In the Search of Optimal Concurrency

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Abstract
Implementing a concurrent data structure typically begins with defining its sequential specification. However, when used as is, a nontrivial sequential data structure, such as a linked list, a search tree, or a hash table, may expose incorrect behavior: lost updates, inconsistent responses, etc. To ensure correctness, portions of the sequential code operating on the shared data must be “protected” from data races using synchronization primitives and, thus, certain schedules of the steps of concurrent operations must be rejected. But can we ensure that we do not “overuse” synchronization, i.e., that we reject a concurrent schedule only if it violates correctness?

In this paper, we treat this question formally by introducing the notion of a concurrency-optimal implementation. A program’s concurrency is defined here as its ability to accept concurrent schedules, i.e., interleavings of steps of its sequential implementation. An implementation is concurrency-optimal if it accepts all interleavings that do not violate the program’s correctness. We explore the concurrency properties of search data structures which can be represented in the form of directed acyclic graphs exporting insert, delete and search operations. We prove, for the first time, that pessimistic (e.g., based on conservative locking) and optimistic serializable (e.g., based on serializable transactional memory) implementations of search data-structures are incomparable in terms of concurrency. Specifically, there exist simple interleavings of sequential code that cannot be accepted by any pessimistic (and resp., serializable optimistic) implementation, but accepted by a serializable optimistic one (and resp., pessimistic). Thus, neither of these two implementation classes is concurrency-optimal.

1 Introduction

Building concurrent abstractions that efficiently exploit multi-processing abilities of modern hardware is a challenging task. One of the most important issues here is the need for synchronization, i.e., ensuring that concurrent processes resolve conflicts on the shared data in a consistent way. Indeed, using, as is, a data structure designed for conventional sequential settings in a concurrent environment may cause different kinds of inconsistencies, and synchronization must be used to avoid them. A variety of synchronization techniques have been developed so far. Conventional pessimistic synchronization conservatively protects shared data with locks before reading or modifying it. Speculative synchronization, achieved using transactional memory (TM) or conditional instructions, such as CAS or LL/SC, optimistically executes memory operations with a risk of aborting them in the future. A programmer typically uses these synchronization techniques to “wrap” fragments of a sequential implementation of the desired abstraction (e.g., operations on the shared data structure) in a way that allows the resulting concurrent execution to “appear” locally sequential, while preserving a global correctness criterion.

It is however difficult to tell in advance which of the techniques will provide more concurrency, i.e., which one would allow the resulting programs to process more executions of concurrent operations without data conflicts. Implementations based on TMs \cite{5,26}, which execute concurrent accesses speculatively, may seem more concurrent than lock-based counterparts whose concurrent accesses are blocking. But TMs conventionally impose serializability \cite{24} or even stronger properties \cite{13} on operations encapsulated within transactions. This may prohibit certain concurrent scenarios allowed by a large class of dynamic data structures \cite{9}.

In this paper, we reason formally about the “amount of concurrency” one can obtain by turning a sequential program into a concurrent one. To enable fair comparison of different synchronization techniques, we (1) define what it means for a concurrent program to be correct regardless of the type of synchronization it uses and (2) define a metric of concurrency. These definitions allow us to compare concurrency properties offered by serializable optimistic and pessimistic synchronization techniques, whose popular examples are, respectively, transactions and conservative locking.
Correctness  Our novel consistency criterion, called locally-serializable linearizability, is an intersection of linearizability and a new local serializability criterion.

Suppose that we want to design a concurrent implementation of a data type $T$ (e.g., integer set), given its sequential implementation $S$ (e.g., based on a sorted linked list). A concurrent implementation of $T$ is locally serializable with respect to $S$ if it ensures that the local execution of reads and writes of each operation is, in precise sense, equivalent to some execution of $S$. This condition is weaker than serializability since it does not require the existence of a single sequential execution that is consistent with all local executions. It is however sufficient to guarantee that executions do not observe an inconsistent transient state that could lead to fatal arithmetic errors, e.g., division-by-zero.

In addition, for the implementation of $T$ to “make sense” globally, every concurrent execution should be linearizable [3][20]: the invocation and responses of high-level operations observed in the execution should constitute a correct sequential history of $T$. The combination of local serializability and linearizability gives a correctness criterion that we call LS-linearizability, where LS stands for “locally serializable”. We show that LS-linearizability, just like linearizability, is compositional [18][20]: a composition of LS-linearizable implementations is also LS-linearizable.

Concurrency metric  We measure the amount of concurrency provided by an LS-linearizable implementation as the set of schedules it accepts. To this end, we define a concurrency metric inspired by the analysis of parallelism in database concurrency control [16][30] and transactional memory [10]. More specifically, we assume an external scheduler that defines which processes execute which steps of the corresponding sequential program in a dynamic and unpredictable fashion. This allows us to define concurrency provided by an implementation as the set of schedules (interleavings of reads and writes of concurrent sequential operations) it accepts (is able to effectively process).

Our concurrency metric is platform-independent and it allows for measuring relative concurrency of LS-linearizable implementations using arbitrary synchronization techniques.

The combination of our correctness and concurrency definitions provides a framework to compare the concurrency one can get by choosing a particular synchronization technique for a specific data type.

Measuring concurrency: pessimism vs. serializable optimism  We explore the concurrency properties of a large class of search concurrent data structures. Search data structures maintain data in the form of a rooted directed acyclic graph (DAG), where each node is a (key, value) pair, and export operations insert(key, value), delete(key), and find(key) with the natural sequential semantics.

A typical sequential implementation of these operations traverses the graph starting from the root in order to locate the “relevant” area corresponding to its key parameter. If the operation is insert or delete and the relevant area satisfies certain conditions, the operation updates it. The class includes many popular data structures, such as linked lists, skiplists, and search trees, implementing various abstractions like sets and multi-sets. Under reasonable assumptions on the representation of data, search structures are “concurrency-friendly”: high-level operations may commute (and, thus, proceed concurrently and independently) if their relevant sets are disjoint. To exploit this potential concurrency, various synchronization techniques can be employed.

In this paper, we compare the concurrency properties of two classes of search-structure implementations: pessimistic and serializable optimistic. Pessimistic implementations capture what can be achieved using classic conservative locks like mutexes, spinlocks, reader-writer locks. In contrast, optimistic implementations, however proceed speculatively and may roll back in the case of conflicts. Additionally, serializable optimistic techniques, e.g., relying on conventional TMs, like TinySTM [8] or NOrec 6 allow for transforming any sequential implementation of a data type to a LS-linearizable concurrent one.

The main result of this paper is that synchronization techniques based on pessimism and serializable optimism, are not concurrency-optimal: we show that no one of their respective set of accepted concurrent schedules include the other.

On the one hand, we prove, for the first time, that there exist simple schedules that are not accepted by any pessimistic implementation, but accepted by a serializable optimistic implementation. Our proof technique, which is interesting in its own right, is based on the following intuitions: a pessimistic implementation has to proceed irrevocably and over-conservatively reject a potentially acceptable schedule, simply because it may result in a data conflict thus leading the data structure to an inconsistent state. However, an optimistic implementation of a search data structure may (partially or completely) restart an operation depending on the current schedule. This way even schedules that potentially lead to conflicts may be optimistically accepted.

On the other hand, we show that pessimistic implementations can be designed to exploit the semantics of the data type. In particular, they can allow operations updating disjoint sets of data items to proceed
independently and preserving linearizability of the resulting history, even though the execution is not serializable. In such scenarios, pessimistic implementations carefully adjusted to the data types we implement can supersede the “semantic-oblivious” optimistic serializable implementations. Thus, neither pessimistic nor serializable optimistic implementations are concurrency-optimal.

Our comparative analysis of concurrency properties of pessimistic and serializable optimistic implementation suggests that combining the advantages of pessimism, namely its semantics awareness, and the advantages of optimism, namely its ability to restart operations in case of conflicts, enables implementations that are strictly better-suited for exploiting concurrency than any of these two techniques taken individually. To the best of our knowledge, this is the first formal analysis of the relative abilities of different synchronization techniques to exploit concurrency in dynamic data structures and lays the foundation for designing concurrent data structures that are concurrency-optimal.

Roadmap We define the class of concurrent implementations we consider in Section 2. In Section 3 we define the correctness criterion and our concurrency metric. Section 4 defines the class of search structures for which our concurrency lower bounds apply. In Section 5 we analyse the concurrency provided by pessimistic and serializable optimistic synchronization techniques in exploiting concurrency w.r.t search structures. Sections 6 and 7 present the related work and concluding remarks respectively.

2 Preliminaries

Sequential types and implementations An type \( \tau \) is a tuple \((\Phi, \Gamma, Q, q_0, \delta)\) where \(\Phi\) is a set of operations, \(\Gamma\) is a set of responses, \(Q\) is a set of states, \(q_0 \in Q\) is an initial state and \(\delta \subseteq Q \times \Phi \times Q \times \Gamma\) is a sequential specification that determines, for each state and each operation, the set of possible resulting states and produced responses [2].

Any type \( \tau = (\Phi, \Gamma, Q, q_0, \delta) \) is associated with a sequential implementation \( IS \). The implementation encodes states in \( Q \) using a collection of elements \( X_1, X_2, \ldots \) and, for each operation of \( \tau \), specifies a sequential read-write algorithm. Therefore, in the implementation \( IS \), an operation performs a sequence of reads and writes on \( X_1, X_2, \ldots \) and returns a response \( \tau \in \Gamma \). The implementation guarantees that, when executed sequentially, starting from the state of \( X_1, X_2, \ldots \) encoding \( q_0 \), the operations eventually return responses satisfying \( \delta \).

Concurrent implementations We consider an asynchronous shared-memory system in which a set of processes communicate by applying primitives on shared base objects [17].

We tackle the problem of turning the sequential algorithm \( IS \) of type \( \tau \) into a concurrent one, shared by \( n \) processes \( p_1, \ldots, p_n \) (\( n \in \mathbb{N} \)). The idea is that the concurrent algorithm essentially follows \( IS \), but to ensure correct operation under concurrency, it replaces read and write operations on \( X_1, X_2, \ldots \) in operations of \( IS \) with their base-object implementations.

Throughout this paper, we use the term operation to refer to high-level operations of the type. Reads and writes implemented by a concurrent algorithm are referred simply as reads and writes. Operations on base objects are referred to as primitives.

We also consider concurrent implementation that execute portions of sequential code speculatively, and restart their operations when conflicts are encountered. To account for such implementations, we assume that an implemented read or write may abort by returning a special response \( \bot \). In this case, we say that the corresponding (high-level) operation is aborted.

Therefore, our model applies to all concurrent algorithms in which a high-level operation can be seen as a sequence of reads and writes on elements \( X_1, X_2, \ldots \) (representing the state of the data structure), with the option of aborting the current operation and restarting it after. Many existing concurrent data structure implementations comply with this model as we illustrate below.

Executions and histories An execution of a concurrent implementation is a sequence of invocations and responses of high-level operations of type \( \tau \), invocations and responses of read and write operations, and invocations and responses of base-object primitives. We assume that executions are well-formed: no process invokes a new read or write, or high-level operation before the previous read or write, or a high-level operation, resp., returns, or takes steps outside its operation’s interval.

Let \( \alpha[p] \) denote the subsequence of an execution \( \alpha \) restricted to the events of process \( p \). Executions \( \alpha \) and \( \alpha’ \) are equivalent if for every process \( p \), \( \alpha[p] = \alpha’[p] \). An operation \( \pi \) precedes another operation \( \pi’ \) in an execution \( \alpha \), denoted \( \pi \rightarrow_\alpha \pi’ \), if the response of \( \pi \) occurs before the invocation of \( \pi’ \). Two operations are concurrent if neither precedes the other. An execution is sequential if it has no concurrent operations.
A sequential execution $\alpha$ is legal if for every object $X$, every read of $X$ in $\alpha$ returns the latest written value of $X$. An operation is complete in $\alpha$ if the invocation event is followed by a matching (non-$\perp$) response or aborted; otherwise, it is incomplete in $\alpha$. Execution $\alpha$ is complete if every operation is complete in $\alpha$.

The history exported by an execution $\alpha$ is the subsequence of $\alpha$ reduced to the invocations and responses of operations, reads and writes, except for the reads and writes that return $\perp$ (the abort response).

**High-level histories and linearizability** A high-level history $\tilde{H}$ of an execution $\alpha$ is the subsequence of $\alpha$ consisting of all invocations and responses of non-aborted operations. A complete high-level history $\tilde{H}$ is linearizable with respect to an object type $\tau$ if there exists a sequential high-level history $S$ equivalent to $\tilde{H}$ such that (1) $\tilde{H} \implies S$ and (2) $S$ is consistent with the sequential specification of type $\tau$. Now a high-level history $\tilde{H}$ is linearizable if it can be completed (by adding matching responses to a subset of incomplete operations in $\tilde{H}$ and removing the rest) to a linearizable high-level history $\tilde{H}$.

**Optimistic and pessimistic implementations** Note that in our model an implementation may, under certain conditions, abort an operation: some read or write return $\perp$, in which case the corresponding operation also returns $\perp$. Popular classes of such optimistic implementations are those based on “lazy synchronization” \cite{15,18} (with the ability of returning $\perp$ and re-invoking an operation) or transactional memory (TM) \cite{3,20}.

In the subclass of pessimistic implementations, no execution includes operations that return $\perp$. Pessimistic implementations are typically lock-based or based on pessimistic TMs \cite{1}. A lock provides exclusive (resp., shared) access to an element $X$ through synchronization primitives $\text{lock}(X)$ (resp., $\text{lock-shared}(X)$), and $\text{unlock}(X)$ (resp., $\text{unlock-shared}(X)$). A process releases the lock it holds by invoking $\text{unlock}(X)$ or $\text{unlock-shared}(X)$. When $\text{lock}(X)$ invoked by a process $p_i$ returns, we say that $p_i$ holds a lock on $X$ (until $p_i$ returns from the subsequent $\text{lock}(X)$). When $\text{lock-shared}(X)$ invoked by $p_i$ returns, we say that $p_i$ holds a shared lock on $X$ (until $p_i$ returns from the subsequent $\text{lock-shared}(X)$). At any moment, at most one process may hold a lock on an element $X$. Note that two processes can hold a shared lock on $X$ at a time. We assume that locks are starvation-free: if no process holds a lock on $X$ forever, then every $\text{lock}(X)$ eventually returns. Given a sequential implementation of a data type, a corresponding lock-based concurrent one is derived by inserting the synchronization primitives (lock and unlock) to protect read and write accesses to the shared data.

### 3 Correctness and concurrency metric

In this section, we define the correctness criterion of locally serializable linearizability and introduce the framework for comparing the relative abilities of different synchronization technique in exploiting concurrency.

#### 3.1 Locally serializable linearizability

Let $H$ be a history and let $\pi$ be a high-level operation in $H$. Then $H|\pi$ denotes the subsequence of $H$ consisting of the events of $\pi$, except for the last aborted read or write, if any. Let $\Sigma IS$ be a sequential implementation of an object of type $\tau$ and $\Sigma IS$, the set of histories of $\Sigma IS$.

**Definition 1** (LS-linearizability). A history $H$ is locally serializable with respect to $\Sigma IS$ if for every high-level operation $\pi$ in $H$, there exists $S \in \Sigma IS$ such that $H|\pi = S|\pi$.

A history $H$ is LS-linearizable with respect to $(\Sigma IS, \tau)$ (we also write $H$ is $(\Sigma IS, \tau)$-LSL) if: (1) $H$ is locally serializable with respect to $\Sigma IS$ and (2) the corresponding high-level history $\tilde{H}$ is linearizable with respect to $\tau$.

Observe that local serializability stipulates that the execution is witnessed sequential by every operation. Two different operations (even when invoked by the same process) are not required to witness mutually consistent sequential executions.

A concurrent implementation $I$ is LS-linearizable with respect to $(\Sigma IS, \tau)$ (we also write $I$ is $(\Sigma IS, \tau)$-LSL) if every history exported by $I$ is $(\Sigma IS, \tau)$-LSL. Throughout this paper, when we refer to a concurrent implementation of $(\Sigma IS, \tau)$, we assume that it is LS-linearizable with respect to $(\Sigma IS, \tau)$.
**Compositionality** We define the composition of two distinct object types \( \tau_1 \) and \( \tau_2 \) as a type \( \tau_1 \times \tau_2 = (\Phi, \Gamma, Q, q_0, \delta) \) as follows: \( \Phi = \Phi_1 \cup \Phi_2 \), \( \Gamma = \Gamma_1 \cup \Gamma_2 \). \( Q = Q_1 \times Q_2 \), \( q_0 = (q_{01}, q_{02}) \), and \( \delta \subseteq Q \times \Phi \times Q \times \Gamma \) is such that \( ((q_1, q_2), \pi, (q'_1, q'_2), r) \in \delta \) if and only if for \( i \in \{1, 2\} \), if \( \pi \in \Phi_i \), then \( (q_i, \pi, q'_i, r) \in \delta_i \cap q_{3-i} = q'_{3-i} \cdot \)

Every sequential implementation \( IS \) of an object \( O_1 \times O_2 \) of a composed type \( \tau_1 \times \tau_2 \) naturally induces two sequential implementations \( IS_1 \) and \( IS_2 \) of objects \( O_1 \) and \( O_2 \), respectively. Now a correctness criterion \( \Psi \) is *compositional* if for every history \( H \) on an object composition \( O_1 \times O_2 \), if \( \Psi \) holds for \( H|O_1 \) with respect to \( IS_1 \), for \( i \in \{1, 2\} \), then \( \Psi \) holds for \( H \) with respect to \( IS = IS_1 \times IS_2 \). Here, \( H|O_i \) denotes the subsequence of \( H \) consisting of events on \( O_i \).

**Theorem 1.** LS-linearizability is compositional.

*Proof.* Let \( H \), a history on \( O_1 \times O_2 \), be LS-linearizable with respect to \( IS \). Let each \( H|O_i \), \( i \in \{1, 2\} \), be LS-linearizable with respect to \( IS_i \). Without loss of generality, we assume that \( H \) is complete (if \( H \) is incomplete, we consider any completion of it containing LS-linearizable completions of \( H|O_1 \) and \( H|O_1 \)).

Let \( \tilde{H} \) be a completion of the high-level history corresponding to \( H \) such that \( \tilde{H}|O_1 \) and \( \tilde{H}|O_2 \) are linearizable with respect to \( \tau_1 \) and \( \tau_2 \), respectively. Since linearizability is compositional \([18, 20]\), \( \tilde{H} \) is linearizable with respect to \( \tau_1 \times \tau_2 \).

Now let, for each operation \( \pi \), \( S_1^0 \) and \( S_2^0 \) be any two sequential histories of \( IS_1 \) and \( IS_2 \) such that \( H|\pi|O_j = S_j^0|\pi, \) for \( j \in \{1, 2\} \) (since \( H|O_1 \) and \( H|O_2 \) are LS-linearizable such histories exist). We construct a sequential history \( S_\pi \) by interleaving events of \( S_1^0 \) and \( S_2^0 \) so that \( S_\pi|O_j = S_j^0 \) for \( j \in \{1, 2\} \). Since each \( S_j^0 \) acts on a distinct component \( O_j \) of \( O_1 \times O_2 \), every such \( S_\pi \) is a sequential history of \( IS \). We pick one \( S_\pi \) that respects the local history \( H|\pi \), which is possible, since \( H|\pi \) is consistent with both \( S_1^0|\pi \) and \( S_2^0|\pi \).

Thus, for each \( \pi \), we obtain a history of \( IS \) that agrees with \( H|\pi \). Moreover, the high-level history of \( H \) is linearizable. Thus, \( H \) is LS-linearizable with respect to \( IS \). \( \Box \)

Note that LS-linearizability is not non-blocking \([18, 20]\): local serializability may prevent an operation in a finite LS-linearizable history from having a completion, e.g., because, it might read an inconsistent system state caused by a concurrent incomplete operation.

**LS-linearizability versus other criteria** LS-linearizability is a two-level consistency criterion which makes it suitable to compare concurrent implementations of a sequential data structure, regardless of synchronization techniques they use. It is quite distinct from related criteria designed for database and software transactions, such as serializability \([21, 29]\) and multilevel serializability \([28, 29]\).

For example, serializability \([24]\) prevents sequences of reads and writes from conflicting in a cyclic way, establishing a global order of transactions. Reasoning only at the level of reads and writes may be overly conservative: higher-level operations may commute even if their reads and writes conflict \([27]\). Consider an execution of a concurrent *list-based set* depicted in Figure 3. We assume here that the set initial state is \( \{1, 3, 4\} \). Operation *contains*(5) is concurrent, first with operation *insert*(2) and then with operation *insert*(5). The history is not serializable: *insert*(5) sees the effect of *insert*(2) because \( R(X_1) \) by *insert*(5) returns the value of \( X_1 \) that is updated by *insert*(2) and thus should be serialized after it. But *contains*(5) misses element 2 in the linked list, but must read the value of \( X_4 \) that is updated by *insert*(5) to perform the read of \( X_5 \), i.e., the element created by *insert*(5). However, this history is LSL since each of the three local histories is consistent with some sequential history of \( LL \).

Multilevel serializability \([28, 29]\) was proposed to reason in terms of multiple semantic levels in the same execution. Being defined for two levels only, does not require a global serialization of low-level operations as 2-level serializability does. LS-linearizability simply requires each process to observe a local serialization, which can be different from one process to another. Also, to make it more suitable for concurrency analysis of a concrete data structure, instead of semantic-based commutativity \([27]\), we use the sequential specification of the high-level behavior of the object \([20]\).

Linearizability \([8, 29]\) only accounts for high-level behavior of a data structure, so it does not imply LS-linearizability. For example, Herlihy’s universal construction \([17]\) provides a linearizable implementation for any given object type, but does not guarantee that each execution locally appears sequential with respect to any sequential implementation of the type. Local serializability, by itself, does not require any synchronization between processes and can be trivially implemented without communication among the processes. Therefore, the two parts of LS-linearizability indeed complement each other.

\(^1\)Here we treat each \( \tau_i \) as a distinct type by adding index \( i \) to all elements of \( \Phi_i, \) \( \Gamma_i, \) and \( Q_i.\)
3.2 Concurrency metric

To characterize the ability of a concurrent implementation to process arbitrary interleavings of sequential code, we introduce the notion of a schedule. Intuitively, a schedule describes the order in which complete high-level operations, and sequential reads and writes are invoked by the user. More precisely, a schedule is an equivalence class of complete histories that agree on the order of invocation and response events of reads, writes and high-level operations, but not necessarily on the responses of read operations or of high-level operations. Thus, a schedule can be treated as a history, where responses of read and high-level operations are not specified.

We say that an implementation $I$ accepts a schedule $\sigma$ if it exports a history $H$ such that $complete(H)$ exhibits the order of $\sigma$, where $complete(H)$ is the subsequence of $H$ that consists of the events of the complete operations that returned a matching response. We then say that the execution (or history) exports $\sigma$. A schedule $\sigma$ is $(IS, \tau)$-LSL if there exists an $(IS, \tau)$-LSL history exporting $\sigma$.

An $(IS, \tau)$-LSL implementation is therefore concurrency-optimal if it accepts all $(IS, \tau)$-LSL schedules.

4 Search data structures

In this section, we introduce a class $D$ of dictionary-search data structures (or simply search structures), inspired by the study of dynamic databases undertaken by Chaudhri and Hadzilacos [4].

**Data representation** At a high level, a search structure is a dictionary that maintains data in a directed acyclic graph (DAG) with a designated root node (or element). The vertices (or nodes) of the graph are key-value pairs and edges specify the traversal function, i.e., paths that should be taken by the dictionary operations in order to find the nodes that are going to determine the effect of the operation. Keys are natural numbers, values are taken from a set $\mathbb{N}$, and the outgoing edges of each node are locally labelled. By a light abuse of notation we say that $G$ contains both nodes and edges. Key values of nodes in a DAG $G$ are related by a partial order $\prec_G$ that additionally defines a property $\mathbb{P}_G$ specifying if there is an outgoing edge from node with key $k$ to a node with key $k'$ (we say that $G$ respects $\mathbb{P}_G$).

If $G$ contains a node $a$ with key $k$, the $k$-relevant set of $G$, denoted $V_k(G)$, is a plus all nodes $b$, such that $G$ contains $(b, a)$ or $(a, b)$. If $G$ contains no nodes with key $k$, $V_k(G)$ consists of all nodes $a$ of $G$ with the smallest $k \prec_G k'$ plus all nodes $b$, such that $(b, a)$ is in $G$. The $k$-relevant graph of $G$, denoted $R_k(G)$, is the subgraph of $G$ that consists of all paths from the root to the nodes in $V_k(G)$.

**Sequential specification** At a high level, every data structure in $D$ exports a sequential specification with the following operations:

- $insert(k, v)$ checks whether a node with key $k$ is already present and, if so, returns false, otherwise it creates a node with key $k$ and value $v$, links it to the graph (making it reachable from the root) and returns true;
- $delete(k)$ checks whether a node with key $k$ is already present and, if so, unlinks the node from the graph (making it unreachable from the root) and returns true, otherwise it returns false;
- $find(k)$ returns the pointer to the node with key $k$ or false if no such node is found.

**Traversals** For each operation $op \in \{ insert(k, v), delete(k), find(k) \}_{k \in \mathbb{N}, v \in V}$, each search structure is parameterized by a (possibly randomized) traverse function $\tau_{op}$. Given the last visited node $a$ and the DAG of already visited nodes $G_{op}$, the traverse function $\tau_{op}$ returns a new node $b$ to be visited, i.e., accessed to get its key and the list of descendants, or $\emptyset$ to indicate that the search is complete.

Figure 1: Three search structures, a binary tree, a linked list and a skip list, whose $k$-relevant set has grey nodes and whose $k$-relevant path is indicated with a dashed arrow.
Find, insert and delete operations Intuitively, the traversal function is used by the operation \( op \) to explore the search structure and, when the function returns \( \emptyset \), the sub-DAG \( G_{op} \) explored so far contains enough information for \( op \) to complete.

If \( op = \text{find}(k) \), \( G_{op} \) either contains a node with key \( k \) or ensures that the whole graph does not contain \( k \). As we discuss below, in sorted search structures, such as sorted linked-lists or skiplists, we can stop as soon as all outgoing edges in \( G_{op} \) belong to nodes with keys \( k' \geq k \). Indeed, the remaining nodes can only contain keys greater than \( k \), so \( G_{op} \) contains enough information for \( op \) to complete.

An operation \( op = \text{insert}(k, v) \), is characterized by an insert function \( \mu_{(k,v)} \) that, given a DAG \( G \) and a new node \( (k, v) \notin G \), returns the set of edges from nodes of \( G \) to \( (k, v) \) and from \( (k, v) \) to nodes of \( G \) so that the resulting graph is a DAG containing \( (k, v) \) and respects \( P_G \).

An operation \( op = \text{delete}(k) \), is characterized by a delete function \( \nu_k \) that, given a DAG \( G \), gives the set of edges to be removed and a set of edges to be added in \( G \) so that the resulting graph is a DAG that respects \( P_G \).

Sequential implementations We make the following natural assumptions on the sequential implementation of a search structure:

- **Traverse-update** Every operation \( op \) starts with the read-only traverse phase followed with a write-only update phase. The traverse phase of an operation \( op \) with parameter \( k \) completes at the latest when for the visited nodes \( G_{op} \) contains the \( k \)-relevant graph. The update phase of a \( \text{find}(k) \) operation is empty.

- **Proper traversals and updates** For all DAGs \( G_{op} \) and nodes \( a \in G_{op} \), the traverse function \( \tau_{op}(a, G_{op}) \) returns \( b \) such that \((a, b) \in G \). The update phase of an \( \text{insert}(k) \) or \( \text{delete}(k) \) operation modifies outgoing edges of \( k \)-relevant nodes.

- **Non-triviality** There exist a key \( k \) and a state \( G \) such that (1) \( G \) contains no node with key \( k \), (2) If \( G' \) is the state resulting after applying \( \text{insert}(k, v) \) to \( G \), then there is exactly one edge \((a, b) \in G' \) such that \( b \) has key \( k \), and (3) the shortest path in \( G' \) from the root to \( a \) is of length at least 2.

The non-triviality property says that in some cases the read-phase may detect the presence of a given key only at the last step of a traverse-phase. Moreover, it excludes the pathological DAGs in which all the nodes are always reachable in one hop from the root. Moreover, the traverse-update property and the fact that keys are natural numbers implies that every traverse phase eventually terminates. Indeed, there can be only finitely many vertices pointing to a node with a given key, thus, eventually a traverse operation explores enough nodes to be sure that no node with a given key can be found.

Examples of search data structures In Figure 1 we describe few data structures in \( D \). A sorted linked list maintains a single path, starting at the root sentinel node and ending at a tail sentinel node, and any traversal with parameter \( k \) simply follows the path until a node with key \( k' \geq k \) is located. The traverse function for all operations follows the only path possible in the graph until the two relevant nodes are located.

A skiplist of \( n \) nodes is organized as a series of \( O(\log n) \) sorted linked lists, each specifying shortcuts of certain length. The bottom-level list contains all the nodes, each of the higher-level lists contains a sublist of the lower-level list. A traversal starts with the top-level list having the longest “hops” and goes to lower lists with smaller hops as the node with smallest key \( k' \geq k \) get closer.

A binary search tree represents data items in the form of a binary tree. Every node in the tree stores a key-value pair, and the left descendant of a non-leaf node with key \( k \) roots a subtree storing all nodes with keys less than \( k \), while the right descendant roots a subtree storing all nodes with keys greater than \( k \). Note that, for simplicity, we do not consider rebalancing operations used by balanced trees for maintaining the desired bounds on the traverse complexity. Though crucial in practice, the rebalancing operations are not important for our comparative analysis of concurrency properties of synchronization techniques.

Non-serializable concurrency There is a straightforward LSL implementation of any data structure in \( D \) in which updates (inserts and deletes) acquire a lock on the root node and are thus sequential. Moreover, they take exclusive locks on the set of nodes they are about to modify (\( k \)-relevant sets for operations with parameter \( k \)).

A find operation uses hand-over-hand shared locking: at each moment of time, the operation holds shared locks on all outgoing edges for the currently visited node \( a \). To visit a new node \( b \) (recall that \( b \) must be a descendant of \( a \)), it acquires shared locks on the new node’s descendants and then releases the
As we show in Section 5, the implementation is however not (safe-strict) serializable.
5 Pessimism vs. serializable optimism

In this section, we show that, with respect to search structures, pessimistic locking and optimistic synchronization providing safe-strict serializability are incomparable, once we focus on LS-linearizable implementations.

5.1 Classes \( \mathcal{P} \) and \( \mathcal{SM} \)

A synchronization technique is a set of concurrent implementations. We define below a specific optimistic synchronization technique and then a specific pessimistic one.

**\( \mathcal{SM} \): serializable optimistic** Let \( \alpha \) denote the execution of a concurrent implementation and \( \text{ops}(\alpha) \), the set of operations each of which performs at least one event in \( \alpha \). Let \( \alpha^k \) denote the prefix of \( \alpha \) up to the last event of operation \( \pi_k \). Let \( \text{Cseq}(\alpha) \) denote the set of subsequences of \( \alpha \) that consist of all the events of operations that are complete in \( \alpha \). We say that \( \alpha \) is strictly serializable if there exists a legal sequential execution \( \alpha' \) equivalent to a sequence in \( \sigma \in \text{Cseq}(\alpha) \) such that \( \rightarrow_\sigma \subseteq \rightarrow_{\alpha'} \).

This paper focuses on optimistic implementations that are strictly serializable and, in addition, guarantee that every operation (even aborted or incomplete) observes correct (serial) behavior. More precisely, an execution \( \alpha \) is safe-strict serializable if (1) \( \alpha \) is strictly serializable, and (2) for each operation \( \pi_k \), there exists a legal sequential execution \( \alpha' = \pi_0 \cdots \pi_i \pi_k \) and \( \sigma \in \text{Cseq}(\alpha^k) \) such that \( \{ \pi_0, \ldots, \pi_i \} \subseteq \text{ops}(\sigma) \) and \( \forall \pi_m \in \text{ops}(\alpha') : a'|m = a^k|m \).

Safe-strict serializability captures nicely both local serializability and linearizability. If we transform a sequential implementation \( IS \) of a type \( \tau \) into a safe-strict serializable concurrent one, we obtain an LSL implementation of \( (IS, \tau) \). Thus, the following lemma is immediate.

**Lemma 3.** Let \( I \) be a safe-strict serializable implementation of \( (IS, \tau) \). Then, \( I \) is LS-linearizable with respect to \( (IS, \tau) \).

Indeed, we make sure that completed operations witness the same execution of \( IS \), and every operation that returned \( \bot \) is consistent with some execution of \( IS \) based on previously completed operations. Formally, \( \mathcal{SM} \) denotes the set of optimistic, safe-strict serializable LSL implementations.

**\( \mathcal{P} \): deadlock-free pessimistic** Assuming that no process stops taking steps of its algorithm in the middle of a high-level operation, at least one of the concurrent operations return a matching response [19]. Note that \( \mathcal{P} \) includes implementations that are not necessarily safe-strict serializable.

5.2 Suboptimality of pessimistic implementations

We show now that for any search structure, there exists a schedule that is rejected by any pessimistic implementation, but accepted by certain optimistic strictly serializable ones. To prove this claim, we derive a safe-strict serializable schedule that cannot be accepted by any implementation in \( \mathcal{P} \) using the non-triviality property of search structures. It turns out that we can schedule the traverse phases of two \( \text{insert}(k) \) operations in parallel until they are about to check if a node with key \( k \) is in the set or not. If it is, both operations may safely return \( \text{false} \) (schedule \( \sigma \)). However, if the node is not in the set, in a pessimistic implementation, both operations would have to modify outgoing edges of the same node \( a \) and, if we want to provide local serializability, both return \( \text{true} \), violating linearizability (schedule \( \sigma' \)).

**Proof intuition.** We first provide an intuition of our results in the context of the integer set implemented as a sorted linked list or binary search tree. The set type is a special case of the dictionary which
The history is not serializable: \(X^0\) sees the effect of \(R\) because \(X^0\) is not stored at node \(X_1\) because \(X^0\) is stored at node \(X_1\). Therefore, a pessimistic implementation, which cannot accept the corresponding schedule \(\sigma\), violates LS-linearizability.

The formal proof follows.

Theorem 4. Any abstraction in \(D\) has a strictly serializable schedule that is not accepted by any implementation in \(\mathcal{P}\), but accepted by an implementation in \(\mathcal{SM}\).

Proof. Consider the key \(k\) and the states \(G\) and \(G'\) satisfying the conditions of the non-triviality property.

Consider the schedule that begins with a serial sequence of operations bringing the search to state \(G\) (executed by a single process). Then schedule the traverse phases of two identical \(insert(k, v)\) operations executed by \(new\) (not yet participating) processes \(p_1\) and \(p_2\) concurrently so that they perform identical steps (assuming that, if they take randomized steps, their coin tosses return the same values). Such an execution \(E\) exists, since the traverse phases are read-only. But if we allow both insert operations to proceed (by deadlock-freedom), we obtain an execution that is not LS-linearizable: both operations update the data structure which can only happen in a successful insert. But, by the sequential specification of \(D\), since node with key \(k\) belongs to \(G\), at least one of the two inserts must fail. Therefore, a pessimistic implementation, since it is not allowed to abort an operation, cannot accept the corresponding schedule \(\sigma\).

Now consider the serial sequence of operations bringing \(D\) to state \(G'\) (executed by a single process) and extend it with traverse phases of two concurrent \(insert(k, v)\) operations executed by new processes \(p_1\) and \(p_2\). The two traverse phases produce an execution \(E'\) which is indistinguishable to \(p_1\) and \(p_2\) from \(E\) up to their last read operations. Thus, if a pessimistic implementation accepts the corresponding schedule \(\sigma\), it must also accept \(\sigma'\), violating LS-linearizability.

Note, however, that an extension of \(E'\) in which both inserts complete by returning false is LS-linearizable. Moreover, any progressive (e.g., using progressive opaque transactional memory) optimistic strictly serializable implementation using will accept \(\sigma'\).

5.3 Suboptimality of serializable optimism

We show below that for any search structure, there exists a schedule that is rejected by any serializable implementation but accepted by a certain pessimistic one (HOH-find, to be concrete).

Proof intuition. We first illustrate the proof in the context of the integer set. Consider a schedule \(\sigma_0\) of a concurrent set implementation depicted in Figure 3. We assume here that the set initial state is \(\{1, 3, 4\}\). Operation \(find(5)\) is concurrent, first with operation \(insert(2)\) and then with operation \(insert(5)\). The history is not serializable: \(insert(5)\) sees the effect of \(insert(2)\) because \(R(X_1)\) by \(insert(5)\) returns the value of \(X_1\) that is updated by \(insert(2)\) and thus should be serialized after it. But \(find(5)\) misses node with value 2 in the set, but must read the value of \(X_1\) that is updated by \(insert(5)\) to perform the read of \(X_5\), i.e., the node created by \(insert(5)\). Thus, \(\sigma_0\) is not (safe-strict) serializable. However, this history is LSL since each of the three local histories is consistent with some sequential history of the integer set. However, there
exists an execution of our HOH-find implementation that exports $\sigma_0$ since there is no read-write conflict on any two consecutive nodes accessed.

To extend the above idea to any search structure, we use the non-triviality property of data structures in $\mathcal{D}$. There exist a state $G'$ in which there is exactly one edge $(a, b)$ in $G'$ such that $b$ has key $k$. We schedule a $op_f = \text{find}(k)$ operation concurrently with two consecutive delete operations: the first one, $op_{d1}$, deletes one of the nodes explored by $op_f$ before it reaches $a$ (such a node exists by the non-triviality property), and the second one, $op_{d2}$ deletes the node with key $k$ in $G'$. We make sure that $op_f$ is not affected by $op_{d1}$ (observes an update to some node $c$ in the graph) but is affected by $op_{d2}$ (does not observe $b$ in the graph). The resulting schedule is not strictly serializable (though linearizable). But our HOH-find implementation in $\mathcal{P}$ will accept it.

**Theorem 5.** For any abstraction in $D \in \mathcal{D}$, there exists an implementation in $\mathcal{P}$ that accepts a non-strictly serializable schedule.

**Proof.** Consider the HOH-find implementation described in Algorithm 1. Take the key $k$ and the states $G$ and $G'$ satisfying the conditions of the non-triviality property.

Now we consider the following execution. Let $op_f = \text{find}(k)$ be applied to an execution resulting in $G'$ (that contains a node with key $k$) and run $op_f$ until it reads the only node $a = (k', v')$ in $G$ that points to a node $b = (k, v)$ in state $G'$. Note that since $R_k(G) = R_k(G')$, the operation cannot distinguish the execution from than one starting with $G$.

The non-triviality property requires that the shortest path from the root to $k$ to $a$ in $R_k(G)$ is of length at least two. Thus, the set of nodes explored by $op_f$ passed through at least one node $c = (k'', v'')$ in addition to $a$. Now we schedule two complete delete operations executed by another process: first $del_1 = \text{delete}(k'')$, which removes $c$, followed by $del_2 = \text{delete}(k)$ which removes $b$. Now we wake up $op_f$ and let it read $a$, find out that no node with key $k$ is reachable, and return false.

Suppose, by contradiction, that the resulting execution is strictly serializable. Since $op_f$ has witnessed the presence of some node $c$ on the path from the root to $a$ in the DAG, $op_f$ must precede $del_1$ in any serialization. Now $del_1$ affected the response of $op_f$, it must precede $op_f$ in any serialization. Finally, $del_2$ precedes $del_1$ in the real-time order and, thus must precede $del_1$ in any serialization. The resulting cycle implies a contradiction.

Since any strictly serializable optimistic implementation only produces strictly serializable executions, from Theorem 5 we deduce that there is a schedule accepted by a pessimistic algorithm that no strictly serializable optimistic one can accept. Therefore, Theorems 4 and 5 imply that, when applied to search structures and in terms of concurrency, the strictly serializable optimistic approach is incomparable with pessimistic locking. As a corollary, none of these two techniques can be concurrency-optimal.

6 Related work

Sets of accepted schedules are commonly used as a metric of concurrency provided by a shared memory implementation. For static database transactions, Kung and Papadimitriou [22] use the metric to capture the parallelism of a locking scheme. While acknowledging that the metric is theoretical, they insist that it may have “practical significance as well, if the schedulers in question have relatively small scheduling times as compared with waiting and execution times.” Herlihy [16] employed the metric to compare various optimistic and pessimistic synchronization techniques using commutativity of operations constituting high-level transactions. A synchronization technique is implicitly considered in [16] as highly concurrent, namely “optimal”, if no other technique accepts more schedules. By contrast, we focus here on a dynamic model where the scheduler cannot use the prior knowledge of all the shared addresses to be accessed. Also, unlike [16,22], we require all operations, including aborted ones, to observe (locally) consistent states.

Gramoli et al. [10] defined a concurrency metric, the input acceptance, as the ratio of committed transactions over aborted transactions when TM executes the given schedule. Unlike our metric, input acceptance does not apply to lock-based programs.

Optimal concurrency is related to the notion of permissiveness [12], originally defined for transactional memory. In [12], a TM is defined to be permissive with respect to serializability if it aborts a transaction only if committing it would violate serializability. Thus, an operation of a concurrent data structure may be aborted only if the execution of the encapsulated transaction is not serializable. In contrast, our framework for analyzing concurrency is independent of the synchronization technique and a concurrency-optimal implementation accepts all correct interleavings of reads and writes of sequential operations (not just high-level responses of the operations).
David et al. [6] devised a pragmatic methodology for evaluating performance of a concurrent data structure via a comparison with the performance of its sequential counterpart. The closer is the throughput of a concurrent algorithm to that of its (inconsistent) sequential variant, the more “concurrent” the algorithm is. In contrast, the formalism proposed in this paper allows for relating concurrency properties of various concurrent algorithms.

Our definition of search data structures is based on the paper by Chaudhri and Hadzilacos [4] who studied them in the context of dynamic databases.

Safe-strict serializable implementations (\(SM\)) require that every transaction (even aborted and incomplete) observes “correct” serial behavior. It is weaker than popular TM correctness conditions like opacity [13] and its relaxations like TMS1 [7] and VWC [21]. Unlike TMS1, we do not require the local serial executions to always respect the real-time order among transactions. Unlike VWC, we model transactional operations as intervals with an invocation and a response and does not assume unique writes (needed to define causal past in VWC). Though weak, \(SM\) still allows us to show that the resulting optimistic LSL implementations reject some schedules accepted by pessimistic locks.

7 Concluding remarks

In this paper, we presented a formalism for reasoning about the relative power of optimistic and pessimistic synchronization techniques in exploiting concurrency in search structures. We expect our formalism to have practical impact as the search structures are among the most commonly used concurrent data structures, including trees, linked lists, skip lists that implement various abstractions ranging from key-value stores to sets and multi-sets.

Our results on the relative concurrency of \(P\) and \(SM\) imply that none of these synchronization techniques might enable an optimally-concurrent algorithm. Of course, we do not claim that our concurrency metric necessarily captures efficiency, as it does not account for other factors, like cache sizes, cache coherence protocols, or computational costs of validating a schedule, which may also affect performance on multi-core architectures. In [11] we already described a concurrency-optimal implementation of the linked-list set abstraction that combines the advantages of \(P\), namely the semantics awareness, with the advantages of \(SM\), namely the ability to restart operations in case of conflicts. We recently observed empirically that this optimality can result in higher performance than state-of-the-art algorithms [14, 15, 23]. Therefore, our findings motivate the search for concurrency-optimal algorithms. This study not only improves our understanding of designing concurrent data structures, but might lead to more efficient implementations.

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