Geometric Random Inner Products: A New Family of Tests for Random Number Generators

Shu-Ju Tu and Ephraim Fischbach

Department of Physics, Purdue University, West Lafayette, Indiana 47907-1396

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We present a new computational scheme, GRIP (Geometric Random Inner Products), for testing the quality of random number generators. The GRIP formalism utilizes geometric probability techniques to calculate the average scalar products of random vectors generated in geometric objects, such as circles and spheres. We show that these average scalar products define a family of geometric constants which can be used to evaluate the quality of random number generators. We explicitly apply the GRIP tests to several random number generators frequently used in Monte Carlo simulations, and demonstrate a new statistical property for good random number generators.

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I. INTRODUCTION

Monte Carlo methods are among the most widely used numerical algorithms in computational science and engineering [1]. The key element in a Monte Carlo calculation is the generation of random numbers. Although a truly random number sequence produced by either a physical process such as nuclear decay, an electronic device etc., or by a computer algorithm, may not actually exist, a new and computationally easy-to-implement scheme to investigate random number generators is always highly desirable.

There have been many proposed schemes for the quality measure of random number generators [2, 3, 4, 5, 6, 7, 8, 9]. These computational tests are based either on probability theory and statistical methods (for example: the $\chi^2$ test, the Smirnov-Kolmogorov test, the correlation test, the spectral test, and the DieHard battery of randomness tests), or on mathematical modeling and simulation for physical systems (for example: random walks and Ising model simulations). These methods also open the door to studying the properties of random number sequences along with the three known qualitative measures the quality of random number generators by explicitly adding a new quantitative property to random number sequences along with the three known qualitative properties summarized in Ref. [10].

In this paper, we study a new method to measure $n$-dimensional randomness which we denote by GRIP (Geometric Random Inner Products). The GRIP family of tests is based on the observation that the average scalar products of random vectors produced in geometric objects (e.g., circles and spheres), define geometric constants which can be used to evaluate the quality of random number generators. After presenting the simplest example of a GRIP test, we exhibit a computational method for implementing GRIP, which is then used to analyze a number of random number generators. We then discuss the GRIP formalism in detail and show how a random number sequence, when converted to random points in a space defined by a geometric object, can produce a series of known geometric constants. Later we introduce additional members and include them within the GRIP family. We then present the computational results for configurations of four, six, and eight random points, along with a consideration of some key issues. Finally, we conclude by discussing how the GRIP test measures the quality of random number generators by explicitly adding a new quantitative property to random number sequences along with the three known qualitative properties summarized in Ref. [10].

II. GENERAL DESCRIPTION OF THE GRIP FORMALISM

The GRIP scheme is derived from the theory of random distance distribution for spherical objects, and can be generalized to other geometric objects with arbitrary densities [10, 11]. First, three random points ($\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$) are independently produced from the sample space defined by a geometric object. We then evaluate the average inner product of $\vec{r}_{12} \cdot \vec{r}_{23}$ from two associated random vectors, $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$ and $\vec{r}_{23} = \vec{r}_3 - \vec{r}_2$. For a geometric object such as an $n$-ball of uniform density with a radius $R$, the analytical result is a geometric constant which can be expressed in terms of the dimensionality $n$ of the space [12, 13]:

$$\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle_n = -\frac{n}{n+2} R^2. \quad (1)$$

A simple derivation of Eq. (1) can be found in the Appendix.

The following procedures are the numerical implementation of our testing programs. A random number sequence produced from a random number generator is used to generate a series of three random points $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, ...
and \( \vec{r}_3 \) such that these random points are uniformly distributed in an \( n \)-dimensional spherical ball \( B \) of radius \( R \), where

\[
B = \{(x_1, x_2, \cdots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2 \}. 
\] (2)

We then compute a series of values for \( \vec{r}_{12} \cdot \vec{r}_{23} \). If \( \vec{r}_{12} \cdot \vec{r}_{23} \) is evaluated \( N \) times, then statistically we expect

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (\vec{r}_{12} \cdot \vec{r}_{23})_i = -\frac{n}{n+2} R^2, 
\] (3)
as predicted by Eq. (1).

## III. RANDOM NUMBER GENERATORS

We now apply the GRIP test to the following random number generators frequently used in Monte Carlo simulations.

1. **RAN0** - a linear congruential generator \( [2, 3] \):

\[
x_n = 16807 \times x_{n-1} \mod 2147483647. 
\] (4)

2. **RAN3** - a lagged Fibonacci generator \( [2, 3] \):

\[
x_n = (x_{n-55} - x_{n-24}) \mod 2^{31}. 
\] (5)

3. **R31** - a generalized feedback shift register (GFSR) generator \( [2, 3] \):

\[
x_n = x_{n-31} \oplus x_{n-3}, 
\] (6)

where \( \oplus \) is the bitwise exclusive OR operator.

4. **durxor** - a generator selected from IBM ESSL (Engineering and Scientific Subroutine Library) \( [19] \).

5. **durand** - a generator selected from IBM ESSL (Engineering and Scientific Subroutine Library) and the sequence period of durand is shorter than durxor \( [18] \).

6. **ran_gen** - one of the subroutines in IMSL libraries from Visual Numeric \( [19] \).

7. **Random** - a Fortran 90/95 standard intrinsic random number generator \( [20] \).

8. **Weyl** - a Weyl sequence generator \( [21, 22] \):

\[
x_n = \{n \alpha \}, 
\] (7)

where \( \{x\} \) is the fractional part of \( x \), and \( \alpha \) is an irrational number such as \( \sqrt{2} \).

9. **NWS** - a nested Weyl sequence generator \( [21, 22] \):

\[
x_n = \{n \alpha \}. 
\] (8)

10. **SNWS** - a shuffled nested Weyl sequence generator \( [21, 22] \):

\[
s_n = M \{n \alpha \} + \frac{1}{2}, 
\] (9)

\[
x_n = \{s_n \alpha \}, 
\] (10)

where \( M \) is a large positive integer.

The computational results obtained from Eq. (3) when \( n = 3 \) and \( n = 9 \) are presented in Table I. Results for random number generators based on other algorithms such as the Data Encryption Standard (DES) \( [2, 3] \) can be found in Ref. \( [23] \) along with the computed results obtained from other geometric objects. We note that both the ran_gen and RAN0 generators perform better overall, while the NWS and Weyl generators (which are based on the Weyl sequence method) are ranked lowest compared to the other generators. The reasons why this is the case will be discussed later.

## IV. GRIP ANALYSIS

In the following, we analyze the relationship between GRIP and a random number sequence, and show how a good random number sequence, when converted to random points in a space defined by a geometric object, can produce a series of known \( n \)-dimensional geometric constants. A random number sequence generated from a random number generator can be written as,

\[
a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} \cdots. 
\] (11)

When the sequence is converted to represent random points in a 2-dimensional geometric object, the random numbers in Eq. (11) can then be grouped in pairs as

\[
(a_1 a_2) (a_3 a_4) (a_5 a_6) a_7 a_8 a_9 a_{10} \cdots, 
\] (12)

where Cartesian coordinates are used. The first set of random points \( \{\vec{r}_1, \vec{r}_2, \vec{r}_3\} \) can thus be identified as

\[
\vec{r}_1 = (a_1, a_2), \quad \vec{r}_2 = (a_3, a_4), \quad \vec{r}_3 = (a_5, a_6). 
\] (13)

GRIP then uses \( \vec{r}_1, \vec{r}_2, \) and \( \vec{r}_3 \) to evaluate the average scalar product which can be computed by rewriting,
TABLE I: Computed results for \( \langle \vec{r}_1 \cdot \vec{r}_2 \rangle \). RNG denotes the specific random number generator defined in the text, and “Expected” is the exact result obtained from Eq. (1). The entries are ranked in terms of their errors, which are the absolute values of the differences between the expected and computed results. For each entry in the table \( N = 10^8 \), and multiple seeds were used where appropriate.

| Rank | RNG       | \( n = 3 \)     | Error         | RNG       | \( n = 9 \)     | Error         |
|------|-----------|-----------------|---------------|-----------|-----------------|---------------|
| 1    | ran_gen   | -0.59999802     | 0.00000198    | RAN0      | -0.81819136     | 0.00000955    |
| 2    | RAN0      | -0.60000722     | 0.00000722    | ran_gen   | -0.81821041     | 0.00002860    |
| 3    | R31       | -0.60005031     | 0.00005031    | Random    | -0.81821550     | 0.00003369    |
| 4    | durand    | -0.59991945     | 0.00008055    | durand    | -0.81821772     | 0.00003591    |
| 5    | duxor     | -0.59991306     | 0.00008694    | duxor     | -0.81822185     | 0.00004044    |
| 6    | RAN3      | -0.59988610     | 0.00011390    | R31       | -0.81824459     | 0.00006278    |
| 7    | Random    | -0.59987912     | 0.00012088    | RAN3      | -0.81827541     | 0.00009360    |
| 8    | SNWS      | -0.59969277     | 0.00030723    | SNWS      | -0.81795246     | 0.00022935    |
| 9    | NWS       | -0.62988317     | 0.02988317    | NWS       | -0.82539808     | 0.00721627    |
| 10   | Weyl      | -1.80809907     | 1.20809907    | Weyl      | -1.05604651     | 0.23786470    |

Expected: -0.60000000

\[
\langle \vec{r}_1 \cdot \vec{r}_2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{2} (a_{6i-4+j} - a_{6i-6+j})(a_{6i-2+j} - a_{6i-4+j}),
\]

where \( N \) is a large positive integer. When the geometric object is a circle of radius \( R \) and uniform density, we expect \( \langle \vec{r}_1 \cdot \vec{r}_2 \rangle \approx -0.5R^2 \) as predicted by Eq. (1).

The analysis for 2-dimensional GRIP can be immediately generalized to the \( n \)-dimensional case. When the sequence in Eq. (11) is used to generate random points in a \( n \)-dimensional spherical object, we can regroup Eq. (11) as follows:

\[
(a_1 \cdots a_k)(a_{k+1} \cdots a_{2k}) (a_{2k+1} \cdots a_{3k}) (\cdots) (\cdots) \cdots.
\]

The average scalar product of \( \vec{r}_1 \cdot \vec{r}_2 \) can then be expressed as

\[
\langle \vec{r}_1 \cdot \vec{r}_2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} (a_{3in-2n+j} - a_{3in-3n+j})(a_{3in-n+j} - a_{3in-2n+j}).
\]

When the geometric object is an \( n \)-ball with a radius \( R = 1 \) and a uniform density, we expect from Eq. (10) that the result of Eq. (16) should be a geometric constant, \(-n/(n+2)\).

V. GRIP MEMBERS

For practical computational purposes, we may wish to transform a random number sequence from a uniform density distribution to one which is non-uniform. One of the most important non-uniform density distributions is the Gaussian (normal) distribution \( P(r) \) with mean zero.
and standard deviation $\sigma$,

$$P(r) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-(1/2)(r^2/\sigma^2)},$$  \hspace{1cm} (17)

Here $\int_0^\infty P(r) \, dr = 1$, $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, and $n$ is the space dimensionality. One can use either the Box-Muller transformation method to generate a random number sequence with a Gaussian density distribution, or use available subroutines from major computational scientific libraries such as IBM ESSL and IMSL \cite{18,19}. By applying the probability density function of the random distance distribution as discussed in Ref. \cite{17}, one can add a new GRIP member to investigate the quality of a Gaussian random number generator, and this new GRIP test can be expressed as:

$$\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle_n = -n\sigma^2.$$

A very common situation arises when one has to produce random points uniformly distributed on the surface of an $n$-sphere of radius $R$. Some general computational techniques for doing this are summarized in Refs. \cite{2,18}. We can then use

$$\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle_n = -R^2,$$

where $2m$ ($m = 2, 3, 4$ etc.) is a positive even number. A derivation of Eq. (20) can be found in the Appendix.

We summarize the computational results for Eq. (20) when $m = 2, 3, 4$ in Tables \ref{table1} \cite{11} and \ref{table4}. A discussion of other results, such as Eqs. (18) and (19), can be found in Ref. \cite{23}.

We observe that all of the generators except NWS and Weyl perform significantly better in $n = 3$ than in $n = 9$ using the GRIP test based on $\langle \langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle \vec{r}_{34} \cdot \vec{r}_{41} \rangle_n$. We also note from Table \ref{table1} and the $n = 9$ results (from R31 to RAN0), that these results are clearly biased to larger numbers compared to the expected value. One interpretation may be that $\langle \langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle \vec{r}_{34} \cdot \vec{r}_{41} \rangle_n$ is a more sensitive and dedicated computational test for investigating random number generators than other GRIP tests. We also note that the results for $n = 9$ are overall worse than $n = 3$, and that the results for $\langle \langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle \vec{r}_{34} \cdot \vec{r}_{41} \rangle_n$ reveal a more significant bias than in any of the other cases. These results suggest that the GRIP test either in higher dimensions (large $n$), or using a configuration of four random points, can serve as a more computationally sensitive test to detect non-random patterns hidden in random number sequences. Finally we note that it is not surprising that the NWS and Weyl generators are ranked worst among all cases in our GRIP test. As reported previously in \cite{23}, these two show unacceptable non-random behavior and strong correlations.

VI. CONCLUSIONS

We have presented a new computational paradigm for evaluating the quality of random number generators. We demonstrate how GRIP helps to understand complexity and randomness by adding a new property, besides three known properties (typical, chaotic, and the stability of frequencies) \cite{10}, for random number sequences. This quantitative feature shows how a random number sequence, when converted to random points in a space defined by a geometric object, can produce a series of known geometric constants. Ten random number generators were selected to run our GRIP tests, and they are ranked based on the errors between the numerical and analytical results. Finally we note that one implication of our work is that computational scientists should test...
TABLE II: Computed results for \(\langle (\vec{r}_{12} \cdot \vec{r}_{23}) (\vec{r}_{34} \cdot \vec{r}_{41}) \rangle_n\). RNG denotes the specific random number generator defined in the text, and “Expected” is the exact result obtained from Eq. (1). The entries are ranked in terms of their errors, which are the absolute values of the differences between the expected and computed results. For each entry in the table the random number generators they use in their simulations, and verify that their random number generators pass as many proposed tests as possible.

| Rank | RNG  | \(n = 3\) | Error | RNG  | \(n = 9\) | Error |
|------|------|-----------|-------|------|-----------|-------|
| 1    | RAN3 | 0.47997937 | 0.0000263 | R31  | 0.74538039 | 0.0015874 |
| 2    | RAN0 | 0.47995440 | 0.00004560 | RAN3 | 0.74567962 | 0.0018779 |
| 3    | durxor | 0.48004715 | 0.00004715 | SNWS | 0.74582736 | 0.0020587 |
| 4    | durand | 0.48006530 | 0.00006530 | ran_gen | 0.74634598 | 0.0025433 |
| 5    | R31  | 0.48002865 | 0.00008265 | durxor | 0.74673990 | 0.00257825 |
| 6    | Random | 0.47990347 | 0.00009653 | Random | 0.74644366 | 0.00264201 |
| 7    | ran_gen | 0.47986482 | 0.00013518 | durand | 0.74646979 | 0.00266814 |
| 8    | SNWS | 0.47975570 | 0.00024430 | RAN0  | 0.74659547 | 0.00279382 |
| 9    | NWS  | 0.55841828 | 0.07842818 | NWS  | 0.69652947 | 0.04727218 |
| 10   | Weyl | 3.31162983 | 2.83162983 | Weyl  | 1.28608478 | 0.54228305 |

TABLE III: Computed results for \(\langle (\vec{r}_{12} \cdot \vec{r}_{23}) (\vec{r}_{34} \cdot \vec{r}_{41}) \rangle_n\). RNG denotes the specific random number generator defined in the text, and “Expected” is the exact result obtained from Eq. (1). The entries are ranked in terms of their errors, which are the absolute values of the differences between the expected and computed results. For each entry in the table the random number generators they use in their simulations, and verify that their random number generators pass as many proposed tests as possible.

| Rank | RNG  | \(n = 3\) | Error | RNG  | \(n = 9\) | Error |
|------|------|-----------|-------|------|-----------|-------|
| 1    | durand | -0.24000387 | 0.0000387 | SNWS | -0.55543884 | 0.0006052 |
| 2    | Random | -0.24001846 | 0.00001846 | R31  | -0.55373136 | 0.00073896 |
| 3    | RAN3  | -0.23997958 | 0.0002042 | ran_gen | -0.55521687 | 0.00074555 |
| 4    | ran_gen | -0.24003206 | 0.0003206 | durxor | -0.55524213 | 0.00077181 |
| 5    | RAN0  | -0.23994286 | 0.0005714 | durand | -0.55528032 | 0.00081000 |
| 6    | durxor | -0.24011639 | 0.00011639 | Random | -0.55533476 | 0.00086714 |
| 7    | SNWS | -0.23964945 | 0.00035055 | RAN0  | -0.55561838 | 0.00114806 |
| 8    | R31  | -0.24070892 | 0.00070892 | RAN3  | -0.55565656 | 0.00148624 |
| 9    | NWS  | -0.28813072 | 0.04813072 | NWS  | -0.48246750 | 0.07200282 |
| 10   | Weyl | -5.78662461 | 5.54662461 | Weyl  | -1.31133451 | 0.7566419 |

APPENDIX: DERIVATION OF \(\langle (\vec{r}_{12} \cdot \vec{r}_{23}) \rangle_n\) AND \(\langle (\vec{r}_{12} \cdot \vec{r}_{23}) (\vec{r}_{34} \cdot \vec{r}_{41}) \rangle_n\)

We derive the analytical result of Eq. (1) for a circle \((n = 2)\) of radius \(R\) and uniform density. The same derivation can be applied to the case of \(n\) dimensions where \(n \geq 3\). We label three independent random points \(\vec{r}_{12}, \vec{r}_{23},\) and \(\vec{r}_{31}\) as 1, 2, and 3 in Fig. 1, and then calculate

\[
\vec{r}_{12} \cdot \vec{r}_{23} = r_{12} r_{23} \cos \theta = -r_{12} r_{23} \cos \alpha,
\]

where \(\alpha + \theta = \pi\). From the triangle formed by the random points, we then have

\[
r_{31}^2 = r_{12}^2 + r_{23}^2 - 2r_{12} r_{23} \cos \alpha.
\]

Extending this 2-dimensional case to the \(n\)-dimensional case, and combining Eqs (A.1) and (A.2), we then evaluate

\[
\langle (\vec{r}_{12} \cdot \vec{r}_{23}) \rangle_n = -\frac{1}{2} \langle r_{12}^2 + r_{23}^2 - r_{31}^2 \rangle_n = -\frac{1}{2} \langle r^2 \rangle_n = -\frac{1}{2} \int_0^{2\pi} P_n(r) r^2 dr = -\frac{n}{n+2} R^2,
\]

where we have utilized the fact that \(\vec{r}_{12}, \vec{r}_{23},\) and \(\vec{r}_{31}\) are three independent random vectors. The functions \(P_n(r)\)
TABLE IV: Computed results for \(\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle \) \(\langle \vec{r}_{34} \cdot \vec{r}_{45} \rangle \) \(\langle \vec{r}_{56} \cdot \vec{r}_{67} \rangle \) \(\langle \vec{r}_{78} \cdot \vec{r}_{81} \rangle \). RNG denotes the specific random number generator defined in the text, and “Expected” is the exact result obtained from Eq. (20). The entries are ranked in terms of the errors, which are the absolute values of the differences between the expected and computed results. For each entry in the table \(N = 10^8\), and multiple seeds were used where appropriate.

| Rank | RNG    | \(n = 3\) Error | RNG    | \(n = 9\) Error |
|------|--------|-----------------|--------|-----------------|
| 1    | ran_gen | 0.00000377      | durrxor| 0.00003778      |
| 2    | durrxor| 0.00009222      | ran_gen| 0.00013053      |
| 3    | RAN3   | 0.00017966      | durand | 0.00015802      |
| 4    | RAN0   | 0.00021044      | Random | 0.00022372      |
| 5    | durand | 0.00028692      | RAN3   | 0.00036773      |
| 6    | Random | 0.00091033      | RAN0   | 0.00050459      |
| 7    | SNWS   | 0.00024771      | SNWS   | 0.00084185      |
| 8    | R31    | 0.00244443      | R31    | 0.00189715      |
| 9    | NWS    | 0.02888766      | NWS    | 0.01713583      |
| 10   | Weyl   | 10.0148479      | Weyl   | 1.09122246      |

**Expected** 0.13440000 **Expected** 0.44873984

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FIG. 1: Three random points configuration in a circle.

\[
\langle \vec{r}_{12} \cdot \vec{r}_{23} \rangle = \frac{\int_0^R dx_1 \int_\sqrt{R^2-x_1^2}^R dy_1 \cdots \int_0^R dx_4 \int_\sqrt{R^2-x_4^2}^R dy_4}{\int_0^R dx_1 \int_\sqrt{R^2-x_1^2}^R dy_1 \cdots \int_0^R dx_4 \int_\sqrt{R^2-x_4^2}^R dy_4} = \frac{3}{8} R^4
\]

where

\[
f_1 = (x_2 - x_1)(x_3 - x_2) + (y_2 - y_1)(y_3 - y_2),
\]

\[
f_2 = (x_4 - x_3)(x_1 - x_4) + (y_4 - y_3)(y_1 - y_4).
\]

A derivation of the general result using the probability density functions \(P_n(r)\) in Eq. (A.3) can be found in Ref. (29).

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