Analytic physical model of anisotropic anomalous diffusion

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Abstract

The temporal Fokker-Planck equation is analytically integrated in an arbitrary number of spatial dimensions but with the 2D and 3D cases highlighted. It is shown that a temporal power-law ansatz for the anisotropic diffusion coefficients leads naturally to a physically reasonable model of normal and anomalous diffusion with drift. An interpretation of the drift and diffusion coefficients is provided and the analytic growth rate of the area and volume of uncertainty is determined. It is shown that the asymptotic growth of the uncertainty in the volume of space about the mean position is proportional to the square-root of time raised to the power of the sum over all exponents of the anisotropic diffusion coefficients.

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I. INTRODUCTION

Normal and anomalous diffusion models have been used in such diverse topics as sub-
marine search [1, 2]; Brownian motion; disease, animal, and cell movement [3]; ballistic or
wavelike movement; turbulent diffusion; and particle acceleration [4]. The current paper
provides an analytic solution to the temporal Fokker-Planck equation and shows how a physical
model of anomalous diffusion naturally arises given an ansatz of temporal power-law growth
in the anisotropic diffusion coefficients.

II. ANALYTIC TEMPORAL FOKKER-PLANCK

For clarity, we start in one spatial dimension, but the results generalize to an arbitrary
number of dimensions as will be shown. The Fokker-Planck (FP) equation in 1D with
spatially constant but temporally varying diffusion and drift coefficients (i.e. the ‘temporal’
FP equation) is given by

\[ \frac{\partial \rho(x, t)}{\partial t} = \frac{D(t)}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} - V(t) \frac{\partial \rho(x, t)}{\partial x}, \]

(1)

where \( \rho(x, t) \) is a normalized probability density of interest (e.g. for a submarine, dust,
biologic, or financial particle):

\[ \int_{-\infty}^{\infty} \rho(x, t) dx = 1, \]

(2)

\( D(t) > 0 \) is the diffusion coefficient, and \( V(t) \in \mathbb{R} \) is the drift coefficient. Both \( D(t) \) and
\( V(t) \) have further temporal integrability conditions as discussed below. Simple dimensional
analysis applied to Eq. (1) gives

\[ [D(t)] = \left[ \frac{\partial x^2}{\partial t} \right] \implies \text{area/time} \]

(3a)

and

\[ [V(t)] = \left[ \frac{\partial x}{\partial t} \right] \implies \text{distance/time}. \]

(3b)

The latter is a drift velocity and the former appears to be an area growth rate. This intuition
is shown to be correct in 2D and the N-dimensional generalization is developed.

With initial conditions at time \( t = t_0 \) set to be a Gaussian with mean position \( x_0 \) and
variance about the mean \( \sigma_{0x}^2 \), a solution (verified analytically below) to Eq. (1) is given by

\[ \rho(x, t) = \mathcal{N} \left[ x; x_0 + I_V(t), \sigma_{0x}^2 + I_D(t) \right], \]

(4)
where $I_V(t)$ and $I_D(t)$ are integrals of the drift and diffusion coefficients respectively:

\begin{align*}
I_V(t) & \equiv \int_{t_0}^t V(t') dt' \\
I_D(t) & \equiv \int_{t_0}^t D(t') dt',
\end{align*}

and $\mathcal{N}[x; a(t), b(t)]$ is standard notation for a normalized Gaussian with mean $a(t)$ and variance $b(t)$:

\begin{equation}
\mathcal{N}[x; a(t), b(t)] = e^{-\frac{(x-a(t))^2}{2b(t)}} \sqrt{\frac{1}{2\pi b(t)}}.
\end{equation}

The position argument $x$ is added for clarity and to allow for easier generalization to higher dimensions; the parameters $a(t)$ and $b(t)$ depend on time but not space. The integrals of Eq. (5) of course have to converge for this solution to make sense and their derivatives give back the respective integrand exactly:

\begin{align*}
\frac{\partial I_V(t)}{\partial t} &= V(t) \quad \text{(7a)} \\
\frac{\partial I_D(t)}{\partial t} &= D(t) \; ; \quad \text{(7b)}
\end{align*}

these relations especially will be used in the analytic proof that follows. Also note that by definition these integral definitions vanish at the initial condition point:

\begin{align*}
I_V(t_0) &= 0, \\
I_D(t_0) &= 0.
\end{align*}

Finally, notice how the units are correct and a time integral of $V(t)$ and $D(t)$ produce quantities of dimension length and area respectively, consistent with Eqs. (3) and (4).

Calculating the mean position and variance about the mean at arbitrary time $t$ gives

\begin{align*}
\langle x \rangle & \equiv \int_{-\infty}^{\infty} \rho(x,t) x \, dx = x_0 + I_V(t) \\
(\Delta x)^2 & \equiv \int_{-\infty}^{\infty} \rho(x,t) (x - \langle x \rangle)^2 \, dx = \sigma_{0x}^2 + I_D(t)
\end{align*}

respectively. At the initial time, $t = t_0$, these collapse to the Gaussian initial conditions, and for all time we see that these integrals of Eq. (5) are the required additions to the exact...
solution of a drifting and diffusing Gaussian. We see that a Gaussian remains a Gaussian for all time here, but the solution is not stationary: both the mean and variance depend on time. This time dependence is modeled as a power law below and this is where the anomalous diffusion lies.

A. Analytic proof

Now we analytically show that \( \rho(x,t) \) of Eq. (4) is an exact solution to Eq. (1). Writing out the solution more explicitly from above gives

\[
\rho(x,t) = e^{-\frac{(x-a(t))^2}{2b(t)}}
\]

where

\[
a(t) = x_0 + I_V(t) ,
\]

\[
b(t) = \sigma_0^2 + I_D(t).
\]

For the proof that follows, note that the exact temporal derivatives of \( a(t) \) and \( b(t) \) are given by

\[
\frac{\partial a(t)}{\partial t} = \frac{\partial I_V(t)}{\partial t} = V(t) \quad (12a)
\]

and

\[
\frac{\partial b(t)}{\partial t} = \frac{\partial I_D(t)}{\partial t} = D(t) \quad (12b)
\]

respectively, where \( V(t) \) and \( D(t) \) are the same coefficients as in Eq. (1). We have not specified the temporal dependence of \( V(t) \) or \( D(t) \) yet, other than requiring that they are both integrable from some initial time \( t_0 \) through an arbitrarily large but finite final time \( t \). However, in the next section their temporal dependence is discussed at length because that is what determines if the diffusion is anomalous or not.

Taking a partial time derivative of Eq. (11) using the product rule of differentiation and the results of Eq. (12) gives

\[
\frac{\partial \rho(x,t)}{\partial t} = \left\{ \frac{V(t) [x - a(t)]}{b(t)} + \frac{D(t) [x - a(t)]^2}{2b^2(t)} - \frac{D(t)}{2b(t)} \right\} \rho(x,t) . \quad (13)
\]

Likewise, the first and second partial spatial derivatives of \( \rho(x,t) \) are

\[
\frac{\partial \rho(x,t)}{\partial x} = -\frac{[x - a(t)]}{b(t)} \rho(x,t) \quad (14)
\]
and
\[ \frac{\partial^2 \rho(x,t)}{\partial x^2} = \left\{ \frac{(x - a(t))^2}{b^2(t)} - \frac{1}{b(t)} \right\} \rho(x,t) \]  

(15)

respectively; multiplying the latter by \( D(t)/2 \) and the former by \(-V(t)\) according to the right-hand side of Eq. (1) gives
\[ \frac{D(t) \frac{\partial^2 \rho(x,t)}{\partial x^2}}{2} - V(t) \frac{\partial \rho(x,t)}{\partial x} = \left\{ \frac{D(t) (x - a(t))^2}{2b^2(t)} - \frac{D(t)}{2b(t)} + \frac{V(t) (x - a(t))}{b(t)} \right\} \rho(x,t) . \]  

(16)

Comparing this with the time-partial result, Eq. (13), shows that they are equivalent and therefore Eq. (1) is satisfied exactly for \( \rho(x,t) \) of Eq. (4)—q.e.d.

B. Multiple spatial dimensions

Now we generalize the analytic solution of the previous section to multiple spatial dimensions. Due to the linearity of Eq. (1), this is easily accomplished with a product of Gaussians as will be shown. The solution easily generalizes to an arbitrary number of spatial dimensions, however the 2D and 3D solutions are emphasized next due to their physical relevance. The N-dimensional results are presented near the end of the paper. We generalize Eq. (1) with the anisotropic application in mind, hence we introduce diffusion coefficients \( D_x(t), D_y(t), \) and \( D_z(t) \). These diffusion coefficients are in general of different magnitudes, especially with the vertical \( D_z(t) \) often smaller or much smaller than its horizontal counterparts often themselves of comparable magnitude: \( D_x(t) \sim D_y(t) \). For completeness note that the drift coefficients, \( V(t) \), are also treated anisotropically in what follows.

1. Analytic anisotropic 2D temporal Fokker-Planck

Next, we generalize the above temporal FP equation to two spatial dimensions and discuss the interpretation of its anisotropic diffusion and drift coefficients. Eq. (1) generalized to 2D with the diffusion and drift coefficients allowed to be anisotropic is
\[ \frac{\partial \rho(x,y,t)}{\partial t} = \frac{1}{2} \left[ D_x(t) \frac{\partial^2}{\partial x^2} + D_y(t) \frac{\partial^2}{\partial y^2} \right] \rho(x,y,t) - \left[ V_x(t) \frac{\partial}{\partial x} + V_y(t) \frac{\partial}{\partial y} \right] \rho(x,y,t) . \]  

(17)
Maintaining the notation defined in the previous section, a solution to this 2D equation is simply a product of normalized Gaussians:

\[
\rho(x, y, t) = N[x; x_0 + I_{V_x}(t), \sigma_{0x}^2 + I_{D_x}(t)] N[y; y_0 + I_{V_y}(t), \sigma_{0y}^2 + I_{D_y}(t)]
\]

(18a)

\[
\equiv N_x N_y .
\]

(18b)

Especially note this definition of \(N_x\) and \(N_y\) in this last line; this is nothing more than a shorthand used below, but the full expression of the previous line is implied. In this solution, the anisotropic 2D drift and diffusion coefficient time integrals are defined by

\[
I_{V_x}(t) \equiv \int_{t_0}^{t} V_x(t')dt' ,
\]

(19a)

\[
I_{V_y}(t) \equiv \int_{t_0}^{t} V_y(t')dt' ,
\]

(19b)

\[
I_{D_x}(t) \equiv \int_{t_0}^{t} D_x(t')dt' ,
\]

(19c)

\[
I_{D_y}(t) \equiv \int_{t_0}^{t} D_y(t')dt' .
\]

(19d)

Due to the product rule of differentiation and the linearity of the derivatives in Eq. (17), this product of Gaussians is easily seen to be a solution. Explicitly we have for the temporal derivative:

\[
\frac{\partial \rho(x, y, t)}{\partial t} = \frac{\partial}{\partial t} (N_x N_y) = \left( \frac{\partial}{\partial t} N_x \right) N_y + N_x \left( \frac{\partial}{\partial t} N_y \right) ,
\]

(20)

and for the spatial derivatives:

\[
\frac{\partial \rho(x, y, t)}{\partial x} = \frac{\partial}{\partial x} (N_x N_y) = \left( \frac{\partial}{\partial x} N_x \right) N_y ,
\]

(21a)

\[
\frac{\partial \rho(x, y, t)}{\partial y} = \frac{\partial}{\partial y} (N_x N_y) = N_x \left( \frac{\partial}{\partial y} N_y \right) ,
\]

(21b)

\[
\frac{\partial^2 \rho(x, y, t)}{\partial x^2} = \frac{\partial^2}{\partial x^2} (N_x N_y) = \left( \frac{\partial^2}{\partial x^2} N_x \right) N_y ,
\]

(21c)

\[
\frac{\partial^2 \rho(x, y, t)}{\partial y^2} = \frac{\partial^2}{\partial y^2} (N_x N_y) = N_x \left( \frac{\partial^2}{\partial y^2} N_y \right) .
\]

(21d)

Thus, combining all the pieces of Eq. (17), this product of Gaussians solution satisfies

\[
N_y \left( \frac{\partial}{\partial t} - \frac{D_x(t)}{2} \frac{\partial^2}{\partial x^2} + V_x(t) \frac{\partial}{\partial x} \right) N_x + N_x \left( \frac{\partial}{\partial t} - \frac{D_y(t)}{2} \frac{\partial^2}{\partial y^2} + V_y(t) \frac{\partial}{\partial y} \right) N_y = 0 ,
\]

(22)

and we see that the equation factorizes into two independent 1D equations. Thus we can use the 1D results of the previous section and Eq. (18) is seen to be a solution of Eq. (17)—q.e.d.
Calculating the 2D mean position and variance about the mean at arbitrary time \( t \), we see that they factorize along with the solution and its norm. Explicitly, we have

\[
1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x, y, t) = \int_{-\infty}^{\infty} N_x dx \int_{-\infty}^{\infty} N_y dy ,
\]

\[
\langle x \rangle \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x, y, t) x = x_0 + I_{V_x}(t) ,
\]

\[
\langle y \rangle \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x, y, t) y = y_0 + I_{V_y}(t) ,
\]

\[
(\Delta x)^2 \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x, y, t) (x - \langle x \rangle)^2 = \sigma_{0x}^2 + I_{D_x}(t) ,
\]

\[
(\Delta y)^2 \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \rho(x, y, t) (y - \langle y \rangle)^2 = \sigma_{0y}^2 + I_{D_y}(t) .
\]

Writing these results in vector notation (here 2-vectors—in the next section, same notation but they will be 3-vectors) gives

\[
\langle x \rangle \equiv \int \rho(x, t) x^2 x = [x_0 + I_{V}(t)]
\]

and

\[
(\Delta x)^2 \equiv \int \rho(x, t) (x - \langle x \rangle)^2 d^2 x = [\sigma_{0x}^2 + I_{D}(t) \cdot (\hat{x} + \hat{y})] ,
\]

for Cartesian unit vectors \( \hat{x} \) and \( \hat{y} \), where the drift and diffusion coefficient time integrals are here 2-vectors:

\[
I_V(t) \equiv [I_{V_x}(t), I_{V_y}(t)] ,
\]

\[
I_D(t) \equiv [I_{D_x}(t), I_{D_y}(t)] ,
\]

and

\[
I_D(t) \cdot (\hat{x} + \hat{y}) \equiv I_{D_x}(t) + I_{D_y}(t) .
\]

Continuing with the interpretation of the 2D drift and diffusion coefficients. First the drift: The drift coefficient vector, \( V(t) \), is equivalent to the instantaneous velocity of the mean position. This follows from taking a derivative of Eq. (24):

\[
\frac{\partial \langle x \rangle}{\partial t} = \frac{\partial I_V(t)}{\partial t} = V(t) = [V_x(t), V_y(t)] .
\]

In summary: the drift coefficient vector \( V(t) \) is equal to the instantaneous velocity of the mean position of the probability density of interest. This interpretation is still valid in an arbitrary number of spatial dimensions with arbitrary time-integrable anisotropic drift coefficients.
Second the diffusion: For the interpretation of the diffusion coefficient vector, first calculate the area of uncertainty (AOU) of the 2D variance about the mean position at arbitrary time \( t \). Using the 1σ ellipse for its definition, we have

\[
AOU(t) = \pi \Delta x \Delta y = \pi \sqrt{\sigma_{0x}^2 + I_{D_x}(t)} \sqrt{\sigma_{0y}^2 + I_{D_y}(t)} .
\]

(28)

Taking a time derivative for the AOU growth rate gives

\[
\frac{\partial AOU(t)}{\partial t} = \frac{\pi}{2} \left[ \frac{D_x(t)}{\sigma_{0x}^2 + I_{D_x}(t)} \sqrt{\sigma_{0y}^2 + I_{D_y}(t)} + \frac{D_y(t)}{\sigma_{0y}^2 + I_{D_y}(t)} \right] ,
\]

(29)

a major result, so we box it. The interpretation of the diffusion coefficient is particularly simple in the 2D isotropic case for then we have \( (D_x(t) = D_y(t) \equiv D(t), \sigma_{0x}^2 = \sigma_{0y}^2 \equiv \sigma_0^2) \):

\[
\left. \frac{\partial AOU(t)}{\partial t} \right|_{\text{isotropic 2D}} = AOU(t) \frac{D(t)}{\sigma_0^2 + I_D(t)} = \pi D(t) .
\]

(30)

Thus, except for the inconsequential factor of \( \pi \), the 2D isotropic diffusion coefficient is equal to the instantaneous AOU growth rate—certainly the units are right as we discussed earlier, and here we showed this intuition to be precisely correct in 2D. Below we see this precise interpretation breaks down in higher spatial dimensions—turns out the 2D case is somewhat of an accident, as will be shown.

2. Analytic anisotropic 3D temporal Fokker-Planck

It is clear that the 3D case (and in general N-dimensional case) of the solution under discussion generalizes just like the 2D case of the previous section. But for completeness, we will write out some of the 3D results explicitly here. First, the anisotropic 3D FP equation with spatially constant but temporally varying diffusion and drift coefficients is

\[
\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2} D(t) \cdot \nabla^2 \rho(x,t) - V(t) \cdot \nabla \rho(x,t) ,
\]

(31)

where the generalized Laplacian is defined by

\[
D(t) \cdot \nabla^2 \equiv D_x(t) \frac{\partial^2}{\partial x^2} + D_y(t) \frac{\partial^2}{\partial y^2} + D_z(t) \frac{\partial^2}{\partial z^2}
\]

(32)
and the drift advection operator is given by
\[ \mathbf{V}(t) \cdot \nabla = V_x(t) \frac{\partial}{\partial x} + V_y(t) \frac{\partial}{\partial y} + V_z(t) \frac{\partial}{\partial z}. \] (33)

Using the notation of the previous section (with 3-vectors instead of 2-vectors), and in particular using the shorthand of Eq. (18) (with the addition of a shorthand for \( N_z \)), the solution, norm, mean, and variance in 3D is given by
\[ \rho(x, t) = N_x N_y N_z, \] (34)
\[ 1 = \int \rho(x, t) d^3x, \] (35)
\[ \langle x \rangle = \int \rho(x, t) x d^3x = x_0 + I_{Vx}(t), \] (36)
\[ (\Delta x)^2 = \int \rho(x, t) (x - \langle x \rangle)^2 d^3x = \sigma_0^2 + I_{Dx}(t) \cdot (\hat{x} + \hat{y} + \hat{z}) \] (37)
respectively. The proof that this is a solution of the above 3D temporal FP equation follows almost identically to the 2D case of the previous section and will not be repeated here.

We close this section by discussing the interpretation of the drift and diffusion coefficients in 3D. First, the drift coefficient vector, \( \mathbf{V}(t) \), follows identically to Eq. (27) of the 2D case except that now it is a 3-vector:
\[ \frac{\partial \langle x \rangle}{\partial t} = \frac{\partial I_{Vx}(t)}{\partial t} = \mathbf{V}(t) = [V_x(t), V_y(t), V_z(t)] . \] (38)

Second, the diffusion coefficient vector, \( \mathbf{D}(t) \), follows similarly to the above except now it is a volume instead of area of uncertainty. (Note for later: In \( N > 3 \) spatial dimensions we keep calling it ‘volume’.) Using the 1\( \sigma \) ellipsoid for the definition of the volume of uncertainty (VOU) in three dimensions, we have
\[ VOU(t) = \frac{4}{3} \pi \Delta x \Delta y \Delta z = \frac{4}{3} \pi \sqrt{\sigma_{0x}^2 + I_{Dx}(t)} \sqrt{\sigma_{0y}^2 + I_{Dy}(t)} \sqrt{\sigma_{0z}^2 + I_{Dz}(t)}. \] (39)

Similar to Eq. (29), but for volume instead of area, taking a time derivative gives the VOU growth rate (skipping similar algebra to the 2D case):
\[ \frac{\partial VOU(t)}{\partial t} = \frac{VOU(t)}{2} \left[ \frac{D_x(t)}{\sigma_{0x}^2 + I_{Dx}(t)} + \frac{D_y(t)}{\sigma_{0y}^2 + I_{Dy}(t)} + \frac{D_z(t)}{\sigma_{0z}^2 + I_{Dz}(t)} \right], \] (40)
the same form as in 2D even with the same factor of 2 in the denominator—this factor of 2 leads to an accident in 2D—but now there are three terms (and in \( N \) dimensions, \( N \)...
terms). As shown below, these $N$ terms lead to an overall multiplicative factor of $N$ in the exponent of the asymptotic temporal growth of the volume uncertainty in the isotropic case. This will become clear below after we discuss the temporal dependence of the diffusion and drift coefficients and show how they relate to the different types of normal and anomalous diffusion.

III. ANISOTROPIC ANOMALOUS DIFFUSION

The analytic solution of the previous section is developed with a power-law ansatz for the diffusion coefficient. Anomalous diffusion is seen to arise and the physical bounds of the diffusion coefficient exponents are discussed. In the final section, the volume growth rate is analyzed in the general case of $N$ spatial dimensions. This generalizes the above 2D result of the AOU growth rate being identified with the diffusion coefficient (modulo a factor of $\pi$).

A. Power-law ansatz

The drift coefficient vector, $V(t)$, is the instantaneous velocity of the mean position as discussed above around Eqs. (27) and (38) in 2D and 3D respectively. The diffusion coefficient vector, $D(t)$, affects the centered variance, $(\Delta x)^2$, according to Eq. (37), but note that the drift $V(t)$ decouples from this diffusion. It is true that the mean squared position does depend on $V(t)$ as follows from Eqs. (36) and (37):

$$\langle x^2 \rangle = (\Delta x)^2 + \langle x \rangle^2 = \sigma_0^2 + \mathbf{I}_D(t) \cdot (\hat{x} + \hat{y} + \hat{z}) + [x_0 + \mathbf{I}_V(t)]^2 . \tag{41}$$

However, the centered variance $(\Delta x)^2$ is what matters with regards to whether the diffusion is anomalous or not as we now show.

To study anomalous diffusion, let the components of $D(t)$ be given by a temporal power law with power $\alpha = 1$ matched to the normal-diffusion case. Thus, in general for anisotropic diffusion, let

$$D(t) \equiv [C_x \alpha_x (t - t_0)^{\alpha_x - 1}, C_y \alpha_y (t - t_0)^{\alpha_y - 1}, C_z \alpha_z (t - t_0)^{\alpha_z - 1}] , \tag{42}$$

for constant vectors $C$ and $\alpha$. Plugging this power-law ansatz into Eq. (37) for the centered
variance, and performing the time integrations gives

\[ (\Delta x)^2 = \sigma_0^2 + C_x (t - t_0)^{\alpha_x} + C_y (t - t_0)^{\alpha_y} + C_z (t - t_0)^{\alpha_z}, \]  

(43)

as long as the exponents are all positive: \( \alpha_x > 0, \alpha_y > 0, \) and \( \alpha_z > 0; \) for zero exponents there is a logarithmic divergence at initial time \( t = t_0 \) and for negative exponents there is a power-law divergence. Thus we see that subdiffusion \( (\alpha < 1) \) is allowed in this model, but the exponents must be positive \( (\alpha > 0) \) in order for the centered variance to be finite as required on physical grounds \cite{5}.

For superdiffusion \( (\alpha > 1) \), the ballistic velocity,

\[ \text{ballistic velocity} \equiv \frac{(\Delta x)^2}{(t - t_0)^2} \approx \frac{C_x}{t_{\geq t_0}} t^\alpha_x - 2 + C_y t^\alpha_y - 2 + C_z t^\alpha_z - 2, \]  

(44)

in the asymptotic large-time limit seems to require \( \alpha_x \leq 2, \alpha_y \leq 2, \) and \( \alpha_z \leq 2 \) for finiteness. However, there is no reason in general to limit this upper bound of the diffusion coefficient exponents: The principle of relativity already naturally bounds the exponents as nicely explained in \cite{4}. Accelerating particles for short times have \( \alpha = 3 \) (turbulent diffusion) or even \( \alpha = 4 \) (nonrelativistic acceleration), however for long times relativity bounds these exponents to the ballistic \( \alpha = 2 \) limit (see \cite{4}).

In summary, the power-law ansatz of Eq. (42) implies Eq. (43) for the centered position variance. For simplicity, in the isotropic case this is \((\Delta x)^2 \sim (t - t_0)^\alpha\), where normal diffusion has \( \alpha = 1 \), subdiffusion has \( 0 < \alpha < 1 \), and superdiffusion has \( \alpha > 1 \) \cite{5}. Figure 1 shows these different phases of anomalous diffusion as follows from integrating the power-law ansatz of the analytic model of the current paper.

B. Volume growth rate in N spatial dimensions

The power law of the previous section is used to discuss the asymptotic large-time growth rate of the N-dimensional ‘volume’ of uncertainty (VOU). Given semi-axis radii of the respective 1σ variance ‘ellipsoid’, the VOU in N spatial dimensions is given by \( VOU = 2a \) for \( N = 1; VOU = \pi ab \) for \( N = 2; VOU = \frac{4}{3} \pi abc \) for \( N = 3; VOU = \frac{\pi^2}{2} abcd \) for \( N = 4; \) etc. Eq. (40) generalized to N spatial dimensions is

\[ \frac{\partial VOU(t)}{\partial t} = \frac{VOU(t)}{2} \sum_{i=1}^{N} \frac{D_{x_i}(t)}{\sigma_{0x_i}^2 + I_{D_{x_i}}(t)}. \]  

(45)
\begin{equation}
(\Delta x)^2 - \sigma_0^2 \\
\begin{align*}
\text{--- } (t-t_0)^{0.1} \\
\text{--- } (t-t_0)^{0.5} \\
(t-t_0) \\
\text{--- } (t-t_0)^{1.5} \\
\text{--- } (t-t_0)^2 \\
(t-t_0)^3 \\
\text{--- } (t-t_0)^4 \\
\end{align*}
\end{equation}

FIG. 1: Variance of the position about the mean from integrating the power-law ansatz of the current paper [see Eqs. (42) and (43)]. The isotropic case is shown for simplicity (in arbitrary units): \((\Delta x)^2 - \sigma_0^2 \sim (t - t_0)^\alpha\). \(\sigma_0^2\) is the variance of the initial Gaussian at time \(t = t_0\). Normal diffusion has \(\alpha = 1\); subdiffusion has \(0 < \alpha < 1\); and superdiffusion has \(\alpha > 1\) \cite{5}. The turbulent diffusion and particle acceleration solutions are explained in \cite{4}.

Plugging in the power-law ansatz of Eqs. (42) and (43), in the large-time asymptotic limit, we see these constant vectors \(C\) cancel in the numerator and denominator and we are left with

\begin{equation}
\frac{\partial VOU(t)}{\partial t} \underset{t \gg t_0}{\sim} \frac{VOU(t)}{2} \sum_{i=1}^{i=N} \frac{C_{x_i} \alpha_{x_i} (t-t_0)^{\alpha_{x_i}-1}}{C_{x_i} (t-t_0)^{\alpha_{x_i}}} \\
\end{equation}

\begin{equation}
\underset{t \gg t_0}{\sim} \frac{VOU(t)}{2 t} \sum_{i=1}^{i=N} \alpha_{x_i} \\
\equiv \frac{VOU(t)}{2 t} \alpha_{\text{Neum}},
\end{equation}

(46)
where $\alpha_{N\text{sum}}$ is a shorthand for the sum of the power-law exponents over all $N$ spatial dimensions. This asymptotic equation is easy to integrate and we are left with

$$VOU(t) \sim t^{\frac{2\alpha_{N\text{sum}}}{2}}.$$

This 2 in the denominator of the exponent in the last formula leads to an accident for spatial dimension $N = 2$, leading to a nice interpretation for the 2D isotropic diffusion coefficient: Eq. (30); however, this interpretation is not valid for $N \neq 2$, and Eq (46) or its integral Eq. (47) must in general be used. Eq. (47) described in words: The uncertainty in the volume of $N$-dimensional space about the mean position grows like the square-root of time raised to the power of the sum over all exponents of the anisotropic diffusion coefficients. This is the same square-root of time as Einstein’s Brownian motion, and the $\alpha_{N\text{sum}}$ generalizes the result to $N$ spatial dimensions with anisotropic anomalous diffusion.

IV. SUMMARY AND DISCUSSION

The temporal Fokker-Planck equation was defined and analytically integrated in an arbitrary number of dimensions, but with the 2D and 3D cases highlighted. It was shown that a power-law ansatz for the anisotropic diffusion coefficients leads to a natural physical model of anomalous diffusion with drift. The volume of uncertainty was determined for an arbitrary number of spatial dimensions $N$ to grow like (e.g. in the isotropic case): $VOU(t) \sim t^{N\alpha/2}$, where $\alpha$ is the temporal exponent of the power-law ansatz for the diffusion coefficient (with $\alpha = 1$ defined to be normal diffusion, i.e. $D(t)$ being a constant).

We can generalize our results further by adding an interaction potential to the right-hand side of our 3D (or $N$-dimensional actually) temporal Fokker-Planck equation:

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2}D(t) \cdot \nabla^2 \rho(x,t) - V(t) \cdot \nabla \rho(x,t) - U(x,t) \rho(x,t).$$

(48)

This is an example of a Euclidean Schrödinger equation of path integral fame (with an additional drift term) which can be matched to a large number of physical diffusive processes. Note that $U(x,t)$ does not depend on spatial derivatives (as in a generalization that includes $\nabla^3$, $\nabla^4$, etc. or other velocity-type operators has not been included), therefore this first term proportional to the anisotropic diffusion coefficient, $D(t)$, is the only dissipative term of the equation. Also note well that this equation is linear and therefore does not suffer from
the sensitivity to initial conditions problem, but yet as shown in the previous sections, intermittent phenomena are contained in solutions to this equation: As the temporal power-law dependence of $D(t)$ is adjusted, sub- and super-diffusive phenomena appear including perhaps surprisingly turbulent diffusion (see Figure 1) even though the defining equation of the analytic model of the current paper is linear [Eq. (31)].

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