Towards Exploring $U_{e3}$

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It is emphasized that resolution of the $\theta_{23}$ ambiguity is important for determination of $\theta_{13}$ if $\sin^2 2\theta_{23} < 1$, and resolution of the $\text{sgn}(\Delta m_{31}^2)$ ambiguity is important for determination of the CP phase $\delta$. I discuss the prospects of resolution of the $\theta_{23}$ ambiguity etc. in the future long baseline experiment after the JPARC experiment measures the oscillation probabilities $P(\nu_\mu \rightarrow \nu_e)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ at $|\Delta m_{31}^2|L/4E = \pi/2$.

1. Introduction

From the recent experiments on atmospheric and solar, and reactor neutrinos, we now know approximately the values of the mixing angles and the mass squared differences of the atmospheric and solar neutrino oscillations: $(\sin^2 2\theta_{12}, \Delta m_{21}^2) \simeq (0.8, 7 \times 10^{-5}\text{eV}^2)$ for the solar neutrino and $(\sin^2 2\theta_{23}, |\Delta m_{31}^2|) \simeq (1.0, 2 \times 10^{-3}\text{eV}^2)$ for the atmospheric neutrino. In the three flavor framework of neutrino oscillations, the quantities which are still unknown to date are the third mixing angle $\theta_{13}$, the sign of the mass squared difference $\Delta m_{31}^2$ of the atmospheric neutrino oscillation, and the CP phase $\delta$. It is expected that these three quantities will be determined by long baseline experiments in the future.

It has been known that even if the values of the oscillation probabilities $P(\nu_\mu \rightarrow \nu_e)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ are exactly given we cannot determine uniquely the values of the oscillation parameters due to parameter degeneracies. There are three kinds of parameter degeneracies: the intrinsic $(\theta_{13}, \delta)$ degeneracy, the degeneracy of $\Delta m_{31}^2 \leftrightarrow -\Delta m_{31}^2$, and the degeneracy of $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$. Each degeneracy gives a twofold solution, so in

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total we have an eightfold solution if all the degeneracies are exact. In this case prediction for physics is the same for all the degenerated solutions and there is no problem. However, at least two out of the three degeneracies are lifted slightly in long baseline experiments, and there are in general eight different solutions. When we try to determine the oscillation parameters, ambiguities arise because the values of the oscillation parameters are slightly different for each solution. In particular, this causes a serious problem in measurement of CP violation, which is expected to be small effect in the long baseline experiments, and we could mistake a fake effect due to the ambiguities for nonvanishing CP violation if we do not treat the ambiguities carefully.

In this talk, assuming the JPARC experiment measures the oscillation probabilities \( P(\bar{\nu}_\mu \to \nu_e) \) and \( P(\bar{\nu}_\mu \to \bar{\nu}_e) \) at the oscillation maximum (i.e., with \( |\Delta m^2_{31}|L/4E = \pi/2 \)), I will discuss the possibilities for an experiment following JPARC to determine \( |U_{e3}| \) and \( \text{arg}(U_{e3}) \). The details of the present discussions and references are found in Ref. 1.

2. \( |U_{e3}| \)

In this section, assuming that the JPARC experiment measures \( P(\nu_\mu \to \nu_e) \) and \( P(\bar{\nu}_\mu \to \bar{\nu}_e) \) at the oscillation maximum \( \Delta \equiv |\Delta m^2_{31}|L/4E = \pi/2 \), I will discuss how the third measurement after JPARC can resolve the ambiguities by using the plot in the \((\sin^2 2\theta_{13}, 1/s^2_{23}\)) plane.

First of all, let me consider the case where experiments are done at the oscillation maximum, i.e., when the neutrino energy \( E \) satisfies \( \Delta = \pi/2 \). In this case, the trajectory of \( P(\nu_\mu \to \nu_e) = P, \ P(\bar{\nu}_\mu \to \bar{\nu}_e) = \bar{P} \) becomes a straight line in the \((X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s^2_{23})\) plane and is given by

\[
Y = \frac{f + \bar{f}}{P/f + P/\bar{f} - C(1/f + 1/\bar{f})} \left( X - \frac{C}{f} \right) \quad (1)
\]

for the normal hierarchy, and

\[
Y = \frac{f + \bar{f}}{P/f + P/\bar{f} - C(1/f + 1/\bar{f})} \left( X - \frac{C}{\bar{f}} \right) \quad (2)
\]

for the inverted hierarchy, where

\[
\left\{ \frac{f}{\bar{f}} \right\} = \pm \frac{\cos(AL/2)}{1 \mp AL/\pi}, \quad C = \left( \frac{\Delta m^2_{32}}{\Delta m^2_{21}} \right)^2 \left[ \frac{\sin(AL/2)}{AL/2\Delta} \right]^2 \sin^2 2\theta_{12},
\]

and \( A \equiv \sqrt{2}G_F N_e \) is the matter effect. Since Eqs. (1) and (2) are linear in \( X \), there is only one solution between them and \( Y=\text{const} \). Thus the ambiguity due to the intrinsic degeneracy is solved by performing experiments...
at the oscillation maximum, although it is then transformed into another ambiguity due to the $\delta \leftrightarrow \pi - \delta$ degeneracy.

![Figure 1](image.png)

Figure 1. The $\theta_{23}$ ambiguity which could arise after the JPARC measurements of $P(\nu_\mu \to \nu_e)$, $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ and $P(\nu_\mu \to \nu_\mu)$ at the oscillation maximum. (a) If $\sin^2 2\theta_{23} \simeq 1.0$ then the values of $\theta_{13}$ and $\theta_{23}$ are close to each other for all the solutions. (b) If $\sin^2 2\theta_{23} < 1$ then the $\theta_{23}$ ambiguity has to be resolved to determine $\theta_{13}$ and $\theta_{23}$ to good precision. (c) Enlarged figure of (b) with four possible values for the CP phase $\delta$ at the oscillation maximum. The solid (dashed) line stands for the normal (inverted) hierarchy.

If $\sin^2 2\theta_{23} \simeq 1$, then all the four solutions are basically close to each other in the $(\sin^2 2\theta_{13}, 1/s^2_{23})$ plane, and the ambiguity due to degeneracies are not serious as far as $\theta_{13}$ and $\theta_{23}$ are concerned (see Fig. 1(a)). On the other hand, if $\sin^2 2\theta_{23}$ deviates fairly from 1, then the solutions are separated into two groups, those for $\theta_{23} > \pi/4$ and those for $\theta_{23} < \pi/4$ in the $(\sin^2 2\theta_{13}, 1/s^2_{23})$ plane, as is shown in Fig. 1(b). In this case resolution of the $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity is necessary to determine $\theta_{13}$, $\theta_{23}$ and $\delta$.

Resolution of the $\theta_{23}$ ambiguity has been discussed by several groups using the disappearance measurement of $P(\bar{\nu}_e \to \bar{\nu}_e)$ at reactors or the silver channel $\nu_e \to \nu_\tau$ at neutrino factories. Here I will discuss the prospects of the channels $\nu_\mu \to \nu_e$, $\bar{\nu}_\mu \to \bar{\nu}_e$ and $\nu_e \to \nu_\tau$.

2.1. $\nu_\mu \to \nu_e$

From the measurements of $P(\nu_\mu \to \nu_e)$ and $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ by JPARC at the oscillation maximum the value of $\delta$ can be deduced up to the eightfold ambiguity ($\delta \leftrightarrow \pi - \delta$, $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$, $\Delta m^2_{31} \leftrightarrow -\Delta m^2_{31}$). As is depicted in Fig. 1(c), depending on whether $s^2_{23} - 1/2$ is positive or negative, I assign the subscript ±, and depending on whether our ansatz for $\text{sgn}(\Delta m^2_{31})$ is correct or wrong, I assign the subscript c or w. Thus the four possible values of $\delta$ for each assumption on the mass hierarchy are given by

$$(\delta_{+c}, \delta_{-c}, \pi - \delta_{+c}, \pi - \delta_{-c}); \quad (\delta_{+w}, \delta_{-w}, \pi - \delta_{+w}, \pi - \delta_{-w}).$$
Now suppose that the third measurement gives the value \( P \) for the oscillation probability \( P(\nu_\mu \rightarrow \nu_e) \). Then there are in general eight lines in the \((X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s^2_{23})\) plane given by

\[
f^2 X = [P - C + 2C \cos^2(\delta + \Delta)] (Y - 1) + P - 2 \cos(\delta + \Delta) \\
\times \sqrt{C(Y - 1)} \sqrt{[P - C \sin^2(\delta + \Delta)] (Y - 1) + P}
\]

for the normal hierarchy, and

\[
f^2 X = [P - C + 2C \cos^2(\delta - \Delta)] (Y - 1) + P - 2 \cos(\delta - \Delta) \\
\times \sqrt{C(Y - 1)} \sqrt{[P - C \sin^2(\delta - \Delta)] (Y - 1) + P}
\]

for the inverted hierarchy, where \( \Delta \equiv |\Delta m^2_{31}| L/4E \) is defined for the third measurement, and \( \delta \) takes one of the four values for each assumption on the mass hierarchy given in Eq. (3).

Let me look at three typical cases: \( L = 295\text{km}, L = 730\text{km}, L = 3000\text{km} \).

The reference values for the oscillation parameters used here are

\[
\sin^2 2\theta_{12} = 0.8, \quad \sin^2 2\theta_{13} = 0.05, \quad \sin^2 2\theta_{23} = 0.96, \\
\Delta m^2_{21} = 7 \times 10^{-5}\text{eV}^2, \quad \Delta m^2_{31} = 2.5 \times 10^{-3}\text{eV}^2 > 0, \quad \delta = \pi/4, \quad (6)
\]

where I am assuming the normal hierarchy for simplicity. Fig. 2 shows the trajectories of \( P(\nu_\mu \rightarrow \nu_e) \) obtained in the third measurement together with the constraint of \( P(\nu_\mu \rightarrow \nu_e) \), \( \bar{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) \) and \( P(\nu_\mu \rightarrow \nu_\mu) \) by JPARC, for \( L = 295\text{km}, L = 730\text{km}, L = 3000\text{km} \), respectively, where \( \Delta \) takes the values \( \Delta = j\pi/8 \ (j = 3, 5, 7) \).

From Eqs. (4) and (5) we see that the only difference of the solutions with \( \delta \) and with \( \pi - \delta \) appears in \( \cos(\delta \pm \Delta) \) or \( \sin(\delta \pm \Delta) \). It turns out that in order to resolve the \( \delta \leftrightarrow \pi - \delta \) ambiguity, it is necessary to perform an experiment at \( \Delta \) which is far away from \( \pi/2 \).

To resolve the \( \Delta m^2_{31} \leftrightarrow -\Delta m^2_{31} \) ambiguity, it is necessary to have a long baseline, as one can easily imagine. What is not trivial to see is that the split of the curves with the different mass hierarchies is larger for lower energy. This can be seen by showing that the ratio of the X-intercept at \( Y = 1 \) for the normal hierarchy to that for the inverted one deviates from one more for larger value of \( \Delta \) as long as \( \Delta < \pi \).

As for resolution of the \( \theta_{23} \leftrightarrow \pi/2 - \theta_{23} \) ambiguity, it turns out that the term \( | \cos(\delta + \Delta) |/f \) has to be small to resolve it. This is because in order for the third measurement curve which goes through the true point to stay away from the fake point, the X-intercept at \( Y = 1 \) of this curve has to be far away from that of the JPARC line, and the difference in the X-intercepts
1.5  2  2.5  3  3.5
1/s
223
sin
2
2
θ
13

E=0.80 GeV, P=0.0253
L=295km
Δ=(3/8)π

E=0.48 GeV, P=0.0206
L=295km
Δ=(5/8)π

E=0.34 GeV, P=0.0020
L=295km
Δ=(7/8)π

E=1.96 GeV, P=0.0277
L=730km
Δ=(3/8)π

E=1.18 GeV, P=0.0258
L=730km
Δ=(5/8)π

E=0.84 GeV, P=0.0043
L=730km
Δ=(7/8)π

E=8.07 GeV, P=0.0388
L=3000km
Δ=(3/8)π

E=4.84 GeV, P=0.0684
L=3000km
Δ=(5/8)π

E=3.46 GeV, P=0.0510
L=3000km
Δ=(7/8)π

Figure 2. The trajectories of $P(\nu_{\mu} \rightarrow \nu_e) = \text{const.}$ of the third experiment at $L = 295\text{km}$, $L = 730\text{km}$, $L = 3000\text{km}$ with $\Delta \equiv |\Delta m^2_{31}|L/4E = (j/8)\pi$ ($j = 3, 5, 7$) after JPARC. The true values are those in Eq. (6). The dashed line is the JPARC result obtained by $P(\nu_{\mu} \rightarrow \nu_e)$ and $P(\bar{\nu}_{\mu} \rightarrow \bar{\nu}_e)$ at the oscillation maximum. The black (grey) solid lines are the trajectories of $P(\nu_{\mu} \rightarrow \nu_e)$ given by the third experiment assuming the normal (inverted) hierarchy, where $\delta$ takes four values for each mass hierarchy. The blob (cross) stands for the true (fake) solution given by the JPARC results on $P(\nu_{\mu} \rightarrow \nu_e)$, $P(\bar{\nu}_{\mu} \rightarrow \bar{\nu}_e)$ and $P(\nu_{\mu} \rightarrow \nu_{\mu})$.

$X_{\text{IPARC}}$ and $X_{3\text{rd}}$ at $Y = 1$ for the JPARC line and the third measurement line is proportional to $|\cos(\delta + \Delta)|/f$. When $AL$ is small, in order for $f$ to be small, $|\Delta - \pi|$ has to be small. Furthermore, $|\cos(\delta + \Delta)|$ has to be large. In real experiments, however, nobody knows the value of the true $\delta$ in advance, so it is difficult to design a long baseline experiment to resolve the $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ ambiguity. If $\delta$ turns out to satisfy $|\cos(\delta + \Delta)| \sim 1$ in the result of the third experiment, then we may be able to resolve the $\theta_{23}$ ambiguity as a byproduct.

2.2. $\bar{\nu}_{\mu} \rightarrow \bar{\nu}_e$

It turns out that the situation does not change very much even if I use the $\bar{\nu}_{\mu} \rightarrow \bar{\nu}_e$ channel in the third experiment. Typical curves are given for $\bar{\nu}_{\mu} \rightarrow \bar{\nu}_e$ in Fig. 3, which are similar to those in Fig. 2. Thus the conclusions
$E=2.39$ GeV, $\bar{P}=0.0051$
$L = 295$ km

$E=5.90$ GeV, $\bar{P}=0.0049$
$L = 730$ km

$E=24.26$ GeV, $\bar{P}=0.0029$
$L = 3000$ km

Figure 3. The trajectories of $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = \bar{P} = \text{const.}$ of the third experiment with
$\Delta \equiv |\Delta m^2_{31}|L/4E = \pi/8$ after JPARC. The behaviors are almost similar to those for $P(\nu_\mu \rightarrow \nu_e) = \text{const.}$ The true values are those in Eq. (6). The blob (cross) stands for the true (fake) solution as in Fig. 2.

$E=24.26$ GeV, $Q=0.0031$
$\Delta = (2/8)\pi$

$E=12.13$ GeV, $Q=0.0125$
$\Delta = (2/8)\pi$

$E=8.09$ GeV, $Q=0.0249$
$\Delta = (3/8)\pi$

Figure 4. The trajectories of $P(\nu_e \rightarrow \nu_\tau) = Q = \text{const.}$ of the third experiment at $L=2810$km with $\Delta \equiv |\Delta m^2_{31}|L/4E = (j/8)\pi$ ($j=1,2,3$) after JPARC. The true values are those in Eq. (6). The blob (cross) stands for the true (fake) solution as in Fig. 2.

drawn on resolution of the ambiguities hold qualitatively in the case of $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ channel.

2.3. $\nu_e \rightarrow \nu_\tau$

The trajectory of $P(\nu_e \rightarrow \nu_\tau) = Q$, where $Q$ is constant, in the $(X \equiv \sin^2 2\theta_{13}, Y \equiv 1/s^2_{23})$ plane is given by

$$X = \frac{Q}{f^2} \left\{ 1 + \frac{2 \cos^2(\delta + \Delta)}{1 - C/Q} \right\} \frac{1 - C/Q}{Y - 1} + 1$$

$$- \frac{2 \cos(\delta + \Delta)}{\sqrt{1 - C/Q}} \sqrt{\left[ 1 + \frac{\cos^2(\delta + \Delta)}{1 - C/Q} \right] \frac{1 - C/Q}{Y - 1} + 1}. \quad (7)$$

Eq. (7) is plotted in Fig. 4 in the case of $L=2810$km. From Fig. 4 we see that the curve $P(\nu_e \rightarrow \nu_\tau) = Q$ intersects with the JPARC dashed line almost perpendicularly and it is experimentally advantageous: Since the lines become thick due to the experimental errors in reality, the allowed region is a small area around the true solution in the $(\sin^2 2\theta_{13}, 1/s^2_{23})$
plane, so that one expects that the fake solution with respect to the \( \theta_{23} \) ambiguity can be excluded. This is in contrast to the case of the \( \nu_{\mu} \rightarrow \nu_{e} \) and \( \bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e} \) channels, in which the slope of the black curves is almost the same as that of the JPARC dashed line and the allowed region can easily contain both the true and fake solutions, so that it becomes difficult to distinguish the true point from the fake one. As in the case of the \( \nu_{\mu} \rightarrow \nu_{e} \) channel, the \( \delta \leftrightarrow \pi - \delta \) ambiguity is expected to be resolved more likely for the larger value of \( |\Delta - \pi/2| \), and the \( \text{sgn}(\Delta m_{31}^{2}) \) ambiguity is resolved easily for larger baseline \( L \) (e.g., \( L \sim 3000\text{km} \)). Thus the measurement of the \( \nu_{e} \rightarrow \nu_{\tau} \) channel is a promising possibility as a potentially powerful candidate to resolve parameter degeneracies in the future.

3. \( \text{arg}(U_{e3}) \)

3.1. \textit{Fake effects on CP violation due to the} \( \text{sgn}(\Delta m_{31}^{2}) \) \textit{ambiguity}

If the true value is \( \delta = 0 \), then the fake value \( \delta' \) with respect to the \( \text{sgn}(\Delta m_{31}^{2}) \) ambiguity in the case of the JPARC experiment is given by

\[
\sin \delta' \simeq -2.2 \sin 2\theta_{13},
\]

which is not negligible unless \( \sin^{2} 2\theta_{13} \ll 10^{-2} \). In Fig.5 the region is depicted in the \((\sin \delta, \sin^{2} 2\theta_{13})\) plane in which CP violation cannot be claimed to be nonzero in the case with the correct (wrong) assumption on the mass hierarchy. Therefore, to determine the CP phase to good precision, it is important to know the sign of \( \Delta m_{31}^{2} \).

3.2. \textit{Fake effects on CP violation due to the} \( \theta_{23} \) \textit{ambiguity}

If the true value \( \delta \) is zero, then the CP phase \( \delta' \) for the fake solution with respect to the \( \theta_{23} \) ambiguity in the case of JPARC is given by

\[
|\sin \delta'| \sim \frac{1}{200} \left| \cot 2\theta_{23} \right| \frac{1}{\sin 2\theta_{13}} \lesssim \frac{1}{500} \frac{1}{\sqrt{\sin^{2} 2\theta_{13}}},
\]

where I have used the bound \( 0.90 \leq \sin^{2} 2\theta_{23} \leq 1.0 \) from the atmospheric neutrino data in the second inequality, so that we see that the ambiguity due to the \( \theta_{23} \) does not cause a serious problem on determination of \( \delta \) for \( \sin^{2} 2\theta_{13} \gtrsim 10^{-2} \).
4. Summary

The two main conclusions are: (1) To determine $\theta_{13}$, it is important to resolve the $\theta_{23}$ ambiguity if $\sin^2 2\theta_{23}$ turns out to deviate fairly from 1; (2) To determine $\delta$, it is important to resolve the $\text{sgn}(\Delta m_{31}^2)$ ambiguity. The possibility to resolve the $\theta_{23}$ ambiguity was discussed in the case of $\nu_\mu \to \nu_e$, $\bar{\nu}_\mu \to \bar{\nu}_e$ and $\nu_e \to \nu_\tau$, using the plot of constant probabilities in the $(\sin^2 2\theta_{13}, 1/s_{23}^2)$ plane. The $\nu_e \to \nu_\tau$ channel seems to be most promising, while other two channels $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ may be useful to resolve the $\theta_{23}$ and $\delta \leftrightarrow \pi - \delta$ ambiguities for $\pi/2 < \Delta \equiv |\Delta m_{31}|L/4E < \pi$. Experiments with longer baselines ($\gtrsim 1000$km) are expected to determine $\text{sgn}(\Delta m_{31}^2)$.

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