Slant helices in Euclidean 4-space $\mathbb{E}^4$

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Abstract

We consider a unit speed curve $\alpha$ in Euclidean four-dimensional space $\mathbb{E}^4$ and denote the Frenet frame by $\{T, N, B_1, B_2\}$. We say that $\alpha$ is a slant helix if its principal normal vector $N$ makes a constant angle with a fixed direction $U$. In this work we give different characterizations of such curves in terms of their curvatures.

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1 Introduction and statement of results

A helix in Euclidean 3-space $E^3$ is a curve whose tangent lines make a constant angle with a fixed direction. A helix curve is characterized by the fact that the ratio $\tau/\kappa$ is constant along the curve, where $\tau$ and $\kappa$ denote the torsion and the curvature, respectively. Helices are well known curves in classical differential geometry of space curves [10] and we refer to the reader for recent works on this type of curves [5, 14]. Recently, Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed direction [6]. They characterize a slant helix if and only if the function $\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'$ is constant. This article motivated generalizations in a twofold sense: first, by increasing the dimension of Euclidean space [8, 12]; second, by considering analogous problems in other ambient spaces, mainly, in Minkowski space $E^n_{1}$ [1, 2, 3, 4, 7, 13].

In this work we consider the generalization of the concept of slant helix in Euclidean 4-space $E^4$. Let $\alpha : I \subset \mathbb{R} \rightarrow E^4$ be an arbitrary curve in $E^4$. Recall that the curve $\alpha$ is said to be of unit speed (or parameterized by arclength function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle , \rangle$ is the standard scalar product in the Euclidean space $E^4$ given by $\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, for each $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in E^4$.

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving frame along $\alpha$, where $T, N, B_1$ and $B_2$ denote the tangent, the principal normal, the first binormal and second binormal vector fields, respectively. Here $T(s), N(s), B_1(s)$ and $B_2(s)$ are mutually orthogonal vectors satisfying

$\langle T, T \rangle = \langle N, N \rangle = \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 1$.

The Frenet equations for $\alpha$ are given by

$$
\begin{bmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa_1 & 0 & 0 \\
-\kappa_1 & 0 & \kappa_2 & 0 \\
0 & -\kappa_2 & 0 & \kappa_3 \\
0 & 0 & -\kappa_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}.
$$

(1)

Recall the functions $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ are called respectively, the first, the second and the third curvatures of $\alpha$. If $\kappa_3(s) = 0$ for any $s \in I$, then $B_2(s)$ is
a constant vector $B$ and the curve $\alpha$ lies in a three-dimensional affine subspace orthogonal to $B$, which is isometric to the Euclidean 3-space $E^3$.

We will assume throughout this work that all the three curvatures satisfy $\kappa_i(s) \neq 0$ for any $s \in I$, $1 \leq i \leq 3$.

**Definition 1.1.** A unit speed curve $\alpha : I \to E^4$ is said to be a generalized helix if there exists a non-zero constant vector field $U$ and a vector field $X \in \{ T, N, B_1, B_2 \}$ such that the function

$$ s \mapsto \langle X(s), U \rangle, \quad s \in I $$

is constant.

Among the possibilities to choose the vector field $X$ we have:

1. If $X$ is the unit tangent vector field $T$, $\alpha$ is called a cylindrical helix. It is known that $\alpha(s)$ is a cylindrical helix if and only if the function

$$ \frac{\kappa_1^2}{\kappa_2^2} + \left[ \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \right) \right]^2 $$

is constant. See [9, 11].

2. If $X$ is the vector field $B_2$, then the curve is called a $B_2$-slant curve. Moreover $\alpha$ is a such curve if and only if the function

$$ \frac{\kappa_3^2}{\kappa_2^2} + \left[ \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right) \right]^2 $$

is constant. See [12].

**Definition 1.2.** A unit speed curve $\alpha : I \to E^4$ is called slant helix if its unit principal normal vector $N$ makes a constant angle with a fixed direction $U$.

Our main result in this work is the following characterization of slant helices.

**Theorem 1.3.** Let $\alpha : I \to E^4$ be a unit speed curve in $E^4$. Then $\alpha$ is a slant helix if and only if the function

$$ \left( \int \kappa_1(s) ds \right)^2 + \left[ \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right)' + \frac{\kappa_2}{\kappa_3} \right]^2 + \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right)^2 $$

is constant. Moreover, this constant agrees with $\tan^2 \theta$, being $\theta$ the angle that makes $N$ with the fixed direction $U$ that determines $\alpha$. 

3
2 Proof of Theorem 1.3

Let $\alpha$ be a unit speed curve in $E^4$. Assume that $\alpha$ is a slant curve. Let $U$ be the direction with which $N$ makes a constant angle $\theta$ (suppose that $\langle U, U \rangle = 1$). Consider the differentiable functions $a_i$, $1 \leq i \leq 4$,

$$U = a_1(s)T(s) + a_2(s)N(s) + a_3(s)B_1(s) + a_4(s)B_2(s), \quad s \in I,$$

that is,

$$a_1 = \langle T, U \rangle, \quad a_2 = \langle N, U \rangle, \quad a_3 = \langle B_1, U \rangle, \quad a_4 = \langle B_2, U \rangle.$$

Because the vector field $U$ is constant, a differentiation in (2) together (1) gives the following ordinary differential equation system

$$\begin{align*}
    a'_1 - \kappa_1 a_2 &= 0, \\
    a'_2 + \kappa_1 a_1 - \kappa_2 a_3 &= 0, \\
    a'_3 + \kappa_2 a_2 - \kappa_3 a_4 &= 0, \\
    a'_4 + \kappa_3 a_3 &= 0.
\end{align*}$$

Then the function $a_2(s) = \langle N(s), U \rangle$ is constant, and it agrees with $\cos \theta$. Then (3) gives

$$\begin{align*}
    a'_1 - \kappa_1 a_2 &= 0, \\
    \kappa_1 a_1 - \kappa_2 a_3 &= 0, \\
    a'_3 + \kappa_2 a_2 - \kappa_3 a_4 &= 0, \\
    a'_4 + \kappa_3 a_3 &= 0.
\end{align*}$$

The first third equations in (4) lead to

$$\begin{align*}
    a_1 &= a_2 \int \kappa_1 ds, \\
    a_3 &= a_2 \kappa_1 \int \kappa_1 ds, \\
    a_4 &= a_2 \left[ \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right)' + \frac{\kappa_2}{\kappa_3} \right].
\end{align*}$$

We do the change of variables:

$$t(s) = \int^s \kappa_3(u)du, \quad \frac{dt}{ds} = \kappa_3(s).$$

In particular, and from (4), we have

$$a'_3(t) = a_4 - a_2 \frac{\kappa_2}{\kappa_3}. $$
As a consequence, if $\alpha$ is a slant helix, the last equation of (4) yields

$$a_4''(t) + a_4(t) - a_2 \frac{\kappa_2(t)}{\kappa_3(t)} = 0.$$  \hspace{1cm} (6)

The general solution of this equation is

$$a_4(t) = a_2 \left[ \left( A - \int \frac{\kappa_2(t)}{\kappa_3(t)} \sin t \, dt \right) \cos t + \left( B + \int \frac{\kappa_2(t)}{\kappa_3(t)} \cos t \, dt \right) \sin t \right],$$  \hspace{1cm} (7)

where $A$ and $B$ are arbitrary constants. Then (7) takes the following form

$$a_4(s) = a_2 \left[ \left( A - \int \kappa_2(s) \sin \int \kappa_3(s) \, ds \right) \cos \int \kappa_3(s) \, ds \right. \left. + \left( B + \int \kappa_2(s) \cos \int \kappa_3(s) \, ds \right) \sin \int \kappa_3(s) \, ds \right].$$  \hspace{1cm} (8)

From (4), the function $a_3$ is given by

$$a_3(s) = a_2 \left[ \left( A - \int \kappa_2(s) \sin \int \kappa_3(s) \, ds \right) \sin \int \kappa_3(s) \, ds \right. \left. - \left( B + \int \kappa_2(s) \cos \int \kappa_3(s) \, ds \right) \cos \int \kappa_3(s) \, ds \right].$$  \hspace{1cm} (9)

From (8), (9) and (5) we have the following two conditions:

$$\frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 \, ds \right)' + \frac{\kappa_2}{\kappa_3} = \left( A - \int \kappa_2(s) \sin \int \kappa_3(s) \, ds \right) \cos \int \kappa_3(s) \, ds$$

$$+ \left( B + \int \kappa_2(s) \cos \int \kappa_3(s) \, ds \right) \sin \int \kappa_3(s) \, ds.$$  \hspace{1cm} (10)

and

$$\frac{\kappa_2}{\kappa_3} \int \kappa_1 \, ds = \left( A - \int \kappa_2(s) \sin \int \kappa_3(s) \, ds \right) \sin \int \kappa_3(s) \, ds$$

$$- \left( B + \int \kappa_2(s) \cos \int \kappa_3(s) \, ds \right) \cos \int \kappa_3(s) \, ds.$$  \hspace{1cm} (11)

The condition (11) can be written as follows:

$$\kappa_1(s) \int \kappa_1(s) \, ds = \left( A - \int \kappa_2(s) \sin \int \kappa_3(s) \, ds \right) \kappa_2(s) \sin \int \kappa_3(s) \, ds$$

$$- \left( B + \int \kappa_2(s) \cos \int \kappa_3(s) \, ds \right) \kappa_2(s) \cos \int \kappa_3(s) \, ds.$$  \hspace{1cm} (12)

If we integrate the above equation we have

$$\left( \int \kappa_1(s) \, ds \right)^2 = C - \left( A - \int \kappa_2(s) \sin \int \kappa_3(s) \, ds \right)^2$$

$$- \left( B + \int \kappa_2(s) \cos \int \kappa_3(s) \, ds \right)^2.$$
where $C$ is a constant of integration. From Equations (10) and (11), we get
\[
\left[ \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right) + \frac{\kappa_2}{\kappa_3} \right]^2 + \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right)^2 = \left( A - \int \left[ \kappa_2(s) \sin \int \kappa_3(s) ds \right] ds \right)^2
+ \left( B + \int \left[ \kappa_2(s) \cos \int \kappa_3(s) ds \right] ds \right)^2.
\]

Now Equations (12) and (13) give
\[
\left( \int \kappa_1(s) ds \right)^2 + \left[ \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right) + \frac{\kappa_2}{\kappa_3} \right]^2 + \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right)^2 = C.
\]

Moreover this constant $C$ calculates as follows. From (14), together the three equations (5) we have
\[
C = \frac{a_1^2 + a_2^2 + a_3^2}{a_2^2} = 1 - \frac{a_2^2}{a_2^2} = \tan^2 \theta,
\]
where we have used (2) and the fact that $U$ is a unit vector field.

We do the converse of the proof. Assume that the condition (14) is satisfied for a curve $\alpha$. Let $\theta \in \mathbb{R}$ be so that $C = \tan^2 \theta$. Define the unit vector $U$ by
\[
U = \cos \theta \left[ \int \kappa_1 ds \mathbf{T} + \mathbf{N} + \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \mathbf{B}_1 + \left[ \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right) + \frac{\kappa_2}{\kappa_3} \right] \mathbf{B}_2 \right].
\]

By taking account (14), a differentiation of $U$ gives that $\frac{dU}{ds} = 0$, which it means that $U$ is a constant vector. On the other hand, the scalar product between the unit principal normal vector field $\mathbf{N}$ with $U$ is
\[
\langle \mathbf{N}(s), U \rangle = \cos \theta.
\]

Thus $\alpha$ is a slant curve. This finishes with the proof of Theorem 1.3.

3 Further characterizations of slant helices

In this section we present two new characterizations of slant helices. The first one is a consequence of Theorem 1.3.

**Theorem 3.1.** Let $\alpha : I \subset R \rightarrow \mathbb{E}^4$ be a unit speed curve in Euclidean space $\mathbb{E}^4$. Then $\alpha$ is a slant helix if and only if there exists a $C^2$-function $f$ such that
\[
\kappa_3 f(s) = \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right) + \kappa_2, \quad \frac{d}{ds} f(s) = -\frac{\kappa_3 \kappa_1}{\kappa_2} \int \kappa_1 ds.
\]
Proof. Let now assume that $\alpha$ is a slant helix. A differentiation of (14) gives

$$
\left( \int \kappa_1(s)ds \right) \left( \int \kappa_1(s)ds \right)' + \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1(s)ds \right) \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1(s)ds \right)' + \frac{1}{\kappa_3} \left[ \frac{1}{\kappa_2} \int \kappa_1(s)ds \right]' + \frac{\kappa_2}{\kappa_3} \left[ \frac{1}{\kappa_2} \int \kappa_1(s)ds \right]' = 0.
$$

(17)

After some manipulations, the equation (17) takes the following form

$$
\frac{\kappa_1 \kappa_3}{\kappa_2} \int \kappa_1(s)ds + \frac{1}{\kappa_3} \left[ \frac{1}{\kappa_2} \int \kappa_1(s)ds \right]' + \frac{\kappa_2}{\kappa_3} \left[ \frac{1}{\kappa_2} \int \kappa_1(s)ds \right]' = 0.
$$

(18)

If we define $f = f(s)$ by

$$
\kappa_3 f(s) = \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1(s)ds \right)' + \kappa_2.
$$

Then Equation (18) writes as

$$
\frac{df}{ds} = -\frac{\kappa_3 \kappa_1}{\kappa_2} \int \kappa_1(s)ds.
$$

Conversely, if (16) holds, we define a unit constant vector $U$ by

$$
U = \cos \theta \left[ \int \kappa_1(s)ds \mathbf{T} + \mathbf{N} + \frac{\kappa_1}{\kappa_2} \int \kappa_1(s)ds \mathbf{B}_1 + f(s) \mathbf{B}_2 \right].
$$

We have that $\langle \mathbf{N}(s), U \rangle = \cos \theta$ is constant, that is, $\alpha$ is a slant helix.

We end giving an integral characterization of a slant helix.

**Theorem 3.2.** Let $\alpha : I \subset R \to \mathbf{E}^4$ be a unit speed curve in Euclidean space $\mathbf{E}^4$. Then $\alpha$ is a slant helix if and only if the following condition is satisfied

$$
\frac{\kappa_1}{\kappa_2} \int \kappa_1(s)ds = \left( A - \int \kappa_2(s) \sin \int \kappa_3(s)ds \right) \sin \int \kappa_3(s)ds - \left( B + \int \kappa_2(s) \cos \int \kappa_3(s)ds \right) \cos \int \kappa_3(s)ds,
$$

(19)

for some constants $A$ and $B$.

Proof. Suppose that $\alpha$ is a slant helix. By using Theorem 3.1, let define $m(s)$ and $n(s)$ by

$$
\phi = \phi(s) = \int^s \kappa_3(u)du,
$$

(20)
\[
m(s) = f(s) \cos \phi + \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right) \sin \phi + \int \left[ \kappa_2 \sin \phi \right] ds,
\]

\[
n(s) = f(s) \sin \phi - \left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right) \cos \phi - \int \left[ \kappa_2 \cos \phi \right] ds.
\]

(21)

If we differentiate Equations (21) with respect to \(s\) and taking into account of (20) and (16), we obtain \(\frac{dn}{ds} = 0\) and \(\frac{dm}{ds} = 0\). Therefore, there exist constants \(A\) and \(B\) such that \(m(s) = A\) and \(n(s) = B\). By substituting into (21) and solving the resulting equations for \(\frac{\kappa_1}{\kappa_2} \int \kappa_1 ds\), we get

\[
\frac{\kappa_1}{\kappa_2} \int \kappa_1 ds = \left( A - \int \left[ \kappa_2(s) \sin \phi \right] ds \right) \sin \phi - \left( B + \int \left[ \kappa_2(s) \cos \phi \right] ds \right) \cos \phi.
\]

Conversely, suppose that (19) holds. In order to apply Theorem 3.1, we define \(f = f(s)\) by

\[
f(s) = \left( A - \int \left[ \kappa_2(s) \sin \phi \right] ds \right) \cos \phi + \left( B + \int \left[ \kappa_2(s) \cos \phi \right] ds \right) \sin \phi,
\]

with \(\phi(s) = \int \kappa_3(u)du\). A direct differentiation of (19) gives

\[
\left( \frac{\kappa_1}{\kappa_2} \int \kappa_1 ds \right)' = \kappa_3 f(s) - \kappa_2.
\]

This shows the left condition in (16). Moreover, a straightforward computation leads to \(f'(s) = -\frac{\kappa_3 \kappa_1}{\kappa_2} \int \kappa_1 ds\), which finishes the proof.

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