Embedding Diagrams - a Hands-on Activity for Understanding Spatial Curvature

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Abstract. Even the basics of General Relativity are considered difficult for students due to its abstract nature and the lack of concrete demonstrations suitable for a typical classroom. One area that offers some possibility of hands-on activities for students are embedding diagrams, the curved surfaces that we use to illustrate spatial curvature. Plastic models of these surfaces can easily be made using a 3D printer, which is a piece of equipment that is rapidly becoming more and more available to schools and even students themselves. In this contribution, we will propose a chain of activities, with some of which the reader might already be familiar, meant for students to help them understand what spatial curvature is and how it relates to the motion of planetary bodies (not only) in our Solar System.

1. Introduction

General Relativity (GR) is a topic rarely discussed in upper secondary education. The obvious reasons for it to be so are the apparent mathematical complexity of GR and a great demand for abstract thinking which is necessary for understanding many of its key concepts. In the case of the author’s home country, the Czech Republic, relativity (both special and general) is not mentioned in the official documents [1] governing the mandatory part of general secondary education. Therefore, relativity is not mandatory and it is up to the schools and teachers whether they wish to include it or not. Indeed, many Czech teachers still teach special relativity (SR) as there is a well-established historic tradition of teaching this subject. However, teaching or even mentioning GR is very rare. We have been looking into why it is so and some preliminary results from our not-yet-published research indicate the two main reasons why GR is not taught in upper secondary schools are lack of suitable teaching materials and the lack of confidence of teachers in their GR knowledge.

In order to help remedy this situation, we have been working on some learning materials and activities that could be used to present upper secondary school students with the basics of GR. One of such activities is a standalone seminar which is a combination of a lecture and a workshop. During the seminar we try to present some difficult-to-understand concepts in as approachable way as possible, including some hands-on activities that hopefully (see section Feedback) help students to come to terms with concepts like curvature, spacetime etc.

2. Relativistic seminar

Our, approximately 2-hour, seminar is meant for any upper secondary students interested in the topic of General Relativity. The participation on the seminar is purely voluntary as it is not our goal to introduce GR as part of normal physics education. We will now describe the key features of our exposition with the emphasis on the hands-on activities.
2.1. Basic outline of the seminar

Our main goal of the seminar is for the students to understand what is gravitation according to the GR. For that purpose we start with the definition inspired by [2]:

"Gravitation is the curvature of spacetime."

Therefore, to understand what gravitation is, we need to understand two concepts, spacetime and curvature. The former is actually defined and motivated by SR. Therefore, in order to devote all of the available time to the exposition of GR, we briefly summarize some key concepts from special relativity, leading to spacetime. This constitutes the biggest downside of our approach. There is no time to talk about time dilation or length contractions, we must (in order not to waste time) take them as facts. However, this is not as big of an obstacle as it might seem. As mentioned, SR is still often taught in Czech upper secondary schools, mostly in the last year or as part of a selectable physics seminar, so some students are already familiar with these concepts. If they are not, we ask them to just take the conclusions of SR as they are because we need them for our present discussion. As we mention in the Feedback section, answers to some follow-up questions reveal that some students do miss more in-depth explanation of the SR claims, which we don’t have time to address during the seminar. We will discuss this issue in the Feedback section.

Broadly speaking, we devote most of our time to explaining what curvature of spacetime is. To help students, we immediately add a more informal description "Time and space sometimes do not behave as we would expect." This, we assume, helps the students to approach the concepts better.

The basic train of thought of our treatment of GR is depicted in Figure 1. We avoid the complexity of spacetime curvature by splitting it into the curvature of time and space and addressing them separately. Curvature of space is then further simplified into curvature of surfaces because these are, unlike curvature of space, possible to visualize and work with. However, at the end of the treatment, we encounter the problem of "why do objects start falling from rest?" This is something the pure curvature of surface/space cannot answer and we must therefore come back to joining the space and time into one entity, thus reinstating the connection between the two.

**Figure 1.** The basic outline of our treatment of spacetime curvature. We first split into time curvature and space curvature. Space curvature is further simplified into surface curvature. Finally, the two concepts are at the end joined back together with the emphasis on them being in reality inseparable.
2.2. Curvature of time

The whole treatment of curvature of time revolves around gravitational time dilation. In other words, time slows down in the presence of gravitation, the stronger the gravitation the more time slows. We start by citing the well-known experiment by Hafele and Keating [3]. This experiment, where atomic clock were flown in planes and their time was compared with that of a stationary clock on the ground, is especially useful because it includes both special relativistic time dilation caused by movement of the planes and the general relativistic one caused by different altitudes of the clocks. Therefore it not only reminds students of the SR part from the beginning, but it also directly relates to the treatment of relativistic corrections for the global navigation satellites (which are also moving compared to an observer on Earth) that we discuss soon after. Before we get into that, we also mention more modern experiments where scientists compared times of highly precise aluminium-based atomic clock less than one meter apart vertically and when moving with relative speed of less than 10 m/s [4]. These experiments further establish the reality of the strange predictions of relativity.

Finally, we discuss the nature of global navigation systems and how relativity is crucial for their function. As an example, we take the European Galileo navigation system, input the parameters of its satellites such as orbital period and average distance from the Earth surface, and using one formula which is given by the GR (i.e. not derived) and some secondary school physics of orbital motion, we calculate by how much the positioning system would differ from reality in one day if it didn’t use relativistic corrections. The approximate answer (about 12 km) shows students the importance of relativity for our everyday life.

2.3. Curvature of space/surface

We move on to the curvature of space. Because such a concept is literally impossible to imagine by our 3D-thinking brains and therefore cannot be grasped without the use of complex mathematics, we rather devote our attention to curvature of surfaces, which is not only possible for a human brain to imagine (as embedded in a flat 3D space) but is also part of our everyday experience. We start by simple examples of a sphere and show that the geometry on a sphere is indeed different from what we learn in school as Euclidean geometry. For example, a triangle on a sphere can have the sum of all its inner angles larger than 180°, two lines that start parallel end up crossing each other, and so on. Without explicitly saying so, we are dealing with inner curvature of the surface. We then introduce the “paper sheet method” to distinguish a surface whose geometry is non-Euclidean. A sheet of paper represents a somewhat flexible Euclidean plane. If we can wrap that sheet around an object without folding or tearing the paper, the object has Euclidean geometry as well. An example of that is a side of a cylinder. Therefore just because a surface is not “straight” doesn’t mean it’s not flat (in a geometrical sense).

2.3.1. Activity one – a cone

Now comes the first activity, a cone. The first part can be found for example in [5]. We take a suitably cut (see Figure 2) piece of paper and draw a straight line on it. This represents a flat piece of spacetime (without gravitation) where a body undisturbed by external forces moves in a straight line (following Newton’s First Law of Motion). We “switch on” gravitation by bending the paper into a cone, thus curving the formerly straight line. We arrive at our first observation: a straight line on a bend surface is no longer straight. The larger the curvature the more the trajectory bends. We can follow up using our paper sheet method to show that a cone cannot be easily wrapped in paper (actually it almost can thanks to the slight flexibility of paper, the problems arise near the tip of the cone). We can then reinforce the idea of curvature bending straight trajectories by preparing a thin long piece of paper, drawing a line on it lengthwise and putting in on the surface of a cone. Again, the originally straight piece of paper is bent into a curved shape.
2.3.2. Activity two – a sphere
The second activity involves a sphere. An ideal aid for representing a large enough sphere is a beach ball. It is easy to carry and deflated takes up very little space. Students react positively when they see it, most likely because it is associated with outside activities. Most of all, beach balls have strongly visible seams that can be used as a natural representation of geographical meridian when talking about the curvature of our own planet (which is sufficiently sphere-like for us to treat it so here). We talk about the closest distance between two points on the sphere. A problem nicely illustrated by trajectories of long-distance flights. Websites such as https://www.flightconnections.com show such trajectories on a map. The fact that the shortest distance between two points on a sphere is not a straight line but a part of a circle can be also very easily demonstrated on the beach ball itself. The “meridian” lines on a beach ball also illustrate well the “story of two travellers”. This illustration of the difference between Newtonian theory of gravitation and GR can be found in [6]. In short, two explorers set out separately from the equator going north at the same time. Even though they start going parallel to each other, soon they realise that the distance between them is decreasing. They might conclude that there must be some mysterious force acting on them, pushing them closer to each other and the force gets stronger the more north they get (the Newtonian approach). However, if we observe the travellers from orbit, we can easily conclude there is no such force, it is the curvature of the planet that is making them closer to each other (Einsteinian view). Finally, we show a simple animation showing the two originally parallel trajectories getting closer on a sphere (both animations used during the activities are described in [7]). It is important to note, however, that we show the animation only after students get the chance to see our reasoning on the beach ball.

2.3.3. Activity three – Flamm’s paraboloid (embedding diagram)
We finally get to the main activity using the so called Flamm’s paraboloid, i.e. the cone-like shape most strongly associated with General Relativity (as a quick internet image search can verify). It is probably the most known one from the large family of embedding diagrams used in relativity and it is meant to visualize spatial geometry of an equatorial plane in the Schwarzschild spacetime. Figure 3 shows a 3D-printed model of the paraboloid.

Of course, for the purposes of our students, it suffices to say that the purpose of such a shape is to visualize spatial curvature of a plane by embedding it in a 3D space and the shape is given by simple-enough equation that is derived from relativistic equations (for more detail see [7]). In our experience, students not only appreciate knowing where this shape comes from (because they have usually already encountered it in some popular depiction of relativity before) but also the fact that this shape is directly derived from ‘complex’ formulas and therefore not ‘just some approximation’. As before, after letting students sufficiently ‘play’ with the shape, we show an interactive animation designed to send ‘marbles’ on such a surface (another option would be for the students to try the animations themselves, of course; however, for this we need to be equipped with enough computers or tablets which is rarely the case). We now finally draw the parallel between the motion of a fictional

Figure 2. The first simple activity showing that curvature bends originally straight lines. We start with a flat piece (left) and introduce curvature by bending it into a cone. The steeper the cone, the stronger bending of the trajectory we see (middle and right).
marble on the curved surface to the motion of planets around a star. Were it not for the presence of a gravitating star, all objects would travel in straight lines. With gravitation present, however, although objects still travel in a straight line locally, their overall trajectory is not straight due to the curvature of space itself.

Figure 3. A 3D-printed Flamm’s paraboloid. By printing these shapes we can easily allow students to apply the same geometrical methods as with the previous shapes on the actual shape that comes from relativistic equations, such as the paper sheet method or the thin piece of paper method.

2.4. Back to spacetime curvature

With spatial curvature explored (of course, in our short time, we are literally scratching the surface, so to speak), we then turn our attention to one final point. If spatial curvature results in changing the trajectory of moving objects, why do objects also fall from rest? We use this example to remind the students that gravitation is the curvature of spacetime, not just space itself.

To better explain the problem, we draw a simple spacetime diagram for a stone initially at rest some height above the surface of Earth. Figure 4 illustrates our reasoning.

Figure 4. Introducing a spacetime diagram for a stone at rest at some height above the ground. Without spacetime curvature, the stone continues to be stationary in space, ‘moving’ only in time (left and middle with coordinate lines). By introducing gravitation, spacetime curves (right – the depicted curves are purely illustrational) and the stone’s ‘motion’ in time inevitably sends it closer to the source of gravitation. The stone falls.

We therefore arrive back at the beginning of our reasoning; gravitation is the curvature of spacetime. Not just time, not just space, spacetime.

If time permits, students are briefly introduced to two so-called classical experiment of GR, the perihelion shift of Mercury and the solar eclipse of 1919 (we consider these experiments to be known enough, kind reader can quickly find enough information about them on the internet). Thus again, reinforcing our exposition with real experiments.

Lastly, at the end of the seminar, students are given references to additional sources of information, mainly popular Youtube channels that deal with relativity, such as PBS Spacetime, Veritasium or VSauce. These videos are well-prepared popular explanations, ideally suited to young students to further develop their interest in the topic.
3. Feedback

Until now, we have organised the seminar two times, on two different secondary schools in Prague, Czech Republic. The first instance was a pilot version for a group of students attending a regular voluntary seminar dedicated to deepening their knowledge of physics. Based on the pilot run, the length and depth of the treatment was revised, activities using geometrical shapes were refined for better clarity. The second run was held at the secondary school where the author teaches. Students from all years (typically 15-19 year-olds) were invited to come purely voluntarily. 23 students came to the seminar. To gather some feedback, a short and simple questionnaire was created to be given to the students after the seminar. The questionnaire included three questions with answers based on the four-point Likert scale ranging from 1- ‘Definitely yes.’ to 4 – ‘Definitely not.’ These question together with the average score can be found in Table 1.

| Question                                                                 | Average score |
|--------------------------------------------------------------------------|---------------|
| Do you think that after the seminar you know more about GR than before?  | 1.04          |
| Do you think that after the seminar you understand the nature of GR more than before? | 1.42          |
| Would you like GR to be taught in school, even though there is very little time to teach it at the moment? | 1.46          |

Of course, such questions can hardly tell us anything about the actual understanding of the students, however they give us a quick and easy probe into their own assessment. The fact that students think that they know and understand more after the seminar is a positive indicator, that the seminar is beneficial to them and that it is reasonably understandable. Another positive indicator is that all the students at the seminar except one (who answered ‘I don’t know.’) answered that they are interested in another seminar focused more on astronomical phenomena such as black holes and gravitation waves. We can take from that answer that GR can be viewed as an interesting field by secondary school students and that it is worthwhile to provide them with an understandable treatment of the subject.

Lastly, students were asked to mention some things from the seminar that they especially liked or they considered well-explained. They were also asked the exact opposite, whether they thought something was explained poorly or insufficiently. As positive, students mentioned the usage of paper models for explanation, 3D-printed models (3D printing is a popular subject among students) and also the above-mentioned animations. Basically, almost all the different pieces of activities were mentioned, which shows that different students prefer different approaches and we should think about the variety of interconnected approaches when preparing such a semi-popularizing lecture, as well as any teaching unit, for that matter.

The prevailing insufficiency written by the students were the already mentioned claims coming from special relativity about time dilation and length contraction that were merely stated and not sufficiently explored to save time. The only students who didn’t mention this were, of course, those who already had SR in their normal lessons (in this case only the last year students). Indeed, it would be beneficial for the students to already encounter SR before this seminar. It is already two hours long and making it longer by incorporating SR at the beginning would most likely be counterproductive.
Also, restricting the potential audience to only the last year students, who might or might have not heard about special relativity (recall that SR is not part of the mandatory curriculum in our country), goes against our aim to spread awareness and understanding of GR among students. On the other hand, all the students indicated their better understanding of GR despite the lack of explanation of the SR prelude, so our situation is not that bad. In conclusion, it is better for the students to encounter SR before but not essential. It is also our hope that younger students might be intrigued by the lack of explanation of SR and look up some information themselves.

4. Conclusion

We have prepared and tested a semi-popular seminar including hands-on activities to promote knowledge and interest in the field of General Relativity. The seminar is being offered to physics teachers from general upper secondary schools. Furthermore, it is our goal to prepare materials for teachers who wish to include GR in their normal lessons (no sufficient materials exist in Czech, so far). Thanks to the mostly positive feedback from our test runs, we conclude that General Relativity is viewed as an interesting subject by some secondary school students and it is therefore worthwhile to organise such voluntary seminars.

5. References

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