_\beta(X)} \right).$$

(4.31)

**Proof.** Observe that by maximum principle we have

$$\|u\|_{C^0(X \times [0,1])} \leq \|f\|_{C^0(X \times [0,1])} + \|u_0\|_{C^0(X)}.$$

Then the estimate (4.31) follows from Corollary 4.7 and a covering argument as in the proof of Corollary 3.4.

□

5. Conical Kähler-Ricci flow

Let $X$ be a compact Kähler manifold and $D = \sum_j D_j$ be a divisor with simple normal crossings. Let $\omega_0$ be a fixed $C^{0,\alpha}_\beta(X)$ conical Kähler metric with cone angle $2\pi \beta$ along $D$ and $\hat{\omega}_t$ be a family of $C^{\alpha',\alpha/2}_\beta$ conical metrics which are uniformly equivalent to $\omega_0$, $\hat{\omega}_0 = \omega_0$ and $\|\hat{\omega}\|_{C^{\alpha',\alpha/2}_\beta(X \times [0,1])} \leq C_0$. We consider the complex Monge-Ampère equation:

$$\begin{cases}
\frac{\partial \varphi}{\partial t} = \log \left( \frac{(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_0^n} \right) + f \\
\varphi|_{t=0} = 0,
\end{cases}$$

(5.1)

where $f \in C^{\alpha',\alpha/2}_\beta(X \times [0,1])$ is a given function. We will use an inverse function theorem argument in [4] which was outlined in [21] to show the short time existence of the flow (5.1).
Theorem 5.1. There exists a small $T = T(n, \beta, \omega_0, f, \alpha, \alpha') > 0$ such that the equation (5.1) admits a unique solution $\varphi \in C^{2+\alpha, 2+\alpha'}_{\beta} (X \times [0, T])$, for any $\alpha < \alpha'$.

Proof. The uniqueness of the solution follows from maximum principle. We will break the proof of short-time existence into three steps.

Step 1. Let $u \in C^{2+\alpha', 2+\alpha'}_{\beta} (X \times [0, 1])$ be the solution to the equation
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta_{g_0} u + f, & \text{in } X \times [0, 1] \\
u|_{t=0} = 0.
\end{cases}
\]
Thanks to Corollary 4.8 such $u$ exists and satisfies the estimate (4.31). We fix an $\varepsilon > 0$ so that as long as $\|\phi\|_{C^{2,\alpha}_{\beta}(X)} \leq \varepsilon$, $\tilde{\omega}_{t, \phi} := \tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \phi$ is equivalent to $\omega_0$, i.e. $C_0^{-1} \omega_0 \leq \omega_{0, \phi} \leq C_0 \omega_0$, and $\|\tilde{\omega}_{t, \phi}\|_{C^{2, \alpha/2}_{\beta}} \leq C_0$.

We claim that for $T_1 > 0$ small enough, $\|u\|_{C^{2+\alpha, (2+\alpha)/2}_{\beta}(X \times [0, T_1])} \leq \varepsilon$. We first observe that by (4.31) that
\[
N := \|u\|_{C^{2+\alpha', (2+\alpha')/2}_{\beta}(X \times [0, 1])} \leq C \|f\|_{C^{2+\alpha', (2+\alpha')/2}_{\beta}(X \times [0, 1])}.
\]
It suffices to show that $[u]_{C^{2+\alpha', (2+\alpha')/2}_{\beta}(X \times [0, T_1])}$ is small since the lower order derivatives are small since $u|_{t=0} = 0$. We calculate for any $t_1, t_2 \in [0, T_1]$
\[
\frac{|T u(x, t_1) - T u(x, t_2)|}{|t_1 - t_2|^{\alpha/2}} + \frac{|\dot{u}(x, t_1) - \dot{u}(x, t_2)|}{|t_1 - t_2|^{\alpha/2}} \leq N |t_1 - t_2|^{(\alpha' - \alpha)/2} \leq \varepsilon/4,
\]
if $N T_1^{(\alpha' - \alpha)/2} < \varepsilon/4$. For any $x, y \in X$ and $t \in [0, T_1]$
\[
\frac{|T u(x, t) - T u(y, t)|}{d_{g_0}(x, y)^\alpha} \leq N \min \left\{ \frac{2 T_1^{\alpha'/2}}{d_{g_0}(x, y)^{\alpha'}}, \frac{d_{g_0}(x, y)^{\alpha'} - \alpha}{d_{g_0}(x, y)^\alpha} \right\} \leq \frac{\varepsilon}{2}.
\]
The claim then follows from triangle inequality.

We define a function
\[
w(x, t) := \frac{\partial u}{\partial t}(x, t) - \log \left( \frac{\tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} u}{\omega_0^n} \right)(x, t) - f(x, t), \forall (x, t) \in X \times [0, T_1].
\]
It is clear that $w(x, 0) \equiv 0$.

Step 2. We consider the small ball
\[
\mathcal{B} = \{ \phi \in C^{2+\alpha, 2+\alpha}_{\beta} (X \times [0, T_1]) | \|\phi\|_{C^{2+\alpha, 2+\alpha}_{\beta}} \leq \varepsilon, \phi(\cdot, 0) = 0 \}
\]
in the space $C^{2+\alpha, 2+\alpha}_{\beta} (X \times [0, T_1])$. $u|_{t \in [0, T_1]} \in \mathcal{B}$ by the discussion in Step 1.

Define the differential map $\Psi : \mathcal{B} \rightarrow C^{\alpha, \alpha/2}_{\beta} (X \times [0, T_1])$ by
\[
\Psi(\phi) = \frac{\partial \phi}{\partial t} - \log \left( \frac{\tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \phi}{\omega_0^n} \right) - f.
\]
The map $\Psi$ is well-defined and $C^1$ with the differential $D \Psi_{\phi}$ at any $\phi \in \mathcal{B}$ is given by
\[
D \Psi_{\phi}(v) = \frac{\partial v}{\partial t} - (\dot{\phi})^j v_j - \frac{\partial v}{\partial t} - \Delta_{\tilde{\omega}_{t, \phi}} v,
\]
for any \( v \in T_\partial B = \{ v \in C^{2+\alpha, 2i\nu/\gamma}_\partial (X \times [0, T_1]) \mid v(\cdot, 0) = 0 \} \), where \((\hat{g}_\phi)^i_j\) denotes the inverse of the metric \( \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi \). As a linear map, \( D\Psi_\phi : T_\partial B \to C^{\alpha, \alpha/2}_\partial (X \times [0, T_1]) \) is injective by maximum principle; is surjective by Corollary 4.8. Thus \( v \) for any \( t < T \) claim that if \( \|v\| \leq \bar{\delta} \) is small enough, then \( \|v\| \leq \bar{\delta} \) for some small \( \delta > 0 \), there exists a unique \( \varphi \in \mathcal{B} \) such that \( \Psi(\varphi) = v \).

**Step 3.** For a small \( T_2 < T_1 \) to be determined, we define a function

\[
\tilde{w}(x, t) = \begin{cases} 0, & t \in [0, T_2] \\ w(x, t - T_2), & t \in [T_2, T_1]. \end{cases}
\]

Since \( u \in C^{2+\alpha, 2i\nu/\gamma}_\partial \), we see that \( w \in C^{\alpha, \alpha/2}_\partial (X \times [0, T_1]) \) with \( M := \|w\|_{C^{\alpha, \alpha/2}_\partial (X \times [0, T_1])} < \infty \). We claim that if \( T_2 \) is small enough, then \( \|w - \tilde{w}\|_{C^{\alpha, \alpha/2}_\partial (X \times [0, T_1])} < \delta \). It is clear from the fact that \( w(\cdot, 0) = 0 \) that \( \|\eta\|_{C^0} \leq \delta/2 \) if \( T_2 \) is small enough.

**Spatial directions:** If \( t < T_2 \) then

\[
\frac{|\eta(x, t) - \eta(y, t)|}{d_{g_\partial}(x, y)^\alpha} = \frac{|w(x, t) - w(y, t)|}{d_{g_\partial}(x, y)^\alpha} \leq M \min \left\{ \frac{2T_2^{\alpha/2}}{d_{g_\partial}(x, y)^\alpha}, d_{g_\partial}(x, y)^{\alpha - \alpha} \right\} \leq 2MT_2^{(\alpha - \alpha)/2},
\]

if \( t \in [T_2, T_1] \) then

\[
\frac{|\eta(x, t) - \eta(y, t)|}{d_{g_\partial}(x, y)^\alpha} = \frac{|w(x, t) - w(y, t) - w(x, t - T_2) + w(y, t - T_2)|}{d_{g_\partial}(x, y)^\alpha} \leq 2M \min \left\{ \frac{T_2^{\alpha/2}}{d_{g_\partial}(x, y)^\alpha}, d_{g_\partial}(x, y)^{\alpha - \alpha} \right\} \leq 2MT_2^{(\alpha - \alpha)/2}.
\]

**Time direction:** If \( t, t' < T_2 \), then

\[
\frac{|\eta(x, t) - \eta(x, t')|}{|t - t'|^{\alpha/2}} = \frac{|w(x, t) - w(x, t')|}{|t - t'|^{\alpha/2}} \leq M|t - t'|^{(\alpha - \alpha)/2} \leq MT_2^{(\alpha - \alpha)/2};
\]

If \( t, t' \in [T_2, T_1] \), then

\[
\frac{|\eta(x, t) - \eta(x, t')|}{|t - t'|^{\alpha/2}} = \frac{|w(x, t) - w(x, t') - w(x, t - T_2) + w(x, t' - T_2)|}{|t - t'|^{\alpha/2}} \leq 2MT_2^{(\alpha - \alpha)/2};
\]

If \( t < T_2 \leq t' \leq T_1 \), then

\[
\frac{|\eta(x, t) - \eta(x, t')|}{|t - t'|^{\alpha/2}} = \frac{|w(x, t) - w(x, t') + w(x, t' - T_2)|}{|t - t'|^{\alpha/2}} \leq 2MT_2^{(\alpha - \alpha)/2}.
\]

Therefore if we choose \( T_2 > 0 \) small so that \( 2MT_2^{(\alpha - \alpha)/2} < \delta/4 \), then we have

\[
\frac{|\eta(x, t) - \eta(x, t')|}{|t - t'|^{\alpha/2}} + \frac{|\eta(x, t) - \eta(y, t)|}{d_{g_\partial}(x, y)^\alpha} \leq \frac{\delta}{2} \forall x \in X, t, t' \in [0, T_1].
\]

It then follows from triangle inequality that

\[
|\eta(x, t) - \eta(y, t')| \leq |\eta(x, t) - \eta(y, t)| + |\eta(y, t) - \eta(y, t')|.
\]
Proof of Corollary 1.3. Recall in (1.13) we write \( \omega_0^\alpha = \frac{\Omega}{\prod_j (\gamma_j)^{1-\beta_j}} \) where \( \Omega \) is a smooth volume form, \( s_j \) and \( h_j \) are holomorphic sections and hermitian metrics of the line bundle associated to the component \( D_j \), respectively. Choose a smooth reference form \( \chi = \sqrt{-1} \partial \bar{\partial} \log \Omega - \sum_j (1 - \beta_j) \sqrt{-1} \partial \bar{\partial} \log h_j \). Define the reference metrics \( \tilde{\omega}_t = \omega_0 + t \chi \) which are \( C^{2+\alpha/2}_\beta \)-conical and Kähler for small \( t > 0 \). Let \( \varphi \) be the \( C^{2+\alpha/2}_\beta \)-solution to the equation (1.11) with \( f \equiv 0 \). Then it is straightforward to check that \( \omega_t = \omega_0 + \tilde{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi \) satisfies the conical Kähler-Ricci flow equation (1.12) and \( \omega \in C^{\alpha/2}_\beta (X \times [0, T]) \) for some small \( T > 0 \).

The smoothness of \( \omega \) in \( X \setminus D \times (0, T] \) follows from the general smoothing properties of parabolic equations (see [37]). Taking \( \frac{\partial}{\partial t} \) on both sides of (1.11) we get

\[
\frac{\partial \varphi}{\partial t} = \Delta_{\omega_t} \varphi + \text{tr}_{\omega_t} \chi, \quad \text{and} \quad \varphi|_{t=0} = 0.
\]

By Corollary 4.8, \( \varphi \in C^{2+\alpha/2}_\beta (X \times [0, T]) \) since \( \text{tr}_{\omega_t} \chi \in C^{\alpha/2}_\beta (X \times [0, T]) \). Therefore the normalized Ricci potential \( \left( \frac{\omega_0^\alpha}{\omega_t^\alpha} \right) \in C^{2+\alpha/2}_\beta (X \times [0, T]) \).

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