Gelfand-Yaglom-Perez Theorem for Generalized Relative Entropies

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Abstract. The measure-theoretic definition of Kullback-Leibler relative-entropy (KL-entropy) plays a basic role in the definitions of classical information measures. Entropy, mutual information and conditional forms of entropy can be expressed in terms of KL-entropy and hence properties of their measure-theoretic analogs will follow from those of measure-theoretic KL-entropy. These measure-theoretic definitions are key to extending the ergodic theorems of information theory to non-discrete cases. A fundamental theorem in this respect is the Gelfand-Yaglom-Perez (GYP) Theorem (Pinsker, 1960, Theorem. 2.4.2) which states that measure-theoretic relative-entropy equals the supremum of relative-entropies over all measurable partitions. This paper states and proves the GYP-theorem for Rényi relative-entropy of order greater than one. Consequently, the result can be easily extended to Tsallis relative-entropy.

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1. Introduction

Rényi [1], by replacing linear averaging in Shannon entropy with Kolmogorov-Nagumo average or quasilinear mean and further imposing the additivity constraint, proposed a one-parameter family of measures of information ($\alpha$-entropies) which is defined as follows:

\[
S_\alpha(p) = \frac{1}{1 - \alpha} \ln \left( \sum_{k=1}^{n} p_k^\alpha \right),
\]

where $p = \{p_k\}_{k=1}^{n}$ is a probability mass function (pmf) and $\alpha \in \mathbb{R}$ and $\alpha > 0$. Rényi entropy [1] is a one-parameter generalization of Shannon entropy in the sense that the limit $\alpha \to 1$ in (1) retrieves Shannon entropy. $S_\alpha$ is referred as the entropy of order $\alpha$. Despite its formal origin, Rényi entropy proved important in a variety of practical applications in coding theory [2], statistical inference [3], quantum mechanics [4], and chaotic dynamical systems [5].

Along similar lines, Rényi defined a one parameter generalization of Kullback-Leibler relative-entropy as [1]

\[
S_\alpha(p||r) = \frac{1}{\alpha - 1} \ln \sum_{k=1}^{n} \frac{p_k^\alpha}{r_k^{\alpha-1}}
\]

for pmfs $p$ and $r$.

On the other hand, though Shannon measure of entropy or information was developed essentially for the case when the random variable takes a finite number of values, in the literature, one often encounters an extension of Shannon entropy in the discrete case to the case of a one-dimensional random variable with density function $p$ in the form (e.g [6, 7])

\[
S(p) = -\int_{-\infty}^{+\infty} p(x) \ln p(x) \, dx.
\]

(3) is known as differential entropy in information theory and Boltzmann H-function in Physics. Indeed, during the early stages of development of information theory, the important paper by Gelfand, Kolmogorov and Yaglom [8] called attention to the case where entropy is defined on an arbitrary measure space $(X, \mathcal{M}, \mu)$. In this respect, Shannon entropy of a probability density function $p : X \to \mathbb{R}^+$ can be defined as

\[
S(p) = -\int_{X} p \ln p \, d\mu,
\]

provided the integral on right exists. One can see from the above definition that the concept of “entropy of a pdf” is a misnomer: there is always another measure $\mu$ in the background. In the discrete case considered by Shannon, $\mu$ is the cardinality measure of [6, pp.19]; in the continuous case considered by both Shannon and Wiener, $\mu$ is the Lebesgue measure cf. [6, pp.54] and [9, pp.61, 62]. All entropies are defined with respect to some $\mu$ counting or cardinality measure $\mu$ on a measurable space $(X, \mathcal{M})$, when is $X$ is a finite set and $\mathcal{M} = 2^X$, is defined as $\mu(E) = \#E$, $\forall E \in \mathcal{M}$. 

\[
\sum_{k=1}^{n} p_k^\alpha.
\]
measure $\mu$, as Shannon and Wiener both emphasized in [6, pp.57, 58] and [9, pp.61, 62] respectively.

This case was studied independently by Kallianpur [10] and Pinsker [11], and perhaps others were guided by the earlier work of Kullback and Leibler [12], where one would define entropy in terms of Kullback-Leibler relative-entropy.

In this respect Gelfand-Yaglom-Perez theorem (GYP-theorem) [13, 14, 15] plays an important role, which equips measure-theoretic KL-entropy with a fundamental definition. The main contribution of this paper is to state and prove GYP-theorem for Rényi relative entropy of order $\alpha > 1$.

We review the measure-theoretic formalisms for classical information measures in §2, where we discuss the relation between Shannon entropy and KL-entropy in the measure-theoretic case. We extend measure-theoretic definitions to generalized information measures in §3. Finally, Gelfand-Yaglom-Perez theorem in the general case is presented in §4.

2. Measure Theoretic Definitions of Classical Information Measures

Let $(X, \mathcal{M}, \mu)$ be a measure space. $\mu$ need not be a probability measure unless otherwise specified. Symbols $P$, $R$ will denote probability measures on measurable space $(X, \mathcal{M})$ and $p$, $r$ denote $\mathcal{M}$-measurable functions on $X$. An $\mathcal{M}$-measurable function $p : X \to \mathbb{R}^+$ is said to be a probability density function (pdf) if $\int_X p \, d\mu = 1$.

In this general setting, entropy $S(p)$ of pdf $p$ defined in (4) can be referred to as the entropy of the probability measure $P$, in the sense that the measure $P$ is induced by $p$, i.e.,

$$P(E) = \int_E p(x) \, d\mu(x) \quad \forall E \in \mathcal{M}. \tag{5}$$

This reference is consistent∥ because the probability measure $P$ can be identified a.e by the pdf $p$. Further, the definition of the probability measure $P$ in (5), allows one to write entropy functional (4) as

$$S(p) = -\int_X \frac{dP}{d\mu} \ln \frac{dP}{d\mu} \, d\mu \quad \tag{6}$$

since (5) implies\¶ $P \ll \mu$, and pdf $p$ is the Radon-Nikodym derivative of $P$ w.r.t $\mu$.

Now we proceed to the definition of Kullback-Leibler relative-entropy or KL-entropy for probability measures.

∥ Say $p$ and $r$ are two pdfs and $P$ and $R$ are corresponding induced measures on measurable space $(X, \mathcal{M})$ such that $P$ and $R$ are identical, i.e., $\int_E p \, d\mu = \int_E r \, d\mu, \forall E \in \mathcal{M}$. Then we have $p \equiv r$ and hence $-\int_X p \ln p \, d\mu = -\int_X r \ln r \, d\mu$.

¶ If a nonnegative measurable function $f$ induces a measure $\nu$ on measurable space $(X, \mathcal{M})$ with respect to a measure $\mu$, defined as $\nu(E) = \int_E f \, d\mu, \forall E \in \mathcal{M}$ then $\nu \ll \mu$. Converse is given by Radon-Nikodym theorem [16] pp.36, Theorem 1.40(b)].
Definition 2.1. Let $P$ and $R$ be two probability measures on measurable space $(X, \mathcal{M})$. Kullback-Leibler relative-entropy of $P$ relative to $R$ is defined as

$$I(P \parallel R) = \begin{cases} \int_X \ln \frac{dP}{dR} dP & \text{if } P \ll R, \\ +\infty & \text{otherwise.} \end{cases} \quad (7)$$

The divergence inequality $I(P \parallel R) \geq 0$ and $I(P \parallel R) = 0$ if and only if $P = R$ can be shown in this case too. Relative-entropy (7) also can be written as

$$I(P \parallel R) = \int_X \frac{dP}{dR} \ln \frac{dP}{dR} dR. \quad (8)$$

Let the $\sigma$-finite measure $\mu$ on $(X, \mathcal{M})$ such that $P \ll R \ll \mu$. Then (7) can be written as

$$I(p \parallel r) = \int_X p(x) \ln \frac{p(x)}{r(x)} d\mu(x), \quad (9)$$

provided the integral on right exists. The pdfs $p(x)$ and $r(x)$ in (9) are the Radon-Nikodym derivatives of $P$ and $R$ with respect to $\mu$, i.e., $p = \frac{dP}{d\mu}$ and $r = \frac{dR}{d\mu}$. Here in the sequel we use the convention

$$\ln 0 = -\infty, \quad \ln \frac{a}{0} = +\infty \text{ for any } a \in \mathbb{R}, \quad 0.(\pm \infty) = 0. \quad (10)$$

Shannon entropy in (6) is defined for a probability measure that is induced by a pdf. By the Radon-Nikodym theorem, one can define Shannon entropy for any arbitrary $\mu$-continuous probability measure as follows.

Definition 2.2. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space. Entropy of any $\mu$-continuous probability measure $P$ ($P \ll \mu$) is defined as

$$S(P) = -\int_X \ln \frac{dP}{d\mu} dP. \quad (11)$$

Properties of entropy of a probability measure in the Definition 2.2 are studied in detail by Ochs [17]. In the literature, one can find notation of the form $S(P|\mu)$ to represent the entropy functional in (11) viz., the entropy of a probability measure, to stress the role of the measure $\mu$ (for example [17], [18]). Since all the information measures we define are with respect to the measure $\mu$ on $(X, \mathcal{M})$, we omit $\mu$ in the entropy functional notation.

By assuming $\mu$ as a probability measure in the Definition 2.2 one can relate Shannon entropy with Kullback-Leibler entropy as

$$S(P) = -I(P \parallel \mu). \quad (12)$$

Note that when $\mu$ is not a probability measure, the divergence inequality $I(P \parallel \mu) \geq 0$ need not be satisfied.

Before we conclude this section, we make a note on the $\sigma$-finiteness of measure $\mu$. In the measure-theoretic definitions of Shannon entropy we assumed that $\mu$ is a
σ-finite measure. This condition was used by Ochs [17], Csiszár [19] and Rosenblatt-Roth [20] to tailor the measure-theoretic definitions. For all practical purposes and for most applications this assumption is satisfied. (See [17] for a discussion on the physical interpretation of measurable space \((X, \mathcal{M})\) with σ-finite measure \(\mu\) for entropic measure of the form (11), and relaxation σ-finiteness condition.) By relaxing this condition, more universal definitions of entropy functionals are studied by Masani [21, 22].

3. Measure-Theoretic Definitions of Generalized Information Measures

We begin with a brief note on the notation and assumptions used. We define all the information measures on the measurable space \((X, \mathcal{M})\), and default reference measure is \(\mu\) unless otherwise stated. To avoid clumsy formulations, we will not distinguish between functions differing on a \(\mu\)-null set only; nevertheless, we can work with equations between \(\mathcal{M}\)-measurable functions on \(X\) if they are stated as valid as being only \(\mu\)-almost everywhere (\(\mu\)-a.e or a.e). Further we assume that all the quantities of interest exist and assume, implicitly, the σ-finiteness of \(\mu\) and \(\mu\)-continuity of probability measures whenever required. Since these assumptions repeatedly occur in various definitions and formulations, these will not be mentioned in the sequel. With these assumptions we do not distinguish between an information measure of pdf \(p\) and of corresponding probability measure \(P\) – hence we give definitions of information measures for pdfs, we use corresponding definitions of probability measures as well, when ever it is convenient or required – with the understanding that \(P(E) = \int_E p\,d\mu\), the converse being due to the Radon-Nikodym theorem, where \(p = \frac{dP}{d\mu}\).

Similar to the definition of Shannon entropy (4) one can extend the Rényi entropy in the discrete case (1) to measure-theoretic case as follows.

**Definition 3.1.** Rényi entropy of a pdf \(p : X \to \mathbb{R}^+\) on \((X, \mathcal{M}, \mu)\) is defined as

\[
S_\alpha(p) = \frac{1}{1 - \alpha} \ln \int_X p(x)^\alpha \,d\mu(x) ,
\]

provided the integral on the right exists and \(\alpha \in \mathbb{R}\) and \(\alpha > 0\).

The same can be written for any \(\mu\)-continuous probability measures \(P\) as

\[
S_\alpha(P) = \frac{1}{1 - \alpha} \ln \int_X \left(\frac{dP}{d\mu}\right)^{\alpha - 1} \,dP
\]

On the other hand, Rényi relative-entropy can be defined as follows.

**Definition 3.2.** Let \(p, r : X \to \mathbb{R}^+\) be two pdfs defined on \((X, \mathcal{M}, \mu)\). Rényi relative-entropy of \(p\) relative to \(r\) is defined as

\[
I_\alpha(p||r) = \frac{1}{\alpha - 1} \ln \int_X \frac{p(x)^\alpha}{r(x)^{\alpha - 1}} \,d\mu(x) ,
\]

provided integral on the right exists.
The same can be written in terms of probability measures as

\[
I_\alpha(P \parallel R) = \frac{1}{\alpha - 1} \ln \int_X \left( \frac{dP}{dR} \right)^{\alpha - 1} dP
\]

\[
= \frac{1}{\alpha - 1} \ln \int_X \left( \frac{dP}{dR} \right)^{\alpha} dR ,
\]

(16)

whenever \( P \ll R; I_\alpha(P \parallel R) = +\infty, \) otherwise. Further if we assume \( \mu \) in (14) is a probability measure then

\[
S_\alpha(P) = I_\alpha(P \parallel \mu) .
\]

(17)

On the other hand, it is well known that unlike Shannon entropy, Kullback-Leibler relative-entropy in the discrete case can be extended naturally to the measure-theoretic case, in the sense that measure-theoretic definitions can be defined as a limit of a sequence of finite discrete entropies of pmfs which approximate the pdfs involved. This fact is shown for Rényi relative-entropy in the continuous valued space \( \mathbb{R} \) by Rényi [1], which can be extended to the measure-theoretic case (see [23]).

4. Gelfand-Yaglom-Perez Theorem in the General Case

In the ergodic approach of information theory, basic definitions of information measures are given for measurable partitions. Before we proceed to the definitions we give our notation. Let \( (X, \mathcal{M}) \) be a measurable space and \( \Pi \) denote the set of all measurable partitions of \( X \). We denote a measurable partition \( \pi \in \Pi \) as \( \pi = \{ E_k \}_{k=1}^m \), i.e, \( \bigcup_{k=1}^m E_k = X \) and \( E_i \cap E_j = \emptyset, i \neq j, i,j = 1, \ldots m \). We denote the set of all simple functions on \( (X, \mathcal{M}) \) by \( L_0^+ \), and the set of all nonnegative \( \mathcal{M} \)-measurable functions by \( L^+ \). The set of all \( \mu \)-integrable functions, where \( \mu \) is a measure defined on \( (X, \mathcal{M}) \), is denoted by \( L^1(\mu) \). Rényi relative-entropy \( I_\alpha(P \parallel R) \) refers to (16), which can be written as

\[
I_\alpha(P \parallel R) = \frac{1}{\alpha - 1} \ln \int_X \varphi^\alpha dR ,
\]

(18)

where \( \varphi \in L^1(R) \) is defined as \( \varphi = \frac{dP}{dR} \).

Let \( P \) and \( R \) be two probability measures on \( (X, \mathcal{M}) \) such that \( P \ll R \). Relative entropy of partition \( \pi \in \Pi \) with \( P \) with respect to \( R \) is defined as

\[
I_P \parallel R(\pi) = \sum_{k=1}^m P(E_k) \ln \frac{P(E_k)}{R(E_k)} .
\]

(19)

Now, the GYP-theorem for KL-entropy states that

\[
I(P \parallel R) = \sup_{\pi \in \Pi} I_P \parallel R(\pi) ,
\]

(20)

where \( I(P \parallel R) \) measure-theoretic KL-entropy defined as in Definition 2.1. When \( P \) is not absolutely continuous with respect to \( R \), GYP-theorem assigns \( I(P \parallel R) = +\infty \). The proof of GYP-theorem given by Dobrushin [15] can be found in [11, pp. 23, Theorem 2.4.2] or in [24, pp. 92, Lemma 5.2.3].
4.1. GYP for Rényi Relative-Entropy

Before we state and prove the GYP-theorem for Rényi relative-entropy of order $\alpha > 1$, we state the following lemma.

**Lemma 4.1.** Let $P$ and $R$ be probability measures on the measurable space $(X, \mathcal{M})$ such that $P \ll R$. Let $\varphi = \frac{dP}{dR}$. Then for any $E \in \mathcal{M}$ and $\alpha > 1$ we have

$$\frac{P(E)^\alpha}{R(E)^{\alpha-1}} \leq \int_E \varphi^\alpha \, dR . \quad (21)$$

**Proof.** Since $P(E) = \int_E \varphi \, dR$, $\forall E \in \mathcal{M}$, by Hölder’s inequality we have

$$\int_E \varphi \, dR \leq \left( \int_E \varphi^\alpha \, dR \right)^{\frac{1}{\alpha}} \left( \int E \, dR \right)^{1-\frac{1}{\alpha}} .$$

That is

$$P(E)^\alpha \leq R(E)^{\alpha(1-\frac{1}{\alpha})} \int_E \varphi^\alpha \, dR ,$$

and hence (21) follows. Since $P \ll R$, it is clear that this inequality reduces to $0 = 0$ if $R(E) = 0$. \qed

First we present our main result in its special case as follows.

**Lemma 4.2.** Let $P$ and $R$ be two probability measures such that $P \ll R$. Let $\varphi = \frac{dP}{dR} \in \mathbb{L}_0^+$. Then for any $0 < \alpha < \infty$, we have

$$I_\alpha(P\|R) = \frac{1}{\alpha - 1} \ln \sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha-1}} , \quad (22)$$

where $\{E_k\}_{k=1}^m \in \Pi$ is the measurable partition corresponding to $\varphi$.

**Proof.** The simple function $\varphi \in \mathbb{L}_0^+$ can be written as $\varphi(x) = \sum_{k=1}^m a_k \chi_{E_k}(x)$, $\forall x \in X$, where $a_k \in \mathbb{R}$, $k = 1, \ldots, m$. Now we have $P(E_k) = \int_{E_k} \varphi \, dR = a_k R(E_k)$, and hence

$$a_k = \frac{P(E_k)}{R(E_k)} , \quad \forall k = 1, \ldots, m . \quad (23)$$

We also have $\varphi^\alpha(x) = \sum_{k=1}^m a_k^\alpha \chi_{E_k}$, $\forall x \in X$ and hence

$$\int_X \varphi^\alpha \, dR = \sum_{k=1}^m a_k^\alpha R(E_k) . \quad (24)$$

Now, from (18), (23) and (24) one obtains (22). \qed

Now we state and prove GYP-theorem for Rényi relative-entropy.
Theorem 4.3. Let \((X, \mathcal{M})\) be a measurable space and \(\Pi\) denote the set of all measurable partitions of \(X\). Let \(P\) and \(R\) be two probability measures. Then for any \(\alpha > 1\), we have

\[
I_\alpha(P\|R) = \sup_{\{E_k\}_{k=1}^m \in \Pi} \frac{1}{\alpha - 1} \ln \sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha - 1}},
\]

(25)

if \(P\|R\), otherwise \(I_\alpha(P\|R) = +\infty\).

Proof. If \(P\) is not absolutely continuous with respect \(R\), Then there exists \(E \in \mathcal{M}\) such that \(P(E) > 0\) and \(R(E) = 0\). Since \(\{E, X - E\} \in \Pi\), \(I_\alpha(P\|R) = +\infty\).

Now, we assume that \(P \ll R\). It is clear that it is enough to prove that

\[
\int_X \varphi dR = \sup_{\{E_k\}_{k=1}^m \in \Pi} \sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha - 1}},
\]

(26)

where \(\varphi = \frac{dP}{dR}\). From Lemma 4.1, for any measurable partition \(\{E_k\}_{k=1}^m \in \Pi\), we have

\[
\sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha - 1}} \leq \sum_{k=1}^m \int_{E_k} \varphi dR = \int_X \varphi dR,
\]

and hence

\[
\sup_{\{E_k\}_{k=1}^m \in \Pi} \sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha - 1}} \leq \int_X \varphi dR.
\]

(27)

Now we shall obtain the reverse inequality to prove \((26)\). That is we shall obtain

\[
\sup_{\{E_k\}_{k=1}^m \in \Pi} \sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha - 1}} \geq \int_X \varphi dR.
\]

(28)

Note that corresponding to any \(\varphi \in L^+\), there exists a sequence of simple functions \(\{\varphi_n\}, \varphi_n \in L^+_0\), which satisfies

\[
0 \leq \varphi_1 \leq \varphi_2 \leq \ldots \leq \varphi
\]

(29)

such that \(\lim_{n \to \infty} \varphi_n = \varphi\) (see \cite{16}, Theorem 1.8(2)). \(\{\varphi_n\}\) induces a sequence of measures \(\{P_n\}\) on \((X, \mathcal{M})\) defined by

\[
P_n(E) = \int_E \varphi_n(x) dR(x), \quad \forall E \in \mathcal{M}.
\]

(30)

We have \(\int_E \varphi_n dR \leq \int_E \varphi dR < \infty, \forall E \in \mathcal{M}\) and hence \(P_n \ll R, \forall n\). From the Lebesgue bounded convergence theorem, we have

\[
\lim_{n \to \infty} P_n(E) = P(E), \quad \forall E \in \mathcal{M}.
\]

(31)

Now, \(\varphi_n \in L^+_0\), \(\varphi_n \leq \varphi_n^\alpha \leq \varphi^\alpha\), \(1 \leq n < \infty\) and \(\lim_{n \to \infty} \varphi_n^\alpha = \varphi^\alpha\) for any \(\alpha > 0\). Hence from Lebesgue monotone convergence theorem \cite{25}, pp.21 we have

\[
\lim_{n \to \infty} \int_X \varphi_n^\alpha dR = \int_X \varphi^\alpha dR.
\]

(32)
The claim is that (32) implies
\[\int \varphi^\alpha dR = \sup \left\{ \int_X \phi dR \mid 0 \leq \phi \leq \varphi^\alpha, \phi \in \mathbb{L}_0^+ \right\} .\] (33)

This can be verified as follows. Denote \(\phi_n = \varphi_n^\alpha\). We have \(0 \leq \phi \leq \varphi^\alpha, \forall n, \phi_n \uparrow \varphi^\alpha\), and
\[\lim_{n \to \infty} \int_X \phi_n dR = \int_X \varphi^\alpha dR .\] (34)

For any \(\phi \in \mathbb{L}_0^+\) such that \(0 \leq \phi \leq \varphi^\alpha\) we have
\[\int_X \phi dR \leq \int_X \varphi^\alpha dR\]
and hence
\[\sup \left\{ \int_X \phi dR \mid 0 \leq \phi \leq \varphi^\alpha, \phi \in \mathbb{L}_0^+ \right\} \leq \int \varphi^\alpha dR .\] (35)

Now we get reverse inequality of (35). If \(\int_X \varphi^\alpha dR < +\infty\), from (34) given any \(\epsilon > 0\) one can find \(0 \leq n_0 < \infty\) such that
\[\int_X \varphi^\alpha dR < \int_X \phi_{n_0} dR + \epsilon\]
and hence
\[\int_X \varphi^\alpha dR < \sup \left\{ \int_X \phi dR \mid 0 \leq \phi \leq \varphi^\alpha, \phi \in \mathbb{L}_0^+ \right\} + \epsilon .\] (36)

Since (36) is true for any \(\epsilon > 0\) we can write
\[\int_X \varphi^\alpha dR \leq \sup \left\{ \int_X \phi dR \mid 0 \leq \phi \leq \varphi^\alpha, \phi \in \mathbb{L}_0^+ \right\} .\] (37)

Now let us verify (37) in the case of \(\int_X \varphi^\alpha dR = +\infty\). In this case, \(\forall N > 0\), one can choose \(n_0\) such that \(\int_X \phi_{n_0} dR > N\) and hence
\[\int_X \varphi^\alpha dR > N \quad (\because 0 \leq \phi_{n_0} \leq \varphi^\alpha)\] (38)
and
\[\sup \left\{ \int_X \phi dR \mid 0 \leq \phi \leq \varphi^\alpha, \phi \in \mathbb{L}_0^+ \right\} > N .\] (39)

Since (38) and (39) are true for any \(N > 0\) we have
\[\int_X \varphi^\alpha dR = \sup \left\{ \int_X \phi dR \mid 0 \leq \phi \leq \varphi^\alpha, \phi \in \mathbb{L}_0^+ \right\} = +\infty\] (40)
and hence (37) is verified in the case of \(\int_X \varphi^\alpha dR = +\infty\). Now (35) and (37) verifies the claim that (32) implies (33). Finally (33) together with the Lemma 4.2 proves (26) and hence the theorem. \[\square\]
4.2. GYP for Tsallis Relative-Entropy

Due to an increasing interest in long-range correlated systems and non-equilibrium phenomena there has recently been much focus on the Tsallis (or nonextensive) entropy. Although, first introduced by Havrda and Charvát \[26\] in the context of cybernetics theory and later studied by Daróczy \[27\], it was Tsallis \[28\] who exploited its nonextensive features and placed it in a physical setting. Tsallis entropy of a pdf $p$ defined on $(X, \mathcal{M}, \mu)$ can be defined as,

$$S_q(p) = \int_X p(x) \ln_q \frac{1}{p(x)} \, d\mu(x) = \frac{1 - \int_X p(x)^q \, d\mu(x)}{q - 1}, \quad (41)$$

provided the integral on the right exists and $q \in \mathbb{R}$, and $q > 0$. $\ln_q x$ is referred to as $q$-logarithm and is defined as $\ln_q x = \frac{x^{1-q} - 1}{1-q} \ (x > 0, q \in \mathbb{R})$. Tsallis entropy too, like Rényi entropy, is a one-parameter generalization of Shannon entropy in the sense that $q \to 1$ in (41) retrieves Shannon entropy. Tsallis entropy can be defined for $\mu$-continuous probability measure $P$ can be written as

$$S_q(P) = \int_X \ln_q \left( \frac{dP}{d\mu} \right)^{-1} dP . \quad (42)$$

In this framework, Tsallis relative-entropy is defined as

$$I_q(p||r) = -\int_X p(x) \ln_q \frac{r(x)}{p(x)} \, d\mu(x) = \int_X \frac{p(x)^q}{r(x)^q} \, d\mu(x) - 1, \quad (43)$$

provided all the integrals mentioned above exist and $q \in \mathbb{R}$, and $q > 0$. The same can be written for two probability measures $P$ and $R$ as

$$I_q(P||R) = -\int_X \ln_q \left( \frac{dP}{dR} \right)^{-1} dP , \quad (44)$$

whenever $P \ll R$; $I_q(P||R) = +\infty$, otherwise. If $\mu$ in (42) is a probability measure then we have

$$S_q(P) = I_q(P||\mu) . \quad (45)$$

Now, from the fact that Rényi and Tsallis relative-entropies (46) and (44) respectively) are monotone and continuous functions of each other, the GYP-theorem presented in the case of Rényi is valid for the Tsallis case too, whenever $q > 1$.

5. Conclusions

Relative-entropy or KL-entropy is an important concept in information theory, since information measures like entropy and mutual information can be formulated as special cases. Further, KL-entropy overcomes the shortcomings of entropy in non-discrete settings. Note that all the above hold even for generalized information measures.

GYP-theorem provides a means to compute KL-entropy and studying its behavior \[24\]. In this paper, we presented the measure-theoretic definitions of
generalized information measures. We stated and proved the GYP-theorem for
generalized relative entropies of order $\alpha > 1$ ($q > 1$ for the Tsallis case). However,
results are yet to be achieved for the case $0 < \alpha < 1$.

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