Canonical factorization and diagonalization of Baxterized braid matrices: Explicit constructions and applications.

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Abstract

Braid matrices \( \hat{R}(\theta) \), corresponding to vector representations, are spectrally decomposed obtaining a ratio \( f_i(\theta)/f_i(-\theta) \) for the coefficient of each projector \( P_i \) appearing in the decomposition. This directly yields a factorization \( (F(-\theta))^{-1}F(\theta) \) for the braid matrix, implying also the relation \( \hat{R}(-\theta)\hat{R}(\theta) = I \). This is achieved for \( GL_q(n), SO_q(2n+1), SO_q(2n), Sp_q(2n) \) for all \( n \) and also for various other interesting cases including the 8-vertex matrix. We explain how the limits \( \theta \to \pm\infty \) can be interpreted to provide factorizations of the standard (non-Baxterized) braid matrices. A systematic approach to diagonalization of projectors and hence of braid matrices is presented with explicit constructions for \( GL_q(2), GL_q(3), SO_q(3), SO_q(4), Sp_q(4) \) and various other cases such as the 8-vertex one. For a specific nested sequence of projectors diagonalization is obtained for all dimensions. In each factor \( F(\theta) \) our diagonalization again factors out all dependence on the spectral parameter \( \theta \) as a diagonal matrix. The canonical property implemented in the diagonalizers is mutual orthogonality of the rows. Applications of our formalism to the construction of \( L \)-operators and transfer matrices are indicated. In an Appendix our type of factorization is compared to another one proposed by other authors.

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1 Introduction

Let $\hat{R}(\theta)$ be a braid matrix Baxterized with a spectral parameter $\theta$ and satisfying, in standard notations,

$$\hat{R}_{12}(\theta)\hat{R}_{23}(\theta + \hat{\theta})\hat{R}_{12}(\theta) = \hat{R}_{23}(\hat{\theta})\hat{R}_{12}(\theta + \hat{\theta})\hat{R}_{23}(\theta)$$  \hspace{1cm} (1.1)

Here, apart from $\theta$, $\hat{R}(\theta)$ can depend on other parameters such as $q$, which will not always be denoted explicitly. Vector representations with $N^2 \times N^2$ braid matrices are implied in all cases. The corresponding $YB$ (Yang-Baxter) matrix is

$$R(\theta) = P\hat{R}(\theta)$$  \hspace{1cm} (1.2)

where the permutation matrix $P$ is defined to be (with $i = (1, 2, \ldots N)$)

$$P = \sum_{ij} E_{ij} \otimes E_{ji}$$  \hspace{1cm} (1.3)

The matrix $E_{ij}$ has zero elements except for a single unit one at $(ij)$.

We assume that the polynomial equation of minimal degree satisfied by $\hat{R}(\theta)$ has distinct roots. When this holds $\hat{R}(\theta)$ can be spectrally decomposed on a basis of projectors $P_i$, satisfying

$$P_i P_j = \delta_{ij}, \quad \sum_i P_i = I_{N^2 \times N^2}$$  \hspace{1cm} (1.4)

Suppressing arguments for the time being, if (with $k_i \neq k_j$ for $i \neq j$)

$$\prod_{i=1}^{p} (\hat{R} - k_i I) = 0$$  \hspace{1cm} (1.5)

then defining

$$P_i = \prod_{j \neq i} \frac{(\hat{R} - k_j I)}{(k_i - k_j)}, \quad (i = 1, 2, \ldots p)$$  \hspace{1cm} (1.6)

the set $P_i$ can be shown to satisfy (1.4) and one obtains

$$\hat{R} = \sum_{i} k_i P_i$$  \hspace{1cm} (1.7)

On the other hand, given (1.7) one obtains (1.5). The $P$’s on the right can, in general, depend on parameters such as $q$. But in all cases they will be independent of the spectral parameter $\theta$. In $R(\theta)$ all $\theta$-dependence is to be found in the coefficients $k_i$. This is consistent with (1.6) and (1.7) and is fundamental for the considerations below.

In all cases to be considered, not only we will obtain explicit spectral decomposition of $\hat{R}(\theta)$, but also a specific factorized form of each $k_i$.
when

\[
R(\theta) = \sum_i k_i(\theta) P_i = \sum_i \frac{f_i(\theta)}{f_i(-\theta)} P_i
\]

This will be our first major step.

The number of projectors and their matrix elements are specific to the case considered. But they always satisfy (1.4). In [1], (1.9) has been obtained explicitly for \(GL_q(n)\), \(SO_q(2n+1)\), \(SO_q(2n)\) and \(Sp_q(2n)\) for all \(n\). The results are recapitulated in Secs. 2, 3. In Secs. 4, 5, 6, 7 we obtain (1.9) for various interesting cases, including the 8–vertex matrix.

An evident, but for us crucial, consequence of (1.4) is that for well-defined and mutually commuting but otherwise arbitrary coefficients \((a_i, b_i)\),

\[
(\sum_i a_i P_i)(\sum_i b_i P_i) = (\sum_i a_i b_i P_i) = (\sum_i b_i P_i)(\sum_i a_i P_i)
\]

Hence, once a spectral decomposition (1.7) has been obtained \(\hat{R}\) can be expressed as a product of arbitrary number of factors

\[
\hat{R} = \prod_i (\sum_i k_i^{(n)} P_i), \quad (\prod_i k_i^{(n)} = k_i)
\]

Of particular interest to us is the factorization

\[
\hat{R}(\theta) = \sum_i \frac{f_i(\theta)}{f_i(-\theta)} P_i = (\sum_i f_i^{-1}(\theta) P_i)(\sum_i f_i(\theta) P_i)
\]

\[
= (F(\theta))^{-1} F(-\theta)
\]

where

\[
F(\theta) = \sum_i f_i(\theta) P_i
\]

This implies the so-called ”unitarity”

\[
\hat{R}(-\theta) \hat{R}(\theta) = I_{N^2 \times N^2}
\]

One obtains from (1.2), since \(P^2 = I\),

\[
R(\theta) = (P(F(\theta))^{-1} P) PF(\theta) = (F_{21}(\theta))^{-1} PF_{12}(\theta)
\]

In (1.12) and (1.15) the key feature is the change of sign of \(\theta\) in \(F^{-1}\).
Other interesting choices are possible. Thus, for example, defining
\[ \hat{F}(\theta) = \sum_i \left( \frac{f_i(\theta)}{f_i(-\theta)} \right)^{\frac{1}{2}} P_i = (\hat{F}(-\theta))^{-1} \]  
(1.16)
one obtains
\[ \hat{R}(\theta) = (\hat{F}(-\theta))^{-1}\hat{F}(\theta) = (\hat{F}(\theta))^2 \]  
(1.17)
and
\[ R(\theta) = (\hat{F}_{21}(-\theta))^{-1}P\hat{F}_{12}(\theta) = (\hat{F}_{21}(\theta))P\hat{F}_{12}(\theta) \]  
(1.18)
Here, even for real \( \hat{R}(\theta) \), for certain domains of \( \theta \) the factor \( \hat{F}(\theta) \) can be complex. Compare (1.15) and (1.18) to a Drinfeld twist [2, 3, 4] of \( P \)
\[ \hat{R}''(\theta) = (F''_{21}(\theta))^{-1}P F''_{12}(\theta) \]  
(1.19)
In (1.15) there is \( (-\theta) \) on the left and in (1.18) there is no inversion of \( \hat{F}_{21}(\theta) \). \( P \) satisfies the \( YB \) equation with the trivial \( \hat{R} = P^2 = I \) for the braid matrix. ( \( P \) also satisfies the braid equation with \( R = P^2 = I \).) The properties of \( R''(\theta) \) will depend on those of \( F''(\theta) \) (such as cocycle conditions). In our case, since one starts from solutions of (1.1) one does not have to verify if \( F(\theta) \) and \( \hat{F}(\theta) \) satisfy suitable constraints, so far as the braid equation is concerned.

The present situation may also be compared to ”contraction” of \( YB \) matrices to non-standard, Jordanian forms. Without even trying to explain the terminology we refer to two [5, 6] of our series of relevant papers (where original sources are cited). We mention this only to point out that, as compared to (1.15), (1.18), (1.19) the role of \( P \) is, so to say, reversed. For the non-standard case \( R \) is a Drinfeld twist of \( I \),
\[ R = (F_{21})^{-1}F, \quad \hat{R} = F^{-1}PF. \]  
(1.20)
The nontrivial matrix \( \hat{R} \) is now ”triangular” since from (1.20)
\[ (\hat{R})^2 = I \]  
(1.21)
The ambiguities arising in factorizing (compare (1.13) and (1.16)), or in defining \( f_i(\theta) \) for a given \( k_i(\theta) \) in (1.8), become particularly relevant in considering the limits \( \theta \to \pm\infty \). From (1.9) one has evidently
\[ \hat{R}(0) = \sum_i P_i = I \]  
(1.22)
It will be seen in the following sections that in each case for \( \theta \to \pm\infty \) one obtains the standard (non-Baxterized) braid matrices (\( \hat{R} \) and the inverse) satisfying
\[ \hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23} \]  
(1.23)
This equation can be considered, consistently with (1.1) as the limiting form (with arguments suppressed) as both \((\theta, \dot{\theta}) \to +\infty\), say. If one denotes

\[
\lim_{\theta \to +\infty} \hat{R}(\theta) = \hat{R}
\] (1.24)

then consistently with (1.14)

\[
\lim_{\theta \to -\infty} \hat{R}(\theta) = \hat{R}^{-1}
\] (1.25)

In these limits special features arise concerning factorizations. It is helpful to consider a simple but frequently encountered example. Suppose that for some \(f_i(\theta)\) one has (dropping the index \(i\) and setting \(q = e^h\))

\[
\frac{f(\theta)}{f(-\theta)} = \frac{\sinh(h - \theta)}{\sinh(h + \theta)}
\] (1.26)

The evident singularity at \(\theta = -h\) can be excluded by definition from the domain of \(\theta\). Now, as \(\theta \to \pm \infty\)

\[
\frac{f(\theta)}{f(-\theta)} \to -q^{\mp 2}
\] (1.27)

But what about the factor \(f(\theta)\)? How does it behave when separated in the factor \(F(\theta)\) or \(\dot{F}(\theta)\) exhibited before?

(1): For the choice

\[
f(\pm \theta) = \sinh(h \mp \theta)
\] (1.28)

separately both \(f(\theta)\) and \(f(-\theta)\) both diverge.

(2): The choice

\[
f(\theta) = \left(\frac{\sinh(h - \theta)}{\sinh(h + \theta)}\right)^\frac{1}{2} = \frac{\sinh(h - \theta)}{(\sinh(h - \theta)\sinh(h + \theta))^{\frac{1}{2}}}
\] (1.29)

gives consistently with (1.27), finite but imaginary limits

\[
f(\theta) \to \pm iq^{\mp 1}, \quad f(-\theta) \to \mp iq^{\pm 1}
\] (1.30)

(3): But more generally than in (1.29) one may choose for any well-defined \(y(\theta)\),

\[
f(\theta) = \frac{\sinh(h - \theta)}{(y(\theta)y(-\theta))^{\frac{1}{2}}}
\] (1.31)

Setting, for example,

\[
f(\theta) = \frac{\sinh(h - \theta)}{(\cosh(h - \theta)\cosh(h + \theta))^{\frac{1}{2}}}
\] (1.32)
one obtains real, finite limits consistent with (1.28)

\[ \theta \to +\infty, \quad f(\pm \theta) \to \mp q^{\pm 1} \]  

with an evident analogous result for \( \theta \to -\infty \).

We will assume that each \( f_i(\theta) \) in \( F(\theta) \) has thus been suitably defined (choosing an appropriate \( y(\theta) \)). Then even \( \hat{R} \) satisfying (1.23) can be considered to be factorized as in (1.12), the implicit spectral parameter not being exhibited in the limits \( \theta \to \pm \infty \). In this sense, the unitarity (1.14) can still be considered to be implicit. Note that even if each \( f_i(\theta) \) has limits analogous to (1.33) with different powers of \( q \), one cannot express the factorization as \( \left( F(-q) \right)^{-1} F(q) \) since the projectors, in general, are \( q \)-dependent (though always independent of \( \theta \)). It is essential to think in terms of \( \theta \) even when it is, in the limits above, implicit.

The implementation of a spectral parameter, the passage from (1.23) to (1.1), renders many aspects more complex. But it also provides an extra margin of manoeuvre, making possible the canonical factorization (1.12) possible whose interest will be studied later on.

Our factorizations are directly based on the resolution (1.9). Other classes of factorizations can also be envisaged. One such class with upper and lower triangular factors for the \( YB \) matrix \( R(\theta) \), leading to interesting properties has been studied by Maillet et al. in a series of papers [7, 8, 9]. This formalism is compared with ours in App.A.

Since all braid matrices studied (Secs.2, 3, 4, 5, 6, 7) are systematically found to lead to spectral decompositions with each coefficient of the form (1.8) and (1.9) a more general study of such forms should be of interest. Here we will limit our observations to the following feature. Let

\[ \hat{R}'(\theta) = \sum_i \frac{g_i(\theta)}{g_i(-\theta)} P_i \]  

where, apart from being well-defined, the \( g \)'s are as yet arbitrary. In general, \( \hat{R}'(\theta) \) does not satisfy (1.1). But defining

\[ \hat{H}(\theta) = \sum_i \frac{g_i(\theta)}{f_i(\theta)} P_i \equiv \sum_i h_i(\theta) P_i \]  

where \( f_i(\theta) \) corresponds to (1.9), a solution \( \hat{R}(\theta) \) of (1.1) on the same basis of projectors

\[ \hat{R}'(\theta) = \left( H(-\theta) \right)^{-1} \hat{R}(\theta) H(\theta) \]  

Any two matrices decomposable on the same spectral basis (satisfying (1.4)) are always related as above.

Substituting in (1.1)

\[ \hat{R}(\theta) = H(-\theta) \hat{R}'(\theta) H(\theta)^{-1} \]  

one can rephrase (1.1) in terms of \( \hat{R}' \) and \( H \). One obtains

\[ \hat{R}'_{12}(\theta) X_1 \hat{R}'_{23}(\theta + \hat{\theta}) X_2 \hat{R}'_{12}(\hat{\theta}) X_3 = X_4 \hat{R}'_{23}(\hat{\theta}) X_5 \hat{R}'_{12}(\theta + \hat{\theta}) X_6 \hat{R}'_{23}(\theta) \]  

5
where
\[ X_1 = (H_{12}(\theta))^{-1}H_{23}(-\theta - \dot{\theta}) \]
and so on.

Now, along with the properties of \( f_i(\theta) \), those of \( g_i(\theta) \) will determine the content of this equation for \( \hat{R}' \). Further study in this direction is beyond the scope of this work.

Our first basic step is the systematic expression of the solutions of (1.1) in the form (1.9). The next major one is the simultaneous diagonalization of each projector \( P_i \) in (1.9) and hence of \( \hat{R}(\theta) \). Our approach is presented step by step in Sec.9. Explicit examples of diagonalizations of lower dimensional cases of Sec.2 and Sec.3 (\( Gl_q(2), Gl_q(3), SO_q(3), SO_q(4), Sp_q(4) \)) are collected together in App.B. At the end of Secs.(4, 5, 6, 7) the diagonalizations are presented explicitly for each case. Our Sec.8 is an exception, where a nested sequence of projectors with simple, attractive features is presented for arbitrary dimensions without constructing explicit solutions of the braid equation. On the contrary, here the diagonalizer is obtained quite simply for arbitrary dimensions.

A canonical feature sought for in our formalism is the mutual orthogonality of the rows of the matrix diagonalizing \( \hat{R}(\theta) \). The elegant and useful consequences of such a feature are pointed out. In the factorized form, our diagonalization factors out again in each factor all \( \theta \)-dependence as a diagonal matrix.

Applications of our spectral decompositions and diagonalizations to the construction of \( L \)-operators and to transfer matrices are discussed respectively in Sec.10 and Sec.11.

## 2 Factorization of braid matrices of \( GL_q(N), SO_q(N) \) and \( Sp_q(N) \):

We recapitulate below the relevant essential results of [1]. The standard \( q \)-dependent \( N^2 \times N^2 \) projectors [10] are assumed to be known. For \( Sp_q \) always \( N = 2n \).

Same notations will be used for projectors in different cases though they are different. The overall normalizing factor for \( \hat{R}(\theta) \) is chosen to obtain 1 for the element (11) at top left. (See however Sec.7.)

For \( GL_q(N) \) one has two projectors \( (P_+, P_-) \) satisfying (1.4). For \( \hat{R}(\theta) \) satisfying (1.1), setting \( h = \ln q \), one obtains

\[
\hat{R}(\theta) = P_+ + \frac{\sinh(h - \theta)}{\sinh(h + \theta)} P_- = (P_+ + (\sinh(h + \theta))^{-1}P_-)(P_+ + \sinh(h - \theta)P_-) \equiv (F(-\theta))^{-1}F(\theta) \]

(2.1)

To illustrate (1.12) we have implemented one simple possible choice for \( F(\theta) \). Ambiguities discussed in Sec.1 (from (1.28) to (1.32)) are always present in this and other examples to follow. This statement will not be repeated in successive sections.
For $SO_q(N)$, for $N = (2n + 1)$ and also for $N = 2n$, one has a basis of three projectors $(P_+, P_-, P_0)$ and two possibilities:

\[
\hat{R}(\theta) = P_+ + \frac{\sinh(h - \theta)}{\sinh(h + \theta)} P_- + \frac{\cosh(\frac{N}{2}h - \theta)}{\cosh(\frac{N}{2}h + \theta)} P_0
\]  

or

\[
\hat{R}(\theta) = P_+ + \frac{\sinh(h - \theta)}{\sinh(h + \theta)} P_- + \frac{\sinh((\frac{N}{2} - 1)h - \theta)\sinh(h - \theta)}{\sinh((\frac{N}{2} - 1)h + \theta)\sinh(h + \theta)} P_0
\]

For $Sp_q(2n)$ one obtains

\[
\hat{R}(\theta) = P_+ + \frac{\sinh(h - \theta)}{\sinh(h + \theta)} P_- + \frac{\sinh((n + 1)h - \theta)}{\sinh((n + 1)h + \theta)} P_0
\]

or

\[
\hat{R}(\theta) = P_+ + \frac{\sinh(h - \theta)}{\sinh(h + \theta)} P_- + \frac{\cosh(nh - \theta)\sinh(h - \theta)}{\cosh(nh + \theta)\sinh(h + \theta)} P_0
\]

The expressions for $F(\theta)$ and $(F(-\theta))^{-1}$ are evident in each case. See however the remarks below (2.1). For each case

\[
\hat{R}(0) = I
\]

For $\theta \to \pm \infty$, carefully taking limits, one respectively obtains:

For $GL_q(N)$

\[
\hat{R} = P_+ - q^{\mp 2} P_-
\]

For $SO_q(N)$

\[
\hat{R} = P_+ - q^{\mp 2} P_- + q^{\mp N} P_0
\]

For $Sp_q(N)$

\[
\hat{R} = P_+ - q^{\mp 2} P_- - q^{\mp (N+2)} P_0
\]

These are the standard (non-Baxterized) braid matrices [10] satisfying (1.23). Concerning factorization see the relevant discussion in Sec.1 (from (1.23) to (1.33)).

At the end of Sec.4 of [1] it has been pointed out that for $q = 1(h = 0)$ all these matrices reduce to ones with constant elements satisfying

\[
\hat{R}^2 = I
\]

They amount to twists of $I$ with constant matrices. This situation is to be contrasted with the corresponding one in Sec.3.
3 A new class of braid matrices for $SO_q(N)$ and $Sp_q(N)$:

This was presented in Sec.4 of [1]. The solution for $SO_q(3)$ appeared already in [11]. The structure (1.9) is again present and hence also the factorization (1.12).

We recapitulate:

Define

$$d = (1 + \epsilon[N - \epsilon])^{-1} = \left(1 + \epsilon\frac{q^{N-\epsilon} - q^{-N+\epsilon}}{q - q^{-1}}\right)^{-1} \quad (3.1)$$

where for $SO_q(N)$

$$\epsilon = 1, \quad N = 3, 4, ...$$

and for $Sp_q(N)$

$$\epsilon = -1, \quad N = 4, 6, ...$$

Define also

$$\tanh \eta = \sqrt{1 - 4d^2} \quad (3.2)$$

The reality of $\eta$ is implied by (3.1) since $4d^2 < 1$. Now with such an $\eta$,

$$\hat{R}(\theta) = P_+ + P_- + \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)}P_0 \quad (3.3)$$

$$= I + \left(\frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} - 1\right)P_0 \quad (3.4)$$

can be shown [1] to satisfy (1.1). The promised structure is explicit in (3.3). One has as usual

$$\hat{R}(0) = I \quad (3.5)$$

and for $\theta \to \pm \infty$

$$\hat{R}(\pm \infty) = P_+ + P_- - e^{\mp 2\eta}P_0 \quad (3.6)$$

$$= I - (1 + e^{\mp 2\eta})P_0 \quad (3.7)$$

where

$$e^{\mp 2\eta} = \frac{1 \mp \sqrt{1 - 4d^2}}{1 \pm \sqrt{1 - 4d^2}} \quad (3.8)$$

These provide a new class of (non-Baxterized) braid matrices $\hat{R}^{\pm 1}$ satisfying (1.23). The $R^{\pm 1} = P\hat{R}^{\pm 1}$ are new solutions of the $YB$ equation for $SO_q$ and $Sp_q$ with $\epsilon$ and $N$ as given below (3.1).
Moreover, from \((3.1)\), for \(q = 1\)

\[
d = \frac{e}{N}
\]

(3.9)

and from \((3.2)\), for \(q = 1\),

\[
(tanh\hat{\eta})_{(q=1)} = \sqrt{1 - \frac{4}{N^2}} \equiv tanh\hat{\eta}
\]

(3.10)

Denoting

\[
(P_0)_{(q=1)} = \hat{P}_0
\]

(3.11)

and so on, one obtains from \((3.3)\) and \((3.7)\) respectively

\[
(\hat{R}(\theta))_{(q=1)} = \hat{P}_+ + \hat{P}_- + \frac{sinh(\hat{\eta} - \theta)}{sinh(\hat{\eta} + \theta)} \hat{P}_0
\]

(3.12)

and

\[
(\hat{R}^{(\pm 1)})_{(q=1)} = \hat{P}_+ + \hat{P}_- - e^{\pm 2\hat{\eta}} \hat{P}_0
\]

(3.13)

The braid matrix \((3.13)\) satisfies a nontrivial Hecke condition

\[
(\hat{R} - I)(\hat{R} + e^{-2\hat{\eta}} I) = 0
\]

(3.14)

and cannot be twisted back to \(I\). This situation should be compared to \((2.10)\) and the remarks that follow \((2.10)\).

Note that we are not expanding in powers of \(h(= lnq)\) to extract the so-called ”classical” \(r\)-matrix. We are directly setting \(q = 1\) and yet getting quite nontrivial results.

4 Two exotic cases \((S03, S14)\):  

Two special braid matrices arising in the classification of \(4 \times 4\) \(YB\) matrices of [12] were Baxterized in [13]. Other aspects were already studied in previous papers of the series [14]. Some ”exotic” features are briefly recapitulated below in the present context:

- Complex projectors for \(S03\) ( for real \(\hat{R}\) )
- Extended freedom of parametrization for \(S14\).

Our solutions presented in \(Sec.3\) can be considered to be an exotic class in arbitrary dimensions \((N^2 \times N^2, N \geq 3)\). For even \(N\) one has two types, exotic orthogonal and exotic symplectic.

\(S03:\)

The braid matrix
\[ \hat{R} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \]  

(4.1)

satisfies

\[ (\hat{R} - (1 + i)I)(\hat{R} - (1 - i)I) = 0 \]  

(4.2)

The corresponding projectors

\[ P_{(\pm)} = \frac{1}{2} (I \pm i(\hat{R} - I)) \]  

(4.3)

provide the spectral decomposition

\[ \hat{R} = (1 - i)P_{(+)} + (1 + i)P_{(-)} \]  

(4.4)

Altering suitably the normalization of [13] gives the Baxterization (with \( z = e^\theta \))

\[ \hat{R}(z) = \left( \frac{f(z)}{f(z^{-1})} \right)^{\frac{1}{2}} P_{(+)} + \left( \frac{f(z^{-1})}{f(z)} \right)^{\frac{1}{2}} P_{(-)} \]  

(4.5)

where

\[ f(z) = (z + z^{-1}) + i(z - z^{-1}) \]  

(4.6)

Thus we obtain the form (1.9) and (1.12) follows.

One can rewrite (4.5) in the explicitly real form

\[ \hat{R}(z) = (z^2 + z^{-2})^{-\frac{1}{8}}((\sqrt{2}z)^{-1}\hat{R} + \sqrt{2}z\hat{R}^{-1}) \]  

(4.7)

and verify again

\[ \hat{R}(z^{-1})\hat{R}(z) = I \]  

(4.8)

The unitary matrix \( M \), where

\[ \sqrt{2}M = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}. \]  

(4.9)

diagonalizes \( P_{(\pm)} \) giving

\[ M\hat{R}M^{-1} = diag(1 - i, 1 - i, 1 + i, 1 + i) \]  

(4.10)
\[(z^2 + z^{-2})^{1/2} M \hat{R}(z) M^{-1} = \frac{1}{\sqrt{2z}} \text{diag}(1 - i, 1 - i, 1 + i, 1 + i) + \frac{z}{\sqrt{2}} \text{diag}(1 + i, 1 + i, 1 - i, 1 - i)\]  

(4.11)

The diagonal elements are complex with real trace.

\[S_{14}:\]

Here

\[\hat{R} = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}.\]

(4.12)

The projectors (three even for a 4 \times 4 \hat{R})

\[P_{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 2P_{(\pm)} = \begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}.\]

(4.13)

give

\[\hat{R} = P_{(0)} + q(P_{(\pm)} - P_{(-)})\]

(4.14)

Baxterization gives

\[\hat{R}(z) = P_{(0)} + v(z)(P_{(\pm)} - P_{(-)})\]

(4.15)

where \(v(z)\) is arbitrary. (See [13] for details.)

One can indeed set (with \(z = e^\theta\), say)

\[v(z) = \frac{f(z)}{f(z^{-1})}, \quad -v(z) = \frac{(z - z^{-1}) f(z)}{(z^{-1} - z) f(z^{-1})}\]

(4.16)

and factorize. But more freedom is present, as compared to (1.1) and all previous examples. Denoting \(\hat{R}(v(z))\) by \(\hat{R}(v)\) one obtains

\[\hat{R}_{12}(v)\hat{R}_{23}(v')\hat{R}_{12}(v'') = \hat{R}_{23}(v'')\hat{R}_{12}(v')\hat{R}_{23}(v)\]

(4.17)

where \((v, v', v'')\) are mutually independent.

Amusingly, \(\hat{R}\) of \(S_{14}\) is diagonalized by (4.1) the \(\hat{R}\) of \(S_{03}\) giving

\[\text{diag}(q, 1, 1, -q)\]

(4.18)
5 Affine $\mathcal{U}_q(\hat{sl}_2)$:

We start below directly with the matrix $\hat{R}_{VV}(z)$ (equations (3.13) and (3.14) of Sec.3.2 of [15]. We obtain the spectral resolution and factorization (finding back the Baxterization of $GL_q(2)$ of Sec.2). Thus, apart from a possible overall factor, the braid matrix of $\mathcal{U}_q(\hat{sl}_2)$ is

$$\hat{R}(z) = P\hat{R}_{VV}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & zc & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (5.1)

where

$$b = \frac{(1-z)q}{(1-q^2z)}, \quad c = \frac{(1-q^2)}{(1-q^2z)}.$$  \hspace{1cm} (5.2)

Define the following basis satisfying (1.4),

$$P_{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (q+q^{-1})P_{(\pm)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{\pm 1} & \pm 1 & 0 \\ 0 & \pm 1 & q^{\mp 1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (5.3)

Note the specific $q$-dependence of $P_{(\pm)}$. Relabelling $(P_{(0)} + P_{(+)})$ as $P_{(+)}$ makes the relation to $GL_q(2)$ clearer. On the other hand (5.3) with $q = 1$ corresponds to the basis for the 6-vertex model (Sec.6).

Setting $q = e^h, z = e^\theta$ one can write (5.1) as

$$\hat{R}(\theta) = P_{(0)} + P_{(+)} + \frac{\sinh(h - \frac{\theta}{2})}{\sinh(h + \frac{\theta}{2})} P_{(-)}$$  \hspace{1cm} (5.4)

where

$$\frac{\sinh(h - \frac{\theta}{2})}{\sinh(h + \frac{\theta}{2})} = \frac{q^2 - z}{(q^2z - 1)} = \frac{z^{-\frac{1}{2}}q - z^{-\frac{1}{2}}q^{-1}}{z^{-\frac{1}{2}}q - z^{-\frac{1}{2}}q^{-1}}.$$  \hspace{1cm} (5.5)

Factorizations of the type (1.12) are now evident. Also evidently from (5.4)

$$\hat{R}(z^{-1})\hat{R}(z) = \hat{R}(-\theta)\hat{R}(\theta) = I$$  \hspace{1cm} (5.6)

Any supplementary overall factor, unless of the form

$$\frac{\rho(z)}{\rho(z^{-1})}$$  \hspace{1cm} (5.7)

will be incompatible with (5.6). The results for $\theta \to \pm \infty$ are displayed below for comparison with the corresponding results for the 6-vertex (Sec.6) and the 8-vertex (Sec.7) to follow.
For $\theta \to \infty$

$$\hat{R}(\theta) \to \hat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^{-2}) & q^{-1} & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.8)$$

For $\theta \to -\infty$

$$\hat{R}(\theta) \to \hat{R}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & (1-q^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.9)$$

For $R = P\hat{R}$ one recognizes the the familiar lower and upper triangular $YB$ matrices of $GL_q(2)$. Thus (5.1) is,indeed, a Baxterized form of (5.8) and (5.9) for a particular choice of basis and parametrization.

The matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.10)$$

diagonalizes each projector giving

$$M\hat{R}(\theta)M^{-1} = \text{diag}(1, 1, \frac{\sinh(h - \frac{q\theta}{2})}{\sinh(h + \frac{q\theta}{2})}, 1) \quad (5.11)$$

We have factorized the basic matrix (5.1). After supplementary quasi-Hopf twists [15] one can seek again a spectral resolution to study analogous possibilities provided that (5.6) is conserved.

### 6 The 6-vertex model:

The more general 8-vertex case is treated in Sec.7. But we introduce already at this stage a basis of projectors, satisfying (1.4), adequate for the 8-vertex matrix.

Define

$$2P_{1(\pm)} = \begin{pmatrix} 1 & 0 & 0 & \pm1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm1 & 0 & 0 & 1 \end{pmatrix}, \quad 2P_{2(\pm)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm1 & 0 \\ 0 & \pm1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6.1)$$

For the 6-vertex we need only the subset
\[ P_{(0)} = P_{1(+)} + P_{1(-)}, \quad P_{(\pm)} = P_{2(\pm)} \]  

(6.2)

Leaving aside all well-known relations, via reparametrizations and limiting processes, to rational affine cases, we illustrate our approach using the trigonometric parametrization and, in particular the "ferroelectric" regime. Extensive discussions and references can be found in the review [16]. (N.B. Our \( \hat{R} \) corresponds to \( R \) in the notation of [16]. See, for example, (2.19) of [16].)

With our standard normalization (Sec.2) in view we define (with \( \gamma > 0, \theta > 0 \)),

\[
x = \frac{\sinh \gamma}{\sinh(\gamma + \theta)}, \quad y = \frac{\sinh \theta}{\sinh(\gamma + \theta)}
\]  

(6.3)

(Though \( \theta > 0 \) for this regime, we will consider later the limits \( \theta \to \pm \infty \).) The braid matrix is

\[
\hat{R}(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & x & y & 0 \\
0 & y & x & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(6.4)

Implementing (6.1) and (6.2)

\[
\hat{R}(\theta) = P_{(0)} + (x + y)P_{(+)} + (x - y)P_{(-)}
\]

\[
= P_{(0)} + \frac{\cosh \frac{1}{2}(\gamma - \theta)}{\cosh \frac{1}{2}(\gamma + \theta)}P_{(+)} + \frac{\sinh \frac{1}{2}(\gamma - \theta)}{\sinh \frac{1}{2}(\gamma + \theta)}P_{(-)}
\]  

(6.5)

We have thus the structure (1.9) and hence the factorization (1.12). From (6.3) as

\[
\theta \to \pm \infty, \quad x \to 0, y \to e^{\mp \gamma}
\]  

(6.6)

Hence the corresponding limits of \( \hat{R}(\theta) \) give respectively (see the discussion starting with (1.23)) for the non-Baxterized \( YB \) matrix

\[
R^{\pm 1} = (P\hat{R})^{\pm 1} = \text{diag}(1, e^{\mp \gamma}, e^{\mp \gamma}, 1)
\]  

(6.7)

This is a special class of even the simplest and the first solution \( H_{3,1} \) in the classification of \( 4 \times 4 \) \( YB \) matrices [12], namely

\[
R = \text{diag}(p, q, r, s)
\]  

(6.8)

In view of the 8-vertex case to follow it is convenient to choose the diagonalizer \( (M = M^{-1}) \) as
\[ \sqrt{2}M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \] (6.9)

Now
\[ M \hat{R}(\theta)M^{-1} = \text{diag}(1, x + y, x - y, 1) \] (6.10)

The crucial difference, for the parametrizations adopted, between (5.1) and (6.4) is the extra factor \( z \) in the second diagonal element of (5.1). This leads to the \( q \)-dependent projectors in (5.3) as compared to the elements \( \pm \frac{1}{2} \) only in (6.1). Finally one is led (for \( \theta \to \pm \infty \)) to triangular and to diagonal \( Y \) matrices in Sec.5 and Sec.6 respectively.

7 The 8-vertex braid matrix:

The braid matrix of the quantum affine algebra \( \mathcal{A}_{q,p}(\hat{sl}_2) \) corresponds to the 8-vertex model. Some references relatively directly relevant to our purpose are [15], [17], [18] and [19]. These cite other basic sources.

Here the symmetrical structure of (6.4) is generalized (with \( z = e^{\theta} \)) to

\[ \hat{R}(z) = \begin{pmatrix} a(z) & 0 & 0 & d(z) \\ 0 & c(z) & b(z) & 0 \\ 0 & b(z) & c(z) & 0 \\ d(z) & 0 & 0 & a(z) \end{pmatrix}. \] (7.1)

There are well-known expressions for the elements in different equivalent forms in terms of elliptic functions. The role of a specific class of overall factors for specific realizations of \( (a, b, c, d) \) will be commented upon later on. Implementing now the full set (6.1) with \( a(z) = a \) and so on

\[ \hat{R}(z) = (a + d)P_{1(+)} + (a - d)P_{1(-)} + (c + b)P_{2(+)} + (c - b)P_{2(-)} \] (7.2)

We have thus a spectral decomposition in terms of the simple basis (6.1) with constant coefficients \( \pm \frac{1}{2} \). The next steps (in order to implement (1.9) and hence (1.12)) consists in explicit constructions of functions \( f_{1(\pm)}(z) \) and \( f_{2(\pm)}(z) \) such that (7.2) satisfies (1.1) for

\[ (a \pm d) = \frac{f_{1(\pm)}(z)}{f_{1(\pm)}(z^{-1})}, \quad (c \pm b) = \frac{f_{2(\pm)}(z)}{f_{2(\pm)}(z^{-1})}. \] (7.3)

These solutions are directly obtained from equations (3.28) and (3.29) of [15] in terms of infinite products

\[ (x; a)_\infty = \prod_{n \geq 0} (1 - xa^n) \] (7.4)
Noting that
\[ q^{1\pm q^{-1}z} = \pm q^{\frac{1}{2}z^{-\frac{1}{2}}} \pm q^{\frac{-1}{2}z^{\frac{1}{2}}} \] (7.5)
one obtains from the results cited above (writing \( z \) for \( \zeta \) and slightly reordering the factors)
\[ a \pm d = \frac{(+p^\frac{1}{2}q^{-1}z;p)_\infty (+p^\frac{1}{2}qz^{-1};p)_\infty}{(+p^\frac{1}{2}q^{-1}z^{-1};p)_\infty (+p^\frac{1}{2}qz;p)_\infty} \] (7.6)
\[ c \pm b = \frac{(q^\frac{1}{2}z^{-\frac{1}{2}} \pm q^{\frac{-1}{2}z^{\frac{1}{2}}})}{(q^\frac{1}{2}z^{\frac{1}{2}} \pm q^{\frac{-1}{2}z^{-\frac{1}{2}}})} \] (7.7)

Our objectives are attained. We have arrived at (1.9) and (1.12). In view of the factored structures of (7.6) and (7.7) the comments (Sec.1) concerning varied possibilities in selecting \( f(\theta) \) are now particularly relevant. Several factors lead to more alternatives.

As \( \theta = lnz \rightarrow \pm \infty \), \( z \rightarrow \infty \) and \( z \rightarrow 0 \) respectively. The extra factor in \((c \pm b)\) contributes
\[ \left( \frac{q^\frac{1}{2}z^{-\frac{1}{2}} \pm q^{\frac{-1}{2}z^{\frac{1}{2}}}}{q^\frac{1}{2}z^{\frac{1}{2}} \pm q^{\frac{-1}{2}z^{-\frac{1}{2}}}} \right)_{z \rightarrow \pm \infty} = \pm q^{\mp 1} \] (7.8)

The ratios of the infinite products (considering the leading term in (7.4) for \( n \leq k \)) give (for \( \theta \rightarrow \pm \infty \) respectively) for both (7.6) and (7.7) a factor
\[ \lim_{k \rightarrow \infty} q^{\mp 2k} \] (7.9)

However, we have not yet implemented our standard normalization, namely obtaining \( 1 \) for the element (11) at top left (Sec.2). This is achieved by a normalizing factor \( a^{-1} \). Such a factor absorbs the limiting factor (7.9). From (7.8) and (7.9), for
\[ (a', b', c', d') = a^{-1}(a, b, c, d) \] (7.10)
\[ \lim_{\theta \rightarrow \pm \infty}(a', b', c', d') = (1, q^{\mp 1}, 0, 0) \] (7.11)

Note that since \( a(z) \) i.e., \( a(\theta) \) is itself of the form \( x(\theta)(x(-\theta))^{-1} \) such a normalization conserves the unitarity (1.14) already satisfied by \((a, b, c, d)\).

However, if one prefers to maintain the simpler symmetry of the parametrization (7.6), (7.7) one may choose to absorb the factors (7.9) by a normalizing factor, say
\[ \frac{(q^2z; 1)_\infty}{(q^2z^{-1}; 1)_\infty} \] (7.12)
This conserves (1.14) and again gives the right hand side of (7.11) as limits (for \((a, b, c, d)\) normalized by (7.12)). This is a particularly simple choice albeit, evidently, not unique. We do not propose to examine here normalizations adopted in the cited sources.

After such a normalization one obtains

$$\lim_{\theta \to \pm\infty} (P\hat{R}(\theta)) = \text{diag}(1, q^{\mp 1}, q^{\mp 1}, 1)$$  \hspace{1cm} (7.13)

Thus indeed one finds again a diagonal \(YB\) matrix. Compare (6.7) and the comments preceding (5.8).

The diagonalizer \(M\) of (6.9) gives now with any normalization factor \(N\) and (7.2)

$$M\hat{R}(\theta)M^{-1} = N\text{diag}(a + d, c + b, c - b, a - d)$$  \hspace{1cm} (7.14)

8 A nested sequence of projectors for higher dimensions:

The 8-vertex matrix has complex features due to the presence of four functions \((a, b, c, d)\) and their realizations in terms of elliptic functions. On the other hand its symmetry permits a spectral resolution on a basis of particularly simple symmetrical projectors with constant elements \((\approx \pm 1)\). For \(N^2 \times N^2\) matrices with \(N > 2\) one can construct different types of generalization of such a basis with constant elements. One example can be easily extracted from the multistate model presented in Sec. 4 of [16] (where original sources are cited). Let us consider the simplest such case \((N = 3)\).

Let \(E_{ij}\) be the matrices defined below (1.3). A set of projectors satisfying (1.4) and suitable for the spectral decomposition of a particular class of \(9 \times 9\) \(\hat{R}(\theta)\) is

\[
2P_{1(\pm)} = (E_{11} + E_{99} \pm E_{19} \pm E_{91}), \quad 2P_{2(\pm)} = (E_{22} + E_{44} \pm E_{24} \pm E_{42})
\]
\[
2P_{3(\pm)} = (E_{33} + E_{77} \pm E_{37} \pm E_{73}), \quad 2P_{4(\pm)} = (E_{66} + E_{88} \pm E_{68} \pm E_{86}), \quad P_{55} = E_{55} \quad (8.1)
\]

Generalizations for \(N > 3\) are not difficult to write down.

Additional simplifications arise in (4.1) of [16] since the functions of \((\gamma, \theta)\) implemented are of the 6-vertex type. One feature should be noted. The action of \(P\) for \(N = 3\) interchanges the rows \((2, 4), (3, 7)\) and \((6, 8)\). Hence when the basis (8.1) is implemented \(\hat{R}\) and \(R(= P\hat{R})\) have fairly analogous structures (as for the \(4 \times 4\) 6-and 8-vertex matrices). Such a feature, though worth noting, is not essential and is indeed not present in the standard cases of Sec. 2. Instead of presenting full details concerning the above-mentioned possibility, we briefly present another one. For \(N = 2\) this coincides with (6.1). This basis does not have (for \(N > 2\)) the simple property of (8.1) and its generalizations for \(N > 3\) under the action of \(P\). But it exhibits a particularly simple canonical nested structure. The prescription for diagonalization is also particularly simple.
For \( n = N^2 = 2l \) define

\[
2P_i(\pm) = (E_{ii} + E_{n-i+1,n-i+1} \pm E_{i,n-i+1} \pm E_{n-i+1,i}), \quad (i = 1, 2, ..., l) \tag{8.2}
\]

For \( n = 2l + 1 \) one has in addition

\[
P_{l+1} = E_{l+1,l+1} \tag{8.3}
\]

To diagonalize this set satisfying (1.4) now define

\[
\sqrt{2}M = \sqrt{2}M^{-1} = \sum_{i=1}^{l} (E_{ii} + E_{i,n-i+1} + E_{n-i+1,i} - E_{n-i+1,n-i+1}) + E_{l+1,l+1} \tag{8.4}
\]

For \( n = 2l \), the last term is absent.

One obtains

\[
MP_i(+)^{-1} = E_{i,i}, \quad MP_i(-)^{-1} = E_{n-i+1,n-i+1}, \quad (i = 1, 2, ..., l) \tag{8.5}
\]

When it is present, \( P_{l+1} \) is already diagonal and commutes with \( M \). Hence, if (with the last term present only for odd \( n \)) and with \( \varepsilon = \pm \),

\[
\hat{R}(z) = \sum_{i=1}^{l} \sum_{\varepsilon} \left( \frac{f_{i(\varepsilon)}(z)}{f_{i(\varepsilon)}(z^{-1})} P_{i(\varepsilon)} \right) + \frac{f_{l+1}(z)}{f_{l+1}(z^{-1})} P_{l+1} \tag{8.6}
\]

\[
M\hat{R}(z)M^{-1} = \sum_{i=1}^{l} \left( \frac{f_{i(+)}(z)}{f_{i(+)}(z^{-1})} E_{ii} + \frac{f_{i(-)}(z)}{f_{i(-)}(z^{-1})} E_{n-i+1,n-i+1} \right) + \frac{f_{l+1}(z)}{f_{l+1}(z^{-1})} E_{l+1,l+1} \tag{8.7}
\]

The two sets, ( the one generalizing (8.1) for all \( n \) and the one given by (8.2), (8.3) ) can be shown to be related through a similarity transformation. But the matrix of conjugation does not possess a tensored structure \( G \otimes G \) ( and hence the tensored components of the base space are not transformed individually ). The question of existence and construction of solutions of (1.1) for the parameterization (8.6) is beyond the scope of this paper.

Let us conclude with a closer look at the simplest nontrivial case. For \( N = 3 \),

With a maximum number of functions in the coefficients and suppressing arguments \( (x(\theta) = x \) and so on )

\[
\hat{R}(\theta) = \sum_{\varepsilon} \left( (x + \varepsilon y)P_1(\varepsilon) + (u + \varepsilon v)P_2(\varepsilon) + (a + \varepsilon b)P_3(\varepsilon) + (c + \varepsilon b)P_4(\varepsilon) \right) + w P_5 \tag{8.8}
\]

The diagonalizer is
\[ \sqrt{2}M = \sqrt{2}M^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}. \] (8.9)

This generalizes (6.9) as one moves up from the 4 \times 4 to the 9 \times 9 case in our sequence. Note the central element ( at (55) ) appearing for odd \( n \)(= 9).

\[ M\tilde{R}(\theta)M^{-1} = \text{diag}((x+y), (u+v), (a+d), (c+b), w, (c-b), (a-d), (u-v), (x-y)) \] (8.10)

If ,say, \( y = 0 \) one can redefine \((P_1(+) + P_1(-))\) as a single projector \(P_1\) and so on ( continuing to satisfy (1.4) ). The number of functions available increases with the number of projectors, but so does the number of constraints due to the braid equation (1.1). Let us note however, that for \( SO_q(N) \) and \( Sp_q(N) \) after fixing the normalization one has only two functions to satisfy four complicated functional equations \([1]\). Yet one emerges with \textit{three} independent solutions in both the cases (\textit{Secs.}2,3). A close study of particular cases in the present context might also lead to interesting possibilities. One recognizes (7.2) and (7.14) as subcases of (8.8) and (8.10) respectively.

\section{Diagonalization and factorization:}

Diagonalization of braid matrices was studied in \([1]\). It was used to elucidate certain aspects of associated noncommutative spaces. Here it will be studied in the context of factorization.

For the \( 4 \times 4 \) matrices (\textit{Secs.}4,5,6,7) the diagonalizer \( M \) has been presented for each case explicitly. In \textit{Sec.}8 \( M \) has been obtained for the nested sequence explicitly for arbitrary dimensions. The results for the lower dimensional cases of \((A,B,C,D)_q\)type algebras are collected in \textit{App.B}. We will see how a striking structure emerges from diagonalization of each factor in (1.12). But to start with it is worthwhile to recapitulate some basic features noted in \([1]\).

\begin{itemize}
\item From (1.6) and (1.7) it is evident that if there exists an invertible matrix \( M \) diagonalizing \( \tilde{R}(\theta) \) it must diagonalize each projector \( P_i \) \textit{separately}.
\item A projector, when diagonalized, can have only +1 or 0 as diagonal elements.
\item The number of unit elements on the on the diagonal is equal to the the trace of the projector, obligatorily a positive integer. For different \( P_i \) in (1.4) these elements can never coincide due to orthogonality. In standard notations \([10]\) one has
\end{itemize}
for $GL_q(N)$

$$P_+(+) + P_-(-) = I_{N^2 \times N^2}$$  \hspace{1cm} (9.1)

with

$$2Tr P_+(\pm) = N(N \pm 1)$$  \hspace{1cm} (9.2)

For $SO_q(N)$ and $Sp_q(N = 2n)$ one has

$$P_+(+) + P_-(-) + P_0 = I_{N^2 \times N^2}$$  \hspace{1cm} (9.3)

and with $\epsilon = \pm 1$ respectively (as below (3.1))

$$2Tr P_+(\pm) = N(N \pm 1) \mp (\epsilon \pm 1), \quad Tr P_0 = 1$$  \hspace{1cm} (9.4)

• If necessary, implementing a simple supplementary conjugation the elements on the diagonal can be reordered. Exploiting this possibility we introduce the following conventions:

For $GL_q(N)$ the unit elements of $P_-(一刀)$ are grouped at the top, followed by those of $P_+(一刀)$. Thus for $GL_q(3)$ and

$$\hat{R} (\theta) = P_+ + vP_-$$  \hspace{1cm} (9.5)

once $M$ is constructed we obtain

$$M \hat{R}(\theta) M^{-1} = diag(v, v, v, 1, 1, 1, 1, 1, 1)$$  \hspace{1cm} (9.6)

For $SO_q$ and $Sp_q$ the chosen ordering is $(P_0, P_-, P_+)$. Thus for $SO_q(3)$ and

$$\hat{R}(\theta) = P_+ + vP_- + wP_0$$  \hspace{1cm} (9.7)

$$M \hat{R}(\theta) M^{-1} = diag(w, v, v, v, 1, 1, 1, 1, 1)$$  \hspace{1cm} (9.8)

Generalizations are evident.

• Since each diagonalized $P_i$ (denoted below by $D_i$) is thus completely fixed beforehand one can (assuming the invertibility of $M$ to be confirmed a posteriori) write separately for each $P_i$ with the same $M$,

$$MP_i = D_i M$$  \hspace{1cm} (9.9)

Here both $P_i$ and $D_i$ are known giving explicit linear constraints on the elements of $M$. One avoids the construction of $M^{-1}$ to start with.

• The block structures in (9.6) and (9.8) and their evident generalizations reveal the extent to which $M$ is arbitrary:

Let $M_i$ denote a matrix of dimension $(Tr P_i \times Tr P_i)$, with a nonzero determinant but with otherwise arbitrary elements. Then, in obvious notations, a supplementary conjugation of (9.6) by a block-diagonal (bd) matrix

$$(M_(-), M_+(刀))(bd)$$  \hspace{1cm} (9.10)
and one of (9.8) by

$$\phi(0, M(-), M(+)) = \phi(0, M(-), M(+))$$

leaves the diagonal forms invariant.

- The arbitrariness thus exhibited, instead of being a source of embarrassment, provides a wide margin of manoeuvre exploitable to select an $M$ with particularly attractive properties. We choose the following canonical feature:

\textit{mutual orthogonality of the rows of $M$.}

(Except for the complex, unitary $M$ of (4.9) for the exotic $S03$ such an orthogonality holds for all the cases we study.)

Agreeable consequences are

1. The inverse of $M$ is obtained effortlessly. The prescription is: Take the transpose $M^T$ of $M$. Normalize each element of the column $j$ of $M^T$ by the same factor $c_j$ such that for each $j$

$$\left( \sum_i M^2_{ij} \right) c_j = 1$$

Thus one obtains $M^{-1}$. Examples can be found in [1].

2. Each row of $M$, transposed to a column provides an eigenvector of $M$ and all together a complete set.

\textbf{Consequences for factorization:}

For

$$\hat{R}(\theta) = \sum_{i} \frac{f_i(\theta)}{f_i(-\theta)} P_i$$

$$M \hat{R}(\theta) M^{-1} = \text{diag}\left( \frac{f_1(\theta)}{f_1(-\theta)}, \ldots, \frac{f_2(\theta)}{f_2(-\theta)}, \ldots \right)$$

Here the multiplicity of $f_i(\theta)$ is equal to $Tr P_i$. Note that $M$ is independent of $\theta$. It diagonalizes each $P_i$ (independent of $\theta$) and hence also $\hat{R}$.

Define

$$D(\theta) = \text{diag}(f_1(\theta), \ldots; f_2(\theta), \ldots)$$

$$M(\theta) = D(\theta) M$$

Now, starting with (1.13),

$$\hat{R}(\theta) = (F(-\theta))^{-1} F(\theta) = (M^{-1} D(-\theta) M)^{-1} (M^{-1} D(\theta) M) = (M(-\theta))^{-1} M(\theta)$$

In each factor all $\theta$-dependence is thus again factorized in a diagonal matrix $D(\theta)$. Some consequences will be studied in the following sections.
10 L-operators:

Here we indicate the general features that arise as one implements our formalism in the construction of L-operators. It is well-known that the FRT definitions [10] (with their \( R^{(+)} = (PRP) \) and with \( L_2^\varepsilon = PL_1^\varepsilon P \))

\[
(PR_1^+P)L_1^\pm L_2^\pm = L_2^\pm L_1^\pm (PR_1^+P), \quad (PR_1^+P)L_1^\pm L_1^\mp = L_2^-L_1^+(PR_1^+P) \tag{10.1}
\]
give in terms of \( \hat{R} = PR \),

\[
\hat{R}L_2^\pm L_1^\pm = L_2^\pm L_1^\pm \hat{R}, \quad \hat{R}L_2^\pm L_1^\pm = L_2^-L_1^+ \hat{R} \tag{10.2}
\]

Taking one more step we define (with \( \varepsilon = \pm \) below)

\[
L_2^\varepsilon P = PL_1^\varepsilon \equiv \hat{L}_\varepsilon \tag{10.3}
\]

when

\[
L_2^\varepsilon L_1^\varepsilon' = L_2^\varepsilon PPL_1^\varepsilon' = \hat{L}_\varepsilon \hat{L}_\varepsilon'
\]

and

\[
\hat{R}L_\varepsilon \hat{L}_\varepsilon = \hat{L}_\varepsilon \hat{L}_\varepsilon \hat{R}, \quad \hat{R}\hat{L}_+ \hat{L}_- = \hat{L}_- \hat{L}_+ \hat{R} \tag{10.4}
\]

All this is before Baxterization. When the spectral parameter is introduced a more general formulation is

\[
\hat{R}(\theta - \theta') \hat{L}_\varepsilon(\theta)\hat{L}_\varepsilon(\theta') = \hat{L}_\varepsilon(\theta')\hat{L}_\varepsilon(\theta)\hat{R}(\theta - \theta'), \quad \hat{R}(\theta - \theta') \hat{L}_+ (\theta)\hat{L}_- (\theta') = \hat{L}_- (\theta')\hat{L}_+ (\theta)\hat{R}(\theta - \theta') \tag{10.5}
\]

(For affine cases extra factors \( q^{\pm c} \) can appear in the argument of \( \hat{R} \) in the last equation. But the above formulation suffices to illustrate our approach.)

One can introduce a development such as

\[
\hat{L}_\varepsilon(\theta) = \frac{1}{\rho(\theta)} \sum_{n \geq 0} \left( \hat{L}_{(\varepsilon,n)} e^{n\theta} + \hat{L}_{(\varepsilon,-n)} e^{-n\theta} \right) \tag{10.6}
\]

But for clarity in our illustrative approach let us concentrate on a particularly simple case [20]. When the spectral basis on the right of (1.9) has only two projectors, \( \hat{R}(\theta) \) can be expressed quite simply in terms of \( \hat{R}^{\pm 1} \). (A more general result is obtained (3.49) of [1].) Thus for \( GL_q(N) \) from (2.1) one obtains

\[
\hat{R}(\theta) = \frac{e^{h+\theta} \hat{R} - e^{-h-\theta} \hat{R}^{-1}}{e^{h+\theta} - e^{-h-\theta}} \tag{10.7}
\]
From (3.3) (redefining \((P_+ + P_-)\) as \(P_1\), say) one obtains for this new class of solutions

\[
\hat{R}(\theta) = P_1 + \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} P_0 = \frac{e^{(\eta+\theta)} \hat{R} - e^{(-\eta-\theta)} \hat{R}^{-1}}{e^{(\eta+\theta)} - e^{(-\eta-\theta)}} \tag{10.8}
\]

Here \(\eta\) is defined for \(SO_q(N)\) and \(Sp_q(N)\) as in (3.1), (3.2). (Setting \(\hat{R}(0) = I\) one obtains the linear relation between \(\hat{R}\) and \(\hat{R}^{-1}\).) For such cases, defining (analogously to (3.5.9) of [20], but in terms of our \(\hat{L}\))

\[
\hat{L}(\theta) \equiv (e^\theta \hat{L}_+ - e^{-\theta} \hat{L}_-)
\]

all the three relations (10.4) can be encapsulated in the single one

\[
\hat{R}(\theta - \theta') \hat{L}(\theta) \hat{L}(\theta') = \hat{L}(\theta') \hat{L}(\theta) \hat{R}(\theta - \theta') \tag{10.10}
\]

(As (10.10) is developed, inserting (10.8) and (10.9), the terms \(\hat{L}_+ \hat{L}_-\) appear in "wrong order", \(\hat{R} \hat{L}_- \hat{L}_+\) and so on. Now, expressing \(\hat{R}\) in terms of \(\hat{R}^{-1}\) and vice versa one can extract the relations (10.4) with different factors depending on arbitrary \((\theta, \theta')\).)

We will use this compact formulation adapted to our special class of braid matrices (Sec.3) to illustrate the consequences of our formalism. For more general cases (see (10.5), (10.6)) the basic features will be analogous along with more elaborate sets of equations. Some indications will be given of such generalizations. Let us however come back to our special case:

Implementing (10.8) in (10.10) one obtains

\[
P_0 \left( \hat{L}(\theta) \hat{L}(\theta') - \hat{L}(\theta') \hat{L}(\theta) \right) P_0 = 0
\]

\[
P_1 \left( \hat{L}(\theta) \hat{L}(\theta') - \hat{L}(\theta') \hat{L}(\theta) \right) P_1 = 0
\]

\[
P_0 \left( (e^{\eta-\theta+\theta'} - e^{-\eta+\theta-\theta'}) \hat{L}(\theta) \hat{L}(\theta') - (e^{\eta+\theta-\theta'} - e^{-\eta-\theta+\theta'}) \hat{L}(\theta') \hat{L}(\theta) \right) P_1 = 0
\]

\[
P_1 \left( (e^{\eta+\theta-\theta'} - e^{-\eta-\theta+\theta'}) \hat{L}(\theta) \hat{L}(\theta') - (e^{\eta-\theta+\theta'} - e^{-\eta+\theta-\theta'}) \hat{L}(\theta') \hat{L}(\theta) \right) P_0 = 0 \tag{10.11}
\]

Here, as in the general case (10.21) below, the constraints are exhaustive due to the resolution of the identity provided by \(\sum P_i = I\). This aspect is evident in the equivalent form obtained below ((10.17), (10.18), (10.19)) via diagonalization.

Here the \(P_i\) do not depend on \(\theta\) but only on \(q\). So now implementing (10.9) dependence on \((\theta, \theta')\) becomes entirely explicit. The coefficients of \(e^{(n\theta+n'\theta')}\) for different \((n, n')\) must vanish separately. Only the factors \(e^{\pm i}\), given by (3.1), (3.2) as
\[
tanh \eta = \sqrt{(1 - 4([N - \epsilon] + \epsilon)^{-2})} \tag{10.12}
\]

and \(q\)-dependent through \([N - \epsilon]\) characterizes the \(L\)-algebra for this specific class of solutions.

As emphasized in Sec.3, this class remains nontrivial even for \(q = 1\). Now for both cases \((\epsilon = \pm 1)\), denoting \(\eta\) as \(\hat{\eta}\) for \(q = 1\),

\[
tanh \hat{\eta} = \pm N^{-1}\sqrt{N^2 - 4} \tag{10.13}
\]

But the projectors \((\hat{P}_0, \hat{P}_1)\) are still different for the two cases \((SO_q, Sp_q)\).

We now present the consequences of diagonalization (Sec.9). Both for \(SO_q(N)\) and \(Sp_q(N)\) (remembering that \(P_1 = P_+ + P_-\)) one obtains

\[
MP_0M^{-1} = \text{diag}(1, 0, ..., 0), \quad MP_1M^{-1} = \text{diag}(0, 1, ..., 1) \tag{10.14}
\]

with

\[
TrP_0 = 1, \quad TrP_1 = N^2 - 1 \tag{10.15}
\]

(Explicit expression for \(M\) are given, in App.B, only for \(SO_q(3), SO_q(4)\) and \(Sp_q(4)\).)

Define

\[
K(\theta) = M\hat{L}(\theta)M^{-1} = e^\theta(M\hat{L}_+M^{-1}) - e^{-\theta}(M\hat{L}_-M^{-1}) \tag{10.16}
\]

Here \(M\) is different for \(SO_q(N)\) and \(Sp_q(N)\). See Sec.9 and our particularly simple prescription for \(M^{-1}\) when the rows of \(M\) are mutually orthogonal. Our diagonalization leads to,

\[
(K(\theta)K(\theta') - K(\theta')K(\theta))_{ij} = 0; \quad (i, j) = (1, 1), (i > 1, j > 1) \tag{10.17}
\]

and for \(j > 1\) to

\[
\left((e^{\eta-\theta+\theta'} - e^{-\eta+\theta+\theta'})K(\theta)K(\theta') - (e^{\eta+\theta-\theta'} - e^{-\eta-\theta+\theta'})K(\theta')K(\theta)\right)_{1j} = 0 \tag{10.18}
\]

\[
\left((e^{\eta+\theta-\theta'} - e^{-\eta-\theta+\theta'})K(\theta)K(\theta') - (e^{\eta-\theta+\theta'} - e^{-\eta+\theta-\theta'})K(\theta')K(\theta)\right)_{j1} = 0 \tag{10.19}
\]

This is the most compact form of the constraints on the \(L\)-operatorrs. Those on the elements of \(\hat{L}(\theta)\) are now obtained from

\[
\hat{L}(\theta) = M^{-1}\hat{K}(\theta)M
\]

Then one can implement Gauss decomposition, if so desired, for the elements of \(L^\pm\) to obtain results more directly comparable to those for standard cases. But all information is
encapsulated in (10.17), (10.18) and (10.19). All \( \theta \)-dependence can be extracted as exponential factors giving the final constraints as coefficients. For \( SO_q(3) \), for example, our equations furnish the 16 ( for \( j = 2, \ldots, 9 \) ) constraints which involve \( \eta \).

For the more general case ((10.5), (10.6), (10.7)), where

\[
\hat{R}(\theta) = \sum_i^p \frac{f_i(\theta)}{f_i(-\theta)} P_i
\]  

(10.20)

the set (10.11) is generalized to the following \( p^2 \) constraints

\[
P_i \left( f_i(\theta - \theta') f_j(-\theta + \theta') \hat{L}(\theta) \hat{L}(\theta') - f_i(-\theta + \theta') f_j(\theta - \theta') \hat{L}(\theta') \hat{L}(\theta) \right) P_j = 0
\]  

(10.21)

where

\[
(\varepsilon, \varepsilon') = (++, (--), (+-)), \quad (i, j = 1, \ldots, p)
\]

Diagonalization and the definition

\[
K_\varepsilon(\theta) = M \hat{L}_\varepsilon(\theta) M^{-1} = \frac{1}{\rho(\theta)} \sum_n \left( M \hat{L}_{\varepsilon, n} M^{-1} e^{n\theta} + M \hat{L}_{\varepsilon, -n} M^{-1} e^{-n\theta} \right)
\]  

(10.22)

reduces (10.21) to

\[
f_i(\theta - \theta') f_j(-\theta + \theta') (K_\varepsilon(\theta) K_{\varepsilon'}(\theta'))_{i', j'} - f_i(-\theta + \theta') f_j(\theta - \theta') (K_{\varepsilon'}(\theta') K_\varepsilon(\theta))_{i', j'} = 0
\]  

(10.23)

Here, for a given \( (i, j) \), the ranges of \( (i', j') \) are fixed by \( TrP_i, TrP_j \) and the order chosen (Sec.9, App.B) for the elements unity in diagonalizing the projectors. A simple example is provided by (10.18) and (10.19). If the expansion (10.22) is a finite series ( (10.9) being an extreme example ) one can extract the limits for \( \theta \) and \( (\theta - \theta') \rightarrow \pm \infty \), since the dependence on these parameters can be made explicit as factored coefficients. But for a correct extraction the functions \( f_i(\theta) \) have to be properly defined ( as noted below (1.33) ).

11 Transfer matrices and diagonalization:

**General formulation** :

We start by introducing notations analogous to those of Sec.10 for the row-to-row transfer matrix \( T^{(L)}(\theta) \), satisfying

\[
\hat{R}(\theta - \theta')(T^{(L)}(\theta) \otimes T^{(L)}(\theta')) = (T^{(L)}(\theta') \otimes T^{(L)}(\theta)) \hat{R}(\theta - \theta')
\]  

(11.1)
Here, apart from evident analogies (since we have again a class of \(L\)-functions) specific features arise concerning the component blocks of \(T^{(L)}\). The dimensions of the blocks increase with the length of the row according to standard prescriptions.

Matrix multiplication for a \(N^2 \times N^2\) matrix \(\hat{R}(\theta)\) is defined by labelling \(T^{(L)}\) for any \(L\) by \(N^2\) blocks. If \(I\) be the \(N \times N\) unit matrix,

\[
T^{(L)}(\theta) \otimes T^{(L)}(\theta') = (T^{(L)}(\theta) \otimes I)(I \otimes T^{(L)}(\theta')) = (P(I \otimes T^{(L)}(\theta))P)(P(T^{(L)}(\theta') \otimes I)P)
\]

\[
= (P(I \otimes T^{(L)}(\theta))((T^{(L)}(\theta') \otimes I)P) = (PT^{(L)}_2(\theta))(T^{(L)}_1(\theta')P) = \hat{T}^{(L)}(\theta)\hat{T}^{(L)}(\theta') \quad (11.2)
\]

where

\[
\hat{T}^{(L)} \equiv PT^{(L)}_2 = T^{(L)}_1 P
\]

are \(N^2 \times N^2\) matrices in terms of blocks of \(T^{(L)}\).

Thus,

\[
\hat{R}(\theta - \theta') (\hat{T}^{(L)}(\theta)\hat{T}^{(L)}(\theta')) = (\hat{T}^{(L)}(\theta)'\hat{T}^{(L)}(\theta))\hat{R}(\theta - \theta') \quad (11.3)
\]

Thus the mixture of matrix multiplication and tensor product in (11.1) has been rephrased as matrix multiplications of \(\hat{R}\) and \(\hat{T}\). Instead of \((T_1, T_2)\) the same \(\hat{T}\) now appears throughout. For (1.9), namely

\[
\hat{R}(\theta) = \sum_i^p \frac{f_i(\theta)}{f_i(-\theta)}P_i \quad (11.4)
\]

one obtains, as in Sec.10, a complete set of \(p^2\) constraints

\[
P_i \left( f_i(\theta - \theta')f_j(-\theta + \theta')\hat{T}^{(L)}(\theta)\hat{T}^{(L)}(\theta') - f_i(-\theta + \theta')f_j(\theta - \theta')\hat{T}^{(L)}(\theta')\hat{T}^{(L)}(\theta) \right)P_j = 0 \quad (11.5)
\]

Using the diagonalizer \(M\) of the braid matrix (Sec.9, App.B) define

\[
\hat{R}_d(\theta) = M\hat{R}(\theta)M^{-1}, \quad \hat{K}^{(L)}(\theta) = M\hat{T}^{(L)}(\theta)M^{-1} \quad (11.6)
\]

one obtains from (11.3), in terms of the diagonal matrix \(\hat{R}_d\)

\[
\hat{R}_d(\theta - \theta') (\hat{K}^{(L)}(\theta)\hat{K}^{(L)}(\theta')) = (\hat{K}^{(L)}(\theta)'\hat{K}^{(L)}(\theta))\hat{R}_d(\theta - \theta') \quad (11.7)
\]

This corresponds to

\[
f_i(\theta - \theta')f_j(-\theta + \theta')(\hat{K}^{(L)}(\theta)\hat{K}(L)(\theta'))_{ij'} - f_i(-\theta + \theta')f_j(\theta - \theta')(\hat{K}^{(L)}(\theta)'\hat{K}^{(L)}(\theta))_{ij'} = 0 \quad (11.8)
\]
We have explained below (10.23) of Sec.10 how the domain of \((i'j')\) depend on the conventions adopted for the diagonalizations of the projectors \(P_i\) and \(P_j\). It was also pointed out before that such a set of constraints is exhaustive.

The elements of \(\hat{K}^{(L)}(\theta)\) are linear combinations of those of \(T^{(L)}(\theta)\), the coefficients being independent of \(\theta\) ( since \(M\) is so ). The bilinear algebraic relations between due to (1.1) between \(T_{ij}^{(L)}(\theta)\) attain their simplest form in (11.8) in terms of these linear combinations. Construction of a ”\(\hat{K}\)-basis” ( a complete set of states specifically adapted to the action of the blocks of \(\hat{K}\) ) would permit a full exploitation of (11.8).

**Particular cases:**

We now consider two particular cases. The first one is chosen because it is familiar and extensively studied. The content of (11.8) for the 6-vertex case can be compared to well-known results. ( See [7, 8, 9, 16] and basic sources cited in these references.) The second one is chosen as a relatively simple but new example of a multistate model. It corresponds to our special class of solutions (Sec.3) for \(SO_q(3)\). This can be compared to a different class of multistate models [16].

- The 6-vertex case:

  Inserting in (11.6) the results of Sec.6 with \(M\) given by (6.9),

  \[ \hat{R}_d(\theta - \theta') = \text{diag}(1, u, v, 1) \]  

  where

  \[ u = \frac{\cosh \frac{1}{2}(\gamma - \theta + \theta')}{\cosh \frac{1}{2}(\gamma + \theta - \theta')} \quad v = \frac{\sinh \frac{1}{2}(\gamma - \theta + \theta')}{\sinh \frac{1}{2}(\gamma + \theta - \theta')} \]  

  ( Other interesting choices of \(M\) are possible. But (6.9) is adequate for our present purposes.) Now let

  \[ \hat{T}^{(L)}(\theta) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]  

  where each entry is a \(2^L \times 2^L\) block obtained according to standard prescriptions ( e.g. Secs.2, 3 of [16] ). From (6.9), (11.2) and (11.6) one obtains

  \[ 2\hat{K}^{(L)}(\theta) = \begin{pmatrix} A + D & B + C & B - C & A - D \\ B + C & A + D & D - A & C - B \\ B - C & A - D & -A - D & -B - C \\ A - D & B - C & B + C & A + D \end{pmatrix}. \]  

  \[ = (A + D) \begin{pmatrix} s_0 & 0 \\ 0 & -s_3 \end{pmatrix} + (A - D) \begin{pmatrix} 0 & -s_2 \\ s_1 & 0 \end{pmatrix} + (B + C) \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} + (B - C) \begin{pmatrix} 0 & s_3 \\ s_0 & 0 \end{pmatrix}. \]
where

\[ s_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

Note that

\[ Tr \hat{K}^{(L)}(\theta) = Tr(A + D) = TrT^{(L)}(\theta) \]

and if \( V \) is an eigenvector of \( \hat{K}(\theta) \) then \( M^{-1}V \) is one of \( \hat{T}^{(L)}(\theta) \).

Now (11.8) reduces to

\[
\left( \hat{K}^{(L)}(\theta) \hat{K}(L)(\theta') - x\hat{K}^{(L)}(\theta')\hat{K}^{(L)}(\theta) \right)_{ij} = 0 \tag{11.14}
\]

where \((u, v)\) being given by (11.10), one obtains \( x \) as follows for values of \((i, j)\) indicated at right:

\[
x = 1, \quad (i, j) = (1, 1), (2, 2), (3, 3), (4, 4), (1, 4), (4, 1);
\]

\[
x = u, \quad (i, j) = (1, 2), (4, 2); \quad x = u^{-1}, \quad (i, j) = (2, 1), (2, 4);
\]

\[
x = v, \quad (i, j) = (1, 3), (4, 3); \quad x = v^{-1}, \quad (i, j) = (3, 1), (3, 4);
\]

\[
x = \frac{u}{v}, \quad (i, j) = (3, 2); \quad x = \frac{v}{u}, \quad (i, j) = (2, 3) \tag{11.15}
\]

Thus we obtain the simplest form of the constraints implied by (11.1) or (11.3).

- A special class of multistate models (\(SO_q(3)\) example):

We consider now the class of braid matrices presented in Sec.3 and explicitly diagonalized for \(SO_q(3)\), as well as for \(SO_q(4)\), in App.B. (We consider only \(SO_q(N)\). For \(Sp_q(N)\) certain states have negative weights.) The precise way in which the model is “nonminimal”, with more than two possible states per link, will be explained at the end by comparing it with another class of models studied in Sec.4 of [16] (where original sources are cited).

We start with (B.8), (B.9) and (B.10). Using the notations of (10.8) the braid matrix is (with \(\eta\) defined in Sec.3)

\[
\hat{R}(\theta) = P_1 + \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} P_0 = I + \left( \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} - 1 \right) P_0 \tag{11.16}
\]

For \(N = 3\) (see Sec.4 of [1]),

\[
(q + q^{-1} + 1) P_0 = q^{-1} E_{11} \otimes E_{33} + q^{-\frac{1}{2}} E_{12} \otimes E_{32} + E_{13} \otimes E_{31} + q^{-\frac{1}{2}} E_{21} \otimes E_{23} + E_{22} \otimes E_{22} + q^{\frac{1}{2}} E_{23} \otimes E_{21} + E_{31} \otimes E_{13} + q^{\frac{1}{2}} E_{32} \otimes E_{12} + q E_{33} \otimes E_{11} \tag{11.17}
\]
Setting $\epsilon = 1$ and $N = 3$ in (3.1) and (3.2)
\[ 2\cosh \eta = (q + q^{-1} + 1) \] (11.18)
and for $M$ given by (B.10)
\[ M \hat{R}(\theta) M^{-1} = \text{diag} \left( \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)}, 1, 1, 1, 1, 1, 1, 1, 1 \right) \] (11.19)
Hence (11.8) reduces to
\[ \left( \hat{K}^{(L)}(\theta) \hat{K}^{(L)}(\theta') - K^{(L)}(\theta') \hat{K}^{(L)}(\theta) \right)_{ij} = 0 \] (11.20)
for $(i, j) = (1, 1)$ and for $(i > 1, j > 1)$. For $j = (2, 3, \ldots, 9)$ one obtains
\[ \left( \sinh(\eta - \theta + \theta') \hat{K}^{(L)}(\theta) \hat{K}^{(L)}(\theta') - \sinh(\eta + \theta - \theta') \hat{K}^{(L)}(\theta') \hat{K}^{(L)}(\theta) \right)_{1j} = 0 \] (11.21)
\[ \left( \sinh(\eta + \theta - \theta') \hat{K}^{(L)}(\theta) \hat{K}^{(L)}(\theta') - \sinh(\eta - \theta + \theta') \hat{K}^{(L)}(\theta') \hat{K}^{(L)}(\theta) \right)_{j1} = 0 \] (11.22)

Thus we obtain the complete set of constraints in the simplest and the most compact form. The foregoing structure is directly generalizable to all $N$. But for $N > 4$ one has either to construct the corresponding $M$ or to use (11.5) with (11.16). A study of the $\hat{K}$-basis adapted to the foregoing set of constraints is beyond the scope of this paper. We conclude with some comments and comparisons.

In 6- or 8-vertex models 2 states are possible for each link. But non-zero Boltzmann weights are associated to a subset of the $2^4$ elements of the braid matrix. When 3 states are possible per link (of a plane lattice) one can implement a $9 \times 9$ matrix (with $3^4$ elements) attributing again non-zero weights to a subset of the possible states, corresponding to the non-zero elements of the matrix.

The number of non-zero elements in (11.16) is $15 (= 3(2.3 - 1))$. For $SO_q(N)$ this number, for our class of solution of Sec.3, is $N(2N - 1)$. In Sec.4 of [16] a class of models is studied where one has precisely the same number of non-zero weights out of $N^4$ elements (with the symbol $q$ for our $N$). Apart from this feature, the block structure in (4.12) of [16] (namely, (11), (1j), (j1), (ij) with $(i > 1, j > 1)$) corresponds also to the structure of our set of constraints ((11.20), (11.21), (11.22)) for $\hat{K}(\theta)$. We will not study here the different possibilities concerning block structures but emphasize that in spite of the foregoing features the two classes are basically different. For $N = 2$ that of [16] reduces to the 6-vertex case, whereas our special class does not exist. Ours is obtained for $SO_q(N)$ and and for all $N \geq 3$ can be diagonalized to the form (11.19) with $(N^2 - 1)$ unit elements.
Finally let us note the situation for \( q = 1 \). As pointed out at the end of Sec. 3, this class of \( \hat{R}(\theta) \) remains nontrivial, even quite interesting, for \( q = 1 \). There is, of course, additional simplicity. Thus denoting \((\hat{P}_0, \hat{\eta})\) for \( q = 1 \) by \((\hat{P}_0, \hat{\eta})\) one obtains for \( SO(3) \), for example,

\[
3\hat{P}_0 = E_{11} \otimes E_{33} + E_{12} \otimes E_{32} + E_{13} \otimes E_{31} \\
+ E_{21} \otimes E_{23} + E_{22} \otimes E_{22} + E_{23} \otimes E_{21} \\
+ E_{31} \otimes E_{13} + E_{32} \otimes E_{12} + E_{33} \otimes E_{11}
\]

(11.23)

and

\[
cosh \hat{\eta} = \frac{3}{2}, \quad \sinh \hat{\eta} = \frac{\sqrt{5}}{2}
\]

(11.24)

It is amusing to note the relation of \( \sinh \hat{\eta} \) with the Golden Mean.

12 Remarks:

The first essential step in our approach has been the spectral decomposition of the braid matrices \( \hat{R}(\theta) \), obtaining the coefficient of the projector \( P_i \) in the form of a ratio \( \frac{f_i(\theta)}{f_i(-\theta)} \).

The next major step was diagonalization. But here again the projectors play a basic role. The fact that the same matrix \( M \) must diagonalize each projector appearing in the decomposition permits a systematic extraction of the necessary constraints on \( M \) and also the exploitation of the remaining freedom in an efficient fashion. This is explained in detail in Sec. 9. Along with diagonalization a remarkable new feature arises in the factorization. In each factor the dependence on the spectral parameter \( \theta \) is again factorized out as a diagonal matrix.

Various directions opened up deserve further exploration. The present work, despite its length, stops short at various points. Applications of our formalism to \( L \)-operators and to transfer matrices have merely been adumbrated. Thus the introduction of basis states specifically adapted to the form of the constraints obtained (”\( \hat{K} \)-basis”) can be particularly helpful. For the braid matrices of Sec. 2 the diagonalizers have been constructed explicitly (App. B) only for lower dimensions. While for \( GL_q(N) \) the general prescription should not be difficult to obtain, for \( SO_q(2n+1) \), \( SO_q(2n) \) and \( Sp_q(2n) \) one has, among other things, to obtain the mutually orthogonal \((2n+1)\)-plets and \(2n\)-plets (App. B). The problem of solving the braid equation (1.1) implementing the nested sequence of projectors of Sec. 8 has not been addressed.

We hope to study elsewhere some of the aspects mentioned above.
13 APPENDIX A: Comparison with triangular factorization

We compare here our factorization schemes with that proposed by Maillet et al. [7, 8]. We start with notations and general features.

For our \( R_q(\theta) \), where \( q = e^h \), define

\[
\begin{align*}
  z_1 &= e^{(h-\theta)}, \\
  z_2 &= e^{(h+\theta)}
\end{align*}
\]  

(A.1)

Then as

\[
\theta \to -\theta, \quad (z_1, z_2) \to (z_2, z_1)
\]  

(A.2)

The unitarity (1.14) is now (in terms of \( R = P\hat{R} \))

\[
R_{21}(z_2, z_1)R_{12}(z_1, z_2) = I
\]  

(A.3)

In [7, 8] the proposed factorization is

\[
R_{12}(z_1, z_2) = (F_{21}(z_2, z_1))^{-1}F_{12}(z_1, z_2)
\]  

(A.4)

where the aim is to obtain lower triangular \( F_{12} \). Our (1.12) corresponds (implicitly with a different \( F \)) to

\[
R_{12}(z_1, z_2) = (F_{21}(z_2, z_1))^{-1}PF_{12}(z_1, z_2)
\]  

(A.5)

Note the presence of \( P \) in (A.5). In a complimentary fashion, for the braid matrix (A.4) leads to

\[
\hat{R}(z_1, z_2) = (F(z_2, z_1))^{-1}PF(z_1, z_2)
\]  

(A.6)

as compared to our (1.12)

\[
\hat{R}(z_1, z_2) = (F(z_2, z_1))^{-1}F(z_1, z_2)
\]  

(A.7)

This last form permits us to fully exploit the spectral decomposition.

Let us note the following features:

- Given a \( R(z_1, z_2) \) one has to extract \( F(z_1, z_2) \) from (A.4). For higher dimensions this (and in particular the explicit construction of \( F^{-1} \)) is difficult. So, assuming invertibility, the authors start from

\[
(F_{21}(z_2, z_1))R_{12}(z_1, z_2) = F_{12}(z_1, z_2)
\]  

(A.8)

For the \( 4 \times 4 \) matrix of the 6-vertex model (see our Sec.6) explicit triangular factors in (A.4) are obtained. For constructing transfer matrices "partial" \( F \)-matrices are defined.
Given our Baxterization, our type of factorization is obtained effortlessly as a byproduct. In [1] the forms (1.9) (reproduced here in Sec.2 and Sec.3) were obtained in a quest for elegance. Factorization was not a goal.

- The limiting cases \( \theta \rightarrow \pm \infty \) corresponds to \((z_1, z_2) \rightarrow (0, \infty), (\infty, 0)\) respectively. We have systematically extracted the standard (non-Baxterized) braid matrices as the corresponding limits of the \( \theta \)-dependent braid matrices (Secs.(2, 3, ..., 7)) and explained in what sense precisely (Sec.1) they can still be considered to be factorized.

- It is instructive to compare different types of factorization explicitly for the simple example of the 6-vertex matrix.

Setting \( \theta = (\lambda - \mu) \) in (69) of [7] one obtains from (89) and (90) of [7], using a block-diagonal notation,

\[
F_{12}(\theta) = (1, B, 1)_{bd}
\]

where the \(2 \times 2\) block \(B\) is

\[
B = \begin{pmatrix}
\frac{1}{\sinh \eta} & 0 \\
\frac{\sinh \theta}{\sinh(\eta + \theta)} & \frac{\sinh \theta}{\sinh(\eta + \theta)}
\end{pmatrix}.
\]

Using the results of Sec.6 one obtains for our \(F\),

\[
F(\theta) = (1, C, 1)_{bd}
\]

where

\[
C = \begin{pmatrix}
&e^{\frac{1}{2}(\gamma - \theta)} & e^{-\frac{1}{2}(\gamma - \theta)} \\
e^{-\frac{1}{2}(\gamma - \theta)} & e^{\frac{1}{2}(\gamma - \theta)}
\end{pmatrix}.
\]

- Define the diagonal matrices

\[
D(\theta) = \text{diag}(1, \cosh \frac{1}{2}(\gamma - \theta), \sinh \frac{1}{2}(\gamma - \theta), 1)
\]

\[
(D(-\theta))^{-1} = \text{diag}(1, (\cosh \frac{1}{2}(\gamma + \theta))^{-1}, (\sinh \frac{1}{2}(\gamma + \theta))^{-1}, 1)
\]

Now, using the \(M\) of (6.9), \(\hat{R}(\theta)\) of (6.5) and inverting (6.10), one can write (with an \(M\) independent of \(\theta\))

\[
\hat{R}(\theta) = (M^{-1}(D(-\theta))^{-1}D(\theta)M) \equiv (M(-\theta))^{-1}M(\theta)
\]

Now in each factor \(M(\theta)\) all \(\theta\)-dependence is again factorized as a diagonal matrix. This is a general feature of this approach.

We have compared different types of factorization. One can hope to implement fruitfully in different contexts their complementary features such as those indicated above.
14 APPENDIX B: Explicit diagoniaztions

In Sec.9 general aspects of diagonalization of braid matrices have been presented. Here we give explicit expressions for matrices $M$ diagonalizing $\hat{R}(\theta)$ for $GL_q(2), GL_q(3), SO_q(3), SO_q(4)$ and $Sp_q(4)$.

The result for $GL_q(2)$ effectively appears in Sec.5 in a form suited to the context. Here we give an equivalent form consistent with the canonical convention of Sec.9 (see (9.5) and (9.6)). Though we stop with $GL_q(3)$, one can see the general structure of $M$ for $GL_q(n)$ emerging. For the orthogonal and the symplectic cases the situation will be discussed at the end.

In each case below the rows of $M$ will be mutually orthogonal.

Hence $M^{-1}$, always given by the prescription (9.12), will not be displayed explicitly. For $GL_q(n)$ we adopt (with the matrices $E_{ij}$ defined below (1.3)) the normalization

$$R_q = \sum_i E_{ii} \otimes E_{ii} + q^{-1} \sum_{i \neq j} E_{ii} \otimes E_{jj} + (1 - q^{-2}) \sum_{j > i} E_{ij} \otimes E_{ji}$$ (B.1)

The, braid matrix is

$$\hat{R}_q = PR_q = P_+ - q^{-2}P_-$$ (B.2)

With the notations of Sec.2,

$$\hat{R}_q(\theta) = P_+ + \frac{\sinh(h - \theta)}{\sinh(h + \theta)}P_- \equiv P_+ + v(\theta)P_-$$ (B.3)

The projectors depend on $q(= e^h)$ but not on $\theta$.

For $n = 2$ and

$$M = (E_{12} - q^{-1}E_{13}) + (E_{22} + qE_{23}) + E_{31} + E_{44}$$ (B.4)

$$M\hat{R}_q(\theta)M^{-1} = diag(v(\theta), 1, 1)$$ (B.5)

For $n = 3$ and

$$M = (E_{12} - q^{-1}E_{14}) + (E_{23} - q^{-1}E_{27}) + (E_{36} - q^{-1}E_{38}) + (E_{42} + qE_{44}) + (E_{53} + qE_{57}) + (E_{66} + qE_{68}) + E_{71} + E_{85} + E_{99}$$ (B.6)

$$M\hat{R}_q(\theta)M^{-1} = diag(v(\theta), v(\theta), v(\theta), 1, 1, 1, 1, 1, 1)$$ (B.7)

The emerging general structure of $M$ for $GL_q(n)$ is as follows:
There are \( \frac{1}{2}n(n-1) \) rows with two nonzero elements \((1, -q^{-1})\), suitably shifted horizontally in successive rows to assure mutual orthogonality. Then there are \( \frac{1}{2}n(n-1) \) rows with two nonzero elements \((1, q)\) in the corresponding columns (as in (B.6)). Then there are \( n \) rows with a single nonzero element 1 in otherwise empty columns.

For \( SO_q(3) \), \( SO_q(4) \) and \( Sp_q(4) \)

\[
\hat{R}_q(\theta) = P_+ + v(\theta)P_- + w(\theta)P_0
\]  

(B.8)

For the orthogonal case the three possibilities for \( v(\theta) \) and \( w(\theta) \) are given (with \( n = 3 \) and \( n = 4 \) respectively) by (2.2), (2.3) and also (3.3) with \( \epsilon = 1 \). For the symplectic case the relevant equations are (2.4), (2.5) (N.B. with \( n = 2 \) there) and (3.3) with \( \epsilon = -1 \).

For \( SO_q(3) \) define

\[
s = -q^{-\frac{1}{2}}(1-q), \quad t = -q^{-\frac{3}{2}}(1+q)
\]  

(B.9)

Now,

\[
M = (E_{13} + q^\frac{1}{2}E_{15} + qE_{17}) + (E_{22} - qE_{24}) + (E_{36} - qE_{38}) + (E_{43} + sE_{45} - E_{47}) + E_{51} + (E_{62} + q^{-1}E_{64}) + (E_{73} + tE_{75} + q^{-2}E_{77}) + (E_{86} + q^{-1}E_{88}) + E_{99}
\]  

(B.10)

gives

\[
M\hat{R}_q(\theta)M^{-1} = diag(w(\theta), v(\theta), v(\theta), v(\theta), v(\theta), 1, 1, 1, 1, 1, 1, 1)
\]  

(B.11)

For \( SO_q(4) \)

\[
M = (E_{14} + qE_{17} + qE_{110} + q^2E_{113}) + (E_{24} + qE_{27} - q^{-1}E_{210} - E_{213}) + (E_{32} - qE_{35}) + (E_{43} - qE_{49}) + ((E_{58} - qE_{514}) + (E_{612} - qE_{615}) + (E_{74} - q^{-1}E_{77} + qE_{710} - E_{713}) + E_{81} + (E_{92} + q^{-1}E_{95}) + (E_{103} + q^{-1}E_{109}) + (E_{11,4} - q^{-1}E_{11,7} - q^{-1}E_{11,10} + q^{-2}E_{11,13}) + (E_{12,8} + q^{-1}E_{12,14}) + (E_{13,12} + q^{-1}E_{13,15}) + E_{14,11} + E_{15,6} + E_{16,16}
\]  

(B.12)

gives

\[
M\hat{R}_q(\theta)M^{-1} = diag(w(\theta), v(\theta), v(\theta), v(\theta), v(\theta), v(\theta), v(\theta), v(\theta), 1, 1, 1, 1, 1, 1, 1, 1)
\]  

(B.13)

For \( Sp_q(4) \)
\[
M = (E_{14} + qE_{17} - q^3E_{1,10} - q^4E_{1,13}) + (E_{24} - q^{-1}E_{27} + qE_{2,10} - E_{2,13}) \\
+ (E_{32} - qE_{35}) + (E_{43} - qE_{49}) + ((E_{58} - qE_{5,14}) + (E_{6,12} - qE_{6,15}) \\
(E_{74} + qE_{77} + q^{-3}E_{7,10} + q^{-2}E_{7,13}) + (E_{84} - q^{-1}E_{87} - q^{-1}E_{8,10} + q^{-2}E_{8,13}) \\
+ (qE_{92} + E_{95}) + (qE_{10,3} + E_{10,9}) + (qE_{11,8} + E_{11,14}) \\
+ (qE_{12,12} + E_{12,15}) + E_{13,1} + E_{14,6} + E_{15,11} + E_{16,16} \quad (B.14)
\]

gives
\[
M \hat{R}_q(\theta)M^{-1} = diag(w(\theta), v(\theta), v(\theta), v(\theta), v(\theta), 1, 1, 1, 1, 1, 1, 1, 1, 1) \quad (B.15)
\]

Note that the multiplicity of \(v(\theta)\) is 6 in \((B.13)\) and 5 in \((B.15)\).

Other examples of \(M\) can be found near the ends of Secs.\((4, 5, 6, 7)\). In Sec.\(8\) \(M\) is obtained for for arbitrary dimensions.

For \(GL_q(n)\) one encounters as elements of different rows , apart from singlets (unity), only the mutually orthogonal doublets

\[
(1, -q^{-1}), \quad (1, q)
\]

But for \(SO_Q(3)\) one has ( implementing \((B.9)\) ) also the mutually orthogonal triplets

\[
(1, q^{\frac{1}{2}}, q), \quad (1, s, -1), \quad (1, t, q^{-2})
\]

For \(SO_q(4)\) and \(Sp_q(4)\), respectively, one similarly encounters the the mutually orthogonal quadruplets

\[
(1, q, q, q^2), (1, q, -q^{-1}, -1), (1, -q^{-1}, q, -1), (1, -q^{-1}, -q^{-1}, q^{-2})
\]

\[
(1, q, -q^3, -q^4), (1, q^{-1}, q, -1), (1, q, q^{-3}, q^{-2}), (1, -q^{-1}, -q^{-1}, q^{-2})
\]

In Sec.\(7\) of [1] the relations of such multiplets with particular types of \(q\)-deformed surfaces ( spheres, hyperboloids) have been pointed out. In constructing \(M\) for \(SO_q(N)\) and \(Sp_q(N)\) for higher dimensions a key feature would be the general sructure of the corresponding \(N\) mutually orthogonal \(N\)-plets.

This paper is dedicated to the memory of P’s elegant participations in my work.
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