A BLOW-UP CRITERION OF STRONG SOLUTIONS TO TWO-DIMENSIONAL NONHOMOGENEOUS MICROPOLAR FLUID EQUATIONS WITH VACUUM

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Abstract. We deal with the Cauchy problem of nonhomogeneous micropolar fluid equations with zero density at infinity in the entire space $\mathbb{R}^2$. We show that for the initial density allowing vacuum, the strong solution exists globally if a weighted density is bounded from above. It should be noted that our blow-up criterion is independent of micro-rotational velocity.

1. Introduction. The three-dimensional (3D for short) nonhomogeneous incompressible micropolar fluid equations (see [16, pp. 145–146]) are given by

$$
\begin{align*}
\rho_t + \text{div} (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) - (\mu_1 + \xi) \Delta \mathbf{u} + \nabla P &= 2\xi \nabla \times \mathbf{w}, \\
(\rho \mathbf{w})_t + \text{div} (\rho \mathbf{u} \otimes \mathbf{w}) + 4\xi \mathbf{w} - \mu_2 \Delta \mathbf{w} - (\mu_2 + \lambda) \nabla \text{div} \mathbf{w} &= 2\xi \nabla \cdot \mathbf{u}, \\
\text{div} \mathbf{u} &= 0,
\end{align*}
$$

(1.1)

where $\rho$, $\mathbf{u}$, $\mathbf{w}$, and $P$ denote the density, velocity, micro-rotational velocity, and pressure of the fluid, respectively. The constants $\mu_1, \mu_2, \lambda, \xi$ stand for the viscosity coefficients of the fluid satisfying $\mu_1, \mu_2, \xi > 0$, $\mu_1 \geq 2\xi$, and $2\mu_2 + 3\lambda \geq 0$.

In the special case, when

$$
\begin{align*}
\rho &= \rho(x_1, x_2, t), \quad \mathbf{u} = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0), \\
P &= P(x_1, x_2, t), \quad \mathbf{w} = (0, 0, w(x_1, x_2, t)),
\end{align*}
$$

the system (1.1) reduces to the 2D nonhomogeneous micropolar fluid equations

$$
\begin{align*}
\rho_t + \text{div} (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) - (\mu_1 + \xi) \Delta \mathbf{u} + \nabla P &= -2\xi \nabla \cdot \mathbf{w}, \\
(\rho \mathbf{w})_t + \text{div} (\rho \mathbf{u} \otimes \mathbf{w}) + 4\xi \mathbf{w} - \mu_2 \Delta \mathbf{w} &= 2\xi \nabla \cdot \mathbf{u}, \\
\text{div} \mathbf{u} &= 0,
\end{align*}
$$

(1.2)

where $\mathbf{u} = (u^1, u^2)$ is a 2D vector with the corresponding scalar vorticity given by

$$
\nabla \cdot \mathbf{u} = \partial_1 u^2 - \partial_2 u^1,
$$

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while \( w \) represents a scalar function with
\[
\nabla^\perp w = (-\partial_2 w, \partial_1 w).
\]
The system (1.2) is supplemented with the initial condition
\[
(\rho, \rho u, \rho w)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 w_0)(x), \quad x \in \mathbb{R}^2,
\]
and the far field behavior
\[
(\rho, u, w)(x, t) \to (0, 0, 0), \quad \text{as } |x| \to +\infty.
\]

The micropolar fluid equations, which were suggested and introduced by Eringen in 1960s (see [7]), are a significant step toward generalization of the Navier-Stokes equations. It is a type of fluids which exhibits micro-rotational effects and micro-rotational inertia, and can be viewed as a non-Newtonian fluid. Physically micropolar fluid may represent fluids that consist of rigid, randomly oriented (or spherical particles) suspended in a viscous medium, where the deformation of fluid particles is ignored. It can describe many phenomena that appear in a large number of complex fluids such as the suspensions, animal blood, liquid crystals which cannot be characterized appropriately by the Navier-Stokes system, and that it is important to the scientists working with the hydrodynamic-fluid problems and phenomena. For more background and applications, we refer to [8,16] and references therein.

When \( \rho \) is constant, which means the fluid is homogeneous, the micropolar fluid equations have been extensively studied. In particular, many authors investigated global existence and regularity of 2D homogeneous micropolar fluid equations with partial dissipation. Dong et al. [4] studied global regularity and large time behavior of solutions to the 2D micropolar fluid equations with only angular viscosity dissipation. Later on, Dong et al. [5] proved global regularity of the two-dimensional fractional micropolar fluid equations with minimal fractional dissipation. Recently, by imposing natural boundary conditions and minimal regularity assumptions on the initial data, the authors [11] established the global existence and uniqueness of solutions to the 2D micropolar fluid equations with only angular velocity dissipation in a smooth bounded domain. On the other hand, by Fourier localization method, Chen and Miao [3] constructed global solutions of the 3D micropolar fluid system in Fourier-Besov spaces with a class of highly oscillating initial data of Cannone’s type, while Zhu [22] investigated ill-posedness of 3D Cauchy problem in Fourier-Besov spaces. For other studies of homogeneous micropolar fluid equations, please refer to [2,6,9,15,18] and references therein.

When \( \rho \) is not constant, the system (1.2) is so-called nonhomogeneous micropolar fluid equations. For the initial density allowing vacuum states, Lukaszewicz [16, Chapter 3] proved the local existence of weak solutions for three-dimensional initial boundary value problem. Imposing a compatibility condition on the initial data, Zhang and Zhu [20] showed the global existence of strong solution in \( \mathbb{R}^3 \) under some smallness condition. Later on, Ye [19] improved their result by removing the compatibility condition and furthermore obtained exponential decay of strong solution. Recently, by some weighted energy estimates, Zhong [21] showed the local existence of strong solutions to the Cauchy problem of (1.2) in \( \mathbb{R}^2 \). In this paper, we will investigate the structure of possible singularities of strong solutions obtained in [21].
Before stating our main result, we first explain the notations and conventions used throughout this paper. For \( r > 0 \), set
\[
B_r \triangleq \{ x \in \mathbb{R}^2 \mid |x| < r \}, \quad \int_{B_r} dx \triangleq \int_{\mathbb{R}^2} dx.
\]
For \( 1 \leq p \leq \infty \) and integer \( k \geq 0 \), the standard Sobolev spaces are denoted by:
\[
L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2), \quad H^k = H^{k,2}(\mathbb{R}^2), \quad D^{k,p} = \{ u \in L^1_{\text{loc}}(\nabla^k u \in L^p) \}.
\]
Now, we wish to define precisely what we mean by strong solutions.

**Definition 1.1** (Strong solutions). \((\rho, u, P, w)\) is called a strong solution to (1.2)–(1.4) in \( \mathbb{R}^2 \times (0, T) \), if for some \( q > 2 \) and \( a > 1 \),
\[
\begin{align*}
\rho &\in C([0, T]; L^1 \cap H^1 \cap W^{1,q}), \\
\rho\bar{x}^a &\in L^\infty(0, T; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt[\rho]{u}, \nabla u, \sqrt[\rho]{\nabla u}, \sqrt[\rho]{\nabla^2 P}, \sqrt[\rho]{\nabla^2 u} &\in L^\infty(0, T; L^2), \\
\sqrt[\rho]{w}, \nabla w, \sqrt[\rho]{\nabla w}, \sqrt[\rho]{\nabla^2 w} &\in L^\infty(0, T; L^2), \\
\nabla u, \nabla w &\in L^2(0, T; H^1) \cap L^{\frac{aq+1}{a-1}} (0, T; W^{1,q}), \\
P &\in L^2(0, T; L^2) \cap L^{\frac{aq+1}{a-1}} (0, T; L^q), \\
\nabla \nabla u, \sqrt[\rho]{\nabla w}, \sqrt[\rho]{\nabla w} &\in L^2(0, T; W^{1,q}), \\
\sqrt[\rho]{u}, \sqrt[\rho]{\nabla u}, \sqrt[\rho]{\nabla w}, \sqrt[\rho]{\nabla w} &\in L^2(\mathbb{R}^2 \times (0, T)),
\end{align*}
\]
and \((\rho, u, P, w)\) satisfies both (1.2) almost everywhere in \( \mathbb{R}^2 \times (0, T) \) and (1.3) almost everywhere in \( \mathbb{R}^2 \). Here
\[
\bar{x} \triangleq (3 + |x|^2)^\frac{1}{2} \log^1 + \eta_0(3 + |x|^2)
\]
and \( \eta_0 \) is a positive number.

Without loss of generality, we assume that the initial density \( \rho_0 \) satisfies
\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]
which implies that there exists a positive constant \( N_0 \) such that
\[
\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int_{\mathbb{R}^2} \rho_0 dx = \frac{1}{2}
\]

Our main result can be stated as follows:

**Theorem 1.1.** In addition to (1.6) and (1.7), assume that the initial data \((\rho_0 \geq 0, u_0, w_0)\) satisfies for any given numbers \( a > 1 \) and \( q > 2 \),
\[
\begin{align*}
\rho_0\bar{x}^a &\in L^1 \cap H^1 \cap W^{1,q}, \quad \sqrt[\rho_0]{u_0} \in L^2, \quad \nabla u_0 \in L^2, \\
\sqrt[\rho_0]{w_0} &\in L^2, \quad \nabla w_0 \in L^2, \quad \text{div} u_0 = 0.
\end{align*}
\]
Let \((\rho, u, P, w)\) be a strong solution to the problem (1.2)–(1.4). If \( T^* < \infty \) is the maximal time of existence for that solution, then for any \( \delta > 0 \), we have
\[
\lim_{T \to T^*} \| \rho\bar{x}^\delta \|_{L^\infty(0, T; L^\infty(\mathbb{R}^2))} = \infty.
\]

**Remark 1.1.** The local existence of an unique strong solution with initial data as in Theorem 1.1 was established in [21]. Hence, the maximal time \( T^* \) is well-defined.
Remark 1.2. The conclusion in Theorem 1.1 is somewhat surprising since the criterion (1.9) is independent of micro-rotational velocity. It indicates that the mechanism of blowup of nonhomogeneous micropolar fluid equations is similar to the nonhomogeneous Navier-Stokes equations [13, Theorem 1.2] and does not depend on further sophistication of the equation (1.2). It is worth mentioning that our Theorem 1.1 holds for arbitrary \( a > 1 \) which is in sharp contrast to Liang [13] where \( a \in (1, 2) \) is required.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries. In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

We begin with the following Gagliardo-Nirenberg inequality (see [17]).

**Lemma 2.1** (Gagliardo-Nirenberg). For \( p \in [2, \infty) \), \( r \in (2, \infty) \), and \( s \in (1, \infty) \), there exists some generic constant \( C > 0 \) which may depend on \( p, r, \) and \( s \) such that for \( f \in H^1(\mathbb{R}^2) \) and \( g \in L^s(\mathbb{R}^2) \cap D^{1, r}(\mathbb{R}^2) \), we have

\[
\|f\|_{L^p(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{p-2}\|\nabla f\|_{L^2(\mathbb{R}^2)}^p,
\]

\[
\|g\|_{C^1(\mathbb{R}^2)} \leq C\|g\|_{L^s(\mathbb{R}^2)}^{(s(r-2))/(2r+s(r-2))}\|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+s(r-2))}.
\]

The following weighted \( L^m \) bounds for elements of the Hilbert space \( \tilde{D}^{1,2}(\mathbb{R}^2) \triangleq \{ v \in H^1_{\text{loc}}(\mathbb{R}^2) | \nabla v \in L^2(\mathbb{R}^2) \} \) can be found in [14, Theorem B.1].

**Lemma 2.2.** For \( m \in [2, \infty) \) and \( \theta \in (1 + \frac{m}{2}, \infty) \), there exists a positive constant \( C \) such that for all \( v \in \tilde{D}^{1,2}(\mathbb{R}^2) \),

\[
\left( \int_{\mathbb{R}^2} \frac{|v|^m}{3 + |x|^2} \left( \log(3 + |x|^2) \right)^{-\theta} dx \right)^{\frac{1}{m}} \leq C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)}.
\]  

(2.1)

The combination of Lemma 2.2 and the Poincaré inequality yields the following useful results on weighted bounds, whose proof can be found in [12, Lemma 2.4].

**Lemma 2.3.** Let \( \tilde{x} \) be as in (1.5). Assume that \( \rho \in L^1(B_1) \cap L^\infty(\mathbb{R}^2) \) is a nonnegative function such that

\[
\|\rho\|_{L^1(B_{N_1})} \geq M_1, \quad \|\rho\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)} \leq M_2,
\]

for positive constants \( M_1, M_2, \) and \( N_1 \geq 1 \). Then for \( \varepsilon > 0 \) and \( \eta > 0 \), there is a positive constant \( C \) depending only on \( \varepsilon, \eta, M_1, M_2, \) and \( N_1 \), such that every \( v \in \tilde{D}^{1,2}(\mathbb{R}^2) \) satisfies

\[
\|v^{\tilde{x} - \eta}\|_{L^{\frac{2+\varepsilon}{2+\varepsilon-\eta}}(\mathbb{R}^2)} \leq C\|\sqrt{\rho}v\|_{L^2(\mathbb{R}^2)} + C\|\nabla v\|_{L^2(\mathbb{R}^2)},
\]  

(2.2)

with \( \tilde{\eta} = \min\{1, \eta\} \).

Finally, we state a critical Sobolev inequality of logarithmic type, which is originally due to Brézis-Wainger [1]. The reader can refer to [13, Lemma 2.5] for the proof.

**Lemma 2.4.** Assume that \( f \in L^2(s, t; D^{1,2} \cap W^{1,q}(\mathbb{R}^2)) \) with some \( q > 2 \) and \( 0 \leq s < t \leq \infty \), then there is a constant \( C > 0 \) independent of \( s \) and \( t \) such that

\[
\|f\|_{L^2(s, t; L^\infty)} \leq C \left( \|\nabla f\|_{L^2(s, t; L^2)} + \|f\|_{L^2(s, t; L^q)} \right) \log(3 + \|f\|_{L^2(s, t; W^{1,q})}^2) + C.
\]  

(2.3)
3. **Proof of Theorem 1.1.** Let \((\rho, u, P, w)\) be a strong solution described in Theorem 1.1. Suppose that (1.9) were false, that is, there exists a constant \(M_0 > 0\) such that

\[
\lim_{T \to T^*} \|\rho x^d\|_{L^\infty(0, T; L^\infty)} \leq M_0 < \infty. \tag{3.1}
\]

We begin with the following standard energy estimate for \((\rho, u, P, w)\) and the estimate on the \(L^p\)-norm of the density.

**Lemma 3.1.** It holds that for any \(T \in [0, T^*)\),

\[
\sup_{0 \leq t \leq T} (\|\rho\|_{L^1 \cap L^\infty} + \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2)
\]

\[
+ \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \, dt \leq C,
\tag{3.2}
\]

where (and in what follows) \(C\) denotes a generic positive constant depending only on \(\mu_1, \mu_2, \xi, \lambda, q, a, \eta_0, M_0, N_0, T^*,\) and the initial data.

**Proof.** 1. Since \(\text{div} \ u = 0\), it is easy to deduce from (1.2) that (see [14, Theorem 2.1]),

\[
\sup_{0 \leq t \leq T} \|\rho\|_{L^1 \cap L^\infty} \leq C. \tag{3.3}
\]

2. Multiplying (1.2) by \(u\), (1.2) by \(w\), and integrating by parts, we obtain that

\[
\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2) + (\mu_1 + \xi) \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 + 4\xi \|w\|_{L^2}^2
\]

\[
= 4\xi \int w \nabla \cdot u \, dx
\]

\[
\leq \xi \|\nabla u\|_{L^2}^2 + 4\xi \|w\|_{L^2}^2,
\]

which gives

\[
\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2) + \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 \leq 0.
\]

Integrating the above inequality over \((0, t)\) leads to

\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \, dt \leq C,
\]

which combined with (3.3) leads to (3.2) and completes the proof of Lemma 3.1. \(\Box\)

**Lemma 3.2.** Under the condition (3.1), it holds that for any \(T \in [0, T^*)\),

\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2) \, dt \leq C. \tag{3.4}
\]

**Proof.** 1. We obtain from [21, Lemma 3.2] and (1.7) that

\[
\sup_{0 \leq t \leq T} \|\rho x^d\|_{L^1} \leq C, \quad \inf_{0 \leq t \leq T} \int_{B_{\eta_0}} \rho \, dx \geq \frac{1}{4}. \tag{3.5}
\]

The combination of (3.5), (3.2), and (2.2) implies that for \(\varepsilon > 0, \eta > 0, v \in D^{1,2}(\mathbb{R}^2),\) and \(\bar{\eta} = \min\{1, \eta\}\)

\[
\|v x^\varepsilon\|_{L^{2+\eta} \eta} \leq C(\varepsilon, \eta) \|\sqrt{\rho} v\|_{L^2}^2 + C(\varepsilon, \eta) \|\nabla v\|_{L^2}^2, \tag{3.6}
\]

\[
\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2) + \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\nabla w\|_{L^2}^2 \leq 0.
\]

Integrating the above inequality over \((0, t)\) leads to

\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \, dt \leq C,
\]

which combined with (3.3) leads to (3.2) and completes the proof of Lemma 3.1. \(\Box\)
which along with (3.2) and (3.5) yields
\[
\|\rho v\|_{L^{\frac{2+p}{3}}_{\rho}} \leq C \|\rho^\frac{3}{2} x \|_{L^{3(2+p)/3}_w} \| v^{\frac{3}{2} x} \|_{L^{3(2+p)/3}_w} \leq C \left( \int \rho^{\frac{4(2+p)\eta}{3}} - 1 \rho^{\eta} dx \right) \| v^{\frac{3}{2} x} \|_{L^{3(2+p)/3}_w} \leq C \|\rho\|_{L^{\infty}} \|\rho^\eta\|_{L^{2+\eta}_w} \left( \|\sqrt{\rho} v\|_{L^2} + \|\nabla v\|_{L^2} \right) \leq C \|\sqrt{\rho} v\|_{L^2} + C \|\nabla v\|_{L^2}. \tag{3.7}
\]

In particular, this together with (3.2) and (3.6) yields
\[
\begin{align*}
\|\rho^\eta u\|_{L^{\frac{2+p}{3}}_{\rho}} + \| u^{\frac{2+p}{3}} \|_{L^{\frac{2+p}{3}}_{\rho}} & \leq C (1 + \|\nabla u\|_{L^2}), \tag{3.8} \\
\|\rho^\eta w\|_{L^{\frac{2+p}{3}}_{\rho}} + \| w^{\frac{2+p}{3}} \|_{L^{\frac{2+p}{3}}_{\rho}} & \leq C (1 + \|\nabla w\|_{L^2}). \tag{3.9}
\end{align*}
\]

2. Multiplying (1.2) by \( u \) and integrating by parts, one has
\[
\frac{\mu_1 + \xi}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \int \rho |u|^2 dt \\
= - \int \rho u \cdot \nabla u \cdot u dx - 2\xi \int \nabla^\perp w \cdot u dx \\
= - \int \rho u \cdot \nabla u \cdot u dx + 2\xi \int \nabla^\perp \cdot u dx \\
= - \int \rho u \cdot \nabla u \cdot u dx + 2\xi \frac{d}{dt} \int \nabla^\perp \cdot u dx - 2\xi \int \nabla^\perp \cdot u dx. \tag{3.10}
\]

Multiplying (1.2) by \( w \) and integrating by parts, we get
\[
\frac{1}{2} \frac{d}{dt} \int (\mu_2 |\nabla w|^2 + 4\xi w^2) dx \int \rho w^2 dx \\
= - \int \rho u \cdot \nabla w \cdot w dx + 2\xi \int \nabla^\perp \cdot w dx, \tag{3.11}
\]

which combined with (3.10) leads to
\[
\frac{1}{2} B'(t) + \int \rho |u|^2 dx + \frac{d}{dt} \int \rho w^2 dx \\
= - \int \rho u \cdot \nabla u \cdot u dx - \int \rho u \cdot \nabla w \cdot u dx, \tag{3.12}
\]

where
\[
B(t) \triangleq \int (\mu_1 + \xi) |\nabla u|^2 + \mu_2 |\nabla w|^2 + 4\xi w^2 - 4\xi \nabla^\perp \cdot u w) dx
\]
satisfies
\[
\mu_1 |\nabla u|^2_{L^2} + \mu_2 |\nabla w|^2_{L^2} \leq B(t) \leq C |\nabla u|^2_{L^2} + C |\nabla w|^2_{L^2} + C \|w\|^2_{L^2}, \tag{3.13}
\]
due to
\[
-4\xi \int \nabla^\perp \cdot u dx \leq 4\xi \|\nabla u\|_{L^2} \|w\|_{L^2} \leq \xi \|\nabla u\|^2_{L^2} + 4\xi \|w\|_{L^2}.
\]

We derive from Cauchy-Schwarz inequality that
\[
| \int \rho u \cdot \nabla u \cdot u dx | \leq \frac{1}{2} \int \rho |u|^2 dx + \frac{1}{2} \int \rho |u|^2 \nabla u|^2 dx.
\]
we then derive from (3.8) and owing to (3.1) and Lemma 2.4. Here
\[ \leq \frac{1}{2} \int \rho |u_t|^2 \, dx + \frac{1}{2} \| \sqrt{\rho} u \|_{L^\infty}^2 \| \nabla u \|_{L^2}^2. \]  
(3.14)
Similarly, one has
\[ - \int \rho u \cdot \nabla \cdot w_t \, dx \leq \frac{1}{2} \int \rho w_t^2 \, dx + \frac{1}{2} \| \sqrt{\rho} u \|_{L^\infty}^2 \| \nabla w \|_{L^2}^2. \]  
(3.15)
Thus, inserting (3.14) and (3.15) into (3.12) gives
\[ B'(t) + \| \sqrt{\rho} u \|_{L^2}^2 + \| \sqrt{\rho} u_t \|_{L^2}^2 \leq \| \sqrt{\rho} u \|_{L^\infty}^2 \left( \| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2 \right). \]  
(3.16)

3. Let
\[ \Phi(t) \triangleq 3 + \sup_{0 \leq s \leq t} (\| \nabla u \|_{L^2}^2 + \| \nabla w \|_{L^2}^2) + \int_0^t (\| \sqrt{\rho} u \|_{L^2}^2 + \| \sqrt{\rho} u_t \|_{L^2}^2) \, dt. \]
Then we obtain from (3.16), (3.13), and Gronwall’s inequality that for \( \delta > 0, \quad q > 2, \) and every \( 0 \leq s \leq T < T^* , \)
\[ \Phi(T) \leq C \Phi(s) \exp \left\{ C \int_s^T \| \sqrt{\rho} u \|_{L^\infty}^2 \, dt \right\} \]
\[ \leq C \Phi(s) \exp \left\{ C \sup_{s \leq t \leq T} \| \rho \tilde{x}^\delta \|_{L^\infty} \int_s^T \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^\infty}^2 \, dt \right\} \]
\[ \leq C \Phi(s) \exp \left\{ C \int_s^T \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^\infty}^2 \, dt \right\} \]
\[ \leq C \Phi(s) \exp \{ CG(s, T) \}, \]  
(3.17)
owing to (3.1) and Lemma 2.4. Here
\[ G(s, T) = \left( \| \nabla (u \tilde{x}^{-\frac{\delta}{2}}) \|_{L^2(s, T; L^2)}^2 + \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2(s, T; L^q)}^2 \right) \log(3 + \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2(s, T; L^q)}^2). \]

Noting that
\[ |\nabla \tilde{x}| \leq (3 + 2\eta_0) \log(3 + |x|^2) \leq C(a, \eta_0) \tilde{x}^{\frac{a+\eta_0}{2}}, \]  
(3.18)
we then derive from (3.8) and \( a > 1 \) that
\[ \| \nabla (u \tilde{x}^{-\frac{\delta}{2}}) \|_{L^2} \leq C \| \nabla u \|_{L^2} + C \| \tilde{x}^{-\frac{\delta}{2} - 1} u \cdot \nabla \tilde{x} \|_{L^2} \]
\[ \leq C \| \nabla u \|_{L^2} + C \| \tilde{x}^{-1} \nabla \tilde{x} \|_{L^\infty} \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2} \]
\[ \leq C \| \nabla u \|_{L^2} + C \| \nabla \tilde{x} \|_{L^\infty} \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2} \]
\[ \leq C \| \nabla u \|_{L^2} + C, \]  
(3.19)
and
\[ \| u \tilde{x}^{-\frac{\delta}{2}} \|_{W^{1, s}} \leq C \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^s} + C \| \nabla (u \tilde{x}^{-\frac{\delta}{2}}) \|_{L^s} \]
\[ \leq C + \| \nabla u \|_{L^2} + C \| \nabla u \|_{L^2} + C \| \nabla \tilde{x} \|_{L^\infty} \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2} \]
\[ \leq C + C \| \nabla u \|_{L^2} + C \| \nabla u \|_{L^2} + C \| \nabla \tilde{x} \|_{L^\infty} \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2} \]
\[ \leq C + C \| \nabla u \|_{L^2} + C \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2}^{1-\frac{\delta}{2}} + C \| \nabla \tilde{x} \|_{L^\infty} \| u \tilde{x}^{-\frac{\delta}{2}} \|_{L^2} \]
\[ \leq C + C \| \nabla u \|_{L^2} + C \| \nabla^2 u \|_{L^2}. \]  
(3.20)
Since \((\rho, u, P, w)\) satisfies the following Stokes system
\[
\begin{align*}
-\nabla P + \nabla \Delta u &= -\mu \Delta u + \nabla \rho \cdot \nabla u - 2\xi \nabla^2 w, & x \in \mathbb{R}^2, \\
\text{div } u &= 0, & x \in \mathbb{R}^2, \\
\mathbf{u}(x) &\to 0, & |x| \to \infty,
\end{align*}
\]
(3.21)
applying regularity theory of Stokes system to (3.21) yields that for any \(p \in [2, \infty)\),
\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq C\|\rho \mathbf{u}\|_{L^p} + C\|\rho \cdot \nabla \mathbf{u}\|_{L^p} + C\|\nabla^2 w\|_{L^p},
\]
(3.22)
which combined with (3.3), (3.8), and Gagliardo-Nirenberg inequality that
\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq C\|\rho \mathbf{u}\|_{L^2} + C\|\rho \cdot \nabla \mathbf{u}\|_{L^2} + C\|\nabla^2 w\|_{L^2},
\]
(3.23)
This together with (3.20) and Young’s inequality leads to
\[
\|u \bar{w}^{-\frac{2}{2}}\|_{L^{1,\infty}} \leq C + C\|\rho \mathbf{u}\|_{L^2} + C\|\nabla^2 \mathbf{u}\|_{L^2} + C\|\nabla w\|_{L^2}.
\]
(3.24)
Inserting (3.19) and (3.24) into (3.17) gives rise to for some \(\tilde{C} > 0\),
\[
\Phi(T) \leq C\Phi(s)(C\Phi(T))^\tilde{C} \int \tilde{C}^2 (1 + \|\nabla \mathbf{u}\|_{L^2}^2) d\tau.
\]
(3.25)
Recalling (3.2), one can choose \(s\) close enough to \(T^*\) such that
\[
\lim_{T \to T^*} \tilde{C} \int_s^T (1 + \|\nabla \mathbf{u}\|_{L^2}^2) d\tau \leq \frac{1}{2},
\]
which along with (3.25) yields
\[
\Phi(T) \leq C\Phi^2(s) < \infty.
\]
The proof of Lemma 3.2 is finished. \(\square\)

**Lemma 3.3.** Under the condition (3.1), it holds that for any \(T \in [0, T^*)\),
\[
\sup_{0 \leq t \leq T} (t\|\sqrt{\rho} \mathbf{u}\|_{L^2}^2 + t\|\nabla w_t\|_{L^2}^2) + \int_0^T (t\|\nabla \mathbf{u}\|_{L^2}^2 + t\|\nabla w_t\|_{L^2}^2) dt \leq C.
\]
(3.26)

**Proof.** 1. It follows from (3.8), (3.9), and (3.4) that for \(\varepsilon > 0, \eta > 0,\) and \(\tilde{\eta} = \min\{1, \eta\}\),
\[
\|\rho^\varepsilon \mathbf{u}\|_{L^{2+\varepsilon}} + \|\mathbf{u} \bar{w}^{-\eta}\|_{L^{2+\varepsilon}} + \|\rho^\eta w\|_{L^{2+\eta}} \bar{w}^{2+\eta} + \|w \bar{w}^{-\eta}\|_{L^{2+\eta}} \bar{w}^{2+\eta} \leq C.
\]
(3.27)
From (3.23) and (3.4), we have
\[
\|\nabla^2 \mathbf{u}\|_{L^2} \leq C + C\|\sqrt{\rho} \mathbf{u}\|_{L^2}.
\]
(3.28)
Employing \(L^2\) theory of elliptic equations, we easily infer from (1.2) and (3.27) that
\[
\|\nabla^2 w\|_{L^2} \leq C\|\rho w_t\|_{L^2} + C\|\rho \mathbf{u} \cdot \nabla w\|_{L^2} + C\|\nabla w\|_{L^2}
\leq C\|\rho\|_{L^\infty}^\frac{1}{2} \|\sqrt{\rho} w_t\|_{L^2} + C\|\rho \mathbf{u}\|_{L^4} \|\nabla w\|_{L^4} + C
\leq C\|\sqrt{\rho} w_t\|_{L^2} + C\|\nabla w\|_{L^2}^\frac{1}{2} \|\nabla^2 w\|_{L^2}^\frac{1}{2} + C
\]
(3.29)
This implies that

\[
\|\nabla^2 w\|_{L^2} \leq C + C\|\sqrt{\rho} w_t\|_{L^2}.
\]

(3.29)

2. Differentiating (1.2)\_2 and (1.2)\_3 with respect to \( t \) and using (1.2)\_1 give rise to

\[
\rho u_{tt} + \rho u \cdot \nabla u_t - (\mu_1 + \xi) \Delta u_t + \nabla P_t
\]
\[
= (u \cdot \nabla \rho)(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - 2\xi \nabla^\perp w_t,
\]

(3.30)

\[
\rho w_{tt} + \rho u \cdot \nabla w_t + 4\xi w_t - \mu_2 \Delta w_t
\]
\[
= (u \cdot \nabla \rho)(w_t + u \cdot \nabla w) - \rho u_t \cdot \nabla w + 2\xi \nabla^\perp \cdot u_t.
\]

(3.31)

Multiplying (3.30) by \( u_t \), (3.31) by \( w_t \), and integrating the resulting equality by parts over \( \mathbb{R}^2 \) and summing them, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \int (\rho |u|^2 + \rho w_t^2) dx + (\mu_1 + \xi) \int |\nabla u_t|^2 dx + \mu_2 \int |\nabla w_t|^2 dx + 4\xi \int w_t^2 dx
\]
\[
= \int ((u \cdot \nabla \rho)(u_t + u \cdot \nabla u) \cdot u_t - \rho u_t \cdot \nabla u \cdot u_t) dx
\]
\[
+ \int ((u \cdot \nabla \rho)(w_t + u \cdot \nabla w) \cdot w_t - \rho u_t \cdot \nabla w \cdot w_t) dx
\]
\[
- 2\xi \int (\nabla^\perp w_t \cdot u_t - \nabla^\perp \cdot u_t w_t) dx \triangleq \sum_{i=1}^{3} I_i.
\]

(3.32)

After integration by parts, we derive from (3.27), (3.7), (3.4), and Gagliardo-Nirenberg inequality that

\[
I_1 \leq C \int \rho |u| \left( |u| |\nabla u_t| + |u| |\nabla^2 u| |u_t| + |u| |\nabla u| |\nabla u_t| + |\nabla u|^2 |u_t| \right) dx
\]
\[
+ \int \rho |u_t|^2 |\nabla u| dx
\]
\[
\leq C \|\sqrt{\rho} u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^6}^{\frac{1}{2}} \|\nabla u\|_{L^6} \left( \|\nabla u_t\|_{L^2} + \|\nabla^2 u\|_{L^2} \right)
\]
\[
+ C \|\rho u\|_{L^{12}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^6} \left( \|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^2} \right)
\]
\[
\leq C \|\sqrt{\rho} u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \left( \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2} \right)
\]
\[
\times \left( \|\nabla u_t\|_{L^2} + \|\nabla^2 u\|_{L^2} \right)
\]
\[
+ C \|\sqrt{\rho} u\|_{L^2} \|\nabla u\|_{L^2} \left( \|\nabla^2 u\|_{L^2} \right)
\]
\[
\leq \frac{\mu_1}{4} \|\nabla u_t\|_{L^2}^2 + C \left( 1 + \|\nabla^2 u\|_{L^2} \right) \left( 1 + \|\sqrt{\rho} u\|_{L^2} \right).
\]

(3.33)

Similarly, one has

\[
I_2 \leq C \int \rho |w| \left( |w_t| |\nabla w_t| + |u| |\nabla^2 w| |w_t| + |u| |\nabla w| |\nabla w_t| + |\nabla u| |\nabla w| |w_t| \right) dx
\]
\[
+ \int \rho |u_t| |w_t| |\nabla w| dx
\]
\[
\leq C \|\sqrt{\rho} w\|_{L^6} \|\sqrt{\rho} w_t\|_{L^6} \|\nabla w_t\|_{L^6} \left( \|\nabla w_t\|_{L^2} + \|\nabla u\|_{L^4} |\nabla w|_{L^4} \right)
\]
\[
+ C \|\rho u\|_{L^{12}} \|\rho w_t\|_{L^6} \|\nabla w_t\|_{L^6} \left( \|\nabla^2 w\|_{L^2} \right)
\]
\[
\leq C \|\sqrt{\rho} w\|_{L^2} \|\sqrt{\rho} w_t\|_{L^2} \|\nabla w_t\|_{L^2} \left( \|\nabla^2 u\|_{L^2} \right).
\]
1. We derive from (3.28) and (3.26) that
\[ \text{The proof of Lemma 3.3 is finished.} \]

Multiplying (3.36) by
\[ \rho \]
Under the condition
\[ \| \mu \|_{L^q}^2 + \| \nabla \|_{L^2}^2 \rho w_t \|
\]
we deduce from (3.3), Gagliardo-Nirenberg inequality (3.40),
\[ \mu \frac{1}{2} \|
\]
Integration by parts, we obtain from Cauchy-Schwarz inequality that
\[ \int \nabla \cdot u_t w_t dx \leq 4 \xi \| w_t \|_{L^2}^2 + \xi \| u_t \|_{L^2}^2. \]

Substituting (3.33)–(3.35) into (3.32), we obtain after using (3.28) and (3.29) that
\[ \frac{d}{dt} \left( \| \mu w_t \|_{L^2}^2 + \mu_1 \| \nabla u_t \|_{L^2}^2 + \mu_2 \| \nabla w_t \|_{L^2}^2 \right)
\]
Multiplying (3.36) by \( t \), we obtain (3.26) after using Gronwall’s inequality and (3.4).

The proof of Lemma 3.3 is finished. \( \square \)

**Lemma 3.4.** Under the condition (3.1), it holds that for any \( T \in [0, T^*) \),
\[ \sup_{0 \leq t \leq T} \left( t \| \nabla^2 u \|_{L^2}^2 + t \| \nabla^2 w \|_{L^2}^2 \right)
\]
\[ + \int_0^T \left( \| \nabla^2 u \|_{L^2}^{q+1} + \| \nabla P \|_{L^q}^{q+1} + t \| \nabla^2 u \|_{L^2}^2 + t \| \nabla P \|_{L^2}^2 \right) dt
\]
\[ + \int_0^T \left( \| \nabla^2 w \|_{L^2}^{q+1} + t \| \nabla^2 w \|_{L^2}^2 \right) dt \leq C. \]

**Proof.** 1. We derive from (3.28) and (3.26) that
\[ \sup_{0 \leq t \leq T} t \| \nabla^2 u \|_{L^2}^2 \leq C. \]

Multiplying (3.29) by \( t \), one gets from (3.26) that
\[ \sup_{0 \leq t \leq T} t \| \nabla^2 w \|_{L^2}^2 \leq C. \]

2. Choosing \( p = q \) in (3.22), we deduce from (3.3), Gagliardo-Nirenberg inequality, (3.27), (3.4), (3.7), (3.28), and (3.29) that
\[ \| \nabla^2 u \|_{L^q}^2 + \| \nabla P \|_{L^q}^2 \]
\[ \leq C \left( \| \mu u_t \|_{L^q}^2 + \| \mu \nabla u \|_{L^2}^2 \right)
\]
\[ \leq C \left( \| \mu u_t \|_{L^q}^2 + \| \mu \nabla u \|_{L^2}^2 \right) + C \left( \| \nabla^2 u \|_{L^2}^{1-\frac{2}{q}} + \| \nabla^2 w \|_{L^2}^{1-\frac{2}{q}} \right)
\]
\[ \leq C \left( \| \mu u_t \|_{L^q}^2 + \| \mu \nabla u \|_{L^2}^2 \right) + C \left( \| \nabla^2 u \|_{L^2}^{1-\frac{2}{q}} + \| \nabla^2 w \|_{L^2}^{1-\frac{2}{q}} \right) + C. \]
which together with (3.4) and (3.26) implies that
\[ \int_0^T \left( \| \nabla^2 u \|_{L^6}^{q+1} + \| \nabla P \|_{L^4}^{q+1} \right) dt \]
\[ \leq C \int_0^T t^{-\frac{q+1}{q-1}} \left( t \| \sqrt{\rho} u \|_{L^2} \right)^{\frac{2}{q-1}} \left( t \| \nabla u \|_{L^2} \right)^{\frac{(q-2)(q+1)}{2(q-2)}} dt + C \int_0^T \left( \| \sqrt{\rho} u \|_{L^2}^{q+1} + \| \sqrt{\rho} w \|_{L^2}^{q+1} \right) dt 
\[ \leq C \sup_{0 \leq t \leq T} \left( t \| \sqrt{\rho} u \|_{L^2}^2 \right)^{\frac{2}{q-1}} \int_0^T t^{-\frac{q+1}{q-1}} \left( t \| \nabla u \|_{L^2}^2 \right)^{\frac{(q-2)(q+1)}{2(q-2)}} dt + C \]
\[ \leq C \left( 1 + \int_0^T \left( t^{-\frac{q+2+a-2}{q+3}} + t \| \nabla u \|_{L^2}^2 \right) dt \right) \]
\[ \leq C, \]

and
\[ \int_0^T \left( \| \nabla^2 u \|_{L^6} + t \| \nabla P \|_{L^4} \right) dt \]
\[ \leq C \int_0^T \left( t \| \sqrt{\rho} u \|_{L^2} + t \| \sqrt{\rho} w \|_{L^2} \right) dt 
\[ + C \int_0^T \left( t \| \sqrt{\rho} u \|_{L^2}^2 \right)^{\frac{2(q-1)}{q+2}} \left( t \| \nabla u \|_{L^2}^2 \right)^{\frac{2}{q+2}} dt + C \]
\[ \leq C + C \sup_{0 \leq t \leq T} \left( t \| \sqrt{\rho} u \|_{L^2}^2 \right)^{\frac{2(q-1)}{q+2}} \int_0^T (1 + t \| \nabla u \|_{L^2}^2) dt \]
\[ \leq C. \]

Similarly, we obtain from (1.2) by $L^q$-estimates to elliptic equations that
\[ \int_0^T \left( \| \nabla^2 w \|_{L^6}^{q+1} + t \| \nabla^2 w \|_{L^4} \right) dt \leq C. \]

This finishes the proof of Lemma 3.4. \hfill \square

**Lemma 3.5.** Under the condition (3.1), it holds that for any $T \in [0, T^*)$,
\[ \sup_{0 \leq t \leq T} \| \rho \bar{x}^\sigma \|_{L^1 \cap H^1 \cap W^{1,q}} \leq C. \]  
(3.41)

**Proof.** 1. A straightforward calculation shows
\[ \bar{x}^{-1} |\nabla \bar{x}| \leq C(3 + |x|^2)^{-\frac{1}{2}}, \]
which combined with Sobolev’s inequality and (3.27) that for any $\sigma > 0$,
\[ \| u \bar{x}^{-\sigma} \|_{L^\infty} \leq C \| u \bar{x}^{-\sigma} \|_{L^3} + C \| \nabla (u \bar{x}^{-\sigma}) \|_{L^3} \]
\[ \leq C + C \| \nabla u \|_{L^3} \| \bar{x}^{-\sigma} \|_{L^\infty} + C \| u \bar{x}^{-\sigma} \|_{L^3} \| \bar{x}^{-1} \nabla \bar{x} \|_{L^\infty} \]
\[ \leq C + \| \nabla u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 \]
\[ \leq C + (1 + \| \nabla^2 u \|_{L^2}). \]  
(3.42)

2. One derives from (1.2) that $\rho \bar{x}^\sigma$ satisfies
\[ \partial_t (\rho \bar{x}^\sigma) + u \cdot \nabla (\rho \bar{x}^\sigma) - \alpha \rho \bar{x}^\sigma \cdot \nabla \log \bar{x} = 0, \]  
(3.43)
which along with (3.42) and (3.4) gives that for any \( r \in [2, q] \),
\[
\frac{d}{dt} \| \nabla (\rho \bar{x}^a) \|_{L^r} \leq C (1 + \| \nabla u \|_{L^q} + \| u \cdot \nabla \log \bar{x} \|_{L^q}) \| \nabla (\rho \bar{x}^a) \|_{L^r} + C \| \rho \bar{x}^a \|_{L^r} \left( \| \nabla u \|_{L^q} \| \nabla \log \bar{x} \|_{L^q} + \| u \|_{L^2} \| \nabla \bar{x} \|_{L^q} \right)
\]
where we have used the following
\[
\| u \cdot \nabla \log \bar{x} \|_{L^q} + \| u \|_{L^2} \| \nabla \bar{x} \|_{L^q} \leq C \| u \bar{x}^{-\frac{q}{q-2}} \|_{L^\infty} \leq C (1 + \| \nabla^2 u \|_{L^2}),
\]
owing to (3.18) and (3.42). Hence, we get the desired (3.41) from (3.44), Gronwall’s inequality, (3.37), (3.28), (3.4), and (3.5). This completes the proof of Lemma 3.5.

With Lemmas 3.1–3.5 at hand, we are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We argue by contradiction. Suppose that (1.9) were false, that is, (3.1) holds. Note that the general constant \( C \) in Lemmas 3.1–3.5 is independent of \( t < T^* \), that is, all the a priori estimates obtained in Lemmas 3.1–3.5 are uniformly bounded for any \( t < T^* \). Hence, the function
\[
(p, u, w)(T^*, x) \triangleq \lim_{t \to T^*} (p, u, w)(t, x)
\]
satisfy the initial condition (1.8) at \( t = T^* \). Therefore, taking \((p, u, w)(T^*, x)\) as the initial data, one can extend the local strong solution beyond \( T^* \), which contradicts the maximality of \( T^* \). Thus we finish the proof of Theorem 1.1.

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