Correction to: The Karoubi envelope and weak idempotent completion of an extriangulated category

Dixy Msapato

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1 Introduction

In the original article [2], we showed that given an extriangulated category \((\mathcal{C}, \mathcal{E}, s)\), its idempotent completion \(\tilde{\mathcal{C}}\) is also an extriangulated category \((\tilde{\mathcal{C}}, \tilde{\mathcal{F}}, \tilde{r})\). An important technical result is [2, Proposition 3.10], which states that the correspondence \(\tilde{r}\) is well-defined. The proof of the proposition given in the original article was incorrect. We will give a correct proof of this statement in this corrigendum. The statement of the proposition is as follows.

**Proposition 3.10** Let \(\delta\) be an extension in \(\mathcal{F}((\mathcal{C}, p), (A, q))\) realised under \(s\) by the following sequences,

\[
\begin{align*}
A & \rightarrow a B \rightarrow b C, \\
A & \rightarrow x Y \rightarrow y C.
\end{align*}
\]

Then given idempotents \(r: B \rightarrow B\) and \(w: Y \rightarrow Y\) such that

\[
a q = r a, \quad p b = b r \quad \text{and} \quad x q = w x, \quad p y = y w
\]

the sequences

\[
\begin{align*}
(A, q) & \rightarrow a q (B, r) \rightarrow p b (C, p), \\
(A, q) & \rightarrow x q (Y, w) \rightarrow p y (C, p).
\end{align*}
\]

are equivalent. That is to say, \(\tilde{r}\) is well-defined.

To prove the equivalence of the sequences (4) and (5) in \(\tilde{\mathcal{C}}\), the strategy used in [2] was to prove that the morphism \(w f r: (B, r) \rightarrow (Y, w)\) is an isomorphism, where \(f: B \rightarrow Y\) is an isomorphism in \(\mathcal{C}\) which gives the equivalence of the sequences (1) and (2) in \(\mathcal{C}\). We claimed
this could be done by first showing that \( wfr = wf \), using the fact that \( wfraq = wfaq \) and hence \( (wfr - wf)aq = 0 \), then employing the fact that \( py \) is a weak cokernel of \( aq \) in \( \tilde{C} \) to further deduce that \( wfr = wf \). However, it is not clear if \( wfr \) is a morphism in \( \tilde{C} \), so we cannot take advantage of the fact that \( py \) is a weak cokernel of \( aq \) in \( \tilde{C} \) in this way.

### 2 Corrigendum

Recall for an extriangulated category \((\mathcal{C}, \mathcal{E}, s)\), we defined the biadditive functor \( F : \tilde{\mathcal{C}}^{\text{op}} \times \tilde{\mathcal{C}} \to \text{Ab} \) on the idempotent completion as follows. Given a pair of objects \((X, p)\) and \((Y, q)\) in \( \tilde{\mathcal{C}} \), we define \( F \) on objects by setting,

\[
F((X, p), (Y, q)) := p^*q_*\mathcal{E}(X, Y) = \{ p^*q_*\delta \mid \delta \in \mathcal{E}(X, Y) \} \subseteq \mathcal{E}(X, Y).
\]

This is just the image of \( \mathcal{E}(X, Y) \) under the group homomorphism \( \mathcal{E}(p, q) \). For morphisms \( f : (A, a) \to (A', a') \) and \( g : (C, c) \to (C', c') \) \( \in \tilde{\mathcal{C}} \), we then defined \( F(f, g) := \mathcal{E}(f, g)_{\mathcal{E}(C, C), (A, A)} \), the restriction of the group homomorphism \( \mathcal{E}(f, g) \) to \( \mathcal{E}((C, C), (A, A)) \). Before we can give the proof of Proposition 3.10, we first need to collect some lemmas which will be needed.

**Lemma 2.1** [1, Lemma 4.1] Let \( \mathcal{C} \) be an additive category with biadditive functor \( \mathcal{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab} \). Let \( X_\bullet = A \xrightarrow{x_1} X \xrightarrow{x_2} C \) and \( Y_\bullet = A \xrightarrow{y_1} Y \xrightarrow{y_2} C \) be a pair of complexes in \( \mathcal{C} \). Suppose that the following sequences of functors are exact,

\[
\begin{align*}
\mathcal{C}(C, -) & \xrightarrow{\mathcal{E}(x_2, -)} \mathcal{C}(X, -) \xrightarrow{\mathcal{E}(x_1, -)} \mathcal{C}(A, -) \\
\mathcal{C}(-, A) & \xrightarrow{\mathcal{E}(-, y_1)} \mathcal{C}(-, X) \xrightarrow{\mathcal{E}(-, y_2)} \mathcal{C}(-, C)
\end{align*}
\]

and likewise for \( Y_\bullet \). Then for any commutative diagram \( f_\bullet = (1_A, f, 1_C) : X_\bullet \to Y_\bullet \)

\[
\begin{array}{ccc}
A & \xrightarrow{x_1} & X & \xrightarrow{x_2} & C \\
\| & & \downarrow f & & \\
A & \xrightarrow{y_1} & Y & \xrightarrow{y_2} & C
\end{array}
\]

the following statements are equivalent.

1. \( f_\bullet \) is a homotopy equivalence.
2. \( f_\bullet \) is an equivalence of the sequences \( X_\bullet \) and \( Y_\bullet \), i.e. \( f \) is an isomorphism and the squares in the above diagram commute.
3. \( f : X \to Y \) is an isomorphism.

**Lemma 2.2** [1, Proposition 2.21] Let \( \delta \in \mathcal{E}(C, A) \) be an extension, and let \( X_\bullet = A \xrightarrow{x_1} X \xrightarrow{x_2} C \) and \( Y_\bullet = A \xrightarrow{y_1} Y \xrightarrow{y_2} C \) be a pair of complexes in \( \mathcal{C} \). Suppose that the following sequences of functors are exact,

\[
\begin{align*}
\mathcal{C}(C, -) & \xrightarrow{\mathcal{E}(x_2, -)} \mathcal{C}(X, -) \xrightarrow{\mathcal{E}(x_1, -)} \mathcal{C}(A, -) \xrightarrow{\delta^\#} \mathcal{E}(C, -) \\
\mathcal{C}(-, A) & \xrightarrow{\mathcal{E}(-, y_1)} \mathcal{C}(-, X) \xrightarrow{\mathcal{E}(-, y_2)} \mathcal{C}(-, C) \xrightarrow{\delta^\#} \mathcal{E}(-, A)
\end{align*}
\]

and likewise for \( Y_\bullet \). Let \( f_\bullet = (1_A, f, 1_C) : X_\bullet \to Y_\bullet \) be a commutative diagram:
Let \( A \xrightarrow{x_1} X \xrightarrow{x_2} C \) and \( A \xrightarrow{y_1} Y \xrightarrow{y_2} C \).

If there exists a commutative diagram \( g_\bullet = (1_A, g, 1_C): Y_\bullet \to X_\bullet \):

\[
A \xrightarrow{y_1} Y \xrightarrow{y_2} C
\]

Then \( f_\bullet \) is a homotopic equivalence.

We also need to strengthen [2, Lemma 3.9] as follows.

**Lemma 2.3** Let \( \delta = p^*q_*\varepsilon \) be an extension in \( \mathbb{F}((Z, p), (X, q)) \) such that

\[
g(p^*q_*\varepsilon) = [X \xrightarrow{X} Y \xrightarrow{Y} Z].
\]

Then the following sequences of functors are exact:

\[
\tilde{C}((Z, p), -) \xrightarrow{\tilde{C}(py, -)} \tilde{C}((Y, r), -) \xrightarrow{\tilde{C}(xq, -)} \tilde{C}((X, q), -) \xrightarrow{\delta^\#} \mathbb{F}((Z, p), -)
\]

\[
\tilde{C}(-, (X, q)) \xrightarrow{\tilde{C}(-, xq)} \tilde{C}(-, (Y, r)) \xrightarrow{\tilde{C}(-, py)} \tilde{C}(-, (Z, p)) \xrightarrow{\delta^\#} \mathbb{F}(-, (X, q))
\]

where \( r: Y \to Y \) is an idempotent morphism such that \( rx = xq \) and \( yr = py \), obtained by an application of [2, Lemma 3.5].

**Proof** We will only show the exactness of the first sequence. The proof of the exactness of the second sequence is dual. Exactness at \( \tilde{C}((Y, r), -) \) is as in [2, Lemma 3.9]. So what is left is to prove exactness at \( \tilde{C}((X, q), -) \). Since \( \tilde{C}(\mathcal{E}, \mathcal{E}, \mathcal{r}) \) is an extriangulated category, the following sequence

\[
\tilde{C}(Z, -) \xrightarrow{\tilde{C}(y, -)} \tilde{C}(Y, -) \xrightarrow{\tilde{C}(x, -)} \tilde{C}(X, -) \xrightarrow{\delta^\#} \mathbb{E}(Z, -)
\]  

(6)

is exact.

Let \( (A, e) \) be an arbitrary object in \( \tilde{C} \). Take any morphism \( f: (Y, r) \to (A, e) \in \tilde{C}((Y, r), (A, e)). \) Then

\[
(\delta^\#_{(A,e)} \circ \tilde{C}(xq, (A, e)))(f) = (f(xq))_{*}\delta = (f(xq))_{*}\delta = (f(rx))_{*}\delta = ((fr)x)_{*}\delta = 0
\]

by the exactness of (6). We conclude that \( \text{im}(\tilde{C}(xq, (A, e))) \subseteq \ker(\delta^\#_{(A,e)}). \)

Now take any morphism \( g: (X, q) \to (A, e) \in \tilde{C}((X, q), (A, e)). \) Recall that this means \( g \) is a morphism \( g: X \to A \) in \( \mathcal{C} \) such that \( gq = eg = g. \) Suppose \( \delta^\#_{(A,e)}(g) = g_{*}\delta = 0. \) Since \( g \) is also a morphism in \( \mathcal{C} \) and \( \delta \) is an \( \mathbb{E} \)-extension, we have by the exactness of (6) that there exists \( h: Y \to A \) such that \( g = hx. \) Now consider the morphism \( h' = ehr: (Y, r) \to (A, e). \)

We have that

\[
h'xq = (ehr)xq = eh(rx)q = e(hx)q = e(g)q = g.
\]

We conclude that \( \ker(\delta^\#_{(A,e)}) \subseteq \text{im}(\tilde{C}(xq, (A, e))). \) Therefore we have exactness at \( \tilde{C}((X, q), -) \) as required. \( \square \)

We are now able to give a proof of Proposition 3.10.
2.1 Proof of Proposition 3.10

Proof Since the sequences $A \xrightarrow{a} B \xrightarrow{b} C$, and $A \xrightarrow{x} Y \xrightarrow{y} C$ both realise $\delta$, they are by definition equivalent in $C$. That is to say we have the following commutative diagram,

$$
\begin{array}{c}
A \xrightarrow{a} B \xrightarrow{b} C \\
\quad \downarrow{f} \\
A \xrightarrow{x} Y \xrightarrow{y} C
\end{array}
$$

(7)

where the morphism $f : B \to Y$ is an isomorphism. Now consider the following diagram.

$$
\begin{array}{c}
(A, q) \xrightarrow{aq} (B, r) \xrightarrow{pb} (C, p) \\
\quad \downarrow{wfr} \\
(A, q) \xrightarrow{xq} (Y, w) \xrightarrow{py} (C, p)
\end{array}
$$

(8)

From the relations in (3) and those arising from the commutative diagram (7), we can observe the following:

$$
wf(ra)q = wf(aq)q = wf(af)q = w(fa)q = w(x)q = (xq)q = xq, \quad (9)
$$

$$
p(yw)f = p(py)f = (py)f = p(yf)r = p(b)r = p(br) = p(pb) = pb. \quad (10)
$$

That is to say, diagram (8) is a commutative diagram. Now consider the following diagram.

$$
\begin{array}{c}
(A, q) \xrightarrow{xq} (Y, w) \xrightarrow{py} (C, p) \\
\quad \downarrow{rfr^{-1}w} \\
(A, q) \xrightarrow{aq} (B, r) \xrightarrow{pb} (C, p)
\end{array}
$$

(11)

From the relations in (3) and those arising from the commutative diagram (7), we can observe the following:

$$
r^{-1}(wx)q = r^{-1}(xq)q = r(f^{-1}x)q = r(a)q = (ra)q = (aq)q = aq, \quad (12)
$$

$$
p(br)f^{-1}w = p(pb)f^{-1}w = p(bf^{-1})w = p(y)w = p(yw) = p(py) = py. \quad (13)
$$

That is to say, diagram (11) is a commutative diagram. By [2, Lemma 3.9] both (4) and (5) are complexes, that is to say $pb \circ aq = 0$ and $py \circ xq = 0$. We apply Lemma 2.2 to conclude that $wfr_\bullet = (1_{(A, q)}, wfr, 1_{(C, p)})$ is a homotopy equivalence, and hence by Lemma 2.1, the morphism $wfr$ is an isomorphism. We conclude that (8) is an equivalence, that is to say $r$ is well-defined. \hfill \qed

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