BIHARMONIC MAPS INTO SOL AND NIL SPACES

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Abstract

In this paper, we study biharmonic maps into Sol and Nil spaces, two model spaces of Thurston’s 3-dimensional geometries. We characterize non-geodesic biharmonic curves in Sol space and prove that there exists no non-geodesic biharmonic helix in Sol space. We also show that a linear map from a Euclidean space into Sol or Nil space is biharmonic if and only if it is a harmonic map, and give a complete classification of such maps.

1. Introduction

In this paper, we work in the category of smooth objects, so manifolds, maps, vector fields, etc., are assumed to be smooth unless it is stated otherwise.

A map \( \varphi : (M, g) \rightarrow (N, h) \) between Riemannian manifolds is called a biharmonic map if \( \varphi|\Omega \) is a critical point of the bienergy

\[
E^2(\varphi, \Omega) = \frac{1}{2} \int_{\Omega} |\tau_2(\varphi)|^2 \, dx
\]

for every compact subset \( \Omega \) of \( M \), where \( \tau_2(\varphi) = \text{Trace}_g \nabla d\varphi \) is the tension field of \( \varphi \). Using the first variational formula (see [13]) one sees that \( \varphi \) is a biharmonic map if and only if its bitension field vanishes identically, i.e.,

\[
\tau^2(\varphi) := -\Delta^\varphi(\tau_2(\varphi)) - \text{Trace}_g R^N(d\varphi, \tau_2(\varphi))d\varphi = 0,
\]

where

\[
\Delta^\varphi = -\text{Trace}_g(\nabla^\varphi)^2 = -\text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_{\nabla^M})
\]

is the Laplacian on sections of the pull-back bundle \( \varphi^{-1}TN \) and \( R^N \) is the curvature operator of \( (N, h) \) defined by

\[
R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.
\]

Note that \( \tau^2(\varphi) = -J^\varphi(\tau_2(\varphi)) \), where \( J^\varphi \) is the Jacobi operator which plays an important role in the study of harmonic maps.

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Clearly, any harmonic map is biharmonic, so harmonic maps are a subclass of biharmonic maps. It is interesting to note that so far, apart from maps between Euclidean spaces (in which case any polynomial map of degree less than four is a biharmonic map), not many example of proper (meaning non-harmonic) biharmonic maps between Riemannain manifolds have been found (see, e.g., [10], [9], [11], and the bibliography of biharmonic maps [8]).

Recently, some work has been done in the study of non-geodesic biharmonic curves in some model spaces. For example, [2] gives a complete classification of non-geodesic biharmonic curves in 3-sphere while in [3] it is proved that a non-geodesic biharmonic curve in Heisenberg group $\mathbb{H}_3$ is a helix and the explicit parametrizations of such curves are given. For the study of biharmonic curves in Berger’s spheres, in Minkowski 3-space, in Cartan-Vranceanu 3-dimensional space, and in contact and Sasakian manifolds see [1], [7], [4], and [6] respectively.

In this paper, we first characterize non-geodesic biharmonic curves in Sol space (Theorem 2.3) and show that there exists no non-geodesic biharmonic helix in Sol space (Theorem 2.7). In the second part of the paper, we study linear biharmonic maps from Euclidean space $\mathbb{R}^m$ into Sol space $(\mathbb{R}^3, g_{Sol})$ and Nil space $(\mathbb{R}^3, g_{Nil})$ using the linear structure of the underlying manifolds of Sol and Nil spaces. We show that such a linear map is biharmonic if and only if it is a harmonic map and give a complete description of such maps (Theorems 3.3 and 3.4).

2. Biharmonic curves in Sol space

Biharmonic curve equation in Frenet Frames. Let $I \subset \mathbb{R}$ be an open interval, and $\gamma : I \rightarrow (N, h)$ be a curve, parametrized by arc length, on a Riemannian manifold. Putting $T = \gamma'$, we can write the tension field of $\gamma$ as $\tau(\gamma) = \nabla_\gamma \gamma'$, and the biharmonic map equation (1) reduces to

$$\nabla^3_T T - R^N(T, \nabla_T T)T = 0.$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve which is recalled in the following

Definition 2.1. (see, for example, [12])

The Frenet frame $\{F_i\}_{i=1,2,...,n}$ associated to a curve $\gamma : I \subset \mathbb{R} \rightarrow (N^n, h)$ parametrized by arc length is the orthonormalisation of the $(n+1)$-tuple

$\{\nabla^{(k)}_t d\gamma(\frac{d}{dt})\}_{k=0,1,2,...,n}$ described by:
\[ F_1 = d\gamma \left( \frac{\partial}{\partial t} \right) \]
\[ \nabla^{(\gamma)} F_1 = k_1 F_2 \]
\[ \nabla^{(\gamma)} F_i = -k_{i-1} F_{i-1} + k_i F_{i+1}, \forall \ i = 2, 3, \ldots, n - 1 \]
\[ \nabla^{(\gamma)} F_n = -k_{n-1} F_{n-1}, \]
where the functions \( \{k_1, k_2, \ldots, k_{n-1}\} \) are called the curvatures of \( \gamma \) and \( \nabla^{\gamma} \) is the connection on the pull-back bundle \( \gamma^{-1}(T N) \). Note that \( F_1 = T = \gamma' \) is the unit tangent vector field along the curve.

With respect to the Frenet frame, the biharmonic curve equation takes the following form.

**Lemma 2.2.** (see, for example, [4]) Let \( \gamma : I \subset \mathbb{R} \rightarrow (N^n, h)(n \geq 2) \) be a curve parametrized by arc length from an open interval of \( \mathbb{R} \) into a Riemannian manifold \( (N^n, h) \). Then \( \gamma \) is biharmonic if and only if:

\[
\begin{align*}
&\begin{align*}
&k_1 k_1' = 0 \\
&k_1' - k_1^3 - k_1 k_2^2 + k_1 R(F_1, F_2, F_1, F_2) = 0 \\
&2k_1' k_2 + k_1 k_3^2 + k_1 R(F_1, F_2, F_1, F_3) = 0 \\
&k_1 k_2 k_3 + k_1 R(F_1, F_2, F_1, F_4) = 0 \\
&k_1 R(F_1, F_2, F_1, F_j) = 0, \quad j = 5, \ldots, n.
\end{align*}
\end{align*}
\]

**Riemannian structure of Sol space.** Sol space, one of Thurston’s eight 3-dimensional geometries, can be viewed as \( \mathbb{R}^3 \) provided with Riemannian metric \( g_{Sol} = ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2 \), where \( (x, y, z) \) are the standard coordinates in \( \mathbb{R}^3 \). Note that the Sol metric can also be written as:

\[ ds^2 = \sum_{i=1}^{3} \omega^i \otimes \omega^i, \]

where

\[ \omega^1 = e^z dx, \quad \omega^2 = e^{-z} dy, \quad \omega^3 = dz, \]

and the orthonormal basis dual to the 1-forms is

\[ e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}. \]

With respect to this orthonormal basis, the Levi-Civita connection and the Lie brackets can be easily computed as:

\[ \nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1 \]
\[ \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -e_2 \]
\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]
We adopt the following notation and sign convention for Riemannian curvature operator.

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla [X, Y] Z,
\]
the Riemannian curvature tensor is given by

\[
R(X, Y, Z, W) = g(R(Y, X) Z, W) = -g(R(X, Y) Z, W).
\]

Moreover we put

\[
R_{abc} = R(e_a, e_b) e_c, \quad R_{abcd} = R(e_a, e_b, e_c, e_d),
\]
where the indices \(a, b, c, d\) take the values 1, 2, 3.

A direct computation using (3), (4), (5), (6), and (7) gives the following non-zero components of Riemannian curvature of Sol space with respect to the orthonomal basis \(\{e_1, e_2, e_3\}\) (we do not list those that can be obtained by symmetric properties of curvature):

\[
R_{121} = -e_2, \quad R_{131} = e_3, \quad R_{122} = e_1,
\]

\[
R_{232} = e_3, \quad R_{133} = -e_1, \quad R_{233} = -e_2,
\]

and

\[
R_{1212} = -g(R(e_1, e_2) e_1, e_2) = -g(-e_2, e_2) = 1,
\]

\[
R_{1313} = -g(R(e_1, e_3) e_1, e_3) = -g(e_3, e_3) = -1,
\]

\[
R_{2333} = -g(R(e_2, e_3) e_2, e_3) = -g(e_3, e_3) = -1.
\]

**Biharmonic curves in Sol space.** Let \(\gamma : I \rightarrow (\mathbb{R}^3, g_{\text{Sol}})\) be a curve on Sol space parametrized by arc length. Let \(\{T, N, B\}\) be the Frenet frame fields tangent to Sol space along \(\gamma\) defined as follows: \(T\) is the unit vector field tangent to \(\gamma\), \(N\) is the unit vector field in the direction of \(\nabla_T T\) (normal to \(\gamma\)), and \(B\) is chosen so that \(\{T, N, B\}\) is a positively oriented orthonormal basis. Then, by Definition (2.1), we have the following Frenet formulas

\[
\begin{align*}
\nabla_T T &= kN \\
\nabla_T N &= -kT + \tau B \\
\nabla_T B &= -\tau N,
\end{align*}
\]
where \(k = |\nabla_T T|\) is the geodesic curvature and \(\tau\) the geodesic torsion of \(\gamma\). With respect to the orthonormal basis \(\{e_1, e_2, e_3\}\) we can write \(T = T_1 e_1 + T_2 e_2 + T_3 e_3\), \(N = N_1 e_1 + N_2 e_2 + N_3 e_3\), \(B = T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3\), and we have
Theorem 2.3. Let $\gamma : I \rightarrow (\mathbb{R}^3, g_{\text{Sol}})$ be a curve parametrized by arc length. Then $\gamma$ is a non-geodesic biharmonic curve if and only if

\begin{equation}
\begin{cases}
k = \text{constant} \neq 0 \\
k^2 + \tau^2 = 2B_3^2 - 1 \\
\tau' = 2N_3B_3.
\end{cases}
\end{equation}

Proof. By (2) of lemma (2.2) we see that $\gamma$ is a biharmonic curve if and only if

\begin{equation}
\begin{cases}
k'k' = 0 \\
k'' - k^3 - k\tau^2 + kR(T, N, T, N) = 0 \\
2k'\tau + k\tau' + kR(T, N, T, B) = 0,
\end{cases}
\end{equation}

which is equivalent to

\begin{equation}
\begin{cases}
k = \text{constant} \neq 0 \\
k^2 + \tau^2 = R(T, N, T, N) \\
\tau' = -R(T, N, T, B)
\end{cases}
\end{equation}

since $k \neq 0$ by the assumption that $\gamma$ is non-geodesic. A direct computation using (3) yields

\begin{align*}
R(T, N, T, N) &= \sum_{i,j,l,p=1}^{3} T_i N_j T_l N_p (R_{lpij}) \\
&= T_1 N_2 T_1 N_2 R_{1212} + T_1 N_2 T_2 N_1 R_{1221} + T_2 N_1 T_2 N_1 R_{2121} + T_2 N_1 T_1 N_2 R_{2112} \\
&+ T_1 N_3 T_1 N_3 R_{1313} + T_1 N_3 T_3 N_1 R_{1331} + T_3 N_1 T_3 N_1 R_{3131} + T_3 N_1 T_1 N_3 R_{3113} \\
&+ T_2 N_3 T_2 N_3 R_{2323} + T_2 N_3 T_3 N_2 R_{2332} + T_3 N_2 T_3 N_2 R_{3232} + T_3 N_2 T_2 N_3 R_{3223} \\
&= T_1^2 N_2^2 - T_1 T_2 N_1 N_2 + T_2^2 N_1^2 - T_1 T_2 N_1 N_2 \\
&- T_2^2 N_3^2 + T_1 T_3 N_1 N_3 - T_3^2 N_1^2 + T_1 T_3 N_1 N_3 \\
&- T_2^2 N_3^2 + T_2 T_3 N_2 N_3 - T_3^2 N_2^2 + T_2 T_3 N_2 N_3 \\
&= (T_1 N_2 - T_2 N_1)^2 - (T_3 N_1 - T_1 N_3)^2 - (T_3 N_2 - T_2 N_3)^2 \\
&= B_3^2 - B_1^2 - B_3^2 = 2B_3^2 - 1 \\
(by \ T \times N = B, T \times B = -N, B_1^2 + B_2^2 + B_3^2 = 1),
\end{align*}
and
\[
R(T, N, T, B) = \sum_{i,l,p=1}^{3} t_{i} N_{p} t_{i} B_{j} R_{i p j}
\]
\[
= t_{1} N_{2} t_{1} B_{2} R_{2121} + t_{1} N_{2} t_{2} B_{1} R_{1221} + t_{2} N_{1} t_{2} B_{1} R_{2112} + t_{1} N_{3} t_{1} B_{3} R_{1313} + t_{1} N_{3} t_{3} B_{1} R_{1311} + t_{3} N_{1} t_{3} B_{1} R_{3113} + t_{2} N_{3} t_{3} B_{3} R_{3223} + t_{2} N_{3} t_{3} B_{2} R_{2323} + t_{3} N_{2} t_{2} B_{2} R_{3223} + t_{3} N_{2} t_{3} B_{3} R_{3223} \]
\[
= T_{1}^{2} N_{2} B_{2} - T_{1} T_{2} N_{1} B_{2} + T_{2}^{2} N_{1} B_{1} - T_{1} T_{2} N_{2} B_{1} - T_{2}^{2} N_{3} B_{3} + T_{1} T_{3} N_{1} B_{3} - T_{3}^{2} N_{1} B_{1} + T_{1} T_{3} B_{1} N_{3} - T_{2}^{2} N_{3} B_{3} + T_{2} T_{3} N_{2} B_{3} - T_{3}^{2} N_{2} B_{2} + T_{2} T_{3} B_{2} N_{3} \]
\[
= (T_{1} B_{2} - T_{2} B_{1})(T_{1} N_{2} - T_{2} N_{1}) - (T_{3} N_{1} - T_{1} N_{3})(T_{3} B_{1} - T_{1} B_{3}) - (T_{3} N_{2} - T_{2} N_{3})(T_{3} B_{2} - T_{2} B_{3}) \]
\[
= - N_{3} B_{3} + N_{2} B_{2} + N_{1} B_{1} = - 2 N_{3} B_{3} \]
\[
(by T \times N = B, T \times B = - N, N_{1} B_{1} + N_{1} B_{2} + N_{1} B_{3} = 0),
\]
these, together with Equation (11), complete the proof of the theorem. \qed

As an immediate consequence we have

**Corollary 2.4.** Let \( \gamma : I \to (\mathbb{R}^{3}, g_{\text{Sol}}) \) be a non-geodesic curve parametrized by arc length. If \( B_{3} = 0 \), then \( \gamma \) is not biharmonic.

**Corollary 2.5.** Let \( \gamma : I \to (\mathbb{R}^{3}, g_{\text{Sol}}) \) be a non-geodesic curve parametrized by arc length. If \( B_{3} \) is constant and \( N_{3} B_{3} \neq 0 \), then \( \gamma \) is not biharmonic.

Similar to the terminology used for curves in \( \mathbb{R}^{3} \), we keep the name helix for a curve in a Riemannian 3-manifold having constant both geodesic curvature and geodesic torsion. With this terminology, we can use Equation (10) to deduce the following

**Corollary 2.6.** Let \( \gamma : I \to (\mathbb{R}^{3}, g_{\text{Sol}}) \) be a non-geodesic biharmonic helix parametrized by arc length, then

\[
\begin{aligned}
B_{3} &= \text{constant} \neq 0 \\
N_{3} &= 0 \\
k^{2} + r^{2} &= 2 B_{3}^{2} - 1.
\end{aligned}
\]

Non-geodesic biharmonic helices in 3-dimensional sphere \( S^{3} \), in Heisenberg group \( H_{3} \), and in Cartan-Vranceanu 3-dimensional space have been studied in \( \mathbb{E} \), \( \mathbb{H} \), and \( \mathbb{I} \) respectively. In contrast to the situations in those 3-dimensional spaces where there are rich examples of such curves our next theorem shows that there exists no such curves in Sol space.
Theorem 2.7. There exists no non-geodesic biharmonic helix in Sol space.

Proof. Suppose that $\gamma : I \rightarrow (\mathbb{R}^3, g_{Sol})$ is a non-geodesic biharmonic helix parametrized by arc length. We shall derive a contradiction by showing that $\gamma$ must be a geodesic. We can use (3) to compute the covariant derivatives of the vector fields $T, N, B$ as:

$$\nabla_T T = (T'_1 + T_1 T'_3) e_1 + (T'_2 - T_2 T'_3) e_2 + (T'_3 - T'_1 + T'_3) e_3$$
$$\nabla_T N = (N'_1 + T_1 N'_3) e_1 + (N'_2 - T_2 N'_3) e_2 + (T_2 N_2 - T_1 N_1 + N'_3) e_3$$
$$\nabla_T B = (B'_1 + T_1 B'_3) e_1 + (B'_2 - T_2 B'_3) e_2 + (T_2 B_2 - T_1 B_1 + B'_3) e_3.\tag{13}$$

It follows that the third components of these vectors are given by

$$\langle \nabla_T T, e_3 \rangle = T'_2 - T'_1 + T'_3$$
$$\langle \nabla_T N, e_3 \rangle = T_2 N_2 - T_1 N_1 + N'_3$$
$$\langle \nabla_T B, e_3 \rangle = T_2 B_2 - T_1 B_1 + B'_3.\tag{14}$$

On the other hand, using Frenet formulas (9) we have,

$$\langle \nabla_T T, e_3 \rangle = k N_3$$
$$\langle \nabla_T N, e_3 \rangle = -k T_3 + \tau B_3$$
$$\langle \nabla_T B, e_3 \rangle = -\tau N_3.\tag{15}$$

Since $\gamma$ is assumed to be a non-geodesic biharmonic helix, we have, by Corollary 2.6, $N_3 = 0$, $B_3 = \text{constant}$. These, together with Equations (14) and (15), give

$$T'_2 - T'_1 + T'_3 = 0$$
$$T_2 N_2 - T_1 N_1 = -k T_3 + \tau B_3$$
$$T_2 B_2 - T_1 B_1 = 0.\tag{16}$$

Noting that $T \times B = -N$ we also have

$$T_2 B_1 - T_1 B_2 = N_3 = 0.\tag{17}$$

Thus, we have

$$\begin{align*}
T_2 N_2 - T_1 N_1 &= -k T_3 + \tau B_3, \tag{1} \\
T_2^2 - T_1^2 + T'_3 &= 0, \tag{2} \\
T_2 B_2 - T_1 B_1 &= 0, \tag{3} \\
T_2 B_1 - T_1 B_2 &= 0. \tag{4}
\end{align*}$$

Case A: $T_1^2 \ne T_2^2$. In this case, Equations (3) and (4) in System (18) viewed as equations in $B_1$ and $B_2$ has a unique solution $B_1 = B_2 = 0$. This implies that $T_3 = \langle N \times B, e_3 \rangle = 0$. Substitute this into (2) of System (18) we have $T_1^2 = T_2^2$, a contradiction. Thus, we must have

Case B: $T_1^2 = T_2^2$. In this case, Equation (2) of System (18) implies that $T_3 = \text{constant}$. To understand the meaning of this, we represent the unit tangent vector
T as $T = \sin \alpha \cos \beta e_1 + \sin \alpha \sin \beta e_2 + \cos \alpha e_3$, where $\alpha = \alpha(s)$, $\beta = \beta(s)$.

With this representation, $T_3 = \text{constant}$ implies that $\cos \alpha = \text{constant}$ and hence $\alpha(s) = \alpha_0$, a constant. This, together with $T_1^2 = T_2^2$, gives

$$\sin \alpha_0 (\cos \beta \pm \sin \beta) = 0.\tag{19}$$

If $\sin \alpha_0 = 0$, then we have $T_1 = T_2 = 0$, and it follows from the first equation of (13) that $\nabla_T T = 0$ which means that $\gamma$ is a geodesic, a contradiction. Thus, we must have $\sin \alpha_0 \neq 0$, which, together with (19), implies that

$$\cos \beta = \pm \sin \beta = \pm \frac{\sqrt{2}}{2},$$

and hence,

$$T_1 = \pm T_2 = \pm \frac{\sqrt{2}}{2} \sin \alpha_0.\tag{20}$$

We use the first equation of (13) again to get

$$\nabla_T T = \sin \alpha_0 \cos \alpha_0 (\pm \frac{\sqrt{2}}{2} e_1 \pm \frac{\sqrt{2}}{2} e_2) = k N$$

which yields

$$N_1 = \pm \frac{\sqrt{2}}{2}, \quad N_2 = \mp \frac{\sqrt{2}}{2}.\tag{21}$$

since $k = |\nabla_T T| = |\sin \alpha_0 \cos \alpha_0|$. By the assumption that $\gamma$ is non-geodesic, we may assume, without loss of generality, that $\sin \alpha_0 \cos \alpha_0 > 0$, so

$$k = \sin \alpha_0 \cos \alpha_0.\tag{22}$$

Using Equations (20), (21) and the fact that $B = T \times N$ we have

$$B_1 = \text{constant}, \quad B_2 = \text{constant}, \quad B_3 = T_1 N_2 - T_2 N_1 = \mp \sin \alpha_0.\tag{23}$$

It follows from (23), (3) of (18), and the third equation of (13) that

$$\tau^2 = |\nabla_T B|^2 = (T_1^2 + T_2^2) B_3^2 = \sin^4 \alpha_0.\tag{24}$$

Substituting (22), (23) and (24) into the third equation in (12) we have

$$\sin^2 \alpha_0 \cos^2 \alpha_0 + \sin^4 \alpha_0 = 2 \sin^2 \alpha_0 - 1,$$

which implies

$$\sin^2 \alpha_0 = 1,$$

and hence

$$\cos^2 \alpha_0 = 0.$$
It follows that \( \cos \alpha_0 = 0 \) from which and (22) we conclude that \( k = 0 \), i.e., \( \gamma \) is a geodesic, a contradiction.

Combining the the results in Cases A and B we complete the proof of the theorem. \( \square \)

3. Linear biharmonic maps into Sol and Nil spaces

In this section, we first derive the biharmonic map equation in local coordinates and then we use it to classify linear biharmonic maps from Euclidean space into Sol and Nil spaces. In the rest of the paper, we adopt the Einstein summation convention that a repeated upper and lower index means the summation of that index over its range that is understood from the context.

3.1 Biharmonic map equation in local coordinates

**Lemma 3.1.** Let \( \varphi : (M^{m}, g) \rightarrow (N^{n}, h) \) with \( \varphi(x^1, \ldots, x^m) = (\varphi^1(x), \ldots, \varphi^n(x)) \) be a map between Riemannian manifolds. With respect to local coordinates \( (x^i) \) in \( M \) and \( (y^\alpha) \) in \( N \), \( \varphi \) is biharmonic if and only if it is a solution of the following system of PDE’s

\[
g^{ij}(\tau_{ij}^\sigma + \tau_{ij}^\sigma \varphi_{i}^\beta \bar{\Gamma}_{\alpha\beta}^\sigma + \frac{\partial}{\partial x^k}(\tau_{ij}^\sigma \varphi_{j}^\beta \bar{\Gamma}_{\alpha\beta}^\sigma) + \tau_{ij}^\sigma \varphi_{i}^\rho \varphi_{j}^\beta \bar{\Gamma}_{\alpha\beta}^\rho \bar{\Gamma}_{\kappa\lambda}^\sigma - \Gamma_{ij}^k(\tau_k^\sigma + \tau_{kj}^\sigma \varphi_{i}^\beta \bar{\Gamma}_{\alpha\beta}^\sigma) - \tau_{ij}^\sigma \varphi_{i}^\rho \varphi_{j}^\beta \bar{R}_{\alpha\beta\kappa\lambda}^\sigma = 0, \quad \sigma = 1, 2, \ldots, n. \tag{25}\]

**Proof.** Let \( g_{ij} \) and \( \Gamma_{ij}^k \) (resp. \( h_{\alpha\beta} \) and \( \bar{\Gamma}_{\alpha\beta}^\sigma \)) denote the components of metric and the connection coefficients in the domain (resp. the target) manifold with respect to the chosen local coordinates and the natural frame \( \{\frac{\partial}{\partial x^i}\} \) (resp. \( \{\frac{\partial}{\partial y^\alpha}\} \)).

The bitension field of \( \varphi \) can be computed as

\[
\tau^2(\varphi) = g^{ij}\left(\nabla^\varphi_{\partial \over \partial x^i} \nabla^\varphi_{\partial \over \partial x^j} - \nabla^\varphi_{\partial \over \partial y^\alpha} \nabla^\varphi_{\partial \over \partial y^\alpha}\right)(\tau(\varphi)) - \text{Trace}_g \bar{R}(d\varphi, \tau(\varphi))d\varphi
\]

\[
= g^{ij}\nabla^\varphi_{\partial \over \partial x^i} \nabla^\varphi_{\partial \over \partial x^j}(\tau(\varphi)) - g^{ij} \Gamma_{ij}^k \nabla^\varphi_{\partial \over \partial x^k}(\tau(\varphi))
\]

\[
- g^{ij}\bar{R}(d\varphi(\partial \over \partial x^i), \tau(\varphi))d\varphi(\partial \over \partial x^j)
\]

\[
= g^{ij}\left(\nabla^\varphi_{\partial \over \partial x^i} \nabla^\varphi_{\partial \over \partial x^j}(\tau(\varphi)) - \Gamma_{ij}^k \nabla^\varphi_{\partial \over \partial x^k}(\tau(\varphi)) - \varphi_{i}^\rho \varphi_{j}^\beta \bar{R}_{\alpha\beta\kappa\lambda}^\sigma(\partial \over \partial y^\alpha, \tau(\varphi))(\partial \over \partial y^\beta)\right).
\tag{26}\]

A direct computation gives

\[
\nabla^\varphi_{\partial \over \partial x^k}(\tau(\varphi)) = \nabla^\varphi_{\partial \over \partial y^\alpha}(\tau^\alpha \partial \over \partial y^\alpha) = \left(\tau_k^\sigma + \tau_{kj}^\sigma \varphi_{i}^\beta \bar{\Gamma}_{\alpha\beta}^\sigma\right) \partial \over \partial y^\alpha,
\tag{27}\]

\[
\nabla^\varphi_{\partial \over \partial y^\alpha}(\tau(\varphi)) = \nabla^\varphi_{\partial \over \partial y^\alpha}(\tau^\beta \partial \over \partial y^\beta) = \left(\tau_k^\beta + \tau_{k\lambda}^\beta \bar{\Gamma}_{\alpha\beta}^\lambda\right) \partial \over \partial y^\alpha,
\tag{28}\]

\[
\nabla^\varphi_{\partial \over \partial x^k}(\tau(\varphi)) = \nabla^\varphi_{\partial \over \partial y^\alpha}(\tau^\alpha \partial \over \partial y^\alpha) = \left(\tau_k^\sigma + \tau_{kj}^\sigma \varphi_{i}^\beta \bar{\Gamma}_{\alpha\beta}^\sigma\right) \partial \over \partial y^\alpha,
\tag{29}\]

\[
\nabla^\varphi_{\partial \over \partial y^\alpha}(\tau(\varphi)) = \nabla^\varphi_{\partial \over \partial y^\alpha}(\tau^\beta \partial \over \partial y^\beta) = \left(\tau_k^\beta + \tau_{k\lambda}^\beta \bar{\Gamma}_{\alpha\beta}^\lambda\right) \partial \over \partial y^\alpha.
\tag{30}\]
\[ \nabla_{\alpha \beta} \cdot \nabla_{\alpha \beta} (\tau(\varphi)) = \nabla_{\alpha \beta} \cdot \nabla_{\alpha \beta} (\partial_{\nu} \varphi) = \partial_{\alpha} \left( \tau_{\alpha}^{\beta} \nabla_{\alpha \beta} \varphi \right) \partial_{\beta}, \]

(28)

and

\[ \tilde{R}(\partial_{\beta} \varphi(\varphi)) \partial_{\gamma} = \tau_{\nu} \tilde{R}_{\beta \alpha} \partial_{\gamma}, \]

(29)

Substitute Equations (27), (28), and (29) into (26) we have

\[ \tau^2(\varphi) = g_{ij} (\tau_{ij} + \tau_{\alpha}^{\beta} \nabla_{\alpha \beta} \Gamma_{ij}^{\sigma} + \partial_{x^i} (\tau_{\alpha}^{\beta} \nabla_{\alpha \beta} \Gamma_{ij}^{\nu}) + \tau_{\alpha}^{\beta} \nabla_{\alpha \beta} \Gamma_{ij}^{\nu}) \partial_{\gamma}, \]

(30)

from which the lemma follows. \( \Box \)

When the domain manifold is a Euclidean space then we have

**Corollary 3.2.** Let \( \varphi : \mathbb{R}^m \rightarrow (N^n, h) \) with \( \varphi(x^1, \ldots, x^m) = (\varphi^1(x), \ldots, \varphi^n(x)) \) be a map from a Euclidean space into a Riemannian manifold. Then \( \varphi \) is biharmonic if and only if it is a solution of the following system of PDE’s

\[ \Delta \tau^\sigma + \left( \nabla \tau^\alpha, \nabla \varphi^\beta \right) \tilde{\Gamma}_{\alpha \beta}^{\sigma} + \left( \nabla \varphi^\beta, \nabla (\tau^\sigma \tilde{\Gamma}_{\alpha \beta}^{\sigma}) \right) \]

\[ + \left( \nabla \varphi^\beta, \nabla \varphi^\rho \right) \tau^\alpha \tilde{\Gamma}_{\alpha \beta \rho}^{\sigma} - \tau^\nu (\nabla \varphi^\alpha, \nabla \varphi^\beta) \tilde{R}_{\beta \alpha \nu}^{\sigma} = 0, \quad \sigma = 1, 2, \ldots, n. \]

**Linear biharmonic maps into Sol space.** Let \((\mathbb{R}^3, g_{\text{Sol}})\) denote Sol space, where the metric can be written as \( g_{\text{Sol}} = e^{2y^3} (dy^1)^2 + e^{-2y^3} (dy^2)^2 + (dy^3)^2 \) with respect to the standard coordinates \((y^1, y^2, y^3)\) in \(\mathbb{R}^3\). Then a direct computation gives the following components of Sol metric and the coefficients of the connection:

\[ g_{11} = e^{2y^3}, \quad g_{22} = e^{-2y^3}, \quad g_{33} = 1, \text{ all other } g_{ij} = 0; \]

\[ g^{11} = e^{-2y^3}, \quad g^{22} = e^{2y^3}, \quad g^{33} = 1, \text{ all other } g^{ij} = 0; \]
only if it is a harmonic map, which is equivalent to either
(i) $A^3 = 0$, $|A^1|^2 = |A^2|^2$, or (ii) $A^3 \neq 0$, $A^1 = A^2 = 0$.

By our convention of curvature operator and the following notation for the components of the Riemannian curvature

\[
\bar{R}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) \frac{\partial}{\partial y^k} = \bar{R}^l_{kij} \frac{\partial}{\partial y^l}
\]

we have

\[
\bar{R}^l_{kij} = \frac{\partial}{\partial y^i} \bar{R}^l_{kj} - \frac{\partial}{\partial y^j} \bar{R}^l_{ki} + \bar{R}^l_{it} \bar{R}^t_{kj} - \bar{R}^l_{jt} \bar{R}^t_{ki}
\]

A straightforward computation using (32) and (34) gives the following components of the Riemannian curvature of Sol space:

\[
\begin{align*}
\bar{R}^1_{221} &= -e^{-2y^3}, \quad \bar{R}^1_{431} = 1, \quad \bar{R}^1_{212} = e^{-2y^3}, \quad \bar{R}^1_{313} = -1; \\
\bar{R}^2_{121} &= e^{2y^3}, \quad \bar{R}^2_{312} = e^{-2y^3}, \quad \bar{R}^2_{332} = 1, \quad \bar{R}^2_{323} = -1; \\
\bar{R}^3_{131} &= -e^{-2y^3}, \quad \bar{R}^3_{232} = -e^{-2y^3}, \quad \bar{R}^3_{113} = e^{2y^3}, \quad \bar{R}^3_{223} = e^{-2y^3}.
\end{align*}
\]

Now we are ready to prove the following classification theorem for linear biharmonic maps into Sol space.

**Theorem 3.3.** Let $\varphi : \mathbb{R}^m \rightarrow (\mathbb{R}^3, g_{Sol})$ with

\[
\varphi(x) = \begin{pmatrix}
    a^1_1 & a^1_2 & \cdots & a^1_m \\
    a^2_1 & a^2_2 & \cdots & a^2_m \\
    a^3_1 & a^3_2 & \cdots & a^3_m
\end{pmatrix}
\begin{pmatrix}
    x^1 \\
    x^2 \\
    \vdots \\
    x^m
\end{pmatrix},
\]

i.e., $\varphi(X) = (A^1X^t, A^2X^t, A^3X^t)$ be a linear map into Sol space, where $A^t$ denotes the row vectors of the representation matrix. Then, $\varphi$ is a biharmonic map if and only if it is a harmonic map, which is equivalent to either (i) $A^3 = 0$, $|A^1|^2 = |A^2|^2$, or, (ii) $A^3 \neq 0$, $A^1 = A^2 = 0$. 
Proof. With respect to the standard Cartesian coordinates \((x^i)\) in \(\mathbb{R}^m\) and \((y^\alpha)\) in \(\mathbb{R}^3\), the tension field of \(\varphi\) is given by

\[
\tau(\varphi) := \text{Trace}_g(\nabla d\varphi) \in \Gamma(\varphi^{-1}TN)
\]

\[
= \tau^\sigma \frac{\partial}{\partial y^\sigma} \\
= g^{ij}(\varphi_i^j - \Gamma^k_{ij}\varphi^\alpha_k + \Gamma^\sigma_{\alpha i} \varphi^\alpha_j \frac{\partial}{\partial y^\sigma}) \\
= (\sum_{i=1}^{m} \Gamma^\sigma_{\alpha i} \varphi^\alpha_j \frac{\partial}{\partial y^\sigma}) \\
= \Gamma^\sigma_{\alpha i} A^\alpha \cdot A^j \frac{\partial}{\partial y^\sigma},
\]

where and in the sequel, \(A^\alpha \cdot A^j\) denotes the inner product and \(|A^\alpha|\) the norm of the vectors in Euclidean space.

Putting \(\tau(\varphi) = \tau^\sigma \frac{\partial}{\partial y^\sigma}\) and substituting (32) into Equation (36) we find the following components of the tension field of \(\varphi\)

\[
\tau^1 = 2A^1 \cdot A^3, \\
\tau^2 = -2A^2 \cdot A^3, \\
\tau^3 = |A^2|^2 e^{-2y^3} - |A^1|^2 e^{2y^3},
\]

where and in the sequel \(y^3 = A^3 X^t\).

A further computation gives

\[
(\Delta \tau^\sigma + \langle \nabla \tau^\alpha, \nabla (\varphi^\beta) \rangle \Gamma^\sigma_{\alpha \beta}) \frac{\partial}{\partial y^\sigma} = \\
(\Delta \tau^\sigma + A^\beta \cdot \nabla \tau^\alpha \Gamma^\sigma_{\alpha \beta}) \frac{\partial}{\partial y^\sigma} =
\]

\[
-2A^1 \cdot A^3(|A^2|^2 e^{-2y^3} + |A^1|^2 e^{2y^3}) \frac{\partial}{\partial y^1} \\
+ 2A^2 \cdot A^3(|A^2|^2 e^{-2y^3} + |A^1|^2 e^{2y^3}) \frac{\partial}{\partial y^2} \\
+ 4|A^3|^2(|A^2|^2 e^{-2y^3} - |A^1|^2 e^{2y^3}) \frac{\partial}{\partial y^3},
\]
\[
\langle \nabla \varphi^\beta, \nabla (\tau^\alpha \bar{\Gamma}^\sigma_{\alpha \beta}) \rangle \frac{\partial}{\partial y^\sigma} = (A^\beta \cdot \nabla \tau^\alpha \bar{\Gamma}^\sigma_{\alpha \beta} + \tau^\alpha A^\beta \cdot \nabla \bar{\Gamma}^\sigma_{\alpha \beta}) \frac{\partial}{\partial y^\sigma} \\
= -2A^1 \cdot A^3 (|A^2|^2 e^{-2y^3} + |A^1|^2 e^{2y^3}) \frac{\partial}{\partial y^1} + 2A^2 \cdot A^3 (|A^2|^2 e^{-2y^3} + |A^1|^2 e^{2y^3}) \frac{\partial}{\partial y^2} + [-4(A^1 \cdot A^3)^2 e^{2y^3} + 4(A^2 \cdot A^3)^2 e^{-2y^3}] \frac{\partial}{\partial y^3},
\]

(39)

\[
\tau^\nu \langle \nabla \varphi^\alpha, \nabla \varphi^\beta \rangle \bar{R}^\alpha_{\beta \alpha \nu} \frac{\partial}{\partial y^\nu} = A^\beta \cdot A^\alpha \tau^\nu \bar{R}^\sigma_{\alpha \beta \nu} \frac{\partial}{\partial y^\sigma} \\
= [-3A^1 \cdot A^3 |A^2|^2 e^{-2y^3} - 2A^2 \cdot A^3 A^1 \cdot A^2 e^{-2y^3} + 2A^1 \cdot A^3 |A^3|^2 e^{2y^3} + 2A^2 \cdot A^3 A^1 \cdot A^2 e^{2y^3} + 3A^2 \cdot A^3 |A^2|^2 e^{-2y^3} - 2A^2 \cdot A^3 A^1 \cdot A^2 e^{2y^3} + 2A^2 \cdot A^3 A^1 \cdot A^2 e^{-2y^3} - 2A^1 \cdot A^3 |A^3|^2 e^{2y^3} + 2A^2 \cdot A^3 A^1 \cdot A^2 e^{-2y^3} - 2A^1 \cdot A^3 |A^3|^2 e^{-2y^3} + 2A^2 \cdot A^3 A^1 \cdot A^2 e^{-2y^3} - 2A^1 \cdot A^3 |A^3|^2 e^{-2y^3}]
\]

(40)

and

It follows from Equations (38), (39), (40), (41) and Corollary 3.2 that the linear map \( \varphi \) is a biharmonic map if and only if

\[
\left\{ \begin{array}{l}
-8A^1 \cdot A^3 |A^1|^2 e^{-2y^3} = 0 \\
8A^2 \cdot A^3 |A^2|^2 e^{-2y^3} = 0 \\
4(|A^2|^2 A^3)^2 (A^2 \cdot A^3)^2 e^{-2y^3} - 4(|A^1|^2 A^3)^2 (A^1 \cdot A^3)^2 e^{2y^3} + 2A^1 |A^3|^4 e^{2y^3} - 2A^2 |A^3|^4 e^{-2y^3} = 0
\end{array} \right.
\]

(42)

Solving System of equations (42) we have either (i) \( A^3 = 0 \), \( |A^1|^2 = |A^2|^2 \), or (ii) \( A^3 \neq 0 \), \( A^1 = A^2 = 0 \). It follows from Equation (57) that in both cases the tension field vanishes identically, i.e., \( \varphi \) is also harmonic. Therefore, we obtain the theorem.

**Linear biharmonic maps into Nil space.** Let \((\mathbb{R}^3, g_{\text{Nil}})\) denote Nil space, where the metric with respect to the standard coordinates \((y^1, y^2, y^3)\) in \(\mathbb{R}^3\) can
be written as \( g_{Nil} = (dy)^2 + (dy^2)^2 + (dy - y^1 dy^2)^2 \). Then an easy computation gives the following components of Nil metric and the coefficients of its connection:

\[
\begin{align*}
g_{11} &= 1, \ g_{12} = g_{13} = 0, \ g_{22} = 1 + (y^1)^2, \ g_{23} = -y^1, \ g_{33} = 1; \\
g^{11} &= 1, \ g^{12} = g^{13} = 0, \ g^{22} = 1, \ g^{23} = y^1, \ g^{33} = 1 + (y^1)^2;
\end{align*}
\]

\[
\begin{align*}
\Gamma^1_{11} &= \Gamma^2_{11} = 0, \ \Gamma^3_{11} = 0; \\
\Gamma^1_{12} &= 0, \ \Gamma^2_{12} = y^1, \ \Gamma^3_{12} = \frac{(y^1)^2 - 1}{2}; \\
\Gamma^1_{13} &= 0, \ \Gamma^2_{13} = -\frac{1}{2}, \ \Gamma^3_{13} = -\frac{y^1}{2}; \\
\Gamma^1_{21} &= 0, \ \Gamma^2_{21} = y^1, \ \Gamma^3_{21} = \frac{(y^1)^2 - 1}{2}; \\
\Gamma^1_{22} &= -y^1, \ \Gamma^2_{22} = 0, \ \Gamma^3_{22} = 0; \\
\Gamma^1_{23} &= \frac{1}{2}, \ \Gamma^2_{23} = 0, \ \Gamma^3_{23} = 0; \\
\Gamma^1_{31} &= 0, \ \Gamma^2_{31} = -\frac{1}{2}, \ \Gamma^3_{31} = -\frac{y^1}{2}; \\
\Gamma^1_{32} &= \frac{1}{2}, \ \Gamma^2_{32} = 0, \ \Gamma^3_{32} = 0; \\
\Gamma^1_{33} &= 0, \ \Gamma^2_{33} = 0, \ \Gamma^3_{33} = 0.
\end{align*}
\]

(43)

A further computation using (34) and (13) gives the following components of the Riemannian curvature of Nil space:

\[
\begin{align*}
\bar{R}^1_{212} &= -\frac{3}{4} + \frac{(y^1)^2}{4}, \ \bar{R}^1_{213} = -y^1, \ \bar{R}^1_{221} = -\frac{3}{4} - \frac{(y^1)^2}{4}, \ \bar{R}^1_{312} = -y^1, \\
\bar{R}^1_{231} &= y^1, \ \bar{R}^1_{313} = \frac{1}{4}, \ \bar{R}^3_{211} = \frac{y^1}{4}, \ \bar{R}^3_{311} = -\frac{1}{4}; \\
\bar{R}^2_{112} &= \frac{3}{4}, \ \bar{R}^2_{121} = -\frac{3}{4}, \ \bar{R}^2_{223} = -y^1, \\
\bar{R}^2_{132} &= \frac{y^1}{4}, \ \bar{R}^2_{323} = \frac{1}{4}, \ \bar{R}^3_{322} = -\frac{1}{4}; \\
\bar{R}^3_{112} &= y^1, \ \bar{R}^3_{113} = -\frac{1}{4}, \ \bar{R}^3_{121} = -y^1, \ \bar{R}^3_{131} = \frac{1}{4}; \\
\bar{R}^3_{232} &= -\frac{(y^1)^2 + 1}{4}, \ \bar{R}^3_{232} = \frac{(y^1)^2 + 1}{4}, \ \bar{R}^3_{323} = \frac{y^1}{4}, \ \bar{R}^3_{332} = -\frac{1}{4}.
\end{align*}
\]

(44)

**Theorem 3.4.** Let \( \varphi : \mathbb{R}^m \rightarrow (\mathbb{R}^3, g_{Nil}) \) with

\[
\varphi(x) = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^m \\ a_2^1 & a_2^2 & \cdots & a_2^m \\ \vdots & \vdots & \ddots & \vdots \\ a_m^1 & a_m^2 & \cdots & a_m^m \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \end{pmatrix},
\]

i.e., \( \varphi(X) = (A^1 X^t, A^2 X^t, A^3 X^t) \) be a linear map into Nil space, where \( A^i \) denotes the row vectors of the representation matrix. Then, \( \varphi \) is a biharmonic map if and only if it is a harmonic map, which is equivalent to either (i) \( A^1 = 0 \), \( A^2 \cdot A^3 = 0 \), or, (ii) \( A^1 = 0 \), \( A^2 = 0 \), or, (iii) \( A^2 = 0 \), \( A^1 \cdot A^3 = 0 \).

**Proof.** Taking the standard Cartesian coordinates \( (x^i) \) in \( \mathbb{R}^m \), \( (y^a) \) in \( \mathbb{R}^3 \) and substituting (13) into (36) we find the tension field of \( \varphi \) to be
\[
\tau(\varphi) = \tau^\sigma \frac{\partial}{\partial y^\sigma}
\]
\[
= [-|A^2|^2 y^1 + A^2 \cdot A^3] \frac{\partial}{\partial y^1} + [A^1 \cdot A^2 y^1 - A^1 \cdot A^3] \frac{\partial}{\partial y^2}
\]
\[
+ [A^1 \cdot A^2 (y^1)^2 - A^1 \cdot A^3 y^1 - A^1 \cdot A^2] \frac{\partial}{\partial y^3},
\]

or,

\[
\tau^1 = -|A^2|^2 y^1 + A^2 \cdot A^3,
\]
\[
\tau^2 = A^1 \cdot A^2 y^1 - A^1 \cdot A^3
\]
\[
\tau^3 = A^1 \cdot A^2 (y^1)^2 - A^1 \cdot A^3 y^1 - A^1 \cdot A^2,
\]

where and in the sequel \(y^1 = A^1 X^1\).

A straightforward computation yields

\[
\Delta \tau^\sigma \frac{\partial}{\partial y^\sigma} = 2 A^1 \cdot A^2 |A^1|^2 \frac{\partial}{\partial y^1},
\]

\[
\langle \nabla \tau^\alpha, \nabla \varphi^\beta \rangle \bar{\Gamma}_{\alpha \beta}^{\gamma} \frac{\partial}{\partial y^\gamma} = A^\beta \cdot \nabla \tau^\alpha \bar{\Gamma}_{\alpha \beta}^{\gamma} \frac{\partial}{\partial y^\gamma}
\]
\[
= [-\frac{1}{2} A^1 \cdot A^2 (|A^1|^2 + |A^2|^2) y^1 + \frac{1}{2} A^1 \cdot A^3 (|A^1|^2 + |A^2|^2)] \frac{\partial}{\partial y^1}
\]
\[
+ [-\frac{1}{2} A^1 \cdot A^2 (|A^1|^2 + |A^2|^2) (y^1)^2 + \frac{1}{2} A^1 \cdot A^3 (|A^1|^2 + |A^2|^2) y^1
\]
\[
+ \frac{1}{2} A^1 \cdot A^2 (|A^2|^2 - |A^1|^2)] \frac{\partial}{\partial y^2},
\]

\[
\langle \nabla \varphi^\beta, \nabla (\tau^\alpha \bar{\Gamma}_{\alpha \beta}^{\gamma}) \rangle \frac{\partial}{\partial y^\gamma}
\]
\[
= (A^\beta \cdot \nabla \tau^\alpha \bar{\Gamma}_{\alpha \beta}^{\gamma} + \tau^\alpha A^\beta \cdot \nabla \bar{\Gamma}_{\alpha \beta}^{\gamma}) \frac{\partial}{\partial y^\gamma}
\]
\[
= [-(-A^1 \cdot A^2)^2 y^1 + (A^1 \cdot A^2)(A^1 \cdot A^3)] \frac{\partial}{\partial y^1}
\]
\[
+ [-\frac{1}{2} A^1 \cdot A^2 |A^2|^2 (y^1)^2 + \frac{1}{2} (A^1 \cdot A^2)(A^2 \cdot A^3) + \frac{1}{2} A^1 \cdot A^3 |A^2|^2] \frac{\partial}{\partial y^2}
\]
\[
+ [\frac{1}{2} A^1 \cdot A^2 |A^2|^2 y^1 + \frac{1}{2} A^1 \cdot A^3 |A^2|^2 - \frac{1}{2} (A^1 \cdot A^3)(A^2 \cdot A^3)] \frac{\partial}{\partial y^3},
\]
By Equations (47), (48), (49), (50), (51) and Corollary 3.2 we conclude that \( \varphi \) is biharmonic if and only if

\[
- (A^1 \cdot A^2)^2 y^1 + |A^2|^4 y^1 \\
+ (A^1 \cdot A^2)(A^1 \cdot A^3) - A^2 \cdot A^3 |A^2|^2 = 0,
\]
and
\begin{equation}
- A^1 \cdot A^2 |A^1|^2 (y^1)^2 - 3 A^1 \cdot A^2 |A^2|^2 (y^1)^2 + A^1 \cdot A^3 |A^1|^2 y^1
\end{equation}
\begin{align*}
+ A^1 \cdot A^3 |A^2|^2 y^1 + 2(A^1 \cdot A^2)(A^2 \cdot A^3)y^1
\end{align*}
\begin{equation}
+ A^1 \cdot A^2 |A^1|^2 + A^1 \cdot A^2 |A^2|^2 = 0.
\end{equation}

To solve the System of biharmonic map equations (52), (53), and (54) we consider the following two cases.

Case I: $A^1 = 0$. Noting that $y^1 = A^1 X^t = 0$ we use Equation (52) to have
\begin{equation*}
- A^2 \cdot A^3 |A^2|^2 = 0,
\end{equation*}
which implies either $A^2 \cdot A^3 = 0$, or $|A^2|^2 = 0$ i.e. $A^2 = 0$.
Substituting $A^1 = 0$, $A^2 \cdot A^3 = 0$ or $A^1 = 0$, $A^2 = 0$ into (53) and (54) we see that they are both solutions of biharmonic map equations.

Case II: $A^1 \neq 0$. Note that in this case, $y^1 = A^1 X^t \neq 0$. We can view system of biharmonic map equations (52), (53), and (54) as a system of polynomial equations in $y^1$. It is not difficult to check that $A^2 = 0$, $A^1 \cdot A^3 = 0$ is the only solution in this case.

Combining Case I and II we obtain the last statement of the theorem. A direct checking using the tension field equation (15) we see that all these maps are also harmonic maps. Thus, we complete the proof of the theorem. \hfill \square

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