A SYMMETRY PROPERTY
OF SOME HARMONIC ALGEBRAIC CURVES

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Abstract. The aim of this note is to give a surprising symmetry property of some harmonic algebraic curves: when all the roots $z_i$ of a complex polynomial $P$ lie on the unit circle $\mathcal{U}$, the points of $\mathcal{U}$ different from the $z_i$, and such that $\text{Arg}(P(z)) = \theta$, form a regular $n$-gon, where $n$ is the degree of $P$.

Let $\mathbf{z} = \{z_1, \ldots, z_n\}$ be a multiset of $n$ points in the complex plane $\mathbb{C}$ and $P$ the monic polynomial with root set $\mathbf{z}$:

$$P(z) = \prod_{i=1}^{n} (z - z_i).$$

For $\theta$ a fixed real number of your choice, consider

$$C_\theta(P) = \{z \in \mathbb{C} : \text{Im}(e^{-i\theta} P(z)) = 0\}.$$

The set $C_\theta(P)$ coincides up to $\mathbf{z}$, to the set $\{z \in \mathbb{C} : \text{Arg}(P(z)) = \theta[\pi]\}$. These curves arise in the Gauss approach to the Fundamental Theorem of Algebra (see e.g. Stillwell [3], and Martin & al. [1]). In their paper Martin & al. [1] and then Savitt [2] initiated the study of the combinatorial topology of the families $C_\theta(P)$. The idea are the following ones: the curves $C_\theta(P)$ have $2n$ asymptotes at angles $(\pi k + \theta)/n$, for $k \in \{0, \ldots, 2n - 1\}$, and form in the generic case $n$ non intersecting curves. This induces a matching: $k$ and $k'$ are matched if and only if the asymptotes $(\pi k + \theta)/n$ and $(\pi k' + \theta)/n$ lie on the same connected component in $C_\theta(P)$. The papers [1] and [2] aim at studying these matchings, and also the properties of the so-called necklaces, formed by the families of matchings obtained when $\theta$ traverses the set $[0, \pi]$.

Let us now state and prove our result. The set $\mathbf{z}$ is clearly included in $C_\theta(P)$. It turns out that when $\mathbf{z}$ is included in the unit circle $\mathcal{U} = \{z : |z| = 1\}$, the set $C_\theta(P) \cap \mathcal{U}$ presents a quite surprising symmetry – illustrated at Figure 1 – that can be stated as follows.

Proposition 1. If $\mathbf{z}$ is a subset of $\mathcal{U}$, then

$$C_\theta(P) \cap \mathcal{U} = \mathbf{z} \cup G(\mathbf{z})$$

where $G(\mathbf{z})$ is the regular $n$-gon on $\mathcal{U}$, with set of vertices $\{e^{i(\Omega+2k\pi/n)} : k = 1, \ldots, n\}$, for

$$\Omega := \frac{2\theta - \sum_{j=1}^{n} \text{Arg}(z_j)}{n} - \pi.$$

There exists a purely geometric proof of this Proposition using that the measure of a central angle is twice that of the inscribed angle intercepting the same arc; we provide below a more compact analytic proof.

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Proof. We will only consider $z \notin \mathbb{Z}$. We have the equivalence:
\[ z \in C_{\theta}(P) \setminus z \iff z \notin \mathbb{Z}, \quad \sum_{i=1}^{n} \text{Arg}(z - z_i) = \theta \ [\pi], \]
where $\text{Arg}(z) \in \mathbb{R}/2\pi\mathbb{Z}$ stands for (any chosen determination of) the argument of $z \neq 0$. Now for any $\nu$ and $\psi$ real numbers,
\[ e^{i\nu} - e^{i\psi} = e^{i\frac{\nu+\psi}{2}} (e^{i\frac{\nu-\psi}{2}} - e^{i\frac{-\nu+\psi}{2}}) = 2i \sin((\nu - \psi)/2)e^{i\frac{\nu+\psi}{2}}. \]
Thus
\[ \text{Arg}(e^{i\nu} - e^{i\psi}) = \frac{\nu + \psi}{2} + \frac{\pi}{2} + \pi \times \text{sgn} \left( \sin((\nu - \psi)/2) \right) \ [2\pi] \]
Hence, $z \in C_{\theta}(P) \setminus z$ is equivalent to:
\[ z \notin \mathbb{Z}, \quad \sum_{j=1}^{n} \left( \frac{\text{Arg}(z) + \text{Arg}(z_j)}{2} + \frac{\pi}{2} \right) = \theta \ [\pi], \]
which leads to the conclusion at once. □

\[ \text{FIGURE 1.} \quad \text{An example where } n = 7, \theta = 0 \text{ and the roots } z_i \text{ randomly chosen.} \]

Note. If $z_i$ is a root of multiplicity $k$ of $P$, and if $z_i$ belongs to $G(z)$, then in the neighborhood of $z_i$, $C_{\theta}(P)$ has $k$ tangents, one of them coinciding with the tangent of the circle at $z_i$. Moreover, it is simple to check that if $z_i$ is not on $G(z)$, then the tangents of $C_{\theta}(P)$ at $z_i$ are not tangent to $\mathcal{U}$.

References

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