WHEN IS A POLYNOMIALLY GROWING AUTOMORPHISM OF $F_n$ GEOMETRIC?

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Abstract. The main result of this paper is an algorithmic answer to the question raised in the title, up to replacing the given $\hat{\varphi} \in \text{Out}(F_n)$ by a positive power.

In order to provide this algorithm, it is shown that every polynomially growing automorphism $\hat{\varphi}$ can be represented by an iterated Dehn twist on some graph-of-groups $\mathcal{G}$ with $\pi_1\mathcal{G} = F_n$. One then uses results of two previous papers [14, 15] as well as some classical results such as the Whitehead algorithm to prove the claim.

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1. Introduction

Let \( \varphi \) be an automorphism of a free group \( F_n \) of finite rank \( n \geq 2 \). One says that \( \varphi \) (or the associated outer automorphism \( \hat{\varphi} \in \text{Out}(F_n) \)) is geometric if there is a positive answer to the following:

**Question:** Does there exist a surface \( S \) with \( \pi_1 S \cong F_n \) and a homeomorphism \( h : S \to S \) with \( h_* = \hat{\varphi} \)?

The main purpose of this paper is to give an algorithmic answer to this question, for the case that \( \varphi \) has polynomial growth, and modulo replacing \( \varphi \) by some higher iterate.

To do this, we describe in this paper several “subalgorithms” which, when properly put together, fulfill this purpose. Some of these sub-algorithms have some interest in themselves, which we describe now:

In section 5 we define iteratively a class of automorphisms of \( F_n \), called iterated Dehn twist automorphisms (of some level \( k \geq 1 \)), which are given by a graph-of-groups \( G \) with trivial edge groups, and an automorphism \( H : G \to G \) which acts trivially on the underlying graph and induces on each vertex group \( G_v \) an iterated Dehn twist automorphisms of level \( k_v \leq k - 1 \). For \( k = 1 \) the edge groups may be cyclic, and the vertex group automorphisms and edge group automorphisms are the identity. Thus for \( k = 1 \) the resulting automorphism is a Dehn twist automorphism in the traditional sense. We show (compare Proposition 5.3):

**Proposition 1.1.** Every polynomially growing automorphism \( \hat{\varphi} \in \text{Out}(F_n) \) has a positive power \( \hat{\varphi}^t \) which is represented by an iterated Dehn twist automorphism \( H \) of some level \( k \geq 1 \).

All the data for \( H \) (including for all lower levels through iteratively passing to the induced vertex group automorphisms) can be derived algorithmically from a relative train track representative of \( \hat{\varphi} \) as given by Bestvina-Feighn-Handel [3].

It can be derived from [3] that the above exponent \( t = t_n \geq 1 \) can be determined depending only on \( n \) and not on the particular choice of \( \hat{\varphi} \).

We then use our results from [14], [15] to derive algorithmically from an iterated Dehn twist representative of level \( k \geq 2 \) either an iterated Dehn twist representative of strictly lower level, or else a conjugacy class \([w] \subset F_n \) that grows at least quadratically under iteration of \( \hat{\varphi} \). Thus we obtain (compare Theorem 5.4 and Corollary 5.5):

**Theorem 1.2.** Every linearly growing automorphism of \( F_n \) has a positive power which is a Dehn twist automorphism.

More precisely: From any iterated Dehn twist representative of some \( \hat{\varphi} \in \text{Out}(F_n) \) one can derive algorithmically either the fact that \( \hat{\varphi} \) has at least quadratic growth, or else all the data of a graph-of-groups decomposition \( F_n \cong \pi_1 G \) as well as a Dehn twist \( H : G \to G \) with \( H_* = \hat{\varphi} \).
In the linearly growing case one then uses work of Cohen-Lustig \cite{7} to derive algorithmically from $H$ and $G$ an \textit{efficient} Dehn twist representative of $\hat{\varphi}$, and derive in section \[6\] from its uniqueness properties (see Theorem \[2.5\]) the following:

\textbf{Proposition 1.3.} An efficient Dehn twist $H : \mathcal{G} \to \mathcal{G}$ represents a geometric automorphism of $F_n$ if and only if for every vertex $v$ of $\mathcal{G}$ the family of “edge generators” $f_{e_i}(g_{e_i})$, where $e_i$ is any edge with terminal vertex $v$ and $g_{e_i}$ is a generator of $G_{e} \cong \mathbb{Z}$, is a boundary family in the vertex group $G_v$.

Here a \textit{boundary family} is a family of elements in a free group $F_m$ which satisfy up to conjugation the sufficient and necessary condition that are well known for elements which represent boundary components of an orientable or non-orientable surface (see Definition \[3.6\]). This condition in turn can be decided algorithmically for any finite family of elements by the Whitehead algorithm on $G_v$, so that we obtain (see section \[6\]):

\textbf{Corollary 1.4.} For any Dehn twist $H : \mathcal{G} \to \mathcal{G}$ on a free group $\pi_1 \mathcal{G} = F_n$ it can be decided algorithmically whether $H_*$ is geometric or not.

Putting all of the above together now gives:

\textbf{Theorem 1.5.} There exists an algorithm which decides, for any given polynomially growing automorphism $\hat{\varphi} \in \text{Out}(F_n)$, whether $\hat{\varphi}^t_n$ is geometric or not.

The concrete terms of this algorithm are presented in detail in the last section of this paper.

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\section{Preliminaries}

Throughout this paper $F_n$ will denote a free group of finite rank $n \geq 2$.

An outer automorphism $\hat{\varphi} \in \text{Out}(F_n)$ is said to be \textit{geometric} if it is induced by a homeomorphism on a surface. In other words, there exist a surface homeomorphism $h : S \to S$ and an isomorphism $\theta : \pi_1(S) \to F_n$ such that the following diagram commutes up to inner automorphisms,

$$
\begin{array}{cccc}
\pi_1(S) & \xrightarrow{h_*} & \pi_1(S) \\
\theta \downarrow & & \theta \downarrow \\
F_n & \xrightarrow{\varphi} & F_n
\end{array}
$$

where $\varphi \in \text{Aut}(F_n)$ is some representative of $\hat{\varphi}$.
2.1. Maps of Graphs and Topological Representatives.

In this paper a graph $\Gamma$ is always assumed to be finite and connected, unless otherwise stated. We denote the set of its vertices by $V(\Gamma)$, and the set of its oriented edges by $E(\Gamma)$. For any edge $e \in E(\Gamma)$ we denote by $\tau \in E(\Gamma)$ the same edge with reverted orientation. Furthermore, $\tau(e)$ denotes the terminal vertex of $e$, and $\iota(e)$ its initial vertex.

A path (or edge path) in $\Gamma$ is either a single vertex, in which case the path is said to be trivial, or a non-empty sequence of edges $e_1e_2 \ldots e_k$ such that $\tau(e_i) = \iota(e_{i+1})$ for $1 \leq i \leq k-1$. A path is reduced if it does not contain a subpath of the form $e\overline{e}$ for some edge $e \in E(\Gamma)$.

A graph map $f : \Gamma \to \Gamma'$ is a map which sends vertices to vertices and edges to edge paths, which may or may not be reduced.

Definition 2.1. A marked graph refers to a pair $(\Gamma, \theta)$ where $\Gamma$ is a graph and the marking $\theta : F_n \to \pi_1 \Gamma$ is an isomorphism.

Definition 2.2 (Topological Representative). Let $\hat{\varphi} \in \text{Out}(F_n)$ be an outer automorphism of $F_n$. A topological representative of $\hat{\varphi}$ with respect to a marking $\theta : F_n \to \pi_1 \Gamma$ is a homotopy equivalence $f : \Gamma \to \Gamma$ which determines the outer automorphism $\hat{\varphi}$ on its fundamental group $\pi_1 \Gamma$ (i.e. $f_* = \theta \hat{\varphi} \theta^{-1}$). Furthermore one requires that $f(e)$ is reduced and not a vertex, for every $e \in E(\Gamma)$.

A filtration for a topological representative $f : \Gamma \to \Gamma$ is an increasing sequence of invariant subgraphs $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_m = \Gamma$. Note that each $\Gamma_i$ is not necessarily connected. Each (possibly non-connected) subgraph $H_i = \text{cl}(\Gamma_i \setminus \Gamma_{i-1})$ is referred to as the $i$-th stratum. We usually assume that the edges of $\Gamma$ are labelled so that those in $H_i$ have smaller label than those in $H_{i+1}$.

Bestvina-Handel [2] (later improved by Bestvina-Feighn-Handel [3]) have shown that every automorphism $\hat{\varphi} \in \text{Out}(F_n)$ has a topological representative with very strong further properties. These relative train track representatives have strata $H_i$ which are either exponentially growing and have a “train track” property, or else they are polynomially growing (relative to lower train track strata). Since in this paper we are only concerned with polynomially growing $\hat{\varphi}$, we restrict ourselves here to quote a special case of their general result.

Theorem 2.3 ([3] Theorem 5.1.5). For every polynomially-growing outer automorphism $\hat{\varphi} \in \text{Out}(F_n)$ one can determine algorithmically for a positive power $\hat{\varphi}^t$ of $\hat{\varphi}$ a topological representative $f : \Gamma \to \Gamma$ with a filtration $V(\Gamma) = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m \Gamma$, which has the following properties:

1. the graph $\Gamma$ has no valence-one vertex;
2. all vertices $v \in V(\Gamma)$ are fixed by the map $f$;
3. every stratum $H_i$ consists of a single edge $e_i$ such that $f(e_i) = e_iu_i$, where $u_i \subset \Gamma_{i-1}$ is a closed path.
It follows from going through the material in \[8\] that the exponent \( t \geq 1 \) in the above theorem can be chosen a priori, i.e. \( t \) does not depend on \( \hat{\varphi} \) but only on the rank \( n \) of the free group \( F_n \), and that this exponent \( t = t_n \) can be determined algorithmically for any \( n \geq 2 \).

### 2.2. Graph-of-groups and Dehn twists.

In this subsection we will briefly recall some basic definitions of graph-of-groups and Dehn twists, which will be used in later sections. We refer the readers to [7], [12], [14], [15] for more detailed information and discussions.

A graph-of-groups \( \mathcal{G} = (\Gamma, (G_v)_{v \in V(\Gamma)}, (G_e)_{e \in E(\Gamma)}, (f_e)_{e \in E(\Gamma)}) \) consists of a finite connected graph \( \Gamma = \Gamma(\mathcal{G}) \), a vertex group \( G_v \) for each vertex \( v \) of \( \mathcal{G} \) (by which we mean “of \( \Gamma(\mathcal{G})^v \)”), an edge group \( G_e \) for each edge \( e \) of \( \mathcal{G} \), and a family of edge monomorphisms \( f_e : G_e \to G_{\tau(e)} \). For every \( e \in E(\Gamma) \), we require \( G_e = G_{\tau(e)} \).

To \( \mathcal{G} \) there is canonically associated a path group \( \Pi(\mathcal{G}) \) which is the free product of all \( G_v \) with the free group over \( E(\Gamma) \), subject to the relations \( t_e = t_e^{-1} \) and \( f_e(g) = t_e f_e(g) t_e^{-1} \) for all \( e \in E(\Gamma) \) and \( g \in G_e \). For any \( v \in V(\Gamma) \) there is a well defined fundamental group \( \pi_1(\mathcal{G}, v) \subset \Pi(\mathcal{G}) \), and they are all naturally conjugate to each other in \( \Pi(\mathcal{G}) \).

A graph-of-groups automorphism \( H : \mathcal{G} \to \mathcal{G} \) is given by a graph automorphism \( H_\Gamma : \Gamma \to \Gamma \), a group isomorphism \( H_v : G_v \to G_{H_\Gamma(v)} \) for each vertex \( v \) of \( \mathcal{G} \), a group isomorphism \( H_e : G_e \to G_{H_\Gamma(e)} \) for each edge \( e \) of \( \mathcal{G} \), and a correction term \( \delta(e) \in G_{\tau(H_e)} \) for every edge \( e \) of \( \mathcal{G} \) which satisfies

\[
H_{\tau(e)} f_e = ad_\delta f_e H_{H(e)}
\]

where \( ad_g : F_n \to F_n \) denotes conjugation with \( g \).

The isomorphism \( H \) induces canonically isomorphisms \( H_\ast : \Pi(\mathcal{G}) \to \Pi(\mathcal{G}) \) and \( H_v : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}, v) \) as well as an outer automorphism \( \hat{H} \) of \( \pi_1 \mathcal{G} \) which is independent of the choice of the base point \( v \). The latter is sometimes also denoted by \( H_\ast \).

A Dehn twist \( D : \mathcal{G} \to \mathcal{G} \) on graph-of-groups \( \mathcal{G} \) (defined in [14] as “general Dehn twist”) is an automorphism of \( \mathcal{G} \) where the graph automorphism \( D_\Gamma \), all vertex groups automorphisms \( D_v \) and all edge groups automorphisms \( D_e \) are the identity. Furthermore, one requires that for each edge \( e \) the correction term \( \delta(e) \) is contained in the centralizer of \( f_e(G_e) \) in \( G_{\tau(e)} \).

For free groups the last condition implies that for a non-trivial correction term \( \delta(e) \) the edge group \( G_e \) must be either trivial or infinite cyclic. Both cases occur when practically working with Dehn twist automorphisms of \( F_n \), i.e. automorphisms \( \hat{\varphi} \in \text{Out}(F_n) \) which satisfy for some identification \( F_n = \pi_1 \mathcal{G} \) that \( \hat{\varphi} = \hat{D} \).

For every edge \( e \) of \( \mathcal{G} \) with \( G_e \cong \mathbb{Z} \) one calls the element \( z_e := g \gamma_e^{-1} \) the twistor of \( e \), where \( \gamma_e \in G_e \) is defined by \( f_e(\gamma_e) = \delta(e) \). If \( G_e \cong \mathbb{Z} \) for all edges \( e \) of \( \mathcal{G} \) then \( D \) is called in [14] a classical Dehn twist. In this case we chose for each edge \( e \) of \( \mathcal{G} \) an edge generator \( g_e \) of the edge group \( G_e \) and
determine a twist exponent \( n(e) \in \mathbb{Z} \) such that \( z(e) = g_e^{n(e)} \). We use the convention that \( g_e \) is always picked so that \( n(e) \geq 0 \).

A partial Dehn twist relative to a subset of vertices \( V \subset V(\Gamma(\mathcal{G})) \) is a graph-of-groups isomorphism \( H : \mathcal{G} \to \mathcal{G} \) on a graph-of-groups \( \mathcal{G} \) with trivial edge groups, which satisfies all conditions of a Dehn twist except that the vertex group automorphisms \( H_v \) for any \( v \in V \) may not be the identity. We also require for such a partial Dehn twist that \( \mathcal{G} \) is minimal, i.e. there is no proper subgraph-of-groups \( \mathcal{G}' \) of \( \mathcal{G} \) where the injection induces an isomorphism \( \pi_1 \mathcal{G}' \cong \pi_1 \mathcal{G} \).

2.3. Efficient Dehn twists.

In [7] Cohen-Lustig introduced for free groups \( F_n \cong \pi_1 \mathcal{G} \) a particular class of Dehn twists which have rather special and nice properties:

**Definition 2.4** (Efficient Dehn twist [7]). A classical Dehn twist \( D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})}) \) is said to be efficient if the following conditions are satisfied:

1. \( \mathcal{G} \) is minimal: if \( v = \tau(e) \) is a valence-one vertex, then the edge homomorphism \( f_e : G_e \to G_v \) is not surjective.
2. No invisible vertex: there is no valence-two vertex \( v = \tau(e_1) = \tau(e_2) \) \((e_1 \neq e_2)\) such that both edge maps \( f_{e_i} : G_{e_i} \to G_v \) \((i = 1, 2)\) are surjective.
3. No unused edge: for every \( e \in E(\Gamma) \) the twistor satisfies \( z_e \neq 1 \) (or equivalently \( \gamma_e \neq \gamma_{e'} \)).
4. No proper power: if \( r^p \in f_e(G_e) \) \((p \neq 0)\) then \( r \in f_e(G_e) \), for all \( e \in E(\Gamma) \).
5. If \( v = \tau(e_1) = \tau(e_2) \), then \( e_1 \) and \( e_2 \) are not positively bonded: for any integers \( n_1, n_2 \geq 1 \) the elements \( f_{e_1}(z_{e_1}^{n_1}) \) and \( f_{e_2}(z_{e_2}^{n_2}) \) are not conjugate in \( G_v \).

The following result of Cohen-Lustig [6, 7] is used crucially in section 6 below.

**Theorem 2.5** ([7]). For two efficient Dehn twists \( D = D(\mathcal{G}, (z_e)_{e \in E(\Gamma)}) \) and \( D' = D(\mathcal{G}', (z_e)_{e \in E(\Gamma')}) \) one has \( \hat{D} = hD'h^{-1} \in \text{Out}(\pi_1(\mathcal{G})) \) for some isomorphism \( h : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{G}') \) if and only if there is a graph-of-groups isomorphism \( H : \mathcal{G} \to \mathcal{G}' \) which induces the isomorphism \( h \) up to inner automorphism and which takes twistors to twistors, i.e. \( H_e(z_e) = z_{H(\Gamma(\mathcal{G}))}^{H(e)} \) for all \( e \in E(\Gamma(\mathcal{G})) \).

\( \square \)

2.4. Partial Dehn twists relative to local Dehn twists.

In [15] the concept of a partial Dehn twist relative to a family of local Dehn twists has been introduced: This is a a partial Dehn twist \( D : \mathcal{G} \to \mathcal{G} \)

\(^1\)Unfortunately this minimality condition was omitted by mistake from Definition 3.7 of [15]
relative to a subset of vertices $V \subset V(\mathcal{G})$ as defined above, with the additional specification that on any $v \in V$ the graph-of-groups automorphism $D$ is given by a Dehn twist automorphism $D_v : G_v \to G_v$.

The following previous result of the author is crucially used in section 5 below:

**Corollary 2.6** (Corollary 1.2 of [15]). Let $\hat{\varphi} \in \text{Out}(F_n)$ be represented by a partial Dehn twist relative to a family of local Dehn twists.

Then either $\hat{\varphi}$ is itself a Dehn twist automorphism, or else $\hat{\varphi}$ has at least quadratic growth.

The proof of this corollary is algorithmic, i.e. it can be effectively decided which alternative of the stated dichotomy holds. Indeed, a more specific statement is given by the following, where we want to stress that it crucially relies on the assumption that the graph $\mathcal{G}$ is minimal, see subsection 2.2:

**Corollary 2.7** (Corollary 7.2 of [15]). Let $\hat{\varphi} \in \text{Out}(F_n)$ be represented by a partial Dehn twist $H : \mathcal{G} \to \hat{\mathcal{G}}$ relative to a family of local Dehn twists. Assume that for some edge $e$ of $\mathcal{G}$ the correction term $\delta(e)$ is not locally zero.

Then $\hat{\varphi}$ has at least quadratic growth. $\square$

### 3. QUOTES

In this section we will assemble material from other sources that will be used crucially later.

#### 3.1. Nielsen-Thurston classification of surface homeomorphisms.

The Nielsen-Thurston’s classification theorem partitions homotopy classes of homeomorphisms $h$ of a compact surface $S$ into three (not mutually exclusive) classes: (i) *periodic*, (ii) *reducible* and (iii) *pseudo-Anosov*. It is shown that in each case $h$ can be improved further by an isotopy so that it becomes geometrically very special, with very strong dynamical properties. However, for the purpose of this paper it suffices to note the following consequence of this theory:

**Theorem 3.1.** Denote $S$ a compact orientable surface. For any homeomorphism $h : S \to S$ there is an integer $m \geq 1$ and a (possibly empty) collection $\mathcal{C}$ of pairwise disjoint essential simple closed curves, such that $h^m : S \to S$ is isotopic to a homeomorphism $h'$ which preserves a decomposition of $S$ into subsurfaces $S_j$ along $\mathcal{C}$, where the restriction $h_j$ of $h'$ to each $S_j$ falls into one of the following three classes:

1. $h_j$ is the identity;
2. $h_j$ is a Dehn twist on an annulus;
3. $h_j$ is a pseudo-Anosov homeomorphism.

**Remark 3.2.** The above classification result extends in its main parts to a non-orientable surfaces $S$ as well, as can be seen directly from lifting the given homeomorphism to the canonical 2-sheeted orientable covering $\hat{S}$ of $S$. 

One then applies the above theorem to get $h' : \tilde{S} \to \tilde{S}$ and uses the essential uniqueness of $h'$ to argue that $h'$ commutes with the covering involution, so that one can "quotient" $h'$ it back to a homeomorphism of $S$.

Here one needs to be a bit careful when it comes to Dehn twists on a curve $\tilde{c}$ in $\tilde{S}$ which is a double cover of a curve $c$ in $S$ with a Moebius band neighborhood $\mathcal{N}(c)$. In this case it turns out that a twist on $\tilde{c}$ would be isotopic to two half-twists on each of the two boundary curves $c_1$ and $c_2$ of $\mathcal{N}(\tilde{c})$ (which is an annulus). However, twisting $n$ times on both, $c_1$ and $c_2$, descends in $\tilde{S}$ to an $n$-fold twist on the boundary curve $c'$ of the Moebius band $\mathcal{N}(c)$ in $S$, and it is easy to see that any such twist (including possible a half-twist) is isotopic to the identity in $\mathcal{N}(c)$.

As a consequence one can impose for any non-orientable surface the additional condition that the collection $\mathcal{C}$ in Theorem 3.1 consists only of curves which have an annulus neighborhood.

If in the situation of Theorem 3.1 (or Remark 3.2) none of the $h_j$ falls into class (3), then $h'$ is a multiple Dehn twist on the collection $\mathcal{C}$ of simple closed curves. It is well-known that Dehn twist homeomorphisms induce linear growth on the conjugacy classes in $\pi_1 S$, while pseudo-Anosov homeomorphisms produce exponential growth. Thus we obtain as direct consequence:

**Corollary 3.3.** (1) A polynomially growing outer automorphism $\hat{\varphi}$ of $F_n$ which has quadratic or higher growth is not geometric.

(2) Any linearly growing $\hat{\varphi}$ which is geometric has a positive power that is induced by a multiple Dehn twist homeomorphism.

### 3.2. Boundary curves of surfaces.

Rather than going directly after the question which automorphism of $F_n$ is geometric, one can first consider the following much easier question:

*When is an outer automorphism $\hat{\varphi}$ of $F_n$ with respect to a fixed identification $F_n = \pi_1 S$ induced by some homeomorphism of $S$?*

This classical question has been answered long time ago by combined work of several well known mathematicians (Dehn, Nielsen, Baer, Fenchel, ...):

**Theorem 3.4** ([16] Theorem 5.7.1 or Theorem 5.7.2). Let $S$ be a (possibly non-orientable) surface with boundary curves $c_1, \ldots, c_k$, and assume $k \geq 1$.

An outer automorphism $\hat{\varphi}$ of $\pi_1(S)$ is induced by a homeomorphism of $S$ if and only if there is a permutation $\sigma$ of $\{1, \ldots, k\}$ and exponents $\varepsilon_i \in \{1, -1\}$ such that $\hat{\varphi}$ maps the conjugacy class determined by $c_i$ to the one determined by $c_{\sigma(i)}^{\varepsilon_i}$.

It remains now to study the possible collections of boundary curves in the fundamental group of a surface. For this purpose one has the following well known result:

**Theorem 3.5.** [16] Let $S$ be a compact surface with boundary. Then there exist a basis $\mathcal{B}$ for the free group $\pi_1 S$ such that the boundary curves of $S$ (up
to reversion of their orientation) determine homotopy classes in $\pi_1 S$ which are given by the following collection $C$ of elements:

1. If $S$ is orientable, then $B = \{s_1, \ldots, s_k, u_1, t_1, \ldots, u_g, t_g\}$ with $k \geq 1$ and $g \geq 0$, and $C = \{s_1, \ldots, s_k, s_1 \cdots s_k \prod_{i=1}^g [u_i, t_i]\}$ (where $[x, y] = xyx^{-1}y^{-1}$).

2. If $S$ is non-orientable, then $B = \{s_1, \ldots, s_k, v_1, \ldots, v_\ell\}$ with $k \geq 1$ and $\ell \geq 0$, and $C = \{s_1, \ldots, s_k, s_1 \cdots s_k v_1^2 \cdots v_\ell^2\}$.

Definition 3.6. Any family $A = \{a_1, \ldots, a_k\}$ of elements $a_i \in F_n$ will be called a boundary family if there is an automorphism of $F_n$ which maps $A$ to a family of elements that are conjugate to the elements of a subset of the collection $C$ as given in case (1) or (2) of the above theorem.

3.3. Whitehead’s Algorithm.

J.H.C. Whitehead invented in the middle of the last century an algorithm which is one of the strongest, of the most interesting, and also one of most studied among all known algorithms. Although originally devised for curves on a handlebody, it was quickly understood that its true character is combinatorial; many improved versions of the algorithm have been published since, but they all rely on Whitehead’s fundamental insights. A relatively moderate version of it is used in this paper:

Theorem 3.7 ([9]). Given two families of elements $A = \{a_1, a_2, \ldots, a_s\}$ and $B = \{b_1, b_2, \ldots, b_s\}$ in $F_n$, it can be decided algorithmically whether or not there exists an outer automorphism $\hat{\varphi} \in \text{Out}(F_n)$ such that $\hat{\varphi}$ maps each conjugacy class $[a_i]$ to $[b_{\sigma(i)}]$, for some permutation $\sigma$.

Combining the above result with Theorem 3.5 we obtain immediately:

Corollary 3.8. There exists an algorithm which decides whether any given finite family of elements of $F_n$ is a boundary family.

4. Special topological representatives

The goal of this section is to derive, for any polynomially growing automorphism $\hat{\varphi} \in \text{Out}(F_n)$, from a relative train track representative of $\hat{\varphi}$ as given by Bestvina-Feighn-Handel, a topological representative with some special properties which are summarized as follows:

Definition 4.1. A self map $f : \Gamma \to \Gamma$ of a graph $\Gamma$ which preserves a filtration $V(\Gamma) = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma$ and induces via some marking isomorphism $F_n \cong \pi_1 \Gamma$ the automorphism $\hat{\varphi} \in \text{Out}(F_n)$ is called a special topological representative of $\hat{\varphi}$ if the following conditions hold:

1. Every connected component of $\Gamma_1$ has non-trivial fundamental group and is pointwise fixed by $f$.

2. Every stratum $H_i = cl(\Gamma_i \setminus \Gamma_{i-1})$ with $i \geq 2$ consists of a single edge $e_i$, with $f(e_i) = w_i e_i u_i$, where $w_i$ and $u_i$ are closed paths in $\Gamma_{i-1}$.
The map $f$ itself will be called a special graph map.

Such a special topological representative can be derived algorithmically from an improved relative train track representative of $\tilde{\varphi}$ given by Bestvina-Handel-Feighn [3]. This will be explained below in detail; we separate the various issues and treat them in disjoint subsections.

4.1. Moving all fixed edges into the bottom stratum.

We first recall from Theorem 2.3 that in [3] it has been shown that for every polynomially-growing outer automorphism $\tilde{\varphi} \in \text{Out}(F_n)$ one can derive algorithmically a topological representative $f : \Gamma \to \Gamma$ with a filtration $V(\Gamma) = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_m = \Gamma$ representing a positive power of $\tilde{\varphi}$ with the following properties:

1. all vertices $v \in V(\Gamma)$ are fixed by the map $f$;
2. every stratum $H_i = \text{cl}(\Gamma_i \setminus \Gamma_{i-1})$ with $i \geq 1$ consists of a single edge $e_i$ such that $f(e_i) = e_iu_i$, where $u_i \subset \Gamma_{i-1}$ is a closed reduced path.

Since we are only interested in the homotopy properties of the filtration and not in the combinatorics, we will now impose the following:

Convention 4.2. We replace the given filtration of $\Gamma$ by a new filtration (denoted homonymously) which has the additional property that all identity edges (i.e. $f(e_i) = e_i$) are assembled in the subgraph $\Gamma_1$, referred from now on to as the bottom subgraph. This is clearly possible simply by relabeling the edge indices, so that we obtain:

2* every stratum $H_i$ with $i \geq 2$ consists of a single edge $e_i$ such that $f(e_i) = e_iu_i$, where $u_i \subset \Gamma_{i-1}$ is a closed non-contractible path;
3. the stratum $H_1$ consists entirely of identity edges.

A connected component of $\Gamma_1$ is called essential if it has non-trivial fundamental group. Otherwise the connected component is contractible and thus called inessential.

4.2. The sliding Operation.

In order to improve the filtered topological representative of $\tilde{\varphi}$ further we first prove in this subsection a general lemma.

Assume $X$ is a topological space, $P, Q, R \in X$ are points in $X$, and $\gamma \subset X$ is a path which joins $Q$ to $R$. Then let $Y = X \cup \{e_1\}$, $Y' = X \cup \{e_2\}$ denote two new topological spaces, where $e_1$ is an edge which joins $P$ to $Q$, $e_2$ is an edge from $P$ to $R$, and both meet $X$ only in their endpoints.

Suppose $f$ and $f'$ are two maps which satisfy the following conditions:

- $f : Y \to Y$ is a map such that $f(X) \subset X$ and $f(e_1) = \beta_0 e_1 \beta_1$, where $\beta_0$ and $\beta_1$ are two paths in $X$.
- $f' : Y' \to Y'$ is a map such that $f'|_X = f$ and $f'(e_2) = \beta_0 e_2 \gamma^{-1} \beta_1 f(\gamma)$.

Lemma 4.3. There is a homotopy equivalence $\kappa : Y \to Y'$ such that the following diagram commutes (up to homotopy):
Proof. Define $\kappa$ as: $\kappa|_X = id_X$, $\kappa(e_1) = e_2\gamma^{-1}$. Let now $\kappa': Y' \to Y$ be a map defined through $\kappa'|_X = id_X$ and $\kappa'(e_2) = e_1\gamma$. Then we can easily verify that $\kappa' \circ \kappa \simeq id_Y$ and $\kappa \circ \kappa' \simeq id_{Y'}$. Thus $Y$ and $Y'$ are homotopy equivalent.

Now we verify the “commutative diagram” $\kappa \circ f \simeq f' \circ \kappa$. The only nontrivial part is to check how they act on $e_1$. We observe:

\[
\begin{align*}
\circ \kappa \circ f(e_1) & \simeq (\beta_0 e_1 \beta_1) \simeq \beta_0 e_2 \gamma^{-1} \beta_1, \\
\circ f' \circ \kappa(e_1) & \equiv f'(e_2\gamma^{-1}) \simeq \beta_0 e_2 \gamma^{-1} \beta_1 f(\gamma) f(\gamma^{-1}) \simeq \beta_0 e_2 \gamma^{-1} \beta_1.
\end{align*}
\]

Thus $\kappa \circ f(e_1) \simeq f' \circ \kappa(e_1)$. Therefore the diagram commutes. \qed

Remark 4.4. In our assumption the edges $e_1$ and $e_2$ are oriented from $P$ to $Q$ and from $P$ to $R$ respectively. However, in the next subsection we also need to work with edges directed conversely, in which case the above proof yields the formulas $\kappa(e_1^{-1}) = \gamma e_2^{-1}$, $f(e_1^{-1}) = \beta_1^{-1} e_1^{-1} \beta_0^{-1}$ and $f'(e_2^{-1}) = f(\gamma)^{-1} \beta_1^{-1} \gamma e_2^{-1} \beta_0^{-1}$.

4.3. Getting rid of the inessential components.

We go now back to our topological representative of $\hat{\varphi}$ as obtained at the end of subsection 4.1, i.e. conditions (1), (2*) and (3) are satisfied. The goal of this subsection is to use the sliding lemma from the previous subsection to get rid of all inessential components in the bottom subgraph $\Gamma_1$ of $\Gamma$.

For this purpose we first note that from condition (2*) it follows for every stratum $H_i$ that the sole edge $e_i$ in $H_i$ is attached at its terminal vertex to an essential component of $\Gamma_{i-1}$. For the initial vertex, however, this is not clear. Let $H_i$ be the lowest stratum such that the initial vertex of $e_i$ is attached to an inessential component $C$ of $\Gamma_{i-1}$.

If no further edge $e_k$ with $k \geq i+1$ is attached to $C$, then we can simply contract $C$ together with $e_i$ to the terminal vertex of $e_i$; the left-over deformation retract is $f$-invariant and hence also a topological representative of $\hat{\varphi}$, with all properties as the original one, but less inessential components in the bottom subgraph.

If any edge $e_k$ with $k \geq i+1$ has its initial vertex attached to $C$, then we define $\gamma$ to be the path in $C \cup e_i$ from the initial vertex of $e_k$ to the terminal vertex of $e_i$. We then use Lemma 4.3 to obtain a homotopy equivalent topological representative of $\hat{\varphi}$ through replacing $e_k$ by an edge $e'_k$ which differs from $e_k$ in that its initial vertex is now equal to the terminal vertex of $e_i$, which lies well in some essential component $C'$. We see from Remark 4.4 that $e'_k$ is mapped to a path $w_k e'_k w_k$, where $w_k = f(\gamma^{-1}) \gamma = u_i^{-1} \gamma^{-1} \gamma$ which reduces to the path $u_i^{-1}$ that is contained in $C'$. After this replacement
we need to adjust all attaching maps of the strata $H_{k'}$ for $k' \geq k + 1$, by
composing the given attaching maps of the edge $e_k'$ in $H_{k'}$ with the homotopy
equivalence $\kappa'$ from Lemma 4.3, followed by a homotopy to guarantee that
the paths $w_{k'}$ and $u_{k'}$ in the obtained image $w_{k'}e_{k'}u_{k'}$ of the edge $e_{k'}$ do not
enter the subgraph $\mathcal{C} \cup e_i$.

We now repeat this operation for any edge with initial vertex in $\mathcal{C}$, until
there is none left over, which allows us to proceed as above in the next to
last paragraph.

After repeating this operation finitely many times no edge will have its
initial edge attached to an inessential component of the bottom subgraph.
Since the homotopy type of the total graph hasn’t changed, and $\Gamma$ was
assumed to be connected, we have proved that in the resulting graph the
bottom subgraph has no inessential connected component. Thus we have
shown:

**Proposition 4.5.** Every polynomially growing automorphism $\hat{\varphi} \in \text{Out}(F_n)$
has a positive power that can be represented by a special topological repre-
sentative.

\[\square\]

## 5. Iterated Dehn twists

Let $\hat{\varphi} \in \text{Out}(F_n)$ be any outer automorphism of $F_n$. The goal of this
section is to derive algorithmically from a special topological representative
of $\hat{\varphi}$ as provided by Proposition 4.5 an iterated Dehn twist representative of
$\hat{\varphi}$. This is a new object which will be defined in this section.

We recall from Definition 4.1 that a special topological representative of
$\hat{\varphi} \in \text{Out}(F_n)$ is given through a graph $\Gamma$ and a filtration $V(\Gamma) = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_m = \Gamma$ and a special graph map $f : \Gamma \to \Gamma$ which preserves
the filtration and induces $f$ via some marking isomorphism $F_n \cong \pi_1 \Gamma$ the
automorphism $\hat{\varphi}$. Here “special graph map” means that furthermore the
following conditions hold:

1. Every connected component of $\Gamma_1$ has non-trivial fundamental group
   and is pointwise fixed by $f$.

2. Every stratum $\Gamma_j \setminus \Gamma_{j-1}$ with $i \geq 2$ consists of a single edge $e_i$, with
   $f(e_i) = u_i e_i v_i$, where $u_i$ and $w_i$ are closed paths in $\Gamma_{j-1}$.

Notice that from condition (1) it follows immediately that $f$ maps every
connected component of any of the invariant subgraphs $\Gamma_i$ of $\Gamma$ to itself.
Notice also that from condition (2) and the connectedness of $\Gamma$ it follows
that $\Gamma_{m-1}$ consists either of a single connected component $\Gamma^0_{m-1}$ or of two
connected components $\Gamma^0_{m-1}$ and $\Gamma^1_{m-1}$. Thus we can always assume that
the initial vertex $v(e_m)$ of the edge $e_m$ is situated in the component $\Gamma^0_{m-1}$.

**Lemma 5.1.** Let $f : \Gamma \to \Gamma$ be a special graph map with respect to a filtra-
tion $V(\Gamma) = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_m = \Gamma$. Assume that for every connected
component $\Gamma^j_{m-1}$ of $\Gamma_{m-1}$ there is given a group $G_j$, a marking isomorphism
\[ \varphi : \pi_1(\Gamma_{m-1}^j, v_j) \to G_j \] (for some vertex \( v_j \in \Gamma_j \)), and a group homomorphism \( \varphi_j : G_j \to G_j \) which satisfies \( \varphi_j = \varphi \circ f_{s,v_j} |_{\pi_1(\Gamma_{m-1}^j, v_j)} \).

Define \( \mathcal{G} \) to be the graph-of-groups which consists of a single edge \( E_m \), as well as a vertex \( V_j \) for each connected component \( \Gamma_{m-1}^j \) of \( \Gamma_{m-1} \), such that the initial or terminal vertex of \( E_m \) is attached to any connected component \( \Gamma_{m-1}^j \) if and only if the corresponding vertex of the only edge \( e_m \) in the stratum \( \Gamma_m \setminus \Gamma_{m-1} \) is attached to the component \( \Gamma_{m-1}^j \). One defines the edge group \( G_{E_m} \) to be trivial, and each vertex group \( G_{V_j} \) to be equal to \( G_j \).

Define a graph-of-groups isomorphism \( H_m : \mathcal{G}_m \to \mathcal{G}_m \) by setting \( H_{E_m} = \text{id} \) and \( H_{V_j} = \varphi_j \), and (after recalling the above convention that \( \iota(e_m) \in \Gamma_{m-1}^j \)) by setting the correction term for the reversed edge \( E_m \) equal to \( \delta(E_m) := \varphi_j(\gamma_{u_m} f(\gamma)) \in G_0 \), where \( \gamma \) denotes a path from \( \tau(\tau_m) \) to the vertex \( v_0 \). The correction term for the edge \( E_m \) is given analogously by \( \delta(E_m) := \varphi_j(\gamma_{w_m} f(\gamma')) \in G_j \) for \( j = 0 \) or \( j = 1 \), where we denote by \( \gamma' \) a path from \( \gamma(e_m) \) to the vertex \( v_j \).

Then the maps \( \psi_0 := \theta_0^{-1} : G_0 \to \pi_1(\Gamma_{m-1}^0, v_0) \subset \pi_1(\Gamma, v_0) \) and \( \psi_1 : G_j \to \pi_1(\Gamma, v_0) \) given by \( \psi_1(g) := \gamma^{-1} \gamma' \theta^{-1}(g) \gamma^{-1} \gamma \), together with the definition \( \psi(t_{E_m}) = \gamma^{-1} \gamma' \) in the case \( V_1 = V_0 \), define an isomorphism \( \psi : \pi_1(\mathcal{G}, V_0) \to \pi_1(\Gamma, v_0) \) which satisfies

\[ f_{v_0} = \psi \circ H_{v_0} \circ \psi^{-1} \]

**Proof.** This is an elementary exercise in chasing through the definition of a graph-of-groups isomorphism (see subsection 2.2) and all the necessary identifications needed there.

\[ \square \]

We note immediately that the graph-of-groups automorphism \( H : \mathcal{G} \to \mathcal{G} \) provided in the last lemma is a partial Dehn twist relative to the full subset \( \mathcal{V} = V(\mathcal{G}) \) of vertices of \( \mathcal{G} \), see subsection 2.2. We’d like to point the reader’s attention here to the fact that the graph-of-groups \( \mathcal{G} \) is indeed minimal: The whole point of introducing “special topological representatives”, and laboring through the previous section in order to get rid of the inessential connected components, is precisely to guarantee this minimality condition.

This gives rise to the following iterative definition:

**Definition 5.2.** We first define an iterated Dehn twist \( D : \mathcal{G} \to \mathcal{G} \) of level \( k = 1 \) to be simply a classical graph-of-groups Dehn twist, see subsection 2.2. For \( k \geq 2 \) we define an iterated Dehn twist \( D : \mathcal{G} \to \mathcal{G} \) of level \( k \) to be a partial Dehn twist relative to \( \mathcal{V} = V(\mathcal{G}) \), where we assume that on each vertex group \( G_v \) of \( \mathcal{G} \) the automorphism \( D_v : G_v \to G_v \) is induced by some iterated Dehn twist \( D^v : G_v \to G_v \) of level \( h_v \leq k-1 \), through some isomorphism \( G_v \cong \pi_1 G_v \).

By formal reasons we count the identity automorphism as iterated Dehn twist of level 0. By definition an iterated Dehn twist of level \( k = 1 \) is precisely
an traditional Dehn twist as defined in subsection 2.2. We see immediately
that an iterated Dehn twist of level $k = 2$ is precisely a partial Dehn twist
relative to a family of local Dehn twists as considered in [15], see subsection
2.4. We can now prove:

**Proposition 5.3.** Every polynomially growing automorphism $\hat{\varphi} \in \text{Out}(F_n)$
has a positive power which can be represented by an iterated Dehn twist $D : \mathcal{G} \to \mathcal{G}$ of some level $k \geq 0$.

The graph-of-groups $\mathcal{G}$ together with all iteratively given data for the vertex
groups as well as the automorphism $D$ can be derived algorithmically from
any special topological representative $f : \Gamma \to \Gamma$, and hence from $\hat{\varphi}$ itself. If $\Gamma$
has $m$ strata, then one obtains $k \leq m - 1$.

**Proof.** The proof of this proposition is done by induction over the number
$m \geq 1$ of strata in the given filtration of the special self map $f : \Gamma \to \Gamma$.

If $m = 1$, then every edge of $\Gamma$ is fixed, so that $\hat{\varphi}$ is the identity automorphism.

We now consider the case $m \geq 2$. As in the situation considered in Lemma
5.1, $\Gamma_{m-1}$ consists of either one or two connected components $\Gamma_m^{j}$, and the
restriction of $f$ to each of them is a special graph map with $m - 1$ or less
strata. Thus we can apply our induction hypothesis to obtain iterated Dehn
 twists $D_j : \mathcal{G}_j \to \mathcal{G}_j$ of level $k_j \leq m - 2$ which represent $f|_{\pi_1 \Gamma_m^{j}}$.

We now apply Lemma 5.1 to derive algorithmically from the edge of the
top stratum of $\Gamma$ together with the iterated Dehn twists $D_j$ for the connected
components of $\Gamma_{m-1}$ the desired partial Dehn twist representative relative
to the family of iterated Dehn twists $D_j : \mathcal{G}_j \to \mathcal{G}_j$ of level $k_j$. This proves
our proposition.

We now come to the main result of this section:

**Theorem 5.4.** Every iterated Dehn twist $D : \mathcal{G} \to \mathcal{G}$ of some level $k \geq 1$
either induces a Dehn twist automorphism on $\pi_1 \mathcal{G}$, or else there are conjugacy
classes in $\pi_1 \mathcal{G}$ which have at least quadratic growth.

This dichotomy can be decided algorithmically.

**Proof.** If $k = 1$, then $D$ is an traditional Dehn twist. For $k \geq 2$ one considers
iteratively any of the vertex groups and descends to some sub-iterated Dehn
twist of level $k = 2$. One then applies the main result of [15] (see Corollary
2.6) which gives precisely the looked-for dichotomy for this sub-iterated Dehn
twist. In case where a conjugacy class with at least quadratic growth is
found, the proof is finished. In the other case one proceeds with the next
sub-iterated Dehn twist of level $k = 2$. If all of them are induce Dehn twist
automorphisms, we can replace all of the given data for the sub-iterated Dehn
twists of level 2 by traditional Dehn twists, i.e. iterated Dehn twists
of level 2. This lowers the level of the total iterated Dehn twist from $k$
to $k - 1$. Hence proceeding iteratively proves the claim.

$\square$
From the last theorem and the previously derived material we obtain:

**Corollary 5.5.** Every polynomially growing automorphism $\widehat{\phi} \in \text{Out}(F_n)$ has either polynomial growth of degree $d \geq 2$ or else has a positive power which is a Dehn twist automorphism. This dichotomy can be decided algorithmically.

Since every Dehn twist automorphism is known to have linear growth, this also shows Corollary [1.4] from the Introduction.

### 6. Surface homeomorphisms

Given any collection $C = \{c_1, \ldots, c_r\}$ of pairwise disjoint non-parallel essential curves $c_i$ with annulus neighborhood in a possibly non-orientable compact surface $S$ with or without boundary, it is well known that $C$ together with the complementary surfaces $S_j \subset S$ define a graph-of-groups $G_C$: the underlying graph $\Gamma(G_C)$ has a vertex $v_j$ for each $S_j$, and for each $c_i$ an edge $e_i$ which can be thought of as “transverse” to $c_i$, in the sense that $e_i$ connects the two vertices $v_j$ and $v_{j'}$ which correspond to the two subsurfaces $S_j$ and $S_{j'}$ adjacent to $c_i$. For the vertex groups one sets canonically $G_{v_j} = \pi_1 S_j$, and for the edge groups $G_{e_i} = \pi_1 c_i \cong \mathbb{Z}$, and the edge injections are induced by the topological inclusions of the $c_i$ as boundary curves of the $S_j$. Van Kampen’s theorem then gives directly:

$$\pi_1 G_C = \pi_1 S$$

For any surface homeomorphism $h : S \to S$ one now uses the groundbreaking Nielsen-Thurston classification for mapping classes (see section 3.1), to obtain a collection $C = C(h)$ of essential simple curves as above, so that (after possibly replacing $h$ by a positive power) every $c_i$ and every $S_j$ is fixed by $h$. More precisely, after improving $h$ by an isotopy, the restriction $h_j : S_j \to S_j$ of $h$ is either the identity homeomorphism, or else it is a pseudo-Anosov automorphism, and furthermore $h$ twists around each $c_i$ an integer number $n_i$ of times. We summarize this in the following well known consequence of Nielsen-Thurston theory:

**Proposition 6.1.** Let $h : S \to S$ be a homeomorphism of a surface $S$ as above. Assume that $S$ has at least one boundary component, and let $\widehat{\phi} \in \text{Out}(F_n)$ be induced by $h$ via some identification isomorphism $\pi_1 S \cong F_n$.

1. If any of the canonical subsurface restrictions $h_j : S_j \to S_j$ of a suitable positive power $h^t$ of $h$ is pseudo-Anosov, then $\widehat{\phi}$ has exponential growth.
2. If none of the $h_j : S_j \to S_j$ is pseudo-Anosov, then $h^t$ is a multiple surface Dehn twist, and $\widehat{\phi}^t$ is a Dehn twist automorphism.

Indeed, in case (2) of the above proposition, the multiple surface Dehn twist $h^t$ gives immediately rise to a graph-of-groups Dehn twist $D_h : G_C(h) \to G_C(h)$ on the above graph-of-groups decomposition of $\pi_1 S$ dual to the collection $C(h)$. In this case we adopt the convention that any of the curves $c_i \in C$
on which $h^t$ doesn’t twist at all is dropped from the collection $\mathcal{C}$, so that the twist exponent of every $c_i$ satisfies $n_i \neq 0$.

We now observe:

**Lemma 6.2.** The above Dehn twist $D_h : \mathcal{G}_{\mathcal{C}(h)} \to \mathcal{G}_{\mathcal{C}(h)}$ is efficient.

**Proof.** We go through the list of properties of an efficient Dehn twist as stated in Definition 2.4:

1. $\mathcal{G}$ is minimal: Suppose by contradiction that there exists a valence 1 vertex $v$ with $\tau(e_i) = v_j$, and a surjective edge homomorphism $f_{e_i} : G_{e_i} \to G_{v_j}$, i.e. we have $G_{v_j} \cong G_{e_i} \cong \mathbb{Z}$. This implies that the vertex $v_j$ corresponds to a subsurface $S_j$ which is an annulus. But then $S_j$ has a second boundary curve, so that $v_j$ would be of valence 2; a contradiction.

2. No invisible vertex: By construction of the graph-of-groups $\mathcal{G}$, an invisible vertex only occurs when there exist two simple closed curves $c_i$ and $c_i'$ are parallel to each other, which is again a contradiction to our assumption.

3. No unused edges: This derives from our above convention that each twist exponent $n_i \neq 0$, since every twistor $z_{e_i}$ is non-trivial.

4. No proper power: Given the fact that each $c_i$ is an essential simple closed curve in $S$, the induced edge homomorphism $f_{e_i}$ must map the generator $g_{e_i}$ of edge group $G_{e_i} = \langle g_{e_i} \rangle$ to an indivisible element in $G_{v_j}$, where $\tau(e_i) = v_j$. The fact that the element $f_{e_i}(g_{e_i})$ is indivisible, i.e. it doesn’t have a proper root in the group $G_{v_j} = \pi_1 S_j$, is a classical fact which can be derived for example from the uniqueness of geodesics in surfaces of constant negative curvature.

5. Whenever two edges $e_i$ and $e_i'$ end at the same vertex $v_j$, then $e_i$ and $e_i'$ are not positively bonded.

Indeed, $e_i$ and $e_i'$ are neither positively nor negatively bonded, as in either case the corresponding boundary curves in the subsurface $S_j$ corresponding to $v_j$ would have to be parallel, which contradicts our assumptions.

As a consequence of the last lemma we can now test efficiently whether a given Dehn twist automorphism $\hat{\varphi} \in \text{Out}(F_n)$ is geometric or not: It suffices to decide whether or not for each vertex group $G_v$ of some efficient Dehn twist representative $D : \mathcal{G} \to \mathcal{G}$ of $\hat{\varphi}$ the family of twistors of the edges adjacent to $v$ define a “boundary family” as has been introduced in Definition 3.6. We recall:

A family $w_1, \ldots, w_r \in F_n$ of elements is called a boundary family if there exists a surface $S$ with boundary curves $c_1, \ldots, c_s$ with $s \geq r$ such that for some identification isomorphism $\theta : F_n \to \pi_1 S$ one has (up to a permutation of the indices of the $c_j$) that every $\theta(w_i)$ determines the conjugacy class given by $c_i$ in $\pi_1 S$.

**Proposition 6.3.** An automorphism $\hat{\varphi} \in \text{Out}(F_n)$ represented by an efficient Dehn twist $D : \mathcal{G} \to \mathcal{G}$ is geometric if and only if for every vertex
group $G_v$ of $\mathcal{G}$ and for the family of edges $e_i \in E(\mathcal{G})$ with terminal vertex $\tau(e_i) = v$ the corresponding elements $f_{e_i}(g_{e_i})$ constitute a boundary family in $G_v$. Here $g_e$ denotes a generator of the cyclic group $G_e$.

Proof. If $\hat{\varphi}$ is geometric, then for some surface $S$, some identification isomorphism $\pi_1 S \simeq F_n$ and some homeomorphism $h : S \to S$ we have $\hat{\varphi} = h_*$. We now apply Proposition [6.1] to $h$. Its alternative (1) is ruled out by our hypothesis that $\hat{\varphi}$ is a Dehn twist automorphism, as those are known to grow linearly (see Theorem [3.1] and Corollary [3.3]). From alternative (2) we obtain a Dehn twist representative $D_h : G_{C(h)} \to G_{C(h)}$ of some positive power of $h$, which by Lemma [6.2] is efficient. From the uniqueness for efficient Dehn twist representatives (see Theorem [2.5]) we obtain a graph-of-groups isomorphism $H : G_{C(h)} \to G$ such that $D_h = H^{-1}D'H$ for some $t \geq 1$. By definition for any vertex group $G_j$ of $G_{C(h)}$ the adjacent edge group generators define a boundary family in $G_j$. Since this property is preserved by the isomorphism $H$, we have shown the “only if” part of the claim.

To show the “if” direction of the claimed equivalence we use, for every vertex $v$ of $\mathcal{G}$, the surface $S_v$ given by the hypothesis on $G_v$ and by Definition [3.6] to construct a surface $S$ by “tubing together” the $S_v$ along annuli with core curve $e_i$ as prescribed by the edges $e_i$ of the underlying graph $\Gamma(\mathcal{G})$. From the choice of the generator of each edge group $G_{e_i} \cong \mathbb{Z}$ we deduce the sign $\varepsilon_i \in \{1, -1\}$, which together with the twist exponents $n(e_i)$ of $D$ define the homeomorphism $h : S \to S$ as multiple Dehn twist which twists at any $e_i$ precisely $\varepsilon_i n(e_i)$ times. It follows directly from this construction that there is a canonical identification isomorphism $\pi_1 S = F_n$ which induces $h_* = \hat{\varphi}$. 

\[ \square \]

From the last proposition we obtain directly:

**Corollary 6.4.** The geometricity question for Dehn twist automorphisms of $F_n$ can be algorithmically decided if for any finite family $W$ of elements in a free group $F_m$ it can be algorithmically decided whether $W$ is a boundary family or not.

\[ \square \]

We now obtain as direct consequence of the last result together with Corollary [3.8] a direct proof of Corollary [1.4] from the Introduction.

7. The Algorithm

In this section we describe concretely the algorithm which decides whether a given automorphism $\hat{\varphi} \in \text{Out}(F_n)$ is of polynomial growth, and whether its power $\hat{\varphi}^t$ is geometric or not.

**Step 1.** We assume that $\hat{\varphi}$ is given as usually through specifying for some basis $B$ of $F_n$ and some representative $\varphi \in \text{Aut}(F_n)$ of $\hat{\varphi}$ the $\varphi$-images of the basis elements as words in $B \cup B^{-1}$. 

It is shown in [3] how to derive from these data an improved relative train track representative \( f : \Gamma \rightarrow \Gamma \) of \( \hat{\varphi} \), which has furthermore the property that either it contains an exponentially growing stratum, or else its \( t_n \)-th power has the conditions specified in Theorem 2.3. Since these conditions are easy to check in finite time, at this point we detect whether \( \hat{\varphi} \) is of polynomial growth or not.

In case of a positive answer, we replace for convenience from now on \( \hat{\varphi} \) by \( \hat{\varphi}^t \).

**Step 2.** We now transform \( f : \Gamma \rightarrow \Gamma \) into a special topological representative of \( \hat{\varphi} \) as specified in Definition 4.1. For this we follow exactly the procedure explained in section 4: One first relabels the edges so that every fixed edge now belongs to the bottom stratum. One then moves up from the bottom through all strata, and each time when a stratum \( H_k \) consists of an edge \( e_i \) which has its initial vertex at some inessential component \( C \) of \( H_{i-1} \), one performs the sliding operation defined in subsection 4.2 to first replace for any \( k \geq i + 1 \) an edge \( e_k \) with initial vertex in \( C \) by an edge \( e'_k \) that has initial vertex in an essential component. As explained in subsection 4.3, after each such replacement one has to adjust the attaching maps of any edge \( e_k' \) with \( k' \geq k + 1 \) by a homotopy, using the data given concretely by the performed sliding operation. As final operation of this step, after having gone through all strata \( H_k \) with \( k \geq i + 1 \), we erase the component \( C \) together with the edge \( e_i \) from the resulting graph.

After finitely many of those operations one has eliminated all inessential components in any subgraph of the given filtration, so that \( f : \Gamma \rightarrow \Gamma \) is now a special topological representative of \( \hat{\varphi} \).

**Step 3.** The next objective would be to derive from the special topological representative \( f : \Gamma \rightarrow \Gamma \) an iterated Dehn twist representative of \( \hat{\varphi} \), as explained in section 5. This is, however, not the most efficient way, from an algorithmic standpoint.

Instead we consider only the subgraph \( \Gamma_2' \) of \( \Gamma \) which is the connected component of \( \Gamma_2 \) that contains the edge \( e_2 \). This subgraph is \( f \)-invariant, and (as shown in the proof of Proposition 5.3) it defines a partial Dehn twist \( D_2 : \mathcal{G}_2 \rightarrow \mathcal{G}_2 \) relative to \( V(\mathcal{G}_2) \), where \( \mathcal{G}_2 \) consists of a single edge \( E_2 \), every vertex group of \( \mathcal{G}_2 \) is given by the fundamental group of the corresponding connected component of \( \Gamma_2' \cap \Gamma_1 \) (there are either one or two such components), and the correction terms of \( E_2 \) and \( \overline{E}_2 \) are given by \( u_2^{-1} \) and \( w_2 \) respectively, for \( f(e_2) = w_2 e_2 u_2 \).

Since \( f \) acts as identity on \( \Gamma_1 \), it follows immediately that \( D_2 \) is an traditional graph-of-groups Dehn twist. At this point we apply the following:

**Subalgorithm I:** In [7] an algorithm is described that transforms any given Dehn twist into an efficient Dehn twist. We apply this to \( D_2 \), so that from now on we can assume that \( D_2 : \mathcal{G}_2 \rightarrow \mathcal{G}_2 \) is efficient.
We now pass to $\Gamma_3$ and proceed precisely as before for $\Gamma_2$: From $\Gamma'_3$ we construct algorithmically a partial Dehn twist $D_3 : G_3 \to G_3$, for which there are two possibilities: If $\Gamma'_3$ and $\Gamma'_2$ are disjoint, then we are exactly in the same situation as before, so that in this case we obtain an efficient Dehn twist $D_3 : G_3 \to G_3$.

If, on the other hand, $\Gamma'_3$ and $\Gamma'_2$ are not disjoint, then $D_3 : G_3 \to G_3$ is an iterated Dehn twist of level 2, or in other words, a partial Dehn twist relative to a family of local Dehn twists. In this case we consider the correction terms of $E_3$ and of $\overrightarrow{E}_3$ in the adjacent vertex groups $G_i$, on which $D_3$ acts as (possibly trivial) efficient Dehn twist $D'_i$. We thus can pass to the following:

**Subalgorithm II ([15])** : For any efficient Dehn twist $D' : G' \to G'$ it can be decided whether (for any vertex $v$ of $G'$) a given element $w \in \pi_1(G',v) \subset \Pi(G')$ is $D'$-conjugate in $\Pi(G')$ to an element of $G'$-length 0.

If one of the correction terms of $D_3$ is not locally zero (i.e. Subalgorithm II gives a negative answer), then it follows from the main result of [15] (see Corollary 2.7) that the automorphism induced by $D_3$ and thus $\tilde{\varphi}$ has at least quadratic growth. In this case we know that $\tilde{\varphi}$ is not geometric (see Corollary 3.3).

(Note that this is the place where we crucially need that there is no inessential component in the bottom subgraph $\Gamma_1$, as otherwise it could happen that $G_3$ is not minimal, and in this case Corollary 2.7 would fail to hold.)

If, on the other hand, all correction terms are locally zero, we can pass to the following:

**Subalgorithm III ([14])**: Every partial Dehn twist $D : G \to G$ relative to a family of local Dehn twists, for which all correction terms are locally zero, induces on $\pi_1G$ an traditional Dehn twist automorphism. A graph-of-groups Dehn twist representative $D'$ of the outer automorphism $\tilde{D}$ can be derived algorithmically from $D$.

Thus we can transform $D_3$ algorithmically first into an traditional Dehn twist, and then apply Subalgorithm I to make it efficient.

We then pass to $\Gamma_4$, and repeat the above procedure iteratively, going through all strata of $\Gamma$. As a result we either obtain that $\tilde{\varphi}$ has at least quadratic growth and hence is not geometric, or else we have derived an efficient Dehn twist representative $D : G \to G$ for $\tilde{\varphi}$.

**Step 4.** We now turn to section [6]. It has been shown in Proposition [6.3] that $\tilde{\varphi}$ is geometric if and only if for every vertex group $G_v$ of $G$ the family of edge group generators for the edges adjacent to the vertex $v$ define a boundary family in $G_v$. This is a question that can be decided algorithmically through the Whitehead algorithm, see Corollary [6.4]. This finishes the algorithm.
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