Reappraisal of the causal interpretation of quantum mechanics and of the quantum potential concept

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Abstract

The causal interpretation of quantum mechanics, as originally stated by deBroglie and Bohm, had several attractive features. Among these is the possibility that it could address some of the most fundamental questions on quantum phenomena. However, subsequent theoretical conjectures, which have now been included in the orthodox view of the deBroglie Bohm theory, are unphysical and have done much to undermine the original theory’s appeal. We, therefore, return to the original theory as our starting point and address one of its perplexing areas: the quantum potential. By avoiding the unphysical conjectures we are led to an understanding of the quantum potential which is distinctly different from that of the orthodox deBroglie Bohm view.

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1 Introduction

The essential postulates of the deBroglie Bohm causal theory [1,2] can be summarized as:

- An individual physical system comprises a wave together with a point particle. The wave and the particle are distinct parts of the same system. The wave and the particle are never separated.
- The wave \( \psi \) is a solution to the Schrödinger equation.
- Particle motion is obtained as the solution to a modified Hamilton - Jacobi equation. The modification to the classical Hamilton - Jacobi equation is
the addition of a term $Q$, the quantum potential. The functional form of $Q$ is determined mathematically by $\psi$.

The causal theory has many attractive features. For example:

- It is deterministic.
- It allows for the possibility of a complete description of a physical system, i.e., it addresses one of the fundamental complaints Einstein had about the Copenhagen interpretation [Ref. 2, pp. 11-15].
- Particle dynamics is written as the sum of classical and quantum terms [Ref. 1, p. 29].
- There is a natural relationship between the classical and quantum regimes [Ref. 2, pp. 270-274].

Together these allow for the possibility that the Causal Theory may be able to address some of the deepest problems associated with the Copenhagen interpretation. The theory may also lead to new physics.

There are, however, weaknesses in the original theory. One of the most obvious of these relates to the quantum potential $Q$: What is its source? Typically in physics a force, and its associated potential, have a source. However, nowhere in the literature is this fundamental question addressed in a physically reasonable way.

To compound the problem, the adherents of the theory have gone further and made conjectures leading to results which are in direct contradiction with experiment.

These conjectures, which we will discuss later, have now become part of the orthodox view of the deBroglie Bohm causal interpretation. Unfortunately, they have done much to undermine the appealing aspects of the original theory.

Our approach is as follows:

1. Return to the original theory as stated by deBroglie and Bohm.
2. Reject all unphysical conjectures.
3. Let experiment guide us as much as possible.
4. For those things about which Copenhagen makes a prediction, our theory must be in agreement.
5. Start with a many-body approach because it is closer to physical reality than a one-body approach.
6. Address two fundamental problems: What is the physical reason for the forces associated with the quantum potential? What is the source of these quantum forces?

Our goal is to then go on in future work to see if the causal theory can be used to address some of the compelling questions of the our day.
2 Basics Of The Causal Theory

We start by considering a physical system consisting of n interacting particles. The time dependent Schrödinger equation for this system is,

\[
\frac{i}{\hbar} \left( \frac{\partial \psi}{\partial t} \right) = \left[ \sum_{i=1}^{n} \left( \frac{-\hbar^2}{2m_i} \nabla_i^2 \right) + V \right] \psi
\] (1)

where

\[
\psi = \psi(x_1, x_2, x_3, \ldots, x_n, t),
\]

\[
V = V(x_1, x_2, x_3, \ldots, x_n, t),
\]

\[
\nabla_i = \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i} \right),
\]

and \((x_1, x_2, x_3, \ldots, x_n)\) provides a set of rectangular Cartesian coordinates of the n particles. Also \(V\) is the classical potential energy which includes the interparticle and external potentials from the traditional forces (gravity, E&M, strong and weak interactions.)

Assume that the energy spectrum is discrete, so that the wavefunction \(\psi\) is localized in configuration space. Without loss of generality, the wavefunction can be written as

\[
\psi = R \exp \left( \frac{S}{\hbar} \right)
\] (2)

where \(R\) and \(S\) are real and

\[
R = R(x_1, x_2, x_3, \ldots, x_n, t)
\]

\[
S = S(x_1, x_2, x_3, \ldots, x_n, t)
\]

Equation (1) is equivalent to the pair of equations

\[
\frac{\partial S}{\partial t} + \sum_{i=1}^{n} (\nabla_i S)^2 (2m_i)^{-1} + Q + V = 0
\] (3)

\[
\frac{\partial R^2}{\partial t} + \sum_{i=1}^{n} \nabla_i \cdot \left( \frac{R^2 \nabla_i S}{m_i} \right) = 0
\] (4)

where

\[
Q = Q(x_1, x_2, x_3, \ldots, x_n, t)
\]

\[
Q = -\sum_{i=1}^{n} \left( \frac{\hbar^2}{2m_i R} \right) \nabla_i^2 R
\] (5)
is defined as the quantum potential [Ref. 2, p. 279]. If it were not for the presence of the quantum potential $Q$, Eq. (3) would be the classical Hamilton-Jacobi equation.

Just as the causal form of the Hamilton-Jacobi equation contains the additional term $Q$, the causal form of Newton’s second law contains an additional term involving $Q$ [Ref. 2, p. 279-280],

$$\frac{dP_i}{dt} = -\nabla_i V - \nabla_i Q \quad (6)$$

here $P_i$ is the momentum of the $i$-th particle, $-\nabla_i V$ is the sum of all the conventional forces on the $i$-th particle, and $-\nabla_i Q$ is interpreted as the quantum force on the $i$-th particle. The total momentum of our $n$-particle system is given by

$$P = \sum_{i=1}^{n} P_i \quad (7)$$

with dynamical equation [Ref. 2, p. 285]

$$\frac{dP}{dt} = -\sum_{i=1}^{n} \nabla_i V - \sum_{i=i}^{n} \nabla_i Q \quad (8)$$

### 3 Studies of the quantum potential

The purpose of this paper is to attempt to answer the following questions. What is the physical reason for the forces associated with the quantum potential? Furthermore, what is the source of these forces? We suggest that $Q$ is the result of a nonholonomic nonlocal constraint on the system, the requirement that the wavefunction $\psi$ not exhibit deterministic chaos (extreme sensitivity to initial conditions.) The forces associated with $Q$ are forces of constraint. And we furthermore suggest that the source of the quantum force on a given particle is the other particles of our $n$ particle system.

Why are we interested in preventing deterministic chaos? In general, quantum mechanical systems localized in configuration space are described by Hermitian Hamiltonians. With such a Hamiltonian, the wavefunction is never subjected to extreme sensitivity to initial conditions. However, removing $Q$ from the causal Hamilton-Jacobi equation (Eq. 3) opens the possibility that the corresponding wavefunction be subjected to extreme sensitivity to initial conditions. All this is shown in Appendix A. In Appendix B ways to detect the presence of deterministic chaos are discussed.

While deterministic chaos of the wavefunction should not occur in a localized quantum system, it is quite possible that such chaos can be associated with the
particle trajectories of the system, as calculated by deBroglie Bohm theory [4]. Furthermore, in a macroscopic quantum system the wavefunction may exhibit deterministic chaos, as is discussed in Appendix C.

We now turn to our second question: What is the source of the quantum forces? On this issue the leading proponents of the causal theory make an assumption (conjecture) which we find truly mysterious. Consider the following quotes:

1. On page 170 of Ref.3, Bohm states that the field $\psi$ “exerts a force on the particle in a way that is analogous to, but not identical with, the way in which an electromagnetic field exerts a force on a charge, and a meson exerts a force on a nucleon$^1$."

2. On page 91 of Ref. 2, Holland states “it should be emphasized that while the quantum field $|\psi>$ does not push on the particle as we might expect a classical wave to, it does nevertheless guide the particle by exerting a direct force on it via Q.”

We infer from these statements that Bohm and Holland make the assumption that the wavefunction, not only is an independent entity distinct from the particle itself, but also is the "source" of the quantum force! Consequently, it is no surprise when Holland goes on to say:

3. Ref. 2, p. 79: “the particle simply responds to local values of the field in the vicinity (via Q) - there is no reciprocal action of the particle on the wave.”

4. Ref. 2, p. 286: “A classically closed system may not be closed quantum mechanically.”

$$\frac{d\mathbf{P}}{dt} = -\sum_{i=1}^{n} \nabla_i Q \neq 0$$

The assumption that the source of the quantum force is the wavefunction itself has led the orthodox proponents of the theory to unphysical results. For example, the third quote above suggests that Newton’s third law (to every action there is an equal and opposite reaction) is violated. Furthermore, the fourth quote allows for the possibility that an isolated system may self-accelerate in the absence of any known force. Not only are these results in disagreement with predictions made using the Copenhagen interpretation of quantum mechanics, but they are also in disagreement with all known experiments.

These unphysical results are a consequence of combining the assumption that “the wavefunction itself exerts a force on the particle” with the fact that the

$^1$It should be noted that Bohm and Hiley [Ref. 1, pp. 29, 30, 37, 38, 40] imply, in contrast, that in our Eq. (6) , $-\nabla_i Q$ is a force being exerted on the i-th particle, but that it is not directly exerted by the wavefunction $\psi$. Rather, the wavefunction controls the action and reaction of the particle with something like the fluctuations of the vacuum, causing “acceleration of the particle in its self-movement.”
Schrödinger equation is homogeneous. Rather than questioning the assumption on the wavefunction, it is apparently the orthodox deBroglie Bohm approach to accept it and instead speculate on the possibility of some inhomogeneous version of the Schrödinger equation\footnote{Bohm \cite[Ref. 3, p. 179]{Bohm} suggests that at a distance less than $10^{-13}$ cm, Schrödinger’s equation becomes inhomogeneous allowing a particle to react back on the field, in analogy with the inhomogeneous Maxwell’s equations allowing a charge to act on the electromagnetic field.}. There is, however, no experimental evidence to support these speculations.

We take a different approach. To obtain a physically reasonable causal theory we add two assumptions to the original deBroglie Bohm postulates:

1. The wavefunction exerts no forces on the particles of the system. Although it is apparent from Eq. (1) through (5) that the wavefunction determines the functional form of the quantum potential, there is no indication that \( \psi \) itself exerts forces on the particles of the system.

2. For an isolated system, the total momentum \( \mathbf{P} \) is a constant of the motion; i.e., it does not change with time. This is consistent with experiment. It is also consistent with the Copenhagen interpretation; because for an isolated system, the wavefunction is an eigenfunction of the total momentum operator.

Using these ideas, and returning to our many-body equations, we are led to a very different result for the source of the quantum force than that in the orthodox deBroglie Bohm view.

Consider the situation in which our \( n \)-particle system is an isolated system. Since the traditional forces obey Newton’s third law we have

\[
-\sum_{i=1}^{n} \nabla_i V = 0
\]

For an isolated system we have

\[
\frac{d\mathbf{P}}{dt} = 0
\]

so that

\[
-\sum_{i=1}^{n} \nabla_i Q = -\sum_{i=1}^{n} \mathbf{F}_i = 0
\]

That is, the sum of all the quantum forces on the \( n \)-particles is zero. From this we have

\[
\mathbf{F}_i = \sum_{j \neq i}^{n} (-\mathbf{F}_j)
\]
of the quantum forces on all the other particles. For \( n = 2 \), this gives \( \mathbf{F}_1 = -\mathbf{F}_2 \), i.e., the quantum force on particle 1 is equal and opposite to the quantum force on particle 2. This strongly suggests that the source of the quantum force on one particle is the other particle. (Which, conveniently, is also consistent with Newton’s Third Law.)

Eq. (9) implies that the net quantum force on the \( i \)-th particle is the result of all the other particles of the system exerting forces on the particle through the intermediary of the quantum potential. If the system is acted on by external forces, the simplest thing to do conceptually is to enlarge the system to include the bodies creating the external forces, so that one has an enlarged isolated system. We see that the source of the quantum force on any particle is nothing more than all the other particles of the system. To us, this seems to be a much more satisfying explanation than is that of Holland or of Bohm and Hiley [Ref. 1, pp. 30, 37, 38, 40]. The nonlocality of the quantum potential discussed by Holland [Ref. 2, p. 282] is the cause of the nonlocal interactions between the particles of the system.

Consider the situation of an isolated hydrogen atom in the ground state. The total force on the atom vanishes. At the same time there are equal and opposite Coulomb forces of attraction on the electron and on the proton. Since the internal wavefunction of the system is real, the causal theory says that the electron is motionless relative to the proton. This results from the fact that there are equal and opposite quantum forces of repulsion between the two particles, such that there is no net force on either particle; and their relative position stays fixed.

With this understanding of the physical reason for, and the source of, the quantum forces, the authors feel that the concept of the quantum potential is less strange and more reasonable than would appear at first sight.

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### 5 Appendix A

When we remove \( Q \) from the Hamilton - Jacobi equation [Eq. (3)], we are adding to the Schrödinger equation the term [Ref. 2, p. 56]

\[
\left( \frac{\hbar^2}{2m} \right) \exp i(S/\hbar) \nabla^2 R = \left( \frac{\hbar^2}{2m} \right) |\psi|^2 \nabla^2 |\psi| = -Q|\psi| \]

so that the effective Hamiltonian becomes
\[ H_{\text{eff}} = -\left( \frac{\hbar^2}{2m} \right) \nabla^2 + V + \left( \frac{\hbar^2}{2m} \right) \left| \psi \right|^{-1} \nabla^2 \left| \psi \right| \] (11)

Note that \( H_{\text{eff}} = H_{\text{eff}}(\psi) \) is a functional of \( \psi \), the state upon which it is operating. This means that the superposition principle no longer holds. When \( \varphi \neq \psi \),

\[ \int (\varphi^* H_{\text{eff}} \psi - \psi H_{\text{eff}}^* \varphi) d\tau \]

\[ = \left( \frac{\hbar^2}{2m} \right) \int \varphi^* \psi \left[ \left| \psi \right|^{-1} \nabla^2 \left| \psi \right| - \left| \varphi \right|^{-1} \nabla^2 \left| \varphi \right| \right] d\tau \]

\[ \neq 0 \]

so that \( H_{\text{eff}} \) is not Hermitian, \( H_{\text{eff}} \neq H_{\text{eff}}^\dagger \). This means that the time development operator \( \exp[(\hbar/2m)H_{\text{eff}}t] \) is not unitary. The time dependent Schrödinger equation is a nonunitary flow. Since

\[ i\hbar \left( \frac{\partial}{\partial t} \right) \psi^* \psi = \psi^* H_{\text{eff}} \psi - \psi H_{\text{eff}}^* \psi \]

we have

\[ i \left( \frac{\partial}{\partial \tau} \right) \int |\psi|^2 d\tau = \]

\[ \left( \frac{\hbar}{2m} \right) \int |\psi|^2 \left[ \left| \psi \right|^{-1} \nabla^2 \left| \psi \right| - \left| \psi \right|^{-1} \nabla^2 \left| \psi \right| \right] d\tau \]

and the normalization of the wavefunction is time independent. When \( \varphi \neq \psi \),

\[ i \left( \frac{\partial}{\partial \tau} \right) \int \varphi^* \psi d\tau = \]

\[ \left( \frac{\hbar}{2m} \right) \int \varphi^* \psi \left[ \left| \psi \right|^{-1} \nabla^2 \left| \psi \right| - \left| \varphi \right|^{-1} \nabla^2 \left| \varphi \right| \right] d\tau \]

\[ = 0 \]

so that
\[
\left( \frac{\partial}{\partial t} \right) \int |\psi - \varphi|^2 \, d\tau \quad (16)
\]
\[
= \left( \frac{\hbar}{2m} \right) \int i[\psi^* \varphi - \varphi^* \psi] \left[ |\psi|^{-1} \nabla^2 |\psi| - |\varphi|^{-1} \nabla^2 |\varphi| \right] \, d\tau 
\]
\[
\neq 0
\]

The normalization of \((\psi - \varphi)\) is real and time dependent.

Consider the case where two initial conditions for the time dependent Schrödinger equation differ only infinitesimally. As time progresses the two corresponding wavefunctions can become quite different. This indicates the possibility of deterministic chaos of the wavefunction (extreme sensitivity to initial conditions.) All this is a consequence of \(H_{\text{eff}}\) being a functional of the state upon which it is acting.

If we remove the term

\[
\left( \frac{\hbar^2}{2m} \right) |\psi|^{-1} \nabla^2 |\psi|
\]

from Eq. (11), we are left with a Hermitian Hamiltonian, as demonstrated by removing the corresponding terms in Eq. (12). Removing the corresponding terms from Eqs. (13) and (14), we see that the normalization of \((\psi - \varphi)\) is time independent. This shows that there can be no deterministic chaos of the wavefunction associated with a Hermitian Hamiltonian.

6 Appendix B

In order to detect the presence of deterministic chaos, it is necessary to transform the time dependent Schrödinger equation so that it looks like a set of classical equations of motion [5]. By expanding the wavefunction \(\psi(t)\) in terms of the members of some orthonormal complete set \(\varphi_i\)

\[
\psi(t) = \sum_j a_j(t) \varphi_j \quad (17)
\]

the Schrödinger equation becomes

\[
\frac{da_j}{dt} = \sum_k M_{jk} a_k \quad (18)
\]

with
\[ M_{jk} \equiv (i\hbar)^{-1} \langle j | H | k \rangle. \] (19)

This is a complex N dimensional flow, where N is the dimension of the Hilbert space appropriate to the problem. (The equivalent real flow is 2N dimensional.) Under ordinary circumstances the flow is linear and there is no reason to expect deterministic chaos. However, if the Hamiltonian is a functional of the state of the system, then \( M_{jk} \) will be a function of the coefficients \( a_i \) and the flow is nonlinear. Thus, if \( 2N > 3 \), the Poincare - Bendixson theorem [6] indicates that deterministic chaos may occur. While numerically integrating Eq. (18), one can simultaneously calculate the Lyapunov exponents appropriate to the solution [7]. If the largest exponent is positive, there is deterministic chaos.

7 Appendix C

Consider the quantum mechanical treatment of a macroscopic system. By definition, such a system is sufficiently large that intensive properties are invariant to size increase. This suggests considering such systems in the limit of infinite size and constant density, know as the thermodynamic limit[8], where phase transitions are sharp (not broadened) and Poincare recurrences do not occur. In the last three decades there has been considerable progress in the algebraic treatment of infinite systems, both with regard to quantum field theory and with regard to quantum statistical mechanics [8,9].

There can be many representations of the quantum mechanical operators corresponding to the observables of a physical system. For a finite system, all representations are unitarily equivalent[10], so that the choice of representation is a matter of convenience. In contrast, for an infinite system there can be many unitarily inequivalent representations. Each representation can have associated with it a complete set of microscopic states and microscopic variables. The various inequivalent representations can be distinguished by the values of intensive macroscopic variables associated with them. For example, in a ferromagnet the components of the magnetization vector are macroscopic variables. There is a three-fold continuous infinity of inequivalent representations associated with the values of these components.

For our purposes, a most important property is that the Hamiltonian is a functional of the representation [ Ref. 9, p. 32]. A classic example is the BCS theory of superconductivity [11], where the so-called reduced Hamiltonian contains the complex superconducting order parameter \( \Delta \), whose magnitude is a measure of the number of Cooper pairs in the system (and thus a function of temperature) and whose phase plays a crucial role in the Josephson effect [12].

Since the Hamiltonian is a functional of the representation, and thus the state of the system, the matrix elements \( M_{jk} \) of Eq. (19) are functions of the coefficients \( a_i \). Thus the equations of motion, Eq. (18), are nonlinear in the coefficients, and deterministic chaos may occur. More specifically, the Hamiltonian \( H \) and the matrix elements \( M_{jk} \) are functions of n macroscopic variables.
indexing the various inequivalent representations, the macroscopic variables in
turn being functions of the $a_i$. Thus the N dimensional flow of Eq. (18) can be
used to generate an n dimensional nonlinear flow involving the n macroscopic
variables. The Poincare-Bendixson theorem [6] requires $n \geq 3$ for the possibility
of chaos.

Since the Hamiltonian is a functional of the state of the system, the super-
position theorem breaks down. This may have something to do with the fact
that no one has ever observed a coherent superposition of the two macroscopic
states of Schrödinger's cat [13].

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