A Slightly Improved Bound for the KLS Constant

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October 10, 2022

Abstract

We refine the recent breakthrough technique of Klartag and Lehec to obtain an improved polylogarithmic bound for the KLS constant.

1 Introduction

The thin-shell constant, $\sigma_n$, and the KLS constant, $\psi_n$, are fundamental parameters of convex sets and log-concave densities in $n$-dimensional Euclidean space. Roughly speaking, the thin-shell constant is the width of an annulus that contains half the measure of a distribution and the KLS constant is the reciprocal of the minimum ratio, over subsets of measure at most half, of the surface measure and measure of the subset. The thin-shell constant is bounded by the KLS constant. The famous KLS conjecture posits that the KLS constant is bounded by a universal constant independent of the dimension for any isotropic log concave density \cite{1}. For background on the conjecture and its myriad connections, see e.g., \cite{2}. In a recent breakthrough paper, Klartag and Lehec established the following bounds.

Theorem 1 (\cite{3}). $\sigma_n \lesssim \log^4 n$, $\psi_n \lesssim \log^5 n$.\footnote{Throughout this paper, we use $a \lesssim b$ to denote $a = O(b)$ and $a \approx b$ to denote $a = \Theta(b)$.}

This improved on previous bounds of Chen \cite{1} and Lee-Vempala \cite{4}. We give the following further improvement.

Theorem 2. $\sigma_n \lesssim \log^{2.226} n$, $\psi_n \lesssim \log^{3.226} n$.

1.1 Background

In this section, we recap relevant definitions and results from \cite{3, 4, 1, 3}. Recall that a density $p$ in $\mathbb{R}^n$ is log-concave if it is nonnegative, has an integral of 1, and its logarithm is concave, i.e., for any $x, y \in \mathbb{R}^n$, $\lambda \in [0, 1]$, we have $p(\lambda x + (1 - \lambda)y) \geq p(x)^\lambda p(y)^{1-\lambda}$. A logconcave density $p$ is said to be $\alpha$-strongly log-concave or $\alpha$-uniformly log-concave if the density can be written as $p(x) = e^{-\alpha\|x\|^2/2}q(x)$ where $q$ is a log-concave function.

Definition 3 (Constants). For a distribution $\mu$ in $\mathbb{R}^n$, we define the following constants

- Thin-shell constant: $\sigma^2_\mu = \frac{\text{Var}_\mu(\|x\|^2)}{n}$.
- $\kappa^2_\mu = \sup_{\|\theta\|=1} \left\| E_{x \sim \mu} \left[ (x, \theta) x^\top \right] \right\|^2_F$.
- KLS constant: $\frac{1}{\psi_\mu} = \inf_{A \subseteq \mathbb{R}^n} \left\{ \frac{\int_A \mu(A) \, d\mu}{\min(\mu(A), 1 - \mu(A))} \right\}$ where the infimum runs over all open sets $A \subseteq \mathbb{R}^n$ with smooth boundary.

We define $\sigma_n$ be the supremum $\sigma_\mu$ over all isotropic log-concave distributions $\mu$ in $\mathbb{R}^n$. We define $\kappa_n$ and $\psi_n$ similarly.

To relate the above constants, Eldan introduced the technique of stochastic localization.

Definition 4 (Stochastic localization). For any log-concave density $p$ in $\mathbb{R}^n$, we define the following stochastic processes on log-concave densities $p_t$ with initial density $p_0 = p$: 
• Eldan’s stochastic localization [3]: \( dp_t(x) = p_t(x - a_t)^\top A_t^{-1} dW_t \) where \( a_t \) and \( A_t \) are the mean and covariance of \( p_t \) respectively.

• Lee-Vempala variant [11]: \( dp_t(x) = p_t(x - a_t)^\top dW_t \).

For brevity, we refer to the first process as the Eldan process and to the second as the LV process. Both processes are martingales in the space of densities.

Eldan showed how to bound \( \psi_n \) by controlling the operator norm of \( A_t \), the covariance of the density \( p_t \) at time \( t \). In particular, he derived the following relation.

**Lemma 5** ([3, Proposition 1.7, Lemma 1.6]).

\[
\psi_n^2 \lesssim \kappa_n^2 \log n \lesssim \sigma_n^2 \log^2 n.
\]

Following [3], Lee and Vempala [11] suggested their variant. The benefit of the LV process is that \( \|A_t\|_{op} \) can be controlled by \( \text{Tr}(A_t^2) \), which in turn can be bound by inspection of its derivatives. Analysis of the LV process crucially involves bounding the following object, which is one of the terms in the derivative of \( \text{Tr}(A_t^2) \).

**Definition 6.** For any log-concave distribution \( \mu \) on \( \mathbb{R}^n \) with mean \( a \) and for any set of symmetric matrices \( A_i \), let

\[
T_\mu(A_1, A_2, A_3) = E_{x,y \sim \mu} \prod_{i=1}^{3}(x - a_1)^\top A_i (y - a)
\]

where \( x \) and \( y \) are independent samples from \( \mu \). We use the same notation \( T_\mu \) for any density \( \mu \).

By considering the case of \( q = 2 \), i.e., \( \text{Tr}(A_t^2) \), [11] gave a bound of \( \psi_n \lesssim n^{1/4} \). Chen gave a substantially better bound of \( \log \psi_n \lesssim \sqrt{\log n} \log \log n \) by choosing \( q = \omega(1) \) and complementing [11]'s analysis with an improved bound on \( T_\mu \) for all \( t \)-strongly log-concave \( \mu \):

**Lemma 7** ([1, Lemma 11]). For any \( \alpha \)-strongly log-concave \( \mu \) with covariance \( A \) and for any \( q \geq 3 \), we have

\[
T_\mu(A^{q-2}, I, I) \lesssim \frac{4}{\alpha} \text{Tr} A^q.
\]

All previous works [3, 11, 1] bound \( \psi_n \) via \( \|A_t\|_{op} \). The key idea of Klartag and Lehec [9] is to bound \( \sigma_n^2 \) via \( \|a_t\|_2^2 \) (where \( a_t \) is the mean of \( p_t \)), and then bound \( \psi_n \) via \( \sigma_n \). The benefit is that \( \|a_t\|_2^2 \) involves a 2-norm and thus possesses favorable smoothness properties compared to \( \text{Tr}(A_t^2) \) considered in prior work. This leads to a polylogarithmic bound on \( \psi_n \). We summarize their technique below and discuss its application in the next section.

**Lemma 8** ([9, Eqn 45, 47]). For any log-concave density \( \rho \) in \( \mathbb{R}^n \), let \( p_t \) be the density given by the LV process starting at \( p \). Let \( a_t \) be the mean of \( p_t \). Then, we have

\[
\sigma_p^2 \lesssim \int_{\lambda_1}^{\infty} \min_{\lambda > 0} \left\{ \frac{1}{n\lambda^2} \mathbb{E} \|a_t\|_2^2 + \frac{1}{\lambda M\lambda} \right\} d\lambda
\]

where \( \lambda_1 = \Omega(\psi_p^{-2}) \) and the expectation is taken over the LV process.

### 1.2 Discussion of the Klartag-Lehec bound

In this section, we explain how Lemma 8 leads to the result \( \sigma_n \lesssim \log^4 n \). The bound on \( \psi_n \) follows from Lemma 5. The intuition of this argument is important for understanding our improvement.

To apply Lemma 8, Klartag and Lehec start with an arbitrary isotropic log-concave density \( \rho \) and apply stochastic localization, which gradually makes the distribution more and more uniformly log-concave by creating a larger and larger Gaussian term in the density. Roughly speaking, the idea then is that if the thin-shell constant \( \sigma_p \) is large, this implies that the squared length of the mean is large at a small time. But the localization process also proves that the squared length of the mean cannot grow quickly — for a small time interval, it in fact remains small. Comparing these bounds allows us to conclude the thin-shell constant cannot be too large.

More precisely, we recall that

\[
\sigma_p^2 \lesssim \int_{\lambda_1}^{\infty} \min_{\lambda > 0} \left\{ \frac{1}{n\lambda^2} \mathbb{E} \|a_t\|_2^2 + \frac{1}{\lambda M\lambda} \right\} d\lambda.
\]
To give some intuition of this integral, for some large enough constant $C$, we set

$$t_\lambda \triangleq \frac{C}{\sigma_p^2} \log^2 \left( \frac{\lambda}{\lambda_1} \right)$$

and let $\lambda_2 \triangleq C\lambda_1 \log^2 n$. Then we have

$$\sigma_p^2 \lesssim \int_{\lambda_1}^{\lambda_2} \frac{\mathbb{E}[|a_{t\lambda}|^2]}{n\lambda^2} d\lambda + \int_{\lambda_1}^{\lambda_2} \frac{\sigma_p^2}{n\lambda^2} d\lambda$$

$$\lesssim \int_{\lambda_1}^{\lambda_2} \frac{\mathbb{E}[|a_{t\lambda}|^2]}{n\lambda^2} d\lambda + \int_{\lambda_1}^{\lambda_2} \frac{\mathbb{E}[|a_{t\lambda}|^2]}{n\lambda^2} d\lambda + \int_{\lambda_1}^{\lambda_2} \frac{\sigma_p^2}{n\lambda^2} d\lambda$$

where the first line uses the definition of $t_\lambda$, the second line splits the integral at $\lambda_2$, and the third line uses the known fact that $\mathbb{E}[|a_t|^2] \leq n$ for all $t^2$ and the inequalities (for sufficiently large constant $C$)

$$\int_{\lambda_1}^{\lambda_2} \frac{1}{C\lambda \log^2 (\lambda/\lambda_1)} d\lambda \lesssim 1 \quad \text{and} \quad \int_{\lambda_1}^{\lambda_2} \frac{\mathbb{E}[|a_{t\lambda}|^2]}{n\lambda^2} d\lambda \lesssim \int_{\lambda_1}^{\lambda_2} \frac{1}{\lambda^2} d\lambda = \frac{1}{\lambda_2} \lesssim \frac{\psi_p^2}{C\log^2 n} \leq \frac{\sigma_n^2}{4}.$$

By picking the worst distribution $p$ such that $\sigma_p = \sigma_n$, we have

$$\sigma_n^2 \lesssim \int_{\lambda_1}^{\lambda_2} \frac{\mathbb{E}[|a_{t\lambda}|^2]}{n\lambda^2} d\lambda. \quad (1.1)$$

Note that the range of $\lambda$ in the integral is quite small: it is between $\lambda_1$ and $O(\log^2 n) \cdot \lambda_1$. Hence, our choice of $t_\lambda$ is at most $C\sigma_n^{-2}(\log \log n)^2$. To complete the analysis, we need to analyze $\mathbb{E}[|a_t|^2]$ for $t$ near $\sigma_n^{-2}$. By Itô’s formula, one can calculate that

$$\frac{d}{dt} \mathbb{E}[|a_t|^2] = \text{Tr} A_t^2 \quad \text{and} \quad \frac{d}{dt} \mathbb{E}[\text{Tr} A_t^2] \leq T_p(I, I, I). \quad (1.2)$$

The key part of their analysis is to bound $T_p(I, I, I)$.

Many quantities in the stochastic localization process (both Eldan and LV versions) such as $a_t$, $A_t$ and $T_p(I, I, I)$ are well-behaved when $\|A_t\|_{op} = O(1)$. Since $\|A_t\|_{op} \leq 2$ for $0 \leq t \leq \frac{1}{c \sigma_n^2 \log n}$ for some universal constant $c$, one can show that $\text{Tr} A_t^2 \lesssim n$ and $\mathbb{E}[|a_t|^2] \lesssim t \cdot n$ for $t$ in this range.

Beyond time $t_1 = \frac{1}{c \sigma_n^2 \log n}$, Chen showed that $\text{Tr} A_t^2$ can only grow polynomially as

$$\text{Tr} A_t^2 \lesssim \left( \frac{t}{t_1} \right)^{O(q^2)} n$$

using Lemma 7. Unfortunately, his proof only works for $q \geq 3$. Although we could indirectly bound $\frac{d}{dt} \mathbb{E}[|a_t|^2]$ by using the inequality $\text{Tr} A_t^2 \leq (\text{Tr} I)^{1/3}(\text{Tr} A_t^2)^{2/3}$, as we will see, the degree of this polynomial growth directly affects the exponent of the logarithm in the final bound. To get an improved bound for $q = 2$, Klartag and Lehec gave a very different proof. They showed

$$T_p(I, I, I) \leq \frac{\gamma}{\alpha} \text{Tr} A^2$$

with $\gamma = 3$ \quad (1.3)

for any $\alpha$-strongly log-concave distribution $\mu$ with covariance $A$. Throughout this paper, we use $\gamma$ to denote the best bound in (1.3). We will later show that $\gamma \leq 2\sqrt{2}$.

Using (1.3) and (1.2),

$$\mathbb{E}[|a_t|^2] \lesssim t \cdot \left( \frac{t}{t_1} \right)^{\gamma} n \quad \text{for } t \geq t_1.$$

With this growth bound on $\mathbb{E}[|a_t|^2]$, using (1.1) we have,

$$\sigma_n^2 \lesssim \int_{\lambda_1}^{\lambda_2} \frac{\mathbb{E}[|a_{t\lambda}|^2]}{n\lambda^2} d\lambda \lesssim \int_{\lambda_1}^{\lambda_2} \frac{\sigma_p^2}{n\lambda^2} d\lambda \lesssim \int_{\lambda_1}^{\lambda_2} \frac{\sigma_p^{2\gamma-2} \log^{2\gamma-2}(\lambda/\lambda_1)}{n\lambda^2} d\lambda \lesssim \sigma_p^{-2\gamma-2} \lambda_1^{1-1}\gamma.$$

\(^2\)To see this, note that $a_0 = 0$, \( \frac{d}{dt} \mathbb{E}[|a_t|^2] = \text{Tr} A_t^2 \geq 0 \), the distribution of $a_t$ tends to $p_0$ as $t \to \infty$, and $\mathbb{E}[p_0(\|x\|^2)] = n.$
Noting $t_1 \approx \frac{1}{\kappa_n \log n}$ and $\psi_n^2 \lesssim \kappa_n^2 \log n \lesssim \sigma_n^2 \log^2 n$, we have

$$t_1^{-\gamma} \lesssim \kappa_n^{2\gamma} \log^{\gamma} n \lesssim \sigma_n^{2\gamma} \log^{2\gamma} n \quad \text{and} \quad \lambda_t^{-1} \lesssim \psi_n^2 \lesssim \sigma_n^2 \log^2 n.$$ 

Substituting the above and choosing constants appropriately yields

$$\sigma_n \lesssim \log^{\gamma+1} n.$$ 

Note that the exponent $\gamma + 1$ above exactly depends on the parameter $\gamma$ defined in (1.3).

Moreover, if there is no gap between the KLS and thin-shell constants (namely, $\psi_n \lesssim \kappa_n \lesssim \sigma_n$ rather than $\psi_n \lesssim \kappa_n \sqrt{\log n} \lesssim \sigma_n \log n$), then by essentially the same proof one can get an improved bound of

$$\sigma_n \lesssim \log^2 n.$$ 

### 1.3 Our contributions

**Unconditional bound on $T_p(I, I, I)$**. Our first contribution is to show $\gamma \leq 2\sqrt{2}$ (Lemma 15). To do this, we first give a refined decomposition of strongly log-concave densities via the Eldan process (Lemma 14). With this decomposition, we follow the argument of Klartag and Lehec (partially inspired by [6]). We think this decomposition may be of independent interest.

**Conditional bound on $T_p(I, I, I)$**. If $\gamma = o(1)$, then we could have an almost logarithmic bound on $\sigma_n$. For one-dimensional log-concave densities, the standard localization lemma together with a computer search suggests that $\gamma \approx 0.37$ (see Appendix A). Our second contribution is to show $\gamma = o(1)$ whenever the covariance matrix $A$ is sufficiently “spiked”. More formally, Lemma 18 proves the following implication:

$$\text{Tr} A^2 = \frac{o(1)}{\alpha} \text{Tr} A^2 \implies T_p(I, I, I) \leq \frac{o(1)}{\alpha} \text{Tr} A^2$$

Curiously, its proof requires the use of the LV process and does not work with the Eldan process.

Next, we prove that the above condition on $\text{Tr} A^2$ is satisfied for (small) positive time beyond the threshold $t_1$. Recall that $t_1 \approx (\kappa_n^2 \log n)^{-1}$ while $t_{1.5} \approx \sigma_n^{-2} \lesssim \kappa_n^{-2} \log n$. Our new bound works well from $t_1$ to $t_{1.5} \approx \kappa_n^{-2}$ and essentially shows that $\|a_t\|^2$ does not grow much in that time period. Beyond this point, we can use the unconditional bound above. Therefore, it effectively “halves” the dependence on $\gamma$. The main lemma of this part (Lemma 21) gives an improved growth bound on the norm of $\|a_t\|^2$ up to a time slightly beyond $t_{1.5}$ to a time we call $t^*$. Our final bound is $\sigma_n \lesssim \log^{2.2226} n$ (Theorem 22).

**Sub-logarithmic bound assuming $\psi_n \approx \sigma_n$.** As discussed in Section 1.2, if there is “no gap” between KLS and thin-shell, i.e., the KLS constant is within a universal constant factor of the thin-shell constant, then the Klartag-Lehec bound becomes $\sigma_n \lesssim \log^2 n$. With our analysis above, we can instead show that “no gap” implies $\sigma_n \lesssim \log^{0.6476} n$, a sub-logarithmic bound, even for the current $\gamma = 2\sqrt{2}$ (Theorem 23).

### 1.4 Preliminaries

In this section, we list various facts and results we use in this paper.

**Lemma 9 ([2, Lemma 2]).** For any $\alpha$-strongly log-concave distribution $\mu$ and any smooth function $f \in L_2(\mu)$ such that $\int f d\mu = \int \nabla f d\mu = 0$, one has

$$\int f^2 d\mu \leq \frac{1}{2\alpha} \int \|\nabla f\|^2 d\mu.$$ 

**Lemma 10** (Properties of the Eldan process). Let $p_t$ given by Eldan’s process. Let $a_t$ and $A_t$ be its mean and covariance. Then, we have

- $da_t = A_t^{1/2} dW_t$ [3, Eqn 16].
- $dA_t = E_{x \sim p_t} (x - a_t) (x - a_t)^\top (x - a_t)^\top A_t^{-1/2} dW_t - A_t dt$ [3, Page 16].
- If the starting distribution $p_0$ is 1-strongly logconcave, then $\|A_t\|_{\text{op}} \leq e^{-t}$ [4, Lemma 6].
Lemma 11 (Properties of the LV process). Let \( p_t \) given by the LV process. Let \( a_t \) and \( A_t \) be its mean and covariance. Then, we have

- \( da_t = A_t dW_t \) [11, Page 12].
- \( dA_t = E_{x \sim p_t} (x - a_t)(x - a_t)^\top dW_t - A_t^2 dt \) [11, Lemma 28].
- \( p_t(x) \propto e^{-\frac{1}{2}A_t^2 \|x\|^2} \) [11, Definition 26].

The next lemma summarizes properties of \( T \); the equivalent isotropic versions are stated in [6].

Lemma 12 (Properties of \( T_\mu \)). For any log-concave distribution \( \mu \) with covariance \( A \) and any symmetric matrices \( M \) and \( N \), we have

- \( T_\mu (M_1, M_2, M_3) \geq 0 \) for any positive definite matrices \( M_1, M_2, M_3 \) [6, Lemma 39].
- \( T_\mu (M, I, I) \preceq \kappa_n^2 \cdot \text{Tr}[A^{1/2}MA^{1/2}] \cdot \|A\|_{\text{op}}^2 \) [6, Lemma 40].
- \( T_\mu (M, N, I) \leq T_\mu ([M]^{1/2}, I, I)^{1/s} \cdot T_\mu ([N]^{1/2}, I, I)^{1/t} \) for \( s, t \geq 1 \) with \((1/s) + (1/t) = 1 \) [6, Lemma 40].
- \( T_\mu (N^{1/2}M \cdot N^{1/2}, N^{1/2}M^{1-\alpha}N^{1/2}, C) \leq T_\mu (N^{1/2}MN^{1/2}, N, C) \) for any PSD matrices \( M, N, C \) and \( \alpha \in [0, 1] \) [6, Lemma 41].

Lemma 13 (Growth of the LV process [9, Lemma 5.2]). Let \( p_t \) given by the LV process with an initial isotropic log-concave distribution \( p_0 \). Let \( A_t \) be its covariance. For \( T \leq \frac{1}{c \log n} \) with large enough constant \( c \), we have

\[
\mathbb{P} \left[ \|A_t\|_{\text{op}} \geq 2 \right. \left. \text{ for } 0 \leq t \leq T \right] \leq \exp(-1/(cT)).
\]

2 Bounding \( T_\mu (I, I, I) \)

In this section, we have two ways to bound \( T_\mu (I, I, I) \). The first way is unconditional and relies on Eldan’s process. The second way is conditional and relies on the LV process. Our overall proof will effectively use both variants of localization.

2.1 Unconditional Bound

By Eldan’s process, we have the following decomposition of 1-strongly log-concave distributions. This lemma is a slightly strengthened version of [9, Lemma 4.1] with an extra conclusion \( \mathbb{E}Q_t^2 = e^{-t}A \). The proof follows from the remark at the end of the proof of [9, Lemma 4.1].

Lemma 14. For any 1-strongly log-concave distribution \( \mu \) with mean 0, we have

\[
\mu \sim \int_0^\infty Q_t dW_t
\]

for positive definite matrices \( Q_t \) with \( 0 \preceq Q_t \preceq e^{-t/2}I \). Moreover, we have

\[
\mathbb{E}Q_t^2 = e^{-t}A
\]

where \( A \) is the covariance of \( \mu \) and the expectation is over the stochastic process generating \( Q_t \).

Proof. We define \( p_t \) according to the Eldan process with initial density given by \( d\mu \). For \( a_t \), the mean of \( p_t \), Lemma 10 shows

\[
da_t = A_t^{1/2} dW_t \quad \text{and} \quad \|A_t\|_{\text{op}} \leq e^{-t}
\]

where \( A_t \) is the covariance of the \( p_t \). Since \( \mathbb{E}p_t = p_0 \) (by definition) and that \( p_\infty \) is a delta measure at \( a_\infty \) (because \( \|A_\infty\|_{\text{op}} = 0 \)), we have

\[
\mu \sim a_\infty = \int_0^\infty A_t^{1/2} dW_t.
\]

This gives the first part with \( Q_t = A_t^{1/2} \).

For the second part, Lemma 10 shows that

\[
dA_t = E_{x \sim p_t} (x - a_t)(x - a_t)^\top A_t^{-1/2} dW_t - A_t dt
\]

Taking expectations, we have \( \frac{d\mu}{dt} A_t = -E A_t \). Solving it, we have \( E A_t = e^{-t}A \). 

\[\square\]
We can use the above decomposition to bound $T_\mu(I,I,I)$ for $\alpha$-strongly log-concave distributions. The proof closely follows that of [9, Lemma 4.2], except that we avoid one Cauchy-Schwarz, and use the stronger bound on the operator norm of $Q_t$ in the decomposition above.

**Lemma 15.** For any $\alpha$-strongly log-concave distribution $\mu$ with covariance $A$, we have

$$T_\mu(I,I,I) \leq \frac{2\sqrt{2}}{\alpha} \text{Tr}A^2.$$

**Proof.** By scaling $\mu$ and shifting, we can assume $\alpha = 1$ and $\mu$ has mean $0$. Lemma 14 shows that $\mu \sim \int_0^\infty Q_s dW_s$. For any $x \sim \mu$, we define the random path $x_t$ by $x_t = \int_0^t Q_s ds$: note that $x_\infty = x$. By Itô’s formula, we have

$$T_\mu(I,I,I) = E_{x,y} (x^\top y)^3 = 3 \int_0^\infty E_{x,y} x_t^\top y \cdot |Q_t y|^2 dt$$

$$\leq 3 \int_0^\infty E_x \sqrt{E_y (x_t^\top y)^2} \cdot \text{Var}|Q_t y|^2 dt.$$

The inequality follows from $E_y x_t^\top y \cdot |Q_t y|^2 = E_y x_t^\top y \left( |Q_t y|^2 - (E|Q_t y|^2)^2 \right)$ (since $E[y] = 0$) and the Cauchy-Schwarz inequality.

Next, since $\mu$ is 1-strongly log-concave and the gradient of $|Q_t y|^2$ has mean 0, Lemma 9 yields

$$\text{Var}|Q_t y|^2 \leq \frac{1}{2} E[2|Q_t y|^2]^2 = 2\text{Tr}Q_t A.$$

Hence, we have

$$T_\mu(I,I,I) \leq 3\sqrt{2} \int_0^\infty E_x \sqrt{E_y (x_t^\top y)^2} \cdot \text{Tr}Q_t A dt$$

$$\leq 3\sqrt{2} \int_0^\infty e^{-t/2} E_x \sqrt{x_t^\top A x_t} \cdot \text{Tr}Q_t A dt$$

where we used $Q_t \leq e^{-t/2}I$ and that $y$ has mean 0 and covariance $A$. By Lemma 14, we have $E Q_s^2 = e^{-t} A$ and

$$E x_t x_t^\top = \int_0^t E Q_s^2 ds = (1 - e^{-t}) A.$$

By another application of Cauchy-Schwarz, we have

$$T_\mu(I,I,I) \leq 3\sqrt{2} \int_0^\infty e^{-t/2} \cdot \sqrt{\text{Tr}E x_t x_t^\top A \cdot \text{Tr}Q_t A} dt$$

$$= 3\sqrt{2} \int_0^\infty e^{-t/2} \cdot \sqrt{(1 - e^{-t}) \cdot e^{-t} dt} \cdot \text{Tr} A^2$$

$$= 2 \sqrt{2} \cdot \text{Tr} A^2. \quad \square$$

### 2.2 Conditional Bound

Our conditional bound also follows the same framework as the previous bound. We first give a stochastic decomposition of $\mu$ based on the LV process. Then we use it to bound $T_\mu(I,I,I)$.

**Lemma 16.** Let $\mu$ be an $\alpha$-strongly log-concave distribution with mean $0$. Then,

$$\mu \sim \int_0^\infty Q_t dW_t$$

for positive definite matrices $Q_t$ with $0 \preceq Q_t \preceq \frac{1}{\alpha+1} I$. Furthermore, we have

$$E Q_t \preceq A$$

where $A$ is the covariance of $\mu$ and the expectation is over the stochastic process generating $Q_t$. 


Proof. We follow the proof of Lemma 14, but define $p_t$ in terms of the LV process. In this case, Lemma 11 shows that $p_t(x) \propto e^{-\frac{1}{2} \|x\|^2} p_0(x)$ and that
\[
d a_t = A_t dW_t
\]
where $A_t$ is the covariance of $p_t$. Since $p_0$ is $\alpha$-strongly log-concave, we have that $p_t$ is $(\alpha + t)$-strongly log-concave, and therefore
\[
\|A_t\|_{\text{op}} \leq \frac{1}{\alpha + t}.
\]
This gives the first part with $Q_t = A_t$. For the second part, Lemma 11 shows that
\[
d A_t = E_{x \sim p_t}(x-a_t)(x-a_t)^\top (x-a_t)^\top dW_t - A_t^2 dt
\]
Taking expectations, we have $\frac{d}{dt} E A_t \leq 0$. Hence, we have $E A_t \leq A$.

This lemma can be used in place of Lemma 14 in the proof of Lemma 15. However, doing so leads to a worse constant because this lemma does not capture the fact that $E Q_t$ is decreasing with $t$. The benefit of this decomposition is that $\text{Tr} Q_t^q$ can be controlled by $\text{Tr} A^q$ throughout the process, as we prove below.

Lemma 17. For any $\alpha$-strongly log-concave distribution $\mu$ with covariance $A$, let $p_t$ be the result of the LV process with initial density $p_0$ given by $\mu$. Let $A_t$ be the covariance of $p_t$. For any $q \geq 3$ we have
\[
E \text{Tr} A_t^q \leq \left(1 + \frac{t}{\alpha}\right)^{q(q-1)/2} \text{Tr} A^q.
\]
In particular, the $Q_t$ defined in Lemma 16 satisfies $E \text{Tr} Q_t^q \leq (1 + \frac{t}{\alpha})^{q(q-1)/2} \text{Tr} A^q$.

Proof. Let $a_t$ be the mean of $p_t$. Lemma 27 with $b = 0$ shows that
\[
\frac{d}{dt} E \text{Tr} A_t^q \leq \frac{q(q-1)}{2} T_{p_t}(A_t^{q-2}, I, I).
\]
Since $p_t$ is $(\alpha + t)$-strongly log-concave (Lemma 11), we can use Lemma 24 to bound the last term to get
\[
\frac{d}{dt} E \text{Tr} A_t^q \leq \frac{q(q-1)}{\alpha + t} \text{Tr} A_t^q
\]
Solving this equation yields the first result. The second result follows from the fact that $Q_t$ in Lemma 16 is exactly $A_t$.

Using the new stochastic decomposition (Lemma 16) and the new property about $\text{Tr} Q_t^q$ (Lemma 17), we have an improved bound on $T_\mu(I, I, I)$.

Lemma 18. Let $\mu$ be an $\alpha$-strongly log-concave distribution with covariance $A$. For $q$ such that $8 \geq q \geq 3$, assume that $\text{Tr} A^q \leq \frac{1}{\alpha^{8/3-2}} \text{Tr} A^2$. Then,
\[
T_\mu(I, I, I) \leq \frac{12}{\alpha^{1/(2(q^2-2))}} \text{Tr} A^2.
\]
Alternatively,
\[
T_\mu(I, I, I) \leq \frac{12}{\alpha^3} \cdot (\alpha^3 \text{Tr} A^q)^c \left(\alpha^3 \text{Tr} A^2\right)^{1-c}
\]
with $c = \frac{1}{2(q^2-2)}$.

Proof. By centering, we can assume $\mu$ has mean 0. Lemma 16 shows that $\mu \sim \int_0^\infty Q_s ds$. For $x \sim \mu$, we define the random path $x_t$ by $x_t = \int_0^t Q_s ds$. Note that $x_\infty = x$. By Itô’s formula and the Cauchy-Schwarz inequality, we have
\[
T_\mu(I, I, I) = E_{x,y \sim \mu} (x^\top y)^3 = 3 \int_0^\infty E_{x,y} x^\top y : |Q_t y|)^2 dt
\]
\[
\leq 3 \int_0^\infty E_x \sqrt{E_y (x^\top y)^2} \cdot \text{Var} |Q_t y|^2 dt.
\]
(2.1)
Since $\mu$ is $\alpha$-strongly log-concave and the gradient of $|Q_t y|^2$ has mean 0, Lemma 9 shows that
\[
\text{Var}|Q_t y|^2 \leq \frac{1}{2\alpha} \mathbb{E}|2Q_t y|^2 = \frac{2}{\alpha} \text{Tr} Q_t^4 A.
\] (2.2)

Up to this point, the proof is the same as that of Lemma 15. Since $x_t$ is a martingale, we have
\[
A = \mathbb{E}_x x_\infty x_\infty^\top = \mathbb{E}_x x_t x_t^\top + \mathbb{E}_x (x_\infty - x_t)(x_\infty - x_t)^\top.
\]
In particular, this shows that $\mathbb{E}_x x_t x_t^\top \preceq A$. Using this, (2.1) and (2.2), we have
\[
T_\mu(I, I, I) \leq 3 \sqrt{\frac{2}{\alpha}} \int_0^\infty \sqrt{\text{Tr} A \cdot \text{Tr} Q_t^4 A dt}.
\] (2.3)

Now we split the proof into two cases for bounding the term $\mathbb{E} \text{Tr} Q_t^4 A$.
In the first case, Lemma 16 shows that $Q_t \preceq \frac{1}{\alpha t} I$, and $\mathbb{E} Q_t \preceq A$. Therefore, we have
\[
\mathbb{E} \text{Tr} Q_t^4 A \leq \frac{1}{(\alpha + t)^3} \mathbb{E} \text{Tr} Q_t A \leq \frac{1}{(\alpha + t)^3} \text{Tr} A^2.
\]

In the second case, Lemma 17 together with the assumption of the current lemma shows that
\[
\text{Tr} Q_t^q \leq \left(1 + \frac{t}{\alpha}\right)^{q(q-1)} \text{Tr} A^q \leq \frac{\alpha^{-q(2)}}{\zeta} \left(1 + \frac{t}{\alpha}\right)^{q(q-1)} \text{Tr} A^2.
\]

Therefore, for $q \leq 8$, we have
\[
\mathbb{E} \text{Tr} Q_t^4 A \begin{cases} 
(i) \leq \frac{\mathbb{E} \text{Tr} Q_t^q A}{(\alpha + t)^{4-q/2}} \leq \frac{\sqrt{\mathbb{E} \text{Tr} Q_t^q A \cdot \text{Tr} A^q}}{(\alpha + t)^{4-q/2}} \leq \frac{\alpha^{1-q/2}(1 + \frac{t}{\alpha})^{q(q-1)/2}}{(\alpha + t)^{4-q/2}\zeta^{1/2}} \text{Tr} A^2. 
(ii) \leq \frac{\int_0^\infty \frac{1}{\alpha^{1/4}} \frac{\text{Tr} A^2}{(\alpha + t)^{q(q-1)/4}} dt}{(\alpha + t)^{4-q/2}\zeta^{1/2}} \text{Tr} A^2.
\end{cases}
\]

Inequality (i) used $Q_t \preceq \frac{1}{\alpha t} I$, (ii) used the Cauchy-Schwarz inequality, and (iii) follows from Lemma 17.

Observe that the above two bounds are equal at $t = \alpha s^*$, where $s^* = \zeta^{1/q-2} - 1$. Thus we have the bound
\[
\mathbb{E} \text{Tr} Q_t^4 A \leq \begin{cases} 
\frac{1}{(\alpha + t)^{0}} \frac{\text{Tr} A^2}{(\alpha + t)^{q(q-1)/4}} \text{Tr} A^2 & \text{if } \frac{1}{\alpha} \geq s^* , \\
\frac{1}{(\alpha + t)^{1/4}} \frac{\text{Tr} A^2}{(\alpha + t)^{q(q-1)/4}} \text{Tr} A^2 & \text{otherwise} .
\end{cases}
\]

Substituting this into 2.3 and splitting the integral yields
\[
T_\mu(I, I, I) \leq \frac{1}{3\alpha^{1/4}} \int_0^{s^*} \frac{(1 + u)^{2-q/2}}{(\alpha + t)^{q(q-1)/4}} du + \frac{1}{\alpha^{1/4}} \int_{s^*}^\infty \left(1 + (1 + u)^{2-q/2}\right) du.
\]
Substituting $u = \frac{t}{\alpha}$, we have
\[
T_\mu(I, I, I) \leq \frac{1}{\alpha^{1/4}} \int_0^{s^*} (1 + u)^{2-q/2} du + \frac{1}{\alpha^{1/4}} \int_{s^*}^\infty \left(1 + (1 + u)^{2-q/2}\right) du
\]
\[
= \frac{4}{(q^2 - 4)\alpha}\left(1 + s^*\right)^{q/4 - 1} \zeta^{1/2} + \frac{2}{\alpha}\left(1 + s^*\right)^{q/4 - 1} \zeta^{1/2}.
\]

Further simplifying, we have
\[
\frac{T_\mu(I, I, I)}{3\sqrt{2\text{Tr} A^2}} \leq \frac{4}{(q^2 - 4)\zeta^{2(q/4-2)}} + \frac{2}{\zeta^{2(q/4-2)}} \leq \frac{2.8}{\zeta^{2(q/4-2)}}.
\]

This gives the first claim. The second claim follows by setting $\zeta = \alpha^{-(q-2)} \frac{\text{Tr} A^2}{\text{Tr} A^*(}\zeta^{1/2}).$
3 Bounding $\text{Tr} A_t^q$

In this section, we derive refined bounds on $\text{Tr} A_t^q$. We will use these to apply our conditional bounds on $T_{\mu}(I, I, I)$. We first bound $\text{Tr} A_t^q$ when $t$ is small. Let $A^+$ denote the restriction of a symmetric matrix $A$ to the span of its positive eigenvectors.

**Lemma 19.** Let $p_t$ be given by the LV process with an initial isotropic log-concave density $p_0$. Assume that $p_0$ is supported on a ball with radius $n$. Let $A_t$ be the covariance of $p_t$, $\mathbf{A}_t = (A_t - I)^+$, and $t_1 = \frac{1}{c n^2 \log n}$ for some large enough constant $c$ depending only on $q$. Then, for any $0 \leq t \leq t_1$ and any $q \geq 3$, we have

$$E \text{Tr} \mathbf{A}_t^q \leq 1 + (C q n^2 t)^{q/2} \cdot n$$

for some universal constant $C > 0$.

**Proof.** Let $\Phi_t = \text{Tr} \mathbf{A}_t^q$, and $E_t$ be the event that $\|A_s\|_{\text{op}} \leq 2$ for all $0 \leq s \leq t$. To bound the trace for the case $\|A_s\|$ is small, we define

$$\Psi_t = E[\Phi_t \cdot 1_{E_t}].$$

Lemma 27 (with $b = 1$) shows that

$$\frac{d}{dt} \Psi_t \leq \frac{q(q - 1)}{2} T_{p_t}(\mathbf{A}_t^{-2}, I, I)$$

where the expectation is over all randomness before time $t$. Lemma 12 followed by Cauchy-Schwarz shows that

$$T_{p_t}(\mathbf{A}_t^{-2}, I, I) \leq \kappa_n^2 \cdot \text{Tr} \mathbf{A}_t^{-2} \cdot \|A_t\|_{\text{op}}^2 \leq \kappa_n^2 \cdot \text{Tr} \mathbf{A}_t^{-2} \cdot \|A_t\|_{\text{op}}^3 \leq \kappa_n^2 \cdot \Phi_t \frac{1}{2} \cdot \frac{n^{3/2}}{n^{3/2}}\cdot \|A_t\|_{\text{op}}^3.$$

Since $f(x) = x^{1-\frac{3}{2}}$ is concave, Jensen’s inequality implies

$$\frac{d}{dt} \Psi_t \leq q^2 \kappa_n^2 \cdot E[\Phi_t^{1-\frac{3}{2}} 1_{E_t}] n^{3/2} \leq q^2 \kappa_n^2 \Psi_t^{1-\frac{3}{2}} n^{3/2}.$$

Hence, we have $\frac{d}{dt} \Psi_t^{2/q} \leq q^2 \kappa_n^2 n^{3/2}$. Since $\Psi_0 = 0$, we have

$$\Psi_t \leq (C q n^2 t)^{q/2}$$

for some universal constant $C$.

Next we bound $E \Phi_t$. By Lemma 13, with $t_1 \approx (q n^2 \log n)^{-1}$, we have

$$P[\|A_t\|_{\text{op}} \geq 2 \text{ for } 0 \leq t \leq t_1] \leq \frac{1}{n^{2q+1}}.$$

Since $p_0$ is supported on a ball with radius $n$, we have $\|A_t\|_{\text{op}} \leq n^2$. Hence, we have

$$E \text{Tr} \mathbf{A}_t^q \leq \Psi_t + P(E_t) n^{2q} \cdot n \leq (C q n^2 t)^{q/2} \cdot n + 1.$$

This finishes the proof.

**Lemma 20.** Under the setting of Lemma 19 with $3 \leq q \leq 4$, for any $t$ with $t_1 \leq t \leq \hat{t} \equiv t_1 \log \frac{1}{t_1} \cdot n$ we have $E \text{Tr} A_t^q \lesssim n$. Further, for any $t \geq \hat{t}$ we have

$$E \text{Tr} A_t^q \lesssim \left(\frac{t}{t_1}\right)^{q(q-1)} \frac{n}{\log^{q/2} n}$$

where $t_1 = \frac{1}{c n^2 \log n}$ for some constant $c$.

**Proof.** Recall from (3.1) that Itô’s formula yields

$$\frac{d}{dt} E \text{Tr} \mathbf{A}_t^q \leq \frac{q(q - 1)}{2} T_{p_t}(\mathbf{A}_t^{-2}, I, I)$$

(3.2)
with \( \Phi_t = \text{Tr} A_t^\alpha \). Now, we bound \( T_{p_t}(A_t^{q-2}, I, I) \) using Lemma 24. Let \( p_t(M) \) be the density of the distribution \( M^{1/2}A_t^{-1/2}(x - a_t) \) where \( x \sim p_t \). Note that \( p_{t,M} \) has mean 0 and covariance \( M \). Using \( A_t \preceq A_t^\alpha + I \) and Lemma 12, we have
\[
T_{p_t}(A_t^{q-2}, I, I) = T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t)
\]
\[
\leq T_{p_t(I)}(A_t^{q-1}, A_t^\alpha + A_t^{-1/2}, A_t + I, A_t + I)
\]
\[
= T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t) + 2T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t)
+ T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t) + T_{p_t(I)}(A_t^{q-2}, I, I)
\]
\[
\leq T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t) + 3T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t) + 3T_{p_t(I)}(A_t^{q-2}, A_t^\alpha, A_t) + T_{p_t(I)}(A_t^{q-2}, I, I).
\]

For the first term, we note that \( p_t, A_t^\alpha \) is \( t \)-strongly log-concave because \( p_t \) is \( t \)-strongly log-concave and that \( A_t^\alpha \preceq A_t \). Hence, Lemma 24 shows that
\[
T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t) = T_{p_t(A_t^\alpha)}(A_t^{q-2}, I, I) \leq \frac{2}{t} \text{Tr} A_t^\alpha.
\]

For other terms, we use Lemma 12 and Hölder’s inequality to get
\[
T_{p_t(I)}(A_t^{q-1}, A_t^\alpha, A_t) \preceq T_{p_t(I)}(A_t^{q-2}, I, I).
\]

Next, as Lemma 12 gives \( T_{p_t}(A_t^{q-2}, I, I) \preceq \kappa_n^2 \text{Tr} A_t^\alpha \), we obtain
\[
T_{p_t}(|A_t - I|^{q-2}, I, I) \leq \frac{2}{t} \text{Tr} A_t^\alpha + O(\kappa_n^2)(\text{Tr} A_t^\alpha + \text{Tr} A_t^{-q})
\]
\[
\leq \left( \frac{2}{t} + O(\kappa_n^2) \right) \text{Tr} A_t^\alpha + O(\kappa_n^2) \cdot \left( \text{Tr} A_t^\alpha \right)^{1 - \frac{q}{2}} n^{\frac{q}{2}}.
\]

Substituting into Equation 3.2 gives
\[
\frac{d}{dt} \mathbb{E} \Phi_t \leq \left( \frac{q(q - 1)}{t} + C\kappa_n^2 \right) \mathbb{E} \Phi_t + C\kappa_n^2 (\mathbb{E} \Phi_t)^{1 - \frac{q}{2}} n^{\frac{q}{2}}
\]
for some universal constant \( C > 0 \). To solve this inequality, we let \( \alpha = q(q - 1) \), \( \beta = C\kappa_n^2 \) and define
\[
\Psi_t = \left( \frac{t}{t_1} \right)^{-\alpha} e^{-\beta(t-t_1)} \mathbb{E} \Phi_t.
\]

Under these substitutions our equation gives
\[
\frac{d}{dt} \Psi_t = - \left( \frac{\alpha}{t} + \beta \right) \Psi_t + \left( \frac{t}{t_1} \right)^{-\alpha} e^{-\beta(t-t_1)} \frac{d}{dt} \mathbb{E} \Phi_t
\]
\[
\leq - \left( \frac{\alpha}{t} + \beta \right) \Psi_t + \left( \frac{t}{t_1} \right)^{-\alpha} e^{-\beta(t-t_1)} \left( \frac{\alpha}{t} + \beta \right) \mathbb{E} \Phi_t + \beta (\mathbb{E} \Phi_t)^{1 - \frac{q}{2}} n^{\frac{q}{2}}
\]
\[
= \beta \left( \frac{t}{t_1} \right)^{-\alpha} e^{-\beta(t-t_1)} \left( \mathbb{E} \Phi_t \right)^{1 - \frac{q}{2}} n^{\frac{q}{2}}
\]
\[
= \beta \left( \frac{t}{t_1} \right)^{-\frac{q}{2} \alpha} e^{-\frac{q}{2} \beta(t-t_1)} \Psi_t^{1 - \frac{q}{2}} n^{\frac{q}{2}}.
\]

Hence,
\[
\frac{d}{dt} \Psi_t^{2/q} \leq \frac{2\beta n^{\frac{q}{2}}}{q} \left( \frac{t}{t_1} \right)^{-\frac{q}{2} \alpha} e^{-\frac{q}{2} \beta(t-t_1)} n^{\frac{q}{2}}.
\]

Integrating between \( t_1 \) and \( t \),
\[
\Psi_t^{2/q} \leq \Psi_{t_1}^{2/q} + \frac{2\beta n^{\frac{q}{2}}}{q} \int_{t_1}^{t} \left( \frac{s}{t_1} \right)^{-\frac{q}{2} \alpha} ds \leq \Psi_{t_1}^{2/q} + \frac{2\beta n^{\frac{q}{2}}}{q} \frac{t_1}{\frac{q}{2} \alpha - 1}.
\]
Using $\alpha = q(q - 1)$ and $q \geq 3$, we have $\Psi_t^{2/q} \leq \Psi_t^{2/q} + \frac{2\kappa t^{2/q}}{q^2}$. Using the definition of $\Psi_t$, we have
\[ E\Phi_t \leq \left( \frac{t}{t_1} \right)^\alpha e^{\beta(t-t_1)} \left( \Psi_t^{2/q} + \frac{2\kappa t^{2/q}}{q^2} \right)^{q/2} = \left( \frac{t}{t_1} \right)^\alpha e^{\beta(t-t_1)} \left( (E\Phi_{t_1})^{2/q} + \frac{2\kappa t^{2/q}}{q^2} t_1 \right)^{q/2}. \]

For all $t \leq \kappa_n^{-2}$, using $q \leq 4$, $t_1 = \frac{1}{\kappa_n^2 \log n}$, $\beta = C \kappa_n^2$ and Lemma 19, we have
\[ E\Phi_t \lesssim \left( \frac{t}{t_1} \right)^\alpha \left( E\Phi_{t_1} + \frac{n}{\log^{q/2} n} \right) \lesssim \left( \frac{t}{t_1} \right)^{q(q-1)} \frac{n}{\log^{q/2} n}. \]

In particular for $t \leq \hat{t}$ this gives
\[ E\Phi_t \lesssim \left( \frac{\log \frac{\hat{t}}{t_1}}{n} \right)^{q(q-1)} \frac{n}{\log^{q/2} n} = n. \]

Thus $E\text{Tr}A_t^q \lesssim 2^q (E\Phi_t + n) \lesssim n$ for $t \leq \hat{t}$. For $t \geq \hat{t}$, we instead use Lemma 17 with initial distribution given by $p_t$. Taking expectations implies
\[ E\text{Tr}A_t^q \lesssim \left( \frac{t}{\hat{t}} \right)^{q(q-1)} E\text{Tr}A_{\hat{t}}^q \lesssim \left( \frac{t}{\hat{t}} \right)^{q(q-1)} n \lesssim \left( \frac{t}{t_1} \right)^{q(q-1)} \frac{n}{\log^{q/2} n} \]
where the last inequality follows from $\hat{t} = t_1 \log^{q/2} n$. \qed

We use the above bound on $\text{Tr}A_t^q$ to bound $E\text{Tr}A_t^q$ and hence $E\|a_t\|^2$.

**Lemma 21.** Under the setting of Lemma 19. For any $3 \leq q \leq 4$, we have
\[ E\|a_t\|^2 \lesssim nt + \frac{t^{q+1}}{(t^*)\gamma} \]
where $t^* \overset{\text{def}}{=} \frac{-2}{\gamma - 3} \log -\frac{2\kappa_n^2 - 2}{2(q-2)} n$ and $\gamma \leq 2\sqrt{2}$.

**Proof.** Using $dA_t = E_{x \sim p_t}(x - a_t)(x - a_t)^\top dW_t - A_t^2 dt$ (11) and Itô’s formula, we have
\[ d\text{Tr}A_t^2 = 2E_{x \sim p_t}(x - a_t)^\top A_t(x - a_t) \cdot (x - a_t)^\top dW_t - 2\text{Tr}A_t^3 dt + T_{\mu_t}(I, I, I) dt. \]

Taking expectation and using Lemma 18, we have
\[ \frac{d}{dt} E\text{Tr}A_t^2 \leq E T_{\mu_t}(I, I, I) \lesssim \frac{1}{t_1^3} E(t^q \text{Tr}A_t^q) - (t^2 \text{Tr}A_t^2)^{1-c} \]
with $c = \frac{1}{2(q-2)}$. Applying $E\text{Tr}A_t^q \lesssim n + \left( \frac{t}{t_1} \right)^{(q-1)} \frac{n}{\log^{q/2} n}$ (Lemma 20), we have
\[ \frac{d}{dt} E\text{Tr}A_t^2 \lesssim t^{q-2+1} n^c (E\text{Tr}A_t^2)^{1-c} + t^{-1+(q-2)c+q(q-1)c} \left( \frac{1}{t_1^{q(q-1)}} \frac{n}{\log^{q/2} n} \right)^c (E\text{Tr}A_t^2)^{1-c}. \]

Rearranging and using $c = \Theta(1)$ (because $3 \leq q \leq 4$), we have
\[ \frac{d}{dt} (E\text{Tr}A_t^2)^c \lesssim t^{q-2+1} n^c + t^{-1+(q-2)c+q(q-1)c} \left( \frac{1}{t_1^{q(q-1)}} \frac{n}{\log^{q/2} n} \right)^c. \]

Integrating from $t_1$ and using $E\text{Tr}A_t^2 \lesssim n$, for all $t \geq t_1$
\[ (E\text{Tr}A_t^2)^c \lesssim n^c + t^{q-2} n^c + t^{q(q-2)c} \left( \frac{1}{t_1^{q(q-1)}} \frac{n}{\log^{q/2} n} \right)^c. \]

Hence, we have
\[ E\text{Tr}A_t^2 \lesssim n + n t^{q-2} + \frac{n^2}{t_1^{q-2}} \frac{n}{\log^{q/2} n}. \] (3.3)
In particular, this shows that for $0 \leq t \leq t^*$ with $t^* \equiv \frac{2^{\gamma+1}}{t^{\gamma+1}} \log^{\frac{2}{\gamma(\gamma-1)}} n$, we have $\text{ETr} A_n^2 \lesssim n$.

For $t \geq t^*$, we switch to the estimate $T_{\mu t}(I, I, I) \leq (\gamma/t)\text{Tr} A_n^2$ with $\gamma \leq 2\sqrt{2}$ (Lemma 15). Applying it to $\frac{d}{dt} \text{ETr} A_n^2 \leq ET_{\mu t}(I, I, I)$, we have

$$\text{ETr} A_n^2 \lesssim \left( \frac{t}{t^*} \right)^{\gamma} \text{ETr} A_n^2 \lesssim \left( \frac{t}{t^*} \right)^{\gamma} n$$

for $t \geq t^*$. Hence, for all $t > 0$, we have

$$\text{ETr} A_n^2 \lesssim \left( 1 + \left( \frac{t}{t^*} \right)^{\gamma} \right) n.$$

Using $da_t = A_t dW_t$ (Lemma 11) and Itô’s formula, we have $\frac{d}{dt} \mathbb{E}\|a_t\|^2 = \text{Tr} A_n^2$. Hence, we have

$$\mathbb{E}\|a_t\|^2 \lesssim \int_0^t \left( 1 + \left( \frac{s}{t^*} \right)^{\gamma} \right) n \cdot ds \lesssim nt + n^{\gamma+1}

$$

with $t^* = \kappa_n \frac{-2^{\gamma+1}}{t^{\gamma+1}} \log^{\frac{2}{\gamma(\gamma-1)}} n = \kappa_n \frac{-2^{\gamma+1}}{t^{\gamma+1}} \log^{\frac{2}{\gamma(\gamma-1)}} n$. This gives the result. \(\square\)

### 4 Improved bounds for thin shell

**Theorem 22.** We have $\sigma_n \lesssim \log^\eta n$ for

$$\eta = \min_{\gamma \leq \eta \leq 4} \left[ \frac{1 + \frac{2^{\gamma+1}}{t^{\gamma+1}} \gamma}{1 + \frac{2^{\gamma+1}}{t^{\gamma+1}} \gamma} \right].$$

**Setting** $\gamma = 2\sqrt{2}$ and $q = \frac{1}{t^*}(112 - 16\sqrt{2} + \sqrt{5630} - 1892\sqrt{2})$, we have

$$\eta \leq \frac{1}{8}(1 + 7\sqrt{2} + \sqrt{53 - 4\sqrt{2}}) \leq 2.2226$$

**Proof.** Pick an isotropic log-concave distribution $p_0$ on $\mathbb{R}^n$ such that $\sigma_{p_0}^2 \geq \frac{1}{2} \sigma_n^2$ and $p_0$ is supported in a ball of radius $n$. Let $p_t$ be given by the LV process with initial distribution $p_0$. By Lemma 8 and Lemma 21, we have

$$\sigma_n^2 \lesssim \int_{\lambda > 0} \left( \frac{1}{\sqrt{\lambda}} \mathbb{E}\|a_{\lambda^2}\|^2 + \frac{1}{\lambda} d\lambda \right),$$

with $\lambda \approx \psi_{p_0}^2$. Picking $\lambda = \min(\sqrt{\lambda}, \lambda^{\frac{1}{\gamma+1}} (t^{\gamma+1})^{-\frac{1}{\gamma+1}})$, we have

$$\sigma_n^2 \lesssim \int_{\lambda} \frac{1}{\lambda^{1.5}} + \frac{1}{\lambda^{1+\frac{\gamma}{1+\gamma+1}} (t^{*+1})^{-\frac{1}{1+\gamma+1}}} d\lambda \lesssim \frac{1}{\sqrt{\lambda}} + \frac{1}{\lambda^{1+\gamma} (t^{*+1})^{-\frac{1}{1+\gamma+1}}}.$$
\[ \eta \leq \frac{\beta}{2 - \alpha} = 1 + \frac{3q^2 - 5q}{\gamma q - 2} \gamma. \]

This shows that \( \sigma_n \lesssim \log^q n \) for the given value of \( \gamma \) and any \( 3 \leq q \leq 4 \). The last claim follows by setting \( \gamma = 2\sqrt{2} \) and optimizing \( q \).

**Theorem 23.** If \( \psi_n \lesssim \sigma_n \), then we have \( \sigma_n \lesssim \log^q n \) with
\[
\eta = \min_{3 \leq q \leq 4} \frac{\frac{3}{4} q^2 q - q}{\gamma q - 2} \gamma.
\]

**Setting** \( \gamma = 2\sqrt{2} \) **and** \( q = 3 \), **we have** \( \eta \leq \frac{63\sqrt{2} - 36}{82} \leq 0.6476 \).

**Proof.** The proof is similar to Theorem 22. The key difference is that we have \( \psi_n^2 \lesssim \sigma_n^2 \) and \( \kappa_n^2 \lesssim \sigma_n^2 \). Hence, the same calculation above shows that
\[
\sigma_n^2 \lesssim \sigma_n + \sigma_n \cdot (\kappa_n^{\frac{2 - q}{q - 2}} \log \frac{2q^2 - 3q}{2(q^2 - 2)} n)^{-\frac{\gamma}{q - 2}}
\]
\[
\lesssim \sigma_n + \sigma_n^{\frac{2 - q}{q - 2}} \cdot \left( \sigma_n^{2 - q} \log \frac{2q^2 - 3q}{2(q^2 - 2)} n \right)^{-\frac{\gamma}{q - 2}}
\]
\[
= \sigma_n \cdot \left( \frac{2q^2 - 3q}{2q^2 - 4q + 2} \right)^{\frac{2 - q}{q - 2}} \left( \log \frac{2q^2 - 3q}{2(q^2 - 2)} n \right)^{-\frac{\gamma}{q - 2}}.
\]

Again, define \( \alpha = 2(\frac{2 - q}{q - 2} \log \frac{2q^2 - 3q}{2(q^2 - 2)} n + \frac{1}{q - 2}) \) and \( \beta = \frac{2q^2 - 3q}{2q^2 - 4q + 2} \). Hence, the above bound yields
\[
\eta \leq \frac{\beta}{2 - \alpha} = \frac{\frac{3}{4} q^2 q - q}{\gamma q - 2} \gamma.
\]

This shows that \( \sigma_n \lesssim \log^q n \) for any \( 3 \leq q \leq 4 \). The last result follows from setting \( \gamma = 2\sqrt{2} \) and \( q = 3 \). We remark that for the current value of \( \gamma \), the value of \( q \) minimizing the above bound is at \( q = 2.4588 \). Unfortunately, Lemma 21 does not apply for \( q < 3 \). \( \square \)

**Acknowledgement.** This work was supported in part by NSF awards CCF-2007443 and CCF-2134105 and a Packard fellowship.

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A Discussion on possible value of $\gamma$

For a 1-dimensional 1-strongly log-concave distribution $\mu$, we have

$$T_\mu(I, I, I) = E_{x,y \sim \mu}(xy)^3 = (E_{x \sim \mu}x^3)^2.$$ 

Therefore, the inequality $T_\mu(I, I, I) \leq \gamma \text{Tr}A^2$ becomes

$$(E_{x \sim \mu}x^3)^2 \leq \gamma (E_{x \sim \mu}x^2)^2.$$ 

Note that $(E_{x \sim \mu}x^3)^2 - \gamma (E_{x \sim \mu}x^2)^2$ is convex in $\mu$. Hence, the localization lemma shows that the 1-strongly log-concave distribution that maximizes $(E_{x \sim \mu}x^3)^2 - \gamma (E_{x \sim \mu}x^2)^2$ is a truncated Gaussian distribution (with variance 1 before truncation).

A computer search suggests that $\gamma \approx 0.37$ with the worst distribution is

$$\exp(-\frac{(x-m)^2}{2})1_{[a,b]}$$

with $a \approx -0.69, b \approx 4.31, m \approx -1.03$.

B Deferred calculations

In this section, we include various deferred calculations. All of them are variants of existing proofs.

The following lemma is an improvement of Lemma 7. It is given in [9, Remark 4.4]. We give a proof here for completeness.

**Lemma 24.** For any $\alpha$-strongly log-concave $\mu$ with covariance $A$ and any $q \geq 3$, we have

$$T_\mu(A^{q-2}, I, I) \leq \frac{2}{\alpha} \text{Tr}A^q.$$ 

**Proof.** By centering, we can assume $\mu$ has mean 0. Let $\eta_i$ and $v_i$ are eigenvalues and eigenvectors of $A$ and hence $A = \sum_{i=1}^d \eta_i v_i v_i^\top$. Let

$$\Delta_i = E_{x \sim \mu}(x^\top A^{-1/2}v_i) \cdot xx^\top.$$ 

Then, we have

$$T_\mu(A^{q-2}, I, I) = E_{x,y \sim \mu}x^\top A^{q-2} y \cdot x^\top y \cdot x^\top y$$

$$= \sum_{i=1}^d \eta_i^{q-1} E_{x,y \sim \mu}x^\top A^{-1/2}v_i A^{-1/2}y \cdot x^\top y \cdot x^\top y$$

$$= \sum_{i=1}^d \eta_i^{q-1} \text{Tr}(\Delta_i^2).$$ 

(B.1)
For the term $\text{Tr}(\Delta^2_i)$, using the fact that $x$ has mean 0 and variance $A$, we have

\[
\text{Tr}(\Delta^2_i) = E_{x \sim \mu} x^\top A^{-1/2} \nu_i \cdot x^\top \Delta_i x
\]

\[
\leq \sqrt{E_{x \sim \mu} (x^\top A^{-1/2} \nu_i)^2} \sqrt{\text{Var}(x^\top \Delta_i x)}
\]

\[
= \sqrt{\text{Var}(x^\top \Delta_i x)}.
\]  

(B.2)

Since $\mu$ is $\alpha$-strongly log-concave and the gradient of $x^\top \Delta_i x$ has mean 0, Lemma 9 shows that

\[
\text{Var}(x^\top \Delta_i x) \leq \frac{1}{2\alpha} \mathbb{E} \|2\Delta_i x\|_2^2 \leq \frac{2}{\alpha} \text{Tr}(A\Delta^2_i).
\]

Using this, (B.1) and (B.2), we have

\[
T_{\mu}(A^{q-2}, I, I) \leq \sum_{i=1}^d \eta_i^{q-1} \sqrt{\frac{2}{\alpha} \text{Tr}(A\Delta^2_i)} \leq \sqrt{\frac{2}{\alpha} \sum_{i=1}^d \eta_i^q} \sqrt{\text{Tr}(A^{q-2}, A, I)} \leq \sqrt{\frac{2}{\alpha} \text{Tr} \sqrt{T_{\mu}(A^{q-2}, I, I)}},
\]

where we used Lemma 12 at the end. This gives the result.

Next, we compute the Hessian of $\text{Tr}(M^{-1/2}A_{ij}M^{-1/2} - I)^q$. The proof is similar to [5, Lemma B.5]. The proof uses the following estimate of the Hessian of matrix functions.

**Lemma 25** ([7, Proposition 3.1]). Let $f$ be a twice differentiable function on $(\alpha, \beta)$ such that for some $\theta, \varphi \in \mathbb{R}$, for all $\alpha \leq a < b < \beta$, we have

\[
\frac{f'(b) - f'(a)}{b - a} \leq \frac{\theta f''(b) + f''(a)}{2} + \varphi.
\]

Then, for any matrix $X$ with eigenvalues lies between $(\alpha, \beta)$, we have

\[
\frac{\partial^2 \text{Tr}(f(X))}{\partial X^2} |_{H,H} \leq \theta \text{Tr}(f''(X)H^2) + \varphi \text{Tr}H^2.
\]

Next we need a simple lemma about the function $(x^+)^q$.

**Lemma 26.** Let

\[
\phi(x) = \begin{cases} 
  x^q & \text{if } x \geq 0 \\
  0 & \text{else}
\end{cases}
\]

with $q \geq 3$. For any $a < b$, we have

\[
\frac{\phi'(b) - \phi'(a)}{b - a} \leq \frac{\phi''(b) + \phi''(a)}{2}.
\]

**Proof.** Case 1) $0 \leq a < b$.

For a fixed $a$, we let $V(b) = \frac{\phi''(b) + \phi''(a)}{2}(b - a) - (\phi'(b) - \phi'(a))$. Note that $V(a) = 0$ and that

\[
V(b) = q(q - 1) \frac{b^{q-2} + a^{q-2}}{2}(b - a) - q(b^{q-1} - a^{q-1}).
\]

Taking derivative with respect to $b$, we have that for all $b$

\[
\frac{2V'(b)}{q(q - 1)(q - 2)} = \frac{q - 3}{q - 2} b^{q-2} + \frac{1}{q - 2} a^{q-2} - b^{q-3} a \geq 0
\]

where we used Young’s inequality at the end. Hence, we have $V'(b) \geq 0$ for all $b > 0$. Hence, $V$ is increasing and that $V(b) \geq V(a) = 0$ for all $b \geq a$.

Case 2) $a \leq 0 < b$.

Since $q \geq 3$, we have

\[
\frac{\phi'(b) - \phi'(a)}{b - a} = \frac{q b^{q-1}}{b} \leq \frac{q(q - 1)}{2} b^{q-2} = \frac{\phi''(b) + \phi''(a)}{2}.
\]

Case 3) $a \leq b \leq 0$.

Both sides are 0.
We can now compute the Hessian.

**Lemma 27.** Let \( \mu \) be a log-concave distribution in \( \mathbb{R}^n \) with covariance \( A \). Let \( p_t \) be the result of the LV process starting with the density of \( \mu \). Let \( A_t \) be the covariance of \( p_t \) and \( \phi \) be as defined in Lemma 26. For any \( q \in \{2\} \cup [3, +\infty) \), we have

\[
d\text{Tr}(\phi(A_t - bI)) = \mathbb{E}_{x \sim p_t} (x - a_t)^\top \phi'(A_t - bI)(x - a_t)(x - a_t)^\top dW_t \\
\quad - \text{Tr}(\phi'(A_t - bI)A_t^2)dt + \frac{1}{2} T_{p_t}(\phi''(A_t - bI), I, I)dt.
\]

In particular,

\[
\frac{d}{dt} \mathbb{E}\text{Tr}(\phi(A_t - bI)) \leq \frac{1}{2} T_{p_t}(\phi''(A_t - bI), I, I)
\]

where the expectation conditional on \( A_t \).

**Proof.** Let \( \Phi_t = \text{Tr}(\phi(A_t - bI)) \). By Lemma 11, we have

\[
dA_t = \mathbb{E}_{x \sim p_t} (x - a_t)(x - a_t)^\top dW_t - A_t^2 dt = \sum_i Z_i \cdot dW_{t,i} - A_t^2 dt
\]

where \( Z_i = \mathbb{E}_{x \sim p_t} (x - a_t)(x - a_t)^\top (x - a_t)_i \). By Itô’s formula, we have

\[
d\Phi_t = \left. \frac{\partial \text{Tr} \Phi_t}{\partial A_t} \right|_{dA_t} + \frac{1}{2} \sum_i \left. \frac{\partial^2 \text{Tr} \Phi_t}{\partial A_t^2} \right|_{Z_i, Z_i} dt.
\]

For the first-order term, we have

\[
\left. \frac{\partial \text{Tr} \Phi_t}{\partial A_t} \right|_{dA_t} = \text{Tr}(\phi'(A_t - bI))dA_t \\
= \mathbb{E}_{x \sim p_t} (x - a_t)^\top \phi'(A_t - bI)(x - a_t)(x - a_t)^\top dW_t - \text{Tr}(\phi'(A_t - bI)A_t^2)dt.
\]

For the second-order term, we use Lemma 25 with \( \theta_+ = 1 \) and \( \varphi_+ = 0 \) to get

\[
\left. \frac{\partial^2 \text{Tr} \Phi_t}{\partial A_t^2} \right|_{Z_i, Z_i} \leq \sum_i \text{Tr}(\phi''(A_t - bI)Z_i^2) = T_{p_t}(\phi''(A_t - bI), I, I).
\]

Combining the first- and second-order terms proves the first claim.

In particular, we have

\[
\frac{d}{dt} \mathbb{E}\text{Tr}(\phi(A_t - bI)) = - \text{Tr}(\phi'(A_t - bI)A_t^2) + \frac{1}{2} T_{p_t}(\phi''(A_t - bI), I, I).
\]

Since \( \phi'(x) \geq 0 \) for all \( x \), we have

\[
\frac{d}{dt} \mathbb{E}\text{Tr}(\phi(A_t - bI)) \leq \frac{1}{2} T_{p_t}(\phi''(A_t - bI), I, I).
\]