A Pointwise Basic Estimate and Hölder Multipliers for the $\overline{\partial}$–Neumann Problem on Pseudoconvex Domains

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Preliminary Report, May 2011

I. Introduction

The most general results in the theory of the $\overline{\partial}$–Neumann problem have been obtained in the classical $L^2$ setting. Estimates are formulated in $L^2$ Sobolev norms. The theory of subelliptic estimates on pseudoconvex domains of finite type is fairly well understood, thanks to the pioneering work of J. J. Kohn (Ko79) and others (e.g., J. D’Angelo (Da82), D. Catlin (Ca87), etc.). In the late 1960s new integral representation formulas were introduced on strictly pseudoconvex domains by G. Henkin and E. Ramirez, independently, which eventually allowed to prove pointwise estimates for solutions of the Cauchy-Riemann equations, such as estimates in supremum norm and in Hölder norms. These new tools also led to proofs of numerous other results involving boundary behavior of holomorphic functions and related objects. (See Ra86 for a systematic exposition.) These techniques made essential use of an explicit holomorphic support function. Besides on strictly pseudoconvex domains, such holomorphic support functions exist also on Euclidean convex domains, but as shown by the 1972 example of Kohn and L. Nirenberg, they do not exist in general for pseudoconvex domains of finite type. Hölder estimates for $\overline{\partial}$ were obtained on certain special convex domains of finite type by R. M. Range (Ra78). The case of general convex domains of finite type was eventually solved in 1997 by A. Cumenge (Cu97), and soon thereafter by K. Diederich and J. E. Fornaess (DF99). In dimension two, where the Levi form is just a scalar function, Hölder estimates were obtained on pseudoconvex domains of finite type by C. Fefferman and J.J. Kohn (FeKo88) by using reduction to the boundary and microlocalization techniques, and shortly thereafter by R. M. Range (Ra90), who used a division theorem of H. Skoda to produce an appropriate holomorphic generating form and integral kernels on $bD \times D$, which however could be estimated in dimension two only. Neither approach could be generalized to higher dimensions, and a more complete understanding of pointwise estimates in the theory of the $\overline{\partial}$–Neumann problem in arbitrary dimensions has remained elusive for quite some time.

In this report we outline a program to obtain new integral representations and certain pointwise estimates for the $\overline{\partial}$–Neumann problem on arbitrary smoothly bounded weakly pseudoconvex domains $D$. In case the boundary $bD$ satisfies suitable additional properties, e.g. finite type, these techniques might eventually lead to obtaining appropriate Hölder estimates in the $\overline{\partial}$–Neumann

\footnote{\textsuperscript{1}Part of this research was done while the author was visiting at the University of Utah in Winter 2011. The author would like to thank the Mathematics Department of that University for the hospitality and support extended to him.}
theory. The principal new ingredients are a generating form $W(\zeta, z)$ for Cauchy-Fantappiè kernels in this general setting which reflects the pseudoconvexity of the domain, and related precise estimates for $W$. This generating form

$$W(\zeta, z) = \sum_{j=1}^{n} \frac{g_j(\zeta, z)d\zeta_j}{\sum_j g_j(\zeta_j - z_j)}$$

is a $(1,0)$ form defined on $bD \times D$ which is constructed explicitly from a particular defining function $r$ for $D$. The new $W$ involves a critical modification (see below) of the classical Henkin-Ramirez holomorphic generating form on strictly pseudoconvex domains. As is well known, in general it is not possible to find support functions and related kernels holomorphic in $z$ on arbitrary smoothly bounded pseudoconvex domains, even in case of domains of finite type.

The new $W$ has the following basic properties:

I.a) While $\Phi = \sum_{j=1}^{n} g_j(\zeta_j - z_j)$ is not holomorphic in $z$, the form $\partial_z \Phi$ has a zero at $z = \zeta$ whose order is carefully controlled;

I.b) $\Phi$ satisfies precise uniform estimates from below somewhat weaker than those familiar in the strictly pseudoconvex case, and which involve explicitly the eigenvalues of the Levi form of the defining function $r$ for $D$;

I.c) In case the domain is strictly pseudoconvex, $W$—while not holomorphic—satisfies the classical estimations known in that case;

I.d) $\Phi$ satisfies the same symmetry properties that have been successfully used on strictly pseudoconvex domains in earlier work.

This generating form $W$ is introduced into the basic integral representation machinery for the $\bar{\partial}$–Neumann problem developed by Lieb and Range in LR83 in place of the standard locally holomorphic generating form for strictly pseudoconvex domains that was used in LR83. The (formal) construction of the relevant integral kernels and integral representations proceeds essentially in the same way as in LR83, except for a technical modification to account for the fact that the differential $dr$ of the chosen defining function $r$ is no longer normalized on the boundary $bD$. The crux in LR83 involved the explicit identification of certain symmetries in the leading terms which led to a cancellation of singularities, and which in turn yielded appropriate Hölder $1/2$ smoothing estimates. These estimates made essential use of strict pseudoconvexity.

Based on I.d), critical symmetries are preserved in some of the leading terms in the general case, while in other terms, e.g., those that required a Levi metric as developed in LR86, there are no cancellations. In particular, the weaker estimates in the general case do no longer yield any Hölder estimates, at least not directly. Instead, a careful analysis of the relevant kernels in the general case (based on I.a, I.b, and I.d ) allows to estimate some of the derivatives, leading to a pointwise analogon of the classical Morrey-Kohn “basic estimate” in the $L^2$ theory on arbitrary weakly pseudoconvex domains. (See below for more details.)
II. The Basic Estimates on Weakly Pseudoconvex Domains

Let us first recall the basic estimate in the $L^2$ theory. For simplicity we consider $(p, q)$ forms in $\mathbb{C}^n$ with the standard Euclidean metric, and with $p = 0$. The generalization to $p > 0$ is standard. By using the techniques developed in LR86, it seems likely that our results carry over to complex manifolds with an arbitrary Hermitian metric.

We fix a smoothly bounded pseudoconvex domain $D$ in $\mathbb{C}^n$ and a point $P \in bD$. As usual, on a sufficiently small neighborhood $U(P)$ of $P$ one introduces a smooth orthonormal frame $\{\omega_1, \omega_2, ..., \omega_n\}$ for $(1, 0)$ forms on $U$, with $\omega_n = \nu\partial_r$ for a suitable defining function $\nu$ for $D$ with $\nu(\zeta) > 0$ on $U$, and the corresponding dual frame $\{L_1, ..., L_n\}$ for $(1, 0)$ vector fields. In the following discussion $q \geq 1$. One defines

$$\mathcal{D}_q(D) = C_{(0, q)}^\infty(D) \cap \text{dom}(\overline{\partial}^*),$$

and denotes by $\mathcal{D}_{qU}$ those forms in $\mathcal{D}_q(D)$ which have compact support in $\overline{D} \cap U$. Then $f \in \mathcal{D}_{qU}$ can be written as $\sum_j f_j\omega^J$, where the summation is over strictly increasing $q$-tuples $J$. One has the following “$L^2$ basic estimate” of Morrey and Kohn. (See FoKo72 or Ko79) There exists a constant $C$ such that

$$\sum_j \|\overline{\partial}_j f_j\|^2 + \sum_K \int_{bD} \mathcal{L}(\zeta; f^K(\zeta))dS(\zeta) \leq C \left[\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2 + \|f\|^2\right]$$

for all $f \in \mathcal{D}_{qU}$. The norms here are the standard $L^2$ norms over $D$. $\mathcal{L}$ is the Levi form of the defining function $\nu$ with respect to the frame $\{L_1, ..., L_n\}$, and for an ordered $(q - 1)$-tuple $K$ the vector field $f^K = (f^K_1, f^K_2, ..., f^K_n)$ is given by $f^K_j = c^K_j f_j$, where $c^K_j$ equals the sign of the permutation which carries the $q$-tuple $jK$ $(j \notin K)$ into the ordered $q$-tuple $J$, and is 0 in all other cases. Since in case $n \in J$ one has $f_j = 0$ on $bD \cap U$ for $f \in \text{dom}(\overline{\partial})$, and hence $f^K = 0$ on $bD$ for all $K$, pseudoconvexity implies that $\mathcal{L}(\zeta; f^K) \geq 0$ on $bD$ for $f \in \mathcal{D}_{qU}$. Furthermore, it readily follows from $f_j|_{bD} = 0$ that one also has the estimate

$$\|f_j\|_1^2 \leq C_1 \|\overline{\partial} f_j\|^2 \leq C_2 \left[\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2 + \|f\|^2\right]$$

if $n \in J$.

Here $\|f_j\|_1$ is the full 1-Sobolev norm, i.e., $\|f_j\|_1^2$ is the sum of the squares of the $L^2$ norms of all first order derivatives of $f_j$.

Proceeding to the pointwise estimates, we define

$$\mathcal{D}^k_q(D) = C_{(0, q)}^k(D) \cap \text{dom}(\overline{\partial}^*)$$

for $k = 1, 2, ..., \infty$, and we denote by $\mathcal{D}^k_{qU}$ those forms in $\mathcal{D}^k_q(D)$ which have compact support in $\overline{D} \cap U$. Vectorfields $V$ act on forms coefficientwise, i.e., if $f = \sum_j f_j\omega^J$, then $V(f) = \sum_j V(f_j)\omega^J$. For a $C^1$ form $f$ of type $(0, q)$ on $\overline{D}$ we define the norm

$$Q_0(f) = |f|_0 + \|\overline{\partial} f\|_0 + |\partial f|_0,$$
where \( \partial \) is the formal adjoint of \( \bar{\partial} \), and \( |\varphi|_0 \) is the sum of the supremum norms over \( D \) of the coefficients of \( \varphi \). For \( 0 < \lambda < 1 \) we denote by \( |\varphi|_\lambda \) the sum of the Hölder \( \lambda \)-norms of the coefficients of \( \varphi \). Similarly, for a positive integer \( k \), \( |\varphi|_k \) denotes the \( C^k(D) \) norm, with the corresponding meaning for \( |\varphi|_{k+\lambda} \).

Our principal general result gives the “pointwise basic estimates” contained in the following theorem.

**Theorem.** Let \( bD \) be (Levi) pseudoconvex in a neighborhood \( U \) of the point \( P \in bD \). If \( U \) is sufficiently small, one has the following uniform estimates for all \( f \in D_q^1 \), and for \( z \in D \cap U \):

\[
\begin{align*}
\text{II.1} & \quad |L_j(f)(z)| \leq C_\alpha \cdot \text{dist}(z, bD)^{-\alpha} \cdot Q_0(f) \quad \text{for } j = 1, \ldots, n - 1 \text{ and any } \alpha > 1/3; \\
\text{II.2} & \quad |T_n(f)(z)| \leq C_\alpha \cdot \text{dist}(z, bD)^{-\alpha} \cdot Q_0(f) \quad \text{for any } \alpha > 0; \\
\text{II.3} & \quad |L_j(f)(z)| \leq C_\alpha \cdot \text{dist}(z, bD)^{-\alpha} \cdot Q_0(f) \quad \text{for } j = 1, \ldots, n - 1 \text{ and any } \alpha > 2/3; \\
\text{II.4} & \quad \text{For the normal components } f_j\omega^j \text{ of } f \text{ with respect to the frame } \{\omega_1, \ldots, \omega_n\} \text{ one has}
\end{align*}
\]

\[
|f_j|_\delta \leq C_\delta Q_0(f) \quad \text{for any } \delta < 2/3 \text{ if } n \in J.
\]

Comparison with the classical \( L^2 \) basic estimate would suggest that II.1 might hold for any \( \alpha > 0 \), and correspondingly, that II.4 might hold for any \( \delta < 1 \). It is not clear at this point whether the techniques and estimations can be improved to yield these stronger results.

So far, I have verified these results in detail in case \( n = 2, q = 1 \). However, the precise estimations of \( W \) involving the eigenvalues of the Levi form (see I.b above, and more details below) provide the necessary ingredients that should give the result for arbitrary \( n \) and \( q \geq 1 \). Note that the top case \( q = n \) is essentially trivial, since the relevant boundary integrals in the standard integral representation formula (Theorem IV.1.7 in Ra86) vanish in this case.

### III. Outline of the Construction of the Generating Form \( W \)

We assume \( D \) has \( C^k \) boundary (\( k \) sufficiently large) and that \( bD \) is pseudoconvex. We fix \( P \in bD \), and for convenience we assume \( P = 0 \). After a linear change of coordinates there is a \( C^k \) defining function \( r \) on a neighborhood \( U(P) \) of the form \( r(z) = x_n - h(z', y_n) \), where \( z' = (z_1, \ldots, z_{n-1}) \), \( z_n = x_n + iy_n \), and \( h(0) = dh_0 = 0 \). Since \( bD \) is pseudoconvex, it follows that for any \( \zeta \in U \) (not just \( \zeta \in bD \)) the Levi form \( \mathcal{L} \) of this particular defining function \( r \) satisfies

\[
\mathcal{L}(r, \zeta; t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial \zeta_j \partial \overline{\zeta}_k}(\zeta)t_j \overline{t_k} \geq 0 \quad \text{for all } t \in \mathbb{C}^n \text{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j}(\zeta)t_j = 0.
\]

Let

\[
F^{(r)}(\zeta, z) = \sum_{j} \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j \partial \overline{\zeta}_k}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k)
\]

\[
= \sum_{j} \frac{\partial r}{\partial \zeta_j}(\zeta)z_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 r}{\partial \zeta_j \partial \overline{\zeta}_k}(\zeta)(z_j - \zeta_j)(z_k - \zeta_k).
\]
be the usual Levi polynomial of \( r \), which is quadratic holomorphic in \( z \). (See Ra86) The following well known equation is a direct consequence of the 2nd order Taylor expansion of \( r(z) \) at \( \zeta \).

\[
2 \Re \left[ F^{(r)}(\zeta, z) - r(\zeta) \right] = -r(\zeta) - r(z) + \mathcal{L}(r, \zeta; \zeta - z) + O(|\zeta - z|^3).
\]

For \( \zeta \in U \) we denote by \( \pi^k_\zeta : \mathbb{C}^n \to T^{1,0}_\zeta(M_\zeta) \subset \mathbb{C}^n \) the orthogonal projection, where \( M_\zeta \) is the level surface of \( r \) through \( \zeta \) and \( T^{1,0}_\zeta(\mathbb{C}^n) \) is identified with \( \mathbb{C}^n \) via the standard basis \( \left\{ \frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_n} \right\} \). We then define

\[
\Phi_K(\zeta, z) = F^{(r)}(\zeta, z) - r(\zeta) + K |\pi^k_\zeta(\zeta - z)|^3,
\]

where \( K > 0 \) is a large constant to be suitably chosen later on.

While \( |\pi^k_\zeta(\zeta - z)|^3 \) is smooth in \( (\zeta, z) \), the term \( |\pi^k_\zeta(\zeta - z)|^3 \) which appears in \( \Phi_K \) is of class \( C^2 \) in general, and smooth only at points \( (\zeta, z) \) with \( \pi^k_\zeta(\zeta - z) \neq 0 \). For \( j = 0, 1, 2, ... \) we introduce the notation \( \mathcal{E}_j \) to denote continuous functions which are locally uniformly bounded and smooth for \( \pi^k_\zeta(\zeta - z) \neq 0 \), and which satisfy an estimate \( |\mathcal{E}_j| \leq \text{const.} |\zeta - z|^j \). Then \( |\pi^k_\zeta(\zeta - z)|^3 = \mathcal{E}_3 \), and if \( D^k \) denotes a partial derivative of order \( k \) with respect to \( \zeta \) and/or \( z \), then \( D^k |\pi^k_\zeta(\zeta - z)|^3 = \mathcal{E}_{3-k} \) for \( k = 1, 2, 3 \).

We now restrict \( \zeta, z \) to \( \overline{D} \cap U \) so that \( -r(\zeta) \geq 0 \) and \( -r(z) \geq 0 \). By choosing \( U \) sufficiently small, fixing \( \zeta \in \overline{D} \cap U \), and carefully estimating with respect to a suitably chosen local holomorphic coordinate system in \( z \) (which depends on the point \( \zeta \)), one obtains the following fundamental estimate for \( \Phi_K(\zeta, z) \) from below for sufficiently large \( K \):

\[
|\Phi_K(\zeta, z)| \geq \text{const. } \left[ \left| \operatorname{Im} F^{(r)}(\zeta, z) \right| + |r(\zeta)| + |r(z)| + \mathcal{L}(r, \zeta; \pi^k_\zeta(\zeta - z)) + \frac{K}{2} |\zeta - z|^3 \right]
\]

for all \( z \in \overline{D} \cap U \), with the constant independent of \( \zeta \) and \( z \).

The estimate holds for any \( K \) chosen so large that the third order remainder term \( O(|\zeta - z|^3) \) of the Taylor expansion of \( r \) at \( \zeta \in U \) is estimated by \( \frac{K}{2} |\zeta - z|^3 \). The error terms that arise when \( |\pi^k_\zeta(\zeta - z)|^3 \) is compared with \( |\zeta - z|^3 \) are absorbed by \( |\Phi_K(\zeta, z)| \) and \( |r(\zeta)| \). Note that by pseudoconvexity and the special choice of the defining function \( r \) one has

\[
\mathcal{L}(r, \zeta; \pi^k_\zeta(\zeta - z)) \geq 0 \text{ for all } \zeta \in \overline{D} \cap U,
\]

so all terms in the estimation of \( |\Phi_K(\zeta, z)| \) are nonnegative!

As in the familiar strictly pseudoconvex case, \( r(\zeta) \) and \( \operatorname{Im} F^{(r)}(\zeta, z) \) can be used as coordinates in a \( C^{k-2} \) real coordinate system in a neighborhood of \( z \).
The crux of the above estimate for $\Phi_K$ is that $\Phi_K$ is of order 1 in the complex normal direction, while the Levi form completely controls $\Phi_K$ from below in the complex tangential directions.

If $D$ is strictly pseudoconvex near $P$, then $\mathcal{L}(r, \zeta; \pi^t_\zeta(\zeta - z)) \geq c \left| \pi^t_\zeta(\zeta - z) \right|^2$ for some $c > 0$, so that

$$|\Phi_K(\zeta, z)| \geq \text{const.} \left[ |\text{Im} \mathcal{F}^{(r)}(\zeta, z)| + |r(\zeta)| + |r(z)| + \bar{c} |\zeta - z|^2 \right]$$

for some $\bar{c} > 0$ and for any $K \geq 0$, provided $|\zeta - z| \leq \varepsilon$ and $\varepsilon$ is chosen sufficiently small.

In the general case one has the following estimates for derivatives of $\Phi_K(\zeta, z)$:

III.1) $\partial_{\zeta} \Phi_K(\zeta, z) = \mathcal{E}_2$;
III.2) $\tilde{L}_{n,z} \Phi_K(\zeta, z) = \mathcal{E}_3$, where $\tilde{L}_{n,z}$ is the standard normal $(0, 1)$ vectorfield acting in $z$;
III.3) $L_{j,z} \Phi_K(\zeta, z) = \mathcal{E}_1$ for $j < n$;
III.4) $\Phi_K(\zeta, z) - \Phi_K(z, \zeta) = \mathcal{E}_3$.

Most importantly, the third order tangential correction term allows to prove the following more delicate estimate which is critical for the estimation of the integral kernels when $n > 2$. Fix $z$, and choose the orthonormal frame $L_1, ..., L_n$ so that $L_1, ..., L_{n-1}$ form an orthonormal basis for $T^{1,0}_{\zeta}(M)$ which diagonalizes the Levi form restricted to $T^{1,0}_{z}(M)$ at the point $\zeta = z$. Note that this is a condition at the single point $\zeta = z$; in general, there is no smooth frame which diagonalizes the Levi form in a neighborhood of $z$. After a unitary change of coordinates in $\zeta_1, ..., \zeta_n$, one can assume that $L_j|_z = \sqrt{2} \frac{\partial}{\partial \zeta_j}|_z$ and hence $L_j|_\zeta = \sqrt{2} \frac{\partial}{\partial \zeta_j}|_\zeta + V_j$, where the coefficients of $V_j$ are smooth and $\mathcal{E}_1$. It then follows that with respect to these particular coordinates one has

$$\mathcal{L}(r, z; \pi^t_\zeta(\zeta - z)) = \sum_{j=1}^{n-1} \lambda_j(z) |\zeta_j - z_j|^2,$$

where the coefficients of the expression on the left are smooth in $z$ and $\zeta$, one obtains

$$\mathcal{L}(r, \zeta; \pi^t_\zeta(\zeta - z)) = \sum_{j=1}^{n-1} \lambda_j(z) |\zeta_j - z_j|^2 + \mathcal{R}(\zeta, z),$$

where the error term $\mathcal{R}(\zeta, z)$ is smooth and $\mathcal{R}(\zeta, z) = \mathcal{E}_3$. Now choose $K$ in the definition of $\Phi_K$ so large that this error term satisfies $|\mathcal{R}(\zeta, z)| \leq \frac{K}{4} |\zeta - z|^3$. It then follows that if $U$ and $\varepsilon$ are sufficiently small, given the fixed point $z$ and the chosen coordinates which depend on $z$, one has the sharper estimate from below.
|Φ_K(ζ, z)| ≥ const. \[\left|\Im F^{(r)}(ζ, z)\right| + |r(ζ)| + |r(z)| + \sum_{j=1}^{n-1} λ_j(z) |ζ_j - z_j|^2 + \frac{K}{4} |ζ - z|^3\].

for ζ, z ∈ \overline{D} ∩ U. All estimates and constants can be chosen to be uniform in ζ, z ∈ \overline{D} ∩ U.

Similarly, with respect to the corresponding dual frame,
\[
\partial r ∧ \overline{∂} r ∧ \partial \overline{∂} r(ζ) = γ(ζ) ω_n ∧ \overline{ω}_n ∧ \left[\sum_{j=1}^{n-1} λ_j(z) ω_j ∧ \overline{ω}_j + Ω_z\right],
\]
where γ(ζ) ≠ 0 and the form Ω_z has smooth \(E_1\) coefficients. It follows that

III.6)
\[
\frac{∂ r ∧ \overline{∂} r ∧ \partial \overline{∂} r(ζ)}{Φ_K(ζ, z)} = ω_n ∧ \overline{ω}_n ∧ \left[\sum_{j=1}^{n-1} A_j ω_j ∧ \overline{ω}_j + \sum_{j,k=1}^{n-1} B_{jk} ω_j ∧ \overline{ω}_k\right],
\]
where
\[
|A_j| \leq \frac{1}{\Im F^{(r)}(ζ, z) + |r(ζ)| + |r(z)| + |ζ_j - z_j|^2 + \frac{K}{4} |ζ - z|^3}
\]
and
\[
|B_{jk}| \leq \frac{|ζ - z|}{\Im F^{(r)}(ζ, z) + |r(ζ)| + |r(z)| + \frac{K}{4} |ζ - z|^3},
\]
for ζ, z ∈ \overline{D} ∩ U.

Remark. The special coordinates in dependence of z introduced above are used only for the estimations III.5—6. The function Φ_K is defined with respect to the fixed standard coordinates of \(\mathbb{C}^n\) introduced at the beginning, and which will continue to be used below.

Next, one verifies that there is a decomposition
\[F^{(r)}(ζ, z) + K |π^t(ζ - z)|^3 = \sum_{j=1}^{n} g_j(ζ, z) (ζ_j - z_j),\]
where the coefficients \(g_j\) are sums of terms holomorphic in z (the ones coming from \(F^{(r)}\)) and terms which are \(E_2\). Furthermore, one has

III.7)
a) \(\overline{∂} g_j = E_1\), and b) \(L_{n,z} g_j = E_2\) for \(j = 1, ..., n\).

We now define the \((1, 0)\) form \(g = \sum_{j=1}^{n} g_j dζ_j\) and set
\[W(ζ, z) = \frac{g}{Φ_K(ζ, z)} \text{ for } (ζ, z) \in (\overline{D} ∩ U) × (\overline{D} ∩ U) - \{(ζ, z) : ζ = z ∈ bD\}.\]
Note that for $\zeta \in bD$ one has
\[
\Phi_K(\zeta, z) = F^{(r)}(\zeta, z) + K |\pi^r(\zeta - z)|^3 = \sum_{j=1}^n g_j(\zeta, z)(\zeta_j - z_j),
\]
so that $< W, \zeta - z > = \sum |g_j/\Phi_K|(\zeta_j - z_j) = 1$ on $bD \cap U \times (D \cap U)$. Thus $W$ is indeed a \textit{generating form} in the terminology of Ra86, and the standard calculus for the Cauchy-Fantappiè forms
\[
\Omega_{n,q}(W) = c_{n,q} W \wedge (\overline{\partial}W)^{n-q-1} \wedge (\overline{\partial}W)^q, \quad 0 \leq q \leq n - 1,
\]
applies for $\zeta \in bD \cap U$.

So far the construction of $W$ is local on $U(P)$, and that is all that is needed for the (local) results discussed in this report. If desired for other applications, the local generating forms can be patched together and extended to $\zeta, z \in D$ by standard techniques to obtain a global form on $\overline{D} \times \overline{D} - \{(\zeta, z) : \zeta = z \in bD\}$.

\section{IV. Hölder Estimates}

Given the general integral representation formulas analogous to those in LR83, the “pointwise basic estimates” on arbitrary pseudoconvex domains contained in II.1—4, and the more refined estimates III.5—6, it is now possible to investigate whether an analogon of Kohn’s techniques of \textit{subelliptic multipliers} (see Ko79) can be used in the integral representation setting to find explicit conditions on the boundary $bD$ near the point $P$ which would imply a Hölder estimate
\section{IV.1) $|T_z(\mu \cdot f)(z)| \leq C_{\delta} |r(z)|^{\delta-1} Q_0(f)$ for all $f \in \mathcal{D}^{1}_{qU}$ and $z \in D \cap U$,}
where $T_z$ is any of the vectorfields $L_1, ..., L_{n-1}, \overline{L_1}, ..., \overline{L_{n-1}}, L_n - \overline{L_n}$ acting in $z$ at the point $\zeta = z$. Note that these $2n - 1$ vectorfields span all vector fields tangential to the level surface $M_z$. Since the system $\overline{D} + \vartheta$ is elliptic, it follows that IV.1 then also holds for $T_z$ equal to $L_n + \overline{L_n}$, that is, for the \textit{real} vector field normal to $bD$. The estimate IV.1 then implies the Hölder estimate
\section{IV.2) $|\mu f|_\delta \leq C_{\delta} Q_0(f)$ for all $f \in \mathcal{D}^{1}_{qU}$.}
on a sufficiently small neighborhood $U$ on which $\mu$ is defined. A germ $\mu$ at $P$ which satisfies IV.2 is called a \textit{q-Hölder multiplier at $P$}. For technical reasons it is better to work with the estimation IV.1 as the defining concept for regularizing multipliers $\mathcal{H}^q(P)$. 

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As in Kohn’s work in the $L^2$ setting, the goal is to find conditions on $bD$ which imply that $1 \in \mathcal{H}^q(P)$, so that a Hölder estimate IV.($\delta$) would hold.

It follows readily that $\mathcal{H}^q(P)$ is an ideal in the ring of germs of $C^\infty$ functions at $P$.

The following two results are also easy consequences of the general integral representation formula and the basic estimates II.1—4 stated above.

IV.A) If $\mu = 0$ on $bD$ near $P$, then $\mu \in \mathcal{H}^q(P)$ for any $q \geq 1$.

IV.B) $1 \in \mathcal{H}^n(P)$, with the estimates IV.1—2 holding for any $\delta < 1$.

Much more delicate is the following result. Recall that $r$ is the suitably chosen defining function for $D$ near the point $P$.

IV.C) If $n = 2$, the eigenvalue of the Levi form (i.e., the coefficient of the $(2,2)$ form $\bar{\partial}r \wedge \partial r \wedge \bar{\partial} \partial r$ is in $\mathcal{H}^1(P)$.

The estimates III.6—7 are critical ingredients for this latter result. The techniques developed so far should allow to prove the corresponding result in arbitrary dimensions as well, as follows.

IV.D) The coefficients of $\bar{\partial}r \wedge \partial r \wedge (\bar{\partial} \partial r)^{n-q}$ are in $\mathcal{H}^q(P)$. (Needs to be confirmed in detail for $n > 2$.)

The results A), B), C) and D) establish in the pointwise estimate setting results that are analogous to parts of Kohn’s theory of subelliptic multipliers (Ko79). Some of the remaining steps in the program involve extending other key parts of Kohn’s theory to the setting of regularizing multipliers. If this program proves successful, combining these results with Kohn’s algorithm and a theorem of K. Diederich and J. E. Fornaess (DF78) would lead to proofs of the following conjectures.

**Conjecture I.** If $bD$ is real analytic and pseudoconvex in a neighborhood of $P$, and if there does NOT exist any germ of a complex subvariety $V \subset bD$ of dimension $q$ with $P \in V$, then $1 \in \mathcal{H}^q(P)$.

**Conjecture II.** Suppose $1 \in \mathcal{H}^q(P)$. Then there exists $\delta > 0$, and for each $k = 0,1,2,...$ there are constants $C_k$ (which depend on $\delta$), such that

$$|f|_{k+\delta} \leq C_k Q_k(f) \text{ for all } f \in \mathfrak{D}^{k+1}_q.$$  

Here $Q_k$ is defined by replacing $| \cdot |_0$ with $| \cdot |_k$ in $Q_0$. For $k = 0$ the estimate essentially just restates the hypotheses. The main point thus is to extend the estimate to arbitrary higher order derivatives.

**Corollary of Conjectures I and II.** If $bD$ is bounded, pseudoconvex, and real analytic, then there exist $\delta > 0$ and $C_k$, $k = 0,1,2,...$, so that if $q \geq 2$ and $f$ is a $\bar{\partial}$—closed $(0,q)$ form in $C^k(D)$, then the “canonical” solution $u = \bar{\partial}^* N_qf$ of $\bar{\partial}u = f$ orthogonal to ker$\bar{\partial}$ in the $L^2$ sense satisfies

$$|\bar{\partial}^* N_qf|_{k+\delta} \leq C_k |f|_k.$$
Remark. These results and techniques might also lead to a proof of the preceding Corollary in case $q = 1$, where $u = \overline{\partial} N_q f$ is a function. At the very least it should be possible to construct a solution operator $S_1$ for $\overline{\partial} u = f$ in case $q = 1$ which satisfies the analogous estimates $|S_1 f|_{k+\delta} \leq C_k |f|_k$ for some $\delta > 0$. Furthermore, if the theory of regularizing multipliers is successful in the pointwise setting, the preceding results would hold if $bD$ is just pseudoconvex and of finite type near $P$. (See Y.-T. Siu, Si10.)

In case $D$ is smoothly bounded and strictly pseudoconvex, and if the metric is a Levi metric, Lieb and Range (LR86) proved—in the present terminology—that $1 \in \mathcal{H}^q(P)$ for any $P \in bD$ and $q \geq 1$, with $\delta = 1/2$. Furthermore, they proved Conjecture II and the Corollary in this case as well, also with $\delta = 1/2$, and they showed that in case $q = 1$ the Corollary holds without any restrictions on the metric. While the precise identification of the principal terms for the various operators in the $\overline{\partial}$–Neumann theory on strictly pseudoconvex domains did require a Levi metric, the techniques developed here might allow to prove the Hölder–1/2 estimates in the strictly pseudoconvex case as in Conjecture II and the Corollary with respect to an arbitrary Hermitian metric. (Note that the case $q = 1$ is already covered by LR86.)

Remark. Aside from the applications to pointwise estimates in the $\overline{\partial}$–Neumann theory outlined in this report, the function $\Phi_K$, the related generating form and integral kernels, and the estimates described above should have other significant applications in complex analysis on (weakly) pseudoconvex domains.

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