Arithmetic progressions in self-similar sets

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Abstract Given a sequence \( \{b_i\}_{i=1}^n \) and a ratio \( \lambda \in (0, 1) \), let \( E = \bigcup_{i=1}^n (\lambda E + b_i) \) be a homogeneous self-similar set. In this paper, we study the existence and maximal length of arithmetic progressions in \( E \). Our main idea is from the multiple \( \beta \)-expansions.

Keywords Self-similar sets, arithmetic progression (AP), \( \beta \)-expansions

MSC 28A80, 28A78, 11A63, 11B25

1 Introduction

An arithmetic progression (AP) in \( \mathbb{R} \) is of the form

\[
P = \{a, a+\delta, \ldots, a+(k-1)\delta\}
\]

for some \( a \in \mathbb{R} \), \( \delta \in \mathbb{R}^+ \), and \( k \in \mathbb{N}^+ \). We say that \( P \) is an arithmetic progression with length \( k \). Finding an arithmetic progression in a subset of \( \mathbb{R} \) is a focus in combinatorial number theory and ergodic theory. Under some conditions, the existence of an arithmetic progression of a set inspires many scholars to investigate. In the setting of discrete case, Erdős and Turán [9] conjectured a subset of natural numbers with positive density necessarily contains arbitrarily long arithmetic progressions. This conjecture was solved by Roth [19] if the length of the arithmetic progression is 3. Szemerédi [22], using the purely combinatorial methods, proved that the Erdős-Turán conjecture holds if the length of the arithmetic progression is 4. Szemerédi [23] extended Roth’s theorem to arbitrarily long arithmetic progressions, and completely addressed the Erdős-Turán conjecture. From then on, many different proofs of Szemerédi’s theorem were found. For instance, Furstenberg [12] proved that Szemerédi’s theorem is equivalent to the multiple recurrence theorem in ergodic theory, and gave a proof of the multiple recurrence theorem. Therefore, he obtained a new proof of Szemerédi’s theorem, for a survey of this topic, see [24]. It is natural to investigate a subset of natural numbers which is of zero density. In this case,
such set may still contain arbitrarily long arithmetic progressions. For instance, Green and Tao [14] proved their celebrated Green-Tao theorem for the primes.

In the setting of continuous case, it follows from Lebesgue's theorem on density points that any set $E \subset \mathbb{R}$ with positive Lebesgue measure must contain arbitrarily long arithmetic progressions. Laba and Pramanik [17], using some techniques from Fourier analysis, proved that if a closed subset $E$ of $\mathbb{R}$ with Hausdorff dimension that is close to one, and $E$ supports a probability measure which obeys appropriate Fourier decay and mass decay, then $E$ contains non-trivial arithmetic progressions with length 3. Shmerkin [20] constructed a Salem set in $\mathbb{R} \setminus \mathbb{Z}$ which does not include any arithmetic progression with length 3. Fraser and Yu [11] proved that for any one-dimensional set, if its Assouad dimension is strictly smaller than 1, then it cannot contain arbitrarily long arithmetic progressions. Li et al. [18] proved that for any Moran set in a special class, it contains arbitrarily long arithmetic progressions if and only if the Assouad dimension of the associated set is 1. Recently, Chaika [3] proved that for a class of middle-$N$th Cantor set, when the contractive ratio tends to $1/2$, the length of arithmetic progressions goes to infinity. Chaika’s main idea to prove the existence of the arithmetic progressions is using the intersection of the Cantor set with its multiple translations. Broderick et al. [2] proved the quantitative result of the length of the arithmetic progressions. They gave the approximated length of the arithmetic progressions when the contractive ratio tends to $1/2$. Their idea was motivated by Schmidt’s game, which is a very useful tool in the setting of Diophantine approximation.

In this paper, we shall consider the arithmetic progressions in self-similar sets. We first review the main result of Broderick et al. [2]. Let $K_\varepsilon$ be the attractor of the iterated function system (IFS):

$$K_\varepsilon = \left\{ f_1(x) = \frac{1 - \varepsilon}{2} x, f_2(x) = \frac{1 - \varepsilon}{2} x + \frac{1 + \varepsilon}{2} \right\}, \quad 0 < \varepsilon < 1.$$  

**Theorem 1** [2] Let $L_{AP}(K_\varepsilon)$ denote the maximal length of an arithmetic progression in $K_\varepsilon$. Then, for all $\varepsilon > 0$ sufficiently small, we have

$$\frac{1}{\varepsilon \log(1/\varepsilon)} \lesssim L_{AP}(K_\varepsilon) \lesssim \frac{1}{\varepsilon} + 1,$$

where $A \lesssim B$ means that there exists a constant $C$ such that $A \leq CB$. Moreover, $L_{AP}(K_\varepsilon) \geq 4$ if and only if $0 < \varepsilon \leq 1/3$.

In this paper, we shall generalize the second result of Theorem 1.

Before we state the main theorem, we introduce some definitions and results. For $b = \{b_i\}_{i=1}^n$ with $b_1 < b_2 < \cdots < b_n$, let

$$E_{\lambda, b} = \bigcup_{i=1}^n (\lambda E_{\lambda, b_i} + b_i)$$

denote the self-similar set with respect to the IFS $\{f_i(x) = \lambda x + b_i\}_{i=1}^n$. For the definition of self-similar sets, see [10,15]. For a general self-similar set, its
IFS may have overlaps, i.e., the IFS does not satisfy the open set condition or strong separation condition (the definitions can be found in [10,15]). However, in this case, it is easy to obtain the following result (the proof is in the next section).

**Proposition 1** If the strong separation condition fails for $E_{\lambda,b}$, then there are three-term APs contained in $E_{\lambda,b}$.

Due to Proposition 1, it is natural to consider the arithmetic progressions in self-similar sets with the strong separation condition.

Note that

$$E_{\lambda,b} = \frac{b_n - b_1}{1 - \lambda} E_{\lambda,\overline{b}} + b_1 \sum_{n=0}^{\infty} \lambda^n,$$

where $\overline{b} = (\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_n)$ satisfies

$$0 = \overline{b}_1 < \overline{b}_2 < \cdots < \overline{b}_n = 1 - \lambda.$$

For a subset $F$ on the line, let

$$L_{\text{AP}}(F) = \sup\{k : \exists \{c_i\}_{i=1}^{k} \text{ is an AP contained in } F\}.$$

Then

$$L_{\text{AP}}(E_{\lambda,b}) = L_{\text{AP}}(E_{\lambda,\overline{b}}).$$

Without loss of generality, we assume that

$$0 = b_1 < b_2 < \cdots < b_n = 1 - \lambda. \quad (1.1)$$

In terms of the following result (the proof is available in the next section), we may assume that $b = \{b_i\}_{i=1}^{n}$ is an arithmetic progression.

**Proposition 2** Suppose that there is no AP in $\{b_i\}_{i=1}^{n}$ satisfying (1.1). If

$$\lambda < \frac{1}{2} \min_{i<j<k} |2b_j - b_i - b_k|,$$

then there is no AP in $E_{\lambda,b}$.

Given an AP $b = \{b_i\}_{i=1}^{n}$ and a ratio $\lambda \in (0, 1/n)$, we obtain a self-similar set $E_{\lambda,b}$ satisfying the strong separation condition. As the discussion above, we assume that

$$b_i = (i-1)(\lambda + \alpha), \quad \alpha = \frac{1 - n\lambda}{n - 1}. \quad (1.2)$$

Now, let $E^{n}_{\lambda} = E_{\lambda,b}$, where $\{b_i\}_{i=1}^{n}$ is defined in (1.2). In particular, for $n = 2$, the self-similar set $E^{n}_{\lambda}$ is the middle-$\alpha$ Cantor set with $\alpha = 1 - 2\lambda$. In this paper, we shall investigate the arithmetic progressions in the attractor $E^{n}_{\lambda}$.

Note that there is a natural AP $\{b_i\}_{i=1}^{n}$ in $E^{n}_{\lambda}$, i.e.,

$$L_{\text{AP}}(E^{n}_{\lambda}) \geq n.$$
An important problem is how about the estimate of $L_{AP}(E^n_{\lambda})$?

Now, we state the main result of this paper.

**Theorem 2** For the self-similar set $E^n_{\lambda}$, the followings are equivalent:

1. $L_{AP}(E^n_{\lambda}) \geq n + 1$;
2. $\lambda \geq 1/(2n - 1)$;
3. $L_{AP}(E^n_{\lambda}) \geq 2n$.

In particular, $L_{AP}(E^n_{\lambda}) = n$ if and only if $\lambda \in (0, 1/(2n - 1))$.

**Remark 1** Shmerkin [2] pointed that $L_{AP}(K_{\varepsilon}) \geq 4$ if and only if $0 < \varepsilon \leq 1/3$, which can be proved by the gap lemma. We, however, will use some basic ideas from $\beta$-expansions [1,4,6–8,13,16,21] to find the arithmetic progressions. Moreover, our proof is constructive. We note that finding the arithmetic progressions in $E^n_{\lambda}$ is essentially a problem in the setting of $\beta$-expansions, i.e., given a point in some interval, then how can we find its expansions in base $1/\lambda$. Nevertheless, for the multiple $\beta$-expansions, to the best of our knowledge, there are few results [5,6]. This is the main reason which makes the constructive proof difficult.

This paper is arranged as follows. In Section 2, we give a proof of Theorem 2. Moreover, we also prove other useful results which estimate the upper and lower bounds of $L_{AP}(E^n_{\lambda})$. In Section 3, we pose a problem.

## 2 Proof of Theorem 2

Before proving Theorem 2, we prove some results concerning with the lower and upper bounds of $L_{AP}(E^n_{\lambda})$.

**Proof of Proposition 1** Suppose

$$f_i(E_{\lambda,b}) \cap f_{i+1}(E_{\lambda,b}) \neq \emptyset$$

with $b_i < b_{i+1}$, we take a point

$$y = f_i(u) = f_{i+1}(v)$$

with $u, v \in E_{\lambda,b}$ in this intersection, and let

$$x = f_i(v), \quad z = f_{i+1}(u).$$

Then

$$x - y = y - z,$$

which means that $\{x, y, z\} (\subset E_{\lambda,b})$ is an AP.

**Proof of Proposition 2** Suppose on the contrary that there exists an AP $x < y < z$ contained in $E_{\lambda,b}$. Let

$$x \in f_i(E_{\lambda,b}), \quad y \in f_j(E_{\lambda,b}), \quad z \in f_k(E_{\lambda,b}).$$
Then
\[ x \in [b_i, b_i + \lambda], \quad y \in [b_j, b_j + \lambda], \quad z \in [b_k, b_k + \lambda], \]
which implies
\[ |2y - x - z| \geq |2b_j - b_i - b_k| - 2\lambda > 0. \]
It is a contradiction. \( \square \)

For the upper bound of \( L_{AP}(E^n_\lambda) \), using the self-similarity, we have the following result.

**Proposition 3** For any \( k \leq L_{AP}(E^n_\lambda) \), we have an AP of length \( k \) with common difference \( d \geq \alpha \). As a result,

\[ L_{AP}(E^n_\lambda) \leq \left[ \frac{1}{\alpha} \right] + 1 = \left[ \frac{1 - n\lambda}{n - 1} \right] + 1. \]

**Remark 2** Proposition 3 was also proved in [2].

We also have another estimate of upper bound of \( L_{AP}(E^n_\lambda) \).

**Proposition 4** Let \( \lambda_{n,m} \in (0, 1) \) be the solution of
\[ 1 = nx + (n - 1)x^m. \]
Then
\[ L_{AP}(E^n_\lambda) \leq n^m, \quad \lambda \leq \lambda_{n,m}. \]
In particular, for \( n = 2 \), we obtain that
\[ L_{AP}(E^2_\lambda) \leq 2^m, \quad \lambda \leq \lambda_{2,m}. \]
Subsequently, we have
\[ L_{AP}(E^2_\lambda) = 4, \quad \frac{1}{3} \leq \lambda < \sqrt{2} - 1. \]

**Proof of Proposition 3** Let \( E^n_\lambda \) be the invariant set of the IFS
\[ \{S_i(x) = \lambda x + (i - 1)(\lambda + \alpha)\}_{i=1}^n. \]
Suppose that \( \{c_i\}_{i=1}^k \subset E^n_\lambda \) is an AP, and
\[ I_{i_1 \ldots i_t} = S_{i_1} \cdots S_{i_t}([0, 1]) \]
is the smallest basic interval containing \( \{c_i\}_{i=1}^k \). Hence, there exist two different \( i_{t+1}, i'_{t+1} \) such that
\[ I_{i_1 \cdots i_{t+1}} \cap \{c_i\}_{i=1}^k \neq \emptyset, \quad I_{i_1 \cdots i_{t+1}} \cap \{c_i\}_{i=1}^k \neq \emptyset. \]
Then \( \{S_{i_1 \cdots i_t}^{-1}(c_i)\}_{i=1}^k \) is also an AP in \( E^n_\lambda \) with common difference equal or greater than
\[ d(I_{i_{t+1}}, I_{i'_{t+1}}) \geq \alpha. \] \( \square \)
Proof of Proposition 4 Suppose on the contrary that $L_{AP}(E^n_\lambda) > n^m$. By the pigeonhole principle, there are two points of the AP lying in a basic interval of length $\lambda^m$.

On the other hand, in the same way as above, the common difference is equal or greater than

$$\alpha = \frac{1 - n\lambda}{n - 1} < \lambda^m$$

if $\lambda \leq \lambda_{n,m}$. Hence, we obtain a contradiction. Combining with Theorem 1 or Theorem 2, we obtain the last statement. \qed

Now, we give a proof of Theorem 2.

**Proof of Theorem 2**

**Step 1** $(3) \Rightarrow (1)$.

It is obvious.

**Step 2** $(1) \Rightarrow (2)$.

Using Proposition 3, we have

$$d \geq \alpha = \frac{1 - n\lambda}{n - 1}.$$

On the other hand, by the pigeonhole principle, there are two points in the AP contained in a same basic interval with length $\lambda$, and then

$$\lambda \geq d \geq \frac{1 - n\lambda}{n - 1}.$$

Hence, $\lambda \geq 1/(2n - 1)$.

**Step 3** $(2) \Rightarrow (3)$.

We will construct an AP $\{c_i\}_{i=1}^{2n}$ such that $\{c_i\}_{i=1}^{2n} \subset E^n_\lambda$. Let

$$k = \left[\frac{1}{\lambda}\right] = \frac{1}{\lambda} - \left(\frac{1}{\lambda}\right),$$

where $[x]$ and $(x)$ are integral and decimal part of $x \geq 0$. Denote

$$\beta = \frac{1}{2} \begin{cases} k, & 2 \mid k, \\ k + 1, & 2 \nmid k. \end{cases}$$

Note that the invariant set of the IFS

$$\{S_i x = \lambda x + i(\lambda + \alpha)\}_{i=0}^{n-1}$$

can be represented as follows:

$$E^n_\lambda = \left\{ (\lambda + \alpha) \sum_{t=1}^{\infty} a_t \lambda^t : a_t \in \{0, 1, \ldots, n - 1\} \right\}.$$
In other words, if \( x \in E^n_\lambda \), then

\[
x = (\lambda + \alpha) \sum_{t=1}^{\infty} a_t \lambda^t
\]

for some \( (a_t) \in \{0, 1, \ldots, n-1\}^\mathbb{N} \). We call \( (a_t) \) a coding of \( x \). For simplicity, we denote

\[
x = a_1 a_2 \cdots.
\]

Suppose that \( \{c_i\}_{i=1}^{2n} \subset E^n_\lambda \) have codings with respect to the IFS as follows:

\[
c_{2i-1} = (i-1)0 c_{2i-1}^{(3)} c_{2i-1}^{(4)} \cdots, \quad c_{2i} = (i-1)\beta c_{2i}^{(3)} c_{2i}^{(4)} \cdots, \quad i = 1, 2, \ldots, n.
\]

That means

\[
c_i = (\lambda + \alpha) \sum_{t=1}^{\infty} c_i^{(t)} \lambda^t \tag{2.1}
\]

with

\[
c_{2i-1}^{(1)} = i-1, \quad c_{2i-1}^{(2)} = 0, \quad c_{2i}^{(1)} = i-1, \quad c_{2i}^{(2)} = \beta,
\]

where

\[
c_i^{(t)} \in \{0, 1, \ldots, n-1\}.
\]

To insure that \( \{c_i\}_{i=1}^{2n} \) is an AP, we only need to show that

\[
\begin{align*}
2c_2 &= c_1 + c_3, \\
2c_3 &= c_2 + c_4, \\
&\quad \ldots, \\
2c_{2n-1} &= c_{2n} + c_{2n-2}.
\end{align*} \tag{2.2}
\]

In fact, for the equation

\[
2c_{2i} = c_{2i-1} + c_{2i+1},
\]

we obtain that

\[
2 \beta \lambda^2 - \lambda = \sum_{t=3}^{\infty} (c_{2i-1}^{(t)} + c_{2i+1}^{(t)} - 2c_{2i}^{(t)}) \lambda^t. \tag{2.3}
\]

In the same way, for

\[
2c_{2i+1} = c_{2i} + c_{2i+2},
\]

we have

\[
\lambda - 2 \beta \lambda^2 = \sum_{t=3}^{\infty} (c_{2i}^{(t)} + c_{2i+2}^{(t)} - 2c_{2i+1}^{(t)}) \lambda^t,
\]

which implies

\[
2 \beta \lambda^2 - \lambda = \sum_{t=3}^{\infty} (2c_{2i+1}^{(t)} - c_{2i}^{(t)} - c_{2i+2}^{(t)}) \lambda^t. \tag{2.4}
\]
Claim 1  There is a sequence \( \{d_t\}_{t=3}^{\infty} \) of integers such that

\[
2\beta \lambda^2 - \lambda = \sum_{t=3}^{\infty} (2d_t) \lambda^t,
\]

where \( d_t \in \{-(n-1), \ldots, -1, 0, 1, \ldots, n-1\} \).

i) We first verify that

\[
|2\beta \lambda^2 - \lambda| \leq 2(n-1) \frac{\lambda^3}{1 - \lambda}.
\]

Case 1  When \( k \) is even, we shall check that

\[
0 \geq 2\beta \lambda - 1 \geq -2(n-1) \frac{\lambda^2}{1 - \lambda},
\]

where

\[
2\beta = \frac{1}{\lambda} - \left(\frac{1}{\lambda}\right).
\]

In fact, since \((1/\lambda) < 1\) and \( \lambda \geq 1/(2n-1) \), we have

\[
2(n-1) \frac{\lambda^2}{1 - \lambda} + 2\beta \lambda - 1 = -\left(\frac{1}{\lambda}\right)\lambda + 2(n-1) \frac{\lambda^2}{1 - \lambda}
\]

\[
\geq \frac{\lambda}{1 - \lambda} ((2n-1)\lambda - 1)
\]

\[
\geq 0.
\]

Case 2  When \( k \) is odd, we need to show that

\[
0 \leq 2\beta \lambda - 1 \leq 2(n-1) \frac{\lambda^2}{1 - \lambda},
\]

where

\[
2\beta = \frac{1}{\lambda} - \left(\frac{1}{\lambda}\right) + 1.
\]

In fact, since \((1/\lambda) \geq 0\) and \( \lambda \geq 1/(2n-1) \), we have

\[
2(n-1) \frac{\lambda^2}{1 - \lambda} - 2\beta \lambda + 1 = 2(n-1) \frac{\lambda^2}{1 - \lambda} - \left(\frac{1}{\lambda} - \left(\frac{1}{\lambda}\right) + 1\right)\lambda + 1
\]

\[
\geq 2(n-1) \frac{\lambda^2}{1 - \lambda} - \lambda
\]

\[
= \frac{\lambda}{1 - \lambda} ((2n-1)\lambda - 1)
\]

\[
\geq 0.
\]

ii) It suffices to verify

\[
J_i \cap J_{i+1} \neq \emptyset,
\]
where \( i \in \{-(n-1), \ldots, -1, 0, 1, \ldots, n-2\} \) and
\[
J_i = 2\left[ i\lambda + \frac{-(n-1)\lambda^2}{1-\lambda}, i\lambda + \frac{(n-1)\lambda^2}{1-\lambda} \right].
\]

In fact, we only need to check that
\[
\lambda + \frac{-(n-1)\lambda^2}{1-\lambda} \leq \frac{(n-1)\lambda^2}{1-\lambda},
\]
i.e.,
\[
\frac{1}{1-\lambda} (\lambda(2n-1) - 1) \geq 0
\]
due to \( \lambda \geq 1/(2n-1) \).

Now, suppose
\[
2\beta\lambda^2 - \lambda = \sum_{t=3}^{\infty} (2d_t)\lambda^t
\]
with \( d_t \in \{-(n-1), \ldots, -1, 0, 1, \ldots, n-1\} \).

(i) When \( 2d_t \geq 0 \) for \( t \geq 3 \), let
\[
c^{(t)}_i = \begin{cases} 
  d_t, & 2 \nmid i, \\
  0, & 2 \mid i.
\end{cases}
\]

(ii) When \( 2d_t < 0 \) for \( t \geq 3 \), let
\[
c^{(t)}_i = \begin{cases} 
  0, & 2 \nmid i, \\
  -d_t, & 2 \mid i.
\end{cases}
\]

Then, by (2.3) and (2.4), for \( c_i \) satisfying (2.1), equations (2.2) hold. The step (2) \( \Rightarrow \) (3) is finished.

3 Problem

We pose the following problem.

**Problem** Whether \( F(\lambda) = L_{AP}(E^2_{\lambda}) \) is an increasing staircase function with respect to \( \lambda \).

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