Quantized Stationary Control Policies in Markov Decision Processes

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Abstract

For a large class of Markov Decision Processes, stationary (possibly randomized) policies are globally optimal. However, in Borel state and action spaces, the computation and implementation of even such stationary policies are known to be prohibitive. In addition, networked control applications require remote controllers to transmit action commands to an actuator with low information rate. These two problems motivate the study of approximating optimal policies by quantized (discretized) policies. To this end, we introduce deterministic stationary quantizer policies and show that such policies can approximate optimal deterministic stationary policies with arbitrary precision under mild technical conditions, thus demonstrating that one can search for $\varepsilon$-optimal policies within the class of quantized control policies. We also derive explicit bounds on the approximation error in terms of the rate of the approximating quantizers. We extend all these approximation results to randomized policies. These findings pave the way toward applications in optimal design of networked control systems where controller actions need to be quantized, as well as for new computational methods for generating approximately optimal decision policies in general (Polish) state and action spaces for both discounted cost and average cost.

Index Terms

Markov decision processes, stochastic control, approximation, quantization, stationary policies.

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I. INTRODUCTION

In the theory of Markov decision processes (MDPs), control policies induced by measurable mappings from state to the action space are called stationary. For a large class of infinite horizon optimization problems, the set of stationary policies is the smallest structured set of control policies in which one can find a globally optimal policy. However, computing an optimal policy even in this class is in general computationally prohibitive for non-finite Polish (that is, complete and separable metric) state and action spaces. Furthermore, in applications to networked control, the transmission of such control actions to an actuator is not realistic when there is an information transmission constraint (imposed by the presence of a communication channel) between a plant, a controller, or an actuator.

Hence, it is of interest to study the approximation of optimal stationary policies. Several approaches have been developed in the literature to tackle this problem, most of which assume finite or countable state spaces, see [4]–[8], [29]. In this paper, we study the following question: for infinite Borel state and action spaces, how much is lost in performance if optimal policy is represented with a finite number of bits? This formulation appears to be new in the networked control literature, where stability properties of quantized control actions have been studied extensively, but the optimization of quantized control actions has not been studied as much in the context of cost minimization.

This paper contains two main contributions: (i) We establish conditions under which quantized control policies are asymptotically optimal; that is, as the accuracy of quantization increases, the optimal cost is achieved as the limit of the cost of quantized policies. (ii) We establish rates of convergence under further conditions; that is, we obtain bounds on the approximation loss due to quantization. These findings are somewhat analogous to results in optimal quantization theory [32].

The rest of the paper is organized as follows. In Section II we review the definition of discrete time Markov decision processes (MDP) in the setting we will be dealing with. In Section II-A we consider the approximation problem for the total and discounted cost cases using strategic measures (that is, measures on the sequence space of states and control actions). In Section II-B a similar approximation result is obtained for the average cost case using ergodic invariant probability measures of the induced Markov chains. In Section IV we derive quantitative
bounds on the approximation error in terms of the rate of the approximating quantizers for both discounted and average costs. In Section V we extend the results of Sections III and IV to approximating randomized stationary policies by randomized stationary quantizer policies. Finally, in Section VI we discuss future research directions.

II. MARKOV DECISION PROCESSES

For a metric space $E$, let $\mathcal{B}(E)$ denote its Borel $\sigma$-algebra. Unless otherwise specified, the term "measurable" will refer to Borel measurability. We denote by $\mathcal{P}(E)$ the set of all probability measures on $E$.

Consider a discrete time Markov decision process (MDP) with state space $X$ and action space $A$, where $X$ and $A$ are complete, separable metric (Polish) spaces equipped with their Borel $\sigma$-algebras $\mathcal{B}(X)$ and $\mathcal{B}(A)$, respectively. For all $x \in X$, we assume that the set of admissible actions is $A$. Let the stochastic kernel $p(\cdot | x, a)$ denote the transition probability of the next state given that previous state-action pair is $(x, a)$ [12]. The probability measure $\mu$ over $X$ denotes the initial distribution.

Define the history spaces $H_n = (X \times A)^n \times X$, $n = 0, 1, 2, \ldots$ endowed with their product Borel $\sigma$-algebras generated by $\mathcal{B}(X)$ and $\mathcal{B}(A)$. A policy is a sequence $\pi = \{\pi_n\}_{n \geq 0}$ of stochastic kernels on $A$ given $H_n$. A policy $\pi$ is said to be deterministic if the stochastic kernels $\pi_n$ are realized by a sequence of measurable functions $\{f_n\}$ from $H_n$ to $A$, i.e., $\pi_n(\cdot | h_n) = \delta_{f_n(h_n)}(\cdot)$ where $f_n : H_n \to A$ is measurable. A policy $\pi$ is called stationary if the stochastic kernels $\pi_n$ depend only on the current state; that is, $\pi_n = \pi_m (m, n \geq 0)$ and $\pi_n$ is a stochastic kernel on $A$ given $X$. A policy $\pi$ that is both deterministic and stationary is called deterministic stationary. Hence, deterministic stationary policies are defined by a measurable function $f : X \to A$. We denote by $S$ the set of deterministic stationary policies.

According to the Ionescu Tulcea theorem [12], an initial distribution $\mu$ on $X$ and a policy $\pi$ define a unique probability measure $P^\pi_\mu$ on $H_\infty = (X \times A)^\infty$, which is called a strategic measure [10]. Thus $P^\pi_\mu$ is symbolically given by

$$P^\pi_\mu(dx_0 da_0 dx_1 da_1 \ldots) := \prod_{n=0}^{\infty} p(dx_n|x_{n-1}, a_{n-1})\pi(da_n|h_n),$$

where $h_n = (x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$ and $p(dx_0|x_{-1}, a_{-1}) = \mu(dx_0)$. The expectation with respect to $P^\pi_\mu$ is denoted by $E^\pi_\mu$. If $\mu = \delta_x$ for some $x \in X$, we write $P^\pi_x$ and $E^\pi_x$ instead of
$P_\delta^\pi$ and $E_{\delta_x}^\pi$, respectively. Hence, given any policy $\pi$ and an initial distribution $\mu$, $\{x_n, a_n\}_{n \geq 1}$ is a $X \times A$-valued stochastic process defined on a probability space $(H_\infty, B(H_\infty), P_\mu^\pi)$ satisfying $P_\mu^\pi(x_0 \in \cdot) = \mu(\cdot)$, $P_\mu^\pi(x_n \in \cdot | x_{n-1}, a_{n-1}) = P_\mu^\pi(x_n \in \cdot | x_{n-1}, a_{n-1}) = p(\cdot | x_{n-1}, a_{n-1})$, and $P_\mu^\pi(a_n \in \cdot | x_n) = \pi_n(\cdot | x_n)$, for all $n$.

Let $c$ and $c_n$, $n = 0, 1, 2, \ldots$, be measurable functions from $X \times A$ to $[0, \infty)$. The cost functions $w$ considered in this paper are the following.

i) Expected Total Cost: $w_t(\pi, \mu) := E_\mu^\pi \left[ \sum_{n=0}^\infty c_n(x_n, a_n) \right]$.

ii) Expected Discounted Cost: $w_\beta(\pi, \mu) := E_\mu^\pi \left[ \sum_{n=0}^\infty \beta^n c(x_n, a_n) \right]$ for some $\beta \in (0, 1)$.

iii) Expected Average Cost: $w_A(\pi, \mu) := \limsup_{N \to \infty} \frac{1}{N} E_\mu^\pi \left[ \sum_{n=0}^N c(x_n, a_n) \right]$.

Note that the expected discounted cost is a special case of the expected total cost. Define $L_{\Delta, \mu} := \{P_\mu^\pi : \pi \in \Delta\}$. Then $L_{\Delta, \mu}$ is the set of all strategic measures with the initial distribution $\mu$. Hence, the cost function $w$ can be viewed as a function from $L_{\Delta, \mu}$ to $[0, \infty]$.

We write $w(\pi, \mu)$ to denote the cost function (either i), ii), or iii)) of the policy $\pi$ for the initial distribution $\mu$. If $\mu = \delta_x$, we write $w(\pi, x)$ instead of $w(\pi, \delta_x)$. A policy $\pi^*$ is called optimal if $w(\pi^*, \mu) = \inf_{\pi \in \Delta} w(\pi, \mu)$ for all $\mu \in \mathcal{P}(X)$. It is well known that the set of deterministic stationary policies is optimal for a large class of infinite horizon discounted cost problems (see, e.g., [12], [26]) and average cost optimal control problems (see, e.g., [1], [26]). For instance, Feinberg et al. [26] (see also [28]) recently showed the existence of an optimal stationary policy for discounted cost under weak continuity of the transition probability and $\mathbb{K}$-inf-compactness of the one-stage cost function, and for average cost with an additional mild assumption.

Throughout the paper, the initial distribution $\mu$ is assumed to be an arbitrary fixed distribution unless otherwise is specified.

A. Notation and Conventions

The set of all bounded measurable real functions and bounded continuous real functions on a metric space $E$ are denoted by $B(E)$ and $C_b(E)$, respectively. For any $\nu \in \mathcal{P}(E)$ and measurable real function $g$ on $E$, define $\nu(g) := \int g d\nu$. Let $E_n = \prod_{i=1}^n E_i$ ($2 \leq n \leq \infty$) be finite or an infinite product space. By an abuse of notation, any function $g$ on $\prod_{i=1}^m E_j$, where $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ ($m \leq n$), is also treated as a function on $E_n$ by identifying it with its natural extension to $E_n$. For any $\pi$ and initial distribution $\mu$, let $\lambda_n^{\pi, \mu}$, $\lambda_n^{\pi, \mu}$, and $\gamma_n^{\pi, \mu}$, respectively, denote the law of $x_n$, $(x_0, \ldots, x_n)$ and $(x_n, a_n)$ for all $n \geq 0$. Hence, for instance, we may write $\lambda_n^{\pi, \mu}(h) =$
\[ \mathcal{F}_n \left( \mathcal{F}_n ( h ) \right) \] where \( h \in B(X^{n+2}) \). Let \( \mathcal{F} \) denote the set of all measurable functions from \( X \) to \( A \).

For any \( g \in B(H_n) \ (n \geq 1) \) and \( f \in \mathcal{F} \), define \( g_f(x_0, \ldots, x_n) := g(x_0, f(x_0), \ldots, f(x_{n-1}), x_n) \).

Hence, when \( c \in B(X \times A) \), \( c_f(x_n) = c(x_n, f(x_n)) \) since \( c \in B(H_{n+1}) \) by our conventions.

\[ \mathcal{F} \]

**B. Problem Formulation**

In this section we give a formal definition of the problems considered in this paper. To this end, we first give the definition of a quantizer.

**Definition 2.1.** A measurable function \( q : X \rightarrow A \) is called a quantizer from \( X \) to \( A \) if the range of \( q \), i.e., \( q(X) = \{ q(x) \in A : x \in X \} \), is finite.

The elements of \( q(X) \) (i.e., the possible values of \( q \)) are called the levels of \( q \). The rate \( R \) of a quantizer \( q \) is defined as the logarithm of the number of its levels: \( R = \log_2 |q(X)| \). Note that \( R \) (approximately) represents the number of bits needed to losslessly encode the output levels of \( q \) using binary codewords of equal length. Let \( \mathcal{Q} \) denote the set of all quantizers from \( X \) to \( A \).

In this paper we introduce a new type of policy called a deterministic stationary quantizer policy. Such a policy is a constant sequence \( \pi = \{ \pi_n \} \) of stochastic kernels on \( A \) given \( X \) such that \( \pi_n(\cdot | x) = \delta_q(x)(\cdot) \) for all \( n \) for some \( q \in \mathcal{Q} \). For any finite set \( \Lambda \subset A \), let \( \mathcal{Q}(\Lambda) \) denote the set of all quantizers having range \( \Lambda \) and let \( S\mathcal{Q}(\Lambda) \) denote the set of all deterministic stationary quantizer policies induced by \( \mathcal{Q}(\Lambda) \).

The principal goal in this paper is to determine conditions on the spaces \( X \) and \( A \), initial distribution \( \mu \), the stochastic kernel \( p \), and the one-stage cost functions \( c, c_n \ (n \geq 0) \) such that there exists a sequence of finite subsets \( \{ \Lambda_k \}_{k \geq 1} \) of \( A \) for which the following statements hold:

**P1** For any \( \pi \in S \) there exists an approximating sequence \( \{ \pi^k \} \) satisfying \( \lim_{k \to \infty} w(\pi^k, \mu) = w(\pi, \mu) \), where \( \pi^k \in S\mathcal{Q}(\Lambda_k) \ (k \geq 1) \).

**P2** For any \( \pi \in S \) the approximating sequence \( \{ \pi^k \} \) in (P1) is such that \( |w(\pi, \mu) - w(\pi^k, \mu)| \) can be explicitly upper bounded by a term depending on the cardinality of \( \Lambda_k \).

Thus (P1) implies the existence of a sequence of stationary quantizer policies converging to an optimal stationary policy, while (P2) implies that the approximation error can be explicitly controlled.
III. APPROXIMATION OF DETERMINISTIC STATIONARY POLICIES

A sequence \( \{ \mu_n \} \) of measures on a measurable space \((E, \mathcal{E})\) is said to converge setwise \([14]\) to a measure \( \mu \) if \( \mu_n(B) \to \mu(B) \) for all \( B \in \mathcal{E} \), or equivalently, \( \mu_n(g) \to \mu(g) \) for all \( g \in B(E) \).

In this section, we will impose the following assumptions:

(a) The stochastic kernel \( p(\cdot \mid x, a) \) is setwise continuous in \( a \in A \), i.e., if \( a_n \to a \), then \( p(\cdot \mid x, a_n) \to p(\cdot \mid x, a) \) setwise for all \( x \in X \).

(b) \( A \) is compact.

Remark 3.1. Note that if \( X \) is countable, then \( B(X) = C_b(X) \) which implies the equivalence of setwise convergence and weak convergence. Hence, results developed in this paper are applicable to the MDPs having weakly continuous, in the action variable, transition probabilities when the state space is countable.

We now define the \( ws^\infty \) topology on \( \mathcal{P}(H_\infty) \) which was first introduced by Schäl in \([15]\).

Let \( C(H_0) = B(X) \) and let \( C(H_n) (n \geq 1) \) be the set of real valued functions \( g \) on \( H_n \) such that \( g \in B(H_n) \) and \( g(x_0, \cdot, x_1, \cdot, \ldots, x_{n-1}, \cdot, x_n) \in C_b(A^n) \) for all \( (x_0, \ldots, x_n) \in X^{n+1} \). The \( ws^\infty \) topology on \( \mathcal{P}(H_\infty) \) is defined as the smallest topology which renders all mappings \( P \mapsto P(g) \), \( g \in \bigcup_{n=0}^\infty C(H_n) \), continuous. Similarly, the weak topology on \( \mathcal{P}(H_\infty) \) can also be defined as the smallest topology which makes all mappings \( P \mapsto P(g) \), \( g \in \bigcup_{n=0}^\infty C_b(H_n) \), continuous \([15, \text{Lemma } 4.1]\]. A theorem due to Balder \([16\text{ page } 149]\) and Nowak \([17]\) states that the weak topology and the \( ws^\infty \) topology on \( L_{\Delta, \mu} \) are equivalent under the assumptions (a) and (b). Hence, the \( ws^\infty \) topology is metrizable with the Prokhorov metric on \( L_{\Delta, \mu} \).

The following theorem is a Corollary of \([18, \text{Theorem } 2.4]\) which will be used in this paper frequently. It is a generalization of the dominated convergence theorem.

**Theorem 3.1.** Let \((E, \mathcal{E})\) be a measurable space and let \( \nu, \nu_n (n \geq 1) \) be measures with the same finite total mass. Suppose \( \nu_n \to \nu \) setwise, \( \lim_{n \to \infty} h_n(x) = h(x) \) for all \( x \in X \), and \( h, h_n \) \((n \geq 1)\) are uniformly bounded. Then, \( \lim_{n \to \infty} \nu_n(h_n) = \nu(h) \).

Let \( d_A \) denote the metric on \( A \). Since the action space \( A \) is compact and thus totally bounded, one can find a sequence of finite sets \( \left(\{a_i\}_{i=1}^{m_k}\right)_{k \geq 1} \) such that for all \( k \),

\[
\min_{i \in \{1, \ldots, m_k\}} d_A(a, a_i) < 1/k \text{ for all } a \in A. \tag{1}
\]
In other words, \( \{a_i\}_{i=1}^{m_k} \) is an \( 1/k \)-net in \( A \). Let \( \Lambda_k := \{a_1, \ldots, a_{m_k}\} \) and for any \( f \in F \) define the sequence \( \{q_k\} \) by letting

\[
q_k(x) := \arg \min_{a \in \Lambda_k} d_A(f(x), a),
\]

where ties are broken so that \( q_k \) are measurable. Note that, \( q_k \in Q(\Lambda_k) \) for all \( k \) and \( q_k \) converges uniformly to \( f \) as \( k \to \infty \). Let \( \pi \in S \) and \( \pi^k \in SQ(\Lambda_k) \) be induced by \( f \) and \( q_k \), respectively. We call each \( \pi_k \) a quantized approximation of \( \pi \). In the rest of this paper, we assume that the sequence \( \{\Lambda_k\} \), as defined above, is fixed.

**Remark 3.2.** Since \( A \) is separable, there exists a totally bounded metric \( \tilde{d}_A \) on \( A \) that is compatible with the original metric structure of \( A \) [30, Corollary 3.41]. Hence, compact action space \( A \) is indeed not necessary for the problem \( (P1) \). However, it is usually necessary to show the existence of an optimal deterministic stationary policy.

**A. Expected Total and Discounted Costs**

Here we consider the first approximation problem \( (P1) \) for the expected total cost criterion and its special case, the expected discounted cost criterion (see Section II). Recall that \( w_t \) and \( w_\beta \) denote the expected total and discounted costs, respectively. We impose the following assumptions in addition to assumptions (a) and (b):

(c) \( c \) and \( c_n \) (\( n \geq 1 \)) are non-negative, bounded measurable functions satisfying \( c(x, \cdot) \),

\[
c_n(x, \cdot) \in C_b(A) \text{ for all } x \in X.
\]

(d) \( \sup_{\tilde{\pi} \in S} \sum_{n=N+1}^{\infty} \gamma_n^\tilde{\pi} \mu(c_n) \to 0 \text{ as } N \to \infty. \)

**Remark 3.3.** We note that all the results in this paper remain valid if it is only assumed that \( c \) and \( c_n \) (\( n \geq 0 \)) are bounded and measurable.

Since the one-stage cost functions \( c_n \) are non-negative, assumption (d) is equivalent to Condition (C) in [15, pg. 349]. Clearly, the expected discounted cost satisfies assumption (d) under assumption (c). We now state our main theorem in this subsection.

**Theorem 3.2.** Suppose assumptions (a), (b), (c) hold. Let \( \pi \in S \) and \( \{\pi^k\} \) be the quantized approximations of \( \pi \). Then, \( w_\beta(\pi^k, \mu) \to w_\beta(\pi, \mu) \) as \( k \to \infty \). The same statement is true for \( w_t \) if we further impose assumption (d).
The proof of Theorem 3.2 requires the following proposition which is proved in Appendix VI-A.

**Proposition 3.1.** Suppose assumptions (a) and (b) hold. Then for any $\pi \in S$, the strategic measures $\{P^\pi_k\}$ induced by the quantized approximations $\{\pi^k\}$ of $\pi$ converge to the strategic measure $P^\pi$ of $\pi$ in the $w_s^\infty$ topology. Hence, $\gamma^\pi_{n+k}(c_n) \to \gamma^\pi_n(c_n)$ as $k \to \infty$ under assumption (c).

**Proof of Theorem 3.2.** Since $w_\beta$ is a special case of $w_t$ and satisfies (d) under assumption (c), it is enough to prove the theorem for $w_t$. By Proposition 3.1, $\gamma^\pi_{n+k}(c_n) \to \gamma^\pi_n(c_n)$ as $k \to \infty$ for all $n$. Then, we have

$$\limsup_{k \to \infty} |w_t(\pi^k, \mu) - w_t(\pi, \mu)| \leq \limsup_{k \to \infty} \sum_{n=0}^{\infty} |\gamma^\pi_{n+k}(c_n) - \gamma^\pi_n(c_n)|$$

$$\leq \lim_{k \to \infty} \sum_{n=0}^{N} |\gamma^\pi_{n+k}(c_n) - \gamma^\pi_n(c_n)| + 2 \sup_{\tilde{\pi} \in S} \sum_{n=N+1}^{\infty} \gamma^\tilde{\pi}_n(c_n)$$

$$= 2 \sup_{\tilde{\pi} \in S} \sum_{n=N+1}^{\infty} \gamma^\tilde{\pi}_n(c_n).$$

Since the last expression converges to zero as $N \to \infty$ by assumption (d), the proof is complete.

**Remark 3.4.** Notice that this proof implicitly shows that $w_t$ and $w_\beta$ are sequentially continuous with respect to the strategic measures in the $w_s^\infty$ topology.

The following is a generic example frequently considered in the theory of Markov decision processes (see [22, p. 496], [21], [13, p. 23]).

**Example 3.1.** Let us consider an additive-noise system given by

$$x_{n+1} = F(x_n, a_n) + v_n, \ n = 0, 1, 2, \ldots$$

where $X = \mathbb{R}^n$ and the $v_n$’s are independent and identically distributed (i.i.d.) random vectors whose common distribution has a continuous, bounded, and strictly positive probability density function. A non-degenerate Gaussian distribution satisfies this condition. We assume that the action space $A$ is a compact subset of $\mathbb{R}^d$ for some $d \geq 1$, the one stage cost functions $c$ and $c_n \ (n \geq 1)$ satisfy assumption (c), and $F(x, \cdot)$ is continuous for all $x \in X$. It is straightforward
to show that assumption (a) holds under these conditions. Hence, under assumption (d) on the
cost functions \( c_n \), Theorem 3.2 holds for this system.

B. Expected Average Cost

In this section we consider the first approximation problem \((P1)\) for the expected average cost
criterion (see Section II). We are still assuming (a), (b), and (c). In contrast to the expected total
and discounted cost criteria, the expected average cost is in general not sequentially continuous
with respect to strategic measures for the \( w_s^{\infty} \) topology under practical assumptions. Instead, we
develop an approach based on the convergence of the sequence of invariant probability measures
under quantized stationary policies.

Recall that \( w_A \) denotes the expected average cost. Observe that any deterministic stationary
policy \( \pi \), induced by \( f \), defines a stochastic kernel on \( X \) given \( X \) via

\[
Q_\pi(\cdot|x) := \lambda^{\pi,x}_1(\cdot) = p(\cdot|x, f(x)).
\]  

Let us write \( Q_\pi g(x) := \lambda^{\pi,x}_1(g) \). If \( Q_\pi \) admits an ergodic invariant probability measure \( \nu_\pi \), then
by Theorem 2.3.4 and Proposition 2.4.2 in \([14]\), there exists an invariant set with full \( \nu_\pi \) measure
such that for all \( x \) in that set we have

\[
w_A(\pi, x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \gamma^{\pi,\mu}_n(c)
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^{\pi,x}_n(c_f) = \nu_\pi(c_f).
\]  

Let \( M_\pi \in \mathcal{B}(X) \) be the set of all \( x \in X \) such that convergence in (4) holds. Hence, \( \nu_\pi(M_\pi) = 1 \)
if \( \nu_\pi \) exists. The following assumptions will be imposed in the main theorem of this section.

(e) For any \( \pi \in S \), \( Q_\pi \) has a unique invariant probability measure \( \nu_\pi \).

(f1) The set \( \Gamma_S := \{ \nu \in \mathcal{P}(X) : \nu Q_\pi = \nu \text{ for some } \pi \in S \} \) is relatively sequentially compact
in the setwise topology.

(f2) There exists \( x \in X \) such that for all \( B \in \mathcal{B}(X) \), \( \lambda^{\pi,x}_n(B) \to \nu_\pi(B) \) uniformly in \( \pi \in S \).

(g) \( M := \bigcap_{\pi \in S} M_\pi \neq \emptyset \).

Theorem 3.3. Let the initial distribution \( \mu \) be concentrated on some \( x \in M \). Let \( \pi \in S \) and \( \{\pi^k\} \)
be the quantized approximations of \( \pi \). Then, \( w_A(\pi^k, \mu) \to w_A(\pi, \mu) \) under the assumptions (e),
(f1) or (f2), and (g).
Proof: See Appendix VI-B.

In the rest of this section we will derive conditions under which assumptions (e), (f1), (f2), and (g) hold. To begin with, assumptions (e), (f2) and (g) are satisfied under any of the conditions $R_i$, $i \in \{0, 1, 1(a), 1(b), 2, \ldots, 6\}$ in [19]. Moreover, $M = X$ in (g) if at least one of the above conditions holds. The next step is to find sufficient conditions for assumptions (e), (f1) and (g) to hold.

Observe that the stochastic kernel $p$ on $X$ given $X \times A$ can be written as a measurable mapping from $X \times A$ to $\mathcal{P}(X)$ if $\mathcal{P}(X)$ is equipped with its Borel $\sigma$-algebra generated by the weak topology [11], i.e., $p(\cdot | x, a) : X \times A \rightarrow \mathcal{P}(X)$. We impose the following assumption:

(e1) $p(\cdot | x, a) \leq \zeta(\cdot)$ for all $x \in X$, $a \in A$ for some finite measure $\zeta$ on $X$.

Proposition 3.2. Suppose (e1) holds. Then, for any $\pi \in S$ induced by $f$, $Q_{\pi}$ has an invariant probability measure $\nu_{\pi}$. Furthermore, $\Gamma_S$ is sequentially relatively compact in the setwise topology. Hence, (e1) implies assumption (f1). In addition, if these invariant measures are unique, then assumptions (e) and (g) also hold with $M = X$ in (g).

Proof: For any $\pi \in S$, define $Q_{\pi, x}^{(N)}(\cdot) := \frac{1}{N} \sum_{n=0}^{N-1} \lambda_n^{\pi, x}(\cdot)$ for some $x \in X$. Clearly, $Q_{\pi, x}^{(N)} \leq \zeta$ for all $N$. Hence, by [14, Corollary 1.4.5] there exists a subsequence $\{Q_{\pi, x}^{(N_k)}\}$ which converges to some probability measure $\nu_{\pi}$ setwise. Following the same steps in [20, Theorem 4.17] one can show that $\nu_{\pi}(g) = \nu_{\pi}(Q_{\pi}g)$, for all $g \in \mathcal{B}(X)$. Hence, $\nu_{\pi}$ is an invariant probability measure for $Q_{\pi}$.

Furthermore, assumption (e1) implies $\nu_{\pi} \leq \zeta$ for all $\nu_{\pi} \in \Gamma_s$. Thus, $\Gamma_s$ is relatively sequentially compact in the setwise topology by again [14, Corollary 1.4.5].

Finally, for any $\pi$, if the invariant measure $\nu_{\pi}$ is unique, then every setwise convergent subsequence of the relatively sequentially compact sequence $\{Q_{\pi, x}^{(N)}\}$ must converge to $\nu_{\pi}$. Hence, $Q_{\pi, x}^{(N)} \rightarrow \nu_{\pi}$ setwise which implies that $w_A(\pi, x) = \limsup_{N \rightarrow \infty} Q_{\pi, x}^{(N)}(c_f) = \lim_{N \rightarrow \infty} Q_{\pi, x}^{(N)}(c_f) = \nu_{\pi}(c_f)$ for all $x \in X$ since $c_f \in \mathcal{B}(X)$. Thus, $M = X$ in (g).

Example 3.2. Let us consider an additive-noise system in Example 3.1 with the same assumptions. Furthermore, we assume $F$ is bounded. Observe that for any $\pi \in S$, if $Q_{\pi}$ has an invariant probability measure, then it has to be unique [14, Lemma 2.2.3] since there cannot exist disjoint invariant sets due to the positivity of $g$. Since this system satisfies (e1) and $R1(a)$ in [19] due to the boundedness of $F$, assumptions (e), (f1), (f2) and (g) hold with $M = X$. This means that
Theorem 3.3 holds for an additive noise system under the above conditions.

IV. Rates of Convergence

In this section we consider the problem (P2) for the discounted and average cost criteria. Let \( \| \cdot \|_{TV} \) denote the total variation distance between measures. We will impose a new set of assumptions in this section:

(h) \( A \) is infinite compact subset of \( \mathbb{R}^d \) for some \( d \geq 1 \).

(j) \( c \) is bounded and \( |c(x, \tilde{a}) - c(x, a)| \leq K_1 d_A(\tilde{a}, a) \) for all \( x \), and some \( K_1 \geq 0 \).

(k) \( \|p(\cdot | x, \tilde{a}) - p(\cdot | x, a)\|_{TV} \leq K_2 d_A(\tilde{a}, a) \) for all \( x \), and some \( K_2 \geq 0 \).

(l) There exists positive constants \( C \) and \( \beta \in (0, 1) \) such that for all \( \pi \in S \), there is a (necessarily unique) probability measure \( \nu_{\pi} \in \mathcal{P}(X) \) satisfying \( \|\lambda_{n}^{\pi,x} - \nu_{\pi}\|_{TV} \leq C \kappa^n \) for all \( x \in X \) and \( n \geq 1 \).

Assumption (l) implies that for any policy \( \pi \in S \), the stochastic kernel \( Q_{\pi} \), defined in (3), has a unique invariant probability measure \( \nu_{\pi} \) and satisfies geometric ergodicity [13]. Note that (l) holds under any of the conditions \( R_i \), \( i \in \{0, 1, 1(a), 1(b), 2, \ldots, 5\} \) in [19]. Moreover, one can explicitly compute the constants \( C \) and \( \kappa \) for certain systems. For instance, consider an additive-noise system in Example 3.1 with Gaussian noise. Let \( X = \mathbb{R} \). Assume \( F \) has a bounded range so that \( F(\mathbb{R}) \subset [-L, L] \) for some \( L > 0 \). Let \( m \) denote the Lebesgue measure on \( \mathbb{R} \). Then, assumption (l) holds with \( C = 2 \) and \( \kappa = 1 - \varepsilon L \), where \( \varepsilon = \frac{1}{\sigma \sqrt{2\pi}} \exp(-(2L)^2/2\sigma^2) \). For further conditions that imply (l) we refer the reader to [19], [13], [2].

Assumptions (h), (j) and (k) will be imposed for both cases, but (l) will only be assumed for the expected average cost. The following example gives the sufficient conditions for the additive noise system under which (j), (k) and (l) hold.

Example 4.3. Consider the additive-noise system in Example 3.1. In addition to the assumptions there, suppose \( F(x, \cdot) \) is Lipschitz uniformly in \( x \in X \) and the common density \( g \) of the \( v_n \) is Lipschitz on all compact subsets of \( X \). Note that a Gaussian density has these properties. Let \( c(x, a) := \|x - a\|^2 \). Under these conditions, assumptions (j) and (k) hold for the additive noise system. If we further assume that \( F \) is bounded, then assumption (l) holds as well.

The following result is a consequence of the fact that if \( A \) is a compact subset of \( \mathbb{R}^d \) then there exist a constant \( \alpha > 0 \) and finite subsets \( \Lambda_k \subset A \) with cardinality \( |\Lambda_k| = k \) such that
\[
\max_{x \in A} \min_{y \in \Lambda_k} d_A(x, y) \leq \alpha (1/k)^{1/d} \text{ for all } k, \text{ where } d_A \text{ is the Euclidean distance on } A \text{ inherited from } \mathbb{R}^d.
\]

**Lemma 4.1.** Let \( A \subset \mathbb{R}^d \) be compact. Then for any measurable function \( f : X \to A \) we can construct a sequence of quantizers \( \{q_k\} \) from \( X \) to \( A \) which satisfy \( \sup_{x \in X} d_A(q_k(x), f(x)) \leq \alpha (1/k)^{1/d} \) for some constant \( \alpha \).

The following proposition is the key result in this section. It is proved in Appendix VI-C.

**Proposition 4.3.** Let \( \pi \in S \) and \( \{\pi^k\} \) be the quantized approximations of \( \pi \). For any initial distribution \( \mu \) we have

\[
\|\lambda_n^{\pi^\mu} - \lambda_n^{\pi^k\mu}\|_{TV} \leq \alpha K_2 (2n - 1)(1/k)^{1/d}
\]

for all \( n \geq 1 \) under assumptions (h), (j), and (k).

**A. Expected Discounted Cost**

The proof of the following theorem essentially follows from Proposition 4.3. The proof is given in Appendix VI-D.

**Theorem 4.1.** Let \( \pi \in S \) and \( \{\pi^k\} \) be the quantized approximations of \( \pi \). For any initial distribution \( \mu \), we have

\[
|w_\beta(\pi, \mu) - w_\beta(\pi^k, \mu)| \leq K(1/k)^{1/d},
\]

where \( K = \frac{\alpha}{1 - \beta}(K_1 - \beta K_2 M + \frac{2\beta M K_2}{1 - \beta}) \) with \( M := \sup_{(x,a) \in X \times A} |c(x,a)| \) under assumptions (h), (j) and (k).

**B. Expected Average Cost Case**

In this section, as in Section III-B we approach the problem by writing the expected average cost as an integral of the one stage cost function with respect to an invariant probability measure for the induced stochastic kernel. This way we obtain a bound on the difference between the actual and the approximated costs. However, the bound for this case will depend both on the rate of the quantizer approximating the actual policy and an extra term which changes with the system parameters. We will show that this extra term goes to zero as \( n \to \infty \).

Note that for any \( \pi \in S \), induced by \( f \), assumption (l) implies that \( \nu_\pi \) is an unique invariant probability measure for \( Q_\pi \) and that \( w_A(\pi, x) = \nu_\pi(c_f) \) for all \( x \) when \( c \) is as in the assumption.
The following theorem basically follows from Proposition 4.3 and the assumption (l). It is proved in Appendix VI-E.

**Theorem 4.2.** Let \( \pi \in S \) and \( \{\pi^k\} \) be the quantized approximations of \( \pi \). Under assumptions (h), (j), (k), and (l), for any \( x \in X \) we have

\[
|w_A(\pi, x) - w_A(\pi^k, x)| \leq 2MC\kappa^n + K_n(1/k)^{1/d} \tag{7}
\]

for all \( n \geq 0 \), where \( K_n = ((2n - 1)K_2\alpha + K_1\kappa) \) and \( M := \sup_{(x, a) \in X \times A} |c(x, a)| \).

Observe that depending on the values of \( C \) and \( \kappa \), we can first make the first term in (7) small enough by choosing sufficiently large \( n \), and then for this \( n \) we can choose \( k \) large enough such that the second term in (7) is small.

**Order Optimality:** The following example demonstrates that the order of approximation errors in Theorems 4.2 and 4.1 cannot be better than \( O((1/k)^{1/d}) \). More precisely, we exhibit a simple standard example where we can lower bound the approximation errors for the optimal stationary policy by \( L(1/k)^{1/d} \), for some positive constant \( L \).

In what follows \( h(\cdot) \) and \( h(\cdot | \cdot) \) denote differential and conditional differential entropies, respectively.

**Example 4.4.** Consider the linear system

\[ x_{n+1} = Ax_n + Ba_n + v_n, n = 0, 1, 2, \ldots, \]

where \( X = A = \mathbb{R}^d \) and the \( v_n \)'s are i.i.d. random vectors whose common distribution has density \( g \). For simplicity suppose that the initial distribution \( \mu \) has the same density \( g \). It is assumed that the differential entropy \( h(g) := -\int_X g(x) \log g(x) dx \) is finite. Let the one stage cost function be \( c(x, a) := \|x - a\| \). Clearly, the optimal stationary policy \( \pi^* \) is induced by the identity \( f(x) = x \), having the optimal cost \( w_i(\pi, \mu) = 0 \), where \( i \in \{\beta, A\} \). Let \( \{\pi^k\} \) be the quantized approximations of \( \pi^* \). Fix any \( k \) and define \( D_n := E^\pi_{\mu} \left[ c(x_n, a_n) \right] \) for all \( n \). Then, by the Shannon lower bound (SLB) [33, p. 12] we have for \( n \geq 1 \)

\[
\log k \geq R(D_n) \geq h(x_n) + \theta(D_n) = h(Ax_{n-1} + Ba_{n-1} + v_{n-1}) + \theta(D_n) \\
\geq h(Ax_{n-1} + Ba_{n-1} + v_{n-1}|x_{n-1}, a_{n-1}) + \theta(D_n) \\
= h(v_{n-1}) + \theta(D_n), \tag{8}
\]
where $\theta(D_n) = -d + \log \left( \frac{1}{d V_d} \right) \left( \frac{d}{D_n} \right)^d$. $R(D_n)$ is the rate-distortion function of $x_n$, $V_d$ is the volume of the unit sphere $S_d = \{x : \|x\| \leq 1\}$, and $\Gamma$ is the gamma function. Here, (8) follows from the independence of $v_{n-1}$ and the pair $(x_{n-1}, a_{n-1})$. Note that $h(v_{n-1}) = h(g)$ for all $n$. Hence, we obtain $D_n \geq L(1/k)^{1/d}$, where $L := \frac{d}{2} \left( \frac{2^h(g)}{d V_d} \right)^{1/d}$. This gives $|w_\eta(\pi^*, \mu) - w_\eta(\pi^k, \mu)| \geq L \frac{1}{1-\beta}(1/k)^{1/d}$ and $|w_A(\pi^*, \mu) - w_A(\pi^k, \mu)| \geq L(1/k)^{1/d}$.

V. APPROXIMATION OF RANDOMIZED STATIONARY POLICIES

In this section, we extend results developed for the deterministic case to randomized stationary policies. This extension is motivated by the facts that: (i) for a large class of average cost optimization problems, it is not known whether one can restrict the optimal policies to deterministic stationary policies, whereas the optimality of possibly randomized stationary policies can be established through the convex analytic method [1], [9], and (ii) randomized stationary policies are necessary in constrained MDPs even for the discounted cost (see e.g. [27]). Throughout this section we skip over all proofs since these follow by applying same steps as in the proofs given in Section [III] and [IV].

Throughout this section, we assume that conditions (a), (b), and (c) hold. Let $\pi \in RS$ be induced by a stochastic kernel $\eta(da|x)$ on $A$ given $X$. By Lemma 1.2 in [24] there exists a measurable function $f : X \times [0, 1] \to A$ such that for any $E \in B(A)$

$$\eta(E|x) = m(\{z : f(x, z) \in E\}),$$

where $m$ is the Lebesgue measure on $[0, 1]$. Equivalently, we can write $\eta(E|x)$ as

$$\eta(E|x) = \int_{[0,1]} \delta_{f(x,z)}(E)m(dz).$$

(9) Hence, $\pi$ can be represented as an (uncountable) convex combination of deterministic stationary policies parameterized by $[0,1]$. For each $z$, let $\{q_k(\cdot, z)\} \in Q(\Lambda_k)$ denote the sequence of quantizers that uniformly converges to $f(\cdot, z)$ defined in Section [III]. Note that such quantizers can be constructed so that the resulting function $q_k(x, z)$ is measurable. Hence, $|q_k(x, z)| = |\Lambda_k|$ for all $z \in [0, 1]$. Let $\{\pi^k\}$ be the sequence of randomized stationary policies induced by the stochastic kernels

$$\eta_k(\cdot|x) := \int_{[0,1]} \delta_{q_k(x,z)}(\cdot)m(dz).$$

(10)
The following assumptions are versions of assumptions imposed in Sections III and IV adapted to randomized stationary policies. They will be imposed as needed throughout this section.

\(\tilde{d}\) \(\sup_{\pi \in RS} \sum_{n=N+1}^{\infty} \int_{\mathcal{H}_\infty} c_n(x_n, a_n) P_{\mu}^\pi \to 0 \) as \(N \to \infty\).

\(\tilde{e}\) For any \(\pi \in RS\), \(Q_\pi\) has a unique invariant probability measure \(\nu_\pi\).

\(\tilde{f}1\) The set \(\Gamma_{RS} := \{\nu \in \mathcal{P}(X) : \nu Q_\pi = \nu \text{ for some } \pi \in RS\}\) is relatively sequentially compact in the setwise topology.

\(\tilde{f}2\) There exists an \(x \in X\) such that for all \(B \in B(X)\), \(Q_\pi^n(B|x) \to \nu_\pi(B)\) uniformly in \(\pi \in RS\).

\(\tilde{g}\) \(M := \bigcap_{\pi \in RS} M_\pi \neq \emptyset\).

\(\tilde{l}\) There exists a positive constant \(C\) and \(\kappa \in (0, 1)\) such that for all \(\pi \in RS\), there is a (necessarily unique) probability measure \(\nu_\pi \in \mathcal{P}(X)\) satisfying

\[\|\lambda_n^{x, \pi} - \nu_\pi\|_{TV} \leq C\kappa^n\] for all \(x \in X\) and \(n \geq 1\).

By adapting the proof of [23, Lemma 3.3] to randomized stationary policies, one can show that assumptions \((\tilde{e})\), \((\tilde{f}2)\), and \((\tilde{g})\) are satisfied under any of the conditions \((i)\), \(i \in \{1, 2, \ldots, 4\}\) in [23] Section 3.3. Moreover, \(M = X\) in \((\tilde{g})\) if at least one of the above conditions holds. Furthermore, the statement in Proposition [3.2] remains true if we replace \(S\) with \(RS\). Hence, assumption \((e1)\) implies \((\tilde{e})\), \((\tilde{f}1)\) and \((\tilde{g})\) with \(M = X\) in \((\tilde{g})\) if the invariant measures are unique.

**Example 5.5.** Let us again consider the additive-noise system of Example 1 with the same assumptions. Recall that boundedness of \(F\) implies assumption \((e1)\). On the other hand, it also implies condition \((2)\) in [23] Section 3.3. Hence, if \(F\) has a bounded range, then \((\tilde{e})\), \((\tilde{f}1)\), \((\tilde{f}2)\) and \((\tilde{g})\) with \(M = X\) hold.

The first result in this section deals with problem \((P1)\) for randomized policies. Recall that \(w_t\), \(w_\beta\) and \(w_A\), respectively, denote the total, discounted, and average costs.

**Theorem 5.1.** Suppose assumptions \((a)\), \((b)\), \((c)\) hold. Let \(\pi \in RS\) and \(\{\pi^k\}\) be the quantized approximations of \(\pi\). Then, \(w_\beta(\pi^k, \mu) \to w_\beta(\pi, \mu)\) as \(k \to \infty\). The same statement is true for \(w_t\) if we further impose assumption \((\tilde{d})\). Furthermore, if \(\mu\) be concentrated on some \(x \in M\), then \(w_A(\pi^k, \mu) \to w_A(\pi, \mu)\) as \(k \to \infty\) under the assumptions \((\tilde{e})\), \((\tilde{f}1)\) or \((\tilde{f}2)\), and \((\tilde{g})\).

The next result deals with problem \((P2)\) in the randomized setting.

**Theorem 5.2.** Let \(\pi \in RS\) and \(\{\pi^k\}\) be the quantized approximations of \(\pi\). Under assumptions
(h), (j) and (k), for any initial distribution $\mu$ we have

$$|w_{\beta}(\pi, \mu) - w_{\beta}(\pi^k, \mu)| \leq \frac{1}{k} \frac{1}{d} K,$$

and on the other hand for all $x \in X$ and all $n \geq 1$

$$|w_A(\pi, x) - w_A(\pi^k, x)| \leq 2MC\kappa^n + K_n(1/k)^{1/d}$$

if we further assume (i). Here, $K$, $K_n$ ($n \geq 1$) and $M$ are as in Theorems 4.1 and 4.2.

VI. Conclusion

In this paper, the problem of approximating deterministic stationary policies in MDPs was considered for total, discounted, and average costs. We introduced deterministic stationary quantizer policies and showed that any deterministic stationary policy can be approximated with an arbitrary precision by such policies. We also found upper bounds on the approximation errors in terms of the rates of the quantizers. These results were then extended to randomized stationary policies.

One direction for future work is to establish similar results for approximations where the set of admissible quantizers has a certain structure, such as the set of quantizers having convex codecells [25], which may give rise to practical design methods. Moreover, if one can obtain further results on the structure of optimal policies (e.g., by showing that an optimal policy satisfies a Lipschitz property with a known bound on the constant), the results in this paper may be directly applied to obtain approximation bounds for quantized policies. As a final remark, since setwise continuity assumption might be too restrictive in certain important cases, it is of interest to study a version of this problem where the setwise continuity assumption is replaced with the weak continuity in the state-action variables.

APPENDIX

A. Proof of Proposition 3.1

We need to prove that $P_{\mu}^{\pi_k}(g) \to P_{\mu}^{\pi}(g)$ for any $g \in \bigcup_{n=0}^{\infty} C(H_n)$. Suppose $g \in C(H_n)$ for some $n$. Then we have $P_{\mu}^{\pi_k}(g) = \lambda_{(n)}^{\pi_k,\mu}(g_k)$ and $P_{\mu}^{\pi}(g) = \lambda_{(n)}^{\pi,\mu}(g_f)$. Note that both $g_f$ and $g_k$ ($k \geq 1$) are uniformly bounded. Since $g$ is continuous in the “a” terms by definition and $g_k$ converges to $f$, we have $g_k \to g_f$. Hence, by Theorem 3.1 it is enough to prove that $\lambda_{(n)}^{\pi_k,\mu} \to \lambda_{(n)}^{\pi,\mu}$ setwise as $k \to \infty.$
We will prove this by induction. Clearly, \( \lambda_{(1)}^{\pi_k,\mu} \to \lambda_{(1)}^{\pi,\mu} \) setwise by assumption (a). Assume the claim is true for some \( n \geq 1 \). For any \( h \in B(\mathcal{X}^{n+2}) \) we can write \( \lambda_{(n+1)}^{\pi_k,\mu}(h) = \lambda_{(n)}^{\pi_k,\mu}(\lambda_{(1)}^{\pi_k,\pi}(h)) \) and \( \lambda_{(n+1)}^{\pi,\mu}(h) = \lambda_{(n)}^{\pi,\mu}(\lambda_{(1)}^{\pi,\pi}(h)) \). Since \( \lambda_{(1)}^{\pi,\pi}(h) \to \lambda_{(1)}^{\pi,\nu}(h) \) for all \( (x_0, \ldots, x_n) \in \mathcal{X}^{n+1} \) by assumption (a) and \( \lambda_{(n)}^{\pi_k,\mu} \to \lambda_{(n)}^{\pi,\mu} \) setwise, we have \( \lambda_{(n+1)}^{\pi_k,\mu}(h) \to \lambda_{(n+1)}^{\pi,\mu}(h) \) by Theorem 3.1 which completes the proof.

B. Proof of Theorem 3.3

Let \( Q_\pi \) and \( Q_{\pi^k} \) be the stochastic kernels, respectively, for \( \pi \) and \( \{\pi^k\} \) defined in (3). By assumption (e), \( Q_\pi \) and \( Q_{\pi^k} \) \( (k \geq 1) \) have unique, and so ergodic, invariant probability measures \( \nu_\pi \) and \( \nu_{\pi^k} \), respectively. Since \( x \in M \), we have \( w_A(\pi^k, \mu) = \nu_{\pi^k}(c_{q_k}) \) and \( w_A(\pi, \mu) = \nu_\pi(c_f) \). Observe that \( c_{q_k}(x) \to c_f(x) \) for all \( x \) by assumption (c). Hence, if we prove \( \nu_{\pi^k} \to \nu_\pi \) setwise, then by Theorem 3.1 we have \( w_A(\pi^k, \mu) \to w_A(\pi, \mu) \). We prove this first under (f1) and then under (f2).

I) Proof under assumption (f1)

We show that every setwise convergent subsequence \( \{\nu_{\pi^k_l}\} \) of \( \{\nu_{\pi^k}\} \) must converge to \( \nu_\pi \). Then, since \( \Gamma_n \) is relatively sequentially compact in the setwise topology, there is at least one setwise convergent subsequence \( \{\nu_{\pi^k_l}\} \) of \( \{\nu_{\pi^k}\} \), which implies the result.

Let \( \nu_{\pi^k_l} \to \nu \) setwise for some \( \nu \in \mathcal{P}(\mathcal{X}) \). We will show that \( \nu = \nu_\pi \) or equivalently \( \nu \) is an invariant probability measure of \( Q_\pi \). For simplicity, we write \( \{\nu_{\pi^l}\} \) instead of \( \{\nu_{\pi^k_l}\} \). Let \( g \in B(\mathcal{X}) \). Then by assumption (e) we have

\[
\nu_{\pi^l}(g) = \nu_{\pi^l}(Q_{\pi^l}g).
\]

Observe that by assumption (a), \( Q_{\pi^l}g(x) \to Q_\pi g(x) \) for all \( x \). Since \( Q_{\pi^l}g(x) \) and \( Q_{\pi^l}g(x) \) \( (l \geq 1) \) are uniformly bounded and \( \nu_{\pi^l} \to \nu \) setwise, we have \( \nu_{\pi^l}(Q_{\pi^l}g) \to \nu_{\pi}(Q_{\pi}g) \) by Theorem 3.1. On the other hand since \( \nu_{\pi^l} \to \nu \) setwise we have \( \nu_{\pi^l}(g) \to \nu(g) \). Thus \( \nu(g) = \nu(Q_{\pi}g) \). Since \( g \) is arbitrary, \( \nu \) is an invariant probability measure for \( Q_\pi \).

II) Proof under assumption (f2)

Observe that for all \( x \in \mathcal{X} \) and all \( n \), \( \lambda_n^{\pi_k,\pi} \to \lambda_n^{\pi,\pi} \) setwise as \( k \to \infty \) since \( P_{\pi^k}^{\infty} \to P_\pi^{\infty} \) in the \( ws^\infty \) topology (see Proposition 3.1). Let \( B \in \mathcal{B}(\mathcal{X}) \) be given and fix some \( \varepsilon > 0 \). By assumption (f2) we can choose \( N \) large enough such that \( |\lambda_{N}^{\pi}(B) - \nu_{\pi}(B)| < \varepsilon/3 \) for all
\( \bar{\pi} \in \{\pi, \pi^1, \pi^2, \ldots\} \). For this \( N \), choose \( K \) large enough such that \( |\lambda_{N}^{x_k} - \lambda_{N}^{x_1}| < \varepsilon/3 \) for all \( k \geq K \). Thus, for all \( k \geq K \) we have
\[
|\nu_{\pi_k}(B) - \nu_{\pi}(B)| \leq |\nu_{\pi_k}(B) - \lambda_{N}^{x_k} - \lambda_{N}^{x_1}| + |\lambda_{N}^{x_k} - \lambda_{N}^{x_1}| + |\lambda_{N}^{x_1} - \nu_{\pi}(B)| < \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, we obtain \( \nu_{\pi_k}(B) \rightarrow \nu_{\pi}(B) \), which completes the proof.

**C. Proof of Proposition 4.3**

We will prove this result by induction. Let \( \mu \) be an arbitrary initial distribution and fix \( k \). For \( n = 1 \) the claim holds by the following argument:
\[
\|\lambda_{1}^{\pi^k} - \lambda_{1}^{\pi^1}\|_{TV} = 2 \sup_{B \in \mathcal{B}(X)} \left| \mu(\lambda_{1}^{\pi^k}(B)) - \mu(\lambda_{1}^{\pi^1}(B)) \right|
\leq \mu\left(\|\lambda_{1}^{\pi^k} - \lambda_{1}^{\pi^1}\|_{TV}\right)
\leq \mu(K_2d_A(\pi, q_k) (by assumption (k))
\leq \sup_{x \in X} K_2d_A(f(x), q_k)) \leq (1/k)^{1/d}K_2\alpha \quad (by \ Lemma \ 4.1).
\]
Observe that the bound \( \alpha K_2(2n - 1)(1/k)^{1/d} \) is independent of the choice of initial distribution \( \mu \) for \( n = 1 \). Assume the claim is true for \( n \geq 1 \). Then we have
\[
\|\lambda_{n+1}^{\pi^k} - \lambda_{n+1}^{\pi^{k+1}}\|_{TV} = 2 \sup_{B \in \mathcal{B}(X)} \left| \lambda_{1}^{\pi^k}(\lambda_{n}^{\pi^x}(B)) - \lambda_{1}^{\pi^{k+1}}(\lambda_{n}^{\pi^{x+1}}(B)) \right|
\leq \lambda_{1}^{\pi^k}(\lambda_{n}^{\pi^x} - \lambda_{n}^{\pi^{x+1}}) + 2\|\lambda_{1}^{\pi^k} - \lambda_{1}^{\pi^{k+1}}\|_{TV}
\leq (1/k)^{1/d}(2n - 1)K_2\alpha + 2(1/k)^{1/d}K_2\alpha
\leq \alpha K_2(2(n + 1) - 1)(1/k)^{1/d}\alpha.
\]
Here (11) follows since
\[
|\mu(h) - \eta(h)| \leq \|\mu - \eta\|_{TV} \sup_{x \in X} |h(x)|
\]
and (12) follows since the bound \( \alpha K_2(2n - 1)(1/k)^{1/d} \) is independent of the initial distribution.
D. Proof of Theorem 4.1

For any fixed $k$, we have

$$|w_{\beta}(\pi) - w_{\beta}(\pi^k)| = \left| \sum_{n=0}^{\infty} \beta^n \lambda_n^{\pi_k}(c_f) - \sum_{n=0}^{\infty} \beta^n \lambda_n^{\pi_k}(c_{q_k}) \right|$$

$$\leq \sum_{n=0}^{\infty} \beta^n \left( |\lambda_n^{\pi_k}(c_f) - \lambda_n^{\pi_k}(c_{q_k})| + |\lambda_n^{\pi_k}(c_{q_k})| \right)$$

$$\leq \sum_{n=0}^{\infty} \beta^n \left( \sup_{x_n \in \mathcal{X}} |c_f - c_{q_k}| + \|\lambda_n^{\pi_k} - \lambda_n^{\pi_k}\|_{TV} M \right)$$

$$\leq \sum_{n=0}^{\infty} \beta^n \left( \sup_{x_n \in \mathcal{X}} d_A(f(x_n), q_k(x_n)) K_1 \right)$$

$$+ \sum_{n=1}^{\infty} \beta^n \left( (1/k)^{1/d}(2n-1)K_2\alpha M \right)$$

$$\leq \sum_{n=0}^{\infty} \beta^n \left( (1/k)^{1/d}\alpha K_1 \right) + \sum_{n=1}^{\infty} \beta^n \left( (1/k)^{1/d}(2n-1)K_2\alpha M \right) \quad \text{(by Lemma 4.1)}$$

$$= (1/k)^{1/d}\alpha(K_1 - \beta K_2 M) \frac{1}{1 - \beta} + (1/k)^{1/d}2K_2\alpha M \frac{\beta}{(1 - \beta)^2}$$

$$= (1/k)^{1/d} \frac{\alpha}{1 - \beta}(K_1 - \beta K_2 M + \frac{2\beta M K_2}{1 - \beta}).$$

Here (13) follows from Assumption (j) and Proposition 4.3. This completes the proof.

E. Proof of Theorem 4.2

For any $k$ and $x \in \mathcal{X}$, we have

$$|w_A(\pi, x) - w_A(\pi^k, x)| = |\nu_\pi(c_f) - \nu_\pi(c_{q_k})|$$

$$\leq |\nu_\pi(c_f) - \nu_\pi(c_{q_k})| + |\nu_\pi(c_{q_k}) - \nu_\pi(c_{q_k})|$$

$$\leq \sup_{x \in \mathcal{X}} |c_f - c_{q_k}| + \|\nu_\pi - \nu_\pi\|_{TV} \sup_{x \in \mathcal{X}} |c_{q_k}|$$

$$\leq \sup_{x \in \mathcal{X}} K_1 d_A(f(x), q_k(x)) + \|\nu_\pi - \nu_\pi\|_{TV} M \quad \text{(by assumption (j))}$$

$$\leq (1/k)^{1/d}K_1 \alpha + (\|\nu_\pi - \lambda_n^{\pi, x}\|_{TV} + \|\lambda_n^{\pi, x} - \lambda_n^{\pi_k, x}\|_{TV} + \|\lambda_n^{\pi_k, x} - \nu_\pi\|_{TV}) M$$

$$\leq (1/k)^{1/d}K_1 \alpha + (2C\kappa^n + (1/k)^{1/d}(2n-1)K_2\alpha M) \quad \text{(14)}$$

$$= 2MC\kappa^n + ((2n-1)K_2\alpha M + K_1 \alpha)(1/k)^{1/d},$$

where (14) follows from assumption (l) and Proposition 4.3.
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