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EXISTENCE OF A WEAK SOLUTION FOR FRACTIONAL EULER-LAGRANGE EQUATIONS.

by

Loïc Bourdin

Abstract. — We derive sufficient conditions ensuring the existence of a weak solution $u$ for fractional Euler-Lagrange equations of the type:

$$
\frac{\partial L}{\partial x}(u, D_0^\alpha u, t) + D_0^\alpha \left( \frac{\partial L}{\partial y}(u, D_0^\alpha u, t) \right) = 0,
$$

(EL$^\alpha$)
on a real interval $[a, b]$ and where $D_0^\alpha$ and $D_0^\alpha$ are the fractional derivatives of Riemann-Liouville of order $0 < \alpha < 1$.

Keywords: Fractional Euler-Lagrange equations; existence; fractional variational calculus.
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1. Introduction

1.1. Context in the fractional calculus. — The mathematical field that deals with derivatives of any real order is called fractional calculus. For a long time, it was only considered as a pure mathematical branch. Nevertheless, during the last two decades, fractional calculus has attracted the attention of many researchers and it has been successfully applied in various areas like computational biology [21] or economy [9]. In particular, the first and well-established application of fractional operators was in the physical context of anomalous diffusion, see [34, 35] for example. Let us mention [23] proving that fractional equations is a complementary tool in the description of anomalous transport processes. We refer to [15] for a general review of the applications of fractional calculus in several fields of Physics.

In a more general point of view, fractional differential equations are even considered as an alternative model to non-linear differential equations, see [5].

For the origin of the calculus of variations with fractional operators, we should look back to 1996-97 when Riewe used non-integer order derivatives to better describe non conservative systems in mechanics [28, 29]. Since then, numerous works on the fractional variational calculus have been made. For instance, in the same spirit, authors of [10, 11] have recently derived fractional variational structures for non conservative equations. Furthermore, one can find a comprehensive literature regarding necessary optimality conditions and Noether’s theorem, see [1, 3, 4, 6, 13, 25]. Concerning the state of the art on the fractional calculus of variations and respective fractional Euler-Lagrange equations, we refer the reader to the
recent book [22].

In the whole paper, we consider $a < b$ two reals, $d \in \mathbb{N}^*$ and the following Lagrangian functional

$$L(u) = \int_a^b L(u, D_+^\alpha u, t) \, dt,$$

where $L$ is a Lagrangian, i.e. a map of the form:

$$L : \mathbb{R}^d \times \mathbb{R}^d \times [a, b] \rightarrow \mathbb{R},$$

where $D_+^\alpha$ is the left fractional derivative of Riemann-Liouville of order $0 < \alpha < 1$ and where the variable $u$ is a function defined almost everywhere (shortly a.e.) on $(a, b)$ with values in $\mathbb{R}^d$. The precise definitions of the fractional operators of Riemann-Liouville will be recalled in Section 2.2. It is well-known that critical points of the functional $L$ are characterized by the solutions of the fractional Euler-Lagrange equation:

$$\frac{\partial L}{\partial x}(u, D_+^\alpha u, t) + D_+^\alpha \left( \frac{\partial L}{\partial y}(u, D_+^\alpha u, t) \right) = 0,$$

where $D_+^\alpha$ is the right fractional derivative of Riemann-Liouville, see detailed proofs in [1, 4] for example. However, as far as the author is aware and despite particular results in [16, 19], no existence result of a solution for $(\text{EL}^\alpha)$ exists in a general case.

The aim of this paper is to derive sufficient conditions on $L$ so that $(\text{EL}^\alpha)$ admits a weak solution.

Let us note that, in a more general setting, existence results for fractional equations is an emerging field. For instance, there are recent results about existence and uniqueness of solution for a class of fractional evolution equations in [32, 33].

1.2. Main result. — We denote by $\| \cdot \|$ the Euclidean norm of $\mathbb{R}^d$ and $\mathcal{C} := \mathcal{C}([a, b]; \mathbb{R}^d)$ the space of continuous functions endowed with its usual norm $\| \cdot \|_{\infty}$.

**Definition 1.** A function $u$ is said to be a weak solution of $(\text{EL}^\alpha)$ if $u \in \mathcal{C}$ and if $u$ satisfies $(\text{EL}^\alpha)$ a.e. on $[a, b].$

Let us enunciate the main result of the paper:

**Theorem 1.** Let $L$ be a Lagrangian of class $\mathcal{C}^1$ and $0 < (1/p) < \alpha < 1$. If $L$ satisfies the following hypotheses denoted by $(H_1)$, $(H_2)$, $(H_3)$, $(H_4)$ and $(H_5)$:

- there exist $0 \leq d_1 \leq p$ and $r_1, s_1 \in \mathcal{C}(\mathbb{R}^d \times [a, b], \mathbb{R}^+)$ such that:
  $$\forall (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a, b], \|L(x, y, t) - L(x, 0, t)\| \leq r_1(x, t)\|y\|^{d_1} + s_1(x, t);$$

- there exist $0 \leq d_2 \leq p$ and $r_2, s_2 \in \mathcal{C}(\mathbb{R}^d \times [a, b], \mathbb{R}^+)$ such that:
  $$\forall (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a, b], \left\| \frac{\partial L}{\partial x}(x, y, t) \right\| \leq r_2(x, t)\|y\|^{d_2} + s_2(x, t);$$

- there exist $0 \leq d_3 \leq p - 1$ and $r_3, s_3 \in \mathcal{C}(\mathbb{R}^d \times [a, b], \mathbb{R}^+)$ such that:
  $$\forall (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a, b], \left\| \frac{\partial L}{\partial y}(x, y, t) \right\| \leq r_3(x, t)\|y\|^{d_3} + s_3(x, t);$$

then there exist $0 \leq d \leq p - 1$ and $r, s \in \mathcal{C}(\mathbb{R}^d \times [a, b], \mathbb{R}^+)$ such that:

$$\forall (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a, b], \left\| \frac{\partial L}{\partial x}(x, y, t) \right\| \leq r(x, t)\|y\|^{d} + s(x, t);$$

and

$$\forall (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a, b], \left\| \frac{\partial L}{\partial y}(x, y, t) \right\| \leq r(x, t)\|y\|^{d} + s(x, t);$$
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- **coercivity condition**: there exist $\gamma > 0$, $1 \leq d_4 < p$, $c_1 \in C([a, b] \times [\gamma, \infty])$, $c_2$, $c_3 \in C([a, b], \mathbb{R})$ such that:
  \[
  \forall (x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a, b], \quad L(x, y, t) \geq c_1(x, t)\|y\|^p + c_2(t)\|x\|^{d_4} + c_3(t); \quad (H_4)
  \]

- **convexity condition**:
  \[
  \forall t \in [a, b], \quad L(\cdot, \cdot, t) \text{ is convex}, \quad (H_5)
  \]

then (EL$^\alpha$) admits a weak solution.

Hypotheses denoted by (H$_1$), (H$_2$), (H$_3$) are usually called regularity hypotheses, see [8, 12]. In Section 5, we prove that Hypothesis (H$_5$) can be replaced by different convexity assumptions.

1.3. **Idea of the proof of Theorem 1.** — In the classical case $\alpha = 1$, $D_1 = -D_+ = d/dt$ and consequently (EL$^\alpha$) is nothing else but the classical Euler-Lagrange equation formulated in the 1750's. In this case, a lot of results of existence of solutions have been already proved. Let us recall that there exist different approaches:

- A first approach is to develop the classical Euler-Lagrange equation in order to obtain an implicit second order differential equation, see [14]. Then, under a hyper regularity or non singularity condition on the Lagrangian $L$, the equation can be written as an explicit second order differential equation and the Cauchy-Lipschitz theorem gives the existence of local or global regular solutions;

- A second approach consists in using the variational structure of the equation, see [12]. Indeed, under some assumptions, the critical points of $L$ correspond to the solutions of the classical Euler-Lagrange equation. The idea is then to prove the existence of critical points of $L$. In this way, author makes some assumptions (like coercivity and convexity of the Lagrangian $L$) ensuring the existence of extrema of $L$. With this second method, author has to use reflexive spaces of functions and consequently, he deals with weak solutions (in a specific sense).

In order to prove Theorem 1, we extend the second approach to the strict fractional case (i.e. $0 < \alpha < 1$). Indeed, although there exist fractional versions of the Cauchy-Lipschitz theorem (see [17, 30]), there is no simple rules for the fractional derivative of a composition and consequently, we can not write (EL$^\alpha$) in a simpler way. Hence, in the strict fractional case, we can not follow the first method.

Theorem 1 is based on the following preliminaries:

- The introduction in Section 3 of an appropriate reflexive separable Banach space $E_{\alpha,p}$ (see (15));
- Assuming Hypotheses (H$_1$), (H$_2$) and (H$_3$). Theorem 2 in Section 4 states that if $u$ is a critical point of $L$, then $u$ is a weak solution of (EL$^\alpha$);
- Assuming additionally Hypotheses (H$_4$) and (H$_5$), Theorem 3 in Section 5 states that $L$ admits a global minimizer.

Hence, the proof of Theorem 1 is complete. Let us note that the method developed in this paper is inspired by:

- the reflexive separable Banach space introduced in [16] allowing to prove the existence of a weak solution for a class of fractional boundary value problems;
– the suitable hypotheses of regularity, coercivity and convexity given in [12] proving the existence of a weak solution for classical Euler-Lagrange equations (i.e. in the case \( \alpha = 1 \)).

1.4. Organisation of the paper. — The paper is organized as follows. In Section 2, some usual notations of spaces of functions are given. We recall the definitions of the fractional operators of Riemann-Liouville and some of their properties. Section 3 is devoted to the introduction and to the study of the appropriate reflexive separable Banach space \( E_{\alpha,p} \). In Section 4, the variational structure of \( (EL^\alpha) \) is considered and we prove Theorem 2. In Section 5, we prove Theorem 3. Then, Section 6 is devoted to some examples. Finally, a conclusion ends this paper.

2. Reminder about fractional calculus

2.1. Some spaces of functions. — For any \( p \geq 1 \), \( L^p := L^p((a,b); \mathbb{R}^d) \) denotes the classical Lebesgue space of \( p \)-integrable functions endowed with its usual norm \( \| \cdot \|_{L^p} \). Let us give some usual notations of spaces of continuous functions defined on \([a,b]\) with values in \( \mathbb{R}^d \):

- \( AC := AC([a,b]; \mathbb{R}^d) \) the space of absolutely continuous functions;
- \( C^\infty := C^\infty([a,b]; \mathbb{R}^d) \) the space of infinitely differentiable functions;
- \( C^\infty_c := C^\infty_c([a,b]; \mathbb{R}^d) \) the space of infinitely differentiable functions and compactly supported in \([a,b]\).

We remind that a function \( f \) is an element of \( AC \) if and only if \( \dot{f} \in L^1 \) and the following equality holds:

\[
\forall t \in [a,b], \quad f(t) = f(a) + \int_a^t \dot{f}(\xi) \, d\xi,
\]

(3) where \( \dot{f} \) denotes the derivative of \( f \). We refer to [20] for more details concerning the absolutely continuous functions.

Finally, we denote by \( C_a \) (resp. \( AC_a \) or \( C^\infty_a \)) the space of functions \( f \in C \) (resp. \( AC \) or \( C^\infty \)) such that \( f(a) = 0 \). In particular, \( C^\infty_c \subset C^\infty_a \subset AC_a \).

Convention: in the whole paper, an equality between functions must be understood as an equality holding for almost all \( t \in (a,b) \). When it is not the case, the interval on which the equality is valid will be specified.

2.2. Fractional operators of Riemann-Liouville. — Since 1695, numerous notions of fractional operators emerged over the year, see [17, 27, 30]. In this paper, we only deal with the fractional operators of Riemann-Liouville (1847) whose definitions and some basic results are reminded in this section. We refer to [17, 30] for the omitted proofs.

Let \( \alpha > 0 \) and \( f \) be a function defined a.e. on \((a,b)\) with values in \( \mathbb{R}^d \). The left (resp. right) fractional integral in the sense of Riemann-Liouville with inferior limit \( a \) (resp. superior limit \( b \)) of order \( \alpha \) of \( f \) is given by:

\[
\forall t \in ]a,b[ , \quad I^\alpha_a f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \xi)^{\alpha-1} f(\xi) \, d\xi ,
\]

(4)
Another classical result is the boundedness of the left fractional integral from $L^1$ concerning the composition between fractional integral and fractional derivative. For any $\alpha > 0$, one can easily deduce the following results from Property 1 and Equalities (6) and (7), respectively:

\[ D^-\alpha f = I^{\alpha - 1} \hat{f} + \frac{f(a)}{(t-a)^\alpha \Gamma(1-\alpha)} \quad \text{and} \quad D^\alpha f = -I^{1-\alpha} \hat{f} + \frac{f(b)}{(b-t)^\alpha \Gamma(1-\alpha)}. \]

In particular, if $f \in AC_\alpha$, then $D^\alpha f = I^{1-\alpha} \hat{f}$.

### 2.3. Some properties of the fractional operators

In this section, we provide some properties concerning the left fractional operators of Riemann-Liouville. One can easily derive the analogous versions for the right ones. Properties 1, 2 and 3 are well-known and can find their proofs in the classical literature on the subject (see [17, Lemma 2.3, p.73], [17, Lemma 2.1, p.72] and [17, Lemma 2.7, p.76] respectively).

The first result yields the semi-group property of the left Riemann-Liouville fractional integral:

**Property 1.** — For any $\alpha, \beta > 0$ and any function $f \in L^1$, the following equality holds:

\[ I^\alpha \circ I^\beta f = I^{\alpha + \beta} f. \]

From Property 1 and Equalities (6) and (7), one can easily deduce the following results concerning the composition between fractional integral and fractional derivative. For any $0 < \alpha < 1$, the following equalities hold:

\[ \forall f \in L^1, \quad D^-\alpha \circ I^-\alpha f = f \quad \text{and} \quad \forall f \in AC, \quad I^-\alpha \circ D^-\alpha f = f. \]

Another classical result is the boundedness of the left fractional integral from $L^p$ to $L^p$:

**Property 2.** — For any $\alpha > 0$ and any $p \geq 1$, $I^\alpha$ is linear and continuous from $L^p$ to $L^p$. Precisely, the following inequality holds:

\[ \forall f \in L^p, \quad \|I^\alpha f\|_{L^p} \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|f\|_{L^p}. \]

The following classical property concerns the integration of fractional integrals. It is occasionally called fractional integration by parts:

**Property 3.** — Let $0 < \alpha < 1$. Let $f \in L^p$ and $g \in L^q$ where $(1/p) + (1/q) \leq 1 + \alpha$ (and $p \neq 1 \neq q$ in the case $(1/p) + (1/q) = 1 + \alpha$). Then, the following equality holds:

\[ \int_a^b I^\alpha f \cdot g \ dt = \int_a^b f \cdot I^\alpha g \ dt. \]

respectively:

\[ \forall t \in [a, b], \quad I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\xi - t)^{\alpha - 1} f(\xi) \ d\xi, \quad (5) \]

where $\Gamma$ denotes the Euler’s Gamma function. If $f \in L^1$, then $I^\alpha f$ and $I_\alpha^\alpha f$ are defined a.e. on $(a,b)$.

Now, let us consider $0 < \alpha < 1$. The left (resp. right) fractional derivative in the sense of Riemann-Liouville with inferior limit $a$ (resp. superior limit $b$) of order $\alpha$ of $f$ is given by:

\[ \forall t \in [a, b], \quad D^-\alpha f(t) := \frac{d}{dt}(I^{1-\alpha}_- f)(t) \quad \text{resp.} \quad \forall t \in [a, b], \quad D^\alpha f(t) := -\frac{d}{dt}(I^{1-\alpha}_+ f)(t). \]

From [17, Corollary 2.2, p.73], if $f \in AC$, then $D^\alpha f$ and $D^-\alpha f$ are defined a.e. on $(a,b)$ and satisfy:

\[ D^\alpha f = I^{1-\alpha}_- \hat{f} + \frac{f(a)}{(t-a)^\alpha \Gamma(1-\alpha)} \quad \text{and} \quad D^-\alpha f = -I^{1-\alpha}_+ \hat{f} + \frac{f(b)}{(b-t)^\alpha \Gamma(1-\alpha)}. \]

The first result yields the semi-group property of the left Riemann-Liouville fractional integral:
This change of side of the fractional integral (from $I^a_-$ to $I^a_+$) is responsible of the emergence of $D^a_\alpha$ in $(EL^\alpha)$ although only $D^a_\alpha$ is involved in the Lagrangian functional $\mathcal{L}$. We refer to Section 4.2 for more details.

The following Property 4 completes Property 2 in the case $0 < (1/p) < \alpha < 1$: indeed, in this case, $I^a_\alpha$ is additionally bounded from $L^p$ to $\mathcal{C}_a$:

**Property 4.** — Let $0 < (1/p) < \alpha < 1$ and $q = p/(p-1)$. Then, for any $f \in L^p$, we have:
- $I^a_\alpha f$ is H"older continuous on $]a,b]$ with exponent $\alpha - (1/p) > 0$;
- $\lim_{t \to a} I^a_\alpha f(t) = 0$.

Consequently, $I^a_\alpha f$ can be continuously extended by 0 in $t = a$. Finally, for any $f \in L^p$, we have $I^a_\alpha f \in \mathcal{C}_a$. Moreover, the following inequality holds:

$$\forall f \in L^p, \|I^a_\alpha f\|_\infty \leq \frac{(b-a)^{\alpha-(1/p)}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|f\|_{L^p}. \tag{12}$$

**Proof.** — Let us note that this result is mainly proved in [16]. Let $f \in L^p$. We first remind the following inequality:

$$\forall \xi_1 \geq \xi_2 \geq 0, \ (\xi_1 - \xi_2)^q \leq \xi_1^q - \xi_2^q. \tag{13}$$

Let us prove that $I^a_\alpha f$ is H"older continuous on $]a,b]$. For any $a < t_1 < t_2 \leq b$, using the H"older’s inequality, we have:

$$\|I^a_\alpha f(t_2) - I^a_\alpha f(t_1)\| = \frac{1}{\Gamma(\alpha)} \left\| \int_a^{t_2} (t_2 - \xi)^{\alpha-1} f(\xi) \, d\xi - \int_a^{t_1} (t_1 - \xi)^{\alpha-1} f(\xi) \, d\xi \right\|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2 - \xi)^{\alpha-1} f(\xi) \, d\xi \right\|$$

$$+ \frac{1}{\Gamma(\alpha)} \left\| \int_a^{t_1} ((t_2 - \xi)^{\alpha-1} - (t_1 - \xi)^{\alpha-1}) f(\xi) \, d\xi \right\|$$

$$\leq \frac{\|f\|_{L^p}}{\Gamma(\alpha)} \left( \int_{t_1}^{t_2} (t_2 - \xi)^{(\alpha-1)q} \, d\xi \right)^{\frac{1}{q}}$$

$$+ \frac{\|f\|_{L^p}}{\Gamma(\alpha)} \left( \int_a^{t_1} ((t_1 - \xi)^{\alpha-1} - (t_2 - \xi)^{\alpha-1})^q \, d\xi \right)^{\frac{1}{q}}$$

$$\leq \frac{\|f\|_{L^p}}{\Gamma(\alpha)} \left( \int_{t_1}^{t_2} (t_2 - \xi)^{(\alpha-1)q} \, d\xi \right)^{\frac{1}{q}}$$

$$+ \frac{\|f\|_{L^p}}{\Gamma(\alpha)} \left( \int_a^{t_1} (t_1 - \xi)^{(\alpha-1)q} - (t_2 - \xi)^{(\alpha-1)q} \, d\xi \right)^{\frac{1}{q}}$$

$$\leq \frac{2\|f\|_{L^p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} (t_2 - t_1)^{\alpha-(1/p)}.$$
The proof is now complete.

3. Space of functions \( E_{\alpha,p} \)

In order to prove the existence of a weak solution of \((\text{EL}^\alpha)\) using a variational method, we need the introduction of an appropriate space of functions. This space has to present some properties like reflexivity, see [12].

For any \(0 < \alpha < 1\) and any \(p \geq 1\), we define the following space of functions:

\[
E_{\alpha,p} := \{ u \in L^p \text{ satisfying } D^-_{\alpha} u \in L^p \text{ and } I^\alpha_{-} \circ D^-_{\alpha} u = u \text{ a.e.} \}.
\]  

We endow \( E_{\alpha,p} \) with the following norm:

\[
\| \cdot \|_{\alpha,p} : E_{\alpha,p} \rightarrow \mathbb{R}^+ \quad u \mapsto \left( \| u \|_{L^p}^p + \| D^-_{\alpha} u \|_{L^p}^p \right)^{1/p}.
\]  

Let us note that:

\[
| \cdot |_{\alpha,p} : E_{\alpha,p} \rightarrow \mathbb{R}^+ \quad u \mapsto \| D^-_{\alpha} u \|_{L^p}
\]

is an equivalent norm to \( \| \cdot \|_{\alpha,p} \) for \( E_{\alpha,p} \). Indeed, Property 2 leads to:

\[
\forall u \in E_{\alpha,p}, \quad \| u \|_{L^p} = \| I^\alpha_{-} \circ D^-_{\alpha} u \|_{L^p} \leq \left( b - a \right)\alpha \Gamma(1 + \alpha) \| D^-_{\alpha} u \|_{L^p}.
\]  

The goal of this section is to prove the following proposition:

**Proposition 1.** — Assuming \(0 < (1/p) < \alpha < 1\), \( E_{\alpha,p} \) is a reflexive separable Banach space and the compact embedding \( E_{\alpha,p} \hookrightarrow \mathcal{C} \alpha \) holds.

Then, in the rest of the paper, we consider:

\[
0 < (1/p) < \alpha < 1 \quad \text{and} \quad q = p/(p - 1).
\]  

Let us detail the different points of Proposition 1 in the following subsections.

3.1. \( E_{\alpha,p} \) is a reflexive separable Banach space. — Let us prove this property. Let us consider \((L^p)^2\) the set \(L^p \times L^p\) endowed with the norm \( \| (u,v) \|_{(L^p)^2} = \| u \|_{L^p}^p + \| v \|_{L^p}^p \)^{1/p}.

Since \(p > 1\), \((L^p, \| \cdot \|_{L^p})\) is a reflexive separable Banach space and therefore, \(( (L^p)^2, \| \cdot \|_{(L^p)^2})\) is also a reflexive separable Banach space.

We define \( \Omega := \{ (u, D^-_{\alpha} u), \ u \in E_{\alpha,p} \}. \) Let us prove that \( \Omega \) is a closed subspace of \(( (L^p)^2, \| \cdot \|_{(L^p)^2})\). Let \((u_n, v_n)_{n \in \mathbb{N}} \subset \Omega \) such that:

\[
(u_n, v_n) \xrightarrow{(L^p)^2} (u, v).
\]  

Let us prove that \((u, v) \in \Omega\). For any \(n \in \mathbb{N}\), \((u_n, v_n) \in \Omega\). Thus, \(u_n \in E_{\alpha,p}\) and \(v_n = D^-_{\alpha} u_n\). Consequently, we have:

\[
u_n \xrightarrow{L^p} u \quad \text{and} \quad D^-_{\alpha} u_n \xrightarrow{L^p} v.
\]  

For any \(n \in \mathbb{N}\), since \(u_n \in E_{\alpha,p}\) and \(I^\alpha_{-}\) is continuous from \(L^p\) to \(L^p\), we have:

\[
u_n = I^\alpha_{-} \circ D^-_{\alpha} u_n \xrightarrow{L^p} I^\alpha_{-} v.
\]
Thus, $u = I_\alpha v$, $D_\alpha u = D_\alpha \circ I_\alpha v = v \in L^p$ and $I_\alpha \circ D_\alpha u = I_\alpha v = u$. Hence, $u \in E_{\alpha,p}$ and $(u, v) = (u, D_\alpha u) \in \Omega$. In conclusion, $\Omega$ is a closed subspace of $((L^p)^2, \| \cdot \|_{(L^p)^2})$ and then $\Omega$ is a reflexive separable Banach space. Finally, defining the following operator:

$$A : E_{\alpha,p} \rightarrow \Omega, \quad u \mapsto (u, D_\alpha u),$$

we prove that $E_{\alpha,p}$ is isometric isomorphic to $\Omega$. This completes the proof of Section 3.1.

3.2. The continuous embedding $E_{\alpha,p} \hookrightarrow C_a$. — Let us prove this result. Let $u \in E_{\alpha,p}$ and then $D_\alpha u \in L^p$. Since $0 < (1/p) < \alpha < 1$, Property 4 leads to $I_\alpha \circ D_\alpha u \in C_a$. Furthermore, $u = I_\alpha \circ D_\alpha u$ and consequently, $u$ can be identified to its continuous representative. Finally, Property 4 also gives:

$$\forall u \in E_{\alpha,p}, \|u\|_\infty = \|I_\alpha \circ D_\alpha u\| \leq \frac{(b - a)^{\alpha - (1/p)}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{1/q}} |u|_{\alpha,p}. \quad (24)$$

Since $\| \cdot \|_{\alpha,p}$ and $| \cdot |_{\alpha,p}$ are equivalent norms, the proof of Section 3.2 is complete.

3.3. The compact embedding $E_{\alpha,p} \hookrightarrow C_a$. — Let us prove this property. Since $E_{\alpha,p}$ is a reflexive Banach space, we only have to prove that:

$$\forall (u_n)_{n \in N} \subset E_{\alpha,p} \text{ such that } u_n \xrightarrow{E_{\alpha,p}} u, \text{ then } u_n \not\hookrightarrow u. \quad (25)$$

Let $(u_n)_{n \in N} \subset E_{\alpha,p}$ such that:

$$u_n \xrightarrow{E_{\alpha,p}} u. \quad (26)$$

Since $E_{\alpha,p} \hookrightarrow C_a$, we have:

$$u_n \not\hookrightarrow u. \quad (27)$$

Since $(u_n)_{n \in N}$ converges weakly in $E_{\alpha,p}$, $(u_n)_{n \in N}$ is bounded in $E_{\alpha,p}$. Consequently, $(D_\alpha u_n)_{n \in N}$ is bounded in $L^p$ by a constant $M \geq 0$. Let us prove that $(u_n)_{n \in N} \subset C_a$ is uniformly lipschitzian on $[a, b]$. According to the proof of Property 4, we have:

$$\forall n \in N, \forall a \leq t_1 < t_2 \leq b, \|u_n(t_2) - u_n(t_1)\| \leq \frac{2\|D_\alpha u_n\|_{L^p}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{1/p}} (t_2 - t_1)^{\alpha - (1/p)}$$

$$\leq \frac{2M}{\Gamma(\alpha)((\alpha - 1)q + 1)^{1/p}} (t_2 - t_1)^{\alpha - (1/p)}. \quad (28)$$

Hence, from Ascoli’s theorem, $(u_n)_{n \in N}$ is relatively compact in $C_a$. Consequently, there exists a subsequence of $(u_n)_{n \in N}$ converging strongly in $C_a$ and the limit is $u$ by uniqueness of the weak limit.

Now, let us prove by contradiction that the whole sequence $(u_n)_{n \in N}$ converges strongly to $u$ in $C_a$. If not, there exist $\varepsilon > 0$ and a subsequence $(u_{n_k})_{k \in N}$ such that:

$$\forall k \in N, \|u_{n_k} - u\|_\infty > \varepsilon > 0. \quad (29)$$

Nevertheless, since $(u_{n_k})_{k \in N}$ is a subsequence of $(u_n)_{n \in N}$, then it satisfies:

$$u_{n_k} \xrightarrow{E_{\alpha,p}} u.$$
In the rest of the paper, we assume that Lagrangian $L$ is of class $\mathcal{C}$ and we define the Lagrangian functional $L$ on $E_{\alpha,p}$ (with $0 < (1/p) < \alpha < 1$). Precisely, we define:

$$L : E_{\alpha,p} \rightarrow \mathbb{R},$$

$$u \mapsto \int_a^b L(u,D_\alpha^a u,t) \, dt.$$
\( \mathcal{L} \) is said to be Gâteaux-differentiable in \( u \in E_{\alpha,p} \) if the map:

\[
DL(u) : E_{\alpha,p} \rightarrow \mathbb{R} \\
v \mapsto DL(u)(v) := \lim_{h \rightarrow 0} \frac{\mathcal{L}(u + hv) - \mathcal{L}(u)}{h}
\]

is well-defined for any \( v \in E_{\alpha,p} \) and if it is linear and continuous. A critical point \( u \in E_{\alpha,p} \) of \( \mathcal{L} \) is defined by \( DL(u) = 0 \).

### 4.1. Gâteaux-differentiability of \( \mathcal{L} \). — Let us prove the following lemma:

**Lemma 1.** — The following implications hold:

- \( L \) satisfies (H1) \( \implies \) for any \( u \in E_{\alpha,p} \), \( L(u, D^\alpha u, t) \in L^1 \) and then \( \mathcal{L}(u) \) exists in \( \mathbb{R} \);
- \( L \) satisfies (H2) \( \implies \) for any \( u \in E_{\alpha,p} \), \( \partial L/\partial x(u, D^\alpha u, t) \in L^1 \);
- \( L \) satisfies (H3) \( \implies \) for any \( u \in E_{\alpha,p} \), \( \partial L/\partial y(u, D^\alpha u, t) \in L^q \).

**Proof.** — Let us assume that \( L \) satisfies (H1) and let \( u \in E_{\alpha,p} \subset \mathcal{C}_a \). Then, \( \|D^\alpha u\|^{d_1} \in L^{p/d_1} \subset L^1 \) and the three maps \( t \rightarrow r_1(u(t), t), s_1(u(t), t), |L(u(t), 0, t)| \in \mathcal{C}([a, b], \mathbb{R}^+) \subset L^\infty \subset L^1 \). Hypothesis (H1) implies for almost all \( t \in [a, b] \):

\[
|L(u(t), D^\alpha u(t), t)| \leq r_1(u(t), t)\|D^\alpha u(t)\|^{d_1} + s_1(u(t), t) + |L(u(t), 0, t)|.
\]

Hence, \( L(u, D^\alpha u, t) \in L^1 \) and then \( \mathcal{L}(u) \) exists in \( \mathbb{R} \). We proceed in the same manner in order to prove the second point of Lemma 1. Now, assuming that \( L \) satisfies (H3), we have \( \|D^\alpha u\|^{d_3} \in L^{p/d_3} \subset L^q \) for any \( u \in E_{\alpha,p} \). An analogous argument gives the third point of Lemma 1.

Let us prove the following result:

**Proposition 2.** — Assuming that \( L \) satisfies Hypotheses (H1), (H2) and (H3), \( \mathcal{L} \) is Gâteaux-differentiable in any \( u \in E_{\alpha,p} \) and:

\[
\forall u, v \in E_{\alpha,p}, DL(u)(v) = \int_a^b \frac{\partial L}{\partial x}(u, D^\alpha u, t) \cdot v + \frac{\partial L}{\partial y}(u, D^\alpha u, t) \cdot D^\alpha v \, dt.
\]

**Proof.** — Let \( u, v \in E_{\alpha,p} \subset \mathcal{C}_a \). Let \( \psi_{u,v} \) defined for any \( h \in [-1, 1] \) and for almost all \( t \in [a, b] \) by:

\[
\psi_{u,v}(t, h) := L(u(t) + hv(t), D^\alpha u(t) + hD^\alpha v(t), t).
\]

Then, we define the following mapping:

\[
\phi_{u,v} : [-1, 1] \rightarrow \mathbb{R} \\\nh \mapsto \int_a^b L(u + hv, D^\alpha u + hD^\alpha v, t) \, dt = \int_a^b \psi_{u,v}(t, h) \, dt.
\]

Our aim is to prove that the following term:

\[
DL(u)(v) = \lim_{h \rightarrow 0} \frac{\mathcal{L}(u + hv) - \mathcal{L}(u)}{h} = \lim_{h \rightarrow 0} \frac{\phi_{u,v}(h) - \phi_{u,v}(0)}{h} = \phi'_{u,v}(0)
\]

exists in \( \mathbb{R} \). In order to differentiate \( \phi_{u,v} \), we use the theorem of differentiation under the integral sign. Indeed, we have for almost all \( t \in [a, b] \), \( \psi_{u,v}(t, :) \) is differentiable on \([-1, 1] \)
Consequently, $D$ and we define similarly $s$. We define:

$$
Hence:

$$
\text{with:}
\begin{align*}
\forall h \in [-1,1], \quad & \frac{\partial \psi_{u,v}}{\partial h}(t,h) = \frac{\partial L}{\partial x}(u(t) + hv(t), D^\alpha u(t) + hD^\alpha v(t), t) \cdot v(t) \\
& + \frac{\partial L}{\partial y}(u(t) + hv(t), D^\alpha u(t) + hD^\alpha v(t), t) \cdot D^\alpha v(t). 
\end{align*}

(43)

Then, from Hypotheses $(H_2)$ and $(H_3)$, we have for any $h \in [-1,1]$ and for almost all $t \in [a,b]$:

$$
|\frac{\partial \psi_{u,v}}{\partial h}(t,h)| \leq \left[ r_2(u(t) + hv(t), t) \|D^\alpha u(t) + hD^\alpha v(t)\|^{d_2} + s_2(u(t) + hv(t), t) \|v(t)\| \right] \\
+ \left[ r_3(u(t) + hv(t), t) \|D^\alpha u(t) + hD^\alpha v(t)\|^{d_3} + s_3(u(t) + hv(t), t) \|D^\alpha v(t)\|. 
\right]
$$

(44)

We define:

$$
r_{2,0} := \max_{(t,h) \in [a,b] \times [-1,1]} r_2(u(t) + hv(t), t)
$$

(45)

and we define similarly $s_{2,0}$, $r_{3,0}$, $s_{3,0}$. Finally, it holds:

$$
\left| \frac{\partial \psi_{u,v}}{\partial h}(t,h) \right| \leq 2^{d_2} r_{2,0} \left( \|D^\alpha u(t)\|^{d_2} + \|D^\alpha v(t)\|^{d_2} \right) \|v(t)\| + s_{2,0} \|v(t)\| \\
+ 2^{d_3} r_{3,0} \left( \|D^\alpha u(t)\|^{d_3} + \|D^\alpha v(t)\|^{d_3} \right) \|D^\alpha v(t)\| + s_{3,0} \|D^\alpha v(t)\|. 
\right.
$$

(46)

The right term is then a $L^1$ function independent of $h$. Consequently, applying the theorem of differentiation under the integral sign, $\phi_{u,v}$ is differentiable with:

$$
\forall h \in [-1,1], \quad \phi'_{u,v}(h) = \int_a^b \frac{\partial \psi_{u,v}}{\partial h}(t,h) \, dt.
$$

(47)

Hence:

$$
DL(u)(v) = \phi'_{u,v}(0) = \int_a^b \frac{\partial \psi_{u,v}}{\partial h}(t,0) \, dt = \int_a^b \frac{\partial L}{\partial x}(u, D^\alpha u, t) \cdot v + \frac{\partial L}{\partial y}(u, D^\alpha u, t) \cdot D^\alpha v \, dt.
$$

(48)

From Lemma 1, it holds:

$$
\frac{\partial L}{\partial x}(u, D^\alpha u, t) \in L^1 \quad \text{and} \quad \frac{\partial L}{\partial y}(u, D^\alpha u, t) \in L^q.
$$

(49)

Since $v \in \mathcal{C}_a \subset L^\infty$ and $D^\alpha v \in L^p$, $DL(u)(v)$ exists in $\mathbb{R}$. Moreover, we have:

$$
|DL(u)(v)| \leq \left\| \frac{\partial L}{\partial x}(u, D^\alpha u, t) \right\|_{L^1} \|v\|_\infty + \left\| \frac{\partial L}{\partial y}(u, D^\alpha u, t) \right\|_{L^q} \|D^\alpha v\|_{L^p} \\
\leq \left( \frac{(b-a)^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \left\| \frac{\partial L}{\partial x}(u, D^\alpha u, t) \right\|_{L^1} + \left\| \frac{\partial L}{\partial y}(u, D^\alpha u, t) \right\|_{L^q} \right) |v|_{\alpha,p}.
$$

Consequently, $DL(u)$ is linear and continuous from $E_{\alpha,p}$ to $\mathbb{R}$. The proof is complete. \hfill \square
4.2. Sufficient condition for a weak solution. — In this section, we prove the following theorem:

**Theorem 2.** — Let us assume that $L$ satisfies Hypotheses (H$_1$), (H$_2$) and (H$_3$). Then:

$$u \text{ is a critical point of } \mathcal{L} \implies u \text{ is a weak solution of } (EL^\alpha).$$

**(50)**

**Proof.** — Let $u$ be a critical point of $\mathcal{L}$. Then, we have in particular:

$$\forall v \in \mathcal{C}^\infty_c, D\mathcal{L}(u)(v) = \int_a^b \frac{\partial L}{\partial x}(u, D^\alpha u, t) \cdot v + \frac{\partial L}{\partial y}(u, D^\alpha u, t) \cdot Dv \, dt = 0. \quad (51)$$

For any $v \in \mathcal{C}^\infty_c \subset AC_a$, $D\alpha + v = I_{1-\alpha} v \in \mathcal{C}^\infty_c$. Since $\partial L/\partial y(u, D^\alpha u, t) \in L^q$, Property 3 gives:

$$\forall v \in \mathcal{C}^\infty_c, \int_a^b \frac{\partial L}{\partial x}(u, D^\alpha u, t) \cdot v + I_{1-\alpha} \left( \frac{\partial L}{\partial y}(u, D^\alpha u, t) \right) \cdot v \, dt = 0. \quad (52)$$

Finally, we define:

$$\forall t \in [a, b], w_u(t) = \int_a^t \frac{\partial L}{\partial x}(u, D^\alpha u, t) \, dt. \quad (53)$$

Since $\partial L/\partial x(u, D^\alpha u, t) \in L^1$, $w_u \in AC_a$ and $\dot{w}_u = \partial L/\partial x(u, D^\alpha u, t)$. Then, an integration by parts leads to:

$$\forall v \in \mathcal{C}^\infty_c, \int_a^b \left( I_{1-\alpha} \left( \frac{\partial L}{\partial y}(u, D^\alpha u, t) \right) - w_u \right) \cdot \dot{v} \, dt = 0. \quad (54)$$

Consequently, there exists a constant $C \in \mathbb{R}^d$ such that:

$$I_{1-\alpha} \left( \frac{\partial L}{\partial y}(u, D^\alpha u, t) \right) = C + w_u \in AC. \quad (55)$$

By differentiation, we obtain:

$$-D^\alpha \left( \frac{\partial L}{\partial y}(u, D^\alpha u, t) \right) = \frac{\partial L}{\partial x}(u, D^\alpha u, t), \quad (56)$$

and then $u \in E_{\alpha, p} \subset \mathcal{C}$ satisfies $(EL^\alpha)$ a.e. on $[a, b]$. The proof is complete. \hfill \Box

Let us note that the use of Property 3 in the previous proof leads to the emergence of $D^\alpha \alpha$ in $(EL^\alpha)$ although $\mathcal{L}$ is only dependent of $D^\alpha u$. This asymmetry in $(EL^\alpha)$ is a strong drawback in order to solve it explicitly. However, from Theorem 1, the existence of a weak solution for $(EL^\alpha)$ will be guarantee.

5. Existence of a global minimizer of $\mathcal{L}$

In this section, under assumptions (H$_4$) and (H$_5$), we prove the existence of a global minimizer $u$ of $\mathcal{L}$, see Theorem 3. Then, $u$ is a critical point of $\mathcal{L}$ and then, according to Theorem 2, $u$ is a weak solution of $(EL^\alpha)$. This concludes the proof of Theorem 1.

As usual in a variational method, in order to prove the existence of a global minimizer of a functional, coercivity and convexity hypotheses need to be added on the Lagrangian. We have already define Hypotheses (H$_4$) (coercivity) and (H$_5$) (convexity) in Section 1.2. In this section, we introduce two different convexity hypotheses (H’$_5$) and (H”$_5$) under which Theorem 1 is still valid:
– Convexity hypothesis denoted by \((H'_5)\):
\[ \forall (x, t) \in \mathbb{R}^d \times [a, b], \ L(x, \cdot, t) \text{ is convex} \]
and \((L(\cdot, y, t))_{(y,t)\in \mathbb{R}^d\times [a,b]}\) is uniformly equicontinuous on \(\mathbb{R}^d\). \((H'_5)\)

We remind that the uniform equicontinuity of \((L(\cdot, y, t))_{(y,t)\in \mathbb{R}^d\times [a,b]}\) has to be understood as:
\[ \forall \varepsilon > 0, \exists \delta > 0, \forall (x_1, x_2) \in (\mathbb{R}^d)^2, \]
\[ \|x_2 - x_1\| < \delta \implies \forall (y, t) \in \mathbb{R}^d \times [a, b], |L(x_2, y, t) - L(x_1, y, t)| < \varepsilon. \] (57)

Let us note that Hypotheses \((H_5)\) and \((H'_5)\) are independent.

– Convexity hypothesis denoted by \((H''_5)\):
\[ \forall (x, t) \in \mathbb{R}^d \times [a, b], \ L(x, \cdot, t) \text{ is convex.} \] \((H''_5)\)

Hypothesis \((H''_5)\) is the weakest. Nevertheless, in this case, the detailed proof of Theorem 3 is more complicated. Consequently, in the case of Hypothesis \((H''_5)\), we do not develop the proof and we use a strong result proved in [12].

Let us prove the following preliminary result:

**Lemma 2.** — Let us assume that \(L\) satisfies Hypothesis \((H_4)\). Then, \(L\) is coercive in the sense that:
\[ \lim_{\|u\|_{\alpha,p} \to +\infty} L(u) = +\infty. \] (58)

**Proof.** — Let \(u \in E_{\alpha,p}\), we have:
\[ L(u) = \int_a^b L(u, D^\alpha u, t) \, dt \geq \int_a^b c_1(u, t)||D^\alpha u||^p + c_2(t)||u||^{d_4} + c_3(t) \, dt. \] (59)

Equation (18) implies that:
\[ ||u||_{L^{d_4}} \leq (b - a)^{1 - \frac{d_4}{p}} ||u||_{L^p}^{d_4} \leq \frac{(b - a)^{\alpha + 1 - \frac{d_4}{p}}}{\Gamma(\alpha + 1)} ||D^\alpha u||_{L^p}^{d_4} = \frac{(b - a)^{\alpha + 1 - \frac{d_4}{p}}}{\Gamma(\alpha + 1)} ||u||_{\alpha,p}^{d_4}. \] (60)

Finally, we conclude that:
\[ \forall u \in E_{\alpha,p}, \ L(u) \geq \gamma ||D^\alpha u||_{L^p}^{p} - ||c_2||_\infty ||u||_{L^{d_4}}^{d_4} - (b - a)||c_3||_\infty \]
\[ \geq \gamma ||u||_{\alpha,p}^{p} - ||c_2||_\infty (b - a)^{\alpha + 1 - \frac{d_4}{p}} \frac{||u||_{\alpha,p}^{d_4} - (b - a)||c_3||_\infty. \] (61)

Since \(d_4 < p\) and since the norms \(|\cdot|_{\alpha,p}\) and \(||\cdot||_{\alpha,p}\) are equivalent, the proof is complete. \(\square\)

Now, we are ready to prove Theorem 3:

**Theorem 3.** — Let us assume that \(L\) satisfies Hypotheses \((H_1)\), \((H_2)\), \((H_3)\), \((H_4)\) and one of Hypotheses \((H'_5)\), \((H''_5)\) or \((H''_5)\). Then, \(L\) admits a global minimizer.
Proof. — Let \((u_n)_{n \in \mathbb{N}}\) be a sequence in \(E_{\alpha,p}\) satisfying:
\[
\mathcal{L}(u_n) \rightarrow \inf_{v \in E_{\alpha,p}} \mathcal{L}(v) =: K.
\] (63)
Since \(L\) satisfies Hypothesis \((H_1)\), \(\mathcal{L}(u) \in \mathbb{R}\) for any \(u \in E_{\alpha,p}\). Hence, \(K < +\infty\). Let us prove by contradiction that \((u_n)_{n \in \mathbb{N}}\) is bounded in \(E_{\alpha,p}\). In the negative case, we can construct a subsequence \((u_{n_k})_{k \in \mathbb{N}}\) satisfying \(\|u_{n_k}\|_{\alpha,p} \rightarrow +\infty\). Since \(L\) satisfies Hypothesis \((H_4)\), Lemma 2 gives:
\[
K = \lim_{k \in \mathbb{N}} \mathcal{L}(u_{n_k}) = +\infty,
\] (64)
which is a contradiction. Hence, \((u_n)_{n \in \mathbb{N}}\) is bounded in \(E_{\alpha,p}\). Since \(E_{\alpha,p}\) is reflexive, there exists a subsequence still denoted by \((u_n)_{n \in \mathbb{N}}\) converging weakly in \(E_{\alpha,p}\) to an element denoted by \(u \in E_{\alpha,p}\). Let us prove that \(u\) is a global minimizer of \(\mathcal{L}\). Since:
\[
u_n \xrightarrow{E_{\alpha,p}} u \quad \text{and} \quad E_{\alpha,p} \hookrightarrow \mathcal{C}_{\alpha},
\] (65)
we have:
\[
u_n \not\to u \quad \text{and} \quad D^\alpha u_n \xrightarrow{L_p} D^\alpha u.
\] (66)
Case \(L\) satisfies \((H_5)\): by convexity, it holds for any \(n \in \mathbb{N}\):
\[
\mathcal{L}(u_n) = \int_a^b L(u_n, D^\alpha u_n, t) \, dt \geq \int_a^b L(u, D^\alpha u, t) \, dt \\
+ \int_a^b \frac{\partial L}{\partial x}(u, D^\alpha u, t) \cdot (u_n - u) \, dt + \int_a^b \frac{\partial L}{\partial y}(u, D^\alpha u, t) \cdot (D^\alpha u_n - D^\alpha u) \, dt.
\] (67)
Since \(L\) satisfies Hypotheses \((H_2)\) and \((H_3)\), \(\partial L/\partial x(u, D^\alpha u, t) \in L^1\) and \(\partial L/\partial y(u, D^\alpha u, t) \in L^q\). Consequently, using (66) and making \(n\) tend to \(+\infty\), we obtain:
\[
K = \inf_{v \in E_{\alpha,p}} \mathcal{L}(v) \geq \int_a^b L(u, D^\alpha u, t) \, dt = \mathcal{L}(u).
\] (68)
Consequently, \(u\) is a global minimizer of \(\mathcal{L}\).
Case \(L\) satisfies \((H_5')\): let \(\varepsilon > 0\). Since \((u_n)_{n \in \mathbb{N}}\) converges strongly in \(\mathcal{C}\) to \(u\), we have:
\[
\exists N \in \mathbb{N}, \forall n \geq N, \|u_n - u\|_\infty < \delta,
\] (69)
where \(\delta\) is given in the definition of \((H_5')\). In consequence, it holds a.e. on \([a,b]\):
\[
\forall n \geq N, \left| L(u_n(t), D^\alpha u_n(t), t) - L(u(t), D^\alpha u_n(t), t) \right| < \varepsilon.
\] (70)
Moreover, for any \(n \geq N\), we have:
\[
\mathcal{L}(u_n) = \int_a^b L(u, D^\alpha u, t) \, dt + \int_a^b L(u_n, D^\alpha u_n, t) - L(u, D^\alpha u_n, t) \, dt \\
+ \int_a^b L(u, D^\alpha u_n, t) - L(u, D^\alpha u, t) \, dt.
\] (71)
Then, for any $n \geq N$, it holds by convexity:

$$L(u_n) \geq \int_a^b L(u, D^\alpha u_n, t) \, dt - \int_a^b |L(u_n, D^\alpha u_n, t) - L(u, D^\alpha u_n, t)| \, dt + \int_a^b \frac{\partial L}{\partial y}(u, D^\alpha u, t) \cdot (D^\alpha u_n - D^\alpha u) \, dt. \quad (72)$$

And, using Equation (70), we obtain for any $n \geq N$:

$$L(u_n) \geq \int_a^b L(u, D^\alpha u, t) \, dt - \varepsilon(b-a) + \int_a^b \frac{\partial L}{\partial y}(u, D^\alpha u, t) \cdot (D^\alpha u_n - D^\alpha u) \, dt. \quad (73)$$

We remind that $\partial L/\partial y(u, D^\alpha u_n, t) \in L^q$ since $L$ satisfies $(H_3)$. Since $(D^\alpha u_n)_{n\in\mathbb{N}}$ converges weakly in $L^p$ to $D^\alpha u$, we obtain by making $n$ tend to $+\infty$ and then by making $\varepsilon$ tend to 0:

$$K = \inf_{v \in E_{\alpha,p}} L(v) \geq \int_a^b L(u, D^\alpha u, t) \, dt = L(u). \quad (74)$$

Consequently, $u$ is a global minimizer of $L$.

**Case $L$ satisfies $(H'_5)$**: we refer to Theorem 3.23 in [12].

Finally, combining Theorems 2 and 3, the proof of Theorem 1 is now complete.

### 6. Examples

Let us consider some examples of Lagrangian $L$ satisfying Hypotheses of Theorem 1. Consequently, the fractional Euler-Lagrange equation (EL$^\alpha$) associated admits a weak solution $u \in E_{\alpha,p}$.

The most classical example is the Dirichlet integral, i.e. the Lagrangian functional associated to the Lagrangian $L$ given by:

$$L(x, y, t) = \frac{1}{2}\|y\|^2. \quad (75)$$

In this case, $L$ satisfies Hypotheses $(H_1)$, $(H_2)$, $(H_3)$, $(H_4)$ and $(H_5)$ for $p = 2$. Hence, the fractional Euler-Lagrange equation (EL$^\alpha$) associated admits a weak solution in $E_{\alpha,p}$ for $(1/2) < \alpha < 1$.

In a more general case, the following Lagrangian $L$:

$$L(x, y, t) = \frac{1}{p}\|y\|^p + a(x, t), \quad (76)$$

where $p > 1$ and $a \in C^1(\mathbb{R}^d \times [a, b], \mathbb{R}^+)$, satisfies Hypotheses $(H_1)$, $(H_2)$, $(H_3)$, $(H_4)$ and $(H'_5)$. Consequently, the fractional Euler-Lagrange equation (EL$^\alpha$) associated to $L$ admits a weak solution in $E_{\alpha,p}$ for any $(1/p) < \alpha < 1$. Let us note that if for any $t \in [a, b]$, $a(\cdot, t)$ is convex, then $L$ satisfies Hypothesis $(H_5)$.

In the unidimensional case $d = 1$, let us take a Lagrangian with a second term linear in its first variable, i.e.:

$$L(x, y, t) = \frac{1}{p}|y|^p + f(t)x, \quad (77)$$
where \( p > 1 \) and \( f \in C^1([a,b],\mathbb{R}) \). Then, \( L \) satisfies Hypotheses (H\(_1\)), (H\(_2\)), (H\(_3\)), (H\(_4\)) and (H\(_5\)). Then, the fractional Euler-Lagrange equation (EL\(^\alpha\)) associated admits a weak solution in \( E_{\alpha,p} \) for any \((1/p)<\alpha<1\).

Theorem 1 is a result based on strong conditions on Lagrangian \( L \). Consequently, some Lagrangian do not satisfy all hypotheses of Theorem 1. We can cite the Bolza’s example in dimension \( d = 1 \) given by:

\[
L(x, y, t) = (y^2 - 1)^2 + x^4. \tag{78}
\]

\( L \) does not satisfy Hypothesis (H\(_4\)) neither Hypothesis (H\(_5\)). Nevertheless, as usual with variational methods, the conditions of regularity, coercivity and/or convexity can often be replaced by weaker assumptions specific to the studied problem. As an example, we can cite [31] and references therein about higher-order integrals of the calculus of variations. Indeed, in this paper, it is proved that calculus of variations is still valid with weaker regularity assumptions.

**Conclusion**

The method developed in this paper gives a framework in order to study the existence of weak solutions for fractional Euler-Lagrange equations. In this paper, we have studied the special case of a Lagrangian functional involving fractional derivatives of Riemann-Liouville. Nevertheless, such a method can also be developed in the case of fractional derivatives of Caputo or Hadamard. Indeed, these operators satisfy similar properties than Riemann-Liouville’s ones, see [17, 30]. In fact, the same method can be developed in the case of general linear operators used in [2, 18, 24, 26]: this is the aim of a forthcoming paper.

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