Pair creation in transport equations using the equal-time Wigner function

Christoph Best and J. M. Eisenberg

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, 69978 Tel Aviv, Israel,

and

Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, 6000 Frankfurt am Main, Germany

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Abstract

Based on the equal-time Wigner function for the Klein-Gordon field, we discuss analytically the mechanism of pair creation in a classical electromagnetic field including back-reaction. It is shown that the equations of motion for the Wigner function can be reduced to a variable-frequency oscillator. The pair-creation rate results then from a calculation analogous to barrier penetration in nonrelativistic quantum mechanics. The Wigner function allows one to utilize this treatment for the formulation of an effective transport theory for the back-reaction problem with a pair-creation source term including Bose enhancement.

1 Introduction

The usage of the equal-time Wigner transform of the two-point correlation function of a quantum field theory has been proposed to investigate nonperturbatively the time evolution of the vacuum state [1, 2]. This approach is
based on the Fourier transform of the two-point correlation function split in the space coordinates but evaluated at equal times. The equation of motion for this object can be formulated for the interaction of the Dirac [1] or the Klein-Gordon field [2] with a classical electromagnetic field and exhibits features of a transport equation. It does not, in particular, entail a constraint equation—as does the covariant Wigner function [3]—and can be computed as an initial-value problem. This makes this approach a good candidate for problems like the formation of the quark-gluon plasma and the back reaction from pair creation in the early universe.

The problem of back-reaction from pair creation in a classical electromagnetic field has been investigated earlier using the field equations directly [4]. Numerical results [5, 6] indicate that the exact solution of the field equations can be very well approximated by a transport theory involving a Schwinger term and Bose enhancement or Pauli blocking, as pertinent. This has raised the question of how to derive such a theory consistently starting from the field equations. While this has been attempted before using the covariant Wigner function [3], we here show how the pair-creation term can be isolated beginning from the equation of motion for the equal-time Wigner function of a Klein-Gordon field interacting with a classical electromagnetic field. Similar approaches based on the equal-time Wigner operator have been taken in [7, 8].

2 Formalism

2.1 The equal-time Wigner function

The Wigner function of the Klein-Gordon field can be defined from the symmetrized two-point function

\[ C_{\alpha\beta}^{\dagger}(q, q'; t) = \langle \Omega | \{ \Phi_\alpha(q, t), \Phi_\beta^+(q', t) \} | \Omega \rangle \]  

(1)
evaluated at equal times \( t \) (\( q \) and \( q' \) are three-dimensional vectors). In order to obtain an evolution equation for the Wigner function, it is necessary to consider the field as a two-component object \( \Phi \) in the Feshbach-Villars representation

\[ \Phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad \phi = \psi + \chi, \quad \left( i \frac{\partial}{\partial t} - eA^0 \right) \phi = m(\psi - \chi), \]  

(2)
where $\phi$ designates the basic Klein-Gordon field obeying the equation of motion in a classical electromagnetic field $A_{\mu}$,

$$
\left( (\partial_{\mu} - ieA_{\mu})(\partial^{\mu} - ieA^{\mu}) + m^2 \right) \phi(x) = 0.
$$

(3)

The Wigner function is defined by

$$
P(q,p) = \int d^3 y \, C^+(q - \frac{1}{2} y, q + \frac{1}{2} y, t) \exp \left( \frac{i}{\hbar} p \cdot y + \frac{ie}{\hbar} \int_{q-y/2}^{q+y/2} A(x) \cdot dx \right),
$$

(4)

where $A(x)$ is the electromagnetic vector potential. The equation of motion of the Wigner function follows from the equation of motion for the Feshbach-Villars field

$$
i \frac{\partial \Phi}{\partial t} = \hat{H} \Phi,
$$

(5)

with the Hamiltonian

$$
\hat{H} = \frac{(\hat{p} - eA)^2}{2m} a + m b + eA^0 \mathbb{I}.
$$

(6)

Here $p$ designates the derivative operator $-i\partial/\partial q$, and $a$ and $b$ are the matrices

$$
a = (\sigma_3 + i\sigma_2) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad b = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(7)

In the free case, (6) has the solutions

$$
\Phi(q,t) = u^{(\pm)}(p) e^{\mp i(E_p t - p \cdot q)},
$$

(8)

with

$$
u^{(\pm)}(p) = \frac{1}{2m \sqrt{E_p}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \pm E_p/m \\ 1 \mp E_p/m \end{pmatrix},
$$

(9)

representing particles of positive and negative energy, respectively.

Applying phase-space calculus [2], the equation of motion for $P$ can be written as a differential equation in phase space,

$$
i \left( \frac{\partial}{\partial t} + eE(q) \cdot \frac{\partial}{\partial p} + \ldots \right) P(q,p)
$$

3
\[
= i \left[ -\frac{p}{2m} \cdot \frac{\partial}{\partial q} + e \left( B(q) \times \frac{p}{m} \right) \cdot \frac{\partial}{\partial p} + \ldots \right] \left( a \cdot P(q,p) + P(q,p) \cdot a^+ \right)
+ \frac{1}{m} \left( p^2 - \frac{1}{4} \frac{\partial^2}{\partial q^2} \right) \left( a \cdot P(q,p) - P(q,p) \cdot a^+ \right)
+ m \left( b \cdot P(q,p) - P(q,p) \cdot b \right),
\]

where the dots indicate quantum corrections involving higher derivatives of \( P \) and of the electric and magnetic field.

The first term on the right-hand side is a (nonrelativistic) flow term. The second and third terms incorporate relativistic effects (manifested by the appearance of \( p^2 \) and \( m^2 \)) and interferences on the Compton scale which are present in any Wigner function and result from the impossibility to measure both position and momentum of a quantum-mechanical particle at the same time. Since the equation of motion for the Wigner function is exact, it incorporates a mechanism to generate these interferences—namely the term \( \partial^2/\partial q^2 \)—which can be safely neglected in the semiclassical limit, i.e., when the spatial variation of the Wigner function is small on the Compton scale. Note that we do not have to make any assumption about the variation in time of the Wigner function. This feature is special to the equal-time Wigner function. This can also be seen from the fact that, after expanding \( P(q,p) \) on a matrix basis (see [3] and section 2.2), \( p^2 \) appears only in the combination

\[
p^2 + m^2 - \frac{1}{4} \frac{\partial^2}{\partial q^2} \approx p^2 + m^2 = E_p^2.
\]

With this approximation, which is exact for a spatially homogeneous problem, eq. (10) contains both flow terms as well as local couplings involving \( m \) and \( p^2 \). In particular, the equations have nonstationary solutions even in the case of a spatially homogeneous Wigner function. These oscillations are due to the internal charge degree of freedom of the Wigner function.

### 2.2 Expansion in a diagonalizing basis

To analyze these local oscillations, we expand \( P \) into eigenoscillations. In the free case, two of the eigenoscillations have the frequencies \( \pm 2E_p \), while two others have zero eigenfrequency and correspond to constant solutions of the equations of motion. The latter can be interpreted as semiclassical
quantities—which later may serve to define the physical phase-space densities of particles and antiparticles—while the former represent interferences.

Expanding into eigenvectors of the local oscillations,

\[ P(q,p) = \sum_{i=1}^{4} f_i \hat{P}_i, \]

with the \( p \)-dependent basis

\[ \hat{P}_1 = \frac{1}{4} \left( \begin{array}{cc} m/E_p + E_p/m & m/E_p - E_p/m \\ m/E_p - E_p/m & m/E_p + E_p/m \end{array} \right) \]

\[ = \frac{1}{2} \left( u^{(+)} \otimes u^{(+)} + u^{(-)} \otimes u^{(-)} \right), \]  

\[ \hat{P}_2 = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \]

\[ = \frac{1}{2} \left( u^{(+)} \otimes u^{(+)} - u^{(-)} \otimes u^{(-)} \right), \]  

\[ \hat{P}_{3/4} = \frac{1}{4} \left( \begin{array}{cc} m/E_p - E_p/m & m/E_p + E_p/m \pm 1/2 \\ m/E_p - E_p/m & m/E_p + E_p/m \mp 1/2 \end{array} \right) \]

\[ = u^{(\pm)} \otimes u^{(\pm)}. \]  

the corresponding semiclassical equations of motion for \( f \) are

\[ \frac{\partial}{\partial t} + e \mathcal{E}(q) \cdot \frac{\partial}{\partial p} f_1 = -\frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_2 + \frac{\mathcal{E}(q) \cdot p}{E_p^2} (f_3 + f_4), \]  

\[ \frac{\partial}{\partial t} + e \mathcal{E}(q) \cdot \frac{\partial}{\partial p} f_2 = -\frac{p}{E_p} \cdot \frac{\partial}{\partial q} (f_1 + f_3 + f_4), \]  

\[ \frac{\partial}{\partial t} + e \mathcal{E}(q) \cdot \frac{\partial}{\partial p} f_3 = -\frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_2 \]

\[ + \frac{\mathcal{E}(q) \cdot p f_1}{E_p^2} + 2iE_p f_3, \]  

\[ \frac{\partial}{\partial t} + e \mathcal{E}(q) \cdot \frac{\partial}{\partial p} f_4 = -\frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_2 \]

\[ + \frac{\mathcal{E}(q) \cdot p f_1}{E_p^2} - 2iE_p f_4. \]  

Here the approximation of eq. (11) has been made, and the higher-order quantum corrections to the electromagnetic interaction have been neglected.
The flow term now has the correct relativistic form. The self-coupling terms that give rise to local oscillations are isolated in the third and fourth equation (which are complex conjugate to each other), rendering $f_3$ and $f_4$ oscillatory even in the case of a homogeneous Wigner function. The expansion into $u^{(\pm)}$ shows that the nonoscillating terms do not mix positive- and negative-energy solutions, while the oscillating ones do. The latter oscillations are therefore caused by Zitterbewegung interferences between particles and antiparticles and would not be present in a nonrelativistic formulation.

The phase-space densities of observables can be expressed in these variables:

\[
\begin{align*}
\text{charge:} & \quad ef_2, \\
\text{energy:} & \quad E_p f_1, \\
\text{current:} & \quad \frac{e p}{E_p} (f_1 + f_3 + f_4), \\
\text{momentum:} & \quad -p f_2 + i p (f_3 - f_4).
\end{align*}
\]

Note that the current involves the highly-oscillatory components $f_3$ and $f_4$. It couples to the electromagnetic field and thus provides a mechanism for the emission of photons in particle annihilation.

A new self-coupling term with the coefficient

\[
\frac{e \mathcal{E}(q) \cdot p}{E_p^2}
\]

has appeared. This term is due to the fact that the basis (13) depends on the coordinate $p$ and therefore generates an additional term when acted upon by a force term. The physical reason for this term is that the local oscillations $f_3$ and $f_4$ have different “directions” in the $f$-space at different momenta of the particle. When a particle changes its momentum because of an external force, its local oscillations will be slightly out of phase and arouse the $f_1$-component, which can be interpreted as pair creation. Inspection of (13) reveals that the basis changes most at low momenta, and thus the coupling term is maximal at $p \ll E$. Physically this means that the pair creation happens at approximately zero momentum of the particles.

The equal-time Wigner function is, by definition, not Lorentz-invariant. In another reference frame, the Wigner function involves the expectation values of the fields at different times. It is not clear that the new function can
be constructed from the Wigner function in the original frame of reference since phase information may be lost in the equal-time Wigner transformation. Moreover, since our description is valid in any inertial system, it leads to the apparent paradox that pair creation happens at approximately zero momentum in any frame of reference. A similar paradox is discussed in [9].

The understanding of these equations can be further facilitated by considering the linear combinations

\[ f_+ = \frac{1}{2}(f_1 + f_2), \quad f_1 = f_+ + f_-, \]
\[ f_- = \frac{1}{2}(f_1 - f_2), \quad f_2 = f_+ - f-, \] (22)

which give the phase-space densities of particles \( f_+ \) and antiparticles \( f_- \). Rewriting the above equations yields

\[
\left( \frac{\partial}{\partial t} + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} + \frac{p}{E_p} \cdot \frac{\partial}{\partial q} \right) f_+ = -\frac{p}{2E_p} \cdot \frac{\partial}{\partial q}(f_3 + f_4) \\
+ e\frac{\mathcal{E} \cdot p}{2E_p^2} (f_3 + f_4),
\]

\[
\left( \frac{\partial}{\partial t} + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} - \frac{p}{E_p} \cdot \frac{\partial}{\partial q} \right) f_- = \frac{p}{2E_p} \cdot \frac{\partial}{\partial q}(f_3 + f_4) \\
- e\frac{\mathcal{E} \cdot p}{2E_p^2} (f_3 + f_4),
\]

\[
\left( \frac{\partial}{\partial t} + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} \right) f_3 = -\frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_2 \\
+ e\mathcal{E} \cdot \frac{p}{E_p^2} \frac{f_1}{2} + 2E_p f_3,
\]

\[
\left( \frac{\partial}{\partial t} + e\mathcal{E}(q) \cdot \frac{\partial}{\partial p} \right) f_4 = -\frac{1}{2} \frac{p}{E_p} \cdot \frac{\partial}{\partial q} f_2 \\
+ e\mathcal{E} \cdot \frac{p}{E_p^2} \frac{f_1}{2} - 2iE_p f_4, \] (23)

which exhibits clearly a flow term for \( f_+ \) and \( f_- \) on the left-hand side. The components \( f_3 \) and \( f_4 \) do not have a flow term for position space since the relevant interferences are not localized. The right-hand side contains quantum effects and, in particular, pair creation. In the nonrelativistic limit, \( f_+ \) and \( f_- \) change on a time scale much larger than the frequency of \( f_3 \) and
\( f_4 \), so that the right-hand sides average out and the equations describe a relativistic free gas of positively and negatively charged particles. Note that \( f_-(p) \) refers to antiparticles of momentum \(-p\) which is consistent with the usual hole interpretation.

### 3 Pair creation

#### 3.1 Reduction to a variable-frequency oscillator

Let us now investigate the case of a homogeneous constant field, that is, the Schwinger mechanism of spontaneous pair creation. Because of the spatial homogeneity, the approximations on local interferences made above in eq. (10) hold exactly. We shall thus be able to derive exact equations for the Wigner function in this case.

We first remove the force terms from the equations of motions by separating out the classical motion using the method of characteristics. Consider a new function \( \tilde{f}(p_0, t) \) defined by

\[
\tilde{f}(p_0, t) = \int dp_0 \tilde{f}(p_0, t) \delta(p - p(p_0, t)),
\]

(24)

where the function \( p(p_0, t) \) describes the classical motion of a particle with initial momentum \( p_0 \),

\[
p(p_0, t) = p_0 + \int_{-\infty}^{t} dt' e\mathcal{E}(t).
\]

(25)

The phase-space evolution of \( f(p, t) \) translates into

\[
\left( \frac{\partial}{\partial t} + e\mathcal{E} \cdot \frac{\partial}{\partial q} \right) f(p, t) = \int dp_0 \frac{\partial \tilde{f}(p_0, t)}{\partial t} \delta(p - p(p_0, t)).
\]

(26)

In this way, the momentum derivative is absorbed into the classical evolution. Here \( \tilde{f}(p_0, t) \) represents the fate of a phase-space element that started at \( p = p_0 \) and moves in phase space according to Newton’s law. It may therefore be called a test particle whose inner degrees of freedom in the Feshbach-Villars sense are governed by

\[
\frac{\partial \tilde{f}_1}{\partial t} = \frac{e\mathcal{E} \cdot p}{E_p^2} (\tilde{f}_3 + \tilde{f}_4),
\]

(27)
\[
\frac{\partial \tilde{f}_2}{\partial t} = 0, \\
\frac{\partial \tilde{f}_3}{\partial t} = \frac{e \mathbf{E} \cdot \mathbf{p}}{E_p^2} \tilde{f}_1 + 2iE_p \tilde{f}_3, \tag{29}
\]
\[
\frac{\partial \tilde{f}_4}{\partial t} = \frac{e \mathbf{E} \cdot \mathbf{p}}{E_p^2} \tilde{f}_1 - 2iE_p \tilde{f}_4, \tag{30}
\]

where \( p = p(p_0, t) \) is now time-dependent. The set of partial differential equations (13) is thus reduced to a set of ordinary differential equations. Instead of solving the flow equation in phase space directly, we have in this way introduced a ‘test particle’, moving in phase space according to Newton’s law and possessing an internal degree of freedom governed by (27)-(30). Accordingly, the equations must be solved for all possible initial values \( p_0 \). The function \( \tilde{f}(p_0, t) \) thus computed can then be assembled using (24) to give the actual solution of the system of partial differential equations (13).

Due to spatial homogeneity, the charge density vanishes. Initially, only \( \tilde{f}_1 \) will be nonzero, but by the action of the electric field, the interference components \( \tilde{f}_3 \) and \( \tilde{f}_4 \) are aroused which in turn couple back to \( \tilde{f}_1 \). The coupling

\[
\frac{e \mathbf{E} \cdot \mathbf{p}(p_0, t)}{E_p^2} \tag{31}
\]

vanishes as \( p \to \infty \). At large times, \( \tilde{f}_3 \) and \( \tilde{f}_4 \) will therefore settle down into steady oscillations of increasing frequency and decouple from \( \tilde{f}_1 \).

The set of ordinary differential equations (27) is equivalent to a single differential equation in a new variable \( \xi(t) \),

\[
\ddot{\xi} + E_p^2(t) \xi = 0, \tag{32}
\]

through the substitution

\[
\tilde{f}_1 = \frac{E_p}{2E_p^{(0)}} \left( |\xi|^2 + \frac{1}{E_p^2} |\dot{\xi}|^2 \right), \tag{33}
\]
\[
\tilde{f}_3 + \tilde{f}_4 = \frac{E_p}{2E_p^{(0)}} \left( |\xi|^2 - \frac{1}{E_p^2} |\dot{\xi}|^2 \right), \tag{34}
\]
\[
i(\tilde{f}_3 - \tilde{f}_4) = \frac{1}{2E_p^{(0)}} (\xi \dot{\xi}^* + \dot{\xi}^* \xi), \tag{35}
\]
where dots denote derivatives with respect to $t$, and $E_p^{(0)}$ is the test-particle energy at some fixed reference time. The relationship between pair creation in a homogeneous field and oscillators of time-dependent frequency has been noticed much earlier from the wave-function approach to pair creation [10, 11, 12, 13]. By using the solutions given in [12], we will be able to give an approximate asymptotic solution for the Wigner function.

### 3.2 Approximate solution

The general solution of the variable frequency oscillator (32) has the form

$$
\xi(t) = \frac{1}{\sqrt{E_p(t)}} C_1 e^{-i \tau} + \frac{1}{\sqrt{E_p(t)}} C_2 e^{i \tau},
$$

where $C_1$ and $C_2$ are slowly-varying functions, and $\tau(t)$ is defined by

$$
\tau(t) = \int_{t_1}^{t} dt' E_p(t'),
$$

with an arbitrarily chosen reference point $t_1$. Making the Liouville substitution

$$
\xi(t) = \frac{1}{\sqrt{E_p}} \chi(\tau(t)),
$$

one obtains the oscillator equation in $\chi(\tau)$

$$
\chi'' + (1 + h(\tau)) \chi = 0,
$$

where primes denote derivatives with respect to $\tau$, and the function $h(\tau)$ is

$$
h(\tau) = \frac{3}{4} \left( \frac{\dot{E}_p}{E_p^2} \right)^2 - \frac{1}{2} \frac{\dddot{E}_p}{E_p^3}.
$$

Equation (38) is the time-dependent Schrödinger equation of a particle in a potential $h(\tau)$. The shape of the potential is determined by $E_p(\tau)$ and thus by the external forces. In particular, $h(\tau)$ vanishes for times $\tau$ when $E_p(\tau)$ is constant, i.e., when the external field is switched off. Even in a constant electric field, $E_p \sim \tau$ for $t \to \pm \infty$, $\tau \sim t^2$ and thus $h(\tau) \sim \tau^{-2}$. The potential is therefore localized, and the problem is equivalent to a barrier-penetration
problem in quantum mechanics. Note that for a constant electric field $\mathcal{E}$, the barrier potential is

$$h(\tau) = \frac{3}{4} \left( \frac{e \mathcal{E} \cdot p}{E_p^2} \right)^2,$$

(41)

in which the quantity $[21]$ resurfaces.

The coefficients $C_1$ and $C_2$ represent the incoming and the reflected waves, respectively. They are asymptotically constant, and the process is characterized by a barrier penetration coefficient (in addition to a phase shift)

$$\rho_0 = \left| \frac{C_2}{C_1} \right|^2, \quad t \to \pm \infty,$$

(42)

the index 0 indicating a transition from the vacuum value. Since this problem is similar to the problem of barrier penetration in ordinary quantum mechanics, the calculation of this parameter is accomplished by well-known methods. The quantity $\rho_0$ will depend on the initial momentum and on the electric field.

Substituting eq. (36) and $\tau = E_p t$ into the relations (33-35), we obtain

$$\tilde{f}_1 = |C_1|^2 + |C_2|^2 = |C_1|^2(1 + \rho_0),$$

(43)

$$\tilde{f}_3 = 2 C_1^* C_2 e^{2iE_p t},$$

(44)

$$\tilde{f}_4 = 2 C_2^* C_1 e^{-2iE_p t}.$$  

(45)

For the vacuum, we choose

$$C_1 = 1, \quad C_2 = 0.$$  

(46)

The relationship between the quantity $\rho$ and the number of particles can be obtained by considering that, due to the reality of the oscillator equation, the form $\xi^* \dot{\xi} - \dot{\xi} \xi^*$ is a constant,

$$\xi^* \dot{\xi} - \dot{\xi} \xi^* = -2iE_p (|C_1|^2 + |C_2|^2) = \text{const} = -2iE_p^{(0)},$$

(47)

where at $t \to -\infty$, $C_1 = 1$ and $C_2 = 0$ have been chosen. From $\rho = |C_2/C_1|^2$, it now follows that

$$|C_1|^2 = \frac{1}{1 - \rho E_p(t)} E_p^{(0)}, \quad |C_2|^2 = \rho \frac{1}{1 - \rho E_p(t)} E_p^{(0)}.$$  

(48)
In particular, we get

\[ |C_1|^2 + |C_2|^2 = \frac{E_p^{(0)}}{E_p(t)} \frac{1 + \rho}{1 - \rho}, \tag{49} \]

which is related to the resulting current density. Substituting (36) into (33), it follows that

\begin{align*}
\tilde{f}_1 &= \frac{E_p}{E_p^{(0)}} \left( |C_1|^2 + |C_2|^2 \right) = \frac{1 + \rho}{1 - \rho}, \tag{50} \\
\tilde{f}_3 + \tilde{f}_4 &= \frac{E_p}{2E_p^{(0)}} \left( C_1^* C_2 e^{-2i\tau} + \text{c. c.} \right), \tag{51} \\
i(\tilde{f}_3 - \tilde{f}_4) &= \frac{E_p}{4E_p^{(0)}} \left( |C_1|^2 + |C_2|^2 - C_1^* C_2 e^{-2i\tau} - \text{c. c.} \right). \tag{52}
\end{align*}

As was to be expected, \( \tilde{f}_1 \) is asymptotically constant, while the other two components oscillate because of interferences. The current density in phase space is

\[ j(q, p) = \frac{e_p}{E_p} (\tilde{f}_1 + \tilde{f}_3 + \tilde{f}_4). \tag{53} \]

The components \( \tilde{f}_3 \) and \( \tilde{f}_4 \) will give rise to fast oscillations. Since the current couples to the electromagnetic field, this may cause the emission of photons (or, in the semiclassical approach, electromagnetic waves of frequency \( 2E_p \)). For our purposes with coupling to a classical electromagnetic field, the oscillations tend to cancel out upon measurement, and the main contribution to the current density in phase space comes from

\[ j(q, p) \approx \frac{e_p}{E_p} \tilde{f}_1 = \frac{e_p}{E_p} \left( 1 + \frac{2\rho}{1 - \rho} \right). \tag{54} \]

The vacuum value of this quantity is \( e_p/E_p \); it represents the charge of the Dirac sea and must be subtracted to yield a measurable quantity. Thus the phase-space density of pairs produced is given by

\[ n = \frac{\rho}{1 - \rho}, \tag{55} \]

in full correspondence with eq. (3.7) of [12].
This yields the following picture of pair creation: Let us assume a constant field $E$ switched on at some time $t_0$. For each initial momentum $p_0$, we have at $t \to -\infty$ $C_1 = 0$, $C_2 = 0$, and thus $\tilde{f}_1 = 1$, $\tilde{f}_3 = \tilde{f}_4 = 0$. When the field is switched on, the particle is accelerated or decelerated. In the latter case it reaches its minimum velocity at some time, corresponding to the barrier in the equivalent quantum mechanical problem. In this moment, $C_1$ and $C_2$ change so that $\tilde{f}_1$, $\tilde{f}_3$, and $\tilde{f}_4$ acquire the values (50)-(52). This transition takes place when $p(p_0, t)$ is minimal, i.e., when its parallel part $p_{\parallel}$ vanishes. Looking at the phase-space distribution $f(p, t)$ we therefore see the characteristic trough found in [1] and which signals the impending pair production at a constant rate per unit space and time: The phase-space density changes at $p_{\parallel} = 0$ which creates the left end of the trough; the change is then transported away by the electric field, and the edge that appeared when the field was switched on constitutes the right end of the trough.

When the field is not constant in time, it may happen that a test particle reaches $p_{\parallel} = 0$ more than once. We then can observe the effects of Bose enhancement (or Pauli blocking for fermions). In a very strong electric field with appreciable pair creation we can assume that the transition takes place during a very small time. This is comparable to solving the barrier penetration problem by gluing together plane-wave segments at the barrier with different amplitudes and phase shifts. The time evolution of $\xi(t)$ can then be thought of as being described by (56) with the coefficients $C_1(t)$ and $C_2(t)$ changing their value abruptly whenever $p(t)$ is small. Let us assume that any one of these transitions can be described by

$$e^{-iE_p t} \to A e^{-i(E_p t + \alpha)} + B e^{i(E_p t + \beta)},$$

where the amplitudes $A$, $B$ and the phases $\alpha$, $\beta$ depend on the field strength at the transition. If a general initial state is given by

$$e^{-iE_p t} + \sqrt{\rho} e^{i(E_p t + \delta)},$$

with some positive-state admixture characterized by $\rho$ and a phase shift $\delta$, this initial state will go into a new state

$$\left(\begin{array}{c} A e^{-i\alpha} + \sqrt{\rho_0} B e^{-i(\beta - \delta)} \end{array}\right) e^{-iE_p t}$$

$$+ \left(\begin{array}{c} B e^{i\beta} + \sqrt{\rho_0} A e^{i(\alpha + \delta)} \end{array}\right) e^{iE_p t}.$$
The new positive-state admixture $\rho'$, defined as the ratio of the square of the amplitudes in front of the oscillatory terms, is accordingly given by

$$\rho' = \frac{\rho + \rho_0 + 2\sqrt{\rho\rho_0} \cos(\alpha + \delta - \beta)}{1 + \rho_0\rho + 2\sqrt{\rho\rho_0} \cos(\alpha + \delta - \beta)} \approx \rho + \rho_0 + 2\sqrt{\rho_0\rho} \cos(\alpha + \delta - \beta), \quad (59)$$

where $\rho_0 = B^2/A^2$ is a characteristic quantity of the transition and will depend on the electric field strength at the time of the transition. In a nearly constant field, $\rho$ will be given by the Schwinger term. As we see, $\rho$ behaves as an additive quantity in the transitions, up to an interference term. The latter can be assumed to average out when different phase-space trajectories are considered.

Because of this, we can assume that the time evolution of the quantity $\rho(t)$ is given by

$$\rho(t) = \int_{-\infty}^{t} dt' \delta(p(p_0, t')) \rho_0(t'), \quad (60)$$

where $p(p_0, t')$ is the momentum of the test particle ($\rho(t)$ depends of course on the parameter $p_0$), and $\rho_0(t')$ is determined by the electric field at the time $t'$. As a result of this assumption, $\rho(t)$ will be a sum of all contributions coming from the individual transitions which occur whenever $p(p_0, t')$ is small. The approximation here lies in neglecting the actual transition process in which $\rho$ changes to $\rho + \rho_0$, since the duration of the transition can be assumed to be quite small in strong fields.

### 3.3 Constructing a flow equation with source term

We have thus derived an expression for the phase-space density of created pairs. Remembering that $\rho = \rho(p_0, t)$ we now consider the phase-space function

$$\tilde{n}(p_0, t) = \frac{\rho(p_0, t)}{1 - \rho(p_0, t)}, \quad (61)$$

where the tilde reminds us that the classical motion of the test particle has been separated out. To transform back to a true phase-space equation, we take the time-derivative of $\tilde{n}(p_0, t)$,

$$\frac{\partial \tilde{n}(p_0, t)}{\partial t} = \frac{1}{(1 - \rho(t))^2} \frac{\partial \rho}{\partial t} = (1 + \tilde{n}(p_0, t))^2 \frac{\partial \rho}{\partial t}. \quad (62)$$
Substituting the time evolution of $\rho(t)$, eq. (60), this is
\[
\frac{\partial \tilde{n}(p_0, t)}{\partial t} = |eE(t)| \delta(p(p_0, t)) \rho_0(t). \tag{63}
\]

Defining the true phase-space function $n(p, t)$ as in (24),
\[
n(p, t) = \int dp_0 \tilde{n}(p_0, t) \delta(p - p(p_0, t)), \tag{64}
\]
the phase-space equation of motion is
\[
\left( \frac{\partial}{\partial t} + eE \cdot \frac{\partial}{\partial q} \right) n(p, t) = (1 + n(p, t))^2 |eE(t)| \times \int dp_0 \rho_0(t) \delta(p(p_0, t)) \delta(p - p(p_0, t))
= (1 + n(p, t))^2 \rho_0(t) \delta(p). \tag{65}
\]
This is a phase-space flow equation with a Bose-enhanced production term on the right-hand side. The quantity $\rho_0(t)$, characterizing the pair-production rate, can be computed either perturbatively (pair production by oscillating fields) or nonperturbatively (Schwinger term), or by some combination, yielding an enhanced estimate of the pair-production rate.

It has been found earlier [5] in the study of the back-reaction problem that pair creation in a spatially homogeneous problem can be surprisingly well described by the source term
\[
(1 + 2n(p, t)) \frac{|eE|}{2\pi} \ln \left[ 1 + \exp \left( -\frac{\pi m^2}{|eE|} \right) \right] \delta(p)
\approx (1 + 2n(p, t)) \frac{|eE|}{2\pi} \exp \left( -\frac{\pi m^2}{|eE|} \right) \delta(p), \tag{66}
\]
the difference in the Bose enhancement factors in Eqs. (65) and (66) stemming from the lack of a mechanism in the former that annihilates pairs in favor of an augmentation of the electric field.

### 4 Summary

We started from the equation of motion for the Wigner function. This equation of motion is a direct consequence of the field equations. We have then
derived an approximate solution in terms of a transport equation involving a Schwinger pair creation and a Bose enhancement term. The Wigner function exhibits an internal structure causing local oscillations, i.e., nonstationary solutions in a spatially homogeneous situation. The form of these oscillations, corresponding to the internal charge degree of freedom of the Klein-Gordon field, are different at different momenta of the particles. Thus in the presence of an external force the oscillations mix with the physically observable phase-space densities of particles and antiparticles and give rise to particle creation. This process is described analogously to a barrier-penetration problem in quantum mechanics. The semiclassical approximation enables us to calculate the pair creation rate and to give an estimate for repeated pair creation involving Bose enhancement.

While it was known earlier that pair creation in an external field can be described by a variable-frequency oscillator from considering the wave functions, it is the advantage of the Wigner function that it allows one to cast this in a phase-space formulation. The use of the equal-time Wigner function is here especially advantageous over the split-time Wigner function since it allows us to exploit the time-dependent problem of the variable-frequency oscillator and thus the known solutions to it in a very natural way.

Some open questions remain: For spatially inhomogeneous fields, we have only shown how the homogeneous-field limit can be recovered. By making a systematic expansion in the field gradient, it should be possible to derive corrections to this result. We have also restricted ourselves to classical electromagnetic fields. In a next step, the electromagnetic field should be treated dynamically. In this approximation, emission of photons, pair annihilation into photons, and particle interaction enter. Finally, even in the semiclassical limit, the theory is, in general, in need of renormalization [1].

In its current form, the equal-time Wigner function is especially suitable for the explicit solution of semiclassical problems like particle creation in definite boundary conditions. The fact that it is not relativistically invariant makes it more difficult to derive generic features of quantum field theories from it. Still, a possible extension could be to consider general \( N \)-particle Wigner functions constructed from \( 2N \) field operators. The resulting hierarchy of equations can then be systematically studied.

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A Appendix: The Schwinger term, obtained by means of the imaginary time method

Since \((32)\) corresponds to the time-independent Schrödinger equation de-
scribing tunneling through a potential barrier, \(\rho\) is the reflection coefficient
for this problem and can be computed by a standard method, namely that
of imaginary times \([12, 13]\). In this case the solution is determined by the
singularities of \(h(\tau)\) in the complex plane: If \(h(\tau)\) has its singularity at \(\tau_0\),
the value of the barrier reflection coefficient is

\[
\rho = 4 \cos^2(\pi/2\alpha) \exp(-4 \text{Im} \tau_0),
\]

where \(\alpha\) is deduced from the behavior of \(E_p(t)\) near the singularity,

\[
\omega(t) \approx \omega_0(t - t_0)^{\alpha-1}.
\]

In the case of a constant field,

\[
E_p(t) = \sqrt{(p + e\mathcal{E}t)^2 + m^2},
\]

and the singularities are at \(E_p = 0\), corresponding to

\[
t_0 = \frac{-p|| \pm im_\perp}{e|\mathcal{E}|},
\]

yielding \(\alpha = 3/2\). To compute the corresponding value of \(\tau_0\), we choose
the integration contour in \((37)\) along the real axis from \(t_1\) to \(\text{Re}t_0\) and then
parallel to the imaginary axis to \(\text{Im}t_0\). No contribution to \(\text{Im} \tau_0\) arises along
the real axis, so the integral to do remains

\[
\text{Im} \tau_0 = \text{Im} \int_0^{\text{Im}t_0} \text{d}(\text{Im} t) \left. E_p(t) \right|_{\text{Re} t = \text{Re}t_0}.
\]
Writing \( x = \text{Im} t_0 \), this is

\[
\text{Im} \tau_0 = \Re \int_0^{\text{Im} t_0} dx \, i e |\mathcal{E}| \times \left[ x^2 + 2i \left( \Re t_0 + \frac{p_\parallel}{e|\mathcal{E}|} \right) - \frac{m^2}{e^2|\mathcal{E}|^2} \right]^{1/2} = \mp \pi \frac{m^2}{4e|\mathcal{E}|},
\]

(72)

The barrier-reflection coefficient is therefore

\[
\rho = \exp \left( -\pi \frac{m^2}{e|\mathcal{E}|} \right),
\]

(73)

which agrees with the lowest-order term of the Schwinger formula.

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