Interface states of quantum spin systems

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Abstract: We review recent results as well as ongoing work and open problems concerning interface states in quantum spin systems at zero and finite temperature.

Keywords: Heisenberg ferromagnet, XXZ model, ground state selection, rigidity of interfaces, 111 interface, quantum SOS model

1 Introduction

Interface states have been extensively studied in rigorous statistical mechanics since Dobrushin’s seminal paper on interfaces Gibbs states in the three-dimensional Ising model \cite{Dobrushin}. Interfaces and surfaces are everywhere in the physical world, and to know their properties is important in understanding phenomena as diverse as droplet evaporation, crystal growth, catalysis, dynamics of magnets, etc. Until recently almost all rigorous work was restricted to classical models, while some of the phenomena are certainly more adequately described by quantum mechanical models. In the past few years, however, several interesting results have been obtained for the quantum Heisenberg and the Falicov-Kimball model. The Falicov-Kimball model is discussed in several other contributions in this volume (\cite{FalicovKimball}). Therefore, we restrict our attention here to the Heisenberg model and quantum spin systems in general. We will also introduce a quantum solid-on-solid (QSOS) model for the 111 interface of the Heisenberg XXZ ferromagnet.

2 Interface ground states of the XXZ ferromagnets

A quantum spin model is defined by specifying a dynamics on an algebra of quasi-local observables. The local structure is given by the finite subsets of the $d-$dimensional lattice $\mathbb{Z}^d$. With each site $x \in \mathbb{Z}^d$, there is associated a copy $\mathcal{A}_x$ of the $n \times n$ matrices with complex entries $M_n(\mathbb{C})$. $\mathcal{A}_x$ is the algebra of observables at the site $x$. For every finite subset $\Lambda \subset \mathbb{Z}^d$, the observables in the volume $\Lambda$ are given by

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_x$$

This is a finite-dimensional $C^*$-algebra, and if $\Lambda_0 \subset \Lambda$, we have the natural embedding

$$\mathcal{A}_{\Lambda_0} = \mathcal{A}_{\Lambda_0} \otimes 1_{\mathcal{I}_{\mathcal{A}_{\Lambda \setminus \Lambda_0}}} \subset \mathcal{A}_\Lambda$$

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The algebra of *local observables* is then defined by

\[ A_{\text{loc}} = \bigcup_{\Lambda \subset \mathbb{Z}^d} A_\Lambda \]

where the union is over all finite subsets of \( \mathbb{Z}^d \). Its completion is the \( C^* \)-algebra of *quasi-local observables*:

\[ A = \overline{A_{\text{loc}}} \]

We will also need the translation automorphisms on \( A \), denoted by \( \tau_x, x \in \mathbb{Z}^d \), canonically mapping \( A_\Lambda \) into \( A_{\Lambda+x} \).

The dynamics is determined by a family of *local Hamiltonians*. For simplicity we will only discuss models with translation invariant finite range interactions. I.e., let \( h = h^* \in A_{\Lambda_0} \), for some finite set \( \Lambda_0 \), and define the local Hamiltonians by

\[ H_\Lambda = \sum_{x: \Lambda_0+x \subseteq \Lambda} \tau_x(h) \]

The generator of the dynamics is the unique closed extension of the derivation

\[ \delta(A) := \lim_{\Lambda \to \mathbb{Z}^d} [H_\Lambda, A], \quad \text{for all } A \in A_{\text{loc}} \]

This generator can be exponentiated to obtain a strongly continuous one-parameter group of \( C^* \)-automorphisms \( \{\alpha_t\}_{t \in \mathbb{R}} \),

\[ \alpha_t(A) := e^{it\delta}(A), \quad A \in A \]

The standard proof can be found in the books by Bratteli and Robinson [5] or Simon [6].

A *state* of the quantum spin system is a linear functional \( \omega \) on \( A \) with the properties:

\[ \omega(A^*A) \geq 0, \quad \text{for all } A \in A \]

\[ \omega(1) = 1 \]

The ground states of a model, by definition, are the solutions of the following set of inequalities:

\[ \omega(A^*\delta(A)) = \lim_{\Lambda \to \infty} \omega(A^*[H_\Lambda, A]) \geq 0, \quad \text{for all } A \in A_{\text{loc}} \]

These inequalities express that all local excitations, i.e., created by a local observable \( A \), raise the energy. This is also called *Local Stability* and we will often refer to this set of inequalities as LS.

For translation invariant states, i.e., states satisfying

\[ \omega \circ \tau_x = \omega, \quad \text{for all } x \in \mathbb{Z}^d \]

the ground states are exactly the states that minimize the energy per site:

\[ \omega(h) = \inf \{ \eta(h) \mid \eta \text{ translation invariant state on } A \} \]

In general, LS has non-translation invariant solutions, e.g., describing domain walls or interfaces. In this paper, our main interest are these non-translation invariant solutions. In one dimension they are often called kink (and antikink) states, or soliton states. Solutions of LS are commonly constructed as limits as \( \Lambda \to \mathbb{Z}^d \), of ground states of the finite-volume Hamiltonians plus boundary terms.

One-dimensional quantum spin models, also called spin chains, are of particular interest for several reasons. First of all it is in one dimension that the most detailed rigorous analysis is possible. This was the main motivation for [7]. More recently it has also become possible to realize quantum spin chains experimentally and to compare theory and experiment in surprising detail. It
is also interesting that there is a special but not so small class of one-dimensional models for which the exact ground states can be given explicitly [8, 9, 10, 11]. And finally there are the integrable quantum spin chains and the rich mathematical structures they exhibit [12].

None of the works cited above, however, deal with non-periodic ground states. Consequently, the problem of characterizing the complete set of solutions of LS, for any model, was, until recently, never dealt with.

The most detailed results have been obtained for the ferromagnetic XXZ Heisenberg model. For $x \in \mathbb{Z}^d$, let $S_x^z, i = 1, 2, 3$, be the standard spin-$S$ matrices, generating a $n = 2S + 1$-dimensional irreducible unitary representation of SU(2), with $S = 1/2, 1, 3/2, \ldots$. The local Hamiltonians of the ferromagnetic XXZ Heisenberg model are given by

$$H_\Lambda = -\sum_{x,y \in \Lambda} \frac{1}{\Delta} (S_x^1 S_y^1 + S_x^2 S_y^2) + S_x^3 S_y^3 \tag{2.1}$$

with $\Delta \geq 1$. $\Delta = +\infty$ is the Ising model. $\Delta = 1$ is the isotropic model, also called the XXX model.

First, we discuss the complete set of ground states, in the sense of LS for the XXZ ferromagnetic chain, i.e., $d = 1$. This class of models depends on two parameters: the dimension of the spin matrices ($2S + 1$, $S = 1/2, 1, 3/2, \ldots$), and the anisotropy $\Delta \geq 1$.

Consider finite volumes of the form $\Lambda = [a, b] \subset \mathbb{Z}$, and denote by $\partial \Lambda = [a, a + r] \cup [b - r, b]$, the boundary of $\Lambda$, where $r \geq 0$ is a suitably chosen integer.

Solutions of the ground state inequalities can be constructed by adding suitable boundary terms $b_\Lambda \in A_{\partial \Lambda}$ to the finite-volume Hamiltonians, and taking limits $\Lambda \uparrow \mathbb{Z}$ of finite volume ground states of the form

$$\omega_\Lambda(A) = \frac{\langle \psi_\Lambda, A \psi_\Lambda \rangle}{\langle \psi_\Lambda, \psi_\Lambda \rangle} \quad \text{for all } A \in A_\Lambda$$

where $\psi_\Lambda \in \otimes_{x \in \Lambda} \mathbb{C}^{2S+1}$, is an eigenvector belonging to the smallest eigenvalue of $H_\Lambda + b_\Lambda$.

For the XXZ chains it suffices to take $r = 1$ and boundary terms of the form

$$b_{[a,b]} = B(S_{a+1}^3 - S_{a-1}^3) \tag{2.2}$$

For convenience we will consider the chain on intervals of the form $[-L, L]$, including the boundary points ($-a = b = L - 1$).

Two translation invariant solutions are trivial to find and have been well-known for a long time: the unique states $\omega_\uparrow$ and $\omega_\downarrow$ determined by

$$\omega_\uparrow(S_x^3) = S, \quad \text{for all } x \in \mathbb{Z}$$

$$\omega_\downarrow(S_x^3) = -S, \quad \text{for all } x \in \mathbb{Z}$$

By taking $B := B(\Delta) \pm S \sqrt{1 - 1/\Delta^2}$, Alcaraz, Salinas, and Wreszinski [13], and independently, Gottstein and Werner [14], found non-translation invariant solutions. A first parameterization of these kink states is by the third component of the total spin, i.e., the eigenvalue $M$ of $\sum_{x=-L}^L S_x^3$, which commutes with the Hamiltonian. The possible values of $M$ are $-S(2L + 1), -S(2L + 1) + 1, \ldots, S(2L + 1)$. For each value of $M$ there is exactly one ground state $\psi_M$. It is useful to define a generating function, $\varphi(z), z \in \mathbb{C}$,

$$\varphi(z) = \sum_M a_M z^M \psi_M \tag{2.3}$$

where the $a_M$ are suitable constants. We refer to [13, 14] for more details. The important fact is that the $a_M$ can be chosen such that $\varphi(z)$ becomes a product:

$$\varphi(z) = \bigotimes_{x=-L}^L \chi(z q^{-x}) \tag{2.4}$$
with
\[ C^{2S+1} \ni \chi(z) = \sum_{m=-S}^{S} z^m q^{S-m} \sqrt{(2S)!/(S-m)!(S+m)!} |m\rangle\]
where \( S^3 |m\rangle = m |m\rangle \), and \( q \in (0, 1) \), such that \( \Delta = (q + q^{-1})/2 \). Note that, in the limit \( L \to \infty \), a translation by one lattice unit transforms \( \varphi(z) \) into \( \varphi(zq) \). Therefore, we can use \( \varphi(z) \) as a parameterization of the kink states by translations. It is then a trivial computation to get the magnetization profile in the states \( \varphi(z) \). E.g., for \( S = 1/2 \), one finds
\[
\omega_{\text{kink}}(S^3) = \frac{1}{2} \tanh((x - a)/\xi)
\]
\[
\omega_{\text{antikink}}(S^3) = -\frac{1}{2} \tanh((x - a)/\xi)
\]
where \( \xi \) and \( a \) are simple functions of \( q \) and \( z \).

The concept of zero-energy states plays an important role in the characterization of the complete set of solution of LS. So, we introduce it here, before we continue our discussion of the XXZ model.

Suppose there exists \( 0 \leq \tilde{h} \in A_{\Lambda_0} \), such that \( H_{\Lambda} + b_{\Lambda} = \tilde{H}_{\Lambda} = \sum_{x: \Lambda_0 + x \subset \Lambda} \tau_x(\tilde{h}) \)
Then, if a state \( \omega \) satisfies \( \omega(\tau_x(\tilde{h})) = 0 \), for all \( x \in \mathbb{Z} \), \( \omega \) is called a zero-energy state. It is easy to show that any zero-energy state satisfies LS.

Gottstein and Werner obtained the complete set of zero-energy states of the anisotropic XXZ chain \( \Delta > 1 \) with the particular boundary terms given in (2.2). They proved that any pure zero-energy state is either one of the two translation invariant ones, or it is a member of a set of mutually equivalent kink states or a set of mutually equivalent antikink states. A state \( \omega \) is called pure, iff for any two states \( \omega_1, \omega_2 \), and \( t \in (0, 1) \), one has
\[ \omega = t \omega_1 + (1-t) \omega_2 \Rightarrow \omega_1 = \omega_2 \]
It is obvious that the solution set of LS is convex, and one can prove it is a face. The same holds for the set of zero-energy states of a fixed \( \tilde{h} \). Therefore, finding the pure solutions is enough.

To which class \( \omega \) belongs is determined by the limits \( \alpha, \beta \in \{\pm S\} \), i.e., its asymptotic behavior:
\[
\alpha := \lim_{x \to -\infty} \omega(S^3), \quad \beta := \lim_{x \to +\infty} \omega(S^3)
\]
The following table summarizes the four types of zero-energy ground states as parametrized by \( \alpha \) and \( \beta \) (\( \Delta > 1 \)):

| \( \alpha \beta \) | type   | dominating configuration |
|------------------|--------|--------------------------|
| ++               | up     | \( \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \cdots \) |
| --               | down   | \( \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \cdots \) |
| --               | kink   | \( \cdots \downarrow \downarrow \downarrow \downarrow \cdots \uparrow \uparrow \uparrow \uparrow \uparrow \cdots \) |
| ++               | antikink | \( \cdots \uparrow \uparrow \uparrow \uparrow \cdots \downarrow \downarrow \downarrow \downarrow \downarrow \cdots \) |

We refer to [13, 14] for more details on the kink and antikink states. The case \( \Delta = 1 \) will be discussed further on.

Let us now return to the question of characterizing all solutions of Local Stability. The first result is due to Matsui [15].
Theorem 2.1 (Matsui, 1996) For the ferromagnetic XXZ chain, with $\Delta > 1$, and $S = 1/2$, the translation invariant states $\omega_\uparrow$ and $\omega_\downarrow$, together with the kink and antikink states described in [14] and discussed above, are the full set of pure solutions of LS.

Note that this theorem does not say anything about the isotropic model ($\Delta = 1$). The isotropic ferromagnetic chains have an infinite family of translation invariant ground states, given by the state $\omega_\uparrow$ and all rotations of it. As expected, this breaking of the rotation symmetry is accompanied by a gapless excitation spectrum. The proof of the above theorem relies on the existence of gap, and, therefore, does not extend to the isotropic case. Recently, Koma and Nachtergaele obtained a proof of the complete set of ground state for all values of $S$ and $\Delta \geq 1$ [16].

Theorem 2.2 (Koma and Nachtergaele, 1997) For the ferromagnetic XXZ chain, with $\Delta > 1$, and $S \geq 1/2$, the translation invariant states $\omega_\uparrow$ and $\omega_\downarrow$, together with the kink and antikink states generalized to arbitrary $S$, as in [13] exhaust the set of pure solutions of LS. If $\Delta = 1$, all solutions are translation invariant.

So, in the isotropic case there are no kink-type ground states. One expects that the same is true for models with a unique translation invariant (or periodic) ground state. This has not yet been proved in general, but Matsui [17] obtained a quite general result in the case of a unique zero-energy state, which we explain next.

Consider an arbitrary spin chain with a translation invariant nearest neighbor interaction, i.e., $h \in \mathcal{A}_{[0,1]}$ :

$$H_{[a,b]} = \sum_{x=a}^{b-1} \tau(h)$$

Assume that $h \geq 0$ and that there is a unique translation invariant state $\omega$ such that

$$\omega(h_{x,x+1}) = 0, \quad \text{for all } x \in \mathbb{Z}$$

One would expect that in this case $\omega$ is the unique solution of LS. Matsui's result, the only result so far, requires an additional assumption on the set of zero-energy states of the the half-infinite chains, i.e., states $\eta$ on $\mathcal{A}_{[1,\infty)}$ ($\mathcal{A}_{(-\infty,0]}$, respectively) such that

$$\eta(h_{x,x+1}) = 0, \quad \text{for all } x > 0 \quad (x < 0, \text{ resp.})$$

Theorem 2.3 (Matsui, 1997) For a spin chain as described above, if $\omega$ is the unique zero-energy ground state and if all zero-energy states of the left and right half-infinite chains are quasi-equivalent, then $\omega$ is the unique solution of LS.

In particular this implies that the AKLT spin 1 chain introduced in [8] has a unique ground state in the sense of LS.

In higher dimensions there also exist non-translation invariant ground states. In finite volume, this was already noted in [13]. Koma and Nachtergaele proved that ground states with a rigid interface in the diagonal (i.e., $11 \cdots 1$) direction exist in all dimensions. They also proved that the excitation spectrum above the ground state with a 11 interface in two dimensions is gapless [13]. Matsui generalized this to arbitrary dimensions $d \geq 2$ [19]. We now give an easy construction of a 111 interface state for the XXZ ferromagnet.

It follows from (2.4) that, for all $z \in \mathbb{C}$,

$$h \chi(z) \otimes \chi(zq^{-1}) = 0,$$

where

$$h := -(S^1 \otimes S^1 + S^2 \otimes S^2) - \Delta(S^2 \otimes S^3 - S^2) + B(\Delta)(S^3 \otimes 1 - 1 \otimes S^3)$$

(2.5)
is the positive definite interaction for which the kink states are the zero-energy ground states. Let $h_{x\rightarrow y}$ denote a copy of $h$ acting on the Hilbert space $\mathcal{H}_{\{x,y\}}$ of the sites $x$ and $y$ in this order. It follows immediately that, for any $\Lambda \subset \mathbb{Z}^d$, $x, y \in \Lambda$, such that $|x - y| = 1$, we have

$$h_{x\rightarrow y}\Omega(z) = 0$$

with

$$\Omega(z) := \bigotimes_{u \in \Lambda} \chi(zq^{-|u|})$$

Hence,

$$H_{\Lambda}\Omega(z) = \sum_{x, y \in \Lambda, |x| < |y|, |x - y| = 1} h_{x\rightarrow y}\Omega(z) = 0 \quad (2.6)$$

With the observation that every interior point in $\Lambda$ is the starting point ($x$) and the end point ($y$) of an equal number of bonds, it is easy to see that $H_{\Lambda}$ is the XXZ Hamiltonian with a boundary field. We conclude that $\Omega(z)$ is a zero-energy ground state for the XXZ model on $\mathbb{Z}^d$ for any $d$.

For concreteness, let us continue the discussion in the two-dimensional case. Generalization to arbitrary dimensions is straightforward. The two-dimensional system can be decomposed into a collection of coupled one-dimensional XXZ models defined the zig-zag chains shown as full lines in Figure 2.1. The bonds coupling two zig-zag chains together are indicated by dashed lines. As $\Omega(z)$ is a zero-energy ground state of the two-dimensional system, its restriction to a zig-zag line must be a zero-energy ground state of the one-dimensional XXZ model. Note that the boundary terms coincide with the ones used before to obtain the kink ground states if we interpret the orientations of the bonds in two dimensions as left-to-right in one dimension. Further inspection shows that the kink states on the zig-zag lines are aligned such that they are all centered on a line in the main diagonal direction (11) of the two-dimensional square lattice. Thus, $\Omega(z)$ describes a 11 · · · 1-interface.
Theorem 2.4 (Koma and Nachtergaele, unpublished; Matsui 1997) If \( d \geq 2 \), \( S \geq 1/2 \), and \( \Delta > 1 \), the family of product states \( \Omega(z), z \in \mathbb{C} \), normalized and extended to infinite volume, are all the pure zero-energy ground states of the interaction defined in (2.5). The excitation spectrum above these ground states is gapless.

We already noticed that \(|z|\) determines the position of the kinks, i.e., a change in \(|z|\) corresponds to a translation of the interface. It is not hard to see that the phase of \(z\) can be changed by applying another symmetry of the Hamiltonian:

\[
e^{i\theta S^3_{\text{tot}}} \Omega(z) = \Omega(e^{i\theta} z)\]

Rotations about the \(z\)-axis are commute with the Hamiltonian, but the states \(\Omega(z)\) are not invariant under it. Therefore, we have identified a broken continuous symmetry that goes along with the breaking of translation invariance in interface states. This is the basis of Matsui’s general proof of the existence of gapless excitations above these interface states of the XXZ model, along the lines of [20].

For sufficiently large \(\Delta\), and \(d \geq 3\), interface ground states (as well as low-temperature states) with the interface along the main coordinate directions (100), have also been shown to exist [21, 22]. In this case, however, there is a gap above the ground state. Another difference is that interfaces in the 100 directions can be understood as perturbations of the same interface states for the Ising model, which describes the limit \(\Delta \rightarrow \infty\). The Ising model does not have a rigid 111 interface. It is easy to see that the ground states with 111 boundary conditions is infinitely degenerate. See [23] for a discussion. For finite \(\Delta\), however, the \(XY\) term in the XXZ Hamiltonian produces a ground state selection effect, i.e., lifts the degeneracy.

3 A quantum SOS model

The next natural question concerns the existence of a rigid 111 interface at non-zero temperature. The structure of a 111 interface in a lattice model on \(\mathbb{Z}^3\) is already quite complicated. In the case of the XXZ model, the presence of gapless excitations in the interface ground states, is additional difficulty. In the past, interesting progress with interface problems has been made by introducing models where the interface is described by a height function. This eliminates most of the complicated geometry of the interfaces, which usually does not substantially affect the most relevant physical properties. Such models are called Solid-On-Solid (SOS) models. In this section, we introduce an quantum SOS (QSOS) model for the 111 interface of the anisotropic Heisenberg ferromagnet. As this model is a quantum system, the most straightforward and reliable way of obtaining an approximation is by restricting the Hilbert space to a suitable subspace. We would also like to retain as much as possible particular features, as, e.g., the gapless excitations of the interface.

The restricted Hilbert space for the QSOS model should at least contain the ground state interface, and capture the most important excitations. In analogy with the classical SOS models we would also like to have the property that each one-dimensional subsystem defined on a line in the 111 direction, i.e., perpendicular to the interface, is found in one of its ground states. In the classical case this would mean that any such line crosses the interface exactly once in any SOS configuration. This implies that the interface can be described by a height function. In our case the one-dimensional subsystems are naturally defined on the zig-zag lines drawn in full in Figure 2.1. This idea can easily be generalized to 3 and more dimensions. For each one-dimensional the natural Hilbert space is then given by

\[
\mathcal{H}_{Hz} = \text{closed linear span of } \{\varphi(z) \mid z \in \mathbb{C}\} \tag{3.1}
\]

By going back to the basis of kink states with fixed magnetization \(M\), one can easily show that this Hilbert space is also given by

\[
\mathcal{H}_{zz} = \text{closed linear span of } \{\varphi(q^n) \mid n \in \mathbb{Z}\} \cong l^2(\mathbb{Z})
\]
The states \( \varphi(q^n), n \in \mathbb{Z} \), are an non-orthogonal basis for \( H_{zz} \), and the index \( n \) can be interpreted as the height function, i.e., the position of the interface. There is a unitary operator \( S \) on \( H_{zz} \) such that
\[
S\varphi(q^n) = \varphi(q^{(n+1)})
\] (3.2)

The Hilbert space for the QSOS model is then a tensor product of copies of \( H_{zz} \), one for each zig-zag line:
\[
H_{QSOS} = \bigotimes_{u \in \Gamma} H_{zz,u}
\]
where \( u \in \Gamma \) label the zig-zag lines. In the case \( d = 2 \), \( \Gamma \) is a simple one-dimensional lattice. For \( d = 3 \), \( \Gamma \) is a two-dimensional triangular lattice.

The next step in the definition of the QSOS model is the calculation of the Hamiltonian. Let \( P_{QSOS} \) denote the orthogonal projection onto \( H_{QSOS} \) considered as a subspace of the original Hilbert space of the \( d \)-dimensional XXZ model. Then, the Hamiltonian of the QSOS model is given by
\[
H_{QSOS} = P_{QSOS} H_{XXZ} P_{QSOS}
\]

Some symmetry properties follow immediately from this definition. The first is invariance under a global shift of the interface. This is expressed by the fact that
\[
[H_{QSOS}, \bigotimes_{u \in \Gamma} S] = 0
\]
where \( S \) is the unitary operator defined by (3.2). Because of this invariance, one needs to consider suitable boundary conditions when taking the thermodynamic limit in the directions parallel to the interface. One can also implement this by defining the infinite-volume model on incomplete tensor product in the sense of Guichardet [24]. Furthermore, from the definition of \( H_{zz} \) it is clear that \( H_{QSOS} \) is invariant under global rotations about the \( z \)-axis. Therefore, the rotation invariance of \( H_{XXZ} \) is inherited by \( H_{QSOS} \). Finally, the translation invariance of the original Hamiltonian, which is also at the origin of the invariance under \( S \) (translations perpendicular to the interface), leads to periodicity or translation invariance of the QSOS model over \( \Gamma \).

There are different possibilities for the concrete representation of \( H_{QSOS} \). In order to represent this Hamiltonian as a matrix in an orthonormal basis one needs to correct for the non-orthogonality of the natural basis \( \{ \varphi(q^n) | n \in \mathbb{Z} \} \) of the \( H_{zz} \). Although there are several examples in the physics literature where in the definition of projected models non-orthogonality of a natural set of basis vectors is ignored, we think it is important to take it into account here. We will elaborate on this in a future publication. We conjecture that the QSOS model, with \( d \geq 3 \), has a rigid phase at sufficiently low temperatures. We will report on partial results on this problem in a future publication [25].

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