Postbuckling Analysis of Functionally Graded Beams

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Abstract. This paper studies the geometrically non-linear bending behavior of functionally graded beams subjected to buckling loads using the finite element method. The computational model is based on an improved first-order shear deformation theory for beams with five independent variables. The abstract finite element formulation is derived by means of the principle of virtual work. High-order nodal-spectral interpolation functions were utilized to approximate the field variables which minimizes the locking problem. The incremental/iterative solution technique of Newton’s type is implemented to solve the nonlinear equations. The model is verified with benchmark problems available in the literature. The objective is to investigate the effect of volume fraction variation in the response of functionally graded beams made of ceramics and metals. As expected, the results show that transverse deflections vary significantly depending on the ceramic and metal combination.

1. Introduction
Functionally graded materials (FGMs) have attracted the attention of the scientific community since their discovery by material scientists in Sendai, Japan [1]. FGMs are manufactured by grading the desired properties of two or more materials, usually ceramics and metals, in the selected direction. They offer great advantages from the materials science point of view because their continuous variation prevents stress concentrations, thus preventing the occurrence of plastic deformations and cracking [2]. Owing to these advantages, a large number of publications based on the analysis of FGM structures exist in the literature. The most relevant contributions to the topic are discussed below.

Kapania and Raciti [3] wrote a detailed review regarding the shear theories used for static cases, vibrations, and buckling for beams and thin shells. Sankar and Tzeng [4] obtained an elastic solution for FGM beams subjected to transverse loads. Kang and Li [5] investigated large deformations in functionally graded beams subjected to transverse loads. Nguyen et al. [6] studied the geometrically non-linear behavior of functionally graded beams and structural frames. Likewise, Arciniega and Reddy [7] studied the geometrically non-linear behavior of shells under different types of loads; their model was validated using different benchmark problems found in the literature.

The behavior of structural members is of equal importance in research and in the industrial field. FGMs were designed to be used under sudden changes in temperature. However, structural members must also be analyzed under any type of conditions, since temperature variations only occur in a short period of time. Therefore, the static behavior of FGM beams must be evaluated in terms of transverse deflection [8]. For example, Reddy [9] proposed a super-convergent finite element Timoshenko model for static problems. Sankar [10] provided an elasticity solution based on the Euler–Bernoulli theory for functionally graded beams subjected to static transverse loads. The static behavior of functionally graded beams using the high-order shear deformation theory has been also studied by Kadoli et al. [11].

This work presents a computational model for the non-linear analysis of functionally graded beams subjected to buckling loads. A theoretical formulation based on the improved first-order shear
deformation theory (FSDT) has been developed; the computational model incorporates finite
deformations under the Lagrangian framework. An abstract finite element formulation with spectral
high-order interpolation functions was then derived. The verification results of the formulation show
that the computational model presented is satisfactory.

2. Beam formulation and finite element model
Let \( \{ x^i \} \) be a set of Cartesian coordinates with orthonormal basis \( \{ e_i \} \). The neutral axis of the beam
is defined by the coordinate \( x^1 \). The displacement vector is assumed to be of the following form
\[
v(x^1, x^3) = u(x^1) + x^3 \phi(x^1) + (x^3)^2 \psi(x^1)
\]  
(1)
where \( u = u, e \) denotes the displacement vector of the neutral axis, \( \phi = \phi, e \) and \( \psi = \psi, e \) are
difference vectors \( (i = 1, 3) \). Equation (1) contains five independent variables. The quadratic term \( \psi \)
is included to avoid the Poisson locking, therefore, no enhanced methods are needed.

For the given displacement field, we define the Green-Lagrange strain tensor as
\[
\varepsilon = \varepsilon^{(0)} + x^3 \varepsilon^{(1)} + (x^3)^2 \varepsilon^{(2)} + (x^3)^3 \varepsilon^{(3)} + (x^3)^4 \varepsilon^{(4)}
\]  
(2)
where high order terms are neglected.

The weak form can be easily constructed from the principle of virtual work. The configuration
solution of the beam is defined by the triplet \( \Phi = (u, \phi, \psi) \). Thus, we obtain
\[
\mathcal{G}(\Phi, \delta \Phi) = \int_{\Omega} \left( \sum_{k=0}^{1} N^{(k)} \cdot \delta \varepsilon^{(k)} \right) dx^1 - \int_{\partial \Omega} \mathbf{p} \cdot \partial u dx^1 = 0
\]  
(3)
where \( \delta \Phi = (\delta u, \delta \phi, \delta \psi) \in \mathbf{v} \) and \( \mathbf{v} \) is the space of admissible variations. The stress resultants are
evaluated from the following equation
\[
\mathbf{N}^{(i)} = \sum_{j=0}^{1} \mathbf{B}^{(i+j)} \varepsilon^{(j)} , \quad i = 0, 1
\]  
(4)
Therefore, the final expression for the weak form will be
\[
\mathcal{G}(\Phi, \delta \Phi) = \int_{\Omega} \sum_{k=0}^{1} \sum_{l=0}^{1} \delta \varepsilon^{(k)} \cdot \mathbf{B}^{(k+j)} \varepsilon^{(j)} dx^1 - \int_{\partial \Omega} \mathbf{p} \cdot \partial u dx^1 = 0
\]  
(5)
The components of the tensor \( \mathbf{B}^{(i)} \) are the material stiffness coefficients and they are given by
\[
\mathbf{B}^{(k)} = \int_{-h/2}^{h/2} (x^3)^k \mathbf{C} dx^3 , \quad k = 0, 1, 2
\]  
(6)
In this two-phase functionally graded materials the properties are assumed to vary through the
thickness of the beam. The materials in the bottom and top surfaces are metal and ceramic respectively.
Therefore, the fourth-order elasticity tensor \( \mathbf{C} \) is a function of the thickness \( x^3 \). Their components are
expressed as
\[
C_{ijkl}(x^3) = C_{ijkl}^c f_c^c + C_{ijkl}^m f_m^m
\]  
(7)
where \( f_c, f_m \) are the volume fractions of the ceramic and metal phases which are computed by
mean of the power law. That is
Let $\mathcal{G}$ be the domain of the neutral axis parametrization where the finite element domain lies in. Recall that $\hat{\mathcal{G}} \equiv [-1,1]$ is a parent domain in $\xi$-space and $x^{1}(\xi) : \hat{\mathcal{G}} \rightarrow \mathcal{G}$. The finite element equations are obtained by interpolating the components of the field variables written in terms of the base vectors. Namely

$$f_{c} = \left( \frac{x}{h} + \frac{1}{2} \right)^{\eta}, \quad f_{m} = 1 - f_{c} \tag{8}$$

The adopted basis functions $\phi^{(i)}$ are $C^{0}$ interpolant polynomials of Gauss–Lobatto-Legendre quadrature points, which are particularly suitable for high-order expansions. Explicitly, the one-dimensional basis functions of order $p = m - 1$ are expressed using the $p$-order Legendre polynomial $P_{p}^{m}$, as shown

$$u^{\varepsilon}(x') = \left[ \sum_{j=1}^{m} h_{x}^{(j)} \phi^{(j)}(\xi) \right] e_{x}, \quad \varphi^{\varepsilon}(x') = \left[ \sum_{j=1}^{m} \phi_{x}^{(j)} \phi^{(j)}(\xi) \right] e_{x}, \quad \psi^{\varepsilon}(x') = \sum_{j=1}^{m} \psi_{x}^{(j)} \phi^{(j)}(\xi) e_{x}, \quad k = 1,3 \tag{9}$$

High-order spectral elements, in contrast to low-order finite elements, do not exhibit locking problems.

Finally, the linearization of the discrete form of equation (5) is carried out leading to a symmetric tangent operator. The resultant set of algebraic nonlinear equations is solved by the incremental-iterative Newton-Raphson method and the predictor-corrector cylindrical arc-length method.

3. Results and discussions

In this section, some results are presented to study the non-linear behavior of FGM beams with compressive loading applied to the neutral axis. To activate the post-buckling equilibrium path, a small perturbation $d$ is applied transverse to the $P$ load of ratio $d/P=0.001$.

The geometry of the beam, based on Massin and Al Mikdad [12], is depicted in figure 1: $L=0.50\,m$, $b=0.075\,m$, $h=0.0045\,m$. The values of Young’s modulus for aluminum and zirconium are given by:

$$E_{m} = 70\,GPa \; (aluminum), \quad E_{c} = 151\,GPa \; (ceramic), \quad v=0.3 \tag{11}$$

Two boundary conditions are analyzed: clamped-free (CF) and simply supported (SS) beams. The example problem is solved using polynomials of fourth order ($p = 4$) with a finite element mesh of ten elements. The shear locking effect was minimized utilizing high-order interpolation functions.

Figure 1. Postbuckling of a plate strip under compressive load.
Figure 2 shows load versus displacement curves for a clamped-free functionally graded beam. Here, we can see how the proposed model describes correctly the non-linear behavior of the beam. Similar results for isotropic cases are shown in reference [13]. The critical load for the example is $P_0 = \pi^2 E_m L^2$.

![Figure 2. Load-displacement curves for clamped-free beam with different volume fraction exponent $n$.](image)

Likewise, figure 3 shows the load-displacement curves at the center of a simply supported beam subjected to buckling load. The critical load is $P_0 = \pi^2 E_m L^2$. This exercise shows the effectiveness of the formulation to describe finite deformations of FGM beams, as depicted by Pagani and Carrera [13].

Finally, figure 4 shows several deformed configurations for the simply supported FGM beam under axial compressive loading and a volume fraction equal to 1.

![Figure 3. Load-displacement curves for simply supported (SS) beams with different volume fraction exponent $n$.](image)
4. Conclusions
A finite element model has been formulated to study the non-linear behavior of FGM beams. The formulation was based on an improved FSDT with five parameters and derived using the principle of virtual work. To assess the formulation, two benchmark problems were analyzed under CF and SS boundary conditions, subjected to buckling loads. The set of nonlinear equations were resolved through the arc-length method. Results for the displacements at the end (CF) and center (SS) of the beam were presented. The effect of the volume fraction exponent on the non-linear behavior of these beams was evaluated. We obtained the following conclusions:

(a) As expected, differences in the displacements for different volume fraction exponents and both cases (CF and SS) were observed.

(b) The proposed tensor formulation provides a robust model for the study of mechanical phenomena of FGM beams undergoing finite deformations.

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