DELTA EDGE-HOMOTOPY INVARIANTS OF SPATIAL GRAPHS VIA DISK-SUMMING THE CONSTITUENT KNOTS

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Abstract. In this paper we construct some invariants of spatial graphs by disk-summing the constituent knots and show the delta edge-homotopy invariance of them. As an application, we show that there exist infinitely many slice spatial embeddings of a planar graph up to delta edge-homotopy, and there exist infinitely many boundary spatial embeddings of a planar graph up to delta edge-homotopy.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $G$ be a finite graph. An embedding of $G$ into the 3-sphere is called a spatial embedding of $G$ or simply a spatial graph. A graph $G$ is said to be planar if there exists an embedding of $G$ into the 2-sphere, and a spatial embedding of a planar graph $G$ is said to be trivial if it is ambient isotopic to an embedding of $G$ into the 2-sphere in the 3-sphere. Note that a trivial spatial embedding of a planar graph is unique up to ambient isotopy [7].

A delta move is a local deformation on a spatial graph as illustrated in Fig. 1.1 which is known as an unknotting operation [8], [12]. A delta move is called a self delta move if all three strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be delta edge-homotopic if they are transformed into each other by self delta moves and ambient isotopies [16]. If the graph is homeomorphic to the disjoint union of 1-spheres, then this equivalence relation coincides with self $\Delta$-equivalence [22] (or delta link homotopy [13]) on oriented links.

For self $\Delta$-equivalence on oriented links, Shibuya proposed the following conjectures in [22] and [23].

Conjecture 1.1. [22] Two cobordant oriented links are self $\Delta$-equivalent.

Conjecture 1.2. [23] Any boundary link is self $\Delta$-equivalent to the trivial link.

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He gave the partially affirmative answers to the conjectures above at the same time. He showed that any ribbon link is self $\Delta$-equivalent to the trivial link [22], and any 2-component boundary link is self $\Delta$-equivalent to the trivial link [23, Theorem 4.6]. But Nakanishi-Shibuya showed that there exists a 2-component link such that it is not self $\Delta$-equivalent but cobordant to the Hopf link [14, Claim 4.5], namely they gave a negative answer to Conjecture [1.1]. Moreover, Nakanishi-Shibuya-Yasuhara showed that there exists a 3-component link such that it is not self $\Delta$-equivalent but cobordant to the Borromean rings [15, Proposition 1]. Note that both the Hopf link and the Borromean rings are not slice. On the other hand, Conjecture [1.2] was solved affirmatively by Shibuya-Yasuhara [24].

On the outcome of the results above, we investigate a more general case. A spatial embedding of a planar graph is said to be slice if it is cobordant to the trivial spatial embedding. A spatial embedding of a graph is called a $\partial$-spatial embedding if all knots in the embedding bound Seifert surfaces simultaneously such that the interiors of the surfaces are mutually disjoint and disjoint from the image of the embedding [19]. If the graph is homeomorphic to the disjoint union of 1-spheres, then this definition coincides with the definition of the boundary link. We note that any non-planar graph does not have a $\partial$-spatial embedding [19, Corollary 1.3]. Then we ask the following questions.

**Question 1.3.** (1) Is any slice spatial embedding of a planar graph delta edge-homotopic to the trivial spatial embedding?
(2) Is any $\partial$-spatial embedding of a graph delta edge-homotopic to the trivial spatial embedding?

In fact, for spatial theta curves, the affirmative answers to Question 1.3 (1) and (2) have already given by the author [17, Corollary 1.3, Corollary 1.5]. But our purpose in this paper is to give the negative answers to the Questions 1.3 (1) and (2) as follows.

**Theorem 1.4.** (1) There exist infinitely many slice spatial embeddings of a graph up to delta edge-homotopy.
(2) There exist infinitely many $\partial$-spatial embeddings of a graph up to delta edge-homotopy.

To accomplish this, we construct some invariants of spatial graphs by considering a disk-summing operation among the constituent knots in a spatial graph in section 2, and show the delta edge-homotopy invariance of them in section 3 (Theorems 2.1 and 2.2). In section 4, we give some remarkable examples which imply Theorem 1.4. Any of those examples is demonstrated by a spatial handcuff graph (see the next section) all of whose constituent links are trivial up to self $\Delta$-equivalence. Therefore our examples also imply that delta edge-homotopy on spatial graphs behaves quite differently than self $\Delta$-equivalence on links.

**Remark 1.5.** (1) Recently, Question 1.3 (1) for oriented links was solved affirmatively by Yasuhara [26, Corollary 1.9].
(2) A sharp move is a local deformation on a spatial oriented graph as illustrated in Fig. 1.2 which is also known as an unknotting operation [11]. A sharp move is called a self sharp move if all four strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be sharp edge-homotopic

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1See [25] for the precise definition of spatial graph-cobordism.
(or self sharp-equivalent [19]) if they are transformed into each other by self sharp moves and ambient isotopies [18]. It is known that two delta edge-homotopic spatial embeddings of a graph are sharp edge-homotopic [18, Lemma 2.1 (2)]. The author showed that two cobordant spatial embeddings of a graph are sharp edge-homotopic [18, Lemma 2.2], and the author and R. Shinjo showed that any $\partial$-spatial embedding of a graph is sharp edge-homotopic to the trivial spatial embedding [19, Theorem 1.5 (1)].

In this section we introduce the invariants of spatial graphs needed later. Let $H_n$ $(n \geq 2)$ be the graph as illustrated in Fig. 2.1. We give the label to each of the edges and give an orientation to each of the loops as presented in Fig. 2.1. A spatial embedding of $H_n$ is called a spatial $n$-handcuff graph, or simply a spatial handcuff graph if $n = 2$. On that occasion, we regard $e_1 \cup e_2$ as an edge of $H_2$ and denote by $e$.

![Figure 1.2.](image)

Figure 1.2.

2. INVARIANTS

In this section we introduce the invariants of spatial graphs needed later. Let $H_n$ $(n \geq 2)$ be the graph as illustrated in Fig. 2.1. We give the label to each of the edges and give an orientation to each of the loops as presented in Fig. 2.1. A spatial embedding of $H_n$ is called a spatial $n$-handcuff graph, or simply a spatial handcuff graph if $n = 2$. On that occasion, we regard $e_1 \cup e_2$ as an edge of $H_2$ and denote by $e$.

![Figure 2.1.](image)

Figure 2.1.

Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ be an ordered and oriented $n$-component link. Let $D$ be an oriented 2-disk and $x_1, x_2, \ldots, x_n$ are mutually disjoint arcs in $\partial D$, where $\partial D$ has the orientation induced by the one of $D$, and these arcs appear along the orientation of $\partial D$ in order and each arc has an orientation induced by the one of $\partial D$. We assume that $D$ is embedded in the 3-sphere so that $D \cap L = x_1 \cup x_2 \cup \cdots \cup x_n$ and $x_i \subset J_i$ with opposite orientations for any $i$. Then we call a knot $K_D^{12 \cdots n} = L \cup \partial D - \cup_{i=1}^n \text{int} x_i$ a $D$-sum of $L$. For a spatial $n$-handcuff graph $f$, we denote $f(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$ by $L_f$ and consider a $D$-sum of $L_f$ so that $f(e_1 \cup e_2 \cup \cdots \cup e_n) \subset D$ and $f(e_i) \cap \partial D = f(e_i \cap \gamma_i) \subset \text{int} x_i$ for any $i$. We call such a $D$-sum of $L_f$ a $D$-sum of $L_f$ with respect to $f$ and denote it by $K_D^{12 \cdots n}(f)$.

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\[2\] This equivalence relation does not depend on the edge orientations.
For a spatial handcuff graph $f$, we define that

$$ n_{12}(f, D) = a_2(K_D^{i_2}(f)) - a_2(f(\gamma_1)) - a_2(f(\gamma_2)) $$

and denote the modulo $lk(L_f)$ reduction of $n_{12}(f, D)$ by $\bar{n}_{12}(f)$, where $lk$ denotes the linking number in the 3-sphere. Then we have the following.

**Theorem 2.1.** If two spatial handcuff graphs $f$ and $g$ are delta edge-homotopic, then $\bar{n}_{12}(f) = \bar{n}_{12}(g)$.

On the other hand, let $f$ be a spatial 3-handcuff graph and $K_{D}^{123}(f)$ a $D$-sum of $L_f$ with respect to $f$. Then, by using the same disk $D$, we can obtain three knots $K_D^{12}(f)$, $K_D^{23}(f)$ and $K_D^{13}(f)$ by forgetting the components $f(\gamma_3)$, $f(\gamma_1)$ and $f(\gamma_2)$, respectively, namely by the $D$-sums of sublinks $f(\gamma_1) \cup f(\gamma_2)$, $f(\gamma_2) \cup f(\gamma_3)$ and $f(\gamma_1) \cup f(\gamma_3)$ of $L_f$. Then we define that

$$ n_{123}(f, D) = -v_3(K_D^{123}(f)) + \sum_{1 \leq i < j \leq 3} v_3(K_D^{ij}(f)) - \sum_{i=1}^3 v_3(f(\gamma_i)), $$

where $v_3(J) = (1/36)V_J^{(3)}(1)$ and $V_J^{(3)}(1)$ denotes the third derivative at 1 of the Jones polynomial\(^3\) of a knot $J$. Assume that $L_f$ is algebraically split, namely all of the pairwise linking numbers of $L_f$ are zero. Then we denote the modulo $\mu_{123}(L_f)$ reduction of $n_{123}(f, D)$ by $\bar{n}_{123}(f)$, where $\mu_{123}$ denotes the triple linking number, namely Milnor’s $\mu$-invariant of length 3 of a 3-component algebraically split link [9], [10]. Then we have the following.

**Theorem 2.2.** Let $f$ and $g$ be two spatial 3-handcuff graphs which are delta edge-homotopic. Assume that both $L_f$ and $L_g$ are algebraically split. Then it holds that $\bar{n}_{123}(f) = \bar{n}_{123}(g)$.

For example, if a spatial handcuff (resp. 3-handcuff) graph $f$ contains a Hopf link (resp. Borromean rings), then our invariants are no use. But if $L_f$ is link-homotopic [9] to the trivial link, then our invariants take effect on our purpose. Because any slice link is link-homotopic to the trivial link [3], [4], and any boundary link is also link-homotopic to the trivial link [1], [2]. We prove Theorems 2.1 and 2.2 in the next section.

### 3. Proofs of Theorems 2.1 and Theorems 2.2

To prove Theorems 2.1 and 2.2 we first recall some results and show a lemma needed later.

**Lemma 3.1.** Let $J_+$ and $J_-$ be two oriented knots and $J_0 = K_1 \cup K_2$ an oriented 2-component link which are identical except inside the depicted regions as illustrated in Fig. 3.1. Then we have that

1. (15 Lemma 5.6]) $a_2(J_+) - a_2(J_-) = lk(J_0)$.

2. (18 Proposition 4.2])

$$ V_{J_+}^{(3)}(1) - V_{J_-}^{(3)}(1) = 36a_2(J_+) + 18\{lk(J_0)\}^2 - 36\{a_2(K_1) + a_2(K_2)\}. $$

\(^3\) We calculate the Jones polynomial of a knot by the skein relation

$$ tv_{J_+}(t) - t^{-1}v_{J_-}(t) = t^{-\frac{1}{2}} - t^{\frac{1}{2}}v_{J_0}(t), $$

where $J_+$ and $J_-$ are two oriented knots and $J_0$ an oriented 2-component link which are identical except inside the depicted regions as illustrated in Fig. 3.1.
Lemma 3.2. Let $K_+$ and $K_-$ be two oriented knots and $K_0$ an oriented 3-component link which are identical except inside the depicted regions as illustrated in Fig. 3.2. Then we have that

$(1)$ ([20, Theorem 1.1]) $a_2(K_+) - a_2(K_-) = 1$.

$(2)$ ([16, Theorem 3.2]) $V_K^{(3)}(1) - V_{K'}^{(3)}(1) = 36\text{Lk}(K_0) - 18$,

where $\text{Lk}(L)$ denotes the total linking number of an oriented link $L$.

Proof. $(1)$ We can see that a self delta move on $f(e)$ is realized by a “doubled-delta move” on $f(\gamma_1)$, see Fig. 3.3. It is easy to see that a doubled-delta move is realized by eight delta moves on the strings in the move and ambient isotopies. Thus we have the result.

$(2)$ We can show in the same way as $(1)$. $\square$

Proof of Theorem 2.1. We first show that $\bar{n}_{12}(f)$ is an ambient isotopy invariant. Let $K_D^{12}(f)$ be a $D$-sum of $L_f$ with respect to $f$ and $K_D^{12}(f)$ another $D'$-sum of $L_f$ with respect to $f$. We may assume that $K_D^{12}(f)$ is obtained from $K_D^{12}(f)$ by a positive full twist of the band corresponding to $f(e)$. Then by Lemma 3.1 (1) we have that

$$n_{12}(f, D') - n_{12}(f, D) = a_2(K_D^{12}(f)) - a_2(K_D^{12}(f)) = \text{lk}(L_f),$$

see Fig. 3.4. This implies that $\bar{n}_{12}(f)$ is an ambient isotopy invariant.

Next we show that $\bar{n}_{12}(f)$ is a delta edge-homotopy invariant. Let $f$ and $g$ be delta edge-homotopic two spatial handcuff graphs. Then, by Lemma 3.3 (1), $g$
Figure 3.3.

Figure 3.4.

is obtained from $f$ by self delta moves on $f(\gamma_i)$ ($i = 1, 2$) and ambient isotopies. Moreover it is known that each of the oriented delta moves can be realized by the one as illustrated in Fig. 3.5 [12, Fig. 1.1]. Hence we may assume that $g$ is obtained from $f$ by a self delta move on $f(\gamma_1)$ as illustrated in Fig. 3.6 without loss of generality. Let $K^{12}_D(f)$ be the $D$-sum of $L_f$ with respect to $f$ as illustrated in Fig. 3.6 and $K^{12}_D(g)$ the $D$-sum of $L_g$ with respect to $g$ by using the same $D$ as illustrated in Fig. 3.6. Namely $K^{12}_D(f)$ and $K^{12}_D(g)$ are identical except the depicted parts which represents the delta move. Note that $f(\gamma_2)$ and $g(\gamma_2)$ are ambient isotopic. Then by Lemma 3.2 (1) we have that

$$n_{12}(f, D) - n_{12}(g, D) = a_2(K^{12}_D(f)) - a_2(K^{12}_D(g)) - \{a_2(f(\gamma_1)) - a_2(g(\gamma_1))\}$$

$$= 1 - 1 = 0.$$

Since a delta move preserves the linking number, we have that $\bar{n}_{12}(f) = \bar{n}_{12}(g)$. This completes the proof.

Figure 3.5.

Remark 3.4. (1) By the first half of the proof of Theorem 2.1 we can also see that the modulo $\text{lk}(L_f)$ reduction of $a_2(K^{12}_D(f))$ is an ambient isotopy invariant of
(2) For a spatial n-handcuff graph $f$ and a $D$-sum $K_{D}^{12 \cdots n}(f)$ of $L_f$ with respect to $f$, we can generalize Theorem 2.1 as follows. Let $l$ be the greatest common divisor of $\text{lk}(f(\gamma_i), L_f - f(\gamma_i))$ ($i = 1, 2, \ldots, n$). Then it can be shown that the modulo $l$ reduction of

$$a_2(K_{D}^{12 \cdots n}(f)) - \sum_{i=1}^{n} a_2(f(\gamma_i))$$

is a delta edge-homotopy invariant of $f$ in the same way as the proof of Theorem 2.1. But this generalized version for $n \geq 3$ is not so strong as we will see in Examples 4.2 and 4.4.

To prove Theorem 2.2, we recall another result. By Polyak’s formula of the triple linking number [21], we have the following.

**Lemma 3.5.** [21] Let $L = J_1 \cup J_2 \cup J_3$ be an ordered and oriented algebraically split 3-component link. Let $K_D$ be a $D$-sum of $L$ and $K_{D}^{12}, K_{D}^{13}$ and $K_{D}^{12}$ three knots obtained from $K_{D}^{123}$ by forgetting the components $J_1, J_2$ and $J_3$, respectively. Then it holds that

$$\mu_{123}(L) = -a_2(K_{D}^{123}) + \sum_{1 \leq i < j \leq 3} a_2(K_{D}^{ij}) - \sum_{i=1}^{3} a_2(J_i).$$

**Proof of Theorem 2.2** We first show that $\tilde{n}_{123}(f)$ is an ambient isotopy invariant. Let $K_{D}^{123}(f)$ be a $D$-sum of $L_f$ with respect to $f$ and $K_{D}^{123}(f)$ another $D'$-sum of $L_f$ with respect to $f$. We may assume that $K_{D}^{123}(f)$ is obtained from $K_{D}^{123}(f)$ by a positive full twist of the band corresponding to $f(\epsilon_1)$, see Fig. 3.7. Then by the
skein relation as illustrated in Fig. 3.8, Lemmas \ref{lem:3.1} (2) and \ref{lem:3.5} we have that

\[
n_{123}(f, D') - n_{123}(f, D) = -\left\{ v_3(K_{123}^{12}(D)) - v_3(K_{123}^{12}(D)) \right\} + \left\{ v_3(K_{123}^{12}(D)) - v_3(K_{123}^{12}(D)) \right\}
\]

\[
= -v_3(K_{123}^{12}(D)) - v_3(K_{123}^{12}(D)) = -a_2(K_{123}^{12}(f)) - \frac{1}{2}\text{lk}(f(\gamma_1) \cup f(\gamma_3)) + \{a_2(f(\gamma_1)) + a_2(K_{123}^{12}(f))\}
\]

\[
+ a_2(K_{12}^{12}(f)) + \frac{1}{2}\text{lk}(f(\gamma_1), f(\gamma_2))^2 - \{a_2(f(\gamma_1)) + a_2(f(\gamma_2))\}
\]

\[
+ a_2(K_{13}^{12}(f)) + \frac{1}{2}\text{lk}(f(\gamma_1), f(\gamma_3))^2 - \{a_2(f(\gamma_1)) + a_2(f(\gamma_3))\}
\]

\[
= -a_2(K_{123}^{12}(f)) + \sum_{1 \leq i < j \leq 3} a_2(K_{ij}^{12}(f)) - \sum_{i=1}^{3} a_2(f(\gamma_i))
\]

\[
= \mu_{123}(L_f).
\]

Hence we have that \( \bar{n}_{123}(f) \) is an ambient isotopy invariant.

Next we show that \( \bar{n}_{123}(f) \) is a delta edge-homotopy invariant. Let \( f \) and \( g \) be delta edge-homotopic two spatial 3-handcuff graphs. Then, by Lemma \ref{lem:3.3} (2), \( g \) is obtained from \( f \) by self delta moves on \( f(\gamma_i) \) \( (i = 1, 2, 3) \) and ambient isotopies. Hence we may assume that \( g \) is obtained from \( f \) by a self delta move on \( f(\gamma_1) \) as illustrated in Fig. 3.9 without loss of generality. Let \( K_{123}^{12}(f) \) be the \( D \)-sum of \( L_f \) with respect to \( f \) as illustrated in Fig. 3.9 and \( K_{123}^{12}(g) \) the \( D \)-sum of \( L_g \) with respect to \( g \) by using the same \( D \) as illustrated in Fig. 3.9. Namely \( K_{123}^{12}(f) \) and \( K_{123}^{12}(g) \) are identical except the depicted parts which represents the delta move. Let \( h \) be the spatial 3-handcuff graph and \( k_1 \) and \( k_2 \) two oriented knots as illustrated in Fig. 3.9 where \( f(H_3) \), \( g(H_3) \) and \( h(H_3) \cup k_1 \cup k_2 \) are identical except the depicted
parts. Let $K_{D}^{123}(h)$ be the $D$-sum of $L_{h}$ with respect to $h$ by using the same $D$ as illustrated in Fig. 3.9. Then by Lemma 3.2 (2) and the homological invariance of the linking number, we have that

$$n_{123}(f, D) - n_{123}(g, D) = -\text{lk}(k_{1}, K_{D}^{123}(h)) - \text{lk}(k_{2}, K_{D}^{123}(h)) - \text{lk}(k_{1}, k_{2}) + \frac{1}{2}$$

$$+ \text{lk}(k_{1}, K_{D}^{12}(h)) + \text{lk}(k_{2}, K_{D}^{12}(h)) + \text{lk}(k_{1}, k_{2}) - \frac{1}{2}$$

$$+ \text{lk}(k_{1}, K_{D}^{13}(h)) + \text{lk}(k_{2}, K_{D}^{13}(h)) + \text{lk}(k_{1}, k_{2}) - \frac{1}{2}$$

$$- \text{lk}(k_{1}, h(\gamma_{1})) - \text{lk}(k_{2}, h(\gamma_{2})) - \text{lk}(k_{1}, k_{2}) + \frac{1}{2}$$

$$= 0.$$  

Note that $L_{f}$ and $L_{g}$ are self $\Delta$-equivalent. Thus they are also link-homotopic, namely $\mu_{123}(L_{f}) = \mu_{123}(L_{g})$. Thus we have that $\bar{n}_{123}(f) = \bar{n}_{123}(g)$. This completes the proof. \hfill \Box

4. Examples

Example 4.1. Let $f_{m}$ be the spatial handcuff graph for $m \in \mathbb{N}$ as illustrated in Fig. 4.1. We can see that $L_{f_{m}}$ is the trivial 2-component link for any $m \in \mathbb{N}$, namely $\text{lk}(L_{f_{m}}) = 0$. We can also see that $f_{m}$ is slice by the hyperbolic transformation on $f_{m}(\gamma_{2})$ along the band $B$ shown in Fig. 4.1.

Now we consider the $D$-sum of $L_{f_{m}}$ with respect to $f_{m}$ as illustrated in Fig. 4.1. Then by a calculation we have that $a_{2}(K_{D}^{123}(f_{m})) = 2m$ and therefore $\bar{n}_{123}(f_{m}) = 2m$. Thus by Theorem 2.1 we have that $f_{m}$ is not delta edge-homotopic to the trivial spatial handcuff graph for any $m \in \mathbb{N}$, and $f_{i}$ and $f_{j}$ are not delta edge-homotopic for $i \neq j$.
Example 4.2. Let \( f_m \) be the spatial 3-handcuff graph for \( m \in \mathbb{N} \) as illustrated in Fig. 4.2. We can see that \( L_{f_m} \) is the trivial 3-component link for any \( m \in \mathbb{N} \), namely \( \mu_{123}(L_{f_m}) = 0 \). We can also see that \( f_m \) is slice by the hyperbolic transformation on \( f_m(\gamma_3) \) along the band \( B \) shown in Fig. 4.2.

Now we consider the \( D \)-sum of \( L_{f_m} \) with respect to \( f_m \) as illustrated in Fig. 4.2 and a skein tree as illustrated in Fig. 4.3. Then we have that

\[
\begin{align*}
 a_2(K_{123}^{D}(f_m)) &= a_2(J_m) - a_2(J) - m - 1 \\
 &= a_2(J_{m-1}) + m + 1 - m - 1 \\
 &= a_2(J_{m-1}) = \cdots = a_2(J_0) = 0.
\end{align*}
\]
Then by Lemma 3.1 and (4.1) we have that

\[ v_3(K_{123} \Delta (f)) = v_3(J_m) = a_2(J_m) + \frac{1}{2}(m + 1)^2 + 1 \]

\[ = \left\{ v_3(J_{m-1}) - a_2(J_{m-1}) - \frac{1}{2}(m + 1)^2 \right\} + a_2(J_m) + \frac{1}{2}(m + 1)^2 + 1 \]

\[ = v_3(J_{m-1}) + 1 = \cdots = v_3(J_0) + m \]

Since \( K_{ij} \Delta (f_m) \) is a trivial knot for any \( 1 \leq i < j \leq 3 \), we have that \( \bar{n}_{123}(f_m) = m \). Thus by Theorem 2.2, we have that \( f_m \) is not delta edge-homotopic to the trivial spatial handcuff graph for any \( m \in \mathbb{N} \), and \( f_i \) and \( f_j \) are not delta edge-homotopic for \( i \neq j \). Note that the generalized version of \( \bar{n}_{12} \) for \( n = 3 \) as mentioned in Remark 3.4 (2) vanishes for \( f_m \) by (4.1).

**Example 4.3.** Let \( f_m \) be the spatial handcuff graph for \( m \in \mathbb{N} \) as illustrated in Fig. 4.4. It is easy to see that \( f_m \) is a \( \partial \)-spatial handcuff graph for any \( m \in \mathbb{N} \). Since \( L_{f_m} \) is a 2-component boundary link, we have that \( L_{f_m} \) is self \( \Delta \)-equivalent to the 2-component trivial link for any \( m \in \mathbb{N} \).

Now we consider the \( D \)-sum of \( L_{f_m} \) with respect to \( f_m \) as illustrated in Fig. 4.4. Then by a calculation we have that \( a_2(K_{12} \Delta (f_m)) = -2m \). Since \( f_m(\gamma_i) \) and \( f_m(\gamma_j) \) are trivial knots, we have that \( \bar{n}_{12}(f_m) = -2m \). Thus by Theorem 2.1, we have that \( f_m \) is not delta edge-homotopic to the trivial spatial handcuff graph for any \( m \in \mathbb{N} \), and \( f_i \) and \( f_j \) are not delta edge-homotopic for \( i \neq j \).

**Example 4.4.** Let \( f \) be the spatial 3-handcuff graph as illustrated in Fig. 4.5. Note that \( f|_{\gamma_i \cup \gamma_j \cup e_i \cup e_j} \) is the trivial spatial handcuff graph for any \( 1 \leq i < j \leq 3 \). It is easy to see that \( f \) is a \( \partial \)-spatial 3-handcuff graph. Since \( L_f \) is a 3-component boundary link, we have that \( \mu_{123}(L_{f_m}) = 0 \) and \( L_f \) is self \( \Delta \)-equivalent to the 3-component trivial link. Note that \( L_f \) is Brunnian, namely any 2-component sublink of \( L_f \) is trivial.

Now we consider the \( D \)-sum of \( L_f \) with respect to \( f \) as illustrated in Fig. 4.5. By a calculation we have that \( a_2(K_{123} \Delta (f)) = 0 \). Since \( f(\gamma_i) \) is a trivial knot for \( i = 1, 2, 3 \), we have that the generalized version of \( \bar{n}_{12} \) for \( n = 3 \) as mentioned in

![Figure 4.2.](image-url)
Remark 3.4 (2) vanishes for $f$. On the other hand, by a calculation we have that

\[ V_{K_D^{123}(f)}(t) = -t^{-12} + 6t^{-11} - 11t^{-10} + t^{-9} + 28t^{-8} - 52t^{-7} \]
\[ + 36t^{-6} + 17t^{-5} - 61t^{-4} + 67t^{-3} - 43t^{-2} + 11t^{-1} \]
\[ + 22 - 57t + 84t^2 - 78t^3 + 32t^4 + 23t^5 - 43t^6 \]
\[ + 24t^7 - 4t^9 - 4t^{10} + 5t^{11} + t^{12} - 3t^{13} + t^{14}, \]

\[ V^{(3)}_{K_D^{123}(f)}(1) = 36. \]
$$\gamma_1 f(\gamma_2) f(\gamma_3) K_{\Delta}^1(f)$$

Figure 4.5.

Since $K_{\Delta}^{ij}(f)$ is also a trivial knot for any $1 \leq i < j \leq 3$, we have that $\bar{n}_{123}(f) = -1$. Thus by Theorem 2.2, we have that $f$ is not delta edge-homotopic to the trivial spatial 3-handcuff graph.

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