LINEAR INDEPENDENCE OF CERTAIN NUMBERS IN THE
BASE-\(b\) NUMBER SYSTEM

Shintaro Murakami ; Yohei Tachiya †

Abstract

Let \((i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 2}\) and \(S_{i,j}\) be an infinite subset of positive integers including all prime numbers in some arithmetic progression. In this paper, we prove the linear independence over \(\mathbb{Q}\) of the numbers

\[1, \sum_{n \in S_{i,j}} \frac{a_{i,j}(n)}{b^{jn}}, \quad (i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 2},\]

where \(b \geq 2\) is an integer and \(a_{i,j}(n)\) are bounded nonzero integer-valued functions on \(S_{i,j}\). Moreover, we also establish a necessary and sufficient condition on the subset \(\mathcal{A}\) of \(\mathbb{N} \times \mathbb{N}_{\geq 2}\) for the numbers

\[1, \sum_{n \in T_{i,j}} \frac{a_{i,j}(n)}{b^{jn}}, \quad (i, j) \in \mathcal{A}\]

to be linearly independent over \(\mathbb{Q}\) for any given infinite subsets \(T_{i,j}\) of positive integers. Our theorems generalize a result of V. Kumar.

Keywords: Linear independence, base-\(b\) number system, primes in arithmetic progression

AMS Subject Classification: 11J72.

1 Introduction

In 1996, Yu. V. Nesterenko [12] showed the lower bound for the transcendence degree of the field generated over \(\mathbb{Q}\) by the values of the Eisenstein series. Nesterenko’s theorem derives a number of remarkable transcendence and algebraic independence results for the values of various modular functions. For example, applying Nesterenko’s theorem, D. Bertrand [2] and independently D. Duverney, Ke. Nishioka, Ku. Nishioka and I. Shiokawa [4] derived algebraic independence results for the values of the Jacobi theta functions.

---

*Hirosaki University, Graduate School of Science and Technology, Hirosaki 036-8561, Japan
e-mail: h20ds203@hirosaki-u.ac.jp

†Hirosaki University, Graduate School of Science and Technology, Hirosaki 036-8561, Japan
e-mail: tachiya@hirosaki-u.ac.jp
In particular, they proved that the number $\sum_{n=1}^\infty \alpha^{n^2}$ is transcendental for any algebraic number $\alpha$ with $0 < |\alpha| < 1$.

On the other hand, we are not aware of the transcendence of the number $\sum_{n=1}^\infty \alpha^{nk}$ for an integer $k \geq 3$.

Recently, V. Kumar [9] obtained the following linear independence result by using the properties of uniform distribution modulo 1 of irrational numbers:

**Theorem 1.1** ([9] Theorem 1). Let $k \geq 2$, $b \geq 2$ and $1 \leq a_1 < a_2 < \cdots < a_m$ be integers such that $\sqrt[a_i]{a_j} \notin \mathbb{Q}$ for any $i \neq j$. Then the numbers

$$1, \quad \sum_{n=1}^\infty \frac{1}{b_{a_1}n^k}, \quad \sum_{n=1}^\infty \frac{1}{b_{a_2}n^k}, \quad \cdots, \quad \sum_{n=1}^\infty \frac{1}{b_{a_m}n^k} \quad (1.1)$$

are linearly independent over $\mathbb{Q}$.

The aim of this paper is to extend Theorem 1.1 by showing linear independence results for certain infinite series. In particular, as an application of our Theorem 1.2, we will generalize Theorem 1.1 even without the condition $\sqrt[a_i]{a_j} \notin \mathbb{Q}$ for any $i \neq j$. This gives a positive answer to the question of V. Kumar [9] Section 4]. Moreover, we establish in Theorem 1.3 a necessary and sufficient condition for a set of certain infinite series to be linearly independent over $\mathbb{Q}$. We will find by Theorem 1.3 that the same conclusion of Theorem 1.1 can be achieved in case where $k \geq 3$ even if the infinite series in (1.1) are replaced by any subseries (Corollary 1.3).

Before stating our Theorem 1.2 we prepare some notations. Let $\mathbb{N}$ denote the set of positive integers. Let $(i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 2}$ and $S_{i,j}$ be an infinite subset of $\mathbb{N}$ including all prime numbers in the arithmetic progression $n \equiv h_{i,j} \pmod{d_{i,j}}$, where $d_{i,j}$ and $h_{i,j}$ are positive integers and relatively prime. Note that Dirichlet’s theorem states that such arithmetic progression contains infinitely many prime numbers. Thus, we can choose various infinite sets $S_{i,j}$ such as the sets of all prime numbers in arithmetic progressions, all prime numbers, all squarefree integers and all positive integers.

Let $S_{i,j}$ be the above infinite set and $a_{i,j}(n)$ be a nonzero integer-valued function on $S_{i,j}$. Define the function

$$f_{i,j}(z) := \sum_{n \in S_{i,j}} a_{i,j}(n)z^{inj}, \quad (1.2)$$

which converges in $|z| < 1$. Since $j \geq 2$, the function (1.2) is not rational. We establish linear independence results for the values of the functions (1.2) at the rational argument $z = 1/b$ with an integer $b \geq 2$. Recall that an infinite set of numbers is called linearly independent if each one of its finite subsets is linearly independent; otherwise it is called linearly dependent. Our results are the following.

**Theorem 1.2.** Let $b \geq 2$ be an integer and $f_{i,j}(z)$ be the functions defined in (1.2). Then the set of the numbers

$$1, \quad f_{i,j}(1/b) = \sum_{n \in S_{i,j}} \frac{a_{i,j}(n)}{b^{inj}}, \quad (i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 2} \quad (1.3)$$

is linearly independent over $\mathbb{Q}$.

Applying Theorem 1.2 with $S_{i,j} := \mathbb{N}$ and $a_{i,j}(n) := (\pm 1)^n$ ($n \geq 1$) for all $(i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 2}$, we have

**Corollary 1.1.** Let $b \geq 2$ be an integer. Then the set of the numbers

$$1, \quad \alpha_{i,j} := \sum_{n=1}^\infty \frac{1}{b^{inj}} \quad (i = 1, 2, \ldots, \ j = 2, 3, \ldots)$$
is linearly independent over \( \mathbb{Q} \). The same holds for the set of the numbers

\[
1, \quad \beta_{i,j} := \sum_{n=1}^{\infty} \frac{(-1)^n}{b^{jn^2}} \quad (i = 1, 2, \ldots, j = 2, 3, \ldots).
\]

Note that the first assertion in Corollary 1.1 gives an unconditional version of Theorem 1.1.

**Remark 1.1.** Let \( \alpha_{i,2} \) and \( \beta_{i,2} \) be as in Corollary 1.1 Then we have the equalities

\[
\alpha_{i,2} = \frac{1 + 2 \sum_{n=1}^{\infty} q^{n^2}}{2}, \quad \beta_{i,2} = \frac{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}}{2},
\]

are the Jacobi theta functions defined in \( |q| < 1 \). Hence, Corollary 1.1 provides linear independence results for the values of the Jacobi theta functions

\[
\theta_\ell(1/b^i), \quad i = 1, 2, \ldots
\]

for each \( \ell \in \{3, 4\} \). As mentioned at the beginning of this section, the numbers in (1.4) are all transcendental (cf. [4, 5]), and hence, so are the numbers \( \alpha_{i,2}, \beta_{i,2} \) \( i = 1, 2, \ldots \). More strongly, any two numbers in the set \( \{\alpha_{i,2}, \beta_{i,2} \mid i = 1, 2, \ldots \} \) are algebraically independent over \( \mathbb{Q} \), while any three are not (cf. [5, 6]).

For example, the explicit algebraic relations among the first three numbers \( \alpha_{i,2} \) \( i = 1, 2, 3 \) and \( \beta_{i,2} \) \( i = 1, 2, 3 \), respectively, can be derived from the results in [5, § 4.1]. Recently, C. Elsner and V. Kumar [7] obtained linear independence results over the field of algebraic numbers for some three numbers in \( \{\alpha_{i,2} \mid i \geq 1\} \).

Moreover, applying Theorem 1.2 with the sets \( S_{i,j} \) of all prime numbers and all squarefree integers, we have

**Corollary 1.2.** Let \( b \geq 2 \) be an integer. Then the set of the numbers

\[
1, \quad \gamma_{i,j} := \sum_{p: \text{prime}} \frac{1}{b^{ip^2}} \quad (i = 1, 2, \ldots, j = 2, 3, \ldots)
\]

is linearly independent over \( \mathbb{Q} \). The same holds for the set of the numbers

\[
1, \quad \sum_{n: \text{squarefree}} \frac{1}{b^{jn^2}} \quad (i = 1, 2, \ldots, j = 2, 3, \ldots).
\]

It should be noted that D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance [11] proved that the numbers \( \alpha_{i,j} \) \( j \geq 3 \) and \( \gamma_{i,j} \) \( j \geq 2 \) are either transcendental or algebraic numbers with degrees at least \( j \) and \( j + 1 \), respectively.

In Theorem 1.2 we can not replace the sets \( S_{i,j} \) by any infinite subsets of positive integers; indeed, the numbers (1.3) can be linearly dependent over \( \mathbb{Q} \) for suitable infinite sets \( S_{i,j} \), e.g. clearly \( \sum_{n \in \mathbb{N}} b^{-n^2} = \sum_{n \in \mathbb{N}} b^{-4n^2} \). The following Theorem 1.3 gives a necessary and sufficient condition on the subset \( A \) of \( \mathbb{N} \times \mathbb{N} \geq 2 \) for the numbers in (1.5) to be linearly independent over \( \mathbb{Q} \) for arbitrary given infinite sets \( T_{i,j} \).
Theorem 1.3. Let $b \geq 2$ be an integer and $A$ be a subset of $\mathbb{N} \times \mathbb{N}_{\geq 2}$. Then for any subsets $T_{i,j} ((i,j) \in A)$ of $\mathbb{N}$ and for any nonzero integer-valued functions $a_{i,j}(n)$ on $T_{i,j}$ the set of the numbers

$$1, \sum_{n \in T_{i,j}} \frac{a_{i,j}(n)}{b^{\nu(n)}}, \quad (i,j) \in A$$

is linearly independent over $\mathbb{Q}$ if and only if the following two conditions are fulfilled.

(i) Let $(i_1,j_1), (i_2,j_2) \in A$ be distinct. Then $i_1 u^{j_1} \neq i_2 v^{j_2}$ for any positive integers $u$ and $v$.

(ii) There exists at most one $(i,j) \in A$ with $j = 2$.

Applying Theorem 1.3, we obtain the following Corollary 1.3 which gives another generalization of Theorem 1.1.

Corollary 1.3. Let $k \geq 3, b \geq 2$ and $1 \leq a_1 < a_2 < \cdots < a_m$ be integers such that $\sqrt[2]{a_i/a_j} \notin \mathbb{Q}$ for any $i \neq j$. Let $T_k (i = 1, 2, \ldots, m)$ be any infinite subsets of $\mathbb{N}$. Then the numbers

$$1, \sum_{n \in T_1} \frac{1}{b^{a_1 n^k}}, \sum_{n \in T_2} \frac{1}{b^{a_2 n^k}}, \cdots, \sum_{n \in T_m} \frac{1}{b^{a_m n^k}}$$

are linearly independent over $\mathbb{Q}$.

The present paper is organized as follows. Theorems 1.2 and 1.3 will be shown in Sections 2 and 3 respectively. In the proofs, we will prove that the base-$b$ representations of nontrivial linear forms over $\mathbb{Z}$ of the numbers (1.3) and (1.5) contain arbitrarily long strings of zero without being identically zero from some point on. To see this, we consider the system of the simultaneous congruences in the proof of Theorem 1.2.

This method is based on elementary arguments used in the papers of S. Chowla [3] and P. Erdős [8]. On the other hand, in the proof of Theorem 1.3 we use Mahler’s result [11] on the finiteness of integer solutions $x$ and $y$ for the Diophantine equation $ax^m + by^n = c$. These approaches are completely different from those used in the papers of V. Kumar [9, 10].

2 Proof of Theorem 1.2

We first show the following lemma.

Lemma 2.1. Let $k \geq 2, u \geq 1$ and $v \neq 0$ be integers. Then there are infinitely many prime numbers $p$ such that the congruence $uX^k + v \equiv 0 \pmod{p^2}$ has an integer solution.

Proof. Let $g(X) := uX^k + v \in \mathbb{Z}[X]$. Then there are infinitely many prime numbers $p$ such that $p$ divides $g(m)$ for some integer $m \geq 1$; in other words, the congruence

$$g(X) \equiv 0 \pmod{p}$$

has an integer solution for infinitely many prime numbers $p$. Let $p$ be such a prime number with $p > \max\{k, u, |v|\}$ and let $x$ be an integer solution of (2.1). Then $g(x) \equiv 0 \pmod{p}$ and $g'(x) = kux^{k-1} \neq 0 \pmod{p}$, since $v \neq 0$. Hence, there exists an integer $y$ satisfying

$$g'(x)y - 1 + g(x)/p \equiv 0 \pmod{p}.$$  \hspace{1cm} (2.2)

By (2.2) we have

$$g(py + x) = u(py + x)^k + v \equiv g'(x)py + g(x) \equiv p \pmod{p^2}.$$  

This shows that $g(X) \equiv 0 \pmod{p^2}$ has an integer solution $X = py + x$ and the proof of Lemma 2.1 is completed. \qed
Proof of Theorem 1.2. Let \( m \geq 2 \) be an arbitrary integer. It suffices to show that the set of the numbers

\[
1, \quad f_{i,j}(1/b) = \sum_{n \in S_{i,j}} a_{i,j}(n) \frac{n}{b^{in^j}} \quad (i = 1, 2, \ldots, m, \quad j = 2, 3, \ldots, m)
\]  

(2.3)

is linearly independent over \( \mathbb{Q} \). Define

\[
A := \{(i, j) \mid i = 1, 2, \ldots, m, \quad j = 2, 3, \ldots, m\}.
\]

We fix \((i_0, j_0) \in A\). Let \( S_{i_0,j_0} \) be an infinite subset of positive integers including all prime numbers \( p \) satisfying \( p \equiv h \pmod{d} \), where \( d := d_{i_0,j_0} \) and \( h := h_{i_0,j_0} \) are positive integers and relatively prime. Let \( N \) be an integer sufficiently large. By Lemma 2.1 for each integer \( \ell = 1, 2, \ldots, N - 1, N + 1, \ldots, 2N - 1 \) there exists a prime number \( p_\ell > N \) such that the congruence

\[
i_0X^{j_0} + \ell - N \equiv p_\ell \pmod{p_\ell^2}
\]

(2.4)

has an integer solution. Since for each \( \ell \) there are infinitely many such prime numbers, we may assume that the prime numbers \( p_\ell \) are distinct. Let \( x_\ell \) be a fixed integer solution of (2.4). Consider the system of the \( 2N - 1 \) simultaneous congruences

\[
\begin{cases}
X \equiv h \pmod{d}, \\
X \equiv x_\ell \pmod{p_\ell^2},
\end{cases} \quad \ell = 1, 2, \ldots, N - 1, N + 1, \ldots, 2N - 1.
\]

(2.5)

We find by the Chinese remainder theorem that there exists the least positive integer solution \( x \) of (2.5) and all integer solutions of (2.5) are given by \( \alpha n + x \ (n \in \mathbb{Z}) \), where

\[
\alpha := d \prod_{\ell \neq N}^N p_\ell^2.
\]

Then the integers \( \alpha \) and \( x \) are relatively prime; indeed, if \( p_\ell \mid x \) for some prime number \( p_\ell \), then by (2.4) and (2.5) we have \( p_\ell \mid \ell - N \), which is impossible by \( 1 \leq |\ell - N| < N < p_\ell \). Moreover, by (2.5) the integers \( d \) and \( x \) are relatively prime, since so are the integers \( d \) and \( h \). Hence, by Dirichlet’s theorem on arithmetic progressions, there exists a positive integer \( n_0 \) such that \( q := \alpha n_0 + x \) is a prime number sufficiently large. Note that \( q \in S_{i_0,j_0} \), since \( q \equiv x \equiv h \pmod{d} \). By (2.4) and (2.5) we have

\[
i_0q^{j_0} + \ell - N \equiv p_\ell \pmod{p_\ell^2}
\]

for each integer \( \ell = 1, 2, \ldots, N - 1, N + 1, \ldots, 2N - 1 \). This implies that the integer \( i_0q^{j_0} + \ell - N \) is divisible by \( p_\ell \) exactly once, and hence, we obtain

\[
i_0q^{j_0} \pm u \notin \{ik^j \mid k \in \mathbb{N}, \ (i, j) \in A\}
\]

(2.6)

for every integer \( u = 1, 2, \ldots, N - 1 \), since \( p_\ell \) is a prime number sufficiently large.

Let \( c \) and \( c_{i,j} \ ((i, j) \in A) \) be integers such that

\[
c + \sum_{(i,j) \in A} c_{i,j} f_{i,j}(1/b) = 0,
\]

(2.7)
where the sum is taken over all \( i, j \) with \((i, j) \in \mathcal{A} \). Let

\[
  f_{i,j}(1/b) = \sum_{n \in S_{i,j}} \frac{a_{i,j}(n)}{b^n} = \sum_{n=1}^{\infty} \frac{e_{i,j}(n)}{b^n},
\]

where

\[
  e_{i,j}(n) := \begin{cases} 
  a_{i,j}(k) & \text{if } n = ik^j \text{ for some integer } k \in S_{i,j}, \\
  0 & \text{otherwise}.
  \end{cases}
\]  

(2.8)

Then by (2.6) and (2.8)

\[
  e_{i,j}(i_0q^j \pm u) = 0, \quad u = 1, 2, \ldots, N - 1
\]

for any \((i, j) \in \mathcal{A}\). Hence, by (2.7) and (2.9) we obtain

\[
  P_N := -c - \sum_{n=1}^{i_0q^j - N} \frac{1}{b^n} \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(n)
  \]

\[
  = \frac{1}{b^{i_0q^j}} \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(i_0q^j) + \sum_{n = i_0q^j + N}^{\infty} \frac{1}{b^n} \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(n).
\]  

(2.10)

Clearly, \(b^{i_0q^j - N}P_N\) is an integer. On the other hand, by (2.10)

\[
  |b^{i_0q^j - N}P_N| \leq b^{-N}C_1 \sum_{(i,j) \in \mathcal{A}} |c_{i,j}| + \sum_{n = i_0q^j + N}^{\infty} b^{-n+i_0q^j} \sum_{(i,j) \in \mathcal{A}} |c_{i,j}|
  \]

\[
  \leq C_2b^{-N} + 2C_2b^{-2N} \to 0 \quad (N \to \infty),
\]

where \(C_1 := \max_{i,j} \{|a_{i,j}(n)| \mid n \in S_{i,j}\} \) and \(C_2 := C_1 \sum_{(i,j) \in \mathcal{A}} |c_{i,j}| \). Thus, we obtain \(P_N = 0\) for every large integer \(N\). Therefore, by (2.10) we find that

\[
  Q_N := \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(i_0q^j) = - \sum_{n = i_0q^j + N}^{\infty} \frac{1}{b^n} \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(n)
  \]

is an integer for every large \(N\) and

\[
  |Q_N| \leq 2C_2b^{-N} \to 0 \quad (N \to \infty).
\]

This implies that

\[
  Q_N = \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(i_0q^j) = 0
\]  

(2.11)

for every large integer \(N\). Let \((i, j) \in \mathcal{A}\) be satisfy \(i_0q^j = ik^j\) for some integer \(k \in S_{i,j}\). Then we obtain \(i \mid i_0\) and \(j \mid j_0\), since \(q\) is a prime number sufficiently large. Hence, the equality (2.11) is written as

\[
  \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(i_0q^j) = 0.
\]  

(2.12)

Now we show \(c_{i,j} = 0\) for any \((i, j) \in \mathcal{A}\) by induction on \(i + j \geq 3\). We first apply the above arguments to the case where \((i_0, j_0) = (1, 2) \in \mathcal{A}\) and the set \(S_{1,2}\). Then there exists a prime number \(q := q_{1,2} \in S_{1,2}\)
satisfying (2.12), so that, \( c_{1,2}e_{1,2}(q^2) = 0\). Since \( e_{1,2}(q^2) = a_{1,2}(q) \neq 0\), we obtain \( c_{1,2} = 0\). Let \( M \geq 3\) be an integer and suppose that \( c_{i,j} = 0\) for any \((i, j) \in \mathcal{A}\) with \( i + j \leq M\). Then, applying the above arguments again to the case where \((i_0, j_0) \in \mathcal{A}\) with \( i_0 + j_0 = M + 1\) and the set \( S_{i_0,j_0}\), we obtain (2.12) for some prime number \( q := q_{i_0,j_0} \in S_{i_0,j_0}\). Hence, by the induction hypothesis

\[
0 = \sum_{(i,j) \in \mathcal{A}} c_{i,j}e_{i,j}(i_0q^{j_0}) = c_{i_0,j_0}e_{i_0,j_0}(i_0q^{j_0}).
\]

Since \( e_{i_0,j_0}(i_0q^{j_0}) = a_{i_0,j_0}(q) \neq 0\), we have \( c_{i_0,j_0} = 0\). Thus, we obtain \( c_{i,j} = 0\) for every \((i, j) \in \mathcal{A}\), and so \( c = 0\) by (2.7). The proof of Theorem [1.2] is completed. 

\[\square\]

3 Proof of Theorem [1.3]

We first show that if both conditions (i) and (ii) do not hold, then the numbers in [1.5] can be linearly dependent over \( \mathbb{Q} \) for suitable infinite sets \( T_{i,j} \) and the constant functions \( a_{i,j}(n) \equiv 1\) \((n \in T_{i,j})\). If condition (i) does not hold, then there exit distinct pairs \((i_1, j_1), (i_2, j_2) \in \mathcal{A}\) and positive integers \( u, v\) such that \( i_1u^{j_1} = i_2v^{j_2}\). Hence, putting \( T_1 := \{u2^{jm} | m \in \mathbb{N}\}\) and \( T_2 := \{v2^{jm} | m \in \mathbb{N}\}\), we obtain the equality \( \sum_{n \in T_1} b^{-i_1n^{j_1}} = \sum_{n \in T_2} b^{-i_2n^{j_2}}\). Next we assume that condition (i) holds while condition (ii) does not. Then there exist \((i_1, 2), (i_2, 2) \in \mathcal{A}\) such that \( i_1i_2\) is not square. Consider the set of positive integer solutions of the Pell equation \( x^2 - i_1i_2y^2 = 1\):

\[
\mathcal{S} := \{(x, y) \in \mathbb{N}^2 | x^2 - i_1i_2y^2 = 1\}.
\]

As is well known, \( \mathcal{S} \) is an infinite set. Hence, noting that \( i_1x^2 - i_2(i_1y)^2 = i_1\) for \((x, y) \in \mathcal{S}\), we have

\[
\sum_{n \in T_1} \frac{1}{b^{i_1n^{j_1}}} = \sum_{n \in T_2} \frac{1}{b^{i_2n^{j_2} + i_1}} = \frac{1}{b} \sum_{n \in T_2} \frac{1}{b^{i_2n^{j_2}}},
\]

where \( T_1 := \{x | (x, y) \in \mathcal{S}\} \) and \( T_2 := \{i_1y | (x, y) \in \mathcal{S}\}\). Thus, our assertion is proved.

In what follows, we assume that both conditions (i) and (ii) hold. We fix \((i_0, j_0) \in \mathcal{A}\) and let \( N \geq 1\) be an integer sufficiently large. Consider the Diophantine equations

\[
i_0x^{j_0} - iy^j = \pm u, \quad (3.1)
\]

where \((i, j) \in \mathcal{A}\) and \( u = 1, 2, \ldots, N - 1\). If \( j_0 = j = 2\), we have \( i_0 = i\) by condition (ii). Then clearly the equation (3.1) has only finitely many integer solutions \( x \) and \( y \). Let \( \max\{j_0, j\} \geq 3\). Then K. Mahler [11] proved that the greatest prime factor of \( i_0x^{j_0} - iy^j\) tends to infinity as \( \max\{|x|, |y|\} \to \infty\) with relatively prime integers \( x \) and \( y \). Hence, also in this case, the equation (3.1) has only finitely many integer solutions \( x \) and \( y \). Thus, there exists a positive integer \( M\) such that

\[
i_0t^{j_0} + u \notin \{ik^j | k \in \mathbb{N}, (i, j) \in \mathcal{A}\} \quad (3.2)
\]

holds for every integer \( t > M\) and for all integers \( u = 1, 2, \ldots, N - 1\).

Let \( c \) and \( c_{i,j} \((i, j) \in \mathcal{A}\) be integers such that

\[
c + \sum_{(i,j) \in \mathcal{A}} c_{i,j} \delta_{i,j} = 0, \quad (3.3)
\]
where
\[ \delta_{i,j} := \sum_{n \in T_{i,j}} a_{i,j}(n) b_{i,j} = \sum_{n=1}^{\infty} e_{i,j}(n) \]
and
\[ e_{i,j}(n) := \begin{cases} a_{i,j}(k) & \text{if } n = ik^j \text{ for some integer } k \in T_{i,j}, \\ 0 & \text{otherwise.} \end{cases} \] (3.4)

Let \( t \in T_{i_0,j_0} (t > M) \) be an integer satisfying (3.2). Then by (3.2) and (3.4)
\[ e_{i,j}(i_0 t^j \pm u) = 0, \quad u = 1, 2, \ldots, N - 1 \]
for any \((i, j) \in A\). Hence, similarly as in the proof of Theorem 1.2, we can derive
\[ \sum_{(i,j) \in A} c_{i,j} e_{i,j}(i_0 t^j) = 0 \] (3.5)
for every large integer \( N \). Clearly, we have \( e_{i_0,j_0}(i_0 t^j) = a_{i_0,j_0}(t) \neq 0 \). Moreover, if \((i, j) \neq (i_0, j_0)\), then by condition (i) we have \( i_0 t^j \neq iv^j \) for every positive integer \( v \), and so \( e_{i,j}(i_0 t^j) = 0 \). Therefore, by (3.5) we obtain \( c_{i_0,j_0} = 0 \). Since the integer pair \((i_0, j_0)\) is chosen arbitrarily, we obtain \( c_{i,j} = 0 \) for every \((i, j) \in A\), and hence, \( c = 0 \) by (3.3). Theorem 1.3 is proved. \( \square \)

Acknowledgements
The authors would like to express their sincere gratitude to Professor Hajime Kaneko for valuable comments and suggestions on this manuscript. The authors are also grateful to Professor Takafumi Miyazaki for pointing out the paper of K. Mahler [11]. This work was partly supported by JSPS KAKENHI Grant Number JP18K03201.

References
[1] D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, On the binary expansions of algebraic numbers, J. Théor. Nombres Bordeaux, 16 (2004), 487–518.
[2] D. Bertrand, Theta functions and transcendence, Ramanujan J. 1 (1997), 339–350.
[3] S. Chowla, On series of the Lambert type which assume irrational values for rational values of the argument, Proc. Natl. Inst. Sci. India Part A 13 (1947), 171–173.
[4] D. Duverney, Ke. Nishioka, Ku. Nishioka and I. Shiokawa, Transcendence of Jacobi’s theta series, Proc. Japan Acad. Ser. A Math. Sci. 72 (1996), 202–203.
[5] C. Elsner, F. Luca and Y. Tachiya, Algebraic results for the values \( \vartheta_3(m\tau) \) and \( \vartheta_3(n\tau) \) of the Jacobi theta-constant, Mosc. J. Comb. Number Theory 8 (2019), 71–79.
[6] C. Elsner, M. Kaneko and Y. Tachiya, Algebraic independence results for the values of the theta-constants and some identities, J. Ramanujan Math. Soc. 35 (2020), 71–80.
[7] C. Elsner and V. Kumar, On linear forms in Jacobi theta-constants, arXiv: 1911.06513.
[8] P. Erdős, On arithmetical properties of Lambert series, J. Indian Math. Soc. (N.S.) 12 (1948), 63–66.
[9] V. Kumar, *Linear independence of certain numbers*, Arch. Math. (Basel) 112 (2019), 377–385.

[10] V. Kumar, *Linear independence of certain numbers*, J. Ramanujan Math. Soc. 35 (2020), 17–22.

[11] K. Mahler, *On the greatest prime factor of $ax^m + by^n$*, Nieuw Arch. Wisk. 1 (1953), 113–122.

[12] Yu. V. Nesterenko, *Modular functions and transcendence questions*, Mat. Sb. 187 (1996) 65–96; English transl. Sb. Math. 187 1319–1348.