Veneziano amplitudes, 
spin chains and 
Abelian reduction of QCD

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Abstract
Although QCD can be treated perturbatively in the high energy limit, lower energies require uses of nonperturbative methods such as ADS/CFT and/or Abelian reduction. These methods are not equivalent. While the first is restricted to supersymmetric Yang-Mills model with number of colors going to infinity, the second is not restricted by requirements of supersymmetry and is designed to work in physically realistic limit of finite number of colors. In this paper we provide arguments in favor of the Abelian reduction methods. This is achieved by further developing results of our recent works reanalyzing Veneziano and Veneziano-like amplitudes and the models associated with these amplitudes. It is shown, that the obtained new partition function for these amplitudes can be mapped exactly into that for the Polychronakos-Frahm (P-F) spin chain model recoverable from the Richardson-Gaudin (R-G) XXX spin chain model originally designed for treatments of the BCS-type superconductivity. Because of this, it is demonstrated that the obtained mapping is compatible with the method of Abelian reduction. The R-G model is recovered from the asymptotic (WKB-type) solutions of the rational Knizhnik-Zamolodchikov (K-Z) equation. Linear independence of these solutions is controlled by determinants whose explicit form (up to a constant) coincides with Veneziano (or Veneziano-like) amplitudes. In the simplest case, the determinantal conditions coincide with those discovered by Kummer in 19-th century. Kummer’s results admit physical interpretation by relating determinantal formula(s) to Veneziano-like amplitudes. Furthermore, these amplitudes can be interpreted as Poisson-Dirichlet distributions playing central role in the stochastic theory of random coagulation-fragmentation processes. Such an interpretation is complementary to that known for the Lund model widely used for description of coagulation-fragmentation processes in QCD.

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1 Introduction

Since times when quantum mechanics (QM) was born (in 1925-1926) two seemingly opposite approaches for description of atomic and subatomic physics were proposed respectively by Heisenberg and Schrödinger. Heisenberg’s approach is aimed at providing an affirmative answer to the following question: Is combinatorics of spectra (of observables) provides sufficient information about microscopic system so that dynamics of such a system can be described in terms of known macroscopic concepts? Schrödinger’s approach is exactly opposite and is aimed at providing an affirmative answer to the question: Using some plausible mathematical arguments is it possible to find equations which under some prescribed restrictions will reproduce the spectra of observables? Although it is widely believed that both approaches are equivalent, already Dirac in his lectures on quantum field theory, Ref.[1], noticed (without much elaboration) that Schrödinger’s description of QM contains a lot of ”dead wood” which can be safely disposed. According to Dirac ”Heisenberg’s picture of QM is good because Heisenberg’s equations of motion make sense”.

To our knowledge, Dirac’s comments were completely ignored, perhaps, because he had not provided enough evidence making Heisenberg’s description of QM superior to that of Schrodinger’s. In recent papers, Ref.[2,3], we found examples supporting Dirac’s claims. From the point of view of combinatorics, there is not much difference in description of QM, quantum field theory or string theory as demonstrated in Ref.[4]. Therefore, in this paper we adopt Heisenberg’s point of view at QCD and string theory using results of our recent works aimed at re analyzing the existing treatments connecting Veneziano (and Veneziano-like) amplitudes with the respective string-theoretic models. As result, we found new tachyon-free models reproducing Veneziano (and Veneziano-like) amplitudes. In this work results of our papers, Ref.s.[5-7], to be called respectively as Part I, Part II and Part III are developed further. This has become possible in view of the work by Reshetikhin and Varchenko, Ref.[8], and by Varchenko summarized in Varchenko’s MIT lecture notes, Ref.[9]. They enabled us to relate Veneziano (and Veneziano-like) amplitudes (e.g. those describing ππ scattering) to Knizhnik-Zamolodchikov (K-Z) equations and, hence, to WZNW models. This is achieved by employing known connections between WZNW models and spin chains. In the present case, between the K-Z equations and the XXX-type Richardson-Gaudin (R-G) spin chains. To keep things in a perspective, we would like to provide some rationale behind the above mentioned connection with spin chains developed in this paper.

As is well known, all information in high energy physics is obtainable through proper interpretation of scattering data. It is believed that for sufficiently high energies such data are well described by the phenomenological Regge theory and can be conveniently summarized with help of Chew-Frautlich (C-F) plots relating masses to spins (angular momenta), e.g. see book by Collins, Ref.[10]. Veneziano amplitudes are by design Regge-behaving. Both Regge theory and Veneziano amplitudes emerged before major developments in QCD took place in 70ies. Once these developments took place, naturally, it was of interest to
recover the Regge theory from QCD. Even though there are many ways of doing so, to our knowledge, the problem is still not solved completely. This is so for the following reasons.

Although the amount of data obtained by perturbative treatments of QCD is quite impressive, e.g., read Ref.[11], these results are not helpful for establishing the Regge-type behavior of QCD. Since such a behavior can be easily established with help of variety of string models, the task lies in connecting these models with QCD. For the sake of space, we do not discuss broader reasons (e.g., quantum gravity) for development of string theory. Recently, another pathways towards establishing Regge-type behavior were explored. For instance, in Ref.[12], in the large N limit (to be defined in the main text) by ignoring masses of quarks the spin zero glueball mass spectrum was obtained analytically for 2+1 dimensional pure Yang-Mills theory\(^2\) which in the high energy limit fits perfectly Regge theory. Although in Ref.[12] string models were not used, the obtained results are in excellent agreement with large N lattice calculations. Given this, they still suffer from several drawbacks. First, the large N limit, even though mathematically convenient, physically is questionable as it will be explained below, in the main text. Second, more physically interesting are the spectra of mesons and baryons. These spectra are traditionally obtained with help of string-theoretic models. Since calculations involving these models are by design made Regge-behaving, the task was (still is) to connect these string models with QCD. It is widely believed that such a connection is achievable via ADS/CFT correspondence [14] between strings living in anti-de Sitter space and \(\mathcal{N} = 4, N\rightarrow \infty\) supersymmetric Yang-Mills (Y-M) model in \(d = 3 + 1\) dimensions. By design, such supersymmetric Y-M model does not contain quark masses\(^3\). Again, such correspondence becomes physically meaningful only if the number of colors N could be made finite and the supersymmetry could be broken. Unfortunately, these requirements are in apparent contradiction with the way the ADS/CFT correspondence was established in the first place thus making such a task very difficult to accomplish.

In 1981 ’t Hooft suggested in Ref.[18] to reduce the non Abelian QCD to Abelian Ginzburg-Landau (G-L) type theory. The rationale for such an Abelian reduction can be traced back to the work of Nambu, Ref.[19]. In his work Nambu superimposed G-L theory with the theory of Dirac monopoles to demonstrate quark confinement for mesons. Incidentally, Veneziano amplitudes are suited the most for describing meson resonances, e.g., see Ref.[10]. These are made of just two quarks: quark and antiquark. Thus, if the existence of Abelian reduction would be considered as proven, this then would be equivalent to the proof of quark confinement. Recent numerical studies have provided convincing evidence supporting the idea of quark confinement through monopole condensation, e.g., see Refs.[20,21]. Since publication of ’t Hooft’s paper many theoretical advancements were made, most notably by Cho, Ref.s[22,23], and Kondo, Ref.s

\(^2\)These results were subsequently extended to 3+1 dimensions in Ref.[13].

\(^3\)The achievements of this method are summarized in excellent recent reviews by Benna and Klebanov, Ref.[15], and Brodsky, Ref.[16]. Our work, Ref.[17], provides an introduction to the ideas and methods of ADS/CFT.
[24-26], whose work was motivated by that by Faddeev and his group. Results of this group are summarized in the recent review by Faddeev, Ref.[27]. From this reference it follows that most of efforts to date were spent on description of the massless version of QCD (just like in the case of ADS/CFT correspondence, but without invoking supersymmetry or requiring $N \to \infty$). Excitation spectrum of solitonic knotted-like structures (admitting interpretation in terms of closed strings) provides the spectrum of glueball masses as demonstrated in Ref.[25].

In a recent paper, Ref.[28], Auckly, Kapitanski and Speight demonstrated how Skyrme model can be obtained from Faddeev model. Since Skyrme model was used for a long time for description of the baryon spectra, e.g. read Ref.[29], and since already Nambu recognized usefulness of the Abelian reduction for description of meson spectra, it follows that the Abelian reduction method is capable of providing sufficient information about QCD in the strong coupling regime. Furthermore, in another paper, Ref.[30], Kapitanski and Auckley obtained result of major importance for this paper. They demonstrated that it is always possible to find such Chern-Simons (C-S) functional which upon minimization will produce the same results as those obtained by minimization of either Skyrme or Faddeev model. Asorey, Falceto and Sierra in Ref.[31] demonstrated how C-S model is related to microscopic BCS model of superconductivity. Since the Abelian reduction of QCD produces G-L-type model (or collection of G-L models), and since the underlying microscopic model is of BCS-type whose elementary excitations are described by the Richardson-Gaudin spin chain model, Ref.[31], the task of this work lies in demonstrating that the combinatorics of scattering processes associated with Veneziano (and/or Veneziano-like) amplitudes leads to the R-G spin chain model of BCS superconductivity. The demonstrated in this work spin chain connection made in the spirit of Heisenberg’s work on quantum mechanics, Ref.[2], favors the Abelian reduction method over ADS/CFT. Unlike other Schrödinger-style papers discussed above-all extracting the Abelian Ginzburg-Landau-type model from the non Abelian QCD- the results of this work use only combinatorics of scattering data as an input to arrive at the same conclusions.

It should be noted that connections between either QCD and spin chains or between strings and spin chains were already discussed in literature for quite some time. Recent paper by Dorey, Ref.[32], contains may references listing these earlier results. Subsequently, they had been replaced by those whose methods are based on ADS/CFT correspondence. From the point of view of this correspondence, connections between strings and QCD also can be made through spin chains as it is demonstrated in the seminal paper by Gubser, Klebanov and Polyakov, Ref.[33]. Their ideas were developed in great detail in the paper by Minahan and Zarembo, Ref.[34]. The spectrum of anomalous dimensions of operators in the $\mathcal{N} = 4$, $N \to \infty$ supersymmetric Yang-Mills ($Y$-
M) model (described in terms of excitation spectrum of the spin chain model) is related to the string spectrum describing hadron masses. To connect these facts with developments in this paper we mention papers by Kruczenski, Ref.[35], and Cotrone et al, Ref.[36]. In both papers spin 1/2 XXX Heisenberg chain was used for description of excitation spectrum. Furthermore, in the paper by Cotrone et al explicit connection with the hadron mass spectrum was made. Both papers invoke ADS/CFT correspondence. In this work, we reobtain these spin chain results using combinatorial arguments following Heisenberg's philosophy. In Section 2 we reobtain Veneziano partition function (derived much more rigorously in Part II). In Section 3 we demonstrate that this partition function coincides with that for the Polychronakos-Frahm (P-F) spin chain model. Although such a model was studied extensively in literature, in Ref.[37] we discuss a variety of new pathways establishing links between the P-F spin chain and many known string-theoretic models, including the most recent ones. For the sake of space, we do not reproduce these results in this work. Instead, in Section 4 we use results of Reshetikhin and Varchenko, Ref.[8], and Varchenko, Ref.[9], in order to make a connection between the P-F and R-G XXX spin chains. We use the results of our Part II in order to demonstrate that the excitation spectrum of new Veneziano model obtained in Part II coincides exactly with that for the R-G spin chain. Since both ADS/CFT and this work point towards the same spin 1/2 XXX chain, this cannot be considered as purely coincidental. Section 5 provides some explanations of noticed coincidence based on independent combinatorial arguments having their origin in the theory of random fragmentation and coagulation processes summarized in Ref.s[38-40]. This theory was adopted for high energy physics by Mekjian, e.g. see Ref.[41] and references therein. A different approach to coagulation-fragmentation processes in QCD is developed by Andersson and collaborators and is known in literature as the Lund model, Ref.[42]. In this work no attempts are made to compare these two approaches. Instead, in Section 5 we argue that in the theory of coagulation-fragmentation processes, Veneziano amplitudes play the central role. In this theory they are known as the Poisson-Diriclet (P-D) probability distributions. The discrete spectra of all exactly solvable quantum mechanical (QM), field and string-theoretic models can be rederived/reobtained in terms of the observables for the P-D stochastic processes. This is so because all exactly solvable QM problems involve some kind of orthogonal polynomials—all derivable from the Gauss hypergeometric function—admitting an interpretation in terms of the P-D process. Since the K-Z equations are essentially the hypergeometric equations of many variables, e.g. see Ref.[9], and since all nontrivial Feynman diagrams of quantum field theory can be looked upon as solutions of these hypergeometric equations as explained in Ref.[4], the processes they describe are also of the P-D type.

Finally, Appendix contains several auxiliary results complementing those presented in Sections 4 and 5.

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7A review of Heisenberg’s arguments resulted in birth of modern quantum mechanics can be found in Ref.[2].
2 Combinatorics of Veneziano amplitudes and spin chains. Qualitative considerations

In Part I, we noticed that the Veneziano condition for the 4-particle amplitude given by
\[ \alpha(s) + \alpha(t) + \alpha(u) = -1, \] (2.1)
where \( \alpha(s), \alpha(t), \alpha(u) \in \mathbb{Z} \), can be rewritten in a more suggestive form. To this purpose, following Ref.[43], we need to consider additional homogenous equation of the type
\[ \alpha(s)m + \alpha(t)n + \alpha(u)l + k \cdot 1 = 0 \] (2.2)
with \( m,n,l,k \) being some integers. By adding this equation to Eq.(2.1) we obtain,
\[ \alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} = \tilde{k} \] (2.3a)
or, equivalently, as
\[ n_1 + n_2 + n_3 = \tilde{N}, \] (2.3b)
where all entries by design are nonnegative integers. For the multiparticle case this equation should be replaced by
\[ n_0 + \cdots + n_k = N \] (2.4)
so that combinatorially the task lies in finding all nonnegative integer combinations of \( n_0, ..., n_k \) producing Eq.(2.4). It should be noted that such a task makes sense as long as \( N \) is assigned. But the actual value of \( N \) is not fixed and, hence, can be chosen quite arbitrarily. Eq.(2.1) is a simple statement about the energy-momentum conservation. Although the numerical entries in this equation can be changed as we just explained, the actual physical values can be subsequently reobtained by the appropriate coordinate shift. Such a procedure should be applied to the amplitudes of conformal field theories (CFT) with some caution since the periodic (or antiperiodic, etc.) boundary conditions cause energy and momenta to become a quasi-energy and a quasi momenta (as it is known from solid state physics).

The arbitrariness of selecting \( N \) reflects kind of a gauge freedom. As in gauge theories, we may try to fix the gauge by using some physical considerations. These include, for example, an observation made in Part I that the four particle amplitude is zero if any two entries into Eq.(2.1) are the same. This fact causes us to arrange the entries in Eq.(2.3b) in accordance with their magnitudes, e.g. \( n_1 \geq n_2 \geq n_3 \). More generally, we can write: \( n_0 \geq n_1 \geq \cdots \geq n_k \geq 1^8 \).

If the entries in this sequence of inequalities are treated as random nonnegative numbers subject to the constraint given by Eq.(2.4), these requirements are necessary and sufficient for recovery of the probability density for such set of random numbers. This density is known in mathematics as the Dirichlet

\[ n_k \geq 1, \] is chosen only for the sake of comparison with the existing literature conventions, e.g. see Ref.[44].
distribution⁹ [38-40,45]. Without normalization, integrals over this distribution coincide with Veneziano amplitudes. Details are given in Section 5. Thus, Veneziano condition leads to Veneziano amplitudes and vice versa. In our work, Ref.[4], we demonstrate that all amplitudes of high energy physics are some linear combinations of Veneziano amplitudes¹⁰.

Provided that Eq.(2.4) holds, we shall call such a sequence a partition and shall denote it as \( n \equiv (n₀, ..., nₖ) \). If \( n \) is partition of \( N \), then we shall write \( n \vdash N \). It is well known, e.g. see Ref.s[46,47], that there is one-to-one correspondence between the Young diagrams and partitions. We would like to use this fact in order to design a partition function associated with Veneziano (and Veneziano-like) amplitudes¹¹. Clearly, such a partition function should also make physical sense. Hence, we would like to provide some qualitative arguments aimed at convincing our readers that such a partition function does exist and is physically sensible.

We begin with observation that there is one-to-one correspondence between the Young tableaux and directed random walks. Furthermore, it is possible to map bijectively such type of random walk back into Young diagram with only two rows, e.g. read Ref.[47], page 5. This allows us to make a connection with spin chains at once. In this work we are not going to use this route to spin chains in view of simplicity of the alternative path described in this section¹². For this purpose we need to consider a square lattice and to place on it the Young diagram associated with some particular partition. Let us choose for this purpose some \( \tilde{n} \times \tilde{m} \) rectangle¹³ so that the Young diagram occupies the left part of this rectangle. We choose the upper left vertex of the rectangle as the origin of \( xy \) coordinate system whose \( y \) axis (South direction) is directed downwards and \( x \) axis is directed Eastwards. Then, the South-East boundary of the Young diagram can be interpreted as directed (that is without self-intersections) random walk which begins at \((0, -\tilde{m})\) and ends at \((\tilde{n}, 0)\). Evidently, such a walk completely determines the diagram. The walk can be described by a sequence of 0’s and 1’s. Say, 0 for the \( x \)-step move and 1 for the \( y \)-step move. The totality \( \mathcal{N} \) of Young diagrams which can be placed into such a rectangle is in one-to-one correspondence with the number of arrangements of 0’s and 1’s whose total number is \( \tilde{m} + \tilde{n} \). Recalling the Fermi statistics, the number \( \mathcal{N} \) can be easily calculated and is given by \( \mathcal{N} = (m + n)!/m!n! \)¹⁴. It

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⁹For reasons explained in Section 5, it is also called the Poisson-Dirichlet (P-D) distribution.

¹⁰This mathematical result was inspired by the observation that, at least perturbatively, all these amplitudes are made of some linear combinations of products of Euler gamma functions (with accuracy up to some logarithmic terms).

¹¹This task should not be confused with the task of connecting the P-D distributions with Veneziano amplitudes to be discussed in Section 5. Alternative pathway through methods of asymptotic combinatorics and representation theory [48] will be treated in a separate publication.

¹²Nevertheless, this option should not be left underappreciated in view of its immediate relevance to Hecke algebra representations, braid groups, etc. e.g. see Ref.s[2] and[49].

¹³Parameters \( \tilde{n} \) and \( \tilde{m} \) will be specified shortly below.

¹⁴We have suppressed the tildas for \( n \) and \( m \) in this expression since these parameters are going to be redefined below anyway.
can be represented in two equivalent ways:

\[
\frac{(m+n)!}{m!n!} = \frac{(n+1)(n+2) \cdots (n+m)}{m!} = \frac{(m+1)(m+2) \cdots (n+m)}{n!}
\]

Let now \( p(N; k, m) \) be the number of partitions of \( N \) into \( \leq k \) nonnegative parts, each not larger than \( m \). Consider the generating function of the following type

\[
F(k, m | q) = \sum_{N=0}^{S} p(N; k, m) q^N,
\]

where the upper limit \( S \) will be determined shortly below. It is shown in Refs.\[44,46\] that \( F(k, m | q) = \left[ k + m \atop m \right]_q \equiv \left[ k + m \atop k \right]_q \) where, for instance, \( \left[ k + m \atop m \right]_q \) is the \( q \)-analog of the binomial coefficient \( \left( k + m \atop m \right) \). In literature \[44,46\] this \( q \)-analog is known as the Gaussian coefficient. Explicitly, it is defined as

\[
\left[ \begin{array}{c} a \\ b \end{array} \right]_q = \frac{ (q^a - 1)(q^{a-1} - 1) \cdots (q^{a-b+1} - 1) }{ (q^b - 1)(q^{b-1} - 1) \cdots (q - 1) }
\]

for some nonegative integers \( a \) and \( b \). From this definition we anticipate that the sum defining generating function \( F(k, m | q) \) in Eq.(2.6) should have only finite number of terms. Eq.(2.7) allows easy determination of the upper limit \( S \) in the sum given by Eq.(2.6). It is given by \( km \). This is just the area of the \( k \times m \) rectangle. In view of the definition of \( p(N; k, m) \), the number \( m = N - k \). Using this fact Eq.(2.6) can be rewritten as \( F(N, k | q) = \left[ \begin{array}{c} N \\ k \end{array} \right]_q \). This expression happens to be the Poincare’ polynomial for the Grassmannian \( Gr(m, k) \) of the complex vector space \( \mathbb{C}^N \) of dimension \( N \) as can be seen from page 292 of the book by Bott and Tu, Ref.[50]. From this (topological) point of view the numerical coefficients, i.e. \( p(N; k, m) \), in the \( q \) expansion of Eq.(2.6) should be interpreted as Betti numbers of this Grassmannian. They can be determined

\[\text{On page 15 of the book by Stanley, Ref.[46], one can find that the number of solutions } N(n, k) \text{ in positive integers to } y_1 + \ldots + y_k = n + k \text{ is given by } \left( \begin{array}{c} n + k - 1 \\ k - 1 \end{array} \right) \text{ while the number of solutions in nonnegative integers to } x_1 + \ldots + x_k = n \text{ is } \left( \begin{array}{c} n + k \\ k \end{array} \right). \text{ Careful reading of Page 15 indicates however that the last number refers to solution in nonnegative integers of the equation } x_0 + \ldots + x_k = n. \text{ This fact was used essentially in Eq.(1.21) of Part I.}\]

\[\text{To make a comparison it is sufficient to replace parameters } t^a \text{ and } n \text{ in Bott and Tu book by } q \text{ and } N.\]
recursively using the following property of the Gaussian coefficients described in Ref.[46], page 26,

\[
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n \\ k+1 \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q ,
\]

provided that \[ \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \]. We refer our readers to Part II for rigorous mathematical proof that \( \mathcal{F}(N,k | q) \) is indeed the Poincare' polynomial for the complex Grassmannian. With this fact proven, we notice that, due to relation \( m = N-k \), it is sometimes more convenient for us to use the parameters \( m \) and \( k \) rather than \( N \) and \( k \). With such a replacement we obtain:

\[
\mathcal{F}(k,m | q) = \begin{bmatrix} k + m \\ k \end{bmatrix}_q = \frac{(q^{k+m} - 1)(q^{k+m-1} - 1) \cdots (q^{m+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}
\]

\[
= \prod_{i=1}^{k} \frac{1 - q^{m+i}}{1 - q^i} .
\]

(2.9)

This result is of central importance. In our work, Part II, considerably more sophisticated mathematical apparatus was used to obtain it (e.g. see Eq.(6.10) of this reference and arguments leading to it).

In the limit \( q \to 1 \) Eq.(2.9) reduces to \( \mathcal{N} \) as required. To make connections with results known in physics literature we need to rescale \( q \)'s in Eq.(2.9), e.g. let \( q = t^{1/2} \). Substitution of such an expression back into Eq.(2.9) and taking the limit \( t \to 1 \) again reproduces \( \mathcal{N} \) in view of Eq.(2.5). This time, however, we can accomplish more. By noticing that in Eq.(2.4) the actual value of \( N \) deliberately is not yet fixed and taking into account that \( m = N-k \), we can fix \( N \) by fixing \( m \). Specifically, we would like to choose \( m = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot k \) and with such a choice we would like to consider a particular term in the product, Eq.(2.9), e.g.

\[
S(i) = \frac{1 - t^{1+m-i}}{1 - t} .
\]

(2.10)

In view of our "gauge fixing" the ratio \( m/i \) is a positive integer by design. This means that we are having a geometric progression. Indeed, if we rescale \( t \) again, e.g. \( t \to t^2 \), we then obtain

\[
S(i) = 1 + t^2 + \cdots + t^{2\hat{m}}
\]

(2.11)

with \( \hat{m} = \frac{m}{t} \). Written in such a form the above sum is just the Poincare' polynomial for the complex projective space \( \mathbb{CP}^{\hat{m}} \). This can be seen by comparing pages 177 and 269 of the book by Bott and Tu, Ref.[50]. Hence, at least for some \( m \)'s, the Poincare' polynomial for the Grassmannian in just the product of the Poincare' polynomials for the complex projective spaces of known dimensionalities. For \( m \) just chosen, in the limit \( t \to 1 \), we reobtain back the number \( \mathcal{N} \) as required. This physically motivating process of gauge fixing we just described.
will be replaced by more rigorous mathematical arguments in the rest of this paper. Rigorous mathematical arguments causing factorization of the Poincare polynomial can be found, for instance, in Ch-3 of lecture notes by Schwartz, Ref.[51]. The relevant physics emerges by noticing that the partition function $Z(J)$ for the particle with spin $J$ is given by, e.g. see Ref.[52],

$$Z(J) = \text{tr}(e^{-\beta H(\sigma)}) = e^{cJ} + e^{c(J-1)} + \cdots + e^{-cJ}$$

$$= e^{cJ}(1 + e^{-c} + e^{-2c} + \cdots + e^{-2cJ}),$$

(2.12)

where $c$ is known constant. Evidently, up to a constant, $Z(J) \simeq S_{J(i)}$. Since mathematically the result, Eq.(2.12), is the Weyl character formula, this fact brings the classical group theory into our discussion. More importantly, because the partition function for the particle with spin $J$ can be written in the language of $N=2$ supersymmetric quantum mechanical model\textsuperscript{17}, as demonstrated by Stone, Ref.[52] and others, Ref.[53], the connection between the supersymmetry and the classical group theory is evident. It was developed to a some extent in Part III.

In view of arguments presented above, the Poincare polynomial for the Grassmannian can be interpreted as a partition function for some kind of a spin chain made of apparently independent spins of various magnitudes\textsuperscript{18}. These qualitative arguments we would like to make more mathematically and physically rigorous. The first step towards this goal is made in the next section.

### 3 Connection with the Polychronakos-Frahm spin chain model

The Polychronakos-Frahm (P-F) spin chain model was originally proposed by Polychronakos and described in detail in Ref.[55]. Frahm [56] motivated by the results of Polychronakos made additional progress in elucidating the spectrum and thermodynamic properties of this model so that it had become known as the P-F model. Subsequently, many other researchers have contributed to our understanding of this exactly integrable spin chain model. Since this paper is not a review, we shall quote only those works on P-F model which are of immediate relevance.

Following Ref.[55], we begin with some description of the P-F model. Let $\sigma^a_i$ ($a = 1, 2, ..., n^2 - 1$) be $SU(n)$ spin operator of $i$-th particle and let the operator $\sigma_{ij}$ be responsible for a spin exchange between particles $i$ and $j$, i.e.

$$\sigma_{ij} = \frac{1}{n} + \sum_a \sigma^a_i \sigma^a_j.$$  

(3.1)

\textsuperscript{17}We hope that no confusion is made about the meaning of $N$ in the present case.

\textsuperscript{18}In such a context it can be vaguely considered as a variation on the theme of the Polyakov rigid string (Grassmann $\sigma$ model, Ref.[54], pages 283-287), except that now it is exactly solvable in the qualitative context just described and, below, in mathematically rigorous context.
In terms of these definitions, the Calogero-type model Hamiltonian can be written as \[57, 58\]
\[ H = \frac{1}{2} \sum_i (p_i^2 + \omega^2 x_i^2) + \sum_{i<j} \frac{l(l - \sigma_{ij})}{(x_i - x_j)^2}, \] (3.2)
where \( l \) is some parameter. The P-F model is obtained from the above model in the limit \( l \to \pm \infty \). Upon proper rescaling of \( H \) in Eq. (3.2), in this limit one obtains:
\[ H_{\text{P-F}} = -\text{sign}(l) \sum_{i<j} \sigma_{ij} \frac{x_i - x_j}{(x_i - x_j)^2}, \] (3.3)
where the coordinate \( x_i \) minimizes the potential for the rescaled Calogero model\(^{19}\), that is
\[ \omega^2 x_i = \sum_{i<j} \frac{2}{(x_i - x_j)^3}. \] (3.4)

It should be noted that \( H_{\text{P-F}} \) is well defined without such a minimization, that is for arbitrary real parameters \( x_i \). This fact will be further explained in Section 4. In the large \( l \) limit the spectrum of \( H \) is decomposable as
\[ E = E_C + lE_{\text{P-F}}, \] (3.5)
where \( E_C \) is the spectrum of spinless Calogero model while \( E_{\text{P-F}} \) is the spectrum of P-F model. In view of such a decomposition, the partition function for the Hamiltonian \( H \) at temperature \( T \) can be written as a product:
\[ Z_H(T) = Z_C(T) Z_{\text{P-F}}(T/l). \] From here, one formally obtains the result:
\[ Z_{\text{P-F}}(T) = \lim_{l \to \infty} \frac{Z_{H(lT)}}{Z_C(T)}. \] (3.6)

It implies that the spectrum of P-F spin chain can be obtained if both the total and the Calogero partition functions can be calculated. In Ref.\(^{55}\) Polychronakos argued that \( Z_C(T) \) is essentially a partition function of \( N \) noninteracting harmonic oscillators. Thus, we obtain:
\[ Z_C(N; T) = \prod_{i=1}^N \frac{1}{1 - q^i}, \quad q = \exp(-\beta), \beta = (k_B T)^{-1}. \] (3.7)

Furthermore, the partition function \( Z_H(T) \) according to Polychronakos can be obtained using \( Z_C(N; T) \) as follows. Consider the grand partition function of the type
\[ \Xi = \sum_{N=0}^{\infty} Z_n(N; T) y^N \equiv \left( \sum_{L=0}^{\infty} Z_C(L; T) y^L \right)^n, \] (3.8)
where \( n \) is the number of flavors\(^{20}\). Using this definition we obtain:
\[ Z_n(N; T) = \sum_{\Sigma, k_i = N} \prod_{i=1}^n Z_C(k_i; T). \] (3.9)

\(^{19}\)The Calogero model is obtainable from the Hamiltonian, Eq. (3.2), if one replaces the spin exchange operator \( \sigma_{ij} \) by 1. Since we are interested in the large \( l \) limit, one can replace the factor \( l(l - 1) \) by \( l^2 \) in the interaction term.

\(^{20}\)That is \( n \) the same number as \( n \) in \( SU(n) \).
Next, Polychronakos identifies $Z_n(N; T)$ with $Z_H(T)$. Then, with help of Eq.(3.6) the partition function $Z_{P-F}(T)$ is obtained straightforwardly as

$$Z_{P-F}(N; T) = \sum_{\sum k_i = N} \prod_{i=1}^{N} \frac{(1 - q^i)^k_i}{\prod_{r=1}^{k_i} (1 - q^r)}. \quad (3.10)$$

Consider this result for a special case $n = 2$. It is convenient to evaluate the ratio first before calculating the sum. Thus, we obtain:

$$\frac{\prod_{i=1}^{N} (1 - q^i)}{\prod_{i=1}^{2k} \prod_{r=1}^{k_i} (1 - q^r)} = \frac{(1 - q) \cdots (1 - q^N)}{(1 - q) \cdots (1 - q^k)(1 - q) \cdots (1 - q^{N-k})} \equiv F(N, k | q), \quad (3.11)$$

where the Poincare' polynomial $F(N, k | q)$ for the Grassmanian of the complex vector space $\mathbb{C}^N$ of dimension $N$ was obtained in the previous section. Indeed, Eq.(3.11) can be trivially brought into the same form as that given in our Eq.(2.9) using the relation $m + k = N$. To bring our Eq.(2.9) in correspondence with Eq.(4.1) of Polychronakos, Ref.[55], we use the second line in Eq.(2.9) in which we make a substitution: $m = N - k$. After this replacement, Eq.(3.10) acquires the form

$$Z_{P-F}^p(N; T) = \sum_{k=0}^{N} \prod_{i=0}^{k} \frac{1 - q^{N-i+1}}{1 - q^i}. \quad (3.12)$$

coinciding with Eq.(4.1) by Polychronakos. This equation corresponds to the ferromagnetic version of the P-F spin chain model. To obtain the antiferromagnetic version of the model requires us only to replace $q$ by $q^{-1}$ in Eq.(3.12) and to multiply the whole r.h.s. by some known power of $q$. Since this factor will not affect thermodynamics, following Frahm, Ref.[56], we shall ignore it. As result, we obtain

$$Z_{P-F}^a(N; T) = \sum_{k=0}^{N} q^{(N/2 - k)^2} \prod_{i=0}^{k} \frac{1 - q^{N-i+1}}{1 - q^i}. \quad (3.13)$$

in accord with Frahm’s Eq.(21). This result is analyzed further in the next section.
4 Connections with WZNW model and XXX s=1/2 Heisenberg antiferromagnetic spin chains

4.1 General remarks

To establish these connections we follow work by Hikami, Ref.[59]. For this purpose, we introduce the notation

\[(q)_n = \prod_{i=1}^{n} (1 - q^i) \quad (4.1)\]

allowing us to rewrite Eq.(3.13) in the equivalent form

\[Z_{af}^{PF}(N; T) = \sum_{k=0}^{N} q^{(N/2-k)^2} \prod_{i=0}^{k} \frac{1 - q^{N-i+1}}{1 - q^i} = \sum_{k=0}^{N} q^{(N/2-k)^2} \frac{(q)_N}{(q)_k (q)_{N-k}}. \quad (4.2)\]

Consider now the limiting case \((N \to \infty)\) of the obtained expression. For this purpose we need to take into account that

\[\lim_{N \to \infty} \left[ \begin{array}{c} N \\ k \end{array} \right]_{q} = \frac{1}{(q)_k}. \quad (4.3)\]

To use this asymptotic result in Eq.(4.2) it is convenient to consider separately the cases of \(N\) being even and odd. For instance, if \(N\) is even, we can write: \(N = 2m\). In such a case we can introduce new summation variables: \(l = k - m\) and/or \(l = m - k\). Then, in the limit \(N \to \infty\) (that is if \(m \to \infty\)) we obtain asymptotically

\[Z_{af}^{PF}(\infty; T) = \frac{1}{(q)_{\infty}} \sum_{i=-\infty}^{\infty} q^{i^2}. \quad (4.4a)\]

in accord with Ref.[59]. Analogously, if \(N = 2m + 1\), we obtain instead

\[Z_{af}^{PF}(\infty; T) = \frac{1}{(q)_{\infty}} \sum_{i=-\infty}^{\infty} q^{(i+\frac{1}{2})^2}. \quad (4.4b)\]

According to Melzer, Ref.[60], and Kedem, McCoy and Melzer, Ref.[61], the obtained partition functions coincide with the Virasoro characters for \(SU_1(2)\) WZNW model describing the conformal limit of the XXX \((s=1/2)\) antiferromagnetic spin chain, e.g. see Ref.[62]. Even though Eqs.(4.4a) and (4.4b) provide the final result, they do not reveal their physical content. This task was accomplished in part in the same papers where connection with the excitation spectrum of the XXX antiferromagnetic chain was made. Hence, at the physical level of rigor the problem of connecting Veneziano amplitudes with physical model can be considered as solved. Nevertheless, below we argue that at the mathematical level of rigor this is not quite so yet. This conclusion concerns not
only problems discussed in this paper but, in general, the connection between the WZNW models, spin chains and K-Z equations.

It is true that K-Z equations and WZNW model are inseparable from each other as explained, for example, in Ref.[62] but the extent to which spin chains can be directly linked to both the WZNW models and K-Z equations still remains to be discussed. For the sake of space, we shall discuss only the most essential facts leaving (with few exceptions) many details and proofs to literature.

Following Varchenko, Ref.[9], we notice that the link between the K-Z equations and WZNW models can be made only with help of the Gaudin model, while the connection with spin chains can be made only by using the quantum version of K-Z equations. Such quantized version of K-Z equations is not immediately connected with the standard WZNW model as discussed in many places, e.g. see Ref.s[9,63]. Therefore, we would like to discuss in some detail the Gaudin model first and only then its relation to P-F spin chain and, accordingly, with the Veneziano model formulated and studied in Part II.

4.2 Gaudin magnets, K-Z equation and P-F spin chains

Although theory of the Gaudin magnets plays an important role in topics such as Langlands correspondence, Hitchin systems, etc., as explained, for instance, in Ref.s[64-66], in this work we do not discuss these topics. Instead, we would like to focus only on issues of immediate relevance to this paper. Gaudin came up with his magnetic chain model in 1976, Ref.[67], being influenced by earlier works of Richardson, Ref.s[68,69] on exact solution of the BCS equations of superconductivity. This connection with superconductivity will play an important role in what follows.

In physics literature all Gaudin-type models are based on $SU(2)$ algebra of spin operators\textsuperscript{21}. Instead of one Hamiltonian, the set of commuting Hamiltonians of the type

$$H_i = \sum_{j(\neq i)=1}^{N} \sum_{\alpha=1}^{3} w_{ij}^{\alpha} \sigma_{i}^{\alpha} \sigma_{j}^{\alpha}$$

is used as discussed in Ref.[71]. In view of the fact that, by construction, $[H_i, H_j] = 0$, and $3N(N - 1)$, the coefficients $w_{ij}^{\alpha}$ should satisfy the following equations

$$w_{ij}^{\alpha} w_{jk}^{\gamma} + w_{ji}^{\beta} w_{ik}^{\gamma} - w_{ik}^{\delta} w_{jk}^{\delta} = 0.$$  

These equations can be solved by imposing the antisymmetry requirement: $w_{ij}^{\alpha} = -w_{ji}^{\alpha}$. It can be satisfied by replacing $w_{ij}^{\alpha}$ by the unknown functions $w_{ij}^{\alpha} = f_{ij}^{\alpha}(z_i - z_j)$ of difference between two new real parameters $z_i$ and $z_j$. It is only natural to make further restrictions based on requirement that the $z$-component of the total spin $S^z = \sum_{i} \sigma_{i}^{z}$ is conserved. This causes us to write

\textsuperscript{21}In mathematics literature to be used below [9,63] the $SL(2, C)$ group is used instead of its subgroup, $SU(2)$ [70].
\[ w_{ij}^1 = w_{ij}^2 \equiv X_{ij}, \text{ and } w_{ij}^3 = Y_{ij} \text{ thus leading to equations} \]
\[ Y_{ij}X_{jk} + Y_{ki}X_{jk} + X_{ki}X_{ij} = 0. \] (4.7)

These constraint equations admit the following sets of solutions:

\[ X_{ij} = \frac{1}{z_i - z_j} \text{ (rational),} \] (4.8a)
\[ X_{ij} = \frac{1}{\sin (z_i - z_j)} \text{, } Y_{ij} = \cos (z_i - z_j) \text{ (trigonometric),} \] (4.8b)
\[ X_{ij} = \frac{1}{\sinh (z_i - z_j)} \text{, } Y_{ij} = \cosh (z_i - z_j) \text{ (hyperbolic).} \] (4.8c)

While the first solution, Eq.(4.8a), to be used in this work, corresponds to the long range analog of the standard XXX spin chain, the remaining two solutions correspond to the long range analogs of the XXZ spin chain.

Following Varchenko, Ref.[9], we are now in the position to write down the K-Z equations. For this purpose we combine Eqs (4.5) and (4.8a) and reintroduce the coupling constant \( g \) (so that \( w_{ij}^\alpha \rightarrow gw_{ij}^\alpha \)) in such a way that the set of K-Z equations acquires the form

\[ (\kappa \frac{\partial}{\partial z_i} - H_i(z_1, \ldots, z_N))\Phi(z_1, \ldots, z_N) = 0, \quad i = 1, \ldots, N \] (4.9)

where \( \kappa = g^{-1} \). This result requires several comments. First, from theory of WZNW models it is known that parameter \( \kappa \) cannot take arbitrary values. For instance, for \( SU_1(2) \) WZNW model \( \kappa = \frac{3}{2} \), e.g. read Ref.[62]. Second, we can always rescale \( z \)-coordinates and to redefine the Hamiltonian to make the constant arbitrary small. Apparently, this is assumed in the asymptotic analysis of K-Z equations described in Ref.s[8,9]. Third, if this is the case, then such analysis (to be used below) differs essentially from other approaches connecting string models with spin chains discussed in the Introduction since such a connection was typically made in the limit \( N \rightarrow \infty \). Since for \( SU(N) \)-type models \( \kappa = \frac{1}{2}(k+N) \), in the limit \( N \rightarrow \infty \) we have \( \kappa \rightarrow \infty \). The WKB-type analysis of K-Z equations of Reshetikhin and Varchenko (to be discussed below) fails exactly in this limit.

With set of K-Z equations defined, we would like now to make a connection between the Gaudin and P-F model. To a large extent this was already accomplished in Ref.[72]. Following this reference, we define the spin Calogero (S-C) model as follows

\[ H_{S-C} = \frac{1}{2} \sum_i (p_i^2 + \omega^2 x_i^2) + g \sum_{i<j} \frac{\hat{\sigma}_i \cdot \hat{\sigma}_j}{(z_i - z_j)^2} \] (4.10)

to be compared with \( \mathcal{H} \) in Eq.(3.2).\(^{22}\) Using the rational form of the Gaudin

\(^{22}\)We added the oscillator-type potential absent in the original work, Ref.[72], for the sake of additional comparisons, e.g. with Eq.(3.4). In what follows such a constraint is not essential and will be ignored.
Hamiltonian this result can be equivalently rewritten as

$$H_{S-C} = \frac{1}{2} \sum_l \left( p_l^2 + \omega^2 x_l^2 + i \frac{g}{2} [p_l, H] \right).$$

(4.11)

That this is indeed the case can be seen by the following chain of arguments.

Consider the strong coupling limit \((g \to \infty)\) of \(H_{S-C}\) so that the kinetic term is a perturbation. Next, we consider the eigenvalue problem for one of the Gaudin’s Hamiltonians, i.e.

$$H_l \Psi(l) = E(l) \Psi(l),$$

(4.12)

and apply the operator \(ip_l\) to both sides of this equation. Furthermore, consider in this limit the combination \(H_{S-C} \Psi(l)\). Provided that the eigenvalue problem, Eq.(4.12), does have a solution, it is always possible to Fourier expand \((ip_l \Psi(l))\) using as basis set \(\Psi(l)\). In such a case we end up with the eigenvalue problem for the P-F spin chain in which the eigenfunctions are the same as those for the Gaudin’s model and the eigenvalues are \(ip_l E(l)\). Physical significance of this result will be discussed in detail below. Before doing so, we have to make a connection between the K-Z, Eq.(4.9), and the Gaudin eigenvalue, Eq.(4.12), problems.

Following Refs.[8,9], we begin by replacing \(SU(2)\) spin operators by the \(SL(2, \mathbb{C}) \equiv sl_2\) operators \(e, f\) and \(h\) obeying commutation relations

$$[h, e] = 2e; \quad [e, f] = h; \quad [h, f] = -2f.$$  

(4.13)

This Lie algebra was discussed extensively in Part II in connection with design of new models reproducing Veneziano amplitudes. In this work, we shall extend already obtained results following ideas of Richardson and Varchenko.

From Ref.[70] it is known that \(SU(2)\) is just a subgroup of \(sl_2\). Introduce the Casimir element \(\Omega \in sl_2 \otimes sl_2\) via

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$$

(4.14)

so that \(\forall x \in sl_2\) it satisfies the commutation relation \([x \otimes 1 + 1 \otimes x, \Omega] = 0\) inside the \(U(sl_2) \otimes U(sl_2)\), where \(U(sl_2)\) is the universal enveloping algebra of \(sl_2\). Consider the vector space \(V = V_1 \otimes V_2 \otimes \cdots \otimes V_N\). An element \(x \in sl_2\) acts on \(V\) as follows: \(x \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes x\). For indices \(1 \leq i < j \leq N\) let \(\Omega^{(i,j)} : V \to V\) be an operator which acts as \(\Omega\) on \(i\)-th and \(j\)-th positions and as identity on all others, then the set of K-Z equations can be written as

$$\kappa \frac{\partial}{\partial z_i} \Phi = \sum_{j \neq i} \frac{\Omega^{(i,j)}}{z_i - z_j} \Phi, \quad i = 1, ..., N.$$  

(4.15)

In the simplest case, this set of equations is defined in the domain \(U = \{(z_1, ..., z_N) \in \mathbb{C}^N \mid z_i \neq z_j\}\).

From now on we shall use Eq.s(4.15) instead of Eq.s(4.9). To connect K-Z equations with the XXX Gaudin magnet we shall use the WKB method.
developed by Reshetikhin and Varchenko, Ref.[8], and summarized in lecture notes by Varchenko, Ref.[9]. Following these authors, we shall look for a solution of Eq.(4.15) in the form \((\kappa \to 0)\) 

\[
\Phi(z, \kappa) = e^{\pm S(z)} \{ f_0(z) + \kappa f_1(z) + \cdots \},
\]

(4.16)

where \(z = \{z_1, \ldots, z_N\}\), \(S(z)\) is some scalar function (to be described below) and \(f_j(z), j = 0, 1, 2, \ldots\), are \(V\)-valued functions. Provided that the function \(S\) is known, \(V\)-valued functions can be recursively determined (as it is done in the WKB analysis). Specifically, given that \(H_i = \sum_{j \neq i} \Omega^{(i,j)}_{z_i - z_j}\), we obtain:

\[
H_i f_0(z) = \frac{\partial S}{\partial z_i} f_0(z),
\]

(4.17)

to be compared with Eq.(4.12). Next, we get

\[
H_i f_1(z) = \frac{\partial S}{\partial z_i} f_1(z) + \frac{\partial f_0}{\partial z_i},
\]

(4.18)

and so on. Since the function \(S(z)\) (the Shapovalov form) plays a significant role in these calculations, we would like to discuss it in some detail now.

### 4.3 The Shapovalov form

Consider the following auxiliary problem. Let \(A(x)\) and \(B(x)\) be some preassigned polynomials of degree \(n\) and \(n - 1\) respectively. Find a polynomial \(C(x)\) of degree \(n - 2\) such that the differential equation

\[
A(x) y''(x) - B(x) y'(x) + C(x) y(x) = 0
\]

(4.19)

has solution which is polynomial of preassigned degree \(k\). Such polynomial solution is called the Lame’ function. Stieltjes, Ref.s[8,9], proved the following

**Theorem 4.1.** Let \(A\) and \(B\) be given polynomials of degree \(n\) and \(n - 1\), respectively so that \(B(x)/A(x) = \sum_{j=1}^{n} \frac{m_j}{x - x_j}\). Then there is a polynomial \(C\) of degree \(n - 2\) and a polynomial solution \(y(x) = \prod_{i=1}^{k} (x - x_i)\) of Eq.(4.19) if and only if \(\mathbf{x} = (x_1, \ldots, x_k)\) is the critical point of the function

\[
\Phi_{k,n}(x_1, \ldots, x_k; z_1, \ldots, z_n) = \prod_{j=1}^{k} \prod_{i=1}^{n} (x_j - z_i)^{-m_i} \prod_{1 \leq i < j \leq k} (x_i - x_j)^2.
\]

(4.20)

**Definition 4.2.** A point \(\mathbf{x}\) is critical for \(\Phi(x)\) if all its first derivatives vanish at it.
We would like now to make a connection between the Shapovalov form $S$ and results just obtained. $S$ is symmetric bilinear form on previously introduced space $V$ such that $S(v, v) = 1$, $S(hx, y) = S(x, hy)$, $S(ex, y) = S(x, fy)$, where $h, e, f$ are defined in Eq.(4.13). Furthermore, $S(\Omega(x_1 \otimes x_2), y_1 \otimes y_2) = S(x_1 \otimes x_2, \Omega(y_1 \otimes y_2)) \forall x_1, y_1 \in V_1$ and $\forall x_2, y_2 \in V_2$. As result, we obtain:

$$S(H, x, y) = S(x, H, y) \forall x, y \in V. \quad (4.21)$$

Next, let $m$ be some nonnegative integer and $V_m$ be the irreducible Verma module with the highest weight $m$ and the highest weight singular vector $v_m$, i.e.

$$hv_m = m v_m, \quad ev_m = 0. \quad (4.22)$$

Consider a tensor product $V \equiv V \otimes^M = V_{m_1} \otimes \cdots \otimes V_{m_n}$ so that $M = (m_1, \ldots, m_n)$. $V_m$ vectors $v_m$, $f v_m$, $f^2 v_m$, $\cdots$, $f^m v_m$ form a basis of $V_m$. \(23\) so that the Shapovalov form is orthogonal with respect to such a basis and is decomposable as $S = S_{m_1} \otimes \cdots \otimes S_{m_n}$. Let, furthermore, $J = (j_1, \ldots, j_n)$ be a set of nonnegative integers such that $j_1 + \cdots + j_n = k$, where $k$ is the same as in Eq.(4.20), and $0 \leq j_i \leq m_i$. This allows us to define the set of vectors $f^J v_M = f^{j_1} v_{m_1} \otimes \cdots \otimes f^{j_n} v_{m_n}$. These vectors $\{ f^J v_M \}$ are by construction orthogonal with respect to the Shapovalov form and provide a basis for the space $V \otimes^M$. Introduce the weight
d of a partition $A$ as $|A| = a_1 + a_2 + \ldots$ then, in view of Eq.(4.22), we define the singular vector $f^J v_M$ via

$$h(f^J v_M) = (|M| - 2 |J|) f^J v_M, \quad e(f^J v_M) = 0 \quad (4.23)$$

doing $|M| - 2 |J|$. \(24\) The Bethe ansatz vectors $V$ for the Gaudin model can be defined now as

$$V(\tilde{x}, z) = \sum_J A_J(\tilde{x}, z) f^J v_M, \quad (4.24)$$

where $\tilde{x}$ is a critical point of $\Phi(x, z)$ defined by Eq.(4.20) while the function $A_J(\tilde{x}, t)$ is defined as follows

$$A_J(\tilde{x}, t) = \sum_{\sigma \in \mathcal{P}(k; J)} \prod_{i=1}^{k} \frac{1}{x_i^j z_{\sigma(i)}} \quad (4.25)$$

with $\mathcal{P}(k; J)$ being the set of maps $\sigma$ from the $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$. Finally, using these definitions it is possible to prove that

$$S(V(\tilde{x}, z), V(\tilde{x}, z)) = \det_{1 \leq i, j \leq k} \frac{\partial^2}{\partial x_i \partial x_j} \ln \Phi_{k, n}(\tilde{x}_1, \ldots, \tilde{x}_k, z_1, \ldots, z_n). \quad (4.26)$$

\(23\) According to Ref.[9] in all subsequent calculations it is sufficient to use the finite Verma module, i.e. $L_m = V_m / < f^{m+1} v_m >$. This restriction is in accord with our previous calculations, e.g. see Part II, Section 8, where such a restriction originates from the Lefschetz isomorphism theorem used in conjunction with supersymmetric model reproducing Veneziano amplitudes.

\(24\) This fact can be easily understood from the properties of $sl_2$ Lie algebra representations since it is known, e.g. see Ref.[9] and Part II, that for the module of highest weight $m$ we have $h(f^k v_m) = (m - 2k)(f^k v_m)$. 
The set of equations determining critical points

\[
\frac{1}{\Phi_{k,n}(x^0, z^0)} \frac{\partial}{\partial x_i} \Phi_{k,n}(x(z), z) \mid_{z=z^0} = 0
\]  

(4.27)

are the Bethe ansatz equations for the Gaudin model. Using these equations the eigenvalue, Eq.(4.17), for the Gaudin model now acquires the following form:

\[
H_i(z^0) \ln(\Phi_{k,n}(x(z), z) \mid_{z=z^0}) = \frac{\partial}{\partial z_i} \ln(\Phi_{k,n}(x(z), z) \mid_{z=z^0}) V(x^0, z^0).
\]

(4.28)

In the next subsection we shall study in some detail the Bethe ansatz Eq.s(4.28). This will allow us to obtain eigenvalues in Eq.(4.28) explicitly.

4.4 Mathematics and physics of Bethe ansatz equations for XXX Gaudin model according to Richardson. Connection with Veneziano model

Using Eq.(4.20) in (4.28) produces the following set of the Bethe ansatz equations:

\[
\sum_{i=1}^{n} \frac{m_i}{x_j - z_i} = \sum_{i=1}^{k} \frac{2}{x_j - x_i}, \quad j = 1, ..., k.
\]

(4.29)

To understand physical meaning of these equations we shall use extensively results of two key papers by Richardson, Ref.s [68,69]. To avoid duplications, and for the sake of space, our readers are encouraged to read thoroughly these papers. Although originally they were written in 60ies having applications to nuclear physics in mind, they are no less significant for condensed matter, Ref.[71], and atomic physics, Ref.[73]. Because of this, only nuclear physics terminology will be occasionally used. At the time of writing of these papers, QCD was still in its infancy. Accordingly, no attempts were made to apply Richardson’s results to QCD. Recently, Ovchinnikov, Ref.[74], conjectured that the Richardson-Gaudin equations can be useful for development of color superconductivity in QCD. A comprehensive review of this topic is given in Ref.[75]. Incidentally, in the same paper, Ref.[75], it is emphasized that such type of superconductivity can exist only if the number of colors is not too large, e.g. \(N_c = 3\). This fact is in accord with the remarks made in Section 4.2. regarding the validity of WKB methods for K-Z equation in the limit \(N \to \infty\). These results clearly favour Abelian reduction over ADS/CFT.

Below, we provide additional mathematically rigorous evidence supporting ‘t Hooft’s idea of Abelian reduction of QCD. These results should be considered as complementary to that presented in Faddeev’s paper, Ref.[27]. For this purpose, following Richardson, Ref.[69], we consider the system of interacting bosons
described by the (pairing) Hamiltonian
\begin{equation}
H = \sum_l \varepsilon_l \hat{n}_l - \frac{g}{2} \sum_{ll'} A_l^+ A_{l'}.
\end{equation}

Here we have \( \hat{n}_l = \sum_{k(\varepsilon_k = \varepsilon_l)} a_k^+ a_k \), \( A_l^+ = \sum_{k(\varepsilon_k = \varepsilon_l)} a_k^+ a_{-k} \) and \( A_l = \sum_{k(\varepsilon_k = \varepsilon_l)} a_{-k} a_k \).

It is assumed that the single-particle spectrum \( \{ \varepsilon_l \} \) is such that \( \varepsilon_l < \varepsilon_{l+1} \) \( \forall l \) and that the degeneracy of \( l \)-th level is \( \Omega_l \) so that the sums (over \( k \) ) each contain \( \Omega_l \) terms. It is assumed furthermore that the system possesses the time-reversal symmetry implying \( \varepsilon_k = \varepsilon_{-k} \). The operators \( a_k^+ \) and \( a_k \) obey usual commutation rules for bosons, i.e. \( [a_k^+, a_{k'}] = \delta_{kk'} \). The sign of the coupling constant, in principle, can be both positive and negative. We shall work, however, with more physically interesting case of negative coupling (so that \( g \) in Eq.(4.30) is actually \( |g| \)).

An easy computation using commutation rule for bosons produces the following results
\begin{align}
[\hat{n}_l, A_{l'}^+] &= 2 \delta_{ll'} A_{l'}^+, \\
[A_l, A_{l'}^+] &= 2 \delta_{ll'} (\Omega_l + 2 \hat{n}_l), \\
[\hat{n}_l, A_{l'}] &= -2 \delta_{ll'} A_{l'}.
\end{align}

If we make a replacement of \( \hat{n}_l \) in Eq.s(4.31a) and (4.31c) by \( \frac{\Omega_l}{4} + \hat{n}_l \equiv \frac{\hat{n}_l}{4} \) and keep the same notation in the r.h.s. of Eq.(4.31b) we shall arrive at the \( sl_2 \) Lie algebra isomorphic to that given in Eq.(4.13). The same Lie algebra was uncovered and used in our Part II for description of new models describing Veneziano amplitudes. Because of this, we would like now to demonstrate that the rest of arguments of Part II can be implemented in the present context thus making the P-F model (which is derivative of the Richardson-Gaudin XXX model) correct model related to Veneziano amplitudes.

Following Richardson, Ref.[69], we notice that the model described by the Hamiltonian, Eq.(4.30), and algebra, Eq.s(4.31), admits two types of excitations: those which are associated with the unpaired particles and those with coupled pairs. The unpaired \( \nu \)-particle state is defined by the following two equations
\begin{align}
\hat{n} \mid \varphi_\nu \rangle &= \nu \mid \varphi_\nu \rangle, \\
A_l \mid \varphi_\nu \rangle &= 0 \forall l.
\end{align}

Here, \( \hat{n} = \sum_l \hat{n}_l \) so that, in fact,
\begin{equation}
\hat{n}_l \mid \varphi_\nu \rangle = \nu_l \mid \varphi_\nu \rangle
\end{equation}
and, therefore, \( \nu = \sum_l \nu_l \). Furthermore,
\begin{equation}
H \mid \varphi_\nu \rangle = \sum_l \varepsilon_l \nu_l \mid \varphi_\nu \rangle.
\end{equation}

\(^{25}\)In the paper with Sherman, Ref.[68], Richardson explains in detail how one can map the fermionic (pairing) system into bosonic.
Following Richardson, we want to demonstrate that parameters \( \varepsilon_l \) in Eq.(4.35) can be identified with parameters \( z_l \) in Bethe Eqs (4.29). Because of this, the eigenvalues for the P-F chain are obtained as described in Section 4.2., that is

\[
E_{l}^{(P-F)} = \frac{\partial}{\partial \varepsilon_{l}} \sum_{l} \varepsilon_{l} u_{l} = \nu_{l}.
\] (4.36)

These are the eigenvalues of \( \hat{n}_l \) defined in Eq.(4.34). Furthermore, this eigenvalue equation is exactly the same as was used in Part II, Section 8, with purpose of reproducing Veneziano amplitudes. Moreover, Eq.s(4.32 ) and (4.33) have the same mathematical meaning as Eq.s (4.23) defining the Verma module. Because of this, we follow Richardson’s paper to describe this module in physical terms. By doing so additional comparisons will be made between the results of Part II and works of Richardson. Since the Hamiltonian, Eq.(4.30), describes two kinds of particles: a) pairs of particles (whose total linear and angular momentum is zero) and, b) unpaired particles (that is single particles which do not interact with just described pairs), the total number of (quasi) particles is 

\[ n = N + \nu. \]

Since we redefined the number operator as 

\[ \Omega_l + \hat{n}_l \equiv \hat{N}_l, \]

we expect that , once the correct state vector describing excitations is found, Eq.(4.30) should be replaced by the analogous equation for \( \hat{N}_l \) whose eigenvalues will be 

\[ \frac{\Omega_l}{2} + \nu.\]

A simple minded way of creating such a state is by constructing the following state vector

\[ A_l^+ \cdots A_N^+ | \varphi_\nu >. \]

This vector does not possess the needed symmetry of the problem. To create the state vector (actually, the Bethe vector of the type given by Eq.(4.24)) of correct symmetry one should introduce a linear combination of \( A_l^+ \) operators according to the following prescription:

\[
B^+_\alpha = \sum_l u_{\alpha}(l) A^+_l, \quad \alpha = 1, \ldots, N
\] (4.37)

with constants \( u_{\alpha}(l) \) to be determined below. The (unnormalized) Bethe-type vectors are given then as \( | \psi > = B^+_1 \cdots B^+_N | \varphi_\nu > \) and, accordingly, instead of Eq.(4.35), we obtain

\[
H | \psi > = (\sum_l \varepsilon_l u_{l}) | \psi > + [H, B^+_1 \cdots B^+_N] | \varphi_\nu >. \] (4.38)

The task now lies in calculating the commutator and to determine the constants \( u_{\alpha}(l) \). Details can be found in Richardson’s paper, Ref.[69]. The final result looks as follows

\[
H | \psi > = -E | \psi >
\]

\[
= \sum_{\alpha=1}^{N} (\prod_{\gamma \neq \alpha} B^+_\gamma) \sum_l A^+_\alpha (2\varepsilon_l - E_{\alpha}) u_{\alpha}(l) + \sum_{l'} (\Omega_{l'} + 2\hat{n}_{l'}) u_{\alpha}(l') + 4g \sum_{\beta (\beta \neq \alpha)} M_{\beta \alpha} | \varphi_\nu >.
\]

\[ \text{In Richardson’s paper we find instead: } n = 2N + \nu. \text{ This is, most likely, a misprint as explained in the text.} \]

\[ \text{These amendments are not present in Richardson’s paper but they are in accord with its content.} \]
By requiring the r.h.s. of this equation to be zero, we arrive at the eigenvalue equation

\[ H \mid \psi > = E \mid \psi > , \]

where \( E = \sum_l \varepsilon_l \nu_l + \sum_{\alpha=1}^{N} E_{\alpha} \). \hspace{1cm} (4.40)

Furthermore, this requirement after several manipulations leads us to the Bethe ansatz equations

\[ \frac{1}{2g} + \sum_{\beta(\beta \neq \alpha)}^{N} \frac{2}{E_{\beta} - E_{\alpha}} - \sum_{l=1}^{L} \frac{\Omega_l/2 + \nu_l}{2\varepsilon_l - E_{\alpha}} = 0, \alpha = 1, ..., N, \hspace{1cm} (4.41a) \]

as well as to the explicit form of coefficients \( u_{\alpha}(l) : u_{\alpha}(l) = 1/(2\varepsilon_l - E_{\alpha}) \) and that for the matrix elements \( M_{\alpha, \beta} \) (since, by construction, \( u_{\alpha}(l)u_{\beta}(l) = M_{\alpha, \beta}u_{\alpha}(l) + M_{\beta, \alpha}u_{\beta}(l) \)). In the limit \( g \to 0 \) we expect \( E_{\alpha} \to 2\varepsilon_l \) and \( \Omega_l \to 0 \) in accord with Eq.s(4.32)-(4.34). Therefore, we conclude that \( \Omega_l^2 + \nu_l^2 \) is an eigenvalue of the operator \( \hat{N}_l \) acting on \( \mid \psi > \) in accord with remarks made before. In the opposite limit: \( g \to \infty \), the system of Eq.s(4.41a) will coincide with Eq.(4.29) upon obvious identifications: \( x_{\alpha} \to E_{\alpha}, 2z_l \to z_l, N \to k, L \to n \) and \( \Omega_l^2/2 + \nu_l \to m_l \).

Next, in view of Eq.s(4.36) and (4.40) we obtain the following result for the occupation numbers:

\[ \tilde{\Omega}_i \equiv E_{i}^{(P-F)} = \frac{\partial}{\partial \varepsilon_i} \left[ \sum_l \varepsilon_l \nu_l + \sum_{\alpha=1}^{N} E_{\alpha} \right] = \nu_i + \sum_{\alpha=1}^{N} \frac{\partial E_{\alpha}}{\partial \varepsilon_i}. \hspace{1cm} (4.42) \]

Based on the results just obtained, it should be clear that, actually, \( E_{i}^{(P-F)} = \nu_i + \Omega_i^2 \) so that \( \frac{\Omega_i^2}{2} = \sum_{\alpha=1}^{N} \frac{\partial E_{\alpha}}{\partial \varepsilon_i} \). Richardson, Ref.[69], cleverly demonstrated that the combination \( \sum_{\alpha=1}^{N} \frac{\partial E_{\alpha}}{\partial \varepsilon_i} \) must be an integer.

Consider now a special case: \( N = 1 \). Evidently, for this case, the derivative \( \frac{\partial E_{\alpha}}{\partial \varepsilon_i} \) should also be an integer. For different \( \varepsilon_i \)'s these may, in general, be different integers. This fact has some physical significance to be explained below.

To simplify matters, by analogy with theory of superconducting grains, Ref.[71], we assume that the energy \( \varepsilon_i \) can be written as \( \varepsilon_i = d(2i - L - 1) \), \( i = 1, 2, ..., L \). The adjustable parameter \( d \) measures the level spacing for the unpaired particles in the limit \( g \to 0 \). With such simplification, we obtain the following BCS-type equation using Eq.s(4.41a) (for \( N = 1 \)):

\[ \sum_{l=1}^{L} \frac{\tilde{\Omega}_l}{2\varepsilon_l - E} = \frac{1}{G}. \hspace{1cm} (4.43) \]

\[ ^{28}\text{It should be noted that in the original paper, Ref. [69], the sign in front of the 3rd term in the l.h.s. is positive. This is because Richardson treats both positive and negative couplings simultaneously. Equation (4.41a) is in agreement with (3.24) of Richardson-Sherman paper, Ref.[68], where the case of negative coupling (pairing) is treated.} \]
where $G$ is the rescaled coupling constant. Such an equation was discussed in the seminal paper by Cooper, Ref.[76], which paved a way to the BCS theory of superconductivity. To solve this equation, let now $F(E) = \sum_{l=1}^{L} \tilde{\Omega}_l (2\varepsilon_l - E)^{-1}$ so that Eq.(4.43) is reduced to

$$F(E) = G^{-1}.$$  \hfill (4.44)

This equation can be solved graphically as depicted below, in Fig.1.

As can be seen from Fig.1, solutions to this equation for $G = \infty$ can be read off from the $x$ axis. In addition, if needed, for any $N \geq 1$ the system of Eq.s (4.41a) can be rewritten in a similar BCS-like form if we introduce the renormalized coupling constant $G_\alpha$ via

$$G_\alpha = G[1 + 2G \sum_{\beta(\beta \neq \alpha)}^{N} \frac{1}{E_\beta - E_\alpha}]^{-1}$$ \hfill (4.45)

so that now we obtain:

$$F(E_\alpha) = G_\alpha^{-1}, \alpha = 1, \ldots, N.$$ \hfill (4.41b)

This system of equations can be solved iteratively, beginning with Eq.(4.44). There is, however, better way of obtaining these solutions. In view of Eq.s(4.19), (4.20) and (4.27) solutions $\{E_\alpha\}$ of Eq.(4.41.b) are the roots of the Lame’—type function which is obtained as solution of Eq.(4.19). Surprisingly, this fact known to mathematicians for a long time has been recognized in nuclear physics literature only very recently, e.g. read Ref.[77].
4.5 Emergence of the Veneziano-like amplitudes as consistency condition for \( N = 1 \) solutions of K-Z equations. Recovery of pion-pion scattering amplitude

Since results for the Richardson-Gaudin (R-G) model are obtainable from the corresponding solutions of K-Z equations, in this subsection we would like to explain why \( N = 1 \) solution of the Bethe-Richardon equations can be linked with the Veneziano-like amplitudes describing the pion-pion scattering. In doing so, we shall by pass the P-F model since, anyway, it is obtainable from the R-G model.

Thus, we begin again with Eq.s (4.14),(4.15). We would like to look at special class of solutions of Eq.(4.15) for which the parameter \( |J| \) in the Verma module, Eq.(4.23), is equal to one. This corresponds exactly to the case \( N = 1 \).

Following Varchenko, Ref.[9], by analogy with Eq.(4.20) we introduce function \( \Phi(z,t) \) via

\[
\Phi(z,t) = \prod_{1 \leq i < j \leq L} (z_i - z_j) \frac{m_im_j}{k} \prod_{l=1}^L (t - z_l) - \frac{m_l}{k}.
\]

(4.46)

It is a multivalued function at the points of its singularities, i.e. at the points \( z_1, ..., z_L \). Using this function, we define the set of 1-forms via

\[
\omega_j = \Phi(z,t) \frac{dt}{t - z_j}, \quad j = 1, ..., L,
\]

(4.47)

and the vector \( \mathbf{I}(\gamma) \) of integrals \( \mathbf{I}(\gamma) = (I_1, ..., I_L) \equiv (\int_\gamma \omega_1, ..., \int_\gamma \omega_L) \) with \( \gamma \) being a particular Pochhammer countour: a double loop winding around any two points \( z_\alpha, z_\beta \) taken from the set \( z_1, ..., z_L \). Details can be found in Ref.s[9,63].

We want now to design the singular Verma module for the K-Z equations using Eq.(4.23) and results just presented. Taking into account the following known relations:

a) \( ef^k v_m = k(m - k + 1) f^{k-1} v_m \), and b) \( h f^k v_m = (m - 2k) f^{k} v_m \)

for the Lie algebra \( sl_2 \), also used in Part II, Section 8, and taking into account that in the present \( (N = 1) \) case the basis vectors \( f^j v_M = f^{j_1} v_{m_1} \otimes \cdots \otimes f^{j_n} v_{m_n} \) acquires the form: \( f^1 v_M = v_{m_1} \otimes \cdots \otimes f v_{m_s} \otimes \cdots \otimes v_{m_n}, s = 1, ..., L \), provided that \( m'_i s \) are the same as in Eq.(4.29) (or (4.46)), the singular vector for such a Verma module is given by

\[
w(\gamma) = \sum_{s=1}^L I_s v_{m_1} \otimes \cdots \otimes f v_{m_s} \otimes \cdots \otimes v_{m_n}.
\]

(4.48)

In view of the Lie algebra relations just introduced, we obtain \( e \cdot w = 0 \) or, explicitly,

\[
\sum_{s=1}^L m_s I_s = 0.
\]

(4.49)
Hence, for a fixed Pochhammer contour $\gamma$ there are $L-1$ independent basis vectors $\{w^i\}$. They represent $L-1$ independent solutions of K-Z equation of the type $k=1$ (or $N=1$). Let now $z_i's$ be ordered in such a way that $z_1 < \cdots < z_L$. Furthermore, in view their physical interpretation described in previous section, these $z_i's$ can be chosen to be equidistant. Consider then a special set of Pochhammer contours $\{\gamma_i\}$ around points $z_i$ and $z_{i+1}$, $i = 1, 2, ..., L-1$, and consider the matrix $M$ made of integrals of the type $M_{ij} = -\frac{m_j}{\kappa} \int_{\gamma_i} \omega_j$ then, any $(k=1)-$ type solution $\phi^i(i = 1, 2, ..., L-1)$ of K-Z equation can be represented as

$$\phi^i = \sum_j M_{ij} w^j, \ i = 1, 2, ..., L-1. \quad (4.50)$$

From linear algebra it is known that in order for these K-Z solutions to be independent we have to require that $\det M \neq 0$. The proof of this fact is given in the Appendix. Calculation of the determinant of $M$ is described in detail in Ref.[9] so that we quote the result:

$$\det M = \pm A \frac{\Gamma(1 - \frac{m_1}{\kappa}) \cdots \Gamma(1 - \frac{m_L}{\kappa})}{\Gamma(1 - \frac{M}{\kappa})}, \quad (4.51)$$

where $\pm A$ is some known constant$^{29}$ and $\Gamma(x)$ is Euler’s gamma function. For $L=2$ without loss of generality one can choose $z_1 = 0$ and $z_2 = 1$, then in the determinant thus obtained one easily can recognize the Veneziano-type $\pi^+ \pi^-$ scattering amplitude used in the work by Lovelace, Ref.[78]. We discussed this amplitude previously in connection with mirror symmetry issues in our work, Ref.[79]. This time, however, we would like to discuss other topics.

In particular, we notice first that all mesons are made of two quarks. Specifically, we have: $ud$ for $\pi^+$, $d\bar{u}$ for $\pi^-$ and $\bar{d}d$ for $\pi^0$. These are very much like the Cooper pairs with $q\bar{q}$ quark pairs contributing to the Bose condensate created as result of spontaneous chiral symmetry breaking. As in the case of more familiar Bose condensate, in addition to the ground state we expect to have a tower of excited states made of such quark pairs. Experimentally, these are interpreted as more massive mesons. Such excitations are ordered by their energies, angular momentum and, perhaps, by other quantum numbers which can be taken into account if needed. Color confinement postulate makes such a tower infinite. Evidently, the Richardson-Gaudin model fits ideally this qualitative picture. Eq.(4.44) describes excitations of such Cooper-like pairs (even in the limit: $G \to \infty$) as can be seen from Fig.1. In the P-F model the factor $\Omega_i$ plays effectively the role of energy as already discussed in this work and in Part II. Therefore, in view of Eq.(4.42), it is appropriate to write: $\Omega_i = f(E_i)$, with $E_i$ being the R-G energies. Although the explicit form of such $f$-dependence may be difficult to obtain, for our purposes it is sufficient only to know that such a

$^{29} \pm A = \prod_{1 \leq i,j \leq L, (i \neq j)} \frac{-(z_i - z_j)}{\kappa}$
dependence does exist. This then allows us to make an identification: \( \tilde{\Omega}_i \equiv \frac{m_i}{\kappa} \) consistent with Varchenko’s results, e.g. compare his Theorem 3.3.5 (page 35) with Theorem 6.3.2. (page 90) of Ref.[9]. But, we already established that \( \tilde{\Omega}_i \) is an integer, therefore, \( \frac{m_i}{\kappa} \) should be also an integer. This creates some apparent problems. For instance, when \( |M| = \kappa \), the determinant, \( \det M \), becomes zero implying that solutions of K-Z equation become interdependent. This fact has physical significance to be discussed below and in Section 5. To do so we use some results from our Part I. In particular, a comparison between

\[
\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \left(\frac{k}{z}\right)\right)(1 + \left(\frac{k}{z}\right)) \tag{4.52}
\]

and

\[
\frac{1}{\Gamma(z)} = ze^{-Cz} \prod_{k=1}^{\infty} \left(1 + \left(\frac{k}{z}\right)\right)e^{-\frac{z}{k}}, \tag{4.53}
\]

where \( C \) is some known constant, tells us immediately that not only \( |M| = \kappa \) will cause \( \det M = 0 \) but also \( |M| = \kappa(k + 1), k = 0, 1, 2, \ldots \). Accordingly, the numerator in Eq.(4.51) will create poles whenever \( \frac{m_i}{\kappa} = 1 \). Existence of independent K-Z solutions is not destroyed if, indeed, such poles do occur. These facts allow us to relabel \( \frac{m_i}{\kappa} \) as \( \alpha(s) \) or \( \alpha(t) \) or \( \alpha(u) \), etc.) as it is done in high energy physics with continuous parameters \( s, t, u, \ldots \) replacing discrete \( i's \), different for different \( \Gamma \) functions in the numerator of Eq.(4.51). In the simplest case, this allows us to reduce the determinant in Eq.(4.51) to the form used by Lovelace, i.e.

\[
\det M = -\lambda \frac{\Gamma(1 - \alpha(s))\Gamma(1 - \alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))}. \tag{4.54}
\]

If, as usual, we parametrize \( \alpha(s) = \alpha(0) + \alpha's \), then equation 1 = \( \alpha(s) + \alpha(t) \) causes the \( \det M \) to vanish. This also fixes the parameter \( \alpha(0) \): \( \alpha(0) = 1/2 \). This result was obtained by Adler long before string theory emerged and is known as Adler’s selfconsistency condition, Ref.[80]. With such “gauge fixing”, one can fix the slope \( \alpha' \) as well if one notices that the experimental data allow us to make a choice: 1 = \( \alpha(m_{\rho}^2) \). This leads to: \( \alpha' = \frac{1}{2m_{\rho}^2} \sim 0.885(Gev^{-2}) \), in accord with observations.

The obtained results are not limited to study of excitations of just one “superconducting” pair of quarks. In principle, any finite amount of such pairs can be studied, e.g. see Ref.[81]. In such a case the result for \( \det M \) is expected to become considerably more complicated but connections with one dimensional magnets still remain unchanged. We plan to discuss these issues in future publications.
5 Random fragmentation and coagulation processes the Poisson-Dirichlet distribution and Veneziano amplitudes

5.1 General facts about the Poisson-Dirichlet distribution

In the Introduction, following Heisenberg, we posed a question: Is combinatorics of observational data sufficient for recovery of underlying unique microscopic model? That is, can we have complete understanding of such a model based on information provided by combinatorics? As we demonstrated, especially in Section 4 and in Ref.[37], this task is impossible to accomplish without imposing additional constraints which, normally, are not dictated by the combinatorics only. Even accounting for such constraints, the obtained results could be in conflict with rigorous mathematics and physical reality. Last but not the least, since Veneziano amplitudes gave birth to string theory one can pose another question: Is these Veneziano (or Veneziano-like) amplitudes, perhaps corrected to account for particles with spin, contain enough information (analytical, number-theoretic, combinatorial, etc.) allowing restoration of the underlying microscopic model uniquely? In the most general case the answer is: No! This happens in spite of the fact that all amplitudes of high energy physics can be made out of linear combination of Veneziano amplitudes (up to logarithmic corrections) as discussed in our recent work, Ref.[4]. In the rest of this section we explain why this is so.

We begin with recalling some known auxiliary facts from the probability theory. For instance, we recall that the stationary Maxwell distribution for velocities of particles in the gas is of Gaussian-type. It can be obtained as a stationary solution of the Boltzmann’s dynamical equation maximizing Boltzmann’s entropy\(^{30}\). The question arises: Is it possible to find (discrete or continuous) dynamical equations which will provide known probability distributions as stable stationary solutions? This task will involve finding of dynamical equations along with the corresponding Boltzmann-like entropies which will reach their maxima at respective equilibria for these dynamical equations. We are certainly not in the position in this work to discuss this problem in full generality. Instead, we focus our attention only on processes described by the so called Dirichlet distributions. These originate from the integral (e.g. see Eq.(2.8) of Part I) attributed to Dirichlet. It is given by

\[
\mathcal{D}(x_1, \ldots, x_{n+1}) = \int \cdots \int_{u_1 \geq 0, \ldots, u_n \geq 0} \int_{u_1 + \cdots + u_n \leq 1} u_1^{-x_1-1} \cdots u_n^{-x_n-1} (1-u_1 - \cdots - u_n)^{x_{n+1}-1} du_1 \cdots du_n.
\]

A random vector \((X_1, \ldots, X_n) \in \mathbb{R}^n\) such that \(X_i \geq 0\ \forall i\) and \(\sum_{i=1}^n X_i = 1\) is said to be Dirichlet distributed with parameters \((x_1, \ldots, x_n; x_{n+1})\), e.g. see Ref.\(^{30}\) As discussed in our work, Ref.[82], on the Poincare\(^7\) and geometrization conjectures.

\(^{30}\)As discussed in our work, Ref.[82], on the Poincare\(^7\) and geometrization conjectures.
[83], if the probability density function for \((X_1, ..., X_n)\) is given by

\[
P_{X_1, ..., X_n}(u_1, ..., u_n) = \frac{\Gamma(x_1 + \cdots + x_{n+1}) u_1^{x_1-1} \cdots u_n^{x_n-1} (1 - \sum_{i=1}^n u_i)^{x_{n+1}-1}}{\Gamma(x_1) \cdots \Gamma(x_{n+1})} \prod_{i=1}^n u_i^{x_i-1}, \quad \text{provided that } u_{n+1} = 1 - u_1 - \cdots - u_n.
\]  

(5.2)

From these results it follows that Veneziano condition, Eq. (2.4), and Veneziano amplitudes are inseparable from each other. Since Veneziano condition is just restatement of energy-momentum conservation, such a requirement should be applicable to whatever amplitude of high energy physics. Not surprisingly, therefore, in Ref.[4], it is demonstrated that this is indeed the case.

It is of interest to mention other uses of Dirichlet distributions beyond that in high energy physics. For this purpose, to get a feeling of just defined distribution, we notice the following peculiar aspects of this distribution. For any discrete distribution, we know that the probability \(p_i\) must be normalized, that is \(\sum p_i = 1\). Thus, the Dirichlet distribution is dealing with averaging of the probabilities! Or, better, is dealing with the problem of effectively selecting the most optimal probability. The most primitive of these probabilities is the binomial probability given by

\[
p_m = \binom{n}{m} p^m (1-p)^{n-m}, \quad m = 0, 1, 2, ..., n.
\]  

(5.3)

If \(X\) is random variable obeying this law of probability then, the expectation \(E(X)\) is calculated as

\[
E(X) = \sum_{m=1}^n mp_m = np \equiv \mu.
\]  

(5.4)

Consider such a distribution in the limit: \(n \to \infty\). In this limit, if we write \(p = \mu/n\), then the Poisson distribution is obtained as

\[
p_m = \frac{\mu^m}{m!} e^{-\mu}.
\]  

(5.5)

Next, we notice that \(m! = \Gamma(m+1)\), furthermore, we replace \(m\) by the real valued variable \(\alpha\) and \(\mu\) by \(x\). This allows us to introduce the gamma distribution with exponent \(\alpha\) whose probability density is

\[
p_X(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}
\]  

(5.6)

for some gamma distributed random variable \(X\). Finally, we would like to demonstrate how the Dirichlet distribution can be represented through gamma
distributions. Since the gamma distribution originates from the Poisson distribution, sometimes in literature the Dirichlet distribution is called the Poisson-Dirichlet (P-D) distribution, Ref.[45]. To demonstrate connection between the Dirichlet and gamma distributions is relatively easy. Following Kingman, Ref.[45], consider a set of positive independent gamma distributed random variables: $Y_1, \ldots, Y_{n+1}$ with exponents $\alpha_1, \ldots, \alpha_{n+1}$. Furthermore, consider $Y = Y_1 + \cdots + Y_{n+1}$ and construct a vector $\mathbf{u}$ with components: $u_i = \frac{Y_i}{Y}$. Then, since $\sum_{i=1}^{n+1} u_i = 1$, the components of this vector are Dirichlet distributed and, in fact, independent of $Y$. Details of the proof are based on results already discussed in Part I and are given in Appendix.

Such described Dirichlet distribution is an equilibrium measure in various fields ranging from spin glasses to computer science, from linguistics to genetics, from forensic science to economics, etc. Many useful references involving these and other applications can be found in Ref.s[38-40]. Furthermore, most of fragmentation and coagulation processes involve the P-D distribution as their equilibrium measure. Some applications of general theory of these processes to to nuclear and particle physics were initiated in a series of papers by Mekjian, e.g. see Ref.[41] and references therein. Alternative approach to the fragmentation-coagulation processes in high energy physics was developed by Andersson, Ref.[42], and is known as the Lund model. As results of our recent work, Ref.[4], indicate, the results of Mekjian, Ref.[41], and that presented in this work are already fully compatible with general theory of coagulation-fragmentation processes discussed in Ref.s[38-40]. The interconnections between the Lund model and general theory of coagulation-fragmentation processes remains to be investigated. In the meantime, we would like to connect general results presented in this subsection with those of Section 4. This is accomplished below.

5.2 Quantum mechanics, hypergeometric functions and P-D distribution

In Ref.[2,3] we provided detailed explanation of the fact that all exactly solvable 2-body quantum mechanical problems involve different kinds of special functions obtainable from the Gauss hypergeometric function whose integral representation is given by

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \quad (5.7)$$

As is well known, the discrete spectrum of all exactly solvable quantum mechanical problems can be obtained only if one can find an appropriate set of orthogonal polynomials related to this spectrum. Since all these orthogonal polynomials are obtainable from Gauss hypergeometric function, the question arises: Under what conditions on coefficients $(a, b$ and $c)$ can infinite hypergeometric series (whose integral representation is given by Eq.(5.7)) be reduced
to a finite (orthogonal) polynomial? This happens, for instance, if we impose the quantization condition: \(-a = 0, 1, 2, \ldots\) In such a case we can write \((1 - zt)^{-a} = \sum_{i=1}^{a} (-1)^i (zt)^i\) and use this finite expansion in Eq.(5.7). In view of Eq.(5.2), we obtain the convergent generating function for the Dirichlet distribution. Hence, all known quantum mechanical problems involving discrete spectrum are effectively examples of the P-D stochastic processes. Furthermore, from this point of view quantum mechanics (also, quantum field theory and string theory, e.g. see Ref.[4]) becomes just an applied theory of the P-D stochastic processes. For hypergeometric functions of multiple arguments this was demonstrated in Ref.[84] only quite recently. Other arguments in favour of such an interpretation are developed in Ref.[4].

Next, we are still interested in the following. Given these observations, can we include the determinantal formula, Eq.(4.51), into emerging quantization scheme? Very fortunately, this can be done as explained in the next subsection.

5.3 Hypergeometric functions, Kummer series and Veneziano amplitudes

In view of just introduced new quantization condition, the question arises: Is this the only condition reducing the hypergeometric function to a polynomial? More broadly: what conditions on coefficients \(a, b\) and \(c\) should be imposed so that the function \(F(a, b, c; z)\) becomes a polynomial? The answer to this question was provided by Kummer in the first half of 19th century as discussed in Ref.[85]. Incidentally, in the case of K-Z equations such a problem was solved only in 2007 in Ref.[86]. We would like to summarize Kummer’s results and to connect them with the determinantal formula, Eq.(4.51).

According to general theory of hypergeometric equations of one variable discussed in Ref.[85], the infinite series for hypergeometric function degenerates to a polynomial if one of the numbers

\[ a, b, c - a \text{ or } c - b \]  

is an integer. This condition is equivalent to the condition that, at least one of eight numbers \(\pm (c - 1) \pm (a - b) \pm (a + b - c)\), is an odd number. According to general theory of hypergeometric functions of multiple arguments discussed in Section 4, the \(k = 1\)-type solutions can be obtained using 1-forms given by Eq.(4.47) accounting for a singular module constraint, Eq.(4.49), in the form given by Eq.(4.46). In the case of Gauss-type hypergeometric functions, relations of the type given by Eq.(4.49) were known already to Kummer. He found 24 interdependent solutions. Evidently, this number is determined by the number of independent Pochhammer contours as explained in Refs.[9, 85]. Therefore, among these he singled out 6 (generating these 24) and among these 6 he established that every 3 of them are related to each other via equation of the type given by Eq.(4.49).

We denote these 6 functions respectively as \(u_1, \ldots, u_6\). Then, we can represent, say, \(u_2\) and \(u_6\) using \(u_1\) and \(u_5\) as the basis set. We can do the same with \(u_1\).
and $u_5$ by representing them through $u_2$ and $u_6$ and, finally, we can connect $u_3$ and $u_4$ with $u_1$ and $u_5$. Hence, it is sufficient to consider, say, $u_2$ and $u_6$. Thus, we obtain:

$$
\begin{pmatrix}
  u_2 \\
  u_6
\end{pmatrix} =
\begin{pmatrix}
  M_1^1 & M_2^1 \\
  M_1^2 & M_2^2
\end{pmatrix}
\begin{pmatrix}
  u_1 \\
  u_5
\end{pmatrix},
$$

(5.9)

with $M_1^1 = \frac{\Gamma(a + b - c + 1)\Gamma(1 - c)}{\Gamma(a + 1 - c)\Gamma(b - c + 1)}$; $M_2^1 = \frac{\Gamma(a + b + 1 - c)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)}$; $M_1^2 = \frac{\Gamma(c + 1 - a - b)\Gamma(1 - c)}{\Gamma(1 - a)\Gamma(1 - b)}$; $M_2^2 = \frac{\Gamma(c + 1 - a - b)\Gamma(c - 1)}{\Gamma(c - a)\Gamma(c - b)}$. The determinant of this matrix becomes zero if either two rows or two columns become the same. For instance, we obtain:

$$
\frac{\Gamma(a)\Gamma(b)}{\Gamma(c - 1)} = \frac{\Gamma(a - c + 1)\Gamma(b - c + 1)}{\Gamma(1 - c)} \quad \text{and} \quad \frac{\Gamma(c - a)\Gamma(c - b)}{\Gamma(c - 1)} = \frac{\Gamma(1 - a)\Gamma(1 - b)}{\Gamma(1 - c)}.
$$

(5.10)

The condition $c = 1$ in Eq.(5.10) causes two solutions of hypergeometric equation to degenerate into one polynomial solution in accord with general theory.

**Appendix**

**A. Linear independence of solutions of K-Z equation**

Linear independence of solutions of K-Z equation is based on the following arguments. Consider change of the basis

$$
\tilde{e}^j = A^i_j e^i, \quad i, j = 1, 2, ..., n
$$

(A.1)

in $\mathbb{R}^n$. Using this result, consider the exterior product

$$
\tilde{e}^1 \wedge \cdots \wedge \tilde{e}^n = [\det A]e^1 \wedge \cdots \wedge e^n.
$$

(A.2)

Next, suppose, that the vectors $\tilde{e}^j$ are linearly-dependent. In particular, this means that

$$
\tilde{e}^n = \alpha_1 \tilde{e}^1 + \cdots + \alpha_{n-1} \tilde{e}^{n-1}
$$

(A.3)

for some nonzero $\alpha_i's$. Using this expansion in Eq.(A.2) we obtain

$$
\tilde{e}^1 \wedge \cdots \wedge \tilde{e}^{n-1} \wedge (\alpha_1 \tilde{e}^1 + \cdots + \alpha_{n-1} \tilde{e}^{n-1}) \equiv 0,
$$

(A.4)

implying $[\det A] = 0$. Conversely, if $[\det A] \neq 0$ then, vectors $\tilde{e}^j$ are linearly independent.

**B. Connections between the gamma and Dirichlet distributions**

Using results of our Part I (especially Eq.(3.27)), such a connection can be easily established. Indeed, consider $n + 1$ independently distributed random
gamma variables with exponents $\alpha_1, ..., \alpha_{n+1}$. The joint probability density for such variables is given by

$$p_{Y_1,...,Y_{n+1}}(s_1,...,s_{n+1}) = \frac{1}{\Gamma(\alpha_1)} \cdots \frac{1}{\Gamma(\alpha_{n+1})} s_1^{\alpha_1-1} \cdots s_{n+1}^{\alpha_{n+1}-1}. \quad (B.1)$$

Let now $s_i = t_i t$, where $t_i$ are chosen in such a way that $\sum_{n=1}^{n+1} t_i = 1$. Then, using such a substitution in Eq.(B.1), we obtain at once:

$$p_{u_1,...,u_{n+1}}(t_1,...,t_{n+1}) = \left[ \int_0^\infty t^{\alpha-1} e^{-t} \right] \frac{1}{\Gamma(\alpha_1)} \cdots \frac{1}{\Gamma(\alpha_{n+1})} t_1^{\alpha_1-1} \cdots t_{n+1}^{\alpha_{n+1}-1} \quad (B.2)$$

Since $\alpha = \alpha_1 + \cdots + \alpha_{n+1}$, we also obtain: $\int_0^\infty t^{\alpha-1} e^{-t} = \Gamma(\alpha_1 + \cdots + \alpha_{n+1})$, implying that the density of probability given by Eq.(B.2) is indeed of Dirichlet-type given by Eq.(5.2) of the main text.
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