HYPERSYMPLECTIC MANIFOLDS AND ASSOCIATED GEOMETRIES

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Abstract. We study an obstruction theory for hypersymplectic manifolds equipped with a free, isometric action of SU(1, 1). When the obstruction vanishes, we show that the manifold is a metric cone over a split 3-Sasakian manifold. Furthermore, if the action of SU(1, 1) is also proper, then we show that the hypersymplectic manifold fibres over a para-quaternionic Kähler manifold. We conclude the article with some examples.

1. Introduction

A hypersymplectic manifold is a 4n-dimensional pseudo-Riemannian manifold, equipped with a metric of neutral signature (2n, 2n), and whose holonomy is contained inside the symplectic group Sp(2n, R). They can be viewed as pseudo-Riemannian analogues of hyperKähler manifolds. The terminology “hypersymplectic” is due to Hitchin [1]. Hypersymplectic geometry appears naturally in the study of integrable systems [2], string theory [3] - where it is also known by Kleinian geometry - and gauge theory [4, 5].

A powerful tool for constructing hypersymplectic manifolds is the hypersymplectic quotient construction, which is an adaptation of the Marden-Weinstein construction in symplectic geometry. However, in contrast with the hyperKähler situation, the hypersymplectic metric can have degeneracies, even if the quotient is a smooth manifold [6].

Another way of obtaining hypersymplectic manifolds is via an adaptation of the Swann’s bundle construction in hyperKähler geometry [7]. Starting with a quaternionic Kähler manifold of positive scalar curvature, say N, Swann’s construction produces a bundle, U(N) → N with a typical fibre \( \mathbb{H}^* / \mathbb{Z}_2 \), whose total space carries a hyperKähler structure. It is possible to carry over this construction to the pseudo-Riemannian case [6]. In order to do so, one needs a para-quaternionic Kähler manifold; i.e., a 4n-dimensional pseudo-Riemannian manifold, whose holonomy is contained inside the group

\[
\text{Sp}(2n, \mathbb{R}) \cdot \text{Sp}(2, \mathbb{R}) = \text{Sp}(2n, \mathbb{R}) \times_{\pm 1} \text{Sp}(2, \mathbb{R}).
\]

They can be thought of as pseudo-Riemannian analogues of quaternionic Kähler manifolds. Starting with a para-quaternionic Kähler manifold N, the construction produces a bundle \( U(N) \rightarrow N \), with typical fibre \( \mathbb{B}^* / \mathbb{Z}_2 \), where \( \mathbb{B}^* \) is the space of non-zero split quaternions with non-zero norm. The total space \( U(N) \) carries a hypersymplectic structure. Both the para-quaternionic Kähler and hypersymplectic geometries turn out to be Ricci-flat and Einstein. The former is characterised by the existence of a closed 4-form, while the latter is equipped with family of symplectic 2-forms.

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In this article, we study a more general picture. Namely, given a hypersymplectic manifold, when does it admit a para-quaternionic Kähler quotient? Our basic observation is that, the total space of a Swann bundle over a para-quaternionic Kähler manifold admits a free, proper, isometric action of \( \text{Sp}(2, \mathbb{R}) \cong \text{SU}(1, 1) \), that “permutates” the hypersymplectic structure. This can be thought of as an analogue of the permuting \( \text{Sp}(1) \)-action on the hyperKähler manifolds. Given a hypersymplectic manifold equipped with a free, permuting action of \( \text{SU}(1, 1) \), we construct two maps \( \rho_2 : M \rightarrow S^4(B) \) - where \( B \) is the standard representation of \( \text{SU}(1, 1) \) on the vector space of split quaternions \( \mathbb{B} \) - and \( \rho_0 : M \rightarrow \mathbb{R}_{>0} \). If \( \rho_2 \) vanishes, we show that \( \rho_0 \) is a hypersymplectic potential. Moreover, the level-sets of \( \rho_0 \) carry a split 3-Sasakian structure and the metrics on different level-sets are homothetic. In particular, the hypersymplectic manifold is a metric cone over a split 3-Sasakian manifold. Split 3-Sasakian structures were introduced by Swann, Jørgensen and Dancer in [6] and have also been studied by Caldarella and Pastore [8], where they are referred to as “mixed 3-Sasakian structures”. The authors show that any split 3-Sasakian structure is necessarily Einstein. Our approach is analogous to that of Boyer, Galicki and Mann [9] for hyperKähler manifolds with permuting \( \text{Sp}(1) \)-action. However, in contrast with the hyperKähler situation, the vanishing of the obstruction is no guarantee that the hypersymplectic manifold in question is the total space of some Swann bundle. One additionally requires that the action of \( \text{SU}(1, 1) \) be proper, in which case, we show that the quotient of a level-set of \( \rho_0 \) produces a para-quaternionic Kähler manifold.

As in the hyperKähler case, a family of hypersymplectic manifolds can be constructed, starting with the flat-space. We show that the Swann-bundle construction commutes with the quotient construction, which produces a family of examples of the theory. The results in this article complement the work of Swann, Jørgensen and Dancer in [6].

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3. Brief introduction to split quaternionic geometry

The space of split quaternions is a 4-dimensional vector space \( \mathbb{B} \), spanned by \((1, i, s, t)\), satisfying the following relations

\[
i^2 = -1, \quad s^2 = 1 = t^2, \quad i \cdot s = t = -s \cdot i, \quad i = t \cdot s.
\]

Like quaternions, the vector space of split quaternions comes equipped with a multiplication operation, which gives it a structure of an associative algebra.

The vector space carries a natural inner product defined by \( \langle p, q \rangle_\mathbb{B} = \text{Re}(\overline{p}q) \), where \( p = p_0 + ip_1 + sp_2 + tp_3 \) and \( \overline{p} = p_0 - ip_1 - sp_2 - tp_3 \). Note that \( \langle p, p \rangle_\mathbb{B} \) is just the norm of \( p \). The norm is multiplicative \( \|pq\|^2 = \|p\|^2 \cdot \|q\|^2 \).

Unlike the quaternionic algebra, the split quaternion algebra contains non-trivial zero divisors. Moreover, the elements \( 1, i, s, t \) are not the only elements with length \( \pm 1 \). It is easily seen that the elements of norm \( 1 \) are parametrized by the 1-sheeted hyperboloid \( x_1^2 - x_2^2 - x_3^2 = 1 \), while those with norm \( -1 \) are parametrized by the 2-sheeted hyperboloid \( x_1^2 - x_2^2 - x_3^2 = -1 \). Any triple \( \{i, s, t\} \) satisfying (3.1) defines a split-quaternionic structure on \( \mathbb{B} \).
Let $U = \{ q \in \mathbb{B} \mid \|q\| \neq 0 \}$ be the set of all units in $\mathbb{B}$. This is clearly a multiplicative group. The subset of $U$ consisting of all elements $q$ with $\|q\| = 1$ forms a non-compact topological group $SU(1, 1)$. This is the special unitary group of all complex $2 \times 2$ matrices $g$ that satisfy

1. the unimodular condition, i.e $\det g = 1$
2. the pseudo-unitary condition, i.e, $g^* J g = J$ where $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

In particular, any element $g \in SU(1, 1)$ has the form $g = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix}$, where $\alpha$ and $\beta$ are complex numbers subject to the condition $|\alpha|^2 - |\beta|^2 = 1$. The Lie algebra of $SU(1, 1)$ is 3-dimensional and spanned by

$$su(1, 1) = \text{Span} \left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right\} \cong \mathfrak{im}(\mathbb{B}).$$

Relation with $SO(1, 2)$: Consider the 3-dimensional Lorentz group $SO(1, 2)$. This is the group of transformations of the 3-dimensional Minkowski space $\mathbb{M}^3$, with determinant 1, that preserves the pseudo-norm. If we divide the Minkowski space into three domains - the 2-sheeted hyperboloid $\|x\| > 0$, the 1-sheeted hyperboloid $\|x\| < 0$, surface of cone $\|x\| = 0$ - then, action of $SO(1, 2)$ is transitive on all the three surfaces. Alternatively, the hyperboloids can be thought of as unit spacelike and timelike vectors in the pseudo-sphere in $\mathbb{M}^3$. The $SO(1, 2)$-action can then be thought of as an analogue of the standard action of $SO(3)$ on the 2-sphere $S^2 \subset \mathbb{R}^3$.

The group $SO(1, 2)$ is disconnected and has two connected components. We denote by $SO^+(1, 2)$ the identity component. Identifying $\mathbb{M}^3$ with the imaginary split-quaternions $\mathfrak{im}(\mathbb{B})$, it is easily seen that the adjoint action of $SU(1, 1)$ on $\mathbb{M}^3$ preserves the pseudo-norm. In particular, the action preserves the 2-sheeted hyperboloid, 1-sheeted hyperboloid, and the surface of the cone. Therefore the linear transformations corresponding to the adjoint action of the elements of $SU(1, 1)$ belong to the identity component $SO^+(1, 2)$. This gives a homomorphism from $SU(1, 1)$ to $SO^+(1, 2)$ with kernel $\pm 1$; i.e., $SU(1, 1)/\pm 1 \cong SO^+(1, 2)$. This is similar to the homomorphism between the groups $SU(2)$ and $SO(3)$.

3.1. Modules over split quaternions. Consider the left $\mathbb{B}$ module $\mathbb{B}^n \cong \mathbb{R}^{4n}$, equipped with the split quaternionic structure $I, S, T$, given by

$$I(q) = q\overline{1}, \quad S(q) = qs, \quad T(q) = qt.$$  \hspace{1cm} (3.2)

The module inherits the natural inner product

$$\langle \alpha, \beta \rangle := \text{Re} \left( \overline{\alpha}^T \beta \right), \quad \alpha, \beta \in \mathbb{R}^{4n}$$

with a signature $(2n, 2n)$. The automorphism group of $\mathbb{B}^n$

$$Sp(n, \mathbb{B}) := \{ A \in M_n(\mathbb{B}) \mid \overline{A}^T A = \text{id} \} \cong Sp(2n, \mathbb{R}),$$

is isomorphic to $Sp(2n, \mathbb{R})$, the automorphism group of the symplectic vector space $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$. The Lie algebra of $Sp(n, \mathbb{B})$ is given by

$$sp(n, \mathbb{B}) := \{ A \in M_n(\mathbb{B}) \mid A + \overline{A}^T = 0 \}.$$
Note that we have the isomorphism $\text{Sp}(1, B) \cong \text{SU}(1, 1) \cong \text{SL}(2, \mathbb{R})$ and so, we can identify the Lie algebra $\mathfrak{sp}(1, B) = \mathfrak{im}(B)$.

Consider the action of the group $\text{Sp}(n, B) \times \text{Sp}(1, B)$ on $B^n$, given by

$$(A, \xi) \cdot q \mapsto A q \xi.$$  

It is easily seen that the action of $\text{Sp}(n, B)$ is isometric and the induced action on the pseudo-sphere of hypersymplectic structures of $B^n$ fixes the hyperboloids and the cone. On the other hand, the $\text{Sp}(1, B)$-action is also isometric, but the induced action on the pseudo-sphere of hypersymplectic structures is nothing but the standard action of $\text{SO}^+(1, 2)$. To see this, let $\mathbb{I} : B \rightarrow \text{End}(B^n)$ denote the algebra homomorphism, given by

$$\mathbb{I}_h := \mathbb{I}(h) = h_0 \text{id} + h_1 I + h_2 S + h_3 T.$$  

For any $\xi \in \text{Sp}(1, B)$,

$$\xi \cdot (I(q)) = \xi \cdot (q \mathbb{I}) = q \mathbb{I} \xi = q \mathbb{I} (\xi \mathbb{I} \xi) = \mathbb{I}_{\text{Ad}_{\xi}^{-1}} I \cdot (\xi \cdot q).$$

Similar argument holds for $S$ and $T$.

### 4. Hypersymplectic Manifolds

Let $(M, g_M, I, S, T)$ be a $4n$-dimensional pseudo-Riemannian manifold, endowed with a triple of endomorphisms $I, S, T$, satisfying the split quaternionic relations (3.1) and a metric of natural signature $(2n, 2n)$, that is compatible with the split quaternionic structure

$$g_M(IX, IY) = g_M(X, Y), \quad g_M(SX, SY) = -g_M(X, Y) = g_M(TX, TY).$$

The split quaternionic structure allows us to define the following 2-forms on $M$

$$\omega_1(X, Y) := g_M(IX, Y), \quad \omega_2(X, Y) := g_M(SX, Y), \quad \omega_3(X, Y) := g_M(TX, Y).$$

If each of the above 2-forms are closed, the manifold $M$ is said to be a \textit{hypersymplectic manifold}. Using Hitchin’s arguments for the hyperKähler manifolds, one can show that the structures $I, S, T$ are integrable; i.e., they are parallel with respect to the Levi-Civita connection. This implies that the holonomy group of $M$ reduces to $\text{Sp}(n, \mathbb{B})$.

The endomorphisms $S$ and $T$ are called \textit{product structures}. This is because the integrability of these structures implies that the manifold $M$ locally looks like a product $M^+ \times M^-$, where $\pm$ denotes the eigenvalues $\pm 1$ of $S$ on $T$ and $TM^\pm$ denotes the corresponding eigenspaces. In fact, every element of the 1-sheeted hyperboloid $x_1^2 - x_2^2 - x_3^2 = 1$ determines a product structure as

$$(x_1, x_2, x_3) \mapsto x_1 I + x_2 S + x_3 T.$$  

Such structures are also known by \textit{paracomplex structures} in literature. On the other hand, $M$ also has an entire family of pseudo-Kähler structures, which are parametrized by the 2-sheeted hyperboloid $y_1^2 - y_2^2 - y_3^2 = -1$ as

$$(y_1, y_2, y_3) \mapsto y_1 I + y_2 S + y_3 T.$$  

Another (unified) way of thinking of these structures is as follows: let $\mathcal{I} \subset \text{End}(TM)$ denote the trivial 3-dimensional sub-bundle spanned by $(I, S, T)$. Then, the hypersymplectic structures can be
thought of as being parametrised by the pseudo-sphere in $I$. The said hyperboloids then correspond to the subsets of spacelike and timelike unit vectors, respectively.

As a matter of convenience, we can think of the endomorphisms $I, S, T$ as a single, covariantly constant endomorphism with values in $\mathfrak{sp}(1, \mathbb{B})^*$, defined by

$$I_h := h_1 I + h_2 S + h_3 T, \quad h \in \mathfrak{sp}(1, \mathbb{B}) = \mathfrak{Im}(\mathbb{B})$$

and similarly, the associated symplectic 2-forms, as a single $(\text{properly elliptic})$ surfaces. Andrada and Salamon, in $I^\text{sp}$ constant endomorphism with values in $I^{1}$-sheeted hyperboloid to a 2-sheeted hyperboloid in $I^\text{sp}$.

$\omega$ is globally defined on $M$ of the group of isometries that preserves each Kähler 2-form $I, S, T$. As a matter of convenience, we can think of the endomorphisms $\omega$ as a single, covariantly constant endomorphism with values in $\mathfrak{sp}(1, \mathbb{B})^*$, defined by

$$\langle \omega, h \rangle := g_M(I_h, \cdot) = h_1 \omega_1 i + h_2 \omega_2 s + h_3 \omega_3 t.$$

Denote by $\mathcal{G} \subset \Lambda^2(M)$ the trivial 3-dimensional sub-space spanned by $\omega_1, \omega_2, \omega_3$. Note that since $I_h^2 = -\|h\|^2$, for any $h$ in the null-cone inside $\mathfrak{Im}(\mathbb{B})$, the endomorphism $I_h$ determines an element of the null-cone in $I$. Similarly, $\omega_h$ will be an element of the null-cone in $\mathcal{G}$. However, $I$ maps the 1-sheeted hyperboloid to a 2-sheeted hyperboloid in $I$ and vice versa.

In some cases, it is possible to explicitly construct a family of examples of hypersymplectic manifolds. Ivanov and Zankovoy [11] constructed a hypersymplectic structure on Kodaira-Thurston (properly elliptic) surfaces. Andrada and Salamon, in [12], show that if there exists a complex product structure on a real Lie algebra $g$; i.e., a pair $I, S$ of complex structure and a product structure, then, it induces a hypersymplectic structure on the complexification $g^\mathbb{C}$. On the other hand, in [10], Ivanov and Tsanov showed that the manifolds underlying the Lie groups $\text{SL}(2m - 1, R)$ and $\text{SU}(m, m - 1)$ carry a complex product structure and therefore, induce hypersymplectic structures on their complexifications.

4.1. Permuting actions. As in the hyperKähler case, consider the fundamental 4-form

$$\Omega = \omega_1 \wedge \omega_1 - \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3.$$  

The form is globally defined on $M$. The stabilizer group $\text{St}_\Omega \in \text{Isom}(M, g_M)$ of $\Omega$ is a sub-group of the group of isometries that preserves each Kähler 2-form $\omega_i$. The induced action of $\text{St}_\Omega$, on the pseudo-sphere of hypersymplectic structures, determines the homomorphism $\text{St}_\Omega \to \text{Sp}(1, \mathbb{B})/ \pm 1 \cong \text{SO}^+(1, 2)$. The kernel of this homomorphism is the group of hypersymplectic isometries, whose induced action on the pseudo-sphere of hypersymplectic structure is trivial; i.e., it fixes the pseudo-sphere.

**Definition 1.** An isometric action of the group $\text{Sp}(1, \mathbb{B})$ is said to be permuting, if the induced action on the pseudo-sphere of hypersymplectic structures is the standard action of $\text{SO}^+(1, 2)$ on the pseudo-sphere in the 3-dimensional Minkowski space $\mathbb{M}^3$.

Henceforth, without loss of generality, we will assume that $M$ admits a free, permuting, effective action of the group $\text{Sp}(1, \mathbb{B})$. The arguments that follow are an adaptation of representation theoretic arguments of [9, 13].

Let $K^M_\xi$ denote the fundamental vector field on $M$ corresponding to $\xi \in \mathfrak{sp}(1, \mathbb{B})$. Define the following operators:

$$\iota : \otimes^p \mathfrak{sp}(1, \mathbb{B})^* \otimes \Omega^q(M) \longrightarrow \mathfrak{sp}(1, \mathbb{B})^* \otimes (\mathfrak{sp}(1, \mathbb{B})^*)^\otimes \otimes \Omega^{q-1}(M), \quad \langle \iota, \alpha \rangle (\xi) = \iota_{K^M_\xi} \alpha$$

and

$$\mathcal{L}_{\mathfrak{sp}(1, \mathbb{B})} : \otimes^p \mathfrak{sp}(1, \mathbb{B})^* \otimes \Omega^q(M) \longrightarrow \mathfrak{sp}(1, \mathbb{B})^* \otimes (\mathfrak{sp}(1, \mathbb{B})^*)^\otimes \otimes \Omega^q(M), \quad \langle \mathcal{L}_{\mathfrak{sp}(1, \mathbb{B})} \alpha, \xi \rangle = \mathcal{L}_{K^M_\xi} \alpha.$$
Then Cartan’s formula $\mathcal{L}_{sp(1,\mathbb{B})} = d \iota_{sp(1,\mathbb{B})} \omega + \iota_{sp(1,\mathbb{B})} d$ is easily verified.

**Lemma 4.1** ([9, 13]). For the $sp(1,\mathbb{B})^*$-valued 2-form $\omega$ we have

$$\mathcal{L}_{sp(1,\mathbb{B})} \omega = 2 \omega. \quad (4.1)$$

**Proof.** We first verify that $\omega$ is $Sp(1,\mathbb{B})$-equivariant. Let $q \in Sp(1,\mathbb{B})$ and $\xi \in Sp(1,\mathbb{B})$. Then for the vector fields $V$, $W$ on $M$

$$\langle q^* \omega, \xi \rangle(V, W) = g_M(\text{Ad}_{\xi} q_s V, q_s W) = g_M(q_s^{-1} \ii_{\xi} q_s V, W) = \langle \omega, \text{Ad}_{q^{-1}}(\xi) (V, W) \rangle.$$

Consider $\xi, \xi' \in sp(1,\mathbb{B})$ such that $\xi \xi' \neq 0 \neq \xi' \xi$. Then, using the identity above, we get:

$$\langle L_{sp(1,\mathbb{B})} \omega, \xi \otimes \xi' \rangle = \mathcal{L}_{\xi'}^M \omega \xi' = \frac{d}{dt} \langle L_{\exp(-t \xi)} \omega \xi' \rangle \bigg|_{t=0} = \frac{d}{dt} \omega_{\exp(t \xi) \xi'} \bigg|_{t=0} = \langle \omega, [\xi, \xi'] \rangle.$$

Using the isomorphism $[\cdot, \cdot]: \Lambda^2 sp(1,\mathbb{B}) \to sp(1,\mathbb{B})$ given by

$$i \wedge s \mapsto [i, s] = 2t, \quad s \wedge t \mapsto [s, t] = 2i, \quad t \wedge i \mapsto [t, i] = 2s,$$

we conclude that $L_{sp(1,\mathbb{B})} \omega = 2 \omega$. \hfill \square

Define the 1-form $\gamma = \frac{1}{2} \iota_{sp(1,\mathbb{B})} \omega \in sp(1,\mathbb{B})^* \otimes sp(1,\mathbb{B})^* \otimes \Omega^1(M)$. More precisely

$$\langle \gamma, \xi \otimes \xi' \rangle = \frac{1}{2} g_M(\ii_{\xi} K^M_{\xi'}, \cdot).$$

The tensor product $sp(1,\mathbb{B})^* \otimes sp(1,\mathbb{B})^*$ splits into a direct sum of sub-representations $S^2 (sp(1,\mathbb{B})^*) \oplus \Lambda^2 (sp(1,\mathbb{B})^*)$. The symmetric part further decomposes into a direct sum of the trace and the traceless component. Consequently,

$$sp(1,\mathbb{B})^* \otimes sp(1,\mathbb{B})^* = \mathbb{R} \oplus \Lambda^2 (sp(1,\mathbb{B})^*) \oplus S_0 (sp(1,\mathbb{B})^*).$$

Correspondingly, the 1-form $\gamma$ decomposes into three components

$$\iota_{sp(1,\mathbb{B})} \omega = (\gamma_0, \gamma_1, \gamma_2). \quad (4.2)$$

From Lemma 4.1, it follows that $d \gamma = 2 \omega$, since $d\omega = 0$. However, note that the right hand side belongs to the $sp(1,\mathbb{B})^* \cong \Lambda^2 (sp(1,\mathbb{B})^*)$. This implies that $d \gamma_0 = 0 = d \gamma_2$ and $d \gamma_1 = 2 \omega$.

**Proposition 4.2.** Let $M$ be a hypersymplectic manifold with a permuting action of the group $Sp(1,\mathbb{B})$. Let $G \subset \Lambda^2(M)$ denote the trivial sub-bundle spanned by $\omega_1, \omega_2, \omega_3$. Then, the de Rham cohomology class of any symplectic 2-form in $G$ vanishes. In particular, $M$ can never be compact.

**Lemma 4.3** ([13]). The map $\gamma_1$ satisfies the following identity

$$\mathcal{L}_{sp(1,\mathbb{B})} \gamma_1 = 2 \gamma_1. \quad (4.3)$$

**Proof.** The proof is identical to that of Lemma 4.1. \hfill \square

Following the approach in [13, 14] for the hyperKähler case, we now show that $\gamma_0$ and $\gamma_2$ are exact. Define $\rho := \iota_{sp(1,\mathbb{B})} \gamma_1 \in \Omega^0(M, sp(1,\mathbb{B})^* \otimes sp(1,\mathbb{B})^*)$. Corresponding to the decomposition $(4.2)$, the map $\rho$ has 3 components:

$$\rho = (\rho_0, \rho_1, \rho_2).$$
Denote by $\text{Alt}$, the projection of $\text{sp}(1, \mathbb{B})^* \otimes \text{sp}(1, \mathbb{B})^*$ to the alternating part $\Lambda^2 \text{sp}(1, \mathbb{B})^*$ and by $\text{Sym}_0$, the projection of $\text{sp}(1, \mathbb{B})^* \otimes \text{sp}(1, \mathbb{B})^*$ to the traceless, symmetric part $S^2 (\text{sp}(1, \mathbb{B})^*)$. Then, the identity (4.3) can be written as
\[
\mathcal{L}_{\text{sp}(1, \mathbb{B})} \gamma_1 = \text{Alt} \left( t_{\text{sp}(1, \mathbb{B})} \omega \right).
\]

Therefore we can write
\[
d t_{\text{sp}(1, \mathbb{B})} \gamma_1 = \mathcal{L}_{\text{sp}(1, \mathbb{B})} \gamma_1 - t_{\text{sp}(1, \mathbb{B})} d \gamma_1 = \text{Alt} \left( t_{\text{sp}(1, \mathbb{B})} \omega \right) - t_{\text{sp}(1, \mathbb{B})} \omega.
\]  

It follows that
\[
d \rho_0 = d \left( \frac{1}{3} \text{tr} \left( t_{\text{sp}(1, \mathbb{B})} \gamma_1 \right) \right) = - \frac{1}{3} \text{tr} \left( t_{\text{sp}(1, \mathbb{B})} \omega \right) = \gamma_0
\]
and
\[
d \rho_2 = d \left( \text{Sym}_0 \left( t_{\text{sp}(1, \mathbb{B})} \gamma_1 \right) \right) = - \text{Sym}_0 \left( t_{\text{sp}(1, \mathbb{B})} \omega \right) = \gamma_2.
\]
In particular, $\gamma_0$ and $\gamma_2$ are exact.

4.1.1. Potentials. Define $\kappa(\xi) := - t_{K_{\xi}^M} \gamma_1(\xi)$ for any $\xi \in \text{sp}(1, \mathbb{B})$ such that $\|\xi\| \neq 0$.

**Case 1:** Suppose that $\xi \in \text{sp}(1, \mathbb{B})$ is such that $\xi^2 = 1$. Then the stabilizer of $I_{\xi}$ is a sub-group $\text{SO}^+(1, 1) \subset \text{Sp}(1, \mathbb{B})$, consisting of $2 \times 2$ real matrices of the form $g = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ such that $a^2 - b^2 = 1$. Its Lie algebra is the vector space of real numbers $(\mathbb{R}, +)$. The group $\text{SO}^+(1, 1)$ preserves the symplectic 2-form $\omega_{\xi}$. Moreover, the associated moment map is given by $\kappa(\xi)$. Indeed, this can be seen as follows:
\[
d \kappa(\xi) = - (d t_{\text{sp}(1, \mathbb{B})} \gamma_1)(\xi, \xi) = (t_{\text{sp}(1, \mathbb{B})} \omega)(\xi, \xi) = t_{K_{\xi}^M} \omega_{\xi}.
\]
Here we have used the identity (4.4).

**Case 2:** Suppose that $\xi \in \text{sp}(1, \mathbb{B})$ is such that $\xi^2 = -1$. Then $I_{\xi}$ defines a complex structure. Let $U(1) \in \text{Sp}(1, \mathbb{B})$ denote the stabilizer of $I_{\xi}$. Then, by the same argument as above, $\kappa(\xi)$ defines the moment map with respect to the $U(1)$-action, for the Kähler 2-form $\omega_{\xi}$.

**Proposition 4.4.** Let $M$ be a hypersymplectic manifold with a permuting $\text{Sp}(1, \mathbb{B})$-action. Let $\xi, \xi' \in \text{sp}(1, \mathbb{B})$ with $\xi \xi' \neq 0 \neq \xi' \xi$. Then the following holds

1. If their squares are $-1$ and they are perpendicular, then, $-\kappa(\xi')$ is the pseudo-Kähler potential for the pseudo-Kähler 2-form $\omega_{\xi}$.
2. If their squares are $-1$ and $1$ respectively, then, $\kappa(\xi')$ is the pseudo-Kähler potential for the pseudo-Kähler 2-form $\omega_{\xi}$.
3. If their squares are $1$ and $-1$ respectively OR both the squares are $1$ and if $\xi$ and $\xi'$ are perpendicular, then, $\kappa(\xi')$ is a “para Kähler potential” for the symplectic 2-form $\omega_{\xi}$.

**Proof.** (1) The proof is the following straight-forward computation
\[
-\frac{1}{2} (d I_{\xi}^* d) (-\kappa(\xi')) = \frac{1}{2} d I_{\xi}^* g_M (I_{\xi'} K_{\xi}^M, \cdot) = \frac{1}{2} d g_M (I_{\xi'} K_{\xi}^M, I_{\xi}(\cdot)) = \frac{1}{2} d g_M (I_{\xi} I_{\xi'} K_{\xi}^M, \cdot) = - \frac{1}{4} d t_{K_{\xi}^M} \omega ([\xi, \xi']) = - \frac{1}{4} L_{K_{\xi}^M} \omega ([\xi, \xi']) = \frac{1}{4} \omega ([\xi', [\xi, \xi']]) = \omega (\xi).
\]
The last equality follows from the fact that $[\xi', [\xi, \xi']] = 4\xi$.

(2) Consider the following computation

$$-\frac{1}{2} (d I_\xi^* d) (\kappa(\xi')) = -\frac{1}{2} d I_\xi^* g_M(\Gamma \xi'^M, \cdot) = \frac{1}{2} d g_M(\Gamma \xi'^M, \Gamma \xi)$$

$$= \frac{1}{2} d g_M(\Gamma \xi I \xi e^M, \cdot) = \frac{1}{4} d I_{\xi e}^* \omega([\xi, \xi'])$$

$$= \frac{1}{4} \mathcal{L}_{\xi e}^* \omega([\xi, \xi']) = -\frac{1}{4} \omega([\xi', [\xi, \xi']]) = \omega(\xi).$$

The last equality follows from the fact that $[\xi', [\xi, \xi']] = -4\xi$. Thus $-\kappa(s)$ and $-\kappa(t)$ are Kähler potentials for $\omega_1$.

(3) By arguments identical to the ones above, we have

$$-\frac{1}{2} (d I_\xi^* d) (\kappa(\xi')) = \omega(\xi).$$

Therefore $\kappa(s)$, $\kappa(t)$ are para Kähler potentials for $\omega_3$ and $\omega_2$, respectively.

Define

$$\mathcal{X} \in \mathfrak{sp}(1, \mathbb{B})^* \otimes \mathfrak{sp}(1, \mathbb{B})^* \otimes \Gamma(M, TM), \quad \mathcal{X}(\xi, \xi') = I_\xi K_{e^M}. $$

Note that if $\|\xi\|^2 = 0$, then $g_M(\mathcal{X}(\xi, \xi'), \mathcal{X}(\xi, \xi')) = 0$. In other words, $\mathcal{X}(\xi, \xi')$ is a vector that lies in the null-cone of the trivial 3-dimensional vector sub-bundle spanned by the fundamental vector fields $\{K_{e^1}^M, K_{e^2}^M, K_{e^3}^M\}$, where $\{e_1, e_2, e_3\}$ is the basis of the Lie algebra $\mathfrak{sp}(1, \mathbb{B})$.

By Clebsch-Gordon decomposition, the map $\mathcal{X}$ splits into three parts: $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2$, given by

$$\mathcal{X}_0 = -\frac{1}{3} \text{tr} \mathcal{X} = -\frac{1}{3} (I K_{e^1}^M + S K_{e^2}^M + T K_{e^3}^M) \in \Gamma(M, TM)$$

$$\mathcal{X}_1([\xi, \xi']) = \frac{1}{2} (I_\xi K_{\xi^e}^M - I_{\xi^e} K_{\xi}^M) \in \mathfrak{sp}(1, \mathbb{B})^* \otimes \Gamma(M, TM)$$

$$\mathcal{X}_2(\xi, \xi') = -\mathcal{X}_0(\cdot, \cdot)\mathbb{B} - \frac{1}{2} (I_\xi K_{\xi^e}^M + I_{\xi^e} K_{\xi}^M) \in S^2_0(\mathfrak{sp}(1, \mathbb{B})^*) \otimes \Gamma(M, TM).$$

Clearly then, $\gamma_i = \frac{1}{g_M(\mathcal{X}_i, \cdot)}$ and therefore $\mathcal{X}_i$ are the gradient vector fields for $\rho_i$.

Lemma 4.5. Let $M$ be a hypersymplectic manifold with a free, permuting $\text{Sp}(1, \mathbb{B})$-action and assume $\mathcal{X}_2 = 0$ (equivalently, $\rho_2 = 0$). Then, $\rho_0$ is the hypersymplectic potential and we have

$$\mathcal{X}_0 = -I_\xi K_{\xi}^M \quad \text{for all} \quad \xi \in \text{Sp}(1, \mathbb{B}), \quad \text{such that} \quad \|\xi\|^2 \neq 0 \quad \text{and} \quad g_M(\mathcal{X}_0, \mathcal{X}_0) = \rho_0 > 0.$$

Moreover, for any $\xi$ with $\|\xi\|^2 = \pm 1$, the vector field $\mathcal{X}_0$ is independent of $\xi$.

Proof. Since $\mathcal{X}_2 = 0$, it implies that $\rho_2$ is constant on connected components. However, since $\rho_2$ is $\text{Sp}(1, \mathbb{B})$-equivariant, this implies that $\rho_2$ must be identically zero on each of the connected components. In particular, it is identically zero. Conversely, if $\rho_2 \equiv 0$, then it follows that $\mathcal{X}_2 = 0$, since $\mathcal{X}_2 = \text{grad}(\rho_2)$. Consequently,

$$\mathcal{X}_2(\xi, \xi) = -\mathcal{X}_0 - \frac{1}{2} (I_\xi K_{\xi}^M + I_{\xi^e} K_{\xi}^M) \implies \mathcal{X}_0 = -I_\xi K_{\xi}^M.$$ 

Clearly, if $\|\xi\| = \pm 1$, the $\mathcal{X}_0$ is independent of $\xi$. Since $\mathcal{X}_0$ is the gradient vector field of $\rho_0$, $g_M(\mathcal{X}_0, \mathcal{X}_0) = \rho_0$. From Proposition 4.4, it follows that $\rho_0 = \kappa(i) = -\kappa(s) = -\kappa(t)$. Thus $\rho_0$ is
the hypersymplectic potential. In particular, \( d\xi d\rho_0 = \epsilon \omega_\xi \), where \( \epsilon = \pm 1 \), according to whether \( \|\xi\| = \mp 1 \).

\[ \square \]

Note. Let \( \{\xi_1, \xi_2, \xi_3\} \) be the basis of \( \mathfrak{sp}(1,\mathbb{B}) \). If \( \mathcal{X}_2 = 0 \), then, the above Lemma says that

\[ \rho_0 = g_M(K_{\xi_1}^M, K_{\xi_1}^M) = -g_M(K_{\xi_2}^M, K_{\xi_2}^M) = -g_M(K_{\xi_3}^M, K_{\xi_3}^M). \]

It is important to note here that since the vector fields \( K_{\xi_i}^M \) generate the free action of \( \text{Sp}(1,\mathbb{B}) \) on \( M \), the norm \( g_M(K_{\xi_1}^M, K_{\xi_1}^M) \) must be positive, while, the norms \( g_M(K_{\xi_2}^M, K_{\xi_2}^M) \) and \( g_M(K_{\xi_3}^M, K_{\xi_3}^M) \) must be negative. Therefore we must have \( \rho_0 = g_M(\mathcal{X}_0, \mathcal{X}_0) > 0 \).

The existence of a hypersymplectic potential on \( M \) implies that the metric \( g_M \) is incomplete. The remainder of the section is dedicated to proving this and a few other consequences of the vanishing of the map \( \rho_2 \).

**Proposition 4.6.** Let \( M \) be a hypersymplectic manifold with a free, permuting \( \text{Sp}(1,\mathbb{B}) \)-action and assume that \( \mathcal{X}_2 = 0 \). Then the following holds

1. \( \gamma_1 = \iota_{\mathcal{X}_0} \omega \)
2. \( \mathcal{L}_{\mathcal{X}_0} \gamma_1 = 2 \gamma_1 \)
3. \( \mathcal{L}_{\mathcal{X}_0} \omega = 2 \omega \)
4. \( \mathcal{L}_{\mathcal{X}_0} \rho_0 = 2 \rho_0 \)
5. \( \mathcal{L}_{\mathcal{sp}(1,\mathbb{B})} \mathcal{X}_0 = 0 \)

**Proof.** First, we make the following observation. Owing to Lemma 4.5, we have

\[ I\mathcal{X}_0 = K_{\xi_1}^M, \quad S\mathcal{X}_0 = -K_{\xi_2}^M, \quad T\mathcal{X}_0 = -K_{\xi_3}^M. \]

(1) Recall that \( \gamma_1 = \text{Alt} (\iota_{\mathcal{sp}(1,\mathbb{B})} \omega) \). Therefore,

\[ \langle \gamma_1, \iota \rangle = \frac{1}{2} \langle \text{Alt} (\iota_{\mathcal{sp}(1,\mathbb{B})} \omega), \iota \rangle = -\frac{1}{2} \langle \iota_{\mathcal{sp}(1,\mathbb{B})} \omega, s \otimes t \rangle = -\iota_{K_{\xi_2}^M} \omega_3 = \iota_{\mathcal{X}_0} \omega_1. \]

The last equality can be seen as follows

\[ \iota_{K_{\xi_2}^M} \omega_3 = g_M(T K_{\xi_2}^M, \cdot) = g_M(IS K_{\xi_2}^M, \cdot) = -g_M(I\mathcal{X}_0, \cdot) = -\iota_{\mathcal{X}_0} \omega_1. \]

Similarly, one can show that

\[ \langle \gamma_1, s \rangle = \iota_{\mathcal{X}_0} \omega_2, \quad \langle \gamma_1, t \rangle = \iota_{\mathcal{X}_0} \omega_3. \]

Therefore, we have \( \gamma_1 = \iota_{\mathcal{X}_0} \omega \).

(2) The second claim follows directly from the first one by observing that

\[ \mathcal{L}_{\mathcal{X}_0} (\iota_{\mathcal{X}_0} \omega) = \iota_{\mathcal{X}_0} \mathcal{L}_{\mathcal{X}_0} \omega = 2 \iota_{\mathcal{X}_0} \omega = 2 \gamma_1. \]

(3) Consider the argument above Lemma 4.1. We have shown that \( d\gamma_1 = 2 \omega \). But from the claim (1), it follows that \( d\gamma_1 = \mathcal{L}_{\mathcal{X}_0} \omega \). In conclusion, \( \mathcal{L}_{\mathcal{X}_0} \omega = 2 \omega \).

(4) Observe that

\[ \mathcal{L}_{\mathcal{X}_0} \rho_0 = \iota_{\mathcal{X}_0} d \rho_0 = d \rho_0 (\mathcal{X}_0) = 2 \rho_0. \]

(5) This follows from the invariance of \( \rho_0 \) under the action of \( \text{Sp}(1,\mathbb{B}) \).

\[ \square \]
**Proposition 4.7.** The gradient vector field of the hypersymplectic potential $\rho_0$ satisfies

$$\nabla X_0 = \text{id}_{TM}$$

where $\nabla$ is the Levi-Civita connection of the metric $g_M$.

To show this, we use the following result by Swann

**Proposition 4.8 ([7]).** Let $(N, I, g_N)$ be a Kähler manifold and $\nabla$ be the associated Chern connection. A smooth function $\rho_0 : N \to \mathbb{R}^+$ is a Kähler potential if and only if

$$\frac{1}{2}(\nabla^2 X_0 Y + \nabla^2 I X_0 I Y) = g_N(X, Y). \quad (4.5)$$

The above statement of the theorem holds even if the metric $g_N$ is a pseudo-Kähler metric.

**Proof of Lemma 4.7.** If we regard $M$ as a pseudo-Kähler manifold with respect to the complex structure $I$, then we have shown that $\rho_0$ is a Kähler potential for $I$. Then, from (4.5) it follows that

$$\nabla(d\rho_0)(X, Y) + \nabla(d\rho_0)(IX, IY) = g_M(X, Y)$$

Now consider

$$g_M(\nabla X_0 Y) = \nabla X(g_M(X_0, Y)) - g_M(X_0, \nabla X Y)
= \nabla X(d\rho_0(Y)) - d\rho_0(\nabla X Y)
= \nabla X(d\rho_0)(Y).$$

Therefore,

$$\nabla X(d\rho_0)(Y) + \nabla IX(d\rho_0)(IY) = g_M(\nabla X X_0 Y) + g_M(\nabla IX X_0 Y)
= g_M(\nabla X X_0 Y) - g_M(IX_0 Y)
= g_M(\nabla X X_0 - IX_0 Y).$$

In Proposition 4.6, we have shown that $\mathcal{L}_{X_0} \omega_1 = 2 \omega_1$ and therefore $\mathcal{L}_{X_0} I = 0$. Using this, we have

$$\nabla IX X_0 = \nabla X_0 IX - \mathcal{L}_{X_0} IX = I\nabla X_0 X - I\mathcal{L}_{X_0} X = I\nabla X X_0.$$  

Plugging this in the previous equation, we get

$$\nabla X(d\rho_0)(Y) + \nabla IX(d\rho_0)(IY) = 2 g_M(\nabla X X_0, Y).$$

From Proposition 4.8, it follows that $g_M(\nabla X X_0, Y) = g_M(X, Y)$, for all $X, Y \in TM$. Therefore, $\nabla X_0 = \text{id}_{TM}$.

**Corollary 4.9.** The hypersymplectic potential $\rho_0$ satisfies

$$\nabla^2 \rho_0 = g_M.$$

There are two consequences of the above corollary. First, the metric on $M$ cannot be complete (see [15]). Second, the metrics from difference level-sets of $\rho_0$ are homothetic.

In the section that follows, we will show that the level-sets of the hypersymplectic potential carry a split 3-Sasakian structure.
5. Split 3-Sasakian geometry

We begin this section by introducing $\varepsilon$-Sasakian manifolds. These are the pseudo-Riemannian analogues of Sasakian manifolds with either a complex or a product structure on the leaves of the 1-dimensional foliation. If the leaves of the foliation are endowed with a complex structure ($\varepsilon = -1$), we call it a pseudo-$\varepsilon$-Sasakian structure. If they are endowed with a product structure ($\varepsilon = 1$), then we call the manifold as para Sasakian manifold. This is slightly different than the conventional terminologies in the literature. However, it allows for a simultaneous treatment of both the cases.

Let $\mathcal{S}$ be a smooth manifold, equipped with a $(1,1)$-tensor $\Phi$, a nowhere vanishing vector field $\xi$ and a 1-form $\eta$ metric dual to $\xi$, satisfying the following relations

$$\Phi \circ \Phi = \varepsilon \text{id} - \varepsilon \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(\Phi(\cdot)) = 0 = \Phi(\xi). \quad (5.1)$$

If $\varepsilon = -1$, then $(\mathcal{S}, \Phi, \eta, \xi)$ is called an almost-contact manifold and if $\varepsilon = 1$, it is called an almost-para contact manifold. To give a simultaneous treatment, we will refer to $(\mathcal{S}, \Phi, \eta, \xi)$ as an $\varepsilon$-almost contact structure. Consider the Nijenhuis tensor of $\Phi$, which is a $(2,1)$-tensor, defined as

$$N_{\Phi}(X,Y) = [\Phi(X), \Phi(Y)] + \Phi^2[X,Y] - \Phi([X, \Phi(Y)]) - \Phi([\Phi(X), Y]).$$

The $(\mathcal{S}, \Phi, \eta, \xi)$ is said to be normal, if $N_{\Phi} = d\eta \otimes \xi$. Suppose now that $\mathcal{S}$ is endowed with a pseudo-Riemannian metric $g_{\xi}$, such that

$$g_{\xi}(\Phi(X), \Phi(Y)) = \varepsilon (-g_{\xi}(X,Y) + \tau \eta(X) \eta(Y)), \quad \text{and} \quad d\eta(X,Y) = \tau g_{\xi}(\Phi(X), Y),$$

where $\tau = g_{\xi}(\xi, \xi)$, then, the $\varepsilon$-almost contact structure is said to be an $\varepsilon$-para contact metric structure and the metric $g_{\xi}$ is said to be compatible with the $\varepsilon$-para contact structure. Additionally, if the structure is normal, the manifold $(\mathcal{S}, g_{\xi}, \Phi, \eta, \xi)$ is said to be a $\varepsilon$-Sasakian manifold. In other words, an $\varepsilon$-Sasakian manifold is an $\varepsilon$-almost contact manifold, endowed with a compatible pseudo-Riemannian metric and the structure is normal.

**Proposition 5.1.** [16] An $\varepsilon$-contact manifold $(\mathcal{S}, g_{\xi}, \Phi, \eta, \xi)$ is $\varepsilon$-Sasakian if and only if the following is satisfied

$$\nabla_X \Phi(Y) = g_{\xi}(\Phi(X), \Phi(Y)) \xi + \tau \eta(Y) \cdot \Phi \circ \Phi(X). \quad (5.2)$$

where $\nabla$ is the Levi-Civita connection of $g_{\xi}$.

5.1. Split 3-Sasakian manifolds. Suppose that $\mathcal{S}$ is a pseudo-Riemannian manifold of dimension $4n + 3$, carrying a metric $g_{\xi}$ of signature $(2n + 1, 2n - 2)$. Additionally, suppose that $\mathcal{S}$ also carries a triple of orthogonal Killing vector fields $(X_1, X_2, X_3)$, of lengths $1, -1, -1$ respectively, satisfying

$$\frac{1}{2}[X_1, X_2] = X_3, \quad \frac{1}{2}[X_2, X_3] = -X_1, \quad \frac{1}{2}[X_3, X_1] = X_2. \quad (5.3)$$

Define the 1-forms $\eta_i(Y) = \tau_i g_{\xi}(X_i, Y)$, where $\tau_i = g_{\xi}(X_i, X_i)$. Then, $\eta_i(X_j) = \delta_{ij}$. Suppose that $\mathcal{S}$ is also endowed with endomorphisms $(\Phi_1, \Phi_2, \Phi_3)$, satisfying

$$g_{\xi}(\Phi_i(X), \Phi_i(Y)) = \varepsilon_i (-g_{\xi}(X,Y) + \tau_i \eta_i(X) \eta_i(Y)), \quad (5.4)$$

where, $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, -1)$. Then, the structure $(\mathcal{S}, g_{\xi}, \{\Phi_i, \eta_i, \xi_i\}_{i=1}^3)$ is said to be an almost split $3$-contact structure. Additionally, if

$$d\eta_i = \tau_i g_{\xi}(\Phi_i(\cdot), \cdot),$$

then the structure is said to be a metric split $3$-contact structure.
Definition 2. A metric split 3-contact manifold $(S, g_S, \{\Phi_i, \eta_i, X_i\}_{i=1}^3)$ is a split 3-Sasakian manifold if the structures are normal; i.e.,

$$N_{\Phi_i} = d\eta_i \otimes X_i, \quad i = 1, 2, 3.$$ 

Equivalently, from Proposition 5.1, we see that $(S, g_S, \{\Phi_i, \eta_i, X_i\}_{i=1}^3)$ is a split 3-Sasakian manifold if

$$\nabla_X \Phi_i(Y) = g_S(\Phi_i(X), \Phi_i(Y)) X_i + \tau_i \eta_i(Y) \cdot \Phi_i \circ \Phi_i(X), \quad X, Y \in TS$$

(5.5)

Note that $(S, g_S, \Phi, \eta, \xi)$ is a pseudo-Sasakian manifold, whereas, for $i = 2, 3$, $(S, g_S, \Phi_i, \eta_i, X_i)$ is a para-Sasakian manifold.

Theorem 5.2 ([8], Thm. 4.1, 5.7 and Prop. 4.2). Let $(S, g_S, \{\Phi_i, \eta_i, \xi_i\}_{i=1}^3)$ be a split 3-contact manifold of dimension $4n + 3$. Then, $S$ is necessarily a split 3-Sasakian manifold. The metric $g_S$ is Einstein and has a constant scalar curvature equal to $(4n + 2)(4n + 3)$.

Remark 1. The split 3-Sasakian structure we discuss below is of the type $(1, -1, -1)$; i.e, the norms of the Reeb vector fields generating the split 3-Sasakian structures are $(1, -1, -1)$. The authors in [8] refers to this as the “negative mixed 3-Sasakian structure”.

The simplest example of a split 3-Sasakian manifold is the positive pseudo-sphere in the split quaternionic module $\mathbb{B}^{n+1}$

$$S_+ = \{ q \in \mathbb{B}^{n+1} \mid \|q\|^2 = 1 \}.$$ 

The manifold $S_+$ carries a pseudo-Riemannian metric of signature $(2n - 1, 2n + 1)$. Moreover, there is an isometric and transitive action of $\text{Sp}(1, \mathbb{B})$ on $S_+$. The Killing vector-fields corresponding to the basis of the 3-dimensional Lie algebra $\mathfrak{sp}(1, \mathbb{B})$ and the restricted metric on $S_+$ determine a split 3-Sasakian structure on $S_+$.

5.2. Level-set of the hypersymplectic potential. Suppose now that $M$ is a hypersymplectic manifold with a free action of $\text{Sp}(1, \mathbb{B})$, such that the obstruction $\mathcal{X}_2$ vanishes. Then we have a canonically defined hypersymplectic potential on $M$. Consider the level-set $S := \rho_0^{-1}(\frac{1}{2})$. Then, $S$ is $\text{Sp}(1, \mathbb{B})$-invariant and the Killing vector fields

$$K^M_{\xi_1} = I \mathcal{X}_0, \quad K^M_{\xi_2} = -S \mathcal{X}_0, \quad K^M_{\xi_3} = -T \mathcal{X}_0$$

can be thought of as vector fields on $S$. We denote their restriction to $S$ by $K^S_{\xi_i}$. Let $g_S := t^* g_M$ denote the restriction of the hypersymplectic metric to the hypersurface $S$. Define

$$\hat{\eta}_i(Y) = \tau_i g_S(K^S_{\xi_i}, Y), \quad Y \in TS,$$ 

where $\tau_i = g_S(K^S_{\xi_i}, K^S_{\xi_i})$.

Note that

$$\hat{\eta}_1 = \tau_1 t^* \langle \gamma_1, i \rangle, \quad \hat{\eta}_2 = \tau_2 t^* \langle \gamma_1, s \rangle, \quad \hat{\eta}_3 = \tau_3 t^* \langle \gamma_1, t \rangle$$ 

and therefore, $d\hat{\eta}_i = \tau_i t^* \omega_i$. (5.6)

Define the 1-forms

$$\hat{\Phi}_1(Y) = IY + \hat{\eta}_1(Y) \mathcal{X}_0, \quad \hat{\Phi}_2(Y) = SY + \hat{\eta}_2(Y) \mathcal{X}_0, \quad \hat{\Phi}_3(Y) = TY + \hat{\eta}_3(Y) \mathcal{X}_0.$$ 

Theorem 5.3. The manifold $(S, g_S, \{K^S_{\xi_i}, \hat{\eta}_i, \hat{\Phi}_i\}_{i=1}^3)$ is a split 3-Sasakian manifold.
Proof. Owing to Theorem 5.2, we need only show that \((S, g_\delta, \{K^{S}_{\xi_i}, \tilde{\eta}_i, \tilde{\Phi}_1\})_{i=1}^{3}\) is a metric split 3-contact manifold. For that, it is enough to show that \((g_\delta, K^{S}_{\xi_2}, \tilde{\eta}_2, \tilde{\Phi}_2)\) is a para contact metric structure. From their definitions, it is clear that

\[
\tilde{\eta}_2 (K^{S}_{\xi_2}) = 1, \quad \tilde{\Phi}_2 \circ \tilde{\Phi}_2 = \text{id}_{TS} - \tilde{\eta}_2 \otimes K^{M}_{\xi_2}, \quad \tilde{\Phi}_2 (K^{S}_{\xi_2}) = 0.
\]

Therefore we need only show that

\[
g_\delta(\tilde{\Phi}_2(X), \tilde{\Phi}_2(Y)) = -g_\delta(X, Y) + \varepsilon_2 \tilde{\eta}_2(X) \tilde{\eta}_2(Y), \quad d\tilde{\eta}_2(X, Y) = g_\delta(\tilde{\Phi}_2(X), Y) \quad X, Y \in TS.
\]

Let \(\eta_\iota(\cdot) = g_M(K^{M}_{\xi_i}, \cdot)\). Then \(\tilde{\eta}_\iota = \iota^* \eta_\iota\). Observe that for any \(X, Y \in TS\),

\[
g_\delta(\tilde{\Phi}_2(X), \tilde{\Phi}_2(Y)) = g_M(SX + \eta_\iota(X)A_0, SY + \eta_\iota(Y)A_0)
\]

\[
= -g_M(X, Y) + \eta_\iota(X)\eta_\iota(Y) g_M(A_0, A_0) + \eta_\iota(X) g_M(SY, A_0) + \eta_\iota(Y) g_M(SX, A_0)
\]

\[
= -g_M(X, Y) + \eta_\iota(X)\eta_\iota(Y) g_M(A_0, A_0) + \eta_\iota(X) g_M(Y, K^{M}_{\xi_2}) + \eta_\iota(Y) g_M(Y, K^{M}_{\xi_2})
\]

\[
= -g_M(X, Y) + 2\rho_0 \eta_\iota(X)\eta_\iota(Y) + \tau_2 \eta_\iota(X)\eta_\iota(Y) + \tau_2 \eta_\iota(X)\eta_\iota(Y)
\]

\[
= -g_M(X, Y) + (2 - 2\rho_0) \tau_2 \eta_\iota(X)\eta_\iota(Y) \quad \text{(since } \tau_2 = -1)\]

\[
= -g_\delta(X, Y) + \tau_2 \tilde{\eta}_2(X) \tilde{\eta}_2(Y)
\]

\[
= \varepsilon_2 (-g_\delta(X, Y) + \tau_2 \tilde{\eta}_2(X) \tilde{\eta}_2(Y)) \quad \text{(since } \varepsilon_2 = 1)\]

We now check the second condition

\[
g_\delta(\tilde{\Phi}_2(X), Y) = g_M(SX + \tilde{\eta}_2(X)A_0, Y)
\]

\[
= g_M(SX, Y) + \tilde{\eta}_2(X) g_M(A_0, Y)
\]

\[
= g_M(SX, Y) = \omega_2(X, Y) = \iota^* \omega_2(X, Y),
\]

for all \(X, Y \in TS\). It follows from eq. (4.1) and eq.(5.6) and the fact that external derivative commutes with the pull-back that \(d\tilde{\eta}_2 = \tau_2 \iota^* \omega_2\). This shows that \(d\eta_\iota(X, Y) = \tau_2 g_\delta(\tilde{\Phi}_2(X), Y)\). Thus \((g_\delta, \xi_2, \tilde{\eta}_2, \tilde{\Phi}_2)\) defines a para contact metric structure on \(S\).

Similar arguments show that \((g_\delta, \xi_1, \tilde{\eta}_1, \tilde{\Phi}_1)\) and \((g_\delta, \xi_3, \tilde{\eta}_3, \tilde{\Phi}_3)\) define a pseudo and a para contact metric structure respectively. Moreover, the vector fields \((K^{S}_{\xi_1}, K^{S}_{\xi_2}, K^{S}_{\xi_3})\) clearly satisfy the split quaternionic relations (3.1). The claim thus follows from the statement of Theorem 5.2. \(\square\)

Note that since the metrics on the level-sets \(S_{\epsilon} := \rho^{-1}_0(\epsilon)\) are homothetic, a split 3-Sasakian structure can be defined on every level-set.

Note. Analogous to the hyperKähler case, it can be easily seen that a metric cone over any split 3-Sasakian manifold is a hypersymplectic manifold, with a free, permuting action of \(\text{Sp}(1, \Bbb{B})\) and a hypersymplectic potential. In particular, the obstruction \(\rho_2\) vanishes. This can be considered as a characterizing property of such hypersymplectic manifolds. In other words, if \(\rho_2\) vanishes, then the hypersymplectic manifold in consideration, can be written as a metric cone over a split 3-Sasakian manifold, as constructed above.
5.3. Para quaternionic Kähler manifolds. A para-quaternionic Kähler manifold is a, pseudo-Riemannian manifold of dimension $4n$, whose holonomy is contained inside the group

$$\text{Sp}(n, \mathbb{B}) \cdot \text{Sp}(1, \mathbb{B}) = \text{Sp}(n, \mathbb{B}) \times \pm 1 \text{Sp}(1, \mathbb{B}).$$

Equivalently, we say that a manifold $N$ is almost para-quaternionic Kähler manifold if there exists a sub-bundle $I' \subset \text{End}(TN)$ which is locally spanned by a tripe $(I, S, T)$ satisfying the split quaternionic relations (3.1). For $n > 1$, the requirement that the holonomy of $N$ be contained inside $\text{Sp}(n, \mathbb{B}) \cdot \text{Sp}(1, \mathbb{B})$ is equivalent to asking the sub-bundle $I'$ being preserved by the Levi-Civita connection. If $n \geq 3$, then, this is equivalent to showing that the globally defined 4-form

$$\Omega = \omega_1 \wedge \omega_1 - \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3$$

is closed. For $n = 1$, we additionally require that the manifold be self-dual and Einstein.

**Theorem 5.4.** [16] Any para-quaternionic Kähler manifold $(N, g_N, \hat{\Omega})$ is Einstein, provided that the dimension of $N$ is greater than 4.

The representation theoretic argument by S. Salamon [17] can also be adapted to the pseudo-Riemannian setting to prove the above theorem.

A wide range of examples of para-quaternionic Kähler manifolds can be constructed by adapting LeBrun’s construction of quaternionic Kähler manifolds [18], to the pseudo-Riemannian case [6]. Starting with a real analytic manifold $F$ of dimension $2n + 1$, endowed with an indefinite metric, it is possible to construct a para-quaternionic Kähler manifold of dimension $4n$. Different manifolds which are conformal to $F$ give rise to distinct para-quaternionic Kähler manifolds. We can thus construct a wide variety of para-quaternionic Kähler manifolds of dimension greater than four.

Looking at Berger’s list [19, 6], one can also construct symmetric para-quaternionic Kähler manifolds of the type $G/H$, where $G$ is semi-simple. Symmetric para-quaternionic Kähler manifolds have been completely classified by D. Alekseevsky and V. Cortés [20].

In the hyperKähler case, the quotient of the 3-Sasakian manifold (a level-set of the hyperKähler potential) by the group $\text{Sp}(1)$ produces a quaternionic Kähler manifold of positive scalar curvature. This, however, cannot be directly carried over to the hypersymplectic situation as the group $\text{SU}(1, 1)$ is non-compact and therefore the quotient may not even be Hausdorff. However, if the $\text{Sp}(1, \mathbb{B})$-action is proper, then, we show that the quotient of the split 3-Sasakian manifold by $\text{Sp}(1, \mathbb{B})$ is a para-quaternionic Kähler manifold. Henceforth, we assume that the $\text{Sp}(1, \mathbb{B})$-action on $M$ proper.

Let $N = S/\text{Sp}(1, \mathbb{B})$ be the quotient of the split 3-Sasakian manifold and consider the diagram,

$$\begin{array}{ccc}
S & \xleftarrow{\iota} & M \\
\pi & & \\
N & \downarrow \pi & \\
\end{array}$$

where $\iota$ is the pseudo-Riemannian embedding and the map $\pi$ is the pseudo-Riemannian principal submersion. The normal bundle $N$ is the 1-dimensional vector bundle $\iota^*\text{span}\{\mathcal{A}_0\} \subset \iota^*TM$. The pull-back bundle $\iota^*TM$ splits into the direct sum $\iota^*TM = N \oplus TS$.

The pull-back metric on $S$ is of signature $(2n - 1, 2n)$. Further, $TS$ splits into a direct sum $V \oplus \mathcal{H}$, where

$$V = \text{Span} \{ K^M_{\xi_1}, K^M_{\xi_2}, K^M_{\xi_3} \} \quad \text{and} \quad \mathcal{H} = \cap_{i=1}^3 \ker \tilde{\eta}_i.$$
In conclusion, the pullback-bundle splits as

\[ \iota^*TM = \mathcal{N} \oplus \mathcal{V} \oplus \mathcal{H}. \]

We will call \( \mathcal{H} \) the “horizontal bundle” of \( TS \). Let \( \beta_i(\cdot, \cdot) = \hat{g} \phi_i(\cdot, \cdot) \) and \( \theta_i := \beta_i + \epsilon_{ijk} \eta_j \wedge \eta_k \). Define the 4-form

\[ \hat{\Omega} = \theta_1 \wedge \theta_1 - \theta_2 \wedge \theta_2 - \theta_3 \wedge \theta_3, \quad i = 1, 2, 3. \]

Then, \( \hat{\Omega} \) is an \( \text{Sp}(1, \mathbb{B}) \)-invariant, horizontal 4-form and therefore it descends to a 4-form \( \Omega_N \) on \( N \) with \( \pi^* \Omega_N = \hat{\Omega} \).

Observe that since \( \rho_2 = 0 \), we have that

\[ K^S_{i} = -\epsilon_i \mathbb{I}_{\xi_i} \mathcal{X}_0, \quad \mathbb{I}_{\xi_i} K^S_{i} = \tau^{ijk} K^S_{i} + \delta_{ij} \epsilon_i \mathcal{X}_0. \]

where \( \tau^{ijk} \) denotes the sign of the permutation \( (i, j, k) \). It follows that \( \mathcal{V} \oplus \mathcal{N} \) is invariant under \( I, S, T \) and therefore, \( \mathcal{H} \) is invariant under \( I, S, T \). We thus get an \( \text{Sp}(1, \mathbb{B}) \)-invariant almost para-quaternionic Kähler structure on \( \mathcal{H} \), which descends to \( N \). The 4-form associated to the almost para-quaternionic Kähler structure is \( \Omega_N \). In order to show that the structure is para-quaternionic Kähler, we need to show that the quotient metric \( g_N \) has holonomy group contained in \( \text{Sp}(n, \mathbb{B}) \cdot \text{Sp}(1, \mathbb{B}) \).

Or equivalently,

\[ \nabla^N \Omega_N = 0, \]

where \( \nabla^N \) is the Levi-Civita connection of the metric \( g_N \). Observe that for any \( W, X, Y, Z \in \mathcal{H} \),

\[ \hat{\Omega}(W, X, Y, Z) = (\iota^* \omega_1 \wedge \iota^* \omega_1 - \iota^* \omega_2 \wedge \iota^* \omega_2 - \iota^* \omega_3 \wedge \iota^* \omega_3)(W, X, Y, Z). \]

The Levi-Civita connection on \( M \) induces a connection \( \nabla^S \) on \( S \), which is precisely the Levi-Civita connection of the pull-back metric \( \iota^* g_M = \hat{g}_S \). It follows that \( \nabla^S \hat{\Omega} = 0 \). From the fact that \( S \to N \) is a pseudo-Riemannian submersion and the standard computation using O’Neil’s formula ([21], Thm. 3.1), we conclude that the holonomy of the quotient metric is a sub-group of \( \text{Sp}(n, \mathbb{B}) \cdot \text{Sp}(1, \mathbb{B}) \).

Thus, \( N \) is a para-quaternionic Kähler manifold. Note that the signature of \( g_N \) is \((2n - 2, 2n - 2)\).

To sum-up

**Theorem 5.5.** Suppose that \( M \) is a hypersymplectic manifold of dimension \( 4n \). Assume that the obstruction \( \rho_2 = 0 \). Then, the quotient of any level set \( \rho_0^{-1}(c)/\text{Sp}(1, \mathbb{B}) \) is a para-quaternionic Kähler manifold of dimension \( 4n - 4 \), endowed with a metric of signature \((2n - 2, 2n - 2)\).

The converse of the above statement is also true. Namely, if the quotient \( \rho_0^{-1}(c)/\text{Sp}(1, \mathbb{B}) \) is para-quaternionic Kähler, then, \( \rho_2 = c \). This follows directly from the arguments in proof of Theorem 2.15 of [9] and Lemmas 4.1, 4.3 and 4.5.

**Swann bundles on para-quaternionic Kähler manifolds.** Going in the the other direction, given a para-quaternionic Kähler manifold, consider its reduced \( \text{Sp}(n, \mathbb{B}) \cdot \text{Sp}(1, \mathbb{B}) \)-frame bundle \( F \). Then, \( \mathcal{S}(N) = F/\text{Sp}(1, \mathbb{B}) \) is a principal \( \text{SO}^+(1, 2) \) bundle. Let \( \mathcal{B}^* \) denote the space of all the divisors in \( \mathbb{B} \) and define \( \mathcal{B}^* := (\mathbb{B} \setminus \{0\}) / \mathcal{B} \). The \( \text{Sp}(1, \mathbb{B}) \)-action on \( \mathcal{B}^* \), given by \((q, h) \mapsto hq, h \in \mathbb{B}, q \in \text{Sp}(1, \mathbb{B})\), descends to an action of \( \text{SO}^+(1, 2) \) on \( \mathcal{B}^*/\mathbb{Z}_2 \). Note that the action is transitive. There is another action of \( \text{Sp}(1, \mathbb{B}) \) on \( \mathcal{B}^* \) by left multiplication, which descends to an action of \( \text{SO}^+(1, 2) \) on \( \mathcal{B}^*/\mathbb{Z}_2 \) and commutes with the first one. However, note that \( \mathcal{B}^* = \text{Sp}(1, \mathbb{B}) \times (\mathbb{R} \setminus \{0\}) \). Therefore, we have \( \mathcal{B}^*/\mathbb{Z}_2 = \text{SO}^+(1, 2) \times \mathbb{R}_{>0} \). Define the bundle

\[ \mathcal{U}(N)_+ := \mathcal{S}(N) \times_{\text{SO}^+(1,2)} (\mathcal{B}^*/\mathbb{Z}_2) \longrightarrow N. \]
This is the positive Swann bundle constructed by Dancer, Jørgensen and Swann [6]. The total space of the bundle $U(N)$ is a hypersymplectic manifold, with the induced (permuting) action of $SO^+(1,2)$ on the fibres. Alternatively, we can write

$$U(N) = S(N) \times \mathbb{R}^>0,$$

and the metric $g_{U(N)} = g_N + g_{SO^+(1,2)}$. In other words, $U(N)$ is a metric cone over $S(N)$, with metric $g_{S(N)} = g_N + g_{SO^+(1,2)}$. This shows that $S(N)$ is a split 3-Sasakian manifold. Note that the hypersymplectic potential on $U(N)$ is just $\rho_0(s, r) = \frac{1}{2} r^2$ and the Euler vector field $X_0 = r \partial/r$.

Using LeBrun’s construction [18], one can construct a para-quaternionic Kähler manifold of dimension $4k$ from a real-analytic, pseudo-Riemannian manifold of dimension $k + 1$ [6]. Moreover, different manifolds, conformal to the real-analytic manifold give rise to distinct para-quaternionic Kähler manifold. In this way, we have a plethora of examples of para-quaternionic Kähler manifolds and therefore also of hypersymplectic manifolds with a permuting $SO^+(1,2)$-action.

6. Examples

6.1. Commutativity of constructions and split quaternionic modules. Many non-trivial examples of hyperKähler manifolds with permuting $Sp(1)$-action are obtained as reductions of the flat-space $\mathbb{H}^n$. Analogously, we use the Marsden-Weinstein construction for constructing non-trivial examples of hypersymplectic manifolds, carrying a permuting $Sp(1,\mathbb{B})$-permuting action, starting with the flat-space $\mathbb{B}^n$.

**Theorem 6.1** (Hypersymplectic reduction, [1]). Suppose that $G$ is a Lie group acting freely and isometrically on a hypersymplectic manifold $(M, g_M, I, S, T)$, preserving the hypersymplectic structure. Let $\mu : M \rightarrow g \otimes sp(1,\mathbb{B})$ be an associated hypersymplectic moment map.

Suppose that $\mu^{-1}(0)$ is a smooth submanifold of $M$, on which $G$ acts freely and properly, so that $M' = \mu^{-1}(0)/G$ is a smooth pseudo-Riemannian submersion. Then, $(M', g_{M'}, I', S', T')$ is again a hypersymplectic manifold, with respect to the induced hypersymplectic structure $(I', S', T')$ and the induced metric $g_{M'}$.

**Note:** In general, although the quotient is a smooth manifold, the hypersymplectic metric has a non-empty de-generacy locus; i.e., the set of all points $p$ such that $g_{M'}(X, X)|_p = 0$. Away from the locus, the quotient is a smooth, hypersymplectic manifold.

Let $(N, g_N, \Omega_N)$ be a para-quaternionic Kähler manifold. Let $(I, S, T)$ denote the local basis for the paraquaternionic Kähler structure and $\eta_I, \eta_S, \eta_T$ denote the corresponding local para-Kähler/pseudo-Kähler 2-forms. Locally, for a Killing vector field $X$, we can define the 1-form

$$\Theta(X) = i \eta_I(X, \cdot) + s \eta_S(X, \cdot) + t \eta_T(X, \cdot).$$

The form is independent of the the choice of the local basis $(I, S, T)$ and is therefore globally defined.

**Theorem 6.2** (Para-quaternionic Kähler reduction, [22] (Thm. 5.2)). Let $G$ be a Lie group acting freely and isometrically on a para-quaternionic Kähler manifold $(N, g_N, \Omega_N)$. Assume that the group action preserves $\Omega_N$. Then, there exists a unique map $\mu : N \rightarrow g^* \otimes sp(1,\mathbb{B})$ such that $d\mu = t_\Theta$.
Suppose that $\mu^{-1}(0)$ is a smooth submanifold of $N$, on which $G$ acts freely and properly, so that $N' = \mu^{-1}(0)/G$ is a smooth pseudo-Riemannian submersion. Then, $(N', g'_{\Omega'}, \Omega'_{N'})$ is again a para-quaternionic Kähler manifold, with respect to the induced para-quaternionic Kähler structure $\Omega'_{N'}$, and the induced metric $g'_{\Omega'}$.

With the above two theorems at hand, we can directly adapt Swann’s arguments in [7] for the hyperKähler situation, to the pseudo-Riemannian setting to show that the quotient construction commutes with reduction. Namely,

**Theorem 6.3.** Let $(N, g_N, \Omega_N)$ be a para-quaternionic Kähler manifold. Suppose that a Lie group $G$ acts isometrically and freely, preserving the para-quaternionic Kähler structure. Then, $G$ induces an isometric action on $U(N)$, which preserves the hypersymplectic structure on $U(N)$. Moreover, the hypersymplectic quotient of $U(N)$ by the $G$ action is the total space of the Swann bundle over the para-quaternionic Kähler quotient of $N$ by $G$.

Let us now consider the split quaternionic module $\mathbb{B}^{n+1}$. Let $\mathbb{B}$ denote the sub-space of all the null-vectors in $\mathbb{B}^{n+1}$ and consider the sub-space of all the space-like vectors (positive norm) $(\mathbb{B}^{n+1})^* := (\mathbb{B}^{n+1} \setminus \{0\}) \setminus \mathbb{B}$. Then $(\mathbb{B}^{n+1})^*$ is a union of two disjoint spaces of space-like and time-like vectors. Let $(\mathbb{B}^{n+1})^*_\perp$ denote the sub-space of space-like vectors. We will show that this is the total space of a Swann bundle over a para-quaternionic Kähler manifold. First, observe that $(\mathbb{B}^{n+1})^*_\perp$ is a hypersymplectic manifold, equipped with a free and proper action of $\text{Sp}(1, \mathbb{B})$, as described in Subsec. 3.1. Moreover, it also carries a homothetic action of $\mathbb{R}^+$ given by $(r, h) \mapsto r \cdot h$. Clearly, we see that the obstruction $\rho_2$ vanishes and the hypersymplectic potential is given by $\rho_0(h) = \frac{1}{2} \|h\|^2$.

Consider the positive sphere

$$S_+ := \rho_0^{-1}(\frac{1}{2}) = \{ q \in (\mathbb{B}^{n+1})^*_\perp \mid \|q\|^2 = 1 \} \cong \frac{\text{Sp}(n+1, \mathbb{B})}{\text{Sp}(n)}.$$

The sphere carries a metric of signature $(2n - 1, 2n + 1)$. Then, clearly, $(\mathbb{B}^{n+1})^*_\perp$ is topologically a metric cone over $S_+$ and so $(\mathbb{B}^{n+1})^* = S_+ \times \mathbb{R}_{>0}$. The hypersymplectic potential is just $\rho_0(s, r) = \frac{1}{2} r^2$. Therefore, $S_+$ is a split 3-Sasakian manifold. The Sp$(1, \mathbb{B})$-action on $(\mathbb{B}^{n+1})^*_\perp$ induces a free, proper and isometric action of Sp$(1, \mathbb{B})$ on $S_+$. By Theorem 5.5, the quotient $S_+/\text{Sp}(1, \mathbb{B})$ is a para-quaternionic Kähler manifold, which is nothing but the para-quaternionic projective space

$$\mathbb{B}P^n = \frac{\text{Sp}(n+1, \mathbb{B})}{\text{Sp}(n) \times \text{Sp}(1, \mathbb{B})}.$$

It follows that $(\mathbb{B}^{n+1})^*_\perp$ is the total space of the Swann bundle over $\mathbb{B}P^n$, i.e, $U(\mathbb{B}P^n)$. This is the **positive Swann bundle** described in [6]. Split quaternionic projective spaces have been studied by Blažič [23] and Wolf [24].

Consider an action of a Lie group $G \subset \text{Sp}(n+1, \mathbb{B})$ on $\mathbb{B}^{n+1}$, that commutes with the permuting Sp$(1, \mathbb{B})$-action by right conjugate multiplication. The induced action on $(\mathbb{B}^{n+1})^*_\perp$ preserves the hypersymplectic structure and therefore also the three symplectic forms. So, there exist three moment maps, which we combine into a single $G \times \text{Sp}(1, \mathbb{B})$-equivariant map

$$\mu : (\mathbb{B}^{n+1})^*_\perp \longrightarrow \mathfrak{sp}(1, \mathbb{B}) \otimes \mathfrak{g}^*, \quad \mu = i \mu_1 + s \mu_2 + t \mu_3.$$
Suppose that 0 is a regular value of $\mu$ and $G$ acts freely and properly on the zero-level-set of the moment map. Additionally, assume that the metric, restricted to the group orbits in $\mu^{-1}(0)$ is non-degenerate. Hitchin’s work [1] now guarantees that the quotient $\tilde{M} = \mu^{-1}(0)/G$ is a hypersymplectic manifold. The permuting action of $\text{Sp}(1, \mathbb{B}) \times \mathbb{R}^+$ on $(\mathbb{B}^{n+1})^+_1$ commutes with the action of $G$ (therefore preserving the zero-level-set of $\mu$) and hence descends to the quotient $\tilde{M}$. In particular the obstruction $\rho_2$ vanishes; i.e., $\tilde{M} = U(N)$ for some para-quaternionic Kähler manifold $N$. Since $\text{Sp}(1, \mathbb{B}) \times \mathbb{R}^+$-action commutes with that of $G$, the latter descends to a para-quaternionic Kähler action on $\mathbb{B}P^n$. By Theorem 6.3, it follows that $N$ is the para-quaternionic Kähler reduction of $\mathbb{B}P^n$ by $G$.

When $G$ is a compact subgroup of $\mathbb{T}^{n+1}$, Dancer and Swann [25] show that the hypersymplectic reduction of $\mathbb{B}^{n+1}$ by $G$ is, a hypersymplectic manifold, with a non-trivial degeneracy locus - the set of all the points where the metric is degenerate along the orbits of the $G$-action. For example, when $G = U(1)$, this is precisely the set of points where $g_M(K^M, K^M) = 0$, where $K^M$ is the fundamental vector field due to $U(1)$-action. Now, the $U(1)$-action descends to an action on $(\mathbb{B}^{n+1})^+_1$ and the moment map is just the restriction of $\mu$. From the discussion above, it follows that the reduced manifold is a smooth hypersymplectic manifold, which is the total space of a Swann bundle over the para-quaternionic Kähler reduction of $\mathbb{B}P^n$ by $U(1)$.

6.2. Moduli spaces of Nahm-Schmid equations. We give an example of a hypersymplectic manifold, with a permuting $\text{SU}(1,1)$-action, for which the obstruction $\rho_2$ does not vanish. For this, we consider the moduli space of Nahm-Schmid equations. We refer to [4] for more details.

Nahm-Schmid equations are pseudo-Riemannian analogues of Nahm’s equations and arise as dimensional reduction of Yang-Mills equations on $\mathbb{R}^{2,2}$. Let $G$ be a compact Lie group and $\mathfrak{g}$ denote its Lie algebra. Let $T_i : \mathbb{R} \rightarrow \mathfrak{g}$ be $C^1$-differentiable maps for $i = 0, 1, 2, 3$. The Nahm-Schmid equations for $\{T_i\}_{i=0}^3$ is a system of equations

$$
\dot{T}_1 + [T_0, T_1] = -[T_2, T_3], \quad \dot{T}_2 + [T_0, T_2] = [T_3, T_1], \quad \dot{T}_3 + [T_0, T_3] = [T_1, T_2].
$$

(6.1)

which are invariant under the action of the gauge group $\mathcal{G} = \{g : \mathbb{R} \rightarrow G\}$. We consider the equations on the bounded interval $[0, 1]$. The equations can be interpreted as the zero-level set of the hypersymplectic moment map for the action of the gauge group $\mathcal{G}_0 = \{g : [0, 1] \rightarrow G \mid g(0) = g(1) = \text{id}\}$, on a suitable space of connections on $\mathbb{R}^{2,2}$. Thus, the moduli space of solutions to (6.1) inherits a hypersymplectic structure (with degeneracies). It is always possible to find a gauge transformation that puts $T_0 = 0$. So, without loss of generality, we can assume that $T_0 = 0$. There is an isometric action of $\text{SU}(1,1)$ on the space of solutions, that rotates the triple $(T_1, T_2, T_3)$, which is analogous to the permuting $\text{SO}(3)$ action in case of Nahm equations. The $\text{SU}(1,1)$-action commutes with that of the gauge group and therefore descends to a permuting action on the moduli space of solutions. It, however, does not extend to a homothetic action of $(\mathbb{B}^*)_+$. 

Suppose now we consider the equations on the half-line $\mathbb{R}^+$. Then, as in the above situation, the moduli space is a (de-generate) hypersymplectic manifold, carrying a permuting action of $\text{SU}(1,1)$. A striking difference between the Nahm equations and the Nahm-Schmid equations is that the latter exist for all time. Indeed, let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ denote a bi-invariant inner product on $\mathfrak{g}$ and let $(T_1, T_2, T_3)$ be a solution to (6.1). Then, the quantity $2\langle T_1, T_1 \rangle_{\mathfrak{g}} + \langle T_2, T_2 \rangle_{\mathfrak{g}} + \langle T_3, T_3 \rangle_{\mathfrak{g}}$ is conserved and the solutions exist for all time. As a result, even in this case, the rotating $\text{SU}(1,1)$-action cannot extend to a homothetic $(\mathbb{B}^*)_+$-action, which is in sharp contrast with the hyperKähler situation.
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