ABSTRACT POLYNOMIAL PROCESSES

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Abstract. We suggest a novel approach to polynomial processes solely based on a polynomial action operator. With this approach, we can analyse such processes on general state spaces, going far beyond Banach spaces. Moreover, we can be very flexible in the definition of what ‘polynomial’ means. We show that ‘polynomial process’ universally means ‘affine drift’. Simple assumptions on the polynomial action operators lead to stronger characterisations on the polynomial class of processes.

In our framework we do not need to specify polynomials explicitly but can work with a general sequence of graded vector spaces of functions on the state space. Elements of these graded vector spaces form the monomials by introducing a sequence of vector space complements. The basic tool of our analysis is the polynomial action operator, which is a semigroup of operators mapping conditional expected values of monomials acting on the polynomial process to monomials of the same or lower grade. Unlike the classical Euclidean case, the polynomial action operator may not form a finite-dimensional subspace after a finite iteration, a property we call locally finite. We study abstract polynomial processes under both algebraic and topological assumptions on the polynomial actions, and establish an affine drift structure. Moreover, we characterize the covariance structure under similar but slightly stronger conditions. A crucial part in our analysis is the use of the (algebraic or topological) dual of the monomials of grade one, which serves as a linearization of the state space of the polynomial process. Our general framework covers polynomial processes with values in Banach spaces recently studied by Cuchiero and Svaluto-Ferro [8].

1. Introduction

A polynomial process \( X \) is characterized by the fact that for any polynomial \( p \) of degree \( m \) there exists another polynomial \( q \) of degree at most \( m \) such that \( \mathbb{E}[p(X(s+h))|\mathcal{F}_s] = q(X_s), \ s, h \geq 0 \). Polynomial processes have been studied on different state spaces as for instance \( \mathbb{R}^d \) or subsets thereof. At the beginning of a study of polynomial processes stands the specification of a space of functions which are considered polynomials. For the real-valued setting for instance, the polynomials up to degree \( m \) are defined by

\[
\mathcal{P}_m := \{ x \mapsto \sum_{k=0}^{m} a_k x^k \}
\]
and form an $m$-dimensional vector space and the space of all polynomials is $\mathcal{P} := \bigcup_{m=0}^{\infty} \mathcal{P}_m$. It is equipped with a degree function $\deg : \mathcal{P} \to \mathbb{N}$ that naturally assigns to each polynomial its degree.

Mimicking such a nested vector space structure, in this paper, we start abstractly with a set of states $E$ (state space) and a sub-vector space $\mathcal{P}$ of the set of functions from $E$ to a field $\mathbb{F}$ together with a degree function $\deg : \mathcal{P} \to \mathbb{N}$. This degree function then defines a gradation sequence $\mathcal{P}_1 \subset \mathcal{P}_2 \ldots$ of sub-vector spaces $\mathcal{P}_n := \{p \in \mathcal{P} : \deg(p) \leq n\}$ and a decomposition into a direct sum $\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{M}_n$ where $\mathcal{M}_n$ is the vector space complement of $\mathcal{P}_{n-1}$ in $\mathcal{P}_n$, and contains the monomials of degree $m$. We then call an $E$-valued process $X(t)_{t \geq 0}$ polynomial if for each function $p \in \mathcal{P}_n$ there exists a function $q \in \mathcal{P}_n$ such that $\mathbb{E}[p(X(s+h))|\mathcal{F}_s] = q(X_s)$, defining a family of semigroups which we call polynomial action operator of the polynomial process.

Our abstract setting allows several major advantages compared to the classical theory: By encoding the polynomial property in the polynomial action operator, we can analyse the processes on general state spaces, going far beyond Banach state spaces. Indeed, set $E$ does not need to be a subset of a topological space but can just be any set to start with. Moreover, we can be very flexible in the definition of what ‘polynomial’ means, as we can include functions which are usually not considered polynomials but still generate an invariant class under conditional expectation. In this abstract setting, we are able to show that ‘polynomial process’ universally means ‘affine drift’. Simple additional assumptions on the polynomial action operators lead to stronger characterisations on the polynomial class of processes, including an understanding of the quadratic covariation.

Related literature: Affine processes, which play a prominent role in finance and economics, have first been analysed systematically in Duffie, Filipović and Schachermayer [9]. Polynomial processes offer a generalization of affine processes — short the additional requirement that they assume finite absolute power moments of any order — and were first introduced in Cuchiero [5]. Since then, polynomial processes received a lot of attention and they have been studied on different state spaces including $\mathbb{R}^n$ and subsets thereof (see Filipović and Larsson [11, 12], for instance). They have also found many applications in financial and insurance mathematics (see Ackerer, Filipović and Pulido [1], Benth and Lavagnini [3], Biagini and Zhang [4], Cuchiero, Keller-Ressel and Teichmann [5], Filipović [10], Filipović, Larsson and Pulido [13], Kleisinger-Yu et al. [15] and Ware [21], for instance). We also would like to mention the linear rational term structure models proposed in Filipović, Larsson and Trolle [14]. Here, the diffusion model has an affine structure of the drift, something we recover under rather mild conditions for our general polynomial processes.

Recently, an infinite-dimensional extension of polynomial processes has been proposed in Cuchiero and Svaluto-Ferro [8] and Benth, Detering and Krühner [2], where, in the latter article, multi-linear maps have been used as the replacement for classical monomials. In particular, in an infinite-dimensional setting some flexibility arises when specifying the class of polynomials. As each of these specifications gives rise to a nested vector space structure considered here, our general analysis is particularly valuable in an infinite-dimensional setting.

Main results and discussion on Cuchiero and Svaluto-Ferro’s paper ‘Infinite-dimensional polynomial processes’ [8]: The paper [8] by Cuchiero and Svaluto-Ferro
has overlap with our paper and we like to discuss their results and differences to the present paper from our perspective. Cuchiero and Svaluto-Ferro [8] consider as space of polynomials the algebra generated by a subset of continuous linear functionals on a Banach space, while we understand by polynomials just any given graded vector space of functions. Recall that classical polynomials as well as the algebra generated by a subset of linear functions has a natural inherent graded structure which is compatible with pointwise multiplication, thus the polynomials considered in [8] are naturally a graded algebra.

Further, in [8] polynomial processes with values in a Banach space are described from the martingale problem perspective, i.e. a polynomial process is described via the action of its generator on polynomials. We do not assume any linear structure for the state space of the polynomial process and start from the above mentioned invariance condition for conditional expectations, i.e. we assume that conditional expectations of polynomials are polynomials again. Cuchiero and Svaluto-Ferro [8] prove that under some technical condition this invariance also holds in their setup while we give natural examples in our setup where the generator description breaks down, cf. Example 6.9 where the constant part of the drift points out of the given Hilbert space while other objects are as regular as one would expect in a stochastic partial differential equation-description for polynomial processes. Allowing polynomial processes to live on non-linear structures has the advantage when studying them, for instance, on Lie-groups where they have a natural interpretation as well — recall that Lévy processes have been studied on Lie groups, cf. Ming [18].

The main results of Cuchiero and Svaluto-Ferro [8, Theorems 3.4 and 3.8] are the dual and bidual moment formulas. The dual moment formula describes the time-evolution of $E[p(X_t)]$ in terms of the generator where $p$ is a polynomial and $X$ a polynomial process. We recover this formula in our setting under some algebraic (Proposition 3.6) and analytic (Lemma 7.4) conditions. An advantage of these formulas in our setup is that polynomials up to order $m$ can be reinterpreted as polynomials of order 1 which allows to apply formulas for first order polynomials to higher order polynomials. The bidual moment formula in Cuchiero and Svaluto-Ferro [8] describes the time evolution of $E[(X_t \otimes^n)_{n=1,\ldots,k}]$, that is, of the expectation of the $k$-th tensor-power of $X$, which is in some situations easier to handle than the specific moments $E[p(X_t)]$ for a given polynomial. In many cases, $E[p(X_t)]$ can be recovered from the expectation of the tensor powers. We divert from this approach and provide instead a drift-martingale-type decomposition of a polynomial process, cf. Theorem 4.5 and Theorem 6.6 which allows to recover $E[p(X_t)]$ directly by applying expectations. Again, by reinterpreting higher order polynomials as first order polynomials and reapplying our results we recover the expectations of the higher tensor powers in the case when we mean by polynomials the algebra generated by linear functionals. If we additionally assume a multiplicative structure on the polynomials, then we can additionally recover the quadratic covariation of the given process, cf. Theorem 5.3 and Theorem 7.5.

1.1. Notations. We denote by $\mathbb{F}$ either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. $\mathbb{N}$ resp. $\mathbb{R}_+$ denote the set of integers resp. real numbers which are greater or equal zero. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ denotes a filtered probability space. For an $\mathbb{F}$-valued function $f$ we denote by $\overline{f}$ its complex conjugate. We will use the letter $I$ for the identity operator on any $\mathbb{F}$-vector space $V$. If $V$ is a vector space over $\mathbb{F}$, then by a semigroup (resp. group) of operators we mean a family $(T_h)_{h \geq 0}$ (resp.
(\(T_h\)) of linear operators on \(V\) with \(T_0 = I\) is the identity on \(V\) and \(T_hT_k = T_{h+k}\) for any \(h, k \geq 0\) (resp. \(h, k \in \mathbb{F}\)). For a set \(A\) we denote by \(1_A\) the functional class which maps any element inside \(A\) to 1 and any other element to 0.

2. Polynomials, Polynomial Actions and Polynomial Processes

In this section we introduce abstract polynomial processes with values on a general non-empty set \(E\). We first recall the classical \(\mathbb{R}^d\)-valued idea behind polynomial processes (see Cuchiero [5]). From this idea a polynomial process \(X\) is a càdlàg process such that for any 'polynomial' \(p\) and any \(h \geq 0\) there is a polynomial \(q\) with degree less or equal than the degree of \(p\) such that for any \(t \geq 0\)
\[\mathbb{E}[p(X_{t+h}|\mathcal{F}_t)] = q(X_t).\]

For this, we need to define what we mean by 'polynomials'. This does, of course, induce an action \(T_h\) on the set of polynomials which with \(h\) fixed produces a polynomial \(q = T_h p\) out of a polynomial \(p\). For this to work well with conditional expectations, this action should be

1. linear,
2. have the semigroup property \(T_h T_k = T_{h+k}\) for \(h, k \geq 0\) and
3. \(h \mapsto T_h p\) be right-continuous.

These properties are motivated by the linearity, the tower property and continuity-properties of the conditional expectation, respectively.

In order to make sense out of the above remark in an abstract setting with states of the process \(X\) in a general set \(E\), we need to:

a) Introduce what we mean by polynomials.
b) Define operators \(T_h\), \(h \geq 0\) on the polynomials with \(T_h T_k = T_{h+k}\) for \(h, k \geq 0\).
c) Say what we mean by \(E\)-valued polynomial processes.

We continue with formalising our ideas. In the rest of the paper, we reserve the notation \(E\) for the state space of our polynomial processes to be defined, and \(\mathcal{F}(E, \mathbb{F})\) is the set of functions from \(E\) to \(\mathbb{F}\).

**Definition 2.1.** By polynomials we mean a sub-vector space \(P \subseteq \mathcal{F}(E, \mathbb{F})\) containing all constant functions, together with a degree-function
\[\text{deg} : P \to \mathbb{N}\]
such that \(\text{deg}(p+q) \leq \max\{\text{deg}(p), \text{deg}(q)\}\) and \(\text{deg}(\lambda p) = \text{deg}(p)\) for any \(p, q \in P\), \(\lambda \in \mathbb{F}\setminus\{0\}\) and \(\text{deg}(p) = 0\) if and only if \(p\) is a constant function.

Note that we use the convention \(\text{deg}(0) = 0\) in the definition above. Allowing for \(\mathbb{C}\)-valued polynomials rather than \(\mathbb{R}\)-valued polynomials does not extend the theory but in some examples it may simplify matters.

We introduce the subset of polynomials of order \(n\) as
\[P_n := \{p \in P : \text{deg}(p) \leq n\}\]
and we fix a vector space complement \(\mathcal{L}\) of the constant functions \(P_0\) in \(P_1\) which we call linear functions. We note that this set can be viewed as a realisation of monomials of order 1, and will play a crucial role in the discussions to come regarding the drift of polynomial processes. We state an example being important for our analysis.
Example 2.2. Fix \( x_0 \in E \). Define
\[
L := \{ p \in \mathcal{P}_1 : p(x_0) = 0 \}.
\]
Then \( L \) is a vector space complement of \( \mathcal{P}_0 \) in \( \mathcal{P}_1 \). This shows that these vector space complements always exist. However, sometimes it is more convenient to work with a different set \( L \), as the one we state here. To illustrate this point let \( E = [1, 2]^d \) and \( \mathcal{P} \) be the set of classical \( d \)-variable polynomials with values in \( \mathbb{R} \) restricted to \( E \). The linear functions \( L \) on \( \mathbb{R}^d \) restricted to \( E \) is a natural choice of complement to the constant polynomials but it is not given via the above construction.

Next, we define polynomial actions:

Definition 2.3. A polynomial action is a family \( (T_h)_{h \geq 0} \) of linear operators on \( \mathcal{P} \) with
\[
\begin{align*}
(1) & \quad \deg(T_hp) \leq \deg(p) \text{ for any } p \in \mathcal{P}, \ h \geq 0, \\
(2) & \quad T_0p = p \text{ for any } p \in \mathcal{P}, \\
(3) & \quad T_hT_k = T_{h+k} \text{ for any } h, k \geq 0 \text{ and} \\
(4) & \quad \mathbb{R}_+ \to \mathbb{C}, h \mapsto (T_hp)(x) \text{ is right-continuous for any } p \in \mathcal{P}, \ x \in E.
\end{align*}
\]
The generator \( \mathcal{G} \) of \( T \) is defined as
\[
\mathcal{G}p : D \to \mathbb{R}, x \mapsto \lim_{h \searrow 0} \frac{T_hp(x) - p(x)}{h}
\]
where
\[
D := \left\{ p \in \mathcal{P} : \exists q \in \mathcal{P} \text{ s.t. } \lim_{h \searrow 0} \frac{T_hp(x) - p(x)}{h} = q(x) \ \forall x \in E \right\}
\]
and we call \( D \) the domain of \( \mathcal{G} \).

Finally, we are ready to introduce what we shall mean by polynomial processes in this paper:

Definition 2.4. A function \( X : \Omega \times \mathbb{R}_+ \to E \) is called an \( E \)-valued \( \mathcal{P} \)-polynomial process with action \( T \) if for any \( p \in \mathcal{P} \):
\[
\begin{align*}
(1) & \quad \mathbb{E}[|p(X_t)|] < \infty, \ t \geq 0, \\
(2) & \quad \mathbb{E}[|p(X_{t+h})|\mathcal{F}_t] = T_hp(X_t), \ t, h \geq 0, \\
(3) & \quad \mathbb{R}_+ \to \mathcal{F}, t \mapsto (p(X_t)) \text{ has càdlàg paths and} \\
(4) & \quad p(X_t) \text{ is } \mathcal{F}_t\text{-measurable for any } t \geq 0.
\end{align*}
\]
If there is no ambiguity, then we sometimes simply refer to \( X \) as a polynomial process.

The càdlàg-property (3) in the definition of polynomial processes above implies that \( p(X) \) is càdlàg for any \( p \in L \) and, hence, we may say that \( X \) is weakly càdlàg.

Remark 2.5. Notice that from the degree reducing property (1) of polynomial actions in Definition 2.3, it follows that \( T_hp \in \mathcal{P}_0 \) whenever \( p \in \mathcal{P}_0 \). I.e., from property (2) in Definition 2.4 linking the polynomial process to the action \( T \), we see that \( T \) is the identity operator on the constant functions, \( T_hp = p, \forall p \in \mathcal{P}_0 \). Hence, the domain \( D \supseteq \mathcal{P}_0 \) contains the constant functions and the generator \( \mathcal{G} \) maps constant functions to 0.

Sometimes it is useful to disregard higher order polynomials. This allows to omit definitions like \( m \)-polynomial processes which appears for instance in Cuchiero [5].
Remark 2.6. Let $n \in \mathbb{N}$. Then $(\mathcal{P}_n, \deg)$ is a graded vector space (in the sense of Bourbaki, N. (1974) Algebra I. Chapter 2, Section 11.) and $X$ is a $\mathcal{P}_n$-polynomial process with action $(T_h|\mathcal{P}_n)_{h \geq 0}$.

We end this section with a number of examples of polynomial processes as we have defined them, with a link to existing theory on polynomial processes.

Example 2.7. Classical polynomial processes in the sense of Cuchiero [5] are of course polynomial in the sense of Definition 2.4. Indeed, let $E \subseteq \mathbb{R}^d$ be closed and $\mathcal{P}$ be the set of classical $d$-variable polynomials restricted to $E$, with the degree function defined as the order of the polynomials.

The next example shows that by taking the ‘right’ set of polynomials many Markov processes can be understood as polynomial processes. Indeed, if $\mathcal{P}$ is the vector space generated by eigenfunctions of the generator, then the Markov process is a $\mathcal{P}$-polynomial process.

Example 2.8. Let $L$ be an $\mathcal{F}$-Lévy process on $E = \mathbb{R}^d$ and $\mathcal{P}$ the vector space generated by the functions $E \to \mathbb{C}$, $x \mapsto e^{iu \cdot x}$ where $u \in \mathbb{R}^d$ and $ux := \sum_{j=1}^d x_j u_j$ denotes the standard scalar product. We define

$$\deg : \mathcal{P} \to \{0, 1\}, p \mapsto 1_{\{p \text{ non-constant}\}}.$$ 

Observe that $\mathcal{P} = \mathcal{P}_1$. Take $L := \{p \in \mathcal{P} : p(0) = 0\}$ to be the linear functions (or, “first order monomials”). Let $\psi$ be the Lévy exponent of $L$, i.e. $\psi : \mathbb{R}^d \to \mathbb{C}$ is such that $\mathbb{E}[e^{iu\cdot X_t}] = \exp(\psi(u))$ for any $t \geq 0$, $u \in \mathbb{R}^d$ and let

$$T_h : \mathcal{P} \to \mathcal{P}, \sum_{n=1}^k c_n e^{iu_n \cdot \cdot \cdot \cdot \cdot} \mapsto \sum_{n=1}^k c_n \exp(h\psi(u_n))e^{iu_n \cdot \cdot \cdot \cdot \cdot}.$$ 

Then $L$ is an $E$-valued $\mathcal{P}$-polynomial process with action $T$.

Another possible choice of polynomials is the set of real-valued continuous functions on the one-point compactification of $E$.

Affine processes are thought to be polynomial processes if they satisfy certain moment conditions [4, Example 4.4.1]. While this is true, they are always polynomial processes when polynomials are interpreted in a different way.

Example 2.9. Let $X$ be an $\mathcal{F}$-affine process with values on $\mathbb{R}^d$, i.e. $X$ is a càdlàg process and there are functions $\psi : \mathbb{R}_+ \times \mathbb{C}^d \to \mathbb{C}$ and $\phi : \mathbb{R}_+ \times \mathbb{C}^d \to \mathbb{C}$ such that

$$\mathbb{E}[e^{iu \cdot X_{t+h}}|\mathcal{F}_t] = \exp(\phi(h, u) + \psi(h, u)X_t)$$

and such that the flow property

$$\phi(h + k, u) = \phi(h, u) + \phi(k, \psi(h, u)), \quad \psi(h + k, u) = \psi(k, \psi(h, u))$$

holds for any $t, h, k \geq 0$, $u \in \mathbb{R}^d$.

We define $(\mathcal{P}, \deg)$ as in Example 2.8 and introduce

$$T_h : \mathcal{P} \to \mathcal{P}, \sum_{n=1}^k c_n e^{iu_n \cdot \cdot \cdot \cdot \cdot} \mapsto \sum_{n=1}^k c_n \exp(\phi(h, u) + \psi(h, u)(\cdot)).$$

Then $X$ is an $E$-valued $\mathcal{P}$-polynomial process with action $T$. For more details on affine processes we refer to Duffie et al. [9].

The next example refers back to a recent study of Cuchiero, Larsson and Svaluto-Ferro [7].
Example 2.10. Let \((S, d)\) be a complete metric space, \(E\) the set of Borel probability measures on \(S\), and \(\mathcal{L}\) the set of functions of the form
\[
E \to \mathbb{R}, \mu \mapsto \int f d\mu.
\]
Here, \(f : S \to \mathbb{R}\) is a continuous function vanishing at infinity, i.e. it has a continuous extension to the one-point compactification of \(S\) with \(f(\Delta) = 0\) where \(\Delta\) is the additional point. Further, let \(\mathcal{P}\) be the algebra generated by \(\mathcal{L}\) and the constant functions and \(\text{deg} : \mathcal{P} \to \mathbb{N}\) with
\[
\text{deg} \left( c_0 + \sum_{i=1}^{n} c_i \prod_{j=1}^{k_i} f_{i,j} \right) = \begin{cases} 0 & \text{if } n = 0, \\ \max\{k_i : i = 1, \ldots, n\} & \text{otherwise.} \end{cases}
\]
where \(n \in \mathbb{N}, k_1, \ldots, k_n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{R}, f_{i,j} \in \mathcal{L}\). \(\mathcal{P}\)-polynomial processes on this structure have been investigated in Cuchiero, Larsson and Svaluto-Ferro [7].

3. Algebraic Structure of Locally Finite Polynomial Actions

We start our analysis of abstract polynomial process \(X\) or rather their corresponding actions defined in Definitions 2.4, 2.3 based on purely algebraic considerations. That is, we will not make any topological assumptions on \(E\), the space \(\mathcal{P}\), or even \(\mathcal{L}\) in this Section. We recall that \(X\) is an \(E\)-valued \(\mathcal{P}\)-polynomial process according to Definition 2.4 for a given set of polynomials \((\mathcal{P}, \text{deg})\) (Definition 2.1) and \(T\) being a polynomial action (Definition 2.3).

The property \(\text{deg}(T_h p) \leq \text{deg}(p)\) of a polynomial action \(T\) on \(p \in \mathcal{P}\) means that \(\mathcal{P}_n\) is \(T\)-invariant, that is, \(T\) respects the gradation \(\text{deg}\) on \(\mathcal{P}\). This has the immediate implication that \(T\) can be understood as a 'triangular' type (matrix) operator on \(\mathcal{P}_n\), where the diagonal elements are operators on vector space complements of \(\mathcal{P}_k\) in \(\mathcal{P}_{k+1}\). To make this precise, we need first to introduce what we shall understand as higher-order monomials in abstract polynomials \(\mathcal{P}\).

For this, let \(\mathcal{M}_n\) be a vector space complement of \(\mathcal{P}_{n-1}\) in \(\mathcal{P}_n\), i.e. \(\mathcal{M}_n\)
\[
\mathcal{P}_n = \mathcal{P}_{n-1} \oplus \mathcal{M}_n
\]
for \(n \geq 2\), \(\mathcal{M}_1 := \mathcal{L}\) and \(\mathcal{M}_0 := \mathcal{P}_0\) be the set of constant functions on \(E\). Further define the linear projections

\[
\Pi_n : \mathcal{P} \to \mathcal{M}_n
\]
with kernel \(\mathcal{P}_{n-1} \oplus \sum_{k=n+1}^{\infty} \mathcal{M}_k\) for \(n \geq 0\) where \(\mathcal{P}_{-1} := \{0\}\). The first order monomials \(\mathcal{M}_1\) will play a prominent role in the construction of the linearization of \(E\) and we thus keep the special notation \(\mathcal{L}\).

The next lemma shows that \(T\) has an upper-block-triangle form relative to the algebraic decomposition \(\mathcal{P} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \ldots\).

Lemma 3.1. It holds that
\[
T_h^{(n)} := \Pi_n T_h |_{\mathcal{M}_n}
\]
is a semigroup of linear operators on \(\mathcal{M}_n\) and
\[
T_h = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \Pi_l T_h \Pi_n = \sum_{n=0}^{\infty} T_h^{(n)} \Pi_n + \sum_{l=0}^{n-1} \Pi_l T_h \Pi_n
\]
where, when inserting a polynomial, at most finitely many summands are non-zero.
and only if the following two conditions hold:

\[ \frac{dX}{dE} \beta, \sigma \] are appropriate integrands. Then

\[ \text{Let us consider the case that} \]

Remark 3.3. polynomials of order 1 is infinite-dimensional and locally finiteness holds.

Remark 2.5 that \[ T \] acts as the identity operator on constants. Rewriting the last equality in Lemma 3.1 as a matrix operator on \( M_0 \oplus M_1 \oplus M_2 \oplus \cdots \) yields

\[
\text{matrix}(T_h) = \begin{pmatrix}
I & \Pi_0 \Pi_0 & \Pi_0 \Pi_0 & \\
0 & T_h^{(1)} & \Pi_1 T_h & \\
0 & 0 & T_h^{(2)} & \\
& & & \ddots
\end{pmatrix}
\]

for any \( h \geq 0 \) where \( I \) denotes the identity operator. We conclude that \( T_h \) can be represented as an upper triangular block matrix.

We will now introduce the following three important properties of the polynomial action \( T \):

**Definition 3.2.** We say

- \( T \) is locally finite if for any \( p \in \mathcal{P} \) there is \( d \in \mathbb{N} \) such that for any \( h > 0 \) the set \( \{ T_h^k p : k = 0, \ldots, d \} \) is linear dependent. Here, \( T_h^k \) means composition of the operator \( T_h \) \( k \)-times.
- \( T \) is reducing if for any \( p \in \mathcal{P}, h \geq 0 \) there is \( c \in \mathbb{F} \) with \( \deg(T_h p - cp) < \max\{1, \deg(p)\} \).
- \( T \) is strongly reducing if there is one \( c \in \mathbb{F} \) such that for any \( p \in \mathcal{P}, h \geq 0 \) one has \( \deg(T_h p - cp) < \max\{1, \deg(p)\} \).

A polynomial process is called locally finite (resp. reducing, resp. strongly reducing) if its action is locally finite (resp. reducing, resp. strongly reducing).

The following remark links the reducing property of \( T \) to the classical finite-dimensional polynomial processes. Example 2.8 shows a process where the set of polynomials of order 1 is infinite-dimensional and locally finiteness holds.

Remark 3.3. Let us consider the case that \( E = \mathbb{R}^d \) and \( \mathcal{P}_n \) is the set of polynomials in \( d \) commuting variables up to degree \( n \). Also we assume that \( X \) is an \( E \)-valued \( \mathcal{P} \)-polynomial process with some action \( T \) and a diffusion process, i.e.

\[ dX_t = \beta_t \, dt + \sigma_t dW_t \]

where \( W \) is a \( d \)-dimensional standard Brownian motion and \( \beta, \sigma \) are appropriate integrands. Then [5, Proposition 4.2.1] yields that \( \beta_t = b(X_t) \) for an affine function \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma_t \sigma_t^T = q(X_t) \) where \( q : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) is a quadratic function.

Since \( \mathcal{P}_n \) is finite dimensional, we know that \( T \) is locally finite. \( T \) is reducing if and only if the following two conditions hold:

1. \( b(x) = b(0) + \lambda_1 x \) for some constant \( \lambda_1 \in \mathbb{R} \)
2. \( q(x) = q(0) + \nabla q(0)x + \lambda_2 (x \otimes x) \) for some constant \( \lambda_2 \in \mathbb{R} \).
$T$ is strongly reducing if and only if $b$ is constant and $q$ is affine.

The characterisations for reducing and strongly reducing will follow easily from Corollaries 3.8 and 3.9 below.

We mention that also the polynomial processes discussed in [2] are strongly reducing.

The following basic relations between reducing and locally finite actions hold:

**Lemma 3.4.** If $T$ is reducing, then it is locally finite. If $P_n$ is finite dimensional for any $n \in \mathbb{N}$, then $T$ is locally finite.

If $T$ is locally finite, then the domain of the generator satisfies $\mathcal{D} = \mathcal{P}$ and $T$ can be extended to a group $(T_h)_{h \in \mathbb{F}}$ of linear operators such that $\deg(T_h p) \leq \deg(p)$ for any $h \in \mathbb{F}$, $p \in \mathcal{P}$ and $\deg(G_p) \leq \deg(p)$ for any $p \in \mathcal{P}$.

**Proof.** We first assume that $T$ is reducing. Let $p \in \mathcal{P}$ and denote $n := \deg(p) + 1$. We claim that $\{T_k^h p: k = 0, \ldots, n\}$ is linear dependent for any $h > 0$. To this end, let $h > 0$. Using the reducing property, we choose recursively $c_0, c_1, \ldots, c_n \in \mathbb{F}$, $q_0, \ldots, q_n \in \mathcal{P}$ such that

$$q_0 := p,$$
$$\deg(T_h q_j - c_j q_j) < \max\{1, \deg(q_j)\},$$
$$q_{j+1} := T_h q_j - c_j q_j$$

for $j = 0, \ldots, n - 1$. Then $\deg(q_0) = n - 1$, $\deg(q_{j+1}) < \max\{1, \deg(q_j)\}$ and, hence, $\deg(q_{n-1}) = 0$. Also observe from this construction that

$$\text{Span}\{T_k^h p: k = 0, \ldots, m\} = \text{Span}\{q_0, \ldots, q_m\}$$

for any $m = 0, \ldots, n$. Since $q_{n-1} \in \mathcal{P}_0$ we find $T_h q_{n-1} = q_{n-1}$ and therefore we may have $c_{n-1} = 1$ and $q_n = 0$. Thus we find

$$\text{Span}\{T_k^h p: k = 0, \ldots, n\} = \text{Span}\{q_0, \ldots, q_n\}$$
$$= \text{Span}\{q_0, \ldots, q_{n-1}\}$$
$$= \text{Span}\{T_k^h p: k = 0, \ldots, n - 1\}$$

which shows that $\{T_k^h p: k = 0, \ldots, n\}$ is linear dependent. Thus, $T$ is locally finite.

Now, we assume that $P_n$ is finite dimensional for any $n \in \mathbb{N}$ and show that $T$ is locally finite. Let $p \in \mathcal{P}$ and define $n := \deg(p)$ and $d := \dim(P_n)$. Then the set $\{T_k^h p: k = 0, \ldots, d\}$ is linear dependent because it is contained in the $d$-dimensional space $P_n$. Hence, $T$ is locally finite.

Finally, we only assume that $T$ is locally finite. We have $\mathcal{D} \subseteq \mathcal{P}$ by definition. Now let $p \in \mathcal{P}$ and choose $n \in \mathbb{N}$ such that $\{T_k^h p: k = 0, \ldots, n\}$ is linear dependent for any $h > 0$. For $h > 0$ we define $P_p^h := \text{Span}\{T_k^h p: k = 0, \ldots, n\}$. Then $P_p^h$ is invariant under $T_h$ due to the linear dependence property and $\dim(P_P^h) \leq n$ for any $h > 0$. From $T_h = T_{h/2}$ and the $T_{h/2}$-invariance property we find that $P_p^{h/2} \supseteq P_p^h$.

Thus,

$$P_p^0 := \bigcup_{l \in \mathbb{N}} P_p^{2^{-l}}$$

contains $P_p^{2^{-l}}$ for any $l \in \mathbb{N}$ and due to the increase of the spaces we find

$$\dim(P_p^0) = \lim_{l \to \infty} \dim(P_p^{2^{-l}}) \leq n.$$
Therefore $\mathcal{P}_p^0$ is a finite dimensional subspace of $\mathcal{P}$ which contains $p$. Moreover, $\mathcal{P}_p^{k2^{-i}} \subset \mathcal{P}_p^{2^{-i}}$ for any $k \in \mathbb{N}$ and thus $\mathcal{P}_p^0$ is $T_h$-invariant for any dyadic number $h \geq 0$.

We now want to see that $\mathcal{P}_p^0$ contains $\mathcal{P}_p^h$ for any $h > 0$. To this end, let $h > 0$ and $(q_n)_{n \in \mathbb{N}}$ be a dyadic decreasing sequence which converges to $h$. By the right-continuity property of $T$ (see Definition 2.3 property (4)), we find $\mathcal{P}_p^h \subseteq \mathcal{P}_p^0$ as required. Thus, $\mathcal{P}_p^0$ is a finite dimensional $T$-invariant subspace of $\mathcal{P}$ which contains $p$. Hence, $T^{(p)} := T|_{\mathcal{P}_p^0}$ defines a semigroup of linear operators such that $h \mapsto T_h^{(p)} q(x)$ is right-continuous for any $x \in E$, $q \in \mathcal{P}_p^0$. $E$ is separating for $\mathcal{P}_p^0$, i.e. for any $q_1, q_2 \in \mathcal{P}_p^0$ there is $x \in E$ with $q_1(x) \neq q_2(x)$ because $\mathcal{P}_p^0$ is a subset of functions from $E$ to $\mathbb{R}$. Thus, $(\delta_x)_{x \in E}$ where $\delta_x : \mathcal{P}_p^0 \to \mathbb{R}$ generates the dual space of $\mathcal{P}_p^0$ and we see that $T^{(p)}$ is a weakly right-continuous $c_0$-semigroup on the finite-dimensional space $\mathcal{P}_p^0$ (Note that due to the finite dimensionality of $\mathcal{P}_p^0$ all norms on it are equivalent and we simply use any norm on it). Hence $T^{(p)}$ is norm-continuous and there is a linear operator $G^{(p)}$ on $\mathcal{P}_p^0$ such that $T_h^{(p)} = \exp(hG^{(p)})$ for any $h > 0$. We find

$$\frac{T_h p - p}{h} = \frac{T^{(p)} h p - p}{h} \xrightarrow{h \to 0} G^{(p)} p$$

and, hence, $p \in \mathcal{D}$ and $G p = G^{(p)} p$. Also, the right-hand side of the representation

$$T_h p = T_h^{(p)} p = \exp(hG^{(p)}) p$$

yields an extension to any $h \in \mathbb{R}$ with the required degree invariance property. □

We next prove a similar result as Lemma 3.1 for the generator $G$ of the action operator.

**Proposition 3.5.** Assume that $T$ is locally finite. Then we have

$$G = \sum_{n=1}^{\infty} \sum_{l=1}^{n} \Pi_l G \Pi_n = \sum_{n=1}^{\infty} \left( G^{(n)} \Pi_n + \sum_{l=1}^{n-1} \Pi_l G \Pi_n \right)$$

where $G^{(n)} := \Pi_n G |_{\mathcal{M}_n}$, and $\mathcal{M}_n$ and $\Pi_n$ are defined in Definition 3. In particular, $\deg(G p) \leq \deg(p)$ for any $p \in \mathcal{P}$.

**Proof.** This is immediate from Lemmas 3.4 and 3.1 after noticing that $\Pi_l \Pi_n p = 0$ when $l < n$. □

Rewriting the last equality in Proposition 3.5 as a matrix operator on $\mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots$ yields

$$\text{matrix}(G) = \begin{pmatrix}
0 & \Pi_0 G & \Pi_0 G & \Pi_0 G & \cdots \\
0 & \Pi_1 G & \Pi_1 G & \Pi_1 G & \cdots \\
0 & \Pi_2 G & \Pi_2 G & \cdots & \\
& \vdots & \ddots & \ddots
\end{pmatrix}.$$
Our next result shows that if $T$ is locally finite, then the semigroup $(T_h)_{h \geq 0}$ can be recovered from its generator $\mathcal{G}$. In principle it shows that $T_h = \exp(h \mathcal{G})$ where $\exp$ is understood as a power-series and powers of $\mathcal{G}$ are meant in terms of composition.

**Proposition 3.6.** Assume that $T$ is locally finite. Then we have

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} \mathcal{G}^n p(x) = T_h p(x)$$

for any $p \in \mathcal{P}$, $x \in E$, $h \geq 0$. Moreover, the extension as a power series converges for any $h \in \mathbb{F}$ and defines in this way an $\mathbb{F}$-parametrised group of operators.

**Proof.** Lemma 3.4 yields that $\mathcal{D} = \mathcal{P}$. Thus $\mathcal{G}$ is a linear operator on $\mathcal{P}$ and it can be applied arbitrary often to an element $p \in \mathcal{P}$. Let $p \in \mathcal{P}$ and define $\mathcal{P}_p := \text{Span}\{\mathcal{G}^n p : n \in \mathbb{N}\}$. Recall from the proof of Lemma 3.4 that $\text{Span}\{T_h^n p : n \in \mathbb{N}\}$ is finite dimensional. Thus, $\mathcal{P}_p$ is finite dimensional. Choose any norm $\| \cdot \|_p$ on $\mathcal{P}_p$ and note that

$$\delta_x : \mathcal{P}_p \to \mathbb{F}, p \mapsto p(x)$$

is a linear map defined on a finite dimensional normed space and hence continuous. On $\mathcal{P}_p$, we have $T_h|_{\mathcal{P}_p} = \exp(h \mathcal{G}|_{\mathcal{P}_p})$ where the exponential function is meant as a power series and the convergence is in operator norm. In particular we find

$$\sum_{n=0}^{k} \frac{h^n}{n!} \mathcal{G}^n p \to T_h p, \quad k \to \infty$$

in the normed space $\mathcal{P}_p$ and continuity of $\delta_x$ yields that

$$\sum_{n=0}^{k} \frac{h^n}{n!} \mathcal{G}^n p(x) \to T_h p(x), \quad k \to \infty$$

in $\mathbb{F}$. □

Proposition 3.6 is a dual-moment formula as in Cuchiero and Svaluto-Ferro [8]. The same is true for Corollary 3.9 below. The difference is that we require in this version locally finiteness but in return we do not need to assume the existence and uniqueness of the corresponding Cauchy problem and no additional technical conditions (in our case, existence and uniqueness follow under locally finiteness).

Since it is possible to recover the polynomial action $(T_h)_{h \geq 0}$ from the generator, it is also possible to construct it in this way.

**Proposition 3.7.** Let $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ be any linear operator such that

1. $\mathcal{G} \mathcal{P}_n \subseteq \mathcal{P}_n$ for any $n \geq 0$ and
2. For any $p \in \mathcal{P}$ there is a finite dimensional subspace $V_p \subseteq \mathcal{P}$ such that $\mathcal{G}(V_p) \subseteq V_p$ and $p \in V_p$.

Define $T_h$ by

$$T_h p(x) := \sum_{n=0}^{\infty} \frac{h^n}{n!} \mathcal{G}^n p(x)$$

for any $p \in \mathcal{P}$, $x \in E$, $h \geq 0$, where absolute convergence of the series is implied by the assumptions. Then $(T_h)_{h \geq 0}$ is a locally finite polynomial action.
Proof. Let \( p \in \mathcal{P} \) and \( V_p \) be finite-dimensional \( \mathcal{G} \)-invariant containing \( p \). Choose any norm \( \| \cdot \|_p \) on \( V_p \). Then \( \mathcal{G}|_{V_p} \) is a continuous linear operator on \( V_p \). Thus, its matrix exponential

\[
\exp(h\mathcal{G}|_{V_p}) = \sum_{n=0}^{\infty} \frac{h^n}{n!}(\mathcal{G}|_{V_p})^n
\]

is well defined, where the convergence is in operator norm, and hence we have

\[
T_hp(x) := \sum_{n=0}^{\infty} \frac{h^n}{n!} \mathcal{G}^n p(x)
\]

is well defined

\[
T_h|_{V_p} = \sum_{n=0}^{\infty} \frac{h^n}{n!}(\mathcal{G}|_{V_p})^n
\]

for any \( h \geq 0 \). It follows straightforwardly from the construction that \( (T_h)_{h \geq 0} \) is a locally finite polynomial action. ∎

If \( T \) is reducing, then the operators \( \mathcal{G}^{(n)} \) appearing in Proposition 3.6 are in fact multiples of the identity. In particular, \( \mathcal{G} \) can be written as an upper triangular matrix.

**Corollary 3.8.** Assume that \( T \) is reducing and recall \( T_h^{(n)} \) defined in Lemma 3.4 and \( \mathcal{G}^{(n)} \) defined in Proposition 3.6. Then there exist constants \( \lambda_n \in \mathbb{F} \) such that

\[
\mathcal{G}^{(n)} f = \lambda_n f,
\]

\[
T_h^{(n)} f = e^{h\lambda_n} f,
\]

\[
\deg(T_h f - e^{h\lambda_n} f) < \max\{1, \deg(f)\},
\]

for any \( h \geq 0 \), \( n \in \mathbb{N} \), \( f \in \mathcal{M}_n \). Note that \( \lambda_0 = 0 \).

**Proof.** Recall from Lemma 3.4 that \( T \) is locally finite whenever it is reducing, and thus \( \mathcal{D} = \mathcal{P} \). Let \( f \in \mathcal{M}_n \). By assumption on \( T \) there are constants \( c_h \in \mathbb{F} \) for any \( h \geq 0 \) such that \( \deg(\Pi_n(T_h f - c_h f)) < \max\{1, \deg(f)\} \). If \( \deg(f) = 0 \), then Proposition 3.6 yields that \( \mathcal{G} f = 0 \) and Proposition 3.6 reveals that \( T_h f = f \) and so \( \lambda_0 = 0 \) satisfies the claim.

Thus we may assume that \( \deg(f) \neq 0 \) and, hence, \( \max\{1, \deg(f)\} = \deg(f) = n \).

Recall from Definition 3.3 the linear projection operator \( \Pi_n : \mathcal{P} \to \mathcal{M}_n \) with kernel \( \mathcal{P}_{n-1} \oplus \sum_{k=n+1}^{\infty} \mathcal{M}_k \) for \( n \geq 0 \), where \( \mathcal{P}_{-1} := \{0\} \). We find that \( \Pi_n(T_h f - c_h f) = 0 \) because \( \deg(\Pi_n(T_h f - c_h f)) < n \). Consequently, we have \( T_h^{(n)} f = \Pi_n T_h f = c_h f \).

By semigroup property, we find \( c_{h+k} f = T_h^{(n)} T_k^{(n)} f = c_h c_k f \), which shows that \( c_{h+k} = c_h c_k \). Since, for any \( x \in E \), it holds,

\[
c_{h+k} f(x) = T_h^{(n)} T_k^{(n)} f(x) = c_h c_k f(x) = c_h f(x)
\]

we can see that \( c \) is a right-continuous semigroup of elements in \( \mathbb{F} \). Hence, there is some \( \lambda_n \in \mathbb{F} \) with \( c_h = \exp(h\lambda_n) \). Moreover, we have for any \( x \in E \)

\[
\mathcal{G}^{(n)} f(x) = \lim_{h \searrow 0} \frac{T_h^{(n)} f(x) - f(x)}{h} = \lambda_n f(x)
\]

and, hence, \( \mathcal{G}^{(n)} f = \lambda_n f \).
Now let $g \in M_n$ and $\psi_n \in \mathbb{F}$ such that $G^{(n)}g = \psi_ng$. We find that there is a constant $\eta_n \in \mathbb{F}$ with $G^{(n)}(f + g) = \eta_n(f + g)$ and, hence,

$$\lambda_n f + \psi_n g = G^{(n)}(f + g) = \eta_n f + \eta_n g.$$ 

This, however, is only possible if $\lambda_n = \psi_n = \eta_n$ which shows that $G^{(n)} = \lambda_n I$ where $I$ is the identity operator on $M_n$. From the exponential formula in Proposition 3.6 we find that

$$T^{(n)}_h f = e^{h\lambda_n} f$$

for any $n \in \mathbb{N}$, $f \in M_n$, $h \geq 0$. The result follows. □

For strongly reducing polynomial processes one can give a nice formula for arbitrary polynomial moments.

**Corollary 3.9.** Assume that $T$ is strongly reducing and let $\lambda_0, \lambda_1 \ldots \in \mathbb{F}$ be as in Corollary 3.8. Then $\lambda_n = 0$ for any $n \geq 0$ and one has $\text{deg}(T_h p - p) < \text{deg}(p)$ for any $p \in P \setminus P_0$. In particular one has

$$T_h p = \sum_{k=0}^{n} \frac{h^k}{k!} G^k p$$

for any $n \geq 0$, $h \geq 0$ and $p \in P_n$.

**Proof.** Corollary 3.8 states that $\lambda_0 = 0$ and $1 = e^{\lambda_0 h} = c_h = e^{\lambda_n h}$ for any $h \geq 0$. Consequently, $\lambda_n = 0$ for any $n \geq 0$. Thus $\text{deg}(G p) < \text{deg}(p)$ for any $p \in P \setminus P_0$ and $G p = 0$ for $p \in P_0$. Thus, we find that $G^{n+1} p = 0$ for any $p \in P_n$. Proposition 3.6 yields the claim. □

Proposition 3.7 revealed how locally finite polynomial actions can be rebuit from their generator. A similar statement can be made for strongly reducing actions and their generators.

**Lemma 3.10.** Let $G : \mathcal{P} \to \mathcal{P}$ be any linear operator such that

1. $G \mathcal{P}_{n+1} \subseteq \mathcal{P}_n$ for any $n \geq 0$ and
2. $G \mathcal{P}_0 = \{0\}$.

Define $T_h$ by

$$T_h p := \sum_{n=0}^{\infty} \frac{h^n}{n!} G^n p$$

for any $p \in \mathcal{P}$, $h \geq 0$ where the summands of the series are only finitely often non-zero. Then $(T_h)_{h \geq 0}$ is a strongly reducing polynomial action.

**Proof.** Let $p \in \mathcal{P}_n$ and observe that $G|\mathcal{P}_n$ is nilpotent of order $n + 1$. Thus,

$$G^k p = 0, \quad k > n$$

which yields the statement about the summands. Moreover,

$$T_h p = \sum_{k=0}^{n} \frac{h^k}{k!} G^k p \subseteq \mathcal{P}_{n-1}$$

if $n > 0$ as claimed. □
4. AFFINE DRIFT FOR LOCALLY FINITE POLYNOMIAL PROCESSES

Recall that \( \mathcal{L} := \mathcal{M}_1 \) is a vector space complement of \( \mathcal{P}_0 \) in \( \mathcal{P}_1 \). The semigroup \( T \) restricted to \( \mathcal{P}_1 = \mathcal{P}_0 \oplus \mathcal{L} \) has a particularly simple structure due to Lemma 3.1,

\[
\text{matrix}(T_h|_{\mathcal{P}_1}) = \begin{pmatrix} I & \Pi_0 T_h \\ 0 & T_h^{(1)} \end{pmatrix}, \quad h \geq 0.
\]

In this section we will mostly work under the following assumption:

**Assumption (F):** For every \( p \in \mathcal{L} \), there is a finite dimensional subspace \( \mathcal{V}_p \subseteq \mathcal{L} \) with \( p \in \mathcal{V}_p \) which is \( T^{(1)} \)-invariant, i.e. \( T_h^{(1)} \mathcal{V}_p \subseteq \mathcal{V}_p \) for any \( h \geq 0 \).

**Assumption (F) means that the smallest \( T^{(1)} \)-invariant vector space containing \( p \) is finite dimensional for any \( p \in \mathcal{L} \).**

**Remark 4.1.** If \( T \) is locally finite, then Assumption (F) holds. If Assumption (F) holds and we forget higher order polynomials (i.e. we assume \( \mathcal{P} = \mathcal{P}_1 \)), then \( T \) is locally finite. This means that Assumption (F) is equivalent to locally finiteness of \( T \) for polynomials up to order 1. This allows, under Assumption (F), to apply most of the preceding statements for polynomials up to order 1.

**Lemma 4.2.** Suppose Assumption (F). Then \( \mathcal{P}_1 \) is the domain of \( \mathcal{G}|_{\mathcal{P}_1} \) and

\[
T_h p(x) = \sum_{n=0}^{\infty} \frac{h^n}{n!} (\mathcal{G}|_{\mathcal{P}_1})^n p(x)
\]

for any \( p \in \mathcal{P}_1, \ h \geq 0 \). Moreover, it has a simple matrix structure relative to the decomposition \( \mathcal{P}_1 = \mathcal{P}_0 \oplus \mathcal{L} \) given by

\[
\text{matrix}(\mathcal{G}|_{\mathcal{P}_1}) = \begin{pmatrix} 0 & \mathcal{G}^{(0,1)} \\ 0 & \mathcal{G}^{(1)} \end{pmatrix}
\]

where \( \mathcal{G}^{(0,1)} : \mathcal{L} \to \mathcal{P}_0 \) and \( \mathcal{G}^{(1)} : \mathcal{L} \to \mathcal{L} \) are linear.

**Proof.** The result is a consequence of Remark 4.1, Propositions 3.5 and 3.6. \( \square \)

Consider the algebraic dual space of \( \mathcal{L} \), which we denote by \( B \). The algebraic dual space \( B \) is understood as a linearisation of \( E \), where we embed the process \( X \) on \( B \) via

(4) \( \tilde{X}_t(p) := p(X_t), \quad p \in \mathcal{L}, t \geq 0. \)

**Remark 4.3.** We believe that it is very natural to assume that \( \mathcal{L} \) is separating for \( E \), i.e. for any \( x, y \in E \) there is \( p \in \mathcal{L} \) with \( p(x) \neq p(y) \). If this is the case, then

\( \delta : E \to B, x \mapsto \delta_x \)

is injective where \( \delta_x : \mathcal{L} \to \mathbb{F}, p \mapsto p(x) \). Injectivity of \( \delta \) means that \( \tilde{X} = \delta(X) \) is a copy of \( X \).

However, our statements in this section do not require this injectivity and are stated without the separation assumption.

Under assumption (F), we show in Theorem 4.5 below that \( \tilde{X} \) is a weak*-martingale driven Ornstein-Uhlenbeck process. This requires a preparatory Lemma where we, in fact, construct the constant and linear part of its drift.
Lemma 4.4. Assume Assumption (F). Let \( U_h := (T_h^{(1)})^* \) for any \( h \geq 0 \). For \( p \in \mathcal{L}, h \geq 0 \) we define \( U^p_h := (T_h^{(1)}|_{V_p})^* \). Then the following diagram commutes for any \( p \in \mathcal{L} \)

\[
\begin{array}{ccc}
B & \xrightarrow{U_h} & B \\
\downarrow & & \downarrow \\
V^*_p & \xrightarrow{U^p_h} & V^*_p
\end{array}
\]

where the mapping from \( B \) to \( V^*_p \) is the restriction of the linear functional on \( \mathcal{L} \) to \( V_p \).

Let \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be continuous. Further, let \( Y \) be a \( B \)-valued weak*-semimartingale, i.e., \( Y(p) \) is a semimartingale for any \( p \in \mathcal{L} \). Then there is a \( B \)-valued process \( Z \) such that for any \( p \in \mathcal{L} \) one has

\[
\int_0^t U^p_{f(t,s)} dY_s|_{V_p} = Z_t|_{V_p}, \quad t \geq 0.
\]

We will also write \( \int_0^t U_{f(t,s)} dY := Z_t \).

Also, \( Z \) is uniquely determined by this property in the sense that if \( \hat{Z} \) is another \( B \)-valued process such that for any \( p \in \mathcal{L} \) one has

\[
\int_0^t U^p_{f(t,s)} dY_s|_{V_p} = \hat{Z}_t|_{V_p}, \quad t \geq 0,
\]

then \( \hat{Z}(p) = Z(p) P \otimes \lambda \text{-a.e.} \), where \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}_+ \).

Proof. We start with the first statement. To this end, let \( y \in B, p \in \mathcal{L} \) and \( q \in V_p \).

We find that \( T_h^{(1)} q \in V_p \) and

\[
(U_h y)|_{V_p}(q) = U_h y(q) = y(T_h^{(1)} q) = y|_{V_p} (T_h^{(1)} q) = (U^p_h (y|_{V_p}))(q).
\]

Thus, \( (U_h y)|_{V_p} = U^p_h (y|_{V_p}) \).

Now, let \( Y \) be a \( B \)-valued weak*-semimartingale. For \( p \in \mathcal{L} \) we have that \( Y|_{V_p} \) is a semimartingale because \( V_p \) is finite-dimensional. Thus, we can define

\[
Z_t(p) := \left( \int_0^t U^p_{f(t,s)} dY_s|_{V_p} \right)(p)
\]

for any \( t \geq 0, p \in \mathcal{L} \). Obviously, \( Z \) has the required properties. \( \square \)

Recall that \( G^{(0,1)} \) in Lemma 4.2 is linear from \( \mathcal{L} \) to \( \mathcal{P}_0 \). Hence, \( (G^{(0,1)})^* \) is linear from \( \mathcal{P}_0^* \) to \( B \), where the former is a one-dimensional space which has a natural unit, namely \( \delta_e : \mathcal{P}_0 \to \mathbb{F}, p \mapsto p(e) \) where \( e \) is any element of \( E \) (indeed, \( \mathcal{P}_0 \) is the set of constant functions from \( E \) to \( \mathbb{F} \)). We will denote this unit simply by \( 1 \).

Theorem 4.5. Suppose Assumption (F) holds. Assume that \( \mathbb{E} \left[ \int_0^t |Gp(X_r)| dr \right] < \infty \) for any \( t \geq 0, p \in \mathcal{L} \) and let \( G^{(0,1)} \) and \( G^{(1)} \) be as in Lemma 4.2. We define \( b := (G^{(0,1)})^* 1, A := (G^{(1)})^* \), \( U_h := (T_h^{(1)})^* \) and

\[
M_t := \hat{X}_t - \hat{X}_0 - \int_0^t (b + A \hat{X}_s) ds, \quad t \geq 0
\]

where the integral is a weak*-integral. Then, \( M \) is a \( B \)-valued weak*-martingale with

\[
M_t(p) = p(X_t) - p(X_0) - \int_0^t Gp(X_s) ds, \quad t \geq 0, p \in \mathcal{L}
\]
and we have
\[
\tilde{X}_t = U_t \tilde{X}_0 + \int_0^t U_{t-s} b ds + \int_0^t U_{t-s} dM_s, \quad t \geq 0
\]
where the first integral is a weak*-integral, the second is the integral discussed in Lemma 4.4 and
\[
\tilde{X}_t : \Omega \times \mathcal{L} \to \mathbb{F}, \tilde{X}_t(p) := p(X_t)
\]
for any \( t \geq 0 \).

In particular, \( p(X) \) is semimartingale for any \( p \in \mathcal{P}_1 \).

Proof. Let \( p \in \mathcal{L} \) and \( 0 \leq s \leq t \). Since \( X \) is a polynomial process we find that
\[
E[p(X_t)|\mathcal{F}_s] = T_{t-s} p(X_s).
\]
Recall that the domain of \( \mathcal{G}|_{\mathcal{P}_1} \) is \( \mathcal{P}_1 \) due to Remark 4.1. By Lemma 4.2 it follows that
\[
\partial_t T_{t-s} p(x) = \mathcal{G} T_{t-s} p(x)
\]
and thus by the Fundamental Theorem of Calculus
\[
T_{t-s} p(X_s) = p(X_s) + \int_s^t \mathcal{G} T_{r-s} p(X_s) dr.
\]
By assumption we may use Fubini’s theorem for the conditional expectation to find that
\[
E[\int_s^t \mathcal{G} p(X_r) dr | \mathcal{F}_s] = \int_s^t \mathcal{G} T_{r-s} p(X_s) dr
\]
and, hence,
\[
E[p(X_t)|\mathcal{F}_s] = p(X_s) + \mathbb{E}[\int_s^t \mathcal{G} p(X_r) dr | \mathcal{F}_s].
\]
Consequently,
\[
M^p_t := p(X_t) - \int_0^t \mathcal{G} p(X_r) dr, \quad t \geq 0
\]
is integrable at any fixed time \( t \geq 0 \) and its conditional expectation is given by
\[
E[M^p_t | \mathcal{F}_s] = M^p_s
\]
which yields that \( M^p \) is a martingale.

By definition of \( b \), we have
\[
b(p) = (\mathcal{G}^{(0,1)})^* 1(p) = 1(\mathcal{G}^{(0,1)} p) = \mathcal{G}^{(0,1)} p(X_u)
\]
Furthermore,
\[
\mathcal{A} \tilde{X}_u(p) = \tilde{X}_u(\mathcal{G}(1)p) = \mathcal{G}(1)p(X_u)
\]
and therefore
\[
b(p) + \mathcal{A} \tilde{X}_u(p) = \mathcal{G} p(X_u).
\]
Thus, we find
\[
M_t(p) = p(X_t) - p(X_0) - \int_0^t (b + \mathcal{A} \tilde{X}_r) p ds
\]
\[
= p(X_t) - p(X_0) - \int_0^t \mathcal{G} p(X_r) dr
\]
\[
= M^p_t - p(X_0)
\]
and consequently, \( M \) is a weak*-martingale.
We finalize the proof by verifying the representation of $\tilde{X}$. To this end, let $p \in \mathcal{L}$ and recall from Proposition 3.6 that $T^{(1)}$, and henceforth $U$, can be extended to groups of operators. We find that

$$Y_t^p := \left( U_t \tilde{X}_0 + \int_0^t U_{t-s} b ds + \int_0^t U_{t-s} dM_s \right)(p)$$

$$= (\tilde{X}_0 + \int_0^t U_{t-s} b ds + \int_0^t U_{t-s} dM_s)(T^{(1)}_t p),$$

where we used Lemma 4.4 for the integrals with respect to $M$. The product formula for semimartingales applied on $V_p$ yields

$$dY_t^p = (b + A\tilde{X}_t)(p) dt + dM_t, \quad t \geq 0$$

and we have $Y_0^p = \tilde{X}_0(p)$. Hence, $Y_t^p = \tilde{X}_t(p)$ as required.

□

If $T$ is locally finite, then the right-hand side of Equation (5) is a martingale for any polynomial $p \in \mathcal{P}$.

Corollary 4.6. Let $T$ be locally finite, $n \in \mathbb{N}$ and assume that $\mathbb{E}[\int_0^t |p(X_r)| dr] < \infty$ for any $t \geq 0$ and any $p \in \mathcal{P}_n$. Then the process

$$M_t^p := p(X_t) - \int_0^t G_p(X_s) ds, \quad t \geq 0$$

is an $\mathbb{F}$-valued martingale for any $p \in \mathcal{P}_n$.

Proof. Define $\mathcal{R}_0 := \mathcal{P}_0$ and $\mathcal{R}_1 := \mathcal{R} := \mathcal{P}_n$. We now consider the polynomials $\mathcal{R}$ where the degree of a polynomial is 1 if it is non-constant. Note that the definition of locally finite does not involve the degrees of the polynomials and, hence, $X$ is an $\mathcal{R}$-polynomial process and $T$ locally finite. Moreover Assumption (F) holds relative to the polynomials $\mathcal{R}$. Thus, Theorem 4.5 states that $M^p$ is a martingale for any $p \in \mathcal{R}_1 = \mathcal{P}_n$.

□

Remark 4.7. Corollary 4.6 establishes that all polynomial processes with locally finite action can be associated with a corresponding martingale problem. Cuchiero and Svaluto-Ferro study polynomial processes with values in Banach spaces from the martingale problem perspective.

Remark 4.8. Considering the decomposition

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t (b + A\tilde{X}_s) ds + M_t$$

given in Theorem 4.5 we see that $\tilde{X}$ is a (weak$^*$-)martingale driven Ornstein-Uhlenbeck process with values in $B$. Recall that a $B$-valued random variable $Z$ has a weak$^*$-expectation $\mathbb{E}[Z]$ if

$$\mathbb{E}[|Z(p)|] < \infty, \quad p \in \mathcal{L},$$

which is the element of $B$ given by

$$\mathbb{E}[Z] : \mathcal{L} \to \mathbb{F}, p \mapsto \mathbb{E}[Z(p)].$$
If $\tilde{X}_t$ has weak*-expectation for every $t \geq 0$ and this expectation can be exchanged with the integral in the decomposition above, then
\[
\mathbb{E}[\tilde{X}_t] = \mathbb{E}[\tilde{X}_0] + \int_0^t (b + A\mathbb{E}[\tilde{X}_s])ds,
\]
i.e. $t \mapsto \mathbb{E}[\tilde{X}_t]$ is a solution to the ODE
\[
u'(t) = b + Au(t), \quad u(0) = \mathbb{E}[\tilde{X}_0].
\]
This recovers the bidual formula given in [8, Theorem 3.8].

5. Covariance Structure for Locally Finite Polynomial Processes

In this section we investigate the covariance structure of polynomial diffusions. This will rely on a multiplicativity structure of the polynomials. We make the following assumption throughout this section:

**Assumption 5.1.** Suppose that, $(T_h)_{h \geq 0}$ is a locally finite polynomial action.

We also recall from the previous section the set $B$ of linear functions from $L$ to $\mathbb{F}$, i.e., the algebraic dual of $L$. For an $E$-valued process $X$ we use (as in the previous section) its $B$-embedded version $\tilde{X}_t$ given by $\tilde{X}_t(p) := p(X_t)$.

**Lemma 5.2.** Let $p \in \mathcal{P}$ with $|p|^2 \in \mathcal{P}$. Then we have $\mathbb{E}[\int_0^t |p(X_r)|dr] < \infty$ and $\mathbb{E}[\int_0^t |p(X_r)|^2dr] < \infty$ for any $t \geq 0$.

**Proof.** Let $t \geq 0$ and define $q := |p|^2 \in \mathcal{P}$. Note that $q(x) \geq 0$ for any $x \in E$. Since $T$ is locally finite, there is a finite dimensional vector space $V_q \subseteq \mathcal{P}$ with $q \in V_q$ which is $T$-invariant. Let $\| \cdot \|_q$ be a norm on $V_q$. Proposition 3.6 yields that $r \mapsto T_rq$ is $\| \cdot \|_q$-continuous. Also, the function
\[
\Gamma : V_q \to \mathbb{F}, f \mapsto \mathbb{E}[f(X_0)]
\]
is linear and defined on a finite dimensional space and, hence, continuous with respect to the operator norm. We find
\[
\left(\mathbb{E}[\int_0^t |p(X_r)|dr]\right)^2 \leq t \int_0^t \mathbb{E}[q(X_r)]dr
\]
\[
= t \int_0^t \mathbb{E}[T_rq(X_0)]dr
\]
\[
= t\Gamma \left( \int_0^t T_rqdr \right) < \infty
\]
where we used Cauchy-Schwarz’ inequality and Tonelli’s theorem for the first inequality, the tower property with conditioning on $\mathcal{F}_0$ for the first equality and linearity and continuity of $\Gamma$ for the last equality and inequality. \qed

Next we state the main result of the section. It identifies the covariance coefficient of a locally finite polynomial process, which exists, and it is a second order polynomial as a function of the state. For our result we assume that the product of two first order polynomials is at most a second order polynomial and we suppose two different sufficient conditions for second order polynomials. Property (2) in the following theorem holds for classical polynomials in $d$ commuting variables and for the polynomials appearing in [8].
Theorem 5.3. Assume that $\mathbb{F} = \mathbb{R}$, $\mathcal{P}_1 \cdot \mathcal{P}_1 \subseteq \mathcal{P}_2$ and at least one of

1. $\mathcal{P}_2 \cdot \mathcal{P}_2 \subseteq \mathcal{P}$
2. Every element in $\mathcal{P}_2$ can be written as a finite linear combination of positive elements in $\mathcal{P}_2$.

Let $X$ be a polynomial process with polynomial action $T$. Let $M$ be the process introduced in Theorem 4.3 (whose requirements are met) and $p, q \in \mathcal{L}$. We define

$$a_{p,q}(x) := \mathcal{G}(pq)(x) - p(x)\mathcal{G}q(x) - q(x)\mathcal{G}p(x), \quad x \in \mathcal{E}.$$ 

Then we have

$$\langle M_{(\cdot)}(p), M_{(\cdot)}(q) \rangle_t = \int_0^t a_{p,q}(X_s)ds$$

for any $t \geq 0$ where $\langle M_{(\cdot)}(p), M_{(\cdot)}(q) \rangle$ denotes the compensator of the quadratic covariation between $M_{(\cdot)}(p)$ and $M_{(\cdot)}(q)$.

Proof. Let $r \in \mathcal{P}_2$.

If (1) holds, then Lemma 5.2 yields that $\mathbb{E}[\int_0^t |r(X_s)|ds] < \infty$.

If (2) holds, then there are positive $q_1, \ldots, q_n \in \mathcal{P}_2$ and constants $c_1, \ldots, c_n \in \mathbb{R}$ such that $r = \sum_{j=1}^n c_j q_j$. We find that

$$\mathbb{E}\left[\int_0^t |r(X_s)|ds\right] = \sum_{j=1}^n c_j \mathbb{E}\left[\int_0^t q_j(X_s)ds\right] \leq \sum_{j=1}^n |c_j| \mathbb{E}\left[\int_0^t |T_s q_j(X_0)| ds\right] < \infty$$

where we used Tonelli’s theorem for the equality and the last inequality follows as in the proof of Lemma 5.2.

Thus, we have $\mathbb{E}[\int_0^t |r(X_s)|ds] < \infty$ for any $r \in \mathcal{P}_2$. Corollary 4.6 yields that

$$M^p_t := p^2(X_t) - p^2(X_0) - \int_0^t \mathcal{G}(p^2)(X_s)ds, \quad t \geq 0,$$

$$M^q_t := q^2(X_t) - q^2(X_0) - \int_0^t \mathcal{G}(q^2)(X_s)ds, \quad t \geq 0$$

are martingales. We have

$$(M^p_t)^2 = p^2(X_t) + p^2(X_0) + \left(\int_0^t \mathcal{G}(p(X_u))du\right)^2$$

$$- 2p(X_t)p(X_0) - 2p(X_t) \int_0^t \mathcal{G}(p(X_u))du + 2p(X_0) \int_0^t \mathcal{G}(p(X_u))du$$

$$= M^p_t + \int_0^t \mathcal{G}(p^2)(X_s)ds + 2p^2(X_0) - 2p(X_t)p(X_0)$$

$$- 2 \int_0^t p(X_u)\mathcal{G}(p(X_u))du - 2 \int_0^t \int_0^u \mathcal{G}(p(X_v))dvdM^p_u$$

$$+ 2p(X_0) \int_0^t \mathcal{G}(p(X_u))du$$

$$= M^p_t + \int_0^t a_{p,p}(X_s)ds + 2p^2(X_0) - 2M^p_t p(X_0) - 2 \int_0^t \int_0^u \mathcal{G}(p(X_v))dvdM^p_u$$

$$= N_t + \int_0^t a_{p,p}(X_s)ds$$
for any $t \geq 0$ where we used the product formula for the second equality, the semimartingale decomposition from above and the fact that $[p(X), \int_0^t G_p(X_u) du]_t = 0$ due to \cite[Proposition 1.4.49(d)]{17} where

$$N_t := M_t^2 + 2p^2(X_0) - 2M_t^p p(X_0) - 2 \int_0^t \int_0^t G_p(X_u) dv M_v^p, \quad t \geq 0$$

is a martingale. Consequently,

$$\langle M(p), M(q) \rangle_t = \int_0^t a_{p,q}(X_s) ds.$$

Now let $q \in P_1$. We have

$$\langle M(p), M(q) \rangle_t = \frac{1}{2} \left( \langle M(p + q), M(p + q) \rangle_t - \langle M(p), M(q) \rangle_t - \langle M(q), M(p) \rangle_t \right)$$

$$= \int_0^t a_{p,q}(X_s) ds$$

for any $t \geq 0$ as claimed. \hfill $\Box$

We can further sharpen our results for $E$-valued diffusions.

**Definition 5.4.** We say that an $E$-valued process $X$ is a diffusion with drift coefficient $\beta : \Omega \times \mathbb{R}_+ \to B$ and diffusion coefficient $\sigma : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ if for any $p \in \mathcal{L}$, $\beta(p) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is progressively measurable and pathwise locally integrable, $\sigma(p) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is progressively measurable and pathwise locally square-integrable, and

$$dp(X_t) = \beta_t(p) dt + \sigma_t(p) dW_t^p$$

for any $p \in \mathcal{P}$, where $W_t^p$ is a standard Brownian motion with values in $\mathbb{R}$.

Under the additional assumptions that a Brownian motion exists on the filtered probability space and that sample paths are continuous, we can show that $X$ is an $E$-valued diffusion in the sense of Definition 5.4.

**Corollary 5.5.** Assume that the requirements of Theorem 5.3 are met, $p(X)$ has continuous sample paths for any $p \in \mathcal{P}$ and that there is some $F$-standard Brownian motion. Then there is an affine function $\beta : B \to B$ such that $X$ is a diffusion with drift coefficient $\beta(\tilde{X}_t)$ and some diffusion coefficient $\sigma$, and

$$[p(X), q(X)]_t = \int_0^t a_{p,q}(X_s) ds, \quad t \geq 0$$

where $[p(X), q(X)]_t$ denotes the quadratic covariation and

$$a : \mathcal{L} \times \mathcal{L} \to \mathcal{P}, (p, q) \mapsto a_{p,q} := G(pq) - pGq - qGp.$$

**Proof.** Recall from Theorem 5.3 $b \in B$, the linear operator $A : B \to B$ as well as the weak*-martingale $M$. We define the affine function $\beta : B \to B, \beta(x) := b + Ax$. By Theorem 5.3 we have that

$$dp(X_t) = \beta(\tilde{X}_t) dt + dM_t(p)$$

for any $p \in \mathcal{P}$. Recall from Theorem 5.3

$$a : \mathcal{L} \times \mathcal{L} \to \mathcal{P}, (p, q) \mapsto a_{p,q} := G(pq) - pGq - qGp.$$
The operator $a$ is bilinear. From Theorem 5.3 we have

$$[p(X), q(X)]_t = [M_{ij}(p), M_{ij}(q)]_t = \int_0^t a_{p,q}(X_s)ds, \quad t \geq 0$$

as required. Jacod [13, Corollaire 14.47(b)] yields that $p(X)$ is a diffusion. □

**Remark 5.6.** Say $dX_t = (\mu + \gamma X_t)dt + \sigma(X_t)dW_t$ for some Brownian motion $W$, $\mu, \gamma \in \mathbb{R}$ and a measurable function $\sigma$ on $\mathbb{R}$ of at most linear growth. One does expect that $X$ has generator

$$\mathcal{G}f(x) = (\mu + \gamma x)f'(x) + \frac{\sigma^2(x)}{2} f''(x)$$

For linear functions $p(x) = \alpha x$, $q(x) = \beta x$ one has

$$a_{p,q}(x) = (\mathcal{G}(pq) - p\mathcal{G}q - q\mathcal{G}p)(x) = \sigma^2(x)\alpha \beta$$

which shows that $a_{p,q}$ is the quadratic covariation coefficient of $p(X)$ and $q(X)$ which appears in Corollary 5.5.

**Example 5.7.** Let $u : (x_0, x_1) \to \mathbb{R}$ be the unique solution of the second order ODE

$$u''(x) = x^2/u(x), \quad x \in (x_0, x_1)$$

with $u(0) = 1$ and $u'(0) = 0$ where $x_1 \in (0, \infty]$ is chosen maximally such that a solution exists, and $x_0 \in [-\infty, 0)$ is chosen minimally such that a solution exists. Observe that $u''$ is positive because $u$ is strictly positive on $(x_0, x_1)$. Thus $u'$ is increasing and, hence, positive on $[0, x_1)$ and negative on $(x_0, 0)$. Consequently, $u$ is increasing on the positive half-line and decreasing on the negative half-line while attaining its minimal value in 0 with $u(0) = 1$. Thus we find that

$$u''(x) \leq x^2, \quad x \in [0, x_1).$$

Consequently, we have $|u'(x)| \leq \frac{|x|^3}{3}$ and $u(x) \leq 1 + \frac{4}{12}$ for any $x \in (x_0, x_1)$. This implies that $x_1 = \infty$ and $x_0 = -\infty$.

Now define $\sigma(x) := \sqrt{u(x)}$ for $x \in \mathbb{R}$. Let $X$ be a solution to the SDE

$$dX_t = \sigma(X_t)dW_t, \quad X_0 = 0$$

For polynomials $\mathcal{P}$ we use the span of $\{1, x, x^2, u(x)\}$, and $\mathcal{L}$ denotes the span of $\{x\}$. Since $\mathcal{P}$ is finite dimensional Proposition 3.7 yields that it suffices to state the generator of the action $T$, which we define by

$$\mathcal{G}p(x) := \frac{1}{2} u(x)p''(x), \quad p \in \mathcal{P}, x \in E := \mathbb{R}.$$

Note that $\mathcal{G}p_1 = \{0\}$, $\mathcal{G}u(x) = \frac{x^2}{2}$ and $\mathcal{G}x^2 = u(x)$ for any $x \in E$ and hence the corresponding action is given by $T_p(x) := \sum_{j=0}^{\infty} \frac{1}{2^j} \mathcal{G}^j p(x)$ for any $p \in \mathcal{P}$, $x \in E$. Moreover,

$$\mathbb{E}[p(X_t) | \mathcal{F}_s] = T_{t-s} p(X_s)$$

and, therefore, $X$ is a $\mathcal{P}$-polynomial process with action $T$. However, $\mathcal{P}_1 : = \mathcal{P}_1$ is a proper subset of $\mathcal{P}$.
6. **Affine Drift for Strongly Continuous Polynomial Processes**

In this section we analyze polynomial processes under the assumptions that the polynomial action is a strongly continuous semigroup. We provide conditions which similar as in Section 4 under algebraic conditions allow to understand the polynomial process $X$ as a process with affine drift. Since $X$ does not take values in a linear space, again we will need to linearise $E$ first and then identify an additive decomposition of $X$ into a process which is 'mean-zero' like and an 'affine drift term'. More precisely, we aim at a decomposition of the following type

$$X_t = U_tX_0 + \int_0^t U_s b \, ds + R_t$$

where the remainder term $R$ is weakly mean-zero, $(U_t)_{t \geq 0}$ is a semigroup of linear operators and $b$ is some constant. In some sense, this means that the drift of $X$ is given as $\int_0^t (b + AX_s)ds$ where $A$ may be thought of as a derivative of $U$ with respect to $t$ at time zero. Several problems will occur, though. First, $b$ can be outside the linearisation of $E$, and possibly further elements have to be added. Second, $U$ does not need to be strongly continuous but only weakly continuous and the meaning of generator is blurred. The second problem is avoided by leaving $U$ as it is and not passing to the generator in case it does not exist. None of these problems occur in the classical case where $P_1$ is finite-dimensional as in the first work on polynomial processes by Cuchiero [4]. Our main result of this section is summarised in Theorem 6.6.

As previously done in Section 4, also in this section we will not make use of the entire structure of $P$, as only $P_0$ and $P_1$ will matter in our analysis. Doing so we might loose the algebraic property of the polynomials which, however, is not needed in this section anyway.

In this section we make the following assumption throughout:

**Assumption 6.1.** There is a norm $\| \cdot \|$ on $P_1$ such that:

1. $T_h|_{P_1}$ has an extension $(T_h)_{h \geq 0}$ to the completion of $(P_1, \| \cdot \|)$ which is a strongly continuous semigroup of bounded operators.
2. The maps $\delta_x : \mathcal{L} \rightarrow \mathbb{F}, f \mapsto f(x)$ are $\| \cdot \|$-continuous for any $x \in E$.
3. $P_1$ is separating for $E$, i.e. for any $x, y \in E$ there is $p \in P_1$ with $p(x) \neq p(y)$.

We recall that $\mathcal{L}$ denotes a closed vector space complement of $P_0$ in $P_1$ (also denoted $\mathcal{M}_1$, the first-order monomials in our setting). We also note in passing that if $(E, \| \cdot \|_E)$ is a Banach space and $P_1$ a subset of linear functionals, then the norm on $P_1$ is naturally chosen to be the operator norm on linear functionals, i.e., the elements in the dual of $E$.

The dual space of $\mathcal{L}$ is denoted by $(B, \| \cdot \|_B)$, i.e. $B$ is the set of $\| \cdot \|$-continuous linear maps on $(\mathcal{L}, \| \cdot \|)$. The space $B$ will play the role as the linearisation of $E$. One sees that the dual space $P_1^*$ of $P_1$ is isomorphic to $\mathbb{F} \times B$ with norm $\|(c, b)||_{\mathbb{F} \times B} := |c| + \|b\|_B$. The extra $\mathbb{F}$-dimension, which is generated by the constant functions, does not play a substantial role in our analysis to come.

The set $E$ has a natural embedding into $B$ because $\delta_x$ as a functional on $\mathcal{L}$ is linear and continuous by property (2) of Assumption 6.1 that is, $\delta_x \in B$. Moreover, $P_1$ is separating for $E$ and, hence, so is $\mathcal{L}$. I.e., $\delta_x \neq \delta_y$ for any $x, y \in E$ with $x \neq y$, which implies that

$$\delta : E \rightarrow B, x \mapsto \delta_x$$
is an injective map. We denote the embedding of \( X \) into \( B \) by
\[
\tilde{X}_t := \delta_X, \quad t \geq 0.
\]
This is how \( B \) will be interpreted as a linearisation of \( E \). We emphasise in passing that in the previous section, \( B \) denoted the algebraic dual, while now it is the space of continuous linear functionals on \( \mathcal{L} \).

**Remark 6.2.** If \((E, \| \cdot \|_E)\) is a reflexive Banach space and \((\mathcal{L}, \| \cdot \|)\) its dual, then \( E \) is isometrically isomorphic to \( B \) where the embedding is \( \delta : E \to B, x \mapsto \delta_x \). Also, note that we assumed \((T_h)_{h \geq 0}\) to be strongly continuous (by Property (1) in Assumption [6.4]). However, its dual semigroup on \( F \times B \) does not need to be strongly continuous. Consider the following example: The left-shift semigroup \( U \) on \( L^1(\mathbb{R}) \) is generated by the weak derivative, i.e. \( \mathcal{G}f = f' \), where \( f \in L^1(\mathbb{R}) \) is absolutely continuous with absolutely continuous derivative \( f' \in L^1(\mathbb{R}) \). The dual of \( L^1(\mathbb{R}) \) is isometric to \( L^\infty(\mathbb{R}) \) with \( \langle g, f \rangle = \int_{\mathbb{R}} f(x)g(x)dx \) for \( f \in L^1(\mathbb{R}) \) and \( g \in L^\infty(\mathbb{R}) \). The dual of the left-shift \( U \) is the right-shift \( R \) on \( L^\infty(\mathbb{R}) \), which is not strongly continuous because \( R_{t \downarrow 0} 1_{[0,1]} \not\to 1_{[0,1]} \) in norm (the norm being the uniform norm) as \( t \downarrow 0 \). \( \mathcal{G}^* = -\partial_x \) on its domain. (This is an example with a strongly continuous group!)

**Remark 6.3.** For any \( p \in \mathcal{P}_1 \), property (3) in Definition [2.4] yields that the \( \mathbb{F} \)-valued process \( p(\tilde{X}) \) has c\'adl\'ag paths. Notice that for \( p \in \mathcal{L} \), we mean by \( p(\tilde{X}_t) = \tilde{X}_t(p) = p(X_t) \). Thus, the notation \( p(\tilde{X}) \) makes use of the identification of \( E \) with \( B \). For general \( p \in \mathcal{P}_1 \), we use that the dual space of \( \mathcal{P}_1 \) is isomorphic to \( \mathbb{F} \times B \), as noted above.

We start by inspecting the structure of the semigroup \( \mathcal{T} \). We use the notations
\[
(7) \quad \overline{T}_h^{(i)} := \Pi_i T_h |_{\overline{\mathcal{P}}}, \quad i = 0, 1,
\]
where \( \Pi_0, \Pi_1 \) are projectors on \( \overline{\mathcal{P}}_1 \) with \( \Pi_0 + \Pi_1 \) equal to the identity and with ranges \( \mathcal{P}_0 \) and \( \mathcal{L} \), respectively. Lemma [3.1] implies on \( \overline{\mathcal{P}}_1 = \mathcal{P}_0 \oplus \mathcal{L} \) that
\[
(8) \quad \text{matrix}(\overline{T}_h) = \begin{pmatrix} I & \overline{T}_h^{(0)} \\ 0 & \overline{T}_h^{(1)} \end{pmatrix}.
\]
This implies that the generator \( \mathcal{G} \) of \( \mathcal{T} \) has the structure
\[
\text{matrix}(\mathcal{G}) = \begin{pmatrix} 0 & \mathcal{G}_0^{(0)} \\ 0 & \mathcal{G}_0^{(1)} \end{pmatrix},
\]
where \( \mathcal{G}_0^{(1)} \) is the generator of \( \overline{T}_h^{(1)} \) and
\[
\mathcal{G}_0^{(0)} := \{(f, c) \in \mathcal{L} \times \mathcal{P}_0 : c = \lim_{h \downarrow 0} \frac{1}{h} \overline{T}_h^{(0)} f \}.
\]
Equation (7) yields that the dual operator \( \overline{U}_h \) of \( \overline{T}_h \) has the presentation
\[
\text{matrix}(\overline{U}_h) = \begin{pmatrix} I & 0 \\ \overline{U}_h^{(0)} & \overline{U}_h^{(1)} \end{pmatrix}
\]
on \( \mathbb{F} \times B \) for any \( h \geq 0 \) where \( \overline{U}_h^{(1)} = (\overline{T}_h^{(1)})^* \) and \( \overline{U}_h^{(0)} = (\overline{T}_h^{(0)})^* \).
Lemma 6.4. Let $\mathcal{D}$ be the domain of $\mathcal{G}$ and $\mathcal{D}^{(1)}$ be the domain of $\mathcal{G}^{(1)}$. Then we have
\[ \mathcal{D} = \mathcal{P}_0 \oplus (\mathcal{D} \cap \mathcal{L}), \quad \mathcal{D} \cap \mathcal{L} \subseteq \mathcal{D}^{(1)} \]
and for any $f = c + \ell \in \mathcal{P}_0 \oplus (\mathcal{D} \cap \mathcal{L})$ we have
\[ \mathcal{G}f = \mathcal{G}^{(0)} \ell + \mathcal{G}^{(1)} \ell. \]

Proof. Clearly, $\mathcal{P}_0 \subset \mathcal{D}$ because for any $c \in \mathcal{P}_0$,
\[ \frac{T_h c - c}{h} = \frac{T_h c - c}{h} = 0, \]
and we find that $\mathcal{G}c = 0$. Thus, we have $\mathcal{D} \supseteq \mathcal{P}_0 \oplus (\mathcal{D} \cap \mathcal{L})$. Let $f \in \mathcal{D}$, $\ell := \Pi_1 f$ and $c := f - \ell \in \mathcal{P}_0$. Since the domain of the generator is a vector space we find that $\ell \in \mathcal{D}$. Consequently, we have
\[ \mathcal{D} = \mathcal{P}_0 \oplus (\mathcal{D} \cap \mathcal{L}). \]

Now, let $\ell \in \mathcal{D} \cap \mathcal{L}$. Then we have
\[ \frac{T_h^{(1)} \ell - \ell}{h} = \frac{\Pi_1 T_h \ell - \ell}{h} \to \Pi_1 \mathcal{G} \ell \]
and hence $\ell \in \mathcal{D}^{(1)}$ and $\mathcal{G}^{(1)} \ell = \Pi_1 \mathcal{G} \ell$. \qed

Remark 6.5. Let $M$ be an $\mathbb{R}^d$-valued martingale, $A \in \mathbb{R}^{d \times d}$ and $b \in \mathbb{R}^d$. Then the process given by
\[ Y_t = Y_0 + \int_0^t b + AY_s ds + M_t, \quad t \geq 0 \]
satisfies
\[ Y_t = V_t Y_0 + \int_0^t V_t b ds \to \int_0^t V_t - s dM_s, \quad t \geq 0 \]
where $V_t = \exp(tA)$ for any $t \geq 0$. These two representations are of course equivalent in finite dimension. On a more general structure, the second representation of $Y$ might not be good enough to recover the first. $V$ might be non-differentiable, and stochastic integration might be ill-defined so that only the process $R_t = \int_0^t V_t - s dM_s$ is available. This is the situation we describe in the next theorem.

We come to our main result in this Section, showing the affine drift of polynomial processes.

Theorem 6.6. There is a Banach space $(B^+, \| \cdot \|_+)$ which contains $B$ as a sub-vector space, and an element $b \in B^+$. Moreover, the dual semigroup of $T^{(1)}$ can be extended uniquely to a semigroup $U$ of bounded operators on $B^+$ such that
\[ \tilde{X}_t = U_t \tilde{X}_0 + \int_0^t U_t b ds + R_t \]
where $R : \Omega \times \mathbb{R}^+ \to B^+$ with $\mathbb{E}[\ell(R_t)] = 0$ for any $\ell \in \mathcal{L}$. The integral in the representation above is understood in a weak*-sense, i.e. $\int_0^t U_t b ds$ is the unique element $c_t \in B$ (which exists) with $p(c_t) = \int_0^t p(U_t b) ds$ for any $p \in \mathcal{C}$ where $\mathcal{C}$ is the domain of $\mathcal{G}$ restricted to $\mathcal{L}$, i.e., $\mathcal{D} \cap \mathcal{L}$ being a dense subspace of $\mathcal{L}$.
If additionally, $\bar{T}$ can be extended to a group of bounded operators, then $M_t := (\bar{T}_t(1))_t^\ast R_t$ is a weak*-martingale, i.e. for any $\ell \in \mathcal{L}$ one has that $\ell(M)$ is an $\mathcal{F}$-valued martingale. In particular, one has
\[
\bar{X}_t = U_t \bar{X}_0 + \int_0^t U_s b ds + U_t M_t, \quad t \geq 0.
\]

If the domain of $\mathcal{G}$ is $\mathcal{L}$, then $\bar{T}$ can be extended to a group of bounded operators and $(B^+, \| \cdot \|_+)$ can be chosen to be equal to $(B, \| \cdot \|_B)$.

**Proof.** **Construction of the extension space $B^+$:** Let $\mathcal{C}$ be the domain of the generator $\mathcal{G}$ restricted to $\mathcal{L}$ and define the norm
\[
\| f \|_\mathcal{C} := \| f \| + \| \mathcal{G} f \|, \quad f \in \mathcal{C}.
\]
Then $(\mathcal{C}, \| \cdot \|_\mathcal{C})$ is a Banach space and we denote its dual space by $(B^+, \| \cdot \|_+)$.

Moreover, Lemma 6.3 yields that $\mathcal{C}$ is $\| \cdot \|_\mathcal{C}$-dense in $\mathcal{L}$. For orientation we have the following diagram:

\[
\begin{array}{ccc}
(\mathcal{L}, \| \cdot \|) & \text{has dual space} & (B, \| \cdot \|_B) \\
\cup & \cap & \cap \uparrow \|

(\mathcal{C}, \| \cdot \|_\mathcal{C}) & \text{has dual space} & (B^+, \| \cdot \|_B+),
\end{array}
\]

We have $\mathcal{C}$ is invariant under $\bar{T}^{(1)}$ and $\bar{T}^{(3)}$ restricted to $\mathcal{C}$ is bounded with respect to the $\| \cdot \|_\mathcal{C}$-operator norm. We denote the dual semigroup of operators of $(\bar{T}_h(1)|c)_{h \geq 0}$ by $U$, i.e. $(U_h)_{h \geq 0}$ is a semigroup of bounded operators on $B^+$. Note that $U_h|_B$ is the dual of $\bar{T}_h(1)$ for any $h \geq 0$.

**Construction of the constant drift part $b \in B^+$:** We have $\mathcal{G}^{(0)}$ is an operator from dom($\mathcal{G}$) to $\mathcal{P}_0$, and therefore find that $\mathcal{G}^{(0)}|_C : C \to \mathcal{P}_0$. Hence, for its dual operator we have $(\mathcal{G}^{(0)}|_C)^* : (\mathcal{P}_0)^* \to B^+$, where $(\mathcal{P}_0)^*$ is the dual of the one-dimensional space $\mathcal{P}_0$. Since $\mathcal{P}_0$ is one-dimensional, there is $1 \in (\mathcal{P}_0)^*$ such that $1(p) = p(x)$ for any $x \in E$. We define $b := (\mathcal{G}^{(0)}|_C)^*1$.

**Existence of the weak*-integral on $B^+$ with values in $B$:** Let $p \in \mathcal{C}$ and $s \geq 0$. We have $\Pi_0 \mathcal{G} T_s p \in \mathcal{P}_0$ and, hence, a constant function. Thus for any $x \in E$ we find that $f_p(s) := (\Pi_0 \mathcal{G} T_s p)(x)$ is its value. By the $c_0$-semigroup property of $T$ we find that $s \mapsto f_p(s)$ is continuous and, hence Lebesgue-integrable. Also, observe that $p(U_t b) = f_p(s)$ by duality.

We have
\[
\int_0^t p(U_t b) ds = \int_0^t f_p(s) ds = \Pi_0 \int_0^t \mathcal{G} T_s p ds(x) = \Pi_0(T_t p - T_0 p)(x) = (T_t^{(0)} - \Pi_0)p(x) = T_t^{(0)} p(x),
\]

(9)

where the integral in the second line is a Bochner-integral in $(\mathcal{L}, \| \cdot \|)$ and the point $x \in E$ is arbitrary because both $T_t^{(0)}$ and $\Pi_0$ map into the constant functions $\mathcal{P}_0$. The last equality holds because simply $\Pi_0(\mathcal{L}) = \{0\}$. The functional $c_0 : \mathcal{L} \to$
\( F, p \mapsto (T^{(0)}_t - \Pi_0)p(x) \) is continuous linear and hence an element of \( B \subseteq B^+ \). From the equation above we find that

\[
\int_0^t p(U_s b)ds = p(c_t)
\]

and therefore \( c_t = \int_0^t U_s b ds \) as a weak*-integral on \( B^+ \).

**Construction of \( R \):** We define \( R_t := \tilde{X}_t - U_t \tilde{X}_0 - \int_0^t U_s b ds \) for any \( t \geq 0 \). Since \( B \) is \( U \)-invariant and by the above argument \( \int_0^t U_s b ds \in B \), we find that \( R_t \in B \) for any \( t \geq 0 \), \( P \)-a.s. We next show that \( \mathbb{E}[|\ell(R_t)|] < \infty \) for \( \ell \in \mathcal{D}_0 \).

Let \( p \in \mathcal{C} \). We have by polynomial property of \( X \)

\[
\mathbb{E}[p(R_t)] = \mathbb{E}\left[p(\tilde{X}_t) - T^{(1)}_t p(\tilde{X}_0) - p\left(\int_0^t U_s b ds\right)\right] = T_t p(\tilde{X}_0) - T^{(1)}_t p(\tilde{X}_0) - p\left(\int_0^t U_s b ds\right) = T_t p(\tilde{X}_0) - T^{(1)}_t p(\tilde{X}_0) - T^{(0)}_t p(\tilde{X}_0) = 0.
\]

Since \( R \) is \( B \)-valued and we have the required identity on \( \mathcal{C} \), we can extend it to \( \mathcal{D}_0 \) by a density argument. Also note that \( R_0 = 0 \).

**Martingale representation when \( T \) can be extended:** We now assume that \( T \) can be extended to a group of bounded operators and let \( p \in \mathcal{C} \). Note that \( (T^{(1)}_{-t})^* \) is the inverse of \( U_t \) and we will simply denoted it as \( U_{-t} \). Obviously, \( (U_t)_{t \in \mathbb{R}} \) is a group of operators on \( B^+ \). We find that

\[
M_t = U_{-t} R_t = U_{-t} \tilde{X}_t - \tilde{X}_0 - U_{-t} \int_0^t U_s b ds.
\]

Moreover,

\[
U_{-t} \int_0^t U_s b ds = \int_0^t U_{s+t} b ds
\]

because for \( q \in \mathcal{C} \) one has \( T^{(1)}_{-t} q \in \mathcal{C} \) and

\[
q(U_{-t} \int_0^t U_s b ds) = T^{(1)}_{-t} q(\int_0^t U_s b ds) = \int_0^t T^{(1)}_{-t} q(U_s b) ds = \int_0^t q(U_{s+t} b) ds.
\]
Also we have by the polynomial property of $X$

$$
E[p(U_{t-s} \tilde{X}_t)|\mathcal{F}_s] = E[T_{t-s}^{(1)} p(\tilde{X}_t)|\mathcal{F}_s]
$$

$$
= T_{t-s} T_{t-s}^{(1)} p(\tilde{X}_s)
= T_{t-s}^{(0)} T_{t-s}^{(1)} p(\tilde{X}_s) + T_{t-s}^{(1)} p(\tilde{X}_s)
= \int_{t-s}^{t} T_{t-s}^{(1)} p(U_t b) dr + p(U_{t-s} \tilde{X}_s)
= \int_{t-s}^{t} p(U_t b) dr + p(U_{t-s} \tilde{X}_s).
$$

In the third equality we used the matrix representation of $T$ in (8), and in the fourth equality we make use of the representation (9). Hence, we find that

$$
\mathbb{E}[p(M_t)|\mathcal{F}_s] = \int_{0}^{t-s} p(U_t b) dr + p(U_{t-s} \tilde{X}_s) - p(\tilde{X}_0) - \int_{0}^{t} p(U_t b) dr
= p(M_s)
$$

as required. Since $M$ is $B$-valued we find that $\mathbb{E}[p(M_t)|\mathcal{F}_s] = p(M_s)$ for any $p \in \mathbb{L}$. The Theorem is proven.

**Corollary 6.7.** Under the assumptions and notations of Theorem 6.6 and for fixed $s \geq 0$ there is a progressively measurable process $(R_t^s)_{t \geq 0}$ with $\mathbb{E}[R_t^s|\mathcal{F}_s] = 0$ for any $t \geq 0$ and

$$
\tilde{X}_t = U_{t-s} \tilde{X}_s + \int_{s}^{t} U_r b dr + R_t^s
$$

for any $t \geq s$.

**Proof.** Define $R_t^s = 0$ for $t \in [0, s]$ and $Y_u := X_{u+s}$ for $u \geq 0$. $Y$ is a polynomial process with action $T$ relative to the filtration $(\mathcal{F}_{u+s})_{u \geq 0}$. According to Theorem 6.6 there is a mean-zero process $(R_u)_{u \geq 0}$ which is progressively measurable with respect to the filtration $(\mathcal{F}_{u+s})_{u \geq 0}$ such that

$$
Y_u = Y_0 + \int_{0}^{u} U_r b dr + R_u
$$

for any $u \geq 0$. Define $R_t^s := R_{t-s}$ for $t > s$. The claim follows. \hfill $\Box$

**Remark 6.8.** Note that the decomposition

$$
(10) \quad \tilde{X}_t = U_t \tilde{X}_0 + \int_{0}^{t} U_r b dr + U_t M_t
$$

which appears in Theorem 6.7 when $T$ can be extended to a $c_0$-group implies that

$$
\tilde{X}_t = U_{t-s} \tilde{X}_s + \int_{0}^{t-s} U_r b dr + U_t (M_t - M_s)
$$

for any $0 \leq s \leq t$.

If the dual $A$ of $\mathcal{T}^{(1)}$ is densely defined and generates a $c_0$-semigroup (which then is $U$), the expression in (10) is the mild solution (see Peszat and Zabczyk [20]) of

$$
d\tilde{X}_t = (b + A \tilde{X}_t) dt + dN_t
$$

where $N_t := \int_{0}^{t} U_{t-s} dM_s$. This holds true whenever we have available a martingale integration theory (see for example van Neerven [19] for stochastic integration on
isomorphism) the dual space of \((\mathcal{L}, \| \cdot \|)\) be the image-wise product. Since \((\mathcal{L}, \| \cdot \|)\) is reflexive, it is (up to isometric isomorphism) the dual space of \((\mathcal{L}, \| \cdot \|)\). In particular, we have \(B \cong E\). Let \(W\) be the Wiener process with covariance operator \(Q\) on \(E\) given by \((Qx)_n := \frac{x_n}{(1+n)^2}\) for \(x \in E\). Let \((Ax)_n := 2\pi inx_n\). \(A\) is a normal operator and it generates the \(c_0\)-group 
\[
(U_t x)_n = e^{2\pi int}x_n.
\]
Moreover, \(A^* = -A\), which generates the \(c_0\)-group \((U_{-t})_{t \in \mathbb{R}}\). Note that if \(x \in \text{dom}(A^*) = \text{dom}(A)\), then \(\sum_{n \in \mathbb{N}} |x_n| < \infty\) because \(Ax \in B\) and, hence, \(x_n = (Ax)_n \frac{1}{2\pi in}\) is the product of two elements in \(B\), from which it follows \(\text{dom}(A) \subseteq l^1(\mathbb{N}, \mathbb{R})\). Now define \(\Gamma : \text{dom}(A^*) \to \mathbb{C}, x \mapsto \sum_{n \in \mathbb{N}} x_n\) which corresponds to the element 
\[
b := (1)_{n \in \mathbb{N}} \in l^1(\mathbb{N}, \mathbb{C}) =: B^+.
\]
\(U\) extends naturally to \(B^+\) and 
\[
\left(\int_0^t U_s b \, ds\right)_n = \frac{e^{2\pi int} - 1}{2\pi in}, \quad t \geq 0, n \in \mathbb{N}, n \geq 1.
\]
Thus, \(\int_0^t U_s b ds \in E\). We now define 
\[
X_t := \int_0^t U_s b ds + \int_0^t U_{t-s} dW_s, \quad t \geq 0
\]
which is an \(E\)-valued polynomial process which can be interpreted as the mild solution to the SPDE 
\[
dX_t = (b + AX_t) dt + dW_t.
\]
We refer to Peszat and Zabczyk \([20]\) for mild solutions of SPDEs.

It is interesting to notice that the state space \(E\) of the polynomial process introduced in the Example above is a Hilbert space. Even for such nice state spaces we may have polynomial processes where the drift \(b\) is outside the state space. On the other hand, this can only happen when the semigroup \(U\) is sufficiently regular.

We close this section with the following corollary.

**Corollary 6.10.** Suppose \((\mathcal{P}_1, \| \cdot \|)\) is a Banach space and \(T\) is locally finite. Then \(T\) extends to a group and we denote \(U\) to be the dual group of \(T|_{\mathcal{L}}\).

Furthermore, there is \(b \in B\) and a weak*-martingale \(M\) such that 
\[
\tilde{X}_t = U_t \tilde{X}_0 + \int_0^t U_s b ds + U_t M_t.
\]

Also, \(U\) is differentiable with respect to \(t\) and its generator \(A := \partial_t U_t|_{t=0} \in L(B)\) is the dual of \(\mathcal{G}\).
Proof. Proposition 3.6 yields that $T$ can be extended to a group of operators on $\mathcal{P}_1$. Also, $T$ is strongly continuous because by assumption for any $p \in \mathcal{P}_1$ there is a finite dimensional space $V_p \subseteq \mathcal{P}_1$ with $p \in V_p$ and $T_t(V_p) \subseteq V_p$ for any $t \geq 0$. Proposition 3.6 yields strong continuity of $T_t|_{V_p}$ and that its generator has domain $V_p$. Consequently, $T$ is strongly continuous and its generator has domain $\mathcal{P}_1$. Then its generator is a bounded operator and

$$T_t = \exp(tG) := \sum_{n=0}^{\infty} \frac{G^n}{n!}.$$  

Its dual group $U$ satisfies

$$U_t = \exp(tA)$$

where $A$ is the dual of $G$ and the claim follows from Theorem 6.6. □

7. Covariance Structure of Polynomial Diffusions

Similar as in Section 5 for locally finite polynomial processes we shall now analyse the covariance structure for polynomial diffusions under continuity assumptions.

Assumption 7.1. There is a complete norm $\| \cdot \|$ on $\mathcal{P}_4$ such that:

1. $(T_h|_{\mathcal{P}_4})_{h \geq 0}$ is a strongly continuous semigroup of bounded operators.
2. The maps $\delta_x : \mathcal{P}_4 \to \mathbb{F}, f \mapsto f(x)$ are $\| \cdot \|$-continuous for any $x \in E$.
3. The linear functional $\Gamma : \mathcal{P}_4 \to \mathbb{F}, p \mapsto \mathbb{E}[p(X_0)]$ is continuous.

Also we assume that:

4. $\mathcal{P}_1$ is separating for $E$, i.e. for any $x, y \in E$ there is $p \in \mathcal{P}_1$ with $p(x) \neq p(y)$.
5. $\mathbb{F} = \mathbb{R}$
6. $\mathcal{P}_1 \cdot \mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{P}_2 \cdot \mathcal{P}_2 \subseteq \mathcal{P}_4$.

Recall that the generator $\mathcal{G}$ had been defined (in Definition 2.3) in a way which does not use the topology on $\mathcal{P}_4$. However, if $p$ is in the domain $\mathcal{C}$ of the generator of the $c_0$-semigroup $(T_h|_{\mathcal{P}_4})_{h \geq 0}$, i.e. there is $q \in \mathcal{P}_4$ such that

$$q = \lim_{h \downarrow 0} \frac{T_hp - p}{h},$$

then $p \in \mathcal{D}$ (which was defined in Definition 2.3) because $\delta_x$ is continuous which yields

$$q(x) = \lim_{h \downarrow 0} \frac{T_hp(x) - p(x)}{h}, \quad x \in E.$$  

In this case we also have $\mathcal{G}p = q$ which reveals that $\mathcal{G}|_{\mathcal{C}}$ is the generator of the $c_0$-semigroup $(T_h|_{\mathcal{P}_4})_{h \geq 0}$. Also, note that $\mathcal{G}p \in \mathcal{P}_n$ for any $p \in \mathcal{C} \cap \mathcal{P}_n$ for $n = 0, \ldots, 4$ because $\mathcal{P}_n$ is a closed $T$-invariant space.

Remark 7.2 (Incomplete norm). In Assumption 7.1 we assumed that the norm is complete. If the norm is not complete one would like to pass to the completion and replace $\mathcal{P}_n$ with its completion for $n = 1, \ldots, 4$.

This is unproblematic for assumption (2) to (5) as $\delta_x$ and $\Gamma$ can be extended to bounded linear maps on the completion. The continuous extension of $T_h$ to the completion is still a bounded operator and the family of continuations is still a semigroup. However, the family of continuations is strongly continuous if and only if $\sup_{h \in [0,1]} \|T_h\|_{\text{op}}$ is bounded. Finally, assumption (6) does not necessarily carry
over to the completion. A sufficient condition to still hold on the completion is that the multiplication from \( P_2 \times P_2 \to P_4 \) is a bounded bilinear map.

Finally we like to note that the degree of some polynomials might be lower after extending to the completion (since it is possible that some elements of \( P_n \) can be approximated by elements in \( P_{n-1} \) for \( n = 2, 3, 4 \)) but this does not pose any problem.

We first state a simple consequence of the fundamental theorem of calculus for the semigroup \( T \).

**Lemma 7.3.** Let \( p \in C \) Then we have
\[
T_t p = T_s p + \int_s^t G_{T_r} p \, dr, \quad t \geq s \geq 0.
\]

**Proof.** We have \( T_t p \in C \) and
\[
\partial_t T_t p = GT_t p
\]
for any \( t \geq 0 \). Since \( G|_C \) is closed we have that
\[
t \mapsto GT_t p
\]
is continuous. The Fundamental Theorem of Calculus yields the claim. \( \square \)

We also have a useful martingale-result for a class of polynomials \( p \in P_2 \):

**Lemma 7.4.** Let \( p \in C \cap P_2 \). Then
\[
M_t^p := p(X_t) - \int_0^t G_p(X_r) \, dr, \quad t \geq 0
\]
defines a martingale with \( E[|M_t^p|^2] < \infty \) for any \( t \geq 0 \).

**Proof.** Define \( q := (Gp)^2 \) and observe that \( q \in P_4 \) by Assumption 7.1(6) with \( q(x) \geq 0 \) for any \( x \in E \). Let \( 0 \leq s \leq t \). We get from Cauchy-Schwarz’ inequality that
\[
E[| \int_0^t G_p(X_r) \, dr |^2] \leq E[ t \int_0^t |G_p(X_r)|^2 \, dr ]
\]
\[
\leq t E[ \int_0^t q(X_r) \, dr ]
\]
\[
= t \int_0^t E[ T_r q(X_0) ] \, dr
\]
\[
= t \Gamma \left( \int_0^t T_r qdr \right) < \infty.
\]
In the first equality above we applied the tower property with conditioning on \( \mathcal{F}_0 \) and in the last bound Assumption 7.1(3). Hence, \( M_t^p \) has finite expectation. Fubini’s theorem for the conditional expectation yields
\[
E[ \int_0^t G_p(X_r) \, dr | \mathcal{F}_s ] = \int_0^s G_p(X_r) \, dr + \int_s^t G T_{r-s} p(X_s) \, dr.
\]

Lemma 7.3 yields
\[
\int_s^t G T_{r-s} p(X_s) \, dr = T_{t-s} p(X_s) - p(X_s).
\]
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Thus,
\[ \mathbb{E}[M^p_t | \mathcal{F}_s] = T_{t-s} p(X_s) - \left( \int_0^s G_p(X_r)dr + T_{t-s} p(X_s) - p(X_s) \right) = M^p_s \]

which shows that \( M^p \) is a martingale.

We have
\[ \mathbb{E}[|M^p_t|^2] \leq 2\mathbb{E}[p^2(X_t)] + 2\mathbb{E} \left[ \left( \int_0^t G_p(X_r)dr \right)^2 \right] < \infty. \]

□

In the previous lemma we made use of a square argument to ensure that \( M^p \) is a martingale with finite second moment. When dealing with classical \( d \)-variable polynomials this can be improved because any \( d \)-variable polynomial of degree at most \( n \) where \( n \) is even can be written as a finite linear combination of positive polynomials of degree \( n \). This, however might fail for abstract polynomials or the infinite dimensional case. For this reason we require the moment condition \( P^2 \cdot P^2 \subset P^4 \) in the last lemma and that \( pq \) is in the domain of \( G \) in the next theorem.

**Theorem 7.5.** Let \( p, q \in \mathcal{C} \cap \mathcal{L} \) with \( pq \in \mathcal{C} \) and define
\[
M^p_t := p(X_t) - \int_0^t G_p(X_r)dr \quad \text{and} \\
M^q_t := q(X_t) - \int_0^t G_q(X_r)dr
\]
for any \( t \geq 0 \) and we define \( a_{p,q} := G(pq) - pGq - qGp \). Recall that Lemma \[7.4\] yields that \( M^p, M^q \) are martingales with finite second moments.

Then the predictable quadratic covariation of \( M^p \) and \( M^q \) is given by
\[ \langle M^p, M^q \rangle_t = \int_0^t a_{p,q}(X_s)ds, \quad t \geq 0. \]

**Proof.** Denote by \( \langle M^p, M^q \rangle \) its predictable quadratic covariation in the sense of [L7 Theorem I.4.2]. Then
\[ M^p_t M^q_t - \langle M^p, M^q \rangle_t, \quad t \geq 0 \]
is a martingale. Since \( pq \in \mathcal{C} \),
\[ M^{pq}_t := (pq)(X_t) - \int_0^t G(pq)(X_r)dr \]
is a martingale due to Lemma \[7.4\]. We have
\[
\int_0^t G_p(X_r)dr \int_0^t G_q(X_r)dr \\
= \int_0^t \left( \int_0^s G_q(X_r)dr G_p(X_s) + \int_0^s G_p(X_r)dr G_q(X_s) \right) ds,
\]
and 
\[
\int_0^t G_p(X_r) dq(X_t) = \int_0^t q(X_r) G_p(X_r) dr + \int_0^t \int_0^r G_p(X_r) dq G_q(X_s) ds + \int_0^t \int_0^r G_p(X_r) dr dM_q^r.
\]

Hence, we find from expanding \(M^p M^q\) and the identities above that \(N\) given by 
\[
N_t := M_t^p M_t^q - \int_0^t a_{p,q}(X_r) dr
\]
\[
= M_t^{pq} - \int_0^t G_p(X_s) ds M_q^s - \int_0^t \int_0^r G_q(X_s) ds dM^p_r
\]
for \(t \geq 0\) is a \(\sigma\)-martingale in the sense of [17, Definition III.6.33] due to [17, Proposition III.6.42]. Consequently, \(N - (M^p M^q - \langle M^p, M^q \rangle_t)\) is a \(\sigma\)-martingale and since 
(11) \(N_t := M_t^p M_t^q - \langle M^p, M^q \rangle_t = \langle M^p, M^q \rangle_t - \int_0^t a_{p,q}(X_r) dr, \ t \geq 0\)

it is predictable and of finite variation. Thus, \(N - (M^p M^q - \langle M^p, M^q \rangle)\) is a special semimartingale in the sense of [17, Definition III.4.21(b)] and, hence, [17, Proposition I.6.35] yields that it is a local martingale. [17, Corollary I.3.16] yields that \(N - (M^p M^q - \langle M^p, M^q \rangle)\) is, in fact, constant 0. Equation (11) yields the claim. \(\square\)

Restricting our attention to diffusions in separable Hilbert spaces, we find the following corollary to our main result:

**Corollary 7.6.** Let \(E\) be a separable Hilbert space with inner product \((\cdot, \cdot)_E\), \(W\) an \(E\)-valued Brownian motion with covariance operator \(Q\) in the sense of [20, Sect. 3.5 and 4.4] and assume that there is a linear subspace \(\mathcal{E}\) of the continuous linear operators from \(E\) to \(\mathbb{R}\) which is contained in \(\mathcal{L} \cap \mathcal{C}\) and such that \(p^2 \in \mathcal{C}\) for any \(p \in \mathcal{E}\). We assume additionally that 
\[
X_t = X_0 + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s, \ t \geq 0
\]
where \(\beta\) is a progressively measurable process, \(E\)-valued and locally integrable and \(\sigma\) is a progressively measurable process, \(L(E)\)-valued and locally square integrable.

We denote the Lebesgue measure restricted to \(\mathbb{R}_+\) by \(\lambda\). Then
\[
p(\beta_t) = G_p(X_t),
\]
\[
(\sigma_t Q \sigma_t^*, p^*)_E = a_{p,q}(X_t)
\]

\(P \otimes \lambda\)-a.s. for any \(p,q \in \mathcal{E}\) where we have that \(pq \in \mathcal{C}\) and \(a_{p,q}\) is defined in Theorem 7.5 and \(p^*\) (resp. \(q^*\)) is the unique element in \(E\) such that \(p = (\cdot, p^*)_E\) (resp. \(q = (\cdot, q^*)_E\)).

Moreover, if \(\mathcal{E}\) is dense in the set of continuous linear operators, then \(G|_{\mathcal{P}_2}\) determines the drift \(\beta\) and the covariance \(\sigma Q \sigma^*\).

**Proof.** Theorem 6.6 yields that
\[
p(X_t) = M_t^p + \int_0^t G_p(X_s) ds, \ t \geq 0
\]
where \( M^p \) is a martingale. Since
\[
p(X_t) = p(X_0) + \int_0^t p\sigma_s dW_s + \int_0^t p(\beta_s) ds
\]
we find the claim for \( \beta \) and see that
\[
M^p_t = p(X_0) + \int_0^t p\sigma_s dW_s, \quad t \geq 0.
\]
Since \( p, q \in \mathcal{E} \) and \( p + q \in \mathcal{E} \) we find that
\[
pq = \frac{1}{2} \((p + q)^2 - p^2 - q^2\) \in \mathcal{C}
\]
Thus \( \langle M^p, M^q \rangle_t = \int_0^t a_{p,q}(X_s) ds \) by Theorem 7.5. On the other hand we have
\[
\langle M^p, M^q \rangle_t = \int_0^t (\sigma_s Q\sigma^*_s, p^*, q^*)_{\mathcal{E}} ds
\]
and the claim follows. \( \square \)

References

[1] D. Ackerer, D. Filipović and S. Pulido (2018). The Jacobi stochastic volatility model. Finance Stoch., 22(3):667–700.
[2] F. E. Benth, N. Detering and P. Krühner (2018). Multilinear processes in Banach space. arXiv:1809.01336 To appear in Stochastics.
[3] F. E. Benth and S. Lavagnini (2019). Correlators of polynomial processes. arXiv: 1906.11320v2
[4] S. Biagini and Y. Zhang (2016). Polynomial diffusion models for life insurance liabilities. Insurance Math. Economics, 71:114–129.
[5] C. Cuchiero and S. Pulido (2020). Markov cubature rules for polynomial processes. Stoch. Proc. Applic., 130(4):1947–1971.
[6] D. Filipović and M. Larsson (2020) Polynomial jump-diffusion models. Stoch. Systems, 10(1):71–97.
[7] D. Filipović, M. Larsson and S. Pulido (2020). Markov cubature rules for polynomial processes. Stoch. Proc. Applic., 130(4):1947–1971.
[8] D. Filipović, M. Larsson and A. B. Trolle (2017). Linear rational term structure models. J. Finance, 72(2):655–704.
[9] X. Kleisinger-Yu and V. Komaric and M. Larsson and M. Regez (2020). A multi-factor polynomial framework for long-term electricity forwards with delivery period. SIAM J. Finan. Math., 11(3):928–957.
[10] J. Jacob (1979). Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Mathematics, vol. 714. Springer Verlag, Berlin.
[11] J. Jacob and A. Shiryaev (2003). Limit Theorems for Stochastic Processes. Second edition. Springer Verlag, Berlin Heidelberg.
[12] M. Liao (2004). Lévy Processes in Lie Groups. Cambridge University Press.
[13] J. van Neerven (2010). Stochastic Evolution Equations, ISEM Lecture Notes 2007/8. Downloaded from http://fa.its.tudelft.nl/~neerven/publications/notes/ISEM.pdf in May 2018.
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[20] Peszat and Zabczyk (2007). Stochastic Equations in Infinite Dimensions. Cambridge University Press.
[21] T. Ware (2019). Polynomial processes for power prices. Appl. Math. Finance, 26(5):453–474.

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