Noise cancellation effect in quantum systems

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(Dated: January 2, 2022)

We consider the time evolution of simple quantum systems under the influence of random fluctuations of the control parameters. We show that when the parameters fluctuate sufficiently fast, there is a cancellation effect of the noise. We propose that such an effect could be experimentally observed by performing a simple experiment with trapped ions. As a byproduct of our analysis, we provide an explanation of the robustness against random perturbations of adiabatic population transfer techniques in atom optics.

PACS numbers: 03.67.Lx

I. INTRODUCTION

In the last ten years the interest around the control and manipulation of quantum systems has grown very fast. The possibility to encode and process information has lead to innovative proposals. The major results have been achieved in Quantum Information Processing (QIP), including both theoretical and experimental ones. Quantum cryptography[11] and information transfer protocols[2] have enhanced our understanding of information processing and this in the near future will presumably lead to a significant technical advancement. Quantum Computation (QC) is still in a initial stage: even if the quantum computers seem to be able to solve quickly some problems which are intractable with classical computers[3], more quantum algorithms are required to extend its applicability.

Unfortunately, the quantum system are very delicate and they are subject to two different kinds of errors. On one hand, there is the loss of information due to decoherence the unavoidable interaction of quantum systems with their environments. This problem has been extensively studied over the past few years and proposals to overcome it have been put forward (and a few have been experimentally tested). These proposals include error avoiding[4], error correcting strategies[5] and decoupling techniques[6]. The other source of errors is the imprecise control of the parameters which perform the quantum operation (e.g. the laser or the magnetic field). How to handle such errors is an open problem, though some progress has been made in the framework of the so called geometrical quantum computation[7,8,9]. The goal of this paper is to approach the second problem in very simple and idealized situations.

A simple way to model the parameter noise is to consider a quantum system subject to a stochastic fluctuating field with zero mean. Such a model has been recently considered in order to study the effect of the noise on holonomic quantum gates: in Ref.[10] it has been shown that there is a cancellation effect for a fast fluctuating stochastic field and shown that such a cancellation is due to the geometrical dependence of the holonomic operator. Recently, the general validity of such cancellation effect has been clarified: in Ref.[11] is has be shown that for sufficient fast fluctuating stochastic field with zero mean the effects of the noise are wiped out.

In this paper we shall study simple quantum systems subjected to stochastic noise and discuss some applications. In Section II we consider random perturbations which are diagonal in the logical basis and propose an experiment to test the cancellation effect of the the noise—a simple modification of the experiment done by Kielpinski et al.[14]. Then we consider a more general noise and compute the fidelity. By elementary perturbation theory, we show that the effects of the noise are wiped out. In Section III we discuss how noise cancellation could be relevant for adiabatic population transfer experiments in atoms optics (in this case, the cancellation effect avoids the break of the adiabatic approximation, and allows for desired transformation, even in presence of fast fluctuating noise).

For sake of simplicity, all the simulations and analytical calculation are done with Gaussian noise distributions but presumably analogous results can be achieved with a generic stochastic noise with zero mean[11].

II. TWO-LEVEL SYSTEMS

Consider a two-level system evolving according to Hamiltonian

\[ H(t) = H_0 + \delta H_1(t). \]  (1)

Suppose that \( H_0 = B_z \sigma_z \) (with \( \sigma_z \) being a Pauli matrix) and that \( \delta H_1(t) \) is of the form

\[ \delta H_1(t) = N \sum_{j=0}^{N} \delta A_j S_j(t), \]  (2)

where \( \delta A_j \) are random 2 \( \times \) 2 matrices and \( S_j(t) \) are “box functions” with time step \( \tau \), i.e., they are functions equal to 1 in the time interval \( (j\tau, (j+1)\tau) \) and zero otherwise. Let \( T \) be some “final” time at which we wish to consider the system, and let \( \tau = T/N; \tau \) can be regarded the correlation time of the noise. Then the evolution operator...
from time zero to time $T$, generated by $H(t)$ with $\delta H(t)$ given by (2), can be written as

$$U(T) = U_N U_{N-1} \cdots U_j \cdots U_2 U_1$$  \hspace{1cm} (3)$$

where

$$U_j = \exp \left[ -i \tau \left( B_j \sigma_z + \delta A_j \right) \right].$$  \hspace{1cm} (4)

A. Diagonal Noise

First, we shall focus on the very simple case corresponding to the choice $\delta A^j = \delta B^j \sigma_z$, where the $\delta B^j$, $j = 1, \ldots, N$ are independent Gaussian distributed random variables with mean zero. In this case the noise is diagonal in the logical basis $\{ |\uparrow\rangle, |\downarrow\rangle \}$ in which $\sigma_z$ is diagonal. Then (3) is trivially computed (due to the commutativity of the $\delta A^j$),

$$U(T) = U^0(T) \delta U(T)$$  \hspace{1cm} (5)$$

where $U^0(T) = \exp(-i T H_0)$ is the evolution generated by $H_0$, and

$$\delta U(T) = \begin{pmatrix} e^{-i \sum_j \frac{\delta B^j}{\sqrt{N}} T} & 0 \\ 0 & e^{i \sum_j \frac{\delta B^j}{\sqrt{N}} T} \end{pmatrix}$$  \hspace{1cm} (6)$$

A standard performance estimator is the fidelity $F_T$, which, in our case, is given by

$$F_T = \frac{\left| \langle \psi(0) | \psi(T) \rangle \right|^2}{\left| \langle \psi(0) | \psi(T) \rangle \right|^2}$$  \hspace{1cm} (7)$$

where $\psi(0)$ is the “ideal” final state evolved according to $H_0$ (that is, when the noise is turned off) and $\psi(T)$ is actual final state (that is, when the noise is turned on). The two final states are generated, of course, by the same (generic) initial state $|\psi(0)\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$. From (5) and (6) one easily obtains

$$F_T^2(\chi, \alpha, \beta) = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \cos \chi,$$  \hspace{1cm} (8)$$

where

$$\chi = 2 T \frac{1}{N} \sum_{j=0}^{N} \delta B^j.$$  \hspace{1cm} (9)$$

Since the random variables $\delta B^j$ in (9) are taken to be independent and identically distributed according to a Gaussian with zero mean and variance $\delta B^2$, the probability distribution of $\chi$ is

$$P(\chi) = \frac{\sqrt{N}}{\sqrt{2 \pi 2 T \delta B^2}} e^{-\frac{N \chi^2}{2(2 T \delta B^2)}},$$  \hspace{1cm} (10)$$

By averaging (8) with respect to (10), we arrive at the mean square fidelity:

$$F_T^2(\alpha, \beta) = \int d\chi P(\chi) F_T^2(\chi, \alpha, \beta) =$$

$$= |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 e^{-\frac{2 T^2 \delta B^2}{N}}$$  \hspace{1cm} (11)$$

As it can be easily seen, for $N \to \infty$ and $\tau \to 0$ (while keeping $N \tau = T$ of order 1), the mean square fidelity $F_T^2(\alpha, \beta)$ approaches 1 (since $|\alpha|^2 + |\beta|^2 = 1$), independently of the initial state. This corresponds to a short correlation time of the noise, that is, to a fast random fluctuating field.

Now we’d like to provide some perspective on such limit behavior. Of course, the limit $\tau \to 0$ is only an idealization for $\tau$ small but finite. The above results are obtained for constant $\delta B^2$, but the Heisenberg uncertainty relation imposes strong constraint on the energy fluctuation happening in such a short time interval, $\delta B^2 \propto 1/\tau$. Note that for $\delta B^2 \propto N$ eq. (10) and (11) describe a system interacting with a white noise environment that is the standard way to model the decoherence effect. So, for very small $\tau$ the variance $\delta B^2$ should not be any more considered constant and our approximation of constant variance breaks down. However, rather than microscopic (environmental) noise, we are interested in modeling the macroscopic (parametric) noise due to imprecision in the control field, and for such our approximation should apply. In this case it is interesting to ask whether the condition of small $\tau$ could be physically relevant. In general, in a quantum evolution the final time $T$ is fixed; the correlation time $\tau$ can be hardly controlled (i.e., stabilizing the control field) and we are not in the condition to have a cancellation effect. In Section III we present an experimental proposal to test the presence of this effect; the experiment realizability lies in the control of the correlation time $\tau$ of the simulated noise. Moreover, there are situations in which this effect can be experimentally relevant: when we have a further degree of freedom and can change the evolution time $T$. In these cases, fixed $\tau$ with the above properties, we can prolong $T$ in order to have $T \gg \tau$ (that is $N \gg 1$) and exploit the cancellation effect. We give an example of such situation in an adiabatic evolution in Section III.

We observe that another interesting feature of eq. (11) is that the cancellation of the noise does not depend on
FIG. 2: The decoherence effect during the evolution for two different value of $N = T/\tau$. Theoretical curves (dashed line) and those obtained by numerical simulations (solid line) are showed. For the theoretical curve (eq. [19]) the decoherence time is $t_{\text{deco}} = \frac{N}{2\pi^2 \delta B^2}$.

the strength $\delta B$: it is always possible to find a suitable $N$ in order to obtain cancellation of the noise—invert equation (14) and express $N$ in terms of $\delta B$ for given $F_T$ and evolution time $T$.

We recall that dependence of the fidelity on a specific choice of the initial state is usually eliminated by averaging $F^2_T(\alpha, \beta)$ over all the possible initial states with respect to the uniform distribution on the unit sphere in the Hilbert space of the system [15] in our case the (projective) sphere $|\alpha|^2 + |\beta|^2 = 1$. Performing this operation yields to

$$F_T^2 \equiv \langle F^2_T(\alpha, \beta) \rangle = \frac{1}{3}(2 + e^{-\frac{2\pi^2 \delta B^2}{\tau}})$$

(12)

The simple model we have been considering here is often used as a toy model for phenomenological decoherence [14, 15]. The relationship between noise cancellation and decoherence is easily seen by considering the time evolution at the discrete times $t = k\tau$, $k \leq N$. By proceeding as before, the average square fidelity at time $t = k\tau$ results

$$F_t^2 = \frac{1}{3}(2 + e^{-\frac{2\pi^2 \delta B^2}{\tau}})$$

(13)

with which it is quite natural to associate a ‘decoherence time’ $t_{\text{deco}} = \frac{N}{2\pi^2 \delta B^2}$ (see Figure 2 [12, 13]).

B. Experimental proposal

In Ref. [14] Kielpinski et al. used the same idea to simulate the decoherence effect due to interaction of the quantum system with the environment degree of freedom. The authors used trapped ions and study the coherence of superposition of quantum state subject to simulated noise. The logical states were the hyperfine states of a trapped Beryllium ion $|F = 2, m_F = \pm 2\rangle$ and $|F = 1, m_F = -1\rangle$ sublevels of the ground state $^2S_{1/2}$. The environment noise was simulated shining the ions with a off-resonant laser with random varying amplitude for the electromagnetic field $E^j$ and random intensity (proportional to $(E^j)^2$). The laser electromagnetic field produce a AC Stark effect on the ions and let one state to acquire a random phase respect to the other. This effect is quadratix (quadratic Stark effect) in the electromagnetic field $E^j$. In fact, the two hyperfine states have the same angular momentum (they have both $L = 0$) and the difference is in the spin part of the wave function. The splitting of the energy level is linear (linear Stark effect) for state with different angular momentum since only these states have non-vanishing matrix element $\langle L = i|\hat{E} \cdot \vec{r}|L = k\rangle$ (with $i \neq k$). Then in the above example we have corrections to the energy level only quadratic and not linear in $E^j$. Because of the quadratic energy shift, we have a random phase difference proportional to $(E^j)^2$ for every $\tau$ interval. Even if this effect is sufficient to produce decoherence effect (as found by the authors), we cannot presumably see the cancellation effect discussed above in a transparent way since our new stochastic variable $\chi \propto \sum (E^j)^2$ has no zero mean.

A small modification of this experiment should allow us to see sharply this cancellation effect. To have an evolution described by Hamiltonian (11) it is sufficient to use states with different angular momentum in order to produce a linear Stark effect. At every time interval $\tau$ the perturbation of the laser produce a shift of the energy levels proportional to the random intensity of the laser $E^j$; this produces an evolution where the phase difference between the states is given by a known dynamical part plus a random phase $\exp(2i\alpha E^j\tau)$ (where $\alpha$ is a proportional constant). In this case, the new stochastic variable $\chi \propto \sum E^j$ has still zero mean and we expect to obtain results shown in the previous section: fixed the evolution time $T$ we should see an increase of $F^2$ as the correlation time decreases (see Figure 1). Moreover, if the environment decoherence does not depend on the simulated noise, we should be able to see effect analogous to the ones in Figure 2. In particular, subtracted the effect of the environment, the decoherence time should increase as $t_{\text{deco}} \propto \frac{1}{\tau}$.

C. Off-Diagonal Noise

We now consider the case of off-diagonal noise, that is, the matrices $\delta A^j$ in (2) are of the form $\delta A^j = \delta B^j_3 \sigma_x + \delta B^j_1 \sigma_y$, with $\delta B^j_3$, $\delta B^j_1$, $j = 1, \ldots, N$ independent Gaussian distributed random variables with mean zero, and $\sigma_x$, $\sigma_y$ being the usual Pauli matrices. Then the one-step evolution operators $U_j$ in (3) are

$$U_j = e^{-i\vec{B}^j \cdot \vec{\sigma} \tau} = \cos(B^j \tau) - i\vec{n}^j \cdot \vec{\sigma} \sin(B^j \tau),$$

where $B^j$ is the modulus of the vector $\vec{B}^j = (\delta B^j_3, \delta B^j_1, B_z)$ and $\vec{n}^j$ is the associated unit vector. By
The matrix elements of $\delta P$ according to (14), this is given by

$$\langle \psi \rangle = \langle \psi | U^0(T) \psi(0) | U(T) \psi(0) \rangle.$$ 

By taking into account that $\sin(B_z \tau) \approx B_z \tau$ and $\exp(i B_z \tau) \approx 1$ for $\tau = T / N \ll 1$, we may finally evaluate the modulus of the scalar product and compute the fidelity. We obtain

$$F_T = 1 - 2 \text{Re} \left( \sum_{j} e^{2iB_z(T-j)} \left( \frac{\delta B_j^x}{N} - i \frac{\delta B_j^y}{N} \alpha^* \beta \right) \right)$$

It is important to note that, also in this case, for $N \to \infty$ and $\tau \to 0$ (while keeping $N \tau = T$ of order 1), the fidelity approaches 1. This is so because $\delta B_j^x$ are independent random variables with mean zero.

So, also in this case there is a noise cancellation effect. This effect is confirmed by the numerical simulations shown in Figure 3.

III. ADIABATIC EVOLUTION

The adiabatic population transfer is an important technique used in atoms optics to achieve population transfer between quantum states of atoms and molecules. We first create coherence between the initial and final state (population trapping) and then produce an adiabatic evolution to transfer the population to the final state. This scheme has seen a great success and has been used in many different areas: chemical reaction, laser-induced cooling, atoms optics, cavity quantum electrodynamics. The wide range of application is due to many advantages of this scheme: it is easy to implement in different system, it has a high rate of population transfer and it is robust respect to variations of field parameters.

Consider two states $|1\rangle$ and $|2\rangle$ coupled to an excited state $|e\rangle$ by two lasers (i.e., a $\Lambda$ system). The states $|1\rangle$ and $|2\rangle$ can be degenerate or quasi-degenerate but it is important that we can address separately both of
them. The Hamiltonian in the rotating frame with resonant laser frequencies is

$$H = - (Ω_1(t)|1⟩⟨e| + Ω_2(t)|2⟩⟨e|) + h.c.$$  \hspace{1cm} (16)

where $Ω_1(t)$ and $Ω_2(t)$ are the time-dependent Rabi frequencies and depend on the parameters of the lasers (amplitude and phase). The diagonalization of (16) gives two eigenstates $|B_±⟩ = \frac{1}{\sqrt{2Ω}}(±|e⟩ + Ω_1(1) + Ω_2(2))$ (called bright states) respectively with eigenvalues $±Ω(t) = ±\sqrt{\sum_{j=1}^{2}|Ω_j|^2}$, and an eigenstate $|D⟩ = 1/Ω(Ω_2|1⟩ − Ω_1|2⟩)$ (called dark state) with zero eigenvalue. In the adiabatic evolution (i.e. when the $Ω_j$’s change slowly and $ΩT ≫ 1$) it follows from the adiabatic theorem \cite{26} that if the system starts at time $t = 0$ in an eigenstate of $H(0)$ (dark or bright state) during all the evolution it will remain in the eigenstate of $H(t)$ with the same eigenvalue.

Now we provide a simple example of the foregoing. Suppose that $Ω_1(t)$ and $Ω_2(t)$ are such that $Ω$ is time-independent and $Ω_1(0) = 0$ and $Ω_2(0) = Ω$. Moreover, suppose that the initial state is $|1⟩$. Then slowly turn on the first Rabi frequency and turn off the second one. The system will always be in the dark state $|D⟩$ and, at the end of the evolution (i.e., when $Ω_1(T) = Ω$ and $Ω_2(T) = 0$), we will be in $|2⟩$ state and have achieved population transfer from $|1⟩$ to $|2⟩$.

Consider now the case in which we have not a complete control of the laser field but the Rabi frequencies can fluctuate $Ω_i → Ω_i + δΩ_i$ (where the $δΩ_i$ are independent and Gaussian distributed with zero mean and variance $σ^2$). This effect can produce errors in population transfer scheme for two reason: in general noisy perturbations may yield to significantly different output state and fast fluctuations could break the adiabatic approximation leading to transition to undesired (bright) states.

Numerical simulations show that, again, in the fast fluctuation regime the noise effects cancel out. In Figure 5 we present the population evolution of $|1⟩$, $|2⟩$ and $|e⟩$ states subject to noisy evolution during the $|1⟩ → |2⟩$ coherent adiabatic transfer \cite{26}. More precisely, we start from $|1⟩$ and, during the evolution, the $|2⟩$ is populated; at the end only the $|2⟩$ is present. The $|e⟩$ is never populated because of the high value of $ΩT$ parameter. These simulations are done using $ΩT = 1000$. Since we are in the adiabatic regime we are sure that the errors present are those induced by the perturbations. In Figure 4 we show the probability evolution when the system is subject to noise with different correlation time $τ$ and different $N = T/τ$. For $N = 10^4$ the transfer operation is not precise (i.e. the $|2⟩$ state is not completely populated) and the $|e⟩$ state is populated; this is a sign of the breaking of the adiabatic approximation due to noise. For $N = 10^5$ the evolution is much more similar to the ideal one: $|2⟩$ is completely populated and $|e⟩$ never appears during the evolution.

The above results can be explained calculating the amplitude transfer from the dark to the bright state in presence of perturbations. The standard rule (see, e.g. \cite{27}) to calculate the probability amplitude of a transition from the state one to the k state ($k ≠ n$) at time $t$ is

$$a_k = \sum_j \int_0^t \frac{∂V_j}{∂t} \exp(i \int_0^t (j + 1) τ) dt$$

where $jτ$ indicates that the term $n = k$ is omitted, $ω_{kn} = E_k − E_n$ and $V_{kn}$ is the matrix element associated to the transition $n → k$. In our case, the initial state is the dark state and the final states are the bright states. The eigenvalues $±Ω$ are constant and then $ω_{kn} = Ω$. The perturbation Hamiltonian for time $jτ ≤ t ≤ (j + 1)τ$ in the $|e⟩$, $|1⟩$, $|2⟩$ basis is $δV_j = \sum_i δΩ_i|^i⟩⟨e| + h.c.$ and in the new (dark-bright states) basis the relevant matrix element are

$$V_{DB_p} = − V_{DB_−} = \sum_j \frac{δΩ_j Ω_1 − δΩ_2 Ω_2}{√2Ω} S((j + 1)τ)$$  \hspace{1cm} (17)

To calculate the matrix element $∂V_j/∂t$ we must take into account that $∂S((j + 1)τ)/∂t = δ(t − jτ) − δ(t − (j + 1)τ)$. Let us focus our attention only on the first term in (17), by differentiating it we obtain

$$1/(2Ω) \sum_j δV_{2j}(δΩ_1/δt)S((j + 1)τ) − δΩ_2 (δΩ_2 δ(t − jτ) − δ(t − (j + 1)τ))$$

The terms $δΩ/δt$ representing the adiabatic driven evolution are very small and can be neglected. Inserting this result in the expression for $a_k$ and performing the integration we have

$$a_k = \frac{1}{√2Ω} \sum_j δΩ_j [a_n(jτ)Ω_1(jτ)e^{(j + 1)τ}]$$

$$− a_n((j + 1)τ)Ω_1((j + 1)τ)e^{(j + 1)τ} + \text{terms with } (Ω_1 ↔ Ω_2)$$  \hspace{1cm} (18)

If $a_n$ and $Ω_i$ change slowly i.e., $a_n((j + 1)τ) ≈ a_n(jτ)$ and $Ω_i((j + 1)τ) ≈ Ω_i(jτ)$. The exponential terms can be simplified to obtain $\exp(δΩ((j + 1)/2)) \sin((Ω/2)/(2τ))$
and eq. (18) for \( \tau = T/N \ll 1 \) gives

\[
a_k = \frac{T}{2\sqrt{2}\Omega} \sum_j e^{i\Omega j \tau} a_n(j \tau) \left[ \Omega_1(j \tau) \frac{\delta \Omega_1}{N} - \Omega_2(j \tau) \frac{\delta \Omega_2}{N} \right]
\]

(19)

The sums of \( \frac{\delta \Omega_i(t)}{N} \) converge to the mean of \( \delta \Omega_i \) that is to zero; the other factors are bounded (0 \( \leq |a_n| \leq 1 \) and 0 \( \leq |\Omega_i| \leq \Omega \)) and by consideration similar to those at the end of section II C (and [18]) we can conclude that \( a_k \to 0 \) for \( N \to \infty \). In our case, \( |n⟩ = |D⟩ \), \( |k⟩ = |D_+⟩ \) and \( |AB_+⟩^2 \to 0 \) as \( N \to \infty \) : the transition from dark to bright states is suppressed in the fast fluctuating regime and the evolution happen in the dark space.

To have a more detailed picture in Figure 5 we show also the trend of the fidelity (upper curve) and the relative average population of the \( |e⟩ \) state (lower curve) as functions of \( T/\tau \). The trends of the two curves are correlated, which is suggesting that the main source of error in the operation is the population of excited state due to loss of the adiabatic approximation. As expected, because of the cancellation effect, for great \( T/\tau \) the fidelity approaches 1 even in presence of strong noise and the \( |e⟩ \) state is not populated.

These results not only can explain why the adiabatic population transfer scheme is robust against field fluctuation but can give information for the experimental set-up. In fact the adiabatic evolution is, in general, arbitrarily long; once the experimental parameters are fixed (as the laser with its proper noise correlation time \( \tau \)) we can prolong the evolution time in order to increase \( T/\tau \) and let the noise average out. As shown before, for every noise strength \( \sigma \) and correlation time \( \tau \) we can find and evolution time \( T \) in order to obtain the desired fidelity: that is, to achieve the population transfer with arbitrary small error.

IV. CONCLUSION

We studied the effect of stochastic noise on several quantum systems. The noise is described by a Gaussian stochastic process with zero mean superposed to the ideal quantum evolution. For all of the systems we found that for fast fluctuating noise a cancellation effect appears: the noise fluctuations average out leading the system to a state near to the ideal one.

We showed by analytical and numerical calculation how this effect can appear in a two level system and propose an experiment to verify the presence of this cancellation regime. The experiment is based on the one performed in Ref. [14] and we think that, with a modification of the experimental set-up, it could be easily performed.

We applied the same model to another important technique in atoms physics: the adiabatic population transfer. We explained how this effect leads to the robustness of the adiabatic process against the perturbation noise. This can be important for the experiments using the adiabatic population transfer.

Acknowledgments

P.S. wishes to thank D. Kielpinski for useful comments.

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[12] Our sampling set of the Bloch sphere (i.e. our initial state space) is given by \( \{ \pm e_i \}_{i=x,y,z} \), \( \{ \pm e_x \pm e_y \}/\sqrt{2}, \{ \pm e_x \pm e_z \}/\sqrt{2}, \{ \pm e_y \pm e_z \}/\sqrt{2} \). Here \( e_i \) denotes the normalized vector of the \( i \)-th direction.

[13] We make simulations with 1000 different realizations and average the resulting square fidelity. Apart from a noisier in the curve in the plots, the results are weakly affected by this parameter so that with 100 realizations we obtain the almost the same results.

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[18] This is an elementary instance of the law of large numbers: Suppose to have a sum written in the form \( \sum_{j=0}^{N} f(\tau_j) \frac{x_j}{\tau} \) where \( x_j \) are independent random variables with zero mean and variance \( \sigma^2 \), \( \tau = T/N \) and \( 0 \leq j \leq T \).

\[ \text{Var}(Y_N) = \frac{\sigma^2}{N^2} \sum_{j=0}^{N} (f(\tau_j))^2 \] is the variance of \( Y_N = \frac{1}{N} \sum_{j=0}^{N} f(\tau_j)x_j \). Thus, if \( f \) is bounded in the interval \([0, T]\), \( \text{Var}(Y_N) \to 0 \) when \( N \to \infty \) and therefore \( Y_N \to 0 \) since the \( x_j \) mean is zero. Since in our case \( f(\tau_j) = \exp(-2Bz\tau_j) \) the second terms in eq. (15) goes to zero and \( F \to 1 \) for \( N \to \infty \) for every initial state (i.e. independently from \( \alpha \) and \( \beta \)).

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