Quantum conditional entropy for infinite-dimensional systems*

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Abstract

In this paper a general definition of quantum conditional entropy for infinite-dimensional systems is given based on recent work of Holevo and Shirokov [3] devoted to quantum mutual and coherent informations in the infinite-dimensional case. The properties of the conditional entropy such as monotonicity, concavity and subadditivity are also generalized to the infinite-dimensional case.

1 The definition

In this paper a general definition of quantum conditional entropy for infinite-dimensional systems is given based on recent work of Holevo and Shirokov [3] devoted to quantum mutual and coherent informations in the infinite-dimensional case. The necessity of such a generalization is clear in particular from the study of Bosonic Gaussian channels and was stressed also in [7]. We refer to [3] for some notations and preliminaries. Let $A, \ldots$ be quantum systems described by the corresponding Hilbert spaces $\mathcal{H}_A, \ldots$, and let $\Phi$ be a channel from $A$ to $B$ with the Stinespring isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$. Let $\rho_{AR}$ be a purification of a state $\rho_A$ with the reference system $R$ and let $\rho_{BRE}$ be the pure state obtained by action of the operator $V \otimes I_R$. The mutual

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information of the quantum channel $\Phi$ at the state $\rho_A$ is defined similarly to finite-dimensional case (cf. [1], [2]) as

$$I(\rho_A, \Phi) = H(\rho_B R \| \rho_B \otimes \rho_R),$$

and coherent information at the state $\rho_A$ with finite entropy $H(\rho_A)$ is defined as follows [3]

$$I_c(\rho_A, \Phi) = I(\rho_A, \Phi) - H(\rho_A).$$

It is then shown that the above-defined quantity satisfies the inequalities

$$-H(\rho_A) \leq I_c(\rho_A, \Phi) \leq H(\rho_A)$$

and the identity

$$I_c(\rho_A, \Phi) + I_c(\rho_A, \tilde{\Phi}) = 0,$$

(1)

where $\tilde{\Phi}$ is the complementary channel.

Let $A, C$ be two (in general, infinite-dimensional) systems and $\rho_{AC}$ a state such that $H(\rho_C) < \infty$. We define the conditional entropy as

$$H(C|A) = H(\rho_C) - H(\rho_{AC} \| \rho_A \otimes \rho_C),$$

(2)

where $H(\cdot \| \cdot)$ is the relative entropy which takes its values in $[0, +\infty]$, so that $H(C|A)$ is well defined as a quantity with values in $[0, +\infty)$. If in addition $H(\rho_A) < \infty$, one recovers the standard formula $H(C|A) = H(AC) - H(A)$.

Now let $B$ be infinite-dimensional system so that arbitrary state $\rho_{BC}$ can be purified to a pure state $\rho_{ABC}$. Consider the channel $\Phi = \text{Tr}_A$ from $AB$ to $B$ so that $\tilde{\Phi} = \text{Tr}_B$, then

$$I_c(\rho_{AB}, \text{Tr}_B) \equiv I(\rho_{AB}, \text{Tr}_B) - H(\rho_{AB}) = H(\rho_{AC} \| \rho_A \otimes \rho_C) - H(\rho_C) = -H(C|A).$$

(3)

Similarly

$$I_c(\rho_{AB}, \text{Tr}_A) \equiv I(\rho_{AB}, \text{Tr}_A) - H(\rho_{AB}) = H(\rho_{BC} \| \rho_B \otimes \rho_C) - H(\rho_C) = -H(C|B).$$

(4)

From the results of [3] concerning the coherent information, we then obtain that under the condition $H(\rho_C) < \infty$ both $H(C|A), H(C|B)$ are well defined and satisfy

$$|H(C|A)| \leq H(\rho_C), \quad |H(C|B)| \leq H(\rho_C),$$

$$H(C|A) + H(C|B) = 0,$$

(5)

(6)
2 Properties of conditional entropy

Proposition 1. Let \( \rho_{AB} \) be such a state that \( H(\rho_A) < \infty \). The function \( H(A|B) \) has the following properties:

1. monotonicity: the inequality
   \[
   H(A|BC) \leq H(A|B). \tag{7}
   \]
   holds for any systems \( A, B, C \).

2. concavity in \( \rho_{AB} \): if \( \rho_{AB} = \alpha \rho_{AB}^1 + (1 - \alpha) \rho_{AB}^2, \alpha \in [0, 1] \), then
   \[
   H(A|B) \geq \alpha H(A^1|B^1) + (1 - \alpha) H(A^2|B^2). \tag{8}
   \]

3. subadditivity: the inequality
   \[
   H(AB|CD) \leq H(A|C) + H(B|D). \tag{9}
   \]
   holds for any systems \( A, B, C, D \) such that \( H(\rho_A) < \infty, H(\rho_B) < \infty \).

Proof. 1. To prove monotonicity we rewrite the inequality (7) using the definition (2), i.e.
   \[
   H(\rho_A) - H(\rho_{ABC} \parallel \rho_{BC} \otimes \rho_A) \leq H(\rho_A) - H(\rho_{AB} \parallel \rho_B \otimes \rho_A).
   \]
   which is equivalent to
   \[
   H(\rho_{AB} \parallel \rho_B \otimes \rho_A) \leq H(\rho_{ABC} \parallel \rho_{BC} \otimes \rho_A),
   \]
   and the last inequality holds by monotonicity of the relative entropy with respect to taking the partial trace.

2. Let \( P_n^A, P_k^B \) be arbitrary increasing sequences of finite rank projectors in the spaces \( \mathcal{H}_A, \mathcal{H}_B \), strongly converging to the operators \( I_A, I_B \) respectively. Consider the sequence of states
   \[
   \rho_{AB}^{nk} = \lambda_{nk}^{-1}(P_n^A \otimes P_k^B \rho_{AB} P_n^A \otimes P_k^B), \lambda_{nk} = \text{Tr}(P_n^A \otimes P_k^B)\rho_{AB},
   \]
   with the partial states \( \rho_A^{nk}, \rho_B^{nk} \).

   Let us show first that
   \[
   \lim_{n,k \to \infty} H(A_{nk}|B_{nk}) = H(A|B). \tag{10}
   \]
By the definition \((2)\) \(H(A_{nk}|B_{nk}) = H(\rho_A^{nk}) - H(\rho_{AB}^{nk} \otimes \rho_A^{nk}).\) By using lower semi-continuity of the von Neumann entropy \([3]\), we obtain
\[
\lim_{n,k \to \infty} \inf H(\lambda_{nk}\rho_A^{nk}) \geq H(\rho_A).
\]

On the other hand, \([5\) lemma 4] implies
\[
H(\lambda_{nk}\rho_A^{nk}) = H(P_n^A(\Tr B I \otimes P_k^B \rho_{AB} I \otimes P_k^B) P_n^A) \leq H(\Tr B I \otimes P_k^B \rho_{AB} I \otimes P_k^B).
\]

Further, by the dominated convergence theorem for entropy \([3]\) we have
\[
\lim_{n,k \to \infty} H(\Tr B I \otimes P_k^B \rho_{AB} I \otimes P_k^B) = H(\rho_A),
\]

since \(\Tr B (I \otimes P_k^B \rho_{AB} I \otimes P_k^B) \leq \rho_A\) and \(H(\rho_A) < \infty\). Thus by the theorem about the limit of the intermediate sequence we obtain
\[
\lim_{n,k \to \infty} H(\lambda_{nk}\rho_A^{nk}) = H(\rho_A), \text{ hence, } \lim_{n,k \to \infty} H(\rho_A^{nk}) = H(\rho_A).
\]

Then we prove that \(\lim_{n,k \to \infty} H(\rho_{AB}^{nk} \| \rho_A^{nk} \otimes \rho_B^{nk}) = H(\rho_{AB} \| \rho_A \otimes \rho_B).\) Consider the following values
\[
H_{nk} = H(\rho_{AB}^{nk} \| \rho_B^{nk} \otimes \rho_A^{nk}) = H(\rho_A^{nk}) + H(\rho_B^{nk}) - H(\rho_{AB}^{nk}),
\]
\[
\tilde{H}_{nk} = H(\rho_{AB}^{nk} \| (\eta^{-1}_k P_k^B \rho_B^P \otimes \mu^{-1}_n P_n^A \rho_A^P)) = -H(\rho_{AB}^{nk}) - \Tr (\rho_A^{nk}) \log (\mu^{-1}_n P_n^A \rho_A^P) - \Tr (\rho_B^{nk}) \log (\eta^{-1}_k P_k^B \rho_B^P),
\]
where \(\mu_n = \Tr P_n^A \rho_A, \eta_k = \Tr P_k^B \rho_B.\)

By using again \([5\) lemma 4] we have
\[
\lim_{n,k \to \infty} \tilde{H}_{nk} = \lim_{n,k \to \infty} H(\rho_{AB} \| (\rho_B^P \otimes \rho_A^P) = H(\rho_{AB} \| \rho_B \otimes \rho_A).
\]

We will prove that \(\lim_{n,k \to \infty} H_{nk} = H(\rho_{AB} \| \rho_A \otimes \rho_B)\) by considering the difference \(H_{nk} - \tilde{H}_{nk}.\) After some calculation we obtain that the difference tends to zero:
\[
\lim_{n,k \to \infty} (H_{nk} - \tilde{H}_{nk}) = \lim_{n,k \to \infty} (H(\rho_A^{nk} \| \mu^{-1}_n P_n^A \rho_A^P) + H(\rho_B^{nk} \| \eta^{-1}_k P_k^B \rho_B^P)) = 0.
\]
The last double limit is equal to zero since it follows from \[5\], lemma 4, that

\[
0 \leq H(\lambda_{nk}\rho_A^{nk} A_{nk}\rho_A^{nk} A_{nk}) = H(P_n^A \Tr B(I_A \otimes P_k^B \rho_{AB} I_A \otimes P_k^B) P_n^A \| P_n^A \rho_A P_n^A) \leq H(\Tr B(I_A \otimes P_k^B \rho_{AB} I_A \otimes P_k^B) \| \rho_A),
\]

and \[3\], lemma 7, implies

\[
\lim_{n,k \to \infty} H(\Tr B(I_A \otimes P_k^B \rho_{AB} I_A \otimes P_k^B) \| \rho_A) = H(\rho_A \| \rho_A) = 0.
\]

Thus, the theorem about the limit of the intermediate sequence implies

\[
\lim_{n,k \to \infty} H(\lambda_{nk}\rho_A^{nk} A_{nk}\rho_A^{nk} A_{nk}) = \lim_{n,k \to \infty} H(\rho_A^{nk} A_{nk}\rho_A^{nk} A_{nk}) = 0.
\]

Similarly we obtain that the second summand of the difference \(H_{nk} - \tilde{H}_{nk}\) also tends to zero:

\[
\lim_{n,k \to \infty} H(\rho_B^{nk} \eta_{nk} P_k^B P_k^B) = 0.
\]

Finally, we have

\[
\lim_{n,k \to \infty} H(\rho_{AB}^{nk} \rho_{AB}^{nk} \otimes \rho_{AB}^{nk}) = H(\rho_{AB} \| \rho_B \otimes \rho_A), \text{ hence } \lim_{n,k \to \infty} H(A_{nk} \| B_{nk}) = H(A \| B).
\]

To prove the concavity of the function \(H(A-B)\) let us consider

\[
\rho_{AB} = \alpha \rho_{AB}^1 + (1-\alpha) \rho_{AB}^2, \alpha \in [0,1].
\]

Also consider again

\[
\rho_{AB}^{nk} = \lambda_{nk}^{-1} P_n^A \otimes P_k^B \rho_{AB} P_n^A \otimes P_k^B =
\]

\[
= \frac{\alpha( P_n^A \otimes P_k^B \rho_{AB}^1 P_n^A \otimes P_k^B) + (1-\alpha)( P_n^A \otimes P_k^B \rho_{AB}^2 P_n^A \otimes P_k^B) }{\alpha \Tr( P_n^A \otimes P_k^B \rho_{AB}^1) + (1-\alpha) \Tr( P_n^A \otimes P_k^B \rho_{AB}^2) } = \frac{\theta_{nk}^{1nk} \rho_{AB}^1 + \theta_{nk}^{2nk} \rho_{AB}^2}{\theta_{nk}^1 + \theta_{nk}^2},
\]

\[
\theta_{nk}^1 = \alpha \Tr( P_n^A \otimes P_k^B \rho_{AB}^1), \theta_{nk}^{1nk} = \frac{\alpha P_n^A \otimes P_k^B \rho_{AB}^1 P_n^A \otimes P_k^B}{\theta_{nk}^1},
\]

\[
\theta_{nk}^2 = (1-\alpha) \Tr( P_n^A \otimes P_k^B \rho_{AB}^2), \theta_{nk}^{2nk} = \frac{(1-\alpha) P_n^A \otimes P_k^B \rho_{AB}^2 P_n^A \otimes P_k^B}{\theta_{nk}^2}.
\]
Due to concavity of the condition information on the finite rank states we can write that
\[ H(A_{nk}|B_{nk}) \geq \frac{\theta^1_{nk}}{\theta^1_{nk} + \theta^2_{nk}} H(A^1_{nk}|B^1_{nk}) + \frac{\theta^2_{nk}}{\theta^1_{nk} + \theta^2_{nk}} H(A^2_{nk}|B^2_{nk}) \]

Taking the limit and using (10) for both parts of inequality we obtain the assertion of (8). Thus, concavity is proved.

3. A direct verification shows that in the finite-dimensional case
\[ H(AB|CD) = H(A|CD) + H(B|CD) - (H(A|CD) - H(A|BCD)), \] (11)
which implies
\[ H(AB|CD) \leq H(A|CD) + H(B|CD), \] (12)
since the value in brackets in (11) is nonnegative. We will prove the inequality (12) in infinite-dimensional case, the subadditivity property (9) can be derived from (12) by using monotonicity of the conditional entropy.

Let \( \rho_{ABCD} = \rho \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D \) be a state with \( H(\rho_A) < \infty, H(\rho_B) < \infty \). Consider an arbitrary increasing sequence of finite rank projectors \( \{P_{iCD}^l\} \) which strongly converges to the operator \( I_{CD} \). Also consider the sequence of states
\[ \rho^l = \tau^{-1}_l (I_A \otimes I_B \otimes P_{iCD}^l) \rho (I_A \otimes I_B \otimes P_{iCD}^l), \tau_l = \text{Tr}(I_A \otimes I_B \otimes P_{iCD}^l) \rho. \]
The relation (11) (which implies (12)) holds for \( \rho^l \) since all the summands in (11) are finite.

We will show that \( H(A^l|CD^l) \rightarrow H(A|CD) \). Actually, using the definition (2) we have
\[ H(A^l|CD^l) = H(\rho^l_A) - H(\rho^l_{ACD}\|\rho^l_{CD} \otimes \rho^l_A). \]
It follows from the dominated convergence theorem for entropy [4] that
\[ \lim_{l \to \infty} H(\rho^l_A) = \lim_{l \to \infty} H(\tau^{-1}_l(I_A \otimes I_B \otimes P_{iCD}^l) \rho) = H(\rho_A). \]

Further, consider the values
\[ H_l = H(\rho^l_{ACD}\|\rho^l_{CD} \otimes \rho^l_A) = -H(\rho^l_{ACD}) + H(\rho^l_A) + H(\rho^l_{CD}). \]
and
\[ \tilde{H}_l = H(\rho^l_{ACD}\|\rho^l_{CD} \otimes \rho_A) = H(\tau^{-1}_l I_A \otimes P_{iCD}^l [\rho_{ACD}] I_A \otimes P_{iCD}^l \| \rho_A \otimes \tau^{-1}_l P_{iCD}^l [\rho_{CD}] P_{iCD}^l) = \]
\[-H(\rho_{ACD}) + H(\rho_{CD}) + \text{Tr} \left( \tau_l^{-1} \text{Tr}_{CD} I_A \otimes P^C_D[\rho_{ACD}] \right)(-\log \rho_A)\].

Then \([5, \text{lemma 4}]\) implies that
\[
\lim_{l \to \infty} \tilde{H}_l = H(\rho_{ACD}\|\rho_A \otimes \rho_{CD}).
\]

On the other hand, after the calculation and using \([3, \text{lemma 7}]\) we obtain
\[
\lim_{l \to \infty} (\tilde{H}_l - H_l) = \lim_{l \to \infty} H \left( \tau_l^{-1} \text{Tr}_{CD} I_A \otimes P^C_D[\rho_{ACD}]\|\rho_A \right) = 0.
\]

This implies that
\[
\lim_{l \to \infty} H(\rho^C_{ACD}\|\rho^C_A \otimes \rho^C_{CD}) = H(\rho_{ACD}\|\rho_A \otimes \rho_{CD}), \text{ hence, } \lim_{l \to \infty} H(A^l|CD^l) = H(A|CD).
\]

In the similar way we obtain \(H(B^l|CD^l) \to H(B|CD)\) and \(H(AB^l|CD^l) \to H(AB|CD)\). Thus the statement \([12]\) is proved in infinite-dimensional case, hence, the subadditivity property holds. \(\Box\)

The following observation is due to M. E. Shirokov.

**Proposition 2.** Let \(A\) be a subset of \(\mathcal{S}(\mathcal{H}_{AC})\) such that the von Neumann entropy is continuous on the set \(A^C = \text{Tr}_A A \subset \mathcal{S}(\mathcal{H}_C)\). Then the function \(\rho_{AC} \mapsto H(C|A)\) is continuous on the set \(A\).

**Proof.** Let \(\{\rho_n\} \subset A\) be a sequence converging to a state \(\rho_0 \in A\). By the well known results of purification theory there exists a corresponding sequence of purifications \(\{\hat{\rho}_n\} \subset \mathcal{S}(\mathcal{H}_{ABC})\) converging to a purification \(\hat{\rho}_0 \in \mathcal{S}(\mathcal{H}_{ABC})\) of the state \(\rho_0\). The sequence \(\{\text{Tr}_C \hat{\rho}_n\}\) converges to \(\text{Tr}_C \hat{\rho}_0\) and \(\lim_{n \to +\infty} H(\text{Tr}_C \hat{\rho}_n) = H(\text{Tr}_C \hat{\rho}_0)\) by the condition (since \(H(\text{Tr}_C \hat{\rho}_n) = H(\text{Tr}_{AB} \hat{\rho}_n)\)). Proposition 4 in \([3]\) implies
\[
\lim_{n \to +\infty} I_c(\text{Tr}_C \hat{\rho}_n, \text{Tr}_B) = I_c(\text{Tr}_C \hat{\rho}_0, \text{Tr}_B).
\]

\(\Box\)

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