LIMITING DYNAMICS FOR SPHERICAL MODELS OF SPIN GLASSES WITH MAGNETIC FIELD

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ABSTRACT. We study the Langevin dynamics for the family of spherical spin glass models of statistical physics, in the presence of a magnetic field. We prove that in the limit of system size $N$ approaching infinity, the empirical state correlation, the response function, the overlap and the magnetization for these $N$-dimensional coupled diffusions converge to the non-random unique strong solution of four explicit non-linear integro-differential equations, that generalize the system proposed by Cugliandolo and Kurchan in the presence of a magnetic field.

We then analyze the system and provide a rigorous derivation of the FDT regime in a large area of the temperature-magnetization plane.

1. Introduction

Many of the unique properties of magnetic systems with quenched random interactions, namely spin glasses, are of dynamical nature (see [15]). Therefore, we would like to understand not only the static properties, but also time dependent features of the spin glass state. This is not an easy task, even for the Sherrington and Kirkpatrick (SK) model.

The extended SK model can be described as follows. Let $\Gamma = \{-1, 1\}$ be the space of spins. Fixing a positive integer $N$ (denoting the system size), define, for each configuration of the spins (i.e. for each $\mathbf{x} = (x_1, \ldots, x_N) \in \Gamma^N$), a random Hamiltonian $H^N_j(\mathbf{x})$, as a function of the configuration $\mathbf{x}$ and of an exterior source of randomness (i.e. a random variable defined on another probability space). For the extended SK model, the mean field random Hamiltonian is defined as:

$$H^N_j(\mathbf{x}) = -m \sum_{p=1}^m \frac{a_p}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq N} J_{i_1 \ldots i_p} x^{i_1} \ldots x^{i_p},$$

where $m \geq 2$, and the disorder parameters $J_{i_1 \ldots i_p} = J_{\{i_1, \ldots, i_p\}}$ are independent (modulo the permutation of the indices) centered Gaussian variables. The variance of $J_{i_1 \ldots i_p}$ is $c(\{i_1, \ldots, i_p\}) N^{-p+1}$, where

$$c(\{i_1, \ldots, i_p\}) = \prod_k l_k!,$$

and $(l_1, l_2, \ldots)$ are the multiplicities of the different elements of the set $\{i_1, \ldots, i_p\}$ (for example, $c = 1$ when $i_j \neq i_{j'}$ for any $j \neq j'$, while $c = p!$ when all $i_j$ values are the same). Denoting by $F^N(\mathbf{x})$ the total magnetization of the system:

$$F^N(\mathbf{x}) = \sum_{i=1}^N x^i,$$

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the Gibbs measure for finitely many spins at inverse temperature $\beta = T^{-1}$ and intensity of the magnetic field $h > 0$ is defined as:

$$
\lambda_{\beta,h,N}^\mathbb{Z}(x) = \frac{1}{Z_{\beta,h,N}^\mathbb{Z}} \exp \left( -\beta H_J^N(x) + h F_N(x) \right) \mathbb{1}_{x \in \mathbb{F}^N}.
$$

where $Z_{\beta,h,N}^\mathbb{Z}$ is a normalizing constant. The propagation of chaos for the dynamics is of much interest. It can be studied from the limit as $N \to \infty$ of the empirical measure:

$$
\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{x^i(t)}
$$

Though the limit was established and characterized in [4] via an implicit non-Markovian stochastic differential equation for the continuous relaxation of the SK model with Langevin dynamics, the complexity of the latter equation prevents it from being amenable to a serious understanding.

*Spherical models* replace the product structure of the configuration space $\Gamma$ by the sphere $S^{N-1}(\sqrt{rN})$ in $\mathbb{R}^N$, for $r = 1$, via imposing the hard constraint $\frac{1}{N} \sum_{i=1}^N x_i^2 = r$. The spherical Gibbs measure is then given by:

$$
\mu_{\beta,h,N}^\mathbb{Z}(dx) = \frac{1}{Z_{\beta,h,N}^\mathbb{Z}} \exp \left( -2\beta H_J^N(x) + 2h F_N(x) \right) \nu_N(dx)
$$

where the measure $\nu_N$ is the uniform measure on the sphere $S^{N-1}(\sqrt{rN})$ (the presence of the extra factor of 2 is just a matter of convenience and is equivalent to the rescaling $\beta \to 2\beta$ and $h \to 2h$). The Langevin dynamics for the normalized spherical mixed spin model (i.e. $r = 1$) without magnetization (i.e. $h = 0$), was rigorously studied in [7] and [14]. The authors have shown that the dynamics of the system can be characterized via two functions, the so called *empirical correlation* and *empirical response* and they have derived the pair of coupled integro-differential equations that characterize them.

Here, we shall first extend their results to allow for a positive magnetic field (i.e. $h > 0$) and any radius of the underlying sphere. Due to the extra complexity introduced in the system via the presence of the magnetic field, that affects the symmetry of the spins, the dynamics will be characterized via a coupled system of four integro-differential equations. We rigorously analyze the behavior of the system in the high temperature regime and derive equations characterizing the phase transition curve. Along the way, we prove (see Theorem 2.4) that the system simplifies dramatically for large radii of the underlying sphere.

To work around the complexity induced by the Langevin dynamics on the sphere, we follow [7], by a further relaxation of the *hard spherical* model, replacing the hard spherical constraint by a *soft* one. Namely, we first replace the uniform measure $\nu_N$ on the sphere $S^{N-1}(\sqrt{rN})$ by a measure on $\mathbb{R}^N$,

$$
\tilde{\nu}_N(dx) = \frac{1}{Z_{N,f}^N} \exp \left( -N f \left( \frac{1}{N} \sum_{i=1}^N x_i^2 \right) \right) dx
$$

where $f$ is a smooth function growing fast enough at infinity. The *soft spherical Gibbs measure* is then given by:

$$
d\tilde{\nu}_{\beta,h,N}^\mathbb{Z}(dx) = \frac{1}{Z_{\beta,h,N}^\mathbb{Z}} \exp \left( -N f \left( \frac{\|x\|^2}{N} \right) \right) dx
$$

Thus, $\tilde{\nu}_{\beta,h,N}^\mathbb{Z}$ is the invariant measure of the randomly interacting particles described by the (Langevin) stochastic differential system:

$$
dx_i^t = dB_i^t - f'(N^{-1}\|x_i\|^2)dx_i^t dt + \beta G^i(x_i) dt + h dt,
$$

where $B = (B^1, \ldots, B^N)$ is an $N$-dimensional standard Brownian motion, independent of both the initial condition $x_0$ and the disorder $J$, and $G^i(x) := -\partial_{x_i} (H_J^N(x))$, for $i = 1, \ldots, N$. In Proposition 2.2 we characterize the long term behavior of the Langevin dynamics of this soft spherical model for a general class
of functions \( f \). We shall then choose an appropriate sequence of functions \( f_n \), satisfying \( \tilde{\mu}^N_{\beta,h,J,v} \rightarrow \mu^N_{\beta,h,J} \), allowing us to derive, in Theorem 2.3, the limiting behavior of the hard spherical model.

We shall first prove that, fixing \( f \), for a.e. disorder \( J \), initial condition \( x_0 \) and Brownian path \( B \), there exists a unique strong solution of (1.6) for all \( t \geq 0 \), whose law we denote by \( \mathbb{P}^N_{\beta,x_0,J} \).

We are interested in the time evolution for large \( N \), of the empirical covariance function:

\[
\text{COV}^N_N(s,t) = \frac{1}{N} \sum_{i=1}^{N} \left[ x^i_s x^i_t - \mathbb{E}_B[x^i_s] \mathbb{E}_B[x^i_t] \right],
\]

where \( \mathbb{E}_B[\cdot] \) represents the expectation with respect to the Brownian motion only (and not with respect to the Gaussian law of the couplings), under the quenched law \( \mathbb{P}^N_{\beta,x_0,J} \), as the system size \( N \rightarrow \infty \). In [7], the authors have formally derived the limiting equations for the empirical state correlation function:

\[
C^N_N(s,t) := \frac{1}{N} \sum_{i=1}^{N} x^i_s x^i_t,
\]

in the absence of a magnetic field (i.e. \( h = 0 \)). The equations characterizing the limit as \( N \rightarrow \infty \) of \( C^N_N(s,t) \) involve the analogous limit for the empirical integrated response function:

\[
\chi^N_N(s,t) := \frac{1}{N} \sum_{i=1}^{N} x^i_s B^i_t,
\]

and the limits are characterized as the unique solution of a system of two coupled integro-differential equations. The presence of the magnetic field requires us to consider also the empirical averaged magnetization:

\[
M^N_N(s) := \frac{1}{N} \sum_{i=1}^{N} x^i_s,
\]

the averaged overlap:

\[
L^N_N(s,t) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_B[x^i_s] \mathbb{E}_B[x^i_t],
\]

and the empirical overlap:

\[
Q^N_N(s,t) := \frac{1}{N} \sum_{i=1}^{N} x^1_s x^2_i,
\]

where \( \{x^k\}_s, k = 1, 2 \) are two independent replicas, sharing the same couplings \( J \), with the noise given by two independent Brownian motions \( \{B^k\}_s \). With these notations, our primary object of study, the empirical covariance can be written as:

\[
\text{COV}^N(s,t) = C^N_N(s,t) - L^N_N(s,t).
\]

The empirical overlap defined in (1.12) is the central quantity in the study of the static properties of the system (see [21] for a comprehensive survey). Its dynamical properties were not rigorously analyzed until now. In the course of our proofs, we show that the limits as \( N \rightarrow \infty \) of \( L^N_N \) (i.e. the averaged overlap - that we need to characterize in order to study the empirical covariance) and of \( Q^N_N \) (i.e. the empirical overlap - that is interesting in its own right), coincide. Also, as opposed to the scenario analyzed in [7] (i.e. \( h = 0 \)), where the authors have characterized the dynamics via a coupled system of two integro-differential equations, the presence of the magnetic field will affect the symmetry of the spins and the dynamics of our system will be characterized via a coupled system of four integro-differential equations.

We shall analyze the solutions of the latter system in a non-perturbative high temperature region of the \((\beta,h)\)-plane, rigorously establishing the existence of the so called FDT regime, where the Frequency Dissipation
Theorem in statistical physics holds. We shall see that the phase plane diagram of the system in \((\beta, h)\) coordinates is the one shown in Figure 1 below.

\[ \text{Figure 1. The Phase Plane Diagram: The hashed region represents the area of applicability of Theorem 2.5, where we can rigorously prove the FDT regime, the light region represents the expected extend of the FDT regime and the red region, past the dynamical phase transition curve, represents the expected extent of the aging regime.} \]

2. Main Results

We shall start by making the same assumptions on the initial conditions as in [7]. Namely, we assume that the initial condition \(x_0\) is independent of the disorder \(J\), and the limits

\[
\lim_{N \to \infty} \mathbb{E}[C_N(0, 0)] = C(0, 0),
\]

and

\[
\lim_{N \to \infty} \mathbb{E}[M_N(0)] = M(0),
\]

exists, and are finite. Further, we assume that the tail probabilities \( \mathbb{P}(|C_N(0, 0) - C(0, 0)| > x) \) and \( \mathbb{P}(|M_N(0) - M(0)| > x) \) decay exponentially fast in \( N \) (so the convergence \( C_N(0, 0) \to C(0, 0) \) and \( M_N(0) \to M(0) \) holds
almost surely), and that for each \( k < \infty \), the sequence \( N \mapsto \mathbb{E}[C_N(0,0)^k] \) and \( N \mapsto \mathbb{E}[M_N(0)^k] \) is uniformly bounded. Also, we will assume that each of the two replicas will have the same (random) initial conditions, hence \( Q_N(0,0) = C_N(0,0) \).

Finally, consider the product probability space \( \mathcal{E}_N = \mathbb{R}^N \times \mathbb{R}^{d(N,m)} \times C([0,T], \mathbb{R}^N) \times C([0,T], \mathbb{R}^N) \) (here \( T \) is a fixed time and \( d(N,m) \) is the dimension of the space of the interactions \( \mathbf{J} \)), equipped with the natural Euclidean norms for the finite dimensional parts, i.e. \( (\mathbf{x}_0, \mathbf{J}) \), and the sup-norm for the Brownian motions \( \mathbf{B}^k \), \( k = 1, 2 \). The space \( \mathcal{E}_N \) is endowed with the product probability measure \( \mathbb{P} = \mu_N \otimes \gamma_N \otimes P_N \otimes P_N \), where \( \mu_N \) denotes the distribution of \( \mathbf{x}_0 \), \( \gamma_N \) is the (Gaussian) distribution of the coupling constants \( \mathbf{J} \), and \( P_N \) is the distribution of the \( N \)-dimensional Brownian motion.

**Hypothesis 2.1.** For \( (\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) \in \mathcal{E}_N \) we introduce the norms

\[
\| (\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) \|^2 = \sum_{i=1}^{N} (x_i^0)^2 + \sum_{p=1}^{m} \sum_{1 \leq i_1 \leq \ldots \leq i_p \leq N} (N^{\frac{p+1}{2}} J_{i_1 \ldots i_p})^2 + \sum_{k=1}^{2} \sup_{0 \leq t \leq T} \sum_{i=1}^{N} (B_{k,i}^t)^2.
\]

We shall assume that \( \mu_N \) is such that the following concentration of measure property holds on \( \mathcal{E}_N \); there exists two finite positive constants \( C \) and \( \alpha \), independent on \( N \), such that, if \( f \) is a Lipschitz function on \( \mathcal{E}_N \), with Lipschitz constant \( K \), then for all \( \rho > 0 \),

\[
\mu_N \otimes \gamma_N \otimes P_N \otimes P_N \| V - \mathbb{E}[V] \| \geq \rho \leq C^{-1} \exp \left( -C \left( \frac{\rho}{K} \right)^{\alpha} \right).
\]

Now, suppose that \( f \) is a differentiable function on \( \mathbb{R}_+ \) with \( f' \) locally Lipschitz, such that

\[
(2.3) \quad \sup_{\rho \geq 0} |f'(\rho)| (1 + \rho)^{-r} < \infty
\]

for some \( r < \infty \), and for some \( A, \delta > 0 \),

\[
(2.4) \quad \inf_{\rho \geq 0} \{ f'(\rho) - A \rho^{m/2 + \delta - 1} \} > -\infty
\]

(typically, \( f(\rho) = \kappa (\rho - 1)^r \) for some \( r > m/2 \) and \( \kappa \gg 1 \)). Then the normalization factor \( Z_{\beta, h, A, f} = \int e^{-\beta H^N_f(s) - N \int (f(N^{-1}\|x\|^2 + h) \mathcal{K}^N_s(x) dx is a.s. finite (by (2.4)).

First, we shall show that, as \( N \to \infty \) the functions \( C_N(s,t), \chi_N(s,t), M_N(s), Q_N(s,t) \) and \( L_N(s,t) \) converge to non-random continuous functions \( C(s,t), \chi(s,t), M(s) \) and \( Q(s,t) = L(s,t) \) that are characterized as the solution of a system of coupled integro-differential equations. We denote by \( \Gamma \) the upper half of the first quadrant, namely:

\[
\Gamma := \{(s,t) \in \mathbb{R}^2 : 0 \leq t \leq s \}
\]

Also, we denote by \( \mathcal{C}^1 \) the class of continuously differentiable symmetric functions of two variables and by \( \mathcal{C}_s \) the class of continuous symmetric functions. These notations will be widely used and will appear through this work.

**Proposition 2.2.** Let \( \psi(r) = \nu'(r) + r \nu''(r) \) and

\[
(2.5) \quad \nu(r) := \sum_{p=1}^{m} \frac{a_p^2}{p!} r^p.
\]

Suppose \( \mu_N \) satisfies hypothesis \( 2.1 \) and \( f \) satisfies \( 2.3 \) and \( 2.4 \). Fixing any \( T < \infty \), as \( N \to \infty \) the random functions \( M_N, \chi_N, C_N, Q_N \) and \( L_N \) converge uniformly on \([0,T]^2\) (or \([0,T]\), whichever applies), almost surely and in \( L^p \) with respect to \( \mathbf{x}_0, \mathbf{J} \) and \( \mathbf{B}^k \), for \( k = 1, 2 \), to non-random functions \( M(s), \chi(s,t) = \int_0^t R(s,u) du, C(s,t), Q(s,t) = Q(s,t) \) and \( L(s,t) = Q(s,t) \). Further, \( R(s,t) = 0 \) for \( t > s \), \( R(s,s) = 1 \), and for \( s > t \) the absolutely continuous functions \( C, R, M, Q, \) and \( K(s) = C(s,s) \) are the unique solution in \( \mathcal{C}^1(\mathbb{R}_+) \times C^1(\Gamma) \times C^1_s(\mathbb{R}_+) \times C^1_s(\mathbb{R}_+) \times C^1(\Gamma) \) of the integro-differential equations:

\[
(2.6) \quad \partial M(s) = -f'(K(s)) M(s) + h + \beta^2 \int_0^s M(u) R(s,u) \nu''(C(s,u)) du, \quad s \geq 0
\]
\[ \partial_t R(s, t) = -f'(K(s))R(s, t) + \beta^2 \int_t^s R(u, t)R(s, u)\nu''(C(s, u))du, \quad s \geq t \geq 0 \]

\[ \partial_t C(s, t) = -f'(K(s))C(s, t) + \beta^2 \int_0^t C(u, t)R(s, u)\nu''(C(s, u))du + \beta^2 \int_0^t \nu'(C(s, u))R(t, u)du + hM(t) + 1_{s \leq t}R(s, t), \quad s, t \geq 0 \]

\[ \partial_t Q(s, t) = -f'(K(s))Q(s, t) + \beta^2 \int_0^t Q(u, t)R(s, u)\nu''(C(s, u))du + \beta^2 \int_0^t \nu'(Q(s, u))R(t, u)du + hM(t), \quad s, t \geq 0 \]

\[ \partial K(s) = -2f'(K(s))K(s) + 1 + 2\beta^2 \int_0^s \psi(C(s, u))R(s, u)du + 2hM(s), \quad s \geq 0 \]

where the initial conditions \( K(0) = C(0, 0) = Q(0, 0) = 0 \) and \( M(0) \) are determined by (2.1) and (2.2), respectively. Moreover, \( C(\cdot, \cdot) \) and \( Q(\cdot, \cdot) \) are non-negative definite kernels, \( K(s) \geq 0, |M(s)| \leq \sqrt{K(s)} \), for all \( s \geq 0 \) and

\[ \left| \int_{t_1}^{t_2} R(s, u)du \right|^2 \leq K(s)(t_2 - t_1), \quad 0 \leq t_1 \leq t_2 \leq s < \infty. \]

For every \( r, L > 0 \), define the function:

\[ f(x) := f_{L,r}(x) = L(x - r)^2 + \frac{1}{4k} \left( \frac{x}{r} \right)^{2k} + \frac{\alpha h x}{r}, \quad k > m/4, k \in \mathbb{Z}, L \geq 0, \]

that is easily seen to satisfy conditions (2.3) and (2.4). We will derive in Section 4 the equations for the hard spherical constraint, by taking the limit \( L \to \infty \). Notice that if there is no magnetic field (i.e. \( h = 0 \)), the equations for the correlation \( C(\cdot, \cdot) \) and the response \( R(\cdot, \cdot) \) will decouple from the magnetization, resulting with the system derived in [14].

**Theorem 2.3.** For every \( r > 0 \), let \( (M_{L,r}, R_{L,r}, C_{L,r}, Q_{L,r}, K_{L,r}) \) be the unique solution of the system (2.6)–(2.10) with potential \( f_{L,r}(\cdot) \) as in (2.12) and initial conditions \( K_{L,r}(0) = Q_{L,r}(0, 0) = r > 0, M_{L,r}(0) = \alpha \sqrt{r}, \alpha \in [0, 1) \) and \( R_{L,r}(t, t) = 1 \) for every \( t \geq 0 \). Then, for any \( T < \infty \), \( (M_{L,r}, R_{L,r}, C_{L,r}, Q_{L,r}, K_{L,r}) \) converges as \( L \to \infty \), uniformly in \( s, t \in [0, T] \), towards \( (M, R, C, Q, K) \) that is the unique solution in \( C^1(\mathbb{R}_+) \times C^1(\Gamma^+) \times C^1(\mathbb{R}_+) \times C^1(\mathbb{R}_+) \times C^1(\mathbb{R}_+) \) of:

\[ \partial M(s) = -\mu(s)M(s) + h_r + \beta^2 \int_0^s M(u)R(s, u)\nu''(C(s, u))du, \quad s \geq 0 \]

\[ \partial_t R(s, t) = -\mu(s)R(s, t) + \beta^2 \int_t^s R(u, t)R(s, u)\nu''(C(s, u))du, \quad s \geq t \geq 0 \]

\[ \partial_t C(s, t) = -\mu(s)C(s, t) + \beta^2 \int_0^t C(u, t)R(s, u)\nu''(C(s, u))du + \beta^2 \int_0^t \nu'(C(s, u))R(t, u)du + h_rM(t), \quad s \geq t \geq 0 \]

\[ \partial_t Q(s, t) = -\mu(s)Q(s, t) + \beta^2 \int_0^s Q(u, t)R(s, u)\nu''(C(s, u))du + \beta^2 \int_0^t \nu'(Q(s, u))R(t, u)du + h_rM(t), \quad s, t \geq 0 \]
Theorem 2.5. Between the critical inverse temperature and the intensity of the field is novel and represents an important
\begin{equation}
\tag{2.18}
\int_{t_1}^{t_2} R(s,u)du \leq r(t_2-t_1), \quad 0 \leq t_1 \leq t_2 \leq s < \infty.
\end{equation}

The predicted structure of the solution is more complicated in the mixed spin case than in the pure spin one. However, we show in Section 5 that as r increases, only the highest level interactions will matter, effectively

\begin{equation}
\tag{2.17}
\mu(s) = \frac{1}{2r} \left( k + 2\beta^2 \int_0^s \psi(C(s,u))R(s,u)du + 2hrM(s) \right)
\end{equation}

satisfying $M(0) = \alpha \sqrt{r}$, $C(t,t) = K(t) = r$, $R(t,t) = 1$, for all $t \geq 0$. Moreover, $C(\cdot, \cdot)$ and $Q(\cdot, \cdot)$ are

non-negative definite kernels, with values in $[0, r]$, $M(s) \in [0, \sqrt{r}]$, for all $s \geq 0$, $R(s,t) \geq 0$ and

$$\int_{t_1}^{t_2} R(s,u)du \leq r(t_2-t_1), \quad 0 \leq t_1 \leq t_2 \leq s < \infty.$$
Equation (2.20) has been analyzed in detail in Proposition 1.4 of [14]. The authors have shown that, for any choice of $\phi(\cdot)$ such that

$$\sup_{x \in [0,1]} \{\phi(x)(1-x)\} \geq \frac{1}{2}$$

the equation has an unique solution in $[0,1]$, that is decreasing, twice differentiable and converges as $s \to \infty$ to $C^\infty = \sup\{x \in [0,1] | \phi(x)(1-x) \geq 1/2\}$. Furthermore, they show that the condition:

$$\phi(C^\infty) > \phi'(C^\infty)(1 - C^\infty),$$

is necessary for the exponential convergence of $C'(s)$ to zero as $s \to \infty$ when $\phi(\cdot)$ is convex.

First, it is easy to check that for our $\phi(x)$ of Theorem 2.5, $\phi(Q^{\text{fdt}})(1 - Q^{\text{fdt}}) = 1/2$, hence (2.21) is satisfied and furthermore, $C^\infty \geq Q^{\text{fdt}}$. Setting $\beta_c(h) \in (0, \infty)$ via

$$\frac{1}{4\beta_c(h)^2} = \sup \left\{\frac{\nu'(x) - \nu'(Q^{\text{fdt}})(1-x)(1-Q^{\text{fdt}})}{x - Q^{\text{fdt}}}: x \in (Q^{\text{fdt}}, 1)\right\},$$

it is easy to check that $C^\infty = Q^{\text{fdt}}$ if $\beta < \beta_c(h)$ whereas $C^\infty > Q^{\text{fdt}}$ for $\beta > \beta_c(h)$. Further, considering $x \to 0$ in (2.23) we find that

$$\frac{1}{4\beta_c(h)^2} \geq \nu''(Q^{\text{fdt}})(1 - Q^{\text{fdt}})^2$$

so, in particular, the condition (2.22) then holds for any $\beta < \beta_c(h)$ (since in this case, as mentioned $C^\infty = Q^{\text{fdt}}$). Furthermore, since $Q^{\text{fdt}}$ is a solution of (2.19), from (2.24) we get $\beta_c(h)^2\nu''(Q^{\text{fdt}}) + h^2 \geq Q^{\text{fdt}}\nu''(Q^{\text{fdt}})$, so:

$$\gamma_c(h)^2 := \left(\frac{\beta_c(h)}{h}\right)^2 \leq \frac{1}{Q^{\text{fdt}}\nu''(Q^{\text{fdt}}) - \nu'(Q^{\text{fdt}})} \to \frac{1}{\nu''(1) - \nu'(1)}$$

This indicates that though the values of $\beta_0(h) \leq \gamma_0 h$ for which we have formally established the FDT regime in Theorem 2.5 are quite small, they should match the predicted dynamical phase transition point $\beta_c(h)$ of our model. Furthermore, $0 < \liminf_{h \to \infty} \gamma_c(h) \leq \limsup_{h \to \infty} \gamma_c(h) < \infty$, indicating that $\beta_c(h)$ will be asymptotically linear in $h$. Figure 1 in the introduction summarizes all the information above.

### 3. Limiting Soft Spherical Dynamics

This section is dedicated to proving Proposition 2.2. The line of proof follows closely [7], and references will be given, when appropriate. First, recall that:

$$G^i(x) := -\partial_{x^i} \left(H^N_{\beta}(x)\right) = \sum_{p=1}^{m} \frac{a_p}{(p-1)!} \sum_{1 \leq i_1, \ldots, i_{p-1} \leq N} J_{i_1 \ldots i_{p-1} i} x^{i_1} \ldots x^{i_{p-1}}.$$
We will start by introducing some notation. For \(q_1, q_2 \in \{1, 2\}\), define
\[
C_{N}^{q_1, q_2}(s, t) := \frac{1}{N} \sum_{i=1}^{N} x_{s}^{q_1,i} x_{t}^{q_2,i}, \\
K_{N}^{q_1, q_2}(s) := C_{N}^{q_1, q_2}(s, s), \\
\chi_{N}^{q_1, q_2}(s, t) := \frac{1}{N} \sum_{i=1}^{N} x_{s}^{q_1,i} B_{t}^{q_2,i}, \\
M_{N}^{q_1}(s) := \frac{1}{N} \sum_{i=1}^{N} x_{s}^{q_1,i}, \\
A_{N}^{q_1, q_2}(s, t) := \frac{1}{N} \sum_{i=1}^{N} G_{s,t}^{q_1} x_{t}^{q_2,i}, \\
F_{N}^{q_1, q_2}(s, t) := \frac{1}{N} \sum_{i=1}^{N} G_{s,t}^{i} (x_{s}^{q_1}) B_{t}^{q_2,i}, \\
R_{N}^{q_1}(s) := \frac{1}{N} \sum_{i=1}^{N} G_{s}^{i} (x_{s}^{q_1}), \\
W_{N}^{q_1}(s) := \frac{1}{N} \sum_{i=1}^{N} B_{s}^{q_2,i},
\]
(3.2)
and
\[
D_{N}^{q_1, q_2}(s, t) := -f'(\mathbb{E}(K_{N}^{q_1, q_2}(t))) C_{N}^{q_1, q_2}(s, t) + A_{N}^{q_1, q_2}(t, s), \\
E_{N}^{q_1, q_2}(s, t) := -f'(\mathbb{E}(K_{N}^{q_1, q_2}(s))) \chi_{N}^{q_1, q_2}(s, t) + F_{N}^{q_1, q_2}(s, t), \\
P_{N}^{q_1}(s) := -f'(\mathbb{E}(K_{N}^{q_1, q_2}(t))) M_{N}^{q_1}(s) + R_{N}^{q_1}(s),
\]
(3.3)
where for \(q = 1, 2\), \(\{B_{t}^{q}\}_{s \geq 0} = \{\{B_{t_1}^{1}, \ldots, B_{t_N}^{q}\}\}_{s \geq 0}\), are two iid \(N\)-dimensional Brownian motions and \(\{x_{s}^{q}\}_{s \geq 0} = \{\{x_{s}^{q_1, 1}, \ldots, x_{s}^{q_2, N}\}\}_{s \geq 0}\) are the two replicas sharing the same frustrations \(J\), with the noise given by the realization of the Brownian motions above. Also, when it is clear from the context that there is only one replica, for simplicity of the notation, the superscripts indicating the replica index will be omitted.

Also, in order to simplify the (already heavy) notations, we will embed the constant \(\beta\) into \(\{a_{p}\}\) resulting with \(\beta G_{1}^{j}(\cdot) \mapsto G_{1}^{j}(\cdot)\) and then having \(\beta = 1\) in the stochastic differential system \((1.6)\). Adopting this convention, we will have from now on \(\beta = 1\).

3.1. **Strong Solutions and Self-Averaging.** First, by similar arguments to the ones employed in Proposition 2.1 of [7], we will show that if \(f'\) is locally Lipschitz, satisfying \((2.4)\), then there exist an unique strong solution to \((1.6)\). Namely, we show:

**Proposition 3.1.** Assume that \(f'\) is locally Lipschitz, satisfying \((2.4)\). Then, for any \(N \in \mathbb{Z}_{+}\), almost any \(J\), initial condition \(x_{0}\) and Brownian path \(B\), there exists a unique strong solution to \((1.6)\). This solution is also unique in law for almost any \(J\), and \(x_{0}\), it is a probability measure on \(\mathcal{C}(\mathbb{R}_{+}, \mathbb{R}^{N})\) which we denote \(\mathbb{P}_{x_{0}, J}\).

Further, with
\[
|J|_{\infty}^{N} = \max_{1 \leq p \leq m} \sup_{1 \leq i \leq N} |u_{i}^{p}| \leq 1, 1 \leq i \leq p
\]
(3.4)
we have for \(\delta > 0\) of \((2.4)\), \(q := m/(2\delta) + 1, \) some \(\kappa < \infty\), all \(N, z > 0\), \(J\), and \(x_{0}\), that
\[
\mathbb{P}_{x_{0}, J}^{N}\left( \sup_{t \in \mathbb{R}^{+}} K_{N}(t) \geq K_{N}(0) + \kappa(1 + |J|_{\infty}^{N})^{2} + Z \right) \leq e^{-zN}.
\]
(3.5)
Consequently, for any \(L > 0\), there exists \(z = z(L) < \infty\) such that
\[
\mathbb{P}\left( \sup_{t \in \mathbb{R}^{+}} K_{N}(t) \geq z \right) \leq e^{-LN}.
\]
(3.6)

**Proof of Proposition 3.1.** The proof follows the same lines as the proof of Proposition 2.1 of [7]. Namely, considering the truncated drift \(b_{M}(u) = \{b_{M}^{1}(u), \ldots, b_{M}^{N}(u)\}\) given by \(b_{M}^{1}(u) = G_{1}^{i}(\phi_{M}(u)) - f'(N^{-1}|u|^{2}) \wedge M)u_{i} + h\), where \(\phi_{M}(x) = x\) when \(|x| \leq \sqrt{NM}\), we see that \(\phi_{M}\) is globally Lipschitz, hence there exist an unique square-integrable strong solution \(u^{(M)}\) for the SDS
\[
du_{i} = b_{M}^{i}(u_{i})dt + dB_{i}^{i}
\]
Fixing $M$ and denoting $x_t = u_t^{(M)}$ and $Z_s = 2N^{-1}\sum_{i=1}^{N} f_{0}^{s\wedge \tau_{M}} x_i dB_t^i$, by applying Itô’s formula for $C_N(t) := N^{-1}||x_t||^2$ we see that

$$C_N(s) \leq C_N(0) + 2\sum_{p=1}^{m} \frac{a_p ||J||_{\infty}^N}{(p-1)!} \int_{0}^{s\wedge \tau_{M}} C_N(t)^{\frac{p}{2}} dt + Z_s + s \wedge \tau_{M}$$

$$-2\int_{0}^{s\wedge \tau_{M}} f'(C_N(t))C_N(t)dt + 2h\int_{0}^{s\wedge \tau_{M}} C_N(t)^\frac{1}{2} dt.$$  

Since $x^1 = \frac{p}{2} f'(x) \to \infty$, it follows from (3.7) that there is an almost surely finite constant $c(||J||_{\infty}, h)$, independent of $M$, such that

$$C_N(s) \leq C_N(0) + c(||J||_{\infty}^N, h)s + Z_s$$

As the quadratic variation of the martingale $Z_s$ is $(4/N) \int_{0}^{s\wedge \tau_{M}} C_N(t)dt \leq 4sN^{-1}M$, applying Doob’s inequality (c.f. [20] Theorem 3.8, p. 13) for the exponential martingale $L_s^\lambda = \exp(\lambda Z_s - 2(\lambda^2/N) \int_{0}^{s\wedge \tau_{M}} C_N(t)dt)$ (with respect to the filtration $\mathcal{H}_t$ of $\mathcal{B}_t$), yields that

$$\mathbb{P}\left(\sup_{s \leq T} L_s^N \geq e^{zN}\right) \leq \mathbb{P}\left(\sup_{s \leq T} L_s^N \geq e^{zN}\right) \leq e^{-zN},$$

for any $z > 0$. Therefore, (3.8) shows that with probability greater than $1 - e^{-zN}$,

$$C_N(s \wedge \tau_{M}) \leq C_N(0) + c(||J||_{\infty}^N, h)T + z + 2\int_{0}^{s\wedge \tau_{M}} C_N(t)dt,$$

for all $s \leq T$, and by Gronwall’s lemma then also

$$\sup_{t \leq T} N^{-1}||u_t^{(M)}||^2 \leq [C_N(0) + c(||J||_{\infty}^N, h)T + z]e^{2T}.$$  

Setting $z = M/3$, for large enough $M$ (depending of $N$, $h$, $x_0$ and $T$ which are fixed here), the right-side of (3.10) is at most $M/2$, resulting with

$$\mathbb{P}(\tau_M \leq T) \leq e^{-MN/3},$$

where $\tau_M = \inf\{t : ||u_t^{(M)}|| \geq \sqrt{M}\}$, and hence that

$$\sup_{M=1}^{\infty} \mathbb{P}(\tau_M \leq T) < \infty,$$

so establishing the existence of the solution after an application of the Borel-Cantelli lemma.

We also have weak uniqueness of our solutions for almost all $J$ since the restriction of any weak solution to the stopped $\sigma$-field $\mathcal{H}_{\tau_M}$ for the filtration $\mathcal{H}_t$ of $\mathcal{B}_t$ is unique. We denote this unique weak solution of (1.6) by $[\mathbb{P}^N_{x_0}, J]$.

Turning to the proof of (3.5), by (2.4) for any $c > 0$ there exists $\kappa < \infty$ such that for all $r, x \geq 0$,

$$2 \left[ f'(x)x - r \sum_{p=1}^{m} \frac{a_p x^{\frac{p}{2}}}{(p-1)!} - h x^{\frac{1}{2}} \right] - 1 \geq c e - \kappa(1 + r + h)^q.$$  

Taking $r = ||J||_{\infty}^N$, we see that by (3.7), for all $N$ and $s \geq 0$,

$$C_N(s \wedge \tau_{M}) \leq C_N(0) - \int_{0}^{s\wedge \tau_{M}} [cC_N(t) - \kappa(1 + ||J||_{\infty}^N + h)^q] dt + Z_s,$$

where $(Z_s)_{s \geq 0}$ is a martingale with bracket $(4N^{-1} \int_{0}^{s\wedge \tau_{M}} C_N(t)dt, s \geq 0)$.
By Doob’s inequality (3.9), with probability at least $1 - e^{-zN}$,

$$\sup_{u \leq s \wedge \tau_M} Z_u \leq 2 \int_0^{s \wedge \tau_M} C_N(t)dt + z,$$

for all $s \geq 0$. Setting $c = 3$ we then have that

$$C_N(s \wedge \tau_M) \leq C_N(0) + z - \int_0^{s \wedge \tau_M} C_N(t)dt + \kappa(1 + \|J\|^N_{\infty} + h)^q(s \wedge \tau_M),$$

so that by Gronwall’s lemma,

$$C_N(s \wedge \tau_M) \leq e^{s \wedge \tau_M} (C_N(0) + z) + \kappa(1 + \|J\|^N_{\infty} + h)^q \int_0^{s \wedge \tau_M} e^{-t}dt,$$

from which the conclusion (3.5) is obtained by considering $M \to \infty$.

In view of the assumed exponential decay of the tail probabilities for $K_N(s)$, and the bound (3.7) of [7] on the corresponding probabilities for $K_N$ we thus get also the bounds of (3.6). 

The next is to extend the arguments in Propositions 2.2 - 2.8 of [7], in order to show that any of the functions $A^{q_1,q_2}, F^{q_1,q_2}, X^{q_1,q_2}, C^{q_1,q_2}, W^q, K_N, M^q_N$, and $L_N$ self-averages for $N$ large. More precisely, we show that:

**Proposition 3.2.** Suppose that $\Psi : \mathbb{R}^\ell \to \mathbb{R}$ is locally Lipschitz with $|\Psi(z)| \leq M\|z\|_k^r$ for some $M, \ell, k < \infty$, and $Z_N \in \mathbb{R}^\ell$ is a random vector, where for $j = 1, \ldots, \ell$, the $j$-th coordinate of $Z_N$ is one of the functions $A^{q_1,q_2}, F^{q_1,q_2}, X^{q_1,q_2}, C^{q_1,q_2}, W^q, M^q_N$ or $L_N$, evaluated at some $(s_j, t_j) \in [0,T]^2$ (or at $s_j \in [0,T]$, whichever applies). Then,

$$\lim_{N \to \infty} \sup_{s_j, t_j} |\mathbb{E}[\Psi(Z_N)] - \Psi(\mathbb{E}[Z_N])| = 0.$$

**Proof of Proposition 3.2** The proof is structured as follows: first we show that $\mathbb{E}[\sup_{s,t \leq T} |U_N(s, t)|^k]$ and $\mathbb{E}[\sup_{s \leq T} |V_N(s)|^k]$ are bounded uniformly in $N$ and also that for any fixed $T < \infty$, the sequences $U_N(s, t)$ and $V_N(s)$ are pre-compact almost surely and in expectation with respect to the uniform topology on $[0,T]^2$, respectively $[0,T]$. Here $U$ is any of the functions $C^{q_1,q_2}, F^{q_1,q_2}, X^{q_1,q_2}, A^{q_1,q_2}$ or $L$ and $V$ is one of the functions $M^q$ or $W^q$. The next step is to establish, similarly to Proposition 2.4 of [7], that all the functions $U$ and $V$ above self-averages, namely:

$$\sum_N \mathbb{P} \left( \sup_{s,t \leq T} |U_N(s, t) - \mathbb{E}[U_N(s, t)]| \geq \rho \right) < \infty$$

$$\sum_N \mathbb{P} \left( \sup_{s \leq T} |V_N(s) - \mathbb{E}[V_N(s)]| \geq \rho \right) < \infty$$

implying by the uniform moment bounds on $\|U_N\|_{\infty}$ and $\|V_N\|_{\infty}$ that we have just established, that:

$$\lim_{N \to \infty} \sup_{s,t \leq T} \mathbb{E} \left[ |U_N(s, t) - \mathbb{E}[U_N(s, t)]|^2 \right] = 0$$

$$\lim_{N \to \infty} \sup_{s \leq T} \mathbb{E} \left[ |V_N(s) - \mathbb{E}[V_N(s)]|^2 \right] = 0$$

The final step is to establish the claim of the proposition, by using (3.14) and the uniform bounds on the moments that we have just established.

By our hypothesis, the mapping $N \mapsto \mathbb{E}[K_N(0)^k]$ is bounded. Since both replicas have the same starting point $K_N^q(0) = K_N(0)$, for $q \in \{1,2\}$. Also, by the estimate (B.6) of Appendix B, of [7],

$$\sup_N \mathbb{E} \left[ \|J\|_{\infty}^k \right] < \infty.$$
for any $k < \infty$, for the norm $||J||_{\infty}^{N}$ of (3.4), the bound (3.5) immediately implies that for each $k < \infty$, and any $q \in \{1, 2\}$ also

\[(3.16) \quad \sup_{N} E \left[ \sup_{t \in \mathbb{R}^+} |K_{N}^{q,k}(t)|^{k} \right] < \infty.\]

Define $||V_{N}||_{\infty} := \sup\{|V_{N}(t) : 0 \leq t \leq T\}$ and $||U_{N}||_{\infty} := \sup\{|U_{N}(s,t) : 0 \leq s, t \leq T\}$. Also let $B_{N}^{q}(t) := \frac{1}{N} \sum_{i=1}^{N} (B_{i}^{q,i})^{2}$, $G_{N}(t) := \frac{1}{N} \sum_{i=1}^{N} (G_{i}^{q,i}(x_{i}^{q}))^{2}$ and $L_{N}(t) := \frac{1}{N} \sum_{i=1}^{N} (E_{B}[x_{i}^{q}])^{2}$. A key result is the bound:

\[(3.17) \quad \sup_{N} E \left[ ||J||_{\infty}^{k} \right] + \sup_{N} E \left[ ||L_{N}||_{k}^{k} \right] + \sup_{N} E \left[ ||K_{N}||_{k}^{k} \right] + \sup_{N} E \left[ ||G_{N}||_{k}^{k} \right] < \infty,
\]

for every fixed $k$, where we have dropped the replica index (since we are taking the expected value anyway). Indeed, the bounds on $||J||_{\infty}^{N}$ and $||K_{N}||_{\infty}^{k}$ are already obtained in (3.15) and (3.16), and by Lemma 2.2 of [7] we have that

\[(3.18) \quad (G_{N}^{q}(t))^{\frac{1}{2}} \leq c ||J||_{k}^{N}[1 + K_{N}^{q,k}(t) \frac{2^{q-1}}{N}],\]

yielding by (3.15) and (3.16) the uniform moment bound on $||G_{N}||_{\infty}$. Also, by Jensen’s inequality, $E[||L_{N}||_{k}^{k}] \leq E[||K_{N}||_{k}^{k}]$ and finally, the exponential tails of $B_{N}^{q}$ (c.f. [7] (2.16)), will provide an uniform bound for each moment of $||B_{N}^{q}||_{k}$, thus concluding the derivation of (3.17).

Similarly, by (3.6), (3.18), the exponential tails of $B_{N}^{q}$ mentioned above and the exponential tails of $||J||_{\infty}^{N}$ (c.f [7] (B.7)), we have for each $L > 0$ the bound:

\[(3.19) \quad \mathbb{P} \left( ||J||_{\infty}^{N} + ||L_{N}||_{k}^{k} + \sum_{q=1}^{2} ||K_{N}^{q,k}||_{k}^{k} + ||G_{N}^{q}||_{k}^{k} \geq M \right) \leq e^{-LN}.\]

will hold for some $M = M(L) < \infty$ and for all $N$. Applying Cauchy-Schwartz inequality to $U_{N}$ and $V_{N}$ and using the estimates (3.17) and (3.19), we see that $E[\sup_{s,t \leq T} |U_{N}(s,t)|^{k}]$ and $E[\sup_{s \leq T} |V_{N}(s)|^{k}]$ are bounded uniformly in $N$. The argument is similar to the one employed in Proposition 2.3 of the cited paper.

With the previous controls on $||U_{N}||_{\infty}$ and $||V_{N}||_{\infty}$ already established, by the Arzela-Ascoli theorem, the pre-compactness of $U_{N}$, respectively $V_{N}$, follows by showing that they are equi-continuous sequences. We notice that such $U_{N}(s,t)$ and $V_{N}(s)$ are all of the form $\frac{1}{N} \sum_{i=1}^{N} a_{s,i}^{q} b_{i}^{q}$ hence,

\[(3.20) \quad |U_{N}(s,t) - U_{N}(s',t')| \leq \frac{1}{N} \sum_{i=1}^{N} |a_{s,i}^{q} - a_{s',i}^{q}| |b_{i}^{q}| + \frac{1}{N} \sum_{i=1}^{N} |a_{s,i}^{q}| |b_{i}^{q} - b_{i'}^{q}| \leq \left[ \frac{1}{N} \sum_{i=1}^{N} |a_{s,i}^{q} - a_{s',i}^{q}|^{2} \right]^{1/2} \left[ \frac{1}{N} \sum_{i=1}^{N} |b_{i}^{q}|^{2} \right]^{1/2} + \left[ \frac{1}{N} \sum_{i=1}^{N} |b_{i}^{q} - b_{i'}^{q}|^{2} \right]^{1/2} \left[ \frac{1}{N} \sum_{i=1}^{N} |a_{s,i}^{q}|^{2} \right]^{1/2},\]

and the same is true also for $|V_{N}(s) - V_{N}(s')|$, where the functions $a_{s}$ and $b_{s}$ are either $x_{s}^{q}$, $B_{s}^{q}$, $G(x_{s}^{q})$, for some $q \in \{1, 2\}$, $E_{B}[x_{s}]$ or 1. So, in view of (3.17) and (3.19), it suffices to show that for any $\epsilon > 0$, some function $L(\delta, \epsilon)$ going to infinity as $\delta$ goes to zero and all $N$,

\[(3.21) \quad \mathbb{P} \left( \sup_{|t-t'|<\delta} \left[ \frac{1}{N} \sum_{i=1}^{N} |b_{i}^{q} - b_{i'}^{q}|^{2} \right] > \epsilon \right) \leq e^{-L(\delta, \epsilon)N},\]

for $b = x^{q}$, $B^{q}$, $G(x^{q})$ and $E_{B}[x]$. Obviously, this holds for $b = B^{q}$. Also, since by (1.6)

\[|x_{s,i}^{q} - x_{s',i'}^{q}| \leq |B_{i}^{q} - B_{i'}^{q}| + ||f'(K_{N}^{q,q})||_{\infty} \int_{t}^{t'} |x_{u,i}^{q} - x_{u,i'}^{q}| du + \int_{t}^{t'} |G'(x_{s}^{q})| du + h(t' - t).\]
we get, by (2.3), for some universal constant \( \rho_1 < \infty \), all \( t, t' \) and \( N \),

\[
\frac{1}{N} \sum_{i=1}^{N} |x_t^{q,i} - x_t'^{q,i}|^2 \leq \frac{4}{N} \sum_{i=1}^{N} |B_t^{q,i} - B_{t'}^{q,i}|^2 + 4|t - t'|^2 \left( \rho_1 (1 + \|K_N^q\|_\infty)^{2r} \|K_N^q\|_\infty + \|G_N^q\|_\infty + h^2 \right)
\]

hence by the bounds established on \( \|G_N^q\|_\infty \) and \( \|K_N^q\|_\infty \), we establish (3.21) for \( \mathbf{b} = \mathbf{x}^q \). An application of Jensen’s inequality will imply the same result for \( \mathbf{b} = \mathbb{E}_B[\mathbf{x}] \). Using the results in Lemma 2.2 of [7], we can now establish (3.21) for \( \mathbf{b} = \mathbb{G}(\mathbf{x}^q) \), thus concluding the equi-continuity of \( U_N \) and \( V_N \), hence the first step of the proof. Note that we have actually shown a stronger result that we will use later, namely that for all \( \epsilon > 0 \) there exists \( \bar{L}(\delta, \epsilon) \to \infty \) for \( \delta \to 0 \), such that for all \( N \),

\[
P \left( \sup_{|s-s'|+|t-t'|<\delta} \left| U_N(s, t) - U_N(s', t') \right| > \epsilon \right) \leq e^{-\bar{L}(\delta, \epsilon)N}
\]

(3.22)

and also

\[
P \left( \sup_{|s-s'|<\delta} \left| V_N(s) - V_N(s') \right| > \epsilon \right) \leq e^{-\bar{L}(\delta, \epsilon)N}
\]

(3.23)

The next step, as mentioned earlier is to establish (3.13) and (3.14). We will use the same approach as in the proof of Proposition 2.4 of [7], by applying the estimate in Lemma 2.5 to \( U_N(s, t) \) and \( V_N(s) \), respectively, for any fixed pair of times \( s, t \). For every \( M < \infty \) and any \( N \), define the subset:

\[
\mathcal{L}_{N,M} = \left\{ (\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) \in \mathcal{E}_N : \|\mathbf{J}\|_\infty + \|L_N\|_\infty + \sum_{q=1}^{2} \|\mathbf{B}_N^q\|_\infty + \|K_N^{q,q}\|_\infty + \|G_N^q\|_\infty \leq M \right\}
\]

of \( \mathcal{E}_N \). For \( M \) sufficiently large, the probability of the complement set \( \mathcal{L}_{N,M}^c \) decays exponentially in \( N \) by (3.19). Since the uniform moment bounds for the functions \( U_N(s, t) \) and \( U_N(s) \) has been established, as well as the stated pointwise bound in \( \mathcal{L}_{N,M} \), the only other ingredient that we need to be able to apply the bound in Lemma 2.5 in the cited paper is the Lipschitz constant of \( U_N \) and \( V_N \) on \( \mathcal{L}_{N,M} \).

To this end, let \( \mathbf{x}^q, \widetilde{\mathbf{x}}^q \) be the two strong solutions of (1.6) constructed from \( (\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) \) and \( (\widetilde{\mathbf{x}}, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}}^1, \widetilde{\mathbf{B}}) \), respectively. If \( (\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) \) and \( (\widetilde{\mathbf{x}}, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}}^1, \widetilde{\mathbf{B}}) \) are both in \( \mathcal{L}_{N,M} \), then

\[
\sup_{t \leq T} \frac{1}{N} \sum_{1 \leq i \leq N} |x_t^{q,i} - \widetilde{x}_t^{q,i}|^2 \leq \frac{D_o(M, T)}{N} ||(\mathbf{x}_0, \mathbf{J}, \mathbf{B}^q) - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}}^q)||^2
\]

(3.24)

\[
\leq \frac{D_o(M, T)}{N} ||(\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}}^1, \widetilde{\mathbf{B}})||^2,
\]

for some \( D_o(M, T) \) independent of \( N \), where the first inequality is due to Lemma 2.6 of [7]. Now, equipped with (3.24), we can easily show the desired Lipschitz estimate for all of the functions of interest \( U_N(s, t) \) and \( V_N(s) \), namely:

\[
\sup_{s, t \leq T} |U_N(s, t) - \widetilde{U}_N(s, t)| \leq \frac{D(M, T)}{\sqrt{N}} ||(\mathbf{x}_0, \mathbf{J}, \mathbf{B}^1, \mathbf{B}^2) - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}}^1, \widetilde{\mathbf{B}})||,
\]

(3.25)
and
\[
\sup_{t \leq T} |V_N(s) - \bar{V}_N(s)| \leq \frac{D(M, T)}{\sqrt{N}} \| (x_0, J, B^1, B^2) - (\bar{x}_0, \bar{J}, \bar{B}^1, \bar{B}^2) \| ,
\]
where the constant \( D(M, T) \) depends only on \( M \) and \( T \) and not on \( N \). Indeed, since every \( U_N(s, t) \) and every \( V_N(s) \) is of the form \( \frac{1}{N} \sum_{i=1}^{N} a_i^T b_i^T \), then \( (3.20) \) will hold, with the functions \( a_i \) and \( b_i \) being one of \( x_i^T, B_i^T, G(x_i^T), E_B[x] \) or 1. By the same proof as the one employed in Lemma 2.7 of [7], we see that:
\[
\left[ \frac{1}{N} \sum_{i=1}^{N} |G(x_i^T) - \tilde{G}(x_i^T)|^2 \right]^{1/2} \leq \frac{C(M, T)}{\sqrt{N}} \| (x_0, J, B^1, B^2) - (\bar{x}_0, \bar{J}, \bar{B}^1, \bar{B}^2) \| .
\]

Also, Jensen's inequality applied to \( (3.24) \) shows:
\[
\sup_{t \leq T} \frac{1}{N} \sum_{i=1}^{N} \left| E_B[x_i^T] - E_B[\tilde{x}_i^T] \right|^2 \leq \frac{D(M, T)}{N} \| (x_0, J, B^1, B^2) - (\bar{x}_0, \bar{J}, \bar{B}^1, \bar{B}^2) \|^2 ,
\]

The last three bounds, together with the \( (3.21) \) plugged into equation \( (3.20) \) and is's analogue for \( V \), will show the Lipschitz bounds \( (3.25) \) and \( (3.26) \), whenever \( (x_0, J, B^1, B^2) \) and \( (\bar{x}_0, \bar{J}, \bar{B}^1, \bar{B}^2) \) are both in \( \mathcal{L}_{N,M} \).

As noticed before, we have all the ingredients for applying Lemma 2.5 of [7] to \( V_N := U_N(s, t) \) and \( V_N := V_N(s) \), for any fixed \( s, t \leq T \), yielding:
\[
P( |V_N - E[V_N]| \geq \rho \) \leq C^{-1} \exp \left( -\frac{\rho}{2D(M(L))} \right)^\alpha \left( N^{\frac{1}{L}} \right) + 4(K + M(L))\rho^{-1} e^{-LN/2} + e^{-NL} .
\]

for constants \( K \) and \( D = D(M(L), T) \) independent of \( s, t, \rho \) and \( N \). Consequently, by the union bound, for any finite subset \( \mathcal{A} \) of \([0, T]^2\) and \( \mathcal{B} \) of \([0, T]\) and any \( \rho > 0 \), the sequences \( \mathcal{N} := \mathbb{P}[\sup_{s, t \in \mathcal{A}} |U_N(s, t) - E[U_N(s, t)]| \geq \rho/3] \) and \( \mathcal{N} := \mathbb{P}[\sup_{s, t \in \mathcal{B}} |V_N(s) - E[V_N(s)]| \geq \rho/3] \) are summable. Recalling \( (3.22) \) and \( (3.23) \), we choose \( \delta > 0 \) small enough so that \( \bar{L}(2\delta, \rho/3) > 3/\rho > 0 \), we thus get \( (3.13) \) by considering the finite subsets \( \mathcal{A} = \{ (i\delta, j\delta) : i, j = 0, 1, \ldots, T/\delta \} \) and respectively \( \mathcal{B} = \{ i\delta : i = 0, 1, \ldots, T/\delta \} \).

Now, we have all the ingredients needed for finalizing the proof. For each \( r \geq R \) let \( c_r \) denote the finite Lipschitz constant of \( \Psi(\cdot) \) (with respect to \( |\cdot|_2 \)), on the compact set \( \Gamma_r := \{ z : |z|_k \leq r \} \). Then,
\[
|E[\Psi(Z_N)] - \Psi(E[Z_N])| \leq E[|\Psi(Z_N) - \Psi(E[Z_N])| 1_{Z_N \in \Gamma_r}] + E[|\Psi(Z_N)| 1_{Z_N \in \Gamma_r, \Psi(Z_N) \neq \Psi(E[Z_N])}] P[Z_N \notin \Gamma_r] \leq c_r E[|Z_N - E[Z_N]|_2^2] + 2L \rho^{-k} \rho^{-1} E[|Z_N|_2^2] .
\]

We have by \( (3.14) \) and the uniform moment bounds of \( U_N(s, t) \) and \( V_N(s) \) that \( \sup_{s, t, N} E[|Z_N - E[Z_N]|_2^2] \to 0 \) as \( N \to \infty \), while \( c' = \sup_{s, t, N} E[|Z_N - E[Z_N]|_2^2] < \infty \), implying that:
\[
\lim_{N \to \infty} \sup_{s, t} E[|\Psi(Z_N) - \Psi(E[Z_N])|] \leq 2c' \ell m r^{-k} ,
\]

which we make arbitrarily small by taking \( r \to \infty \).

Notice that, since \( L_N(s, t) \) and \( Q_N(s, t) \) have the same first moment, for every \( s \) and \( t \), the above proposition implies that any limit point of \( L_N(s, t) \) is also a limit point of \( Q_N(s, t) \).
3.2. Getting the Limiting Equations. The key step of the proof of Proposition 2.2 is summarized by Proposition 3.3. Fixing any $T < \infty$, any limit point of the sequences $E[M_N]$, $E[\chi_N]$, $E[C_N]$ and $E[Q_N] = \mathbb{E}[C_N^{1,2}]$ with respect to uniform convergence on $[0, T]^2$, satisfies the integral equations
\begin{align*}
(3.28)\quad M(s) &= M(0) + hs + \int_0^s P(u)du, \\
(3.29)\quad \chi(s, t) &= s \wedge t + \int_0^s E(u, t)du, \\
(3.30)\quad C(s, t) &= C(s, 0) + \chi(s, t) + \int_0^t D(s, u)du + htM(s), \\
(3.31)\quad Q(s, t) &= Q(s, 0) + \int_0^t H(s, u)du + htM(s), \\
(3.32)\quad P(t) &= -f'(C(t, t))M(t) + \nu'(C(t, t))M(t) - \nu'(C(0, t))M(0) \\
&\quad - \int_0^t \nu'(C(t, u))P(u)du - \int_0^t M(u)\nu''(C(t, u))D(u, t)du \\
&\quad - h \left[ M(t) \int_0^t M(u)\nu''(C(t, u))du + \int_0^t \nu'(C(t, u))du \right], \\
(3.33)\quad E(s, t) &= -f'(C(s, s))\chi(s, t) + \chi(s, t)\nu'(C(s, s)) - hQ(s) \int_0^s \nu''(C(s, u))\chi(u, t)du \\
&\quad - \int_0^s \chi(u, t)\nu''(C(s, u))D(s, u)du - \int_0^{t \wedge s} \nu'(C(s, u))du - \int_0^s \nu'(C(s, u))E(u, t)du, \\
(3.34)\quad D(s, t) &= C(s, t \vee s)\nu'(C(t \wedge s)) - C(s, 0)\nu'(C(0, t)) - f'(C(t, t))C(t, s) \\
&\quad - \int_0^{t \wedge s} \nu'(C(t, u))D(s, u)du - \int_0^{t \wedge s} C(s, u)\nu''(C(t, u))D(u, t)du \\
&\quad - h \left[ M(t) \int_0^{t \wedge s} C(s, u)\nu''(C(t, u))du + M(s) \int_0^{t \wedge s} \nu'(C(t, u))du \right], \\
(3.35)\quad H(s, u) &= -f'(C(t, t))Q(t, s) + X(s, u) + Y(s, u) \\
X(s, t) &= Q(s, t \vee s)\nu'(C(t \wedge s)) - Q(s, 0)\nu'(C(0, t)) \\
&\quad - \int_0^{t \wedge s} \nu'(C(t, u))H(s, u)du - \int_0^{t \wedge s} Q(s, u)\nu''(C(t, u))D(t, u)du \\
&\quad - h \left[ M(t) \int_0^{t \wedge s} Q(s, u)\nu''(C(t, u))du + M(s) \int_0^{t \wedge s} \nu'(C(t, u))du \right],
\end{align*}
and $Y(s, y)$ is defined similarly to $X(s, t)$, with the roles of $C$ and $Q$ and respectively $D$ and $H$ reversed, in the space of bounded continuous functions on $[0, T]^2$, subject to the symmetry conditions $C(s, t) = C(t, s)$ and $Q(s, t) = Q(t, s)$ and the boundary conditions $E(s, 0) = 0$ for all $s$, and $E(s, t) = E(s, s)$ for all $t \geq s$.

We will then show in Lemma 3.4 that every solution of (3.28)-(3.36) is necessarily a solution of (2.6)-(2.10), thus allowing us to conclude the proof of Proposition 2.2 upon showing, in Proposition 3.5, the uniqueness of the solution of (2.6)-(2.10).

Lemma 3.4. Fixing $T < \infty$, suppose $(M, \chi, C, Q, D, E, P, H)$ is a solution of the integral equations (3.28)-(3.36) in the space of continuous functions on $[0, T]^2$ subject to the symmetry conditions $C(s, t) = C(t, s)$ and $Q(s, t) = Q(t, s)$ and the boundary conditions $E(s, 0) = 0$ for all $s$, and $E(s, t) = E(s, s)$ for all $t \geq s$. Then, $\chi(s, t) = \int_0^t R(s, u)du$ where $R(s, t) = 0$ for $t > s$, $R(s, s) = 1$ and for $T \geq s > t$, the bounded
and absolutely continuous functions $M, C, R, Q$ and $K(s) = C(s, s)$ necessarily satisfy the integro-differential equations (2.6)-(2.10).

**Proposition 3.5.** Let $T \geq 0$. There exists at most one solution $(M, R, C, Q, K)$ in $C^{1}(\mathbb{R}_+^+) \times C^{1}(\Gamma) \times C^{1}(\mathbb{R}_+^+ \times \mathbb{R}_+^+) \times C^{1}(\mathbb{R}_+^2) \times C^{1}(\mathbb{R}_+^+) \times C^{1}(\mathbb{R}_+^+)$ to (2.6)-(2.10) with $R(s, s) = 1$, $C(s, s) = K(s)$, $\forall s \geq 0$, $C(0, 0) = Q(0, 0) = K(0)$ and $M(0)$ known.

We will now change the notations in (7), denoting in short $\check{U}_{N}^{q_1, q_2} := \mathbb{E}[U_{N}^{q_1, q_2}]$, whenever $U$ is one of the functions of interest $A, C, F, K, \chi, D, E$ and respectively, $\check{V}_{N}^{q} := \mathbb{E}[V_{N}^{q}]$, whenever $V$ is one of the functions $M, P$ or $R$. As before, when there is only one replica present, we will drop the index superscript (for example $\hat{C} = \hat{C}_{1,1}$).

Recall the integrated form of the equation (1.6), for $C_{N}, (\hat{\chi}_{N})^{2}$ whose polynomial growth is guaranteed by our assumption (2.3), we deduce that:

\[
x_{s}^{q,i} = x_{0}^{q,i} + B_{s}^{q,i} - \int_{0}^{s} f'(K_{N}^{q,i}(u))x_{u}^{q,i} du + \int_{0}^{s} G'(x_{u}^{q,i}) du + hs
\]

From now on, we will write $X \equiv Y$ whenever the random variables $X$ and $Y$ have the same law and $a \sim b$ when $a \cdot \to b$ or $a \cdot \to b$ whenever $a(\cdot) \to b(\cdot)$ as $N \to \infty$, uniformly on $[0, T]^2$ (or $[0, T]$, whichever applies). Let us denote by $\hat{Q}_{N}(s, t) := C_{N}^{1,2}(s, t) = C_{N}^{2,1}(s, t)$ (since $C_{N}^{1,2}(s, t) \equiv C_{N}^{2,1}(s, t)$). Applying Proposition 3.2 (for $\Psi(z) = z_{1}f'(z_{2})$ whose polynomial growth is guaranteed by our assumption (2.3)), we deduce that:

\[
\mathbb{E}[f'(K_{N}^{q_1, q_2}(u))U_{N}^{q_1, q_2}(u, t)] \approx f'(\hat{K}_{N}^{q_1, q_2}(u))\hat{U}_{N}^{q_1, q_2}(u, t)
\]

and

\[
\mathbb{E}[f'(K_{N}(u))M_{N}(u)] \approx f'(\hat{K}_{N}(u))\hat{M}_{N}(u)
\]

whenever $U$ is one of the functions $C$ or $\chi$. Hence, upon multiplying (3.37) with $x_{s}^{q,i}, B_{s}^{q,i}, x_{s}^{3-q,i}$ and 1, respectively, followed by averaging over $i$ and taking the expected value, we get that for any $s, t \in \mathbb{R}_+$,

\[
\hat{M}_{N}(s) \approx \hat{M}_{N}(0) + hs - \int_{0}^{s} f'(\hat{K}_{N}(u))\hat{M}_{N}(u) du + \int_{0}^{s} \hat{R}_{N}(u) du
\]

\[
\hat{\chi}_{N}(s, t) \approx \hat{\chi}_{N}(0, t) + t \wedge s - \int_{0}^{s} f'(\hat{K}_{N}(u))\hat{\chi}_{N}(u, t) du + \int_{0}^{s} \hat{F}_{N}(u, t) du
\]

\[
\hat{C}_{N}(s, t) \approx \hat{C}_{N}(0, t) + \hat{\chi}_{N}(s, t) - \int_{0}^{s} f'(\hat{K}_{N}(u))\hat{C}_{N}(u, t) du + \int_{0}^{s} \hat{A}_{N}(u, t) du + hs\hat{M}_{N}(t)
\]

\[
\hat{Q}_{N}(s, t) \approx \hat{Q}_{N}(0, t) - \int_{0}^{s} f'(\hat{K}_{N}(u))\hat{Q}_{N}(u, t) du + \int_{0}^{s} \hat{A}_{N}^{1,2}(u, t) du + hs\hat{M}_{N}(t)
\]

In the following proposition, we will approximate the terms $\hat{R}_{N}, \hat{F}_{N}, \hat{R}_{N}$ and $\hat{A}_{N}^{1,2}$, in order to compute the limits of (3.38)-(3.41) as $N \to \infty$.

**Proposition 3.6.** We have that

\[
\hat{A}_{N}(s, t) \approx \nu'(\hat{C}_{N}(t, u)\hat{C}_{N}(s, t))\hat{C}_{N}(s, t) - \nu'(\hat{C}_{N}(t, u))\hat{C}_{N}(s, t)
\]

\[
- \int_{0}^{s} \nu'(\hat{C}_{N}(t, u))\hat{C}_{N}(s, u)\hat{D}_{N}(t, u) du - \int_{0}^{s} \nu'(\hat{C}_{N}(t, u))\hat{D}_{N}(s, u) du
\]

\[
- h \left[ \hat{M}_{N}(t) \int_{0}^{s} \nu'((\hat{C}_{N}(t, u))\hat{C}_{N}(s, u) du + \hat{M}_{N}(s) \int_{0}^{s} \nu'((\hat{C}_{N}(t, u)) du \right],
\]

\[
\hat{A}_{N}^{1,2}(s, t) \approx \sum_{r=1}^{2} \left[ \nu'(\hat{C}_{N}^{1,r}(t, u))\hat{C}_{N}^{2,r}(s, t) - \nu'(\hat{C}_{N}^{1,r}(t, u))\hat{C}_{N}^{2,r}(s, t) \right]
\]

\[
- \int_{0}^{s} \nu'(\hat{C}_{N}^{1,r}(t, u))\hat{D}_{N}^{2,r}(s, t) du + \int_{0}^{s} \nu'(\hat{C}_{N}^{1,r}(t, u))\hat{D}_{N}^{2,r}(s, t) du \hat{D}_{N}^{1,r}(t, u) du
\]
and integers. Keeping for simplicity the implicit notation \( k \) integrable finite variation part. In doing so, recall that by (3.46), each
\[
(3.44) \quad \tilde{F}_N(s, t) \approx \tilde{x}_N(s, t \wedge s) \nu'(\tilde{C}_N(s, s)) - \int_0^s \nu'(\tilde{C}_N(s, u)) \tilde{E}_N(u, t \wedge u) du - \int_t^{t \wedge s} \nu'(\tilde{C}_N(s, u)) \tilde{D}_N(s, u) du - h \tilde{M}_N(s) \int_0^s \nu''(\tilde{C}_N(s, u)) \tilde{x}_N(u, t \wedge u) du,
\]
and
\[
(3.45) \quad \tilde{R}_N(t) \approx \nu'(\tilde{C}_N(t, t) \tilde{M}_N(t) - \nu'(\tilde{C}_N(0, t)) \tilde{M}_N(0) - \int_0^t \tilde{M}_N(u) \nu''(\tilde{C}_N(t, u)) \tilde{D}_N(u, t) du - \int_0^t \nu'(\tilde{C}_N(t, u)) \tilde{P}_N(u) du - h \int_0^t \tilde{M}_N(t) \int_0^t \nu''(\tilde{C}_N(t, u)) du + \int_0^t \nu'(\tilde{C}_N(t, u)) du.
\]

It is clear that using the results in Proposition 3.6 in formulas (3.38)-(3.41), we have proved Proposition 3.3. We shall start by developing the tools needed to conclude the proof of Proposition 3.6. To begin, we first prove a slightly more general version of Lemma 3.2 of [7]. The proof is essentially the same, replacing \( x^i_t \) by \( x^{q_1,i}_t \) and \( x^i_s \) by \( x^{q_2,i}_s \), respectively and will not be repeated.

**Lemma 3.7.** Let \( E_J \) denotes the expectation with respect to the Gaussian law \( \mathbb{P}_J \) of the disorder \( J \). Then, for the continuous paths \( x^{q_1} \in C(\mathbb{R}^+, \mathbb{R}^N), q \in \{q_1, q_2\} \), and all \( s, t \in [0, T] \) and \( i, j \in \{1, \ldots, N\} \),
\[
(3.46) \quad k^{q_1, q_2, i,j}_{ts} (x) = \frac{x^{q_1,i}_t x^{q_2,j}_s}{N} \int \nu''(C^{q_1,q_2}(s, t)) \mathbb{1}_{i,j} \nu'(C^{q_2,q_1}(s, t))
\]
where \( k^{q_1, q_2, i,j}_{ts} (x) := E_J[G^t(x^{q_1}_t)G^s(x^{q_2}_s)] \).

Fixing continuous paths \( x^{q_1} \), let \( k^{q_1, q_2}_{ts} \) denote the operator on \( L^2(\{1, \ldots, N\} \times [0, t]) \) with the kernel \( k = k^{q_1, q_2}(x) \) of (3.46). That is, for \( f \in L^2(\{1, \ldots, N\} \times [0, t]) \), \( u \leq t, \ i \in \{1, \ldots, N\} \)
\[
(3.47) \quad [k^{q_1, q_2}_{ts} f]_u^i = \sum_{j=1}^N \int_0^t k^{q_1, q_2, i,j}_{us} f^j_v dv,
\]
which is clearly also in \( L^2(\{1, \ldots, N\} \times [0, t]) \). We next extend the definition (3.47) to the stochastic integrals of the form
\[
[k^{q_1, q_2}_{ts} \circ dZ]_u^i = \sum_{j=1}^N \int_0^t k^{q_1, q_2, i,j}_{us} dZ^j_v,
\]
where \( Z^j_v \) is a continuous semi-martingale with respect to the filtration \( \mathcal{F}_t = \sigma(x^{q_1}_u, x^{q_2}_u : 0 \leq u \leq t) \) and
is composed for each \( j \), of a squared-integrable continuous martingale and a continuous, adapted, squared-integrable finite variation part. In doing so, recall that by (3.46), each \( k^{q_1, q_2}_{ts} (x) \) is the finite sum of terms such as \( x^{q_1,i_1}_u \ldots x^{q_1,i_a}_u x^{q_2,j_1}_v \ldots x^{q_2,j_b}_v \), where in each term \( a, b \) and \( i_1, \ldots, i_a, j_1, \ldots, j_b \) are some non-random integers. Keeping for simplicity the implicit notation \( \int_0^t k^{q_1, q_2, i,j}_{ts} dZ^j_v \) we thus adopt hereafter the convention of accordingly decomposing such integral to a finite sum, taking for each of its terms the variable \( x^{q_1,i_1}_u \ldots x^{q_1,i_a}_u \) outside the integral, resulting with the usual Itô adapted stochastic integrals. The latter are well defined, with \( [k^{q_1, q_2}_{ts} \circ dZ]_u^i \) being in \( L^2(\{1, \ldots, N\} \times [0, t]) \).

Our next step is to generalize Proposition C.1 of [7]:
Proposition 3.8. Let \( m \in \mathbb{Z}_+ \) and suppose under the law \( \mathbb{P} \) we have a finite collection \( \mathbf{J} = \{ J_\alpha \}_\alpha \) of non-degenerate, independent, centered Gaussian random variables, and \( G^{i,j}_s = \sum_\alpha J_\alpha L^{i,j}_u(\alpha) \), for \( q = 1, \ldots, m \), for all \( s \in [0, \tau] \) and \( i \leq N \), where for each \( \alpha \) the coefficients \( L^{i,j}_s \) which are independent of \( \mathbf{J} \) and also of each other, for different \( q \)'s, are in \( L^2([1, \ldots, N] \times [0, \tau]) \). Suppose further that \( U^{q,i}_s \) are continuous semi-martingales, independent of \( \mathbf{J} \) and such that for each \( \alpha \) and \( q \), the stochastic integral

\[
\mu^q_\alpha := \sum_{i=1}^N \int_0^\tau L^{q,i}_u(\alpha) d\mathbb{U}^{q,i}_u,
\]

is well defined and almost surely finite. Let \( \mathbb{P}^* \) denote the law of \( \mathbf{J} \) such that \( \mathbb{P}^* = \prod_{q=1}^m L^q_\tau / \mathbb{E} \left( \prod_{q=1}^m L^q_\tau \right) \mathbb{P} \), where

\[
L^q_\tau = \exp \left\{ \sum_{i=1}^N \int_0^\tau G^{q,i}_s d\mathbb{U}^{q,i}_s - \frac{1}{2} \sum_{i=1}^N \int_0^\tau (G^{q,i}_s)^2 ds \right\}.
\]

Let \( V^{q,i}_s = \mathbb{E}(G^{q,i}_s) \), \( L^{q,i,\alpha q}_s = \mathbb{E}(G^{q,i}_t G^{\alpha q}_s) \) and \( \Gamma^{q,i,\alpha q}_s \) \( \text{ and } \Gamma^{q,i,\alpha q}_s = \mathbb{E}\left[ (G^{q,i}_t - V^{q,i}_t)(G^{\alpha q}_s - V^{\alpha q}_s) \right] \). Then, for any \( s \leq \tau, i \leq N \) and \( q \in \{1, \ldots, m\} \),

\[
V^{q,i}_s + \sum_{r=1}^m [k^{q,i}_r V^{r}_s]_t = \sum_{r=1}^m [k^{q,i}_r \circ d\mathbb{U}^r_s],
\]

and for any \( s, t \leq \tau, i, l \leq N \) and \( q_1, q_2 \in \{1, \ldots, m\} \)

\[
\sum_{r=1}^N \sum_{j=1}^N \int_0^\tau k^{q_1,\alpha q}_s \circ d\mathbb{U}^{q_1,\alpha q}_s + \Gamma^{q_2,\alpha q}_s = k^{q_1,\alpha q}_s \circ d\mathbb{U}^{q_2,\alpha q}_s.
\]

Proof of Proposition 3.8 Let \( v_\alpha = \mathbb{E}(J^2_\alpha) > 0 \) denote the variance of \( J_\alpha \) and

\[
R^{q}_\alpha := \sum_{i=1}^N \int_0^\tau L^{q,i}_u(\alpha) L^{q,i}(\gamma) du,
\]

observing that

\[
L^q_\tau = \exp \left\{ \sum_\alpha J_\alpha \mu^q_\alpha - \frac{1}{2} \sum_\alpha J_\alpha J_\beta \Gamma^{q,\gamma}_\alpha \right\}.
\]

With \( \mathbf{D} = \text{diag}(v_\alpha) \) a positive definite matrix and \( \mathbf{R} := \sum_{q=1}^m R^q = \{ \sum_{q=1}^m R^q \} \) positive semi-definite, it follows from this representation of \( L^q_\tau \), that under \( \mathbb{P}^* \) the random vector \( \mathbf{J} \) has a Gaussian law with covariance matrix \( (\mathbf{D}^{-1} + \sum_{q=1}^m R^q)^{-1} \) and mean vector \( \mathbf{w} = (w_\alpha) = (\mathbf{D}^{-1} + \sum_{q=1}^m R^q)^{-1}(\sum_{q=1}^m \mu^q) \). Hence, for any \( \alpha \),

\[
\sum_{q=1}^m \left( \sum_{q=1}^m R^q \right)_{\alpha q} \gamma = v_\alpha \sum_{q=1}^m \mu^q_\alpha.
\]

As \( k^{q_1,\alpha q,\alpha q}_s = \sum_\alpha L^{q_1,\alpha q}_s(\alpha) v_\alpha L^{\alpha q,\alpha q}_s(\alpha) \), it is not hard to check that

\[
[k^{q_1,\alpha q,\alpha q}_s \circ d\mathbb{U}^{q_1,\alpha q}_s]_t := \sum_{j=1}^N \int_0^\tau k^{q_1,\alpha q,\alpha q}_s d\mathbb{U}^{q_1,\alpha q}_s = \sum_\alpha L^{q_1,\alpha q}_s(\alpha) v_\alpha \sum_{j=1}^N \int_0^\tau L^{q_1,\alpha q}_s(\alpha) d\mathbb{U}^{q_1,\alpha q}_s = \sum_\alpha L^{q_1,\alpha q}_s(\alpha) v_\alpha \mu^q_\alpha.
\]

Obviously,

\[
V^{q,i}_s = \sum_\alpha L^{q,i,\alpha q}_s w_\alpha.
\]
and also,
\[
[k^{q_1,q_2}_u V^{q_2,j}] := \sum_{j=1}^N \int_0^\tau k^{q_1,q_2,j}_u V^{q_2,j}_u du = \sum_{\alpha,\gamma} L^{q_1,j}_u(\alpha)v_\alpha w_\gamma \sum_{j=1}^N \int_0^\tau L^{q_2,j}_u(\alpha)L^{q_2,j}(\gamma)du
\]
\[
= \sum_{\alpha,\gamma} L^{q_1,j}_u(\alpha)v_\alpha R^{q_2}_\gamma w_\gamma,
\]
so we get (3.49) out of (3.52), with the last identity due to (3.51). Turning to prove (3.50), since $\Gamma^{q_1,q_2,j}_u$ is the covariance of $G^{q_1,j}_u$ and $G^{q_2,j}_t$ under the tilted law $\mathbb{P}_t^\nu$, we have that
\[
\Gamma^{q_1,q_2,j}_u = \sum_{\alpha,\gamma} L^{q_1,j}_u(\alpha) \left[ (\mathbf{D}^{-1} + \sum_{q=1}^m \mathbf{R}^q)^{-1} \right] L^{q_2,j}(\gamma),
\]
and hence by (3.51) we see that
\[
\sum_{j=1}^N \int_0^\tau k^{q_1,q_2,j}_u V^{q_2,j}_u du = \sum_{j=1}^N \int_0^\tau k^{q_1,q_2,j}_u V^{q_2,j}_u du = \sum_{\alpha,\gamma} L^{q_1,j}_u(\alpha) \left[ (\mathbf{D}^{-1} + \sum_{q=1}^m \mathbf{R}^q)^{-1} \right] L^{q_2,j}(\gamma)du
\]
\[
= \sum_{\sigma,\alpha,\gamma} L^{q_1,j}_u(\sigma)v_\sigma R^{q_2}_\alpha \left[ (\mathbf{D}^{-1} + \sum_{q=1}^m \mathbf{R}^q)^{-1} \right] L^{q_2,j}(\gamma)du
\]
\[
= \sum_{\sigma,\gamma} L^{q_1,j}_u(\sigma)v_\sigma [\mathbf{R}^\nu \left[ (\mathbf{D}^{-1} + \sum_{q=1}^m \mathbf{R}^q)^{-1} \right]]_{\sigma\gamma} L^{q_2,j}(\gamma)du
\]
With $\mathbf{D} = \text{diag}(v_\alpha)$ we easily get (3.50) out of the matrix identity:
\[
\left( \mathbf{I} + \mathbf{D} \left( \sum_{q=1}^m \mathbf{R}^q \right) \right) \left( \mathbf{D}^{-1} + \sum_{q=1}^m \mathbf{R}^q \right)^{-1} = \mathbf{D}.
\]

Now, the same proof as in Lemma 3.2 of [7], with $L^N_\nu$ replaced by:
\[
L^N_\nu = \exp \left\{ \sum_{q=1}^m \sum_{i=1}^N \int_0^\tau G^i(x^q_s)dU^{q,i}_s(x) - \frac{1}{2} \sum_{i=1}^N \int_0^\tau (G^i(x^q_s))^2 ds \right\}
\]
and using Proposition 3.8 above instead of Proposition C.1 of [7], will show:

**Lemma 3.9.** Let $m \in \mathbb{Z}_+$ and consider $m$ replicas $\{x^q\}_s$, for $q = 1, \ldots, m$, sharing the same couplings $\mathbf{J}$, with the noise given by $m$ independent $N$-dimensional Brownian motions $\{B^q\}_s$. Fixing $r \in \mathbb{R}_+$ and denoting $x = (x^1, \ldots, x^m)$, let $V^{q,i}_u(x) = \mathbb{E}[G^i(x^q)^2|F_u]$ and $Z^{q,i}_u(x) = \mathbb{E}[B^{q,i}_u|F_u]$ for $s \in [0, r]$. Then, under $\mathbb{P}_r^\nu \otimes \mathbb{P}_x^{N^0}$ we can choose a version of these conditional expectations such that the stochastic processes
\[
U^{q,i}_s(x) := x^{q,i}_s - x^{q,i}_0 + \int_0^s f(K^{q,i}_N(u))x^{q,i}_u du - hs
\]
\[
Z^{q,i}_s(x) := U^{q,i}_s(x) - \int_0^s V^{q,i}_u(x^q)du,
\]
are both continuous semi-martingales with respect to the filtration \( \mathcal{F}_t = \sigma(x^k_u : 0 \leq u \leq t, 1 \leq k \leq m) \), composed of squared-integrable continuous martingales and finite variation parts. Moreover, such choice satisfies for any \( i, q \) and \( s \in [0, \tau] \),

\[
\tag{3.55}
V^{q,i}_s = \sum_{r=1}^{m} \int_0^s [k^{r,q}_t \circ dU^i_t] \, dt,
\]

and \( V^{q,i}_s = \sum_{r=1}^{m} [k^{r,q}_t \circ dZ^i_t] \) for any \( i, q \) and all \( s \leq \tau \). Further, for any \( u, v \in [0, \tau] \) and \( i, j \leq N \), let

\[
\tag{3.56}
\Gamma^{q_1,q_2,ij}_{uv} := \mathbb{E} \left[ (G^i(x^{q_1}_u) - V^{q_1,i}_u)(G^j(x^{q_2}_v) - V^{q_2,j}_v) | \mathcal{F}_t \right].
\]

Further, we can choose a version of \( \Gamma^{q_1,q_2,il}_{uv} \) such that for any \( s, v \leq \tau, \) any \( q_1, q_2 \in \{1, \ldots, m\} \) and all \( i, l \leq N \),

\[
\tag{3.57}
\sum_{r=1}^{m} \sum_{j=1}^{N} \int_0^\tau k^{r,q_1,j}_{st} \nu^{r,j}_s \, d\nu^{q_2,i}_s = k^{q_1,q_2,il}_{st}.
\]

**Proof of Proposition 3.6** We first apply (3.55) to derive (3.43). Fix \( s, t \in [0, T]^2 \), let \( \tau = t \vee s \) and define:

\[
a^{q_1,q_2}_N(t, s) = \frac{1}{N} \sum_{i=1}^{N} V^{q_1,i}_t(x) x^{q_2,i}_s.
\]

Since \( x^{q_1}_s \) is measurable on \( \mathcal{F}_t \), \( q = 1, 2 \), we see that:

\[
\tilde{A}^{q_1,q_2}_N(t, s) = \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[G^i(x^{q_1}_t) x^{q_2,i}_s | \mathcal{F}_t] \right] = \mathbb{E}[a^{q_1,q_2}_N(t, s)] = \tilde{a}^{q_1,q_2}_N(t, s).
\]

Hence, with \( t \leq \tau \), combining (3.55) and (3.33), and suppressing in the notation the dependence of \( k^{q_1,q_2,ij}_{tu} \) and \( V^{q_1,q_2}_u \) of \( x \), we get:

\[
\tag{3.58}
a^{1,2}_N(t, s) = \sum_{r=1}^{2} \left[ \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau f'(K^{r,ij}_N(u)) x^{2,i}_s k^{1,1,rij}_{tu} V^{r,j}_u \, du + \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau x^{2,i}_s k^{1,1,rij}_{tu} d\nu^{r,j}_s \right]
\]

Using the explicit expression of \( k^{q_1,q_2,ij}_{tu} \) from Lemma 3.7 and collecting terms while changing the order of summation and integration, we arrive at the identity:

\[
\tag{3.59}
a^{1,2}_N(t, s) = -\sum_{r=1}^{2} \left[ \int_0^\tau C^{2,r}_N(s, u) \nu''(C^{r,1}_N(t, u)) a^{r,1}_N(u, t) \, du + \int_0^\tau \nu'(C^{r,1}_N(t, u)) d\nu^{2,1}_N(u, s) \, du \right]
\]

\[
- \frac{1}{2} \sum_{r=1}^{2} \left[ \int_0^\tau M^3_N(t) C^{2,r}_N(s, u) \nu''(C^{r,1}_N(t, u)) \, du + \int_0^\tau M^2_N(t) \nu'(C^{r,1}_N(t, u)) \, du \right]
\]

\[
+ \sum_{r=1}^{2} \left[ \int_0^\tau f'(K^{r}_N(u)) C^{2,r}_N(s, u) \nu''(C^{r,1}_N(t, u)) \, du \right]
\]

\[
+ \sum_{r=1}^{2} \left[ \int_0^\tau f'(K^{r}_N(u)) \nu'(C^{r,1}_N(t, u)) C^{r,2}_N(u, s) \, du \right]
\]

\[
+ \sum_{r=1}^{2} \left[ \int_0^\tau C^{2,r}_N(s, u) \nu'(C^{r,1}_N(t, u)) d\nu^{r,1}_N(u, t) + \int_0^\tau \nu'(C^{r,1}_N(t, u)) d\nu^{r,2}_N(u, s) \right].
\]
Applying Lemma A.1 of [7] for the semi-martingales \( x = x^t, y = x^1, z = x^2 \) and polynomials \( P(x) = x \) and \( Q(x) = \nu'(x) \), the stochastic integrals in the last line of (3.59) can be replaced with:

\[
\sum_{r=1}^{2} \left[ \nu'(C^r_{N}(t, u))C^{2r}_{N}(t, s) - \nu'(C^r_{N}(0, t))C^{2r}_{N}(0, s) \right]
- \sum_{r=1}^{2} \left[ \frac{1}{2N} C^{1r}_{N}(t, t) \int_{0}^{\tau} \nu''(C^r_{N}(u, t))C^{2r}_{N}(u, s)du + \frac{1}{N} C^{1r}_{N}(s, t) \int_{0}^{\tau} \nu'(C^r_{N}(u, t))du \right].
\]

Now, it is easy to see that since \( E[\sup_{s,t \leq T} |A^{q_1, q_2}_{N}(s, t)|] \) is uniformly bounded in \( N \) (see the discussion prior to Proposition 3.2), then the same is true for \( A^{q_1, q_2}_{N}(t, s) \), hence the terms in the second line (3.60) above will converge almost surely to 0, as \( N \to \infty \). Furthermore, \( A^{q_1, q_2}_{N}(t, s) = E[A^{q_1, q_2}_{N}(t, s) \mathcal{F}_r] \) inherits the self-averaging property from \( A^{q_1, q_2}_{N} \), hence, we can apply Corollary 3.2 with possibly \( a^{r, q_2}_{N} \) as one of the arguments of the locally Lipschitz function \( \Psi(z) \) of at most polynomial growth at infinity. Doing so for the functions \( z_1 z_2 \nu''(z_3) \), and \( z_1 \nu'(z_2) \) and applying Proposition 3.2 also for \( f'(z_1) z_2 \nu''(z_3) z_3 \) and \( f'(z_1) \nu'(z_2) z_3 \), we deduce from (3.59) and (3.60) that

\[
\tilde{A}^{1, 2}_{N}(t, s) \simeq \sum_{r=1}^{2} \left[ \int_{0}^{\tau} \tilde{C}^{2r}_{N}(t, u)\nu''(\tilde{C}^{r+1}_{N}(t, u))\tilde{A}^{1r}_{N}(u, t)du + \int_{0}^{\tau} \nu'(\tilde{C}^{r+1}_{N}(t, u))\tilde{C}^{2r}_{N}(u, s)du \right]
- \sum_{r=1}^{2} \left[ \int_{0}^{\tau} \tilde{M}_{N}(t)\tilde{C}^{2r}_{N}(t, u)\nu''(\tilde{C}^{r+1}_{N}(t, u))du + \int_{0}^{\tau} \tilde{M}_{N}(s)\nu'(\tilde{C}^{r+1}_{N}(t, u))du \right]
+ \sum_{r=1}^{2} \left[ \nu'(\tilde{C}^{r+1}_{N}(t, u))\tilde{C}^{2r}_{N}(t, s) - \nu'(\tilde{C}^{r+1}_{N}(0, t))\tilde{C}^{2r}_{N}(0, s) \right].
\]

Finally, recalling that

\[
\tilde{A}^{q_1, q_2}_{N}(t, s) = \tilde{D}^{q_1, q_2}_{N}(t, s) + f'(\tilde{K}_{N}(t))\tilde{C}^{q_1, q_2}_{N}(t, s),
\]

and noting that \( \tilde{K}^{r, r}_{N}(t) = \tilde{K}_{N}(t) \), for all \( t \) and \( r \), setting \( \tau = t \lor s \), we indeed arrive at:

\[
\tilde{A}^{1, 2}_{N}(s, t) \simeq \sum_{r=1}^{2} \left[ \int_{0}^{\tau} \tilde{C}^{2r}_{N}(t, u)\nu''(\tilde{C}^{r+1}_{N}(t, u))\tilde{D}^{1r}_{N}(t, u)du + \int_{0}^{\tau} \nu'(\tilde{C}^{r+1}_{N}(t, u))\tilde{A}^{2r}_{N}(s, u)du \right]
- \sum_{r=1}^{2} \left[ \int_{0}^{\tau} \tilde{M}_{N}(t)\tilde{C}^{2r}_{N}(t, u)\nu''(\tilde{C}^{r+1}_{N}(t, u))du + \int_{0}^{\tau} \tilde{M}_{N}(s)\nu'(\tilde{C}^{r+1}_{N}(t, u))du \right]
+ \sum_{r=1}^{2} \left[ \nu'(\tilde{C}^{r+1}_{N}(t \lor s, t))\tilde{C}^{2r}_{N}(t \lor s, s) - \nu'(\tilde{C}^{r+1}_{N}(0, t))\tilde{C}^{2r}_{N}(0, s) \right].
\]

that is (3.43).

For deriving (3.42) next, the single-replica equivalent of (3.43), we can apply the same strategy as above. Namely, defining:

\[
a_{N}(t, s) = \frac{1}{N} \sum_{i=1}^{N} V^{i}(x)x^{i}_{s},
\]
we see that $a_N(t, s)$ has the same first moment with $A_N(t, s)$. Furthermore, since $F_\tau$ is generated only by the realization of one replica up to time $\tau$, (3.55) will imply that:

$$a_N(t, s) + \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau x_s k_{tu}^i V_u^j du + h \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau x_s^i k_{tu}^j x_u^j du$$

$$= \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau f'(K_N(u)) x_s^i k_{tu}^j x_u^j du$$

Note that the above equation is indeed the one-dimensional version of (3.58) (without the sums and the replica indices), so we would expect the results to be similar. Indeed, using the explicit expression of $k_{tu}^i$ from Lemma 3.7 we arrive at the identity:

$$(3.61) \quad a_N(t, s) = - \int_0^\tau C_N(s, u) \nu''(C_N(t, u)) a_N(u, t) du - \int_0^\tau \nu'(C_N(t, u)) a_N(u, s) du$$

$$- \left[ M_N(t) C_N(s, u) \nu''(C_N(t, u)) du + \int_0^\tau M_N(s) \nu'(C_N(t, u)) du \right]$$

$$+ \int_0^\tau f'(K_N(u)) C_N(s, u) \nu''(C_N(t, u)) C_N(t, u) du$$

$$+ \int_0^\tau C_N(s, u) \nu''(C_N(t, u)) du C_N(u, t) + \int_0^\tau \nu'(C_N(t, u)) du C_N(u, s)$$

Applying again Lemma A.1 of [7], this time for the semi-martingales $x = y = z = w = x$ and polynomials $P(x) = x$ and $Q(x) = \nu'(x)$, the stochastic integrals in the last line of (3.61) can be replaced with:

$$\nu'(C_N(\tau, t)) C_N(\tau, s) - \nu'(C_N(0, t)) C_N(0, s)$$

$$- \left[ \frac{1}{2N} C_N(\tau, t) \int_0^\tau C_N(s, u) \nu''(C_N(\tau, u)) du + \frac{1}{N} C_N(s, t) \int_0^\tau \nu'(C_N(t, u)) du \right].$$

As before, the terms in the second line above will converge to 0 as $N \to \infty$, and, $a_N(t, s) = E[A_N(t, s)|F_\tau]$ inherits the self-averaging property from $A_N$. Hence applying Corollary 3.2 with possibly $a_N$ as one of the arguments of the locally Lipschitz function $\Psi(z)$, setting $\tau = t \vee s$ and recalling that $\tilde{A}_N(t, s) = \tilde{D}_N(t, s) + f'(\tilde{K}_N(t)) \tilde{C}_N(s, t)$, we arrive at (3.43).

Now, for (3.45), denoting $r_N(s) = \frac{1}{N} \sum_{i=1}^{N} V_i^i(x)$ and we easily see that $\tilde{r}_N(s) = \tilde{R}_N(s)$, so by (3.55) and (3.53) we get that:

$$r_N(t) + \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau k_{tu}^{ij} V_u^j du = \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau f'(K_N(u)) k_{tu}^{ij} x_u^j du + \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau k_{tu}^{ij} x_u^j du - h \frac{1}{N} \sum_{i,j=1}^{N} \int_0^\tau k_{tu}^{ij} du$$

So, as before, using the explicit expression of $k_{tu}^{ij}$, we get to:

$$(3.62) \quad r_N(t) = - \int_0^\tau M_N(u) \nu''(C_N(t, u)) a_N(u, t) du - \int_0^\tau \nu'(C_N(t, u)) r_N(u) du$$

$$- h \left[ \int_0^\tau M_N(t) M_N(u) \nu''(C_N(t, u)) du + \int_0^\tau \nu'(C_N(t, u)) du \right]$$

$$+ \int_0^\tau f'(K_N(u)) M_N(u) \nu''(C_N(t, u)) C_N(u, t) du$$
we get, for $s$

\[\text{The right hand side of (3.64) was evaluated in the proof of Proposition 3.1 of [7]. Using their result into (3.64), hence by (3.37) and (3.46):}\]

As before, the terms in the second line above will converge to 0 as $N \to \infty$, and, $r_N(t) = E[R_N(t)|\mathcal{F}_t]$ inherits the self-averaging property from $R_N$. Hence applying Corollary 3.2 with possibly $a_N$ and $r_N$ as some of the arguments of $\Psi(z)$ and recalling that $\tilde{P}_N(t) = \tilde{R}_N(t) + f'(\tilde{K}_N(t))M_N(t)$, we arrive at (3.45).

Now the derivation of (3.44) is similar to the derivation of its analogue in the proof of Proposition 3.1 in [7]. Namely, since:

\[\text{(3.63)}\]

\[E[G_s^i B_t^i] + E\left[\int_0^t \Gamma_{sv}^i dv\right] = E\left[[k_s \circ dZ_i^i, Z_i^i]\right].\]

the equation (3.2) implies:

\[\tilde{F}_N(s, t) = E\left[\frac{1}{N} \sum_{i=1}^N E\left[[k_s \circ dB_i^i|\mathcal{F}_s]\right] B_t^i\right] - E\left[\frac{1}{N} \sum_{i=1}^N \int_0^t \Gamma_{sv}^i dv\right]\]

hence by (3.37) and (3.46):

\[\text{(3.64)}\]

\[\tilde{F}_N(s, t) + h \frac{1}{N} \sum_{i=1}^N E\left[[k_s^i|\mathcal{F}_s] B_t^i\right] = \frac{1}{N} \sum_{i=1}^N \left(E\left[E\left[[k_s \circ d\mathbf{x}_s]^i + [k_s f'(K_N)\mathbf{x}_s]^i - [k_s G_s^i|\mathcal{F}_s] B_t^i\right] - E\left[\int_0^t \Gamma_{sv}^i dv\right]\right)\right),\]

The right hand side of (3.64) was evaluated in the proof of Proposition 3.1 of [7]. Using their result into (3.64), we get, for $s \geq t$:

\[\tilde{F}_N(s, t) + h \frac{1}{N} \sum_{i=1}^N E\left[[k_s^i|\mathcal{F}_s] B_t^i\right] \simeq \tilde{\chi}_N(s, t)\nu'(\tilde{C}_N(s, s)) - \int_0^{t \wedge s} \nu'(\tilde{C}_N(s, u))du\]

\[- \int_0^s \nu'(\tilde{C}_N(s, u))\tilde{E}_N(u, t \wedge u)du - \int_0^{t \wedge s} \nu'(\tilde{C}_N(s, u))du\]

\[- \int_0^s \tilde{\chi}_N(u, t \wedge u)\nu'(\tilde{C}_N(s, u))\tilde{D}_N(s, u)du\]

Now, using the explicit formula for $k_s$ to compute the remaining term:

\[h \frac{1}{N} \sum_{i=1}^N E\left[[k_s^i|\mathcal{F}_s] B_t^i\right] = h \frac{1}{N} \sum_{i=1}^N E\left[\left(M_N(s) \int_0^s \nu'(C_N(s, u))x^i du + \int_0^s \nu'(C_N(s, u))du\right) B_t^i\right]\]

\[= h E\left[\int_0^s M_N(s)\nu'(C_N(s, u))\tilde{\chi}_N(s, u)du + \int_0^s \nu'(C_N(s, u))W_N(t)du\right]\]

\[\simeq h \tilde{M}_N(s) \int_0^s \nu'(\tilde{C}_N(s, u))\tilde{\chi}_N(s, u)du\]
Proof of Lemma 3.4. We shall show that every solution of (3.28)-(3.36) is necessarily a solution of (2.6) with (2.10), where \( \chi(s,t) = \int_0^t R(s,u)du \).

First, the same argument as in the beginning of Lemma 5.1 of [7] applied to

\[
h(s,t) := -f'(C(s,s))\chi(s,t) - \int_0^s \chi(u,t)\nu''(C(s,u))D(s,u)du + \chi(s,t)\nu'(C(s,s)) - \int_0^{\tau_v s} \nu'(C(s,u))du - hM(s) \int_0^s \nu''(C(s,u))\chi(u,t)du
\]

will show that \( t \mapsto \chi(s,t) \) is continuously differentiable on \( s \geq t \), with \( \chi(s,t) = \int_0^t R(s,u)du \), where \( R(s,s) = 1 \) for all \( s \) and \( \chi(s,t) = \chi(s,s) \) for \( t > s \), implying that \( R(s,t) = 0 \), for \( t > s \).

From (3.30) we have that \( C(s,t) - \chi(s,t) \) is differentiable with respect to its second argument \( t \), hence \( \partial_2 C(s,t) = D(s,t) + R(s,t) + hM(s) \) Further, \( C(s,t) = C(t,s) \) implying that \( \partial_1 C(s,t) = \partial_2 C(t,s) = D(t,s) + R(t,s) + hM(t) \) on \([0,T]^2\). Thus, combining the identity

\[
C(s,t) - C(s,0) = \int_0^{\tau_v s} \nu'(C(s,u))\partial_2 C(s,u)du + \int_0^{\tau_v s} C(s,u)\nu''(C(t,u))\partial_2 C(t,u)du,
\]

with (3.34) we have that for all \( t, s \in [0,T]^2 \),

\[
(3.65) \quad D(s,t) = -f'(K(t))C(t,s) + \int_0^{\tau_v s} \nu'(C(t,u))R(s,u)du + \int_0^{\tau_v s} C(s,u)\nu''(C(t,u))R(t,u)du.
\]

Interchanging \( t \) and \( s \) in (3.65) and adding \( R(t,s) = 0 \) when \( s > t \), results for \( s > t \) with

\[
\partial_1 C(s,t) = -f'(K(s))C(s,t) + \int_0^s \nu'(C(s,u))R(t,u)du + \int_0^s C(t,u)\nu''(C(s,u))R(s,u)du + hM(t),
\]

which is (2.8) for \( \beta = 1 \).

Now, from (3.28), \( M(\cdot) \) is differentiable and \( M'(t) = h + P(t) \), hence combining the identity

\[
M(t)\nu'(C(t,t)) - M(0)\nu'(C(t,0)) = \int_0^t \nu'(C(t,u))M'(u)du + \int_0^t M(u)\nu''(C(t,u))\partial_2 C(t,u)du,
\]

with (3.32) we have that for all \( t \in [0,T] \),

\[
(3.66) \quad P(t) = -f'(K(t))M(t) + \int_0^t M(u)\nu''(C(t,u))R(t,u)du.
\]

thus showing is (2.9) for \( \beta = 1 \).

Also, since from (3.31), \( \partial_2 Q(s,t) = H(s,t) + hM(s) \), from the identity

\[
Q(s,t)\nu'(C(t,t)) - Q(s,0)\nu'(C(t,0)) = \int_0^{\tau_v s} \nu'(C(t,u))\partial_2 Q(s,u)du + \int_0^{\tau_v s} Q(s,u)\nu''(C(t,u))\partial_2 C(t,u)du,
\]

where the last line is obtained by two applications of Proposition 3.2 (eventually with the zero mean random variable \( W_X(t) = \frac{1}{\sqrt{N}} \sum_{i=1}^N B_i^t \) as one of its arguments), hence concluding the proof of (3.44). \( \square \)
with (3.36) we have that for all \( t \in [0, T] \),

\[
X(s, t) = \int_0^{t \vee s} \nu''(C(t, u))R(t, u)Q(s, u)du.
\]

Similarly,

\[
Y(s, t) = \int_0^{t \vee s} \nu'(Q(t, u))R(s, u)du.
\]

thus showing is (2.9) for \( \beta = 1 \).

Since \( K(s) = C(s, s) \), with \( C(s, t) = C(t, s) \) and \( \partial_2 C(t, s) = D(t, s) + R(t, s) + hM(t) \), it follows that for all \( k > 0 \),

\[
K(s) - K(s - k) = \int_{s-k}^s (D(s, u) + R(s, u) + hM(s))du + \int_k^s (D(s - k, u) + R(s - k, u) + hM(s - k))du.
\]

Recall that \( R(s, u) = 0 \) for \( u > s \), hence, dividing by \( k \) and taking \( k \downarrow 0 \), we thus get by the continuity of \( D \) and that of \( R \) for \( s \geq t \) that \( K(\cdot) \) is differentiable, with \( \partial_x K(s) = 2D(s, s) + R(s, s) + 2hM(s) = 2D(s, s) + 1 + 2hM(s) \), resulting by (3.65) with (2.10) for \( \beta = 1 \).

Further, it follows from (3.29) that \( \partial_t \chi(u, t) = E(u, t) + 1_{u < t} \). Hence, combining the identity

\[
\chi(s, t)^\nu'(C(s, s)) - \chi(0, t)^\nu'(C(s, 0)) = \int_0^s \nu'(C(s, u))\partial_t \chi(u, t)du + \int_0^s \chi(u, t)^\nu''(C(s, u))\partial_2 C(s, u)du,
\]

with (3.33) we have that for all \( T \geq s \geq t \),

\[
E(s, t) = -f'(K(s))\chi(s, t) + \int_0^s \chi(u, t)\nu''(C(s, u))R(s, u)du
\]

(recall that \( \chi(0, t) = \chi(0, 0) = 0 \)). Let

\[
g(s, t) := -f'(K(s))R(s, t) + \int_0^s R(u, t)^\nu''(C(s, u))R(s, u)du,
\]

for \( s, t \in [0, T]^2 \). Recall that \( \chi(s, t) = \int_0^t R(s, v)dv \), so by Fubini’s theorem, (3.69) amounts to \( E(s, t) = \int_0^t g(s, v)dv \) for all \( s \geq t \). Further, with \( E(s, t) = E(s, s) \) when \( t > s \), it follows that

\[
E(s, t) = \int_0^{t \vee s} g(s, v)dv
\]

for all \( s, t \leq T \). Putting this into (3.29) we have by yet another application of Fubini’s theorem that

\[
\int_0^t R(s, u)du = \chi(s, t) = t + \int_0^s \int_0^t g(u, v)dvdu = t + \int_t^s \int_v^s g(u, v)du dv,
\]

for any \( s \geq t \). Consequently, for every \( t \leq s \),

\[
R(s, t) = 1 + \int_t^s g(u, v)du,
\]

implying that \( \partial_t R = g \) for a.e. \( s > t \), which in view of (3.70) gives (2.7) for \( \beta = 1 \), thus completing the proof of the lemma. \( \square \)

**Proof of Lemma 3.5** We shall show that the system (2.6)–(2.10) with initial conditions \( C(t, t) = K(t) \), \( R(t, t) = 1 \), \( M(0) = \alpha \) and \( Q(0, 0) = K(0) = C(0, 0) = 0 \) admits at most one bounded solution \( (M, R, C, Q, K) \) on \([0, T] \times [0, T]^2 \times \{0\} \times [0, T]^2 \times [0, T]^2 \). First notice that if we denote \( D(t) := Q(t, t) \), by the symmetry of \( Q \), we have \( \partial D(t) = 2\partial_1 Q(t, t) \). Now consider the difference between the integrated form of (2.6)–(2.9) for two
such solutions \((M, R, C, Q, K, D)\) and \((\bar{M}, \bar{R}, \bar{C}, \bar{Q}, \bar{K}, \bar{D})\) and define the functions \(\Delta V(s, t) = |V(s, t) - \bar{V}(s, t)|\), when \(V\) is one of the functions \(C, R\) or \(Q\) and \(\Delta U(s) = U(s) - \bar{U}(s)\), when \(U\) is \(M, D\) or \(K\). Then, since \(\nu''\) is uniformly Lipschitz on any compact interval and \(C, Q, C, Q\) are continuous, hence bounded on \([0, T]^2\), we have, for \(0 \leq t \leq s \leq T\),

\[
\begin{align*}
\Delta M(t) & \leq \kappa_1 \int_0^s \Delta M(v)dv + \int_0^s h(v)dv \\
\Delta R(s, t) & \leq \kappa_1 \int_t^s \Delta R(v, t)dv + \int_t^s h(v)dv \\
\Delta C(s, t) & \leq \kappa_1 \int_t^s \Delta C(v, t)dv + \int_t^s h(v)dv + \Delta M(t) + \Delta K(t) + h(t) \\
\Delta Q(s, t) & \leq \kappa_1 \int_t^s \Delta Q(v, t)dv + \int_t^s h(v)dv + \Delta M(t) + \Delta D(t) + h(t) \\
\Delta K(t) & \leq \kappa_1 \int_t^s \Delta K(v)dv + \int_t^s h(v)dv + \Delta M(t) + h(t) \\
\Delta D(t) & \leq \kappa_1 \int_t^s \Delta D(v)dv + \int_t^s h(v)dv + \Delta M(t) + h(t)
\end{align*}
\]

where \(h(v) := \int_0^v [\Delta R(v, \theta) + \Delta C(v, \theta) + \Delta Q(v, \theta) + \Delta M(\theta) + \Delta D(\theta) + \Delta K(\theta)]d\theta\) and \(\kappa_1 < \infty\) depends on \(T, \beta, \nu(\cdot)\) and the maximum of \(|M|, |R|, |C|, |Q|, |\bar{M}|, |\bar{R}|, |\bar{C}|, |\bar{Q}|\) on \([0, T]^2\). Integrating \((3.71)-(3.76)\) over \(t \in [0, s]\), since \(\Delta R(v, u) = 0\) for \(u \geq v\), \(\Delta C(v, u) = \Delta C(u, v)\) and \(\Delta Q(v, u) = \Delta Q(u, v)\), we find that

\[
\begin{align*}
\int_0^s \Delta M(t)dt & \leq \kappa_2 \int_0^s h(v)dv , \\
\int_0^s \Delta R(s, t)dt & \leq \kappa_2 \int_0^s h(v)dv , \\
\int_0^s \Delta C(s, t)dt & \leq \kappa_2 \int_0^s h(v)dv , \\
\int_0^s \Delta Q(s, t)dt & \leq \kappa_2 \int_0^s h(v)dv , \\
\int_0^s \Delta K(t)dt & \leq \kappa_2 \int_0^s h(v)dv , \\
\int_0^s \Delta D(t)dt & \leq \kappa_2 \int_0^s h(v)dv ,
\end{align*}
\]

for some finite constant \(\kappa_2\) (of the same type of dependence as \(\kappa_1\)). Summing the last three inequalities, we see that for all \(s \in [0, T]\),

\[
0 \leq h(s) \leq \kappa_3 \int_0^s h(v)dv.
\]

where \(\kappa_3 = 6 \max \{\kappa_1, \kappa_2\}\). Further, \(h(0) = 0\), so by Gronwall’s lemma \(h(s) = 0\) for all \(s \in [0, T]\). Plugging this result back into \((3.71)-(3.76)\) and observing that \(\Delta R(t, t) = \Delta K(0) = \Delta M(0) = \Delta D(0) = 0, \Delta C(t, t) = \Delta K(t)\) and \(\Delta Q(t, t) = \Delta D(t)\), we deduce that \(\Delta R(s, t) = \Delta C(s, t) = \Delta M(t) = \Delta Q(s, t) = \Delta D(t) = \Delta K(s) = 0\) for all \(0 \leq t \leq s \leq T\), hence, by symmetry, the stated uniqueness.

\[
\square
\]

4. LIMITING HARD SPHERICAL DYNAMICS

Through this section, we will fix \(r > 0\) and, for convenience of notation, suppress the \(r\) dependence in the subscripts.
The uniform bounds on the moments of $K_N(s)$ used to establish Proposition 4.1 (namely equation (4.10)), will show that $\sup_{t \geq 1} K(t) < \infty$. Further, as $C(s, t)$ is the limit of $C_N(s, t) = \frac{1}{N} \sum_{i=1}^{\infty} x_i^1 x_i^2$, it is a non-negative definite kernel on $\mathbb{R}_+ \times \mathbb{R}_+$ and in particular, $C(s, t)^2 \leq K(s) K(t)$ and $C(t, t) \geq 0$. Also, since $Q(s, t)$ is the limit of $Q_N(s, t) = \frac{1}{N} \sum_{i=1}^{\infty} x_i^1 x_i^2$, for two iid replicas $x_1^1$ and $x_2^1$, by the Cauchy-Schwartz inequality and then taking the limit as $N \to \infty$, we have $Q(s, t)^2 \leq K(s) K(t)$.

To complete the proof of Theorem 4.1, we first prove that any solution $(M, R, C, Q, K)$ of (2.6)–(2.10) consists of positive functions, a key fact in our forthcoming analysis.

Lemma 4.1. For any $f: \mathbb{R}_+ \to \mathbb{R}$ whose derivative is bounded above on compact intervals and any $K(0) > 0$, $M(0) > 0$, a solution $(M, R, C, Q, K)$ to (2.6)–(2.10), if it exists, is positive at all times. Furthermore, $\tilde{C}(s, t) := C(s, t) - M(s) M(t)$ is also non-negative.

Proof of Lemma 4.1. By definition $K(t) \geq 0$ for all $t \in \mathbb{R}_+$. Define

$$S_1 = \inf \{ u \geq 0 : C(u, t) \leq 0 \} \text{ for some } t \leq u \}.$$ 

and

$$S_2 = \inf \{ u \geq 0 : M(u) \leq 0 \}.$$ 

and suppose that $S = \min\{S_1, S_2\} < \infty$. By continuity of $(C, K, Q)$, since $K(0) > 0$ and $M(0) > 0$, also $S_1, S_2 > 0$, hence $S > 0$. Set $L(s, t) = \exp(- \int_s^t u(\nu(du)) > 0$ for $\mu(\nu) = f(K(u))$ which is bounded above on compact intervals, and $R(s, t) = L(s, t) H(s, t)$. Then, by [16], for $s \geq t$,

$$H(s, t) = 1 + \sum_{n \geq 1} \beta^{2n} \sum_{\sigma \in NC_n} \int_{t \leq t_1 \leq \cdots \leq t_{2n} \leq s} \prod_{i \in \sigma(i)} \nu''(C(t_i, t_{\sigma(i)})) \prod_{j=1}^{2n} dt_i$$

where NC$_n$ denotes the set of involutions of $\{1, \cdots, 2n\}$ without fixed points and without crossings and $\text{cr}($ is defined to be the set of indices $1 \leq i \leq 2n$ such that $i < \sigma(i)$. Consequently,

$$R(s, t) \geq L(s, t) > 0 \text{ for } t \leq s \leq S,$$

and thus, (2.8) implies that

$$C(s, t) \geq K(t) L(s, t) > 0 \text{ for } t \leq s \leq S.$$ 

Also, (2.6) implies that

$$M(s) \geq M(0) L(s, 0) > 0 \text{ for } 0 \leq s \leq S.$$ 

Note that in the last two estimates we used the fact that $\nu'(\cdot)$ and $\nu''(\cdot)$ are non-negative on $\mathbb{R}_+$. Similarly, from the equation (2.10) we see that $\partial [L(s, 0)^{-2} K(s)] \geq L(s, 0)^{-2}$ for all $s \leq S$ resulting with

$$K(s) \geq K(0) L(0, 0)^{-2} + \int_0^s L(s, v)^{-2} dv > 0$$

Hence, the continuous functions $R(s, t), C(s, t)$ and $M(s)$ are bounded below by a strictly positive constant for $0 \leq t \leq s \leq S$ in contradiction with the definition of $S$. We thus deduce that $S = \infty$, hence $S_1 = S_2 = \infty$ and by the preceding argument and the symmetry of $C$, the functions $R(s, t), C(s, t)$ and $M(s)$ are positive.

Similarly, let $S_3 = \inf \{ u \geq 0 : Q(u, t) \leq 0 \text{ for some } t \leq u \}$ and assume $S_3 < \infty$. Then, from the symmetry of $Q(s, t) = Q(t, s)$, defining $D(t) := Q(t, t)$, we have $\partial D(t) = 2 \partial \tilde{Q}(t, t)$, hence by (2.9) we have:

$$D(s) \geq D(0) L^2(s, 0) > 0 \text{ for } 0 \leq s \leq S_3.$$ 

and hence, using again (2.9):

$$Q(s, t) \geq Q(t, t) L(s, t) = D(t) L(s, t) > 0 \text{ for } t \leq s \leq S_3.$$ 

Hence the continuous function $Q$ is bounded below by a positive constant on $0 \leq t \leq s \leq S_3$, contradiction to the definition of $S_3$. Hence $S_3 = \infty$ and by the symmetry of $Q$, it is positive on $\mathbb{R}_+^2$. This concludes our proof that $M, R, C, Q, K$ are all positive functions.
Furthermore, from (2.6) and (2.8), we know that $\tilde{C}(s,t) = C(s,t) - M(s)M(t)$ satisfies:
\[
\partial_t \tilde{C}(s,t) = -f'(K(s))\tilde{C}(s,t) + \beta^2 \int_0^t \tilde{C}(u,t)R(s,u)\nu''(C(s,u))du
\]
\[
+ \beta^2 \int_0^t \nu'(C(s,u))R(t,u)du
\]

hence
\[
\tilde{C}(s,t) \geq \tilde{C}(t,t)L(s,t) \geq 0 \quad \text{for} \quad t \leq s \leq S
\]
since $\tilde{C}(t,t) = K(t) - M^2(t) \geq 0$.

We next show that if $(M_{L,r}, R_{L,r}, C_{L,r}, Q_{L,r}, K_{L,r})$ are solutions of the system (2.6)-(2.10) with potential $f_{L,r}(\cdot)$ as in (2.12), then $K_{L,r}(s) \to r$ as $L \to \infty$, uniformly over compact intervals. Specifically,

**Lemma 4.2.** Assuming $K_L(0) = r$, there exist $L_0 > 0$ such that $K_L(s) \geq r - B_0L^{-1}$, for some $B_0 > 0$, for all $L > L_0$ and $s \geq 0$. Further, for any $T$ finite there exists $B(T) < \infty$ (depending on $r$), such that $K_L(s) \leq r + B(T)L^{-1}$ for all $s \leq T$ and $L \geq \max\{B(T), L_0\}$.

**Proof of Lemma 4.2.** We first deal with the lower bound on $K_L(\cdot)$. Fix $L > 0$ and let $g_L(x) := 1 - 2xf'_L(x) + 1 + 4Lx(r - x) = \left(\frac{r}{x}\right)^2 - 2ahx \leq 0$. Let $x_L$ be the largest root of $g_L(x)$ smaller than $r$. It is easy to see that $g_L(r) < 0$ and also that $\lim_{L \to \infty} g_L(r/2) > 0$, so there exist $L_0 > 0$ such that $x_L > r/2$ whenever $L > L_0$.

Furthermore,
\[
L(r - x_L) = -\frac{1}{4x_L} + \left(\frac{x_L}{r}\right)^2 \frac{1}{4x_L} + \frac{2ahx_L}{r} \leq B_0
\]
for $B_0 = 4r^{-1} + 2oh$. By Lemma 4.1, we know that the functions $R_L(\cdot, \cdot)$, $C_L(\cdot, \cdot)$ and $M_L(\cdot)$ are non negative, as is $\psi(x)$ for $x \geq 0$, so from (2.10) we get the lower bound $\partial K_L(s) \geq g_L(K_L(s))$. Since $K_L(0) = r$, it follows that $K_L(s) \geq x_L$, for all $x \geq 0$, so $K_L(s) \geq r - B_0L^{-1}$, for $L \geq L_0$.

Turning now to the complementary upper bound, recall that $\psi(x)$ is a polynomial of degree $m - 1$, hence there exists $\kappa < \infty$ such that $\psi(ab) \leq \kappa(1 + a^2)^{m/2}(1 + b^2)^{m/2}$ for all $a, b$. Thus, by (2.11), the monotonicity of $\psi(x)$ on $\mathbb{R}_+$ and the non-negative definiteness of $C_L(s, u)$ we have that for any $s, t, u \geq 0$,
\[
\psi(C_L(s, u)) \leq \kappa(1 + K_L(u))^{\frac{m}{2}} (1 + K_L(s))^{\frac{m}{2}}
\]
and
\[
\int_0^t R_L(s, u)du \leq \sqrt{tK_L(s)}, \quad K_L(t) \leq \sqrt{K_L(t)}
\]
and from (2.10) we find that
\[
(4.2) \quad \partial K_L(s) \leq g(K_L(s)) + 2\beta^2 \kappa \left(1 + \sup_{u \leq s} K_L(u)\right)^m \sqrt{K_L(s)} \sqrt{s} + 2h \sqrt{K_L(s)}.
\]

Setting now $B(T) = \frac{1}{2r} \left(1 + 2\sqrt{r + 1} \beta^2 \kappa(r + 2)^m \sqrt{T} + 2\sqrt{r + 1}h\right)$ and fixing $T < \infty$ and $L \geq \max\{L_0, B(T)\}$, let
\[
\tau := \inf\{u \geq 0 : K_L(u) \geq r + B(T)L^{-1}\}.
\]
By the continuity of $K_L(\cdot)$ and the fact that $K_L(0) = r < r + B(T)L^{-1}$, we have that $\tau > 0$ and further, if $\tau < \infty$ then necessarily
\[
K_L(\tau) = \sup_{u \leq \tau} K_L(u) = r + B(T)L^{-1} \leq r + 1.
\]
Recall that $g_L(x) \leq 1 + 4Lx(r - x)$, whereas from (4.2) we see that if $\tau < \infty$ then
\[
\partial K_L(\tau) \leq 1 - 4K_L(\tau)B(T) + 2\sqrt{r + 1} \beta^2 \kappa(r + 2)^m \sqrt{T} + 2\sqrt{r + 1}h
\]
\[
= 2rB(\tau) - 4K_L(\tau)B(T) \leq 2rB(\tau) - 2rB(T).
\]
where the last inequality holds since $L \geq L_0$ implies $K_L(s) \geq r/2$, as previously shown. Recall the definition of $\tau < \infty$ implying that $\partial K_L(\tau) \geq 0$. Hence the above inequality implies $B(\tau) \geq B(T)$, hence $\tau > T$, for our choice of $B = B(T)$. That is, $K_L(s) \leq r + B(T)L^{-1}$ for all $s \leq T$ and $L \geq \max\{B(T), B_0\}$, as claimed. □

Let $\mu_L(s) = f^*_L(K_L(s), h_L(s) = \partial K_L(s))$. Fixing hereafter $T < \infty$ (recall $r > 0$ is fixed) and denoting $\tilde{L} = \max\{L_0, B(T)\}$, we next prove the equi-continuity and uniform boundedness of $(M_L, R_L, C_L, Q_L, K_L, \mu_L, h_L)$, en-route to having limit points for $(M_L, R_L, C_L, Q_L, K_L, \mu_L)$.

**Lemma 4.3.** The continuous functions $M_L(s), K_L(s), \mu_L(s), h_L(s)$ and their derivatives are bounded uniformly in $L \geq \tilde{L}$ and $0 \leq s \leq T$. The same is true for $C_L(s,t), Q_L(s,t)$ in $L \geq \tilde{L}$ and $0 \leq s, t \leq T$ and also for $R_L(s,t)$ in $L \geq \tilde{L}$ and $0 \leq t \leq s \leq T$.

**Proof of Lemma 4.3**. Recall that by Lemma 4.2 for any $L \geq \tilde{L}$,

$$\sup_{s \leq T} |K_L(s) - r| \leq \frac{\tilde{B}}{L},$$

where $\tilde{B} = \max\{B(T), B_0\}$. Consequently, the collections $\{C_L(s,t), 0 \leq s, t \leq T, L \geq \tilde{L}\}$ and $\{Q_L(s,t), 0 \leq s, t \leq T, L \geq \tilde{L}\}$ are uniformly bounded (since both $|C_L(s,t)|$ and $|Q_L(s,t)|$ are bounded above by $\sqrt{K_L(s)K_L(t)}$) and also $\{M_L(s), 0 \leq s \leq T, L \geq \tilde{L}\}$ (since $M_L(s) \leq \sqrt{K_L(s)}$). By (4.3) and our choice of $f^*_L(r)$, we have that

$$|\mu_L(s)| \leq 2L|K_L(s) - r| + \left(\frac{K_L(s)}{r}\right) 2^{k-1} \leq 2\tilde{B} + \left(\frac{r + 1}{r}\right) 2^{k-1}, \quad \forall L \geq \tilde{L}, s \leq T.$$

By (4.1), the collection $\{H_L(s,t), 0 \leq t \leq s \leq T, L \geq \tilde{L}\}$ is also uniformly bounded and since $R_L(s,t) = H_L(s,t) \exp\left(-\int_t^s \mu_L(u)du\right)$, the collection $\{R_L(s,t), 0 \leq t \leq s \leq T, L \geq \tilde{L}\}$ is also uniformly bounded. Further, since by (2.10):

$$(4.4) \quad h_L(s) = 1 - 2K_L(s)\mu_L(s) + 2\beta^2 \int_0^s \psi(C_L(s,u))R_L(s,u)du + 2hM_L(s),$$

it follows from the uniform boundedness of $K_L, M_L, \mu_L, C_L$ and $R_L$ that $\{h_L(s), s \in [0,T], L \geq \tilde{L}\}$ is also uniformly bounded. By the same reasoning, from (2.6), (2.7), (2.8), and (2.9), we deduce that $\partial_M(s)$, $\partial C_L(s,t), \partial_t R_L(s,t), \partial_t Q_L(s,t)$ and $\partial D L(s,t)$ are bounded uniformly in $L \geq \tilde{L}$ and $s, t \in [0,T]$.

Next, differentiating the identity (4.1) with respect to $t$, we get for $f = f_L$ that

$$\partial_t H_L(s,t) = \sum_{n \geq 1} \beta_{2n} \sum_{\sigma \in NC_n} \int_{t=t_1 \leq t_2 \leq \cdots \leq t_{2n} \leq s} \prod_{i \in \sigma} \nu''(C_L(t_i, t_{\sigma(i)})) \prod_{j=2}^{2n} dt_j,$$

where $NC_n$ denotes the finite set of non-crossing involutions of $\{1, \ldots, 2n\}$ without fixed points. With the Catalan number $|NC_n|$ bounded by $4^n$, and since $C_L(t_i, t_{\sigma(i)}) \in [0, r + 1]$ for $t_i, t_{\sigma(i)} \leq T, L \geq \tilde{L}$, we thus deduce by the monotonicity of $x \mapsto \nu''(x)$ that

$$0 \leq \partial_t H_L(s,t) \leq \sum_{n \geq 1} \frac{\beta_{2n}}{(2n - 1)!} 4^n (\nu''(r + 1))^n (s-t)^{2n-1},$$

so $\partial_t H_L(s,t)$ is finite and bounded uniformly in $L \geq \tilde{L}$ and $0 \leq s \leq t \leq T$. Since

$$\partial_t R_L(s,t) = \mu_L(t)R_L(s,t) + e^{-\int_t^s \mu_L(u)du} \partial_t H_L(s,t),$$

we thus have that $|\partial_t R_L(s,t)|$ is also bounded uniformly in $L \geq B(T)$ and $0 \leq t \leq s \leq T$.

Also, due to the symmetry of $C_L, \partial_2 C_L(s,t) = \partial_1 C_L(t,s)$, hence $\partial_2 C_L(s,t)$ is also bounded uniformly in $L \geq \tilde{L}$ and $0 \leq s, t \leq T$. The same argument applied to $Q$, will show that $\partial_2 Q_L(s,t)$ is also bounded uniformly in $L \geq \tilde{L}$ and $0 \leq s, t \leq T$. 
Turning to deal with $\partial h_L(s)$, setting $g_L(x):=|f'_L(x)x'| - 2rL = 4L(x-r) + \frac{k}{r} \left(\frac{x}{r}\right)^{2k-1} + \frac{ah}{r}$, we deduce from (4.3) that $|g_L(K_L(s))| \leq 4B + \frac{k}{r} \left(\frac{r}{L}\right)^{2k-1} + \frac{ah}{r}$ for any $s \leq T$ and $L \geq \bar{L}$. Differentiating (4.4) we find that $\partial h_L(s) = -4rLh_L(s) + \kappa_L(s)$ for

$$\kappa_L(s) = -2g_L(K_L(s))h_L(s) + 2\beta^2 \frac{\partial}{\partial s} \left( \int_0^s \psi(C_L(s,u))R_L(s,u)du \right) + 2h\partial M_L(s),$$

which is thus bounded uniformly in $L \geq B(T)$ and $s \leq T$ (in view of the uniform boundedness of $h_L$, $C_L$, $R_L$, $\partial_1 C_L$, $\partial_1 R_L$ and $\partial M_L$). Further, recall that $K_L(0) = r$, so by (2.10) and our choice of $f_L(\cdot)$ we have that $h_L(0) = 1 - 2rf'_L(r) + 2ha = 0$, resulting with

$$h_L(s) = \int_0^s e^{-4rL(s-u)}\kappa_L(u)du.$$

hence for $L \geq \bar{L}$,

$$\sup_{s \leq T} |h_L(s)| \leq \sup\{|\kappa_L(u)| : L \geq \bar{L}, u \leq T\} = \frac{A(T)}{4Lr} < \infty,$$

where $A(T) := \sup\{|\kappa_L(u)| : L \geq \bar{L}, u \leq T\} < \infty$ and the uniform boundedness of $|\partial h_L(s)|$ follows.

Finally, by definition, $\partial \mu_L(s) = f''_L(K_L(s))h_L(s)$, yielding for our choice of $f_L$ that

$$|\partial \mu_L(s)| \leq \left(2L + \frac{2k-1}{2r^2} \left(\frac{r+1}{r}\right)^{2k-2}\right)|h_L(s)|, \quad \forall L \geq \bar{L}, s \leq T,$$

which by (4.5) provides the uniform boundedness of $|\partial \mu_L(s)|$.

**Proof of Theorem 2.3.** In Lemma 4.3 we have established that the functions $(M_L(s), R_L(s,t), C_L(s,t), Q_L(s,t))$, $L \geq \bar{L}$ are equi-continuous and uniformly bounded on their respective domains for $0 \leq s, t \leq T$. Further, $(K_L(s), \mu_L(s), h_L(s))$ are equi-continuous and uniformly bounded on $s \in [0, T]$. By the Arzela-Ascoli theorem, the collection $(M_L, R_L, C_L, Q_L, K_L, \mu_L, h_L)$ has a limit point $(M, R, C, Q, K, \mu, h)$ with respect to uniform convergence on $[0, T] \times (\Gamma \cap [0, T]^2) \times [0, T]^2$.

By Lemma 4.2 we know that the limit $K(s) = r$ for all $s \leq T$, whereas by (4.5) we have that $h(s) = 0$ for all $s \leq T$. Consequently, considering (4.4) for the subsequence $L_n \to \infty$ for which $(M_{L_n}, R_{L_n}, C_{L_n}, Q_{L_n}, K_{L_n}, \mu_{L_n}, h_{L_n})$ converges to $(M, R, C, Q, K, \mu, h)$ we find that the latter must satisfy (2.17) for $k = 1$. Further, recalling that $R_L(t, t) = 1$, $C_L(t, t) = K_L(t)$, integrating (2.6), (2.7), (2.8) and (2.9) we find that $M_L(s) = M_L(0) + \int_0^s \tilde{M}_L(\theta)d\theta$ and $V_L(s, t) = V_L(t, t) + \int_t^s \tilde{V}_L(\theta, t)d\theta$, for $V$ any of the functions $R, C$ or $Q$, where:

$$\tilde{M}_L(\theta) = -\mu_L(\theta)M_L(\theta) + \beta^2 \int_0^\theta M_L(u)R_L(\theta, u)\nu''(C_L(\theta, u))du + h$$

$$\tilde{R}_L(\theta, t) = -\mu_L(\theta)R_L(\theta, t) + \beta^2 \int_t^\theta R_L(u, t)R_L(\theta, u)\nu''(C_L(\theta, u))du,$$

$$\tilde{C}_L(\theta, t) = -\mu_L(\theta)C_L(\theta, t) + \beta^2 \int_0^\theta C_L(u, t)R_L(\theta, u)\nu''(C_L(\theta, u))du + \beta^2 \int_0^t \nu'(C_L(\theta, u))R_L(t, u)du + hM_L(t),$$

$$\tilde{Q}_L(\theta, t) = -\mu_L(\theta)Q_L(\theta, t) + \beta^2 \int_0^\theta Q_L(u, t)R_L(\theta, u)\nu''(C_L(\theta, u))du + \beta^2 \int_0^t \nu'(Q_L(\theta, u))R_L(t, u)du + hM_L(t)$$.
The system (2.13)-(2.17) thus becomes:

\[ \mu(M, R, C, Q, \mu), \]

and the initial conditions satisfy \( (M, R, C, Q, \mu) \) of (2.13)–(2.17), as claimed. □

5. Convergence to the Pure Spin Model

Let \((M_r, R_r, C_r, Q_r)\) be the solution of (2.13)-(2.17), for \( h_r = h r^{m/2} \) and the initial conditions \( R_r(t, t) = 1, C_r(t, t) = 0 \), \( M_r(0) = 0 \). Set:

\[ \tilde{\mu}_r(s) = \frac{\mu(s r^{1-m/2})}{r^{m/2-1}}. \]

and recall the definitions used in Theorem 2.4:

\[ M_r(s) = \frac{M_r(s r^{1-m/2})}{r^{m/2-1}}, \]

\[ R_r(s, t) = R_r(s r^{1-m/2}, t r^{1-m/2}) \]

\[ C_r(s, t) = C_r(s r^{1-m/2}, t r^{1-m/2}) \]

\[ Q_r(s, t) = Q_r(s r^{1-m/2}, t r^{1-m/2}) \]

The system (2.13)-(2.17) thus becomes:

\[ \begin{align*}
& (5.1) \quad \partial_t \tilde{M}_r(s) = -\tilde{\mu}_r(s) \tilde{M}_r(s) + h + \beta^2 \int_0^s \tilde{M}_r(u) \tilde{R}_r(s, u) \frac{\mu''(r \tilde{C}_r(s, u))}{r^{m-2}} du, \quad s \geq 0 \\
& (5.2) \quad \partial_t \tilde{R}_r(s, t) = -\tilde{\mu}_r(s) \tilde{R}_r(s, t) + \beta^2 \int_t^s \tilde{R}_r(u, t) \tilde{R}_r(s, u) \frac{\mu''(r \tilde{C}_r(s, u))}{r^{m-2}} du, \quad s \geq t \geq 0 \\
& (5.3) \quad \partial_t \tilde{C}_r(s, t) = -\tilde{\mu}_r(s) \tilde{C}_r(s, t) + \beta^2 \int_0^s \tilde{C}_r(u, t) \tilde{R}_r(s, u) \frac{\mu''(r \tilde{C}_r(s, u))}{r^{m-2}} du \\
& \quad + \beta^2 \int_0^t \frac{\nu'(r \tilde{C}_r(s, u))}{r^{m-1}} R_r(t, u) du + h \tilde{M}_r(t), \quad s \geq t \geq 0 \\
& (5.4) \quad \partial_t \tilde{Q}_r(s, t) = -\tilde{\mu}_r(s) \tilde{Q}_r(s, t) + \beta^2 \int_0^s \tilde{Q}_r(u, t) \tilde{R}_r(s, u) \frac{\mu''(r \tilde{C}_r(s, u))}{r^{m-2}} du \\
& \quad + \beta^2 \int_0^t \frac{\nu'(r \tilde{Q}_r(s, u))}{r^{m-1}} R_r(t, u) du + h \tilde{M}_r(t), \quad s, t \geq 0
\end{align*} \]

where

\[ \tilde{\mu}_r(s) = \frac{1}{2 r^{m/2}} + \beta^2 \int_0^s \frac{\psi(r \tilde{C}_r(s, u))}{r^{m-1}} \tilde{R}_r(s, u) du + h \tilde{M}_r(s). \]

and \( \tilde{C}_r(t, t) = \tilde{R}_r(t, t) = \tilde{Q}_r(0, 0) = 1, \tilde{M}_r(0) = \alpha, \tilde{C}_r(t, s) = \tilde{C}_r(s, t) \) and \( \tilde{Q}_r(t, s) = \tilde{Q}_r(s, t) \).

Fixing \( T < \infty \), the first step of the proof is to establish, in Lemma 5.1 that the function \( \tilde{M}_r, \tilde{R}_r, \tilde{C}_r, \tilde{Q}_r \) and \( \tilde{\mu}_r \) are equi-continuous and uniformly bounded. Then we will be able to use Arzela-Ascoli theorem to establish the desired limits.
Lemma 5.1. The continuous functions $\tilde{M}_r(s), \tilde{\mu}_r(s), \tilde{C}_r(s,t)$ and $\tilde{Q}_r(s,t)$ and their derivatives are uniformly bounded in $r \geq 1$ and $0 \leq s, t \leq T$. The same is true for $\tilde{R}_r(s,t)$ in $r \geq 1$ and $0 \leq s, t \leq T$.

Proof of Lemma 5.1. Recall Theorem 2.3 implies that $C_r(s,t), Q_r(s,t), M_r^2(s) \in [0,r]$, for all $0 \leq s, t \leq T$. Hence, by construction, $\tilde{C}_r(s,t), \tilde{Q}_r(s,t)$ and $\tilde{M}_r^2(s)$ take values in the interval $[0,1]$, for every $r > 0$, thus showing the uniform boundedness of $\tilde{C}_r(s,t), \tilde{Q}_r(s,t)$ and $\tilde{M}_r(s)$ on $0 \leq s, t \leq T$ and $r \geq 1$.

Also notice that $\tilde{R}_r(s,t) = \tilde{H}_r(s,t) \exp(-\int_s^t \tilde{\mu}_r(u)du)$, for $s \geq t$, where, by (5.1), $\tilde{H}_r(s,t)$ satisfies:

\begin{equation}
\tilde{H}_r(s,t) = 1 + \sum_{n \geq 1} \beta^{2n} \sum_{\sigma \in NC_n} \int_{t \leq t_1 \leq \cdots \leq t_{2n} \leq s} \prod_{i \in \sigma} \overline{\nu''(r\tilde{C}_r(t_i, t_{\sigma(i)}))} \left( \sum_{j=1}^{2n} dt_j \right).
\end{equation}

Since $\nu''(x)$ is a polynomial of degree $m-2$, there exist an universal constant $K_1$ (depending on $\nu''$) such that, for any $r \geq 1$, and $x \in [0,1]$, $\frac{\nu''(r x)}{r^{m-2}} < K_1$. Hence the collection $\{\tilde{H}_r(s,t), 0 \leq t \leq s \leq T, r \geq 1 \}$ is also uniformly bounded (since $\tilde{C}_r(t_i, t_{\sigma(i)}) \in [0,1]$). Since $\tilde{\mu}_r(s,t)$ is uniformly bounded, $\{\tilde{\mu}_r(s,t), 0 \leq s \leq t \leq T, r \geq 1 \}$ is uniformly bounded.

Since $\psi(x)$ is a polynomial of degree $m-1$, there exist an universal constant $K_2$ (depending on $\psi$) such that, for any $r \geq 1$, and $x \in [0,1]$, $\frac{\psi(x)}{r^{m-2}} < K_2$. Since in addition $\tilde{\mu}_r(s,t) \geq 0$, (5.5) implies that the family $\{\tilde{\mu}_r(s,t), 0 \leq s \leq T, r \geq 1 \}$ is uniformly bounded.

Moving over to the partial derivatives, since by (5.1):

\begin{equation}
\partial_r \tilde{M}_r(s) = -\tilde{\mu}_r(s) \tilde{M}_r(s) + \beta^2 \int_0^s \tilde{M}_r(u) \tilde{R}_r(s,u) \frac{\nu''(r\tilde{C}_r(s,u))}{r^{m-2}} du + h,
\end{equation}

it follows from the uniform boundedness of $\tilde{\mu}_r, \tilde{M}_r, \tilde{R}_r$ and $\tilde{C}_r$ that the family $\{\partial_r \tilde{M}_r(s), 0 \leq s \leq T, r \geq 1 \}$ is also uniformly bounded. By similar reasoning, using (5.2), (5.3) and (5.4), we show that $\partial_t \tilde{R}_r(s,t), \partial_t \tilde{C}_r(s,t)$ and $\partial_t \tilde{Q}_r(s,t)$ are uniformly bounded in $r \geq 1$ and $0 \leq t \leq s \leq T$ (or $s,t \in [0,T]$), whichever is relevant.

Now, differentiating the identity (5.6) with respect to $t$, we get

\begin{equation}
\partial_2 \tilde{H}_r(s,t) = \sum_{n \geq 1} \beta^{2n} \sum_{\sigma \in NC_n} \int_{t \leq t_1 \leq \cdots \leq t_{2n} \leq s} \prod_{i \in \sigma} \overline{\nu''(r\tilde{C}_r(t_i, t_{\sigma(i)}))} \left( \sum_{j=2}^{2n} dt_j \right),
\end{equation}

where $NC_n$ denotes the finite set of non-crossing involutions of $\{1, \ldots, 2n\}$ without fixed points. With the Catalan number $|NC_n|$ bounded by $4^n$, and since $0 \leq \frac{\nu''(r\tilde{C}_r(t_i, t_{\sigma(i)}))}{r^{m-2}} \leq K_1$ for $0 \leq t_i, t_{\sigma(i)} \leq T$ and $r \geq 1$, we thus deduce that

\begin{equation}
0 \leq \partial_2 \tilde{H}_r(s,t) \leq \sum_{n \geq 1} \frac{\beta^{2n}}{(2n-1)!} (4K_1)^n (s-t)^{2n-1},
\end{equation}

so $\partial_2 \tilde{H}_r(s,t)$ is finite and bounded uniformly when $r \geq 1$ and $0 \leq t \leq s \leq T$. Since

\begin{equation}
\partial_2 \tilde{R}_r(s,t) = \tilde{\mu}_r(t) \tilde{R}_r(s,t) + e^{-\int_t^s \tilde{\mu}_r(u)du} \partial_2 \tilde{H}_r(s,t),
\end{equation}

we thus have that $|\partial_2 \tilde{R}_r(s,t)|$ is also bounded uniformly in $r \geq 1$ and $0 \leq t \leq s \leq T$. Also, since $\tilde{C}_r$ is symmetric, $\partial_2 \tilde{C}_r(s,t) = \partial_1 \tilde{C}_r(t,s)$, hence $\partial_2 \tilde{C}_r(s,t)$ is also bounded uniformly in $r \geq 1$ and $s \leq t \leq T$. Since $\tilde{Q}_r$ is also symmetric, we derive the same conclusion about $\partial_2 \tilde{Q}_r(s,t)$.

Finally, by (5.5),

\begin{equation}
\partial \tilde{\mu}_r(s) = \beta^2 \int_0^s \left[ \frac{\psi'(r\tilde{C}_r(s,u))}{r^{m-2}} \partial_1 \tilde{C}_r(s,u) \tilde{R}_r(s,u) + \frac{\psi(r\tilde{C}_r(s,u))}{r^{m-1}} \partial_1 \tilde{R}_r(s,u) \right] du + \frac{\psi(r)}{r^{m-1}} + h \partial \tilde{M}_r(s).
\end{equation}
and since $\psi(x)$ is a polynomial of order $m - 1$, it follows that $\psi(rx)/r^{m-1}$ and $\psi'(rx)/r^{m-2}$ are uniformly bounded in $r \geq 1$, $x \in [0,1]$. Since $\partial_t \tilde{C}_r$, $\partial_t \tilde{R}_r$, $\partial_t \tilde{M}_r$, and $\tilde{R}_r$ are uniformly bounded and $\tilde{C}_r(s,u) \in [0,1]$, it follows that the functions $\partial_t \tilde{\mu}_r(s)$ are uniformly bounded on $0 \leq s \leq T$ and $r \geq 1$, thus concluding the proof.

**Proof of Theorem 2.4.** In Lemma 5.1 we have established that the functions $\tilde{M}_r(s)$, $\tilde{R}_r(s,t)$, $\tilde{C}_r(s,t)$, $\tilde{Q}_r(s,t)$ and $\tilde{\mu}_r(s)$ are equi-continuous and uniformly bounded for $r \geq 1$. By the Arzela-Ascoli theorem, the collection $(\tilde{M}_r, \tilde{R}_r, \tilde{C}_r, \tilde{Q}_r, \tilde{\mu}_r)$ has a limit point $(M, R, C, Q, \mu)$ with respect to uniform convergence on $C^1[0,T] \times C^1([0,T]^2) \times C^1([0,T]^2) \times C^1([0,T]^2) \times C^1[0,T]$. Let $r_n$ be an increasing sequence going to infinity, such that $(M_{r_n}, \tilde{R}_{r_n}, \tilde{C}_{r_n}, \tilde{Q}_{r_n}, \tilde{\mu}_{r_n})$ converges uniformly to $(M, R, C, Q, \mu)$.

Now, since $\tilde{C}_r(\theta, u) \in [0,1]$, for all $r \geq 1$ and $\theta, u \geq 0$, the same is true for its limit point $C(\theta, u)$. Since $\nu(\cdot)$ is a polynomial of degree $m$ with the dominant coefficient $a_m$, $\psi(x) = \nu'(x) + 2x\nu''(x)$ is a degree $m - 1$ polynomial with dominant coefficient $a_m - \frac{a_m}{m-1} + \frac{a_m}{m-2}$. Recalling that $\tilde{\psi}(x) = \left[\frac{a_m}{m-1} + \frac{a_m}{m-2}\right] x^{m-1}$, we can easily see that there exist constant $K_3$ (depending only on $\nu(\cdot)$), such that

$$\sup_{0 \leq \theta, r \leq T} \left| \frac{\psi(r\tilde{C}_r(\theta, u))}{r^{m-1}} - \frac{\psi(C(\theta, u))}{r^{m-1}} \right| \leq K_3 \frac{\tilde{C}_r(\theta, u) - C(\theta, u)}{r} \leq K_4 \tilde{C}_r(\theta, u),$$

for every $r$. Altogether, we have shown that:

$$\frac{\psi(r_n\tilde{C}_{r_n}(\theta, u))}{r_n^{m-1}} \xrightarrow{n \to \infty} \tilde{\psi}(C(\theta, u)),$$

and the convergence is uniform on $[0,T]^2$. Using this result, together with the uniform convergence of $\tilde{C}_{r_n}, \tilde{R}_{r_n}$ and $\tilde{M}_{r_n}$, we conclude that $\tilde{\mu}_{r_n}(s)$, as it is defined in (5.5), converges to the right hand side of (2.17) for $k = 0$ and the convergence is uniform on $[0,T]$.

Furthermore, since $\tilde{R}_r(t,t) = 1$, $\tilde{C}_r(t,t) = 1$, $\tilde{M}_r(0) = \alpha$ and $\tilde{Q}_r(0,0) = 1$ integrating (5.1), (5.2), (5.3) and (5.4) we find that $\tilde{M}_r(s) = \alpha + \int_0^s \tilde{M}_r(\theta)d\theta$ and $\tilde{V}_r(s,t) = \tilde{V}_r(t,t) + \int_t^s \tilde{V}_r(\theta,t)d\theta$, for $V$ any of the functions $\tilde{R}$, $\tilde{C}$ or $\tilde{Q}$, for:

$$\tilde{M}_r(\theta) = - \tilde{\mu}_r(\theta)\tilde{M}_r(\theta) + \beta^2 \int_0^\theta \tilde{M}_r(u,\theta)\tilde{R}_r(u,\theta)\psi''(\tilde{C}_r(u,\theta))\frac{1}{r^{m-2}} du + h,$$

$$\tilde{R}_r(\theta,t) = - \tilde{\mu}_r(\theta)\tilde{R}_r(\theta,t) + \beta^2 \int_0^\theta \tilde{R}_r(u,t,\theta)\tilde{R}_r(\theta,u)\psi''(\tilde{C}_r(u,\theta))\frac{1}{r^{m-2}} du,$$

$$\tilde{C}_r(\theta,t) = - \tilde{\mu}_r(\theta)\tilde{C}_r(\theta,t) + \beta^2 \int_0^\theta \tilde{C}_r(u,t,\theta)\tilde{R}_r(\theta,u)\psi''(\tilde{C}_r(u,\theta))\frac{1}{r^{m-2}} du + \beta^2 \int_0^\theta \tilde{C}_r(u,t,\theta)\tilde{R}_r(u,\theta)\psi''(\tilde{C}_r(u,\theta))\frac{1}{r^{m-2}} du + \tilde{M}_r(t),$$

$$\tilde{Q}_r(\theta,t) = - \tilde{\mu}_r(\theta)\tilde{Q}_r(\theta,t) + \beta^2 \int_0^\theta \tilde{Q}_r(u,t,\theta)\tilde{R}_r(\theta,u)\psi''(\tilde{C}_r(u,\theta))\frac{1}{r^{m-2}} du + \beta^2 \int_0^\theta \tilde{Q}_r(u,t,\theta)\tilde{R}_r(u,\theta)\psi''(\tilde{C}_r(u,\theta))\frac{1}{r^{m-2}} du + \tilde{M}_r(t).$$
Similar arguments as employed earlier will show that:

\[ \frac{\nu''(r\tilde{C}_{r_n}(\theta, u))}{r_n^m}, \quad \frac{\nu'(r\tilde{C}_{r_n}(\theta, u))}{r_n^{m-1}} \xrightarrow{n \to \infty} \tilde{\nu}'(C(\theta, u)), \]

and the same for \( \nu''(Q(\theta, u)) \) and \( \nu'(Q(\theta, u)) \), where the convergence is uniform on \([0,T]^2\). Using this result, together with the uniform convergence of the quad-uple \((M_{r_n}, \tilde{R}_{r_n}, \tilde{C}_{r_n}, \tilde{\mu}_{r_n})\), we conclude that \( M_{r_n}(s) \) converges to the right hand side of \((2.13)\) and the convergence is uniform on \([0,T]\).

Similarly we show that \( \tilde{R}_{r_n}(s, t), \tilde{C}_{r_n}(s) \) and \( \tilde{Q}_{r_n}(s, t) \) converge uniformly on \((s, t) \in [0,T]^2\) to the right hand sides of \((2.14), (2.15)\) and \((2.16)\), respectively. Thus, we see that for each limit point \((M, R, C, Q, \mu)\), the functions \( M(s), \tilde{R}(s, t), \tilde{C}(s, t) \) and \( Q(s, t) \) are differentiable in \(s\) on \(0 \leq s, t \leq T\) and all limit points satisfy the equations \((2.13)-(2.17)\). Since \( \tilde{R}_{r_n}(t, t) = 1 \) and the functions \( \tilde{Q}_{r_n} \) and \( \tilde{C}_{r_n} \) are non-negative definite symmetric kernels, the same applies for any limit point \( R, Q \) or \( C \).

Finally, using a Gronwell-type argument similar to the one employed in Lemma \(\ref{lemma:uniform_conv} \), we show that there exist at most one bounded solution \((M, R, C, Q, \mu)\) on \(C^1([0,T]) \times C^1([0,T]^2) \times C^1([0,T]^2) \times C^1([0,T]^2)\) to the system \((2.13)-(2.17)\), with initial conditions \(C(t, t) = R(t, t) = Q(0, 0) = 1\) and \(M(0) = \alpha \in (0, 1)\).

In conclusion, when \(r \to \infty\) the collection \((M_{r_n}, \tilde{R}_{r_n}, \tilde{C}_{r_n}, \tilde{Q}_{r_n}, \tilde{\mu}_{r_n})\) converges towards the unique solution \((M, R, C, Q, \mu)\) of \((2.13)-(2.17)\), as claimed.

6. FDT regime

6.1. Proof Preliminaries. The arguments that are used for the cases \(\beta\) and \(h\) small and, respectively, \(\gamma\) small, are very similar, and we will be treating them in parallel. On the high level, we will use a perturbation argument based on the stability of linear and respectively Riccati differential equations. From now on, we will refer to the case when \(\gamma = \frac{\beta}{h}\) is small as the first case and when both \(\beta\) and \(h\) are small as the second case.

First, notice that, since \(r = 1\), making the substitution \(V_h(s, t) = U(s/h, t/h)\), for \(U\) any of \(R, C\) or \(Q\) and \(V_h(s) = V(s/h)\) for \(V\) any of \(M\) or \(D\), the equations \((2.13)-(2.17)\) are transformed to:

\[
\begin{align*}
\partial M_h(s) &= -\mu_h(s)M_h(s) + 1 + \gamma^2 \int_0^s M_h(u)R_h(s, u)\nu''(C_h(s, u))du, \quad s \geq 0 \\
\partial R_h(s, t) &= -\mu_h(s)R_h(s, t) + \gamma^2 \int_t^s R_h(u, t)R_h(s, u)\nu''(C_h(s, u))du, \quad s \geq t \geq 0 \\
\partial C_h(s, t) &= -\mu_h(s)C_h(s, t) + \gamma^2 \int_0^s C_h(u, t)R_h(s, u)\nu''(C_h(s, u))du \\
&\quad + \gamma^2 \int_0^t \nu'(C_h(s, u))R_h(t, u)du + M_h(t), \quad s \geq t \geq 0 \\
\partial Q_h(s, t) &= -\mu_h(s)Q_h(s, t) + \gamma^2 \int_0^s Q_h(u, t)R_h(s, u)\nu''(C_h(s, u))du \\
&\quad + \gamma^2 \int_0^t \nu'(Q_h(s, u))R_h(t, u)du + M_h(t), \quad s, t \geq 0
\end{align*}
\]

with

\[
\mu_h(s) = \frac{1}{2h} + M_h(s) + \gamma^2 \int_0^s \psi(C_h(s, u))R_h(s, u)du.
\]

From now on, we will be interested in the behavior of the functions \(C(s, t)\) and \(Q(s, t)\) only for \(s \geq t\) (the rest of the plane will be automatically given, by symmetry). We do need, however, to specify initial conditions for \(Q(\cdot, \cdot)\). Defining \(D(s) := Q(s, s)\), due to the symmetry of \(Q\), the function \(D\) will satisfy \(\partial D(s) = 2\partial Q(s, s),\)
hence:
\[ \frac{\partial D(s)}{2} = -\mu(s)D(s) + hM(s) + \beta^2 \int_s^\infty Q(u,s)R(s,u)\nu''(C(s,u))du \]
+ \beta^2 \int_0^s \nu'(Q(s,u))R(s,u)du \]

hence it’s time transform, \( D_h(s) := Q_h(s,s) \) solves:
\[ \frac{\partial D_h(s)}{2} = -\mu_h(s)D_h(s) + M_h(s) + \gamma^2 \int_0^s Q_h(u,s)R_h(s,u)\nu''(C_h(s,u))du \]
+ \gamma^2 \int_0^s \nu'(Q_h(s,u))R_h(s,u)du, \]

In the course of the proof, we will establish that, when either \( \gamma \) is small or both \( \beta \) and \( h \) are small, the limits:
\[ M^{\text{fict}} = \lim_{t \to \infty} M(t) , \]
\[ R^{\text{fict}}(\tau) = \lim_{t \to \infty} R(t + \tau, t) , \]
\[ C^{\text{fict}}(\tau) = \lim_{t \to \infty} C(t + \tau, t) , \]
\[ Q^{\text{fict}}(\tau) = \lim_{t \to \infty} Q(t + \tau, t) , \]

are well-defined for \( \tau \geq 0 \) and, furthermore, that \( R^{\text{fict}} \) decays to 0 exponentially fast (i.e. \( 0 \leq R^{\text{fict}}(\tau) \leq K_1 e^{-K_2 \tau} \)), for some positive constants \( K_1 \) and \( K_2 \) depending on \( \beta \), \( h \) and \( \alpha = M(0) \).

Notice that if the FDT limits exist for the functions \( M_h, R_h, C_h \) and \( Q_h \), the same is true for the functions \( M, R, C \) and \( Q \). We will establish (6.8)-(6.11) for \( (M_h, R_h, C_h, Q_h) \), in the first case, and for \( (M, R, C, Q) \) in the second. Also, until further notice, we will drop the \( h \) subscript in the regime when \( \gamma \) is small (i.e. the first case).

Recalling our notation \( \Gamma = \{ (s, t) : 0 \leq t \leq s \} \subset \mathbb{R}_+ \times \mathbb{R}_+ \), consider the maps \( \Psi_i : (M, R, C, Q) \mapsto (\tilde{M}_i, \tilde{R}_i, \tilde{C}_i, Q_i), i = 1, 2 \), on
\[ A = \{ (M, R, C, Q) \in C^1(\mathbb{R}_+) \times C^1(\Gamma) \times C^1(\mathbb{R}_+^2) \times C^1(\mathbb{R}_+^2) \mid M(0) = \alpha \in (0, 1], \]
\[ R(t, t) = C(t, t) = Q(0, 0) = 1, \ C(s, t) = C(t, s), \ Q(s, t) = Q(t, s) \}, \]

such that for \( s \geq 0 \),
\[ \partial \tilde{M}_1(s) = -\left( \frac{1}{2h} + \tilde{M}_1(s) \right) \tilde{M}_1(s) + 1 \]
+ \( \gamma^2 \left( \int_0^s M(u)R(s,u)\nu''(C(s,u))du - M(s)\int_0^s \psi(C(s,u))R(s,u)du \right) \]
\[ \partial \tilde{M}_2(s) = -\mu_2(s)\tilde{M}_2(s) + \tilde{h}^2 \int_0^s M(u)R(s,u)\nu''(C(s,u))du \]

and for \( s \geq t \geq 0 \):
\[ \partial_t \tilde{R}_1(s, t) = -\mu_i(s)\tilde{R}_1(s, t) + \epsilon_i\int_t^s \tilde{R}_1(u,t)\tilde{R}_1(s,u)\nu''(C(s,u))du, \]
\[ \partial_t \tilde{C}_1(s, t) = -\mu_i(s)\tilde{C}_1(s, t) + k_i\tilde{M}_1(t) \]
+ \( \epsilon_i \left( \int_0^s C(u,t)R(s,u)\nu''(C(s,u))du + \int_0^t \nu'(C(s,u))R(t,u)du \right) , \]
\[ \partial_t \tilde{Q}_i(s, t) = -\mu_i(s) \tilde{Q}_i(s, t) + k_i \tilde{M}_i(t) + \epsilon_i^2 \left( \int_0^s Q(u, t) R(s, u) \nu''(C(s, u)) du + \int_t^s \nu'(Q(s, u)) R(t, u) du \right), \]

with initial conditions \( \tilde{R}_i(t, 0) = \tilde{C}_i(t, 0) = \tilde{Q}_i(0, 0) = 1 \), \( \tilde{D}_i(t) := \tilde{Q}_i(t, t) \) and symmetry conditions \( \tilde{C}_i(t, s) = \tilde{C}_i(s, t) \) and \( \tilde{Q}_i(t, s) = \tilde{Q}_i(s, t) \), where \( \tilde{D}_i \) satisfies:

\[ \frac{\partial \tilde{D}_i(s)}{2} = -\mu_i(s) \tilde{D}_i(s) + k_i \tilde{M}_i(t) + \epsilon_i^2 \left( \int_0^s Q(u, s) R(s, u) \nu''(C(s, u)) du + \int_0^s \nu'(Q(s, u)) R(s, u) du \right) \]

and

\[ \begin{align*}
\mu_1(s) &= \omega_1(s) + \gamma^2 \int_0^s \psi(C(s, u)) R(s, u) du = \frac{1}{2h} + \tilde{M}_1(s) + \gamma^2 \int_0^s \psi(C(s, u)) R(s, u) du \\
\mu_2(s) &= \omega_2(s) + \beta^2 \int_0^s \psi(C(s, u)) R(s, u) du = \frac{1}{2} + hM(s) + \beta^2 \int_0^s \psi(C(s, u)) R(s, u) du
\end{align*} \]

and \( k_1 = 1, k_2 = h, \epsilon_1 = \gamma, \epsilon_2 = \beta \) and the functions \( \omega_1(s), \omega_2(s) \) are defined implicitly above.

Assuming \( (M, R, C, Q) \in \mathcal{A} \), then both the Ricatti equation, (6.12) and the linear one, (6.13) have unique solutions in \( C(\mathbb{R}_+) \) for the initial conditions \( \tilde{M}_i(0) = \alpha \). Thus, \( \mu_i(s) \) are continuous and further, by (16) there exists a unique non-negative solution \( \tilde{R}_i(s, t) \) of (6.14) which is continuous on \( \Gamma \) (see for example (4.1) for existence, uniqueness and non-negativity of the solution, and the proof of Lemma 4.3 for the differentiability, hence continuity of \( \tilde{R}_i(s, t) \)). With \( C, R \) and \( \tilde{M}_i \) continuous, clearly there is also a unique solution \( \tilde{C}_i(s, t) \) to (6.15) which is continuous on \( \Gamma \) and due to the boundary condition \( \tilde{C}_i(t, t) = 1 \), its symmetric extension to \( \mathbb{R}_+ \times \mathbb{R}_+ \) remains continuous. By the same reasoning, there exist an unique solution \( \tilde{D}_i(s, t) \) to (6.17), hence also an unique solution \( \tilde{Q}_i(s, t) \) to (6.16) defined on \( \Gamma \) with boundary condition \( \tilde{Q}_i(t, t) = \tilde{D}_i(t) \). Furthermore, by the boundary conditions, its symmetric extension to \( \mathbb{R}_+ \times \mathbb{R}_+ \) is differentiable, hence continuous. Thus, \( \Psi \) is well-defined and \( \Psi_i(\mathcal{A}) \subseteq \mathcal{A} \).

Notice that the solution \( (M^h, R^h, C^h, Q^h) \) of (6.1)-(6.5) is a fixed point of the mapping \( \Psi_1 \) and also that the solution \( (M, R, C, Q) \) of (2.13)-(2.17) is a fixed point of the mapping \( \Psi_2 \). We will show that, for sufficiently small \( \gamma = \frac{\beta}{2} \), any fixed point of \( \Psi_1 \) is in the space \( S(\delta, \rho, a, d) \) and also, for sufficiently small \( \beta \) and \( h \), any fixed point of \( \Psi_2 \) is in the same space, for a suitable choice of constants \( \delta, \rho, a, d \), independent of \( \beta \) and \( h \). Here:

\[ S(\delta, \rho, a, d) = \{(M, R, C, Q) \in \mathcal{B}(\delta, \rho, a, d) : \forall \tau \geq 0, \exists R^{\text{fkt}}(\tau) = \lim_{t \to \infty} R(t + \tau, t), \exists C^{\text{fkt}}(\tau) = \lim_{t \to \infty} C(t + \tau, t), \exists Q^{\text{fkt}}(\tau) = \lim_{t \to \infty} Q(t + \tau, t), \exists M^{\text{fkt}} = \lim_{t \to \infty} M(t), \exists Q^\infty = \lim_{t \to \infty} Q(t, 0) \}, \]

and

\[ \mathcal{B}(\delta, \rho, a, d) = \{(M, R, C, Q) \in \mathcal{A} : 0 \leq C(s, t), Q(s, t) \leq d, 0 \leq R(s, t) \leq q e^{-\delta(s-t)}, 0 \leq Q(s, s) \leq \frac{d}{2}, 0 \leq M(s) \leq a, \text{ for all } s \geq t \}. \]
This of course will imply that the FDT limits {6.8} exist and are in the space:

\[ D(\delta, \rho, a, d) = \{ (M, R, C, Q) : R, C, Q : \mathbb{R} \to \mathbb{R}, M \in \mathbb{R}_+, \]
\[ C(\tau) = C(-\tau), \quad Q(\tau) = Q(-\tau), \quad 0 \leq C(\tau), Q(\tau) \leq d, \]
\[ 0 \leq R(\tau) \leq pe^{-\delta h \tau}, \quad 0 \leq Q(0) \leq \frac{d}{2}, \quad 0 \leq M \leq a, \quad \text{for all } \tau \geq 0 \}. \]

6.2. Invariant Spaces. We will begin by finding constants \((\delta, \rho, a, d)\) such that \(\mathcal{S}(\delta, \rho, a, d)\) is invariant under the mapping \(\Psi_i\).

**Proposition 6.1.** There exist \(\gamma_1, \beta_1\) and \(h_1\), depending only on \(\alpha\), and a positive, universal constant \(c_1\), such that for our choice of constants \(a = \sqrt{\frac{7}{4}}, b = \min \{\alpha, \frac{1}{2}\}, \rho = c_1, \delta = \frac{b}{2}\) and \(d = 2 \max \left\{1 + \frac{2(a+1)}{b^2}, \frac{a+1}{b}\right\}\), if \(\gamma := \frac{b}{h} < \gamma_1\) and \(i = 1\) or \(\beta < \beta_1\), \(h < h_1\) and \(i = 2\), then

\[ \Psi_i(\mathcal{B}(\delta, \rho, a, d)) \subset \mathcal{B}(\delta, \rho, a, d), \]

and

\[ \Psi_i(\mathcal{S}(\delta, \rho, a, d)) \subset \mathcal{S}(\delta, \rho, a, d). \]

Furthermore, under the same conditions, if \((M, R, C, Q) \in \mathcal{B}(\delta, \rho, a, d)\), then for every \(s \geq 0\):

\[ \mu_i(s) \geq \omega_i(s) \geq b > 0 \]

**Proof of Proposition 6.1** We will start by verifying that (6.20) holds.

We will first be dealing with the bounds on \(\tilde{M}_i\). Here, due to the different nature of the equations (6.12) and (6.13) (Ricatti, respectively linear), our analysis will be different. Indeed (6.12) is equivalent to:

\[ \partial \tilde{M}_1(s) = -\left(\tilde{M}_1(s)\right)^2 - \frac{\tilde{M}_1(s)}{2h} + 1 + \gamma^2 (I_0(s) - I_1(s)) \]

for

\[ I_0(s) = \int_0^s M(u)R(s,u)\nu''(C(s,u))du \]
\[ I_1(s) = M(s) \int_0^s \psi(C(s,u))R(s,u)du \]

Since \((M, R, C, Q) \in \mathcal{B}(\delta, \rho, a, d)\), then we have the bounds:

\[ \gamma^2 |I_0(s) - I_1(s)| \leq \gamma^2 (|I_0(s)| + |I_1(s)|) \leq \gamma^2 \left[ \frac{a\nu''(d)\rho}{\delta} + \frac{a\psi(d)\rho}{\delta} \right] \leq \frac{3}{4} \]

for \(\gamma\) sufficiently small. For \(k = 1, 2\), define \(M_{1,k}(\cdot)\), to be the unique solutions to the Ricatti differential equations:

\[ \partial M_{1,k}(s) = -(M_{1,k}(s))^2 - \frac{M_{1,k}(s)}{2h} + 1 + (-1)^k \frac{3}{4}, \quad M_{1,k}(0) = \alpha \]

Since \(\partial M_{1,1}(s) \leq \partial M(s) \leq \partial M_{1,2}(s)\), for every \(s\) and all three functions start at the same point, we can sandwich \(\tilde{M}_i(\cdot)\) between \(M_{1,1}(\cdot)\) and \(M_{1,2}(\cdot)\), hence:

\[ \inf_{s \in [0, \infty)} M_{1,1}(s) \leq M_{1,1}(s) \leq \tilde{M}_1(s) \leq M_{1,2}(s) \leq \sup_{s \in [0, \infty)} M_{1,2}(s) \]

Define the polynomial \(P_2(x) = -(x^2 + \frac{x}{2k} - \frac{7}{4})\). Since its only positive root is \(x_2 = -\frac{1}{2k} + \sqrt{\frac{1}{4} + \frac{2}{2k} - \frac{7}{4}} < \sqrt{\frac{7}{4}}\) and \(M_{1,2}(0) = \alpha \in (0, 1)\), a sign analysis of \(\partial M_{1,2}\) will show that \(M_{1,2}\) must be monotonic on \([0, \infty)\) and
\[
\lim_{t \to \infty} M_{1,2}(t) = x_2. \text{ So}
\]
\[
\sup_{s \in [0, \infty)} M_{1,2}(s) \leq \max\{\alpha, x_2\} \leq \max\left\{\alpha, \sqrt{\frac{7}{4}}\right\} \leq \sqrt{\frac{7}{4}}
\]
which, together with (6.26) establishes the upper bound on \(\tilde{M}_1\):
\[
(6.27) \qquad \tilde{M}_1(s) \leq \sqrt{\frac{7}{4}} = a
\]

Define also the polynomial \(P_1(x) = -(x^2 + \frac{x}{2\pi} - \frac{1}{4})\). Again, its only positive root is \(x_1 = -\frac{1}{4\pi} + \sqrt{\frac{1}{(4\pi)^2} + \frac{1}{4}} > 0\) and since \(M_{1,1}(0) = \alpha \in (0, 1)\), analyzing the sign of \(\partial M_{1,1}\), we conclude that \(M_{1,1}\) is monotonic on \([0, \infty)\) and \(\lim_{t \to \infty} M_{1,1}(t) = x_1\), so:
\[
\inf_{\sigma \in [0, \infty)} M_{1,1}(s) \geq \min\{\alpha, x_2\} > 0
\]

Combining the above inequality with (6.26) and (6.27) we will finish establishing the desired bounds on \(\tilde{M}_1\):
\[
(6.28) \qquad 0 \leq \tilde{M}_1(s) \leq a
\]
The bound on \(\omega_1\) will follows suit:
\[
(6.29) \qquad \omega_1(s) = \frac{1}{2h} + \tilde{M}_1(s) \geq \frac{1}{2h} + \inf_{s \in [0, \infty)} M_{1,1}(s)
\]
\[
\geq \min\left\{\alpha + \frac{1}{2h}, \frac{1}{4h} + \sqrt{\frac{1}{(4h)^2} + \frac{1}{4}}\right\} > \min\left\{\alpha, \frac{1}{2}\right\} = b
\]

Furthermore, since \(C\) and \(R\) are positive, we are done proving (6.22): for \(i = 1, 2\):
\[
(6.30) \qquad L_i(s, t) = e^{-\int_s^t \mu(u) du} \geq 0,
\]
Solving the linear equation (6.13) (recall \(\tilde{M}_2(0) = \alpha\)), we obtain:
\[
(6.31) \qquad \tilde{M}_2(s) = \alpha L_2(s, 0) + \beta^2 \int_0^s I_0(u) L_2(s, u) du + h \int_0^s L_2(s, u) du
\]
with \(I_0\) defined in (6.23). Since \(\alpha > 0\) and \(M, R, C\) are positive, then the RHS above is positive, hence \(\tilde{M}_2(s) \geq 0\). This implies \(\mu_2(s) \geq \omega_2(s) \geq \frac{1}{2} \geq b\), proving (6.22) for \(i = 2\) and consequently \(L_i(s, t) \leq \exp(-b(s - t))\). Also, since \((M, R, C, Q) \in B(\delta, \rho, a, d)\), \(I_0(u)\) is positive and bounded above uniformly by \(\frac{\rho''(d)\rho}{4}\), hence recalling that \(\alpha < 1\), we obtain the desired upper bound on \(\tilde{M}_2\):
\[
\tilde{M}_2(s) \leq 1 + \beta^2 \frac{\rho''(d)\rho}{b\delta} \leq \frac{7}{4} = a
\]
holding for \(h, \beta\) small enough, as claimed.

Considering next the functions \(\tilde{H}_i\), let \(\tilde{H}_i(t, s) = L_i(s, t)\tilde{H}_i(s, t)\), where \(L_i\) is defined as in (6.30), with \(\tilde{H}_i(t, t) = 1\). Further, from [16] we have that for any \((s, t) \in \Gamma\),
\[
(6.32) \qquad \tilde{H}_i(s, t) = 1 + \sum_{n \geq 1} \sum_{\sigma \in N\phi_n} \int_{t \leq t_1 \leq \ldots \leq t_{2n} \leq s} \prod_{k \in \pi_r(\sigma)} \nu''(C(t_k, t_{\sigma_k})) \prod_{j=1}^{2n} dt_j.
\]
Consequently, since \(|NC_n| = (2\pi)^{-1} \int_{-2}^{2} x^{2n} \sqrt{4 - x^2} dx\) and \(C(u, v) \in [0, d]\), by the definition of \(B(\delta, \rho, a, d)\), we can bound \(\tilde{H}_i\):

\[
\tilde{H}_i(s, t) \leq \sum_{n \geq 0} \left( \epsilon_i^2 v''(d) \right)^n \sum_{\sigma \in NC_n} \int_{t \leq t_1 \leq \cdots \leq t_{2n} \leq s} \prod_{j=1}^{2n} dt_j

= \sum_{n \geq 0} \left( \epsilon_i^2 v''(d) \right)^n (s-t)^{2n} \left( \frac{2\pi}{2n!} \right)^{-1} \int_{-2}^{2} x^{2n} \sqrt{4 - x^2} dx

= (2\pi)^{-1} \int_{-2}^{2} e^{\epsilon_i \sqrt{v''(d)(s-t)}} \sqrt{4 - x^2} dx.
\]

It is well known (see for example [5, (3.8)]) that for some universal constant \(1 \leq c_1 < \infty\) and all \(\theta\),

\[
(2\pi)^{-1} \int_{-2}^{2} e^{\theta x} \sqrt{4 - x^2} dx \leq c_1(1 + |\theta|)^{-3/2} e^{2|\theta|},
\]

from which we thus deduce that:

\[
\tilde{H}_i(s, t) \leq c_1 \left( 1 + \epsilon_i \sqrt{v''(d)(s-t)} \right)^{-3/2} e^{2\epsilon_i \sqrt{v''(d)(s-t)}} \leq c_1 e^{2\epsilon_i \sqrt{v''(d)(s-t)}}.
\]

Further, since \((M, R, C, Q) \in B(\delta, \rho, a, d)\) and \(L_i(s, t) \leq e^{-b(s-t)}\), then for \(\epsilon_i \leq \frac{b}{4 \sqrt{v''(d)}}\) and for our choice of \(\rho = c_1\), and \(\delta = \frac{b}{2} \leq \delta - 2\epsilon_i \sqrt{v''(d)}\), we can establish the desired upper bound on \(\tilde{R}_i\):

\[
\tilde{R}_i(s, t) \leq c_1 e^{-b} e^{-\delta(s-t)} \leq \rho e^{-\delta(s-t)}.
\]

Finally, since \(L_i > 0\) and \(\tilde{H}_i > 0\) (since \(C \geq 0\)), the lower bound on \(\tilde{R}_i\) follows:

\[
\tilde{R}_i(s, t) \geq 0.
\]

Considering next the function \(\tilde{C}_i\), recall that \(\tilde{C}_i(t, t) = 1\), hence solving the linear equation \((6.15)\), we get, for \((s, t) \in \Gamma\):

\[
\tilde{C}_i(s, t) = L_i(s, t) + \epsilon_i^2 \int_t^s L_i(s, v) I_2(v, t) dv + \epsilon_i^2 \int_t^s L_i(s, v) I_3(v, t) dv + k_i \tilde{M}_i(t) \int_t^s L_i(s, v) dv
\]

where

\[
I_2(v, t) = \int_0^t C(u, t) R(v, u) v''(C(v, u)) du
\]

\[
I_3(v, t) = \int_0^t \nu'(C(v, u)) R(t, u) du
\]

Since \(L_i, C, R\) and \(\tilde{M}_i\) are positive, then \(I_2(v, t), I_3(v, t) \geq 0\) (recall \(\nu\) is a polynomial with positive coefficients). Hence the lower bound on \(\tilde{C}_i\) follows easily from \((6.37)\):

\[
\tilde{C}_i(s, t) \geq 0.
\]

Now, for the upper bound, since \((M, R, C, Q) \in B(\delta, \rho, a, d)\), \(I_2\) and \(I_3\) are bounded above, uniformly by \(\frac{dv''(d)\rho}{\delta}\) and \(\frac{\nu'(d)\rho}{\delta}\), respectively, hence, \((6.37)\) implies:

\[
\tilde{C}_i(s, t) \leq e^{-b(s-t)} + \int_t^s e^{-b(s-v)} dv \left[ \epsilon_i^2 \left( \frac{dv''(d)\rho}{\delta} + \frac{\nu'(d)\rho}{\delta} \right) + k_ia \right]

\leq 1 + \frac{1}{b} \left[ \epsilon_i^2 \left( \frac{dv''(d)\rho}{\delta} + \frac{\nu'(d)\rho}{\delta} \right) + ak_i \right] \leq 1 + \frac{a + 1}{b} < d
\]
whenever \( \epsilon_i^2 \left( \frac{d\nu^\prime(d)\rho}{\delta} + \frac{\nu^\prime(d)\rho}{\delta} \right) < 1 \) and \( k_i \leq 1 \) (i.e. \( \gamma \) is small enough for \( i = 1 \) and \( \beta \) is small enough and \( h \leq 1 \), for \( i = 2 \), respectively).

Now, for \( \tilde{D}_i(s) = \tilde{Q}_i(s, s) \), recalling that \( \tilde{D}_i(0) = 1 \), solving (6.17) we get:

\[
(6.42) \quad \tilde{D}_i(s) = L_i^2(s, 0) + 2\epsilon_i^2 \int_0^s L_i^2(s, v)I_4(v, 0)dt + 2\epsilon_i^2 \int_0^s L_i^2(s, v)I_5(v, 0)dt + 2k_i\tilde{M}_i(0) \int_0^s L_i^2(s, v)dv
\]

where

\[
(6.43) \quad I_4(v, t) = \int_0^v Q(u, t)R(v, u)\nu^\prime(C(v, u))du
\]

\[
(6.44) \quad I_5(v, t) = \int_0^v \nu^\prime(Q(v, u))R(t, u)du
\]

Notice that \( I_4 \) and \( I_5 \) share the same uniform bounds as \( I_2 \) and \( I_3 \), respectively. Recalling that \( L_i(s, t) \leq \exp(-b(s - t)) \), we establish the bound:

\[
(6.45) \quad 0 \leq \tilde{D}_i(s) \leq 1 + \frac{2}{b^2} \left[ \epsilon_i^2 \left( \frac{d\nu^\prime(d)\rho}{\delta} + \frac{\nu^\prime(d)\rho}{\delta} \right) + ak_i \right] \leq 1 + \frac{2(a + 1)}{b^2} \leq \frac{d}{2}
\]

for \( \gamma \) small for \( i = 1 \) and for \( \beta, h \) small for \( i = 2 \).

Moving over to \( \tilde{Q}_i(s, t) \), since \( \tilde{Q}_i(s, s) = \tilde{D}_i(s) \), we can solve the linear equation (6.16):

\[
(6.46) \quad \tilde{Q}_i(s, t) = \tilde{D}_i(t)L_i(s, t) + \epsilon_i^2 \int_t^s L_i(s, v)I_4(v, t)dt + \epsilon_i^2 \int_t^s L_i(s, v)I_5(v, t)dt + k_i\tilde{M}_i(t) \int_t^s L_i(s, v)dv ,
\]

where \( I_4 \) and \( I_5 \) are defined by (6.43) and (6.44), respectively. Using the same bounds on \( L_i, I_4 \) and \( I_5 \) as above, as well as the controls on \( D_i \) provided by (6.45), we show that:

\[
0 \leq \tilde{Q}_i(s, t) \leq \tilde{D}_i(t) + \frac{1}{b} \left[ \epsilon_i^2 \left( \frac{d\nu^\prime(d)\rho}{\delta} + \frac{\nu^\prime(d)\rho}{\delta} \right) + ak_i \right] \leq 1 + \frac{2(a + 1)}{b^2} + \frac{a + 1}{b} \leq d
\]

thus concluding the proof.

So, indeed, for our choices of \( a, \rho, \delta, d \) and \( b, (\tilde{M}_i, \tilde{R}_i, \tilde{C}_i, \tilde{Q}_i) \in \mathcal{B}(\delta, \rho, a, d) \), for sufficiently small \( \gamma = \beta \) \((i = 1)\) or sufficiently small \( \beta \) and \( h \) \((i = 2)\), thus showing (6.20). Furthermore, \( \mu_i(s) \geq \omega_i(s) \geq b \), hence (6.22) is true, under the same regime as above, as claimed.

Our next task is to verify that (6.21), the second statement of the theorem, holds. Namely, assuming that \((M, R, C, Q) \in \mathcal{S}(\delta, \rho, a, d)\) we are to show that the limits \((\hat{M}_i^\text{fit}, \hat{R}_i^\text{fit}, \hat{C}_i^\text{fit}, \hat{Q}_i^\text{fit})\) exist for the solution \((\hat{M}_i, \hat{R}_i, \hat{C}_i, \hat{Q}_i)\) of (6.14)–(6.19). The main idea used in this section of the proof is to use the exponential decay of \( R \) and \( L_i \) to bound all the relevant integrals by \( L^1 \) functions and then apply dominated convergence theorem in order to show the existence of the desired limits. To this end, recall that by (6.12), (6.31), (6.37),...
for any $t \geq 0$ and $\tau \geq v \geq 0$,

\[
\partial \tilde{M}_1(s) = -\left(\tilde{M}_1(s)\right)^2 - \frac{\tilde{M}_1(s)}{2h} + 1 + \gamma^2 \left(I_0(s) - I_1(s)\right) + \beta^2 \int_0^s I_0(u)\tilde{L}_2(s, u)du + h \int_0^s \tilde{L}_2(s, u)du
\]

\[
\tilde{M}_2(s) = \alpha L_2(s, 0) + \beta^2 \int_0^s I_0(u)\tilde{L}_2(s, u)du
\]

\[
\tilde{C}_i(t + \tau, t) = L_i(t + \tau, t) + c_i^2 \int_0^\tau L_i(t + \tau, t + v)I_2(t + v, t)dv 
\]

\[
+ c_i^2 \int_0^\tau L_i(t + \tau, t + v)I_3(t + v, t)dv + k_i\tilde{M}_i(t) \int_0^\tau L_i(t + \tau, t + v)dv 
\]

\[
\tilde{Q}_i(t + \tau, t) = \tilde{D}_i(t)L_i(t + \tau, t) + c_i^2 \int_0^\tau L_i(t + \tau, t + v)I_4(t + v, t)dv 
\]

\[
+ c_i^2 \int_0^\tau L_i(t + \tau, t + v)I_5(t + v, t)dv + k_i\tilde{M}_i(t) \int_0^\tau L_i(t + \tau, t + v)dv 
\]

\[
\tilde{D}_1(t) = L_1^2(t, 0) + 2c_i^2 \int_0^t L_i^2(t, v)I_4(v, 0)dv + 2c_i^2 \int_0^t L_i^2(t, v)I_5(v, 0)dv + 2k_i\tilde{M}_1(t) \int_0^t L_i^2(t, v)dv 
\]

\[
\tilde{R}_i(t + \tau, t) = L_i(t + \tau, t)\tilde{H}_i(t + \tau, t) 
\]

\[
\tilde{H}_i(t + \tau, t) = 1 + \sum_{n \geq 1} \beta^{2n} \sum_{\sigma \in \mathcal{N}_{C_n}} \int_{0 \leq \theta_1 \leq \cdots \leq \theta_{2n} \leq \tau} \prod_{i \in \alpha_{\sigma}(\sigma)} \nu''(C(t + \theta_i, t + \theta_{2n}(\sigma))) \prod_{j=1}^{2n} d\theta_j 
\]

\[
L_1(t + \tau, t + v) = \exp \left(-\frac{\tau - v}{2h} - I_0(t + \tau, t + v) - \gamma^2 \int_v^\tau I_7(t + u, t)du\right) 
\]

\[
L_2(t + \tau, t + v) = \exp \left(-\frac{\tau - v}{2h} - hI_8(t + \tau, t + v) - \beta^2 \int_v^\tau I_7(t + u, t)du\right) 
\]

where $I_0$ and $I_1$ are given by (6.23) and (6.24), respectively and:

\[
I_2(t + \tau, t) = \int_{-t}^\tau C(t + u, t)R(t + \tau, t + u)v''(C(t + \tau, t + u))du 
\]

\[
I_3(t + \tau, t) = \int_{-t}^0 \nu'(C(t + \tau, t + u))R(t, t + u)du 
\]

\[
I_4(t + \tau, t) = \int_{-t}^\tau Q(t + u, t)R(t + \tau, t + u)v''(C(t + \tau, t + u))du 
\]

\[
I_5(t + \tau, t) = \int_{-t}^0 \nu'(Q(t + \tau, t + u))R(t, t + u)du 
\]

\[
I_6(t + \tau, t + v) = \int_{-t}^\tau \tilde{M}_1(t + u)du 
\]

\[
I_7(t + \tau, t) = \int_{-t}^\tau \psi(C(t + \tau, t + u))R(t + \tau, t + u)du 
\]

\[
I_8(t + \tau, t + v) = \int_{-t}^\tau M(u + t)du 
\]
We will show that the limits \( \lim_{s \to -\infty} I_k(s) \) exist for \( k = 1, 2 \) and also that \( \lim_{s \to -\infty} I_k(\tau) \) exist, for \( k = 3, \ldots, 8 \). For \( I_0 \), begin by dividing the integral into two parts:

\[
(6.54) \quad I_0(s) = \int_{-s/2}^{s/2} M(u)R(s, u)\nu''(C(s, u))du + \int_{-s/2}^{0} M(s + u)R(s, s + u)\nu''(C(s, s + u))du
\]

Since \( \nu''(\cdot) \) is continuous and \( (M, R, C, Q) \in S(\delta, \rho, a, d) \), as \( s \to \infty \) the bounded integrand in the second integral above converges pointwise to the corresponding expression for \( (R_{\text{frt}}, C_{\text{frt}}, M_{\text{frt}}) \). Further, by the exponential tails of \( R \) the afore-mentioned integrands are uniformly in \( s \) bounded by \( f(\theta) := a\nu''(d)e^{\theta t} \), which is integrable on \( (-\infty, \theta) \). Thus, by dominated convergence theorem, we deduce that

\[
\lim_{s \to -\infty} \int_{-s/2}^{0} M(s + u)R(s, s + u)\nu''(C(s, s + u))du = \int_{0}^{\infty} M_{\text{frt}}R_{\text{frt}}(u)\nu''(C_{\text{frt}}(u))du
\]

The first integral in (6.54) is bounded above by \( \rho\delta^{-1}\nu''(s)(e^{-\delta s/2} - e^{-\delta s}) \) that converges to 0 as \( s \to \infty \), hence:

\[
(6.55) \quad \tilde{I}_0 := \lim_{s \to -\infty} I_0(s) = M_{\text{frt}}\int_{0}^{\infty} R_{\text{frt}}(u)\nu''(C_{\text{frt}}(u))du
\]

Applying a similar argument to \( I_1 \), we conclude that:

\[
(6.56) \quad \tilde{I}_1 := \lim_{s \to -\infty} I_1(s) = M_{\text{frt}}\int_{0}^{\infty} R_{\text{frt}}(u)\psi(C_{\text{frt}}(u))du
\]

Now, due to the above limits, for any \( 0 < \epsilon < \frac{1}{8h} \) there exist \( s_\epsilon > 0 \) such that if \( s > s_\epsilon \), \( |I_0(s) - I_1(s)| - |\tilde{I}_0 - \tilde{I}_1| < \epsilon \). Recalling the Ricatti equation (6.12) that characterizes \( \tilde{M}_1 \), we can sandwich \( \tilde{M}_1 \) between the functions \( M_{1.3} \) and \( M_{1.4} \) that are defined for \( s \geq s_\epsilon \) as the unique solutions of the differential equations:

\[
\frac{\partial M_{1,k}(s)}{\partial s} = -(M_{1,k}(s))^2 - \frac{M_{1,k}(s)}{2h} + 1 + \gamma^2 \left( (\tilde{I}_0 - \tilde{I}_1) + (-1)^k \epsilon \right),
\]

while for \( s \leq s_\epsilon \), \( M_{1.3}(s) = M_{1.4}(s) = \tilde{M}_1(s) \). Using the joint bound on \( I_0 \) and \( I_1 \) provided by (6.25) and observing that our choice of \( \epsilon \) guarantees \( \gamma^2 \epsilon < \frac{1}{8} \), we can conclude that the polynomials \( P_k(X) = -X^2 - \frac{X}{2h} + 1 + \gamma^2 \left( (\tilde{I}_0 - \tilde{I}_1) + (-1)^k \epsilon \right) \), for \( k = 3, 4 \), have exactly one positive root and one negative root.

Furthermore, denoting with \( x_k(\epsilon) \) the afore-mentioned positive roots, it is easy to see that:

\[
\lim_{t \to \infty} M_{1,k}(t) = x_k(\epsilon) = -\frac{1}{4h} + \sqrt{\frac{1}{(4h)^2} + 1 + \gamma^2 \left( (\tilde{I}_0 - \tilde{I}_1) + (-1)^k \epsilon \right)}
\]

Recalling that \( \tilde{M}_1 \) is bounded above by \( M_{1.4} \) and below by \( M_{1.3} \), we obtain:

\[
x_3(\epsilon) \leq \liminf_{t \to \infty} \tilde{M}_1(s) \leq \limsup_{t \to \infty} \tilde{M}_2(s) \leq x_4(\epsilon)
\]

Since \( \lim_{\epsilon \to 0} x_3(\epsilon) = \lim_{\epsilon \to 0} x_4(\epsilon) = -\frac{1}{4h} + \sqrt{\frac{1}{(4h)^2} + 1 + \gamma^2 \left( (\tilde{I}_0 - \tilde{I}_1) + (-1)^k \epsilon \right)} \) we can conclude that:

\[
(6.57) \quad \tilde{M}_{1_{\text{frt}}} := \lim_{t \to \infty} \tilde{M}_1(s) = -\frac{1}{4h} + \sqrt{\frac{1}{(4h)^2} + 1 + \gamma^2 \left( (\tilde{I}_0 - \tilde{I}_1) \right)}
\]

Consequently, applying again dominated convergence theorem, this time to (6.51), we show:

\[
(6.58) \quad \tilde{I}_0(\tau, v) := \lim_{t \to \infty} I_0(t + \tau, t + v) = (\tau - v)\tilde{M}_{1_{\text{frt}}}
\]

Also, since \( M(s) \) converges as \( s \to \infty \):

\[
(6.59) \quad \tilde{I}_0(\tau, v) := \lim_{t \to \infty} I_0(t + \tau, t + v) = (\tau - v)M_{1_{\text{frt}}}
\]
Since $\psi(\cdot)$, $\nu''(\cdot)$ and $\nu'(\cdot)$ are continuous and $(M, R, C, Q) \in S(\delta, \rho, a, d)$, as $t \to \infty$ the bounded integrands in (6.47), (6.48), (6.49), (6.50) and (6.52) converge pointwise to the corresponding expression for $(\hat{M}^{\text{dft}}, \hat{R}^{\text{dft}}, \hat{C}^{\text{dft}}, \hat{Q}^{\text{dft}})$. Further, by the exponential tail of $R$, the integrals over $[-t,-m]$ in afore-mentioned formulas, are bounded uniformly in $t$ by $\rho \delta^{-1} \psi(d)e^{-\delta m}$. Thus, applying dominated convergence theorem for the integrals over $[-m,v]$, then taking $m \to \infty$, we deduce that for each fixed $v \geq 0$,

(6.60)  \[
\tilde{I}_2(t) := \lim_{t \to \infty} I_2(t + \tau, t) = \int_0^\infty C^{\text{dft}}(\tau - \theta)R^{\text{dft}}(\theta)\nu''(C^{\text{dft}}(\theta))d\theta,
\]

(6.61)  \[
\tilde{I}_3(t) := \lim_{t \to \infty} I_3(t + \tau, t) = \int_\tau^\infty \nu'(C^{\text{dft}}(\theta))R^{\text{dft}}(\theta - \tau)d\theta,
\]

(6.62)  \[
\tilde{I}_4(t) := \lim_{t \to \infty} I_4(t + \tau, t) = \int_0^\infty Q^{\text{dft}}(\tau - \theta)R^{\text{dft}}(\theta)\nu''(C^{\text{dft}}(\theta))d\theta,
\]

(6.63)  \[
\tilde{I}_5(t) := \lim_{t \to \infty} I_5(t + \tau, t) = \int_\tau^\infty \nu'(Q^{\text{dft}}(\theta))R^{\text{dft}}(\theta - \tau)d\theta,
\]

(6.64)  \[
\tilde{I}_7 := \lim_{t \to \infty} I_7(t + \tau, t) = \int_0^\infty \psi(C^{\text{dft}}(\theta))R^{\text{dft}}(\theta)d\theta,
\]

hence also:

(6.65)  \[
\tilde{L}_1(\tau - v) := \lim_{t \to \infty} L_1(t + \tau, t + v) = \exp\left(-\frac{1}{2} + \frac{\tilde{M}^{\text{dft}}}{2} + \gamma^2 \tilde{I}_7\right)
\]

(6.66)  \[
\tilde{L}_2(\tau - v) := \lim_{t \to \infty} L_2(t + \tau, t + v) = \exp\left(-\frac{1}{2} + hM^{\text{dft}} + \beta^2 \tilde{I}_7\right)
\]

with $\tilde{\omega}_i = -\frac{\log \tilde{L}_i(t)}{t} \geq b > 0$.

Moving over to $\tilde{M}_2$, we first split each integral from the right hand side of (6.31) into $[0,s/2]$ and $[s/2,s]$. Since the integral over $[0,s/2]$ is bounded below by 0 and above by $\exp(-bs/2) - \exp(-bs)\alpha\rho\eta''(d)\delta^{-1}$, it converges to 0 as $s \to \infty$. The integrand over $[s/2,s]$ is dominated by the integrable function $\exp(-bs)\alpha\rho\eta''(d)\delta^{-1}$ hence we can and will apply dominated convergence theorem, concluding:

(6.67)  \[
\tilde{M}_2 := \lim_{s \to \infty} \tilde{M}_2(s) = \frac{\beta^2 \tilde{I}_0 + h}{\tilde{\omega}_2}
\]

A similar argument will show that

(6.68)  \[
\tilde{I}_4 := \lim_{s \to \infty} I_4(s,0) = \tilde{Q}_4 = \int_0^\infty R^{\text{dft}}(u)\nu''(C^{\text{dft}}(u))du
\]

Since trivially $\tilde{I}_5 := \lim_{t \to \infty} I_5(t,0) = 0$, similar arguments applied to the integrals in (6.42) and (6.37) will show:

(6.69)  \[
\tilde{D}_i := \lim_{t \to \infty} D_i(t) = \frac{\tilde{Q}_i + k_i\alpha}{\tilde{\omega}_i} = \tilde{Q}_i := \lim_{t \to \infty} \tilde{Q}_i(t,0)
\]

By the preceding discussion we also know that for all $v, t \geq 0$ and $i \in \{2, 3, 4, 5, 7\}$, $0 \leq I_i(t+v,t) \leq \rho\psi(d)\delta^{-1}$ and the same bound holds for $I_0(t)$, uniformly in $t$. Since $0 \leq L_i(t+\tau, t+v) \leq \exp(-b(\tau-v))$, we can bound all the integrands in the right hand sides of (6.37) and (6.46) by the integrable function $\rho\psi(d)\delta^{-1}\exp(-bx)$.
and then apply dominated convergence theorem, concluding:

\[ \bar{C}^{\text{sd}}_i(\tau) := \lim_{t \to \infty} \bar{C}_i(t + \tau, t) = \hat{L}_i(\tau) + c_i^2 \int_0^\tau \hat{L}(\tau - v) \hat{L}_2(v) dv 
+ c_i^2 \int_{0}^{\tau} \hat{L}_i(\tau - v) \hat{L}_3(v) dv + k_i \bar{M}^{\text{sd}}_1 \int_{0}^{\tau} \hat{L}_i(v) dv . \]

\[ \bar{Q}^{\text{sd}}_i(\tau) := \lim_{t \to \infty} \bar{Q}_i(t + \tau, t) = \bar{D}_1^{\text{sd}} \hat{L}_i(\tau) + c_i^2 \int_0^\tau \hat{L}(\tau - v) \hat{L}_4(v) dv 
+ c_i^2 \int_{0}^{\tau} \hat{L}_i(\tau - v) \hat{L}_3(v) dv + k_i \bar{M}^{\text{sd}}_1 \int_{0}^{\tau} \hat{L}_i(v) dv . \]

We also have that for any \( n \in \mathbb{Z}_+ \), all \( \sigma \in \mathcal{N}_n \) and each fixed \( \theta_1, \ldots, \theta_{2n} \geq 0 \),

\[ \lim_{t \to \infty} \prod_{i \in \mathcal{C}(\sigma)} \nu''(C(t + \theta_1, t + \theta_{\sigma(i)})) = \prod_{i \in \mathcal{C}(\sigma)} \nu''(C^{\text{sd}}(\theta_1 - \theta_{\sigma(i)})) , \]

By dominated convergence, the corresponding integrals over \( 0 \leq \theta_1 \leq \cdots \leq \theta_{2n} \leq \tau \) converge. Further, the non-negative series \((6.32)\) is dominated in \( t \) by a summable series (see \((6.33)\)), so by dominated convergence,

\[ \bar{H}^{\text{sd}}_i(\tau) := \lim_{t \to \infty} \bar{H}_i(t + \tau, t) = 1 + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{N}_n} \int_{0 \leq \theta_1 \leq \cdots \leq \theta_{2n} \leq \tau} \prod_{i \in \mathcal{C}(\sigma)} \nu''(C^{\text{sd}}(\theta_1 - \theta_{\sigma(i)})) \prod_{j=1}^{2n} d\theta_j . \]

It thus follows that

\[ \bar{R}^{\text{sd}}_i(t + \tau, t) = \lim_{\tau \to \infty} \bar{R}_i(t + \tau, t) = \hat{L}_i(t) \bar{H}^{\text{sd}}_i(t) , \]

exists for each \( \tau \geq 0 \), which establishes our claim \((6.21)\) (we have already shown that \( \bar{M}^{\text{sd}}, \bar{C}^{\text{sd}}(\tau), \bar{Q}^{\text{sd}}(\tau) \) and \( \bar{Q}^{\infty}_i \) exists).

6.3. **Contraction Mapping.** The next step in our proof is to establish that the mappings \( \Psi_i \) are contractions on \( \mathcal{S}(\delta, \rho, a, d) \). Thus we will be able to conclude that their unique fixed point, that coincides with the solution of our system, will be stationary in the limit, hence the FDT limits \((6.8)-(6.11)\) are well-defined.

**Proposition 6.2.** For \( \delta, \rho, a, b, d, \gamma_1, h_1, \beta_1 \) of Proposition \((6.1)\), there exist \( 0 < \gamma_2 \leq \gamma_1, 0 < \beta_2 \leq \beta_1 \) and \( 0 < h_2 \leq h_1 \), such that the mappings \( \Psi_i \) are contractions on \( \mathcal{S}(\delta, \rho, a, d) \), equipped with the norm

\[ \| (M, R, C, Q) \| := \sup_{s \in \mathbb{R}_+} |M(s)| + \sup_{s, t \in \mathbb{R}_+} |Q(s, t)| + \sup_{s, t \in \mathbb{R}_+} |C(s, t)| + \sup_{s, t \in \mathbb{R}_+} |R(s, t)| e^{\xi(s-t)} , \]

whenever \( \gamma \in [0, \gamma_2] \) (for \( i = 1 \)) or \( \beta \in [0, \beta_2] \) and \( h \in [0, h_2] \) (for \( i = 2 \)), for \( \xi = \frac{b}{\delta} > 0 \). Also the solution \((M, R, C, Q)\) of \((2.13)-(2.17)\) is also the unique fixed point of \( \Psi_1 \) in \( \mathcal{S}(\delta h, \rho, a, d) \) and of \( \Psi_2 \) in \( \mathcal{S}(\delta, \rho, a, d) \). Consequently, the functions \( \bar{M}^{\text{sd}}, \bar{R}^{\text{sd}}, \bar{C}^{\text{sd}} \) and \( \bar{Q}^{\text{sd}} \) of \((6.8)-(6.11)\) are then the unique solution in \( \mathcal{D}(\delta h, \rho, a, d) \), respectively \( \mathcal{D}(\delta, \rho, a, d) \) of the FDT equations

\[ (6.75) \quad 0 = -\mu M + h + \beta^2 M \int_{0}^{\infty} R(\theta) \nu''(C(\theta)) d\theta , \]

\[ (6.76) \quad R'(\tau) = -\mu R(\tau) + \beta^2 \int_{0}^{\tau} R(\tau - \theta) R(\theta) \nu''(C(\theta)) d\theta , \]

\[ (6.77) \quad C'(\tau) = -\mu C(\tau) + \beta^2 \int_{0}^{\infty} C(\tau - \theta) R(\theta) \nu''(C(\theta)) d\theta + \beta^2 \int_{0}^{\infty} \nu'(C(\theta)) R(\theta - \tau) d\theta + h M , \]

\[ (6.78) \quad Q'(\tau) = -\mu Q(\tau) + \beta^2 \int_{0}^{\infty} Q(\tau - \theta) R(\theta) \nu''(C(\theta)) d\theta + \beta^2 \int_{0}^{\infty} \nu'(Q(\theta)) R(\theta - \tau) d\theta + h M , \]
where

\[
\mu = \frac{1}{2} + \beta^2 \int_0^\infty \psi(C(\theta)) R(\theta) d\theta + hM ,
\]

with initial conditions \(D(0) = R(0) = 1\) and \(Q'(0) = 0\).

**Proof of Proposition 6.2.** Keeping \(\delta, \rho, a, b\) and \(d\) as in Proposition 6.1, we will show that \(\Psi_i\) is a contraction on \(S(\delta, \rho, a, d)\) equipped with the uniform norm \(\|((M, R, C, Q))\|\) of (6.74), for any \(\gamma\) small enough (\(i = 1\) or \(\beta, h\) small enough (\(i = 2\)). We will first recall that in Proposition 6.1 we have shown that if \((M, R, C, Q) \in S(\delta, \rho, a, d)\), then \(\omega_i(s) \geq b\), for all \(s \geq 0\), a critical fact that we will use in our upcoming proof. For simplicity of notation, we will denote by \(E(s, t) = R(s, t)e^{\xi(s-t)}\).

Consider a pair of elements in \(S(\delta, \rho, a, d)\), \((M_k, R_k, C_k, Q_k)\) for \(k = 1, 2\) and consider their images through \(\Psi_i\), namely \((M_{i,k}, R_{i,k}, C_{i,k}, Q_{i,k}) = \Psi_i(M_k, R_k, C_k, Q_k)\) for \(i = 1, 2\). We will also use the already established notation \(D_k(s) := Q_k(s, s)\). We will denote hereafter in short \(\Delta f(s, t) = f_1(s, t) - f_2(s, t)\) and \(\Delta f(s) = \sup_{0 \leq u \leq s} \|\Delta f(u, u)\|\) when \(f\) is one of the functions of interest to us, such as \(Q, C, R, E, L\) or \(H\). A similar notation will be used for functions \(f\) of only one variable, for example \(M\) or \(D\), namely \(\Delta f(s) = f_1(s) - f_2(s)\) and \(\Delta f(s) = \sup_{0 \leq u \leq s} |\Delta f(u)|\).

Denoting by \(\vartheta_1 = \gamma^2\) and \(\vartheta_2 = \beta^2 + h\), we shall show that for \(i = 1, 2\), there exist finite positive constants \(L_{M,i}, L_{E,i}, L_{C,i}\) and \(L_{Q,i}\) depending on \(\delta, \rho, a, b\) and \(d\), such that for any finite \(s \geq 0\),

\[
\begin{align*}
\Delta M_i(s) & \leq \vartheta_i L_{M,i}[\Delta M(s) + \Delta E(s) + \Delta C(s) + \Delta Q(s)], \\
\Delta E_i(s) & \leq \vartheta_i L_{E,i}[\Delta M(s) + \Delta E(s) + \Delta C(s) + \Delta Q(s)], \\
\Delta C_i(s) & \leq \vartheta_i L_{C,i}[\Delta M(s) + \Delta E(s) + \Delta C(s) + \Delta Q(s)], \\
\Delta Q_i(s) & \leq \vartheta_i L_{Q,i}[\Delta M(s) + \Delta E(s) + \Delta C(s) + \Delta Q(s)],
\end{align*}
\]

whenever \(\gamma \in [0, \gamma_1]\) and \(i = 1\) or \(h \in [0, h_1], \beta \in [0, \beta_1]\) and \(i = 2\). Here \(\gamma_1, h_1, \beta_1, a, d, \rho, \delta\) and \(b\) are the ones of Proposition 6.1.

So, if \(\vartheta_i\) is small enough (i.e. \(\vartheta_i \leq \min\{1/(5L_{M}), 1/(5L_E), 1/(5L_{C}), 1/(5L_{Q})\}\), \(\vartheta_1 \leq \gamma_1^2\) and \(\vartheta_2 \leq \beta_1^2 + h_1\), then from (6.80) to (6.83) we deduce that

\[
\| (\Delta M_i, \Delta R_i, \Delta C_i, \Delta Q_i) \| = \sup_{s \geq 0} \Delta M_i(s) + \sup_{s \geq 0} \Delta E_i(s) + \sup_{s \geq 0} \Delta C_i(s) + \sup_{s \geq 0} \Delta Q_i(s)
\]

\[
\leq \frac{4}{5} \left[ \sup_{s \geq 0} \Delta M(s) + \sup_{s \geq 0} \Delta E(s) + \sup_{s \geq 0} \Delta C(s) + \sup_{s \geq 0} \Delta Q(s) \right]
\]

\[
= \frac{4}{5}\| (\Delta M, \Delta R, \Delta C, \Delta Q) \|.
\]

In conclusion, the mapping \(\Psi_i\) is then a contraction on \(B(\delta, \rho, a, d)\), since

\[
\| \Psi_i(M_1, R_1, C_1, Q_1) - \Psi_i(M_2, R_2, C_2, Q_2) \| \leq \frac{4}{5}\| (M_1, R_1, C_1, Q_1) - (M_2, R_2, C_2, Q_2) \|,
\]

whenever \((M_k, R_k, C_k, Q_k) \in B(\delta, \rho, a, d)\), for \(k = 1, 2\).

From now until the end of the proof, for simplifying the notations, we will denote:

\[
\Delta(s) := \Delta M(s) + \Delta E(s) + \Delta C(s) + \Delta Q(s)
\]

Before we start, recall that \(I_{0,k}\) is defined by (6.23) for \((M_k, C_k, R_k, Q_k)\) and \(I_{1,k}\) is defined by (6.24). Notice that for every \(i \in \{0, 1\}\) and \(k \in \{1, 2\}\), that \(I_{i,k}\) is of the form \(\int_0^\infty R_k(s, u)T_{k,i}(u; s, t) du\), where \(T_{k,i}(u; s, t)\) are polynomial function depending only on \(C_k(\theta_1, \theta_2), M_k(\theta_1)\) and \(Q_k(\theta_1, \theta_2)\), for \(\theta_1, \theta_2 \in \{s, t, u\}\). By the definition of \(S(\delta, \rho, a, d)\) the family \(\{\int_0^\infty R_k(s, u) du\}_{s \geq 0}\) is uniformly bounded above by \(\rho\delta^{-1}\), hence:

\[
0 \leq I_{i,k}(s) \leq K_i, \quad i = 0, 1
\]
for $K_i = \frac{\xi}{3} \phi(d) \max\{a, d\}$. Similar arguments will show also that:

\begin{equation}
0 \leq I_{i,k}(s, t) \leq K_i \quad i = 2, 3, 4, 5, 7
\end{equation}

where $I_{2,k}$, $I_{3,k}$, $I_{4,k}$, $I_{5,k}$ and $I_{7,k}$ are defined by (6.38), (6.39), (6.43), (6.44) and (6.52), respectively, for $(M, R, C, Q) = (M_k, R_k, C_k, Q_k)$. Now consider the difference between $I_{0,1}$ and $I_{0,2}$. Since $\int_0^s |\Delta R(t, u)|du \leq \frac{\Delta \xi(t)}{\xi}$ (by the definition of $E_k$), the difference between $I_{0,1}$ and $I_{0,2}$ can be controlled, yielding:

\[ \Delta I_0(s) \leq L_{I_0} [\Delta M(s) + \Delta E(s) + \Delta C(s) + \Delta Q(s)] = L_{I_0} \bar{\Delta}(s) \]

for $L_{I_0} = \max\left\{ \frac{\alpha''(d)}{\delta}, \frac{\alpha''(d)}{\xi}, \frac{a \alpha''(d)}{\delta} \right\}$. In a similar manner, we obtain analogous bounds for $\Delta I_i(s)$, for $i = 1, 2, 3, 4, 5, 7$, for positive and finite constants $L_{I_i}$, depending only on $a$, $d$, $\rho$, $\delta$ and $\xi$. Hence, defining $L_I := \max_{r \in \{0,1,2,3,4,5,7\}} L_{I_r}$, we establish an uniform Lipschitz control on $I$:

\begin{equation}
\Delta I_i(s) \leq L_I \bar{\Delta}(s), \quad i = 0, 1, 2, 3, 4, 5, 7
\end{equation}

- **The Lipschitz bound** (6.80) on $\tilde{M}_1$. Recall that $\tilde{M}_{1,k}$ satisfies:

\begin{equation}
\partial \tilde{M}_{1,k}(s) = - \left( \tilde{M}_{1,k}(s) \right)^2 - \frac{\tilde{M}_{1,k}(s)}{2h} + 1 + \gamma^2 (I_{0,k}(s) - I_{1,k}(s))
\end{equation}

Let $\Theta(s, t) = \exp \left( - \int_s^t \left( \tilde{M}_{1,1}(\theta) + \tilde{M}_{1,2}(\theta) + \frac{1}{2h} \right) d\theta \right)$. In Proposition 6.1 we have shown that if $(M_k, R_k, C_k, Q_k) \in B(\rho, a, d)$ then both $\tilde{M}_{1,k}(s) \geq 0$ and $\omega_{1,k}(s) = \tilde{M}_{1,k}(s) + \frac{1}{2h} \geq b$ are true, hence $0 \leq \Theta(s, t) \leq \exp(-(t-s)b)$. Now, considering the difference between the realizations of (6.88) for $k = 1$ and $k = 2$, respectively, we get:

\[ \partial \Delta \tilde{M}_1(s) = - \Delta \tilde{M}_1(s) \left( \tilde{M}_{1,1}(\theta) + \tilde{M}_{1,2}(\theta) + \frac{1}{2h} \right) + \gamma^2 (\Delta I_0(s) - \Delta I_1(s)) \]

and since $\Delta \tilde{M}_1(0) = 0$ we get:

\[ \Delta \tilde{M}_1(s) = \gamma^2 \int_0^s (\Delta I_{0,k}(u) - \Delta I_{1,k}(u)) \Theta(u, s) du \]

hence:

\begin{equation}
\Delta \tilde{M}_1(s) \leq \gamma^2 [\Delta I_0(s) + \Delta I_1(s)] \int_0^s e^{-(s-u)b}du \leq L_{M,1} \gamma^2 \bar{\Delta}(s)
\end{equation}

with $L_{M,1} = \frac{2h}{b}$, where in the last inequality we have used the Lipschitz bound on $I_i$’s established in (6.87).

- **The Lipschitz bound** (6.80) on $\tilde{M}_2$. We will first establish the Lipschitz bounds on $\mu_i$ and $L_i$, $i = 1, 2$, that will be needed later. Namely, for $i = 1$, from (6.18):

\[ |\Delta \mu_1(v)| \leq |\Delta \tilde{M}_1(v)| + \gamma^2 |\Delta I_2(v, 0)| \leq (L_I + K_{M,1}) \gamma^2 \bar{\Delta}(s) \]

where in the last inequality we have used the bounds in (6.87) and (6.80) for $i = 1$. Since $|e^{-x} - e^{-y}| \leq |x - y|$ for all $x, y \geq 0$ and $\mu_{k,i}(s) \geq b$, $i, k = 1, 2$, denoting $K_4 := L_I + K_{M,1}$, we get that:

\begin{equation}
|\Delta L_1(s, t)| \leq e^{-(s-t)b} \int_t^s |\Delta \omega_1(v)| dv \leq \left[ K_4 e^{-b(s-t)}(s-t) \right] \gamma^2 \bar{\Delta}(s)
\end{equation}

Similarly, for $i = 2$, we get from (6.19):

\[ |\Delta \mu_2(v)| \leq h |\Delta M(v)| + \beta^2 |\Delta I_2(v, 0)| \leq (L_I + 1)(h + \beta^2) \bar{\Delta}(s) \]

and a similar argument as above, for $K_5 := L_I + 1$, will establish:

\begin{equation}
|\Delta L_2(s, t)| \leq \left[ K_5 e^{-b(s-t)}(s-t) \right] (\beta^2 + h) \bar{\Delta}(s)
\end{equation}
Hence, from (6.90) and (6.91) we establish the Lipschitz bound for $L_i$:

\[
\Delta L_i(s) \leq K_6 \vartheta_i \tilde{\Delta}(s)
\]

with $K_6 := \max\{K_4, K_5\} \sup_{\theta \geq 0} (\theta e^{-b\theta})$ and also:

\[
\int_0^s |\Delta L_i(s, u)| du \leq K_7 \vartheta_i \tilde{\Delta}(s)
\]

where $K_7 := \max\{K_4, K_5\} \sup_{\theta \geq 0} (e^{-b\theta} \theta^2)$.

We can now establish the Lipschitz bound (6.80) for $\tilde{M}_2$. Recalling that $\tilde{M}_{2,k}$ satisfies (6.31), we get:

\[
|\Delta \tilde{M}_2(s)| \leq \alpha |\Delta L_2(s, 0)| + \beta^2 \int_0^s |\Delta L_2(s, u)| du + h \int_0^s |\Delta L_2(s, u)| du
\]

\[
\leq \alpha |\Delta L_2(s)| + \beta^2 \bar{\Delta} L_0 + (\beta^2 K_I + h) \int_0^s |\Delta L_2(s, u)| du
\]

\[
\leq K_8 (\beta^2 + h) \Delta(s) = K_8 \vartheta_2 \Delta(s)
\]

with $K_8 := \alpha K_0 + \frac{\beta}{b} K_I + K_7 (\bar{\beta}^2 K_I + h_1)$, where in the last line of the derivation above we have used the bounds in (6.85), (6.87), (6.92) and (6.93).

**The Lipschitz bound (6.81) on $\tilde{e}$**. We rely on the formulas (6.32) and $\tilde{R}_{i,k}(s, t) = \tilde{H}_{i,k}(s, t) L_{i,k}(s, t)$. Indeed, since $C_1$ and $C_2$ are $[0, d]$-valued symmetric functions, $t_i \in [0, s]$, and both $\nu''(\cdot)$ and $\nu'''(\cdot)$ are non-negative and monotone non-decreasing, it follows that for any $n, t_{2n} \leq s$ and $\sigma \in NC_n$,}

\[
\left| \prod_{i \in \pi(\sigma)} \nu''(C_1(t_{i}, t_{\sigma})) - \prod_{i \in \pi(\sigma)} \nu''(C_2(t_{i}, t_{\sigma})) \right| \leq n \nu''(d)^{n-1} \nu'''(d) \Delta C(s).
\]

Thus we easily deduce from (6.32) that

\[
|\Delta \tilde{H}_i(s, t)| \leq 4c_2 \nu'''(d)(s-t)^2 \sum_{n \geq 1} n(2n!)^{-1} [2r(s-t)]^{2(n-1)} \Delta C(s)
\]

\[
\leq c_2^2 K_9 (s-t)^2 \epsilon \sqrt{\nu''(d)(s-t)} \Delta C(s).
\]

for $K_9 = 2 \nu'''(d)$. Recalling that $\epsilon_i^2 \leq \vartheta_i$ and since $\tilde{E}_{i,k}(s, t) = \tilde{R}_{i,k}(s, t) e^{c \epsilon_i^2}$, we now obtain from (6.34), (6.94), (6.90) and (6.91) that:

\[
\Delta \tilde{E}_i(s, t) \leq e^{c \epsilon_i^2} \left[ L_{i,1}(s, t) \Delta \tilde{H}_i(s, t) + \tilde{H}_{i,2}(s, t) \Delta L_i(s, t) \right]
\]

\[
\leq \vartheta_i e^{(-b + c_1 + 2c_1) \epsilon_i^2} (s-t) [K_9 (s-t)^2 + c_1 (K_4 + K_5) (s-t)] \Delta(s)
\]

\[
\leq \vartheta_i e^{(-b/3) \epsilon_i^2} [K_9 (s-t)^2 + c_1 (K_4 + K_5) (s-t)] \Delta(s)
\]

\[
\leq \vartheta_i L_{E,i} \Delta(s)
\]

for $\epsilon_i < \frac{b}{6 \sqrt{\nu''(d)}}$ and for the finite positive constant

\[
L_{E,i} := \sup_{\theta \geq 0} e^{-b\theta/3} \left[ K_9 \theta^2 + c_1 (K_4 + K_5) \theta \right].
\]
• The Lipschitz bounds (6.82) and (6.83) on $\tilde{C}$ and $\tilde{Q}$, respectively. Recalling the solution (6.37) of $C_{i,k}$, we have:

$$
\Delta \tilde{C}_i(s, t) = \Delta L_i(s, t) + \epsilon_i^2 \int_t^s \Delta (L_i(s, v) I_7(v, t)) dv \\
+ \epsilon_i^2 \int_t^s \Delta (L_i(s, v) I_5(v, t)) dv + k_i \int_t^s \Delta (\bar{M}_i(t) L_i(s, v)) dv
$$

Using the Lipschitz bounds in (6.87), (6.92) and (6.93), the first two integrals above are each bounded by:

$$
(\vartheta_i K_7 K_I + L_I) \bar{\Delta}(s)
$$

while by (6.80), the last one is bounded by:

$$
\vartheta_i \left( \frac{L_{M,i}}{b} + a K_6 \right) \bar{\Delta}(s)
$$

Wrapping all together, we get:

$$
|\Delta \tilde{C}_i(s)| \leq \vartheta_i L_{C,i} \bar{\Delta}(s)
$$

for $L_{C,i} = (K_5 + K_6) + 2((\beta_1 + \gamma_1) K_7 K_I + L_I) + \left( \frac{L_{M,i}}{b} + a K_6 \right) (h_1 + 1)$ and consequently, (6.82) holds.

Similarly, by the solution (6.42) of $D_{i,k}(s) := Q_{i,k}(s, s)$, we have:

$$
\Delta \tilde{D}_i(s) = \Delta (L_i^2(s, 0)) + 2\epsilon_i^2 \int_t^s \Delta (L_i^2(s, v) I_4(v, 0)) dv + 2\epsilon_i^2 \int_t^s \Delta (L_i^2(s, v) I_5(v, 0)) dv \\
+ 2\alpha k_i \int_t^s \Delta (L_i^2(s, v)) dv
$$

Since $L_i(s, t) \in [0, 1]$, then $|\Delta L_i^2(s, t)| \leq 2|\Delta L_i(s, t)|$, hence similarly as above, we get:

$$
|\Delta \tilde{D}_i(s)| \leq L_{D,i} \vartheta_i \bar{\Delta}(s)
$$

for $L_{D,i} = 4L_{C,i}$. Moving over to $\tilde{Q}_{i,k}$, since:

$$
\Delta \tilde{Q}_i(s, t) = \Delta (\tilde{D}_i(t) L_i(s, t)) + \epsilon_i^2 \int_t^s \Delta (L_i(s, v) I_4(v, t)) dv \\
+ \epsilon_i^2 \int_t^s \Delta (L_i(s, v) I_5(v, t)) dv + k_i \int_t^s \Delta (\bar{M}_i(t) L_i(s, v)) dv
$$

using the Lipschitz bound on $\tilde{D}_i$ and similar reasonings as above, we get:

$$
|\Delta \tilde{Q}_i(s)| \leq L_{Q,i} \vartheta_i \bar{\Delta}(s)
$$

for $L_{Q,i} = L_{D,i} + \max\{d, 1\} L_{C,i}$, thus concluding the argument that $\Psi_i$ is a contraction.

Now suppose that, for a choice of parameters $\beta$ and $h$, the constants $d, \rho, a, b$ and $d$ are such that $\Psi_i$ is a contraction on $B(\delta, \rho, a, d)$, hence also on its non-empty subset $S(\delta, \rho, a, d)$. Proposition 6.1 shows that both $B(\delta, \rho, a, d)$ and $S(\delta, \rho, a, d)$ are invariant under $\Psi_i$. We start at some $S_{i,0} = (M_0, R_0, C_0, Q_0) \in S(\delta, \rho, a, d)$ and construct recursively the sequences $S_{i,k} = \Psi_i(S_{i,k-1})$ for $k = 1, 2, \ldots$, in $S(\delta, \rho, a, d)$. For $i = 1, 2$, since $\Psi_i$ is a contraction, clearly $\{S_{i,k}\}_{k \in \mathbb{Z}_+}$ is a Cauchy sequence for the uniform norm $\| \|$ of (6.74). Hence, $S_{i,k} \rightarrow S_{i,\infty} = (M_{i,\infty}, R_{i,\infty}, C_{i,\infty}, Q_{i,\infty})$ in the Banach space $(\mathcal{C}(\mathbb{R}_+) \times \mathcal{C}(\mathbb{R}) \times \mathcal{C}(\mathbb{R}_+) \times \mathcal{C}(\mathbb{R}_+))$. Note that $B(\delta, \rho, a, d)$ is a closed subset of this Banach space, so $S_{i,\infty} \in B(\delta, \rho, a, d)$. Further, fixing $\tau \geq 0$, since $S_{i,k} \in S(\delta, \rho, a, d)$ we have that

$$
\lim_{T \rightarrow \infty} \sup_{t, t' \geq T} \|C_{i,\infty}(t + \tau, t) - C_{i,\infty}(t' + \tau, t')\| \\
\leq 2\|C_{i,\infty} - C_{i,k}\| + \lim_{T \rightarrow \infty} \sup_{t, t' \geq T} \|C_{i,k}(t + \tau, t) - C_{i,k}(t' + \tau, t')\| = 2\|S_{i,\infty} - S_{i,k}\|.
$$
Taking $k \to \infty$ we deduce that, for any $\tau \geq 0$, $t \mapsto C_{i,\infty}(t + \tau, t)$ is a Cauchy function from $\mathbb{R}_+$ to $[0, d]$, hence $C_{i,\infty}(t + \tau, t)$ converges as $t \to \infty$. A similar bounding procedure as above will show that the same is true for $E_{i,\infty}$ and $Q_{i,\infty}$ and will also show that $M_{i,\infty}(t)$ converges as $t \to \infty$. Now, since, by definition, $R_{i,\infty}(s, t) = E_{i,\infty}(s, t)e^{-t(s-t)}$, then $R_{i,\infty}$ will inherit the limiting property from $E_{i,\infty}$. Hence $S_{i,\infty} \in S(\delta, \rho, a, d)$ and further $S_{i,\infty}$ is the unique fixed point of the contraction $\Psi_i$ on the metric space $(S(\delta, \rho, a, d), \| \cdot \|)$. 

By our construction of $\Psi_i$, it follows that $(M_{i,\infty}, R_{i,\infty}, C_{i,\infty}, Q_{i,\infty})$ satisfies \((6.1)-(6.5)\) and also that $(M_{2,\infty}, R_{2,\infty}, C_{2,\infty}, Q_{2,\infty})$ satisfies \((2.13)-(2.17)\). Recalling that any solution of \((6.1)-(6.5)\) is a solution of \((2.13)-(2.17)\) that has been time-scaled by a factor of $h$, we can conclude that the unique solution of \((2.13)-(2.17)\) is in $S(\delta h, \rho, a, d)$, for $\gamma \in [0, \gamma_2]$ and in $S(\delta, \rho, a, d)$, respectively, for $\beta \in [0, \beta_2]$ and $h \in [0, h_2]$. As noted above, this shows that the FDT limits $M^{\text{fdt}}, R^{\text{fdt}}(\tau), C^{\text{fdt}}(\tau)$ and $Q^{\text{fdt}}(\tau)$ exist, for the unique solution of \((2.13)-(2.17)\) and furthermore, $(M^{\text{fdt}}, R^{\text{fdt}}, C^{\text{fdt}}, Q^{\text{fdt}}) \in D(\delta h, \rho, a, d)$ if $\gamma \leq \gamma_2$ and $(M^{\text{fdt}}, R^{\text{fdt}}, C^{\text{fdt}}, Q^{\text{fdt}}) \in D(\delta, \rho, a, d)$ if $\beta \leq \beta_2$ and $h \leq h_2$.

In order to conclude the proof, we show that $M^{\text{fdt}}, R^{\text{fdt}}(\cdot), C^{\text{fdt}}(\cdot)$ and $Q^{\text{fdt}}(\cdot)$ are the unique solution in $D(\delta h, \rho, a, d)$, respectively $D(\delta, \rho, a, d)$, of \((6.79)-(6.79)\). While proving Proposition $6.1$ we found that on $S(\delta, \rho, a, d)$, the mapping $\Psi_i$ induces a mapping $\Psi_i^{\text{fdt}} : (M^{\text{fdt}}, R^{\text{fdt}}, C^{\text{fdt}}, Q^{\text{fdt}}) \to (\tilde{M}_i^{\text{fdt}}, \tilde{R}_i^{\text{fdt}}, \tilde{C}_i^{\text{fdt}}, \tilde{Q}_i^{\text{fdt}})$ such that

\begin{align*}
0 &= -\left(\tilde{M}_i^{\text{fdt}}\right)^2 - \frac{\tilde{M}_i^{\text{fdt}}}{2h} + 1 + \gamma^2 \left(\tilde{I}_0 - \tilde{I}_1\right) \\
0 &= -\tilde{\omega}_2 \tilde{M}_i^{\text{fdt}} + h + \beta^2 \tilde{I}_0 \\
\tilde{R}_i^{\text{fdt}}(\tau) &= \tilde{L}_i(\tau) \sum_{n \geq 0} \epsilon_i^{2n} \int_{\sigma \in NC_n} \prod_{i \in \epsilon_i(\sigma)} \nu''(C^{\text{fdt}}(\theta_i - \theta_{\sigma(i)})) \int_{\tau_{i-1} \leq \tau_i} \nu''(\tilde{C}_i^{\text{fdt}}(\theta_i)) \\
\tilde{C}_i^{\text{fdt}}(\tau) &= \tilde{L}_i(\tau) + \epsilon_i^2 \int_0^\tau \tilde{L}_i(\tau - v) \tilde{I}_2(v) dv + \epsilon_i^2 \int_0^\tau \tilde{L}_i(\tau - v) \tilde{I}_3(v) dv + k_i \tilde{M}_i^{\text{fdt}} \int_0^\tau \tilde{L}_i(v) dv, \\
\tilde{Q}_i^{\text{fdt}}(\tau) &= \tilde{D}_i^{\text{fdt}} \tilde{L}_i(\tau) + \epsilon_i^2 \int_0^\tau \tilde{L}_i(\tau - v) \tilde{I}_4(v) dv + \epsilon_i^2 \int_0^\tau \tilde{L}_i(\tau - v) \tilde{I}_5(v) dv + k_i \tilde{M}_i^{\text{fdt}} \int_0^\tau \tilde{L}_i(v) dv, \\
0 &= -\tilde{\omega}_3 \tilde{I}_3(0) + \epsilon_i^2 \tilde{I}_4(0) + \epsilon_i^2 \tilde{I}_5(0) + k_i \tilde{M}_i^{\text{fdt}},
\end{align*}

where $\tilde{I}_0, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4$ and $\tilde{I}_5$ are given by \((6.55)-(6.56), (6.60), (6.61), (6.62)\) and \((6.63)\), respectively. In particular, $C^{\text{fdt}}, R^{\text{fdt}}$ and $Q^{\text{fdt}}$ are differentiable on $\mathbb{R}_+$, and, for $\tau \geq 0,$

\begin{align*}
(6.95) &
0 = -\left(\tilde{M}_i^{\text{fdt}}\right)^2 - \frac{\tilde{M}_i^{\text{fdt}}}{2h} + 1 + \gamma^2 \int_0^\infty R^{\text{fdt}}(\theta) \nu''(C^{\text{fdt}}(\theta)) d\theta \\
(6.96) &
0 = -\tilde{\omega}_2 \tilde{M}_i^{\text{fdt}} + \beta^2 \int_0^\infty R^{\text{fdt}}(\theta) \nu''(C^{\text{fdt}}(\theta)) d\theta + h \\
(6.97) &
\partial \tilde{R}_i^{\text{fdt}}(\tau) = -\tilde{\omega}_1 \tilde{R}_i^{\text{fdt}}(\tau) + \epsilon_i^2 \int_0^\tau \tilde{R}_i^{\text{fdt}}(\tau - \theta) \tilde{R}_i^{\text{fdt}}(\theta) \nu''(C^{\text{fdt}}(\theta)) d\theta, \\
(6.98) &
\partial \tilde{C}_i^{\text{fdt}}(\tau) = -\tilde{\omega}_1 \tilde{C}_i^{\text{fdt}}(\tau) + \epsilon_i^2 \int_0^\infty C^{\text{fdt}}(\tau - \theta) \tilde{R}_i^{\text{fdt}}(\theta) \nu''(C^{\text{fdt}}(\theta)) d\theta \\
& \quad + \epsilon_i^2 \int_\tau^\infty \nu''(C^{\text{fdt}}(\theta)) R^{\text{fdt}}(\theta - \tau) d\theta + k_i \tilde{M}_i^{\text{fdt}} \\
(6.99) &
\partial \tilde{Q}_i^{\text{fdt}}(\tau) = -\tilde{\omega}_1 \tilde{Q}_i^{\text{fdt}}(\tau) + \epsilon_i^2 \int_0^\infty Q^{\text{fdt}}(\tau - \theta) \tilde{R}_i^{\text{fdt}}(\theta) \nu''(C^{\text{fdt}}(\theta)) d\theta
\end{align*}
\[ + \epsilon_i^2 \int_{-\tau}^{\infty} \nu'(Q_f^\text{fdd}(\theta)) R_f^\text{fdd}(\theta - \tau) d\theta + k_i \tilde{M}_i^\text{fdd} \]

with \( \tilde{R}_i^\text{fdd}(0) = 1 \), \( \tilde{C}_i^\text{fdd}(0) = 1 \), \( \partial \tilde{Q}_i^\text{fdd}(0) = 0 \) and

\[
(6.100) \quad \tilde{\omega}_1 = \frac{1}{2h} + \gamma^2 \int_{0}^{\infty} \psi(C_i^\text{fdd}(\theta)) R_i^\text{fdd}(\theta) d\theta + \tilde{M}_i^\text{fdd} \\
(6.101) \quad \tilde{\omega}_2 = \frac{1}{2} + \beta^2 \int_{0}^{\infty} \psi(C_i^\text{fdd}(\theta)) R_i^\text{fdd}(\theta) d\theta + hM_i^\text{fdd}
\]

where in the derivation of (6.97) we have used the results in [16].

Recall that if the functions \( M, R, C \) and \( Q \) solve (2.13)-(2.17), then the functions \( M_h, R_h, C_h \) and \( Q_h \) are the unique solution of (6.1)-(6.5), hence the unique fixed point of \( \Psi \). Then, by (6.95)-(6.101) the corresponding quad-uple \((M_h, R_h, C_h, Q_h)\) is a fixed points of \( \Psi \). Then \( M_h^\text{fdd}, R_h^\text{fdd}(\tau) := R_h^\text{fdd}(h\tau), C_h^\text{fdd}(\tau) := C_h^\text{fdd}(h\tau) \) and \( Q_h^\text{fdd}(\tau) := Q_h^\text{fdd}(h\tau) \) satisfy the FDT equations (6.75)-(6.79). Noticing that the quad-uple \((M_h^\text{fdd}, R_h^\text{fdd}, C_h^\text{fdd}, Q_h^\text{fdd})\) that we have just defined coincide with the FDT limits of the original \((M, R, C, Q)\), we have established that, for \( \gamma \in [0, \gamma_2] \), \((M_h^\text{fdd}, R_h^\text{fdd}, C_h^\text{fdd}, Q_h^\text{fdd})\) satisfy (6.75)-(6.79).

Also, if \((M, R, C, Q)\) is the unique solution of (2.13)-(2.17), it is the unique fixed point of \( \Psi_2 \), hence \((M^\text{fdd}, R^\text{fdd}, C^\text{fdd}, Q^\text{fdd})\) is a fixed point of \( \Psi^\text{fdd} \), hence it satisfies (6.75)-(6.79).

Now, denoting by \( E^\text{fdd}(\tau) = e^{\tau h} R^\text{fdd}(\tau) \), by the same arguments as in the Lipschitz estimates (6.80)-(6.83) of Proposition 6.2, we show that:

\[
\begin{align*}
\Delta \tilde{M}_i^\text{fdd} & \leq \vartheta_i L_{M,i} \{ \Delta M^\text{fdd} + \Delta E^\text{fdd}(\infty) + \Delta C^\text{fdd}(\infty) + \Delta Q^\text{fdd}(\infty) \}, \\
\Delta \tilde{L}_i^\text{fdd}(\tau) & \leq \vartheta_i L_{E,i} \{ \Delta M^\text{fdd} + \Delta E^\text{fdd}(\tau) + \Delta C^\text{fdd}(\tau) + \Delta Q^\text{fdd}(\tau) \}, \\
\Delta \tilde{C}_i^\text{fdd}(\tau) & \leq \vartheta_i L_{C,i} \{ \Delta M^\text{fdd} + \Delta E^\text{fdd}(\tau) + \Delta C^\text{fdd}(\tau) + \Delta Q^\text{fdd}(\tau) \}, \\
\Delta \tilde{Q}_i^\text{fdd}(\tau) & \leq \vartheta_i L_{Q,i} \{ \Delta M^\text{fdd} + \Delta E^\text{fdd}(\tau) + \Delta C^\text{fdd}(\tau) + \Delta Q^\text{fdd}(\tau) \}
\end{align*}
\]

for all \( \tau < \infty \), where \( \Delta f(s) = \sup_{0 \leq u \leq s} |f_1(u) - f_2(u)| \) when \( f \) is one of the function of interest \( E, C \) or \( Q \), and \( \Delta M = |M_1 - M_2| \), thus showing that the mappings \( \Psi_{fdd} \) are also contractions, they have unique fixed points in \( D(h, \rho, a, d) \) and \( D(\delta, \rho, a, d) \), respectively. So (6.75)-(6.79) have an unique solution in \( D(h, \rho, a, d) \), for \( \gamma \in [0, \gamma_2] \) and in \( D(\delta, \rho, a, d) \), for \( \beta \in [0, \beta_2] \) and \( h \in [0, h_2] \), as claimed. \( \square \)

### 6.4. Exponential Decay of the Covariance

One consequence of Proposition 6.2 is that if either \( \gamma \) is small or both \( \beta \) and \( h \) are small, the response function is positive and decays to 0 exponentially fast. In the next proposition we will establish an analogous result for the covariance. Namely, we show:

**Proposition 6.3.** For \( \gamma_2, \beta_2, h_2 > 0 \) of Proposition 6.2, if \( \gamma \in [0, \gamma_2] \) or \( \beta \in [0, \beta_2] \) and \( h \in [0, h_2] \) there exist \( M = M(\beta, h, \alpha) > 0 \) and \( \eta = \eta(\beta, h, \alpha) \) such that for every \( s \geq t \geq 0 \):

\[
(6.102) \quad |C(s, t) - Q(s, t)| \leq M e^{-(s-t)\eta}
\]

**Proof of Proposition 6.3.** Let \( COV(s, t) := C(s, t) - Q(s, t) \) and respectively \( COV_h(s, t) := C_h(s, t) - Q_h(s, t) \), with \( U_h(s, t) := U(s/h, t/h) \), whenever \( U \) is one of \( C \) or \( Q \). Subtracting (2.16) from (2.15), we get:

\[
(6.103) \quad \partial_t COV(s, t) = -\mu(s) COV(s, t) + \beta^2 \int_0^s COV(u, t) R(s, u) \nu''(C(s, u)) du \\
+ \beta^2 \int_0^t COV(s, u) P(C(s, u), Q(s, u)) R(t, u) du, \quad s \geq t \geq 0
\]

for the multivariate polynomial \( P(X, Y) = \frac{\nu'(X) - \nu'(Y)}{X - Y} \), where \( \mu \) is defined by (2.17), hence

\[
(6.104) \quad COV(s, t) = L(s, t) + \beta^2 \int_s^t L(s, v) I_0(v, t) dv + \beta^2 \int_t^s L(s, v) I_{10}(v, t) dv
\]
with \( L(s, v) = \exp(-\int_v^s \mu(u)du) \),

\[
(6.105) \quad I_0(v, t) = \int_0^v \text{COV}(u, t)R(v, u)\nu''(C(v, u))du, \\
(6.106) \quad I_1(v, t) = \int_0^t \text{COV}(v, u)P(C(v, u), Q(v, u))R(t, u)du.
\]

By Proposition 6.2 we know that, whenever \( \beta < \beta_2 \) and \( h < h_2, R(s, t) \leq pe^{-(s-t)\delta} \) and \( \mu(s) \geq b \) implying \( L(s, v) \leq e^{-b(s-v)} \). Also, Theorem 2.3 shows \( C(s, t), Q(s, t) \in [0, 1] \), implying \( P(C(s, t), Q(s, t)) \leq \nu''(1) \), since \( \nu(\cdot) \) is a polynomial with positive coefficients. So, we get:

\[
|I_0(v, t)| \leq \nu''(1) \int_0^v |\text{COV}(u, t)|pe^{-(v-u)\delta}du \leq \nu''(1)p e^{-(v-t)\delta}, \\
|I_1(v, t)| \leq \nu''(1)p \delta^{-1}\sup_{u \leq t} |\text{COV}(u, v)|.
\]

and hence, with the symmetric function \( \Delta(t, s) := \sup_{u \leq t} |\text{COV}(u, v)| \) we deduce from (6.104) that for \( s \geq t \geq 0 \),

\[
\Delta(t, s) \leq e^{-b(s-t)} + \beta^2 \nu''(1)p \int_t^s e^{-b(s-v)} \left[ \int_0^v e^{-(v-u)\delta}du + \delta^{-1}\Delta(t, v) \right]dvdu \\
\leq e^{-b(s-t)} + \beta^2 \rho \nu''(1)\int_t^s e^{-b(s-v)} \int_0^t e^{-(v-u)\delta}dudv \\
+ \beta^2 \rho \nu''(1)\int_t^s \Delta(t, v)[\delta^{-1}e^{-b(s-v)} + \int_t^v e^{-b(s-v) - \delta(v-u)\delta}du]dv
\]

Since for any \( \delta \in (0, b/2) \) and \( s \geq t \),

\[
(6.107) \quad \int_t^s e^{-b(s-v) - \delta(v-u)\delta}dv \leq 2b^{-1}e^{-\delta(s-t)}
\]

and with \( \delta \in (0, b) \) we thus obtain for \( s \geq t \) the bound

\[
\Delta(t, s) \leq M e^{-\delta s} + A \int_t^s \Delta(t, v)e^{-\beta(s-v)\delta}dv,
\]

with \( M = 1 + 2\beta^2 \rho \nu''(1)(b\delta)^{-1} \) and \( A = \beta^2 \rho \nu''(1)\delta^{-1}(1 + 2b^{-1}) \). Therefore, fixing \( t \geq 0 \), the function \( h_t(s) = e^{\delta(s-t)} \Delta(t, s) \) satisfies

\[
h_t(s) \leq M + A \int_t^s h_t(v)dv, \quad s \geq t,
\]

and so by Gronwall’s lemma \( h_t(s) \leq Me^{A(s-t)} \). We therefore conclude that for any \( s \geq t \),

\[
|C(s, t) - Q(s, t)| \leq Me^{-(\delta-A)(s-t)},
\]

which proves the lemma in this case, since for \( \beta \to 0 \) we have that \( A = A(\beta) \to 0 \) (and so \( \eta = \delta - A > 0 \) for any \( \beta > 0 \) small enough).

Similarly, from (6.4) from (6.3), we get:

\[
(6.108) \quad \partial_t \text{COV}_h(s, t) = -\mu_h(s)\text{COV}_h(s, t) + \gamma^2 \int_t^s \text{COV}_h(u, t)R_h(s, u)\nu''(C_h(s, u))du \\
+ \gamma^2 \int_0^t \text{COV}_h(s, u)P(C_h(s, u), Q_h(s, u))R_h(t, u)du, \quad s \geq t \geq 0
\]
where \( \mu_h \) is defined by (6.5). Recalling that if \( \gamma \leq \gamma_2, \mu_h(s) \geq b \), the same argument as before, with \( \gamma \) in the place of \( \beta \), will show that \( \Delta_h(s, t) := \Delta(s/h, t/h) \leq M e^{-(\delta - A)(s-t)} \), that is equivalent to:

\[
|C(s, t) - Q(s, t)| \leq M e^{-(\delta - A)(s-t)}
\]

hence concluding out proof. \( \square \)

6.5. Simplifying the FDT System. The final step of the proof is to relate the solutions of the limiting equations (6.75)-(6.79) to the FDT equations (2.20) and (2.19), hence concluding the proof of Theorem 2.5.

Proposition 6.4. There exist \( \gamma_3, \beta_3, h_3 > 0 \) such that whenever \( \gamma \in [0, \gamma_3] \) or \( \beta \in [0, \beta_3] \) and \( h \in [0, h_3] \), the equations (2.20) and (2.19) have unique solutions \( C(t) \) and \( Q(t) \). Furthermore, the quadruple \( (M, C, R, Q) \), where \( R(t) := -2\partial C(t) \) and \( Q(t) := Q(t) \) solves the system (6.75)-(6.79) with initial conditions \( C(0) = R(0) = 1, Q'(0) = 0 \). Furthermore, \( R(t) \) is positive and decays exponentially fast to 0 and \( C(t) \) is positive and bounded, converging to \( Q \) as \( t \to \infty \).

Proof of Proposition 6.4 Consider the function \( f(x) = 4(x - 1)^2(\beta^2 \nu'(x) + h^2) - x \). Since for any \( h > 0, f(1 - (2h)^{-1}) > 0 \) and \( f(1) < 0 \) and also \( f(0) > 0 \), there exist at least a solution to \( f(x) = 0 \) in \( [1 - (2h)^{-1}] \cup [0, 1] \). By definition, any of these solutions satisfies (2.19). Fix \( Q \) to be one of them.

Let \( C \) be the unique \([0, 1]\)-valued solution of (2.20) for \( \phi(x) = \frac{1}{2} - 2\beta^2 Q \nu'(x) + 2h^2(1 - Q) + 2\beta^2 \nu'(x) \) (see Proposition 1.4 of [14] for existence and uniqueness of the solution). Also, since \( Q \in [1 - (2h)^{-1}] \cup [0, 1] \), it is easy to see that for small enough \( \gamma \), the following bound holds:

\[
2\beta^2 (\nu'(1) - \nu'(Q)) \geq \beta \gamma \nu''(1) \geq 2\sqrt{\beta^2 \nu'(1)}
\]

and if \( \beta \) is small enough, then:

\[
\frac{1}{2} \geq 2\sqrt{\beta^2 \nu'(1)}
\]

thus concluding that in both scenarios, \( \phi(1) > 2\sqrt{b \nu'(1)} \), hence, according to the above-mentioned result, \( C' \)

decays exponentially to 0 with some positive exponent (it is easy to see that \( \phi \) is convex, so the conditions in the quoted proposition are satisfied).

Moreover, by the same result, \( C \) converges as \( t \to \infty \) to

\[
C_{\infty} := \sup \left\{ x \in [0, 1] : \phi(x)(1 - x) \geq \frac{1}{2} \right\}
\]

Now, from the definition of \( Q \), it is easy to see that \( \phi(Q)(1 - Q) = 1/2 \) and since \( Q \in [0, 1], C_{\infty} \geq Q \). Also, for \( \gamma \) sufficiently small, for \( x \in [Q, 1],

\[
2\beta^2 \left( \frac{\nu'(x) - \nu'(Q)}{x - Q} \right) \leq 2\gamma^2 h^2 \nu''(1) < 4h^2 \leq \frac{1}{(1 - Q)(1 - x)}
\]

hence \( \phi(x)(1 - x) < 1/2 \) for \( x \in [Q, 1] \), implying \( C_{\infty} = Q \). Similarly, for \( \beta \) small,

\[
2\beta^2 \left( \frac{\nu'(x) - \nu'(Q)}{x - Q} - (1 - Q)(1 - x) \right) \leq 2\beta^2 \nu''(1) < 1
\]

so \( \phi(x)(1 - x) < 1/2 \) for \( x \in [Q, 1] \), hence \( C_{\infty} = Q \).

Now, denoting by \( R(t) := -2\partial C(t) \), and \( Q(t) \equiv Q \), since \( Q = \lim_{t \to \infty} C(t) \), some simple algebra will show that \( (M, R, C, Q) \) satisfy (6.75)-(6.79) with initial conditions \( C(0) = 1, R(0) = 1, Q'(0) = 0 \), if and only if:

\[
(6.109) \quad 0 = -\mu M + h + 2\beta^2 M (\nu'(1) - \nu'(Q))
\]

\[
(6.110) \quad 0 = -\mu Q + 2\beta^2 (Q \nu'(1) - 2Q \nu'(Q) + \nu'(Q)) + h M
\]

with

\[
\mu = \frac{1}{2} + 2\beta^2 (\nu'(1) - Q \nu'(Q)) + h M
\]
It's easy to check that $M := 2h(1-Q)$ and $Q$ are a solution to \([6.109]-[6.110]\), hence $(M, R, C, Q)$ satisfy \([6.75]-[6.79]\). Furthermore, $M, Q \in [0, 1]$, as needed.

Now, for every root of \(2.19\) we can use the same procedure as above to construct a quaduple $(M, R, C, Q)$, that solves the system. Since $Q \in [(1 - (2h)^{-1}) \wedge 0, 1]$, the same arguments as above will conclude that $(M, R, C, Q)$ are positive, $C(\cdot)$ is bounded and $R(\cdot)$ decays to 0 exponentially fast. Since according to Proposition \(6.2\), the system \([6.75]-[6.79]\) has a unique solution with these properties, the injectivity of the mapping $Q \mapsto (M, R, C, Q)$ shows that \(2.19\) has a unique root in $[(1 - (2h)^{-1}) \wedge 0, 1]$, thus concluding the proof. □

Now we have all the ingredients we need to finalize the proof of our theorem:

**Proof of Theorem 2.5.** Fix $\gamma_0 = \min\{\gamma_i : i = 1, 2, 3\}$, $\beta_0 = \min\{\beta_i : i = 1, 2, 3\}$ and $h_0 = \min\{h_i : i = 1, 2, 3\}$, for $\gamma_i, \beta_i, h_i$ of Proposition \(6.1\), $\gamma_2, \beta_2, h_2$ of Proposition \(6.2\) and $\gamma_3, \beta_3, h_3$ of Proposition \(6.3\). Then, according to Proposition \(6.2\), the FDT limits \([6.8]-[6.11]\) exist and are the unique solution of \([6.75]-[6.79]\) with initial conditions $C(0) = R(0) = 1, Q'(0) = 0$, in the space of positive functions such that $C(\cdot), Q(\cdot)$ are bounded above and $R(\cdot)$ decays exponentially to 0.

By Proposition \(6.3\), for the same possible values of the parameters $\beta$ and $h$, $C(\cdot), R(\cdot) := -2\partial C(\cdot), Q(\cdot) := -Q$ and $M := 2h(1-h)$ are a solution of \([6.75]-[6.79]\) and furthermore, $R$ decays exponentially fast to 0 and $0 \leq M, Q(\cdot), C(\cdot) \leq 1$, so, by the aforementioned uniqueness result, they are indeed the solution of \([6.75]-[6.79]\), thus concluding the proof. □

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