Căldăraru’s conjecture on abelian categories of twisted sheaves

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Abstract

We use an idea of Rosenberg to prove a reconstruction theorem for abelian categories of $\alpha$-twisted quasi-coherent sheaves on quasi-compact and quasi-separated schemes $X$ when $\alpha \in \text{Br}(X)$. By applying the work of Toën on derived Azumaya algebras, we give a proof of Căldăraru’s conjecture.

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1 Introduction

We prove the following result, a generalization of Căldăraru’s conjecture [2, Conjecture 4.1].

Theorem 1.1 (Căldăraru’s conjecture). Let $X$ and $Y$ be quasi-compact and quasi-separated schemes over a commutative ring $R$ (for instance over $\mathbb{Z}$), and fix $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. Suppose that there is an equivalence $\text{QCoh}(X, \alpha) \simeq \text{QCoh}(Y, \beta)$ of $R$-linear abelian categories of quasi-coherent twisted sheaves. Then, there is an isomorphism $f : X \rightarrow Y$ of $R$-schemes such that $f^*(\beta) = \alpha$.

We also prove an analogous theorem for noetherian schemes $X$ and $Y$ where we only require $\alpha \in \text{Br}^e(X)$ and $\beta \in \text{Br}^e(Y)$. This latter theorem relies on a reconstruction theorem of Perego [8].

The theorem has been established previously by Canonaco and Stellari [3] when $X$ and $Y$ are smooth projective varieties over an algebraically closed field. The methods of Canonaco and Stellari use Fourier-Mukai functors and pass through derived categories. Our methods also use derived categories, or rather their dg categorical enhancements.

The proof has three main steps. First, we show that $X$ as in the theorem can be reconstructed from the abelian category $\text{QCoh}(X, \alpha)$. This is an extension of a theorem of Rosenberg [9] and has its roots in Gabriel’s thesis [5], where the statement for abelian categories of untwisted quasi-coherent sheaves was proved for noetherian schemes. Second, we show that the reconstruction theorem results in an isomorphism $f : X \rightarrow Y$ and an equivalence $\mathcal{O}\text{Coh}(\alpha) \simeq \mathcal{O}\text{Coh}(f^*(\beta))$ of Zariski stacks of abelian categories on $X$. It remains to prove that the existence of this equivalence of stacks implies that $f^*(\beta) = \alpha$. We
do this by inducing an equivalence \( D(\alpha) \simeq D(f^*(\beta)) \) of étale stacks of dg categories. But, Toën showed that the derived Brauer group \( \text{dBr}(X) = H^1_{\text{ét}}(X, \mathbb{Z}) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \) classifies stacks of dg categories that are étale locally derived Morita equivalent to the base. Since \( \text{Br}(X) \subseteq \text{dBr}(X) \), the theorem follows. Instead of using Toën, we could also use stable \( \infty \)-categories and the work of [1].

We find the passage through derived algebraic geometry to be a beautiful application of that new field to ordinary algebraic geometry.

## 2 Background on abelian categories

In this section, we review some basic concepts from the theory of abelian categories. These ideas are due to Gabriel, Grothendieck, and Serre. For details and proofs, see [5] or [6].

We consider two types of abelian categories, those that behave like the abelian category of quasi-coherent sheaves \( \text{QCoh}(X) \) when \( X \) is quasi-compact and quasi-separated, and those that behave like the abelian category of coherent sheaves \( \text{Coh}(X) \) when \( X \) is noetherian. These are “big” and “small” abelian categories, respectively.

**Definition 2.1.** Let \( A \) be an abelian category. A non-empty full subcategory \( B \subseteq A \) is thick (or épaisse) if it is closed under taking subobjects, quotients, and extensions; in other words, if for every exact sequence

\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]

in \( A \), we have \( M \in B \) if and only if \( M' \) and \( M'' \) are in \( B \).

If \( B \subseteq A \) is thick, then following Serre we can often define a quotient abelian category \( A/B \), which has as objects the same objects as \( A \) and where

\[
\text{Hom}_{A/B}(M, N) = \text{colim} \text{Hom}_A(M', N/N')
\]

where the colimit is over all \( M' \subseteq M \) and \( N' \subseteq N \) where \( M/M' \in B \) and \( N' \in B \) and is taken in the abelian category of abelian groups. In order for the colimit to exist in \( \text{Mod}_\mathbb{Z} \) (in our fixed universe), we need to know that the colimit diagram is essentially small. This is guaranteed if either \( A \) is essentially small (as is the case for \( \text{Coh}(X) \) when \( X \) is abelian) or is a Grothendieck abelian category (such as \( \text{QCoh}(X) \) for \( X \) quasi-compact and quasi-separated), a concept we now define.

**Definition 2.2.** An abelian category \( A \) is said to satisfy AB3 if it has all small coproducts (this implies that \( A \) has all colimits). An abelian category \( A \) satisfies AB5 if it satisfies AB3 and if the following condition holds: whenever \( M = \bigcup_{i=1}^\infty M_i \) where \( M_1 \subseteq M_2 \subseteq \cdots \), the natural map \( \text{Hom}(M, N) \rightarrow \text{lim} \text{Hom}(M_i, N) \) is an isomorphism. A generator of an abelian category is an object \( U \) such that if \( N \subseteq M \) is a proper subobject, there is a morphism \( U \rightarrow M \) that does not factor through \( N \). An abelian category is called a Grothendieck abelian category if it satisfies AB5 and has a generator.

If \( A \) possesses a generator, then every object of \( A \) has a set of subobjects and quotient objects. It follows that for any thick subcategory \( B \subseteq A \), the colimit appearing in (1) is over a small diagram, and hence exists in \( \text{Mod}_\mathbb{Z} \). This will guarantee that all quotients taken in this paper are well-defined without expanding the universe.

**Proposition 2.3.** Let \( B \subseteq A \) be a thick subcategory, and assume that \( B/A \) exists. Then,
1. $A/B$ is abelian;

2. the natural map $A \to A/B$ is exact.

**Definition 2.4.** A thick subcategory $B \subseteq A$ is localizing if the functor $j : A \to A/B$ admits a right adjoint $j_*$.

**Lemma 2.5.** If $B \subseteq A$ is localizing, then the right adjoint $j_*$ is fully faithful and left exact.

**Proof.** The fully faithfulness of $j_*$ can be seen by applying (1) to two objects in the image of $j_*:$ they have no non-zero subquotients contained in $B$. Since $j_*$ is a right adjoint, it preserves limits that exist. Since $A$ and $A/B$ have finite limits, it preserves finite limits. But, kernels are finite limits, hence $j_*$ is left exact. ■

The following proposition is very useful for checking that a subcategory of a Grothendieck abelian category is localizing.

**Proposition 2.6.** Let $A$ be a Grothendieck abelian category, and let $B \subseteq A$ be a thick subcategory. Then, the following are equivalent:

1. $B \subseteq A$ is localizing;

2. the inclusion $B \to A$ admits a right adjoint;

3. the inclusion $B \to A$ preserves colimits;

4. every object $M$ of $A$ contains a maximal subobject contained in $B$.

**Proof.** This is left to the reader. We remind them of the adjoint functor theorem and the fact that left adjoints preserve colimits. For details, see Gabriel [5, Section III]. ■

### 3 Abelian categories of twisted sheaves

Let $X$ be a scheme and let $\alpha \in H^2(X, G_m)$ be represented by a $G_m$-gerbe $\mathcal{X} \to X$. Then, we write $\text{QCoh}(X, \alpha)$ for the abelian category of $\mathcal{X}$-twisted sheaves $\text{QCoh}^{\text{tw}}(\mathcal{X})$, defined for instance in Lieblich [7]. An $\mathcal{X}$-twisted sheaf is a quasi-coherent sheaf on $\mathcal{X}$ such that the inertial action of $G_m$ on the left agrees with the action through $\mathcal{O}_\mathcal{X}$. If $\alpha$ is the Brauer class of an Azumaya algebra $A$ on $X$, then $\text{QCoh}(X, \alpha) \simeq \text{QCoh}(X, A)$, where $\text{QCoh}(X, A)$ denotes the abelian category of quasi-coherent left $A$-modules on $X$. If $f : Y \to X$ is an $X$-scheme, write $\text{QCoh}(Y, \alpha)$ for $\text{QCoh}(Y, f^*(\alpha))$. This defines a prestack $\mathcal{Q}\text{Coh}(\alpha)$ of abelian categories over $X$.

**Proposition 3.1.** The prestack $\mathcal{Q}\text{Coh}(\alpha)$ of $\mathcal{X}$-twisted sheaves forms a stack of abelian categories on the big étale site over $X$.

**Proof.** See [7, Proposition 2.1.2.3]. ■

**Proposition 3.2.** The abelian category $\text{QCoh}(X, \alpha)$ is a Grothendieck abelian category when $X$ is quasi-compact and quasi-separated.
Proof. This follows because the abelian category of all quasi-coherent modules on $X$ is a Grothendieck abelian category. The main task is to produce a generator, but this can be done by étale descent. Or, using the equivalence $\text{QCoh}(X, \alpha) \simeq \text{QCoh}(X, A)$, we can produce a generating set by taking a representative set of all $A$-modules of finite type. The direct sum of the elements of this set will be a generator.

**Remark 3.3.** The stack $\mathcal{Q}\mathcal{Coh}(\alpha)$ is a stack of $\mathcal{O}_X$-linear abelian categories, in the sense that for every $U \subseteq X$ there is the structure of $\Gamma(U, \mathcal{O}_X)$-linear abelian category on $\mathcal{Q}\mathcal{Coh}(U, \alpha)$, and these are compatible with restriction.

4 Quasi-coherent reconstruction

In this section, we prove the first part of the reconstruction theorem.

**Theorem 4.1.** Suppose that $X$ and $Y$ are quasi-compact and quasi-separated schemes over a commutative ring $R$ with Brauer classes $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. If there is an equivalence of $R$-linear abelian categories $\text{QCoh}(X, \alpha) \simeq \text{QCoh}(Y, \beta)$, then there is an isomorphism $f : X \to Y$ of $R$-schemes and a natural equivalence of stacks of $\mathcal{O}_X$-linear abelian categories $\mathcal{Q}\mathcal{Coh}(\alpha) \simeq \mathcal{Q}\mathcal{Coh}(f^*(\beta))$ on $X$.

**Corollary 4.2.** Suppose that $X$ and $Y$ are quasi-compact and quasi-separated schemes such that $\text{QCoh}(X) \simeq \text{QCoh}(Y)$ as abelian categories. Then, $X \cong Y$.

**Remark 4.3.** The corollary is originally due to Rosenberg [9], although there are some problems with the published exposition, as it claims to give the result for arbitrary schemes.

To prove the theorem we introduce an intermediary construction, the spectrum of $\mathcal{Q}\text{Coh}(X, \alpha)$, following Rosenberg [9]. While our definition differs from Rosenberg’s, the main ideas of the proof of the critical Proposition 4.7 below are due to Rosenberg, with some important adjustments to take into account the twisting.

**Definition 4.4.** If $A$ is an abelian category and $M \in A$, let $\text{add} M$ denote the full subcategory of $A$ consisting of all subquotients of finite direct sums of the object $M$. Call an abelian category $A$ quasi-local if it has a quasi-final object, which is an object contained in $\text{add} M$ for all non-zero objects $M$ of $A$.

**Definition 4.5.** If $A$ is an abelian category, let $\text{Sp} A$ denote the class of localizing subcategories $B \subseteq A$ such that $A/B$ is quasi-local. Denote by $S_B$ a quasi-final object of $A/B$, which we often view as an object of $A$ via the right adjoint to $A \to A/B$.

**Lemma 4.6.** Suppose that $j : A \to A/C$ is an exact localization and let $B$ be a localizing subcategory in $\text{Sp} A$. Then, either $j(S_B) = 0$ or $B/B \cap C \in \text{Sp} A/C$.

**Proof.** We assume that $j(S_B) \neq 0$. Note that the localization $A \to (A/C)/(B/B \cap C)$ kills every object of $B$. Therefore, it factors through $A \to A/B$. Thus, the image of $S_B$ in $(A/C)/(B/B \cap C)$ is quasi-final.

The first result is that there is a bijection between $\text{Sp} \text{QCoh}(X, \alpha)$ and $X$. In particular, it is a set rather than a proper class. We will use implicitly throughout the equivalence of $\text{QCoh}(X, \alpha)$ and $\text{QCoh}(X, A)$ when $A$ is an Azumaya algebra with class $\alpha$.
Proposition 4.7. If $X$ is quasi-compact and quasi-separated, $α ∈ Br(X)$, and $A$ is any Azumaya algebra representing $α$, there are inverse bijections

$$\text{Sp} \, \text{QCoh}(X, α) \xrightarrow{\psi} X \xrightarrow{φ} \text{Supp}(S_B)$$

defined by

$$\psi(B) = \text{the generic point of supp}(S_B),$$

$$φ(x) = \{M ∈ \text{QCoh}(X, α) | A(x) ∉ \text{add} \, M\},$$

where $A(x) = A/p(x)$ and $p(x)$ is the kernel of $A → A ⊗_{O_X} k(x)$. Moreover, $φ(x)$ does not depend on the Azumaya algebra $A$.

Proof. Note that we compute $\text{supp}(S_B)$ by viewing $S_B$ as an object of $\text{QCoh}(X, α)$ via the fully faithful right adjoint $\text{QCoh}(X, α)/B → \text{QCoh}(X, α)$. The main work of the proof is to show that both $ψ$ and $φ$ are in fact well-defined.

To see that $ψ$ is well-defined, let $B$ be a localizing subcategory of $\text{QCoh}(X, α)$. To check that $\text{supp}(S_B)$ is irreducible, we can assume that $X = \text{Spec} \, R$ by Lemma 4.6. Then, it suffices to check that $\text{ann}_R(S_B)$ is prime. Suppose that $a, b ∈ R$ are elements such that $ab ∈ \text{ann}_R(S_B)$. Consider the submodules $aS_B$ and $bS_B$ of $S_B$. Assume that $aS_B$ is non-zero. Then, $b ∈ \text{ann}_R(aS_B)$. Since $S_B$ is in $\text{add}(aS_B)$ as $S_B$ is quasi-final, it follows that $b ∈ \text{ann}_R(S_B)$. So, $\text{ann}_R(S_B)$ is prime. Finally, it is clear than any two quasi-final objects in $\text{QCoh}(X, α)/B$ have the same support. Thus, $ψ$ is well-defined.

The map $φ(x)$ defines a full subcategory of $A$, but it is not immediately clear that it is thick much less localizing and contained in $\text{Sp} \, \text{QCoh}(X, α)$. Let $x ∈ X$, and take $A(x)$ as above. Then, $A(x)$ is the quotient of $A$ by a sheaf of two-sided prime ideals. Let $N ⊆ A(x)$ be a non-zero left submodule. We want to show that $A(x) ∈ \text{add} \, N$.

To do this, we can assume that $N$ is of finite type. Let $j : U ⊆ X$ be an affine open with $j$ quasi-compact and such that $x ∈ U$ and $N|_U$ is non-zero. Then, $N|_U ⊆ A(x)|_U$. The ideal

$$\text{ann}_{A(x)(U)}(N(U)) = \{x ∈ A(x)(U) | xN(U) = 0\}$$

is a 2-sided ideal. Since $A(x) ⊗_{O_X} k(y)$ is a central simple algebra for $y ∈ \overline{x}$, it follows that $\text{ann}_{A(x)(U)}(N(U)) ⊗_{O_X} k(y) = 0$ for all such $y$. Since $N$ is of finite type, so is $\text{ann}_{A(x)(U)}(N(U))$. Nakayama’s lemma now implies that the annihilator $\text{ann}_{A(x)(U)}(N(U))$ vanishes. Pick generators $a_1, \ldots, a_n$ of $N(U)$. Then, the natural map

$$A(x)(U) → \bigoplus_{i=1}^n A(x)(U)/\text{ann}_{A(x)(U)}(a_i)$$

is injective. But, each quotient $A(x)(U)/\text{ann}_{A(x)(U)}(a_i)$ is contained in $\text{add} \, (N(U))$ (viewed as a subcategory of the abelian category $\text{Mod}_{A(x)(U)}$). It follows that $A(x)(U)$ is contained in $\text{add}(N(U))$. This means that $A(x)|_U ∈ \text{add} \, N|_U$ in $\text{QCoh}(U, α)$. By adjunction, $j_*A(x)|_U ∈ \text{add} \, N$.

Now, we claim that $A(x)$ is a subsheaf of a direct sum

$$j_i_*A(x)|_{U_i}$$
for $j_i : U_i \to X$ finitely many quasi-compact open immersions. This will show that $A(x)$ is contained in $\text{add} N$. Write $V$ for the support of $A(x)$. That is $V = \{x\}$. Then, the inclusion $i : V \to X$ is quasi-compact. It follows that there are finitely many open affines $U_i$ of $X$ each intersecting $V$ that cover $V$ and such that $j_i : U_i \to X$ is quasi-compact. The claim follows, and we have proven that $A(x) \in \text{add} N$, as desired.

Now, it follows that $\phi(x)$ is thick. Indeed, let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence. If $A(x) \notin \text{add} M$, then it is clearly not in $\text{add} M'$ or in $\text{add} M''$. Conversely, if it is not in $\text{add} M'$ nor in $\text{add} M''$, then it cannot be in $\text{add} M$. Otherwise, $A(x)$ would be a subquotient of $M^\oplus n$ for some $n$. Either an object of $\text{add} M'$ has a non-zero map to $A(x)$ in which case $A(x) \in \text{add} M'$ by the previous paragraph, or there is no such map, in which case $A(x) \in \text{add} M''$. Therefore, $\phi(x)$ is thick. If $M = \text{colim}_i M_i$ for some objects $M_i \in \phi(x)$ and if $A(x) \in \text{add} M$, then some object of $\text{add} M_i$ must map to $A(x)$ with non-zero image by axiom AB5. Hence, $\phi(x)$ is a localizing subcategory of $\text{QCoh}(X, \alpha)$ by Proposition 2.6. To see that the quotient is quasi-local, note that the image of $A(x)$ in the quotient $\text{QCoh}(X, \alpha)/\phi(x)$ is quasi-final by the definition of $\phi(x)$.

Now that we have seen that the maps are well-defined, we show that they are mutual inverses. It is clear that $\psi(\phi(x)) = x$. So, fix $B \in \text{Sp \text{QCoh}}(X, \alpha)$. Then, $\phi(\psi(B))$ consists of $M \in \text{QCoh}(X, \alpha)$ such that $A(x) \notin \text{add} M$ where $x$ is the generic point of the support of $S_B$. Clearly, $B \subseteq \phi(\psi(B))$. Fix a quasi-final object $S'$ of $\text{QCoh}(X, \alpha)/\phi(\psi(B))$ (viewed as usual as an object of $\text{QCoh}(X, \alpha)$). Let $M \in \phi(\psi(B))$ be an object not contained in $B$. Then, by definition, $S' \notin \text{add} M$, and moreover, $S_B \in \text{add} M$. Now, by construction of $\phi$ and $\psi$, the image of $S_B$ in $\text{QCoh}(X, \alpha)/\phi(\psi(x))$ is non-zero. By quasi-finality of $S'$, we have that $S' \in \text{add} S_B$. But, this means that $S' \in \text{add} M$, in contradiction to our choice of $M$. Therefore, $B = \phi(\psi(B))$.

Finally, that $\phi(x)$ does not depend on the choice of Azumaya algebra $A$ with Brauer class $\alpha$ follows from the fact that $\phi$ is the inverse to $\psi$, and hence is unique. 

**Remark 4.8.** Let $X$ be an arbitrary quasi-compact and quasi-separated scheme and $\alpha \in \text{Br}^r(X) = H^2_{\text{et}}(X, G_m)_{\text{tors}}$ a cohomological Brauer class. One might ask whether the theorem extends to this case. At the moment, we are not certain, although we can say the following. Note that $\psi_X$ is well-defined regardless. Let $U \subseteq X$ be an affine open subset. Then, by the theorem of Gabber and de Jong [4], the restriction of $\alpha$ to $U$ is represented by an Azumaya algebra. Looking at the commutative diagrams

$$\begin{array}{ccc}
\text{Sp \text{QCoh}}(U, \alpha) & \xrightarrow{\psi_U} & U \\
\downarrow & & \downarrow \\
\text{Sp \text{QCoh}}(X, \alpha) & \xrightarrow{\psi_X} & X
\end{array}$$

for all such $U$, we see that $\psi_X$ is surjective. The difficulty seems to be in constructing the inverse to $\psi_X$ and there we needed, or at least used, an Azumaya algebra.

The proposition says that the set of points of $X$ can be recovered from $\text{QCoh}(X, \alpha)$. We go even farther, showing that $X$ can be recovered as a ringed space from $\text{QCoh}(X, \alpha)$. To do this, we first introduce a topology on $\text{Sp \text{QCoh}}(X, \alpha)$. 

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Definition 4.9. If $A$ is a Grothendieck abelian category, and if $M$ is an object of $A$, define $\operatorname{supp}_{\text{Sp}}(M) \subseteq \text{Sp} A$ to be the set of all $B \in \text{Sp} A$ such that the image of $M$ in $A/B$ is non-zero. Give $\text{Sp} A$ the topology generated by the basis of closed sets

$$\{\operatorname{supp}_{\text{Sp}}(M) | M \in A \text{ is finitely presented}\}.$$  

Recall that $M$ is finitely presented if the functor $\operatorname{Hom}_A(M, -)$ commutes with arbitrary coproducts. Because finite direct sums of finitely presented objects are finitely presented, the set is closed under finite unions.

Proposition 4.10. The maps

$$\text{Sp QCoh}(X, \alpha) \xrightarrow{\phi} X$$

are inverse homeomorphisms.

Proof. It suffices to show that $\psi$ induces a bijection between elements of a basis for the topologies of $\text{Sp QCoh}(X, \alpha)$ and $X$. Let $V \subseteq X$ be a closed subset defined by a sheaf of ideals $I_V \subseteq \mathcal{O}_X$ of finite type. Then, $\mathcal{O}_V$ is a finitely presented $\mathcal{O}_X$-module. Write $A_V$ for $A \otimes_{\mathcal{O}_X} \mathcal{O}_V$. It is a left $A$-module of finite presentation in $\text{QCoh}(X, \alpha)$. It follows that $\operatorname{supp}_{\text{Sp}}(\mathcal{O}_V)$ as defined above is a closed subset of $\text{Sp QCoh}(X, \alpha)$. We show that $\psi(\operatorname{supp}_{\text{Sp}}(A_V)) = \operatorname{supp}(A_V) = V$. Since the closed subsets defined by a finitely generated sheaf of ideals $I_V$ of finite type form a basis for $X$, the proposition will follow. Now, if $B \in \operatorname{supp}_{\text{Sp}}(A_V)$, then $S_B \in \text{add} A_V$, where $S_B$ is a quasi-final object of the quotient $\text{QCoh}(X, \alpha)/B$. It follows that $\operatorname{supp}(S_B) \subseteq \operatorname{supp}(A_V) = V$. Hence, $\psi(\operatorname{supp}_{\text{Sp}}(A_V)) \subseteq V$.

On the other hand, if $x \in V$, then recall that $A(x)$ is precisely $A_{\{x\}}$, where $\overline{\{x\}}$ is the closure of $\{x\}$. Since $\overline{\{x\}}$ is a closed subscheme of $V$, it follows that $A(x)$ is a quotient of $A_V$. That is, $A(x) \subseteq \text{add} A$. Therefore, $\psi(x)$ does not contain $A_V$. But, this means that $\phi(x) \in \text{supp}_{\text{Sp}}(A_V)$. Since $\phi$ and $\psi$ are inverse bijections, it follows that $\psi(\operatorname{supp}_{\text{Sp}}(A_V)) = V$. ■

Definition 4.11. The canonical prestack on $\text{Sp} A$ is the prestack $\text{St}^p_A$ of abelian categories given by sending an open set $U \subseteq \text{Sp} A$ to

$$\text{St}^p_A(U) = A / \cap_{B \in U} B,$$

where the intersection $\cap_{B \in U} B$ is localizing as each $B$ is localizing. Recall that any abelian category $A$ has a center $C(A)$, which is the commutative ring of endomorphisms of the identity functor of $A$. By taking the center of these categories, we obtain a presheaf of commutative rings $\mathcal{O}^p_A$, which has $\mathcal{O}^p_A(U) = C(\text{St}_A(U))$. Write $\mathcal{O}_A$ for the sheafification of $\mathcal{O}^p_A$. Similarly, write $\text{St}_A$ for the stackification of $\text{St}^p_A$. Note that $\text{St}_A$ is naturally an $\mathcal{O}_A$-linear stack.

Proposition 4.12. If $U \subseteq \text{Sp QCoh}(X, \alpha)$ is a quasi-compact immersion, there is a natural equivalence

$$\text{QCoh}(X, \alpha)/ \cap_{B \in U} B \simeq \text{QCoh}(\psi(U), \alpha).$$

Proof. It suffices to check that $\cap_{B \in U} B$ is equal to $\text{QCoh}_Z(X, \alpha)$ where $Z = X - \psi(U)$. Indeed, since $j : \psi(U) \to X$ is quasi-compact, there is a pushforward $j_* : \text{QCoh}(U, \alpha) \to$
We remark that the following theorem holds, where the assumption that \( \mathcal{A}(x) \) is a quasi-final object of \( \text{QCoh}(X, \alpha) / \phi(x) \). In particular, \( \mathcal{A}(x) \notin \text{add} M \) for any \( x \in \psi(U) \). This implies that \( M \) must be supported on \( Z \). On the other hand, clearly if \( M \) is supported on \( Z \), then \( \mathcal{A}(x) \notin \text{add} M \) for \( x \in \psi(U) \). Hence, \( M \in \cap_{B \in U} B \), as desired.

**Corollary 4.13.** The map \( \phi : X \to \text{Sp QCoh}(X, \alpha) \) induces an equivalence

\[
\Omega \text{-cof}(\alpha) \simeq \phi^* \text{St}_{\text{QCoh}(X, \alpha)}
\]

of Zariski stacks of \( \mathcal{O}_X \)-linear abelian categories.

**Proof.** This follows from the proposition as the open subschemes with quasi-compact inclusion morphism \( U \subseteq X \) form a basis for the topology on \( X \), as \( X \) is quasi-compact and quasi-separated.

**Proof of Theorem 4.1.** Suppose now that \( X \) and \( Y \) are quasi-compact and quasi-separated schemes with \( \alpha \in \text{Br}(X) \) and \( \beta \in \text{Br}(Y) \), and suppose that there is an equivalence \( \sigma_* : \text{QCoh}(X, \alpha) \simeq \text{QCoh}(Y, \beta) \) of \( R \)-linear abelian categories. Then, \( \sigma \) induces a homeomorphism

\[
\sigma : \text{Sp QCoh}(X, \alpha) \simeq \text{Sp QCoh}(Y, \beta),
\]

as the definitions of \( \text{Sp A} \) are purely categorical. Moreover, \( \sigma_* \) behaves well with respect to taking centers and canonical stacks. It follows that \( \sigma \) is an isomorphism of \( R \)-schemes and that, by definition, there is an equivalence

\[
\text{St}_{\text{QCoh}(X, \alpha)} \simeq \sigma^* \text{St}_{\text{QCoh}(Y, \beta)}
\]

of stacks of \( \mathcal{O}_{\text{QCoh}(X, \alpha)} \)-abelian categories on \( \text{Sp QCoh}(X, \alpha) \).

Write \( \phi_X \) and \( \psi_X \) for the maps of Proposition 4.7 on \( X \), and similarly for \( \phi_Y \) and \( \psi_Y \). The previous proposition implies that there is an equivalence

\[
\text{St}_{\text{QCoh}(Y, \beta)} \simeq \psi_Y^* \Omega \text{-cof}(\beta)
\]

of stacks of \( \mathcal{O}_{\text{QCoh}(Y, \beta)} \)-linear abelian categories on \( \text{Sp QCoh}(Y, \beta) \), where we view \( \Omega \text{-cof}(\beta) \) as a Zariski stack on \( Y \). Similarly, \( \phi_X \) induces an equivalence

\[
\Omega \text{-cof}(\alpha) \simeq \phi_X^* \text{St}_{\text{QCoh}(X, \alpha)}
\]

of stacks of \( \mathcal{O}_X \)-linear abelian categories on \( X \).

Stringing these equivalences together, we obtain an equivalence \( \Omega \text{-cof}(\alpha) \simeq f^* \Omega \text{-cof}(\beta) \). If \( U \subseteq X \) is an open subscheme, then \( (f^* \Omega \text{-cof}(\beta))(U) = \Omega \text{-cof}(f(U), \beta) \). Thus, by definition, \( f^* \Omega \text{-cof}(\beta) \simeq \Omega \text{-cof}(f^*(\beta)) \). The theorem is proved.

## 5 Noetherian reconstruction

We remark that the following theorem holds, where the assumption that \( \alpha \in \text{Br}(X) \) is dropped. In the statements below, \( \text{Coh}(X, \alpha) \) is the abelian category of \( \alpha \)-twisted coherent \( \mathcal{O}_X \)-modules.
Theorem 5.1. Suppose that \( X \) and \( Y \) are noetherian schemes over a commutative ring \( R \) with cohomological Brauer classes \( \alpha \in \text{Br}'(X) \) and \( \beta \in \text{Br}'(Y) \). If there is an equivalence of \( R \)-linear abelian categories \( \text{Coh}(X, \alpha) \cong \text{Coh}(Y, \beta) \), then there is an isomorphism \( f : X \to Y \) of \( R \)-schemes and a natural equivalence of stacks of \( \mathcal{O}_X \)-linear abelian categories \( \text{QCoh}(\alpha) \cong \text{QCoh}(f^*(\beta)) \) on \( X \).

The theorem follows immediately from the constructions of Perego [8]. As a consequence, we have the following corollary.

Corollary 5.2. Suppose that \( X \) and \( Y \) are noetherian schemes over a commutative ring \( R \) with cohomological Brauer classes \( \alpha \in \text{Br}'(X) \) and \( \beta \in \text{Br}'(Y) \). If there is an equivalence of \( R \)-linear abelian categories \( \text{Coh}(X, \alpha) \cong \text{Coh}(Y, \beta) \), then there is an isomorphism \( f : X \to Y \) of \( R \)-schemes and a natural equivalence of stacks of \( \mathcal{O}_X \)-linear abelian categories \( \text{QCoh}(\alpha) \cong \text{QCoh}(f^*(\beta)) \) on \( X \).

Proof. This follows from the theorem, since the ind-completion of \( \text{Coh}(X, \alpha) \) is precisely \( \text{QCoh}(X, \alpha) \). Recall that the ind-completion of \( \text{Coh}(X, \alpha) \) is the abelian category of exact functors \( \text{Fun}^{\text{ex}}(\text{Coh}(X, \alpha)^{\text{op}}, \text{Mod}_Z) \). As ind-completion is purely categorical, the equivalence at the level of stacks of coherent twisted sheaves induces an equivalence of stacks of quasi-coherent twisted sheaves, by stackifying the pointwise ind-completion of \( \text{Coh}(X, \alpha) \) and \( \text{Coh}(Y, \beta) \).

\[ \square \]

6 Căldăraru’s conjecture

Căldăraru’s conjecture now follows from the following theorem, which we prove using results of Toën [11], and hence using derived algebraic geometry.

Theorem 6.1. Suppose that \( X \) is a quasi-compact and quasi-separated scheme. Suppose that \( \alpha \) and \( \beta \) are in \( H^2_{\text{ét}}(X, \mathbb{G}_m) \), and suppose that there is an equivalence \( \text{QCoh}(X, \alpha) \cong \text{QCoh}(X, \beta) \) of Zariski stacks of \( \mathcal{O}_X \)-linear abelian categories. Then, \( \alpha = \beta \).

Proof. The basic idea of the proof is that we use the equivalence to induce an equivalence of étale stacks of locally presentable dg categories on \( X \). The conclusion will then follow from [11, Corollary 3.12]. For background on stacks of dg categories, see [11, Definition 3.6].

If \( A \) is a Grothendieck abelian category, let \( D_{dg}(A) \) denote the dg category of chain complexes on \( A \). The derived category of \( D_{dg}(A) \) is the triangulated category \( D(A) \) of unbounded chain complexes in \( A \). In this case, \( D_{dg}(A) \) is a locally presentable dg category, which means that (1) the derived category \( D_{dg}(A) \) has hom sets; (2) there is a \( \kappa \)-compact generator for some regular cardinal \( \kappa \) of the derived category, and (3) the derived category has all small coproducts.

Let \( D_{\text{Zar}}(\alpha) \) denote the sheafification of the prestack of locally presentable dg categories

\[ U \subseteq X \mapsto D_{dg}(\text{QCoh}(U, \alpha)), \]

where \( U \to X \) ranges over all open affine subschemes. The equivalence in the hypothesis induces an equivalence of Zariski stacks of locally presentable dg categories \( D_{\text{Zar}}(\alpha) \cong D_{\text{Zar}}(\beta) \).
Over an affine scheme $U = \text{Spec} S \subseteq X$, it follows from Yoneda that giving an étale stack of locally presentable dg categories is the same as giving a locally presentable $S$-linear dg category $E$ with étale hyperdescent. By definition, there is an equivalence between $D$ of dg categories of complexes of $S$-algebras and their hypercovers, but this is not an important point for our purposes, and it holds below regardless.)

It is known from [11, Proposition 3.7] that a compactly generated locally presentable $S$-linear dg category satisfies étale hyperdescent. Since $D_{\mathcal{O}}(\text{QCoh}(U, \alpha))$ is compactly generated for $U$ affine (this follows for instance from [11, Theorem 4.7]), it follows that $D_{\mathcal{O}}(\alpha)|_U$ determines naturally an étale stack $D_{\mathcal{O}}(\alpha)|_U$. These are moreover compatible, in the sense that if $U$ and $V$ are two affine open subschemes of $X$, then there is a natural equivalence between $D_{\mathcal{O}}(\alpha)|_U(U \cap V)$ and $D_{\mathcal{O}}(\alpha)|_V(U \cap V)$.

Using the fundamental fact [11, Theorem 3.4] that stacks of locally presentable dg categories themselves form a stack, it follows that we can glue the $D_{\mathcal{O}}(\alpha)|_U$ together to form an étale stack $D_{\mathcal{O}}(\alpha)$ on $X$. We do this for $D_{\mathcal{O}}(\beta)$ as well, and the equivalence of the hypothesis continues to give $D_{\mathcal{O}}(\alpha) \simeq D_{\mathcal{O}}(\beta)$.

On the other hand, from Toën’s paper, for $\alpha \in \text{Br}(X)$, there is a uniquely determined étale stack of locally presentable dg categories on $X$, denoted by $L_\alpha$; it is precisely the stack of dg categories of complexes of $\alpha$-twisted sheaves with quasi-coherent cohomology. By definition, there is an equivalence between $D_{\mathcal{O}}(\alpha)$ and the restriction of $L_\alpha$ to the small Zariski site on $X$. The recipe above for using affine open subschemes of $X$ to create $D_{\mathcal{O}}(\alpha)$ extends by definition to an equivalence $D_{\mathcal{O}}(\alpha) \simeq L_\alpha$. So, since $D_{\mathcal{O}}(\alpha) \simeq D_{\mathcal{O}}(\beta)$, there is an equivalence of étale stacks $L_\alpha \simeq L_\beta$. By the uniqueness of $L_\alpha$ from [11, Corollary 3.12], it follows that $\alpha = \beta$. 

**Corollary 6.2.** Suppose that $X$ and $Y$ are quasi-compact and quasi-separated schemes over a commutative ring $R$ with $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. If $\text{QCoh}(X, \alpha) \simeq \text{QCoh}(Y, \beta)$ as $R$-linear abelian categories, then there is an isomorphism $f : X \to Y$ of $R$-schemes such that $f^*(\beta) = \alpha$.

**Corollary 6.3.** Suppose that $X$ and $Y$ are noetherian schemes over a commutative ring $R$ with $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. If $\text{Coh}(X, \alpha) \simeq \text{Coh}(Y, \beta)$ as $R$-linear abelian categories, then there is an isomorphism $f : X \to Y$ of $R$-schemes such that $f^*(\beta) = \alpha$.

**Remark 6.4.** We can take $R = \mathbb{Z}$ in the above statements, in which case the condition is simply that the abelian categories involved be equivalent as abelian categories.

## 7 Concluding remarks

1. We do not know whether to expect the main theorem to hold for $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. It might even be possible for it to hold for $\alpha \in H^2_{\mathcal{O}}(X, \mathbb{G}_m)$ and $\beta \in H^2_{\mathcal{O}}(Y, \mathbb{G}_m)$. That is, for arbitrary $\mathbb{G}_m$-gerbes. This would amount to a reconstruction theorem for $\mathbb{G}_m$-gerbes.
2. The passage through the work of Toën while very satisfying seems to come out of the blue. It would be nice to have a theory of Morita theory of stacks of abelian categories that is internal in some sense to the theory of abelian categories.

3. At the moment, it seems like a much more difficult question to determine when $D_{\text{qc}}(X, \alpha) \simeq D_{\text{qc}}(X, \beta)$ as $R$-linear triangulated categories. Examples are given in [2] where $\alpha$ and $\beta$ are not related by any automorphism of $X$. However, as in the proof above, the natural context for Morita theory of Azumaya algebras is that of Morita equivalence of stacks. Toën’s theorem says exactly that two Azumaya algebras have the same Brauer class if and only if the stacks of dg categories of modules over $X$ are derived Morita equivalent as stacks. Asking when $D_{\text{qc}}(X, \alpha) \simeq D_{\text{qc}}(X, \beta)$ is like asking when two sheaves of $\mathcal{O}_X$ modules have isomorphic $\Gamma(X, \mathcal{O}_X)$-modules of global sections, a mostly unnatural question.

References

[1] B. Antieau and D. Gepner, Brauer groups and étale cohomology in derived algebraic geometry, ArXiv e-prints (2012), available at http://arxiv.org/abs/1210.0290.

[2] A. Căldăraru, Derived categories of twisted sheaves on elliptic threefolds, J. Reine Angew. Math. 544 (2002), 161–179.

[3] A. Canonaco and P. Stellari, Twisted Fourier-Mukai functors, Adv. Math. 212 (2007), no. 2, 484–503.

[4] A. J. de Jong, A result of Gabber, available at http://www.math.columbia.edu/~dejong/.

[5] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323–448.

[6] A. Grothendieck, Sur quelques points d’algèbre homologique, Tôhoku Math. J. (2) 9 (1957), 119–221.

[7] M. Lieblich, Moduli of twisted sheaves, Duke Math. J. 138 (2007), no. 1, 23–118.

[8] A. Perego, A Gabriel theorem for coherent twisted sheaves, Math. Z. 262 (2009), no. 3, 571–583.

[9] A. L. Rosenberg, The spectrum of abelian categories and reconstruction of schemes, Rings, Hopf algebras, and Brauer groups (Antwerp/Brussels, 1996), Lecture Notes in Pure and Appl. Math., vol. 197, Dekker, New York, 1998, pp. 257–274.

[10] G. Tabuada, Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories, C. R. Math. Acad. Sci. Paris 340 (2005), no. 1, 15–19.

[11] B. Toën, Derived Azumaya algebras and generators for twisted derived categories, Invent. Math. 189 (2012), no. 3, 581–652.