THE CLASSIFICATION OF SEPARABLE SIMPLE C*-ALGEBRAS WHICH ARE INDUCTIVE LIMITS OF CONTINUOUS-TRACE C*-ALGEBRAS WITH SPECTRUM HOMEOMORPHIC TO THE CLOSED INTERVAL [0,1]

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Abstract. A classification is given of certain separable nuclear C*-algebras not necessarily of real rank zero, namely, the class of separable simple C*-algebras which are inductive limits of continuous-trace C*-algebras whose building blocks have spectrum homeomorphic to the closed interval [0, 1], or to a disjoint union of copies of this space. Also, the range of the invariant is calculated.

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1. Introduction

It is shown in [23] that an important class of separable simple crossed product C*-algebras are approximately subhomogeneous. Recall that a C*-algebra is said to be subhomogeneous if it is isomorphic to a sub-C*-algebra of $M_n(C_0(X))$ for some natural number $n$ and for some locally compact Hausdorff space $X$. An approximately subhomogeneous C*-algebra, abbreviated ASH algebra, is an inductive limit of subhomogeneous algebras.

This article contains a partial result in the direction of classifying all simple ASH algebras by their Elliott invariant.

The first result on the classification of C*-algebras not of real rank zero was the classification by G. Elliott of unital simple approximate interval algebras, abbreviated AI algebras (see [12]). This result was extended to the non-unital case independently by I. Stevens ([30]) and K. Thomsen ([34]). Also, an interesting partial extension of this result to the non-simple case was given by K. Stevens ([32]). It is worth mentioning that all these algebras are what are referred to as approximately homogeneous algebras, abbreviated AH algebras, and that the most general classification result for simple AH algebras was obtained by Elliott, Gong and Li in [16].

One of the first isomorphism results for ASH algebras was the proof by H. Su of the classification of C*-algebras of real rank zero which are inductive limits of matrix algebras over non-Hausdorff graphs; see [33]. The classification of ASH algebras was also considered in [19], [27] and [29]. (This list of contributions is intended to be representative rather than complete for the classification of ASH algebras.)

An important work on the classification of ASH algebras not of real rank zero, and in fact one of the first ones, is due to I. Stevens ([31]). The main result of the present paper is a substantial extension of Stevens's work, to the class consisting of all simple C*-algebras which are inductive limits of continuous-trace C*-algebras with spectrum homeomorphic to the closed interval $[0,1]$ (or to a finite disjoint union of closed intervals). In particular, the spectra of the building blocks considered here are the same as for those
considered by Stevens. The building blocks themselves are more general.

The isomorphism theorem is proved by applying the Elliott intertwining argument.

Inspired by I. Stevens’s work, the proof proceeds by showing an Existence Theorem and a Uniqueness Theorem for certain special continuous trace C*-algebras. (As can be seen from the proofs, it is convenient to have a special kind of continuous trace C*-algebra as the domain algebra in both these theorems. By special we mean having finite dimensional irreducible representations and such that the dimension of the representation, as a function on the interval, is a finite (lower semicontinuous) step function.)

The present Existence Theorem, Theorem 5.1, differs in an important way from that of [31], Theorem 29.4.1. In fact Theorem 29.4.1 of [31] is false, as is shown in Section 5.1 below.

The proof of the present Existence Theorem is an eigenvalue pattern perturbation, as shown in Section 5, which is similar to the approach used in [31]. (Indeed, once the statement of Theorem 29.4.1 of [31] is corrected, the argument given in [31] does not need to be essentially changed.)

The proof of the present Uniqueness Theorem is different from the one in [31]. It uses the finite presentation of special continuous trace C*-algebras that was given in [17] and [18]. Also the present Uniqueness Theorem has the advantage that both the statement and the proof are intrinsic, i.e., there is no need to say that the building blocks are hereditary sub-C*-algebras of interval algebras as in [31].

In order to apply the Existence and Uniqueness Theorems, it is necessary to approximate the general continuous trace C*-algebras appearing in a given inductive limit decomposition by special continuous trace C*-algebras, as described in [18], Theorem 4.15. This is admissible since in [18] (and also more generally in [17]), it is shown that these special C*-algebras are weakly semiprojective, i.e., have stable relations. (A result of T. Loring, Lemma 15.2.2, [24], allows one to conclude that the original inductive limit decomposition can be replaced by an inductive limit of special continuous trace
An important step of the proof of the isomorphism theorem is the pulling back of the invariant from the inductive limit to the finite stages. The invariant has roughly two major components: a stable part and a non-stable part. The pulling back of the stable part is contained in [12] or [31] and is performed in the present situation with respect to the unital hereditary sub-C*-algebras. The intertwining which is obtained at the level of the stable invariant will approximately respect the non-stable part of the invariant on finitely many elements, as pointed out in [31]. To be able to apply the Existence Theorem it is crucial to ensure that the non-stable part of the invariant is exactly preserved on finitely many elements (actually, just a single element). It is possible to obtain an exact preservation of the non-stable invariant on finitely many elements because one can change the given finite stage algebras in the inductive limit decomposition in such a way that a non-zero gap arises at the level of the affine function spaces; see Section 8 below. It is this non-zero gap that will ultimately guarantee (after passing to subsequences in a convenient way) the exact intertwining on finite sets of the non-stable invariant, as shown in Section 9. It is worth mentioning that in the pulling back of the stable invariant, we must ensure, at the same time that the maps at the affine function space level are given by eigenvalue patterns. This is necessary in order to apply the Existence Theorem and is possible by the Thomsen-Li theorem.

Now all the hypotheses of the Elliott intertwining argument are fulfilled and in this way the proof of the isomorphism Theorem 3.1 is completed.

I. Stevens’s description of the range of the invariant is also extended to include the case of unbounded traces (Theorem 3.2).

To conclude, the class of simple inductive limits of continuous-trace C*-algebras under consideration is compared with the class of simple AI algebras.
2. The invariant

The invariant is similar to the invariant I. Stevens has used in [31], usually summed up as the Elliott invariant, namely, \((K_0(A), \text{Aff} T^+ A, \text{Aff}' A)\), where \(K_0(A)\) is a partially ordered abelian group, \(\text{Aff} T^+ A\) is a partially ordered vector space consisting of linear and continuous functions defined on the cone of traces \(T^+ A\), \(\text{Aff}' A\) is a certain special subset of \(\text{Aff} T^+ A\). The special subset \(\text{Aff}' A\) is the most important part of the invariant for our purposes, and in an informal way it might be said to be the non-stable part of \(\text{Aff} T^+ A\). Formally, the special subset \(\text{Aff}' A\) is the convex set obtained as the closure of \(\{ \hat{a} \in \text{Aff} T^+ A \mid a \geq 0, a \in \text{Ped}(A) \text{ and } ||a|| \leq 1 \} \) inside \(\text{Aff} T^+ A\), with respect to the topology naturally associated to a full projection. Here \(\hat{a}\) is the linear and continuous function defined by the positive element \(a\) from the Pedersen ideal by \(\hat{a}(\tau) = \tau(a)\) where \(\tau \in T^+ A\). As shown in [31], Remark 30.1.1 and Remark 30.1.2, the information given by \(\text{Aff}' A\) is equivalent with that given by the trace-norm map, which is a lower semicontinuous function \(\mu : T^+ A \to \mathbb{R}, \mu(\tau) = ||\tau|| \) and \(\infty\) if \(\tau\) is unbounded.

It is a crucial fact that the trace-norm map is equivalent to the dimension function in the case of a building block algebra, cf. Section 4 below. The dimension function of a building block (i.e. the function that assigns to each point in the spectrum of the building block the dimension of the irreducible representation) can be viewed as a lower semicontinuous function on the extreme traces normalized on minimal projections in primitive quotiens and hence we can compare it with functions from \(\text{Aff} T^+ A\). Then the subset \(\text{Aff}' A\) is the closure of the set of all affine functions smaller than the dimension function. Conversely by taking the supremum over all elements of \(\text{Aff}' A\) we recover the dimension function in the case of the building blocks.

3. The results

Using the invariant described above it is possible to prove a complete isomorphism theorem, namely,
Theorem 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-unital simple $C^*$-algebras which are inductive limits of continuous-trace $C^*$-algebras with spectrum homeomorphic to $[0,1]$. Assume that

1. there is a order preserving isomorphism $\psi_0 : K_0(\mathcal{A}) \to K_0(\mathcal{B})$,
2. there is an isomorphism $\psi_T : \text{Aff}T^+ \to \text{Aff}T^+\mathcal{B}$, such that
   $$\psi_T(\text{Aff}'\mathcal{A}) \subseteq \text{Aff}',$$
3. the two isomorphisms are compatible:
   $$\psi_0([p]) = \psi_T([p]), \ [p] \in K_0(\mathcal{A}).$$

Then there is an isomorphism of the algebras $\mathcal{A}$ and $\mathcal{B}$ that induces the given isomorphism at the level of the invariant.

A description is given of the range of the invariant. More precisely, the following theorem is proved:

Theorem 3.2. Suppose that $G$ is a simple countable dimension group and $V$ is the cone associated to a metrizable Choquet simplex. Let $\lambda : V \to \text{Hom}^+(G,R)$ be a continuous affine map which takes extreme rays into extreme rays. Let $f : V \to [0, +\infty]$ be an affine lower semicontinuous map, zero at zero and only at zero. Then $(G, V, \lambda, f)$ is the invariant of some simple non-unital inductive limit of continuous-trace $C^*$-algebras whose spectrum is the closed interval $[0,1]$.

4. Special continuous trace $C^*$-algebras with spectrum the interval $[0,1]$

In this section we will introduce some terminology. A very important piece of data that we shall consider is a map that assigns, to each class of irreducible
representations, the dimension of a representation from that class. Roughly speaking, the dimension function can be thought of as the non-stable part of the invariant when restricted to the building blocks.

**Definition 4.1.** Let $A$ be a $C^*$-algebra and let $\hat{A}$ denote the spectrum of $A$. Then the *dimension function* is the map from $\hat{A}$ to $\mathbb{R} \cup +\infty$,

$$\pi \mapsto \dim(H_\pi),$$

where by $\dim(H_\pi)$ we mean the dimension of the irreducible representation $\pi$.

It was shown in [18], Theorem 4.13, that the dimension function is a complete invariant for continuous trace $C^*$-algebras with spectrum the closed interval $[0,1]$. Also concrete examples were constructed for each given dimension function, cf. Section 7 of [18].

Therefore given a lower semicontinuous integer valued (i.e., a “dimension function”) which is finite-valued and bounded we can exhibit a continuous trace $C^*$-algebra

$$\begin{pmatrix}
C_0(A_n) & C_0(A_n) & C_0(A_n) & \ldots & C_0(A_n) \\
C_0(A_n) & C_0(A_{n-1}) & C_0(A_{n-1}) & \ldots & C_0(A_{n-1}) \\
C_0(A_n) & C_0(A_{n-1}) & C_0(A_{n-2}) & \ldots & C_0(A_{n-2}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_0(A_n) & C_0(A_{n-1}) & C_0(A_{n-2}) & \ldots & C[0,1] 
\end{pmatrix} \subseteq M_n \otimes C[0,1].$$

whose dimension function is the given function. Here $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq [0,1]$ and each $A_i$ is an open subset of $[0,1]$. Moreover any trace on such an algebra is of the form $tr \otimes \nu$, where $tr$ is the usual trace normalized on minimal matrix projections and $\nu$ is a finite measure on $[0,1]$. The extreme traces are parameterized by $t \in [0,1]$, and are given as $(tr \otimes \delta_t)_{t \in [0,1]}$, where
\( \delta_t \) is the normalized point mass at \( t \). Then the trace norm map is equal to the dimension function when restricted to the extreme traces. To see that the trace norm map is equivalent to the special subset \( \text{Aff}'() \) of the affine function space \( \text{Aff}T^+() \) we repeat the proof of I. Stevens from \cite{31}, Remark 30.1.1 and Remark 30.1.2.

Inspired by a construction of I. Stevens in \cite{31} we make

**Definition 4.2.** A continuous-trace \( C^* \)-algebra whose spectrum is \([0, 1]\) will be called a special continuous-trace \( C^* \)-algebra if its dimension function is a finite-valued finite step function: there is a partition of \([0, 1]\) into a finite union of intervals such that the dimension function is finite and constant on each such subinterval.

**Remark 4.1.** Let \( A \) be a continuous trace \( C^* \)-algebra with spectrum \([0, 1]\) and with dimension function \( d : [0, 1] \to \mathbb{N} \cup \{+\infty\} \). There exists a projection-valued function that if composed with the rank function gives rise to the dimension function \( d \). To see this first we notice that because the Dixmier-Douady invariant of \( A \) is trivial, the \( C^* \)-algebra \( A \) is a continuous field of elementary \( C^* \)-algebras over \([0, 1]\), where the fibers are hereditary sub-\( C^* \)-algebras of the algebra of compact operators. Then take the unit of the hereditary sub-\( C^* \)-algebra in each fiber. In this way we construct a projection-valued function which is lower semicontinuous. By composing this constructed projection-valued function with the rank function we get the dimension function \( d \).

**Remark 4.2.** A priori our definition for a special sub-\( C^* \)-algebra is more general than I. Stevens’s definition. As it is shown in \cite{18}, any special sub-\( C^* \)-algebra in our sense is isomorphic to a special sub-\( C^* \)-algebra in I. Stevens’s sense.

**Remark 4.3.** It was shown in \cite{18} that special continuous trace \( C^* \)-algebras are finite presented and weakly semiprojective. Also a stronger result was proven in \cite{8}, namely that special continuous trace \( C^* \)-algebras are strongly semiprojective.
5. Balanced inequalities and the Existence Theorem

The proof of the isomorphism theorem 3.1 is based on the Elliott intertwining argument. Among the main ingredients of this procedure are the Existence Theorem that will be described below as well as the Uniqueness Theorem that is presented in Section 6.

It is worth noticing that for the Existence Theorem and the Uniqueness Theorem we require that the inequalities are balanced, i.e., independent of the choice we make for the normalization of the affine function space. We normalize the affine function spaces with respect to a full projection. Even though we fix a projection in the domain algebra for both the Existence Theorem and the Uniqueness Theorem, this choice does not make any difference when we apply the theorems to obtain an approximate commuting diagram. As was pointed out to us by Andrew Toms, we only need to consider a compatible family of projections when we go through the whole proof, provided that a corresponding projection is chosen in the codomain algebra. In fact, we can state the theorems without mentioning the choices of the projections as long as their $K_0$-classes are compatible with respect to the $K_0$-map under consideration even though they exist and some choices of them will be used during the proof.

To be able to focus on the new aspects of the present Existence Theorem as opposed to the Existence Theorem for unital continuous trace C*-algebras proved by Elliott in [12], we will both state the theorem and prove it in terms of so-called eigenvalue pattern maps. In our situation an eigenvalue pattern map is a positive unital map from $C([0, 1])$ to $C([0, 1])$ which is a finite sum of *-homomorphisms from $C([0, 1])$ to $C([0, 1])$. Using the Gelfand theory each such *-homomorphism is given by a continuous function from $[0, 1]$ to $[0, 1]$. As follows from the intertwining of the invariant and will be explained below, Section 9, one can always obtain a (non-necessarily compatible) eigenvalue patterns maps.

The proof of the Existence Theorem is obtained by perturbing an eigenvalue pattern map between the affine function spaces in a such a way that it
defines an algebra map between the building blocks.

**Theorem 5.1.** Let $A$ be a special building block and by $d_A$ denote the dimension function of $A$. Let a finite subset $F$ contained in $\text{Aff}^+ A$, and $\epsilon > 0$ be given. There is $f' \in \text{Aff}' A$ such that for any special building block $B$ with dimension function $d_B$, and maps $k : D(A) \to D(B)$ and $T : \text{Aff}^+ A \to \text{Aff}^+ B$ verifying the conditions

1. $k$ has multiplicity $M_k$,
2. $T$ is given by an eigenvalue pattern and has the property
   
   $$T(f') \leq d_B,$$
3. $k$ and $T$ are exactly compatible, i.e.,
   
   $$\hat{k}([r]) = T([\hat{r}]),$$

there is a homomorphism $\psi : A \to B$ such that $k = \psi_0$ and

$$|| (T - \psi_T) a ||_{\hat{k}(p)} \leq \epsilon ||a||_{\hat{p}}, \ a \in F.$$

**Remark 5.2.** Recall that $\text{Aff}^+ A$ is a Banach space with a norm given by $||f||_p = \sup\{|f(\tau)| \mid \tau(p) = 1, \ \tau \in T^+ A\}$, where $f \in \text{Aff}^+ A$ and $p$ is a fixed full projection of $A$. In addition, using the norm we just defined, $\text{Aff}^+ A$ is identified with $C([0, 1])$. This identification allows us to compare in the supremum norm the dimension function and elements of $\text{Aff}^+ A$. Also the norm of $\text{Aff}^+ B$ is defined with respect to a projection from $B$ which is Murray-von Neumann equivalent to $k(p)$. Since our inequalities at the level of the affine function spaces are balanced, which is the only theorem that makes sense, in particular they are independent of the choice of the projection $p$.

**Proof.** The idea of the proof is to choose in a clever way a function $f'$ and then change within the given tolerance the eigenvalue functions that appear in the
eigenvalue pattern $T$ so that the image of the dimension function $d_A$ under the new eigenvalue pattern is smaller than or equal the dimension function of the algebra $B$, as desired.

Let $\epsilon > 0$ and a finite set $F \subset \text{Aff}T^A$ be given.

As already mentioned it is a crucial step how $f'$ is chosen. There is no loss in generality if we assume that the dimension function $d_A$ has only one discontinuity point, $t_0 \in [0, 1]$.

Choose $f'$ to be a continuous function such that $f'(t) = d_A(t)$ for $t \in [0, t_0 - \delta] \cup [t_0 + \delta, 1]$, $f'(t) \leq d_A(t)$ for $t \in [0, 1]$, and $f'(t_0) = d_A(t_0)$, where $\delta \leq \frac{\epsilon}{2M_k}$. Hence $f'$ is a continuous function defined on the interval $[0, 1]$ which approximate $d_A$, namely $f'$ is equal to $d_A$ except on a small neighbourhood around the discontinuity point.
Next we proceed by showing how to change the eigenfunctions such that a desired eigenvalue pattern is obtained. We will carry out this procedure in a very special case, namely all the eigenfunctions are assumed to be the identity function.

In the above picture we have the original eigenvalue function \( \lambda \) which is the identity map. We define a new eigenvalue function as the picture shows below, Figure 4. More precisely the new eigenvalue function \( \hat{\lambda} : [0, 1] \to [0, 1] \), \( \hat{\lambda}(t) = t \) for \( t \in [0, t_0 - \delta) \cup (t_0 + \delta + t_0, 1] \), \( \hat{\lambda}(t) = t_0 - \delta \) for \( t \in [t_0 - \delta, t_0 + \delta] \),
the linear map $\hat{\lambda}(t) = t_0 - \delta + (t - t_0 - \delta)\frac{2\delta + \delta t_0}{\delta t_0}$ for $t \in [t_0 + \delta, t_0 + \delta + \delta \rho]$, where $\delta t_0$ is a strictly positive number such that $t_0 + \delta + \delta t_0 \leq 1$.

A short computation or a geometric argument shows that the difference $||\lambda - \hat{\lambda}||_\infty = 2\delta$.

Moreover the dimension function $d_A$ evaluated on the perturbed eigenvalue $\hat{\lambda}$ is smaller then $f'$ evaluated on the given eigenvalue $\lambda$

$$d_A(\hat{\lambda}(t)) \leq f'(\lambda(t)).$$

Hence by hypothesis 2 we have

$$\sum_{i=1}^{M_k} d_A \circ \hat{\lambda} \leq \sum_{i=1}^{M_k} f' \circ \lambda \leq d_B.$$ 

Here we say that one dimension function is smaller than another one if the relation holds pointwise.

The change of the eigenvalues is small because of the choice of $\delta$

$$||(T_\lambda - T)(a)||_{k(p)} = \sum_{i=1}^{M_k} ||a \circ (\hat{\lambda}_i - \lambda_i)||_{k(p)}$$
\[ \sum_{i=1}^{M_k} \sup \{ |a \circ (\hat{\lambda}_i - \lambda_i)(\tau)| \mid \tau(k(p)) = 1, \tau \in T^+A \} \]

\[ = \sum_{i=1}^{M_k} \sup \left\{ M_k \left| a \circ (\hat{\lambda}_i - \lambda_i) \left( \frac{1}{M_k} \tau \right) \right| \mid \tau(p) = M_k, \tau \in T^+A \right\} \]

\[ = \sum_{i=1}^{M_k} M_k \|a \circ (\hat{\lambda}_i - \lambda_i)\|_{\tilde{p}} \leq 2\delta M_k^2 \|a\|_{\tilde{p}} \leq \epsilon \|a\|_{\tilde{p}}, \ a \in F. \]

To obtain the inequality above we used the linearity of the function \(a \circ (\hat{\lambda}_i - \lambda_i)\) and that an extreme trace \(\tau\) in \(T^+A\) has the property that \(\tau(k(p)) = 1\) if and only if \(\tau(p) = M_k\).

We claim that the argument for the special case shown above can be extended to the case of piecewise linear eigenfunctions which is known to be equivalent to the general case of continuous eigenfunctions that arise in the inductive limits of interval algebras (see for instance [12]).

\[ \square \]

5.1. **An exact inequality is necessary between the non-stable part of the invariant.** As mentioned in the introduction, the Theorem 29.4.1 of [31] is false. To prove the Existence Theorem it is fundamental to have an exact inequality between the non-stable part of the invariant at the level of the affine function space, i.e., \(T(f) \leq d_B\) for some continuous affine function \(f \leq d_A\). A weaker inequality is required in the statement of the Existence Theorem of [31], Theorem 29.4.1, i.e., \(T(f) \leq d_B(1 + \delta)\) for some small \(\delta > 0\). Therefore it is possible to construct a counterexample to the I. Stevens Existence Theorem. This counterexample is already assuming that the positive linear map \(T\) is given by an eigenvalue pattern. To reduce the proof of Theorem 29.4.1 of [31] to an eigenvalue pattern problem, one needs an extra assumption in hypothesis 2, for instance a positive gap \(\eta > 0\) in the other side of the inequality described above \(T(f) + \eta \leq d_B(1 + \delta)\).
Next we describe the counterexample. Let $d_A$ be the lower semicontinuous function defined on $[0, 1]$ which is equal to 2 on the subintervals $[0, 1/2)$ and $(1/2, 1]$, and equal to 1 at $1/2$. Let $\epsilon_0$ be such that $0 < \epsilon_0 < 1/4$ and $F = \{a_1(t) = t\}$. Let $f$ be a continuous function which approximates $d_A$. Since they can not be equal everywhere around $1/2$, we can assume that $f(t) < 2 = d_A(t)$ for all $t$ in $(1/2 - \eta, 1/2 + \eta)$, where $\eta > 0$ can be chosen as small as needed.

Let $\delta > 0$ be given. There exists a positive integer $M_k$ such that $\frac{1}{2M_k - 1} < \delta$. Then choose $T$ to be defined by $M_k$ eigenvalue functions $(\lambda_i)_{i=1,\ldots,M_k}$, all being the identity functions, $\lambda_i(t) = t$, for all $i = 1, \ldots, M_k$. Next choose $B$ to be a continuous trace C*-algebra with dimension function constant equal to $2M_k - 1$.

Note that the hypothesis 2 of the Existence Theorem 29.4.1 from [31] holds

$$T(f)(t) = \sum_{i=1}^{M_k} f \circ \lambda_i(t) \leq 2M_k \leq (1 + \delta)d_B(t).$$

Now we claim that among all perturbations of $T$ which are within the given $\epsilon_0$ with respect to the finite set $F$, the particular one $P$ which is given by the continuous eigenfunctions $(\mu_i)_{i=1,\ldots,M_k}$ that have the property $\mu_i(t) = 1/2$ for $t \in (1/2 - \eta, 1/2 + \eta)$, is the smallest in the sense that the value of $P(d_A)$ is the smallest. Here it is important to notice that because $\epsilon_0 < 1/4$ it forces that $(\mu_i)_i(t) = \lambda(t) = t$ for $t$ close to 0 and 1 including 0 and 1. In particular we have $(\mu_i)(0) = \lambda_i(0) = 0$. Therefore

$$P(d_A)(0) = \sum_{i=1}^{M_k} d_A(\mu_i(0)) = 2M_k > 2M_k - 1 = d_B(0).$$
Therefore we cannot perturb the eigenfunctions to obtain a compatible eigenvalue pattern and the Existence Theorem as stated in [31] cannot be proved.

6. Uniqueness Theorem

It is important to notice that the conclusion of the Existence Theorem is part of the hypothesis of the Uniqueness Theorem; this makes sense since all inequalities are balanced (i.e. independent of the choice of projection with respect to which the normalization is done).

**Theorem 6.1.** Let $A$ be a special continuous-trace C*-algebra, $F \subset A$ a finite subset and $\epsilon > 0$. Let $B$ be a special continuous-trace C*-algebra and $\psi, \varphi : A \to B$ be maps with the following properties:

1. $\varphi_0 = \psi_0 : K_0(A) \to K_0(B)$,

2. $\psi$ and $\varphi$ have at least the fraction $\delta$ of their eigenvalues in each of the $d$ consecutive subintervals of length $\frac{1}{d}$ of $[0, 1]$, for some $d > 0$ such that for $\hat{r}_i$ the functions equal to 0 from 0 to $\frac{i}{d}$, equal to 1 on $[\frac{i+1}{d}, 1]$ and linear in between, for each $0 \leq i \leq d$, $\|((\varphi_T - \psi_T)(\hat{r}_i))\|_{K_p} < \delta \|\hat{r}_i\|_p$, with respect to the norm of $\text{Aff}T^+B$,

Then there is an approximately inner automorphism of $B$, $f$, such that

$$\|((\psi - f\varphi)(a))\| < \epsilon, \quad a \in F$$

*Proof.* Because of the isomorphism theorem 4.13 from [18], there is no loss of generality to assume that our building blocks are in a very special form.
\[
A \cong \begin{pmatrix}
C_0(A_1) & C_0(A_1) & C_0(A_1) & \cdots & C_0(A_1) \\
C_0(A_1) & C_0(A_2) & C_0(A_2) & \cdots & C_0(A_2) \\
C_0(A_1) & C_0(A_2) & C_0(A_3) & \cdots & C_0(A_3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_0(A_1) & C_0(A_2) & C_0(A_3) & \cdots & C[0,1]
\end{pmatrix}
\]

Notice that the cancellation property holds for the unital sub-C*-algebra of \( A \) and any projection of \( A \) is Murray-von Neumann equivalent to a projection inside of the unital sub-C*-algebra. Therefore the cancellation property holds for \( A \). A similar argument shows that the cancellation property holds for any continuous-trace C*-algebra with the spectrum the closed interval \([0,1]\).

Since \( \varphi_0 = \psi_0 \), we can assume that \( \varphi(p) = \psi(p) \), where \( p \) is the unit of the sub-C*-algebra \( C([0,1]) \) of \( A \). In other words the restrictions of the maps to the unital subalgebra share the same unit.

The stable part of the Elliott invariant (i.e., the \( K_0 \) group and the affine function space \( \text{Aff}T^+ \)) of \( A \) and of \( C([0,1]) \) is the same. Let us restrict the two maps \( \varphi \) and \( \psi \) to the unital sub-C*-algebra \( C([0,1]) \). The image of \( C([0,1]) \) under \( \varphi \) and \( \psi \) is up to a unitary a full matrix algebra over the interval. Then using assumptions 1 and 2 we notice that the hypotheses of the Elliott Uniqueness Theorem ([12], Theorem 6), are fulfilled. Hence we get a partial isometry \( V \) of \( B \) (a unitary inside of the full matrix sub-C*algebra of \( B \)) such that

\[
||\varphi(f_{A_i} \otimes e_{nn}) - V\psi(f_{A_i} \otimes e_{nn})V^*|| \leq \epsilon, \quad i \in \{1, \ldots, n\}.
\]

We want this relation to hold for the case when the domain is \( A \). We follow a strategy already present in the case of full matrix over the interval. An
important data that we will use is that the domain algebra \( A \) has a finite presentation. In fact we will use the concrete description of this presentation that was given in [18, Section 8]. The set of generators consists of elements of the form \( f_{A_i} \otimes e_{in} \) which are certain positive functions tensor the matrix units.

For each \( i \) let \( u_i \) be a continuous function defined on \([0, 1]\) which is equal to 1 on \( A_i \) except near the end points of each open subinterval of \( A_i \) and 0 otherwise. One can think of \( u_i \) as an approximate unit of the functions \( f_{A_i}, i \in \{1, \ldots, n\} \) and later estimates depend on the size of the subset of \( A_i \) where \( u_i \) is not equal to 1.

Define

\[
\mathcal{V} = \sum_{i=1}^{n} \varphi(u_i \otimes e_{ni})^*V \psi(u_i \otimes e_{ni}).
\]

Then

\[
\mathcal{V} \psi(f_{A_i} \otimes e_{ni}) \mathcal{V}^* = \\
= (\sum_{k=1}^{n} \varphi(u_k \otimes e_{nk})^*V \psi(u_k \otimes e_{nk}) \psi(f_{A_i} \otimes e_{ni})(\sum_{l=1}^{n} \psi(u_l \otimes e_{ln})^* \varphi(u_l \otimes e_{nl})) = \\
= \varphi(u_n \otimes e_{nn})V \psi(f_{A_i} \otimes e_{ni})(\sum_{l=1}^{n} \psi(u_l \otimes e_{ln})^* \varphi(u_l \otimes e_{nl})) = \\
= \varphi(u_n \otimes e_{nn})V \psi(f_{A_i} \otimes e_{ni})(\sum_{l=1}^{n} \psi(u_l \otimes e_{nl})V^* \varphi(u_l \otimes e_{nl})) = \\
= \varphi(u_n \otimes e_{nn})V \psi(f_{A_i} \otimes e_{ni}) \psi(u_i \otimes e_{in})V^* \varphi(u_i \otimes e_{ni}) = \\
= \varphi(u_n \otimes e_{nn})V \psi(f_{A_i} \otimes e_{ni})V^* \varphi(u_i \otimes e_{ni}).
\]

Now we have that

\[
\varphi(f_{A_i} \otimes e_{ni}) = \varphi(u_n \otimes e_{nn}) \varphi(f_{A_i} \otimes e_{nn}) \varphi(u_i \otimes e_{ni}).
\]
Therefore
\[ ||\varphi(f_{A_i} \otimes e_{ni}) - \mathcal{V}\psi(f_{A_i} \otimes e_{ni})\mathcal{V}^*|| = \]
\[ ||\varphi(u_n \otimes e_{nn})(\varphi(f_{A_i} \otimes e_{nn}) - V\psi(f_{A_i} \otimes e_{nn})\mathcal{V}^*)\varphi(u_i \otimes e_{ni})|| \leq \]
\[ \leq ||\varphi(u_n \otimes e_{nn})||\epsilon||\varphi(u_i \otimes e_{ni})|| \]
i.e. it can be made as small as needed.

We want to argue that \( \mathcal{V} \) gives rise to a partial isometry. Let us calculate
\[ \mathcal{V}^*\mathcal{V} = \]
\[ = \sum_{l=1}^{n} \psi(u_l \otimes e_{nl})^*\mathcal{V}^*\varphi(u_l \otimes e_{nl})\sum_{i=1}^{n} \varphi(u_i \otimes e_{ni})^*\mathcal{V}\psi(u_i \otimes e_{ni}) = \]
\[ = \sum_{l=1}^{n} \psi(u_l \otimes e_{ln})^*\mathcal{V}^*\varphi(u_l \otimes e_{nl})\sum_{i=1}^{n} \varphi(u_i \otimes e_{in})\mathcal{V}\psi(u_i \otimes e_{ni}) = \]

Assuming that each \( u_i \) is equal to 1 on the open intervals \( A_i \) except small neighbourhood around the end points of \( A_i \) we get
\[ = \sum_{i=1}^{n} \psi(u_i \otimes e_{in})\mathcal{V}^*\varphi(u_i \otimes e_{nn})\mathcal{V}\psi(u_i \otimes e_{ni}) \]
which is very close to
\[ \sum_{i=1}^{n} \psi(u_i \otimes e_{in})\psi(u_l \otimes e_{nn})\psi(u_i \otimes e_{ni}) = \]
\[ = \sum_{i=1}^{n} \psi(u_i \otimes e_{ii}) \]
which is the value of the projection-valued map of the hereditary sub-C*-algebra generated by \( \psi(A) \) inside \( B \). In other words \( \mathcal{V}^*\mathcal{V} \) is as close as we want to be a projection. It is important to notice that this is true if we are not in a small neighbourhood of the singularity points of the dimension function of the hereditary sub-C*-algebra generated by \( \psi(A) \) (i.e. whenever
Similarly $VV^*$ is almost equal to the $\sum_{i=1}^n \varphi(u_i \otimes e_{ii})$ if we are not in a small neighbourhood of the singularity points of the dimension function of the hereditary sub-C*-algebra generated by $\varphi(A)$. Notice that any singularity point $y_0$ of the dimension function of the hereditary sub-C*-algebra generated by $\varphi(A)$ or $\psi(A)$ has the property that there is an eigenfunction $\lambda_i$ such that $\lambda_i(y_0)$ is a singularity point of the dimension function $d_A$ of $A$. In addition $\lambda_i$ is uniform continuous function from $[0, 1]$ to $[0, 1]$. Hence small neighbourhoods of $y_0$ correspond to small neighbourhoods of some singularity point of $d_A$.

From the polar decomposition $V = W|V|$ we get a partial isometry $W$. We claim that $W$ still intertwines approximately the two maps $\varphi$ and $\psi$, i.e.,

$$||\varphi(f_{A_i} \otimes e_{ni}) - W\psi(f_{A_i} \otimes e_{ni})W^*|| < 3\epsilon,$$

$$||W^*\varphi(f_{A_i} \otimes e_{ni})W - \psi(f_{A_i} \otimes e_{ni})|| < 3\epsilon.$$

This is true because

$$|||\varphi(f_{A_i} \otimes e_{ni}) - W\psi(f_{A_i} \otimes e_{ni})W^*|| =$$

$$= |||\varphi(f_{A_i} \otimes e_{ni}) - V\psi(f_{A_i} \otimes e_{ni})V^* + W|V|\psi(f_{A_i} \otimes e_{ni})|V|W^* - W\psi(f_{A_i} \otimes e_{ni})W^*|| \leq$$

$$\leq ||\varphi(f_{A_i} \otimes e_{ni}) - V\psi(f_{A_i} \otimes e_{ni})V^*|| + ||W|V|\psi(f_{A_i} \otimes e_{ni})|V|W^* - W\psi(f_{A_i} \otimes e_{ni})W^*|| \leq$$

$$\leq \epsilon + |||V|\psi(f_{A_i} \otimes e_{ni})|V| - \psi(f_{A_i} \otimes e_{ni})|| \leq$$

$$\leq \epsilon + |||V|\psi(f_{A_i} \otimes e_{ni})|V| - |V|\psi(f_{A_i} \otimes e_{ni}) + |V|\psi(f_{A_i} \otimes e_{ni}) - \psi(f_{A_i} \otimes e_{ni})|| \leq$$

$$\leq \epsilon + \epsilon + \epsilon = 3\epsilon.$$

and similarly we get the other desired inequality.

Hence we have constructed a family of partial isometries $W$ from the hereditary sub-C*-algebra generated by $\varphi(A)$ to the hereditary sub-C*-algebra
generated by $\psi(A)$. In addition $\mathcal{W}$ induces an isomorphism between the two above mentioned hereditary sub-C*-algebras. In particular it implies that the two hereditary sub-C*-algebras have the same dimension function.

Next we will show how to approximate $\mathcal{W}$ with a unitary in the unitization of the codomain algebra.

Let us start by applying Theorem 4.12 of [18] to the projection-valued function corresponding to the hereditary sub-C*-algebra generated by $\varphi(A)$. Hence we get a decomposition, possibly infinite, in terms of functions each of which is projection-valued of rank 1 on a certain open subset of $[0, 1]$ and zero otherwise. Notice that the discontinuity points of the dimension function of the hereditary sub-C*-algebra generated by $\varphi(A)$ correspond to the discontinuity points of the functions appearing in the decomposition and the open sets are increasing in a suitable sense.

Next we apply Lemma 6.2 for each point at singularity in the interval $[0, 1]$, or, in other words, to each function appearing on the decomposition. Thus, we have a family of unitaries that preserves the continuity of the continuous elements of the hereditary sub-C*-algebra $\varphi(A)$ and at the same time has the property that it still intertwines the two maps.

In the following lemma the hereditary sub-C*-algebras $H_1$ and $H_2$ are assumed to be continuous bundles over $[0, 1]$ (for more details about continuous bundles of C*-algebras see [20]).

If $A$ is a continuous bundle of C*-algebras over $[0, 1]$ then $A^t$ stands for the fiber of $A$ over $t$.

**Lemma 6.2.** Let $H_1$ and $H_2$ be hereditary sub-C*-algebra of $M_2(C[0, 1])$ with the same spectrum $[0, 1]$ and identical dimension function equal to 1 on the closed interval $[0, t_0]$ and equal to 2 on the half-open interval $(t_0, 1]$, $t_0 \in (0, 1)$. Let $\mathcal{W} = (W(t))_{t \in [0, 1]}$ be a family of partial isometries indexed by the points of $[0, 1]$. For each $t \in [0, 1]$, $W_t : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ such that $W(t)W(t)^* = \text{the unit of } H_1^t$ and $W(t)^*W(t) = \text{the unit of } H_2^t$. Then there exists a family
\( \mathcal{W}^\perp \) of partial isometries indexed by \([0, 1]\) such that \( \mathcal{W} + \mathcal{W}^\perp \) is a unitary inside of \( M_2(C[0, 1]) \) and \((W + W^\perp)_t(f)(t) = W_t(f)(t)\) for any continuous function \( f \in H_1 \) and \( t \in [0, 1] \).

**Proof.** Diagrammatically the dimension function of \( H_1 \) and \( H_2 \) can be pictured as follows.

![Figure 5. Dimension function of \( H_1 \) and \( H_2 \).](image)

We construct the family \( \mathcal{W}^\perp = (\mathcal{W}_t^\perp)_{t \in [0, 1]} \) as follows. Fix a \( t \) in \([0, 1]\), \( t \leq t_0 \). \( W(t) \) is a partial isometry on some dimension-one subspace of \( M_2(\mathbb{C}) \). Hence \( W_t(M) = c(M)M_t \) where \( c(M) \) is a constant depending on \( M \) and \( M_t \) is a projection matrix in \( M_2(\mathbb{C}) \). Let \( W^\perp_t = c(M)(I_2 - M_t) \). Notice that \( W_t + W^\perp_t \) is a unitary operator on \( M_2(\mathbb{C}) \). If \( t > t_0 \) then \( W^\perp_t = 0 \).

The family of unitaries \((W_t + W^\perp_t)_{t \in [0, 1]}\) is continuous except at the point \( t_0 \). Our work below shows that this family can be modified to be continuous overall \([0, 1]\).

Extend \((W_t)_{t \in [0, t_0]}\) to be a continuous family \((W^\perp_t)_{t \in [0, 1]}\) of partial isometries on dimension-one subspaces of \( M_2(\mathbb{C}) \). \( W^\perp_{t_0} \) and \( \lim_{t \to t_0, t > t_0} (W_t - W^\perp_t) \) are two partial isometries on the same dimension one subspace of \( M_2(\mathbb{C}) \), hence they
differ by a constant of absolute value one, i.e.

$$W_{t_0}^\perp = c \lim_{t \to t_0, t > t_0} (W_t - W_1^t).$$

Define the continuous family of unitaries $(U_t)_{t \in [0,1]}$ to be $U_t = W_t + W_{t_0}^\perp$ if $t \leq t_0$ and $U_t = W_1^t + c(W_t - W_1^t)$ if $t > t_0$.

Hence the continuous family of unitaries $W_t$ is given by $U_t$ and $(U_t(f)(t) = W_t(f)(t)$ for any continuous function $f \in H_1$ and $t \in [0,1]$.

$$\square$$

7. **Inductive limits of special continuous trace C*-algebras**

Next let us show that the Existence Theorem and the Uniqueness Theorem presented above can be applied, i.e., that the hypotheses of the theorems can be fulfilled. As a first step in this direction let us show that an inductive limit of continuous-trace C*-algebras with spectrum $[0,1]$ (or disjoint unions of closed intervals) is isomorphic to an inductive limit of special continuous-trace C*-algebras.

The basic tools in establishing this step are the fact that special continuous trace C*-algebras are semiprojective (cf. [18], Theorem 6.5) and a result by T. Loring ([24], Lemma 15.2.2) which for the convenience of the reader we state below:

Suppose that $A$ is a C*-algebra containing a (not necessarily nested) sequence of sub-C*-algebras $A_n$ with the property that for all $\epsilon > 0$ and for any finite number of elements $x_1, \ldots, x_k$ of $A$, there exist an integer $n$ such that

$$\{x_1, \ldots, x_k\} \subset \epsilon A_n.$$
If each $A_n$ is weakly semiprojective and finitely presented, then

$$A \cong \lim_{\to} (A_{n_k}, \gamma_k)$$

for some subsequence of $(A_n)$ and some maps $\gamma_k : A_{n_k} \to A_{n_{k+1}}$.

**Proposition 7.1.** Let $A$ be a simple inductive limit of continuous-trace C*-algebras whose building blocks have their spectrum homeomorphic to $[0,1]$. Then $A$ is an inductive limit of direct sums of special continuous-trace C*-algebras with spectrum $[0,1]$.

**Proof.** In Proposition 5.4 and Theorem 6.5 of [18] it is proved that the class of special continuous trace C*-algebras with spectrum $[0,1]$ are finitely presented and have weakly stable relations. Each building block from the inductive limit decomposition of $A$ can be approximated by special continuous trace C*-algebras (cf. Theorem 6.14 of [18]). Then $A$ satisfies Loring’s hypothesis where the sequence of semiprojective algebras is given by the special algebras from the approximation of the building blocks. Thus the Loring’s lemma implies that $A$ is an inductive limit of special continuous trace C*-algebras. □

8. **Getting a non-zero gap at the level of affine function spaces**

To be able to exactly intertwine the non-stable part of the invariant it is useful to know that the dimension function of any building block $A_m$ or $B_m$ is taken by the homomorphism $\phi_{m,m+1}$ respectively $\psi_{m,m+1}$ into a function smaller than or equal to the dimension function of $A_{m+1}$ or $B_{m+1}$ such that a non-zero gap arises. In other words we want to exclude the possible cases when the dimension function is taken into the next stage dimension function such that equality holds at a point or at more points. We shall show this in the following lemma. Recall that because of Proposition 7.1, the algebras that we want to classify can be assumed to be inductive limits of special
continuous trace C*-algebras with spectrum [0, 1], i.e., $A \cong \lim\to (A_n, \phi_{nm})$ and $B \cong \lim\to (B_n, \psi_{nm})$, where $A_n, B_n$ are special continuous trace C*-algebras.

**Lemma 8.1.** Let $A = \lim\to (A_n, \phi_{nm})$ be a simple C*-algebra, where each $A_n$ is a special continuous trace C*-algebra with spectrum the closed interval [0, 1] and the dimension function assumed to be a finite-valued bounded function. Then there exists $\delta_1 > 0$, a subsequence $(A_{n_i})_{n_i \geq 0}$ of $(A_n)_n$ and a sequence of maps $\phi_i : A_{n_i} \to A_{n_i+1}$ such that

1. $A \cong \lim\to (A_{n_i}, \phi_{n_i,m_i})$,
2. $(\phi_{n_i,n_2})_T(\hat{P}_{A_{n_1}}) + \delta_1 < \hat{P}_{A_{n_2}}$,

where the inequality holds pointwise, $(\phi_{nm})_T$ is the induced map at the level of the affine function spaces, $P_{A_{n_1}}$ and $P_{A_{n_2}}$ are the units of the biduals of $A_{n_1}$ and $A_{n_2}$, and $\hat{P}_{A_{n_1}}$ and $\hat{P}_{A_{n_2}}$ denote the corresponding lower semicontinuous functions.

**Proof.** Let $A$ be equal to $\lim\to A_n$ with maps $\phi_{n,m} : A_n \to A_m$.

The plan is to keep the same building blocks and to change slightly the maps with respect to some given finite sets such that the desired property holds. To do this we use the property that the building blocks that appear in the inductive limit decomposition are weakly semiprojective.

Assume that the dimension function of $\phi_{12}(A_1)$ equals the dimension function of $A_2$ at some point or even everywhere and let $\epsilon > 0$, $F_1 \subset A_1$ be given. Because the largest value of the dimension function of the hereditary sub-C*-algebra generated by $\phi_{12}(A_1)$ inside $A_2$ is attained on an open subset $U$ of [0, 1], let us construct another dimension function as follows: shrink one of the open intervals of the open set $U$ to get $U'$ and in exchange enlarge the interval adjacent to that discontinuity point. $U'$ is constructed in such a way that is as close as necessary to the given $U$.

In this manner we find a sub-C*-algebra $B$ which is as close as we want to the hereditary sub-C*-algebra generated by $\phi_{12}(A_1)$ inside of $A_2$. Next we use that $A_1$ is weakly semiprojective to find another *-homomorphism $\rho_1 : A_1 \to B$ which is close within the given $\epsilon$ on the given finite set $F_1$.
Then there exists some open interval between the dimension function of $A_2$ and the dimension function of the $B$. This open interval corresponds to a non-zero ideal $I_1$ inside of $A_2$. Now the image of $I_1$ in the inductive limit is also a non-zero ideal. Since the inductive limit is simple, it implies that the ideal is the whole algebra. We know that there are full projections in the inductive limit. Therefore there is a finite stage in the inductive limit of the ideals coming from $I_1$ that has a full projection. Assume that the finite stage is inside of $A_k$. This means that at that stage the image of the ideal $I_1$ is $A_k$. Pick a strictly positive element $a_1$ in $I_1$. Then the image of $a_1$ in $A_k$ will be strictly positive at each point from $[0, 1], k > 1$. This shows that the image of the dimension function $d_B$ inside the dimension of $A_k$ has a gap of at least 1 everywhere in $[0, 1]$.

Because of the normalizations of the affine function, this gap of size 1 will correspond to some strictly non-zero $\delta_1$. To complete the proof we relabel $B$ as $A_{n_1}, A_k$ as $A_{n_2}$ etc.

□

Corollary 8.2. Let $A = \lim(A_n, \phi_{n,m})$ be a simple C*-algebra. Then there exists a sequence $(\delta_i)_{i \geq 1}, \delta_i > 0$, a subsequence of algebras $(A_{n_i})_{i \geq 1}$ of $(A_i)_{i \geq 1}$ and a sequence of maps $\phi : A_{n_i} \to A_{n_{i+1}}$ such that:

1. $A \cong \lim(A_{n_i}, \phi_{n_i,m_i}),$
2. $\phi_{T_{n_i, n_{i+1}}}(\hat{P}_{A_{n_i}}) + \delta_i < \hat{P}_{A_{n_{i+1}}}.$

Proof. Follows by successively applying the previous lemma. □

9. Pulling back of the isomorphism between inductive limits at the level of the invariant

Step 1 The intertwining between the stable part of the invariant

With no loss of generality we assume that the building blocks have the
following concrete representation

\[
\begin{pmatrix}
C_0(A_1) & C_0(A_1) & C_0(A_1) & \ldots & C_0(A_1) \\
C_0(A_1) & C_0(A_2) & C_0(A_2) & \ldots & C_0(A_2) \\
C_0(A_1) & C_0(A_2) & C_0(A_3) & \ldots & C_0(A_3) \\
\vdots & \vdots & \vdots & \ddots & \\
C_0(A_1) & C_0(A_2) & C_0(A_3) & \ldots & C[0,1]
\end{pmatrix}
\]

One can distinguish a full unital hereditary sub-C*-algebra

\[
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & \ddots & 0 & \ldots & 0 \\
0 & \ldots & C[0,1] & \ldots & C[0,1] \\
\vdots & \vdots & \vdots & \ddots & \\
0 & \ldots & C'[0,1] & \ldots & C'[0,1]
\end{pmatrix}
\]

The unital hereditary sub-C*-algebra has the same stable invariant (i.e., $K_0$, $\text{AffT}^+$ and the pairing) as the given C*-algebra. Moreover the unital hereditary sub-C*-algebra is a full matrix algebra over the closed interval $[0,1]$. Using this fact we derive an intertwining between the stable invariant, as is shown in [31] or originally in [12].

It is important to mention the method of normalizing the affine function spaces. Pick a full projection $p_1 \in A_1$. Normalize the affine space $\text{AffT}^+ A_1$ with respect to $p_1$. Next consider a image of $p_1$ in $A_2$ under the map at the dimension range level, call it $p_2$. Normalize $\text{AffT}^+ A_2$ with respect to $p_2$. Note that the map which is induced at the affine level is a contraction. Continue in this way so that we obtain an inductive limit sequence at the level of the
affine spaces, with all the maps being contractions:

\[
\text{Aff}^T A_1 \to \text{Aff}^T A_2 \to \cdots \to \text{Aff}^T A.
\]

Let \( p_\infty \) denote the image of \( p_1 \) in the inductive limit \( A \) and denote by \( q_\infty \) a representative of \( \phi_0(p_\infty) \) in \( B \). Then there exists \( q_1 \in B_1 \) such that the image of \( q_1 \) is \( q_\infty \) in the inductive limit. Normalize the \( \text{Aff}^T B_1 \) with respect to \( q_1 \), \( \text{Aff}^T B_2 \) with respect to a image of \( q_1 \) in \( B_2 \) and so on. Hence we obtain another inductive limit of affine spaces with contractions maps

\[
\text{Aff}^T A_1 \to \text{Aff}^T A_2 \to \cdots \to \text{Aff}^T A
\]

\[
\text{Aff}^T B_1 \to \text{Aff}^T B_2 \to \cdots \to \text{Aff}^T B
\]

As already mentioned above, we pull back the invariant for the unital hereditary sub-C*-algebras (i.e. full matrix algebras or the stable invariant). This will give rise to an exact commuting diagram at the \( K_0 \)-level, an approximate commuting diagram at the affine function spaces level and an exact pairing. The compatibility can be made exact as shown in [11] by noticing that, because of simplicity, non-zero positive elements in both \( K_0 \) and \( \text{Aff}^T \) are sent into strictly positive elements and then normalize the affine function spaces in a suitable way.

To summarize, we now have a commutative diagram

\[
\begin{array}{cccccccc}
C[0, 1] & \overset{\phi_{12}}{\longrightarrow} & C[0, 1] & \overset{\phi_{23}}{\longrightarrow} & \cdots & \longrightarrow & (\text{Aff}^T A, \text{Aff}' A) \\
\downarrow \tau_1 & \nearrow \tau'_1 & \downarrow \tau_2 & \nearrow \tau'_2 & \nearrow \tau'_3 & \cup & \\
C[0, 1] & \overset{\psi_{12}}{\longrightarrow} & C[0, 1] & \overset{\psi_{23}}{\longrightarrow} & \cdots & \longrightarrow & (\text{Aff}^T B, \text{Aff}' B)
\end{array}
\]

where \( \text{Aff}^T A_i \) and \( \text{Aff}^T B_i \) are identified with \( C([0, 1]) \) and each finite stage algebra \( A_i \) and \( B_i \) is assumed to have only one direct summand.

For us it is very important to study the pulling back of the non-stable part
Step 2. The intertwining of the non-stable part of the invariant

As I. Stevens mentioned in [31], at this moment we know that the non-stable part of the invariant is only approximately mapped at a later stage into the non-stable part of the invariant.

To be able to apply the Existence Theorem 5.1, one needs to check that hypothesis 2 can be ensured. Otherwise, a counterexample can be given to the Existence Theorem, as shown in Section 5.1 above. The special assumption from the hypothesis of the isomorphism theorem, $\phi_T(\text{Aff}'A) \subseteq \text{Aff}'B$, as well as Corollary 8.2 will be used to prove the above mentioned claim.

By applying Corollary 8.2 to the given inductive limits $A = \lim_\to(A_n, \phi_{n,m})$, $B = \lim_\to(B_n, \phi_{n,m})$ we get two sequences $(\delta_i)_{i \geq 1}, \delta_i > 0$ and $(\delta'_i)_{i \geq 1}, \delta'_i > 0$ respectively, and two subsequences of algebras such that after relabeling, we can assume that $\phi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}, \psi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}, \phi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}$ and $\psi_{ii+1}(\hat{P}_{A_i}) + \delta_i < \hat{P}_{A_{i+1}}$ for all $i \geq 1$.

Reworking the intertwining of the stable invariant for the new sequences of algebras and the new maps that have gaps $\delta_i$ we obtain the following intertwining

$$C([0, 1]) \xrightarrow{\phi_{12}} C([0, 1]) \xrightarrow{\phi_{23}} \ldots \xrightarrow{\phi_{T}} (\text{Aff}^+T, \text{Aff}'A)$$

$$\downarrow \tau_1 \xrightarrow{\tau'_1} \downarrow \tau_2 \xrightarrow{\tau'_2} \ldots \xrightarrow{\tau'_T}$$

$$C[0, 1] \xrightarrow{\psi_{12}} C[0, 1] \xrightarrow{\psi_{23}} \ldots \xrightarrow{\psi_{T}} (\text{Aff}^+T, \text{Aff}'B)$$

As a consequence of the Thomsen-Li theorem, which in the present case states that the closed convex hull of the set of all unital *-homomorphisms of $C([0, 1])$ in the strong operator topology is exactly the set of positive of unital operators on $C([0, 1])$, we can assume that all the maps $\phi_{ii+1}, \psi_{ii+1}, \tau_i, \tau'_i$ are
given by eigenvalue patterns. Because each such map takes the unit, say $\hat{p}$, into the unit, $\hat{K}(p)$, it follows that each map is an average of the eigenvalues, i.e., $\phi_{i,i+1}(f) = \sum_{i=1}^{N_i} \frac{f \circ \lambda}{N_i}$, etc.

Let $\hat{P}_{A_1}$ be the image in the affine function space of the unit in the bidual of $A_1$. Take a continuous function $f$ smaller than $\hat{P}_{A_1}$. It is important to say that there are no extra conditions on $f$, i.e., $f$ can be any element of the special set Aff$^\prime A_1$. Then there exists $\delta_1 > 0$ such that

$$\phi_{12}(\hat{P}_{A_1}) + \delta_1 < \hat{P}_{A_2}.$$ 

Since $\phi_{12}(f) \leq \phi_{12}(\hat{P}_{A_1})$ we have

$$\phi_{12}(f + \delta_1) \leq \phi_{12}(\hat{P}_{A_1} + \delta_1) < \hat{P}_{A_2}.$$ 

Since $\phi_T(Aff' A) \subseteq Aff' B$, it follows that there exists a large $N$ and $\epsilon_N \leq \delta_1$ such that

$$\tau_N \circ \phi_{N-2N-1} \circ \ldots \phi_{12}(f + \delta_1) < \hat{P}_{B_N} + \epsilon_N.$$ 

It is important to say that a different choice for $f$ will give rise to possibly different $N$. This is not a difficulty because we can always pass to subsequence. Equivalently we have

$$\tau_N \circ \phi_{N-2N-1} \circ \ldots \phi_{12}(f) + \delta_1 < \hat{P}_{B_N} + \epsilon_N.$$ 

Using $\delta_1 \geq \epsilon_N$ we conclude

$$\tau_N \circ \phi_{N-2N-1} \circ \ldots \phi_{12}(f) < \hat{P}_{B_N},$$

which is the desired strict inequality from the hypothesis 2 of the Existence Theorem 5.1.
10. The Isomorphism Theorem

To complete the proof of the Isomorphism Theorem 3.1 for the algebras 
\( \lim_{\to} A_i = A \) and \( \lim_{\to} B_i = B \), we have to construct an approximate commutative diagram at the algebra level in the following sense, as was defined by Elliott in [11], “for any fixed element in any \( A_i \) (or \( B_i \)), the difference of the images of this element along two different paths in the diagram, starting at \( A_i \) (or \( B_i \)) and ending at the same place, converges to zero as the number of steps for which the two paths coincide, starting at the beginning, tends to infinity.”

At this stage because of Step 2 of the previous section, Section 9, we can apply the Existence Theorem to generate a sequence of algebra homomorphisms \( \nu_1, \nu_2, \ldots \) and \( \nu'_1, \nu'_2, \ldots \) such that \( \|\tau_i(f) - \nu_{i*}(f)\| \leq \frac{\varepsilon}{2^i} \) and \( \|\tau'_i(f) - \nu'_{i*}(f)\| \leq \frac{\varepsilon}{2^i} \) for \( f \in F_i \) and \( g \in G_i \), where \( \nu_{i*}, \nu'_{i*} \), are the induced affine maps by algebra maps \( \nu_i, \nu'_i \), and \( F_i \) and \( G_i \) are finite sets.

After relabeling the indices of the inductive limit systems we now have a (not necessarily approximately commutative) diagram of algebra homomorphisms

\[
\begin{array}{cccc}
A_1 & \phi_{12} & A_2 & \phi_{23} & \ldots & A \\
\downarrow \tau_1 & \uparrow \tau'_1 & \downarrow \tau_2 & \uparrow \tau'_2 & \\
B_1 & \psi_{12} & B_2 & \psi_{23} & \ldots & B
\end{array}
\]

that induces an approximately commutative diagram at the level of the invariant.

This will be done with respect to given arbitrary finite sets \( F_i \subset A_i \) and \( G_i \subset B_i \).

To make the diagram approximately commuting we modify the diagonal maps by composing with approximately inner automorphisms and this will
be done with respect to a given arbitrary finite sets \( F_i \subset A_i \) and \( G_i \subset B_i \) with dense union in \( A \) and \( B \) respectively.

Here we notice that we can apply the Uniqueness Theorem to the data obtained from the Existence Theorem because our inequalities are balanced.

For every \( \epsilon > 0 \) we find an increasing sequence of integers \( 1 = M_0 < L_1 < M_2 < L_2 < \ldots \) and unitaries \( (U_{M_i+1}) \in A^+_{M_i+1}, (V_{L_i}^n) \in B^+_{L_i} \) such that for \( f \in F_{M_i} \) and \( g \in G_{L_i} \) we have

\[
\frac{||U_{M_i+1}(\tau_{M_i}(f))U^*_{M_i+1} - \phi_{M_i,M_{i+1}}(f)||}{||f||} < \frac{\epsilon}{2^i},
\]

\[
\frac{||V_{L_i+1}(\tau_{L_i}(g))V^*_{L_i+1} - \phi_{L_i,L_{i+1}}(g)||}{||g||} < \frac{\epsilon}{2^i}.
\]

In other words passing to suitable subsequences of algebras, it is possible to perturb each of the homomorphisms obtained in the Existence Theorem by an approximately inner automorphism, in such a way that the diagram becomes an approximate intertwining, in the sense of Theorem 2.1, [11].

Therefore, by the Elliott approximate intertwining theorem (see [11], Theorem 2.1), the algebras \( A \) and \( B \) are isomorphic.

11. The Range of the Invariant

In this section we prove Theorem 3.2 which answers the question what are the possible values of the invariant from the isomorphism theorem 3.1. It is useful to notice that the invariant consists of two parts. One part is the stable part, i.e., \( K_0, \text{Aff}T^+, \lambda : T^+ \leftrightarrow S(K_0) \) which was shown by K. Thomsen in [34] to be necessary if one wants to construct an AI-algebra, and the other part which one may call the non-stable part, namely \( \text{Aff}' \) or equivalently, as shown in [31], Remark 30.1.1 and Remark 30.1.2, the trace norm map. It is the non-stable part of the invariant that one needs to investigate in its full generality. Next the definition of the trace norm map is introduced.
Definition 11.1. Let $\mathcal{A}$ be a sub-C*-algebra of a $C^*$-algebra $\mathcal{B}$. The trace norm map associated to $\mathcal{A}$ is a function $f : T^+(\mathcal{A}) \to (0, \infty]$ such that $f(\tau) = ||\tau|_{\mathcal{A}}||$, $\infty$ if $\tau$ is unbounded.

Recall that:

Definition 11.2. $T^+(\mathcal{A})$ is the cone of positive trace functionals on $\mathcal{A}$ with the inherited $w^*$-topology.

Remark 11.1. The trace norm map is a lower semicontinuous affine map (being a supremum of a sequence of continuous functions).

Remark 11.2. The dimension range can be determined using the values of the trace norm map $f$, the simplex of tracial states $S$ and dimension group $G$. A formula for the dimension range $D$ is:

$$D = \{x \in G/v(x) < f(v), v \in S, v \neq 0\}$$

I. Stevens has constructed a hereditary sub-C*-algebra of a simple (unital) AI-algebra which is obtained as an inductive limit of hereditary sub-C*-algebras of interval algebras, and has as a trace norm map any given affine continuous function; cf. [31], Proposition 30.1.7. Moreover she showed that any lower semicontinuous map can be realized as a trace norm map in a special case. Our result is a generalization to the case of unbounded trace norm map when restricted to the base of the cone. It is worth mentioning that our approach gives another proof in the case of any lower semicontinuous map as a trace norm map. Still our approach is using the I. Stevens’s proof for the case of continuous trace norm map.

Theorem 3.2 Suppose that $G$ is a simple countable dimension group, $V$ is the cone associated to a metrizable Choquet simplex. Let $\lambda : V \to Hom^+(G, R)$ be a continuous affine map and taking extreme rays into extreme rays. Let $f : V \to [0, \infty]$ be an affine lower semicontinuous map, zero at zero and only at zero. Then $[G, V, \lambda, f]$ is the Elliott invariant of some simple non-unital
inductive limit of continuous trace $C^*$-algebras whose spectrum is the closed interval $[0, 1]$ or a finite disjoint union of closed intervals.

Proof. The proof is based on I. Stevens’s proof in a special case and consists of several steps.

Step 0

We start by constructing a simple stable AI algebra $\mathcal{A}$ with its Elliott invariant: $[(G, D), V, \lambda]$. We know that this is possible (see [30]). By tensoring with the algebra of compact operators we may assume $\mathcal{A}$ is a simple stable AI algebra.

Step 1

We restrict the map $f$ to some base $S$ of the cone $T^+(\mathcal{A})$, where the cone $V$ is naturally identified with $T^+(\mathcal{A})$. Since any lower semicontinuous affine map $f : S \to (0, +\infty]$ is a pointwise limit of an increasing sequence of continuous affine positive maps, (see [2]), we can choose $f = \lim f_n$, where $f_n$ are continuous affine and strictly positive functions.

Moreover by considering the sequence of functions $g_n = f_{n+1} - f_n$ if $n > 1$ and $g_1 = f_1$ we get that:

$$\sum_{n=1}^{\infty} g_n = f$$

Step 2

Next we use the results of Stevens ([31], Prop. 30.1.7), to realize each such continuous affine map $g_n$ as the norm map of a hereditary sub-$C^*$-algebra $\mathcal{B}_n$ (which is an inductive limit of special algebra) of the AI algebra $\mathcal{A}$ obtained at Step 0.

Consider the $L^\infty$ direct sum $\bigoplus \mathcal{B}_i$ as a sub-$C^*$-algebra of $\mathcal{A}$. The trace norm map of the sub-$C^*$-algebra $\bigoplus \mathcal{B}_i$ of $\mathcal{A}$ is equal to $\sum_{i=1}^{\infty} g_n = f$. 
To see that $\oplus \mathcal{B}_i$ is a sub-$C^*$-algebra of $\mathcal{A}$ we use that $\mathcal{A}$ is a stable $C^*$-algebra:

$$\oplus \mathcal{B}_i = \begin{pmatrix} \mathcal{B}_1 & 0 \\ \mathcal{B}_2 & \ddots \\ 0 & \ddots \end{pmatrix} \subseteq \mathcal{A} \otimes \mathbb{K} \cong \mathcal{A}.$$  

Next denote with $\mathcal{H}$ the hereditary sub-$C^*$-algebra generated by $\oplus \mathcal{B}_i$ inside of $\mathcal{A}$.

To prove that the trace norm map of $\mathcal{H}$ is $f$ is enough to show that the norm of a trace on $\oplus \mathcal{B}_i$ is the same as on $\mathcal{H}$.

It suffices to prove that an approximate unit of the sub-$C^*$-algebra $\oplus \mathcal{B}_i$ is still an approximate unit for the hereditary sub-$C^*$-algebra $\mathcal{H}$.

We shall prove first that the hereditary sub-$C^*$-algebra generated by $\oplus \mathcal{B}_i$ coincides with the hereditary sub-$C^*$-algebra generated by one of its approximate units. Let $(u_\lambda)_\lambda$ be an approximate unit of $\oplus \mathcal{B}_i$. Denote by $\mathcal{U}$ the hereditary sub-$C^*$-algebra of $\mathcal{H}$ generated by $\{(u_\lambda)_\lambda\}$. We want to prove that $\mathcal{U}$ is equal with $\mathcal{H}$.

Since $(u_\lambda)_\lambda$ is a subset of $\oplus \mathcal{B}_i$ we clearly have

$$\mathcal{U} \subseteq \mathcal{H}.$$  

For the other inclusion, one can observe that

for all $b \in \oplus \mathcal{B}_i : \ b = \lim_{\lambda \to \infty} u_\lambda b u_\lambda.$

Now each $u_\lambda b u_\lambda$ is an element of the hereditary sub-$C^*$-algebra generated by $(u_\lambda)_\lambda$ and hence $b \in \mathcal{U}$. Therefore $\oplus \mathcal{B}_i \subset \mathcal{U}$ which implies $\mathcal{H} \subseteq \mathcal{U}$.

We conclude that $\mathcal{H} = \mathcal{U}$ and hence the trace norm map of $\mathcal{H}$ is $f$. Therefore
Remark 11.3. The approximate unit $(u_\lambda)_\lambda$ of $\bigoplus B_i$ is still an approximate unit for the hereditary sub-C*-algebra $U$. To see why this is true let us consider the sub-C*-algebra of $A$ defined as follows: $\{ h \in A \mid h = \lim_{\lambda \to \infty} u_\lambda h \}$.

This sub-C*-algebra of $A$ is a hereditary sub-C*-algebra. Indeed let $0 \leq k \leq h$ with $h = \lim_{\lambda \to \infty} u_\lambda h$. We want to prove that $k = \lim_{\lambda \to \infty} u_\lambda k$.

Consider the hereditary sub-C*-algebra $\overline{hAh}$ of $A$ which clearly contains $h$ (because $h^2 = \lim_{\lambda \to \infty} hu_\lambda h$). Therefore $k \in \overline{hAh}$.

Since $h = \lim_{\lambda \to \infty} u_\lambda h$ we obtain that $u_\lambda$ is an approximate unit for $\overline{hAh}$. In particular

$$k = \lim_{\lambda \to \infty} u_\lambda k$$

and hence $\{ h \in A \mid h = \lim_{\lambda \to \infty} u_\lambda h \}$ is a hereditary sub-C*-algebra of $A$. Since $U$ is the smallest hereditary containing $(u_\lambda)_\lambda$ we get that

$$U \subseteq \{ h \in A \mid h = \lim_{\lambda \to \infty} u_\lambda h \}$$

and $u_\lambda$ is an approximate unit for $U$.

12. Non-AI algebras which are inductive limits of continuous-trace C*-algebras

In this section we present a necessary and sufficient condition on the invariant for the algebra to be AI. We shall use this in the next section to construct an inductive limit of continuous trace C*-algebras with spectrum $[0, 1]$ which is not an AI algebra.

With $[G, V, \lambda, f]$ as before we observe that for an AI algebra with Elliott invariant canonically isomorphic to the given invariant the following equality
always holds:

$$f(v) = \sup\{v(g) : g \in D\},$$

where $D$ is the dimension range. This is seen by simply using the fact that any AI algebra has an approximate unit consisting of projections.

Therefore a sufficient condition imposed on the invariant in order to get an inductive limit of continuous trace C*-algebra with spectrum $[0,1]$ but not an AI algebra is

$$f(v) \neq \sup\{v(g) : g \in D\}.$$

This condition is also necessary. Namely assume that we have $f(v) = \sup\{v(g) : g \in D\}$ and we have constructed a simple C*-algebra $\mathcal{A}$ which is an inductive limit of continuous trace C*-algebras with spectrum $[0,1]$ and with the invariant canonically isomorphic with the tuple $[G,V,\lambda,f]$. Consider $D = \{x \in G : v(x) < f(v), v \in S, v \neq 0\}$, where $S$ is a base of the cone $V$. For the tuple $[G,D,V,S,\lambda]$ we can build (via the range of the invariant for simple AI algebras, [30]) a simple AI-algebra $\mathcal{B}$ with the invariant naturally isomorphic with the given tuple.

Note that the trace norm map which is defined starting from the tuple $[K_0(\mathcal{B}), D(\mathcal{B}), T^+\mathcal{B}, \lambda_\mathcal{B}]$ is exactly $f$ because of the equality

$$f(v) = \sup\{v(g) : g \in D\}$$

and $\mathcal{B}$ is an AI algebra.

It is clear that $\mathcal{B}$ is an inductive limit of continuous trace C*-algebras with spectrum $[0,1]$ and hence by the isomorphism theorem 2.1 we conclude that $\mathcal{A}$ isomorphic to $\mathcal{B}$. Hence $\mathcal{A}$ is a simple AI algebra as desired and we have proved the following theorem:
Theorem 12.1. Let $\mathcal{A}$ be a simple $C^*$-algebra which is an inductive limit of continuous-trace $C^*$-algebras whose spectrum is homeomorphic to $[0, 1]$. A necessary and sufficient condition for $\mathcal{A}$ to be a simple AI algebra is

$$f(v) = \sup\{v(g) : g \in D\}.$$ 

13. The class of simple inductive limits of continuous trace $C^*$-algebras with spectrum $[0, 1]$ is much larger than the class of simple AI algebras

To see this consider the simple AI algebra necessarily not of real rank zero with scaled dimension group $(\mathbb{Q}, \mathbb{Q}^+)$ and cone of positive trace functionals a 2-dimensional cone; see [30]. Then the set of possible stably AI algebras, or equivalently the set of possible trace norm maps, may be represented as the extended affine space shown in the following schematic diagram:

![Figure 6](image)

Each off-diagonal point in the diagram is the trace norm map of one of I. Stevens’s algebras. The boundary points of the first quadrant are removed (dotted lines) and the points with infinite coordinates are allowed. The dimension range is embedded in a canonical way in the extended affine space as the main diagonal consisting of the points with rational coordinates.

The two bold lines represent the cases of inductive limits of continuous
trace C*-algebras with unbounded trace norm map (points on these two lines have at least one coordinate infinity).

If the point is off the diagonal and in the first quadrant, by Theorem 12.1 we get that the corresponding C*-algebra is an inductive limit of continuous trace C*-algebras which is not AI-algebra. It is clear that the size of the set of points off the diagonal is much larger then the size of the set of points on the diagonal. (For instance in terms of the Lebesgue measure.)

This picture shows that the class of simple AI algebras sits inside the class of inductive limits of continuous trace C*-algebras in the same way that the main diagonal sits inside the first quadrant.

References

[1] C. A. Akemann, G. K. Pedersen, Ideal perturbation of elements in C*-algebras, Math. Scand. 41 (1997), 117–139.
[2] E. M. Alfsen, Compact convex sets and boundary integrals, Springer-Verlag, New York, 1971.
[3] L. G. Brown, Stable isomorphism of hereditary subalgebras of C*-algebras, Pacific J. Math. 71, (1977), 335–348.
[4] L. G. Brown, P. Green and M. A. Rieffel, Stable isomorphism and strong Morita equivalence, Pacific J. Math. 71 (1977), 349–363.
[5] L. G. Brown, G. K. Pedersen, C*-algebras of real rank zero, J. Funct. Anal. 99 (1991), 131–149.
[6] M. D. Choi, G. A. Elliott, Density of the selfadjoint elements with finite spectrum in an irrational rotation C*-algebra, Math. Scand. 67 (1990), 73–86.
[7] J. Dixmier, Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.
[8] S. Eilers, T. Loring, G. K. Pedersen, Stability of anticommutation relations: an application of noncommutative CW complexes, J. Reigne. Angew. Math. 499 (1998), 101–143.
[9] G. A. Elliott, Ideal preserving automorphisms of postliminary C*-algebras, Proc. Amer. Math. Soc. 27 (1971), 107–109.
[10] G. A. Elliott, Automorphisms determined by multipliers on ideals of a C*-algebra, J. Funct. Anal. 23 (1976), 1–10.
[11] G. A. Elliott, *On the classification of $C^*$-algebras of real rank zero*. J. Reine Angew. Math. 443 (1993), 179–219.

[12] G. A. Elliott, *A classification of certain simple $C^*$-algebras*, Quantum and Non-Commutative Analysis (Kyoto, 1992), Math. Phys. Stud., 16, Kluwer Acad. Publ., Dordrecht, 1993, 373–385.

[13] G. A. Elliott, *The classification problem for amenable $C^*$-algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1,2 (Zurich, 1994), Birkhäuser, Basel, 1995, 922–932.

[14] G. A. Elliott, *An invariant for simple $C^*$-algebras*, Canadian Mathematical Society, 1945–1995, Vol.3, Canadian Math. Soc., Ottawa, 1996, 61–90.

[15] G. A. Elliott, G. Gong, *On the classification of $C^*$-algebras of real rank zero. II*, Ann. of Math. 144 (1996), 497–610.

[16] G. A. Elliott, G. Gong, L. Li, *On the classification of simple inductive limit $C^*$-algebras, II: The isomorphism theorem*, preprint.

[17] C. Ivanescu, *On the classification of simple $C^*$-algebras which are inductive limits of continuous-trace $C^*$-algebras with spectrum the closed interval $[0, 1]$*, Ph.D. thesis, University of Toronto, 2004.

[18] C. Ivanescu, *On the classification of continuous-trace $C^*$-algebras with spectrum the closed interval $[0, 1]$*, to appear in ”The Proceedings of the International Conference on Operator Algebras and Mathematical Physics”, Sinaia, 2003.

[19] X. Jiang, H. Su, *A classification of simple limits of splitting interval algebras*, J. Funct. Anal. 151 (1997), 50–76.

[20] E. Kirchberg, S. Wassermann, *Operations on continuous bundles of $C^*$-algebras*, Mathematische Annalen 303 (1995), 677–697.

[21] L. Li, *Classification of simple $C^*$-algebras: inductive limits of matrix algebras over 1-dimensional spaces*, Ph.D. thesis, University of Toronto, 1995.

[22] L. Li, *Simple inductive limit $C^*$-algebras: Spectra and approximations by interval algebras*, J. Reine Angew. Math. 507 (1999), 57–79.

[23] Q. Lin, N. C. Phillips, *Direct limit decomposition for $C^*$-algebras of minimal diffeomorphisms*, Operator algebras and applications, 107–133, Adv. Stud. Pure Math., 38, Math. Soc. Japan, Tokyo, 2004.

[24] T. Loring, *Lifting solutions to perturbing problems in $C^*$-algebras*, Fields Institute Monographs, Vol. 8, American Mathematical Society, Providence, RI, 1997.
[25] G. K. Pedersen, *C*-algebras and Their Automorphism Groups*, Academic Press, London/New York/San Francisco, 1979, MR 81e:46037.

[26] G. K. Pedersen, *Real rank of C*-algebras*, Proceedings of the Satellite Conference of ICM-90 113 (1988), 659–700. MR 89b:58085.

[27] S. Razak, *On the classification of simple stably projectionless C*-algebras*, Canad. J. Math. 54 (2002), 138–224.

[28] M. Rordam, *Classification of nuclear C*-algebras. Entropy in operator algebras*, 1–145, Encyclopedia Math. Sci., 126, Springer, Berlin, (2002).

[29] J. Mygind, *Classification of certain simple C*-algebras with torsion in K1*, Canad. J. Math. 53 (2001), 1223–1308.

[30] I. Stevens, *Simple approximate circle algebras*, II, Operator Algebras and their Applications, Fields Institute Communications, Vol. 20, American Mathematical Society, Providence, RI, 1998, 97–104.

[31] I. Stevens, *Hereditary subalgebras of certain simple non real rank zero C*-algebras*, Lectures on Operator Theory, Fields Institute Communications, Vol. 13, American Mathematical Society, Providence, RI, 1999, 209–241.

[32] K. Stevens, *The classification of certain non-simple approximate interval algebras*, Ph. D. thesis, 1994; Fields Institute Communications, 20, American Mathematical Society, Providence, RI, 1998, 105–148.

[33] H. Su, *On the classification of C*-algebras: Inductive limits of matrix algebras over non-Hausdorff graphs*, Memoirs Amer. Math. Soc. Number 547 114 (1995).

[34] K. Thomsen, *Inductive limits of interval algebras: The tracial state space*, Amer. J. Math. 116 (1994), 605–620.

[35] J. Villadsen, *The range of the Elliott invariant*, J. Reine Angew. Math. 462 (1995), 31–55.

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