Super fidelity and related metrics

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We report a new metric of quantum states. This metric is build up from super-fidelity, which has deep connection with the Uhlmann-Jozsa fidelity and plays an important role in quantifying entanglement. We find that the new metric possess some interesting properties.

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I. INTRODUCTION

In quantum information theory, a fundamental task is to distinguish two quantum states. One of the main tools used in distinguishability theory is trace metric, another closed related tool is quantum fidelity [1, 2]. Both are widely used by the quantum information science community and have been found applications in a number of problems such as quantifying entanglement [3, 4], quantum error correction [5], quantum chaos [6], and quantum phase transitions [7].

Suppose ρ and σ are two quantum states, then the Uhlmann-Jozsa fidelity [1, 2] between them is given by

\[ F(\rho, \sigma) = |\text{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}|^2 \] (1)

We know that for the case of qubits, Uhlmann-Jozsa fidelity has a simple form. From the Bloch sphere representation of quantum states, a qubit is described by a density matrix as:

\[ \rho(u) = \frac{1}{2}(I + \sigma \cdot u) \] (2)

where I is the 2 × 2 unit matrix and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices. Assume \( \rho(u) \) and \( \rho(v) \) are two states of one qubit, then they can be represented by two vectors u and v in the Bloch sphere. The Uhlmann-Jozsa fidelity for qubits has an elegant form:

\[ F(\rho(u), \rho(v)) = \frac{1}{2}(1 + u \cdot v + \sqrt{1-|u|^2}\sqrt{1-|v|^2}) \] (3)

where \( u \cdot v \) is the inner product of u and v, and \( |u| \) is the magnitude of u.

We know that for general quantum states, the Uhlmann-Jozsa fidelity has no simple form like the case of qubits. To use the simple form of fidelity, we note that in [8], the authors introduce a new fidelity, called super-fidelity, defined as

\[ G(\rho_1, \rho_2) := \text{Tr}\rho_1\rho_2 + \sqrt{(1 - \text{Tr}\rho_1^2)(1 - \text{Tr}\rho_2^2)} \] (4)

and it was proved that when \( \rho_1 \) and \( \rho_2 \) are two qubits, super-fidelity \( G(\rho_1, \rho_2) \) coincides with Uhlmann-Jozsa fidelity \( F(\rho_1, \rho_2) \).

The super-fidelity \( G(\rho_1, \rho_2) \) has some appealing properties[8, 9, 10]. Let \( \rho_0 = \frac{1}{N}(I + \sqrt{N(N-1)}\sum\lambda_i \rho_i) \) be the density matrix of a qubit(N × N quantum state), where I is the N × N unit matrix, \( \sum\lambda_i \rho_i \) are the generators of SU(N), and u is the (\( N^2 - 1 \))-dimensional Bloch vector. Then super-fidelity can be rewritten as \( G(\rho_0, \rho_i) = \frac{1}{N}[1 + (N-1)u \cdot v + (N-1)\sqrt{1-|u|^2}(1-|v|^2)] \). This shows that super-fidelity only depends on the magnitudes of u, v and the angle between them (that is, u, v). This property make super-fidelity easy to calculate, and has a clear geometrical interpretation.

Moreover, very recently, it was found that super-fidelity play an important role in quantifying entanglement [11]. So it is natural to study the property of super-fidelity in further step.

Recall that super-fidelity by itself is not a metric. It is a measure of the “closeness” of two states. If we say a function \( d(x, y) \) defined on the set of quantum states is a metric, it should satisfies the following four axioms:

(M1). \( d(x, y) \geq 0 \) for all states \( x \) and \( y \);

(M2). \( d(x, y) = 0 \) if and only if \( x = y \);

(M3). \( d(x, y) = d(y, x) \) for all states \( x \) and \( y \);

(M4). The triangle inequality: \( d(x, y) \leq d(x, z) + d(y, z) \) for all states \( x, y \) and \( z \).

For super-fidelity, one can define the following three functions[10]:

\[ A(\rho, \sigma) := \arccos \sqrt{G(\rho, \sigma)}, \] (5)

\[ B(\rho, \sigma) := \sqrt{2 - 2\sqrt{G(\rho, \sigma)}}, \] (6)

\[ C(\rho, \sigma) := \sqrt{1 - G(\rho, \sigma)}, \] (7)

It was proved in [10] that \( C(\rho, \sigma) \) is a genuine metric, that is, it satisfying the axioms M1-M4, while \( A(\rho, \sigma) \) and \( B(\rho, \sigma) \) do not preserve the metric properties.
The purpose of this paper is to introduce a novel method to define metric of quantum states based on super-fidelity. Surprising, we find the metric induced by the new method coincides with the metric introduced in [10] for the qubits case, and the new metrics have deep connection with spectral metric. Also we find the new metrics possess some appealing properties which make the metrics very useful in quantum information theory. The paper is organized as follows: In Sec. II, two new metrics were defined, and the metric character of the metrics were established. In Sec. III, intrinsic properties of the two metrics were discussed. Conclusion and discussion were made in the last section.

II. METRIC INDUCED BY SUPER-FIDELITY

The most widely used metric may be trace metric, which was defined as

$$D_{Tr}(\rho, \sigma) = \frac{1}{2} \text{Tr}|\rho - \sigma|$$

(8)

On the other hand, one can define other types of distance measures for quantum states, and these measures also have their own advantages, see [1, 8, 10, 11, 12, 13, 14, 15, 16, 17].

Let us define a new metric of states as follows:

$$D_G(\rho, \sigma) = \max_{\tau} |G(\rho, \tau) - G(\sigma, \tau)|$$

(9)

where the maximization is taken over all quantum states $\tau$ (mix or pure). We call this metric $D_G(\rho, \sigma)$ as the G-metric, and the state $\tau$ that attained the maximal is called the optimal state for the metric $D_G(\rho, \sigma)$.

The above definition of metric may be not easy to calculate. So we can change its definition slightly. If $\tau$ is a pure state, then super-fidelity can be simplified as $G(\rho, \tau) = \text{Tr}(\rho \tau)$, hence one can define another version of metric as follows:

$$D_{PG}(\rho, \sigma) = \max_{\tau} |G(\rho, \tau) - G(\sigma, \tau)|$$

(10)

where the maximization is taken over all pure states $\tau$. We call this metric $D_{PG}(\rho, \sigma)$ as the PG-metric, and call the pure state $\tau$ that attained the maximal as the optimal pure state.

First we consider the case of qubits.

**Proposition 1** [17]. For the qubit case, $D_{PG}(\rho, \sigma)$ equals to the trace metric, namely $D_{PG}(\rho, \sigma) = D_{Tr}(\rho, \sigma) = \frac{1}{2} \text{Tr}|\rho - \sigma|$. We can connect our metric with the metric introduced in [10] as following:

**Proposition 2** [17]. For the qubit case, $D_G(\rho, \sigma) = G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$.

Now we come to discuss the case of qunit (i.e., $N \times N$ quantum states). In this case, if $\tau$ is a pure state, then the super-fidelity has a simple form: $G(\rho, \tau) = \text{Tr}(\rho \tau)$, this make the PG-metric easy to study. So we first show the metric character of $D_{PG}(\rho, \sigma)$, where the optimal state $\tau$ is restricted to pure state, and then turn to show the metric character of $D_G(\rho, \sigma)$.

We need the following concepts: For two quantum state $\rho$ and $\sigma$, let $\lambda_i$ ($i = 1, 2, 3, ..., n$), be all eigenvalues of $\rho - \sigma$, and arranged as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Define $E(\rho, \sigma) := \max \lambda_i$. We can give an interpretation of $E(\rho, \sigma)$ as follows: Let $\rho$ and $\sigma$ be two quantum states, then the following is well known (for example, see [20]):

$$E(\rho, \sigma) = \max_{\tau} \text{Tr}[\tau(\rho - \sigma)],$$

(11)

where the maximization is taken over all pure states $\tau$.

Note that generally $E(\rho, \sigma)$ is not a metric, since $E(\rho, \sigma)$ may not equal to $E(\sigma, \rho)$, but we can symmetrize it as:

$$D_S(\rho, \sigma) := \max[E(\rho, \sigma), E(\sigma, \rho)] = \max |\lambda_i|$$

(12)

where $|\lambda_i|$ is the absolute value of $\lambda_i$. From the knowledge of matrix analysis, we get that $D_S(\rho, \sigma)$ equal to the spectral metric between $\rho$ and $\sigma$, which was defined as the largest singular value of $\rho - \sigma$, hence we know that $D_S(\rho, \sigma)$ is in fact the spectral metric. Moreover, we have the following:

**Proposition 3** [17]. For quantum states $\rho$ and $\sigma$, $D_{PG}(\rho, \sigma) = D_S(\rho, \sigma)$, that is, the PG-metric is nothing but the spectral metric.

Now we know that the PG-metric is in fact the spectral metric, so it is a true metric. In the following we shall prove that the G-metric is also a true metric.

**Theorem 1.** The T-metric $D_G(\rho, \sigma)$ as shown in Eq. (9) is truly a metric, i.e., it satisfies conditions M1-M4.

**Proof.** From the definition, it is easy to prove conditions M1 and M3 hold. What we need to do is to prove conditions M2 and M4. If $\rho = \sigma$, then of course $D_G(\rho, \sigma) = 0$. If $D_G(\rho, \sigma) = 0$, we will prove $\rho = \sigma$. From the definition, we know that $D_G(\rho, \sigma) \geq D_{PG}(\rho, \sigma)$, so we get $D_{PG}(\rho, \sigma) = 0$, since $D_{PG}(\rho, \sigma)$ is a true metric, we get $\rho = \sigma$. Now we come to prove M4, the triangle inequality $D_G(\rho, \sigma) \leq D_G(\rho, \tau) + D_G(\sigma, \tau)$. $D_G(\rho, \sigma) = \max_{\tau} |G(\rho, \tau) - G(\sigma, \tau)|$, and suppose $\tau$ is the optimal state that attains the maximal, so $D_G(\rho, \sigma) = |G(\rho, \tau) - G(\sigma, \tau)|$. Assume that $|G(\rho, \tau) - G(\sigma, \tau)| = G(\rho, \tau) - G(\sigma, \tau)$, then we get $G(\rho, \tau) - G(\sigma, \tau) \geq |G(\rho, \tau) - G(\sigma, \tau)| \leq |G(\rho, \tau) - G(\sigma, \tau)|$. Thus one finally has $D_G(\rho, \sigma) \leq D_G(\rho, \sigma) + D_G(\sigma, w)$. Theorem is proved.

III. PROPERTIES OF $D_G$ AND $D_{PG}$

We know that for qubits, $D_G$ has a clear form as: $D_G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$, how about higher dimension? For the qunit case, one does not have the relation $D_G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$ as in Proposition 2.
However, the following upper bound holds: For qudits \( \rho \) and \( \sigma \), the following relation holds:

\[
D_G(\rho, \sigma) \leq \sqrt{\frac{2 \times (N-1)}{N}} \times \sqrt{1 - G(\rho, \sigma)}
\]  

(13)

Proof: Let \( \rho = \rho(u) \), \( \sigma = \sigma(v) \) and \( \tau = \tau(w) \), where \( u, v, w \) are the corresponding Bloch vectors of the states \( \rho, \sigma, \tau \), then one obtains

\[
|G(\rho, \tau) - G(\sigma, \tau)| \times \frac{N}{N-1}
\]

\[
= \left| (u - v) \cdot w + \sqrt{1 - |w|^2} \left( \sqrt{1 - |u|^2} - \sqrt{1 - |v|^2} \right) \right|
\]

\[
\leq |u - v||w| + \sqrt{1 - |w|^2} \left( \sqrt{1 - |u|^2} - \sqrt{1 - |v|^2} \right)
\]

\[
\leq \sqrt{|u - v|^2 + \left( \sqrt{1 - |u|^2} - \sqrt{1 - |v|^2} \right)^2}
\]

\[
= \sqrt{\frac{2 - 2u \cdot v - 2\sqrt{1 - |u|^2} \sqrt{1 - |v|^2}}{N-1}}
\]

\[
= \sqrt{\frac{2 \times N}{N-1}} \times \sqrt{1 - G(\rho(u), \sigma(v))}.
\]

Now we will discuss the inequality (13) in more detail. When \( N = 2 \), i.e., in the case of qubits, we get that inequality (13) becomes equality, that is, \( D_G(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)} \). But for higher dimension, the equality sign does not hold in general. Why?

The reason is subtle. When the equality sign holds, i.e., \( D_G(\rho(u), \sigma(v)) = \sqrt{\frac{2 \times (N-1)}{N}} \times \sqrt{1 - G(\rho(u), \sigma(v))} \), then the inequality (14) need to be equality, that means the optimal state \( \tau = \tau(w) \) is always attained, where \( w \) is a vector that parallels to \( u - v \), and \( |w| = \frac{\sqrt{2 - 2u \cdot v - 2\sqrt{1 - |u|^2} \sqrt{1 - |v|^2}}}{\sqrt{2 \times N \sqrt{1 - G(\rho(u), \rho(v))}}} \), but in fact we can not always get such optimal state. Because such an operator \( \tau(w) \) may not be a density operator! We will explain it in the following.

It is well known that every \( N \times N \) density matrix can be represented by the \((N^2 - 1)\)-dimensional Bloch vector as: \( \rho(u) = \frac{1}{N}(I + \sqrt{\frac{N(N-1)}{2}} \lambda \cdot u) \), but the converse is not true, i.e., not all operator of the form \( \frac{1}{N}(I + \sqrt{\frac{N(N-1)}{2}} \lambda \cdot u) \) is a density matrix, where \( u \) is an arbitrary \((N^2 - 1)\)-dimensional Bloch vector. Note that a density matrix must satisfy three conditions: (a). Trace unity, \( \text{Tr}(\rho(u)) = 1 \). (b). Hermitian, \( \rho(u)^+ = \rho(u) \); and (c). positivity, i.e., all eigenvalues of \( \rho(u) \) are non-negative.

Indeed, the operator \( \frac{1}{N}(I + \sqrt{\frac{N(N-1)}{2}} \lambda \cdot u) \) automatically satisfies the conditions (a) and (b). However, not every vector \( u \), \( |u| \leq 1 \), allows \( \rho(u) \) satisfies the positive condition (c), for example, see [18].

To get that inequality (14) becomes equality, we need that the optimal state \( \tau = \tau(w) \) is a density matrix, where \( w \) is a vector that parallels to \( u - v \), and \( |w| = \frac{\sqrt{N - |u - v|}}{\sqrt{2N \sqrt{1 - G(\rho(u), \rho(v))}}} \), but this is not always true in general. So we can only get the inequality (13).

The following counterexample will show that strict inequality will occur.

Example 1. Let \( |\psi\rangle = \frac{1}{\sqrt{2}}(00) + \frac{1}{\sqrt{2}}(11) \), \( |\phi\rangle = \frac{1}{\sqrt{2}}(00) + \frac{1}{\sqrt{2}}(11) \). Define \( \rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi| \), then we get that \( D_G(\rho, \sigma) = \frac{1}{2} \), while \( \sqrt{\frac{2 \times (3)}{4}} \times \sqrt{1 - G(\rho, \sigma)} = \frac{\sqrt{2}}{2} > \frac{1}{2} \).

Now we will study the intrinsic properties of the G-metric \( D_G \) and PG-metric \( D_{PG} \). We are interested in the following properties:

**Property 1: contractive under quantum operation.** suppose \( T \) is a quantum operation, i.e., a completely positive trace preserving (CPT) map, and \( \rho, \sigma \) are density operators, we say a metric \( D(\rho, \sigma) \) is contractive under quantum operation, if the following holds:

\[
D(T(\rho), T(\sigma)) \leq D(\rho, \sigma)
\]

(15)

Why we study the property of contractive under quantum operation? It has a physical interpretation [14]: a quantum process acting on two quantum states can not increase their distinguishability.

**Property 2: joint convex property.** we say that the metric \( D(\rho, \sigma) \) has the convex property, if \( p_j \) are probabilities, then

\[
D(\sum_j p_j \rho_j, \sum_j p_j \sigma_j) \leq \sum_j p_j D(\rho_j, \sigma_j)
\]

(16)

The joint convex property also has a physical interpretation [14]: the distinguishability between the states \( \sum_j p_j \rho_j \) and \( \sum_j p_j \sigma_j \), where \( p_j \) is not known, can never be greater than the average distinguishability when \( p_j \) is known.

We know that the Uhlmann-Jozsa fidelity \( F(\rho, \sigma) \) has the CPT expansive property: If \( \rho \) and \( \sigma \) are density matrices, \( \Phi \) is a CPT map, then

\[
F(\Phi(\rho), \Phi(\sigma)) \geq F(\rho, \sigma)
\]

(17)

We may guess that the super-fidelity \( G(\rho, \sigma) \) also has the CPT expansive property, the following counterexample shows that this property does not holds.

**Example 2** [10]. Let

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

Define \( \Phi(\gamma) = A\gamma A^+ + B\gamma B^+ \), where \( \gamma \) is an arbitrary density operator, then we defined a completely positive trace preserving map.
Let $\rho$ and $\sigma$ be the density operators defined by
\[
\rho = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \sigma = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{pmatrix},
\]
Then
\[
\Phi(\rho) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \Phi(\sigma) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

One then easily obtains $G(\rho, \sigma) > G(\Phi(\rho), \Phi(\sigma))$, which shows that the CP expansive property property does not hold for super-fidelity.

So we get that the metric $C(\rho, \sigma) = \sqrt{1 - G(\rho, \sigma)}$ introduced in [10] is not contractive under quantum operation.

However, we can prove the following:

**Theorem 2.** The PG-metric $D_{PG}(\rho, \sigma)$ is contractive under quantum operation, that is, $D_{PG}(\Phi(\rho), \Phi(\sigma)) \leq D_{PG}(\rho, \sigma)$.

**Proof.** Suppose $\gamma$ is the optimal pure state for quantum states $\phi(\rho), \phi(\sigma)$, so we get $D_{PG}(\phi(\rho), \phi(\sigma)) = |G(\phi(\rho), \gamma) - G(\phi(\sigma), \gamma)| = |\langle Tr\phi(\rho)\gamma \rangle - \langle Tr\phi(\sigma)\gamma \rangle|$

Let $\phi$ be a quantum operation, and denote $\gamma := \phi^*(\gamma)$. Then we have
\[
D_{PG}(\phi(\rho), \phi(\sigma)) = |\langle Tr\phi(\rho)\gamma \rangle - \langle Tr\phi(\sigma)\gamma \rangle|
\]
\[
= |\langle Tr\rho \phi^*(\gamma) \rangle - \langle Tr\sigma \phi^*(\gamma) \rangle|
\]
\[
\leq D_{PG}(\rho, \sigma)
\]
Theorem is proved.

Note that the PG-metric is in fact the spectral metric, and it was proved in [22] that spectral metric is contractive under quantum operation, here we give an elementary proof, our method is quite different from that of [22].

How about the G-metric? Numerical experiment shows that the G-metric $D_{G}(\rho, \sigma)$ is not contractive under quantum operation.

Now we discuss the joint convex property.

**Proposition 4.** (joint convexity of the PG-metric): Let $(p_i)$ be probability distributions over an index set, let $\rho_i$ and $\sigma_i$ be density operators with the indices from the same index set. Then
\[
D_{PG}(\sum_{i} p_i \rho_i, \sum_{i} p_i \sigma_i) \leq \sum_{i} p_i D_{PG}(\rho_i, \sigma_i) \quad (18)
\]

We know that $D_{PG}(\rho, \sigma) = D_S(\rho, \sigma) = \max (E(\rho, \sigma), E(\sigma, \rho))$, so we only need to prove the following holds:
\[
E(\sum_{i} p_i \rho_i, \sum_{i} p_i \sigma_i) \leq \sum_{i} p_i E(\rho_i, \sigma_i)
\]
since $E(\rho, \sigma) = \max_{\gamma} \text{Tr}(\gamma(\rho - \sigma))$, where the maximization in the right hand is taken over all pure states $\gamma$, then there exists a pure state $\gamma$ such that
\[
E(\sum_{i} p_i \rho_i, \sum_{i} p_i \sigma_i) = \sum_{i} p_i \text{Tr}(\gamma(\rho_i - \sigma_i)) \leq \sum_{i} p_i E(\rho_i, \sigma_i).
\]
The proof is complete.

We also find that, the metric $D_G$ is not joint convex. However, numerical experiment shows that its square is joint convex, that is, the following holds:
\[
D^2_{G}((\lambda \rho_1 + (1-\lambda) \rho_2), \sigma) \leq \lambda D^2_{G}(\rho_1, \sigma) + (1-\lambda) D^2_{G}(\rho_2, \sigma)
\]

**IV. CONCLUSION**

In summary, we have introduced a new way to define metric of quantum states from super-fidelity. We find that, for qubit case, our metric $D_G$ coincides with the metric $C(\rho, \sigma)$ introduced in [10]. We proved that the metric $D_{PG}$ is contractive under quantum operation, while the metric $D_G$ does not behave monotonically under quantum operation. Also, we rigorously proved that $D_{PG}$ is joint convex, and numerically proved that the square of $D_G$ is joint convex. All these show that the metric $D_G$ is worthwhile studying.

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