Series solution of the $1+2$ continuous Toda chain

D.B Fairlie*† A.N.Leznov ‡§ and R.Torres-Cordoba

Abstract

A way to obtain the series solutions of the $1+2$ dimensional continuous Toda chain is presented.

1 Introduction

The usual form of the equation under consideration is the following one;

$$\rho_{y,x} = (e^\rho)_{z,z}, \quad (\ln u)_{y,x} = u_{z,z}, \quad u \equiv e^\rho \quad (1)$$

Here $\rho(x, y, z)$ is an unknown function of three independent variables. This equation arises as a reduction of the Plebansky equation [1] describing self-dual four dimensional $(0+4), (2+2)$ gravity. In this connection it was considered in [2]. Equation (1) can also be obtained as a limit of the discrete Toda chain

$$\rho_{y,x} = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}}$$

under appropriate rescaling ($n \to z$). This is the reason for the title of the present paper. A solution of the two dimensional reduction of (1) $\rho = \rho(z, y + x)$ was found in implicit form in [3]. Infinite series solutions of the symmetry equation corresponding to (1) were found in [4]. But the

* e-mail david.fairlie@durham.ac.uk
† Durham University, Durham, England.
‡ e-mail: andrey@buzon.uaem.mx
§ Universidad Autonoma del Estado de Morelos, CIICAp, Cuernavaca, Mexico
connection between series solutions of the symmetry equation with the solution of the initial system \( \text{(1)} \) has not been found. But general theory gives a guarantee that with each solution of symmetry equation is connected with an analytical solution of the initial system in explicit or implicit form. The goal of the present paper is fill this gap and demonstrate a way how an analytical solution of \( \text{(1)} \) is connected with the solution constructed in \( \text{(1)} \) of the symmetry equation. In \( \text{(5)} \), the Plebansky equation was represented in the form of two equations of the first order for two unknown functions. One of which satisfies the Plebansky equation by itself, the second one satisfies the corresponding symmetry equation. It is possible to find by an independent way the solution of symmetry equation in recurrence form. As was remarked the above equation under consideration in the present paper is a reduction of Plebansky equation and so it is possible to try to solve it by the same methods.

2 Preliminary manipulations. Short excursion into \( \text{(4)} \)

Let us rewrite \( \text{(1)} \) in the form of a system of two equations of the first degree

\[
(\ln u)_y = T_z, \quad u_z = T_x, \quad (\ln u)_{x,y} = u_{zz}, \quad \left( \frac{T_x}{u} \right)_y = T_{zz}
\]

or as the initial equation is symmetrical with respect to exchange of the variables \( x, y \), the following is also a possible form;

\[
(\ln u)_x = w_z, \quad u_z = w_y, \quad (\ln u)_{x,y} = u_{zz}, \quad \left( \frac{w_y}{u} \right)_x = w_{zz}
\]

The symmetry equation arises from the initial one after differentiation by some arbitrary parameter and considering this derivative as a new unknown function. In the case under consideration this equation is:

\[
\dot{u} = S, \quad \left( \frac{S}{u} \right)_{x,y} = S_{zz}, \quad S = T_x(w_y), \quad \left( \frac{T_x}{u} \right)_y = T_{zz}, \quad \left( \frac{w_y}{u} \right)_x = w_{zz}
\]

Thus we see that the linear system of equations of the first few lines of this section is connected with the symmetry equation of the initial system. In
we have obtained series solutions of the symmetry equation in integro-differential terms of the function $u$. And thus it is possible use these expressions in the system $T, u$ and obtain two self consistent equation instead of only one equation for the function $u$. It is obvious that in this way we will not be able to obtain a general solution for the equation for $u$ but only its partial soliton like series solutions. Resolving the second equation of $u = \theta_x, T = \theta_z$ we rewrite (1) in the form

$$\theta_{y,x} = \theta_x \theta_{z,z}, \quad w_{y,x} = w_y w_{z,z}$$

In [4] it was shown that solution of symmetry equation $T$ may be obtained in terms of $r \alpha_n$ functions which satisfy the following recurrence relations

$$\alpha_{n-1} = \int dy \frac{(u^{n+1} \alpha_n)_z}{(n+1)u^{n+1}}, \quad \frac{(u^{n+1} \alpha_n)_x}{u^{n+1}} = (n+1)\alpha_{n-1}z$$

Eliminating $\alpha_{n-1}$ from both equations we arrive at the equation for $\alpha_n$ function in the form

$$\left(\frac{(u^{n+1} \alpha_n)_z}{u^n}\right)_z = \left(\frac{(u^{n+1} \alpha_n)_x}{u^{n+1}}\right)_y, \quad (u^{n+2} \alpha^n_z)_z = (u^{n+1} \alpha^n_y)_x$$

The left and right equations are the same. From these expressions it follows that there exists an obvious solution $\alpha^n = 1$ which leads to a finite solution for $T$. The solution for $T$ becomes

$$T = u \alpha^0 = u \int dy \frac{(u^2 \alpha^1)_z}{2u}, \quad \frac{\theta_y - \theta^2_z}{2} = \frac{u^2 \alpha_1}{2}$$

The second equality is obtained from the first one after the substitution $T = \theta_z, u = \theta_x$ and differentiation of the subsequent $\frac{\theta_y}{\theta_x}$ with respect to the argument $y$ and integration once over $z$.

3 Generalization of R. Ward’s solution.

This section explains why analytic solutions of Ward exist at all. The simplest solution of the symmetry equation is a linear combination of derivatives of the functions $u \ S = w_y = u_z = au_x + bu_y + cu_z$. The solution of Richard Ward corresponds to choosing $c = 1, a = -b$. In the case $c$ is not equal to zero we
have \( u_z = Au_x + Bu_y \) and the second system under this additional condition becomes

\[
\begin{align*}
\quad & u_x = uw_z, \quad u_z = w_y, \quad u_x = u(Aw_x + Bw_y), \quad Au_x + Bu_y = w_y
\end{align*}
\]

Let us seek a solution of this system in the form

\[
\begin{align*}
\quad & x = \theta(u, w), \quad y = \sigma(u, w)
\end{align*}
\]

The system of equations defining derivatives of \( u, w \) with respect to space coordinates \((x, y)\) is the following one;

\[
\begin{align*}
1 = \theta_u u_x + \theta_w w_x, & \quad 0 = \sigma_u u_x + \sigma_w w_x \\
0 = \theta_u u_y + \theta_w w_y, & \quad 1 = \sigma_u u_y + \sigma_w w_y.
\end{align*}
\]

After solving the last system,

\[
\begin{align*}
\quad & u_x = \frac{\sigma_w}{D}, \quad w_x = -\frac{\sigma_u}{D}, \quad u_y = -\frac{\theta_w}{D}, \quad w_y = \frac{\theta_u}{D}
\end{align*}
\]

and substitution into the previous one we arrive at a linear system of equations for \( \theta(u, w) \) and \( \sigma(u, w) \).

\[
\begin{align*}
\sigma_w = u(-A\sigma_u + B\theta_u), \quad A\sigma_w - B\theta_w = \theta_u
\end{align*}
\]

The last system after eliminating (for instance) the function \( \sigma \) leads to an equation of the second order with separable variables

\[
\begin{align*}
\quad & u\theta_{u,u} + \frac{1}{A}\theta_{w,u} + \frac{B}{A}\theta_{w,w} = 0
\end{align*}
\]

The case considered by Ward corresponds to the limiting case \( A \to \infty, B \to \infty, \frac{B}{A} \to -1 \).

## 4 The zero order term of series solution to the symmetry equation

In the case \( \alpha_0 = 1 \) from the general formula it follows that \( T = u \) or \( w = u \) and from the corresponding formulas of the previous section we obtain \( u_x = u_z \) or \( u_y = u_z \). These are particular cases of the generalized Ward construction of the previous section. The first equations in this cases lead \( u_y = uu_z \). This well the known Monge equation (the equation of Hamilton-Jacobi for free motion in one dimension) with general solution \( z + y + ux = F(u) \) or \( z + x + uy = F(u) \). It is not difficult to connect these solutions with the generalized Ward solution of the previous section.
The first term of the symmetry equation series solution

In the case $\alpha_1 = 1$ in connection with the recurrence procedure we obtain

$$T = u(\int dyu_z), \quad U = \int dyu, \quad u = U_y, \quad T = U_yU_z$$

and the equations which are necessary to resolve are the following;

$$U_{y,z} = (U_yU_z)_x, \quad (\ln U_y)_y = (U_yU_z)_z, \quad (\ln U_y)_x = U_z,z$$

Taking into account the last equation the first one leads to a relation between the derivatives (after integration once with respect to the argument $z$) in the form $\ln U_y = \frac{1}{2} U_z^2 + U_x, \quad U_y = e^{\frac{1}{2} U_z^2 + U_x}$. Let us seek a solution of these equations using the following parametrisation

$$x = X(U_x, U_z, y), \quad z = Z(U_x, U_z, y), \quad U_{x,x} = \frac{Z_\gamma}{D}, \quad U_{x,z} = -\frac{Z_\beta}{D}, \quad U_{z,z} = \frac{X_\beta}{D}$$

where indices $\beta, \gamma$ denote respectively the first and second arguments of the functions $X, Z$. $X = W_\beta, Z = W_\gamma$ and equation transform to a linear equation of second order with separable variables

$$-2W_{\beta,\gamma} + W_{\gamma,\gamma} = W_{1,1}, \quad W = e^{k\beta U(\gamma)}, \quad -2kU_\gamma + U_{\gamma,\gamma} = k^2 U$$

It is possible obtain the dependence of the function $W$ after solution of two equations which arise after differentiation of the previous equations by the argument $y$

$$X_\beta u_{x,y} + X_\gamma u_{z,y} + X_y = 0, \quad Z_\beta u_{x,y} + Z_\gamma u_{z,y} + Z_y = 0,$$

remembering that $U_y = e^{\frac{1}{2} U_z^2 + U_x}$. After trivial manipulations we obtain for $W$

$$W = -ye^{\frac{1}{2} U_z^2 + U_x} + W^L, \quad x = W_\beta = ye^{\frac{1}{2} U_z^2 + U_x} + W^L_\beta \quad x = W_1 = y2e^{\frac{1}{2} U_z^2 + U_x} + W^L_\beta$$

where $W^L$ is solution of the linear equation obtained above. Solution of Toda chain of the beginning of this paper is given by connection $u = U_y = e^{\frac{1}{2} U_z^2 + U_x}$. 5
5.1 One simple example

It is easy to check that $W = 2e^{-\beta}$ is an explicit solution of the linear equation and thus $W = ye^{\frac{1}{2}U_x^2 + U_x} + 2e^{-\beta}$. The implicit form of the solution is given by

$$x = W_\beta = -ye^{\frac{1}{2}U_x^2 + U_x} - 2e^{-\beta}, \quad z = W_\gamma = -y2e^{\frac{1}{2}U_x^2 + U_x} + e^{-\beta}, \quad U_z = \gamma, U_x = \beta$$

From these expressions immediately follows the equation $U_{z,z} + 2U_zU_{z,x} - U_{x,x} = 0$, which is equivalent to our linear system above. With the help of this equation it is not difficult to check that $T = U_yU_z = 2e^{\frac{1}{2}U_x^2 + U_x}, \quad U_y = e^{\frac{1}{2}U_x^2 + U_x}$ satisfy the equations

$$U_{y,z} = T_x, \quad (\ln U_y)_y = T_z$$

and thus $u = U_y$ is a solution of the equation of title of this paper. More intriguing and interesting are the following comments. We rewrite equations which define an implicit solution in a form $u = U_y = e^{\frac{1}{2}U_x^2 + U_x} \equiv e^{\frac{1}{2}\gamma^2 + 1}$ in equivalent form

$$x = -yu - \frac{1}{u}e^{\frac{1}{2}\gamma^2}, \quad z = -2yu + \frac{1}{u}e^{\frac{1}{2}\gamma^2}$$

After excluding terms containing $y$ we arrive at a quadratic equation for determining the variable $\gamma$

$$\gamma^2 + \frac{z}{yu}\gamma + \frac{v}{yu} + 1 = 0$$

Substituting the solution of this equation into the first or second equations we come to equation determining in implicit form the function $u$. It is obvious that to obtain this equation is not a very simple problem. In Appendix we present an alternative method of solution of the problem of this section.

6 Second step

In the case where $\alpha_2 = 1$ in connection with the recurrence procedure we obtain $\alpha_1 = \int dyu = U_z, \alpha_0 = \int dyu^2\alpha_1 = \int dy(U_{y,z}U_z + \frac{1}{2}U_yU_{z,z}) = \frac{1}{2}(U_z^2 + U_x)$ and the solution for $T$ takes the form

$$T = \frac{1}{2}U_y(U_z^2 + U_x), \quad U = \int dyu, \quad u = U_y$$

(4)
The equations which are necessary to resolve are the following ones:

\[ U_{y,z} = (T)_x = \frac{1}{2}[U_{y,x}(U_z^2 + U_x) + 2U_zU_{z,x} + U_{x,x}], \quad (\ln U_y)_y = (T)_z, \quad U_{y,x} = U_yU_{z,z} \]

Let us introduce notations

\[ U_y = e^c, \quad U_x = a, \quad U_z = b, \quad \alpha = \frac{1}{2}(b^2 + a) \]

Equations above together with introduced notations lead to the following system of equations

\[
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_z =
\begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    \frac{1}{2} & b & \alpha
\end{pmatrix}
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_x
\equiv L
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_x,
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_y =
\begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix}
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_z
\equiv e^cL
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_z
\]

As in the cases above we will seek solution of these equations by implicit substitution

\[ x = X(a, b, c), \quad z = Z(a, b, c), \quad y = Y(a, b, c) \]

After differentiation all these equalities with respect to independent arguments of problem and introduction matrix \( V = \begin{pmatrix} X_a & X_b & X_c \\ Z_a & Z_b & Z_c \\ Y_a & Y_b & Y_c \end{pmatrix} \) we have

\[
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_x = V^{-1}
\begin{pmatrix}
    1 \\
    0 \\
    0
\end{pmatrix},
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_z = V^{-1}
\begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix},
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}_y = V^{-1}
\begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix}
\]

Substituting these expressions in linear system equations of the first order we obtain

\[
\begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix} = VLV^{-1}
\begin{pmatrix}
    1 \\
    0 \\
    0
\end{pmatrix},
\begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix} = e^cVLV^{-1}
\begin{pmatrix}
    0 \\
    1 \\
    0
\end{pmatrix}
\]

The last equations allow to reconstruct explicit form matrix \( VLV^{-1} \)

\[
VLV^{-1} =
\begin{pmatrix}
    0 & 0 & \frac{1}{2}e^c \\
    1 & 0 & be^c \\
    0 & e^{-c} & \alpha
\end{pmatrix}
= 
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & e^{-c}
\end{pmatrix}
LT
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & e^c
\end{pmatrix}
\]

7
The first two columns are direct consequence of equations above. The last column arises from the fact $\text{Trace}(V L V^{-1})^n = \text{Trace}L^n$. Now we come to linear system of equations for determining $X, Z, Y$ functions

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^c
\end{pmatrix} V L = L^T \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^c
\end{pmatrix} V
$$

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$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^c
\end{pmatrix} V L = L^T \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^c
\end{pmatrix} V
$$

$e^c Y$ we will denote by $Y$. System of 9 equations are the following one

$$
\begin{pmatrix}
\frac{1}{2} (Y_a - X_c) & \frac{1}{2} Y_b - X_a - bX_c & \frac{1}{2} (Y_c - Y) - X_b - \alpha X_c \\
X_a + bY_a - \frac{1}{2} Z_c & X_b + bY_b - Z_a - bZ_c & X_c + b(Y_c - Y) - Z_b - \alpha Z_c \\
Z_a + \alpha Y_a - \frac{1}{2} (Y_c - Y) & Z_b + \alpha Y_b - Y_a - b(Y_c - Y) & Z_c - Y_b
\end{pmatrix} = 0
$$

Elements $M_{1,1}$ and $M_{3,3}$ lead to a parametrization $X = R_a, Y = R_c, Z = R_b + f(a, b)$. Elements $M_{2,1}$ and $M_{1,2}$ both lead to equation $(R_a + bR_c)_a = \frac{1}{2} R_{b,c}$. Element $M_{2,2}$ allow to conclude function $f$ depend only from one argument $b$. Elements $M_{3,1}$ and $M_{1,3}$ are the same and lead to equation $(R_b + \alpha R_c)_a = \frac{1}{2} R_{c,c}$. And at last elements $M_{3,2}$ and $M_{2,3}$ pass to a third equation in the form $(R_b + f(b) + \alpha R_c)_b = (R_a + bR_c)_c$. Thus we have three equations which it is necessary to resolve

$$
R_a + bR_c)_a = \frac{1}{2} R_{b,c}, \quad (R_a + \alpha R_c)_a = \frac{1}{2} R_{c,c}, \quad (R_b + f(b) + \alpha R_c)_b = (R_a + bR_c)_c
$$

or

$$
R_a + bR_c = W_b, \quad \frac{1}{2} R_c = W_a, \quad R_b + f(b) + \alpha R_c = W_c
$$

8
For further calculations it will be more suitable variables \( b, \alpha = \frac{1}{2}(a + b), c \). In these variables the system equations above looks as

\[
R_c = W_\alpha, \quad \frac{1}{2}R_\alpha + bR_c = W_b + bW_\alpha, \quad R_b + bR_\alpha + \alpha R_c = W_c
\]

First equation give \( R = Q_\alpha, W = Q_c \). Substituting both others equation we pass to system of two equations

\[
Q_{\alpha,\alpha} = 2Q_{b,c}, \quad Q_{\alpha,b} + bQ_{\alpha,\alpha} + \alpha Q_{\alpha,c} = Q_{c,c}
\]

Let us seek solution of this linear system above in Laplace-Furier form

\[
Q = \int dk dp e^{k\alpha + pb + \frac{k^2}{2p}c} f(p, k)
\]

The first equation is satisfied automatically. The second one leads to a differential equation of the first order in partial derivatives for the determination of the under the integral function \( f(k, p) \), which for the function \( F \equiv k^3 f \) is

\[
2p \frac{\partial F}{\partial p} + k \frac{\partial F}{\partial k} = \left( \frac{2p^2}{k} - \frac{k^2}{2p} \right) F
\]

with the obvious solution

\[
F = k^3 f = e^{\frac{k^2}{2p} + \frac{1}{2} \ln k \frac{k^2}{p}} \phi \left( \frac{k^2}{p} \right)
\]

(5)

where \( \phi \) is an arbitrary function of the argument \( \frac{k^2}{p} \).

7 Outlook

We have presented a new idea, unknown up to now to the best of our knowledge in theory of integrable systems connected with the symmetry equation of the initial system. We have also presented some non-trivial solutions of the 2 + 1 continu Toda chain.

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