Quantum Mechanical Properties of Bessel Beams

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(Dated: August 14, 2018)

Abstract

Bessel beams are studied within the general framework of quantum optics. The two modes of the electromagnetic field are quantized and the basic dynamical operators are identified. The algebra of these operators is analyzed in detail; it is shown that the operators that are usually associated to linear momentum, orbital angular momentum and spin do not satisfy the algebra of the translation and rotation group, and that this algebra is not closed. Some physical consequences of these results are examined.

PACS numbers: 42.50.Vk, 32.80.Lg
I. INTRODUCTION

The increasing use of light to control the motion of atomic systems and microparticles has renewed the interest in the mechanical properties of electromagnetic (EM) beams. The realization that light carries energy as well as linear and angular momenta was essential in the development of the classical theory of electromagnetism; nowadays, the interchange of such quantities with matter is a well-established fact. In most cases, the interaction of light with matter can be satisfactorily described by taking the EM field as a superposition of idealized plane waves; in this simple picture, each normal mode carries momenta in the direction of the propagation vector, and the angular momentum is directly related to the states of polarization.

However, the decomposition of angular momentum into an orbital and spin part has some ambiguities within the framework of quantum optics, as was recognized in classical papers by Darwin [1], Humblet [2], de Broglie [3], and many others. Recent interest in defining angular momentum in optics can be traced back to the works of Lenstra and Mandel [4], Allen et al. [5], and van Enk and Nienhuis [6, 7]. Lenstra and Mandel considered periodic boundary conditions that limit the isotropy of the EM field and thus affect the angular momentum. Allen et al. showed that angular momentum can indeed be decomposed into orbital and spin parts for Laguerre-Gaussian modes, i.e., paraxial elementary waves with cylindrical symmetry. Finally, van Enk and Nienhuis [7] studied this decomposition and showed that neither the spin nor the orbital quantum operators of the electromagnetic field satisfy the commutation relations of angular momentum; they made the assumption that the total angular momentum obtained from Noether theorem is given directly as a sum of these operators; however, this is not the case in many situations where boundary conditions give rise to important surface effects.

In general, orbital angular momentum (OAM) is identified with the part of the total angular momentum that depends explicitly on position (with respect to an origin of coordinates). When OAM is evaluated with respect to the axis of symmetry, it turns out that Laguerre-Gaussian modes do carry OAM in integer multiples of $\hbar$ [5]. However, the formal decomposition into OAM and spin angular momentum does not appear to be natural beyond the paraxial approximation [8]. Nevertheless, recent experiments have shown that angular momentum do indeed possess orbital and spin parts [10, 11, 12, 13, 14], the former having
an intrinsic and extrinsic nature with direct physical consequences [15].

Bessel beams have interesting properties that make them especially attractive: they propagate with an intensity pattern invariant along a given axis [16] and carry angular momentum along that axis. Experimental realizations of such beams and their mechanical effects on atoms and microparticles are the subject of many current investigations [13]. The purpose of the present paper is to study the mechanical properties of Bessel beams within the general formalism of quantum optics. In particular, we study the standard decomposition of the total angular momentum of these beams into orbital and spin parts, and obtain explicit expressions for these observables as quantum operators. It turns out, however, that they do not satisfy the usual commutation relations for angular momentum vectors and that, contrary to van Enk and Nienhuis [7] results, the algebra does not even close; we argue that this discrepancy is due to the boundary conditions. We also show that the superposition of Bessel modes, as well as their polarization states, can be characterized by a set of operators that appears naturally within our formulation and can be measured in principle. Some predictions that could eventually be tested experimentally are discussed in the conclusions.

The organization of the article is as follows. In section II the electromagnetic Bessel beams are expressed in terms of Hertz potentials. In Section III the fields are quantized and explicit expressions for the operators are given, relating them to their main mechanical properties. Section IV is devoted to the study of the algebraic properties of the operators that are usually identified with orbital and spin angular momentum. Finally, a brief discussion of the results and their experimental implications is presented in Section V. Some useful formulas are given in Appendix A, and expressions relating Bessel modes with plane and spherical EM modes are given in Appendix B.

II. ELECTROMAGNETIC BESSEL MODES

An electromagnetic field with cylindrical symmetry can be conveniently described in the terms of Hertz potentials \( \Theta_1 \) and \( \Theta_2 \) [17]. In cylindrical coordinates \( \{\rho, \phi, z\} \), the electromagnetic potentials are given by

\[
\Phi = -\frac{\partial}{\partial z}\Theta_1, \quad (1)
\]
\[ A = \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \Theta_2, -\frac{\partial}{\partial \rho} \Theta_2, \frac{\partial}{\partial t} \Theta_1 \right\}, \]  

(2)

and satisfy the Lorentz gauge condition. Then, the electric field \( E \) is:

\[ E_\rho = \frac{\partial^2}{\partial z \partial \rho} \Theta_1 - \frac{1}{\rho \cot \partial \rho} \Theta_2, \]  

(3)

\[ E_\phi = \frac{1}{\rho \cot \partial \phi} \Theta_1 + \frac{\partial^2}{\partial z \partial \rho} \Theta_2, \]  

(4)

\[ E_z = -\frac{\partial^2}{c^2 \partial t^2} \Theta_1 + \frac{\partial^2}{\partial z^2} \Theta_1, \]  

(5)

and the magnetic field \( B \) is:

\[ B_\rho = \frac{1}{\rho \cot \partial \phi} \Theta_1 + \frac{\partial^2}{\partial z \partial \rho} \Theta_2, \]  

(6)

\[ B_\phi = -\frac{\partial^2}{c^2 \partial t^2} \Theta_2 + \frac{1}{\rho \cot \partial \phi} \Theta_2, \]  

(7)

\[ B_z = -\frac{\partial^2}{c^2 \partial t^2} \Theta_2 + \frac{\partial^2}{\partial z^2} \Theta_2. \]  

(8)

Both Hertz potentials \( \Theta_i \), \( (i = 1, 2) \), satisfy the equations:

\[ -\frac{\partial^2}{c^2 \partial t^2} \Theta_i + \frac{1}{\rho \cot \partial \rho} \left( \rho \frac{\partial}{\partial \rho} \Theta_i \right) + \frac{1}{\rho^2 \partial \phi^2} \Theta_i + \frac{\partial^2}{\partial z^2} \Theta_i = 0. \]  

(9)

Any solution of this equation that is regular at the origin can be written as a linear combination of the functions

\[ \Theta_i = C_i J_m(k_\perp \rho) \exp\{-i\omega t + ik_z z + im \phi\}, \]  

(10)

where \( J_m \) is the Bessel function of order \( m \), \( C_i \) are constants, and \( k_\perp = \sqrt{(\omega/c)^2 - k_z^2} \). Bessel functions form a complete orthogonal basis as follows from Eq. (73) in Appendix A.

An electromagnetic mode is associated to each Hertz potential, \( \Theta_1 \) and \( \Theta_2 \), via Eqs. (5-8), giving rise to transverse magnetic and electric modes respectively. In the following we will occasionally denote them by the superscripts \( TM \) and \( TE \).

It is convenient to make a further gauge transformation to the transverse gauge with \( \Phi = 0 \). This can be achieved with the transformations \( \Phi \rightarrow \Phi - \partial \Lambda/c \partial t \) and \( A \rightarrow A + \nabla \Lambda \), and taking \( \Lambda = (k_z c/\omega) \Theta_1 \).
The electromagnetic vectors and potential can be decomposed in terms of their basic modes. First, we define the vectors:

\[
M(\mathbf{r}, t; K) = \frac{\omega}{ck_z} \left[ m \frac{J_m(k_{\perp} \rho)}{k_{\perp} \rho} \mathbf{e}_\rho + i J'_m(k_{\perp} \rho) \mathbf{e}_\phi \right] e^{-i\omega t + im\phi + ik_z z} \tag{11}
\]

and

\[
N(\mathbf{r}, t; K) = \left[ i J'_m(k_{\perp} \rho) \mathbf{e}_\rho - \frac{m}{k_{\perp} \rho} J_m(k_{\perp} \rho) \mathbf{e}_\phi + \frac{k_{\perp}}{k_z} J_m(k_{\perp} \rho) \mathbf{e}_z \right] e^{-i\omega t + im\phi + ik_z z}. \tag{12}
\]

Here and in the following, the set of quantum numbers \( \{k_{\perp}, m, k_z\} \) are denoted by the generic symbol \( K \) whenever no confusion can arise.

Accordingly, we obtain the following forms for the modes of the electromagnetic potential:

\[
A^{(TM)}(\mathbf{r}, t; K) = \frac{c}{i\omega} \mathcal{E}^{(TM)}_m(k_{\perp}, k_z) N(\mathbf{r}, t; K) \tag{13}
\]

and

\[
A^{(TE)}(\mathbf{r}, t; K) = -\frac{c}{i\omega} \mathcal{E}^{(TE)}_m(k_{\perp}, k_z) M(\mathbf{r}, t; K), \tag{14}
\]

where the functions \( \mathcal{E}^{(i)}_m(k_{\perp}, k_z) \) \((i = TM, TE)\) refer to the amplitudes of the transverse electric and magnetic modes.

The electric field for each mode is now given by:

\[
E^{(i)}(\mathbf{r}, t; K) = \frac{i\omega}{c} A^{(i)}(\mathbf{r}, t; K), \tag{15}
\]

and the magnetic field is:

\[
B^{(TM)}(\mathbf{r}, t; K) = \mathcal{E}^{(TM)}_m(k_{\perp}, k_z) M(\mathbf{r}, t; K), \tag{16}
\]

\[
B^{(TE)}(\mathbf{r}, t; K) = \mathcal{E}^{(TE)}_m(k_{\perp}, k_z) N(\mathbf{r}, t; K). \tag{17}
\]

Finally, we notice that the vector \( N \) can also be written in the form:

\[
N(\mathbf{r}, t; K) = -\frac{i}{2} \left[ J_{m+1}(k_{\perp} \rho) e^{i(m+1)\phi} \mathbf{e}_- - J_{m-1}(k_{\perp} \rho) e^{i(m-1)\phi} \mathbf{e}_+ \right] e^{-i\omega t + ik_z z}
\]

\[
+ \frac{k_{\perp}}{k_z} J_m(k_{\perp} \rho) e^{i m \phi} \mathbf{e}_3, \tag{18}
\]

with

\[
\mathbf{e}_\pm = \mathbf{e}_1 \pm i \mathbf{e}_2, \tag{19}
\]
and that \( ck_z \mathbf{M} = \omega \mathbf{N} \times \mathbf{e}_3 \). Some useful formulas involving the vectors \( \mathbf{M} \) and \( \mathbf{N} \) are given in the Appendices.

Notice also that the electromagnetic field described by the above expressions is purely transverse, in the sense that \( \nabla \cdot \mathbf{E} = 0 \).

For the sake of comparison, we have included an appendix with the expansion of the Bessel modes in terms of the more common plane and spherical waves.

### III. QUANTIZATION AND DYNAMICAL VARIABLES.

In the formalism of the previous section, the electromagnetic field is described in terms of two independent sets of modes. This representation has the advantage that the field can be quantized without further complications, even though it is a vector field with an additional freedom of gauge. The field operator \( \hat{\mathbf{A}}(r, t) \) takes the explicit form:

\[
\hat{\mathbf{A}}(r, t) = \sum_{i=1,2} \sum_{m=-\infty}^{\infty} \int_0^\infty dk_\perp \int_{-\infty}^{\infty} dk_z \left[ \hat{a}_{m}^{(i)}(k_z, k_\perp) \mathbf{A}^{(i)}(r, t; K) + \hat{a}_{m}^{(i)}(k_z, k_\perp) \mathbf{A}^{(i)*}(r, t; K) \right],
\]

where the annihilation and creation operators satisfy the usual commutation relations

\[
[\hat{a}_{m}^{(i)}(k_\perp, k_z), \hat{a}_{m'}^{(i')}\dagger(k'_\perp, k'_z)] = \delta^{(i,i')}\delta_{m,m'}\delta(k_\perp - k'_\perp)\delta(k_z - k'_z),
\]

the index \( i \) referring to the two modes of the electromagnetic field, that is, the \( TM(i = 1) \) and \( TE(i = 2) \) modes.

The modes should be so normalized that each photon of frequency \( \omega = c \sqrt{k_z^2 + k_\perp^2} \) carries an energy \( \hbar \omega \) given by

\[
\mathcal{E}(K) = \frac{1}{8\pi} \int \left[ |\mathbf{E}(r, t; K)|^2 + |\mathbf{B}(r, t; K)|^2 \right] dV.
\]

This condition can be satisfied provided the integral is well defined. For Bessel modes, with the volume of integration taken as the whole space, the normalization condition must be generalized to:

\[
\frac{1}{4\pi} \int \left[ \mathbf{E}^{(i)*}(r, t; K) \cdot \mathbf{E}^{(i)}(r, t; K') + \mathbf{B}^{(i)*}(r, t; K) \cdot \mathbf{B}^{(i)}(r, t; K') \right] dV
\]
\[ \hbar \omega \delta_{m,m'} \delta(k_\perp - k'_\perp) \delta(k_z - k'_z), \quad (23) \]

which is equivalent to choosing amplitudes \( \mathcal{E}_m^{(TE)}(k_\perp, k_z) = \mathcal{E}_m^{(TM)}(k_\perp, k_z) = k_z c \sqrt{\hbar k_\perp / 2 \pi \omega} \) for each mode.

Defining now the generalized number operator:

\[ \hat{N}_m^{(i)} = \frac{1}{2} \left( \hat{a}_m^{(i)\dagger} \hat{a}_m^{(i)} + a_m^{(i)} \hat{a}_m^{(i)\dagger} \right), \quad (24) \]

the quantum energy operator takes the form:

\[ \hat{\mathcal{E}} = \hbar \sum_{i,m} \int d k_\perp d k_z \omega \hat{N}_m^{(i)}(k_\perp, k_z), \quad (25) \]

as it should be.

For time independent states, the expectation value of the energy operator, integrated over a certain volume, will remain constant as long as the total energy flux over a surface enclosing that volume is zero. If the volume is the whole space, this condition is satisfied provided the field is localized, that is, its expectation value decays to zero sufficiently fast at infinity.

We now turn our attention to other dynamical variables. The general expression for the momentum operator is [18]:

\[ \hat{P}(t) = \frac{1}{8 \pi c} \int \left[ \hat{\mathcal{E}}(r,t) \times \hat{\mathcal{B}}(r,t) - \hat{\mathcal{B}}(r,t) \times \hat{\mathcal{E}}(r,t) \right] dV. \quad (26) \]

For the Bessel modes under consideration, it takes the form:

\[
\hat{P} = \hbar \sum_{i,m} \int d k_\perp d k_z \left[ i k_\perp \hat{a}_m^{(i)\dagger} \hat{a}_m^{(i)} e_- - i k_\perp \hat{a}_m^{(i)} \hat{a}_{m-1}^{(i)\dagger} e_+ + k_z \hat{N}_m^{(i)} e_3 \right] \\
= \hbar \sum_i \int d k_\perp d k_z \left[ k_\perp \hat{\Pi}_+^{(i)} e_- + k_\perp \hat{\Pi}_-^{(i)} e_+ + k_z \hat{\Pi}_3^{(i)} e_3 \right], \quad (27)
\]

where the operators \( \hat{\Pi}_{\pm,3}^{(i)}(k_\perp, k_z) \) are defined as

\[ \hat{\Pi}_+^{(i)} = i \sum_m \hat{a}_m^{(i)\dagger} \hat{a}_m^{(i)}, \quad (28) \]
\[ \hat{\Pi}_-^{(i)} = -i \sum_m \hat{a}_m^{(i)} \hat{a}_m^{(i)\dagger}, \quad (29) \]
\[ \hat{\Pi}_3^{(i)} = \sum_m \hat{N}_m^{(i)}. \quad (30) \]
Notice that the $z$ component of the momentum is diagonal in this basis, just as for plane waves, but this is not the case for the other components. Nevertheless, Eq. (27) shows that Bessel beams with $k_\perp \neq 0$ may carry linear momentum in the plane perpendicular to the propagation direction $e_3$. This is the case, for instance, for a field described by two mode coherent states such as $|\alpha\rangle_{i,k_\perp,m,k_z}|\alpha'\rangle_{i,k_\perp,m\pm 1,k_z}$, where as usual,

$$\hat{a}_m^{(i)}(k_z,k_\perp)|\alpha\rangle_{i,K} = \alpha|\alpha\rangle_{i,K}. \quad (31)$$

From Noether theorem and the isotropy of space, the following definition of the field angular momentum in a volume $V$ and around a point $r_0$ is obtained [18]:

$$J(r_0) = \frac{1}{4\pi c} \int_V (r - r_0) \times [E(r,t) \times B(r,t)] dV \quad (32)$$

$$= J(0) - r_0 \times P. \quad (33)$$

Due to the presence of the position vector $r$, this integral may diverge if taken over the whole space; this may be the case even if the fields are localized and the integrated energy and linear momentum are finite.

Using Maxwell equations, the total angular momentum can also be written as [9]:

$$J(r_0) = \frac{1}{4\pi c} \int_V E_i[(r - r_0) \times \nabla] A_i \ dV + \frac{1}{4\pi c} \int_V E \times A \ dV$$

$$- \frac{1}{4\pi c} \oint_S E[(r - r_0) \times A] \cdot ds, \quad (34)$$

where summation over repeated indices is implicit and $S$ is the surface enclosing $V$. The first integral involves the differential operator $(r - r_0) \times \nabla$, which is usually associated to the orbital angular momentum; thus, it is customary to identify

$$L(r_0) = \frac{1}{4\pi c} \int_V E_i[(r - r_0) \times \nabla] A_i \ dV, \quad (35)$$

with the OAM of the field [9]. On the other hand,

$$S = \frac{1}{4\pi c} \int_V E \times A \ dV \quad (36)$$

is independent of the choice of origin and is identified with the spin of the field.

However, one should be careful with the above identification of spin and orbital terms because they depend on the chosen gauge, whereas physically observable quantities should not. This difficulty is commonly avoided using the transverse gauge, $\nabla \cdot A = 0$; if each
mode of the EM field has a well defined frequency, that is \( \mathbf{E}_{\omega}(\mathbf{r}, t) = \mathbb{R} \mathbf{E}_0(\mathbf{r}) e^{-i\omega t + \phi} \), then, in this gauge, \( \mathbf{A}_{\omega}(\mathbf{r}, t) = (-i/\omega)\mathbf{E}_{\omega}(\mathbf{r}, t) \) for each mode, and \( \mathbf{L} \) and \( \mathbf{S} \) can be written in an apparently gauge independent form. Once the EM field is quantized, the results obtained in the transverse gauge turn out to be consistent with the expected values \( \pm \hbar \) of the spin in plane and spherical symmetries, but this is not the case for other gauge selections \([2, 3]\). The former consistency is also obtained for the angular momentum flux in the transverse gauge \([20]\).

Notice also that, in general, the intrinsic angular momentum of a massless particle cannot be defined in an unambiguous way. Instead, the relevant dynamical variable is the helicity (see, e. g., the discussion in Ref. \([21]\)) and it is actually this quantity that Beth measured in his classical experiment \([22]\).

Finally, it is important to notice that the integral associated to \( \mathbf{L} \) in Eq. \((34)\) is well defined only if the electromagnetic field vanishes faster than \( r^{-2} \). Van Enk and Nienhuis \([7]\) studied the consequences of quantizing the electromagnetic field in terms of creation and annihilation operators related to such localized electromagnetic modes: they have shown that even under this boundary condition the corresponding operators \( \hat{\mathbf{L}} \) and \( \hat{\mathbf{S}} \) cannot be identified with angular momentum operators because they satisfy a closed but different algebra. However, if the electric and magnetic fields are written using a basis formed by non localized modes, there is no natural separation of spin and orbital momentum since the integrals are not well defined.

This kind of difficulties is manifest for Bessel beams. One possible way to overcome the problem is to impose boundary conditions on a given surface (a cylinder in this case), but such a restriction would break isotropy \([4]\). The other possibility is to take Eqs. \((35)\) and \((36)\) as definitions of angular and spin operators, and to carry on the calculations in order to study the properties of the resulting operators. We will use the latter approach in the following.

The result for the “orbital” angular momentum quantum operator using the basis of Bessel modes turns out to be:

\[
\hat{\mathbf{L}}(0) = \hbar \sum_{i,m} \int dk_{\perp} dk_z \left[ i \frac{k_z}{k_{\perp}} (m - \frac{1}{2}) \hat{a}_{m-1}^{(i)\dagger} \hat{a}_{m}^{(i)} \mathbf{e}_- - \frac{k_z}{k_{\perp}} (m - \frac{1}{2}) \hat{a}_{m}^{(i)\dagger} \hat{a}_{m-1}^{(i)} \mathbf{e}_+ + m \hat{N}_m \mathbf{e}_3 \right]
\]

\((37)\)
\[ \sum_i \int dk_\perp dk_z \left[ \frac{k_z}{k_\perp} \hat{\Lambda}_+^{(i)} e_- + \frac{k_z}{k_\perp} \hat{\Lambda}_+^{(i)} e_+ + \hat{\Lambda}_3^{(i)} e_3 \right], \]

where the operators \( \hat{\Lambda}_{\pm,3}(k_\perp, k_z) \) are given by

\[ \hat{\Lambda}_+^{(i)} = i \sum_m (m - \frac{1}{2}) \hat{a}_m^{(i)\dagger} (k_\perp, k_z) \hat{a}_m^{(i)} (k_\perp, k_z), \quad (38) \]

\[ \hat{\Lambda}_-^{(i)} = -i \sum_m (m - \frac{1}{2}) \hat{a}_m^{(i)\dagger} (k_\perp, k_z) \hat{a}_{m-1}^{(i)} (k_\perp, k_z), \quad (39) \]

\[ \hat{\Lambda}_3^{(i)} = \sum_m m \hat{N}_m^{(i)} (k_\perp, k_z). \quad (40) \]

The above relations have a more complex structure than the one obtained for spherical vectors (see Appendix B), but this is to be expected since the latter are explicitly constructed to describe the orbital angular momentum.

Notice also that \( \hat{L}_z(0) \) is invariant under Lorentz transformations along the \( z \) axis as expected; it can be interpreted as an intrinsic operator since it does not depend explicitly on \( k_\perp \). On the other hand, \( \hat{L}_{x,y}(0) \) are highly dependent on quantum numbers \( \{k_\perp, m, k_z\} \); moreover, if we define \( \hat{L}_\pm \equiv \hat{L}_x \pm i \hat{L}_y \), then \( \hat{L}_+ (\hat{L}_-) \) acts as a lowering (rising) operator that changes \( m \to m - 1 \) (\( m \to m + 1 \)).

For the helicity operator \( \hat{S} \), we find:

\[ \hat{S} = \hbar \sum_m \int dk_\perp dk_z \frac{c}{2\omega} \left[ k_\perp (\hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)} - \hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)}) e_- + k_\perp (\hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)} - \hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)}) e_+ + i k_z (\hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)} - \hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)}) e_3 \right] \]

\[ = \hbar \int dk_\perp dk_z \frac{c}{\omega} \left[ k_\perp \hat{\Sigma}_- e_- + k_\perp \hat{\Sigma}_+ e_+ + k_z \hat{\Sigma}_3 e_3 \right], \quad (42) \]

where the operators \( \hat{\Sigma}_{\pm,3}(k_\perp, k_z) \) are defined as

\[ \hat{\Sigma}_+ = \frac{1}{2} \sum_m (\hat{a}_m^{(2)\dagger} \hat{a}_m^{(1)} - \hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)}) \]

\[ \hat{\Sigma}_- = \frac{1}{2} \sum_m (\hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)} - \hat{a}_m^{(2)\dagger} \hat{a}_m^{(1)}) \]

\[ \hat{\Sigma}_3 = i \sum_m (\hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)} - \hat{a}_m^{(1)\dagger} \hat{a}_m^{(2)\dagger}) \]

\[ 10 \]
IV. ALGEBRAIC PROPERTIES OF THE DYNAMICAL OPERATORS.

The basic dynamical quantities can in general be identified by their algebraic properties. Thus, for instance, the components of the linear momentum operator must commute among themselves; a direct calculation shows that this is indeed the case: the operators $\Pi_n$ defined in Eq. (27) do commute, $[\Pi_i, \Pi_j] = 0$, and therefore $[\hat{P}_i, \hat{P}_j] = 0$ as expected.

However, the components of $\hat{L}$ and $\hat{S}$ do not satisfy the commutation relations of angular momentum. In fact, it can be seen that:

$$[\hat{L}_+ + \hat{L}_-, \hat{L}_3] = \hbar \hat{L}_3,$$  \hspace{1cm} (46)

$$[\hat{L}_-, \hat{L}_3] = -\hbar \hat{L}_-, \hspace{1cm} (47)$$

$$[\hat{L}_+, \hat{L}_-] = 2\hbar^2 \sum_i \int dk_\perp dk_z \frac{k_z^2}{k_\perp^2} \hat{\Lambda}_3.$$

(48)

On the other hand, all the components of the operator $\Sigma$ commute among themselves: $[\hat{\Sigma}_i, \hat{\Sigma}_j] = 0$, so that

$$[\hat{S}_i, \hat{S}_j] = 0,$$  \hspace{1cm} (49)

and it can also be shown that they commute with the momentum operator:

$$[\hat{P}_i, \hat{S}_j] = 0.$$  \hspace{1cm} (50)

These properties are compatible with the identification of $\hat{S}$ as an helicity operator.

Furthermore, $\hat{L}_3$ commutes with the z-component of the linear momentum operator $\hat{P}$,

$$[\hat{L}_3, \hat{P}_3] = 0.$$  \hspace{1cm} (51)

while

$$[\hat{L}_3, \hat{P}_-] = \hbar \hat{P}_-,$$  \hspace{1cm} (52)

$$[\hat{L}_+, \hat{P}_-] = \hbar \hat{P}_3,$$  \hspace{1cm} (53)

$$[\hat{L}_+, \hat{P}_3] = 0,$$  \hspace{1cm} (54)

$$[\hat{L}_+, \hat{P}_+] = \hbar^2 \sum_{i,m} \int dk_\perp dk_z \frac{k_z}{k_\perp^2} \hat{a}_{m-1}^{(i)\dagger} \hat{a}_{m+1},$$  \hspace{1cm} (55)

and therefore the algebra of these operators does not close.

Finally, $\hat{S}_3$ commutes with $\hat{L}$,

$$[\hat{S}_3, \hat{L}] = 0,$$  \hspace{1cm} (56)
while

\[
\begin{align*}
[\hat{S}_+, \hat{L}_+] &= -\hbar \hat{S}_z \quad (57) \\
[\hat{S}_+, \hat{L}_z] &= -\hbar^2 \int dk_\perp dk_\omega \frac{ck_\perp}{\omega} \Sigma_+ \quad (58) \\
[\hat{S}_+, \hat{L}_-] &= -i\hbar^2 \int dk_\perp dk_\omega \frac{ck_\perp}{\omega} (a_{m-1}^{(1)} a_{m+1}^{(2)\dagger} - a_{m-1}^{(2)} a_{m+1}^{(1)\dagger}) \quad (59)
\end{align*}
\]

Summing up, the components of the momentum operator \( P \) commute among themselves, as it should be, but the algebra they generate with the other two operators \( \hat{L}(0) \) and \( \hat{S} \) is \textit{not} the standard one for the translation and rotation group. This is not unexpected since, according to our previous discussion, there is an ambiguity with the decomposition of the total angular momentum into spin and orbital parts; and moreover, the “spin” is rather the helicity.

The polarization state of a plane wave with propagation vector \( \mathbf{k} = k_3 \mathbf{e}_z \) is completely characterized by its Stokes parameters. Their quantum counterparts \[19\] are given by the operators

\[
\hat{\sigma}_1 = \hat{a}^{(x)\dagger} \hat{a}^{(y)} + \hat{a}^{(y)\dagger} \hat{a}^{(x)}, \\
\hat{\sigma}_2 = i(\hat{a}^{(y)\dagger} \hat{a}^{(x)} - \hat{a}^{(x)\dagger} \hat{a}^{(y)}), \\
\hat{\sigma}_3 = \hat{a}^{(x)\dagger} \hat{a}^{(x)} - \hat{a}^{(y)\dagger} \hat{a}^{(y)}, \\
\hat{\sigma}_0 = \hat{a}^{(x)\dagger} \hat{a}^{(x)} + \hat{a}^{(y)\dagger} \hat{a}^{(y)},
\]

where the indices \( x \) and \( y \) refer to linearly polarized plane waves in the corresponding directions. The operators \( \{\sigma_1, \sigma_2, \sigma_3\} \) satisfy the algebra of the rotation group: \([\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\) up to a factor 2. One can readily extend these definitions to Bessel beams identifying the indices with the TE and TM superscripts. Clearly, \( \sigma_2 \) is the elementary operator appearing in \( \hat{S}_3 \). Thus, measurements of \( \hat{\sigma} \) for Bessel beams should yield important information about their polarization states, just as in the case of plane waves.

Since \( \hat{E}, \hat{P}_3, \hat{L}_3(0) \), and \( \hat{S}_3 \) commute among themselves, they can be simultaneously diagonalized. This can be done by introducing the operators,

\[
\hat{a}^{(\pm)}_m = \frac{1}{\sqrt{2}} \left( \hat{a}^{(1)}_m \pm i\hat{a}^{(2)}_m \right),
\]

which corresponds to a new basis

\[
A^{(\pm)}_m = \frac{1}{\sqrt{2}} \left( A^{(TM)}_m \pm iA^{(TE)}_m \right)
\]
for the fundamental modes.

At this stage, it is important to compare our results with an alternative selection of basis modes that appears in the literature [8, 23]. Namely, the following modes

\[
A_m^{(R)}(r, t; k_\perp, k_z) = A_0^{(R)} \left[ e^{-\psi_m} + \frac{i}{2} \frac{k_\perp}{k_z} \psi_{m-1} e_3 \right],
\]

\[
A_m^{(L)}(r, t; k_\perp, k_z) = A_0^{(L)} \left[ e^{\psi_m} - \frac{i}{2} \frac{k_\perp}{k_z} \psi_{m+1} e_3 \right],
\]

where \( \psi_m(r, t; k_\perp, k_z) = J_m(k_\perp \rho) \exp \{-i\omega t + ik_z z + im\phi\} \). They are considered to be the analogues of right (R) and left (L) polarized plane wave modes [8, 23]. Their superpositions \( A_m^{(R)} \pm A_m^{(L)} \) define linearly polarized modes, and they can be written as linear combinations of elementary TE and TM modes:

\[
A_m^{(R)} = A_0^{(R)} \left( A_{m-1}^{(TM)} + \frac{ck_z}{\omega} A_{m-1}^{(TE)} \right),
\]

\[
A_m^{(L)} = A_0^{(L)} \left( A_{m+1}^{(TM)} - \frac{ck_z}{\omega} A_{m+1}^{(TE)} \right).
\]

Within the quantization scheme, this change of basis corresponds to the following definition of the annihilation operators:

\[
\hat{a}_m^{(R)} = \frac{1}{\sqrt{1 + (ck_z/\omega)^2}} \left( \hat{a}_m^{(1)} + \frac{ck_z}{\omega} \hat{a}_m^{(2)} \right),
\]

\[
\hat{a}_m^{(L)} = \frac{1}{\sqrt{1 + (ck_z/\omega)^2}} \left( \hat{a}_m^{(1)} - \frac{ck_z}{\omega} \hat{a}_m^{(2)} \right).
\]

Now, the point is that, although the helicity operator \( \hat{S}_3 \) is diagonal in this basis:

\[
\hat{S}_3 = \hbar \sum_m \int dk_\perp dk_z \frac{1 + (\omega/ck_z)^2}{2} \left( \hat{N}_m^{(R)} - \hat{N}_m^{(L)} \right),
\]

the operators \( \hat{E}, \hat{P}_3 \) and \( \hat{L}_3 \) are not diagonal. This can be seen from the fact that:

\[
\hat{a}_m^{(1)\dagger} \hat{a}_m^{(1)} + \hat{a}_m^{(2)\dagger} \hat{a}_m^{(2)} = \frac{1}{4} \left[ 1 + (ck_z/\omega)^2 \right] \left( \hat{a}_{m-1}^{(R)\dagger} \hat{a}_{m-1}^{(R)} + \hat{a}_{m+1}^{(L)\dagger} \hat{a}_{m+1}^{(L)} \right)
\]

\[
+ \left[ 1 - (\omega/ck_z)^2 \right] \left( \hat{a}_{m+1}^{(R)\dagger} \hat{a}_{m-1}^{(R)} - \hat{a}_{m-1}^{(R)\dagger} \hat{a}_{m+1}^{(R)} \right),
\]

as follows with some straightforward algebra. It should be noticed that it is only in the paraxial approximation, \( k_z \sim \omega/c \), that the second term in this last equation, which is non diagonal, does vanish.
V. DISCUSSION AND CONCLUSIONS

Let us summarize the main results obtained with the quantization of Bessel beams. The proper values of the set of observables \( \{ \hat{E}, \hat{P}_z, \hat{L}_3(0), \hat{S}_3 \} \) define the possible quantum numbers that characterize the Bessel photons: \( \{ \omega, \hbar k_z, m\hbar, \pm \hbar k_z c / \omega \} \); their physical interpretations are clear: for instance, \( \hat{S}_3 \) is the helicity operator. We have also analyzed the role of all the dynamical operators appearing within the quantization scheme. It turned out that the three components of the orbital angular momentum \( \{ \hat{L}_+ , \hat{L}_-, \hat{L}_3 \} \) do not satisfy a closed algebra, despite the fact that \( \hat{L}_3 \) is related to a spatial rotation around the \( z \) axis; in fact, this algebra is not the same as obtained for localized fields in Ref. [7]. This is the price we have to pay for not taking the surface terms in Eq. (34) into consideration.

Now, the algebra of the local operators \( \hat{E} \) and \( \hat{B} \) is independent of the gauge and the basis set. Accordingly, global bilinear operators of the electromagnetic fields such as \( \hat{P} \) and \( \hat{S} \) have commutation relations among all their components that are also independent of the basis set; this is guaranteed by the normalization condition that each photon carries an energy \( \hbar \omega \). However, the global operators \( \hat{J} \) and \( \hat{L} \) are defined in terms not only of \( \hat{E} \) and \( \hat{B} \), but also of the position vector \( r \); this term induces a strong dependence of \( \hat{J} \) and \( \hat{L} \) on the boundary conditions satisfied by \( \hat{E} \) and \( \hat{B} \). This fact is illustrated in Appendix B, where the equivalent \( \hat{L}(0) \) operator is given in terms of the spherical vector basis, and it is shown that it does satisfy the standard algebra. In any case, the algebraic properties of the dynamical operators and their commutation relations have physical consequences because they imply, for instance, specific uncertainty relations that could be verified experimentally.

It is also worth mentioning that all the dynamical operators we have studied in this paper correspond to global observable quantities. A further analysis of local dynamical quantities, such as the tensor \( M_{ij} \) describing the angular momentum flux, could elucidate the difference between “spin” and “orbital” angular momentum. In fact, it was shown by Barnett [20] that in the classical case, there is a natural separation into spin and orbital parts for the \( z \) component of this flux, \( M_{zz} \). However, a quantum description should also include the full commutation relations of the appropriate separated parts of this tensor. This is particularly relevant in the light of recent experiments measuring the rates of spin and orbital rotation of trapped particles at different distances from the beam axis [14].

It is now well established that Bessel beams induce the rotation of microparticles trapped
in an optical tweezers \[13, 15\]. The experiments described by O’Neil et al. \[15\] use Laguerre-Gaussian waves that are circularly polarized in the sense of our Eq. (66). When the beams are converted into linearly polarized waves by a birefringent trapped particle, the particle spins around its own axis in a direction determined by the handedness of the circular polarization, while small particles trapped off the beam axis rotate around that axis in a direction determined by the handedness of the helical phase fronts \[15\]. Now, according to our results, a similar experiment with Bessel beams that are superpositions of elementary TE and TM modes, \( \mathbf{A}(K) = \mathbf{A}^{(TM)}(K) \pm i\mathbf{A}^{(TE)}(K) \), should also induce the spinning of a trapped particle around its axis; but, since there is a relation \( S_3 = \pm \vec{k}_z c/\omega \) for each photon, the angular momentum should exhibit a linear dependence on \( k_z \) for a fixed beam intensity. This prediction could be tested experimentally.

In a future publication, we will investigate the quantum electromagnetic interaction of atoms with Bessel beams using the formalism developed in this paper. Particular emphasis will be given to further clarifying the role of spin and orbital angular momentum of light. A detailed analysis of this interaction should explain why the spontaneous emission of Bessel photons by atoms is a strongly inhibited process, as experiments have shown so far.

**Acknowledgements**

We acknowledge very stimulating discussions with Karen Volke-Sepúlveda. This work was partially supported by PAPIIT IN-103103.

**Appendix A. Some useful equations.**

The following formulas are used in order to perform the integrations of terms involving Bessel functions. They can be easily obtained from the Hankel transform and anti-transform formulas (see, e. g., \[24\]). Namely:

\[
\int_0^\infty J_m(k\rho)J_m(k'\rho)\rho d\rho = \frac{1}{k} \delta(k - k'),
\]

(73)

and

\[
\int_0^\infty J_m(k\rho)J'_m(k'\rho)\rho^2 d\rho = -\frac{1}{k^2} \delta'(k - k') - \frac{1}{k^2} \delta(k - k').
\]

(74)
From these last expressions and using the standard recurrence relations for Bessel functions, it also follows that

\[
\int_0^\infty \left[ \frac{m^2}{kk'} J_m(k\rho) J_m(k'\rho) + \rho J'_m(k\rho) J'_m(k'\rho) \right] d\rho = \frac{1}{k} \delta(k - k'),
\]

(75)

and

\[
\int_0^\infty J_m(k\rho) J_{m+1}(k'\rho) \rho^2 d\rho = \frac{1}{k'} \delta'(k - k') + \frac{m + 1}{k^2} \delta(k - k').
\]

(76)

Using the above formulas, we can obtain several typical integrals that are used in Section III. Using the shorthand notation \( M = M_m(x^\mu; k_\perp, k_z) \), \( M' = M_{m'}(x^\mu; k'_\perp, k'_z) \), etc., it can be shown that for the scalar products:

\[
\int M \cdot M'^* dV = \int N \cdot N'^* dV = (2\pi)^2 \frac{\omega^2}{c^2 k_\perp k_z^2} \delta_{m,m'} \delta(k_\perp - k'_\perp) \delta(k_z - k'_z),
\]

(77)

and

\[
\int M \cdot N'^* dV = \int N \cdot M'^* dV = 0.
\]

(78)

Also:

\[
\int M \cdot M' dV = -\int N \cdot N' dV = -(2\pi)^2 \frac{\omega^2}{k_\perp k_z^2} \delta_{m,-m'} \delta(k_\perp - k'_\perp) \delta(k_z + k'_z) e^{-2i\omega t},
\]

(79)

and

\[
\int M \cdot N' dV = \int N \cdot M' dV = 0.
\]

(80)

Similarly for the vector products:

\[
-\int M \times N'^* dV = \int N \times M'^* dV = (2\pi)^2 \frac{\omega}{k_z^2} \left\{ \frac{i}{2} \left[ \delta_{m+1,m'} e_- - \delta_{m-1,m'} e_+ \right] + \frac{k_z}{k_\perp} \delta_{m,m'} e_3 \right\} \delta(k_\perp - k'_\perp) \delta(k_z - k'_z),
\]

(81)

and

\[
\int M \times M'^* dV = \int N \times N'^* dV = 0.
\]

(82)

And also

\[
\int (M \times N' - N \times M') dV = 0.
\]

(83)
Define now the operator $L_{\pm} = L_x \pm iL_y$, with $\vec{L} = -i\mathbf{r} \times \nabla$. Then:

$$L_{\pm} = e^{\pm i\phi} \left[ z \left( \frac{\partial}{\partial \rho} \pm \frac{i}{\rho} \frac{\partial}{\partial \phi} \right) - \rho \frac{\partial}{\partial z} \right]. \quad (84)$$

It then follows that:

$$\int \mathbf{M}' \cdot (\mathbf{L}_+ \mathbf{M}) dV =$$

$$i(2\pi)^2 \frac{\omega'}{k_\perp k_z k_z'} e^{i(\omega - \omega')t} \delta_{m,m'+1} \left[ \frac{k_\perp}{\partial k_z} - \frac{k_z}{\partial k_\perp} + m \frac{k_z}{k_\perp} \right] \delta(k_\perp - k_\perp') \delta(k_z - k_z'), \quad (85)$$

and

$$\int \mathbf{M}^* \cdot (\mathbf{L}_+ \mathbf{M}) dV =$$

$$i(2\pi)^2 \frac{\omega'}{k_\perp k_z k_z'} e^{i(\omega - \omega')t} \delta_{m,m'-1} \left[ \frac{k_\perp}{\partial k_z} - \frac{k_z}{\partial k_\perp} - m \frac{k_z}{k_\perp} \right] \delta(k_\perp - k_\perp') \delta(k_z - k_z'). \quad (86)$$

Also:

$$\int \mathbf{M}' \cdot (\mathbf{L}_+ \mathbf{M}) dV = 0, \quad (87)$$

using the fact that in this formula all Dirac deltas appear multiplied by their arguments, and $x \delta(x) = 0$.

### Appendix B. Comparison with plane and spherical vector EM modes.

Using the formula

$$J_m(k_\perp \rho) e^{im\phi} = \frac{(-i)^m}{2\pi} \int_{-\pi}^{\pi} d\varphi_k e^{im\varphi_k} e^{i k_\perp \rho [\cos \varphi_k \cos \phi + \sin \phi \sin \varphi_k]}, \quad (88)$$

it can be seen that the vectors $\mathbf{M}$ and $\mathbf{N}$, given by Eqs. (11) and (12), and that determine the TE and TM Bessel vector potentials, can also be written in the form:

$$\mathbf{M}(\mathbf{r}, t; \kappa) = (-i)^m \int d^3k' e^{ik' \cdot \mathbf{r}} \delta(k_z - k_z') \delta(k_\perp - k_\perp') e^{im\varphi_k} e^{\frac{\omega}{ck_z k_\perp} \varphi_k}, \quad (89)$$

$$\mathbf{N}(\mathbf{r}, t; \kappa) = -(i)^m \int d^3k' e^{ik' \cdot \mathbf{r}} \delta(k_z - k_z') \delta(k_\perp - k_\perp') e^{im\varphi_k} e^{\frac{ck_z}{\omega k_\perp} \varphi_k}, \quad (90)$$

where

$$\hat{\theta}_k = \cos \theta_k \cos \varphi_k \hat{e}_1 + \cos \theta_k \sin \varphi_k \hat{e}_2 - \sin \theta_k \hat{e}_3 \quad (91)$$
and
\[ \hat{\phi}_k = -\sin \varphi_k \hat{e}_1 + \cos \varphi_k \hat{e}_2 \] (92)
are the spherical unitary vectors associated to the angular coordinate \( \theta_k \) and \( \varphi_k \) in the space of the propagator vectors \( k \). These expressions show explicitly the transverse nature of the electromagnetic Bessel modes. They can be considered expansions of Bessel modes in terms of plane waves that, as it is well known, diagonalize the momentum operator. Eqs. (89-90) permit to evaluate the expressions for Bessel modes in terms of the spherical vectors.

Spherical vectors form a complete basis for transverse electromagnetic fields in free space. They are defined by (see, e.g., [21])
\[ A^{(i)}_{\omega jm}(r) = \frac{1}{(2\pi)^3} \int d^3k \tilde{A}^{(i)}_{\omega jm}(k) e^{i k r} , \] (93)
where
\[ \tilde{A}^{(i)}_{\omega jm}(k) = \frac{4 \pi^2 e^2 \hbar^{1/2}}{\omega^{3/2}} \delta(|k| - \omega) Y^{(i)}_{jm}(\hat{n}) , \quad \hat{n} = \frac{k}{|k|} . \] (94)
In these equations, the superscript specifies the electric (E) and magnetic (B) modes, and
\[ Y^{(E)}_{jm}(\theta_\hat{n}, \varphi_\hat{n}) = \frac{1}{j(j+1)} \nabla_\hat{n} Y_{jm}(\theta_\hat{n}, \varphi_\hat{n}) , \] (95)
\[ Y^{(M)}_{jm}(\theta_\hat{n}, \varphi_\hat{n}) = \hat{n} \times Y^{(E)}_{jm}(\theta_\hat{n}, \varphi_\hat{n}) , \] (96)
with
\[ \nabla_\hat{n} = \hat{\theta}_k \frac{\partial}{\partial \theta_k} + \hat{\varphi}_k \frac{1}{\sin \theta_k} \frac{\partial}{\partial \varphi_k} ; \] (97)
\( Y_{jm}(\theta_\hat{n}, \varphi_\hat{n}) \) are the spherical harmonics. When the electromagnetic field is properly quantized in terms of spherical vectors (SV) the corresponding angular momentum operator \( \hat{L}^{(SV)} \) takes the form
\[ \hat{L}^{(SV)}(0) = \hbar \sum_{i,j,m} \int d\omega \left[ \frac{1}{2} \sqrt{(j-m)(j+m+1)} \hat{a}_{\omega,j,m}^{(i)\dagger} \hat{a}_{\omega,j,m}^{(i)} e_- + \frac{1}{2} \sqrt{(j+m)(j-m+1)} \hat{a}_{\omega,j,m}^{(i)\dagger} \hat{a}_{\omega,j,m}^{(i)} e_+ + m \hat{N}_{\omega,j,m}^{(i)} e_3 \right] , \] (98)
with the associated number operator:
\[ \hat{N}_{\omega jm}^{(i)} = \frac{1}{2} \left( \hat{a}_{\omega jm}^{(i)\dagger} \hat{a}_{\omega jm}^{(i)} + \hat{a}_{\omega jm}^{(i)} \hat{a}_{\omega jm}^{(i)\dagger} \right) . \] (99)
A direct calculation shows that, in these case, the standard commutation relations are obtained: \[ [\hat{L}_i^{(SV)}, \hat{L}_j^{(SV)}] = i\hbar \epsilon_{ijk} \hat{L}_k^{(SV)} . \]
From a straightforward calculation it follows that:

\[
N_{k_\perp k_z m}(r) = \sum_{j=1}^{\infty} \sum_{-m_j}^{m_j} \int d\omega u(k_\perp, k_z, m; \omega, j, m_j)(A^{(E)}_{\omega jm_j}(r) + A^{(M)}_{\omega jm_j}(r)),
\]

(100)

\[
M_{k_\perp k_z m}(r) = \sum_{j=1}^{\infty} \sum_{-m_j}^{m_j} \int d\omega v(k_\perp, k_z, m; \omega, j, m_j)(A^{(E)}_{\omega jm_j}(r) - A^{(M)}_{\omega jm_j}(r)),
\]

(101)

with

\[
u(k_\perp, k_z, m; \omega, j, m_j) = \int d^3r N_{k_\perp k_z m} \cdot A^{(E)}_{\omega jm_j}
\]

\[
= \int d^3r N_{k_\perp k_z m} \cdot A^{(M)}_{\omega jm_j}
\]

\[
= -4\pi^2(-1)^{m+(m+|m|)/2}(i)^{m+j}\delta(|k| - \omega)\delta_{m,m_j} \frac{c^{1/2}k_\perp}{k_z \omega^{1/2}}
\]

\[
\times \sqrt{\frac{(2j+1)(j-|m|)!}{4\pi(j+|m|)!}} \frac{\partial}{\partial k_z} \mathcal{P}_j^{[m]} \left( \frac{ck_z}{\omega} \right)
\]

(102)

and

\[
v(k_\perp, k_z, m; \omega, j, m_j) = \int d^3r M_{k_\perp k_z m} \cdot A^{(E)}_{\omega jm_j}
\]

\[
= -\int d^3r M_{k_\perp k_z m} \cdot A^{(M)}_{\omega jm_j}
\]

\[
= 4\pi^2(-1)^{m+(m+|m|)/2}(i)^{m+j}\delta(|k| - \omega)\delta_{m,m_j} \frac{c^{1/2}k_\perp}{k_z \omega^{1/2}}
\]

\[
\times \sqrt{\frac{(2j+1)(j-|m|)!}{4\pi(j+|m|)!}} \frac{im\omega}{ck_L} \mathcal{P}_j^{[m]} \left( \frac{ck_z}{\omega} \right),
\]

(103)

where \( \mathcal{P}_j^{[m]} \) are the standard Laguerre polynomials. Thus, as expected, Bessel modes are an infinite superposition of spherical waves with different orbital angular momentum \( j \).

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