Chiral perturbation theory for $K^+ \to \pi^+\pi^0$ decay in the continuum and on the lattice

Maarten Golterman$^a$ and Ka Chun Leung$^a$ *

$^a$Department of Physics, Washington University, St. Louis, MO 63130, USA.

We use one-loop chiral perturbation theory (ChPT) to compare lattice results for the $K^+ \to \pi^+\pi^0$ decay amplitude with the experimental value. Three systematic effects: quenching, finite-volume effects, and the use of unphysical values of quark masses and pion external momenta can be investigated. We find that the corrections help in explaining the discrepancy between lattice and experimental results. We also discuss the relation to $B_K$.

1. Basic theory

In the Standard Model, the decay $K^+ \to \pi^+\pi^0$ is induced by the four-fermion operator

$$O_4 = s_L\gamma^\mu d_L\bar{u}_L\gamma^\mu u_L + s_L\gamma^\mu d_L\bar{u}_L\gamma^\mu u_L - s_L\gamma^\mu d_L\bar{d}_L\gamma^\mu d_L.$$  

In ChPT, $O_4$ is represented by

$$O_4 = \alpha_2 r_{kl}^{ij}(\Sigma\partial^\mu\Sigma^\mu),$$  

at $O(p^2)$ with the tensor $r_{kl}^{ij}$ projecting out the $\Delta S = 1$, $\Delta I = 3/2$ components, and a large number of $O(p^2)$ operators, each with its associated coefficient, constructed from the unitary $\Sigma$-field and the quark-mass matrix $M$. A similar representation can be constructed for $O' = s_L\gamma^\mu d_L\bar{u}_L\gamma^\mu u_L$, which is related to $\Sigma^\dagger - K^0$ mixing. At $O(p^2)$, this amounts to just changing the tensor $r_{kl}^{ij}$.

The matrix element $\langle \pi^+\pi^0|O_4|K^+\rangle$ to one loop can be obtained by calculating loop diagrams from the $O(p^4)$ operator, which gives rise to chiral logarithms with known coefficients, and which depend on a cutoff $\Lambda$, and tree-level contributions from $O(p^4)$ operators. Since we do not have enough information to determine the $O(p^4)$-operator coefficients, we will estimate the size of one-loop corrections by setting all these coefficients to zero and choosing $\Lambda$ to be 770 MeV or 1 GeV, which is generally believed to be the energy scale below which physical effects of more massive hadrons can be absorbed into the $O(p^4)$ coefficients. The sensitivity of physical quantities to these different values of the cutoff is taken as an estimate of the systematic error associated with the lack of knowledge of the $O(p^4)$ coefficients.

2. Continuum result

The one-loop calculation of $\langle \pi^+\pi^0|O_4|K^+\rangle$ in ChPT (for $m_\pi = 0$) was first undertaken in [3]. Numerically, the real-world result for $\langle \pi^+\pi^0|O_4|K^+\rangle$, with $m_u = m_d \neq m_s$, and $m_\pi = 136$ MeV, $m_K = 496$ MeV and $f_\pi = 132$ MeV, is

$$\langle \pi^+\pi^0|O_4|K^+\rangle = \frac{12i\alpha_{\pi^+}}{\sqrt{2}f_\pi} (m_K^2 - m_s^2) \times \left(1 + \frac{0.63}{0.36}, \Lambda = 1 \text{ GeV}, \Lambda = 770 \text{ MeV}\right).$$

We see that the one-loop correction is fairly large.

3. Lattice results

It is possible to extract a related matrix element on a lattice with spatial volume $L^3$ from a computation of the correlation function

$$C(t_2, t_1) \equiv \langle 0|\pi^+(t_2)\pi^0(t_2)O_4(t_1)K^- (0)|0\rangle \langle 0|\pi^+(t_2)\pi^0(t_2)O_4(t_1)K^- (0)|K^-|0\rangle \times$$

$$\langle \pi^+\pi^0|\pi^+\pi^0\rangle \langle \pi^+\pi^0|O_4|K^+\rangle \langle K^+|K^-|0\rangle,$$

where $\pi^+(t) = \sum_\mathbf{x} \pi^+(\mathbf{x}, t)$, $\pi^+ = \pi^+(0)$, etc. and $E_{2\pi}$ is the energy of a state with two pions at rest in a finite volume. (One also needs
kaon and two-pion correlation functions.) Except for $m_K = 2m_\pi$, an “unphysical” matrix element is obtained since all external mesons are at rest. Also, in current lattice computations \cite{4}, unphysical masses (degenerate quark masses which are heavier than real-world ones), the quenched approximation and, of course, finite volume are used. We will use quenched ChPT \cite{6} to calculate $\langle \pi^+ \pi^0 | O_I | K^+ \rangle^{\text{quenched}}$ to one loop. The result can then be compared to eq. (2) to get an estimate of all these systematic effects. It should be noted that the coefficients $\alpha_{27}$ and $\alpha_{27}'$, the tree-level meson decay constants $f$ and $f_q$ (for quenched), etc., are in principle not equal. As for finite volume, we have: 1. the difference between finite-volume and continuum values for operator coefficients are exponentially small in $L$ (ESL) \cite{3}; 2. spatial momentum integrals $\int \frac{d^3k}{(2\pi)^3} f(k^2, m_\pi)$ are replaced by discrete sums $\frac{1}{L^3} \sum_{\vec{k} = \vec{a}/L} f(k^2, m_\pi)$, $\vec{a} \in \mathbb{Z}^3$ (for periodic boundary conditions). If the integrand is regular, the sums are equal to the continuum integrals with corrections ESL. Otherwise, there are additional power corrections in $L^{-1}$ \cite{3}:

$$\frac{1}{L^3} \sum_{k \neq 0} \frac{f(k^2, m_\pi)}{k^2} = \int \frac{d^3k}{(2\pi)^3} \frac{f(k^2, m_\pi)}{k^2} \tag{4}$$

$$-0.22578 \frac{L}{f(0, m_\pi)} - \frac{1}{L^2} \frac{df}{dk^2}(0, m_\pi) + \text{ESL}.$$  

We will not consider ESL corrections. We calculated the unphysical amplitude for degenerate quark masses, in which case $O_4$ and $O'$ do not couple to the $\eta'$-meson, and hence there are no contributions from the “$\eta'$-parameters” $\delta$ and $\alpha$ \cite{3}.

Details of the calculation can be found in \cite{3}. The diagram in which the two pions produced from the $K^+$-decay strongly rescatter leads to power-like finite-volume effects. Suppose the strong-interaction vertex acts at $t_s$ with $t_1 < t_s < t_2$. The essential part of the expression for this diagram is $L^{-3} \sum_{\vec{k}} f(t_1) dt_s \exp[-2(\sqrt{k^2 + m_\pi^2} - m_\pi)(t_s - t_1)]$. For $\vec{k} = 0$, this gives $(t_2 - t_1)/L^3$ which can be resummed into the tree-level result, and thus produces $E_{2\pi} = 2m_\pi + 1/(2L^3f_\pi^2)$ in the exponent in eq. (3). For $\vec{k} \neq 0$, we get terms $\propto 1/(\sqrt{m_\pi^2 + \vec{k}^2 - m_\pi}) = (\sqrt{m_\pi^2 + \vec{k}^2 + m_\pi})/\vec{k}^2$. These give rise to sums like eq. (4), which results in $L^{-1}$ and $L^{-3}$ corrections. We will discard any excited-state contributions \cite{3}.

The collected one-loop corrections for the unphysical matrix element $\langle \pi^+ \pi^0 | O_4(0) | K^+ \rangle$ are

$$\frac{m_\pi^2}{(4\pi f)^2} \left(-6 \log \frac{m_\pi^2}{\Lambda^2} + F(m_\pi L)\right) \tag{5}$$

relative to the tree-level value $24i\alpha_{27}m_\pi^2L^3/\sqrt{2}f^3$ for the full theory, and

$$\frac{m_\pi^2}{(4\pi f_q)^2} \left(-3 \log \frac{m_\pi^2}{\Lambda^2} + F(m_\pi L)\right) \tag{6}$$

relative to the tree-level value $24ia_{27}'m_\pi^2L^3/\sqrt{2}f^3$ for the quenched theory.

$$F(m_\pi L) = 17.827/(m_\pi L) + 12\pi^2/(m_\pi L)^3 \tag{7}$$

is the finite-volume correction.

4. Numerical examples

To one loop in ChPT the physical matrix element and the unphysical one from quenched lattice computations (after extrapolation to the continuum limit) are related by

$$\langle \pi^+ \pi^0 | O_4(0) | K^+ \rangle_{\text{phys}} = Y \frac{\alpha_{27}}{\alpha_{27}'} (\frac{f_q}{f'})^3 \times \frac{m_K^2 - m_\pi^2}{2M_\pi^2} \langle \pi^+ \pi^0 | O_4(0) | K^+ \rangle_{\text{unphys}}, \tag{8}$$

with

$$Y = \frac{1 + 0.089, \Lambda = 1 \text{ GeV}}{1 + M_\pi^2/(4\pi f^2)^2 \left[-3 \log \frac{M_\pi^2}{\Lambda^2} + F(M_\pi L)\right]} \tag{9}.$$  

The conversion factor $Y$ embodies all the one-loop corrections. The $\alpha_{27}$ and $f$ ratios in the prefactor are not known. We will arbitrarily set them equal to one. It remains one of the uncertainties that cannot be resolved within ChPT. $M_\pi$ and $F$ refer to values computed on the lattice while $m_K$ and $m_\pi$ refer to real-world values. When we apply the formula to the lattice data of \cite{3} (in which the mass-squared-ratio prefactor
was already taken into account), we get values which are shown in fig. 1, along with the original data (for which the error bars are statistical only). The error bars on our points come from varying $\Lambda$ and $\Lambda_q$ independently, and do not contain the statistical errors. We have eliminated points with smaller physical volume or at which $M_\pi > 770$ MeV $\approx m_\rho$. At all points, $Y < 1$ and each “corrected” amplitude lies below the corresponding original one. The one-loop results reduce the discrepancy between the lattice data and the experimental result. However, one-loop effects are rather substantial, and two-loop corrections can probably not be neglected. For more discussion of all uncertainties involved in these estimates, we refer to [3].

\begin{equation}
\mathcal{R} = f_K \frac{\langle \pi^+ \pi^0 | O_4 | K^+ \rangle}{\langle K^0 | O_4^4 | K^0 \rangle} \tag{12}
\end{equation}

in which $\alpha_{27}$ drops out is of some interest. In the physical case, the one-loop correction is 60% to 95% of the tree-level value in magnitude, depending on the values of the cutoff. This calls into question the reliability of ChPT in this case. In the unphysical case,

\begin{equation}
\mathcal{R}^{\text{full}}_{\text{unphys}} = \mathcal{R}^{\text{quenched}}_{\text{unphys}} = \frac{3i}{\sqrt{2}} \left[ 1 + \frac{M_\pi^2}{(4\pi F_\pi)^2} \left( 3 \log \frac{M_\pi^2}{\Lambda^2} + F(M_\pi L) \right) \right]. \tag{13}
\end{equation}

The typical magnitude of the one-loop corrections for the data of [3] is 10–15%.

We would like to thank Claude Bernard and Steve Sharpe for discussions. This work was supported in part by the DOE.

**REFERENCES**

1. J. Kambor, J. Missimer, D. Wyler, Nucl. Phys. B346, 17 (1990).
2. J. Bijnens, H. Sonoda, M. Wise, Phys. Rev. Lett. 53 (1984) 2367.
3. M. Golterman, K.C. Leung, Phys. Rev. D56 (1997) 2950.
4. N. Ishizuka for JLQCD, these proceedings.
5. C. Bernard, M. Golterman, Phys. Rev. D45 (1992) 853.
6. J. Gasser, H. Leutwyler, Phys. Lett. B184 (1987) 83.
7. M. Lüscher, Comm. Math. Phys. 105 (1986) 153.
8. C. Bernard, A. Soni, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 495.