Research Article

On Hardy–Knopp Type Inequalities with Kernels via Time Scale Calculus

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In this paper, we study the inequalities of Hardy–Knopp type with kernel functions which have two nonnegative different weighted functions in two different spaces, in a general domain called a time scale calculus. A time scale calculus is considered as a unification of the continuous calculus and the discrete calculus. We will prove these inequalities in a time scale calculus to avoid proving them twice once in the continuous case and the second in the discrete case. Also, as special cases of the time scale calculus, we can prove some new inequalities in new different domains. Our results will be proved by using the definition of a general Hardy operator on time scale. These inequalities (when \( T = \mathbb{N} \)) are essentially new.

1. Introduction

In [1], Hardy showed that

\[
\sum_{h=1}^{\infty} \left( \frac{1}{h} \sum_{j=1}^{h} b(j) \right)^{\eta} \leq \left( \frac{\eta}{\eta - 1} \right)^{\eta} \sum_{h=1}^{\infty} b^\eta(h), \quad \eta > 1,
\]

(1)

where \( b(h) \geq 0 \) for \( h \geq 1 \) and \( \sum_{h=1}^{\infty} b^\eta(h) < \infty \).

In [2], Hardy established the continuous case of (1), which says that for \( w \geq 0 \) and integrable over \((0, s)\) where \( s \in (0, \infty) \), then

\[
\int_{0}^{s} \left( \frac{1}{x} \int_{0}^{x} w(\tau) \, d\tau \right)^{\eta} \, dx \leq \left( \frac{\eta}{\eta - 1} \right)^{\eta} \int_{0}^{s} w^{\eta}(s) \, ds, \quad \eta > 1.
\]

(2)

The constant \((\eta/(\eta - 1))^\eta\) in (1) and (2) is the finest possible.

In [3], Kajiser et al. generalized (2) with a convex function \( \Psi \geq 0 \) and demonstrated the general the Hardy–Knopp inequality:

\[
\int_{0}^{s} \Psi \left( \frac{1}{s} \int_{0}^{s} w(\tau) \, d\tau \right) \frac{ds}{s} \leq \int_{0}^{s} \Psi \left( w(s) \right) \frac{ds}{s},
\]

(3)

In [4], Čižmešija et al. established a generalization of (3) with two different weighted functions. Specifically, it was demonstrated that if \( 0 < x \leq \infty, p: (0, x) \to \mathbb{R} \) is a nonnegative function such that the function \( s \to p(s)/s^2 \) is locally integrable on \((0, x)\) and \( \Psi \) is convex on \((y, z), \) where \( -\infty \leq y < z \leq \infty \), the following inequality:

\[
\int_{0}^{x} p(s) \Psi \left( \frac{1}{s} \int_{0}^{s} w(\tau) \, d\tau \right) \frac{ds}{s} \leq \int_{0}^{x} q(s) \Psi \left( w(s) \right) \frac{ds}{s},
\]

(4)

holds for all integrable functions \( w: (0, x) \to \mathbb{R} \) s.t. \( w(s) \in (y, z) \forall s \in (0, x) \) and
\[
q(\tau) = \tau \int_{r}^{\tau} \frac{p(s)}{s} \, ds, \quad \text{for } \tau \in (0, x). \tag{5}
\]

In [5], Kajser et al. generalized (2) as follows: if \(0 < x \leq \infty\), \(p : (0, x) \to \mathbb{R}\), \(h : (0, x) \times (0, x) \to \mathbb{R}\) are nonnegative functions, and

\[
q(s) = \int_{s}^{x} p(\tau) \frac{h(\tau, s)}{H(\tau)} \, d\tau < \infty, \quad s \in (0, x), \quad \tag{6}
\]

then

\[
\int_{0}^{x} p(s) \Psi(A_{h}w(s)) \, ds \leq \int_{0}^{x} q(s) \Psi(w(s)) \, ds, \tag{7}
\]

where \(\Psi\) is convex on \(I \subseteq \mathbb{R}\), \(w : (0, x) \to \mathbb{R}\) is a function with values in \(I\), and

\[
A_{h}w(s) = \frac{1}{H(s)} \int_{s}^{x} h(s, \zeta)w(\zeta) \, d\zeta, \quad H(s) = \int_{s}^{x} h(s, \zeta) \, d\zeta, \quad s \in (0, x). \tag{8}
\]

Also, in [5], it is established that if \(1 < \eta \leq t < \infty\), \(\lambda \in (1, \eta)\) and \(0 < x < \infty\). Furthermore, assume that \(\Psi\) is a convex and strictly monotone function on \((y, z)\), \(-\infty < y < z < \infty\), and assume that the general Hardy operator \(A_{h}\) defined by (8) and \(p(s)\) and \(q(s)\) are nonnegative weighted functions. Then,

\[
\left( \int_{0}^{\infty} [\Psi(A_{h}w(s))]^{t} p(s) \, ds \right)^{1/t} \leq C \left( \int_{0}^{\infty} \Psi^{\eta}(w(s))q(s) \, ds \right)^{1/n}, \quad \tag{9}
\]

holds for the nonnegative function \(w(s)\), \(y < w(s) < z\), \(s \in [0, x]\), and \(C > 0\) if

\[
B(s) = \sup_{0 < \zeta < \infty} \left[ Q(\zeta) \right]^{1 \over t-1} \left( \int_{\zeta}^{x} \frac{h(s, \zeta)}{H(s)} \right)^{1/t} \left[ Q(s) \right]^{(t-\lambda)/\eta} p(s) \, ds^{1/t} < \infty, \quad \tag{10}
\]

where

\[
Q(\zeta) = \int_{0}^{\zeta} \left[ q(\tau) \right]^{-1} \left( \tau^{1-\lambda} \right) \, d\tau. \tag{11}
\]

For the reader, we introduce some related papers like [6–9]. In the last decades, a new theory has been discovered to unify the continuous calculus and discrete calculus. It is called a time scale theory. Many authors established dynamic inequalities and generalized them on time scales. For

\[
D = \sup_{x < \tau < y} \left( \int_{x}^{\tau} p(\tau) \Delta \tau \right)^{1/\eta} \left( \int_{x}^{\tau} q^{-1} \left( \tau_{-}^{\alpha} (\tau) \Delta \tau \right)^{1/\lambda^{*}} \right)^{1/\lambda^{*}} < \infty, \quad \lambda^{*} = \frac{\lambda}{\lambda - 1}, \quad \tag{13}
\]

holds. Moreover, the estimate for the constant \(C > 0\) in (12) is

\[
D \leq C \left( 1 + \frac{\eta}{\lambda^{*}} \right)^{1/\eta} \left( 1 + \frac{\lambda^{*}}{\eta} \right)^{1/\lambda^{*}} D, \quad \tag{14}
\]

where the functions \(p\) and \(q\) which are nonnegative rd-continuous functions are called the weighted functions and condition (13) gives the characterization of these two functions which leads to the validation of inequality (12).

In [19], Özkan et al. showed that if \(0 \leq x < y \leq \infty\), \(p \in C_{rd}([x, y], \mathbb{R})\) is a nonnegative function such that \(\int_{x}^{y} p(s) \, ds \) exists and

\[
q(\tau) = (\tau - x) \int_{\tau}^{y} \frac{p(s)}{(s - x)(\sigma(s) - x)} \, ds, \quad \tau \in [x, y]. \tag{15}
\]

Furthermore, if \(\Psi : (c, d) \to \mathbb{R}, c, d \in \mathbb{R}\) is continuous and convex, then

\[
\int_{x}^{y} p(s) \Psi \left( \frac{1}{\sigma(s) - x} \int_{x}^{\sigma(s)} w(\tau) \, d\tau \right) \Delta s \leq \int_{x}^{y} q(s) \Psi(w(s)) \Delta s, \quad \tag{16}
\]

holds for \(w \in C_{rd}([x, y], \mathbb{R})\) s.t. \(w(s) \in (c, d)\). Also, they proved that if \(p \in C_{rd}([y, \infty), \mathbb{R})\) is a nonnegative function and
\[ q(r) = \frac{1}{r} \int_y^r p(s) \Delta s, \quad r \in [y, \infty), \] \tag{17} \]

then the inequality
\[ \int_x^y p(s) \Psi \left( \frac{1}{\sigma(s) - x} \int_x^{\sigma(s)} w(\tau) \Delta \tau \right) \frac{\Delta s}{s-x} \leq \int_x^y q(s) \Psi (w(s)) \frac{\Delta s}{s-x}, \] \tag{18} \]
holds for \( w \in C_{rd}([y, \infty), \mathbb{R}) \) s.t. \( w(s) \in (c,d) \forall s \in [y, \infty) \).

In [19], the authors proved inequality (2) for several functions on time scales and showed that if \( x \in \mathbb{T} \) and \( \omega_h, h = 1,2,\ldots,n, \) are nonnegative delta integrable functions, then
\[ \int_x^{\infty} \left( \frac{w_h^p(\tau) \tau \tau_{\mathcal{H}}^p}{(\sigma(\tau) - x)^n} \right)^{\frac{1}{p}} \Delta \tau \leq \int_x^y q(s) \Psi (w(s)) \frac{\Delta s}{s-x}, \] \tag{19} \]
where
\[ w_h(\tau) := \int_x^\tau w_h(s) \Delta s. \] \tag{20} \]

Furthermore, if \( \Psi : (c,d) \to \mathbb{R} \) is continuous and convex, then
\[ \int_x^y p(s) \Psi \left( \frac{1}{n} \left( \int_{h=1}^n w_h^p(s) \right)^{\frac{1}{n}} \right) \frac{\Delta s}{s-x} \leq \int_x^y q(s) \Psi \left( \frac{1}{n} \left( \sum_{h=1}^n w_h(s) \right)^{\frac{1}{n}} \right) \frac{\Delta s}{s-x}. \] \tag{22} \]

In [20], Özkam and Yildirim generalized (16) with kernels and proved that if \( \Psi : (c,d) \to \mathbb{R} \) s.t. \( c,d \in \mathbb{R} \) is continuous and convex, \( h(s,\zeta) \in C_{rd}([x,y] \times [x,y], \mathbb{R}) \) and \( p \in C_{rd}([x,y], \mathbb{R}) \) are nonnegative functions and
\[ q(\zeta) = (\zeta - x) \int_x^\zeta h(s,\zeta) \frac{\Delta s}{s-x}, \quad \zeta \in [x,y]. \] \tag{23} \]

Then, the inequality
\[ \int_x^y p(s) \Psi (A_h w(\sigma(s), s)) \frac{\Delta s}{s-x} \leq \int_x^y q(s) \Psi (w(s)) \frac{\Delta s}{s-x}, \] \tag{24} \]
satisfies for all delta integrable functions \( w \in C_{rd}([x,y], \mathbb{R}) \) s.t. \( w(s) \in (c,d) \), where
\[ A_h w(\sigma(s), s) := \frac{1}{H(\tau,s)} \int_x^\tau h(s,\zeta) w(\zeta) \Delta \zeta, \] \tag{25} \]
holds for \( w \in C_{rd}([y, \infty), \mathbb{R}) \) s.t. \( w(s) \in (c,d) \forall s \in [y, \infty) \).

The aim of this manuscript is to establish some new characterizations for dynamic inequalities of Hardy–Knopp type with kernels in different spaces to prove the following inequality:
\[ \left( \int_{s_0}^x \left( \frac{\sigma(\zeta) - s}{\sigma(\zeta) - x_0} \right)^{\frac{1}{q}} \right)^{q/(1-q)} \leq C \left( \int_{s_0}^x \frac{\Phi(\omega(s))}{\sigma(\zeta) - x_0} \Delta \zeta \right)^{1/q}, \] \tag{26} \]
where \( 1 < \eta \leq q < \infty \).

The paper is coordinated as follows: in Section 2, we show some basics on \( \mathbb{T} \) calculus and some theorems needed in Section 3, where we prove the main outcomes. Our key conclusions (when \( \mathbb{T} = \mathbb{R} \)) give the characterizations of inequality (9) proved by Kajser, Nikolova, Persson, and Wedestig. Also, we give some new characterizations of weights for new general theorems.

## 2. Preliminaries

A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \). The forward jump operator defined by \( \sigma(\tau) = \inf \{ \eta \in \mathbb{T} : \eta > \tau \} \). For any function \( \Phi : \mathbb{T} \to \mathbb{R} \), the notation \( \Phi^\sigma(\tau) \) denotes \( \Phi(\sigma(\tau)) \). The set of all such rd-continuous functions is denoted by \( C_{rd}([\mathbb{T}, \mathbb{R}] \cup \mathbb{R} \cup \mathbb{R} \cup \mathbb{R} \cup \mathbb{R} \) for \( 0 < q \leq 1 \).

The derivative of \( \Phi \omega \) and \( \Phi \omega^\lambda \) (where \( \omega \omega^\lambda \neq 0 \) of two differentiable functions \( \Phi \) and \( \omega \) is
\[ (\Phi \omega)^\lambda = \Phi^\lambda \omega + \Phi \omega^\lambda = \Phi_{\omega^\lambda} + \Phi_{\omega^\lambda \omega}, \quad \left( \frac{\omega^\lambda}{\omega} \right) = \Phi^\lambda \omega - \Phi \omega^\lambda. \] \tag{27} \]

The integration by parts formula on \( \mathbb{T} \) is
\[ \int_{t_0}^t \lambda(\tau) \psi(\tau) \Delta \tau = [\lambda(\tau) \psi(\tau)]_{t_0}^t - \int_{t_0}^t \lambda^\Delta(\tau) \psi^\sigma(\tau) \Delta \tau. \] \tag{28} \]

The time scales chain rule is
\[ (\omega^\lambda)^\lambda = \omega^{\lambda^2} (\Phi(\omega)) \right) \quad \text{where} \quad \omega \in [\tau, \sigma(\tau)], \] \tag{29} \]
such that \( \omega : \mathbb{R} \to \mathbb{R} \) is continuously differentiable and \( \varphi : \mathbb{T} \to \mathbb{R} \) is delta differentiable. The Hölder inequality on \( \mathbb{T} \) is
\[ \int_{t_0}^t |\lambda(\tau) \varphi(\tau)| \Delta \tau \leq \left( \int_{t_0}^t |\lambda(\tau)|^{\mu} \Delta \tau \right)^{\frac{1}{\mu}} \left( \int_{t_0}^t |\varphi(\tau)|^{\nu} \Delta \tau \right)^{\frac{1}{\nu}}, \] \tag{30} \]
where \( \zeta \in \mathbb{T} \), \( \lambda, \varphi \in C_{rd}([1, \mathbb{R}], \mathbb{R}), \gamma > 1, \) and \( 1/\gamma + 1/\nu = 1. \)
Theorem 1. (Jensen’s inequality). Assume that $\xi_0, \xi \in \mathbb{T}$ and $r_0, r \in \mathbb{R}$. If $\lambda \in C_{r, d}( [\xi_0, \xi]^T, \mathbb{R})$, $\varphi \in C_{r, d}( [\xi_0, \xi]^T, (r_0, r))$ and $\Psi: (r_0, r) \rightarrow \mathbb{R}$ is continuous and convex, then

$$
\Psi\left( \frac{1}{\int_{r_0}^{\xi} \lambda(\tau) \Delta r} \int_{r_0}^{\xi} \lambda(\tau) \varphi(\tau) \Delta r \right) \\
\leq \frac{1}{\int_{r_0}^{\xi} \lambda(\tau) \Delta r} \int_{r_0}^{\xi} \lambda(\tau) \Psi(\varphi(\tau)) \Delta r.
$$

(31)

Direction of (31) will be reversed if $\Psi$ is a concave function. Assume $(\mathbb{X}, \mathbb{R}, \mu_\lambda)$ and $(\mathbb{X}, \mathbb{R}, \lambda_\lambda)$ are finite dimensional time scale measure spaces. We define $(\mathbb{X} \times \mathbb{R}, \mathbb{R} \times \mathbb{R}, \mu_\lambda \times \lambda_\lambda)$ such that $\mathbb{R} \times \mathbb{R}$ is the product $\sigma$-algebra by $\{G \times H: G \in \mathbb{R}, H \in \mathbb{R}\}$ and $(\mu_\lambda \times \lambda_\lambda)(G \times H) = \mu_\lambda(G)\lambda_\lambda(H)$.

Theorem 2. (Minkowski’s inequality [11]). Assume that $p, q, r$ and $w$ are nonnegative functions on $\mathbb{X}$, $\Pi$ and $\mathbb{X} \times \Pi$, respectively. If $a \geq 1$, then

$$
\left( \int_{\mathbb{X}} p(s) \left( \int_{\Pi} w(s, \xi) q(\xi) \Delta \xi \right)^a \right)^{1/a} \\
\leq \int_{\Pi} q(\xi) \left( \int_{\mathbb{X}} w^a(s, \xi) p(s) \Delta s \right)^{1/a} \Delta \xi.
$$

(32)

3. Main Results

In what follows, we define the general Hardy operator $A_h$ as follows:

$$
A_h w(r, s) = \frac{1}{H(r, s)} \int_{x_s}^r h(s, \xi) w(\xi) \Delta \xi, H(r, s)
$$

(33)

where $r, s > x_0$ and $w \in C_{r, d}( [x_0, x]^T, \mathbb{R})$ is a delta integrable and $h(s, \xi) \in C_{r, d}( [x_0, x]^T \times [x_0, x]^T, \mathbb{R})$ is a delta integrable and nonnegative function. Throughout this section, we will assume that the functions are nonnegative rd-continuous and the right-hand sides of the inequalities converge if the left sides converge. Now, we are ready to state and prove our main results.

Theorem 3. Assume that $x_0, x \in \mathbb{T}$, $1 < \eta < q < \infty$, and $r \in (1, \eta)$. Let $\phi$ be a nonnegative and convex function on $(c, d)$, $-\infty < c < d < \infty$, and $u$ and $v$ are nonnegative weighted functions. Then,

$$
\left( \int_{x_0}^x \phi\left( A_h w(\sigma(s), s) \right) \Delta s \right)^{1/q} \leq C \left( \int_{x_0}^x \phi^\eta\left( w(\xi) \right) \Delta s \right)^{1/\eta},
$$

(34)

holds for $w \geq 0$ and $C > 0$, if

$$
A_h = \sup_{\xi \in [x_0, x]} \left[ V^{\sigma-1/\eta}(\xi) \left( \int_{\xi}^x \left( \frac{h(\xi, \xi)}{H(\sigma(s), s)} \right)^q \right)^{1/\eta} \left( \frac{u(s)}{\sigma(s) - x_0} \right)^{q/\eta} < \infty,
$$

(35)

where

$$
V(\xi) = \int_{x_0}^\xi [v(\tau)]^{-1/\eta} (\sigma(\tau) - x_0)^{1/\eta-1} \Delta \tau.
$$

(36)

Proof. By applying (31), we have

$$
\int_{x_0}^x \left[ \phi\left( A_h w(\sigma(s), s) \right) \frac{u(s)}{\sigma(s) - x_0} \right] \frac{\Delta s}{\sigma(s) - x_0}
$$

$$
= \int_{x_0}^x \left[ \phi\left( \frac{1}{H(\sigma(s), s)} \int_{x_0}^{\sigma(s)} h(\xi, \xi) w(\xi) \Delta \xi \right) \right] \frac{u(s)}{\sigma(s) - x_0} \Delta s
$$

(37)

Define a function $g$ such that

$$
\phi^\eta\left( w(\xi) \right) \frac{v(\xi)}{\sigma(\xi) - x_0} = \phi\left( g(\xi) \right),
$$

(38)

and then we have
\[
\int_{x_0}^{\sigma(s)} h(s, \zeta) \phi (w(\zeta)) \Delta \zeta
\]
\[
= \int_{x_0}^{\sigma(s)} h(s, \zeta) \phi^{1/\eta} (g(\zeta)) [V^\sigma (\zeta)]^{r - 1/\eta} [V^\sigma (\zeta)]^{1 - r/\eta} [v(\zeta)]^{-1/\eta} (\sigma(\zeta) - x_0)^{1/\eta} \Delta \zeta. \tag{39}
\]

Applying (30) with \( \eta > 1 \) and \( \eta/(\eta - 1) \) on the term,
\[
\int_{x_0}^{\sigma(s)} h(s, \zeta) \phi^{1/\eta} (g(\zeta)) [V^\sigma (\zeta)]^{r - 1/\eta} [V^\sigma (\zeta)]^{1 - r/\eta} [v(\zeta)]^{-1/\eta} (\sigma(\zeta) - x_0)^{1/\eta} \Delta \zeta,
\]
we get
\[
\leq \left( \int_{x_0}^{\sigma(s)} h^\eta (s, \zeta) \phi (g(\zeta)) [V^\sigma (\zeta)]^{r - 1} \Delta \zeta \right)^{1/\eta}
\times \left( \int_{x_0}^{\sigma(s)} [V^\sigma (\zeta)]^{1 - r/\eta - 1} [v(\zeta)]^{-1/\eta - 1} (\sigma(\zeta) - x_0)^{1/\eta - 1} \Delta \zeta \right)^{\eta - 1/\eta}. \tag{40}
\]

Substituting (41) into (39), we obtain
\[
\int_{x_0}^{\sigma(s)} h(s, \zeta) \phi (w(\zeta)) \Delta \zeta
\]
\[
\leq \left( \int_{x_0}^{\sigma(s)} h^\eta (s, \zeta) \phi (g(\zeta)) [V^\sigma (\zeta)]^{r - 1} \Delta \zeta \right)^{1/\eta}
\times \left( \int_{x_0}^{\sigma(s)} [V^\sigma (\zeta)]^{1 - r/\eta - 1} [v(\zeta)]^{-1/\eta - 1} (\sigma(\zeta) - x_0)^{1/\eta - 1} \Delta \zeta \right)^{\eta - 1/\eta}. \tag{42}
\]

And then we have from (37) that
\[
\int_{x_0}^{x} [\phi (A_h w(\sigma(s), s))]^q \mu(s) \frac{\Delta s}{\sigma(s) - x_0}
\]
\[
\leq \int_{x_0}^{x} \left( \int_{x_0}^{\sigma(s)} h^\eta (s, \zeta) \phi (g(\zeta)) [V^\sigma (\zeta)]^{r - 1} \Delta \zeta \right)^{q/\eta}
\times \left( \int_{x_0}^{\sigma(s)} [V^\sigma (\zeta)]^{1 - r/\eta - 1} [v(\zeta)]^{-1/\eta - 1} (\sigma(\zeta) - x_0)^{1/\eta - 1} \Delta \zeta \right)^{q(q - 1)/\eta}
\frac{\mu(s)}{H^q (\sigma(s), s) (\sigma(s) - x_0)^\Delta s}. \tag{43}
\]
Since
\[
V(\zeta) = \int_{x_0}^{\zeta} [v(\tau)]^{-1/\eta - 1} (\sigma(\tau) - x_0)^{1/\eta - 1} \Delta \tau,
\]
then
\[
V^\Delta (\zeta) = [v(\zeta)]^{-1/\eta - 1} (\sigma(\zeta) - x_0)^{1/\eta - 1} > 0. \tag{44}
\]
Therefore, the function $V$ is increasing. By applying (29) on the term $V^{1−(r−1)/\eta−1}(\zeta)$, we obtain

\[
[V^{1−(r−1)/\eta−1}(\zeta)]^\Delta = [V^{\eta−r/\eta−1}(\zeta)]^\Delta \\
= \left(\frac{\eta - r}{\eta - 1}\right) V^{−(r−1)/\eta−1}(\lambda)V^\Delta(\zeta),
\]

where $\lambda \in [\zeta, \sigma(\zeta)]$. Thus, by substituting (45) into (46), we get

\[
[V^{\eta−r/\eta−1}(\zeta)]^\Delta = \left(\frac{\eta - r}{\eta - 1}\right) V^{−(r−1)/\eta−1}(\lambda)[v(\zeta)]^{-1/\eta−1}(\sigma(\zeta) - \sigma(0))^{1/\eta−1}. 
\]

Since $\lambda \leq \sigma(\zeta)$ and $V$ is increasing, we have

\[
V(\lambda) \leq V^\sigma(\zeta). 
\]

Using the relation $1 < r < \eta$, we see

\[
V^{−(r−1)/\eta−1}(\lambda) \geq [V^\sigma(\zeta)]^{−(r−1)/\eta−1}. 
\]

Substituting (49) into (47), we get (note $\eta - r > 0$ and $\eta - 1 > 0$ that

\[
\int_{x_0}^{\sigma(\zeta)} [V^{\eta−r/\eta−1}(\zeta)]^\Delta \zeta \geq \left(\frac{\eta - r}{\eta - 1}\right) \int_{x_0}^{\sigma(\zeta)} [V^{\sigma}(\zeta)]^{−(r−1)/\eta−1} [v(\zeta)]^{-1/\eta−1}(\sigma(\zeta) - \sigma(0))^{1/\eta−1} \Delta \zeta. 
\]

Thus,

\[
\int_{x_0}^{\sigma(\zeta)} [V^\sigma(\zeta)]^{−(r−1)/\eta−1} [v(\zeta)]^{-1/\eta−1}(\sigma(\zeta) - \sigma(0))^{1/\eta−1} \Delta \zeta \\
\leq \left(\frac{\eta - 1}{\eta - r}\right) \int_{x_0}^{\sigma(\zeta)} [V^{\eta−r/\eta−1}(\zeta)]^\Delta \zeta \\
= \left(\frac{\eta - 1}{\eta - r}\right) [V^\sigma(s)]^{\eta−r/\eta−1},
\]

and then we have from (52) that

\[
\int_{x_0}^{\sigma(\zeta)} [V^\sigma(\zeta)]^{1−r/\eta−1} [v(\zeta)]^{-1/\eta−1}(\sigma(\zeta) - \sigma(0))^{1/\eta−1} \Delta \zeta \\
\leq \left(\frac{\eta - 1}{\eta - r}\right) [V^\sigma(s)]^{\eta−r/\eta−1}. 
\]

Substituting (53) into (43), we get

\[
\int_{x_0}^{\sigma(\zeta)} [\phi(A_\delta w(\sigma(s), s))]^q \mu(s) \frac{\Delta s}{\sigma(s) - x_0} \\
\leq \left(\frac{\eta - 1}{\eta - r}\right)^{q(\eta−1)/\eta} \int_{x_0}^{\sigma(\zeta)} \int_{x_0}^{\sigma(\zeta)} h^q(s, \zeta)\phi(g(\zeta)) [V^\sigma(\zeta)]^{r−1} \Delta \zeta \\
\times \left(\frac{[V^\sigma(s)]^{\eta−r/\eta−1}}{H(\sigma(s), s)} \frac{\mu(s)}{\sigma(s) - x_0} \Delta s \right). 
\]

From (30) and the definition of $g$ in (38), we obtain
Applying (32) on the term

\[
\int_{x_0}^{x} \left( \int_{x_0}^{s} h^\eta (s, \xi) \phi^\eta (\omega (\xi)) [V^\eta (\zeta)]^{-1} \frac{\nu (\xi)}{\sigma (s) - x_0} \Delta \xi \right)^{\eta \eta} \times \left( \frac{[V^\eta (s)]^{\eta - r/q}}{H (\sigma (s), s)} \right)^{\eta} \frac{u (s)}{\sigma (s) - x_0} \Delta s,
\]

with \( q/\eta > 1 \), we observe

\[
\int_{x_0}^{x} \left( \int_{x_0}^{s} h^\eta (s, \xi) \phi^\eta (\omega (\xi)) [V^\eta (\zeta)]^{-1} \frac{\nu (\xi)}{\sigma (s) - x_0} \Delta \xi \right)^{\eta \eta} \times \left( \frac{[V^\eta (s)]^{\eta - r/q}}{H (\sigma (s), s)} \right)^{\eta} \frac{u (s)}{\sigma (s) - x_0} \Delta s
\]

Substituting (57) into (55), we get

\[
\int_{x_0}^{x} \left[ \int_{x_0}^{s} \phi^\eta (\omega (\xi)) [V^\eta (\zeta)]^{-1} \frac{\nu (\xi)}{\sigma (s) - x_0} \Delta \xi \right]^{\eta \eta} \times \left( \frac{[V^\eta (s)]^{\eta - r/q}}{H (\sigma (s), s)} \right)^{\eta} \frac{u (s)}{\sigma (s) - x_0} \Delta s \Delta \zeta.
\]

which is (34) with \( C = (\eta - 1/\eta - r)^{\eta - 1/q} A (r) \).

\[ \square \]

Remark 1. If \( T = \mathbb{R} \), \( \sigma (s) = s \), and \( x_0 = 0 \), then (34) reduces to (9), established in [5].

Corollary 1. If \( T = \mathbb{N} \), \( \sigma (s) = s + 1 \), and \( x_0 = 1 \), then (34) reduces to
\[
\left( \sum_{n=1}^{N} \left[ \phi \left( \frac{1}{\sum_{m=1}^{n} h(n,m) \sum_{m=1}^{n} h(n,m) w(m)} \right) \right]^{\eta} \frac{u(n)}{n} \right)^{1/q} \leq C \left[ \sum_{n=1}^{N} \phi^{\gamma}(w(m)) \frac{v(m)}{m} \right]^{1/q}, N \in \mathbb{N},
\]

with \( C = (\eta - 1/\eta - r)^{q-1/q} A(r) \), such that

\[
A(r) = \sup_{n \in \{1, N\}} [V(n)]^{-1/q} \left( \sum_{m=n}^{N} \frac{h(m,n)}{H(m,m)} \right)^{q} \left( \frac{V(m)}{V(n)} \right)^{q(q-1)/q} u(m) \frac{1}{m} < \infty,
\]

where

\[
V(m) = \sum_{\tau=1}^{m} [V(\tau)]^{-1/q} \tau^{1/q-1}, H(m,n) = \sum_{\tau=1}^{m} h(n,\tau).
\]

**Theorem 4.** Let \( x_{n}, x \in \mathbb{T}, 1 < \eta \leq q < \infty \), and \( r \in (1, \eta) \). Furthermore, assume that \( \phi, \psi \) are nonnegative functions on \((c,d), -\infty < c < d < \infty\), and \( \psi \) is a convex function such that

\[
A \psi \leq \phi \leq D \psi,
\]

where \( A \) and \( D \) are constants and \( u \) and \( v \) are nonnegative weighted functions. Then, the inequality

\[
\left( \int_{x_{0}}^{x} \left[ \phi(A_{h}w(\sigma(s),s)) \right]^{q} u(s) \frac{\Delta s}{\sigma(s) - x_{0}} \right)^{1/q}
\]

\[
\leq C \left( \int_{x_{0}}^{x} \phi^{\gamma}(w(\zeta)) \frac{v(\zeta)}{\sigma(\zeta) - x_{0}} \Delta \zeta \right)^{1/q},
\]

holds for the nonnegative function \( w \) and \( C > 0 \), if

\[
A(r) = \sup_{\zeta \in [x_{0}, x], \tau} [V^{\alpha}(\zeta)]^{-1/q} \left( \int_{\tau}^{x} u(s) [V^{\alpha}(\zeta)]^{q(q-1)/q} \frac{h(s,\zeta)}{H(\sigma(s),\zeta)} \frac{\Delta s}{\sigma(s) - x_{0}} \right)^{1/q} < \infty.
\]

**Proof.** From (33) and by applying (31), we have

\[
\int_{x_{0}}^{x} \left[ \phi(A_{h}w(\sigma(s),s)) \right]^{q} u(s) \frac{\Delta s}{\sigma(s) - x_{0}}
\]

\[
\leq D \int_{x_{0}}^{x} \left[ \psi(A_{h}w(\sigma(s),s)) \right]^{q} u(s) \frac{\Delta s}{\sigma(s) - x_{0}}
\]

\[
= D \int_{x_{0}}^{x} \left[ \psi \left( \frac{1}{H(\sigma(\tau),s)} \int_{x_{0}}^{\tau} h(s,\zeta) \phi(\zeta) \Delta \zeta \right) \right]^{q} u(s) \frac{\Delta s}{\sigma(s) - x_{0}}
\]

\[
\leq D \int_{x_{0}}^{x} \left[ \left( \frac{1}{H(\sigma(\tau),s)} \int_{x_{0}}^{\tau} h(s,\zeta) \psi(\zeta) \Delta \zeta \right) \right]^{q} u(s) \frac{\Delta s}{\sigma(s) - x_{0}}
\]

\[
= D \int_{x_{0}}^{x} \frac{u(s)}{H^{\alpha}(\sigma(s),s)} \frac{\Delta s}{\sigma(s) - x_{0}}
\]
where

\[ J(s) = \int_{x_0}^{\sigma(s)} h(s, \zeta) \psi(w(\zeta)) d\zeta. \]  \hfill (67)

Define a function \( g \) such that

\[ \psi^\theta(w(\zeta)) \frac{v(\zeta)}{\sigma(\zeta) - x_0} = \psi(g(\zeta)). \]  \hfill (68)

Substituting (68) into (67), we obtain

\[ J(s) = \int_{x_0}^{\sigma(s)} h(s, \zeta) \psi^{1/\eta}(g(\zeta)) \left[ v(\zeta) \right]^{-1/\eta} (\sigma(\zeta) - x_0)^{1/\eta} d\zeta. \]  \hfill (69)

Note that

\[ J(s) = \int_{x_0}^{\sigma(s)} h(s, \zeta) \psi^{1/\eta}(g(\zeta)) \left[ V^\sigma(\zeta) \right]^{-1/\eta} \]  \hfill (70)

\[ \cdot \left[ V^\sigma(\zeta) \right]^{1-1/\eta} (\sigma(\zeta) - x_0)^{1/\eta} d\zeta. \]

Applying (30) with \( \eta > 1 \) and \( \eta/(\eta-1) \) on the term

\[ \int_{x_0}^{\sigma(s)} h(s, \zeta) \psi^{1/\eta}(g(\zeta)) \left[ V^\sigma(\zeta) \right]^{-1/\eta} \]  \hfill (71)

\[ \cdot \left[ V^\sigma(\zeta) \right]^{1-1/\eta} (\sigma(\zeta) - x_0)^{1/\eta} d\zeta, \]

we get

\[ \int_{x_0}^{\sigma(s)} h(s, \zeta) \psi^{1/\eta}(g(\zeta)) \left[ v(\zeta) \right]^{-1/\eta} (\sigma(\zeta) - x_0)^{1/\eta} d\zeta. \]

Substituting (72) into (70), we observe

\[ J(s) \leq \left( \int_{x_0}^{\sigma(s)} h^n(s, \zeta) \psi(g(\zeta)) \left[ V^\sigma(\zeta) \right]^{-1} d\zeta \right)^{1/\eta} \]  \hfill (73)

\[ \times \left( \int_{x_0}^{\sigma(s)} \left[ V^\sigma(\zeta) \right]^{1-1/\eta-1} (\sigma(\zeta) - x_0)^{1/\eta-1} d\zeta \right)^{\eta-1/\eta}. \]

Substituting (73) into (66), we have

\[ \int_{x_0}^{x} \left[ \phi(A_n w(\sigma(s), ts)) \right] q u(s) \frac{\Delta s}{\sigma(s) - x_0} \]  \hfill (74)

\[ \leq D^\zeta \int_{x_0}^{x} u(s) H^n(\sigma(s), s)(\sigma(s) - x_0) \left( \int_{x_0}^{\sigma(s)} h^n(s, \zeta) \psi(g(\zeta)) \left[ V^\sigma(\zeta) \right]^{-1} d\zeta \right)^{\eta/\eta} \]  \hfill (74)

\[ \times \left( \int_{x_0}^{\sigma(s)} \left[ V^\sigma(\zeta) \right]^{1-1/\eta-1} (\sigma(\zeta) - x_0)^{1/\eta-1} d\zeta \right)^{(\eta-1)/\eta} d\zeta. \]

Since

\[ V(\zeta) = \int_{x_0}^{\zeta} \left[ v(\tau) \right]^{-1/\eta-1} (\sigma(\tau) - x_0)^{1/\eta-1} d\tau, \]  \hfill (75)

then

\[ V^\Delta(\zeta) = \left[ v(\zeta) \right]^{-1/\eta-1} (\sigma(\zeta) - x_0)^{1/\eta-1} > 0. \]  \hfill (76)

Therefore, the function \( V \) is increasing. Applying (29) on the term \( V^{1-(r-1)/\eta-1}(\zeta) \), we obtain
\[ [\nu_1^{1-(r-1)/\eta - 1}(\zeta)]^\Delta = [\nu^\eta/r^{1/\eta - 1}(\zeta)]^\Delta \]
\[ = \left(\frac{\eta - r}{\eta - 1}\right) V^{-(r-1)/\eta - 1}(\zeta)^\Delta. \]  
(77)

where \( \lambda \in [\zeta, \sigma(\zeta)] \). Using the relation \( 1 < r < \eta \) and substituting (76) into (77), we see that

\[ V^{-(r-1)/\eta - 1}(\zeta) \leq V^{\sigma(\zeta)}^{-(r-1)/\eta - 1}. \]  
(80)

Since \( \lambda \leq \sigma(\zeta) \) and \( V \) is increasing, we have
\[ V(\lambda) \leq V^{\sigma(\zeta)}. \]  
(79)

Using the relation \( 1 < r < \eta \), we get

\[ [\nu_1^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta \geq \left(\frac{\eta - r}{\eta - 1}\right)^\Delta \nu^{\eta/r^{1/\eta - 1}}(\zeta)^\Delta \]
\[ \geq \left(\frac{\eta - r}{\eta - 1}\right)^\Delta \nu^{\sigma(\zeta)}^{\eta/r^{1/\eta - 1}}(\zeta)^\Delta \]  
(81)

and then

\[ \int_{x_0}^{\sigma(s)} [\nu_1^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta \geq \left(\frac{\eta - r}{\eta - 1}\right) \int_{x_0}^{\sigma(s)} [\nu^{\sigma(\zeta)}^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta \]
\[ \geq \left(\frac{\eta - r}{\eta - 1}\right) \int_{x_0}^{\sigma(s)} [\nu^{\sigma(\zeta)}^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta \]
\[ = \left(\frac{\eta - r}{\eta - 1}\right) [\nu^{\sigma(\zeta)}^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta. \]  
(82)

Thus,
\[ \int_{x_0}^{\sigma(s)} [\nu^{\sigma(\zeta)}]^{(r-1)/\eta - 1} [\nu(\zeta)]^{1/\eta - 1}(\sigma(\zeta) - x_0)^{1/\eta - 1} \Delta \zeta \]
\[ \leq \left(\frac{\eta - 1}{\eta - r}\right) \int_{x_0}^{\sigma(s)} [\nu^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta \]
\[ = \left(\frac{\eta - 1}{\eta - r}\right) [\nu^{\sigma(\zeta)}^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta. \]  
(83)

and then we have
\[ \int_{x_0}^{\sigma(s)} [\nu^{\sigma(\zeta)}]^{(r-1)/\eta - 1} [\nu(\zeta)]^{1/\eta - 1}(\sigma(\zeta) - x_0)^{1/\eta - 1} \Delta \zeta \]
\[ \leq \left(\frac{\eta - 1}{\eta - r}\right) [\nu^{\sigma(\zeta)}^{\eta/r^{1/\eta - 1}}(\zeta)]^\Delta d\zeta. \]  
(84)

Substituting (84) into (74), we get

\[ \int_{x_0}^{x} [\phi(A_{\theta}, \nu(\sigma(s), s))]^\eta u(s) \frac{\Delta s}{\sigma(s) - x_0} \]
\[ \leq D^\eta \left(\frac{\eta - 1}{\eta - r}\right)^{\eta/\eta - 1} \int_{x_0}^{\sigma(s)} \left(\int_{x_0}^{\sigma(s)} h^\eta(s, \zeta) \psi(\sigma(\zeta)) [\nu^{\sigma(\zeta)}]^{1/\eta - 1} \Delta \zeta \right)^{\eta/\eta - 1} \]
\[ \times [\nu^{\sigma(\zeta)}]^{\eta/\eta - 1} u(s) \Delta s \frac{\Delta \zeta}{H^\eta(s, \zeta)(\sigma(s) - x_0)^{1/\eta - 1}} \]  
(85)

Applying (32) on the term
\[
\int_{x_0}^{x} \left( \int_{x_0}^{\sigma(s)} h^n(s, \zeta) \psi(g(\zeta)) [V^\sigma(\zeta)]^{-1} \Delta \zeta \right)^{\eta/q} \left[ V^\sigma(s) \right]^{\eta(q-\eta)/\eta} \frac{u(s)\Delta s}{H^q(\sigma(s), s)(\sigma(s) - x_0)}.
\]

with \( q/\eta > 1 \), we observe

\[
\int_{x_0}^{x} \left( \int_{x_0}^{\sigma(s)} h^n(s, \zeta) \psi(g(\zeta)) [V^\sigma(\zeta)]^{-1} \Delta \zeta \right)^{\eta/q} \left[ V^\sigma(s) \right]^{\eta(q-\eta)/\eta} \frac{u(s)\Delta s}{H^q(\sigma(s), s)(\sigma(s) - x_0)}
\]

Substituting (87) into (85), we obtain

\[
\int_{x_0}^{x} [\phi(A_{\infty} w(\sigma(s), s))]^\eta u(s) \frac{\Delta s}{\sigma(s) - x_0}
\]

\[
\leq D^\eta \left( \frac{\eta - 1}{\eta - r} \right)^{\eta(q-1)/\eta} \left[ \int_{x_0}^{x} \psi(g(\zeta)) [V^\sigma(\zeta)]^{-1} \Delta \zeta \right]^{\eta/\eta}
\]

Using assumption (65) and the definition of \( g \) in (68), we have from (63) and (88) that

\[
\left( \int_{x_0}^{x} [\phi(A_{\infty} w(\sigma(s), s))]^\eta u(s) \frac{\Delta s}{\sigma(s) - x_0} \right)^{1/\eta}
\]

\[
\leq D \left( \frac{\eta - 1}{\eta - r} \right)^{\eta(q-1)/\eta} A^\eta(r) \left[ \int_{x_0}^{x} \psi^\eta(\zeta) \frac{v(\zeta)}{\sigma(\zeta) - x_0} \Delta \zeta \right]^{\eta/\eta}
\]

and then

\[
\left( \int_{x_0}^{x} [\phi(A_{\infty} w(\sigma(s), s))]^\eta u(s) \frac{\Delta s}{\sigma(s) - x_0} \right)^{1/\eta}
\]

which is (64) with the constant

\[ C = D/A(\eta - 1/\eta - r)^{\eta(q-1)/\eta} A(r). \]

Corollary 2. When \( T = \mathbb{R} \), then \( \sigma(s) = s \), and inequality (64) in Theorem 3 reduces to the following inequality:

\[
\left( \int_{x_0}^{x} [\phi(A_{\infty} w(\sigma(s), s))]^\eta u(s) \frac{\Delta s}{\sigma(s) - x_0} \right)^{1/\eta}
\]

\[
\leq D \left( \frac{\eta - 1}{\eta - r} \right)^{\eta(q-1)/\eta} A^\eta(r) \left[ \int_{x_0}^{x} \psi^\eta(\zeta) \frac{v(\zeta)}{\sigma(\zeta) - x_0} \Delta \zeta \right]^{\eta/\eta},
\]
\[
\left( \int_{x_0}^{x} \left[ \phi \left( A(s) \sigma(s) (s) \right) \right]^q u(s) \frac{ds}{\sigma(s) - x_0} \right)^{1/q} \leq C \left( \int_{x_0}^{x} \phi^q (w(\xi)) \frac{v(\xi) d\xi}{\phi(\xi) - x_0} \right)^{1/q},
\]
which holds for the nonnegative function \( w \) and \( C = D/A (\eta - 1/\eta - r)^{q-1/q} A(r) > 0 \), if
\[
A(r) = \sup_{t \in [x_0, x]} [V(t)]^{-1/q} \left( \int_{x_0}^{x} u(s) [V(s)]^{q(q-r)/q} \left( \frac{h(s, \xi)}{H(s, s)} \right)^q \frac{ds}{s - x_0} \right)^{1/q} < \infty,
\]
where \( 1 < \eta \leq q < \infty, r \in (1, \eta), u \) and \( v \) are nonnegative weighted functions, \( \phi \) and \( \psi \) are nonnegative functions on \((c, d), -\infty < c < d < \infty\), and \( \psi \) is a convex function such that \( A \psi \leq \phi \leq D \psi \), \( A, D \) are constants.

**Remark 2.** As a special case of Corollary 3, if \( A = D = 1 \) and \( x_0 = 0 \), then we have (9) established in [5].

**Corollary 3.** If \( T = \mathbb{N}, \sigma(n) = n + 1 \), and \( x_0 = 1 \), we obtain a new discrete inequality:
\[
A(r) = \sup_{n \in [1, N]} [V(n)]^{-1/q} \left( \sum_{n=1}^{N} \left( \frac{h(n, n)}{H(n, n)} \right)^q [V(n)]^{q(q-r)/q} \frac{u(n)}{m} \right)^{1/q} < \infty,
\]
where
\[
V(n) = \sum_{r=1}^{n} [v(r)]^{-1/q-1} r^{1/q-1}, H(n, n) = \sum_{r=1}^{n} h(n, r).
\]

**4. Conclusion and Future Work**

In this paper, we present some new inequalities of Hardy–Knopp type with kernel functions which have two nonnegative different weighted functions in two different spaces \( L^q \) and \( L^p \) for \( 1 < \eta \leq q < \infty \). Also, we established some inequalities using convex functions and applying general Hardy operator on time scale. In the future, we may generalize these results to be with multidiagonal time scale. Also, some new dynamic inequalities of Hardy–Knopp type with kernel functions in two different weighted functions in two different spaces \( L^q \) and \( L^p \) for \( \eta \leq q < \infty \) and \( 0 < \eta < 1 \). We hope general studies about a generalized Hardy operator on time scale and its application for some dynamic inequalities on time scales.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.

**Authors’ Contributions**

H. M. Rezk, G. AlNemer, and A I. Saied provided the software and wrote the original draft. H. M. Rezk and M. Zakarya reviewed and edited the manuscript. All authors have read and agreed to the published version of the manuscript.

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