Refractive index in holographic superconductors

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Abstract: With the probe limit, we investigate the behavior of the electric permittivity and effective magnetic permeability and related optical properties in the s-wave holographic superconductors. In particular, our result shows that unlike the strong coupled systems which admit a gravity dual of charged black holes in the bulk, the electric permittivity and effective magnetic permeability are unable to conspire to bring about the negative Depine-Lakhtakia index at low frequencies, which implies that the negative phase velocity does not appear in the holographic superconductors under such a situation.
1. Introduction

Since the advent of AdS/CFT correspondence, not only do the main efforts include the examination of its validity case by case, but also involve its applications in various fields, in which condensed matter physics has recently received a special attention. The motives are twofold. On the one hand, there exists many strong coupled systems in condensed matter physics, which are intractable by the traditional approaches. While AdS/CFT correspondence, as kind of strong/weak duality, can provide a powerful tool to attack them. On the other hand, unlike other disciplines such as particle physics and cosmology, condensed matter physics allows one to engineer matter such that various vacuum states and phases can be created in the laboratory, which may in turn provide the first experimental evidence for AdS/CFT correspondence. For a review of this subject, please refer to [1, 2, 3, 4].

In particular, most recently such a bulk/boundary duality has been applied to explore the optical properties of strong coupled media which admit charged black holes as a dual gravitational description and it is shown that this type of media generally have a negative refractive index at low frequencies[5]. Along this line, this paper is a first attempt to investigate how the refractive index and related optical properties behave in the holographic superconductors and to see whether the negative refractive index shows up at low frequencies.
The gravity dual of superconductors was firstly established to model the s-wave superconductors[6, 7]. Later both the p-wave and d-wave holographic superconductors were also realized[8, 9, 10, 11, 12, 13]. Please refer to [14] and references therein for a review of various properties related to such holographic superconductors.

In the next section, we shall recall the holographic model of s-wave superconductors, where to make our life easier we would like to work in the probe limit and focus on the special case in which the complex scalar field is massless. After the setup, in Section 3 we will present our numerical results for the behavior of the relevant optical quantities in such holographic superconductors. Conclusions and discussions will be addressed in the end.

2. Holographic model of superconductors

2.1 Abelian-Higgs model and holographic dictionary

The bulk dual contents of a holographic superconductor with a s-wave order parameter in 4+1 dimensions include the gravitational field, the \( U(1) \) gauge field, and the complex scalar field of mass \( m \) and charge \( q \). The corresponding bulk action is given by[11]

\[
S_{\text{bulk}} = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{12}{L^2} \right) - \frac{1}{q^2} \left( \frac{1}{4} F_{ab} F^{ab} + |D_a \Phi|^2 + m^2 |\Phi|^2 \right) \right],
\]

where \( F \equiv dA \), and \( D \equiv \nabla - iA \) where \( \nabla \) is the covariant derivative compatible with the metric. In what follows, we will work in the probe limit, i.e., \( \kappa q L \to 0 \) in which gravity decouples from the matter fields and the backreaction from the matter fields can be ignored. Thus a solution to Einstein equation is a black brane, i.e.,

\[
ds^2 = \frac{L^2}{u^2} \left[ - f(u) dt^2 + dx^2 + dy^2 + dz^2 + \frac{du^2}{f(u)} \right],
\]

where \( f(u) = 1 - (\frac{u}{u_h})^4 \) with \( u_h > 0 \) the horizon. Obviously, in the limit \( u \to 0 \), such a solution asymptotically becomes anti-de Sitter. Next around this background the equations of motion for the matter field are given by

\[
0 = D_a D^a \Phi - m^2 \Phi = \frac{1}{\sqrt{-g}} \left( \partial_a - i A_a \right) \left[ \sqrt{-g} g^{ab} \left( \partial_b - i A_b \right) \Phi \right] - m^2 \Phi,
\]

\[
0 = \nabla_a F^{ab} - i \left( \Phi D^b \Phi - \Phi D^b \Phi \right) = \frac{1}{\sqrt{-g}} \partial_a \left[ \sqrt{-g} F^{ab} \right] - i \left[ \Phi \left( \partial^b - i A^b \right) \Phi - \Phi \left( \partial^b + i A^b \right) \Phi \right].
\]

By the holographic dictionary, the boundary value of the metric \( g_{\mu \nu} \) acts as the source for the energy momentum stress tensor \( T^{\mu \nu} \) on the boundary, the bulk gauge field \( A_\mu \)
evaluated at the boundary serves as the source for a conserved current $J^\mu$ associated with a global $U(1)$ symmetry, and the near boundary data of the scalar field $\Phi$ sources a scalar operator $O$ with the conformal scaling dimension $\Delta$ satisfying $\Delta(\Delta - 4) = m^2L^2$. In what follows, we shall be focusing on the particular case, i.e., $m^2 = 0$, which yields an operator of dimension four by normalizability. Whence the asymptotic solution of $\Phi$ and $A_\mu$ can be expanded as

$$
\Phi = \frac{g}{\sqrt{L^3}}(\Phi^{(0)} + \Phi^{(1)}u^4 + \cdots),
$$

$$
A_\mu = A^{(0)}_\mu + \frac{g^2}{2L}[A^{(1)}_\mu u^2 - g^{(0)\rho\sigma}(\partial_\rho A^{(0)}_\sigma - \partial_\sigma A^{(0)}_\rho)u^2 \ln \frac{u}{\epsilon}] + \cdots
$$

(2.4)

with $\epsilon \ll 1$ a UV cutoff and $g^0_{\mu\nu} = \lim_{u \to 0} \frac{u^2}{L^2}g_{\mu\nu}$. Then it follows from the holographic dictionary that the expectation value of the corresponding boundary quantum field theory operators $\langle O \rangle$ and $\langle J^\mu \rangle$ can be obtained by variations of the renormalized action with respect to the sources, i.e.,

$$
\langle O \rangle = \lim_{u \to 0} \frac{1}{\sqrt{-g^{00}}} \frac{\delta S}{\delta \Phi^{(0)}} = \Phi^{(1)},
$$

$$
\langle J^\mu \rangle = \lim_{u \to 0} \frac{1}{\sqrt{-g^{00}}} \frac{\delta S}{\delta A^{(0)}_\mu} = g^{(0)\mu\nu}[A^{(1)}_\nu + cg^{(0)\rho\sigma}(\partial_\rho A^{(0)}_\sigma - \partial_\sigma A^{(0)}_\rho)],
$$

(2.5)

where the constant $c$ is renormalization scheme dependent, to be fixed in the later discussions.

### 2.2 Holographic normal phase and superconducting phase

With the following ansatz, i.e.,

$$
\Phi = \Phi(z), A_t = \phi(z), A_x = 0, A_y = 0, A_z = 0,
$$

(2.6)

the equations of motion can be reduced to

$$
0 = \Phi'' + \left(\frac{f'}{f} - \frac{3}{u}\right)\Phi' + \frac{\phi^2}{f^2}\Phi,
$$

$$
0 = \phi'' - \frac{1}{u}\phi' - \frac{2L^2\phi^2}{fu^2}\phi,
$$

(2.7)

where the prime denotes the differentiation with respect to $u$ and $\Phi$ has been taken to be real by taking into account the fact that the $u$ component of Maxwell equations implies that the phase of $\Phi$ is independent of $u$. Note that the regularization condition requires $\phi = 0$ on the horizon. Then multiplying the first equation in (2.7) by $f$, one
can find $\Phi' = 0$ on the horizon. Thus for the above two second differential equations, we have only a two parameter family of solutions, which can be labeled by $\Phi(u_h)$ and $\phi'(u_h)$. For convenience, we will set $q = 1$, $L = 1$, $u_h = 1$ in the later calculations. The result for any other $u_h$ can be easily restored by the following scaling rules, i.e.,

\[
\begin{align*}
\Phi^{(0)}(u_h) &= \Phi^{(0)}(1), \\
\Phi^{(1)}(u_h) &= \frac{1}{u_h}\Phi^{(1)}(1), \\
A^{(0)}_{\mu}(u_h) &= \frac{1}{u_h}A^{(0)}_{\mu}(1), \\
A^{(1)}_{\mu}(u_h) &= \frac{1}{u_h^3}A^{(1)}_{\mu}(1), \\
T(u_h) &= \frac{1}{u_h \pi}.
\end{align*}
\] (2.8)

Firstly, it is easy to find the trivial solution as

\[
\Phi = 0, \phi = C(1 - u^2),
\] (2.9)

which corresponds to the normal phase on the boundary with the chemical potential $C$. On the other hand, according to the numerical calculations, by fixing the charge density or chemical potential the non-trivial solution emerges below a critical temperature.

**Figure 1:** The condensate as a function of charge density, where the temperature is fixed to be 1 as the reference scale.
Figure 2: The condensate as a function of chemical potential, where the temperature is fixed to be 1 as the reference scale.

$T_c$, and the condensate increases when the temperate is lowered\[16\]. Its boundary correspondence is the superconducting phase.

By the above scaling rules, such a non-trivial solution can also be thought of as the superconducting phase emerging above a critical charge density $\rho_c$ or a chemical potential $\Sigma_c$ if the temperature is fixed. We would like to plot the corresponding condensate as a function of charge density and chemical potential in Fig.1 and Fig.2 respectively.

3. Refractive index in holographic superconductors

3.1 Holographic setup

Before proceeding, we would like to introduce some quantities we want to calculate and their relations without going into details. In the linear response theory, the induced electromagnetic current $J$ is related to the external vector potential $A$ as $J_i(\omega,k) = G_{ij}(\omega,k)A_j(\omega,k)$, where $G$ is the retarded Green function. On the other hand, in the Laudau-Lifshitz approach to electrodynamics of continuous media\[17, 18, 19\], for the isotropic media, the transverse part of the dielectric tensor is determined by the
transverse retarded Green function as follows\(^{1}\)\(^{20}\)

\[
\epsilon_T(\omega, k) = 1 + \frac{4\pi}{\omega^2} G_T(\omega, k),
\]

from which the corresponding electric permittivity and effective magnetic permeability can be expressed as

\[
\epsilon(\omega) = 1 + \frac{4\pi}{\omega^2} G_{T0}(\omega),
\]

\[
\mu(\omega) = \frac{1}{1 - 4\pi G_{T2}(\omega)}.
\]

Here \(G_{T0}(\omega)\) and \(G_{T2}(\omega)\) are the expansion coefficients of the transverse retarded Green function in \(k\), i.e.,

\[
G_T(\omega, k) = G_{T0}(\omega) + k^2 G_{T2}(\omega) + \cdots
\]

with \(G_T(\omega, k) = (\delta^{ij} - \frac{k_i k_j}{k^2}) G_{ij}(\omega, k)\). With this, the refractive index can be given by

\[
n^2(\omega) = \epsilon(\omega)\mu(\omega).
\]

But to identify whether our holographic superconductors display the opposite phase velocity to the power flow, we shall appeal to the Depine-Lakhtakia index, which is defined as

\[
n_{DL}(\omega) = |\epsilon(\omega)| Re[\mu(\omega)] + |\mu(\omega)| Re[\epsilon(\omega)].
\]

As shown in \(^{21}\), the phase velocity is opposite to the power flow if and only if \(n_{DL} < 0\).

Now let us move on to the strategy to calculate these quantities by holography. By symmetry, in order to consider the refractive index in the above holographic superconductors, it is enough to consider the transverse fluctuations of the field \(A\) as follows

\[
\delta A_x = a_x(u)e^{-i\omega t + iky}.
\]

Note that such a fluctuation decouples and has no back reaction, satisfying the equation of motion

\[
0 = a''_x + \left(\frac{f'}{f} - \frac{1}{u}\right) a'_x + \left(\frac{\omega^2}{f^2} - \frac{k^2}{f} - \frac{2\Phi^2}{f u^2}\right) a_x.
\]

Whence \(a_x\) goes like \((1-u)^{\pm i\Phi}\) near the horizon, corresponding to the outgoing and ingoing boundary conditions respectively. We here choose the ingoing boundary condition, which will lead to the retarded Green function in the dual field theory\(^{22}\). Speaking specifically, with such an ingoing boundary condition, we assume the corresponding asymptotic expansion \(a_x\) goes like

\[
a_x = a_x^{(0)} + \frac{1}{2} [a_x^{(1)} u^2 - (\omega^2 - k^2) a_x^{(0)} u^2 \ln\frac{u}{\epsilon}] + \cdots
\]

\(^{1}\)Note that the expression in \(^5\) is different from ours due to the different conventions used for the retarded Green function.
where we have used Eq.(3.5) and the second equation in (3.7). By Eq.(2.5), the induced current is given by

$$J_x = a^{(1)}_x + c(\omega^2 - k^2)a^{(0)}_x. \quad (3.8)$$

Whence the retarded transverse Green function can be obtained as

$$G_T(\omega, k) = \frac{a^{(1)}_x}{a^{(0)}_x} + c(\omega^2 - k^2). \quad (3.9)$$

An analytic solution of Eq.(3.6) does not appear to be available. So in the subsequent section, we shall resort to Mathematica for numerical calculation, where in order to obtain $G_{T0}(\omega)$ and $G_{T2}(\omega)$, it is advisable to adopt an alternative approach by firstly expanding $a_x$ in series of $k$, i.e.,

$$a_x = a_{x0} + k^2 a_{x2} + \cdots, \quad (3.10)$$

from which Eq.(3.6) becomes

$$0 = a''_{x0} + \left(\frac{f'}{f} - \frac{1}{u}\right)a'_{x0} + \left(\frac{\omega^2}{f^2} - \frac{2\Phi^2}{fu^2}\right)a_{x0},$$

$$0 = a''_{x2} + \left(\frac{f'}{f} - \frac{1}{u}\right)a'_{x2} + \left(\frac{\omega^2}{f^2} - \frac{2\Phi^2}{fu^2}\right)a_{x2} - \frac{1}{u}a_{x0}. \quad (3.11)$$

Near the boundary, the asymptotic expansion for $a_{x0}$ and $a_{x2}$ can be read out of Eq.(3.7) as

$$a_{x0} = a^{(0)}_{x0} + \frac{1}{2} \left(\frac{a^{(1)}_{x0}}{u^2} - \omega^2 a^{(0)}_{x0} u^2 \ln \frac{u}{\epsilon}\right) + \cdots$$

$$a_{x2} = a^{(0)}_{x2} + \frac{1}{2} \left(\frac{a^{(1)}_{x2}}{u^2} - \omega^2 a^{(0)}_{x2} - a^{(0)}_{x0}\right) u^2 \ln \frac{u}{\epsilon} + \cdots. \quad (3.12)$$

Whence we have

$$G_{T0}(\omega) = \frac{a^{(1)}_{x0}}{a^{(0)}_{x0}} + c\omega^2,$$

$$G_{T2}(\omega) = \frac{a^{(1)}_{x2}}{a^{(0)}_{x2}} \left[\frac{a^{(1)}_{x0}}{a^{(0)}_{x0}} - \frac{a^{(0)}_{x2}}{a^{(0)}_{x0}}\right] - c, \quad (3.13)$$

where $c$ can be fixed by requiring that $G_{T0}$ approaches zero at large frequencies as the system has no time to respond to the rapid variation of external fields.
Figure 3: The imaginary part of the $\epsilon_T$ for real $\omega$ and $k$ at temperature $\frac{T}{T_c} = 0.85$.

Figure 4: The imaginary part of the $\epsilon_T$ for real $\omega$ and $k$ at temperature $\frac{T}{T_c} = 0.25$.

3.2 Numerical results

Here we plot all the results by fixing the charge density to be 1 but varying the temperature with respect to the critical temperature and consequently the condensate.

As explained in [5], firstly we have to ensure that the system is in thermodynamical
Figure 5: The real part of the electric permittivity as a function of frequency with various temperatures.

Figure 6: The imaginary part of the electric permittivity as a function of frequency with various temperatures.

equilibrium, i.e., the imaginary part of $G_T$, or equivalently the imaginary part of $\epsilon_T$ should be greater than zero for real $\omega$ and $k$. This is actually guaranteed by the holographic duality, as the bulk background is static with the black brane. For illustration, we would like to plot the imaginary part of $\epsilon_T$ in Fig.3 and Fig.4 to demonstrate that this is the case.

Then we plot the real and imaginary parts of the electric permittivity in Fig.3
Figure 7: The real part of the effective magnetic permeability as a function of frequency with various temperatures.

and Fig.6 respectively. By the scaling rules, to lower the temperature is equivalent to increase the charge density and chemical potential. Therefore the electric permittivity here demonstrates the similar behavior at low frequencies as that considered in [5]. In particular, the larger are the charge density and chemical potential, the larger is the regime for the negative real part of the electric permittivity. In addition, below the critical temperature, both of the real and imaginary parts of the electric permittivity diverge at $\omega = 0$. By the fact that the electric permittivity is related to the conductivity as $\epsilon(\omega) = 1 + i\frac{4\pi}{\omega}\sigma(\omega)$, the real part of the electric permittivity diverges as $-\frac{1}{\omega^2}$, and accordingly the imaginary part blows up like $\delta(\omega)$ due to the Kramers-Kronig relations from causality [14].

Next we plot the real and imaginary parts of the effective magnetic permeability in Fig.7 and Fig.8 respectively. The real part of the effective magnetic permeability still displays the similar behavior at low frequencies as that in [3], while the imaginary part shows a totally different behavior, i.e., there is no blow-up at $\omega = 0$.

Such a difference eventually leads to the conclusion that at low frequencies the negative phase velocity does not show up in the holographic superconductors at least within the accuracy of our numerics, which is illustrated by the non-negative Depine-Lakhtakia index in Fig.4. In hindsight, the reason may arise in the fact that the bulk background is essentially kind of neutral black hole although the electromagnetic field has an influence on the fluctuation equation (3.6) indirectly through the condensate.

At last, we would like to plot the real and imaginary parts of the refractive index in Fig.10 and Fig.11 respectively. Note that the ratio of the imaginary part of the
refractive index to its real part describes the ratio between dissipation and propagation. We see that there exists a characteristic frequency, below which light can not propagate in our holographic superconductors while above which light can propagate without dissipation.
Figure 10: The real part of the refractive index as a function of frequency with various temperatures.

Figure 11: The imaginary part of the refractive as a function of frequency with various temperatures.

4. Conclusions

We have carried out the numerical analysis of the optical properties of the s-wave
holographic superconductors in the probe limit. In particular, we have calculated the electric permittivity, effective magnetic permeability, refractive index, and Depine-Lakhtakia index by the holographic duality, which is generically difficult to be achieved by other approaches. As a result, the electric permittivity and effective magnetic permeability conspire to make our holographic superconductors robust against a negative phase velocity at low frequencies.

We conclude with some generalizations of our work in various directions. Firstly, although the cases for other scalar masses are expected to demonstrate the qualitatively similar properties as the massless case, it is interesting to investigate how the specific optical properties of holographic superconductors depend quantitatively on the mass $m$. Secondly, it is natural to extend our work to the 2+1 dimensional holographic superconductors. More importantly, as the superconductors are believed to have the potential to support low losses, which is critical for many applications like super-resolution imaging, it is rewarding to search for the negative phase velocity in other situations. In particular, it is worthwhile to go beyond the probe limit to explore the full behavior of the theory, where the negative refractive index is expected to appear at low frequencies because the background is essentially kind of charged black hole. In addition, as suggested by the early experimental implementation of negative refractive index in the ferromagnet-superconductors, it is highly possible to see the negative refractive show up when the external magnetic field is added to our holographic superconductors. Furthermore, it is also intriguing to extend our analysis to both the p-wave and d-wave holographic superconductors, where some new features should come out due to the nonisotropy of the dual media. Finally, it is also tempted to generalize our work to some non-minimal models of holographic superconductors.

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References

[1] S. A. Hartnoll, Class. Quant. Grav. 26, 224002(2009).
[2] C. P. Herzog, J. Phys. A42, 343001(2009).
[3] J. McGreevy, arXiv:0909.0518[hep-th].
[4] S. Sachdev, arXiv:1002.2947[hep-th].
[5] A. Amariti, D. Forcella, A. Mariotti, and G. Policastro, arXiv:1006.5714[hep-th].
[6] S. S. Gubser, Phys. Rev. D78, 065034(2008).
[7] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. 101, 031601(2008).
[8] S. S. Gubser, Phys. Rev. Lett. 101, 191601(2008).
[9] S. S. Gubser and S. Pufu, JHEP 0811, 033(2008).
[10] J. W. Chen et al., Phys. Rev. D81, 106008(2010).
[11] C. P. Herzog, Phys. Rev. D81, 126009(2010).
[12] F. Benini, C. P. Herzog, and A. Yarom, arXiv:1006.0731[hep-th].
[13] F. Benini et al., arXiv:1007.1981[hep-th].
[14] G. T. Horowitz, arXiv:1002.1722[hep-th].
[15] K. Skenderis, Class. Quant. Grav. 19, 5849(2002).
[16] G. T. Horowitz and M. M. Roberts, Phys. Rev. D78, 126008(2008).
[17] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media(Pergamon Press, Oxford, 1984).
[18] D. B. Melrose and R. C. Mcphedran, Electromagnetic Processes in Dispersive Media(Cambridge University Press, Cambridge, 1991).
[19] V. M. Agranovich and Y. N. Gartstein, Metamaterials 3, 1(2009).
[20] M. Dressel and G. Gruner, Electrodynamics of Solids(Cambridge University Press, Cambridge, 2002).
[21] R. A. Depine and A. Lakhtakia, Microwave and Optical Technology Letters 41, 315(2004).
[22] D. T. Son and A. O. Starinets, JHEP 0209, 042(2002).
[23] X. Gao and H. Zhang, Work in progress.

[24] A. Pimenov, A. Loidl, P. Przyslupski, and B. Dabrowski, Phys. Rev. Lett. 95, 247009(2005).

[25] S. Franco, A. Garcia-Garcia and D. Rodriguez-Gomez, JHEP 1004, 092(2010).

[26] F. Aprile and J. G. Russo, Phys. Rev. D81, 026009(2010).

[27] Q. Y. Pan et al., Phys. Rev. D81, 106007(2010).

[28] Q. Y. Pan and B. Wang, arXiv:1005.4743[hep-th].

[29] Y. Liu and Y. W. Sun, arXiv:1006.2726[hep-th].

[30] R. G. Cai, Z. Y. Nie, and H. Q. Zhang, arXiv:1007.3321[hep-th].