Nonlinear Phase Modification of the Schrödinger Equation

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Abstract

A nonlinear modification of the Schrödinger equation is proposed in which the Lagrangian density for the Schrödinger equation is extended by terms polynomial in $\Delta^m \ln (\Psi^*/\Psi)$ multiplied by $\Psi^*\Psi$. This introduces a homogeneous nonlinearity in a Galilean invariant manner through the phase $S$ rather than the amplitude $R$ of the wave function $\Psi = R \exp (iS)$. From this general scheme we choose the simplest minimal model defined in some reasonable way. The model in question offers the simplest way to modify the Bohm formulation of quantum mechanics so as to allow a leading phase contribution to the quantum potential and a leading quantum contribution to the probability current removing asymmetries present in Bohm’s original formulation. It preserves most of physically relevant properties of the Schrödinger equation including stationary states of quantum-mechanical systems. It can be thought of as the simplest model of nonlinear quantum mechanics of extended objects among other such models that also emerge within the general scheme proposed. The extensions of this model to $n$ particles and the question of separability of compound systems are studied. It is noted that there exists a weakly separable extension in addition to a strongly separable one. The place of the general modification scheme in a broader spectrum of nonlinear modifications of the Schrödinger equation is discussed. It is pointed out that the models it gives rise to have a unique definition of energy in that the field-theoretical energy functional coincides with the quantum-mechanical one. It is found that the Lagrangian for its simplest variant represents the Lagrangian for a restricted version of the Doebner-Goldin modification of this equation. It is also noted that a large class of particular models generated by this scheme contradict the thesis that the homogeneity of a nonlinear Schrödinger equation automatically entails its weak separability.

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1 Introduction

Several years ago Staruszkiewicz \[1\] put forward new, and what at that time seemed to be unique \[1\] a way of modifying the Schrödinger equation by adding to its action density a term $a(\Delta S)^2$, where $S$ is the quantum-mechanical phase, historically motivated by his theory of free electromagnetic phase \[3\]. It is a peculiarity of this modification that the constant $a$ is dimensionless in natural units, however its dimensionless character is restricted to the three-dimensional space. In general, in the same system of units, $a$ has the dimensions of meter$^{3-d}$, where $d$ is the dimension of space in which to construct a theory. One observes that $a(\Delta S)^2 = -a [\Delta \ln (\Psi^*/\Psi)]^2 / 4$, where $\Psi = R \exp (iS)$. It is this density that together with the Lagrangian density for the Schrödinger equation,

$$L_{SE}(\vec{r}, t) = \frac{i \hbar}{2} \left( \Psi^* \frac{\partial \Psi^*}{\partial t} - \frac{\partial \Psi^*}{\partial t} \Psi \right) - \frac{\hbar^2}{2m} \vec{\nabla} \Psi^* \vec{\nabla} \Psi - \Psi^* V \Psi, \tag{1}$$

constitutes the integrand of the complete action for the Schrödinger equation with a potential $V$ in the modification proposed by Staruszkiewicz.

The purpose of the present paper is to suggest yet another possible way of modifying the Schrödinger equation which seems to have some advantages of greater universality and generality compared to Staruszkiewicz’s version and rather insignificant shortcomings. One can view our proposal as a variation of sorts on his modification which preserves its main theme in that it grants the phase an important dynamical role in the modified equations of motion and also in that it employs the nonlinearity of the $\Delta S$ term. Nevertheless, it is a completely distinct construction. Recently, there has been a considerable interest in nonlinear modifications of quantum mechanics spurned largely by Weinberg’s proposal \[4, 5\] of a relatively general framework for nonlinear quantum mechanics. However, in spite of this, with the exception of the Staruszkiewicz modification that precedes Weinberg’s, we have not found in the literature a construction similar to ours.

This paper is organized as follows. We introduce our modification in the next section where we formulate its general scheme, discuss distinct classes of particular modifications that emerge within this scheme, and single out two examples of the simplest nonlinear Schrödinger equations characteristic of these classes. These two models are discussed in more detail throughout the rest of the paper. In the subsequent section, we present the simplest minimal version of the modification. In section 4, as one more way of demonstrating its peculiar properties, we confront the modification with some notable nonlinear generalizations of the Schrödinger equation put forward earlier. This section is followed by conclusions where also the motivations for studying the modification proposed and its conceivable applications are elaborated on.

2 The Modification

The basic idea of our modification is to supplement the Lagrangian density of the Schrödinger equation by terms involving the phase in explicit yet a Galilean-invariant manner by means of the $\ln (\Psi^*/\Psi)$

\[^{1}\] As shown in \[2\], the original Staruszkiewicz modification can be extended in a manner that preserves its characteristic features.

\[^{2}\] It should be noted that here $S$ is dimensionless for it represents the angle. This differs from a more common convention in which $\Psi = R \exp (iS/\hbar)$ so that $S$ has the dimensions of action. We will adhere to this convention throughout the most of this paper.
contribution. For this reason, the terms containing \( \vec{\nabla} \ln (\Psi^*/\Psi) \) have to be excluded as breaking this invariance. Indeed, since under the Galilean transformation of coordinates, \( \vec{x} = \vec{x}' + \vec{v}t' \), \( t' = t \), the phase changes as

\[
S' = S - m\vec{v} \cdot \vec{x} + \frac{1}{2} m\vec{v}^2 t,
\]

the lowest order operator that complies with the condition of Galilean invariance is \( \Delta \). We also require that the modification be independent of the dimensionality of space in which to develop it. For this to be accomplished one needs to multiply the \( [\Delta^k \ln (\Psi^*/\Psi)]^n \) terms by \( R^2 = \Psi^*\Psi \), \( n \) and \( k \) being positive integers. Therefore, the modified Lagrangian density we propose consists, in addition to (1), of terms

\[
L_{kn} = c_{kn} \Psi^* [\Delta^k \ln (\Psi^*/\Psi)]^n \Psi = C_{kn} (\Delta^k S)^n R^2,
\]

where \( c_{kn} \) and \( C_{kn} \) are certain presumably small dimensional constants, complex (in general) and real, respectively.

Now, the most general Lagrangian density for the modified Schrödinger equation is \( L(R, S) = L_{SE}(R, S) + L_{NL}(R, S) \), where, after (1),

\[
-L_{SE}(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m} \left[ (\vec{\nabla} R)^2 + R^2 (\vec{\nabla} S)^2 \right] + R^2 V,
\]

and

\[
L_{NL}(R, S) = - \sum_{k,n} C_{kn} (\Delta^k S)^n R^2.
\]

In principle, \( k \) and \( n \) run from 1 to infinity, but for practical purposes such a structure cannot be deemed very attractive and one needs either to terminate this series at some point or to choose only some terms out of it. Before we proceed to the modified Schrödinger equation, for the sake of further discussion, let us first write down the linear Schrödinger equation in the nonlinear Madelung representation [4],

\[
\hbar \frac{\partial R^2}{\partial t} + \frac{\hbar^2}{m} \vec{\nabla} \cdot (R^2 \vec{\nabla} S) = 0,
\]

\[
\frac{\hbar^2}{m} \Delta R - 2\hbar R \frac{\partial S}{\partial t} - 2RV - \frac{\hbar^2}{m} R (\vec{\nabla} S)^2 = 0.
\]

The modified Schrödinger equation in the same formulation is equivalent to a system of two nonlinear equations that derive from \( L(R, S) \),

\[
\hbar \frac{\partial R^2}{\partial t} + \frac{\hbar^2}{m} \vec{\nabla} \cdot (R^2 \vec{\nabla} S) - \sum_{k,n} nC_{kn} \Delta^k \left[ (\Delta^k S)^{n-1} R^2 \right] = 0,
\]

\[
\frac{\hbar^2}{m} \Delta R - 2\hbar R \frac{\partial S}{\partial t} - 2RV - \frac{\hbar^2}{m} R (\vec{\nabla} S)^2 - 2\sum_{k,n} C_{kn} (\Delta^k S)^n R = 0,
\]

where the first of these equations is the continuity equation for the probability density \( \rho = R^2 \) and the current

\[
\vec{j} = \frac{\hbar^2}{m} R^2 \vec{\nabla} S - \sum_{k,n} nC_{kn} \vec{\nabla} \left[ \Delta^{k-1} \left( (\Delta^k S)^{n-1} R^2 \right) \right]
\]
Due to the higher order derivatives occurring in the Lagrangian densities (3) and (5), one obtains the above equations from the principle of least action assuming that not only variations of $S$ and $R$ vanish on the boundary in the spatial infinity but also variations of their derivatives, such as, for instance, $\tilde{\nabla} S$ and $\tilde{\nabla} R$. The energy functionals attributed to these Lagrangian densities are given by the spatial integrals of these densities and since they are homogeneous in $R^2$ one expects the total “modified” energy to be finite for the configurations that satisfy the normalization condition $\int d^3x R^2 = 1$. For the very same reason one can weaken our condition on the variations of derivatives by admitting arbitrary such variations. The boundary terms will then not contribute to the variations of the total action for their integrals vanish in the spatial infinity as $R$ vanishes there. It should be noted that the total energy functional,

$$E = \int d^3x \left\{ \frac{\hbar^2}{2m} \left[ (\tilde{\nabla} R)^2 + R^2 (\tilde{\nabla} S)^2 \right] + \sum_{k,n} C_{kn} \left( \Delta^k S \right)^n R^2 + V R^2 \right\},$$

(11)

which derives within the Lagrangian field-theoretical framework as a constant of motion for configurations that do not depend explicitly on time (as, for instance, when $V \neq V(t)$) coincides with the quantum-mechanical one that represents the expectation value of the Hamiltonian operator for the modification in question [7]. Since such a property is relatively rare among the nonlinear modifications of the Schrödinger equation [7], it certainly adds to the uniqueness of this class of modifications and the consistency of their formulation in a manner similar to the linear Schrödinger equation which shares with them the property in question. One can formulate the modified equation in terms of the entire wave function $\Psi$ and its complex conjugate $\Psi^*$. We will do this in the next section for a special version of the modification proposed.

Let us now comment on some particular cases of the modifications that derive from $L_{kn}$ of (3) as a function of $k$ and $n$. Let us note that the dimensions of coupling constants associated with $L_{kn}$ and $L_{nk}$ are the same. However, in general, $L_{kn}$ and $L_{nk}$ do not lead to the same type of modifications. It is so only when $k \geq 1$ and $n > 1$ in which case the modifications that stem from these Lagrangians describe time-reversible systems. As seen from equations (6) and (7), for the free linear Schrödinger equation to be invariant under the time-reversal transformation, $S$ must change the sign. This change of sign is not the symmetry of a modification if $n = 1$ independently of the value of $k$. Such modifications describe irreversibility in quantum systems. The simplest example of a nonlinear part of the Lagrangian for these type of modifications is provided by $L_{11}$. We will denote the total Lagrangian for this modification, i.e., including also its linear part, by $L_1$. The simplest examplification of a nonlinear part of the Lagrangian for the former type of modifications is given by $L_{12}$. Let us denote the total Lagrangian for this modification by $L_2$. In what follows, we will concentrate on these Lagrangians as representatives of the discussed classes of modifications. To conlude this part, mixing $L_{kl}$ and $L_{1n}$ for $k \geq n$ and an arbitrary $l$ will lead to equations of motion that are time-irreversible. It is also for this reason that the general equations (8-9), without imposing any constraints, describe irreversible systems.

One can approach the problem of irreversibility from completely opposite angles. One can accept it arguing that it is only due to the smallness of the coupling constant that the violation of time-reversal has not been observed yet. We note that one obtains in this way a simple mechanism to generate the time asymmetry on the quantum level. Or, one can consider the time-reversal violation a blemish or an unnecessary ingredient of theory in which case terms that cause it should not be allowed in the construction of a modified Lagrangian density. We will call a version that does not admit these terms
a minimal phase extension of the Schrödinger equation for it entails the smallest departure from the properties of the free linear Schrödinger equation.

The linear Schrödinger equation involves only the mass of a quantum system \( m \) and the Planck constant. Unless some other dimensional parameters are present one cannot define the characteristic energy scale of a system or theory. These parameters could be related to the coupling constants of potentials (or their concomitants, e.g., the frequency \( \omega \) in the potential of harmonic oscillator) or their range, as, for example, is the case for an infinitely deep potential well of some width \( d \). The modification under discussion does contain such a parameter, the coupling constant \( C_{kn} \). The dimensional analysis shows that in a theory defined by \( m, \hbar, \) and \( C_{kn} \) the energy scales as \( E_{kn} = \left( \hbar^{2kn} / |C_{kn}| m^{kn} \right)^{1/(kn-1)} (n \cdot k \neq 1) \) which in the limit corresponding to linear quantum mechanics \( (C_{kn} \rightarrow 0) \) produces infinity! Let us note though that the characteristic length of such a theory is given by \( l_{kn}^2 = \left( |C_{kn}| m/\hbar^2 \right)^{1/(kn-1)} \). Consequently,

\[
|C_{kn}| = \frac{\hbar^2 l_{kn}^{2(kn-1)} m}{q(kn-1)} = \frac{q(kn-1) \hbar^2}{m^{2kn-1} c^{2(kn-1)}},
\]

where \( \lambda_c = \hbar/mc \) is the Compton wavelength and \( q = l_{kn}^2/\lambda_c^2 \) is some dimensionless number that we will call the Compton quotient. Now, since \( E_{kn} = \hbar^2 / mL_{kn}^2 \), we see that the most natural way to avoid the infinity in question is to assume that \( l_{kn}^2 \) is proportional to \( L^2 \) which represents an intrinsic property of the system and as such, similarly as the mass of a quantum-mechanical system, is never equal zero. This leads to a novel type of theory which can be thought of as quantum mechanics of extended objects of some characteristic size \( l_c \), the simplest model being furnished by strings, and which is necessarily nonlinear. The classical limit of this theory does exist when \( \hbar \) tends to zero, but nevertheless on the level of equations the theory cannot be reduced to linear quantum mechanics, similarly as the latter does not boil down to any physically meaningful theory when the mass of a quantum particle becomes zero. These largely dimensional arguments do not apply to the modification that derives from \( L_1 \) for which no fundamental length exists; the dimension of length cannot be expressed as a function of dimensions of \( m, \hbar, \) and \( C_1 \) in parallel to linear quantum mechanics. The same holds true for the energy. The simplest of the modifications that can be interpreted in the way discussed are provided by \( L_{12} \) or \( L_{12} \) or a combination of these. Their coupling constants are proportional to \( |C_2| = \hbar^2 L_{0}^2 / m \).

From now on, we will focus our attention on the simplest extensions of our general scheme which, as pointed out above, stem from Lagrangians \( L_1 \) and \( L_2 \). These are the leading terms in the scheme. As in any field-theoretical construction, also here we settle for the simplest models. Only if compelling reasons arise to experiment with higher order terms one finds doing so justifiable. Hence, our selection at this point is based mainly on the principle of simplicity which, even if proved extremely useful in physics, belongs to the realm of aesthetics or methodology rather than physics proper. Other, physical, principles that one would like to use in order to discriminate between these two models or argue for their uniqueness have either already been invoked or still will. It is also the principle of simplicity that ensures the smallest departure of the discussed models from the linear Schrödinger equation. In what follows, the general scheme we have introduced will be used only in a few circumstances in order to make some more general observations.

We discuss the modification that derives from \( L_2 \) in the next section. Since \( L_1 \) gives rise to the modification that constitutes a part of a more general scheme already proposed by Doebner and Goldin [5, 6], we elaborate on this modification in the context of Doebner-Goldin proposal in section 4.
3 The Simplest Minimal Phase Extension (SMPE)

The simplest of minimal phase extensions stems from $L_2$ introduced in the previous section. It involves the minimal number of derivatives of the lowest possible order while preserving the basic features of the free linear Schrödinger equation such as, apart from the Galilean invariance, the invariance under the space and time reflections. Other features of this equation may however be compromised. This, as we will see, is the case for the weak separability of composite systems [10]. The total Lagrangian for this modification is

\[ -L_{SMPE}(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m} \left[ (\nabla R)^2 + R^2 (\nabla S)^2 \right] + CR^2 (\Delta S)^2 + R^2 V, \]  

where as argued before one can identify the coupling constant $C$ with $\hbar^2 l_c^2 / m$, $l_c$ being the characteristic size of an extended particle-system. The complete energy functional for the SMPE derived from this Lagrangian,

\[ E = \int d^3x \left\{ \frac{\hbar^2}{2m} \left[ (\nabla R)^2 + R^2 (\nabla S)^2 \right] + CR^2 (\Delta S)^2 + VR^2 \right\}, \]

clearly exhibits good convergent properties for square integrable wave functions. The equations of motion that derive from (13) read in the hydrodynamic formulation

\[ \hbar \frac{\partial R^2}{\partial t} + \frac{\hbar^2}{m} \nabla \cdot (R^2 \nabla R) - 2C \Delta (R^2 \Delta S) = 0, \]

\[ \frac{\hbar^2}{m} \Delta R - 2R \frac{\partial S}{\partial t} - 2RV - \frac{\hbar^2}{m} R (\nabla S)^2 - 2CR (\Delta S)^2 = 0. \]

One observes that any stationary solution to the linear Schrödinger equation is also a solution to these equations. Indeed, since for such solutions $S = -Et/\hbar + \text{const}$, the above equations reduce in this case to the Schrödinger-Madelung equations. There may however exist other stationary solutions to (15-16); they are supposed to satisfy the condition $\partial R^2 / \partial t = 0$. It is only for the Schrödinger equation that this condition ensures that the phase is a unique linear function of time only. Unlike the discussed version of the modification, its variant generated by $L_1$ does not allow the stationary solutions for which $S = -Et/\hbar + \text{const}$ and therefore affects rather dramatically the stationary states of known physical systems such as the hydrogen atom or harmonic oscillator.

One can also put the modified Schrödinger equation in terms of $\Psi$ and $\Psi^*$. To this end one expresses $L_{SMPE}$ in terms of these variables and derives from it the Euler-Lagrange equation for $\Psi^*$. The Lagrangian in question reads

\[ L_{SMPE}(\Psi, \Psi^*) = i\hbar \left( \Psi^* \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^*}{\partial t} \Psi \right) - \frac{\hbar^2}{2m} \nabla \Psi^* \nabla \Psi - \Psi^* V \Psi + \frac{C}{4} P^2 \Psi \Psi^*, \]

where the factor $C/4$ was chosen so as to reproduce the equations of motion in the hydrodynamic form of (15-16). The result of the derivation turns out to be

\[ i\hbar \frac{\partial \Psi}{\partial t} = H_{SMPE} \Psi = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \Psi - \frac{C}{4} G[\Psi, \Psi^*] \Psi, \]

where

\[ G[\Psi, \Psi^*] = P^2 + \frac{2\Delta (\Psi \Psi^* P)}{\Psi \Psi^*} \]
and
\[ P = \Delta \ln(\frac{\Psi^*}{\Psi}). \] (20)

The classical limit of this modification in the sense of the Ehrenfest theorem may not always exist since the standard Ehrenfest theorem of linear quantum mechanics is altered by nonlinear corrections. Let us now work out these corrections. For a general observable \( A \) one finds that
\[
\frac{d}{dt} \langle A \rangle = \frac{d}{dt} \langle A \rangle_L + \frac{d}{dt} \langle A \rangle_{NL},
\] (21)
where the nonlinear contribution due to \( H_{NL}[\Psi, \Psi^*] = H_R + iH_I \) can be expressed as
\[
\frac{d}{dt} \langle A \rangle_{NL} = \langle [A, H_I] \rangle - i \langle [A, H_R]\rangle,
\] (22)
with \( H_R \) and \( H_I \) being the real and imaginary part of \( H_{NL} = -CG/4 \), correspondingly. The brackets \(<\cdot\cdot>\) denote the mean value of the quantity embraced, \([\cdot,\cdot]\) and \(\{\cdot,\cdot\}\) denote commutators and anti-commutators, respectively. Specifying \( A \) for the position and momentum operators, one obtains the general form of the modified Ehrenfest relations
\[
m \frac{d}{dt} \langle \vec{r} \rangle = \langle \vec{p} \rangle + I_1,
\] (23)
\[
\frac{d}{dt} \langle \vec{p} \rangle = -\langle \vec{\nabla}V \rangle + I_2,
\] (24)
where
\[
I_1 = \frac{2m}{\hbar} \int d^3x \vec{r} R^2 H_I,
\] (25)
\[
I_2 = \int d^3x R^2 \left( 2H_I \vec{\nabla}S - \vec{\nabla}H_R \right) - i \int d^3x \vec{\nabla} \left( R^2 H_I \right).
\] (26)
The imaginary term in the last formula can be discarded for homogeneous modifications such as the one in question. For this type of modifications, \( H_I = f(R^2)/R^2 \), where \( f(R^2) \) is a certain operator acting on \( R^2 \) that can also involve the phase. Now, for square integrable wave functions for which \( R \) vanishes sufficiently fast in the infinity, we have \( \int d^3x \vec{\nabla} \left( R^2 H_I \right) = \int d^2x \vec{n} f(R^2) = 0 \), where \( \vec{n} \) is the unit vector normal to the boundary in the infinity. In the case under study, \( H_R = C (\Delta S)^2 \) and \( H_I = C \Delta (\Delta SR^2) / R^2 \), therefore the Ehrenfest relations for the SMPE read
\[
m \frac{d}{dt} \langle \vec{r} \rangle = \langle \vec{p} \rangle + \frac{2Cm}{\hbar} \int d^3x \vec{r} \Delta (\Delta SR^2),
\] (27)
\[
\frac{d}{dt} \langle \vec{p} \rangle = -\langle \vec{\nabla}V \rangle + C \int d^3x \left[ 2\vec{\nabla}S \Delta (\Delta SR^2) - R^2 \vec{\nabla} (\Delta S)^2 \right].
\] (28)
The Ehrenfest relations are Galilean invariant as are their nonlinear contributions for the wave functions that satisfy the equations of motion.

We see that, in general, the nonlinear corrections to the Ehrenfest relations do not vanish, leading to a different classical limit than in the linear theory. This feature of the modification is shared by other nonlinear generalizations of the Schrödinger equation, the only notable exception from this rule being the Bialynicki-Birula and Mycielski modification, for which \( H_I = 0 \) and \( \vec{\nabla} H_R = \vec{\nabla} R^2 / R^2 \) so that
both \( I_1 \) and \( I_2 \) vanish, the first identically and the second one for normalizable wave functions. Some of these modifications do possess the Ehrenfest limit, but only for certain values of their parameters. This, for instance, applies to the Doebner-Goldin modification. That the modification discussed does not have such a limit suggests that its equations are not linearizable, i.e., they cannot be transformed into the form of the linear Schrödinger equation, and thus are likely to contain some new physics that cannot be described by linear theory. This observation is corroborated by the fact that the Doebner-Goldin equations in their Ehrenfest domain are linearizable. It is probably justified to expect that a linearizable modification possesses the Ehrenfest limit. The converse may not be true, that is, a modification that has the classical limit does not have to be linearizable as seems to be the case for the Bialynicki-Birula and Mycielski modification. The modified Ehrenfest relations may still allow for the classical limit for some special wave functions for which both \( I_1 \) and \( I_2 \) vanish. On the other hand, perhaps due to its simplicity, the relevance of the Ehrenfest theorem tends to be overestimated.

As shown in [11], this theorem is neither sufficient nor necessary to characterize the classical regime of quantum theory. It is easy to convince oneself that in one dimension as long as \( \Delta S \) is an arbitrary function of time the Ehrenfest theorem holds also for the modification under study. The same holds true in higher dimensions for factorizable wave functions.

The modification of ours when cast in the Bohmian framework of quantum mechanics offers interesting and rather a logical extension of it. Let us recall that in Bohm’s approach to quantum mechanics a part of the linear Schrödinger equation (7) can be put as:

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \nabla S \right)^2 + V_C + V_Q^{Sch} = 0,
\]

where \( V_Q^{Sch} = V_R = -\hbar^2 \Delta R / 2mR \) represents the quantum potential of Bohm for the Schrödinger equation [12]. Without the external “classical” potential \( V_C \) this equation reduces to the ordinary Hamilton-Jacobi equation for the phase \( S \) in the quantum potential \( V_Q^{Sch} \). It is somewhat puzzling that this potential does not depend on the phase itself, as in general one would expect the evolution of a quantum system driven by the very potential to depend not only on the amplitude of its wave function but the phase as well. In the proposed modification the quantum potential \( V_Q^{Sch} \) is supplemented by a term that involves also the phase, which eliminates the asymmetry in question. As a result of this contribution, \( V_S = C (\Delta S)^2 / 2m \hbar^2 \), the quantum potential becomes \( V_Q = V_R + V_S \). One can interpret this additional term as a backreaction term, due to the reaction of the phase on the evolution in the potential \( V_R \). Because of the presence of \( \hbar^2 \) in the denominator of \( V_S \), this component of the quantum potential would dominate the other term of \( V_Q \), were it not, as argued earlier, for the smallness of \( C = q_\pm \hbar^4 / m^3 c^2 \), where \( q_\pm \) is a real number whose absolute value equals the Compton quotient but whose sign is undetermined by the theory as signalled by its subscript. This enables one to write the quantum potential in a more succinct way,

\[
V_Q = \hbar^2 \left[ \frac{q_\pm}{2m^3 c^2} (\Delta S)^2 - \frac{1}{2m} \frac{\Delta R}{R} \right].
\]

In the Bohmian framework of linear Schrödinger equation the continuity equation (6) looses any trace of the Planck constant, which reveals an even greater asymmetry of this approach. As opposed to equation (7) that contains “quantum” contributions, i.e., terms proportional to \( \hbar^2 \), equation (6) looks

\footnote{To arrive at this form of quantum potential, one needs to restore \( \hbar \) in the phase of the wave function so that \( \Psi = R \exp(iS/\hbar) \), where now \( S \) has the dimensions of action. This is formally equivalent to changing \( S \to S/\hbar \).}
like a classical equation for a “classical” current $\vec{j}_C = R^2 \vec{\nabla} S/m$. This situation is “corrected” in the proposed modification where the probability current contains a quantum contribution as well,

$$\vec{j}_Q = \frac{2q_\pm h^2}{m^3 c^2} \vec{\nabla} (R^2 \Delta S),$$

and the total current reads

$$\vec{j} = \frac{1}{m} R^2 \vec{\nabla} S + \frac{2q_\pm h^2}{m^3 c^2} \vec{\nabla} (R^2 \Delta S).$$

One can now write equations (15-16) in a very compact and symmetrical way,

$$\frac{\partial R^2}{\partial t} + \vec{\nabla} \cdot (\vec{j}_C + \vec{j}_Q) = 0,$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\vec{\nabla} S)^2 + V_C + V_Q = 0,$$

where both $\vec{j}_Q$ and $V_Q$ contain terms proportional to $\hbar^2$. The SMPE offers the simplest way to extend the Bohm approach to quantum mechanics by incorporating a quantum phase contribution to the quantum potential and a quantum contribution to the probability current. Let us note that these new terms are by no means negligible being of the same order of magnitude in terms of $\hbar$ and $c$ as the spin-orbit coupling responsible for the fine structure of atomic spectra. It is therefore conceivable even if speculative that the discussed terms describe some physically relevant “hydrodynamic” fine structure of the quantum world that owing to the elusive nature of the phase has somehow managed to escape our attention.

Let us note that the coupling constant $C$ has the same dimensions as the coupling constant that emerges in the leading relativistic approximation to the Schrödinger equation. One obtains this approximation from the relativistic relation between the energy $E$ and the momentum $p$ of a single particle of mass $m$ truncated to

$$E = \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2},$$

which upon the first quantization leads to the modified Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \Delta - \frac{\hbar^4}{8m^3 c^2} \Delta^2 \right) \Psi.$$

Despite the same dimensions of these coupling constants, the last equation does not coincide with our modification in the hydrodynamic representation, even in the Galilean limit.

The continuity equation for this modification can also be cast in the form of the generalized Fokker-Planck equation

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x^i} (D_i W) + \frac{\partial}{\partial x^j} D_{ij} W = 0,$$

where $D_i$ and $D_{ij}$ are some vector and tensor object, respectively, and the summation over repeated indices is assumed. This is not unlike in the other particular version of our general scheme that stems from $L_1$, originally proposed by Doebner and Goldin by adopting the simplest form of continuity equation of the Fokker-Planck type. One easily identifies $W$ with $R^2$, $mD_i$ with $\partial S/\partial x^i$, and $D_{ij}$ with $-2C\delta_{ij}\Delta S$. On the other hand, the energy equation (16) when written in the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\vec{\nabla} S)^2 + B (\Delta S)^2 - \frac{\hbar^2}{2m} \frac{\Delta R}{R} + V = 0,$$
can be thought of as a generalization of the Hamilton-Jacobi equation with \( B = C/\hbar^2 \).

It is a general feature of nonlinear wave mechanics that in a system consisting of two particles the very existence of one of them affects the wave function of the other one. Thus, even in the absence of forces the rest of the world influences the behavior of an isolated particle. As emphasized in [10], this is not necessarily a physically sound situation. A way to avoid it that have come to be known as the weak separability [13] was suggested by Białynicki-Birula and Mycielski [10]. As we will demonstrate it, in its most straightforward multi-particle extension, the SMPE does not permit the weak separability of composite systems. To explain why this is so, let us first demonstrate the weak separability of the linear Schrödinger equation in the hydrodynamic formulation.

We are considering a quantum system made up of two noninteracting subsystems in the sense that

\[ V(x_1, x_2, t) = V_1(x_1, t) + V_2(x_2, t). \]  

We will show that a solution of the Schrödinger equation for this system can be put in the form of the product of wave functions for individual subsystems for any \( t > 0 \), that is, \( \Psi(x_1, x_2, t) = \Psi_1(x_1, t) \Psi_2(x_2, t) = R_1(x_1, t) R_2(x_2, t) \exp \{i(S_1(x_1, t) + S_2(x_2, t)) \} \) and that this form entails the separability of the subsystems. The essential element here is that the subsystems are initially uncorrelated which is expressed by the fact that the total wave function is the product of \( \Psi_1(\vec{x}_1, t) \) and \( \Psi_2(\vec{x}_2, t) \) at \( t = 0 \). The assumption that the compound wave function is the product one is what defines this form of separability. What we will show then is that the subsystems remain uncorrelated during the evolution and that, at the same time, they also remain separated. It is the additive form of the total potential that guarantees that no interaction between the subsystems occurs, ensuring that they remain uncorrelated during the evolution. However, such an interaction may, in principle, occur in nonlinear modifications of the Schrödinger equation even if the form of the potential itself does not imply that. This is due to a coupling that the nonlinear term of the equation usually causes between \( \Psi_1(\vec{x}_1, t) \) and \( \Psi_2(\vec{x}_2, t) \).

The Schrödinger equation for the total system, assuming for simplicity that the subsystems have the same mass \( m \), reads now

\[
\hbar \frac{\partial R_1^2 R_2^2}{\partial t} + \hbar^2 \left\{ \left( \vec{\nabla}_1 + \vec{\nabla}_2 \right) \cdot \left[ R_1^2 R_2^2 \left( \vec{\nabla}_1 S_1 + \vec{\nabla}_2 S_2 \right) \right] \right\} = hR_2^2 \frac{\partial R_1}{\partial t} + hR_1^2 \frac{\partial R_2}{\partial t}
\]

\[
+ \frac{\hbar^2}{m} R_1 \vec{\nabla}_1 \cdot \left( R_1^2 \vec{\nabla}_1 S_1 \right) + \frac{\hbar^2}{m} R_2 \vec{\nabla}_2 \cdot \left( R_2^2 \vec{\nabla}_2 S_2 \right) = R_1^2 R_2 \left\{ \left[ \hbar \frac{1}{R_1^2} \frac{\partial R_1^2}{\partial t} \right] + \frac{\hbar^2}{m} \vec{\nabla}_1 \cdot \left( R_1^2 \vec{\nabla}_1 S_1 \right) \right\}
\]

and

\[
\frac{\hbar^2}{m} \left( \Delta_1 + \Delta_2 \right) R_1 R_2 - 2\hbar R_1 R_2 \frac{\partial (S_1 + S_2)}{\partial t} - \frac{\hbar^2}{m} R_1 R_2 \left( \vec{\nabla}_1 S_1 + \vec{\nabla}_2 S_2 \right)^2 -
\]

\[
(V_1 + V_2) R_1 R_2 = \frac{\hbar^2}{m} R_2 \Delta_1 R_1 + \frac{\hbar^2}{m} R_1 \Delta_2 R_2 - 2\hbar R_1 R_2 \frac{\partial S_1}{\partial t} - 2\hbar R_1 R_2 \frac{\partial S_2}{\partial t}
\]

\[
+ \frac{\hbar^2}{m} R_1 R_2 \left( \vec{\nabla}_1 S_1 \right)^2 + \frac{\hbar^2}{m} R_1 R_2 \left( \vec{\nabla}_2 S_2 \right)^2 - V_1 R_1 R_2 - V_2 R_1 R_2 =
\]
\[ \frac{R_1 R_2}{m} \left\{ \left[ \frac{\hbar^2 \Delta_1 R_1}{m} - 2h \frac{\partial S_1}{\partial t} + \frac{\hbar^2}{m} \left( \vec{\nabla}_1 S_1 \right)^2 - V_1 \right] + \left[ \frac{\hbar^2 \Delta_2 R_2}{m} - 2h \frac{\partial S_2}{\partial t} + \frac{\hbar^2}{m} \left( \vec{\nabla}_2 S_2 \right)^2 - V_2 \right] \right\} = 0. \]  

(41)

Implicit in the derivation of these equations is the fact that \( \vec{\nabla}_1 f_1 \cdot \vec{\nabla}_2 g_2 = 0 \), where \( f_1 \) and \( g_2 \) are certain scalar functions defined on the configuration space of particle 1 and 2, correspondingly. What we have obtained is a system of two equations, each consisting of terms (in square brackets) that pertain to only one of the subsystems. By dividing the first equation by \( R_1^2 R_2^2 \) and the second one by \( R_1 R_2 \), one completes the separation of the Schrödinger equation for the compound system into the equations for the subsystems. Moreover, we have also showed that indeed the product of wave functions of the subsystems evolves as the wave function of the total system. This is, however, not so for the discussed version of our modification as seen from (15-16). For instance, the second of these equations contains the term \( R (\Delta S)^2 \) which is nonseparable due to the coupling \( \Delta_1 S_1 \Delta_2 S_2 \). It is due to similar couplings that other particular modifications that emerge from our general scheme for \( n \geq 2 \) also violate the weak separability of compound systems.

The simplest way to avoid the nonseparability is to change the Lagrangian (or the equations) for the multi-particle case. The equations for two particles that we considered above stem from the general \( N \)-particle Lagrangian,

\[ - L_n(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \left[ \left( \vec{\nabla}_i R \right)^2 + R^2 \left( \vec{\nabla}_i S \right)^2 \right] + CR^2 \left( \sum_{i=1}^{N} \Delta_i S \right)^2 + R^2 V, \]  

(42)

which is nonseparable because of the discussed coupling of terms, \( \Delta_i S \Delta_j S \). However, if instead of this Lagrangian we use

\[ - L_n(R, S) = \hbar R^2 \frac{\partial S}{\partial t} + \sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \left[ \left( \vec{\nabla}_i R \right)^2 + R^2 \left( \vec{\nabla}_i S \right)^2 \right] + CR^2 \sum_{i=1}^{N} \left( \Delta_i S \right)^2 + R^2 V, \]  

(43)

we obtain perfectly weakly separable equations. Moreover, the coupling constant \( C \) can be now made particle-dependent, i.e., \( C = C_i \). The most natural way to do this is by employing the characteristic size of the particle \( l_c \) using the relation \( C = \pm \hbar^2 l_c^2 / m \). It is in this theory that the characteristic size of the particle retains its physical meaning in the case of many particles. Both Lagrangians reduce to the same expression for one particle, but for many particles they describe different theories. If one wants to have a theory that does not rule out separability, one should choose the Lagrangian given by (43). This, however, does not guarantee that more general compound systems described by nonfactorizable wave functions will not turn out to be nonseparable. This weakly separable extension is truly unique in that it does not split the total wave function into separate one-particle wave functions as would be the case for the cubic nonlinear equation \([3]\). Such an extension is possible only for homogeneous nonlinear modifications and, in particular, it can be performed for all the models of the general modification scheme defined by the Lagrangian (4-5).

On the other hand, as asserted by Czachor \([13]\), this modification is separable for any class of quantum-mechanical systems including also entangled systems in a novel alternative approach of strong separability (see also \([14, 15]\) where this approach originated). This particular formalism of strong separability that chooses as its starting point the nonlinear von Neumann equation for density matrices \([10]\) admits a broader family of nonlinearities than the weak separability which we used for the above demonstration.
4 Comparison with Other Nonlinear Modifications

To this day a number of other nonlinear generalizations of the Schrödinger equation have been proposed. In this section we will discuss some of them stressing similarities and differences with our proposal as yet another way of presenting its characteristic properties.

To this end, let us start from the Staruszkiewicz modification, the closest in spirit to the one presented here. In contradistinction to our generalization, Staruszkiewicz’s is nonhomogeneous and, consequently, the dimensions of its coupling constant depend on the dimensionality of space-time. It is the presence of the homogeneous $\Psi^\dagger\Psi$ terms that makes the coupling constants in our scheme independent of this dimensionality, which certainly gives it some greater universality in a fashion similar to that of the original Schrödinger equation. Moreover, it is due to the same reason that in the Staruszkiewicz modification the field-theoretical definition of energy as a certain conserved quantity differs from the quantum-mechanical one that treats energy as the expectation value of the Hamiltonian operator \[ H \]. It is also because of the nonhomogeneity that the Staruszkiewicz modification is not weakly separable and cannot be made weakly separable in a way that would comply with the strong separability in the fundamentalist approach \[ 2 \] to this issue. However, it can be made strongly separable in Czachor’s effective approach \[ 13 \].

Arguably, in the recent years, the most studied nonlinear modifications of the Schrödinger equation have been the modification proposed by Białynicki-Birula and Mycielski \[ 10 \] and that of Doebner and Goldin\[ 4 \]. The basic physical condition that selects the form of nonlinear terms in each of these modifications is that of weak separability of composite systems.\[ 5 \] It is this property of the separability that lead Białynicki-Birula and Mycielski to their model of nonlinearity in which no correlations are introduced by the nonlinear term. However, this choice would not be unique without another postulate that greatly limits the class of nonlinear terms. Namely, it is also stipulated in \[ 10 \] that the only nonlinear terms of the Lagrangian density for this modification are potential terms that do not contain any derivatives. By weakening this condition one would end up with a multitude of admissible terms not unlike in the general scheme of our modification. Indeed, the possible terms would then include not only the unique nonlinear potential $V_{\text{mod}} = b \ln (a |\Psi|^2)$, where $a$ and $b$ are the only undetermined dimensional constants, but also any term of the form $LV_{\text{mod}}$, where $L$ is an arbitrary scalar linear operator, as for example $\Delta$ or $\Delta^2$. One should note that this stipulation does not carry any physical contents, except for ensuring the standard Ehrenfest limit whose physical relevance could be limited even in the linear theory as pointed out earlier \[ 11 \]. It is just an additional simplifying assumption without which the unique pick of the physically desirable nonlinearity is not attainable. As we have demonstrated it in the preceding section, our modification does possess a similar unique construction, based on some reasonable postulates, and it involves only one free parameter. Nevertheless, the fact that the Białynicki-Birula and Mycielski modification does not alter the standard Ehrenfest relations makes it exceptional among nonlinear modifications of the Schrödinger equation. Other proposals either admit the Ehrenfest theorem for certain values of their free parameters only or for wave functions of particular properties. It has been found that the term $b \ln (a |\Psi|^2)$ can be given meaningful physical interpretations. One can see it as the effect of statistical uncertainty in the form of the potential \[ 21 \] or as the potential energy associated with the information encoded or stored in

\[ \text{As shown in } [20], \text{ the modifications in question can be viewed as belonging to the same family that also includes the Kostin modification } [17], \text{ and are related by some generalized nonlinear gauge transformation.} \]

\[ \text{As recently demonstrated by Lücke } [18, 19], \text{ this by no means ensures the separability in a more general sense, that is, when a compound system is described by a nonfactorizable wave function.} \]
the distribution of matter described by the probability density $|\Psi|^2$ [22]. Although the modification in question was not originally intended to apply to quantum systems of finite size, as argued in [23], if reinterpreted as a theory of extended objects it might be applicable to the nuclear realm. The modification itself does not imply such an interpretation unlike the SMPE. One observes that in the modification of Bialynicki-Birula and Mycielski the nonlinear term contains no phase contribution in which it is complementary to the modification of ours that introduces nonlinearity only through the phase. What they have in common is the use of a logarithmic nonlinearity.

The modification derived from $L_1$ has the same continuity equation as the Doebner-Goldin modification. In fact, it constitutes a special case of this modification. As opposed to the Bialynicki-Birula and Mycielski, the Staruszkiewicz, and the SMPE modifications, in its full-fledged form, the Doebner-Goldin generalization was intended to describe only a certain domain of quantum realm, irreversible and dissipative quantum systems. A strong argument in favor of this model of nonlinearity is lent by group theoretical analysis of the representations of the $\text{Diff}(\mathbb{R}^3)$ group which was proposed as a “universal quantum kinematical group” [24]. In the quantum-mechanical context, particular terms of the general Doebner-Goldin scheme had appeared well before the fully developed scheme was put forward. To the best of our knowledge, as a way to modify the Schrödinger equation, a homogeneous term of the type considered by Doebner and Goldin, was first explicitly employed by Rosen [26] in 1965 but conceived by him even earlier [25]. This term, $\Delta R/R$ in our notation, subsequently appears in the work of other authors guided by rather diverse motivations. In [28, 29, 30, 32, 34], it appears solo while in [31, 33, 35] as part of larger combination. Moreover, a different term of the general scheme in question, which does not preserve the Galilean invariance of the modified Schrödinger equation, was used in [27].

One can easily extend the Lagrangian density $L_1$ to the density from which a more general form of the Doebner-Goldin modification emerges, but which still constitutes a restricted variant of the full-fledged modification in question. To demonstrate this, let us start from the Doebner-Goldin modification in the form that is a slight variation on its original form [66],

\[
i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \Psi - \frac{i\hbar D}{2} F_{\{a\}} [\Psi, \Psi^*] \Psi + \hbar D F_{\{b\}} [\Psi, \Psi^*] \Psi, \tag{44}
\]

where

\[
F_{\{x\}} [\Psi, \Psi^*] = \sum_{i=1}^{n} x_i F_i [\Psi, \Psi^*] \tag{45}
\]

and $x_i$ are some dimensionless coefficients that form a generic array $\{x\}$ while $F_i [\Psi, \Psi^*]$ are functionals of $\Psi$ and $\Psi^*$ homogeneous of degree zero in these functions. The coupling constant $D$ has the dimensions of the diffusion coefficient, meter$^2$/second$^{-1}$. In the hydrodynamic representation, the general form of the functional employed by Doebner and Goldin is [66]

\[
F_{\{x\}}^{DG} [\rho, S] = x_1 \Delta S + x_2 \nabla \cdot \left( \frac{\nabla \rho}{\rho} \right) + x_3 \frac{\Delta \rho}{\rho} + x_4 \left( \frac{\nabla \rho}{\rho} \right)^2 + x_5 \left( \nabla S \right)^2, \tag{46}
\]

where $\rho = R^2$. The imaginary part of the Schrödinger equation is supposed to give a continuity equation. The standard way to obtain it is to multiply both sides of the Schrödinger equation by $\Psi^*$ and to take the imaginary part of the ensuing expression. Now, if $\rho F_{\{a\}}$ is to form the divergence of some current, one can show that two terms emerge to play this role: $\nabla \cdot (\rho \nabla S)$ and $\Delta \rho$. One obtains
these in a unique way by putting $a_1 = a_2 = a$ and $a_4 = a_5 = 0$. Renaming $a_3 D \to D'$ and $a D \to D$ allows us to write the modified continuity equation as

$$\frac{\partial \rho}{\partial t} + \frac{\hbar}{m} \hat{\nabla} \cdot (\rho \hat{\nabla} S) + D \hat{\nabla} \cdot (\rho \hat{\nabla} S) + D' \Delta \rho = 0. \quad (47)$$

It is now straightforward to check that the following Lagrangian density

$$L_{DG}^r(\Psi, \Psi^*) = L_{SE}(\Psi, \Psi^*) + c_1 \Psi^* \Psi \Delta \ln \frac{\Psi^*}{\Psi} + c_2 \Psi^* \Psi \Delta \ln a' \Psi^* \Psi + c_3 \Psi^* \Psi (\nabla \ln b' \Psi^* \Psi)^2 + c_4 \Psi^* \Psi (\nabla \ln b' \Psi^* \Psi) \cdot \nabla \ln \frac{\Psi^*}{\Psi} + c_5 \Psi^* \Psi \left( \nabla \ln \frac{\Psi^*}{\Psi} \right)^2 \quad (48)$$

leads to a restricted version of the DG modification characterized by $b_2 = 2b_4 + b_3 = 0$. This becomes clear when $L_{DG}^r$ is put in the hydrodynamic form

$$L_{DG}^r(R, S) = -L_{SE}(R, S) + d_1 R^2 \Delta S + d_2 (\nabla R)^2 + d_3 R^2 (\nabla S)^2 \quad (49)$$

that derives from the previous expression for $L_{DG}$ after dropping total derivatives like $\nabla \cdot (R^2 \nabla S)$ and $\Delta R^2$, and noting that $2\Delta R$, which one obtains from $(\nabla R)^2$, is the same as $\left[ \Delta \rho/\rho - \left( \nabla \rho/\rho \right)^2 / 2 \right] R$. Here $c_i$’s and $d_i$’s are arbitrary dimensional coupling constants, complex (in general) and real, respectively. The Lagrangian formulation of the entire Doebner-Goldin modification has not been found yet, and it is conceivable that it does not exist. The Lagrangian for the restricted variant in question, (47) or (48), has not been presented in the literature before. We also note that the nonlinear part of Lagrangian (48) represents a special case of the general nonlinear Lagrangian for which the field-theoretical energy functional is identical to its quantum-mechanical one.

As is well known, the Doebner-Goldin modification is both weakly separable and homogeneous of degree one in $\Psi$, in which it differs from other special versions of our modification for $n \geq 2$. The straightforward multi-particle extensions of these versions being homogeneous are not weakly separable. This is at variance with the hope expressed by Weinberg that the homogeneity of nonlinear modifications of the Schrödinger equation, that plays a crucial role in his scheme of nonlinear quantum mechanics, guarantees the weak separability of such systems. As stated in [1]: “The problem of dealing with separated systems has led other authors to limit possible nonlinear terms in the Schrödinger equation to a logarithmic form;” and “The homogeneity assumption (2) makes this unnecessary.” Even if this holds true in the Weinberg modification of nonlinear quantum mechanics, the example of the discussed versions of our modification shows that this is not always the case. In general, the separability is independent of homogeneity. This was explicitly demonstrated in the previous section for a particular case of the modification originating from $L_2$. However, as we also demonstrated it in there, it is possible to construct a unique weakly separable multi-particle extension of these versions.

Out of the modifications mentioned, only the Bia³ynicki-Birula and Mycielski and Weinberg ones have been confronted with some sort of experimental data and thereby upper bounds on their parameters have been imposed. It seems that the former, to quote [30], “has been practically ruled out by

\[\text{The term associated with } D \text{ does not appear in the continuity equation of the original Doebner-Goldin modification.}
\]
\[\text{To restore the correspondence with the Doebner-Goldin modification one needs to discard } d_3 R^2 (\nabla S)^2 \text{ in the Lagrangian (49), which will lead to even a more restricted version of this modification.}\]
The measurements in question [37] (see also [38, 39] for a more comprehensive discussion of these and other relevant techniques and experiments) have established an upper limit on the only nonlinear parameter of the modification to be $3.3 \times 10^{-15}$ eV. The Weinberg general scheme leads to a number of particular models of nonlinear Schrödinger equation. Several experiments [40, 41, 42, 43], based on different ideas and using different techniques, were carried out to test the most basic of these models in some simple physical situations. For instance, in one of them, the transitions between two atomic levels were examined to see if the resonant absorption frequencies of these transitions were affected by the nonlinearity of the models [4]. The experiments implied an upper limit for the nonlinearity parameter of the models to be of the order of $|1 \times 10^{-20}|$ eV [43, 40] which is the most stringent upper bound on nonlinear corrections to quantum mechanics yet to date.

The Doebner-Goldin modification has not been subjected to any experimental verification yet. As shown in [44], one can, in principle, test this modification in some experimental setup by measuring the component of the electric current parallel to a uniform electric field which is crossed with a magnetic field. This current is perpendicular to both fields when all nonlinear parameters of the theory vanish. Some indirect evidence for the validity of the modification in question can also be gained by extending it as proposed in [45] and studying the response of a two-level system to a monochrome excitation. The nontrivial predictions regarding the dependence of the response function on the frequency of excitation can be experimentally verified, and if found correct, they would put the Doebner-Goldin modification on a stronger foothold.

As far as SMPE is concerned, it remains to be investigated whether it would be possible to determine the upper limit for the only parameter of this modification for which we can conveniently choose the Compton quotient, the pure number $q$, and how this modification could manifests itself in physical phenomena. This number can, in principle, vary from one class of objects to another.

5 Conclusions

We presented a possible nonlinear modification of the Schrödinger equation utilizing the phase of the wave function to introduce the nonlinearity in question. Although a variation on the original proposal of Staruszkiewicz, it does possess a number of distinctive features that in our opinion make it deserve a more elaborate investigation, especially as far as the simplest minimal phase extension is concerned. Unlike Staruszkiewicz’s theoretical construction, ours is independent of the space-time dimensionality. It modifies the original Schrödinger equation by admitting corrections involving both the phase and the amplitude of the wave function. The unlimited, in principle, number of terms with dimensional coefficients that the general scheme of the modification introduces can be easily handled by restricting our attention to the simplest models. Only such models ensure a reasonably small departure from the Schrödinger equation. This leads us to the SMPE, a phase analogue of the unique nonlinear extension of Bialynicki-Birula and Mycielski in the realm of the amplitude. This variant of the modification preserves all physically relevant features of the Schrödinger equation, compromising only the weak separability of composed systems in its most straightforward multiparticle extension. It is, however, possible to construct the unique multiparticle extension that is weakly separable. The strong separability which, as opposed to the weak one, applies also to entangled states is maintained in the modification in question, at least in its effective formulation [43]. One could avoid this compromise by breaking the time-reversal symmetry of the free Schrödinger equation, which would lead us to the
Doebner-Goldin modification or higher order irreversible modifications that emerge from Lagrangian densities $L_{k1}$. However, this would also result in giving up stationary states of quantum-mechanical systems well established by the standard linear theory and confirmed experimentally. We find this alternative too limiting. Instead, the SMPE preserves these states, suggesting in addition that there may exist stationary states beyond those predictable by linear quantum mechanics. The modification discussed also offers an attractive minimal way of supplementing the quantum potential of Bohm by the term that depends on the phase of wave function. This removes a striking asymmetry of this potential as a functional of the amplitude only. Moreover, it too removes the asymmetry between the continuity equation and the other equation in the Bohm-Madelung formulation by furnishing the probability current with a “quantum” component of the same order in the Planck constant as the quantum potential. In general, the SMPE does not possess the classical limit in the sense of the Ehrenfest theorem, which suggests that it is not linearizable and thus can describe some new phenomena that cannot be captured by the linear theory. We have argued that the most natural way to interpret this modification is as the simplest model of quantum mechanics of extended objects of mass $m$ and some characteristic size $l_c$ to which the coupling constant of the modification is proportional. Other modifications that derive from the Lagrangians of $n \geq 2$ can be ascribed the same interpretation.

An alternative more traditional interpretation of the SMPE in terms of a universal coupling constant that does not emphasize $l_c$ is also possible. The general scheme of the proposed modification gives rise to a large class of particular models, the SMPE being only one of them, which demonstrate that the homogeneity of nonlinear Schrödinger equation is not sufficient to ensure the weak separability of composite systems contrary to the belief expressed in the literature \cite{1, 2}. Yet, it is possible to formulate multiparticle extensions of these models that are weakly separable. Last but not least, all the models of the modification possess rather an exceptional property among nonlinear modifications of this equation in that each model’s field-theoretical and quantum-mechanical energy functional are one and the same thing \cite{3}.

There are several goals one would like to achieve through the study of the modification proposed, the SMPE in the first place. One is a better understanding of possible ways in which the phase of wave function can manifest itself in quantum-mechanical systems. Due to a pervasive role the phase plays in quantum mechanics, it is rather hard to overestimate the importance of this question. In fact, this aspect has attracted a considerable attention since the discovery of the Aharonov-Bohm effect \cite{4} and the Berry phase \cite{5} subsequently generalized by Aharonov and Anandan \cite{6}. As shown in these papers (see also \cite{7} for a more comprehensive review), the phase can have a nontrivial impact on the evolution of quantum systems. One can use the models presented in this work as a laboratory for exploring new conceivable ways of how the phase can affect the evolution in question.

In this context, let us note that the solutions to the models mentioned do not contain ordinary Gaussian wave packets even though the plane wave solution is supported by all the modifications in our general scheme. This fact is rather generic for nonlinear generalizations of the Schrödinger equation. As is well known, the wave packets in standard quantum mechanics have one disturbing property: they spread beyond limit implying that there may exist macroscopically extended quantum objects, contrary to the experimental evidence gathered so far. Therefore, the exclusion of these solutions should by no means be construed as a defect of the modification, but perhaps rather as a desirable feature of it. It is also for this reason that the wave packets of linear quantum theory can serve only as very crude models of particle-like configurations. The localization of wave packets, however poor it may be, is due to the interference of elementary waves and so the phase plays a crucial role in it. One can thus wonder if the nonlinearity of the modified Schrödinger equation can work
in synergy with this localizing effect of the phase. We find this rather plausible. We believe that since the SMPE can be thought of as a quantum theory of extended particles, it is very natural to expect that it possesses localized solutions that would describe such particles. In fact, without such solutions the very idea of the SMPE as a theory of extended objects appears to be rather vacuous. Their existence would provide a viable particle representation of the wave-particle duality embodied in quantum mechanics. As of now, it is only the wave representation of this duo that is supported by the current understanding of quantum theory. Although it is premature to speculate on it, it is not entirely out of the question that the SMPE as a theory of extended objects might be useful in the description of some classes of such objects as, for instance, nuclei.

Another general and important motivation to study this modification comes from the long-standing problem of the collapse of wave function. The linear Schrödinger equation is incomplete in the sense that it does not describe this process. Therefore, it has been conceived in the literature that introduction of nonlinear terms can solve this problem. Various particular models incorporating such terms have been proposed, to name just a few of them [51, 52, 53, 54, 55] (for a review see [56, 57]). Since it is certainly reasonable to expect that the phase plays a significant role in this process, we believe that a further exploration of the modification presented could cast more light on possible mechanisms of this phenomenon.

As noted in [10], the Schrödinger equation is one of rare examples of fundamental equations of physics that are apparently linear. One can however entertain the thought that this so only in some leading approximation to a more general nonlinear equation that still awaits its discovery. No fundamental proof to the contrary has been provided yet as emphasized recently once more by Czachor [13] and Jordan [15]. In fact, nonlinear extensions of this equation have found some applications, typically in the description of collective phenomena. Of particular use here has been the cubic nonlinear Schrödinger equation that naturally emerges in the mean-field approach to many-body problems in quantum mechanics. For instance, this equation appears in the theoretical treatment of Bose-Einstein condensation which, as emphasized in [58], “is a common phenomenon occurring in physics on all scales from condensed matter to nuclear, elementary particle and astrophysics.” In this case, the nonlinear Schrödinger equation is often referred to as the Gross-Pitaevskii equation [59, 60, 61]. It describes a macroscopic population of particles, be it atoms, kaons or excitons in semiconductor heterostructures, condensed in the ground state of a system at the absolute zero temperature and interacting weakly through a short range potential modeled by a Dirac delta term. Yet another application for this equation has been found in nuclear physics where it was invoked to account for the compressibility of nuclear matter in elastic scattering of heavy ions [62]. Interestingly, this equation also tends to appear in schemes involving gravity. In such situations [63, 64, 65], the nonlinear term emerges as a correction to the linear equation dictated by the effect of the gravitational field of the particle on its own wave function, i.e., via the backreaction. Nevertheless, due to the smallness of the Newton constant, the gravitationally induced correction is practically negligible. As these examples demonstrate, even if the Schrödinger equation is never found truly fundamentally nonlinear, the quantum-mechanical picture of reality can equally well be found incomplete without nonlinearities invoked to account for particular physical effects, especially that some of these effects, even though negligibly small, appear as approximations in a grander scheme of things, as in the framework of general relativity. It would be interesting to find whether the SMPE can be an approximation to a more all-encompassing theoretical model.

Such solutions have indeed been found in the form of Gaussian solitons.
However, from a fundamental point of view, as accentuated by Weinberg [4], nonlinear generalizations of the Schrödinger equation can play a useful role as a kind of foil against which to test the linearity of quantum mechanics. Such a role has already been successfully played by the Białynicki-Birula and Mycielski modification and even to a greater extent by a particular model of Weinberg’s proposal. It is in this context that one can envisage another use of the modification proposed. More studies are needed to examine whether, and if so, how the SMPE could be employed to set up a bound on the nonlinearity parameter it introduces, similarly as it was done in the modifications just mentioned.

Yet another major goal of studying this modification is a more profound examination of the original phase modification of Staruszkiewicz, the modification of Doebner and Goldin whose restricted but fairly general version appears as a rather natural extension of a particular model of our general scheme, and its generalization recently proposed by this author [66]. Although these modifications differ from the SMPE that we consider the most interesting proposal presented here, one can expect that some insight gained from the study of this modification may also be applicable to them. Since, unlike the other modifications, the Staruszkiewicz modification is nonhomogeneous in the wave function, one hopes that comparative studies of these modifications could cast more light on the issue of physical consequences of nonhomogeneity in a modified Schrödinger equation.

Further investigations of the SMPE including in particular some physically interesting solutions will be reported elsewhere [50, 67].

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