On the Frame-Stewart Conjecture

Youngjin Bae

Abstract

The Frame-Stewart conjecture states the least number of moves to solve a generalized Tower of Hanoi problem, of n disks and p pegs. In this paper, we prove several weaker versions of the Frame-Stewart conjecture.

Contents

1 Introduction 2
2 The Frame-Stewart Conjecture 4
3 Preliminary Facts 4
4 Main Results 6
1 Introduction

The generalized Tower of Hanoi problem can be formally stated as following.

**Definition 1.1** Let $n$ and $p \geq 3$ be natural numbers. Then a generalized Tower of Hanoi problem is a problem of moving $n$ ordered disks (we may number those disks from 1 to $n$) from an initial peg to another one, satisfying following conditions:

1. No larger disk can be on top of a smaller one
2. A disk can be moved from one peg to another peg only when no other disks are on top of it.

We simply call a generalized Tower of Hanoi problem with $n$ disks and $p$ pegs as $(n, p)$-problem.

**Definition 1.2** For $n$ and $p \geq 3$, $M(n, p)$ is the least number of moves needed to solve $(n, p)$-problem.

**Theorem 1.1 (The original Tower of Hanoi problem)** For natural number $n$,

$$M(n, 3) = 2^n - 1$$

**Theorem 1.2 (A.A.K.Majumdar)** For $n, p$, there exist an unique natural number $r$ satisfying

$$\binom{p + r - 3}{p - 2} \leq n < \binom{p + r - 2}{p - 2}$$

and

$$M(n, p) \leq \sum_{t=0}^{r-1} 2^t \binom{p + t - 3}{p - 3} + 2^r (n - \binom{p + r - 3}{p - 2})$$

**Proof.** (See [2], for example) The existence of $r$ is clear. Define $K(n, p) = \sum_{t=0}^{r-1} 2^t \binom{p + t - 3}{p - 3} + 2^r (n - \binom{p + r - 3}{p - 2})$. We will prove the inequality by explicitly...
showing that it is possible to solve the \((n,p) - problem\) with exactly \(K(n,p)\) times of move. \((1)\)

We use induction on \(p\) and then on \(n\). First, for \(p = 3\), we have \(M(n,3) = 2^n - 1 = K(n,3)\) for all \(n\). Assume that it is possible to solve the \((n,p) - problem\) with \(K(n,p)\) times of move for \(3 \leq p \leq q - 1\). For \(p = q\), we use induction on \(n\). For \(n = 1\), \(K(1,q) = 1\) and it is indeed possible to move a single disk with 1 move. Assume that (1) holds for \(n \leq m - 1\). For \(n = m\), let \(m = \binom{q+r-3}{q-3} + \alpha\) where \(0 \leq \alpha < \binom{q+r-3}{q-3}\). Since \(\binom{q+r-3}{q-3} = \binom{q+r-4}{q-4} + \binom{q+r-4}{q-3}\), there are natural numbers \(\beta, \gamma\) such that \(\alpha = \beta + \gamma\) and \(\beta < \binom{q+r-4}{q-3}\) and \(\gamma < \binom{q+r-4}{q-4}\). We call the peg on where every disks are at the beginning as initial peg (\(I\) for short) and the peg on where every disks are at the end as final peg (\(F\) for short). Also, since \(p \geq 3\), we can pick a peg different from \(I\) and \(F\) and call it middle peg (\(M\) for short). Note that \(m - k = \binom{q+r-4}{q-3} + \gamma\). Then, we move \(m\) pegs through the following process:

1. Move disks 1 to \(k\) from \(I\) to \(M\) with \(K(k,q)\) moves.
2. Move disks \(k+1\) to \(m\) from \(I\) to \(F\) with \(K(m-k,q-1)\) moves. (Note that we do not use the peg \(M\) here.)
3. Move disks 1 to \(k\) from \(M\) to \(F\) with \(K(k,q)\) moves.

So far, we have moved the \(m\) disks with \(2K(k,q) + K(m-k,q-1)\) moves. Now it is enough to check that

\[
K(m,q) = 2K(k,q) + K(m-k,q-1)
\]

This can be shown by calculation:

We have

\[
K(k,q) = \sum_{t=0}^{r-2} 2^t \binom{q+t-3}{q-3} + 2^{r-1} \beta
\]
and

\[ K(m - k, q - 1) = \sum_{t=0}^{r-1} 2^t \left( \frac{q + t - 4}{q - 4} \right) + 2^r \gamma \]

Thus,

\[ 2K(k, q) + K(m - k, q - 1) = \sum_{t=0}^{r-2} 2^{t+1} \left( \frac{q + t - 3}{q - 3} \right) + 2^{r-1} + \sum_{t=0}^{r-1} 2^t \left( \frac{q + t - 4}{q - 4} \right) + 2^r (\beta + \gamma) \]

\[ = \sum_{t=0}^{r-1} 2^t \left( \frac{q + t - 4}{q - 3} \right) + \left( \frac{q + t - 4}{q - 4} \right) + 2^r \alpha = K(m, q) \]

Which finishes the proof. Note that the proof works for every possible \( \beta \) and \( \gamma \) satisfying the conditions, which implies that the minimal solution might not be unique.

2 The Frame-Stewart Conjecture

The Frame-Stewart Conjecture states that the DP-algorithm in the proof of previous theorem is actually optimal and thus \( M(n, p) = K(n, p) \).

Conjecture 2.1 (Frame-Stewart Conjecture) For \( n, p \), \( M(n, p) = K(n, p) \) holds.

The conjecture indeed holds for \( p = 3 \).

3 Preliminary Facts

For natural number \( x \), we define \( \bar{x} := \{1, 2, ..., x\} \).

Definition 3.1 Given \( n, p \), a state of the \((n, p)\) – problem (\((n, p)\) – state in short) is \( n \) disks being allocated on \( p \) pegs. Formally, a state is equivalent to a function \( f : \bar{n} \to \bar{p} \) We define the set of all states of the \((n, p)\) – problem as

\[ X(n, p) := \{ f : \bar{n} \to \bar{p} \} \]
Definition 3.2 Given \( n,p \) and two states \( f,g \) of the \((n,p)\) problem, a path connecting \( f \) and \( g \) is a finite sequence of \((n,p)\) states such that the initial term of the sequence is \( f \) and the final term is \( g \). If \( P = \{ P_0 = f, P_1, \ldots, P_k = g \} \) is a path connecting (between) \( f \) and \( g \), define length of the path as \( |P| := k \).

Definition 3.3 Let \( f,g \) be \((n,p)\) states. Define \( P(f,g) \) as the set of all paths connecting \( f \) and \( g \). A path between \( f \) and \( g \) is a shortest path if its length is minimal among \( P(f,g) \). A length 1 path is called move from \( f \) to \( g \). We formally write a shortest path between \( f \) and \( g \) as \( f - g \). It is obvious that \( |f - g| = |g - f| \) for any given \( f \) and \( g \). Note that shortest path between \( f \) and \( g \) may not unique and \( f - g \) is not well-defined. Still, \( |f - g| \) is well-defined.

Definition 3.4 Let \( f,g \) be \((n,p)\) states and \( \psi \) be a path between \( f \) and \( g \). If \( X \) is a subset of \( \bar{n} \), we define \( |\psi|_X \) be the number of moves of disks in \( X \) while \( \psi \).

Example 3.1 \((n,p)\) problem can be demonstrated as finding shortest path between two distinct constant states (i.e. constant function) \( f \) and \( g \).

We introduce a notation by Roberto Demontis and a notion of demolishing sequence. The triple \((j,i,t)\) with \( 1 \leq j < i \leq \infty \) and \( j < t \leq \infty \), denotes that the disk \( j \) moves from being on the disk \( i \) to be placed on the disk \( t \). We write \( i = \infty \) when there was no disk under \( j \) before it moves onto \( t \). Similarly, we write \( t = \infty \) when disk \( j \) moves to an empty peg.

Definition 3.5 A path \( P \) between \( f \) and \( g \) is said to be demolishing sequence if

1. \( f \) is a constant state
2. The last move of \( P \) is \((n,\infty,\infty)\)
3. The move \((n,\infty,\infty)\) appears exactly once in \( P \).

We call the final state of a minimal demolishing sequence as middle state.

Definition 3.6 Let \( P \) and \( Q \) be sequences satisfying \( P|_P = Q_0 \). Define \( P + Q \) be a sequence concatenate \( P \) and \( Q \). \( |P + Q| = |P| + |Q| \) holds.
Theorem 3.1 (Roberto Demontis)  Given $f \equiv I_f, g \equiv I_g$ be two distinct constant states and $S := f - g$. Assume $f - h$ be a minimal demolishing sequence of moves. Then, $|S| = 2|f - h| + 1$ holds.

Proof Since $f$ and $g$ are two distinct constant states, there must be at least one $(n, \infty, \infty)$ move in $S$, which we will call $\psi$. Let $P$ be a subsequence of $S$ from the beginning to the last move before $\psi$. Then, $S = P + \psi + Q$. Define $P^r$ be a sequence which is reverse of $P$ but $I_f$ and $I_g$ are switched. If $|P| > |Q|$, we have $|S| = |P + \psi + Q| < |P + \psi + P^r|$, which contradicts to the minimality of $S$. Similarly, if $|P| < |Q|$, we have $|S| = |P + \psi + Q| < |Q^r + \psi + Q|$, also contradiction. Thus, we have $|P| = |Q|$ and $|S| = 2|Q| + 1 = 2|f - h| + 1$ since both $Q$ and $f - h$ are minimal demolishing sequences.

By the theorem above, it is enough to find minimal demolishing sequence instead of the whole $(n,p) - problem$.

Theorem 3.2 (Roberto Demontis)  Let $S$ be a minimal demolishing sequence of $(n,p) - problem$. Suppose that the disks have been arranged on $r \leq p - 1$ stacks at the end of $S$. Let $n, n - 1$ and $j_1 < ... < j_{r-2}$ be the disks at the bottom of the $r$ stacks at the end of $S$. Then during the demolishing phase, no disk $y > j_1$ has arranged on the peg on which the disk $j_1$ will be stacked at the end of $S$.

Proof See [1].

4 Main Results

Definition 4.1  Let $\mu$ be a middle state of a solution $S$ of $(n,p) - problem$. Assume that $k < n$ be the largest disk which is not stacked on $n - 1$ at $\mu$. We define $B(S) = n - k - 1$ the base of $S$.

The above definition implies that every disks $j$ of $k + 1 \leq j < k + 1$ are stacked on $n - 1$ at $\mu$. 

6
Definition 4.2 For a sequence $P$ and a state $\mu \in P$, let $\mu^+$ be the next state of $\mu$ and $\mu^-$ be the state before $\mu$.

Definition 4.3 \( \chi \) be a sequence of \((n,p)\)–problem and \( A \subset \bar{n} \). Let \( \chi|_A \) be a sequence of \((A,p)\)–problem such that disks of \( \bar{n}\setminus A \) are removed from \( \chi \). \( \chi|_A \) is called restriction of \( \chi \) on \( A \).

Lemma 4.1 Let \( n, p \) are natural numbers satisfying \( \binom{r+p-3}{p-2} \leq n < \binom{r+p-2}{p-2} \). Then, \( K(n,p) - K(n-1,p) \) is either \( 2^r \) or \( 2^{r-1} \).

Proof If \( n = \binom{r+p-3}{p-2} \), then \( n-1 = \binom{r+p-3}{p-2} - 1 \).

We have

\[
K(n,p) = \sum_{t=0}^{r-1} 2^t \binom{p+t-3}{p-3}
\]

and

\[
K(n-1,p) = \sum_{t=0}^{r-2} \binom{p+t-3}{p-3} + 2^{r-1} \left( \binom{p+r-3}{p-2} - 1 - \binom{p+r-4}{p-2} \right)
\]

Thus

\[
K(n,p) - K(n-1,p) = 2^{r-1}
\]

Otherwise, \( \binom{p+r-3}{p-2} < n < \binom{p+r-2}{p-2} \) holds. It is obvious that \( K(n,p) - K(n-1,p) = 2^r \).

Lemma 4.2 For a sequence \( \chi \), \( |\chi| \geq |\chi_0 - \chi|_\chi| \) holds. Equality holds when \( \chi \) is minimal.

Theorem 4.1 For a solution \( S \) of \((n,p)\)–problem, if \( B(S) \geq r \) where \( r \) is the unique natural number satisfying \( \binom{r+p-3}{p-2} \leq n < \binom{r+p-2}{p-2} \), \( |S| \geq K(n,p) \).

In other words, if there is a shorter solution \( S \) which contradicts to the Frame-Stewart conjecture, than it must satisfy \( B(S) < r \).
Proof Let \( j \) be the initial state and \( \mu \) be the middle state of \( S \). Define \( \nu \) be the state when the \( n-1\backslash k \) tower completes, i.e. the state right after the last \((k + 1, *, k + 2)\) move between \( j \) and \( \mu \).

First, in case of \( |j - \nu|_k \geq \frac{K(n,p)}{2} - K(B(S), 3) \), we have

\[
M(n, p) = |S| = 2|\mu| + 1 = 2(|j - \mu|_k + |j - \mu|_{n-1\backslash k}) + 1 \geq 2|j - \nu|_k + 2|j - \mu|_{n-1\backslash k} + 1
\]

By Theorem 3.2, through the sequence \( j - \mu \), any disks in \( n-1\backslash k \) have not placed on any other pegs than the initial peg, \( \mu(n-1) \) and \( \mu^+(n) \) where \( \mu^+ \) is the state right after \( \mu \). Therefore, we have

\[
|j - \mu|_{n-1\backslash k} \geq K(B(S), 3)
\]

Thus, in this case, \( 2|j - \nu|_k + 2|j - \mu|_{n-1\backslash k} + 1 \geq 2\left(\frac{K(n,p)}{2} - K(B(S), 3)\right) + 2K(B(S), 3) + 1 \geq K(n, p) \) and \( |S| \geq K(n, p) \).

Otherwise,

\[
|j - \nu|_k < \frac{K(n,p)}{2} - K(B(S), 3)
\]

holds. Define \( T := j - \mu \) and we have \( |S| = 2|T| + 1 \). Since \( \nu \) is the state right after the \( n-1\backslash k \) tower has completed, no disks of \( n-1\backslash k \) has moved after \( \nu \). Let \( \chi \) be a sequence such that \( \chi_0 = \nu \), \( \chi_1 = \mu(n-1) \) and \( \chi_{|\chi|} = \nu_{|k|} - \nu(n-1) \) (i.e. \( \chi \) is a sequence beginning from \( \nu \) and moving disks of \( k \) onto \( n-1 \) minimally, instead of end up with \( \mu \).) Note that \( |\chi| \leq |j - \nu|_k \) holds. This is because \( \chi_0|_{k} = \nu|_{k}, \chi_{|\chi|}|_{k} = j|_{k} \) and Lemma 4.2. However, we also have the sequence \( (j - \nu) + \chi \), which begins with \( j \) and end up with complete \( n-1 \) tower. This gives the following:

\[
|(j - \nu) + \chi| \geq K(n - 1, p)
\]

Thus we get

\[
|T| + |j - \nu|_k \geq |j - \nu| + |\chi| \geq K(n - 1, p)
\]
and

\[ |S| = 2|T|+1 \geq 2(K(n-1, p)-|j-\nu|)+1 \geq 2K(n-1, p)-2\left(\frac{K(n, p)}{2}\right)-K(B(S), 3)-1)+1 \]

\[ = 2K(n-1, p) - K(n, p) + 2K(B(S), 3) + 3 \]

By Lemma 4.1, \( 2K(n-1, p) - K(n, p) \geq K(n, p) - 2^{r+1} \) and thus

\[ 2K(n-1, p) - K(n, p) + 2K(B(S), 3) + 3 \geq K(n, p) + 2K(B(S), 3) + 3 - 2^{r+1} \]

\[ = K(n, p) + 2(2^{B(S)} - 2^r) + 1 \geq K(n, p) \]

which finishes the proof.

The only case left is \( B(S) \leq t \) for proving the Frame-Stewart Conjecture.

**Conjecture 4.1** For \( n, p \) such that \( \left(\frac{r+p-3}{p-2}\right) \leq n < \left(\frac{r+p-2}{p-2}\right) \) and \( S \) be a solution of \((n, p)\) problem. \( |S| \geq K(n, p) \) holds for \( B(S) < r \)
References

[1] Roberto Demontis *What is the least number of moves needed to solve the k-peg Tower of Hanoi Problem?*

[2] A.K.A. Majumdar *Generalized Multi-Peg Tower of Hanoi Problem.* J. Austral.Math.Soc.Ser.B38(1996),201-208