POLYCHROMATIC COLORINGS OF 1-REGULAR AND 2-REGULAR SUBGRAPHS OF COMPLETE GRAPHS

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Abstract. If $G$ is a graph and $\mathcal{H}$ is a set of subgraphs of $G$, we say that an edge-coloring of $G$ is $\mathcal{H}$-polychromatic if every graph from $\mathcal{H}$ gets all colors present in $G$ on its edges. The $\mathcal{H}$-polychromatic number of $G$, denoted $\text{poly}_\mathcal{H}(G)$, is the largest number of colors in an $\mathcal{H}$-polychromatic coloring. In this paper we determine $\text{poly}_\mathcal{H}(G)$ exactly when $G$ is a complete graph on $n$ vertices, $q$ is a fixed nonnegative integer, and $\mathcal{H}$ is one of three families: the family of all matchings spanning $n-q$ vertices, the family of all 2-regular graphs spanning at least $n-q$ vertices, and the family of all cycles of length precisely $n-q$. There are connections with an extension of results on Ramsey numbers for cycles in a graph.

1. Introduction

If $G$ is a graph and $\mathcal{H}$ is a set of subgraphs of $G$, we say that an edge-coloring of $G$ is $\mathcal{H}$-polychromatic if every graph from $\mathcal{H}$ has all colors present in $G$ on its edges. The $\mathcal{H}$-polychromatic number of $G$, denoted $\text{poly}_\mathcal{H}(G)$, is the largest number of colors in an $\mathcal{H}$-polychromatic coloring. If an $\mathcal{H}$-polychromatic coloring of $G$ uses $\text{poly}_\mathcal{H}(G)$ colors, it is called an optimal $\mathcal{H}$-polychromatic coloring of $G$.

Alon et. al. [1] found a lower bound for $\text{poly}_\mathcal{H}(G)$ when $G = Q_n$, the $n$-dimensional hypercube, and $\mathcal{H}$ is the family of all subgraphs isomorphic to $Q_d$, where $d$ is fixed. Offner [12] showed this lower bound is, in fact, the exact value for all $d$ and sufficiently large $n$. Bialostocki [3] showed that if $d = 2$, then the polychromatic number is 2 and that any optimal coloring uses each color about half the time. Goldwasser et. al. [10] considered the case when $\mathcal{H}$ is the family of all subgraphs isomorphic to $Q_d$ minus an edge or $Q_d$ minus a vertex.

Bollobas et. al. [4] treated the case where $G$ is a tree and $\mathcal{H}$ is the set of all paths of length at least $r$, where $r$ is fixed. Goddard and Henning [9] considered vertex colorings of graphs such that each open neighborhood gets all colors.

For large $n$, it makes sense to consider $\text{poly}_\mathcal{H}(K_n)$ only if $\mathcal{H}$ consists of sufficiently large graphs. Indeed, if the graphs from $\mathcal{H}$ have at most a fixed number $s$ of vertices, then $\text{poly}_\mathcal{H}(K_n) = 1$ for sufficiently large $n$ by Ramsey’s theorem, since even with only two colors there exists a monochromatic clique with $s$ vertices.

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Axenovich et. al. [2] considered the case where \( G = K_n \) and \( \mathcal{H} \) is one of three families of spanning subgraphs: perfect matchings (so \( n \) must be even), 2-regular graphs, and Hamiltonian cycles. They determined \( \text{poly}_{\mathcal{H}}(K_n) \) precisely for the first of these and to within a small additive constant for the other two. In this paper, we determine the exact \( \mathcal{H} \)-polychromatic number of \( K_n \), where \( q \) is a fixed nonnegative integer and \( \mathcal{H} \) is one of three families of graphs: matchings spanning precisely \( n - q \) vertices, \((n-q)\)-cycles, and 2-regular graphs spanning at least \( n - q \) vertices (so \( q = 0 \)) gives the results of Axenovich et. al. in [2] without the constant.

This paper is organized as follows. We give a few definitions and state the main results in Section 2. We give some more definitions in Section 3. The optimal polychromatic colorings in this paper are all based on a type of ordering, and in Section 4 we state and prove the technical ordering lemmas we will need. In Section 5 we describe precisely the various ordered optimal polychromatic colorings of \( K_n \). In Section 6 we prove Theorem 2.1, a result about matchings. In Section 7 we use some classical results on Ramsey numbers for cycles to take care of polychromatic numbers 1 and 2 for cycles. In Section 8 we prove Theorem 2.6, a result about coloring cycles, and use some results on long cycles in the literature to prove a lemma we need. In Section 9 we give the rather long proofs of the three main lemmas we need. In Section 10 we show how our results can be reconstituted in a context which generalizes the classical results on Ramsey numbers of cycles presented in Section 7. In Section 11 we state a general conjecture of which most of our results are special cases.

2. Main Results

We call an edge coloring \( \varphi \) of \( K_n \) ordered if there exists an ordering \( v_1, v_2, \ldots, v_n \) of \( V(K_n) \) such that \( \varphi(v_i v_j) = \varphi(v_i v_m) \) for all \( 1 \leq i < j < m \leq n \). Moreover this coloring is simply-ordered if for all \( i < j < m \), \( \varphi(v_i v_m) = \varphi(v_j v_m) = a \) implies that \( \varphi(v_i v_m) = a \) for all \( i \leq t \leq j \). Simply-ordered colorings play a fundamental role in this paper. An ordered edge coloring \( \varphi \) induces a vertex coloring \( \varphi' \) on \( V(K_n) \) called the \( \varphi \)-inherited coloring, defined by \( \varphi'(v_i) = \varphi(v_i v_m) \) for \( i < m \leq n \) and \( \varphi'(v_n) = \varphi'(v_{n-1}) \). We can represent the induced vertex coloring \( \varphi' \) by the sequence \( c_1, c_2, \ldots, c_n \) of colors, where \( c_i = \varphi'(v_i) \) for each \( i \). A block in this sequence is a maximal set of consecutive vertices of the same color. If \( \varphi \) is simply-ordered then the vertices in each color class appear in a single block, so in that case, the number of blocks equals the number of colors.

Let \( q \) be a fixed nonnegative integer. We define four families of subgraphs of \( K_n \) as follows.

1. \( F_q(n) \) is the family of all matchings in \( K_n \) spanning precisely \( n - q \) vertices (so \( n - q \) must be even).
2. \( C_q(n) \) is the family of all cycles of length precisely \( n - q \).
3. \( R_q(n) \) is the family of all 2-regular subgraphs spanning at least \( n - q \) vertices.
4. \( C_q^∗(n) \) is the family of all cycles of length precisely \( n - q \) where \( n \) and \( q \) are such that \( \text{poly}_{C_q(n)}(K_n) \geq 3 \).
Further, let $\varphi_{F_q}(n) = \text{poly}_{F_q}(K_n)$, $\varphi_{C_q}(n) = \text{poly}_{C_q}(K_n)$, and $\varphi_{R_q}(n) = \text{poly}_{R_q}(K_n)$. Our main result is that for $F_q(n)$, $R_q(n)$, and $C_q^*(n)$ there exist optimal polychromatic colorings which are simply ordered, or almost simply ordered (except for $C_q(n)$ if $\varphi_{C_q}(n) = 2$). Once we know there exists an optimal simply ordered (or nearly simply ordered) coloring, it is easy to find it and to determine a formula for the polychromatic number. Our main results are the following.

**Theorem 2.1.** For all integers $q$ and $n$ such that $q$ is nonnegative and $n - q$ is positive and even, there exists an optimal simply-ordered $F_q$-polychromatic coloring of $K_n$.

**Theorem 2.2.** [2] If $n \geq 3$, then there exist optimal $R_0$-polychromatic and $C_0$-polychromatic colorings of $K_n$ which can be obtained from simply-ordered colorings by recoloring one edge.

**Theorem 2.3.** If $n \geq 4$, then there exist optimal $R_1$-polychromatic and $C_1$-polychromatic colorings of $K_n$ which can be obtained from simply-ordered colorings by recoloring two edges.

**Theorem 2.4.** Let $q \geq 2$ be an integer. If $n \geq q + 3$, then there exists an optimal simply-ordered $R_q$-polychromatic coloring of $K_n$. If $n \geq q + 4$, then there exists an optimal simply-ordered $C_q$-polychromatic coloring except if $n \in [2q + 2, 3q + 2]$ and $n - q$ is odd.

**Theorem 2.5.** Suppose $q \geq 2$ and $n \geq 6$

a) If $n - q$ is even then there exists a $C_q$-polychromatic 2-coloring of $K_n$ if and only if $n \geq 3q + 3$.

b) If $n - q$ is odd then there exists a $C_q$-polychromatic 2-coloring of $K_n$ if and only if $n \geq 2q + 2$.

Theorem 2.5 follows from results of Bondy and Erdős [5] and Faudree and Schelp [8].

The following result, which is needed for the proof of Theorem 2.4, may be of independent interest, so we state it as a theorem:

**Theorem 2.6.** Let $n$ and $j$ be integers with $4 \leq j \leq n$, and let $\varphi$ be an edge-coloring of $K_n$ with at least three colors so that every $j$-cycle gets all colors. Then every cycle of length at least $j$ gets all colors under $\varphi$.

The statements about cycles in Theorems 2.2, 2.5 can be used to get an extension of the result of Faudree and Schelp [8] in the following manner. Let $s$ and $t$ be integers with $t \geq 2$, $s \geq 3$, and $s \geq t$. The $t$-polychromatic cyclic Ramsey number $PR_t(s)$ is the smallest integer $N \geq s$ such that in any $t$-coloring of the edges of $K_N$ there exists an $s$-cycle whose edges do not contain all $t$ colors. Note that in the special case $t = 2$, this is the classical Ramsey number for cycles, the smallest integer $N$ such that in any 2-coloring of the edges of $K_N$ there exists a monochromatic $s$-cycle. These numbers were determined for all $s$ by Faudree and Schelp [8], confirming a conjecture of Bondy and Erdős [5].
Theorem 2.7. Let $\text{PR}_t(s)$ be the smallest integer $n \geq s \geq 3$ such that in any $t$-coloring of the edges of $K_n$ there exists an $s$-cycle whose edges do not contain all $t$ colors. If $t \geq 3$,

$$\text{PR}_t(s) = \begin{cases} 
    s, & \text{if } 3 < s \leq 3 \cdot 2^{t-3} \\
    s + 1, & \text{if } s \in [3 \cdot 2^{t-3} + 1, 5 \cdot 2^{t-2} - 2] \\
    s + 2, & \text{if } s \in [5 \cdot 2^{t-2} - 1, 5 \cdot 2^{t-1} - 4] \\
    s + \text{Round} \left( \frac{s-2}{2^{t-2}} \right), & \text{if } s \geq 5 \cdot 2^{t-1} - 3 
\end{cases}$$

where $\text{Round} \left( \frac{s-2}{2^{t-2}} \right)$ is the closest integer to $\frac{s-2}{2^{t-2}}$, rounding up if it is $\frac{1}{2}$ more than an integer.

### 3. Definitions

Recall that if $\varphi$ is an ordered edge coloring of $K_n$ with respect to the ordering $v_1, \ldots, v_n$ of its vertices, we say that $\varphi'$ is the $\varphi$-inherited coloring (or just inherited coloring) if it is the vertex coloring of $K_n$ defined by $\varphi'(v_i) = \varphi(v_i)$ for $1 \leq i < j \leq n$ and $\varphi'(v_n) = \varphi'(v_{n-1})$. Given an ordering of $V(K_n)$, any vertex coloring $\varphi'$ such that $\varphi'(v_{n-1}) = \varphi'(v_n)$ uniquely determines a corresponding ordered coloring. We define a color class $M_i$ of color $i$ to be the set of all vertices $v$ where $\varphi'(v) = i$. In this paper, we shall always think of the ordered vertices as arranged on a horizontal line with $v_i$ to the left of $v_j$ if $i < j$. We say that an edge $v_iv_m$, $i < m$ goes from $v_i$ to the right and from $v_m$ to the left. If $X$ is a (possibly empty) subset of $V(K_n)$, we say that the edge-coloring $\varphi$ of $K_n$ is

- **X-constant** if for any $v \in X$, $\varphi(vu) = \varphi(vw)$ for all $u, w \in V \setminus X$.
- **X-ordered** if it is X-constant and the vertices of $X$ can be ordered $x_1, \ldots, x_m$ such that for each $i = 1, \ldots, m$, $\varphi(x_ix_p) = \varphi(x_iw) = \varphi(x_i)$ for all $i < p \leq m$ and all $w \in V \setminus X$,.

If $Z$ is a nonempty subset of $V(K_n)$ we say $\varphi$ is

- **Z-quasi-ordered** if
  - (1) $\varphi$ is X-constant
  - (2) Each vertex $v_i$ in $Z$ is incident to precisely $n - 2$ edges of one color, which we call the main color of $v_i$, and one edge $v_iv_j$ of another color, where $v_j \in Z$. If that other color is $t$, then $v_j$ is incident to precisely $n - 2$ edges of color $t$.

It is not hard to show that there are only two possibilities for the set $Z$ in a Z-quasi-ordered coloring:

1. $|Z| = 3$, the three vertices in $Z$ have different main colors, and there is one edge in $Z$ of each of these colors
2. $|Z| = 4$, with two vertices $u, v$ in $Z$ with one main color, say $i$ and two vertices $y, z$ in $Z$ with another main color, say $j$, and $\varphi(uv) = \varphi(uy) = \varphi(vz) = i, \varphi(yz) = \varphi(yv) = \varphi(zu) = j$.

- **quasi-ordered** if it is Z-quasi-ordered and $\varphi$ restricted to $V \setminus Z$ is ordered
- **quasi-simply ordered** if it is Z-quasi-ordered and $\varphi$ restricted to $V \setminus Z$ is simply ordered.
• nearly X-ordered if it is Z-quasi-ordered and the restriction of $\varphi$ to $V(K_n) \setminus Z$ is $T$-ordered for some (possibly empty) subset $T$ of $V(K_n) \setminus Z$ and $X = Z \cup T$. (If $\varphi$ is nearly X-ordered then one or two edges could be recolored to get an X-ordered coloring.)

It is easy to check that if $\varphi$ is quasi-ordered (quasi-simply-ordered) for some set $Z$ then if $|Z| = 3$ one edge can be recolored, and if $|Z| = 4$, then two edges can be recolored to get an ordered (simply-ordered) coloring.

The maximum monochromatic degree of an edge coloring of $K_n$ is the maximum number of edges of the same color incident with a single vertex. If the maximum monochromatic degree of a coloring is $d$, and the vertex $v$ is incident with $d$ edges of color $t$, and the other $n - 1 - d$ edges incident with $v$ have color $s$, we say $v$ is a $t$-max vertex and also a $(t, s)$-max vertex with majority color $t$ and minority color $s$.

We extend the notion of inherited to quasi-ordered colorings as follows. If $\varphi$ is a quasi-ordered coloring with $\psi$ the ordered coloring which is a restriction of $\varphi$ to $V \setminus Z$, we define $\varphi'$, the $\varphi$-inherited coloring, by letting $\varphi'(x)$ equal the main color of $x$ if $x \in Z$ and $\varphi(y) = \psi'(y)$ if $y \not\in Z$. We think of the vertices in $Z$ preceding those not in $Z$, in the order left to right, and if $|Z| = 4$ we list two vertices in $Z$ with the same main color first, then the other two vertices with the same main color.

4. Ordering Lemmas

Let $\varphi$ be an ordered edge coloring of $K_n$ with vertex order $v_1, v_2, \ldots, v_n$, colors $1, \ldots, k$, and $\varphi'$ be the inherited coloring of $V(K_n)$. For each $t \in [k]$ and $j \in [n]$, let $M_t$ be a color class $t$ of $\varphi'$ and $M_t(j) = M_t \cap \{v_1, v_2, \ldots, v_j\}$. The next Lemma is a key structural lemma that characterizes ordered polychromatic colorings.

Lemma 4.1. Let $\varphi : E(K_n) \to [k]$ be an ordered or quasi-ordered coloring with vertex order $v_1, v_2, \ldots, v_n$.

Then the following statements hold:

(I) $\varphi$ is $F_q$-polychromatic $\iff \forall t \in [k] \exists j \in [n]$ such that $|M_t(j)| > \frac{i+q}{2}$,

(II) $\varphi$ is $C_q$-polychromatic $\iff \forall t \in [k]$ either

(a) $\exists j \in [q + 1, n - 1]$ such that $|M_t(j)| \geq \frac{i+q}{2}$ or

(b) $q = 0$, $\varphi$ is $Z$-quasi-ordered with $|Z| = 3$ and $t$ is the color of some edge in $Z$ or

(c) $q = 1$, $\varphi$ is $Z$-quasi-ordered with $|Z| = 4$ and $t$ is the color of some edge in $Z$.

(III) $\varphi$ is $R_q$-polychromatic $\iff \forall t \in [k]$ either

(a) $\exists j \in [n]$ such that

(i) $|M_t(j)| > \frac{i+q}{2}$ or

(ii) $|M_t(j)| = \frac{i+q}{2}$ and $j \in \{2 + q, n - 2\}$ or

(iii) $|M_t(j)| = \frac{i+q}{2}$ and $|M_t(j + 2)| = \frac{i+q+2}{2}$ where $j \in [4 + q, n - 3]$.

(b) $q = 0$, $\varphi$ is $Z$-quasi-ordered and $t$ is the color of some edge in $Z$

(c) $q = 1$, $\varphi$ is $Z$-quasi-ordered with $|Z| = 3$ and $t$ is the color of some edge in $Z$.
Let $x$ there is an edge of $H$ edge in $Z$ is not $m$ with an endpoint in $H \in \{Z\}$.

**Proof.** Note that to prove the lemma, it is sufficient to consider an arbitrary color $t$ and show for $H \in \{F_q, C_q, R_q\}$ and for each $H \in H$, that the given respective conditions are equivalent to $H$ containing an edge of color $t$.

**(I)** Let $j$ be an index such that $|M_t(j)| = m_j > (j + q)/2$ and let $H$ be a 1-factor. Let $x_1, \ldots, x_m$ be the vertices of $M_t$ in order and let $y_1, \ldots, y_{m-j}$ be the other vertices of $\{v_1, v_2, \ldots, v_j\}$ in order. Since $j - m_j < \frac{j-q}{2}$ and $m_j - q > \frac{j-q}{2}$, then at least one edge of $H$ with an endpoint in $M_t(j)$ must go to the right, and thus, have color $t$.

On the other hand, by way of contradiction, assume that for each $j \in [n]$, $|M_t(j)| \leq (j + q)/2$. Letting $m = |M_t|$, we have $m \leq (n + q)/2$. Consider a 1-factor that spans all vertices except for $q$ vertices in $M_t$. Let $x_1, \ldots, x_{m-q}$ be the $m - q$ vertices remaining from $M_t$ in order and let $y_1, \ldots, y_{n-m}$, be the vertices outside of $M_t$ in order. Note that since $m \leq (j + q)/2$, it follows that $n - m \geq m - q$ since if $n - m < m - q$ then $n < 2m - q$ and so $j > n$ which is impossible. Now, let $H$ consist of the edges $x_1 y_1, x_2 y_2, \ldots, x_{m-q} y_{m-q}$ and a perfect matching on $\{y_{m-q+1}, \ldots, y_{n-m}\}$ (if this set is non-empty). We will show that $y_i$ precedes $x_i$ in the order $v_1, v_2, \ldots, v_n$ for each $i \in [m - q]$, so $H$ has no edge of color $t$.

By way of contradiction, assume $x_i$ precedes $y_i$ for some $i \in [m - q]$. Letting $j = 2i + 1 + q$, $y_i$ cannot be among the first $j$ vertices in the order $v_1, v_2, \ldots, v_n$, because if it were there would be at least $i + q$ vertices of color $t$ among these $j$ vertices, so a total of at least $2i + q > j$ vertices. Hence

$$\frac{j + q}{2} = \frac{2i + 2q - 1}{2} < i + q \leq |M_t(j)| \leq \frac{j + q}{2}$$

which is impossible. Hence $y_i$ precedes $x_i$ for each $i$ and $\varphi$ is not $F_q$ polychromatic.

**(II)** If $t$ is a color such that (a) holds with strict inequality, the argument in (I) shows there is an edge of $H$ with color $t$. If $|M_t(j)| = \frac{j+q}{2}$ for some $j \in [q + 1, n - 1]$ and every edge in $H$ incident to a vertex in $M_t(j)$ goes to the left then, since each of these edges has its other vertex not in $M_t(j)$, $H$ contains $\frac{j+q}{2}$ vertices in $M_t(j)$ and the same number not in $M_t(j)$. If $\frac{j+q}{2} = 1$, then the vertex in $M_t(j)$ is incident with at least one edge which goes to the right, and if $\frac{j+q}{2} > 1$ then $H$ contains a 2-regular subgraph, which is impossible because an $n - q$ cycle can’t have a 2-regular subgraph on less than $n - q$ vertices.

If $t$ is such that (b) holds, then note that $t$ must be the main color of a vertex in $Z$ and that the cycle must contain 2 edges incident with each vertex in $Z$. Any choice of these edges will contain an edge of color $t$ since only one edge incident with each vertex in $Z$ is not the main color of that vertex.

If $t$ is such that (c) holds, then note that $t$ must be the main color of a vertex in $Z$ and any cycle on $n - 1$ vertices must contain 2 edges incident with at least three of the four vertices in $Z$. Any choice of these edges will contain an edge of color $t$ since only one edge incident with each vertex in $Z$ is not the main color of that vertex.

On the other hand, suppose that for each $j \in [q + 1, n - 1]$, $|M_t(j)| = m < \frac{j+q}{2}$ and $\varphi$ is not $Z$-quasi-ordered with $t$ a main color. In particular, when $j = n - 2$, we have that $|M_t(j)| = m < \frac{j+q}{2} - 1$. Consider a cycle that spans all vertices except for $q$ vertices in $M_t$. Let $x_1, \ldots, x_{m-q}$ be the other $m - q$ vertices in $M_t$ in order and $y_1, \ldots, y_{n-m}$ be the vertices outside of $M_t$ in order. Note that if $m < \frac{j+q}{2}$, then $n - m > m - q$ since $n - m \leq m - q \implies j > n$.
which is impossible. Consider the cycle \( y_1x_1y_2x_2 \cdots y_{m-q}x_{m-q}y_{m-q+1} \cdots y_{n-m}y_1 \). Suppose \( y_i \) is to the right of \( x_i \) for some \( i \). Then at most \( i \) of the first \( j = 2i + q \) vertices are not in \( M_t(j) \), so \( |M_t(j)| \geq i + q = \frac{j+q}{2} \), which is impossible. Hence \( y_i \) and \( y_{i+1} \) are to the left of \( x_i \) for each \( 1 \leq i \leq m \), all edges of \( H \) incident to \( M_t \) go to the left, and thus are not of color \( t \).

**Observation.** If \( H \) is a 2-regular subgraph that has no edge of color \( t \), and \( M \) is any subset of \( M_t \), then all edges of \( H \) incident to \( M \) go to the left, so at most half the vertices in \( H \) are in \( M_t \) and if \( |M_t(j)| = \frac{j+q}{2} \), then of the first \( j \) vertices, precisely \( j - q \) are in \( H \), precisely half of these in \( M_t \), and if \( j - q \geq 4 \) then these \( j - q \) vertices induce a 2-regular subgraph of \( H \).

(iii) Let \( j \) be an index such that (iii)a(i), (ii), or (iii) holds. Assume first that (i) holds, i.e., that \( |M_t(j)| > \frac{j+q}{2} \) and let \( H \) be a 2-factor. Then the argument given in (I) shows that at least one edge of \( H \) with an endpoint in \( M_t(j) \) must go to the right, and thus, have color \( t \). Assume that (ii) holds. If \( j = 2 + q \), then \( M_t \) contains \( q + 1 \) of the first \( q + 2 \) vertices, so \( H \) contains a vertex in \( M_t \) which has an edge that goes to the right, so there is an edge of color \( t \) in \( H \). If \( j = n - 2 \) and \( H \) has no edges of color \( t \), then (by the previous observation) the subgraph of \( H \) induced by \([n - 2]\) is a 2-factor. Since the remaining two vertices do not form a cycle, \( H \) is not a 2-factor, a contradiction. Finally, assume that (iii) holds. If \( H \) does not have an edge of color \( t \), then by the previous observation, \( H \) has a 2-regular subgraph spanning \( j - q + 2 \) vertices, which has a 2-regular subgraph spanning \( j - q \) vertices, which is impossible.

If (iii)b or (iii)c holds, by the argument for (II), \( H \) has an edge of color \( t \).

On the other hand, suppose that none of (iii)a, (iii)b, or (iii)c hold. We shall construct a 2-factor that does not have an edge of color \( t \). If \( |M_t(j)| < \frac{j+q}{2} \) for each \( j \in [q + 1, n - 1] \), then there is a cycle with no color \( t \) edge as described in (II). If not, let \( i_1, i_2, \ldots, i_k \) be the values of \( j \) in \([4 + q, n - 3]\) for which \( |M_t(j)| = \frac{j+q}{2} \). Since (iii)a(iii) is not satisfied, \( i_{q+1} - i_q \) is at least 4 and even for \( q = 1, 2, \ldots, k - 1 \). As before, suppose there are \( m \) vertices of color \( t \). Let \( x_1, x_2, \ldots, x_{m-q} \) be the last \( m - q \) of these, in order, and let \( y_1, y_2, \ldots, y_{n-m} \) be the other vertices, in order. Note that since \( m \leq \frac{n-q}{2} \) we have \( m - q \leq \frac{n-q}{2} \) and \( n - m \geq \frac{n-q}{2} \). For each \( q \) in \([1, k - 1]\), moving left to right within the interval \([i_q + 1, i_{q+1}]\), there are always more \( y \)'s than \( x \)'s (except an equal number of each at the end of the interval), since otherwise there would have been another value of \( j \) between \( i_q \) and \( i_{q+1} \) where \( |M_t(j)| = \frac{j+q}{2} \). Form an \((i_{q+1} - i_q)\)-cycle by alternately taking \( y \)'s and \( x \)'s, starting with the \( y \) with the smallest subscript. Also form an \( i_1 - q \) cycle using the first \( \frac{i_1 - q}{2} \) \( y \)'s and the same number of \( x \)'s, and an \( n - i_k \) cycle at the end, first alternating the \( y \)'s and \( x \)'s, putting any excess \( y \)'s at the end.

**Lemma 4.2.** Let \( \mathcal{H} \in \{F_q, R_q, C_q\} \). If there exists an ordered (quasi-ordered) \( \mathcal{H} \)-polychromatic coloring of \( K_n \) with \( k \) colors, then there exists one which is simply-ordered (quasi-simply-ordered) with \( k \) colors.

**Proof.** Let \( V(K_n) = [n] \) with the natural order. If \( c' \) is a coloring of \([n]\), a **block of** \( c' \) is a maximal interval of integers from \([n]\) which all have the same color. So a simply-ordered \( k \)-polychromatic coloring has precisely \( k \) blocks. We define a **block shift operation** as follows.
Assume that $t \in [k]$ is a color for which there are at least 2 blocks. Let $j(t) = j$ be the smallest integer so that $M_t(j) > (j + q)/2$ if such exists. If there is a block $[m, s]$ in $M_t$ where $m > j$, delete this block, then take the color of the last vertex in the remaining sequence, and add $s - m + 1$ more vertices with this color at the end of the sequence. If each block of color $t$ has its smallest element less than or equal to $j$, consider the block $B$ of color $t$ that contains $j$ and consider another block $B_1$ of color $t$ that is strictly to the left of $B$. Form a new coloring by “moving” $B_1$ next to $B$. We see that the resulting coloring has at least one less block.

Let $c$ be an ordered (quasi-ordered) $F_q$-polychromatic coloring of $K_n$ on vertex set $[n]$ with $k$ colors such that the inherited vertex coloring $c'$ has the smallest possible number of blocks. Assume that color $t$ has at least 2 blocks. Let $j(t) = j$ be the smallest integer so that $M_t(j) > (j + q)/2$. Such $j$ exists by Lemma 4.1(I) and the color of $j$ is $t$. Apply the block shifting operation. The condition from part 1 of Lemma 4.1 is still valid for all color classes, so the new coloring is $F_q$-polychromatic using $k$ colors. This contradicts the choice of $c$ having the smallest number of blocks.

If $c$ is an ordered (quasi-ordered) $C_q$-polychromatic coloring of $K_n$, an argument very similar to the one above shows if $[\text{I.I}][\text{a}][\text{b}][\text{c}]$ or $[\text{a}][\text{b}][\text{c}]$ hold, there exists a simply-ordered (quasi-simply-ordered) coloring that uses the same number of colors and that is $C_q$-polychromatic.

Finally, let $c$ be an ordered $(X$-quasi-ordered) $R_q$-polychromatic coloring of $K_n$ on vertex set $[n]$ with $k$ colors such that the inherited vertex coloring $c'$ has the minimum possible number of blocks. Assume that $t \in [k]$ is a color for which there are at least 2 blocks. If $[\text{III}][\text{b}]$ or $[\text{III}][\text{c}]$ hold, then the block shifting operation gives a coloring that is still $R_q$-polychromatic with the same number of colors and fewer blocks.

Thus, by Lemma 4.1(III) there exists $j$ such that

1. $|M_t(j)| > (j + q)/2$ or
2. $M_t(2 + q) = 1 + q$ or
3. $|M_t(n - 2)| = (n + q - 2)/2$ or
4. $|M_t(n - 1)| = (n + q - 1)/2$ or
5. $|M_t(j)| = (j + q)/2$ and $|M_t(j + 2)| = (j + q + 2)/2$ and $4 + q \leq j \leq n - 3$.

If $[1]$ holds, then we apply the block shifting operation and observe, as in the case of $F_q$, that the resulting coloring is still $R_q$-polychromatic with the same number of colors and fewer blocks. The case when $[2]$ applies is similar.

Assume neither $[1]$ nor $[2]$ holds. If $[3]$ holds then, since $c'(v_{n-1}) = c'(v_n)$, neither $v_{n-1}$ nor $v_n$ can have color $t$. Hence there is another block of color $t$ vertices to the left of the one containing $v_{n-2}$, so we can do a block shift operation to reduce the number of blocks, a contradiction.

The same argument works if $[4]$ holds.

Finally, assume that none of $[1][4]$ holds, but $[5]$ holds. This implies that $c'(j) = c'(j + 2) = t$ and $c'(j + 1) = u \neq t$. Now define $c''$ by $c''(i) = c'(i)$ if $i \not\in \{j + 1, j + 2\}$, $c''(j + 1) = t$, and $c''(j + 2) = u$. Clearly $c''$ has at least one fewer block than $c'$. Since $j + q + 1$ is odd, the
only situation where \( c'' \) would not be \( R_q \)-polychromatic is if \( M_u(j + 1) > \frac{j + q + 1}{2} \). However, then \( |M_u(j - 1)| = |M_u(j + 1)| - 1 > \frac{j + q - 1}{2} \), so \( c'' \) is \( R_q \)-polychromatic after all.

5. Optimal Polychromatic Colorings

The seven following colorings are all optimal \( F_q, R_q \), or \( C_q \) polychromatic colorings for various values of \( q \) and \( n \). Each of them is simply-ordered or quasi-simply-ordered. We describe the color classes for each, and give a formula for the polychromatic number \( k \) in terms of \( q \) and \( n \).

5.1. \( F_q \)-polychromatic coloring \( \varphi_{F_q} \) of \( E(K_n) \) (even \( n - q \geq 2 \)). Let \( q \) be nonnegative and \( n - q \) positive and even with \( k \) a positive integer such that

\[
(q + 1)(2^k - 1) \leq n < (q + 1)(2^{k+1} - 1).
\]

Let \( \varphi_{F_q} \) be the simply-ordered edge \( k \)-coloring with colors 1, 2, \ldots, \( k \) and inherited vertex \( k \) coloring of \( \varphi'_{F_q} \) with successive color classes \( M_1, M_2, \ldots, M_k \), moving left to right such that \( |M_i| = 2^{i-1}(q + 1) \) if \( i < k \) and \( |M_k| = n - \sum_{i=1}^{k-1} |M_i| = n - (2^{k-1} - 1)(q + 1) \). We have \( k \leq \log_2 \frac{n + q + 1}{q + 1} < k + 1 \) so \( \varphi_{F_q} = k = \left\lfloor \log_2 \frac{n + q + 1}{q + 1} \right\rfloor \).

5.2. \( R_q \)-polychromatic coloring \( \varphi_{R_q} \) (\( q \geq 2 \)). If \( q \geq 2 \), \( n \geq q + 3 \) and \( n \) and \( k \) are such that (5.1) is satisfied, we let \( \varphi_{R_q} = \varphi_{F_q} \) (same color classes), giving us the same formula for \( k \) in terms of \( n \).

5.3. \( C_q \)-polychromatic coloring \( \varphi_{C_q} \), (\( q \geq 2 \)). If \( q \geq 2 \), \( n \geq q + 3 \) and

\[
(2^k - 1)q + 2^{k-1} < n \leq (2^{k+1} - 1)q + 2^k
\]

let \( \varphi_{C_q} \) be the simply-ordered edge \( k \)-coloring with colors 1, 2, \ldots, \( k \) and inherited vertex \( k \) coloring \( \varphi'_{C_q} \) with successive color classes \( M_1, M_2, \ldots, M_k \) of sizes given by:

\[
|M_1| = q + 1
\]

\[
|M_i| = 2^{i-1}q + 2^{i-2} \text{ if } i \in [2, k - 1]
\]

\[
|M_k| = n - \sum_{i=1}^{k-1} |M_i| = n - 2^{k-1}q - 2^{k-2}
\]

From equation (5.2) we get \( \varphi_{C_q} = k = \left\lfloor \log_2 \frac{2(n + q - 1)}{2q + 1} \right\rfloor \).
5.4. \( R_0 \)-polychromatic coloring \( \varphi_{R_0} \) \((q = 0)\). If \( n \geq 3 \) and \( 2^{k-1} - 1 \leq n < 2^{k-1} \) let \( \varphi_{R_0} \) be the quasi-simply-ordered coloring with \( |X| = 3 \) and color class sizes \( |M_1| = |M_2| = 1 \) and \( |M_3| = n - 2 \) if \( 3 \leq n \leq 6 \), and if \( n \geq 7 \):

\[
|M_1| = |M_2| = |M_3| = 1 \\
|M_i| = 2^{i-2} \text{ if } i \in [4, k-1] \\
|M_k| = n - \sum_{i=1}^{k} i |M_i| = n - 2^{k-2} + 1
\]

From this, we get \( P_{R_0} = k = 1 + |\log_2(n+1)| \) where \( n \geq 3 \).

5.5. \( C_0 \)-polychromatic coloring \( \varphi_{C_0} \) \((q = 0)\). If \( n \geq 3 \) and \( 3 \cdot 2^{k-3} < n \leq 3 \cdot 2^{k-2} \) let \( \varphi_{C_0} \) be the quasi-simply-ordered coloring with \( |X| = 3 \) and color class sizes \( |M_1| = |M_2| = 1 \) and \( |M_3| = n - 2 \) if \( 3 \leq n \leq 6 \), and if \( n \geq 7 \):

\[
|M_1| = |M_2| = |M_3| = 1 \\
|M_i| = 3 \cdot 2^{i-4} \text{ if } i \in [4, k-1] \\
|M_k| = n - \sum_{i=1}^{k-1} i |M_i| = n - 3 \cdot 2^{k-4}
\]

From this, we get \( P_{C_0} = k = \left\lfloor \log_2 \frac{8(n-1)}{3} \right\rfloor \) where \( n \geq 4 \).

5.6. \( R_1 \)-polychromatic coloring \( \varphi_{R_1} \) \((q = 1)\). If \( n \geq 4 \) and \( 3 \cdot 2^{k-1} - 2 \leq n < 3 \cdot 2^{k-2} \) let \( \varphi_{R_1} \) be the quasi-simply-ordered coloring with \( |X| = 4 \) and color class sizes \( |M_1| = 2 \) and \( |M_2| = n - 2 \) if \( 4 \leq n \leq 9 \), and if \( n \geq 10 \):

\[
|M_1| = |M_2| = 2 \\
|M_i| = 3 \cdot 2^{i-2} \text{ if } i \in [3, k-1] \\
|M_k| = n - \sum_{i=1}^{k-1} i |M_i| = n - 3 \cdot 2^{k-2} + 2
\]

From this, we get \( P_{R_1} = k = \left\lfloor \log_2 \frac{2(n+2)}{3} \right\rfloor \) where \( n \geq 4 \).

5.7. \( C_1 \)-polychromatic coloring \( \varphi_{C_1} \) \((q = 1)\). If \( n \geq 4 \) and \( 5 \cdot 2^{k-2} \leq n < 5 \cdot 2^{k-1} \) let \( \varphi_{C_1} \) be the quasi-simply-ordered coloring with \( |X| = 4 \) and color class sizes \( |M_1| = |M_2| = 2 \) and \( |M_3| = n - 4 \) if \( 4 \leq n \leq 9 \) and change every edge of color 3 to color 2, and if \( n \geq 10 \):
\[ |M_1| = |M_2| = 2 \]
\[ |M_i| = 5 \cdot 2^{i-3} \text{ if } i \in [3,k-1] \]
\[ |M_k| = n - \sum_{i=1}^{k-1} |M_i| = n - 5 \cdot 2^{k-3} + 1 \]

From this, we get \( P_{C_1} = k = \lfloor \log_2 \frac{4n}{5} \rfloor \) where \( n \geq 4 \).

6. PROOF OF THEOREM 2.1 ON MATCHINGS

We prove Theorem 2.1. This proof is similar to the proof of Theorem 1 in [2]. Let \( k = \varphi_{F_q}(n) \) be the polychromatic number for 1-factors spanning \( n-q \) vertices in \( G = K_n = (V,E) \). Among all \( F_q \)-polychromatic colorings of \( K_n \) with \( k \) colors we choose ones that are \( X \)-ordered for a subset \( X \) (possibly empty) of the largest possible size, and, of these, choose a coloring \( c \) whose restriction to \( V \setminus X \) has the largest possible maximum monochromatic degree. Let \( v \) be a vertex of maximum monochromatic degree, \( r \), in \( c \) restricted to \( G[V \setminus X] \), let the majority color on the edges incident to \( v \) in \( V \setminus X \) be color 1. By the maximality of \( |X| \), there is a vertex \( u \) in \( V \setminus X \) such that \( c(uv) \neq 1 \). Assume \( c(uv) = 2 \). If every 1-factor spanning \( n-q \) vertices containing \( uv \) had another edge of color 2, then the color of \( uv \) could be changed to 1, resulting in a \( F_q \)-polychromatic coloring where \( v \) has a larger maximum monochromatic degree in \( V \setminus X \), a contradiction. Hence, there is a 1-factor \( F \) spanning \( n-q \) vertices in which \( uv \) is the only edge with color 2 in \( c \).

Let \( c(vy_i) = 1 \), \( y_i \in V \setminus X \), \( i = 1, \ldots, r \). Note that for each \( k \in [r] \), \( y_k \) must be in \( F \). If not, then \( F - uv + vy_k \) is a 1-factor spanning \( n-q \) vertices with no edge of color 2 (since \( uv \) was the unique edge of color 2 in \( F \) and \( vy_k \) is color 1). For each \( i \in [r] \), let \( y_iw_i \) be the edge of \( F \) containing \( y_i \) (perhaps \( w_i = y_j \) for some \( j \neq i \)). See Figure 1. We can get a different 1-factor \( F_i \) by replacing the edges \( uv \) and \( y_iw_i \) in \( F \) with edges \( vy_i \) and \( uw_i \). Since \( F_i \) must have an edge of color 2 and \( c(vy_i) = 1 \), we must have \( c(uw_i) = 2 \) for each \( i \in [r] \).

\[ \text{Figure 1. Maximum polychromatic degree in an } F_q \text{-polychromatic coloring} \]
If \( w_i \in X \) for some \( i \) then, since \( c \) is \( X \)-constant, \( c(w_iy_i) = c(w_iu) = 2 \), so \( y_iw_i \) and \( uv \) are two edges of color 2 in \( F \), a contradiction. So, \( w_i \in V \setminus X \). Thus \( c(uw_i) = c(uv) = \cdots = 2 \), and the monochromatic degree of \( u \) in \( V \setminus X \) is at least \( r + 1 \), larger than that of \( v \), a contradiction. Hence \( X = V \), \( c \) is ordered, and, by Lemma 4.2, there exists a simply-ordered \( F_1 \)-polychromatic coloring \( c_s \) with \( k \) colors. By Lemma 4.1 if \( M_1, M_2, \ldots, M_k \) are the successive color classes, moving left to right, of the inherited vertex coloring \( c'_{s} \), then \( |M_t| \geq 2^{t-1}(q + 1) \) for \( t = 1, 2, \ldots, k \). Since this inequality holds with equality for \( t = 1, 2, \ldots, k - 1 \) for the inherited vertex-coloring \( \varphi_{F_q} \), the number of color classes of \( c_s \) cannot be greater than that of \( \varphi_{F_q} \), so \( k \leq \left\lfloor \log_2 \frac{n+q+1}{q+1} \right\rfloor \).

7. \( C_q \)-polychromatic Numbers 1 and 2

The following theorem is a special case of a theorem of Faudree and Schelp.

**Theorem 7.1.** Let \( s \geq 5 \) be an integer and let \( c(s) \) denote the smallest integer \( n \) such that in any 2-coloring of the edges of \( K_n \) there is a monochromatic \( s \)-cycle. Then \( c(s) = 2s - 1 \) if \( s \) is odd and \( c(s) = \frac{3}{2}s - 1 \) if \( s \) is even.

Faudree and Schelp actually determined all values of \( c(r, s) \), the smallest integer \( n \) such that in any coloring of the edges of \( K_n \) with red and blue, there is either a red \( r \)-cycle or a blue \( s \)-cycle. Their theorem extended partial results and confirmed conjectures of Bondy and Erdős [5] and Chartrand and Schuster [6] (who showed \( c(3) = c(4) = 6 \)). The coloring of \( K_{2s-2} \) to prove the lower bound for \( s \) odd is a copy of \( K_{s-1,s-1} \) of red edges with all other edges blue, while for \( s \) even it’s a red \( K_{\frac{s}{2}-1,s-1} \) with all other edges blue.

**Proof of Theorem 2.5.** By Theorem 7.1 if \( s \geq 5 \) is odd then there is a polychromatic 2-coloring of \( K_n \) if and only if \( n \leq 2s - 2 = 2(n - q) - 2 \), so if and only if \( n \geq 2q + 2 \). If \( s \geq 5 \) is even then there is a polychromatic 2-coloring if and only if \( n \leq \frac{3}{2}s - 2 = \frac{3}{2}(n - q) - 2 \), so if and only if \( n \geq 3q + 4 \). Hence if \( n \in [2q + 2, 3q + 2] \) then \( \varphi_{C_q}(n) = 1 \) if \( n - q \) is even and \( \varphi_{C_q}(n) = 2 \) if \( n - q \) is odd. The smallest value of \( n \) for which there is a simply ordered \( C_q \)-polychromatic 2-coloring is \( n = 3q + 3 \), so there does not exist one if \( n - q \) is odd and \( n \leq 3q + 2 \).

We remark that the only values for \( q \geq 2 \) and \( n \) such that there is no optimal simply-ordered \( C_q(n) \)-polychromatic coloring of \( K_n \) are the ones given in Theorem 2.5 (\( n \in [2q + 2, 3q + 2] \) and \( n - q \) is odd), and \( q = 2, n = 5 \) (two monochromatic \( C_5 \)'s is a coloring of \( K_5 \) with no monochromatic \( C_5 \)'s).

8. Proofs of Theorem 2.6 and Lemmas on Long Cycles

We will need some results on the existence of long cycles in bipartite graphs.

**Theorem 8.1.** (Jackson [11].) Let \( G \) be a connected bipartite graph with bipartition \( V(G) = S \cup T \) where \( |S| = s, |T| = t \), and \( s \leq t \). Let \( m \) be the minimum degree of a vertex in
S and p be the minimum degree of a vertex in T. Then G has a cycle with length at least min\{2s, 2(m + p − 1)\}.

**Theorem 8.2** (Rahman, Kaykobad, Kaykobad [13]). Let G be a connected m-regular bipartite graph with 4m vertices. Then G has a Hamiltonian cycle.

**Lemma 8.3.** Let B be a bipartite graph with vertex bipartition S, T where |S| = s, |T| = t, and s ≤ t. Suppose each vertex in T has degree m and each vertex in S has degree t − m. Then B has a 2s-cycle unless s = t = 2m and B is the disjoint union of two copies of K_{m,m}.

**Proof.** Suppose s < t. Summing degrees in S and T gives us s(t − m) = tm, so

\[
m = \frac{st}{s + t} > \frac{st}{2t} = \frac{s}{2}
\]

so B is connected. By Theorem 8.1 B has a 2s-cycle, since 2|m+(t−m)−1| = 2(t−1) ≥ 2s. If s = t, then B is an m-regular graph with 4m vertices. If B is connected then, by Theorem 8.2 it has a 2s-cycle. If B is not connected then clearly it is the disjoint union of two copies of K_{m,m}.

We say that a cycle H' of length n − q is obtained from a cycle H of length n − q by a twist of disjoint edges e_1 and e_2 of H if E(H) \{ e_1, e_2 \} ⊆ E(H'), i.e. we remove e_1, e_2 from H and introduce two new edges to make the resulting graph a cycle. Note that the choice of the two edges to add is unique (due to connectedness), however, both choices would result in a 2-regular subgraph.

One main difference between the definitions of C_q(n) and R_q(n) is that for the former, we consider only cycles of length precisely n − q, whereas, in the latter, we consider all 2-regular subgraphs spanning at least n − q vertices. This is because we can prove Theorem 2.7 for cycles, however, a similar result for 2-regular subgraphs remains elusive (see Conjecture 11.1).

**8.1. Proof of Theorem 2.6.** Suppose not. Let m be an integer in [j, n − 1] such that every m-cycle gets all colors but there is an (m + 1)-cycle H, v_1v_2, ..., v_{m+1}v_1 which does not have an edge of color t. Then c(v_i v_{i+2}) = t for all i, where the subscripts are read mod (m + 1), because otherwise, there is an m-cycle with no edge of color t.

**Case 1.** If m + 1 is odd, then v_1v_3v_5 \cdots v_{m+1}v_2v_4 \cdots v_{m-2}v_1 is an m-cycle with at most two colors, since all edges except possibly v_{m-2}v_1 have color t. This is impossible.

**Case 2.** Suppose m + 1 is even. Then c_E = v_2v_4 \cdots v_{m+1}v_2 and c_O = v_1v_3 \cdots v_m v_1 are \frac{m+1}{2}-cycles with all edges of color t. Suppose H has a chord v_jv_{j+r} with color t for some j and odd integer r in [3, m − 2]. Then v_{j+2}v_{j+4} \cdots v_{j-2}v_jv_{j+r}v_{j+r+2} \cdots v_{j+r-4} is a path with m vertices (missing v_{j+r−2}) and all edges of color t, so there is an m-cycle with at most two colors, which is impossible. Hence if v_i is a vertex in c_E and v_j is a vertex in c_O, then v(v_i v_j) ≠ t.

We claim that for each j and even integer s, c(v_j v_{j+s}) = t. If not, then v_{j}v_{j+s}v_{j+s+1} \cdots v_{j-3}v_{j-2}v_{j+s−1}v_{j+s−2} \cdots v_{j+1}v_j is an m-cycle (missing v_{j−1}) with no edge of color t (note c(v_{j−2}v_{j+s−1}) ≠ t because j − 2 and j + s − 1 have different parities). Hence, the vertices
of $c_E$ and $c_O$ each induce a complete graph with $\frac{m+1}{2}$ vertices and all edges of color $t$, and there are no other edges of color $t$ in $K_n$.

If there is a color $w$, different than $t$, such that there exist two disjoint edges of color $w$, then it is easy to find an $m$-cycle with two edges of color $w$ and the rest of color $t$. If there do not exist two such edges of color $w$, then all edges of color $w$ are incident to a single vertex $x$, so any $m$-cycle with $x$ incident to two edges of color $t$ does not contain an edge of color $w$ (these exist since $\frac{m+1}{2} \geq 3$).

We remark that the statement in Theorem 2.6 would be false without the requirement that there be at least three colors. If $m \geq 3$ is odd, then two vertex disjoint complete graphs each with $\frac{m+1}{2}$ vertices and all edges of color $t$ with all edges between them of color $w$ has an $(m+1)$-cycle with all edges of color $w$, while every $m$-cycle has edges of both colors. This is the reason for the difference between odd and even values of $n-q$ in Theorem 2.5. The statement would also be false with three colors if $j=3$ and $n=4$.

9. Main Lemmas and Proofs of Theorems

We now state and prove the three main lemmas needed for the proofs of Theorems 2.2, 2.3, and 2.4.

Lemma 9.1.

(a) Let $\mathcal{H} \in \{R_q(n), C_q^*(n)\}$. Of all optimal $\mathcal{H}$-polychromatic colorings, let $\varphi$ be one which is $X$-ordered on a (possibly empty) subset $X$ of $V(K_n)$ of maximum size and, of these, such that $G_m = K_n[Y]$ has a vertex $v \in Y$ of maximum possible monochromatic degree $d$ in $G_m$ where $Y = V(K_n) \setminus X$, $|Y| = m$, and $d < (m-1)$. If $v$ is incident in $G_m$ to $d$ edges of color 1 and $u \in Y$ is such that $\varphi(vu) = 2$, then $v$ is a $(1,2)$-max vertex in $G_m$ and $u$ is a $(2,t)$-max vertex in $G_m$ for some color $t$ (possibly $t=1$).

(b) The same is true if $X \neq \emptyset$ and $\varphi$ is nearly $X$-ordered.

Proof of (a). Let $y_1, y_2, \ldots, y_d \in Y$ be such that $\varphi(vy_i) = 1$. Let $H \in C_q^*$ or $H \in R_q$ be such that $uv$ is the only edge of color 2. There must be such an $H$ otherwise we could change the color of $uv$ from 2 to 1, giving an $\mathcal{H}$-polychromatic coloring with monochromatic degree greater than $d$ in $G_m$. Orient the edges of $H$ to get a directed cycle or 2-regular graph $H'$ where $uv$ is an arc.

If $y_i \in H'$ then the predecessor $w_i$ of $y_i$ in $H'$ must be such that $\varphi(w_iu) = 2$, because otherwise we can twist $uv$ and $w_iy_i$ to get an $(n-q)$-cycle (if $H \in C_q^*$) or a 2-regular graph (if $H \in R_q$) with no edge of color 2. Note that $w_i$ must be in $Y$ because otherwise, since $\varphi$ is $X$-constant, $\varphi(w_iu) = \varphi(w_iy_i) = 2$, contradicting the assumption that $uv$ is the only edge in $H$ of color 2.

Suppose $y_i \notin H$ for some $i \in [d]$. If $\varphi(y_iu) \neq 2$, then $J = (H \setminus \{uv\}) \cup \{vy_i, y_iu\}$ has no edge of color 2. This is impossible if $H \in R_q$, because $J$ is a 2-regular graph spanning $n-q+1$ vertices. If $H \in C_q^*$, then $J$ is an $(n-q+1)$-cycle with no edge of color 2, so by Theorem 2.6 since the polychromatic number of $H$ is at least 3, there exists an $(n-q)$-cycle which is not polychromatic, a contradiction. Hence $\varphi(y_iu) = 2$ in either case.
Thus, for each $i \in [d]$, either $y_i \notin H$ and $\varphi(y_i u) = 2$, or $y_i \in H$ and $\varphi(w_i u) = 2$ where $w_i$ is the predecessor of $y_i$ in $H'$. That gives us $d$ edges in $G_m$ of color 2 which are incident to $u$. Since $v$ has maximum monochromatic degree in $G_m$, it follows that $v = w_i$ for some $i$ (otherwise $wv$ is a different edge of color 2 incident to $u$) and it also follows that no edge in $G_m$ incident to $v$ can have color $t$ where $t \notin \{1, 2\}$. This is because if $vz$ were such an edge, as shown above, then either $z \in H$ and $\varphi(w'u) = 2$ where $w'$ is the predecessor of $z$ in $H'$, or $z \notin H$ and $\varphi(zu) = 2$. In either case we get $d + 1$ edges of color 2 in $G_m$ incident to $u$, a contradiction. So $v$ is a $(1, 2)$-max-vertex and $u$ is a $(2, t)$-max-vertex for some color $t$.

The proof of \([b]\) is exactly the same.

**Lemma 9.2.** Let $n \geq 7$ and $H \in \{R_q(n), C_q(n)\}$. If there does not exist an optimal $H$-polychromatic coloring of $K_n$ with maximum monochromatic degree $n - 1$, then one of the following holds.

\begin{itemize}
  \item[a)] $H = C_q(n)$, $n - q$ is odd and $n \in [2q + 2, 3q + 2]$ \((\text{and } \varphi_{C_q}(n) = 2)\).
  \item[b)] $q = 0$ and there exists an optimal $H$-polychromatic coloring which is $Z$-quasi-ordered with $|Z| = 3$.
  \item[c)] $q = 1$ and there exists an optimal $H$-polychromatic coloring which is $Z$-quasi-ordered with $|Z| = 4$.
\end{itemize}

**Proof.** First assume that $H = C_q(n)$ and that $q \geq 2$ and $n$ are such that $\varphi_{C_q}(n) \leq 2$. If $n - q$ is even then, by Theorem 2.5, there is a $C_q$-polychromatic 2-coloring if and only if $n \geq 3q + 3$. Since $3q + 3$ is the smallest value of $n$ such that the simply-ordered $C_q$-polychromatic coloring $\varphi_{C_q}$ uses two colors, if $\varphi_{C_q}(n) \leq 2$ and $n - q$ is even, then there is an optimal simply-ordered $C_q$-polychromatic coloring, and this coloring has a vertex (in fact $q + 1$ of them) with monochromatic degree $n - 1$.

If $n - q$ is odd then, by Theorem 2.5, there is a $C_q$-polychromatic 2-coloring if and only if $n \geq 2q + 2$. Since there is a simply-ordered $C_q$-polychromatic 2-coloring if $n \geq 3q + 3$, that means that if $n - q$ is odd, $\varphi_{C_q}(n) \leq 2$ and $n \notin [2q + 2, 3q + 2]$ then there is a simply-ordered $C_q$-polychromatic coloring. Thus if $\varphi_{C_q}(n) \leq 2$, there is an optimal simply-ordered $C_q$-polychromatic coloring, and hence one with maximum monochromatic degree $n - 1$, unless $n - q$ is odd and $n \in [2q + 2, 3q + 2]$, which are the conditions for \([a]\).

Now let $H \in \{R_q(n), C_q(n)\}$ and suppose there does not exist an optimal $H$-polychromatic coloring of $K_n$ with maximum monochromatic degree $n - 1$. Of all optimal $H$-polychromatic colorings of $K_n$, let $\varphi$ be the one with maximum possible monochromatic degree $d$ (so $d < n - 1$).

**Claim 1.** $d > \frac{n-1}{2}$.

**Proof.** Since there are only two colors at a max-vertex, certainly $d \geq \frac{n-1}{2}$. Assume $d = \frac{n-1}{2}$ (so $n$ is odd) and that $x$ is a max-vertex where colors $i$ and $j$ appear. Then $x$ is both an $i$-max and $j$-max vertex so, by Lemma 9.1, each vertex in $V$ is a max-vertex.

Suppose there are more than 3 colors, say colors $i, j, s, t$ are all used. If $i$ and $j$ appear at $x$ then no vertex $y$ can have colors $s$ and $t$, because there is no color for $xy$. So the sets of colors on the vertices is an intersecting family of 2-sets. Since there are at least 4 colors, the
only way this can happen is if some color, say $i$, appears at every vertex. Let $n_{ij}, n_{is},$ and $n_{it}$ be the number of $(i,j)$-max, $(i,s)$-max, and $(i,t)$-max vertices with $n_{ij} \leq n_{is} \leq n_{it}$. Then $n_{ij} < \frac{n}{3}$ (in fact, $n_{ij} \leq \frac{n}{4}$). If $x$ is an $(i,j)$-max vertex and $y$ is an $(i,s)$-max vertex, then $c(xy) = i$. Hence the number of edges of color $j$ incident to $x$ is at most $n_{ij} - 1 < \frac{n^2}{2} < d$, a contradiction.

Now suppose there are precisely 3 colors. Let $A, B, C$ be the set of all $(1,2)$-max, $(2,3)$-max, and $(1,3)$-max vertices, respectively, with $|A| = a, |B| = b,$ and $|C| = c$. All edges from a vertex in $A$ to a vertex in $B$ have color 2, from $B$ to $C$ have color 3, from $A$ to $C$ have color 1; internal edges in $A$ have color 1 or 2, in $B$ have color 2 or 3, in $C$ have color 1 or 3. We clearly cannot have $a, b,$ or $c$ greater than $\frac{n-1}{2}$ so, without loss of generality, we can assume $a \leq b \leq c \leq \frac{n-1}{2}$ and $a + b + c = n$.

Consider the graph $F$ formed by the edges of color 1 or 2. Vertices of $F$ in $B$ or $C$ have degree $\frac{n-1}{2}$, while vertices in $A$ have degree $n - 1$. Since $a \leq c$ we have $a \leq \frac{n-b}{2}$. The internal degree in $F$ of each vertex in $B$ is $\frac{n-1}{2} - a \geq \frac{n-1}{2} - \frac{n-b}{2} = \frac{b-1}{2}$. As is well known (Dirac’s theorem), that means there is a Hamiltonian path within $B$. Similarly there is one within $C$. If $a \geq 2$, that makes it easy to construct a Hamiltonian cycle in $F$. If $a = 1$ we must have $b = c = \frac{n-1}{2}$, so $F$ is two complete graphs of size $\frac{n+1}{2}$ which share one vertex. This graph has a spanning 2-regular subgraph if $n \geq 7$ (a 3-cycle and a 4-cycle if $n = 7$), so no $R_q$-polychromatic coloring with 3 colors for any $q \geq 0$ if $n \geq 7$.

If $a = 1$ and $b = c = \frac{n-1}{2}$ consider the subgraph of all edges of colors 1 or 3. It consists of a complete bipartite graph with vertex parts $A \cup B$ and $C$, with sizes $\frac{n+1}{2}$ and $\frac{n-1}{2}$, plus internal edges in $C$. Clearly this graph has an $(n - 1)$-cycle, but no Hamiltonian cycle. Hence there can be a $C_q$-polychromatic coloring only if $q = 0$. However, the $C_0$-polychromatic coloring $\varphi_{C_0}$ uses at least 4 colors if $n \geq 7$, so there is no optimal one with maximum monochromatic degree $\frac{n-1}{2}$.

**Claim 2.** If $q = 0$, then, up to relabeling the colors, there is a $(1,2)$-max-vertex, a $(2,3)$-max-vertex and a $(3,1)$-max-vertex.

**Proof.** Assume that every max-vertex has majority color either 1 or 2. Then $u$ must be a $(2,1)$-max-vertex. This is because by Lemma 9.1 if it were a $(2,t)$-max-vertex for some third color $t$, and $c(uz) = t$, then $z$ would have to be a $t$-max-vertex, a contradiction. Hence, every max-vertex is either a $(1,2)$-max-vertex or a $(2,1)$-max-vertex. Let $S$ be the set of all $(1,2)$-max-vertices, $T$ be the set of all $(2,1)$-max-vertices, and $W = V \setminus (S \cup T)$. Edges within $S$ and from $S$ to $W$ must have color 1 (because any minority color edge at a max-vertex is incident to a max-vertex of that color), edges within $T$ and from $T$ to $W$ must have color 2, and all edges between $S$ and $T$ must have color 1 or 2. If $|S| = s$ and $|T| = t$ and $m = n - 1 - d$, then each vertex in $S$ is adjacent to $m$ vertices in $T$ by edges of color 2 (and adjacent to $t - m$ vertices in $T$ by edges of color 1), and each vertex in $T$ is adjacent to $m$ vertices in $S$ by edges of color 1.

Suppose $s < t$ and consider any edge $ab$ from $S$ to $T$ of color 2. As before, there is an $H \in H$ which contains $ab$, but no other edges of color 2. Hence $H$ has no edges from $T$ to $W$. Since $s < t$ there must be an edge of $H$ with both vertices in $T$, so it does have
another edge of color 2 after all, a contradiction. The same argument works if \( t < s \) with an edge with color 1. To avoid this, we must have \( s = t = 2m \). If there is an edge from \( S \) to \( W \) then, again, \( H \) has an internal edge in \( T \), which is impossible. Hence if \( H = C_0^* \) then \( W = \emptyset \) and every edge has color 1 or 2, which is impossible since \( H \) has at least 3 colors. If \( H = R_0 \) then the subgraph of \( H \) induced by \( S \cup T \) is the union of cycles. If \( m = 1 \) then \( S \cup T \) induces a 4-cycle in \( H \), two edges of each color, so \( ab \) is not the only edge with color 2. If \( m \geq 2 \) then two applications of Hall’s Theorem gives two disjoint perfect matchings of edges of color 1 between \( S \) and \( T \), whose union is a 2-factor of edges of color 1 spanning \( S \cup T \), which together with the subgraph of \( H \) induced by \( W \), produces a 2-factor \( H' \in R_0 \) with no edge of color 2.

We have shown that \( u \) is not a \((2,1)\)-max vertex, so it must be a \((2,3)\)-max vertex for some other color 3. Say \( \varphi(uz) = 3 \). Then, by Lemma 9.1, \( z \) is a 3-max vertex. If \( \varphi(vz) = 2 \), then \( z \) would be a 2-max vertex. So \( z \) would be both a 2-max and a 3-max vertex, and so \( d = \frac{n-1}{2} \), a contradiction to Claim 1. Hence \( \varphi(vz) = 1 \), which means \( z \) must be a \((3,1)\)-max vertex.

**Claim 3.** If \( q = 0 \) then \( V \) can be partitioned into sets \( A, B, D, E \) where the following properties hold (see Figure 3).

1. All vertices in \( A \) are \((1,2)\)-max-vertices.
2. All vertices in \( B \) are \((2,3)\)-max-vertices.
3. All vertices in \( D \) are \((3,1)\)-max-vertices.
4. No vertex in \( E \) is a max-vertex.
5. All edges within \( A \), from \( A \) to \( D \), and from \( A \) to \( E \) are color 1.
6. All edges within \( B \), from \( B \) to \( A \), and from \( B \) to \( E \) are color 2.
7. All edges within \( D \), from \( D \) to \( B \), and from \( D \) to \( E \) are color 3.
8. \(|A| = |B| = |D| = m = n - 1 - d|.

**Proof.** Let \( A = \{x : x \text{ is a } (1,2)\text{-max vertex}\}, B = \{x : x \text{ is a } (2,3)\text{-max vertex}\}, D = \{x : x \text{ is a } (3,1)\text{-max vertex}\} \) and \( E = V \setminus (A \cup B \cup D) \). Let \( x \in A \). If \( y \in A \), then \( \varphi(xy) = 1 \) because if \( \varphi(xy) = 2 \), then \( y \) would be a 2-max vertex. If \( y \in B \), then \( \varphi(xy) = 2 \) because that is the only possible color for an edge incident to \( x \) and \( y \) and, similarly, if \( y \in D \), then \( \varphi(xy) = 1 \).

Suppose \( w \) is a max-vertex in \( E \). Then the two colors on edges incident to \( w \) must be a subset of \( \{1,2,3\} \), because, otherwise, it would be disjoint from \( \{1,2\}, \{2,3\}, \) or \( \{1,3\} \), so there would be an edge incident to \( w \) for which there is no color. Say 1 and 2 are the colors at \( w \). Since \( w \notin A \), \( w \) is a \((2,1)\)-max vertex. Let \( z \) be a \((3,1)\)-max vertex. Then the edge \( wz \) must have color 1 so, by Lemma 9.1, \( z \) is a 1-max vertex, a contradiction. We have now verified (1)-(4). If \( x \in A \) and \( w \in E \) then \( \varphi(xw) = 1 \) because if \( \varphi(xw) = 2 \) then \( w \) would be a 2-max vertex. Similar arguments show that if \( y \in B \) then \( \varphi(yw) = 2 \) and if \( y \in D \) then \( \varphi(yw) = 3 \). We have now verified (1)-(7).

We have shown that if \( x \) is in \( A \) then \( \varphi(xy) = 2 \) if and only if \( y \in B \). That means \(|B| = m| \), and by the same argument \(|A| = |C| = m| \) as well, completing the proof of Claim 3.
Claim 4. If $\mathcal{H} \in \{C_0^*, R_0\}$, and there exists an optimal $\mathcal{H}$-polychromatic coloring satisfying (1)–(8) with $m > 1$, then there exists one with $m = 1$, i.e. one that is $Z$-quasi-ordered with $|Z| = 3$.

Proof. Let $A = \{a_i : i \in [m]\}, B = \{b_i : i \in [m]\}, D = \{d_i : i \in [m]\}$. Define an edge coloring $\gamma$ by

$$
\gamma(a_1b_i) = 1 \text{ if } i > 1 \\
\gamma(b_1d_i) = 2 \text{ if } i > 1 \\
\gamma(d_1a_i) = 3 \text{ if } i > 1 \\
\gamma(uv) = \varphi(uv) \text{ for all other } u, v \in V.
$$

It is easy to check that $\gamma$ has the structure described above with $m = 1$. We have essentially moved $m - 1$ vertices from each of $A$, $B$, and $D$, to $E$. Since $a_1, b_1$, and $c_1$ each have monochromatic degree $n - 2$, any 2-factor must have edges of colors 1,2, and 3 under the coloring $\gamma$, so if it had all colors under $\varphi$, it still does under $\gamma$. \hfill \blacksquare

We remark that the coloring $\gamma$ with $m = 1$ in Claim 4 is $Z$-quasi-ordered with $|Z| = 3$. As we have shown, if there exists such an $R_0$-polychromatic coloring $\varphi$ with $m > 1$, then there exists one with $m = 1$. However, if $m > 1$ and $n > 6$, a coloring $\varphi$ satisfying properties (1)–(8) might not be $R_0$-polychromatic. This is because if $E$ has no internal edges with color 1, then any 2-factor with a $2m$-cycle consisting of alternating vertices from $A$ and $B$ has no...
edge with color 1. However, the modified coloring $\gamma$ (with $m = 1$) is an $R_0$-polychromatic coloring because then colors 1, 2, and 3 must appear in any 2-factor.

**Claim 5.** If $q \geq 1$ then, up to relabelling colors, every max vertex is a $(1, 2)$-max vertex or a $(2, 1)$-max vertex.

**Proof.** As before, we assume $v$ is a $(1, 2)$-max vertex, that $\varphi(uv) = 2$ and that $H \in R_q$ (or $H \in C_q^*$) is such that $uv$ is the only edge of color 2. We know that $u$ is a $(2, t)$-max vertex for some color $t$. By way of contradiction, suppose $u$ is a $(2, 3)$-max vertex. Then we have the configuration of Figure 2 with $|A| = |B| = |D| = m$. If $uw$ is also an edge of $H$ then $w \in D$, since otherwise $\varphi(uw) = 2$. Let $Q$ be the set of vertices not in $H$ (so $|Q| = q > 0$) and suppose $p \in Q$ but $p \notin B$. Then we can replace $u$ in $H$ with $p$ to get a 2-regular graph (cycle) with no edge of color 2. Hence $Q \subseteq B$. Orient the edges of $H$ to get a directed graph $H'$ where $\overrightarrow{uw}$ is an arc. Since $|B \setminus Q| < |D|$, and every vertex in $D$ appears in $H'$, for some $d \in D$ and $e \notin B$, $\overrightarrow{de}$ is an arc in $H'$. Since $\varphi(du) = 3$ and $\varphi(ev) = 1$, when you twist $uv$ and $de$ you get a 2-regular graph (cycle) with no edge of color 2, a contradiction. Hence every max-vertex is a $(1, 2)$-max vertex or $(2, 1)$-max vertex.

**Claim 6.** If $q = 1$ then, up to relabelling colors, the vertex set can be positioned into $S, T, W$ such that

1. $S$ is the set of all $(1, 2)$-max vertices
2. $T$ is the set of all $(2, 1)$-max vertices
3. $W$ has no max vertices
4. All internal edges in $S$ and all edges from $S$ to $W$ have color 1; all internal edges in $T$ and all edges from $T$ to $W$ have color 2
5. The edges of color 1 between $S$ and $T$ form two disjoint copies of $K_{m,m}$, as do the edges of color 2 (so $|S| = |T| = 2m$, where $n - m - 1$ is the maximum monochromatic degree)

**Proof.** By Claim 5 if $q \geq 1$, then every max vertex is a $(1, 2)$ or $(2, 1)$-max vertex.

Let $S$ be the set of all $(1, 2)$-max vertices and $T$ be the set of all $(2, 1)$-max vertices, with $|S| = s$ and $|T| = t$, $s \leq t$, and let $m$ be the maximum monochromatic degree. Let $W = V(G) \setminus (S \cup T)$ and let $B$ be the complete bipartite graph with vertex bipartition $S, T$ and edges colored as they are in $G$. So each vertex of $B$ in $S$ is incident with $m$ edges of color 2 and $t - m$ edges of color 1, and each vertex of $B$ in $T$ is incident with $m$ edges of color 1 and $s - m$ edges of color 2. All edges of $G$ within $S$ and between $S$ and $W$ have color 1 (otherwise there would be a $(2, 1)$-max vertex not in $T$) and all edges within $T$ and between $T$ and $W$ have color 2.

We note that the edges of color 1 in $B$ satisfy the conditions of Lemma 8.3 so $B$ has a $2s$-cycle of edges of color 1 unless $s = t = 2m$ and the edges of color 1 (and those of color 2) form two disjoint copies of $K_{m,m}$.

Again, let $v \in S$ and $u \in T$ be such that $c(uv) = 2$, and let $H \in C_q^*(n)$ (or $H \in R_q(n)$), $q \geq 1$, be such that $uv$ is the only edge of color 2. If $uw$ is also an edge of $H$ then $w \in S$, because otherwise $c(uw) = 2$. Hence if $z$ is a vertex of $G$ not in $H$ then $z \in T$, because
otherwise we can replace \( u \) with \( z \) in \( H \) to get \( H'' \in C^*_q(n) \) (or \( H'' \in R_q(n) \)) with no edge of color 2. That means that if \( Q \) is the set of vertices of \( G \) not in \( H \), then \( Q \subseteq T \). Since \( uv \) is the only edge in \( H \) with color 2, each vertex in \( T \setminus Q \) is adjacent in \( H \) to two vertices in \( S \), so there are \( 2(t - q) \) edges in \( H \) between \( S \) and \( T \), where \( q = |Q| \geq t - s \).

Let \( M \) be the subgraph of \( H \) remaining when the \( 2(t - q) \) edges in \( H \) between \( S \) and \( T \) have been removed (along with any remaining isolated vertices). If \( q = t - s \) then, since every edge in \( H \) incident to a vertex in \( T \) goes to \( S \), either \( H \) is a 2s-cycle and \( W = \emptyset \) (if \( H \in C^*_q(n) \)) or the union of the components of \( H \) which have a vertex in \( T \) is a 2-regular graph spanning \( S \) and \( s = t - q \) vertices in \( T \). In either case, since \( s < t \), we can replace the components of \( H \) which intersect \( T \) with the 2s-cycle of edges of color 1 promised by Theorem 8.1, to get an \( H'' \in C^*_q(n) \) (or \( H'' \in R_q(n) \)) with no edge of color 2. Hence \( q > t - s \).

Each component of \( M \) is a path with at least one edge, both endpoints in \( S \) with interior points in \( S \) or \( W \). If a component has \( j > 2 \) vertices in \( S \), we split it into \( j - 1 \) paths which each have their endpoints in \( S \) with all interior points in \( W \). If a vertex of \( S \) is an interior point in a component then it is an endpoint of two of these paths. The number of such paths is \( 2(s - (t - q)) = s - (t - q) > 0 \).

We denote the paths by \( P_1, P_2, \ldots, P_t \) where \( r = s - (t - q) \). For each \( i \) in \( [r] \) where \( P_i \) has more than 2 vertices, we remove the edges containing the two endpoints (which are both in \( S \)), leaving a path \( W_i \) whose vertices are all in \( W \) (the union of the vertices in all the \( W_i \)'s is equal to \( W \)).

We will now show that there cannot be a 2s-cycle of edges of color 1 in \( B \). Suppose \( J \) is such a 2s-cycle. Let \( R = \{x_1, x_2, \ldots, x_r\} \) be the set of any \( r \) vertices in \( T \cap V(J) \) and let \( K \) be the subgraph of \( J \) obtained by removing the \( r \) vertices in \( R \). For each \( i \in [r] \) let \( y_{ia} \) and \( y_{ib} \) be the vertices adjacent to \( x_i \) in \( J \). Both are in \( S \) and possibly \( y_{ib} = y_{ja} \) if \( i \neq j \). Now, for each \( i \in [r] \), attach \( W_i \) to \( y_{ia} \) and \( y_{ib} \) \((R_i \text{ can be oriented either way})\). More precisely, if \( W_i \) is the path \( w_{i1}, w_{i2}, \ldots, w_{id} \) in \( W \), we attach it to \( K \) by adding the edges \( y_{ia}w_{i1} \) and \( y_{ib}w_{id} \), while if \( W_i \) is empty (meaning the \( i \)th component of \( M \) has only two vertices, so none in \( W \)) we add the edge \( y_{ia}y_{ib} \). The resulting graph \( H'' \) has no edge of color 2, since we constructed it using only edges from \( J \) and edges from \( H \) within \( S \cup W \). Since \( V(H'') = V(G) \setminus R \), \( H'' \) has \( n - q \) vertices. Clearly \( H'' \) is 2-regular and, if \( H \) is a cycle, so is \( H'' \) (if \( H \) is not a cycle, \( H'' \) will still be a cycle if \( H \) does not have any components completely contained in \( W \)). Thus \( H'' \in R_q(n) \) \((H'' \in C^*_q(n))\) and has no edge of color 2, a contradiction. Hence there is no 2s-cycle of edges of color 1 in \( B \).

By Lemma 8.3 it follows that \( s = t = 2m \) with the edges of color 1 forming two vertex-disjoint copies of \( K_{m,m} \). (If these two disjoint copies have vertex sets \( S_1 \cup T_1 \) and \( S_2 \cup T_2 \), where \( S_1 \cup S_2 = S \) and \( T_1 \cup T_2 = T \), then \( S_1 \cup T_2 \) and \( S_2 \cup T_1 \) are the vertex sets which induce two disjoint copies of \( K_{m,m} \) with edges of color 2.) We have now verified that properties (1)–(5) hold if \( q \geq 1 \). We will now show we get a contradiction if \( q \geq 2 \).

Assume \( q \geq 2 \). Let \( T_1 \) and \( T_2 \) be the sets of vertices in \( T \) in the two \( s \)-cycles of edges of color 1 \((|T_1| = |T_2| = \frac{s}{2}, T_1 \cup T_2 = T)\). Recall that \( v \in S \), \( u \in T \), and \( uv \) is the only edge of \( H \) of color 2. The subgraph \( M \) of \( H \) defined earlier still consists of paths which can be
split into paths $P_1, P_2, \ldots, P_q$ (since $r = s - t + q = q$) with endpoints in $S$ and interior points in $W$. Let $J$ be the union of the two $s$-cycles of edges of color 1. Choose the subset $Q$ of size $q$ so that it has at least one vertex in each of $T_1$ and $T_2$, say $Q = \{x_1, x_2, \ldots, x_q\}$ where $x_1 \in T_1$ and $x_q \in T_2$. Again, let $K$ be the subgraph obtained from $J$ by removing the vertices in $Q$. Then, as before, the paths $W_1, W_2, \ldots, W_q$ (perhaps some of them empty) can be stitched into $W$. We attach $W_i$ to $y_{ia}$ and $y_{ib}$ if $i \in [2, q - 1]$ (just adding the edge $y_{ia}y_{ib}$ if $W_i$ is empty). We attach $W_1$ to $y_{1a}$ and $y_{1b}$ and $W_q$ to $y_{q1}a$ and $y_{qa}$, creating an $(n - q)$-cycle if no component of $H$ is contained in $W$, and a 2-regular graph spanning $n - q$ vertices if $H$ has a component contained in $W$. There is no edge of color 2 in this graph contradicting the assumption that if $q \geq 2$ and $H \in \{R_q(n), C_q^*(n)\}$ then the maximum monochromatic degree in all optimal $\mathcal{H}$-polychromatic colorings is less than $n - 1$.

**Claim 7.** If $\mathcal{H} \in \{C_1^*, R_1\}$ and there exists an $\mathcal{H}$-polychromatic coloring satisfying (1)–(5) in Claim 6 with $m > 1$, then there exists one with $m = 1$, i.e. one that is $Z$-quasi-ordered with $|Z| = 4$.

**Proof.** Assume there is an $R_1$-polychromatic coloring ($C_1^*$-polychromatic coloring) $c$ with $q = 1$ satisfying (1) – (5) of Claim 6 where $s = t > 2$. Let $v$ and $x$ be vertices in $S$ and $u$ and $y$ be vertices in $T$ such that $c(vu) = c(xy) = 2$ and $c(xu) = c(vy) = 1$. Let $c'$ be the coloring obtained from $c$ by recoloring the following edges (perhaps they are recolored the same color they had under $c$):

$$
\begin{align*}
    c'(vp) &= 1 \text{ for all } p \in T \setminus \{u, y\} \\
    c'(xp) &= 1 \text{ for all } p \in T \setminus \{u, y\} \\
    c'(zu) &= 2 \text{ for all } z \in S \setminus \{v, x\} \\
    c'(zy) &= 2 \text{ for all } z \in S \setminus \{v, x\} \\
    c'(zp) &= 3 \text{ for all } p \in T \setminus \{u, y\} \text{ and } z \in S \setminus \{v, x\}
\end{align*}
$$

Since all but one edge incident to $v$ and $x$ have color 1 under $c'$, certainly every $(n - 1)$-cycle contains an edge of color 1. Similarly for $u$ and $y$ and edges of color 2. Every edge which was recolored had color 1 or 2 under $c$, so $c'$ must be a polychromatic coloring with the same number of colors. It has the desired form with $|S| = |T| = 2$, so, in fact, is $Z$-quasi-ordered with $Z = \{v, x, u, y\}$. \qed

We remark that a coloring $c$ satisfying properties (1)–(5) of Claim 6 with $s = t > 2$ is actually not $R_1$-polychromatic. To see this, let $S_1 \cup T_1$ and $S_2 \cup T_2$ be the vertex sets of the two copies of $K_{m, m}$ of edges of color 1 ($S_1 \cup S_2 = S$, $T_1 \cup T_2 = T$) where $v \in S_1$, $u \in T_2$ and $uv$ is the only edge of color 2 in $H \in R_1$. The subgraph $M$ of $H$ in the proof of Claim 6 has only one component (since $s - (t - q) = 1$), a path $dw_1w_2 \ldots w_ez$ where $d \in S_1$, $z \in T_1$, and $\{w_1, w_2, \ldots, w_e\} \subseteq W$. To construct a 2-regular subgraph with no edges of color 2 spanning $n - 1$ vertices, remove a vertex $x$ in $T_2$ from one of the two $s$-cycles of edges of color 1. If $y_a$ and $y_b$ are the two vertices in $S_2$ adjacent to $x$ in the $s$-cycle, attach the path $w_1w_2 \ldots w_e$ to $y_a$ and $y_b$ to get a 2-regular subgraph with no edge of color 2 spanning $n - 1$ vertices. However, this construction cannot be done when $m = 1$, so in this case you do get an $R_1$-polychromatic coloring.

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Lemma 9.3. Let $\mathcal{H} \in \{R_q(n),C_q^*(n)\}$.

a) Suppose for some $X \neq \emptyset$ there exists an optimal $X$-ordered $\mathcal{H}$-polychromatic coloring of $K_n$. Then there is one which is ordered.

b) Suppose there exists an optimal $Z$-quasi-ordered $\mathcal{H}$-polychromatic coloring of $K_n$. Then there is one which is quasi-ordered.

Proof. Among all such $\mathcal{H}$-polychromatic colorings we assume $\varphi$ is one such that

a) if $\varphi$ is $X$-ordered then $X$ has maximum possible size

b) if $\varphi$ is $Z$-quasi-ordered then the restriction of $\varphi$ to $V(K_n) \setminus Z$ is $T$-ordered for the largest possible subset $T$ of $V(K_n) \setminus Z$. In this case, we let $X = Z \cup T$ so $\varphi$ is nearly $X$-ordered (one or two edges could be recolored to make it $X$-ordered).

For both [a] and [b] we assume that $\varphi$ is such that its restriction to $G_m = K_n[Y]$ has a vertex $v$ of maximum possible monochromatic degree in $G_m$, where $Y = V(K_n) \setminus X$, $|Y| = m$, and the degree of $v$ in $G_m$ is $d < m - 1$ (if $d = m - 1$ then $|X|$ is not maximal).

Since $v$ has maximum monochromatic degree $d$ in $G_m$, by Lemma 9.1 it is a $(1,2)$-max vertex in $G_m$, for some colors 1 and 2, and if $u \in Y$ is such that $\varphi(uv) = 2$, then $u$ is a $(2,t)$-max vertex for some color $t$ (perhaps $t = 1$).

As before, let $y_1, y_2, \ldots, y_d$ be vertices in $Y$ such that $c(vy_i) = 1$ for $i = 1, 2, \ldots, d$. As before, let $H \in \mathcal{H}$ be such that $uv$ is its only edge with color 2. Let $H'$ be a cyclic orientation of the edges of $H$ such that $\bar{uv}$ is an arc, and let $w_i$ be the predecessor of $y_i$ in $H'$ for $i = 1, 2, \ldots, d$. As shown before, $c(w_iu) = 2$ for $i = 1, 2, \ldots, d$.

Suppose there is an edge of $H$ which has one vertex in $X$ and one in $Y$. Then there exist $w \in Y$ and $x \in X$ such that $\bar{wx} \in H'$. Certainly $w$ is not the predecessor in $H'$ of any $y_i$ in $Y$. Since $\varphi$ is $X$-constant and $uv$ is the only edge of color 2 in $H$, $\varphi(xv) = \varphi(xw) \neq 2$. Now twist $xw, uv$ in $H$. Since $\varphi(xv) \neq 2$, we must have $\varphi(wu) = 2$, so $u$ is incident in $G_m$ to at least $d + 1$ vertices of color 2, a contradiction. Hence $H$ cannot have an edge with one vertex in $X$ and one in $Y$.

Now suppose $x \in X$ and $x \notin H$. If $\varphi(xv) = \varphi(xu) \neq 2$ then $H \setminus \{uv\} \cup \{ux, xv\}$ is an $(n - q + 1)$-cycle with no edge of color 2, which is clearly impossible if $\mathcal{H} = R_q(n)$, and is impossible if $\mathcal{H} = C_q^*(n)$ by Theorem 2.6. Hence $\varphi(xv) = \varphi(xu) = 2$ for each $x \in X$.

Since $u$ is a $(2,t)$-max vertex for some color $t \neq 2$, we can repeat the above argument with $u$ in place of $v$. That shows that $\varphi(xv) = \varphi(xu) = t$ for each $x \in X$, which is clearly impossible.

It remains to consider the possibility that $\mathcal{H} = R_q(n)$ and $X$ is spanned by a union of cycles in $H$. Suppose $xz$ is an edge of $H$ contained in $X$. Then we can twist $xz$ and $uv$ to get another subgraph in $R_q$ and, unless either $x$ or $z$ has main color 2, this subgraph has no edge of color 2. Hence at least half the vertices in $X$ have main color 2 (and more than half would if $H$ had an odd component in $X$).

The above argument can be repeated with $u$ in place of $v$. If $u$ is a $(2,t)$-max vertex then that would show that at least half the vertices in $X$ have main color $t \neq 2$. So each vertex in $X$ has main color 2 or $t$. Since $\varphi$ is $X$-ordered or nearly $X$-ordered, some vertex $x \in X$
has monochromatic degree $n - 2$ or $n - 1$ and the main color of $x$ must be 2 or $t$. Assume it is 2. Then every cycle containing $x$ has an edge with color 2, contradicting the assumption that $H$ has only one edge with color 2. Similarly, we get a contradiction if the main color of $x$ is $t$. We have shown there is no vertex $v$ with monochromatic degree $d < m - 1$, so $\varphi$ is ordered or quasi-ordered. ■

Now there is not much left to do to prove Theorems 2.2, 2.3, and 2.4.

9.1. Proof of Theorem 2.4. Theorem 2.5 takes care of the case of $C_q$-polychromatic colorings when $q \geq 2$ and $n \in [2q + 2, 3q + 2]$. The smallest value of $n$ for which there is a simply-ordered $C_q$-polychromatic 2-coloring is $n = 3q + 3$ (the coloring $\varphi_{C_q}$ in Section 5.3). Hence if $q \geq 2$ and $\varphi_{C_q} \leq 2$ then there exists an optimal simply-ordered $C_q$-polychromatic coloring except if $n - q$ is odd and $n \in [2q + 2, 3q + 2]$, or if $q = 2$ and $n = 5$ (the coloring of $K_5$ with two monochromatic 5-cycles has no monochromatic 3-cycle). So we need only consider $H \in \{R_q(n), C_q(n)\}$ (when $q \geq 2$). Since [a] is not satisfied in Lemma 9.2 there exists an optimal $H$-polychromatic coloring with maximum monochromatic degree $n - 1$. That means it is $X$-ordered, for some nonempty set $X$, so by Lemma 9.3 there exists one which is quasi-ordered and then, by Lemma 4.2, one which is quasi-simply-ordered with $|Z| = 3$, so recoloring one edge would give a simply-ordered coloring. ■

9.2. Proof of Theorem 2.2. If $H \in \{R_0(n), C_0(n)\}$ then, by Lemma 9.2 there exists an optimal $H$-polychromatic coloring which is $Z$-quasi-ordered with $|Z| = 3$. Then, by Lemma 9.3 there exists one which is quasi-ordered and then, by Lemma 4.2 one which is quasi-simply-ordered with $|Z| = 3$, so recoloring one edge would give a simply-ordered coloring.

9.3. Proof of Theorem 2.3. Exactly the same as the proof of Theorem 2.2 except now $|Z| = 4$, so two edges need to be recolored to get a simply-ordered coloring.

10. Polychromatic cyclic Ramsey numbers

Let $s, t$, and $j$ be integers with $t \geq 2, s \geq 3, s \geq t$, and $1 \leq j \leq t - 1$. We define $CR(s, t, j)$ to be the smallest integer $n$ such that in any $t$-coloring of the edges of $K_n$ there exists an $s$-cycle that uses at most $j$ colors. Erdős and Gyárfás [7] defined a related function for cliques instead of cycles. So $CR(s, t, 1)$ is the classical $t$-color Ramsey number for $s$-cycles and $CR(s, 2, 1) = c(s)$, the function in Theorem 7.1. While it may be difficult to say much about the function $CR(s, t, j)$ in general, if $j = t - 1$ we get $CR(s, t, t - 1) = PR_t(s)$ the smallest integer $n \geq s$ such that in any $t$-coloring of $K_n$ there exists an $s$-cycle that does not contain all $t$ colors. This is the function of Theorem 2.7 if $t \geq 3$, while $PR_2(s) = c(s)$.

10.1. Proof of Theorem 2.7. Let $q \geq 0, s \geq 3$, and $n$ be integers with $n = q + s$. Assume $q \geq 2$. By Theorem 2.4 and the properties of the coloring $\varphi_{C_q}$ (see Section 5.3), there exists
a $C_q$-polychromatic $t$-coloring of $K_n$ if and only if

\[ q + s = n \geq (2^t - 1)q + 2^{t-1} + 1, \]
\[ s \geq (2^t - 2)q + 2^{t-1} + 1, \]
\[ q \leq \frac{s - 2^{t-1} - 1}{2^t - 2} = \frac{s - 2}{2^t - 2} - \frac{1}{2} \]

Since $q \geq 2$, we want to choose $s$ so that the right-hand side of the last inequality is at least 2, so

\[ s - 2 \geq \frac{5}{2}(2^t - 2) = 5 \cdot 2^{t-1} - 5 \]
\[ s \geq 5 \cdot 2^{t-1} - 3 \]

So if $s \geq 5 \cdot 2^{t-1} - 3$, then the smallest $n$ for which there does not exist a $C_q$-polychromatic $k$-coloring is $n = q + s$ where $q > \frac{s-2}{2^{t-2}} - \frac{1}{2}$, so $n = s + \left\lfloor \frac{s-2}{2^{t-2}} + \frac{1}{2} \right\rfloor = s + \text{Round} \left( \frac{s-2}{2^{t-2}} \right)$.

We note that if $3s \geq 5 \cdot 2^{t-1} - 3$ then Round $\left( \frac{s-2}{2^{t-2}} \right) \geq \text{Round} \left( \frac{s}{2} \right) = 3$, so PR$_t(s) \geq s + 3$ if $s \geq 5 \cdot 2^{t-1} - 3$.

Now we assume that PR$_t(s) = s + 2$. So $s + 2$ is the smallest value of $n$ for which in any $t$-coloring of the edges of $K_n$ there is an $s$-cycle which does not have all colors, which means there is a polychromatic $t$-coloring when $n = s + 1$. Since $q = 1$ in such a coloring, by Theorem 2.3 and the properties of the coloring $\varphi_{C_1}$, $n \geq 5 \cdot 2^{t-2}$. Hence if $s \in [5 \cdot 2^{t-2} - 1, 5 \cdot 2^{t-1} - 4]$, then PR$_t(s) = s + 2$.

Now we assume that PR$_t(s) = s + 1$. So $n - s$ is the largest value of $n$ such that in any $t$-coloring of $K_n$, every $s$-cycle gets all colors. So $q = n - s = 0$ and, by Theorem 2.2 and properties of the coloring $\varphi_{C_0}$, $n \geq 3 \cdot 2^{t-3} + 1$.

Finally, since the $t$-coloring $\varphi_{C_0}$ requires $n \geq 3 \cdot 2^{t-3} + 1$ where $t \geq 4$ if $n \leq 3 \cdot 2^{t-3}$ and $t \geq 4$, then in any $t$-coloring of $K_n$, some Hamiltonian cycle will not get all colors, so PR$_t(s) = s$ if $3 < s \leq 3 \cdot 2^{t-3}$.

11. Conjectures

We mentioned that we have been unable to prove a result for 2-regular graphs analogous to Theorem 2.6 for cycles. In fact we think it even holds for two colors, except for a few cases with $j$ and $n$ small.

**Conjecture 11.1.** Let $n \geq 6$ and $j$ be integers such that $3 \leq j < n$, and if $j = 5$ then $n \geq 9$, and let $\varphi$ be an edge-coloring of $K_n$ so that every 2-regular subgraph spanning $j$ vertices gets all colors. Then every 2-regular subgraph spanning at least $j$ vertices gets all colors under $\varphi$.

This does not hold for $j = 3, n = 4$, and 3 colors; $n = 5, j = 3$, and 2 colors.

We can extend the notions of $Z$-quasi-ordered, quasi-ordered, and quasi-simply-ordered to sets $Z$ of larger size, allowing a main color to have degree less than $n - 2$. Let $q \geq 0$ and $r \geq 1$ be integers such that $q \leq 2r - 3$. Hence $\frac{2r-2}{q+1} \geq 1$, and we let $k = \left\lceil \frac{2r-2}{q+1} \right\rceil + 1 \geq 2$ and $z = k(q + 1)$. Let $Z$ be a set of $z$ vertices. We define a *seed-coloring* $\varphi$ with $k$ colors on the
edges of the complete graph $K_z$ with vertex set $Z$ as follows. Partition the $z$ vertices into $k$ sets $S_1, S_2, \ldots, S_k$ of size $q + 1$. For $j = 1, 2, \ldots, k$, all edges within $S_j$ have color $j$, all edges between $S_j$ and $S_i$ ($i \neq j$) have color $i$ or $j$, and for each $j$ and each vertex $v$ in $S_j$, $v$ is incident to $\left\lceil \frac{(q+1)(k-1)}{2} \right\rceil$ or $\left\lfloor \frac{(q+1)(k-1)}{2} \right\rfloor$ edges with colors other than $j$ (so, within round off, half of the edges from each vertex in $S_j$ to vertices in other parts have color $j$). We say each vertex in $S_j$ has main color $j$.

If $n \geq z$, we get a $Z$-quasi-ordered coloring $c$ of $K_n$ which is an extension of the coloring $\varphi$ on $Z$ if for each $j$ and each $v \in S_j$, $c(vy) = j$ for each $y \in V(K_n) \setminus Z$. If $c$ is $Z$-quasi-ordered then it is quasi-ordered if $c$ restricted to $V(K_n) \setminus Z$ is ordered, and quasi-simply-ordered if $c$ restricted to $V(K_n) \setminus Z$ is simply-ordered.

If $r > 0$ and $q \geq 0$ are integers we let $\mathcal{R}(n, r, q)$ be the set of all $r$-regular subgraphs of $K_n$ spanning precisely $n - q$ vertices (assume $n - q$ is even if $r$ is odd, so the set is nonempty), and if $r \geq 2$ let $\mathcal{C}(n, r, q)$ be the set of all such subgraphs which are connected.

Since $k - 1 = \left\lfloor \frac{2r-2}{q+1} \right\rfloor \leq \frac{2r-2}{q+1}$, we have $r \geq \frac{(q+1)(k-1)}{2} + 1 > \left\lfloor \frac{(q+1)(k-1)}{2} \right\rfloor$. So if $H$ is in $\mathcal{R}(n, r, q)$ or $\mathcal{C}(n, r, q)$, then $H$ contains an edge with each of the $k$ colors on edges within $Z$, because it contains at least one vertex in $S_j$ for each $j$, and fewer than $r$ of the edges incident to this vertex have colors other than $j$. We can get an $\mathcal{R}(n, r, q)$-polychromatic or $\mathcal{C}(n, r, q)$-polychromatic quasi-simply-ordered coloring of $K_n$ with $m > k$ colors by making the color classes $M_t$ on the vertices in $V(K_n) \setminus Z$ for $t = k+1, k+2, \ldots, m$ sufficiently large. If $H \in \mathcal{R}(n, r, q)$, for each $t \in [k+1, m]$ we will need the size of $M_t$ to be at least $q + 1$ more than the sum of the sizes of all previous color classes, while if $H \in \mathcal{C}(n, r, q)$ we will need the size of $M_t$ to be at least $q$ more than the sum of the sizes of all previous classes, with an extra vertex in $M_m$. To try to get optimal polychromatic colorings we make the sizes of these color classes as small as possible, yet satisfying these conditions.

For example, if $r = 2$ and $q = 0$ then $k = \left\lceil \frac{2r-2}{q+1} \right\rceil + 1 = 3$ and $z = k(q+1) = 3$, and we get the quasi-simply-ordered colorings $\varphi_{R_0}$ and $\varphi_{C_0}$ with $|Z| = 3$ of Theorem 2.2 If $r = 2$ and $q = 1$ then $k = 2$ and $z = 4$, and we get the colorings $\varphi_{R_1}$ and $\varphi_{C_1}$ with $|Z| = 4$ of Theorem 2.3.

**Example 1** ($r = 3, q = 0$, so $k = 5, z = 5$). Let $\varphi$ be the edge coloring obtained where $\{v_1, v_2, v_3, v_4, v_5\} = Z$ such that $v_iv_{i+1}$ and $v_iv_{i+2}$ (mod 5) have color $i$. The edges connecting $v_i$ to the remaining vertices in $V(K_n) \setminus Z$ are color $i$. See Figure 3.

**Example 2** ($r = 3, q = 3, k = 2, z = 8$). $Z$ has two color classes, 4 vertices in each. The complete bipartite graph between these two sets of vertices could have two vertex disjoint copies of $K_{2,2}$ of one color and also of the other color, or could have an 8-cycle of each color.

**Example 3** ($r = 4, q = 2, k = 3, z = 9$). So $S_1, S_2, S_3$ each have size $q + 1 = 3$. One way to color the edges between parts is for $j = 1, 2, 3$, each vertex in $S_j$ is incident with 2 edges of color $j$ to vertices in $S_{j+1}$ and 1 edge of color $j$ to a vertex in $S_{j-1}$ (so is incident with one edge of color $j+1$ and two edges of color $j-1$, cyclically). The smallest value of $n$ for which this seed can generate a quasi-simply-ordered $\mathcal{R}(n, 4, 2)$-polychromatic coloring with 5-colors

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is \( n = 45 \) (the 4\(^{th}\) and 5\(^{th}\) color classes would have sizes \( 9 + 2 + 1 = 12 \) and \( 21 + 2 + 1 = 24 \) respectively), while to get a simply-ordered \( R(n, 4, 2) \)-polychromatic coloring with 5 colors you would need \( n \geq 69 \) (color class sizes 3, 3, 9, 18, 36 works).

**Conjecture 11.2.** Let \( r \geq 1 \) and \( q \geq 0 \) be integers such that \( q \leq 2r - 3 \). Let \( k = \left\lfloor \frac{2r-2}{q+1} \right\rfloor + 1 \geq 2 \) and \( z = k(q + 1) \). If \( n \geq z \) and \( n - q \) is even if \( r \) is odd, then there exist optimal quasi-simply-ordered \( R(n, r, q) \) and \( C(n, r, q) \)-polychromatic colorings with seed \( Z \) with parameters \( r, q, k, z \).

It is not hard to check that each of these quasi-simply-ordered colorings does at least as well as a simply-ordered coloring for those values of \( r \) and \( q \). The only question is whether some other coloring does better and the conjecture says no.

What if \( \frac{2r-2}{q+1} < 1 \)? Then \( k = \left\lfloor \frac{2r-2}{q+1} \right\rfloor + 1 = 1 \), which seems to be saying no seed \( Z \) exists with at least 2 colors.

**Conjecture 11.3.** Let \( r \geq 1 \) and \( q \geq 0 \) be integers with \( q \geq 2r - 2 \), \( n \geq q + r + 1 \), and not both \( r \) and \( n - q \) are odd. Then there exists an optimal simply-ordered \( R(n, r, q) \)-polychromatic coloring of \( K_n \). If \( r \geq 2 \) then there exists a \( C(n, r, q) \)-polychromatic coloring of \( K_n \) (unless \( r = 2 \), \( q \geq 2 \), \( n - q \) is odd, and \( n \in [2q + 2, 3q + 1] \)).

Theorem 2.1 says this conjecture is true for \( r = 1 \). Theorem 2.4 says it is true for \( C(n, r, q) \) for \( r = 2 \) and that it would be true for \( R(n, r, q) \) for \( r = 2 \) if Theorem 2.6 held for 2-regular graphs.

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