Consistent Selection of the Number of Groups in Panel Models via Sample-Splitting

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Abstract

Group number selection is a key question for group panel data modelling. In this work, we develop a cross validation method to tackle this problem. Specifically, we split the panel data into a training dataset and a testing dataset on the time span with temporal structure preserved. We first use the training dataset to estimate the parameters and group memberships. Then we apply the fitted model to the testing dataset and then the group number is estimated by minimizing certain prediction error on the testing dataset. We design the loss functions for panel data models either with or without fixed effects. The proposed method has two advantages. First, the method is totally data-driven thus no further tuning parameters are involved. Second, the method can be flexibly applied to a wide range of panel data models. Theoretically, we establish the estimation consistency by taking advantage of the optimization property of the estimation algorithm. Experiments on a variety of synthetic and empirical datasets are carried out to further illustrate the advantages of the proposed method.

KEY WORDS: Cross validation; Group number estimation; Panel data; Sample-splitting.

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1 Introduction

Panel data modelling is one of the most important fields in statistics and econometrics (Arellano, 2003; Ke et al., 2015, 2016; Ando and Bai, 2017; Bretó et al., 2019; Hsiao, 2022; Gao et al., 2023; Lumsdaine et al., 2023), which characterizes information from both time and cross-sectional dimension. A critical problem in panel data modelling is to quantify the individual heterogeneity resulting from distinct backgrounds and individual characteristics (Bai and Li, 2014; Ke et al., 2015; Li et al., 2016; Fan et al., 2018; Xiao et al., 2021; Wang and Zhu, 2022; Liu, 2023; Hong et al., 2023). In this regard, recently, latent group structure is widely specified and receives enormous attention (Ke et al., 2015,?; Su et al., 2019; Gu and Volgushev, 2019; Zhu et al., 2023; Mehrabani, 2023; He et al., 2023). The key assumption in the group panel data modelling is that the individuals within the same group share the same set of model coefficients (Ke et al., 2015; Su et al., 2016; Fang et al., 2023). Researches have shown that the latent group structure can capture flexible unobserved heterogeneity of panel data while remaining parsimonious model form and desirable statistical efficiency.

While group panel data models prove to be useful, their practical performances are significantly contingent on the specification of the group number. If the group number is under-specified, one will end up with an underfitted model with inferior performance. On the other hand, over-specification of the group number may result in an overly fitted model with suboptimal estimation efficiency. Consequently, a critical problem for the group panel data models is how to determine the number of groups.

In the related fields to selecting number of groups (clusters) in statistical models, two common wisdoms exist. The first involves the utilization of an information criterion (IC) that integrates both model fitness and model complexity. Methods based on information criteria are extensively utilized in the literature for tuning parameter selection in the tasks of model selection problems (Schwarz, 1978; Hurvich et al., 1998; Wang et al., 2009; Zhang et al., 2010). For instance, Naik et al. (2007) extended the Akaike information criterion to mixture regression models for selecting number of
mixture components. Li et al. (2016) employed a specific information criterion to determine the number of structural breaks. Hu et al. (2020) proposed a corrected BIC to determine the number of communities in the community detection task. In the analysis of grouped panel models, Lin and Ng (2012) proposed a modified BIC for linear panel model. Bonhomme and Manresa (2015) used a BIC based criterion for grouped fixed-effects model. Liu et al. (2020) designed a PC criterion for panel models with individual fixed effects. Theoretically, the IC based methods can consistently select the true group number under certain conditions. However, to implement the IC based method, one still needs to specify certain tuning parameters, which may vary from the model and error distribution specifications, making it subjective and unstable. The second one is hypothesis-testing based method (Tibshirani et al., 2001; Onatski, 2009; Choi et al., 2017). This type of methods formulates the group number estimation problem into a sequential hypothesis testing procedure. For example, Hardy (1996) summarized the Wolfe’s test (Wolfe, 1970) for the task of selecting the cluster number in clustering analysis. Lo et al. (2001) utilized a likelihood ratio statistic to test the number of components in a Gaussian mixture model. Bickel and Sarkar (2016) proposed a hypothesis testing approach to determine the community numbers in community detection task. In the investigation of panel models with group structures, Lin and Ng (2012) used a test to test the homogeneity of the data. Lu and Su (2017) proposed a residual-based Lagrange multiplier-type test to determinate the group number for linear group panel data model. However, the testing based methods are usually restricted to the linear panel models and cannot provide a unified group number estimation framework with general model forms.

In this work, we propose a unified group number estimation method via sample splitting for general panel data models. The method allows for entirely data-driven implementation, eliminating the need for specifying tuning parameters. Specifically, we first split the panel data on the time span and use the first half time periods for model training and the rest for model testing. This splitting method effectively maintains the temporal integrity of the panel data. Subsequently, we estimate the group structure and model parameters based on the training data given a specified group number }
Then, we evaluate the prediction error with the testing data by approximating the loss with a local quadratic function. Finally, the determination of the group number is achieved by minimizing the prediction error.

The idea is in spirit similar to the $r$-fold cross validation (CV) method, which is widely used to evaluate the prediction performances of regression and classification methods. Existing theoretical properties have shown that the $r$-fold CV method tends to select an overfitting model when applied to model selection tasks (Shao, 1993; Wang et al., 2007). Noteworthily, the consistency of group number estimation can be established with our panel data splitting procedure, attributed to the unique temporal structure preservation approach in the splitting procedure. Although contradicting the intuition, we illustrate the merit of the proposed method by carefully examining the selection criterion and show its connection to the IC based methods (Wang et al., 2007; Su et al., 2016, 2019; Liu et al., 2020). Similar procedure has been adopted by Zou et al. (2020) to select the number of change points in the change point detection area. See also Wang (2010); Chen and Lei (2018); Lei (2020); Rabinowicz and Rosset (2022) for the recent literature using CV methods for consistent model selection and relevant inference tasks. However, to our best knowledge, the data splitting method has not been introduced to model selection in the group panel data models.

In particular, our method can be applied to a wide range of panel models, including linear panel models (Lin and Ng, 2012) and nonlinear panel models, such as Probit panel model (Su et al., 2016), Logit panel model (Liu et al., 2020), as long as the corresponding loss function can be approximated with a local quadratic function form (Wang and Leng, 2007; Zou et al., 2020; Zhu et al., 2021). Consequently, it provides a flexible and unified parametric solution for selecting the group number in group panel data models. Our theoretical framework establishes the inner-connection between the estimation consistency and widely used $k$-means type optimization algorithm (Lin and Ng, 2012; Bonhomme and Manresa, 2015; Liu et al., 2020). Moreover, the proposed framework exhibits considerable potential for extension to more generalized model settings with minor adaptations. To validate the superiority of our method, we conduct
a comprehensive set of numerical studies, demonstrating its advantages over existing approaches.

The rest of the article is organized as follows. In Section 2, we provide an introduction to the group panel model. Additionally, we propose a selection criterion for group number estimation, offering a detailed exposition of the procedure employed for the estimation. Theoretical properties on asymptotic selection consistency are established in Section 3. Subsequently, Section 4 extends the proposed selection criterion to the group panel model with fixed effects. Simulations and real data analysis are presented in Sections 5 and 6, respectively. Finally, we conclude the article with a discussion in Section 7. All proofs and technique lemmas can be found in the Appendices A–C.

2 Selection with Cross-Validation

2.1 Model and Notations

Let $Y_{it}$ be a response variable and $x_{it} \in \mathbb{R}^p$ be the associated $p$-dimensional covariates. The data $z_{it} = (Y_{it}, x_{it}^\top)^\top$ is collected from the $i$th ($1 \leq i \leq N$) individual at the $t$th ($1 \leq t \leq T$) time point. Suppose that the $N$ individuals are divided into $G$ groups, where $G$ is predetermined and individuals within the same group share the same regression coefficients. Specifically, for each individual $i$, let $g_i \in \{1, 2, \ldots, G\}$ denote its corresponding group membership. Accordingly, let $\beta_{g_i} \in \mathbb{R}^p$ be the group specific regression coefficient. Let $C_g = \{i : g_i = g\}$, $G_G = \{C_1, C_2, \ldots, C_G\}$ and $G = (g_1, \cdots, g_N)^\top \in \mathbb{R}^N$ be the membership vector for all individuals. In addition, define the model parameters as $\beta = (\beta_1^\top, \cdots, \beta_G^\top)^\top \in \mathbb{R}^{G \times p}$. Then we obtain the parameter estimation $\{\hat{\beta}, \hat{G}\}$ by minimizing the loss function $\mathcal{L}(Z; \beta, G)$, i.e.,

$$\{\hat{\beta}, \hat{G}\} = \arg \min_{\beta, G} \mathcal{L}(Z; \beta, G) = \arg \min_{\beta, G} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathcal{L}(z_{it}; \beta_{g_i}). \quad (2.1)$$
Given a pre-specified group number $G$, the model parameter and group memberships can be estimated by using an iterative estimation method (Liu et al., 2020), which is given in Algorithm 1.

In practice, the true group number $G_0$ is unknown in advance, and therefore needs to be pre-determined. However, accurately identifying the parameter $G$, (i.e., $G = G_0$), is challenging due to the unobservability of the group patterns. To select the number of groups, a popular way is to adopt the information criterion methods. However, this type of methods still relies on a constant which needs to be pre-specified (Su et al., 2016; Liu et al., 2020). The selected group number $\hat{G}$ can be sensitive to the user-chosen constant, whereas the user-chosen constant may vary from the model and error distribution specifications, making it less robust in practical applications. Thus, we aim to develop a data-driven procedure to estimate $G_0$.

**General Notations.** We use $\| \cdot \|_2$ to denote the Euclidean norm of a vector. For any matrix $M$, $\|M\|_F = \sqrt{\text{tr}(M^T M)}$ denotes the Frobenius norm of $M$. In addition, for any symmetric matrix $M$, let $\lambda_{\text{max}}(M)$ and $\lambda_{\text{min}}(M)$ denote the maximum and minimum eigenvalues of $M$, respectively. Moreover, for any vector $v \in \mathbb{R}^n$, let $v^{(k)}$ denote its $k$th element and define $\|v\|_M = (v^T M v)^{1/2}$ for any positive definite matrix $M$. Similarly, for any two vectors $v_1, v_2 \in \mathbb{R}^n$, define $\Xi_M(v_1, v_2) = v_1^T M v_2$ for any positive definite matrix $M$ and $\Xi(v_1, v_2) = v_1^T v_2$. For a sequence $a_n > 0$, we denote $X_n \gtrsim a_n$ if there exists a constant $C > 0$ such that $X_n \geq Ca_n$ as $n \to \infty$ with probability tending to one. In addition, we denote $X_n \gg a_n$ if $X_n/a_n \to \infty$ as $n \to \infty$ with probability tending to one.

### 2.2 Quadratic Approximation to Objective Function

To estimate the number of groups in a data-driven manner, we present a selection criterion with cross-validation (CV) method. Following Zou et al. (2020), the selection criterion is designed based on a quadratic approximation to the loss function $L(z; \beta)$. 
To motivate the idea, we first define the node-wise loss function as

\[ \L_i(Z; \beta) = \frac{1}{T} \sum_{t=1}^{T} L(z_{it}; \beta). \]  

(2.2)

Subsequently, we could obtain a node-wise estimator by minimizing \( \L_i(Z; \beta) \) as \( \hat{\beta}_i = \arg \min_{\beta} \L_i(Z; \beta) \). Then we can approximate \( \L_i(Z; \beta) \) by using a Taylor’s expansion in the neighbourhood of \( \hat{\beta}_i \) as

\[ \L_i(Z; \beta) \approx \L_i(Z; \hat{\beta}_i) + \frac{1}{2T} \sum_{t=1}^{T} (\beta - \hat{\beta}_i) \top \hat{L}(z_{it}; \hat{\beta}_i)(\beta - \hat{\beta}_i), \]

where the first term is not related to \( \beta \). Define \( \hat{W}_i = T^{-1} \sum_{t=1}^{T} \hat{L}(z_{it}; \hat{\beta}_i) \) and \( s(z_{it}; \beta) = \hat{L}(z_{it}; \beta) \). We note that \( \hat{W}_i(\beta - \hat{\beta}_i) \approx T^{-1} \sum_{t=1}^{T} s(z_{it}, \beta) \), then we have

\[ \frac{1}{T} \sum_{t=1}^{T} (\beta - \hat{\beta}_i) \top \hat{L}(z_{it}; \hat{\beta}_i)(\beta - \hat{\beta}_i) = (\beta - \hat{\beta}_i) \top \hat{W}_i(\beta - \hat{\beta}_i) \]

\[ \approx \left\{ \frac{1}{T} \sum_{t=1}^{T} s(z_{it}; \beta) \right\} \top \hat{W}_i^{-1} \left\{ \frac{1}{T} \sum_{t=1}^{T} s(z_{it}; \beta) \right\}. \]  

(2.3)

For convenience, we consider the following quadratic approximation to the loss function in (2.1) as

\[ Q(Z; \beta, G) = \frac{1}{N} \sum_{g=1}^{G} \sum_{i \in C_g} \left\{ \frac{1}{T} \sum_{t=1}^{T} s(z_{it}; \beta_g) \right\} \top \hat{W}_i^{-1} \left\{ \frac{1}{T} \sum_{t=1}^{T} s(z_{it}; \beta_g) \right\} \]

\[ \overset{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \overline{Q}_i(Z; \beta_{g_i}) \]  

(2.4)

by ignoring the constants. We remark that we utilize the quadratic approximation to mainly simplify our theoretical analysis. In practice, we can also work with the loss function \( \L(Z; \beta, G) \) as the selection criterion.
2.3 Selection Criterion with Cross-Validation

Based on the least squares approximation (2.4), we present a selection criterion using a 2-fold cross-validation method. The main idea is to split the data $Z$ into a training dataset $Z_{tr}$ and a testing dataset $Z_{te}$. Specifically the splitting is conducted on the time dimension. We first split $1 \leq t \leq T$ into training and testing sets in the middle of the time span as $T_{tr} = \{t : 1 \leq t \leq \lfloor T/2 \rfloor \}$ and $T_{te} = \{t : \lfloor T/2 \rfloor + 1 \leq t \leq T \}$, where $[m]$ denotes the integer part of $m$. Then we denote $Z_{tr} = (z_{it} : 1 \leq i \leq N, t \in T_{tr})^\top$ and $Z_{te} = (z_{it} : 1 \leq i \leq N, t \in T_{te})^\top$. We note that with this data splitting scheme, the individual group memberships and the parameters can be preserved for both training and testing datasets.

To conduct the CV method, we first estimate the parameters and the group memberships using the training dataset $Z_{tr}$, denoted as

$$\{\hat{\beta}^{tr}, \hat{G}^{tr}\} = \arg\min_{\beta, G} \mathcal{L}(Z_{tr}; \beta, G).$$

Then we apply $\{\hat{\beta}^{tr}, \hat{G}^{tr}\}$ to the testing dataset $Z_{te}$ and obtain the out-sample loss function value as $Q(Z_{te}; \beta^{tr}, \hat{G}^{tr})$. Similarly we can use $Z_{te}$ to estimate the parameters $\{\hat{\beta}^{te}, \hat{G}^{te}\}$ and obtain $Q(Z_{tr}; \beta^{te}, \hat{G}^{te})$. Lastly, we select $G$ to minimize the following CV based criterion,

$$\hat{G} = \arg\min_{G} \left\{ Q(Z_{te}; \beta^{tr}, \hat{G}^{tr}) + Q(Z_{tr}; \beta^{te}, \hat{G}^{te}) \right\}.$$

(2.5)

The selection procedure is summarized in Algorithm 2. We explain the rationality of (2.5) by decomposing $Q(Z_{te}; \beta^{tr}, \hat{G}^{tr})$. Specifically, we first define $\bar{s}_i(Z_{te}; \beta) = |T_{te}|^{-1} \sum_{t \in T_{te}} s(z_{it}; \beta)$ and $\Delta_i(Z_{te}; \beta_1, \beta_2) = \bar{s}_i(Z_{te}; \beta_1) - \bar{s}_i(Z_{te}; \beta_2)$ with any given $\beta, \beta_1, \beta_2$. Then we have

$$Q(Z_{te}; \beta^{tr}, \hat{G}^{tr}) = \frac{1}{N} \sum_{g=1}^{G} \sum_{i \in \hat{C}_g} \left\| \Delta_i(Z_{te}; \hat{\beta}^{tr}, \hat{\beta}^{te}) \right\|_{\tilde{W}_i}^2 + \frac{1}{N} \sum_{g=1}^{G} \sum_{i \in \hat{C}_g} \left\| \bar{s}_i(Z_{te}; \hat{\beta}^{te}) \right\|_{\tilde{W}_i}^2.$$
This yields
\[
2015; Su et al., 2016; Liu et al., 2020). The first term in (2.7) is used to evaluate
to the IC based method for determining the group number (Bonhomme and Manresa,

...tion, we use the least squares objective function, i.e.,
\[
\varepsilon
\]
where recall that \( \|v\|_W^2 = v^T W v \) and \( \Xi_W(v_1, v_2) = v_1^T W v_2 \) for \( v, v_1, v_2 \in \mathbb{R}^p \), \( W \in \mathbb{R}^{p \times p} \). The third term is a cross term which is dominated by the first two terms under both underfitting and overfitting cases. In the underfitting case (i.e., \( G < G_0 \)), \( D(Z_{te}; \hat{\beta}^{te}) \) will dominate with large prediction error on the testing dataset. In the overfitting case (i.e., \( G > G_0 \)), we can show that \( S(Z_{te}; \hat{\beta}^{tr}, \hat{\beta}^{te}) \) will dominate by taking consideration of the optimization strategy of the algorithm (Zou et al., 2020). Consequently, \( S(Z_{te}; \hat{\beta}^{tr}, \hat{\beta}^{te}) \) plays the role of the penalty term in the BIC or AIC selection while using a data-driven strategy. This avoids setting a user-determined constant in information criteria. In addition, we also remark that we split the data over the time span with a 2-fold cross validation. One can also consider an \( m \)-fold cross-validation method with \( m > 2 \) to obtain a reasonable result.

**Remark 1.** We consider a simple univariate model for illustration, i.e., \( Z_{it} = \beta_{gi} + \varepsilon_{it} \), where \( \varepsilon_{it} \) is the independent noise term with mean 0 and variance 1. For estimation, we use the least squares objective function, i.e., \( N^{-1} |T_{tr}|^{-1} \sum_{t=1}^{N} \sum_{t \in T_{tr}} (Z_{it} - \beta_{gi})^2 \). Suppose the group structure is estimated using the training dataset \( Z_{tr} \) as \( \hat{C}_g \) for \( g = 1, \cdots, G \). Then we have \( \hat{\beta}_g^{tr} = |\hat{C}_g|^{-1} |T_{tr}|^{-1} \sum_{i \in \hat{C}_g} \sum_{t \in T_{tr}} Z_{it} \) and \( \hat{\beta}_g^{te} = |\hat{C}_g|^{-1} |T_{te}|^{-1} \sum_{i \in \hat{C}_g} \sum_{t \in T_{te}} Z_{it} \). In this case, we have \( \bar{s}_i(Z_{te}; \beta) = |T_{te}|^{-1} \sum_{t \in T_{te}} (\beta - Z_{it}) = \beta - |T_{te}|^{-1} \sum_{t \in T_{te}} Z_{it} \) and \( \Delta_i(Z_{te}; \hat{\beta}_g^{tr}, \hat{\beta}_g^{te}) = \bar{s}_i(Z_{te}; \hat{\beta}_g^{tr}) - \bar{s}_i(Z_{te}; \hat{\beta}_g^{te}) = \hat{\beta}_g^{tr} - \hat{\beta}_g^{te} \). This yields

\[
Q(Z_{te}, \hat{\beta}^{tr}, \hat{\beta}^{te}) = \frac{1}{N} \sum_{g=1}^{G} \sum_{i \in \hat{C}_g} \left( \frac{1}{|T_{te}|} \sum_{t \in T_{te}} Z_{it} - \hat{\beta}_g^{te} \right)^2 + \frac{1}{N} \sum_{g=1}^{G} |\hat{C}_g| (\hat{\beta}_g^{tr} - \hat{\beta}_g^{te})^2, \quad (2.7)
\]
and \( R(Z_{te}; \hat{\beta}^{tr}, \hat{\beta}^{te}) = 0 \) in this case since \( \sum_{i \in \hat{C}_g} \bar{s}_i(Z_{te}; \hat{\beta}_g^{te}) = 0 \). The form is similar to the IC based method for determining the group number (Bonhomme and Manresa, 2015; Su et al., 2016; Liu et al., 2020). The first term in (2.7) is used to evaluate
the fitness level and the second term plays the role of penalty for model complexity as in the IC method. In contrast to the user-specified tuning parameter typically involved in the second penalty term for the IC method, the amount of “penalty” in (2.7) can be totally determined by the data information. As a consequence, our data-splitting method is more user-friendly. Lastly, using (2.5) in the practical implementation can help to improve the robustness of the proposed method.

3 Theoretical Properties

In this section, we establish the selection consistency of the proposed data splitting method. Consequently, it guarantees that we can identify the true group number with probability tending to one with the proposed procedure. In the following, we first present the technical conditions, then we give theoretical properties regarding the selection consistency result.

3.1 Technical Conditions

To establish the theoretical properties, the following conditions are required.

(C1) (PARAMETER SPACE) Suppose there exists a constant $R > 0$ such that $\max_{g \in G} \| \beta_g \|_2 \leq R$.

(C2) (DISTRIBUTION) The individuals are dependent across time and mutually independent with each other.

(C2.1) (TIME DEPENDENCE) For each $i \in [N]$, the process $\{z_{it} : t \in [T]\}$ is stationary and $\beta$-mixing with mixing coefficients $\beta_i(\cdot)$. Moreover, $\beta(\tau) \overset{\text{def}}{=} \sup_{N \geq 1} \max_{1 \leq i \leq N} \beta_i(\tau)$ satisfies $\beta(\tau) \leq 2 \exp \left(-C_0 \tau^{b_0} \right)$ for all $\tau \geq 0$ and some constants $C_0, b_0 > 0$.

(C2.2) (INDIVIDUAL INDEPENDENCE) $\{z_{it}, t \in [T]\}$ are mutually independent of each other across $i \in [N]$. 

(C3) (Smoothness) There exists a non-negative function $K(z_{it})$, such that for all $\beta, \beta' \in \mathbb{R}^p$, it holds

$$|\mathcal{L}(z_{it}; \beta) - \mathcal{L}(z_{it}; \beta')| \leq K(z_{it})\|\beta - \beta'\|_2. \quad (3.1)$$

In addition we have $|\mathcal{L}(z_{it}; \beta)| \leq K(z_{it})$ for all $\beta \in \mathbb{R}^p$. There exist $b_1 > 0$ and $B_1 > 0$ such that

$$\sup_{N \geq 1} \sup_{i \leq N} P(K(z_{it}) > v) \leq \exp \left\{ 1 - (v/B_1)^{b_1} \right\}, \quad \text{for all} \ v > 0. \quad (3.2)$$

Define $\mathcal{L}^{(k)}(z_{it}; \beta)$ as the $k$th order derivative with respect to $\beta$. Suppose (3.1) and (3.2) also hold for $\|\mathcal{L}^{(k)}(z_{it}; \beta) - \mathcal{L}^{(k)}(z_{it}; \beta')\|_F$ with $k = 1, 2$. Furthermore, there exists a function $M(z_{it})$ such that $\|\mathcal{L}^{(3)}(z_{it}; \beta)\|_{\infty} \leq M(z_{it})$ for $\|\beta\|_2 \leq R$ and it holds $E(M(z_{it})) < \infty$.

(C4) (Bounded Moments) There exist a function $M^*(z_{it})$ and two integers $0 < m_1 \leq 4$ and $m_2 \geq 4$, such that, for all $\beta$ with $\|\beta\|_2 \leq R$, the following condition holds almost surely,

$$\left| \frac{\partial^{m_1} \mathcal{L}(z_{it}; \beta)}{\partial \beta^{m_1}} \right| \leq M^*(z_{it}) \quad \text{and} \quad \sup_{i \in [N]} E[\{M^*(z_{it})\}^{m_2}] < \infty. \quad (3.3)$$

(C5) (Convexity) Define $H_g^i(\beta) = E\{\mathcal{L}^{(2)}(z_{it}; \beta)\}$ for $i \in C_g$. There exists positive constants $c_0$ and $c_1$, such that $c_0 \leq \min_g \lambda_{\min}(H_g^i(\beta_0^g)) \leq \max_g \lambda_{\max}(H_g^i(\beta_0^g)) \leq c_1$.

(C6) (Identification) Let $L_i^*(\beta) = E\{\bar{L}_i(Z; \beta)\}$, then

$$\inf_{N \geq 1} \inf_{1 \leq i \leq N} \inf_{\|\beta - \beta_0^g\|_2 \geq \epsilon} \left[ L_i^*(\beta) - L_i^*(\beta_0^g) \right] \geq C \epsilon. \quad \forall C, \ \epsilon > 0.$$

Moreover, the condition also holds for $Q_i^*(\beta) = E\{\bar{s}_i(Z; \beta)\}^2_{W_{-1}}$.

(C7) (Sample Size) Let $d = b_0b_1/(b_0 + b_1)$ and $d \in (0, 1)$, then $\log N = o[T^{d/(2(1+d)^2)}]$. 

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(C8) (Group Size) For all \( g \in [G_0] \), there exists a positive constant \( \pi_g \) such that \( N_g/N \to \pi_g \) as \( N \to \infty \).

(C9) (Group Separation) Let \( \beta^0_g \) be the true model parameter in the \( g \)th group. Then let \( d_0 \overset{\text{def}}{=} \min_{g' \neq g} \| \beta^0_{g'} - \beta^0_g \|_2 > 0 \).

We explain the rationality of the technical conditions in details as follows. First, Condition (C1) assumes that the parameter space is compact, which is a regular condition in literature (Arellano et al., 2007; Lin and Ng, 2012; Bonhomme and Manresa, 2015; Su et al., 2016; Liu et al., 2020). Condition (C2) specifies the dependence structure of the observations. Specifically, (C2.1) assumes that for each individual, weak time dependence is allowed across the time dimension. The corresponding mixing coefficient decays at an exponential rate as the time gap between two observations increases. (C2.2) assumes cross-sectional independence that the individuals are mutually independent. This dependence assumption is widely specified in the panel data literature (Gao, 2007; Bonhomme and Manresa, 2015; Sarafidis and Weber, 2015; Su et al., 2016; Liu et al., 2020). Condition (C3) assumes a certain extent of smoothness for the loss function as well as its derivatives. In addition, the exponential bound is imposed on the tail probability of the related function, and similar conditions are assumed by Bonhomme and Manresa (2015) and Liu et al. (2020). Condition (C4) basically requires a bounded moment condition for the \( m \)th moment of the derivatives, which is also assumed in relevant literature to ensure the grouping accuracy (Hahn and Newey, 2004; Su et al., 2016; Liu et al., 2020).

Next, Condition (C5) assumes convexity within the interested area, and Condition (C6) is an identification condition imposed on each individual to ensure the local minimum at the true value. Condition (C7) specifies the relationship between \( N \) and \( T \). The larger sample size of \( N \) with respect to \( T \) is allowed. The corresponding constant \( d \) is related to the mixing coefficient of \( z_{it} \) and its distribution properties. Condition (C8) assumes that the group size \( N_g \) should diverge in the same speed with the total number of individuals \( N \). Lastly, Condition (C9) imposes a sufficient large gap between the true group coefficients, which is necessary for group membership estimation. By Conditions
(C1)–(C9), we can obtain the group estimation consistency result when $G \geq G_0$ (Liu et al., 2020, Theorem 2). Specifically, for each estimated group $\hat{C}_g = \{i : \hat{g}_i = g\}$, the group estimation consistency implies that there exists a true group $C_g = \{i : g_i^0 = g\}$ such that $\lim_{(N,T) \to \infty} P(\hat{C}_g \subseteq C_g) = 1$.

### 3.2 Selection Consistency

Let $G^0 = (g_1^0, g_2^0, \cdots, g_N^0)^\top \in \mathbb{R}^N$ be the true group membership vector. Define $G_{G_0} = \{C_1^0, C_2^0, \cdots, C_{G_0}^0\}$ as the true group partition, where recall that $C_g^0 = \{i : g_i^0 = g\}$. Recall that $\hat{G}_G = \{\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_G\}$ is the estimated group partition when $G$ groups are specified, where $\hat{C}_g = \{i : \hat{g}_i = g\}$. Furthermore, we denote $\hat{G}_{G}^* = \{\hat{C}_{gg}^* : g \in [G_0], g' \in [G]\}$ as a cross group partition by $G_{G_0}$ and $\hat{G}_{G}$, where $\hat{C}_{gg'}^* = \{i : g_i^0 = g, \hat{g}_i = g'\}$.

We first show that the selection consistency can be obtained if a critical condition is satisfied in the following Theorem 1. Then in following analysis, we verify the critical condition by using the properties of certain optimization algorithms.

**Theorem 1.** Suppose conditions (C1)-(C9) hold. Further assume for $G > G_0$,

\[
Q(Z_{tr}; \hat{\beta}_{G_{G_0}}; G^0) - Q(Z_{tr}; \hat{\beta}_{\hat{G}_G}; G_{\hat{G}_G}) \gtrsim T^{-1}.
\]  

(3.4)

Then we have $\lim_{\min\{N,T\} \to \infty} P(\hat{G} = G_0) = 1$.

The proof of Theorem 1 is provided in Appendix A.2. We remark that (3.4) is an important condition and we will verify it in the subsequent Theorem 2. The condition (3.4) means that the reduction in the approximated loss function due to further partitioning $G_{G_0}$ by $\hat{G}_G$ (with $G > G_0$) should be at least equal to the rate $T^{-1}$. This condition can be guaranteed by the optimization algorithm applied on the training dataset. In the $k$-means type algorithm, it searches the optimal partition which minimizes the loss function given a specified group number. Using this property, we are able to establish the following result in Theorem 2, which is verified to satisfy the condition (3.4).
**Theorem 2.** Suppose conditions (C1)-(C9) hold. Then the condition (3.4) is valid for the k-means type algorithm described in Algorithm 1.

The proof of Theorem 2 is provided in Appendix A.3. The conclusion of Theorem 2 makes (3.4) automatically satisfied. Immediately by Theorem 1, we can conclude that \( \lim_{\min\{N, T\} \to \infty} P(\hat{G} = G_0) = 1 \). The main proof is splitted into two major parts, which are the underfitting part \( (G < G_0) \) and overfitting part \( (G > G_0) \) respectively. For the underfitting part, the main idea follows the strategy of Liu et al. (2020) and shows that there exists at least one estimated \( \hat{\beta}_g \), which will be sufficiently distant from any of the true parameters \( \{\beta_0^g : g \in [G_0]\} \). This makes the resulting Q-function in (2.4) sufficiently large than its value at the true parameters. For the overfitting part, we borrow the idea from Zou et al. (2020), which discusses consistent selection of the number of change-points. Following the similar routine, we show that the lower bound in (3.4) can be guaranteed by optimization procedure with the k-means type algorithm. Subsequently, we extend our data splitting method to more general model settings with fixed effects in the following section.

4 Group Panel Data Model with Fixed Effects

4.1 Panel Data Estimation with Fixed Effects

In this section, we further discuss the group estimation based on data splitting method for nonlinear panel data models with fixed effects. In practice, individual leveled heterogeneity may exist and the data splitting method needs to be revised to circumvent the individual specific heterogeneity. Suppose the fixed effect of the individual \( i \) is denoted as \( \alpha_i \), which characterizes individual leveled heterogeneity. We can obtain parameter estimation by minimizing the loss function \( \mathcal{L}(Z; \beta, \alpha, G) \), i.e.,

\[
\{\hat{\beta}, \hat{\alpha}, \hat{G}\} = \arg \min_{\beta, \alpha, G} \mathcal{L}(Z; \beta, \alpha, G) = \arg \min_{\beta, \alpha, G} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathcal{L}(z_{it}; \beta_{gi}, \alpha_i),
\]
Specifically, we use \( \alpha = (\alpha_i, 1 \leq i \leq N)^T \). Here we abuse the notation slightly by using the same \( \mathcal{L}(\cdot) \). In practice, we can use a profile objective function for estimation. Specifically, define \( \hat{\alpha}_i(\beta) = \arg \min_{\alpha} T^{-1} \sum_{t=1}^T \mathcal{L}(z_{it}; \beta, \alpha_i) \) and then the profile objective function is given by

\[
\mathcal{L}^P(Z; \beta, G) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \mathcal{L}^P(z_{it}; \beta_{gi}) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \mathcal{L}(z_{it}; \beta_{gi}, \hat{\alpha}_i(\beta_{gi})). \tag{4.1}
\]

A feasible iterative estimation method for obtaining \( \{\hat{\beta}, \hat{G}\} = \arg \min_{\beta, G} \mathcal{L}^P(Z; \beta, G) \) is summarized in Algorithm 3. Subsequently, we discuss estimation of \( G_0 \) in presence of the fixed effects \( \{\alpha_i : 1 \leq i \leq N\} \) based on the profile objective function. We define

\[
\mathcal{L}^\beta(z_{it}, \beta, \alpha_i) = \partial \mathcal{L}(z_{it}, \beta, \alpha_i)/\partial \beta, \quad \text{and} \quad \mathcal{L}^\alpha(z_{it}, \beta, \alpha_i) = \partial \mathcal{L}(z_{it}, \beta, \alpha_i)/\partial \alpha_i.
\]

Similarly, define \( \mathcal{L}^{\alpha\beta}(z_{it}, \beta, \alpha_i) \) and \( \mathcal{L}^{\alpha\alpha}(z_{it}, \beta, \alpha_i) \) and let

\[
\bar{\mathcal{L}}^\beta_i(\beta, \alpha_i) = T^{-1} \sum_{t} \mathcal{L}^\beta(z_{it}, \beta, \alpha_i)
\]

and

\[
\bar{\mathcal{L}}^\alpha_i(\beta, \alpha_i) = T^{-1} \sum_{t} \mathcal{L}^\alpha(z_{it}, \beta, \alpha_i).
\]

Following the literature (Liu et al., 2020), denote

\[
U_i(z_{it}, \beta, \alpha_i) = \mathcal{L}^\beta(z_{it}, \beta, \alpha_i) - \mathcal{L}^\alpha(z_{it}, \beta, \alpha_i) \bar{\mathcal{L}}^\alpha_i(\beta, \alpha_i) \bar{\mathcal{L}}^\beta_i(\beta, \alpha_i). \tag{4.2}
\]

\[
U_i^{\beta}(z_{it}, \beta, \alpha_i) = \frac{\partial U_i(z_{it}, \beta, \alpha_i)}{\partial \beta}, \quad U_i^{\beta\beta}(z_{it}, \beta, \alpha_i) = \frac{\partial^2 U_i(z_{it}, \beta, \alpha_i)}{\partial \beta^2},
\]

\[
U_i^{\alpha}(z_{it}, \beta, \alpha_i) = \frac{\partial U_i(z_{it}, \beta, \alpha_i)}{\partial \alpha}, \quad U_i^{\alpha\alpha}(z_{it}, \beta, \alpha_i) = \frac{\partial^2 U_i(z_{it}, \beta, \alpha_i)}{\partial \alpha^2},
\]

\[
U_i^{\alpha\beta}(z_{it}, \beta, \alpha_i) = \frac{\partial^2 U_i(z_{it}, \beta, \alpha_i)}{\partial \alpha \partial \beta}.
\]

Denote \( U_i(z_{it}) = U_i(z_{it}, \beta^0_{gi}, \alpha_i^0) \). In addition, define \( U_i^{\beta}(z_{it}) = \partial U_i(z_{it}, \beta_{gi}^0, \alpha_i^0)/\partial \beta \) and \( U_i^{\alpha}(z_{it}), U_i^{\beta\beta}(z_{it}), U_i^{\alpha\alpha}(z_{it}), \bar{L}_i(z_{it}) \) in the same way at the true value. Specifically, we use \( \bar{U}_i(\beta, \alpha_i) \) to denote \( T^{-1} \sum_{t=1}^T U_i(z_{it}, \beta, \alpha_i) \) and use \( \bar{U}_i \) to denote \( T^{-1} \sum_{t=1}^T U_i(z_{it}, \beta_{gi}^0, \alpha_i^0) \). Similarly, define \( \bar{L}_i^\alpha, \bar{L}_i^{\alpha\alpha}, \bar{L}_i^{\alpha\beta}, \bar{L}_i^{\beta\beta} \) in the same way at the true value. Given the group memberships \( G \), we can verify

\[
\sum_{i \in C_0} \sum_{t=1}^T \frac{\partial \mathcal{L}^P(z_{it}; \beta_{gi})}{\partial \beta_{gi}} = \sum_{i \in C_0} \sum_{t=1}^T U_i(z_{it}; \beta_{gi}, \hat{\alpha}_i(\beta_{gi})).
\]

To derive a simple yet effective loss function when the fixed effects are presented,
we need the following result on the estimation properties. The technical conditions are listed in Appendix B.2.

**Proposition 1.** Assume conditions \((C1^*)-(C6^*)\) and \((C7)-(C9)\) hold. In addition, define \(V_i = E\{\partial U_i(z_{it}; \beta_0^g, \alpha_i^0) / \partial \beta^\top_g\}\). Then we have

\[
U_i(\beta_g, \alpha_i(\beta_g)) = V_i(\beta_g - \beta_0^g) + \tilde{U}_i + R_i + o_p(T^{-1}) + o_p(\|\beta_g - \beta_0^g\|_2),
\]

(4.3)

for any \(\beta_g\) satisfying \(\|\beta_g - \beta_0^g\|_2 = o_p(1)\), where

\[
R_i \overset{\text{def}}{=} \left[ \frac{\tilde{L}_i^\alpha}{E(L_i^\alpha \alpha^\top)} \right] \left[ \frac{E(\tilde{U}_i^\alpha \alpha) \tilde{L}_i^\alpha}{2E(L_i^\alpha \alpha)} - \tilde{U}_i^\alpha \right].
\]

(4.4)

The proof of Proposition 1 is given in Appendix B.5. First, the leading term involves both \(\tilde{U}_i\) and \(R_i\). Particularly, the \(R_i\) term is an extra bias term caused by the individual leveled fixed effects, which is in the order of \(O_p(T^{-1})\). It cannot be reduced by using aggregated information of all individuals. The bias term will disappear when the fixed effects are not presented. Second, as suggested by the expansion of \(U_i(\beta_g, \alpha_i(\beta_g))\), an individual weighting matrix \(V_i\) is involved on the linear leading term \(\beta_g - \beta_0^g\). The weighting matrix is variant across \(i\) due to the existence of the fixed effects. Motivated by this fact, we consider a revised weighted quadratic objective function as

\[
Q(Z; \beta, \mathbb{G}) = \frac{1}{N} \sum_{g=1}^{G} \sum_{i \in C_g} U_i(\beta_g, \tilde{\alpha}_i(\beta_g))^\top \hat{V}_i^{-2} U_i(\beta_g, \tilde{\alpha}_i(\beta_g)).
\]

(4.5)

The \(Q\)-function in (4.5) is actually a reweighted loss function after adjusting to the individual leveled heterogeneous weighting matrix \(\hat{V}_i\). In practice, \(\hat{V}_i\) can be obtained as follows. Specifically, in the first step, we obtain \(\{\tilde{\beta}^{(1)}, \mathbb{G}\}\) using Algorithm 3. Subsequently, we construct the \(\hat{V}_i\) by

\[
\hat{V}_i = \left. \frac{\partial U_i(\beta_g, \alpha_i)}{\partial \beta^\top_g} \right|_{\beta_g = \tilde{\beta}_g^{(1)}, \alpha_i = \tilde{\alpha}_i^{(1)}} \quad \text{and} \quad \hat{\alpha}_i^{(1)} = \hat{\alpha}_i(\tilde{\beta}_g^{(1)}).
\]

(4.6)

In the second step, we obtain the final estimator \(\hat{\beta}\) by minimizing (4.5) and fixing \(\mathbb{G}\),
i.e., \( \hat{\beta} = \arg\min_{\beta} Q(Z; \beta, \hat{G}) \). The estimation procedure is summarized in Algorithm 4.

The following Proposition establishes the asymptotic expansion of \( \hat{\beta} \) when the true groups are partitioned into subgroups.

**Proposition 2.** Assume conditions \((C1^*)-(C6^*)\) and \((C7)-(C9)\) hold. Suppose the \( g \)th true group is partitioned into \( H \) subgroups and we have \( C_g^0 = C_{g1} \cup C_{g2} \cup \cdots \cup C_{gH} \). Denote \( \hat{\beta}_{gh} = \arg\min_{\beta} Q_{gh}(Z; \beta) \), where \( Q_{gh}(Z; \beta) = T^{-1} \sum_{i \in C_{gh}} U_i^\top (\beta, \hat{\alpha}_i(\beta))^\top \hat{V}_i^{-2} U_i (\beta, \hat{\alpha}_i(\beta)) \).

Then we have

\[
\hat{\beta}_{gh} - \beta_g^0 = -\frac{1}{N_{gh}} \sum_{i \in C_{gh}} V_i^{-1} (\hat{U}_i + R_i) + o_p \left( \| \hat{\beta}_{gh} - \beta_g^0 \|_2 \right) + o_p (T^{-1}),
\]

where \( R_i \) is defined in \((4.4)\), and \( N_{gh} = |C_{gh}| \). Furthermore, we have \( \hat{\beta}_{gh} - \beta_g^0 = O_p (T^{-1}) + O_p (N_{gh}^{-1/2} T^{-1/2}) \).

The proof of Proposition 2 is given in Appendix B.5. As shown by \((4.7)\), the leading term of the expansion for \( \hat{\beta}_{gh} - \beta_g^0 \) takes a simple weighted average of \( \hat{U}_i + R_i \). Implied by Theorem 2 of Liu et al. (2020), we can conclude that when \( G \geq G_0 \), we have the true groups are partitioned into subgroups with probability tending to 1. Consequently, the expansion of \((4.7)\) is critical to investigate the properties when the model is overfitted. Thanks to the simple average form of \((4.7)\), we are able to establish the lower bound as in \((4.9)\), which is critical for the consistent estimation of \( G_0 \) using the data splitting method.

### 4.2 Group Number Estimation using Data Splitting Method

Based on the objective function \( Q(Z; \beta, G) \), we then use the data splitting method for the estimation of group number. Specifically, the estimated \( \hat{G} \) is given by

\[
\hat{G} = \arg\min_{\hat{G}} \left\{ Q(Z_{te}; \hat{\beta}^{te}, \hat{G}^{te}) + Q(Z_{tr}; \hat{\beta}^{te}, \hat{G}^{te}) \right\},
\]

(4.8)
where \( \{\widehat{\beta}_{tr}, \widehat{G}_{tr}\} \) and \( \{\widehat{\beta}_{te}, \widehat{G}_{te}\} \) are estimators obtained using Algorithm 4 from training data and testing data respectively. The comprehensive selection procedure is summarized in Algorithm 5. Subsequently, we establish the estimation consistency of \( \widehat{G} \) under a pivotal condition (i.e., (4.9)) in Theorem 3 and verify the condition in Theorem 4.

**Theorem 3.** Suppose conditions (C1*)-(C7*) and (C8)-(C9) hold. Further assume for \( G > G_0 \),

\[
Q(Z_{tr}; \widehat{\beta}_{G_0}, G_0^0) - Q(Z_{tr}; \widehat{\beta}_{\widehat{G}_G}, G_{\widehat{G}_G}) \gtrsim T^{-1}.
\]  

(4.9)

Then we have \( \lim_{\min\{N,T\} \to \infty} P(\widehat{G} = G_0) = 1 \).

The proof of Theorem 3 is provided in Appendix B.3. The condition (4.9) means that the reduction in the \( Q \)-function due to further partitioning \( G_0 \) by \( \widehat{G}_G \) (with \( G > G_0 \)) should also be at least equal to the rate \( T^{-1} \), which is the same as (3.4). In the following, we show that the lower bound condition (4.9) can be satisfied by using the revised \( Q \)-function.

**Theorem 4.** Suppose conditions (C1*)-(C7*) and (C8)-(C9) hold. Then the condition (4.9) is valid for the two-step algorithm described in Algorithm 4.

The proof of Theorem 4 is provided in Appendix B.4. The conclusion Theorem 4 makes (4.9) automatically satisfied. Immediately by Theorem 3, we can conclude that \( \lim_{\min\{N,T\} \to \infty} P(\widehat{G} = G_0) = 1 \) still holds for group panel data model with fixed effects. Following the similar routine of the overfitting part in Theorem 2, we show that the lower bound in (4.9) can be guaranteed by optimization procedure with the Algorithm 4. Though the term \( R_i \) is introduced into the expansion of the \( Q \)-function due to the fixed effects, we find that its order is \( O_p(T^{-1}) \), which enables us to obtain the same lower bound as (3.4) and select the true group number for the model with fixed effects consistently. Next, we present a set of numerical studies to illustrate the usefulness of the proposed method.
5 Numerical Studies

5.1 Simulation Models and Selection Criterions

To evaluate the finite sample performances of the CV method, we conduct a number of simulation studies in this section. For comparison, we include the information criterion (IC) based methods (Bonhomme and Manresa, 2015; Su et al., 2016; Liu et al., 2020) and hypothesis testing based method (Lu and Su, 2017) in the following. First, the IC based methods minimizes the criterion in the following form, i.e.,

\[
\text{IC}(G) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathcal{L}(z_{it}; \hat{\beta}_{gi}) + \rho_{\lambda NT}(G),
\]

(5.1)

where \(\rho_{\lambda}(G)\) is the penalty function with tuning parameter \(\lambda_{NT}\). We evaluate various forms of the penalty functions in Bonhomme and Manresa (2015), Su et al. (2016), and Liu et al. (2020). Specifically, Bonhomme and Manresa (2015) used a BIC based criterion for linear panel data models with

\[
\rho_{\lambda NT}(G) = \hat{\sigma}^2 \frac{GT + N + p}{NT} \log(NT),
\]

where \(\hat{\sigma}^2 = (NT - G_{\text{max}}T - N - p)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathcal{L}(z_{it}; \hat{\beta}_{gi}^{(G_{\text{max}})})\). Here \(G_{\text{max}}\) is a predetermined maximum number of groups and \(\hat{\beta}_{gi}^{(G_{\text{max}})}\) denotes the corresponding estimator. The BIC criterion can only be applied to linear panel data models, however, no theoretical guarantees are provided. Next, Su et al. (2016), and Liu et al. (2020) set \(\rho_{\lambda NT}(G) = p\lambda_{NT}G\) and \(\rho_{\lambda NT}(G) = \lambda_{NT}G\) respectively. Su et al. (2016) recommend to use \(\lambda_{NT} = 2/3(NT)^{-1/2}\) for linear panel model and \(\log\{\log T/(4T)\}\) for probit panel data model. Liu et al. (2020) recommend to use \(\lambda_{NT} = 1/\{5\log(T)T^{1/8}\}\) and \(\lambda_{NT} = \log(N)^{1/8}/\{5(\log T)T^{1/8}\}\) for linear and probit panel models respectively. We refer to the information criteria given by Su et al. (2016) and Liu et al. (2020) as LIC (Lasso-based information criterion) and PC (penalty criterion) respectively in our following simulation studies. In addition to the IC based methods, we also consider the
method based on hypothesis testing. Specifically, we utilize the residual-based hypothesis testing proposed by Lu and Su (2017) and refer to the method as HT. To estimate $G_0$, we conduct the following hypothesis testing

$$H_0(K_0) : G = G_0 \quad \text{versus} \quad H_1(G_0) : G_0 < G \leq G_{\text{max}},$$

where the $G_0$ and $G_{\text{max}}$ are pre-specified. The test statistic is a residual-based LM-type test statistic and follows the standard normal distribution under the null hypothesis. We conduct the hypothesis testing from 1 to $G_{\text{max}}$ until the null hypothesis is rejected at $G^*$ and then we conclude that the number of groups is $G^*$. The method is designed for the linear models. Therefore, we only include HT method for linear panel models (i.e., DGP 1. and DGP 2.) in the following examples.

We consider four data generating processes (DGPs) including linear panel of static and dynamic models and a non-linear panel model. The sample sizes are given as $N = 200, 400$ and $T = 100, 125, 150$. Next, we set $G_0 = 4$ with equal group sizes as $N/G_0$. The true parameters for each DGP are given in Table 1. The details are presented as follows.

| DGP 1. | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ |
|-------|-----------|-----------|-----------|-----------|
| DGP 1. | (0.5, 0.5) | (0.56, 0.44) | (0.44, 0.56) | (0.38, 0.62) |
| DGP 1. (with $\alpha_i$) | (1.4, 0.6) | (1.0, 1.0) | (0.6, 1.4) | (0.2, 1.8) |
| DGP 2. | (0.1, 0.3, 0.7) | (0.1, 0.5, 0.5) | (0.1, 0.7, 0.3) | (0.1, 0.1, 0.9) |
| DGP 2. (with $\alpha_i$) | (0.1, 1.4, 0.6) | (0.1, 1.0, 1.0) | (0.1, 0.6, 1.4) | (0.1, 0.2, 1.8) |
| DGP 3. | (0.1, 0.3, 1.7) | (0.1, 0.6, 1.4) | (0.1, 0.9, 1.1) | (0.1, 1.2, 0.8) |
| DGP 4. | (0.35, 0.65) | (0.5, 0.5) | (0.2, 0.8) | (0.65, 0.35) |

**DGP 1. (Linear Panel Model)** We first consider the linear panel models as $Y_{it} = \beta_{yt}^{\top} x_{it} + \alpha_i + \epsilon_{it}$, which is also included in Su et al. (2016) and Liu et al. (2020). We consider two specifications for the fixed effects. In the first specification, we set $\alpha_i = 0$ across $i \in [N]$. In this case, the model does not include the fixed effects as we
considered in Section 2 and Section 3. In the second specification, the fixed effects $\alpha_i$ is generated independently from a uniform distribution $U[-1, 1]$. Next, the explanatory variable is generated as $x_{it} = (0.2\alpha_i + e_{it,1}, 0.2\alpha_i + e_{it,2})^T$, where $e_{it,1}, e_{it,2} \sim i.i.d N(0, 1)$. Lastly, we generate $\epsilon_{it}$ independently from standard normal distribution $N(0, 1)$ across $i \in [N], t \in [T]$.

DGP 2. (Linear Dynamic Panel Model) The linear dynamic panel model takes the form as $Y_{it} = \gamma g_i Y_{i,t-1} + \beta^\top g_i x_{it} + \alpha_i + \epsilon_{it}$. The model is also referred to as “Linear panel AR(1) model” in Su et al. (2016) since the lagged response is included. The fixed effects, covariate $x_{it}$, and $\epsilon_{it}$ are set in the same way as DGP 1.

DGP 3. (Dynamic Probit Panel Model) Beyond the linear panel data models included in DGPs 1 and 2, we subsequently consider a nonlinear panel model. Following Liu et al. (2020), we include the probit panel model as $Y_{it} = 1(\gamma g_i Y_{i,t-1} + \beta^\top g_i x_{it} \geq \epsilon_{it})$, where $\epsilon_{it}$ is the idiosyncratic error generated from standard normal distribution $N(0, 1)$. The probit panel model is widely used to model binary responses for classification problems.

DGP 4. (Static Poisson Panel Model) Lastly, to further illustrate the applicability of our method, we consider another nonlinear panel model, which is poisson panel model. The model is used to describe dynamic counted variables using Poisson distribution (Wooldridge, 2005). Specifically, we set $\lambda_{it} = \exp(\beta^\top g_i x_{it})$ and generate $Y_{it}$ from a Poisson distribution with mean $\lambda_{it}$. Since the selection criterion is not recommended for this case, we set the tuning parameter the same as in DGP 3.

5.2 Simulation Results

The random experiments are repeated with $R = 500$ times. To measure the finite sample performance, we show the distribution of $\hat{G} - G_0$ under DGPs 1-4 in Figure 1. Suppose the estimated group number is $\hat{G}^{(r)}$ in the $r$th replicate. We further report the mean estimation bias as $\text{Bias} = R^{-1} \sum_{r=1}^{R} (\hat{G}^{(r)} - G_0)$ and root mean squared error as $\text{RMSE} = \left\{ R^{-1} \sum_{r=1}^{R} (\hat{G}^{(r)} - G_0)^2 \right\}^{1/2}$. The results are given in Table 2.
Figure 1: Distribution of $\hat{G} - G_0$ for the PC, LIC, BIC, HT and CV methods. The DGP 1 is displayed in the top panel, where the case without fixed effects is given in the top left panel and the other one is in top right. The middle panel shows the performances under DGP 2 without fixed effects (middle left panel) and with fixed effects (middle right panel) respectively. The DGP 3 and DGP 4 are presented in bottom left and right panel respectively.
Table 2: Simulation results for all DGPs with 500 replications. The numerical performances are evaluated for different sample sizes $N$ and $T$. The Bias and RMSE are reported for all estimators under different DGPs.

| $N$ | $T$ | Method | Bias 1 | RMSE 1 | Bias 2 | RMSE 2 | Bias 3 | RMSE 3 | Bias 4 | RMSE 4 |
|-----|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 100 | 100 | BIC    | -2.00  | 2.00   | -1.91  | 1.93   | -2.00  | 2.00   | -1.95  | 1.96   |
|     |     | PC     | -2.00  | 2.00   | -0.17  | 0.41   | -2.00  | 2.00   | -0.15  | 0.39   |
|     |     | LIC    | -1.00  | 1.00   | 0.18   | 0.44   | -1.00  | 1.00   | 0.14   | 0.39   |
|     |     | HT     | 3.00   | 3.00   | 2.76   | 2.81   | 3.00   | 3.00   | 3.00   | 3.00   |
|     |     | CV     | -0.03  | 0.69   | 0.24   | 0.64   | -0.76  | 0.89   | 0.15   | 0.43   |
| 200 | 125 | BIC    | -2.00  | 2.00   | -1.98  | 1.98   | -2.00  | 2.00   | -1.99  | 1.99   |
|     |     | PC     | -2.00  | 2.00   | -0.22  | 0.47   | -2.00  | 2.00   | -0.22  | 0.47   |
|     |     | LIC    | -1.00  | 1.00   | 0.24   | 0.53   | -0.99  | 0.99   | 0.23   | 0.51   |
|     |     | HT     | 3.00   | 3.00   | 2.98   | 2.99   | 3.00   | 3.00   | 3.00   | 3.00   |
|     |     | CV     | 0.16   | 0.59   | 0.26   | 0.64   | -0.30  | 0.57   | 0.20   | 0.55   |
| 150 | 150 | BIC    | -2.00  | 2.00   | -1.99  | 1.99   | -2.00  | 2.00   | -2.00  | 2.00   |
|     |     | PC     | -2.00  | 2.00   | -0.25  | 0.52   | -2.00  | 2.00   | -0.22  | 0.51   |
|     |     | LIC    | -1.00  | 1.00   | 0.28   | 0.55   | -0.93  | 0.97   | 0.26   | 0.53   |
|     |     | HT     | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   |
|     |     | CV     | 0.27   | 0.65   | 0.36   | 0.75   | -0.06  | 0.32   | 0.29   | 0.63   |
| 400 | 125 | BIC    | -2.00  | 2.00   | -0.17  | 0.41   | -2.00  | 2.00   | -0.13  | 0.36   |
|     |     | PC     | -2.00  | 2.00   | -0.16  | 0.40   | -2.00  | 2.00   | -0.12  | 0.35   |
|     |     | LIC    | -1.00  | 1.00   | 0.17   | 0.43   | -0.78  | 0.88   | 0.13   | 0.36   |
|     |     | HT     | 3.00   | 3.00   | 2.99   | 2.99   | 3.00   | 3.00   | 3.00   | 3.00   |
|     |     | CV     | -0.13  | 0.60   | 0.06   | 0.28   | -0.61  | 0.79   | 0.04   | 0.21   |
| 150 | 150 | BIC    | -2.00  | 2.00   | -0.20  | 0.44   | -2.00  | 2.00   | -0.21  | 0.46   |
|     |     | PC     | -2.00  | 2.00   | -0.24  | 0.49   | -2.00  | 2.00   | -0.20  | 0.45   |
|     |     | LIC    | -1.00  | 1.00   | 0.27   | 0.57   | -0.20  | 0.45   | 0.22   | 0.49   |
|     |     | HT     | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   |
|     |     | CV     | 0.09   | 0.46   | 0.09   | 0.35   | -0.09  | 0.30   | 0.04   | 0.20   |
| 500 | 150 | BIC    | -2.00  | 2.00   | -0.27  | 0.52   | -2.00  | 2.00   | -0.25  | 0.50   |
|     |     | PC     | -2.00  | 2.00   | -0.25  | 0.51   | -2.00  | 2.00   | -0.25  | 0.50   |
|     |     | LIC    | -0.96  | 0.98   | 0.29   | 0.58   | -0.01  | 0.11   | 0.26   | 0.53   |
|     |     | HT     | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   | 3.00   |
|     |     | CV     | 0.22   | 0.50   | 0.14   | 0.44   | 0.00   | 0.09   | 0.11   | 0.36   |
First, we can observe that as the sample sizes $N$ and $T$ increase, the estimation accuracy increases as well. For instance, using the CV method, we can observe that the RMSE of $\hat{G}$ drops from 1.32 to 0.91 for the DGP 3 when $(N, T)$ increases from $(200, 100)$ to $(400, 150)$. Next, for the linear panel models without fixed effects (i.e., DGP 1), we observe that the IC based methods tend to underestimate the group number while the HT method tends to overestimate. That is mainly because the true group parameters are set to be close to each other, which makes it difficult to distinguish individuals from different groups. For the linear model case with fixed effects, the performances of the IC based methods and the CV method are comparable, while the finite sample performance of the CV method is slightly better. Similar patterns are observed for the two nonlinear panel data models (DGPs 3 and 4). Particularly, the advantage of the proposed CV method is more obvious for the Poisson panel model (in DGP 4). The other methods are more sensitive to tuning parameter specification while the CV method is more robust and can be applied to various models.

6 Empirical Studies

In this section, we conduct two empirical studies with the proposed methodology. The first study investigates the savings behavior across different countries and the second models the relationship between income and democracy. Specifically, we compare the performance of the proposed CV method with the information criterion based methods (Bonhomme and Manresa, 2015; Su et al., 2016; Liu et al., 2020) and hypothesis testing method (Lu and Su, 2017).

6.1 Savings Rate Modelling

We first investigate the savings behavior of different countries utilizing the data provided by World Bank-World Development Indicators. The dataset includes the savings data of 56 countries from 1995 to 2010. Following Edwards (1996) and Su et al. (2016),
we employ the following dynamic linear model to study the savings behavior as

\[ y_{it} = \gamma g_i y_{i,t-1} + \beta I_{i,g} I_{it} + \beta R_{i,g} R_{it} + \beta G_{i,g} G_{it} + \alpha_i + \epsilon_{it}, \]

where \( y_{it} \) is the ratio of savings to GDP, \( I_{it} \) is the CPI-based inflation rate, \( R_{it} \) is the real interest rate, \( G_{it} \) is the per capita GDP growth rate, \( \alpha_i \) is a fixed effect, and \( \epsilon_{it} \) is an error term.

Subsequently, we apply the proposed CV method, the information criterion based methods (LIC, BIC, PC) and the hypothesis testing method (HT) for model and group number estimation. The tuning parameters are specified the same as in Section 5.1. All the methods suggest \( \hat{G} = 2 \). To illustrate the rationality of the result, we calculate the prediction errors for \( G = 2, \ldots, 6 \). Specifically, we use \( T_{tr} = 8 \) years for model training and the subsequent \( T_{pr} = 7 \) years for model prediction. To evaluate the prediction accuracy, the mean square predicted error is calculated as

\[ \text{MPSE} = \frac{1}{NT_{pr}} \sum_{t=T_{tr}+1}^{T_{pr}} (\hat{y}_{it} - y_{it})^2, \]

where \( \hat{y}_{it} \) is the predicted response. This leads to the left panel of Figure 2. One can observe a sharp increase in the prediction error from \( G = 2 \) to \( G = 3 \). This implies that \( G = 2 \) is sufficient to capture the heterogeneity of the savings data.

Next, we study the sensitivity of the IC based methods to the specification of the tuning parameters. Take LIC as an example and recall that its \( \rho_{\lambda NT}(G) \) in (5.1) is specified as \( \rho_{\lambda NT}(G) = p\lambda_{NT} G \) where \( \lambda_{NT} = C\sqrt{NT} \) for linear panel model. We set \( C = 1/10, 1/6, 2/3 \) to evaluate the group number estimation result. The results are shown in the right panel of Figure 2. It indicates that the IC values can vary in different patterns for different tuning parameters. Specifically, it estimates \( \hat{G} = 2 \) for \( C = 2/3 \), \( \hat{G} = 3 \) for \( C = 1/6 \), and \( \hat{G} = 6 \) for \( C = 1/10 \). This suggests that the IC based methods could be sensitive to the tuning parameters.
6.2 Income and Democracy Modelling

In this study, we investigate the relationship between income and democracy for different countries. The income data is provided by the Penn World Tables while the measure of democracy is obtained from Freedom House. Following Lu and Su (2017), we construct a balanced panel data of 74 countries over the years 1961 to 2000. Specifically, the years 1961-2000 is split into $T = 8$ periods and each period refers to a five-years interval, (for example, $t = 1$ means 1961-1965). Following Bonhomme and Manresa (2015), we consider the following dynamic linear model to establish the regression relationship,

$$
Dem_{it} = \gamma_i Dem_{i,t-1} + \beta_{Inc,g_i} Inc_{i,t} + \alpha_i + \epsilon_{it},
$$

where $Dem_{it}$ is the measures of democracy and $Inc_{it}$ is the real GDP per capita of country $i$ during the $t$th period, $\alpha_i$ is the individual fixed effect and $\epsilon_{it}$ is the error term.

Next, we apply the proposed CV method, the information criterion based methods (LIC, BIC, PC) and the hypothesis testing method (HT) for model and group number estimation. The tuning parameters are specified the same as in Section 5.1. All the

*All the data are directly from AJRY: [http://economics.mit.edu/faculty/acemoglu/data/ajry2008](http://economics.mit.edu/faculty/acemoglu/data/ajry2008).
information based methods suggest $\hat{G} = 2$ while the HT suggests $\hat{G} = 6$. However, the proposed CV method suggests $\hat{G} = 3$, which is the same as Lu and Su (2017). Subsequently, we still calculate the MPSE for different group number to elucidate the rationality of the results. Specifically, we use $T_{tr} = 4$ years for model training and the following $T_{pr} = 2$ years for model prediction. The prediction is made for every year and the overall MPSE is given in the left panel of Figure 3. One can observe a rapid decrease in the prediction error from $G = 2$ to $G = 3$ and a sharp increase from $G = 3$ to $G = 4$. This indicates the $G = 2$ cannot capture the heterogeneity of the income and democracy data while $G = 4$ will result in a higher prediction error. Consequently, $\hat{G} = 3$ serves as a proper estimation. Next, we study the sensitivity of the IC based methods to the specification of the tuning parameters. Similarly, take LIC as an example and set $C = 1/6, 2/3, 3/4$ to evaluate the group number estimation result. The results are shown in the right panel of Figure 3. It also indicates that the IC values can vary in different patterns for different tuning parameters. Specifically, it estimates $\hat{G} = 2$ for $C = 2/3$, $\hat{G} = 6$ for $C = 1/6$, and $\hat{G} = 5$ for $C = 1/10$. This also suggests that the IC based methods could be sensitive to the tuning parameters.

Figure 3: Group estimation result for the income and democracy dataset. Left panel: the MPSE of the proposed CV method. Right panel: the IC curves for $C = 1/10, 1/6, 2/3$ of LIC method.
7 Conclusion

In this work, we propose a cross validation method to tackle the group number estimation problem for a variety of grouped panel models. Specifically, we split the panel data on the time span and estimate the group number by minimizing certain loss functions on the testing data. We provide the theoretical guarantee for the estimation consistency by utilizing the optimization properties of the grouping algorithm. Our proposed CV method has two major merits. First, the method is totally data-driven thus no further tuning parameters are involved. Second, the method can be flexibly applied to a wide range of panel data models.

To conclude the article, we provide several topics for future studies. First, we can apply the proposed method to panel data models with time fixed effect (Lu and Su, 2017) other than the individual fixed effects. In this case, the time fixed effects should be dealt with more carefully with our data splitting procedure. Next, it is interesting to extend the proposed method to a diverging group number. The theoretical property can be discussed and established accordingly. Third, in addition to the estimation consistency, the statistical inference procedure should be designed and the corresponding theoretical properties should be discussed for the group number estimation method.

Supplemental Material

Appendices A - C: proofs of Theorem 1 and Theorem 2; proofs of Theorem 3 and Theorem 4; preliminary lemmas.
Algorithm 1 A k-means type algorithm algorithm for group panel models
Input: The number of groups $G$; The dataset $Z$
Output: The estimation $\hat{\beta}$ and $\hat{G}$

1: Choose the initial estimators $\hat{\beta}(0) = (\hat{\beta}^\top_{(0)1}, \ldots, \hat{\beta}^\top_{(0)G})^\top \in \mathbb{R}^{G \times p}$.
2: **Update group membership:** for each $i \in \{1, \ldots, N\}$, in the $(s + 1)$th iteration, find
   \[
   \hat{g}_{(s+1)i} = \arg\min_{g \in [G]} \frac{1}{T} \sum_{t=1}^{T} L(z_{it}; \hat{\beta}_{(s)\hat{g}_{(s)i}}).
   \]
   Then update the group membership $\hat{G}_{(s+1)} = (\hat{g}_{(s+1)1}, \ldots, \hat{g}_{(s+1)N})$.
3: **Update coefficients:** given group membership $\hat{G}_{(s+1)}$, solve
   \[
   \hat{\beta}_{(s+1)} = \arg\min_{\beta} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} L(z_{it}; \beta_{\hat{g}_{(s+1)i}}).
   \]
4: Repeat steps 2–3 until the convergence criterion is met.
5: Return $\hat{\beta}$ and $\hat{G}$.

Algorithm 2 Group number selection for group panel models
Input: The maximum possible number of groups $G_{\text{max}}$; The dataset $Z$
Output: The estimation $\hat{G}$

1: Split the dataset $Z$ into $Z_{\text{tr}}$ and $Z_{\text{te}}$ on the middle of the time span.
2: for $G = 1, 2, \ldots, G_{\text{max}}$ do
3: Use Algorithm 1 on $Z_{\text{tr}}$ and $Z_{\text{te}}$ separately to obtain $\{\hat{\beta}_{\text{tr}}, \hat{G}_{\text{tr}}\}$ and $\{\hat{\beta}_{\text{te}}, \hat{G}_{\text{te}}\}$.
4: Calculate the validation loss
   \[
   \overline{Q}(G) = Q(Z_{\text{te}}; \hat{\beta}_{\text{tr}}, \hat{G}_{\text{tr}}) + Q(Z_{\text{tr}}; \hat{\beta}_{\text{te}}, \hat{G}_{\text{te}}).
   \]
5: end for
6: Select the number of groups $\hat{G}$ by
   \[
   \hat{G} = \arg\min_{1 \leq G \leq G_{\text{max}}} \overline{Q}(G).
   \]
Algorithm 3 A \( k \)-means type algorithm for group panel models with fixed effects

**Input:** The number of groups \( G \) and the dataset \( Z \).

**Output:** The estimation \( \hat{\beta} \) and \( \hat{G} \)

1: Set the initial estimators \( \hat{\beta}(0) = (\hat{\beta}_{(0)1}^T, \ldots, \hat{\beta}_{(0)G}^T)^T \in \mathbb{R}^{G \times p} \).
2: **Update group membership:** for each \( i \in \{1, \ldots, N\} \), in the \((s+1)\)th iteration, find
   \[
   \hat{g}(s+1)i = \arg \min_{g \in [G]} \frac{1}{T} \sum_{t=1}^{T} \mathcal{L}(z_{it}; \hat{\beta}(s)_{\hat{g}(s)i}, \alpha_i).
   \]

   Then update the group membership \( \hat{G}(s+1) = (\hat{g}(s+1)1, \ldots, \hat{g}(s+1)N) \).
3: **Update coefficients:** given group membership \( \hat{G}(s+1) \), solve
   \[
   \hat{\beta}(s+1) = \arg \min_{\beta} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathcal{L}(z_{it}; \beta_{\hat{g}(s+1)i}).
   \]
4: Repeat steps 2–3 until the convergence criterion is met.
5: Return \( \hat{\beta} \) and \( \hat{G} \).

Algorithm 4 A two-step algorithm for group panel models with fixed effects

**Input:** The number of groups \( G \); The dataset \( Z \)

**Output:** The estimation \( \hat{\beta} \) and \( \hat{G} \)

1: Use Algorithm 3 to obtain the estimation \( \{\hat{\beta}^{(1)}, \hat{G}\} \).
2: Use \( \{\hat{\beta}^{(1)}, \hat{G}\} \) to construct \( \hat{V} \) by (4.6).
3: Given group membership \( \hat{G} \), solve \( \hat{\beta} = \arg \min_{\beta} \mathcal{Q}(Z; \beta, \hat{G}) \) where \( \mathcal{Q}(Z; \beta, \hat{G}) \) is defined in (4.5).
4: Return \( \hat{\beta} \) and \( \hat{G} \).

Algorithm 5 Group number selection for group panel models with fixed effects

**Input:** The maximum possible number of groups \( G_{\text{max}} \); The dataset \( Z \)

**Output:** The estimation \( \hat{G} \)

1: Split the dataset \( Z \) into \( Z_{tr} \) and \( Z_{te} \) on the middle of the time span.
2: **for** \( G = 1, 2, \ldots, G_{\text{max}} \) **do**
3: Use Algorithm 4 on \( Z_{tr} \) and \( Z_{te} \) separately to obtain \( \{\hat{\beta}_{tr}^{(1)}, \hat{G}_{tr}^{(1)}\} \) and \( \{\hat{\beta}_{te}^{(1)}, \hat{G}_{te}^{(1)}\} \).
4: Calculate the validation loss
   \[
   \mathcal{Q}(G) = \mathcal{Q}(Z_{te}; \hat{\beta}_{tr}^{(1)}, \hat{G}_{tr}^{(1)}) + \mathcal{Q}(Z_{tr}; \hat{\beta}_{te}^{(1)}, \hat{G}_{te}^{(1)}).
   \]
5: **end for**
6: Select the number of groups \( \hat{G} \) by
   \[
   \hat{G} = \arg \min_{1 \leq G \leq G_{\text{max}}} \mathcal{Q}(G).
   \]
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