Disk-Jet Coupling in Black Hole Accretion Systems I: General Relativistic Magnetohydrodynamical Models

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ABSTRACT

General relativistic numerical simulations of magnetized accretion flows around black holes show a disordered electromagnetic structure in the disk and corona and a highly relativistic, Poynting-dominated funnel jet in the polar regions. The polar jet is nearly consistent with the stationary paraboloidal Blandford-Znajek model of an organized field threading the polar regions of a rotating black hole. How can a disordered accretion disk and corona lead to an ordered jet? We show that the polar jet is associated with a strikingly simple angular-integrated toroidal current distribution $dI_\phi/dr \propto r^{-5/4}$, where $I_\phi(r)$ is the toroidal current enclosed inside radius $r$. We demonstrate that the poloidal magnetic field in the simulated jet agrees well with the force-free field solution for a non-rotating thin disk with an $r^{-5/4}$ toroidal current, suggesting rotation leads to negligible self-collimation. We find that the polar field is confined/collimated by the corona. We also study the properties of the bulk of the simulated disk, which contains a turbulent magnetic field locked to the disk's Keplerian rotation except for rapidly rotating prograde black holes ($a/M \gtrsim 0.4$) for which within $r \lesssim 3GM/c^2$ the field locks to roughly half the black hole spin frequency. The electromagnetic field in the disk also scales as $r^{-5/4}$, which is consistent with some Newtonian accretion models that assume rough equipartition between magnetic and gas pressure. However, the agreement is accidental since toward the black hole the magnetic pressure increases faster than the gas pressure. This field dominance near the black hole is associated with magnetic stresses that imply a large effective viscosity parameter $\alpha \sim 1$, whereas the typically assumed value of $\alpha \sim 0.1$ holds far from the black hole.

Key words: accretion disks, black hole physics, galaxies: jets, gamma rays: bursts, X-rays: bursts

1 INTRODUCTION

Black hole accretion is one of the most powerful sources of energy in the universe. A substantial fraction of the gravitational binding energy of the accreting gas is released within tens of gravitational radii from the black hole, and this energy supplies the power for a variety of astrophysical systems including active galactic nuclei, X-ray binaries, and gamma-ray bursts. Elucidating the processes that take place in the central regions of black hole disks is obviously crucial if we wish to understand the physics of these energetic objects.

Magnetized, differentially-rotating accretion disks exhibit the magneto-rotational instability (MRI) and magnetohydrodynamic turbulence (Balbus & Hawley 1991, 1998), which generate large spatio-temporal variations in all fluid quantities and strong correlations between fluid quantities. Recent general relativistic magnetohydrodynamic (GRMHD) simulations of black hole accretion systems have begun to resolve these processes and have revealed a flow structure that can be decomposed into a disk, corona, disk wind, and highly magnetized polar region that contains a jet (De Villiers, Hawley, & Krolik 2003; McKinney & Gammie 2004). As expected, the simulations show complex time-dependent behavior in the disk, corona, and wind. Surprisingly, however, the polar regions of the flow are found to have a simple structure that can be decomposed into a disk, corona, disk wind, and highly magnetized polar region that contains a jet (De Villiers, Hawley, & Krolik 2003; McKinney & Gammie 2004). As expected, the simulations show complex time-dependent behavior in the disk, corona, and wind. Surprisingly, however, the polar regions of the flow are found to have a simple structure that can be decomposed into a disk, corona, disk wind, and highly magnetized polar region that contains a jet (McKinney & Gammie 2004). The numerical solution here is quantitatively consistent with the relativistic force-free model proposed by Blandford & Znajek (1977), hereafter BZ (McKinney & Gammie 2004).
The primary question this paper explores is the following: how can a turbulent accretion disk lead to a nearly stationary, collimated and ordered Poynting-dominated jet? In simple force-free models of the disk-jet coupling, like the one developed by BZ, one finds stationary solutions for a fixed toroidal current and angular velocity of the disk (see, e.g. Blandford & Znajek 1977). Since in GRMHD simulations the poloidal field threading the black hole is simple and nearly stationary, this implies that the toroidal current must also be simple and nearly stationary. However, it has not yet been known or understood what radial dependence would be chosen by turbulent accretion flows driven by the MRI. In order to place the GRMHD simulation results in the context of analytical models of jets and winds that treat the disk as a equatorial boundary condition, our first objective is to determine the radial dependence of the toroidal current, angular velocity of the plasma, and angular velocity of the magnetic field.

Given the radial distribution of the toroidal current and angular velocities of the plasma and field, one can generate force-free models of the jet that approximate the disk as an infinitely thin rotating conductor. General relativistic force-free solutions of this kind are obtained and then compared to GRMHD simulations and the models of BZ in a followup paper (McKinney & Narayan 2006).

Our second objective in this paper is to determine the angular distribution of toroidal currents and the flow pattern of poloidal currents in the accretion flow. The location of the toroidal currents helps identify the role played by the weakly magnetized corona in confining the highly magnetized jet. For example, the corona provides forces that balance the forces due to the strong poloidal field gradients (associated with strong toroidal currents) at the boundary between the magnetized jet and corona. Hence, the corona can be understood as required to confine the magnetized jet. Also, a stationary model must have poloidal currents that close like a circuit, yet it has not been known where such poloidal currents flow in MHD turbulent disks around black holes. We study the distribution of poloidal currents that are associated with the outgoing power of the jet/wind in order to establish where the poloidal currents flow and to resolve the issue of current closure.

Our third objective is to check if either of the two magnetic field geometries described by BZ, viz., the split-monopole and the paraboloidal geometries (originally discovered by Michel 1973 and Blandford 1976 respectively), is a good description of the jets found in our GRMHD simulations. We find that neither model is satisfactory. Instead we identify a third model, in between the two and close to the paraboloidal model, which agrees surprisingly well with the simulations as long as the jet is nearly force-free.

The toroidal current in the split-monopole and paraboloidal solutions scale with radius as $dl_{\phi}/dr \propto r^{-2}$, $r^{-1}$, respectively, whereas the toroidal current in the GRMHD simulations is found to scale as $r^{-3/4}$. Interestingly, the latter scaling is identical to that proposed by Blandford & Payne (1982) (BP), who developed a non-relativistic self-similar magnetohydrodynamic (MHD) model of disk winds by assuming that the sound speed and Alfvén speed in the disk scale similarly with radius. Our fourth objective in this paper is to establish how the disk magnetic field strengths, sound speed, Alfvén speed, plasma speed depend on radius in order to determine whether the agreement between the GRMHD simulations and the BP model has a deep physical significance or is merely a coincidence. The answer appears to be the latter in the sense that the assumptions made by BP are broken near the black hole. More importantly, the field threading the disk is disorganized and the disk wind is thermally-driven instead of behaving like a “bead on a wire” as in the BP model.

Prior studies of the BZ power output suggested that the black hole power output should be too small compared to the disk power output and too small to account for the most powerful radio sources (Ghosh & Abramowicz 1997; Livio, Oueliev, & Pringle 1999). Such studies assumed that the effective viscosity parameter $\alpha$ was as determined in non-relativistic simulations and assumed the field strength near the black hole was set by sub-equipartition arguments. Our final objective is to determine the magnetic $\alpha$ viscosity parameter as a function of radius within the disk. Both of their assumptions end up not applying near the black hole.

PAPER OUTLINE

In section 2 we discuss the origin of the ordered poloidal field in GRMHD simulations. We show that the angular-integrated toroidal currents in the turbulent accretion disk follow a simple power-law behavior. We discuss the angular structure of the toroidal currents and the flow of poloidal currents in the accretion flow. We discuss the field angular velocity in the transition region between the accretion disk and black hole. In section 3 we study the GRMHD accretion flow in order to extract other electromagnetic properties, such as the magnetic field strength as a function of radius. We test the assumptions of BP against our GRMHD numerical models and study the electromagnetic stress that leads to an enhanced angular momentum transport near the black hole. In section 4 we discuss the limitations of our calculations. Finally, in section 5 we discuss our results and conclude.

In appendix A we summarize the GRMHD equations of motion and point out the reduction to the force-free set of equations. In appendix B we show how to obtain force-free solutions in Schwarzschild and flat spacetimes for an arbitrary current sheet at the equator, and we discuss how the disk currents are integrated to obtain a toroidal current density as a function of radius. We also give three example solutions corresponding to the split-monopole, paraboloidal, and our new self-similar solution.

UNITS AND NOTATION

The units in this paper have $GM = c = 1$, which sets the scale of length ($r_g \equiv GM/c^2$) and time ($t_g \equiv GM/c^3$). The horizon is located at $r = r_s \equiv r_g(1 + \sqrt{1 - (a/M)^2})$. For a black hole with angular momentum $J = aGM/c$, $a/M$ is the dimensionless Kerr parameter with $-1 \leq a/M \leq 1$. In order to obtain a density for a given mass accretion rate, one requires the field as a function of black hole spin given by GRMHD models such as described in McKinney (2005b, 2006c). The mass scale is determined by setting the observed
The mass accretion rate ($\dot{M}_0$) equal to the accretion rate through the black hole horizon as measured in a simulation. So the mass scale is set by the mass accretion rate ($\dot{M}_0$) at the horizon, such that $p_{\text{disk}} \equiv \dot{M}_0[r = r_g] / r_g^3$ and the mass scale is then just $m \equiv p_{\text{disk}} r_g^3 = \dot{M}_0[r = r_g] / r_g^3$.

The results of the simulations can be applied to any astrophysical system once the value of $p_{\text{disk}}$ is estimated. For example, a collapsar model with $M = 0.1 M_\odot \text{ s}^{-1}$ and $M \approx 3 M_\odot$ has $p_{\text{disk}} \approx 3.4 \times 10^{10} \text{ g cm}^{-3} \text{ s}^{-1}$ (MacFadyen & Woosley 1999). M87 has a mass accretion rate of $\dot{M} \approx 10^{-2} M_\odot \text{ yr}^{-1}$ and a black hole mass of $M \approx 3 \times 10^7 M_\odot$ (Ho 1999; Reynolds et al. 1996) giving $p_{\text{disk}} \approx 10^{-16} \text{ g cm}^{-3}$. GRS 1915+105 has a mass accretion rate of $\dot{M} \approx 7 \times 10^{-7} M_\odot \text{ yr}^{-1}$ (Mirabel & Rodriguez 1994; Mirabel & Rodríguez 1999; Fender & Belloni 2004) with a mass of $M \sim 14 M_\odot$ (Greiner et al. 2001) (but see Kaiser et al. 2004). This gives $p_{\text{disk}} \sim 3 \times 10^{-4} \text{ g cm}^{-3}$.

The notation follows Misner et al. (1973) and the signature of the metric is $\text{+}++$. Tensor components are given in a coordinate basis. The components of the tensors of interest are given by $\hat{g}_{\mu \nu}$ for the metric, $F^{\mu \nu}$ for the Faraday tensor, $\hat{F}^{\mu \nu}$ for the dual of the Faraday, and $T^{\mu \nu}$ for the stress-energy tensor. The determinant of the metric is given by $\sqrt{-g} \equiv \text{Det}(g_{\mu \nu})$. The field angular frequency is $\Omega \equiv \Omega_r = \Omega_\phi$. The coordinate current density is given by the current density $J$ and a enclosed (from the pole to some point) poloidal current ($B_\phi \equiv F_{\phi \theta}$). The electromagnetic luminosity is $L \equiv -2 \pi \int d\Omega T^{(EM)} r^2 \sin \theta$. See Gammie et al. (2003a); McKinney & Gammie (2004); McKinney (2004); McKinney & Gammie (2005a); McKinney & Gammie (2005b); McKinney & Gammie (2005c); McKinney & Gammie (2006a) for details on this standard notation.

2 THE ORGANIZED POLAR FIELD

In this section we discuss the origin and nature of the organized field threading the black hole. We first review some relevant results from GRMHD simulations of accretion flows. We then demonstrate that at large radii the jet from the black hole is electromagnetically pure, while the disk wind is dirty. Next, we show that the angular-integrated toroidal current over the accretion flow is simple and has a power-law radial dependence, which is associated with the simple organized poloidal field. We analyze the angular structure of these toroidal currents to locate the toroidal currents (or equally the large poloidal field gradients). We show how the jet and disk wind are associated with poloidal currents that close in an electric circuit. Next, we determine the radial dependence of the disk-averaged plasma and field angular velocities, which have a simple behavior. This also elucidates how strongly the field in the disk couples to the rotation of the black hole. Finally, we compare the polar field in the GRMHD simulations to the non-rotating force-free field solution that emerges from the same power-law toroidal current as in the GRMHD simulations. The agreement found between these models demonstrates that the polar jet is accurately described by a force-free model and that rotation plays a negligible role in self-collimating the polar jet.

2.1 Review of Prior GRMHD Simulation Results

In this section, we review some relevant results from GRMHD simulations, where we start the discussion with the fiducial model of McKinney & Gammie (2004). Figure 1 shows the time-averaged field geometry for this fiducial model. The black hole has a spin of $a/M \approx 0.9375$, which is close to the equilibrium value of $a/M \approx 0.92$ (Gammie, Shapiro, & McKinney 2004).

The simulated accretion flow is dominated by a turbulent hydromagnetic dynamo driven by the MRI, which drives an increase in poloidal field strength whose saturated magnitude is insensitive to the initial poloidal field strength (De Villiers, Hawley, & Krolik 2003; McKinney & Gammie 2004). The dynamo eventually dies out unless modelled in three dimensions (Cowling 1934). For the two-dimensional simulations of McKinney & Gammie (2004), measurements are made only during the turbulent period and their results are consistent with three-dimensional simulations.

The polar region contains a well-ordered field whereas the disk appears to have a turbulent and disordered field (McKinney & Gammie 2004; Hirose et al. 2003; McKinney 2005a). Between these two regions is the corona which, in a time-averaged sense, contains only weak disordered fields. In the figure, the field that appears to come from the disk does not reach large distances, whereas the organized field in the polar region reaches large radii (McKinney 2006d).

It is important to note that the organized polar field geometry shown in Figure 1 is generic for simulations that are initialized with a well-organized poloidal field with or without net flux. Here an “organized” poloidal field means a field geometry with few poloidal polarity changes, but it could be initially contained entirely within the disk. The quasi-stationary structure of the accretion flow and flux threading the black hole or disk otherwise depends little on the initial field strength or geometry (McKinney & Gammie 2004; Hirose et al. 2004).

Cases when the black hole does not end up with an organized field include an initially purely toroidal field (no poloidal currents so no poloidal field) (De Villiers et al. 2005a) or highly tangled field (McKinney & Gammie, in prep.). With a purely toroidal field, no jet is produced. With a highly tangled field, the Poynting-dominated jet does not form since it is continuously contaminated with disk material, and so the corona and coronal wind dominate the entire polar region. Even when the disk is initially threaded by net flux, the quasi-stationary corona still only contains weak disordered field.

The lack of organized field in the corona or threading

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1 The codes used in the studies mentioned above preserve the solenoidal constraint ($\nabla \cdot \mathbf{B} = 0$) to machine accuracy, and so the codes preserve the net poloidal flux in the system. Hence, the net radial flux remains constant and the $\theta$ flux only changes by losing flux into the black hole or through the outer radial boundary. Codes that violate this property might generate large artificial net flux and then regions of organized flux could not be trusted.
the disk might be understandable as a result of the inability of flux to be simply advected radially inward if the ratio of viscosity to magnetic diffusivity (magnetic Prandtl number \( P_m \)) is such that \( P_m \sim 1 \). In thin turbulent-driven accretion flows, the magnetic Prandtl number is order unity, and so one does not expect to be able to simply accrete net flux from large radii. On the other hand, thick advection-dominated accretion flows are expected to be able to advect a nonnegligible flux. In three-dimensional models, the advection of flux may occur in isolated flux tubes and still build near the black hole (Spruit & Uzdensky 2005). The physics of how large-scale flux can be accreted through disks remains an open issue (see, e.g., Reynolds et al. 2006; Contopoulos et al. 2006).

So in the GRMHD simulations, how does the black hole become threaded with flux and how is the magnitude of the field maintained? Does large-scale flux simply advect inward despite the above-mentioned issues? In the simulations, the initial large-scale flux around the black hole is created as a result of the accretion of equatorial field loops that do not thread vertically through the disk, thus bypassing the problem of advecting large-scale flux threading the entire disk. The long-term flux is maintained by the 2D axisymmetric dynamo that drives single polarity poloidal loops (corresponding to single polarity toroidal currents) to twist in an axisymmetric sense, break through reconnection, and interchange around each other within the disk allowing the black hole to accrete an arbitrary poloidal polarity. Near the black hole the electromagnetic field dominates and this allows the magnetic flux threading the horizon to grow by the attraction of similarly-signed toroidal currents and the repulsion of oppositely-signed toroidal currents – a classical electrodynamical phenomena.

The magnetic field strength near the black hole saturates when material forces of the disk+corona can just support the magnetic pressure of the polar field. Thus one expects the magnitude of the magnetic pressure to be in near equipartition with the magnitude of the gas pressure fairly close to the horizon, which is the case (McKinney & Gammie 2004). This balance is qualitatively similar to the balance that is associated with the “magnetospheric radius” in accreting neutron star systems.

Also important is the accretion of large-scale polarity changes that can overwhelm any prior field built-up around the black hole. The relevance of accreting large-scale opposite polarizations probably depends on the astrophysical system (see, e.g., Narayan et al. 2003). Reconnection tends to convert the pre-existing flux into thermal energy and erase the organization of the polar field. When the reconnection time-scale is long, the field energy associated with the flux is difficult to remove once the flux is in place.

The magnetized jet is produced as the black hole spin and disk rotation create a large toroidal field whose gradient launches a significant fraction of accreted flux out along the poles. This process effectively leads to net flux threading the black hole since the flux that reaches large distances becomes causally disconnected from the disk (McKinney 2006c). The stronger the black hole spin, the stronger is this effect. In a stationary state, the organized polar field threads the black hole horizon, but not the inner disk (Hirose et al. 2004; McKinney 2005a).

In summary, the black hole naturally becomes threaded by organized field when the disk contains a field with few large-scale poloidal polarity changes. Early in the simulation, the field threading the black hole is advected through the equatorial region rather than being advected as large-scale flux threading the entire disk. The magnetic field strength near the black hole saturates when a balance is reached between the polar magnetic pressure and the disk+corona pressure. The disk and black hole rotations drive the flux near the black hole to large radii, and this effectively leads to a net flux threading the horizon. In a quasi-stationary state, large-scale field threads the hole but not the inner disk.

2.2 Power Output of Black Hole and Disk

In principle, a significant fraction of the electromagnetic power may come from a disk wind (Blandford & Znajek 1977) and may even dominate the black hole power output (Ghosh & Abramowicz 1997; Livio, Ogilvie, & Pringle 1999). However, the lack of an organized field threading the disk suggests that the electromagnetic output of the disk may be seriously compromised compared to those simple...
models that have treated the disk as just a boundary condition. To test these ideas, we compare the electromagnetic power output of the black hole and the disk in the fiducial \((a/M = 0.9375)\) model shown in Figure 1.

Let us define the time-averaged angular density of the electromagnetic power output as

\[
\langle \frac{dP}{d\theta} \rangle = 2\pi r^2 \langle -T^{(EM)}_r \rangle, \tag{1}
\]

where \(-T^{(EM)}_r\) is the radial electromagnetic flux written in a suitable coordinate basis (either Boyer-Lindquist or Kerr-Schild coordinates). The time-averaging is performed over approximately 8 orbital periods at \(r = 10r_g\) once the flow reaches a quasi-stationary, turbulent state. Figure 2 shows \(\langle dP/d\theta \rangle\) as a function of \(\theta\) at four radii: the horizon at \(r = r_h\), the inner-most stable circular orbit (ISCO) at \(r = r_{\text{ISCO}}\), \(r = 10r_g\), and \(r = 40r_g\). Notice how smooth and well-behaved the power is near the poles and how erratic and disordered it is away from the poles.

The total electromagnetic power at any radius \(r\) is given by

\[
P(r) = \int_0^\pi \frac{dP}{d\theta} \sin(\theta) d\theta. \tag{2}
\]

Table 1 lists values of \(P\) at the same four radii shown in Figure 2. For comparison with the force-free models discussed in McKinney 
& Narayan (2006), the power is normalized such that

\[
P(r) \rightarrow \frac{P(r)}{(B^r)^2 |_{\theta=0} / 4\pi r_g^2 c}. \tag{3}
\]

where the field is in Gaussian units and is measured on the horizon at the poles at the final time of the simulation.

At the horizon the electromagnetic power is that from the black hole with most of the net electromagnetic power coming from the polar regions. With increasing radius, electromagnetic power from the disk is added. However, this power is steadily converted into matter energy so that, by \(r = 40r_g\), most of it is no longer in electromagnetic form. This explains why the power at \(r = 40r_g\) in Table 1 is not much larger than the power at \(r = r_h\).

Table 1 also gives the Lorentz factor \(\Gamma\) of the jet far from the black hole and the half-opening angle \(\theta_j\) of the jet (defined as the angle at which \(dP/d\theta\) is maximum). These results are from McKinney (2006c).

In summary, we find that the electromagnetic power at the poles of the black hole remains undiminished out to the largest radius shown, whereas the electromagnetic power from the disk region decreases outward as it is efficiently converted into kinetic and thermal energy of the plasma. Simple estimates of the power output of the black hole and disk did not consider the conversion of electromagnetic power into thermal and material power in the corona and disk wind (Ghosh & Abramowicz 1997; Livio, Ogilvie, & Pringle 1998), and so they may have seriously overestimated the power from the disk. Thus, the black hole may dominate the electromagnetic power output of accretion systems. Also, when the disk is turbulent, the black hole remains the only possible clean \((b^2/(\rho c^2) \gg 1)\) source of electromagnetic power.

2.3 A Power-Law Radial Distribution of Toroidal Current

For stationary flows, the toroidal current directly leads to the poloidal field structure. Since the Poynting-dominated jet has a simple, stationary poloidal structure, there must be simple toroidal currents to support the field. However, given the turbulence in the disk and the efficient conversion of electromagnetic power from the disk into material power, one might assume that the electromagnetic properties of the disk would be complex and hard to relate in any simple way to the organized polar flux. This is certainly suggested by Figures 1 and 2.

As discussed in the introduction, the motivation for studying the toroidal current comes from simplified force-free models that include the accretion disk as an equatorial boundary condition, such as the paraboloidal BZ model (Blandford 
& Znajek 1977). In general, models of winds and jets typically specify the toroidal current in the disk (or stellar surface) and find the corresponding solution (see, e.g. Michel 1973; Okamoto 1974; 1978; Blandford 1974; Blandford 
& Payne 1982; Lovelace et al. 1984; Heyvaerts 
& Norman 1984; Nitta et al. 1991; Li et al. 1992; Appel 
& Camenzind 1992; 1993; Beskin 
& Pariev 1993; Contopoulos 1994; Contopoulos 
& Lovelace 1994; Contopoulos 1995a,b). We stress that for a stationary solution the toroidal current and poloidal field are just different languages for the same physics, but our motivation is to see if a turbulent GRMHD disk can be related to the vast array of models that describe the disk as simply a current sheet.

The relevant questions are: 1) Are such simple models applicable to thick, turbulent, magnetized accretion flows?
2) What radial dependence of the toroidal current is the “correct” choice? ; and 3) Does the angular-integrated toroidal current predict the shape of the organized polar field?

Let us consider the angular-integrated (over all angles) toroidal current that for a stationary flow corresponds to a line integral of the magnetic field around a closed poloidal loop. We integrate over all angles to capture currents that sometimes rise into the corona and to capture variations in the disk+corona thickness in time and as a function of radius. We consider the angular distribution of toroidal currents in the next section.

The current density (as given by equation (A11)) can be integrated to determine the net toroidal current enclosed within the volume between $r_0$ and $r$. This invariant current is

$$I_\phi = \int_{r_0}^{r} \int_{0}^{\pi} (J^\mu d\Sigma_\mu)$$

$$= \int_{r_0}^{r} \int_{0}^{\pi} \sqrt{-g} \left( J^\phi dr' d\theta' \right)$$

$$\equiv \int_{r_0}^{r} \left( \frac{dI_\phi}{dr'} \right),$$

(4)

where $d\Sigma_\mu \equiv \epsilon_{\mu\nu\alpha\beta} t^\nu d^\alpha d^\beta$, $t^\nu = \{1, 0, 0, 0\}$ is the time-like Killing vector, and $dI_\phi/dr$ is the toroidal current per unit radial distance. Note that the magnitude of the enclosed current is set by the value of $I_\phi(r_0)$, which is an arbitrary constant and set to be the enclosed current in the numerical grid of size $dr_0$ at $r_0$. Only the magnitude of the current density has physical significance. Notice that

$$\frac{dI_\phi}{dr} \equiv \int_{\theta=0}^{\pi} \sqrt{-g} J^\phi d\theta,$$

(5)

where $\sqrt{-g} \approx r^2 \sin \theta$ far from the black hole.

In the following we are particularly interested in models with a power-law dependence of the current density, i.e.,

$$\frac{dI_\phi}{dr} \propto \frac{1}{r^\nu}.$$  

(6)

We are motivated by the fact that the split-monopole and paraboloidal force-free models both have currents of this form, with $\nu = 0$ and 1, respectively. Example solutions are given in appendix [B].

The solid line in Figure 3 shows the enclosed toroidal current $I_\phi(r)$ at a very early time ($t = 500t_g$) of the fiducial simulation described in McKinney & Gammie (2004). At this time, the system has hardly deviated from the initial conditions and there is very little accretion taking place. Correspondingly, the current has a fairly complicated dependence, which primarily reflects the particular initial conditions chosen for this simulation.

Figure 4 shows the enclosed toroidal current at a later time ($t = 1000t_g$) when the accretion flow is highly turbulent and has reached a quasi-steady state. We see that the current distribution has changed dramatically from its initial distribution. More importantly, the current profile looks smooth and simple. Figure 5 shows the enclosed current as time-averaged over the period $t = 500t_g$ to $1500t_g$ (roughly a turbulent time scale at $r = 40r_g$). The result is very similar to that shown in Figure 3 except that the current looks even smoother because of the averaging.

In Figures 4-6 we show for comparison the enclosed currents corresponding to the split-monopole ($\nu = 0$, dotted line) and paraboloidal ($\nu = 1$, long-dashed line) models. It is clear that neither of the power-law models is a good representation of the enclosed current in the steady state GRMHD model. On the other hand, the short-dashed lines, which correspond to a power-law model with $\nu = 3/4$, describe the quasi-stationary GRMHD results surprisingly well. This particular model is associated with a radial dependence of the current density of the form $dI_\phi/dr \propto r^{-5/4}$.

A few interesting conclusions can be reached from these results: 1) The GRMHD model is nearly coincident with the $\nu = 3/4$ model; 2) The GRMHD model is not consistent with the split-monopole or paraboloidal models; 3) Despite the complicated nonlinear turbulence, the currents in the

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### Table 1. Electromagnetic power (per unit $(B^r)^2$ on horizon at poles), Lorentz factor, and half-opening angle.

| $a$     | $P[r_0]$ | $P[r_{ISCO}]$ | $P[r = 10r_g]$ | $P[r = 40r_g]$ | $P[r = 5 \times 10^3r_g]$ | $P[r = 5 \times 10^5r_g]$ | $\Gamma[r = 5 \times 10^3r_g]$ | $\theta_1[r = 5 \times 10^5r_g]$ |
|---------|----------|----------------|----------------|----------------|----------------------------|-----------------------------|-----------------------------|--------------------------------|
| 0.9375  | 0.324    | 0.407          | 0.595          | 0.374          | 10                         | 5°                          | 5°                          | 5°                             |

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![Figure 3](image-url)
The simple $\nu = 3/4$ current distribution described above is entirely consistent with the smooth field distribution seen in the evacuated polar region in Figure 1. It is also consistent with the fact that the Poynting-dominated jet is nearly stationary and nearly resembles BZ’s paraboloidal solution [McKinney & Gammie 2004; McKinney 2006c]. Furthermore, the fact that the best fit is obtained for $\nu = 3/4$ rather than $\nu = 1$ explains why the large-scale field lines in the jet are nearly paraboloidal but somewhat less collimated in the GRMHD simulations [McKinney 2005a, 2006c]. Section 2.7 discusses this point further.

The above results are roughly independent of the initial field geometry assumed in the GRMHD simulations. McKinney & Gammie (2004) considered different initial conditions such as multiple magnetic loops in the initial torus, a net vertical field, and loops of alternating poloidal direction. Once these simulations have reached a quasi-steady state, each closely follows a power-law toroidal current density with $\nu = 3/4$. This is despite the fact that the particular angular structure of the toroidal current varies considerably. For example, for thin disks with alternating polarity of multiple field loops, there is no strong poloidal field in the funnel, but the toroidal current still maintains the $\nu = 3/4$ power-law dependence.

We also investigated the dependence of the toroidal current distribution on the black hole spin. Once again we find that, for all $a/M$ ranging from $-0.999$ to $+0.999$, the toroidal current settles down to a $\nu = 3/4$ power-law distribution once the flow reaches quasi-steady state. This is despite significant changes in the enclosed poloidal current as shown in Figure 4 since for slowly spinning black holes there is negligible poloidal current flowing in the funnel region.

How robust are these results to changes in the mass distribution and disk thickness? The fiducial model studied has an initial mass distribution of a hydrostatic equilibrium torus with a constant specific angular momentum such that $H/R \sim 0.3$. An alternative initial mass distribution we tried has a quasi-equilibrium Keplerian disk with a Gaussian vertical mass distribution with $H/R \sim 0.3$. Once the turbulence reaches the nonlinear phase, this alternative model also has a toroidal current closely matching the $\nu = 3/4$ profile.

We have also studied a thin (i.e. small $H/R$) Keplerian disk with an initial Gaussian vertical distribution with an ad hoc cooling model to keep $H/R \sim 0.05$, and we find that this model also obeys the $\nu = 3/4$ toroidal current distribution. The only qualitative change is that there is a larger variance around the $\nu = 3/4$ solution. However, the solution is still quite different from the monopole or paraboloidal solutions.

2.4 Angular Structure of Toroidal Current

Of course, the angular-integrated toroidal current does not reveal the angular location of the currents. By integrating equation (4) over a finite radial range for each $\theta$, one can identify structures in the accretion flow with the toroidal currents. While the instantaneous value of $I_\phi(\theta)$ is very oscillatory and difficult to interpret, the running integral ($\int_0^\theta I_\phi(\theta') d\theta'$) is relatively smooth so we focus on this quantity.
We consider two radial sections of the accretion flow. One radial range considered is \( r = r_+ \) to \( r = 3r_+ \), while the other radial range considered is \( r = r_+ \) to \( r = 10r_+ \). Figure 6 shows the radially integrated, angular running integral of \( I_\phi(\theta) \) for the fiducial simulation in [McKinney & Gamman (2004)]. For this GRMHD simulation, the funnel jet extends from the poles inward by about 1/2 to 1 radian depending upon the radius. Just beyond this range at the corona-funnel interface, the toroidal current changes sign and nearly annihilates itself in an integral sense. This toroidal current is associated with the polar jet. In the corona there is no organized field and the magnetic field is in equipartition with the gas pressure [McKinney & Gamman (2004)]. We find that across this coronal-funnel interface there is force balance between the organized field in the polar region and the equipartition, disorganized, weakly magnetized, hot plasma in the coronal region. If self-collimation is weak within the polar jet, then the corona is responsible for confining/collimating the polar jet. The importance of rotation of the polar field is tested in later sections and in McKinney & Narayan (2006).

The next jump in the toroidal current is at the coronadisk interface. Within the disk the toroidal current oscillates with a slightly non-zero toroidal current. For the integrated radial range of \( r = r_+ \) to \( r = 10r_+ \), the angular integrated distribution of the toroidal current is roughly given by

\[
I_\phi(\theta) d\theta' \propto (1 - \cos(\theta)),
\]

although there is significant sub-structure. This fit is shown in Figure 6.

While changes in the initial field geometry do not affect the radial distribution of toroidal currents with \( dI_\phi/dr \propto r^{-5/4} \), the angular location of the toroidal currents (and so the location of poloidal field gradients) depends sensitively on the initial field geometry. Simulations that start with a highly disorganized field with many loop of different polarity lead to no organized flux in the poles and then the toroidal currents are primarily located inside the disk-corona interface.

2.5 Poloidal Currents and Current Closure

For stationary flows, the toroidal field is equally described by the poloidal current, and these are related to the power output of the jet. At large radii, the radial electromagnetic power output is given by \( \dot{E} \propto \Omega_F B_0 B^2 \), where \( \Omega_F \) is the field rotation frequency, \( B^2 \) is the radial field strength, and \( B_0 \) is the “polar enclosed poloidal current” (\( B_0 \approx RB^2 \)). In a stationary, axisymmetric flow, \( \Omega_F \) and \( B_0 \) are constant along field lines labelled by the vector potential \( A_\phi \) (also referred to as the flux function \( \Psi \) or stream function). Thus, the poloidal current is an indicator of the electromagnetic power output per unit poloidal field strength, and the closure of this poloidal current is best considered in a plot of \( B_0 \) vs. \( A_\phi \).

Figure 6 shows the time-averaged value of \( B_0 \) vs. the time-averaged value of \( A_\phi \) at \( r = 10r_+ \). The values within \( A_\phi < 0.2 \) correspond to the region inside the funnel that contains the Poynting-dominated jet. The value of \( A_\phi \approx 0.2 \) corresponds to the transition between the force-free funnel and the corona. The value of \( A_\phi \approx 0.21 \) is at the transition between the corona and disk, where the increase in the poloidal current is largest. Within the disk the enclosed current remains constant until the equator is reached and there is a strong return current at \( A_\phi \approx 0.26 \). Notice that both hemispheres have been shown to demonstrate that despite the time-averaging, the equatorial region is still asymmetric while the jet region remains symmetric. Such a plot can be used to compare GRMHD accretion systems to pulsar systems (see, e.g., Contopoulos et al. 1999, McKinney 2006) and similar black hole force-free systems (McKinney & Narayan 2006). Despite the turbulence, the dependence of \( B_0 \) on \( A_\phi \) remain surprisingly simple. Simulations that start with a highly disorganized field lead to a more complicated poloidal electric circuit that is distributed over all angles.

2.6 Turbulence Leads to Simple Field Angular Velocity

For stationary flows, the angular velocity of the field lines (\( \Omega_F \)) is an important quantity that determines the toroidal field geometry and determines the electromagnetic power output of a jet or wind. Also, a comparison of \( \Omega_F \) and the fluid angular velocity \( \Omega \) can reveal how well-coupled the matter is to the field. In principle, large scale fields can develop in the disk and the plasma might slip arbitrarily along the field lines, leading to \( \Omega_F \neq \Omega \). However, if there is strong turbulence, it would lead to a significant random component.
to the field and the plasma would be unable to slip as much. Simple models of the accretion disk assume an arbitrary value of \( \Omega_F \) in the disk, while here we establish which radial dependence of \( \Omega_F \) is motivated by GRMHD simulations of turbulent accretion flows near rotating black holes.

We first investigate the behavior of \( \Omega_F \) and \( \Omega \) as a function of radius in the fiducial model of McKinney & Gammie (2004). One needs to choose some method to space- and time-average \( \Omega_F \) in order to obtain its radial distribution. A poor choice would be to directly average \( \Omega_F \equiv E_\theta/B' \) itself because it is highly oscillatory and is undefined at positions where the radial component of the field \( B' \) momentarily vanishes. Thus, we consider the ratio of space- and volume-averaged quantities to obtain a mean angular velocity \( \langle \Omega_F \rangle \equiv \langle E_\theta \rangle/\langle B' \rangle \). For each radial shell this quantity is volume-averaged over a disk scale-height and time-averaged over approximately 8 orbital periods at \( r = 10r_s \) once the flow has reached a quasi-stationary, turbulent state. An alternative time-averaging is performed using the absolute value of each composite quantity (\( |E_\theta| \) and \( |B'| \)) giving \( \langle |\Omega_F| \rangle \equiv \langle |E_\theta| \rangle/\langle |B'| \rangle \).

Figure 8 shows the radial profiles of both forms of \( \Omega_F \) normalized by the local Keplerian angular velocity \( \Omega_K \). Both methods of averaging \( \Omega_F \) lead to similar results. The plot also shows the angular velocity of the plasma per unit Keplerian (\( \Omega/\Omega_K \)) and the angular velocity of the zero angular momentum observer (ZAMO) per unit Keplerian (\( \Omega_{\text{ZAMO}}/\Omega_K \)). We use Boyer-Lindquist coordinates, with \( \Omega_K = 1/(r^{3/2}+a) \). Note that \( \Omega_{\text{ZAMO}} = \Omega_H = a/(2r_s) \) on the horizon.

As Figure 8 shows, \( \Omega \approx \Omega_F \approx \Omega_K \) over much of the disk for radii \( r \gtrsim 2r_s \). At these radii, there are no large-scale fields for the plasma to slip along, so that the plasma and field are locked together in a turbulent mixture. However, \( \Omega_F \) becomes somewhat sub-Keplerian near the horizon as the field lines become more ordered. Thus, while the plasma is forced to corotate with the black hole at the horizon, the field lines rotate slower with \( \Omega_F \gtrsim \Omega_H/2 \). As discussed in McKinney & Gammie (2004), this behavior for \( \Omega_F \) is consistent with the Gammie (1999) inflow model of the plunging region. For \( r \gtrsim 20r_s \), the angular velocities deviate from a simple behavior because the solution still depends on the initial conditions.

As in the case of the toroidal current distribution, the results described here for the angular velocity of the field lines and the plasma are quite robust and are independent of the assumed initial field geometry, mass distribution, or disk thickness. However, the black hole spin has a dramatic qualitative effect.

For spins from \( a = -0.999 \) to \( a = 0.999 \), all the models have the plasma locked to the field at large radii. However, close to the black hole there is a qualitative change in the results around \( a/M \sim 0.4 \). For \( a/M \lesssim 0.4 \) we find that \( \Omega_F \approx \Omega_K \approx \Omega_H \) even at the horizon because the disk dominates over the black hole. However, for \( a/M \gtrsim 0.4 \) we find that \( \Omega_F \approx \Omega_H/2 \) near the horizon as the black hole dominates over the disk. This transition at \( a/M \approx 0.4 \) is consistent with the fact that there is a qualitative change in the energy output of a black hole-disk-jet system at \( a/M \approx 0.36 \).
when the Keplerian angular velocity of gas at the ISCO is equal to the angular velocity of the black hole. It is also consistent with the fact that there is negligible (or negative) electromagnetic energy extracted from the black hole for spins \( a/M \lesssim 0.4 \) for thick disks (McKinney & Gammie 2004; McKinney 2005a). Since \( dP/d\theta \propto \Omega F (\Omega_F - \Omega_H) \) on the horizon, we expect \( dP/d\theta \sim 0 \) since \( \Omega_F \sim \Omega_H \) when \( a/M \lesssim 0.4 \).

### 2.7 Comparison between GRMHD field and \( \nu = 3/4 \) force-free field

We have shown above that the toroidal current \( I_\phi \) and the field angular velocity \( \Omega_F \) in GRMHD simulations are well-behaved and so easy to model. One expects the current distribution to be consistent with the nearly force-free funnel field geometry if the funnel field is setup directly by those currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents. Here we test the robustness of the association between the power-law index of the angle-integrated toroidal currents.

We consider a simplified problem in which neither the black hole nor the disk rotates. We replace the disk with a current sheet at the equatorial plane and assume that the rest of the volume is in force-free equilibrium. In Appendix B, we describe how to obtain force-free solutions in Schwarzschild and flat spacetimes for an arbitrary current sheet at the equator with no rotation.

Figure 9 shows the field geometry from a time-dependent GRMHD numerical model overlaid with the force-free field corresponding to \( \nu = 3/4 \). We see that there is a reasonably good agreement between the two models in the funnel region, where the GRMHD solution is Poynting-dominated and force-free. The small differences between the models could be due to (i) residual weak time-dependence in the GRMHD solution, and (ii) additional collimation in the inner regions of the jet due to either rotation (which is ignored in the force-free solution plotted here) or coronal pressure (see McKinney & Narayan 2006). Also, at very small angles the jet is more paraboloidal. However, the differences are small. In fact, the agreement between the GRMHD numerical models and the \( \nu = 3/4 \) force-free model is found to be good out as far as \( r \sim 10^3 r_g \) in the large scale simulations of McKinney (2006). Beyond this radius the inertia of the matter becomes nonnegligible in the GRMHD model and the force-free approximation is no longer applicable.

Although the overlay comparison shown in Figure 9 is impressive, we caution that the force-free field shown here is for the non-rotating case whereas the GRMHD solution corresponds to both a spinning black hole and a spinning disk. We discuss the effect of rotation on force-free solutions in McKinney & Narayan (2006).

### 3 Electromagnetic Properties of the Disk

In McKinney (2006), we presented a detailed study of the electromagnetic properties of the Poynting-dominated jet by providing both a qualitative description of the jet and many quantitative measures of the jet. For example, radial scalings within the jet helped isolate the mechanism of jet formation and jet stability.

In this section we first determine the radial scalings of the magnetic field components in the disk+corona. We connect these scalings to quasi-analytic models of the accretion flow (McKee & Gammie 1994) that describe the disk inside the ISCO. Given the simplicity of the time-averaged fields in the disk, we suggest that such models might be extendable to include the disk beyond the ISCO. Next, we compare the GRMHD simulations to simple Newtonian disk models that have a disk wind (such as that by Blandford & Payne 1982). While such models have a similar magnetic field scaling in the disk, the assumptions they make do not hold in the GRMHD simulations. Finally, we discuss the role of magnetic stresses near the black hole and compute the effective magnetic \( \alpha \) viscosity parameter, which near the black hole rises to order unity.

#### 3.1 Radial Dependence of Disk Magnetic Field

In this section, we focus on the radial dependence of the electromagnetic properties in the bulk of the disk+corona. Thus, this study excludes the direct properties within the jet. However, the mechanisms for disk-jet coupling may be better constrained by knowing the radial scalings within the disk+corona.

We consider the space-time average of the absolute magnitude of the field strengths. The volume-averaged value is found for each radial shell over the disk+corona, which includes only the flow that has \( b^2/\rho_0 \lesssim 1 \) (McKinney & Gammie 2004). Thus the highly magnetized
jet is explicitly excluded. For any quantity $B$ within the disk+corona we compute

$$\bar{B} = \frac{\int B \sqrt{g} d\theta d\phi}{\int \sqrt{g} d\theta d\phi} = \frac{\int B \sin \theta d\theta}{\sin \theta d\theta},$$

which clearly preserves the radial dependence of the quantity in question.

As before, the time-average is computed over the turbulent period of accretion. The field strengths are given in Gaussian units and normalized by the rest-mass density within the disk as done in McKinney (2006d), such that given an estimate of the density of the disk near the black hole or the mass accretion rate near the black hole one can convert to physical units. Figure 10 shows these field strengths as a function of radius; where for $r \gtrsim 10r_g$ the initial conditions still contribute to the solution and so the field there is not included in the fitting procedure.

For models with any black hole spin, we find that the radial dependence of the comoving field and toroidal strengths roughly follow power-law behaviors close to the black hole. The solution at $r = 0$ is in Gaussian units and normalized by the rest-mass density of the disk near the black hole or the mass accretion rate near the black hole. The field strengths are given in the core of the disk. Equations (9-11) show that the toroidal field dominates the other field components within the disk.

This type of solution is similar to the model proposed by Gammie (1999) who described a solution such that within the plunging region the dimensionless radial magnetic flux

$$\tilde{F}_{\theta\phi} \equiv \frac{F_{\theta\phi}}{\sqrt{-g} F_M} = \frac{r^2 B_r}{\sqrt{-2\pi r^2 \rho_0 u'}} \left(\frac{3}{2}\right),$$

is a constant function of radius, where we have temporarily reintroduced $GM$ and $c$, $B_r$ is in Gaussian units, and $u'$ is the radial 4-velocity. Here we report that this parameter is nearly constant throughout the entire accretion flow out to $r \sim 40r_g$ with a value of

$$\tilde{F}_{\theta\phi} \approx 1.09,$$

which is the same as found by McKinney & Gammie (2004) for the plunging region in GRMHD simulations. The constancy of this parameter is consistent with a disk containing a radial field that is nearly monopolar per unit mass flux, as envisioned in the Gammie inflow model.

This behavior of the accretion flow is consistent with the fact that the radial dependence of $b^\phi$ within the GRMHD plunging region follows the Gammie inflow solution (McKinney & Gammie 2004). As described in McKinney & Gammie (2004), the thin disk Gammie solution does well to model the GRMHD flow apart from the lack of modelling the effects of pressure that lead to a non-zero radial velocity across the ISCO and the lack of a feature in the flow near the ISCO. One primary difference is that the Gammie (1999) model assumes $\Omega_F$ is also constant along the radial field lines, while in the turbulent disk $\Omega_F \sim \Omega_K$ in the outer disk.

In summary, despite the obvious turbulence in the bulk of the disk and the weak-disordered field in the corona, the time-averaged magnetic field averaged over the disk+corona has simple radial scalings with a strong similarity to the model of Gammie (1999). That model might be extendable to apply to the entire accretion flow within a disk scale height.

### 3.2 Comparison to the BP and ADAF Models

The $r^{-5/4}$ power-law scaling for the electromagnetic field is of particular interest because it occurs rather naturally in certain self-similar models in the literature. Here we check whether the assumptions made by these models applies to the GRMHD simulations.

A popular model for disk winds is the Blandford & Payne (1982) (BP) self-similar MHD wind...
model in which they assumed that the Alfvén speed at the equatorial plane scaled as the local Keplerian speed. Coupling this with the additional assumption that the density scales as \( \rho \sim r^{-3/2} \), they found that the magnetic field should scale as \( |b| \propto r^{-5/4} \), which is consistent with the GRMHD flow as given by equation (9). The BP model also assumes that the disk is threaded by a large-scale organized field. Clearly this latter assumption is broken, as shown in figure 11 and as discussed in Hirose et al. (2004); McKinney & Gammie (2004). This is also consistent with the fact that within the disk the ingoing fast magnetosonic surface is located at \( r \approx 1.8r_g \), located between the ISCO and the marginally bound orbit (MBO).

One important feature of black hole space-times is the ISCO, which for the simplest viscous thin disks demarcates where the fluid plunges into the black hole. For our simulations of moderately thick disks with \( H/R \) \( \sim 0.05 \), we find that near the black hole within \( r \lesssim 3r_g \) the fluid plums into the black hole with a power-law radial 4-velocity,

\[
u' \approx -0.3 \left( \frac{r}{r_+} \right)^{-2},\]

where \( u' \) is written in Boyer-Lindquist coordinates. For the fiducial model with \( H/R \approx 0.26 \) and \( a/M = 0.9375 \), this power-law plunging starts at approximately

\[
R_{\text{plunge}} \approx R_{\text{ISCO}} \left( 1 + \frac{H}{R} \right)^{0.5}.
\]

This scaling is also consistent with the other \( H/R \approx 0.05 \) model we studied and of course is trivially consistent with thin disk theory for which a thin Keplerian disk has the ISCO located at \( R_{\text{ISCO}} = 2.044r_g \) for \( a/M = 0.9375 \). Further studies can determine how general this fit is. However, notice that the fluid begins to plunge toward the black hole at around \( r \approx 6r_g \), significantly further out than the ISCO. Thus there appears to be a factor of 2 ambiguity in identifying the location where material plunges into the black hole. This GRMHD result has some not-well-defined bearing on recent measurements of black hole spin that used the observed spectra to estimate \( R_{\text{ISCO}} \) and so estimate \( a/M \) (Shafee et al. 2006; McClintock et al. 2006). The ingoing slow magnetosonic and Alfvén surfaces are at \( r \approx 5r_g \) just inside where the fluid begins to plunge into the black hole.

Within a scale-height and for all radii throughout the disk the mass accretion rate obeys

\[
\dot{M}[r] \approx \text{Const},\]

which indicates that for this particular model with \( a/M = 0.9375 \) that mass-loss is not significant to the properties of the bulk flow. For more rapidly rotating black holes the mass-loss can become significant (Hawley & Krolik 2006), but the radial dependence in the disk appears non-trivial. Since \( \dot{M}[r] \propto r^2 \rho u' \), then one expects a roughly constant proper density and indeed

\[
\frac{\rho_0}{\rho_{0,\text{disk}}} \approx 1 \left( \frac{r}{r_+} \right)^{0.5}\]

and the Keplerian speed is given by

\[
v_K \approx \frac{r}{r^{3/2} + a}.
\]

Let us focus on the region interior to \( r \approx 10r_g \), which has reached a quasi-steady-state (the region farther out is still sensitive to the initial conditions). Clearly, as Figure 11 shows, \( v_0 \) is not simply related to either \( v_K \) or \( c_s \). Thus, neither the BP nor ADAF assumptions are satisfied. This is perhaps not surprising since self-similar models assume a Newtonian gravity and so only apply far from the horizon. Figure 11 shows that the magnetic field dominates over the matter near the horizon, consistent with the results in De Villiers, Hawley, & Krolik (2003) and McKinney & Gammie (2004). This is also consistent with the fact that within the disk the ingoing fast magnetosonic surface is located at \( r \approx 1.8r_g \), located between the ISCO and the marginally bound orbit (MBO).

**Figure 11.** Radial profiles of the plasma rotational speed (\( \nu^\phi \), solid line), the magnetosonic speed (\( c_{ms} \), dotted line), the Alfvén speed (\( v_A \), long-dashed line), and the radial 4-velocity (\( u' \), dot-short dash line) for a GRMHD accretion disk simulation with \( a/M = 0.9375 \). All speeds are plotted in units of the Keplerian speed (\( v_K \)). The angular and Keplerian speeds are in Boyer-Lindquist coordinates, while the others are given in comoving coordinates. The vertical lines indicate the radii of the ISCO and MBO. Note that the equipartition assumption, viz., \( v_0 \sim c_s \sim v_K \), is strongly violated. The solution at \( r \gtrsim 20r_g \) is still dependent on initial conditions and is excluded from the analysis.
instead of $\rho_0 \propto r^{-3/2}$ as one would expect in the BP or ADAF model. Notice that this defines $\rho_{0,\text{disk}}$.

The growth in the Alfvén speed is consistent with the fact that within $r \lesssim 10r_g$ the rest-mass density is nearly constant and the gas pressure is small and quickly diminishes at larger radii, following

$$\frac{p}{\rho_{0,\text{disk}}c^2} \approx 0.01 \left(\frac{r}{r_+}\right)^{-2},$$

(20)

which implies that the enthalpy is small compared to the rest-mass density. An equally good fit has $p \propto r^{-1.5}$. The comoving field energy is small compared to the rest-mass density, so that

$$\langle \nu_a \rangle \propto \langle |b| \rangle \propto r^{-5/4},$$

(21)

for $r \lesssim 10r_g$. Interestingly, the de-correlated average

$$\langle \langle \nu_a \rangle \rangle \equiv \frac{\langle |b| \rangle}{\sqrt{\langle \rho_0 \rangle + \langle u \rangle + \langle p \rangle + \langle b^2 \rangle}}$$

(22)

follows this dependence even more strictly than the direct space-time average of $\nu_a$. This shows that space-time correlations are mild between the various sources of pressure.

We thus conclude that it is purely an accident that the field and the current in the GRMHD solutions scale exactly as in the BP and ADAF models. Clearly the fact that $v_a \propto r^{-5/4}$ and $\nu_K \propto r^{-1/2}$ means that $v_a \sim v_K$ is not held and so the BP/ADAF assumptions are violated in this region close to the black hole. This is found to be true for many models of the disk and a large range of black hole spins. It remains an open question as to what mechanism enforces the $\nu = 3/4$ toroidal current density associated with the polar field.

3.3 Magnetic $\alpha$ viscosity parameter

Accretion flows without magnetic fields are often assumed to be free of dissipation and torques within the ISCO. However, it has long been understood that magnetic fields can drastically violate this assumption through the action of extended fields that generate torques across the ISCO even without turbulent dissipation. While equation (17) demonstrates that the gas pressure scale-height is related the effective location of the ISCO, magnetic stresses may play some role in setting this scaling. Magnetic stresses may also play independent roles not at all modelled by a scale-height-based argument.

An important parameter in understanding the presence of stresses within the ISCO is the magnitude and radial scaling of the $\alpha$ viscosity parameter. Early estimates of the BZ power output depended upon the argument that $\alpha$ was as determined in local shearing box calculations (Ghosh & Abramowicz 1995). They also assumed that the field threading the hole was determined by a sub-equipartition between the field and gas pressure in the disk (Ghosh & Abramowicz 1997; Livio, Ogilvie, & Pringle 1994). Finally, they assumed that the electromagnetic power from the disk is not converted into material and thermal forms. Based upon these assumptions, they concluded that BZ power output was negligible and certainly weaker than the electromagnetic power of the disk. Here we check whether $\alpha$ behaves the same near the black hole as in local non-relativistic simulations.

Local shearing box simulations of a small section of the accretion flow have suggested that $v_a \approx 2\sqrt{c_{\text{ac}}}$, where $\alpha \sim 0.01 - 0.1$ is the usual dimensionless viscosity in standard accretion disk models (Hawley et al. 1993). Global pseudo-Newtonian simulations have shown that $\alpha$ rises sharply inside the ISCO (Hawley & Krolik 2001). In our GRMHD simulations, the radial transport of angular momentum can be investigated by measuring the effective magnetic $\alpha$

$$\alpha_{\text{mag}} \approx \left(\frac{-b^\mu b_{\mu} \phi^\nu}{P}\right),$$

(23)

where $\phi^\nu = \{0, 0, 0, 1\}$ is the $\phi$ Killing vector associated with the axisymmetry of the system, $b^\nu$ is the comoving radial field strength, and $b_{\mu}$ is the covariant comoving 4-field. This form of $\alpha_{\text{mag}}$ is independent of the coordinate system for axisymmetric space-times. We choose $P$ to be either the total pressure or just the magnetic pressure $b^2/2$. From the high-resolution fiducial model studied in McKinney & Gammie (2004), we compute the radial dependence of $\alpha_{\text{mag}}$, integrated over a disk scale height ($H/R \approx 0.26$) and over the turbulent period of accretion. The angular integration is confined to the disk, so that the result is not influenced by the corona above the disk. The result is shown in Figure 12. We see that $\alpha_{\text{mag}}$ rises toward the horizon, which is consistent with the non-relativistic results of Hawley & Krolik (2001). The large $\alpha_{\text{mag}}$ is associated with a flux of angular momentum from inside the ISCO (McKinney & Gammie 2004).
The plot also shows the uncorrelated time-average of
\[ \alpha_{\text{mag, uncorrelated}} = -\left( \frac{(b^\prime)(b_\phi)}{(b^2/2)} \right), \]
where the brackets denote a time-average. This quantity is nearly constant within the entire flow, showing that temporal correlations between the quantities are significant.

Since the value of \( \alpha \) rises near the black hole, one might expect that the increased stress would be associated with an increased energy flux from the black hole and an associated increased luminosity from the disk near the ISCO compared to standard thin disk theory (Krolik 1999; Gammie 1999). However, for these GRMHD models, which correspond to radiatively inefficient accretion flows (RIAFs), the energy per baryon accreted is similar to that for a thin disk with a fixed \( \alpha \) and no torques inside the ISCO (McKinney & Gammie 2004).

The fact that \( \alpha \sim 1 \) near the black hole violates the assumptions of Ghosh & Abramowicz (1997); Livio, Ogilvie, & Pringle (1999). Also, black hole spin plays a nontrivial role in enhancing the magnetic field threading the black hole horizon (McKinney 2005a) and as discussed in section 4.2, the field and gas pressure can be near equipartition near the black hole. Finally, as discussed in section 4.2, a significant amount of the disk’s electromagnetic power output is lost to material and thermal power. Thus, the BZ mechanism may still be responsible for the most of the electromagnetic power output from accretion systems and may still account for the most powerful radio sources. These effects should be considered when comparing the BZ and disk electromagnetic powers.

4 LIMITATIONS

The primary limitation of the present study is that the numerical models are axisymmetric. A 3D model may show that \( \nu = 3/4 \) is not generally chosen by the system. Comparisons between 2D and 3D GRMHD simulations have shown reasonable qualitative and quantitative consistency (De Villiers, Hawley, & Krolik 2003; McKinney & Gammie 2004). In particular, both show that the flow partitions into a disk, corona, disk wind, and magnetized funnel region. Both give similar accretion rates of energy and angular momentum per unit baryon. The primarily problem with axisymmetric simulations is that turbulence decays after some length of time. We avoid this problem by only making measurements during the turbulent period of accretion, so our results are unlikely to significantly change in 3D simulations.

In the regime where radiation determines the energy balance in the disk, and so determines the disk thickness, radiative cooling should make little difference to these results since models with disk thicknesses between \( H/R \sim 0.05 \) to \( H/R \sim 0.3 \) show the same behavior. However, radiative effects could also be dynamically important, such as through the photon bubble instability (Arons 1992; Gammie 1998), and it is unknown how this would change our results.

The particular field geometry accreted can significantly change the mass-loading of the Poynting-dominated jet. Accreting a more random field leads to a larger coronal region that can extend all the way to the poles around the black hole, and no Poynting-dominated jet would form unless the black hole spin were sufficiently large that the toroidal magnetic pressure exceeded the ram pressure of the coronal material. Preliminary models of Keplerian disks, with cooling to keep the thickness fixed, suggest that the accretion of an irregular field leads to no Poynting-jet formation for at least \( a/M \lesssim 0.94 \) (McKinney & Gammie, in prep.). Despite the absence of the Poynting-jet, the toroidal current within the disk is still well-modelled by the \( \nu = 3/4 \) solution. In the tangled field case, the corona simply fills the region that would otherwise have been occupied by the Poynting-dominated jet.

All the models studied have disks that only extend out to \( r \sim 40r_g \) with a similar circularization radius. The results of this paper, such as the radial scalings, only directly apply within \( r \lesssim 10r_g \) where general relativistic effects of the space-time play an important role. The results found here may not hold far from the black hole. For example, \( \alpha_{\text{mag}} \approx \text{Const.} \sim 0.1 \) is expected far from the black hole. Also, while a more extended disk is not expected to affect the flow properties near the black hole, an extended disk may affect the Poynting-dominated jet at large radii. Future studies should investigate a more extended disk to model systems that have a large circularization radius.

5 DISCUSSION AND CONCLUSIONS

We have shown in this paper that GRMHD numerical simulations of magnetized accretion flows lead to a simple angular-integrated toroidal current density of the form \( dI_d/dr \sim r^{-5/4} \), even though the accretion disk, corona, and outflowing wind are highly turbulent and chaotic. We showed that the poloidal field distribution in the jet is consistent with the force-free field solution for an \( r^{-5/4} \) current distribution in a non-rotating equatorial disk. The fact that we obtained good agreement even with a highly simplified non-rotating model suggests that self-collimation by hoop-stresses may not be required within the jet. If hoop-stresses are ineffective, then force-balance requires the strongly magnetized jet to be confined by the weakly magnetized corona and wind (see Lynden-Bell 2006 for a discussion of pressure confinement). By measuring forces across the corona-funnel interface, we found that force balance primarily requires the coronal pressure indicating that the corona is required to confine/collapse the polar jet. Future work should study by what mechanism the confining coronal pressure generates a force-free field consistent with the \( r^{-5/4} \) toroidal current density.

Because of the strong turbulence, at large radii the plasma and the magnetic field in the disk are locked together so that both rotate around the black hole at roughly the local Keplerian angular frequency \( (\Omega_r) \). For \( r \lesssim 3r_g \), the behavior of the plasma and field qualitatively changes and depends strongly on whether the black hole spin is larger or smaller than \( a/M \sim 0.4 \). Near the horizon, the field angular velocity asymptotes to \( \Omega_r \approx \Omega_H/2 \) in the case of rapidly spinning holes with \( a/M \gtrsim 0.4 \), where \( \Omega_H \) is the angular frequency of the hole, and to \( \Omega_r \sim \Omega_H \) for \( a/M \lesssim 0.4 \).

Although the disk is turbulent, the average magnetic field strength in the disk varies smoothly as \( |b| \sim r^{-5/4} \), and the individual components of the field \( (B^\prime, B^\phi, \text{and } B^z) \) also have simple power-law scalings. The scaling of \( |b| \) is similar...
to that assumed in the Blandford & Payne (1982) and ADAF (Narayan & Yi 1995b) models, but we showed that the similarity is purely accidental. Those models assume Newtonian gravity and that gas and magnetic pressures scale similarly with radius. Gravity near the black hole is obviously not Newtonian, and in the simulations the magnetic pressure rises more rapidly than the gas pressure toward the black hole. Thus, the physical reason for the particular scaling of $r$ rises more rapidly than the gas pressure toward the black hole. The increased magnetic stress leads to enhanced angular momentum transport, but the energy per baryon accreted remains similar to that from thin disk theory (see also Gammie & Mckinney 2004). Prior studies that estimated the BZ power of black holes and the disk power output assumed an $\alpha$ based upon local shearing-box models that obtained $\alpha \sim 0.1$ (Ghosh & Abramowicz 1997). Livio, Ogilvie, & Pringle 1999). These studies also assumed that the field strength near the black hole would be set by sub-equipartition arguments, but we find that the field near the black hole is a non-trivial function of black hole spin (McKinney 2005a). They also assumed that the disk was threaded by an organized field, but we find that turbulence disorganizes any ordered field threading the disk. 

The basic picture of the disk-jet coupling that emerges is that turbulence driven by the MRI leads to toroidal currents that in a force-free state would drive an organized poloidal field from the disk. However, turbulence dominates the accretion flow and large-scale modes lead to angular momentum transport that continuously advects equatorially field through the disk into the hole and transports field at higher latitudes to larger radii. This keeps the corona free of ordered fields. The disk-corona interface is a site for dissipation of disk field through reconnection. The energy from the dissipation drives a hot coronal wind off the disk. The coronal wind is thus quite different from MHD models of disk winds (e.g., Blandford & Payne 1982), and is more akin to a thermally driven wind. The pressure of the corona confines the Poynting-dominated jet, since without the corona the field gradients at the funnel-corona interface could not be supported.

The geometry of the accreted magnetic field plays a crucial role in controlling the production of the Poynting-dominated jet. Quite similar jets are obtained in GRMHD simulations that start from a variety of different initial configurations of the magnetic field: uniform vertical field, poloidal loop of magnetic field in the disk, multiple loops of alternating poloidal direction. Even though the latter two models have zero net vertical field, nevertheless, they end up with a net vertical flux through the black hole and jet. The mechanism by which this happens is described in the discussion of Model B in Igumenshchev et al. (2003) and see Narayan et al. (2003). However, for models initialized with a mostly disorganized field, the Poynting-dominated jet is weaker or absent because mass continuously loads the polar region. Despite the lack of a simple Poynting-dominated jet in such models, there is still a disk wind containing a disorganized field and the currents and field strengths within the disk still follow the same power-law dependencies.

Although we obtain good agreement between the magnetic field geometry in a simple force-free current sheet model and the poloidal magnetic field configuration in the Poynting-dominated jet in GRMHD models, we note that we have not solved for the force-free toroidal structure of the field in the jet. Since our force-free model has no rotation, $B^\phi = 0$ and so there is no Poynting flux. Also, the Lorentz factor cannot be estimated from our non-rotating force-free solution. To model these quantities we must consider force-free models in which the disk and field are rotating (e.g., with a Keplerian profile). It would be interesting to see how well simple rotating force-free models can reproduce the toroidal structure and acceleration of Poynting-dominated jets found in GRMHD simulations. This is the topic of a companion paper (McKinney & Narayan 2006).

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APPENDIX A: EVOLUTION EQUATIONS

The GRMHD equations of motion are used to study magnetized accretion disks in the gravitational field of rapidly rotating black holes described by the Kerr metric using the HARM code (Gammie et al. 2003a) with improvements described in McKinney (2006c); Noble et al. (2006). The Kerr metric is written in Kerr-Schild coordinates, such that the inner-radial computational boundary can be placed inside the horizon and so out of causal contact with the flow. The Kerr metric in Kerr-Schild coordinates and the Jacobian transformation to Boyer-Lindquist coordinates are given in McKinney & Gammie (2004).

Boyer-Lindquist coordinates are not chosen because it is difficult to avoid interactions between the inner-radial computational boundary and the jet. The coordinate singularity at the event horizon in Boyer-Lindquist can be avoided by placing the inner-radial computational boundary outside the horizon. However, Poynting-dominated flows have waves that propagate outward even arbitrarily close to the event horizon. Using Boyer-Lindquist coordinates can lead to excessive variability in the jet since the ingoing superfast transition is not on the computational grid, and then the details of the boundary condition can significantly impact the jet. Numerical models of viscous flows have historically had related issues (see discussion in, e.g., McKinney & Gammie 2002).
A1 GRMHD Equations of Motion

The GRMHD notation follows Misner et al. [1973], hereafter MTW. A single-component MHD approximation is assumed such that particle number is conserved,

\[(\rho_0 u^\mu)_{,\mu} = 0,\]  

(\text{A1})

where \(\rho_0\) is the rest-mass density and \(u^\mu\) is the 4-velocity. A 4-velocity with a spatial drift is introduced that is unique by always being related to a physical observer for any space-time and has well-behaved spatially interpolated values, which is useful for numerical schemes. This 4-velocity is

\[\bar{u}^\mu \equiv u^\mu - \gamma \eta^\mu,\]  

(\text{A2})

where \(\gamma = -u^\mu u_\mu\). The additional term represents the spatial drift of the zero angular momentum (ZAMO) frame definition equation is

\[\text{a}_\text{ed} = \text{b}_\text{ed}\]  

(\text{A3})

where \(\text{a}^\mu\) is the energy-momentum conservation equation is

\[T_{\text{MA}}^{\mu\nu} = (\rho_0 + u_\eta)u^\mu u^\nu + p_\eta g_{\mu\nu},\]  

(\text{A4})

with a relativistic ideal gas pressure \(p_\eta = (\gamma - 1)u_\eta\), where \(u_\eta\) is the internal energy density and the projection tensor is \(g_{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu\), which projects any 4-vector into the comoving frame (i.e. \(P^{\mu\nu}u_\mu = 0\)).

In terms of the Faraday (or electromagnetic field) tensor \(F^{\mu\nu}\),

\[T_{\text{EM}}^{\mu\nu} = F^{\mu\gamma} F^{\nu\gamma} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta},\]  

(\text{A5})

which is written in Heaviside-Lorentz units such that a factor of 4π is absorbed into the definition of \(F^{\mu\nu}\), where the Gaussian unit value of the magnetic field is obtained by multiplying the Heaviside-Lorentz value by \(\sqrt{4\pi}\). The induction equation is given by the space components of \(F^{\mu\nu}_{,\nu} = 0\), where \(F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\delta} F_{\lambda\delta}\) is the dual of the Faraday tensor (Maxwell tensor), and the time component gives the no-monopoles constraint. Here \(\epsilon\) is the Levi-Civita tensor, where \(\epsilon^{\mu\nu\lambda\delta} = -\frac{1}{\sqrt{\eta}} [\mu\nu\lambda\delta] \) and \([\mu\nu\lambda\delta]\) is the completely antisymmetric symbol. The comoving electric field is defined as

\[e^{\mu} \equiv u^\mu F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\delta} u_\nu F_{\lambda\delta} = \eta j^\mu,\]  

(\text{A6})

where \(\eta\) corresponds to a scalar resistivity for a comoving current density \(j^\mu = j_\eta F^{\mu\nu}\). The comoving magnetic field is defined as

\[b^\mu \equiv u_\mu F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\delta} u_\nu F_{\lambda\delta}.\]  

(\text{A7})

The ideal MHD approximation, \(\eta = e^{\mu} = 0\), is assumed, and so the invariant \(e^{\mu} b_\mu = 0\). Since the Lorentz acceleration on a particle is \(F_{\mu\nu} = ge^{\mu\nu}\), then this implies that the Lorentz force vanishes on a particle in the ideal MHD approximation. Since \(e^{\mu} u_\mu = b^\mu u_\mu = 0\), they each have only 3 independent components. One can show that

\[F^{\mu\nu} = b^\mu u^\nu - b^\nu u^\mu,\]  

(\text{A8})

and

\[F^{\mu\nu} = e^{\mu\sigma\nu} u_\sigma b_\nu,\]  

(\text{A9})

so that the electromagnetic part of the stress-energy tensor can be written as

\[T^{\mu\nu}_{\text{EM}} = \frac{1}{2} (u^\nu u^\nu + F^{\mu\nu}) - b^\mu b^\nu.\]  

(\text{A10})

The other Maxwell equations,

\[J^\mu = F^{\mu\nu}_{,\nu},\]  

(\text{A11})

define the current density, \(J^\mu\), but are not needed in the ideal MHD approximation for the evolution of the matter or the magnetic field.

For numerical simplicity, another set of field vectors are introduced, such that \(B^\mu = T^{\mu
nu}_{\text{EM}}\) and \(E^\mu = F^{\mu\nu}_{,\nu}\). The two 4-vectors \(e^\mu\) and \(b^\mu\) and the 3-vectors \(B^\mu\) and \(E^\mu\) are just different ways of writing the independent components of the Faraday or Maxwell tensors. Equation (A10) \(implies \(b^\mu = B^\mu u_\mu\) and \(b^\nu = \left(B^\mu + u^\mu b^\nu\right)/u^\nu\). Then the non-monopoles constraint becomes

\[(\sqrt{-g} B^\mu)_i = 0,\]  

(\text{A12})

and the magnetic induction equation becomes

\[(\sqrt{-g} B^\mu)_,t = - (\sqrt{-g} (b^\mu u^\nu - b^\nu u^\mu)),\]  

(\text{A13})

where \(v^\nu = u^\nu/u^t\), \(E_i = e_i = -e_{ijk} v^k B^k = -v \times B\) is the EMF, and \(\epsilon_{ijk}\) is the spatial permutation tensor. The above set of equations are those that are solved. A more complete discussion of the relativistic MHD equations can be found in Anile [1989].

A2 Stationary, Axisymmetric Constraints

We now write down the Faraday tensor in terms of a vector potential \(A_\mu\), where \(F^{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}\). If the field is axisymmetric \((\partial_\theta \to 0)\) and stationary \((\partial_t \to 0)\), then evaluating the condition \(F^{\mu\nu} F_{\mu\nu} = 0\) one finds that

\[A_\phi,\theta A_{t,r} - A_t,\theta A_\phi,r = 0.\]  

(\text{A14})

It follows that one may write

\[\frac{A_{t,\theta}}{A_{\phi,\theta}} = \frac{A_{t,r}}{A_{\phi,r}} \equiv -\Omega_F (r, \theta),\]  

(\text{A15})

where \(\Omega_F (r, \theta)\) is an as-yet-unspecified function. It is usually interpreted as the “rotation frequency” of the electromagnetic field (this is Ferraro’s law of isorotation; see e.g. Frank, King, & Raine 2002, §9.7 in a nonrelativistic context). This can also be written as \(\Omega_F \equiv F_{\phi t}/F_{\phi r} \equiv F_{\phi t}/F_{\phi r}\). As shown in McKinney & Gammie 2004, one can then write \(F_{\mu\nu}\) in terms of the three free functions \(\Omega_F, A_\phi, A_\phi\), and \(B^\phi\), the toroidal magnetic field:

\[F_{tr} = -F_{rt} = \Omega_F A_{\phi, r}\]  

(\text{A16})

\[F_{\theta r} = -F_{r \theta} = \Omega_F A_{\phi, \theta}\]  

(\text{A17})

\[F_{\phi t} = -F_{t \phi} = \sqrt{-g} B^\phi\]  

(\text{A18})

\[F_{\phi r} = -F_{r \phi} = A_{\phi, r}\]  

(\text{A19})

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with all other components zero. Written in this form, the electromagnetic field automatically satisfies Maxwell’s
source-free equations. Notice that \( A_{\phi,\theta} = \sqrt{-g} B^\theta \) and \( A_{\phi,r} = -\sqrt{-g} B^r \).

**APPENDIX B: REDUCING THE GRMHD SOLUTION TO A CURRENT SHEET**

Once we have trivialized the full GRMHD accretion disk into a toroidal current sheet, then we can obtain the effective
stationary, axisymmetric poloidal magnetic field from equation (A11). For an axisymmetric, stationary solution
equation (A11) gives that

\[
-\sqrt{-g} J^\phi = (F^r \phi \sqrt{-g})_r + (F^{\theta \phi} \sqrt{-g})_\theta \quad (B1)
\]

where \( F_\phi = A_{\phi,\theta} = \sqrt{-g} B^\theta \), \( F_\theta = A_{\phi,r} = -\sqrt{-g} B^r \), \( F^{\phi \phi} = g^{\phi \phi} \theta^r F_{\mu \nu} \). The equation in Boyer-Lindquist becomes

\[
\sqrt{-g} J^\phi = (|g| g^{rr} B^\theta (\Omega g_{\phi \phi} + g_{\theta \phi})),_r + (|g| g^{\phi \phi} B (\Omega g_{\phi \phi} + g_{\theta \phi})),_\theta \quad (B2)
\]

or in terms of the vector potential and \( \Omega_F \),

\[
\sqrt{-g} J^\phi = (\sqrt{-g} g^{rr} A_{\phi,r}, (\Omega g_{\phi \phi} + g_{\theta \phi})),_r + (\sqrt{-g} g^{\phi \phi} A_{\phi,\theta}, (\Omega g_{\phi \phi} + g_{\theta \phi})),_\theta. \quad (B3)
\]

In either Boyer-Lindquist or Kerr-Schild coordinates, the linear partial differential equation (PDE) for \( a/M = 0 \) is

\[
-\rho^2 \sin \theta J^\phi = \left( \frac{r-2M}{r \sin \theta} A_{\phi,r} \right)_r + \left( \frac{1}{r^2 \sin \theta} A_{\phi,\theta} \right)_\theta \quad (B4)
\]

and for a current sheet,

\[
-\rho^2 K^\phi \delta(\theta - \pi/2) = \left( \frac{r-2M}{r \sin \theta} A_{\phi,r} \right)_r + \left( \frac{1}{r^2 \sin \theta} A_{\phi,\theta} \right)_\theta \quad (B5)
\]

This equation is an elliptic partial differential equation for \( r > 2M \) and is hyperbolic for \( r < 2M \). The quantity \( K^\phi \) is the surface current density on the equatorial current sheet.

**B1 Vacuum Solutions in Schwarzschild space-time**

Equation (B5) can be solved to find the complementary solution by setting the quantity on the right to 0 everywhere except on the current sheet at \( \theta = \pi/2 \) and assuming that the solution is separable such that

\[
A_{\phi}(r, \theta) = R(r) \Theta(\theta). \quad (B6)
\]

Since the PDE is second order, there are in general two free functions. The radial ordinary differential equation (ODE)
can be written in closed form in terms of generalized hypergeometric function or solved numerically for \( M \neq 0 \). In the
limit of \( x \to \infty \), one finds a complementary function of

\[
R(r) = C_0 r^{l+1} + C_1 r^{l+1}. \quad (B7)
\]

where \( C_0 \) and \( C_1 \) are arbitrary constants and \( l \geq -1/2 \). These are the standard spherical radial eigenfunctions. The
angular complementary function can be written in terms of
generalized hypergeometric functions or alternatively written
in terms of the associated Legendre functions of the first
(\( P \)) and second (\( Q \)) kind, where

\[
\Theta(\theta) = D_0 |\sin \theta| P_{\ell+1}^1 (\cos \theta) + D_1 |\sin \theta| Q_{\ell+1}^1 (\cos \theta), \quad (B8)
\]

where \( D_0 \) and \( D_1 \) are different arbitrary constants and where \( l \geq -1/2 \) are linearly independent (i.e. \( P^\mu\nu_{\ell+1} = P^\mu\nu_{\ell+1} \)). This
form is a mixture of two hypergeometric functions for each \( l \) (see, e.g., chpt. 8 in [Abramowitz & Stegun 1972] and in
[Gradsteyn & Ryzhik 1994]). These functions are just one of the vector spherical harmonics.

**B2 Constraints**

This general solution is a sum of any coefficients for all \( l \), but has no constraints to avoid divergences (e.g. monopoles or
divergences in the physical field strength) on the coordinate
singularities. The only constraint required is that \( F^{\mu \nu} F_{\mu \nu} \)
remains finite, where in Boyer-Lindquist coordinates with
\( a/M = 0 \) and \( B^\theta = 0 \),

\[
F^{\mu \nu} F_{\mu \nu} = 2((F^{r \phi})^2 g_{rr} g_{\phi \phi} + (F^{\theta \phi})^2 g_{\theta \theta} g_{\phi \phi}). \quad (B9)
\]

and so avoid divergences at the polar axes one requires \( A_\phi \propto \theta^{2+n} + \text{Const.} \) for \( n \geq 0 \). There is no requirement on the
horizon and no constraint on divergence need be placed at \( r = 0 \) since it is a physical singularity.

The solution given by equation (B5) with \( \sin(\theta) \) multiplied by the associated Legendre functions of the first kind
gives \( A_\phi \propto \theta^2 + O(\theta^4) \) for any \( l \), while the term with the second kind gives \( A_\phi \propto \text{Const.} + O(\theta^2) \) for all \( l \). Hence, the only valid solutions are of the first kind with any radial
solution, except for angular solutions combined with a
radial solution that does not depend on radius. The single
nontrivial example of this exception is the monopole solution
with \( l = -1 \) and \( C_0 = D_0 = 0 \) giving \( A_\phi = -\cos \theta \).

The remaining solutions require the first kind with \( D_1 = 0 \).
The monopole solution is only a result of the limit to the open
set around \( l = -1 \) for \( C_1 \neq 0 \) or \( l = 0 \) for \( C_0 \neq 0 \) for the associated Legendre function of the first kind.
This issue regarding the monopole solution can be considered a
pathology of using the Legendre functions that is not man-
ifested in the hypergeometric form of the solution, where \( l = 0 \) naturally generates the monopole and paraboloidal
type solutions.

**B3 Currents**

The current density given by equation (A11) can be integrated to determine the net toroidal current enclosed within
the volume between \( r_0 \) and \( r \), as given by equation (A4) For a model with a current sheet located at \( \theta = \pi/2 \) with

\[
J^\phi = K^\phi \delta(\theta - \pi/2), \quad (B10)
\]

where \( K^\phi \) is the surface current density, then

\[
\frac{dI_\phi}{dr} = \sqrt{-g} \Theta(\theta, \pi/2) K^\phi. \quad (B11)
\]

For a general power-law toroidal current per unit radius with

\[
\frac{dI_\phi}{dr} = C/r^{n+1} = C/r^{2-n}. \quad (B12)
\]
the enclosed current (for \( n \neq 0 \)) is
\[
I_\phi(r) \equiv I_\phi(r_0) + (I_\phi(r_2) - I_\phi(r_0)) \left( \frac{1}{r_0} - \frac{1}{r_2} \right),
\]
where \( r_2 \) denotes some radius for which \( r_0 < r < r_2 \) and for which the enclosed toroidal current \( I_\phi(r_2) \) is known. If \( a/M = 0 \) and in spherical polar coordinates, then \( \sqrt{-g} \hat{\theta} = \pi/2 = r^2 \), so that the toroidal surface current density is
\[
K^\phi = \frac{C}{r^{n+3}} = \frac{C}{r^{4-n}}.
\]
(B14)

### B4 Split-Monopole Field

The split-monopole solution for \( a/M = 0 \) is a solution where the current sheet at \( \theta = \pi/2 \) has
\[
J^\phi = \frac{C}{r^4} \delta(\theta - \pi/2),
\]
so that
\[
\frac{dI_\phi}{dr} = \frac{C}{r^2},
\]
which gives a total enclosed current of
\[
I_\phi = \frac{C}{r_0 - r},
\]
(B17)

where \( r_0 < r \) are the inner- and outer- radial positions. The split-monopole has \( n = 1 \) and \( \nu = 0 \).

The linearly independent particular solution for the split-monopole vector potential (for any \( M \)) is
\[
A^{(\text{split})}_\mu = \begin{cases} 
- C \cos \phi \phi_\mu & \theta < \pi/2 \\
+ C \cos \phi \phi_\mu & \theta > \pi/2,
\end{cases}
\]
where \( \phi_\mu = \{0, 0, 0, 1\} \) is the \( \phi \) Killing vector. The above gives that \( B^\phi = B^\theta = 0 \) and that
\[
B^r = \begin{cases} 
+ \frac{C}{r^2} & \theta < \pi/2 \\
- \frac{C}{r^2} & \theta > \pi/2,
\end{cases}
\]
(B19)

This solution has field lines that have an opening angle following
\[
\theta_j \propto r^{-0.5}.
\]
(B20)

### B5 Paraboloidal Field

The paraboloidal field solution for \( a/M = 0 \) is a solution where the current sheet at \( \theta = \pi/2 \) has
\[
J^\phi = \frac{C}{r^3} \delta(\theta - \pi/2),
\]
so that
\[
\frac{dI_\phi}{dr} = \frac{C}{r^2},
\]
(B22)

Notice that the total enclosed toroidal current would diverge for a disk of infinite radial extent, i.e.,
\[
I_\phi = C \log \left( \frac{r}{r_0} \right),
\]
where \( r_0 < r \) are the inner- and outer- radial positions. The paraboloidal solution has \( n = 0 \) and \( \nu = 1 \).

The linearly independent particular solution for the BZ paraboloidal field vector potential is
\[
A^{(\text{para})}_\mu = \begin{cases} 
g(r, \theta) \phi_\mu & \theta < \pi/2 \\
g(r, \pi - \theta) \phi_\mu & \theta > \pi/2,
\end{cases}
\]
(B24)

where \( g(r, \theta) \equiv \frac{C_1}{r^2} (r f_\perp + 2 M f_\perp (1 - \ln f_\perp)) f_\perp = 1 + \cos \theta \), \( f_\perp = 1 - \cos \theta \), and \( M \) is the mass of the black hole. Thus, \( B^\theta = 0 \) and
\[
B^r = \begin{cases} 
+ \frac{C}{r^2} (r + 2 M f_\perp) & \theta < \pi/2 \\
- \frac{C}{r^2} (r + 2 M f_\perp) & \theta > \pi/2,
\end{cases}
\]
and
\[
B^\theta = \begin{cases} 
+ \frac{C \tan(\theta/2)}{r^4} & \theta < \pi/2 \\
+ \frac{C \tan(\theta/2)}{r^4} & \theta > \pi/2,
\end{cases}
\]
(B26)

This solution has field lines that have an opening angle that at large radii approximately obeys
\[
\theta_j \propto r^{-0.5}.
\]
(B27)

### B6 Constructing Current Sheets by Splicing Source-Free Solutions

While in general it is difficult to find \( A_\phi \) for arbitrary source functions, one can construct source functions corresponding to equatorial current sheets by splicing a single equatorially asymmetric vector potentials \( G(r, \theta) \). Then the general solution with a current sheet is
\[
A_\phi(\theta) = G(\theta)(1 - H(\theta - \pi/2)) + G(\pi - \theta) H(\theta - \pi/2),
\]
for either \( G \) in the range of \( 0 \leq \theta < \pi/2 \) or \( \pi/2 < \theta \leq \pi \), where \( H \) is the Heaviside function.

First, the simple \( a/M = M = 0 \) equations are considered. To construct a radial power-law current density of the form given by equation (B11) so that the source function is \( \sqrt{-g} J^\phi \propto r^{\nu - 2} \), the radial complementary or particular functions must satisfy
\[
R(r) \propto r^\nu,
\]
(B29)

at large radii \( (r \gg M) \) as demonstrated by plugging this form of \( A_\phi \) into equation (B11). This means that one must choose either
\[
l = -\nu
\]
with \( C_0 \neq 0 \) and \( C_1 = 0 \) or one must choose
\[
l = \nu - 1
\]
with \( C_1 \neq 0 \) and \( C_0 = 0 \), where \( l \geq -1/2 \). Hence, for \( \nu > 1/2 \), only the second choice leads to a real solution. For \( \nu < 1/2 \), only the first choice leads to a real solution. For \( \nu = 1/2 \) both give \( l = -1/2 \). Thus, for fixed \( \nu \), there is a unique \( l \) that gives a single allowed \( R(r) \propto r^\nu \) and \( \Theta(\theta) \).

For example, for \( \nu = 1 \) one has that \( l = 0 \), and then the most general solution for \( M = 0 \) is
\[
A_\phi = (c_0 + c_1 r)(d_1 + d_2 \cos \theta),
\]
(B32)

which after forcing \( A_\phi \propto \theta^{3-\nu} + \text{Const.} \), for \( \nu \leq 1 \) near \( \theta = 0 \), one has that
\[
A_\phi = (c_0 + c_1 r)(\cos \theta - 1),
\]
(B33)

which for \( c_0 = 0 \) gives paraboloidal solution in the upper
hemisphere, $c_1 = 0$ gives the monopole solution, while combinations give a mixture of monopole and paraboloidal solutions. Another example is $C_0 = D_1 = 0$ and $l = 1/2$ giving a decollimating field geometry.

The GRMHD numerical solutions are associated with models with $\nu = 3/4$, for which one must choose $l = -1/4$ and $C_0 = 0$. For the solution to satisfy regularity on the axis near $\theta = 0$, one must set

$$D_0 \neq 0 \quad \text{and} \quad D_3 = 0.$$  \hspace{1cm} (B34)

This solution is quite similar to the paraboloidal solution, but slightly less collimated, as expected. This solution has field lines that have an opening angle that approximately follows

$$\theta_j \propto r^{-0.375}. \quad \hspace{1cm} (B35)$$

This force-free model is also used in McKinney & Narayan (2006) to find force-free solutions with arbitrary $M$ and $a/M$ using a general relativistic force-free electrodynamic code (McKinney et al. 2006).

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