Abstract. Let $K$ be a compact Lie group, endowed with a bi-invariant Riemannian metric, which we denote by $\kappa$. The complexification $K^C$ of $K$ inherits a Kähler structure having twice the kinetic energy of the metric as its potential; let $\varepsilon$ denote the symplectic volume form. Left and right translation turn the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon)$ of square-integrable holomorphic functions on $K^C$ relative to a suitable measure written as $e^{-\kappa/t} \varepsilon$ into a unitary $(K \times K)$-representation; here $\eta$ is an additional term coming from the metaplectic correction, and $t > 0$ is a real parameter. In the physical interpretation, this parameter amounts to Planck’s constant $\hbar$.

We establish the statement of the Peter-Weyl theorem for the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon)$ to the effect that

(i) $\mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon)$ contains the vector space of representative functions on $K^C$ as a dense subspace and that

(ii) the assignment to a holomorphic function of its Fourier coefficients yields an isomorphism of Hilbert algebras from the convolution algebra $\mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon)$ onto an algebra of the kind $\oplus \text{End}(V)$. Here $V$ ranges over the irreducible rational representations of $K^C$ and $\oplus$ refers to a suitable completion of the direct sum algebra $\oplus \text{End}(V)$.

Consequences are:

(i) the existence of a uniquely determined unitary isomorphism between $L^2(K, dx)$ (where $dx$ refers to Haar measure on $K$) and the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon)$, and

(ii) a proof that this isomorphism coincides with the Blattner-Kostant-Sternberg pairing map from $L^2(K, dx)$ to $\mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon)$, multiplied by $(4\pi t)^{-\dim(K)/4}$.

Among our crucial tools is Kirillov’s character formula. Our methods are geometric, rely on the orbit method, and are independent of heat kernel harmonic analysis, which is used by B. C. Hall to obtain many of these results [J. of Funct. Anal. 122 (1994), 103–151], [Comm. in Math. Physics 226 (2002), 233–268].

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Introduction

Let $K$ be a compact Lie group and let $K^C$ be its complexification. Given a finite dimensional rational representation $V$ of $K^C$, the familiar assignment to

$$\varphi \otimes w \in V^* \otimes V \cong \text{End}(V)$$

of the representative function $\Phi_{\varphi,w}$ given by $\Phi_{\varphi,w}(q) = \varphi(qw)$ ($q \in K^C$) yields an embedding of $\text{End}(V)$ into the Hopf algebra $C[K^C]$ of representative functions on $K^C$, the diagonal map of $C[K^C]$ being induced by the group multiplication. As $V$ ranges over the finite dimensional irreducible representations of $K^C$, these maps assemble to an isomorphism

$$(0.1) \quad \bigoplus \text{End}(V) \rightarrow C[K^C]$$

of $(K^C \times K^C)$-representations. Restriction to $K$ induces an isomorphism from the Hopf algebra $C[K^C]$ onto the Hopf algebra $R(K)$ of complex representative functions of $K$ and induces an isomorphism

$$(0.2) \quad \bigoplus \text{End}(V) \rightarrow R(K)$$

of $(K \times K)$-representations. Integration over $K$ relative to Haar measure induces inner products on the left- and right-hand sides of (0.2), the completion of the right-hand side is the Hilbert space $L^2(K, dx)$ relative to Haar measure $dx$ which, as a unitary $K$-representation, is the regular representation, and the completion

$$(0.3) \quad \hat{\bigoplus} \text{End}(V) \rightarrow L^2(K, dx)$$

of the isomorphism (0.2) yields the familiar Peter-Weyl theorem for the compact group $K$. Here and below the notation $\hat{\bigoplus}$ refers to a Hilbert space direct sum involving infinitely many summands. Moreover, integration over $K$ relative to Haar measure, suitably normalized, induces a convolution product on $R(K)$ and on $L^2(K, dx)$. Relative to this convolution product, when $\bigoplus \text{End}(V)$ is endowed with the obvious algebra structure having the $\text{End}(V)$’s as minimal two-sided ideals, (0.2) is an isomorphism of algebras (without 1 unless $K$ is a finite group), and (0.3) is an isomorphism of Hilbert algebras and hence yields in particular the decomposition of the convolution algebra $L^2(K, dx)$ into minimal two-sided topological ideals. Furthermore, a choice of bi-invariant Riemannian metric on $K$ determines a Laplace-Beltrami operator on $K$ which admits a unique extension to an (unbounded) self-adjoint operator on $L^2(K, dx)$, and the decomposition on the left-hand side of (0.3) is precisely the standard refinement of the spectral decomposition of this operator. This extension of the Laplace-Beltrami operator on $K$ to an operator on $L^2(K, dx)$, multiplied by $-1/2$, is the (quantum mechanical) energy operator on $L^2(K, dx)$ associated with the metric; indeed, vertical dequantization of this operator or, equivalently, the operation of passing to the symbol, yields the energy associated with the Riemannian metric on $K$, viewed as a function on the total space $T^*K$ of the cotangent bundle of $K$.

Since $K^C$ is reductive, the coordinate ring of $K^C$ coincides with the algebra of representative functions on $K^C$, and hence the Hilbert space $L^2(K, dx)$ contains the complex vector space underlying the coordinate ring of $K^C$. However, a more natural
the Hilbert space containing the coordinate ring would be a Hilbert space of holomorphic functions on \( K^C \). This raises the question whether \( K^C \) carries a suitable measure such that, after completion relative to this measure, (0.1) yields an isomorphism of Hilbert spaces between a Hilbert space of the kind \( \oplus \text{End}(V) \) and a Hilbert space of holomorphic functions on \( K^C \), and what the spectral decomposition of the energy operator might correspond to in terms of \( K^C \). Actually, we were led to these questions by the observation that Kähler quantization is, perhaps, better suited to explore quantization in the presence of singularities than ordinary Schrödinger quantization. We shall comment on this motivation below.

The desired measure on \( K^C \) is provided for by the measure coming from half-form quantization on \( K^C \), though the construction of the measure itself is independent of the program of geometric quantization: A choice of bi-invariant Riemannian metric on \( K \) and the polar decomposition map of \( K^C \) determine a diffeomorphism between \( T^*K \) and \( K^C \) and, via this diffeomorphism, \( T^*K \) and \( K^C \) both become Kähler manifolds where the requisite symplectic structure is the cotangent bundle structure on \( T^*K \). Since \( K^C \) is parallelizable, it admits a metaplectic structure for trivial reasons, and the bundle of holomorphic half-forms on \( K^C \) furnishes a measure on \( K^C \) having the desired properties. This measure can be written in the form \( e^{-\kappa/\eta \varepsilon} \); here \( \kappa \) is the metric, written as a function on \( T^*K \), \( t > 0 \) is a real parameter which, in the physical interpretation, amounts to Planck’s constant \( h \), \( \varepsilon \) is the Liouville volume measure, and \( \eta \) is a suitable function coming from the metaplectic correction and closely related with the familiar function coming into play in Kirillov’s character formula and commonly written as \( j \) [4], [21].

We shall establish a holomorphic version of the Peter-Weyl theorem to the effect that the following hold: (i) The Hilbert space \( \mathcal{H}L^2(K^C, e^{-\kappa/\eta \varepsilon}) \) of holomorphic functions that are square integrable relative to the measure \( e^{-\kappa/\eta \varepsilon} \) contains the vector space \( \mathbb{C}[K^C] \) of representative functions on \( K^C \) as a dense subspace in such a way that the decomposition (0.1) induces the decomposition of \( \mathcal{H}L^2(K^C, e^{-\kappa/\eta \varepsilon}) \), viewed as a unitary \((K \times K)\)-representation, into its isotypical summands; and (ii) the assignment to a holomorphic function of its Fourier coefficients yields an isomorphism of Hilbert algebras from \( \mathcal{H}L^2(K^C, e^{-\kappa/\eta \varepsilon}) \), made into an algebra via the convolution product, onto an algebra of the kind \( \oplus \text{End}(V) \) where \( V \) ranges over the irreducible rational representations of \( K^C \) and where \( \oplus \) refers to a suitable completion of the direct sum algebra \( \oplus \text{End}(V) \); a precise statement is given as Theorem 1.14 below. We then refer to the resulting decomposition as the holomorphic Peter-Weyl decomposition of the Hilbert space \( \mathcal{H}L^2(K^C, e^{-\kappa/\eta \varepsilon}) \). The algebraic decomposition (0.1) itself into isotypical \((K^C \times K^C)\)-summands is commonly interpreted as an algebraic Peter-Weyl theorem.

Consequences of the holomorphic Peter-Weyl theorem are the existence of a uniquely determined unitary \((K \times K)\)-equivariant isomorphism between \( L^2(K, dx) \) (where \( dx \) refers to Haar measure on \( K \)) and the Hilbert space \( \mathcal{H}L^2(K^C, e^{-\kappa/\eta \varepsilon}) \), given in Theorem 5.3 below, and a holomorphic Plancherel theorem, given as Corollary 5.4 below. In Section 6 we shall then show that the isomorphism between the two Hilbert spaces coincides with the Blattner-Kostant-Sternberg pairing map from the Hilbert space \( L^2(K, dx) \) to the Hilbert space \( \mathcal{H}L^2(K^C, e^{-\kappa/\eta \varepsilon}) \), multiplied by \((4\pi t)^{-\dim(K)/4}\). However the abstract unitary isomorphism between the two Hilbert spaces is independent of the Blattner-Kostant-Sternberg pairing. The identification
of the two Hilbert spaces implies, in particular, that the spectral decomposition of the energy operator on $H^2(\mathbb{H}^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$ associated with the metric refines to the holomorphic Peter-Weyl decomposition of this Hilbert space in the usual manner and thus yields the decomposition of $H^2(\mathbb{H}^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$ into irreducible isotypical $(\mathbb{K} \times \mathbb{K})$-representations; this will be explained in Section 7.

A crucial step towards the holomorphic Peter-Weyl theorem consists in proving that the representative functions on $\mathbb{K}^\mathbb{C}$ are square integrable relative to the measure $e^{-\kappa/t}\eta\varepsilon$ on $\mathbb{K}^\mathbb{C}$ and, furthermore, in actually calculating their square integrals. We do these calculations by means of Kirillov’s character formula; we could as well have taken Weyl’s character formula, but then the calculations would be somewhat more involved. Our argument establishing the completeness of the representative functions, given in Section 4 below, is geometric and is guided by the principle that quantization commutes with reduction.

An obvious question emerges here: How is the Hilbert space $H^2(\mathbb{K}^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$ of holomorphic functions that are square integrable relative to the measure $e^{-\kappa/t}\eta\varepsilon$ on $\mathbb{K}^\mathbb{C}$ related to the other Hilbert spaces? In a final section we shall show that, indeed, as a unitary $(\mathbb{K} \times \mathbb{K})$-representation, this Hilbert space is unitarily equivalent to the Hilbert space $H^2(\mathbb{K}^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$ in an obvious manner.

We now relate the present paper with what we know to be already in the literature. Since the measure $e^{-\kappa/t}\eta\varepsilon$ involves a Gaussian constituent, the square-integrability of the representative functions relative to the corresponding measure can also be established directly, and Lemma 10 in [7], combined with the observation just made, entails that the representative functions are dense in $H^2(\mathbb{K}^\mathbb{C}, e^{-\kappa/t}\eta\varepsilon)$. The fact that the BKS-pairing map between the two Hilbert spaces, multiplied by a suitable constant, is a unitary isomorphism, has been established by B. Hall [9]. In that paper, the pairing map is shown to coincide, up to multiplication by a constant, with a version of the Segal-Bargmann coherent state transform developed, in turn, over Lie groups admitting a bi-invariant Riemannian metric, in a sequence of preceding papers [7]–[9]. The main technique in those papers is heat kernel harmonic analysis and, in fact, in [9], Hall derives the unitarity of the pairing map by identifying the measure on $\mathbb{K}^\mathbb{C}$ coming from the half-form bundle with an appropriate heat kernel measure which, in turn, he has shown in the preceding papers to furnish a unitary transform.

This approach in terms of the Segal-Bargmann transform, combined with the ordinary Peter-Weyl theorem, also entails the statement of the holomorphic Peter-Weyl theorem. A version of the holomorphic Plancherel Theorem may be found in [22] as well as in Lemmata 9 and 10 of [7]. In [7] (Section 8), the completeness of the representative functions is established by analytical considerations. A number of results in Section 10 of [7] are actually independent of heat kernel methods in the sense that they are valid for more general measures than that coming from heat kernel analysis and, when these results are applied to the heat kernel measure, evaluation of coefficients is possible in terms of the eigenvalues of the Laplacian. The uniform convergence, on compact sets of a group of the kind $\mathbb{K}^\mathbb{C}$, of what we refer to as the holomorphic Fourier series may be found already in Proposition 12 of [3]. See also Remark 5.5 below.

Our methods are direct and independent of heat kernels and of the Laplacian, involve little analysis, if any, and imply, in particular, that the unitary isomorphism between the two Hilbert spaces is independent of heat kernels; see Remarks 6.8
and 7.4 below. Thus, our approach answers a question raised in [9], see Remark 5 in (2.5) of [9]; it also paves the way towards exploring the unitarity issue of the BKS-pairing map over homogeneous spaces, to which we will come back elsewhere.

The Segal-Bargmann transform on a symmetric space of compact type has been studied in [29] but the question how this transform is related with the corresponding BKS-pairing has not been investigated in that paper. We plan to show at another occasion that the description of the pairing maps in terms of heat kernels is a direct consequence of our method. For comparison of the two approaches with the BKS-pairing, see the identity (7.2) below. It is, perhaps, also worthwhile pointing out that the space $H^L_2(K^C,e^{-\kappa/t}\eta\varepsilon)$ is a weighted Bergman space but we shall not use the theory of general Bergman spaces.

This paper was written during a stay at the Institute for Theoretical Physics at the University of Leipzig. This stay was made possible by the German Research Council (Deutsche Forschungsgemeinschaft) in the framework of a Mercator visiting professorship, and I wish to express my gratitude to this organization. It is a pleasure to acknowledge the stimulus of conversation with G. Rudolph and M. Schmidt at Leipzig. The paper is part of a research program aimed at exploring quantization on classical phase spaces with singularities [11]–[17], in particular on classical lattice gauge theory phase spaces. Details for the special case of a single spatial plaquette where $K = SU(2)$ are worked out in [19]. The precise information needed for this research program is the equivalence of the two Hilbert spaces spelled out in Theorem 5.3 below.

I am indebted to R. Szőke, B. Hall and M. Lassalle for discussion and for having provided important information which helped placing the paper properly in the literature.

1. The Peter-Weyl decomposition of the half-form Hilbert space

Let $K$ be a compact Lie group and $K^C$ its complexification, and let $\mathfrak{k}$ and $\mathfrak{k}^C$ be the Lie algebras of $K$ and $K^C$, respectively. Choose an invariant inner product $\cdot: \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$ on $\mathfrak{k}$, and endow $K$ with the corresponding bi-invariant Riemannian metric. Using the metric, we identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ and the total space $TK$ of the tangent bundle with the total space $T^*K$ of the cotangent bundle, and we will denote by $|\cdot|$ the resulting norms on $\mathfrak{k}$ and on $\mathfrak{k}^*$.

Consider the polar decomposition map

$$ (1.1) \quad K \times \mathfrak{k} \longrightarrow K^C, \quad (x, Y) \mapsto x \cdot \exp(iY), \quad (x, Y) \in K \times \mathfrak{k}. $$

The composite of the inverse of left trivialization with (1.1) identifies $T^*K$ with $K^C$ in a $(K \times K)$-equivariant fashion. Then the induced complex structure on $T^*K$ combines with the symplectic structure to a $K$-bi-invariant (positive) Kähler structure. Indeed, the real analytic function

$$ (1.2) \quad \kappa: K^C \to \mathbb{R}, \quad \kappa(x \cdot \exp(iY)) = |Y|^2, \quad (x, Y) \in K \times \mathfrak{k}, $$

on $K^C$ which is twice the kinetic energy associated with the Riemannian metric, is a (globally defined) $K$-bi-invariant Kähler potential; in other words, the function $\kappa$ is strictly plurisubharmonic and (the negative of the imaginary part of) its Levi form yields (what corresponds to) the cotangent bundle symplectic structure, that is, the
tautological cotangent bundle symplectic structure on $T^*K \cong K^C$ is given by $i\bar{\partial} \kappa$. An explicit calculation which establishes this fact may be found in [9]. For related questions see [24], [30].

We now introduce an additional real parameter $t > 0$; in the physical interpretation, this parameter amounts to Planck’s constant $\hbar$. For ease of comparison with the heat kernel measure, cf. the identity (7.3) below, we prefer the notation $t$ rather than $\hbar$. Half-form Kähler quantization, cf. e. g. [32] (chap. 10), applied to $K^C$ relative to the tautological cotangent bundle symplectic structure on $K^C$, multiplied by $1/t$, is accomplished by means of a certain Hilbert space of holomorphic functions on $K^C$ which we now recall, for ease of exposition; see [9] for details. For the sake of brevity, we do not spell out the half-forms explicitly.

Let $\varepsilon$ be the symplectic (or Liouville) volume form on $T^*K \cong K^C$; this form induces the Liouville volume measure, and we will refer to $\varepsilon$ as Liouville (volume) measure as well. Further, let $dx$ denote the volume form on $K$ yielding Haar measure, normalized so that it coincides with the Riemannian volume measure on $K$, and let $dY$ be the volume form inducing Lebesgue measure on $\mathfrak{k}$, normalized by the inner product on $\mathfrak{k}$. In terms of the polar decomposition (1.1), we then have the identity $\varepsilon = dx dY$. We prefer not to normalize the inner product on $\mathfrak{k}$ since this inner product yields the kinetic energy.

Define the function $\eta: K^C \to \mathbb{R}$ by

$$\eta(x, Y) = \left( \det \left( \frac{\sin(\text{ad}(Y))}{\text{ad}(Y)} \right) \right)^{\frac{1}{2}}, \ x \in K, \ Y \in \mathfrak{k};$$

this yields a non-negative real analytic function on $K^C$ which depends only on the variable $Y \in \mathfrak{k}$ and, for $x \in K$ and $Y \in \mathfrak{k}$, we will also write $\eta(Y)$ instead of $\eta(x, Y)$. The function $\eta^2$ is the density of Haar measure relative to the Liouville volume measure on $K^C$, cf. [8] (Lemma 5). Both measures are $K$-bi-invariant; in particular, as a function on $\mathfrak{k}$, $\eta$ is $\text{Ad}(K)$-invariant. For later reference we point out that, with the notation

$$j(Y) = \det \left( \frac{\sinh(\text{ad}(Y/2))}{\text{ad}(Y/2)} \right)^{\frac{1}{2}}, \ Y \in \mathfrak{g},$$

where $\mathfrak{g}$ is a general Lie algebra, $j(iY) = \eta(Y/2)$ ($Y \in \mathfrak{k}$). The notation $j$ is due to [4] and [21] ((2.3.6) p. 459). We also note that a variant of the function $\eta$ is known in the literature as the van Vleck-Morette determinant. On the space of holomorphic functions on $K^C$, we will denote by $\langle \cdot, \cdot \rangle_{t,K^C}$ the normalized inner product given by

$$\langle \Phi, \Psi \rangle_{t,K^C} = \frac{1}{\text{vol}(K)} \int_{K^C} \overline{\Phi} \Psi e^{-\kappa/t} \eta \varepsilon,$$

and we denote by $\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)$ the resulting Hilbert space of holomorphic functions that are square integrable with respect to the measure $e^{-\kappa/t} \eta \varepsilon$. This Hilbert space is intrinsically a Hilbert space of holomorphic half-forms on $K^C$, cf. [9], [27], [32]. It is, furthermore, a unitary $(K \times K)$-representation in an obvious fashion.

Given two holomorphic functions $\Phi$ and $\Psi$ on $K^C$, we define their convolution $\Phi \ast \Psi$ by

$$\langle \Phi \ast \Psi \rangle(q) = \frac{1}{\text{vol}(K)} \int_K \Phi(x) \Psi(x^{-1}q) dh_K(x), \ q \in K^C;$$
since $K$ is compact, the convolution $\Phi \ast \Psi$ is a holomorphic function on $K^C$, indeed the unique extension to a holomorphic function on $K^C$ of the convolution $(\Phi|_K) \ast (\Psi|_K)$ of the restrictions to $K$. Since restriction to $K$ yields an isomorphism from $\mathbb{C}[K^C]$ onto the space $R(K)$ of representative functions on $K$ and since the operation of convolution turns $R(K)$ into an algebra, indeed, a topological algebra relative to the inner product determined by Haar measure on $K$, the operation of convolution turns the vector space $\mathbb{C}[K^C]$ of representative functions on $K^C$ into an algebra. We will refer to the vector space $\mathbb{C}[K^C]$ of representative functions on $K^C$, turned into an algebra via the convolution product, as the convolution algebra of representative functions on $K^C$.

Let $T$ be a maximal torus in $K$, $t$ its Lie algebra, $T^C \subseteq K^C$ the complexification of $T$, $t^C$ the complexification of $t$, and let $W$ denote the Weyl group. Choose a dominant Weyl chamber $C^+$, and let $R^+$ be the corresponding system of positive real roots. Here and below the convention is that, given $Z \in t$ and an element $A$ of the root space $t_\alpha$ associated with the root $\alpha$, the bracket \([Z,A]\) is given by \([Z,A] = i \alpha(Z)A\) so that, in particular, $\alpha$ is a real valued linear form on $t$. In [2] (V.1.3 on p. 185) these $\alpha$’s are called infinitesimal roots. Relative to the chosen dominant Weyl chamber, let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, so that $2\rho$ is the sum of the positive roots.

We will denote by $\hat{K}^C$ the set of isomorphism classes of irreducible rational representations of $K^C$. As usual, we identify $K^C$ with the space of highest weights relative to the chosen dominant Weyl chamber. For a highest weight $\lambda$, we denote by $T_\lambda: K^C \to \text{End}(V_\lambda)$ a representation in the class of $\lambda$ and by $d_\lambda$ the dimension of $V_\lambda$.

Let $\lambda$ be a highest weight. For $\psi \in V_\lambda^*$ and $w \in V_\lambda$, the function $\Phi_{\psi,w}$ given by

$$\Phi_{\psi,w}(q) = \psi(qw), \quad q \in K^C,$$

is a representative function on $K^C$, and the assignment to $\psi \otimes w \in V_\lambda^* \otimes V_\lambda$ of the representative function $\Phi_{\psi,w}$ yields a morphism

$$\iota_\lambda: V_\lambda^* \otimes V_\lambda \to \mathbb{C}[K^C]$$

of $(K^C \times K^C)$-representations, necessarily injective since $V_\lambda^* \otimes V_\lambda$ is an irreducible $(K^C \times K^C)$-representation. We will write

$$V_\lambda^* \otimes V_\lambda = \iota_\lambda(V_\lambda^* \otimes V_\lambda) \subset \mathbb{C}[K^C].$$

Given an $L^2$-function $f$ on $K$ and the irreducible representation $T_\lambda: K \to \text{End}(W_\lambda)$ of $K$ associated with $\lambda$, following one of the possible conventions, we define the Fourier coefficient $\hat{f}_\lambda \in \text{End}(W_\lambda)$ of $f$ relative to $\lambda$ by

$$\hat{f}_\lambda = \frac{1}{\text{vol}(K)} \int_K f(x)T_\lambda(x^{-1})dx.$$

Given a holomorphic function $\Phi$ on $K^C$ and the irreducible rational representation $T_\lambda: K^C \to \text{End}(V_\lambda)$ of $K^C$ associated with $\lambda$, we define the Fourier coefficient $\hat{\Phi}_\lambda \in \text{End}(V_\lambda)$ of $\Phi$ relative to $\lambda$ to be the Fourier coefficient of the restriction of $\Phi$
to $K$. The notational distinction between $V_\lambda$ and $W_\lambda$ will be justified in Section 5 below.

Let

$$C_{t,\lambda} = (t\pi)^{\dim(K)/2}e^{t|\lambda|^2}.$$  \hspace{1cm} (1.11)

The precise significance of the real constant $C_{t,\lambda}$ will be explained in Lemma 3.3 below. On $\text{End}(V_\lambda)$, we take the standard inner product $\langle \cdot, \cdot \rangle_\lambda$ given by

$$\langle A,B \rangle_\lambda = \text{tr}(A^*B), \ A,B \in \text{End}(V_\lambda),$$  \hspace{1cm} (1.12)

the adjoint $A^*$ of $A$ being computed as usual with respect to a $K$-invariant inner product on $V_\lambda$. We endow $\bigoplus_{\lambda \in \mathcal{C}} \text{End}(V_\lambda)$ with the inner product which, on the summand $\text{End}(V_\lambda)$, is given by

$$\frac{d_\lambda}{C_{t,\lambda}} \langle \cdot, \cdot \rangle_\lambda;$$  \hspace{1cm} (1.13)

then $\hat{\bigoplus}_{\lambda \in \mathcal{C}} \text{End}(V_\lambda)$ refers to the completion relative to this inner product. Thus, up to a constant, the resulting norm on each $\text{End}(V_\lambda)$ is the familiar Hilbert-Schmidt norm.

**Theorem 1.14.** [Holomorphic Peter-Weyl theorem]

(i) The Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon)$ contains the vector space $\mathbb{C}[K^C]$ of representative functions on $K^C$ as a dense subspace and, as a unitary $(K \times K)$-representation, $\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon)$ decomposes as the direct sum

$$\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon) = \hat{\bigoplus}_{\lambda \in \mathcal{C}} V_\lambda^* \otimes V_\lambda$$  \hspace{1cm} (1.14.1)

into $(K \times K)$-isotypical summands.

(ii) The operation of convolution induces a convolution product $\ast$ on $\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon)$ and, relative to this convolution product, as $\lambda$ ranges over the irreducible rational representations of $K^C$, the assignment to a holomorphic function $\Phi$ on $K^C$ of its Fourier coefficients $\hat{\Phi}_\lambda \in \text{End}(V_\lambda)$ yields an isomorphism

$$\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon) \rightarrow \hat{\bigoplus}_{\lambda \in \mathcal{C}} \text{End}(V_\lambda)$$  \hspace{1cm} (1.14.2)

of Hilbert algebras, where each summand $\text{End}(V_\lambda)$ is endowed with its obvious algebra structure.

The decomposition (1.14.1) of $\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon)$ is the Peter-Weyl decomposition of this Hilbert space alluded to earlier.

2. The convolution algebra of representative functions

For ease of exposition we recall the familiar decomposition into minimal two-sided ideals of the convolution algebra of representative functions on $K^C$.

The operation

$$L_x : \mathbb{C}[K^C] \rightarrow \mathbb{C}[K^C], \ (L_x(\Phi))(q) = \Phi(x^{-1}q), \ x,q \in K^C, \ \Phi \in \mathbb{C}[K^C],$$
of left translation on $K^C$ and the operation
$$R_y: \mathbb{C}[K^C] \rightarrow \mathbb{C}[K^C], \quad (R_y(\Phi))(q) = \Phi(qy), \quad y, q \in K^C, \quad \Phi \in \mathbb{C}[K^C],$$
of right translation on $K^C$ are well known to turn $\mathbb{C}[K^C]$ into an algebraic $(K^C \times K^C)$-representation in such a way that the operations of left and right translation commute. Furthermore, the assignment to the two representative functions $f$ and $h$ on $K^C$ of $\langle f, h \rangle = f \ast h(\epsilon)$ yields a complex symmetric $K^C$-invariant bilinear form
$$\langle \cdot, \cdot \rangle: \mathbb{C}[K^C] \otimes \mathbb{C}[K^C] \rightarrow \mathbb{C}$$
on $\mathbb{C}[K^C]$, cf. e.g. [28].

Let $\lambda$ be a highest weight. We endow $V^*_\lambda \otimes V_\lambda \cong V^*_\lambda \otimes V_\lambda$ with the obvious complex symmetric bilinear form coming from the evaluation mapping. By construction, this form coincides with the restriction of the complex symmetric bilinear form (2.1) to $V^*_\lambda \otimes V_\lambda$ whence this restriction is non-degenerate, that is, a complex inner product. The operation of convolution is defined on $\mathbb{C}[K^C]$ and, relative to the convolution product on $\mathbb{C}[K^C]$, the assignment to $\Phi \in \mathbb{C}[K^C]$ of its Fourier coefficient $\hat{\Phi}_\lambda \in \text{End}(V_\lambda)$ induces a surjective morphism of algebras
$$F_\lambda: \mathbb{C}[K^C] \rightarrow \text{End}(V_\lambda)$$
where $\text{End}(V_\lambda)$ carries its obvious algebra structure, and this morphism has the property that, for every $x, y \in K^C$ and every $w \in V_\lambda$,
$$\langle L_x L_y(\Phi)(w) = T_\lambda(x)(\Phi((T_\lambda(y^{-1})w))), \quad \Phi \in \mathbb{C}[K^C].$$
Furthermore, the composite
$$F_\lambda \circ \iota_\lambda: V^*_\lambda \otimes V_\lambda \rightarrow \text{End}(V_\lambda)$$
is the canonical isomorphism.

We will use the notation $\alpha = (\alpha_\lambda) \in \oplus \text{End}(V_\lambda)$ and $T = (T_\lambda: K^C \rightarrow \oplus \text{End}(V_\lambda))$, as $\lambda$ ranges over the highest weights. In terms of this notation, the obvious action of $K^C \times K^C$ on $\oplus \text{End}(V_\lambda)$ is given by the association
$$(x, y, \alpha) \mapsto T(x) \circ \alpha \circ T(y^{-1}), \quad x, y \in K^C.$$We now recall the algebraic analogue of the Peter-Weyl theorem; see e.g. Section 5 of [28] for details.

**Proposition 2.5.** (i) The complex vector space $\mathbb{C}[K^C]$ of representative functions decomposes as the direct sum
$$\mathbb{C}[K^C] = \oplus_\lambda V^*_\lambda \otimes V_\lambda$$
of $(K^C \times K^C)$-representations and, relative to the complex symmetric bilinear form (2.1), the decomposition is orthogonal.
(ii) For each $\lambda \in \hat{K}^C$, the summand $V^*_\lambda \otimes V_\lambda$ is the isotypical summand of $\mathbb{C}[K^C]$ determined by $\lambda$, and the restriction of the complex symmetric bilinear form (2.1) to this summand is non-degenerate.
(iii) Relative to the convolution product on $\mathbb{C}[K^C]$, the induced morphism
$$F_\lambda: \mathbb{C}[K^C] \rightarrow \oplus_\lambda \text{End}(V_\lambda)$$
of algebras is an isomorphism of $(K^C \times K^C)$-representations and yields the decomposition of the convolution algebra $\mathbb{C}[K^C]$ into minimal two-sided ideals. □
3. The square integrability of the representative functions

The aim of the present section is to establish the square-integrability of the representative functions on $K^C$ and to reduce the calculation of the requisite integrals over $K^C$ to integrals over $K$.

**Lemma 3.1.** Each representative function on $K^C$ is square integrable relative to the measure $e^{-\kappa/\eta} \eta \varepsilon$.

We shall exploit the following integration formula

$$\int f(Y) dY = \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_{K/T} f(Ad_{y}(Y)) d(yT) \right\} dY,$$

valid for any integrable continuous function $f$ on $K$. For the special case where $\mathfrak{k} = su(2)$, the formula comes essentially down to integration on $\mathbb{R}^3$ in ordinary spherical polar coordinates. For the general case, see e.g. [10] (Theorem I.5.17, p. 195) or [5] ((3.14.2) on p. 185 combined with (3.14.4) on p. 187).

Let $\lambda$ be a highest weight. We will use the notation $\varphi^C$ etc. for representative functions on $K^C$ in the isotypical summand $V^*_\lambda \otimes V^*_\lambda$ of $\mathbb{C}[K^C]$ associated with $\lambda$ and, accordingly, we will denote the restriction of $\varphi^C$ to $K$ by $\varphi$; then $\varphi$ is necessarily a representative function on $K$ which lies in the isotypical summand of $L^2(K, dx)$ associated with $\lambda$ by virtue of the ordinary Peter-Weyl theorem. Lemma 3.1 is implied by the following.

**Lemma 3.3.** Given the representative function $\varphi^C$ on $K^C$ in the isotypical summand $V^*_\lambda \otimes V^*_\lambda$ of $\mathbb{C}[K^C]$ associated with the highest weight $\lambda \in \hat{K}^C$, 

$$\int_{K^C} \varphi^C \varphi^C e^{-\kappa/\eta} \eta \varepsilon = C_{t,\lambda} \int_K \varphi \varphi dx, \quad C_{t,\lambda} = (t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2}.$$

To prepare for the proof, we will denote by $\chi^C_\lambda$ the holomorphic character of $K^C$ associated with the highest weight $\lambda$ and, accordingly, we denote by $\chi_\lambda$ the restriction of $\chi^C_\lambda$ to $K$; this is plainly the irreducible character of $K$ associated with $\lambda$.

**Lemma 3.4.** The character $\chi^C_\lambda$ of the irreducible representation $T_\lambda: K^C \to \text{End}(V_\lambda)$ of $K^C$ associated with the highest weight $\lambda$ satisfies the identity

$$\int_{K^C} \|\chi^C_\lambda\|^2 e^{-\kappa} \eta \varepsilon = \frac{1}{d_\lambda} \int_{K^C} \|T_\lambda\|^2 e^{-\kappa} \eta \varepsilon$$

where, as before, $d_\lambda = \dim(V_\lambda)$.

To prepare for the proof of this Lemma recall that, given an $L^2$-function $f$ on $K$, the appropriate version of the Plancherel theorem says that a function $f$ on $K$ satisfying suitable hypotheses, e.g. ‘$f$ smooth’ suffices, admits the Fourier decomposition

$$f(x) = \sum_{\theta} d_\theta \text{tr}(\hat{f}_\theta T_\theta(x)), \quad x \in K,$$
where $\theta$ ranges over the highest weights; see e. g. [21] (2.3.10). Furthermore, one version of the Plancherel formula takes the form

\begin{equation}
\frac{1}{\text{vol}(K)} \int_K |f(x)|^2dx = \sum_{\theta} d_{\theta} ||\hat{f}_{\theta}||^2;
\end{equation}

see e. g. [21] (2.3.11).

For $f = \chi_{\lambda}$, the only non-zero Fourier coefficient equals $\hat{f}_{\lambda} = \frac{1}{d_{\lambda}} \text{Id}_{V_{\lambda}}$, and the Fourier decomposition of the character $\chi_{\lambda}$ takes the form

$$\chi_{\lambda}(x) = d_{\lambda} \text{tr}(\hat{f}_{\lambda} T_{\lambda}(x)), \ x \in K.$$ 

**Proof of Lemma 3.4.** Because the measure $\xi = e^{-\kappa \eta \varepsilon}$ is $K$-bi-invariant it is in particular invariant under right translation by elements of $K$. Hence, for every function $f$ on $K^C$ which is square integrable relative to this measure, for each $x \in K$,

$$\int_{K^C} ||f(y)||^2d\xi(y) = \int_{K^C} ||f(yx)||^2d\xi(y).$$

Integrating this identity over $K$ yields

$$\text{vol}(K) \int_{K^C} ||f(y)||^2d\xi(y) = \int_{K^C} \int_K ||f(yx)||^2dxd\xi(y).$$

Given $y \in K^C$, the Fourier coefficient $\hat{f}_{\lambda}^y$ of the function $f^y$ on $K$ defined by

$$f^y(x) = \chi_C(x^y) = \text{tr}(T_{\lambda}(y)T_{\lambda}(x)), \ x \in K,$$

is given by

$$\hat{f}_{\lambda}^y = T_{\lambda}(y)\hat{f}_{\lambda},$$

and this is the only non-zero coefficient. Hence, given $y \in K^C$, applying the Plancherel formula (3.6) on $K$ to the function $f^y$, we find

$$\frac{1}{\text{vol}(K)} \int_K ||f^y(x)||^2dx = d_{\lambda} ||T_{\lambda}\hat{f}_{\lambda}||^2 = d_{\lambda} \frac{1}{d_{\lambda}} ||T_{\lambda}(y)||^2 = \frac{1}{d_{\lambda}} ||T_{\lambda}(y)||^2.$$ 

Consequently

$$\int_{K^C} ||\chi_C(x^y)||^2d\xi(y) = \frac{1}{\text{vol}(K)} \int_{K^C} \int_K ||f^y(x)||^2dxd\xi(y) = \frac{1}{d_{\lambda}} \int_{K^C} ||T_{\lambda}(y)||^2d\xi(y)$$

as asserted. \qed

**Proof of Lemma 3.3.** We establish the statement of the Lemma for the special case where $t = 1$. The general case is reduced to the special case by a change of variables.

As a $(K \times K)$-representation, the isotypical summand $V_{\lambda}^* \otimes V_{\lambda}$ is generated by the character $\chi_{\lambda}^C$. Hence it suffices to establish the assertion for $\varphi^C = \chi_{\lambda}^C$. By Lemma 3.4, it suffices to compute the integral $\int_{K^C} ||T_{\lambda}||^2e^{-\kappa \eta \varepsilon}$. To compute this integral, let $y = x \exp(iY)$ where as before $x \in \hat{K}$ and $Y \in \mathfrak{k}$. Let $T_{\lambda}^t: \mathfrak{t}^C \to \text{End}(V_{\lambda})$ denote
the corresponding Lie algebra representation and let $A(Y) \in \text{End}(V)_{\lambda}$ be given by $A(Y) = iT'_\lambda(Y)$. Then

$$T_\lambda(y) = T_\lambda(x)T_\lambda(\exp(iY)) = T_\lambda(x)\exp(iT'_\lambda(Y)) = T_\lambda(x)e^{A'(Y)}$$

and

$$T'_\lambda(y)T_\lambda(y) = \left(e^{A(Y)}\right)^* \left(e^{A(Y)}\right).$$

Since the endomorphism $T'_\lambda(Y)$ is skew-hermitian, the endomorphism $A(Y)$ is hermitian, that is, $A(Y)^* = A(Y)$ whence

$$\left(e^{A(Y)}\right)^* \left(e^{A(Y)}\right) = e^{A(Y)^*}e^{A(Y)} = e^{2A(Y)}$$

and thence

$$||e^{A(Y)}||^2 = \text{tr}(e^{2A(Y)}) = \text{tr}(e^{A(Y)}) = \text{tr}(e^{iT'_\lambda(Y)}) = \chi^C_\lambda(\exp(2iY)).$$

View $\lambda + \rho$ as a point of $\mathfrak{k}^*$ via the orthogonal decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{q}^+$ where $\mathfrak{q}^+$ is the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$, and let $\Omega_{\lambda + \rho}$ be the coadjoint orbit generated by $\lambda + \rho$. Given $Y \in \mathfrak{k}$, Kirillov’s character formula, evaluated at the point $\exp(2iY)$, yields the identity

$$\text{vol}(\Omega_{\rho})j(2iY)\chi^C_\lambda(\exp(2iY)) = \text{vol}(\Omega_{\rho})\eta(Y)\chi^C_\lambda(\exp(2iY)) = \int_{\Omega_{\lambda + \rho}} e^{-\partial Y} d\sigma(\partial),$$

cf. [20], [21]. Here $\partial$ refers to the variable on $\Omega_{\lambda + \rho}$ and $d\sigma$ denotes the symplectic volume form on $\Omega_{\lambda + \rho}$. Using the diffeomorphism from $K/T$ onto $\Omega_{\lambda + \rho}$ which sends $yT$ ($y \in K$) to $(\text{Ad}_y)^{-1}(\lambda + \rho)$, we rewrite the integral in the form

$$\int_{\Omega_{\lambda + \rho}} e^{-\partial Y} d\sigma(\partial) = \frac{\text{vol}(\Omega_{\lambda + \rho})}{\text{vol}(K/T)} \int_{K/T} e^{-(\text{Ad}_y)^{-1}(\lambda + \rho)(2Y)} d(yT) = \frac{d_{\lambda}\text{vol}(\Omega_{\rho})}{\text{vol}(K/T)} \int_{K/T} e^{-(\lambda + \rho)(\text{Ad}_y(Y))} d(yT).$$

Hence

$$\eta(Y)||e^{A(Y)}||^2 = \eta(Y)\text{tr}(e^{iT'_\lambda(Y)}) = \frac{d_\lambda}{\text{vol}(K/T)} \int_{K/T} e^{-2(\lambda + \rho)(\text{Ad}_y(Y))} d(yT)$$

whence, in view of the integration formula (3.2),

$$\int_\mathfrak{k} ||T_\lambda(x \exp(iY))||^2 e^{-|Y|^2} \eta(Y) dY = \int_\mathfrak{k} ||e^{A(Y)}||^2 e^{-|Y|^2} \eta(Y) dY$$

$$= d_\lambda \int_{C_+} \prod_{\alpha \in \mathfrak{R}^+} \alpha(Y)^2 \left\{ \int_{K/T} e^{-2(\lambda + \rho)(\text{Ad}_y(Y))} d(yT) \right\} e^{-|Y|^2} dY$$

$$= d_\lambda \int_{C_+} e^{-2(\lambda + \rho)(|Y|)^2} dY = d_\lambda \pi^{\dim(K)/2} e^{\lambda + \rho}.$$
Consequently
\[
\int_{K^c} ||\chi_\lambda||^2 e^{-\kappa \eta} \, d\lambda = \frac{1}{d\lambda} \int_{K^c} ||T_\lambda||^2 e^{-\kappa \eta} \, = \pi^{\dim(K)/2} e^{\lambda + \rho} \text{vol}(K)
\]
as asserted. In particular, \(C_{1,\lambda} = \pi^{\dim(K)/2} e^{\lambda + \rho} \). □

4. The constituents given by integral forms

Left and right translation turn \(\mathcal{H}L^2(K^c, e^{-\kappa/t} \eta)\) into a unitary \((K \times K)\)-representation. To establish the statement (i) of the holomorphic Peter-Weyl theorem, it remains to show that the decomposition (2.5.1) of the vector space of representative functions on \(K^c\) into isotypical summands determines the decomposition of \(\mathcal{H}L^2(K^c, e^{-\kappa/t} \eta)\) into isotypical summands, that is to say:

**Proposition 4.1.** There is no irreducible \((K \times K)\)-summand in \(\mathcal{H}L^2(K^c, e^{-\kappa/t} \eta)\) beyond those which come from the decomposition (2.5.1).

We shall establish this fact via a geometric argument which is guided by the principle that quantization commutes with reduction. Our argument relies on the familiar complete reducibility of a continuous unitary representation of a compact Lie group on a Hilbert space, see e. g. Theorem III.5.10 on p. 142 of [2].

We now begin with the preparations for the proof of Proposition 4.1. For intelligibility, we will first recall a few standard facts. Let \(B\) be a Borel subgroup (maximal solvable subgroup) of \(K^c\) containing \(T^c\). Let \(\lambda \in \mathfrak{t}^* = \text{Hom}(t, \mathbb{R})\) be an integral form and let \(\vartheta_\lambda; T^c \rightarrow \mathbb{C}^*\) be the corresponding algebraic character of the complexification \(T^c\) of \(T\). Thus \(\vartheta_\lambda\) is given by the formula

\[
\vartheta_\lambda(\exp(w)) = e^{i\lambda(w)}, \quad w \in \mathfrak{t}^c,
\]
and the derivative of the restriction of \(\vartheta_\lambda\) to the maximal torus \(T\) coincides with \(\lambda\). The corresponding algebraic character of \(B\) is given by the composite of \(\vartheta_\lambda; T^c \rightarrow \mathbb{C}^*\) with the projection from \(B\) to \(T^c\), and we denote this character by \(\vartheta_\lambda; B \rightarrow \mathbb{C}^*\) as well. The \(\mathbb{C}\)-linear subspace

\[
\lambda \mathbb{C}[K^c] = \{ \phi \in \mathbb{C}[K^c]; \phi(qy) = \vartheta_\lambda(y)^{-1} \phi(q), \quad q \in K^c, y \in B \}
\]
of \(\mathbb{C}[K^c]\) inherits an algebraic \(K^c\)-action in an obvious fashion, the \(K^c\)-action being given by the assignment to \((x, \phi) \in K^c \times \lambda \mathbb{C}[K^c]\) of \(x\phi \in \lambda \mathbb{C}[K^c]\) where \((x\phi)(q) = \phi(x^{-1}q) \quad (q \in K^c)\). Let \(\lambda \mathbb{C}\) denote the 1-dimensional complex vector space of complex numbers, viewed as a 1-dimensional rational representation of \(B\) via \(\vartheta_\lambda\), more precisely, as a 1-dimensional rational left \(B\)-module. In terms of this notation, \(\lambda \mathbb{C}[K^c]\) is the \(K^c\)-representation \(\text{Ind}_{B}^{K^c} \lambda \mathbb{C}\), the rational \(K^c\)-representation which is induced from \(\lambda \mathbb{C}\). Likewise the \(\mathbb{C}\)-linear subspace

\[
\mathbb{C}[K^c]_\lambda = \{ \psi \in \mathbb{C}[K^c]; \psi(qy) = \vartheta_\lambda(y)^{-1} \psi(q), \quad q \in K^c, y \in B \}
\]
of \(\mathbb{C}[K^c]\) inherits the algebraic \(K^c\)-action given by the assignment to \((x, \phi) \in K^c \times \mathbb{C}[K^c]_\lambda\) of \(x\phi \in \mathbb{C}[K^c]_\lambda\) where \((x\phi)(q) = \phi(qx) \quad (q \in K^c)\). With respect to the 1-dimensional rational right \(B\)-module \(\mathbb{C}_\lambda\) which is the vector space of complex
numbers, made into a rational $B$-representation via $\vartheta_\lambda$, $\mathbb{C}[K^C]_\lambda$ amounts to the $K^C$-representation $\text{Ind}_B^K \mathbb{C}_\lambda$, the rational $K^C$-representation which is induced from $\mathbb{C}_\lambda$. The inversion mapping $x \mapsto x^{-1}$ on $K^C$ which sends $x \in K^C$ to $x^{-1}$ induces an isomorphism

$$\lambda \mathbb{C}[K^C] \longrightarrow \mathbb{C}[K^C]_{-\lambda}$$

of $K^C$-representations.

The choice of dominant Weyl chamber $C^+$ in $\mathfrak{t}$ determines a Borel subgroup of $K^C$ which we denote by $B^+$. Throughout, highest weights will be understood relative to this Weyl chamber. Given the integral form $\lambda \in \mathfrak{t}^*$, we denote the corresponding algebraic character of $B^+$ by $\vartheta_\lambda^+: B^+ \rightarrow \mathbb{C}^*$, the resulting algebraic $K^C$-representation (4.3.left) by $\lambda \mathbb{C}[K^C]^+$, and the representation (4.3.right) by $\mathbb{C}[K^C]^+_{\lambda}$. Let $C^-$ be the Weyl chamber in $\mathfrak{t}$ which is opposite to $C^+$, that is, the Weyl chamber arising from interchanging positive and negative roots, and let $B^-$ be the corresponding Borel subgroup of $K^C$ containing $T^C$. Given the integral form $\lambda \in \mathfrak{t}^*$, let $\vartheta^-_\lambda: B^- \rightarrow \mathbb{C}^*$ denote the corresponding algebraic character of $B^-$ which is the composite of $\vartheta_\lambda: T^C \rightarrow \mathbb{C}^*$ with the projection from $B^-$ to $T^C$, and denote the resulting algebraic $K^C$-representation (4.3.left) by $\lambda \mathbb{C}[K^C]^-$ and the corresponding representation (4.3.right) by $\mathbb{C}[K^C]_{\lambda}^-$. By construction, precisely when $\lambda$ lies in the dominant Weyl chamber $C^+$, the complex vector spaces $\lambda \mathbb{C}[K^C]^+$ and $\mathbb{C}[K^C]_{\lambda}^-$ are non-zero and the resulting representations are irreducible algebraic $K^C$-representations. Below, to establish Proposition 4.1, we shall take $V_\lambda = \lambda \mathbb{C}[K^C]^+$ as $\lambda$ ranges over the highest weights.

**Proposition 4.5.** Let $\lambda$ be a highest weight (relative to $C^+$). The assignment to $(\psi, \phi) \in \mathbb{C}[K^C]_{\lambda}^- \times \lambda \mathbb{C}[K^C]^+$ of $\langle \psi, \phi \rangle (= (\psi \ast \phi)(e))$, cf. (2.1), yields a perfect pairing

$$\mathbb{C}[K^C]_{\lambda}^- \otimes \mathbb{C}[K^C]^+ \longrightarrow \mathbb{C}$$

which induces isomorphisms

$$\mathbb{C}[K^C]_{\lambda}^- \longrightarrow (\lambda \mathbb{C}[K^C]^+)^* = \text{Hom}(\lambda \mathbb{C}[K^C]^+, \mathbb{C})$$

$$\lambda \mathbb{C}[K^C]^+ \longrightarrow (\mathbb{C}[K^C]_{\lambda}^-)^* = \text{Hom}(\mathbb{C}[K^C]_{\lambda}^-, \mathbb{C})$$

of algebraic $K^C$-representations. □

As a side remark we note that, in view of the Borel-Weil theorem, the inclusions of the spaces $\lambda \mathbb{C}[K^C]^+$ and $\mathbb{C}[K^C]_{\lambda}^-$ into the corresponding spaces of holomorphic functions on $K^C$ come down to identity mappings, that is, there is no difference between algebraic and holomorphic functions at this point.

To recall the familiar descriptions of the spaces $\lambda \mathbb{C}[K^C]^+$ and $\mathbb{C}[K^C]_{\lambda}^-$ in terms of complex line bundles, let for the moment $\lambda \in \mathfrak{t}^*$ be a general integral form. Consider the complex line bundle

$$\lambda \beta^\pm: K^C \times_{B^\pm} \lambda \mathbb{C} \rightarrow K^C/B^\pm$$

on $K^C/B^\pm$. By construction, the assignment to $\phi \in \lambda \mathbb{C}[K^C]^+$ of the induced algebraic section $s_\phi$ of $\lambda \beta^+$ yields an isomorphism

$$\lambda \mathbb{C}[K^C]^+ \rightarrow \Gamma_{\text{alg}}(\lambda \beta^+)$$
of complex vector spaces, and in this fashion $\Gamma_{\text{alg}}(\lambda^{\beta \pm})$ acquires the structure of an algebraic $K^C$-representation. In view of the Borel-Weil theorem, $\Gamma_{\text{alg}}(\lambda^{\beta \pm})$ is non-zero when $\lambda$ lies in the dominant Weyl chamber $C^+$. Likewise, given the general integral form $\lambda$, consider the complex line bundle

\[ \beta^\pm : \mathbb{C} \times_{B^\pm} K^C \to B^\pm \setminus K^C \]

on $B^\pm \setminus K^C$. By construction, the assignment to $\psi \in \mathbb{C}[K^C]_\lambda^-$ of the induced algebraic section $s_\psi$ of $\beta^-_\lambda$ yields an isomorphism

\[ \mathbb{C}[K^C]^-_\lambda \to \Gamma_{\text{alg}}(\beta^-_\lambda) \]

of complex vector spaces and, in this fashion, the space $\Gamma_{\text{alg}}(\beta^-_\lambda)$ of algebraic sections of $\beta^-_\lambda$ acquires the structure of an algebraic $K^C$-representation. For a general integral form $\lambda$, the algebraic mapping

\[ K^C \times_{B^\pm} \chi \to \mathbb{C} \times_{B^\pm} K^C, \quad (x, v) \mapsto (v, x^{-1}) \]

where $x \in K^C$ and $v \in \mathbb{C}$, induces an isomorphism

\[ -\lambda \beta^\pm \to \beta^\pm_\lambda \]

of algebraic line bundles which, on the bases, is the algebraic isomorphism

\[ K^C / B^\pm \to B^\pm \setminus K^C, \quad xB^\pm \mapsto B^\pm x^{-1}, \quad x \in K^C, \]

and this isomorphism induces an isomorphism of algebraic $K^C$-representations

\[ \Gamma_{\text{alg}}(-\lambda \beta^\pm) \to \Gamma_{\text{alg}}(\beta^\pm_\lambda) \]

which is plainly compatible with the isomorphism (4.4) of algebraic $K^C$-representations between $-\lambda \mathbb{C}[K^C]^\pm$ and $\mathbb{C}[K^C]_\lambda^\pm$. In particular, by the Borel-Weil theorem, $\Gamma_{\text{alg}}(\beta^-_\lambda)$ is non-zero precisely when $-\lambda$ lies in the Weyl chamber corresponding to $B^-$, that is, when $\lambda$ lies in the dominant Weyl chamber determined by $B^+$ and, in this case, $\mathbb{C}[K^C]^-_\lambda \cong \Gamma_{\text{alg}}(\beta^-_\lambda)$ is an irreducible $K^C$-representation with highest weight $w(-\lambda + \rho) - \rho$ where $w$ is the unique element of the Weyl group such that $w(-\lambda + \rho) - \rho$ lies in the interior of the dominant Weyl chamber $C^+$ where as before $\rho$ refers to one half the sum of the positive roots.

For $\nu \in \mathfrak{t}^*$, let $O_\nu = K\nu \subseteq \mathfrak{t}^*$ be the coadjoint orbit generated by $\nu$. For completeness, we recall that the $K$-action on $\mathfrak{t}^*$ is given in the standard way, that is, by means of the association

\[ K \times \mathfrak{t}^* \to \mathfrak{t}^*, \quad (x, \chi) \mapsto x\chi = \text{Ad}_x^* (\chi), \quad x \in K, \chi \in \mathfrak{t}^*. \]

Let $\lambda$ be an integral form in $C^+$. Then $\lambda + \rho$ lies in the interior of the dominant Weyl chamber, and the coadjoint orbit $O_{\lambda + \rho}$ of $\lambda + \rho$ in $\mathfrak{t}^*$ has maximal dimension, whether or not the orbit of $\lambda$ has maximal dimension, that is, the stabilizer of the point $\lambda + \rho$ of $\mathfrak{t}^*$ is minimal and coincides with the maximal torus $T$; likewise the stabilizer of the point $-(\lambda + \rho)$ of $\mathfrak{t}^*$ is minimal and coincides with the maximal torus.
$T$. Since the inclusion mapping $K \subseteq K^C$ induces a diffeomorphism $K/T \to K^C/B^+$, the assignment to $x \in K$ of

$$(4.10+)\quad x(\lambda + \rho) = \text{Ad}_{x}^\star(\lambda + \rho) \in \mathfrak{k}^*$$

induces an embedding

$$(4.11+)\quad \lambda \mu^+: K^C/B^+ \to \mathfrak{k}^*$$

of the homogeneous space $K^C/B^+$ into $\mathfrak{k}^*$ which induces a $K$-equivariant diffeomorphism from $K^C/B^+$ onto the coadjoint orbit $O_{\lambda+\rho}$. It is well known that the Kirillov-Kostant-Souriau symplectic structure $\sigma_{\lambda+\rho}$ on $O_{\lambda+\rho}$ combines with the complex structure on $K^C/B^+$ to a positive $K$-invariant Kähler structure on both $O_{\lambda+\rho}$ and $K^C/B^+$ in such a way that $\lambda \mu^+$ identifies the two resulting Kähler manifolds in a $K$-equivariant fashion and such that $\lambda \mu^+$ is a $K$-equivariant momentum mapping. Furthermore, relative to the additional structure on $K^C/B^+$, the line bundle $\lambda \beta^+$ is positive, in fact, a prequantum bundle, by construction necessarily $K$-equivariant, the unique hermitian connection being the requisite connection. Likewise the inclusion mapping $K \subseteq K^C$ induces a diffeomorphism $T \setminus K \to B^- \setminus K^C$, and the assignment to $y \in K$ of

$$(4.10-)\quad -y^{-1}(\lambda + \rho) = -\text{Ad}_y^\star(\lambda + \rho) \in \mathfrak{k}^*$$

induces an embedding

$$(4.11-)\quad \mu^-: B^- \setminus K^C \to \mathfrak{k}^*$$

of the homogeneous space $B^- \setminus K^C$ into $\mathfrak{k}^*$. For convenience, we convert the obvious $K^C$-action on the right of $B^- \setminus K^C$ in the standard way to a left action via the association

$$(4.12)\quad K^C \times (B^- \setminus K^C) \to B^- \setminus K^C, \ (y, B^- x) \mapsto B^- xy^{-1}, \ x, y \in K^C.$$

With this convention, (4.11-) induces a $K$-equivariant diffeomorphism from $B^- \setminus K^C$ onto the coadjoint orbit $O_{-(\lambda+\rho)}$ in such a way that (i) the Kirillov-Kostant-Souriau symplectic structure $\sigma_{-(\lambda+\rho)}$ on $O_{-(\lambda+\rho)}$ combines with the complex structure on $B^- \setminus K^C$ to a positive $K$-invariant Kähler structure on both $O_{-(\lambda+\rho)}$ and $B^- \setminus K^C$, that (ii) $\mu^-_\lambda$ identifies the two resulting Kähler manifolds in a $K$-equivariant fashion, that (iii) $\mu^-_\lambda$ is a $K$-equivariant momentum mapping, and such that (iv) relative to the additional structure on $B^- \setminus K^C$, the line bundle $\beta^-_\lambda$ is positive, in fact, a prequantum bundle, by construction necessarily $K$-equivariant, the requisite connection being the unique hermitian connection.

As before, we consider $T^* K$ as a Hamiltonian $(K \times K)$-space relative to the $(K \times K)$-action which arises from the lifts of the left translation and of the right translation action on $K$. The momentum mapping

$$\mu^{K \times K}: T^* K \to \mathfrak{k}^* \times \mathfrak{k}^*$$
for this \((K \times K)\)-action on \(T^* K \cong K^C\) is well known to be given by the association

\[
T^* K \ni \alpha_x \longmapsto (\alpha_x \circ R_x, \alpha_x \circ L_x) \in \mathfrak{t}^* \times \mathfrak{t}^*, \ x \in K, \ \alpha_x \in T_x^* K,
\]

where \(R_x : \mathfrak{t} = T_e K \to T_x K\) and \(L_x : \mathfrak{t} = T_e K \to T_x K\) refer to the operations of left- and right translation, respectively, by \(x \in K\). With an abuse of notation, we denote the corresponding momentum mapping on \(K^C\) by

\[
\mu^{K \times K} : K^C \to \mathfrak{t}^* \times \mathfrak{t}^*
\]

as well, and we denote the symplectic structure on \(K^C\) by \(\sigma_K\).

Consider the product manifold

\[
N^\times = K^C \times (K^C / B^+) \times (B^- \backslash K^C),
\]

endowed with the product Kähler structure. Let \(\sigma^\times\) be the resulting product symplectic structure which underlies the product Kähler structure, essentially the sum of \(\sigma_K\), \(\sigma_{\lambda+\rho}\), and \(\sigma_{-(\lambda+\rho)}\). The group \(K^C \times K^C\) acts on \(N^\times\) in the obvious fashion, that is, the action on \(K^C\) is given by left- and right translation, that on \(K^C / B^+\) by the projection to the first factor \(K^C\), followed by the \(K^C\)-action on \(K^C / B^+\), and that on \(B^- \backslash K^C\) by the projection to the second factor \(K^C\), followed be the \(K^C\)-action on \(B^- \backslash K^C\). Furthermore, by construction, the symplectic structure \(\sigma^\times\) is \((K \times K)\)-invariant. Let

\[
\mu^\times : N^\times \to \mathfrak{t}^* \times \mathfrak{t}^*
\]

be the \((K \times K)\)-momentum mapping for the \((K \times K)\)-action on \(N^\times\) relative to the symplectic structure \(\sigma^\times\). This momentum mapping is essentially the sum of the momentum mapping \(\mu^{K \times K}\) and the two momentum mappings \((4.11+)\) and \((4.11-)\). The \((K \times K)\)-reduced space \((\mu^\times)^{-1} (0,0)/(K \times K)\) at the point zero of \(\mathfrak{t}^* \times \mathfrak{t}^*\) boils down to a single point.

We will denote the complex vector space of holomorphic functions on \(K^C\) by \(\mathcal{H}(K^C)\). Left and right translation on \(K^C\) turn \(\mathcal{H}(K^C)\) into a holomorphic \((K^C \times K^C)\)-representation. By construction, the product line bundle

\[
\beta^\times = \beta_\lambda \beta^+ \times \beta^-_\lambda
\]

is a holomorphic \((K^C \times K^C)\)-equivariant line bundle and, in view of the isomorphisms \((4.7+)\) and \((4.7-)\), since the complex vector spaces \(\lambda \mathbb{C}[K^C]^+\) and \(\mathbb{C}[K^C]^-\) are finite-dimensional, as a \((K^C \times K^C)\)-representation, the space of holomorphic sections of this line bundle amounts to the tensor product

\[
\lambda \mathbb{C}[K^C]^+ \otimes \mathbb{C}[K^C]^-_\lambda \otimes \Gamma_{\text{hol}}(\beta) \cong \lambda \mathbb{C}[K^C]^+ \otimes \mathbb{C}[K^C]^-_\lambda \otimes \mathcal{H}(K^C)
\]

of representations. In view of the isomorphisms \((4.5.2)\), as a \((K^C \times K^C)\)-representation, this tensor product may be written as

\[
\text{Hom}_\mathbb{C} \left( \mathbb{C}[K^C]^-_\lambda \otimes \lambda \mathbb{C}[K^C]^+, \mathcal{H}(K^C) \right).
\]
By construction, the product line bundle $\beta^\times$ is a holomorphic $(K \times K)$-equivariant prequantum bundle on the Kähler manifold $N^\times$. The $(K \times K)$-reduced space $(\mu^\times)^{-1}(0,0)/(K \times K)$ at the point zero of $\mathfrak{k}^* \times \mathfrak{k}^*$ is a single point. Indeed, consider the point $(e, B^+, B^-)$ of $N^\times$. This point lies in $(\mu^\times)^{-1}(0,0)$, and the $(K \times K)$-orbit of this point is the entire zero locus $(\mu^\times)^{-1}(0,0)$. This observation implies at once that the $(K \times K)$-reduced space $(\mu^\times)^{-1}(0,0)/(K \times K)$ at the point zero of $\mathfrak{k}^* \times \mathfrak{k}^*$ is a single point. For completeness we note that the stabilizer of the point $(e, B^+, B^-)$ of $(\mu^\times)^{-1}(0,0)$ is a copy of the maximal torus $T$. With these preparations out of the way, we conclude that the space of $(K \times K)$-invariant holomorphic sections of the product line bundle $\beta^\times$ is at most 1-dimensional, that is, the space

\[(4.13) \quad \text{Hom}_C \left( \mathbb{C}[K^C]_\lambda \otimes \mathbb{C}[K^C]^+, \mathcal{H}(K^C) \right)^{K \times K}\]

is at most 1-dimensional.

We explain briefly under somewhat more general circumstances how one arrives at the last conclusion: Let $G$ be a compact Lie group, let $N$ be a $G$-Hamiltonian Kähler manifold, with $G$-equivariant momentum mapping $\mu: N \to \mathfrak{g}^*$, and suppose that $G$ preserves the complex structure on $N$. Then $G$ preserves the associated Riemannian metric as well, and the $G$-action extends canonically to a holomorphic $G^C$-action on $N$. The saturation of the zero locus $\mu^{-1}(0)$ is the subspace $G^C \mu^{-1}(0) \subseteq N$, and the inclusion $\mu^{-1}(0) \subseteq G^C \mu^{-1}(0)$ induces a homeomorphism from the reduced space $N_0 = \mu^{-1}(0)/G$ onto the $G^C$-quotient $G^C \mu^{-1}(0)/G^C$. In this fashion, the reduced space $N_0$ acquires a complex analytic structure. Let $\zeta: E \to N$ be a $G$-invariant prequantum bundle, and let

$$\zeta^0: E|_{G^C \mu^{-1}(0)} \to G^C \mu^{-1}(0)$$

be the restriction of $\zeta$ to $G^C \mu^{-1}(0) \subseteq N$. Passing to $G^C$-quotients, we obtain the coherent analytic sheaf $\zeta_0: E_0 \to N_0$ on $N_0$, not necessarily an ordinary line bundle. The canonical morphism $\pi: \Gamma(\zeta_0^0)^G \to \Gamma(\zeta_0)$ of complex vector spaces is plainly injective (even an isomorphism, but we do not need this fact): A $G$-equivariant section of $\zeta_0^0$ inducing the zero section of $\zeta_0$ is manifestly the zero section. Furthermore, the restriction mapping from $\Gamma(\zeta_0^0)^G$ onto $\Gamma(\zeta_0^0)^G$ is an isomorphism. Applying this reasoning to $N = N^\times$ and $G = K \times K$, we conclude that the vector space $(4.13)$ is at most 1-dimensional as asserted. For intelligibility we note that, as a space, the saturation

$$(K^C \times K^C)(\mu^\times)^{-1}(0,0) \subseteq N^\times$$

amounts to a homogeneous space of the kind $(K^C \times K^C)/T^C$, the complexification $T^C$ of the maximal torus $T$ of $K$ being suitably embedded into $K^C \times K^C$, but we shall not need this fact. However, this observation shows that, for topological reasons, the saturation cannot be all of $N^\times$.

We now take $V^\times_\lambda = \lambda \mathbb{C}[K^C]^+$. Then, in view of Proposition 4.5, $V^\times_\lambda \cong \mathbb{C}[K^C]^-$. Since we already know that, in the decomposition (2.5.1) of the vector space $\mathbb{C}[K^C]$ of representative functions on $K^C$, $V^\times_\lambda \circ V_\lambda$ is the isotypical summand corresponding to $\lambda$, and since, by virtue of Lemma 3.1, $V^\times_\lambda \circ V_\lambda$ is actually a subspace of the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon)$, we conclude that the space

$$\text{Hom}_C \left( \mathbb{C}[K^C]_\lambda \otimes \lambda \mathbb{C}[K^C]^+, \mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon) \right)^{K \times K}$$

is manifestly the zero section. Furthermore, consider the product line bundle $\beta^\times$ to $\mathfrak{g}^* \times \mathfrak{g}^*$, and since, by virtue of Lemma 3.1, $V^\times_\lambda \circ V_\lambda$ is actually a subspace of the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon)$, we conclude that the space

$$\text{Hom}_C \left( \mathbb{C}[K^C]_\lambda \otimes \lambda \mathbb{C}[K^C]^+, \mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon) \right)^{K \times K}$$

is manifestly the zero section. Furthermore, consider the product line bundle $\beta^\times$ to $\mathfrak{g}^* \times \mathfrak{g}^*$, and since, by virtue of Lemma 3.1, $V^\times_\lambda \circ V_\lambda$ is actually a subspace of the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon)$, we conclude that the space

$$\text{Hom}_C \left( \mathbb{C}[K^C]_\lambda \otimes \lambda \mathbb{C}[K^C]^+, \mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon) \right)^{K \times K}$$

is manifestly the zero section. Furthermore, consider the product line bundle $\beta^\times$ to $\mathfrak{g}^* \times \mathfrak{g}^*$, and since, by virtue of Lemma 3.1, $V^\times_\lambda \circ V_\lambda$ is actually a subspace of the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon)$, we conclude that the space

$$\text{Hom}_C \left( \mathbb{C}[K^C]_\lambda \otimes \lambda \mathbb{C}[K^C]^+, \mathcal{H}L^2(K^C, e^{-\kappa/|t|} \eta \varepsilon) \right)^{K \times K}$$
is 1-dimensional. However, this vector space is that of morphisms of \((K \times K)\)-representations from \(\mathbb{C}[K^C]^- \otimes \lambda \mathbb{C}[K^C]^+\) to \(\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\). Since this space is 1-dimensional, it is generated by a single such morphism, and this morphism picks out the \((K \times K)\)-irreducible constituent \(\mathbb{C}[K^C]^- \otimes \lambda \mathbb{C}[K^C]^+ \cong \text{End}_{\mathbb{C}}(\lambda \mathbb{C}[K^C]^+)\) from \(\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\). In other words, \(V^*_\lambda \otimes V_\lambda\) is the isotypical summand in \(\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\) determined by \(\lambda\).

These observations imply that the vector space \(\mathbb{C}[K^C]\) of representative functions on \(K^C\) is dense in the Hilbert space \(\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\). This proves Proposition 4.1 and hence establishes statement (i) of the holomorphic Peter-Weyl theorem, Theorem 1.14.

5. The abstract identification with the vertically polarized Hilbert space

The vertically polarized Hilbert space arising from geometric quantization on \(T^*K\) is a Hilbert space of half forms. Haar measure \(dx\) on \(K\) then yields a concrete realization of this Hilbert space as \(L^2(K, dx)\). In this section we will compare the Hilbert space \(\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\) with the vertically polarized Hilbert space. This will in particular provide a proof of statement (ii) of the holomorphic Peter-Weyl theorem.

Let \(\lambda\) be a highest weight. Let \(W_\lambda\) denote the space of complex representative functions on \(K\) which arise by restriction to \(K\) of the holomorphic functions in \(V_\lambda\). Since a holomorphic function on \(K^C\) is determined by its values on \(K\), this restriction mapping is the identity mapping of complex vector spaces, in fact, of \(K\)-representations. To justify the distinction in notation, we note that the embedding \(\iota_\lambda\) given as (1.8) above yields an embedding
\[
\iota_\lambda: W^*_\lambda \otimes W_\lambda \to R(K) = \mathbb{C}[K^C]
\]
and, maintaining the notation \(\otimes\) introduced in Section 2, we write
\[
W^*_\lambda \otimes W_\lambda = \iota_\lambda(W^*_\lambda \otimes W_\lambda) \subseteq R(K).
\]

The \(K\)-representation \(W^*_\lambda \otimes W_\lambda\) inherits a \(K\)-invariant inner product from the embedding into \(L^2(K, dx)\). On the other hand, \(V^*_\lambda \otimes V_\lambda\) acquires an inner product from its embedding into the Hilbert space \(\mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\) induced by (1.8) which turns \(V^*_\lambda \otimes V_\lambda\) into a unitary \(K\)-representation, but the relationship between the inner products on \(V^*_\lambda \otimes V_\lambda\) and \(W^*_\lambda \otimes W_\lambda\) is not a priori clear. We therefore distinguish the resulting unitary \(K\)-representations \(W_\lambda\) and \(V_\lambda\) in notation as indicated.

Let \(\langle \cdot, \cdot \rangle_K\) denote the normalized inner product on \(L^2(K, dx)\) given by
\[
\langle f, h \rangle_K = \frac{1}{\text{vol}(K)} \int_K \mathcal{F} f h dx.
\]

As usual, we endow \(\bigoplus_{\lambda \in K^C} \text{End}(W_\lambda)\) with the inner product which, on the summand \(\text{End}(W_\lambda)\), is given by
\[
d_\lambda(\cdot, \cdot) = \langle \cdot, \cdot \rangle_\lambda.
\]

This inner product differs from the inner product (1.13); see the completion of the proof of the holomorphic Peter-Weyl theorem given below for an explanation. Then \(\bigoplus_{\lambda \in K^C} \text{End}(W_\lambda)\) refers to the completion relative to this inner product. As in the situation of the inner product (1.13), up to a constant, the resulting norm on each \(\text{End}(W_\lambda)\) coincides with the Hilbert-Schmidt norm. For ease of exposition, we spell out the ordinary Peter-Weyl theorem in the following form.
Proposition 5.2. (i) The space \( R(K) \) of representative functions on \( K \) is dense in \( L^2(K, dx) \) and, as a unitary \((K \times K)\)-representation, \( L^2(K, dx) \) decomposes as the direct sum

\[
L^2(K, dx) = \bigoplus_{\lambda} (W_\lambda^* \otimes W_\lambda) \cong \bigoplus_{\lambda} \text{End}(W_\lambda)
\]

into \((K \times K)\)-isotypical summands as \( \lambda \) ranges over the highest weights.

(ii) Relative to the convolution product \(*\) on \( L^2(K, dx) \), as \( \lambda \) ranges over the highest weights, the assignment to an \( L^2 \)-function \( f \) on \( K \) of its Fourier coefficients \( \hat{f}_\lambda \in \text{End}(W_\lambda) \) yields an isomorphism

\[
L^2(K, dx) \longrightarrow \bigoplus_{\lambda} \text{End}(W_\lambda)
\]

of Hilbert algebras where \( L^2(K, dx) \) is endowed with the normalized inner product \( \langle \cdot, \cdot \rangle_K \).

The following is an immediate consequence of the ordinary and the holomorphic Peter-Weyl theorem, combined with the explicit determination of the constants \( C_{t,\lambda} \) for the highest weights \( \lambda \) given in Lemma 3.3, viz. \( C_{t,\lambda} = (t\pi)^{\dim(K)/2}e^{t|\lambda+\rho|^2/2} \).

Theorem 5.3. The association

\[
V_\lambda^* \otimes V_\lambda \ni \varphi^C \mapsto \hat{C}_{t,\lambda}^{1/2} \varphi = (t\pi)^{\dim(K)/4}e^{t|\lambda+\rho|^2/2} \varphi \in W_\lambda^* \otimes W_\lambda,
\]

as \( \lambda \) ranges over the highest weights, induces a unitary isomorphism

\[
H_t: \mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon) \longrightarrow L^2(K, dx)
\]

of unitary \((K \times K)\)-representations.

Completion of the proof of the holomorphic Peter-Weyl theorem. Let

\[
H_{t}^{\text{End}}: \bigoplus_{\lambda \in \widetilde{K}^C} \text{End}(V_\lambda) \longrightarrow \bigoplus_{\lambda \in \widetilde{K}^C} \text{End}(W_\lambda)
\]

be the obvious unitary isomorphism of \((K \times K)\)-representations which, restricted to the summand \( \text{End}(V_\lambda) \), is given by multiplication by \( C_{t,\lambda}^{1/2} \), as \( \lambda \) ranges over the highest weights. By construction, the diagram

\[
\begin{array}{ccc}
\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon) & \overset{H_t}{\longrightarrow} & L^2(K, dx) \\
\bigoplus_{\lambda \in \widetilde{K}^C} \text{End}(V_\lambda) & \overset{H_{t}^{\text{End}}}{\longrightarrow} & \bigoplus_{\lambda \in \widetilde{K}^C} \text{End}(W_\lambda)
\end{array}
\]

is commutative where the unlabelled vertical arrows are given by the assignment to a function of its Fourier coefficients. Moreover, in view of Theorem 5.3, the upper horizontal arrow is an isomorphism of unitary \((K \times K)\)-representations, the lower horizontal arrow is such an isomorphism as just pointed out and, by virtue of the ordinary Peter-Weyl theorem, the right-hand vertical arrow is an isomorphism of Hilbert algebras. In view of the algebraic version of the Peter-Weyl theorem,
Proposition 2.5 above, we conclude that the convolution product on the algebra $\mathbb{C}[K^C]$ extends to a convolution product on $\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon)$ and that the left-hand vertical arrow is an isomorphism of Hilbert algebras as asserted. □

As a consequence of the holomorphic Peter-Weyl theorem, we will now spell out a holomorphic version of the Plancherel theorem. Given the holomorphic function $\Phi$ on $K^C$, we refer to the series $\sum \lambda d_{\lambda} \text{tr} \left( T_{\lambda}(y)\hat{\Phi}_{\lambda} \right)$ in the variable $y \in K^C$ as the holomorphic Fourier series of $\Phi$. Up to a change of variable, the holomorphic Fourier series of $\Phi$ coincides with the ordinary Fourier series of the restriction of $\Phi$ to $K$. We will denote by $\| \cdot \|_{t,K^C}$ the norm associated with the inner product (1.5).

**Corollary 5.4.** [Holomorphic Plancherel theorem] The holomorphic Fourier series of a holomorphic function $\Phi$ on $K^C$ that is square integrable relative to the measure $e^{-\kappa/t}\eta\varepsilon$ converges to $\Phi$ in $H\mathcal{L}^2(K^C, e^{-\kappa/t}\eta\varepsilon)$ and hence converges to $\Phi$ pointwise as well. Furthermore, given the family $(c_{\lambda})_{\lambda \in \mathbb{C}}$ where $c_{\lambda} \in \text{End}(V_{\lambda})$, the series $\sum \lambda d_{\lambda} \text{tr} (T_{\lambda}(y)c_{\lambda})$ furnishes a holomorphic function on $K^C$ which is square-integrable relative to the measure $e^{-\kappa/t}\eta\varepsilon$ if and only if the series $\sum \lambda d_{\lambda} C_{t,\lambda} ||c_{\lambda}||^2$ converges; if this happens to be the case, when $\hat{\Phi}$ denotes the resulting holomorphic function, the Plancherel formula takes the form

\[
(5.4.1) \quad \|\Phi\|_{t,K^C}^2 = \frac{1}{\text{vol}(K)} \int_{K^C} |\Phi|^2 e^{-\kappa}\eta\varepsilon = \sum \lambda d_{\lambda} C_{t,\lambda} ||c_{\lambda}||^2.
\]

**Proof.** Let $\lambda$ be a highest weight, let $T_{\lambda}: K^C \to \text{End}(V_{\lambda})$ be the associated irreducible rational representation of $K^C$, and let

\[
T_{\lambda}^C = \frac{T_{\lambda}}{C_{t,\lambda}^{1/2}}, \quad \hat{\Phi}_{\lambda}^C = C_{t,\lambda}^{1/2} \hat{\Phi}_{\lambda}.
\]

Then, with the obvious extension of the notation $\langle \cdot, \cdot \rangle_{t,K^C}$, we have

\[
\hat{\Phi}_{\lambda}^C = \langle T_{\lambda}^C, \Phi \rangle_{t,K^C} = \frac{1}{\text{vol}(K)} \int_{K^C} T_{\lambda}^C(e^{-\kappa}\eta\varepsilon),
\]

that is, $\hat{\Phi}_{\lambda}^C$ is the Fourier coefficient of $\Phi$ relative to $\lambda$, calculated with respect to the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t}\eta\varepsilon)$. Thus the holomorphic Fourier series of $\Phi$ can be written in the form

\[
\sum \lambda d_{\lambda} \text{tr} \left( T_{\lambda}^C(y)\hat{\Phi}_{\lambda}^C \right) = \sum \lambda d_{\lambda} \text{tr} \left( T_{\lambda}(y)\frac{C_{t,\lambda}^{1/2}}{C_{t,\lambda}^{1/2}} \hat{\Phi}_{\lambda} \right).
\]

Given the representative function $\varphi^C$ in $V_{\lambda}^* \otimes V_{\lambda}$, under the isomorphism (5.3.1), the representative function $\varphi^C/C_{t,\lambda}^{1/2}$ on $K^C$ goes to the restriction $\varphi$ of $\varphi^C$ to $K$. These observations imply the assertions. □

**Remark 5.5.** A version of the holomorphic Plancherel Theorem may be found in Lemmata 9 and 10 of [7]. According to [3] (Proposition 12), the holomorphic Fourier series (referred to in [3] as a Fourier-Laurent series) of a general holomorphic function
on $\mathbb{K}^C$, not necessarily square integrable relative to the measure $e^{-\kappa/t}\eta\varepsilon$, converges uniformly on compact sets. This fact has been extended to holomorphic functions on the complexification of a general symmetric space of a compact Lie group in [23] (Theorem 3 in Subsection 5.5). The statement of Corollary 5.4 can, perhaps, be deduced from the estimates given in [22] but we do not know whether this has been worked out in the literature. Corollary 5.4 includes the convergence in the Hilbert space $\mathcal{H}L^2(\mathbb{K}^C, e^{-\kappa/t}\eta\varepsilon)$; for the convergence in this Hilbert space, see also Lemma 10 in [7]. In Theorem 9 (iii) of [7] a formula similar to (5.4.1) above is given, valid relative to any sufficiently regular bi-invariant measure.

6. The Blattner-Kostant-Sternberg pairing

In this section we will show that the isomorphism (5.3.1) is realized by the Blattner-Kostant-Sternberg-pairing, multiplied by a global constant; see e. g. [27], [32] for details on the BKS-pairing. We maintain the notation $\langle \cdot, \cdot \rangle_{t,K^C}$ for the normalized inner product on $\mathcal{H}L^2(\mathbb{K}^C, e^{-\kappa/t}\eta\varepsilon)$ induced by the measure $e^{-\kappa/t}\eta\varepsilon$.

Let $\Phi$ be a holomorphic function on $\mathbb{K}^C$ which is square integrable relative to $e^{-\kappa/t}\eta\varepsilon$ and let $F$ be a square integrable function on $\mathbb{K}$; the ordinary BKS-pairing $\langle \cdot, \cdot \rangle_{\text{BKS}}$ between the two half-form Hilbert spaces $\mathcal{H}L^2(\mathbb{K}^C, e^{-\kappa/t}\eta\varepsilon)$ and $L^2(\mathbb{K}, dx)$ assigns the integral

$$\langle \Phi, F \rangle_{\text{BKS}} = \frac{1}{\text{vol}(\mathbb{K})} \int_{\mathbb{K}} \int_{\mathfrak{k}} \Phi(x \exp(iY)) F(x) e^{-\frac{|Y|^2}{2t} \eta(Y/2)} dY dx$$

to $\Phi$ and $F$ provided this integral exists. The requisite calculation which yields the explicit form (6.1.1) of the BKS-pairing under the present circumstances is given in the appendix of [9], where the notation $\zeta(Y) = \eta(Y/2)$ is used ($Y \in \mathfrak{k}$).

We will now show that (6.1.1) extends to a pairing which is defined everywhere, that is, to a pairing of the kind

$$\langle \cdot, \cdot \rangle_{\text{BKS}} : \mathcal{H}L^2(\mathbb{K}^C, e^{-\kappa/t}\eta\varepsilon) \otimes L^2(\mathbb{K}, dx) \longrightarrow \mathbb{C}.$$ We do not assert that the integral is absolutely convergent for every $\Phi$ and $F$, though. To begin with we note that is manifest that, given a holomorphic function $\Phi$ on $\mathbb{K}^C$ which is square integrable relative to $e^{-\kappa/t}\eta\varepsilon$, when the complex function $F_\Phi$ on $\mathbb{K}$ given by the expression

$$F_\Phi(x) = \int_{\mathfrak{k}} \Phi(x \exp(iY)) e^{-\frac{|Y|^2}{2t} \eta(Y/2)} dY, \quad x \in \mathbb{K},$$

is well defined, that is, when the integral exists for every $x \in \mathbb{K}$,

$$\langle \Phi, F \rangle_{\text{BKS}} = \langle F_\Phi, F \rangle_K.$$

**Lemma 6.4.** Let $\lambda$ be a highest weight, let $\varphi^C$ be a representative function on $\mathbb{K}^C$ in the isotypical summand $V^*_\lambda \otimes V_\lambda$ of $\mathcal{H}L^2(\mathbb{K}^C, e^{-\kappa/t}\eta\varepsilon)$ associated with $\lambda$ and, as before, let $\varphi$ denote the restriction of $\varphi^C$ to $\mathbb{K}$, necessarily a representative function on $\mathbb{K}$ which lies in the isotypical summand $W^*_\lambda \otimes W_\lambda$ of $L^2(\mathbb{K}, dx)$. Then the integral (6.2) exists for every $x \in \mathbb{K}$, and the resulting function $F_{\varphi^C}$ on $\mathbb{K}$ is given by

$$F_{\varphi^C} = D_{t,\lambda} \varphi, \quad D_{t,\lambda} = (2t\pi)^{\dim(K)/2} e^{t|\lambda+\rho|^2/2}.$$
Proof. We establish the statement of the Lemma for the special case where \( t = 1 \). The general case is reduced to the special case by a change of variables.

As a \((K \times K)\)-representation, the isotypical summand \( V^\chi_\lambda \otimes V_\lambda \) associated with \( \lambda \) is spanned by the character \( \chi^C_\lambda \) of \( K^C \) associated with the highest weight \( \lambda \). Thus it suffices to establish the claim for \( \varphi^C = \chi^C_\lambda \), and we will now do so:

In view of the integration formula (3.2), given \( x \in K \),

\[
F_{\chi^C_\lambda}(x) = \int_T \chi^C_\lambda(x \exp(iY)) e^{-|Y|^2/2} \eta(Y/2) dY
\]

\[
= \frac{1}{\text{vol}(T)} \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_K \chi^C_\lambda(x \exp(\text{Ad}_y(iY))) dy \right\} e^{-|Y|^2/2} \eta(Y/2) dY
\]

\[
= \frac{1}{\text{vol}(T)} \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_K \chi^C_\lambda(y^{-1}x \exp(iY)) dy \right\} e^{-|Y|^2/2} \eta(Y/2) dY.
\]

Let \( x \in K \) and \( Y \in \mathfrak{k} \); using the formula

\[
\int_K \chi^C_\lambda(y^{-1}x \exp(iY)) dy = \frac{\text{vol}(K)}{d_\lambda} \chi_\lambda(x) \chi^C_\lambda(\exp(iY))
\]

where, as before, \( d_\lambda \) denotes the dimension of the irreducible representation associated with \( \lambda \), we conclude

\[
F_{\chi^C_\lambda}(x) = \frac{\text{vol}(K/T)}{d_\lambda} \chi_\lambda(x) \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \chi^C_\lambda(\exp(iY)) e^{-|Y|^2/2} \eta(Y/2) dY.
\]

Given \( Y \in \mathfrak{k} \), Kirillov’s character formula, cf. [20], [21], evaluated at the point \( \exp(iY) \), yields the identity

\[
\text{vol}(\Omega_\rho) j(iY) \chi^C_\lambda(\exp(iY)) = \text{vol}(\Omega_\rho) \eta(Y/2) \chi^C_\lambda(\exp(iY)) = \int_{\Omega_{\lambda + \rho}} e^{-\vartheta(Y)} d\sigma(\vartheta).
\]

Now, as in the proof of Lemma 3.3, using the diffeomorphism from \( K/T \) onto \( \Omega_{\lambda + \rho} \) which sends \( yT \) (\( y \in K \)) to \( (\text{Ad}_y)^{-1}(\lambda + \rho) \), we rewrite the integral as an integral over \( K/T \) and obtain the identity

\[
\eta(Y/2) \chi^C_\lambda(\exp(iY)) = \frac{d_\lambda}{\text{vol}(K/T)} \int_{K/T} e^{-(\lambda + \rho)(\text{Ad}_\psi(Y))} d(yT).
\]

Hence

\[
F_{\chi^C_\lambda}(x) = \frac{\text{vol}(K/T)}{d_\lambda} \chi_\lambda(x) \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \chi^C_\lambda(\exp(iY)) e^{-|Y|^2/2} \eta(Y/2) dY
\]

\[
= \chi_\lambda(x) \int_{C^+} \prod_{\alpha \in R^+} \alpha(Y)^2 \left\{ \int_{K/T} e^{-(\lambda + \rho)(\text{Ad}_\psi(Y))} d(yT) \right\} e^{-|Y|^2/2} dY
\]

\[
= \chi_\lambda(x) \int_{\mathfrak{k}} e^{-(\lambda + \rho)(Y)} |Y|^2/2 dY = (2\pi)^{\dim(K)/2} e^{(\lambda + \rho)^2/2} \chi_\lambda(x)
\]

whence, in particular, \( D_{1,\lambda} = (2\pi)^{\dim(K)/2} e^{(\lambda + \rho)^2/2} \chi_\lambda(x) \) as asserted. \( \square \)
Theorem 6.5. The BKS-pairing (6.1.1) extends to a (non-degenerate) \((K \times K)\)-invariant pairing of the kind (6.1.2). Furthermore, the assignment to a representative function \(\Phi\) on \(K^C\) of the function \(F_\Phi\) on \(K\) induces a bounded \((K \times K)\)-equivariant operator

\[\Theta_t: \mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon) \to L^2(K, dx)\]

such that

\[\langle \Phi, F \rangle_{\text{BKS}} = \langle \Theta_t(\Phi), F \rangle_K\]

and such that, when \(\varphi^C\) is a member of the isotypical summand \(V_\lambda^* \circ V_\lambda\),

\[\Theta_t(\varphi^C) = F_{\varphi^C} = D_{t, \lambda} \varphi,\]

where as before \(\varphi\) refers to the restriction of \(\varphi^C\) to \(K\). Finally, the operator

\[(4t\pi)^{-\dim(K)/4} \Theta_t: \mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon) \to L^2(K, dx)\]

sends a representative function \(\varphi^C \in V_\lambda^* \circ V_\lambda\) to \(C_{t, \lambda}^{1/2} \varphi = (t \pi)^{\dim(K)/4} e^{t|\lambda + \rho|^2/2} \varphi\) and thus coincides with the unitary isomorphism (5.3.1) of \((K \times K)\)-representations.

Proof. Let \(\lambda_1\) and \(\lambda_2\) be two highest weights, let \(\varphi^C\) be a representative function on \(K^C\) which is a member of the isotypical summand \(V_{\lambda_1}^* \circ V_{\lambda_1}\) associated with the highest weight \(\lambda_1\), and let \(\psi\) be a representative function on \(K\) which is a member of the isotypical summand \(W_{\lambda_2}^* \circ W_{\lambda_2}\) associated with the highest weight \(\lambda_2\). In view of the identity (6.3) and Lemma 6.4,

\[\langle \varphi^C, \psi \rangle_{\text{BKS}} = \langle F_{\varphi^C}, \psi \rangle_K = D_{t, \lambda_1} \langle \varphi, \psi \rangle_K.\]

Hence, by virtue of the ordinary Peter-Weyl theorem and of the holomorphic Peter-Weyl theorem, the BKS-pairing (6.1.2) is everywhere defined. By construction, the pairing is \(K\)-bi-invariant.

Let \(\varphi^C\) be a representative function on \(K^C\) which is a member of the isotypical summand \(V_\lambda^* \circ V_\lambda\) associated with the highest weight \(\lambda\). Since

\[
\frac{C_{t, \lambda}}{D_{t, \lambda}} = (t \pi)^{\dim(K)/4} e^{t|\lambda + \rho|^2/2} = ((4t\pi)^{-\dim(K)/4})^2,
\]

by virtue of Lemma 3.3 and Lemma 6.4,

\[
\int_{K^C} \overline{\varphi^C} \varphi^C e^{-\kappa/t} \eta \varepsilon = \frac{C_{t, \lambda}}{D_{t, \lambda}} \int_K F_{\varphi^C} F_{\varphi^C} dx = ((4t\pi)^{-\dim(K)/4})^2 \int_K F_{\varphi^C} F_{\varphi^C} dx.
\]

In view of the ordinary Peter-Weyl theorem and of the holomorphic Peter-Weyl theorem, this identity implies the remaining assertions of Theorem 6.5. \(\square\)

Let \(\Theta_t^*: L^2(K, dx) \to \mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon)\) be the adjoint of \(\Theta_t\). Let \(\lambda\) be a highest weight, let \(\varphi^C \in V_\lambda^* \circ V_\lambda\) and let \(\varphi \in W_\lambda^* \circ W_\lambda\) be the restriction of \(\varphi^C\) to \(K\). Define the number \(A_{t, \lambda}\) by \(\Theta_t^*(\varphi) = A_{t, \lambda} \varphi^C\). Then

\[D_{t, \lambda} \langle \varphi, \varphi \rangle_K = \langle \Theta_t(\varphi^C), \varphi \rangle_K = \langle \varphi^C, \Theta_t^* \varphi \rangle_{t, K^C} = A_{t, \lambda} \langle \varphi^C, \varphi^C \rangle_{t, K^C} = A_{t, \lambda} C_{t, \lambda} \langle \varphi, \varphi \rangle_K\]

whence \(A_{t, \lambda} = \frac{D_{t, \lambda}}{C_{t, \lambda}} = 2^{\dim(K)/2} e^{-t|\lambda + \rho|^2/2}\). Hence

\[\Theta_t^*(\varphi) = 2^{\dim(K)/2} e^{-t|\lambda + \rho|^2/2} \varphi^C.\]
Corollary 6.7. The resulting operator

\[ (6.7.1) \quad (4t\pi)^{-\dim(K)/4}\Theta_t^\ast: L^2(K, dx) \to \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t^2}e^{-\eta^2/4}) \]

is unitary and coincides with the inverse of the isomorphism (5.3.1).

Remark 6.8. As explained already in the introduction, the unitarity of the BKS-pairing map, multiplied by a suitable constant, has been established in [9] by means of the heat kernel techniques developed in [7]. In [25], using the very same heat kernel methods, the authors have shown that the unitarity of the BKS-pairing map can be reduced to a computation on matrix entries. Likewise, the proof of Theorem 5.3 reduces the abstract unitary equivalence between the two Hilbert spaces involved to inspection of certain square integrals of representative functions. However, the proof of Theorem 5.3 is direct and independent of heat kernel techniques, and in fact the statement of Theorem 5.3 is at first independent of the BKS-pairing map as well.

7. The spectral decomposition of the energy operator

Let \( \Delta_K \) denote the Casimir operator on \( K \) associated with the bi-invariant Riemannian metric on \( K \). When \( X_1, \ldots, X_m \) is an orthonormal basis of \( \mathfrak{k} \),

\[ \Delta_K = X_1^2 + \cdots + X_m^2 \]

in the universal algebra \( U(\mathfrak{k}) \) of \( \mathfrak{k} \), cf. e. g. [26] (p. 591). The Casimir operator depends only on the Riemannian metric, though. Since the metric on \( K \) is bi-invariant, so is the operator \( \Delta_K \); hence, by Schur’s lemma, each isotypical summand \( W_\lambda^* \otimes W_\lambda \subseteq L^2(K, dx) \) is an eigenspace, whence the representative functions are eigenfunctions for \( \Delta_K \). The eigenvalue of \( \Delta_K \) corresponding to the highest weight \( \lambda \) is known to be given explicitly by \( -\varepsilon_\lambda \) where

\[ \varepsilon = (|\lambda + \rho|^2 - |\rho|^2), \]

cf. e. g. [10] (Ch. V.1 (16) p. 502). The present sign is dictated by the interpretation in terms of the energy given below. Thus \( \Delta_K \) acts on each isotypical summand \( W_\lambda^* \otimes W_\lambda \) as scalar multiplication by \( -\varepsilon_\lambda \). The Casimir operator is known to coincide with the nonpositive Laplace-Beltrami operator associated with the (bi-invariant) Riemannian metric on \( K \), see e. g. [31] (A 1.2). In the Schrödinger picture (vertical quantization on \( T^*K \)), the unique extension \( \hat{E}_K \) of the operator \( -\frac{1}{2}\Delta_K \) to an unbounded self-adjoint operator on \( L^2(K, dx) \) is the quantum mechanical energy operator associated with the Riemannian metric, whence the spectral decomposition of this operator refines in the standard manner to the Peter-Weyl decomposition of \( L^2(K, dx) \) into isotypical \( (K \times K) \)-summands.

Via the embedding of \( \mathfrak{k} \) into \( \mathfrak{k}^\mathbb{C} \), the operator \( \Delta_K \) is a differential operator on \( K^\mathbb{C} \). In view of the holomorphic Peter-Weyl theorem, the unitary transform (5.3.1) (or, equivalently, (6.7.1),) is compatible with the operator \( \Delta_K \). Consequently, in the holomorphic quantization on \( T^*K \cong K^\mathbb{C} \), via the transform (5.3.1) (or, equivalently, via the BKS-pairing map (6.5.1) multiplied by \( (4t\pi)^{-\dim(K)/4} \)), the operator \( \hat{E}_{K^\mathbb{C}} \) which arises as the unique extension of the operator \( -\frac{1}{2}\Delta_K \) on \( \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/t^2}e^{-\eta^2/4}) \) to an
unbounded self-adjoint operator is the quantum mechanical energy operator associated with the Riemannian metric, and the spectral decomposition of this operator refines to the holomorphic Peter-Weyl decomposition of $\mathcal{H}L^2(K^C, e^{-\kappa/t\varepsilon})$ into isotypical $(K \times K)$-summands.

Finally, we note that, in terms of the Casimir operator $\Delta_K$, the identity (6.6) may plainly be written in the form

$$\Theta^*_t(\varphi) = 2^{\dim(K)/2} e^{-t|\rho|^2/2} e^{-t\Delta_K/2} \varphi^C,$$

where $\varphi$ is any representative function on $K$. In this description of the operator $\Theta^*_t$, the highest weights, present in the description (6.6), no longer appear explicitly. Consequently, for any smooth function $f$ on $K$, $\Theta^*_t(f)$ is the unique holomorphic function on $K^C$ whose restriction to $K$ is given by

$$\Theta^*_t(f)|_K = 2^{\dim(K)/2} e^{-t|\rho|^2/2} e^{t\Delta_K/2} f.$$

In Theorem 2.6(1) of [9], this operator $\Theta^*_t$ is written as $\Pi_\hbar$, where the parameter $\hbar$ corresponds to the present notation $t$. The value $e^{t\Delta_K/2} f$ is also given by

$$(e^{t\Delta_K/2} f)(y) = \int_K p_t(yx^{-1}) f(x) dx = (p_t * f)(y), \quad y \in K,$$

where $p_t$ is the fundamental solution of the heat equation $\frac{du}{dt} = \frac{1}{2} \Delta_K(u)$ on $K$, subject to the initial condition that $p_0$ be the Dirac distribution supported at $e \in K$ [9], [26] (Section 8).

**Remark 7.4.** With some computational effort, the numerical values of the eigenvalues $-\varepsilon \lambda$ of the Laplace operator being known, the abstract isomorphism between the two Hilbert spaces spelled out in Theorem 5.3 above can also be derived from Theorem 10 in [7] which, in turn, is proved via heat kernel techniques. Needless, perhaps, to repeat again: Our approach to the abstract isomorphism between the two Hilbert spaces spelled out in Theorem 5.3 is independent of heat kernel techniques.

8. Relationship with the naive Hilbert space

We refer to the Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t\varepsilon})$ of holomorphic functions that are square-integrable relative to the measure $e^{-\kappa/t\varepsilon}$ as the naive Hilbert space. We content ourselves with the following simplified version of the corresponding holomorphic Peter-Weyl theorem.

**Proposition 8.1.** The Hilbert space $\mathcal{H}L^2(K^C, e^{-\kappa/t\varepsilon})$ contains the vector space which underlies the algebra $C[K^C]$ of representative functions on $K^C$ as a dense subspace.

**Proof.** Since the measure is Gaussian, standard arguments involving the appropriate estimates show that each representative function is square-integrable relative to the measure $e^{-\kappa/t\varepsilon}$. The reasoning which establishes Proposition 4.1 is also valid for the naive Hilbert space. This completes the proof. $\Box$

For the highest weight $\lambda$, define the constant $\tilde{C}_{t,\lambda}$ by the identity

$$\int_{K^C} ||\chi^C_\lambda||^2 e^{-\kappa\varepsilon} = \tilde{C}_{t,\lambda} \text{vol}(K).$$

Analogously to Theorem 5.3, we now have the following.
Theorem 8.2. The association
\[ V^*_\lambda \otimes V_{\lambda} \ni \varphi \mapsto \tilde{C}_{t,\lambda}^{1/2} \varphi \in W^*_\lambda \otimes W_{\lambda}, \]
as \( \lambda \) ranges over the highest weights, induces a unitary isomorphism
\[ (8.2.1) \quad \tilde{H}_t: \mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon) \to L^2(K, dx) \]
of unitary \((K \times K)\)-representations. □

Consequently the two Hilbert spaces \( \mathcal{H}L^2(K^C, e^{-\kappa/t} \eta \varepsilon) \) and \( \mathcal{H}L^2(K^C, e^{-\kappa/t} \varepsilon) \) are unitarily equivalent as \((K \times K)\)-representations. However we do not know how to compute the values of the constants \( \tilde{C}_{t,\lambda} \). A tool like Kirillov's character formula does not seem to be available for this case. Furthermore, we do not know whether there is a candidate for a pairing inducing the equivalence between the two Hilbert spaces.

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