HAMILTON–JACOBI EQUATIONS FOR NONSYMMETRIC MATRIX INFERENCE

HONG-BIN CHEN

Courant Institute of Mathematical Sciences, New York University

Abstract. We study the high-dimensional limit of the free energy associated with the inference problem of a rank-one nonsymmetric matrix. The matrix is expressed as the outer product of two vectors, not necessarily independent. The distributions of the two vectors are only assumed to have scaled bounded supports. We bound the difference between the free energy and the solution to a suitable Hamilton–Jacobi equation in terms of two much simpler quantities: concentration rate of this free energy, and the convergence rate of a simpler free energy in a decoupled system. To demonstrate the versatility of this approach, we apply our result to the i.i.d. case and the spherical case. By plugging in estimates of the two simpler quantities, we identify the limits and obtain convergence rates.

1. Introduction

Recovering a matrix from a noisy observation is a basic problem in statistical inference. Our setting is in the high-dimensional regime. For \( n \in \mathbb{N} \), let \( m = m(n) \in \mathbb{N} \) be a function of \( n \) satisfying \( \lim_{n \to \infty} m(n) = \infty \). Let \( X = (X_1, X_2, \ldots, X_m) \in \mathbb{R}^m \) and \( Y = (Y_1, Y_2, \ldots, Y_n) \in \mathbb{R}^n \) be two random vectors with joint law \( P_{X,Y}^n \), which are not necessarily independent. We assume that

\[
|X| \leq \sqrt{m}, \quad |Y| \leq \sqrt{n}, \quad \text{a.s. } \forall n.
\]

Let \( N = N(n) = \sqrt{mn} \) be the geometric mean of the sizes \( m \) and \( n \).

The noisy observation is given by

\[
Z = \sqrt{2t} \frac{1}{N} XY^\top + W
\]

where \( 2t \geq 0 \) is interpreted as the signal-to-noise ratio, and \( W = (W_{ij}) \) is an \( m \times n \) matrix with independent standard Gaussian entries. The goal of inference is to recover information about the matrix \( XY^\top \) from the observation \( Z \).

An important object is the limit of the mutual information \( I(X,Y;Z) \) between the observation \( Z \) and the unknown vectors \( X,Y \) as \( n \to \infty \), as well as the associated minimum mean square error. This limit allows to locate the critical \( t_c \) at which a phase transition occurs. The inference task is theoretically possible due to low noise level for \( t < t_c \), and becomes impossible once above this threshold. It is well known

E-mail address: hbchen@cims.nyu.edu.
Date: March 6, 2021.
that $-\frac{1}{N}I(X,Y;Z)$ is equal to the free energy $F^o_n(t)$ associated with the system up to a trivial additive term (see [1]). The exact formula of the free energy is given by

$$F^o_n(t) = \frac{1}{N} \mathbb{E} \log \int_{\mathbb{R}^m \times \mathbb{R}^n} e^{H^o_n(t,x,y)} P_n^{X,Y} (dx, dy)$$

where

$$H^o_n(t,x,y) = \sqrt{\frac{2t}{N}} x \cdot W y + \frac{2t}{N} xy^T \cdot XY^T - \frac{t}{N} |xy^T|^2$$

is the Hamiltonian corresponding to the law of $(X,Y)$ conditioned on observing $Z$. Indeed, for any bounded measurable function $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$, by Bayes’ rule, one can compute that

$$\mathbb{E} \left[ g(X,Y) \bigg| Z \right] = \frac{\int_{\mathbb{R}^m \times \mathbb{R}^n} g(x,y) e^{H^o_n(t,x,y)} P_n^{X,Y} (dx, dy)}{\int_{\mathbb{R}^m \times \mathbb{R}^n} e^{H^o_n(t,x,y)} P_n^{X,Y} (dx, dy)}.$$

To study the limit of $F^o_n$, this work follows the Hamilton–Jacobi equation approach in [19, 20, 21, 23, 22]. It has already been known in physics literature that finite volume free energies satisfy approximate Hamilton–Jacobi equations [11, 10, 7, 6]. We seek to identify the limit of an enriched version of the free energy $F^o_n$ as the unique solution to a suitable Hamilton–Jacobi equation. In addition to $Z$, we introduce an additional observation, for $h_1, h_2 \geq 0$,

$$Z = \left( \sqrt{2h_1} X + U, \sqrt{2h_2} Y + V \right)$$

where $U \in \mathbb{R}^m$ and $V \in \mathbb{R}^n$ are standard Gaussian vectors independent from each other and from $W$. Corresponding to the law of $(X,Y)$ conditioned on observing $Z$ and $Z$, the enriched Hamiltonian is given by

$$H_n(t,h,x,y) = H^o_n(t,x,y) + \sqrt{2h_1} U \cdot x + 2h_1 X \cdot x - h_1 |x|^2 + \sqrt{2h_2} V \cdot y + 2h_2 Y \cdot y - h_2 |y|^2,$$

for $(t,h) \in [0, \infty) \times [0, \infty)^2$. The associated free energy is

$$F_n(t,h) = \frac{1}{N} \log \int e^{H_n(t,h,x,y)} P_n^{X,Y} (dx, dy),$$

and its expectation is written as $\overline{F}_n(t,h) = \mathbb{E} F_n(t,h)$.

Note that $\overline{F}_n(0,\cdot)$ can be viewed as the free energy of a decoupled system, which is much simpler to analyze. In addition to assumption (1.1), we also assume that $\overline{F}_n(0,\cdot)$ converges to a function $\psi$. Due to the absence of coupled interaction in $\overline{F}_n(0,\cdot)$, this assumption should, in general, be relatively easy to check. Moreover, explicit estimates on the speed of convergence should also be attainable.

Under these assumptions, we will show that $\overline{F}_n$ satisfies an approximate Hamilton–Jacobi equation, whose limiting equation is

$$\partial_t f - (\partial_{h_1} f)(\partial_{h_2} f) = 0, \quad \text{in } [0, \infty) \times [0, \infty)^2,$$

with initial condition $f(0,\cdot) = \psi$. We adopt the notion of weak solutions advocated in [20], for this notion allows a simpler way to study the convergence of $\overline{F}_n$. Let
us rewrite the nonlinear term in (1.6) as \( (\partial_{h_1} f)(\partial_{h_2} f) = H(\nabla f) \) where \( H : \mathbb{R}^2 \to \mathbb{R} \) is given by \( H(p) = p_1 p_2 \). Different from [19] and [20] which studied the symmetric matrix inference problem, in the nonsymmetric setting, the function \( H \) is not convex. As in [5] and [15], the existence of solution is ensured and can be expressed by a variational formula if the initial condition is convex. For the uniqueness, a partial convexity condition (see (4) of Definition 4.1) and the nonnegativity of all entries in the Hessian of \( H \) are sufficient.

The limit of free energy related to inference models was first computed in [13]. The inference of symmetric matrices, in the rank-one case or more general, has been extensively studied. We refer to [14] for a description of these results. For the nonsymmetric matrix inference as is concerned here, the problem has been studied in [18] for the case where \( X \) and \( Y \) have i.i.d. entries. In particular, a variational formula for the limit of the free energy is obtained. Most recently, employing the adaptive interpolation method (introduced in [2, 3]), [16] established results in the case where \( X \) and \( Y \) are uniformly distributed on properly scaled spheres. More generally, results have been extended to rank-one tensor inference in [4] and [12]. The inference of second-order matrix tensor product is recently studied in [24] using a more general adaptive interpolation method.

Our main contribution is to provide a different approach, which is conceptually simpler in some aspect. The main result bounds the local \( L^\infty L^1 \) norm of \( F_n - f \) in terms of two quantities: 1) the concentration rate of \( F_n \) towards \( \overline{F}_n \) and 2) the convergence rate of \( F_n(0, \cdot) \). These two quantities are much simpler to study, and there is a plethora of tools and techniques to obtain good estimates. Convergence in the local uniform topology can also be obtained as discussed in Remark 2.4. As an application of the main result, we estimate the two quantities for the i.i.d. case and the spherical case. We recovered known results on the limit of the free energy for these two cases simultaneously. In addition, convergence rates are also obtained.

The rest of the paper is organized as follows. In Section 2, we state the main results. We also include results for the i.i.d. case and spherical case. Then, we show \( \overline{F}_n \) satisfies an approximate Hamilton–Jacobi equation, and list a few basic estimates in Section 3. In Section 4, we define the notion of weak solutions, prove the uniqueness of solutions, and describe conditions for the existence of solutions. Section 5 contains the proof of the main results. Lastly, in Section 6, we collect estimates needed to derive convergence results for the two special cases. In addition, we briefly describe possible modifications of the current approach to study the sparse model in Section 6.4.

Throughout the paper, we write \( \mathbb{R}_+ = [0, \infty) \). The symbol \( C \) denotes a positive absolute constant which may vary from instance to instance.

Acknowledgements. I am grateful to Jean-Christophe Mourrat for introducing me to this subject and for many helpful insights. I would like to thank Jiaming Xia for helpful comments.

2. SETTINGS AND MAIN RESULTS

2.1. General setting. We assume (1.1) and the existence of \( \alpha > 0 \) such that

\[
\lim_{n \to \infty} \frac{m(n)}{n} = \alpha > 0.
\]
Let us define the following quantities, for $M > 0$, $n \in \mathbb{N}$ and $\psi : \mathbb{R}_+^2 \to \mathbb{R}$,

\[ K_{M,n} = \left( \mathbb{E} \sup_{(t,h) \in [0,M]^2} |F_n - \overline{F}_n(t)|^2 \right)^{1/2}, \]

\[ L_{\psi,M,n} = \sup_{h \in [0,M]^2} |\overline{F}_n(0,h) - \psi(h)|. \]

**Theorem 2.1.** Suppose that $X$ and $Y$ are independent for all $n$, and that there is a function $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ such that

\[ \overline{F}_n(0, \cdot) \to \psi, \quad \text{pointwise as } n \to \infty. \]

Then

1. There is a unique weak solution $f$ to the Hamilton–Jacobi equation

\[ \begin{cases} \partial_t f - (\partial_{h_1} f)(\partial_{h_2} f) = 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+^2, \\ f(0, \cdot) = \psi, & \text{in } \mathbb{R}_+^2; \end{cases} \]

and $f$ admits a variational representation known as the Hopf formula

\[ f(t,h) = \sup_{z \in \mathbb{R}_+^2} \inf_{v \in \mathbb{R}_+^2} \left\{ z \cdot (h - y) + \psi(y) + tz_1z_2 \right\}; \]

2. There is a constant $C > 0$ such that the following holds for all $M \geq 1$ and all $n \in \mathbb{N}$:

\[ \sup_{t \in [0,M]} \int_{[0,M]^2} |\overline{F}_n(t,h) - f(t,h)| \, dh \leq CM^2 \left( L_{\psi,CM,n} + n^{-1} + (K_{CM,n})^2 + K_{CM,n} \right). \]

This theorem is proved in Section 5.2.

**Remark 2.2.** Due to (1.5), the initial free energy $\mathbb{E}F_n^\phi (t)$ associated with the inference task (1.2) converges to $f(t,0)$ as $n \to \infty$.

**Remark 2.3.** When $X$ and $Y$ are not independent, a similar result can be obtained. To guarantee the existence of a weak solution, we need to further assume the decay of $L_{\psi,M,n}$ and $K_{M,n}$ as $n \to \infty$. This result is recorded in Theorem 5.2. However, it is not proven that, in the dependent case, the unique weak solution is given by the Hopf formula. See Section 6.3 for a more detailed discussion.

**Remark 2.4.** Assuming that the right hand side of the inequality in the second part of Theorem 2.1 converges to $0$ as $n \to \infty$, we can also obtain local uniform convergence by utilizing the fact $f$ is Lipschitz and $\overline{F}_n$ is Lipschitz uniformly in $n$ (see Definition 4.1 and (3.15)). We briefly sketch the argument. Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth radial bump function supported on the unit disk and satisfy $0 \leq \xi \leq 1$ and $\int \xi = 1$. For $\epsilon \in (0,2)$, set $\xi_\epsilon(x) = \epsilon^{-2}\xi(\epsilon^{-1}x)$. Notice that for any Lipschitz $g : \mathbb{R}^2 \to \mathbb{R}$ and $A \subset \mathbb{R}^2$, we have

\[ \|g\|_{L^\infty(A)} \leq \|g * \xi_\epsilon\|_{L^\infty(A)} + \|g - g * \xi_\epsilon\|_{L^\infty(A)} \leq \epsilon^{-2}\|g\|_{L^1(A_\epsilon)} + \epsilon \|\nabla g\|_{L^\infty(\mathbb{R}^2)}, \]

where $A_\epsilon$ is the $\epsilon$-neighbourhood of $A$. Extend $\overline{F}_n(t,\cdot)$, $f(t,\cdot)$ symmetrically to $\mathbb{R}^2$. For any compact $B \subset \mathbb{R}^2$, we choose $M$ large so that $B_1 \subset [-M,M]^2$. Fix any $t \in [0,M]$ and set $g_n = \overline{F}_n(t,\cdot) - f(t,\cdot)$. By our assumption at the beginning
of this remark, we have \( \lim_{n \to \infty} \| g_n \|_{L^1([-M,M]^2)} = 0. \) Then for large \( n \), we have
\[
\epsilon = (\| g_n \|_{L^1([-M,M]^2)} \| \nabla g_n \|_\infty)^{1/2} < 1.
\]
Plug this \( \epsilon \) to the above display to see
\[
\| g_n \|_{L^\infty(B)} \leq C \| g_n \|_{L^1([-M,M]^2)}^2 \| \nabla g_n \|_\infty^2 \leq C \| g_n \|_{L^1([-M,M]^2)}^7.
\]
Here the last equality is due to the Lipschitzness of \( f \) and \( F_n \) (uniformly in \( n \)).

2.2. Special cases. We apply Theorem 2.1 to the i.i.d. case and the spherical case. We also present a simple example where \( X \) and \( Y \) are dependent in Section 2.2.3 as an application of Theorem 5.2. To quantify the rate of convergence in (2.1), we set
\[
\beta(n) = \left| \frac{m(n)}{n} - \alpha \right|.
\]
The proofs are collected in Section 6.

2.2.1. The i.i.d. case. Let \( P_X^1 \) and \( P_Y^1 \) be supported on \([-1,1]\). Recall that \( m = m(n) \) is a function of \( n \). For \( n \in \mathbb{N} \), we set
\[
P_n^{X,Y} = (P_X^1)^{\otimes m} \otimes (P_Y^1)^{\otimes n},
\]
where \( \otimes \) denotes the product of measures. Hence, \( X \) and \( Y \) have i.i.d. entries, distributed according to \( P_X^1 \) and \( P_Y^1 \), respectively.

Proposition 2.5 (Convergence for the i.i.d. case). There is a constant \( C > 0 \), such that the following holds for all \( M \geq 1 \) and \( n \in \mathbb{N} \):
\[
\sup_{t \in [0,M]} \int_{[0,M]^2} | F_n(t,h) - f(t,h) | dh \leq CM^3 \left( \beta(n) + (n^{-1} \log n)^{1/2} \right).
\]
Here \( f \) is given by (2.6) with
\[
\psi(h) = \left( \alpha m(1) \right)^{1/2} F_1(0,h_1,0) + \left( \alpha^{-1} m(1) \right)^{1/2} F_1(0,0,h_2).
\]
In (2.8), \( m(1) \) is the evaluation of \( m = m(n) \) at \( n = 1 \).

2.2.2. The spherical case. For \( k \in \mathbb{N} \), let \( \mathcal{U}_k \) be the uniform measure on the centered sphere with radius \( \sqrt{k} \), denoted as \( \sqrt{k} \mathbb{S}^{k-1} \). Consider the joint distribution of \( X \) and \( Y \) given by
\[
P_n^{X,Y}(dx,dy) = \mathcal{U}_m(dx) \otimes \mathcal{U}_n(dy).
\]
In particular, \( X \) and \( Y \) are independent.

Proposition 2.6 (Convergence for the spherical case). There is a constant \( C > 0 \), such that the following holds for all \( M \geq 1 \) and \( n \in \mathbb{N} \):
\[
\sup_{t \in [0,M]} \int_{[0,M]^2} | F_n(t,h) - f(t,h) | dh \leq CM^3 \left( \beta(n) + (n^{-1} \log n)^{1/2} \right).
\]
Here \( f \) is given by (2.6) with
\[
\psi(h) = \alpha \left( h_1 - \frac{\log(1 + 2h_1)}{2} \right) + \alpha^{-1} \left( h_2 - \frac{\log(1 + 2h_2)}{2} \right).
\]
2.2.3. An example where $X$ and $Y$ are dependent. Let $m(n) = n$ and $Y = \sqrt{\nu}X + \sqrt{1-\nu}X'$ where $\nu \in [0,1]$ and $X'$ is an independent copy of $X$. Assume that $X$ has i.i.d. entries and the law $P$ of $X_1$ is supported on $[-1,1]$. We can verify that $F_n(0,h) = \psi(h)$ for all $n$ where

\begin{equation}
\psi(h) = \mathbb{E}\log \int e^{H_1(0,h,x_1,\sqrt{\nu}x_1 + \sqrt{1-\nu}x'_1)} P(dx_1)P(dx'_1).
\end{equation}

Proposition 2.7. There is a constant $C > 0$, such that the following holds for all $M \geq 1$ and $n \in \mathbb{N}$:

$$\sup_{t \in [0,M)} \int_{[0,M]^2} |\nabla F_n(t,h) - f(t,h)| dh \leq CM^3 (n^{-1}\log n)^{\frac{3}{2}}.$$ 

Here $f$ is the unique weak solution to (2.5) with initial condition $\psi$ given in (2.10).

We emphasize again that it is not proven in this work that $f$ is given by the Hopf formula (2.6) when $X$ and $Y$ are dependent, although we expect this to hold. A detailed discussion on this issue is in Section 6.3.

3. APPROXIMATE HAMILTON–JACOBI EQUATIONS

In this section, we show that $F_n$ satisfies an approximate Hamilton–Jacobi equation as stated in Lemma 3.1, under the notation and assumptions introduced in the previous two sections.

Lemma 3.1. For each $n \in \mathbb{N}$, the function $F_n: \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\left| \partial_t F_n - (\partial_{h_1} F_n)(\partial_{h_2} F_n) \right| \leq \frac{1}{2N} \Delta F_n + \frac{1}{2} \mathbb{E} |\nabla (F_n - F_n)|^2.$$ 

The Laplacian $\Delta$ and the gradient $\nabla$ are all carried out in the variable $h$. Note that the relation $F_n$ satisfies in this lemma bears a strong resemblance to the Hamilton–Jacobi equation (2.5). The Laplacian term above can be seen as the vanishing viscosity. The last term has a flavor of concentration. These suggest that $F_n$ should satisfy (2.5) asymptotically.

3.1. Proof of Lemma 3.1. Let us introduce the notation $\langle \cdot \rangle$ for the Gibbs measure. For a measurable function $f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, we write

$$\langle f(x,y) \rangle = (\mathcal{Z}(t,h))^{-1} \int_{\mathbb{R}^m \times \mathbb{R}^n} f(x,y)e^{H_n(t,h,x,y)} P_n^X P_n^Y (dx,dy),$$

with the normalizing factor

$$\mathcal{Z}(t,h) = \int_{\mathbb{R}^m \times \mathbb{R}^n} e^{H_n(t,h,x,y)} P_n^X P_n^Y (dx,dy).$$ 

In view of $H_n$ defined in (1.3), the probability measure $\langle \cdot \rangle$ is random, and depends on $t$ and $h$. For simplicity of notation, such dependence is suppressed. We also consider independent copies of $(x, y)$ with respect to this Gibbs measure. They are called “replicas” and denoted as $(x', y')$, $(x'', y'')$, etc.

Two important tools in our computations are the Nishimori identity and the Gaussian integration by parts. The Nishimori identity can be stated as the following: for a bounded measurable $f: (\mathbb{R}^m \times \mathbb{R}^n)^2 \to \mathbb{R}$,

\begin{equation}
\mathbb{E}\langle f(x,y, X, Y) \rangle = \mathbb{E}\langle f(x,y, x', y') \rangle,
\end{equation}

where $X$ and $Y$ are independent copies of $(x', y')$. Note that $X$ and $Y$ are also dependent, although we expect this to hold. A detailed discussion on this issue is in Section 6.3.
where \((x',y')\) is replica of \((x,y)\). To show this, the key step is to verify the following identity based on Bayes’ rule

\[
\langle f(x,y) \rangle = \mathbb{E}[f(X,Y)|Z]
\]

where, with \(Z\) given in (1.2),

\[
Z = (Z, \sqrt{2h_1}X + U, \sqrt{2h_2}Y + V).
\]

An obvious extension of (3.1) holds when more replicas are involved. In other words, the Nishimori identity allows us to replace one replica by \((X,Y)\) inside \(\mathbb{E}\{\cdot\}\). For more details, one can see [20, Section 3.1] in a slightly different setting.

The other ingredient, the Gaussian integration by parts, is the fact that, for any smooth \(f: \mathbb{R} \to \mathbb{R}\) with polynomial growth,

\[
\int_{\mathbb{R}} xf(x)e^{-x^2/2}dx = \int_{\mathbb{R}} f'(x)e^{-x^2/2}dx.
\]

Recall the definition of \(F_n\) in (1.4). We start to compute the derivatives of \(F_n\) and \(\overline{F}_n\). The first order derivatives of \(F_n\) are given by

\[
\partial_t F_n(t,h) = \frac{1}{N} \left\langle \frac{1}{\sqrt{2N t}} y \cdot W x + \frac{2}{N} x y^T \cdot X Y^T - \frac{1}{N^2} |xy|^2 \right\rangle,
\]

\[
\partial_h F_n(t,h) = \frac{1}{N} \left\langle \partial_h H_n \right\rangle = \frac{1}{N} \left\langle \frac{1}{\sqrt{2h_1}} U \cdot x + 2x \cdot X - |x|^2 \right\rangle,
\]

\[
\partial_{h_2} F_n(t,h) = \frac{1}{N} \left\langle \partial_{h_2} H_n \right\rangle = \frac{1}{N} \left\langle \frac{1}{\sqrt{2h_2}} V \cdot y + 2y \cdot Y - |y|^2 \right\rangle.
\]

After taking expectations, we apply the Nishimori identity and the Gaussian integration by parts to obtain

\[
\partial_t \overline{F}_n(t,h) = \frac{1}{N^2} \mathbb{E}\langle (x \cdot x')(y \cdot y') \rangle,
\]

\[
\partial_h \overline{F}_n(t,h) = \frac{1}{N} \mathbb{E}\langle \partial_h H_n \rangle = \frac{1}{N} \mathbb{E}\langle x \cdot x' \rangle,
\]

\[
\partial_{h_2} \overline{F}_n(t,h) = \frac{1}{N} \mathbb{E}\langle \partial_{h_2} H_n \rangle = \frac{1}{N} \mathbb{E}\langle y \cdot y' \rangle.
\]

Therefore, we have

\[
\partial_t \overline{F}_n - \langle \partial_{h_1} \overline{F}_n \rangle \langle \partial_{h_2} \overline{F}_n \rangle = \frac{1}{N^2} \left( \mathbb{E}\langle (x \cdot x')(y \cdot y') \rangle - \mathbb{E}\langle x \cdot x' \rangle \mathbb{E}\langle y \cdot y' \rangle \right)
\]

\[
= \frac{1}{N^2} \mathbb{E}\left( (x \cdot x' - \mathbb{E}(x \cdot x'))(y \cdot y' - \mathbb{E}(y \cdot y')) \right).
\]

After an application of the Cauchy–Schwarz inequality, it becomes

\[
\left| \partial_t \overline{F}_n - \langle \partial_{h_1} \overline{F}_n \rangle \langle \partial_{h_2} \overline{F}_n \rangle \right| \leq \frac{1}{2N^2} \left( \mathbb{E}\langle (x \cdot x' - \mathbb{E}(x \cdot x'))^2 \rangle + \mathbb{E}\langle (y \cdot y' - \mathbb{E}(y \cdot y'))^2 \rangle \right).
\]

We will show the following upper bound

\[
\mathbb{E}\langle (x \cdot x' - \mathbb{E}(x \cdot x'))^2 \rangle \leq N \partial_{h_1}^2 \overline{F}_n + N^2 \mathbb{E}(\partial_{h_1} F_n - \partial_{h_1} \overline{F}_n)^2.
\]

A similar upper bound for the second term on the right of (3.8) can be obtained. Lemma 3.1 follows from these two upper bounds and (3.8).
3.1.1. **Proof of (3.9).** Using the expression of $\partial_{h_1} F_n$ in (3.3), we can calculate

$$N \partial^2_{h_1} F_n = \langle (\partial_{h_1} H_n)^2 \rangle - \langle \partial_{h_1} H_n \rangle^2 - \frac{1}{(2h_1)^2} \langle U \cdot x \rangle \tag{3.10}$$

Apply the Gaussian integration by parts to obtain

$$N \partial^2_{h_1} F_n = E(\langle \partial_{h_1} H_n \rangle^2) - N^2 E(\partial_{h_1} F_n)^2 - \frac{1}{2h_1} E \langle \langle x \rangle^2 \rangle + \frac{1}{2h_1} E \langle \langle x \rangle^2 \rangle. \tag{3.11}$$

The key computation is the following

$$E(\langle \partial_{h_1} H_n \rangle^2) \geq E(\langle x \rangle^2) + \frac{1}{2h_1} E \langle \langle x \rangle^2 \rangle. \tag{3.12}$$

This will be done slightly later. Now insert (3.12) into (3.11) to see

$$N \partial^2_{h_1} F_n \geq E(\langle x \rangle^2) - N^2 E(\partial_{h_1} F_n)^2.$$  

Finally, by (3.6), we have

$$E(\langle x \cdot x' - \langle x \rangle^2 \rangle) = E(\langle x \cdot x' \rangle^2) - (E(\langle x \cdot x' \rangle)^2) = E(\langle x \rangle^2) - N^2(\partial_{h_1} F_n)^2.$$  

From the above two displays, we can deduce (3.9).

Lastly, let us derive (3.12). Using the expression of $\partial_{h_1} H_n$ in (3.3), we have

$$E(\langle \partial_{h_1} H_n \rangle^2) = E\left( \left( \frac{1}{\sqrt{2h_1}} U \cdot x + 2x \cdot X - \langle x \rangle^2 \right)^2 \right) = E\left( \frac{1}{2h_1}(U \cdot x)^2 + 4(x \cdot X)^2 + |x|^4 + \frac{4}{\sqrt{2h_1}} (U \cdot x)(x \cdot X) - \frac{2}{\sqrt{2h_1}} (U \cdot x)|x|^2 - 4(x \cdot X)|x|^2 \right) \tag{3.13}$$

Let us rewrite the first term in the above display as

$$E\left( \frac{1}{2h_1}(U \cdot x)^2 \right) = \sum^{m}_{i,j=1} \frac{1}{2h_1} E\langle U_i U_j x_i x_j \rangle.$$  

If $i \neq j$, using the Gaussian integration by parts, we have

$$\frac{1}{2h_1} E\langle U_i U_j x_i x_j \rangle = E(x_i x_j (x_i - x'_j)(x_j + x'_j - 2x''_j)).$$  

If $i = j$, we have

$$\frac{1}{2h_1} E\langle U_i U_i x_i x_i \rangle = E(x_i x_i (x_i - x'_i)(x_i + x'_i - 2x''_i)) + \frac{1}{2h_1} E\langle x_i^2 \rangle.$$  

These three displays combined yield

$$E\left( \frac{1}{2h_1}(U \cdot x)^2 \right) = E(|x|^4 - 2|x|^2(x \cdot x') - (x \cdot x')^2 + 2(x \cdot x')(x \cdot x'') + \frac{1}{2h_1} E\langle x^2 \rangle.$$  

Other terms can be computed using the Nishimori identity and the Gaussian integration by parts. We shall omit the details but only list the results:

\[
\mathbb{E}\langle (x \cdot X)^2 \rangle = \mathbb{E}\langle (x \cdot x')^2 \rangle,
\]
\[
\mathbb{E}\left( \frac{1}{\sqrt{2h_1}} (U \cdot x)(x \cdot X) \right) = \mathbb{E}\langle |x|^2 (x \cdot x') + (x \cdot x')^2 - 2(x \cdot x')(x' \cdot x'') \rangle,
\]
\[
\mathbb{E}\left( \frac{1}{\sqrt{2h_1}} (U \cdot x)|x|^2 \right) = \mathbb{E}\langle |x|^4 - |x|^2 (x \cdot x') \rangle,
\]
\[
\mathbb{E}\langle (x \cdot X)|x|^2 \rangle = \mathbb{E}\langle |x|^2 (x \cdot x') \rangle.
\]

Inserting these computations into (3.13) yields

\[
\mathbb{E}\langle (\partial_{h_i} H_n)^2 \rangle = \mathbb{E}\langle (x \cdot x')^2 \rangle + 6\mathbb{E}\langle (x \cdot x') (x \cdot x'') \rangle + \frac{1}{2h_1}\mathbb{E}\langle |x|^2 \rangle.
\]

Apply the Cauchy–Schwarz inequality and the symmetry of replicas to see

\[
\mathbb{E}\langle (x \cdot x')(x \cdot x'') \rangle \leq \frac{1}{2}\mathbb{E}\langle (x \cdot x')^2 \rangle + \frac{1}{2}\mathbb{E}\langle (x \cdot x'')^2 \rangle = \mathbb{E}\langle (x \cdot x')^2 \rangle.
\]

These two displays imply (3.12).

3.2. Basic estimates. We end this section by proving basic estimates for derivatives of \( F_n \) and \( \overline{F}_n \).

**Lemma 3.2.** There is a constant \( C > 0 \) such that the following hold for all \( n \in \mathbb{N} \) and all \((t, h) \in \mathbb{R}^2_+\):

\[
|\overline{F}_n| \leq C(t + |h|); \tag{3.14}
\]
\[
|\partial_{h_i} \overline{F}_n|, \quad |\nabla \overline{F}_n| \leq C; \tag{3.15}
\]
\[
\partial_{h_i}^2 \overline{F}_n, \quad \partial_{h_i} \overline{F}_n, \quad \partial_{h_i} \partial_{h_j} \overline{F}_n \geq 0, \quad \forall i, j \in \{1, 2\}; \tag{3.16}
\]
\[
|\partial_t \overline{F}_n| \leq C \left( 1 + \frac{|W|_{op}}{\sqrt{n t}} \right), \quad |\nabla \overline{F}_n| \leq C \left( 1 + \frac{|U| + |V|}{\sqrt{n |h|}} \right); \tag{3.17}
\]
\[
\partial_{h_{1i}}^2 \overline{F}_n \geq -C|U|n^{-\frac{1}{2}}h_1^{-\frac{3}{2}}, \quad \partial_{h_{2i}}^2 \overline{F}_n \geq -C|V|n^{-\frac{1}{2}}h_2^{-\frac{3}{2}}. \tag{3.18}
\]

In (3.17), we use \(| \cdot |_{op}\) to denote the operator norm, namely,

\[
|W|_{op} = \sup_{x \in \mathbb{R}^m, \ |x| \leq 1} |Wx|.
\]

**Proof of (3.14).** Note that \( \overline{F}_n(0, 0) = 0 \) for all \( n \). Hence (3.14) follows from (3.15).

**Proof of (3.15).** This estimate follows from the formulae in (3.5)-(3.7) and the assumption (1.1) on the boundedness of \( X \) and \( Y \).

**Proof of (3.16).** Due to the independence of replicas, we have \( \mathbb{E}\langle (x \cdot x') \rangle = \mathbb{E}\langle |x|^2 \rangle \). Hence, it is evident from (3.6)-(3.7) that \( \partial_{h_i} \overline{F}_n \geq 0 \), for \( i = 1, 2 \).

Then, we compute \( \partial_{h_i} \partial_{h_j} \overline{F}_n \) for \( i, j \in \{1, 2\} \). Recall the definition of \( H_n \) in (1.3) and expressions of first order derivatives in (3.6)-(3.7). Using these and the Nishimori
identity, we compute
\[ \partial_{h_1} \partial_{h_2} \overline{F}_n = \partial_{h_2} \left( N^{-1} \mathbb{E} \langle x, x' \rangle \right) \]
\[ = N^{-1} \mathbb{E} \left( 2x \cdot x' ((2h_2)^{-\frac{1}{2}} V \cdot y + 2y \cdot y'' - y \cdot y) \right. \]
\[ \left. - 2x \cdot x' ((2h_2)^{-\frac{1}{2}} V \cdot y'' + 2y'' \cdot y'' - y'' \cdot y'') \right) \].

Apply the Gaussian integration to get
\[ \partial_{h_1} \partial_{h_2} \overline{F}_n = 2N^{-1} \mathbb{E} \left( x \cdot x' ((y + y' - 2y'') \cdot y + 2y \cdot y'' - y \cdot y) \right. \]
\[ \left. - x \cdot x' ((y + y' + y'' - 3y'''') \cdot y'' + 2y'' \cdot y''' - y'' \cdot y''') \right) .
\]

Collecting terms and using the symmetry of replicas, we arrive at
\[ \partial_{h_1} \partial_{h_2} \overline{F}_n = 2N^{-1} \mathbb{E} \left( (x \cdot x')(y \cdot y') - 2(x \cdot x')(y \cdot y'') + (x \cdot x')(y'' \cdot y''') \right) \]
\[ = 2N^{-1} \mathbb{E} \left( \langle xy \rangle - \langle y \rangle \langle y \rangle \right)^2 \geq 0 .
\]

To compute \( \partial_{h_1}^2 \overline{F}_n \), we repeat the above calculation with \( h_2, y, V \) replaced by \( h_1, x, U \). In a similar way, we can also treat \( \partial_{h_2}^2 \overline{F}_n \). These calculations yield
\[ \partial_{h_1}^2 \overline{F}_n = 2N^{-1} \mathbb{E} \left( \langle xy \rangle - \langle x \rangle \langle y \rangle \right)^2 \geq 0 , \]
\[ \partial_{h_2}^2 \overline{F}_n = 2N^{-1} \mathbb{E} \left( \langle yy \rangle - \langle y \rangle \langle y \rangle \right)^2 \geq 0 .
\]

Lastly, we compute \( \partial_t^2 \overline{F}_n \). Recall the formula of \( \partial_t \overline{F}_n \) in (3.5). Take one more derivative in \( t \) and we have
\[ \partial_t^2 \overline{F}_n = \frac{2}{N^3} \mathbb{E} \left( (x \cdot x')(y \cdot y') \left( \sqrt{\frac{N}{2t}} x \cdot W y + 2x y \cdot X Y \cdot |xy|^2 \right. \right. \]
\[ \left. \left. - \sqrt{\frac{N}{2t}} x' \cdot W y' - 2x' y' \cdot X Y \cdot |x'y'|^2 \right) \right) .
\]

Using the Gaussian integration by parts and the Nishimori identity, we obtain, after collecting terms,
\[ \partial_t^2 \overline{F}_n = \frac{2}{N^3} \mathbb{E} \left( (x \cdot x')(y \cdot y') \left( (x \cdot x')(y \cdot y') - 2(x \cdot x')(y \cdot y'') + (x' \cdot x') (y'' \cdot y''') \right) \right) \]
\[ = \frac{2}{N^3} \sum_{i,j,k,l} \mathbb{E} \left( \langle x_i y_j x_k y_l \rangle - \langle x_i y_j \rangle \langle x_k y_l \rangle \right)^2 \geq 0 .
\]

Proof of (3.17). This result is a consequence of (3.2)-(3.4) and the boundedness assumption (1.1).

Proof of (3.18). Recognizing a variance term on the right hand side of (3.10), we have
\[ \partial_{h_1}^2 F_n \geq - \frac{1}{N(2h_1)^2} \langle U \cdot x \rangle .
\]

By the same argument, a similar lower bound also holds for \( \partial_{h_2}^2 F_n \).
4. Weak solutions of Hamilton–Jacobi equations

In this section, we study a slightly more general version of the Hamilton–Jacobi equation (2.5). We will define the notion of weak solutions and prove the uniqueness of solutions. Under additional assumptions on the initial condition $\psi$, we verify that the Hopf formula gives a weak solution.

Let us describe the setting. Let $d \in \mathbb{N}$, $\psi : \mathbb{R}^d_+ \to \mathbb{R}$ be a continuous function, and $H : \mathbb{R}^d \to \mathbb{R}$ be smooth and satisfy

$$\partial^2_{ij} H(p) \geq 0, \quad \forall p \in \mathbb{R}^d_+, \forall i, j.$$  

We investigate the following equation

$$\begin{aligned}
\left\{ \begin{array}{c}
\partial_t f(t,x) - H(\nabla f(t,x)) = 0, \\
(0,x) \in \mathbb{R}^d_+ \times \mathbb{R}^d_+\end{array} \right. \\
f(0,x) = \psi(x), \quad x \in \mathbb{R}^d_+. 
\end{aligned}$$

Now, we give the precise definition of a weak solution.

**Definition 4.1.** A function $f : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}$ is a weak solution to (4.2)-(4.3) if

1. $f$ is Lipschitz and satisfies (4.2) almost everywhere;
2. $f(0,x) = \psi(x)$ for all $x \in \mathbb{R}^d_+$;
3. for each $t \geq 0$, $f(t,\cdot)$ is nondecreasing;
4. for all $(t,x) \in \mathbb{R}^{d+1}_+$, all $\lambda \geq 0$, and all $i, j \in \{1, 2, \ldots, d\}$, it holds that

$$f(t,x + \lambda e_i + \lambda e_j) + f(t,x) - f(t,x + \lambda e_i) - f(t,x + \lambda e_j) \geq 0.$$  

Here, a function $u : \mathbb{R}^d_+ \to \mathbb{R}$ is called nondecreasing provided $u(x) - u(x') \geq 0$ if $x - x' \in \mathbb{R}^d_+$. This monotonicity condition serves as a Neumann type boundary condition. In (4), $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$. Part (4) can be interpreted as a partial convexity condition. Indeed, this condition implies that, for mollifiers $\xi$ of the type introduced in Remark 2.4,

$$\partial^2_{ij}(f * \xi) \geq 0, \quad \forall i, j.$$  

Below are our results on the uniqueness and the existence of solutions.

**Proposition 4.2** (Uniqueness). The equation (4.2)-(4.3) has at most one weak solution.

**Proposition 4.3** (Hopf formula). Suppose $H$ is given by

$$H(p) = \prod_{i=1}^d p_i, \quad p \in \mathbb{R}^d.$$  

In addition, suppose, for $1 \leq i \leq d$, there are Lipschitz, convex and nondecreasing functions $\psi_i : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\psi(x) = \sum_{i=1}^d \psi_i(x_i), \quad x \in \mathbb{R}^d_+.$$  

Then, there exits a unique weak solution $f$ to (4.2)-(4.3) given by the Hopf formula:

$$f(t,x) = \sup_{z \in \mathbb{R}^d_+} \inf_{y \in \mathbb{R}^d_+} \left\{ z \cdot (x - y) + \psi(y) + tH(z) \right\}.$$
The conditions in Proposition 4.3 might be restrictive, but are sufficient enough for our purpose. We prove the uniqueness in Section 4.1 and show that the Hopf formula is a weak solution in Section 4.2. Before proceeding to proofs, let us mention a quick remark.

**Remark 4.4** (Comparison principle and Lipschitz coefficients). By a slight modification to the proof of Proposition 4.2 (instead of (4.17), let \(\phi(z) = 0\) for \(z \leq \delta(\|f\|_{Lip} + \|g\|_{Lip})\) and positive otherwise), we can obtain the following comparison principle: suppose \(f\) and \(g\) are two weak solutions with \(f(0, \cdot) \leq g(0, \cdot)\), then \(f \leq g\) almost everywhere. Using this principle and comparing \(f\) with \(f(\cdot, \cdot + h)\) for \(h \in \mathbb{R}^2_+\), we can obtain

\[
\|f(t, \cdot)\|_{Lip} = \|f(0, \cdot)\|_{Lip}, \quad \forall t \geq 0.
\]

Writing \(\psi = f(0, \cdot)\) and using (4.2), we can conclude

\[
\|f\|_{Lip} \leq \|\psi\|_{Lip} \vee \left(\sup_{\|p\| \leq \|\psi\|_{Lip}} |H(p)| \right).
\]

### 4.1. Proof of Proposition 4.2.

The idea of this proof can be seen in [9, Section 3.3.3]. The difference is that here \(H\) is not convex. Instead, we will utilize the fact that all entries in the Hessian of \(H\) are nonnegative as in (4.1), and the partial convexity (4.5).

Let \(f\) and \(g\) be weak solutions to (4.2). Set \(w = f - g\). Then we have

\[
\partial_t w = H(\nabla f) - H(\nabla g) = b \cdot \nabla w.
\]

where the vector \(b\) is given by

\[
b = \int_0^1 \nabla H(r \nabla f - (1 - r) \nabla g) \, dr.
\]

Take \(v = \phi(w)\) for some smooth function \(\phi : \mathbb{R} \to \mathbb{R}_+\) to be chosen later. Hence, we have

\[
\partial_t v = b \cdot \nabla v.
\]

To proceed, we regularize \(f, g\) and \(b\). Note that they are defined for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\). Recall the mollifier \(\xi_\epsilon\) introduced in Remark 2.4, and extend it to \(\mathbb{R}^d\) in the obvious way. Let \(f_\epsilon = f * \xi_\epsilon, g_\epsilon = g * \xi_\epsilon\) and

\[
b_\epsilon = \int_0^1 \nabla H(r \nabla f_\epsilon + (1 - r) \nabla g_\epsilon) \, dr.
\]

Here \(*\) denotes the convolution in \(x\). Note that these regularized versions are well-defined for \((t, x) \in \mathbb{R}_+ \times \mathbb{R}_\epsilon^d\), where \(\mathbb{R}_\epsilon = [\epsilon, \infty)\).

On \(\mathbb{R}_+ \times \mathbb{R}_\epsilon^d\), the equation (4.9) can be expressed as

\[
\partial_t v = \text{div}(vb_\epsilon) - \text{div}b_\epsilon + (b - b_\epsilon) \cdot \nabla v.
\]

Before proceeding further, we need to estimate some terms in this display.

Due to (4.5) and the fact that \(f\) and \(g\) are nondecreasing, we have, for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d_\epsilon\) and all \(1 \leq i, j \leq d,\)

\[
\partial_i f_\epsilon(t, x), \partial_i g_\epsilon(t, x), \partial^2_{ij} f_\epsilon(t, x), \partial^2_{ij} g_\epsilon(t, x) \geq 0.
\]
Using (4.1) and the above display, we obtain, for \((t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\),
\[
\text{div} b_\epsilon = \int_0^1 \nabla^2 H(r
abla f_\epsilon + (1-r)\nabla g_\epsilon) \cdot (r\nabla^2 f_\epsilon + (1-r)\nabla^2 g_\epsilon) \, dr \geq 0.
\] (4.12)
Here \(\nabla^2\) stands for the Hessian. By the definitions of \(f_\epsilon\) and \(g_\epsilon\), we also have
\[
|\nabla f_\epsilon| \leq \|f\|_{\text{Lip}}, \quad |\nabla g_\epsilon| \leq \|g\|_{\text{Lip}}.
\] (4.13)

Let us set
\[
R = 1 + \sup \{|\nabla H(p)| : p \in \mathbb{R}^d_+ \not\in \mathbb{N}\}, \quad |p| \leq \|f\|_{\text{Lip}} + \|g\|_{\text{Lip}}.
\] (4.14)
Fix any \(T, \eta > 0\) and define, for \(t \in [0,T]\),
\[
D_t = \{x \in \mathbb{R}^d_+ : |x| \leq R(T - t)\} \cap [\eta, \infty)^d,
\] (4.15)
\[
\Gamma_t = \partial D_t \cap \{|x| = R(T - t)\}.
\]

Now, let us introduce
\[
J(t) = \int_{D_t} v(t,x) \, dx.
\]
Let \(\epsilon < \eta\) to ensure \(D_\epsilon \subset \mathbb{R}^d_+\). Using (4.11) and integration by parts, we can compute
\[
\frac{d}{dt} J(t) = \int_{D_t} \partial_t v - R \int_{\Gamma_t} v
= \int_{\Gamma_t} (\mathbf{n} \cdot b_\epsilon - R) v + \int_{\partial D_t \setminus \Gamma_t} (\mathbf{n} \cdot b_\epsilon) v + \int_{D_t} v (-\text{div} b_\epsilon) + \int_{D_t} (b_\epsilon - b_\epsilon) \cdot \nabla v,
\]
where \(\mathbf{n}\) stands for the outer normal vector. We treat the integrals after the second equality separately. Due to (4.10), (4.13) and (4.14), the first integral is nonpositive.

For the second integral, note that on \(\partial D_t \setminus \Gamma_t\), we have \(-\mathbf{n} \in \mathbb{R}^d_+\). By (4.6) and \(f, g\) being nondecreasing, we can infer from the definition of \(b_\epsilon\) that \(b_\epsilon \in \mathbb{R}^d_+\) on \(\partial D_t \setminus \Gamma_t\).

In view of (4.12), the third integral is again nonpositive, while the last one is \(o_\epsilon(1)\).

Therefore, taking \(\epsilon \to 0\), we conclude that
\[
\frac{d}{dt} J(t) \leq 0.
\] (4.16)

Since \(w(0, x) = f(0, x) - g(0, x) = 0\), for each \(\delta > 0\), we have \(\|w(\delta, \cdot)\|_{\infty} \leq \delta(\|f\|_{\text{Lip}} + \|g\|_{\text{Lip}})\). Let us choose \(\phi\) to satisfy
\[
\begin{cases}
\phi(z) = 0, & \text{if } |z| \leq \delta(\|f\|_{\text{Lip}} + \|g\|_{\text{Lip}}), \\
\phi(z) > 0, & \text{otherwise}.
\end{cases}
\] (4.17)
Therefore, due to \(v = \phi(w)\), we have
\[
J(\delta) = \int_{D_\delta} v(\delta, x) \, dx = \int_{D_\delta} \phi(w(\delta, x)) \, dx = 0.
\]
Since \(J(t)\) is nonnegative, (4.16) implies that \(J(t) = 0\) for all \(t \in [\delta, T]\). This together with the definition of \(\phi\) guarantees that
\[
|f(t, x) - g(t, x)| \leq \delta(\|f\|_{\text{Lip}} + \|g\|_{\text{Lip}}), \quad \forall x \in D_t, \forall t \in [\delta, T].
\]
Recall the definition of \(D_t\) in (4.15) which depends on \(T\) and \(\eta\). Taking \(\delta \to 0\), \(\eta \to 0\) and \(T \to \infty\), we conclude that \(f = g\).
4.2. Proof of Proposition 4.3. Since (4.6) satisfies (4.1), uniqueness is ensured by Proposition 4.2.

Let us rewrite the Hopf formula as
\[
 f(t, x) = \sup_{z \in \mathbb{R}_+^d} \inf_{y \in \mathbb{R}_+^d} \left\{ z \cdot (x - y) + \psi(y) + tH(z) \right\}
\]
(4.18)

\[
 = \sup_{z \in \mathbb{R}_+^d} \left\{ z \cdot x - \psi^*(z) + tH(z) \right\}
\]

\[
= (\psi^* - tH)^*(x).
\]

Here the superscript * denotes the Fenchel transformation over \(\mathbb{R}_+^d\), namely,
\[
u^*(x) = \sup_{y \in \mathbb{R}_+^d} \{ y \cdot x - u(y) \}, \quad x \in \mathbb{R}_+^d.
\]
(4.19)

In the following, we verify (4.18) is a weak solution. Since the supremum in (4.18) is taken over \(\mathbb{R}_+^d\), it is clear that \(f(t, \cdot)\) is nondecreasing. We will check \(f\) satisfies the following in order: initial condition, semigroup property (or dynamic programming principle), Lipschitzness, satisfying (4.2) almost everywhere, and partial convexity.

4.2.1. Verification of the initial condition. The goal is to show
\[
\psi(x) = \sup_{z \in \mathbb{R}_+^d} \inf_{y \in \mathbb{R}_+^d} \left\{ z \cdot (x - y) + \psi(y) \right\} = \psi^*(x), \quad x \in \mathbb{R}_+^d.
\]
(4.20)

Due to the assumption on \(\psi\), (4.20) follows from the following lemma.

Lemma 4.5 (Fenchel–Moreau identity). Let \(u : \mathbb{R}_+^d \to (-\infty, +\infty]\) be a function not identically equal to \(+\infty\). Then, \(u^{**} = u\) if and only if \(u\) is convex, l.s.c. (lower semi-continuous), and nondecreasing.

Proof. Let \(u^{**} = u\). Note that for any function \(v\), we know \(v^*\) is convex and l.s.c. Since now the supremum is taken over \(\mathbb{R}_+^d\), we have \(v^*\) is nondecreasing. Hence, we must have \(u\) is convex, l.s.c., and nondecreasing.

Let \(u\) be convex, l.s.c and nondecreasing, and we want to show \(u^{**} = u\). Extend the domain of \(u\) to \(\mathbb{R}^d\) by setting \(u(x) = \infty\) for \(x \notin \mathbb{R}_+^d\). Let \(\otimes\) stand for the Fenchel transformation over \(\mathbb{R}^d\), that is,
\[
u^{\otimes}(x) = \sup_{y \in \mathbb{R}^d} \{ y \cdot x - u(y) \}.
\]

The convexity and lower semi-continuity of \(u\) yields
\[
u(x) = u^{\otimes\otimes}(x) = \sup_{z \in \mathbb{R}^d} \inf_{y \in \mathbb{R}^d} \left\{ z \cdot (x - y) + u(y) \right\} = \sup_{z \in \mathbb{R}^d} \left\{ z \cdot x - u^{\otimes}(z) \right\}.
\]
(4.21)

The goal is to show that \(\mathbb{R}^d\) in the above display can be replaced by \(\mathbb{R}_+^d\) whenever \(x \in \mathbb{R}_+^d\). By our extension of \(u\), we must have
\[
u^{\otimes}(z) = \sup_{y \in \mathbb{R}_+^d} \{ y \cdot z - u(y) \}.
\]
(4.22)

It remains to verify that \(z \in \mathbb{R}^d\) in (4.21) can be replaced by \(z \in \mathbb{R}_+^d\) whenever \(x \in \mathbb{R}_+^d\).

We claim that
\[
u^{\otimes}(z) = u^{\otimes}(z \lor 0), \quad \forall z \in \mathbb{R}^d,
\]
(4.23)
where \( z \lor 0 = (z_1 \lor 0, z_2 \lor 0, \ldots, z_d \lor 0) \). For two vectors \( x, x' \in \mathbb{R}^d \), we write \( x \geq x' \) if \( x - x' \in \mathbb{R}^d_+ \). Indeed, if (4.23) is true, then due to
\[
x \cdot z \leq x \cdot (z \lor 0), \quad \forall x \in \mathbb{R}^d_+,
\]
we have
\[
\sup_{z \in \mathbb{R}^d} \{ x \cdot z - u^\wedge(z) \} \leq \sup_{z \in \mathbb{R}^d} \{ x \cdot (z \lor 0) - u^\wedge(z \lor 0) \} = \sup_{z \in \mathbb{R}^d_+} \{ x \cdot z - u^\wedge(z) \}.
\]
From this, (4.21) and (4.22), we can deduce that \( u = u^{**} \).

Now, let us verify (4.23). Let \( z \in \mathbb{R}^d_+ \). For \( y \in \mathbb{R}^d_+ \), we write
\[
\tilde{y} = (y_1 \mathbb{1}_{\{z_1 \geq 0\}}, y_2 \mathbb{1}_{\{z_2 \geq 0\}}, \ldots, y_d \mathbb{1}_{\{z_d \geq 0\}}).
\]
Using this notation, we have
\[
(4.24) \quad z \cdot y \leq (z \lor 0) \cdot y = z \cdot \tilde{y}, \quad \forall y \in \mathbb{R}^d_+.
\]
Since \( u \) is non-decreasing, we also have \( u(y) \geq u(\tilde{y}) \). Using this and (4.24) repeatedly, we can obtain
\[
\sup_{y \in \mathbb{R}^d_+} \{ z \cdot y - u(y) \} \leq \sup_{y \in \mathbb{R}^d_+} \{ (z \lor 0) \cdot y - u(y) \}
\leq \sup_{y \in \mathbb{R}^d_+} \{ z \cdot \tilde{y} - u(\tilde{y}) \} \leq \sup_{y \in \mathbb{R}^d_+} \{ z \cdot y - u(y) \}.
\]
Here in the last inequality we used the fact that \( \{ \tilde{y} : y \in \mathbb{R}^d_+ \} \subset \mathbb{R}^d_+ \). From the above display, we can deduce
\[
\sup_{y \in \mathbb{R}^d_+} \{ z \cdot y - u(y) \} = \sup_{y \in \mathbb{R}^d_+} \{ (z \lor 0) \cdot y - u(y) \}.
\]
This display along with (4.22) implies (4.23). \( \square \)

4.2.2. Lipschitzness. Since \( \psi \) is Lipschitz, we have \( \psi^*(z) = \infty \) outside the compact set \( \{ |z| \leq \| \psi \|_{\text{Lip}} \} \). This together with (4.18) implies that for each \( x \in \mathbb{R}^d_+ \), there is \( z \in \mathbb{R}^d_+ \) with \( |z| \leq \| \psi \|_{\text{Lip}} \) such that
\[
f(t, x) = z \cdot x - \psi^*(z) + tH(z).
\]
This yields that, for any \( x' \in \mathbb{R}^d_+ \),
\[
f(t, x) - f(t, x') \leq z \cdot (x - x') \leq \| \psi \|_{\text{Lip}} |x - x'|.
\]
By symmetry, we conclude that \( f \) is Lipschitz in \( x \), and the Lipschitz coefficient is uniform in \( t \).

To show Lipschitzness in \( t \), we fix any \( x \in \mathbb{R}^d_+ \). Then, we have, for some \( z \in \mathbb{R}^d_+ \) with \( |z| \leq \| \psi \|_{\text{Lip}} \),
\[
f(t, x) = z \cdot x - \psi^*(z) + tH(z) \leq f(t', x) + (t - t')H(z)
\leq f(t', x) + |t' - t| \left( \sup_{|z| \leq \| \psi \|_{\text{Lip}}} |H(z)| \right) \leq f(t', x) + \| \psi \|_{\text{Lip}}^d |t' - t|.
\]
Here in the last inequality, we used the expression of \( H \) in (4.6). Again by symmetry, Lipschitzness in \( t \) is obtained.
4.2.3. Hopf formula satisfies (4.2). Due to Rademacher’s theorem, Lipschitzness of $f$ implies that $f$ is differentiable almost everywhere.

We want to verify that (4.18) satisfies (4.2) almost everywhere. Let $(t, x)$ be a point at which $f$ is differentiable. We can assume $(t, x)$ to satisfy $t, x_1, x_2, \ldots, x_d > 0$, because otherwise $(t, x)$ belongs to a Lebesgue measure zero set. Since $f(t, \cdot)$ is Lipschitz, we know that outside a compact set $f^*(t, \cdot)$ is infinity. Therefore, there is $\tilde{z} \in \mathbb{R}_+^d$ such that

$$f(t, x) = \tilde{z} \cdot x - \psi^*(\tilde{z}) + tH(\tilde{z}).$$

Then using (4.18), we have, for $s \geq 0$ and $h \in \mathbb{R}^d$ sufficiently small,

$$f(t, x) \leq f(t - s, x + h) - \tilde{z} \cdot h + sH(\tilde{z}).$$

Set $s = 0$ and vary $h$ to see

$$\tilde{z} = \nabla f(t, x).$$

Then, we set $h = 0$ in (4.26), take $s \to 0$ and insert (4.27) to obtain

$$\partial_t f(t, x) \leq H(\nabla f(t, x)).$$

To verify the other direction, we use (4.18) and (4.25) to see that, for $s \geq 0$,

$$f(t + s, x) \geq \tilde{z} \cdot x - \psi^*(\tilde{z}) + (t + s)H(\tilde{z}) = f(t, x) + sH(\tilde{z}).$$

Send $s \to 0$ and use (4.27) to see

$$\partial_t f(t, x) \geq H(\nabla f(t, x)).$$

4.2.4. Partial convexity. The case $i = j$ can be deduced from the convexity of $f$ which is evident from the Hopf formula (4.18).

Now consider the case $i \neq j$. By relabeling, we may assume $i = 1$ and $j = 2$.

The Lipschitzness of $\psi$ implies that $\psi^*$ is infinite outside a compact set. Due to (4.18), there are $z, z'$ such that

$$f(t, x + \lambda e_1) = z \cdot (x + \lambda e_1) - \psi^*(z) + tH(z),
\quad f(t, x + \lambda e_2) = z' \cdot (x + \lambda e_2) - \psi^*(z') + tH(z').$$

Case 1: $(z_1, z_2) \leq (z'_1, z'_2)$ or $(z_1, z_2) \geq (z'_1, z'_2)$. Let us only treat the latter case. The other case can be done in an analogous way. Using (4.18), we have

$$f(t, x + \lambda e_1 + \lambda e_2) \geq z \cdot (x + \lambda e_1 + \lambda e_2) - \psi^*(z) + tH(z),
\quad f(t, x) \geq z' \cdot x - \psi^*(z') + tH(z').$$

This along with (4.28) implies that the left hand side of (4.4) is bounded below by

$$\lambda z \cdot e_2 - \lambda z' \cdot e_2 = \lambda (z_2 - z'_2) \geq 0.$$

Case 2: neither $(z_1, z_2) \leq (z'_1, z'_2)$ nor $(z_1, z_2) \geq (z'_1, z'_2)$. This condition implies that

$$(z_1 - z'_1)(z_2 - z'_2) < 0.$$  

Let $\bar{z} = (z_1, z'_2, z_3, \ldots, z_d)$ and $\bar{z}' = (z'_1, z_2, z'_3, \ldots, z'_d)$. In other words, $\bar{z}$ is obtained from $z$ through replacing $z_2$ by $z'_2$, and similarly for $\bar{z}'$. By (4.18), for each $\delta > 0$, there are $y, y' \in \mathbb{R}_+^d$ such that

$$f(t, x + \lambda e_1 + \lambda e_2) \geq \bar{z} \cdot (x + \lambda e_1 + \lambda e_2 - y) + \psi(y) + tH(\bar{z}) - \delta,$n
\quad f(t, x) \geq \bar{z}' \cdot (x + y') + \psi(y') + tH(\bar{z}') - \delta.$$
Using the same construction for $\tilde{z}, \tilde{z}'$, we set
\[
\tilde{y} = (y_1, y_2, y_3, \ldots, y_d), \quad \tilde{y}' = (y'_1, y'_2, y'_3, \ldots, y'_d).
\]
Note that
\[
(4.31) \quad \tilde{z} \cdot y + \tilde{z}' \cdot y' - z \cdot \tilde{y} - z' \cdot \tilde{y}' = 0.
\]
From (4.28), we also have
\[
(4.32) \quad f(t, x + \lambda e_1) \leq z \cdot (x + \lambda e_1 - \tilde{y}) + \psi(\tilde{y}) + tH(z),
\]
\[
\quad f(t, x + \lambda e_2) \leq z' \cdot (x + \lambda e_2 - \tilde{y}') + \psi(\tilde{y}') + tH(z').
\]
To lower bound the left hand side of (4.4), we start by observing that, due to (4.31),
\[
\tilde{z} \cdot (x + \lambda e_1 + \lambda e_2 - y) + \tilde{z}' \cdot (x - y') - z \cdot (x + \lambda e_1 - \tilde{y}) - z' \cdot (x + \lambda e_2 - \tilde{y}')
\]
\[
= (\tilde{z} + \tilde{z}' - z - z') \cdot x - (\tilde{z} \cdot y + \tilde{z}' \cdot y' - z \cdot \tilde{y} - z' \cdot \tilde{y}') + \lambda(z_1 + z'_2 - z_1 - z'_2) = 0.
\]
This along with (4.30) and (4.32) implies that the left hand side of (4.4) can be bounded below by
\[
\psi(\tilde{y}) + \psi(\tilde{y}') - \psi(\tilde{y}) - \psi(\tilde{y}') + t(H(\tilde{z}) + H(\tilde{z}')) - H(z) - H(z') - 2\delta.
\]
From (4.7), we can see
\[
\psi(\tilde{y}) + \psi(\tilde{y}') = \psi(\tilde{y}) + \psi(\tilde{y}').
\]
Lastly, due to (4.29) and the definition of $H$ in (4.6), we can compute
\[
H(\tilde{z}) + H(\tilde{z}') - H(z) - H(z') = -(z_1 - z'_1)(z_2 - z'_2)z_3 \ldots z_d \geq 0.
\]
The above three displays imply that the left hand side of (4.4) is bounded below by $-2\delta$. The desired result follows by setting $\delta \to 0$.

5. Convergence of the Free Energy

The goal is to prove Theorem 2.1 and other convergence results to be stated. The method is similar to the one employed in Section 4.1.

For $c \geq 0$, define $\mathcal{A}(c)$ to be the class of all functions $\psi : \mathbb{R}^2_+ \to \mathbb{R}$ satisfying the following properties:

- $\psi(0) = 0$ and $\|\psi\|_{\text{Lip}} \leq c$;
- there is a unique weak solution $f$ to the Hamilton–Jacobi equation (2.5) with initial condition $\psi$.

Recall the assumption on the law of $X$ and $Y$ in (1.1), and the definitions in (2.2)-(2.3). The following results bounds difference between $F_n$ and $f$ in terms of those objects, where $f$ is a solution with initial condition $\psi$. Note that convergence is not asserted.

**Proposition 5.1.** Let $c \geq 0$. Under assumptions (1.1) and (2.1), there is $C > 0$ such that the following holds for all $M \geq 1$, all $n \in \mathbb{N}$ and all $\psi \in A(c)$,
\[
\sup_{t \in [0,M]} \int_{[0,M]^2} |F_n(t, h) - f(t, h)| dh \leq CM^2 \left(L_{\psi,CM,n} + n^{-1} + (K_{CM,n})^3 + K_{CM,n}\right),
\]
where $f$ is the unique weak solution to (2.5) with $f(0, \cdot) = \psi$. 
In order to derive a convergence result using this proposition, we set $\psi$ to be the limit of $F_n(0, \cdot)$ and verify that $\psi$ belongs to $A(c)$ for some $c$. In other words, we need to show that $\psi$ is Lipschitz and that there is a solution $f$ with initial condition $\psi$. Note that the Lipschitzness is automatic due to (3.15) and the nontrivial part is the existence of $f$. In the proof of Theorem 2.1 below in Section 5.2, we will show that $f$ is given explicitly by the Hopf formula through verifying the conditions in Proposition 4.3. We emphasize that the independence between $X$ and $Y$, as assumed in Theorem 2.1, plays an important role in the verification of those conditions.

When the independence is absent, the existence of $f$ can still be obtained via a limiting procedure under additional assumptions on the decay of $K_{M,n}$ and $L_{\psi,M,n}$ as $n \to \infty$. This is summarized in the following theorem.

**Theorem 5.2.** Under assumptions (1.1) and (2.1), suppose $\lim_{n \to \infty} K_{M,n} = 0$ for all $M \geq 1$ and that there is a $\psi : \mathbb{R}_+^2 \to \mathbb{R}$ such that $\lim_{n \to \infty} L_{\psi,M,n} = 0$ for all $M \geq 1$. Then, (2.5) admits a unique weak solution $f$ with $f(0, \cdot) = \psi$. Furthermore, there is $C > 0$ such that the following holds for all $M \geq 1$ and all $n \in \mathbb{N}$:

$$\sup_{t \in [0,M]} \int_{[0,M]^2} |F_n(t, h) - f(t, h)|dh \leq CM^2 \left( L_{\psi,CM,n} + n^{-1} + K_{CM,n} \right).$$

The first subsection is devoted to the proof of Proposition 5.1. After that, we prove Theorem 2.1 and Theorem 5.2. Results on Hamilton–Jacobi equations in Section 4 are needed. Some arguments in the proof of Proposition 5.1 will be reused to prove Theorem 5.2.

### 5.1. Proof of Proposition 5.1

Let $w_n = F_n - f$ and

$$r_n = \partial_t F_n - (\partial_{h_1} F_n)(\partial_{h_2} F_n).$$

Then, we have

$$\partial_t w_n = a_n \cdot \nabla w_n + r_n$$

where

$$a_n = (a_{n,1}, a_{n,2}) = (\partial_{h_2} f, \partial_{h_1} F_n).$$

For $\delta > 0$, let $\phi_\delta : \mathbb{R} \to [0, \infty)$ be given by

$$\phi_\delta(x) = (\delta + x^2)^{\frac{1}{2}},$$

which serves as a smooth approximation of the absolute value. Take $v_n = \phi_\delta(w_n)$ and multiply both sides of (5.2) by $\phi_\delta'(w_n)$ to see

$$\partial_t v_n = a_n \cdot \nabla v_n + \phi_\delta'(w_n)r_n.$$

Recall the mollifier $\xi$ given in Remark 2.4. Let us regularize $a_n$ by setting $a_{n,i} = a_{n,i} \cdot \xi_\epsilon$, with the convolution taken in $h$. Note that $a_{n,i}^\epsilon$ is well-defined for $(t, h) \in \mathbb{R}_+ \times \mathbb{R}_\epsilon^2$ where $\mathbb{R}_\epsilon = [\epsilon, \infty)$. For $(t, h) \in \mathbb{R}_+ \times \mathbb{R}_\epsilon^2$, we can rewrite the above display as

$$\partial_t v_n = \text{div}(v_n a_{n}^\epsilon) - v_n \text{div} a_{n}^\epsilon + (a_n - a_{n}^\epsilon) \cdot \nabla v_n + \phi_\delta'(w_n)r_n.$$

Let us derive a few estimates related to $a_{n}^\epsilon$. Since $\psi \in A(c)$, we know that $\|\psi\|_{\text{Lip}} \leq c$. By (4.8), there is $C > 0$ such that

$$\sup_{\psi \in A(c)} \|f\|_{\text{Lip}} \leq C.$$
where $f$ is a weak solution with initial condition $f(0, \cdot) = \psi$. By this, (3.15) and (5.3), there is $C > 0$ such that the following holds for all $n$, all $\epsilon \in (0, 1)$ and all $(t, h) \in \mathbb{R}_+ \times \mathbb{R}_r^2$,

\begin{align*}
(5.7) & \quad \|a_n - a_n^\epsilon\|_\infty = o_\epsilon(1); \\
(5.8) & \quad \|a_n^\epsilon\|_\infty \leq \|a_n\|_\infty \leq C.
\end{align*}

Using (3.16) and (4) in Definition 4.1, we also have, for $(t, h) \in \mathbb{R}_+ \times \mathbb{R}_r^2$,

\begin{equation}
(5.9) \quad \text{div} a_n^\epsilon = \partial_{h_1} \partial_{h_2} (f * \xi_\epsilon) + \partial_{h_1} \partial_{h_2} (F_n * \xi_\epsilon) \geq 0.
\end{equation}

Choose $R = 1 + \sup_{n, \epsilon} \|a_n^\epsilon\|_\infty$. Let $T \geq 1$ and $\eta > 0$ be specified later. Consider the following sets, indexed by $t \in [0, T]$,

\begin{equation}
(5.10) \quad D_t = \{ h \in \mathbb{R}_r^2 : h_1, h_2 \geq \eta, |h| \leq R(T-t) \}, \\
\Gamma_t = \{ h : |h| = R(T-t) \} \cap D_t.
\end{equation}

Let us consider the object

\begin{equation}
(5.11) \quad J_\delta(t) = \int_{D_t} v_n(t, h) dh = \int_{D_t} \phi_\delta(w_n(t, h)) dh.
\end{equation}

Let $\epsilon < \eta$, which guarantees $D_t \subset \mathbb{R}_r^2$. Differentiate $J_\delta(t)$ in $t$ and use (5.5) to see

\[
\frac{d}{dt} J_\delta(t) = \int_{D_t} \partial_t v_n - R \int_{\Gamma_t} v_n \\
= \int_{\Gamma_t} (a_n^\epsilon \cdot n - R) v_n + \int_{\partial D_t \setminus \Gamma_t} (a_n^\epsilon \cdot n) v_n + \int_{D_t} (- v_n \text{div} a_n^\epsilon + (a_n - a_n^\epsilon) \cdot \nabla v_n + \phi_\delta'(w_n)n_n).
\]

Here in the second identity, we used integration by parts. The first integral above is nonpositive due to the choice of $R$. The second integral is also nonpositive due to the direction of $n$ on $\partial D_t \setminus \Gamma_t$, and the fact that $f$ and $\nabla F_n$ are nondecreasing. Applying (5.7)–(5.9) to the last integral and sending $\epsilon \to 0$, we obtain

\begin{equation}
(5.12) \quad \frac{d}{dt} J_\delta(t) \leq \int_{D_t} \phi_\delta'(w_n)n_n \leq \int_{D_t} |n_n|.
\end{equation}

Here, in the last inequality, we used $\|\phi_\delta\|_\infty \leq 1$ which is evident from (5.4). Lemma 3.1 gives an upper bound for $|n_n|$. Hence, we have

\begin{equation}
(5.13) \quad \int_{D_t} |n_n| \leq \left( \frac{C}{n} \int_{D_t} \Delta F_n \right) + \left( \frac{1}{2} \mathbb{E} \int_{D_t} |\nabla (F_n - \bar{F}_n)|^2 \right).
\end{equation}

In view of (3.15), after integration by parts, the first term can be bounded by $CTn^{-1}$. Here and henceforth, we absorb $R$ into $C$. To avoid heavy notation, let us write

\begin{equation}
(5.14) \quad K = K_{RT,n}, \quad L = L_{\psi,RT,n}.
\end{equation}

For the last integral in (5.13), we will show that

\begin{equation}
(5.15) \quad \mathbb{E} \int_{D_t} |\nabla (F_n - \bar{F}_n)|^2 \leq CT(1 + \eta^{-\frac{1}{2}}) K.
\end{equation}

These estimates imply that

\begin{equation}
(5.16) \quad \int_{D_t} |n_n| \leq CT \left( n^{-1} + (1 + \eta^{-\frac{1}{2}}) K \right).
\end{equation}
This along with (5.12) implies that
\[ J_\delta(t) \leq J_\delta(0) + CT^2\left(n^{-1} + (1 + \eta^{-\frac{1}{2}})K\right), \quad t \in [0, T]. \]

Recall definitions (2.3), (5.4) and (5.11). Hence, for \( t = 0 \), we have
\[ \lim_{\delta \to 0} J_\delta(0) = \int_{D_0} |\overline{F}_n(0, h) - f(0, h)|dh \leq CT^2L. \]

Sending \( \delta \to 0 \), from the above two displays, we derive that
\[ \sup_{t \in [0, T]} \int_{\{|h| \leq R(T-t)\} \setminus D_t} |\overline{F}_n(t, h) - f(t, h)|dh \leq \int_{\{|h| \leq R(T-t)\} \setminus D_t} CT \leq CT^2\eta. \]

Therefore, we obtain
\[ \sup_{t \in [0, T]} \int_{\{|h| \leq R(T-t)\} \setminus D_t} |\overline{F}_n(t, h) - f(t, h)|dh \leq \int_{\{|h| \leq R(T-t)\} \setminus D_t} CT \leq CT^2\eta, \]

Let us now choose proper values for \( T \) and \( \delta \). Set \( T = \sqrt{2}(1 + R)M/R \) to ensure \([0, M]^3 \subset \{(t, h) : t \in [0, T], |h| \leq R(T-t)\} \). Inserting this \( T \) and \( \eta = K^\frac{1}{2} \) into the above display to see
\[ \sup_{t \in [0, T]} \int_{\{|h| \leq R(T-t)\} \setminus D_t} |\overline{F}_n(t, h) - f(t, h)|dh \leq CM^2(L + n^{-1} + K^\frac{1}{2} + K). \]

Recall our notation (5.14). This gives the desired result.

It remains to verify (5.15).

5.1.1. Proof of (5.15). Using integration by parts, we have
\[ \int_{D_t} |\nabla(F_n - \overline{F}_n)|^2 = \int_{\partial D_t} (F_n - \overline{F}_n)\nabla(F_n - \overline{F}_n) \cdot n - \int_{D_t} (F_n - \overline{F}_n)\Delta(F_n - \overline{F}_n) \]
\[ \leq \|F_n - \overline{F}_n\|_{L^\infty([0,RT]^3)}\left(\int_{\partial D_t} |\nabla(F_n - \overline{F}_n)| + \int_{D_t} |\Delta(F_n - \overline{F}_n)| \right). \]

Let us estimate the last integral. The lower bound (3.16) shows \( \Delta \overline{F}_n \geq 0 \), and the lower bound (3.18) implies that
\[ \Delta F_n + Cn^{-\frac{1}{2}}(h_1^{-\frac{3}{2}}|U| + h_2^{-\frac{3}{2}}|V|) \geq 0. \]

These yield
\[ \int_{D_t} |\Delta(F_n - \overline{F}_n)| \leq \int_{D_t} |\Delta F_n| + |\Delta \overline{F}_n| \]
\[ \leq \int_{D_t} (\Delta F_n + \Delta \overline{F}_n) + \int_{D_t} 2Cn^{-\frac{1}{2}}(h_1^{-\frac{3}{2}}|U| + h_2^{-\frac{3}{2}}|V|). \]
Applying integration by parts to the first integral and the definition of $D_t$ to the second integral, we can see
\[
\int_{D_t} |\Delta(F_n - \bar{F}_n)| \leq \int_{\partial D_t} |\nabla F_n| + |\nabla \bar{F}_n| \, dh + CT n^{-\frac{1}{2}} \eta^{-\frac{1}{2}} (|U| + |V|).
\]
Due to (3.15) and (3.17), we can obtain
\[
\int_{\partial D_t} |\nabla F_n| + |\nabla \bar{F}_n| \leq CT \left(1 + n^{-\frac{1}{2}} \eta^{-\frac{1}{2}} (|U| + |V|)\right).
\]
This display also serves as a bound for the first integral in (5.17). Insert the above two displays into (5.17) to get
\[
\int_{D_t} |\nabla(F_n - \bar{F}_n)|^2 \leq CT \|F_n - \bar{F}_n\|_{L^\infty([0,RT]^3)} \left(1 + n^{-\frac{1}{2}} \eta^{-\frac{1}{2}} (|U| + |V|)\right).
\]
Recall the definition of $K_{M,n}$ in (2.2). Take expectations on both sides of this inequality and invoke the Cauchy–Schwarz inequality to conclude (5.15).

5.2. **Proof of Theorem 2.1.** We view (2.5) as a particular case of the equation (4.2)-(4.3) with $d = 2$ and $\psi$ given in (2.4). Proposition 4.2 guarantees the uniqueness of solutions. We want to verify conditions imposed on $\psi$ in Proposition 4.3 are satisfied, in order to ensure the existence and represent the solution by the Hopf formula (2.6). This is where the assumption on the independence of $X$ and $Y$ is needed.

Due to the independence, we can see that, for all $n \in \mathbb{N}$,
\[
\bar{F}_n(0, h) = \bar{F}_n(0, h_1, 0) + \bar{F}_n(0, 0, h_2)
\]
By the assumption (2.4), we have $\psi(h) = \psi_1(h_1) + \psi_2(h_2)$ where
\[
\psi_1(h_1) = \lim_{n \to \infty} \bar{F}_n(0, h_1, 0), \quad \psi_2(h_2) = \lim_{n \to \infty} \bar{F}_n(0, 0, h_2).
\]
This display along with (3.15) and (3.16) implies that $\psi_1$ and $\psi_2$ are Lipschitz, convex and nondecreasing. Therefore, this allows us to apply Proposition 4.3, which along with the uniqueness result implies the first part of Theorem 2.1. Due to $\bar{F}_n(0, 0) = 0$, we also have $\psi(0) = 0$. The second part of Theorem 2.1 is now a direct consequence of Proposition 5.1.

5.3. **Proof of Theorem 5.2.** Note that once the existence of solutions is shown, Theorem 5.2 easily follows from Proposition 5.1 and Proposition 4.2. To obtain the existence, the plan is to first show that $\{F_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the local uniform topology and then verify that the limit, denoted as $f$, is a weak solution.

5.3.1. **The sequence $\{\bar{F}_n\}_{n=1}^\infty$ is Cauchy.** For this, we will need the assumptions $\lim_{n \to \infty} K_{M,n} = 0$ and $\lim_{n \to \infty} L_{\psi,M,n} = 0$.

We proceed similarly as in the proof of Proposition 5.1. Recall the definitions of $r_n$ in (5.1) and $\phi_\delta$ in (5.4). Let $n, n' \in \mathbb{N}$. We take $w = \bar{F}_n - \bar{F}_{n'}$, $v = \phi_\delta(w)$, $a = (\partial_h \bar{F}_n, \partial_h \bar{F}_{n'})$ and $a^* = a \ast \xi$ where $\xi$ is the mollifier in Remark 2.4. Similar to the derivation of (5.5), we have
\[
\partial_t v = \text{div}(va^*) - v\text{div}a^* + (a - a^*) \cdot \nabla + \phi'_\delta(w)(r_n + r_{n'}).
\]
The only difference is that we have an additional $r_{n'}$ in the last term.
Since $\mathcal{F}_n$ is Lipschitz uniformly in $n$ due to (3.15), we can fix $R = 1 + \sup_{n,\epsilon} \|a'\|_{\infty}$. Set $D_t$ as in (5.10), and similarly take

$$J_\delta(t) = \int_{D_t} v(t, h)dh = \int_{D_t} \phi_\delta(w(t, h))dh.$$  

By similar treatments used to obtain (5.12), we have

$$\frac{d}{dt} J_\delta(t) \leq \int_{D_t} \phi'_\delta(w)(r_n + r_n') \leq \int_{D_t} |r_n| + |r_n'|.$$  

The rest follows the exact same path after (5.12) in the proof of Proposition 5.1. The only difference is that we have more terms due to $\mathcal{F}_n'$, but they are treated in the same way as $\mathcal{F}_n$. One can see that, instead of the estimate in Theorem 2.1, we obtain

$$\sup_{t \in [0, M]} \int_{[0, M]^2} |\mathcal{F}_n(t, h) - \mathcal{F}_n'(t, h)| dh \leq CM^2 \left(L_{CM,n} + n^{-1} + (K_{CM,n})^{\frac{3}{2}} + K_{CM,n}ight)$$

$$+ L_{CM,n'} + (n')^{-1} + (K_{CM,n'})^{\frac{3}{2}} + K_{CM,n'}).$$

Hence, by the assumptions on the decay of $K_{M,n}$ and $L_{\psi, M, n}$, we know that $\mathcal{F}_n$ is Cauchy in local $L_t^\infty L_h^1$. Due to the argument in Remark 2.4, we can upgrade this to $\mathcal{F}_n$ being Cauchy locally uniformly. Let us denote the limit by $f$.

5.3.2. Verify that $f$ is a weak condition. We check that each property listed in Definition 4.1 is satisfied by $f$.

Since $\mathcal{F}_n$ is nondecreasing due to (3.16) and Lipschitz uniformly in $n$ due to (3.15), we can conclude that $f$ is nondecreasing and Lipschitz. Due to $\lim_{n \to \infty} L_{\psi, M, n} = 0$, we have $f(0, \cdot) = \psi$ verifying (2) of Definition 4.1. By (3.16), property (4) of the definition also holds.

It remains to verify that $f$ satisfies (4.2) almost everywhere (a.e.). By (3.16), we know that, along each coordinate, both $\mathcal{F}_n$ and $f$ are convex. It is well known that convexity implies convergence of derivatives at each point of differentiability. The Lipschitzness of $f$ and Rademacher’s theorem imply that $f$ is differentiable a.e. Hence, we deduce that $\partial_t \mathcal{F}_n - (\partial_{h_1} \mathcal{F}_n)(\partial_{h_2} \mathcal{F}_n)$ converges to $\partial_t f - (\partial_{h_1} f)(\partial_{h_2} f)$ pointwise a.e.

Since $\mathcal{F}_n$ is Lipschitz uniformly in $n$, the bounded convergence theorem implies that, for any compact $B \in (0, \infty)^2$ and t a.e.,

$$\int_B |\partial_t f - (\partial_{h_1} f)(\partial_{h_2} f)| dt dh = \lim_{n \to \infty} \int_B |\partial_t \mathcal{F}_n - (\partial_{h_1} \mathcal{F}_n)(\partial_{h_2} \mathcal{F}_n)| dt dh.$$  

We want to show the right hand side is zero. Recall the definition of $D_t$ in (5.10). By choosing $T$ and $\delta$ in $D_t$ suitably, we can ensure $B \subset D_t$. Then, by (5.1), (5.16) and $\lim_{n \to \infty} K_{M,n} = 0$, we conclude that the right hand side of the above display is zero. Since $B$ and $t$ are arbitrary, we conclude that $\partial_t f - (\partial_{h_1} f)(\partial_{h_2} f) = 0$ a.e.

6. Application to special cases

Recall the settings for the i.i.d. case in Section 2.2.1, the spherical case in Section 2.2.2 and a dependent case in Section 2.2.3. In order to apply Theorem 2.1 or Theorem 5.2 to these cases, we need to identify $\psi$ in (2.4) and obtain estimates for $K_{M,n}$ and $L_{\psi, M, n}$ defined in (2.2)-(2.3). In this section, we establish these results, which are stated below.
Lemma 6.1. In all three cases, there is $C > 0$ such that the following holds for all $M \geq 1$ and $n \in \mathbb{N}$,

$$K_{M,n} \leq Cn^{-\frac{1}{2}}(M + \sqrt{\log n}).$$

Recall $\beta(n)$ in (2.7).

Lemma 6.2. In the i.i.d. case, the function $\psi$ in (2.4) is given by (2.8). There is $C > 0$ such that, for all $M > 0$ and all $n \in \mathbb{N}$,

$$L_{\psi,M,n} \leq CM\beta(n).$$

Lemma 6.3. In the spherical case, the function $\psi$ in (2.4) is given by (2.9). There is $C > 0$ such that, for all $M > 0$ and all $n \in \mathbb{N}$,

$$L_{\psi,M,n} \leq CM\left(n^{-\frac{1}{2}} + \beta(n)\right).$$

As a consequence, Proposition 2.5 and Proposition 2.6 follow from these lemmas and Theorem 2.1. Lastly, in the dependent example in Section 2.2.3, we can verify that $F_n(0, \cdot) = \psi$ for all $n$ and thus $L_{\psi,M,n} = 0$. This along with Theorem 5.2 and Lemma 6.1 yields Proposition 2.7.

6.1. Proofs of Lemma 6.2 and Lemma 6.3. Due to the independence of $X$ and $Y$, we write $P_{n,Y}(dx,dy) = P_{m}(dx) \otimes P_{n}(dy)$. Then, we have

$$\overline{F}_n(0,h) = \overline{F}_n(0,h_1,0) + \overline{F}_n(0,0,h_2)$$

with

$$\overline{F}_n(0,h_1,0) = \frac{1}{N}E \log \int e^{\sqrt{2\pi y^2}U + 2y_1X - h_1|X|}P_{m}(dx),$$

$$\overline{F}_n(0,0,h_2) = \frac{1}{N}E \log \int e^{\sqrt{2\pi y^2}U + 2y_2Y - h_2|Y|}P_{n}(dy).$$

Recall $N = \sqrt{mn}$. By (3.14), there is $C > 0$ such that

$$|\overline{F}_n(0,h_1,0)|, |\overline{F}_n(0,0,h_2)| \leq C|h|, \quad \forall n \in \mathbb{N}. \tag{6.1}$$

6.1.1. The i.i.d case. By the entry-wise independence, we can compute

$$\overline{F}_n(0,h_1,0) = \sqrt{\frac{m(n)m(1)}{n}}F_1(0,h_1,0), \quad \overline{F}_n(0,0,h_2) = \sqrt{\frac{nm(1)}{m(n)}}F_1(0,0,h_2).$$

Due to (2.1), and the above displays, we have that $\overline{F}_n(0,h)$ converges locally uniformly to

$$\psi(h) = \sqrt{am(1)}F_1(0,h_1,0) + \sqrt{am^2(1)}F_1(0,0,h_2).$$

Due to (2.7) and (6.1), for any $M > 0$, we have the following convergence rate estimate

$$\sup_{h \in [0,M]} |\overline{F}_n(0,h) - \psi(h)| \leq CM\beta(n).$$

The above two displays are exactly the content of Lemma 6.2.
6.1.2. The spherical case. As in Section 2.2.2, we denote the uniform measure on $\sqrt{k}S^{k-1}$ by $\mathcal{U}_k$. Note that
\[
\sqrt{\frac{n}{m}}F_1(0, h_1, 0) = \frac{1}{m} \mathbb{E} \log \int e^{\sqrt{2n}U \cdot x + 2h_1X \cdot x - h_1|x|^2} \mathcal{U}_m(dx)
\]
is the free energy associated with the Hamiltonian $x \mapsto \sqrt{2h_1U \cdot x + 2h_1X \cdot x - h_1|x|^2}$.

The following estimate can be computed using the standard interpolation method. There is $C > 0$ such that, for all $n \in \mathbb{N}$ and all $h_1 \in \mathbb{R}_+$,
\[
(6.2) \quad \left| \sqrt{\frac{n}{m}}F_1(0, h_1, 0) - \left( h_1 - \frac{\log(1 + 2h_1)}{2} \right) \right| \leq C \frac{h_1}{\sqrt{m}}.
\]

For completeness, we briefly sketch the key steps in the proof of (6.2) and omit some computations. Details can be seen in [16, Appendix A].

Sketch of proof. Let $\tilde{X}$ and $\tilde{U}$ be two standard Gaussian vectors in $\mathbb{R}^m$, independent from each other and from other randomness. Let $P^\mathcal{U}_m$ be the law of $\tilde{X}$. Note that $\sqrt{m} \tilde{X}/|\tilde{X}|$ has the same distribution as $\mathcal{U}_m$. For $s \in [0, 1]$, we introduce an interpolating Hamiltonian
\[
\mathcal{H}_s(h_1, x) = \left( \sqrt{2h_1(1-s)mU \cdot \frac{x}{|x|}} + 2h_1(1-s)m \tilde{X} \cdot \frac{x}{|x|} - h_1(1-s)m \frac{|x|^2}{|x|} \right) + \left( \sqrt{2h_1s\tilde{U} \cdot x + 2h_1s\tilde{X} \cdot x - h_1s|x|^2} \right).
\]
and define, for fixed $m$ and $h_1$,
\[
f(s) = \frac{1}{m} \mathbb{E} \log \int e^{\mathcal{H}_s(h_1, x)}P^\mathcal{U}_m(dx).
\]

It is immediate that $\sqrt{n/m}F_1(0, h_1, 0) = f(0)$. For $s = 1$, by computing a Gaussian integration, we can see $f(1) = h_1 - \frac{1}{2} \log(1 + 2h_1)$. Since $|f(1) - f(0)| \leq \sup_{s \in [0, 1]} |f'(s)|$, the next step is to estimate $|f'(s)|$. After some computation (see [16, (30)-(31)]), we can obtain
\[
|f'(s)| \leq 2 \frac{h_1}{\sqrt{m}} \left( \mathbb{E}(|\tilde{X}| - \sqrt{m})^2 \right)^{\frac{1}{2}}.
\]

Using the standard estimate of $\tilde{X}$ concentrating near $\sqrt{m}S^{m-1}$ as $m \to \infty$ (see [25, Theorem 3.1.1]), we can bound the above display by $Ch_1/\sqrt{m}$, achieving the result.

Similarly, we also have
\[
\left| \sqrt{\frac{m}{n}}F_1(0, 0, h_2) - \left( h_2 - \frac{\log(1 + 2h_2)}{2} \right) \right| \leq C \frac{h_2}{\sqrt{n}}.
\]

Let us set
\[
\psi(h) = a \left( h_1 - \frac{\log(1 + 2h_1)}{2} \right) + a^{-1} \left( h_2 - \frac{\log(1 + 2h_2)}{2} \right).
\]

These displays along with (6.2), (2.7) and (6.1) imply that, for some $C > 0$,
\[
\sup_{h \in [0, M]^2} |F_n(0, h) - \psi(h)| \leq CM(n^{-\frac{1}{2}} + \beta(n)).
\]
This completes the proof of Lemma 6.3.
6.2. **Proof of Lemma 6.1.** The plan is to first obtain an estimate of \( \mathbb{E} e^{\lambda^2 n|F_n - F|^2} \) for small \( \lambda > 0 \) pointwise at each \( (t, h) \in [0, M]^3 \). Then, we use an \( \varepsilon \)-net argument to bound \( \mathbb{E} \sup_{(t, h) \in [0, M]^3} e^{\lambda^2 n|F_n - F|^2} \). The desired result follows from Jensen’s inequality.

6.2.1. **Pointwise estimate.** Let \((t, h) \in [0, M]^3\). Denote by \( G = (W, U, V) \) the Gaussian vector consisting of all Gaussian random variables in \( F_n \). We also write \( \mathbb{E}_G, \mathbb{E}_X, \mathbb{E}_Y \) as the expectation integrating over \( G, X, Y \), respectively. Let \( \lambda > 0 \) be chosen later. Using Hölder’s inequality, we have

\[
\mathbb{E} e^{\lambda|F_n - F|^2} \leq \mathbb{E} \left( e^{\lambda|F_n - F|^2} \mathbb{E}_X e^{\lambda|F_n - F|^2} \mathbb{E}_X e^{\lambda|F_n - F|^2} \right)^{\frac{1}{2}} \leq \left( \mathbb{E} e^{3\lambda|F_n - F|^2} \right)^{\frac{1}{3}} \left( \mathbb{E} e^{3\lambda|F_n - F|^2} \right)^{\frac{1}{3}} \left( \mathbb{E} e^{3\lambda|F_n - F|^2} \right)^{\frac{1}{3}}.
\]

(6.3)

To treat the last term, we will use the Gaussian concentration inequality. We start by computing

\[
\partial_{W_i} F_n = \frac{1}{N} \sqrt{\frac{2t}{N}} (x_i y_j), \quad \partial_{U_i} F_n = \frac{1}{N} \sqrt{2h_1 (x_i)}, \quad \partial_{V_j} F_n = \frac{1}{N} \sqrt{2h_2 (y_j)}.
\]

Therefore, by (1.1), we have

\[
|\nabla_G F_n|^2 = \sum_{i,j} |\partial_{W_i} F_n|^2 + \sum_i |\partial_{U_i} F_n|^2 + \sum_j |\partial_{V_j} F_n|^2
\]

\[
= \frac{2t}{N^3} \langle (x' \cdot y') \rangle + \frac{2h_1}{N^2} \langle x' \cdot y' \rangle + \frac{2h_2}{N^2} \langle y' \cdot y' \rangle \leq CMn^{-1}.
\]

Invoking [8, Theorem 5.5], we obtain

\[
\mathbb{E} e^{\lambda|F_n - F|^2} \leq e^{C\lambda^2 M n^{-1}}.
\]

(6.4)

Then, we treat the first two terms in (6.3). In the i.i.d. case, due to the boundedness assumption \( |X_1|, |Y_1| \leq 1 \), we have

\[
|\partial_X F_n| = \left| \frac{1}{N} \left( \sum_j 2t y_j x_j + 2h_1 x_i \right) \right| \leq CMn^{-1}.
\]

(6.5)

Using the boundedness again and [8, Theorem 6.2] (see the penultimate display in its proof), we obtain

\[
\mathbb{E} e^{\lambda|F_n - F|^2} \leq e^{C\lambda^2 M^2 n^{-1}}.
\]

(6.6)

Applying the same argument to the second term in (6.3), we have

\[
\mathbb{E} e^{\lambda|F_n - F|^2} \leq e^{C\lambda^2 M^2 n^{-1}}.
\]

(6.7)

Similar, the above estimates also hold for the example in Section 2.2.3.

Now, let us turn to the spherical case. Since \( |X|, |Y| \leq C\sqrt{n} \), using the expression of \( \partial_X F_n \) in (6.5), we have

\[
|\nabla_X F_n| \leq CMn^{-\frac{1}{2}}.
\]

We need Levy’s inequality stated below (see [17, Corollary 5.4])

**Lemma 6.4.** Let \( f : \mathbb{S}^{n-1} \rightarrow \mathbb{R} \) be Lipschitz with Lipschitz constant \( L \), and let \( \mathcal{X} \) be distributed uniformly on \( \mathbb{S}^{n-1} \). Then there are constants \( C, c > 0 \) such that

\[
\mathbb{P} \{ |f(\mathcal{X}) - \mathbb{E} f(\mathcal{X})| \geq Lt \} \leq Ce^{-ct^2}
\]
We claim that, for small $\lambda \sqrt{n}$, the inequality (3.17) holds. This immediately gives

$$\mathbb{P}_X[|F_n - \mathbb{E}_X F_n| \geq t] \leq C e^{-\varepsilon n M^{-2} t^2}.$$  

Now apply [25, Proposition 2.5.2] to see that (6.6) holds, and so does (6.7) via the same method.

In conclusion, (6.3), (6.4), (6.6) and (6.7), with $\lambda$ replaced by $\lambda \sqrt{n}$ yield

$$\mathbb{E} e^{\lambda \sqrt{n} |F_n - \mathbb{F}_n|} \leq C e^{C \lambda^2 M^2}.$$  

[25, Proposition 2.5.2] implies that, for $\lambda$ sufficiently small,

$$\mathbb{E} e^{\lambda^2 |F_n - \mathbb{F}_n|^2} \leq C e^{C \lambda^2 M^2}.$$  

6.2.2. Application of an $\varepsilon$-net argument. The goal is upgrade (6.9) to a bound on $\mathbb{E} \sup_{(t,h) \in [0,M]^3} e^{\lambda^2 |F_n - \mathbb{F}_n|^2}$. The estimate (3.17) implies that, for $|t - t'| + |h - h'| \leq 1$,

$$|F_n(t, h) - F_n(t', h')| \leq C \left(1 + n^{-1/2} (|W|_{op} + |U| + |V|) \right) \left(|t - t'|^{1/2} + |h - h'|^{1/2}\right).$$

For $\varepsilon \in (0, 1]$, let us introduce the $\varepsilon$-net

$$A_{\varepsilon} = \{\varepsilon, 2\varepsilon, 3\varepsilon \ldots \}^3 \cap [0, M]^3.$$  

Hence, for $\lambda$ small, we have

$$\mathbb{E} \sup_{(t,h) \in [0,M]^3} e^{\lambda^2 |F_n - \mathbb{F}_n|^2} \leq \mathbb{E} \left( \sup_{(t,h) \in A_{\varepsilon}} e^{\lambda^2 |F_n - \mathbb{F}_n|^2} \right) \exp \left( C \lambda^2 \varepsilon \left(n + |W|_{op}^2 + |U|^2 + |V|^2\right) \right) \leq \left( \mathbb{E} \sup_{(t,h) \in A_{\varepsilon}} e^{2\lambda^2 |F_n - \mathbb{F}_n|^2} \right)^{1/2} \left( \mathbb{E} \exp \left(C \lambda^2 \varepsilon \left(n + |W|_{op}^2 + |U|^2 + |V|^2\right) \right) \right)^{1/2}.$$  

where we used the Cauchy–Schwarz inequality in the second inequality. Since $|A_{\varepsilon}| \leq (M/\varepsilon)^3$. Using the union bound and (6.9), we have

$$\left( \mathbb{E} \sup_{(t,h) \in A_{\varepsilon}} e^{2\lambda^2 |F_n - \mathbb{F}_n|^2} \right)^{1/2} \leq C(M/\varepsilon)^{3/2} e^{C \lambda^2 M^2}, \quad \lambda \in \mathbb{R}.$$  

Set $\varepsilon = C^{-1} n^{-1}$ in (6.10) with $C$ therein, and use (6.11) to see

$$\mathbb{E} \sup_{(t,h) \in [0,M]^3} e^{\lambda^2 |F_n - \mathbb{F}_n|^2} \leq C(Mn)^{3/2} e^{C \lambda^2 M^2} \left( \mathbb{E} \exp \left( \lambda^2 \left(1 + n^{-1} (|W|_{op}^2 + |U|^2 + |V|^2) \right) \right) \right)^{1/2}.$$

We claim that, for small $\lambda > 0$,

$$\mathbb{E} \exp \left( \lambda^2 \left(1 + n^{-1} (|W|_{op}^2 + |U|^2 + |V|^2) \right) \right) \leq C.$$  

This immediately gives

$$\mathbb{E} \sup_{(t,h) \in [0,M]^3} e^{\lambda^2 |F_n - \mathbb{F}_n|^2} \leq C(Mn)^{3/2} e^{C \lambda^2 M^2}.$$
Finally, using Jensen’s inequality, we conclude that
\[
\mathbb{E} \sup_{(t,h) \in [0,M]^2} |F_n - \mathcal{F}_n|^2 \leq \lambda^{-2} n^{-1} \log \left( \mathbb{E} \sup_{(t,h) \in [0,M]^2} e^{\lambda n |F_n - \mathcal{F}_n|^2} \right) 
\leq C n^{-1} (M^2 + \log n),
\]
as desired. The proof will be complete once (6.12) is verified.

6.2.3. Proof of (6.12). We want to bound exponential moments of $|W|_{op}$, $|U|$ and $|V|$. Using the fact that $U$ and $V$ are standard Gaussian in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, we have, for $\lambda$ small,
\[
\mathbb{E} e^{\lambda^2 n^{-1} |U|^2}, \quad \mathbb{E} e^{\lambda^2 n^{-1} |V|^2} \leq C.
\]

Then, we try to bound $\mathbb{E} e^{\lambda^2 |W|_{op}^2}$. For $x \in \mathbb{R}^m$ with $|x| \leq 1$, $\{ (Wx)_i \}_{i=1}^n$ is a vector in $\mathbb{R}^n$ consisting of independent centered Gaussian entries with variance $|x|^2 \leq 1$. Hence, there are $C, c > 0$ such that for $\lambda$ small,
\[
\mathbb{E} e^{\lambda^2 |Wx|^2} \leq C e^{c \lambda^2 n},
\]
which by Chebyshev’s inequality implies that
\[
\mathbb{P} \{ \lambda^2 |Wx|^2 \geq t \} \leq C e^{-t + c \lambda^2 n}.
\]

Now let us consider a finite set $B \subset \{ x \in \mathbb{R}^m : |x| \leq 1 \}$ with the following properties:

- for any distinct $x, y \in B$, we have $|x - y| > \frac{1}{2}$;
- for every $x \in \{ x \in \mathbb{R}^m : |x| \leq 1 \}$, there is $y \in B$ such that $|x - y| \leq \frac{1}{2}.$

By the first property, all balls with radius $\frac{1}{2}$ and centered at a point in $B$ are disjoint. Since the union of these balls is contained in a ball of radius 2, the size of $B$ satisfies $(1/2)^m |B| \leq 2^m$. Also recall (2.1). Hence, there is $a > 0$ such that,
\[
|B| \leq 4^m \leq a^n.
\]

The second property of $B$, along with the fact $|Wx - Wy| \leq |W|_{op} |x - y|$, implies that
\[
|W|_{op} = \sup_{|x| \leq 1} |Wx| \leq \sup_{x \in B} |Wx| + \frac{1}{2} |W|_{op}.
\]
Therefore, we have $|W|_{op} \leq 2 \sup_{x \in B} |Wx|$. This along with (6.14) and (6.15) yields
\[
\mathbb{P} \{ e^{\lambda^2 n^{-1} |W|_{op}^2} \geq t \} \leq C a^n \exp \left( - c' n \log t + c n \lambda^2 \right) \leq C \left( \frac{ae^c}{t^{c' \lambda^2}} \right)^n,
\]
for an additional constant $c' > 0$. Writing $b = (ae^c)^{\frac{1}{c' \lambda^2}}$, we have, for $n$ large,
\[
\mathbb{E} e^{\lambda^2 n^{-1} |W|_{op}^2} = \int_0^\infty \mathbb{P} \{ e^{\lambda^2 n^{-1} |W|_{op}^2} \geq t \} dt \leq b + \int_b^\infty C \left( \frac{b}{t} \right)^{c'n} dt = b + \frac{Cb}{c'n - 1} \leq C.
\]
This and (6.13) imply (6.12).
6.3. Discussion on the case where \( X \) and \( Y \) are dependent. When \( X \) and \( Y \) are not independent, under certain conditions, Theorem 5.2 yields the convergence of \( \overline{F}_n \) to the unique weak solution \( f \) of the equation (2.5) with \( f(0, \cdot) = \psi \) being the limit of \( \overline{F}_n(0, \cdot) \). However, when the independence is absent, \( \psi \) may not be of the form (4.7) and thus Proposition 4.3 cannot be applied to give a variational representation of \( f \). For instance, in the example considered in Section 2.2.3, the function \( \psi \) given in (2.10) does not split as in (4.7) in general.

Essentially, the issue is that it is not straightforward to verify that the Hopf formula is a weak solution, in particular, (4) of Definition 4.1, when \( \psi \) does not satisfy (4.7). By contrast, if solutions are defined in the viscosity sense, then it is easy to verify that the Hopf formula is a viscosity solution for any Lipschitz convex initial condition. Hence, one possible remedy is to explore and use the connection between these two notions of solutions. In this work, we do not pursue this direction, because viscosity solutions require more involved treatments. We postpone the resolution of this issue to future investigations.

6.4. Discussion on the sparse model. Since one of the goals of this work is to convey the versatility of our approach, we briefly discuss possible modifications of our arguments to study the sparse model, described below.

Let \( P_1 \) and \( P_2 \) be probability distributions supported on \([-1,1]\). Consider two sequences \( \{\rho_{1,n}\}_{n=1}^{\infty} \) and \( \{\rho_{2,n}\}_{n=1}^{\infty} \) of real numbers in \([0,1]\). They are interpreted as the sparsity parameters. Typically, at least one of the parameters goes to zero as \( n \to \infty \). Let \( X \) and \( Y \) be distributed according to

\[
P_n^{X,Y} = \left( \rho_{1,n}P_1 + (1 - \rho_{1,n})\delta_0 \right)^{\otimes m} \otimes \left( \rho_{2,n}P_2 + (1 - \rho_{2,n})\delta_0 \right)^{\otimes n}
\]

where \( \delta_0 \) is the Dirac measure at 0. In other words, \( X \) and \( Y \) have i.i.d. entries and their distributions have certain degrees of sparsity. To compensate for this sparsity of signal, it is natural to enlarge the signal-to-noise ratio correspondingly. Namely, we introduce another sequence \( \{t_n\}_{n=1}^{\infty} \) with \( \lim_{n \to \infty} t_n = \infty \).

For each \( n \in \mathbb{N} \), let \( f_n \) be the weak solution to (2.5) with initial condition \( \psi_n = \overline{F}_n(0, \cdot) \). We are interested in the asymptotics of \( \overline{F}_n(t_n, \cdot) - f_n(t_n, \cdot) \) as \( n \to \infty \). Recall the definition of \( \mathcal{A}(c) \) introduced above Proposition 5.1. It can be checked that all \( \psi_n \) belong to \( \mathcal{A}(c) \) for some \( c > 0 \). Hence, Proposition 5.1 is applicable here. However, the issue is that \( t_n \) now diverges to \( \infty \), while the estimate in Proposition 5.1 depends polynomially on \( t_n \). Therefore, more effort is needed to get meaningful estimates.

Since we are mostly interested in asymptotics when \( h \) is small, one place to start is to choose a better region \( A_n = [0, t_n] \times [0, a_n] \times [0, b_n] \) instead of blindly applying Proposition 5.1 to the region \([0, t_n] \times [0, t_n]^2\). Examining its proof, we can notice that \( M \) is linked to the choice of \( R \) defined after (5.9). In its current form, we artificially add 1 to the definition of \( R \), but actually it can be made to have the same order as \( \| \overline{F}_n \|_{\text{Lip}} + \| f_n \|_{\text{Lip}} \). Even further, instead of the circular region (5.10), we can choose a rectangular region with two sides varying differently, the orders of which depend on \( \| \partial_{h_1} \overline{F}_n \|_{\infty} \) and \( \| \partial_{h_1} f_n \|_{\infty} \). These quantities can be bounded in terms of powers of \( \rho_{k,n} \). From these considerations, also taking into account the specific orders of \( t_n \) and \( \rho_{k,n} \), we can determine a proper \( A_n \).

In the proof of Proposition 5.1, we used several estimates from Lemma 3.2. In the sparsity case, we can improve these estimates by obtaining decay rates in terms of the
sparsity parameters. Finally, after we have chosen a better region $A_n$, the quantities $K$ and $L$ in (2.2)-(2.3) can be redefined accordingly. Again, we can use the sparsity to improve many related estimates.

REFERENCES

[1] J. Barbier, M. Dia, N. Macris, F. Krzakala, T. Lesieur, and L. Zdeborová. Mutual information for symmetric rank-one matrix estimation: A proof of the replica formula. In Advances in Neural Information Processing Systems (NIPS), volume 29, pages 424–432, 2016.

[2] J. Barbier and N. Macris. The adaptive interpolation method: a simple scheme to prove replica formulas in bayesian inference. Probability Theory and Related Fields, 174(3-4):1133–1185, 2019.

[3] J. Barbier and N. Macris. The adaptive interpolation method for proving replica formulas. applications to the curie–weiss and wigner spike models. Journal of Physics A: Mathematical and Theoretical, 52(29):294002, 2019.

[4] J. Barbier, N. Macris, and L. Miolane. The layered structure of tensor estimation and its mutual information. In 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pages 1056–1063. IEEE, 2017.

[5] M. Bardi and L. C. Evans. On Hopf's formulas for solutions of Hamilton-Jacobi equations. Nonlinear Analysis: Theory, Methods & Applications, 8(11):1373–1381, 1984.

[6] A. Barra, G. Dal Ferraro, and D. Tantari. Mean field spin glasses treated with PDE techniques. The European Physical Journal B, 86(7):1–10, 2013.

[7] A. Barra, A. Di Biasio, and F. Guerra. Replica symmetry breaking in mean-field spin glasses through the Hamilton–Jacobi technique. Journal of Statistical Mechanics: Theory and Experiment, 2010(09):P09006, 2010.

[8] S. Boucheron, G. Lugosi, and P. Massart. Concentration inequalities: A nonasymptotic theory of independence. Oxford university press, 2013.

[9] L. C. Evans. Partial Differential Equations, volume 19. American Mathematical Soc., 2010.

[10] G. Genovese and A. Barra. A mechanical approach to mean field spin models. Journal of Mathematical Physics, 50(5):053303, 2009.

[11] F. Guerra. Sum rules for the free energy in the mean field spin glass model. Fields Institute Communications, 30(11), 2001.

[12] J. Kadmon and S. Ganguli. Statistical mechanics of low-rank tensor decomposition. In Advances in Neural Information Processing Systems, pages 8201–8212, 2018.

[13] S. B. Korada and N. Macris. Exact solution of the gauge symmetric p-spin glass model on a complete graph. Journal of Statistical Physics, 136(2):205–230, 2009.

[14] M. Lelarge and L. Miolane. Fundamental limits of symmetric low-rank matrix estimation. Probability Theory and Related Fields, 173(3-4):859–929, 2019.

[15] P.-L. Lions and J.-C. Rochet. Hopf formula and multitime Hamilton-Jacobi equations. Proceedings of the American Mathematical Society, 96(1):79–84, 1986.

[16] C. Luneau, N. Macris, and J. Barbier. High-dimensional rank-one nonsymmetric matrix decomposition: the spherical case. arXiv preprint arXiv:2004.06975, 2020.

[17] E. S. Meckes. The random matrix theory of the classical compact groups, volume 218. Cambridge University Press, 2019.

[18] L. Miolane. Fundamental limits of low-rank matrix estimation: the non-symmetric case. arXiv preprint arXiv:1702.00473, 2017.

[19] J.-C. Mourrat. Hamilton-Jacobi equations for mean-field disordered systems. arXiv preprint arXiv:1811.01432, 2018.

[20] J.-C. Mourrat. Hamilton-Jacobi equations for finite-rank matrix inference. arXiv preprint arXiv:1904.05294, 2019.

[21] J.-C. Mourrat. Parisi’s formula is a Hamilton-Jacobi equation in Wasserstein space. arXiv preprint arXiv:1906.08471, 2019.

[22] J.-C. Mourrat. Nonconvex interactions in mean-field spin glasses. arXiv preprint arXiv:2004.01679, 2020.

[23] J.-C. Mourrat and D. Panchenko. Extending the Parisi formula along a Hamilton-Jacobi equation. Electronic Journal of Probability, 25, 2020.
[24] G. Reeves. Information-theoretic limits for the matrix tensor product. arXiv preprint arXiv:2005.11273, 2020.

[25] R. Vershynin. High-dimensional probability: An introduction with applications in data science, volume 47. Cambridge university press, 2018.