A VON NEUMANN ALGEBRA APPROACH TO QUANTUM METRICS

GREG KUPERBERG AND NIK WEAVER

Abstract. We propose a new definition of quantum metric spaces, or W*-metric spaces, in the setting of von Neumann algebras. Our definition effectively reduces to the classical notion in the atomic abelian case, has both concrete and intrinsic characterizations, and admits a wide variety of tractable examples. A natural application and motivation of our theory is a mutual generalization of the standard models of classical and quantum error correction.

It has proven to be fruitful in abstract analysis to think of various structures connected to Hilbert space as “noncommutative” or “quantum” versions of classical mathematical objects. This point of view has been emphasized in [8] (see also [34]). For instance, it is well established that C*-algebras and von Neumann algebras can profitably be thought of as quantum topological and measure spaces, respectively. The use of the word “quantum” is called for if, for example, the structures in question play a role in modelling quantum mechanical systems analogous to the role played by the corresponding classical mathematical structures in classical physics.

Basic examples in noncommutative geometry such as the quantum tori [21] clearly exhibit a metric aspect in that they carry a natural noncommutative analog of the algebra of bounded scalar-valued Lipschitz functions on a metric space. However, the general notion of a quantum metric has been elusive. Possible definitions have been proposed by Connes [9], Rieffel [22], and Weaver [31]. Connes’ definition involves his notion of spectral triples and is patterned after the Dirac operator on a Riemannian manifold. Possibly the right interpretation of this definition is “quantum Riemannian manifold” rather than “quantum metric space”. Weaver proposed a definition involving unbounded derivations of von Neumann algebras into dual operator bimodules. This definition neatly recovers classical Lipschitz algebras in the abelian case, but it has not led to a deeper structure theory. Rieffel’s definition, which is also called a C*-metric space [23], generalizes the classical Lipschitz seminorm on functions on a metric space. This definition has attracted the most interest recently; among other interesting properties, it leads to a useful model of Gromov-Hausdorff convergence.

We introduce a new definition of a quantum metric space. To distinguish between our model and that of Rieffel, it can also be called a W*-metric space. Recall that an operator system is a linear subspace of $\mathcal{B}(H)$ that is self-adjoint and contains the identity operator. We say that a $W^*$-filtration of $\mathcal{B}(H)$ is a one-parameter family of weak* closed operator systems $\mathcal{V}_t$, $t \in [0, \infty)$, such that

(i) $\mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}$ for all $s, t \geq 0$

(ii) $\mathcal{V}_t = \bigcap_{s \geq t} \mathcal{V}_s$ for all $t \geq 0$. 

Date: May 3, 2010.
Notice that $\mathcal{V}_0$ is automatically a von Neumann algebra, since the filtration condition (i) implies that it is stable under products. We define a $W^*$-metric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ to be a $W^*$-filtration $\{\mathcal{V}_t\}$ such that $\mathcal{V}_0$ is the commutant of $\mathcal{M}$. Since we interpret a $W^*$-metric as a type of quantum metric, and since it is the main type that we will consider in this article, we will also just call it a quantum metric.

We will justify this definition with various constructions and results. We can begin with a correspondence table between the usual axioms of a metric space and some of our conditions:

\[
\begin{align*}
    d(x,x) &= 0 \iff I \subseteq \mathcal{V}_0 \\
    d(x,y) &= d(y,x) \iff \mathcal{V}_t^* = \mathcal{V}_t \\
    d(x,z) &\leq d(x,y) + d(y,z) \iff \mathcal{V}_s\mathcal{V}_t \subseteq \mathcal{V}_{s+t}
\end{align*}
\]

The rough intuition is that $\mathcal{V}_t$ consists of the operators that do not displace any mass more than $t$ units away from where it started.

We will show that quantum metrics on $\mathcal{M}$ do not depend on the representation of $\mathcal{M}$ (Theorem 2.24). We will also show that any quantum metric on $\mathcal{M}$ yields a $C^*$-algebra $UC(\mathcal{M})$ of uniformly continuous elements and an algebra $\text{Lip}(\mathcal{M}) \subseteq UC(\mathcal{M})$ of (commutation) Lipschitz elements that are both weak* dense in $\mathcal{M}$ (Proposition 5.13 and Theorem 4.26).

One motivation for our approach is the standard model of quantum error correction. Classical error correction is a theory of minimum-distance sets in metric spaces: if $X$ is a metric space and $C \subseteq X$ is a subset with minimum distance $t$ (i.e., $\inf \{d(x,y) : x, y \in C, x \neq y\} = t$), then $C$ is said to be a code that detects errors of size less than $t$ and corrects errors of size less than $t/2$. In particular, the Hamming metric on the space $X = \{0,1\}^n$ of $n$-bit words is defined by letting the distance between two words be the number of bits that differ. In quantum information theory, the quantum Hamming metric is a quantum metric in our sense on $n$ qubits. Here a qubit is a quantum system with von Neumann algebra $M_2(\mathbb{C})$; thus $M_{2^n}(\mathbb{C}) \cong M_2(\mathbb{C})^\otimes n$ is the von Neumann algebra of $n$ qubits. The filtration of the quantum Hamming metric is

\[
\mathcal{V}_t = \text{span}\{A_1 \otimes \cdots \otimes A_n : A_i \in M_2(\mathbb{C}) \text{ and } A_i = I_2 \text{ for all but at most } t \text{ values of } i\} \subseteq M_{2^n}(\mathbb{C}).
\]

This filtration models the error operators that corrupt at most $t$ qubits for some $t$. (Note that an operator $C \in \mathcal{V}_t$ can be a linear combination, or quantum superposition, of operators that corrupt different sets of $t$ or fewer qubits.) There is a natural definition of a code subspace $C \subseteq (\mathbb{C}^2)^\otimes n$ of minimum distance $t$, which is then a quantum code. Quantum codes both resemble classical codes and are used for the same purposes. Various generalizations of the quantum Hamming metric on qubits have been studied; for instance, it is routine to replace qubits by qudits with algebra $M_d(\mathbb{C})$. Knill, Laflamme, and Viola considered a general operator system as an error model [17]; this is equivalent to a quantum graph metric with $\mathcal{M} = \mathcal{B}(H)$ (see Section 3.2). However, more general $W^*$-metrics, even with $\mathcal{M} = \mathcal{B}(H)$, have not previously been studied to our knowledge.

A second motivation is the intermediate model of a measurable metric space due to Weaver [29], in which the metric set $X$ is replaced by a measure space $(X, \mu)$. If $\mu$ is atomic, so that $\mathcal{M} = l^\infty(X)$, then a quantum metric on $\mathcal{M}$ is equivalent to a classical metric on $X$ (Proposition 2.5). In the general measurable
setting, a measurable metric on \((X, \mu)\) is equivalent to a reflexive quantum metric on \(L^\infty(X, \mu)\) (Theorem 2.22).

Our new definition is related to the other models of quantum metric spaces mentioned above. First, every spectral triple in Connes’ sense yields a W*-metric (Definition 3.23). This can be seen as encoding the purely metric features of the spectral triple, as opposed to its Riemannian or spinorial structure. Second, every W*-metric yields a Leibniz Lipschitz seminorm in Rieffel’s sense (Definition 4.19). Third, as anticipated in Weaver’s earlier work, every W*-metric yields a Lipschitz algebra that is the domain of a W*-derivation (Definition 4.20). One twist is that in the noncommutative case, the classical Lipschitz condition \(|f(x) - f(y)| \leq C \cdot d(x, y)|\) splits into two distinct conditions, a commutation condition and a spectral condition. The commutation version is the one with good algebraic properties, but it is the spectral version that admits an elegant abstract axiomatization (Definition 4.14).

We establish several equivalent definitions of a W*-metric space. Our main definition is the one stated above in terms of W*-filtrations. This is trivially equivalent to a displacement gauge on \(B(H)\) (Definition 2.1 (b)). A much deeper result gives an intrinsic characterization of quantum metrics in terms of quantum distance functions defined on pairs of projections in \(\mathcal{M} \otimes B(l^2)\) (Definition 2.7/Theorem 2.45). This characterization can be attractively recast in terms of quantum Lipschitz gauges on the self-adjoint part of \(\mathcal{M} \otimes B(l^2)\) (Definition 4.14/Corollary 4.17).

We wish to thank David Blecher, Chris Bumgardner, Renato Feres, Jerry Kaminker, Michael Kapovich, Nets Katz, Greg Knese, Emmanuel Knill, John McCarthy, Stephen Power, Marc Rieffel, Zhong-Jin Ruan, David Sherman, and András Vasy for helpful conversations. We also thank the referee for suggesting many minor improvements.

We work with complex scalars throughout. “Projection” always means “orthogonal projection”.

**Contents**

1. Measurable and quantum relations 4
2. Quantum metrics 6
   2.1. Basic definitions 7
   2.2. More definitions 10
   2.3. The abelian case 17
   2.4. Reflexivity and stabilization 19
   2.5. Constructions with quantum metrics 21
   2.6. Intrinsic characterization 29
3. Examples 30
   3.1. Operator systems 31
   3.2. Graph metrics 32
   3.3. Quantum metrics on \(M_2(\mathbb{C})\) 33
   3.4. Quantum Hamming distance 33
   3.5. Quantum tori 36
   3.6. Hölder metrics 40
   3.7. Spectral triples 41
4. Lipschitz operators 44
   4.1. The abelian case 45
1. MEASURABLE AND QUANTUM RELATIONS

It is convenient to begin with a brief summary of basic results about measurable and quantum relations. This material will be used sporadically in subsequent sections. For a fuller treatment see [35]. The reader is encouraged to skip this chapter and refer back to it as needed.

We first state the definition of a measurable relation. A measure space \((X, \mu)\) is finitely decomposable if it can be partitioned into a possibly uncountable family of finite measure subspaces \(X_\lambda\) such that a set \(S \subseteq X\) is measurable if and only if its intersection with each \(X_\lambda\) is measurable, in which case \(\mu(S) = \sum \mu(S \cap X_\lambda)\) ([32], Definition 6.1.1). Finitely decomposable measures generalize both \(\sigma\)-finite measures and counting measures. The significance of the condition is that it ensures \(L^\infty(X, \mu) \cong L^1(X, \mu)^*\), and hence that the projections in \(L^\infty(X, \mu)\) (equivalently, the measurable subsets of \(X\) up to null sets) constitute a complete Boolean algebra.

**Definition 1.1.** ([35], Definition 1.2) Let \((X, \mu)\) be a finitely decomposable measure space. A measurable relation on \(X\) is a family \(\mathcal{R}\) of ordered pairs of nonzero projections in \(L^\infty(X, \mu)\) such that
\[
\left( \bigvee p_\lambda, \bigvee q_\kappa \right) \in \mathcal{R} \iff \text{some } (p_\lambda, q_\kappa) \in \mathcal{R}
\]
for any pair of families of nonzero projections \(\{p_\lambda\}\) and \(\{q_\kappa\}\).

Now let \(H\) be a complex Hilbert space, not necessarily separable. Recall ([27], Definition II.2.1) that the weak* (or \(\sigma\)-weak operator) topology on \(B(H)\) is the weak topology arising from the pairing \(\langle A, B \rangle \mapsto \text{tr}(AB)\) of \(B(H)\) with the trace class operators \(\mathcal{T}C(H)\); that is, it is the weakest topology that makes the map \(A \mapsto \text{tr}(AB)\) continuous for all \(B \in \mathcal{T}C(H)\). The weak* topology is finer than the weak operator topology but the two agree on bounded sets.

An operator algebra is a linear subspace of \(B(H)\) that is stable under products. A subspace of \(B(H)\) is self-adjoint if it is stable under adjoints and unital if it contains the identity operator \(I\). A von Neumann algebra is a weak* closed self-adjoint unital operator algebra. We will refer to [27] for standard facts about von Neumann algebras. For example, the commutant of a von Neumann algebra \(\mathcal{M}\) is the von Neumann algebra
\[
\mathcal{M}' = \{ A \in B(H) : AB = BA \text{ for all } B \in \mathcal{M} \}
\]
and von Neumann’s double commutant theorem states that every von Neumann algebra equals the commutant of its commutant, \(\mathcal{M} = \mathcal{M}''\) ([27], Theorem II.3.9).

A dual operator space is a weak* closed subspace \(V\) of \(B(H)\); it is a \(W^*\)-bimodule over a von Neumann algebra \(\mathcal{M} \subseteq B(H)\) if \(\mathcal{M}V\mathcal{M} \subseteq V\). A dual operator system is a self-adjoint unital dual operator space.
Definition 1.2. (Definition 2.1) A quantum relation on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \) is a W*-bimodule over its commutant \( \mathcal{M}' \), i.e., it is a weak* closed subspace \( \mathcal{V} \subseteq \mathcal{B}(H) \) satisfying \( \mathcal{M}' \mathcal{V} \mathcal{M}' \subseteq \mathcal{V} \).

Quantum relations are effectively representation independent.

Theorem 1.3. (Theorem 2.7) Let \( H_1 \) and \( H_2 \) be Hilbert spaces and let \( \mathcal{M}_1 \subseteq \mathcal{B}(H_1) \) and \( \mathcal{M}_2 \subseteq \mathcal{B}(H_2) \) be isomorphic von Neumann algebras. Then any isomorphism induces a 1-1 correspondence between the quantum relations on \( \mathcal{M}_1 \) and the quantum relations on \( \mathcal{M}_2 \), and this correspondence respects the conditions \( \mathcal{V} \subseteq \mathcal{W} \), \( \mathcal{V} = \mathcal{M}' \), \( \mathcal{V}^* = \mathcal{W} \), and \( \mathcal{W}^{wk^*} = \mathcal{W} \).

If \( H_2 = K \otimes H_1 \) then the 1-1 correspondence is given by \( \mathcal{V} \leftrightarrow \mathcal{B}(K) \otimes \mathcal{V} \), where \( \otimes \) is the normal spatial tensor product, i.e., the weak* closure of the algebraic tensor product in \( \mathcal{B}(K \otimes H) \). Checking this case suffices to establish the result, because any two faithful normal unital representations of a von Neumann algebra become spatially equivalent when each is tensored with a large enough Hilbert space (Theorem IV.5.5).

Quantum relations effectively reduce to classical relations in the atomic abelian case. Let \( V_{xy} \) be the rank one operator \( V_{xy} : g \mapsto \langle g, e_y \rangle e_x \) on \( l^2(X) \). Here \( \{e_x\} \) is the standard basis of \( l^2(X) \).

Proposition 1.4. (Proposition 2.2) Let \( X \) be a set and let \( \mathcal{M} \equiv l^\infty(X) \) be the von Neumann algebra of bounded multiplication operators on \( l^2(X) \). If \( R \) is a relation on \( X \) then

\[
\mathcal{V}_R = \{ A \in \mathcal{B}(l^2(X)) : (x, y) \notin R \Rightarrow \langle Ae_y, e_x \rangle = 0 \}
\]

is a quantum relation on \( \mathcal{M} \); conversely, if \( \mathcal{V} \) is a quantum relation on \( \mathcal{M} \) then

\[
R_\mathcal{V} = \{ (x, y) \in X^2 : \langle Ae_y, e_x \rangle \neq 0 \text{ for some } A \in \mathcal{V} \}
\]

is a relation on \( X \). The two constructions are inverse to each other.

Before we state the fundamental result relating measurable relations on \( (X, \mu) \) and quantum relations on \( L^\infty(X, \mu) \), we need the notion of (operator) reflexivity:

Definition 1.5. (Definition 2.14) A subspace \( \mathcal{V} \subseteq \mathcal{B}(H) \) is (operator) reflexive if

\[
\mathcal{V} = \{ B \in \mathcal{B}(H) : PVQ = 0 \Rightarrow PBQ = 0 \},
\]

with \( P \) and \( Q \) ranging over projections in \( \mathcal{B}(H) \).

A simple observation is that if \( \mathcal{V} \) is a quantum relation on \( \mathcal{M} \) then \( P \) and \( Q \) can be restricted to range over projections in \( \mathcal{M} \) in the preceding definition (Proposition 2.15).

For \( f \in L^\infty(X, \mu) \) let \( M_f \in \mathcal{B}(L^2(X, \mu)) \) be the corresponding multiplication operator, \( M_f : g \mapsto fg \).

Theorem 1.6. (Theorem 2.9/Corollary 2.16) Let \( (X, \mu) \) be a finitely decomposable measure space and let \( \mathcal{M} \equiv L^\infty(X, \mu) \) be the von Neumann algebra of bounded multiplication operators on \( L^2(X, \mu) \). If \( R \) is a measurable relation on \( X \) then

\[
\mathcal{V}_R = \{ A \in \mathcal{B}(L^2(X, \mu)) : (p, q) \notin R \Rightarrow M_p AM_q = 0 \}
\]
is a quantum relation on $\mathcal{M}$; conversely, if $\mathcal{V}$ is a quantum relation on $\mathcal{M}$ then

$$\mathcal{R}_\mathcal{V} = \{(p, q) : M_p A M_q \neq 0 \text{ for some } A \in \mathcal{V}\}$$

is a measurable relation on $X$. We have $\mathcal{R} = \mathcal{R}_{\mathcal{V}_\mathcal{R}}$ for any measurable relation $\mathcal{R}$ on $X$ and $\mathcal{V} \subseteq \mathcal{V}_{\mathcal{R}_\mathcal{V}}$ for any quantum relation $\mathcal{V}$ on $\mathcal{M}$, with equality if and only if $\mathcal{V}$ is reflexive.

The following basic tool will be used repeatedly.

**Lemma 1.7.** ([35], Lemma 2.8) Let $\mathcal{V}$ be a quantum relation on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and let $A \in \mathcal{B}(H) - \mathcal{V}$. Then there is a pair of projections $P$ and $Q$ in $\mathcal{M} \otimes \mathcal{B}(l^2) \subseteq \mathcal{B}(H \otimes l^2)$ such that

$$P(A \otimes I)Q \neq 0$$

but

$$P(B \otimes I)Q = 0$$

for all $B \in \mathcal{V}$.

We conclude this brief review of quantum relations with a result that characterizes them intrinsically. Denote the range projection of $A$ by $[A]$.

**Definition 1.8.** ([35], Definition 2.24) Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M} \otimes \mathcal{B}(l^2)$, equipped with the restriction of the weak operator topology. An *intrinsic quantum relation* on $\mathcal{M}$ is an open subset $\mathcal{R} \subset \mathcal{P} \times \mathcal{P}$ satisfying

(i) $(0, 0) \notin \mathcal{R}$

(ii) $(\bigvee P_\lambda, \bigvee Q_\kappa) \in \mathcal{R} \Leftrightarrow \text{some } (P_\lambda, Q_\kappa) \in \mathcal{R}$

(iii) $(P, [BQ]) \in \mathcal{R} \Leftrightarrow ([B^* P], Q) \in \mathcal{R}$

for all projections $P, Q, P_\lambda, Q_\kappa \in \mathcal{P}$ and all $B \in I \otimes \mathcal{B}(l^2)$.

**Theorem 1.9.** ([35], Theorem 2.32) Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{P}$ be the set of projections in $\mathcal{M} \otimes \mathcal{B}(l^2)$. If $\mathcal{V}$ is a quantum relation on $\mathcal{M}$ then

$$\mathcal{R}_\mathcal{V} = \{(P, Q) \in \mathcal{P}^2 : P(A \otimes I)Q \neq 0 \text{ for some } A \in \mathcal{V}\}$$

is an intrinsic quantum relation on $\mathcal{M}$; conversely, if $\mathcal{R}$ is an intrinsic quantum relation on $\mathcal{M}$ then

$$\mathcal{V}_\mathcal{R} = \{A \in \mathcal{B}(H) : (P, Q) \notin \mathcal{R} \Rightarrow P(A \otimes I)Q = 0\}$$

is a quantum relation on $\mathcal{M}$. The two constructions are inverse to each other.

2. Quantum Metrics

In this chapter we state our new definition of quantum metric spaces, present some related definitions, and develop their basic properties. The principal difference between our approach here and earlier work on quantum metrics is that we do not attempt to directly model a noncommutative version of the Lipschitz functions on a metric space, in the way that C*-algebras and von Neumann algebras generalize $C(X)$ and $L^\infty(X, \mu)$ spaces (though we do eventually attain this goal in Corollary 4.17). Instead, our definition is based on the idea of mass transport. Operators on $L^2(X, \mu)$ are graded by the maximum distance that they displace mass supported in localized regions of $X$, and it is this notion of displacement that replaces Lipschitz number as the fundamental quantity. Our motivation comes from quantum
information theory, where one wants to recover quantum mechanically encoded information that may have been corrupted, i.e., displaced from its original state by the introduction of errors. (We discuss quantum information theory in Section 3.4 and Lipschitz numbers in Chapter 4.)

Operators that displace mass only a maximum distance have been widely used in various parts of mathematics, often under the name of finite propagation operators. Some representative references are [6, 24, 26]. It is possible that our approach could shed new light on some of this work, or that it could point the way to noncommutative generalizations.

2.1. Basic definitions. We adopt the convention that metric spaces can have infinite distances. Thus, a metric on a set \( X \) is a function \( d : X^2 \to [0, \infty] \) such that

\[
d(x, y) = 0 \iff x = y,
\]

\[
d(x, y) = d(y, x),
\]

\[
d(x, z) \leq d(x, y) + d(y, z),
\]

for all \( x, y, z \in X \), and a pseudometric is defined similarly with the first condition weakened to \( d(x, x) = 0 \) for all \( x \in X \). If \( d(x, y) < \infty \) for all \( x \) and \( y \) then we say that all distances are finite.

We model quantum metrics using a special class of filtrations of \( \mathcal{B}(H) \). Recall that a dual operator system is a weak* closed self-adjoint unital subspace of \( \mathcal{B}(H) \).

**Definition 2.1.** (a) A \( \text{W}^* \)-filtration of \( \mathcal{B}(H) \) is a one-parameter family of dual operator systems \( V = \{ V_t \} \), \( t \in [0, \infty) \), such that

\[
(i) \ V_s V_t \subseteq V_{s+t} \text{ for all } s, t \geq 0
\]

\[
(ii) \ V_t = \bigcap_{s>t} V_s \text{ for all } t \geq 0.
\]

(b) A displacement gauge on \( \mathcal{B}(H) \) is a function \( D : \mathcal{B}(H) \to [0, \infty] \) such that

\[
(i) \ D(I) = 0
\]

\[
(ii) \ D(aA) \leq D(A) \text{ for } a \in \mathbb{C}
\]

\[
(iii) \ D(A + B) \leq \max\{D(A), D(B)\}
\]

\[
(iv) \ D(A^*) = D(A)
\]

\[
(v) \ D(AB) \leq D(A) + D(B)
\]

\[
(vi) \ A_\lambda \to A \text{ weak operator implies } D(A) \leq \lim\inf D(A_\lambda)
\]

for all \( A, B, A_\lambda \in \mathcal{B}(H) \) with \( \sup \| A_\lambda \| < \infty \).

In part (a) the appropriate convention for \( t = \infty \) is \( V_\infty = \mathcal{B}(H) \).

The notions of \( \text{W}^* \)-filtration and displacement gauge are equivalent:

**Proposition 2.2.** If \( V \) is a \( \text{W}^* \)-filtration of \( \mathcal{B}(H) \) then

\[
D_V(A) = \inf\{ t : A \in V_t \}
\]

(with \( \inf \emptyset = \infty \)) is a displacement gauge on \( \mathcal{B}(H) \). Conversely, if \( D \) is a displacement gauge on \( \mathcal{B}(H) \) then \( V_D = \{ V_D^t \} \) with

\[
V_D^t = \{ A \in \mathcal{B}(H) : D(A) \leq t \}
\]

is a \( \text{W}^* \)-filtration. The two constructions are inverse to each other.

The proof of this proposition is straightforward. (Recall from the Krein-Smulian theorem that a subspace of \( \mathcal{B}(H) \) is weak* closed if and only if it is boundedly weak operator closed.) The equivalence between \( \text{W}^* \)-filtrations and displacement gauges is not technically substantial, but we nonetheless find it convenient to be able to pass between the two concepts. Broadly speaking, \( \text{W}^* \)-filtrations tend to be formally simpler but displacement gauges may be more intuitive.
For the sake of notational simplicity we will generally suppress the subscript and simply write \( D \) for the displacement gauge associated to a W*-filtration \( V \).

We now present our definition of quantum metrics.

**Definition 2.3.** A quantum pseudometric on a von Neumann algebra \( M \subseteq B(H) \) is a W*-filtration \( V \) of \( B(H) \) satisfying \( M' \subseteq V_0 \); it is a quantum metric if \( V_0 = M' \).

Note that \( V_0 \) is automatically a von Neumann algebra, so that any W*-filtration \( V \) is a quantum metric on \( M = V_0' \). If \( V \) is a quantum pseudometric on a von Neumann algebra \( M \) then \( V_0' \) is a von Neumann algebra contained in \( M \), and passing from \( M \) to \( V_0' \) is the quantum version of factoring out null distances in order to turn a pseudometric into a metric.

Also, observe that the filtration condition implies that each \( V_t \) is a bimodule over \( V_0 \). Thus if \( V \) is a quantum pseudometric on \( M \) then each \( V_t \) is a quantum relation on \( M \) (Definition 1.2). We can say: a quantum pseudometric on \( M \) is a one-parameter family of quantum relations on \( M \) which satisfy conditions (i) and (ii) in Definition 2.1 (a). This allows us to immediately deduce from Theorem 1.3 the fact, which we record now, that quantum pseudometrics are representation independent. We order quantum pseudometrics by inclusion and write \( V \leq W \) if \( V_t \subseteq W_t \) for all \( t \).

**Theorem 2.4.** Let \( H_1 \) and \( H_2 \) be Hilbert spaces and let \( M_1 \subseteq B(H_1) \) and \( M_2 \subseteq B(H_2) \) be isomorphic von Neumann algebras. Then any isomorphism induces an order preserving 1-1 correspondence between the quantum (pseudo)metrics on \( M_1 \) and the quantum (pseudo)metrics on \( M_2 \).

We will give intrinsic characterizations of quantum pseudometrics in Definition 2.7/Theorem 2.15 and Definition 4.14/Corollary 4.17 below.

The interpretation of the \( V_t \) as quantum relations corresponds to the classical fact that a metric \( d \) on a set \( X \) gives rise to a family of relations

\[
R_t = \{(x, y) \in X^2 : d(x, y) \leq t\},
\]

one for each value of \( t \in [0, \infty) \). The usual metric axioms can be recast as properties of this family of relations:

(i) \( R_0 \) is the diagonal relation \( \Delta \)
(ii) \( R_t = R_t^T \) for all \( t \)
(iii) \( R_s R_t \subseteq R_{s+t} \) for all \( s \) and \( t \)

corresponding to the metric axioms \( d(x, y) = 0 \iff x = y, d(x, y) = d(y, x), d(x, z) \leq d(x, y) + d(y, z) \). (In (ii), \( R^T \) is the transpose of \( R \).) The relations are also nested such that \( R_t = \bigcap_{s \geq t} R_s \) for all \( t \). Conversely, it is easy to check that any family of relations with the preceding properties arises from a unique metric defined by

\[
d(x, y) = \inf\{t : (x, y) \in R_t\}.
\]

Pseudometrics are characterized similarly, with condition (i) weakened to

(i') \( R_0 \) contains the diagonal relation \( \Delta \).

Thus classical metrics and pseudometrics have an alternative axiomatization as one-parameter families of relations satisfying the above conditions. A moment’s thought shows that replacing classical relations with quantum relations yields our definitions of quantum metrics and pseudometrics, the only (inessential) difference being that we do not explicitly specify that \( V_t \) be a bimodule over \( M' \subseteq V_0 \), because this follows anyway from the filtration property.
In light of Proposition 1.4, the above implies that Definition 2.1 should reduce to the classical notions of pseudometric and metric in the atomic abelian case. The proof of the following proposition is essentially just this observation. Recall that $V_{xy} \in \mathcal{B}(l^2(X))$ is the rank one operator $V_{xy} : g \mapsto \langle g,e_y \rangle e_x$.

We emphasize that in the following result, although $X$ is in effect given the discrete topology, the metric $d$ is completely arbitrary. There is no restriction on the metrics which can be handled by our theory.

**Proposition 2.5.** Let $X$ be a set and let $\mathcal{M} \cong l^\infty(X)$ be the von Neumann algebra of bounded multiplication operators on $l^2(X)$. If $d$ is a pseudometric on $X$ then $\nu_d = \{V_d\}$ with
\[
\nu_t^d = \{A \in \mathcal{B}(l^2(X)) : d(x,y) > t \implies \langle Ae_y,e_x \rangle = 0\}
\]
\[
= \overline{\operatorname{span}}\{V_{xy} : d(x,y) \leq t\}
\]
$t \in [0,\infty)$) is a quantum pseudometric on $\mathcal{M}$; conversely, if $\nu$ is a quantum pseudometric on $\mathcal{M}$ then
\[
d(\nu(x,y)) = \inf\{t : \langle Ae_y,e_x \rangle \neq 0 \text{ for some } A \in \nu_t\}
\]
(with $\inf \emptyset = \infty$) is a pseudometric on $X$. The two constructions are inverse to each other, and this correspondence between pseudometrics and quantum pseudometrics restricts to a correspondence between metrics and quantum metrics.

**Proof.** Let $d$ be a pseudometric on $X$ and for each $t \in [0,\infty)$ let $R_t = \{(x,y) \in X^2 : d(x,y) \leq t\}$. Then $\nu_t^d = \nu_{R_t}$, the quantum relation associated to $R_t$ as in Proposition 1.4. It follows from Proposition 1.4 that the two expressions for $\nu_t^d$ agree and that each $\nu_t^d$ is a weak* closed linear subspace of $\mathcal{B}(l^2(X))$ that contains $\mathcal{M}$, and since $R_t^\prime = R_t$ it follows that $\nu_t^d$ is self-adjoint. Thus each $\nu_t^d$ is a dual operator system and $\nu_0^d$ contains $\mathcal{M} = \mathcal{M}'$. Since $R_s R_t \subseteq R_{s+t}$ (the classical triangle inequality), it follows that $\nu_{s+t}^d \subseteq \nu_s^d \nu_t^d$. Finally, $\nu_t^d = \bigcap_{s \geq t} \nu_s^d$ because $d(x,y) \leq t \iff d(x,y) \leq s$ for all $s > t$. Thus $\nu_t^d$ is a quantum pseudometric on $\mathcal{M}$. If $d$ is a metric then $R_0$ is the diagonal relation, hence $\nu_0^d = \mathcal{M}'$, hence $\nu_t^d$ is a quantum metric.

Next let $V$ be a quantum pseudometric on $\mathcal{M}$. Then $I \in \nu_0$ and $\langle Ie_y,e_x \rangle \neq 0$ imply $d_V(x,x) = 0$, for any $x \in X$. We have $d_V(x,y) = d_V(y,x)$ because each $\nu_t$ is self-adjoint and
\[
\langle Ae_y,e_x \rangle \neq 0 \iff \langle A^\ast e_x,e_y \rangle \neq 0.
\]
The triangle inequality holds by the following argument. Suppose $d_V(x,y) < s$ and $d_V(y,z) < t$. Then there exist $A \in \nu_s$ and $B \in \nu_t$ such that $\langle Ae_y,e_x \rangle \neq 0$ and $\langle Be_z,e_y \rangle \neq 0$. Since $\nu_t$ and $\nu_t$ are bimodules over $\mathcal{M}$ we then have $M_{e_x} A M_{e_y} \in \nu_s$ and $M_{e_y} B M_{e_z} \in \nu_t$. These are nonzero scalar multiples of the rank one operators $V_{xy}$ and $V_{yz}$, so $V_{xy} \in \nu_s$ and $V_{yz} \in \nu_t$. Then $V_{xz} = V_{xy} V_{yz} \in \nu_{s+t}$, which implies that $d_V(x,z) \leq s + t$. Taking the infimum over $s$ and $t$ yields $d_V(x,y) \leq d_V(x,y) + d_V(y,z)$. So $d_V$ is a pseudometric. By similar reasoning, if $d_V(x,y) = 0$ then $V_{xy} \in \nu_s$ for all $s > 0$ and therefore $V_{xy} \in \nu_0$. So if $V$ is a quantum metric, i.e., $\nu_0 = \mathcal{M}$, then
\[
d_V(x,y) = 0 \implies V_{xy} \in \nu_0 \implies x = y,
\]
and hence $d_V$ is a metric.

Now let $d$ be a pseudometric on $X$, let $V = V_d$, and let $d = d_V$. Then $\nu_t$ is the quantum relation $V_{R_t}$ associated to the relation $R_t = \{(x,y) \in X^2 : d(x,y) \leq t\}$.
and \( R_t = \{(x, y) \in X^2 : \tilde{d}(x, y) \leq t\} \) is the relation associated to \( \mathcal{V}_t \), both as in Proposition 1.4. So \( R_t = R_t \) for all \( t \) by Proposition 1.4 and we conclude that \( d = \tilde{d} \).

Finally, let \( V \) be a quantum pseudometric on \( M \), let \( d = d_V \), and let \( \tilde{V} = V_d \). Then \( R_t = \{(x, y) \in X^2 : d(x, y) \leq t\} \) is the relation associated to \( \mathcal{V}_t \) and \( \tilde{V}_t \) is the quantum relation associated to \( R_t \), both as in Proposition 1.3. So \( \tilde{V}_t = \mathcal{V}_t \) for all \( t \) by Proposition 1.3 and we conclude that \( \tilde{V} = V \).

\[ \text{Pseudometrics on } X \text{ correspond to displacement gauges on } B(l^2(X)) \text{ satisfying } D(A) = 0 \text{ for all } A \in M \cong l^\infty(X) \text{ via the formulas} \]
\[ D(A) = \sup\{d(x, y) : \langle Ae_y, e_x \rangle \neq 0\} \]
and
\[ d(x, y) = \inf\{D(A) : \langle Ae_y, e_x \rangle \neq 0\}. \]

2.2. More definitions. The next definition, of distances between pairs of projections in \( \mathcal{M} \otimes B(l^2) \), is fundamental for later work. To some extent it replaces classical distances between points. In the atomic abelian case it corresponds to the usual notion of minimal distance between sets, \( d(S, T) = \inf\{d(x, y) : x \in S, y \in T\} \). There is a version of the triangle inequality which holds in this setting; see Definition 2.7 (v) below.

We give basic properties of the distance function in Proposition 2.8; we will show later (Theorem 2.45) that these properties characterize quantum distance functions.

**Definition 2.6.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( M \subseteq B(H) \). We define the distance between any projections \( P \) and \( Q \) in \( M \otimes B(l^2) \) by
\[ \rho_V(P, Q) = \inf\{t : P (A \otimes I) Q \neq 0 \text{ for some } A \in \mathcal{V}_t\} = \inf\{D(A) : A \in B(H) \text{ and } P (A \otimes I) Q \neq 0\} \]
(with \( \inf \emptyset = \infty \)).

By identifying \( M \) with \( M \otimes I \subseteq M \otimes B(l^2) \) we can consider the restriction of \( \rho_V \) to projections in \( M \). Equivalently, for projections \( P \) and \( Q \) in \( M \) we have
\[ \rho_V(P, Q) = \inf\{t : PAQ \neq 0 \text{ for some } A \in \mathcal{V}_t\} = \inf\{D(A) : A \in B(H) \text{ and } PAQ \neq 0\}. \]

For the sake of notational simplicity we will generally suppress the subscript and simply write \( \rho \) for the distance function associated to a quantum pseudometric \( V \).

It will be convenient later (in Theorem 2.45) to introduce the following terminology. Recall that \([A] \) denotes the range projection of \( A \).

**Definition 2.7.** Let \( M \) be a von Neumann algebra and let \( P \) be the set of projections in \( M \otimes B(l^2) \). A quantum distance function on \( M \) is a function \( \rho : P^2 \rightarrow [0, \infty] \) such that

(i) \( \rho(P, 0) = \infty \)
(ii) \( PQ \neq 0 \Rightarrow \rho(P, Q) = 0 \)
(iii) \( \rho(P, Q) = \rho(Q, P) \)
(iv) \( \rho(P \lor Q, R) = \min\{\rho(P, R), \rho(Q, R)\} \)
(v) \( \rho(P, R) \leq \rho(P, Q) + \sup\{\rho(Q, R) : QQ \neq 0\} \)
(vi) \( \rho([P, BQ]) = \rho([B^*P], Q) \)
(vii) if \( P_\lambda \to P \) and \( Q_\lambda \to Q \) weak operator then \( \rho(P,Q) \geq \limsup \rho(P_\lambda,Q_\lambda) \)

for all projections \( P,Q,R,P_\lambda,Q_\lambda \in \mathcal{M} \otimes \mathcal{B}(l^2) \) and all \( B \in I \otimes \mathcal{B}(l^2) \). In (v) we take the supremum over all projections \( Q \) such that \( QQ \neq 0 \).

Property (iv) can be strengthened to \( \rho(\vee P_\lambda,\vee Q_\lambda) = \inf \rho(P_\lambda,Q_\lambda) \). This can easily be proven directly for \( \rho_V \), or it can be deduced from the stated properties. (Property (iv) implies that \( \rho \) is monotone in the sense that \( P \leq \tilde{P} \) implies \( \rho(\tilde{P},Q) \leq \rho(P,Q) \); this plus (iii) yields the inequality \( \leq \). For the reverse inequality, first check that \( \rho(\vee P_\lambda,\vee Q_\lambda) = \inf \rho(P_\lambda,Q_\lambda) \) holds for finite joins using (iii) and (iv), and then take limits using (vii) to pass to arbitrary joins.)

**Proposition 2.8.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). Then \( \rho_V \) is a quantum distance function.

**Proof.** Verification of properties (i) through (iv) is easy. Property (vi) holds because

\[
P(A \otimes I)[BQ] \neq 0 \iff P(A \otimes I)BQ \neq 0 \iff PB(A \otimes I)Q \neq 0 \iff [B^*P](A \otimes I)Q \neq 0.
\]

One can check property (vii) using the fact that \( P_\lambda \to P \) and \( Q_\lambda \to Q \) weak operator implies \( P_\lambda \to P \) and \( Q_\lambda \to Q \) strong operator (so \( P_\lambda(A \otimes I)Q_\lambda = 0 \) for all \( \lambda \) implies \( P(A \otimes I)Q = 0 \)).

For (v) assume \( \rho(P,Q) < \infty \), let \( \epsilon > 0 \), and find \( A \in \mathcal{B}(H) \) such that \( P(A \otimes I)Q \neq 0 \) and \( D(A) \leq \rho(P,Q) + \epsilon \). Then \( QQ \neq 0 \) where \( \tilde{Q} \) is the projection onto the closure of

\[
(M' \otimes I)(\text{ran}(A^* \otimes I)P)
\]

and \( \tilde{Q} \in \mathcal{M} \otimes \mathcal{B}(l^2) \) since its range is invariant for \( \mathcal{M} \otimes I = (\mathcal{M} \otimes \mathcal{B}(l^2))' \). Now assume \( \rho(\tilde{Q}, R) < \infty \) and find \( C \in \mathcal{B}(H) \) such that \( \tilde{Q}(C \otimes I)R \neq 0 \) and \( D(C) \leq \rho(\tilde{Q},R) + \epsilon \). It follows that \( R(C^* \otimes I)\tilde{Q} \neq 0 \), so \( R(C^*B^*A^* \otimes I)P \neq 0 \) for some \( B \in \mathcal{M}' \), and hence \( P(ABC \otimes I)R \neq 0 \) for some \( B \in \mathcal{M}' \). Then

\[
\rho(P,R) \leq D(ABC) \leq D(A) + D(B) + D(C) \leq \rho(P,Q) + \rho(\tilde{Q},R) + 2\epsilon.
\]

Taking \( \epsilon \to 0 \) yields the desired inequality. \( \square \)

We now show that distance between projections is representation independent and that the W*-filtration \( V \) can be recovered from the quantum distance function \( \rho_V \). Generally speaking, this means that any representation independent notion defined in terms of \( V \) will have an equivalent definition in terms of \( \rho_V \).

**Proposition 2.9.** Let \( \pi_1 : \mathcal{M} \to \mathcal{B}(H_1) \) and \( \pi_2 : \mathcal{M} \to \mathcal{B}(H_2) \) be faithful normal unital representations of a von Neumann algebra \( \mathcal{M} \), let \( V_1 \) be a quantum pseudometric on \( \pi_1(\mathcal{M}) \), and let \( V_2 \) be the corresponding quantum pseudometric on \( \pi_2(\mathcal{M}) \) as in Theorem 2.4. Then the quantum distance functions \( \rho_{V_1} \) and \( \rho_{V_2} \) on projections in \( \mathcal{M} \otimes \mathcal{B}(l^2) \) associated to \( V_1 \) and \( V_2 \) are equal.

**Proof.** As in the proof of Theorem 2.4 it is sufficient to consider the case where \( \pi_1 = \text{id}, H_2 = K \otimes H_1 \), and \( \pi_2 : A \to I_K \otimes A \). Given projections \( P,Q \in \mathcal{M} \otimes \mathcal{B}(l^2) \), the corresponding projections in \( \pi_2(\mathcal{M}) \otimes \mathcal{B}(l^2) \) are \( I_K \otimes P \) and \( I_K \otimes Q \). Also, if \( V_1 = \{ V_1^2 \} \) then \( V_2 = \{ V_2^2 \} \) with \( V_2^2 = B(K) \otimes V_1^2 \). Now if \( P(A \otimes I_2)Q \neq 0 \) for some \( A \in V_1^2 \) then \( I_K \otimes A \in V_2^2 \) and

\[
(I_K \otimes P)(I_K \otimes A \otimes I_2)(I_K \otimes Q) \neq 0;
\]
conversely, if \( P(A \otimes I_2)Q = 0 \) for all \( A \in \mathcal{V}_1^1 \) then
\[(I_K \otimes P)(B \otimes A \otimes I_2)(I_K \otimes Q) = 0\]
for all \( A \in \mathcal{V}_1^1 \) and all \( B \in \mathcal{B}(K) \) and hence
\[(I_K \otimes P)(\tilde{A} \otimes I_2)(I_K \otimes Q) = 0\]
for all \( \tilde{A} \in \mathcal{V}_1^2 \). So \( \rho_{\mathcal{V}_1}(P,Q) = \rho_{\mathcal{V}_2}(I_K \otimes P, I_K \otimes Q) \).
\( \square \)

**Proposition 2.10.** Let \( \mathcal{V} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). Then
\[
\mathcal{V}_t = \{ A \in \mathcal{B}(H) : \rho(P,Q) > t \implies P(A \otimes I)Q = 0 \}
\]
for all \( t \in [0,\infty) \), with \( P \) and \( Q \) ranging over projections in \( \mathcal{M} \otimes \mathcal{B}(l^2) \).

**Proof.** Let \( \tilde{\mathcal{V}}_1 = \{ A \in \mathcal{B}(H) : \rho(P,Q) > t \implies P(A \otimes I)Q = 0 \} \). Then \( \mathcal{V}_t \subseteq \tilde{\mathcal{V}}_t \) is immediate from the definition of \( \rho \). Conversely, let \( A \in \mathcal{B}(H) \setminus \mathcal{V}_t \); then we must have \( A \notin \mathcal{V}_s \) for some \( s > t \) and by Lemma 1.7 there exist projections \( P,Q \in \mathcal{M} \otimes \mathcal{B}(l^2) \) such that \( P(B \otimes I)Q = 0 \) for all \( B \in \mathcal{V}_s \), and hence \( \rho(P,Q) \geq s > t \), but \( P(A \otimes I)Q \neq 0 \). This shows that \( A \notin \mathcal{V}_t \). We conclude that \( \mathcal{V}_t = \tilde{\mathcal{V}}_t \). \( \square \)

A \( \text{W}^* \)-filtration \( \mathcal{V} \) generally cannot be recovered from the restriction of \( \rho_{\mathcal{V}} \) to projections in \( \mathcal{M} \); see Example 3.1.

Next we introduce a von Neumann algebra which detects the existence of infinite distances.

**Definition 2.11.** Let \( \mathcal{V} = \{ \mathcal{V}_t \} \) be a \( \text{W}^* \)-filtration. For \( t \in (0,\infty] \) we define
\[
\mathcal{V}_{< t} = \bigcup_{s < t} \mathcal{V}_s^{w^{s^*}}.
\]
In particular,
\[
\mathcal{V}_{< \infty} = \bigcup_{t \geq 0} \mathcal{V}_t = \{ A : D(A) < \infty \}^{w^{k^*}}.
\]

Note that \( \mathcal{V}_{< \infty} \) is a von Neumann algebra that contains \( \mathcal{M}' \subseteq \mathcal{V}_0 \). Thus, it is the commutant of a von Neumann subalgebra of \( \mathcal{M} \). We show next that in the atomic abelian case \( \mathcal{V}_{< \infty} \) corresponds to the equivalence relation \( x \sim y \iff d(x,y) < \infty \) on \( X \). The condition \( \mathcal{V}_{< \infty} = \mathcal{B}(H) \) is the quantum equivalent of all distances being finite.

**Proposition 2.12.** Let \( X \) be a set and let \( \mathcal{M} \equiv l^\infty(X) \) be the von Neumann algebra of bounded multiplication operators on \( l^2(X) \). Also let \( d \) be a pseudometric on \( X \) and define \( \mathcal{V}_d \) as in Proposition 2.6. Then \( \mathcal{V}_d^{< \infty} = \mathcal{M}' \) where
\[
\mathcal{M}_\infty = \{ M_f : f \in l^\infty(X) \} \quad \text{and} \quad d(x,y) < \infty \iff f(x) = f(y) \subseteq \mathcal{M}.
\]
In particular, all distances in \( X \) are finite if and only if \( \mathcal{V}_{< \infty} = \mathcal{B}(l^2(X)) \).

**Proof.** \( \mathcal{M}'_\infty \) is the von Neumann algebra generated by the rank one operators \( V_{xy} \) such that \( d(x,y) < \infty \). Since \( \mathcal{V}_d^{< \infty} = \text{span}^{w^{s^*}} \{ V_{xy} : d(x,y) \leq t \} \) it follows that \( \mathcal{V}_{< \infty} = \bigcup_{t \in (0,\infty]} \mathcal{V}_d^{< s^*} \) is also generated by \( \{ V_{xy} : d(x,y) < \infty \} \). So \( \mathcal{V}_d^{< \infty} = \mathcal{M}'_\infty \). \( \square \)
The condition \( V_{<\infty} = \mathcal{B}(H) \) can also be characterized in terms of distances between projections (and hence it is representation independent). Say that the projections \( P \) and \( Q \) in \( M \otimes \mathcal{B}(l^2) \) are unlinked if there exist projections \( \tilde{P}, \tilde{Q} \in I \otimes B(l^2) \) with \( P \leq \tilde{P}, Q \leq \tilde{Q} \), and \( \tilde{P}\tilde{Q} = 0 \); otherwise they are linked. The motivation for these terms comes from the following proposition, which shows that \( P \) and \( Q \) are unlinked if and only if there is an operator \( A \in B(H) \) such that \( (A \otimes I)v \) is not orthogonal to the range of \( P \), for some vector \( v \) in the range of \( Q \).

**Proposition 2.13.** Let \( \mathbf{V} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq B(H) \). Two projections \( P \) and \( Q \) in \( \mathcal{M} \otimes \mathcal{B}(l^2) \) are linkable if and only if there exists \( A \in B(H) \) such that \( P(A \otimes I)Q \neq 0 \). If \( P \) and \( Q \) are unlinked then \( \rho(P,Q) = \infty \). The following are equivalent:

(i) \( V_{<\infty} = \mathcal{B}(H) \)

(ii) \( \rho(P,Q) < \infty \) for any linked projections \( P \) and \( Q \) in \( \mathcal{M} \otimes \mathcal{B}(l^2) \)

(iii) \( \rho(P,Q) < \infty \) for any nonzero projections \( P \) and \( Q \) in \( \mathcal{M} \).

**Proof.** Suppose \( P \leq \hat{P} \) and \( Q \leq \hat{Q} \) where \( \hat{P}, \hat{Q} \) are projections in \( I \otimes B(l^2) \) which satisfy \( \tilde{P}\tilde{Q} = 0 \). Then for any \( A \in B(H) \) we have \( A \otimes I \in B(H) \otimes I = (I \otimes B(l^2))' \) and therefore

\[
P(A \otimes I)Q = P(\tilde{P}(A \otimes I)\tilde{Q})Q = P((A \otimes I)\tilde{P}\tilde{Q})Q = 0.
\]

Conversely, if \( P(A \otimes I)Q = 0 \) for all \( A \in B(H) \) then the projections \( \tilde{P} \) and \( \tilde{Q} \) onto the closures of \( (B(H) \otimes I)(\text{ran}(P)) \) and \( (B(H) \otimes I)(\text{ran}(Q)) \) satisfy \( P \leq \tilde{P}, Q \leq \tilde{Q}, \tilde{P}\tilde{Q} = 0 \), and

\[
\tilde{P}, \tilde{Q} \in (B(H) \otimes I)' = I \otimes B(l^2),
\]

so \( P \) and \( Q \) are unlinked.

It immediately follows that \( \rho(P,Q) = \infty \) for any unlinked projections \( P \) and \( Q \) in \( \mathcal{M} \otimes \mathcal{B}(l^2) \).

(i) \( \Rightarrow \) (ii): Suppose \( V_{<\infty} = \mathcal{B}(H) \) and \( \rho(P,Q) = \infty \). Then \( P(A \otimes I)Q = 0 \) for all \( A \in B(H) \) with \( D(A) < \infty \), and hence, by taking weak* limits, for all \( A \in B(H) \). Thus \( P \) and \( Q \) are unlinked.

(ii) \( \Rightarrow \) (iii): Trivial.

(iii) \( \Rightarrow \) (i): If \( V_{<\infty} \neq \mathcal{B}(H) \) then there exists a nontrivial projection \( P \) in its commutant. Then \( P \) and \( I - P \) are both nonzero but \( \rho(P,I-P) = \infty \). \( \square \)

Next we present a variety of basic definitions that can be made directly in terms of the \( W^* \)-filtration \( \mathbf{V} \), followed by a series of propositions giving basic facts about them.

**Definition 2.14.** Let \( \mathbf{V} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq B(H) \).

(a) The **diameter** of \( \mathbf{V} \) is the quantity

\[
diam(\mathbf{V}) = \inf\{ t : \mathcal{V}_t = \mathcal{B}(H) \} = \sup\{ D(A) : A \in B(H) \}
\]

(with \( \inf\emptyset = \infty \)).

(b) For \( \epsilon > 0 \) and \( P \) a projection in \( \mathcal{M} \), the **open \( \epsilon \)-neighborhood of \( P \)** is the projection \( (P)_\epsilon \) onto

\[
\mathcal{V}_{<\epsilon}(\text{ran}(P)) = \bigvee_{t<\epsilon} \mathcal{V}_t(\text{ran}(P)).
\]

Equivalently, \( (P)_\epsilon = \bigvee\{ |AP| : D(A) < \epsilon \} \).
(c) The closure of a projection \( P \in \mathcal{M} \) is the projection \( \overline{P} \) onto
\[
\bigcap_{t > 0} \mathcal{V}_t(\text{ran}(P))
\]
and \( P \) is closed if \( P = \overline{P} \).

(d) \( \mathcal{V} \) is uniformly discrete if there exists \( t > 0 \) such that \( \mathcal{V}_t = \mathcal{V}_0 \), or equivalently there exists \( t > 0 \) such that
\[
D(A) > 0 \quad \Rightarrow \quad D(A) \geq t.
\]

(e) \( \mathcal{V} \) is a path quantum pseudometric if
\[
\bigcap_{t > 0} \mathcal{V}_{s+t}^{\text{wk}^*} = \mathcal{V}_{s+t}
\]
for all \( s, t \geq 0 \).

We will show below (Propositions \ref{prop:Hausdorff-distance} and \ref{prop:Gromov-Hausdorff-distance}) that \((P)_\epsilon\) and \( \overline{P} \) belong to \( \mathcal{M} \).

The notion of an open \( \epsilon \)-neighborhood immediately suggests a definition of Hausdorff distance: if \( P \) and \( Q \) are projections in \( \mathcal{M} \), we can define their Hausdorff distance to be
\[
\inf\{\epsilon : P \leq (Q)_\epsilon \quad \text{and} \quad Q \leq (P)_\epsilon\}.
\]
Observe that this is an actual pseudometric; it satisfies the triangle inequality because \((P)_\epsilon) \subseteq (P)_{\epsilon+\delta} \). If \( \mathcal{V} \) and \( \mathcal{W} \) are quantum pseudometrics on von Neumann algebras \( \mathcal{M} \subseteq \mathcal{B}(H) \) and \( \mathcal{N} \subseteq \mathcal{B}(K) \) then we can define their Gromov-Hausdorff distance to be the infimum of the Hausdorff distance between \( I_H \) and \( I_K \) over all quantum pseudometrics on \( \mathcal{M} \oplus \mathcal{N} \) that restrict to \( \mathcal{V} \) on \( \mathcal{M} \) and \( \mathcal{W} \) on \( \mathcal{N} \), i.e., \( \mathcal{W}^*-\)filtrations of \( \mathcal{B}(H \oplus K) \) whose intersection with \( \mathcal{B}(H) \subseteq \mathcal{B}(H \oplus K) \) equals \( \mathcal{V} \) and whose intersection with \( \mathcal{B}(K) \subseteq \mathcal{B}(H \oplus K) \) equals \( \mathcal{W} \). (Cf. Definition \ref{def:W*-filtration} (b) and the discussion following it, which shows that the Gromov-Hausdorff distance is always at most \( \max\{\text{diam}(\mathcal{V}), \text{diam}(\mathcal{W})\}/2 \). This too is a pseudometric, the key observation here being that if we are given a \( \mathcal{W}^*-\)filtration on \( \mathcal{B}(H \oplus H') \) which restricts to \( \mathcal{V} \) and \( \mathcal{V}' \), and a \( \mathcal{W}^*-\)filtration on \( \mathcal{B}(H' \oplus H'') \) which restricts to \( \mathcal{V}' \) and \( \mathcal{V}'' \), then after embedding both into \( \mathcal{B}(H \oplus H' \oplus H'') \) their meet (see Definition \ref{def:W*-filtration} (c) below) restricts to a \( \mathcal{W}^*-\)filtration on \( \mathcal{B}(H \oplus H'') \) which restricts to \( \mathcal{V} \) and \( \mathcal{V}'' \). However, generally speaking this does not seem to be a good tool for analyzing convergence of quantum metrics (e.g., the analog of Theorem \ref{thm:local-convergence} below fails). A better candidate may be the notion of local convergence introduced below in Definition \ref{def:local-convergence}.

We first observe that the preceding definitions reduce to the corresponding classical notions in the atomic abelian case. All parts of the next proposition are straightforward consequences of the characterization \( \mathcal{V}_d^t = \text{span} \{\mathcal{V}_{xy} : d(x, y) \leq t\} \) (Proposition \ref{prop:characterization}). Denote the characteristic function of the set \( S \) by \( \chi_S \).

**Proposition 2.15.** Let \( X \) be a set and let \( \mathcal{M} \cong l^\infty(X) \) be the von Neumann algebra of bounded multiplication operators on \( l^2(X) \). Also let \( d \) be a pseudometric on \( X \) and define \( \mathcal{V}_d \) as in Proposition \ref{prop:characterization}.

(a) \( \text{diam}(\mathcal{V}_d) = \sup\{d(x, y) : x, y \in X\} \).

(b) For any \( S \subseteq X \) we have \( (M_{\chi_S})_\epsilon = M_{\chi_{N_\epsilon(S)}} \) where \( N_\epsilon(S) = \{x \in X : d(x, S) < \epsilon\} \).

(c) For any \( S \subseteq X \) the closure of \( M_{\chi_S} \) is \( M_{\chi_{S^p}} \).
(d) \( V_\delta \) is uniformly discrete if and only if there exists \( t > 0 \) such that \( d(x, y) > 0 \Rightarrow d(x, y) \geq t \).

(e) \( V_\delta \) is a path quantum pseudometric if and only if \( d(x, y) \) is the infimum of the lengths of paths from \( x \) to \( y \) in the completion of \( X \), for all \( x, y \in X \).

Now we establish some basic properties of the concepts introduced in Definition 2.14. In particular, we provide some alternative characterizations in terms of projection distances.

**Proposition 2.16.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). Then \( \text{diam}(V) = \sup\{\rho(P, Q) : P \text{ and } Q \text{ are linkable projections in } \mathcal{M} \otimes \mathcal{B}(l^2)\} \).

**Proof.** Let \( t \geq 0 \) and suppose \( V_\epsilon \neq \mathcal{B}(H) \). Let \( A \in \mathcal{B}(H) - V_\epsilon \); then by Lemma 1.7 there exist projections \( P \) and \( Q \) in \( \mathcal{M} \otimes \mathcal{B}(l^2) \) such that \( P(A \otimes I)Q \neq 0 \) but \( P(B \otimes I)Q = 0 \) for all \( B \in V_\epsilon \). Thus \( \rho(P, Q) \geq t \), and \( P(A \otimes I)Q \neq 0 \) implies that \( P \) and \( Q \) are linkable (Proposition 2.13). Since \( \text{diam}(V) = \sup\{t : V_\epsilon \neq \mathcal{B}(H)\} \), taking the supremum over \( t \) yields the inequality \( \leq \). For the reverse inequality, let \( P \) and \( Q \) be linkable projections in \( \mathcal{M} \otimes \mathcal{B}(l^2) \). Then there exists \( A \in \mathcal{B}(H) \) such that \( P(A \otimes I)Q \neq 0 \) by Proposition 2.18, and we have \( \rho(P, Q) \leq D(A) \leq \text{diam}(V) \). Taking the supremum over \( P \) and \( Q \) yields the inequality \( \geq \). \( \square \)

Restricting the supremum in Proposition 2.16 only to nonzero projections in \( \mathcal{M} \) would not suffice in general; see Example 3.31.

**Proposition 2.17.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \), let \( \epsilon > 0 \), and let \( P \) be a projection in \( \mathcal{M} \). Then

\[
(P)_\epsilon = I - \bigvee\{Q : \rho(P, Q) \geq \epsilon\} \in \mathcal{M}
\]

with \( \epsilon \) ranging over projections in \( \mathcal{M} \). The open \( \epsilon \)-neighborhood of the join of any family of projections in \( \mathcal{M} \) equals the join of their open \( \epsilon \)-neighborhoods.

**Proof.** First we check that \( (P)_\epsilon \in \mathcal{M} \). For each \( t < \epsilon \) the subspace \( V_\epsilon(\text{ran}(P)) \subseteq H \) is invariant for \( \mathcal{M}' \subseteq \mathcal{V}_\epsilon \). Hence the join of these subspaces, which is the range of \( (P)_\epsilon \), is also invariant for \( \mathcal{M}' \), and this shows that \( (P)_\epsilon \in \mathcal{M} \).

Define \( \bar{P} = I - \bigvee\{Q \in \mathcal{M} : Q \text{ is a projection and } \rho(P, Q) \geq \epsilon\} \). If \( \rho(P, Q) \geq \epsilon \) and \( D(A) < \epsilon \) then \( QAP = 0 \); it follows that \( Q(P)_\epsilon = 0 \), and this shows that \( (P)_\epsilon \leq \bar{P} \). Conversely, let \( Q = I - (P)_\epsilon \). Then \( QAP = 0 \) for all \( A \in \mathcal{B}(H) \) with \( D(A) < \epsilon \), so \( \rho(P, Q) \geq \epsilon \), and this plus the result of the last paragraph shows that \( Q \) belongs to the join used to define \( \bar{P} \). Thus \( I - (P)_\epsilon \leq I - \bar{P} \). We conclude that \( (P)_\epsilon = \bar{P} \).

The last assertion follows from the fact that

\[
\left[ A : \bigvee P_\lambda \right] = \bigvee [AP_\lambda]
\]

for any family of projections \( \{P_\lambda\} \) and any \( A \in \mathcal{B}(H) \). Taking the join over \( D(A) < \epsilon \) yields the open \( \epsilon \)-neighborhood of \( \bigvee P_\lambda \) on the left and the join of the open \( \epsilon \)-neighborhoods of the \( P_\lambda \) on the right. \( \square \)

**Proposition 2.18.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \) and let \( P \) be a projection in \( \mathcal{M} \). Then

\[
\overline{P} = \bigwedge_{\epsilon > 0} (P)_\epsilon = I - \bigvee\{Q : \rho(P, Q) > 0\} \in \mathcal{M}
\]
with \( Q \) ranging over projections in \( \mathcal{M} \). We have \( \rho(P, Q) = \rho(\overline{P}, Q) \) for any projection \( Q \) in \( \mathcal{M} \). The projection \( \overline{P} \) is closed, as is \( I - (P)_{\epsilon} \) for any \( \epsilon > 0 \). The meet of any family of closed projections is closed.

Proof. For any \( \epsilon > 0 \) we have

\[
\bigvee_{t < \epsilon} \mathcal{V}_t(\operatorname{ran}(P)) \leq \mathcal{V}_\epsilon(\operatorname{ran}(P)) \leq \bigvee_{\epsilon < 2\epsilon} \mathcal{V}_\epsilon(\operatorname{ran}(P));
\]

taking the meet over \( \epsilon > 0 \), the first inequality yields \( \bigwedge_{\epsilon > 0}(P)_{\epsilon} \leq P \) and the second yields \( P \leq \bigwedge_{\epsilon > 0}(P)_{\epsilon} \). So \( \overline{P} = \bigwedge_{\epsilon > 0}(P)_{\epsilon} \). Then by Proposition 2.17

\[
I - \overline{P} = I - \bigwedge_{\epsilon > 0}(P)_{\epsilon}
\]

\[
= \bigvee_{\epsilon > 0} (I - (P)_{\epsilon})
\]

\[
= \bigvee_{\epsilon > 0} \{Q : \rho(P, Q) \geq \epsilon\}
\]

\[
= \bigvee_{\epsilon > 0} \{Q : \rho(P, Q) > 0\},
\]

which proves the second formula for \( \overline{P} \).

It is clear that \( \rho(\overline{P}, Q) \leq \rho(P, Q) \) since \( P \leq \overline{P} \). To prove the reverse inequality suppose \( QA \overline{P} \neq 0 \) and let \( \delta > 0 \). Since \( \overline{P} \leq (P)_\delta \) there must exist \( B \in \mathcal{V}_{2\delta} \) such that \( QABP \neq 0 \). But \( D(AB) \leq D(A) + 2\delta \), so taking the infimum over \( A \) and letting \( \delta \to 0 \) yields \( \rho(P, Q) \leq \rho(\overline{P}, Q) \), as desired. We conclude that \( \rho(\overline{P}, Q) = \rho(P, Q) \).

It follows that

\[
\overline{P} = I - \bigvee \{Q : \rho(\overline{P}, Q) > 0\} = I - \bigvee \{Q : \rho(P, Q) > 0\} = \overline{P},
\]

so that \( \overline{P} \) is closed.

Next let \( \epsilon > 0 \); then \( I - (P)_{\epsilon} = \bigvee \{Q : \rho(P, Q) \geq \epsilon\} \) implies \( \rho(I - (P)_{\epsilon}, P) \geq \epsilon \), and hence \( \rho(I - (P)_{\epsilon}, P) \geq \epsilon \), so \( I - (P)_{\epsilon} \) belongs to the join that defines \( I - (P)_{\epsilon} \). Since \( I - (P)_{\epsilon} \leq I - (P)_{\epsilon} \) is trivial, this shows that \( I - (P)_{\epsilon} = I - (P)_{\epsilon} \), i.e., \( I - (P)_{\epsilon} \) is closed.

Finally, let \( \{P_\lambda\} \) be any family of closed projections in \( \mathcal{M} \) and let \( P = \bigwedge P_\lambda \). Let

\[
\hat{Q} = \bigvee \{Q : \rho(P_\lambda, Q) > 0 \text{ for some } \lambda \},
\]

Every \( Q \) that contributes to this join satisfies \( \rho(P, Q) > 0 \) and hence is orthogonal to \( P \). So \( \overline{P} \leq I - \hat{Q} \). However, since each \( P_\lambda \) is closed we have \( I - P_\lambda = \bigvee \{Q : \rho(P_\lambda, Q) > 0 \} \leq \hat{Q} \) for all \( \lambda \), and hence \( I - \hat{Q} \leq \bigwedge P_\lambda = P \). Thus \( \overline{P} \leq P \), and we conclude that \( P \) is closed.

\[ \tag*{\Box} \]

Proposition 2.19. Let \( \mathbf{V} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). Then \( \mathbf{V} \) is uniformly discrete if and only if there exists \( t > 0 \) such that

\[
\rho(P, Q) > 0 \quad \Rightarrow \quad \rho(P, Q) \geq t,
\]

with \( P \) and \( Q \) ranging over projections in \( \mathcal{M} \subseteq \mathcal{B}(I^2) \).

Proof. If \( \mathcal{V}_0 = \mathcal{V}_t \) for some \( t > 0 \) then it immediately follows from the definition of \( \rho(P, Q) \) that it cannot lie in the interval \( (0, t) \). Conversely, suppose \( \mathcal{V}_0 \neq \mathcal{V}_t \) for all \( t > 0 \) and fix \( t \). Then there must exist \( s \in (0, t) \) such that \( \mathcal{V}_s \neq \mathcal{V}_t \) because otherwise \( \mathcal{V}_0 = \bigcap_{s > 0} \mathcal{V}_s \) would equal \( \mathcal{V}_t \). Letting \( A \in \mathcal{V}_t - \mathcal{V}_s \), Lemma 1.17 then
yields the existence of projections $P, Q \in \mathcal{M} \otimes \mathcal{B}(l^2)$ such that $P(A \otimes I)Q \neq 0$ but $P(B \otimes I)Q = 0$ for all $B \in \mathcal{V}_\kappa$. We conclude that $0 < s \leq \rho(P, Q) \leq t$. Since $t > 0$ was arbitrary, we have shown that $\inf \{\rho(P, Q) : \rho(P, Q) > t\} = 0$.

2.3. The abelian case. Measurable metric spaces were introduced in [29] and have subsequently been studied in connection with derivations [30, 31, 33] and local Dirichlet forms [13, 14, 15]. We recall the basic definition:

**Definition 2.20.** ([32], Definition 6.1.3) Let $(X, \mu)$ be a finitely decomposable measure space and let $\mathcal{P}$ be the set of nonzero projections in $L^\infty(X, \mu)$. A measurable pseudometric on $(X, \mu)$ is a function $\rho : \mathcal{P}^2 \to [0, \infty]$ such that

(i) $\rho(p, p) = 0$
(ii) $\rho(p, q) = \rho(q, p)$
(iii) $\rho(\bigvee p_\lambda, \bigvee q_\lambda) = \inf \lambda, \kappa \rho(p_\lambda, q_\kappa)$
(iv) $\rho(p, r) \leq \sup_{q' \leq q} (\rho(p, q') + \rho(q', r))$

for all $p, q, r, p_\lambda, q_\kappa \in \mathcal{P}$. It is a measurable metric if for all disjoint $p$ and $q$ there exist nets $\{p_\lambda\}$ and $\{q_\kappa\}$ such that $p_\lambda \to p$ and $q_\kappa \to q$ weak* and $\rho(p_\lambda, q_\kappa) > 0$ for all $\lambda$.

If either $p$ or $q$ is (or both are) the zero projection then the appropriate convention is $\rho(p, q) = \infty$. (Note that in the measurable triangle inequality, property (iv), $q'$ ranges over nonzero projections.) Basic properties of measurable metrics are summarized in Section 1.5 of [35].

In the atomic case measurable metrics reduce to pointwise metrics in the expected way:

**Proposition 2.21.** ([32], Proposition 6.1.4) Let $\mu$ be counting measure on a set $X$. If $d$ is a pseudometric on $X$ then

$$\rho_d(\chi_S, \chi_T) = \inf \{d(x, y) : x \in S, y \in T\}$$

is a measurable pseudometric on $X$; conversely, if $\rho$ is a measurable pseudometric on $X$ then

$$d_\rho(x, y) = \rho(e_x, e_y)$$

is a pseudometric on $X$. The two constructions are inverse to each other, and this correspondence between pseudometrics and measurable pseudometrics restricts to a correspondence between metrics and measurable metrics.

We omit the easy proof. (If $d$ is a metric, we show that $\rho_d$ is a measurable metric by approximating disjoint positive measure subsets $S, T \subseteq X$ by finite subsets.)

The relation between quantum metrics and measurable metrics is explained in the following theorem.

**Theorem 2.22.** Let $(X, \mu)$ be a finitely decomposable measure space and let $\mathcal{M} \cong L^\infty(X, \mu)$ be the von Neumann algebra of bounded multiplication operators on $L^2(X, \mu)$. If $\rho$ is a measurable pseudometric on $X$ then $\mathcal{V}_\rho = \{V_\rho^p\}$ with

$$V_\rho^p = \{A \in \mathcal{B}(L^2(X, \mu)) : \rho(p, q) > t \Rightarrow M_p A M_q = 0\}$$

is a quantum pseudometric on $\mathcal{M}$; conversely, if $\mathcal{V}$ is a quantum pseudometric on $\mathcal{M}$ then

$$\rho_{\mathcal{V}}(p, q) = \inf \{D(A) : M_p A M_q \neq 0\}$$

is a measurable pseudometric on $\mathcal{M}$. We have $\rho = \rho_{\mathcal{V}}$, for any measurable pseudometric $\rho$ on $X$ and $\mathcal{V} \leq V_{\rho_{\mathcal{V}}}$ for any quantum pseudometric $\mathcal{V}$ on $\mathcal{M}$, with equality
if and only if each $V_t$ is reflexive (Definition 1.3). A measurable pseudometric $\rho$ is a measurable metric if and only if $V_\rho$ is a quantum metric.

**Proof.** Let $\rho$ be a measurable pseudometric. It follows from Definition 2.20 (i) and (iii) that $pq \neq 0$ implies $\rho(p, q) = 0$ (since $p = p\vee pq$ and $q = q\vee pq$). Thus $\rho(p, q) > t \Rightarrow M_pM_qM_q = M_p$ for all $f \in L^\infty(X, \mu)$, which shows that $M \subseteq V_\rho^0$ for all $t$. Each $V_t^p$ is self-adjoint because $\rho$ is symmetric and is clearly weak operator, and hence weak*, closed. So each $V_t^p$ is a dual operator system and $M \subseteq V_0^\rho$. Condition (ii) in the definition of a $W^*$-filtration (Definition 2.1) is easy. For condition (i) let $R_t$ be the measurable relation $R_t = \{(p, q) : \rho(p, q) < t\}$ (35), Lemma 1.16 and observe that the corresponding quantum relations $\mathcal{V}_{R_t} = \{A \in B(L^2(X, \mu)) : \rho(p, q) \geq t \Rightarrow M_pAM_q = 0\}$ satisfy $\mathcal{V}_t = \bigcap_{t > t} \mathcal{V}_{R_t}$. Now let $V_{s,t,\epsilon} = \mathcal{V}_{R_s,\epsilon} \cup \mathcal{V}_{R_s,\epsilon}$. We have $\mathcal{V}_{s,t,\epsilon} \subseteq \mathcal{V}_{s,t,\epsilon} \cup \mathcal{V}_{R_s,\epsilon} \subseteq \mathcal{V}_{R_s,\epsilon+\epsilon}$, so

$$V_s^pV_t^p \subseteq \mathcal{V}_{s,t,\epsilon} \cup \mathcal{V}_{R_s,\epsilon} \subseteq \mathcal{V}_{R_s,\epsilon+\epsilon} \subset V_{s,t,\epsilon+\epsilon};$$

intersecting over $\epsilon > 0$ yields $V_s^pV_t^p \subseteq \mathcal{V}_{s,t,\epsilon+\epsilon}$. This completes the proof that $V_\rho$ is a quantum pseudometric on $M$.

Next, let $V$ be a quantum pseudometric on $M$. Verification of conditions (i), (ii), and (iii) of Definition 2.20 for $\rho$ is straightforward. For (iv), let $p, q,$ and $r$ be nonzero projections in $L^\infty(X, \mu)$ and let $\epsilon > 0$. We may assume $\rho\nu(q, r) < \epsilon$. Find $A \in B(L^2(X, \mu))$ such that $D(A) \leq \rho\nu(q, r) + \epsilon$ and $M_qAM_q \neq 0$. Then

$$Q = \sqrt{\{[M_qAM_q] : f \in L^\infty(X, \mu)\}}$$

is invariant for $M$ and hence $Q = M_{q'}$ for some nonzero $q' \leq q$. We may assume $\rho(p, q') < \infty$. Now find $B \in B(L^2(X, \mu))$ such that $D(B) \leq \rho\nu(p, q') + \epsilon$ and $M_qBM_q \neq 0$, so that $M_qBM_q \neq 0$ for some $f \in L^\infty(X, \mu)$. Then

$$\rho\nu(p, r) \leq D(BM_qA) \leq D(B) + D(A) \leq \rho\nu(p, q') + \rho\nu(q', r) + 2\epsilon$$

since $\rho\nu(q, r) \leq \rho\nu(q', r)$. Taking the infimum over $\epsilon$ yields

$$\rho\nu(p, r) \leq \sup_{q' \leq q} (\rho\nu(p, q') + \rho\nu(q', r)).$$

This completes the proof that $\rho\nu$ is a measurable metric.

Now let $\rho$ be a measurable pseudometric, let $V = V_\rho$, and let $\tilde{\rho} = \rho\nu$. Applying the formula $R = R_{V_\rho}$ (Theorem 1.6) to $R_t = \{(p, q) : \rho(p, q) < t\}$ yields

$$R_t = \{(p, q) : (\exists A \in B(L^2(X, \mu)))(\rho(p', q') \geq t \Rightarrow M_pAM_q = 0 \quad \text{and} \quad M_qAM_q \neq 0)\}.$$ 

Letting

$$\tilde{R}_t = \{(p, q) : \tilde{\rho}(p, q) < t\} = \{(p, q) : (\exists A \in B(L^2(X, \mu)))(D(A) < t \quad \text{and} \quad M_qAM_q \neq 0)\},$$

we then have

$$\tilde{R}_t \subseteq R_t \subseteq \tilde{R}_{t+\epsilon}$$

for all $t$ and all $\epsilon > 0$. This shows that $\rho = \tilde{\rho}$.

Next let $V$ be any quantum pseudometric on $M$, let $\rho = \rho\nu$, and let $\tilde{V} = V_\rho$. The inequality $V \leq \tilde{V}$ is straightforward. Conversely, let

$$W_t = \{A \in B(L^2(X, \mu)) : M_pV_tM_q = 0 \Rightarrow M_pAM_q = 0\},$$

then
so that \( \mathcal{V}_t \subseteq \mathcal{W}_t \) and \( \mathcal{V}_t \) is reflexive if and only if \( \mathcal{V}_t = \mathcal{W}_t \). We have

\[ \mathcal{V}_t = \{ A \in \mathcal{B}(L^2(X, \mu)) : M_p\mathcal{V}_{t+\epsilon}M_q = 0 \text{ for some } \epsilon > 0 \Rightarrow M_pAM_q = 0 \}, \]

and

\[ \mathcal{W}_t \subseteq \tilde{\mathcal{V}}_t \subseteq \mathcal{W}_{t+\epsilon} \]

for any \( \epsilon > 0 \). Thus if each \( \mathcal{V}_t \) is reflexive then

\[ \mathcal{V}_t = \mathcal{W}_t \subseteq \tilde{\mathcal{V}}_t \subseteq \bigcap_{s \succ t} \mathcal{W}_s = \bigcap_{s \succ t} \mathcal{V}_s = \mathcal{V}_t \]

for all \( t \), so that \( \mathcal{V}_t = \tilde{\mathcal{V}}_t \), and if some \( \mathcal{V}_t \) is not reflexive then

\[ \mathcal{V}_t \nsubseteq \mathcal{W}_t \subseteq \tilde{\mathcal{V}}_t \]

for that \( t \), so that \( \mathcal{V}_t \neq \tilde{\mathcal{V}}_t \). So \( \mathbf{V} = \tilde{\mathbf{V}} \) if and only if each \( \mathcal{V}_t \) is reflexive.

Finally, let \( \rho \) be a measurable metric. If \( A \in \mathcal{V}_{0}^\rho \) then \( \rho(p, q) > 0 \) implies \( M_pAM_q = 0 \), so the measurable metric condition implies that \( M_pAM_q = 0 \) for any disjoint projections \( p \) and \( q \) in \( \mathcal{M} \). But this implies that \( A \) belongs to \( \mathcal{M} \), so we have shown that if \( \rho \) is a measurable metric then \( \mathcal{V}_{0}^\rho = \mathcal{M} \), i.e., \( \mathbf{V}_\rho \) is a quantum metric. For the converse, let the closure of \( q \) be \( \bar{q} = X - \sqrt{\{ p : \rho(p, q) > 0 \}} \). If \( \rho \) is not a measurable metric then the closed projections in \( L^\infty(X, \mu) \) do not generate \( L^\infty(X, \mu) \) as a von Neumann algebra (see Section 1.5 of [35]). There must therefore exist an operator \( A \in \mathcal{B}(L^2(X, \mu)) \) that commutes with \( M_q \) for every closed projection \( q \) in \( L^\infty(X, \mu) \) but does not belong to \( \mathcal{M} \). Now if \( A \notin \mathcal{V}_{0}^\rho \) then there exist projections \( p, q \in L^\infty(X, \mu) \) with \( \rho(p, q) > 0 \) and \( M_pAM_q \neq 0 \), but then \( A \) cannot commute with \( M_q \), a contradiction. So we conclude that \( A \in \mathcal{V}_{0}^\rho \), and this shows that \( \mathcal{V}_{0}^\rho \neq \mathcal{M} \). So if \( \rho \) is not a measurable metric then \( \mathbf{V}_\rho \) is not a quantum metric.

\[ \square \]

2.4. Reflexivity and stabilization. We have seen the value of working with projections in \( \mathcal{M} \otimes \mathcal{B}(l^2) \), and we will give simple examples in Section 3.1 showing that projections in \( \mathcal{M} \) generally do not suffice in the basic results of the theory. However, by inflating \( \mathcal{M} \) to \( \bar{\mathcal{M}} = \mathcal{M} \otimes \mathcal{B}(l^2) \) and \( \{ \mathcal{V}_t \} \) to \( \{ \mathcal{V}_t \otimes I \} \) we can ensure that projections in \( \mathcal{M} \) do suffice for the basic theory. This is a consequence of the general principle that projections in \( \mathcal{M} \) suffice if \( \mathbf{V} \) is reflexive.

Definition 2.23. Let \( \mathbf{V} = \{ \mathcal{V}_t \} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \).

(a) \( \mathbf{V} \) is reflexive if each \( \mathcal{V}_t \) is reflexive (Definition 1.5).

(b) The stabilization of \( \mathbf{V} \) is the quantum pseudometric \( \mathbf{V} \otimes I = \{ \mathcal{V}_t \otimes I \} \) on the von Neumann algebra \( \mathcal{M} \otimes \mathcal{B}(l^2) \).

We just give one illustration of the sufficiency of projections in \( \mathcal{M} \) when \( \mathbf{V} \) is reflexive; cf. Proposition 2.10 and Example 3.1. The reader will not have any difficulty in supplying analogous versions of, e.g., Propositions 2.16 and 2.19. See also Proposition 2.30 below.

Proposition 2.24. Let \( \mathbf{V} \) be a reflexive quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). Then

\[ \mathcal{V}_t = \{ A \in \mathcal{B}(H) : \rho(P, Q) > t \quad \Rightarrow \quad PAQ = 0 \} \]

for all \( t \in [0, \infty) \), with \( P \) and \( Q \) ranging over projections in \( \mathcal{M} \).
Proof. Fix $t$ and let $\hat{V}_t = \{ A \in B(H) : \rho(P, Q) > t \Rightarrow PAQ = 0 \}$. Then $V_t \subseteq \hat{V}_t$ follows immediately from the definition of $\rho$ (Definition 2.6). Conversely, let $A \in \hat{V}_t$. For any $s > t$ we have

$$PV_t Q = 0 \Rightarrow \rho(P, Q) \geq s \Rightarrow PAQ = 0$$

for any projections $P, Q \in M$. By reflexivity we conclude that $A$ belongs to $V_s$ for all $s > t$, and hence that $A$ belongs to $V_t$. Thus $V_t = \hat{V}_t$. \qed

Next we observe that reflexivity can always be achieved by stabilization.

Proposition 2.25. Let $V$ be a quantum pseudometric on a von Neumann algebra $M \subseteq B(H)$. Then $V \otimes I$ is a reflexive quantum pseudometric on $M \otimes B(l^2)$ and $V$ is a quantum metric if and only if $V \otimes I$ is a quantum metric.

The first assertion follows from ([35], Proposition 2.20) and the second is easy.

We can transfer results and constructions from $M$ to $M \otimes B(l^2)$. For example, given a quantum pseudometric $V$ on a von Neumann algebra $M \subseteq B(H)$ and $\epsilon > 0$ we define the open $\epsilon$-neighborhood of a projection $P \in M \otimes B(l^2)$ to be the projection onto

$$\overline{(V_\epsilon \otimes I)(\text{ran}(P))} = \bigcup_{t < \epsilon} (V_t \otimes I)(\text{ran}(P)),$$

which is just its open $\epsilon$-neighborhood in $M \otimes B(l^2)$ relative to the quantum pseudometric $V \otimes I$. We similarly define the closure of $P$ to be the projection onto

$$\bigcap_{t > 0} (V_t \otimes I)(\text{ran}(P)),$$

which again is just its closure in $M \otimes B(l^2)$ relative to the quantum pseudometric $V \otimes I$. We still say that $P$ is closed if it equals its closure. Note that Propositions 2.17 and 2.18 hold for projections in $M \otimes B(l^2)$. Thus the above concepts all have manifestly representation independent reformulations in terms of projection distances in $M \otimes B(l^2)$.

This notion of closure in $M \otimes B(l^2)$ can be used to give an intrinsic characterization of the metric/pseudometric distinction.

Proposition 2.26. Let $V$ be a quantum pseudometric on a von Neumann algebra $M \subseteq B(H)$. Then $V$ is a quantum metric if and only if the closed projections in $M \otimes B(l^2)$ generate $M \otimes B(l^2)$ as a von Neumann algebra.

Proof. Let $N \subseteq M \otimes B(l^2)$ be the von Neumann algebra generated by the closed projections. We will show that $N^\prime = V_0 \otimes I$; thus $N = V_0^\prime \otimes B(l^2)$, so that $N = M \otimes B(l^2)$ if and only if $V_0 = M^\prime$, i.e., $V$ is a quantum metric.

Observe first that every projection in $I \otimes B(l^2)$ is closed. Thus $N^\prime \subseteq (I \otimes B(l^2))^\prime = B(H) \otimes I$. Now if $A \in V_0$ then the range of any closed projection is clearly invariant for $A \otimes I$. Since $A^*$ also belongs to $V_0$ it follows that $A \otimes I$ commutes with every closed projection, and therefore $V_0 \otimes I \subseteq N^\prime$. Conversely, let $A \in B(H) - V_0$. Then there exists $t > 0$ such that $t \notin V_t$, and by Lemma 1.7 we can then find projections $P, Q \in M \otimes B(l^2)$ such that $P(B(l^2)Q = 0$ for all $B \in V_t$ but $P(A \otimes I)Q \neq 0$. It follows that $P$ is orthogonal to $Q$, and thus the range of $Q$ cannot be invariant for $A \otimes I$. So $A \otimes I$ does not commute with $Q$, and hence $A \otimes I \notin N^\prime$. This completes the proof that $N^\prime = V_0 \otimes I$. \qed
2.5. **Constructions with quantum metrics.** In this section we describe some simple constructions that can be performed on quantum metrics. We start by identifying the appropriate morphisms in the category.

**Definition 2.27.** Let $V$ and $W$ be quantum pseudometrics on von Neumann algebras $M$ and $N$. A *co-Lipschitz morphism* from $M$ to $N$ is a unital weak* continuous $*$-homomorphism $\phi : M \to N$ for which there exists a number $C \geq 0$ such that

$$\rho(P, Q) \leq C \cdot \rho((\phi \otimes \text{id})(P), (\phi \otimes \text{id})(Q))$$

for all projections $P, Q \in M \otimes B(l^2)$. The minimum value of $C$ is the *co-Lipschitz number* of $\phi$, denoted $L(\phi)$, and $\phi$ is a co-contraction morphism if $L(\phi) \leq 1$. It is a co-isometric morphism if it is surjective and

$$\rho(\tilde{P}, \tilde{Q}) = \sup\{\rho(P, Q) : (\phi \otimes \text{id})(P) = \tilde{P}, (\phi \otimes \text{id})(Q) = \tilde{Q}\}$$

for all projections $\tilde{P}, \tilde{Q} \in N \otimes B(l^2)$. We set $L(\phi) = \infty$ if $\phi$ is not co-Lipschitz. Thus

$$L(\phi) = \sup_{P, Q} \frac{\rho(P, Q)}{\rho((\phi \otimes \text{id})(P), (\phi \otimes \text{id})(Q))}$$

with $P$ and $Q$ ranging over projections in $M \otimes B(l^2)$ and using the convention $\frac{0}{0} = \infty = 0$.

We immediately record the most important property of co-Lipschitz morphisms, which follows directly from their definition:

**Proposition 2.28.** Let $V_1$, $V_2$, and $V_3$ be quantum pseudometrics on von Neumann algebras $M_1$, $M_2$, and $M_3$ and let $\phi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ be co-Lipschitz morphisms. Then $\psi \circ \phi : M_1 \to M_3$ is a co-Lipschitz morphism and $L(\psi \circ \phi) \leq L(\psi)L(\phi)$.

**Definition 2.27** is motivated by the atomic abelian case, where the unital weak* continuous $*$-homomorphisms from $l^\infty(X)$ to $l^\infty(Y)$ are precisely the maps given by composition with functions from $Y$ to $X$. If $X$ and $Y$ are pseudometric spaces, let $L(f)$ denote the Lipschitz number of $f : Y \to X$,

$$L(f) = \frac{\sup_{x' , y' \in Y} \frac{d_X(f(x'), f(y'))}{d_Y(x', y')}}{d_Y(x', y')}$$

(with the convention $\frac{0}{0} = 0$).

**Proposition 2.29.** Let $X$ and $Y$ be pseudometric spaces and equip $l^\infty(X)$ and $l^\infty(Y)$ with the corresponding quantum pseudometrics (Proposition 2.7). If $f : Y \to X$ is a Lipschitz function then $\phi : g \mapsto g \circ f$ is a co-Lipschitz morphism from $l^\infty(X)$ to $l^\infty(Y)$, and $L(\phi) = L(f)$. Every co-Lipschitz morphism from $l^\infty(X)$ to $l^\infty(Y)$ is of this form.

**Proof.** Let $f$ be any function from $Y$ to $X$ and let $\phi : l^\infty(X) \to l^\infty(Y)$ be composition with $f$. The projections in $l^\infty(X) \otimes B(l^2)$ can be identified with projection-valued functions from $X$ into $B(l^2)$, and similarly for $Y$. Taking $P = e_x \cdot I$ and $Q = e_y \cdot I$ for $x, y \in X$, we have $(\phi \otimes \text{id})(P) = \chi_{f^{-1}(x)} \cdot I$ and $(\phi \otimes \text{id})(Q) = \chi_{f^{-1}(y)} \cdot I$. So

$$\frac{d(x, y)}{d(f^{-1}(x), f^{-1}(y))} = \frac{\rho(P, Q)}{\rho((\phi \otimes \text{id})(P), (\phi \otimes \text{id})(Q))} \leq L(\phi)$$
continuous

M ⊆ B

Proposition 2.30.

M

l

x

for all

P

the latter. For the reverse inequality, fix projections

supremum defining the former is effectively contained in the supremum defining

L

Conversely, let

P

may assume that

ρ

ranging over projections in

In that case taking

ǫ

22 GREG KUPERBERG AND NIK WEAVER

Let ˜

P

with

Then

We may assume that

ρ

φ

∈

X

y

ρ

have nonzero product and

d

ρ

ρ

(φ ⊗ id)(P), (φ ⊗ id)(Q)) < ∞. Given

P

ρ

(φ ⊗ id)(P), (φ ⊗ id)(Q)) > ϵ > 0 then

and taking

ϵ → 0 and the supremum over

P

Q

Y

such that

(φ ⊗ id)(P)(x') = P(f(x'))

and

(φ ⊗ id)(Q)(y') = Q(f(y'))

have nonzero product and

d

ρ((φ ⊗ id)(P), (φ ⊗ id)(Q)) + ϵ. If

d

ρ

ρ

P

Q

does not contribute to

L(φ), or else

ρ

P

Q

> 0 and the above implies

In that case taking

ϵ → 0 yields

L(f) = ∞, which again implies

L(φ) ≤ L(f).

So we have shown that

L(φ) = L(f), and hence

φ

is co-Lipschitz if and only if

P

Q

is Lipschitz. Since every unital weak* continuous ∗-homomorphism from

l∞(X)

to

l∞(Y)

is given by composition with a function

f : Y → X,

every co-Lipschitz morphism must arise in the above manner.

□

Under the assumption of reflexivity the co-Lipschitz number can be computed
using only projections in

M.

The proof of this result is notable for its use of the

hard direction of Theorem

L9

Proposition 2.30. Let

V

and

W

be quantum pseudometrics on von Neumann algebras

M ⊆ B(H)

and

N ⊆ B(K)

and let

φ : M → N

be a unital weak*

continuous ∗-homomorphism. Suppose

V

is reflexive (Definition

22.24

(a)). Then

L(φ) = \sup_{P,Q} \frac{\rho(P,Q)}{\rho(\phi(P), \phi(Q))},

with

P

and

Q

ranging over projections in

M.

Proof. Let

\hat{L}(φ) = \sup \rho(P,Q)/\rho(\phi(P), \phi(Q))

be the supremum with

P

and

Q

ranging over projections in

M.

Then it is immediate that

\hat{L}(φ) ≤ L(φ)

since the supremum defining the former is effectively contained in the supremum defining

the latter. For the reverse inequality, fix projections

P

and

Q

in

M ⊆ B(F). We may assume that

ρ(P,Q) > 0 and

ρ((φ ⊗ id)(P), (φ ⊗ id)(Q)) < ∞, so let

0 < s < ρ(P,Q)

and let

t > ρ((φ ⊗ id)(P), (φ ⊗ id)(Q)). Then

the pair

((φ ⊗ id)(P), (φ ⊗ id)(Q))

belongs to the intrinsic quantum relation

R_{\mathcal{W}_t}

(Definition

13.8

associated to the quantum relation

\mathcal{W}_t

(Theorem

L9

and hence the pair

(P,Q)

belongs to its pullback

R = \phi^*(R_{\mathcal{W}_t})

(35, Proposition 2.25 (b)). Let

V = \mathcal{V}_R

be the quantum
relation associated to $\mathcal{R}$; then since $(P,Q) \in \mathcal{R} = \mathcal{R}_V$ (Theorem 1.9) there exists $A \in \mathcal{V}$ such that $P(A \otimes I)Q \neq 0$. But $\rho(P,Q) > s$, so $A \notin \mathcal{V}_s$, and this shows that $V \not\subseteq V_s$. Since $V_s$ is reflexive we can then find projections $\tilde{P}, \tilde{Q} \in \mathcal{M}$ such that $\tilde{P}V_s \tilde{Q} = 0$ but $\tilde{P}V \tilde{Q} \neq 0$. Then $\rho(\tilde{P}, \tilde{Q}) \geq s$ but $(\tilde{P}, \tilde{Q}) \in \mathcal{R}$ so $\rho(\phi(\tilde{P}), \phi(\tilde{Q})) \leq t$, so we have

$$\frac{\rho(\tilde{P}, \tilde{Q})}{\rho(\phi(\tilde{P}), \phi(\tilde{Q}))} \geq \frac{s}{t}.$$  

Taking $s \to \rho(P,Q)$ and $t \to \rho((\phi \otimes \text{id})(P), (\phi \otimes \text{id})(Q))$ shows that

$$\frac{\rho(P,Q)}{\rho((\phi \otimes \text{id})(P), (\phi \otimes \text{id})(Q))} \leq \tilde{L}(\phi)$$

and taking the supremum over $P$ and $Q$ finally yields $L(\phi) \leq \tilde{L}(\phi)$. $\square$

The co-Lipschitz number is formulated in terms of the projection distances introduced in Definition 2.6 but it has an equivalent version in terms of W*-filtrations. We use the general form of a unital weak* continuous *-homomorphism $\phi : \mathcal{M} \to \mathcal{N}$ ([27], Theorem IV.5.5) which states that every such map can be expressed as an inflation followed by a restriction followed by an isomorphism. Since this expression is not unique, if we defined $L(\phi)$ in the concrete way indicated below then the definition would appear to be ambiguous. But the fact that this definition is equivalent to the intrinsic one given above means that there is actually no real ambiguity.

**Proposition 2.31.** Let $\mathbf{V}$ and $\mathbf{W}$ be quantum pseudometrics on von Neumann algebras $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ and let $\phi : \mathcal{M} \to \mathcal{N}$ be a unital weak* continuous *-homomorphism. Let $\tilde{K}$ be a Hilbert space, $R$ a projection in $\mathcal{B}(\tilde{K}) \otimes \mathcal{M}'$, and $U$ an isometry from $K$ to $\text{ran}(R)$ such that $\phi(A) = U^*(I_{\tilde{K}} \otimes A)U$ for all $A \in \mathcal{M}$. Then

$$L(\phi) = \inf\{ C \geq 0 : \mathcal{W}_t \subseteq U^*(\mathcal{B}(\tilde{K}) \bar{\otimes} \mathcal{V}_t)U \text{ for all } t \geq 0 \}$$

(with $\inf \emptyset = \infty$).

**Proof.** Let $\tilde{L}(\phi) = \inf\{ C \geq 0 : \mathcal{W}_t \subseteq U^*(\mathcal{B}(\tilde{K}) \bar{\otimes} \mathcal{V}_t)U \text{ for all } t \geq 0 \}$. Let $P$ and $Q$ be projections in $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2)$ and set

$$\hat{P} = (\phi \otimes \text{id})(P) = (U^* \otimes I_{l^2})(I_{\tilde{K}} \otimes P)(U \otimes I_{l^2})$$

and

$$\hat{Q} = (\phi \otimes \text{id})(Q) = (U^* \otimes I_{l^2})(I_{\tilde{K}} \otimes Q)(U \otimes I_{l^2}),$$

both in $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2)$.

Assuming $\rho(\hat{P}, \hat{Q}) < \infty$, let $t > \rho(\hat{P}, \hat{Q})$ and find $A \in \mathcal{W}_t$ such that $\hat{P}(A \otimes I_{l^2})\hat{Q} \neq 0$. Then $A \in U^*(\mathcal{B}(\tilde{K}) \bar{\otimes} \mathcal{V}_{L(\phi)t})U$, so

$$U\mathcal{A}\mathcal{U}^* \in R(\mathcal{B}(\tilde{K}) \bar{\otimes} \mathcal{V}_{L(\phi)t})R \subseteq \mathcal{B}(\tilde{K}) \bar{\otimes} \mathcal{V}_{L(\phi)t}$$

(since $\mathcal{M} \bar{\otimes} \mathcal{V}_{L(\phi)t}, \mathcal{M}' \subseteq \mathcal{V}_{L(\phi)t}$), and

$$(R \otimes I_{l^2})(I_{\tilde{K}} \otimes P)(U\mathcal{A}\mathcal{U}^* \otimes I_{l^2})(I_{\tilde{K}} \otimes Q)(R \otimes I_{l^2}) = (U \otimes I_{l^2})\hat{P}(A \otimes I_{l^2})\hat{Q}(U^* \otimes I_{l^2}) \neq 0.$$  

Thus $\rho(P,Q) \leq \tilde{L}(\phi)t$, and taking $t \to \rho(\hat{P}, \hat{Q})$ and then taking the supremum over $P$ and $Q$ shows that $L(\phi) \leq \tilde{L}(\phi)$. $\square$
Conversely, assume \( \hat{L}(\phi) > 0 \), let \( C < \hat{L}(\phi) \), and find \( t \) such that \( \mathcal{W}_t \subsetneq U^* (B(\hat{K}) \otimes \mathcal{V}_{Ct}) U \), or equivalently, such that \( U\mathcal{W}_t U^* \subsetneq B(\hat{K}) \otimes \mathcal{V}_{Ct} \). By Lemma 1.7, with \( I_K \otimes \mathcal{M} \) in place of \( \mathcal{M} \) we can find projections \( P, Q \in \mathcal{M} \otimes B(\hat{F}) \) such that

\[
(I_K \otimes P)(U A U^* \otimes I_2)(I_K \otimes Q) \neq 0
\]

for some \( A \in \mathcal{W}_t \), and hence \( \hat{P}(A \otimes I_2) \hat{Q} \neq 0 \) with \( \hat{P} = (\phi \otimes \text{id})(P) \) and \( \hat{Q} = (\phi \otimes \text{id})(Q) \) as in the first part of the proof, but

\[
(I_K \otimes P)(B \otimes I_2)(I_K \otimes Q) = 0
\]

for all \( B \in B(\hat{K}) \otimes \mathcal{V}_{Ct} \). It follows that \( \rho(P, Q) \geq Ct \) but \( \rho(\hat{P}, \hat{Q}) \leq t \). So \( L(\phi) \geq C \), and we conclude that \( \hat{L}(\phi) \leq L(\phi) \). Thus we have shown that \( L(\phi) = \hat{L}(\phi) \). \( \square \)

The formula for \( L(\phi) \) in Proposition 2.31 may be more transparent if the map \( \phi \) is explicitly decomposed as follows. Let \( \mathcal{M}_1 = I_K \otimes \mathcal{M} \subseteq B(\hat{K} \otimes H) \), \( \mathcal{M}_2 = \mathcal{M}_1 R \), and \( \mathcal{N}_1 = U \mathcal{N} \subseteq B(\mathcal{R}) \), equipped with quantum pseudometrics \( \mathcal{V}_1 = \{ \mathcal{V}_1^t \} \), \( \mathcal{V}_2 = \{ \mathcal{V}_2^t \} \), and \( \mathcal{W}_1 = \{ \mathcal{W}_1^t \} \) where

\[
\mathcal{V}_1^t = B(\hat{K}) \otimes \mathcal{V}_1 \quad \mathcal{V}_2^t = R \mathcal{V}_1^t R \quad \mathcal{W}_1^t = U \mathcal{W}_1 U^*.
\]

Then \( \phi = \phi_1 \circ \phi_2 \circ \phi_3 \) where \( \phi_1 : \mathcal{M} \to \mathcal{M}_1 \), \( \phi_2 : \mathcal{M}_1 \to \mathcal{M}_2 \), \( \phi_3 : \mathcal{M}_2 \to \mathcal{N}_1 \), and \( \phi_4 : \mathcal{N}_1 \to \mathcal{N} \) are defined by \( \phi_1 : A \mapsto I_K \otimes A \), \( \phi_2 : A \mapsto A R \), \( \phi_3 = \text{id} \), and \( \phi_4 : A \mapsto U^* A U \). Unless some degeneracy occurs such that \( L(\phi) = L(\phi_i) = 0 \) for some \( i \), we have

\[
L(\phi_1) = \inf \{ C \geq 0 : \mathcal{V}_1^t \subseteq B(\hat{K}) \otimes \mathcal{V}_{Ct} \text{ for all } t \geq 0 \} = 1
\]

\[
L(\phi_2) = \inf \{ C \geq 0 : \mathcal{V}_2^t \subseteq R \mathcal{V}_1^t R \text{ for all } t \geq 0 \} = 1
\]

\[
L(\phi_3) = \inf \{ C \geq 0 : \mathcal{W}_1^t \subseteq \mathcal{V}_2^t \text{ for all } t \geq 0 \} = 1
\]

\[
L(\phi_4) = \inf \{ C \geq 0 : \mathcal{W}_1^t \subseteq U^* \mathcal{W}_1^t U \text{ for all } t \geq 0 \} = 1
\]

and \( L(\phi) = L(\phi_1) \).

Next we present three easy constructions.

**Definition 2.32.** (a) Let \( \mathbf{V} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq B(H) \) and let \( C \geq 0 \). Then the **truncation** of \( \mathbf{V} \) to \( C \) is the quantum pseudometric \( \hat{\mathbf{V}} = (\hat{\mathbf{V}}_t) \) defined by

\[
\hat{\mathbf{V}}_t = \begin{cases} 
\mathbf{V}_t & \text{if } t < C \\
B(H) & \text{if } t \geq C.
\end{cases}
\]

(b) Let \( \mathbf{V} \) and \( \mathbf{W} \) be quantum pseudometrics on von Neumann algebras \( \mathcal{M} \subseteq B(H) \) and \( \mathcal{N} \subseteq B(K) \). Their **direct sum** is the von Neumann algebra \( \mathcal{M} \oplus \mathcal{N} \subseteq B(H \oplus K) \) equipped with the quantum pseudometric \( \mathbf{V} \oplus \mathbf{W} = \{ \mathbf{V}_t \oplus \mathbf{W}_t \} \).

(c) Let \( \{ \mathbf{V}_\lambda \} \) with \( \mathbf{V}_\lambda = \{ \mathbf{V}_\lambda^t \} \) be a family of quantum pseudometrics on a von Neumann algebra \( \mathcal{M} \subseteq B(H) \). Their **meet** is the quantum pseudometric \( \bigwedge \mathbf{V}_\lambda = \{ \bigwedge \mathbf{V}_\lambda^t \} \).

In the atomic abelian case truncations reduce to the classical construction

\[
\hat{d}(x, y) = \min \{ d(x, y), C \},
\]

direct sums reduce to the disjoint union construction \( X \bigsqcup Y \) with \( d(x, y) = \infty \) for all \( x \in X \) and \( y \in Y \), and meets reduce to the supremum of a family of pseudometrics. In the case of direct sums note that if \( \text{diam}(\mathbf{V}), \text{diam}(\mathbf{W}) \leq C \) then we can truncate their disjoint union to \( C \) without affecting the embedded copies of \( \mathbf{V} \) and \( \mathbf{W} \). This corresponds to setting \( d(x, y) = C \) for all \( x \in X \) and all \( y \in Y \) in the classical case.
More generally, for any \( r \geq \max\{\text{diam}(V), \text{diam}(W)\}/2 \) we can replace \( \mathcal{V}_t \oplus \mathcal{W}_t \) with
\[
\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A \in \mathcal{V}_t, B \in \mathcal{B}(K, H), C \in \mathcal{B}(H, K), D \in \mathcal{W}_t \right\}
\]
for all \( t \geq r \). This corresponds to setting \( d(x, y) = r \) for all \( x \in X \) and all \( y \in Y \) in the classical case.

The meet construction in general is not obtained at the level of projections by setting \( \rho(P, Q) = \sup \rho_\lambda(P, Q) \); see Example 3.2. However, truncations and direct sums do satisfy the obvious formulas at the projection level.

**Proposition 2.33.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( M \subseteq B(H) \) and let \( V \) be its truncation to \( C \geq 0 \). Then
\[
\rho_V(P, Q) = \min\{\rho_V(P, Q), C\}
\]
for all linkable projections \( P, Q \in \mathcal{M} \otimes \mathcal{B}(I^2) \).

(Recall from Proposition 2.13 that the distance between unlinkable projections is always \( \infty \).)

**Proposition 2.34.** Let \( V \) and \( W \) be quantum pseudometrics on von Neumann algebras \( M \subseteq B(H) \) and \( N \subseteq B(K) \) and let \( P = P_1 \oplus P_2 \) and \( Q = Q_1 \oplus Q_2 \) be projections in \( (M \oplus N) \otimes \mathcal{B}(I^2) \cong (M \otimes \mathcal{B}(I^2)) \oplus (N \otimes \mathcal{B}(I^2)) \). Then
\[
\rho_{V \otimes W}(P, Q) = \min\{\rho_V(P_1, Q_1), \rho_W(P_2, Q_2)\}.
\]

The proofs are straightforward. In the proof of Proposition 2.34 we use the fact that if \( P \) and \( Q \) are linkable then there exists \( A \in B(H) \) such that \( P(A \otimes I)Q \neq 0 \) (Proposition 2.13).

We now turn to quotients, subobjects, and products. Quotients are simplest. If \( \phi : M \to N \) is a surjective unital weak* continuous \(*\)-homomorphism then \( \ker(\phi) \) is a weak* closed ideal of \( M \), and hence \( \ker(\phi) = RM \) for some central projection \( R \in M \). Thus \( M = RM \oplus (I - R)M \) with \( (I - R)M \cong N \). So metric quotients are modelled by von Neumann algebra direct summands.

**Definition 2.35.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( M \subseteq B(H) \). A metric quotient of \( M \) is a direct summand \( N = RM \subseteq B(K) \) of \( M \), where \( R \) is a central projection in \( M \) and \( K = \text{ran}(R) \), together with the quantum pseudometric \( W = \{ W_t \} \) on \( N \) defined by
\[
W_t = RVtR \subseteq B(K).
\]

In this definition note that \( W_t \subseteq V_t \) because \( R \in M' \) and \( V_t \) is a bimodule over \( M' \subseteq V_0 \).

For example, in Definition 2.32 (b) \( M \) and \( N \) are metric quotients of \( M \oplus N \), and this remains true after truncation to \( \max\{\text{diam}(V), \text{diam}(W)\} \).

**Proposition 2.36.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( M \subseteq B(H) \) and let \( N = RM \) be a metric quotient of \( M \) with quantum pseudometric \( W \). Then the quantum distance function \( \rho_W \) on \( N \) is the restriction of the quantum distance function \( \rho_V \) on \( M \) to \( N \otimes \mathcal{B}(I^2) \subseteq M \otimes \mathcal{B}(I^2) \).

This proposition follows from the observation that if \( P \) and \( Q \) are projections in \( N \otimes \mathcal{B}(I^2) \subseteq M \otimes \mathcal{B}(I^2) \) and \( A \in V_t \) then \( RAR \in W_t \subseteq V_t \) and
\[
P(A \otimes I)Q \neq 0 \quad \Leftrightarrow \quad P(RAR \otimes I)Q \neq 0.
\]
Corollary 2.37. Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$ and let $\mathcal{N} = R\mathcal{M}$ be a metric quotient of $\mathcal{M}$ with quantum pseudometric $W$. Then the map $\phi : A \mapsto AR$ is a co-isometric morphism from $\mathcal{M}$ to $\mathcal{N}$. Up to isomorphism of the range every co-isometric morphism has this form.

Proof. Let $\tilde{P}$ and $\tilde{Q}$ be projections in $\mathcal{N} \otimes B(l^2)$. The projections in $\mathcal{M} \otimes B(l^2) \cong (R\mathcal{M} \otimes B(l^2)) \oplus ((I - R)\mathcal{M} \otimes B(l^2))$ that map to $\tilde{P}$ and $\tilde{Q}$ are just those of the form $\tilde{P} \oplus P$ and $\tilde{Q} \oplus Q$ for arbitrary projections $P$ and $Q$ in $(I - R)\mathcal{M} \otimes B(l^2)$. We have

$$\rho_V(\tilde{P} \oplus P, \tilde{Q} \oplus Q) \leq \rho_V(\tilde{P} \oplus 0, \tilde{Q} \oplus 0) = \rho_W(\tilde{P}, \tilde{Q}),$$

with equality if $P = Q = 0$. So $\phi$ is a co-isometric morphism.

Any co-isometric morphism is a surjective weak* continuous $^*$-homomorphism and hence up to isomorphism of the range is of the form $\phi : A \mapsto AR$ from $\mathcal{M}$ to $\mathcal{N} = R\mathcal{M}$ where $R$ is a central projection in $\mathcal{M}$. The condition

$$\rho_W(\tilde{P}, \tilde{Q}) = \sup \rho_V(\tilde{P} \oplus P, \tilde{Q} \oplus Q) = \rho_V(\tilde{P} \oplus 0, \tilde{Q} \oplus 0)$$

then implies that $\mathcal{N}$ is a metric quotient of $\mathcal{M}$ by Proposition 2.36. □

Next we consider subobjects. Even in the classical setting the dual construction is slightly subtle; this is the metric quotient, discussed in Section 1.4 of [32].

Definition 2.38. Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$. A metric subobject of $\mathcal{M}$ is a unital von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ together with the quantum pseudometric $V_N = \bigwedge \{ W : V \leq W \text{ and } N' \subseteq W_0 \}$ where $W$ ranges over $W^*$-filtrations of $B(H)$. In other words, $V_N$ is the meet of all quantum pseudometrics on $\mathcal{N}$ that dominate $V$.

Metric subobjects have an obvious universal property:

Proposition 2.39. Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$ and let $\mathcal{N} \subseteq \mathcal{M}$ be a metric subobject of $\mathcal{M}$.

(a) The inclusion map $\iota : \mathcal{N} \rightarrow \mathcal{M}$ is a co-contraction morphism (equipping $\mathcal{N}$ with the quantum pseudometric $V_N$).

(b) If $W$ is any quantum pseudometric on $\mathcal{N}$ which makes the inclusion map a co-contraction morphism then the identity map from $\mathcal{N}$ to itself is a co-contraction morphism from the pseudometric $W$ to the pseudometric $V_N$.

This holds because $L(\iota) \leq 1$ if and only if $V \leq W$, by Proposition 2.31. However, even basic questions such as “Under what conditions will a metric subobject of a quantum metric be a quantum metric (not just a pseudometric)?” in general have no simple answer. But this is already true in the classical case for the dual question about quotients of metric spaces.

In order to define products of quantum metrics we need to be able to take tensor products of dual operator systems. The most useful generalization of the spatial tensor product of von Neumann algebras to dual operator spaces is the normal Fubini tensor product [12]. If $V \subseteq B(H)$ and $W \subseteq B(K)$ are dual operator spaces then their normal Fubini tensor product can be defined concretely as

$$V \otimes_F W = (V \otimes B(K)) \cap (B(H) \otimes W)$$
where $\otimes$ is, as before, the normal spatial tensor product (i.e., the weak* closure of the algebraic tensor product). Abstractly, $V \otimes_F W$ is characterized as the dual of the projective tensor product of the preduals of $V$ and $W$ \cite{25}:

$$V \otimes_F W \cong (V_* \otimes W_*)^*.$$  

The normal spatial tensor product is always contained in the normal Fubini tensor product but this inclusion may be strict. Thus, the equality $V \otimes_F W = (V \otimes B(K)) \cap (B(H) \otimes W)$, which is crucial for the following proof, fails in general for the normal spatial tensor product.

**Proposition 2.40.** Let $V = \{V_t\}$ and $W = \{W_t\}$ be $W^*$-filtrations of $B(H)$ and $B(K)$, respectively. Then $V \otimes_F W = \{V_t \otimes W_t\}$ is a $W^*$-filtration of $B(H \otimes K)$.

**Proof.** We have

$$(V_s \otimes F W_s)(V_t \otimes F W_t) = ((V_s \otimes B(K)) \cap (B(H) \otimes W_s))((V_t \otimes B(K)) \cap (B(H) \otimes W_t)) \subseteq (V_s \otimes B(K))(V_t \otimes B(K)) \cap ((B(H) \otimes W_s) (B(H) \otimes W_t)) \subseteq (V_{s+t} \otimes B(K)) \cap (B(H) \otimes W_{s+t}) = V_{s+t} \otimes F W_{s+t}$$

and

$$\bigcap_{s > t} V_s \otimes F W_s = \bigcap_{s > t} ((V_s \otimes B(K)) \cap (B(H) \otimes W_s)) = \left(\bigcap_{s > t} V_s \otimes B(K)\right) \cap \left(\bigcap_{s > t} B(H) \otimes W_s\right) = (V_t \otimes B(K)) \cap (B(H) \otimes W_t) = V_t \otimes F W_t,$$

so both conditions of Definition 2.11 (a) are satisfied. \( \square \)

**Definition 2.41.** Let $V$ and $W$ be quantum pseudometrics on von Neumann algebras $\mathcal{M} \subseteq B(H)$ and $\mathcal{N} \subseteq B(K)$. Their **metric product** is the von Neumann algebra $\mathcal{M} \otimes \mathcal{N} \subseteq B(H \otimes K)$ equipped with the quantum pseudometric $V \otimes_F W = \{V_t \otimes W_t\}$.

**Proposition 2.42.** Let $V$ and $W$ be quantum pseudometrics on von Neumann algebras $\mathcal{M} \subseteq B(H)$ and $\mathcal{N} \subseteq B(K)$.

(a) The **metric product** $V \otimes_F W$ is the meet of the quantum pseudometrics $V \otimes W_0 = \{V_t \otimes W^0_t\}$ and $V_0 \otimes W = \{V^0_t \otimes W_t\}$ where $V_0 = \{V^0_t\}$ and $W_0 = \{W^0_t\}$ are the trivial quantum pseudometrics with $V^0_t = B(H)$ and $W^0_t = B(K)$ for all $t$.

(b) The natural embeddings $i_1 : A \mapsto A \otimes I_K$ and $i_2 : B \mapsto I_H \otimes B$ of $\mathcal{M}$ and $\mathcal{N}$ into $\mathcal{M} \otimes \mathcal{N}$ realize $\mathcal{M}$ and $\mathcal{N}$ as metric subobjects of the metric product.

(c) $V \otimes_F W$ is a quantum metric if and only if both $V$ and $W$ are quantum metrics.

**Proof.** (a) Trivial.

(b) By symmetry it is enough to consider the embedding of $\mathcal{M}$ into $\mathcal{M} \otimes \mathcal{N}$. The quantum pseudometric on $\mathcal{M} \otimes I \subseteq B(H \otimes K)$ corresponding to $V$ is $V \otimes W_0$ where $W_0$ is the trivial quantum pseudometric as in part (a) (Theorem 1.3), so we have to show that

$$V \otimes W_0 = \bigwedge \{W : V \otimes_F W \leq W \text{ and } \mathcal{M} \otimes B(K) \subseteq W_0\}$$

where $W$ ranges over $W^*$-filtrations of $B(H \otimes K)$. It is easy to check that $V \otimes W_0$ belongs to the meet on the right, which verifies the inequality $\leq$. For the reverse
inequality, let \( \hat{W} \) be any \( W^* \)-filtration satisfying \( V \otimes F W \leq \hat{W} \) and \( M \otimes B(K) \subseteq W_0 \). Then \( V_t \otimes I \subseteq \hat{W}_t \) for all \( t \) and \( I \otimes B(K) \subseteq W_0 \). Since \( \hat{W}_t \otimes W_t \subseteq \hat{W}_t \), it follows that \( V_t \otimes B(K) \subseteq \hat{W}_t \) for all \( t \), i.e., that \( V \otimes F W \leq \hat{W} \). This verifies the inequality \( \geq \).

(c) If either \( V \) or \( W \) is not a quantum metric then either \( M' \subseteq V_0 \) or \( N' \subseteq W_0 \), and it follows that \( M \otimes N' \subseteq V_0 \otimes F W_0 \), so that \( V \otimes F W \) is not a quantum metric. The reverse implication follows from the fact that

\[
M \otimes N' = M' \otimes F N'
\]

since \( M' \) and \( N' \) are von Neumann algebras \[11\].

The definition of the metric product can be varied. For instance, an \( L^p \) product \( (1 \leq p < \infty) \) could be defined as the smallest \( W^* \)-filtration \( \hat{W} = \{ \hat{W}_t \} \) of \( B(H \otimes K) \) satisfying

\[
V_s \otimes F W_t \subseteq \hat{W}_{(s+t)/p}
\]

for all \( s, t \geq 0 \). This product also has the properties proven for the metric product in Proposition 2.42 (b) and (c); the first holds by essentially the same proof given for metric products, and the second follows from the fact that the \( L^p \) product \( W^* \)-filtration is contained in the metric product \( W^* \)-filtration. However, it is not clear that the \( L^p \) product \( W^* \)-filtration has any more explicit description than the one just given.

Finally, we note that the quotient, subobject, and product constructions discussed above reduce to the standard notions in the atomic abelian case.

**Proposition 2.43.** Let \( X \) and \( Y \) be pseudometric spaces with pseudometrics \( d \) and \( d' \) and let \( M \cong l^\infty(X) \) and \( N \cong l^\infty(Y) \) be the von Neumann algebras of bounded multiplication operators on \( l^2(X) \) and \( l^2(Y) \), equipped with the corresponding quantum pseudometrics (Proposition 2.4).  

(a) For any subset \( X_0 \) of \( X \) with inherited pseudometric \( d_0 \), the von Neumann algebra

\[
M_0 = \{ M_f : \text{supp}(f) \subseteq X_0 \} \subseteq M
\]

equipped with the quantum pseudometric \( V_{d_0} \) (Proposition 2.3) is a metric quotient of \( M \). Every metric quotient of \( M \) is of this form.

(b) For any equivalence relation \( \sim \) on \( X \) with quotient pseudometric \( \hat{d} \) (32, Definition 1.4.2), the von Neumann algebra

\[
\hat{M} = \{ M_f : f \in l^\infty(X), \quad x \sim y \Rightarrow f(x) = f(y) \} \subseteq M
\]

equipped with the quantum pseudometric \( V_{\hat{d}} \) is a metric subobject of \( M \). Every metric subobject of \( M \) is of this form.

(c) The metric product of \( M \) and \( N \) is the von Neumann algebra \( M \otimes N \cong l^\infty(X \times Y) \) equipped with the quantum pseudometric \( V_{d_{X \times Y}} \) associated to the pseudometric

\[
d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d'(y_1, y_2)\}.
\]

The proof is straightforward. In part (b) we show that \( V_{\hat{d}} \) is the metric subobject quantum pseudometric by observing that \( \hat{d} \) has the universal property stated in Proposition 2.39 see (32, Proposition 1.4.3).
2.6. **Intrinsic characterization.** We show that quantum pseudometrics can be characterized intrinsically in terms of quantum distance functions. But first we observe that in finite dimensions quantum pseudometrics can be encoded as positive operators in $\mathcal{M} \otimes \mathcal{M}^{\text{op}}$.

**Proposition 2.44.** Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a finite dimensional von Neumann algebra and let $\mathbf{V}$ be a quantum pseudometric on $\mathcal{M}$. Then there is a positive operator $X$ in $\mathcal{M} \otimes \mathcal{M}^{\text{op}}$ such that

$$\mathcal{V}_t = \{ B \in \mathcal{B}(H) : \Phi_{P_{t,\infty}}(X)(B) = 0 \}$$

for all $t \geq 0$, where $P_{t,\infty}(X)$ is the spectral projection of $X$ and $\Phi$ is the action of $\mathcal{M} \otimes \mathcal{M}^{\text{op}}$ on $\mathcal{B}(H)$ defined by

$$\Phi_{A \otimes C}(B) = ABC.$$ 

**Proof.** For each $t$ the set

$$\mathcal{I}_t = \{ Y \in \mathcal{M} \otimes \mathcal{M}^{\text{op}} : \Phi_Y(B) = 0 \text{ for all } B \in \mathcal{V}_t \}$$

is a left ideal of $\mathcal{M} \otimes \mathcal{M}^{\text{op}}$, and hence is of the form $(\mathcal{M} \otimes \mathcal{M}^{\text{op}})P_t$ for some projection $P_t \in \mathcal{M} \otimes \mathcal{M}^{\text{op}}$. Then $P_t \in \mathcal{I}_t$, so $\Phi_{P_t}(B) = 0$ for all $B \in \mathcal{V}_t$. Conversely, $Y = YP_t$ for any $Y \in \mathcal{I}_t$, so that $\Phi_{P_t}(B) = 0$ implies $\Phi_Y(B) = \Phi_Y \Phi_{P_t}(B) = 0$ for all $Y \in \mathcal{I}_t$ implies $B \in \mathcal{V}_t$. Thus $\mathcal{V}_t = \{ B : \Phi_{P_t}(B) = 0 \}$. Finally, the $P_t$ for $t \geq 0$ constitute a decreasing right continuous one-parameter family of projections in $\mathcal{M} \otimes \mathcal{M}^{\text{op}}$ and hence are the spectral projections $P_{t,\infty}(X)$ for some positive operator $X \in \mathcal{M} \otimes \mathcal{M}^{\text{op}}$. \hfill \Box

Now we proceed to our main result which gives a general intrinsic characterization of quantum pseudometrics. Recall the abstract notion of a “quantum distance function” from Definition 2.7. Also recall that $D = D_{\mathbf{V}}$ is the displacement gauge associated to $\mathbf{V}$ (Proposition 2.22). We will give a different intrinsic characterization in Corollary 4.17.

**Theorem 2.45.** Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. If $\mathbf{V}$ is a quantum pseudometric on $\mathcal{M}$ then

$$\rho_{\mathbf{V}}(P,Q) = \inf \{ D(A) : A \in \mathcal{B}(H) \text{ and } P(A \otimes I)Q \neq 0 \}$$

(with $\inf \emptyset = \infty$) is a quantum distance function on $\mathcal{M}$; conversely, if $\rho$ is a quantum distance function on $\mathcal{M}$ then $\mathbf{V}_\rho = \{ \mathcal{V}_\rho^t \}$ with

$$\mathcal{V}_\rho^t = \{ A \in \mathcal{B}(H) : \rho(A) > t \implies P(A \otimes I)Q = 0 \}$$

is a quantum pseudometric on $\mathcal{M}$. The two constructions are inverse to each other.

**Proof.** Let $\mathbf{V}$ be a quantum pseudometric on $\mathcal{M}$. The fact that $\rho_{\mathbf{V}}$ is a quantum distance function was proven in Proposition 2.8. Now let $\rho$ be any quantum distance function. We have $\mathcal{M}' \subseteq \mathcal{V}_\rho^t$ for all $t$ by property (ii) of Definition 2.7 since

$$\rho(P,Q) > t \implies PQ = 0 \implies P(A \otimes I)Q = (A \otimes I)PQ = 0$$

for all $A \in \mathcal{M}'$. Also $\mathcal{V}_\rho^t$ is self-adjoint by property (iii) of Definition 2.7 and it is weak* closed because

$$P(A \otimes I)Q = 0 \iff \langle (A \otimes I)w,v \rangle = 0 \text{ for all } v \in \text{ran}(P), w \in \text{ran}(Q).$$

So each $\mathcal{V}_\rho^t$ is a dual operator system and $\mathcal{V}_\rho^0$ contains $\mathcal{M}'$. The fact that $\mathcal{V}_\rho^t = \bigcap_{s \geq t} \mathcal{V}_s^\rho$ for all $t$ is easy. For the filtration condition, let $A \in \mathcal{V}_s^\rho$, $B \in \mathcal{V}_t^\rho$, and
$P, R \in \mathcal{P}$ and suppose $P(AB \otimes I)R \neq 0$. We must show that $\rho(P, R) \leq s + t$. Let $Q$ be the projection onto the closure of 

$$(\mathcal{M}' \otimes I)(\text{ran}((B \otimes I)R)).$$

The range of $Q$ is invariant for $\mathcal{M}' \otimes I$ and hence $Q$ belongs to $(\mathcal{M}' \otimes I)' = \mathcal{M} \square \mathcal{B}(l^2)$. We have $P(A \otimes I)Q \neq 0$ since $(B \otimes I)R \leq Q$ and we have $\tilde{Q}(B \otimes I)R \neq 0$ for any $\tilde{Q} \in \mathcal{P}$ such that $QQ \neq 0$ because 

$$\tilde{Q}(B \otimes I)R = 0 \Rightarrow \tilde{Q}(CB \otimes I)R = 0 \text{ for all } C \in \mathcal{M}' \Rightarrow \tilde{Q}Q = 0.$$ 

Since $A \in \mathcal{V}_\rho$ and $B \in \mathcal{V}_\rho$, the above implies that $\rho(P, R) \leq s + t$ by property (v) of Definition 2.7. This shows that $AB \in \mathcal{V}_{s+t}$, and we conclude that $\mathcal{V}_\rho \mathcal{V}_\rho \subseteq \mathcal{V}_{s+t}$. This completes the proof that $\mathcal{V}_\rho$ is a quantum pseudometric on $\mathcal{M}$.

Now let $\mathbf{V}$ be any quantum pseudometric on $\mathcal{M}$, let $\rho = \rho_\mathbf{V}$, and let $\tilde{\mathbf{V}} = \mathbf{V}_\rho$. The fact that $\mathbf{V} = \tilde{\mathbf{V}}$ is just the content of Proposition 2.10.

Finally, let $\rho$ be any quantum distance function, let $\mathbf{V} = \mathbf{V}_\rho$, and let $\tilde{\rho} = \rho_\mathbf{V}$. Fix $t > 0$ and define 

$$\mathcal{R}_t = \{(P, Q) \in \mathcal{P}^2 : \rho(P, Q) < t\}$$ 

and 

$$\tilde{\mathcal{R}}_t = \{(P, Q) \in \mathcal{P}^2 : \tilde{\rho}(P, Q) \leq t\}.$$ 

Then $\mathcal{R}_t$ is an open subset of $\mathcal{P}^2$ because its complement is closed by property (vii) of Definition 2.7. We have $(0, 0) \notin \mathcal{R}_t$ by property (i) of Definition 2.7. $(\bigvee P_A, \bigvee Q_A) \in \mathcal{R}_t$ if and only if $(P_A, Q_A) \in \mathcal{R}_t$ by the comment following Definition 2.7, and $(P, [BQ]) \in \mathcal{R}_t$ if and only if $([B^*P], Q) \in \mathcal{R}_t$ for all $B \in I \otimes \mathcal{B}(l^2)$ by property (vi) of Definition 2.7. Thus $\mathcal{R}_t$ is an intrinsic quantum relation (Definition 1.8) and we therefore have 

$$\mathcal{R}_t = \{(P, Q) \in \mathcal{P}^2 : (\exists A \in \mathcal{B}(H)) (\rho(P', Q') \geq t \Rightarrow P'(A \otimes I)Q' = 0 \text{ and } P(A \otimes I)Q \neq 0)\}$$

by Theorem 1.9. Comparing this with the definition of $\tilde{\mathcal{R}}_t$ then shows that $\mathcal{R}_t \subseteq \tilde{\mathcal{R}}_t$ for all $\epsilon > 0$. It follows that $\rho(P, Q) = \tilde{\rho}(P, Q)$ for all $P$ and $Q$, i.e., $\rho = \tilde{\rho}$. □

3. Examples

Our new definition of quantum metrics supports a wide variety of examples. This is also true of the previously proposed definitions mentioned in the introduction, and indeed the main classes of examples in the different cases substantially overlap. If anything, a complaint could be made that the previous definitions are too broad. In contrast, our definition is sufficiently rigid to permit, for example, a simple classification of all quantum metrics on $M_2(\mathbb{C})$ (Proposition 3.6) and a general analysis of translation-invariant quantum metrics on quantum tori (Theorem 3.10).

The metric aspect of error correcting quantum codes provided the original motivation behind our new approach. This connection is explained in Section 3.4. We are able to present a natural common generalization of basic aspects of classical and quantum error correction. Our theory also encompasses mixed classical/quantum settings.

We present a small variety of interesting classes of examples. The list could obviously be greatly expanded, but we have tried to give a fair representation of the principal methods of construction.
3.1. **Operator systems.** We begin our survey of examples with possibly the simplest natural class. For any dual operator system $\mathcal{A} \subseteq \mathcal{B}(H)$ define

$$
V^A_t = \begin{cases} 
CI & \text{if } 0 \leq t < 1 \\
\mathcal{A} & \text{if } 1 \leq t < 2 \\
\mathcal{B}(H) & \text{if } t \geq 2.
\end{cases}
$$

The verification that $V_\mathcal{A} = \{V^A_t\}$ is a quantum metric on $\mathcal{M} = \mathcal{B}(H)$ is trivial. Despite their simplicity, the $V^A_t$ are already good for producing easy counterexamples.

**Example 3.1.** A quantum metric for which $V_t \neq \{A \in \mathcal{B}(H) : \rho(P, Q) > t \Rightarrow PAQ = 0\}$, with $P$ and $Q$ ranging over projections in $\mathcal{M}$ (cf. Proposition 2.10), and furthermore $\text{diam}(V) > \sup\{\rho(P, Q) : P$ and $Q$ are nonzero projections in $\mathcal{M}\}$ (cf. Proposition 2.16). Let $\mathcal{A}$ be a dual operator system properly contained in $\mathcal{B}(H)$ such that for any nonzero $v, w \in H$ there exists $A \in \mathcal{A}$ with $\langle Aw, v \rangle \neq 0$. For instance, we could take

$$A = \{A \in \mathcal{B}(H) : \text{tr}(AB) = 0\}$$

where $B$ is any nonzero traceless self-adjoint trace class operator. (Suppose $\langle Aw, v \rangle = 0$ for all $A \in \mathcal{A}$, with $v$ and $w$ nonzero. Then $A \mapsto \langle Aw, v \rangle$ and $A \mapsto \text{tr}(AB)$ are nonzero linear functionals with the same kernel, hence they are scalar multiples of each other. This implies that $B$ is a scalar multiple of the trace class operator $u \mapsto \langle u, v \rangle w$, which contradicts the assumption that it is traceless and self-adjoint.) Then $V_\mathcal{A}$ has diameter 2 but $\rho(P, Q) = 1$ for any nonzero projections $P, Q \in \mathcal{B}(H)$.

**Example 3.2.** A pair of quantum metrics $V_\mathcal{A}$ and $V_\mathcal{B}$ on $\mathcal{B}(H)$ such that the formula

$$\rho_{V_\mathcal{A} \land V_\mathcal{B}}(P, Q) = \max\{\rho_{V_\mathcal{A}}(P, Q), \rho_{V_\mathcal{B}}(P, Q)\}$$

fails (cf. Propositions 2.33 and 2.34 where analogous formulas hold). Fix a pair of orthogonal unit vectors $v$ and $w$ and find self-adjoint operators $A$ and $B$ such that

$$\langle Aw, v \rangle \neq 0 \neq \langle Bw, v \rangle$$

but $\mathcal{A} \cap \mathcal{B} = CI$ where $\mathcal{A} = \text{span}\{A, I\}$ and $\mathcal{B} = \text{span}\{B, I\}$. It is clear that $\mathcal{A}$ and $\mathcal{B}$ are dual operator systems, and letting $P$ and $Q$ be the projections onto $Cv$ and $Cw$, we have $\rho_{V_\mathcal{A}}(P, Q) = \rho_{V_\mathcal{B}}(P, Q) = 1$ but $\rho_{V_\mathcal{A} \land V_\mathcal{B}}(P, Q) = 2$.

It is worth noting that every quantum metric on $\mathcal{B}(H)$ for which the range of the quantum distance function on linkable projections is contained in $\{0, 1, 2\}$ is of the form $V_\mathcal{A}$ for some dual operator system $\mathcal{A}$. This follows from an easy and more general fact:

**Proposition 3.3.** Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$, let

$$L = \{0\} \cup \{t \in (0, \infty) : V_t \neq V_t\}$$

(taking $V_\infty = \mathcal{B}(H)$), and let

$$R = \{t \in [0, \infty) : s > t \Rightarrow V_t \neq V_s\}.$$
Then the range of \( \rho \) restricted to linkable pairs of projections in \( \mathcal{M} \otimes \mathcal{B}(l^2) \) equals \( L \cup R \).

**Proof.** We proved in Proposition 2.13 that \( \mathcal{V}_{< \infty} \neq \mathcal{V}_\infty \), i.e., \( \infty \in L \), if and only if \( \rho(P, Q) = \infty \) for some pair of linkable projections \( P \) and \( Q \). This settles the case \( t = \infty \); for the rest of the proof assume \( t \) is finite.

Suppose \( t \notin L \cup R \). Then \( \mathcal{V}_t = \mathcal{V}_{< t} = \mathcal{V}_{t+\epsilon} \) for some \( \epsilon > 0 \). For any pair of projections \( P, Q \in \mathcal{M} \otimes \mathcal{B}(l^2) \), if \( P(A \otimes I)Q = 0 \) for all \( A \in \mathcal{V}_t \) then \( P(A \otimes I)Q = 0 \) for all \( A \in \mathcal{V}_{t+\epsilon} \) and hence \( \rho(P, Q) \geq t + \epsilon \). On the other hand, if \( P(A \otimes I)Q = 0 \) for all \( A \in \mathcal{V}_s \), for all \( s < t \), then \( P(A \otimes I)Q = 0 \) for all \( A \in \bigcup_{s < t} \mathcal{V}_s = \mathcal{V}_t \); so \( P(A \otimes I)Q \neq 0 \) for some \( A \in \mathcal{V}_t \) implies \( P(A \otimes I)Q \neq 0 \) for some \( A \in \mathcal{V}_s \), for some \( s < t \). So in either case \( \rho(P, Q) \neq t \), and we conclude that \( t \) is not in the range of \( \rho \). This proves one inclusion.

Finally, let \( t \in R \) and let \( n \in \mathbb{N} \). Then \( \mathcal{V}_t \subseteq \mathcal{V}_{t+1/n} \), so by Lemma 1.7 we can find projections \( P_n, Q_n \in \mathcal{M} \otimes \mathcal{B}(l^2) \) such that \( P_n(A \otimes I)Q_n \neq 0 \) for some \( A \in \mathcal{V}_t \) but \( P_n(B \otimes I)Q_n = 0 \) for all \( B \in \mathcal{V}_t \). This implies that \( t \leq \rho(P_n, Q_n) \leq t + 1/n \). Taking countable direct sums, we get \( \bigoplus P_n, \bigoplus Q_n \in \bigoplus \mathcal{M} \otimes \mathcal{B}(l^2) \cong \mathcal{M} \otimes \bigoplus \mathcal{B}(l^2) \subseteq \mathcal{M} \otimes \mathcal{B}(l^2) \) and

\[
\rho \left( \bigoplus P_n, \bigoplus Q_n \right) = \inf \{ \rho(P_n, Q_n) \} = t.
\]

This shows that \( R \) is contained in the range of \( \rho \). \( \square \)

### 3.2. Graph metrics.

Let \( \mathcal{M} \subseteq \mathcal{B}(H) \) be a von Neumann algebra and let \( \mathcal{V} \) be a dual operator system that is a bimodule over \( \mathcal{M} \). (Thus \( \mathcal{V} \) is a *quantum graph* according to Definition 2.6 (d) of [35]. For \( \mathcal{M} \) a matrix algebra this definition appeared in [10].) Then set \( \mathcal{V}_t = \overline{\mathcal{V}}^{w^*} \cdot \cdots \cdot \overline{\mathcal{V}}^{w^*} \), the weak* closure of the algebraic product taken \( t \) times, where \( \lceil t \rceil \) is the greatest integer \( \leq t \) and with the convention that the empty product is \( \mathcal{M} \). We call \( \mathcal{V} = \{ \mathcal{V}_t \} \) the *quantum graph metric* associated to \( \mathcal{V} \). We are most interested in the case where \( \mathcal{M} = \mathcal{B}(H) \), \( \mathcal{V}_0 = \mathcal{C}I \), and \( \mathcal{V} \) is any dual operator system in \( \mathcal{B}(H) \).

**Proposition 3.4.** Let \( \mathcal{M} \subseteq \mathcal{B}(H) \) be a von Neumann algebra and let \( \mathcal{V} \) be a dual operator system that is a bimodule over \( \mathcal{M} \). Then the quantum graph metric is the smallest quantum metric \( \mathcal{V} \) on \( \mathcal{M} \) such that \( \mathcal{V}_t = \mathcal{V} \).

**Proposition 3.5.** Let \( X \) be a set and let \( \mathcal{M} \cong l^\infty(X) \) be the von Neumann algebra of bounded multiplication operators on \( l^2(X) \). Also let \( R \) be a reflexive, symmetric relation on \( X \), let \( \mathcal{V} = \mathcal{V}_R \) (Proposition 1.4), and let \( \mathcal{V} \) be the quantum graph metric on \( \mathcal{M} \) associated to \( \mathcal{V} \). Then \( d\mathcal{V} \) (Proposition 2.3) is the graph metric associated to \( R \).

**Proof.** \( R \) defines a graph with vertex set \( X \) in the obvious way. Now\[
\mathcal{V} = \overline{\text{span}}^{w^*} \{ V_{xy} : (x, y) \in R \}
\]
and
\[
\mathcal{V}^n = \overline{\text{span}}^{w^*} \{ V_{xy} : (x, y) \in R^n \} = \overline{\text{span}}^{w^*} \{ V_{xy} : \text{there is a path of length } \leq n \text{ from } x \text{ to } y \}.
\]
Thus $d_{\mathcal{V}}(x, y)$ is the length of the shortest path from $x$ to $y$ (or $\infty$ if there is no such path), as desired.

3.3. Quantum metrics on $M_2(\mathbb{C})$. We analyze the possible quantum metrics on $\mathcal{M} = M_2(\mathbb{C}) = B(\mathbb{C}^2)$. Let $\sigma_x$, $\sigma_y$, and $\sigma_z$ be the Pauli spin matrices

$$
\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.
$$

The only one dimensional operator system in $M_2(\mathbb{C})$ is $\mathcal{C}I$. Any two dimensional operator system $\mathcal{V}$ in $M_2(\mathbb{C})$ must contain $I$ and some non-scalar self-adjoint matrix $A$ and hence must equal $\mathcal{C}I + \mathcal{C}A$. We can then choose an orthonormal basis $\{f_1, f_2\}$ of $\mathbb{C}^2$ so that $A$ is diagonalized; then $\mathcal{V}$ will consist of all diagonal matrices, $\mathcal{V} = \mathcal{C}I + \mathcal{C}\sigma_x$. Now let $\mathcal{W}$ be a three dimensional operator system. It contains a two dimensional operator system, without loss of generality (by the preceding case) the diagonal matrices. It also contains a non-diagonal self-adjoint operator $B$, without loss of generality zero on the diagonal (since we can subtract off its diagonal part), and then by replacing $f_2$ with $\alpha f_2$ for a suitable complex number $\alpha$ of modulus 1 we can take $B$ to be a real scalar multiple of $\sigma_y$. So we have $\mathcal{W} = \mathcal{C}I + \mathcal{C}\sigma_x + \mathcal{C}\sigma_y$. Finally, the only four dimensional operator system in $M_2(\mathbb{C})$ is $M_2(\mathbb{C}) = \mathcal{C}I + \mathcal{C}\sigma_x + \mathcal{C}\sigma_y + \mathcal{C}\sigma_z$ itself. We can now characterize all quantum metrics on $M_2(\mathbb{C})$.

**Proposition 3.6.** Let $0 \leq a \leq b \leq c \leq \infty$ satisfy $c \leq a + b$ and let $\mathcal{V} = \{V_t\}$ where

$$
V_t = \begin{cases} 
\mathcal{C}I & \text{if } 0 \leq t < a \\
\mathcal{C}I + \mathcal{C}\sigma_x & \text{if } a \leq t < b \\
\mathcal{C}I + \mathcal{C}\sigma_x + \mathcal{C}\sigma_y & \text{if } b \leq t < c \\
\mathcal{C}I + \mathcal{C}\sigma_x + \mathcal{C}\sigma_y + \mathcal{C}\sigma_z & \text{if } t \geq c.
\end{cases}
$$

Then $\mathcal{V}$ is a quantum pseudometric on $M_2(\mathbb{C})$. It is a quantum metric if and only if $a > 0$ and it is reflexive if and only if $b = c$. Up to a change of orthonormal basis every quantum pseudometric on $M_2(\mathbb{C})$ is of this form.

**Proof.** It is elementary to check that $\mathcal{V}$ is a quantum pseudometric, that it is a quantum metric if and only if $a > 0$, and that $\mathcal{C}I$ and $\mathcal{C}I + \mathcal{C}\sigma_x$ are reflexive but $\mathcal{C}I + \mathcal{C}\sigma_x + \mathcal{C}\sigma_y$ is not, so that $\mathcal{V}$ is reflexive if and only if $b = c$. Now let $\mathcal{W}$ be any quantum pseudometric on $M_2(\mathbb{C})$. Then the discussion before the proposition shows that after a change of basis $\mathcal{W}$ must have the given form for some $0 \leq a \leq b \leq c \leq \infty$. Furthermore, we must have $c \leq a + b$ because

$$
\mathcal{V}_a \mathcal{V}_b = (\mathcal{C}I + \mathcal{C}\sigma_x)(\mathcal{C}I + \mathcal{C}\sigma_x + \mathcal{C}\sigma_y) \subseteq \mathcal{V}_{a+b}
$$

and $\sigma_x \sigma_y = \sigma_z$, so that $V_c \subseteq \mathcal{V}_{a+b}$. "

3.4. Quantum Hamming distance. Fix a natural number $n$ and let $H = \mathbb{C}^{2^n} \cong \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. If $\{e_0, e_1\}$ is the standard orthonormal basis of $\mathbb{C}^2$ then

$$
\{e_i \otimes \cdots \otimes e_{i_n} : \text{each } i_k = 0 \text{ or } 1\}
$$

is an orthonormal basis for $H$. These basis vectors correspond to binary strings of length $n$. Thus the information represented by such a string can be encoded in an appropriate physical system as the state modelled by the corresponding basis vector. For example, a single photon has two basis polarization states, so a binary string of length $n$ could be encoded as the polarization of an array of $n$ photons. The
basic difference with classical information is the physical possibility of superposing base states. Thus $C^2$ models not a single bit of information, but rather a single “quantum bit” or “qubit”.

In quantum error correction one is concerned with the possibility that information encoded in this way could be corrupted. The quantum Hamming metric measures the number of errors that might be introduced. Recall that $[t]$ denotes the greatest integer $\leq t$.

\textbf{Definition 3.7.}  The quantum Hamming metric on $M_{2^n}(C)$ is the quantum metric $V_{\text{Ham}} = \{V_{\text{Ham}}^t\}$ where

$$V_{\text{Ham}}^t = \text{span}\{A_1 \otimes \cdots \otimes A_n : each \ A_i \in M_2(C) \\
\text{and} \ A_i = I_2 \text{ for all but at most } [t] \text{ values of } i\}.$$  

The quantum Hamming metric is a quantum graph metric (Section 3.2). It is also the $n$-fold $l_1$ product (see the comment following Proposition 2.42) with itself of the quantum metric $V_t = \{C I \text{ if } 0 \leq t < 1 \text{ and } M_2(C) \text{ if } t \geq 1\}$ on $M_2(C)$. The rough idea is that operators in $V_{\text{Ham}}^t$ can corrupt at most $[t]$ qubits of an $n$-qubit string. In more detail, if a basis state $e_{i_1} \otimes \cdots \otimes e_{i_n}$ is acted on by an operator in $V_{\text{Ham}}^t$ and the resulting vector is subjected to a measurement that projects it into a basis state, then the final state will differ from the initial state in at most $[t]$ factors.

A quantum code is a subspace $C$ of $H$. It corrects up to $k$ errors if $PAP$ is a scalar multiple of $P$ for all $A \in V_{\text{Ham}}^k$, where $P$ is the projection onto $C$. Equivalently, $PAv$ is a scalar multiple of $v$ for all $v \in C$. If the scalar is nonzero, this means that any state in $C$ can be recovered by projecting onto $C$.

The volume bound is a simple upper bound on the dimension of a quantum code that corrects up to $k$ errors. For such a code write $PAP = \varepsilon(A)P$ for all $A \in V_{\text{Ham}}^k$. Then

$$\langle A, B \rangle = \varepsilon(B^*A)$$

is a positive semidefinite sesquilinear form on $V_{\text{Ham}}^k$, and if $K$ is the Hilbert space formed by factoring out null vectors then we have an embedding of $K \otimes C$ into $H$ defined by

$$A \otimes v \mapsto Av.$$  

Thus

$$\dim(C) \leq \dim(H)/\dim(K).$$

We generalize the above to arbitrary quantum metrics. One obvious application would be to $M = \bigoplus_{i=1}^k M_{2^n_i}(C) \subseteq M_{2^n_1 + \cdots + 2^n_k}(C)$ equipped with the quantum metric defined by letting $V_i$ be the span of the operators

$$(B_1^i \otimes \cdots \otimes B_{n_1}^1) \oplus \cdots \oplus (B_1^k \otimes \cdots \otimes B_{n_k}^k)$$

such that $B_j^i = I_2$ for all but at most $[t]$ values of $i$ and $j$. This generalizes the quantum Hamming metric to a mixed classical/quantum system in which information is encoded in $k$ disentangled quantum packets.

However, we can actually state a version of the volume bound for arbitrary quantum metrics. A classical code that corrects up to $k$ errors is a subset of the
space of all strings any two distinct elements of which are more than 2k units apart; we need a general version of this condition. First we state it in its most useful form, and then we prove that our definition is equivalent to two other possibly more intuitive characterizations.

**Definition 3.8.** Let $\mathbf{V}$ be a quantum metric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and let $P$ be a projection in $\mathcal{M}$.

(a) The **minimum distance** in $P$ is the value

$$\delta(P) = \sup \{ t \geq 0 : \mathcal{P} \mathcal{V}_t P = PM\mathcal{V}_0 P \}.$$  

(b) The **induced quantum pseudometric** on $PM\mathcal{P}$ is the smallest quantum pseudometric $\mathbf{V} = \{ \mathcal{V}_t \}$ on $PM\mathcal{P} \subseteq \mathcal{B}(K)$, where $K = \mathcal{R}(P)$, such that $PM\mathcal{V}_t P \subseteq \mathcal{V}_t$ for all $t$.

(The definition of an induced quantum pseudometric generalizes our definition of metric quotients in Definition 2.35.)

**Proposition 3.9.** Let $\mathbf{V}$ be a quantum metric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and let $P$ be a projection in $\mathcal{M}$. Suppose $\delta(P) > 0$. Then

$$\delta(P) = \inf \{ \rho(P_1, P_2) : P_1, P_2 \leq P \otimes I \text{ and } \rho(P_1, P_2) > 0 \}$$

where $\mathbf{V}$ is the induced pseudometric on $PM\mathcal{P}$ and $P_1$ and $P_2$ range over projections in $\mathcal{M} \otimes \mathcal{B}(l^2)$.

**Proof.** Let $K = \mathcal{R}(P)$. Observe first that $\mathcal{V}_0 = \mathcal{M}'$ implies $PM\mathcal{V}_0 P = PM\mathcal{M}'$, and that $PM\mathcal{M}' = (PM\mathcal{P})'$ where the commutant on the right is taken in $\mathcal{B}(K)$.

(The inclusion $\subseteq$ is easy and the inclusion $\supseteq$ follows from the double commutant theorem.)

Now

$$\mathbf{V}_t = \begin{cases} 
\mathbf{V}_0 P & \text{if } 0 \leq t < \delta(P) \\
\mathcal{B}(K) & \text{if } t \geq \delta(P)
\end{cases}$$

is a quantum pseudometric on $PM\mathcal{P}$ with $PM\mathcal{V}_t P = PM\mathcal{V}_0 P \subseteq \mathbf{V}_t$ for all $t < \delta(P)$, which shows that $\mathbf{V}_t \subseteq \mathbf{V}_t \mathcal{K}$ for all $t$ and hence that $\mathbf{V}_t = PM\mathcal{V}_t P = \mathcal{V}_t$ for all $t < \delta(P)$. Conversely, if $t > \delta(P)$ then

$$\mathbf{V}_0 = PM\mathcal{V}_0 P \subseteq PM\mathcal{V}_1 P \subseteq \mathbf{V}_t,$$

so $\mathbf{V}_0 \neq \mathbf{V}_t$. This proves the first equality.

For the second equality first let $P_1$ and $P_2$ be any two projections in $\mathcal{M} \otimes \mathcal{B}(l^2)$ with $P_1, P_2 \leq P \otimes I$ and suppose $\rho(P_1, P_2) < \delta(P)$. Then there exists $A \in \mathcal{B}(H)$ such that $P_1(A \otimes I)P_2 \neq 0$ and $\mathcal{D}(A) < \delta(P)$. But then $PAP = PBP$ for some $B \in \mathcal{V}_0$ and we have

$$P_1(B \otimes I)P_2 = P_1(PBP \otimes I)P_2 = P_1(PAP \otimes I)P_2 = P_1(A \otimes I)P_2 \neq 0$$

so that $\rho(P_1, P_2) = 0$. Conversely, if $s < \delta(P) < t$ then $PM\mathcal{V}_s P = PM\mathcal{V}_t P$ and so by Lemma 1.7 there exist $P_1, P_2 \in PM\mathcal{B}(l^2)$ satisfying $P_1(PBP \otimes I)P_2 \neq 0$ for some $A \in \mathcal{V}_s$ but $P_1(BBP \otimes I)P_2 = 0$ for all $B \in \mathcal{V}_s$. Regarding $P_1$ and $P_2$ as elements of $\mathcal{M} \otimes \mathcal{B}(l^2)$, we then have $P_1, P_2 \leq P \otimes I$ and $P_1(A \otimes I)P_2 \neq 0$ for some $A \in \mathcal{V}_s$ but $P_1(B \otimes I)P_2 = 0$ for all $B \in \mathcal{V}_s$, so that $s \leq \rho(P_1, P_2) \leq t$. This suffices to prove the second equality.
The condition that $\delta(P) > 0$ in Proposition 3.9 is necessary. Even if $P V_0 P \neq P V_1 P$ for all $t > 0$ we could still have $V_0 = V_1$ for some $t > 0$ because the condition $V_0 = \bigcap_{t>0} V_1$ could force $P V_0 P \subsetneq V_0$.

Now let $V$ be a quantum metric on a von Neumann algebra $\mathcal{M}$ and let $P$ be a projection in $\mathcal{M}$ with minimum distance $\delta(P) > 0$. We have a positive semidefinite sesquilinear form on $V_{\delta(P)/2}$ with values in $P V_0 P = P \mathcal{M}' = (P \mathcal{M} P)'$ defined by

$$\langle A, B \rangle = PB^* A P$$

for $A, B \in V_{\delta(P)/2}$. Let $E$ be the vector space formed by factoring out null vectors. Now $V_{\delta(P)/2}$ is a right $\mathcal{M}'$-module, and this descends to a right action of $P \mathcal{M}'$ on $E$. So $E$ is a right pre-Hilbert $P \mathcal{M}'$-module. Letting $K = \text{ran}(P)$, we can then define an inner product on $E \otimes_{P \mathcal{M}'} K$ by

$$\langle A \otimes v, B \otimes w \rangle = \langle \langle A, B \rangle v, w \rangle.$$

Let $E_{\otimes_{P \mathcal{M}'} K}$ denote the completion of $E \otimes_{P \mathcal{M}'} K$ for this inner product. The following result gives a general version of the volume bound.

**Theorem 3.10.** Let $V$ be a quantum metric on a von Neumann algebra $\mathcal{M}$, let $P$ be a projection in $\mathcal{M}$ with minimum distance $\delta(P) > 0$, let $E$ be the pre-Hilbert $P \mathcal{M}'$-module formed from $V_{\delta(P)/2}$ as above, and let $K = \text{ran}(P)$. Then the map

$$\tilde{A} \otimes v \mapsto A v$$

extends to an isometric isomorphism of $E_{\otimes_{P \mathcal{M}'} K}$ onto $\text{ran}((P)_{\delta(P)/2}) \subseteq H$, where $(P)_{\delta(P)/2}$ is the open $\delta(P)/2$-neighborhood of $P$ (Definition 2.14 (b)).

**Proof.** The essential computation is

$$\langle \tilde{A} \otimes v, \tilde{B} \otimes w \rangle = \langle \langle A, B \rangle v, w \rangle = \langle PB^* A P v, w \rangle = \langle Av, B w \rangle$$

for $A, B \in V_{\delta(P)/2}$ and $v, w \in \text{ran}(P)$. Taking linear combinations shows that the map $\tilde{A} \otimes v \mapsto A v$ is well-defined and isometric on the uncompleted version of the construction, and taking completions then yields an isometry from the completed tensor product onto $V_{\delta(P)/2} \overline{\text{ve}}(\text{ran}(P)) = \text{ran}((P)_{\delta(P)/2})$. $\square$

3.5. Quantum tori. We formulate a notion of translation invariant quantum pseudometrics on quantum tori. Using the analysis of translation invariant quantum relations on quantum tori developed in Section 2.7 of [35], it is then straightforward to deduce strong structural information about translation invariant quantum pseudometrics.

Quantum tori are the simplest examples of noncommutative manifolds. They are related to the quantum plane, which plays the role of the phase space of a spinless one-dimensional particle. The classical version of such a system has phase space $\mathbb{R}^2$, with the point $(q, p) \in \mathbb{R}^2$ representing a state with position $q$ and momentum $p$, so that the position and momentum observables are just the coordinate functions on phase space. When such a system is quantized the position and momentum observables are modelled by unbounded self-adjoint operators $Q$ and $P$ satisfying $QP - PQ = i\hbar I$. Polynomials in $Q$ and $P$ can then be seen as a quantum analog of polynomial functions on $\mathbb{R}^2$. The quantum analog of the continuous functions on the torus — equivalently, the $(2\pi, 2\pi)$-periodic continuous functions on the plane — is the C*-algebra generated by the unitary operators $e^{iQ}$ and $e^{iP}$, which satisfy...
the commutation relation $e^{itQ}e^{iP} = e^{-it}e^{iP}e^{iQ}$. For more background see [21] or Sections 4.1, 4.2, 5.5, and 6.6 of [54].

Let $T = \mathbb{R}/2\pi\mathbb{Z}$ and fix $h \in \mathbb{R}$. Let $\{e_{m,n}\}$ be the standard basis of $l^2(\mathbb{Z}^2)$. We model the quantum tori on $l^2(\mathbb{Z}^2)$ as follows.

**Definition 3.11.** Let $U_h$ and $V_h$ be the unitaries in $\mathcal{B}(l^2(\mathbb{Z}^2))$ defined by

\[
U_h e_{m,n} = e^{-imn/2}e_{m+1,n}, \\
V_h e_{m,n} = e^{imn/2}e_{m,n+1}.
\]

The quantum torus von Neumann algebra for the given value of $h$ is the von Neumann algebra $W^*(U_h, V_h)$ generated by $U_h$ and $V_h$.

The commutant of $W^*(U_h, V_h)$ is $W^*(U_{-h}, V_{-h})$ ([35], Corollary 2.38).

If $h$ is an irrational multiple of $\pi$ then $W^*(U_h, V_h)$ is a hyperfinite $II_1$ factor. We will not need this fact.

Conjugating $U_h$ and $V_h$ by the Fourier transform $\mathcal{F}: L^2(T^2) \rightarrow l^2(\mathbb{Z}^2)$ yields the operators

\[
\hat{U}_h f(x, y) = e^{ix}f(x, y - \frac{h}{2}) \\
\hat{V}_h f(x, y) = e^{iy}f(x + \frac{h}{2}, y)
\]

on $L^2(T^2)$, with $W^*(\hat{U}_h, \hat{V}_h)$ reducing to the algebra of bounded multiplication operators when $h = 0$. However, for our purposes the $l^2(\mathbb{Z}^2)$ picture is more convenient.

**Definition 3.12.** Let $A \in \mathcal{B}(l^2(\mathbb{Z}^2))$.

(a) For $x, y \in T$ define

\[
\theta_{x,y}(A) = M_{e^{i(mx+ny)}}AM_{e^{-i(mx+ny)}}.
\]

(b) For $k, l \in \mathbb{Z}$ define

\[
A_{k,l} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-ikx+ily}\theta_{x,y}(A) \, dx \, dy.
\]

We call $A_{k,l}$ the $(k, l)$ Fourier term of $A$.

(c) For $k, l \in \mathbb{N}$ define

\[
S_{k,l}(A) = \sum_{|k'| \leq k, |l'| \leq l} A_{k',l'}
\]

and for $N \in \mathbb{N}$ define

\[
\sigma_N(A) = \frac{1}{N^2} \sum_{0 \leq k, l \leq N-1} S_{k,l}(A).
\]

In the $L^2(T^2)$ picture the operator $M_{e^{i(mx+ny)}}$ on $l^2(\mathbb{Z}^2)$ becomes translation by $(-x, -y)$, so that $\theta_{x,y}$ is conjugation by a translation.

The integral used to define $A_{k,l}$ can be understood in a weak sense: for any vectors $v, w \in l^2(\mathbb{Z}^2)$ we take $(A_{k,l}w, v)$ to be $\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-ikx+ily}(\theta_{x,y}(A)w, v) \, dx \, dy$. In particular, if $w = e_{m,n}$ and $v = e_{m',n'}$ then we have

\[
(A_{k,l}e_{m,n}, e_{m',n'}) = \begin{cases} 
(A_{m+k,n}, e_{m',n'}) & \text{if } m' = m + k \text{ and } n' = n + l \\
0 & \text{otherwise}. 
\end{cases} \tag{\ast}
\]
The \( A_{k,l} \) are something like Fourier coefficients, the \( S_{k,l}(A) \) like partial sums of a Fourier series, and the \( \sigma_N(A) \) like Cesàro means.

**Definition 3.13.** Let \( \mathcal{M} \equiv l^\infty(\mathbb{Z}^2) \) be the von Neumann algebra of bounded multiplication operators in \( \mathcal{B}(l^2(\mathbb{Z}^2)) \).

(a) For any weak* closed subspace \( \mathcal{E} \) of \( \mathcal{M} \) that is invariant under the natural action of \( \mathbb{Z}^2 \), define \( \mathcal{V}_\mathcal{E} \) by

\[
\mathcal{V}_\mathcal{E} = \{ A \in \mathcal{B}(l^2(\mathbb{Z}^2)) : A_{k,l} \in \mathcal{E} \cdot U^{-k} U^{-l} V \text{ for all } k, l \in \mathbb{Z} \}.
\]

(b) For any closed subset \( S \subseteq T^2 \) define

\[
\mathcal{E}_0(S) = \{ M_f : f \in l^\infty(\mathbb{Z}^2) \text{ and } \sum f \hat{g} = 0 \text{ for all } g \in l^1(\mathbb{Z}^2) \text{ such that } \hat{g}|_S = 0 \} = \text{span}^{\text{wk}} \{ M_{s,n} : (x, y) \in S \}
\]

and

\[
\mathcal{E}_1(S) = \{ M_f : f \in l^\infty(\mathbb{Z}^2) \text{ and } \sum f \hat{g} = 0 \text{ for all } g \in l^1(\mathbb{Z}^2) \text{ such that } \hat{g}|_T = 0 \text{ for some neighborhood } T \text{ of } S \}
\]

where \( \mathcal{E}_0(S) \) is the open \( \epsilon \)-neighborhood of \( S \).

In part (a), \( \mathcal{V}_\mathcal{E} \) is a quantum relation on \( W^*(U_\hbar, V_\hbar) \) satisfying \( \theta_{x,y}(\mathcal{V}_\mathcal{E}) = \mathcal{V}_\mathcal{E} \) for all \( x, y \in T \) by (35), Theorem 2.41. This suggests the following definition.

**Definition 3.14.** A quantum pseudometric \( \mathbf{V} = \{ \mathcal{V}_t \} \) on \( W^*(U_\hbar, V_\hbar) \) is translation invariant if \( \theta_{x,y}(\mathcal{V}_t) = \mathcal{V}_t \) for all \( x, y \in T \) and all \( t \in [0, \infty) \).

We can characterize translation invariance intrinsically as follows.

**Proposition 3.15.** Let \( \mathbf{V} \) be a quantum pseudometric on \( W^*(U_\hbar, V_\hbar) \) and let \( \rho \) be the associated quantum distance function (Definition 2.9). Then \( \mathbf{V} \) is translation invariant if and only if

\[
\rho(P, Q) = \rho((\theta_{x,y} \otimes 1)(P), (\theta_{x,y} \otimes 1)(Q))
\]

for all \( x, y \in T \) and all projections \( P, Q \in W^*(U_\hbar, V_\hbar) \otimes \mathcal{B}(l^2) \).

**Proof:** The forward implication follows from (35), Proposition 2.40) and the fact that

\[
\rho(P, Q) = \inf \{ t : (P, Q) \in \mathcal{R}_t \}
\]

where \( \mathcal{R}_t \) is the intrinsic quantum relation corresponding to \( \mathcal{V}_t \) (Theorem 1.9). For the converse, suppose \( \mathbf{V} \) is not translation invariant and find \( t \in [0, \infty) \), \( x, y \in T \), and \( A \in \mathcal{V}_t \) such that \( \theta_{x,y}(A) \notin \mathcal{V}_t \). Then \( \theta_{x,y}(A) \notin \mathcal{V}_s \) for some \( s > t \), and by Lemma 1.7 we can find projections \( P, Q \in W^*(U_\hbar, V_\hbar) \otimes \mathcal{B}(l^2) \) such that \( P((\theta_{x,y}(A) \otimes 1)Q) \neq 0 \) but \( P((B \otimes 1)Q) = 0 \) for all \( B \in \mathcal{V}_s \). Then \( (P, Q) \notin \mathcal{R}_s \) but \((\theta_{x,-y} \otimes 1)(P), (\theta_{x,-y} \otimes 1)(Q)) \in \mathcal{R}_t \) because

\[
(\theta_{x,-y} \otimes 1)(P)(A \otimes 1)(\theta_{x,-y} \otimes 1)(Q) = (\theta_{x,-y} \otimes 1)(P(\theta_{x,y}(A) \otimes 1)Q) \neq 0.
\]

Thus

\[
\rho((\theta_{x,-y} \otimes 1)(P), (\theta_{x,-y} \otimes 1)(Q)) \leq t < s \leq \rho(P, Q).
\]

This proves the reverse implication. \( \square \)
Say that a pseudometric $d$ on $\mathbb{T}^2$ is closed if

$$\mathcal{N}_\epsilon(x, y) = \{(x', y') \in \mathbb{T}^2 : d((x, y), (x', y')) \leq \epsilon\}$$

is closed in the usual topology, for every $(x, y) \in \mathbb{T}^2$ and every $\epsilon > 0$. The following basic structural result for translation invariant quantum pseudometrics on quantum tori follows immediately from Theorem 2.41 and Corollary 2.42 of [35].

**Theorem 3.16.** Let $\mathcal{M} \cong l^\infty(\mathbb{Z}^2)$ be the von Neumann algebra of bounded multiplication operators in $\mathcal{B}(l^2(\mathbb{Z}^2))$.

(a) If $\mathcal{V}_0 = \{\mathcal{V}^0_t\}$ is a translation invariant quantum pseudometric on $W^*(U_0, V_0) \cong L^\infty(\mathbb{T}^2)$ then $\mathcal{V}_h = \{\mathcal{V}^h_t\}$ is a translation invariant quantum pseudometric on $W^*(U_h, V_h)$, where $\mathcal{V}^h_t = \mathcal{V}_{t_1}$ with $t_1 = t_0 \cap \mathcal{M}$. Every translation invariant quantum pseudometric on $W^*(U_h, V_h)$ is of this form.

(b) Let $\mathcal{V} = \{\mathcal{V}_t\}$ be a translation invariant quantum pseudometric on $W^*(U_h, V_h)$. Then

$$d((x, y), (x', y')) = \inf\{t : M_{\mathcal{H}_{\mathcal{V}_0}((x-x')^2 + (y-y')^2)} \in \mathcal{V}_t\}$$

is a closed translation invariant pseudometric on $\mathbb{T}^2$ and $\mathcal{V}_0 = \{\mathcal{V}^0_t\}$ and $\mathcal{V}_1 = \{\mathcal{V}^1_t\}$ are translation invariant quantum pseudometrics on $W^*(U_h, V_h)$ where

$$\mathcal{V}_t^0 = \mathcal{V}_{\mathcal{E}_0(S_t)}$$

$$\mathcal{V}_t^1 = \mathcal{V}_{\mathcal{E}_1(S_t)}$$

with $S_t = \{(x, y) \in \mathbb{T}^2 : d((0, 0), (x, y)) \leq t\}$. We have $\mathcal{V}_0 \leq \mathcal{V} \leq \mathcal{V}_1$.

We would like to say that the $W^*$-filtration $\mathcal{V}_h$ in Theorem 3.16 (a) converges to the $W^*$-filtration $\mathcal{V}_r$ as $h \to r$. It is easy to see that convergence does not occur in the Gromov-Hausdorff sense (see the comment following Definition 2.14). The right notion of convergence seems to be the following. Denote the closed unit ball of any Banach space $\mathcal{V}$ by $[\mathcal{V}]_1$.

**Definition 3.17.** Let $\{\mathcal{V}_\lambda\}$ be a net of $W^*$-filtrations of $\mathcal{B}(H)$. We say that $\{\mathcal{V}_\lambda\}$ locally converges to a $W^*$-filtration $\mathcal{V}$ of $\mathcal{B}(H)$ if for every $0 \leq s < t$ and every weak* open neighborhood $U$ of $0 \in \mathcal{B}(H)$ we eventually have

$$[\mathcal{V}_\lambda]_1 \subseteq [\mathcal{V}]_1 + U \quad \text{and} \quad [\mathcal{V}_\lambda]_1 \subseteq [\mathcal{V}]_1 + U.$$

Equivalently, for any $\epsilon > 0$ and any vectors $v_1, \ldots, v_n, w_1, \ldots, w_n \in H$ the sets $[\mathcal{V}_s]_1$ and $[\mathcal{V}_\lambda]_1$ are, respectively, eventually within the $\epsilon$-neighborhoods of the sets $[\mathcal{V}]_1$ and $[\mathcal{V}]_1$ for the seminorm

$$\|A\| = \sum |\langle Aw_i, v_i \rangle|.$$ 

Indeed, it would be enough to show this with the $v_i$ and $w_i$ ranging over a spanning subset of $H$. The next result is an easy consequence of this characterization.

**Proposition 3.18.** Let $\{d_\lambda\}$ be a net of pseudometrics on a set $X$. Then the corresponding quantum pseudometrics $\mathcal{V}_\lambda$ (Proposition 2.9) locally converge to the quantum pseudometric $\mathcal{V}$ corresponding to a pseudometric $d$ on $X$ if and only if $d_\lambda(x, y) \to d(x, y)$ for all $x, y \in X$.

**Proof.** Suppose $d_\lambda(x, y) \not\to d(x, y)$ for some $x, y \in X$. Then either $\limsup d_\lambda(x, y) > d(x, y)$ or $\liminf d_\lambda(x, y) < d(x, y)$. Suppose the former and let $s = d(x, y)$ and $s < t < \limsup d_\lambda(x, y)$. Also let $v_1 = e_x$ and $w_1 = e_y$. Then the rank one operator $V_{xy}$ belongs to $[\mathcal{V}_s]_1$, but for any $\lambda$ with $d_\lambda(x, y) > t$ we have $\langle A e_y, e_x \rangle = 0$ for all
\[ a \in \mathcal{V}_x^A; \text{this shows that } V_{xy} \text{ is frequently not approximated by operators in } \mathcal{V}_x^A \text{ for}
\]
the seminorm \(|(Ax_1, v_1)|\). So \( \mathcal{V}_x \) does not locally converge to \( \mathcal{V} \). In the second case (\( \liminf d(x, y) < d(x, y) \)) the same proof works, now interchanging the roles of \( \mathcal{V} \) and \( \mathcal{V}_x \).

Conversely, suppose \( d(x, y) \to d(x, y) \) for all \( x, y \in X \). We verify the condition stated just before the proposition with the vectors \( v_i \) and \( w_i \) ranging over the standard basis \( \{e_x\} \) of \( l^2(X) \). So let \( v_1, \ldots, v_n, w_1, \ldots, w_n \) be basis vectors and find a finite set \( S \subseteq X \) on which they are all supported. Fix \( 0 \leq s < t \). Then eventually we have \( d(x, y) \leq t \) for all \( x, y \in S \) with \( d(x, y) \leq s \), so that if \( A \in [\mathcal{V}_S]_1 \) then \( M_{x_s} AM_{x_s} \in [\mathcal{V}_S^A]_1 \), and \( \|A - M_{x_s} AM_{x_s}\| = 0 \) for the seminorm \( \| \cdot \| \). A similar argument shows that \( [\mathcal{V}_S]_1 \) is eventually within the \( \epsilon \)-neighborhood of \( [\mathcal{V}_S]_1 \) for the seminorm \( \| \cdot \| \), for all \( \epsilon > 0 \). We conclude that \( \mathcal{V}_x \) locally converges to \( \mathcal{V} \). \( \Box \)

Now we show that translation invariant quantum pseudometrics on quantum torus von Neumann algebras converge as the parameter \( h \) varies.

**Theorem 3.19.** Let \( \mathcal{V}_0 \) be a translation invariant quantum pseudometric on \( W^*(U_0, V_0) \) and for each \( h \in \mathbb{R} \) let \( \mathcal{V}_h \) be the corresponding translation invariant quantum pseudometric on \( W^*(U_h, V_h) \) (Theorem 3.16 (a)). Let \( r \in \mathbb{R} \). Then \( \mathcal{V}_h \) locally converges to \( \mathcal{V}_r \) as \( h \to r \).

**Proof.** We use the alternative characterization of local convergence given following Definition 3.17 with the \( v_i \) and \( w_i \) ranging over the standard basis \( \{e_m, n\} \) of \( l^2(\mathbb{Z}^2) \). We will show that \( [\mathcal{V}_r]_1 \) is eventually within the \( \epsilon \)-neighborhood of \( [\mathcal{V}_S]_1 \) for the seminorm \( \| \cdot \| \); the corresponding assertion with \( r \) and \( h \) switched is proven similarly.

Let \( v_1, \ldots, v_n, w_1, \ldots, w_n \) be basis vectors and find a finite subset \( S \subseteq \mathbb{Z}^2 \) on which they are all supported. Then the inner products \( \langle U_h^k V_h^l, w_i, v_i \rangle \) will converge to the inner products \( \langle U_r^k V_r^l, w_i, v_i \rangle \) as \( h \to r \), uniformly in \( k \) and \( l \) since these inner products are zero for sufficiently large \( k \) and \( l \).

Next let \( \epsilon > 0 \) and observe that the inner products \( \langle \sigma_N(A) w_i, v_i \rangle \) converge to the inner products \( \langle Aw_i, v_i \rangle \) as \( N \to \infty \), uniformly in \( A \in [\mathcal{V}_S]_1 \). Also, \( A \in [\mathcal{V}_S]_1 \) implies \( \sigma_N(A) \in [\mathcal{V}_S]_1 \) by (33), Proposition 2.35. By the preceding observation, if we write \( \sigma_N^h(A) = \sum a_{k,l} U_h^k V_h^l - A \) where \( \sigma_N(A) = \sum a_{k,l} U_r^k V_r^l \), then we will have

\[ \|\sigma_N^h(A) - A\| \leq \|\sigma_N(A) - A\| + \|\sigma_N(A) - A\| \]

with the first term on the right going to zero as \( h \to r \) and the second going to zero as \( N \to \infty \), both uniformly in \( A \in [\mathcal{V}_S]_1 \). We just need \( \limsup \|\sigma_N^h(A)\| \leq 1 \), uniformly in \( A \in [\mathcal{V}_S]_1 \). This follows from the fact that the \( C^* \)-algebras generated by \( U_h \) and \( V_h \) form a continuous field \( [20] \), so that the norms of polynomials in \( U_h \) and \( V_h \) vary continuously in \( h \). \( \Box \)

**3.6. Hölder metrics.** Let \( \mathcal{V} = \{\mathcal{V}_t\} \) be any quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \) and let \( 0 < \alpha < 1 \). Then \( \mathcal{V}_t^\alpha = \{\mathcal{V}_t^{\alpha} \} \) where \( \mathcal{V}_t^{\alpha} = \mathcal{V}_{t^{\alpha}}^{\alpha} \) is also a quantum pseudometric on \( \mathcal{M} \); the filtration condition follows from the inequality \( s^{1/\alpha} + t^{1/\alpha} \leq (s + t)^{1/\alpha} \) for \( s, t \geq 0 \). We call \( \mathcal{V}_t^\alpha \) a Hölder or snowflake quantum pseudometric. It is easy to see that this construction reduces to the usual one in the atomic abelian case:

**Proposition 3.20.** Let \( X \) be a set and let \( \mathcal{M} \cong l^\infty(X) \) be the von Neumann algebra of bounded multiplication operators on \( l^2(X) \). If \( d \) is a pseudometric on \( X \) with corresponding quantum pseudometric \( \mathcal{V}_d \) (Proposition 2.24) and \( 0 < \alpha < 1 \),
then the quantum pseudometric $V_{d^\alpha}$ corresponding to the pseudometric $d^\alpha$ equals $V_d^\alpha$.

As in the classical case, Hölder metrics have some pathological qualities. For example, they are essentially never path metrics (Definition 2.32 (e)).

**Proposition 3.21.** Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M}$ and let $0 < \alpha < 1$. Suppose that $\mathcal{V}_t \neq \mathcal{V}_0$ for some $t > 0$. Then $V^\alpha$ is not a path quantum pseudometric.

**Proof.** Choose $t > 0$ such that $\mathcal{V}_t \neq \mathcal{V}_0$ and let $t_0 = \inf\{s : \mathcal{V}_s = \mathcal{V}_t\}$. So $t_0 > 0$ and $\mathcal{V}_{t_0} \neq \mathcal{V}_s$ for any $s < t_0$. Now

$$V_{t_0/2}^\alpha V_{t_0/2}^\alpha = V_{(t_0/2)^{1/\alpha}}(t_0/2)^{1/\alpha} \subseteq V_{2(t_0/2)^{1/\alpha}} = V_{2^{1-1/\alpha}t_0}$$

and $2^{1-1/\alpha}t_0 < t_0$, so for sufficiently small $\epsilon > 0$ we will have

$$V_{(t_0/2)+\epsilon}^\alpha V_{(t_0/2)+\epsilon}^\alpha \subseteq \mathcal{V}_s$$

for some $s < t_0$. Thus

$$V_{(t_0/2)+\epsilon}^\alpha \subseteq \mathcal{V}_s \subseteq V_{t_0} = V_{t_0}^\alpha$$

and this shows that $V^\alpha$ is not a path quantum pseudometric. □

The following result gives a more general version of the Hölder construction.

**Proposition 3.22.** Let $V = \{V_t\}$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$ and let $f : [0, \infty) \to [0, \infty]$ be a right continuous nondecreasing function such that $f(s) + f(t) \leq f(s + t)$ for all $s, t \geq 0$. Then $V^f = \{V^f_t\}$ where $V^f_t = V_{f(t)}^f$ is a quantum pseudometric on $\mathcal{M}$ (taking $\mathcal{V}_\infty = B(H)$). If $\rho$ is the quantum distance function associated to $V$ and $\rho^f$ is the quantum distance function associated to $V^f$ then we have

$$\rho^f(P, Q) = f(\rho(P, Q))$$

for all projections $P, Q \in \mathcal{M}\overline{\bigotimes}_{\infty}B(l^2)$.

The proof is trivial. The truncation of a quantum pseudometric (Definition 2.32 (a)) is also a special case of this construction. (Define $f(t) = t$ for $t < C$ and $f(t) = \infty$ for $t \geq C$.)

3.7. **Spectral triples.** A spectral triple is a triple $(\mathcal{A}, H, D)$ consisting of a Hilbert space $H$, a unital $*$-algebra $\mathcal{A} \subseteq B(H)$, and a self-adjoint operator $D$ with compact resolvent, such that $[D, \mathcal{A}]$ is bounded for all $A \in \mathcal{A}$ (Definition IV.2.11).

We will not actually use the hypothesis that $D$ has compact resolvent, and indeed there are natural examples where this fails. (For example, $H = L^2(\mathbb{R})$, $\mathcal{A} = \text{Lip}(\mathbb{R})$ acting by multiplication on $H$, and $D = \text{id}/dx$.) So this requirement can be dropped in the following discussion.

**Definition 3.23.** Let $(\mathcal{A}, H, D)$ be a spectral triple, possibly minus the assumption that $D$ has compact resolvent. The quantum pseudometric associated to $(\mathcal{A}, H, D)$ is the smallest quantum pseudometric $V$ on $\mathcal{A}''$ such that $e^{itD} \in \mathcal{V}_t$ for all $t > 0$.

In other words, $V$ is the smallest $\text{W}^*$-filtration such that $\mathcal{A}' \subseteq \mathcal{V}_0$ and $e^{itD} \in \mathcal{V}_t$ for all $t > 0$. This definition makes sense because arbitrary meets of quantum pseudometrics exist (Definition 2.32 (c)).

We can also characterize the quantum pseudometric associated to a spectral triple internally:
Lemma 3.24. Let $V = \{V_t\}$ be the quantum pseudometric associated to a spectral triple $(A,H,D)$. For each $t > 0$ let $W_t$ be the weak* closure of the span of the operators of the form

$$e^{is_1 D}A_1e^{is_2 D}A_2\cdots e^{is_n D}A_n$$

with $n \in \mathbb{N}$, each $A_i \in \mathcal{A}'$, each $s_i \in \mathbb{R}$, and $\sum |s_i| \leq t$. Then $V_t = \bigcap_{s \geq t} W_s$ for all $t \geq 0$.

Proof. Since $e^{\pm is D}A$ must belong to $V_t$ for all $t \geq 0$ and all $A \in \mathcal{A}'$, it follows from the filtration property that $W_t \subseteq V_t$ for all $t$, and then the fact that $V_t = \bigcap_{s \geq t} V_s$ shows that $\bigcap_{s \geq t} W_s \subseteq V_t$ for all $t$. It is evident that $\mathcal{A}' \subseteq \bigcap_{s \geq t} W_s$ and $e^{it D} \in \bigcap_{s \geq t} W_s$ for all $t \geq 0$. To complete the proof we must check that $V_t = \bigcap_{s \geq t} W_s$ defines a quantum pseudometric. Condition (i) of Definition 2.1 holds because $W_s W_t \subseteq W_{s+t}$ for all $s$ and $t$, and condition (ii) is easy. \hfill $\square$

Next we note that quantum pseudometrics associated to spectral triples are always path quantum pseudometrics (Definition 2.14(e)).

Proposition 3.25. The quantum pseudometric associated to any spectral triple is a path quantum pseudometric.

Proof. Let $V$ be the quantum pseudometric associated to the spectral triple $(A,H,D)$. Fix $s, t \geq 0$. For any $\epsilon > 0$ we have $V_{s+t} V_{t+\epsilon} \subseteq V_{s+t+2\epsilon}$, and it follows that

$$\bigcap_{\epsilon > 0} V_{s+t} V_{t+\epsilon}^{wk*} \subseteq V_{s+t}.$$  

(This will be true of any quantum pseudometric.) For the reverse inclusion, by Lemma 3.24 it will be enough to show that any operator of the form $e^{is_1 D}A_1e^{is_2 D}A_2\cdots e^{is_n D}A_n$ with $n \in \mathbb{N}$, each $A_i \in \mathcal{A}'$, each $s_i \in \mathbb{R}$, and $\sum |s_i| \leq s + t + \epsilon$, belongs to $V_{s+t}\mathcal{V}_{t+\epsilon}$. This is true because we can write

$$e^{is_1 D}A_1e^{is_2 D}A_2\cdots e^{is_n D}A_n = (e^{is_1 D}A_1\cdots e^{is_{j-1} D}A_{j-1}e^{is_j D}A_j e^{is_{j+1} D}A_{j+1} \cdots e^{is_n D}A_n)$$

with $s_j' + s_j'' = s_j$, $|s_1| + \cdots + |s_j'| \leq s$, and $|s_j'| + \cdots + |s_n| \leq t + \epsilon$. \hfill $\square$

We will now show that the construction in Definition 3.23 recovers the standard quantum metric for spectral triples involving the Hodge-Dirac operator (see [7]) on a complete connected Riemannian manifold with positive injectivity radius (i.e., the infimum of the injectivity radius over all points in the manifold is nonzero; see [10]).

Let $M$ be a complete connected Riemannian manifold equipped with volume measure. It carries an intrinsic metric $d$ and a corresponding intrinsic measurable metric defined by

$$\rho(\chi S, \chi T) = \inf \{d(x,y) : x \text{ is a Lebesgue point of } S \text{ and } y \text{ is a Lebesgue point of } T \}.$$  

It is straightforward to verify that $\rho$ is indeed a measurable metric. (To verify the join condition, Definition 2.20(iii), use the fact that if $(X, \mu)$ is $\sigma$-finite then any join $\bigvee_{\lambda} p_{\lambda}$ of projections in $L^\infty(X, \mu)$ equals the join of some countable subcollection.)

In Theorem 2.22 we identified a quantum metric $\mathbf{V}_\rho$ corresponding to $\rho$ which lives on the Hilbert space $L^2(M, \mu)$. Here we need to work on $H =$ the complexification of the space of $L^2$ differential forms on $M$, with $L^\infty(M, \mu)$ acting by
Theorem 3.26. Let $M$ be a complete connected Riemannian manifold with positive injectivity radius, let $\mu$ be volume measure on $M$, let $H$ be the complexification of the space of $L^2$ differential forms, and let $A = \mathcal{C}_c^\infty(M)$, acting on $H$ by multiplication. Let $D = d + d^*$ be the Hodge-Dirac operator. Then the quantum pseudometric $\mathbf{V}$ associated to $(A, H, D)$ (Definition 3.23) equals the quantum metric $\tilde{\mathbf{V}}_\rho$ associated to the intrinsic measurable metric $\rho$.

Proof. Note that $D$ has compact resolvent if $M$ is compact, but not in general. The inequality $\mathbf{V} \leq \tilde{\mathbf{V}}_\rho$ is easy. Let $S$ and $T$ be positive measure subsets of $M$ and suppose $\rho(\chi_S, \chi_T) > t$; we must show that $M_{\chi_S}AM_{\chi_T} = 0$ for any operator $A \in \mathcal{V}_t$. By Lemma 3.24 it is enough to show this for operators $A$ of the form $e^{i(s_1)D}A_1e^{i(s_2)D}A_2 \cdots e^{i(s_n)D}A_n$ with each $A_i \in \mathcal{A}'$ and $\sum|s_i| \leq t$. We may assume that $S$ and $T$ are open; otherwise, replace them with the open $\epsilon$-neighborhoods of their Lebesgue sets for some $\epsilon < (\rho(\chi_S, \chi_T) - t)/2$. Since the $A_i$ belong to $\mathcal{A}'$, it will suffice to show that if $f \in \mathcal{H}$ is supported in $S$ then $e^{isD}f$ is supported in the $s$-neighborhood of $S$. By continuity we may assume $f$ is smooth. The desired conclusion now follows from (24), Proposition 5.5). Thus, we have shown that $\mathbf{V} \leq \tilde{\mathbf{V}}_\rho$ (without assuming positive injectivity radius).

For the reverse inclusion, we need to assume that $M$ has positive injectivity radius $\delta$. We first show that $\tilde{\mathbf{V}}_\rho$ is a path quantum metric. Let $s, t \geq 0$ and let $A \in \tilde{\mathbf{V}}_{t+\epsilon}^\rho$. Fix $\epsilon > 0$ and let $\delta' = \min(\delta, \epsilon)$; we want to show that $A$ is in the weak* closure of $\tilde{\mathbf{V}}_{t+\epsilon}^\rho \tilde{\mathbf{V}}_{t+\delta'}^\rho$. Let $\{x_n\}$ be a $\delta'$-net in $M$ and let $\{S_n\}$ be a measurable partition of $M$ with $S_n \subseteq \text{ball}(x_n, \delta')$; then $A = \sum_{m,n} M_{\chi_{S_m}}AM_{\chi_{S_m}}$, and each summand belongs to $\tilde{\mathbf{V}}_{t+\epsilon}^\rho$, so we may restrict attention to a single summand and assume $A = M_{\chi_{S_m}}AM_{\chi_{S_m}}$. Now if $A \neq 0$ then $d(x_m, x_n) < s + t + 2\epsilon$, so let $\gamma: [0, r] \to M$ be a unit speed geodesic from $x_n$ to $x_m$ with $r < s + t + 2\epsilon$, let $y = \gamma(s + \epsilon)$, and let $V \in \mathcal{B}(H)$ be the partial isometry from $M_pH$ to $M_qH$, where $p = \chi_{\text{ball}(x_m, \delta')}$ and $q = \chi_{\text{ball}(y, \delta')}$, induced via weighted composition from the homeomorphism of ball$(x_m, \delta')$ with ball$(y, \delta')$ arising from the exponential maps. Then $d(x_m, y) < t + \epsilon$ and $A = (V^*)^n(VA) \in \tilde{\mathbf{V}}_{t+3\epsilon}^\rho \tilde{\mathbf{V}}_{t+3\epsilon}^\rho$. This suffices to establish that $\tilde{\mathbf{V}}_\rho$ is a path quantum metric.

To verify $\tilde{\mathbf{V}}_\rho \leq \mathbf{V}$ it will now be enough to show that $\tilde{\mathbf{V}}_t^\rho \subseteq \mathcal{V}_t$ for $t < \delta$. Fix $t < \delta$ and let $x, y \in M$ satisfy $d(x, y) < t$; we will show that there exist neighborhoods $S$ and $T$ of $x$ and $y$ respectively such that any operator $A \in \mathcal{B}(H)$ satisfying $A = M_{\chi_S}AM_{\chi_T}$ belongs to $\mathcal{V}_t$. By an argument similar to the one given in the last paragraph, this will suffice.

Observe that $\mathcal{A}' = L^\infty(M, \mu')$ can be identified with the bounded measurable sections of $\text{End}(\Lambda^*T^*M)$. Identify $L^2(M, \mu)$ with the $L^2$ 0-forms as a subspace of $H$. It will be enough to find neighborhoods $S$ and $T$ of $x$ and $y$ respectively such that every operator $A \in \mathcal{B}(L^2(M, \mu)) \subseteq \mathcal{B}(H)$ satisfying $A = M_{\chi_S}AM_{\chi_T}$ belongs
to $V_i$; the corresponding statement will then also be true for arbitrary operators in $\mathcal{B}(H)$ because $V_i$ is a bimodule over $\mathcal{A}'$.

The Hodge Laplacian is $\Delta = dd^* + d^*d$. $L^2(M, \mu)$ is an invariant subspace for $\Delta$ and $\Delta_0 = \Delta|_{L^2(M, \mu)}$ is the scalar Laplacian. Also note that since the power series expansion of $\cos z$ involves only even powers of $z$, we have $\cos(sD) = \cos(s\sqrt{\Delta})$.

Thus for any $\phi \in L^1(\mathbb{R})$ the operator

$$\int_0^t \phi(s) \cos(s\sqrt{\Delta}) \, ds = \frac{1}{2} \int_{-t}^t \phi(|s|)e^{isD} \, ds$$

belongs to $V_i$, as does $\int_0^t \phi(s) \cos(s\sqrt{\Delta_0}) \, ds = P \int_0^t \phi(s) \cos(s\sqrt{\Delta}) \, ds$ where $P \in \mathcal{A}' \subseteq \mathcal{B}(H)$ is the projection onto $L^2(M, \mu)$.

In the remainder of the proof we work with scalar functions on $M$. For $0 \leq s \leq t$ let $u_x(s)$ be the distribution $u_x(s) = \cos(s\sqrt{\Delta_0})\delta_x$ and note that it satisfies the wave equation with initial conditions $u_x(0) = \delta_x$ and $u'_x(0) = 0$. By examining the wavefront set (see Theorem 4 of [28]) we see that $y$ is in the support of $u_x(s)$ for $s = d(x, y)$. Then let $\phi \in C^\infty(\mathbb{R})$ satisfy $\phi = 1$ in an interval about $s = d(x, y)$ and $\phi = 0$ outside a slightly larger interval, and let $B = \int_0^t \phi(s) \cos(s\sqrt{\Delta_0}) \, ds \in V_i$. Then $B$ is an integral operator with continuous kernel $k(x', y')$, and if $\phi$ is suitably chosen there will exist neighborhoods $S$ and $T$ of $x$ and $y$ such that $k(x', y') \neq 0$ for all $x' \in S$ and $y' \in T$. It follows that $M_fBM_g$ belongs to $V_i$ for all $f \in C_c(S)$ and $g \in C_c(T)$; this is the integral operator with kernel $k(x', y')f(x')g(y')$. By the Stone-Weierstrass theorem we conclude that $V_i$ contains all integral operators with continuous kernel supported in $S \times T$, and hence, by weak* closure, all operators $A \in \mathcal{B}(L^2(M, \mu))$ satisfying $A = M_{\chi_S}AM_{\chi_T}$, as desired. \qed

In the above case the quantum pseudometric associated to the spectral triple $(\mathcal{A}, H, D)$ is in fact a quantum metric, not just a quantum pseudometric, on $\mathcal{A}'$. A good problem for future work will be to identify natural conditions that ensure the quantum pseudometric associated to a spectral triple is a quantum metric.

4. Lipschitz Operators

The algebra Lip$(X)$ of bounded scalar-valued Lipschitz functions on a metric space $X$ has been studied extensively (see [22]). Under appropriate hypotheses one can recover the space $X$ as the normal spectrum of Lip$(X)$, with metric inherited from the dual of Lip$(X)$; moreover, basic metric properties of $X$ are reflected in algebraic properties of Lip$(X)$ (closed subsets correspond to weak* closed ideals, etc.). Thus the relation between metric spaces and Lipschitz algebras is strongly analogous to the relation between topological and measure spaces and the algebras $C(X)$ and $L^\infty(X, \mu)$.

We find that there are two distinct, but related, versions of the notion of a “Lipschitz operator” associated to a quantum pseudometric. There is a spectral version with good lattice properties but poor algebraic properties and a commutation version with the opposite qualities. (Something similar happens with lower semicontinuity, which bifurcates in the $C^*$-algebra setting into two notions, “$\text{isc}$” and “$\text{q-isc}$”, with, respectively, good algebraic and lattice properties [1].)

We can abstractly characterize the spectral Lipschitz gauge (Theorem 4.16), which provides another intrinsic characterization of quantum pseudometrics, perhaps more elegant than the one given in Theorem 2.4. Our other main result,
Theorem 4.23 states that commutation Lipschitz number is always less than or equal to spectral Lipschitz number. We prove this using a powerful result of Browder and Sinclair ([11, Corollary 26.6]) which equates the norm and spectral radius of a Hermitian element of a complex unital Banach algebra. Our inequality is valuable because we have general techniques for constructing operators with finite spectral Lipschitz number (see Lemma 4.12).

4.1. The abelian case. We start by reviewing the measurable version of Lipschitz number. Say that the essential range of a function \( f \in L^\infty(X,\mu) \) is the set of all \( a \in \mathbb{C} \) such that \( f^{-1}(U) \) has positive measure for every open neighborhood \( U \) of \( a \). Equivalently, it is the spectrum of the operator \( M_f \in B(L^2(X,\mu)) \). If \( p \in L^\infty(X,\mu) \) is a projection then we denote the essential range of \( f|_{\text{supp}(p)} \) by \( \text{ran}_p(f) \).

**Definition 4.1.** ([32, Definition 6.2.1]) Let \( (X,\mu) \) be a finitely decomposable measure space and let \( \rho \) be a measurable pseudometric on \( X \). The Lipschitz number of \( f \in L^\infty(X,\mu) \) is the quantity

\[
L(f) = \sup \left\{ \frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p,q)} : p,q \in L^\infty(X,\mu) \text{ and } \rho(p,q) \neq 0 \right\},
\]

where the supremum is taken over all nonzero projections \( p,q \in L^\infty(X,\mu) \) and we use the convention \( \frac{0}{0} = 0 \). (Note that line 4 of Definition 6.2.1 of [32] should say “essential infimum of the function \( |f(p) - f(q)|^2 \).” Here \( d \) is the minimum distance between compact subsets of \( \mathbb{C} \). We call \( L \) the Lipschitz gauge associated to \( \rho \) and we define \( \text{Lip}(X,\mu) = \{ f \in L^\infty(X,\mu) : L(f) < \infty \} \).

We first observe that this definition generalizes the atomic abelian case.

**Proposition 4.2.** Let \( \mu \) be counting measure on a set \( X \), let \( d \) be a pseudometric on \( X \), and let \( f \in L^\infty(X) \). Let \( \rho \) be the associated measurable pseudometric (Proposition 2.27). Then

\[
L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x,y \in X, d(x,y) > 0 \right\}.
\]

**Proof.** The inequality \( \geq \) follows immediately from the definition of \( L(f) \) in Definition 4.1 by taking \( p = \chi_{\{x\}}, q = \chi_{\{y\}} \). For the reverse inequality let \( p, q \in L^\infty(X) \), say \( p = \chi_S \) and \( q = \chi_T \), with \( S \) and \( T \) both nonempty. Find sequences \( \{x_n\} \subseteq S \) and \( \{y_n\} \subseteq T \) such that \( d(x_n,y_n) \to \rho(p,q) \); then either \( \lim \inf |f(x_n) - f(y_n)| = 0 \), in which case \( d(\text{ran}_p(f), \text{ran}_q(f)) = 0 \), or else

\[
\frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p,q)} \leq \lim \inf \frac{|f(x_n) - f(y_n)|}{\rho(p,q)} \leq \sup \frac{|f(x_n) - f(y_n)|}{d(x_n,y_n)}.
\]

This proves the inequality \( \leq \). \( \square \)

Next, in anticipation of our axiomatization of quantum Lipschitz gauges in Section 4.2 we give an alternative axiomatization of measurable pseudometrics in terms of Lipschitz numbers. Many versions of this result have appeared in the literature; probably the closest is Example 6.2.5 of [32].

**Definition 4.3.** Let \( (X,\mu) \) be a finitely decomposable measure space. An abstract Lipschitz gauge on \( L^\infty(X,\mu) \) is a function \( \mathcal{L} \) from the real part of \( L^\infty(X,\mu) \) to \([0,\infty)\) satisfying.
for any $a \in \mathbb{R}$ and $f, g, f_\lambda \in L^\infty(X, \mu)$ real-valued with $\sup \|f_\lambda\|_\infty < \infty$.

**Theorem 4.4.** Let $(X, \mu)$ be a finitely decomposable measure space. If $\rho$ is a measurable pseudometric on $X$ then the restriction of the associated Lipschitz gauge $L$ to the real part of $L^\infty(X, \mu)$ is an abstract Lipschitz gauge. If $\mathcal{L}$ is an abstract Lipschitz gauge then

$$\rho_{\mathcal{L}}(p, q) = \sup \{d(\text{ran}_p(f), \text{ran}_q(f)) : \mathcal{L}(f) \leq 1\}$$

is a measurable pseudometric on $X$. The two constructions are inverse to each other.

**Proof.** Let $\rho$ be a measurable pseudometric on $X$ and let $L$ be the associated Lipschitz gauge. The fact that $L$ is an abstract Lipschitz gauge follows from ([35], Corollary 1.21). We verify that $\rho = \rho_L$. The inequality $\rho(p, q) \geq \rho_L(p, q)$ for all $p, q$ follows immediately from the definition of $L$. Conversely, if $\rho(p, q) < \infty$ then let $c = \rho(p, q)$ and define

$$f = \bigvee \min\{\rho(r, q), c\} \cdot r,$$

taking the join in $L^\infty(X, \mu)$ over all nonzero projections $r$. We have $\text{ran}_q(f) = \{0\}$ and $\text{ran}_q(f) = \{c\}$, so that $d(\text{ran}_p(f), \text{ran}_q(f)) = c = \rho(p, q)$. Also, $L(f) \leq 1$ by ([35], Lemma 1.22). This proves the inequality $\rho(p, q) \leq \rho_L(p, q)$. If $\rho(p, q) = \infty$ then take $c \to \infty$ in the preceding argument.

Now let $\mathcal{L}$ be an abstract Lipschitz gauge. We will show that $\rho_{\mathcal{L}}$ is a measurable pseudometric and that $\mathcal{L}(f) = L(f)$ for all real-valued $f \in L^\infty(X, \mu)$, where $L$ is the Lipschitz gauge associated to $\rho_{\mathcal{L}}$.

We verify the conditions in Definition [2.20]. Conditions (i) and (ii) are trivial. The inequality $\leq$ in condition (iii) is easy; for the reverse inequality suppose $\rho_{\mathcal{L}}(p_\lambda, q_\kappa) > a$ for all $\lambda, \kappa$ and find $f_{\lambda\kappa} \in L^\infty(X, \mu)$ real-valued such that $\mathcal{L}(f_{\lambda\kappa}) \leq 1$ and $d(\text{ran}_{p_\lambda}(f_{\lambda\kappa}), \text{ran}_{q_\kappa}(f_{\lambda\kappa})) \geq a$ for all $\lambda, \kappa$. Then define

$$g_{\lambda\kappa} = \bigwedge_{c \in \text{ran}_{p_\lambda}(f_{\lambda\kappa})} (|f_{\lambda\kappa} - c \cdot 1| \wedge a \cdot 1);$$

we still have $\mathcal{L}(g_{\lambda\kappa}) \leq 1$, and $\text{ran}_{p_\lambda}(g_{\lambda\kappa}) = \{0\}$ and $\text{ran}_{q_\kappa}(g_{\lambda\kappa}) = \{a\}$. So

$$g = \bigwedge_{\lambda} \bigvee_{\kappa} g_{\lambda\kappa}$$

covers $L(g) \leq 1$, $\text{ran}_{p_\lambda}(g_{\lambda\kappa}) = \{0\}$, and $\text{ran}_{q_\kappa}(g_{\lambda\kappa}) = \{a\}$. Thus

$$\rho_{\mathcal{L}}(\bigvee_{p_\lambda}, \bigvee_{q_\kappa}) \geq a,$$

which is enough.

To verify condition (iv), let $p, q, r \in L^\infty(X, \mu)$ be nonzero projections and let $f \in L^\infty(X, \mu)$ be real-valued and satisfy $\mathcal{L}(f) \leq 1$. Given $\epsilon > 0$, choose $a \in \text{ran}_q(f)$ and find $q' \leq q$ such that

$$\text{ran}_{q'}(f) \subseteq [a - \epsilon, a + \epsilon].$$

Then

$$d(\text{ran}_p(f), \text{ran}_r(f)) \leq d(\text{ran}_p(f), \text{ran}_{q'}(f)) + d(\text{ran}_{q'}(f), \text{ran}_r(f)) + 2\epsilon,$$

$$\leq \rho_{\mathcal{L}}(p, q') + \rho_{\mathcal{L}}(q', r) + 2\epsilon.$$
and taking $\epsilon \to 0$ and the supremum over $f$ yields the measurable triangle inequality. So $\rho_L$ is a measurable pseudometric.

Now let $f \in L^\infty(X,\mu)$ be real-valued. First suppose $\mathcal{L}(f) < \infty$ and let $L(f)$ be the Lipschitz number of $f$ with respect to the pseudometric $\rho_L$. For any $0 < a < 1/\mathcal{L}(f)$, we then have $\mathcal{L}(a f) < 1$ and so

$$\rho_L(p, q) \geq d(\text{ran}_p(a f), \text{ran}_q(a f)) = a \cdot d(\text{ran}_p(f), \text{ran}_q(f))$$

for any $p$ and $q$. Taking $a \to 1/\mathcal{L}(f)$, this shows that

$$\frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho_L(p, q)} \leq \mathcal{L}(f),$$

and taking the supremum over $p$ and $q$ yields $L(f) \leq \mathcal{L}(f)$. For the reverse inequality, suppose $L(f) < 1$; we will show that $\mathcal{L}(f) \leq 1$. For any $a, b \in \mathbb{R}$ with $a < b$ let $p = \chi_{f^{-1}(-\infty,a]}$ and $q = \chi_{f^{-1}[b,\infty)}$ and find $f_{ab}$ such that $\mathcal{L}(f_{ab}) \leq 1$ and

$$d(\text{ran}_p(f_{ab}), \text{ran}_q(f_{ab})) = d(\text{ran}_p(f), \text{ran}_q(f)) \geq b - a.$$

We can do this because $d(\text{ran}_p(f), \text{ran}_q(f)) \leq L(f)\rho_L(p, q) < \rho_L(p, q)$. Now define

$$g_{ab} = \bigwedge_{c \in \text{ran}_p(f_{ab})} (|f_{ab} - c \cdot 1| \land (b - a) \cdot 1);$$

then $\mathcal{L}(g_{ab}) \leq 1$, $\text{ran}_p(g_{ab}) = \{0\}$, and $\text{ran}_q(g_{ab}) = \{b - a\}$. So

$$f = \bigwedge_{|a| \leq \|f\|_\infty} \bigvee_{b > a} (g_{ab} + a \cdot 1),$$

and this shows that $\mathcal{L}(f) \leq 1$. We conclude that $\mathcal{L}(f) \leq L(f)$. This completes the proof. \qed

Lemma 6.2.4 of [32] can be used to prove a similar result with $\mathcal{L}$ defined on all of $L^\infty(X,\mu)$. It is interesting to note that although Lipschitz numbers do satisfy the seminorm condition in Definition 4.3 (iii), it is only used in the proof to ensure that $\mathcal{L}(f + c \cdot 1) = \mathcal{L}(f)$. In the noncommutative setting only this weaker version holds (see Definition 4.14 and Example 4.18).

### 4.2. Spectral Lipschitz numbers

Now we introduce the spectral Lipschitz number of a self-adjoint operator in a von Neumann algebra $\mathcal{M}$ equipped with a quantum pseudometric, or more generally a self-adjoint operator in $\mathcal{M} \otimes \mathcal{B}(l^2)$. Recall that $P_S(A)$ denotes the spectral projection of the self-adjoint operator $A$ for the Borel set $S \subseteq \mathbb{R}$.

**Definition 4.5.** Let $\rho$ be a quantum distance function on a von Neumann algebra $\mathcal{M}$ (Definition 2.7) and let $A \in \mathcal{M} \otimes \mathcal{B}(l^2)$ be self-adjoint. The spectral Lipschitz number of $A$ is the quantity

$$L_s(A) = \sup \left\{ \frac{b - a}{\rho(P_{(-\infty,a]}(A), P_{[b,\infty)}(A))} : a, b \in \mathbb{R}, a < b \right\},$$

with the convention $\frac{a}{0} = 0$, and $A$ is spectrally Lipschitz if $L_s(A) < \infty$. We call the function $L_s$ the spectral Lipschitz gauge.

We immediately note an alternative formula for $L_s(A)$.
Proposition 4.6. Let $\rho$ be a quantum distance function on a von Neumann algebra $\mathcal{M}$ and let $A \in \mathcal{M} \otimes S(l^2)$ be self-adjoint. Then

$$L_s(A) = \sup \left\{ \frac{d(S, T)}{\rho(P_S(A), P_T(A))} : S, T \subseteq \mathbb{R} \text{ Borel} \right\},$$

with the convention $0 \cdot 0 = 0$.

Proof. Here $d(S, T) = \inf \{d(x, y) : x \in S, y \in T\}$. The inequality follows by taking $S = (-\infty, a]$ and $T = [b, \infty)$ for arbitrary $a, b \in \mathbb{R}$, $a < b$. For the reverse inequality, let $S, T \subset \mathbb{R}$ be Borel and (assuming $d(S, T) > 0$, otherwise the pair makes no contribution to the supremum) partition them as $S = \bigcup S_i$, $T = \bigcup T_j$ such that each $S_i$ and each $T_j$ has diameter at most $d(S, T)$. We can do this so that $P_{S_i}(A) = P_{T_j}(A) = 0$ for all but finitely many $i$ and $j$, since the interval $[-\|A\|, \|A\|]$ has finite length. Then $P_S(A) = \bigvee P_{S_i}(A)$ and $P_T(A) = \bigvee P_{T_j}(A)$ implies that we must have $\rho(P_{S_i}(A), P_{T_j}(A)) = \rho(P_{S_i}(A), P_T(A))$ for some $i$ and $j$. Since $d(S_i, T_j) \geq d(S, T) \geq \max\{\text{diam}(S_i), \text{diam}(T_j)\}$, without loss of generality we may assume that $a < b$ where $a = \sup S_i$ and $b = \inf T_j$ for this choice of $i$ and $j$. We then have

$$\frac{d(S, T)}{\rho(P_S(A), P_T(A))} \leq \frac{d(S_i, T_j)}{\rho(P_{S_i}(A), P_{T_j}(A))} \leq \frac{b-a}{\rho(P_{[-\infty, a]}(A), P_{[b, \infty]}(A))} \leq L_s(A).$$

Taking the supremum over $S$ and $T$ yields the desired inequality.

Spectral Lipschitz numbers generalize measurable Lipschitz numbers (Definition 4.1).

Proposition 4.7. Let $(X, \mu)$ be a finitely decomposable measure space, let $\rho$ be a measurable pseudometric on $X$, and let $V_\rho$ be the associated quantum pseudometric on the von Neumann algebra $\mathcal{M} \cong L^\infty(X, \mu)$ of bounded multiplication operators on $L^2(X, \mu)$ (Theorem 2.22). Then for any real-valued $f \in L^\infty(X, \mu)$ we have $L_s(M_f) = L(f)$.

Proof. Let $\hat{\rho}$ be the quantum distance function associated to $V_\rho$ (Definition 2.6) and recall that we have $\hat{\rho}(M_p, M_q) = \rho(p, q)$ for all nonzero projections $p, q \in L^\infty(X, \mu)$ (Theorem 2.22). Now the inequality $L_s(M_f) \leq L(f)$ is proven by taking $p = \chi_{f^{-1}([-\infty, a])}$ and $q = \chi_{f^{-1}([b, \infty]}$ for arbitrary $a, b \in \mathbb{R}$, $a < b$ (so that $P_{[-\infty, a]}(M_f) = M_p$ and $P_{[b, \infty]}(M_f) = M_q$) in Definition 4.1 since $b-a \leq d(\text{ran}_p(f), \text{ran}_q(f))$ and therefore

$$\frac{b-a}{\hat{\rho}(P_{[-\infty, a]}(M_f), P_{[b, \infty]}(M_f))} = \frac{b-a}{\rho(p, q)} \leq \frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p, q)} \leq L(f).$$

Taking the supremum over $a$ and $b$ shows that $L_s(M_f) \leq L(f)$. For the reverse inequality, let $p, q \in L^\infty(X, \mu)$ be nonzero projections. If $\epsilon = d(\text{ran}_p(f), \text{ran}_q(f)) = 0$ then the pair makes no contribution to $L(f)$, so assume $\epsilon > 0$. Now partition $p$ and $q$ as $p = \sum p_i$ and $q = \sum q_j$ such that $\text{ran}_p(f)$ and $\text{ran}_q(f)$ have diameter at most $\epsilon$ for all $i$ and $j$. Then for some choice of $i$ and $j$ we have $\rho(p_i, q_j) = \rho(p, q)$, and as in the proof of Proposition 4.6 we may assume $a < b$ where $a = \sup \text{ran}_{p_i}(f)$ and $b = \inf \text{ran}_{q_j}(f)$. Thus $M_{p_i} \leq P_{[-\infty, a]}(M_f)$ and $M_{q_j} \leq P_{[b, \infty]}(M_f)$, so that

$$\frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p, q)} \leq \frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p_i, q_j)} \leq L(f).$$


Proposition 4.10. Let $L$ we have

Proposition 4.9. also point to the following easy result. numbers in the abelian case is evidence that Definition 4.5 is reasonable. We can

isomorphism and equip $X$ of $A$ algebra

where

spectrum of $A$ and let

Together with Proposition 4.2 this implies that the spectral Lipschitz number reduces to the classical Lipschitz number in the atomic abelian case.

Corollary 4.8. Let $d$ be a pseudometric on a set $X$ and equip the von Neumann algebra $M \cong l^\infty(X)$ of bounded multiplication operators on $l^2(X)$ with the associated quantum pseudometric (Proposition 2.5). Then for any real-valued $f \in l^\infty(X)$ we have $L_a(M_f) = L_a(f)$.

The fact that spectral Lipschitz numbers effectively reduce to classical Lipschitz numbers in the abelian case is evidence that Definition 4.5 is reasonable. We can also point to the following easy result.

Proposition 4.9. Let $\rho$ be a quantum distance function on a von Neumann algebra $M$, let $A \in M \otimes B(l^2)$ be self-adjoint and spectrally Lipschitz, and let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then $f(A)$ is spectrally Lipschitz and $L_a(f(A)) \leq L(f) L_a(A)$.

Proof. Let $a, b \in \mathbb{R}$, $a < b$, and let $S = f^{-1}([-\infty, a])$ and $T = f^{-1}([b, \infty))$. Then

$$P_{[-\infty, a]}(f(A)) = P_S(A) \quad \text{and} \quad P_{[b, \infty)}(f(A)) = P_T(A).$$

Also $b - a \leq L(f) \cdot d(S, T)$, so

$$\frac{b - a}{\rho(P_{[-\infty, a]}(f(A)), P_{[b, \infty)}(f(A)))} \leq L(f) \cdot \frac{d(S, T)}{\rho(P_S(A), P_T(A))} \leq L(f) \cdot L_a(A)$$

by Proposition 4.10. Taking the supremum over $a$ and $b$ proves that $f(A)$ is spectrally Lipschitz and yields the stated inequality. \qed

We will give other basic properties of $L_a$, in particular its compatibility with spectral joins and meets, in Lemma 4.15 and Theorem 4.16.

The spectral Lipschitz condition can be related to the notion of co-Lipschitz number introduced in Definition 2.27 (and consequently Corollary 4.8) can also be deduced from Proposition 2.29. Recall that if $A \in M$ is self-adjoint then the von Neumann algebra $W^*(A)$ it generates is *-isomorphic to $L^\infty(X, \mu)$ where $X$ is the spectrum of $A$ and $\mu$ is some finite measure on $X$ (27, Theorem III.1.22). Let $\phi : L^\infty(X, \mu) \cong W^*(A) \subseteq M$ be such an isomorphism and define a measurable metric $\tilde{\rho}$ on $X$ by setting

$$\tilde{\rho}(p, q) = d(\text{ran}_p(\zeta), \text{ran}_q(\zeta))$$

where $\zeta = \phi^{-1}(A)$.

Proposition 4.10. Let $\rho$ be a quantum distance function on a von Neumann algebra $M$ and let $A \in M$ be self-adjoint. Let $\phi : L^\infty(X, \mu) \cong W^*(A)$ be a *-isomorphism and equip $X$ with the quantum metric $V_{\tilde{\rho}}$ (Theorem 2.22) associated to the measurable metric $\tilde{\rho}$ defined above. Then the spectral Lipschitz number $L_a(A)$ of $A$ equals the co-Lipschitz number $L(\phi)$ of $\phi$. 
Proof. One inequality is easy: given \( a, b \in \mathbb{R} \) take \( p = \phi^{-1}(P_{[\infty,a]}(A)) \) and \( q = \phi^{-1}(P_{[b,\infty)}(A)) \) in Definition 2.22. Then \( \tilde{\rho}(p, q) \geq b - a \) and so

\[
\frac{b - a}{\rho(P_{[\infty,a]}(A), P_{[b,\infty)}(A))} \leq \frac{\tilde{\rho}(p, q)}{\rho(\phi(p), \phi(q))} \leq L(\phi).
\]

Taking the supremum over \( a \) and \( b \) yields \( L_s(A) \leq L(\phi) \). For the reverse inequality, since \( \nabla_{\tilde{\rho}} \) is reflexive Proposition 2.30 implies that we can restrict attention to projections in \( L^\infty(X, \mu) \) when evaluating \( L(\phi) \). So let \( p, q \in L^\infty(X, \mu) \) be projections. As in the proof of Proposition 4.7 we can find projections \( p' \leq p \) and \( q' \leq q \) such that \( \rho(\phi(p'), \phi(q')) = \rho(\phi(p), \phi(q)) \) and \( \text{ran}_{\phi}(\zeta) \) and \( \text{ran}_{\phi}(\zeta') \) have diameter less than \( \tilde{\rho}(p, q) \), where \( \zeta = \phi^{-1}(A) \). Without loss of generality \( a < b \) where \( a = \sup \text{ran}_{\phi}(\zeta) \) and \( b = \inf \text{ran}_{\phi}(\zeta) \). Then \( \phi(p') \leq P_{[\infty,a]}(A) \) and \( \phi(q') \leq P_{[b,\infty)}(A) \), so that

\[
\frac{\tilde{\rho}(p, q)}{\rho(\phi(p), \phi(q))} \leq \frac{\rho(p', q')}{\rho(\phi(p'), \phi(q'))} \leq \frac{b - a}{\rho(P_{[\infty,a]}(A), P_{[b,\infty)}(A))} \leq L_s(A).
\]

Taking the supremum over \( p \) and \( q \) yields \( L(\phi) \leq L_s(A) \). \( \square \)

By stabilization (see Section 2.4) we can therefore equate the spectral Lipschitz number of any self-adjoint operator \( A \in \mathcal{M} \otimes \mathcal{B}(l^2) \) with the co-Lipschitz number of a \(*\)-isomorphism \( \phi : L^\infty(X, \mu) \cong W^*(A) \subseteq \mathcal{M} \otimes \mathcal{B}(l^2) \). We can also prove a counterpart to Proposition 4.9.

**Corollary 4.11.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras equipped with quantum distance functions, let \( A \in \mathcal{M} \otimes \mathcal{B}(l^2) \) be self-adjoint and spectrally Lipschitz, and let \( \psi : \mathcal{M} \to \mathcal{N} \) be a co-Lipschitz morphism. Then \( (\psi \otimes I)(A) \) is spectrally Lipschitz and \( L_s((\psi \otimes I)(A)) \leq L(\psi)L_s(A) \).

**Proof.** Let \( \phi : L^\infty(X, \mu) \cong W^*(A) \) be a \(*\)-isomorphism as in Proposition 4.10. Then \( (\psi \otimes I) \circ \phi \) restricts to an isomorphism from \( L^\infty(S, \mu|_S) \) to \( W^*((\psi \otimes I)(A)) \) for some \( S \subseteq X \), and we can take this to be the \(*\)-isomorphism used in Proposition 4.10 for the operator \( (\psi \otimes I)(A) \). So

\[
L_s((\phi \circ I)(A)) = L(\psi \circ \phi|_S) \leq L(\psi)L(\phi|_S) \leq L(\psi)L_s(A)
\]

by Proposition 2.28. \( \square \)

Next we want to show that the quantum distance function \( \rho \) associated to a quantum pseudometric can be recovered from the spectral Lipschitz gauge. In order to prove this we need to show that there are sufficiently many spectrally Lipschitz operators. The basic tool is the following analog of (35, Lemma 1.22).

The spectral join of a bounded family of self-adjoint operators \( \{A_\lambda\} \) is the self-adjoint operator \( \bigvee A_\lambda \) whose spectral projections satisfy

\[
P_{[a,\infty)} \left( \bigvee A_\lambda \right) = \bigvee \lambda P_{[a,\infty)}(A_\lambda)
\]

for all \( a \in \mathbb{R} \). Their spectral meet \( \bigwedge A_\lambda \) has spectral projections

\[
P_{[a,\infty)} \left( \bigwedge A_\lambda \right) = \bigwedge \lambda P_{[a,\infty)}(A_\lambda).
\]

Equivalently, \( \bigwedge A_\lambda = -\bigvee (-A_\lambda) \).
Lemma 4.12. Let \( \rho \) be a quantum distance function on a von Neumann algebra \( \mathcal{M} \), let \( R \in \mathcal{M} \otimes \mathcal{B}(l^2) \) be a nonzero projection, and let \( c > 0 \). Then

\[
\min \{ \rho(P,R), c \} \cdot P,
\]

taking the spectral join over all projections \( P \) in \( \mathcal{M} \otimes \mathcal{B}(l^2) \), has spectral Lipschitz number at most 1.

Proof. Let \( A \) be this spectral join. Then \( P_{(a,\infty)}(A) \) is the join of the projections whose distance from \( R \) is greater than \( a \), for any \( a < c \). Now let \( a, b \in [0, \infty) \), \( a < b \); we must show that \( b - a \leq \rho(P_{[-\infty,a]}(A), P_{[b,\infty]}(A)) \). We may assume that \( b \leq c \) as otherwise \( P_{[b,\infty]}(A) = 0 \) and so the right side is infinite. Let \( \epsilon > 0 \), \( P = P_{(b-\epsilon,\infty)}(A) \), and \( Q = P_{[-\infty,a]}(A) \) and observe that if \( Q \) is any projection such that \( QQ \neq 0 \) then \( \rho(Q,R) \leq a \) (as otherwise we would have \( \hat{Q} \leq P_{(a,\infty)}(A) \) by the definition of \( A \)). Also \( \rho(P,R) \geq b - \epsilon \) since \( P \) is a join of projections whose distance from \( R \) is greater than \( b - \epsilon \). Thus

\[
b - \epsilon \leq \rho(P,R) \leq \rho(P,Q) + \sup \{ \rho(\hat{Q},R) : QQ \neq 0 \} \leq \rho(P,Q) + a
\]

by Definition 2.7 (v). Thus \( b - a \leq \rho(P_{[-\infty,a]}(A), P_{[b,\infty]}(A)) + \epsilon \), and taking \( \epsilon \to 0 \) yields the desired inequality. \( \square \)

This lets us recover the quantum distance function from the spectral Lipschitz gauge.

Theorem 4.13. Let \( \rho \) be a quantum distance function on a von Neumann algebra \( \mathcal{M} \). Then

\[
\rho(P,Q) = \sup \{ a \geq 0 : \text{some self-adjoint } A \in \mathcal{M} \otimes \mathcal{B}(l^2) \text{ satisfies } L_s(A) \leq 1, P \leq P_{[-\infty,0]}(A), \text{ and } Q \leq P_{(a,\infty)}(A) \}
\]

for any projections \( P, Q \in \mathcal{M} \otimes \mathcal{B}(l^2) \).

Proof. Let \( \hat{\rho}(P,Q) \) be the displayed supremum. Given \( a \) and \( A \) satisfying the conditions in the definition of \( \hat{\rho}(P,Q) \), we must have

\[
a \leq \rho(P_{[-\infty,0]}(A), P_{[a,\infty]}(A)) \leq \rho(P,Q)
\]

(the first inequality because \( L_s(A) \leq 1 \), the second because \( P \leq P_{[-\infty,a]}(A) \) and \( Q \leq P_{[a,\infty]}(A) \)). So \( \hat{\rho}(P,Q) \leq \rho(P,Q) \). Conversely, suppose \( \rho(P,Q) < \infty \) and let \( A \) be the operator defined in Lemma 4.12 with \( R = P \) and \( c = \rho(P,Q) \). Then it is clear that \( Q \leq P_{[c]}(A) \), and we have \( P \leq P_{[0]}(A) \) since \( P \) is orthogonal to any projection whose distance from \( P \) is nonzero (Definition 2.7 (ii)). According to Lemma 4.12 we have \( L_s(A) \leq 1 \), so this shows that \( \rho(P,Q) = c \leq \hat{\rho}(P,Q) \). Thus \( \hat{\rho}(P,Q) = \rho(P,Q) \). If \( \rho(P,Q) = \infty \) then take \( c \to \infty \) in the preceding argument. \( \square \)

We now proceed to the main result of this section, which abstractly characterizes spectral Lipschitz gauges. Because of the mutual recoverability of spectral Lipschitz numbers and quantum distance functions and the equivalence of quantum distance functions with quantum pseudometrics (Theorem 2.45), this gives us a second intrinsic characterization of quantum pseudometrics.

In order to state the relevant definition we need the following notion. Let \( \mathcal{M} \) be a von Neumann algebra and let \( A, B \in \mathcal{M}, A \geq 0 \). Then \( A_{[B]} \in \mathcal{M} \) will denote the positive operator with spectral subspaces

\[
P_{(a,\infty)}(A_{[B]}) = [BP_{(a,\infty)}(A)]
\]
for $a > 0$, where the bracket on the right side denotes range projection.

**Definition 4.14.** A quantum Lipschitz gauge on a von Neumann algebra $\mathcal{M}$ is a function $\mathcal{L}$ from the self-adjoint part of $\mathcal{M} \otimes \mathcal{B}(l^2)$ to $[0, \infty]$ which satisfies

1. $\mathcal{L}(A + I) = \mathcal{L}(A)$
2. $\mathcal{L}(aA) = |a|\mathcal{L}(A)$
3. $\mathcal{L}(A \vee \hat{A}) \leq \max\{\mathcal{L}(A), \mathcal{L}(\hat{A})\}$
4. $\mathcal{L}(A \uparrow B) \leq \mathcal{L}(A)$ if $A \geq 0$
5. if $A \rightarrow A$ weak operator then $\mathcal{L}(A) \leq \sup \mathcal{L}(A_{\lambda})$

for any $a \in \mathbb{R}$, any self-adjoint $A, \hat{A}, A_{\lambda} \in \mathcal{M} \otimes \mathcal{B}(l^2)$ with $\sup \|A_{\lambda}\| < \infty$, and any $B \in I \otimes \mathcal{B}(l^2)$. (In (i), $I$ is the unit in $\mathcal{M} \otimes \mathcal{B}(l^2)$; in (iii), $A \vee \hat{A}$ is the spectral join of $A$ and $\hat{A}$.)

We emphasize that we do not assume $\mathcal{L}(A + \hat{A}) \leq \mathcal{L}(A) + \mathcal{L}(\hat{A})$; see Example 4.15 below.

**Lemma 4.15.** Let $\mathcal{L}$ be a quantum Lipschitz gauge on a von Neumann algebra $\mathcal{M}$ and let $\{A_{\lambda}\}$ be a bounded family of self-adjoint operators in $\mathcal{M} \otimes \mathcal{B}(l^2)$. Then

$$\mathcal{L}\left(\bigvee A_{\lambda}\right), \mathcal{L}\left(\bigwedge A_{\lambda}\right) \leq \sup \mathcal{L}(A_{\lambda}).$$

(Again, $\bigvee A_{\lambda}$ and $\bigwedge A_{\lambda}$ are the spectral join and meet. The desired inequality holds for finite joins by property (iii), and then for arbitrary joins by property (v), taking the weak operator limit of the net of finite joins; the inequality for meets then follows from the identity $\bigwedge A_{\lambda} = -\bigvee -A_{\lambda}$.)

**Theorem 4.16.** Let $\mathcal{M}$ be a von Neumann algebra. If $\rho$ is a quantum distance function on $\mathcal{M}$ (Definition 2.7) then the associated spectral Lipschitz gauge $L_s$ is a quantum Lipschitz gauge. Conversely, if $\mathcal{L}$ is a quantum Lipschitz gauge then

$$\rho_\mathcal{L}(P, Q) = \sup\{a \geq 0 : \text{some self-adjoint } A \in \mathcal{M} \otimes \mathcal{B}(l^2) \text{ satisfies } L_s(A) \leq 1, P \leq P_{(-\infty, 0]}(A), \text{ and } Q \leq P_{[a, \infty)}(A)\}$$

is a quantum distance function. The two constructions are inverse to each other.

**Proof.** Let $\rho$ be a quantum distance function on $\mathcal{M}$ and let $L_s$ be the associated spectral Lipschitz gauge. We verify properties (i) – (v) of Definition 4.13. Properties (i) and (ii) are easy. For (iii), let $\epsilon > 0$ and let $a, b \in \mathbb{R}$, $a < b$, and observe that

$$P_{-\epsilon, a}(A \vee \hat{A}) = P_{-\epsilon, a}(A) \wedge P_{-\epsilon, a}(\hat{A})$$

and

$$P_{b, \infty}(A \vee \hat{A}) \leq P_{b, \epsilon, \infty}(A \vee \hat{A}) = P_{b, \epsilon, \infty}(A) \wedge P_{b, \epsilon, \infty}(\hat{A}).$$

So

$$\rho(P_{-\epsilon, a}(A \vee \hat{A}), P_{b, \infty}(A \vee \hat{A}))$$

$$\geq \min\{\rho(P_{-\epsilon, a}(A \vee \hat{A}), P_{b, \epsilon, \infty}(A)), \rho(P_{-\epsilon, a}(A \vee \hat{A}), P_{b, \epsilon, \infty}(\hat{A}))\}$$

$$\geq \min\{\rho(P_{-\epsilon, a}(A), P_{b, \epsilon, \infty}(A)), \rho(P_{-\epsilon, a}(A), P_{b, \epsilon, \infty}(\hat{A}))\},$$

and hence

$$\frac{b - a}{\rho(P_{-\epsilon, a}(A \vee \hat{A}), P_{b, \infty}(A \vee \hat{A}))} \leq \max \left\{ \frac{b - a}{\rho(P_{-\epsilon, a}(A), P_{b, \epsilon, \infty}(A))}, \frac{b - a}{\rho(P_{-\epsilon, a}(A), P_{b, \epsilon, \infty}(\hat{A}))} \right\}.$$
This completes the proof that $L \geq 0$. To verify property (iv), as in the proof of property (iii) it will suffice to show that

$$\rho(P(-\infty,a)(A_{[B]}), P_{(b,\infty)}(A_{[B]})) \geq \rho(P(-\infty,a)(A_{[B]}), P_{(b,\infty)}(A_{[B]})).$$

Thus let $P = P_{(-\infty,a)}(A)$ and $Q = P_{(b,\infty)}(A)$ and observe that $P_{(b,\infty)}(A_{[B]}) = [BQ]$ and

$$P_{(-\infty,a)}(A_{[B]}) = I - P_{(\infty,a)}(A_{[B]})$$

$$= I - [BP_{(\infty,a)}(A)]$$

$$= I - [B(I - P)].$$

We claim that $[B^*(I - [B(I - P)])] \leq P$. To see this suppose $v \perp \text{ran}(P)$. Then $Bv \in \text{ran}(B(I - P))$, so that $\langle Bv, w \rangle = 0$ for any $w \in \text{ran}(I - [B(I - P)])$. That is, $(v, B^*w) = 0$, so we have shown that $v \perp \text{ran}(B^*(I - [B(I - P)]))$. This proves the claim. It now follows from Definition 2.7 (vi) that

$$\rho(P_{(-\infty,a)}(A_{[B]}), P_{(b,\infty)}(A_{[B]})) = \rho(I - [B(I - P)], [BQ])$$

$$= \rho([B^*(I - [B(I - P)])), Q)$$

$$\geq \rho(P, Q),$$

as desired.

Finally, to prove property (v), suppose $A_\lambda \to A$ boundedly weak operator and let $a, b \in \mathbb{R}$, $a < b$. Let $\epsilon > 0$. Then applying (ii) to $(A - aI) \oplus (bI - A) \in (\mathcal{M} \otimes \mathcal{B}(l^2)) \oplus (\mathcal{M} \otimes \mathcal{B}(l^2))$, we get nets of projections $\{P_\kappa\}$ and $\{Q_\kappa\}$ in $\mathcal{M} \otimes \mathcal{B}(l^2)$ such that $P_\kappa \to P_{(-\infty,a]}(A)$ and $Q_\kappa \to P_{[b,\infty)}(A)$ for some $\lambda$. By Definition 2.7 (vii)

$$\rho(P_{(-\infty,a]}(A), P_{[b,\infty)}(A)) \geq \inf_\kappa \rho(P_\kappa, Q_\kappa)$$

$$\geq \inf_\lambda \rho(P_{(-\infty,a+\epsilon]}(A_\lambda), P_{[b,\infty)}(A_\lambda))$$

and so

$$\frac{b - a}{\rho(P_{(-\infty,a]}(A), P_{[b,\infty)}(A))} \leq \sup_\lambda \frac{b - a}{\rho(P_{(-\infty,a+\epsilon]}(A_\lambda), P_{[b,\infty)}(A_\lambda))}$$

$$\leq \frac{b - a}{b - a - 2\epsilon} \sup \lambda L_s(A_\lambda).$$

Taking $\epsilon \to 0$ and the supremum over $a$ and $b$ then yields $L_s(A) \leq \sup L_s(A_\lambda)$. This completes the proof that $L_s$ is a quantum Lipschitz gauge.

Next let $L$ be any quantum Lipschitz gauge. We verify that $\rho_L$ is a quantum distance function. Property (i) follows by taking $A = 0$ in the definition of $\rho_L$, property (ii) is immediate, and property (iii) follows from the fact that $L(aI - A) = L(A)$. For property (iv), suppose there exist self-adjoint operators $A, \hat{A} \in \mathcal{M} \otimes \mathcal{B}(l^2)$ such that $L(A), L(\hat{A}) \leq 1, R \leq P_{(-\infty,0]}(A), P \leq P_{(a,\infty)}(A), R \leq P_{(-\infty,0]}(\hat{A}),$ and $Q \leq P_{(a,\infty)}(\hat{A})$. Then we have $L(A \vee \hat{A}) \leq 1,$

$$R \leq P_{(-\infty,0]}(A \vee \hat{A})$$

and

$$P \vee Q \leq P_{(a,\infty)}(A \vee \hat{A}),$$

as desired.
and taking the supremum over \( a \) yields 
\[
\rho_L(P \lor Q, R) \geq \min\{\rho_L(P, R), \rho_L(Q, R)\}.
\]
The reverse inequality is trivial, so this verifies property (iv). For property (v) let \( \epsilon > 0 \) and find a self-adjoint operator \( A \in M \mathcal{B}(\mathbb{F}) \) such that \( \mathcal{L}(A) \leq 1 \), \( P \leq P_{(-\infty,0]}(A) \), and \( R \leq P_{[b,\infty)}(A) \) where \( b = \rho_L(P, R) - \epsilon \). Let \( a \in \mathbb{R} \) be the largest value such that \( Q \leq P_{[a,\infty)}(A) \). Then \( \rho_L(P, Q) \geq a \), and letting \( \tilde{Q} = P_{(-\infty,a+\epsilon]}(A) \) we have \( QQ \neq 0 \) and \( \rho_L(\tilde{Q}, R) \geq b - a - \epsilon \). So
\[
\rho_L(P, R) = b + \epsilon \leq \rho_L(P, Q) + \rho_L(\tilde{Q}, R) + 2\epsilon,
\]
and taking the supremum over \( \tilde{Q} \) and \( \epsilon \to 0 \) yields property (v).

To establish property (vi), let \( \epsilon > 0 \) and find a self-adjoint operator \( A \in M \mathcal{B}(\mathbb{F}) \) such that \( \mathcal{L}(A) \leq 1 \), \( [B^*P] \leq P_{(-\infty,0]}(A) \), and \( Q \leq P_{[a,\infty)}(A) \) where \( a = \rho_L([B^*P], Q) - \epsilon \). By replacing \( A \) with \( A^+ = A \lor 0 \) we may assume it is positive. We now have \( \mathcal{L}(A_{[B]}(A) \leq 1 \) and \( [BQ] \leq P_{[a,\infty)}(A_{[B]}) \). We claim that \( P \leq P_{(-\infty,0]}(A_{[B]}) \), that is, \( P \) is orthogonal to \( P_{[a,\infty)}(A_{[B]}) = [BP_{[a,\infty)}(A)] \). For if \( v \in \text{ran}(P) \) and \( w \in \text{ran}(P_{[a,\infty)}(A)) \) then \( \langle v, Bw \rangle = \langle B^*v, w \rangle = 0 \) because \([B^*P] \leq P_{(-\infty,0]}(A) \), and this proves the claim. Thus
\[
\rho_L(P, [BQ]) \geq a = \rho_L([B^*P], Q) - \epsilon,
\]
and taking \( \epsilon \to 0 \), we conclude that \( \rho_L(P, [BQ]) \geq \rho_L([B^*P], Q) \). The reverse inequality follows by symmetry (property (iii)), interchanging \( P \) with \( Q \) and \( B \) with \( B^* \). For property (vii), suppose \( P_\lambda \to P \) and \( Q_\lambda \to Q \) and let \( a < \lim sup \rho_L(P_\lambda, Q_\lambda) \). As we frequently have \( \rho_L(P_\lambda, Q_\lambda) > a \), find self-adjoint operators \( A_\lambda \in M \mathcal{B}(\mathbb{F}) \) such that \( \mathcal{L}(A_\lambda) \leq 1 \), \( P_\lambda \leq P_{(-\infty,0]}(A_\lambda) \), and \( Q_\lambda \leq P_{[a,\infty)}(A_\lambda) \). Replacing \( A_\lambda \) with \( (A_\lambda \lor 0) \land aI \), we may assume \( 0 \leq A_\lambda \leq aI \).

Now pass to a weak operator convergent subnet and let \( A = \lim A_\lambda \). Then \( A \) is positive, \( P_\lambda \leq P_{[0]}(A_\lambda) \), and for any \( v \in \text{ran}(P) \) we have \( \langle P_\lambda v, v \rangle \to \langle Pv, v \rangle = \|v\|^2 \), which implies that \( \langle Av, v \rangle = \lim(A_\lambda v, v) = 0 \) and hence that \( Av = 0 \). This shows that \( P \leq P_{[0]}(A) \), and applying the same argument to \( aI - A \) yields \( Q \leq P_{[a]}(A) \).

So \( \rho_L(P, Q) \geq a \), and taking \( a \to \lim sup \rho_L(P_\lambda, Q_\lambda) \) yields the desired inequality. This completes the proof that \( \rho_L \) is a quantum distance function.

Now let \( \rho \) be any quantum distance function, let \( L_s \) be the associated spectral Lipschitz gauge, and let \( \rho_{L_s} \) be the quantum distance function derived from \( L_s \). Then \( \rho = \rho_{L_s} \) by Theorem 4.1.

Finally, let \( \mathcal{L} \) be any quantum Lipschitz gauge and let \( L_s \) be the spectral Lipschitz gauge associated to \( \rho_L \). We immediately have that \( \mathcal{L}(A) \leq 1 \) implies \( \rho_L(P_{(-\infty,a]}(A), P_{[b,\infty)}(A)) \geq b - a \) for any \( a, b \in \mathbb{R} \), \( a < b \), and hence \( L_s(A) \leq 1 \); this shows that \( L_s(A) \leq \mathcal{L}(A) \) for all \( A \). Conversely, suppose \( L_s(A) < 1 \). For each \( a, b \in \mathbb{R} \), \( a < b \), we have \( \rho_L(P_{(-\infty,a]}(A), P_{[b,\infty)}(A)) > b - a \) so we can find a self-adjoint operator \( A_{ab} \in M \mathcal{B}(\mathbb{F}) \) such that \( \mathcal{L}(A_{ab}) \leq 1 \), \( P_{(-a,0]}(A) \leq P_{(-\infty,a]}(A_{ab}) \), and \( P_{[b,\infty)}(A) \leq P_{[b,\infty)}(A_{ab}) \). Then
\[
A = \bigwedge_{|a| \leq \|A\|, a < b \leq \|A\|} \left( (A_{ab} \lor aI) \land bI \right),
\]
which shows that \( \mathcal{L}(A) \leq 1 \). We conclude that \( \mathcal{L}(A) \leq L_s(A) \) for all \( A \), and hence the two are equal.
Corollary 4.17. Let \( \mathcal{M} \subseteq \mathcal{B}(H) \) be a von Neumann algebra. If \( \mathcal{V} \) is a quantum pseudometric on \( \mathcal{M} \) then \( L_\varepsilon \), defined by

\[
L_\varepsilon(A) = \sup \left\{ \frac{b-a}{t} : a, b \in \mathbb{R}, a < b, \mathcal{P}_{(-\infty,a]}(A)(\mathcal{V}_t) \mathcal{P}_{[b,\infty)}(A) \neq 0 \right\}
\]

(for \( A \in \mathcal{M} \otimes \mathcal{B}(H) \) self-adjoint), is a quantum Lipschitz gauge on \( \mathcal{M} \). Conversely, if \( \mathcal{L} \) is a quantum Lipschitz gauge on \( \mathcal{M} \) then \( \mathcal{V} = \{ \mathcal{V}_t \} \) with

\[
\mathcal{V}_t = \{ B \in \mathcal{B}(H) : \mathcal{P}_{(-\infty,a]}(A)(\mathcal{V}_t) \mathcal{P}_{[a,\infty)}(A) = 0 \text{ for all } a > t \text{ and all self-adjoint } A \in \mathcal{M} \otimes \mathcal{B}(l^2) \text{ with } \mathcal{L}(A) \leq 1 \}
\]

is a quantum pseudometric on \( \mathcal{M} \). The two constructions are inverse to each other.

The corollary follows straightforwardly from Theorems 2.45 and 4.16.

As we mentioned in Section 4.1, the spectral Lipschitz gauge is not a seminorm in general (although it is in the abelian case by Proposition 4.7 and ([35], Corollary 1.21)). We conclude this section with a simple example which demonstrates this; in fact, we show that a sum of spectrally Lipschitz operators need not be spectrally Lipschitz.

Example 4.18. Let \( \mathcal{M} = M_2(\mathbb{C}) \) and let \( n \in \mathbb{N} \). Define a quantum metric on \( \mathcal{M} \) by letting \( a = 2/n \) and \( b = c = 1 \) in Proposition 3.6. Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \frac{1}{n} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). Then \( A \) has eigenvalues 0 and 1 and the distance between the corresponding spectral subspaces is 1, so \( L_\varepsilon(A) = 1 \). The operator \( B \) has eigenvalues 0 and 2/n and the distance between the corresponding spectral subspaces is 2/n, so \( L_\varepsilon(B) = 1 \). But the operator

\[
A + B = \frac{1}{n} \cdot \begin{pmatrix} n+1 & 1 \\ 1 & 1 \end{pmatrix}
\]

has eigenvalues \((n + 2 \pm \sqrt{n^2 + 4})/2n \) and the distance between the corresponding spectral subspaces is \( 2/n \), so \( L_\varepsilon(A + B) = \sqrt{n^2 + 4}/2 \). This witnesses the failure of the seminorm property \( L(A + B) \leq L(A) + L(B) \). Moreover, by taking the \( l^\infty \) direct sum of this sequence of examples as \( n \) ranges over \( \mathbb{N} \), we obtain operators \( A \) and \( B \) such that \( L_\varepsilon(A) = L_\varepsilon(B) = 1 \) and \( L_\varepsilon(A + B) = \infty \). Thus a sum of two spectrally Lipschitz operators need not be spectrally Lipschitz.

4.3. Commutation Lipschitz numbers. We have just seen that spectral Lipschitz gauges are algebraically very poorly behaved. However, there is a related alternative notion that has good algebraic properties. Recall that \( |\mathcal{V}|_1 \) denotes the closed unit ball of the Banach space \( \mathcal{V} \).

Definition 4.19. Let \( \mathcal{V} \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). We define the commutation Lipschitz number of \( A \in \mathcal{M} \) to be

\[
L_c(A) = \sup \left\{ \frac{||[A,C]||}{t} : t \geq 0, C \in |\mathcal{V}|_1 \right\},
\]

where \([A,C] = AC - CA\) and with the convention that \( \frac{0}{0} = 0 \). We say that \( A \) is commutation Lipschitz if \( L_c(A) < \infty \) and we call \( L_c \) the commutation Lipschitz gauge. We define

\[
\text{Lip}(\mathcal{M}) = \{ A \in \mathcal{M} : L_c(A) < \infty \}
\]

and equip \( \text{Lip}(\mathcal{M}) \) with the norm \( \|A\|_L = \max\{\|A\|, L_c(A)\} \).
Note that taking $t = 0$ shows that $L_c(A) < \infty$ implies $A \in \mathcal{V}_0 \subseteq \mathcal{M}$. This is a quantum version of the fact that Lipschitz functions on a pseudometric space respect the equivalence relation which makes points equivalent if their distance is zero.

We can define the commutation Lipschitz number of any operator in $\mathcal{M} \otimes \mathcal{B}(l^2)$ by stabilization (see Section 2.4). Explicitly, we set
\[
L_c(A) = \sup \left\{ \frac{\| [A, C \otimes I] \|}{t} : t \geq 0, C \in [\mathcal{V}_1]_1 \right\}
\]
for $A \in \mathcal{M} \otimes \mathcal{B}(l^2)$.

There is an analogue of the measurable de Leeuw map ([35], Definition 1.19) for commutation Lipschitz operators. We use it to establish the basic properties of commutation Lipschitz numbers.

**Definition 4.20.** Let $\mathcal{V}$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. The operator de Leeuw map is the map $\Phi: A \mapsto \bigoplus_{t,C} \frac{1}{t}[A,C]$, where $\lambda$ ranges over all pairs $(t,C)$ such that $t \geq 0$ and $C \in [\mathcal{V}_1]_1$, from Lip($\mathcal{M}$) into the $l^\infty$ direct sum $\bigoplus \mathcal{B}(H)$. Also define $\pi: \mathcal{M} \to \bigoplus \mathcal{B}(H)$ by $\pi(A) = \bigoplus A$.

**Proposition 4.21.** Let $\mathcal{V}$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and let $\Phi$ be the operator de Leeuw map.

(a) For all $A \in \text{Lip}(\mathcal{M})$ we have $L_c(A) = \| \Phi(A) \|$.

(b) $\Phi$ is linear and we have $\Phi(AB) = \pi(A)\Phi(B)+\Phi(A)\pi(B)$ for all $A,B \in \text{Lip}(\mathcal{M})$.

(c) The graph of $\Phi$ is weak* closed in $\mathcal{M} \otimes \bigoplus \mathcal{B}(H)$.

**Proof.** Parts (a) and (b) are straightforward. For part (c), let $\{A_\lambda\}$ be a net in $\text{Lip}(\mathcal{M})$ and suppose $A_\lambda \bigoplus \bigoplus [A_\lambda, C] \to A \oplus B$ weak*; we must show that $A \in \text{Lip}(\mathcal{M})$ and $\Phi(A) = B$. But
\[
\langle [A_\lambda, C]w, v \rangle = \langle A_\lambda Cw, v \rangle - \langle A_\lambda w, C^*v \rangle \\
= \langle ACw, v \rangle - \langle Aw, C^*v \rangle \\
= \langle [A, C]w, v \rangle
\]
for all $C$ and all $v, w \in H$, and this implies that $B = \bigoplus [A, C]$. Thus $A \in \text{Lip}(\mathcal{M})$ and $\Phi(A) = B$, as desired. \hfill $\square$

**Corollary 4.22.** Let $\mathcal{V}$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$.

(a) $L_c(aA) = |a| \cdot L_c(A)$, $L_c(A^*) = L_c(A)$, $L_c(A + B) \leq L_c(A) + L_c(B)$, and $L_c(AB) \leq \|A\|L_c(B) + \|B\|L_c(A)$ for all $A, B \in \text{Lip}(\mathcal{M})$ and $a \in \mathcal{C}$.

(b) If $\{A_\lambda\} \subseteq \mathcal{M}$ is a net that converges weak* to $A \in \mathcal{M}$ then $L_c(A) \leq \sup L_c(A_\lambda)$.

(c) Lip($\mathcal{M}$) is a self-adjoint unital subalgebra of $\mathcal{M}$. It is a dual Banach space for the norm $\| \cdot \|_L$.

**Proof.** Part (a) is straightforward. For part (b) we use the fact that $A_\lambda \to A$ weak* implies $[A_\lambda, C] \to [A, C]$ weak* (and that weak* limits cannot increase norms). The fact that Lip($\mathcal{M}$) is a unital subalgebra of $\mathcal{M}$ follows from part (a), and the fact that it is a dual space follows from Proposition 4.21(c) since the map $A \mapsto A \oplus \Phi(A)$ is an isometric isomorphism between Lip($\mathcal{M}$) and the graph of $\Phi$. \hfill $\square$

The operator de Leeuw map does not respect adjoints. In order to ensure $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{M}$ we could change the definition to a direct sum
of derivations into $\mathcal{B}(H \oplus H)$ of the form

$$A \mapsto \begin{bmatrix} 0 & i[A, C] \\ i[A, C^*] & 0 \end{bmatrix}.$$ 

Proposition 4.21 would still hold and $\Phi$ would then be a $W^*$-derivation in the sense of (32, Definition 7.4.1) or (34, Definition 10.3.7).

We now prove our main result about commutation Lipschitz numbers, which relates them to spectral Lipschitz numbers.

**Theorem 4.23.** Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. Let $A \in \mathcal{M}$ be self-adjoint.

(a) Let $C \in \mathcal{B}(H)$. If $P_{(-\infty, a]}(A)CP_{b, \infty}(A) = 0$ for all $a, b \in \mathbb{R}$, $a < b$, such that $b - a > t$, then $||[A, C]|| \leq t||C||$.

(b) $L_c(A) \leq L_s(A)$.

*Proof.* Part (b) follows from part (a) because we have $P_{(-\infty, a]}(A)CP_{b, \infty}(A) = 0$ for all $t > 0$, all $C \in [\chi_1, 1]$, and all $a, b \in \mathbb{R}$, $a < b$, such that $b - a > tL_s(A)$ (since the latter implies $\rho(P_{(-\infty, a]}(A), P_{b, \infty}(A)) > t$). So part (a) allows us to infer that $||[A, C]|| \leq tL_s(A)$ for all $t > 0$ and all $C \in [\chi_1, 1]$. This shows that $L_c(A) \leq L_s(A)$.

We prove part (a). Since $A$ is self-adjoint, we may suppose $H = L^2(\mathbb{X}, \mu)$ and $A = M_f$ for some real-valued $f \in L^\infty(\mathbb{X}, \mu)$. Let $\epsilon > 0$ and find a real-valued simple function $g \in L^\infty(\mathbb{X}, \mu)$ such that $\|f - g\|_\infty \leq \epsilon$; then we still have

$$P_{(-\infty, a]}(M_f)CP_{b, \infty}(M_g) = 0$$

when $b - a > t + 2\epsilon$ because $P_{(-\infty, a]}(M_g) \leq P_{(-\infty, a+\epsilon]}(M_f)$ and $P_{b, \infty}(M_g) \leq P_{b-\epsilon, \infty}(M_f)$. Since $[M_g, C] \to [M_f, C]$ as $\epsilon \to 0$, it will suffice to show that $||[M_g, C]|| \leq (t + 2\epsilon)||C||$.

Let

$$\mathcal{V} = \{B \in \mathcal{B}(H) : P_{(-\infty, a]}(M_g)BP_{b, \infty}(M_g) = 0 \text{ for all } a, b \in \mathbb{R}, a < b, \text{ such that } b - a > t + 2\epsilon\},$$

observe that $\mathcal{V}$ is a $W^*$-bimodule over the von Neumann algebra of bounded multiplication operators, and define $\Phi : \mathcal{V} \to \mathcal{V}$ by $\Phi(B) = [M_g, B]$. Say $g = \sum_{i=1}^k a_i \chi_{S_i}$ such that the $S_i$ partition $\mathbb{X}$ and write $P_i = M_{\chi_{S_i}}$. Then

$$\Phi^n(B) = \sum_{i,j=1}^k (a_i - a_j)^n P_i BP_j,$$

so if we define $e^{is\Phi}$ by a power series we get

$$e^{is\Phi}(B) = \sum_{i,j=1}^k e^{is(a_i - a_j)} P_i BP_j = M_{e^{isg}}BM_{e^{-isg}}.$$}

Thus $\|e^{is\Phi}(B)\| = \|B\|$ for all $s \in \mathbb{R}$, which implies that $\Phi$ is a “Hermitian” operator (3, Definition 5.1) on the complex Banach space $\mathcal{V}$ by (3, Lemma 5.2). It then follows from Corollary 26.6 of [4] that the norm of $\Phi$ equals its spectral radius $\lim ||\Phi^n||^{1/n}$.

Since $|a_i - a_j| > t + 2\epsilon$ implies $P_i BP_j = 0$ the expression

$$\Phi^n(B) = \sum_{i,j=1}^k (a_i - a_j)^n P_i BP_j$$

yields the estimate

$$\|\Phi^n\| \leq k^2(t + 2\epsilon)^n.$$
Thus \( \| \Phi \| = \lim \| \Phi^n \|^{1/n} \leq t + 2\epsilon \), and we conclude that \( \| [M_g, C] \| = \| \Phi(C) \| \leq (t + 2\epsilon) \| C \| \), as desired. \( \square \)

**Corollary 4.24.** Let \( V \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). Then any spectrally Lipschitz self-adjoint element of \( \mathcal{M} \) is commutation Lipschitz.

The converse fails: in general not every commutation Lipschitz operator is spectrally Lipschitz. This follows from Example 4.18 since the operators \( A \) and \( B \) in that example will both be commutation Lipschitz by the preceding corollary, and hence their sum will be too by Corollary 4.22 (a).

However, in the measure theory setting the spectral and commutation Lipschitz gauges agree.

**Corollary 4.25.** Let \((X, \mu)\) be a finitely decomposable measure space, let \( p \) be a measurable pseudometric on \( X \), and let \( V_p \) be the associated quantum pseudometric on \( \mathcal{M} \cong L^\infty(X, \mu) \) (Theorem 2.22). Then for any real-valued \( f \in L^\infty(X, \mu) \) we have \( L_\epsilon(M_f) = L(M_f) = L(f) \).

**Proof.** The equality \( L_\epsilon(M_f) = L(f) \) was Proposition 4.17 and we have \( L_\epsilon(M_f) \leq L_\epsilon(M_f) \) by Theorem 4.23. For the reverse inequality let \( p, q \in L^\infty(X, \mu) \) be nonzero projections and let \( \epsilon > 0 \). Say \( p = \chi_S \), \( q = \chi_T \) and apply the equality \( R = R_{V_p} \) in Theorem 1.16 to the measurable relation \( R = \{ (p', q') : \rho(p, q) < \rho(p, q) + \epsilon \} \) to find \( C \in \mathcal{B}(L^2(X, \mu)) \) such that \( M_p C M_q \neq 0 \) but \( M_p C M_q' = 0 \) whenever \( \rho(p', q') > \rho(p, q) + \epsilon \).

Decompose \( p \) and \( q \) as \( p = \sum p_i \) and \( q = \sum q_j \) so that \( \text{ran}_{p_i}(f) \) and \( \text{ran}_{q_j}(f) \) have diameter at most \( \epsilon \) for all \( i \) and \( j \). Fix values of \( i \) and \( j \) such that \( B = M_p C M_q \neq 0 \). We may assume that \( \| B \| = 1 \). Let \( a \in \text{ran}_{p_i}(f) \) and \( b \in \text{ran}_{q_j}(f) \). Then \( B \in V_{\rho(p, q) + \epsilon} \) and

\[
\| [M_f, B] - (a - b)B \| \leq \| (M_f - aI) M_p, B \| + \| BM_{g_j}(M_f - bI) \| \leq 2\epsilon
\]

so

\[
L_c(M_f) \geq \frac{\| [M_f, B] \|}{\rho(p, q) + \epsilon} \geq \frac{|a - b| - 2\epsilon}{\rho(p, q) + \epsilon} \geq \frac{d(\text{ran}_{p_i}(f), \text{ran}_{q_j}(f)) - 2\epsilon}{\rho(p, q) + \epsilon}.
\]

Taking \( \epsilon \to 0 \) and the supremum over \( p \) and \( q \) then yields \( L(f) \leq L_c(M_f) \). \( \square \)

Theorem 4.23 is nontrivial even in the atomic abelian case. For example, let \( H = L^2(\mathbb{Z}) \) and let \( U \in \mathcal{B}(L^2(\mathbb{Z})) \) be the bilateral shift. Let \( f(z) = \sum_{k=-n}^n a_k e^{ikz} \) be a trigonometric polynomial of degree \( n \), let \( C = \sum_{k=-n}^n a_k U^k \) be the corresponding polynomial in \( U \), and for \( N \in \mathbb{N} \) define

\[
g_N(k) = \begin{cases} N & \text{if } k > N \\ k & \text{if } -N \leq k \leq N \\ -N & \text{if } k < -N \end{cases}.
\]

Then giving \( Z \) the standard metric, we have \( L_c(M_{g_N}) = L(g_n) = 1 \) and \( D(C) = n \). So Theorem 4.23 (b) implies that \( \| [M_{g_N}, C] \| \leq n \| C \| \). Taking inner products against standard basis vectors shows that the weak operator limit of \( [M_{g_N}, C] \) as \( N \to \infty \) is the operator \( \hat{C} = \sum_{k=-n}^n k a_k U^k \). Thus we conclude that \( \| \hat{C} \| \leq n \| C \| \).

Taking the Fourier transform, we get

\[
\| \hat{f} \|_\infty = \| \hat{C} \| \leq n \| C \| = n \| f \|_\infty.
\]
This is Bernstein’s inequality from classical complex analysis (see [5]).

We include one more general result about Lip($\mathcal{M}$) which states that it is weak* dense in $\mathcal{M}$ if $V$ is a quantum metric. This is not surprising, but the proof is interesting because it uses some of the machinery that we have built up in the last two sections.

**Proposition 4.26.** Let $V$ be a quantum metric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$. Then Lip($\mathcal{M}$) is weak* dense in $\mathcal{M}$.

**Proof.** The weak* closure of Lip($\mathcal{M}$) is a von Neumann subalgebra of $\mathcal{M}$ by Corollary 4.22 (c). By the double commutant theorem, to prove equality we must show that Lip($\mathcal{M}$)' $\subseteq \mathcal{M}'$. Thus let $C \in B(H) - \mathcal{M}'$; we must find an operator in Lip($\mathcal{M}$) that does not commute with $C$.

Since $V$ is a quantum metric we have $C \notin V_0$. Thus $D(C) > 0$ and so by Proposition 2.10 there exist projections $P, Q \in \mathcal{M} \overline{\otimes} B(l^2)$ such that $\rho(P, Q) > 0$ and $P(C \otimes I)Q \neq 0$. Now let $A$ be the spectral join in $\mathcal{M} \overline{\otimes} B(l^2)$ defined in Lemma 4.12 with $R = P$ and $c = \rho(P, Q)$. Then $PA = 0$ and $AQ = \rho(P, Q) \cdot Q$ so

$$P[A, C \otimes I]Q = -\rho(P, Q)P(C \otimes I)Q \neq 0.$$  

Since $[A, C \otimes I]$ is nonzero there exists a rank one projection $P_0 \in B(l^2)$ such that if $B = (I \otimes P_0)A(I \otimes P_0)$ then

$$[B, C \otimes I] = (I \otimes P_0)[A, C \otimes I](I \otimes P_0) \neq 0.$$  

Say $B = B_0 \otimes P_0$; then $[B, C \otimes I] = [B_0, C] \otimes P_0$ and this implies that $[B_0, C] \neq 0$. Finally, for any $t$ and any $D \in [V_t]_1$ we have

$$\| [B_0, D] \| = \| [B, D \otimes I] \| = \| (I \otimes P_0)[A, D \otimes I](I \otimes P_0) \| \leq \| [A, D \otimes I] \| \leq t$$  

by Theorem 4.23 (b) since $L_c(A) \leq 1$ (relative to the quantum pseudometric $V \otimes I$ on $\mathcal{M} \overline{\otimes} B(l^2)$). This shows that $L_c(B_0) \leq 1$, so that $B_0 \in$ Lip($\mathcal{M}$). Thus we have found an operator in Lip($\mathcal{M}$) that does not commute with $C$. \hfill $\Box$

Finally, we relate Lip($\mathcal{M}$) to $C^*(U_h, V_h)$ in the quantum tori.

**Proposition 4.27.** Let $h \in \mathbb{R}$ and let $d$ be a translation invariant metric on $T^2$ that is quasi-isometric (i.e., homeomorphic via a bijection which is Lipschitz in both directions) to the standard metric. Equip $W^*(U_h, V_h)$ with the quantum metric $V_0$ defined in Theorem 4.20 (b). Then Lip($W^*(U_h, V_h)$) is (operator norm) densely contained in the $C^*$-algebra $C^*(U_h, V_h)$ generated by $U_h$ and $V_h$.

**Proof.** If $A \in W^*(U_h, V_h)$ is commutation Lipschitz then

$$\| A - \theta_{x,y}(A) \| = \| [A, M_{x,(m_x+n_y)}] \| \to 0$$  

as $(x, y) \to (0, 0)$, which implies that $A \in C^*(U_h, V_h)$ by (24, Proposition 6.6.5). This shows that Lip($W^*(U_h, V_h)$) $\subseteq C^*(U_h, V_h)$.

For density, it will be enough to prove that $U_h$ and $V_h$ belong to Lip($W^*(U_h, V_h)$) since this will entail that every polynomial in $U_h, V_h, U_h^{-1} = U_h^*, V_h^{-1} = V_h^*$ is in Lip($W^*(U_h, V_h)$) by Corollary 4.22 (c). We will prove that the real and imaginary parts of $U_h$ (actually, any self-adjoint Lipschitz element of $C^*(U_h) \cong C(T)$) are spectrally Lipschitz, and hence commutation Lipschitz by Corollary 4.24. This implies that $U_h$ is commutation Lipschitz by Corollary 4.22 (c). The analogous statements for $V_h$ are proven similarly.
Identify $C^*(U_h)$ with $C(T)$, let $A \in C^*(U_h)$ be self-adjoint, and suppose $A \in \text{Lip}(T) \subset C(T)$. Recall (Definition 3.13) that $V_t = \mathcal{E}_t(S_t)$ consists of the operators whose $(k,l)$ Fourier term belongs to $\mathcal{E}_t(S_t) \cdot U_h^k V_h^l$, for all $k$ and $l$, where $\mathcal{E}_t(S_t)$ is the weak* closed span of the operators $M_{\epsilon t(mx+ny)}$ with $(x,y) \in S_t$.

Now conjugate all operators in $B(L^2(T^2))$ by the unitary $M_{\epsilon t}$ (an easy computation directly from Definition 3.11), $\mathcal{E}_t(S_t)$ is unaffected, and $U_h$ becomes the shift $e_{m,n} \mapsto e_{m+1,n}$. In the $L^2(T^2)$ picture, $A$ now becomes multiplication by a Lipschitz function in the first variable and the operators $M_{\epsilon t(mx+ny)}$ with $(x,y) \in S_t$, which generate $\mathcal{E}_t(S_t)$, become translations by vectors of length at most $t$. Thus if $A = M_f, f \in \text{Lip}(T)$, then for any $t > 0$ we have

$$P_{(-\infty,a]}(A)\mathcal{E}_t(S_t/L(f))P_{[b,\infty)}(A) = 0$$

for any $a, b \in \mathbb{R}$, $a < b$, such that $b - a > t$. But since the spectral projections of $A$ commute with $U_{-\epsilon}$ and $V_{-\epsilon}$, this implies that

$$P_{(-\infty,a]}(A)V_t/L(f)P_{[b,\infty)}(A) = 0$$

for any $a, b \in \mathbb{R}$, $a < b$, such that $b - a > t$. So $\rho(P_{(-\infty,a]}(A), P_{[b,\infty)}(A)) \geq (b-a)/L(f)$, and we conclude that $L_s(A) \leq L(f)$. Thus, we have shown that $A$ is spectrally Lipschitz, as claimed.

4.4. Little Lipschitz spaces. Classically, little Lipschitz functions are Lipschitz functions which satisfy a kind of “local flatness” condition (see Chapter 3 of [92]). On nice spaces like connected Riemannian manifolds the only little Lipschitz functions are constant functions, but on totally disconnected or Hölder spaces they are abundant.

We can formulate spectral and commutation versions of the little Lipschitz condition in our setting.

**Definition 4.28.** Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$ and let $\rho$ be the associated quantum distance function.

(a) A self-adjoint operator $A \in \mathcal{M}$ is **spectrally little Lipschitz** if it is spectrally Lipschitz and for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho(P_{(-\infty,a]}(A), P_{[b,\infty)}(A)) \leq \epsilon$$

for any $a, b \in \mathbb{R}$, $a < b$, such that $b - a > t$. So $\rho(P_{(-\infty,a]}(A), P_{[b,\infty)}(A)) \leq \delta$.

(b) An operator $A \in \mathcal{M}$ is **commutation little Lipschitz** if it is commutation Lipschitz and for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\|[A,C]\|}{t} \leq \epsilon$$

whenever $t \leq \delta$ and $C \in [V_t]$. We let $\text{lip}(\mathcal{M})$ be the set of elements of $\text{Lip}(\mathcal{M})$ that are commutation little Lipschitz, equipped with the inherited norm $\|\cdot\|_L$.

This generalizes the atomic abelian case; see Corollary 4.34 below.

**Proposition 4.29.** Let $V$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq B(H)$. Then $\text{lip}(\mathcal{M})$ is a closed unital self-adjoint subalgebra of $\text{Lip}(\mathcal{M})$.

**Proof.** All of the assertions follow from the observation that $A \in \text{Lip}(\mathcal{M})$ belongs to $\text{lip}(\mathcal{M})$ if and only if $\Phi_\alpha(A) \to 0$ as $t \to 0$, where $\alpha$ ranges over all pairs $(t,C)$ such that $t > 0$ and $C \in [V_t]$, and $\Phi_\alpha(A) = \frac{1}{t}[A,C]$. □
We omit the proofs of the next two results; they are straightforward adaptations of the proofs of Propositions 4.16 and 4.19.

**Proposition 4.30.** Let $\rho$ be a quantum distance function on a von Neumann algebra $\mathcal{M}$ and let $A \in \mathcal{M}$ be self-adjoint and spectrally little Lipschitz. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{d(S,T)}{\rho(P_{S}(A), P_{T}(A))} \leq \epsilon$$

for any Borel sets $S, T \subseteq \mathbb{R}$ such that $\rho(P_{S}(A), P_{T}(A)) \leq \delta$.

**Proposition 4.31.** Let $\rho$ be a quantum distance function on a von Neumann algebra $\mathcal{M}$, let $A \in \mathcal{M}$ be self-adjoint and spectrally little Lipschitz, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz. Then $f(A)$ is spectrally little Lipschitz.

**Proposition 4.32.** Let $\rho$ be a quantum distance function on a von Neumann algebra $\mathcal{M}$ and let $A, \hat{A} \in \mathcal{M}$ be self-adjoint and spectrally little Lipschitz. Then their spectral join and meet are also spectrally little Lipschitz.

**Proof.** The statement about joins follows from the inequality

$$\frac{b-a}{\rho(P_{(−∞,a]}(A \vee A), P_{[b,∞)}(A \vee A))} \leq \max \left\{ \frac{b-a}{\rho(P_{(−∞,a]}(A), P_{[b,∞)}(A))}, \frac{b-a}{\rho(P_{(−∞,a]}(A), P_{[b,c,∞)}(A))} \right\}$$

established in the course of showing that $L_{\alpha}$ satisfies property (iii) of Definition 4.14 in the proof of Theorem 4.16. (Whichever term on the right dominates the left side must have a smaller denominator, so the spectral little Lipschitz condition can be applied.) The statement about meets can either be proven similarly or reduced to the statement about joins via the identity $A \wedge \hat{A} = -(A \vee -\hat{A})$. □

**Proposition 4.33.** Let $\mathcal{V}$ be a quantum pseudometric on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. Then any spectrally little Lipschitz self-adjoint element of $\mathcal{M}$ is commutation little Lipschitz.

**Proof.** Let $A \in \mathcal{M}$ be self-adjoint and spectrally little Lipschitz. Given $\epsilon > 0$, find $\delta > 0$ witnessing the spectral little Lipschitz condition. Fix $0 \leq t \leq \delta$ and $C \in |\mathcal{V}|_{1}$. Suppose $P_{(−∞,a]}(A)CP_{[b,∞)}(A) \neq 0$; then $\rho(P_{(−∞,a]}(A), P_{[b,∞)}(A)) \leq t \leq \delta$, so the spectral little Lipschitz condition implies that $t \geq \rho(P_{(−∞,a]}(A), P_{[b,∞)}(A)) \geq (b-a)/\epsilon$. This shows that if $b-a > \epsilon t$ then $P_{(−∞,a]}(A)CP_{[b,∞)}(A) = 0$, and so Theorem 4.23 (a) yields $\|A,C\| \leq \epsilon t$. We conclude that $A$ is commutation little Lipschitz. □

**Corollary 4.34.** Let $X$ be a set, let $d$ be a pseudometric on $X$, let $\mathcal{M} \cong l^{\infty}(X)$ be the von Neumann algebra of bounded multiplication operators on $l^{2}(X)$, and let $\mathcal{V}_{d}$ be the quantum pseudometric on $\mathcal{M}$ corresponding to $d$ (Proposition 4.27). If $f \in l^{\infty}(X)$ is real-valued then $M_{f}$ is spectrally little Lipschitz if and only if $M_{f}$ is commutation little Lipschitz if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x,y) \leq \delta \quad \Rightarrow \quad \frac{|f(x) - f(y)|}{d(x,y)} \leq \epsilon.$$
Proof. Let \( f \in l^\infty(X) \) be real-valued. By Corollary 4.25 we may assume \( f \) is Lipschitz. First, suppose \( f \) is little Lipschitz, i.e., it satisfies the \( \epsilon, \delta \) condition stated in the proposition. Let \( \epsilon > 0 \) and find \( \delta > 0 \) satisfying this condition. Then if \( a, b \in \mathbb{R}, a < b \), satisfy \( \rho(P_{[-\infty,a]}(M_f), P_{[b,\infty]}(M_f)) < \delta \), we can find sequences \( \{x_n\} \subseteq f^{-1}((-\infty, a]) \) and \( \{y_n\} \subseteq f^{-1}([b, \infty)) \) such that \( d(x_n, y_n) \to \rho(P_{[-\infty,a]}(M_f), P_{[b,\infty]}(M_f)) \) and \( d(x_n, y_n) \leq \delta \) for all \( n \). Then \( |f(x_n) - f(y_n)| \geq b - a \) for all \( n \), so

\[
\frac{b - a}{\rho(P_{[-\infty,a]}(M_f), P_{[b,\infty]}(M_f))} \leq \limsup \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)} \leq \epsilon.
\]

This verifies the spectral little Lipschitz condition for \( M_f \).

Next, if \( M_f \) is spectrally little Lipschitz then it is commutation little Lipschitz by Proposition 4.33.

Finally, suppose \( M_f \) is commutation little Lipschitz, let \( \epsilon > 0 \), and find \( \delta > 0 \) satisfying the commutation little Lipschitz condition. For any \( x, y \in X \) with \( t = d(x, y) \leq \delta \), the operator \( V_{xy} \) then belongs to \([V_t]_t\) with \( t \leq \delta \), so

\[
\frac{|f(x) - f(y)|}{d(x, y)} = \frac{\|[M_f, V_{xy}]\|}{t} \leq \epsilon.
\]

This shows that \( f \) is little Lipschitz. \( \square \)

Finally, we note that just as in the abelian case, little Lipschitz operators are abundant when the underlying metric space is Hölder. This result is a straightforward consequence of the definitions of spectral and commutation little Lipschitz operators, together with the fact that if \( 0 < \alpha < 1 \) then \( t/t^\alpha \to 0 \) as \( t \to 0 \).

Proposition 4.35. Let \( V \) be a quantum pseudometric on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \) and let \( 0 < \alpha < 1 \). Let \( \text{lip}^\alpha(\mathcal{M}) \) denote the little Lipschitz space relative to the Hölder quantum pseudometric \( V^\alpha \) (Section 3.7). Then \( \text{Lip}(\mathcal{M}) \subseteq \text{lip}^\alpha(\mathcal{M}) \). Any self-adjoint element of \( \mathcal{M} \) that is spectrally Lipschitz relative to \( V \) will be spectrally little Lipschitz relative to \( V^\alpha \).

The most significant substantive result about little Lipschitz spaces states that \( \text{lip}^\alpha(X)^{**} \cong \text{Lip}^\alpha(X) \) for any compact metric space \( X \) and any \( 0 < \alpha < 1 \). We conjecture that this remains true for general quantum metrics, with the hypothesis “\( X \) is compact” modified to “the closed unit ball of \( \text{Lip}^\alpha(\mathcal{M}) \) is compact for the operator norm topology”.

5. QUANTUM UNIFORMITIES

In this brief final chapter we propose a quantum analog of the notion of a uniform space. A classical uniformity on a set can be defined in terms of a family of relations called “entourages”. We can give a natural quantum generalization of this definition which is representation independent (Theorem 5.3) and effectively reduces to the classical definition in the atomic abelian case (Proposition 5.4). We find that the basic theory of uniformities, including their presentability in terms of families of pseudometrics, generalizes to the quantum setting (Theorem 5.6), and we also develop some basic material on quantum uniform continuity (which, like the Lipschitz condition, bifurcates into two distinct but related notions). However, we do not attempt to mine the subject in detail.
5.1. Basic results. We start with our definition of a quantum uniformity. It is not overtly expressed in terms of dual operator bimodules, but we immediately show that there is an equivalent reformulation in these terms.

Definition 5.1. A quantum uniformity is a family $U$ of dual operator systems contained in some $B(H)$ that satisfies the following conditions:

(i) any dual operator system that contains a member of $U$ belongs to $U$
(ii) if $U, \tilde{U} \in U$ then $U \cap \tilde{U} \subseteq U$
(iii) for every $U \in U$ there exists $\tilde{U} \in U$ such that $\tilde{U}^2 \subseteq U$.

The elements of $U$ are quantum entourages. $U$ is a quantum uniformity on the von Neumann algebra $M \subseteq B(H)$ if $M' \subseteq \bigcap U$, and it is Hausdorff if $M' = \bigcap U$. A subfamily $U_0 \subseteq U$ generates $U$ if every member of $U$ contains some member of $U_0$.

Equivalently, we could work with dual unital operator spaces and require that $U \in U \Rightarrow U \cap U^* \in U$.

Note that the intersection $\bigcap U$ is always a von Neumann algebra. (It is clearly a dual operator system, and it is an algebra by property (iii).) So if $U$ is a quantum uniformity on the von Neumann algebra $M$ then $\bigcap U$ is a von Neumann algebra containing $M'$ and we can ensure the Hausdorff property by passing from $M$ to the commutant of $\bigcap U$ (a possibly smaller von Neumann algebra).

Proposition 5.2. Let $U$ be a quantum uniformity on a von Neumann algebra $M \subseteq B(H)$. Then $U$ is generated by the subfamily

$U_0 = \{U \in U : U$ is a quantum relation on $M\}$.

Proof. Let $U \in U$ and apply property (iii) of Definition 5.1 twice to obtain $\tilde{U} \in U$ such that $\tilde{U}^2 \subseteq U$. Then $M'U'M' \subseteq \bigcap U$ is a quantum relation on $M$ that contains $\tilde{U}$, and hence is a quantum entourage, and it is contained in $\tilde{U}^2 \subseteq U$. □

Thus, we could just as well define a quantum uniformity on $M$ to be a family $U$ of quantum relations on $M$ such that

(i) $M' \subseteq U = U^*$ for all $U \in U$
(ii) any quantum relation $U$ that contains a member of $U$ and satisfies $U = U^*$ belongs to $U$
(iii) if $U, \tilde{U} \in U$ then $U \cap \tilde{U} \subseteq U$
(iv) for every $U \in U$ there exists $\tilde{U} \in U$ such that $\tilde{U}^2 \subseteq U$.

Given the preceding, the next two results follow from, respectively, Theorem 1.3 and Proposition 1.4. Order the quantum uniformities on a von Neumann algebra by inclusion.

Theorem 5.3. Let $H_1$ and $H_2$ be Hilbert spaces and let $M_1 \subseteq B(H_1)$ and $M_2 \subseteq B(H_2)$ be isomorphic von Neumann algebras. Then any isomorphism induces an order preserving 1-1 correspondence between the quantum uniformities on $M_1$ and the quantum uniformities on $M_2$.

Proposition 5.4. Let $X$ be a set and let $M \cong l^\infty(X)$ be the von Neumann algebra of bounded multiplication operators on $l^2(X)$. If $\Phi$ is a uniformity on $X$ then

$U_\Phi = \{U : V_R \subseteq U$ for some $R \in \Phi\}$
(\mathcal{Y}_R \text{ as in Proposition 1.4, } \mathcal{U} \text{ ranging over dual operator systems}) \text{ is a quantum uniformity on } \mathcal{M}; \text{ conversely, if } \mathcal{U} \text{ is a quantum uniformity on } \mathcal{M} \text{ then}

\Phi_\mathcal{U} = \{ U \subseteq X^2 : R_\mathcal{U} \subseteq U \text{ for some } \mathcal{U} \in \mathcal{U} \}

(R_\mathcal{U} \text{ as in Proposition 1.4}) \text{ is a uniformity on } X. \text{ The two constructions are inverse to each other.}

Finally, we show that every quantum uniformity arises from a family of quantum pseudometrics.

**Definition 5.5.** Let \( \{ \mathcal{V}_\lambda \} \) with \( \mathcal{V}_\lambda = \{ \mathcal{V}_t^\lambda \} \) be a family of quantum pseudometrics on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \). \text{The associated quantum uniformity on } \mathcal{M} \text{ is the family of dual operator systems } \mathcal{U} \subseteq \mathcal{B}(H) \text{ such that}

\[ \mathcal{V}_t^{\lambda_1} \cap \cdots \cap \mathcal{V}_t^{\lambda_n} \subseteq \mathcal{U} \]

for some \( \epsilon > 0 \), some \( n \in \mathbb{N} \), and some \( \lambda_1, \ldots, \lambda_n \). Thus, it is the smallest quantum uniformity that contains \( \mathcal{V}_t^\lambda \) for every \( \lambda \) and every \( t > 0 \).

**Theorem 5.6.** Every quantum uniformity on a von Neumann algebra \( \mathcal{M} \subseteq \mathcal{B}(H) \) is the quantum uniformity associated to some family of quantum pseudometrics on \( \mathcal{M} \).

**Proof.** Let \( \mathcal{U} \) be a quantum uniformity on \( \mathcal{M} \) and let \( \mathcal{F} \) be the family of all quantum pseudometrics \( \mathcal{V} \) on \( \mathcal{M} \) with the property that \( \mathcal{V}_t \) is a quantum entourage for all \( t > 0 \). We claim that \( \mathcal{U} \) is the quantum uniformity associated to \( \mathcal{F} \). The inclusion \( \supseteq \) is easy because \( \mathcal{V}_t^{\lambda_1} \cap \cdots \cap \mathcal{V}_t^{\lambda_n} \) is a quantum entourage for all \( \epsilon > 0 \) and all \( \mathcal{V}_{\lambda_1}, \ldots, \mathcal{V}_{\lambda_n} \in \mathcal{F} \).

To prove the reverse inclusion, let \( U_t \in \mathcal{U} \); we will find a quantum pseudometric \( \mathcal{V} \in \mathcal{F} \) such that \( \mathcal{V}_t \subseteq U_t \) for some \( t > 0 \). To do this, first find a sequence \( \{ U_n \} \) of quantum entourages such that \( U_{t+1}^2 \subseteq U_n \) for all \( n \). For each \( s > 0 \) define

\[ W_s = \text{span} \{ A_1 \cdots A_k : k \in \mathbb{N} \text{ and } A_i \in U_{n_i} \text{ for some } i \leq k \text{ where } \sum_{i=1}^k 2^{-n_i} \leq s \} \]

and then define a \( W^k \)-filtration \( \mathcal{V} \) of \( \mathcal{B}(H) \) by setting \( \mathcal{V}_t = \bigcap_{s \geq t} W_s \) for all \( t \geq 0 \). It is straightforward to check that \( \mathcal{V} \) is a quantum pseudometric on \( \mathcal{M} \). We claim that \( W_{2^{-n}} = U_n \) for all \( n \). It is clear that \( U_n \subseteq W_{2^{-n}} \). For the reverse inclusion, fix \( A_1 \cdots A_k \in W_{2^{-n}} \); we want to show that \( A_1 \cdots A_k \in U_n \). If \( k = 1 \) the assertion is trivial, so we may inductively assume it holds for all \( n \) and all smaller values of \( k \). Suppose \( k \geq 2 \) and split the product up into three segments \( A_1 \cdots A_{j_1}, A_{j_1+1} \cdots A_{j_2}, \text{ and } A_{j_2+1} \cdots A_k \) such that the corresponding sums \( \sum_{i=1}^{j_1} 2^{-n_i}, \sum_{j_1+1}^{j_2} 2^{-n_i}, \text{ and } \sum_{j_2+1}^k 2^{-n_i} \) are each at most \( 2^{-n-1} \). Then each of the three subproducts is in \( U_{n+1} \) by the induction hypothesis, and hence the entire product is in \( U_{n+1}^{2^{-n}} \subseteq U_n \). This completes the proof of the claim.

It follows that \( \mathcal{V} \in \mathcal{F} \) (since for each \( t > 0 \), \( \mathcal{V}_t \) contains \( W_{2^{-n}} = U_n \) for any \( n \) such that \( 2^{-n} \leq t \)) and that \( \mathcal{V}_t \subseteq W_1 \subseteq U_1 \) for any \( t < 1 \), as desired. \( \square \)

5.2. **Uniform continuity.** The natural morphisms between uniform spaces are the uniformly continuous maps. As with the Lipschitz condition, in the quantum setting we have both a spectral version and a commutator version of this notion.
Definition 5.7. Let $\mathcal{U}$ be a quantum uniformity on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. 

(a) A self-adjoint operator $A \in \mathcal{M}$ is spectrally uniformly continuous if for every $\epsilon > 0$ there exists $\mathcal{U} \in \mathcal{U}$ such that

$$P_{(-\infty, a]}(A)\mathcal{U}P_{[b, \infty)}(A) = 0$$

for all $a, b \in \mathbb{R}$, $a < b$, such that $b - a > \epsilon$.

(b) An operator $A \in \mathcal{M}$ is commutation uniformly continuous if for every $\epsilon > 0$ there exists $\mathcal{U} \in \mathcal{U}$ such that

$$\|\mathcal{U}C\| \leq \epsilon$$

for every $C \in [\mathcal{U}]_1$. We let $UC(\mathcal{M})$ be the set of commutation uniformly continuous operators in $\mathcal{M}$, with the inherited operator norm.

This generalizes the atomic abelian case; see Corollary 5.10 below.

For quantum uniformities arising from quantum pseudometrics we can characterize spectral and commutation uniform continuity directly in terms of the $\mathcal{W}^*$-filtration.

Proposition 5.8. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra equipped with a quantum pseudometric $V$, and let $\mathcal{U}$ be the quantum uniformity generated by the quantum relations $\mathcal{V}_t$ for $t > 0$.

(a) A self-adjoint operator $A \in \mathcal{M}$ is spectrally uniformly continuous relative to $\mathcal{U}$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho(P_{(-\infty, a]}(A), P_{[b, \infty)}(A)) \geq \delta$$

for every $a, b \in \mathbb{R}$, $a < b$, with $b - a > \epsilon$.

(b) An operator $A \in \mathcal{M}$ is commutation uniformly continuous relative to $\mathcal{U}$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathcal{U}C\| \leq \epsilon$ for every $C \in [\mathcal{V}_0]_1$.

The proof of this proposition is straightforward.

Next we observe that, just as for Lipschitz conditions, spectral uniform continuity is stronger than commutation uniform continuity. This result follows immediately from Theorem 5.8.3 (a).

Theorem 5.9. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra equipped with a quantum uniformity $\mathcal{U}$ and let $A \in \mathcal{M}$ be self-adjoint. If $A$ is spectrally uniformly continuous then it is commutation uniformly continuous.

Corollary 5.10. Let $X$ be a set, let $\Phi$ be a uniformity on $X$, let $\mathcal{M} \cong l^\infty(X)$ be the von Neumann algebra of bounded multiplication operators on $l^2(X)$, and let $\mathcal{U}_\Phi$ be the quantum uniformity on $\mathcal{M}$ corresponding to $\Phi$ (Proposition 5.4). If $f \in l^\infty(X)$ is real-valued then $M_f$ is spectrally uniformly continuous if and only if $M_f$ is commutation uniformly continuous if and only if for every $\epsilon > 0$ there exists $R \in \Phi$ such that $(x, y) \in R$ implies $|f(x) - f(y)| \leq \epsilon$.

Proof. Suppose that $f$ is uniformly continuous in the sense stated in the corollary, let $\epsilon > 0$, and find an entourage $R$ witnessing uniform continuity of $f$. Then $\mathcal{V}_R \in \mathcal{U}_\Phi$, and $(x, y) \in R$ implies $|f(x) - f(y)| \leq \epsilon$, so that $P_{(-\infty, a]}(M_f)\mathcal{V}_R P_{[b, \infty)}(M_f) = 0$ whenever $b - a > \epsilon$, for every $(x, y) \in R$. Since $\mathcal{V}_R$ is generated by $\{V_{xy} : (x, y) \in R\}$, this shows that $P_{(-\infty, a]}(M_f)\mathcal{V}_R P_{[b, \infty)}(M_f) = 0$ whenever $b - a > \epsilon$, and this demonstrates that $M_f$ is spectrally uniformly continuous.
Spectral uniform continuity implies commutation uniform continuity by Theorem 5.9.

Finally, if \( f \) is not uniformly continuous then there exists \( \epsilon > 0 \) such that for every entourage \( R \in \Phi \) there is a pair \( (x, y) \in R \) with \( |f(x) - f(y)| > \epsilon \). Then the operator \( V_{xy} \) belongs to \( [V_R]_1 \), and we have \( \|M_f, V_{xy}\| = |f(x) - f(y)| > \epsilon \). Since every quantum entourage contains a quantum entourage of the form \( V_R \), this shows that \( M_f \) is not commutation uniformly continuous. So commutation uniform continuity of \( M_f \) implies uniform continuity of \( f \). \( \square \)

Next we look at algebra and lattice properties of spectral and commutation uniform continuity.

**Proposition 5.11.** Let \( M \subseteq \mathcal{B}(H) \) be a von Neumann algebra equipped with a quantum uniformity \( U \) and let \( A, \tilde{A} \in M \) be self-adjoint and spectrally uniformly continuous. Then their spectral join and meet are also spectrally uniformly continuous.

**Proof.** Let \( \epsilon > 0 \) and find quantum entourages \( U \) and \( \tilde{U} \) such that
\[
P_{(-\infty, a]}(A)UP_{(b, \infty)}(A) = P_{(-\infty, a]}(\tilde{A})\tilde{U}P_{(b, \infty)}(\tilde{A}) = 0
\]
for all \( a, b \in \mathbb{R} \) with \( b - a > \epsilon \). Then
\[
P_{(-\infty, a]}(A \vee \tilde{A}) = P_{(-\infty, a]}(A) \wedge P_{(-\infty, a]}(\tilde{A})
\]
and
\[
P_{(b, \infty)}(A \vee \tilde{A}) = P_{(b, \infty)}(A) \vee P_{(b, \infty)}(\tilde{A}),
\]
so
\[
P_{(-\infty, a]}(A \vee \tilde{A})(U \cap \tilde{U})P_{(b, \infty)}(A \vee \tilde{A}) = 0.
\]
This shows that \( A \vee \tilde{A} \) is uniformly continuous. The fact that \( A \wedge \tilde{A} \) is uniformly continuous can either be proven analogously or inferred from the equality \( A \wedge \tilde{A} = (\neg A \vee (\neg \tilde{A})) \). \( \square \)

**Proposition 5.12.** Let \( M \subseteq \mathcal{B}(H) \) be a von Neumann algebra equipped with a quantum uniformity \( U \). Then \( UC(M) \) is a unital C*-algebra.

The proof of this proposition is routine.

The sum of two spectrally uniformly continuous operators need not be spectrally uniformly continuous. Indeed, this is the case for the operators constructed in Example 4.18 as one can easily check using the characterization of spectral uniform continuity given in Proposition 5.8. Since \( A \) and \( B \) are both spectrally Lipschitz, it follows that they are spectrally uniformly continuous, but their sum fails spectrally uniform continuity because \( \rho(P_{(-\infty, 1/2]}(\tilde{A} + \tilde{B}), P_{(1, \infty)}(\tilde{A} + \tilde{B})) \neq 0 \).

Recall that a quantum uniformity \( U \) on a von Neumann algebra \( M \) is Hausdorff if \( M = \bigcap U \) (Definition 5.1). We now show that under this hypothesis \( UC(M) \) is weak* dense in \( M \); this result is analogous to, and easily deduced from, the corresponding result about weak* density of \( \text{Lip}(M) \) in \( M \) (Proposition 4.26).

**Proposition 5.13.** Let \( U \) be a Hausdorff quantum uniformity on a von Neumann algebra \( M \subseteq \mathcal{B}(H) \). Then \( UC(M) \) is weak* dense in \( M \).
Proof. We must show that $UC(M) \subseteq M'$. Thus let $C \in \mathcal{B}(H) - M'$; we must find an operator in $UC(M)$ that does not commute with $C$.

Since $U$ is Hausdorff and $C \notin M'$, we must have $C \notin U$ for some quantum entourage $U$. By Theorem 5.14 there is a quantum pseudometric $V$ on $M$ such that every $V_t$ is a quantum entourage and $V_t \subseteq U$ for some $t > 0$. Then $C \notin V_0$, so by Proposition 4.26 there is an operator $B \in V_0^c \subseteq M$ that is commutation Lipschitz relative to $V$, and hence commutation uniformly continuous relative to $U$, and does not commute with $C$. □

We conclude with a simple result about commutation uniform continuity in the quantum tori. Recall that on a compact space every continuous function is uniformly continuous.

**Proposition 5.14.** Let $h \in \mathbb{R}$ and let $d$ be a translation invariant metric on $T^2$ that is equivalent to the flat Euclidean metric. Equip $W^*(U_h, V_h)$ with the quantum metric $V_0$ defined in Theorem 3.16 (b). Then $UC(W^*(U_h, V_h)) = C^*(U_h, V_h)$.

**Proof.** The proof is similar to, but slightly simpler than, the proof of Proposition 4.27. If $A \in W^*(U_h, V_h)$ is commutation uniformly continuous then we must have

$$\|A - \theta_{x,y}(A)\| = \|[A, M_{e^{i(mx+ny)}}]\| \to 0$$

as $(x, y) \to (0, 0)$, and this implies that $A \in C^*(U_h, V_h)$ by (34, Proposition 6.6.5).

Conversely, by Proposition 5.12 to establish that every operator in $C^*(U_h, V_h)$ is commutation uniformly continuous it will suffice to show this for $U_h$ and $V_h$. We will prove that the real and imaginary parts of $U_h$ (actually, any self-adjoint element of $C^*(U_h)$) are spectrally uniformly continuous, and hence commutation uniformly continuous by Theorem 5.9. The analogous statements for $V_h$ are proven similarly.

Let $A \in C^*(U_h)$ be self-adjoint. As in the proof of Proposition 4.27, $V_t = V_{E_0(S_t)}$ consists of the operators whose $(k, l)$ Fourier term belongs to $E_0(S_t) \cdot U_h^k V_h^l$, for all $k$ and $l$.

Now conjugate $B(l^2(Z^2))$ by the operator $M_{e^{ibm+n/2}}$. Then $U_h$ will still commute with both $U_{-h}$ and $V_{-h}$. $E_0(S_t)$ is unaffected, and $U_h$ becomes the shift $e_{m,m} \mapsto e_{m+1,n}$. In the $L^2(T^2)$ picture, $A$ now becomes multiplication by a continuous (hence uniformly continuous) function in the first variable and the operators $M_{e^{i(mx+ny)}}$ with $(x, y) \in S_t$, which generate $E_0(S_t)$, become translations by vectors of length at most $\ell$. Thus given $\epsilon > 0$ we can find $\delta > 0$ such that

$$P_{(-\infty,a]}(A)E_0(S_\delta)P_{[b,\infty)}(A) = 0$$

for any $a, b \in \mathbb{R}$, $a < b$, such that $b - a > \epsilon$. But since the spectral projections of $A$ commute with $U_{-h}$ and $V_{-h}$, this implies that

$$P_{(-\infty,a]}(A)V_hP_{[b,\infty)}(A) = 0$$

for any $a, b \in \mathbb{R}$, $a < b$, such that $b - a > \epsilon$. Thus, we have shown that $A$ is spectrally uniformly continuous, as claimed. □
REFERENCES

[1] C. Akemann, The general Stone-Weierstrass problem, J. Funct. Anal. 4 (1969), 277-294.
[2] D. P. Blecher, Tensor products of operator spaces II, Canad. Math. J. 44 (1992), 75-90.
[3] F. F. Bonsall and J. Duncan, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, Cambridge University Press, 1971.
[4] ———, Numerical Ranges II, Cambridge University Press, 1973.
[5] R. B. Burckel, An Introduction to Classical Complex Analysis, vol. I, 1979.
[6] J. Cheeger, M. Gromov, and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differential Geo. 17 (1982), 15-53.
[7] J. Cnops, An Introduction to Dirac Operators on Manifolds, 2002.
[8] A. Connes, Noncommutative Geometry, Academic Press, 1994.
[9] A. Connes and J. Lott, The metric aspect of noncommutative geometry, in New Symmetry Principles in Quantum Field Theory (Cargèse, 1991), Plenum Press (1992), 54-93.
[10] R. Duan, S. Severini, and A. Winter, Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovász function, arXiv:1002.2514.
[11] E. G. Effros and Z.-J. Ruan, On approximation properties for operator spaces, Internat. J. Math. 1 (1990), 163-187.
[12] M. Hamana, Tensor products for monotone complete C*-algebras I, Japan J. Math. 8 (1982), 259-283.
[13] F. Hirsch, Intrinsic metrics and Lipschitz functions, J. Evol. Equ. 3 (2003), 11-25.
[14] ———, Measurable metrics, intrinsic metrics and Lipschitz functions, Current trends in potential theory (2005), 47-61.
[15] ———, Measurable metrics and Gaussian concentration, Forum Math. 18 (2006), 345-363.
[16] W. Klingenberg, Riemanian Geometry (second edition), 1995.
[17] E. Knill, R. Laflamme, and L. Viola, Theory of quantum error correction for general noise, Phys. Rev. Lett. 84 (2000), 2525-2528, arXiv:quant-ph/9908066.
[18] W. Page, Topological Uniform Structures, 1978.
[19] T. Palmer, Banach Algebras and the General Theory of ∗-Algebras, Vol. 2, Cambridge University Press, 2001.
[20] M. Rieffel, Continuous fields of C*-algebras coming from group cocycles and actions, Math. Ann. 283 (1989), 631-643.
[21] ———, Noncommutative tori—a case study of noncommutative differentiable manifolds, in Geometric and Topological Invariants of Elliptic Operators (1990), 191-211.
[22] ———, Gromov-Hausdorff Distance for Quantum Metric Spaces, Mem. Amer. Math. Soc. 168, 2004, arXiv:math.OA/0011063.
[23] ———, Leibniz seminorms for “Matrix algebras converge to the sphere”, manuscript, arXiv:math.OA/0707.3229.
[24] J. Roe, Elliptic Operators, Topology and Asymptotic Methods, Longman, 1998.
[25] Z.-J. Ruan, On the predual of dual algebras, J. Operator Theory 27 (1993), 179-192.
[26] R. Schrader, Finite propagation speed and causal free quantum fields on networks, J. Phys. A: Math. Theor. 42 (2009), 495401.
[27] M. Takesaki, Theory of Operator Algebras I, Springer, 1979.
[28] A. Vasy, Geometric optics and the wave equation on manifolds with corners, in Recent Advances in Differential Equations and Mathematical Physics (2006), 315-333.
[29] N. Weaver, Nonatomic Lipschitz spaces, Studia Math. 115 (1995), 277-289.
[30] ———, Weak*-closed derivations from C[0,1] into L∞[0,1], Canad. Math. Bull. 39 (1996), 367-375.
[31] ———, Lipschitz algebras and derivations of von Neumann algebras, J. Funct. Anal. 139 (1996), 261-300.
[32] ———, Lipschitz Algebras, World Scientific, 1999.
[33] ———, Lipschitz algebras and derivations II: exterior differentiation, J. Funct. Anal. 178 (2000), 64-112, arXiv:math.FA/9807096.
[34] ———, *Mathematical Quantization*, CRC Press, 2001.

[35] ———, Quantum relations, manuscript, arXiv:math.OA/1005.0354.

Department of Mathematics, University of California, Davis, CA 95616

Department of Mathematics, Washington University in Saint Louis, Saint Louis, MO 63130

E-mail address: greg@math.ucdavis.edu, nweaver@math.wustl.edu