Spin chains in magnetic field, non-skew-symmetric classical $r$-matrices and BCS-type integrable systems

T. Skrypnyk

International School for Advanced Studies, via Beirut 2-4, 34014 Trieste, Italy
Bogoliubov Institute for Theoretical Physics, Metrologichna street 14-b, Kiev 03143, Ukraine

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Abstract

We construct generalized Gaudin systems in an external magnetic field corresponding to arbitrary $so(3)$-valued non-skew-symmetric $r$-matrices with spectral parameters and non-homogeneous external magnetic fields. In the case of $r$-matrices diagonal in the $sl(2)$ basis we calculate the spectrum and the eigen-values of the corresponding generalized Gaudin hamiltonians using the algebraic Bethe ansatz. We explicitly consider several one-parametric families of non-skew-symmetric classical $r$-matrices and the corresponding generalized Gaudin systems in a magnetic field. We apply these results to fermionic systems and obtain a wide class of new integrable fermionic BCS-type hamiltonians.

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1. Introduction

During the last years there arised a new interest to quantum integrable Gaudin spin chains [1,2]. The simplest case of Gaudin model is described by the following commuting quantum spin hamiltonians:

$$\hat{H}_G = \sum_{k=1, k\neq l}^{N} \sum_{a=1}^{3} \frac{\hat{S}^a_k \hat{S}^a_l}{v_k - v_l}, \quad l \in 1, 2, \ldots, N,$$

(1)

E-mail address: skrypnyk@sissa.it.
where \( \hat{S}_k^a, k \in \{1, 2, \ldots, N\}, a \in \{1, 2, 3\} \) are the orthonormal basic elements of \( N \) copies of “spin” Lie algebra \( \text{so}(3) \) in some irreducible representation, \( v_k, k \in \{1, 2, \ldots, N\} \) are the coordinates of the sites of spins in the chain. A great interest is attracted also by the slight modification of the Gaudin model, namely by “Gaudin model in an external magnetic field”, which in the same simplest case is described by the following hamiltonians [3):

\[
\hat{H}_{G,c}^l = c \hat{S}_3^l + \sum_{k=1, k\neq l}^{N} \sum_{a=1}^{3} \frac{\hat{S}_k^a \hat{S}_l^a}{v_k - v_l}, \quad l \in \{1, 2, \ldots, N\}. \tag{2}
\]

This interest is explained by the fact that several new connections between Gaudin magnets and other models of mathematical physics were discovered. The most important of them is a connection [4] between Gaudin spin chains in external magnetic field and “reduced” BCS or Richardson model [5,6] described by the following fermionic hamiltonian:

\[
\hat{H}_{\text{BCS}} = \sum_{l=1}^{N} \epsilon_l \sum_{\epsilon \in +,-} c_{l,\epsilon}^\dagger c_{l,\epsilon} + g \sum_{m,l=1}^{N} c_{m,+,\epsilon}^\dagger c_{m,-,\epsilon}^\dagger c_{l,-,\epsilon} c_{l,+,\epsilon}, \tag{3}
\]

where \( c_{m,+,\epsilon}^\dagger, c_{l,+,\epsilon}, c_{m,-,\epsilon}, c_{l,-,\epsilon} \) are fermion creation–annihilation operators corresponding to the two (time-reversed) states labeled by the energies \( \epsilon_m \) and indices +, −. It turned out [4] that after introducing the so-called “pseudo-spin” operators that are quadratic in fermionic creation–annihilation operators the reduced BCS hamiltonian can be expressed as a function of the Gaudin hamiltonians in the external magnetic field (2).

The interpretation of the hamiltonian \( \hat{H}_{\text{BCS}} \) in terms of the rational Gaudin model gives a clue for the construction of its integrable generalization. Indeed, using the trigonometric Gaudin model in an external magnetic field instead of rational one it is possible to construct new integrable BCS-type hamiltonians with non-uniform coupling constants [7–10] of the following form:

\[
\hat{H}_{\text{BCS}} = \sum_{l=1}^{N} \epsilon_l \sum_{\epsilon \in +,-} c_{l,\epsilon}^\dagger c_{l,\epsilon} + \sum_{m,l=1}^{N} g_{ml} c_{m,+,\epsilon}^\dagger c_{m,-,\epsilon}^\dagger c_{l,-,\epsilon} c_{l,+,\epsilon} + \sum_{m,l=1}^{N} U_{ml} \sum_{\epsilon,\epsilon' \in +,-} c_{m,\epsilon}^\dagger c_{m,\epsilon'}^\dagger c_{l,\epsilon} c_{l,\epsilon'}, \tag{4}
\]

where \( g_{ml}, U_{ml} \) are coefficients of pairing and electrostatic interactions that are not arbitrary but are expressed via parameters of the trigonometric Gaudin model [7–10].

Nevertheless the approach of [7–10] to the integrability of the BCS-type hamiltonians based on the standard Gaudin models has strong limitations. Indeed, it is known that ordinary Gaudin models [1,2] are connected with the skew-symmetric classical \( r \)-matrices [11]. On the other hand there exist only rational, trigonometric and elliptic classical skew-symmetric \( r \)-matrices [12]. The rational Gaudin models are connected with the standard Richardson model, the trigonometric Gaudin models have been exploited in the context of the BCS models in [7–10] and in the case of the elliptic \( r \)-matrices and elliptic Gaudin model there is no possibility to introduce external magnetic field without spoiling integrability of the model. That is why, in order to obtain further examples of integrable hamiltonians of the type (4) (i.e., another types of the coupling constants \( g_{ml}, U_{ml} \) when hamiltonian (4) is integrable) it is necessary to use more general integrable spin models than ordinary Gaudin models.
In our previous papers [13–16] such a generalization of standard Gaudin models was explicitly constructed. Like the ordinary Gaudin models [2] our models exist for any semisimple (reductive) Lie algebra $g$. The main difference with the standard Gaudin models is that we use non-skew-symmetric classical $r$-matrices instead of skew-symmetric ones that are used in the standard Gaudin models [1,2]. The variety of non-skew-symmetric $r$-matrices is much more wider than the variety of skew-symmetric $r$-matrices. The non-skew-symmetric $r$-matrices satisfy the “generalized” or “permuted” classical Yang–Baxter equation [20–22] instead of ordinary classical Yang–Baxter equation [11,12]. That is why they drop out of the Belavin–Drienfield classification and do not fall into the above three classes (rational, trigonometric and elliptic). Hence, the set of the generalized Gaudin models associated with the non-skew-symmetric classical $r$-matrices $r(u,v)$ is larger then the set of ordinary Gaudin models associated with skew-symmetric ones.

It is necessary also to emphasize that contrary to the classical skew-symmetric $r$-matrices non-skew-symmetric $r$-matrices are not connected with the quantum groups and related structures, in general. That is why in a general case our models, contrary to ordinary Gaudin models, cannot be derived as a limits of Heisenberg-type spin chains.

In order to obtain a BCS-type integrable fermionic Hamiltonian it is necessary to use a generalization of the Gaudin spin chains in a magnetic field. In paper [17] we proposed a general way of introducing of an external magnetic field into the previously constructed generalized Gaudin models without spoiling their integrability. The key observation is that only a special type of an external magnetic field (non-homogeneous in general), form of which substantially depends on the corresponding classical $r$-matrix, will lead to the integrability of the corresponding generalized Gaudin systems in a magnetic field. The role of such a magnetic field is played by the “shift elements”—$g$-valued functions $c(u)$ of the spectral parameter which do not depend on dynamical variables.

In the present paper we consider the generalized Gaudin systems in a magnetic field in the most physically important case $g = so(3) \simeq sl(2)$, find their spectrum and apply the obtained results to the integrable BCS-type models. In more details, we consider the case of the general “diagonal” in the $sl(2)$ basis $r$-matrix $r(u,v)$ and the “diagonal” shift elements (see Section 2 for the exact definitions). We explicitly construct diagonal shift elements using components of the $r$-matrix $r(u,v)$. In the result we obtain a wide class of the Gaudin-type models labeled by the three components of classical $r$-matrix $r(u,v)$, three components of its regular part—matrix $r_0(u,v)$ and one component of the shift function $c(u)$. Each of these ingredients appear in the quantum spin Hamiltonian $\hat{H}_l$ as follows: components of the $r$-matrix give quadratic spin–spin interactions, components of $r_0$-matrix give one site quadratic spin–spin couplings and a shift function $c(u)$ gives the non-homogeneous magnetic interaction linear in spin variables. It turned out that algebraic Bethe ansatz is applicable in the case of the “diagonal” non-skew-symmetric $r$-matrices like in the usual skew-symmetric case. Using it we calculate the spectrum of the corresponding generalized quantum Gaudin Hamiltonians in a magnetic field $\hat{H}_l$ and explicit form of the corresponding Bethe equations.

We use the obtained results in order to construct generalized BCS Hamiltonians of the type (4) and find their spectrum. The most general BCS-type Hamiltonian obtained in the framework of our construction is: $\hat{H} = \sum_{l=1}^{N} \eta_l \hat{H}_l$, where $\eta_l$, $l = 1, 2, \ldots, N$, are arbitrary complex constants and spin operators are expressed in the terms of fermion creation–annihilation operators in the standard way [4,7,9]. In the case of the diagonal $r$-matrices the Hamiltonian $\hat{H}$ has the form (4) with the coefficients $\epsilon_l$, $g_{ml}$, $U_{ml}$ being expressed via constants $\eta_l$, $\nu_k$, the components of the $r$-matrix, its regular part—matrix $r_0$ and shift function $c$. The spectrum of this Hamiltonian is obtained using the spectrum and Bethe equations of the constructed generalized Gaudin systems.
in magnetic field in the case of the special choice of the highest weights of irreducible representation of $so(3)^{\oplus N}$: $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \frac{1}{2}$.

It is necessary to emphasize that we give general answers that hold true for any form of dependence of the components of the $r$-matrix $r(u, v)$ on the spectral parameters $u$ and $v$. Nevertheless, in order to make our results more concrete we explicitly consider several classes of the non-skew-symmetric classical $r$-matrices that are diagonal in $sl(2)$ basis and write down the spectrum and the Bethe equations of the corresponding Gaudin-type systems using obtained in this paper general formulas. The considered examples of non-skew symmetric $r$-matrices fall into three classes: “shifted” $r$-matrices [18], “anisotropic” $r$-matrices [13] and “$K$-twisted” classical $r$-matrices [19]. In the result we explicitly obtain several new classes of the integrable BCS-type hamiltonians (4) with anisotropic couplings.

The structure of the present paper is the following. In Section 2 we introduce the generalized Gaudin models and generalized Gaudin models in an external magnetic field. In Section 3 we diagonalize the constructed models by means of the Bethe ansatz technique. In Section 4 we consider several families of examples of non-skew-symmetric classical $r$-matrices and the corresponding generalized Gaudin systems in magnetic fields. At last in Section 5 we apply the obtained results to the generalized BCS models.

2. Quantum integrable systems and classical $r$-matrices

2.1. Definitions and notations

Let $g = so(3)$ be the Lie algebra of the three-dimensional rotation group over the field of complex numbers. Let $X_\alpha$, $\alpha = 1, 2, 3$, be a basis in $so(3)$ with the commutation relations:

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{3} c^\gamma_{\alpha\beta} X_\gamma,$$

where the structure constants $c^\gamma_{\alpha\beta}$ depend on the chosen basis. In the present paper we will use mainly the “root” basis following from the isomorphism $so(3) \simeq sl(2)$.

**Definition 1.** A function of two complex variables $r(u_1, u_2)$ with values in the tensor square of the algebra $so(3)$ is called a classical $r$-matrix if it satisfies the following “generalized” classical Yang–Baxter equation [20–22]:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)],$$

where

$$r_{12}(u_1, u_2) \equiv \sum_{\alpha, \beta=1}^{3} r^{\alpha\beta}(u_1, u_2) X_\alpha \otimes X_\beta \otimes 1,$$

$$r_{13}(u_1, u_3) \equiv \sum_{\alpha, \beta=1}^{3} r^{\alpha\beta}(u_1, u_3) X_\alpha \otimes 1 \otimes X_\beta,$$

etc., and $r^{\alpha\beta}(u, v)$ are matrix elements of the $r$-matrix $r(u, v)$.
Remark 1. In the case of skew-symmetric \( r \)-matrices, i.e. when \( r_{12}(u_1, u_2) = -r_{21}(u_2, u_1) \) the “generalized” classical Yang–Baxter equation passes to the usual classical Yang–Baxter equation:

\[
[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] + [r_{23}(u_2, u_3), r_{13}(u_1, u_3)].
\] (7)

Due to the fact that each solution of Eq. (7) is skew [23], the set of solutions of (7) is a subset in the set of solutions of (6).

We will be interested only in the meromorphic \( r \)-matrices that possess the following decomposition:

\[
r(u, v) = \Omega_{u-v} + r_0(u, v),
\] (8)

where \( r_0(u, v) \) is a holomorphic function with values in \( so(3) \otimes so(3) \), \( \Omega \in so(3) \otimes so(3) \) is the tensor Casimir: \( \Omega = \sum_{\alpha, \beta} g^{\alpha \beta} X_{\alpha} \otimes X_{\beta} \), \( g^{\alpha \beta} \) is the nondegenerate invariant metric on \( so(3) \).

We will also need the following definitions.

Definition 2. We will call the \( r \)-matrix \( r(u, v) \) diagonal if there exists a basis \( \{X_{\alpha}\} \) with the dual basis \( \{X^\alpha\} \) such that:

\[
r(u, v) = \sum_{\alpha, \beta=1}^3 r^{\alpha \beta}(u, v) X_{\alpha} \otimes X_{\beta} = \sum_{\alpha, \beta=1}^3 g^{\alpha \beta} r^{\beta \alpha}(u, v) X_{\alpha} \otimes X_{\beta},
\]

where \( g^{\alpha \beta} = (X^\alpha, X^\beta) = \frac{1}{2} \text{Tr}(X^\alpha X^\beta) \) is the Killing–Cartan form on \( so(3) \).

Definition 3. A constant (i.e., not depending on dynamical variables) \( so(3) \)-valued function \( c(u) = \sum_{\alpha=1}^3 c^{\alpha}(u) X_{\alpha} \) is called a “generalized shift element” if it solves the following equation:

\[
[r_{12}(u, v), c_1(u)] - [r_{21}(v, u), c_2(v)] = 0.
\] (9)

Remark 2. Note, that due to the linearity of Eq. (9) shift elements constitute a linear space, in particular, if \( c(u) \) is a shift element then \( kc(u) \), \( k \in \mathbb{C} \) is also a shift element.

2.2. Algebra of Lax operators

Using a classical \( r \)-matrix \( r(u, v) \) it is possible to define in the space of certain \( so(3) \)-valued functions of \( u \) with the operator coefficients \( \hat{L}(u) = \sum_{\alpha=1}^3 \hat{L}^{\alpha}(u) X_{\alpha} \) the “tensor” Lie bracket:

\[
[\hat{L}_1(u), \hat{L}_2(v)] = [r_{12}(u, v), \hat{L}_1(u)] - [r_{21}(v, u), \hat{L}_2(v)],
\] (10)

where \( \hat{L}_1(u) = \hat{L}(u) \otimes 1 \), \( \hat{L}_2(v) = 1 \otimes \hat{L}(v) \), \( r_{21}(u, v) = P_{12} r_{12}(u, v) P_{12} \), and \( P_{12} \) is the operator which interchanges the first and second spaces in the tensor product.

A tensor bracket (10) between the Lax operators \( \hat{L}_1(u) \) and \( \hat{L}_2(v) \) yields the following expression of the Lie brackets between their matrix elements:

\[
[\hat{L}^\alpha(u), \hat{L}^\beta(v)] = \sum_{\gamma, \delta=1}^3 (c^{\alpha}_{\gamma \delta} r^{\gamma \beta}(u, v) \hat{L}^\delta(u) - c^{\beta}_{\gamma \delta} r^{\gamma \alpha}(v, u) \hat{L}^\delta(v)),
\] (11)
where the components of the Lax operator $\hat{L}^\alpha(u)$ depend on an auxiliary complex parameter $u$ and the non-commuting quantum dynamical variables. The form of such a dependence is determined by a concrete physical model.

**Example 1.** Let $\hat{S}^i_\alpha, \alpha = 1, 2, 3, i = 1, \ldots, N$, be linear operators in some Hilbert space that constitute a Lie algebra isomorphic to $so(3) \oplus N$ with the commutation relations:

$$[\hat{S}^i_\alpha, \hat{S}^j_\beta] = \delta^{ij} \sum_{\gamma=1}^{3} c_{\alpha\gamma}^\beta \hat{S}^\gamma_{\alpha}.$$

Using generalized classical Yang–Baxter equations it is possible to show that it satisfies the commutation relations (10). The Lax operator (13) is the Lax operator of a generalized Gaudin system. In the case of the ordinary skew-symmetric $r$-matrices it coincides with the Lax operator of the usual Gaudin system.

Let us explain the role of the “shift elements” in the algebra of Lax operators. The following proposition holds true:

**Proposition 2.1.** Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (10) and $c(u)$ be the “shift element” satisfying Eq. (9). Then the operator $\hat{L}^c(u) = \hat{L}(u) + c(u)$ also satisfies the commutation relations (10).

(Proof of this proposition follows from the explicit form of commutation relations (10), definition (9) and the fact that $c^\alpha(u)$ are $c$-numbers and, hence, $[c^\alpha(u), c^\beta(v)] = 0.$)

**Example 2.** As it follows from the above proposition the matrix $c(u)$ satisfies condition (40) and it is possible to consider the following “shifted” quantum Lax operators:

$$\hat{L}^c(u) = \hat{L}(u) + c(u) = \sum_{\beta=1}^{3} (\hat{L}^\beta(u) + c^\beta(u)) X_\beta$$

$$= \sum_{k=1}^{N} \sum_{\alpha, \beta=1}^{3} r^{\alpha,\beta}(v_k, u) \hat{S}^k_\alpha X_\beta + \sum_{\alpha=1}^{3} c^\alpha(u) X_\alpha.$$

As we will show below, Lax operator (14) is the Lax operator of the generalized Gaudin system in an external magnetic field. In the case of the ordinary skew-symmetric $r$-matrices it coincides with the Lax operator of the usual Gaudin system in a magnetic field.
2.3. Quantum integrals

In this subsection we will explain the connection of classical non-skew-symmetric $r$-matrices with quantum integrability. It was shown in our previous paper [16] that, just like in the case of classical $r$-matrix Lie–Poisson brackets [20,21,24,25], the Lie bracket (11) leads to an algebra of mutually commuting quantum integrals.

In more details, let us consider the following quadratic in generators of the Lax algebra operators:

$$\hat{\tau}(u) = \text{Tr}(\hat{L}(u))^2 = \sum_{\alpha\beta=1}^{3} g_{\alpha\beta} \hat{L}^\alpha(u) \hat{L}^\beta(u),$$  \hspace{1cm} (15)

here $g_{\alpha\beta} = (X_\alpha, X_\beta)$ and $(,)$ is an invariant scalar product on $so(3)$: $(X_\alpha, X_\beta) = \frac{1}{2} \text{Tr}(X_\alpha X_\beta)$.

Now, in order to obtain quantum integrable systems one has to show, that

$$[\hat{\tau}(u), \hat{\tau}(v)] = 0.$$  \hspace{1cm} (17)

This equality does not follows directly from the corresponding classical Poisson commutativity of $\tau(u)$ and $\tau(v)$ with respect to the corresponding Lie–Poisson brackets because of the problem of ordering of quantum operators.

Nevertheless, the following theorem is true [16]:

**Theorem 2.1.** Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (10). Assume that in some open region $U \times U \subset \mathbb{C}^2$ the function $r(u, v)$ is meromorphic and possesses the following decomposition:

$$r(u, v) = \frac{\Omega}{u - v} + r_0(u, v),$$  \hspace{1cm} (16)

where $r_0(u, v)$ is a holomorphic function with values in $so(3) \otimes so(3)$, $\Omega \in so(3) \otimes so(3)$ is the tensor Casimir: $\Omega = \sum_{\alpha,\beta} g_{\alpha\beta} X_\alpha \otimes X_\beta$, $g_{\alpha\beta}$ is the nondegenerate invariant metric on $so(3)$: $\sum_{\gamma=1}^{3} g^{\alpha\gamma} g_{\gamma\beta} = \delta_{\alpha\beta}$. Then the operator-valued function $\hat{\tau}(u)$ is a generator of a commutative algebra, i.e.:

$$[\hat{\tau}(u), \hat{\tau}(v)] = 0.$$  \hspace{1cm} (17)

From the above theorem follows, in particular, the next corollary:

**Corollary 2.1.** The operators of the form $\hat{H}^l = \frac{1}{2} \text{res}_{u=v_l} \hat{\tau}(u)$ commute for all $l \in 1, 2, \ldots, N$.

**Example 3.** Let us apply the above corollary to the Lax operators of the generalized Gaudin systems. The direct calculation gives us the following explicit form of $\hat{H}^l$:

$$\hat{H}^l = \sum_{\alpha,\beta=1}^{3} \sum_{k \neq l}^{N} \kappa^{\alpha\beta}(v_k, v_l) S^k_{\alpha} \hat{S}^l_{\beta} + \frac{1}{2} \sum_{\alpha,\beta=1}^{3} \kappa^{\alpha\beta}(v_l, v_l) (\hat{S}^l_{\alpha} \hat{S}^l_{\beta} + \hat{S}^l_{\beta} \hat{S}^l_{\alpha}).$$  \hspace{1cm} (18)

The hamiltonians $\hat{H}^l$ are the generalized Gaudin hamiltonians obtained in the papers [14,15] for an arbitrary semisimple Lie algebra $\mathfrak{g}$ by another method. It is easy to show that they coincide with the ordinary Gaudin hamiltonians in the case of skew-symmetric $r$-matrices.

Combining Theorem 2.1 with Proposition 2.1 we obtain the following:
Corollary 2.2. Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (10). Let $c(u)$ be the “shift element” satisfying Eq. (9). Then the operator-valued function $\hat{\tau}^c(u) = \text{Tr}(\hat{L}(u) + c(u))^2$ generates a commutative algebra:

$$[\hat{\tau}^c(u), \hat{\tau}^c(v)] = 0.$$  

Remark 3. The connection between the functions $\hat{\tau}^c(u), \hat{\tau}(u)$ is the following:

$$\hat{\tau}^c(u) = \hat{\tau}(u) + 2\text{Tr}(c(u)L(u)) + \text{Tr}c(u)^2. \quad (19)$$

We will use this connection while diagonalizing the “shifted” generating function $\hat{\tau}^c(u)$.

From the above corollary follows, in its turn, the next corollary:

Corollary 2.3. The operators of the form $\hat{H}_l^c = \frac{1}{2} \text{res}_{u=\nu_l} \hat{\tau}^c(u)$ commute for all $l \in 1, 2, \ldots, N$.

Example 4. Let us apply the above corollary to the Lax operators of the generalized Gaudin systems in an external magnetic field. The direct calculation gives us the following explicit form of $\hat{H}_l^c$:

$$\hat{H}_l^c = \frac{3}{2} \sum_{k=1}^N \sum_{\alpha, \beta=1}^3 r^{ab}_{0}(\nu_k, \nu_l)S^k_\alpha S^k_\beta + \frac{1}{2} \sum_{\alpha, \beta=1}^3 r^{ab}_{0}(\nu_l, \nu_l)S^l_\alpha S^l_\beta + \sum_{\alpha=1}^3 c^\alpha(\nu_l)S^l_\alpha. \quad (20)$$

The hamiltonian $\hat{H}_l^c$ may be interpreted as an energy of the spinning particle living in the site $l$, that interact with the spins living in the other sites and with the external magnetic field $\tilde{B}$ with the values $\tilde{B}(\nu_l) = c(\nu_l)$ in the points $\nu_l$.

2.4. Shift elements for the diagonal $r$-matrices

In this subsection we will explicitly construct generalized shift elements for all the “diagonal” in $sl(2)$ basis classical $r$-matrices. They will be used below in an explicit construction of the generalized Gaudin systems in an external magnetic field and generalized BCS fermionic systems.

Let us consider the $sl(2)$-basis in the Lie algebra $so(3, \mathbb{C}) \simeq sl(2, \mathbb{C})$: $\{X_3, X_+, X_-\}$, where $X_3 = ix_3$, $X_\pm = i(x_1 \pm ix_2)$, and $x_i$ is orthonormal basis of $so(3)$: $(x_i, x_j) = -\delta_{ij}$. The commutation relations are:

$$[X_3, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = 2X_3.$$

In this basis we have: $(X_\pm, X_\mp) = 2$, $(X_3, X_3) = 1$, $X^3 = X_3$, $X^{\pm} = \frac{1}{2}X_\mp$.

A non-skew-symmetric diagonal in this basis $r$-matrix has the following form:

$$r(u, v) = \left(\frac{1}{2}r^-(u, v)X_+ \otimes X_- + \frac{1}{2}r^+(u, v)X_- \otimes X_+ + r^3(u, v)X_3 \otimes X_3\right). \quad (21)$$

The Lax matrices in this basis are written as follows:

$$L(u) = L^3(u)X_3 + L^+(u)X_+ + L^-(u)X_-.$$
In the subsequent exposition we will deal with the “dual” components \( L_\alpha(u) = g_{\alpha\beta}L_\beta(u) \):

\[
L_+(u) = 2L_-(u), \quad L_- = 2L_+ \quad L_3(u) = L_3^3(u).
\]

Let us now pass to the problem of an explicit construction of “shift elements” for the diagonal in \( sl(2) \) basis \( r \)-matrices (21). The generic shift element in \( sl(2) \) basis has the form:

\[
c(u) = c_+(u)X_+ + c_-(u)X_- + c_3(u)X_3.
\]

We will be more specific and consider the “diagonal” shift elements such that \( c_+(u) = c_-(u) = 0 \), i.e. \( c(u) = c_3(u)X_3 \). For such the shift elements we will solve the condition (9) and find the explicit expression for \( c_3(u) \). The direct calculation gives:

\[
[r_{12}(u,v),c_1(u)] = r_+^+(u,v)c_3(u)X_+ \otimes X_+ - r^-_-(u,v)c_3(u)X_- \otimes X_-, \quad [r_{21}(v,u),c_2(v)] = r_+^+(v,u)c_3(v)X_+ \otimes X_- - r_-^-(v,u)c_3(v)X_- \otimes X_+.
\]

In the result condition (9) yields the following equations for \( c_3(u) \):

\[
\begin{align*}
(22a) & \quad r^+_+(u,v)c_3(u) + r^-_- (v,u)c_3(v) = 0, \\
(22b) & \quad r^+_+(v,u)c_3(v) + r^-_- (u,v)c_3(u) = 0.
\end{align*}
\]

The second of Eqs. (22) follows from the first one after the substitution \( u \leftrightarrow v \). In order to solve first equation we will use the following proposition:

**Proposition 2.2.** For an arbitrary \( r \)-matrix of the form (21) satisfying (8) the following identities hold:

\[
\begin{align*}
(23) & \quad r^3(u,v)r^+_+(u,v) - r^3(u,v)r^-_-(v,u) = -r^+_-(v,u)\left(2r^3_0(v,v) - (r^+_0(v,v) + r^-_0(v,v))\right), \\
(24) & \quad r^3(u,v)r^+_+(u,v) - r^3(u,v)r^-_-(v,u) = r^+_+(v,u)\left(2r^3_0(u,u) - (r^+_0(u,u) + r^-_0(u,u))\right).
\end{align*}
\]

(See Appendix A for the detailed proof of the proposition.)

Adding Eqs. (23) and (24) we obtain the following important corollary:

**Corollary 2.4.** The function \( c_0(u) = (r^3_0(u,u) - \frac{1}{2}(r^+_0(u,u) + r^-_0(u,u))) \) solves Eq. (22a).

From this corollary and Remark 2 it follows that for \( \forall k \in \mathbb{C} \) the following element

\[
c_3(u)X_3 = kc_0(u)X_3 \equiv k\left(r^3_0(u,u) - \frac{1}{2}(r^+_0(u,u) + r^-_0(u,u))\right)X_3
\]

is a correctly defined shift element for the generic diagonal in \( sl(2) \) basis \( r \)-matrix.

**Remark 4.** Note, that in a general case shift elements \( c(u) \) given by (25) do not exhaust the variety of all shift elements. For example, in the case of ordinary skew-symmetric \( r \)-matrices we have:

\[
r^3_0(u,u) = r^-_0(u,u) + r^+_0(u,u) = 0.
\]

Hence in this case formula (25) gives \( c(u) = 0 \). On the other hand, in the case of skew-symmetric \( r \)-matrices we have: \( r^-_-(v,u) = -r^+_+(u,v) \) and equality (22a) acquires the form: \( r^+_+(u,v)(c_3(u) - c_3(v)) = 0 \), i.e. there exists the constant diagonal shift element: \( c_3(u) = c \).
3. Diagonalization of quantum hamiltonians

Let us consider the “generalized Gaudin system” in the case of the diagonal $r$-matrix in details. It will be convenient to re-write the corresponding spin algebra $so(3)^{\otimes N}$ in the $sl(2)^{\otimes N}$ basis. Let $\hat{S}_+^i, \hat{S}_-^i, \hat{S}_3^i, i = 1, \ldots, N$, be linear operators in some Hilbert space that constitute a Lie algebra isomorphic to $so(3)^{\otimes N} \simeq sl(2)^{\otimes N}$ with the commutation relations:

\[
\begin{align*}
\{\hat{S}_+^i, \hat{S}_-^j\} &= 2\delta_{ij} \hat{S}_+^j, & \{\hat{S}_+^i, \hat{S}_3^j\} &= -\delta_{ij} \hat{S}_3^j, & \{\hat{S}_-^i, \hat{S}_3^j\} &= \delta_{ij} \hat{S}_-^j, \\
\{\hat{S}_+^i, \hat{S}_+^j\} &= \{\hat{S}_-^i, \hat{S}_-^j\} &= 0.
\end{align*}
\]

(26)

Let us consider the “shifted” Lax matrix in the $sl(2)$ basis:

\[
\hat{L}^c(u) = \hat{L}_3(u)X_3 + \frac{1}{2}\hat{L}_-(u)X_+ + \frac{1}{2}\hat{L}_+(u)X_- + c_3(u)X_3.
\]

(28)

It corresponds to the diagonal in this basis $r$-matrix and $c_3(u)$ is a “shift function” satisfying Eq. (22). In particular, $c_3(u) = k \lambda_0(u)$.

The nontrivial commutation relations among $\hat{L}_3(u), \hat{L}_\pm(u)$ are the following:

\[
\begin{align*}
[\hat{L}_+(u), \hat{L}_3(v)] &= -(r^3(u, v)\hat{L}_+(u) + r^-(v, u)\hat{L}_+(v)), \\
[\hat{L}_-(u), \hat{L}_3(v)] &= (r^3(u, v)\hat{L}_-(u) + r^+(v, u)\hat{L}_-(v)), \\
[\hat{L}_+(u), \hat{L}_-(v)] &= 2(r^+(u, v)\hat{L}_3(u) + r^-(v, u)\hat{L}_3(v)).
\end{align*}
\]

(29a, 29b, 29c)

The Gaudin hamiltonians in a magnetic field corresponding to Lax operator (28) are:

\[
\begin{align*}
\hat{H}_l^l &= \sum_{k=1,k\neq l}^N \left( r^3(v_k, v_l)\hat{S}_+^k \hat{S}_3^l + \frac{1}{2} r^+(v_k, v_l)\hat{S}_-^k \hat{S}_+^l + \frac{1}{2} r^-(v_k, v_l)\hat{S}_+^k \hat{S}_-^l \right) \\
&\quad + \left( r^3(v_l, v_l)\hat{S}_3^l \hat{S}_3^l + \frac{1}{2} r^+(v_l, v_l)\hat{S}_-^l \hat{S}_+^l + \frac{1}{4} (r_0^+ v_l, v_l)\hat{S}_-^l \hat{S}_+^l \right) + c_3(v_l)\hat{S}_3^l.
\end{align*}
\]

(30)

The Casimir operators in these coordinates have the form: $C_2^l = (\hat{S}_3^l)^2 + \frac{1}{2}(\hat{S}_+^l \hat{S}_3^l + \hat{S}_3^l \hat{S}_+^l)$.

Remark 5. Note, that subtracting from the hamiltonian (30) operator proportional to Casimir operator one may simplify hamiltonian (30) to the following form:

\[
\begin{align*}
\hat{H}_l' &= \sum_{k=1,k\neq l}^N \left( r^3(v_k, v_l)\hat{S}_+^k \hat{S}_3^l + \frac{1}{2} r^+(v_k, v_l)\hat{S}_-^k \hat{S}_+^l + \frac{1}{2} r^-(v_k, v_l)\hat{S}_+^k \hat{S}_-^l \right) \\
&\quad + c_0(v_l)(\hat{S}_3^l)^2 + c_3(v_l)\hat{S}_3^l.
\end{align*}
\]

(31)

Note also that in the case of the choice of the representation of the algebra $sl(2)^{\otimes N}$ with the highest weight $\lambda_l = \frac{l}{2}, l = 2, \ldots, N$ one has $(\hat{S}_3^l)^2 = \frac{1}{4}$ and hamiltonian (31) becomes similar to the standard Gaudin hamiltonians in the external magnetic field, but with the other functional dependence of $r^3(v_k, v_l), r^\pm(v_k, v_l)$ and $c_3(v_l)$ on $v_k$ and $v_l$. 

Let us also note, that except for the hamiltonians that can be obtained using the generating function $\hat{\tau}^c$, there exist an additional integral commuting with $\hat{H}_l$.

The following proposition holds true:

**Proposition 3.1.** Let the $r$-matrix $r(u, v)$ be diagonal in $\text{sl}(2)$-basis. Let the shift element $c(u)$ be also diagonal. Then the operator: $\hat{S}_3 \equiv \sum_{j=1}^N \hat{S}_3^j$ is a quantum integral of motion, i.e.:

$$[\hat{S}_3, \hat{\tau}^c] = 0.$$

Now let us consider a finite-dimensional irreducible representation of the algebra $\text{so}(3)^{\otimes N}$ in some space $\mathcal{H}$. Due to the fact that any irreducible representation of the direct sum of the Lie algebras is a tensor product of irreducible representations of their components, we will have $\mathcal{H} = V^{\lambda_1} \otimes V^{\lambda_2} \otimes \ldots \otimes V^{\lambda_N}$, where $V^{\lambda_k}$ is an irreducible finite-dimensional representation of the $k$th copy of $\text{so}(3)$ with the spin $\lambda_k$, where $\lambda_k \in \frac{1}{2}\mathbb{N}$.

Each representation $V^{\lambda_k}$ contains the highest weight vector $v_{\lambda_k}$ such that

$$\hat{S}_+ v_{\lambda_k} = 0, \quad \hat{S}_-^k v_{\lambda_k} = \lambda_k v_{\lambda_k}, \quad \hat{S}_3 v_{\lambda_k} = \lambda_k v_{\lambda_k},$$

and the whole space $V^{\lambda_k}$ is spanned by $v_{\lambda_k}^m = (\hat{S}_3^k)^m v_{\lambda_k}$, $m = 0, \ldots, 2\lambda_k$.

The Casimir function $\hat{C}_2^k$ acts on each vector $v_{\lambda_k}^m \in V^{\lambda_k}$ in the usual way:

$$\hat{C}_2^k v_{\lambda_k}^m = \lambda_k (\lambda_k + 1) v_{\lambda_k}^m.$$

Let us consider the following “vacuum” vector in the space $\mathcal{H}$: $|0\rangle = v_{\lambda_1} \otimes v_{\lambda_2} \otimes \ldots \otimes v_{\lambda_N}$. We have that $\hat{L}_+(u)|0\rangle = 0$, due to the definition of $\hat{L}_+(u)$ and equality (32). It is easy to show, that the vector $|0\rangle$ is an eigen-vector for the generating function of the quantum hamiltonians:

$$(\hat{\tau}^c(u) - c_3^2(u))|0\rangle = (\Lambda^c_0(u) + \partial_u A_3(u) + 2c_3^2(u) + r_0^+(u, u) A_3(u))|0\rangle,$$

where $A_3(u) = \sum_{k=1}^N r^3(v_k, u)\lambda_k$ is an eigenvalue of $\hat{L}_3(u)$ on the vector $|0\rangle$, and we used that

$$[\hat{L}_+, \hat{L}_-] = 2(\partial_u \hat{L}_3(u) + (r_0^-(u, u) + r_0^+(u, u)) \hat{L}_3(u)).$$

Let us now construct other eigen-vectors of $\hat{\tau}^c(u)$ using the Bethe ansatz technique.

The following theorem holds true:

**Theorem 3.1.** Let us consider the following Bethe-type vectors:

$$|v_1 v_2 \ldots v_M\rangle = \hat{L}_-(v_1) \hat{L}_-(v_2) \ldots \hat{L}_-(v_M)|0\rangle,$$

where the complex parameters $v_i$ satisfy the following Bethe-type equations:

$$\sum_{k=1}^N r^3(v_k, u)\lambda_k - \sum_{j=1, j \neq i}^M r^3(v_j, v_i) = c_0(v_i) - c_3(v_i), \quad i = 1, \ldots, M, \quad (34)$$

$c_0(v) = r_0^3(v, v) - \frac{1}{2}(r_0^+(v, v) + r_0^-(v, v))$ and $c_3(v)$ is a solution of Eq. (22).

Then the vectors $|v_1 v_2 \ldots v_M\rangle$ are eigen-vectors of the generating function of the quantum hamiltonians $\hat{\tau}^c(u): \hat{\tau}^c(u)|v_1 v_2 \ldots v_M\rangle = (\Lambda^c(u)|v_i\rangle + c_3^2(u)|v_1 v_2 \ldots v_M\rangle$ with the following
eigen-values:

\[ \Lambda^c(u | \{v_i\}) = \left( A_3(u) - \sum_{i=1}^{M} r^3(v_i, u) \right)^2 - \sum_{i=1}^{M} r^+(v_i, u)r^-(v_i, u) + \partial_u A_3(u) + \left( r_0^- (u, u) + r_0^+ (u, u) \right) A_3(u) + 2c_3(u) \left( A_3(u) - \sum_{i=1}^{M} r_3(v_i, u) \right), \]

where \( A_3(u) = \sum_{k=1}^{N} r^3(v_k, u) \lambda_k. \) (35)

**Remark 6.** In terms of spin operators Bethe vectors (33) are written as follows:

\[ |v_1 v_2 \cdots v_M\rangle = \sum_{k_1, k_2, \ldots, k_M=1}^{N} r^+(v_{k_1}, v_1)r^+(v_{k_2}, v_2) \cdots r^+(v_{k_M}, v_M) \hat{S}_{k_1} \hat{S}_{k_2} \cdots \hat{S}_{k_M} |0\rangle. \]

(See Appendix B for the proof of the theorem.)

Using Theorem 3.1 and the definition of the generalized Gaudin hamiltonian in a magnetic field \( \hat{H}^l_c \), we finally obtain the following corollary:

**Corollary 3.1.** The spectrum of the hamiltonians \( \hat{H}^l_c \) on the Bethe-type vectors (33) has the following form:

\[ \hat{H}^l_c |v_1 v_2 \cdots v_M\rangle = h^l_c(\{v_i\}) |v_1 v_2 \cdots v_M\rangle, \]

where

\[ h^l_c(\{v_i\}) = \lambda_l \left( \sum_{m=1, m\neq l}^{N} r^3(v_m, v_l) \lambda_m - \sum_{i=1}^{M} r^3(v_i, v_l) + r_0^3(v_i, v_l) \lambda_l \right) + \frac{1}{2} (r_0^+ (v_i, v_l) + r_0^- (v_i, v_l)) + c_3(v_i). \] (36)

where \( r^3(u, v), r^\pm(v, u) \) are components of the diagonal \( r \)-matrix, \( r_0^3(u, v), r_0^\pm(u, v) \) are their regular parts, \( c_3(u) \) is a shift function, \( v_l, l = 1, 2, \ldots, N \), are the cites of the spin chain, \( \lambda_l, l = 1, 2, \ldots, N \) are the highest weights of the irreducible representation of \( sl^{\otimes N}(2) \) and \( v_i \) are solutions of Bethe-type equations (34).

Let us calculate also the spectrum of the additional integral \( \hat{S}_3 \) on the Bethe vectors \( |v_1 v_2 \cdots v_M\rangle \). By direct calculation, using the commutation relations (26) and definition of the Bethe vectors it is easy to prove the following proposition:

**Proposition 3.2.** The spectrum of the hamiltonians \( \hat{S}_3 \) on the Bethe-type vectors (33) has the following form:

\[ \hat{S}_3 |v_1 v_2 \cdots v_M\rangle = \lambda_3(M) |v_1 v_2 \cdots v_M\rangle = \left( \sum_{i=1}^{N} \lambda_i - M \right) |v_1 v_2 \cdots v_M\rangle. \] (37)
4. Examples

In this section we will consider three classes of non-skew-symmetric classical $r$-matrix, find their subclasses that are diagonal in $sl(2)$ basis and then, applying the results of the previous section, will calculate the spectrum of the corresponding Gaudin-type systems in an external magnetic field. At first we will consider a class of the non-skew-symmetric classical $r$-matrices that are maximally close to the skew-symmetric ones.

4.1. “Shifted” non-skew-symmetric classical $r$-matrices

Let $g_0$ be a Lie subalgebra of $so(3)$ (in particular, $g_0$ may coincide with $so(3)$ itself). We call $so(3) \otimes so(3)$-valued $r$-matrix $r_{12}(u,v)$ to be $g_0$-invariant if

$$[r_{12}(u,v), X \otimes 1 + 1 \otimes X] = 0, \quad \forall X \in g_0.$$ 

As it was shown in [18] if, moreover, $r_{12}(u,v)$ possesses an additional property:

$$(38) r_{12}(u,v) + r_{21}(v,u) = \alpha \Omega^0_{12},$$

then function

$$(40) r^c_{12}(u,v) = r_{12}(u,v) + c_{12}$$

satisfies the generalized Yang–Baxter equation (6) on $so(3)$. We will consider two simplest examples of the $r$-matrix (40) corresponding to the simplest possible solutions $c_{12}$ of Eq. (39).

1. Let us consider skew-symmetric $r$-matrix of Yang:

$$(41) r_{12}(u-v) = \frac{1}{u-v} X_3 \otimes X_3 + \frac{1}{2(u-v)} (X_+ \otimes X_- + X_- \otimes X_+).$$

In this case $r_{12}(u-v) + r_{21}(v-u) = 0$, hence $\alpha = 0$ and any tensor $c_{12}$ that solve constant generalized Yang–Baxter equation will give us a shifted $r$-matrix (40). Due to the evident fact that any one-dimensional subalgebra of $so(3)$ is abelian, and Eq. (39) is satisfied for any tensor $c_{12}$ on the abelian subalgebra, one may put $c_{12} = c X_3 \otimes X_3$. That is why, any tensor of the form:

$$(42) r^c_{12}(u,v) = \left( \frac{1}{u-v} + c \right) X_3 \otimes X_3 + \frac{1}{2(u-v)} (X_+ \otimes X_- + X_- \otimes X_+),$$

where $c$ is an arbitrary constant, satisfies generalized Yang–Baxter equation. In the case at hand we also have that

$$(r^c)^3 = \frac{1}{u-v} + c, \quad (r^c)^\pm = \frac{1}{u-v}, \quad (r^c)_{0}^3 = c, \quad (r^c_0)^\pm = 0.$$ (43)

Let us take the diagonal shift element $c(u) = c_3(u) X_3$ with the shift function $c_3(u) = kc_0(u)$. The corresponding generalized Gaudin hamiltonians in an external magnetic field have the following form:
It is possible to re-write this Hamiltonian in the following form:

\[
\hat{H}_c^l = \sum_{k=1, k \neq l}^{N} \left( \frac{1}{v_k - v_l} + c \right) \hat{S}_3^k \hat{S}_3^l + \frac{1}{2(v_k - v_l)} \left( \hat{S}_3^k \hat{S}_3^l + \hat{S}_3^l \hat{S}_3^k \right)
\]

\[
+ c \left( \hat{S}_3^l \right)^2 + k' \hat{S}_3^l \quad (k' = kc).
\]

(44)

where, as it was defined before \( \hat{S}_3 = \sum_j^{N} \hat{S}_3^j \). In such a form this Hamiltonian coincide with the usual “rational ” Gaudin Hamiltonian in magnetic field, but the “magnetic field” in this case depend additionally on the total spin of the system \( \hat{S}_3 \).

Let us consider the explicit form of the Bethe equations and a spectrum of the generalized Gaudin systems in the magnetic field for the case of the classical \( r \)-matrix (42) and the chosen shift function \( c_3(u) \). Substituting expressions (42) and (43) into the formula (34) we obtain the following Bethe equations for the diagonalization of Gaudin hamiltonians (44) in an external magnetic field:

\[
\sum_{j=1}^{N} \frac{\lambda_j}{v_j - v_i} - \sum_{j=1, j \neq i}^{M} \frac{1}{v_j - v_i} = -c \left( k + \sum_{j=1}^{N} \lambda_j - M \right), \quad i = 1, \ldots, M.
\]

(46)

The spectrum of the generalized Gaudin Hamiltonians (44) has the following explicit form:

\[
h_c^l(\{v_i\}) = -\lambda_l \left( \sum_{i=1}^{M} \frac{1}{v_i - v_l} - \sum_{m=1, m \neq l}^{N} \frac{\lambda_m}{v_m - v_l} - c \left( k + \sum_{j=1}^{N} \lambda_j - M \right) \right).
\]

(47)

(2) Let us consider the non-skew-symmetric solution of the generalized classical Yang–Baxter equation on \( so(3) \simeq sl(2) \) of the following explicit form (see e.g. [22]):

\[
\r_{12}(u, v) = \frac{v^2}{u^2 - v^2} X_3 \otimes X_3 + \frac{uv}{2(u^2 - v^2)} \left( X_+ \otimes X_- + X_- \otimes X_+ \right).
\]

(48)

It is easy to see that \( \r_{12}(u, v) + \r_{21}(v, u) = X_3 \otimes X_3 \equiv \Omega_{12}^{0} \), where \( g_0 \equiv C X_3 \). Due to the evident fact that in the case at hand subalgebra \( g_0 \) is abelian, Eq. (39) is satisfied for any tensor \( c_{12} \) having the form \( c_{12} = c X_3 \otimes X_3 \). That is why, any tensor of the form:

\[
\r_{12}^c(u, v) = \left( \frac{v^2}{u^2 - v^2} + c \right) X_3 \otimes X_3 + \frac{uv}{2(u^2 - v^2)} \left( X_+ \otimes X_- + X_- \otimes X_+ \right),
\]

(49)

where \( c \) is an arbitrary constant, satisfies generalized a Yang–Baxter equation. In the case at hand we also have that

\[
(r^c)^3 = \frac{v^2}{u^2 - v^2} + c, \quad (r^c)^\pm = \frac{uv}{u^2 - v^2}, \quad (r^c)^3_0 = c - \frac{1}{2}, \quad (r^c)_0^\pm = 0.
\]

(50)

Let us take the diagonal shift element \( c(u) = c_3(u) X_3 \) with the shift function \( c_3(u) = kc_0(u) \). The corresponding generalized Gaudin Hamiltonians in an external magnetic field have the following explicit form:
\[ \hat{H}_c^l = \sum_{k=1, k \neq l}^N \left( \frac{v_j^2}{v_k^2 - v_l^2} + c \right) \hat{S}_j^k \hat{S}_l^k \left( \frac{v_k v_l}{2(v_k^2 - v_l^2)} (\hat{S}_k^l \hat{S}_l^k + \hat{S}_l^k \hat{S}_k^l) \right) + \left( c - \frac{1}{2} \right) (\hat{S}_3^l)^2 + k' \hat{S}_3^l, \quad k' = k \left( c - \frac{1}{2} \right). \]  

It is possible to re-write this hamiltonian in the following form:

\[ \hat{H}_c^l = \sum_{k=1, k \neq l}^N \left( \frac{v_j^2 + v_k^2}{2(v_k^2 - v_j^2)} \hat{S}_j^k \hat{S}_l^k + \frac{v_k v_l}{2(v_k^2 - v_l^2)} (\hat{S}_k^l \hat{S}_l^k + \hat{S}_l^k \hat{S}_k^l) \right) + \left( c - \frac{1}{2} \right) (\hat{S}_3 + k) \hat{S}_3^l. \]

In such a form this hamiltonian almost coincide with the usual “trigonometric” Gaudin hamiltonian in magnetic field, but the “magnetic field” in this case depend additionally on the total spin of the system \( \hat{S}_3 \).

Let us consider the explicit form of the Bethe equations and a spectrum of the generalized Gaudin hamiltonians (51) in an external magnetic field:

\[ \sum_{j=1}^N \frac{v_j^2 + v_i^2}{2(v_j^2 - v_i^2)} \lambda_j \lambda_i = - \sum_{j=1, j \neq i}^M \frac{v_j^2 + v_i^2}{2(v_j^2 - v_i^2)} \left( c - \frac{1}{2} \right) \left( k + \sum_{j=1}^N \lambda_j - M \right), \quad i = 1, \ldots, M. \]  

The spectrum of the generalized Gaudin hamiltonians (51) has the following explicit form:

\[ h_c^l(\{v_i\}) = -\lambda_i \left( \sum_{i=1}^M \frac{v_i^2 + v_j^2}{2(v_j^2 - v_i^2)} - \sum_{m=1, m \neq l}^N \frac{(v_m^2 + v_l^2)\lambda_m}{2(v_m^2 - v_l^2)} - \left( c - \frac{1}{2} \right) \left( k + \sum_{j=1}^N \lambda_j - M \right) \right). \]  

**Remark 7.** Note that, as it was noticed before, the spectrum of \( \hat{S}_3 \) on the Bethe vectors \( |v_1 v_2 \cdots v_M \rangle \) is \( \left( \sum_{j=1}^N \lambda_j - M \right) \). That is why the additional summands in spectrum of the hamiltonians (44) and (51) and in the right-hand-side of the Bethe equations (46) and (52) are due to the presence in these hamiltonians of the “magnetic type terms” depending on the total spin \( \hat{S}_3 \).

**Remark 8.** The non-skew-symmetric classical \( r \)-matrices (42) and (49) are a simplest possible generalization of the skew-symmetric rational and trigonometric \( r \)-matrices and are equivalent to them in the special partial cases \( c = 0 \) and \( c = \frac{1}{2} \), respectively. Thus the generalized Gaudin hamiltonians corresponding to the \( r \)-matrices (42) and (49) constitute one-parametric families of the integrable deformations of the usual rational and trigonometric Gaudin hamiltonians [1]. In the next subsection we will consider the case of the other non-skew-symmetric classical \( r \)-matrices and their generalized Gaudin hamiltonians which are more substantially different from the standard skew-symmetric \( r \)-matrices and standard Gaudin hamiltonians.

### 4.2. “Anisotropic” \( r \)-matrices

Let us now consider the case of the so-called “anisotropic” non-skew-symmetric classical \( r \)-matrices [13]. Let \( X_{ik} = \epsilon_{ijk} x_i \), \( i \in \{1, 2, 3\} \), be a matrix basis in the Lie algebra \( so(3) \). Let \( A \in gl(3, \mathbb{C}) \) be an arbitrary symmetric matrix. Let us also define the following matrix: \( A(u) \equiv u 1_3 - A \), where \( 1_3 \) is a unit matrix. Then [13] the following irrational function of two complex
variables:
\[ r_A(u, v) = \frac{1}{(u - v)} \sum_{i,j=1}^{n} A(u)^{1/2} X_{ij} A(u)^{1/2} \otimes A(v)^{-1/2} X_{ji} A(v)^{-1/2} \] (54)

satisfies the generalized classical Yang–Baxter equation (6) on \( so(3) \), i.e., is a classical \( r \)-matrix. Let us consider the case of the diagonal matrices \( A: A = \text{diag}(a_1, a_2, a_3) \). In this case \( r \)-matrix (54) has the following explicit form:
\[ r_A(u, v) = \frac{1}{(u - v)} \sum_{i,j=1}^{3} \frac{\sqrt{(u - a_i)(u - a_j)}}{(v - a_i)(v - a_j)} X_{ij} \otimes X_{ij}. \] (55)

Remark 9. In the limit \( a_i \rightarrow 0 \) the \( r \)-matrix (55) acquires the following form:
\[ r(u, v) = \frac{u}{v(u - v)} \sum_{i,j=1}^{3} X_{ij} \otimes X_{ij}. \] (56)

After the multiplication by \( v^2 \) (the multiplication by the arbitrary function of \( v \) is an equivalence in the class of the non-skew-symmetric \( r \)-matrices [17]) and after the substitution of variables \( u \rightarrow u^{-1}, v \rightarrow v^{-1} \) it coincides with the skew-symmetric rational \( r \)-matrix of Yang.

In the present paper we are mainly interested in the diagonal in \( \mathfrak{sl}(2) \) basis \( r \)-matrices. In order for the \( r \)-matrix (55) to be diagonal in \( \mathfrak{sl}(2) \)-basis one should put \( a_1 = a_2 \). Moreover, due to possibility to shift spectral parameters one may put \( a_1 = a_2 = 0 \). In the result we obtain the following explicit form of the corresponding \( r \)-matrix:
\[ r_A(u, v) = \frac{1}{(u - v)} \left( \frac{u}{v} X_3 \otimes X_3 + \frac{\sqrt{u(u - a)}}{2\sqrt{v(v - a)}} (X_+ \otimes X_- + X_+ \otimes X_-) \right). \] (57)

where \( a \equiv a_3 \), and we have used that \( X_3 = i x_3 \), \( X_\pm = i (x_1 \pm i x_2) \), and \( X_{jk} = \epsilon_{ijk} x_i \).

In this case we have:
\[ (r_A)^3(u, v) = \frac{u}{v(v - u)}, \quad (r_A)^\pm(u, v) = \frac{1}{(v - u)} \frac{\sqrt{u(u - a)}}{2\sqrt{v(v - a)}}, \] (58)
\[ (r_A)^0_v(v, v) = \frac{1}{v}, \quad (r_A)^\pm_0(v, v) = \frac{1}{2} \left( \frac{1}{v} + \frac{1}{v - a} \right). \] (59)

Let us take the diagonal shift element \( c(u) = c_3(u) X_3 \) with the shift function \( c_3(u) = kc_0(u) \). The corresponding generalized Gaudin hamiltonians in an external magnetic field (30) have the following explicit form:
\[ \hat{H}_l^c = \sum_{k=1, k \neq l}^{N} \left( \frac{v_k}{v_l(v_k - v_l)} \delta_{3k}^l \delta_{3l}^k + \frac{\sqrt{v_k(v_k - a)}}{2\sqrt{v_l(v_l - a)}} \left( \delta_{3k}^l \delta_{3l}^k + \delta_{3l}^k \delta_{3k}^l \right) \right) \\
+ \left( \frac{1}{v_l} (\delta_{3k}^l)^2 + \frac{1}{4} \left( \frac{1}{v_l} + \frac{1}{v_l - a} \right) \left( \delta_{3k}^l \delta_{3l}^k + \delta_{3l}^k \delta_{3k}^l \right) \right) + \frac{k}{2} \left( \frac{1}{v_l} - \frac{1}{v_l - a} \right) \delta_{3k}^l. \] (60)

Let us consider the explicit form of the Bethe equations and spectrum of the generalized Gaudin systems in a magnetic field for the case of the “anisotropic” classical \( r \)-matrix and the chosen shift function \( c_3(u) \). Substituting expressions (58) and (59) into the formula (34) we obtain the
following Bethe-type equations for the “rapidities” $v_i$ that are used to diagonalize Hamiltonians (60):

$$
\sum_{j=1}^{N} \frac{v_j}{v_j - v_i} \lambda_j + \sum_{j=1, j \neq i}^{M} \frac{v_j}{v_j - v_i} = \frac{(1 - k)}{2} \left(1 - \frac{v_i}{v_i - a}\right), \quad i \in 1, \ldots, M.
$$

Substituting the expressions (58) and (59) into the formula (36) we obtain the explicit form of the spectrum of the “anisotropic” Gaudin-type Hamiltonians (60):

$$
h_l^j \left(\{v_i\}\right) = -\frac{\lambda_l}{v_l} \left(\sum_{i=1}^{M} \frac{v_i}{v_i - v_l} - \sum_{m=1, m \neq l}^{N} \frac{\lambda_m v_m}{v_m - v_l} - \lambda_j \right) - \frac{1}{2} \left(1 + \frac{v_l}{v_l - a}\right) - \frac{k}{2} \left(1 - \frac{v_l}{v_l - a}\right),
$$

where it is assumed that the cites of the spin chain are chosen in such a way that $v_l \neq v_m \neq a$.

4.3. $K$-twisted non-skew-symmetric $r$-matrices

Let us consider the case of the “$K$-twisted” non-skew-symmetric $r$-matrices. This class of the non-skew-symmetric $r$-matrices is connected with the so-called “reflection equations” [26]. Let us briefly remind the explicit construction of such the $r$-matrices [19]. Let us use the isomorphism $so(3) \simeq sl(2)$ and consider our algebra to be the algebra of traceless two by two matrices. Let $r(u - v)$ be the skew-symmetric classical $sl(2)$-valued $r$-matrix satisfying usual classical Yang–Baxter equation. Let $K(u)$ is a $GL(2)$-valued meromorphic function of $u$ satisfying the following equation:

$$
r_{12}(u - v)K_1(u)K_2(v) + K_1(u)K_2(v)r_{12}(v - u)
= K_1(u)r_{12}(-u - v)K_2(v) + K_2(v)r_{12}(u + v)K_1(u),
$$

where $K_1(u) = K(u) \otimes 1$, $K_2(v) = 1 \otimes K(v)$, then it is possible to define with the help of $r_{12}(u - v)$ and $K(u)$ new non-skew-symmetric $r$-matrix $r^K_{12}(u, v)$ of the following form [19]:

$$
r^K_{12}(u, v) = r_{12}(u - v) - K_2(v)r_{12}(u + v)K^{-1}_2(v).
$$

We call this $r$-matrix the “$K$-twisted” non-skew-symmetric $r$-matrix.

Let us consider several examples of $K$-twisted $r$-matrices. For this purpose, let us at first consider the classical skew-symmetric elliptic $r$-matrix of Sklyanin [27]:

$$
r(u - v) = \sum_{k=1}^{3} r_k(u - v)x_k \otimes x_k,
$$

where $r_k(u)$ are expressed via Jacobi functions:

$$
r_1(u) = \frac{1}{sn(u)}, \quad r_2(u) = \frac{dn(u)}{sn(u)}, \quad r_3(u) = \frac{cn(u)}{sn(u)}.
$$

It is easy to see that $r_{12}(u - v) = -r_{12}(v - u)$ due to the fact that functions $r_k(u)$ are odd.
There are two important limiting cases of the above elliptic $r$-matrix (which will mainly be used below), namely its trigonometric and rational limits given by the formulas:

$$r_1(u) = r_2(u) = \frac{1}{\text{sh}(u)}, \quad r_3(u) = \frac{\text{ch}(u)}{\text{sh}(u)}.$$

(67)

It is possible to show that the matrix

$$K(u) = K(u, \xi) = \begin{pmatrix} k_1(u) & k_2(u) \\ k_3(u) & k_1(u) \end{pmatrix}$$

satisfies Eq. (63) for the elliptic $r$-matrix (65), (66) and arbitrary complex parameter $\xi$. The formula (68) for $K(u)$ holds true also for the trigonometric and rational limits. In these cases we have

$$K(u) = \begin{pmatrix} \text{sh}(u) \text{ch}(u) + \text{sh}(\xi) \text{ch}(\xi) \\ \text{sh}(\xi) \text{ch}(\xi) - \text{sh}(u) \text{ch}(u) \end{pmatrix}$$

or

$$K(u) = \begin{pmatrix} u + \xi \\ \xi - u \end{pmatrix}$$

respectively. Let us note, that only in the rational and trigonometric cases the proposed $r$-matrix (64) will be diagonal in the $sl(2)$ basis. That is why hereafter we will restrict ourselves to the consideration of these two cases. The corresponding non-symmetric rational and trigonometric $r$-matrices have the following form:

$$r^K(u, v) = r^3(u, v)X_3 \otimes X_3 + \frac{1}{2}r^-(u, v)X_+ \otimes X_- + \frac{1}{2}r^+(u, v)X_- \otimes X_+,$$

(69)

where the corresponding coefficients are given by the following explicit formulas:

$$r^-(u, v) = r_1(u - v) - r_1(u + v)\frac{k_2(v)}{k_1(v)},$$

$$r^+(u, v) = r_1(u - v) - r_1(u + v)\frac{k_1(v)}{k_2(v)},$$

$$r^3(u, v) = r_3(u - v) - r_3(u + v).$$

(70)

Here

$$k_1(u) = \frac{\text{sh}(u)}{\text{ch}(u)} + \frac{\text{sh}(\xi)}{\text{ch}(\xi)}, \quad k_2(u) = \frac{\text{sh}(\xi)}{\text{ch}(\xi)} - \frac{\text{sh}(u)}{\text{ch}(u)}$$

in the trigonometric case, $k_1(u) = u + \xi$, $k_2(u) = \xi - u$ in the rational case, and we have taken into account that in the both cases $r_1(u) = r_2(u)$. For the considered cases of the $K$-twisted classical $r$-matrices we have:

$$r_0^-(u, u) = -r_1(2u)\frac{k_2(u)}{k_1(u)}, \quad r_0^+(u, u) = -r_1(2u)\frac{k_1(u)}{k_2(u)}, \quad r_0^3(u, u) = -r_3(2u),$$

(72)

that is why we obtain that

$$c_0(u) = -\left(r_3(2u) - r_1(2u)\frac{k_2^2(u) + k_1^2(u)}{2k_1(u)k_2(u)}\right).$$

Let us take the diagonal shift element with the shift function $c_3(u) = k c_0(u)$. The corresponding generalized Gaudin Hamiltonians in an external magnetic field (30) have the following explicit form:
\[
\hat{H}_i^l = \sum_{k=1, k \neq l}^{N} \left[ (r_3(v_k - v_j) - r_3(v_k + v_j)) \hat{S}_3^k \hat{S}_3^l \right.
\]
\[
+ \frac{1}{2} \left( r_1(v_k - v_j) - \frac{k_2(v_j)}{k_1(v_j)} r_1(v_k + v_j) \right) \hat{S}_3^k \hat{S}_+^l
\]
\[
+ \frac{1}{2} \left( r_1(v_k - v_j) - \frac{k_1(v_j)}{k_2(v_j)} r_1(v_k + v_j) \right) \hat{S}_3^k \hat{S}_-^l
\]
\[
- \left( r_3(2v_j)(\hat{S}_3^l)^2 + r_1(2v_j) \frac{k_2^2(v_j) + k_1^2(v_j)}{4k_1(v_j)k_2(v_j)} (\hat{S}_-^l \hat{S}_+^l + \hat{S}_+^l \hat{S}_-^l) \right)
\]
\[
- k \left( r_3(2v_j) - r_1(2v_j) \frac{k_2^2(v_j) + k_1^2(v_j)}{2k_1(v_j)k_2(v_j)} \right) \hat{S}_3^l.
\]

(73)

Hence we have obtained two one-parametric families of the \(K\)-twisted Gaudin hamiltonians in the external magnetic field corresponding to the \(K\)-twisted rational and trigonometric \(r\)-matrices labeled by the parameter \(\xi\).

Let us consider the Bethe equations for “rapidities” \(v_i\) for the cases of the Gaudin-type hamiltonians (73). Substituting expressions (71) and (72) into the formula (34) obtain the following Bethe-type equations for the diagonalization of \(K\)-twisted Gaudin hamiltonians in the external magnetic field:

\[
\sum_{j=1}^{N} \lambda_j \left( r_3(v_j - v_i) - r_3(v_j + v_i) \right) - \sum_{j=1, j \neq i}^{M} \left( r_3(v_j - v_i) - r_3(v_j + v_i) \right)
\]
\[
= -(1 - k) \left( r_3(2v_i) - r_1(2v_i) \frac{k_2^2(v_i) + k_1^2(v_i)}{2k_1(v_i)k_2(v_i)} \right).
\]

(74)

Finally, the substituting expressions (71) and (72) into the formula (36) we obtain the explicit form of the spectrum of the “\(K\)-twisted” Gaudin hamiltonians in an external magnetic field:

\[
h^l_i(\{v_i\}) = -\lambda_i \left( \sum_{m=1}^{M} \left( r_3(v_m - v_i) - r_3(v_m + v_i) \right) 
\right.
\]
\[
- \sum_{m=1, m \neq l}^{N} \lambda_m \left( r_3(v_m - v_i) - r_3(v_m + v_i) \right)
\]
\[
+ r_3(2v_i)\lambda_i + r_1(2v_i) \frac{k_2^2(v_i) + k_1^2(v_i)}{2k_1(v_i)k_2(v_i)}
\]
\[
+ k \left( r_3(2v_i) - r_1(2v_i) \frac{k_2^2(v_i) + k_1^2(v_i)}{2k_1(v_i)k_2(v_i)} \right).
\]

(75)

Here it is implied that parameters \(\xi\) and \(v_i \neq \pm v_j\) are chosen in such a way that \(\xi \neq \pm v_l\).

5. Integrable BCS-type models and \(r\)-matrices

Having obtained quantum integrable spin system it is possible to derive, using them, integrable fermionic system. For this purpose it is necessary to consider realization of the corresponding spin operators in terms of fermionic creation–annihilation operators. We will consider here only
simplest “fermionization” of the Lie algebra so(3) \( \simeq sl(2) \) corresponding to the case of the representation of this Lie algebra with the spin \( \lambda = \frac{1}{2} \).

In more details, let \( c_{j,e}^\dagger, c_{i,e}, i, j \in 1, 2, \ldots, N, \epsilon, \epsilon' \in \{+, -\} \), be fermionic creation–annihilation operators, i.e:

\[
\begin{align*}
c_{j,e}^\dagger c_{i,e} + c_{i,e}^\dagger c_{j,e} &= \delta_{\epsilon \epsilon'} \delta_{ij}, \\
c_{i,e} c_{j,e}^\dagger + c_{j,e}^\dagger c_{i,e} &= 0,
\end{align*}
\]

(76)

It is well known that the following formulas:

\[
\hat{S}^+_j = c_{j,\epsilon} c_{j,\epsilon}, \quad \hat{S}^-_j = c_{j,\epsilon}^\dagger c_{j,\epsilon'},
\]

(77)

\[
\hat{S}^3_j = \frac{1}{2} (1 - c_{j,\epsilon} c_{j,\epsilon'} + c_{j,\epsilon'} c_{j,\epsilon}), \quad i, j \in 1, 2, \ldots, N, \quad \epsilon \neq \epsilon'
\]

provide realization of the Lie algebra \( sl(2) \oplus N \) with the highest weight \( \lambda_1 = \lambda_2 = \cdots = \lambda_N = \frac{1}{2} \).

Now, let us obtain integrable fermionic hamiltonian using the realization (77) and the constructed in the previous sections integrable spin chains in a magnetic field. Let us consider general quadratic in spin variables hamiltonians of such the system:

\[
\hat{H} \equiv \sum_{l=1}^{N} \eta_l \hat{H}_c^l,
\]

(78)

where the coefficients \( \eta_k \) are in the general case arbitrary complex numbers or may be specially chosen relying on some physical considerations (see Remark 10 below).

In the case of the diagonal in \( sl(2) \) basis \( r \)-matrices and for the diagonal shift elements we obtain the following simple expressions for the hamiltonians (78):

\[
\begin{align*}
\hat{H} &= \sum_{l=1}^{N} \eta_l \left( c_3(v_l) + \frac{1}{2} (r_0^-(v_l, v_l) + r_0^+(v_l, v_l)) \right) \hat{S}_3^l \\
&+ \frac{1}{2} \sum_{m,l=1}^{N} (\tilde{r}^+(v_m, v_l) \eta_l + \tilde{r}^-(v_l, v_m) \eta_m) \hat{S}_3^m \hat{S}_3^l \\
&+ \sum_{m,l=1}^{N} \eta_l \tilde{r}^3 (v_m, v_l) \hat{S}_3^m \hat{S}_3^l,
\end{align*}
\]

(79)

where \( \tilde{r}^\alpha (v_m, v_l) \equiv r^\alpha (v_m, v_l) \) if \( m \neq l \), \( \tilde{r}^\alpha (v_m, v_l) \equiv r_0^\alpha (v_m, v_l) \) if \( m = l \) and \( \alpha \in \{3, +, -\} \).

In the terms of the fermionic operators we obtain the following BCS-type hamiltonian:

\[
\begin{align*}
\hat{H} &= \sum_{l=1}^{N} \epsilon_l \sum_{\epsilon, \epsilon' \in +, -} c_{l,\epsilon}^\dagger c_{l,\epsilon'} + \sum_{m,l=1}^{N} g_{ml} c_{m,\epsilon}^\dagger c_{l,\epsilon'} c_{l,\epsilon}^\dagger c_{m,\epsilon} \\
&+ \sum_{m,l=1}^{N} U_{ml} \sum_{\epsilon, \epsilon' \in +, -} c_{m,\epsilon}^\dagger c_{m,\epsilon'} c_{l,\epsilon}^\dagger c_{l,\epsilon'} + E_0,
\end{align*}
\]

(80)

where

\[
\epsilon_l = -\frac{1}{4} \left( \eta_l (2c_3(v_l) + r_0^-(v_l, v_l) + r_0^+(v_l, v_l)) + \sum_{m=1}^{N} (\tilde{r}^3 (v_m, v_l) \eta_l + \tilde{r}^3 (v_l, v_m) \eta_m) \right),
\]
\[ g_{ml} = \frac{1}{2}(\eta_l r^+(v_m, v_l) + \eta_m r^-(v_l, v_m)), \quad U_{ml} = \frac{\eta_l}{4} r^3(v_m, v_l), \]

\[ E_0 = \frac{1}{4} \sum_{l=1}^{N} \eta_l \left( 2c_3(v_l) + r_0^-(v_l, v_l) + \sum_{m=1}^{N} r^3(v_m, v_l) \right). \]

As it follows from the results of the previous section, the spectrum of the hamiltonian \( \hat{H} \) on the Bethe-type vectors (34) has the following explicit form:

\[ h(v_1, ..., v_M) = \sum_{l=1}^{N} \eta_l h_l \left( \{v_i\} \right), \]

where \( h_l \left( \{v_i\} \right) \) are given by the formula (36) with \( \lambda_l = \frac{1}{2} \) and the “rapidities” \( v_i \) are defined as solutions of the following Bethe-type equations:

\[ \frac{1}{2} \sum_{m=1}^{N} r^3(v_m, v_i) - \sum_{j=1, j \neq i}^{M} r^3(v_j, v_i) = c_0(v_i) - c_3(v_i), \quad i = 1, ..., M, \tag{81} \]

where \( c_0(v) = r_0^+(v, v) - \frac{1}{2}(r_0^+(v, v) + r_0^-(v, v)) \) and \( c_3(v) \) is any shift function satisfying equation (22) (in particular \( c_3(v) = k c_0(v) \)).

**Remark 10.** Note that we have \( 2N \) parameters of the theory \( \eta_l, v_l, \lambda \in 1, N \), connected with the “free energies” \( \epsilon_l \). In some special cases one can simplify substantially the corresponding hamiltonians (79) and (80) by the special choices of \( \eta_l \) or \( v_l \). In particular, in the simplest case of the classical skew-symmetric \( r \)-matrix of Yang one has that \( r^3(v_l, v_m) = r^\pm(v_l, v_m) = (v_l - v_m)^{-1} \), \( r_0^+(v_l, v_l) = r_0^-(v_l, v_l) = c_0(v_l) = 0 \), \( c_3(v) = \text{const} = g^{-1} \). Putting \( \eta_l = v_l = -\epsilon_l \) one obtains that the hamiltonian (79) acquires the form:

\[ \hat{H} = g^{-1} \sum_{l=1}^{N} (-\epsilon_l) \hat{S}_l^3 - \frac{1}{2} \sum_{m=1, m \neq l}^{N} \hat{S}_m^3 \hat{S}_l^3 - \frac{1}{2} \sum_{m=1, m \neq l}^{N} \hat{S}_m^\pm \hat{S}_l^\pm, \tag{82} \]

Subtracting from this hamiltonian sum of the Casimir operators \( \frac{1}{2} \sum_{l=1}^{N} \hat{C}_l^2 \), using the fact that in the skew-symmetric case \( \sum_{l=1}^{N} \hat{S}_l^3 = g \sum_{l=1}^{N} \hat{H}_l^3 \) is an integral of motion, we subtract it and add its second power (multiplied by \( \frac{1}{2} \)) to the integral (82) and obtain (after rescaling) the following hamiltonian [7–10]:

\[ \hat{H}_R = \sum_{l=1}^{N} 2(-\epsilon_l) \hat{S}_l^3 - g \sum_{m=1}^{N} \hat{S}_m^3 \hat{S}_l^3, \]

or, in terms of the fermionic creation–annihilation operators reduced BCS hamiltonian:

\[ \hat{H}_{BCS} = \sum_{l=1}^{N} \epsilon_l (\epsilon_{l,+}^\dagger c_{l,+} + \epsilon_{l,-}^\dagger c_{l,-}) - g \sum_{m,l=1}^{N} \epsilon_{m,+}^\dagger c_{m,-}^\dagger c_{l,-} c_{l,+}. \tag{83} \]

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Appendix A. Proof of Proposition 2.2

Let us consider the generalized classical Yang–Baxter equation in a component form:

\[ C^x_{xk} r^{\delta \beta} (u_1, u_2) r^{xy} (u_1, u_3) - C^y_{xk} r^{\delta \gamma} (u_2, u_3) r^{ax} (u_1, u_2) + C^y_{xk} r^{\delta \beta} (u_3, u_2) r^{ax} (u_1, u_3) = 0. \]  (A.1)

For an arbitrary \( r \)-matrix of the form (21), we have that

\[ r^{11} (u_1, u_2) = r^{22} (u_1, u_2) = r^{13} (u_1, u_2) = r^{31} (u_1, u_2) = r^{32} (u_1, u_2) = r^{23} (u_1, u_2) = 0, \]  (A.2)

\[ r^{12} (u_1, u_2) = \frac{1}{2} r^+ (u_1, u_2), \quad r^{21} (u_1, u_2) = \frac{1}{2} r^- (u_1, u_2), \]

\[ r^{33} (u_1, u_2) = r^3 (u_1, u_2), \]  (A.3)

where index 1 corresponds to \( X_- \), index 2 corresponds to \( X_+ \) and index 3 to \( X_3 \). Eq. (A.1) gives us for \( \alpha = 2, \beta = 3, \gamma = 1 \) the equality:

\[ r^{33} (u_1, u_2) r^{21} (u_1, u_3) - 2 r^{21} (u_2, u_3) r^{21} (u_1, u_2) - r^{33} (u_3, u_2) r^{21} (u_1, u_3) = 0. \]

Taking the limit \( u_1 \to u_2 \) in this equation we obtain that:

\[ r^{33} (u_2, u_2) r^{21} (u_2, u_3) - 2 r^{21} (u_2, u_3) r^{21} (u_2, u_2) - r^{33} (u_3, u_2) r^{21} (u_2, u_3) = -\partial u_2 r^{21} (u_2, u_3). \]  (A.4)

Deriving formula (A.4) we have used that near the "diagonal" \( u_2 = u_1 \) we have the following expansion of \( r^{\alpha \beta} (u_1, u_2) \) in the Laurent power series:

\[ r^{\alpha \beta} (u_1, u_2) = \frac{g^{\alpha \beta}}{u_1 - u_2} + r^{\alpha \beta}_0 (u_2, u_2) + \sum_{k=1}^{\infty} (u_1 - u_2)^k \partial u_2^k r^{\alpha \beta}_0 (u_2, u_2), \]  (A.5)

\[ r^{\alpha \beta} (u_1, u_3) = r^{\alpha \beta} (u_2, u_3) + \sum_{k=1}^{\infty} (u_1 - u_2)^k \partial u_2^k r^{\alpha \beta} (u_2, u_3), \]  (A.6)

where we have also used that \( r^{\alpha \beta}_0 (u_1, u_2) \) and \( r^{\alpha \beta} (u_1, u_3) \) are regular in the first argument in the vicinity of \( u_1 = u_2 \). We have also taken into account that in our basis \( g^{12} = 1/2, \ g^{33} = 1 \).

In an analogous way, using Eq. (A.1) for \( \beta = 2, \alpha = 3, \gamma = 1 \), we derive the identity:

\[ -2 r^{12}_0 (u_2, u_2) r^{21} (u_2, u_3) + r^{21} (u_2, u_3) r^{33}_0 (u_2, u_2) + r^{12} (u_3, u_2) r^{33} (u_2, u_3) = \partial u_2 r^{21} (u_2, u_3). \]  (A.7)

Adding Eqs. (A.4) and (A.7), using identities (A.3) and setting \( u_2 \equiv v, u_3 \equiv u \) we come to Eq. (23).

Eq. (24) is derived in the analogous way using classical Yang–Baxter equation in a component form for the components \( \alpha = 1, \beta = 3, \gamma = 2 \) and \( \alpha = 3, \beta = 1, \gamma = 2 \).

Proposition is proved.

Appendix B. Proof of Theorem 3.1

To find the spectrum of \( \hat{c}^\varepsilon (u) \) or \( (\hat{c}^\varepsilon (u) - c_2^\varepsilon (u)) \) we have to calculate \( \hat{c}^\varepsilon (u) |v_1 v_2 \cdots v_M \rangle \) or \( (\hat{c}^\varepsilon (u) - c_2^\varepsilon (u)) |v_1 v_2 \cdots v_M \rangle \). We have:
\[
(\hat{\tau}^c(u) - c_3^2(u))|v_1 v_2 \cdots v_M\rangle = \Lambda_0^c(u)|v_1 v_2 \cdots v_M\rangle + [\hat{\tau}^c(u) - c_3^2(u), \hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)]|0\rangle, \tag{B.1}
\]

where \(\Lambda_0^c(u)\) is the eigenvalue of \(\hat{\tau}^c(u) - c_3^2(u)\) corresponding to the eigen-vector \([0]\). Hence, in order to describe the spectrum, we have to find \([\hat{\tau}^c(u) - c_3^2(u), \hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)]\).

On the other hand, \([c_3^2(u), \hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)] = 0\) and, as it was shown in our previous paper [16], the following commutation relations holds:

\[
[\hat{\tau}(u), \hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)] = -2\left(\sum_{i=1}^{M} r^3(v_i, u)\right)\hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)\hat{L}_3(u) \\
+ \left(\sum_{i=1}^{K} r^3(v_i, u)\right)^2 - \sum_{i=1}^{M} r^-(v_i, u)r^+(v_i, u)\hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_K) \\
+ 2\left(\sum_{i=1}^{M} r^-(v_i, u)\hat{L}_-(v_1)\cdots\hat{L}_-(v_{i-1})\hat{L}_-(u)\hat{L}_-(v_{i+1})\cdots\hat{L}_-(v_M)\hat{L}_3(v_i)\right) \\
- 2\left(\sum_{i=1}^{M} r^-(v_i, u)\left(c_0(v_i) + \sum_{j=1,j \neq i}^{M} r^3(v_j, v_i)\right) \right) \\
\times \hat{L}_-(v_1)\cdots\hat{L}_-(v_{i-1})\hat{L}_-(u)\hat{L}_-(v_{i+1})\cdots\hat{L}_-(v_M) \tag{B.2}
\]

where, as before, \(c_0(v_i) = r_0^3(v_i, v_i) - \frac{1}{2}(r_0^+(v_i, v_i) + r_0^-(v_i, v_i))\).

Let us now take into account, that \(\hat{\tau}^c(u) = \hat{\tau}(u) + 2c_3(u)\hat{L}_3(u) + c_3^2(u)\) and, hence,

\[
[\hat{\tau}^c(u) - c_3^2(u), \hat{L}_-(v_1)\cdots\hat{L}_-(v_M)] = [\hat{\tau}(u), \hat{L}_-(v_1)\cdots\hat{L}_-(v_M)] + 2c_3(u)[\hat{L}_3(u), \hat{L}_-(v_1)\cdots\hat{L}_-(v_M)].
\]

The direct computation gives:

\[
[\hat{L}_3(u), \hat{L}_-(v_1)\cdots\hat{L}_-(v_M)] = -\left(\sum_{i=1}^{M} r^3(v_i, u)\right)\hat{L}_-(v_1)\cdots\hat{L}_-(v_M) \\
- \sum_{i=1}^{M} r^+(u, v_i)\hat{L}_-(v_1)\cdots\hat{L}_-(v_{i-1})\hat{L}_-(u)\hat{L}_-(v_{i+1})\cdots\hat{L}_-(v_M).
\]

By virtue of the fact, that \(c_3(u)\) is a shift element we have: \(r^+(u, v_i)c_3(u) = -r^-(v_i, u)c_3(v_i)\).

Using all above we finally obtain:

\[
[\hat{\tau}^c(u) - c_3^2(u), \hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)] = -2\left(\sum_{i=1}^{M} r^3(v_i, u)\right)\hat{L}_-(v_1)\hat{L}_-(v_2)\cdots\hat{L}_-(v_M)\hat{L}_3(u)
\]
\[ + \left( \sum_{i=1}^{M} r^{3}(v_{i}, u) \right)^{2} - \sum_{i=1}^{M} r^{-}(v_{i}, u)r^{+}(v_{i}, u) \]

\[ + 2c_{3}(u) \left( - \sum_{i=1}^{M} r^{3}(v_{i}, u) \right) \hat{L}_{-}(v_{1}) \hat{L}_{-}(v_{2}) \cdots \hat{L}_{-}(v_{M}) \]

\[ + 2 \left( \sum_{i=1}^{M} r^{-}(v_{i}, u) \hat{L}_{-}(v_{1}) \cdots \hat{L}_{-}(v_{i-1}) \hat{L}_{-}(u) \hat{L}_{-}(v_{i+1}) \cdots \hat{L}_{-}(v_{M}) \right) \hat{L}_{3}(v_{i}) \]

\[ - 2 \left( \sum_{i=1}^{M} r^{-}(v_{i}, u) \left( c_{0}(v_{i}) - c_{3}(v_{i}) + \sum_{j=1, j \neq i}^{M} r^{3}(v_{j}, v_{i}) \right) \right) \]

\[ \times \hat{L}_{-}(v_{1}) \cdots \hat{L}_{-}(v_{i-1}) \hat{L}_{-}(u) \hat{L}_{-}(v_{i+1}) \cdots \hat{L}_{-}(v_{M}) \]  \hspace{1cm} (B.3)\[ \]

From Eq. (B.3) we see that the Bethe vector \(|v_{1}v_{2}\cdots v_{M}\rangle\) is an eigen-vector for \(\hat{\tau}^{c}(u) - c_{3}^{2}(u)\) if and only if the summands of the type

\[ |v_{1}\cdots v_{i-1}u v_{i+1}v_{M}\rangle = \hat{L}_{-}(v_{1}) \cdots \hat{L}_{-}(v_{i-1}) \hat{L}_{-}(u) \hat{L}_{-}^{u}(v_{i+1}) \cdots \hat{L}_{-}^{u}(v_{M})|0\rangle \]

vanish in the resulting formula for all \(i \in 1, \ldots, M\). This happens when the coefficients before these vectors are zero. But this is true if the “generalized Bethe equations” (34) are satisfied. In this case, the direct calculation of the corresponding eigen-value of \(\hat{\tau}^{c}(u)\) using expressions (B.3) and (B.1) gives formula (35).

Theorem is proved.

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