Initial value problem of heat flow in non-homogeneous materials

G Britto Antony Xavier¹, S John Borg² and S Jaraldpushparaj³

¹, ², ³Department of Mathematics, Sacred Heart College, Tirupattur, Tamil Nadu, S.India.
E-mail: ¹email: brittoshc@gmail.com

Abstract. The scope of this paper is to analyse transfer of heat in a rod made of multiple materials stacked together using partial difference equation. The methodology of induction is employed to guide us reach the destination. With Newton’s law of cooling as the basis, the equation for heat transfer of the rod made of four different materials is formulated as the preliminary case. The solution arrived at for the above problem is generalized for the case of the rod with multiple materials. The results put forth in this research work are validated by numerical examples.

Keywords: Discrete heat equation, Initial value problem and Partial difference equation.
AMS Subject classification: 80A20, 34A12, 39A70, 39A10.

1. Introduction
The heat diffusion problems arise in numerous applications of Science and Engineering [7, 11]. This study of heat transfer occupy a unique position in modern mathematical physics. The research regarding this had induced the mathematical formulation of many other physical processes in terms of diffusion. The credit go to Fourier for making an innovative attempt to make in roads into this field in 1807. He formulated heat conduction in terms of partial differential equation [5]. In the last two decades this problem had attracted great attention and numerous paper. The authors here in this work try to approach this problem of heat transfer from the background of difference equations and its operators.

The field of difference equation came in to existence with the initiation of the difference operator $\Delta$ in 1984 by Jerzy Popenda [6]. This was further enriched by Miller and Rose by the introduction of the Riemann-Liouville fractional derivative termed as inverse fractional difference operator $\Delta^{-1}_h$ [9]. This operator helps in drawing the higher partial sums on arithmetic, geometric progression and products of n-consecutive terms of arithmetic progression [10].

A better progression in this field was given by Maria Susai Manuel et.al., in 2011 with the extension of $\Delta$ to $\alpha$-difference operator $\Delta_{\alpha}$, where $u(k + \ell) - \alpha u(k) = \Delta_{\alpha} u(k), \ell > 0$. The developments in this research area were further complemented by the works of Britto Antony Xavier et.al., [1, 2]. The initial study of finding the solutions of heat flow in a homogeneous material was achieved in the literature [3, 4, 13, 12]. This work focuses on the more complex field of heat flow in a non-homogeneous materials of a rod made of multiple materials stacked together. Using the method of induction, the solutions are derived.
2. Finding heat conductivity of a rod made of four materials

Let \( v(k_1, k_2) \) be the temperature at position \( k_1 \) and at time \( k_2 \) of a rod made of non-homogenous materials. Let \( \ell_1, \ell_2 \) be the shift values of \( k_1 \) and \( k_2 \) respectively. Let \( v(j_1, j_2) \) be the temperature at initial position \( j_1 \) and at initial time \( j_2 \) and \( \alpha(k_1) \) be the heat conductivity of rod at \( k_1 \).

By Newton’s law of cooling, the partial difference equation of the heat flow in the rod made of four material is controlled by the equation

\[
v(k_1, k_2) = \sum_{r=-2}^{1} \alpha(k_1 + r\ell_1)v(k_1 + r\ell_1, k_2 - \ell_2). \tag{1}
\]

By Replacing \( k_2 \) by \( k_2 - q\ell_2 \), \( q = 1, 2, ..., n \) in (1) and again substituting these in (1), we get

\[
v(k_1, k_2) = \alpha^n(k_1)v(k_1, k_2 - n\ell_2) + \sum_{q=1}^{n} \sum_{r=-2}^{1} \alpha^{q-1}(k_1)\alpha(k_1 + r\ell_1)v(k_1 + r\ell_1, k_2 - q\ell_2). \tag{2}
\]

Taking \( k_2 - n\ell_2 = j_2 \) and then \( n - s = q \) the equation (2) becomes

\[
v(k_1, j_2 + n\ell_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{s=0}^{n-1} \sum_{r=-2}^{1} \alpha^{n-1-s}(k_1)\alpha(k_1 + r\ell_1)v(k_1 + r\ell_1, j_2 + s\ell_2),
\]

which can be also expressed as

\[
v(k_1, k_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{s=1}^{n} \sum_{r=-2}^{1} \alpha^{n-s}(k_1)\alpha(k_1 + r\ell_1)v(k_1 + r\ell_1, j_2 + (s - 1)\ell_2). \tag{3}
\]

The values of heat conductivities \( \alpha(k_1 + r\ell_1) \) for \( r = -2, -1, 0 \) and 1 are obtained by solving four linear equations obtained by applying \( n = 1, 2, 3, 4 \) in (3).

3. Illustration for finding heat conductivities

To illustrate the method of finding rates of heat conductivity namely \( \alpha(k_1), \alpha(k_1 + \ell_1), \alpha(k_1 - \ell_1) \), we assume the following initial values

\[
\begin{align*}
\alpha(k_1, j_2) &= 2, \quad \alpha(k_1 + \ell_1, j_2) = 1, \quad \alpha(k_1 - \ell_1, j_2) = 2, \quad \alpha(k_1 - 2\ell_1, j_2) = 4, \quad \alpha(k_1 + \ell_1, j_2 + \ell_2) = 3, \\
\alpha(k_1 - \ell_1, j_2 + \ell_2) &= 2, \quad \alpha(k_1, j_2 + \ell_2) = 4, \quad \alpha(k_1 - 2\ell_1, j_2 + \ell_2) = 4, \quad \alpha(k_1 + \ell_1, j_2 + 2\ell_2) = 4, \\
\alpha(k_1 - \ell_1, j_2 + 2\ell_2) &= 3, \quad \alpha(k_1, j_2 + 2\ell_2) = 5, \quad \alpha(k_1 - 2\ell_1, j_2 + 2\ell_2) = 6, \quad \alpha(k_1 + \ell_1, j_2 + 3\ell_2) = 5, \\
\alpha(k_1 - \ell_1, j_2 + 3\ell_2) &= 6, \quad \alpha(k_1, j_2 + 3\ell_2) = 6, \quad \alpha(k_1 - 2\ell_1, j_2 + 3\ell_2) = 5.
\end{align*}
\]

Since the rod is made of four material consecutively, we assume that, for \( t \in Z \),

\[
\alpha(k_1 + (4t + 0)\ell_1) = x_0, \quad \alpha(k_1 + (4t + 1)\ell_1) = x_1, \quad \alpha(k_1 + (4t - 1)\ell_1) = x_2, \quad \alpha(k_1 + (4t - 2)\ell_1) = x_3.
\]

Taking \( n = 1, 2, 3, 4 \) in (3) and substituting the above values in the corresponding equations, we arrive at a system of linear equations as given below:

(i) \( 2x_0 + x_1 + 2x_2 + 4x_3 = 4 \), (ii) \( 4x_0 + 3x_1 + 2x_2 + 4x_3 = 5 \),

(iii) \( 5x_0 + 4x_1 + 3x_2 + 6x_3 = 6 \) and (iv) \( 6x_0 + 5x_1 + 4x_2 + 5x_3 = 7 \).

Solving (i), (ii), (iii) and (iv) gives \( x_1 = -2, \ x_3 = 0, \ x_0 = \frac{5}{2}, \ x_2 = \frac{1}{2} \).

4. Solution of IVP of heat flow in rod made of four materials

To illustrate the method for finding the solution of IVP of heat flow in a rod made of four materials assume that the initial temperatures \( v(j_1, j_2), \ v(j_1 + \ell_1, j_2), \ v(j_1 + 2\ell_1, j_2), \ v(j_1 + 3\ell_1, j_2),..., v(j_1 + p\ell_1, j_2) \) are known and each value is equal to \( 2 \). Using the method given in section (3), after finding the heat conductivity rates \( \alpha(k_1 + r\ell_1) \) for \( r = -2, -1, 0 \) and
and by taking \( n = 5 \) in (3), we arrive
\[
v(j_1, j_2 + 5\ell_2) = x_0^2(2) + x_0^3x_2(2) + x_0^3[x_1(2) + x_2(2) + x_3(2)] \\
+ x_0^3[x_1(2) + x_2(2) + x_3(2)] + x_0[x_1(2) + x_2(2) + x_3(2)] + x_1(2) + x_2(2) + x_3(2)
\]
\[
= \frac{3125}{16} - \frac{625}{4} + \frac{625}{16} + 0 + \frac{125}{8}(-4 + 1) + \frac{25}{3}(-3) + \frac{5}{2}(-3) - 4 + 1 = 2.
\]

Similarly, we are able to find the value of \( v(j_1 + p\ell_1, j_2 + q\ell_2) \) by measuring the initial temperature at all positions. Thus, we obtain the solution of the heat equation of the rod made of four materials as
\[
v(j_1 + p\ell_1, j_2 + q\ell_2) = v(k_1, k_2) = \alpha^q(j_1 + p\ell_1)v(j_1 + p\ell_1, j_2)
\]
\[
+ \sum_{n=0}^{q-1} \alpha^n(j_1 + p\ell_1)\left[\alpha(j_1 + (p + 1)\ell_1)v(j_1 + (p + 1)\ell_1, j_2 - (n - (q - 1))\ell_2)
\right]
\]
\[
+ \alpha(j_1 + (p - 1)\ell_1)v(j_1 + (p - 1)\ell_1, j_2 - (n - (q - 1))\ell_2)
\]
\[
+ \alpha(j_1 + (p - 2)\ell_1)v(j_1 + (p - 2)\ell_1, j_2 - (n - (q - 1))\ell_2)\]
\[
\text{where } p \text{ and } q \geq 0.
\]

The relation (4) gives the temperature of the rod at positions and at all times by varying the values \( p \) and \( q \).

5. Solution of IVP of rod made of multiple materials

Consider a rod of non homogenous materials with parameter \( k_1, k_2, \ell_1 \) and \( \ell_2 \), where \( k_1 \) be the position of rod, \( \ell_1 \) be the shift value of \( k_1 \), \( k_2 \) denotes the time and \( \ell_2 \) be the shift value of \( k_2 \). Let \( v(j_1, j_2) \) be the temperature at initial position \( j_1 \), at initial time \( j_2 \), and \( \alpha(k_1) \) denotes the heat conductivity rate of rod at \( k_1 \).

Since the rod is made of non-homogeneous material, the corresponding partial difference equation of heat flow can be expressed as
\[
v(k_1, k_2) = \sum_{i \in S} \alpha(k_1 - i\ell_1)v(k_1 - i\ell_1, k_2 - \ell_2),
\]
(5)

where \( S = S_e = \{-m, -m + 1, \ldots, 0, 1, 2, 3, \ldots, m - 1\} \) for the rod made of \( 2m \) materials or \( S = S_o = \{-m, -m + 1, \ldots, 0, 1, 2, 3, \ldots, m\} \) for the rod made of \( 2m + 1 \) materials.

By replacing \( k_2 \) by \( k_2 - \ell_2, k_2 - 2\ell_2, \ldots, k_2 - n\ell_2 \) and applying these values in (5), we obtain
\[
v(k_1, k_2) = \alpha^n(k_1)v(k_1, k_2 - n\ell_2) + \sum_{q=1}^{n} \sum_{i \in S - \{0\}} \alpha^{q-1}\alpha(k_1 + i\ell_1)v(k_1 + i\ell_1, k_2 - q\ell_2).
\]
(6)

By taking \( k_2 - n\ell_2 = j_2 \) in (6), we arrive at
\[
v(k_1, k_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{q=1}^{n} \sum_{i \in S - \{0\}} \alpha^{q-1}\alpha(k_1 + i\ell_1)v(k_1 + i\ell_1, j_2 + (n - q)\ell_2).
\]

Taking \( n - s = q \) yields
\[
v(k_1, j_2 + n\ell_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{s=1}^{n} \sum_{i \in S - \{0\}} \alpha^{n-s}\alpha(k_1 + i\ell_1)v(k_1 + i\ell_1, j_2 + (s - 1)\ell_2).
\]

Considering the experimental values for the given period, the heat conductivity rates can be obtained by using the following equation:

Case (i). When the rod is stacked with even numbers of materials, the solution of (5) is
\[
v(j_1 + p\ell_1, j_2 + q\ell_2) = \alpha^q(j_1 + p\ell_1)v(j_1 + p\ell_1, j_2)
\]
\[
+ \sum_{n=0}^{q-1} \sum_{r \in S_e \atop r \neq 0} \alpha^n(j_1 + p\ell_1)\alpha(j_1 + (p + r)\ell_1)v(j_1 + (p + r)\ell_1, j_2 - (n - (q - 1))\ell_2).
\]
Case (ii). When the rod is stacked with odd number of materials, the solution of (5) is
\[ v(j + p\ell_1, j_2 + q\ell_2) = \alpha^q(j + p\ell_1)v(j_1 + p\ell_1, j_2) + \sum_{n=0}^{q-1} \sum_{r \in \mathbb{Z} - \{0\}} \alpha^n(j_1 + p\ell_1)\alpha(j_1 + (p + r)\ell_1)v(j_1 + (p + r)\ell_1, j_2 - (n - (q - 1))\ell_2). \]

Case (iii). When the rod consists of infinitely many materials arranged consequently then the solution of (5) is
\[ v(j + p\ell_1, j_2 + q\ell_2) = \alpha^q(j + p\ell_1)v(j_1 + p\ell_1, j_2) + \sum_{n=0}^{q-1} \sum_{r \in \mathbb{Z} - \{0\}} \alpha^n(j_1 + p\ell_1)\alpha(j_1 + (p + r)\ell_1)v(j_1 + (p + r)\ell_1, j_2 - (n - (q - 1))\ell_2). \]

By taking \( k_1 = j_1 + p\ell_1 \) and \( k_2 = j_2 + q\ell_2 \), the temperature \( v(k_1, k_2) \) throughout the rod is obtained by varying the values of \( p \) and \( q \).

**Remark 5.1** The numerical solution for rod made of multiple materials as mentioned in (5) can be found by applying the method explained in the sections (3) and (4).

6. Conclusion

This paper presents an innovative approach to study the flow of heat in a long rod made of multiple materials stacked together using partial difference equations. The method presented here is very convenient for solving the heat equation and determining the temperature for all periods by having the knowledge of initial temperatures. Additionally, the method also provides an useful tool to measure rate of heat conductivity. The proposed method is efficient and thus can be used to solve the thermal conductivity problems of composite materials.

**Compliance with ethical standards**

**Conflict of interest** No potential conflict of interest from the authors.

**References**

[1] Agarwal R P 2000 Difference Equations and Inequalities Theory Methods and Applications 2/e Revised and Expanded Marcel Dekker New York
[2] Britto Antony Xavier G, John Borg S and Meganathan M 2017 Discrete heat equation model with shift values Applied Mathematics 8 pp 1343-1350.
[3] Britto Antony Xavier G, John Borg S and Jaraldpushparaj S 2018 Discrete Heat Equation for Thin Plate and Medium by Second Order Fibonacci Difference Operator Online International Interdisciplinary Research Journal 08(04) pp 15-21
[4] Britto Antony Xavier G, John Borg S and Jaraldpushparaj S 2018 Discrete heat equation model for Rod by Partial Fibonacci Difference operator, Mathematical sciences International Research Journal 7(1) pp 188-191
[5] Fourier J B J 1807 Théorie de la propagation de la chaleur dans les solides Unpublished manuscript submitted to Institut de France
[6] Jerzy Popenda and Blażej Szmanda 1984 On the Oscillation of Solutions of Certain Difference Equations Demonstratio Mathematica XVII(1) pp 153-164
[7] Khosro Sayevand 2014 Convergence and stability analysis of modified backward time centered space approach for non-dimensionalizing parabolic equation J. Nonlinear Sci. Appl. 7 pp 11-17
[8] Maria Susai Manuel M, Chandrasekar V, Britto Antony Xavier G 2011 Solutions and Applications of Certain Class of \( \alpha \)-Difference Equations, International Journal of Applied Mathematics 24(6) pp 943-954
[9] Miller K S and Rose B 1989 Fractional Difference Calculus in Univalent Functions Horwood, Chichester, UK pp 129-152
[10] Maria Susai Manuel M, Chandrasekar V and Britto Antony Xavier G 2012 Some Applications of the Generalized Difference Operator of the \( n \)th Kind, Far East Journal of Applied Mathematics 66(2) 107 - 126
[11] Ramandeep Kaur 2013 Numerical Solutions and Stability of Some Partial Differential Equations Using Finite Difference Methods Roll No: 301103015 MSc Thesis, Thapar University, Patiala-147004 (Punjab) INDIA.
[12] Xavier G B A, Borg S J and Meganthan M 2018 Discrete Heat equation with shift values, S. Pinelas et.al (eds),
Differential and Difference equations with application, Springer proceedings in Mathematics and Statistics 230, Springer International Publishing 2, 13-19, https://doi.org/10.1007/978-3-319-75647-9

[13] Xavier G B A, Borg S J, Govindan B, et.al, 2018 Journal of Analysis, Springer Singapore, https://doi.org/10.1007/s41478-018-0113-6