Stable SIP Discontinuous Galerkin Approximations of the Hydrostatic Stokes Equations

F. Guillén González, M.V. Redondo Neble, J.R. Rodríguez Galván

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Abstract

We propose a Discontinuous Galerkin (DG) scheme for the numerical solution of the Hydrostatic Stokes equations in Oceanography. This new scheme is based on the introduction of the symmetric interior penalty (SIP) technique for the Hydrostatic Stokes mixed variational formulation. Recent research showed that stability of the mixed formulation of Primitive Equations requires LBB (Ladyzhenskaya–Babuška–Brezzi) inf-sup condition and an extra hydrostatic inf-sup restriction relating the pressure and the vertical velocity. This hydrostatic inf-sup condition invalidates usual Stokes continuous finite elements like Taylor-Hood $P_2/P_1$ or bubble $P_{1b}/P_1$. Here we consider $P_k/P_k$ discontinuous finite elements and, using adequate LBB-like and hydrostatic discrete inf-sup conditions we can demonstrate stability of the SIP DG scheme in the natural energy norm for this problem. Finally, according numerical tests are provided.

1 Introduction

In this work we delve into the stability of a discrete Discontinuous Galerkin formulation for the Hydrostatic Stokes equations (or Primitive Equations of the ocean), where a penalization of interior jumps of velocity, based on the symmetric interior penalty (SIP) technique [Arn82], is introduced. We show that this new formulation allows writing the equations as a mixed (Stokes-like) problem which satisfies the well-known LBB (Ladyzhenskaya–Babuška–Brezzi) condition and also the hydrostatic inf-sup restriction which has been observed in the Hydrostatic Stokes model [Azé00, Azé96, Azé94, GGRG16, GGRG15a, GGRG15b]. Thus, the Hydrostatic Stokes equations can be approximated in standard Finite Element (FE) meshes without vertical integration (customary in most ocean and atmosphere models).

The equations of geophysical fluid dynamics governing the motion of the ocean and atmosphere are derived from the conservation laws from physics. In the case of large scale ocean (see e.g. [CB09]), the resulting system is too complex and, from a practical point of view, numerous simplifications are introduced, including the “small layer” hypothesis:

$$\varepsilon = \frac{\text{vertical scale}}{\text{horizontal scale}} \text{ is very small,}$$

for example a few Kms over some thousand Kms, that is $\varepsilon \approx 10^{-3}, 10^{-4}$.

Variables like temperature and salinity will not be considered, so that constant density is assumed (although this work could be extended to the general variable-density case in future works). Thus we can focus on the momentum law, leading to the Navier-Stokes equations.
The anisotropic domain, after a vertical scaling, is transformed into the following isotropic or adimensional (independent of $\varepsilon$) domain

$$\Omega = \{(x, z) \in \mathbb{R}^3 / x = (x, y) \in S, -D(x) < z < 0\},$$

where $S \subset \mathbb{R}^2$ is the surface domain and $D = D(x)$ is the bottom function. Here, the rigid lid hypothesis has been assumed (no vertical displacements of the free surface of the ocean). We decompose the boundary into three parts: the surface, $\Gamma_s = S \times \{0\}$, the bottom, $\Gamma_b = \{(x, -D(x)) / x = (x, y) \in S\}$, and the talus or lateral walls, $\Gamma_t = \{(x, z) / x \in \partial S, -D(x) < z < 0\}$.

Finally, a $\varepsilon$-dependent scaling of vertical velocity is introduced (see [AG01]), leading to the following equations in the time-space domain $(0, T) \times \Omega$ (called Anisotropic or Quasi-Hydrostatic Navier-Stokes Equations and, for the limit case $\varepsilon = 0$, Hydrostatic Navier-Stokes or Primitive Equations) where we denote $\varepsilon = \nu \varepsilon^2$:

$$\partial_t u + (u \cdot \nabla u) + v \partial_z u - \nu \Delta u + \nabla x p = f,$$

$$\varepsilon \left(\partial_t v + (u \cdot \nabla u) v + v \partial_z v - \nu \Delta v \right) + \partial_z p = -g,$$

$$\nabla x \cdot u + \partial_z v = 0.$$

Here $\nabla_x = (\partial_x, \partial_y)^T$, $\nabla_x \cdot u = \partial_x u_1 + \partial_y u_2$ and $\nu$ is the (adimensional kinematic) viscosity. The unknowns are the 3D velocity field, $(u, v) : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ and the pressure, $p : \Omega \times (0, T) \rightarrow \mathbb{R}$. The term $f = (f_1, f_2)^T$ models a given horizontal force while $g$ involves the force due to gravity, which can be written in a potential form and incorporated to the pressure term, hence it can be assumed $g = 0$ in (2). Other phenomena like the effects due to the Coriolis acceleration are not considered because they are linear terms not affecting to the results presented in this work. The system is endowed with initial values for the velocity field, $(u, v)|_{t=0} = (u_0, v_0)$ and adequate boundary conditions, for instance:

$$\nu \partial_z u|_{\Gamma_s} = g_s, \quad v|_{\Gamma_s} = 0,$$

$$u|_{\Gamma_s \cup \Gamma_t} = 0, \quad v|_{\Gamma_b} = 0,$$

$$\varepsilon \nabla x v \cdot n_x|_{\Gamma_t} = 0,$$

where $g_s$ represents the wind stress, $n_x$ is the horizontal part of the normal vector.

The limit of the Hydrostatic Equations (1)–(3) when $\varepsilon \rightarrow 0$ is studied on rigorous mathematical grounds in [BL92] (for the stationary case) and [AG01] (for the evolutive case). Most of existence and regularity results (see e.g. [CG00, CR05, CT07, CB09, GR17]) for (1)–(3), and also the major part of the associated numerical schemes (see e.g. [CG00, CR05]) are based on the introduction of an equivalent integral-differential problem, by doing a vertical integration of the vertical momentum equation (2). From the numerical point of view, this idea has advantages (it is only necessary to compute a 2D pressure, only defined in the surface $S$) but also some drawbacks (for instance, standard FE in unstructured meshes, variable density and non-hydrostatic cases are difficult to handle).

In this work we are concerned on the linear steady model related to (1)–(3) and in the less favorable limit case, $\varepsilon = 0$. Results shall be extended to the non-hydrostatic case $\varepsilon > 0$ in further works. The case $\varepsilon = 0$ is known as Hydrostatic Stokes equations and its mixed
variational formulation reads: find \((u, v, p) \in U \times V \times P\) such that
\[
\nu(\nabla u, \nabla u) - (p, \nabla \cdot u) = (f, u) + (g_s, u)|_{\Gamma_s} \quad \forall u \in U, \quad (\text{7})
\]
\[
(p, \partial_z v) = 0 \quad \forall v \in V, \quad (\text{8})
\]
\[
(\nabla \cdot (u, v), p) = 0 \quad \forall p \in P. \quad (\text{9})
\]

Here \((\cdot, \cdot)\) is the \(L^2(\Omega)\) scalar product, \((\cdot, \cdot)_{\Gamma_s}\) is the \(L^2(\Gamma_s)\) scalar product and we define
\[
U = H^1_{b,t}(\Omega) = \left\{ \begin{array}{l}
u \in H^1(\Omega)^2 \mid \nu|_{\Gamma_b \cup \Gamma_t} = 0 \end{array} \right\},
\]
\[
V = H^1_{z,0}(\Omega) = \left\{ v \in L^2(\Omega) \mid \partial_z v \in L^2(\Omega), \; v|_{\Gamma_s \cup \Gamma_b} = 0 \right\},
\]
\[
P = L^2_0(\Omega) = \left\{ p \in L^2(\Omega) \mid \int_\Omega p = 0 \right\}.
\]

The space \(U\) is endowed with the norm \(\|\nabla u\|\) (hereafter \(\|\cdot\|\) denotes the \(L^2(\Omega)\)-norm) while in \(V\) we consider \(\|\partial_z v\|\), which is a norm owing to the homogeneous Dirichlet condition \(v|_{\Gamma_s \cup \Gamma_b} = 0\) and a vertical Poincaré inequality.

As stated in [GGRG15a, Azé94], well-posedness of (7)–(9) hinges on the following inf-sup conditions:
\[
\beta_p \|p\| \leq \sup_{0 \neq (u, v) \in U \times V} \frac{(\nabla \cdot (u, v), p)}{\|\nabla (u, \partial_z v)\|} \quad \forall p \in P, \quad (IS)^P
\]
\[
\|\partial_z v\| \leq \sup_{0 \neq p \in P} \frac{(\partial_z v, p)}{\|p\|} \quad \forall v \in V, \quad (IS)^V
\]
where \(\|\nabla (u, \partial_z v)\|\) is the norm of \(U \times V\). Note that \((IS)^P\) is basically the well-known LBB condition while \((IS)^V\) is a new hydrostatic restriction.

In the discrete setting, it was shown in [GGRG15a] (see also [Azé94]) that the discrete counterpart of inf-sup condition \((IS)^P\) is no longer sufficient for stability of standard conforming FE approximations of (7)–(9), because it is also necessary to choose FE spaces satisfying the discrete counterpart of \((IS)^V\). Unfortunately, standard Stokes FE like Taylor-Hood \(P_2-P_1\) or \((P_1+\text{bubble})-P_1\) do not satisfy \((IS)^V\). Thus different FE must be considered (for instance, by approximation of vertical velocity in a space other than horizontal velocity, see [GGRG15a, GGRG16]).

A different idea was introduced in [GGRG15b], where discrete \((IS)^V\) is avoided by adding a consistent stabilizing term to the vertical momentum equation (8). In this way, the stability for Stokes-LBB FE combinations is shown and error estimates are provided for Taylor-Hood \(P_2-P_1\) FE and mini-element \((P_1+\text{bubble})-P_1\) approximations, showing optimal convergence order in the \(P_2-P_1\) case.

The current work introduces a third approach that, until now, has not been explored: using Discontinuous Galerkin (DG) methods, we can define approximations that, without any stabilization, satisfy (in some sense) both \((IS)^V\) and \((IS)^P\) restrictions.

DG methods, which are well suited for the construction of stable discretizations of compressible (advection-dominated) flows and in general for hyperbolic operators, have been extended also for incompressible flows (for a review, see e.g. [CKS11, ABCM02, DPE12] and references therein) and in general for elliptic operators. More in detail, DG methods for second order elliptic operators can be split roughly into two groups: first, the so called Local
Discontinuous Galerkin (LDG) schemes, where the operator is converted into a system of first order equations and numerical fluxes are devised as in hyperbolic equations [ABCM02]. On the other hand, the schemes augmenting the elliptic operator by penalizing the discontinuities of the shape functions [DD76, Arn82]. These latter schemes are known as Interior Penalty (IP) DG method (SIP DG methods in the usual symmetric case).

In the same way, DG schemes for compressible (and for incompressible) flows can be split into two groups: some of them are based in the LDG schemes [CKSS02] while other discretizations are based on the IP method [HL02, DPE12]. The scheme presented here is based in the latter methods and our main contribution is in the design of a SIP DG scheme where, somehow, the Hydrostatic restriction (IS)\(^V\) is verified, in addition to the LBB-like restriction (IS)\(^P\).

This paper is structured as follows: in Section 2 we fix notation and introduce some useful results from SIP DG approximation of diffusion equations. In Section 3 we introduce a SIP DG approximation for (7)–(9) where the velocity field and the pressure unknowns are defined by the same \(k\)-degree discontinuous \(P_k\) polynomials. In Section 4 we show well-posedness for this approximation of the Hydrostatic Equations and in Section 5 some numerical tests are shown which agree with the theory.

## 2 SIP DG Approximation of Diffusion Equations

We start fixing notations and collecting some results which shall be useful in following sections. Let us denote by \(\mathcal{T}_h\) a family of meshes of the domain \(\Omega \subset \mathbb{R}^d\) \((d = 2 \text{ or } 3 \text{ in practice})\) into non-degenerate disjoint simplicial elements \(K\) satisfying usual regularity assumptions[Cia78]:

\[
\exists \rho > 0 \ / \ \rho h_K \leq r_K \quad \forall K \in \mathcal{T}_h,
\]

where \(h_K\) is the diameter of \(K\) and \(r_K\) is the radius of the largest ball inscribed in \(K\). The number of edges (faces) of the elements is denoted as \(N_{\partial}\). Note that more general meshes can also be handled, specifically elements \(K\) are not required to be simplicial elements and \(\mathcal{T}_h\) can be any shape and contact-regular mesh, see e.g. [DPE12], Section 1.4.

We associate to each triangulation \(\mathcal{T}_h\) the set of interior faces (edges in 2D) \(\mathcal{E}_h^0\) and the set of boundary faces \(\mathcal{E}_h^\partial\), defined as follows: \(e \in \mathcal{E}_h^0\) if there are two polyhedra \(K^+\) and \(K^-\) \(\in \mathcal{T}_h\) such that \(e = K^+ \cap K^-\) and \(e \in \mathcal{E}_h^\partial\) if there is \(K \in \mathcal{T}_h\) such that \(e = \partial K \cap \partial \Omega\). We define \(\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial\).

Let \(u\) be a scalar-valued function on \(\Omega\) and assume that \(u\) is smooth enough to admit on all \(e \in \mathcal{E}_h^\partial\) a (possibly two-valued) trace. We define the jump and the average of \(v\) on \(e \in \mathcal{E}_h^0\), denoted respectively as \([u]\) and \(\{u\}\), as follows: if \(e = K^+ \cap K^-\), then

\[
[u]_e = u|_{K^+} - u|_{K^-} \quad \text{and} \quad \{u\}_e = \frac{1}{2}(u|_{K^+} + u|_{K^-}).
\]

If \(e \in \mathcal{E}_h^\partial\), we define \([u] = \{u\} = u|_e\).

Let us define the following broken discrete Sobolev space, for each \(m \geq 0\),

\[
H^m(\mathcal{T}_h) = \{ u \in L^2(\Omega) \mid u \in H^m(K) \ \forall K \in \mathcal{T}_h \},
\]

the broken gradient operator \(\nabla_h\) for each \(u \in H^1(\mathcal{T}_h)\),

\[
(\nabla_h u)|_K = \nabla (u|_K) \quad \forall K \in \mathcal{T}_h
\]
and the following finite-dimensional subspace of $H^m(\mathcal{T}_h)$, composed of polynomials of degree no more than $k$ in each element:

\[
\mathcal{P}_h^k := \left\{ u \in L^2(\Omega) \mid u \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h \right\}.
\]

The following discrete trace inequality shall be useful: for all $u_h \in \mathcal{P}_h^k$ and $K \in \mathcal{T}_h$,

\[
h_K^{1/2} \| u_h | K \|_{L^2(\partial K)} \leq C_{tr} \| u_h \|_{L^2(K)} ,
\]

(10)

where $C_{tr}$ is a constant independent of $h$ and $K$ (and depending on $k$, $d$ and $\rho$). See e.g. [DPE12], Lemma 1.46 and Remark 1.47, for details. From this inequality, one has the following technical result.

**Lemma 1.** For every $p_h \in \mathcal{P}_h^k$, the following inequalities are satisfied, for constant $C > 0$ independent of $h$ (and dependent on $k$, $d$ and $\rho$):

\[
\left( \sum_{e \in \mathcal{E}_h} h_e \int_e \{ p_h \}^2 \right)^{1/2} \leq C \| p_h \|_{L^2(\Omega)},
\]

(11)

\[
\left( \sum_{e \in \mathcal{E}_h} h_e \int_e [ p_h ]^2 \right)^{1/2} \leq C \| p_h \|_{L^2(\Omega)}.
\]

(12)

**Proof.** First we observe that, for all $e \in \mathcal{E}_h^0$ with $e = \partial K_1 \cap \partial K_2$, one has $[ p_h ]^2 \leq 2(p_i^2 + p_j^2)$ and $\{ p_h \}^2 \leq (p_i^2 + p_j^2)/2$, where $p_i = p_h | K_i$, $i \in \{1, 2\}$. Also if $e \in \mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial \Omega$, by definition, $[ p_h ]^2 = p_i^2$ and $\{ p_h \}^2 = p_i^2$. In any case, we can write

\[
[p_h ]^2 \leq C_s(p_i^2 + p_j^2) \quad \text{and} \quad \{ p_h \}^2 \leq C_s(p_i^2 + p_j^2),
\]

for some constant $C_s > 0$. Therefore

\[
\sum_{e \in \mathcal{E}_h} h_e \int_e \{ p_h \}^2 \leq C_s \sum_{e \in \mathcal{E}_h} h_e \int_e (p_i^2 + p_j^2)
\]

\[
\leq 2C_s \sum_{K \in \mathcal{T}_h} h_K \sum_{e \in \mathcal{E}_{K^\partial} \cap \partial K} \int_e p_h^2 \leq 2N \rho C_s \sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} p_h^2 \leq C \| p_h \|^2,
\]

where $C = 2N \rho C_s C_{tr}$ and $C_{tr}$ is the constant introduced in (10). Inequality (12) can be shown similarly. \hfill \square

Let us consider the following symmetric interior penalty (SIP) bilinear form for discontinuous FE approximation of second order elliptic and parabolic equations:

\[
d_h^{\text{sip}}(u, \overline{u}) = \int_\Omega \nabla_h u \cdot \nabla_h \overline{u} - \sum_{e \in \mathcal{E}_h} \int_e \left( \{ \nabla_h u \} \cdot \mathbf{n}_e [ \overline{\pi} ] + [ u ] \{ \nabla_h u \} \cdot \mathbf{n}_e \right) + \eta \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e [ u ] [ \overline{\pi} ],
\]

(13)

for each $u, \overline{u} \in \mathcal{P}_h^k$. Here $\mathbf{n}_e = (\mathbf{n}_x, \mathbf{n}_z) \in \mathbb{R}^d$ denotes the normal vector (in a fixed chosen sense) across the edge or face $e$, $h_e$ is the diameter of $e$ and $\eta > 0$ is a constant. The second term at RHS of (13) arises for consistency and symmetry, while the last one introduces a
penalization on interior faces and boundary faces which enforces coercivity. Also boundary values are penalized which, assuming Dirichlet boundary conditions, is used impose weakly these conditions. Indeed, one has the following coercivity result in $\mathcal{P}^k_h$ (see e.g. [DPE12], Lemma 4.12) for the norm

$$\|u\|_{\text{sip}} = \left( \|\nabla_h u\|^2 + |u|^2_{U^*} \right)^{1/2},$$
where $|u|_U = \left( \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e [u]^2 \right)^{1/2}$.

**Lemma 2** (Coercivity for $a^\text{sip}_{\eta}(\cdot,\cdot)$). Let us denote $\eta_* = C^2_{tr}$, with $C_{tr}$ given in (10). For all $\eta > \eta_*$, one has

$$a^\text{sip}_{\eta}(u_h, u_h) \geq C(\eta) \|u_h\|^2_{\text{sip}}, \quad \forall u_h \in \mathcal{P}^k_h,$$

where $C(\eta) = (\eta - \eta_*)/(1 + \eta)$.

One also has boundedness on $\mathcal{P}^k_h$ (see e.g. [DPE12], Lemmas 4.16 and 4.20):

**Lemma 3** (Boundedness). There is $C_{\text{bnd}} > 0$ independent of $h$ (and depending on $\eta$) such that

$$a^\text{sip}_{\eta}(u_h, \bar{u}_h) \leq C_{\text{bnd}} \|u_h\|_{\text{sip}} \|\bar{u}_h\|_{\text{sip}}, \quad \forall u_h, \bar{u}_h \in \mathcal{P}^k_h.$$

### 3 SIP DG Discretization of the Hydrostatic Stokes Equations

In this section we introduce the discrete variational formulation for the Hydrostatic problem (7)–(9) based on an SIP DG approximation. For simplicity, homogeneous Dirichlet boundary conditions are considered (in particular we take $g_s = 0$) although Neumann conditions can also be imposed in practice, as outlined in Section 5.

Here we introduce the same polynomial order for the velocity field $w_h = (u_h, v_h)$ and the pressure $p_h$ spaces:

$$U_h = (\mathcal{P}^k_h)^{d-1}, \quad V_h = \mathcal{P}^k_h, \quad W_h = U_h \times V_h = (\mathcal{P}^k_h)^d,$$
$$P_h = \mathcal{P}^k_h.$$

For each $w_h = (u_h, v_h)$ and $\bar{w}_h = (\bar{u}_h, \bar{v}_h) \in W_h$, with $u_h = (u_i)_{i=1}^{d-1}$ and $\bar{u}_h = (\bar{u}_i)_{i=1}^{d-1}$, we define the following bilinear form associated to (7)–(9):

$$a_h(w_h, \bar{w}_h) = \nu \left( \sum_{i=1}^{d-1} a^\text{sip}_{\eta}(u_i, \bar{u}_i) + \eta \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e [v_h n_z] \|[\bar{v}_h n_z]\] \right),$$

where SIP bilinear form for vertical velocity is not introduced (due to the lack of diffusive terms in vertical momentum equations) although a penalization term for $v_h$, in vertical direction, is present in interior and boundary faces.

The next step consists in introducing a suitable norm on $w_h = (u_h, v_h) \in W_h$ for which a generalized coercivity result can be obtained. Note that using Lemma 2 we have only

$$a_h(w_h, w_h) \geq \nu \left( C(\eta) \|u_h\|^2_{\text{sip}} + \eta |v_h|^2_{V^*} \right),$$

where $C(\eta) = (\eta - \eta_*)/(1 + \eta)$. 


with
\[ \| \mathbf{u}_h \|_{\text{sip}}^2 = \sum_{i=1}^{d-1} \| u_i \|_{\text{sip}}^2, \quad |v_h|_{\text{V}}^2 = \sum_{e \in E_h} \frac{1}{h_e} \int_e \| v_h n_z \|^2. \]

If we define the following “isotropic” velocity norm
\[ \| \mathbf{w}_h \|_{\text{iso}} = (\| \mathbf{u}_h \|_{\text{sip}}^2 + \| v_h \|_{\text{sip}}^2)^{1/2}, \]
then no control for \( \| v_h \|_{\text{sip}}^2 \) can be obtained. In order to avoid this obstacle, we introduce the following anisotropic or hydrostatic velocity norm:
\[ \| \mathbf{w}_h \|_{\text{anis}} = (\| \mathbf{u}_h \|_{\text{sip}}^2 + \| \partial_{z,h} v_h \|^2 + |v_h|_{\text{V}}^2)^{1/2}, \]
where \( \partial_{z,h} \) is the broken vertical derivative (which is defined similarly to \( \nabla_h \)). Although inequality (16) does not allow to infer the coercivity of \( a_h(\cdot,\cdot) \) for \( \| \cdot \|_{\text{anis}} \) (due to the lack of control for \( \partial_{z,h} v_h \)), one has the following inf-sup bound for \( \| \partial_{z,h} v_h \| \) in terms of \( P_h \):

**Lemma 4** (Stability for \( \partial_{z,h} v_h \)). It holds
\[ \| \partial_{z,h} v_h \| = \sup_{\mathbf{p}_h \in P_h} \frac{\int_{\Omega} \mathbf{p}_h \cdot \partial_{z,h} v_h}{\| \mathbf{p}_h \|} \quad \forall \mathbf{v}_h \in V_h. \] (17)

**Proof.** Given \( \mathbf{v}_h \in V_h \), it suffices to note that the supremum of (17) is reached for \( \mathbf{p}_h = \partial_{z,h} v_h \in P_h \subset L^2(\Omega) \).

**Remark 1.** In general, \( \partial_{z,h} v_h \notin L^2(\Omega) \), thus previous result is not clear taking supreme on zero-mean discrete pressures \( P_h \cap L^2(\Omega) \).

At this point, well-posedness of the discrete problem hinges on a bound of \( \| p_h \| \). The problem is that, for general (non zero-mean) pressures, it cannot be obtained by the well-known discrete inf-sup (or LBB) condition. For this reason, a specific DG Galerkin inf-sup condition for bounding \( \| p - \langle p \rangle_\Omega \| \) is now introduced (where \( \langle p \rangle_\Omega \) denotes the mean of \( p \) in \( \Omega \)). Let us define the following discrete bilinear form:
\[ b_h(\mathbf{w}_h, p_h) = -\int_{\Omega} \mathbf{p}_h \nabla_h \cdot \mathbf{w}_h + \sum_{e \in E_h} \int_e [\mathbf{w}_h] \cdot \mathbf{n}_e \langle \mathbf{p}_h \rangle \]
where \( \nabla_h \) is the “broken” divergence operator (defined on each \( K \in T_h \)). It is not difficult to show continuity for \( b_h(\mathbf{w}_h, p_h) \) with \( \| \mathbf{w}_h \|_{\text{anis}} \) and \( \| p_h \| \) in \( W_h \times P_h \). The following property is also satisfied:
\[ b_h(\mathbf{w}_h, 1) = 0 \quad \forall \mathbf{w}_h \in W_h. \] (18)

Let us consider the following pressure seminorm in \( H^1(T_h) \supset P_h^k \):
\[ |p|_P = \left( \sum_{e \in E_h^0} h_e \| [p] \|^2_{L^2(e)} \right)^{1/2}. \]

**Lemma 5** (Stability for \( p_h \)). There exists \( \gamma_p > 0 \) independent of \( h \), such that
\[ \gamma_p \| p_h - \langle p_h \rangle_\Omega \| \leq \sup_{\mathbf{w}_h \in W_h \setminus \{0\}} \frac{b_h(\mathbf{w}_h, p_h)}{\| \mathbf{w}_h \|_{\text{anis}}} + |p_h|_P, \quad \forall p_h \in P_h. \] (19)
Proof. Let \( p_h \in P_h \). It is known (see e.g. [DPE12], Lemma 6.10) that inequality (19) holds for zero-mean pressures if \( \| w_h \|_{\text{anis}} \) is replaced by \( \| w_h \|_{\text{iso}} \). On the other hand, for all \( v_h \in V_h \), we have \( \| \partial_z v_h \|^2 \leq \| \nabla v_h \|^2 \) and then \( \| w_h \|_{\text{anis}} \leq \| w_h \|_{\text{iso}} \) for all \( w_h \in W_h \). Therefore, exists \( \gamma_p > 0 \) such that

\[
\gamma_p \| p_h - \langle p_h \rangle_\Omega \| \leq \sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(w_h, p_h - \langle p_h \rangle_\Omega)}{\| w_h \|_{\text{iso}}} + |p_h - \langle p_h \rangle_\Omega|_P
\]

\[
\leq \sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(w_h, p_h)}{\| w_h \|_{\text{anis}}} + |p_h|_P.
\]

To conclude this section, let us formulate the following DG discretization of the Hydrostatic-Stokes problem (7)–(9): find \((w_h, p_h) \in W_h \times P_h\) such that

\[
\begin{cases}
a_h(w_h, w_h) + b_h(w_h, p_h) = \int_\Omega f \cdot \bar{u}_h, & \forall \bar{w}_h = (\bar{u}_h, v_h) \in W_h, \\
-b_h(w_h, \bar{p}_h) + s_h(p_h, \bar{p}_h) + \delta_p \int_\Omega p_h \bar{p}_h = 0 & \forall \bar{p}_h \in P_h,
\end{cases}
\]

(20)

where the stabilization bilinear form

\[
s_h(p_h, \bar{p}_h) = \sum_{e \in E_h^0} h_e \int_e \dual{n} \bar{p}_h
\]

is introduced to control the \( L^2 \)-norm of \( p_h - \langle p_h \rangle_\Omega \) (by Lemma 5) and \( \delta_p > 0 \) is a small penalization parameter. Note that the choice \( \delta_p = 0 \) leads to a ill-posed system, due to the fact that

\[
s_h(p_h, 1) = 0 & \forall p_h \in P_h
\]

(21)

which, together with (18), means that if \( \delta_p = 0 \) and \((w_h, p_h) \in W_h \times P_h\) is a solution to (20), then \((w_h, p_h + C)\) is also in \( W_h \times P_h \) and it solves (20).

4 Well-Posedness of the Discrete Problem

The discrete formulation (20) can be rewritten in a vectorial form as follows: find \((w_h, p_h) \in W_h \times P_h\) such that

\[
c_h((w_h, p_h), (\bar{w}_h, \bar{p}_h)) = \int_\Omega f \cdot \bar{u}_h, & \forall (\bar{w}_h, \bar{p}_h) \in W_h \times P_h,
\]

(22)

where

\[
c_h((w_h, p_h), (\bar{w}_h, \bar{p}_h)) = a_h(w_h, \bar{w}_h) + b_h(w_h, p_h)
\]

\[
-b_h(w_h, \bar{p}_h) + s_h(p_h, \bar{p}_h) + \delta_p \int_\Omega p_h \bar{p}_h.
\]

(23)
We consider the following norm in $X_h = W_h \times P_h$:

$$
\|(w_h, p_h)\|_{X_h} = \left(\|w_h\|_{\text{anis}}^2 + \|p_h\|^2 + |p_h|^2_P\right)^{1/2}
= \left(\|u_h\|_{\text{lip}}^2 + \|\partial_{z,h}v\|^2 + |v|^2_V + \|p_h\|^2 + |p_h|^2_P\right)^{1/2}.
$$

According to Banach-Necas-Bařuška theorem (see e.g. [EG04]) well-posedness of discrete problem (20) hinges on the following discrete stability result for $c_h(\cdot, \cdot)$.

**Theorem 6** (Discrete inf-sup stability). Assume that the penalty parameter $\eta$ in $\delta_h^{\text{lip}, \eta}(\cdot, \cdot)$ is such that $\eta > \eta_\ast$, with $\eta_\ast$ defined in Lemma 2. Then, there is $\gamma > 0$ independent of $h$ and $\delta_p$ such that, for all $(w_h, p_h) \in X_h = W_h \times P_h$, one has

$$
\sqrt{\delta_p} \|p_h - \langle p_h \rangle_{\Omega}\| + \gamma \|(w_h, p_h - \langle p_h \rangle_{\Omega})\|_{X_h}
\leq \sup_{(w_h, \bar{p}_h) \in X_h \setminus \{0\}} \frac{c_h((w_h, p_h), (\bar{w}_h, \bar{p}_h)) + \delta_p \|\bar{p}_h\|^2_{\Omega}}{\|(\bar{w}_h, \bar{p}_h)\|_{X_h}}. \quad (24)
$$

**Corollary 7.** If $(w_h, p_h) \in X_h = W_h \times P_h$ is a solution of scheme (20), in particular $\langle p_h \rangle_{\Omega} = 0$ and then (24) implies

$$
\gamma \|(w_h, p_h)\|_{X_h} \leq \sup_{(w_h, \bar{p}_h) \in X_h \setminus \{0\}} \frac{c_h((w_h, p_h), (\bar{w}_h, \bar{p}_h))}{\|(\bar{w}_h, \bar{p}_h)\|_{X_h}}.
$$

Consequently, scheme (20) is well-posed.

**Proof of Theorem 6.** Let $(w_h, p_h) \in X_h$, let $S(w_h, p_h)$ denote the supreme on the right hand side of (24) and let us introduce the following notation: $\Phi \lesssim \Psi$ if $\Phi \leq C \Psi$ for some constant $C > 0$ independent of $h$. Owing to (16) and also to (18) and (21),

$$
c_h((w_h, p_h), (w_h, p_h - \langle p_h \rangle_{\Omega})) = a_h(w_h, w_h) + s_h(p_h, p_h) + \delta_p \int_{\Omega} p_h (p_h - \langle p_h \rangle_{\Omega})
\geq \|u_h\|_{\text{lip}}^2 + |v|^2_V + |p_h|^2_P + \delta_p \|p_h - \langle p_h \rangle_{\Omega}\|^2,
$$

where we applied the following property: $\int_{\Omega} p_h (p_h - \langle p_h \rangle_{\Omega}) = \|p_h - \langle p_h \rangle_{\Omega}\|^2$. Therefore

$$
\|u_h\|_{\text{lip}}^2 + |v|^2_V + |p_h|^2_P + \delta_p \|p_h - \langle p_h \rangle_{\Omega}\|^2 \lesssim S(w_h, p_h) \|(w_h, p_h - \langle p_h \rangle_{\Omega})\|_{X_h}. \quad (25)
$$

The rest of the proof is divided into 3 steps:

1) to estimate $\|p_h - \langle p_h \rangle_{\Omega}\|$ uniformly on $\delta_p$,

2) to estimate $\|\partial_{z,h}v\|$, and

3) to collect estimates and apply Young’s inequality.

**Step 1:** Estimate of $\|p_h - \langle p_h \rangle_{\Omega}\|$. It can be obtained arguing as in the Stokes framework. Specifically, definition of $c_h(\cdot, \cdot)$ means that, for all $\bar{w}_h \in W_h$,

$$
b_h(\bar{w}_h, p_h) = c_h((w_h, p_h), (\bar{w}_h, 0)) - a_h(w_h, \bar{w}_h),
$$
then inf-sup condition (19) imply
\[ \gamma_p \| p_h - \langle p_h \rangle \| \leq \sup_{w_h \in W_h} \left( \frac{-a_h(w_h, \bar{w}_h)}{\|w_h\|_{\text{anis}}} + \frac{c_h((w_h, p_h), (\bar{w}_h, 0))}{\|w_h, 0\|_{\text{anis}}} \right) + |p_h|_P, \]
\[ \leq \sup_{w_h \in W_h} \frac{-a_h(w_h, \bar{w}_h)}{\|w_h\|_{\text{anis}}} + S(w_h, p_h) + |p_h|_P. \]
Boundedness of \( a_h(\cdot, \cdot) \) for \( \| \cdot \|_{\text{anis}} \), follows from Lemma 3, namely
\[ a_h(w_h, \bar{w}_h) \lesssim \sum_{i=1}^{d-1} \| u_i \|_{\text{sip}} \| \bar{u}_i \|_{\text{sip}} + \eta \sum_{e \in E_h} \frac{1}{h_e} \int_e [v_h n_z] [\bar{v}_h n_z], \]
so that
\[ \| p_h - \langle p_h \rangle \| \lesssim \| u_h \|_{\text{sip}} \| \bar{u}_h \|_{\text{sip}} + |v_h|_V |\bar{v}_h|_V, \]
(26)
Note that former bound depends on \( \| u_h \|_{\text{sip}} + |v_h|_V \) and not on \( \| \partial_{x,h} v \| \), what now allows bounding \( \partial_{x,h} v \) in terms of pressure.

**Step 2: Estimate of \( \| \partial_{x,h} v \| \):**

Definition of \( c_h(\cdot, \cdot) \) yields, for all \( \bar{p}_h \in P_h \),
\[ -b_h(w_h, \bar{p}_h) = c_h((w_h, p_h), (0, \bar{p}_h)) - s_h(p_h, \bar{p}_h) - \delta_p \int_\Omega p_h \bar{p}_h. \]
Therefore, from the definition of \( b_h(\cdot, \cdot) \):
\[ \int_\Omega \bar{p}_h \partial_{x,h} v_h = - \int_\Omega \bar{p}_h \nabla_{x,h} \cdot u_h + \sum_{e \in E_h} \int_e [w_h] \cdot n_e \{\bar{p}_h\} + c_h((w_h, p_h), (0, \bar{p}_h)) - s_h(p_h, \bar{p}_h) - \delta_p \int_\Omega p_h \bar{p}_h. \]
And inf-sup condition (17) imply
\[ \| \partial_{x,h} v_h \| \leq S_1 + S_2 + S(w_h, p_h) + S_3 + S_4 \]
where we define
\[ S_1 = \sup_{\bar{p}_h \in P_h} \frac{\int_\Omega \bar{p}_h \nabla_{x,h} \cdot u_h}{\|\bar{p}_h\|}, \quad S_2 = \sup_{\bar{p}_h \in P_h} \frac{\sum_{e \in E_h} \int_e [w_h] \cdot n_e \{\bar{p}_h\}}{\|\bar{p}_h\|}, \]
\[ S_3 = \sup_{\bar{p}_h \in P_h} \frac{s_h(p_h, \bar{p}_h)}{\|\bar{p}_h\|}, \quad S_4 = \sup_{\bar{p}_h \in P_h} \frac{\delta_p \int_\Omega p_h \bar{p}_h}{\|\bar{p}_h\|} = \delta_p \|p_h\|. \]
For \( S_1 \), it is easy to see that
\[ S_1 = \| \nabla_{x,h} \cdot u_h \| \lesssim \| \nabla_{x,h} u_h \|. \]
For \( S_2 \), applying Cauchy-Schwarz inequality:
\[ \sum_{e \in E_h} \int_e [w_h] \cdot n_e \{\bar{p}_h\} \leq \left( \sum_{e \in E_h} \frac{1}{h_e} \int_e \| [w_h] \cdot n_e \|^2 \right)^{1/2} \left( \sum_{e \in E_h} h_e \int_e \{\bar{p}_h\}^2 \right)^{1/2} := I_1 \cdot I_2. \]
One has:

\[ I_1 \leq \left( \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e \left( \sum_{i=1}^{d-1} \left[ u_i \right]_e^2 + \left[ w_i \right]_e \right) \right)^{1/2} \]

\[ \lesssim \left( \sum_{i=1}^{d-1} \sum_{e \in \mathcal{E}_h} \left( \frac{1}{h_e} \left[ u_i \right]_e \right)^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \left[ w_i \right]_e ^2 \right)^{1/2} = \left( \sum_{i=1}^{d-1} \left[ u_i \right]^2 + \left[ w_i \right]^2 \right)^{1/2} \]

On the other hand, using (11) one has \( I_2 \lesssim \| \bar{p} \| \). Therefore

\[ S_2 \lesssim \left( \sum_{i=1}^{d-1} \left[ u_i \right]^2 + \left[ w_i \right]^2 \right)^{1/2} \]

To bound \( S_3 \), we apply (21), then Cauchy-Schwarz and inequality (12):

\[ s_h(p_h, \bar{p}_h) = \sum_{e \in \mathcal{E}_h^0} h_e \int_e \left[ p_h - \langle p_h \rangle_\Omega \right] \left[ \bar{p}_h \right] \]

\[ \leq \left( \sum_{e \in \mathcal{E}_h^0} h_e \int_e \left[ p_h - \langle p_h \rangle_\Omega \right]^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h^0} h_e \int_e \left[ \bar{p}_h \right]^2 \right)^{1/2} \]

\[ \lesssim \| p_h - \langle p_h \rangle_\Omega \| \| \bar{p}_h \| \]

Therefore

\[ S_3 \lesssim \| p_h - \langle p_h \rangle_\Omega \| \]

Finally,

\[ S_4 = \delta_p \| p_h \| \leq \delta_p \| p_h - \langle p_h \rangle_\Omega \| + \delta_p \| \langle p_h \rangle_\Omega \| = \delta_p \| p_h - \langle p_h \rangle_\Omega \| + \delta_p |\Omega| \| p_h \|_\Omega^2 \]

Summarizing:

\[ \| \partial_{z,h}v_h \| \lesssim \| \nabla_{x,h}u_h \| + \left( \sum_{i=1}^{d-1} \left[ u_i \right]^2 + \left[ v_h \right]^2 \right)^{1/2} + S(w_h, p_h) \]

\[ + \| p_h - \langle p_h \rangle_\Omega \| + \delta_p |\Omega| \| p_h \|_\Omega^2 \]

that is

\[ \| \partial_{z,h}v_h \| \lesssim \| u_h \|_\text{lip} + \| v_h \|_V + S(w_h, p_h) + \| p_h - \langle p_h \rangle_\Omega \| + \delta_p |\Omega| \| p_h \|_\Omega^2 \]  \hspace{1cm} (27) \]

Step 3:

Taking into account (25) and the above estimates (26) and (27), one has

\[ \| (w_h, p_h - \langle p_h \rangle_\Omega) \|_X_h^2 \lesssim A^{u,v,p} + B^p + C^{\partial_{z,v}} \]

where \( A^{u,v,p} \), \( B^p \) and \( C^{\partial_{z,v}} \), are defined and bounded as follows:

\[ A^{u,v,p} = \| u_h \|_\text{lip}^2 + \| v_h \|_V^2 + \| p_h \|_p^2 + \delta_p \| p_h - \langle p_h \rangle_\Omega \|^2 \lesssim S(w_h, p_h) \| (w_h, p_h - \langle p_h \rangle_\Omega) \|_X_h \]

\[ B^p = \| p_h - \langle p_h \rangle_\Omega \|_{L^2}^2 \lesssim \left( \| u_h \|_\text{lip} + \| v_h \|_V + S(w_h, p_h) + \| p_h \|_p \right)^2 \]

\[ \lesssim S(w_h, p_h) \| (w_h, p_h - \langle p_h \rangle_\Omega) \|_X_h + S(w_h, p_h)^2 \]

\[ C^{\partial_{z,v}} = \| \partial_{z,h}v \|^2_{L^2} \lesssim (\| u_h \|_\text{lip} + \| v_h \|_V + \| p_h - \langle p_h \rangle_\Omega \|_{L^2} + S(w_h, p_h) + \delta_p |\Omega| \| p_h \|_\Omega^2)^2 \]

\[ \lesssim S(w_h, p_h) \| (w_h, p_h - \langle p_h \rangle_\Omega) \|_X_h + S(w_h, p_h)^2 + (\delta_p |\Omega| \| p_h \|_\Omega^2)^2 \].
Therefore,
\[
\|(w_h, p_h - \langle p_h \rangle_\Omega)\|^2_{X_h} \lesssim S(w_h, p_h) \|(w_h, p_h - \langle p_h \rangle_\Omega)\|_{X_h} + S(w_h, p_h)^2 + (\delta_p |\Omega| \langle p_h \rangle_\Omega^2)^2
\]
and the conclusion follows from Young’s inequality.

5 Numerical Tests

We have developed some qualitative numerical tests which are agree with previous theoretical results. Specifically, we were able to program a standard lid driven cavity test for the discrete formulation (20) using FreeFem++ [Hec12], a high level PDE language and solver which makes simple to develop variational formulations. In the first test, we used discontinuous \( P_1/P_1 \) for velocity and pressure and introduced the following parameters: \( \Omega = (0,1)^2 \subset \mathbb{R}^2 \), unstructured mesh with \( h \approx 1/30 \), horizontal viscosity \( \nu = 1 \), SIP penalization \( \eta = 10^2 \), pressure penalization \( \delta_p = 10^{-12} \).

The following Dirichlet boundary are defined. On surface, \( \Gamma_s = \{(x,1), x \in (0,1)\} \): \( u(x,y) = x(1-x), v = 0 \). On on bottom, \( \Gamma_b = \{(x,0), x \in (0,1)\} \): \( u = 0, v = 0 \). And on sidewalls, \( \Gamma_l = \{(x,y) \in \mathbb{R}^2, x \in \{0,1\}, y \in (0,1)\} \): \( u = 0 \). We introduce homogeneous Neumann boundary condition for \( v \) on \( \Gamma_l \): \( \partial_x v = 0 \).

Former boundary conditions are fixed weakly. Specifically, the SIP bilinear form (13), utilized in (15) for horizontal components of velocity, and also the jump bilinear term introduced for \( v_h \) in (15), are modified as follows: for each term regarding to a Dirichlet boundary edge, a corresponding term is introduced in the right hand side linear form. As all boundary conditions are zero except \( u|_{\Gamma_s} \), the only additional terms correspond to:

\[
\sum_{e \in E_h \cap \Gamma_s} \int_e x(1-x) \left( \nabla u_h \cdot \mathbf{n} + \frac{\eta}{h_e} u_h - n_x p_h \right) ds.
\]

This expression can be simplified even more, considering that \( n_x = 0 \) on \( \Gamma_s \). On the other hand, terms related to Neumann boundary edges are eliminated in (15) and a corresponding term is introduced in the RHS as usual in Neumann boundary conditions. In our case, we have only a null Neumann condition for \( v|_{\Gamma_l} \).

Resulting velocity field and pressure iso-values (figure 1) reproduce the expected behavior: velocity recirculation and hydrostatic (vertical) pressure iso-values. These results are improved for higher polynomial order approximation, specifically for discontinuous \( P_2/P_2 \) velocity approximation (figure 2). In this case, a higher SIP penalization parameter, \( \eta = 10^4 \), must be introduced.

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Velocity field. Pressure iso-values.

Figure 1: Cavity test, $P_1/P_1$ SIP DG

Velocity field. Pressure iso-values.

Figure 2: Cavity test, $P_2/P_2$ SIP DG
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