THE RIESZ CAPACITY IN METRIC SPACES

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ABSTRACT. We study a capacity theory based on a definition of a Riesz potential in metric spaces with a doubling measure. In this general setting, we study the basic properties of the Riesz capacity, including monotonicity, countable subadditivity and several convergence results. We define a modified version of the Hausdorff measure and provide lower bound and upper bound estimates for the capacity in terms of the modified Hausdorff content. We also study isocapacitary inequalities and boundedness of the Riesz potential.

1. Introduction

In this paper, we study a theory of capacity based on a metric version of the Riesz potential in the setting of a general metric space \((X, d)\) equipped with a doubling measure \(\mu\). We define a related Hausdorff measure and study the connections between the Riesz capacity and the Hausdorff measure. With our definitions and results, we extend the classical Riesz capacity theory from the Euclidean space, with the Lebesgue measure, to the setting of a general metric measure space. In \(\mathbb{R}^n\), the capacity theory for the Riesz potential was introduced by Meyers in [30] and can also be found for example in [3], [4] and [31]. During the past twenty years, different capacities have been studied in metric measure spaces for example in [16], [18], [21], [22], [23], [24] and [25]. Also, a part of the theory for Riesz capacity follows from general results in [13] and [33]. Here, we formulate the theory explicitly and state the results to keep the paper self-contained.

We define a metric version of the Riesz potential of order \(\gamma\), where \(0 < \gamma < 1\), as

\[
I_\gamma f(x) = \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} \, d\mu(y).
\]

One can find a similar definition for the Riesz potential in the works of Kairema and Sjödin (see [20], [32] and [33]). In the definition, there appears only the measure of balls in the Riesz kernel. Another definition for a metric version of the Riesz potential is such that it also has the distance function as a part of the kernel. This version of the Riesz potential can be found for example in [17], [19] and [27]. Also, other

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Riesz potentials and fractional integral operators have been studied in the metric setting for example in [14], [15] and [28]. We emphasize that throughout the paper, we do not assume any type of Ahlfors $Q$-regularity on the measure $\mu$, which would give uniform lower bounds or upper bounds for the measure of balls in terms of the radii. In this generality, our definition of the Riesz potential, with no distance function as a part of the kernel, works better.

In Section 3, we define a metric version of the Riesz capacity $C_{\gamma,p}$ and show that it satisfies the basic properties of capacity. These properties include monotonicity, countable subadditivity and several convergence results. In particular, we show that the Riesz capacity is a Fatou capacity. This lower semicontinuity property of capacity is an analogue of Fatou’s lemma. We also study the capacitability of sets and show that the Riesz capacity is a so called Choquet capacity. This means that the capacity of a Borel set can be obtained by approximating with compact sets from the inside and open sets from the outside. We finish the section by briefly studying capacitary distributions and the dual Riesz capacity.

In Section 4, we prove an estimate that gives an upper bound for the capacity of balls in terms of the measure. This upper bound estimate leads us to define a modified version of the standard Hausdorff measure. For the Hausdorff measure, we provide two results that relate it to the Riesz capacity. One can find a related Hausdorff measure in a paper by Sjödin [32]. As a special case, our results give the main result there (see [32, Theorem 2.2]). Here, we give direct proofs to more general results that apply not only for compact sets. In particular, we do not need to use Frostman’s lemma to obtain the results.

Finally, in Section 5, we study isocapacitary inequalities and boundedness of the Riesz potential. In this section, we assume our measure $\mu$ to be a Radon measure. In [20], it is shown that our metric version of the Riesz potential is bounded as an operator from $L^p(X)$ to $L^q(X)$, if and only if a certain restriction is placed on the order $\gamma$ of the Riesz potential $I_\gamma$. This is an analogous result in the metric case for the well-known Hardy-Littlewood-Sobolev theorem in the Euclidean space. In Theorem 5.2, we give three equivalent conditions to the boundedness of $I_\gamma$ from $L^p(X)$ to $L^q(X)$. In particular, an isocapacitary inequality characterizes the boundedness of the Riesz potential. Similar result in the Euclidean space can be found in the paper by Adams (see [2]). However, in the metric setting our results are completely new. We also prove a lemma, which shows that the Riesz potential is bounded from $L^p(X)$ to a Lorentz capacitary space $L^{p,q}(C_{\gamma,p})$.

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2. Notation and preliminaries

2.1. **Riesz potential.** We assume that $X = (X, d, \mu)$ is a locally compact metric measure space equipped with a metric $d$ and a Borel regular, doubling outer measure $\mu$. The doubling property means that there is a fixed constant $c_d \geq 1$, called the doubling constant of $\mu$, such that

$$\mu(B(x, 2r)) \leq c_d \mu(B(x, r))$$

for every ball $B(x, r) = \{y \in X : d(y, x) < r\}$. We also assume that the measure of each open ball is positive and finite. The doubling condition implies that

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left(\frac{r}{R}\right)^Q$$

for every $0 < r \leq R$ and $y \in B(x, R)$ for some $C > 0$ and $Q > 0$ that only depend on $c_d$. In fact, we may take $Q = \log_2 c_d$ and $C = c_d^{-2}$ (see [6]). In addition, we assume that spheres are of measure zero, i.e.

$$\mu(\{y \in X : d(x, y) = r\}) = 0,$$

for $x \in X$ and $B(x, r)$. This assumption is needed for the Riesz potential, defined below, to satisfy lower semicontinuity properties (see Remark 3.3) that are required for the capacity theory.

**Definition 2.1.** Let $0 < \gamma < 1$. The Riesz potential of order $\gamma$ of a measurable function $f$ is

$$I_\gamma f(x) = \int_X \frac{f(y)}{\mu(B(x, d(x, y)))}^{1-\gamma} d\mu(y).$$

**Remark 2.2.** (i) To be precise, we would need to define the kernel separately for the cases $x \neq y$ and $x = y$. However, we assume our space $X$ to be such that $\mu$ vanishes on sets which consist of a single point. Then the domain of integration $X \setminus \{x\}$ can be replaced by $X$ (see [20]). Since we have a doubling metric measure space, this is equivalent to the condition that there are no isolated points in our space $X$.

(ii) In the Euclidean space, with the $n$-dimensional Lebesgue measure, we have, by the notation $\alpha = \gamma n \in (0, n)$, the usual Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

of order $\alpha$ on $\mathbb{R}^n$ (up to a dimensional constant).

Another way to define a Riesz potential in a metric space, as in [17] and [19], is

$$\tilde{I}_\gamma f(x) = \int_X \frac{f(y)d(x, y)^\gamma}{\mu(B(x, d(x, y)))} d\mu(y).$$
If the measure $\mu$ is (Ahlfors) $Q$-regular, that is, there exists a constant $C > 1$ such that
\begin{equation}
C^{-1}r^Q \leq \mu(B(x, r)) \leq Cr^Q
\end{equation}
for every $x \in X$ and $0 < r < \text{diam}(X)$, then $I_\gamma f$ and $\tilde{I}_{\gamma Q} f$ are comparable in the sense that there exists a constant $C \geq 1$ such that
\begin{equation}
C^{-1}I_\gamma f \leq \tilde{I}_{\gamma Q} f \leq CI_\gamma f.
\end{equation}

In the next sections, we do not assume the (Ahlfors) $Q$-regularity or any other estimates that would give uniform lower bounds or upper bounds for the measure of balls in terms of the radii. We assume only the doubling property (2.1) and develop the theory of Riesz capacity based on the definition (2.4) of the Riesz potential. In particular, this definition works better for our purposes in Section 4, where we define a modified version of the standard Hausdorff measure and prove two results that relate the Riesz capacity and the Hausdorff measure.

2.2. Function spaces and capacities. We have by Cavalieri’s principle that $L^p(X) = L^p(X, \mu)$ is the space of all $\mu$-measurable functions $f$ in $X$ such that
\begin{equation}
\|f\|_{L^p(X)} = \left( \int_0^\infty pt^{p-1}\mu(\{z \in X : |f(z)| > t\}) dt \right)^{1/p} < \infty,
\end{equation}
which is a Banach space when $1 \leq p < \infty$. The weak $L^p$-space $L^{p,\infty}(X)$ is defined by the condition
\begin{equation}
\|f\|_{L^{p,\infty}(X)} := \sup_{t > 0} t\mu(\{z \in X : |f(z)| > t\})^{1/p} < \infty.
\end{equation}
We denote by $L^p_+(X)$ the subset of $L^p(X)$ of non-negative functions.

Definition 2.3. We define, on the family of $\mu$-measurable subsets of $X$, a capacity to be a non-negative set function $C$, which has the following properties:

(a) $C(\emptyset) = 0$,
(b) If $A \subset B$, then $C(A) \leq C(B)$,
(c) $C(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty C(A_i)$.

A capacity $C$ is called a Fatou capacity if $C(A_i) \to C(A)$, whenever $A_1 \subset A_2 \subset \cdots$ are subsets of $X$ and $A = \bigcup_{i=1}^\infty A_i$. We also say that a property holds $C$-q.e. on $X$ if it holds for all $x \in X$ except those in a set $E$ with $C(E) = 0$.

The capacitary Lorentz spaces $L^{p,q}(C)$, $p, q > 0$, are defined by the condition
\begin{equation}
\|f\|_{L^{p,q}(C)} := \left( \int_0^\infty t^{q-1}C(\{z \in X : |f(z)| > t\})^{q/p} dt \right)^{1/q} < \infty,
\end{equation}
when \( q < \infty \), and in the case of \( q = \infty \) by
\[
\|f\|_{L^p,\infty}(C) := \sup_{t > 0} t C(\{z \in X : |f(z)| > t\})^{1/p} < \infty.
\]
The space \( L^{p,\infty}(C) \) is called the weak capacitary \( L^p \)-space. For the general facts and properties of the capacitary Lorentz spaces, we refer to [7], [8] and [9]. Throughout the paper, we denote the characteristic function of a set \( E \subset X \) by \( \chi_E \). In general, \( C \) will denote a positive constant whose value is not necessarily the same at each occurrence.

3. Riesz Capacity

**Definition 3.1.** Let \( 1 < p < \infty \) and \( 0 < \gamma < 1 \). The Riesz \((\gamma,p)\)-capacity of a set \( E \subset X \) is the number
\[
C_{\gamma,p}(E) = \inf_{f \in \mathcal{A}(E)} \|f\|_{L^p(X)}^p,
\]
where
\[
\mathcal{A}(E) = \{ f \in L^p_+(X) : I_x f \geq 1 \text{ on } E \}.
\]

If \( \mathcal{A}(E) = \emptyset \), we set \( C_{\gamma,p}(E) = \infty \). Functions belonging to \( \mathcal{A}(E) \) are called admissible functions or test functions for \( E \). From now on, we always assume in this section that \( 1 < p < \infty \) and \( 0 < \gamma < 1 \).

In the Euclidean space, with the Lebesgue measure, one can find the basic properties of the Riesz capacity for example in [3], [4], [30] and [31]. In the metric case, assuming only the doubling property from the Borel regular measure \( \mu \), we begin by showing that the Riesz \((\gamma,p)\)-capacity is an outer measure. This means that the Riesz capacity satisfies the properties of Definition 2.3.

**Theorem 3.2.** The Riesz \((\gamma,p)\)-capacity is an outer measure.

*Proof.* Clearly \( C_{\gamma,p}(\emptyset) = 0 \), since 0 is an admissible function. The definition of the capacity also implies monotonicity, since if \( E_1 \subset E_2 \), then \( \mathcal{A}(E_2) \subset \mathcal{A}(E_1) \).

To prove the countable subadditivity, let \( \{A_i\}_{i=1}^\infty \) be a sequence of sets in \( X \) and let \( A = \bigcup_{i=1}^\infty A_i \). We may assume that \( \sum_{i=1}^\infty C_{\gamma,p}(A_i) < \infty \). Then, \( C_{\gamma,p}(A_i) < \infty \) for all \( i \in \mathbb{N} \). Let \( \epsilon > 0 \), and for each \( i \in \mathbb{N} \), let \( f_i \in \mathcal{A}(A_i) \) be such that
\[
\|f_i\|_{L^p(X)}^p < C_{\gamma,p}(A_i) + \epsilon 2^{-i}.
\]
We define \( f(x) := \sup_{i \in \mathbb{N}} f_i(x) \). We have that \( f(x)^p \leq \sum_{i=1}^\infty f_i(x)^p \), which implies
\[
\|f\|_{L^p(X)}^p \leq \sum_{i=1}^\infty \|f_i\|_{L^p(X)}^p \leq \sum_{i=1}^\infty (C_{\gamma,p}(A_i) + \epsilon 2^{-i}) = \sum_{i=1}^\infty C_{\gamma,p}(A_i) + \epsilon.
\]
Moreover, we have that \( I_x f(x) \geq I_x f_i(x) \), since \( f(x) \geq f_i(x) \) for all \( x \in X \) and \( i \in \mathbb{N} \). Let \( x \in A \). Then there exists \( j \in \mathbb{N} \) such that
\( x \in A_j \) and hence \( I_j f(x) \geq I_\gamma f_j(x) \geq 1 \). Thus \( f \) is an admissible function for \( A = \bigcup_{i=1}^{\infty} A_i \). Now
\[
C_{\gamma,p} \left( \bigcup_{i=1}^{\infty} A_i \right) \leq \|f\|_{L^p(X)}^p \leq \sum_{i=1}^{\infty} C_{\gamma,p}(A_i) + \epsilon,
\]
and the claim follows by letting \( \epsilon \to 0 \).

\[\square\]

Remark 3.3. The Riesz potential, as defined in (2.4), is lower semi-continuous. For our purposes, it is enough to prove the lower semi-continuity for functions \( f \in L^p_+(X) \). Let \( x_0 \in X \).

\[
I_\gamma f(x_0) = \int_X \frac{f(y)}{\mu(B(x_0, d(x_0, y)))^{1-\gamma}} d\mu(y).
\]

We need to show that
\[
I_\gamma f(x_0) \leq \liminf_{x \to x_0} I_\gamma f(x),
\]
when \( x \to x_0 \). Let \( x \in X \). Since for any \( y \in X \)
\[
B(x, d(x, y)) \subset B(x_0, d(x, y) + d(x_0, x)),
\]
we have by the monotonicity of \( \mu \) that
\[
\mu(B(x, d(x, y))) \leq \mu(B(x_0, d(x, y) + d(x_0, x))).
\]
The above inequality and equality (2.3) imply that
\[
\liminf_{x \to x_0} \mu(B(x, d(x, y))) \leq \lim_{x \to x_0} \mu(B(x_0, d(x, y) + d(x_0, x)))
\]
\[
= \mu(B(x_0, d(x_0, y))).
\]
Now, for any \( y \in X \),
\[
\frac{f(y)}{\mu(B(x_0, d(x_0, y)))^{1-\gamma}} \leq \liminf_{x \to x_0} \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}}
\]
and, by using Fatou's lemma, we get
\[
I_\gamma f(x_0) = \int_X \frac{f(y)}{\mu(B(x_0, d(x_0, y)))^{1-\gamma}} d\mu(y)
\]
\[
\leq \int_X \liminf_{x \to x_0} \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} d\mu(y)
\]
\[
\leq \liminf_{x \to x_0} \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} d\mu(y)
\]
\[
= \liminf_{x \to x_0} I_\gamma f(x).
\]

In addition, we get the following lower semicontinuity property of the Riesz potential as an operator
\[
I_\gamma f \leq \liminf_{i \to \infty} I_\gamma f_i,
\]
when \( f_i \to f \) weakly in \( L^p_+(X) \). The weak convergence implies that \( f_i \mu \to f \mu \) converge weakly as measures with the vague topology of [13].
Section 1.1. Also, because of equality \([2.3]\) in the previous section, we have that our Riesz kernel is continuous and hence lower semicontinuous. The result then follows from \([13\, \text{Lemma } 2.2.1.b)] and \([31\, \text{Theorem } 1.2. p.58}\).

Using the fact that the Riesz potential of a function \(f\) is lower semicontinuous, we show that the Riesz capacity is an outer capacity. This means that the capacity of a set \(E \subset X\) can be obtained by approximating with open sets from the outside.

**Theorem 3.4.** \(C_{\gamma,p}\) is an outer capacity, that is,

\[
C_{\gamma,p}(E) = \inf \{C_{\gamma,p}(O) : O \supset E, O \text{ open}\}.
\]

**Proof.** By the monotonicity,

\[
C_{\gamma,p}(E) \leq \inf \{C_{\gamma,p}(O) : O \supset E, O \text{ open}\}.
\]

To prove the inequality to the reverse direction, we may assume that \(C_{\gamma,p}(E) < \infty\). Let \(0 < \epsilon < 1\) and let \(f \in L^p_+(X)\) be a function such that \(I_\gamma f \geq 1\) on \(E\) and

\[
\|f\|_{L^p(X)}^p < C_{\gamma,p}(E) + \epsilon.
\]

We define

\[
f_\epsilon := \frac{1}{1 - \epsilon} f
\]

and

\[
G := \{x \in X : I_\gamma f_\epsilon(x) > 1\}.
\]

Since \(I_\gamma f_\epsilon\) is lower semicontinuous, \(G\) is an open set. We also have that \(f_\epsilon(x) > f(x)\) for all \(x \in X\), since \(0 < \epsilon < 1\). Now, if \(x \in E\), then \(I_\gamma f_\epsilon(x) \geq 1\) and hence \(I_\gamma f_\epsilon(x) > 1\). Thus we have that \(x \in G\) and \(E \subset G\). Moreover, \(f_\epsilon\) is admissible for \(C_{\gamma,p}(G)\) and

\[
C_{\gamma,p}(G) \leq \|f_\epsilon\|_{L^p(X)}^p = \left(\frac{1}{1 - \epsilon}\right)^p \|f\|_{L^p(X)}^p \leq C_{\gamma,p}(E)(1 - \epsilon)^{-p} + \epsilon(1 - \epsilon)^{-p}.
\]

Since we have that \(\inf \{C_{\gamma,p}(O) : O \supset E, O \text{ open}\} \leq C_{\gamma,p}(G)\), letting \(\epsilon \to 0\) yields the inequality to the other direction. \(\square\)

The next capacitary weak type lemma shows in particular that the Riesz potential \(I_\gamma f\) of a nonnegative \(L^p\)-function \(f\) belongs to the weak capacitary \(L^p\)-space.

**Lemma 3.5.** If \(f \in L^p_+(X)\), then the capacitary weak type estimate

\[
C_{\gamma,p}(\{x \in X : I_\gamma f(x) > a\}) \leq a^{-p}\|f\|_{L^p(X)}^p
\]

holds for each \(0 < a < \infty\). Moreover, \(I_\gamma\) is bounded from \(L^p_+(X)\) to \(L^{p,\infty}(C_{\gamma,p})\).
Proof. Let $f \in L^p_+(X)$ and $0 < a < \infty$. We define

$$f_a := \frac{f}{a}$$

and

$$F := \{ x \in X : I_\gamma f(x) > a \}.$$ 

Since $I_\gamma f_a = I_\gamma (\frac{f}{a}) \geq 1$ on $F$, $f_a$ is admissible for $F$ and

$$C_{\gamma,p}(F) \leq \|f_a\|_{L^p_+(X)}^p = a^{-p} \|f\|_{L^p_+(X)}^p.$$ 

Moreover, the capacitary weak type estimate implies that

$$\|I_\gamma f\|_{L^p,\infty(C_{\gamma,p})} = \sup_{t>0} t C_{\gamma,p}(\{ x \in X : I_\gamma f(x) > t \})^{1/p} \leq \|f\|_{L^p_+(X)},$$

and the second claim follows. □

We can use the capacitary weak type estimate to prove the next result, which in particular says that the Riesz potential of a function $f \in L^p_+(X)$ is finite $C_{\gamma,p}$-a.e. As a corollary we get that the Riesz potential, as an operator, is linear outside a set of capacity zero.

**Theorem 3.6.** Let $E \subset X$. Then $C_{\gamma,p}(E) = 0$ if and only if there exists a nonnegative function $f \in L^p(X)$ such that $I_\gamma f(x) = \infty$ for all $x \in E$.

**Proof.** If $C_{\gamma,p}(E) = 0$, then for any integer $j$, we can find an admissible function $f_j \in A(E)$ such that

$$\|f_j\|_{L^p_+(X)}^p = \int_X f_j(y)^p d\mu(y) < 2^{-j}.$$

Then the function $f := \sum_{j=1}^{\infty} f_j$ belongs to $L^p(X)$ and

$$I_\gamma f(x) = \int_X \frac{\sum_{j=1}^{\infty} f_j(y)}{\mu(B(x, d(x,y)))^{1-\gamma}} d\mu(y) = \sum_{j=1}^{\infty} I_\gamma f_j(x) = \infty$$

for all $x \in E$, since $I_\gamma f_j \geq 1$ on $E$ for each $j$.

Conversely, if there exists a nonnegative function $f \in L^p(X)$ such that $I_\gamma f = \infty$ on $E$, then by the capacitary weak type estimate

$$C_{\gamma,p}(E) \leq C_{\gamma,p}(\{ x \in X : I_\gamma f(x) > a \}) \leq a^{-p} \|f\|_{L^p_+(X)}^p$$

for every $a > 0$. By letting $a \to \infty$, we see that $C_{\gamma,p}(E) = 0$. □

**Corollary 3.7.** Let $f_1$, $f_2$, $f \in L^p_+(X)$. Then

$$I_\gamma(f_1 + f_2) = I_\gamma(f_1) + I_\gamma(f_2), \quad C_{\gamma,p}$-q.e.

and

$$I_\gamma(af) = aI_\gamma(f), \quad C_{\gamma,p}$-q.e.,$$

where $a$ is any finite constant.
Proof. If each term on the right side of the above equalities is finite at a point \( x \in X \), then the equalities hold at such point by the definition of the Riesz potential. By Theorem 3.6, the sets where the equalities can fail are of capacity zero.

□

Next, we are going to prove several convergence results. We start by defining the convergence of a sequence of functions in capacity.

Definition 3.8. We say that a sequence \( \{f_i\} \) converges in capacity to \( f \), denoted \( f_i \to f \) in \( \mathcal{C}_{\gamma,p} \), if for every \( \epsilon > 0 \)

\[
\lim_{i \to \infty} \mathcal{C}_{\gamma,p}(\{x \in X : |f(x) - f_i(x)| > \epsilon\}) = 0.
\]

We show that the \( L^p \)-convergence of functions implies that the corresponding sequence of the Riesz potentials converges in capacity. Also, for a subsequence, we have pointwise convergence except for a set of capacity zero.

Theorem 3.9. Let \( \{f_i\} \subset L^p_+(X) \) and \( f \in L^p_+(X) \). Each of following statements is a consequence of the previous one.

(i) \( f_i \to f \) in \( L^p(X) \)
(ii) \( I_\gamma f_i \to I_\gamma f \) in \( \mathcal{C}_{\gamma,p} \)
(iii) There exists a subsequence \( \{f_{i_j}\} \) of \( \{f_i\} \) such that \( I_\gamma f_{i_j} \to I_\gamma f \) uniformly \( \mathcal{C}_{\gamma,p} \)-a.e.

Proof. We show first that (i) implies (ii). By Theorem 3.6, the potentials \( I_\gamma f_i \) and \( I_\gamma f \) are finite \( \mathcal{C}_{\gamma,p} \)-a.e. Then, we have by Corollary 3.7 and by Lemma 3.5 that

\[
\mathcal{C}_{\gamma,p}(\{x \in X : |I_\gamma f_i(x) - I_\gamma f(x)| > \epsilon\}) \leq \epsilon^{-p}||f_i - f||_{L^p(X)}^p,
\]

which proves the claim.

Next, we assume that (ii) holds and show that it implies (iii). Let \( \epsilon > 0 \). There exists a subsequence \( \{f_{i_j}\} \) and sets \( A_j \), where \( \mathcal{C}_{\gamma,p}(A_j) \leq \epsilon 2^{-j} \), for which

\[
|I_\gamma(f_{i_j})(x) - I_\gamma f(x)| \leq \frac{1}{j} \quad \text{except on } A_j.
\]

Therefore \( I_\gamma(f_{i_j}) \to I_\gamma f \), as \( j \to \infty \), uniformly in \( X \setminus \bigcup_j A_j \), where \( \mathcal{C}_{\gamma,p}(\bigcup_j A_j) \leq \epsilon \). Then, a simple diagonalization argument proves the claim.

□

By using the above theorem, we can strengthen the lower semicontinuity property of \( I_\gamma \) from Remark 3.3, at least outside a set of capacity zero.

Theorem 3.10. Let \( \{f_i\} \subset L^p(X) \) and \( f \in L^p(X) \).

(i) If \( f_i \to f \) weakly in \( L^p(X) \), then

\[
\liminf_{i \to \infty} I_\gamma f_i \leq I_\gamma f \leq \limsup_{i \to \infty} I_\gamma f_i, \quad \mathcal{C}_{\gamma,p} \text{-a.e.}
\]
(ii) If $f_i \rightarrow f$ weakly in $L^p_+(X)$, then

$$I_\gamma f \leq \liminf_{i \rightarrow \infty} I_\gamma f_i \text{ everywhere}$$

and

$$I_\gamma f = \liminf_{i \rightarrow \infty} I_\gamma f_i \quad \mathcal{C}_{\gamma,p}\text{-q.e.}$$

**Proof.** We prove first the claim (i). By the Banach-Saks Theorem (see [5]), there exists a subsequence \{f'_i\} such that a sequence \{g_j\}, where

$$g_j = j^{-1} \sum_{i=1}^{j} f'_i,$$

converges to $f$ in $L^p(X)$. Then, by Theorem 3.9, there exists a subsequence \{g'_j\} such that

$$I_\gamma f = \lim_{j \rightarrow \infty} I_\gamma g'_j \quad \mathcal{C}_{\gamma,p}\text{-q.e.}$$

Then, the left inequality in (i) follows due to the fact that

$$I_\gamma f = \lim_{j \rightarrow \infty} I_\gamma g'_j \geq \liminf_{i \rightarrow \infty} I_\gamma f'_i \geq \liminf_{i \rightarrow \infty} I_\gamma f_i \quad \mathcal{C}_{\gamma,p}\text{-q.e.}$$

The right inequality in (i) follows by replacing $f_i$ and $f$ by $-f_i$ and $-f$ in the previous argument.

If $f_i \rightarrow f$ weakly in $L^p_+(X)$, then by the lower semicontinuity of $I_\gamma(\cdot)$ (see Remark 3.3), we have that

$$I_\gamma f \leq \liminf_{i \rightarrow \infty} I_\gamma f_i \quad \text{everywhere}$$

and it follows by (i) that

$$I_\gamma f = \liminf_{i \rightarrow \infty} I_\gamma f_i \quad \mathcal{C}_{\gamma,p}\text{-q.e.}$$

We prove two more convergence results for the Riesz capacity. As a corollary of Theorem 3.12, we get a lower semicontinuity property for the capacity that is an analogue of Fatou’s lemma.

**Theorem 3.11.** If $X \supset K_1 \supset K_2 \cdots$ are compact sets and $K = \bigcap_{i=1}^{\infty} K_j$, then

$$\lim_{i \rightarrow \infty} C_{\gamma,p}(K_i) = C_{\gamma,p}(K).$$

**Proof.** Clearly, by the monotonicity, $\lim_{j \rightarrow \infty} C_{\gamma,p}(K_j) \geq C_{\gamma,p}(K)$. On the other hand, let $O$ be an open set containing $K$. By the compactness of $K$, we have that $K_j \subset O$ for all sufficiently large $j$. Then $\lim_{j \rightarrow \infty} C_{\gamma,p}(K_j) \leq C_{\gamma,p}(O)$. Finally, since $C_{\gamma,p}$ is an outer capacity by Theorem [3.4]

$$\lim_{j \rightarrow \infty} C_{\gamma,p}(K_j) \leq \inf \{C_{\gamma,p}(O) : O \supset K, O \text{ open}\} = C_{\gamma,p}(K).$$

□
Theorem 3.12. If $A_1 \subset A_2 \subset \cdots$ are subsets of $X$ and $A = \bigcup_{i=1}^{\infty} A_i$, then

$$\lim_{i \to \infty} C_{\gamma,p}(A_i) = C_{\gamma,p}(A),$$

that is, the Riesz capacity $C_{\gamma,p}$ is a Fatou capacity.

Proof. We may assume that $\lim_{i \to \infty} C_{\gamma,p}(A_i) = l < \infty$. Let $f_i \geq 0$ be a test function for $C_{\gamma,p}(A_i)$ such that

$$\|f_i\|_{L^p(X)} \leq C_{\gamma,p}(A_i) + \frac{1}{i}. \quad (3.1)$$

Then, the sequence $\{f_i\}$ is bounded in $L^p(X)$ and there exists a subsequence $\{f_{i_j}\}$ that converges weakly to a function $f \geq 0$ in $L^p(X)$. By Theorem 3.10 (ii), we have that

$$I_{\gamma} f \geq 1 \text{ on } A_{i_j} \text{ } C_{\gamma,p}\text{-q.e.}$$

and hence

$$I_{\gamma} f \geq 1 \text{ on } A \text{ } C_{\gamma,p}\text{-q.e.}$$

Let $B$ be the subset of $A$, where the previous inequality holds. Then, by (3.1) and the weak convergence of the functions

$$C_{\gamma,p}(A) = C_{\gamma,p}(B) \leq \|f\|_{L^p(X)}^p \leq \lim_{j \to \infty} \|f_{i_j}\|_{L^p(X)}^p \leq l,$$

from which the result follows. Here, we also used the the lower semi-continuity of $\| \cdot \|_p^p$ (see [4, Lemma 3.1.2. p.109]). \qed

Corollary 3.13. If $(A_i)_{i=1}^{\infty}$ is a sequence of sets in $X$, then

$$C_{\gamma,p}(\liminf_{i \to \infty} A_i) \leq \liminf_{i \to \infty} C_{\gamma,p}(A_i).$$

Proof. Let $B := \liminf_{i \to \infty} A_i = \bigcup_{j} \bigcap_{k \geq j} A_k$ and $B_i := \bigcup_{j=1}^{i} \bigcap_{k \geq j} A_k$. Then $B_i \subset B_{i+1} \subset \cdots$ and $B = \bigcup_{i=1}^{\infty} B_i$, and by Theorem 3.12

$$C_{\gamma,p}(B) = \lim_{i \to \infty} C_{\gamma,p}(B_i) \leq \liminf_{i \to \infty} C_{\gamma,p}(A_i).$$

\qed

The next definition extends the outer capacity property of Theorem 3.4 to the case where the Riesz capacity of a set $E \subset X$ can also be obtained by approximating with compact sets from the inside. By Theorem 3.15, we have this inner capacity property for the Riesz capacity, when considering analytic sets (for the definition of analytic sets, we refer to e.g. [10], [29]).

Definition 3.14. A set $E \subset X$ is called $C_{\gamma,p}$-capacitable, if

$$C_{\gamma,p}(E) = \sup \{ C_{\gamma,p}(K) : K \subset E, \text{ } K \text{ compact} \} = \inf \{ C_{\gamma,p}(O) : O \supset E, \text{ } O \text{ open} \}.$$
Capacitability has been studied in a very general context by Choquet in [10]. Other references are [1], [3], [12], [29] and [30], and the references therein. Theorem 4 in [10] together with Theorems 3.11 and 3.12 give the next theorem, which says that all analytic sets are $C_{\gamma,p}$-capacitable. In particular, we have that all Borel sets are $C_{\gamma,p}$-capacitable, which means that the Riesz capacity is a so called Choquet capacity.

**Theorem 3.15.** All analytic sets, and hence all Borel sets, are $C_{\gamma,p}$-capacitable.

For the following variational problem

\begin{equation}
\min \left\{ \|f\|_{L^p(X)}^p : f \in L^p(X), I_\gamma f \geq 1 \text{ on } A C_{\gamma,p}-q.e. \right\},
\end{equation}

we call a solution $f$ a $C_{\gamma,p}$-capacitary distribution of $A$ and $I_\gamma f$ a $C_{\gamma,p}$-capacitary potential of $A$.

**Theorem 3.16.** If $C_{\gamma,p}(A) < \infty$, then $A$ has a unique $C_{\gamma,p}$-capacitary distribution $f$ for which $f \in L^p_+(X)$, $\|f\|_{L^p(X)}^{p} = C_{\gamma,p}(A)$ and

\[ \int_X f(x)^{p-1} g(x) d\mu(x) \geq 0 \]

for all $g \in L^p(X)$ such that $I_\gamma g \geq 0$ on $A C_{\gamma,p}$-q.e.

**Proof.** Using Clarkson’s inequality for $L^p$-norms, the theorem follows as in [30, Theorem 9] due to previous results in the paper. □

**Dual Riesz capacity.** Let $\nu$ be a positive measure on $X$, $1 < p < \infty$ and $A \in \mathcal{F}$, where $\mathcal{F}$ is the $\sigma$-algebra of sets which are $\nu$-measurable for all positive measures $\nu$ with finite total variation in $X$. The total variation of any such $\nu$ in $X$ is

\[ \|\nu\| = |\nu|(X) = \sup \left\{ \nu(A) : A \subset X, A \text{ is measurable} \right\}. \]

In the case of a measure of this type, we define

\[ I_\gamma \nu(x) = \int_X \frac{d\nu(y)}{\mu(B(x, d(x, y))))^{1-\gamma}}, \]

and following [30] we introduce a capacity, which uses measures as test elements

\[ c_{\gamma,p}(A) = \sup \left\{ \|\nu\| : \nu \text{ is a positive measure in } X \text{ that is supported on } A, \|\nu\| < \infty \text{ and } \|I_\gamma \nu\|_{L^p(X)} \leq 1 \right\}, \]

where

\[ \|I_\gamma \nu\|_{L^p(X)} = \left( \int_X \left( \int_X \frac{d\nu(y)}{\mu(B(x, d(x, y))))^{1-\gamma}} \right)^{p'} d\mu(x) \right)^{1/p'} \]
and \( \frac{1}{p} + \frac{1}{p'} = 1 \).

With the same techniques as in [30, Theorem 12, Theorem 14] we see that \( c_{\gamma,p} \) is an inner capacity on \( \mathcal{F} \) that satisfies

\[
(3.3) \quad c_{\gamma,p}(A) = C_{\gamma,p}(A)^{1/p}
\]

and that all analytic sets are \( c_{\gamma,p} \)-capacitable on \( \mathcal{F} \).

Indeed, for the inequality \( c_{\gamma,p}(A) \leq C_{\gamma,p}(A)^{1/p} \), let \( f \in L^p_+(X) \) be the unique \( C_{\gamma,p} \)-capacitary distribution on \( A \). Then by Hölder’s inequality

\[
\nu(A) \leq \int_A I_\gamma f(x) d\nu(x) = \int_X \int_A \frac{1}{\mu(B(x, d(x, y)))^{1-\gamma}} d\nu(x) f(y) d\mu(y)
\]

\[
\leq \left( \int_X \left( \int_A \frac{1}{\mu(B(x, d(x, y)))^{1-\gamma}} d\nu(x) \right)^{p'} d\mu(y) \right)^{\frac{1}{p'}} \left( \int_X f(y)^p d\mu(y) \right)^{\frac{1}{p}}
\]

\[
\leq \|f\|_{L^p(X)} \|I_\gamma \nu\|_{L^{p'}(X)} \leq \|f\|_{L^p(X)},
\]

and hence

\[
c_{\gamma,p}(A) \leq C_{\gamma,p}(A)^{1/p}.
\]

Now, one can show that (see [30])

\[
C_{\gamma,p}(A)^{-1/p} = \sup \{ \inf_{x \in A} I_\gamma f(x) : f \in L^p_+(X), \|f\|_{L^p(X)} \leq 1 \},
\]

and by definition (we may restrict \( \|\nu\| \) to one), we have

\[
c_{\gamma,p}(A) = \left( \inf_{\|\nu\| = 1} \|I_\gamma \nu\|_{L^{p'}(X)} \right)^{-1} = \left( \inf_{\|\nu\| = 1} \sup_{f \in L^p_+(X)} \left| \int_X (I_\gamma \nu) f d\mu \right| \right)^{-1},
\]

where \( \nu \) is a positive measure supported on \( A \) and \( \|f\|_{L^p(X)} \leq 1 \). Therefore, as in [30, Theorem 14], with the use of the Minimax Theorem (see [11]), Theorem 3.15 and also the capacitability of \( c_{\gamma,p} \), equality (3.3) follows.

### 4. Capacity estimates and Hausdorff measure

In this section, we define a Hausdorff measure based on the upper bound estimate for the capacity of balls. We provide an upper bound estimate for the capacity of a set \( E \subset X \) in terms of the Hausdorff content and also show that the Hausdorff content, satisfying a condition placed by \( \gamma p \), is zero if the capacity of the set is zero. For the latter result, we need to assume our space to be connected in order to get inequality (4.1). Similar results for compact sets can be found in [32, Theorem 2.2], where the assumption of connectedness is replaced by a density condition that gives an inequality similar to (4.1). In this section, we give direct and shorter proofs to results that are more general and apply not only for compact sets. In particular, we do not need to use a version of Frostman’s lemma in our proofs. Also, unlike in [32], we do not assume our space to be complete. We start by showing that the Riesz capacity of a ball is bounded from above by constant times the measure of the ball to the power \( 1 - \gamma p \).
Lemma 4.1. Let $1 < p < \infty$ and $0 < \gamma < 1$ be such that $\gamma p < 1$. Then
\[
C_{\gamma, p}(B(x, r)) \leq C \mu(B(x, r))^{1-\gamma p}.
\]

Proof. Choose
\[
g = c^2 3^Q \frac{\chi_{B(x, r)}}{\mu(B(x, r))^\gamma},
\]
where $c > 0$ and $Q > 0$ are some constants, for which the inequality (2.2) holds. For each $z \in B(x, r)$, we have
\[
I_\gamma g(z) = \int_X \frac{1}{\mu(B(z, d(z, y)))^{1-\gamma}} \cdot c^2 3^Q \frac{\chi_{B(x, r)}(y)}{\mu(B(x, r))^\gamma} d\mu(y)
= \frac{c^2 3^Q}{\mu(B(x, r))^\gamma} \int_{B(x, r)} \frac{1}{\mu(B(z, d(z, y)))^{1-\gamma}} d\mu(y).
\]
For each $y \in B(x, r)$, we have that $d(z, y) \leq 2r$, since $z \in B(x, r)$. Now
\[
\mu(B(z, d(z, y))) \leq \mu(B(z, 2r)) \leq \mu(B(x, 3r)),
\]
for each $y \in B(x, r)$. Then
\[
I_\gamma g(z) \geq \frac{c^2 3^Q}{\mu(B(x, r))^\gamma} \int_{B(x, r)} \frac{1}{\mu(B(x, 3r))^{1-\gamma}} d\mu(y)
= c^2 3^Q \frac{\mu(B(x, r))^{1-\gamma}}{\mu(B(x, 3r))^{1-\gamma}}
\geq c^2 3^Q \frac{1}{c^2} \left( \frac{r}{3r} \right)^{Q(1-\gamma)} = 1,
\]
where the last inequality follows by (2.2). Thus $g$ is admissible and we get the upper bound
\[
C_{\gamma, p}(B(x, r)) \leq ||g||_{L^p(X)}^p
= c^2p 3^Q \frac{\mu(B(x, r))}{\mu(B(x, r))^{\gamma p}}
= C \mu(B(x, r))^{1-\gamma p}.
\]
\[\square\]

Lemma 4.1 leads us to define a modified version of the Hausdorff measure that works in our generality. In this section, (and throughout the paper) we do not assume the doubling measure $\mu$ to satisfy the regularity (2.6) or any other estimates that would give uniform lower bounds or upper bounds for the measure of balls in terms of the radii. Recall that the usual definition for the $\lambda$-Hausdorff content of a set $E \subset X$, for $0 < r_1 \leq \infty$, is
\[
\mathcal{H}_\lambda^\mu(E) = \inf \left\{ \sum_{i=1}^\infty r_i^\lambda : E \subset \bigcup_{i=1}^\infty B(x_i, r_i), x_i \in E, r_i \leq r \right\},
\]

A semilinear parabolic equation is a differential equation of the form
\[
u_t + a_\gamma(x, \nabla G) = |\nabla u|^p + f,
\]
where $a_\gamma$, $G$, and $f$ are given functions, and $u$ is the unknown function. The operator $\Delta_G$ is defined as
\[
\Delta_G u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} G_{ij}(x, \nabla u),
\]
where $G_{ij}(x, \nabla u)$ are the entries of the Hessian matrix of $G$. The functions $a_\gamma$, $G$, and $f$ are assumed to be measurable and satisfy certain conditions to ensure the well-posedness of the equation. The operator $\Delta_G$ is a nonlinear operator, and the parabolic equation is a nonlinear partial differential equation. The study of such equations involves techniques from the theory of partial differential equations, functional analysis, and geometric measure theory.
and the $\lambda$-Hausdorff measure of $E$ is $\mathcal{H}^{\lambda}(E) = \lim_{r \to 0} \mathcal{H}^{\lambda}_{r}(E)$. The Hausdorff dimension of $E$ is the number
\[
\dim(E) = \inf \{ \lambda > 0 : \mathcal{H}^{\lambda}(E) = 0 \}.
\]
Let $1 < p < \infty$, $0 < \gamma < 1$ and $\gamma p < 1$. In our case, we define the Hausdorff content of a set $E \subset X$, for $0 < r \leq \infty$, as
\[
\tilde{\mathcal{H}}^{\gamma,p}_{r}(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(B(x_i, r_i))^{1-\gamma p} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), x_i \in E, r_i \leq r \right\}.
\]
Then, the Hausdorff measure is
\[
\tilde{\mathcal{H}}^{\gamma,p}(E) = \lim_{r \to 0} \tilde{\mathcal{H}}^{\gamma,p}_{r}(E).
\]
Note that if the measure is $Q$-regular, then
\[
\tilde{\mathcal{H}}^{\gamma,p}_{r}(E) \approx \mathcal{H}^{Q}_{r}(1-\gamma p).
\]
In the next theorem, we show that the Riesz capacity of a set $E \subset X$ is bounded from above by constant times the (modified) Hausdorff content of the set $E$. In particular, this implies that compact sets with positive capacity have positive Hausdorff measure (see [32, Theorem 2.2]).

**Theorem 4.2.** Let $1 < p < \infty$ and $0 < \gamma < 1$ be such that $\gamma p < 1$. Then $C_{\gamma,p}(E) \leq C \tilde{\mathcal{H}}^{\gamma,p}_{\infty}(E)$ for each $E \subset X$, where $C$ is the same constant as in Lemma 4.1.

**Proof.** Suppose that $\tilde{\mathcal{H}}^{\gamma,p}_{\infty}(E) < \infty$, otherwise the claim is obvious. For $\epsilon > 0$, there is a countable covering $\{B(x_i, r_i)\}$ of $E$ such that
\[
\sum_{i=1}^{\infty} \mu(B(x_i, r_i))^{1-\gamma p} < \tilde{\mathcal{H}}^{\gamma,p}_{\infty}(E) + \epsilon.
\]
Now, by the monotonicity and Theorem 4.1
\[
C_{\gamma,p}(E) \leq \sum_{i=1}^{\infty} C_{\gamma,p}(B(x_i, r_i)) \\
\leq C \sum_{i=1}^{\infty} \mu(B(x_i, r_i))^{1-\gamma p} \\
< C(\tilde{\mathcal{H}}^{\gamma,p}_{\infty}(E) + \epsilon).
\]
The claim follows by letting $\epsilon \to 0$. \qed

For the proof of the next theorem, we need an opposite inequality to (2.2) which is true in connected spaces. Indeed, if $X$ is connected then by [6, Corollary 3.7] there exist constants $C > 0$ and $s > 0$ such that for all balls $B(y, R)$ in $X$, all $z \in B(y, R)$ and all $0 < r \leq R$,
\[
(4.1) \quad \frac{\mu(B(z, r))}{\mu(B(y, R))} \leq C \left( \frac{r}{R} \right)^s.
\]
Note that inequality (4.1) given by the connectedness (or uniform perfectness) of the space \( X \) is essentially the same as the density condition assumed in [32]. The proof of the next theorem is direct and we do not need to use Frostman’s lemma to obtain the result. In the Euclidean space, with the Lebesgue measure, we use the notation \( \alpha = \gamma n \) and consider the usual Riesz potential of order \( \alpha \) from Remark 2.2 (ii).

Our result implies the classical result that if the Riesz capacity of a set \( E \) is zero, then \( E \) has Hausdorff dimension at most \( n - \alpha p \), where \( \alpha p < n \) (see e.g. [31, Section 5.2, Theorem 2.3]).

**Theorem 4.3.** Assume that \( X \) is connected. Let \( 1 < p < \infty \), \( 1 < \frac{\gamma}{p} < \infty \), \( 0 < \gamma < 1 \) and \( 0 < \frac{\gamma}{p} < 1 \) be such that \( \gamma p < 1 \), \( \frac{\gamma}{p} < 1 \) and \( \frac{\gamma}{p} < \gamma p \).

If \( C_{\gamma,p}(E) = 0 \), then \( H_{\frac{\gamma}{p}}^\infty(E) = 0 \).

**Proof.** We may assume that the set \( E \) is compact. Indeed, by Theorem 3.15, the Riesz capacity of an analytic set can be obtained by approximating with compact sets from the inside. With the same techniques as in Section 3, we can see that the same is true also for the Hausdorff content of the set \( E \).

For \( \epsilon > 0 \), there is an admissible function \( f \geq 0 \) such that

\[
||f||_{L^p(X)} < \epsilon.
\]

For such a function \( f \), at each point \( x \in E \),

\[
1 \leq \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} d\mu(y).
\]

Let \( x_0 \in E \). We choose \( R_0 > \text{diam}(E) \) large enough such that \( E \subseteq B(x_0, R_0) \) and that the integral below is more than one half. Notice that we can always find such a radius \( R_0 \) but the selection depends on the set \( E \). Define \( R = 2R_0 \) and \( r_i = 2^{-i}R \), for \( i \in \mathbb{N} \). For each point \( x \in E \)

\[
\frac{1}{2} \leq \int_{B(x_0, R_0)} \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} d\mu(y)
\]

\[
\leq \int_{B(x, R)} \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} d\mu(y).
\]
and hence
\[
1 \leq 2 \int_{B(x,R)} \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\gamma}} \, d\mu(y)
\]
\[
= 2 \sum_{i=0}^{\infty} \int_{B(x,r_i) \setminus B(x,r_{i+1})} \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\gamma}} \, d\mu(y)
\]
\[
\leq 2 \sum_{i=0}^{\infty} \int_{B(x,r_i) \setminus B(x,r_{i+1})} \frac{f(y)}{\mu(B(x,r_{i+1}))^{1-\gamma}} \, d\mu(y)
\]
\[
\leq 2 \sum_{i=0}^{\infty} \frac{1}{\mu(B(x,r_{i+1}))^{1-\gamma}} \int_{B(x,r_i)} f(y) \, d\mu(y).
\]

Using Hölder’s inequality and the doubling condition, we get
\[
1 \leq C \sum_{i=0}^{\infty} \mu(B(x,r_i))^{\gamma-1+\frac{1}{p}} \left( \int_{B(x,r_i)} f(y)^p \, d\mu(y) \right)^{1/p}.
\]

Next, we use the fact that for any \(\delta > 0\) there is a constant \(C > 0\) such that
\[
1 = C \sum_{i=0}^{\infty} 2^{-i\delta} = C \sum_{i=0}^{\infty} \left( \frac{r_i}{R} \right)^\delta.
\]

Since \(X\) is connected, inequality (4.1) gives
\[
\sum_{i=0}^{\infty} \left( \frac{\mu(B(x,r_i))}{\mu(B(x,R))} \right)^{\delta/s} \leq C \sum_{i=0}^{\infty} \left( \frac{r_i}{R} \right)^\delta = 1.
\]

Now, by putting the measure of the ball \(B(x_0,R)\) as part of the constant, we have that
\[
C \sum_{i=0}^{\infty} \mu(B(x,r_i))^{\delta/s} \leq \sum_{i=0}^{\infty} \mu(B(x,r_i))^{(\gamma-1)+\frac{1}{p}} \left( \int_{B(x,r_i)} f(y)^p \, d\mu(y) \right)^{1/p},
\]
where the constant \(C\) depends on \(R\). For \(\delta > 0\), there exists at least one index \(i_x \in \mathbb{N}\) such that
\[
\mu(B(x,r_{i_x}))^{(\gamma-1)+\frac{1}{p}} \left( \int_{B(x,r_{i_x})} f(y)^p \, d\mu(y) \right)^{1/p} \geq C \mu(B(x,r_{i_x}))^{\delta/s}
\]
and, by raising both sides to power \(p\), we get
\[
\int_{B(x,r_{i_x})} f(y)^p \, d\mu(y) \geq C \mu(B(x,r_{i_x}))^{\delta/p+(\gamma-1)p-p+1}
\]
\[
= C \mu(B(x,r_{i_x}))^{\delta/p-\gamma p+1}.
\]

We choose
\[
\delta = \frac{\gamma p - \gamma p^+}{p} \cdot s,
\]
which is positive, when $\gamma p > \tilde{\gamma} \tilde{p}$. We obtain for each $x \in E$ a ball $B(x, r_x) = B_x$ such that

\begin{equation}
\mu(B_x)^{1-\tilde{\gamma} \tilde{p}} \leq C \int_{B_x} f(y)^p \, d\mu(y).
\end{equation}

By using the basic $5r$-covering theorem (see e.g. [19]), we obtain countably many points $x_j \in E$, such that the balls $B_j = B_{x_j}$ are pairwise disjoint and $E \subset \bigcup_{j=1}^{\infty} 5B_j$. Using the estimate (4.2), the doubling property of measure $\mu$ and the pairwise disjointness of the balls $B_j$, we get

$$
\tilde{\mathcal{H}}_{\tilde{\gamma}, \tilde{p}}(E) \leq \sum_{j=1}^{\infty} \mu(5B_j)^{1-\tilde{\gamma} \tilde{p}} \leq C \sum_{j=1}^{\infty} \mu(B_j)^{1-\tilde{\gamma} \tilde{p}} 
\leq C \sum_{j=1}^{\infty} \int_{B_j} f(y)^p \, d\mu(y) \leq C \int_X f(y)^p \, d\mu(y)
= C \|f\|_{L^p(X)}^p < C \epsilon.
$$

Letting $\epsilon \to 0$ yields the claim. \hfill \Box

We can see that in the metric space, with a doubling measure $\mu$, the Hausdorff content and the Hausdorff measure have the same null sets. This relation has been studied in [22, Section 7] for slightly different versions of the Hausdorff content and the Hausdorff measure. For our definitions, the result of [22, Lemma 7.9] follows without any extra assumptions. By the previous two theorems, we get as a special case the main result of [32]. In our case, the next result is true not only for compact sets, but for Borel sets as well.

**Corollary 4.4.** Let $1 < p < \infty$, $1 < \tilde{p} < \infty$, $0 < \gamma < 1$ and $0 < \tilde{\gamma} < 1$ be such that $\gamma p < 1$, $\tilde{\gamma} \tilde{p} < 1$ and $\tilde{\gamma} \tilde{p} < \gamma p$. Then

$$
\mathcal{C}_{\gamma, p}(E) > 0 \text{ implies that } \tilde{\mathcal{H}}^{\tilde{\gamma}, \tilde{p}}(E) > 0.
$$

If we also assume our space $X$ to be connected, then

$$
\tilde{\mathcal{H}}^{\tilde{\gamma}, \tilde{p}}(E) > 0 \text{ implies that } \mathcal{C}_{\gamma, p}(E) > 0.
$$

5. **Isocapacitary inequalities and boundedness of the Riesz potential**

In the Euclidean space, with the $n$-dimensional Lebesgue measure, we have by the notation $\alpha = \gamma n \in (0, n)$ the usual Riesz potential $I_{\alpha}$ of order $\alpha$. The well-known Hardy-Littlewood-Sobolev Theorem states that if $I_{\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $p$ and $q$, then the exponents are related by $1/p - 1/q = \alpha/n$, i.e. $q = \frac{np}{n-\alpha p}$. This condition is also sufficient to have a bounded operator. In particular, when $q > p$, operator $I_{\alpha}$ improves the integrability of a function. In [20], there is an analogous result in a doubling metric measure space.
for the operator $I_\gamma$ (see [20 Proposition 4.1]). If $\mu(X) = \infty$, then $I_\gamma$
is bounded from $L^p(X)$ to $L^q(X)$, if and only if $1/p - 1/q = \gamma$. If $\mu(X) < \infty$, then $I_\gamma$
is bounded from $L^p(X)$ to $L^q(X)$, if and only if $1/p - 1/q \leq \gamma$. In Theorem 5.2, we give three equivalent conditions
to the condition that Riesz potential $I_\gamma$ maps $L^p(X)$ to $L^q(X)$. In particular, an isocapacitary inequality characterizes the boundedness
of $I_\gamma$. Note that, by Lemma 3.5, for any nonnegative $f \in L^p(X)$
$$\|I_\gamma f\|_{L^{p,\infty}(C_{\gamma,p})} = \sup_{t>0} t C_{\gamma,p} \left\{ x \in X : |I_\gamma f(x)| > t \right\}^{1/p} \leq \|f\|_{L^p(X)},$$
which says that $I_\gamma$ is bounded from $L^p_\ast(X)$ to $L^{p,\infty}(C_{\gamma,p})$. In Lemma 5.3,
we show that $I_\gamma$ is bounded from $L^p(X)$ to $L^{p,q}(C_{\gamma,p})$. This is an improvement
to the previous boundedness, since $L^{p,q}(C_{\gamma,p}) \subset L^{p,\infty}(C_{\gamma,p})$.
Throughout the section, we denote the conjugate exponents of $p$ and $q$ by $p'$ and $q'$. From now on, we also assume that the measure $\mu$ is a Radon measure. First, we give a definition for the relative Riesz
capacity, which is needed in the proof of Lemma 5.3.

**Definition 5.1.** Let $1 < p < \infty$ and $0 < \gamma < 1$. The relative Riesz
$(\gamma,p)$-capacity of a set $E \subset X$ with respect to an open set $A \subset X$ is the number
$$C_{\gamma,p}(E; A) = \inf \left\{ \|f\|_{L^p(X)}^p : f \in L^p_\ast(X), I_\gamma f \geq 1 \text{ on } E, \text{ supp}(f) \subset A \right\},$$
where supp$(f)$ is the support of $f$. If such an $f$ does not exist, we set
$C_{\gamma,p}(E; A) = \infty$.

Note that the relative Riesz capacity of a set $E$ with respect to the whole space $X$ is the Riesz capacity $C_{\gamma,p}(E)$ we defined in the beginning of Section 3. Let $A \subset B \subset X$. It follows immediately from the definition that for each $E \subset X$,
$$C_{\gamma,p}(E) \leq C_{\gamma,p}(E; B) \leq C_{\gamma,p}(E; A) .$$
The next result is new in the setting of a metric measure space. In the Euclidean case, a similar result can be found in [2]. We use the notation $\mu_K = \mu|_K$ i.e. $\mu_K(A) = \mu(K \cap A)$.

**Theorem 5.2.** Let $1 < p \leq q < \infty$ and $0 < \gamma < 1$. The following properties are equivalent:

(i) For each compact subset $K$, $\mu(K) \leq A_4 C_{\gamma,p}(K)^{q/p}$.
(ii) For each $f \in L^p(X)$,
$$\|I_\gamma f\|_{L^q(X)} \leq A_2 \|f\|_{L^p(X)}.$$
(iii) For each $f \in L^p(X)$,
$$\|I_\gamma f\|_{L^{p,\infty}(X)} \leq A_3 \|f\|_{L^p(X)}.$$
(iv) $\|I_\gamma \mu_K\|_{L^{p'}(X)} \leq A_4 \mu(K)^{1/q'}$, for each compact subset $K$. 
Proof. Since $\|I_\gamma f\|_{L^q(X)} \leq \|I_\gamma f\|_{L^p(X)}$, it is immediate that (ii) implies (iii). We show that (iii) implies (iv). We have for any compact set $K \subset X$ and any $\delta > 0$ that

$$\int_X I_\gamma |f| \, d\mu_K = \int_0^\infty \mu_K (\{I_\gamma |f| > t\}) \, dt$$

$$= \int_0^\delta \mu_K (\{I_\gamma |f| > t\}) \, dt + \int_\delta^\infty \mu_K (\{I_\gamma |f| > t\}) \, dt$$

$$\leq \mu(K) \int_0^\delta \, dt + \int_\delta^\infty \mu (\{I_\gamma |f| > t\}) \, dt$$

$$\leq \mu(K) \delta + \int_\delta^\infty t^{-q} \sup_{t \geq \delta} t^q \mu (\{I_\gamma |f| > t\}) \, dt$$

$$\leq \mu(K) \delta + \int_\delta^\infty t^{-q} \|I_\gamma |f|\|_{L^q(X)}^q \, dt$$

$$\leq \mu(K) \delta + \int_\delta^\infty t^{-q} A_3^q \|f\|_{L^p(X)}^q \, dt$$

$$\leq \mu(K) \delta + A_3^q \frac{\delta^{1-q}}{q-1} \|f\|_{L^p(X)}^q.$$

Also, by using Fubini’s theorem and the doubling condition, we have that

$$\int_X f I_\gamma \mu_K \, d\mu = \int_X f(x) \int_X \frac{d\mu_K(y)}{\mu(B(x, d(x, y))^{1-\gamma})} \, d\mu(x)$$

$$= \int_X \int_X \frac{f(x) \, d\mu(x)}{\mu(B(x, d(x, y))^{1-\gamma})} d\mu_K(y)$$

$$\leq \int_X \int_X \frac{c_4^{-1-\gamma} |f(x)| \, d\mu(x)}{\mu(B(x, 2d(x, y))^{1-\gamma})} d\mu_K(y)$$

$$\leq \int_X \int_X C \frac{|f(x)| \, d\mu(x)}{\mu(B(y, d(y, x))^{1-\gamma})} d\mu_K(y)$$

$$= C \int_X I_\gamma |f| \, d\mu_K.$$

If $\delta = A_3 \frac{\|f\|_{L^p(X)}}{\mu(K)^{1/q}}$, then by the previous estimates

$$\left| \int_X f I_\gamma \mu_K \, d\mu \right| \leq C \int_X I_\gamma |f| \, d\mu_K \leq C \left( \mu(K) \delta + A_3^q \frac{\delta^{1-q}}{q-1} \|f\|_{L^p(X)}^q \right)$$

$$\leq C \left[ \mu(K) A_3 \left( \frac{\|f\|_{L^p(X)}}{\mu(K)^{1/\gamma}} \right)^{1-q} \frac{1}{q-1} \|f\|_{L^p(X)}^q \right]$$

$$= C \mu(K)^{1-1/\gamma} A_3 \|f\|_{L^p(X)} \left( 1 + \frac{1}{q-1} \right) = C q' A_3 \mu(K)^{1-1/\gamma} \|f\|_{L^p(X)}.$$
Taking the supremum over all $f \in L^p(X)$ satisfying $\|f\|_{L^p(X)} \leq 1$, we have by duality that

$$\|I_{1,\mu}K\|_{L^p(X)} = \sup_{\|f\|_{L^p(X)} \leq 1} \left| \int_X f I_{1,\mu}Kd\mu \right| \leq Cq' A_3 \mu(K)^{1-1/q}.$$ 

This gives (iv) with $A_4 = Cq'A_3$.

We show next that (iv) implies (i). Let $f \in A(K)$. Then, as in (5.1), we get that

$$\mu(K) = \mu_K(K) \leq \int_X I_{1,\mu}Kd\mu \leq C \int_X f I_{1,\mu}Kd\mu \leq C\|f\|_{L^p(X)}\|I_{1,\mu}K\|_{L^p(X)} \leq C\|f\|_{L^p(X)} A_4 \mu(K)^{1-1/q}$$

and hence

$$\mu(K) \leq Cq\|f\|_{L^p(X)}^{q} A_4^q = A_4^q C q (\|f\|_{L^p(X)}^p)^{q/p}.$$ 

Now, taking infimum over such functions $f$ yields

$$\mu(K) \leq A_4^q C q C_{\gamma,p}(K)^{q/p}.$$ 

Finally, (ii) follows from (i) by the following Lemma 5.3. Indeed, since $\mu$ is Radon measure and the Riesz capacity satisfies inner regularity, we can approximate from the inside by compact sets and we have by (i) and Lemma 5.3 that

$$\|I_{1,\mu}f\|_{L^q(X)} = \left( \int_0^\infty q^{p-1} \mu(\{x \in X : |I_{1,\mu}f(x)| > t\}) dt \right)^{1/q} \leq \left( \int_0^\infty q^{p-1} A_4^q C q C_{\gamma,p} (\{x \in X : |I_{1,\mu}f(x)| > t\})^{q/p} dt \right)^{1/q} \leq A_4 C (2^q - 1)^{1/q} \|f\|_{L^p(X)}.$$ 

The following lemma also improves the boundedness of the Riesz potential from the weak type inequality of Lemma 3.5, since we have that

$$\|I_{1,\mu}f\|_{L^{p,q}(C_{\gamma,p})} = \left( \int_0^\infty q^{p-1} C_{\gamma,p} (\{x \in X : |I_{1,\mu}f(x)| > t\})^{q/p} dt \right)^{1/q}$$

and $L^{p,q}(C_{\gamma,p}) \subset L^{p,\infty}(C_{\gamma,p})$.

**Lemma 5.3.** Let $1 < p \leq q < \infty$ and $0 < \gamma < 1$. If $f \in L^p(X)$, then

$$\left( \int_0^\infty q^{t^{q-1}} C_{\gamma,p} (\{x \in X : |I_{1,\mu}f(x)| > t\})^{q/p} dt \right)^{1/q} \leq (2^{q-1})^{1/q} \|f\|_{L^p(X)}.$$
Proof. Let \( f \in L^p(X) \). It follows by monotonicity and the definition of the relative Riesz capacity that

\[
\int_0^\infty q^{q-1} C_{\gamma,p}(\{|I_\gamma f| > t\})^{q/p} dt \\
\leq \int_0^\infty q^{q-1} C_{\gamma,p}(\{|I_\gamma f| > t\}; \{I_\gamma |f| \leq 2t\})^{q/p} dt \\
\leq \int_0^\infty q^{q-1} C_{\gamma,p}(\{|I_\gamma f| > t\}; \{I_\gamma |f| \leq 2t\})^{q/p} dt \\
= \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} q^{q-1} C_{\gamma,p}(\{|I_\gamma f| > 2^{j-1}\}; \{I_\gamma |f| \leq 2^j\})^{q/p} dt \\
\leq \sum_{j=-\infty}^{\infty} 2^{(j-1)q} (2^q - 1) C_{\gamma,p}(\{|I_\gamma f| > 2^{j-1}\}; \{I_\gamma |f| \leq 2^j\})^{q/p}.
\]

We estimate the relative Riesz capacity by function \( \frac{|f|}{2^{j-1}} \chi_{\{2^{j-1} < |f| \leq 2^j\}} \), which is admissible, and obtain

\[
\sum_{j=-\infty}^{\infty} 2^{(j-1)q} (2^q - 1) C_{\gamma,p}(\{|I_\gamma f| > 2^{j-1}\}; \{I_\gamma |f| \leq 2^j\})^{q/p} \\
\leq (2^q - 1) \sum_{j=-\infty}^{\infty} 2^{(j-1)q} \left\| \frac{|f|}{2^{j-1}} \chi_{\{2^{j-1} < |f| \leq 2^j\}} \right\|_{L^q(X)}^q \\
= (2^q - 1) \sum_{j=-\infty}^{\infty} \left\| f \chi_{\{2^{j-1} < |f| \leq 2^j\}} \right\|_{L^q(X)}^q.
\]

Now,

\[
\left( \int_0^\infty q^{q-1} C_{\gamma,p}(\{|x_0| > t\})^{q/p} dt \right)^{1/q} \\
\leq (2^q - 1)^{1/q} \left( \sum_{j=-\infty}^{\infty} \left\| f \chi_{\{2^{j-1} < |f| \leq 2^j\}} \right\|_{L^q(X)}^q \right)^{1/q}.
\]

When \( p = q \), we get that
\[
(2^q - 1)^{1/q} \left( \sum_{j=-\infty}^{\infty} \|f| \chi_{\{2^{j-1} < |f| \leq 2^j \}} \|_{L^p(X)}^q \right)^{1/q}
\]
\[
= (2^p - 1)^{1/p} \left( \sum_{j=-\infty}^{\infty} \int_X |f|^p \chi_{\{2^{j-1} < |f| \leq 2^j \}}(y) \, d\mu(y) \right)^{1/p}
\]
\[
= (2^p - 1)^{1/p} \left( \int_X \sum_{j=-\infty}^{\infty} |f|^p \chi_{\{2^{j-1} < |f| \leq 2^j \}}(y) \, d\mu(y) \right)^{1/p}
\]
\[
= (2^p - 1)^{1/p} \|f\|_{L^p(X)}.
\]

This computation holds for any disjoint elements, which says that the space \( L^p(X) \) satisfies a lower \( p \)-estimate (with constant one), see [26, Definition 1.f.4]). Then, using [26, Theorem 1.f.7.], we have that \( L^p(X) \) satisfies also a lower \( q \)-estimate, for every \( p \leq q < \infty \). This implies that

\[
\left( \int_0^\infty qt^{q-1} C_{\gamma,p}(\{x \in X : |I_{\gamma} f(x)| > t\})^{q/p} \, dt \right)^{1/q} \leq (2^q - 1)^{1/q} \|f\|_{L^p(X)},
\]

for every \( p \leq q < \infty \).

\[\square\]

**References**

[1] D.R. Adams, *Choquet integrals in potential theory*, Publ. Mat. 42 (1998), 3–66.
[2] D.R. Adams, *On the existence of capacitary strong type estimates in \( \mathbb{R}^n \)*, Arkiv för Matematik, 14 (1976), 125–140.
[3] D.R. Adams and L.I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag (1996).
[4] H. Aikawa, M.R. Essen, *Potential theory: selected topics no.1633*, Springer (1996).
[5] S. Banach, S. Saks, *Sur la convergence forte dans les champs \( L^p \)*, Studia Mathematica 2 (1930) 51–57.
[6] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics 17, European Mathematical Society (EMS), Zürich (2011).
[7] C. Bennett and B. Sharpley, *Interpolation of Operators*, Academic Press Inc. (1988).
[8] J. Cerdà, *Lorentz capacitary spaces*, AMS, Contemporary Mathematics 445 (2007), 49-55.
[9] J. Cerdà, J. Martín and P. Silvestre, *Capacitary function spaces*, Collectanea Math. 62 (2011), 95–118.
[10] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) 5 (1953-54), 131–295.
[11] K. Fan, *Minimax theorems*, Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 42–47.
[12] R.A. Fefferman, *A theory of entropy in Fourier analysis*, Adv. in Math. 30 (1978), 171–201.
[13] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. 103 (1960), 139–215.
[14] A.E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, Analysis and Partial Differential Equations, C. Sadosky (ed.), Dekker (1990), 171–216.

[15] A.E. Gatto, C. Segovia and S. Vági, *On fractional differentiation and integration on spaces of homogeneous type*, Rev. Mat. Iberoamericana 12 (1996), 111–145.

[16] V. Gol’dshtein and M. Troyanov, * Capacities in metric spaces*, Integral Equ. Oper. Theory 44 (2002), 212–242.

[17] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000).

[18] H. Hakkarainen and J. Kinnunen, *The BV-capacity in metric spaces*, Manuscripta Math. 132 (2010), 369–390.

[19] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag New York (2001).

[20] A. Kairema, *Sharp weighted bounds for fractional integral operators in a space of homogeneous type*, Mathematica Scandinavica 114 (2014) no 2, 226–253.

[21] S. Kallunki and N. Shanmugalingam, *Modulus and continuous capacity*, Ann. Acad. Sci. Fenn. Math. 26 (2001), 455–464.

[22] J. Kinnunen, R. Korte, N. Shanmugalingman and H. Tuominen, *Lebesgue points and capacities via the boxing inequality in metric spaces*, Indiana Univ. Math. J. 57 (2008), 401–430.

[23] J. Kinnunen and O. Martio, *The Sobolev capacity on metric spaces*, Ann. Acad. Sci. Fenn. Math. 21 (1996), 367–382.

[24] J. Kinnunen and O. Martio, *Choquet property for the Sobolev capacity in metric spaces*, In: Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), 285–290. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Math., Novosibirsk (2000).

[25] J. Lehrbäck, *Neighbourhood capacities*, Ann. Acad. Sci. Fenn. Math. 37 (2012), 35–51.

[26] J. Lindestrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Springer-Verlag, A Series of Modern Surveys in Mathematics (1979).

[27] J. Maly, *Coarea integration in metric spaces*, Nonlinear Analysis, Function Spaces and Applications, Proceedings of the Spring School held in Prague, July 17-22, 2002, Czech Academy of Sciences, Mathematical Institute, Praha, 7 (2003), 149–192.

[28] J. Maly and L. Pick, *The sharp Riesz potential estimates in metric spaces*, Indiana Univ. Math. J., 51 (2002), 251–268.

[29] V.G. Maz’ya, *Sobolev Spaces* (translated from the Russian by T. O. Shaposhnikova), Springer-Verlag (1985).

[30] N.G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand., 103 (1970), 255–292.

[31] Y. Mizuta, *Potential Theory in Euclidean Spaces*, Gakuto International Series Mathematical Sciences and Applications Volume 6 (1996).

[32] T. Sjödin, *A Note on Capacity and Hausdorff Measure in Homogeneous Spaces*, Potential Analysis 6 (1997), 87–97.

[33] T. Sjödin, *Polar sets and capacitary potentials in homogeneous spaces*, Ann. Acad. Sci. Fenn. Math. 38 (2013), 771–783.

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