ON THE POSITIVITY OF CERTAIN THETA KERNELS

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Abstract. In this work we first show that the theta kernel \( \Theta_f(z) \) of certain genus 0 entire function \( f(z) \) is positive on \((0, \infty)\) if it is eventually logarithmic concave on \((0, \infty)\), and its zeros are real-part dominating. This proof depends on an observation that the Laplace transforms of \( \phi(t) \) and \( t^{\alpha-1} \phi(1/t) \) are related through a Hankel-type transform when \(-1 < \alpha < \frac{1}{2}\). Then we show that \( f(z) \) has only negative zeros by applying the generalized Laguerre polynomial \( L_n^{(\alpha)}(x) \) and one of its connections to the Bessel function \( J_\nu(z) \). By applying our results to \( f(z) = \xi(1/2 + \sqrt{x}) \) we may have essentially solved a problem posted by K. Gröchenig indirectly.

1. Introduction

Given an entire function \( f(x) \) of genus 0, we define its associated theta kernel to be \( \Theta_f(z) = \sum_{n=0}^{\infty} f(n) e^{zn} \) whenever it converges. It is known that a Hankel-type transform can connect the Laplace transforms of a function \( \phi(t) \) and its modified reversed function \( t^{\alpha-1} \phi(1/t) \), \([7]\), which is a kind of generalization of the Jacobi inversions for the classical Jacobi theta functions, \([2, 5, 12]\). In this work we first apply this connection and the properties of the Bessel function \( J_\nu(z) \) to establish that the theta kernel \( \Theta_f(z) \) of a genus 0 entire function \( f(z) \) is positive if its zeros are real-part dominating, and \( f(x) \) is also logarithmic concave on a tail of \((0, \infty)\). This positivity implies that both \( 1/f(x) \) and \( f'(x)/f(x) \) are completely monotonic on \((0, \infty)\). We apply this result to the entire function \( f(z) = \xi(1/2 + \sqrt{x}) \) to prove that both

\[
\frac{\xi'(1/2 + \sqrt{x})}{\sqrt{x}\xi(1/2 + \sqrt{x})}, \quad \frac{1}{\xi(1/2 + \sqrt{x})}
\]

are completely monotonic on \((0, \infty)\) where \( \xi(s) \) is the Riemann xi function \([2, 5, 6, 11]\). Then the Hausdorff–Bernstein–Widder theorem implies that \( 1/\xi(1/2 + \sqrt{x}) \) must be a Laplace transform of a finite nonnegative Borel measure on \([0, \infty)\), and it partially settles a question posted by K. Gröchenig in \([11]\) whether \( 1/\xi(1/2 + \sqrt{x}) \) can be expressed as a Laplace transform of a totally positive function \( \Lambda(y) \),

\[
\frac{1}{\xi(1/2 + \sqrt{x})} = \int_0^\infty \Lambda(y)e^{-xy}dy, \quad x \geq 0.
\]

which is equivalent to the celebrated Riemann hypothesis. Then we apply the generalized Laguerre polynomial \( L_n^{(\alpha)}(x) \) and one of its connections to the Bessel

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function $J_\nu(z)$ to prove that $f(z)$ can have only negative zeros, which essentially settles Gröchenig’s question without verifying the total positivity of $\Lambda(y)$.

According to the late Hungarian mathematician Paul Turán, special functions are the mathematical functions that appear often in mathematics, science and technology. Many interesting special functions are entire functions of genus at most 1. An entire function $f(z)$ of genus at most 1 with $f(0) \neq 0$ has the Hadamard factorization, \[ f(z) = e^{az + b} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{\mu_n} \right) \exp \left( \frac{z}{\mu_n} \right), \] where $a, b$ are complex numbers, $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ are roots of $f(z)$ with $\sum_{n=1}^{\infty} \frac{1}{|\mu_n|^2} < \infty$.

If the parameters satisfy
\[ \sum_{n=1}^{\infty} \frac{1}{|\mu_n|} < \infty, \quad a = -\sum_{n=1}^{\infty} \frac{1}{\mu_n}, \]
then $f(z)$ is a genus 0 entire function with the infinite product expansion,
\[ f(z) = f(0) \prod_{n=1}^{\infty} \left( 1 - \frac{z}{\mu_n} \right). \]

Many well-known classical special functions are entire functions of genus 1. For example, \[ J_\nu(z), \quad K_{iz}(a), \quad \Xi(z), \quad \frac{1}{\Gamma(z + \nu + 1)}, \quad z \in \mathbb{C}, \]
where $\nu > -1, \quad a > 0 \quad J_\nu(z)$ is the Bessel function of the first kind, $K_{iz}(a)$ the modified Bessel function of the second kind, $\Gamma(z)$ the Euler Gamma function, and $\Xi(z)$ the Riemann Xi function. Most of the popular $q$-special functions are genus 0 entire functions, \[ E_q(x), \quad A_q(z), \quad \frac{J_{\nu}^{(2)}(z; q)}{z^\nu}, \quad \frac{J_{\nu}^{(3)}(z; q)}{z^\nu}; \quad 0 < q < 1. \]

Notice that we can also produce genus 0 entire functions from even entire functions of genus at most 1. Let $f(z)$ be an even entire function of genus at most 1 with $f(0) \neq 0$, then
\[ g(z) = f \left( iz^{1/2} \right) \]
is an entire function of genus 0, for example, $K_{z^{1/2}}(a), \quad \Xi \left( iz^{1/2} \right)$ are entire functions of genus 0. Clearly, here the entire function $f(z)$ has only nonzero real roots is the same as that $g(z)$ has only negative roots.

2. Preliminaries

We shall use the usual notations: $\mathbb{C}$ for complex numbers, $\mathbb{R}$ for real numbers, $\mathbb{Z}$ for all integers, $\mathbb{N}_0$ for all nonnegative integers and $\mathbb{N}$ for positive integers. If $z \in \mathbb{C}$, then $\Re(z), \quad \Im(z)$ denote the real and imaginary part of $z$ respectively, and $f^{(n)}(z)$ represents the $n$-th derivative of $f(z)$ with respect to $z$. 
Definition 1. \cite{15} Let $F : (0, \infty) \mapsto \mathbb{R}$ be a $C^\infty(0, \infty)$ function such that $F(x) \geq 0$ on $(0, \infty)$. Function $F(x)$ is completely monotonic if $(-1)^n F^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}_0$ and $x > 0$, and it is a Bernstein function if $(-1)^{n-1} F^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}$ and $x > 0$.

Clearly, the Bernstein and completely monotonic functions are closely connected through their definitions. The derivative of a Bernstein function is completely monotonic, whereas the primitive of a completely monotonic function is a Bernstein function if it is also nonnegative on $(0, \infty)$. Furthermore, the function $F$ is a Bernstein function if, and only if, for every completely monotonic function $G$ the composition $G \circ F$ is completely monotonic, \cite{15} Theorem 3.7]. The Hausdorff–Bernstein–Widder theorem states that a function $G(x)$ is completely monotonic if and only if there exists a finite nonnegative Borel measure $d\mu(t)$ on $[0, \infty)$ such that

\begin{equation}
G(x) = \int_0^\infty e^{-xt} d\mu(t), \quad x > 0.
\end{equation}

When the measure is absolutely continuous,

\begin{equation}
d\mu(t) = \varphi(t) dt, \quad \varphi \geq 0, \text{ a. e.},
\end{equation}

then

\begin{equation}
G(x) = \int_0^\infty e^{-xt} \varphi(t) dt, \quad x > 0.
\end{equation}

Given a general Laplace integral $G(x)$ without the positivity restrictions on $d\mu(t)$ and $\varphi(t)$, the Laplace inversion is the process to find $d\mu(t)$ in (2.1) or $\varphi(t)$ in (2.3). There are many Laplace inversion methods known today, both real and complex. One of the real methods is the popular Post-Widder Laplace inversion theorem, \cite{19}. The following special case in \cite{18} Theorem 4] will be suffice for our purpose.

Lemma 2. Let

\begin{equation}
f(x) = \int_0^\infty e^{-xt} \varphi(t) dt,
\end{equation}

where $\varphi(t)$ is Lebesgue integrable and uniformly bounded in the interval $(0, \infty)$. Then

\begin{equation}
\lim_{k \to \infty} \frac{(-1)^k f^{(k)}(k/t) k^{k+1}}{k! t^{k+1}} = \varphi(t)
\end{equation}

almost everywhere for $t \in (0, \infty)$.

Let $J_\alpha(z)$ be the Bessel function of the first kind defined by, \cite{5} (10.2.2)]

\begin{equation}
J_\alpha(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left( \frac{z}{2} \right)^{2n+\alpha}, \quad z \in \mathbb{C}, \quad \alpha > -1,
\end{equation}

where $\Gamma(z)$ is the Euler gamma function defined as in \cite{5} (5.2.1)]. Then for $\alpha > -1$, a version of Hankel transform is defined by \cite{5} (10.22.76)]

\begin{equation}
g(y) = \int_0^\infty f(x) J_\alpha(xy) (xy)^\alpha dx, \quad y > 0.
\end{equation}
The following lemma connects the Laplace transforms of a function $\varphi(t)$ and its modified reversed function $t^{\alpha-1}\varphi(1/t)$, and in this work we shall apply the case $\alpha = 0$, which reminds us of the Jacobi inversion transform for the classical theta functions. \cite[(20.7.30-33)]{5}.

**Lemma 3.** Assume that $\varphi(t) \not\equiv 0$ is continuous and uniformly bounded on $(0, \infty)$ such that there exists a positive number $c$ such that

\begin{equation}
\varphi(t) = O(e^{-ct}), \quad t \to +\infty.
\end{equation}

Let

\begin{equation}
u_{\alpha}(x) = \int_{0}^{\infty} e^{-xt}t^{\alpha-1}\varphi(t^{-1})dt, \quad x > 0
\end{equation}

exists. Furthermore, for $\frac{1}{2} > \alpha > -1$, $t^{\alpha-1}\varphi(t^{-1})$ is continuous and uniformly bounded on $(0, \infty)$, and its Laplace transform

\begin{equation}
v_{\alpha}(x) = \int_{0}^{\infty} e^{-xt}t^{\alpha-1}\varphi(t^{-1})dt, \quad x > 0
\end{equation}

also exists. Then by the Hausdorff–Bernstein–Widder theorem $u(x)$ is completely monotonic on $(0, \infty)$ if and only if $\nu_{\alpha}(x)$ is.

Moreover, for any $x > 0$ and $k \in \mathbb{N}_0$,

\begin{equation}
(-1)^{k} \frac{d^{k}v_{\alpha}(x)}{dx^{k}} = \int_{0}^{\infty} \frac{J_{\alpha+k}(2\sqrt{xy})}{(\sqrt{xy})^{(\alpha+k)}} y^{\alpha}u(y)dy.
\end{equation}

Consequently, $\nu_{\alpha}(x)$ is completely monotonic if and only if

\begin{equation}
\int_{0}^{\infty} J_{\alpha+k}(\sqrt{xy})y^{(\alpha-k)/2}u(y)dy > 0
\end{equation}

for all $x > 0$ and all $k \in \mathbb{N}_0$.

**Proof.** Since $\varphi(t)$ is continuous and uniformly bounded on $(0, \infty)$, then by (2.8) we see that $u(x)$ exists for all $x \geq 0$. Similarly, they also imply that $t^{\alpha-1}\varphi(t^{-1})$ on $(0, \infty)$ is continuous and uniformly bounded on $(0, \infty)$. This can be seen from

\begin{equation}
t^{\alpha-1}\varphi(t^{-1}) = O((t^{\alpha-1}), \quad t \to +\infty
\end{equation}

and

\begin{equation}
t^{\alpha-1}\varphi(t^{-1}) = O\left(t^{\alpha-1}e^{-c/t}\right), \quad t \to +0.
\end{equation}

Therefore, for any real number $\alpha$ with $-1 < \alpha < \frac{1}{2}$, the Laplace transform

\begin{equation}
v_{\alpha}(x) = \int_{0}^{\infty} e^{-xt}t^{\alpha-1}\varphi(t^{-1})dt, \quad x > 0
\end{equation}

also exists. Then by the Hausdorff–Bernstein–Widder theorem $u(x)$ is completely monotonic on $(0, \infty)$ if and only if $\varphi(t)$ is nonnegative on $(0, \infty)$, and if only if $v_{\alpha}(x)$ is completely monotonic.

For $x \geq 0$, let

\begin{equation}
w(x) = \int_{0}^{\infty} e^{-xt} |\varphi(t)| dt.
\end{equation}
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Since $\varphi(x)$ is continuous and uniformly bounded on $(0, \infty)$ and satisfying (2.8), then we have

\begin{equation}
0 < w(x) \leq w(0) = \int_0^\infty |\varphi(t)| \, dt < \infty, \quad x \geq 0,
\end{equation}

which gives

\begin{equation}
\int_0^1 y^\alpha w(y) \, dy \leq w(0) \int_0^1 y^\alpha \, dy = \frac{w(0)}{1 + \alpha} < \infty.
\end{equation}

Notice that for $-1 < \alpha < \frac{1}{2}$ we have $1/4 < (\alpha - 1/2)/2 + 1 < 1$ and

\begin{align*}
&\int_1^\infty y^{\alpha/2-1/4} w(y) \, dy = \int_0^\infty \int_1^\infty y^{\alpha/2-1/4} e^{-yt} \, dy \, |\varphi(t)| \, dt \\
&= \int_0^\infty \left( \int_1^\infty y^{\alpha/2-1/4} e^{-yt} \, dy \right) |\varphi(t)| \, dt \\
&\leq \Gamma \left( \frac{\alpha - 1/2}{2} + 1 \right) \int_0^\infty |\varphi(t)| \, dt \frac{1}{t^{(\alpha-1/2)/2+1}} < \infty.
\end{align*}

Then we combine (2.18) and (2.19) to obtain that

\begin{equation}
\int_0^1 y^\alpha w(y) \, dy + \int_1^\infty y^{\alpha/2-1/4} w(y) \, dy < \infty.
\end{equation}

For any fixed $x > 0$ and $\alpha$ with $-1 < \alpha < \frac{1}{2}$, by (2.10), it is clear that

\begin{align*}
&\int_0^\infty \int_0^\infty |J_\alpha(2\sqrt{xy})| y^{\alpha/2} e^{-yt} \, |\varphi(t)| \, dt \, dy \\
&= \int_0^\infty |J_\alpha(2\sqrt{xy})| y^{\alpha/2} \left( \int_0^\infty e^{-yt} \, |\varphi(t)| \, dt \right) \, dy \\
&= \int_0^1 |J_\alpha(2\sqrt{xy})| y^{\alpha/2} w(y) \, dy + \int_1^\infty |J_\alpha(2\sqrt{xy})| y^{\alpha/2} w(y) \, dy \\
&= O \left( \int_0^1 y^\alpha w(y) \, dy + \int_1^\infty y^{\alpha/2-1/4} w(y) \, dy \right) < \infty,
\end{align*}

where we have applied that for $\alpha > -1$ and $\Re(z) > 0$, \[2\] [3]

\begin{equation}
J_\alpha(z) = O(z^\alpha), \quad z \to +0
\end{equation}

and

\begin{equation}
J_\alpha(z) = O(z^{-1/2}), \quad z \to +\infty.
\end{equation}

Then for any $x, t > 0$, $\frac{1}{2} > \alpha > -1$, by \[3\] (10.22.51) we get

\begin{equation}
x^{-\alpha/2} \int_0^\infty J_\alpha(2\sqrt{xy}) y^{\alpha/2} e^{-yt} \, dy = \frac{e^{-x/t}}{t^{\alpha+1}}.
\end{equation}
Then we apply (2.24) and the Fubini's theorem to obtain
\[
v_\alpha(x) = \int_0^\infty e^{-xt}t^{\alpha-1}\varphi(t^{-1})dt = \int_0^\infty e^{-x/t}t^{\alpha-1}\varphi(t)dt
\]
(2.25)
\[
v_\alpha(x) = \int_0^\infty \left(x^{-\alpha/2}\int_0^\infty J_\alpha(2\sqrt{xy})y^{\alpha/2}e^{-yt}dy\right)\varphi(t)dt
\]
\[
v_\alpha(x) = x^{-\alpha/2}\int_0^\infty J_\alpha(2\sqrt{xy})y^{\alpha/2}\left(\int_0^\infty e^{-yt}\varphi(t)dt\right)dy
\]
\[
v_\alpha(x) = x^{-\alpha/2}\int_0^\infty J_\alpha(2\sqrt{xy})y^{\alpha/2}u(y)dy.
\]

Since for \(z \in \mathbb{C}\setminus(-\infty,0], \alpha > -1, \) and \(k \in \mathbb{N}_0, [5, (10.6.6)], [10, 7.3(17)]\)
\[
(\frac{1}{z}d\frac{d}{dz})^k \left(\frac{J_\alpha(z)}{z^\alpha}\right) = (-1)^k \frac{J_{\alpha+k}(z)}{z^{\alpha+k}},
\]
and for \(x > 0,\)
\[
\frac{df(x)}{dx} = 2 \left(\frac{1}{2\sqrt{x}}\frac{d}{d(2\sqrt{x})}\right) f(x),
\]
we get for \(x, y > 0, \alpha > -1, \) and \(k \in \mathbb{N}_0\)
\[
(-1)^k \frac{d^k}{dx^k} \left(\frac{J_\alpha(2\sqrt{xy})}{(2\sqrt{xy})^\alpha}\right) = \frac{J_{\alpha+k}(2\sqrt{xy})}{(xy)^{(\alpha+k)/2}}.
\]

Then for \(k \in \mathbb{N}_0, -1 < \alpha < \frac{1}{2}, \) and \(x, y > 0,\)
\[
\int_0^\infty |J_{\alpha+k}(2\sqrt{xy})| y^{(\alpha-k)/2} |u(y)| dy
\]
\[
\leq \int_0^\infty |J_{\alpha+k}(2\sqrt{xy})| y^{(\alpha-k)/2} w(y) dy
\]
\[
= \int_0^1 |J_{\alpha+k}(2\sqrt{xy})| y^{(\alpha-k)/2} w(y) dy
\]
\[
+ \int_1^\infty |J_{\alpha+k}(2\sqrt{xy})| y^{(\alpha-k)/2} w(y) dy
\]
(2.29)
\[
= \mathcal{O}\left(\int_0^1 y^\alpha w(y)dy + \int_1^\infty y^{\alpha/2-1/4-k/2} w(y)dy\right)
\]
\[
= \mathcal{O}\left(\int_0^1 y^\alpha w(y)dy + \int_1^\infty y^{\alpha/2-1/4} w(y)dy\right) < \infty,
\]
where we applied \(y^{-k/2} \leq 1\) for \(y \geq 1, k \geq 0.\) Thus,
\[
(-1)^k \frac{d^k v_\alpha(x)}{dx^k} = \int_0^\infty (-1)^k \frac{d^k}{dx^k} \left(\frac{J_\alpha(2\sqrt{xy})}{(2\sqrt{xy})^\alpha}\right) y^{\alpha} u(y) dy
\]
(2.30)
\[
= \int_0^\infty J_{\alpha+k}(2\sqrt{xy}) y^{\alpha} u(y) dy = \frac{2^{(\alpha+k)}}{(2\sqrt{x})^{\alpha+k}} \int_0^\infty J_{\alpha+k}(\sqrt{4xy}) y^{\alpha-k} u(y) dy.
\]

On the other hand, if (2.12) is true, then apply the definition of a completely monotonic function to (2.11) we see that \(v_\alpha(x)\) is completely monotonic. Then by (2.15) and the Hausdorff–Bernstein–Widder theorem we conclude that \(t^{\alpha-1}\varphi(t^{-1})\) is nonnegative for \(t > 0,\) which implies that \(\varphi(t)\) is nonnegative for \(t > 0.\) Then
by (2.10) and the Hausdorff–Bernstein–Widder theorem we have proved that \( u(x) \) is completely monotonic for \( x > 0 \).

\[ \Box \]

3. Main Results

**Theorem 4.** Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_0 > 0, \quad a_n \geq 0, \quad n \in \mathbb{N} \]

be a genus 0 entire function such that all its roots \( \{-\lambda_n | n \in \mathbb{N}\} \) are real-part dominating, i.e. satisfy

\[ \Re(\lambda_n) \geq C |\lambda_n|, \quad n \in \mathbb{N} \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^\rho} < \infty, \]

where \( \rho, C \in (0, 1) \) are two positive constants depending on \( f(z) \) only.

Then the theta kernel

\[ \Theta(t) = \Theta(t|f) = \sum_{n=1}^{\infty} e^{-\lambda_n t}, \quad t > 0 \]

is a \( C^\infty(0, \infty) \) function such that for each \( k \in \mathbb{N}_0 \),

\[ \Theta^{(k)}(t) = \mathcal{O}(t^{-\rho-k}), \quad t \to 0^+ \]

and

\[ \Theta^{(k)}(t) = \mathcal{O}(e^{-ct}), \quad t \to +\infty, \]

where \( c \) is any real number \( c < C \inf \{|\lambda_n| | n \in \mathbb{N}\} \). Furthermore, it satisfies

\[ \frac{d\log f(x)}{dx} = \frac{f'(x)}{f(x)} = \int_0^\infty e^{-xt}\Theta(t)dt, \quad x > 0. \]

If additionally \( f(x) \) is eventually logarithmic concave, i.e. there exists a real number \( \beta \geq 0 \) such that

\[ \frac{d^2 \log f(x)}{dx^2} = \frac{d}{dx} \left( \frac{f'(x)}{f(x)} \right) \leq 0, \quad x \geq \beta, \]

then \( \Theta(t) \geq 0 \) for \( t > 0 \). In this case \( \log \frac{f(x)}{f(0)} \) is a Bernstein function, both \( f'(x)/f(x) \) and \( 1/f(x) \) are completely monotonic on \( (0, \infty) \).

**Proof.** Since for any \( t > 0 \) and \( k \in \mathbb{N}_0 \),

\[ e^t > \frac{t^{k+1}}{(k+1)!}, \quad t^k e^{-t} < \frac{(k+1)!}{t}, \]
then,
\[
\sum_{n=1}^{\infty} |\lambda_n^k e^{-\lambda_n t}| = \sum_{n=1}^{\infty} |\lambda_n|^k e^{-\Re(\lambda_n)t}
\]
(3.10)
\[
\leq \sum_{n=1}^{\infty} |\lambda_n|^k e^{-C|\lambda_n|t} \leq \frac{(k+1)t}{(tC)^{k+1}} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty,
\]
which converges uniformly on any compact subset of \(t \in (0, \infty)\). Thus, \(\Theta(t) \in C^\infty(0, \infty)\) and
\[
(-1)^k \Theta^{(k)}(t) = \sum_{n=1}^{\infty} \lambda_n^k e^{-\lambda_n t}.
\]
(3.11)
It is clear that for any \(c < C \inf \left\{ |\lambda_n| \mid n \in \mathbb{N} \right\}\), by (3.10) and the Lebesgue’s dominated convergent theorem we get (3.6).

Observe that for any \(k \in \mathbb{N}_0\) we have
\[
\int_0^{+\infty} t^{\rho+k-1} |\Theta^{(k)}(t)| \, dt \leq \sum_{n=1}^{\infty} \int_0^{+\infty} t^{\rho+k-1} |\lambda_n|^k e^{-\Re(\lambda_n)t} \, dt
\]
(3.12)
\[
\leq \sum_{n=1}^{\infty} \int_0^{+\infty} t^{\rho+k-1} |\lambda_n|^k e^{-C|\lambda_n|t} \, dt = \frac{\Gamma(\rho + k)}{C^{\rho+k}} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^\rho} < \infty,
\]
particularly,
\[
\int_0^{+\infty} t^{\rho+k} |\Theta^{(k+1)}(t)| \, dt \leq \frac{\Gamma(\rho + k + 1)}{C^{\rho+k+1}} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^\rho} < \infty.
\]
(3.13)
Since by (3.10),
\[
\lim_{t \to +\infty} t^{\rho+k} \Theta^{(k)}(t) = 0,
\]
(3.14)
then for \(t > 0\),
\[
t^{\rho+k} \Theta^{(k)}(t) = - \int_t^{+\infty} \left( x^{\rho+k} \Theta^{(k)}(x) \right)' \, dx
\]
(3.15)
\[
= - \int_t^{+\infty} \left( (\rho + k)x^{\rho+k-1} \Theta^{(k)}(x) + x^{\rho+k} \Theta^{(k+1)}(x) \right) \, dx,
\]
then for all \(k \in \mathbb{N}_0\),
\[
\left| t^{\rho+k} \Theta^{(k)}(t) \right| \leq \int_t^{+\infty} \left( (\rho + k)x^{\rho+k-1} \left| \Theta^{(k)}(x) \right| + x^{\rho+k} \left| \Theta^{(k+1)}(x) \right| \right) \, dx
\]
(3.16)
\[
\leq \int_0^{+\infty} \left( (\rho + k)x^{\rho+k-1} \left| \Theta^{(k)}(x) \right| + x^{\rho+k} \left| \Theta^{(k+1)}(x) \right| \right) \, dx < \infty,
\]
which proves (3.5).

Since for each \(k \in \mathbb{N}_0\),
\[
\sum_{n=1}^{\infty} \int_0^{+\infty} t^{\rho} e^{-\Re(\lambda_n)t} \, dt \leq \int_0^{+\infty} \sum_{n=1}^{\infty} t^{\rho} e^{-\Re(\lambda_n)t} \, dt
\]
(3.17)
\[
= \sum_{n=1}^{\infty} \frac{k!}{(x + C|\lambda_n|)^{k+1}} \leq \sum_{n=1}^{\infty} \frac{k!}{(C|\lambda_n|)^{k+1}} < \infty,
\]
then
\[(3.18) \quad \int_0^\infty t^k e^{-xt} |\Theta(t)| \, dt \leq \sum_{n=1}^\infty \int_0^\infty t^k e^{-(x+\Re(\lambda_n))t} \, dt \leq \sum_{n=1}^\infty \frac{k!}{(C|\lambda_n|)^{k+1}} < \infty.\]

For \(x > 0\) and \(k \in \mathbb{N}_0\) we also trivially have
\[(3.19) \quad \sum_{n=1}^\infty \frac{1}{|x+\lambda_n|^{k+1}} \leq \sum_{n=1}^\infty \frac{1}{(x+\Re(\lambda_n))^{k+1}} \leq \frac{1}{C^{k+1}} \sum_{n=1}^\infty \frac{1}{|\lambda_n|^{k+1}} < \infty.\]

For \(x > 0\) by (3.18) and (3.19) and the Fubini’s theorem to obtain the following identity,
\[(3.20) \quad \frac{f'(x)}{f(x)} = \sum_{n=1}^\infty \frac{1}{x+\lambda_n} = \sum_{n=1}^\infty \int_0^\infty e^{-(x+\lambda_n)t} \, dt = \int_0^\infty e^{-xt\Theta(t)} \, dt.\]

Clearly, if \(\Theta(t) \geq 0\) for \(t > 0\), then \(\frac{f'(x)}{f(x)}\) is completely monotonic on \((0, \infty)\) by the Hausdorff–Bernstein–Widder theorem.

On the other hand, if \(\frac{f'(x)}{f(x)}\) is completely monotonic, so is
\[(3.21) \quad -\frac{d}{dx} \left( \frac{f'(x)}{f(x)} \right) = \int_0^\infty e^{-xt\Theta(t)} \, dt, \quad x > 0.\]

Then by (3.18), (3.20) and (3.21) we have
\[(3.22) \quad \int_0^\infty |\Theta(t)| (t+1) \, dt < \infty, \quad \sup_{t>0} t|\Theta(t)| < \infty.\]

By Lemma 2 we get
\[(3.23) \quad \Theta(t) = \lim_{k \to \infty} \frac{k^{k+1}}{k!} \left( (-1)^{k+1} \frac{d^{k+1}}{dx^{k+1}} \frac{f'(x)}{f(x)} \right)_{x=k/t} \geq 0, \quad t > 0,\]
which shows that \(\Theta(t) \geq 0\) for \(t > 0\).

Now we prove the last assertion. Let \(g(x) = f(x + \beta)\), since by (3.7),
\[(3.24) \quad \frac{g'(x)}{g(x)} = \frac{f'(x + \beta)}{f(x + \beta)} = \int_0^\infty e^{-xt} e^{-\beta t\Theta(t)} \, dt, \quad x > 0.\]

Clearly, \(e^{-\beta t\Theta(t)} \geq 0, \quad t > 0\) if and only if \(\Theta(t) \geq 0, \quad t > 0\). Thus without losing any generality we may assume \(\beta = 0\) throughout the remainder of this proof.

For \(x > 0\), let
\[(3.25) \quad u(x) = -\frac{d}{dx} \left( \frac{f'(x)}{f(x)} \right) = -\left( \log f(x) \right)^{(2)},\]
then by the logarithmic concavity of \(\log(f(x))\) (3.27),
\[(3.26) \quad u(x) = \int_0^\infty e^{-xt} t\Theta(t) \, dt \geq 0, \quad x > 0.\]

Then for any \(x > 0\), by (3.18),
\[(3.27) \quad |u(x)| \leq \int_0^\infty t|\Theta(t)| \, dt < \infty.\]
and by (2.4),
\[
0 \leq \int_0^\infty u(x)\,dx = \int_0^\infty \left( \int_0^\infty e^{-xt} t\,\Theta(t)\,dt \right)\,dx = \int_0^\infty \left| \int_0^\infty e^{-xt} t\,\Theta(t)\,dt \right|\,dx
\]
(3.28) \leq \int_0^\infty \left( \int_0^\infty e^{-xt} |\Theta(t)|\,dt \right)\,dx = \int_0^\infty \left( \int_0^\infty e^{-xt} dx \right) t |\Theta(t)|\,dt
\]
(3.29) = \int_0^\infty |\Theta(t)|\,dt \leq \sum_{n=1}^{\infty} \int_0^\infty e^{-R(\lambda_n)t}\,dt \leq \sum_{n=1}^{\infty} e^{-C\lambda_n|t|}\,dt < \infty.

Then by (2.6), (3.5), (2.10), (2.11) and the special case \( \alpha = 0 \) in Lemma 3 we get
(3.30) \quad v_0(x) = \int_0^x e^{-x^2 t} t^{-2} \Theta(t^{-1})\,dt, \quad x > 0

and for all \( x > 0 \) and \( k \in \mathbb{N}_0 + 4, \)
(3.31) \quad \frac{(-1)^k}{k!} \frac{d^k v_0(x)}{dx^k} = \int_0^\infty \frac{J_k(2\sqrt{xy})}{(\sqrt{xy})^k} u(y)\,dy.

Then by (2.6) and (3.5), the \( C^{\infty}(0, \infty) \) function \( t^{-2}\Theta(t^{-1}) \) satisfies
(3.32) \quad t^{-2}\Theta(t^{-1}) = \mathcal{O} \left( t^{-2+\nu} \right), \quad t \to +\infty

and
(3.33) \quad \sup_{t>0} t^{-2} |\Theta(t^{-1})| < \infty, \quad \int_0^\infty t^{-2} |\Theta(t^{-1})|\,dt < \infty.

Then by Lemma 2
(3.34) \quad t^{-2}\Theta(t^{-1}) = \lim_{k \to \infty} \frac{(-1)^k v_0(k/t)k^{k+1}}{k!t^{k+1}}, \quad t > 0.

On the other hand, by Lemma 3
(3.35) \quad (-1)^k v_0(k/t) = \int_0^\infty \frac{J_k(2\sqrt{yk/t})}{(\sqrt{yk/t})^k} u(y)\,dy, \quad t > 0.

We notice that for \( t > 0, k \in \mathbb{N} \) and \( \nu = k, z = 2x > 0 \) by inequality [5] (10.14.4) we obtain
(3.36) \quad \frac{J_k(2x)}{x^k} \geq \frac{1}{k!}, \quad k \in \mathbb{N}, \quad x > 0.

Since by (3.20) \( u(y) \geq 0 \) on \( (0, \infty) \), then
(3.37) \quad \frac{(-1)^k v_0(k/t)k^{k+1}}{k!t^{k+1}} \geq -\frac{k^{k+1}}{(k!)^2t^{k+1}} \int_0^\infty u(y)\,dy.

For each fixed \( t > 0 \), by the Stirling’s formula we get [5] (5.11.10),
(3.38) \quad \frac{k^{k+1}}{(k!)^2t^{k+1}} = \mathcal{O} \left( \left( \frac{e^2}{k!} \right)^k \right), \quad k \to +\infty.
Then for each fixed \( t > 0 \), by (3.37) and (3.38),

\[
(3.39) \quad t^{-2} \Theta(t^{-1}) = \lim_{k \to \infty} \frac{(-1)^k e^{(k)}(k/t)k^{k+1}}{k!k^{k+1}} \geq \lim_{k \to \infty} \left( -\frac{k^{k+1}}{(k!)^2k^{k+1}} \int_0^\infty u(y)dy \right) = 0,
\]

which is the same as \( \Theta(t) \geq 0 \) for \( t > 0 \). By the Hausdorff–Bernstein–Widder theorem, (3.39) and (3.7) we have shown that

\[
(3.40) \quad \exp \left( -\log \frac{f(x)}{f(0)} \right) = \frac{f(0)}{f(x)},
\]

which is equivalent to that \( \frac{1}{f(x)} \) is also completely monotonic on \((0, \infty)\). \( \square \)

**Lemma 5.** The sequence \( \{\lambda_n\}_{n=1}^\infty \) in Theorem 7 is positive if and only if the theta kernel \( \Theta(t) \) is completely monotonic on \((0, \infty)\).

**Proof.** By (3.11) the theta kernel \( \Theta(t) \) is completely monotonic on \((0, \infty)\) is equivalent to that for each \( k \in \mathbb{N}_0 \),

\[
(3.41) \quad (-1)^k \Theta^{(k)}(t) = \sum_{n=1}^\infty \lambda_n^k e^{-\lambda_n t} \geq 0, \quad t > 0.
\]

Since the sufficiency is trivial, we only need to prove the necessity. If \( \Theta(t) \) is completely monotonic on \((0, \infty)\), then by the Hausdorff–Bernstein–Widder theorem there exists a finite nonnegative Borel measure \( d\mu(t) \) on \([0, \infty)\) such that

\[
(3.42) \quad \Theta(t) = \sum_{n=1}^\infty e^{-\lambda_n t} = \int_0^\infty e^{-yt}d\mu(y), \quad t > 0.
\]

For \( x > 0 \) we multiply \( e^{-xt}dt \) both sides of (3.42) and integrate them from 0 to \( \infty \), similar to the proof of (3.7), we get

\[
(3.43) \quad \frac{f'(x)}{f(x)} = \int_0^\infty e^{-xt}\Theta(t)dt = \int_0^\infty \frac{d\mu(y)}{x+y}, \quad x > 0.
\]

By Fatou's Lemma and (3.5) and (3.6) we get

\[
(3.44) \quad 0 < \int_0^\infty \frac{d\mu(y)}{y} = \int_0^\infty \lim_{x \downarrow 0} \frac{d\mu(y)}{x+y} = \int_0^\infty \liminf_{x \downarrow 0} \frac{d\mu(y)}{x+y}
\leq \liminf_{x \downarrow 0} \int_0^\infty \frac{d\mu(y)}{x+y} = \liminf_{x \downarrow 0} \frac{f'(x)}{f(x)} = \liminf_{x \downarrow 0} \frac{f'(x)}{f(0)} = f'(0) < \infty.
\]
Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \geq \Re(\lambda_1) > 0$, $\beta \neq 0$, then for any $\frac{\alpha}{\beta} > \epsilon > 0$ we have

$$\int_0^\infty \frac{d\mu(y)}{|(y-\alpha) + i\beta|} = \int_0^{\alpha+\epsilon} \frac{d\mu(y)}{|y-\alpha|} + \int_{\alpha+\epsilon}^\infty \frac{d\mu(y)}{|y-\alpha|}$$

$$\leq \frac{1}{\beta} \int_0^{\alpha+\epsilon} \frac{y d\mu(y)}{y} + \int_{\alpha+\epsilon}^\infty \frac{y d\mu(y)}{|y-\alpha|}$$

$$\frac{\alpha + \epsilon}{\beta} \int_0^{\alpha+\epsilon} \frac{d\mu(y)}{y} + \int_{\alpha+\epsilon}^\infty \frac{y - \alpha + \alpha}{y-\alpha} d\mu(y)$$

$$\leq \frac{\alpha + \epsilon}{\beta} \int_0^{\alpha+\epsilon} \frac{d\mu(y)}{y} + \left(1 + \frac{\alpha}{\beta}\right) \int_{\alpha+\epsilon}^\infty \frac{d\mu(y)}{y} < \infty.$$ (3.45)

Since the right-hand side of (3.43) defines an analytic function on $\mathbb{C}\setminus(-\infty, 0]$ and the left-hand side series defines an meromorphic function on $\mathbb{C}\setminus\{\lambda_n\}_{n=1}^\infty$ and they agree on $x \in (0, \infty)$ via (3.43), then on $\mathbb{C}\setminus\{\lambda_n\} \cup (-\infty, 0]$ both sides must be the same analytic function by the principle of analytic continuation. If there exists any pole $-\lambda_n$ of the left-hand side is not in $(-\infty, 0]$, by the assumption that $\lambda_n$ has a positive real part, then we must have $\Im(\lambda_n) \neq 0$. Now we take the limit $z \to -\lambda_n$, the left-hand side of (3.43) is infinity since it has $-\lambda_n$ as a pole, while the right-hand side of (3.43) remains bounded because of (3.45), which is impossible. This contradiction proves that $\lambda_n \in [0, \infty)$, but $\lambda_n$ is already assumed to have positive real part, then $\lambda_n$ must be positive.

**Lemma 6.** Assume that $\phi(t)$ is a $C^\infty((0, \infty))$ function satisfying (3.3) and (3.6), then it is completely monotonic on $(0, \infty)$ if and only if that for each $k \in \mathbb{N}_0$, there exists a positive integer $\ell_k > k$ such that

$$\int_0^\infty t^m e^{-xt} L_0^{(m)}(xt) \phi(t) dt \geq 0$$

holds for all integers $m \geq \ell_k$ and $x > 0$. Furthermore, let

$$\Phi(x) = \int_0^\infty e^{-xt} \phi(t) dt, \quad x > 0,$$

then the condition (3.40) is equivalent to that for each $k \in \mathbb{N}_0$ there exists a positive integer $\ell_k > k$ such that

$$(-1)^m \sum_{j=0}^k \binom{k}{j} x^j \Phi(m+j)(x) \geq 0$$

for all $m \geq \ell_k$ and $x \in (0, \infty)$.

**Proof.** Because the $C^\infty((0, \infty))$ function $\phi(t)$ satisfies (3.3) and (3.6), then it is completely monotonic if and only if for each $k \in \mathbb{N}_0$ we have $(-1)^k \phi^{(k)}(t) \geq 0$ on $t \in (0, \infty)$, which is equivalent to that for any positive integer $\ell_k > k$ such that

$$t^{\ell_k+k} \left((-1)^k \phi^{(k)}(t)\right) \geq 0, \quad t > 0,$$

it is in turn equivalent to

$$\int_0^\infty e^{-xt} t^{\ell_k+k} \left((-1)^k \phi^{(k)}(t)\right) dt$$

(3.49)

(3.50)
is completely monotonic on $x \in (0, \infty)$ by the Hausdorff–Bernstein–Widder theorem. Then (3.50) is completely monotonic on $x \in (0, \infty)$ if and only if for all $n \geq 0$ and $x > 0$ we have

$$0 \leq (-1)^n \frac{d^n}{dx^n} \int_0^\infty e^{-xt}t^k e^{t^k+n} \left((-1)^k \phi^{(k)}(t)\right) dt$$

(3.51)\hfill (3.52)

then we have

$$k! \int_0^\infty t^m e^{-xt} L_k^{(m)}(xt) \phi(t) dt$$

(3.53)

where we have applied, [2, (6.2.1), (6.2.18)] and integration by parts. By taking $m = \ell_k + n$ [4.49] is proved.

By [2] (6.2.2),

$$k! L_k^{(m)}(x) = (m+1)_k \sum_{j=0}^k \frac{(-k)_j x^j}{(m+1)_j j!},$$

(3.52)

then we have

$$k! \int_0^\infty t^m e^{-xt} L_k^{(m)}(xt) \phi(t) dt$$

$$= (m+1)_k \sum_{j=0}^k \frac{(-k)_j x^j}{(m+1)_j j!} \int_0^\infty t^{m+j} e^{-xt} \phi(t) dt$$

$$= (1)^m (m+1)_k \sum_{j=0}^k \frac{(-1)^j x^j}{(m+1)_j j!} \Phi^{(m+j)}(x)$$

(3.53)

$$= (1)^m (m+1)_k \sum_{j=0}^k \frac{(-1)^j x^j}{(m+1)_j j!} \Phi^{(m+j)}(x).$$

\[\square\]

**Theorem 7.** Let $C, p, \{\lambda_n\}_{n \in \mathbb{N}}$, $\Theta(x)$ and $f(z)$ be defined as in (3.1) to (3.4) in Theorem 4. Then $\{\lambda_n\}_{n=1}^\infty$ is a positive sequence if and only if for each $k \in \mathbb{N}_0$ there exists an integer $\ell_k > k$ such that for all integers $m \geq \ell_k$,

$$\int_0^\infty t^m e^{-xt} L_k^{(m)}(xt) \Theta(t) dt \geq 0, \quad x > 0,$$

(3.54)

where $L_k^{(\alpha)}(x)$ is the $k$-th generalized Laguerre polynomial with $\alpha > -1$, $k \in \mathbb{N}_0$, [2, 5, 13, 10]. Furthermore, the condition (3.54) is equivalent to that for each $k \in \mathbb{N}_0$ there exists an integer $\ell_k > k$ such that for all $m \geq \ell_k$ and $x > 0$,

$$S_{m,k}(x) = (1)^m \sum_{j=0}^k \frac{k^j}{(m+1)_j} \frac{x^j}{f(x)} (f'(x))^{(m+j)} \geq 0.$$

(3.55)

If additionally $f(x)$ is eventually logarithmic concave, i.e. it satisfies (3.8), then all its zeros are negative.

**Proof.** Since by Lemma 5 $\{\lambda_n\}_{n=1}^\infty$ is a positive sequence if and only if $\Theta(t)$ is completely monotonic. Then the inequalities (3.54) and (3.55) are obtained by taking $\phi(t) = \Theta(t)$ and $\Phi(x) = f'(x)/f(x)$.
Assume that if additionally $f(x)$ is eventually logarithmic concave, i.e. it satisfies (3.8), then the $C^\infty(0, \infty)$ function $\Theta(t)$ is a satisfying (3.5), (3.6) and $\Theta(t) \geq 0$ on $(0, \infty)$. Then to show $f(z)$ has only negative zeros it is sufficient to show that $\Theta(t)$ is completely monotonic on $(0, \infty)$. By (3.54) it is enough to show that for each $k \in \mathbb{N}_0$ and $\ell_k = 2k + 1$,

$$\int_0^\infty t^m e^{-xt} L_k^{(m)}(xt) \Theta(t) dt \geq 0$$

holds for all integers $m \geq \ell_k$ and $x > 0$. Recall that for $n \in \mathbb{N}_0$, $\alpha > -1$ and $x > 0$, by [2, (6.2.15)],

$$e^{-x^n n! L_n^{(n)}(x)} = \int_0^\infty t^n e^{-t} L_n^{(n)}(2\sqrt{xt}) dt$$

to get

$$e^{-xt} (xt)^m k! L_k^{(m)}(xt) = \int_0^\infty y^k e^{-y} (xyt)^{m/2} J_m(2\sqrt{xyt}) dy. \tag{3.58}$$

Then (3.56) is equivalent to

$$k! x^m \int_0^\infty t^m e^{-xt} L_k^{(m)}(xt) \Theta(t) dt = \int_0^\infty e^{-xt} (xt)^m k! L_k^{(m)}(xt) \Theta(t) dt \tag{3.59}$$

where is the exchange order of integrations can be easily verified by applying the Fubini’s theorem and the asymptotic behaviors of $J_m(x)$. For $\nu > -1, \mu + \nu > 0$ and $x > 0$, by [5] (10.22.54), (13.2.2-13.2.4), [16] pp.442(66) or by (3) on page 394 of [17] to get

$$\int_0^\infty J_\nu(2\sqrt{xy}) \exp(-y) y^{\mu/2 - 1} dy \tag{3.60}$$

$$= \frac{\Gamma(\frac{\nu + \mu}{2}) x^{\nu/2}}{\Gamma(\nu + 1)} \exp(-x) \gamma F_1\left(\frac{\nu - \mu}{2} + 1; \nu + 1; x\right).$$

Since for $\nu > -1, \mu > 1, \nu - \mu \geq -2$ and $x > 0$

$$\frac{\Gamma(\frac{\nu + \mu}{2}) x^{\nu/2}}{\Gamma(\nu + 1)} \exp(-x) \gamma F_1\left(\frac{\nu - \mu}{2} + 1; \nu + 1; x\right) \geq 0,$$

then

$$\int_0^\infty J_\nu(2\sqrt{xy}) \exp(-y) y^{\mu/2 - 1} dy \geq 0. \tag{3.62}$$

For any $k \in \mathbb{N}_0$, taking $\mu = k + \frac{m}{2} + 1, \nu = m \geq 2k + 1$ we get

$$\int_0^\infty y^k e^{-y} (xy)^{m/2} J_m(2\sqrt{xy}) dy \geq 0. \tag{3.63}$$
Then by (3.59) to obtained that for each 
\[ k \in \mathbb{N}_0, \; m \geq 2k + 1 \] and \( x > 0 \),
\[
(3.64) \quad k! x^m \int_0^\infty t^m e^{-xt} L_k^{(m)}(xt) \Theta(t) dt = \int_0^\infty \left( \int_0^\infty y^{k+m/2} e^{-y} J_m(2\sqrt{xyt}) dy \right) (xt)^{m/2} \Theta(t) dt \geq 0.
\]

\[ \square \]

4. Riemann \( \xi(s) \) Function

In this section we present an application of Theorem \[4\] and Theorem \[7\] to the Riemann Xi function \( \Xi(s) \) which is essentially the Riemann xi function \( \xi(s) \). It is clear that our approach can solve the similar problems of \( L(\chi, s) \) such that
\[
(4.1) \quad \int_{-\infty}^{\infty} t^{2n} \varphi(t; \chi, a) dt > 0, \quad n \in \mathbb{N}_0,
\]
where
\[
(4.2) \quad \varphi(t; \chi, a) = 2e^{(a+1/2)t} \sum_{n=1}^{\infty} n^a \chi(n) e^{-n^2 \pi e^{2t}/q}
\]
and \( \chi \) is a real primitive character modulo \( q \in \mathbb{N} + 1 \) with the convention that \( a = 0 \) if \( \chi \) is even and \( a = 1 \) if \( \chi \) is odd. \[4\]. The verifications are essentially the same as the one we will perform on the Riemann \( \xi(s) \) function.

Let \( s = \sigma + it, \; \sigma, t \in \mathbb{R} \), the Riemann \( \xi \)-function is defined by \[2, 6, 8, 9, 3\]
\[
(4.3) \quad \xi(s) = \pi^{-s/2} (s - 1) \Gamma \left( 1 + \frac{s}{2} \right) \zeta(s),
\]
then the entire function \( \xi(s) \) satisfies
\[
(4.4) \quad \xi(s) = \xi(1 - s).
\]
The Riemann \( \Xi(s) \) function is an even entire function defined by
\[
(4.5) \quad \Xi(s) = \xi \left( \frac{1}{2} + is \right) = \xi \left( \frac{1}{2} - is \right) = \Xi(-s).
\]
It is known that \[8, 9\]
\[
(4.6) \quad \Xi(s) = \int_{-\infty}^{\infty} \Phi(u) e^{isu} du = 2 \int_0^{\infty} \Phi(u) \cos(us) du,
\]
where
\[
(4.7) \quad \Phi(u) = \sum_{n=1}^{\infty} \left( 4n^4 \pi^2 e^{9u/2} - 6n^2 \pi e^{5u/2} \right) e^{-n^2 \pi e^{2u}}.
\]
It is also known that \( \Phi(u) > 0, \; u \in \mathbb{R} \), and \( \Xi(s) \) is of genus 1.

Furthermore,
\[
(4.8) \quad \xi \left( \frac{1}{2} + s \right) = \sum_{n=0}^{\infty} a_n s^{2n},
\]
where \( a_0 = \xi(1/2) \approx 0.497121 > 0 \) and \( a_n > 0 \) for \( n \in \mathbb{N} \).
Let
\[ f(z) = \xi \left( \frac{1}{2} + \sqrt{z} \right) = \sum_{n=0}^{\infty} a_n z^n, \]
then \( f(z) \) is an entire function of genus 0 such that \( \Xi(s) \) has only nonzero real roots if and only if \( f(z) \) has only negative roots.

Let \( \{ \pm s_n \}_{n=1}^{\infty} \) be the set of all zeros of \( \Xi(s) \) such that \( \{ s_n \}_{n=1}^{\infty} \) with positive real part arranged in the nondecreasing order,
\[ 14 < \Re(s_1) < \Re(s_2) < \cdots \leq \Re(s_{n+1}) \leq \ldots. \]
All of \( \{ s_n \}_{n=1}^{\infty} \) are inside the horizontal half-strips bounded by \( |t| < \frac{1}{2} \) and \( |\sigma| > 14.1347, [6] \). Therefore, all the non-zeros roots of \( f(z) \) are \( \{ -s_n^2 \}_{n=1}^{\infty} \),
\[ \Re(s_n^2) = |s_n|^2 - 2 (\Im(s_n))^2 \geq |s_n|^2 - \frac{1}{2} \]
\[ \geq |s_n|^2 - \frac{|s_n|^2}{2 \cdot 14^2} = \frac{391}{392} |s_n|^2. \]
For any given \( \epsilon > 0 \) by the theorem [6, Section 2.5],
\[ \sum_{n=1}^{\infty} \frac{1}{|s_n^2|^{(1+\epsilon)/2}} < \infty. \]
Since for \( x > 1 \) each factor in
\[ f(x) = \pi^{-\frac{x}{4}} \sqrt{x} \left( \sqrt{x} - \frac{1}{2} \right) \Gamma \left( \frac{\sqrt{x}}{2} + \frac{5}{4} \right) \zeta \left( \sqrt{x} + \frac{1}{2} \right) \]
is positive. Then for \( x > 1 \),
\[ \log f(x) = \log \left( \zeta \left( \sqrt{x} + \frac{1}{2} \right) \right) \]
\[ + \log \left( \pi^{-\frac{x}{4}} \sqrt{x} \left( \sqrt{x} - \frac{1}{2} \right) \Gamma \left( \frac{\sqrt{x}}{2} + \frac{5}{4} \right) \right) \]
and
\[ (\log f(x))^{(2)} = \left( (\log \left( \zeta \left( \sqrt{x} + \frac{1}{2} \right) \right))^{(2)} \right) \]
\[ + \left( (\pi^{-\frac{x}{4}} \sqrt{x} \left( \sqrt{x} - \frac{1}{2} \right) \Gamma \left( \frac{\sqrt{x}}{2} + \frac{5}{4} \right))^{(2)} \right). \]
By [6, 12] and [3] (27.4.12),
\[ \left( \log \left( \zeta \left( \sqrt{x} + \frac{1}{2} \right) \right) \right)^{(2)} = \frac{1}{4x} \sum_{n=2}^{\infty} \Lambda(n) \log n \left( \frac{1}{\sqrt{x} \log n} + 1 \right) \]
\[ = O \left( \frac{1}{x} \sum_{n=2}^{\infty} \frac{\log^2 n}{n^{\sqrt{x}+1/2}} \right) = O \left( \frac{1}{x^{2\sqrt{x}/2}} \right), \quad x \to +\infty, \]
where \( \Lambda(n) \) is the von Mangoldt function.
On the other hand, by [5, (5.15.8)] we get
\begin{equation}
(4.17)
(\log \left( \pi^\frac{1}{4} \frac{\sqrt{x}}{\sqrt{x^2 - 1}} \right)^2 \Gamma \left( \frac{\sqrt{x}}{2} + \frac{3}{4} \right))^{(2)}
= \log(4e^2\pi^2) - \log x + O(x^{-2}), \quad x \to +\infty.
\end{equation}

Therefore, as $x \to +\infty$,
\begin{equation}
(4.18)
(\log f(x))^{(2)} = \frac{\log(4e^2\pi^2) - \log x}{16\sqrt{x^3}} + O(x^{-2}),
\end{equation}
which clearly shows that there exists a large positive number $\beta$ with $(\log f(x))^{(2)} \leq 0$ for $x \geq \beta$. Then by Theorems 4 and 7 we have proved the following:

**Corollary 8.** Let $\{s_n\}_{n=1}^{\infty}$ be the zeros of $\Xi(s)$ satisfying (4.10). Then
\begin{equation}
(4.19)\Theta(x) = \sum_{n=1}^{\infty} e^{-s_n^2 x} \geq 0, \quad x > 0,
\end{equation}
and the functions
\begin{equation}
(4.20)\frac{\xi'(\frac{1}{2} + \sqrt{x})}{\sqrt{x} \xi(\frac{1}{2} + \sqrt{x})} \frac{1}{\xi'(\frac{1}{2} + \sqrt{x})}
\end{equation}
are completely monotonic on $(0, \infty)$.

Furthermore, the entire function $\xi \left( \frac{1}{2} + \sqrt{z} \right)$ has only negative zeros, or equivalently, the Riemann $\Xi(s)$ has only real zeros.

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