Topological charge pumping in excitonic insulators

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We show that in excitonic insulators with s-wave electron-hole pairing, an applied electric field (either pulsed or static) can induce a p-wave component to the order parameter, and further drive it to rotate in the $s+i\rho$ plane, realizing a Thouless charge pump. In one dimension, each cycle of rotation pumps exactly two electrons across the sample. Higher dimensional systems can be viewed as a stack of one dimensional chains in momentum space in which each chain crossing the fermi surface contributes a channel of charge pumping. Physics beyond the adiabatic limit, including in particular dissipative effects is discussed.

Controlling many-body systems, and in particular using appropriately applied external fields to ‘steer’ order parameters of symmetry broken phases, has emerged as a central theme in current physics \cite{1-8}. The excitonic insulator (EI) is state of matter first proposed in the 1960s \cite{9–11} with an order parameter defined as a condensate of bound electron hole pairs that activates a hybridization between two otherwise (in the simplest case) decoupled bands and opens a gap in the electronic spectrum. Several candidate materials including electron-hole bilayers \cite{12–14}, Ta$_2$NiSe$_5$ \cite{15–20} and 1$T$-TiSe$_2$ \cite{21–24} are objects of current intensive study; recent work \cite{14, 25–29} has pointed out possible topological energy components, as shown in Fig. 1(a), leading to a realization that their energies are equal and opposite ($\xi$-band) and the bands do not dispers in a high symmetric s-wave type, realistic interactions also allow for pairing in sub-dominant channels including p-wave (inversion-odd) ones. In equilibrium, the s-wave ground state is favored, with the potential for p-wave order revealed by its fluctuations accompanied by dipole moment oscillations: the ‘Bardasis-Schrieffer’ collective mode \cite{30}.

In this paper we show that applied electric fields can steer the order parameter to rotate in the space of s and p symmetry components, as shown in Fig. 1(a), leading to a realization of the ‘Thouless charge pump’ \cite{31–34}, providing quantized charge transport across an insulating sample.

The minimal model of an excitonic insulator involves two electron bands shown in Fig. 1(b): a valence band with energy $\xi_v(k)$ that disperses downwards from a high symmetry point (taken to have zero momentum) and a conduction band ($\xi_c$) that disperses upwards. For simplicity we assume that their energies are equal and opposite ($\xi_c = -\xi_v = \xi$). Relaxing this assumption does not change our results in an essential way. Defining the overlap $G = 2\xi_v(0)$, we distinguish the ‘BCS’ case $G > 0$ where the two bands cross at a fermi wavevector $k_F$ with fermi velocity $v_F$ as shown by the dashed lines, leading to electron and hole pockets, and the ‘BEC’ case where $G < 0$ and the bands do not cross. Excitonic order corresponds to the spontaneous formation of a hybridization between the two bands due to the electron-electron interaction $V$, leading to an order parameter $\Delta(k) = \sum_{\nu} V_{kk\nu} (\psi_{c,k}^\dagger \psi_{v,k\nu} + c.c.)$, where $\psi_{c/v}$ is the electron annihilation operator of the conduction/valence band. The s-wave order parameter $\Delta_s(k)$ is invariant under crystal symmetry operations while p-wave order parameters are odd under inversion: $\Delta_p(k) = -\Delta_p(-k)$, and often transform as a multi-dimensional representation of the crystal symmetry group. For simplicity we neglect the $k$-dependence of $\Delta_s$, and define $\Delta_p(k) = \Delta_p f_k$ where the pairing function $f_k$ carries the momentum dependence and satisfies $\max(|f_{k\pm}|) = 1$.

We focus here on the $p_x$ pairing channel, which is induced by the $x$-direction electric fields we consider here.

Writing the partition function $Z$ as a path integral over fermion fields $\psi \equiv (\psi_c, \psi_p)$, performing a Hubbard-Stratonovich transformation of the interaction term in the excitonic pairing channel and subsuming the intraband interaction into $\xi$ one obtains (see SI section I)

$$S = \int d\tau dt \{ \psi^\dagger (\partial_t + H_m) \psi + \frac{1}{g_s} |\Delta_s|^2 + \frac{1}{g_p} |\Delta_p|^2 \}$$

and the partition function is $Z = \int D\psi D[\Delta] e^{-S}$. For physically reasonable interactions such as the screened Coulomb interaction, the s-wave pairing interaction $g_s$ is typically the strongest while $g_p$ is the leading subdominant one. We may write the mean field Hamiltonian as $\int d\tau \psi^\dagger H_m \psi = \sum_k \psi^\dagger_k H_m^k \psi_k$ with

$$H_m^k |\Delta_s, \Delta_p\rangle = \xi_k \sigma_3 + \Delta_s \sigma_1 + \Delta_p f_k \sigma_2$$

where $\sigma_j$ are the Pauli matrices acting in the c/v band space. The electromagnetic field $A$ enters Eq. (2) through the minimal coupling $k \rightarrow k - A$ required by local gauge invariance.
and we set electron charge and speed of light to be one. Interband dipolar couplings could also occur [6] but do not affect our results. Since the global phase is not important, we choose the s-wave order parameter to be real. As we will show, the system develops an electrical polarization as a p-wave component π/2 out of phase with the equilibrium \( \Delta_s \) is introduced and applied electric fields create \( \Delta_p \) primarily in this channel in the BCS weak coupling case (see Ref. [30] and SI section VI), so we choose p-wave pairing in the \( \sigma_2 \) channel. The quasiparticle spectrum is \( E_k = \pm \sqrt{\varepsilon_k^2 + \Delta_s^2 + \Delta_p f_k^2} \) as shown by Fig. 1(c) for two dimension (2D). In the pure p-wave state (\( \Delta_s = 0 \)), the spectrum will have gapless points (nodes) at \( (k_x, k_y) = (0, \pm k_F) \).

**Charge pump**—Spatially uniform changes in \( \Delta_{s,p} \) produce uniform currents \( j = \sum_k \bar{\partial}_k \mathcal{H}_m \) (see SI section II), whose time integral from the initial \( (\Delta_s, \Delta_p) = (\Delta, 0) \) to the final point then gives the pumped charge \( P \). In the limit of slow order parameter dynamics, \( P \) is difference in the polarization of the final state and the initial state and has a geometrical meaning [33, 35] in terms of the flux of the Berry curvature 2-form \( B \) through the 2D surface \( S \) defined in the abstract space spanned by \( \Delta_s, \Delta_p \) and the one dimensional (1D) crystal momentum \( k \) by the trajectory in \( \Delta_{s,p} \) and the occupied momenta, or alternatively by the line integral of the Berry connection \( \alpha_{\mu} = i \langle \psi | \partial_{\mu} | \psi \rangle \) around the boundary of \( S \):

\[
P = \frac{1}{2\pi} \int_S dS \cdot B = \frac{1}{2\pi} \oint \mathbf{dl} \cdot \alpha
\]

where \( \mu = (k, \Delta_s, \Delta_p) \) (see Fig. 2(a)).

The Berry curvature \( B \) is sourced by monopoles which for the Hamiltonian Eq. (2) are the points \( \xi = \Delta_s = \Delta_p = 0 \), i.e. the points \( (k, \Delta_s, \Delta_p) = (\pm k_F, 0, 0) \) each of which has monopole charge 1. If the order parameter evolution completes a full cycle on the \( s+i \) plane, \( S \) becomes the surface of the 2-torus shown in Fig. 2(a) and the net charge pumped is the total flux from the enclosed monopoles, which is an integer \( N = 2 \) in the present case. This quantized change in the polarization is known as the Thouless pump [31], a topological phenomenon immune to disorder. Note that the monopoles exist only for the ‘BCS’ (\( G > 0 \), band inversion) case, while in the ‘BEC’ case \( \xi(k) \neq 0 \) for all \( k \) and there are no monopoles enclosed in \( S \) (see SI section II.C).

To compute the polarization for the case the order parameter does not complete a full cycle, we use the line integral representation; an explicit expression for the valence band wave function from (2) at \( (k, \Delta_s, \Delta_p) \) is

\[
|\psi\rangle = (u^*, u) = \frac{1}{\sqrt{2E(E - \xi)}} (1 \pm \frac{\xi}{\sqrt{2}}) \xi - E, \Delta^* \rangle
\]

where \( \Delta = \Delta_s + i\Delta_p f_k \equiv \Delta \delta e^{i\theta} \) and \( \xi = |u|^2(v^2) \) is given by \( \frac{1}{2}(1 \pm \frac{\xi}{\sqrt{2}}) \). The Berry connection \( \alpha_{\mu} = |u|^2 \partial_{\mu} \phi \) has singularities associated with the Dirac strings, the intersections of which with \( S \) (marked by crosses in Fig. 2(b)] must be correctly treated in the evaluation of the line integral. Noting that in the weak coupling BCS limit \( |u|^2 \to 0 \) deep inside the fermi sea and

![Fig. 2. (a) The surface \( S \) in the \((k, \Delta_s, \Delta_p)\) space used to calculate the flux of the Berry curvature for a 1D excitonic insulator for which the order parameter evolution completes a full cycle in the \( s+i \) plane. The left and right ends of the cylinder are identified so that \( S \) is a 2-torus. In the BCS case \((G > 0)\), there are two Berry curvature monopoles located at \( \pm k_F \) labeled by the blue dots. 'Dirac strings' are shown by the red dashed lines with direction shown by black arrows. (b) The surface of the torus shown in (a) parametrized by \( k \) and \( \theta \) and with \( k = \pm \pi \) and \( \theta = 0 \) identified. The contour integral of the Berry connection around the blue lines yields the charge pumped during a full cycle, with the only contributions from the vortices at \( k = \pi/a \) due to intersections with the Dirac strings. The red rectangles are used to compute the flux for a partial cycle in the BCS limit.](image)
Lagrangian dynamics, we use a low energy effective Ginzburg-Landau Lagrangian
\begin{equation}
L(\Delta_s, \Delta_p; E) = F - K + L_{\text{drive}}
\end{equation}

obtained by interpreting the action as the Lagrangian for semiclassical fields \(\Delta_s, \Delta_p\). The dynamics is given by the standard Euler-Lagrange equation \(\frac{d}{dt} \frac{\delta L}{\delta \Delta_i} = \frac{\delta L}{\delta \dot{\Delta}_i}\) and is that of a point particle moving in the landscape defined by \(F\), with kinetic energy \(K\) and driven by an electric field through \(L_{\text{drive}}\). We find
\begin{equation}
L_{\text{drive}} = -P(\theta) E^2 - s(\Delta_s, \Delta_p) E^3 + O(E^4)
\end{equation}

where \(P\) is the adiabatic polarization in Eqs. (5) or (6), \(s = \lim_{\omega \to 0} \sigma(\omega)/(2i\omega)\) and \(\sigma(\omega)\) is the optical conductivity from virtual interband excitations (see SI section IV). It is natural that electric field couples to the polarization and therefore acts to rotate the order parameter in the \(\Delta_s, \Delta_p\) plane.

\[ F(\Delta_s, \Delta_p) \] gives the potential landscape in which the dynamics takes place; it has the anisotropic 'Mexican hat' form shown in Fig. 4(a). For (quasi) 1D systems in the weak coupling BCS limit:

\begin{equation}
F = -\nu \left( \frac{\Delta_s^2 + \Delta_p^2}{\Delta_s^2 + \Delta_p^2} \right) \ln \left( \frac{2\Lambda}{\Delta_s^2 + \Delta_p^2} \right) + \frac{1}{8s} \Delta_s^2 + \frac{1}{8p} \Delta_p^2
\end{equation}

where \(\nu\) is density of states in the normal phase and
$\Lambda$ is an UV cutoff \cite{10}. The first term becomes $-\nu \int \frac{d\theta}{2\pi} \left( \Delta_s^2 + \Delta_p^2 \cos^2 \theta \right) \ln \left( \frac{2\Lambda}{\sqrt{\Delta_s^2 + \Delta_p^2 \cos^2 \theta}} \right)$ for a 2D isotropic Fermi surface and $\frac{d\theta}{2\pi} - \frac{\sin \theta d\theta \cdot d^2 \theta}{4\pi^2}$ for 3D. The landscape has a local maximum at $R = 0$ surrounded by a trough at $R(\theta)$ of lower values of $F$. The ground state minima are at $(\pm \Delta, 0)$ and the pure p-wave phases at $(0, \pm \Delta_p)$ are saddle points with energy higher by $F_b = \nu(\Delta^2 - c\Delta_p^2)/2$ where $c$ is a constant depending on the space dimension.

We may estimate the minimal electric field required to drive the system from the minimum through the p-wave saddle point by equating the potential energy barrier to the work $E\theta(\theta = \pi/2) + \theta(E^2)$ done by the electric field, obtaining

$$E_c \approx \frac{\nu}{1 - \frac{1}{4} \frac{\Delta_p}{\Delta_s}} \frac{\Delta_s}{\nu F}$$

(12)

where $\xi_0 = \nu F/\Delta$ is the coherence length (electron hole pair size), $\kappa = \frac{1}{2} (1 - \Delta^2 / \Delta_s^2)$ in 1D and $\kappa = \frac{1}{2} - \frac{1}{4} \frac{\Delta_p}{\Delta_s}$ in 2D, and $E_0$ is at the order of the dielectric breakdown field. For $\nu F = 10^{10} \text{m/s}, \Delta = 10 \text{meV}$ and $\Delta_p \ll \Delta$, such as the case of electron hole bilayers, the threshold field is $E_c \sim 10^3 \text{V/cm}$ which can be easily achieved by modern optical technique. For a 100meV gap such as that in Ta$_2$NiSe$_5$ \cite{15,16} (assuming it is in the BCS regime), the threshold field is about $10^3 \text{V/cm}$. At such large field, $O(E^2)$ terms in the Lagrangian will be important, which pushes the order parameter closer to zero but does not destroy the qualitative dynamics in the transient regime. (See SI section IV D)

The dynamical term $K$ has a relatively simple form if the gap never closes on the Fermi surface and the order parameter variation timescale is long compared to the inverse of the gap. For example for (quasi) 1D

$$K \approx \nu (\tilde{R}^2/R^2 + 3\tilde{\theta}^2) / 12$$

(13)

to lowest order in time derivatives. For higher dimensions with closed Fermi surfaces, there are $O(1)$ changes to the coefficients and, crucially, dissipation and time non-locality arises from quasiparticle excitations near the nodes of the p-wave gap when $\Delta_s$ passes zero. This dissipation also brings a correction to the pumped charge: $P = P_{2D} + P_{\text{dis}}$. To estimate $P_{\text{dis}}$, we observe that as the order parameter passes this gapless regime with a velocity $\Delta_s$, the probability for the spinor at $k$ to be excited to the high energy state is given by the Landau-Zener formula \cite{37}: $P_k = e^{-2\pi \delta^2 / 12(\Delta_2 \Delta_3)}$ where $\delta^2 = (\Delta_s^2 + \Delta_p^2 - 1/2\nu F$). It $\Delta_s^2 + \Delta_p^2 / 2\nu F$ is its minimal energy splitting during the dynamics. In 2D, summing over momenta, one obtains the number of excited quasi particles $N = \frac{k_F}{2\pi^2 \nu F} \left( \frac{\Delta_s}{\nu F} \right)^2$ and the non-adiabatic correction to the pumped charge

$$P_{\text{dis}} = -P_{2D} \frac{1}{8\pi^2} \left( \frac{\Delta_s}{\nu F} \right)^2$$

(14)

valid if $\left| \Delta_s \right| \ll \left| \Delta_p \right|$ (see SI section VI B).

**Numerics and Experiment**—We numerically solved the mean field dynamics implied by Eq. (1) in the weak coupling BCS limit and driven by a train of widely separated electric field pulses (Fig. 4(c)). A static electric field in the DC transport regime could also drive such an order parameter rotation but heating effects beyond the scope of this paper would have to be considered.

To perform the computations we make the mean field approximation that each momentum state evolves in the time dependent mean field ($\Delta_s, \Delta_p \int_{k=\frac{\pi}{2}}^{\pi/2} \xi_{k=\frac{\pi}{2}}$) with $\Delta_s, \Delta_p$ determined self consistently by the gap equation, neglecting any spatial fluctuations, and including a weak phenomenological damping $\gamma$ to represent energy loss caused e.g. by a phonon bath (see SI section VI). Each pulse drives the order parameter along the trajectory shown as the black dashed line in Fig. 4(a), advancing it by $\theta = \pi$ to stabilize the system in the other s-wave ground state. The total duration of the evolution from one minimum to the next is $T_s = 10/\Delta$ and the amount of charge pumped is $WP/2$ where $W$ is the width of the sample. Using a train of pulses with inter pulse separation $T_0 \gg T_s$ such that the order parameter is stabilized before next pulse arrives, each pulse will induce such a dynamics and charge pumping, and a quasi steady current $I_0 = eWk_F / (\pi T_0)$ is generated. For a $10 \mu$m wide sample with normal state carrier density of $10^{12} \text{cm}^{-2}$, and inter pulse time $T_0 = 1 \text{ns}$, the current is $I_0 = 255 \text{mA}$ considering spin degeneracy.

A minimum field strength $\sim I_0$ is required as the maximum electric field $E_{\text{max}}$ of the pulse is increased beyond the threshold, the charge pumping (DC current) will onset sharply, as shown in Fig. 4(c). As $E_{\text{max}}$ further increases, each pulse induces a rotation of more cycles and would pump more charge, giving rise to the step structure. Deviations from perfect quantization arise from fast order parameter dynamics caused by the short duration pulse. A precisely engineered long duration pulse can substantially reduce these deviations; see SI section V.

**Discussion**—In summary, we have shown that applied electric fields can reveal a p-type order in an otherwise s-type excitonic insulator and can drive a Thouless charge pump, if the difference between s and p wave coupling constants is not too large. Similar dynamics and charge pumping can happen in general when the ground state order parameter and the sub dominant one have different parities under inversion. Observation of the charge pumping would provide both a verification of order parameter steering and a probe of the excitonic insulating state, in particular, distinguishing BCS and BEC states. Study of the charge pumping in the vicinity of the BCS-BEC crossover is of interest. The photo induced dynamics in Fig. 4(a) switches the system between the two degenerate states with $\pm \Delta_s$, and can be viewed as writing a memory storage device. This memory is read-in systems where the two states can be distinguished, such as those with interband hybridization (leading to ferroelectricity \cite{38}) or coupling to the lattice that breaks the U(1) invariance \cite{18,20,39}.

In the special case of $g_s = g_p$ in 1D, the free energy landscape Eq. (11) is rotationally symmetric on the $s + ip$ plane with degenerate minima along the circle $R = \Delta$. Exact mean field dynamics predicts that an electric field pulse establishes...
a dissipation-less rotation which persists with a ‘supercurrent’ flowing (see SI section VI A). Further investigation beyond mean field and BCS weak coupling limit is interesting.

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Supplemental Material for ‘Topological charge pumping in excitonic insulators’

I. THE HAMILTONIAN

We base our discussion on the two-band spinless Fermion Hamiltonian that is a minimal model for excitonic insulators:

\[ H = \int dr \left[ \psi^\dagger \left( \xi(p - A)\sigma_3 + \varphi \right) \psi + \int dr dr' V(r - r')\psi^\dagger(r)\psi(r)\psi^\dagger(r')\psi(r') \right] \]

(S1)

where \( \psi^\dagger = (\psi^\dagger_c, \psi^\dagger_v) \) is the two component electron creation operator with c/v labeling the conduction/valence band, \( \xi(p) = e(p) - \mu \) is the kinetic energy, \( p = -i\hbar \nabla \) are the Pauli matrices in c-v space, \( (\varphi, A) \) is the EM potential and we have set \( e = c = 1 \). In the non interacting case, the overlap of the bands gives rise to an electron and a hole pocket, each with the Fermi momentum \( k_F \), Fermi velocity \( v_F \), Fermi level density of states \( \nu = \frac{1}{2\pi} k_F / v_F \) in 2D and carrier density \( n/2 \) of electrons in the conduction band and holes in the valence band.

The repulsive interaction \( V \) between electrons is attractive between electrons and holes and can induce pairing in several angular momentum channels, in formal analogy to pairing in superconductors. We write the model as a fermionic path integral so the partition function is \( Z = \int D[\bar{\psi}, \psi]e^{\frac{\hbar}{2}\int d\theta \cos(\theta) V(2k_F \sin(\theta/2))} \) and decouple the interaction in the pairing channel: \( Z = \int D[\bar{\psi}, \psi]D[\Delta, \Delta^\dagger]e^{-\frac{\hbar}{2}(\psi^\dagger \psi)} \). The Hubbard-Stratonovich fields then represent the order parameters.

We resolve the Hubbard-Stratonovich fields \( \Delta \) into basis functions \( f_i \) of the point symmetry of the material as \( \Delta(p) = \sum f_i f_i^\dagger(p) \). We assume for notational simplicity that the excitonic effects occur near a high symmetry point so lattice effects are unimportant and that the interaction effects may be restricted to the Fermi surface. In this case the \( l \) become the usual d-dimensional rotational harmonics and the interaction is parameterized by one momentum transfer connecting two points on the Fermi surface. We focus on s symmetry \( f_s(k) \) has the full point symmetry of the lattice; we take \( f_s = 1 \) and \( p_s \) symmetry \( f_p = k_s / k_F \). Projecting the interaction onto these channels defines the coupling constants \( g_l = \frac{1}{2\pi} \int d\theta \cos(\theta) V(2k_F \sin(\theta/2)) \).

Note that for \( l = 0 \), the \( 1/2\pi \) factor should be changed to \( 1/4\pi \). For Thomas-Fermi screening interaction \( V(q) = \frac{2\pi}{c(q + \pi F)} \) in 2D where \( q_{TF} / (2k_F) = \alpha = e^2 / (\epsilon h v_F) \) and \( \epsilon \) is the dielectric constant of the environment, the s-wave pairing strength is

\[ v_{g_s} = \frac{\alpha}{\sqrt{1 - \alpha^2}} \frac{2\pi}{\pi} \frac{1}{\tan^{-1}\left(\sqrt{1 - \alpha^2}\right)} \]

(S2)

and the p-wave one is

\[ v_{g_p} = \frac{1}{2\pi} \int d\theta \frac{\cos \theta}{2k_F \sin(\theta/2) + q_{TF}} \]

\[ = \alpha \left[ -\frac{4}{\pi} + 2\alpha + \frac{4}{\alpha} \frac{1 - 2\alpha^2}{\pi \sqrt{1 - \alpha^2}} \left( \tan^{-1}\left(\sqrt{1 - \alpha^2}\right) - \tan^{-1}\left(\frac{\sqrt{1 - \alpha^2}}{1 + \alpha}\right) \right) \right] \]

(S3)

where \( v = k_F / (\pi h v_F) \) is the normal state density of state without spin degeneracy and \( \alpha = e^2 / (\epsilon h v_F) \) is the ‘fine structure constant’ in this system. These equations were previously given [30] and are reproduced here for convenience.

The pairing interactions are shown in Fig. S1 for the screened Coulomb interaction in 2D. To obtain a substantial \( g_p / (2g_s) \), one needs the high density case where the fermi velocity is large so that the Thomas fermi wave vector is smaller than the fermi momentum: \( q_{TF} / (2k_F) = \alpha = e^2 / (\epsilon h v_F) \ll 1 \). Stronger dielectric screening of the environment can further reduce \( \alpha \) and increase \( g_p / (2g_s) \). A non-negligible interlayer distance \( a \) changes the bare electron-hole Coulomb attraction into \( V(r) = 1/\sqrt{r^2 + a^2} \), making it more nonlocal and thus can lead to a larger \( g_p / (2g_s) \). Other types of interactions such as nearest neighbor Hubbard interaction (although originating from Coulomb) could give very strong \( g_p \), so that the band overlapping is suitable.

We further observe that the overall phase of the Hubbard Stratonovich field is not relevant for our considerations, so we choose \( \Delta_i \) to be real. As we will see, the main effect of an electric field is to induce a p-wave field that is \( \pi/2 \) out of phase with \( \Delta_i \); we restrict attention to this case. The result is

\[ S[\psi, \Delta_x, \Delta_p, A] = \int d\tau dr \left\{ \psi^\dagger (\partial_t + H_m) \psi + \frac{1}{g_s} |\Delta_x|^2 + \frac{1}{g_p} |\Delta_p|^2 \right\} \]

(S4)

The mean field Hamiltonian is \( H_m = \xi k \sigma_3 + \Delta_x \sigma_3 + \Delta_p f_k \sigma_2 \) and the EM field enters as \( k \rightarrow k - A \). Note that minimal coupling substitution is also applied to the p-wave decoupling term: \( \Delta_p f_k - A \), although this term comes from the electron-electron interaction that contains no EM field. We discuss this choice here in terms of local gauge invariance.

In the full functional integral, the general gauge invariant form of the decoupling term is \( e^{-\int \frac{1}{4 \pi^2} d^2 \theta(\theta) |\Delta(1_1, r_2)\psi(1_1)\psi(1_2)|^2} \), which preserves its form under the usual local gauge transformation \( U_g : \psi(r) \rightarrow \psi(r) e^{i\theta(r)} \), \( A_\mu \rightarrow A_\mu + \partial_\mu \theta(r) \). We write
FIG. S1. (a) The s,p,d-wave components of the screened Coulomb interaction in 2D and the ‘fine structure constant’ \( \alpha = \frac{e^2}{\epsilon \hbar v_F} = q_{TF}/k_F \) as functions of electron density \( n_i = \frac{m^2 v^2_f}{4\pi \hbar^2} \) computed from Eqs. (S2) and (S3) using \( m = 0.05 m_e \) and \( \epsilon = 10 \). (b) The ratio \( g_p/(2g_s) \) as a function of \( \alpha = q_{TF}/(2k_F) \). For \( \alpha \ll 1 \), i.e., in the high density case, \( g_p/(2g_s) \) becomes considerable and approaches one in the high density limit. Spin degeneracy is neglected.

\[ \Delta(r_1, r_2) = |\Delta(r_1, r_2)| e^{i \phi(r_1, r_2)} \]
both the amplitude \( |\Delta(r_1, r_2)| \) and the phase \( \phi(r_1, r_2) \) are dynamical variables. The dependence on ‘center of mass’ coordinate \( r = (r_1 + r_2)/2 \) gives the spatial variation of the order parameter while the dependence on \( r_1 - r_2 \) gives the internal structure of the electron hole pair (the momentum dependence of the pairing function). Writing the phase degrees of freedom as \( \phi(r_1, r_2) = \phi_0(r) + (r_1 - r_2)\alpha(r) \) in the slow varying limit, the \( \phi_0 \) is the usual order parameter phase and the combination \( \alpha + A \) enters structure of pairing wave function as \( \tilde{f}_{k-A-A} \). In the full long wavelength theory one should track the dynamics of \( \alpha \). If the dynamics driven by external electric fields does not significantly change the internal structure of the electron-hole pair (as is the case in the weak coupling BCS limit) then we may neglect the dynamics of \( \alpha \).

Even in the general case when the time dependence of \( \alpha \) must be considered at intermediate stages of the dynamics, the initial and final values remain the same and the amount of charge pumping during a full cycle is still the quantized value given that the system finally returns to its initial state, as shown in general by Thouless [31].

If the two bands are formed by atomic orbitals having different parities, e.g., p and d orbitals, an interband dipolar moment term \( D\sigma_1 \) can also occur. This term also contributes to the EM response due to change of inter orbital hybridization. However, for a full cycle of order parameter dynamics, the amount of pumped charge won’t be affected since the initial and final states have the same inter orbital hybridization. The dynamics itself won’t be qualitatively affected if the interband dipole \( D \) is not large compared to the dipole formed between \( s \) and \( p \)-symmetry electron and hole bound states that produce the order parameters. This is true in the BCS case since the former is proportional to the size of atomic orbitals while the latter is the size \( \xi \) of the extended electron hole bound state.

II. COMPUTATION OF THE POLARIZATION

In this section, we explicitly derive the charge pumping in an 1D excitonic insulator by computing the polarization \( P \) (charge pumped) as a time integral of the current \( J \) induced by adiabatic changes to the order parameter over a time interval from 0 to \( t \) and comparing the result to the formula in terms of Berry curvature, consistent with previous results [33].

A. Polarization

For convenience we reproduce the mean field Hamiltonian \( H \) and current operator \( J \) here:

\[ H_k = \xi_k \sigma_3 + \Delta_s \sigma_1 + \Delta_p f_k \sigma_2, \quad j_k = \psi^\dagger (v_k \sigma_3 + \Delta_p \partial_k f_k \sigma_2) \psi, \quad J = \sum_k j_k \nabla \phi_k \]  \hspace{1cm} (S5)
where \( v_k = \partial_k \xi \) and the energy eigenvalues are \( E_k = \pm \sqrt{\xi_k^2 + \Delta_0^2 + \Delta_p^2 f_k^2} \). The current \( \delta J \) in response to a change in order parameter \( \delta \Delta = \delta \Delta_s(t) \sigma_1 + f_k \delta \Delta_p(t) \sigma_2 \) is

\[
\delta J(\Omega_n) = -T \sum \sum \int \left[ J_k(i\omega_n + i\Omega_n - H) \delta \Delta(i\omega - H) \right] \frac{\delta \Delta_s(\omega)}{(\omega_n + \Omega_n)^2 + E_k^2_0} \left( \omega_n^2 + E_k^3 \right) \frac{\delta \Delta_p(\omega)}{\omega_n + E_k^2} \]

in frequency representation. Carrying out the trace over band indices, performing the frequency summation at \( T = 0 \) and analytically continuing \( i\Omega_n \) to \( \omega \), one obtains

\[
\delta J(\omega) = \frac{i\omega}{2} \left( \sum \frac{v_k f_k - \xi_k \partial_k f_k}{E_k^3} \Delta_p \delta \Delta_s(\omega) - \sum \frac{v_k f_k}{E_k^3} \sum \frac{\Delta_p \delta \Delta_p(\omega)}{\omega_n + E_k^2} \right) \]

(S7)

In the adiabatic limit we may neglect the \( \omega^2 \) in the denominators; then transforming to the time domain we obtain

\[
J(t) = -\frac{1}{2} \left[ \sum \frac{v_k f_k - \xi_k \partial_k f_k}{E_k^3} \Delta_p \partial \Delta_s \frac{dt}{dt} - \sum \frac{v_k f_k}{E_k^3} \Delta_p \frac{dt}{dt} \right] \]

(S8)

Integrating in time gives the change in polarization:

\[
P = \int \left( \Delta_p \left( \frac{v_k f_k - \xi_k \partial_k f_k}{2E_k^3} \right) \frac{d \Delta_s dk}{2\pi} + \frac{\Delta_s v_k f_k d \Delta_p dk}{2E_k^3} \right) \]

(S9)

### B. Berry Connection and Berry Curvature

The Berry connection \( \mathcal{A}_\mu \) is given in terms of the change in wave function under infinitesimal variation of the parameters \( \mu = (k, \Delta_s, \Delta_p) \) as \( \mathcal{A}_\mu = i \psi | \partial_\mu \psi \rangle \). Defining \( \Delta = \Delta_s + i \Delta_p f_k \equiv |\Delta| e^{i\phi} \) we may write the valence band wave function as

\[
|\psi \rangle = (-v^*, u^*) = \frac{1}{\sqrt{2E(E - \xi)}} (\xi - E, \Delta^*) \]

(S10)

implying \( \mathcal{A}_\mu = |u|^2 \partial_\mu \phi \) where \( |u|^2 = \frac{1}{2} \left( 1 + \frac{\xi}{E} \right) \). Explicitly,

\[
\left( \mathcal{A}_{\Delta_s}, \mathcal{A}_{\Delta_p}, \mathcal{A}_k \right) = |u|^2 \left( -\frac{f_k \Delta_p}{\Delta_s^2 + f_k^2 \Delta_p^2}, \frac{f_k \Delta_s}{\Delta_s^2 + f_k^2 \Delta_p^2}, \frac{\Delta_s \Delta_p \partial_k f_k}{\Delta_s^2 + f_k^2 \Delta_p^2} \right) \]

(S11)

Note that \( \mathcal{A} \) has singularities (“Dirac strings”) along the line \( \Delta_s = \Delta_p = 0 \) and also, for a closed Fermi surface, along the line \( \Delta_s = k_x = 0 \). These are shown as dashed lines in Fig. S2(a).

The Berry curvature \( B = dA \) is then

\[
\left( B_{\Delta_s}, B_{\Delta_p}, B_k \right) = -\left( \frac{\Delta_s v_k f_k}{2 E_k^3}, \frac{\Delta_p v_k f_k - \xi \partial_k f_k}{2 E_k^3}, \frac{\Delta_s \Delta_p \partial_k f_k}{2 E_k^3} \right) \]

(S12)

Considering now the flux of \( B \) through a surface element of an oriented 2D manifold in \( \Delta_s, \Delta_p, k \) space defined by a function \( S(\Delta_s, \Delta_p) \) = constant and choosing the orientation to be pointing ‘inside’ the cylinder in Fig. S2(a) we see by comparison to Eqs. S9 that the flux through the surface is just the polarization. This conclusion is independent of the choice of coordinate.

### C. BCS-BEC crossover

If the numbers of electrons and holes are separately conserved, the total number \( n_s = (n_{\text{e}lectron} + n_{\text{hole}}) = -\langle \sigma_3 \rangle + n_0 \) is also conserved where \( n_0 \) is the particle number of a completely occupied band. \( n_s \) is the analogy to the total charge in a superconductor, and gives the constraint that shifts \( G \) from positive to negative as interaction becomes stronger such that the system crossovers from a BCS to a BEC type condensate. This is the situation in electron hole bilayers with no interlayer tunneling. Moreover, \( n_s \) can also be approximately fixed by gate voltage. For natural crystals, \( n_s \) is not fixed since there are always interband conversion mechanisms breaking this \( U(1) \) symmetry. Hartree terms due to Coulomb repulsion between a/b orbitals will shift up \( G \) and induce such a crossover in this case.
In the BEC case ($G < 0$, no band inversion), there are no monopoles and the Dirac string structure looks like that in Fig. S2, rendering zero pumped charge. Intuitively, the excitons in the BEC state are tightly bound electron hole pairs that don’t overlap with other, and can be viewed as charge neutral point particles. Thus no charge transport can occur.

Therefore, there is a topological transition at $G = 0$ during the BCS-BEC crossover, and the charge pumping $P$ can be viewed as an ‘order parameter’ that separate these two regimes, as shown in Fig. S2(c). However, we focused on the dynamics in the BCS limit in this paper, and it is interesting to investigate similar dynamics in the crossover regime.

## D. Pumped charge for arbitrary rotation angle

In this section we provide the details leading to Eqs. (5), (6) of the main text.

Parameterizing $S$ using $k$ and the angle $\theta \equiv \arg(\Delta_s, \Delta_p)$ defined in Fig. 1(a), the Berry connection and curvature can be projected onto the $(k, \theta)$ space. In other words, The wave function can now be viewed as a function of $(k, \theta)$ and $\Delta(k, \theta) = \Delta_s + i \Delta_p f_k = |\Delta(k, \theta)| e^{i \phi(k, \theta)}$ is the pairing field at $(k, \theta)$. Note that $f_k$ has a sign that depends on the direction of $k$, and that $|u|^2 = \frac{1}{2} \left( 1 + \frac{\phi}{\pi} \right)$ is nearly zero deep inside the fermi sea and $|u|^2 - 1$ outside when $\phi \gg |\Delta|$.

The Berry curvature is concentrated in the region $|\ell| \leq \Delta$ and thus the net density of pumped charge during a full cycle is

$$\int_{\ell = 0}^{\pi} \left| \frac{\partial \phi}{\partial \theta} \right| dt_{\ell} = \frac{4\pi}{2\pi} \sum_{\ell = 0}^{\Delta} (\ell + 1)$$

for an 1D excitonic insulator. This result may also be understood by noting that the low energy physics around $\pm k_F$ is of two massive Dirac models, each of which realizes a Goldstone-Wilczek [36] mechanism of charge pumping.

In a 2D system one has two momenta, which we choose to be parallel $(k_x)$ and antiparallel $(k_y)$ to the direction defined by the antisymmetry of $\Delta_p$. The net charge pumped is then an integral over $k_y$ of the previously obtained formula. The only change is that now $\xi(k_x) \rightarrow \xi(k_x, k_y)$ and it may be that for some values of $k_y$, the sign of $\xi$ does not change, meaning that for these $k_y$ the monopoles lie outside the torus of integration so no charge pumping occurs. In the weak coupling limit the issue may be discussed in terms of the Fermi surface of the disordered $(\Delta = 0)$ phase. If the fermi surface is open (fermi crossings for each $k_y$ as $k_x$ is varied) the density of transferred charge is $2/\pi a_y$ during a full cycle where $a_y$ is the lattice constant in $y$ direction; if the fermi surface is closed, then only the range of $k_y$ where crossings occur gives rise to a charge pumping; thus the net density of pumped charge during a full cycle is $2k_F a_y / \pi$ where $k_F$ is the maximum extent of the fermi surface in the $y$ direction.

For an incomplete cycle with arbitrary $\theta$, note that each 1D momentum chain crossing the fermi surface at $\left( \pm \sqrt{k_F^2 - k_y^2}, k_y \right) = k_F (\pm \cos \theta_0, \sin \theta_0)$ contributes a charge pumping channel described by Eq. (S13), with effective rotation angle $\phi(k_y) = \tan^{-1}(\cos \theta_0 \tan \theta)$. Summing over all the chains, one obtains

$$P = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk_y \frac{\phi(k_y)}{\pi} = \frac{k_F}{2\pi} \int_{-1}^{1} dt \tan^{-1} \left( \sqrt{1 - t^2} \tan \theta \right) = \frac{k_F}{2\pi} \tan \frac{\theta}{2}$$

for $0 < \theta < \pi/2$. Extending the above integral to higher angles, one obtains Eq. (6) of the main text.

## E. Current response in time domain

In this section, we try to expand Eq. (S6) to higher orders in frequency and show that this won’t give corrections to the adiabatic result. We focus on the nodes at $k = (0, \pm k_F)$ in 2D when the system is close to pure $p_x$-wave order. In 3D, the nodes become a nodal line and the result stays the same up to some $O(1)$ constants. We assume the order parameter passes the point $(0, \Delta_p)$ with nearly constant velocity $\Delta_s$. Close to $(0, \Delta_p)$, since the trajectory of motion is nearly along the $\Delta_s$
where \( \Delta_\mu \) is the adiabatic current leading to Eq. \( \text{(S14)} \) and
\[
\Delta_\theta(\theta, 0) = \sum_k v_k f_k \frac{\Delta_\mu}{E_k} \frac{-2i\omega}{\omega^2 - 4E_k^2} = C_0(\Delta_\mu, \Delta_\sigma)(-i\omega) + C_1(\Delta_\mu, \Delta_\sigma)(-i\omega)^2 + O(\omega^3)
\] (S15)

where
\[
C_0 = \frac{-1}{2} \Delta_\mu \sum_k v_k f_k \frac{1}{E_k} = -\Delta_\mu v \int d\theta \frac{1}{2\pi} \frac{1}{\sqrt{\Delta_\mu^2 \cos^2 \theta + \Delta_\sigma^2}} = -\nu v F \frac{1}{\Delta_\mu} = -\frac{1}{2\pi} \frac{k_F}{\Delta_\mu}
\] (S16)
is the adiabatic current leading to Eq. \( \text{(S14)} \) and \( C_1 \) is a dissipative term that arises from quasiparticles excitations. At exactly \((0, \Delta_\mu)\), it is
\[
C_1 = \frac{1}{-i\omega} \frac{\pi}{\Delta_\mu} \sum_k v_k f_k \frac{1}{E_k} \frac{2E_k}{\omega^2 - 4E_k^2} \frac{2E_k}{\omega^2 - 4E_k^2} (\delta(\omega - 2E_k) - \delta(\omega + 2E_k)) \approx \frac{1}{8} \frac{k_F}{\Delta_\mu^2}
\] (S17)

Another source for dissipative current is the quasiparticle contributed optical conductivity from the node:
\[
\sigma_{xx} = \frac{i}{\omega} \chi_0 \Delta_\mu \delta_{kk_\sigma \sigma_2} \frac{i}{k_F^2} \chi_0 \sigma_{2 \sigma_2} = \frac{i}{\omega} \frac{\Delta_\mu^2}{k_F^2} \frac{\Delta_\sigma^2}{\omega^2 - 4E_k^2} \frac{\Delta_\mu}{8 \frac{k_F}{\Delta_\mu}} + \nu \text{Im}[\sigma_{xx}]
\] (S18)

Its real part is suppressed by the small number \( \Delta_\mu / k_F \) and can thus be neglected.

It appears from Eq. \( \text{(S17)} \) that there is a correction to the pumped charge as \( \delta P = \int d\tau C_1 \partial_\tau \Delta_\sigma \). However, if one includes higher order terms in frequency, the current response from Eq. \( \text{(S15)} \) can be written in time domain:
\[
j(t) = \int_{-\infty}^t dt' \chi(t - t', t') \partial_\tau \Delta_\sigma
\] (S19)

where
\[
\chi(t, t') = \sum_k v_k f_k \frac{\Delta_\mu}{E_k^2} \sin(2E_k t) = \nu \frac{1}{2\pi} \int d\xi \text{Im} \Delta_\mu v_F \cos^2 \theta \frac{\sin(2\xi t)}{\xi^2 + \Delta_\sigma^2 + \Delta_\mu^2}
\]
\[
= \text{node contribution + high energy state contribution}
\]
\[
\approx \nu \frac{2}{2\pi} \Delta_\mu^2 v_F \int_0^{\Delta_\mu} d\xi \cos^2 \theta \frac{\sin(2\xi t)}{\xi^2 + \Delta_\sigma^2} + \nu \frac{2\pi}{2\pi} \Delta_\mu v_F \int_0^{\Delta_\mu} d\xi \sin(2\xi t)
\]
\[
= \nu \Delta_\mu^2 v_F \int_0^{\Delta_\mu} d\xi \cos^2 \theta \frac{\sin(2\xi t)}{\xi^2 + \Delta_\sigma^2} + \nu \frac{2\pi}{2\pi} \Delta_\mu v_F \int_0^{\Delta_\mu} d\xi \sin(2\xi t)
\]
\[
= \nu \Delta_\mu^2 v_F \int_0^{\Delta_\mu} d\xi \cos^2 \theta \frac{\sin(2\xi t)}{\xi^2 + \Delta_\sigma^2} + \nu \frac{2\pi}{2\pi} \Delta_\mu v_F \int_0^{\Delta_\mu} d\xi \sin(2\xi t)
\]
\[
= -\nu v F \Delta_\mu^2 \left( \partial_\tau + 4\Delta_\sigma^2 \int dt \right) \frac{1}{\nu} \left( \sin(2\Delta_\mu t) - \sin(2\Delta_\sigma t) \right)
\] (S20)
is the response kernel which is time dependent due to the fact that $\Delta_0, \Delta_p$ changes with time. If one uses their values at $t'$ and evaluate the polarization at $t \to \infty$ by $\int dt \, j(t)$, one recovers exactly the adiabatic current and the topological charge pumping. Therefore, the non-adiabatic correction is beyond the scope of Eq. (S20), but lies in the fact that the state at $t'$ is not the ground state of the instantaneous mean field Hamiltonian, as assumed here. We will show that this physics can be addressed in terms of exact dynamics of pseudo-spins.

III. EDGE STATES

In this section we analyse the behavior of edge states. For simplicity we focus on the weak coupling BCS limit of Eq. (S4) with open boundary condition. Linearizing the Hamiltonian near the two fermi points $\pm k_F$, we find that an edge state wave function may be written

$$\psi(x) = \phi_1 e^{-ik_F x + k_0 x} + \phi_2 e^{i k_F x + k_0 x}$$

with energy $E^2 = \Delta^2 - v_F^2 k_0^2$ where $\Delta^2 = \Delta_0^2 + \Delta_p^2 f(k_F)^2 = \Delta_0^2 + \Delta_p^2$ and we have made use of our convention $f(k_F) = 1$. The spinor part of the wave function is

$$\phi_1 = (\Delta_0 + i \Delta_p, -i v_F k_0 + E), \quad \phi_2 = (\Delta_0 - i \Delta_p, i v_F k_0 + E).$$

To satisfy the open boundary condition $\psi(0) = 0$, one requires $\phi_1 + \phi_2 = 0$ which yields

$$\frac{\Delta_0 + i \Delta_p}{\Delta_0 - i \Delta_p} = \frac{E - i v_F k_0}{E + i v_F k_0}.$$

This and the relation $E^2 = \Delta^2 - v_F^2 k_0^2$ is satisfied by two solutions: $(k_0, E_+) = (\Delta_p/v_F, \Delta_0)$ and $(k_0, E-) = (\Delta_p/v_F, -\Delta_0)$. The corresponding wave functions are

$$\psi_{\pm} = \frac{1}{C_{\pm}} (1, \pm 1) \sin(k_F x) e^{\mp i k_0 x/v_F}.$$

Note that the subscript $\pm$ tracks each wave function smoothly as $\theta$ varies, but does not specify either the energy or the side where the state is localized at. They are determined by the sign of their energies and the exponential factors.

The relation between the two edge states follows from symmetries. One may define two unitary operations, the ‘phase rotated inversion’ $\hat{P} : (\psi_a(x), \psi_b(x)) \mapsto (\psi_a(-x), -\psi_b(-x))$ and the $\hat{T} : (\psi_a(x), \psi_b(x)) \mapsto (\psi_b(-x), -\psi_a(-x))$. Both operators inter convert the two edge states. $\hat{P}$ is a symmetry of the mean field Hamiltonian $H$ in Eq. (2) if the system is in a pure $p$-wave state while $\hat{T}$ always anti commutes with $H$. Therefore, $\psi_{\pm}$ have opposite energies and will be at zero energy in a pure $p$-wave state.

Note that in open 1D wires connecting two reservoirs, although the edge states seem to be responsible for the charge pumping, the actual carriers are all electrons in the valence band moved by continuous deformation of their wave functions, which is a bulk property. Indeed, in macroscopically long wires, the expansion and shrinking of edges states happen only in a tiny vicinity of $\theta = 0, \pi$, while the charge pumping is a continuous process as $\theta$ varies. For example in the $\theta = 0$, state, although there is an occupied edge state localized on the right, the other electrons in the valence band form a density distribution that has a ‘hole’ on the right, such that the total polarization is still nearly zero. In the $\theta \to \pi/2$ state, the background density distribution has a ‘half’ hole on each edge. Together with the occupied edge state on the right, it look like there is a half charge on the right edge and a half hole on the left, so that the polarization $\hat{P} = 1/2$.

IV. THE GINZBURG-LANDAU ACTION

In this section we present the derivation of the semiclassical action used in the main text to discuss the dynamics. We interpret the action as the Lagrangian for the order parameter fields moving in the presence of an externally applied electric field $E$. We write the Lagrangian

$$L(\Delta_0, \Delta_p; E) = F - K - L_{\text{dis}} + L_{\text{drive}}$$

as the sum of four terms: the static free energy landscape $F$, the ‘Kinetic energy’ $K$, and dissipation and drive terms. We consider each in turn.
A. Static free energy landscape

By integrating out the fermions for time independent values of the Hubbard stratonov parameters we obtain $F = \text{Tr} \ln \left[ i \omega_n + \xi_k \sigma_3 + \Delta_s \sigma_1 + \Delta_p f_k \sigma_2 \right] + \frac{\Delta_s^2}{g_s} + \frac{\Delta_p^2}{g_p}$. Explicitly evaluating the Trln we find for (quasi) 1D systems in the BCS limit:

$$F = -\nu \left( \Delta_s^2 + \Delta_p^2 \right) \ln \frac{2\Lambda}{\sqrt{\Delta_s^2 + \Delta_p^2}} + \frac{1}{g_s} \Delta_s^2 + \frac{1}{g_p} \Delta_p^2$$  \hspace{1cm} (S26)

where $\Lambda$ is a UV cutoff determined by both the fermi energy and the Thomas-Fermi screening length [10]. For a 2D isotropic Fermi surface, the first term is replaced by

$$-\nu \int \frac{d\theta_k}{2\pi} \left( \Delta_s^2 + \Delta_p^2 \cos^2 \theta_k \right) \ln \frac{2\Lambda}{\sqrt{\Delta_s^2 + \Delta_p^2}} \cos^2 \theta_k$$  \hspace{1cm} (S27)

and $\frac{d\theta_k}{2\pi} \rightarrow \frac{\sin \theta_k d\theta_k d\phi}{4\pi}$ for 3D. In 2D, as long as $g_p < 2g_s$, the $s$-wave phase at $\Delta = 2\Lambda e^{-\frac{1}{g_p}}$ is the ground state with energy $-\nu \Delta^2/2$ while the $p$-wave phase at $\Delta_{p0} = 4\Lambda e^{-1/\sqrt{g_p}}$ is a saddle point that has energy $-\nu \Delta_{p0}^2/4$.

B. Kinetic Energy

The action for order parameter fluctuations is obtained by expanding

$$S = \text{Tr} \ln \left[ \partial_t \mathbf{1} - \epsilon_{\mu} \sigma_3 - \Delta_s \cdot \sigma_1 - \Delta_p \cdot \sigma_2 \right] + \frac{(\delta \Delta_s)^2}{g_s} + \frac{\sum_k (\delta \Delta_p)^2}{g_p}.$$  \hspace{1cm} (S28)

around the mean field minimum to second order in $\Delta_s, \Delta_p$. We assume spatially uniform, time-dependent order fluctuations $\Delta(i\Omega_n) = (\Delta_s + \delta \Delta_s(i\Omega_n)) \sigma_1 + (\Delta_p + \delta \Delta_p(i\Omega_n)) \sigma_2$ and find

$$S_2 = \frac{1}{2} \delta \Delta \frac{\delta \Delta_s}{g_s} + \frac{\delta \Delta_p}{g_p}.$$  \hspace{1cm} (S29)

(Here for convenience we include the $f_k$ in the definition of $\Delta_p$ and its fluctuation). Evaluating the frequency integral at $T = 0$, taking the trace explicitly, rearranging and keeping only the terms with $\Omega$ dependence gives

$$S_2(\Omega) - S_2(\Omega = 0) = -\sum_k \frac{\Omega_k^2}{4} \left( \frac{(\delta \Delta_s)^2}{g_s} + (\delta \Delta_p)^2 \right) + \frac{(\delta \Delta_s \Delta_p + \delta \Delta_p \Delta_p)^2}{g_p}.$$  \hspace{1cm} (S30)

Writing $\Sigma_k = N_0 \int d\epsilon_k d\Omega_k$ with $\Omega_k$ the angular coordinates on the contours of constant energy, one obtains

$$S_2(\Omega) - S_2(\Omega = 0) = \nu \int d\Omega_k \int d\epsilon_k \left( \frac{\Omega_k^2}{4} \left( \frac{(\delta \Delta_s)^2}{g_s} + (\delta \Delta_p)^2 \right) + \frac{(\delta \Delta_s \Delta_p + \delta \Delta_p \Delta_p)^2}{g_p} \right) \hspace{1cm} (S31)$$

Defining $\epsilon = \Delta \tan \psi$ we find for the energy integral

$$\frac{1}{2} \int d\Psi \frac{\cos \Psi}{1 + \frac{\Omega^2}{4\Delta^2} \cos^2 \psi} = \int_0^1 d\sin \psi \frac{1 + \frac{\Omega^2}{4\Delta^2} \sin^2 \psi}{1 + \frac{\Omega^2}{4\Delta^2} \sin^2 \psi} = \frac{1}{2} \frac{2\Delta}{\Omega} \ln \frac{\sqrt{1 + \frac{\Omega^2}{4\Delta^2}} + \frac{\Omega}{\Delta}}{\sqrt{1 + \frac{\Omega^2}{4\Delta^2}} - \frac{\Omega}{\Delta}}.$$  \hspace{1cm} (S32)

In the adiabatic limit (lowest order in $\omega$ expansion), the kinetic energy is thus

$$K = \nu \int d\Omega_k \frac{1}{12 \Delta^2} \left( 3 \Delta^2 \left( \delta \partial_t \Delta_s \right)^2 \right) - 2 \left( \Delta_s \delta \partial_t \Delta_s + \Delta_p \delta \partial_t \Delta_p \right)^2.$$  \hspace{1cm} (S33)

In 1D, writing $\Delta_s + i \Delta_p = R e^{i\theta}$, the kinetic term becomes

$$K = \nu \frac{1}{12 R^2} \left( (\partial_t R)^2 + 2R^2 (\delta \partial_t \theta)^2 \right).$$  \hspace{1cm} (S34)

If $\Delta_s$ is very small and the system has a closed Fermi surface in $d = 2$ or $d = 3$ then the adiabatic expansion breaks down in the regions where the gap vanishes. In this case the operator $K$ becomes nonlocal in time, and the physics is most efficiently treated directly from the action Eq. (S4).
C. Dissipative terms

At zero temperature, the correlation function reads

\[
\chi_{s,s'}(0, q) = \frac{1}{2} \sum_k \frac{1}{\omega^2 - (E + E')^2} \left\{ (E + E') \text{Tr} \left[ \sigma_i \sigma_j - \frac{H_k \sigma_i H_k \sigma_j}{EE'} \right] + \omega \text{Tr} \left[ \sigma_i H_k \sigma_j \frac{1}{E'} - \frac{H_k \sigma_i \sigma_j}{E} \right] \right\}
\]

where \( H_k = \xi_k \sigma_3 + \Delta_1 \sigma_1 + \Delta_p f_k \sigma_2 \). To repeat the previous section, we drive the kinetic terms by expanding the order parameter correlation functions in frequency:

\[
S = \sum_\omega \left( \Delta_s(-\omega) \Delta_p(-\omega) \left[ \frac{1}{\bar{g}_s} + \chi_{s,s'}(0, 0) \right] \chi_{s,s'}(0, 0) \right) \left( \Delta_s(\omega) \Delta_p(\omega) \right).
\]

The \( \Delta_s^2 \) term is

\[
\chi_{\Delta_s,\Delta_s}(0, 0) = 4 \sum_k \frac{\xi_k^2 + \Delta_p^2 f^2(k)}{(\omega^2 - 4E_k^2)E_k} = -\frac{1}{\bar{g}_s} \int \frac{d\theta}{\Omega_D} (\omega^2 - 4\Delta_s^2) F(\Delta_s, \omega) = \chi_{\Delta_s,\Delta_s}(0, 0) - \omega^2 \varphi \left\{ \frac{1 - \Delta_s^2}{\bar{g}_s^2} \sum_{D=1}^2 \varphi \right\} D = 1 + O(\omega^4)
\]

where \( \Delta_\theta^2 = \Delta_s^2 + \Delta_p^2 f^2(k_\theta), \Delta_s^2 = \Delta_s^2 + \Delta_p^2 f^2(\theta) \).

The \( \Delta_p^2 \) term is

\[
\chi_{\Delta_p,\Delta_p}(0, 0) = 4 \sum_k \frac{f^2(k) \xi_k^2 + \Delta_s^2}{E(\omega^2 - 4E_k^2)} = -\frac{1}{\bar{g}_s} \int \frac{d\theta}{\Omega_D} (\omega^2 - 4\Delta_p^2 f^2(\theta)) F(\Delta_p, \omega)
\]

The above expansions in \( \omega \) fail as \( \omega \sim \Delta_s \), the minimal gap around the fermi surface, especially when \( \Delta_s = 0 \) such that there are nodes at \( k = (0, \pm k_F) \) to 2D. We next evaluate the kernels in the pure \( p \)-wave case \( \Delta_s = 0 \) to gain a rough idea of the crossover of dynamical behavior. The dissipative part of \( \Delta_s \) kernel is

\[
\text{Im} \left[ \chi_{\Delta_s,\Delta_s}(0, 0) \right] = \text{Im} \left[ 4 \sum_k \frac{E}{(\omega + i\eta)^2 - 4E^2} \right] = -\pi \sum_k \left( \delta(\omega + 2E) - \delta(\omega - 2E) \right) \frac{\omega \varphi_{\Delta_p, \varphi}}{2E} \frac{\omega^3}{\bar{g}_s^2} \Delta_p^2
\]

where we have made use of the quasi-particle density of states due to the nodes: \( g(E) = \frac{1}{2\bar{g}_s} k_F E (\nu_F \Delta_p) \). The linear in frequency dissipation continues with a cutoff of about \( \Delta_p \) beyond which it scales as a constant. Kramers-Kronig relation implies that

\[
\chi_{\Delta_s,\Delta_s}(0, 0) \approx \frac{1}{2} \text{v} \left( \frac{\omega}{\bar{g}_s} \right)
\]

The dissipative part of \( \Delta_p \) kernel is

\[
\text{Im} \left[ \chi_{\Delta_p,\Delta_p}(0, 0) \right] = \text{Im} \left[ 4 \sum_k \frac{f^2(k) \xi_k^2}{E^2} \frac{E}{(\omega + i\eta)^2 - 4E^2} \right] = -\pi \sum_k \frac{f^2(k) \xi_k^2}{E^2} \left( \delta(\omega + 2E) - \delta(\omega - 2E) \right) \frac{\omega \varphi_{\Delta_p, \varphi}}{2E} \frac{\omega^3}{\bar{g}_s^2} \Delta_p^2
\]

and the cubic behavior has the cutoff \( \Delta_p \). This together with the \( \Delta_s = 0 \) limit of Eq. (S38) gives

\[
\chi_{\Delta_p,\Delta_p}(0, 0) \approx \frac{\pi}{2\text{v}} \left( \frac{\omega^3}{\Delta_p^2} \right)
\]
I. In time domain

With the adiabatic approximation so at time \( t_0 \) we write \( \Delta = \Delta^0(t_0) + \delta \Delta(t_0 + t) \), the action reads

\[
S = \int dtV[\Delta] + \frac{1}{2} \int dt dt' \frac{\partial \delta \Delta}{\partial t} M^R(t - t') \frac{\partial \delta \Delta}{\partial t'}
\]

(S44)

so the instantaneous (force) term in the Euler-Lagrange equations comes from the equal time correlator (potential) and the dynamics comes from expanding in derivatives, in other words

\[
\frac{\delta V}{\delta \Delta} = \partial_t \int^t dt' M^R(t - t') \partial_t \delta \Delta(t')
\]

(S45)

Noting that \( M^R(0) = 0 \) we have

\[
\frac{\delta V}{\delta \Delta} = \int^t dt' \partial_t M^R(t - t') \partial_t \delta \Delta(t')
\]

(S46)

The adiabatic approximation is reasonable if the change in \( \Delta \) over a time corresponding to the range of \( M \) is small \( (\partial_t \Delta / \Delta) << 1 \), so we can evaluate \( M \) at fixed \( \Delta \). If we have a fully gapped configuration (open Fermi surface or \( \Delta_s \) not small), \( M \) decays on times larger than \( |\Delta|^{-1} = 1/\sqrt{\Delta_s^2 + \Delta_p^2} \) we can shift the derivative to the \( t' \) and integrate by parts to get

\[
\frac{\delta V}{\delta \Delta} = \int^t dt' M^R(t - t') \partial^2_{t'} \delta \Delta(t') \rightarrow M \partial^2_{t'} \Delta
\]

(S47)

with \( M = \int^t dt' M^R(t - t') \). However, for closed Fermi surfaces, the vanishing of \( \Delta_p(k) \) at some Fermi surface points means that when \( \Delta_s \) is small \( M \) has a part that decays slowly, actually on the time-scale of \( 1/\Delta_s \) and a more careful analysis is needed. In the isotropic 2D case, we have

\[
T = \frac{1}{2} \int d_1 d_2 \left( \partial_t \delta \Delta_s(t_1) \partial_t \delta \Delta_p(t_1) \right) M_R(t_1 - t_2) \left( \partial_t \delta \Delta_s(t_2) \partial_t \delta \Delta_p(t_2) \right)
\]

(S48)

and the (retarded) correlator is given by

\[
M_R(t) = \Theta(t) \sum_k \sin^2 E_k t \left( \begin{array}{c} \epsilon^2_k + \Delta_p^2 \\ -\Delta_s \Delta_p f_k \\ (\epsilon^2_k + \Delta_s^2) f^2(k) \end{array} \right)
\]

(S49)

Performing the integral over momentum, one obtains the low energy kernel

\[
\partial_t M^{11}_R(t) \approx \Theta(t) \frac{v}{2 \Delta_p} \int_{\Delta_s}^{\Delta_p} 2 \pi \left( 1 - \frac{\Delta_s^2}{v^2} \right) \cos 2vt + \text{high energy contribution}
\]

(S50)

\[
= \Theta(t) \frac{v}{2 \Delta_p} \left[ \sin 2\Delta_p t - \sin 2\Delta_s t \right] + \Delta_s \left[ 2\Delta_p t - \cos 2\Delta_s t \right] \cos 2vt + \frac{v}{6(\Delta_s^2 + \Delta_p^2)} \partial_t \delta(t)
\]

\approx \Theta(t) \frac{v}{2 \Delta_p} \sin 2\Delta_p t - \sin 2\Delta_s t + \frac{v}{6(\Delta_s^2 + \Delta_p^2)} \partial_t \delta(t),
\]

\[
\partial_t M^{22}_R(t) \approx \frac{v}{6(\Delta_s^2 + \Delta_p^2)} \partial_t \delta(t)
\]

The off diagonal terms don’t affect the qualitative dynamics which we neglect. At small \( \Delta_s \) we can neglect the second term of \( \partial_t M^{11} \). Therefore, in 2D, a smooth crossover between non dissipative and dissipative behaviors during the swiping across \( \theta = \pi/2 \) can be described by the retarded kinetic kernel

\[
S_{\text{dis}} = \frac{1}{2} \int dt dt' \Delta_s(t) M^R(t - t') \Delta_s(t'), \quad M^R(t) \approx \frac{v}{2 \Delta_p} \int_0^t dt' \sin 2\Delta_p t' - \sin 2\Delta_s t'.
\]

(S51)

Eq. (S51) implies the equation of motion

\[
\frac{\delta V}{\delta \Delta_i} = \frac{v}{6(\Delta_s^2 + \Delta_p^2)} \partial_t^2 \Delta_i + \frac{v \delta i, s}{2 \Delta_p} \int_{-\infty}^t dt' \frac{\sin [2\Delta_p (t - t')] - \sin [2\Delta_s (t - t')]}{t - t'} \Delta_i(t')
\]

(S52)

which describes the crossover behavior when \( \Delta_s \) crosses zero during the dynamics.
D. The drive term

In the drive term $L_{\text{drive}} = -P(\theta)E - s(\Delta_i, \Delta_p)E^2 + O(E^3)$, the linear coupling of electric field to the polarization is obvious. We derive the second term in this section. The kernel of the $O(A^2)$ term is \[ K_{ij}(\omega) = \left( \frac{n}{m} + \chi_{j,i}(\omega) \right) \delta_{ij} \] (S53)

where $j$ is the current operator in Eq. (S5). Since the second term in the current in Eq. (S5) is suppressed by the factor $\Delta_p/\epsilon_F$ in the BCS limit, its contribution can be neglected. In 1D, the current correlation function is thus

\[ \chi_{j,i}(\omega) = \chi_{o_3, o_3}(\omega) = -4v_F^2 \Delta^2 F(\Delta, \omega) = -v_F^2 \sqrt{\omega} \left( 1 + \frac{2}{3} \left( \frac{\omega}{2\Delta} \right)^2 + O\left( \left( \frac{\omega}{2\Delta} \right)^4 \right) \right) \] (S54)

where $\Delta^2 = \Delta_p^2 + \Delta^2$ and $F(\omega) = \sum_k \frac{1}{2\omega} \sin^{-1} \left( \frac{\omega}{2\Delta} \right) = \frac{\sqrt{\omega}}{2\Delta} \left( 1 + \frac{\omega}{2\Delta} \right)^2 + O\left( \left( \frac{\omega}{2\Delta} \right)^4 \right)$. The constant term cancels the diamagnetic contribution $\eta/m$ and what remains in the kernel is the $O(\omega^2)$ term that corresponds to the static polarizability from ‘scattering states’ of the electron hole pair. In 2D, the current correlator up to $O(\omega^2)$ is

\[ \chi_{j,i}(\omega) = -\delta_{ij} \frac{1}{d} v_F^2 \sqrt{\omega} \int \frac{d\theta}{2\pi} \left( 1 + \frac{2}{3} \frac{\omega^2}{4(\Delta_p^2 + \Delta^2 \cos^2 \theta)} \right) = -\delta_{ij} \frac{1}{d} v_F^2 \sqrt{\omega} \left( 1 + \frac{\omega^2}{6(\Delta_p^2 + \Delta^2) + |\Delta| \sqrt{\Delta_p^2 + \Delta^2}} \right) \] \hspace{1cm} (S55)

Therefore, the $O(E^2)$ term in the action reads

\[ L_2 = \frac{1}{\omega^2} K_{ij} E_i E_j = -\frac{1}{6} v \Delta^2 \left( \frac{E}{E_0} \right)^2 \left\{ \frac{\Delta^2}{\Delta_p^2 + \Delta^2} \right\} \] (1D)

\[ \frac{\Delta^2}{\Delta_p^2 + |\Delta| \sqrt{\Delta_p^2 + \Delta^2}} \] (2D) \hspace{1cm} (S56)

where the coefficient can be interpreted as $s = \lim_{\omega \to 0} \sigma(\omega)/(2i\omega)$. The higher order terms is $E$ are in higher powers of $\left( \frac{E}{E_0} \right)^2$ $\frac{\Delta^4}{\Delta_p^2 + \Delta^2}$.

V. THE ADIABATIC TRANSPORT SCHEME

A. Description

If the sin pulse is wide enough in time, it is possible to make the dynamics perfectly adiabatic since the system simply follows the instantaneous minimum on the free energy landscape. As the field increases, the minimum shifts away from $(\Delta, 0)$ counter-clockwise while the maximum at $(0, \Delta_p, 0)$ shifts clockwise. The maximum field needed is simply that making the instantaneous minimum and maximum coincide. In 1D, this field can be computed analytically:

\[ E_m(g_p) = 2t \sqrt{1 - x^2} e^{-1/2 - t + \sqrt{t^2 + 1/4}} \] \hspace{1cm} (S57)

where $x = (-1/t + \sqrt{1/t^2 + 4})/2$ and $t = (1/(v g_p)) - 1/(v g_s)$. After reaching the maximum value (a little higher than that), the field starts to decrease, shifting back the two extrema. The order parameter is moved to the immediate left of the maximum, which gradually shifts back to $(0, \Delta_p, 0)$ as the field decreases to zero. The second half of the sin pulse would therefore transport the order parameter to the minimum at $(\Delta, 0)$, completing a half cycle. However, if the decreasing field phase of the pulse is too slow, unstable fluctuations of order parameter tend to grow exponentially[8] and get comparable to its mean field value within the ‘spinodal time’ $\left\{ \frac{1}{|\Delta|} \ln \frac{1}{G} \right\}$ where $G \sim \frac{|\Delta|}{\epsilon_F} \ll 1$ is the Ginzburg parameter of the landau theory. Therefore, the time scale of the pulse has to be smaller than the spinodal time.

If $g_p$ is too small, the requires maximum field is so large that the $O(E^2)$ term $L_2 = -\frac{1}{6} v E^2 \frac{\Delta^4}{\Delta_p^2 + \Delta^2}$ would pull the order parameter to the origin and destroy the above adiabatic trajectory. This imposes a lower bound for the $p$-wave pairing strength $g_{pc} = g_s/(1 + \sqrt{3}/8v g_s)$. For the adiabatic transport scheme to work, $g_p$ has to be larger than $g_{pc}$. These conclusions apply qualitatively to higher dimensions.

If the adiabatic scheme is realized, experimental measurement of the threshold electric field gives the estimation of $g_p$ through Eq. (S57). In the fast scheme described in the main text, if the full frequency spectrum of the current can be measured, it is possible to reconstruct the angular dynamics through, e.g., Eq. (6) for 2D.
The parameters are curves are other. If they successfully meet each other at certain to be closed, i.e., for the minima in \( \theta = 0, \pi/2 \), the pure \( p \)-wave order line, one obtains the minimal \( E_c' \) for \( g_p \to 0 \), as shown in Fig. S3(b).

The \( O(E^2) \) term in Eq. (S58) lowers the energy dramatically close to \( r = 0 \), and therefore tends to pull the system to the zero order state. As a result, the free energy has a maximum in the \( r \) direction, followed by the minimum as \( r \) increases. Thus the \( r_1 \) curve has two branches: the left one has \( \partial_r^2 f < 0 \) (maxima) while the right one has \( \partial_r^2 f > 0 \) (minima).

B. Derivation

In 1D, incorporating the effect of a static electric field up to \( O(E^2) \), the free energy is

\[
f(\Delta_s, \Delta_p) = v \left( -r^2 \ln \frac{2\Lambda}{r} + \frac{1}{v g_s} r^2 + \left( \frac{1}{v g_p} - \frac{1}{v g_s} \right) r^2 \sin^2 \theta - \frac{1}{2} \Delta_s^2 E' \theta - \frac{1}{6} E^2 \Delta_s^4 \right) \tag{S58}
\]

where the ‘polar’ coordinate is defined as \( (\Delta_s, \Delta_p) = r(\cos \theta, \sin \theta) \), the dimensionless electric field is \( E' = E/E_0 \), \( E_0 = \Delta_0^2/v_F \) and \( \Delta_0 = \Delta \). We look for saddle points on the free energy landscape within the domain \( \theta \in [0, \pi/2] \). There are two curves defined by \( \partial_r f = 0 \) and \( \partial_\theta f = 0 \) respectively, whose solutions read

\[
r_1(\theta) = \Lambda e^{-\frac{1}{2} \sin^2 \theta - \frac{1}{2} t}, \quad r_2(\theta) = \sqrt{\frac{1}{2 t \sin 2\theta} \Delta_0^2 E'} \tag{S59}
\]

where \( t = \frac{1}{v g_p} - \frac{1}{v g_s} \), as shown in Fig. S3(a). Note that we temporarily neglected the \( O(E^2) \) terms in the free energy. The intersections of the two curves are the saddle points. At zero field, the two saddle point are just the two minima at \( (\theta, r) = (0, \Delta_0), (\pi/2, \Delta_p) \). For weak field, the two saddle points shift towards each other in angular direction. As the field further increases to the critical value \( E_m \), the two saddle point meet which means the two lines are tangent to each other: \( r_1 = r_2, \partial_\theta r_1 = \partial_\theta r_2 \) is satisfied at the intersection. This condition gives the angle at intersection as \( \cos (2\theta_m) = \frac{1}{2} \left( -\frac{1}{4} + \sqrt{\frac{1}{4} + 4} \right) \) and critical field

\[
E_m = 2 t \sin (2\theta_m) e^{-1/2 - t + \sqrt{t^2 + 4}} \tag{S60}
\]

It increases from zero as \( g_p \) decreases from \( g_s \), and diverges as \( 1/\sqrt{g_p} \) as \( g_p \to 0 \), as shown in Fig. S3(b).

The \( O(E^2) \) term in Fig. S3(a) but with the \( O(E^2) \) terms taken into account. The \( r_2 \) curves are not affected by the \( O(E^2) \) terms while \( r_1 \) curves are deformed. Each \( r_1 \) curve can be separated into two branches: the left branch has \( \partial_r^2 f < 0 \) (maxima) while the right branch has \( \partial_r^2 f > 0 \) (minima).
can happen that \( A \) annihilates with another intersection on the left branch of \( r_1 \). In this situation, the order parameter will be transported to zero order instead of to the \( p \)-wave state. The critical \( p \)-wave pairing strength can be estimated roughly in this way: the summit of \( r_1 \) collides with the left most point of \( r_2 \) as field increases. This condition leads to the equality \( r^2 = \sqrt{2/3} \Delta_0^2 E_l^2 = \frac{1}{2\pi} \Delta_0^2 E_l^2 \) which renders \( g_{pc} = g_s/(1 + \sqrt{3} 8 \nu g_s) \).

### VI. EXACT MEAN FIELD DYNAMICS

The mean field dynamics is described by the rotation of the Anderson pseudo spins in the time dependent self consistent mean field:

\[
\mathbf{s}_k = (\mathbf{b}_k - \gamma \mathbf{b}_k \times \mathbf{s}_k) \times \mathbf{s}_k, \quad \mathbf{b}_k \equiv \left( \frac{g_s}{2} \sum_{k'} s_{1k'} + \frac{g_p}{2} \sum_{k'} f(k') s_{1k'}, \frac{g_s}{2} \sum_{k'} s_{2k'} + \frac{g_p}{2} \sum_{k'} f(k') s_{2k'}, \xi(k) \right)
\]

where the EM vector potential \( A(t) \) enters by \( k \rightarrow k - A(t) \) and we use a phenomenological damping \( \gamma \) to account for the effect of energy loss due to e.g., the phonon bath. The current \( j = \sum_k \{ v_k s_3 + \Delta_p \delta_f f_k s_2 \} \) is evaluated and integrated over time during the dynamics to obtain the pumped charge. Some numerical solutions to Eq. (S62) are shown in Figs. S4 and S5.

To see why the order parameter dynamics is restricted within the \( s + ip \) plane in the BCS weak coupling limit, we prove that the pseudo-spin \( s' \) at \( k_F + \delta k \) and the other spin \( s' \) at \(-k_F + \delta k \) are always related to each other by the mirror operation \( M \) with respect to ‘\( y - z \)’ plane: \( s_{1p}^*, s_{2p}^*, s_{3p}^* = s_{1p}^*, s_{2p}^*, s_{3p}^* \). This is obviously true for the initial ground state. Since the coupling to vector potential \( A(t) \) through \( \Delta_p f(k - A(t)) \) is suppressed by the small number \( \Delta_p/\varepsilon_F \), it can be neglected in the weak coupling limit. As a result, the ‘magnetic field’ \( \mathbf{b}_{\parallel //} \) on the two pseudo-spins are also mirror image of each other under \( M \), so are \( \mathbf{b} \times \mathbf{s} \). Therefore, this relation is sustainable during the dynamics, which guarantees that the order parameter lies on the \( s + ip \) plane through the gap equation (S62).

#### A. ‘Super-current’ in 1D systems

The solution is trivial in the degenerate case \( g_s = g_p \) in the BCS limit where the effect of the pairing function \( f_k \) is captured by \( f_k = \pm 1 \) on the right/left fermi point. We start from a ground state \( \langle \Delta_p, \Delta_p \rangle = (\Delta, 0) \) where all spins are pointing in \( xz \) plane: \( \mathbf{s}_k = (\Delta, 0, \xi_k)/E_k \). The electric field pulse at \( t = 0 \) is applied through \( A = A_0 \Theta(t) \). The leading driving term due to electric field is \( \mathbf{b}_k = (0, 0, v_k A) \) where \( v_k = \pm v_F \) around the right/left fermi point. The diamagnetic term \( \sim A^2 \) is subleading in driving the spinor dynamics but contributes a magnetic current which we will discuss in the end. After the kick, the spinors start to rotate around \( z \) with angular frequency \( \omega = 2v_F A_0 \). The mean field rotates at the same speed: \( \langle \Delta_p, \Delta_p \rangle = \Delta(\cos(\omega t), \sin(\omega t)) \) such that \( \mathbf{b}_k = (0, 0, v_A A) \) is always parallel to each spinor, not affecting the spin rotation. Thus the solution is that each spin synchronize and keeps rotating around \( z \) with angular frequency \( v_F A_0 \). Now we evaluate the current \( j = j_p + j_D \). The paramagnetic current \( j_p = \sum_k (v_k \delta_3) \) vanishes in this state. The diamagnetic current is \( j_D = \frac{2}{\pi} v_F A_0 = 2f = 2\omega/(2\pi) \). Therefore, the system behaves like a ‘superconductor’ with the superfluid density \( n \).
FIG. S5. Order parameter dynamics of a 2D excitonic insulator subject to a pump pulse described by the vector potential $A(t) = -E_{\text{max}} w \left( \tanh \left( \frac{t - t_0}{\Delta/2} \right) + 1 \right)$. Left panel is the trajectory on the free energy landscape plotted on the $s + ip$ plane for $E_{\text{max}} = 0.686 E_0$. Middle panel is the polarization as a function of time. Right panel is the pumped charge as a function of $E_{\text{max}}$. The parameters are $w = 1/(2\Delta)$, $g_s \nu = 0.3$, $g_p \nu = 0.5$, $\Delta = 2 A e^{-1/(g_s \nu)} = 0.071 \Lambda$, $\gamma = 0.07 \Delta$. The grid in time direction is $10^4$.

B. Dynamics of the node: Landau-Zener formula

The node contribution to the polarization is captured by the Dirac Hamiltonian with time dependent gap:

$$H_k(t) = \frac{\Delta_p}{k_F v_F} v_F k_x \sigma_2 + v_F k_y \sigma_3 + \Delta_s(t) \sigma_1$$

(S63)

which is an approximation to Eq. (2) of the main text around $k_0 = (0, k_F)$, and is valid for $k_x, k_y \ll k_F$. Taking into account the higher order term $k^2 \sigma_3$ in the Hamiltonian, the current in $x$ direction is

$$j_x = \frac{v_F}{k_F} k_x \sigma_3 + \frac{\Delta_p}{k_F} \sigma_2.$$  

In the second quantized language, each momentum $k$ labels two single particle states, while mean field dynamics here implies that the total occupation number at $k$ is always one (the number operator $\sigma_0$ is conserved), restricting to a two dimensional state space which can be mapped to an Anderson pseudo spin.

In the BCS limit we are concerned here, during the dynamics, the spinor at $(k_x, k_y)$ is always the mirror image of that at $(-k_x, -k_y)$ with respect to the $-2 - 3$ plane (note that the spinors are axial vectors, thus the mirror operation $\hat{M} = \sigma_1$ transforms the spins as $\sigma_3 \rightarrow -\sigma_3$). Therefore, the $\sigma_2$ contributions to the current will always sum to zero, and it is enough to consider $j_x = \frac{v_F}{k_F} k_x \sigma_3$.

Define the energy variables $k'_x = \frac{\Delta_p}{k_F} k_x$, $k'_y = v_F k_y$, the Hamiltonian becomes

$$H_k(t) = k'_x \sigma_2 + k'_y \sigma_3 + \Delta_s(t) \sigma_1$$

(S64)

and the current reads $j_x = \frac{v_F}{\Delta_p} k_x \sigma_3$. We now use Eq. (S64) to study the spinor dynamics and the current generated.

As the order parameter passes the $(0, \Delta_p)$ point with nearly constant velocity, the nodal gap $\Delta_s$ changes sign. For the spinor at certain $k$, as $\Delta_s$ swaps from the positive value $\Delta$ at time $-t_0$ to negative value $-\Delta$ at time $t_0$, the energy splitting starts from $\sqrt{\delta^2_k + \Delta^2}$, passes through the minimal splitting $|\delta_k| = |k'|$, and ends up with $\sqrt{\delta^2_k + \Delta^2}$. If the initial state is the low energy state, the probability of finally tunneling into the high energy state is given by the Landau-Zener formula [37]:

$$P_k = e^{-2\pi \frac{\delta^2_k}{(\Delta/\Delta_p)}}$$

(S65)

which is exact if $\Delta \gg \delta_k$. Therefore, the tunneling probability is unity at the node and decays to zero away from the node within a range of $\sim \sqrt{\delta_k \Delta_s}$. Considering there are two nodes, the total number of quasiparticles excited is thus

$$N = 2 \sum_k P_k = \frac{2}{4\pi^2} \frac{k_F}{v_F \Delta_p} \int dk'_x dk'_y e^{-\pi \frac{\delta^2_k}{(\Delta/\Delta_p)}} = \frac{2}{4\pi^2} \frac{k_F}{v_F \Delta_p} \frac{\pi}{\pi} = \frac{1}{2\pi^2} \frac{k_F}{v_F} \frac{|\delta_k \Delta_s|}{\Delta_p} = \frac{k_F^2}{2\pi^2} \frac{1}{k_F v_F} \frac{|\delta_k \Delta_s|}{\Delta_p}.$$  

(S66)

Since we have assumed Dirac dispersion in the integral, Eq. (S66) is accurate if $\sqrt{|\delta_k \Delta_s|} \ll \Delta_p$.
1. The pumped charge around the node

We now compute the pumped charge, which reads

\[ P = 2 \sum_k \int dt \langle j_x(k) \rangle_t = \frac{2}{4\pi^2} \frac{k_F}{\Delta_p} \int dk'_x dk'_y dt k'_x (\sigma_3)_{k,t} = P_0 + P_{dis}. \]  
\[ (S67) \]

The integral is completely determined by the dynamics governed by Eq. (S64), the evaluation of which requires more detailed analysis of the time evolution of each spinor. Before that, we can guess the result simply from dimensional analysis. The nonadiabatic correction \( P_{dis} \) comes from spinors with \( \delta_k \ll \Delta \), and the contribution arises during the anti crossing regime when \( \Delta_\ell(t) \) is not much larger than \( \delta_k \). Therefore, neither the momentum cutoff nor the maximum value of \( \Delta_\ell \) should enter the result. The only remaining energy scale in Eq. (S64) is provide by \( \partial_t \Delta_\ell \) which has the unit of energy\(^2\). Since the integral in \( P_{dis} \) has the unit of energy\(^2\), one obtains \( P_{dis} = k \frac{k_F}{2\pi^2} \frac{|\partial_\ell \Delta_t|}{\Delta_p} \) where \( k \) is a universal \( O(1) \) constant.

Now we compute \( P_{dis} \) exactly. It is more convenient to perform a permutation of the Pauli matrices: \( (\sigma_2, \sigma_3, \sigma_1) \rightarrow (\sigma_1, \sigma_2, \sigma_3) \) such that the node Hamiltonian reads

\[ H_2(t) = k'_x \sigma_1 + k'_y \sigma_2 + \Delta_\ell(t) \sigma_3 \]  
\[ (S68) \]

and the current becomes \( j_x = \frac{\mu_F}{\Delta_p} k'_x \sigma_2 \). The dynamics of the pseudo spin at \( k' \) is a Landau-Zener problem [37]. At time \( -t_0 \), we have \( \Delta_\ell = \Delta \gg k' \) and the spin is in the ground state: \( \psi = (0,1)^T \). The time evolution can be written as \( \psi = (A(t)e^{i\phi(t)}, B(t)e^{i\phi(t)})^T \) where \( \phi(t) = \int dt \Delta_\ell(t) \). The Schrodinger equation for the amplitudes reads

\[ \partial_t A = -i(k'_x - i k'_y) B e^{i\phi}, \quad \partial_t B = -i(k'_x + i k'_y) A e^{-i\phi} \]  
\[ (S69) \]

which leads to

\[ \partial_t^2 A - i 2\Delta_\ell(t) \partial_t A + k'^2 A = 0, \quad \partial_t^2 B + i 2\Delta_\ell(t) \partial_t B + k'^2 B = 0. \]  
\[ (S70) \]

The current involves the expectation value of \( \sigma_2 \):

\[ \langle \sigma_2 \rangle = i \left[ B^* A e^{i\theta} - c.c. \right] = - \left( \frac{1}{k'_x - i k'_y} B \partial_t B^* + c.c. \right) \]  
\[ (S71) \]

whose time integral gives the charge:

\[ \int dt \langle \sigma_2 \rangle = - \text{Re} \left[ \frac{1}{k'_x - i k'_y} \right] (|B(t_0)|^2 - |B(-t_0)|^2) - i \text{Im} \left[ \frac{1}{k'_x - i k'_y} \right] \int dt |B(t)|^2 \partial_t \ln \left( \frac{B^*}{B} \right). \]  
\[ (S72) \]

It can be seen from Eq. (S70) that the time dependent wave function is the same between the spins at \( (k'_x, k'_y) \) and \( (-k'_x, k'_y) \). Since the current is \( j_x = \frac{\mu_F}{\Delta_p} k'_x \sigma_2 \), the second term in Eq. (S72) will be canceled out by the two spins. The first term just needs the initial and final state information:

\[ \int dt \langle \sigma_2 \rangle = - \text{Re} \left[ \frac{1}{k'_x - i k'_y} \right] (|B(t_0)|^2 - |B(-t_0)|^2) = \text{Re} \left[ \frac{1}{k'_x - i k'_y} \right] (1 - P_k). \]  
\[ (S73) \]

which is provide by the Landau-Zener formula. Summing over all the spins, the pumped charge reads

\[ P = \frac{2}{4\pi^2} \frac{k_F}{\Delta_p} \int dk'_x dk'_y \text{Re} \left[ \frac{1}{k'_x - i k'_y} \right] (1 - P_k) = P_0 + P_{dis} \]  
\[ (S74) \]

where the nonadiabatic correction is identified as

\[ P_{dis} = \frac{2}{4\pi^2} \frac{k_F}{\Delta_p} \int dk'_x dk'_y \text{Re} \left[ \frac{1}{k'_x - i k'_y} \right] P_k \]
\[ = - \frac{2}{4\pi^2} \frac{k_F}{\Delta_p} \int dk'_x dk'_y \frac{k'^2_F}{k'^2} e^{-\pi \frac{k'^2_F}{\Delta_p^2}} = - \frac{2}{4\pi^2} \frac{k_F}{\Delta_p} \int dk' \partial \theta \cos^2 \theta e^{-\pi \frac{k'^2_F}{\Delta_p^2}} = - \frac{k_F}{8\pi^2} \frac{|\partial_\ell \Delta_\ell|}{\Delta_p^2}. \]  
\[ (S75) \]

Due to the negative relative sign of the non-adiabatic correction to the adiabatic one, we conclude that

\[ P_{dis} = - \frac{k_F}{8\pi^2} \frac{|\partial_\ell \Delta_\ell|}{\Delta_p^2} = - P_0 \frac{1}{8\pi^2} \frac{|\partial_\ell \Delta_\ell|}{\Delta_p^2}. \]  
\[ (S76) \]

Therefore, Eq. (S76) gives the non adiabatic correction during each half cycle of order parameter rotation, which is valid if \( \sqrt{|\partial_\ell \Delta_\ell|} \ll \Delta_p \). This formula is nonperturbative in the swiping speed in the sense that, it can not be obtained by integrating over instantaneous linear or nonlinear current response functions perturbatively over the time evolution.