Optimal control of coupled forward-backward stochastic system with jumps and related Hamilton-Jacobi-Bellman equations

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Abstract: In this paper we investigate a kind of optimal control problem of coupled forward-backward stochastic system with jumps whose cost functional is defined through a coupled forward-backward stochastic differential equation with Brownian motion and Poisson random measure. For this end, we first study the regularity of solutions for this kind of forward-backward stochastic differential equations. We obtain that the value function is a deterministic function and satisfies the dynamic programming principle for this kind of optimal control problem. Moreover, we prove that the value functions is a viscosity solutions of the associated Hamilton-Jacobi-Bellman equations with integral-differential operators.

Keywords: Optimal control; Forward-backward stochastic differential equations; Hamilton-Jacobi-Bellman equations; Dynamic programming principle; Poisson random measure.

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) in the framework of Brownian motion were introduced by Pardoux and Peng [10]. Since this pioneering work, the theory of BSDEs has been developing quickly and dynamically, and it has become a powerful tool in the study of partial differential equations (PDEs in short), stochastic control, stochastic differential games and mathematical finance.

Forward backward stochastic differential equations (FBSDEs in short) in the framework of Brownian motion were first studied by Antonelli [1] using contraction mapping on a small time interval. By virtue of a four step scheme, Ma, Protter and Yong [8] studied the solvability of FBSDEs with deterministic coefficients over an arbitrarily time duration, in which they obtained that the backward components of the solution are determined explicitly by the forward component of the solution by means of the solution of PDEs. But they required that the forward equation is non-degenerate. Hu and Peng [4], Peng and Wu [12] and Yong [17] investigated the solvability of FBSDEs on an arbitrarily time duration via method of continuation. This method allows the coefficients to be random and the forward equation to be degenerate. But they required some monotonicity conditions on the coefficients. For more details of the theory of FBSDEs, we refer to Ma and Yong [9] and the references therein.

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By virtue of BSDE methods Peng [11] investigated stochastic optimization problem of decoupled FBSDEs. Wu and Yu [16] studied stochastic control problem of decoupled FBSDEs with reflection in the framework of Brownian motion. Li and Peng [5] investigated stochastic control problem of decoupled FBSDEs in the framework of Brownian motion and Poisson random measure. Wu and Yu [16] and Li and Peng [5] generalize the results in Peng [11]. In Peng [11], Wu and Yu [16] and Li and Peng [5], they supposed that the coefficients are Hölder continuous with respect to the control variable. Recently, Li and Wei [7] and Li and Peng [5] study optimal control problems of coupled FBSDEs in the framework of Brownian motion. Lin [7] study Nash equilibrium payoffs for stochastic differential games with jumps and coupled nonlinear cost functionals.

Motivated by the above mentioned papers, the objective of this paper is to investigate one kind of optimal control problem of coupled forward-backward stochastic system with jumps whose cost functional is defined by a coupled FBSDE in the framework of Brownian motion and Poisson random measure. Wu and Yu [15] first studied stochastic control problem of decoupled FBSDEs with Brownian motion and Poisson random measure. Lin [7] investigated stochastic control problem of decoupled FBSDEs with jumps, which also has its own importance. Our main results is that $W$ is a deterministic function and satisfies the dynamic programming principle.

The objective of this paper is to study the value function $W$. For this, we first investigate the regularity for solutions of coupled FBSDEs with jumps, which also has its own importance. Our main results is that $W$ is a deterministic function and satisfies the dynamic programming principle.
The other main result is that we give a probabilistic interpretation of a class of Hamilton-Jacobi-Bellman equations with integral-differential operators, i.e., the value function $W$ is a viscosity solution of the following Hamilton-Jacobi-Bellman equations with integral-differential operators:

$$
\begin{cases}
\frac{\partial}{\partial t}W(t, x) + H(t, x, W, DW, D^2W) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n, \\
W(T, x) = \Phi(x), & x \in \mathbb{R}^n.
\end{cases}
$$

where for $(t, x, u) \in [0, T] \times \mathbb{R}^n \times U,$

$$H(t, x, W, DW, D^2W) = \sup_{u \in U} H_0(t, x, W, DW, D^2W, u)$$

and

$$H_0(t, x, W, DW, D^2W, u) = \frac{1}{2} tr(\sigma \sigma^T(t, x, W(t, x), u)D^2W(t, x))$$

$$+ DW(t, x) \cdot b(t, x, W(t, x), DW(t, x) \cdot \sigma(t, x, W(t, x), u),$$

$$\int_E (W(t, x + g(t, x, W(t, x), u, e) - W(t, x))\lambda(de), u)$$

$$+ \int_E (W(t, x + g(t, x, W(t, x), u, e) - W(t, x) - DW(t, x) \cdot g(t, x, W(t, x), u, e))\lambda(de)$$

$$+ f(t, x, W(t, x), DW(t, x) \cdot \sigma(t, x, W(t, x), u),$$

$$\int_E (W(t, x + g(t, x, W(t, x), u, e)) - W(t, x))\lambda(de), u).$$

For the proof of the above probabilistic interpretation, we adapt the original ideas from Peng [11] for the optimal control problem of decoupled forward-backward stochastic system in the framework of Brownian motion, and the developed ideas from Li and Wei [6] for the optimal control problem of coupled forward-backward stochastic system in the framework of Brownian motion to our framework. In particular, in comparison with Li and Wei [6], we make a short proof (see the proof of Lemma 5.5).

We investigate optimal control problem of coupled forward-backward stochastic system without supposing the coefficients to be Hölder continuous with respect to the control variable, while Peng [11], Wu and Yu [16] and Li and Peng [5] supposed that the coefficients are Hölder continuous with respect to the control variable. On the other hand, in comparison with Li and Peng [5], we investigate optimal control problem of coupled forward-backward stochastic system in the framework of Brownian motion and Poisson random measure, while Li and Peng [5] studied the decoupled forward-backward stochastic system in the framework of Brownian motion and Poisson random measure.

Li and Wei [6] studied the optimal control problem of coupled forward-backward stochastic system in the framework of Brownian motion. They studied two cases of the diffusion coefficient $\sigma$, i.e., (i) $\sigma$ does not depend on $z$, but depends on the control $u$; (ii) $\sigma$ depends on $z$, but does not depend on the control $u$. In this paper, we generalize the result of the case (i) to the forward-backward system in the framework of Brownian motion and Poisson random measure.

Li and Wei [6] obtained the regularity of coupled FBSDEs without jumps which plays an important role when $Y$ is one dimensional and the coefficients are the same. In comparison with Li
and Wei [6], we use a simple method to get the regularity of coupled FBSDEs with jumps when $Y$

is multidimensional and the coefficients can be different. Moreover, the presence of jumps in this

paper brings us much difficulty and adds us a supplementary complexity.

This paper is organized as follows: Sections 2 recalls some notations and preliminaries. In

Section 3 we study the regularity for solutions of coupled FBSDEs with jumps, which is useful

in what follows. Section 4 studies optimal control of our forward-backward stochastic control

system. We prove that the value function $W$ is deterministic and satisfies the dynamic programming

principle. In Section 5, we give a probabilistic interpretation of a class of Hamilton-Jacobi-Bellman

equations with integral-differential operators.

2 Preliminaries

In this section, we present some preliminaries. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the

completed product of the Wiener space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and the Poisson space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. In the

Wiener space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$: $\Omega_1 = C_0(\mathbb{R}; \mathbb{R}^d)$ is the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$ with

value zero at 0, endowed with the topology generated by the uniform convergence on compacts.

Moreover, $\mathcal{F}_1$ is the Borel $\sigma$-algebra over $\Omega_1$, completed by the Wiener measure $\mathbb{P}_1$ under which the

dimensional coordinate processes $B_s(\omega) = \omega_s$, $s \in \mathbb{R}_+$, $\omega \in \Omega_1$, and $B_{-s}(\omega) = \omega(-s)$, $s \in \mathbb{R}_+$, $\omega \in \Omega_1$, are two independent $d$-dimensional Brownian motions. We denote by $\{F^B_s, s \geq 0\}$

the natural filtration generated by $B$ and augmented by all $\mathbb{P}_1$-null sets, i.e.,

$$F^B_s = \sigma\{B_r, r \in (-\infty, s]\} \lor \mathcal{N}_{\mathbb{P}_1}, s \geq 0.$$  

Let us now introduce the Poisson space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. We denote by $E = \mathbb{R}^l \setminus \{0\}$ endowed the

space $E$ with its Borel $\sigma$-field $\mathcal{B}(E)$. We consider a point function $p$ on $E$, i.e., $p : D_p \subset \mathbb{R} \rightarrow E$

where the domain $D_p$ is a countable subset of the real line $\mathbb{R}$. The point function $p$ defines on $\mathbb{R} \times E$

the counting measure $N(p, dtde)$ by the following

$$N(p, (s, t] \times \Delta) = \sharp\left\{r \in D_p \cap (s, t] : p(r) \in \Delta\right\}, \Delta \in \mathcal{B}(E), s, t \in \mathbb{R}, s < t.$$  

We denote by $\Omega_2$ the collection of all point functions $p$ on $E$, and $\mathcal{F}_2$ be the smallest $\sigma$-field

on $\Omega_2$ with respect to which all mappings $p \rightarrow N(p, (s, t] \times \Delta), s, t \in \mathbb{R}, s < t, \Delta \in \mathcal{B}(E)$ are measurable. On the measurable space $(\Omega_2, \mathcal{F}_2)$ let us consider the probability measure $\mathbb{P}_2$ under which $N(p, dtde)$ becomes a Poisson random measure with Lévy measure $\lambda$. The compensator of $N$

is $\check{N}(dtde) = dt\lambda(de)$ and $\{\check{N}((s, t] \times A) = (N - \check{N})((s, t] \times A)\}_{s \leq t}$ is a martingale for any $A \in \mathcal{B}(E)$
satisfying $\lambda(A) < \infty$, where $\lambda$ is an arbitrarily given $\sigma$-finite Lévy measure on $(E, \mathcal{B}(E))$, i.e., a

measure on $(E, \mathcal{B}(E))$ with the property that $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$. The filtration $(F^N_t)_{t \geq 0}$ is generated by $N$ by the following

$$\mathcal{F}^N_t = \sigma\left\{N((s, r] \times \Delta) : -\infty < s \leq r \leq t, \Delta \in \mathcal{B}(E)\right\}, t \geq 0,$$

and taking the right-limits $\mathcal{F}^N_t = \left(\bigcap_{s \leq t} \mathcal{F}^N_s\right) \lor \mathcal{N}_{\mathbb{P}_2}, t \geq 0$, augmented by the $\mathbb{P}_2$-null sets. Let us

put $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$, where $\mathcal{F}$ is completed with respect to $\mathbb{P}$, and the

filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is generated by

$$\mathcal{F}_t := \mathcal{F}^B_t \otimes \mathcal{F}^N_t, \quad t \geq 0,$$

augmented by all $\mathbb{P}$-null sets.
Let $T > 0$ be an arbitrarily fixed time horizon. For any $n \geq 1$, we denote by $|z|$ the Euclidean norm of $z \in \mathbb{R}^n$. We introduce the following spaces of stochastic processes.

- $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n) = \left\{ \xi \mid \xi : \Omega \to \mathbb{R}^n \text{ is an } \mathcal{F}_T\text{-measurable random variable such that} \right\}$
  \[ \mathbb{E}[|\xi|^2] < +\infty \],

- $S^2(0, T; \mathbb{R}) = \left\{ \varphi \mid \varphi : \Omega \times [0, T] \to \mathbb{R} \text{ is an } \mathbb{F}\text{-adapted càdlàg process such that} \right\}$
  \[ \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\varphi_t|^2 \right] < +\infty \],

- $\mathcal{H}^2(0, T; \mathbb{R}^d) = \left\{ \varphi \mid \varphi : \Omega \times [0, T] \to \mathbb{R}^d \text{ is an } \mathbb{F}\text{-predictable process such that} \right\}$
  \[ \mathbb{E}\int_0^T |\varphi_t|^2 dt < +\infty \],

- $\mathcal{K}_2^2(0, T; \mathbb{R}^d) = \left\{ k \mid k : \Omega \times [0, T] \to \mathbb{R}^d \text{ is an } \mathbb{F}\text{-predictable process such that} \right\}$
  \[ \mathbb{E}\int_0^T \int_E |k_t(e)|^2 \lambda(de) dt < +\infty \}.

Let us consider the following BSDE with data $(f, \xi)$:

\[ y_t = \xi + \int_t^T f(s, y_s, z_s, k_s) ds - \int_t^T z_s dB_s - \int_t^T \int_E k_s(e) d\tilde{N}(de, ds), \quad 0 \leq t \leq T. \tag{2.1} \]

Here $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is $\mathbb{F}$-predictable and satisfies the following assumptions:

(H2.1) (Lipschitz condition): There exists a positive constant $C$ such that, for all $(t, y_i, z_i, k_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $i = 1, 2$,

\[ |f(t, y_1, z_1, k_1) - f(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |k_1 - k_2|). \]

(H2.2) $f(\cdot, 0, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$.

The following existence and uniqueness theorem of BSDE (2.1) was first established in Tang and Li [13].

**Lemma 2.1.** Under assumptions (H2.1) and (H2.2), for all $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, BSDE (2.1) has a unique solution $(y, z, k) \in S^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d) \times \mathcal{K}_2^2(0, T; \mathbb{R})$.

For some $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfying (H2.1) and (H2.2), let us put

\[ f_i(s, y_s^i, z_s^i, k_s^i) = f(s, y_s^i, z_s^i, k_s^i) + \varphi_i(s), \quad i = 1, 2, \]

where $\varphi_i \in \mathcal{H}^2(0, T; \mathbb{R})$. If $\xi_1$ and $\xi_2$ are in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, then we have the following lemma. For its proof, the reader can refer to Barles, Buckdahn and Pardoux [2].
Lemma 2.2. Let us denote by \((y^1, z^1, k^1)\) and \((y^2, z^2, k^2)\) the solutions of BSDE (2.1) with the data \((\xi_1, f_1)\) and \((\xi_2, f_2)\), respectively. Then there exists a positive constant \(C\) such that, for all \(t \in [0, T]\),
\[
|y_t^1 - y_t^2|^2 + \frac{1}{2} \mathbb{E} \left[ \int_t^T (|y^1_s - y^2_s|^2 + |z^1_s - z^2_s|^2) ds \right] + \mathbb{E} \int_t^T |k^1_s(e) - k^2_s(e)|^2 \lambda(de) ds |F_t| \leq C \mathbb{E} |\xi_1 - \xi_2|^2 |F_t| + C \mathbb{E} \left[ \int_t^T |\varphi_1(s) - \varphi_2(s)|^2 ds \right], \quad \mathbb{P} - a.s.
\]

3 Regularity for solutions of coupled FBSDEs with jumps

The objective of this section is to investigate regularity for solutions of coupled FBSDEs with jumps, which is very useful in what follows and also has its own importance.

Let us consider the following coupled FBSDE with data \((b, \sigma, g, f, \zeta, \Phi)\):
\[
\begin{cases}
  dX_s^{t, \zeta} = b(s, X_s^{t, \zeta}, Y_s^{t, \zeta}, Z_s^{t, \zeta}, K_s^{t, \zeta})ds + \sigma(s, X_s^{t, \zeta}, Y_s^{t, \zeta}, Z_s^{t, \zeta}, K_s^{t, \zeta})dB_s \\
  -dY_s^{t, \zeta} = f(s, X_s^{t, \zeta}, Y_s^{t, \zeta}, Z_s^{t, \zeta}, K_s^{t, \zeta})ds - Z_s^{t, \zeta}dB_s - \int_E K_s^{t, \zeta}(e) \tilde{N}(de, ds), \\
  X_t^{t, \zeta} = \zeta, \\
  Y_T^{t, \zeta} = \Phi(X_T^{t, \zeta}),
\end{cases}
\]

where
\[
\begin{align*}
  b : & \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \to \mathbb{R}^n, \\
  \sigma : & \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \to \mathbb{R}^{n \times d}, \\
  g : & \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \times E \to \mathbb{R}^n, \\
  f : & \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \to \mathbb{R}^m, \\
  \Phi : & \Omega \times \mathbb{R}^n \to \mathbb{R}^m
\end{align*}
\]

are \(F_t\)-progressively measurable processes, and \(\zeta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)\). For a given \(m \times n\) full-rank matrix \(G\) let us define:
\[
v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A(t, v, k) = \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix},
\]

where \(G^T\) is transpose matrix of \(G\). We use the usual inner product and Euclidean norm in \(\mathbb{R}^n, \mathbb{R}^m\) and \(\mathbb{R}^{m \times d}\), respectively.

Let us make the following assumptions:

(H3.1) (i) \(b, \sigma, g\) and \(f\) are uniformly Lipschitz with respect to \((x, y, z, k)\), and \((b(\cdot, 0, 0, 0, 0), \sigma(\cdot, 0, 0, 0, 0), f(\cdot, 0, 0, 0, 0)) \in H^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}), g(\cdot, 0, 0, 0, 0) \in \mathcal{K}^2_\lambda(0, T; \mathbb{R}^n)\); (ii) \(\Phi(x)\) is uniformly Lipschitz with respect to \(x \in \mathbb{R}^n\), and for any \(x \in \mathbb{R}^n, \Phi(x) \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^m)\).
In order to simplify the notations we denote by $\Phi(x) - \Phi(\bar{x}), G(x - \bar{x}) \geq \mu_1 |G\bar{x}|^2,$ for all $v = (x, y, z), \overline{v} = (\overline{x}, \overline{y}, \overline{z}), \bar{v} = y - \bar{y}, \bar{z} = z - \bar{z}, \bar{g} = g(v, k) - g(\overline{v}, \overline{\bar{v}})$, where $\alpha, \beta, \mu_1$ are nonnegative constants with $\alpha + \beta > 0, \beta + \mu_1 > 0.$ Moreover, we have $\alpha > 0, \mu_1 > 0$ (resp. $\beta > 0$), when $m > n$ (resp. $m < n$).

We have the following existence and uniqueness of solutions of FBSDE (3.1). For the proof the reader can refer to Wu [14].

**Lemma 3.1.** Under assumptions (H3.1) and (H3.2), for any $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, FBSDE (3.1) has a unique adapted solution $(X^{t, \zeta}, Y^{t, \zeta}, Z^{t, \zeta}, K^{t, \zeta}).$

If $\zeta_1$ and $\zeta_2$ are in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, then we have the following lemma.

**Proposition 3.2.** We suppose that $(b_i, \sigma_i, g_i, f_i, \zeta_i, \Phi_i)$, for $i = 1, 2$, satisfy (H3.1) and (H3.2). Let $(X^i_{s}, Y^i_{s}, Z^i_{s}, K^i_{s})_{t \leq s \leq T}$ be the solution of FBSDE (3.1) associated to $(b_i, \sigma_i, g_i, f_i, \zeta_i, \Phi_i).$ Then the following holds: for all $t \in [0, T],$

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |X^1_{s} - X^2_{s}|^2 + \sup_{s \in [t, T]} |Y^1_{s} - Y^2_{s}|^2 + \int_t^T (|Y^1_{s} - Y^2_{s}|^2 + |Z^1_{s} - Z^2_{s}|^2) ds |\mathcal{F}_t \right] \\
\quad + \mathbb{E} \left[ \int_t^T \int_E |K^1_{s}(e) - K^2_{s}(e)|^2 \lambda(e) ds |\mathcal{F}_t \right] \\
\leq C|\zeta_1 - \zeta_2|^2 + C \mathbb{E} \left[ \int_t^T |b_1(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s}) - b_2(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s})|^2 ds |\mathcal{F}_t \right] \\
\quad + C \mathbb{E} \left[ \int_t^T |\sigma_1(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s}) - \sigma_2(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s})|^2 ds |\mathcal{F}_t \right] \\
\quad + C \mathbb{E} \left[ \int_t^T |f_1(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s}) - f_2(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s})|^2 ds |\mathcal{F}_t \right] \\
\quad + C \mathbb{E} \left[ \int_t^T \int_E |g_1(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s}(e)) - g_2(s, X^1_{s}, Y^1_{s}, Z^1_{s}, K^1_{s}(e))|^2 \lambda(de) ds |\mathcal{F}_t \right] \\
\quad + C \mathbb{E} \left[ |\Phi_1(X^1_{T}) - \Phi_2(X^2_{T})|^2 ds |\mathcal{F}_t \right], \mathbb{P} - a.s.,
\]

and

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |X^1_{s}|^2 + \sup_{s \in [t, T]} |Y^1_{s}|^2 + \int_t^T (|Y^1_{s}|^2 + |Z^1_{s}|^2) ds |\mathcal{F}_t \right] \\
\quad + \mathbb{E} \left[ \int_t^T \int_E |K^1_{s}(e)|^2 \lambda(e) ds |\mathcal{F}_t \right] \leq C(1 + |\zeta_1|^2), \mathbb{P} - a.s.
\]

**Proof.** We only consider the case when $m > n.$ For the other case we can use a similar argument. In order to simplify the notations we denote by

\[
\hat{v} = v^1 - v^2 = (\hat{X}, \hat{Y}, \hat{Z}) = (X^1 - X^2, Y^1 - Y^2, Z^1 - Z^2), \hat{k} = K^1_{s}(e) - K^2_{s}(e),
\]
and the following holds:

\[
\begin{aligned}
\left\{ \begin{array}{l}
d\hat{X}_s &= [b_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - b_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)]ds \\
                  &+ \int_E [g_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - g_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)]d\widetilde{B}_s \\
-d\hat{Y}_s &= f_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - f_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)ds - \tilde{Z}_s d\widetilde{B}_s - \int_E \tilde{K}_s(e)\tilde{N}(de, ds), \\
\hat{X}_t &= \zeta_1 - \zeta_2, \\
\hat{Y}_T &= \Phi_1(X_T^1) - \Phi_2(X_T^2).
\end{array} \right.
\]

From (H3.1) and standard arguments for SDEs it follows that

\[
\begin{align*}
\mathbb{E}[\sup_{s \in [t,T]} |\hat{X}_s|^2 |\mathcal{F}_t] &\leq C|\zeta_1 - \zeta_2|^2 + \mathbb{E}[\int_t^T (|\hat{Y}_s|^2 + |\tilde{Z}_s|^2)ds|\mathcal{F}_t] + \mathbb{E}[\int_t^T \int_E |\tilde{K}_s(e)|^2\lambda(e)ds|\mathcal{F}_t] \\
&\quad + C\mathbb{E}[\int_t^T |b_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - b_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)|^2ds|\mathcal{F}_t] \\
&\quad + C\mathbb{E}[\int_t^T |\sigma_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - \sigma_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)|^2ds|\mathcal{F}_t] \\
&\quad + C\mathbb{E}[\int_t^T \int_E |g_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - g_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)|^2\lambda(de)ds|\mathcal{F}_t].
\end{align*}
\quad (3.2)
\]

Applying Itô’s formula to \(e^{\beta t}\hat{Y}_s^2\) and taking \(\beta\) big enough, by virtue of (H3.1), (H3.2) and standard arguments for BSDEs we get

\[
\begin{align*}
|\hat{Y}_t|^2 &+ \mathbb{E}[\int_t^T (|\hat{Y}_s|^2 + |\tilde{Z}_s|^2)ds|\mathcal{F}_t] + \mathbb{E}[\int_t^T \int_E |\tilde{K}_s(e)|^2\lambda(e)ds|\mathcal{F}_t] \\
&\leq C\mathbb{E}[|\Phi_1(X_T^1) - \Phi_2(X_T^2)|^2] + C\mathbb{E}[|\hat{X}_T^2|] + C\mathbb{E}[\int_t^T |\hat{X}_s|^2ds|\mathcal{F}_t] \\
&\quad + C\mathbb{E}[\int_t^T |f_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - f_2(s, X_s^2, Y_s^2, Z_s^2, K_s^2)|^2ds|\mathcal{F}_t].
\end{align*}
\quad (3.3)
\]

Using Itô’s formula to \(\langle G\hat{X}_s, \hat{Y}_s \rangle\) we deduce

\[
\begin{align*}
\mathbb{E}[\langle (G\hat{X}_T, \Phi_1(X_T^1)) - \Phi_2(X_T^2) \rangle|\mathcal{F}_t] &= \mathbb{E}[\langle G(\zeta_1 - \zeta_2), \hat{Y}_t \rangle] \\
&= \mathbb{E}[\int_t^T \langle A_2(s, v_s^1, K_s^1) - A_2(s, v_s^2, K_s^2), \hat{v}_s \rangle ds|\mathcal{F}_t] \\
&\quad + \mathbb{E}[\int_t^T \langle G(1_s, v_s^2, K_s^2(e)) - g_1(s, v_s^2, K_s^2(e)), \tilde{K}_s(e) \rangle ds|\mathcal{F}_t] \\
&\quad + \mathbb{E}[\int_t^T \langle \hat{Y}_s, -G^T(f_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1)) - f_2(s, X_s^2, Y_s^1, Z_s^1, K_s^1))ds|\mathcal{F}_t] \\
&\quad + \mathbb{E}[\int_t^T \langle \tilde{Z}_s, G(\sigma_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - \sigma_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1))ds|\mathcal{F}_t] \\
&\quad + \mathbb{E}[\int_t^T \langle \tilde{K}_s(e), G^T(g_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e)) - g_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e))))\lambda(de)ds|\mathcal{F}_t].
\end{align*}
\]
where

\[ A_2(t, v, K) = \begin{pmatrix} -G^T f_2 \\ Gb_2 \\ G\sigma_2 \end{pmatrix} (t, v, K). \]

From (H3.2) and Young inequality it follows that there exists a positive constant \( C_2 \) such that

\[
\beta_2 \mathbb{E}\left[ \int_t^T (|\tilde{Y}_s|^2 + |\tilde{Z}_s|^2) ds|\mathcal{F}_t \right] + \beta_2 \mathbb{E}\left[ \int_t^T |\tilde{K}_s(e)|^2 \lambda(e) ds|\mathcal{F}_t \right]
\]

\[
+ \beta_1 \mathbb{E}\left[ \int_t^T |\tilde{X}_s|^2 ds|\mathcal{F}_t \right] + \mu_1 \mathbb{E}[|G\tilde{X}_T|^2|\mathcal{F}_t] - \langle G(\zeta_1 - \zeta_2), \tilde{Y}_t \rangle
\]

\[
\leq C_2 \mathbb{E}[|\Phi_1(X_t) - \Phi_2(X_T)|^2|\mathcal{F}_t] + \varepsilon \mathbb{E}\left[ \int_t^T (|\tilde{Y}_s|^2 + |\tilde{X}_s|^2 + |\tilde{Z}_s|^2) ds + \int_t^T |\tilde{K}_s(e)|^2 \lambda(e) \right] ds|\mathcal{F}_t]
\]

\[
+ C_2 \mathbb{E}\left[ \int_t^T |b_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - b_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds|\mathcal{F}_t \right]
\]

\[
+ C_2 \mathbb{E}\left[ \int_t^T |\sigma_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - \sigma_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds|\mathcal{F}_t \right]
\]

\[
+ C_2 \mathbb{E}\left[ \int_t^T |f_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - f_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds|\mathcal{F}_t \right]
\]

\[
+ C_2 \mathbb{E}\left[ \int_t^T \int_E |g_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e)) - g_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e))|^2 \lambda(de) ds|\mathcal{F}_t \right],
\]

(3.4)

where

\[
\varepsilon = \min\left(\frac{1}{3}, \frac{1}{3C_1T}, \frac{\beta_1|G^T G|}{3}, \frac{\mu_1|G^T G|}{3}\right).
\]

From (3.2), (3.3) and Young inequality there exists a constant \( C_3 \) such that

\[
\langle G(\zeta_1 - \zeta_2), \tilde{Y}_t \rangle
\]

\[
\leq C_3 |\zeta_1 - \zeta_2|^2 + \frac{1}{3} \mathbb{E}\left[ \int_t^T (|\tilde{Y}_s|^2 + |\tilde{Z}_s|^2) ds + \int_t^T |\tilde{K}_s(e)|^2 \lambda(e) \right] ds|\mathcal{F}_t]
\]

\[
+ C_3 \mathbb{E}\left[ \int_t^T |b_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - b_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds|\mathcal{F}_t \right]
\]

\[
+ C_3 \mathbb{E}\left[ \int_t^T |\sigma_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - \sigma_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds|\mathcal{F}_t \right]
\]

\[
+ C_3 \mathbb{E}\left[ \int_t^T |f_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - f_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds|\mathcal{F}_t \right]
\]

\[
+ C_3 \mathbb{E}\left[ \int_t^T \int_E |g_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e)) - g_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1(e))|^2 \lambda(de) ds|\mathcal{F}_t \right],
\]

(3.5)

Combining (3.2), (3.3), (3.4) with (3.5) yields

\[
\mathbb{E}\left[ \sup_{s \in [t, T]} |X_s^1 - X_s^2|^2 + \sup_{s \in [t, T]} |Y_s^1 - Y_s^2|^2 + \int_t^T (|Y_s^1 - Y_s^2|^2 + |Z_s^1 - Z_s^2|^2) ds|\mathcal{F}_t \right]
\]

\[
+ \mathbb{E}\left[ \int_t^T \int_E |K_s^1(e) - K_s^2(e)|^2 \lambda(e) ds|\mathcal{F}_t \right]
\]

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\[ \leq C|\zeta_1 - \zeta_2|^2 + C\mathbb{E}\left[\int_t^T |b_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - b_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds | \mathcal{F}_t \right] + C\mathbb{E}\left[\int_t^T |\sigma_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - \sigma_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds | \mathcal{F}_t \right]
\]
\[ + C\mathbb{E}\left[\int_t^T |f_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1) - f_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1)|^2 ds | \mathcal{F}_t \right]
\]
\[ + C\mathbb{E}\left[\int_t^T \int_E |g_1(s, X_s^1, Y_s^1, Z_s^1, K_s^1(\epsilon)) - g_2(s, X_s^1, Y_s^1, Z_s^1, K_s^1(\epsilon))|^2 \lambda(\epsilon) ds | \mathcal{F}_t \right]
\]
\[ + C\mathbb{E}[|\Phi(X_T) - \Phi_2(X_T)|^2 ds | \mathcal{F}_t], \]

from which we get the desired result. We can use the above argument to get the second inequality. The proof is complete. \(\square\)

By virtue of the above Proposition we have the following corollary.

**Corollary 3.3.** There exists a positive constant \(C\) such that

\[ |Y_t^{t,\zeta}| \leq C(1 + |\zeta|); \quad |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'|, \quad \mathbb{P} - \text{a.s.} \]

Let us now introduce the random field:

\[ u(t, x) = Y_{s=t}^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \]

where \(Y_{s=t}^{t,x}\) is the solution of FBSDE (3.1) with \(\zeta = x\).

From Corollary 3.3 it follows that, for all \(t \in [0, T]\), \(x, y \in \mathbb{R}^n\),

\[ |u(t, x) - u(t, y)| \leq C|x - y|, \quad |u(t, x)| \leq C(1 + |x|). \quad (3.6) \]

**Proposition 3.4.** Let assumptions (H3.1) and (H3.2) hold. Then, for any \(t \in [0, T]\) and \(\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)\), we have

\[ u(t, \zeta) = Y_t^{t,\zeta}, \quad \mathbb{P}\text{-a.s.} \]

**Proof.** Let us first consider the case when \(\zeta\) is a simple random variable of the form

\[ \zeta = \sum_{i=1}^N x_i I_{A_i}, \]

where \(\{A_i\}_{i=1}^N\) is a finite partition of \((\Omega, \mathcal{F}_t)\) and \(x_i \in \mathbb{R}^n\), for \(1 \leq i \leq N\). Let us denote by

\[ (X, Y, Z, K) = (X^{t,\zeta}, Y^{t,\zeta}, Z^{t,\zeta}, K^{t,\zeta}). \]

Then \((X, Y, Z)\) is the solution of the following FBSDE:

\[ \left\{ \begin{array}{ll}
X_r & = \zeta + \int_t^r b(s, X_s, Y_s, Z_s, K_s) ds + \int_t^r \sigma(s, X_s, Y_s, Z_s, K_s) dB_s \\
& \quad + \int_t^r \int_E g(s, X_s, Y_s, Z_s, K_s) N(ds, de) ds,
Y_r & = \Phi(X_T) + \int_t^r f(s, X_s, Y_s, Z_s, K_s) ds - \int_t^r Z_s dB_s - \int_t^r \int_E K_s(e) N(ds, de),
\end{array} \right. \quad (3.7) \]
For each \(i\), we denote by \((X^i_s, Y^i_s, Z^i_s, K^i_s) = (X^{1,s,x}_s, Y^{1,s,x}_s, Z^{1,s,x}_s, K^{1,s,x}_s)\). Then \((X^i, Y^i, Z^i)\) is the solution of the following FBSDE:

\[
\begin{align*}
X^i_t &= x_i + \int_t^r b(s, X^i_s, Y^i_s, Z^i_s, K^i_s)\,ds + \int_t^r \sigma(s, X^i_s, Y^i_s, Z^i_s, K^i_s)\,dB_s \\
Y^i_t &= \Phi(X^i_T) + \int_t^r f(s, X^i_s, Y^i_s, Z^i_s, K^i_s)\,ds - \int_t^r Z^i_s\,dB_s - \int_t^r \int_E K^i_s(e)\,\tilde{N}(de, ds),
\end{align*}
\]

From \(\sum_{i=1}^N \varphi(x_i) 1_{A_i} = \varphi(\sum_{i=1}^N x_i 1_{A_i})\) it follows that

\[
\begin{align*}
\sum_{i=1}^N I_{A_i} X^i_t &= x_i + \int_t^r b(s, \sum_{i=1}^N I_{A_i} X^i_s, \sum_{i=1}^N I_{A_i} Y^i_s, \sum_{i=1}^N I_{A_i} Z^i_s, \sum_{i=1}^N I_{A_i} K^i_s)\,ds \\
&\quad + \int_t^r \sigma(s, \sum_{i=1}^N I_{A_i} X^i_s, \sum_{i=1}^N I_{A_i} Y^i_s, \sum_{i=1}^N I_{A_i} Z^i_s, \sum_{i=1}^N I_{A_i} K^i_s)\,dB_s \\
&\quad + \int_t^r \int_E g(s, \sum_{i=1}^N I_{A_i} X^i_s, \sum_{i=1}^N I_{A_i} Y^i_s, \sum_{i=1}^N I_{A_i} Z^i_s, \sum_{i=1}^N I_{A_i} K^i_s)\,\tilde{N}(de, ds),
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^N I_{A_i} Y^i_t &= \Phi(\sum_{i=1}^N I_{A_i} X^i_T) + \int_t^r f(s, \sum_{i=1}^N I_{A_i} X^i_s, \sum_{i=1}^N I_{A_i} Y^i_s, \sum_{i=1}^N I_{A_i} Z^i_s, \sum_{i=1}^N I_{A_i} K^i_s)\,ds \\
&\quad - \int_t^r \int_E \sum_{i=1}^N I_{A_i} K^i_s(e)\,\tilde{N}(de, ds).
\end{align*}
\]

By virtue of the uniqueness solution of FBSDE (3.7) we have

\[
X^{t,\zeta}_s = \sum_{i=1}^N I_{A_i} X^i_s, \quad Y^{t,\zeta}_s = \sum_{i=1}^N I_{A_i} Y^i_s, \quad Z^{t,\zeta}_s = \sum_{i=1}^N I_{A_i} Z^i_s, \quad s \in [t, T].
\]

Since \(u(t, x_i) = Y^i_t, 1 \leq i \leq N\), we deduce that

\[
Y^{t,\zeta}_t = \sum_{i=1}^N I_{A_i} Y^i_t = \sum_{i=1}^N I_{A_i} u(t, x_i) = u(t, \sum_{i=1}^N I_{A_i} x_i) = u(t, \zeta).
\]

Therefore, for simple random variables, we have the desired result.

For any \(\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)\) we can find a sequence of simple random variables \(\{\zeta_n\}\) in \(L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)\) such that \(\lim_{n \to \infty} \mathbb{E}[|\zeta_n - \zeta|^2] = 0\). Then from (3.6) we see that

\[
\mathbb{E}[|Y^{t,\zeta_n}_t - Y^{t,\zeta}_t|^2] \leq C \mathbb{E}[|\zeta_n - \zeta|^2] \to 0, \quad \text{as} \quad n \to \infty,
\]

\[
\mathbb{E}[|u(t, \zeta_n) - u(t, \zeta)|^2] \leq C \mathbb{E}[|\zeta_n - \zeta|^2] \to 0, \quad \text{as} \quad n \to \infty.
\]

Thanks to \(Y^{t,\zeta_n}_t = u(t, \zeta_n), n \geq 1\), we have

\[
Y^{t,\zeta}_t = u(t, \zeta),
\]

from which we conclude the proof. \(\square\)
We make the following assumption.

(H3.3) $\sigma$ and $g$ are independent of $(z,k)$.

**Proposition 3.5.** Let assumptions (H3.1), (H3.2) and (H3.3) hold. Then for all $p \geq 2$, $\zeta \in L^2(\Omega, F_t, \mathbb{P}; \mathbb{R}^n)$, there exists a positive constant $0 \leq \delta_0 \leq T - t$ such that for all $0 \leq \delta \leq \delta_0$,

$$
\mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |X_s^{t,\zeta}|^p + \sup_{s \in [t, t+\delta]} |Y_s^{t,\zeta}|^p + \mathbb{E} \left[ \left( \int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds \right)^{\frac{p}{2}} \right] F_t \right]
$$

$$
+ \mathbb{E} \left[ \int_t^{t+\delta} |K_s^{t,\zeta}(e)|^2 \lambda(e) ds \right]^{\frac{p}{2}} \mathbb{P} - a.s.,
$$

$$
\mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |X_s^{t,\zeta} - x|^p \right] \leq C_p \delta_0^{\frac{p}{2}} (1 + |\zeta|^p), \mathbb{P} - a.s.
$$

**Proof.** From the uniqueness of solutions of FBSDE (3.1) and Proposition 3.4, it is easy to check that

$$
u(s, X_s^{t,\zeta}) = Y_s^{t,\zeta} = Y_s^{t,\zeta}.$$

Consequently, from (3.6) it follows that

$$|Y_s^{t,\zeta}| \leq C(1 + |X_s^{t,\zeta}|). \quad (3.8)$$

By (3.1) we have

$$
\int_t^s Z_r^{t,\zeta} dB_r + \int_t^s \int_E K_r^{t,\zeta}(e) \tilde{N}(de, dr) = Y_r^{t,\zeta} - Y_t^{t,\zeta} + \int_t^s f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}) dr,
$$

Therefore, BDG inequality yields

$$
\mathbb{E} \left[ \int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds + \int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(e) ds \right]^{\frac{p}{2}} \mathbb{P} - a.s.,
$$

$$
\leq C_p \mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |Y_s^{t,\zeta}|^p + \int_t^{t+\delta} f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}, K_r^{t,\zeta}) dr | F_t \right]
$$

$$
\leq C_p \mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |Y_s^{t,\zeta}|^p + \left( \int_t^{t+\delta} (1 + |X_r^{t,\zeta}| + |Y_r^{t,\zeta}| + |Z_r^{t,\zeta}| + \int_E |K_r^{t,\zeta}(e)\lambda(de)) dr | F_t \right]
$$

$$
\leq C_p \mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |Y_s^{t,\zeta}|^p + \sup_{s \in [t, t+\delta]} |X_s^{t,\zeta}|^p + \sup_{s \in [t, t+\delta]} |Y_s^{t,\zeta}|^p
$$

$$
+ \delta_0 \left( \int_t^{t+\delta} (|Z_r^{t,\zeta}|^2 + \int_E |K_r^{t,\zeta}(e)|^2 \lambda(de)) dr \right) \mathbb{P} - a.s.,
$$

$$
\leq C_p \mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |Y_s^{t,\zeta}|^p | F_t \right] + C_p \mathbb{E} \left[ \sup_{s \in [t, t+\delta]} |X_s^{t,\zeta}|^p | F_t \right]
$$

$$
+ C_p \delta_0 \mathbb{E} \left[ \int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds + \int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(e) ds \right]^{\frac{p}{2}} F_t
$$

Let us choose $0 \leq \tilde{\delta}_0 \leq T - t$ such that $1 - C_p \tilde{\delta}_0^{\frac{p}{2}} > 0$. Therefore, for all $0 \leq \delta \leq \tilde{\delta}_0$

$$
\mathbb{E} \left[ \int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds + \int_t^{t+\delta} \int_E |K_s^{t,\zeta}(e)|^2 \lambda(e) ds \right]^{\frac{p}{2}} F_t
$$

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\[ \leq C_p \delta^\frac{p}{2} + C_p (1 + \delta^\frac{p}{2}) \mathbb{E} \left[ \sup_{s \in [t, t + \delta]} |Y_s^{t, \zeta} - \zeta|^p \mid \mathcal{F}_t \right] + C_p \delta^\frac{p}{2} \mathbb{E} \left[ \sup_{s \in [t, t + \delta]} |X_s^{t, \zeta} - \zeta|^p \mid \mathcal{F}_t \right]. \]  

From (3.1) and (3.8) we see that

\[
\mathbb{E} \left[ \sup_{s \in [t, t + \delta]} |X_s^{t, \zeta} - \zeta|^p \mid \mathcal{F}_t \right] \leq C_p \delta^\frac{p}{2} (1 + |\zeta|^p) + C_p \delta^\frac{p}{2} \mathbb{E} \left[ \left( \int_t^{t \pm \delta} |Z_s^{t, \zeta}|^2 \, ds + \int_E \left( |K_s^{t, \zeta}(e)|^2 \lambda(e) \, ds \right)^\frac{p}{2} \right] \mid \mathcal{F}_t \right].
\]  

(3.10)

Combining the above inequality with (3.9) yields

\[
\mathbb{E} \left[ \left( \int_t^{t \pm \delta} |Z_s^{t, \zeta}|^2 \, ds + \int_E \left( |K_s^{t, \zeta}(e)|^2 \lambda(e) \, ds \right)^\frac{p}{2} \right] \mid \mathcal{F}_t \right] \leq C_p + C_p |\zeta|^p + C_p \delta^\frac{p}{2} (1 + |\zeta|^p) + C_p \delta^\frac{p}{2} (1 + \delta^p) \mathbb{E} \left[ \left( \int_t^{t \pm \delta} |Z_s^{t, \zeta}|^2 \, ds + \int_E \left( |K_s^{t, \zeta}(e)|^2 \lambda(e) \, ds \right)^\frac{p}{2} \right] \mid \mathcal{F}_t \right].
\]

We can choose \( 0 \leq \delta_0 \leq \hat{\delta}_0 \) such that \( 1 - C_p \delta^\frac{p}{2} (1 + \delta^p) > 0 \). Consequently, for all \( 0 \leq \delta \leq \delta_0 \),

\[
\mathbb{E} \left[ \left( \int_t^{t \pm \delta} |Z_s^{t, \zeta}|^2 \, ds + \int_E \left( |K_s^{t, \zeta}(e)|^2 \lambda(e) \, ds \right)^\frac{p}{2} \right] \mid \mathcal{F}_t \right] \leq C_p (1 + |\zeta|^p)
\]

Then, by (3.10) we conclude that

\[
\mathbb{E} \left[ \sup_{s \in [t, t + \delta]} |X_s^{t, \zeta} - \zeta|^p \mid \mathcal{F}_t \right] \leq C_p \delta^\frac{p}{2} (1 + |\zeta|^p)
\]

which together with (3.8) yields

\[
\mathbb{E} \left[ \sup_{s \in [t, t + \delta]} |X_s^{t, \zeta} - \zeta|^p \mid \mathcal{F}_t \right] \leq C_p (1 + |\zeta|^p).
\]

The proof is complete. \( \square \)
In the following of this section we investigate properties of the following FBSDEs with jumps, which are very useful in what follows.

For $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, let us consider the following coupled FBSDE with data $(b, \sigma, g, f, \zeta, \Phi)$:

$$
\begin{cases}
  dX^t_s = b(s, X^t_s, Y^t_s, Z^t_s, K^t_s)ds + \sigma(s, X^t_s, Y^t_s)dB_s \\
  -dY^t_s = f(s, X^t_s, Y^t_s, Z^t_s, K^t_s)ds - Z^t_s dB_s - \int_E K^t_s(e)\tilde{N}(de, ds), \\
  X^t_t = \zeta, \quad s \in [t, t+\delta], \\
  Y^t_{t+\delta} = \Phi(X^t_{t+\delta}),
\end{cases}
$$

(3.11)

where

$$
\begin{align*}
  b &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^n, \\
  \sigma &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^{n \times d}, \\
  g &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^n, \\
  f &: [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}, \\
  \Phi &: \mathbb{R}^n \rightarrow \mathbb{R}
\end{align*}
$$

are continuous with respect to $(t, u)$.

We also make the following assumptions:

(H3.4) (i) $b, \sigma, g$ and $f$ are uniformly Lipschitz with respect to $(x, y, z, k)$;

(ii) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^n$;

(iii) (Monotonicity conditions)

\[
\langle A(t, v, k) - A(t, \overline{v}, \overline{k}), v - \overline{v} \rangle + \int_E (G\tilde{g}, \hat{k}(e))\lambda(de) \\
\leq -\beta_1|G\overline{x}|^2 - \beta_2(|G^T\tilde{y}|^2 + |G^T\hat{\zeta}|^2 + \int_E |G^T\hat{\tilde{k}}(e)|^2\lambda(de)),
\]

for all $v = (x, y, z), \overline{v} = (\overline{x}, \overline{y}, \overline{z}), \overline{\overline{x}} = x - \overline{x}, \overline{\tilde{y}} = y - \overline{y}, \overline{\hat{\zeta}} = z - \overline{z}, \overline{\tilde{y}} = g(v, k) - g(\overline{v}, \overline{k})$, where $\beta_1, \beta_2$ and $\mu_1$ are nonnegative constants with $\beta_1 + \beta_2 > 0, \beta_2 + \mu_1 > 0$. Moreover, we have $\beta_2 > 0$ when $n > 1$.

Similar to Ma and Yong [9] one can show the following lemma.

**Lemma 3.6.** Under the assumption (H3.4), there exists a constant $0 \leq \delta' \leq T - t$ such that, for all $0 \leq \delta \leq \delta'$ FBSDE (3.11) has a unique solution on the interval $[t, t + \delta]$.

We also have the following propositions. For their proofs are similar to that in Propositions 3.2 and 3.5, we omit them here.

**Proposition 3.7.** For $i = 1, 2$, we suppose that $(b_i, \sigma_i, g_i, f_i, \zeta_i, \Phi_i)$ satisfies (H3.4). Let $(X^i, Y^i, Z^i, K^i)$ be the solution of FBSDE (3.11) associated to $(b_i, \sigma_i, g_i, f_i, \zeta_i, \Phi_i)$. There exists a constant $0 \leq \delta' \leq T - t$ such that, for all $0 \leq \delta \leq \delta'$,

\[
\mathbb{E}[\sup_{s \in [t, t+\delta]} |X^1_s - X^2_s|^2 + \sup_{s \in [t, t+\delta]} |Y^1_s - Y^2_s|^2 + \int_t^{t+\delta} (|Y^1_s - Y^2_s|^2 + |Z^1_s - Z^2_s|^2)ds] |\mathcal{F}_t]
\]
Proposition 3.8. Let \((b, \sigma, g, f, \zeta, \Phi)\) satisfy (H3.4). Then for all \(p \geq 2\), there exists a positive constant \(0 \leq \delta_0 \leq T - t\) such that for all \(0 \leq \delta \leq \delta_0\),

\[
\mathbb{E}\left[ \sup_{s \in [t, t+\delta]} |X^{t,x}_s|^p + \sup_{s \in [t, t+\delta]} |Y^{t,x}_s|^p + \mathbb{E}\left[ \int_{t}^{t+\delta} |Z^{t,x}_s|^2 ds \right] \right] + \mathbb{E}\left[ \int_{t}^{t+\delta} \int_{E} |K^{t,x}_s(e)|^2 \lambda(e) ds \right] \leq C|\lambda|^{\frac{p}{2}}(1 + |x|^p), \quad \mathbb{P} - a.s.,
\]

\[
\mathbb{E}\left[ \sup_{s \in [t, t+\delta]} |X^{t,x}_s - x|^p |\mathcal{F}_t\right] \leq C\delta^p(1 + |x|^p), \quad \mathbb{P} - a.s.
\]

In order to get the comparison theorem for solutions of FBSDEs (3.11) we make the following assumption.

(H3.5) There exists a constant \(K > -1\) such that, for all \((s, y, z, k_1, k_2) \in [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^2\),

\[f(s, x, y, z, k_1) - f(s, x, y, z, k_2) \geq K(k_1 - k_2)\]

The following comparison theorem for solutions of FBSDEs (3.11) follows from a similar argument as in Wu [15].

Lemma 3.9. For \(i = 1, 2\), we suppose that \((b, \sigma, g, f_i, x, \Phi_i)\) satisfy (H3.4). Let \((X^i, Y^i, Z^i, K^i)\) be the solution of FBSDE (3.11) associated to \((b, \sigma, g, f_i, x, \Phi_i)\). There exists a constant \(0 \leq \delta' \leq T - t\) such that, for all \(0 \leq \delta \leq \delta'\), \(f_i\) satisfies (H3.5) and the following holds:

(i) \(f^1(s, X^1_s, Y^1_s, Z^1_s, K^1_s) \geq f^2(s, X^2_s, Y^1_s, Z^1_s, K^1_s)\), for all \(s \in [t, t + \delta]\),

(ii) \(\Phi^1(X^1_{t+\delta}) \geq \Phi^2(X^1_{t+\delta})\), or \(\Phi^1(X^1_{t+\delta}) \geq \Phi^2(X^2_{t+\delta})\).

Then, we have \(Y^1_t \geq Y^2_t\), a.s.

4 Optimal control of coupled forward-backward stochastic system with jumps

Our objectives of this section is to investigate optimal control problem of coupled forward-backward stochastic system with jumps. We obtain that the value function satisfies the dynamic programming principle.
Let us suppose that the control state space $U$ is a compact metric space. The set of admissible controls $\mathcal{U}$ is the set of all $U$-valued $\{\mathcal{F}_t\}$-predictable processes.

For a given admissible control $u(\cdot) \in \mathcal{U}$, the initial time $t \in [0, T]$ and the initial state $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, let us consider the following coupled forward-backward stochastic control system:

\[
\begin{align*}
\text{(i)} & \quad \frac{dX_s^{t,\zeta, u}}{dt} = b(s, X_s^{t,\zeta, u}, Y_s^{t,\zeta, u}, Z_s^{t,\zeta, u}, K_s^{t,\zeta, u}, u_s)ds + \sigma(s, X_s^{t,\zeta, u}, Y_s^{t,\zeta, u}, u_s)dB_{s, u}, \\
\text{(ii)} & \quad -dY_s^{t,\zeta, u} = f(s, X_s^{t,\zeta, u}, Y_s^{t,\zeta, u}, Z_s^{t,\zeta, u}, K_s^{t,\zeta, u}, u_s)ds - Z_s^{t,\zeta, u}dB_s - \int_E K_s^{t,\zeta, u}(e)\tilde{N}(de, ds), \\
\text{(iii)} & \quad X_T^{t,\zeta, u} = \zeta, \quad s \in [t, T], \\
\text{(iv)} & \quad Y_T^{t,\zeta, u} = \Phi(X_T^{t,\zeta, u}),
\end{align*}
\]

(4.1)

where

\[
\begin{align*}
& b : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U \longrightarrow \mathbb{R}^n, \\
& \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \longrightarrow \mathbb{R}^{n \times d}, \\
& g : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times E \times U \longrightarrow \mathbb{R}^n, \\
& f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U \longrightarrow \mathbb{R}, \\
& \Phi : \mathbb{R}^n \longrightarrow \mathbb{R}
\end{align*}
\]

are continuous with respect to $(t, u)$.

We also make the following assumptions:

(H4.1) (i) $b, \sigma, g$ and $f$ are uniformly Lipschitz with respect to $(x, y, z, k)$.

(ii) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^n$.

(iii) There exists a constant $K > -1$ such that, for all $(s, y, z, k_1, k_2) \in [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U$,

\[ f(s, x, y, z, k_1, u) - f(s, x, y, z, k_2, u) \geq K(k_1 - k_2). \]

(iv) (Monotonicity conditions)

\[
\langle A(t, v, k) - A(t, \overline{v}, \overline{k}), v - \overline{v} \rangle + \int_E \langle G\tilde{g}, \tilde{k}(e) \rangle \lambda(de) \leq -\beta_1|G\overline{x}|^2 - \beta_2(|G^T\overline{y}|^2 + |G^T\overline{z}|^2 + \int_E |G^T\tilde{k}(e)|^2 \lambda(de)),
\]

\[ \langle \Phi(x) - \Phi(\overline{x}), G(x - \overline{x}) \rangle \geq \mu_1|G\overline{x}|^2, \]

for all $v = (x, y, z), \overline{v} = (\overline{x}, \overline{y}, \overline{z}), \overline{x} = x - \overline{x}, \overline{y} = y - \overline{y}, \overline{z} = z - \overline{z}, \tilde{g} = g(v(k) - g(\overline{v}(k)), \beta_1, \beta_2$ and $\mu_1$ are nonnegative constants with $\beta_1 + \beta_2 > 0, \beta_2 + \mu_1 > 0$. Moreover, we have $\beta_2 > 0$ when $n > 1$.

From Corollary 3.3 we know that

\[
(i) \quad |Y_t^{t,\zeta, u}| \leq C(1 + |\zeta'|); \quad (ii) \quad |Y_t^{t,\zeta, u} - Y_t^{t,\zeta', u}| \leq C|\zeta - \zeta'|. \tag{4.2}
\]

Let us now introduce a subspace of admissible controls.
Step 1: For any $u(\cdot) \in \mathcal{U}_{t,T}$, let us introduce the following associated cost functional:

$$ J(t, x; u) := Y_{s}^{t,x,u} \big|_{s=t}, \quad (t, x) \in [0, T] \times \mathbb{R}^{n}, $$

where the process $Y_{s}^{t,x,u}$ is defined by FBSDE (4.1) with $\zeta = x$.

Let us now define the value function of the stochastic control problem:

$$ W(t, x) := \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u). $$

Under the assumption (H4.1), the value function $W(t, x)$ is well defined and a bounded $\mathcal{F}_{t}$-measurable random variable. But we have the following proposition which shows that $W$ is a deterministic function.

**Proposition 4.2.** Under the assumption (H4.1), for any $(t, x) \in [0, T] \times \mathbb{R}^{n}$, we have $W(t, x) = \mathbb{E}[W(t, x)]$, $\mathbb{P}$-a.s.

The following lemma will show that $W$ is invariant by a large class of transformations on $\Omega$. Together with the lemma, we can prove the above proposition by a similar argument as in [3].

**Lemma 4.3.** Under the assumption (H4.1), for $(t, x) \in [0, T] \times \mathbb{R}^{n}$, $\tau : \Omega \to \Omega$ is an invertible $\mathcal{F} - \mathcal{F}$ measurable transformation such that

i) $\tau$ and $\tau^{-1}$ : $\Omega \to \Omega$ are $\mathcal{F}_{t} - \mathcal{F}_{t}$ measurable;

ii) $(B_{s} - B_{t}) \circ \tau = B_{s} - B_{t}$, $s \in [t, T]$,

$$ N((t, s] \times A) \circ \tau = N((t, s] \times A), \quad s \in [t, T], A \in \mathcal{B}(E); $$

iii) the law $\mathbb{P} \circ [\tau]^{-1}$ of $\tau$ is equivalent to the underlying probability measure $\mathbb{P}$,

then $W(t, x) \circ \tau = W(t, x)$, $\mathbb{P}$-a.s.

**Proof.** The proof is divided into three steps:

**Step 1:** For any $u \in \mathcal{U}_{t,T}$, $J(t, x; u) \circ \tau = J(t, x; u(\tau))$, $\mathbb{P}$-a.s.

Using the transformation $\tau$ to FBSDE (4.1) (with $\zeta = x$), we compare the obtained FBSDE with the FBSDE (4.1) in which $u$ is replaced by $u(\tau)$. By virtue of the uniqueness of the solution of FBSDE (4.1) we deduce

$$ X_{s}^{t,x,u(\tau)} = X_{s}^{t,x,u(\tau)}, \quad \text{for all } s \in [t, T], \mathbb{P}\text{-a.s.}, $$

$$ Y_{s}^{t,x,u(\tau)} = Y_{s}^{t,x,u(\tau)}, \quad \text{for all } s \in [t, T], \mathbb{P}\text{-a.s.}, $$

$$ Z_{s}^{t,x,u(\tau)} = Z_{s}^{t,x,u(\tau)}, \quad dsd\mathbb{P}\text{-a.e. on } [0, T] \times \Omega, $$

$$ K_{s}^{t,x,u(\tau)} = K_{s}^{t,x,u(\tau)}, \quad ds\lambda(de)\mathbb{P}\text{-a.e. on } [0, T] \times \Omega. $$

It follows that, in particular,

$$ J(t, x; u)(\tau) = J(t, x; u(\tau)), \mathbb{P}\text{-a.s.} $$
Step 2: We now show that
\[
\left\{ \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) \right\}(\tau) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u)(\tau) \right\}, \mathbb{P}\text{-a.s.}
\]

Let us denote by \( I(t, x) = \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) \). Then, for all \( u \in \mathcal{U}_{t,T} \), \( I(t, x) \geq J(t, x; u) \). Thus, \( I(t, x)(\tau) \geq J(t, x; u)(\tau) \). Consequently,
\[
\left\{ \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) \right\}(\tau) \geq \text{esssup}_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u)(\tau) \right\}, \mathbb{P}\text{-a.s.} \tag{4.3}
\]

For all \( u \in \mathcal{U}_{t,T} \) and random variable \( \xi \) with \( \xi \geq J(t, x; u)(\tau) \), we see that \( \xi(\tau^{-1}) \geq J(t, x; u), \mathbb{P}\text{-a.s.} \). Then, \( \xi(\tau^{-1}) \geq I(t, x), \mathbb{P}\text{-a.s.} \), and thus, \( \xi \geq I(t, x)(\tau), \mathbb{P}\text{-a.s.} \). Consequently,
\[
\text{esssup}_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u)(\tau) \right\} \geq \left\{ \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) \right\}(\tau), \mathbb{P}\text{-a.s.},
\]
from which and (4.3) we conclude that
\[
\left\{ \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) \right\}(\tau) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u)(\tau) \right\}, \mathbb{P}\text{-a.s.}
\]

Step 3: Let us now show that \( W(t, x) \) is invariant with respect to \( \tau \), i.e.,
\[
W(t, x)(\tau) = W(t, x), \mathbb{P}\text{-a.s.}
\]

The above two steps yield
\[
W(t, x)(\tau) = \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u)(\tau) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \left\{ J(t, x; u)(\tau) \right\} = \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u(\tau)) = \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u(\tau)) = W(t, x), \mathbb{P}\text{-a.s.},
\]
where we have used \( \{ u(\tau) \mid u(\cdot) \in \mathcal{U}_{t,T} \} = \mathcal{U}_{t,T} \). The proof is complete. \( \square \)

The following property of the value function \( W \) follows from Corollary 3.3.

Lemma 4.4. There exists a constant \( C > 0 \) such that, for all \( 0 \leq t \leq T, x, x' \in \mathbb{R}^n \),
\[
(i) \quad |W(t, x) - W(t, x')| \leq C|x - x'|, \quad (ii) \quad |W(t, x)| \leq C(1 + |x|).
\]

We now investigate the dynamic programming principle for the optimal control of coupled forward-backward stochastic system (4.1). Let us first define the backward stochastic semigroups for the coupled forward-backward stochastic system. The notion of stochastic backward semigroup was first introduced by Peng [11] to study the optimal control of decoupled forward-backward stochastic system. The interested reader can also refer to Wu and Yu [16], Li and Peng [5] and Li and Wei [6].
For \((t, x) \in [0, T] \times \mathbb{R}^n, 0 \leq \delta \leq T - t, u(\cdot) \in \mathcal{U}_{t, t+\delta}\) and a real-valued \(\mathcal{F}_{t+\delta} \otimes \mathcal{B}(\mathbb{R}^n)\) measure function \(\psi : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}\), let us define

\[
C_{s, t+\delta}^{t, x; u}[\psi(t + \delta, \bar{X}_{t+\delta})] := \bar{Y}_{s \wedge (t + \delta)}^{t, x; u}, \quad s \in [t, t + \delta],
\]

where \((\bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, \bar{Z}_{s}^{t, x; u}, \bar{K}_{s}^{t, x; u})\) is the solution of the following FBSDE with data \((b, \sigma, g, f, x, \psi)\):

\[
\begin{align*}
d\bar{X}_{s}^{t, x; u} & = b(s, \bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, \bar{Z}_{s}^{t, x; u}, \bar{K}_{s}^{t, x; u}, u_s)ds + \sigma(s, \bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, u_s)dB_s \\
- d\bar{Y}_{s}^{t, x; u} & = f(s, \bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, \bar{Z}_{s}^{t, x; u}, \bar{K}_{s}^{t, x; u}, u_s)ds - \bar{Z}_{s}^{t, x; u}dB_s \\
\bar{X}_{t}^{t, x; u} & = x, \\
\bar{Y}_{t+\delta}^{t, x; u} & = \psi(t + \delta, \bar{X}_{t+\delta}).
\end{align*}
\]

By Lemma 3.6 we know that when \(\delta\) is sufficiently small, the above solution has a unique solution. Moreover, we have the following dynamic programming principle.

**Theorem 4.5.** Let the assumption (H4.1) hold. Then the following dynamic programming principle holds: there exists a sufficiently small \(\delta_0\) such that for any \(0 \leq \delta < \delta_0, 0 \leq T - \delta, x \in \mathbb{R}^n\),

\[
W(t, x) = \text{esssup}_{u \in \mathcal{U}_{t, t+\delta}} C_{s, t+\delta}^{t, x; u}[W(t + \delta, \bar{X}_{t+\delta})],
\]

where \((\bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, \bar{Z}_{s}^{t, x; u}, \bar{K}_{s}^{t, x; u})\) is the solution of the following FBSDE:

\[
\begin{align*}
d\bar{X}_{s}^{t, x; u} & = b(s, \bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, \bar{Z}_{s}^{t, x; u}, \bar{K}_{s}^{t, x; u}, u_s)ds + \sigma(s, \bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, u_s)dB_s \\
- d\bar{Y}_{s}^{t, x; u} & = f(s, \bar{X}_{s}^{t, x; u}, \bar{Y}_{s}^{t, x; u}, \bar{Z}_{s}^{t, x; u}, \bar{K}_{s}^{t, x; u}, u_s)ds - \bar{Z}_{s}^{t, x; u}dB_s \\
\bar{X}_{t}^{t, x; u} & = x, \\
\bar{Y}_{t+\delta}^{t, x; u} & = W(t + \delta, \bar{X}_{t+\delta}).
\end{align*}
\]

**Proof.** In order to simplify notations let us denote by

\[
W_\delta(t, x) = \text{esssup}_{u \in \mathcal{U}_{t, t+\delta}} G_{s, t+\delta}^{t, x; u}[W(t + \delta, \bar{X}_{t+\delta})].
\]

Then it follows that \(W_\delta(t, x)\) is deterministic by a similar argument as in the proof of Proposition 4.2. The proof follows by the following two lemmas.

**Lemma 4.6.** \(W_\delta(t, x) \leq W(t, x)\).

**Proof.** Let us denote by

\[
I_\delta(t, x; u) := C_{s, t+\delta}^{t, x; u}[W(t + \delta, \bar{X}_{t+\delta})].
\]
Then there exists a sequence \( \{u_1^i, i \geq 1\} \subset U_{t, t+\delta} \), such that
\[
W_\delta(t, x) = \text{esssup}_{u_1 \in U_{t, t+\delta}} \ I_\delta(t, x; u_1) = \sup_{i \geq 1} I_\delta(t, x; u_1^i), \ \mathbb{P} \text{-a.s.}
\]

For any \( \varepsilon > 0 \), let us set
\[
\Gamma_1 := \left\{ W_\delta(t, x) \leq I_\delta(t, x; u_1^1) + \varepsilon \right\} \in \mathcal{F}_t,
\]
and
\[
\Gamma_i := \left\{ W_\delta(t, x) \leq I_\delta(t, x; u_1^1) + \varepsilon, W_\delta(t, x) > I_\delta(t, x; u_1^1) + \varepsilon, j \leq i - 1 \right\} \in \mathcal{F}_t, \ i \geq 2.
\]

Then \( \{\Gamma_i\}_{i \geq 1} \) is an \((\Omega, \mathcal{F}_t)\)-partition, and
\[
u_1^i := \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} u_1^i \in U_{t, t+\delta}.
\]

From the uniqueness of the solution of the coupled FBSDE it follows that
\[
I_\delta(t, x; u_1^1) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t, x; u_1^i), \ \mathbb{P} \text{-a.s.}
\]
Therefore,
\[
W_\delta(t, x) \leq \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t, x; u_1^i) + \varepsilon = I_\delta(t, x; u_1^1) + \varepsilon
\]
\[
= \mathcal{G}_{t, t+\delta}^{t, x; u_1^1} [W(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1})] + \varepsilon.
\]

From (4.2) and Lemma 4.4 we know that there exists a constant \( C \in \mathbb{R} \) such that
\[
(i) \quad |J(t + \delta, y; u_2) - J(t + \delta, y'; u_2)| \leq C|y - y'|, \ \mathbb{P} \text{-a.s., for all } u_2 \in U_{t+\delta, T};
\]
\[
(ii) \quad |W(t + \delta, y) - W(t + \delta, y')| \leq C|y - y'|, \ \text{for all } y, y' \in \mathbb{R}^n,
\]
from which and by approximating \( \bar{X}_{t+\delta}^{t, x; u_1^1} \) we conclude that
\[
W(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1}) \leq \text{esssup}_{u_2 \in U_{t+\delta, T}} J(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1}; u_2) + \varepsilon, \ \mathbb{P} \text{-a.s.}
\]

There exists a sequence \( \{u_2^j, j \geq 1\} \subset U_{t+\delta, T} \) such that
\[
\text{esssup}_{u_2 \in U_{t+\delta, T}} J(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1}; u_2) = \sup_{j \geq 1} J(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1}; u_2^j), \ \mathbb{P} \text{-a.s.}
\]
Let us set
\[
\tilde{\Lambda}_j := \left\{ \text{esssup}_{u_2 \in U_{t+\delta, T}} J(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1}; u_2) \leq J(t + \delta, \bar{X}_{t+\delta}^{t, x; u_1^1}; u_2^j) + \varepsilon \right\} \in \mathcal{F}_{t+\delta}, \ j \geq 1,
\]
and put
\[
\Lambda_1 := \tilde{\Lambda}_1, \Lambda_j := \tilde{\Lambda}_j \setminus \bigcup_{l=1}^{j-1} \tilde{\Lambda}_l \in \mathcal{F}_{t+\delta}, \ j \geq 2.
\]
Then, \( \{ \Lambda_j \}_{j \geq 1} \) is an \( (\Omega, \mathcal{F}_{t+\delta}) \)-partition and \( u_2^\varepsilon := \Sigma_{j \geq 1} 1_{\Lambda_j} u_j^2 \in \mathcal{U}_{t+\delta,T} \). Using the uniqueness of the solution of the coupled FBSDE we deduce

\[
J(t + \delta, \bar{X}_{t+\delta}^{t,x;u_1^\varepsilon}; u_2^\varepsilon) = Y_{t+\delta}^{t+\delta,\bar{X}_{t+\delta}^{t,x;u_1^\varepsilon}; u_2^j} = \sum_{j \geq 1} 1_{\Lambda_j} J(t + \delta, \bar{X}_{t+\delta}^{t,x;u_1^\varepsilon}; u_2^j)
\]

Consequently,

\[
W(t + \delta, \bar{X}_{t+\delta}^{t,x;u_1^\varepsilon}) = \text{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t + \delta, \bar{X}_{t+\delta}^{t,x;u_1^\varepsilon}; u_2) + \varepsilon
\]

from which, (4.6) and Lemma 3.9 it follows that

\[
W_\delta(t,x) \leq g_{t+\delta}[J(t + \delta, \bar{X}_{t+\delta}^{t,x;u_1^\varepsilon}; u_2) + \varepsilon] + \varepsilon
\]

where \( \delta = u_1^\varepsilon \oplus u_2^\varepsilon \in \mathcal{U}_{t,T} \), and \( (\bar{X}_{t,x}^{t,x;u}, \bar{Y}_{t,x}^{t,x;u}, \bar{Z}_{t,x}^{t,x;u}, \bar{R}_{t,x}^{t,x;u}) \) is the solution of the following equation:

\[
\begin{align*}
\frac{d\bar{X}_{t,x}^{t,x;u}}{dt} & = b(s, \bar{X}_{s}^{t,x;u}, \bar{Y}_{s}^{t,x;u}, \bar{Z}_{s}^{t,x;u}, \bar{R}_{s}^{t,x;u}, u_s) ds + \sigma(s, \bar{X}_{s}^{t,x;u}, \bar{Y}_{s}^{t,x;u}, u_s) dB_s + \int_{E} g(s, \bar{X}_{s}^{t,x;u}, \bar{Y}_{s}^{t,x;u}, \bar{Z}_{s}^{t,x;u}, \bar{R}_{s}^{t,x;u}, e, u_s) \bar{N}(de, ds), \\
-d\bar{Y}_{t,x}^{t,x;u} & = f(s, \bar{X}_{s}^{t,x;u}, \bar{Y}_{s}^{t,x;u}, \bar{Z}_{s}^{t,x;u}, \bar{K}_{s}^{t,x;u}, u_s) ds - \bar{Z}_{t,x}^{t,x;u} dB_s - \int_{E} \bar{K}_{s}^{t,x;u}(e) \bar{N}(de, ds), \\
\bar{X}_{t,x}^{t,x;u} & = x, \quad s \in [t, t+\delta], \\
\bar{Y}_{t+\delta,x}^{t,x;u} & = J(t + \delta, \bar{X}_{t+\delta}^{t,x;u}; u_2) + \varepsilon.
\end{align*}
\]

Letting \( \varepsilon \downarrow 0 \) we get \( W_\delta(t,x) \leq W(t,x) \).

\[\square\]

**Lemma 4.7.** \( W(t,x) \leq W_\delta(t,x) \).

**Proof.** From (4.5) it follows that

\[
W_\delta(t,x) \geq g_{t+\delta}[W(t + \delta, \bar{X}_{t+\delta}^{t,x;u_1})], \quad \mathbb{P}\text{-a.s., for all } u_1 \in \mathcal{U}_{t,T}.
\]

The definition of \( W(t + \delta, y) \) yileds, for all \( y \in \mathbb{R}^n \) and \( u_2 \in \mathcal{U}_{t+\delta,T} \),

\[
W(t + \delta, y) \geq J(t + \delta, y; u_2), \quad \mathbb{P}\text{-a.s.}
\]
Let $u \in U_{t,T}$ be arbitrarily chosen and decomposed into $u_1 = u|_{(t,t+\delta)} \in U_{t,t+\delta}$, and $u_2 = u|_{(t+\delta,T]} \in U_{t+\delta,T}$. From the above inequalities and Lemma 3.9 we conclude that, for all $t \in U_{t,T}$,

$$W_\delta(t,x) \geq G_{t,t+\delta}^{t,x;u_1}[J(t+\delta, X_{t+\delta}^{t,x;u_1}; u_2)]$$

$$= G_{t,t+\delta}^{t,x;u_2}[Y_{t+\delta}^{t,x;u_2}]$$

$$= Y_{t}^{t,x;u}, \ P\text{-a.s.}$$

Therefore,

$$W_\delta(t,x) \geq \operatorname{esssup}_{u \in U_{t,T}} J(t,x;u) = W(t,x), \ \mathbb{P}\text{-a.s.}$$

The proof is complete.

By virtue of Theorem 4.5 we now can show that $W(t,x)$ has the continuity with respect to $t$.

**Proposition 4.8.** Under the assumption (H4.1), the value function $W(t,x)$ is continuous in $t$.

**Proof.** Let $\delta > 0$ be sufficiently small such that $0 < \delta \leq T - t$. We shall show that the following holds: for $(t,x) \in [0,T] \times \mathbb{R}^n$

$$-C(1 + |x|)\frac{1}{\delta} \leq W(t,x) - W(t+\delta,x) \leq C(1 + |x|)\frac{1}{\delta}$$

We only give the proof of the second inequality. A similar argument will show that the first one holds.

From the proof of Theorem 4.5 we know that there exists $u^\varepsilon \in U_{t,T}$ such that

$$G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, \tilde{X}_{t+\delta}^{t,x;u^\varepsilon})] + \varepsilon \geq W(t,x) \geq G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, \tilde{X}_{t+\delta}^{t,x;u^\varepsilon})].$$

Therefore,

$$W(t,x) - W(t+\delta,x) \leq G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, \tilde{X}_{t+\delta}^{t,x;u^\varepsilon})] - W(t+\delta,x) + \varepsilon = I_1^\varepsilon + I_2^\varepsilon + \varepsilon,$$

where

$$I_1^\varepsilon = G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, \tilde{X}_{t+\delta}^{t,x;u^\varepsilon})] - G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta,x)],$$

$$I_2^\varepsilon = G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta,x)] - W(t+\delta,x).$$

Let us recall

$$G_{s,t+\delta}^{t,x;u}[W(t+\delta, \tilde{X}_{t+\delta}^{t,x;u})] := \tilde{Y}_{s}^{t,x;u}, \ s \in [t,t+\delta],$$

where $(\tilde{X}_{s}^{t,x;u}, \tilde{Y}_{s}^{t,x;u}, \tilde{Z}_{s}^{t,x;u}, \tilde{K}_{s}^{t,x;u}, t \leq s \leq t+\delta)$ is the solution of the following FBSDE:

$$\begin{cases}
  d\tilde{X}_{s}^{t,x;u} = b(s, \tilde{X}_{s}^{t,x;u}, \tilde{Y}_{s}^{t,x;u}, \tilde{Z}_{s}^{t,x;u}, \tilde{K}_{s}^{t,x;u}, u_s)ds + \sigma(s, \tilde{X}_{s}^{t,x;u}, \tilde{Y}_{s}^{t,x;u}, e, u_s)d\mathcal{N}(de, ds), \\
  -d\tilde{Y}_{s}^{t,x;u} = f(s, \tilde{X}_{s}^{t,x;u}, \tilde{Y}_{s}^{t,x;u}, \tilde{Z}_{s}^{t,x;u}, \tilde{K}_{s}^{t,x;u}, u_s)ds - \tilde{Z}_{s}^{t,x;u}d\mathcal{N}(de, ds), \\
  \tilde{X}_{t}^{t,x;u} = x, \\
  \tilde{Y}_{t+\delta}^{t,x;u} = W(t+\delta, \tilde{X}_{t+\delta}^{t,x;u}).
\end{cases}$$
and
\[ C^d_{t,x;u} W(t + \delta, x) := \bar{Y}_{t}^{d,x;u}, \ s \in [t, t + \delta], \]
where \((\bar{X}_{t}^{d,x;u}, \bar{Y}_{t}^{d,x;u}, \bar{Z}_{t}^{d,x;u}, \bar{K}_{t}^{d,x;u}, u_s)_{t \leq s \leq t + \delta}\) is the solution of the following FBSDE:
\[
\left\{
\begin{aligned}
\frac{d\bar{X}_{t}^{d,x;u}}{dt} &= b(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, \bar{Z}_{s}^{d,x;u}, \bar{K}_{s}^{d,x;u}, u_s) ds + \sigma(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, u_s) dB_s \\
-\frac{d\bar{Y}_{t}^{d,x;u}}{dt} &= f(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, \bar{Z}_{s}^{d,x;u}, \bar{K}_{s}^{d,x;u}, u_s) ds - \bar{Z}_{s}^{d,x;u} dB_s \\
\bar{X}_{t}^{d,x;u} &= x, \\
\bar{Y}_{t}^{d,x;u} &= W(t + \delta, x).
\end{aligned}
\]

From the above equations and Proposition 3.7 it follows that there exists some positive constant \( C \) independent of the control \( u \) such that
\[
|I_\delta^1|^2 \leq C E[|W(t + \delta, \bar{X}_{t+\delta}^{d,x;u}) - W(t + \delta, x)|^2 | \mathcal{F}_t] + C E[\int_t^{t+\delta} |\bar{X}_{s}^{d,x;u} - \bar{X}_{s}^{d,x;u}|^2 ds | \mathcal{F}_t].
\]

By virtue of Lemma 4.4 and Proposition 3.5 we get
\[
|I_\delta^1|^2 \leq C E[|\bar{X}_{t+\delta}^{d,x;u} - x|^2 | \mathcal{F}_t] + C \delta E[\sup_{s \in [t, t+\delta]} |\bar{X}_{s}^{d,x;u} - x|^2 | \mathcal{F}_t] \\
+ C \delta E[\sup_{s \in [t, t+\delta]} |\bar{X}_{s}^{d,x;u} - x|^2 | \mathcal{F}_t]
\leq C (1 + |x|) \delta.
\]

From the definition of \( G_{t,x,t+\delta}^d[.] \) it follows that
\[
I_\delta^2 = E[W(t + \delta, x) + \int_t^{t+\delta} f(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, \bar{Z}_{s}^{d,x;u}, \bar{K}_{s}^{d,x;u}, u_s) ds | \mathcal{F}_t] - W(t + \delta, x) \\
= E[\int_t^{t+\delta} f(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, \bar{Z}_{s}^{d,x;u}, \bar{K}_{s}^{d,x;u}, u_s) ds | \mathcal{F}_t].
\]
Consequently,
\[
|I_\delta^2| \leq E[\int_t^{t+\delta} |f(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, \bar{Z}_{s}^{d,x;u}; u_s)| ds | \mathcal{F}_t] \\
\leq \delta^{1/2} \left( E[\int_t^{t+\delta} |f(s, \bar{X}_{s}^{d,x;u}, \bar{Y}_{s}^{d,x;u}, \bar{Z}_{s}^{d,x;u}; u_s)|^2 ds | \mathcal{F}_t] \right)^{1/2} \\
\leq C \delta^{1/2} E[\int_t^{t+\delta} (1 + |\bar{X}_{s}^{d,x;u}|^2 + |\bar{Y}_{s}^{d,x;u}|^2 + |\bar{Z}_{s}^{d,x;u}|^2 + \int_{E} |\bar{K}_{s}^{d,x;u}(e)|^2 \lambda(de) ds | \mathcal{F}_t]^{1/2} \\
\leq C \delta^{1/2} (1 + |x|).
\]
From the above inequalities we have
\[
W(t, x) - W(t + \delta, x) \leq C (1 + |x|) \delta^{1/2} + \varepsilon.
\]
Letting \( \varepsilon \downarrow 0 \), we get the desired result. The proof is complete. \( \square \)
5 Hamilton-Jacobi-Bellman equations with integral-differential operators

The section is devoted to giving a probabilistic interpretation of a class of Hamilton-Jacobi-Bellman equations with integral-differential operators, i.e., the value function $W$ is a viscosity solution of the following Hamilton-Jacobi-Bellman equations with integral-differential operators:

\[
\begin{aligned}
\frac{\partial}{\partial t} W(t,x) + H(t,x,W,DW,D^2W) &= 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \\
W(T,x) &= \Phi(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\]

where for $(t,x,u) \in [0,T] \times \mathbb{R}^n \times U$,

\[ H(t,x,W,DW,D^2W) = \sup_{u \in U} H_0(t,x,W,DW,D^2W,u) \]

and

\[ H_0(t,x,W,DW,D^2W,u) = \frac{1}{2} \text{tr}(\sigma \sigma^T(t,x,W(t,x),u)) D^2W(t,x) + DW(t,x) \cdot b(t,x,W(t,x),DW(t,x) \cdot \sigma(t,x,W(t,x),u), \]

\[ + \int_E \left( (W(t,x + g(t,x,W(t,x),u,e) - W(t,x)) - DW(t,x) \cdot g(t,x,W(t,x),u,e)) \lambda(de) \right) + f(t,x,W(t,x),DW(t,x) \cdot \sigma(t,x,W(t,x),u), \]

\[ + \int_E \left( (W(t,x + g(t,x,W(t,x),u,e) - W(t,x)) - W(t,x)) \lambda(de) \right). \]

We now introduce the notion of a viscosity solution of (5.1), which is similar to that in Barle, Buckdah and Pardoux [2]. Let us denote by $C^3_{l,b}([0,T] \times \mathbb{R}^n)$ the set of real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.

**Definition 5.1.** A real-valued continuous function $\Psi \in C([0,T] \times \mathbb{R}^n)$ is called

(i) a viscosity subsolution of equation (5.1) if $\Psi(T,x) \leq \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C^3_{l,b}([0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T] \times \mathbb{R}^n$ such that $\Psi - \varphi$ attains a global maximum at $(t,x)$,

\[
\frac{\partial \varphi}{\partial t}(t,x) + \sup_{u \in U} \left\{ D\varphi(t,x) \cdot b(t,x,\varphi(t,x),D\varphi(t,x) \cdot \sigma(t,x,\varphi(t,x),u), A^u \varphi(t,x),u) \right. \]

\[ + \frac{1}{2} \text{tr}(\sigma \sigma^T(t,x,\varphi(t,x),u)D^2\varphi(t,x)) + B^u \varphi(t,x) \]

\[ + f(t,x,\varphi(t,x),D\varphi(t,x) \cdot \sigma(t,x,\varphi(t,x),u), A^u \varphi(t,x),u) \right\} \geq 0, \]

where

\[ A^u \varphi(t,x) = \int_E \left( \varphi(t,x + g(t,x,\varphi(t,x),u,e) - \varphi(t,x)) \right) \lambda(de), \]
\[ B^u \varphi(t, x) = \int_E \left( \varphi(t, x + g(t, x, \varphi(t, x), u, e) - \varphi(t, x) - D\varphi(t, x) \cdot g(t, x, \varphi(t, x), u, e) \right) \lambda(de) \]

(ii) a viscosity supersolution of equation (5.1) if \( \Psi(T, x) \geq \Phi(x) \), for all \( x \in \mathbb{R}^n \), and if for all functions \( \varphi \in C^2_{loc}([0, T] \times \mathbb{R}^n) \) and \( (t, x) \in [0, T] \times \mathbb{R}^n \) such that \( \Psi - \varphi \) attains a global minimum at \( (t, x) \),

\[
\frac{\partial \varphi}{\partial t}(t, x) + \sup_{u \in U} \left\{ D\varphi(t, x) \cdot b(t, x, \varphi(t, x), D\varphi(t, x) \cdot \sigma(t, x, \varphi(t, x), u), A^u \varphi(t, x), u) \right. \\
+ \frac{1}{2} \text{tr} (\sigma \sigma^T(t, x, y, u) D^2 \varphi(t, x)) + B^u \varphi(t, x) \\
+ f(t, x, \varphi(t, x), D\varphi(t, x) \cdot \sigma(t, x, \varphi(t, x), u), A^u \varphi(t, x), u) \left. \right\} \geq 0,
\]

(iii) a viscosity solution of equation (5.1) if it is both a viscosity sub- and a supersolution of equation (5.1).

**Theorem 5.2.** Let the assumption (H4.1) hold. Then the value function \( W \) is a viscosity solution of (5.1).

**Proof.** We only show that \( W \) is a viscosity subsolution of (5.1). We can use a similar argument to show that \( W \) is a viscosity supersolution of (5.1). Note that \( W(T, x) = \Phi(x) \), for \( x \in \mathbb{R}^n \). For \( (t, x) \in [0, T] \times \mathbb{R}^n \) and \( \varphi \in C^2_{loc}([0, T] \times \mathbb{R}^n) \), without loss of generality let us suppose that \( W \leq \varphi \) and \( W(t, x) = \varphi(t, x) \), and we define, for \( (s, x, y, z, k, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times U \).

\[
F(s, x, y, z, k, u) = \frac{\partial \varphi}{\partial s}(s, x) + \frac{1}{2} \text{tr} (\sigma \sigma^T(s, x, y + \varphi(s, x), u) D^2 \varphi(s, x)) \\
+ D\varphi(s, x) \cdot b(s, x, y + \varphi(s, x), z + D\varphi(s, x) \cdot \sigma(s, x, y + \varphi(s, x), u), k + \int_E (\varphi(s, x + g(s, x, y + \varphi(s, x), u, e) - \varphi(s, x)) \lambda(de), u) \\
+ \int_E (\varphi(s, x + g(s, x, y + \varphi(s, x), u, e) - \varphi(s, x) - D\varphi(s, x) \cdot g(s, x, y + \varphi(s, x), u, e)) \lambda(de) \\
+ f(s, x, y + \varphi(s, x), z + D\varphi(s, x) \cdot \sigma(s, x, y + \varphi(s, x), u), k + \int_E (\varphi(s, x + g(s, x, y + \varphi(s, x), u, e)) - \varphi(s, x)) \lambda(de), u).
\]

From Lemma 3.6 it follows that there exists a sufficiently small \( 0 < \delta' \leq T - t \) such that for all \( 0 < \delta \leq \delta' \) the following coupled FBSDE has a unique solution.

\[
\begin{align*}
\frac{dX^u_s}{ds} &= b(s, X^u_{s+}, Y^u_s, Z^u_s, K^u_s, u_s)ds + \sigma(s, X^u_s, Y^u_s, e, u_s)dB_s + \int_E g(s, X^u_s, Y^u_s, e, u_s)N(de, ds), \\
-dY^u_s &= f(s, X^u_s, Y^u_s, Z^u_s, K^u_s, u_s)ds - Z^u_s dB_s - \int_E K^u_s(e)N(de, ds), \\
X^u_t &= x, \\
Y^u_{t+\delta} &= \varphi(t + \delta, X^u_{t+\delta}).
\end{align*}
\] (5.2)

Moreover, from Proposition 3.5 the following estimates hold, for all \( p \geq 2 \),

\[
\mathbb{E}[ \sup_{s \in [t, t+\delta]} |X^u_s|^p + \sup_{s \in [t, t+\delta]} |Y^u_s|^p + \mathbb{E}[\int_t^{t+\delta} |Z^u_s|^2 ds]^\frac{p}{2} |\mathcal{F}_t]
\]

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By the definition of backward stochastic semigroup we have

\[
+ \mathbb{E}\left( \int_t^{t+\delta} \int_E |K_s^u(e)|^2 \lambda(e) ds \right)^{\frac{p}{2}} |F_t| \leq C |x|^p, \quad \mathbb{P} - a.s.,
\]  

(5.3)

and

\[
\mathbb{E}\left[ \sup_{s \in [t,t+\delta]} |X_s^u - x|^p |F_t| \right] \leq C \delta^{\frac{p}{2}} (1 + |x|^p), \quad \mathbb{P} - a.s.
\]

(5.4)

By the definition of backward stochastic semigroup we have

\[
Y_s^u = \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{s,t+\delta}^{t,x,u}[\varphi(t+\delta, X_{t+\delta}^u)], \quad s \in [t,t+\delta].
\]

From Theorem 4.5 it follows that

\[
\varphi(t,x) = W(t,x) = \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x,u}[W(t+\delta, \tilde{X}_{t+\delta}^{t,x,u})],
\]

where \((\tilde{X}^{t,x,u}, \tilde{Y}^{t,x,u}, \tilde{Z}^{t,x,u}, \tilde{K}^{t,x,u})\) is the solution of the following FBSDE:

\[
\begin{aligned}
&d\tilde{X}_{s}^{t,x,u} = b(s, \tilde{X}_{s}^{t,x,u}, \tilde{Y}_{s}^{t,x,u}, \tilde{Z}_{s}^{t,x,u}, \tilde{K}_{s}^{t,x,u}, u_s) ds + \sigma(s, \tilde{X}_{s}^{t,x,u}, \tilde{Y}_{s}^{t,x,u}, \tilde{Z}_{s}^{t,x,u}, e, u_s) \tilde{N}(de, ds), \\
&-d\tilde{Y}_{s}^{t,x,u} = f(s, \tilde{X}_{s}^{t,x,u}, \tilde{Y}_{s}^{t,x,u}, \tilde{Z}_{s}^{t,x,u}, \tilde{K}_{s}^{t,x,u}, u_s) ds - \tilde{Z}_{s}^{t,x,u} dB_s - \int_E \tilde{K}_{s}^{t,x,u}(e) \tilde{N}(de, ds), \\
&\tilde{X}^{t,x,u}_{t+\delta} = x, \quad s \in [t,t+\delta], \\
&\tilde{Y}^{t,x,u}_{t+\delta} = W(t+\delta, \tilde{X}^{t,x,u}_{t+\delta}).
\end{aligned}
\]

Since \(W \leq \varphi\), from Lemma 3.9 we get

\[
\text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x,u}[\varphi(t+\delta, X_{t+\delta}^u)] - \varphi(t,x) \geq 0, \quad \mathbb{P}\text{-a.s.}
\]

(5.5)

We apply Itô’s formula to \(\varphi(s, X_s^u)\), then from (5.2) the following lemma holds.

**Lemma 5.3.** For every \(u(\cdot) \in \mathcal{U}_{t,t+\delta}\), \(\left( Y_s^{u, s} - \varphi(s, X_s^u), Z_s^{u, s} - D \varphi(s, X_s^u) \sigma(s, X_s^u, Y_s^u, u_s), K_s^u(e) - \varphi(s, X_s^u + g(s, X_s^u, Y_s^u, u_s)) + \varphi(s, X_s^u) \right)_{s \in [t,t+\delta]}\) is the solution of the following BSDE:

\[
\begin{aligned}
&-dY_{s}^{1,u} = F(s, X_{s}^{u, s}, Y_{s}^{1,u}, Z_{s}^{1,u}, K_{s}^{1,u}, u_s) ds - Z_{s}^{1,u} dB_s - \int_E K_{s}^{1,u}(e) \tilde{N}(de, ds), \\
&Y_{t+\delta}^{1,u} = 0, \quad s \in [t,t+\delta].
\end{aligned}
\]

(5.6)

We now give some estimates for solutions of the above equation.

**Lemma 5.4.** For every \(u \in \mathcal{U}_{t,t+\delta}\), there exists a \(0 \leq \delta_1 \leq \delta'\) such that for all \(0 \leq \delta \leq \delta_1\)

\[
|Y_{s}^{1,u}| \leq C \delta^{\frac{1}{2}} (1 + |X_s^u|), \quad \mathbb{P}\text{-a.s.,}
\]

\[
|Y_s^{1,u}|^2 + \mathbb{E}\left( \int_s^{t+\delta} |Z_s^{1,u}|^2 + \int_E |K_r^{1,u}(e)|^2 \lambda(de) dr |F_s| \right) \leq C \delta^{\frac{3}{2}}, \quad \mathbb{P}\text{-a.s.,}
\]

\[
\mathbb{E}\left( \int_s^{t+\delta} (|Y_r^{1,u}| + |Z_r^{1,u}| + \int_E |K_r^{1,u}(e)| \lambda(de) dr |F_r| \right) \leq C \delta^{\frac{3}{2}}, \quad \mathbb{P}\text{-a.s.}
\]

where the constant \(C\) is independent of the control \(u\) and \(\delta\).
Proof. For all $s \in [t, t + \delta]$, since $Y^{1,u}_s = Y^u_s - \varphi(s, X^u_s)$, we have

$$Y^{1,u}_s = \varphi(t + \delta, X^{1,u}_{t+\delta}) - \varphi(s, X^u_s) + \int_s^{t+\delta} f(r, X^u_r, Y^u_r, \epsilon, K^u_r, u_r)dr$$

$$- \int_s^{t+\delta} Z^u_r dB_r - \int_s^{t+\delta} \int_E K^u_r(e) \tilde{N}(de, dr),$$

from which, (5.3) and (5.4) it follows that

$$|Y^{1,u}_s| \leq E[|\varphi(t + \delta, X^{1,u}_{t+\delta}) - \varphi(s, X^u_s)|] + \int_s^{t+\delta} E[|f(r, X^u_r, Y^u_r, \epsilon, K^u_r, u_r)|]dr |\mathcal{F}_s]$$

$$\leq E[|\varphi(t + \delta, X^{1,u}_{t+\delta}) - \varphi(s, X^u_s)|] + \int_s^{t+\delta} (1 + |X^u_r| + |Y^u_r| + |Z^u_r| + \int_E |K^u_r(\epsilon)|\lambda(de))dr |\mathcal{F}_s]$$

$$\leq C\delta^{\frac{2}{4}} + C^2 E[|X^{1,u}_{t+\delta} - X^u_s| |\mathcal{F}_s]$$

$$\quad + C\delta^{\frac{2}{4}} (E\int_s^{t+\delta} (1 + |X^u_r|^2 + |Y^u_r|^2 + |Z^u_r|^2 + \int_E |K^u_r(\epsilon)|\lambda(de))dr |\mathcal{F}_s))^{\frac{1}{2}}$$

$$\leq C\delta^{\frac{2}{4}} (1 + |X^u_s|).$$

Using Itô’s formula to $|Y^{1,u}_s|^2$ we get

$$|Y^{1,u}_s|^2 + E[\int_s^{t+\delta} (|Z^{1,u}_r|^2 + \int_E |K^1_r(\epsilon)|^2 \lambda(de))dr |\mathcal{F}_s]$$

$$= 2E[\int_s^{t+\delta} Y^{1,u}_s F(r, X^u_r, Y^{1,u}_r, Z^{1,u}_r, K^{1,u}_r, u_r)dr |\mathcal{F}_s]$$

$$\leq C\int_s^{t+\delta} |Y^{1,u}_r|(1 + |X^u_r|^2 + |Y^u_r|^2 + |Z^u_r|^2 + \int_E |K^1_r(\epsilon)|\lambda(de))dr |\mathcal{F}_s]$$

$$\leq C\delta^{\frac{4}{2}} E[\int_s^{t+\delta} (1 + |X^u_r|^2 + |X^u_r|^3)dr |\mathcal{F}_s] + C\delta^{\frac{4}{2}} E[\int_s^{t+\delta} (|Z^{1,u}_r|^2 + \int_E |K^{1,u}_r(\epsilon)|^2 \lambda(de))dr |\mathcal{F}_s].$$

We choose $0 \leq \delta_1 \leq \delta'$ such that $1 - C\delta_1^{\frac{4}{2}} > 0$. Therefore, for all $0 \leq \delta \leq \delta_1$ we have

$$|Y^{1,u}_s|^2 + E[\int_s^{t+\delta} (|Z^{1,u}_r|^2 + \int_E |K^1_r(\epsilon)|^2 \lambda(de))dr |\mathcal{F}_s]$$

$$\leq C\delta^{\frac{4}{2}} E[\int_s^{t+\delta} (1 + |X^u_r|^2 + |X^u_r|^3)dr |\mathcal{F}_s]$$

$$\leq C\delta^{\frac{2}{2}}.$$

Consequently,

$$E[\int_t^{t+\delta} (|Y^{1,u}_r| + |Z^{1,u}_r| + \int_E |K^{1,u}_r(\epsilon)|\lambda(de))dr |\mathcal{F}_t]$$

$$\leq C\delta^{\frac{2}{2}} + C\delta^{\frac{2}{2}} \left( E[\int_s^{t+\delta} (|Z^{1,u}_r|^2 + \int_E |K^{1,u}_r(\epsilon)|^2 \lambda(de))dr |\mathcal{F}_s) \right)^{\frac{1}{2}}$$

$$\leq C\delta^{\frac{2}{2}}.$$
Let us define
\[
F_1(s, x, y, k, u) = \frac{\partial \varphi}{\partial s}(s, x) + \frac{1}{2} tr(\sigma \sigma^T(s, y + \varphi(s, x), u)D^2 \varphi(s, x)) \\
+ D \varphi(s, x) \cdot b(s, x, y + \varphi(s, x), Z_s^{1, u} + D \varphi(s, x) \cdot \sigma(s, x, Y_s^{1, u} + \varphi(s, x), u), \\
k + \int_E (\varphi(s, x + g(s, x, y + \varphi(s, x), u, e) - \varphi(s, x))\lambda(de), u) \\
+ \int_E (\varphi(s, x + g(s, x, y + \varphi(s, x), u, e) - \varphi(s, x) - D \varphi(s, x) \cdot g(s, x, y + \varphi(s, x), u, e))\lambda(de) \\
+ f(s, x, y + \varphi(s, x), Z_s^{1, u} + D \varphi(s, x) \cdot \sigma(s, x, Y_s^{1, u} + \varphi(s, x), u), u), \\
k + \int_E (\varphi(s, x + g(s, x, y + \varphi(s, x), u, e)) - \varphi(s, x))\lambda(de), u).
\]

It is easy to check that the following BSDE has a unique solution.
\[
\begin{align*}
-dY_{s}^{2, u} &= F_1(s, x, 0, 0, u)ds - Z_s^{2, u}dB_s - \int_E K_s^{2, u}(e)\tilde{N}(de, ds), \\
Y_{t+\delta}^{2, u} &= 0, s \in [t, t + \delta].
\end{align*}
\tag{5.7}
\]

We have the following lemma.

**Lemma 5.5.** For every \(u \in U_{t, t+\delta}\), we have
\[
|Y_{t}^{1, u} - Y_{t}^{2, u}| \leq C\delta^{\frac{2}{3}}, \quad \mathbb{P}\text{-a.s.,}
\]
where \(C\) is independent of the control process \(u\).

**Proof.** By equations (5.6) and (5.7) we see that
\[
|Y_{t}^{1, u} - Y_{t}^{2, u}| = |\mathbb{E}[(Y_{t}^{1, u} - Y_{t}^{2, u})|\mathcal{F}_t]| \\
= \mathbb{E}\left[\int_t^{t+\delta} (F(s, X_s^{1, u}, Y_s^{1, u}, Z_s^{1, u}, K_s^{1, u}, u_s) - F_1(s, x, 0, 0, u_s))ds|\mathcal{F}_t]\right] \\
\leq C\mathbb{E}\left[\int_t^{t+\delta} (1 + |x| + |X_s^{1, u}| + Y_s^{1, u}) + \int_E |K_s^{1, u}(e)|\lambda(de)(|X_s^{1, u} - x| + |Y_s^{1, u}|)ds|\mathcal{F}_t\right] \\
+ C\mathbb{E}\left[\int_t^{t+\delta} (|X_s^{1, u} - x| + |X_s^{1, u} - x|^2)(1 + |Z_s^{u}| + |x|)ds|\mathcal{F}_t\right].
\]

Thanks to Lemma 5.4 we have
\[
\mathbb{E}\left[\int_t^{t+\delta} (1 + |x| + |X_s^{u}| + |Y_s^{1, u}| + \int_E |K_s^{1, u}(e)|\lambda(de)(|X_s^{1, u} - x| + |Y_s^{1, u}|)ds|\mathcal{F}_t\right] \leq C\delta^{\frac{2}{3}},
\]
and
\[
\mathbb{E}\left[\int_t^{t+\delta} |X_s^{u} - x|(1 + |Z_s^{u}| + |x|)ds|\mathcal{F}_t\right] \\
\leq C\mathbb{E}\left[\int_t^{t+\delta} |X_s^{u} - x|ds|\mathcal{F}_t\right] + \left(\mathbb{E}\left[\int_t^{t+\delta} |X_s^{u} - x|^2ds|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_t^{t+\delta} |Z_s^{u}|^2ds|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
\leq C\delta^{\frac{2}{3}},
\]

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\[ \mathbb{E} \left[ \int_t^{t+\delta} \left| X^u_s - x \right|^2 (1 + |Z^u_s| + |x|) ds \right] \mathcal{F}_t \leq C \delta^{\frac{5}{4}}. \]

Therefore,

\[ |Y^{1,u}_t - Y^{2,u}_t| \leq C \delta^{\frac{5}{4}}, \quad \mathbb{P}\text{-a.s.} \]

We also consider the following BSDE:

\[
\begin{cases}
\begin{align*}
-dY^3_{s,u} &= F(s, x, 0, 0, u_s) ds - Z^3_{s,u} dB_s - \int_{E} K^{3,u}_s (\epsilon) \tilde{N}(de, ds), \\
Y^3_{t+\delta} &= 0, \quad s \in [t, t+\delta].
\end{align*}
\end{cases}
\] (5.8)

**Lemma 5.6.** For every \( u \in \mathcal{U}_{t,t+\delta} \), we have

\[ |Y^{2,u}_t - Y^{3,u}_t| \leq C \delta^{\frac{5}{4}}, \quad \mathbb{P}\text{-a.s.}, \]

where \( C \) is independent of the control process \( u \).

**Proof.** From equations (5.6) and (5.7) we have

\[
|Y^{2,u}_t - Y^{3,u}_t| = \mathbb{E} \left[ |Y^{2,u}_t - Y^{3,u}_t| | \mathcal{F}_t \right] = \mathbb{E} \left[ \int_t^{t+\delta} (F_1(s, x, 0, 0, u_s) - F(s, x, 0, 0, u_s)) ds | \mathcal{F}_t \right] \leq C \mathbb{E} \left[ \int_{t}^{t+\delta} (|Y^{1,u}_s| + |Z^{1,u}_s|) ds | \mathcal{F}_t \right].
\]

From Lemma 5.4 it follows that

\[ |Y^{2,u}_t - Y^{3,u}_t| \leq C \delta^{\frac{5}{4}}. \]

**Lemma 5.7.** Let \( Y^4 \) be the solution of the following ordinary differential equation:

\[
\begin{cases}
\begin{align*}
-\dot{Y}^4_s &= F_0(s, x, 0, 0, 0), \quad s \in [t, t+\delta] \\
Y^4_{t+\delta} &= 0,
\end{align*}
\end{cases}
\]

where the function \( F_0 \) is defined by

\[ F_0(s, x, 0, 0, 0) = \sup_{u \in \mathcal{U}} F(s, x, 0, 0, u) . \]

Then,

\[ Y^4_t = \text{esssup}_{u \in \mathcal{U}_{t,t+\delta}} Y^{3,u}_{t}, \quad \mathbb{P} - \text{a.s.} \]
Proof. From the definition of $F_0(s,x,0,0,0)$ it follows that

$$F_0(s,x,0,0,0) \geq F(s,x,0,0,0,u), \quad \text{for any } u \in \mathcal{U}_{t,t+\delta}.$$ 

By virtue of the comparison theorem of BSDEs we get

$$\tilde{Y}_s^4 \geq Y_s^3,u, a.s., \quad \text{for any } s \in [t,t+\delta],$$

where $(\tilde{Y}_s^4, \tilde{Z}_s^4)$ is the solution of the following BSDE:

$$\begin{cases}
-d\tilde{Y}_s^4 = F_0(s,x,0,0,0)ds - \tilde{Z}_s^4dB_s - \int_E \tilde{K}_{s}^4(e)\tilde{N}(de,ds), \ s \in [t,t+\delta] \\
\tilde{Y}_{t+\delta}^4 = 0,
\end{cases}$$

In fact, $(\tilde{Y}_s^4, \tilde{Z}_s^4, \tilde{K}_s^4) = (Y_s^4,0,0)$. Therefore, $Y_t^4 \geq Y_t^3,u, a.s.,$ for all $u \in \mathcal{U}_{t,t+\delta}$.

On the other hand, from the definition of $F_0(s,x,0,0,0)$ it follows that there exists $\tilde{u} : (s,x) \rightarrow \mathcal{U}$, such that $F(s,x,0,0,0,\tilde{u}(s,x)) = F_0(s,x,0,0,0)$. Let us define $\tilde{u}_s = \tilde{u}(s,x)$, $s \in [t,t+\delta]$. Then $\tilde{u}_s^0 \in \mathcal{U}_{t,t+\delta}$, and

$$F_0(s,x,0,0,0) = F(s,x,0,0,0,\tilde{u}_s^0), \ s \in [t,t+\delta]$$

Therefore, by virtue of the uniqueness of the solution of the BSDE we know that $Y_t^4 = Y_t^3,\tilde{u}_s^0$, $\mathbb{P}$-a.s. Consequently,

$$Y_t^4 = \text{esssup}_{u \in \mathcal{U}} Y_t^3,u.$$ 

\[\square\]

We now come to the proof of Theorem 5.2. From (5.5) it follows that

$$\text{esssup}_{u \in \mathcal{U}} Y_t^{1,u} \geq 0, \ \mathbb{P}$-a.s.,$$

which together with Lemmas 5.5 and 5.6 yields

$$\text{esssup}_{u \in \mathcal{U}} Y_t^{3,u} \geq C\delta^\frac{4}{5}, \ \mathbb{P}$-a.s.,$$

from which it follows that

$$C\delta^\frac{4}{5} \geq \frac{1}{\delta} Y_t^4 \rightarrow \sup_{u \in \mathcal{U}} F(t,x,0,0,0,u)$$

as $\delta \rightarrow 0$. Therefore,

$$\sup_{u \in \mathcal{U}} F(t,x,0,0,0,u) \geq 0.$$ 

From the definition of $F$ it now follows that $W$ is a viscosity subsolution of (5.1). We conclude the proof. 

\[\square\]
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