Analytic Origin of Advection of Magnetic Fields by Diffusivity Gradients

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Received ______________; accepted ______________
ABSTRACT

We derive the advection of magnetic fields due to gradients in magnetic diffusivity, starting from the magnetic induction equation. We discuss physical examples, and compare our results to those in the literature. Our induction and diffusion equations may be useful in solar dynamo studies, and for space and laboratory plasmas.

Subject headings: magnetic, advection, solar, diffusivity, dynamo
1. Introduction

Dynamic magnetic fields have complex and interesting relationships with a physical property of plasmas, the magnetic diffusivity, $\eta$, defined as the ratio of the electrical resistivity to the magnetic permeability. The usual form of the induction equation derived from Maxwell’s equations and Ohm’s law:

$$\frac{dB}{dt} = \nabla \times [(v \times B) - \eta \nabla \times B]$$

(1)

shows that both diffusivity $\eta$ and flows $v$ constrain the rate of change of magnetic fields $B$.

Magnetic diffusivity can be enhanced by plasma turbulence (Leighton 1964), and strong magnetic fields can suppress turbulence (Rüdiger & Kitchatinov 1993; Tobias 1996). Therefore magnetic fields can suppress (or “quench”) turbulence-enhanced diffusivity (Gilman & Rempel 2005), which can then permit fields to amplify locally, within limits. Nonlinear interactions between diffusivity quenching and magnetic fields evolution are being studied numerically in the context of the solar dynamo (Guerrero et al. 2009).

In this paper we are primarily interested in an analytic relationship between dynamic magnetic fields $B$ and $\eta$. We find that gradients in magnetic diffusivity $\eta$ can cause the magnetic fields $B$ to advect, adding a new flow term to the induction equation. Our new advection term can ultimately restructure both $\eta$ (which generally varies in space as well as in time) and the magnetic fields. Such processes may have important physical implications for magnetohydrodynamics in the convection zone and for the solar dynamo, among other systems.

Our original motivation in this inquiry was to better understand the fundamental source of the “diamagnetic pumping velocity,” $U^{\text{dia}} = f(\nabla \eta_T)$ described as driving magnetic advection in the context of the mean field dynamo equation

$$\frac{d\mathbf{B}}{dt} = \nabla \times \left[ (\mathbf{v} \times \mathbf{B}) + \alpha \cdot \mathbf{B} - \sqrt{\eta_T} \nabla \times (\sqrt{\eta_T} \mathbf{B}) \right]$$

(2)
where $\eta_T$ is the linear turbulent magnetic diffusivity (Rüdiger & Hollerbach 2004), Ch.4. In the process of investigating this relation, we rigorously derived expressions for magnetic field advection, as described below.

2. Derivation of magnetic advection from diffusivity gradients in the induction equation

We derive magnetic advection explicitly from diffusivity gradients, using the induction equation (1). In an effort to understand Rüdiger’s equation (2), we note the presence of $\sqrt{\eta}$ terms and seek to relate them directly to $\nabla \eta$. Since $\frac{d}{dx} \sqrt{\eta} = \frac{1}{2} \sqrt{\eta} \frac{d}{dx} \eta$, we can rewrite this as

$$\nabla \sqrt{\eta} = \frac{\nabla \eta}{2 \sqrt{\eta}},$$

(3)

assuming that diffusivity gradients are only in the radial direction. It will turn out to be useful to expand a term similar to the diffusivity term in the induction equation (1), using the result above and standard vector identities. Letting $C = \nabla \times B$, we find that

$$\nabla \times [\sqrt{\eta} \nabla \times B] = \nabla \times [\sqrt{\eta} C]$$

$$= \sqrt{\eta} \nabla \times C + \nabla \sqrt{\eta} \times C$$

$$= \sqrt{\eta} \nabla \times (\nabla \times B) + \nabla \sqrt{\eta} \times (\nabla \times B).$$

Using (3) in the last term, this becomes

$$\nabla \times [\sqrt{\eta} \nabla \times B] = \sqrt{\eta} \nabla \times (\nabla \times B) + \frac{\nabla \eta}{2 \sqrt{\eta}} \times (\nabla \times B).$$

(4)

Recalling the identity $\nabla \times \nabla \times B = -\nabla^2 B$, and multiplying by $\sqrt{\eta}$, we find

$$\sqrt{\eta} (\nabla \times [\sqrt{\eta} \nabla \times B]) = -\eta \nabla^2 B + \frac{1}{2} \nabla \eta \times (\nabla \times B).$$

(5)
The last term above can be rewritten using the vector identity

\[ \mathbf{D} \times (\nabla \times \mathbf{B}) = \nabla (\mathbf{D} \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{D}) - (\mathbf{D} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{D} \]  \hspace{1cm} (6)

and letting \( \mathbf{D} = \nabla \eta \):

\[ \nabla \eta \times (\nabla \times \mathbf{B}) = \nabla (\nabla \eta \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \nabla \eta) - (\nabla \eta \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \nabla \eta. \]  \hspace{1cm} (7)

Keeping only first-order differentials in \( \eta \), we have

\[ \nabla \eta \times (\nabla \times \mathbf{B}) = -(\nabla \eta \cdot \nabla) \mathbf{B}. \]  \hspace{1cm} (8)

Combining this result with (5) yields

\[ \sqrt{\eta} (\nabla \times [\sqrt{\eta} \nabla \times \mathbf{B}]) = -\eta \nabla^2 \mathbf{B} - \frac{1}{2} (\nabla \eta \cdot \nabla) \mathbf{B}. \]  \hspace{1cm} (9)

Similarly, we can rewrite the exact diffusivity term in the induction equation (1), using \( \mathbf{C} = \nabla \times \mathbf{B} \), the same vector identities, and (8):

\[ \nabla \times [\eta \nabla \times \mathbf{B}] = \eta \nabla \times (\nabla \times \mathbf{B}) + \nabla \eta \times (\nabla \times \mathbf{B}) \]

\[ = -\eta \nabla^2 \mathbf{B} - (\nabla \eta \cdot \nabla) \mathbf{B}. \]

Halving the last term in this result, we obtain two terms as in the right-hand side of (9):

\[ \nabla \times [\eta \nabla \times \mathbf{B}] = -\eta \nabla^2 \mathbf{B} - \frac{1}{2} (\nabla \eta \cdot \nabla) \mathbf{B} - \frac{1}{2} (\nabla \eta \cdot \nabla) \mathbf{B}. \]  \hspace{1cm} (10)

Comparing (9) and (10), we find

\[ \nabla \times [\eta \nabla \times \mathbf{B}] = \sqrt{\eta} [\nabla \times (\sqrt{\eta} \nabla \times \mathbf{B})] - \frac{1}{2} (\nabla \eta \cdot \nabla) \mathbf{B}. \]  \hspace{1cm} (11)

Using (3), we can rewrite

\[ \frac{1}{2} (\nabla \eta \cdot \nabla) = \nabla \sqrt{\eta} \cdot (\sqrt{\eta} \nabla). \]
Then (11) becomes
\[ \nabla \times [\eta \nabla \times B] = \sqrt{\eta} [\nabla \times (\sqrt{\eta} \nabla \times B)] - \nabla \sqrt{\eta} \cdot (\sqrt{\eta} \nabla B). \] (12)

A more enlightening form can be obtained by letting \( \nabla' = \sqrt{\eta} \nabla \), and \( \mathbf{U}_\eta \equiv \nabla \sqrt{\eta} : \)
\[ \nabla \times [\eta \nabla \times B] = \nabla' \times \nabla' \times B - (\mathbf{U}_\eta \cdot \nabla' B), \] (13)
where \( \mathbf{U}_\eta \) is the magnetic field advection velocity due to diffusivity gradients, and the gradient operator \( \nabla' \) is also rescaled by the square root of the diffusivity. Note that (13) has the form of a diffusion equation itself, with constant diffusion coefficient equal to 1, which may be particularly easy to simulate. Finally, the induction equation becomes
\[ \frac{d\mathbf{B}}{dt} = \nabla \times [(\mathbf{v} \times \mathbf{B}) - \eta \nabla \times \mathbf{B}] = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \eta \cdot (\sqrt{\eta} \nabla \mathbf{B}) - \sqrt{\eta} [\nabla \times (\sqrt{\eta} \nabla \times \mathbf{B})] \]
or
\[ \frac{d\mathbf{B}}{dt} = \nabla \times (\mathbf{v} \times \mathbf{B}) + (\mathbf{U}_\eta \cdot \nabla' \mathbf{B}) - \nabla' \times \nabla' \times \mathbf{B}. \] (14)

3. Physical considerations

It is well known that magnetic fields diffuse more readily into (or out of) regions of higher magnetic diffusivity (depending on relative magnetic field concentrations), and that they tend to "freeze" into regions of low diffusivity (high conductivity). The new "flow" term in our induction/diffusion equation, \( (\mathbf{U}_\eta \cdot \nabla' \mathbf{B}) \), takes the form of a convective derivative term, with field advection velocity \( \mathbf{U}_\eta \equiv \nabla \sqrt{\eta} \). The last term \(-\nabla' \times \nabla' \times \mathbf{B}\) in (14) is analogous to that in a heat flow equation; here diffusion depends on the \( \eta \)-scaled gradient,
Fig. 1.— Model diffusivity gradients in the solar convection zone. Parallel radial profiles used in solar dynamo simulations. Tachocline is 0.7 r/R, photosphere at 1.0 (r=R). Gradients are radially outward (to the right). Curve (a), dotted line: steep diffusivity gradient above the tachocline. Curve (d), solid line: steep diffusivity gradient below the tachocline.

\[ \nabla' = \sqrt{\eta} \nabla \] (Fig.1). In nonlinear dynamic systems, the relationship may not be so simple, as the diffusivity itself can vary in response to field changes. Whether fields are advected into or out of regions of changing magnetic diffusivity (or field strength) also depends on what other effects are operating (e.g. magnetic buoyancy, magnetic reconnection, etc.), and with what relative strengths.

For example, we have observed in flux-transport dynamo simulations that, when magnetic fields encounter steep diffusivity gradients leading into regions of very low diffusivity, e.g. near the tachocline, it is possible for these fields to get trapped in high-conductivity regions, where they are prevented from further participation in meridional circulation (Fig.2). In some cases, this trapping can quench the solar dynamo altogether.
Diffusion gradients due to magnetic reconnection have been shown to modify the effective convection of the magnetic field, in simulations of the photospheric meridional flow (Cohen et al. 2006). It has been shown (Boyd & Sanderson 2003) that gradients in resistivity (or diffusivity) can provide a driving force for resistive tearing modes in regions of antiparallel magnetic fields. This has been demonstrated in laboratory plasmas such as the Reversed-Field Pinch (Sarff et al. 2005), and inferred in solar and galactic dynamics (Parker 1970, 1971). Magnetic quenching of turbulence has also been studied in supernova remnants (Ziegler 1996).

4. Comparison to Mean Field Theory

In equation (2), as usual in mean field theory (MFT), one considers two components of the total magnetic field ($B = \overline{B} + b'$). The mean field $\overline{B}$ varies on length scales much longer than those of the field fluctuations $b'$. Similarly, the total velocity $v = \overline{v} + v'$ has mean ($\overline{v}$) and fluctuating ($v'$) components. The expansion of the general induction equation (1) in mean field theory

$$\frac{d}{dt}(\overline{B} + b') = \nabla \times \left[ (\overline{v} + v') \times (\overline{B} + b') - \eta \times \nabla \times (\overline{B} + b') \right]$$

is generally simplified to

$$\frac{d\overline{B}}{dt} = \nabla \times \left[ \overline{v} \times \overline{B} - \eta \times \nabla \times \overline{B} + \mathcal{E} \right]. \quad (15)$$

One general result of mean field theory is that the $\langle$ volume average $\rangle$ of the cross product between small fluctuations in flows and small fluctuations in fields can yield a mean electromotive force,

$$\mathcal{E} = \langle v' \times b' \rangle = \alpha \cdot \overline{B} + \gamma \times \overline{B}, \quad (16)$$
even if the local volume average of each fluctuating term vanishes. These can produce significant emfs $\mathcal{E}$, which can create and/or sustain mean magnetic fields. For example, in toroidal Reversed Field Pinch (RFP) plasma experiments (Sarff et al. 2005), the fluctuations $< v_r b_z - v_z b_r >$ create an $\mathcal{E}_\theta$ which supports the mean axial $B_z$ well beyond the predicted lifetimes of these high magnetic energy density devices, which operate outside the Kruskal-Shafranov limit and tend to spontaneously relax to a dynamic force-free state (Taylor 1974, 1986). The $\alpha$ and $\gamma$ terms in (16) are an alternate way of looking at the mean-field emf, using the diagonal and off-diagonal terms of the turbulent velocity tensor, respectively (Krause & Rädler 1980). To complicate matters, Ziegler (1996) notes that each component of these tensors should have its own quenching factor (Ferriere & Schmidt 2000, p.139).

Returning to our original inquiry: can our derivation help us understand the analytic origin of equation (2)? The diamagnetic pumping velocity in that relation, which “will dominate the transport of mean magnetic fields” (p.132), is proportional to diffusivity gradients in the limiting cases of both weak and strong fields (p.114): $U^{\text{dia}} = -\frac{1}{2} \nabla \eta_T$ when $\mathbf{B} = 0$, and $U^{\text{dia}} = -\Phi^{\text{dia}}(\Omega, B) \nabla \eta$ for strong $\mathbf{B}$, where $\Phi^{\text{dia}}(\Omega, B) > 0$ is a function of rotation and field strength. This differs in general from our advection velocity $U_\eta \equiv \nabla \sqrt{\eta}$.

Though Rüdiger’s MFT equation (2)

$$\frac{d\mathbf{B}}{dt} = \nabla \times \left[ \mathbf{\nabla} \times \mathbf{B} + \alpha \cdot \mathbf{B} - \sqrt{\eta_T} \nabla \times \left( \sqrt{\eta_T} \mathbf{B} \right) \right]$$

appears similar to our induction equation

$$\frac{dB}{dt} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \eta \cdot (\sqrt{\eta} \nabla \mathbf{B}) - \sqrt{\eta} \left[ \nabla \times (\sqrt{\eta} \nabla \times \mathbf{B}) \right],$$

in fact these are quite independent. The former is from MFT and the latter is derived from MHD. The advection velocity is field-dependent and linear with diffusivity gradients.
in MFT; while the advection velocity is field-independent and nonlinear with $\nabla \eta$ in MHD. Though we have not met our original goal, we have answered a potentially interesting new question.

5. Summary and Future Work

A magnetohydrodynamic analysis motivated by a question from mean field theory has produced a new magnetic field diffusion equation (13) and a new form of the induction equation (14). These specify the role of magnetic diffusivity in changing magnetic fields. Magnetic field advection by gradients in turbulent diffusivity is relevant to solar dynamo theory, and may be fruitfully applied to space and laboratory plasmas.

6. Acknowledgments

Zita is grateful to Tom Bogdan for insightful discussions of the mathematical derivation in Section 2, and to Mausumi Dikpati and Gustavo Guerrero for a discussion about mean-field dynamo theory.

This work was supported by NSF grant 0807651, and partly by the Sponsored Research Program of The Evergreen State College.
Fig. 2.— Results of dynamo simulation using profile in Fig.1a, a steep diffusivity gradient above the tachocline. Normal butterfly diagram (left) and normal toroidal flux patterns in convection zone (right) indicate normal solar dynamo. (b) Results of simulation using profile in Fig.1d, a steep diffusivity gradient below the tachocline. Pathological butterfly diagram (left) and trapped toroidal flux near the tachocline (right) indicate abnormal dynamo.
REFERENCES

Boyd, T.J.M., & Sanderson, J.J. 2003 The Physics of Plasmas, Cambridge, 150

Cohen, O., Fisk, L. A., Roussev, I. I., Toth, G., Gombosi, T. I. 2006, ApJ, 645, 1537 (doi:10.1086/504402)

Ferriere K. & Schmitt D. 2000, A&A, 358, 125 (http://adsabs.harvard.edu/abs/2000A&A...358..125F)

Gilman, P.A. & Rempel M. 2005, ApJ, 630, 615 (doi:10.1086/431929)

Guerrero, G., Dikpati, M., dal Pino, E.M. 2009, ApJ, 701,725 (doi:10.1088/0004-637X/701/1/725)

Krause, F. & Rädler, K.H. 1980 Mean-Field Magnetohydrodynamics and Dynamo Theory, Pergamon: Oxford

Leighton, R.B. 1964, ApJ, 140, 1547 (doi:10.1086/148058)

Parker, E.N. 1970, ApJ, 162, 665 (doi:10.1086/150697)

Parker, E.N. 1971, ApJ, 163, 255 (doi:10.1086/150765)

Rüdiger, G. & Kitchatinov, L.L. 1993, A&A, 269, 581 (http://adsabs.harvard.edu/abs/1993A&A...269..581R)

Rüdiger, G. & Hollerbach, R. 2004, The Magnetic Universe

Sarff, J., Almagri, A., Anderson, J., Brower, D., Craig, D., Deng, B., den Hartog, D., Ding, W., Fiksel, G., Forest, C., Mirnov, V., Prager, S., Svidzinski, V. 2005 “Magnetic reconnection and self-organization in Reversed Field
Pinch laboratory plasmas”, in The Magnetized Plasma in Galaxy Evolution, ed. K.T.Chyzy, K.Otmianowska-Mazur, M.Soida & R.-J.Dettmar, 48-55, (http://adsabs.harvard.edu/abs/2005mpge.conf...48S)

Taylor, J.B. 1986 Phys. Rev. Lett., 33, 1139 (doi:10.1103/PhysRevLett.33.1139)

Taylor, J.B. 1986 Rev.Mod.Phys., 58, 741 (doi:10.1103/RevModPhys.58.741)

Tobias, S.M., 2006, ApJ, 467, 870 (doi:10.1086/177661)

Ziegler, U., 1996, A&A, 313, 448 (1996A &A...313..448Z)

Zita, E.J., 2010 astro-ph (2010arXiv1009.5965Z)

This manuscript was prepared with the AAS \LaTeX macros v5.2.
Btor: $\iota = 2$, $dp = 0.30$, $pc = 0.70$, $\maxbt = 98$
