Strangeness degree of freedom, when trapped in a bound nuclear system, affects every physical observable, starting from deuteron to neutron stars. It induces subtle distortions in the system even with the presence of a single hyperon \[1\]. The hypernuclear systems with one or two quanta of strangeness in a finite nucleus and also with the bulk of it as in an infinite-body hyperon star offer an unique opportunity to enrich our knowledge about the role of strangeness in a nuclear medium of different densities. Study of such systems may provide useful informations about baryon-baryon and three-baryon forces. Notable advancements have been made on theoretical as well as on experimental frontiers of the subject.

The recent unambiguous observation of six-baryon double-Λ hypernucleus \(\Lambda^6\) in the Japanese high energy laboratory (KEK) hybrid experiment \(E373\) and the evidence \(3\) for a bound four-baryon double-Λ hypernucleus \(\Lambda^4\) as suggested by the Brookhaven alternating-gradient synchrotron (BNL-AGS) experiment E906, have given fresh impetus to the physics of \(S = -2\) sector. Besides \(\Lambda^4\), we have three well-established double-Λ hypernuclear species \((\Lambda^6\text{He, NAGARA event}), (\Lambda^6\text{Be, }10\text{Be}),\) and \((\Lambda^6\text{C})\) \(2, 3, 4\), whose realistic study along with many single-Λ hypernuclei having rich experimental statistics with a wide range of baryon numbers and orbital angular momentum \(\ell_A \leq 3\) is a major thrust area of research.

A theoretical study requires a realistic Hamiltonian and a good wave function (WF) that includes all dynamical correlations induced by the potentials of the Hamiltonian. However, when the same is employed, computational complexities increase with increasing mass number \(A\), irrespective of the many-body technique involved. The Faddeev-Yakubovsky (FY) calculations \(7\) have not been extended to \(A \geq 5\) baryon hypernuclei, because calculational dimensions expand to an unmanageable situation. However, in the cluster Faddeev-Yakubovsky (CFY) approach, calculations have been made possible to \(A = 5\) and \(6\) baryon hypernuclei \(8, 9\). Along with these, the fully coupled channel stochastic variational (FCCSV) approach \(8\) and the fully correlated variational Monte Carlo (FCVMC) study \(9, 10\) have pushed ahead the frontiers of the subject. The \(\Lambda N\) space exchange correlation (SEC) is built into all the above studies except for CFY. Being an important correlation, SEC affects every physical observable in \(\Lambda^4\) and also in \(\Lambda^6\).

A bound state for the recently conjectured \(\Lambda^4\) is not found over a wide range of \(\Lambda\) strengths in the FY search \(11\). But, a recent FCCSV search \(12\), in contrast to it, predicts a bound \(\Lambda^4\). Though, the search is not free of uncertainties. Various couplings of the \(4 \times 4\) Hamiltonian matrix used as basic inputs in this approach are uncertain, specially those falling in the \(S = -2\) sector for which there is no direct information from experiments in free space. The many-body effects on these strengths in a nuclear medium constitute another important issue, which modify the free space results. The strong \(\Lambda N - \Sigma N\) transition potential in the \(S = -1\) sector also lacks complete preciseness.

Potentials related to these couplings have got strong tensorial dependence, therefore, are sensitive to other operators of the WF, specially to the tensor operator. The expectation value of tensor operator in a Jastrow WF for a closed-shell nucleus is zero, whereas expectation value for its square is nonzero. This may lead to a quadratic dependence of the expectation value of the potentials with respect to their strengths. A slight variation in these may offset the variational WF and may result in an appreciable change in the energy breakdown and also in the total ground-state energy. Uncertainty in the strengths (basic input) leads to the uncertainty in the results as well as in the consistency of calculations.

These factors should be addressed carefully. It would be useful to investigate how results and consistency of the calculations are affected with the variations in the strengths and how do they interplay. We explore these important issues in this letter and investigate the role of every potential strength that matter which is inevitable for precise determination of strengths, for the resolution of \(A = 5\) anomaly and for an authentic search of \(\Lambda^4\), whose binding is a subtle issue.
We follow a different approach other than the coupled channel formalism. One may always project out $\Sigma$, $\Delta$, etc., degrees of freedom from this formalism. In the $S=-1$ sector, this would result in a three-baryon $\Lambda NN$ potential. It is written as a sum of two-pion exchange (TPE) attractive term and a repulsive term [13, 14, 15] like its non-strange counterpart, the $NNN$ potential. The repulsive term is suggested by the suppression mechanism due to $\Lambda N-\Sigma N$ coupling [16, 17, 18, 19], which is a medium effect. For the $S=-2$ sector, we may use simulated Nijmegen potential models [7]. Thus, our basic ingredients are two-baryon and three-baryon potentials. We then use a fully correlated WF as in Ref. [9] written for all the s-shell single and double-$\Lambda$ hypernuclei. With such a Hamiltonian and WF, a microscopic study of six-baryon double-$\Lambda$ hypernucleus is easily manageable in the framework of cluster Monte Carlo technique [20, 21].

Variations in the strengths directly affect the expectation values of the respective potentials and also the WF as correlations are nothing but the solutions of these potentials. Moreover, there are sensitivities among various terms of the Hamiltonian and the operators of the WF. Thus, a change in any of the strengths may affect the complete energy breakdown. Bearing these factors in mind, we proceed in a systematic way. We perform a FCVMC study [9] of $^3\text{He}$ hypernucleus, using a realistic Hamiltonian and a fully correlated WF. It suggests that a study ignoring SEC would be misleading. In a subsequent study, we calculate $\Lambda$-separation energy ($B_{\Lambda} = E_{4\text{He}} - E_{3\text{He}}$) and obtain solutions of all the strengths for reproduce experimental $B_{\Lambda}^{\text{exp}}=3.12(2)$ MeV for the range of the strengths [22]. In order to know how do behaviours change with two quanta of strangeness, we extend our study to $^6\Lambda^6\text{He}$ hypernucleus [10], where SEC effects are found more evident because of the presence of a pair of $\Lambda$ hyperons. We then aim to study the behaviour of all the strengths for this hypernucleus herein.

For the $S=-2$ sector, we use various Nijmegen models representing $^1S_0$ $\Lambda\Lambda$ potential, which are simulated to a phase equivalent three-range Gaussian form [7, 23, 24]

$$v_{\Lambda\Lambda}(r) = 9324 \exp \left(-\frac{r^2}{0.352}\right) - 379.1 \gamma \exp \left(-\frac{r^2}{0.7712}\right) - 21.49 \exp \left(-\frac{r^2}{1.3422}\right).$$

The dimensionless quantity $\gamma$ distinguishes amongst various Nijmegen potential models. For example, NSC97e, ND, and NEC00, which are represented by $\gamma=0.5463, \gamma=1.0$ and $\gamma=1.2044$, respectively. The $\Lambda\Lambda$-separation energy, ($B_{\Lambda\Lambda} = E_{4\text{He}} - E_{3\text{He}}$), therefore, depends on the choice of the potential, which should be taken with caution. The expectation value of the $\Lambda\Lambda$ potential would depend upon the choice of $\gamma$. It may affect the WF through its self induced correlation due to same reason mentioned above. For the $S=-1$ sector, we use charge symmetric $\Lambda N$ potential [25]

$$v_{\Lambda N}(r) = [v_c(r) - \tau T^2_{\pi}(r)][(1 - \varepsilon + \varepsilon P_x) + \frac{v_{\pi}}{4} T^2_{\pi}(r)]\sigma_{\Lambda} \cdot \sigma_N.$$  

Here, $\varepsilon$ determines the odd-state potential, which is the strength of the space-exchange potential relative to the direct potential. $v_c(r)$ is the Saxon-Woods core and $T_{\pi}(r)$ is the one-pion tensor shape factor. The constants, $\tau$ and $v_{\pi}$, are respectively the spin-average and spin-dependent strengths.

The Hamiltonian for the $A$-baryon double-$\Lambda$ hypernucleus reads as

$$H = H_{NC} + H_{\Lambda_1} + H_{\Lambda_2} + v_{\Lambda_1\Lambda_2}$$

$$H_{NC} = T_{NC} + \sum_{i<j} A_{-2} v_{ij} + \sum_{i<j<k} A_{-2} V_{ijk},$$

$$H_{\Lambda_n} = T_{\Lambda_n} + \sum_{i} A_{-2} v_{\Lambda_n i} + \sum_{i<j} A_{-2} V_{\Lambda_n ij}.$$  

$H_{NC}$ is the nuclear core (NC) Hamiltonian and $H_{\Lambda_n}$ is the Hamiltonian arising due to an individual $\Lambda_n$. Subscripts $i$, $j$ and $k$ refer to nucleons. Obviously, $H_{NC} + H_{\Lambda_n}$ is the Hamiltonian for the $A-1$ baryon single-$\Lambda$ hypernucleus. For the $S=0$ sector, we use well established Argonne $v_{18}$ [26] NN potential and Urbana type $NNN$ potential [27]. The $\Lambda NN$ potential is the sum of a repulsive dispersive term [13]

$$V_{Aij}^P(W_{\pi}) = W_{\pi} T^2_{\pi}(r_{\Lambda}) T^2_{\pi}(r_{\Lambda})[1 + \sigma_{\Lambda} \cdot (\sigma_i + \sigma_j)/6]$$

and a TPE term for P- and S-wave $\pi-N$ scatterings [13]

$$V_{Aij}^S(C^P) = - (C^P/6) \tau_i \cdot \tau_j \{X_{i\Lambda}, X_{Aij}\}$$

and

$$V_{Aij}^S(C^S) = C^S Z(r_{\Lambda}) Z(r_{\Lambda}) \sigma_{\Lambda} \cdot \hat{r}_{i\Lambda} \sigma_{\Lambda} \cdot \hat{r}_{j\Lambda} \tau_i \cdot \tau_j$$

with

$$X_{i\Lambda} = (\sigma_{\Lambda} \cdot \sigma_i) Y_\pi(r_{\Lambda}) + S_{i\Lambda} T_{\pi}(r_{\Lambda})$$

and

$$Z(r) = \frac{m_{\pi} r}{3} [Y_\pi(r) - T_{\pi}(r)].$$

Here, $W^D, C^P$ and $C^S$ are the strengths and $Y_\pi(r)$ is the Yukawa function.

In order to know the role of strange sector potential strengths and the various sensitivities among them, we
must analyse our present findings in the light of the findings of previous studies \cite{9, 10, 20, 28}. A couple of simplifications arise: (i) due to strong suppression of S-wave ΛNN potential as its non-strange counterpart S-wave NNN potential \cite{5} (ii) and also due to weak spin part of the ΛN potential for the spin-zero core nucleus. Therefore, variations in the strengths of these potentials, $C^S$ and $v_\sigma$, are hardly noteworthy. The $C^S$ is, however, reasonably fixed to 1.5 MeV as in Ref. \cite{22}. Therefore, strengths which matter are $\pi$, $\varepsilon$, $C^P$ and $W^D$.

The TPE potential has a generalised tensor type structure. It is sensitive to operators and specially to tensor operator, hence to its self induced correlation as mentioned before with reason. Variation in its strength, $C^P$, offsets the WF, hence complete energy breakdown. On the other hand, variation in $\varepsilon$ affects the baryon density profiles through the repulsive central ΛN correlations and $SEC$ \cite{4} as they are solutions of the Schrödinger equation involving $\varepsilon$ in the ΛN potential. The ΛN central correlation pushes the nucleons at the centre and towards the periphery, however, $SEC$ weakens this effect \cite{3, 4}. This leads to modifications in densities with changing $\varepsilon$, which affects the complete energy breakdown, even its central pieces. The density effect directly appears in the TPE potential through $T_\varepsilon(r)$ and $\gamma(r)$ functions, which in turn affects the WF through its sensitivity with operators. Thus, $C^P$ is correlated with $\varepsilon$.

The WF remains unaffected with the variations in $W^D$. Therefore, a linear relationship between $W^D$ and the expectation value of its potential is observed. A change in density profiles due to variations in $\varepsilon$ affects the expectation value, hence the slope of this relationship. The $v_{\Lambda i}$, for spin-zero core nucleus, is almost a central quantity due to weak spin part. Its expectation value depends about 0.87. Hence $v_{\Lambda i}$ (being a negative quantity) decreases with decreasing $\varepsilon$. The $v_{\Lambda i}$ falls in the same line. The $V_{\Lambda i j}$, which is the sum of TPE and dispersive potentials, has got an opposite trend. This with the NC part of the energy ($E_{NC}$) resists any change in $v_{\Lambda i}$ and $v_{\Lambda A}$ caused due to variation in $\varepsilon$.

A linear relationship between $W^D$ and $\varepsilon$ is found to be existing for both the hypernuclei: $^6\Lambda\Lambda$He \cite{9} and $^6\Lambda\Lambda$He \cite{10}, but with a different slope. For the $^6\Lambda\Lambda$He, the relationship has been observed for a range of $C^P$ \cite{22}. Thus, we may always find a suitable $W^D$ to reproduce $B^\Lambda_\Lambda^{exp}$ against any change in $\varepsilon$. As expected, results vary with the variations in $\pi$ at any value of $C^P$. The above mentioned slope is significantly affected against any change in $\pi$ in case of $^6\Lambda\Lambda$He, which was of lesser importance in case of $^6\Lambda\Lambda$He. As a result, different combinations of $\varepsilon$ and $W^D$ that reproduce $B^\Lambda_\Lambda^{exp}$, do not converge at the same $B^\Lambda_\Lambda$ except for $\pi=6.15$ MeV. At this value of $\pi$, a change in $v_{\Lambda i}$ and $v_{\Lambda A}$ is balanced by an opposite effect in $V_{\Lambda i j}$ and $E_{NC}$ giving $\partial B^\Lambda_\Lambda/\partial \varepsilon \approx 0$, which is an accident. With decreasing $\pi$ we notice an increase in $\partial B^\Lambda_\Lambda/\partial \varepsilon$, because changes in $V_{\Lambda i j}$ and $E_{NC}$ win over the changes in $v_{\Lambda i}$ and $v_{\Lambda A}$.

We first reproduce $B^\Lambda_\Lambda^{exp}(\pi, \varepsilon, C^P, W^D)$ for the range of the strengths. Thereafter, using the same strengths,
we plot a sensitivity graph between $B_{AA}(\tau, \varepsilon, C^P, W^D, \gamma)$ and the only free strength $\gamma$. This we report in Fig. 11. The $B_{AA}$ increases with increasing value of $\gamma$. However, all the above mentioned features remain the same. We notice an important result at $C^P=0.75$ MeV that the sets of strengths even with different $\tau$, which reproduce $B_{AA}^{exp}$, yield the same value of $B_{AA}$ at $\varepsilon \approx 0.18$ irrespective of the value of $\gamma$. Thus, by adjusting $\gamma$, we may reproduce $B_{AA}^{exp} = 7.25(19)$ MeV for the same strengths that reproduce $B_{AA}^{exp}$. Therefore, $\varepsilon \approx 0.18$ turns out to be the only condition of consistency. Millener [22] too has suggested a small value for $\varepsilon$. We observe that $0.85 < \gamma < 0.95$ reproduces $B_{AA}^{exp}$ at this value of $\varepsilon$.

As variations in $C^P$ leads to an appreciable change in the above mentioned slope [22] and in the energy breakdown, we double the value of $C^P$ from 0.75 to 1.5 MeV and repeat our calculations with a suitable $W^D$ that reproduces $B_{AA}^{exp}$. Using same strengths, we perform calculations for $B_{AA}$. Results for both the values of $C^P$ are presented in Table I for $\tau = 6.10$ MeV and $\varepsilon = 0.2$. This value of $\varepsilon$ is very close to the value of consistency $(\varepsilon \approx 0.18)$. Although, we observe a significant effect in every energy piece as $C^P$ offsets the WF, we find close predictions for $B_{AA}$ for both the values of $C^P$. The total energy remains almost the same because of the change in $NC$ part of the energy is balanced by an opposite change in $\Lambda$ part of the energy. Therefore, condition of consistency remains unaltered irrespective of the value of $C^P$.

We notice more than three fold increase in the expectation value of TPE potential corresponding to two fold increase in its strength which suggests a quadratic behaviour. Strong effect in the $NC$ part of the energy ($E_{NC}$) is observed. As a result, $NC$ is more polarised for higher value of $C^P$. Nemura et al. [20] too have noticed strong sensitivity between $E_{NC}$ and the tensorial $\Lambda N - \Sigma N$ transition potential.

A similar study of $^4$H, $^2$H$^*$ and $^3$H hypernuclei may decide the strengths of strange sector potentials including $\gamma$ in a single shot, and hence may resolve the $A = 5$ anomaly [14, 51] with no additional effort. This investigation is under consideration, which would be followed by an authentic search of $^4$H. The knowledge would be helpful to bridge the gap in our fundamental understanding of baryon-baryon forces.

Some improvements in the Hamiltonian may be suggested. One may also adopt Green’s function Monte Carlo method. But these would be mere refinements.

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* Electronic address: anisul@iucaa.ernet.in

References

[1] B. F. Gibson and E. V. Hungerford III, Phys. Rep. 257, 349 (1995).
[2] H. Takahashi et al., Phys. Rev. Lett 87, 121502 (2001).
[3] K. Ahn et al., Phys. Rev. Lett 87, 123504 (2001).
[4] M. Danyus et al., Nucl. Phys. 49, 121 (1963); Phys. rev. Lett. 11, 29 (1963); R. H. Dalitz, D. H. Davis, P. H. Fowler, A. Montwill, J. Pniewski and J. A. Zakrzewski, Proc. R. Soc. London, Ser. A 246, 1 (1989); S. Aoki et al., Prog. Theor. Phys. 85, 1287 (1991); C. B. Dover, D. J. Millener, A. Gal and D. H. Davis, Phys. Rev. C 44, 1905 (1991).
[5] A. Nogga, H. Kamada and W. Gloeckle, Phys. Rev. Lett. 88, 172501 (2002).
[6] I. N. Filikhin, A. Gal and V. M. Suslov, Phys. Rev. C 68, 024002 (2003); I. N. Filikhin and A. Gal, Nucl. Phys. A707, 491 (2002).
[7] I. N. Filikhin and A. Gal, Phys. Rev. C 65, 041001(R) (2002); E. Hiyama, M. Kamimura, T. Motoba, T. Yamada and Y. Yamamoto, Phys. Rev. Lett. 89, 142508 (2002); Phys. Rev. C66, 024007 (2002).
[8] H. Nemura, S. Shimamura, Y. Akaishi and K. S. Myint, Phys. Rev. Lett. 94, 202502 (2005).
[9] A. A. Usmani, Phys. Rev. C 73, 011302(R) (2006).
[10] A. A. Usmani and Z. Hasan, Phys. Rev. C 74, 034320 (2006).
[11] I. N. Filikhin and A. Gal, Phys. Rev. Lett. 89, 172502 (2002).
[12] H. Nemura, Y. Akaishi and K. S. Myint, Phys. Rev. C 67, 051001(R) (2003).
[13] A. R. Bodmer and Q. N. Usmani, Nucl. Phys. A477, 621 (1988).
[14] A. Gal, Adv. Nucl. Phys. 8, 1 (1975).
[15] R. K. Bhaduri, B. A. Loiseau and Y. Nogami, Anns. Phys. (N. Y) 44, 57 (1967).
[16] Q. N. Usmani and A. R. Bodmer, Phys. Rev. C60, 055215 (1999).
[17] A. R. Bodmer and D. M. Rote, Nucl. Phys. A169, 1 (1971).
[18] J. Rozynek and J. Dabrowski, Phys. Rev. C20, 1612 (1979); J. Dabrowski and J. Rozynek, ibid. 23, 1706(1981); Y. Yamamoto and H. Bando, Suppl. Prog. Theor. Phys. Suppl. 81, 9(1985); Y. Yamamoto, Nucl. Phys. A450, 275c (1986).
[19] A. R. Bodmer, D. M. Rote and A. L. Mazza, Phys. Rev. C2, 1623 (1970).
[20] A. A. Usmani, S. C. Pieper and Q. N. Usmani, Phys. Rev. C51, 2347 (1995).
[21] S. C. Pieper, R. B. Wingard and V. R. Pandharipande, Phys. Rev. C46, 1741 (1992).
[22] A. A. Usmani and F. C. Khanna, submitted to Phys. Rev. C.
[23] Th. A. Rijken et al., Phys. Rev C59, 21 (1999).
[24] E. Hiyama, M. Kamimura, T. Motoba, I. Yamada and I. Yamamoto, Prog. Theor. Phys. 97, 881 (1997).
[25] A. R. Bodmer and Q. N. Usmani and J. Carlson, Phys. Rev. C29, 684 (1984); I. E. Lagaris and V. R. Pandharipande, Nucl. Phys. A359, 331 (1981).
[26] R. B. Wingard, V. G. J. Stoks, and R. Schiavilla, Phys. Rev. C51, 38 (1995).
[27] B. S. Pudliner, V. R. Pandharipande, J. Carlson and R. B. Wingard, Phys. Rev. Lett. 74, 4396 (1995); J. Carlson, V. R. Pandharipande and R. B. Wingard, Nucl. Phys. A401, 59 (1983).
[28] A. A. Usmani and S. Murtaza, Phys. Rev. C68, 024001 (2003); A. A. Usmani, Phys. Rev. C52, 1773 (1995).
[29] D. J. Millener, Nucl. Phys. A691, 93c (2001).
[30] H. Nemura, Y. Akaishi and Y. Suzuki, Phys. Rev. Lett. 89, 142504 (2002).
[31] R. H. Dalitz, R. C. Herndon, and Y. C. Tang, Nucl. Phys. B47, 109 (1972); E. V. Hungerford and L. C. Biedenhorn, Phys. Lett. 142B, 232 (1984).