Random thoughts about Complexity, Data and Models

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Abstract

Data Science and Machine learning have been growing strong for the past decade. We argue that to make the most of this exciting field we should resist the temptation of assuming that forecasting can be reduced to brute-force data analytics. This owes to the fact that modelling, as we illustrate below, requires mastering the art of selecting relevant variables.

More specifically, we investigate the subtle relation between “data and models” by focussing on the role played by algorithmic complexity, which contributed to making mathematically rigorous the long-standing idea that to understand empirical phenomena is to describe the rules which generate the data in terms which are “simpler” than the data itself.

A key issue for the appraisal of the relation between algorithmic complexity and algorithmic learning is to do with a much needed clarification on the related but distinct concepts of compressibility, determinism and predictability. To this end we will illustrate that the
evolution law of a chaotic system is compressible, but a generic initial condition for it is not, making the time series generated by chaotic systems incompressible in general. Hence knowledge of the rules which govern an empirical phenomenon are not sufficient for predicting its outcomes. In turn this implies that there is more to understanding phenomena than learning – even from data alone – such rules. This can be achieved only in those cases when we are capable of “good modelling”.

Clearly, the very idea of algorithmic complexity rests on Turing’s seminal analysis of computation. This motivates our remarks on this extremely telling example of analogy-based abstract modelling which is nonetheless heavily informed by empirical facts.

KEYWORDS Models, algorithmic complexity, explanation, computation, big data.

1 Introduction

Suppose that we could find a finite system of rules which enabled us to say whether any given formula was demonstrable or not. This system would embody a theorem of metamathematics. There is of course no such theorem and this is very fortunate, since if there were we should have a mechanical set of rules for the solution of all mathematical problems, and our activities as mathematicians would come to an end. [Hardy 1929]

Is data science proving what Hardy feared, but lightheartedly dismissed as obviously false? For if we take Hardy’s “finite system of rules” to compress algorithmically empirical data, and we take machine learning to be capable of using this data to produce scientific knowledge, isn’t there reason to believe that Hardy was indeed wrong?

It has been argued time and again in science and philosophy that the aim of science is to organise in the most economical fashion the data collected from experiments. Mach [Mach 1907 Mach 2014] championed this view, which turned out to be compelling to many scientists:

The so-called descriptive sciences must chiefly remain content with reconstructing individual facts . . . . But in sciences that
are more highly developed, rules for the reconstruction of great numbers of facts may be embodied in a single expression.

It is remarkable that recently such an approach has been reconsidered in the framework of algorithmic complexity [Li and Vitányi 2009] from researchers without specific philosophical interests. For instance, Solomonoff, one of the fathers of the theory of the algorithmic complexity, identifies a scientific law with an algorithm for compressing the results of experiments [Solomonoff 1964]:

The laws of science that have been discovered can be viewed as summaries of large amounts of empirical data about the universe. In the present context, each such law can be transformed into a method of compactly coding the empirical data that gave rise to the law.

Similar opinions are shared by [Barrow 2007], a scientist well know for his popular books:

The intelligibility of the world amounts to the fact that we find it to be algorithmically compressible. We can replace sequences of facts and observational data by abbreviated statements which contain the same information content. These abbreviations we often call laws of Nature. If the world were not algorithmically compressible, then there would exist no simple laws of Nature.

This view of the science, which puts at its center information and algorithms, has unsurprisingly gained much interest and broad success in the last decades. And this certainly created a favourable framework for the recent rise of datacentric enthusiasm. We have put forward some methodological words of caution about the temptation of replacing the art of scientific modelling with the machine-aided analysis of time series [Coveney et al. 2016], more generally we have argued [Hosni and Vulpiani 2018] that the accuracy of forecasts need not be monotonic in the amount of data, – indeed the opposite is true in paramount examples from dynamical systems.

Our present concern is to do with a rather subtle but frequent conceptual confusion which is likely to arise in many scientific contexts which refer to algorithmic complexity, information and chaos. Specifically it is a confusion on the relation among the determinism, predictability, and the view according
to which the scientific understanding of a phenomenon can be accounted for in terms of algorithmic compression.

A telling example is provided for instance, by Davies [Davies 1990]:

there is a wide class of physical systems, the so-called chaotic ones, which are not algorithmically compressible.

This sounds intuitively appealing, but it is misleading. As we will see in Section 3 the evolution law of a chaotic system is compressible. What is not compressible is the time sequence generated by chaotic systems, and this, as argued in [Chibbaro and Vulpiani 2017], is due to the non-compressibility of a generic initial condition.

Hence, the aim of the present paper is to assess in some detail the methodological scope and relevance for scientific enquiry of the concepts of compression and algorithmic complexity. Before tackling our key question, namely do scientific laws compress empirical data?, it is worth recalling that the concepts of compression and algorithmic complexity are direct offsprings of the visionary intuitions which led Alan Mathison Turing to provide an abstract and very general model of computation, which in addition is capable of identifying the bounds of and computability. Reviewing, if very quickly, the path followed by Turing will give us precious insights in the subtle art of modelling and how this depends on the careful selection of relevant data.

2 Alan Turing’s model of computation

The early 1930s have been dotted with deep results on the powers and limitations of formal mathematical reasoning. Chief among them is Kurt Gödel’s proof of the existence of undecidable sentences in first order languages capable of expressing arithmetic. This marked a distinction between what can be seen as mathematically true and what is mathematically provable: if proofs are to be consistent, then there exists true statements which can neither be proved nor disproved. And if this was not enough, one such undecidable statement expresses the consistency of that very system of proof.

Gödel’s incompleteness theorems left logically open the possibility that the Entscheidungsproblem could be solved positively for first order logic. As put by Hilbert and Ackermann
The Entscheidungsproblem is solved if one knows a procedure which will permit one to decide, using a finite number of operations, on the [satisfiability] of a given logical expression.

Of course the incompleteness result cast serious doubts on a positive solution, but to prove that the Entscheidungsproblem was indeed unsolvable one first needed to state it in a mathematically rigorous way rather than resting on an intuitive understanding of a “procedure” or “algorithm”. This was the task that Turing gave to computing machines in [Turing 1936] – which have been called ever since Turing Machines – by means of which it became possible to suppose that the informal concept of “effective procedure” (or “algorithm”) coincides with the class of functions computable by means of a Turing Machine. Armed with this formalisation, it was possible for Turing to prove that the Entscheidungsproblem was indeed unsolvable. We are not, at present, so much interested in either the proof or the consequences of this result, which the interested reader can appreciate e.g. in [Davis 2000]. Rather we are interested in recalling Turing’s initial motivation and formulation for the definition of algorithmic procedure in [Turing 1936]. In other words, how Turing constructed his model of computability. The reader who wishes to find out more is referred to the unsurpassed analysis provided in [Gandy 1995] by Turing’s only student at Manchester.

Turing arrives at his model of computation in four steps:

1. Finds out what is the right question to ask;

2. Carries out an analogy with the human activity of computing and observes the cognitive limitations of humans as they carry out computations;

3. Claims that the actions and operations which define a Turing Machine are sufficient to account for the concept of “effective procedure” i.e. for a mathematical definition of algorithms.

Step 1) is of fundamental importance. The question as to whether all solvable problems can be solved algorithmically is clearly too vast to be tackled. But Turing’s modelling genius allows him to see that there was no loss of generality in asking which numbers are computable, i.e. “real numbers whose expressions as a decimal are calculable by finite means”. Then in step 2) Turing imagines a human “computor” tackling this task and strips away all the irrelevant features from the picture. So to compute, one needs to read,
write and erase symbols on a large enough physical support (tape). And to
do so purposefully, the computer must perform those actions according to
the instructions provided by their state of mind. What cannot be stripped
away are the limitations of the human computer: there is only a limited
number of symbols that can be read, erased and written at the same time,
and the state of mind of the computer may contain only a finite number
of instructions. Those abstractions and constraints give Turing a model of com-
puting behaviour which relied only on finite means and a definite procedure.
Then, step 3) is the claim that the class of numbers (functions etc.) that
can be computed mechanically coincides with that computable by means of
a suitably defined Turing Machine. Within this model of computability Tur-
ing proved the undecidability of the Halting Problem, which establishes the
existence of well defined problems which cannot be solved algorithmically.

A far reaching consequence of this is that not all data can be compressed
algorithmically. If scientific understanding were adequately modelled by algo-
ithmic compressibility, this would imply that there are unknowable scientific
facts. But what would that mean?

3 Do scientific laws compress empirical data?

Let us begin by noting the rather obvious fact that once a scientific law has
been established, one has reached some sort compression. But care must
be placed in handling this conclusion. Especially when chaos enters the
picture. So let us follow Turing in delimiting the scope of our question to two
rather difficult and important problems which are useful to clarify the role
of chaos, initial conditions and algorithmic complexity in scientific practice:
first chaotic deterministic systems and second the features of fully developed
turbulence (FDT) [Frisch 1995].

3.1 Classical mechanics and chaos

As first we have to stress that there is a persistent confusion about determin-
ism, chaos and predictability. It is possible to understand that determinism
and predictability are completely unrelated. In few words we can say that
determinism is ontic and has to do with how Nature behaves, while pre-
ddictability is epistemic and is related to what the human beings are able to
compute. On the other hand a careful analysis of chaotic systems in terms of
the Lyapunov exponents and the Kolmogorov-Sinai entropy show how deterministic chaos, although with an epistemic character, is non subjective at all, for a detailed discussion see (Atmanspacher 2002, Chibbaro et al 2014).

It is well known that from the Newton equation and the gravitation law one can derive many important astronomical facts, for instance the Kepler’s laws. From this, however it is not completely correct to conclude that the Newton equation and the gravitation law are capable of compressing all astronomical behavior. Indeed, after the seminal contribution of Poincaré, we know that a system of three bodies interacting with the gravitational force, usually is chaotic.

Rather than developing this difficult question in full detail, let us consider a one dimensional map:

$$x_{t+1} = 2x_t \mod 1.$$  \hspace{1cm} (1)

This model, called Bernoulli shift, is known to be chaotic, i.e. a small initial uncertainty increases exponentially, namely

$$\delta x_t \sim 2^t \delta x_0.$$ \hspace{1cm} (2)

In spite of its (apparent) simplicity the Bernoulli shift shares with the three body problem many central features, so let us analyse the problem of the compression, with accuracy $\Delta$, of a sequence $x_t$, $0 < t < T$, generated by the rule (1). In the parlance of algorithmic information theory, this is equivalent to the problem of transmitting the sequence to say, a friend. At first glance, the problem seems quite simple: we could opt for transmitting $x_0$ and the rule (1), which costs a number of bits independent of $T$. Our friend would then be left with the task of generating the sequence $x_1, x_2, ..., x_T$. However, we must also choose the number of bits to which $x_0$ should be specified. From (2), the accuracy $\Delta$ at time $T$ requires accuracy $\delta_0 \sim 2^{-T} \Delta$ for $x_0$, hence that the number of bits specifying $x_0$ grows with $T$.

Writing the initial condition in the forms

$$x_0 = \sum_{n=1}^{\infty} \frac{a_n}{2^n} = (a_1, a_2, ....)$$ \hspace{1cm} (3)

we understand that the evolution law of (1) is nothing but a shift of the binary point of the sequence $\{a_1, a_2, ....\}$. So we have to tackle the problem of the complexity of a sequence of symbols, $\{a_0, a_1, ...\}$.

Recall that the evolution of $x_0$ is regular (e.g. periodic) if its sequence $\{a_1, a_2, ....\}$ is not complex, while it is irregular if $\{a_1, a_2, ....\}$ cannot be compressed.
Therefore both in systems with regular behavior (e.g. the pendulum) and in chaos (e.g. the three body or the Bernoulli’s shift) the evolution law is straightforward to compress. The difference between the two classes of systems lies with the outputs: which are always regular in the pendulum, whereas in the Bernoulli’s shift can be regular or irregular at varying initial conditions.

The conclusion is that, in a deterministic system, the details of the time evolution are well hidden in the initial condition which is, typically incompressible. This follows from an important result of Martin-Löf who showed that almost all infinite binary sequences, which express the real numbers in [0, 1], are complex \[\text{Martin-Löf 1966}\]. Note that the identification randomness with incompressibility put forward by Martin-Löf (and further developed by Kolmogorov and Chaitin) follows rather closely the modelling footsteps of Turing’s model of computation. Indeed it has become standard to refer to the claim that “incompressibility” is the mathematically rigorous counterpart of “random” as the Martin-Löf Thesis \[\text{Li and Vitányi 2009}\].

3.2 Turbulence

Turbulent flows, a paradigmatic case of complex system, are governed by the Navier-Stokes equations (NSE) which can be written in two lines. So, naively, one could conclude that, since we know the equation for the time evolution of the velocity field, somehow, the phenomenon of turbulence has been compressed. The study of some specific aspects of the turbulence allows for the understanding of the precise meaning of such a conclusion. As first we consider the initial conditions, of course in any experiment they are necessarily known with a limited precision. A rather severe limitation is due to the fact that in the limit of very large Reynolds numbers \(R_e\) for a proper description of the turbulent velocity field it is necessary to consider a huge number of variables which increases quickly with \(R_e\). Therefore for the typical values of \(R_e\) in the limit of fully developed turbulence (FDT), because of the gigantic amount of data necessary to describe the involved degrees of freedom, we have an obvious impossibility to access to the initial conditions with the proper accuracy.

In addition at large \(R_e\) the NSE are chaotic: the distance between two initially close initial conditions increase very fast. Therefore as consequence of the practical impossibility to access the initial conditions with high accuracy, and the presence of deterministic chaos, even a very powerful computer and
accurate numerical algorithms is not possible to perform a simulation of the NSE for a long term and compare the prediction with the experimental results.

Since the practical impossibility to compare the experimental results with the numerical computation of the velocity field, we cannot say that the NSE are able to compress the turbulent behaviors. Nevertheless there is a general consensus on the validity of the NSE for the FDT. We can mention at least four items supporting the opinion that NSE are able to describe FDT, the agreement of the results observed in FDT and those obtained by the NSE for:

a) short time prediction of the velocity field;

b) long time prediction of averaged (e.g. spatially coarse grained) quantities;

c) the scaling laws, and more generally, the statistical features;

d) the qualitative and quantitative spatio-temporal features (e.g. large scale coherent structures).

4 Science only from data?

The NSE have been derived on a theoretical basis using the Newton equations and assuming the hypothesis of the continuity of matter plus some thermodynamic considerations. In other words, by *modelling*. One can wonder about the possibility to obtain the NSE just looking directly at experimental data.

A less ambitious task, but with the same conceptual aspects, is to build models in finite dimension on the basis of experimental data [Gershenfeld and Weigend 1994]. And again for sake of simplicity only, let us focus on the discrete time case. The most favourable case is clearly the one in which we know that the state of the system at time $k$ is a finite dimensional vector $x_k$. So let us start from here and consider the problem of making predictions from the available data, i.e. a long time sequence.

The natural approach is to search for a past state similar to the present state of a given phenomenon of interest, then, looking at the sequence of events that followed the past state, one may infer by analogy the evolution that will follow the present state [Cecconi et al. 2012]. In other words, given a known sequence of *analogues*, i.e. of past states $x_1, ..., x_M$ which resemble each other closely in pairs, so that $|x_k - x_M| < \epsilon$ with $\epsilon$ reasonably small, one makes the approximate prediction:

$$x_{M+1} = x_{k+1}$$
if $x_k$ is an analogue of $x_M$.

In the case the above protocol can be used, one may then proceed to build a model of the phenomenon, i.e. to determine a function $f(x)$ such that the sequences of states is well approximated by the dynamical system

$$x_{k+1} = f(x_k).$$

The application of this method requires knowledge of at least one analogue. It is possible to realise, – and this is the Kac lemma – that such knowledge requires sufficiently long sequences, at least of duration of order $T_R \sim (L/\epsilon)^D$, where, if the system is dissipative, $D$ is the dimension of the attractor, basically $D$ is the number of relevant variables.

The exponential growth of $T_R$ as a function of $D$ has a severe impact on our ability to make predictions, and build the evolution law (4), solely relying on previously acquired data. One can say that $D$ larger than 6 renders the approach described here useless, because it makes it practically impossible to observe the ”same” state twice, i.e. within an acceptable accuracy $\epsilon$.

On the other hand typically the state of the system, i.e. the variables which describe the phenomenon under investigation, is not known. Therefore an unavoidable technical aspect is the determination of proper state of the system from the study of a time series $\{u_1, u_2, ..., u_M\}$, where $u$ is an observable. The most relevant result for such a problem is due to Takens [Takens 1981] who has been able to show that, at least from a mathematical point of view, it is possible (if we know that the system is deterministic and is described by a finite dimensional vector) to determine the proper variable $X$. In a nutshell: there is a finite integer $m$ such that the delay coordinate vector (of dimension $m$)

$$\mathbf{y}_k^{(m)} = (u_k, u_{k-1}, ..., u_{k-m+1})$$

can faithfully reconstruct the properties of the underlying dynamics, from heuristic arguments one can expect $m = [D] + 1$.

Of course the practical limitation due to the exponential increasing of $T_R$ as function of $D$, is present also in the Takens’s method; therefore we have rather severe practical limitations.

A way out of this practical limitation, as we argued in [Cecconi et al. 2012, Hosni and Vulpiani 2018] consists a careful selection of the relevant features of the phenomenon of interest, that is to say, in modelling. Weather forecasting provides a very telling example of this, one which clarifies that the main
limit to predictions based on analogs is not the sensitivity to initial conditions, typical of chaos. But, as first realized by Lorenz [Cecconi et al. 2012] the key difficulty lies with the problem of finding good analogues.

The first modern steps in weather forecasting are due to Richardson [Lynch 2006] who, in his visionary work, introduced many of the ideas on which modern meteorology is based. His approach was, to a certain extent, in line with genuine reductionism, and may be summarised as follows: the atmosphere evolves according to the hydrodynamic (and thermodynamics) equations for the velocity, the density, and so on. Therefore, future weather can be predicted, in principle at least, by solving the proper partial differential equations, with initial conditions given by the present state of the atmosphere.

The key idea by Richardson to forecast the weather was correct, but in order to put it in practice it was necessary to introduce one further ingredient that he could not possibly have known [Lynch 2006]. After few decades von Neumann and Charney noticed that the equations originally proposed by Richardson, even though correct, are not suitable for weather forecasting [Lynch 2006, Hosni and Vulpiani 2018]. The apparently paradoxical reason is that they are too accurate: they also describe high-frequency wave motions that are irrelevant for meteorology. So it is necessary to construct effective equations that get rid of the fast variables.

The effective equations have great practical advantages, e.g. it is possible to adopt large integration time steps making the numerical computations satisfactorily efficient. Even more importantly, they are able to capture the essence of the phenomena of interest, which could otherwise be hidden in too detailed a description, as in the case of the complete set of original equations. It is important to stress that the effective equations are not mere approximations of the original equations, and they are obtained with a subtle mixture of hypotheses, theory and observations [Chibbaro and Vulpiani 2017]. Just like the cognitive limitations of any human computer provided Turing with constraints which shaped crucially his model of computability.

5 Final remarks and conclusion

Let us add to the examples discussed in Sections 3 and 4 two further remarks, which will lend themselves to a more general conclusion.

The relevance of the scale resolution is linked with the effective variables
describing a phenomenon. One understands such a topic considering a fluid which can be described in terms of its molecules; in such an approach the correct variables are the positions and momenta of the molecules. So we have a very accurate description containing a lot of informations. On the other hand, often this approach is not interesting, for instance in engineering (or geophysical) problems we describe a fluid in terms of few fields (for velocity, temperature and so on). Therefore one has a huge decreasing of the amount of information and an increasing of the possibility to compress data.

Second, a remark on the qualitative aspects of science, which, sometimes, are considered to be less important than the quantitative ones. Some results cannot be expressed in terms of numerical sequences, but they can be interesting and rigorous. We can mention the Lotka-Volterra like equations: it is not possible to find the explicit solution, but one can show that the time behavior is periodic. In a similar way, in some celestial mechanics problems, it is enough to be sure that the motion (e.g. of an asteroid) remain in a bounded region [Cencini et al. 2010]. The previous qualitative results, although cannot be formalized in terms of algorithmic compression (which involves sequences) are genuine forms of compression of information.

So, here is, in conclusion, the gist of our argument. The claim that the world is comprehensible because it is algorithmically compressible is, in our opinion, a truism, which is equivalent to saying that laws of nature exist. Like many truisms, it may well have heuristic value. For instance in the familiar situations which present a series of numbers which we are invited to continue appropriately. Guessing rightly the intended continuation of the series, by all means, is to understand how the series is generated, i.e. the law which governs it. But, as we argued above, there are serious limitations in generalising this approach to the wider and multifarious field of scientific knowledge. The fact that we know the law ruling a certain phenomena does not imply that we are able to control (e.g. predict) the system. This has been discussed for a simple example of chaotic system (Bernoulli shift) and for the Navier-Stokes equation for fluids.

Disregarding the distinction between initial conditions and laws of nature can lead to great confusion. For instance Davies 1990 claims that chaotic systems are not algorithmically compressible. On the other hand, as discussed in Sect. 2, the evolution law ruling chaotic systems can be trivially compressible (in the sense that it is easy to write down the evolution laws, as, e.g., for the Bernoulli’s shift). What can be not compressible is the output and this is related to the complexity of the sequence associated to the initial
condition.

We have pointed out the importance of the concept of state of the system, i.e. in mathematical terms, the variables which describe the phenomenon under investigation, which is often overlooked and its relevance underestimated. Only in few cases, e.g. in mechanical systems, it is easy to identify the variables which describe the phenomenon. On the contrary, in a generic case, there are serious difficulties; we can say that the main effort in building a theory on nontrivial phenomena concerns the identification of the appropriate variables.

This difficulty is well-known in the context of statistical physics: Onsager and Machlup, in their seminal work on fluctuations and irreversible processes, stressed the problem with the caveat [Onsager and Machlup 1953].

\[ \text{how do you know you have taken enough variables, for it to be Markovian?} \]

In a similar way, [Ma 1985] notes that

\[ \text{the hidden worry of thermodynamics is: we do not know how many coordinates or forces are necessary to completely specify an equilibrium state.} \]

Unfortunately, usually we have no definitive method for selecting the proper variables and only a deep theoretical understanding can suggest the “good ones”. We have recalled that choosing the right question to ask and cutting down the number of variables according to the relevant practical limitations, were two of the key steps which led Alan Turing to his model of effective computation. Nearly a century on, the very same ideas stand still as guidelines for the subtle art of (algorithmic) modelling.

On the other hand, if the laws are not known and we have just the possibility to study time series, the scenario is quite pessimistic. If the effective dimensionality is (relatively) large, even in the most simple case of deterministic system, it is not possible to find the evolution laws and therefore to perform an explicit compression [Chibbaro and Vulpiani 2017].

The above difficulties have been often underestimated by the supporters of a science based only on data and algorithms, for a recent detailed critics of such an approach which tries to avoid the use of any theory, see [Mezard 2018, Buchanan 2019].
References

[Atmanspacher 2002] Atmanspacher, H. (2002) Determinism is ontic, determinability is epistemic, In: Atmanspacher, H. and Bishop, R. (eds.) Between Chance and Choice, Imprint Academic.

[Barrow 2007] Barrow, J.D. (2007). New Theories of Everything. Oxford University Press

[Buchanan 2019] Buchanan, M, (2019) The limits of machine prediction Nature Physics 15., 304.

[Cecconi et al. 2012] Cecconi, F., Cencini, M., Falcioni, M., Vulpiani, A. (2012) The prediction of future from the past: an old problem from a modern perspective American Journal of Physics 80, 11. 1001-1008.

[Cencini et al. 2010] Cencini, M. ,Cecconi, F., and Vulpiani, A. (2010). Chaos. WorldScientific, Singapore.

[Chibbaro and Vulpiani 2017] Chibbaro, S., Vulpiani, A. (2017) Compressibility, Laws of Nature, Initial Conditions and Complexity Foundation of Physics 47. 1368.

[Coveney et al. 2016] P. V. Coveney, E. R. Dougherty, and R.R. Highfield. (2016) Big data need big theory too Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 280, 374, 1-11.

[Davies 1990] Davies, P.C.W. (1990) Why is the physical world so comprehensible In: Zurek, W.H. (ed.) Complexity, Entropy and the Physics of Information. Addison-Wesley, Boston.

[Davis 2000] Davis, M. (2000) The Universal Computer. The Road from Leibniz to Turing. W. W. Norton Company

[Frisch 1995] Frisch, U. (1995) Turbulence: The Legacy of A.N. Kolmogorov Cambridge University Press.

[Gandy 1995] Gandy, R. (1995) The confluence of ideas in 1936. In The universal Turing machine (2nd ed.), Rolf Herken (Ed.). Springer-Verlag New York, Inc., Secaucus, NJ, USA 51-102.
[Gershenfeld and Weigend 1994] Gershenfeld, N.A., Weigend, A.S. (eds.) (1994) *Time Series Prediction: Forecasting the Future and Understanding the Past*. Addison-Wesley, Reading.

[Hardy 1929] Hardy, G.H. (1929). Mathematical proof *Mind*, 38, 149.

[Hosni and Vulpiani 2018] Hosni, H., Vulpiani, A. (2018) Forecasting in Light of Big Data *Philosophy and Technology*. 31, 557.

[Li and Vitányi 2009] Li, M. and Vitányi, P. (2009) *An Introduction to Kolmogorov Complexity and Its Applications*. Springer-Verlag, Berlin.

[Lynch 2006] Lynch, P. (2006) *The Emergence of Numerical Weather Prediction: Richardson’s Dream* Cambridge University Press, Cambridge.

[Ma 1985] Ma, S. K. (1985). *Statistical Mechanics* World Scientific, Singapore.

[Mach 2014] Mach, E. (2014) *On the Economical Nature of Physical Inquiry* Cambridge University Press.

[Mach 1907] Mach, E. (1907) *The Science of Mechanics: A Critical and Historical Account of Its Development*. Open Court Publishing Company, Chicago.

[Martin-Löf 1966] Martin-Löf, P. (1966) The definition of random sequences. *Inf. Control* 9,602.

[Mezard 2018] Mezard, M. (2018) Artificial intelligence and its limits *Europhysics News* 49, 26.

[Onsager and Machlup 1953] Onsager L. and Machlup S. (1953) Fluctuations and irreversible processes *Physical Review* 91, 1505-1512.

[Solomonoff 1964] Solomonoff, R.J.(1964). A formal theory of inductive inference. Part I-II. *Inf. Control* 7, 1.

[Takens 1981] Takens, F. (1981) Detecting strange attractors in turbulence In: D. Rand, L.-S. Young (Ed.s), Dynamical Systems and Turbulence, *Lecture Notes in Mathematics*, 898, 366–381.
[Turing 1936] Turing, A.M. (1936) On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*. 42,1, 230-265.