Lipschitz neural networks are dense in the set of all Lipschitz functions

Stephan Eckstein∗

September 30, 2020

Abstract

This note shows that, for a fixed Lipschitz constant $L > 0$, one layer neural networks that are $L$-Lipschitz are dense in the set of all $L$-Lipschitz functions with respect to the uniform norm on bounded sets.

Keywords: Feedforward neural networks, universal approximation theorem, Lipschitz continuity

1 Introduction and main result

Let $d \in \mathbb{N}$, $K \subset \mathbb{R}^d$ be bounded and $L > 0$. We fix a norm $\| \cdot \|$ on $\mathbb{R}^d$ and for a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we recall the uniform norm on $K$ given by $\|f\|_{\infty,K} = \sup_{x \in K} |f(x)|$. Let Lip$_{L,K}$ be the set of all functions mapping from $\mathbb{R}^d$ to $\mathbb{R}$ that are $L$-Lipschitz on $K$, i.e., all functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in K$.

We further fix an activation function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and define the set $N^m$ of all one layer neural networks with layer-width $m \in \mathbb{N}$ mapping $\mathbb{R}^d$ to $\mathbb{R}$, i.e., $f_m \in N^m$ can be written as

$$f_m(x) = b + \sum_{i=1}^{m} a_i \varphi\left(\sum_{j=1}^{d} w_{i,j} x_j + c_i\right)$$

for all $x \in \mathbb{R}^d$,

where $b, a_1, \ldots, a_m, w_{1,1}, \ldots, w_{m,d}, c_1, \ldots, c_m \in \mathbb{R}$ are the parameters of the network $f_m$.

Approximation properties of the set $N^m$ are well studied (see, e.g., [8, 15]). In this note however, we study approximation properties of the set Lip$_{L,K}^m$ := Lip$_{L,K}$ $\cap N^m$. We consider the question of approximating functions in Lip$_{L,K}$ by networks in Lip$_{L,K}^m$. Related questions were studied in [1, 9] and working with neural networks under a Lipschitz constraint occurs in many problems related to Wasserstein distances (see, e.g., [2, 13]) and regularization and adversarial robustness (see, e.g., [4, 6, 16]). Even though in practice, enforcing a

∗Department of Mathematics, University of Konstanz, Universitätsstrasse 10, 78464 Konstanz, Germany, stephan.eckstein@uni-konstanz.de
Lipschitz constraint for neural networks has to rely on either penalization methods (see, e.g., [7, 14]) or special architectures or weight restrictions (see, e.g., [1, 2, 12]), the set $\text{Lip}_{L,K}^m$ can be regarded as an idealized version of working with neural networks under a Lipschitz constraint. This note shows, under mild assumptions on the activation function, that the addition of a Lipschitz constraint does not inhibit the expressiveness of neural networks. The main result is the following:

**Theorem 1.** Let $\varphi$ be one time continuously differentiable and not polynomial, or let $\varphi$ be the ReLU. Then it holds:

For any $\varepsilon > 0$, there exists some $m = m(\varepsilon) \in \mathbb{N}$ so that

$$\sup_{f \in \text{Lip}_{L,K}} \inf_{f_m \in \text{Lip}_{L,K}^m} \| f - f_m \|_{\infty,K} \leq \varepsilon.$$  

The proof of Theorem 1 relies on existing work on neural network approximations of functions and their derivatives. The references are [15] for the case of a continuously differentiable activation functions, and [10] for the ReLU. Instead of the ReLU, other weakly differentiable activation functions could be considered which satisfy the assumptions of [10] Theorem 4.1, 4.2 or 4.3.

The usual methods apply when transitioning from shallow networks (with one hidden layer) to many-layer networks. The result still holds, since the later layers can approximate the identity function under a Lipschitz constraint up to arbitrary accuracy.

### 2 Proof of Theorem 1

For the proof of Theorem 1, we will first show in Subsection 2.2 that a simpler statement holds, where the size $m = m(\varepsilon, f)$ of the network may depend on the Lipschitz function $f \in \text{Lip}_{L,K}$ to be approximated. The general case is a simple consequence and is shown in Subsection 2.3. First, we state simplifications which will be used in the first part of the proof.

#### 2.1 Scaling and simplifications

We only show the statements for $L = 1$. This may be done since neural networks can be multiplied by a constant. Thus, instead of approximating $f \in \text{Lip}_{L,K}$ up to accuracy $\varepsilon$, one may approximate the function $\frac{f}{L}$ up to accuracy $\frac{\varepsilon}{L}$ and then scale the approximating networks by the factor $L$.

Analogously to the Lipschitz constant, we assume that the considered norms are normalized to $\max_{x \in [0,1]^d} \|x\| = 1$, which means in particular that $\|x\|_1 = \frac{1}{d} \sum_{i=1}^d |x_i|.$

We also assume that any function to be approximated is normalized to $f(0) = 0$. This is not a restriction, since neural networks can be shifted by constants, and hence one can first approximate the function $f - f(0)$ and then shift the neural network by the constant $f(0)$.
Further, we assume without loss of generality that a function $f \in \text{Lip}_{1,K}$ is 1-Lipschitz on the whole domain $\mathbb{R}^d$ and bounded. Formally, for $f \in \text{Lip}_{1,K}$, by \cite[Theorem 1]{11}, there exists a function $\tilde{f} \in \text{Lip}_{1,\mathbb{R}^d}$ with $\tilde{f}(x) = f(x)$ for all $x \in K$ and $\sup_{x \in \mathbb{R}^d} |\tilde{f}(x)| = \sup_{x \in K} |f(x)|$. Since for the statement of Theorem \cite{11} only the values of $f$ on $K$ are of interest, one can replace $f$ by $\tilde{f}$ and approximate $\tilde{f}$ instead.

Finally, we work with $K = (0,1)^d$ which can be done without loss of generality, the reason being as follows: Suppose the statements hold for $K = (0,1)^d$ and we want to prove them for general $K$: Take $l_i := \inf \{x_i : x \in K\}$ and $u_i := \sup \{x_i : x \in K\}$ for $i = 1, \ldots, d$ and set $l = (l_1, \ldots, l_d)$ and $M := \max \{u_i - l_i : i \in \{1, \ldots, d\}\}$. Take any $\tilde{f} \in \text{Lip}_{1,K}$ and $\varepsilon > 0$ and define

$$f(x) := \frac{\tilde{f}(Mx - l)}{M} \quad \text{for } x \in \mathbb{R}^d.$$ 

Then $f \in \text{Lip}_{1,K}$ (where we already used that $\tilde{f}$ is assumed to be 1-Lipschitz on $\mathbb{R}^d$). By approximating $f$ by a function $f_m \in \text{Lip}_{1,K}^m$ on $K$ up to accuracy $\varepsilon/M$ and setting $\tilde{f}_m(x) := Mf_m((x + l)/M)$, we get $\tilde{f}_m \in \text{Lip}_{1,K}^m$ and the desired approximation of $\tilde{f}$ by $\tilde{f}_m$.

### 2.2 Proof of Theorem \cite{11}: first part

Fix $f \in \text{Lip}_{1,K}$ and $\varepsilon > 0$. We will show that there exists some $m \in \mathbb{N}$ and $f_m \in \text{Lip}_{1,K}^m$ such that $\|f - f_m\|_{\infty,K} \leq \varepsilon$.

Define $\hat{f} := (1 - \varepsilon/2)f$ and note $\sup_{x \in K} |\hat{f}(x) - f(x)| \leq \varepsilon/2$ (where we used the normalization of $\|\cdot\|$) and w.l.o.g. $\hat{f} \in \text{Lip}_{1-\varepsilon/2,\mathbb{R}^d}$. By \cite[Theorem 1]{3} there is a smooth (i.e., $C^\infty$) function $\tilde{f} \in \text{Lip}_{1-\varepsilon/4,\mathbb{R}^d}$ that satisfies $\|\tilde{f} - \hat{f}\|_{\infty} < \varepsilon/4$. Hence also $\|\hat{f} - f\|_{\infty,K} \leq 3\varepsilon/4$.

We next approximate $\tilde{f}$ and its first partial derivatives by a function $f_m \in \mathbb{N}^m$. The desired accuracy depends on the norm $\|\cdot\|$. Since all norms on $\mathbb{R}^d$ are equivalent, we can find a constant $C > 0$ such that $\|\cdot\|_1 \leq C\|\cdot\|$. Set $\delta := \min\{\varepsilon/4, \varepsilon/(4dC)\}$ and find a function $f_m \in \mathbb{N}^m$ which satisfies

$$\left\|\frac{\partial f_m}{\partial x_i} - \frac{\partial \tilde{f}}{\partial x_i}\right\|_{\infty,K} \leq \delta \quad \text{for all } i \in \{1, \ldots, d\},$$

$$\|f_m - \tilde{f}\|_{\infty,K} \leq \delta.$$  

(2.1)

(2.2)

This can be done by \cite[Theorem 4.1]{15} for the case of a continuously differentiable activation function, and by \cite[Theorem 4.3]{10} for the case of the ReLU \cite{8}.

It then holds

$$\|f_m - f\|_{\infty,K} \leq \|f_m - \tilde{f}\|_{\infty,K} + \|\tilde{f} - f\|_{\infty,K} \leq \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} = \varepsilon.$$  

\footnote{In case of the ReLU, $\frac{\partial f_m}{\partial x}$ is understood in the weak sense. To apply \cite[Theorem 4.3]{10} to the ReLU, note that $G(x) = \max\{0, x\} - 2\max\{0, x + 1\} + \max\{0, x + 2\}$ gives the desired linear combination of scaled shifted rotations of the ReLU.}
It remains to show that \( f_m \in \text{Lip}_{1,K} \). First, we consider the case where the activation function is continuously differentiable and hence so is \( f_m \). We use Lemma 2 in the appendix and show that \( f_m \) satisfies part (i) of the lemma. For \( x \in K, v \in \mathbb{R}^d \) it holds

\[
|Df_m(x) \cdot v| \leq |Df_m(x) \cdot v - D\tilde{f}(x) \cdot v| + |D\tilde{f}(x) \cdot v|
\]

\[
\leq \left| \sum_{i=1}^{d} \left( \frac{\partial f_m}{\partial x_i}(x) - \frac{\partial \tilde{f}}{\partial x_i}(x) \right) v_i \right| + (1 - \frac{\varepsilon}{4}) \|v\|
\]

\[
\leq \sup_{\hat{x} \in K, i \in \{1, \ldots, d\}} \left| \frac{\partial f_m}{\partial x_i}(\hat{x}) - \frac{\partial \tilde{f}}{\partial x_i}(\hat{x}) \right| \left\|v\right\|_1 + (1 - \frac{\varepsilon}{4}) \|v\|
\]

\[
\leq \delta \|D\tilde{f}\| + (1 - \frac{\varepsilon}{4}) \|v\|
\]

where we used Lemma 2 for \( \tilde{f} \). This shows \( f_m \in \text{Lip}_{1,K} \).

We now consider the case of the ReLU. We choose a standard mollifier \( \eta_\kappa \) for \( \kappa > 0 \). We define \( f_{m,\kappa} := f_m * \eta_\kappa \) and \( \tilde{f}_\kappa := \tilde{f} * \eta_\kappa \). Note that there exists \( \lambda(\kappa) > 0 \) with \( \lambda(\kappa) \to 0 \) for \( \kappa \to 0 \) such that \( \sup_{i \in \{1, \ldots, d\}} \sup_{x \in K} |\frac{\partial \tilde{f}_\kappa}{\partial x_i}(x) - \frac{\partial \tilde{f}}{\partial x_i}(x)| \leq \lambda(\kappa) \) and \( \|f_m - f_{m,\kappa}\|_{\infty,K} \leq \lambda(\kappa) \). Further, we note that for \( i \in \{1, \ldots, d\} \) and \( x \in K \) it holds

\[
\left| \frac{\partial \tilde{f}_\kappa}{\partial x_i}(x) - \frac{\partial f_{m,\kappa}}{\partial x_i}(x) \right| = \left| \int_{B(0,\kappa)} \left( \frac{\partial \tilde{f}}{\partial x_i}(x-y) - \frac{\partial f_m}{\partial x_i}(x-y) \right) \eta_\kappa(y) \, dy \right|
\]

\[
\leq \sup_{\hat{x} \in (-\kappa,1+\kappa)^d} \left| \frac{\partial \tilde{f}}{\partial x_i}(\hat{x}) - \frac{\partial f_m}{\partial x_i}(\hat{x}) \right|.
\]

In the following, we will assume w.l.o.g. that \( \sup_{\hat{x} \in (-\kappa,1+\kappa)^d} \left| \frac{\partial \tilde{f}}{\partial x_i}(\hat{x}) - \frac{\partial f_m}{\partial x_i}(\hat{x}) \right| \leq \delta \) holds for all \( \kappa < 1 \). The reason we can make this assumption without loss of generality is that the approximations in Equation (2.1) may be taken for \( K = (-1,3)^d \), since \( f \) (and hence \( \tilde{f} \)) can be assumed to be Lipschitz on \( \mathbb{R}^d \) as argued in Subsection 2.1. It then holds for \( x \in K, v \in \mathbb{R}^d \),

\[
|Df_{m,\kappa}(x) \cdot v|
\]

\[
\leq |Df_m(x) \cdot v - Df_{m,\kappa}(x) \cdot v| + |Df_m(x) \cdot v - D\tilde{f}(x) \cdot v| + |D\tilde{f}(x) \cdot v|
\]

\[
\leq d \|v\|_1 \left( \sup_{i \in \{1, \ldots, d\}} \sup_{\hat{x} \in K} \left| \frac{\partial \tilde{f}_\kappa}{\partial x_i}(\hat{x}) - \frac{\partial f_{m,\kappa}}{\partial x_i}(\hat{x}) \right| + \sup_{i \in \{1, \ldots, d\}} \sup_{\hat{x} \in K} \left| \frac{\partial \tilde{f}}{\partial x_i}(\hat{x}) - \frac{\partial f_m}{\partial x_i}(\hat{x}) \right| \right) + (1 - \frac{\varepsilon}{4}) \|v\|
\]

\[
\leq d C \|v\| \left( \sup_{i \in \{1, \ldots, d\}} \sup_{\hat{x} \in (-\kappa,1+\kappa)^d} \left| \frac{\partial \tilde{f}}{\partial x_i}(\hat{x}) - \frac{\partial f_m}{\partial x_i}(\hat{x}) + \lambda(\kappa) \right| + (1 - \frac{\varepsilon}{4}) \|v\|
\]

\[
\leq d C \|v\| \delta + d C \|v\| \lambda(\kappa) + (1 - \frac{\varepsilon}{4}) \|v\|
\]

\[
\leq (1 + d C \lambda(\kappa)) \|v\|
\]

\footnote{See, e.g., [3]. The mollifier is taken w.r.t. the euclidean norm, and \( B(x, \kappa) \) denotes the open ball around \( x \) of radius \( \kappa \) w.r.t. the euclidean norm.}
and hence \( f_{m,\kappa} \) is \( (1 + dC \lambda(\kappa)) \)-Lipschitz on \( K \) according to Lemma 2. Thus, for all \( x, y \in K \), we have
\[
|f_m(x) - f_m(y)| \leq |f_m(x) - f_{m,\kappa}(x) + f_{m,\kappa}(x) - f_{m,\kappa}(y) + f_{m,\kappa}(y) - f_m(y)| \\
\leq 2\lambda(\kappa) + (1 + dC \lambda(\kappa))\|x - y\|,
\]
and taking \( \kappa \to 0 \) yields \( f_m \in \text{Lip}_{1,K} \). The first part of the proof is complete.

### 2.3 Proof of Theorem 1: second part

We prove that the size \( m \) of the networks may be chosen only depending on \( \varepsilon \), but independently of \( f \). We still assume that any Lipschitz function satisfies \( f(0) = 0 \), since shifting neural network functions by constants does not affect their size. We choose some compact set \( \hat{K} \subset \mathbb{R}^d \) with \( K \subset \hat{K} \). We set \( \mathcal{F} := \{ g : \hat{K} \to \mathbb{R} : g(0) = 0 \text{ and } g \text{ is } L\text{-Lipschitz} \} \). Since \( \mathcal{F} \) is bounded, convex, closed and equicontinuous, by the Arzelà-Ascoli theorem \( \mathcal{F} \) is compact with respect to the uniform norm.

Hence, for a given \( \varepsilon > 0 \) we can find \( g_1, ..., g_n \in \mathcal{F} \) such that
\[
\sup_{g \in \mathcal{F}} \inf_{i \in \{1, ..., n\}} \|g - g_i\|_{\infty, \hat{K}} \leq \frac{\varepsilon}{2}.
\]
We can approximate \( g_1, ..., g_n \) (respectively their extensions to the whole domain \( \mathbb{R}^d \)) as in Subsection 2.2 up to accuracy \( \frac{\varepsilon}{2} \) by functions \( g_1^{m_1}, ..., g_n^{m_n} \) and set \( m := \max\{m_i : i \in \{1, ..., n\}\} \) so that \( g_i^{m_i} \in \text{Lip}_{m_i,K} \) for all \( i = 1, ..., n \). Then, for any \( f \in \text{Lip}_{L,K} \), choose an extension \( \tilde{f} \in \text{Lip}_{L,\mathbb{R}^d} \) by [11, Theorem 1] and set \( g := \tilde{f}_{|\hat{K}} \in \mathcal{F} \) to be the restriction of \( \tilde{f} \) to \( \hat{K} \). Choose \( i \in \{1, ..., n\} \) such that \( \|g_i - g\|_{\infty, \hat{K}} \leq \frac{\varepsilon}{2} \) and obtain the desired approximation of \( f \) by \( g_i^{m_i} \). The proof of Theorem 1 is complete.

### A Lipschitz continuity and directional derivatives

The following lemma is a slight simplification of [5, Section 5.8, Theorem 4].

**Lemma 2.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be continuously differentiable, fix \( L > 0 \) and assume that \( K \) is open and convex. Then the following are equivalent:

(i) For all \( x \in K \) and \( v \in \mathbb{R}^d \) it holds \( |Df(x) \cdot v| \leq L\|v\| \)

(ii) \( f \in \text{Lip}_{L,K} \)

**Proof.** Assume (i) holds and take \( x, y \in K \). Then it holds
\[
|f(x) - f(y)| = \left| \int_0^1 Df(tx + (1-t)y) \cdot (x-y) \, dt \right| \leq \int_0^1 |Df(tx + (1-t)y) \cdot (x-y)| \, dt \leq L\|x - y\|
\]
since by convexity \( tx + (1-t)y \in K \) for all \( t \in (0,1) \). Thus \( f \in \text{Lip}_{L,K} \).
Conversely, assume (ii) holds. For $x \in K, v \in \mathbb{R}^d$ it holds

$$|Df(x) \cdot v| = \left| \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} \right| \leq \lim_{h \to 0} \frac{L\|hv\|}{h} = L\|v\|$$

since $x + hv \in K$ for $h$ small enough since $K$ is open. This shows (i). \hfill \square

Acknowledgments

The author thanks Jonas Blessing, Luca De Gemmaro Aquino, Marlene Koch and Michael Kupper for helpful discussions and remarks on an earlier version of this note.

References

[1] C. Anil, J. Lucas, and R. Grosse. Sorting out lipschitz function approximation. In *International Conference on Machine Learning*, pages 291–301, 2019.

[2] M. Arjovsky, S. Chintala, and L. Bottou. Wasserstein generative adversarial networks. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 214–223, 2017.

[3] D. Azagra, J. Ferrera, F. López-Mesas, and Y. Rangel. Smooth approximation of lipschitz functions on riemannian manifolds. *Journal of Mathematical Analysis and Applications*, 326(2):1370–1378, 2007.

[4] M. Cisse, P. Bojanowski, E. Grave, Y. Dauphin, and N. Usunier. Parseval networks: improving robustness to adversarial examples. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 854–863, 2017.

[5] L. C. Evans. Partial differential equations, ams. *Graduate Studies in Mathematics*, 19, 2002.

[6] H. Gouk, E. Frank, B. Pfahringer, and M. Cree. Regularisation of neural networks by enforcing lipschitz continuity. *arXiv preprint arXiv:1804.04368*, 2018.

[7] I. Gulrajani, F. Ahmed, M. Arjovsky, V. Dumoulin, and A. C. Courville. Improved training of wasserstein gans. In *Advances in neural information processing systems*, pages 5767–5777, 2017.

[8] K. Hornik. Approximation capabilities of multilayer feedforward networks. *Neural networks*, 4(2):251–257, 1991.

[9] T. Huster, C.-Y. J. Chiang, and R. Chadha. Limitations of the lipschitz constant as a defense against adversarial examples. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 16–29. Springer, 2018.

[10] Y. Ito. Differentiable approximation by means of the radon transformation and its applications to neural networks. *Journal of computational and applied mathematics*, 55(1):31–50, 1994.

[11] G. J. Minty. On the extension of lipschitz, lipschitz-hölder continuous, and monotone functions. *Bulletin of the American Mathematical Society*, 76(2):334–339, 1970.

[12] T. Miyato, T. Kataoka, M. Koyama, and Y. Yoshida. Spectral normalization for generative adversarial networks. In *International Conference on Learning Representations*, 2018.
[13] S. Ozair, C. Lynch, Y. Bengio, A. Van den Oord, S. Levine, and P. Sermanet. Wasserstein dependency measure for representation learning. In Advances in Neural Information Processing Systems, pages 15604–15614, 2019.

[14] H. Petzka, A. Fischer, and D. Lukovnikov. On the regularization of wasserstein gans. In International Conference on Learning Representations, 2018.

[15] A. Pinkus. Approximation theory of the mlp model in neural networks. Acta numerica, 8(1):143–195, 1999.

[16] Y. Tsuzuku, I. Sato, and M. Sugiyama. Lipschitz-margin training: Scalable certification of perturbation invariance for deep neural networks. In Advances in neural information processing systems, pages 6541–6550, 2018.