The gravitational field outside a spatially compact stationary source in the most general class of fourth-order theories of gravity

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In the most general class of fourth-order theories of gravity, namely \( F(X,Y,Z) \) gravity, the metric for the external gravitational field of a spatially compact stationary source is provided, where \( X := R \) is Ricci scalar, \( Y := R_{\mu\nu} R^{\mu\nu} \) is Ricci square, and \( Z := R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) is Riemann square. A new type of gauge condition is proposed so that the linearized gravitational field equations of \( F(X,Y,Z) \) gravity are greatly simplified, and then, the stationary metric in the region exterior to the source is derived. The metric consists of General Relativity-like part and \( F(X,Y,Z) \) part, where the latter is the correction to the former. As with the previous one for a point-like source in references, the metric is characterized by two characteristic lengths, but since it can indicate the influence of the size and shape of a realistic source on the external gravitational field, it will have a wider range of applications. With this result, the corresponding metrics in various sub-models of \( F(X,Y,Z) \) gravity can also be deduced, and by such approach, the metrics in General Relativity, \( f(R) \) gravity, and \( f(R,G) \) gravity are all presented.

I. INTRODUCTION

Although General Relativity (GR) survives in many tests \cite{1–3}, it still faces many challenges, and one of the typical examples is that GR can not give a widely accepted explanation to the cosmic acceleration without introduction of dark energy \cite{4}. Introduction of the alternative theories of gravity is one approach to handling these difficulties \cite{3,5}, and thus, identifying the correct theory of gravity is a crucial issue of modern physics. In this work, we shall focus our attention on the fourth-order theories of gravity in the metric formalism, which modifies the Einstein-Hilbert action by adopting a general function of curvature invariants in the gravitational Lagrangian. Since derivatives of curvature invariants like \( \Box R \) and the parity-odd Chern-Simons invariant \cite{6} that enters at the same order in curvatures and derivatives are not considered, the gravitational field equations of such theories are fourth-order in derivatives of the metric tensor \cite{7,8}.

\( f(R) \) gravity \cite{9–12} is one of the simplest fourth-order theories of gravity. Besides the Ricci scalar \( R \), there are several other curvature invariants that one can construct from the metric, and all of them are the combinations of contractions of the Riemann tensor one or more times with itself and the metric \cite{13}. The most general class of fourth-order theories of gravity, namely \( F(X,Y,Z) \) gravity, is obtained by replacing the Einstein-Hilbert action in the gravitational Lagrangian by a general function \( F \) of curvature invariants \( X,Y, \) and \( Z \) \cite{8,14}, where \( X := R, \ Y := R_{\mu\nu} R^{\mu\nu}, \) and \( Z := R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) with \( R_{\mu\nu} \) as the Ricci tensor and \( R_{\mu\nu\rho\sigma} \) as the Riemann tensor. \( F(X,Y,Z) \) gravity contains a large number of sub-models, such as GR, Starobinsky gravity, \( F(R) \) gravity, and \( f(R,G) \) gravity (\( G \) is the Gauss-Bonnet scalar), etc., and they all have a wide range of applications in gravitational physics \cite{9–12,15–20}. It could be expected that if a certain result of some specific physical phenomenon is given in \( F(X,Y,Z) \) gravity, the corresponding one in its any sub-model can also be derived. As a result, \( F(X,Y,Z) \) gravity actually provides a general theoretical framework for fourth-order theories of gravity.

In order to deal with various phenomena within \( F(X,Y,Z) \) gravity, the metric for the external gravitational field of a gravitating source should be deduced. However, since the gravitational field equations of \( F(X,Y,Z) \) gravity are exceedingly complicated, one usually has to adopt the weak-field approximation to simplify them. Even so, these equations are still difficult to be disposed of, which results in that the external metric for a spatially compact stationary source has not been obtained yet. We will make an attempt to solve this problem in this paper, and the key point is how to simplify the linearized gravitational field equations of \( F(X,Y,Z) \) gravity. Motivated by the method in Refs. \cite{21–24}, a new type of gauge condition is proposed so that the linearized gravitational field equations of \( F(X,Y,Z) \) gravity can be successfully transformed into D’Alembert equation and Klein-Gordon equations with external sources. Then, with the help of the symmetric and trace-free (STF) formalism in terms of the irreducible

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Cartesian tensors, developed by Thorne [25], Blanchet, Damour, and Iyer [26–28], the stationary metric in the region exterior to the source is presented in the form of multipole expansion.

The metric consists of the GR-like part and the $F(X,Y,Z)$ part, where the former is exactly the result in GR when $F(X,Y,Z)$ gravity reduces to GR, and the latter is the correction to the former in $F(X,Y,Z)$ gravity. As with the previous one for a point-like source in Refs. [8, 14], the $F(X,Y,Z)$ part of the metric is also characterized by two characteristic lengths depending on the value of derivatives of $F$ with respect to curvature invariants, which implies that in $F(X,Y,Z)$ gravity, there are two massive propagations in general. Moreover, it should be noted that since the metric obtained in the present paper is applicable for the external gravitational field of any spatially compact stationary source, it is suitable to be used to explore phenomena happening in the gravitational field outside a realistic gravitating source because the effect of the size and shape of the source is extremely crucial. As a consequence, it is certain that our result will make $F(X,Y,Z)$ gravity have a wider range of applications.

As mentioned before, a large number of fourth-order theories of gravity are sub-models of $F(X,Y,Z)$ gravity, and therefore, the metrics for the external gravitational field of a spatially compact stationary source in these models can be directly obtained from the one in $F(X,Y,Z)$ gravity under certain conditions. As examples, the corresponding metrics in GR, $f(R)$ gravity, and $f(R, \mathcal{G})$ gravity are all presented in this paper. It is shown that when $F(X,Y,Z) \rightarrow f(R)$ or $f(R, \mathcal{G})$, one characteristic length of the metric in $F(X,Y,Z)$ gravity disappears, and the metric in $f(R, \mathcal{G})$ gravity is the same as that in $f(R)$ gravity, which is consistent with the fact that the Gauss-Bonnet scalar $\mathcal{G}$, being a topological invariant, has no contribution to the gravitational field dynamics. Furthermore, when $F(X,Y,Z)$ gravity reduces to GR, both the characteristic lengths disappear, and the metric recovers the classical one in GR. The increment or reduction of the characteristic lengths in different sub-models of $F(X,Y,Z)$ gravity implies that the theoretical predictions of these models for a physical phenomenon are completely different. Hence, if the metrics in these sub-models of $F(X,Y,Z)$ gravity are applied to dealing with some specific phenomenon, by comparing the resulting theoretical results with the experimental or observational data, the constraints on the coefficients of curvature invariants in the gravitational Lagrangians of these models may be obtained, and as a consequence, by such approach, a large class of fourth-order theories of gravity could be assessed in detail.

II. NOTATION AND RELEVANT FORMULAS IN THE STF FORMALISM [21–24, 28]

Throughout the paper, the following notation and rules are adopted:

- The international system of units is employed;
- The Greek letters, representing the spacetime indices, range from 0 to 3, and the Latin letters, representing the space indices, range from 1 to 3;
- Repeated indices appearing within a term indicate that the sum should be taken over;
- The signature of the spacetime metric $g_{\mu\nu}$ is $(-, +, +, +)$;
- In the linearized gravity theory, the coordinates $(x^\mu) = (ct, x_i)$ are treated as the Minkowskian coordinates;
- The spherical coordinate system $(ct, r, \theta, \varphi)$ is given by
  \[ x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta; \]
- The radial vector in the flat space is $\mathbf{x} = x_i \partial_i$ with $x_i$ as the components and $\partial_i := \partial/\partial x_i$ as the coordinate basis vectors. Let $r = |\mathbf{x}|$ be the length of $\mathbf{x}$, and then, $n = \mathbf{x}/r = n_i \partial_i$ is the unit radial vector with $n_i = x_i/r$.

The STF part of a Cartesian tensor $B_{i_1 i_2 \cdots i_l}$ is defined by
  \[ \hat{B}_{i_1 i_2 \cdots i_l} := B_{<i_{l+1}<} = B_{<i_1 i_2 \cdots i_l>} := \sum_{k=0}^{l} b_k \delta_{(i_1 i_2} \cdots \delta_{i_{2k+1} i_{2k+2} \cdots i_l)} g_{i_{2k+1} \cdots i_l} a_1 a_2 \cdots a_{2k}; \]

where
  \[ b_k := (-1)^k \frac{(2l - 2k - 1)!!}{(2l - 1)!!} \frac{l!}{(2k)!(l - 2k)!}, \]
and
\[ S_l := B_{(l)} = B_{(i_1i_2\cdots i_l)} := \frac{1}{l!} \sum_{\sigma} B_{i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(l)}} \] (2.4)
is its symmetric part with \( \sigma \) running over all permutations of \((12\cdots l)\). The quantities
\[ X_l = X_{i_1i_2\cdots i_l} := x_{i_1}x_{i_2}\cdots x_{i_l}, \] (2.5)
\[ N_l = N_{i_1i_2\cdots i_l} := n_{i_1}n_{i_2}\cdots n_{i_l} \] (2.6)
are used to denote the tensor products of \( l \) radial and unit radial vectors, respectively, and they satisfy
\[ X_l = r^l N_l. \] (2.7)

Other relevant formulas in the STF formalism include:
\[ \dot{N}_l = \sum_{k=0}^{[\frac{l}{2}]} b_k \delta_{i_1i_2}\cdots \delta_{i_{l-2k-1}i_{l-2k}} N_{i_{l-k+1}\cdots i_l}, \] (2.8)
\[ \dot{F}_l = \sum_{k=0}^{[\frac{l}{2}]} b_k \delta_{i_1i_2}\cdots \delta_{i_{l-2k-1}i_{l-2k}} \partial_{i_{l-k+1}\cdots i_l} (\nabla^2)^k, \] (2.9)
\[ \dot{\rho}_l \left( \frac{F(r)}{r} \right) = \dot{N}_l \sum_{k=0}^{l} \frac{(l+k)!}{(-2)^k k!(l-k)!} \frac{(\partial_r F(r))^{l-k}}{r^{k+1}}, \] (2.10)
where \( \nabla^2 = \partial_\mu \partial^\mu \) is the Laplace operator in flat space, \( \partial_{i_1i_2\cdots i_l} := \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l} \), and \( \partial_r^{l-k} \) is the \((l-k)\)-th derivative with respect to \( r \).

### III. Metric for the External Gravitational Field of a Spatially Compact Stationary Source in \( F(X, Y, Z) \) Gravity

The action of \( F(X, Y, Z) \) gravity [8, 14] is
\[ S = \frac{1}{2\kappa c^4} \int dx^4 \sqrt{-g} F(X, Y, Z) + S_M(g^{\mu\nu}, \psi), \] (3.1)
where \( \kappa = 8\pi G/c^4 \) with \( G \) as the gravitational constant, \( c \) is the velocity of light in vacuum, \( g \) is the determinant of metric \( g_{\mu\nu} \), and \( S_M(g^{\mu\nu}, \psi) \) is the matter action. In the metric formalism, the gravitational field equations and the corresponding trace equation of \( F(X, Y, Z) \) gravity are given by varying the above action with respect to \( g^{\mu\nu} \),
\[ H_{\mu\nu} = \kappa T_{\mu\nu}, \quad H = \kappa T, \] (3.2)
where
\[ H_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} F + (R_{\mu\nu} + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F_X - 2 \nabla^\lambda \nabla_\mu (F_Y R_{\lambda\nu}) + g_{\mu\nu} \nabla_\alpha \nabla_\beta (F_Y R_{\alpha\beta}) + \Box (F_Y R_{\mu\nu}) + 2 F_Y R_{\mu\rho} R^\rho_{\nu} + 4 \nabla_\mu \nabla_\nu \nabla^\rho (F_Z R_{\mu\nu\rho}) + 2 F_Z R_{\mu}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} + \Box F_X + \Box (F_Y X) + 2 \nabla^\rho \nabla_\rho (F_Y R_{\mu\nu}) + 4 \nabla^\rho \nabla_\rho (F_Z R_{\mu\nu}) \] (3.3)
with \( F_X := \partial F/\partial X, \ F_Y := \partial F/\partial Y, \) and \( F_Z := \partial F/\partial Z \), and \( T_{\mu\nu} \) is the energy-momentum tensor of matter with \( T = g_{\mu\nu} T^{\mu\nu} \) as its trace. As in Refs. [8, 14], we assume that \( F(X, Y, Z) \) is expressed as a power series
\[ F(X, Y, Z) = F_0 + F_1 X + F_2 Y + F_3 Z + \frac{1}{2} (F_4 X^2 + F_2 Y^2 + F_3 Z^2 + 2 F_{12} XY + 2 F_{13} XZ + 2 F_{23} YZ) + \cdots, \] (3.5)
where the dimensions of the coefficients \( F_2, F_3, F_{11}, F_{22} \cdots \) are \([X]^{-1}, [X]^{-1}, [X]^{-1}, [X]^{-3}, \cdots \), respectively. Such models are physically interesting and allow to recover the results of GR and the correct boundary and asymptotic conditions in general.
Equations (3.2)–(3.4) show that the gravitational field equations of $F(X,Y,Z)$ gravity are exceedingly complicated, so as usual, we will first adopt the weak-field approximation to simplify them. Let $\eta^{\mu\nu}$ be the Minkowskian metric in a fictitious flat spacetime, and the gravitational field amplitude $h^{\mu\nu}$ is defined as

$$h^{\mu\nu} := \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}.$$  

(3.6)

Just as in GR [29, 30], $h^{\mu\nu}$, being a perturbation in the linearized framework of the weak-field approximation, satisfies

$$|h^{\mu\nu}| \ll 1,$$

(3.7)

and then, the linear parts (denoted by the superscript (1)) of the Riemann tensor, the Ricci tensor and the Ricci scalar are, respectively,

$$R^{\mu\nu\rho\sigma(1)} = -\frac{1}{2}(\partial^\rho \partial^\sigma h^{\mu\nu} - \partial^\sigma \partial^\rho h^{\mu\nu} + \partial^\mu \partial^\rho h^{\nu\sigma} - \partial^\mu \partial^\sigma h^{\rho\nu})$$

$$-\frac{1}{4}(\eta^{\rho\sigma} \partial^\mu h^{\nu\lambda} + \partial^\mu \eta^{\rho\sigma} \partial^\nu h^{\rho\lambda} - \eta^{\mu\rho} \partial^\sigma \partial^\nu h^{\rho\lambda}),$$

(3.8)

$$R^{\mu\nu(1)} = \frac{1}{2} \Box h^{\mu\nu} - \frac{1}{4} \eta^{\mu\nu} \Box h - \frac{1}{2} (\partial_\lambda \partial_\mu h^{\nu\lambda} + \partial_\lambda \partial_\nu h^{\mu\lambda}),$$

(3.9)

$$X^{(1)} = R^{(1)} = -\frac{1}{2} \Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}.$$  

(3.10)

with $\partial^\alpha := \eta^{\alpha\sigma} \partial_\sigma = \eta^{\alpha\sigma} \partial / \partial x^\sigma$, $h = \eta_{\mu\nu} h^{\mu\nu}$, and $\Box := \eta^{\mu\nu} \partial_\mu \partial_\nu$. Thus, from Eq. (3.5), $F, F_X, F_Y,$ and $F_Z$ can be written as

$$\begin{align*}
F &= X^{(1)} + o(h^{\mu\nu}), \\
F_X &= 1 + F_{11} X^{(1)} + o(h^{\mu\nu}), \\
F_Y &= F_2 + F_{12} X^{(1)} + o(h^{\mu\nu}), \\
F_Z &= F_3 + F_{13} X^{(1)} + o(h^{\mu\nu}),
\end{align*}$$

(3.11)

where $o(h^{\mu\nu})$ is the higher order terms of $h^{\mu\nu}$. By virtue of Eqs. (3.6)–(3.11), the linearized gravitational field equations and the corresponding trace equation of $F(X,Y,Z)$ gravity can be derived from Eqs. (3.2)–(3.4),

$$H^{\mu\nu(1)} = R^{\mu\nu(1)} + (F_2 + 4F_3) \Box h^{\mu\nu(1)} - \frac{1}{2} \eta^{\mu\nu} X^{(1)} + \left(F_{11} + \frac{F_2}{2}\right) \eta^{\mu\nu} \Box h^{\mu\nu(1)}$$

$$-\left(F_{11} + F_2 + 2F_3\right) \partial^\rho \partial^\sigma X^{(1)} = \kappa T^{\mu\nu},$$

(3.12)

$$H^{(1)} = - X^{(1)} + (3F_{11} + 2F_2 + 2F_3) \Box h^{\mu\nu} X^{(1)} = \kappa T.$$  

(3.13)

It should be noted that $T^{\mu\nu}$ and $T$ in the above two equations are the energy-momentum tensor of matter living in Minkowski spacetime and its trace, respectively. One may observe that these linearized equations are still difficult to be disposed of, which implies that we need to find a new method to further simplify them. A basic fact is that since the Lagrangian of $F(X,Y,Z)$ gravity is a function of the gauge-invariant curvature scalars, its gravitational field equations are gauge-invariant, and as a result, we have the freedom to perform a gauge transformation [31]. Motivated by the approach in Refs. [21–24, 31], we construct the effective gravitational field amplitude of the linearized $F(X,Y,Z)$ gravity as

$$\tilde{h}^{\mu\nu} := h^{\mu\nu} + a \eta^{\mu\nu} X^{(1)} + b R^{\mu\nu(1)},$$

(3.14)

where $a$ and $b$ are the unspecified parameters. In this paper, the gauge condition

$$\partial_\mu \tilde{h}^{\mu\nu} = 0$$

(3.15)

will be imposed, and one will see that it is due to this condition that the linearized gravitational field equations of $F(X,Y,Z)$ gravity can be transformed into D’Alembert equation and Klein-Gordon equations with external sources. Firstly, by inserting Eqs. (3.14) and (3.15) into Eqs. (3.9) and (3.10), $R^{\mu\nu(1)}$ and $X^{(1)}$ are reexpressed as

$$\begin{align*}
R^{\mu\nu(1)} &= \frac{1}{2} \Box \tilde{h}^{\mu\nu} - \frac{1}{4} \eta^{\mu\nu} \Box \tilde{h} + \frac{2a + b}{4} \eta^{\mu\nu} \Box h^{\mu\nu} X^{(1)} - \frac{b}{2} \Box \eta^{\mu\nu} R^{\mu\nu(1)} + \left(\frac{a + b}{2}\right) \partial^\mu \partial^\nu X^{(1)}, \\
X^{(1)} &= - \frac{1}{2} \Box \tilde{h} + (3a + b) \Box h^{\mu\nu} X^{(1)},
\end{align*}$$

(3.16)

(3.17)
and then, substituting them in Eqs. (3.12) and (3.13), one gets

\[ H^{\mu\nu(1)} = \frac{1}{2} \Box_{\eta} \hat{h}^{\mu\nu} + \left( F_{11} + \frac{F_{2}}{2} - a - \frac{b}{4} \right) \eta^{\mu\nu} \Box_{\eta} X^{(1)} + \left( F_{2} + 4F_{3} - \frac{b}{2} \right) \Box_{\eta} R^{\mu\nu(1)} \]

\[ - \left( F_{11} + F_{2} + 2F_{3} - a - \frac{b}{2} \right) \partial_\mu \partial_\nu X^{(1)} = \kappa T^{\mu\nu}, \]  

(3.18)

\[ H^{(1)} = \frac{1}{2} \Box_{\eta} \hat{h} + (3F_{11} + 2F_{2} + 2F_{3} - 3a - b) \Box_{\eta} X^{(1)} = \kappa T. \]  

(3.19)

If we pick

\[ \begin{cases} a = F_{11} - 2F_{3}, \\ b = 2F_{2} + 8F_{3}, \end{cases} \]  

(3.20)

Eqs. (3.18) and (3.19) reduce to

\[ \Box_{\eta} \hat{h}^{\mu\nu} = 2\kappa T^{\mu\nu}, \]

(3.21)

\[ \Box_{\eta} \hat{h} = 2\kappa T. \]  

(3.22)

Obviously, the effective gravitational field amplitude \( \hat{h}^{\mu\nu} \) satisfies D’Alembert equation, and therefore, it should behave just as its counterpart in GR. The physical meaning of the gauge condition (3.15) can be found from Eq. (3.20). By plugging Eq. (3.20) into Eq. (3.14) and then making use of the linearized Bianchi’s identity \( \partial_\mu R^{\mu\nu(1)} = \partial_\nu X^{(1)}/2 \), one can acquire

\[ \partial_\mu h^{\mu\nu} = -(F_{11} + F_{2} + 2F_{3}) \partial_\nu X^{(1)}, \]

which explicitly shows that the gauge condition (3.15) is no longer the harmonic gauge condition \( \partial_\mu h^{\mu\nu} = 0 \) [3]. Now, if we substitute Eq. (3.20) back in Eqs. (3.16) and (3.17) and introduce the quantities

\[ m_1^2 := \frac{1}{3F_{11} + 2F_{2} + 2F_{3}}, \]

(3.23)

\[ m_2^2 := -\frac{1}{F_{2} + 4F_{3}}, \]  

(3.24)

the differential equations fulfilled by \( R^{\mu\nu(1)} \) and \( X^{(1)} \) are obtained,

\[ R^{\mu\nu(1)} = \kappa \left( T^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} T \right) + \frac{1}{m_2^2} \Box_{\eta} R^{\mu\nu(1)} \]

\[ + \frac{1}{3} \left( \frac{1}{m_1^2} - \frac{1}{m_2^2} \right) \left( \frac{1}{2} \eta^{\mu\nu} \Box_{\eta} X^{(1)} + \partial_\mu \partial_\nu X^{(1)} \right), \]  

(3.25)

\[ X^{(1)} = -\kappa T + \frac{1}{m_1^2} \Box_{\eta} X^{(1)}. \]  

(3.26)

Equation (3.26) is actually Klein-Gordon equation with an external source, namely

\[ \Box_{\eta} X^{(1)} - m_1^2 X^{(1)} = m_1^2 \kappa T, \]  

(3.27)

and in order to get a physically meaningful solution [32, 33], we constrain \( F(X, Y, Z) \) such that \( m_1^2 > 0 \). Thus, as implied in Ref. [14], it is clearly shown that the Ricci scalar \( X^{(1)} \) presents a massive propagation. Equation (3.25) is very complicated so that seeking its solution is a challenging task, which is the key reason why the metric for the external gravitational field of a realistic source has not been obtained yet. In fact, with the help of Eq. (3.27), Eq. (3.25) can be greatly simplified. From Eq. (3.27), \( \eta^{\mu\nu} \Box_{\eta} X^{(1)} \) and \( \partial_\mu \partial_\nu X^{(1)} \) in Eq. (3.25) have the following decompositions:

\[ \begin{cases} \eta^{\mu\nu} \Box_{\eta} X^{(1)} = d_1 \eta^{\mu\nu} m_1^2 X^{(1)} + (1 - d_1) \eta^{\mu\nu} \Box_{\eta} X^{(1)} + d_1 \eta^{\mu\nu} m_1^2 \kappa T, \\ \partial_\mu \partial_\nu X^{(1)} = \frac{d_2}{m_1^2} \partial_\mu \partial_\nu \left( \Box_{\eta} X^{(1)} \right) + (1 - d_2) \partial_\mu \partial_\nu X^{(1)} - d_2 \partial_\mu \partial_\nu (\kappa T), \end{cases} \]  

(3.28)
where \(d_1\) and \(d_2\) are arbitrary two parameters, and by applying (3.28), one is able to successfully rewrite Eq. (3.25) as

\[ \Box_g P^{\mu \nu} - m_2^4 P^{\mu \nu} = -m_2^2 \kappa S^{\mu \nu} \]  
(3.29)

with

\[ P^{\mu \nu} = R^{\mu \nu} \eta^{(1)} - \frac{1}{6} \eta^{\mu \nu} X^{(1)} - \frac{1}{3m_1^2} \partial^\mu \partial^\nu X^{(1)}, \]
(3.30)

\[ S^{\mu \nu} = T^{\mu \nu} - \frac{1}{3} P^{\mu \nu} T + \frac{1}{3m_2^2} \partial^\mu \partial^\nu T. \]
(3.31)

Equation (3.29) explicitly indicates that \(P^{\mu \nu}\) also satisfies Klein-Gordon equation with an external source, and if \(m_2^2 > 0\), the components of the tensor \(P^{\mu \nu}\) also present massive propagations in \(F(X, Y, Z)\) gravity. As in Refs. [8, 14], we will always choose \(m_2^2 > 0\) and \(m_2^2 > 0\) in this paper so as to obtain physically meaningful solutions to Eqs. (3.27) and (3.29). From the above process, we observe that by imposing the gauge condition (3.15), the original linearized gravitational field equations (3.12) of \(F(X, Y, Z)\) gravity have indeed been converted to D’Alembert equation (3.21) and Klein-Gordon equations (3.27) and (3.29). Since these equations are easy to handle, the following tasks are straightforward.

Next, we shall seek to find the solutions to Eqs. (3.21), (3.27), and (3.29) for a spatially compact stationary source with the help of the STF formalism presented in Sec. II. Since the source is time-independent, Eqs. (3.21), (3.27), and (3.29) reduce to

\[ \nabla^2 \tilde{h}^{\mu \nu} = 2 \kappa T^{\mu \nu}, \]
(3.32)

\[ \nabla^2 X^{(1)} - m_2^4 X^{(1)} = m_2^2 \kappa T, \]
(3.33)

\[ \nabla^2 P^{\mu \nu} - m_2^4 P^{\mu \nu} = -m_2^2 \kappa S^{\mu \nu}. \]
(3.34)

According to relevant results in Ref. [21], the solution to Eq. (3.32) for a spatially compact source is

\[
\begin{align*}
\tilde{h}^{00}(x) &= -\frac{4G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_l \partial_l \left( \frac{1}{r} \right), \\
\tilde{h}^{0i}(x) &= -\frac{4G}{c^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \epsilon_{iab} \hat{S}_{ab} t_{l-1} \partial_t \frac{1}{r}, \\
\tilde{h}^{ij}(x) &= 0
\end{align*}
\]
(3.35)

with

\[
\begin{align*}
\hat{M}_l &= \frac{1}{c^2} \int d^3x' \hat{X}'_l \left( T^{00}(x') + T^{aa}(x') \right), \\
\hat{S}_l &= \frac{1}{c} \int d^3x' \epsilon_{ab<l} \hat{X}'_{ab} t_{l+1} T^{0b}(x'), \quad l \geq 1
\end{align*}
\]
(3.36)

as the mass-type and current-type source multipole moments, where \(\epsilon_{iab}\) is the totally antisymmetric Levi-Civita symbol and \(\hat{X}'_l\) is the STF part of \(X'_l := x'_{i_1} x'_{i_2} \cdots x'_{i_l}\). By following the method in Ref. [21], the solutions to Eqs. (3.33) and (3.34) for a spatially compact source are, respectively,

\[ X^{(1)}(x) = -\frac{m_2^2 \kappa}{4\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{Q}_l \partial_l \left( \frac{e^{-m_2 r}}{r} \right), \]
(3.37)

\[ P^{\mu \nu}(x) = \frac{m_2^2 \kappa}{4\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} F_{l>l}^{\mu \nu} \partial_l \partial_l \left( \frac{e^{-m_2 r}}{r} \right) \]
(3.38)

with

\[ \hat{Q}_l = \frac{(2l+1)!!}{m_2^l} \int \hat{X}'_l \left[ \left( \frac{d}{r' dr'} \right)^l \frac{\sinh(m_1 r')}{m_1 r'} \right] T(x') d^3 x', \]
(3.39)

\[ F_{l>l}^{\mu \nu} = \frac{(2l+1)!!}{m_2^l} \int \hat{X}'_l \left[ \left( \frac{d}{r' dr'} \right)^l \frac{\sinh(m_2 r')}{m_2 r'} \right] S^{\mu \nu}(x') d^3 x'. \]
(3.40)
In order to acquire the gravitational field amplitude $h^{\mu \nu}$, one needs to resort to Eq. (3.14), and by further employing Eqs. (3.20), (3.23), (3.24), and (3.30), $h^{\mu \nu}$ can be expressed in terms of $\tilde{h}^{\mu \nu}$, $P^{\mu \nu}$, and $X^{(1)}$, namely

$$h^{\mu \nu} = \tilde{h}^{\mu \nu} + \frac{2}{m_2^2} P^{\mu \nu} + \frac{2}{3m_1^2m_2^2} \partial^{\mu} \partial^{\nu} X^{(1)} - \frac{1}{3} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \eta^{\mu \nu} X^{(1)}. \tag{3.41}$$

Then, inserting Eqs. (3.37) and (3.38),

$$h^{\mu \nu} = \tilde{h}^{\mu \nu} + \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left( \hat{Q}_{l l} \left( \frac{m_1^2 + m_2^2}{6m_2^2} \eta^{\mu \nu} - \frac{1}{3m_2^2} \partial^{\mu} \partial^{\nu} \right) \hat{\delta}_{l l} \left( \frac{e^{-m_1 r}}{r} \right) + F_{<l,l>^{\mu \nu}} \hat{\partial}_{l l} \left( \frac{e^{-m_2 r}}{r} \right) \right). \tag{3.42}$$

As previously noted, $\tilde{h}^{\mu \nu}$ is the counterpart of $h^{\mu \nu}$ in GR, and the second term is the correction to $\tilde{h}^{\mu \nu}$ in $F(X, Y, Z)$ gravity. Remember that the dimensions of the coefficients $F_2, F_3$, and $F_{11}$ are all $[X]^{-1}$, so the above correction to $\tilde{h}^{\mu \nu}$ shows a Yukawa-like dependence on two characteristic lengths $m_1^{-1}$ and $m_2^{-1}$. On the basis of Eqs. (3.23) and (3.24), when $F(X, Y, Z) \rightarrow f(R)$, there are $m_1^2 \rightarrow 1/(3F_{11})$ and $m_2 \rightarrow +\infty$, and then, Eq. (3.42) recovers the corresponding result in $f(R)$ gravity [21]. As to the metric for the gravitational field, it is given by [24]

$$g_{\mu \nu} = \eta_{\mu \nu} - \tilde{\eta}_{\mu \nu} + o(h^{\mu \nu}) \tag{3.43}$$

with

$$\tilde{\eta}_{\mu \nu} := h_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} h. \tag{3.44}$$

Thus, the metric for the gravitational field outside a spatially compact stationary source in $F(X, Y, Z)$ gravity is

$$\begin{align*}
g_{00}(x) &= -1 + \frac{2}{c^2} U(x) - \frac{4}{c^2} V(x) + o(h^{\mu \nu}), \\
g_{0i}(x) &= -\frac{4}{c^2} U^i(x) + \frac{4}{c^2} V^i(x) + o(h^{\mu \nu}), \\
g_{ij}(x) &= \delta_{ij} \left( 1 + \frac{2}{c^2} U(x) \right) + \frac{4}{c^2} V^{ij}(x) + o(h^{\mu \nu}),
\end{align*} \tag{3.45}$$

where the potentials $U(x), V(x), U^i(x), V^i(x)$, and $V^{ij}(x)$ are, respectively, defined as

$$\begin{align*}
U(x) &= G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_l \hat{\delta}_{l l} \left( \frac{1}{r} \right), \\
V(x) &= \frac{G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{1}{6} \hat{Q}_{l l} \hat{\delta}_{l l} \left( \frac{e^{-m_1 r}}{r} \right) + F_{<l,l>^{00}} \hat{\partial}_{l l} \left( \frac{e^{-m_2 r}}{r} \right), \\
U^i(x) &= G \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \epsilon_{iab} \hat{\delta}_{a l l} \hat{\delta}_{b l l} \left( \frac{1}{r} \right), \\
V^i(x) &= \frac{G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} F_{<l,l>^{0i}} \hat{\partial}_{l l} \left( \frac{e^{-m_2 r}}{r} \right), \\
V^{ij}(x) &= \frac{G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left( \hat{Q}_{l l} \left( \frac{1}{6} \delta_{ij} + \frac{1}{3m_1^2} \partial^i \partial^j \right) \hat{\delta}_{l l} \left( \frac{e^{-m_1 r}}{r} \right) \right) - F_{<l,l>^{ij}} \hat{\partial}_{l l} \left( \frac{e^{-m_2 r}}{r} \right) \tag{3.46}
\end{align*}$$

with

$$\begin{align*}
F_{<l,l>^{00}} &= \frac{(2l + 1)!!}{m_2^2} \int \hat{X}^l \left[ \left( \frac{d}{r^3 dr^l} \right)^l \left( \sinh \left( \frac{m_2 r'}{m_2^l} \right) \right) \right] \left( T_{00}(x') + \frac{1}{3} T(x') \right) d^3 x', \\
F_{<l,l>^{0i}} &= \frac{(2l + 1)!!}{m_2^2} \int \hat{X}^l \left[ \left( \frac{d}{r^3 dr^l} \right)^l \left( \sinh \left( \frac{m_2 r'}{m_2^l} \right) \right) T_{0i}(x') d^3 x', \\
F_{<l,l>^{ij}} &= \frac{(2l + 1)!!}{m_2^2} \int \hat{X}^l \left[ \left( \frac{d}{r^3 dr^l} \right)^l \left( \sinh \left( \frac{m_2 r'}{m_2^l} \right) \right) \right] \left( T^{ij}(x') - \frac{1}{3} \delta_{ij} T(x') + \frac{1}{3m_2^2} \partial x_i \partial x_j \right) d^3 x'. \tag{3.47}
\end{align*}$$
From Eqs. (3.23) and (3.24), we observe that when $F(X,Y,Z)$ gravity reduces to GR, both $m_1$ and $m_2$ tend to positive infinity, which means that the potentials $V(x)$, $V^i(x)$, and $V^{ij}(x)$ vanish, and thus,

\[
\begin{align*}
\begin{cases}
  g^{GR}_{00}(x) = -1 + \frac{2}{c^2}U(x) + o(h^{\mu\nu}), \\
g^{GR}_{0i}(x) = -\frac{4}{c^3}U^i(x) + o(h^{\mu\nu}), \\
g^{GR}_{ij}(x) = \delta_{ij}\left(1 + \frac{2}{c^2}U(x)\right) + o(h^{\mu\nu})
\end{cases}
\end{align*}
\]  

(3.48)

is exactly the metric in GR. Therefore, in the expression (3.45), the terms, not related to the potentials $V(x)$, $V^i(x)$, and $V^{ij}(x)$, constitute the GR-like part, and the remaining terms, being the correction to the GR-like part in $F(X,Y,Z)$ gravity, constitute the $F(X,Y,Z)$ part. Obviously, the $F(X,Y,Z)$ part is characterized by the two characteristic lengths $m_1^{-1}$ and $m_2^{-1}$, which are dependent on the value of derivatives of $F$ with respect to curvature invariants. In fact, the similar conclusion has been reported in Refs. [8, 14] by means of the metric at Newtonian order.

When the gravitational field is generated by a ball-like source and the distance from the center-of-mass of the source is much less than the scale associated with the volume occupied by the source, we are allowed to ignore all characteristic lengths $\nu$, and $V^{\mu\nu}$ is exactly the metric in GR. Therefore, in the expression (3.45), the terms, not related to the potentials $V(x)$, $V^i(x)$, and $V^{ij}(x)$, constitute the GR-like part, and the remaining terms, being the correction to the GR-like part in $F(X,Y,Z)$ gravity, constitute the $F(X,Y,Z)$ part. Obviously, the $F(X,Y,Z)$ part is characterized by the two characteristic lengths $m_1^{-1}$ and $m_2^{-1}$, which are dependent on the value of derivatives of $F$ with respect to curvature invariants. In fact, the similar conclusion has been reported in Refs. [8, 14] by means of the metric at Newtonian order.

As previously mentioned, $F(X,Y,Z)$ gravity, being the most general class of fourth-order theories of gravity, contains a large number of sub-models, such as GR, $f(R)$ gravity, and $f(R,G)$ gravity, etc., and as a consequence, the metrics for the external gravitational field of a spatially compact stationary source in these models could be directly obtained from Eq. (3.45) under certain conditions. By this means, the metric (3.48) in GR has already been derived, and it is
observed that both characteristic lengths \( m_1^{-1} \) and \( m_2^{-1} \) disappear in GR. As to \( f(R) \) gravity, when \( F(X, Y, Z) \rightarrow f(R) \), the power series (3.5) reduces to

\[
f(R) = R + \frac{1}{2} f_{11} R^2 + \cdots ,
\]

and then, from Eqs. (3.23) and (3.24), there are \( m_1^2 \rightarrow 1/(3 f_{11}) \) and \( m_2 \rightarrow +\infty \). Thus, inserting them into Eqs. (3.45) and (3.46), one can deduce the metric in \( f(R) \) gravity,

\[
\begin{aligned}
&g_{00}^{(R)}(x) = -1 + \frac{2}{c^2} U(x) - \frac{4}{c^2} V_{f(R)}(x) + o(h^{\mu \nu}), \\
&g_{0i}^{(R)}(x) = -\frac{4}{c^2} U^i(x) + o(h^{\mu \nu}), \\
&g_{ij}^{(R)}(x) = \delta_{ij} \left( 1 + \frac{2}{c^2} U(x) + \frac{4}{c^2} V_{f(R)}(x) \right) + o(h^{\mu \nu})
\end{aligned}
\]

with

\[
\begin{aligned}
&V_{f(R)}(x) := G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{Q}_l \hat{R}_l \left( \frac{e^{-m_i r}}{r} \right), \\
&V_{f(R)}(x) := 0, \\
&V_{f(R)}^{ij}(x) := \delta_{ij} V_{f(R)}(x),
\end{aligned}
\]

which is exactly the same as that in Ref. [21]. These expressions imply that since the characteristic length \( m_2^{-1} \) vanishes, there is only one massive propagation with mass \( m_1 \) in \( f(R) \) gravity. Now, let us examine \( f(R, \mathcal{G}) \) gravity. We postulate that

\[
F(X, Y, Z) = f(R, \mathcal{G}) := R + f_2 \mathcal{G} + \frac{1}{2} \left( f_{11} R^2 + 2 f_{12} R \mathcal{G} + f_{22} \mathcal{G}^2 \right) + \cdots ,
\]

and due to \( \mathcal{G} = X^2 - 4Y + Z \), there are

\[
\begin{aligned}
F_{11} = f_{11} + 2 f_2, \\
F_2 = -4 f_2, \\
F_3 = f_2.
\end{aligned}
\]

Substituting these expressions in Eqs. (3.23) and (3.24), we acquire that \( m_1^2 = 1/(3 f_{11}) \) and \( m_2 = +\infty \), which explicitly indicates the metric in \( f(R, \mathcal{G}) \) gravity is also Eq. (3.56). Therefore, it is concluded that the metric in \( f(R, \mathcal{G}) \) gravity is the same as that in \( f(R) \) gravity, which is consistent with the fact that the Gauss-Bonnet scalar \( \mathcal{G} \), being a topological invariant, has no contribution to the gravitational field dynamics. In the future, when one studies some specific phenomena within the framework of \( F(X, Y, Z) \) gravity, the increment or reduction of the characteristic lengths in its various sub-models will have a great influence on final theoretical results. Thus, by comparing these theoretical results with the experimental or observational data, the constraints on the coefficients of curvature invariants in the gravitational Lagrangians of these models may be obtained, and in this manner, a large class of fourth-order theories of gravity could be assessed in detail.

**IV. CONCLUSIONS**

In this paper, \( F(X, Y, Z) \) gravity, the most general class of fourth-order theories of gravity involving curvature invariants \( X = R \), \( Y = R_{\mu \nu} R^{\mu \nu} \), and \( Z = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \), has been considered. Conducting a study on \( F(X, Y, Z) \) gravity is of great importance because it provides a general theoretical framework for many famous gravitational models, such as GR, Starobinsky gravity, \( f(R) \) gravity, and \( f(R, \mathcal{G}) \) gravity, etc. In order to deal with various phenomena within \( F(X, Y, Z) \) gravity, the first thing we need to do is to deduce the metric for the external gravitational field of a gravitating source. However, since the gravitational field equations of \( F(X, Y, Z) \) gravity are exceedingly complicated, the external metric for a spatially compact stationary source has not been obtained yet.

By applying the weak-field approximation, the linearized gravitational field equations of \( F(X, Y, Z) \) gravity are obtained firstly, and how to further deal with them and find their solutions is the main task of the present paper.
Following the method in Refs. [21–24], we propose a new type of gauge condition in \( F(X,Y,Z) \) gravity, and after imposing it, the gravitational field amplitude \( h^{\mu \nu} \), associated with the metric \( g_{\mu \nu} \), is successfully expressed in terms of the effective gravitational field amplitude \( \tilde{h}^{\mu \nu} \), the tensor \( P^{\mu \nu} \) relevant to the linearized Ricci tensor \( R^{\mu \nu} \), and the linearized Ricci scalar \( X^{(1)} \), where \( \tilde{h}^{\mu \nu} \) satisfies D’Alembert equation and both \( P^{\mu \nu} \) and \( X^{(1)} \) satisfy Klein-Gordon equations with external sources. By invoking the stationary solutions to D’Alembert equation and Klein-Gordon equation in Ref. [21], the metric, presented in the form of multipole expansion, for the gravitational field outside a spatially compact stationary source in \( F(X,Y,Z) \) gravity is derived.

Since the effective gravitational field amplitude \( \tilde{h}^{\mu \nu} \), satisfying the wave equation, behaves just as its counterpart in GR, the terms related to \( \tilde{h}^{\mu \nu} \) in the expression of the metric constitute the GR-like part, and the remaining terms constitute the \( F(X,Y,Z) \) part, where the latter is the correction to the former in \( F(X,Y,Z) \) gravity. In Refs. [8, 14], the metric for point-like source in \( F(X,Y,Z) \) gravity is provided, and it is shown that in this case, the metric is characterized by two characteristic lengths depending on the value of derivatives of \( F \) with respect to curvature invariants. The metric obtained in this paper also shares the same feature, which implies that in \( F(X,Y,Z) \) gravity, there are two massive propagations in general. Moreover, since this metric is applicable for the external gravitational field of any spatially compact stationary source, it is suitable to be used to explore phenomena happening in the gravitational field outside a realistic gravitating source because the effect of the size and shape of the source is extremely crucial. As a consequence, it is certain that our result will make \( F(X,Y,Z) \) gravity have a wider range of applications.

Finally, in GR, \( f(R) \) gravity, and \( f(R,G) \) gravity, the metrics for the external gravitational field of a spatially compact stationary source are derived from the one in \( F(X,Y,Z) \) gravity because these theories are all the sub-models of \( F(X,Y,Z) \) gravity. When \( F(X,Y,Z) \) gravity reduces to \( f(R) \) or \( f(R,G) \) gravity, it is proved that one characteristic length of the metric in \( F(X,Y,Z) \) gravity disappears, and the metric in \( f(R,G) \) gravity is identical to that in \( f(R) \) gravity, which confirms the fact that the Gaussian-Bonnet scalar \( G \), as a topological invariant, has no contribution to the gravitational field dynamics. When \( F(X,Y,Z) \) gravity further reduces to GR, both the characteristic lengths of the metric disappear, and the GR-like part is exactly the result in GR. It could be expected that the increment or reduction of the characteristic lengths in these sub-models of \( F(X,Y,Z) \) gravity will have a great impact on their theoretical predictions for a physical phenomenon.

As noted above, \( F(X,Y,Z) \) gravity is the most general class of fourth-order theories of gravity, and it contains a large number of sub-models, and for these models, the metric obtained in the present paper is a universal outcome, which actually provides a powerful tool for us to analyze the gravitational phenomena happening in the gravitational field outside a realistic stationary source in these models. Although only the metric for stationary source is derived, it is sufficient to be employed to explain many phenomena, such as the light bending. One can also seek to find the external metric for non-stationary source under the weak-field and slow-motion approximation, but a basic fact is that such metric is only valid within the near zone of the source [30], so it seems that our present result will make \( F(X,Y,Z) \) gravity have a wider range of applications. In the future, if the metrics in various sub-models of \( F(X,Y,Z) \) gravity are applied to dealing with some specific phenomenon, by comparing the resulting theoretical result with the experimental or observational data, the constraints on the coefficients of curvature invariants in the gravitational Lagrangians of these models may be obtained, and therefore, our result can also be used to assess in detail a large class of fourth-order theories of gravity.

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