Relation between quantum tomography and optical Fresnel transform

Hong-Yi Fan\textsuperscript{1,2} and Li-yun Hu\textsuperscript{1*} \\
\textsuperscript{1}Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China \textsuperscript{2}Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, China

February 2, 2008

Abstract

Corresponding to optical Fresnel transformation characteristic of ray transfer matrix elements \((A,B,C,D)\), \(AD - BC = 1\), there exists Fresnel operator \(F(A,B,C,D)\) in quantum optics, we show that under the Fresnel transformation the pure position density \(\langle x | x \rangle\) becomes the tomographic density \(\langle x \rangle_{s,r,s,r} \langle x \rangle\), which is just the Radon transform of the Wigner operator, i.e. 

\[
F \langle x | x \rangle F^\dagger = \langle x \rangle_{s,r,s,r} \langle x \rangle = \int_{-\infty}^{\infty} dp' dx' \delta \left[ x - (Dx' - Bp') \right] \Delta (x',p'),
\]

where \(s,r\) are the complex-value expression of \((A,B,C,D)\). So the probability distribution for the Fresnel quadrature phase is the tomography (Radon transform of Wigner function), and the tomogram of a state \(|\psi\rangle\) is just the wave function of its Fresnel transformed state \(F^\dagger |\psi\rangle\), i.e. 

\[
\langle x | \psi \rangle = \langle x | F^\dagger |\psi\rangle.
\]


1 Introduction

In quantum optics theory, all possible linear combination of quadratures \(X\) and \(P\) of the oscillator field mode \(a\) and \(a^\dagger\) can be measured by the homodyne measurement just by varying the phase of the local oscillator. The average of the random outcomes of the measurement, at a given local oscillator phase, is connected with the marginal distribution of Wigner function, thus the homodyne measurement of light field permits the reconstruction of the Wigner function of a quantum system by varying the phase shift between two oscillators. In Ref. \[1\] Vogel and Risken pointed out that the probability distribution for the rotated quadrature phase \(X_\theta \equiv (a^\dagger \exp(i\theta) + a \exp(-i\theta))/\sqrt{2}, [a,a^\dagger] = 1\), which depends on only one \(\theta\) angle, can be expressed in terms of Wigner function, and that the reverse is also true (named as Vogel-Risken relation), i.e., one can obtain the Wigner distribution by tomographic inversion of a set of measured probability distributions, \(P_\theta (x_\theta)\), of the quadrature amplitude. Smithey, Beck and Raymer \[2,3,4\] also pointed out that once the distribution \(P_\theta (x_\theta)\) are obtained, one can use the inverse Radon transformation familiar in tomographic imaging to obtain the Wigner distribution and density matrix. The Radon transform of the Wigner function is closely related to the expectation values or densities formed with the eigenstates to the rotated canonical observables. The field of problems of the reconstruction of the density operator from

\*Corresponding author. Email: hlyun@sjtu.edu.cn

PACS numbers: 03.65.-w, 42.30.Kq
such data is called quantum tomography. (Optical tomographic imaging techniques derive two-
dimensional data from a three-dimensional object to obtain a slice image of the internal structure
and thus have the ability to peer inside the object noninvasively.) The theoretical development in
quantum tomography in the last decade has progressed in the direction of determining more physical
relevant parameters of the density from tomographic data \([5, 6, 7]\).

In \([8, 9]\) the Radon transform of Wigner function which depends on two continuous parameters is
introduced. In this Letter we extend the rotated quadrature phase \(X_\theta\) to \(X_F \equiv (s^*a + ra^\dagger + sa^\dagger + r^*a) / \sqrt{2}\), where

\[ |s|^2 - |r|^2 = 1, \ (s, r) \text{ are related to a classical ray transfer matrix } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ by}
\]

\[ s = \frac{1}{2} [A + D - i (B - C)], \ r = -\frac{1}{2} [A - D + i (B + C)], \ AD - BC = 1. \tag{1} \]

We shall show that the \((D, B)\) related Radon transform of the Wigner operator \(\Delta (x, p)\) is just the
pure state density operator \(|x\rangle_{s,r,s,r} \langle x|\) formed with the eigenstates belonging to the quadrature
\(x_F\). We name \(x_F\) the Fresnel transformed canonical observable, or Fresnel transformed quadrature
phase, so the probability distribution for the Fresnel quadrature phase is the Radon transform of
Wigner function. These can be expressed neatly by

\[ F \ |x\rangle \langle x | F^\dagger = |x\rangle_{s,r,s,r} \langle x| = \int \int_{-\infty}^{\infty} dx' dp' \delta \left[ x - (Dx' - Bp') \right] \Delta (x', p'), \]

\[ D = \frac{1}{2} (s + s^* + r + r^*), \ B = \frac{1}{2i} (s^* - s + r^* - r), \tag{2} \]

where \(X_F |x\rangle_{s,r,s,r} = x |x\rangle_{s,r,s,r}\), and \(|x\rangle_{s,r} = F |x\rangle, \ |x\rangle\) is the eigenvector of \(X_{\theta=0}\), and \(F\) is the Fresnel
operator corresponding to classical Fresnel transform in optical diffraction theory which we shall
describe in the following. We name \(|x\rangle_{s,r,s,r} \langle x|\) the tomographic density operator. While the \((A, C)\)
related Radon transform of \(\Delta (x, p)\) is just the pure state density operator \(|p\rangle_{s,r,s,r} \langle p|\) formed with the
eigenstates belonging to the conjugate quadrature of \(X_F\),

\[ F \ |p\rangle \langle p | F^\dagger = |p\rangle_{s,r,s,r} \langle p| = \int \int_{-\infty}^{\infty} dx' dp' \delta \left[ p - (Dx' - Bp') \right] \Delta (x', p'), \]

\[ A = \frac{1}{2} (s^* - r^* + s - r), \ C = \frac{1}{2i} (s - r + s^* + r^*). \tag{3} \]

Through \(2\) and \(3\) one can see how the quantum tomography is related to optical Fresnel transform.

The optical diffraction transform is described by Fresnel integration whose parameters \((A, B, C, D)\)
are elements of a ray transfer matrix \(M\) describing optical systems, \(AD - BC = 1\), \(M\) belongs to
the unimodular symplectic group, the input light field \(f (x)\) and output light field \(g (x')\) are related to
each other by Fresnel integration \([10, 11]\)

\[ g (x') = \frac{1}{\sqrt{2\pi i B}} \int_{-\infty}^{\infty} \exp \left[ \frac{i}{2B} \left( Ax^2 - 2x'x + Dx'^2 \right) \right] f (x) \ dx. \tag{4} \]

In order to find the quantum correspondence (Fresnel operator) of Fresnel transform, the coherent state
\([12, 13]\) \(|x, p\rangle = \exp \left[ -\frac{1}{4} (p^2 + x^2) + (x + ip) a^\dagger / \sqrt{2} \right] |0\rangle\) is a good candidate for providing
with classical phase-space description of quantum systems, thus we construct the following ket-bra
projection operator

\[ \sqrt{\frac{1}{2} [A + D - i (B - C)]} \int_{-\infty}^{\infty} dx dp \left| \begin{array}{c} A \\ C \\ \end{array} \right\rangle \left. \begin{array}{c} x \\ p \end{array} \right\rangle = F, \tag{5} \]

as the FO, where the factor \(\sqrt{\cdot}\) is attached for anticipating the unitarity of the operator \(F\). In
\([5]\) the symplectic transformation \(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \left( \begin{array}{c} x \\ p \end{array} \right)\) mapping onto an Fresnel operator (FO) in
Hilbert space is manifestly shown through the coherent state basis. Using the notation of \( |z\rangle = \exp \left[ -\frac{1}{2} |z|^2 + za^\dagger \right] |0\rangle \), \( z = (x + ip)/\sqrt{2} \), and introducing complex numbers
\[
s = \frac{1}{2} [A + D - i (B - C)], \quad r = -\frac{1}{2} [A - D + i(B + C)], \quad |s|^2 - |r|^2 = 1,
\]
such that
\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \left( \begin{array}{c} x \\ p \end{array} \right) = \left( \begin{array}{cc} s & -r \\ s^* & z^* \end{array} \right) \left( \begin{array}{c} z \\ z^* \end{array} \right) = |sz - rz^*\rangle
\]
\[
= \exp \left[ -\frac{1}{2} |sz - rz^*|^2 + (sz - rz^*)a^\dagger \right] |0\rangle,
\]
so (5) becomes (14)
\[
F(s, r) = \sqrt{s} \int \frac{d^2 z}{\pi} |sz - rz^*\rangle \langle z|,
\]
Using the vacuum-state projector \(|0\rangle\langle 0|\) in normal ordering of boson operators
\[
|0\rangle\langle 0| = : e^{-a^\dagger a} :,
\]
and the technique of integration within an ordered product (IWOP) of operators \([15, 16, 17]\) we can directly perform the integration in (8) and obtain
\[
F(s, r) = \sqrt{s} \int \frac{d^2 z}{\pi} \exp \left[ -|s|^2 |z|^2 + sza^\dagger + z^* (a - ra^\dagger) + \frac{r^* s}{2} z^2 - \frac{r s^*}{2} z^* z^2 - a^\dagger a \right] 
\]
\[
= \frac{1}{\sqrt{s^*}} \exp \left( -\frac{r}{2s^*} a^\dagger a \right) : \exp \left( \frac{1}{s^*} - 1 \right) a^\dagger a : \exp \left( \frac{r^*}{2s^*} a^\dagger a \right),
\]
Note that this can be identified as a generalized squeezing operator with three real parameters \([18, 19, 20]\). It then follows
\[
\langle z | F(s, r) | z' \rangle = \frac{1}{\sqrt{s^*}} \exp \left( -\frac{|z'|^2}{2} - \frac{|z|^2}{2} - \frac{r}{2s^*} z^* z' - \frac{r^*}{2s^*} z'^* z + \frac{z'^*}{s^*} z \right).
\]
Then using the overlap between the coordinate eigenvector and the coherent state
\[
\langle x | z \rangle = \pi^{-1/4} \exp \left( -\frac{x^2}{2} + \sqrt{2}xz - \frac{z^2}{2} - \frac{|z|^2}{2} \right),
\]
and the completeness relation of coherent state we obtain the matrix element of \( F(s, r) \) in coordinate representation,
\[
\langle x^' | F(s, r) | x \rangle = \int \frac{d^2 z}{\pi} \langle x^' | z \rangle \langle z | F_1(s, r) \int \frac{d^2 z'}{\pi} |z'\rangle \langle z' | x \rangle
\]
\[
= \frac{1}{\sqrt{2\pi B}} \exp \left[ i \frac{B}{2} (Ax^2 - 2x^' x + Dx'^2) \right],
\]
which is just the kernel of generalized Fresnel transform in (14).

Now if we define \( g(x') = \langle x^' | g \rangle, f(x) = \langle x | f \rangle \) and using Eq. (13), we can rewrite Fresnel transform in Eq. (14) as
\[
\langle x^' | g \rangle = \int_{-\infty}^{\infty} dx \langle x^' | F(A, B, C) | x \rangle \langle x | f \rangle = \langle x^' | F(A, B, C) | f \rangle.
\]
Therefore, the 1-dimensional FT in classical optics corresponds to the 1-mode FO \( F(A, B, C) \) operating on state vector \( |f\rangle \) in Hilbert space, i.e. \( |g\rangle = F(A, B, C) |f\rangle \). Using the Fock representation of \( |x\rangle \),
\[
|x\rangle = \pi^{-1/4} \exp \left\{ -\frac{1}{2} x^2 + \sqrt{2} x a^\dagger - \frac{a^2}{2} \right\} |0\rangle,
\]
and (8) as well as (12) we calculate

\[ F(s, r) |x\rangle = \sqrt{s} \int \frac{d^2 z}{\pi} |sz - rz^*\rangle \langle z| x\rangle \]

\[ = \pi^{-1/4} \sqrt{s} \int \frac{d^2 z}{\pi} \exp \left[ -\frac{1}{2} (sz - rz^*)^2 + (sz - rz^*)a^\dagger \right] |0\rangle \]

\[ \exp \left( -\frac{x^2}{2} + \sqrt{2}xz^* - \frac{z^2}{2} - |z|^2 \right) = |x\rangle_{s,r}, \]

where

\[ |x\rangle_{s,r} \equiv \frac{\pi^{-1/4}}{\sqrt{s^* + r^*}} \exp \left\{ -\frac{sz^* - rz^*}{s^* + r^*} - \frac{r + s}{s^* + r^*}a^\dagger \right\} |0\rangle. \]

We can see that \( |x\rangle_{s,r} = F(s, r) |x\rangle \) is the eigenstate of Fresnel transformed quadrature phase, because from

\[ a |x\rangle_{s,r} = \left( \frac{\sqrt{2}x}{s^* + r^*} - \frac{s + r}{s^* + r^*}a^\dagger \right) |x\rangle_{s,r}, \]

so

\[ [(s^* + r^*) a + (s + r) a^\dagger] |x\rangle_{s,r} = \sqrt{2}(x |x\rangle_{s,r}. \]

This can be further confirmed by examining its completeness relation. Using the IWOP technique and (11) as well as \( \frac{1}{|s + r|^2} = \frac{s^* - r^*}{2(s^* + r^*)} + \frac{1}{2(s^* + r^*)} \), we can prove that \( |x\rangle_{s,r} \) make up a complete set (named as the tomography representation),

\[ \int_{-\infty}^{\infty} dx \left| x\right\rangle_{s,r} \left\langle x\right| = \int_{-\infty}^{\infty} \frac{dx}{|s + r|^2} \exp \left\{ -\frac{1}{D^2 + B^2} \left( x - \frac{D}{D^2 + B^2} \right)^2 \right\} \]

then using (13) and \( X = \frac{a + a^\dagger}{\sqrt{2}}, P = \frac{a^\dagger - \frac{B}{\sqrt{2}}}{\sqrt{2}} \) as well as \( s^* + r^* = D + iB, s^* - r^* = A - iC \), we can reform (20) as

\[ 1 = \frac{1}{\sqrt{D^2 + B^2}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} \exp \left\{ -\frac{1}{D^2 + B^2} \left( x - DX + BP \right)^2 \right\} \]

and \( |x\rangle_{s,r} \) is express as

\[ |x\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{D + iB}} \exp \left\{ -\frac{A - iC}{D + iB} x^2 - \frac{\sqrt{2}x}{D + iB} a^\dagger - \frac{D - iB a^\dagger}{D + iB} \right\} |0\rangle. \]

and (19) becomes

\[ (DX - BP) |x\rangle_{s,r} = x |x\rangle_{s,r}. \]

According to the Weyl quantization scheme [21]

\[ H(X, P) = \int \int_{-\infty}^{\infty} dp dx \Delta(x, p) h(x, p), \]

where \( h(x, p) \) is the Weyl correspondence of \( H(X, P) \), \( \Delta(x, p) \) is the Wigner operator [22] [23].

\[ \Delta(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{iup} \left| x + \frac{u}{2} \right\rangle \langle x - \frac{u}{2} \right| \]

(25)
we know that the classical Weyl correspondence of the projection operator \( |x\rangle x,rs,r \langle x| \) is
\[
2\pi Tr \left[ \Delta (x',p') |x\rangle x,rs,r \langle x| \right] = s,r \langle x| \int_{-\infty}^{\infty} du e^{ip'u} \left| x' + \frac{u}{2} \right| \langle x' - \frac{u}{2} | x \rangle s,r
\]
\[
= \frac{1}{2\pi B} \int_{-\infty}^{\infty} du \exp \left[ ip'u + \frac{i}{B} u (x - Dx') \right]
\]
\[
= \delta [x - (Dx' - Bp')].
\]
(26)

From we see that under the Fresnel transformation \( |x\rangle \langle x| \) becomes the tomographic density \( |x\rangle x,rs,r \langle x| \), and further from \( (24) \) and \( (26) \) we see that it is just the Radon transform of the Wigner operator, i.e.
\[
F |x\rangle \langle x| F^\dagger = |x\rangle x,rs,r \langle x| = \int \int_{-\infty}^{\infty} dx'dp' \delta [x - (Dx' - Bp')] \Delta (x',p').
\]
(27)

Therefore, the probability distribution for the Fresnel quadrature phase is the Radon transform of Wigner function
\[
| (x) F^\dagger | \psi \rangle |^2 = |s,r \langle x| \psi \rangle |^2 = \int \int_{-\infty}^{\infty} dx'dp' \delta [x - (Dx' - Bp')] \langle \psi | \Delta (x',p') | \psi \rangle .
\]
(28)

Moreover, the tomogram of quantum state \( |\psi \rangle \) is just the wave function of its Fresnel transformed state \( F^\dagger |\psi \rangle \), i.e. \( s,r \langle x| \psi \rangle = (x) F^\dagger |\psi \rangle \). \( (27) \) and \( (28) \) are the new relation between quantum tomography and optical Fresnel transform, which may provide experimentalists to figure out new approach for generating tomography.

Similarly, we find that for momentum density,
\[
F |p\rangle \langle p| F^\dagger = |p\rangle x,rs,r \langle p| = \int \int_{-\infty}^{\infty} dx'dp' \delta [p - (Ap' - Cx') \Delta (x',p'),
\]
(29)

where
\[
F |p\rangle = |p\rangle s,r = \frac{\pi^{-1/4}}{\sqrt{A - iC}} \exp \left\{ -\frac{D + iB p^2}{A - iC} 2 + \frac{\sqrt{2} ip}{A - iC} a^\dagger + \frac{A + iC a^2}{A - iC} 2 \right\} |0\rangle .
\]
(30)

As an application of the relation \( (27) \), we recall that the Fresnel operator \( F(r,s) \) makes up a loyal representation of the symplectic group, i.e.
\[
F(r,s)F^\dagger (r',s') = \sqrt{ss'} \int \frac{d^2z}{\pi} \int \frac{d^2z'}{\pi} |sz - rz'\rangle \langle z' |s'z' - r'z''\rangle (z')
\]
\[
= \frac{1}{\sqrt{ss'rr''}} \exp \left( -\frac{rr''}{2ss'rr''} a^2 \right) : \exp \left( \frac{1}{s' s''} - 1 \right) a^\dagger a : \exp \left( \frac{rr''}{2ss'rr''} a^2 \right) = F(r'', s''),
\]
(31)

where
\[
\left( \begin{array}{c}
\frac{s}{s' r''}
\frac{r}{s' r''}
\end{array} \right) = \left( \begin{array}{c}
r
s
\end{array} \right) \left( \begin{array}{c}
r'
\frac{s'}{r'}
s'
\end{array} \right), \ |s' s''|^2 - |r' r''|^2 = 1,
\]
(32)

or
\[
\left( \begin{array}{c}
A''
B''
C''
D''
\end{array} \right) = \left( \begin{array}{c}
A
B
C
D
\end{array} \right) \left( \begin{array}{c}
A'
B'
C'
D'
\end{array} \right) = \left( \begin{array}{c}
A A' + B C' + A B' + B D'
A' C + C' D + A' C + C' D
\end{array} \right).
\]
(33)

It then follows from \( (27) \) and \( (31) \) and that
\[
F^\dagger (r',s') F(r,s) |x\rangle \langle x| F^\dagger (r',s') \]
\[
= \int \int_{-\infty}^{\infty} dx'dp' \delta [x - ((B'C + D'D') x' - (AB' + BD') p')] \Delta (x',p')
\]
\[
= \int \int_{-\infty}^{\infty} dx'dp' \delta [x - (D'' x' - B'' p')] \Delta (x',p'),
\]
(34)
in this way the complicated Radon transform of tomography can be viewed as the sequential operation of two Fresnel transforms.

In sum, we have revealed the new relation connecting optical Fresnel transformation to Radon transformation in quantum tomography, i.e. the probability distribution for the Fresnel quadrature phase is the tomography (Radon transform of Wigner function). The tomography representation \( s_r \langle x \rangle \) is set up, based on which the tomogram of a state \( |\psi\rangle \) is just the wave function of its Fresnel transformed state \( F^\dagger |\psi\rangle \), i.e. \( s_r \langle x | \psi \rangle = \langle x | F^\dagger |\psi\rangle \). The group property of Fresnel operators help us to analyze complicated Radon transforms in terms of some sequential Fresnel transformations. The new relation may provide experimentalists to figure out new approach for realizing tomography.

We would like to acknowledge support from the National Natural Science Foundation of China under Grant Nos. 10775097 and 10475056.

References

[1] K. Vogel and H. Risken, Phys. Rev. A 40, 2847 (1989).
[2] D. T. Smithey, M. Beck, M. G. Raymer and A. Faridani, Phys. Rev. Lett. 70, 1244 (1993).
[3] D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani, Phys. Scr. 48, 35 (1993).
[4] D. T. Smithey, M. Beck, J. Cooper, and M. G. Raymer, Phys. Rev. A 48, 3159 (1993).
[5] Y. C. Wei, J. Hsieh and G. Wang, Phys. Rev. Lett. 95, 258102 (2005)
[6] G. R. Myers, S. C. Mayo, T. E. Gureyev, D. M. Paganin and S. W. Wilkins, Phys. Rev. A 76, 045804 (2007)
[7] M. Asorey, P. Facchi, V. I. Manko, G. Marmo, S. Pascazio, and E. G. Sudarshan, Phys. Rev. A 76, 012117 (2007)
[8] A. Wünsche, J. Mod. Opt. 44, 2293 (1997).
[9] A. Wünsche, Phys. Rev. A 54, 5291 (1996).
[10] Kogelnik H, Appl. Opt. 4, 1562 (1965).
[11] J. W. Goodman, Introduction to Fourier Optics, McGraw-Hill, New York, 1972.
[12] J.R. Klauder and B.-S. Skagerstam, Coherent States, World Scientific, Singapore (1995).
[13] R. J. Glauber Phys. Rev. 131, 2766 (1963); Phys. Rev. 130, 2529 (1963).
[14] Hong-yi Fan and Hai-liang Lu, Opt. Commun. 258, 51 (2006)
[15] Hong-yi Fan, H.Z. Zaidi and J.R. Klauder, Phys. Rev. D 35, 1831 (1987); Hong-yi Fan and J.R. Klauder, Phys. Rev. A 49, 704 (1994); Hong-yi Fan, J. Opt B: Quantum Semiclass. Opt. 5, R147 (2003).
[16] A. Wünsche, J Opt B: Quantum Semiclass. Opt. 1, R11 (1999).
[17] Hong-yi Fan, Hai-liang Lu and Yue Fan, Ann. Phys. 321, 480 (2006)
[18] D. F. Walls, Nature 324, 210 (1986).
[19] See e.g., G. M. D’Ariano, M. G. Rassetti, J. Katriel and A. I. Solomon, Squeezed and Nonclassical Light ed P Tombesi and E R Pike (New York:Plenum 1989).
[20] V. Bužek, J. Mod.Opt. 37, 303 (1990); R. Loudon and P. L. Knight, J. Mod.Opt. 34, 709 (1987); V. V. Dodonov, J. Opt. B: Quantum Semiclass. Opt. 4, R1 (2002).
[21] H. Weyl, Z. Phys. 46, 1 (1927).

[22] Hong-yi Fan and Yue Fan, Mod. Phys. Lett. A 12, 2325 (1997).

[23] E. Wigner, Phys. Rev. 40, 749 (1932); R. F. O’Connell and E. P. Wigner, Phys. Lett. A 83, 145 (1981); M. Hillery, R. F. O’Connell, M. O. Scully and E. P. Wigner, Phys. Rep. 106 121 (1984); G. S. Agawal and E. Wolf, Phys. Rev. D 2, 2161 (1972); Phys. Rev. D 2, 2187 (1972); Phys. Rev. D 2, 2206 (1972).