A Measurable Angular Distribution for $\bar{B} \to D^* \tau^- \bar{\nu}_\tau$ Decays

Bhubanjyoti Bhattacharya, a Alakabha Datta, b Saeed Kamali b and David London c

a Department of Natural Sciences, Lawrence Technological University, Southfield, MI 48075, USA
b Department of Physics and Astronomy, 108 Lewis Hall, University of Mississippi, Oxford, MS 38677-1848, USA,
c Physique des Particules, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, QC, Canada H3C 3J7
E-mail: bbhattach@ltu.edu, datta@phy.olemiss.edu, skamali@go.olemiss.edu, london@lps.umontreal.ca

Abstract: At present, the measurements of $R_{D^{(s)}}$ and $R_{J/\psi}$ hint at new physics (NP) in $b \to c \tau^- \bar{\nu}_\tau$ decays. The angular distribution of $\bar{B} \to D^* (\to D \pi) \tau^- \bar{\nu}_\tau$ would be useful for getting information about the NP, but it cannot be measured. The reason is that the three-momentum $\vec{p}_\tau$ cannot be determined precisely since the decay products of the $\tau^-$ include an undetected $\nu_\tau$. In this paper, we construct a measurable angular distribution by considering the additional decay $\tau^- \to \pi^- \nu_\tau$. The full process is $\bar{B} \to D^* (\to D \pi') \tau^- (\to \pi^- \nu_\tau) \bar{\nu}_\tau$, which includes three final-state particles whose three-momenta can be measured: $D$, $\pi'$, $\pi^-$. The magnitudes and relative phases of all the NP parameters can be extracted from a fit to this angular distribution. One can measure CP-violating angular asymmetries. If one integrates over some of the five kinematic parameters parametrizing the angular distribution, one obtains (i) familiar observables such as the $q^2$ distribution and the $D^*$ polarization, and (ii) new observables associated with the $\pi^-$ emitted in the $\tau$ decay: the forward-backward asymmetry of the $\pi^-$ and the CP-violating triple-product asymmetry.
1 Introduction

At the present time, there are discrepancies with the predictions of the standard model (SM) in the measurements of some observables in a number of B decays. These include $R_D^{(*)} \equiv \frac{\mathcal{B}(\bar{B} \to D^{(*)} \tau^- \bar{\nu}_\tau)}{\mathcal{B}(\bar{B} \to D^{(*)} \ell^- \bar{\nu}_\ell)}$ ($\ell = e, \mu$) [1–10] and $R_{J/\psi} \equiv \frac{\mathcal{B}(B_c^+ \to J/\psi \tau^+ \nu_\tau)}{\mathcal{B}(B_c^+ \to J/\psi \mu^+ \nu_\mu)}$ [11]. The experimental results are shown in Table 1. The values of the SM predictions for $R_D$ and $R_{D^*}$, as well as their experimental measurements, are the average values used by the Heavy Flavor Averaging Group (HFLAV) [12]. They find that the deviation from the SM in $R_D$ and $R_{D^*}$ (combined) is $3.1\sigma$.\footnote{However, we note that this is not completely settled: for example, a more recent analysis finds $(R_{D^*})_{SM} = 0.250 \pm 0.003$ [13]. With this value, not included in the HFLAV average, the deviation from the SM prediction is larger than $3.1\sigma$.} For $R_{J/\psi}$, the discrepancy with the SM is $1.7\sigma$ [14]. These measurements suggest the presence of new physics (NP) in $b \to c\tau^-\bar{\nu}$ decays.

A great many papers have examined the question of what type of NP is required to explain the above anomalies. These include both model-independent [14, 16–28] and model-dependent analyses [29–66]. Clearly there are many possibilities for the NP. In order...
Table 1. Measured values of observables that suggest NP in $b \rightarrow c\tau^{-}\bar{\nu}_{\tau}$.

| Observable     | SM Prediction | Measurement          |
|---------------|---------------|----------------------|
| $R_{D}^{||/\ell}$ | $0.258 \pm 0.005$ [12] | $0.295 \pm 0.011 \pm 0.008$ [12] |
| $R_{D}^{||/\ell}$ | $0.299 \pm 0.003$ [12] | $0.340 \pm 0.027 \pm 0.013$ [12] |
| $R_{D}^{||/\mu}$ | $0.283 \pm 0.048$ [14] | $0.71 \pm 0.17 \pm 0.18$ [11] |
| $R_{D}^{||/e}$  | $\sim 1.0$     | $1.04 \pm 0.05 \pm 0.01$ [15] |

The above observables are all CP-conserving. But one can also consider CP-violating observables in $\bar{B} \rightarrow D^{*}\tau^{-}\bar{\nu}_{\tau}$ [87–90]. All CP-violating effects require the interference of two amplitudes with different weak (CP-odd) phases. Since the SM has only one amplitude, the observation of CP violation in this decay would be a smoking-gun signal of NP.

In Ref. [91], we began to explore the prospects for measuring CP-violating effects in $\bar{B} \rightarrow D^{*}\tau^{-}\bar{\nu}_{\tau}$. There, we noted that, since $\bar{B} \rightarrow D^{*}$ is the only hadronic transition in this decay, all amplitudes will have the same strong (CP-even) phase. As a result, the direct CP asymmetry is expected to be very small. The main CP-violating effects appear as CP-violating asymmetries in the angular distribution. These are kinematical observables, and require that the two interfering amplitudes have different Lorentz structures. This fact allows us to distinguish different NP explanations. We demonstrated this by constructing the angular distribution for the decay $\bar{B} \rightarrow D^{*}\mu^{-}\bar{\nu}_{\mu}$, and showing that one could extract the different NP contributions from an analysis of the CP-violating angular asymmetries.

The reason we did not apply this to $\bar{B} \rightarrow D^{*}\tau^{-}\bar{\nu}_{\tau}$ is that the construction of the angular distribution requires the knowledge of the three-momentum $\vec{p}_{\tau}$. But since the $\tau$ decays to final-state particles that include $\nu_{\tau}$, which is undetected, $\vec{p}_{\tau}$ is largely unknown. As a result, the full angular distribution in $\bar{B} \rightarrow D^{*}(\rightarrow D\pi)\tau^{-}\bar{\nu}_{\tau}$ cannot be measured.

In this paper, we construct a measurable angular distribution in $\bar{B} \rightarrow D^{*}(\rightarrow D\pi)\tau^{-}\bar{\nu}_{\tau}$. This is obtained by considering the additional decay $^{2}$ $\tau^{-} \rightarrow \pi^{-}\nu_{\tau}$. Now there are three final-state particles whose three-momenta can be measured: the $D$ and $\pi$ (from $D^{*}$ decay), and the $\pi^{-}$ (from $\tau$ decay). The new angular distribution is given in terms of five kinematic parameters: $q^{2}$, $\theta^{*}$ (describing $D^{*} \rightarrow D\pi$), and three quantities describing the $\pi^{-}$, $E_{\pi}$, $\theta_{\pi}$ and $\chi_{\pi}$. It includes CP-violating angular asymmetries, which can be measured and used to extract information about the NP.

But the angular distribution yields even more information. Some of the CP-even angular functions vanish in the SM; if they are found to be nonzero, this will indicate NP. In addition, while the distribution depends on only five kinematic parameters, it contains a great many terms, proportional to the various helicity amplitudes. As a result, all the NP parameters can be extracted from a fit to the full distribution, with a multitude of cross-

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2We note in passing that the decay $\tau^{-} \rightarrow \pi^{-}\nu_{\tau}$ has been used in the context of a proposed method for measuring the $\tau$ polarization in $B \rightarrow D\tau^{-}(\rightarrow \pi^{-}\nu_{\tau})\bar{\nu}_{\tau}$ [92, 93].
checks. It is also possible to integrate over one or more of the five parameters. When one
does this, all the familiar observables that have been proposed to distinguish NP models,
such as the $q^2$ distribution and the $D^*$ polarization, are reproduced. But there are also
new observables that depend on the kinematic angles associated with the $\pi^-$ emitted in
the $\tau$ decay, $\theta_{\pi}$ and $\chi_{\pi}$. These include the forward-backward asymmetry of the $\pi^-$, and
the CP-violating triple-product asymmetry.

We begin in Sec. 2 with the derivation of the angular distribution of $B \to D^*(\to D\pi)\ell^-\bar{\nu}_\ell$. Here, some information is given in the Appendices. In Sec. 3, we
discuss the NP signals, both CP-conserving and CP-violating, in the differential decay
rate. Observables obtained by integrating this rate over one or more of the kinematical
variables are described in Sec. 4. We conclude in Sec. 5.

2 Angular Distribution

We begin by describing our method of calculating the angular distribution of $B \to D^*(\to D\pi')\tau(\to \pi\nu_{\tau})\bar{\nu}_{\tau}$. (Note that this section is somewhat technical. The reader wishing to
simply see the results may skip to the next section.)

2.1 Structure of the new angular distribution

Consider first the angular distribution of the decay $\bar{B} \to D^*(\to D\pi)\ell^-\bar{\nu}_\ell$. This is obtained
as follows. Assuming only left-handed (LH) neutrinos, the decay is parametrized as $\bar{B} \to D^*N^{*-}(\to \ell^-\bar{\nu}_\ell)$, where $N = S - P$, $V - A$, $T$ represent LH scalar, vector and tensor
interactions, respectively. For $\ell = \mu, e$, there is no NP, so that $N = W$ and the coupling is
$V - A$. But for $\ell = \tau$, all couplings are allowed. The full amplitude is then squared, and
can be expressed as a function of the final-state momenta. These momenta are defined in
terms of the three helicity angles of Fig. 1, $\theta_{\ell}$, $\theta^*$ and $\chi$. In this way, one produces a set of
angular functions whose coefficients are different combinations of the helicity amplitudes.
This is the angular distribution [91].
We now consider the case where the final-state lepton is $\ell = \tau$. The $\tau$ is not directly detected in experiments; instead, it is detected through its decay products. We choose to study the simplest possible hadronic decay of the $\tau$, $\tau \to \pi \nu_\tau$. While NP in the $\tau$ decay is a possibility, in this analysis we restrict ourselves to NP only in the $B$ decay. As we will show, even using this simple two-body decay of the $\tau$, one can extract a great deal of information about this NP.

Once we let the $\tau$ decay, the process $B \to D^*(\to D\pi')\tau(\to \pi\nu_\tau)\bar{\nu}_\tau$ has five particles in the final state. This decay can be broken down into four successive quasi-two-body decays of the $B$ meson and three intermediate states. The five-body phase space for the decay of a massive spinless particle, such as the $B$ meson, depends on 8 independent parameters: five helicity angles and the invariant squares of the masses of the three intermediate particles. Since two of these intermediates – the $D^*$ and the $\tau$ – can go on shell, two of the three invariant mass parameters are given by $m_{D^*}$ and $m_\tau$. Thus, this decay depends on six independent parameters: five helicity angles and $q^2$, the invariant mass-squared of the $\tau\bar{\nu}_\tau$ pair. In the following, given that it could be NP that couples to $\tau\bar{\nu}_\tau$, we will refer to the center-of-momentum frame of the $\tau\bar{\nu}_\tau$ pair as the $N^*$ rest frame.

Now, the helicity angles are typically defined in the rest frames of the corresponding intermediate states. Following this procedure, we define (i) $\theta^*$ as the polar angle of the $D$-meson three-momentum in the rest frame of its parent $D^*$ meson, (ii) $\theta_\tau$ and $\chi_\tau$ as the polar and azimuthal angles, respectively, of the $\tau$ three-momentum in the $N^*$ rest frame, and finally (iii) $\theta$ and $\chi$ as the polar and azimuthal angles, respectively, of the $\pi$ three momentum in its parent $\tau$ rest frame.

However, this leads to a problem. Although one can in principle theoretically define all five helicity angles, most of them are of no practical use. To be specific, since the $\tau$ lepton is not directly observed in experiments, the angles either associated with its three momentum or defined in its rest frame are not measurable. Thus, four of the five helicity angles ($\theta(\tau), \chi(\tau)$) are of no use to us. This problem can be remedied (at least partially) through a convenient change of variables.

### 2.2 New parameters

Since we do not have experimental access to the $\tau$ rest frame, in our analysis we choose to express the $\tau \to \pi \nu_\tau$ phase space in the $N^*$ rest frame (this frame can be easily determined from information about the hadronic side of the $B$ decay). Since the pion three-momentum can be precisely measured in this frame, we consider three new variables. $E_\pi$, $\theta_\pi$ and $\chi_\pi$ represent the pion energy, polar and azimuthal angles, respectively, defined in this frame. (The new helicity angles are shown in Fig. 2.) These three variables replace three of the unmeasurable helicity angles. The fourth unmeasurable angle is an azimuthal angle and is easily integrated over. We describe below the mathematical method for this transformation.

Let us consider the product, $d^4I$, of the quasi-two-body phase spaces for $N^* \to \tau\bar{\nu}_\tau$ ($\phi_{N^*}$) and $\tau \to \pi \nu_\tau$ ($\phi_\tau$). (The $d^4$ serves as a reminder that this phase-space factor ultimately depends on four independent kinematic variables.) Each phase-space factor is evaluated in the corresponding parent rest frame, and is expressed in terms of the four unmeasurable helicity angles. However, since each individual phase-space factor is Lorentz
Figure 2. Definition of the angles in the $B \rightarrow D^* (\rightarrow D\pi) \tau^- (\rightarrow \pi^- \nu_\tau) \bar{\nu}_\tau$ distribution.

Invariant, we can write this entire product in the measurable $N^*$ rest frame:

$$d^4 I = \int d\phi_{N^*} (p_\tau, p_{\bar{\nu}_\tau}) \int d\phi_{\tau^-} (p_\pi, p_{\nu_\tau}) ,$$

$$= \frac{1}{(4\pi)^4} \int \frac{d^3 p_\tau d\phi_{\tau^-}}{E_\tau E_{\bar{\nu}_\tau}} \delta^4 (q - p_\tau - p_{\bar{\nu}_\tau}) \int \frac{d^3 p_\pi d\phi_{\tau^-}}{E_\pi E_{\nu_\tau}} \delta^4 (p_\tau - p_\pi - p_{\nu_\tau}) ,$$

$$= \frac{1}{(4\pi)^4} \int \frac{d^3 p_\tau d\phi_{\tau^-}}{E_\tau E_{\bar{\nu}_\tau}} \frac{\delta (\sqrt{q^2} - E_\tau - E_{\bar{\nu}_\tau})}{E_\pi E_{\nu_\tau}} \delta^3 (\vec{\tau} - \vec{\tau}_\pi - \vec{\bar{\nu}}_{\bar{\nu}_\tau}) \int \frac{d^3 p_\pi d\phi_{\tau^-}}{E_\pi E_{\nu_\tau}} \frac{\delta (E_\tau - E_\pi - E_{\bar{\nu}_\tau})}{E_\pi E_{\nu_\tau}} \delta^3 (\vec{\tau}_\pi - \vec{\bar{\nu}}_{\bar{\nu}_\tau} - \vec{\nu}_\tau) , \quad (2.1)$$

where, in the final line, $E_x$ and $\vec{p}_x$ respectively represent the energy and three-momentum of the particle $x$ in the $N^*$ rest frame. Performing the integrals over the $\nu_\tau$ and $\bar{\nu}_\tau$ three-momenta, and neglecting neutrino masses, we find

$$d^4 I = \frac{1}{(4\pi)^4} \int \frac{d^3 p_\tau}{E_\tau |\vec{p}_\tau|} \delta (\sqrt{q^2} - E_\tau - |\vec{p}_\tau|) \int \frac{d^3 p_\pi}{E_\pi |\vec{p}_\pi|} \delta (E_\tau - E_\pi - |\vec{p}_\tau - \vec{p}_\pi|) . \quad (2.2)$$

Without loss of generality, we now choose to write the $\tau$ and $\pi$ three-momentum integral measures such that the associated polar angle can be determined, at least theoretically. In the case of $d^3 p_\pi$, clearly the polar and azimuthal angles of the pion three-momentum relative to the $N^*$ direction, $\theta_\pi$ and $\chi_\pi$ respectively, are measurable. Here, $\theta_\pi$ is defined using three-momenta evaluated in the $N^*$ rest frame,

$$\cos \theta_\pi = - \frac{\vec{p}_{D^*} \cdot \vec{p}_\pi}{|\vec{p}_{D^*}| |\vec{p}_\pi|} , \quad (2.3)$$

while $\chi_\pi$ is defined using three-momenta evaluated in the $B$ rest frame,

$$\sin \chi_\pi = \frac{[(\vec{p}_{\pi'} \times \vec{p}_{D^*}) \times (\vec{p}_{D^*} \times \vec{p}_\pi)] \cdot \vec{p}_{D^*}}{|\vec{p}_{\pi'} \times \vec{p}_{D^*}||\vec{p}_{D^*} \times \vec{p}_\pi||\vec{p}_{D^*}|} . \quad (2.4)$$

Since $\vec{p}_{D^*} = \vec{p}_D + \vec{p}_{\pi'}$, one can easily verify that $\sin \chi_\pi$ is proportional to the scalar triple product $(\vec{p}_{\pi'} \times \vec{p}_D) \cdot \vec{p}_\pi$. 

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In the case of $d^4p_\tau$, the polar angle of the $\tau$ direction relative to the pion direction, $\theta_{\tau\pi}$, can be theoretically determined. The fourth angle – the corresponding azimuthal angle $\chi_{\tau\pi}$ – cannot be determined. However, at a later stage we will eliminate this angle by integrating over it. After appropriately transforming the delta functions, and writing the phase space in terms of the above new variables ($\theta_\pi$, $\chi_\pi$, $\theta_{\tau\pi}$ and $\chi_{\tau\pi}$), we find

$$d^4I = \frac{1}{(4\pi)^4} \int \frac{d|\vec{p}_\tau|}{\sqrt{q^2}} d\cos \theta_{\tau\pi} d\chi_{\tau\pi} dE_\pi d\cos \theta_\pi d\chi_\pi$$

$$\delta \left( |\vec{p}_\tau| - \frac{q^2 - m_\pi^2}{2\sqrt{q^2}} \right) \delta \left( \cos \theta_{\tau\pi} - \frac{2E_\pi E_\pi - m_\tau^2 - m_\pi^2}{2|\vec{p}_\tau||\vec{p}_\pi|} \right), \quad (2.5)$$

Expressed in the above form, it is clear that the remaining two delta functions can be used to remove the two variables $|\vec{p}_\tau|$ and $\cos \theta_{\tau\pi}$. We are thus left with a phase-space factor that depends only on four variables ($\chi_{\tau\pi}$, $E_\pi$, $\theta_\pi$ and $\chi_\pi$), as expressed below:

$$d^4I = \frac{1}{(4\pi)^3} \frac{1}{\sqrt{q^2}} d\chi_{\tau\pi} dE_\pi d\cos \theta_\pi d\chi_\pi,$$  \quad (2.6)

where the following replacements in the squared invariant amplitude of the decay ($|M|^2$) are understood:

$$E_\pi \rightarrow \frac{q^2 + m_\pi^2}{2\sqrt{q^2}}, \quad |\vec{p}_\pi| \rightarrow \frac{q^2 - m_\pi^2}{2\sqrt{q^2}}, \quad \cos \theta_{\tau\pi} \rightarrow \frac{2E_\pi E_\pi - m_\tau^2 - m_\pi^2}{2|\vec{p}_\tau||\vec{p}_\pi|}. \quad (2.7)$$

Using the above choice of kinematic parameters we may now express the differential decay rate for the full process as follows:

$$\frac{d^5\Gamma}{dq^2 d\cos \theta^* dE_\pi d\cos \theta_\pi d\chi_{\tau\pi}} = \frac{|\vec{p}_{D^*}| |\vec{p}_D|}{2^{15}\pi^7 m_D^2 m_{D^*} \sqrt{q^2}} \int d\chi_{\tau\pi} \frac{dp_{D^*}^2}{2\pi} \frac{dp_{D}^2}{2\pi} |M|^2. \quad (2.8)$$

Here $|\vec{p}_{D^*}| = \sqrt{\lambda(m_{D^*}^2; q^2, m_D^2, m_{D^*}^2)/(2m_B)}$ and $|\vec{p}_D| = \sqrt{\lambda(m_{D^*}^2; m_D^2, m_{\pi}^2)/(2m_{D^*})}$, where

$$\lambda(a; b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca. \quad (2.9)$$

The right-hand side of Eq. (2.8) contains integrals over three independent variables (out of the eight variables discussed in the previous subsection). We will see in the following subsection that these integrals can be performed quite simply once we express $|M|^2$ as an explicit function of these variables.

### 2.3 Calculating $|M|^2$

The next step is to calculate $|M|^2$, appropriately summed over spins and polarizations. In Ref. [91], we derived the angular distribution for $B \rightarrow D^* \mu \bar{\nu}_\mu$. In the presence of NP, the relevant two-body processes are $\bar{B} \rightarrow D^* N^* \rightarrow \mu^- \bar{\nu}_\mu$, where $N = S - P, V - A, T$ represent left-handed scalar, vector and tensor interactions, respectively. These are labeled $SP$, $VA$ and $T$. (The $VA$ contribution includes that of the SM.) For each of the leptonic $SP$, $VA$ and $T$ Lorentz structures, the hadronic piece (the $b \rightarrow c$ transition) also has a NP contribution. The effective Hamiltonian is
\[ \mathcal{H}_{\text{eff}} = \frac{G_F V_{cb}}{\sqrt{2}} \left\{ [(1 + g_L) \bar{c} \gamma_\mu (1 - \gamma_5) b + g_R \bar{c} \gamma_\mu (1 + \gamma_5) b] \gamma^\mu (1 - \gamma_5) \nu_\mu + [g_S \bar{c} b + g_P \bar{c} \gamma_5 b] \bar{\mu} \gamma^\mu (1 - \gamma_5) \nu_\mu + g_T \bar{c} \sigma^{\mu\nu} (1 - \gamma_5) b \bar{\ell} \sigma_{\mu\nu} (1 - \gamma_5) \nu_\ell + \text{h.c.} \right\}. \]

The decay amplitude is then written as the product of a hadronic piece \( \mathcal{H}_{D^*} \), a leptonic piece \( \mathcal{L}^{N^*} \), and a helicity amplitude piece \( \mathcal{M}^{N^*} \), appropriately summed over helicities labeled by \( m, n, \) and \( p \).

This all applies to the decay \( B \to D^* \tau \bar{\nu}_\tau \), except that now one must also include the decay \( \tau \to \pi \nu_\tau \). In addition to numerical factors and factors of \( f_\pi |V_{ud}| \) coming from the \( \tau \to \pi \nu_\tau \) transition, the leptonic piece changes. Representing the new leptonic pieces by \( \tilde{\mathcal{L}}^{N^*} \), the spin-summed squared invariant amplitude for the full 5-body decay can now be expressed as

\[
|\mathcal{M}|^2 = \frac{96 \pi G_F^2 |V_{cb}|^2 m_{D^*} m_{D^*} \Gamma_{D^*} \mathcal{B}(D^* \to D \pi^\prime) m_\tau \Gamma_\tau \mathcal{B}(\tau \to \pi \nu_\tau)}{\tilde{p}_{D^*}^3 (m_\tau^2 - m_\mu^2)^2 (p_{D^*}^2 - m_\tau^2)^2 + m_{D^*}^2 \Gamma_{D^*}^2 (p_\tau^2 - m_\tau^2)^2 + m_\tau^2 \Gamma_{D^*}^2} \left| \sum_{m=\pm 0} \mathcal{H}_{D^*}(m) \left( \mathcal{M}^{SP}_{(m)} \tilde{\mathcal{L}}^{SP}(m) + \sum_{n=t,\bar{t}} g_{n,n} \mathcal{M}^{VA}_{(m,n)} \tilde{\mathcal{L}}^{VA}(n) \right) \right|^2,
\]

In the above, the new leptonic pieces are of the form

\[
\tilde{\mathcal{L}}^{SP} = m_\tau \bar{u}(\nu_\tau) \bar{\psi}_n (1 - \gamma_5) v(\bar{\nu}_\tau),
\]

\[
\tilde{\mathcal{L}}^{VA}(n) = \epsilon_{V_A}^\beta(n) \left[ \bar{u}(\nu_\tau) \bar{\psi}_n \gamma_\beta (1 - \gamma_5) v(\bar{\nu}_\tau) \right],
\]

\[
\tilde{\mathcal{L}}^T(n,p) = -i m_\tau \epsilon_{\tau}^\beta(n) \epsilon_{\tau}^\delta(p) \left[ \bar{u}(\nu_\tau) \bar{\psi}_n \gamma_\delta (1 - \gamma_5) v(\bar{\nu}_\tau) \right],
\]

and we have used the SM expressions for the branching fractions \( \mathcal{B}(D^* \to D \pi^\prime) \), and \( \mathcal{B}(\tau \to \pi \nu_\tau) \):

\[
\mathcal{B}(\tau \to \pi \nu_\tau) = \frac{G_F^2 |V_{ud}|^2 f_{D^*}^2 (m_\tau^2 - m_\mu^2)^2}{16 \pi m_\tau \Gamma_\tau} \left( m_{D^*}^2 - m_\mu^2 \right)^2, \quad \mathcal{B}(D^* \to D \pi^\prime) = \frac{|\tilde{p}_{D^*}|^3}{6 \pi m_{D^*}^2 \Gamma_{D^*}}.
\]

The hadronic pieces, \( \mathcal{H}_{D^*} \), and the helicity amplitude pieces \( \mathcal{M}^{N^*} \) are the same as those obtained in our earlier work, Ref. [91]. For completeness, we have provided this information in Appendix A.

We now see that the dependence of \( |\mathcal{M}|^2 \) on the variables \( \tilde{p}_{D^*}^3 \), and \( p_\tau^2 \) appears only through the propagators of the corresponding intermediate particles. Since both of these particles – the \( D^* \) and the \( \tau \) – go on shell, we can apply the narrow-width approximation to replace these propagators with delta functions, making the corresponding integrals simple. Under the narrow-width approximation, one can show that

\[
\int \frac{dp_\tau^2}{2 \pi} \frac{m_\Gamma \mathcal{B}}{(p_\tau^2 - m_\tau^2)^2 + m_\tau^2 \Gamma^2} \to \frac{\mathcal{B}}{2},
\]
Furthermore, the dependence of $|\mathcal{M}|^2$ on the unmeasurable azimuthal angle $\chi_{\tau\pi}$ is a result of fermionic traces over products of the leptonic pieces ($\mathcal{E}^{N'}$). This dependence turns out to be combinations of simple trigonometric functions, such as $\sin \chi_{\tau\pi}$ and $\cos \chi_{\tau\pi}$. It is therefore straightforward to integrate over $\chi_{\tau\pi}$.

After integrating over the three variables $p_D^2, p_T^2$ and $\chi_{\tau\pi}$, the full five-body differential decay rate is given by

$$\frac{d^6\Gamma}{dq^2 dE_\pi d\cos \theta^* d\cos \theta_{\pi} d\chi_{\tau\pi}} = \frac{3|V_{cb}|^2 G_F^2|qD| (q^2)^{3/2}}{2^{11}\pi^4 m_{\pi}^2 (m_{\pi}^2 - m_\tau^2)^2} B(D^* \to D\pi') B(\tau \to \pi\nu_\tau)$$

$$\times \sum_{i,j} (N_i^S \bar{A}_i |^2 + N_j^R \text{Re}[A_i A_j^*] + N_{ijk} \text{Im}[A_i A_j^*]) , \quad (2.15)$$

where $i, j = t, 0, \perp, |, SP, (0, T), (\perp, T), (||, T)$. Here the $A_i$ represent the helicity amplitudes that contain crucial physics information that can be extracted from this study, while the $\lambda_i^{S,R,I}$ represent functions of the five independent kinematic variables of interest to us ($q^2, \theta^*, E_\pi, \theta_{\pi}$, and $\chi_{\tau\pi}$).

In a standard angular distribution, such as that in Eq. (2.26) of Ref. [91], the differential decay rate is expressed as a sum over a product of angular functions and helicity amplitudes. However, since here we let the $\tau$ decay, and we focus only on measurable quantities in the $\tau$ decay, the $\lambda_i^{S,R,I}$ functions in Eq. (2.15) are no longer purely angular functions, but depend also on $q^2$ and $E_\pi$.

In Table 2, we present the information relevant for the $N_i^S |A_i|^2$ pieces of Eq. (2.15). The first column contains the various $|A_i|^2$ helicities. The third column contains the associated $N_i^S$ terms. In these terms, we have separated out the parts that depend on $q^2$ and $E_\pi$, and put them into the $S_i$ factors. The expressions for the $S_i$ are given in Appendix B. Finally, in the second column, we indicate which NP terms of Eq. (2.10) contribute to each of the $N_i^S |A_i|^2$ pieces. The information relevant for the $N_i^R \text{Re}[A_i A_j^*]$ and $N_{ijk} \text{Im}[A_i A_j^*]$ pieces is given in Tables 3 and 4, respectively. The expressions for the $R_i$ and $I_i$ are also given in Appendix B.

### 3 Differential Decay Rate: New-Physics Signals

In the previous section, we described the angular distribution of the decay $B \to D^* (\to D\pi') \tau^-(\to \pi^-\nu_\tau)\bar{\nu}_\tau$. This is measurable. The question now is: what can we learn from it? That is, what are the signals of NP? This is discussed in the present section.

In Eq. (2.10), there are five NP parameters: $g_L, g_R, g_S, g_P$ and $g_T$. Of these, $g_S$ does not contribute to this decay. Furthermore, the Lorentz structure associated with $g_L$ is $(V - A) \times (V - A)$, as in the SM. For this reason, the quantity $1 + g_L$ appears in the angular distribution, where the 1 is due to the SM. Thus, the angular distribution involves $1 + g_L, g_R, g_P$ and $g_T$. As we will see below, it is possible to extract $g_R, g_P$ and $g_T$. If these are found to be zero, this indicates that the only NP present is $g_L$; the value of $|1 + g_L|$ can be found from the measurement of $R_{D(\tau)}$.

$g_L, g_R, g_P$ and $g_T$ are complex quantities. In principle, they may have both weak (CP-odd) and strong (CP-even) phases. However, as argued in Ref. [91] (and summarized...
As indicated by the second column of Tables 2 and 3, some of the couplings $g_L$, $g_R$, $g_P$ and $g_T$ are present in each helicity amplitude. Therefore, a straightforward way of observing NP is to measure all possible helicity amplitudes and compare them with the predictions of the SM. Particularly interesting here are the helicity amplitudes $A_{SP}$ and $A_{(0,\perp,\|)},T$, as these are absent in the SM. Any observation of these helicity amplitudes, however weak, will indicate the presence of NP.

The angular distribution is written in terms of functions of the three angles, $\theta^*$, $\theta_\pi$ and $\chi_\pi$. However, some of these functions do not appear in the SM distribution. For example, in Table 2, if we set all the NP parameters to zero, the angular functions $\cos 2\theta_\pi \sin^2 \theta^*$ and $\cos 2\chi_\pi \sin^2 \theta_\pi \sin^2 \theta^*$ do not appear. (They are found only in the last two entries of this Table.) If a full fit found that one of the coefficients of these angular functions was
| Amplitude          | Coupling                                                                 | $N_{i,j}^{R}$                  |
|-------------------|--------------------------------------------------------------------------|--------------------------------|
| $\text{Re}[A_{i}A_{0}]$ | $|1 + g_{L} - g_{R}|^{2}$                                                | $R_{00} \cos \chi_{\pi} \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{i}A_{0}^{*}]$ | $|1 + g_{L} - g_{R}|^{2}$                                                | $R_{10} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0}A_{0}^{*}]$ | $|1 + g_{L} - g_{R}|^{2}$                                                | $R_{00} \cos \chi_{\pi} \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{0}A_{0}^{*}]$ | $(1 + g_{L} - g_{R})(1 + g_{L} + g_{R})^{*}$                            | $R_{0 \perp} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{i}A_{i}^{*}]$ | $(1 + g_{L} - g_{R})(1 + g_{L} + g_{R})^{*}$                            | $R_{1 \perp} \cos \chi_{\pi} \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{SP}A_{SP}^{*}]$ | $(1 + g_{L} - g_{R})^{*}$                                               | $R_{SP \perp} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{SP}A_{SP}^{*}]$ | $g_{P}(1 + g_{L} - g_{R})^{*}$                                          | $R_{SP0} \cos \chi_{\pi} \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{SP}A_{SP}^{*}]$ | $g_{P}(1 + g_{L} - g_{R})^{*}$                                          | $R_{SP0} \cos \chi_{\pi} \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{SP}A_{SP}^{*}]$ | $(1 + g_{L} - g_{R})^{*}$                                               | $R_{SP \parallel} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{SP}A_{SP}^{*}]$ | $g_{P}g_{T}^{*}$                                                        | $R_{SP0T} \cos \chi_{\pi} \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{0,T}A_{i,T}^{*}]$ | $g_{T}(1 + g_{L} - g_{R})^{*}$                                          | $R_{0TT} \cos \chi_{\pi} \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $g_{T}(1 + g_{L} - g_{R})^{*}$                                          | $[R_{0T0,1} + R_{0T0,2} \cos 2\theta_{\pi}] \cos^{2} \theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $g_{T}(1 + g_{L} + g_{R})^{*}$                                          | $R_{0T \perp} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $g_{T}(1 + g_{L} - g_{R})^{*}$                                          | $R_{0T \perp} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{0T \perp T} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{0T \perp T} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{0T \perp T} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{0T \perp T} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{0,T}A_{0,T}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{0T \perp T} \cos \chi_{\pi} \sin \theta_{\pi} \sin 2\theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $g_{T}(1 + g_{L} + g_{R})^{*}$                                          | $[R_{1 \perp \perp,1} + R_{1 \perp \perp,2} (\cos 2\theta_{\pi} + 2 \cos 2\chi_{\pi} \sin^{2} \theta_{\pi})] \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $g_{T}(1 + g_{L} - g_{R})^{*}$                                          | $R_{1 \perp \perp} \cos \theta_{\pi} \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{1 \perp \perp} \cos \theta_{\pi} \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{1 \perp \perp} \cos \theta_{\pi} \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{1 \perp \perp} \cos \theta_{\pi} \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{1 \perp \perp} \cos \theta_{\pi} \sin^{2} \theta^{*}$ |
| $\text{Re}[A_{1 \perp}A_{1 \perp}^{*}]$ | $|g_{T}|^{2}$                                                           | $R_{1 \perp \perp} \cos \theta_{\pi} \sin^{2} \theta^{*}$ |

Table 3. Contributions to the $N_{ij}^{R}$ $\text{Re}[A_{i}A_{j}^{*}]$ pieces of Eq. (2.15). The coefficients $R_{i}$ depend on the kinematic parameters $q^{2}$ and $E_{\pi}$, and are listed in Appendix B.

nonzero, that would be a smoking-gun signal for NP. A similar analysis can be done for Table 3.

The angular distribution presented here is nonstandard: not only does it include functions of angles, but also functions of $q^{2}$ and $E_{\pi}$. However, one may perform a fit to extract the coefficients of the functions $N_{i,j}^{S,R}$. We see that fitting only to the $N_{ij}^{S}$ functions will yield $|1 + g_{L} - g_{R}|$, $|1 + g_{L} + g_{R}|$, $|g_{P}|$ and $|g_{T}|$. Since several helicity amplitudes depend on the same coefficients, the comprehensive measurement of the distribution involves an overcomplete set of observables, leading to a great deal of redundancy, and providing an opportunity for cross-checks.

Cross-checks will also come from several coefficients of the $N_{ij}^{R}$ functions presented in
Table 4. Contributions to the $N^I_j$, Im[$A_i A^*_j$] pieces of Eq. (2.15). The coefficients $I_i$ depend on the kinematic parameters $q^2$ and $E_\pi$, and are listed in Appendix B.

Table 3. The helicity amplitude coefficients of the $N^R$ functions are of the form Re[$A_i A^*_j$], where $i$ and $j$ represent different helicities. In general, these terms are sensitive to the phase difference between different NP couplings. However, since there is only one NP coefficient of the tensor type, $g_T$, interference between two tensor helicities does not provide us any new information. Instead these can be used as further cross-checks for $|g_T|$.

In the case of vector helicities, there are effectively two couplings with different complex phases. These are $1 + g_L + g_R$ (contributes to $A_\perp$, $A_0$ and $A_\parallel$), and $1 + g_L - g_R$ (contributes to $A_\perp$). Thus, interferences among the vector helicities $A_\perp$, $A_0$ and $A_\parallel$ do not provide any new information – these can be used as cross-checks. Interferences between $A_\perp$ and another vector helicity can be used to extract the phase difference between $1 + g_L + g_R$ and $1 + g_L - g_R$. Together with the information about their magnitudes, one may then separately obtain $|1 + g_L|$, $|g_R|$, and their relative phase.

Finally, all other interferences between helicity amplitudes of different types can be used to determine phase differences between the corresponding NP couplings. Here as well, there are several angular functions that depend on the same combinations of NP coefficients, hence providing a way to cross-check the determination of the relative phases between $1 + g_L, g_R, g_P$ and $g_T$. In this way the magnitudes of the four NP couplings and their corresponding complex-phase differences can be determined from Tables 2 and 3.

### 3.2 CP-violating angular terms

Above, we argued that the strong-phase differences between the various amplitudes are expected to be very small. This implies that all direct CP-violating effects are also expected to be tiny. Even so, CP-violating effects can be present in the angular distribution. To be specific, the coefficients of certain angular terms are related to triple-products (TPs) of the form $\vec{p}_1 \cdot (\vec{p}_2 \times \vec{p}_3)$, where the $\vec{p}_i$ are the final-state momenta. As we will see below, TP
asymmetries do not require a strong-phase difference between the interfering amplitudes. Indeed, they are maximal when this strong-phase difference vanishes. In the decay $B \to D^*(\to D\pi')\tau(\to \pi\nu_\tau)\bar{\nu}_\tau$, the 3-momenta of the final-state particles $D$, $\pi$ and $\pi'$ can be measured. From these, a TP can be constructed; all the entries of Table 4 are proportional to this TP.

Now, all entries are also proportional to $\text{Im}[A_iA_j^*]$, where $A_i$ and $A_j$ are the two interfering helicity amplitudes. Writing

$$A_i = |A_i|e^{i\phi_i}e^{i\delta_i}, \quad A_j = |A_j|e^{i\phi_j}e^{i\delta_j}.$$  \hspace{1cm} (3.1)

where $\phi_{i,j}$ ($\delta_{i,j}$) are the weak (strong) phases, we see that

$$\text{Im}[A_iA_j^*] = |A_i||A_j|\sin(\phi_i - \phi_j + \delta_i - \delta_j).$$  \hspace{1cm} (3.2)

If, as we have assumed, the strong-phase difference is negligible, the TP is proportional to $\sin(\phi_i - \phi_j)$. This is a CP-violating quantity. On the other hand, if the strong-phase difference is not negligible, the TP can be nonzero even if the weak-phase difference vanishes. That is, this is not CP-violating (it is known as a “fake TP”). To obtain a true CP-violating term, this must be compared to the TP in the CP-conjugate process. In the CP-conjugate process, the weak phases change sign, but the strong phases do not. But there is an additional change. Each $\mathcal{N}_I$ in Table 4 is proportional to $\sin\chi_\pi$, so that these functions are parity odd. This means that, in going from process to CP-conjugate process, there is an additional minus sign [94], so that the CP-conjugate TP is proportional to

$$-\text{Im}[\bar{A}_i\bar{A}_j^*] = |A_i||A_j|\sin(\phi_i - \phi_j - \delta_i + \delta_j).$$  \hspace{1cm} (3.3)

The true, CP-violating effect is then found by adding the TPs in process and CP-conjugate process [95], so that it remains even in an untagged data sample.\(^3\)

The key point is that, in the SM, CP-violating effects are absent. Thus, the observation of a nonzero entry in Table 4 would be a smoking-gun signal of NP. And the information obtained from the analysis of Tables 2 and 3 provides a cross-check.

4 Integrated Observables

The full differential decay rate for $B \to D^*(\to D\pi')\tau(\to \pi\nu_\tau)\bar{\nu}_\tau$ depends on the five kinematic parameters $q^2$, $E_\pi$, $\theta^\tau$, $\theta_\pi$, and $\chi_\pi$. While a complete study of the decay distribution as a function of all five parameters can reveal NP effects, a full experimental analysis may be statistics limited. Effects of NP can still be studied through “integrated observables,” obtained by integrating the differential decay rate over one or more of the kinematic parameters.

We separate the integrated observables into two types. The first type is found by integrating over all three of the lepton-side parameters ($E_\pi$, $\theta_\pi$, $\chi_\pi$). Such observables are

\(^3\)Whether to add or subtract individual angular terms for the construction of a true CP-violating effect depends on the sign convention used to define the azimuthal angle. Theory sign conventions for the decay $B \to K^*\mu^+\mu^-$, which our discussion follows for the $B \to D^*\tau^-\bar{\nu}_\tau$, can be found in Ref. [96]. Ref. [97] presents detailed comparisons between sign conventions used in $B \to K^*\mu^+\mu^-$ theory versus experiment.
functions of $q^2$, and are independent of the dynamics of the lepton decay. They can, therefore, be used to study lepton-flavor universality. Observables, such as the longitudinal and transverse $D^*$ polarizations ($F_{L,T}^{D^*}$), fall in this category. The second type of observables are constructed by integrating over the hadron-side parameter, $\theta^*$, and either of the parameters $\theta_\pi$, and $\chi_\pi$. These observables explicitly depend on the effects from the $\tau \rightarrow \pi \nu_\tau$ decay. Since lighter leptons cannot decay to a pion, this second type of observables appears only when the intermediate lepton is a $\tau$.

4.1 Lepton flavor universality

Here we consider observables constructed from the differential decay distribution by integrating over $E_\pi$, $\theta_\pi$, $\chi_\pi$. The resulting distribution in $q^2$ and $\theta^*$ can be expressed as

$$\frac{d^2\Gamma}{dq^2 \ d\cos \theta^*} = \frac{3}{2} \frac{d\Gamma}{dq^2} \frac{a(q^2) + c(q^2) \cos^2 \theta^*}{3a(q^2) + c(q^2)} ,$$

where $F_{L}^{D^*}(q^2)$ and $F_{T}^{D^*}(q^2) = 1 - F_{L}^{D^*}(q^2)$ are the longitudinal and transverse polarization fractions of the $D^*$. The functions $a(q^2)$ and $c(q^2)$ are given by

$$a(q^2) = 2 \left( 1 + \frac{m_\tau^2}{2q^2} \right) (|A||^2 + |A|^2) + 16 \left( 1 + \frac{2m_\tau^2}{q^2} \right) (|A_{||},T|^2 + |A_{\perp},T|^2)$$

$$- \frac{24m_\tau}{\sqrt{q^2}} \left( \text{Re}[A_{||},A_{||,T}^*] + \text{Re}[A_{\perp},A_{\perp,\perp}^*] \right) ,$$

$$c(q^2) = 2 \left( 1 + \frac{m_\tau^2}{2q^2} \right) \left( 2|A_0|^2 - |A||^2 - |A|^2 \right) + 6 \left( \frac{m_\tau}{\sqrt{q^2}} A_t + A_{\Delta} \right)^2$$

$$+ 16 \left( 1 + \frac{2m_\tau^2}{q^2} \right) \left( 2|A_{0,T}|^2 - |A_{||,T}|^2 - |A_{\perp,T}|^2 \right)$$

$$- \frac{24m_\tau}{\sqrt{q^2}} \left( 2\text{Re}[A_{0,0}^*] - \text{Re}[A_{||}A_{||,T}^*] - \text{Re}[A_{\perp}A_{\perp,\perp}^*] \right) .$$

The longitudinal and transverse polarization fractions $F_{L,T}^{D^*}$ can be obtained from Eq. (4.1):

$$F_{L}^{D^*} = \frac{a(q^2) + c(q^2)}{3a(q^2) + c(q^2)} , \quad F_{T}^{D^*} = \frac{2a(q^2)}{3a(q^2) + c(q^2)} .$$

Further integration over $\cos \theta^*$ gives us the decay distribution as a function of $q^2$:

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{cb}|^2 |p_{D^*}|^2}{128 m_B^2 \pi^3} \left( 1 - \frac{m_\tau^2}{q^2} \right)^2 B(D^* \rightarrow D\pi^0) B(\tau \rightarrow \pi \nu_\tau) \left( a(q^2) + \frac{c(q^2)}{3} \right) .$$

The integrated observables constructed above are not affected by the dynamics of the $\tau$ decay, since the relevant kinematic parameters have been integrated over. Indeed, the expressions for these observables agree with those found elsewhere in the literature (apart from the factor $B(\tau \rightarrow \pi \nu_\tau)$ in Eq. (4.5)). The comparison of the measured values of these observables with those found in decays involving the light leptons, taking into account the larger $\tau$ mass and the associated kinematic differences, provides a test of lepton flavor universality.
4.2 Lepton-side observables

Here we discuss observables obtained by integrating the full differential distribution over $\theta^*$ and either (or both) of $\theta_\tau$ and $\chi$. These observables depend on at least one kinematic parameter associated with the decay of the $\tau$, $E_\pi$. Therefore, these observables can only be constructed in the $\tau$ lepton case, and specifically for the decay $\tau \to \pi \nu_\tau$.

The first step is to integrate the full differential decay rate of Eq. (2.15) over $\theta^*$. The $\theta^*$ dependence of the full angular distribution is hidden in the $\mathcal{N}_{i,j}^{(R,S,I)}$ functions listed in Tables 2, 3 and 4. These functions are proportional to one of three forms – $\cos^2 \theta^*$, $\sin^2 \theta^*$ and $\sin 2\theta^*$, The integral over $\theta^*$ eliminates all helicity amplitude combinations proportional to $\sin 2\theta^*$, but keeps the other two. The remaining expression is long and may not carry any more insight than the full angular distribution itself. We therefore proceed one step further and integrate over $\chi$. The left-over differential decay rate is a function of $q^2$, $E_\pi$ and $\theta_\tau$, and is given by

$$\frac{d^3\Gamma}{dq^2dE_\pi d\cos \theta_\tau} = \frac{3}{2} \frac{d^3\Gamma}{dq^2dE_\pi} a_\pi + b_\pi \cos \theta_\tau + c_\pi \cos^2 \theta_\tau,$$

where the coefficients $a_\pi$, $b_\pi$ and $c_\pi$ are functions of $q^2$ and $E_\pi$:

$$a_\pi = (S_{0,1} - S_{0,2})|A_0|^2 + (S_{0T,1} - S_{0T,2})|A_{0,T}|^2 + 2S_{SP}|A_{SP}|^2 + 2|A_t|^2$$
$$+ (2S_{||,1} + S_{||,2} + S_{||,3})|A_{||}|^2 + 2(S_{||,1} - S_{||,2})|A_{||}|^2 + 2(S_{\perp,1} - S_{\perp,2})|A_{\perp}|^2$$
$$+ 2(S_{\perp,T,1} - S_{\perp,T,2})|A_{\perp,T}|^2 + (R_{0T0,1} - R_{0T0,2}) \text{Re}[A_{0,T}A_{0}^*] + R_{SPR} \text{Re}[A_{SP}A_{\pi}^*]$$
$$+ 2(R_{\perp,T,1} - R_{\perp,T,2}) \text{Re}[A_{\perp,T}A_{\pi}^*] + 2(R_{\perp,T,1} - R_{\perp,T,2}) \text{Re}[A_{\perp,T}A_{\pi}^*]$$

$$b_\pi = R_{0T1} \text{Re}[A_{0,T}A_{\pi}^*] + R_{SP0} \text{Re}[A_{SP}A_{0}^*] + R_{SP0T} \text{Re}[A_{SP}A_{0,T}] + R_{0o} \text{Re}[A_{0,T}]$$
$$+ 2R_{\perp,T} \text{Re}[A_{\perp,T}A_{\pi}^*] + 2R_{\perp,T} \text{Re}[A_{\perp,T}A_{\pi}^*] + 2R_{\perp,T} \text{Re}[A_{\perp,T}A_{\pi}^*]$$
$$+ 2R_{\perp,T} \text{Re}[A_{\perp,T}A_{\pi}^*];$$

$$c_\pi = 2S_{0,2}|A_0|^2 + 2S_{0T,2}|A_{0,T}|^2 + (S_{||,2} - S_{||,3})|A_{||}|^2$$
$$+ 4S_{||,1}|A_{||}|^2 + 4S_{||,2}|A_{||}|^2 + 4S_{\perp,T,2}|A_{\perp,T}|^2$$
$$+ 2R_{0T0,1} \text{Re}[A_{0,T}A_{0}^*] + 4R_{\perp,T,2} \text{Re}[A_{\perp,T}A_{\pi}^*] + 4R_{\perp,T,1} \text{Re}[A_{\perp,T}A_{\pi}^*].$$

Further integrating over $\theta_\tau$ gives us the decay distribution as a function of $q^2$ and $E_\pi$:

$$\frac{d^2\Gamma}{dq^2dE_\pi} = \frac{3}{2} \frac{d^2\Gamma}{dq^2} a_\pi + c_\pi$$

At this stage, we can perform an asymmetric integral over $\cos \theta_\tau$, to find the forward-backward asymmetry in the distribution of the $\pi$ coming from the $\tau$ decay. This is done by integrating the differential decay rate with a uniform negative weight for the positive
values of \( \cos \theta_\pi \), and subtracting this from a similar integral with a uniform positive weight for the negative values of \( \cos \theta_\pi \). Appropriately normalizing this function, we can define the forward-backward asymmetry \( (A_{FB}) \) as follows:

\[
A_{FB}(q^2, E_\pi) = \frac{\int_0^1 dq^2 dE_\pi d\cos \theta_\pi \cos \theta_\pi - \int_0^1 dq^2 dE_\pi d\cos \theta_\pi}{\int_0^1 dq^2 dE_\pi d\cos \theta_\pi},
\]

\[
= -\frac{3}{2} \frac{b_\pi}{3 a_\pi + c_\pi}.
\]  

(4.11)

As can be seen from the form of \( b_\pi \) [Eq. (4.8)], \( A_{FB} \) is nonzero in the SM. In order to see if NP is present, one must combine this measurement with that of other observables, or of other terms in the angular distribution. With enough independent measurements of functions of the helicity amplitudes, it is possible to determine if some NP amplitudes must be nonzero.

Changing the order of integrals over \( \chi_\pi \) and \( \theta_\pi \) can yield valuable complementary information. In the preceding discussion we obtained observables by first integrating over \( \chi_\pi \) and then over \( \theta_\pi \). If instead the integral over \( \theta_\pi \) is performed first, all but the helicity-amplitude combinations proportional to \( \cos^2 \theta_\pi \) and \( \sin^2 \theta_\pi \) are removed. The left-over differential decay rate, a function of \( q^2 \), \( E_\pi \) and \( \theta_\pi \), is found to be

\[
\frac{d^3 \Gamma}{dq^2 dE_\pi d\chi_\pi} = \frac{d^2 \Gamma}{dq^2 dE_\pi} \frac{B_1 + B_2 \cos 2\chi_\pi + B_3 \sin 2\chi_\pi}{2\pi B_1},
\]

(4.12)

where the coefficients \( B_i \) can be expressed in terms of helicity amplitudes and functions of \( q^2 \) and \( E_\pi \) as follows:

\[
B_1 = (3S_{0,1} - S_{0,2})|A_0|^2 + (3S_{0,T,1} - S_{0,T,2})|A_{0,T}|^2 + 3S_{SP}|A_{SP}|^2 + 3S_{T}|A_t|^2 \\
+ 2(3S_{||,1} + 2S_{||,2})|A_{||}|^2 + 2(3S_{T,1} - S_{T,2})|A_{T}|^2 + 2(3S_{L,1} - S_{L,2})|A_L|^2 \\
+ 2(3R_{T,1} - S_{T,2})|A_{T}\rangle^2 + (3R_{T,0,1} - R_{T,0,2})Re[A_{0,T}A_0^\ast] + 3R_{SP}Re[A_{SP}A_t^\ast] \\
+ 2(3R_{L,1} - R_{L,2})Re[A_{T}A_{T}^\ast] + 2(3R_{L,1} - R_{L,2})Re[A_{L}A_{L}^\ast],
\]

(4.13)

\[
B_2 = 2(S_{||,3} - S_{||,2})|A_{||}|^2 - 8S_{T,2}|A_{T}\rangle^2 + 8S_{L,2}|A_L|^2 + 8S_{T,2}|A_{T}|^2 \\
- 8R_{T,2}Re[A_{T}\rangle A_{T}^\ast] + 8R_{L,2}Re[A_{L}A_{L}^\ast],
\]

(4.14)

\[
B_3 = 4 \left( I_{||,1} Im[A_{||}A_{T}^\ast] + I_{T,1} Im[A_{T}A_{T}^\ast] + I_{T,1} Im[A_{T}A_{T}^\ast] \right).
\]

(4.15)

Note that the coefficient \( B_1 \) is related to the coefficients \( a_\pi \) and \( c_\pi \):

\[
B_1 = 3a_\pi + c_\pi.
\]

(4.16)

An asymmetric integral over \( \chi_\pi \) can now isolate an observable that is nonzero only if true CP-violating TP asymmetries are present. This new observable, \( A_{TP} \), can be defined
as

\[ A_{TP}(q^2, E_\pi) = \left( \frac{d^2 \Gamma}{dq^2 dE_\pi} \right)^{-1} \left( \frac{\pi/2}{\pi/2} + \frac{3\pi/2}{\pi} - \frac{2\pi}{3\pi/2} \right) \frac{d^3 \Gamma}{dq^2 dE_\pi d\chi_\pi} d\chi_\pi = \frac{B_3}{2\pi B_1}. \]  

(4.17)

From Eq. (4.15), we see that \( B_3 \) vanishes in the absence of weak-phase differences. This shows that \( A_{TP} \) is a CP-violating observable.

Above, \( A_{TP} \) is defined as a function of \( q^2 \) and \( E_\pi \). However, one can further integrate this function over both of these variables. The resulting integrated observable can be directly compared to an experimental event analysis in which one obtains the asymmetry between the number of events with \( \sin 2\chi_\pi > 0 \) and \( \sin 2\chi_\pi < 0 \).

We note that \( A_{TP} \) involves interferences of vector-vector and vector-tensor type. The only way to generate a nonzero value of \( A_{TP} \) is if there is a nonzero weak phase in at least one of \( g_R \) and \( g_T \). Thus, if \( A_{TP} \) (and/or its form integrated over \( q^2 \) and \( E_\pi \)) is found to be nonzero, this will be an unmistakeable sign of CP-violating NP.

## 5 Conclusions

At the present time, the measurements of \( R_{D^{(*)}} \equiv B(\bar{B} \to D^{(*)}\tau^-\bar{\nu}_\tau)/B(\bar{B} \to D^{(*)}\ell^-\bar{\nu}_\ell) \) (\( \ell = e, \mu \)) and \( R_{J/\psi} \equiv B(B_c^+ \to J/\psi\tau^+\nu_\tau)/B(B_c^+ \to J/\psi\mu^+\nu_\mu) \) disagree with the predictions of the SM, hinting at NP in \( b \to c\tau\bar{\nu} \) decays. There are many possibilities for this NP. A variety of observables have been proposed to distinguish the various NP explanations: the \( q^2 \) distribution, the \( D^* \) polarization, the \( \tau \) polarization, etc.

Another potential way of distinguishing the NP explanations involves CP violation. Within the SM, there are no CP-violating effects in \( \bar{B} \to D^*\tau^-\bar{\nu}_\tau \), so that any observation of CP violation in this decay would be a smoking-gun signal of NP. Here, the main CP-violating effects appear as CP-violating asymmetries in the angular distribution. However, this is problematic. The construction of the angular distribution requires the knowledge of the three-momentum \( \vec{p}_\tau \). But this cannot be measured precisely, since the \( \tau \) decays to final-state particles that include \( \nu_\tau \), which is undetected. The result is that the full angular distribution in \( \bar{B} \to D^* \to D\pi\tau^-\bar{\nu}_\tau \) cannot be measured.

In this paper, we construct a measurable angular distribution by considering the additional decay \( \tau^- \to \pi^-\nu_\tau \). The full process then is \( \bar{B} \to D^*(\to D\pi')\tau^-\to(\pi^-\nu_\tau)\bar{\nu}_\tau \). Here there are three final-state particles whose three-momenta can be measured: the \( D \) and \( \pi' \) (from \( D^* \) decay), and the \( \pi^- \) (from \( \tau \) decay). The new angular distribution is given in terms of five kinematic parameters: \( q^2, \theta^* \) (describing \( D^* \to D\pi \)), and three quantities describing the \( \pi^- \), \( E_\pi \), \( \theta_\pi \) and \( \chi_\pi \). It includes CP-violating angular asymmetries, which can be measured and used to extract information about the NP.

But much more information can be extracted from the angular distribution. In the most general case, the angular distribution involves the couplings \( 1 + g_L, g_R, g_P \), and \( g_T \), where \( g_L, g_R, g_P \) and \( g_T \) are the NP parameters. The magnitudes and relative phases of all four couplings can be extracted from a fit to the full distribution. This will go a long way towards identifying the NP.
It is also possible to integrate over one or more of the five kinematic parameters. If one integrates over the lepton-side parameters $E_\pi, \theta_\pi$ and $\chi_\pi$, all the familiar observables that have been proposed to distinguish NP models are reproduced. These include the $q^2$ distribution and the $D^*$ polarization. And if one integrates over the hadron-side quantities, one obtains new observables that depend on the kinematic angles associated with the $\pi^-$ emitted in the $\tau$ decay, $\theta_\pi$ and $\chi_\pi$. These include the forward-backward asymmetry of the $\pi^-$, and the CP-violating triple-product asymmetry.

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A Hadronic and Helicity Amplitude Pieces

The differential decay rate for the process $B \to D^*(\to D\pi')\tau(\to \pi\nu_\tau)\bar{\nu}_\tau$ has been written in terms of a collection of hadronic pieces $H_{D^*}$, helicity amplitude pieces $M_{N^*}$, and leptonic pieces $\tilde{L}_{N^*}$ in Section 2.3. While the leptonic pieces are new in this analysis, the hadronic and helicity amplitudes were presented in Ref. [91]. For convenience, below we summarize these relationships.

The hadronic pieces, $H_{D^*}$, can be expressed as

$$H_{D^*}(m) = \epsilon_{D^*}(m) \cdot p_D, \quad m = 0, \pm,$$

(A.1)

where $p_D$ represents the four-momentum of the $D$ meson, and $\epsilon_{D^*}(m)$ represents the polarization of the $D^*$ meson. We follow the convention of expressing the four-momentum and polarizations of the $D^*$ meson in the $B$-meson rest frame as follows:

$$p_D^\mu = (k_0, 0, 0, k_z), \quad \epsilon_D^\mu(\pm) = (0, 1, \pm i, 0)/\sqrt{2}, \quad \epsilon_D^\mu(0) = (k_z, 0, 0, k_0)/m_{D^*}.$$

(A.2)

In addition to the hadronic pieces corresponding to the three well-defined helicities of the on-shell $D^*$ meson, we make use of a fourth timelike helicity for an off-shell particle, defined such that $H_{D^*}(t) = H_{D^*}(0)$.

The helicity amplitude pieces, $M_{N^*}$, also depend on the helicities of the intermediate particles. These are of the scalar-pseudoscalar ($SP$), the vector-axialvector ($VA$), and the tensor ($T$) types. Furthermore, since the decaying $B$ meson is spinless, only certain helicity combinations survive. The list of non-zero components of the helicity amplitude pieces are listed below:

$$M_{SP}^{(0)}(B \to D^*SP^*) = A_{SP},$$
$$M_{VA}^{(+;+)}(B \to D^*VA^*) = A_+, \quad M_{VA}^{(-;-)}(B \to D^*VA^*) = A_-, \quad M_{VA}^{(0;0)}(B \to D^*VA^*) = A_0,$$
Based on the above limits, the normalized parameters are limited to values between 0 and 1. Below we express the amplitudes in terms of these normalized parameters.

\[
\begin{align*}
\mathcal{M}^{VA}_{(0;0)}(B \to D^*VA^*) &= A_t , \\
\mathcal{M}^{T}_{(+;+;0)}(B \to D^*T^*) &= \mathcal{M}^{T}_{(+;+;t)}(B \to D^*T^*) = A_{+,T} , \\
\mathcal{M}^{T}_{(0;-,t)}(B \to D^*T^*) &= \mathcal{M}^{T}_{(0;0,t)}(B \to D^*T^*) = A_{0,T} , \\
\mathcal{M}^{T}_{(-;-,0)}(B \to D^*T^*) &= \mathcal{M}^{T}_{(-;-,t)}(B \to D^*T^*) = A_{-,T} .
\end{align*}
\]

\[\text{(A.3)}\]

As seen in the above, there are a total of 8 independent helicity amplitudes: one of type SP, four of type VA, and three independent amplitudes of type T. Using the definitions for \(B \to D^*\) form factors given in Refs. [34, 98], we can further represent each helicity amplitude as follows:

\[
\begin{align*}
A_{SP} &= -g_P \frac{\sqrt{\lambda(m_B^2, m_{D^*}^2, q^2)} A_0(q^2)}{m_b + m_c} , \\
A_0 &= -\frac{(1 + g_L - g_R)(m_B + m_{D^*})}{2m_{D^*} \sqrt{q^2}} \left( (m_B^2 - m_{D^*}^2 - q^2)A_1(q^2) + \frac{\lambda(m_B^2, m_{D^*}^2, q^2)}{(m_B + m_{D^*})^2} A_2(q^2) \right) , \\
A_t &= -(1 + g_L - g_R) \frac{\sqrt{\lambda(m_B^2, m_{D^*}^2, q^2)}}{\sqrt{q^2}} A_0(q^2) , \\
A_{\pm} &= (1 + g_L - g_R)(m_B + m_{D^*})A_1(q^2) \mp (1 + g_L + g_R) \frac{\sqrt{\lambda(m_B^2, m_{D^*}^2, q^2)}}{m_B + m_{D^*}} V(q^2) , \\
A_{0,T} &= \frac{g_T}{2m_{D^*}} \left( (m_B^2 + 3m_{D^*}^2 - q^2)T_2(q^2) - \frac{\lambda(m_B^2, m_{D^*}^2, q^2)}{m_B^2 - m_{D^*}^2} T_1(q^2) \right) , \\
A_{\pm,T} &= g_T \frac{\sqrt{\lambda(m_B^2, m_{D^*}^2, q^2)} T_1(q^2) \pm (m_B^2 - m_{D^*}^2) T_2(q^2)}{\sqrt{q^2}} ,
\end{align*}
\]

where \(\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca\).

Finally, the amplitudes for the vector and tensor types can be expressed in the transversity basis (using \(\perp, ||\)) instead of the helicity basis (using \(\pm\)), using the following relationships,

\[
\begin{align*}
A_{\perp,(T)} &= (A_{+(T)} + A_{-(T)})/\sqrt{2} , \\
A_{\perp,(T)} &= (A_{+(T)} - A_{-(T)})/\sqrt{2} .
\end{align*}
\]

\[\text{(A.5)}\]

### B Expressions for the \(S_i, R_i\) and \(I_i\) Factors

The kinematics of the five-body decay restricts the range of values that the parameters \(q^2\) and \(E_\pi\) can take.

\[
\begin{align*}
\frac{m_\pi^2}{2} &\leq q^2 \leq (m_B - m_{D^*})^2 , \\
\frac{m_\pi^2 + m_\pi^2 q^2}{2m_\pi^2 \sqrt{q^2}} &\leq E_\pi \leq \frac{q^2 + m_\pi^2}{2\sqrt{q^2}} .
\end{align*}
\]

\[\text{(B.1)}\]

We define the normalized parameters \(\rho_\pi \equiv m_\pi^2/\sqrt{q^2}, \rho_\pi \equiv m_\pi/\sqrt{q^2},\) and \(E_\pi \equiv E_\pi/\sqrt{q^2}\). Based on the above limits, the normalized parameters are limited to values between 0 and 1. Below we express \(S_i, R_i,\) and \(I_i\) in terms of these normalized parameters.
The expressions for the $S_i$ factors are

$$S_i = \rho_i^2 S_{SP} = 16\rho_i^2 \left(2\varepsilon_\pi \rho_i^2 - \rho_i^4 - \rho_\pi^2\right),$$

$$S_{0,1} = \frac{1}{\varepsilon_\pi^2 - \rho_\pi^2} \left(4\rho_i^2 \left(2\varepsilon_\pi \rho_i^2 \left(1 - \rho_\pi^2\right) + \rho_i^4 + 3\rho_\pi^2\right)
+ 2\varepsilon_\pi^2 \left(1 - 2\rho_\pi^2 - \rho_i^4\right) - 4\varepsilon_\pi^3 \left(1 - \rho_\pi^2\right)
+ \rho_i^4 \left(-1 + \rho_\pi^2\right) - 3\rho_\pi^2 - \rho_i^4\right),$$

$$S_{0,2} = -4S_{1,2} = \frac{1}{\varepsilon_\pi^2 - \rho_\pi^2} \left(4\rho_i^2 \left(2\varepsilon_\pi \left(1 + \rho_\pi^2\right) \left(3\rho_\pi^2 + \rho_i^2\right)
- 2\varepsilon_\pi^2 \left(1 + 6\rho_\pi^2 - \rho_i^4 + 2\rho_\pi^2\right) + 4\varepsilon_\pi^3 \left(1 + \rho_\pi^2\right)
- \rho_i^4 \left(3 + \rho_\pi^2\right) - \rho_\pi^2 - \rho_i^4\right),$$

$$S_{1,1} = \frac{1}{\varepsilon_\pi^2 - \rho_\pi^2} \left(-\rho_i^2 \left(2\varepsilon_\pi \left(1 + 3\rho_\pi^2\right) + \rho_i^4 + 5\rho_\pi^2\right)
- 2\varepsilon_\pi^2 \left(3 + 2\rho_\pi^2 - \rho_i^4 + 2\rho_\pi^2\right) + 4\varepsilon_\pi^3 \left(3 - \rho_\pi^2\right)
- \rho_i^4 \left(1 + 3\rho_\pi^2\right) + 5\rho_\pi^2 + 3\rho_i^4\right),$$

$$S_{||,1} = 8\rho_i^2 \left(1 - 2\varepsilon_\pi + \rho_\pi^2\right),$$

$$S_{||,2} = \frac{(-4)\rho_i^2 \left(1 - 2\varepsilon_\pi + \rho_\pi^2\right) \left(2\varepsilon_\pi \rho_i^2 - \rho_i^4 - \rho_\pi^2\right)}{\varepsilon_\pi^2 - \rho_\pi^2},$$

$$S_{||,3} = \frac{(-8\rho_i^2) \left(2\varepsilon_\pi \left(1 + \rho_\pi^2\right) - \rho_i^2 - \rho_\pi^2\right) \left(2\varepsilon_\pi - \rho_i^2 + \rho_\pi^2\right)}{\varepsilon_\pi^2 - \rho_\pi^2},$$

$$S_{0T,1} = \frac{1}{\varepsilon_\pi^2 - \rho_\pi^2} \left(-64 \left(2\varepsilon_\pi \rho_i^2 \left(1 - \rho_\pi^2\right) + 3\rho_\pi^2\right)
+ 2\varepsilon_\pi^2 \rho_i^2 \left(1 + 2\rho_\pi^2 - \rho_i^4\right) - 4\varepsilon_\pi^3 \rho_i^2 \left(1 - \rho_\pi^2\right)
+ \rho_i^4 \rho_\pi^2 \left(1 + 3\rho_\pi^2\right) - \rho_i^4 + \rho_\pi^6\right),$$

$$S_{0T,2} = -4S_{\perp T,2} = -4S_{||T,2}
= \frac{1}{\varepsilon_\pi^2 - \rho_\pi^2} \left(-64 \left(2\varepsilon_\pi \rho_i^2 \left(1 + \rho_i^2\right) \left(\rho_i^2 + 3\rho_\pi^2\right)
- 2\varepsilon_\pi^2 \left(6\rho_i^2 \rho_\pi^2 + \rho_i^4 \left(2 + \rho_\pi^2\right) + \rho_\pi^2\right) + 4\varepsilon_\pi^3 \rho_i^2 \left(1 + \rho_\pi^2\right)
+ \rho_i^4 \rho_\pi^2 \left(1 - \rho_\pi^2\right) - \rho_i^4 - 3\rho_\pi^6\right),$$

$$S_{\perp T,1} = S_{||T,1} = \frac{1}{\varepsilon_\pi^2 - \rho_\pi^2} \left(-16 \left(2\varepsilon_\pi \rho_i^2 \left(3 + \rho_\pi^2\right) - 5\rho_i^4 + \rho_\pi^2\right)
- \varepsilon_\pi^2 \left(4\rho_i^2 \rho_\pi^2 + \rho_i^4 \left(4 + 6\rho_\pi^2\right) - 2\rho_\pi^2\right) - 4\varepsilon_\pi^3 \rho_i^2 \left(1 - 3\rho_\pi^2\right)
+ \rho_i^4 \rho_\pi^2 \left(3 + 5\rho_\pi^2\right) - 3\rho_i^4 - \rho_\pi^6\right).$$

The expressions for the $R_i$ factors are

$$R_{00} = 2\sqrt{2}R_{\|} = \rho_i R_{SP0} = 2\sqrt{2}\rho_i R_{SP\|} = \frac{-32\rho_i^2 \left(1 - \varepsilon_\pi\right) \left(2\varepsilon_\pi \rho_i^2 - \rho_i^4 - \rho_\pi^2\right)}{\varepsilon_\pi^2 - \rho_\pi^2},$$

$$R_{0\|} = \frac{2\sqrt{2}\rho_i^2 \left(2\varepsilon_\pi \left(1 + \rho_\pi^2\right) \left(3\rho_\pi^2 + \rho_i^2\right) - 2\varepsilon_\pi^2 \left(1 + 6\rho_\pi^2 + \rho_i^4 + 2\rho_\pi^2\right)}
And finally the expressions for the \( I \) factors are

\[
I_{\perp \perp} = \rho_{\perp} I_{SP \perp} = \frac{8 \sqrt{2} \rho_{\perp}^2 (1 - \rho_{\perp}^2) (2 \rho_{\perp}^2 - \rho_{\perp}^4 - \rho_{\perp}^4)}{\sqrt{\rho_{\perp}^2 - \rho_{\perp}^4}},
\]

\[
I_{\parallel \parallel} = \sqrt{2} I_{\parallel 0 \parallel} = \frac{-4 \sqrt{2} \rho_{\parallel}^2 \left( 2 \rho_{\parallel}^2 + \rho_{\parallel}^4 \right) (\rho_{\parallel}^2 - 3 \rho_{\parallel}^2 + 2 \rho_{\parallel}^4)}{\sqrt{\rho_{\parallel}^2 - \rho_{\parallel}^4}},
\]

\[
I_{0 \parallel} = -I_{\parallel \perp} = \frac{-16 \sqrt{2} \rho_{\parallel} (1 - 2 \rho_{\parallel}^2) \rho_{\perp}^2 (\rho_{\perp}^2 - \rho_{\perp}^4)}{\sqrt{\rho_{\perp}^2 - \rho_{\perp}^4}},
\]

\[
I_{\perp \parallel} = \rho_{\perp} I_{SP \parallel} = \frac{32 \sqrt{2} \rho_{\parallel} (\rho_{\perp}^2 - \rho_{\perp}^4) (2 \rho_{\perp}^2 - \rho_{\perp}^4 - \rho_{\perp}^4)}{\sqrt{\rho_{\perp}^2 - \rho_{\perp}^4}},
\]
\[ I_{\perp T0} = -I_{0T\perp} = \frac{1}{\sqrt{2}} I_{\perp T} = -\frac{1}{\sqrt{2}} I_{\parallel T\perp} \]
\[ = \frac{8\sqrt{2} \rho_\tau}{\xi_\tau^2 - \rho_\tau^2} \left( \xi_\pi \left( 8\rho_\tau^2 \rho_\pi^2 + 3\rho_\pi^4 (1 + \rho_\pi^2) + 3\rho_\pi^2 + 3\rho_\pi^4 \right) \right. \]
\[ - 4\xi_\tau^2 (1 + \rho_\pi^2) \left( \rho_\pi^2 + \rho_\pi^2 \right) + 4\xi_\tau^3 \rho_\pi^2 - 2\rho_\pi^2 (1 + \rho_\pi^2) \left( \rho_\pi^2 + \rho_\pi^2 \right) \right), \]
\[ I_{\parallel T\perp} = \sqrt{2} I_{0T\perp} = \frac{-16 \rho_\tau}{\xi_\tau^2 - \rho_\tau^2} \left( \xi_\pi \left( 8\rho_\tau^2 \rho_\pi^2 + 3\rho_\pi^4 (1 + \rho_\pi^2) + 3\rho_\pi^2 + 3\rho_\pi^4 \right) \right. \]
\[ - 4\xi_\tau^2 (1 + \rho_\pi^2) \left( \rho_\pi^2 + \rho_\pi^2 \right) + 4\xi_\tau^3 \rho_\pi^2 - 2\rho_\pi^2 (1 + \rho_\pi^2) \left( \rho_\pi^2 + \rho_\pi^2 \right) \right). \quad (B.4) \]

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