Mathematical modeling of heating temperature mode for a heat exchange system of the type "pipe in pipe"

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Abstract. A mathematical model of heating of inhomogeneous medium “TEH-sand-air” was constructed and investigated as an initial-boundary value problem for the heat equation in polar coordinate system with boundary conditions that take into account the peculiarities of thermal processes at the boundaries of the inhomogeneous medium. A solution is given to the initial-boundary problem and an algorithm is proposed for calculating the thermal characteristics of the heating process and regular temperature conditions. The study is relevant for development of a computer virtual model of the heating process of an inhomogeneous medium in order to clearly demonstrate the heating process over a short period of time, as well as to calculate the corresponding thermal characteristics based on real experimental data.

1. Introduction
At present, the theory of modeling the thermal state of bodies of various shapes describing real objects for various technological processes is widely developed and used [1-12]. Quite a large number of variants of formulation of the heat conduction problem for bodies of various shapes are given in the works of A.V. Lykov, which considers the solution of the unsteady heat conduction equation (semi-infinite body, unbounded plate, solid cylinder, ball, hollow cylinder) by several methods (separation of variables, operational, Fourier and Hankel integral transforms). However, the mathematical description of thermal state of body for specific production conditions has significant differences in determining the shape of the body and the boundary conditions.

This article is devoted to study of temperature during the process of heating an inhomogeneous medium consisting of a thermoelectric heater (TEH) placed in a steel cylindrical pipe, sand which fills the space between the inner steel pipe and the outer copper pipe, and the air surrounding the outer pipe. This study is relevant for the studies described in articles [13-14] on development of a computer virtual model of heating process of an inhomogeneous medium in order to clearly demonstrate the heating process over a short period of time, as well as to calculate the corresponding thermal characteristics based on real experimental data.
In order to calculate the temperature mode in the process of heating the inhomogeneous medium “TEH-sand-air”, three experiments were carried out with different TEH electric capacities, and temperature was measured using a thermocouple at four points (on the surface of the outer and inner pipes and at two internal points) at successive points in time (every 5 milliseconds) from the beginning of the process until the temperature stabilizes. The thermocouple readings were recorded by an analog-to-digital converter and processed by the computer program Table Curve 2D.

This raises the following questions:
1. To find relationships between coefficients of the obtained formulas and the thermal characteristics of process: coefficients of thermal conductivity, heat capacity, and heat transfer.
2. Whether it is possible to find heat transfer coefficients from TEH to sand and heat transfer of sand to air.
3. What is the temperature dynamics in space and in time.

To answer these questions, the methodology of work [15] is used, consisting of the following steps:
I. Construction of a mathematical model of heating process in the form of an initial-boundary problem for the heat equation with boundary conditions that take into account the inhomogeneity of the medium.
II. The solution of the boundary problem by the method of separation of variables, where it is necessary to construct a basis of eigenfunctions that satisfy the boundary conditions.
III. Calculation of thermal characteristics of the process based on experimental data using the formulas of the stationary and regular modes.

The “TEH-sand-air” heating process under study differs qualitatively from the process considered in [13]. As a consequence, we obtain another initial-boundary value problem, where the basis of eigenfunctions cannot be found explicitly. The existence and completeness of the eigenfunctions of the corresponding Sturm-Liouville problem was proved in [16]. With its help, a formula for solving an initial-boundary value problem, and, in particular, a regular-mode formula, is derived. The regular mode determines the “acceleration curve” which is the temperature change curve from the beginning of the heating process until the moment of stabilization.

2. Construction of mathematical model
To build a mathematical model of the heating process of an inhomogeneous medium "TEH-sand-air" we give the following assumptions and notations.
1. Temperature $U(t, x)$ is a function of variables $t$ and $x$, where $t$ is time $t \geq 0$, $x$ is distance from the center of the inner pipe to the point where the temperature is measured $r \leq x \leq R$, $r > 0$, $r$ is the radius of the inner pipe, $R$ is the radius of the outer pipe; in the conducted experiments $r = 10$ mm, $R = 50$ mm. Therefore, the temperature depends only on the distance to the center of the inner pipe and it is constant along any cylindrical surface concentrically located relative to the inner pipe.
2. We denote the temperature on the surface of the inner pipe at time $t$ as $U(t, r)$, $U(t, r + 0)$ is the temperature of the sand at the boundary with the inner pipe, $U(t, x)$ is the temperature of the sand at the inner point $x$, $r < x < R$, $U(t, R - 0)$ is the temperature of sand at the border with the outer pipe, $U(t, R)$ is the air temperature near the outer pipe. In this case, the temperature of the copper pipe is neglected due to the large value of copper coefficient of thermal conductivity.
3. An inhomogeneous medium is characterized by the following constants: $c_\rho$ is specific heat capacity of the inner (steel) pipe, $c_R$ specific heat capacity of the outer (copper) pipe, $Q$ is amount of heat released from a unit area of the inner pipe in a unit period of time, $h_r$ is heat transfer coefficient from the inner pipe to sand, $h_R$ is heat transfer coefficient of sand to air, $\alpha_R$ is air heat loss coefficient near the outer pipe, $k$, $\rho$, $c$ is heat conductivity, density and specific heat of sand, $a^2 = \frac{k}{c\rho}$ thermal diffusivity of sand.
The mathematical model of the heating process will consist of the balance equations in the inner space of sand, on the inner surface of pipe, at the boundaries of “TEH-sand”, “sand-air”, in the air near the external pipe.

The equation of heat balance in the internal space of sand, according to the Fourier law, [13], has the form:

\[
\frac{1}{a^2} \frac{\partial u}{\partial t}(t, x) = \left( \frac{\partial^2 u}{\partial x^2}(t, x) + \frac{1}{x} \frac{\partial u}{\partial x}(t, x) \right), \quad t > 0, \quad r < x < R,
\]

where the right-hand side is the Laplace operator in the polar coordinate system with the radius-vector x.

On the surface of the inner pipe, we set the heat balance as

\[
c_r \frac{\partial u}{\partial t}(t, r) - k \frac{\partial u}{\partial x}(t, r + 0) = Q, \quad t > 0,
\]

where \(c_r \frac{\partial u}{\partial t}(t, r)\) is the amount of heat for inner pipe warming, \(-k \frac{\partial u}{\partial x}(t, r + 0)\) is the amount of heat given to sand.

At the “TEH-sand” boundary, the heat balance condition is:

\[-k \frac{\partial u}{\partial x}(t, r) = h_r [U(t, r) - U(t, r + 0)], \quad t > 0\]

At the “sand-air” boundary we have:

\[-k \frac{\partial u}{\partial x}(t, R - 0) = h_R [U(t, R - 0) - U(t, R)], \quad t > 0\]

The heat balance in the air near the outer pipe is set by condition

\[-k \frac{\partial u}{\partial x}(t, R - 0) = c_R \frac{\partial u}{\partial t}(t, R) + \alpha_R U(t, R),\]

where \(c_R \frac{\partial u}{\partial t}(t, R)\) is the amount of heat for warming of the outer pipe, \(\alpha_R U(t, R)\) is the amount of heat, lost by air near the outer pipe.

At the initial moment, the inhomogeneous medium has an ambient temperature:

\[U(0, x) = u_o, \quad r \leq x \leq R\]

Thus, the mathematical model of heating process is an initial-boundary problem for the differential heat conduction equation (1) in polar coordinate system with boundary conditions (2), (3) on the inner boundary, (4), (5) on the outer boundary, and the initial condition (6). By solution \(U(t, x), t \geq 0, r \leq x \leq R\) of the initial-boundary problem (1) - (6) a temperature mode is defined during the heating of the inhomogeneous medium “TEH-sand-air”. The function \(U(t, x)\) at each \(t > 0\) have discontinuities in points \(x = r\) and \(x = R\).

2.1 Stationary mode

Stationary mode, as in [15], is called stabilization of temperature during heating:

\[\bar{U}(x) = \lim_{t \to +\infty} U(t, x), \quad r \leq x \leq R,\]

where \(\bar{U}(x)\) is temperature at distance x from the center of TEH when stabilized. In (1) - (5), passing to the limit at \(t \to +\infty\), we derive that the function \(\bar{U}(x)\) should meet the following conditions:

\[\frac{d^2 \bar{u}(x)}{dx^2} + \frac{1}{x} \frac{d \bar{u}(x)}{dx} = 0, \quad r < x < R,\]
\[ \begin{align*}
-k \frac{d\tilde{u}}{dx}(r + 0) = Q, \quad &-k \frac{d\tilde{u}}{dx}(r + 0) = h_r [\tilde{U}(r) - \tilde{U}(r + 0)], \\
-k \frac{d\tilde{u}}{dx}(R - 0) = \alpha_R \tilde{U}(R), \quad &-k \frac{d\tilde{u}}{dx}(R - 0) = h_R [\tilde{U}(R - 0) - \tilde{U}(R)].
\end{align*} \]

From these expressions we obtain \( \tilde{U}(x) \):
\[ \tilde{U}(x) = \begin{cases} 
\tilde{U}(r + 0) + \frac{Q}{h_r}, & x = r, \\
\left(1 + \frac{\alpha_R}{h_R}\right) \frac{rQ}{\alpha_R} + \frac{rQ}{k} \ln \frac{R}{x}, & r < x < R, \\
\frac{rQ}{\alpha_R}, & x = R.
\end{cases} \]

Using the formula (7), basing on the experimental data, one can obtain the values for thermal constants \( \alpha_R, k, h_r, h_R \). Namely, we assume that the given values are temperatures in four measurement points at stabilization: \( \tilde{U}(r), \tilde{U}(r_2), \tilde{U}(r_3), \tilde{U}(R) \), where \( r < r_2 < r_3 < R \). By assuming \( \tilde{U}(r) = \tilde{U}(r), \tilde{U}(r_2) = \tilde{U}(r), \tilde{U}(r_3) = \tilde{U}(r_3), \tilde{U}(R) = \tilde{U}(R) \), in relation to the unknown values \( \alpha_R, k, h_r, h_R \) we obtain the following system of linear equations:
\[ \begin{align*}
&\left(1 + \frac{\alpha_R}{h_R}\right) \frac{rQ}{\alpha_R} + \frac{rQ}{k} \ln \frac{R}{r} + \frac{Q}{h_r} = \tilde{U}(r), \\
&\left(1 + \frac{\alpha_R}{h_R}\right) \frac{rQ}{\alpha_R} + \frac{rQ}{k} \ln \frac{R}{r_2} = \tilde{U}(r_2), \\
&\left(1 + \frac{\alpha_R}{h_R}\right) \frac{rQ}{\alpha_R} + \frac{rQ}{k} \ln \frac{R}{r_3} = \tilde{U}(r_3), \\
&\frac{rQ}{\alpha_R} = \tilde{U}(R).
\end{align*} \]

Further we find the unknown values:
\[ \begin{align*}
\alpha_R &= \frac{rQ}{\tilde{U}(r)}, \\
k &= \frac{rQ \ln \frac{r_3}{r_2} - \tilde{U}(r_2) - \tilde{U}(r_3)}{\tilde{U}(r_2) - \tilde{U}(r_3)}, \\
h_R &= \left(r^2 - \frac{\tilde{U}(r_2) \ln \frac{r_3}{r_2} - \tilde{U}(r_3)}{\ln \frac{r_3}{r_2} - \frac{1}{\alpha_R}}\right)^{-1}, \\
h_r &= \left(\frac{\tilde{U}(r)}{r} - \frac{1}{\alpha_R} \ln \frac{r_3}{r_2} - \frac{r \ln \frac{R}{r}}{\frac{1}{h_R} + \frac{r \ln \frac{R}{r}}{r}}\right)^{-1}.
\end{align*} \]

The solution of the problem (1) - (6) we find in the following form
\[ U(t, x) = \tilde{U}(x) + \sum_{i=1}^{\infty} A_i V_i(x) e^{-\lambda_i a^2 t}, \]
where \( 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \) and functions are defined as eigenvalues and eigenfunctions of the Sturm-Liouville problem in the form of Kneser [13]:
\[ \begin{align*}
\frac{d^2V}{dx^2}(x) + \frac{1}{x} \frac{dV}{dx}(x) + \lambda V(x), & \quad r < x < R, \\
\lambda c_r a^2 V(r) + k \frac{dV}{dx}(r + 0) = 0, \quad &-k \frac{dV}{dx}(r + 0) = h_r (V(r) - V(r + 0)), \\
(-\lambda c_r a^2 + \alpha_R)V(R) = -k \frac{dV}{dx}(R - 0), \quad &-k \frac{dV}{dx}(R - 0) = h_R (V(R - 0) - V(R)).
\end{align*} \]
\[ H = \left\{ f(x) : \int_{r}^{R} f^2(x) \, dx < +\infty, \ |f(r)| + |f(R)| < \infty \right\} \]

with dot product
\[
\langle f, g \rangle = \int_{r}^{R} x f(x) g(x) \, dx + \frac{r a^2 c_r}{k} f(r) g(r) + \frac{R a^2 c_R}{k} f(R) g(R). 
\]

In other words, any function \( f(x) \in H \) is uniquely representable as a series in eigenfunctions of the problem (10):
\[
f(x) = \sum_{i=1}^{\infty} B_i V_i(x),
\]
where \( B_i = \langle f, V_i \rangle, \langle V_i, V_j \rangle = 1, i = 1, 2, \ldots \) and \( \langle V_i, V_j \rangle = 0 \) at \( i \neq j \). At the same time, for the first eigenvalue \( \lambda_1 \) there exists an inequality:
\[
\frac{a}{c_R} \leq \lambda_1 \leq \frac{4a}{c_R}.
\]

By assuming \( f(x) = u_0 - \bar{U}(x) \), we find the coefficients \( A_i, i = 1, 2, \ldots \) of the representation (9):
\[
A_i = \langle u_0 - \bar{U}, V_i \rangle, \quad i = 1, 2, \ldots 
\]

In this case the series in the right part of the formula (9) at each \( t > 0 \) converges in the common sense.

2.2 Regular mode

According to the terminology of [17], the regular temperature regime is determined by the approximate formula
\[
U(t, x) \approx \bar{U}(x) + A_1 V_1(x) e^{-\lambda_1 a^2 t},
\]
where \( \lambda_1 \) is the first eigenvalue, and \( V_1(x) \) is the corresponding normalized eigenfunction of the Sturm-Liouville problem (10); they can be found using approximately numerical-analytical methods. The coefficient \( A_1 \) is determined by the formula \( A_1 = \langle u_0 - \bar{U}, V_1 \rangle \).

Thus, you can set the following algorithm for calculating the regular temperature mode during the process of heating the inhomogeneous medium "TEH-sand-air":

1. From experimental data \( \bar{U}(r), \bar{U}(r_2), \bar{U}(r_3), \bar{U}(R) \), obtained during temperature measurements at four points during stabilization, we find the approximate values of thermal constants using formulas (8).
2. Using numerical-analytical methods, we find the approximate values of the first eigenvalue \( \lambda_1 \) of the Sturm-Liouville problem (10) and the corresponding normalized eigenfunction \( V_1(x) \).
3. We find the stationary mode \( \bar{U}(x) \) from the formula (7).
4. We calculate the coefficient \( A_1 \) from the formula
\[
A_1 = \int_{r}^{R} x(u_0 - \bar{U}(x)) V_1(x) \, dx + \frac{r a^2 c_r}{k} (u_0 - \bar{U}(r)) V_1(r) + \frac{R a^2 c_R}{k} (u_0 - \bar{U}(R)) V_1(R).
\]
5. The regular mode is calculated from the formula
\[
U(t, x) \approx \bar{U}(x) + A_1 V_1(x) e^{-\lambda_1 a^2 t}, \quad r \leq x \leq R, \quad t \geq 0
\]
3. Conclusions

Thus, we constructed and investigated a mathematical model of the process of heating an inhomogeneous medium “TEH-sand-air” in the form of an initial-boundary value problem for the heat equation in a polar coordinate system with boundary conditions which take into account the peculiarities of thermal processes at the boundaries of an inhomogeneous medium was. We present a solution to the initial-boundary problem and propose an algorithm for calculating the thermal characteristics of the heating process and a regular temperature regime using numerical-analytical methods for solving the Sturm-Liouville problem in the form of Kneser.

This study is relevant for the development of a computerized virtual model of the heating process of an inhomogeneous medium in order to clearly demonstrate the heating process in a short period of time, as well as to calculate the corresponding thermal characteristics based on actual experimental data.

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