Analytical Mechanics With Quasi-Velocities

Farhad Aghili

Abstract

This paper presents a formulation of Lagrangian dynamics of constrained mechanical systems in terms of reduced quasi-velocities and quasi-forces that can be used for simulation, analysis, and control purposes. In this formulation, Cholesky decomposition of the mass matrix in conjunction with adequate orthogonal matrices are used to define reduced-quasi velocities, input quasi-forces, and constraint quasi-forces which possess natural metric. The new state and input variables always have homogeneous units despite the generalized coordinates may involve in both translational and rotational components and the constraint wrench may involve in both force and moment components. Therefore, this formulation is inherently invariant with respect to changes in dimensional units without requiring weighting matrices. Moreover, in this formulation the equations of motion are completely decoupled from those of constrained force. This allows the possibility of a simple force control action that is totally independent of the motion control action facilitating a hybrid force/motion control. The properties of the new dynamics formulation are investigated and subsequently, force/motion tracking control and regulation of constrained multibody systems based on quasi-velocities and quasi-forces are presented.

1 Introduction

A unifying idea for most modelling techniques used for multibody system (MBS) dynamics is to describe the equations of motion in terms of generalized coordinates and generalized velocities. In classical mechanics of constrained systems, a generalized velocity is taken to be an element of tangential space of configuration manifold, and a generalized force is taken to be the cotangent space. However, neither space possesses a natural metric as the generalized coordinates or the constrains may have a combination of rotational and translational components. As a result, the corresponding dynamic formulation is not invariant and a solution depends on measure units or a weighting matrix selected [1-9].

Transformation of the Lagrange dynamics into quasi-Lagrange dynamics, and with feedback force/position control, have been established in the literature [10,12]. Mathematical models for constrained robot dynamics, incorporating the effects of constraint force required to maintain satisfaction of the constraints, and tracking using feedback control were presented in [10]. The problem of the control of a class of mechanical systems with a finite number of degrees-of-freedom, subject to unilateral constraints on the position is presented in [11]. Various switching control strategies are analysed in this work based on a nonsmooth dynamics formulation. The

*email: faghili@encs.concordia.ca
use of kinetic quasi-velocities was also developed for a particular case in [13,14]. Alternatively, the dynamics of a multibody system can be formulated in terms of the vector of quasi-velocities, i.e., a vector whose Euclidean norm is proportional to the square root of the system’s kinetic energy. It is known that this formulation can lead to simplification of the equations of unconstrained MBSs [9,15-33,33]. In short, the square-root factorization of the mass matrix is used as a transformation to obtain the quasi-velocities, which are a linear combination of the velocity and the generalized coordinates [22,25,33,34].

In the literature, the concept of quasi-velocities has been used for dynamics modelling of unconstrained MBSs. The differential variational principle of Jourdain was extended in [19] to cover the dynamics of impulsive motion formulated in terms of quasi-velocities. It was shown by Kodistchek [15] that if the square-root factorization of the inertia matrix is integrable, then the dynamics can be significantly simplified. In such a case, transforming the generalized coordinates to quasi-coordinates by making use of the integrable factorization modifies the dynamics to the system of a double integrator. It was later realized by Gu et al. [16] that such a transformation is a canonical transformation because it satisfies Hamilton’s equations. Rather than deriving the mass matrix of MBS first and then obtaining its factorization, Rodriguez et al. [18] derived the closed-form expressions of the mass matrix factorization of a MBS and its inverse directly from the link geometric and inertial parameters. This eliminates the need for the matrix inversion required to compute the forward dynamics. The interesting question of when the factorization of the inertia matrix is integrable, i.e., the factorization being the Jacobian of some quasi-coordinates, was addressed independently in [17] and [32]. It was shown that Riemannian manifold defined by the inertia matrix should be locally flat. The advantages of using the notion quasi-velocities for control of unconstrained manipulators have been recognized by many researchers and various setpoint PD controllers based on the quasi-velocities feedbacks have been proposed [20,24,35,36]. A closed inverse dynamic formulation by the Lagrangian approach in terms of quasi-coordinates for the general Stewart platform manipulator is presented in [37]. The quasi-Lagrange equations are derived with and without friction in [12] based on quasi-velocities computed with the kinetic metric of a Lagrangian system. The quasi-Lagrange dynamics involves the mass matrix inversion that allows for a clear splitting between normal and tangential dynamics. Generalization of Lagrangian dynamics equations by taking into account the unilateral and bilateral contacts as well as friction can be also found in [14]. In spite of different quasi-Lagrange dynamics formulations proposed in the literature, preservation of homogeneous units yet needs to be rendered in these formulations.

A problem that often arises in motion/force control of MBSs with minimum solution to joint rate or force, is that generalized coordinate may have a combination of rotational and translational components that can be even compounded by having combination of rotational and translational constraints [38]. This may lead to inconsistent results unless adequate weighting matrices are used [2,6,38-40]. For example, the minimum joint rate rates or minimum norm force are not meaningful quantities if the MBS has both revolute and prismatic joints [38]. The contribution of this paper is to extend the concept of square-root factorization of inertia matrix to define homogeneous vectors of quasi-velocities and quasi-forces for dynamics formulation of constrained MBS that can be used for simulation, analysis, and control purposes. In this paper, we introduce new state and input variables comprising of reduced quasi-velocities, input quasi-forces, and constraint quasi-forces by making use of Cholesky decomposition and adequate orthogonal matrices in order to derive Lagrangian dynamics of constrained mechanical
systems. The advantages of the square–root factorization based formulation of the constrained Lagrangian dynamics is that every vectors of quasi-velocities, input quasi-forces, or constraint quasi-forces all have the same physical units. Therefore, unlike other approaches \cite{6,7,11,42}, this formulation does not require any weighting matrix when the generalized coordinates or the constraints have both translational and rotational components. Furthermore, the equations of motions and the equation of constraints are decoupled in such a way that separate control inputs are associated to each set of equations, which facilitates motion/force control of constrained systems such as robotic manipulators. This paper is organized as follows: Section 2 presents the derivation of constrained Lagrangian dynamics formulation based on the notion of reduced quasi-velocities and decoupling quasi-forces. Properties of the dynamic formulation that could be useful for control purposes are presented in Sections 2.1. Finally, Section 3 is devoted to force/motion control based on reduced quasi-velocities and quasi-forces.

2 Quasi-Variables Transformation

The kinetic energy of a MBS has the following quadratic form:

\[
K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q},
\]

where vectors \( q \in \mathbb{R}^n \) and \( \dot{q} \in \mathbb{R}^n \) are the generalized coordinates and generalized velocities, and \( M(q) \) is the generalized inertia matrix, which is \textit{symmetric and positive definite} for all \( q \). According to the \textit{Cholesky decomposition}, the symmetric and positive-definite matrix \( M \) can be decomposed into

\[
M = QQ^T,
\]

where \( Q \) is a lower–triangular matrix with strictly positive-diagonal elements; \( Q \) is also called the \textit{Cholesky triangle}. The following formula can be used to obtain the Cholesky triangle through some elementary operations

\[
Q_{ii} = (M_{ii} - \sum_{k=1}^{i-1} Q_{ik}^2)^{1/2} \quad \forall i = 1, \cdots, n
\]

\[
Q_{ji} = (M_{ji} - \sum_{k=1}^{i-1} Q_{jk} Q_{ik})/Q_{ii} \quad \forall j = i + 1, \cdots, n
\]

Since \( Q \) is a lower-triangular matrix, its inverse can be simply computed by the back substitution technique.

The dynamics equations of a constrained MBS with kinetic energy \( K \) can be derived by the \textit{Euler–Lagrange} equations

\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} = \tau - A^T \lambda \quad (4a)
\]

\[
A(q) \dot{q} = 0 \quad (4b)
\]

Here, \( \tau = \tau_a + \tau_f + \tau_p \) is the generalized forces containing all applied loads including the actuator forces applied to the joints \( \tau_a \) and the joint friction \( \tau_f \) plus the conservative forces \( \tau_p = -\partial P/\partial q \)
owing to gravitational energy, vector $\lambda \in \mathbb{R}^m$ represents the generalized Lagrangian multipliers, and matrix $A \in \mathbb{R}^{m \times n}$ is the corresponding constraint matrix associated with the constraints imposed on the generalized velocities. Equations (4b) can be representation of holonomic or non-holonomic constraints. Also, it should be pointed out that $A$ is not necessarily a full-rank matrix because of the possible redundant constraints. Substituting (2) into (1) and then applying (4a) yields

$$
\tau - A^T \lambda = \frac{d}{dt}(QQ^T \dot{q}) - \frac{1}{2} \left( \frac{\partial}{\partial q} \|Q^T(q)\dot{q}\|^2 \right)^T
$$

$$= Q\frac{d}{dt}(Q^T \dot{q}) + \dot{Q}Q^T \dot{q} - \frac{\partial (Q^T \dot{q})}{\partial q} Q^T \dot{q}
$$

$$= Q\frac{d}{dt}(Q^T \dot{q}) + (\dot{Q} - \frac{\partial (Q^T \dot{q})}{\partial q}) Q^T \dot{q}
$$

Define the vectors of quasi-velocities and input quasi-forces as follows:

$$v \triangleq Q^T(q) \dot{q}, \quad \text{(6a)}$$

$$u \triangleq Q^{-1}(q) \tau, \quad \text{(6b)}$$

**Remark 1** Since $\det Q = \sqrt{\det M} \neq 0$, matrix $Q$ is always full rank and thus $Q^{-1}$ always exists. Therefore, (6) implies that there are one-to-one relationships between the set of generalized velocity and generalized force $\{\dot{q}, \tau\}$ on one hand and the set of quasi-velocities and quasi-forces $\{v, u\}$ on the other hand.

Now, pre-multiplying both sides of (5) by $Q^{-1}$ and then substituting (6) into the resultant equation, we arrive at the equations of mechanical systems expressed by the quasi-variables:

$$\dot{v} + \Gamma v = u - \Lambda^T \lambda \quad \text{(7a)}$$

$$\Lambda v = 0 \quad \text{(7b)}$$

where

$$\Gamma \triangleq Q^{-1} \left( \dot{Q} - \frac{\partial v}{\partial q} \right) \quad \text{(8)}$$

$$\Lambda \triangleq AQ^{-T} \quad \text{(9)}$$

Alternatively, matrix $\Gamma$ can be described by $\Gamma = Q^{-1} \Psi$ where the $ij$th entries of matrix $\Psi$ can be calculated through the following partial derivative equations

$$\Psi_{ij} = \sum_k \left( \frac{\partial Q_{ij}}{\partial q_k} - \frac{\partial Q_{kj}}{\partial q_i} \right) \dot{q}_k. \quad \text{(10)}$$

It is worth noting that the constraint equations (4b) are imposed on the quasi-velocities analogous to constraint equations (4b) imposed on the generalized velocities. Since matrix $Q$ is always full-rank, we can say $\text{rank}(\Lambda) = \text{rank}(A) = r$, where $r \leq m$ is the number of independent
constraints. Then, according to the singular value decomposition (SVD) there exist unitary (orthogonal) matrices $U = [U_1 \ U_2] \in \mathbb{R}^{m \times m}$ and $V = [V_1 \ V_2] \in \mathbb{R}^{n \times n}$ (i.e., $U^T U = U U^T = I_m$ and $V^T V = V V^T = I_n$) such that

$$\Lambda = U \Sigma V^T$$

where

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

and $S = \text{diag}(\sigma_1, \ldots, \sigma_r)$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$ being the non-zero singular values\[\text{[43,44]}\]. The unitary matrices are partitioned so that the dimensions of the submatrices $U_1$ and $V_1 \in \mathbb{R}^{n \times r}$ are consistent with those of $S$. That is the columns of $U_1$ and $V_2 \in \mathbb{R}^{n \times (n-r)}$ are the corresponding sets of orthonormal eigenvalues which span the range space and the null space of $\Lambda$, respectively\[\text{[45]}\]. Thus

$$\Lambda V_2^T = 0$$

Define reduced order quasi-velocities or independent quasi-velocities $v_r \in \mathbb{R}^{n-r}$ as follow:

$$v_r = V_2^T v = V_2^T Q^T \dot{q}$$

Since $V_2 V_2^T$ is a projection matrix corresponding to the kernel of matrix $\Lambda$, we have $V_2 V_2^T v = v$. Therefore, we can readily obtain the reciprocal of (13) by per-multiplying both sides of the latter equation by $V_2$, i.e.,

$$v = V_2 v_r$$

The time-derivative of the reduced quasi-velocities can be expressed by

$$\dot{v}_r = V_2^T \dot{v} + V_2^T V_2 v_r$$

It can be inferred from (6a) and (14) that the generalized velocities can be constructed from the reduced quasi-velocities via the following mapping

$$\dot{q} = Q^{-T} V_2 v_r$$

Finally, pre-multiplying both sides of (17a) by $V_2^T$ and then using identities (12), (14), and (15), we arrive at

$$\dot{v}_r = \Gamma_v v_r + u_v$$

where

$$\Gamma_v = V_2^T \Gamma V_2 + \dot{V}_2^T V_2$$

$$u_v = V_2^T Q^{-1} \tau$$

Note that matrix $\dot{V}_2^T$ in the RHS of (18a) can be obtained from time-derivative of (12) as follows: $\Lambda \dot{V}_2^T = -\dot{\Lambda} V_2^T$, and thus

$$\dot{V}_2^T = -V \Sigma^+ U^T \dot{\Lambda} V_2^T$$

where $\Sigma^+$ contains the inverse of non-zero singular values. On the other hand, using the orthogonality property $V_1^T V_2 = 0$ and pre-multiplying both sides of (13) by $V_1^T$, one can
conclude that $V_1^T$ indeed acts an annihilator for the quasi-velocities, i.e., $V_1^T v = 0$. Thus, time-derivative of the latter identity gives us

$$V_1^T \dot{v} + \dot{V}_1^T v = 0 \quad (19)$$

Pre-multiplying both sides of (7a) by $V_1^T$ and then using identity (19), we arrive at

$$\xi = \Gamma \xi v_r + u_\xi \quad (20)$$

where

$$\Gamma \xi = -V_1^T \Gamma V_2 + \dot{V}_1^T V_2, \quad (21a)$$

and $\xi$ and $u_\xi$ are, respectively, the constraint quasi-forces and the corresponding input quasi-forces defined by

$$\xi = V_1^T Q^{-1} A^T \lambda \quad (21b)$$

$$u_\xi = V_1^T Q^{-1} \tau \quad (21c)$$

Equations (17) and (20) completely characterize the dynamics behaviour of a constrained MBS in terms of quasi variables. It should be noted that that the corresponding input quasi-forces for the equations of motion and the constraint forces, i.e., $u_r$ and $u_\xi$ are naturally decoupled. The decoupling of the equations of motions and constraint forces allows the development of independent motion and force controllers without any need to compensate for the cross-coupling terms. It should be also pointed out that the conventional transformation used in quasi-Lagrangian dynamics allows for splitting between the normal and tangential dynamics [12]. However, the normal dynamics is eliminated from the above equations, which are expressed in terms of the reduced quasi-velocities.

### 2.1 Properties of the System in Terms of Quasi-Velocities and Quasi-Forces

In the following analysis, we explore some properties of the quasi-variable formulation (17) and (20) that will be useful in control design purposes.

i. Kinetic energy

$$K = \frac{1}{2} \parallel v_r \parallel^2. \quad (22)$$

ii. Skew-symmetric property

$$v_r^T \Gamma v_r = 0 \quad \forall v_r \in \mathbb{R}^{n-r}. \quad (23)$$

iii. Boundedness of matrices $\Gamma v$ and $\Gamma \xi$, i.e.,

$$\parallel \Gamma v \parallel, \parallel \Gamma \xi \parallel \leq \gamma \parallel v_r \parallel \quad \exists \gamma > 0. \quad (24)$$

The kinetic energy is trivially given by

$$K = \frac{1}{2} \parallel v \parallel^2 = \frac{1}{2} v_r V_2^T V_2 v_r = \frac{1}{2} \parallel v_r \parallel^2, \quad (25)$$
and hence (22) is proven. Furthermore, in the absence of any external active force, the principle of conservation of kinetic energy dictates that the kinetic energy of mechanical system is bound to be constant, i.e., $u_v = 0 \implies \dot{K} = 0$. On the other hand, the zero-input response of a mechanical system is $\dot{v} = -\Gamma_r v_r$. Substituting the latter equation in the time-derivative of (25) gives $v^T_r \Gamma_r v_r = 0$, which implies (23). Assume that $c_m$ denote the minimum eigenvalue of $M$ for all configurations $q$, that is, $c_m I \leq M(q)$. Then, using the norm properties leads to

$$\|Q^{-1}\| \leq \frac{1}{\sqrt{c_m}}. \quad (26)$$

In view of (16) and (26) and knowing that $\|V_2\| = 1$, we can say

$$\|\dot{q}\| \leq \|Q^{-T}\| \|V_2\| \|v_r\| \leq \frac{\|v_r\|}{\sqrt{c_m}}. \quad (27)$$

Moreover, if the factorization $Q(q)$ is a sufficiently smooth function, then all partial-derivative terms in (10) are bounded and hence there exists a finite $c_\psi > 0$ such that $\|\Psi\| \leq c_\psi \|\dot{q}\|$, and hence we can say

$$\|\Gamma\| \leq \frac{c_\psi}{c_m} \|v_r\|. \quad (28)$$

Also, the entries of the time-derivative of matrix $V$ can be written as

$$\dot{V}_{ij} = \sum_k \frac{\partial V_{ij}}{\partial q_k} \dot{q}_k \quad (29)$$

Consequently, we can say $\|\dot{V}\| \leq c_\psi \|\dot{q}\| \leq c_\psi/c_m \|v_r\|$. Finally, knowing that $\|V_1\| = \|V_2\| = 1$ and $\|\dot{V}_1\|, \|\dot{V}_2\| \leq \|\dot{V}\|$, one can infer (24) from the RHS expressions in (18a) and (21a) where $\gamma = (c_\psi + c_m)/c_m$.

### 2.2 Natural Metric

A problem that often arises in robotics, namely hybrid control or the minimum solution to joint rate or force is that generalized coordinate $q$ may have a combination of rotational and translational components that can be even compounded by having combination of rotational and translational constraints [38]. This may lead to inconsistent results, i.e., results that are invariant with respect to changes in dimensional units unless adequate weighting matrices are used [26, 38, 40]. For example, the minimum joint rate rates, $\min \|\dot{q}\|$, or minimum norm force, $\min \|\tau\|$, are not meaningful quantities if the robot has both revolute and prismatic joints [38].

**Remark 2** It is also important to note the important property of the reduced quasi-velocities and quasi-forces is that they always have homogeneous units. The expression of kinetic energy (22) implies $\|v_r\| = \|v\| = \sqrt{2K}$ meaning that all elements of the vector of quasi-velocities $v$ or $v_r$ must have a homogeneous unit $[\sqrt{\text{kgm/s}}]$. This is true even if the vector of the generalized coordinate or the constraints have combinations of rotational and translational components. Similarly, one can argue from (17) and (20) that the elements of the quasi-forces $u_v$ and $\xi$ have always identical unit $[\sqrt{\text{kgm/s}^2}]$, regardless of the units of the generalized force or the constraint wrench.
Therefore, minimization of $\|v\|$ or $\min\|u\|$ is legitimate because the latter vectors have always homogeneous units. Moreover, the selection matrices which are often needed in hybrid position-force control of manipulators when both translational and rotational constraints are involved between its end effector and its environment \[10\] becomes a non-issue here.

### 2.3 Existence of Quasi-Coordinates

It should be pointed out that despite the one-to-one correspondence between velocity coordinate $\dot{q}$ and the quasi-velocities $v$, they are not synonymous. This is because the integration of the former variable leads to the generalized coordinate, while integration of the latter variable does not always lead to a meaningful vector describing the configuration of the mechanical system. Now let us assume that $\dot{\phi} = v$, where $\phi$ is called quasi-coordinates. For $\phi$ to be an explicit function of $q$, i.e., $\phi = \phi(q)$, it must be the gradient of a scalar function meaning that $\phi$ is a conservative vector field. In that case, (6a) implies that $Q^T(q)$ is actually a Jacobian as

$$Q_{ij} = \frac{\partial \phi_j}{\partial q_i}. \quad (30)$$

If $\phi(q)$ exists and it is a smooth function, then we can say

$$\frac{\partial Q_{ij}}{\partial q_k} - \frac{\partial Q_{kj}}{\partial q_i} = \frac{\partial^2 \phi_j}{\partial q_i \partial q_k} - \frac{\partial^2 \phi_i}{\partial q_k \partial q_i} = 0 \quad (31)$$

Under this circumstance, the expression in the parenthesis of the right-hand side of (10) vanishes, i.e., $\Psi \equiv 0$ and hence $\Gamma \equiv 0$. Therefore, the equations of motion become a simple integrator system and hence $\dot{\phi}$ and $v$ are indeed alternative possibilities for generalized coordinates and generalized velocities. Technically speaking, a necessary and sufficient condition for the existence of the quasi-coordinates, $\phi$, is that the Riemannian manifold defined by the inertia matrix $M(q)$ be locally flat—by definition, a Riemannian manifold that is locally isometric to Euclidean manifold is called a locally flat manifold \[17\]. However, that has been proven to be a very stringent condition \[32\]. Nevertheless, state variables $\{q, v_r\}$ are sufficient to describe completely the states of MBS. Setting (16) and (17) in state space form gives

$$\frac{d}{dt} \begin{bmatrix} q \\ v_r \end{bmatrix} = \begin{bmatrix} Q^{-T}V_2 \\ -\Gamma_v \end{bmatrix} v_r + \begin{bmatrix} 0 \\ I \end{bmatrix} u_r. \quad (32)$$

It is interesting to note that dynamics system (32) is in the form of the so-called second-order kinematic model of constrained mechanism, which appears in kinematics of nonholonomic systems. This is the manifestation of the fact that the integration of quasi-velocities, in general, does not lead to quasi-coordinates.

### 3 Force/Motion Control Based on Quasi-Velocities and Quasi-Forces

In this section, we use dynamics formulation (17) and (20) for tracking control and regulation control of constrained MBSs. Due to presence of only $r$ independent constraints, the actual
number of degrees of freedom of the system is reduced to \( n - r \). Consequently, there must be \( n - r \) independent variables \( \theta(q) \in \mathbb{R}^{n-r} \), which is also called a **minimal set of generalized coordinates**. Thus, we can say

\[
\dot{\theta} = \left( \frac{\partial \theta}{\partial q} \right) \dot{q}
\]  

(33)

Substituting \( \dot{q} \) from (16) into (33) yields

\[
\dot{\theta} = D(\theta)v_r, \quad \text{where} \quad D \triangleq \left( \frac{\partial \theta}{\partial q} \right) Q^{-T}V_2.
\]  

(34)

Since both variables \( v_r \) and \( \dot{\theta} \) are with the same dimension, the reciprocal of mapping (33) must uniquely exist, i.e., \( D^{-1} \) is always well-defined. Now we adopt a Lyapunov-based control scheme [46, p.74] for designing a feedback control in terms of quasi-velocities. Define the composite error

\[
\epsilon \equiv \epsilon_v + \epsilon_\theta, \quad \text{where} \quad \epsilon_v = v_r - v_{rd}, \quad \text{and} \quad \epsilon_\theta = \theta - \theta_d.
\]

(35)

where \( k_p > 0, \epsilon_v = v_r - v_{rd}, \) and \( \epsilon_\theta = \theta - \theta_d \). Also, define the auxiliary variable \( \sigma = v_r - \epsilon = v_{rd} - k_p D^{-1} \epsilon_\theta \), which is used in the following control law:

\[
u_v = \sigma + \Gamma_v v_r - k_d \epsilon_v,
\]  

(36)

where \( k_d > 0 \). Applying control law (36) to system (7) gives the dynamics of the error \( \epsilon \) in terms of the following first-order differential equation:

\[
\dot{\epsilon} = -k_d \epsilon.
\]  

(37)

In other words, the composite error \( \epsilon \) is exponentially stable

\[
\epsilon = \epsilon(0)e^{-k_d t}.
\]  

(38)

Pre-multiplying both sides of (35) by \( D(\theta) \), the resultant equation can be rearranged to the following differential equation

\[
\dot{\epsilon}_\theta = -k_p \epsilon_\theta + D \epsilon.
\]  

(39)

Now, it remains to show that the solution of the above non-autonomous system converges to zero. First, notice from definition (34) that \( D \) is the product of three bounded matrix and thus it should be a bounded matrix too. That is because \( \|Q^{-T}\| \leq 1/\sqrt{c_m} \) according to (26) and \( V_2 \) is a unitary matrix meaning that \( \|V_2\| \leq 1 \). Thus, there exists scalar \( c_d > 0 \) such that

\[
D(\theta) \leq c_d I.
\]  

(40)

where \( c_d = \|\partial Q / \partial q\|/\sqrt{c_m} \). One can show that the solution of (39) satisfies

\[
\|\epsilon_\theta\| \leq \left( \|\epsilon_\theta(0)\| + c_d \|\epsilon(0)\| \right)e^{-k_d t},
\]

(41)

which implies exponential stability of the tracking error; see Appendix for details. Tracking of the desired constraint force \( \xi_d = \Lambda^T r \lambda_d \) can be simply achieved by compensating the velocity perturbation term in (20), i.e.,

\[
u_\xi = \xi_d + \Gamma_\xi v_r \quad \Longrightarrow \quad \xi = \xi_d.
\]  

(42)
From definitions of \( \mathbf{u}, \mathbf{u}_v, \) and \( \mathbf{u}_\xi \) in (6b), (18b) and (21c), one can write the following relationship

\[
\begin{bmatrix} \mathbf{u}_\xi \\ \mathbf{u}_v \end{bmatrix} = \mathbf{V}^T \mathbf{u}
\]

and thus \( \mathbf{u}_\xi^T \mathbf{u}_\xi + \mathbf{u}_v^T \mathbf{u}_v = \mathbf{u}^T \mathbf{V} \mathbf{V}^T \mathbf{u} \). Since \( \mathbf{V} \) is a unitary matrix, the latter identity is equivalent to \( \| \mathbf{u}_v \|^2 + \| \mathbf{u}_\xi \|^2 = \| \mathbf{u} \|^2 \). It is worth noting that in view of the latter norm identity, we can say that

\[
\min \| \mathbf{u} \| \leftarrow \mathbf{u}_\xi \equiv 0
\]

That is tantamount to minimization of weighted norm of the generalized forces where the weight matrix is the inverse of the inertia matrix because

\[
\| \mathbf{u}_v \|^2 + \| \mathbf{u}_\xi \|^2 = \| \mathbf{u} \|^2.
\] (43)

In other word, the kinetic metric of the generalized force in minimized.

### 3.1 Setpoint Control

In this section, we extend such a feedback control for hybrid motion/force control of constrained robotic systems. Consider the following control law for system (7)

\[
\mathbf{u}_v = -k_d \mathbf{v}_r - k_p \mathbf{\epsilon}_\theta,
\] (44)

where \( \mathbf{\epsilon}_\theta = \mathbf{\theta} - \mathbf{\theta}_d \), and \( k_d, k_p > 0 \). Then, the dynamics of the closed-loop system becomes

\[
\dot{\mathbf{v}}_r = -\Gamma \mathbf{v}_r - k_d \mathbf{v}_r - k_p \mathbf{\epsilon}_\theta.
\] (45)

Choose the following standard Lyapunov function

\[
V = \frac{1}{2} \| \mathbf{\epsilon}_v \|^2 + \frac{1}{2} k_p \| \mathbf{\epsilon}_\theta \|^2.
\] (46)

Then, using Property (23) in the time-derivative of (46) along (45) yields

\[
\dot{V} = -\mathbf{v}_r^T \mathbf{K}_d \mathbf{v}_r \leq 0,
\]

which is negative-semidefinite. Therefore, according to LaSalle’s Global Invariant Set Theorem \cite{lasalle1961invariance}, the solution of system (45) asymptotically converges to the invariant set \( \mathcal{S} = \{ \mathbf{v}_r, \mathbf{\epsilon}_\theta : \mathbf{v}_r = 0, \mathbf{\epsilon}_\theta = 0 \} \), i.e., \( \mathbf{\theta} \to \mathbf{\theta}_d \) as \( t \) goes to infinity. Define \( \mathbf{\xi}_d = \mathbf{A}_d^T \mathbf{\lambda}_d \) where is the desired value of the Lagrangian multiplier. Then the force control law can be simply given by

\[
\mathbf{u}_\xi = \mathbf{\xi}_d.
\] (47)

Substituting (47) into (20) and using Property (24), we get

\[
\| \mathbf{\xi} - \mathbf{\xi}_d \| \leq \gamma \| \mathbf{v}_r \|^2.
\]

Since \( \mathbf{v}_r \to 0 \), then \( \mathbf{\xi} \to \mathbf{\xi}_d \) as \( t \) goes to infinity.
4 Conclusions

A consistent formulation for modelling of constrained MBSs using the concept of quasi-velocities and quasi-forces has been presented. The main advantage of this formulation is that it does not require adequate weighting matrices when the generalized coordinates involve in both translational and rotational components and/or the generalized force or constraint wrench involve in both force and moment components. It has been also shown that using the Cholesky decomposition of the mass matrix and the unitary transformation corresponding to the kernel of the Pfaffian constraints of the quasi-velocities led to the decoupling of motion and force control inputs. This allowed the possibility to develop a simple force control action that is totally independent of the motion control action. Some properties of the constrained Lagrangian dynamics formulation based on the quasi-variables were presented that could be useful for control purposes. It followed by the development of the force/motion control of constrained MBSs based on the quasi-velocities and quasi-forces.

Appendix

Consider the following positive–definite function

\[ V = \frac{1}{2} \| \epsilon_\theta \|^2, \]  

(48)

whose time-derivative along (39) gives

\[ \dot{V} = -k_p \| \epsilon_\theta \|^2 + \epsilon_\theta^T D(\theta) \epsilon. \]

From (38) and (40), one can find a bound on \( \dot{V} \) as

\[ \dot{V} \leq -2k_p V + c_d \| \epsilon(0) \| \| \epsilon_\theta \| e^{-k_p t}, \]  

(49)

which is in the form of a Bernoulli differential inequality. The above nonlinear inequality can be linearized by the following change of variable \( U = \sqrt{V} \), i.e.,

\[ \dot{U} \leq -k_p U + \frac{c_d \| \epsilon(0) \|}{\sqrt{2}} e^{-k_p t}. \]  

(50)

In view of the comparison lemma [48, p. 222] and (38), one can show that the solution of (50) must satisfy

\[ U \leq \left( U(0) + \frac{c_d \| \epsilon(0) \|}{\sqrt{2}} \right) e^{-k_p t}, \]

which is equivalent to (41).

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