Interface resistances and AC transport in a Luttinger liquid.

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(Dated: November 1, 2018)

Abstract

We consider a Luttinger liquid (LL) connected to two reservoirs when the two sample-reservoir interface resistances \( R_S \) and \( R_D \) are arbitrary (not necessarily quantized at half-the-quantum of resistance). We compute exactly the dynamical impedance of a Luttinger liquid and generalize earlier expressions for its dynamical conductance in the following situations. (i) We first consider a gated Luttinger liquid. It is shown that the Luttinger liquid parameters \( u \) and \( K \) and the interface resistances \( R_S \) and \( R_D \) can be experimentally determined by measuring both the dynamical conductance and impedance of a gated wire at second order in frequency. The parallel law addition for the charge relaxation resistance \( R_q \) is explicitly recovered for these non-trivial interface resistances as \( R_q^{-1} = R_S^{-1} + R_D^{-1} \). (ii) We discuss the AC response when only one electrode is connected to the LL. (iii) Thirdly we consider application of an arbitrary AC electric field along the sample and compute the dynamical response of the LL with arbitrary interface resistances. The discussion is then specialized to the case of a uniform electric field.
I. INTRODUCTION.

The Luttinger liquid (LL) is one of the best understood strongly correlated system and departs strikingly from the more familiar Landau Fermi liquid with features such as spin-charge separation and charge fractionalization [1]. Of interest is the exploration of Luttinger physics in a mesoscopic context: several materials at the mesoscopic scale such as the quantum wires or the carbon nanotubes have been indeed proposed as realizations of a Luttinger liquid [2]. In this regard AC transport probes are an important tool because they allow access to the non-equilibrium physics of the LL.

In this paper we propose to discuss AC transport in a Luttinger liquid resistively connected to two reservoirs through arbitrary interface resistances (not necessarily quantized at half a quantum of resistance): we will consider in turn (i) a gated Luttinger liquid, (ii) the response when one disconnects one of the reservoirs and (iii) application of an arbitrary AC electric field along the sample.

The first calculation for settings (i) and (iii) was done by Ponomarenko using the inhomogeneous Luttinger liquid model, where one models the reservoirs as 1D non-interacting Fermi systems [3]. His results for the gated wire were later recovered using RPA by Blanter, Hekking and Büttiker [4] and using the inhomogeneous LL model and then a boundary conditions formalism by Safi [5]; the case of a constant electric field was also considered by Sablikov and Shchamkhalova with results consistent with Ponomarenko’s; appealing to Shockley’s theorem they however claim that the real current measured at an electrode is not given by the electron current but add a displacement current contribution due to a charging of the reservoir caused by the charging of the wire itself: for a uniform electric field this results in a net current measured equal to a spatial average of the current through the wire [6]. Another approach has been advocated by Cuniberti, Sassetti and Kramer who consider an infinite system with long-range interactions and compute an absorptive conductance which has the advantage of being measurable by absorption of electromagnetic radiations without application of voltage or current probes [7]. An interesting development in several of these groups has been a focus on both capacitive and inductive effects with consideration of the kinetic inductance of a LL [7, 8]. A transmission line approach to AC transport in a LL has also been proposed by Burke [9] to investigate the plasmon physics of LL based on earlier works on single-walled [10] and multi-walled carbon nanotubes [11]; the LL is modelled as
a RL line coupled capacitively to a ground voltage. Additionally Burke discusses plasmon damping, a topic rather unexplored so far in that context. Inclusion of Coulomb interactions has also been considered in several papers.

In the case of short-range interactions (the pure LL) it is noteworthy that the DC limit of earlier calculations corresponds to contact resistances quantized at $R_0 = e^2/h$ the quantum of resistance (or equivalently to interface sample-electrode resistances $R_0/2$). We consider in this paper a more general situation by allowing for interface resistances distinct from $R_0/2$: while leaving open the experimental possibility that interface resistances are quantized at $R_0/2$, this permits dirty contacts to electrodes, which a priori is a not too unreasonable assumption.

We generalize earlier expressions for the dynamical conductance in situations (i) and (iii) above [3, 4, 5], and additionally compute the dynamical impedance. Case (ii) where only one electrode is connected is considered because it is a setting paradigmatic of time-dependent transport where the role of displacement currents is especially clear. For case (i) we show explicitly that to second order in frequency the LL can be represented by an equivalent electrical circuit comprised of interface resistances $R_S$ and $R_D$ connected in series to an intrinsic inductance (per unit length) $L = \frac{h}{2u Ke^2}$ (which is not purely kinetic but includes the effect of interactions), the whole being capacitively coupled to the ground through an intrinsic LL capacitance (per unit length) $C = \frac{2Ke^2}{\hbar u}$ (see Figure 1). This shows explicitly from first principles the validity of the transmission line analogy considered by Burke up to order two in a low-frequency expansion [9] (our results however do not assume Galilean invariance which implies in turn the relation $v_F = u K$ for the Fermi velocity).

This in turn shows that AC measurements of both the dynamical gate conductance $G_{33}$ and the impedance of the system up to order two in frequency allow full determination of the Luttinger liquid parameters $u$ and $K$ (the plasmon velocity and interaction strength) and of the interface resistances $R_S$ and $R_D$. In particular the expected parallel law addition for the charge relaxation resistance is explicitly recovered as $R_q^{-1} = R_S^{-1} + R_D^{-1}$.

The paper is organized as follows: in Section II, we introduce a boundary condition formalism which allows for the modelling of reservoirs resistively contacted to the Luttinger liquid. III. We discuss the gated Luttinger liquid computing both dynamical conductance and impedance matrices, as well as the LL connected to a single reservoir. IV. We impose an AC electric field.
II. VOLTAGE DROPS AT INTERFACES: MODELLING CONTACT RESISTANCES THROUGH BOUNDARY CONDITIONS.

A. Chiral chemical potential operators.

We consider the standard Luttinger Hamiltonian for a wire of length \( L = 2a \).

\[
H = \int_{-a}^{a} dx \frac{\hbar u}{2K} \left( \rho_+^2 + \rho_-^2 \right) + eV_3 (\rho_+ + \rho_-)
\]

\( V_3 \) is a gate potential which controls the Fermi level of the LL, \( \rho_+ \) and \( \rho_- \) are chiral particle densities which obey the relation \( \rho_\pm (x, t) = \rho_\pm (x \mp ut) \). Their sum is just the total particle density \( \rho - \rho_0 \) while the electrical current is simply \( i(x, t) = eu (\rho_+ - \rho_-) \).

We now define the following operators:

\[
\mu_\pm (x, t) = \frac{\delta H}{\delta \rho_\pm (x, t)}
\]

Physically they correspond to (canonical) chemical potential operators: their average value yields the energy needed to add a particle locally at position \( x \) to the chiral density: \( \rho_\pm \rightarrow \rho_{\pm} + \delta(x) \). Similar operators have been introduced in Safi’s boundary conditions formalism\[5\]: the main difference being that we consider chiral chemical potentials linked to the eigenmodes of the Luttinger liquid while she defines chemical potentials related to the left or right moving (bare) electrons. Such chiral operators are much more convenient for calculations since they are directly related to the LL eigenmodes.

From their definition it follows that:

\[
\mu_\pm (x, t) = \frac{\hbar u}{K} \rho_\pm (x, t) + eV_3 (t)
\]

and therefore:

\[
i(x, t) = K \frac{e}{\hbar} \left( \mu_+ (x, t) - \mu_- (x, t) \right)
\]

It is convenient to redefine the chemical potentials by taking \( V_3 \) as reference:

\[
\mu'_\pm (x, t) = \mu_\pm (x, t) - eV_3 (t)
\]

and using the fact that these shifted operators have a chiral time evolution:

\[
\mu'_\pm (x, t) = \mu'_\pm (x \mp ut)
\]
it follows immediately that:

\[
\begin{align*}
\mu'_+(a, \omega) &= \exp i\phi \mu'_+(-a, \omega), \\
\mu'_-(a, \omega) &= \exp -i\phi \mu'_-(-a, \omega), \\
\phi &= \omega \frac{2a}{u}.
\end{align*}
\]

(3)

B. Interface resistances as boundary conditions.

Up to now the Luttinger liquid is free standing. In real experimental settings coupling to probes is however unavoidable but in the absence of an exact solution of the problem of a mesoscopic LL wire coupled to many electrodes we decide to model the coupling to reservoirs through boundary conditions imposed on the otherwise free standing Luttinger liquid.

To enforce that we appeal to Sharvin-Imry contact resistance\[12\]: at the interface between a reservoir and a ballistic wire there is a voltage drop between the electrode voltage and the mean chemical potential within the wire; for a two-terminal geometry this in turn implies the existence of a contact resistance which can be viewed as the series addition of two interface resistances. For constrictions adiabatically connected to the reservoirs the contact resistance is quantized as \(R_0 = \frac{\hbar}{e^2}\). But in general it need not be; as shown by Büttiker incoherent transport through barriers can affect quantization\[13\].

For the LL we therefore make the hypothesis that as far as transport is concerned the resistive coupling to reservoirs can be modelled as:

\[
\begin{align*}
i(-a, t) &= \frac{1}{R_S} \left(V_S(t) - \frac{\mu_+(-a, t) + \mu_-(-a, t)}{2e}\right) \\
i(a, t) &= \frac{1}{R_D} \left(\frac{\mu_+(a, t) + \mu_-(a, t)}{2e} - V_D(t)\right)
\end{align*}
\]

In the above equations we have considered two electrodes connected at the boundaries of the LL, the left electrode being a source at voltage \(V_S(t)\) and the right electrode being a drain at voltage \(V_D(t)\). Currents are oriented from left to right.

We stress that these relations are operator ones: we work therefore in the Heisenberg representation. For computation of noise properties it is indeed crucial that these relations
are enforced at the operator level and not as average; knowledge of a current average is insufficient to specify fluctuation properties.

These relations extend an earlier formalism developed by the author and collaborators [14]: the main difference is that earlier we considered the chemical potentials as uniform (as is the case in a DC context) in a grand-canonical approach while here we work in a canonical setting with local potentials, which is more suitable to the AC context.

We note in passing that such relations can be derived explicitly in several exactly solvable models: for instance for the inhomogeneous Luttinger liquid with interface resistances \( R_S = R_D = R_0/2 \); for a chiral Luttinger liquid connected by a point contact to a Fermi liquid with \( R = R_0/2 \) or more generally for a reservoir which is a LL with LL parameter \( K_{\text{res}} \) also connected through a point contact to the sample, the interface resistance is \( R = R_0/2K_{\text{res}} \). Safi’s boundary conditions [5] \( V_{S/D}(t) = \delta H \delta \rho_0^{\pm}(x,t) \) where \( \rho_0^{\pm} \) are the electron densities at the right and left Fermi points (and not the chiral densities) coincide with our boundary conditions for the special values \( R_S = R_D = R_0/2 \). For a detailed discussion of these relations we refer to our earlier work [14].

Simple though these relations may seem they permit to go beyond earlier AC results found by using for instance the inhomogeneous LL model as will next be shown.

III. DYNAMICAL RESPONSE OF A GATED WIRE.

A. Dynamical impedance.

We now consider time-varying voltage sources \( V_S(t) = V_1 \exp i\omega t \) and \( V_D(t) = V_2 \exp i\omega t \) and a gate voltage \( V_3(t) = V_3 \exp i\omega t \) and compute the dynamical impedance and conductance matrices of the LL. The currents at the boundaries of the system then acquire the same time dependence; we define currents as entering the system:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = \begin{pmatrix}i(-a,\omega) \\
-i(a,\omega)
\end{pmatrix}.
\]

To enforce current conservation there will in general be a displacement current corresponding to the charging of the sample. In that section we fix the currents at the bound-
aries as \( i_1 = i_1^0 \exp i\omega t \) and \( i_2 = i_2^0 \exp i\omega t \). Therefore the voltages \( V_S(t) = V_1 \exp i\omega t \) and \( V_D(t) = V_2 \exp i\omega t \) can be viewed as responses to the currents.

The boundary conditions are therefore rewritten as:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{R_S} \left( V_1 - \frac{\mu_+(a) + \mu_-(a)}{2e} \right) \\
\frac{1}{R_D} \left( V_2 - \frac{\mu_+(a) + \mu_-(a)}{2e} \right)
\end{pmatrix}
\]

(4)

Using eq. (4) and eq. (3) it follows that:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix}
= \frac{Ke}{\hbar} \begin{pmatrix} 1 & -\exp i\phi \\ -\exp i\phi & 1 \end{pmatrix} \begin{pmatrix} \mu'_+(-a) \\ \mu'_-(a) \end{pmatrix}
\]

(5)

Defining the vector \( \vec{\mu} \) as:

\[
\vec{\mu} = \begin{pmatrix} \mu'_+(-a) \\ \mu'_-(a) \end{pmatrix}
\]

(6)

and using eq. (3, 4, 2) there follows:

\[
\begin{pmatrix} V_1 - V_3 \\ V_2 - V_3 \end{pmatrix} = \begin{pmatrix} \mu'_+(-a) + \mu'_-(a) \\ \mu'_+(-a) + \mu'_-(a) \end{pmatrix} + \begin{pmatrix} R_S & 0 \\ 0 & R_D \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}
\]

\[
= \frac{1}{e} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \exp i\phi \\ \frac{1}{2} \exp i\phi & \frac{1}{2} \end{pmatrix} \vec{\mu} + \begin{pmatrix} R_S & 0 \\ 0 & R_D \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}
\]

Inserting eq. (5), there comes:

\[
\begin{pmatrix} V_1 - V_3 \\ V_2 - V_3 \end{pmatrix} = \frac{1}{e} \begin{pmatrix} \frac{1}{2} + K\overline{R}_S & \exp i\phi \left( \frac{1}{2} - K\overline{R}_S \right) \\ \exp i\phi \left( \frac{1}{2} - K\overline{R}_D \right) & \frac{1}{2} + K\overline{R}_D \end{pmatrix} \vec{\mu}
\]

(7)

where \( \overline{R}_D = R_D/R_0 \) and \( \overline{R}_S = R_S/R_0 \) are resistances measured against the quantum of resistance \( R_0 = \frac{\hbar}{e^2} \). Inverting now eq. (5) and inserting it in eq. (7) one gets:

\[
\begin{pmatrix} V_1 - V_3 \\ V_2 - V_3 \end{pmatrix} = \frac{h}{Ke^2} \begin{pmatrix} \frac{1}{2} + K\overline{R}_S & \exp i\phi \left( \frac{1}{2} - K\overline{R}_S \right) \\ \exp i\phi \left( \frac{1}{2} - K\overline{R}_D \right) & \frac{1}{2} + K\overline{R}_D \end{pmatrix} \begin{pmatrix} 1 & -\exp i\phi \\ -\exp i\phi & 1 \end{pmatrix}^{-1} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}
\]

Defining the dynamical impedance matrix as:

\[
\begin{pmatrix} V_1 - V_3 \\ V_2 - V_3 \end{pmatrix} = Z \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}
\]

one finally finds:

\[
Z = \begin{pmatrix} R_S + i\frac{R_0}{2K} \cot \phi & i\frac{R_0}{2K} \sin \phi \\ i\frac{R_0}{2K} \sin \phi & R_D + i\frac{R_0}{2K} \cot \phi \end{pmatrix}
\]

(8)

where \( \phi = \frac{L}{u} \) (\( L \) is the length of the system, and \( u \) is the plasmon velocity). This is the main result of this sub-section.
B. Intrinsic inductance of the Luttinger liquid.

We now consider the following experimental arrangement in order to measure the impedance of the LL:

\[ i_1 = -i_2 = i_0 \exp i \omega t \]

The impedance of the system is therefore related to the matrix elements of the full impedance matrix by:

\[ Z = \frac{V_1 - V_2}{i} = Z_{11} + Z_{22} - Z_{12} - Z_{21} \]

and therefore:

\[ Z = R_S + R_D - iR_0 \frac{K}{2} \tan \left( \frac{\phi}{2} \right) \]

That especially simple formula admits as low frequency limit:

\[ Z = R_S + R_D - iR_0 \omega \left( \frac{L}{2u} \right) + O(\omega^3) \]

where \( L = 2a \) is the size of the system.

Comments:

(i) This shows firstly that the total contact resistance results as it should be from a series addition of the two interface resistances \( R_S \) and \( R_D \).

(ii) Secondly, since \( Z = R_S + R_D - i \omega (L \ell) + O(\omega) \) there appears an inductance per unit length:

\[ \mathcal{L} = \frac{h}{2u K e^2} \]

This is as it should be; indeed direct inspection of the Luttinger Hamiltonian shows that the Luttinger liquid must have an inductance precisely set at that value. Indeed:

\[ H = \int_{-a}^{a} dx \frac{h u}{4K} \rho^2 + \frac{h u K}{4} J^2 \]

where \( J = \rho_+^0 - \rho_-^0 \) is the difference between bare right and left electron densities (at right and left Fermi points \( \pm k_F \)). Rewriting the Hamiltonian in terms of charge density and current:

\[ \rho_e = e \rho; \quad j_e = e u K \bar{J} \]

8
(the last expression follows from charge conservation and the equations of motion) there follows:

\[ H = \int_{-a}^{a} dx \frac{\hbar u}{4Ke^2 \rho_e^2} + \frac{\hbar}{4u Ke^2} j_e^2. \]

This shows indeed an inductance per unit length \( \mathcal{L} = \frac{\hbar}{2u Ke^2} \) while the zero mode of the first term yields \( \frac{\hbar u}{4K e^2 L} Q^2 \) which shows there is a capacitance per unit length:

\[ C = \frac{2Ke^2}{\hbar u}. \]

While there has been ample emphasis on the intrinsic capacitance of the Luttinger liquid, the fact that the LL possesses an intrinsic inductance is less well stressed: see however. We note in passing that the term \( \frac{\hbar}{4u Ke^2} j_e^2 \) in the Hamiltonian results from both kinetic energy and interactions: the inductance \( \mathcal{L} = \frac{\hbar}{2u Ke^2} \) has therefore a mixed origin and is not merely contrary to what Burke argues a kinetic inductance: this point is somewhat obscured by the fact that in Galilean invariant systems \( v_F = u K \) which implies then that the intrinsic LL inductance assumes exactly the same value as in a non-interacting system; however since Galilean invariance is in general not realized the previous identity does not hold and a renormalization by the interactions of the kinetic inductance should follow. At any rate whether experiments can provide independent measurements of both \( u \) and \( K \): therefore one need not assume that \( v_F = u K \), since the validity or non-validity of that relation can be checked.

(iii) For carbon nanotubes assuming a length \( L \sim 1 \mu m \) and a plasmon velocity of the order of \( v_F = 10^5 ms^{-1} \) means that each successive term in the low-frequency expansion of the impedance goes as \( R_0 \left( \frac{\omega}{10^{6} Hz} \right)^n \). This implies that at already a frequency of about \( 10kHz \) the inductive correction is \( \delta Z/Z = 10^{-7} \). While the first order correction is quite measurable the next order (three) correction is much less accessible unless one goes to the \( GHz \) range.

C. Dynamical conductance.

We now fix voltages; therefore the relation \( i_1 = -i_2 \) is not valid any more. As amply stressed by Büttiker there is a displacement current \( i_3 \) due to the charging of the system. Current conservation is enforced only if one considers that additional current.
One now inverts the relation \( \begin{pmatrix} V_1 - V_3 \\ V_2 - V_3 \end{pmatrix} = Z \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \); whence the upper 2 \times 2 part of the dynamical conductance matrix:

\[
\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \frac{KG_0}{\left( \frac{1}{2} + K R_S \right) \left( \frac{1}{2} + K R_D \right) - \exp i2\phi \left( \frac{1}{2} - K R_S \right) \left( \frac{1}{2} - K R_D \right)} \times \\
\begin{pmatrix} \frac{1}{2} + K R_D + \exp i2\phi \left( \frac{1}{2} - K R_D \right) & -\exp i\phi \\ -\exp i\phi & \frac{1}{2} + K R_S + \exp i2\phi \left( \frac{1}{2} - K R_S \right) \end{pmatrix} \begin{pmatrix} V_1 - V_3 \\ V_2 - V_3 \end{pmatrix}
\]

Using current conservation \( \sum_k i_k = 0 \) the full conductance matrix follows:

\[
\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = G \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}
\]

with matrix elements:

\[
G_{11} = \frac{KG_0 \left[ \frac{1}{2} + K R_D + \exp i2\phi \left( \frac{1}{2} - K R_D \right) \right]}{\left( \frac{1}{2} + K R_S \right) \left( \frac{1}{2} + K R_D \right) - \exp i2\phi \left( \frac{1}{2} - K R_S \right) \left( \frac{1}{2} - K R_D \right)}
\]

\[
G_{12} = G_{21} = \frac{-KG_0 \exp i\phi}{\left( \frac{1}{2} + K R_S \right) \left( \frac{1}{2} + K R_D \right) - \exp i2\phi \left( \frac{1}{2} - K R_S \right) \left( \frac{1}{2} - K R_D \right)}
\]

\[
G_{22} = \frac{KG_0 \left[ \frac{1}{2} + K R_S + \exp i2\phi \left( \frac{1}{2} - K R_S \right) \right]}{\left( \frac{1}{2} + K R_S \right) \left( \frac{1}{2} + K R_D \right) - \exp i2\phi \left( \frac{1}{2} - K R_S \right) \left( \frac{1}{2} - K R_D \right)}
\]

\[
G_{13} = G_{31} = -G_{11} - G_{12}
\]

\[
G_{23} = G_{32} = -G_{22} - G_{21}
\]

\[
G_{33} = G_{11} + G_{22} - G_{12} - G_{21} = \frac{KG_0 \left[ 1 + K \left( R_S + R_D \right) - 2 \exp i\phi + \exp i2\phi \left( 1 - K \left( R_S + R_D \right) \right) \right]}{\left( \frac{1}{2} + K R_S \right) \left( \frac{1}{2} + K R_D \right) - \exp i2\phi \left( \frac{1}{2} - K R_S \right) \left( \frac{1}{2} - K R_D \right)}
\]

A useful check is to set the interface resistances to \( R_S = R_D = R_0/2 \); one recovers immediately eq.(10-11) of Blanter et al.\[4\].

We now expand the gate conductance \( G_{33} \):

\[
G_{33} = -iC\omega + \omega^2 (CL)^2 R_q + i\omega^3 (CL)^3 R_q^2 \left( 1 + \frac{R_q^2}{4K^2 R_q R_C} - \frac{R_0^2}{12K^2 R_q^2} \right) + O(\omega^3) \quad (9)
\]

where \( C = \frac{2Ke^2}{\hbar u}, \quad R_q = (R_S R_D)/(R_S + R_D) \) and \( R_C = R_S + R_D \) is the contact resistance.
D. Discussion.

The previous expression shows:

(i) firstly that the capacitance per unit length $C = \frac{2Ke^2}{h u}$ is independent of the coupling to the reservoirs: this is quite sensible; its value is just that expected from a direct inspection of the Luttinger Hamiltonian (see above). Measurement of both $\mathcal{L} = \frac{h}{2uKe^2}$ and $C = \frac{2Ke^2}{h u}$ therefore provides a direct way to get the values of $u$ and $K$. As already noticed by Burke using the telegraphist equation [9], the plasmon velocity is just $\sqrt{\mathcal{L}C} = \frac{1}{u}$, an identity well-known to electrical engineers while the transmission line impedance is just: $Z_0 = \sqrt{\mathcal{L}/C} = \frac{1}{2K}$.

This justifies a posteriori the transmission line analogy proposed by Burke. Note however that our results show that the transmission line analogy is valid only up to order two in a low frequency expansion.

(ii) That capacitance $C$ is fully chemical and corresponds to (the inverse of) the density of states: as stressed by Büttiker [16] in a general experimental setting one has to add a geometrical capacitance describing the coupling to a wire so that the total capacitance is $C_{tot}^{-1} = (\mathcal{L}C)^{-1} + C_{geom}^{-1}$.

(iii) There appears a charge relaxation resistance $R_q$ which obeys a parallel addition law: $R_q^{-1} = R_S^{-1} + R_D^{-1}$. This is quite sensible because relaxation probabilities should add for independent relaxation processes. The charge relaxation resistance is distinct from the contact resistance in that it corresponds to an $RC$ time for the discharging of a system and not (directly) to energy dissipation [16]. We also observe that by measuring both impedance and gate conductance up to order two in frequency one can directly measure both interface resistances: there is therefore no need to assume that they are a priori set at $R_S = R_D = R_0/2$ since this can be checked experimentally.

(iv) Several approaches have been advocated for determining the Luttinger parameters using AC measurements. Ponomarenko [3], Sablikov and Shchamkhalova [6] proposed to measure the period of the conductance since they are oscillating functions of the parameter $\phi = \omega \frac{L}{u}$. This has been criticized by Blanter et al. who argue that the frequency is quite high (GHz range) [4]. Safi proposed to measure the conductance at low frequency and measure its deviation to the DC limit [5], by showing that at low frequency for a symmetric electric field arrangement one can neglect the displacement current so that $G_{12} = -G_{11} = G_0(1+i\omega \frac{L}{2uK})$: this allows access to the product $uK$. Blanter et al. argued that such deviations are hard
to identify and proposed to measure the gate conductance up to order three to determine the values of $u$ and $K$.

As is apparent from our discussion of the inductance of the Luttinger liquid a joint measurement of both impedance and gate conductance circumvents the need to go to order three in frequency: it is sufficient to go to order one, which means that measurements at the kHz range should be enough rather than the 100 GHz range. That measurement of the impedance is equivalent to the measurement of $G_{12}$ proposed by Safi with the advantage that fixing the currents as in an impedance measurement avoids the complications of displacement current. In addition note that our derivation does not require a symmetric configuration for the electrodes.

(v) But if conversely one is able to make measurements up to order three (i.e. up to $G_0\left(\frac{\omega}{100\text{GHz}}\right)^3$, or at GHz range), the order third term instead of providing a mere fit of the LL theory to experiments now constitutes a distinct non-trivial prediction of the theory. But as is obvious from eq.(9) instead of the simpler expression $i\omega^3 (CL)^3 R_q^2 \left(1 - \frac{1}{3K^2}\right)$ given by Blanter et al., charge relaxation and contact resistances enter in an intricate way if they do not take the trivial values $R_q = \hbar/4e^2$ and $R_C = \hbar/e^2$. The same comment applies to the dynamical impedance:

$$Z = R_S + R_D - i\omega LL + i\omega^3 L^3 \frac{L^2C}{12} + O(\omega^3).$$

We note in passing that the appearance of $K^2$ in the formula can be understood quite simply as the translation of inductive corrections since $L/C = \frac{R^2}{4K^2}$.

E. AC response of a LL connected to a single reservoir.

That experimental setting is especially interesting because in the DC limit there is no current. The dynamics within an AC experiment is wholly governed by the charge dynamics within the sample and illustrates nicely the role of the displacement current as stressed by Büttiker.

To the author’s knowledge such a setting for a LL has not been treated in the literature even in the case of $R_S = R_0/2$. Yet in our formalism that situation is quite straightforwardly described: it suffices to take the limit of infinite interface resistance $R_D$ if one wants for instance to disconnect the drain electrode. No current can flow into the drain and the current
$i_2$ is therefore zero. The only non-zero matrix elements of the dynamical conductance are $G_{11}, G_{13}, G_{33}, G_{31}$ and are determined by a single number:

$$G_{11} = G_{33} = \frac{K G_0 \left( 1 - \exp i2\phi \right)}{\left( \frac{1}{2} + K R S \right) + \exp i2\phi \left( \frac{1}{2} - K R S \right)},$$

$$G_{13} = G_{31} = -G_{11}.$$

In that situation the incoming current charges the Luttinger liquid and therefore $i_1 + i_3 = 0$ the displacement current compensates exactly the charge current. The impedance $Z = \frac{V_1 - V_3}{i_1}$ is the inverse of $G_{11}$:

$$Z = R_S + \frac{i}{2K} \cot \phi$$

$$= \frac{1}{-iCL\omega} + R_S + O(1).$$

The low-frequency expansion of the conductance is:

$$G_{33} = -iCL\omega + \omega^2 (CL)^2 R_S + i\omega^3 (CL)^3 R_S^2 \left( 1 - \frac{R_0^2}{12K^2R_S^2} \right).$$

In such a setting the inductive effects are much more difficult to see: one must go to order three in the conductance to observe them.

**IV. DYNAMICAL RESPONSE OF A LL TO AN AC ELECTRIC FIELD.**

**A. Equations of motion and boundary conditions.**

We now apply an AC electric field along the sample and give therefore a spatial dependence to $V_3$:

$$V_3(x,t) = V_3(x) e^{i\omega t}.$$  

The Luttinger Hamiltonian can be rewritten in terms of the phase field $\theta$ conjugate to the density as (we set $\hbar = 1$):

$$H = \int_a^a dx \frac{\pi u}{2K} \rho^2 + \frac{u}{2\pi} (\partial_x \theta)^2 + eV_3\rho$$

The equations of motion in the Heisenberg representation for the density and the particle current are:

$$\partial_t^2 \rho - u^2 \partial_x^2 \rho = \frac{uKe}{\pi} \partial^2_x V_3$$

$$\partial_t^2 j - u^2 \partial_x^2 j = -\frac{uKe}{\pi} \partial_{x,t}^2 V_3.$$
where the particle current is (as can be checked from the current conservation equation):

\[ j = -u K \frac{\partial}{\pi} \theta \]

These operators can therefore be written as:

\[ \rho(x, t) = \rho_+(t - \frac{x}{u}) + \rho_-(t + \frac{x}{u}) + \rho_0(x) e^{i\omega t} \]
\[ j(x, t) = j_+(t - \frac{x}{u}) + j_-(t + \frac{x}{u}) + j_0(x) e^{i\omega t} \]

where \( \rho_0(x) e^{i\omega t} \) and \( j_0(x) e^{i\omega t} \) are arbitrary particular solutions of the equations of motion. One can choose \( \rho_0(x) \) to be proportional to \( j_0(x) \). Indeed using current conservation \( \partial_t \rho + \partial_x j = 0 \) it follows immediately that:

\[ \partial_x j = -\partial_t \rho = -\partial_t (\rho_+ + \rho_-) - i\omega \rho_0(x) e^{i\omega t} \]
\[ = u \partial_x (\rho_+ - \rho_-) - i\omega \rho_0(x) e^{i\omega t} \]

which implies that we can set:

\[ j_\pm = \pm u \rho_\pm, \quad \rho_0 = \frac{-1}{i\omega} \partial_x j_0. \]

The chiral chemical potential operators are now:

\[ \mu_\pm(x, t) = 2\pi u \frac{K}{\rho_\pm(x, t) + \rho_0(x)} + \frac{\pi u}{K} \rho_0 \pm \frac{\pi}{K} j_0. \]

It is readily checked that the current operator is given by eq. (10):

\[ i(x, t) = e j(x, t) = K e \left( \mu_+(x, t) - \mu_-(x, t) \right). \]

Again it is convenient to shift the chemical potential operators to have operators which have a purely chiral time-evolution:

\[ \mu'_\pm(x, t) = \mu_\pm(x, t) - e V_3(t) \frac{\pi u}{K} \rho_0 \pm \frac{\pi}{K} j_0 \].

We now consider the following boundary conditions:

\[ e j(-a, t) = \frac{1}{R_S} \left( V_3(-a, t) - \frac{\mu_+(a, t) + \mu_-(a, t)}{2e} \right), \]
\[ e j(a, t) = \frac{1}{R_D} \left( \frac{\mu_+(a, t) + \mu_-(a, t)}{2e} - V_3(a, t) \right). \]

These boundary conditions correspond to source and drain voltages set to the ground (zero voltage): therefore only the potential \( V_3 \) appears; it corresponds to the energy gained due to the initial (or final) acceleration given by the applied electric field. In the previous section one did not have to take it into account since no electric field was applied.
B. Dynamical response.

Defining again the currents as entering the system:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = \begin{pmatrix} e^{j(-a,t)} \\ -e^{j(a,t)} \end{pmatrix},
\]

the boundary conditions are rewritten as:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = \begin{pmatrix} \frac{1}{R_S} \left( -\frac{\pi u}{eK} \rho_0(-a) - \frac{\mu'_+(a)+\mu'_-(a)}{2e} \right) \\ \frac{1}{R_D} \left( \frac{\pi u}{eK} \rho_0(a) - \frac{\mu'_+(a)+\mu'_-(a)}{2e} \right) \end{pmatrix}.
\]

Therefore:

\[
\begin{pmatrix}
-\frac{\pi u}{eK} \rho_0(-a) \\
\frac{\pi u}{eK} \rho_0(a)
\end{pmatrix} = \begin{pmatrix} \frac{\mu'_+(a)+\mu'_-(a)}{2e} \\ \frac{\mu'_+(a)+\mu'_-(a)}{2e} \end{pmatrix} + \begin{pmatrix} R_S & 0 \\ 0 & R_D \end{pmatrix} \begin{pmatrix} i_1 \\
i_2
\end{pmatrix} (11)
\]

\[
\begin{pmatrix}
\frac{1}{e} \left( \frac{1}{2} \exp i\phi \right) \\
\frac{1}{2} \exp i\phi
\end{pmatrix} \mu = \begin{pmatrix} R_S & 0 \\ 0 & R_D \end{pmatrix} \begin{pmatrix} i_1 \\
i_2
\end{pmatrix}
\]

where the vector \( \mu \) is defined as above (eq. (8)). But according to eq. (1) the current is rewritten as:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = Ke \left( \begin{pmatrix} 1 & -\exp i\phi \\ -\exp i\phi & 1 \end{pmatrix} \mu + \begin{pmatrix} e^{j_0(-a)} \\ -e^{j_0(a)} \end{pmatrix} \right).
\]

Therefore substitution of the previous equation in eq. (11) yields:

\[
\begin{pmatrix}
-\frac{\pi u}{eK} \rho_0(-a) \\
\frac{\pi u}{eK} \rho_0(a)
\end{pmatrix} = \frac{1}{e} \begin{pmatrix} \frac{1}{2} + K R_S & \exp i\phi \left( \frac{1}{2} - K R_S \right) \\ \exp i\phi \left( \frac{1}{2} - K R_D \right) & \frac{1}{2} + K R_D \end{pmatrix} \mu + \begin{pmatrix} e^{j_0(-a)} R_S \\ -e^{j_0(a)} R_D \end{pmatrix} (12)
\]

Elimination of \( \mu \) then yields:

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = Z^{-1} \begin{pmatrix} \frac{1}{e} \left( -\frac{\pi u}{eK} \rho_0(-a) - e^{j_0(-a)} R_S \right) \\ \frac{1}{e} \left( \frac{\pi u}{eK} \rho_0(a) + e^{j_0(a)} R_D \right) \end{pmatrix} + \begin{pmatrix} e^{j_0(-a)} \\ -e^{j_0(a)} \end{pmatrix} (13)
\]

where the matrix \( Z \) is the same dynamical impedance matrix found above in eq. (8):

\[
Z = \begin{pmatrix} R_S + i \frac{R_0}{2K} \cot \phi & i \frac{R_0}{2K \sin \phi} \\ i \frac{R_0}{2K \sin \phi} & R_D + i \frac{R_0}{2K} \cot \phi \end{pmatrix}. (14)
\]

\( Z^{-1} \) is just the 2 \( \times \) 2 upper restriction of the dynamical conductance matrix \( G \) found for the gated LL. Equation (13) is the main result of this sub-section. It can be rewritten as:
\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - Z^{-1} \begin{pmatrix} R_S + i \frac{1}{\omega} \partial_x & 0 \\ 0 & R_D + i \frac{1}{\omega} \partial_x \end{pmatrix} \right] \begin{pmatrix} e j_0(-a) \\ -e j_0(a) \end{pmatrix}
\]
where \( C = \frac{2 Ke^2}{h u} \) is the intrinsic LL capacitance and where for instance
\[
j_0(x) = \frac{i \omega e K}{u \pi} \int_0^x dy \int_0^y dz E(z)e^{i \frac{\pi}{4}(x+z-2y)}.
\]
It is easily checked that the current response is independent of the particular solution \( j_0 \) chosen: shifting \( j_0 \) and \( \rho_0 = \frac{1}{i \omega} \partial_x j_0 \) by either chiral currents \( \delta j_+ \) or \( \delta j_- \) gives contributions which cancel each other.

Even for the subcase \( R_S = R_D = R_0/2 \) this matrix equation does not seem to appear in the literature: for instance Ponomarenko writes the current at a given position in the inhomogeneous LL model for an arbitrary electric field as a Green function convolution but that simple matrix relation between the response of a gated wire and the response to an arbitrary electric field is not explicitly written. Joint measurements in both context would be interesting to reveal such relations between the dynamical responses.

**C. Uniform electric field.**

We now specialize the discussion to a uniform electric field \( E = -\partial x V_3 \) so that:
\[
j_0 = -\frac{i u Ke}{\omega \pi} E = -i 2 G_0 u Ke E
\]
The current response is therefore:
\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = e j_0 \left[ Z^{-1} \begin{pmatrix} R_S \\ -R_D \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]
\]
\[
= -i E \frac{G_0 u K}{\omega} \begin{pmatrix} \frac{1}{2} + K R_S & \frac{1}{2} - K R_D \exp i 2 \phi - 2 K R_D \exp i \phi \ \\ \frac{1}{2} + K R_D & \frac{1}{2} - K R_S \exp i 2 \phi - 2 K R_S \exp i \phi \end{pmatrix}
\]
If \( R_S = R_D = R_0/2 \), this yields:
\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} = -i E \frac{2 G_0 u K}{\omega} \frac{-i \sin \frac{\phi}{2}}{K \cos \frac{\phi}{2} - i \sin \frac{\phi}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]
which is exactly the expressions found by Sablikov et al. and Ponomarenko.

Sablikov and Shchamkhalova argue that due to a charging of the reservoirs the real current measured in an AC experiment is not \( i_1 \) but that one must add a displacement current \( \frac{dQ_S}{dt} \) where \( Q_S \) is the charge appearing at the source. Appealing to a result initially derived by Shockley, using Laplace equation they find that:

\[
\frac{dQ_S}{dt} = -i_1 + \frac{1}{L} \int_{-L/2}^{L/2} i(x) \, dx,
\]

and therefore the current measured at the left electrode is:

\[
i_{mes} = \frac{1}{L} \int_{-L/2}^{L/2} i(x) \, dx,
\]

for a uniform electric field and plane electrodes orthogonal to the wire.

That point of view is however valid only if one does not take into account relaxation processes in the reservoir: the charging of the reservoir must be taken into account only for frequencies \( \omega \gg 1/\tau_{rel} \) where \( \tau_{rel} \) is the relaxation time of the reservoir, i.e. the inverse of the plasma frequency \( \omega_P = 1/\tau_{rel} \sim 10^{15} \text{Hz} \). For optical processes this becomes relevant but not for the transport experiments one considers here.

It is quite easy to extract the distribution of current and charge in the sample:

\[
i(x, \omega) = K \frac{e}{\hbar} \left( \mu'_+(x, \omega) - \mu'_-(x, \omega) \right) + ej_0(x).
\]

Since:

\[
\mu'_+(x, \omega) = \exp i\frac{\phi}{2} \exp i\frac{\omega x}{u} \mu'_+(-a, \omega), \\
\mu'_-(x, \omega) = \exp i\frac{\phi}{2} \exp -i\frac{\omega x}{u} \mu'_-(a, \omega),
\]

it follows:

\[
i(x) = K \frac{e}{\hbar} \exp i\frac{\phi}{2} \left( \exp i\frac{\omega x}{u}, \exp -i\frac{\omega x}{u} \right) \cdot \vec{\mu} + ej_0(x).
\]

Using the relation between \( \vec{\mu} \) and \( j_0 \) (eq. [12] above where one takes \( \rho_0 = 0 \) because the electric field is uniform) one easily finds:

\[
i(x) = ej_0 \left[ 1 - \frac{K \left( \cos(\frac{\omega x}{u}) (RS + RD) \left( \cos \frac{\phi}{2} - i4KR_0 \sin \frac{\phi}{2} \right) - \sin(\frac{\omega x}{u}) \sin \frac{\phi}{2} (RS - RD) \right)}{\left( \frac{1}{2} + KR_S \right) \left( \frac{1}{2} + KR_D \right) - \exp i2\phi \left( \frac{1}{2} - KR_S \right) \left( \frac{1}{2} - KR_D \right)} \right]
\]

\[17\]
For the symmetric case $R_S = R_D = R_0/2$ this reduces to:

$$i(x) = e j_0 \left[ 1 - \frac{K \cos \left( \frac{\omega x}{u} \right)}{K \cos \frac{\phi}{2} - i \sin \frac{\phi}{2}} \right],$$

which is also found by Sablikov et al. [6].

The density is then easily found as $\rho(x) = -\frac{1}{iu} \partial_x i(x)$.

V. CONCLUSIONS.

In this paper we have discussed consequences on AC transport of the inclusion of arbitrary interface resistances $R_S$ and $R_D$ between the sample and the source and drain electrodes. The resistive coupling of the Luttinger liquid to the electrodes is described using a boundary condition formalism.

We considered a gated two-port Luttinger liquid which enabled us to generalize expressions of the dynamical conductance matrix. By considering the dynamical impedance we were able in particular to show that in the low-frequency limit the Luttinger liquid can be modelled as an electrical circuit comprising an inductance per unit length $\mathcal{L} = \frac{h}{2au Ke^2}$ in series with the interface resistances, the whole being capacitively coupled to the ground with intrinsic conductance per unit length $\mathcal{C} = \frac{2Ke^2}{hu}$. That transmission line analogy is however invalid beyond order two in frequency.

Focusing in the impedance response of the LL we showed that a joint measurement of both dynamical impedance and gate conductance $G_{33}$ up to order one in frequency is sufficient to extract the Luttinger parameters. A measurement up to order two allows extraction of the interface resistances whose quantization can therefore be checked (or disproved).

We then considered the application of an arbitrary AC electric field along the sample; we then discussed the case of a uniform electric field generalizing earlier results valid only for $R_S = R_D = R_0/2$. The author acknowledges useful discussions with C. Texier and Hélène Bouchiat’s group.

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FIG. 1: Electrical circuit equivalent to the Luttinger liquid. The inductance per unit length is $\mathcal{L} = \frac{\hbar}{2uKc^2}$ and the capacitance per unit length is $C = \frac{2Kc^2}{\hbar u}$. 