The internal nonlocality in general dilated Hermiticity

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\(\mathcal{PT}\)-symmetric systems can be viewed as effective models in the sense of open systems. This allows people to simulate a \(\mathcal{PT}\)-symmetric system by dilating it to a global Hermitian one. Recently, the internal nonlocality in such dilated Hermitian systems was discussed for a special case with some two fold structure. In this paper, we extend such a discussion to the general case, by utilizing a general property of Hermitian dilation Hamiltonians. Different correlation pictures are proposed and the corresponding correlation bounds are obtained.

I. INTRODUCTION

In recent years, researchers have witnessed a growing interest in discussing parity-time (\(\mathcal{PT}\))-symmetric systems. Since Bender and his colleagues' discussion of \(\mathcal{PT}\)-symmetry in 1998 \cite{Bender1}, lots of theoretical and experimental applications of \(\mathcal{PT}\)-symmetric systems were found \cite{Kovacic2012, Alba2019}. The concept of \(\mathcal{PT}\)-symmetry was also generalized to the pseudo-Hermiticity \cite{Bender2007, Albeverio2010}, and anti-\(\mathcal{PT}\)-symmetry \cite{Bender2007, Almeida2008}. Recently, the discussion of \(\mathcal{PT}\)-symmetry extended to the field of dynamics and topology \cite{Pereira2015}.

Similar to the Feshbach formalism dealing with an effective description \cite{Bender2007}, \(\mathcal{PT}\)-symmetric systems can be viewed as effective models in the sense of open systems. In 2008, Günther and Samsonov showed that a class of unbroken two-dimensional \(\mathcal{PT}\)-symmetric Hamiltonians can always be dilated to some four-dimensional Hermitian ones \cite{Guenther2008}. In fact, by using the dilation techniques, one can simulate any finite dimensional unbroken \(\mathcal{PT}\)-symmetric systems in dilated Hermitian systems \cite{Guenther2008, Albeverio2010}. By evolving states under the Hermitian dilation Hamiltonians, it is always possible to simulate the evolution of unbroken \(\mathcal{PT}\)-symmetric Hamiltonians in subspaces. On the other hand, for broken \(\mathcal{PT}\)-symmetric systems, the simulation of their evolutions can also be simulated by utilizing time dependent Hermitian Hamiltonians \cite{Guenther2008}.

In the simulation of \(\mathcal{PT}\)-symmetric systems, the Hermitian dilation Hamiltonians play an important role, which governs a composite system. By projecting the Hermitian dilation Hamiltonians to some subsystems, the effect of \(\mathcal{PT}\)-symmetric Hamiltonians can be realized \cite{Guenther2008}. Owing to the non-Hermiticity of \(\mathcal{PT}\)-symmetric systems, the Hermitian dilation Hamiltonians usually bring nonlocal correlations between the subsystems. Recently, by proposing different correlation pictures, the internal nonlocality of Hermitian dilation Hamiltonians were discussed \cite{Guenther2008}. By evaluating the correlations with local measurements in three different pictures, the resulting different expectations of the Bell operator reveal the distinction of the internal non-locality. Such a result provides the figure of merit to test the reliability of the simulation, as well as to verify a \(\mathcal{PT}\)-symmetric (sub)system. However, the discussions mainly focus on Günther and Samsonov's special example and the discussion relies on the two fold structure of the dilation Hamiltonian. The problem is, a two fold structure may not exist in generic Hermitian dilation Hamiltonian. Then how can we discuss the internal nonlocality in the general case?

In this paper, we propose a generalization of the scenario in \cite{Guenther2008}. Different correlation pictures are proposed. The expectations of the Bell operator and their bounds are obtained. In particular, it is shown that the local Hermitian picture has two natural generalisations and the correlation behaviours are more complex and have new features. The remainder of this paper is organized as follows. In Sec. II, we introduce the preliminaries on the related notions of \(\mathcal{PT}\)-symmetric systems, the CHSH (Clauser, Horne, Shimony, and Holt) scenario and the results in \cite{Guenther2008}. In Sec. III, we propose different correlation pictures for general Hermitian dilations. The expectations of the Bell operator and their bounds are obtained. In section IV, some discussions are made. Finally, we conclude our results in Sec. V.

II. PRELIMINARIES

A. Basic notions of \(\mathcal{PT}\) symmetry

A parity operator \(\mathcal{P}\) is a linear operator such that \(\mathcal{P}^2 = \mathcal{I}_d\), where \(\mathcal{I}_d\) is the identity operator on \(\mathbb{C}^d\).

A time reversal operator \(\mathcal{T}\) is an anti-linear operator such that \(\mathcal{T}^2 = \mathcal{I}_d\). Moreover, \(\mathcal{PT} = \mathcal{T}\mathcal{P}\).

A linear operator \(\mathcal{H}\) is said to be \(\mathcal{PT}\)-symmetric if \(\mathcal{H}\mathcal{PT} = \mathcal{PT}\mathcal{H}\). Moreover, a \(\mathcal{PT}\)-symmetric operator \(\mathcal{H}\)
leading to a neat and more symmetric form of $\hat{H}$.

Apparently, there exists some freedom in the choice of $\tau_{18}$. The final operator is said to be unbroken if it is similar to a real Hamiltonian. By dilating $H$ to a Hermitian Hamiltonian, we can find some Hermitian operator $\hat{H} = \begin{bmatrix} H_1 & H_2 \\ H_2^\dagger & H_4 \end{bmatrix}$ and $\tau$ such that for any vector $\psi$,

$$\begin{bmatrix} i\psi' \\ i(\tau\psi)' \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_2^\dagger & H_4 \end{bmatrix} \begin{bmatrix} \psi \\ \tau\psi \end{bmatrix} = \begin{bmatrix} H\psi \\ \tau H\psi \end{bmatrix}. \quad (1)$$

Note that for the first component, $i\psi' = H\psi$. According to the Schrödinger equation,

$$\psi(t) = e^{-iHt}\psi(0),$$

the effect of a $\mathcal{PT}$-symmetric systems is realized. If we further denote

$$\eta = I + \tau^\dagger\tau,$$

then it follows that

$$H^\dagger\eta = \eta H. \quad (3)$$

Usually, $\eta$ is called a metric operator and $\hat{H}$ is a Hermitian dilation Hamiltonian of $H$. Once the metric operator $\eta$ is given, one can construct the operator $\tau$. For convenience, $\tau$ is often chosen to be Hermitian, that is, $\tau = \sqrt{\eta - T}$.

For an unbroken $\mathcal{PT}$-symmetric Hamiltonian $H$, it can be proved that such metric operators and dilation Hamiltonians always exist and are generally not unique. In fact, Eq. (11) shows that

$$\begin{align*}
H_2 &= (H - H_1)\tau^{-1}, \\
H_4 &= (\tau H - H_1^\dagger)\tau^{-1}.
\end{align*} \quad (4-5)$$

Apparently, there exists some freedom in the choice of $H_1$, whose only constraint is Hermiticity. One may take

$$H_1 = (H\tau^{-1} + \tau H)(\tau^{-1} + \tau)^{-1}. \quad (6)$$

It can be verified that $H_1$ is Hermitian and $H_3 = H_4$, leading to a neat and more symmetric form of $\hat{H}$:

$$\begin{align*}
\hat{H} &= I_2 \otimes \Lambda + i\sigma_y \otimes \Omega, \\
\Lambda &= (H\tau^{-\frac{1}{2}} + \tau\frac{1}{2}H)(\tau^{-\frac{1}{2}} + \tau\frac{1}{2})^{-1}, \\
\Omega &= (H - \tau\frac{1}{2}H\tau^{-\frac{1}{2}})(\tau^{-\frac{1}{2}} + \tau\frac{1}{2})^{-1},
\end{align*} \quad (7-9)$$

in which the details about $\tau$ can be referred to Refs. 16-18.

C. Two dimensional model

A widely used two dimensional $\mathcal{PT}$-symmetric Hamiltonian is as follows [16, 21],

$$H = E_0I_2 + s \begin{bmatrix} i\sin\alpha & 1 \\
1 & -i\sin\alpha \end{bmatrix}. \quad (10)$$

The corresponding eigenvalues for this two dimensional non-Hermitian system are $\lambda_{\pm} = E_0 \pm s\cos\alpha$. Moreover, there exists an exceptional point when $\alpha = \pm \frac{\pi}{2}$, in which case the Hamiltonian is no longer diagonalized. When $\alpha \neq \pm \frac{\pi}{2}$, the Hamiltonian $\hat{H}$ has real eigenvalues and can be diagonalized. Hence, $\mathcal{PT}$-symmetry is unbroken. In particular, when $\alpha = 0$, the Hamiltonian is also Hermitian.

For the $\mathcal{PT}$-symmetric Hamiltonian in Eq. (10), a typical Hermitian dilation $\hat{H}$ is

$$\hat{H} = I_2 \otimes H_1 + i\sigma_y \otimes H_2, \quad (11)$$

where

$$\begin{align*}
H_1 &= E_0I_2 + \frac{\omega_0}{2}\cos\alpha\sigma_x, \\
H_2 &= \frac{i\omega_0}{2}\sin\alpha\sigma_z, \\
\omega_0 &= 2s\cos\alpha,
\end{align*} \quad (12-14)$$

and

$$\tau = \frac{1}{\cos\alpha} \begin{bmatrix} 1 & -i\sin\alpha \\
i\sin\alpha & 1 \end{bmatrix} \quad (15)$$

[16, 17]. It can be verified that the above example is a special case of Eqs. [16, 17]. According to Eq. (11), the Hermitian dilation Hamiltonian $\hat{H}$ is inseparable. That is, $\hat{H}$ cannot be written as a tensor product of two local operators. As a consequence, such a global Hamiltonian $\hat{H}$ can bring nonlocal correlations to the subsystems. Another interesting observation is that $\hat{H}$ is isospectral to $H$. It means that the eigenvalues of $\hat{H}$ and $H$ are the same, but the Hermitian dilation Hamiltonian has a two fold spectra, i.e., the multiplicities of eigenvalues of $\hat{H}$ are two. Such a property implicitly allows us to use the measurements on the large space to simulate the measurements of the $\mathcal{PT}$-symmetric system.

Briefly speaking, by measuring the Hermitian dilation Hamiltonian $\hat{H}$, one can read out the eigenvalues of the $\mathcal{PT}$-symmetric system.

D. CHSH Scenario

Here we briefly recall the elements of CHSH inequality [22, 23]. In a standard Bell’s test on non-locality, two (sub)systems shared by Alice and Bob are spatially separated. By performing local measurements, Alice obtains several possible outcomes from her subsystem, denoted
as $a$, with the outcomes denoted as $b$ from Bob’s measurements on his subsystem. The CHSH scenario focuses on the correlations of Alice’s and Bob’s outcomes.

Suppose Alice can perform two local measurements denoted as $A_i, i \in \{0, 1\}$; while Bob can also perform two local measurements $B_j, j \in \{0, 1\}$. The possible outcomes of $A_i$ and $B_j$ have two values labeled $a, b \in \{+1, -1\}$. Now, let $\langle A_i B_j \rangle = \sum_{a,b} ab p(ab|ij)$ be the expectation value of the product $ab$ for given measurements $A_i B_j$. Here, $A_i B_j$ is often called the correlation function. With these notions, one can further define the following Bell operator,

$$S = A_0 B_0 + A_1 B_0 + A_0 B_1 - A_1 B_1. \quad (16)$$

Under the assumption of classical(local) correlations, one can obtain the following inequality

$$\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle \leq 2, \quad (17)$$

which is known as the (classical) CHSH inequality.

However, the two subsystems measured by Alice and Bob can also be quantum. Suppose their subsystems are two qubits in the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, where $|0\rangle$ and $|1\rangle$ are the eigenstates of $\sigma_z$ for the eigenvalues of $+1$ and $-1$. Suppose that the $A_0$ and $A_1$ correspond to the measurements of spin in the orthogonal directions $e_0$ and $e_1$, respectively. Similarly, $B_0$ and $B_1$ correspond to the measurements in the directions $-\frac{1}{\sqrt{2}}(e_0 + e_1)$ and $\frac{1}{\sqrt{2}}(-e_0 + e_1)$. Then it follows that

$$\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2}, \quad (18)$$

as $\langle A_0 B_0 \rangle = \langle A_1 B_0 \rangle = \langle A_0 B_1 \rangle = -\langle A_1 B_1 \rangle = \frac{1}{\sqrt{2}}$. Eq. (18) is actually the quantum bound and is larger than the classical bound, showing the difference between the classical(local) and nonlocal correlations.

E. The internal non-locality in simulating PT-symmetric systems

Inspired by the CHSH scenario, one can propose a CHSH-like discussion on the nonlocal correlations introduced by the Hermitian dilatlon Hamiltonian of Eq. (11). Since the nonlocal correlations come from the global Hamiltonian rather than the states, hence we call it internal non-locality to distinguish it from the standard CHSH scenario. The three pictures to discuss internal non-locality are as follows [20].

The Simulation picture

In the simulation picture, the Hermitian dilatlon Hamiltonian $\hat{H}$ is assumed to be shared by Alice and Bob. Since we are now discussing the correlations introduced by the Hamiltonian, Alice and Bob will use local states instead of local measurements to obtain outcomes. Suppose Alice makes two local measurements denoted as $A_0$ and $A_1$; while Bob also makes two local measurements $B_0$ and $B_1$. Let Alice have the local state $\{ |u_+ \rangle = u(0) + v|1\rangle \}$ for $A_0$ and $\{ |u_- \rangle = |\pi(0) - \pi|1\rangle \}$ for $A_1$; while Bob have two local states $\{ |0\rangle \}$ and $\{ |1\rangle \}$ for $B_0$ and $B_1$ respectively. Then the expectations of $B_i A_j$ can be calculated as follows:

$$\langle B_0 A_0 \rangle = Tr(|0\rangle \langle 0| \otimes |u_+\rangle \langle u_+|)\hat{H}, \quad (19)$$
$$\langle B_1 A_0 \rangle = Tr(|1\rangle \langle 1| \otimes |u_+\rangle \langle u_+|)\hat{H}, \quad (20)$$
$$\langle B_0 A_1 \rangle = Tr(|0\rangle \langle 0| \otimes |u_-\rangle \langle u_-|)\hat{H}, \quad (21)$$
$$\langle B_1 A_1 \rangle = Tr(|1\rangle \langle 1| \otimes |u_-\rangle \langle u_-|)\hat{H}. \quad (22)$$

Now, one can further consider the expectation value of the Bell operator:

$$\langle B_0 A_0 \rangle + \langle B_0 A_1 \rangle + \langle B_1 A_0 \rangle - \langle B_1 A_1 \rangle$$
$$= Tr(2|0\rangle \langle 0| \otimes (\hat{H}_1 + \hat{H}_2) + |u_+\rangle \langle u_+| u_+ \rangle - |u_-\rangle \langle u_-| u_- \rangle)$$
$$= 2E_0 + \omega_0 \sqrt{2} \cos \alpha. \quad (24)$$

For the last term shown in Eq. (24), we have

$$|\omega_0 + \sqrt{2} \cos \alpha| \leq 2s \cos^2 \alpha, \quad (25)$$

The Classical picture

The classical picture means that one skips the details of quantum mechanics but only considers a classical description of what Alice and Bob do. There are several key points in such a classical description. Firstly, the classical picture should be consistent with the simulation picture. It requires that Alice has a “PT-symmetric like” subsystem and the joint measurements of Alice and Bob depict the characteristics of measuring the global Hamiltonian $\hat{H}$. A natural consequence is to assume the measurement results of $A_j$ are just $A_{\pm}$, namely the eigenvalues of the $\mathcal{T}$-symmetric Hamiltonian $\hat{H}$. Moreover, note that the Hermitian dilatlon Hamiltonian $\hat{H}$ has the same eigenvalues as the $\mathcal{T}$-symmetric Hamiltonian $\hat{H}$ but with a multiplicity of two. Hence the results of $B_i$ should be 1, such that the correlation functions $B_i A_j$ trivially give the eigenvalues of $\hat{H}$. Secondly, the “results” of Alice and Bob are independent, leading to a classical (nonlocal) correlation. In fact, since Bob’s results always give 1, apparently the two observers’ results and the corresponding probability distributions are independent. Thus, we do have a classical local picture.

Now the expectation of Bell operator is as follows,

$$\langle B_0 A_0 \rangle + \langle B_0 A_1 \rangle + \langle B_1 A_0 \rangle - \langle B_1 A_1 \rangle$$
$$= \int |B_0(\nu)\langle A_0 + A_1\rangle(\nu) + B_1(\nu)\langle A_0 - A_1\rangle(\nu)|d\nu$$
$$= \int |(A_0 + A_1)(\nu) + (A_0 - A_1)(\nu)|d\nu$$
$$= 2E_0 + \omega_0 (p_+ - p_-). \quad (26)$$
where $p_{\pm}$ are the probabilities corresponding to the situations when the results of $A_0$ are $\lambda_{\pm}$.

Local Hermitian picture

In this picture, we try to give a description of what Alice and Bob do by some Hermitian Hamiltonian $\hat{H}'$, which is in a tensor product form of two local Hermitian Hamiltonians. In contrast to $\hat{H}$, the form of $\hat{H}'$ implies that it will not introduce nonlocal correlations between the subsystems. To have this local Hermitian picture being consistent with the simulation, one can assume that $\hat{H}'$ has the same eigenvalues as $\hat{H}$ and one of the local Hamiltonians has the same eigenvalues as $\hat{H}$. Hence, we have $\hat{H}' = I \otimes H_h$, where $H_h = \lambda_+ |s_+\rangle \langle s_+| + \lambda_- |s_-\rangle \langle s_-|$ and $|s_{\pm}\rangle$ are two orthogonal states.

Again, by substituting the $\hat{H}'$ in the local Hermitian picture to Eqs. (19)-(22), the expectation of the Bell operator is

\[
(B_0 A_0) + \langle B_1 A_0 \rangle + \langle B_0 A_1 \rangle - \langle B_1 A_1 \rangle = \text{Tr}(I \otimes |u_+\rangle \langle u_+|)(I \otimes H_h) + \text{Tr}([|0\rangle \langle 0| - |1\rangle \langle 1|] \otimes |u_+\rangle \langle u_+|)(I \otimes H_h),
\]

which can be further reduced to

\[
2\langle u_+|H_h|u_+\rangle = 2\lambda_+ \langle |u_+|s_+\rangle|^2 + 2\lambda_- \langle |u_+|s_-\rangle|^2.
\]

As $\lambda_{\pm} = E_0 \pm \frac{\alpha}{\sqrt{2}}$, we can denote $p_{\pm} = \langle |u_+|s_{\pm}\rangle|^2$ and reach

\[
2E_0 + \omega_0 (p_+ - p_-). \tag{29}
\]

By comparing Eq. (29) with Eq. (21) and Eq. (23), all the expectations in the three pictures contain two terms. The common term $2E_0$ is the sum of the two eigenvalues $\lambda_+$ and $\lambda_-; while the other one represents a deviation term. This deviation term is the same for the classical and local Hermitian pictures. Moreover, we also have

\[
|\omega_0 (p_+ - p_-)| = |2s(p_+ - p_-)\cos\alpha| \leq |2s\cos\alpha|, \tag{30}
\]

which means that these two pictures give a larger value of the upper bound than that obtained in the simulation picture. Such a result can help to distinguish the Hermitian dilation Hamiltonian.

In the above discussions, the special form of the Hermitian dilation Hamiltonian implicitly plays a key role. In particular, $\hat{H}$ has completely the same eigenvalues as $\hat{H}$, which makes it possible to readout the eigenvalues of the $\mathcal{PT}$-symmetric Hamiltonian by measuring the Hermitian dilation Hamiltonian. This renders a reasonable and relatively direct way to establish the connections between the global Hermitian system and the $\mathcal{PT}$-symmetric subsystem. Such a point is well illustrated in proposing a classical picture. One can assume the measurement results of Alice’s measurements $A_1$ are the eigenvalues of the $\mathcal{PT}$-symmetric Hamiltonian $\hat{H}$. In addition, Bob’s results should be 1, such that the correlation functions $B_1 A_1$ trivially give the eigenvalues of $\hat{H}$. Based on this, one can discuss the correlations that the Hermitian dilation Hamiltonian bring to the subsystems [20].

III. THE GENERAL CASE

Now if we are considering the general case, $\hat{H}$ may not have such a special structure as Eq. (11) . In particular, for a general Hermitian dilation $\hat{H}$, it can have eigenvalues different from $H$. Then can we discuss the internal non-locality for a general $\hat{H}$? Next we show that the property of dilation actually allows for a more general discussion.

Such a generalization is based on the following observation: If $\hat{H}$ is a Hermitian dilation of a $\mathcal{PT}$-symmetric Hamiltonian $H$ (i.e. Eq. (1) is valid for $H$ and $\tau$), then we have

\[
\begin{pmatrix}
H_1 & H_2 \\
H_2^\dagger & H_4
\end{pmatrix}
\begin{pmatrix}
-\tau^\dagger \\
I
\end{pmatrix}
= \begin{pmatrix}
-\tau^\dagger H_1^\perp \\
H_4
\end{pmatrix}, \tag{31}
\]

where

\[
H_4 = -H_2^\dagger \tau^\dagger + H_4. \tag{32}
\]

In fact, utilizing Eq. (2), one can verify Eq. (31) through direct calculations. This observation, together with Eq. (1), shows that when confined to the subsystems, the effect of Hermitian dilation Hamiltonian $\hat{H}$ can be represented by two Hamiltonians, one is the $\mathcal{PT}$-symmetric Hamiltonian $H$, the other is $H^\perp$. Eq. (31) also shows that the eigenvalues of the Hamiltonian $H^\perp$ is just the eigenvalues of the Hermitian dilation Hamiltonian $\hat{H}$. Hence the eigenvalues of $H^\perp$ are also real.

In particular, it can be verified that if $H_1$ the same as Eq. (6), then $H_1^\perp = H$ and the Hermitian dilation Hamiltonian $\hat{H}$ reduce to the special case of Eq. (1). Note that in the special case of Eq. (7), we build up connections between $\hat{H}$ and $H$ to discuss the internal non-locality. For the general case, both $H$ and $H^\perp$ will be connected with $\hat{H}$. To be more precise, one can assume that Alice is either measuring the Hamiltonian $H$ or $H^\perp$ and the joint measurements of Alice and Bob depict the characteristics of the global Hamiltonian $\hat{H}$. Similar to the discussions in section II, it is possible to reconstruct the different correlation pictures in the general case.

The simulation picture

As an example, let $H$ be the $\mathcal{PT}$-symmetric Hamiltonian in Eq. (10). Moreover, $\hat{H}$ is any general in Hermitian dilation Hamiltonian of $H$. Similar to section II, Alice and Bob can make measurements and calculate the expectation values of the Bell operator in different pictures.

In the simulation picture, the expectation of Bell operator is still given by Eq. (29). However, a general Hermitian dilation Hamiltonian $\hat{H}$ usually has a numerical value different from Eq. (24). Calculations show that the numerical value of the expectation is (see the appendix
for details),
\[
2E_0 + 2a + (\pi v + \nu w) \omega_0 \cos \alpha \\
+ \frac{i(b + b \sin^2 \alpha + 2a \sin \alpha)}{\cos^2 \alpha} \\
- \frac{u}{2} \left( \frac{b + b \sin^2 \alpha + 2a \sin \alpha}{\cos^2 \alpha} \right)^2 
\]
where \( a \) and \( b \) are parameters characterizing the difference between the general Hermitian dilation Hamiltonians and the special one in Eq. (1).

Comparing Eq. (33) with Eq. (24), one can see that now the expectation of the Bell operator has an energy shift \( 2a \). According to the Schwartz inequality, the derivation term is less than
\[
\sqrt{4s^2 \cos^4 \alpha + \frac{(b + b \sin^2 \alpha + 2a \sin \alpha)^2}{\cos^4 \alpha}} 
\]
By comparing it with Eq. (25), we see that this term is enlarged.

The classical picture

To have a classical picture, suppose Alice can make two measurements \( A_0 \) and \( A_1 \). The results of \( A_0 \) are the eigenvalues of \( H \) and the results of \( A_1 \) are the eigenvalues of \( H^\perp \). However, Alice only knows that one of \( A_1 \) outputs the eigenvalues of \( H \) and the other outputs the eigenvalues of \( H^\perp \). Moreover, we assume that Alice makes measurements in some black box, that is, she is unaware of which measurement she has conducted. Briefly speaking, Alice can only obtain the measurement results but cannot distinguish between \( A_0 \) and \( A_1 \). As for Bob, his results are always 1. Thus, their results are independent, implying the correlations are classical (local).

Denote the eigenvalues of \( H \) by \( \lambda_\alpha \) and the eigenvalues of \( H^\perp \) by \( \lambda'_\alpha \). The expectation of Bell operator is as follows,
\[
\langle B_0A_0 \rangle + \langle B_0A_1 \rangle + \langle B_1A_0 \rangle - \langle B_1A_1 \rangle \\
= \int \langle B_0(\nu)(A_0 + A_1)(\nu) + B_1(\nu)(A_0 - A_1)(\nu) \rangle d\nu \\
= \int \langle (A_0 + A_1)(\nu) + (A_0 - A_1)(\nu) \rangle d\nu.
\]
Since Alice cannot distinguish between the measurements \( A_0 \) and \( A_1 \), the expectation value of Bell operator is (see appendix for details)
\[
E_0 + E'_0 + \frac{1}{2} [\omega_0(p_+ - p_-) + \omega'_0(p'_+ - p'_-)] ,
\]
where \( E_0' \) is the mean value of the eigenvalues of \( H^\perp \) and \( p'_\pm \) are the probabilities that the measurement results are \( \lambda'_\alpha \).

The local Hermitian and genuine local Hermitian picture

There are different approaches to generalizing the local Hermitian picture in [20], which we call local Hermitian and genuine local Hermitian pictures.

In the local Hermitian picture, the global Hamiltonian is
\[
\hat{H}' = |0\rangle\langle 0| \otimes H_h + |1\rangle\langle 1| \otimes H_h^\perp. 
\]
where \( H_h \) and \( H_h^\perp \) are Hermitian Hamiltonians having the same eigenvalues as \( H \) and \( H^\perp \), respectively. The implication behind this picture is that when the ancillary system is in the state \(|0\rangle \) or \(|1\rangle \), the effect of the Hermitian dilation Hamiltonian can be characterized by \( H \) or \( H^\perp \). This is similar to the Hermitian dilation Hamiltonian.

Suppose that \( H_h = \lambda_+ |s_+\rangle\langle s_+| + \lambda_- |s_-\rangle\langle s_-| \) and \( |s_\pm\rangle \) are two orthogonal states. Similarly, \( H_h^\perp = \lambda'_+ |s'_+\rangle\langle s'_+| + \lambda'_- |s'_-\rangle\langle s'_-| \) are two orthogonal states. Substituting \( \hat{H}' \) into Eqs. (19-22), the expectation of the Bell operator is
\[
\langle B_0A_0 \rangle + \langle B_1A_0 \rangle + \langle B_0A_1 \rangle - \langle B_1A_1 \rangle \\
= Tr(|0\rangle\langle 0| \otimes H_h + |1\rangle\langle 1| \otimes 2(\pi v|0\rangle\langle 1| + \nu w|1\rangle\langle 0|)H_h^\perp).
\]
Direct calculations show that the local Hermitian bound is (see appendix)
\[
Tr(H) + \langle u_+|H_h^\perp|u_+\rangle - \langle u_-|H_h^\perp|u_-\rangle 
= 2E_0 + \omega_0(p'_+ - p'_-),
\]
\[
\omega_0 = \lambda'_+ - \lambda'_- = 2 \sqrt{4s^2 \cos^2 \alpha + \frac{4c^2}{\cos^2 \alpha} + \frac{4(b + a \sin \alpha)^2}{\cos^2 \alpha}}.
\]
Comparing Eq. (40) with Eq. (24), we see that when there is no energy shift (i.e. \( a = 0 \)), the derivation term gives a larger bound than the simulation picture.

In the genuine local Hermitian picture, the global Hamiltonian is
\[
\hat{H}_g' = \frac{1}{2} I \otimes (H_h + H_h^\perp). 
\]
Substituting \( \hat{H}_g' \) into Eqs. (19-22), the expectation of the Bell operator is (see appendix)
\[
\langle B_0A_0 \rangle + \langle B_1A_0 \rangle + \langle B_0A_1 \rangle - \langle B_1A_1 \rangle \\
= \langle u_+|H_h|u_+\rangle + \langle u_-|H_h^\perp|u_-\rangle \\
= E_0 + E'_0 + \frac{1}{2} [\omega_0(p_+ - p_-) + \omega'_0(p'_+ - p'_-)],
\]
where \( p_\pm = |\langle u_+|s_\pm\rangle|^2 \) and \( p'_\pm = |\langle u_+|s'_\pm\rangle|^2 \).

Direct calculations show that the derivation term is smaller than
\[
\sqrt{\frac{(\omega_0)^2}{2} + \frac{(\omega'_0)^2}{2} + 2\frac{\omega_0\omega'_0}{2} \cos 2\delta},
\]
where \( \delta \) is some parameter related to the angle of \( |s_\pm\rangle \) and \( |s'_\pm\rangle \). In particular, when \( \delta = 0 \), the above bound reduces to \( \frac{1}{2}(\omega_0 + \omega'_0) \). Since \( \omega'_0 \geq \omega_0 \), we see that the derivation is larger than the simulation picture.
IV. DISCUSSIONS

Here, we discuss the physical implications behind the general results by contrasting them with the special case in [20].

A significant difference exists in deriving the classical pictures. To obtain a classical picture for the general case, one needs an extra assumption that Alice cannot distinguish between the measurements $A_i$. Such an assumption is not needed in the special case [20]. However, in the general case, due to the fact that $H$ and $H^\perp$ usually have different eigenvalues, there are four outcomes of measurements, which is quite different from the usual CHSH scenario. If we calculate the usual expectation of the Bell operator, that is, if Alice is aware of the details of his measurements, then she can get an expectation value of Eq. (31), irrelevant to $H^\perp$. Intuitively, such a biased value is not suitable for investigating the properties of the global Hamiltonian $H$. For this reason, the extra assumption is needed and gives a more reasonable expectation value. Moreover, such an assumption is implicitly valid for the special Hermitian dilation Hamiltonian in Eq. (11), in which case $H^\perp = H$. Only with the same measurement results, Alice cannot distinguish between $A_0$ and $A_1$. Hence such an assumption is natural and the classical picture will reduce to the special one in section II.

In the special case of Eq. (11), the classical and local Hermitian pictures have the same expectation of the Bell operator. In the general case, one can see they are usually different by comparing Eq. (39) with Eq. (41). The reason is that the form of $H'$ is not a tensor product. Although we still use the term local Hermitian picture by comparison with the [20], it is actually non-local, which cannot be described by a classical picture. In fact, the classical picture is derived by an assumption that Alice cannot distinguish between the measurements $A_0$ and $A_1$. However, in the local Hermitian picture, $H' = |0\rangle\langle 0| \otimes H_h + |1\rangle\langle 1| \otimes H_h^\perp$ implies that Alice can distinguish between $A_0$ and $A_1$ ($|0\rangle$ and $|1\rangle$) by simply reading out the eigenvalues of $H$ and $H^\perp$, contradicting with the classical picture.

On the other hand, by comparing Eq. (31) with Eq. (42), we see that the classical and genuine local Hermitian pictures still have the same expectations of the Bell operator. To see why this is valid, we try to find the physical realization of the classical picture. Note that if Alice can distinguish between the measurements $A_0$ and $A_1$, then the Hamiltonian she uses should be

$$H_0 = |0\rangle\langle 0| \otimes H_h + |1\rangle\langle 1| \otimes H_h^\perp,$$

which is the same as the local Hermitian picture. According to the form of the above Hamiltonian, a natural interpretation is when the measurement is $A_0$ (i.e. Alice uses state $|0\rangle$), the result of Alice is the eigenvalues of $H$. Similarly, the measurement result of $A_1$ is the eigenvalues of $H^\perp$. However, if Alice mistaken $A_1$ for $A_0$, then the Hamiltonian should be

$$H_1 = |0\rangle\langle 0| \otimes H_h^\perp + |1\rangle\langle 1| \otimes H_h.$$

Now, since Alice cannot distinguish between $A_1$, the Hamiltonian realizing the classical should be $\frac{1}{2}(H_0 + H_1)$, which is just the Hamiltonian $H'_g$ in the genuine local Hermitian picture. Hence it is natural that the classical and genuine local Hermitian pictures have the same expectations. Here we also see a subtle difference in using the term measurement from the usual sense. As was mentioned in the classical picture, the results of Alice’s local measurements are the eigenvalues of $H$ and $H^\perp$ and Bob’s result is always 1. However, in the usual sense, the joint measurement of Alice and Bob should give the eigenvalues of $H'_g$, which is not the product of Alice and Bob’s local results. Such a contradiction is because that the global Hamiltonian $H'_g$ is the average of $H_0$ and $H_1$. By only considering the Hamiltonian $H_0$ or $H_1$, Alice’s measurement results will be the eigenvalues of $H$ and $H^\perp$. However, the eigenvalues of $H'_g$ will be changed when summing up $H_0$ and $H_1$.

In [20], it is shown that the expectations in different correlation pictures can help to distinguish the special Hermitian dilation Hamiltonian in Eq. (11) and have some potential applications. Similar discussions can be made in the general case. The first result is that one can distinguish between general Hermitian dilation Hamiltonians and the special case in Eq. (11) by making measurements and reading out the eigenvalues. As was shown in the appendix, for a general Hermitian dilation Hamiltonian $H$, $H^\perp$ has the same eigenvalues as $H$ if and only if $H$ has the same form as Eq. (11). In this sense, the two fold structure in Eq. (11) is unique and it is easy to distinguish it from the general Hermitian dilation Hamiltonians. One can also distinguish a general Hermitian dilation Hamiltonian $H$ from the Hamiltonians $H'$ and $H''$ in Eq. (40) and (41). First, note that the simulation bound is smaller than the local Hermitian bound, since $\omega_0 \geq \omega_h$. Hence by calculating the expectations of the Bell operator, one can immediately know whether the Hamiltonian is a Hermitian dilation or that in Eq. (50). On the other hand, the expectations of the Bell operator...
sometimes cannot help in distinguishing a Hermitian dilation \( H' \) from the Hamiltonian \( H' \) in the genuine local Hermitian picture. As was mentioned in the appendix, the expectations of the Bell operator can sometimes have the same range for the two types of Hamiltonians. However, in this case, one can distinguish between \( H \) and \( H' \) by measuring them and reading out the eigenvalues. Apparently, the Hamiltonian \( H' \) has only two eigenvalues and the multiplicity of each eigenvalue is two. However, as was mentioned above, a general Hermitian dilation has four eigenvalues. Hence one can distinguish the two types of Hamiltonians simply by measurement. To sum up, for general Hermitian dilation Hamiltonians, one can distinguish it from those Hamiltonians in other correlation pictures. Such a result provides the figure of merit to test the reliability of the simulation and verifies a general Hermitian dilation Hamiltonian, which might have potential applications \[20\].

V. CONCLUSION

In this paper, we show that besides the \( \mathcal{PT} \)-symmetric Hamiltonian \( H \), the effect of a general Hermitian dilation Hamiltonian can also be characterized by another Hamiltonian \( H^\perp \). Based on this observation, we extend the discussion of internal non-locality in \[21\] to the general case. Different correlation pictures are proposed and the corresponding correlation bounds are obtained. In particular, it is shown that the local Hermitian picture has two natural generalisations. Different from \[20\], the expectations of the simulation and genuine local Hermitian pictures can have the same range. It is shown that the two fold structure of the special Hermitian dilation Hamiltonian is unique. In addition, it is shown that by calculating the expectations of the Bell operator or measuring the eigenvalues, one can distinguish a general Hermitian dilation Hamiltonian from the special case in \[20\] and those in the local Hermitian and genuine local Hermitian pictures.

VI. APPENDIX

In this part, we show how the expectations in different correlation pictures are obtained. For the convenience of calculations, we denote a general Hermitian dilation Hamiltonian by \( \begin{bmatrix} H_1' \ H_2' \\ H_2' \ H_1' \end{bmatrix} \) and the special Hermitian dilation Hamiltonian in Eq. \[11\] still by \( \begin{bmatrix} H_1 \ H_2 \\ H_2 \ H_1 \end{bmatrix} \). Moreover, assume that \( H_1' = H_1 + H_1'' \) and

\[
H_1'' = \begin{bmatrix} a + c & ib \\ -ib & a - c \end{bmatrix}.
\]

Now we see the effect of \( H_1'' \) when \( \tau \) is fixed. Note that \( H_1' = H_1 + H_1'' \). Then

\[
\begin{align*}
H_2' &= (H - H'_1)\tau^{-1} \\
&= (H - H_1)\tau^{-1} - H''_1\tau^{-1} \\
&= H_2 - H''_1\tau^{-1}.
\end{align*}
\]

On the other hand,

\[
\begin{align*}
H_4' &= (\tau H - H'_1)\tau^{-1} + \tau^{-1}H''_1\tau^{-1} \\
&= H_4 + \tau^{-1}H''_1\tau^{-1}.
\end{align*}
\]

Moreover, \( H^\perp = -(H'_2)\tau + H_4' \)

\[
= -H''_1\tau + H_4 + \tau^{-1}H''_1(\tau + \tau^{-1}),
\]

where \( \tau = \frac{1}{\cos \alpha} \begin{bmatrix} 1 & -i \sin \alpha \\ i \sin \alpha & 1 \end{bmatrix} \), just the same as Eq. \[15\]. Note that \( H^\perp \) in the above equation is actually \( (H^\perp)' \), as we have added a prime symbol to distinguish the general Hermitian dilation Hamiltonian from the special case. However, we write \( H^\perp \) for simplicity and it will not cause confusion.

Direct calculations show that

\[
\begin{align*}
\tau^{-1}H''_1\tau^{-1} &= \begin{bmatrix} a + c + 2b \sin \alpha + (a - c) \sin^2 \alpha & \frac{i(b + b \sin^2 \alpha + 2a \sin \alpha)}{\cos^2 \alpha} \\
\frac{-i(b + b \sin^2 \alpha + 2a \sin \alpha)}{\cos^2 \alpha} & a - c + 2b \sin \alpha + (a + c) \sin^2 \alpha \end{bmatrix},
\end{align*}
\]

and

\[
\begin{align*}
\tau^{-1}H''_1(\tau + \tau^{-1}) &= \begin{bmatrix} 2(a + c + b \sin \alpha) & 2i(a + b \sin \alpha - c \sin \alpha) \\
-2i(b + a \sin \alpha + c \sin \alpha) & \frac{2(a - c + b \sin \alpha)}{\cos^2 \alpha} \end{bmatrix}.
\end{align*}
\]

Denote

\[
\begin{align*}
A_1 &= \frac{2(a + b \sin \alpha)}{\cos^2 \alpha}, \\
A_2 &= \frac{2(b + a \sin \alpha)}{\cos^2 \alpha}, \\
C_1 &= \frac{2c}{\cos^2 \alpha}, \\
C_2 &= \frac{2c \sin \alpha}{\cos^2 \alpha}.
\end{align*}
\]

Note that \( -H''_1\tau + H_4 = H \), hence we have

\[
H^\perp = H + \tau^{-1}H''_1(\tau + \tau^{-1}).
\]

The eigenvalues of \( H^\perp \) are

\[
\lambda_+ = E_0 + A_1 + \sqrt{(C_1 + i \sin \alpha)^2 + (s - iC_2)^2 + A_2^2},
\]

\[
\lambda_- = E_0 + A_1 - \sqrt{(C_1 + i \sin \alpha)^2 + (s - iC_2)^2 + A_2^2}.
\]
According to Eqs. (18) and (19),
\[ \omega_0' = \lambda_+ - \lambda_- = 2 \sqrt{s^2 \cos^2 \alpha + \frac{4c^2}{\cos^2 \alpha} + \frac{4(b + a \sin \alpha)^2}{\cos^4 \alpha}}. \]

Noting that \( H_b \) and \( H_b^\perp \) have the same eigenvalues as \( H \) and \( H^\perp \), one can obtain the local Hermitian and genuine Hermitian bounds in section III.

Using the above results, one can also prove an interesting result: for a general Hermitian dilation Hamiltonian \( \hat{H} \), \( H \) and \( H^\perp \) have the same eigenvalues if and only if \( \hat{H} \) takes the same form of Eq. (11). In fact, if \( H \) and \( H^\perp \) have the same eigenvalues, then \( \lambda_+ + \lambda_- = 2E_0 \). It follows that \( a = -b \sin \alpha \). By substituting it into Eq. (50), one can verify that \( a = b = c = 0 \), in which case the \( \hat{H} \) is just the Hamiltonian in Eq. (11).

We can also calculate the simulation bound. By substituting a general Hermitian dilation Hamiltonian
\[ \begin{pmatrix} H_{11} & H_{12} \\ (H_{21})^\dagger & H_{22} \end{pmatrix} \]
onto Eqs. (19)-(22), the simulation bound is given by
\[ Tr[|0\rangle\langle 0| \otimes H_{11}^\dagger H_0 + |1\rangle\langle 1| \otimes 2(u\overline{v} |0\rangle\langle 1| + \overline{v} u |1\rangle\langle 0|)H_{22}]. \]

Direct calculations show that the numerical value of the simulation bounds is
\[ 2E_0 + 2a + 2(\overline{v} u + u\overline{v}) \omega_0 \cos \alpha + i(b + b \sin^2 \alpha + 2a \sin \alpha) \]
\[ -i(b + b \sin^2 \alpha + 2a \sin \alpha). \]
From the above equation, we see there is an energy shift \( 2\alpha \) and the derivation term is less than
\[ \sqrt{4s^2 \cos^4 \alpha + \frac{(b + b \sin^2 \alpha + 2a \sin \alpha)^2}{\cos^4 \alpha}}. \]

For the classical picture, note that if Alice can distinguish between the measurements \( A_1 \), then the expectation value should be
\[ 2E_0 + \omega_0(p_+ - p_-). \]

However, Alice is unaware of the details of the measurements, hence she may mistaken \( A_1 \) for \( A_0 \), then the the expectation value will be
\[ 2E'_0 + \omega_0'(p_{+}' - p_{-}'). \]
where \( E_0' \) is the mean value of the eigenvalues of \( H^\perp \) and \( p_{+}' \) are the probabilities that the results are \( \lambda_{+}' \).

Now the best Alice can do is to calculate the mean value of the above two results, that is
\[ E_0 + E'_0 + \frac{1}{2}[\omega_0(p_+ - p_-) + \omega_0'(p_{+}' - p_{-}')], \]
which is just Eq. (35).

To see the bound of the derivation term, note that it is independent of the choice of the basis. Hence we may assume that \( |s_+\rangle = |0\rangle \) and \( |s_-\rangle = |1\rangle \). Moreover, assume that
\[ |u_+\rangle = \begin{bmatrix} \cos \alpha \\ e^{i\Delta} \sin \alpha \end{bmatrix}, \]
\[ |s_+\rangle = \begin{bmatrix} \cos \delta \\ e^{i\Delta'} \sin \delta \end{bmatrix}, \]
where \( \Delta, \Delta' \), \( \alpha \) and \( \delta \) are parameters. Note that \( \langle s_\pm | s_\pm' \rangle = \cos \delta \), hence \( \delta \) is a parameter characterizing the angle between \( |s_\pm\rangle \) and \( |s_\pm'\rangle \). Then direct calculations show that
\[ \frac{1}{2}[\omega_0(p_+ - p_-) + \omega_0'(p_{+}' - p_{-}')]
= \frac{\omega_0}{2}(\cos^2 \alpha - \sin^2 \alpha) + \frac{\omega_0'}{2}(\cos^2(\alpha + \delta) - \sin^2(\alpha + \delta) + \sin 2\alpha \sin 2\delta + \cos(-\Delta + \Delta') \sin 2\alpha \sin 2\delta)
= \cos 2\alpha(\frac{\omega_0}{2} + \frac{\omega_0'}{2} \cos 2\delta) + (\frac{\omega_0}{2} \sin 2\delta \cos(-\Delta + \Delta')) \sin 2\alpha
\leq \sqrt{(\frac{\omega_0}{2} + \frac{\omega_0'}{2} \cos 2\delta)^2 + (\frac{\omega_0}{2} \sin 2\delta \cos(-\Delta + \Delta'))^2}
\leq \sqrt{(\omega_0^2 + \omega_0'^2 + 2 \omega_0 \omega_0' \cos 2\delta).} \]
When \( \delta = 0 \) or \( \pi \), Eq. (54) saturates its largest value. That is, when \( H \) and \( H^\perp \) are commutative, or they have the same eigenstates, then one can obtain the largest bound \( \frac{1}{2}(\omega_0 + \omega_0') \).

In addition, the genuine local Hermitian bound can coincide with the simulation bound. In fact, if we assume that \( \Delta = 0 \), \( b = -a \sin \alpha \) and \( c = \frac{1}{2} \sqrt{4s^2 \cos^2 \alpha - 4s^2 \cos^4 \alpha + a^2 \sin^2 \alpha} \), then direct calculations show that the expectations of the Bell operator have the same range.

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