Non-orthogonal Fusion Frames and the Sparsity of Fusion Frame Operators

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Abstract Fusion frames have become a major tool in the implementation of distributed systems. The effectiveness of fusion frame applications in distributed systems is reflected in the efficiency of the end fusion process. This in turn is reflected in the efficiency of the inversion of the fusion frame operator $S_{W}$, which in turn is heavily dependent on the sparsity of $S_{W}$. We will show that sparsity of the fusion frame operator naturally exists by introducing a notion of non-orthogonal fusion frames. We show that for a fusion frame $\{W_{i}, v_{i}\}_{i \in I}$, if dim$(W_{i}) = k_{i}$, then the matrix of the non-orthogonal fusion frame operator $S_{W}$ has in its corresponding location at most a $k_{i} \times k_{i}$ block matrix. We provide necessary and sufficient conditions for which the new fusion frame operator $S_{W}$ is diagonal and/or a multiple of an identity. A set of other critical questions are also addressed. A scheme of multiple fusion frames whose corresponding fusion frame operator becomes a diagonal operator is also examined.

Keywords Frames · Fusion frames · Sparsity of the fusion frame operator · Sensor network · Data fusion · Distributed processing · Parallel processing

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1 Introduction

Fusion frames were introduced in [5] (under the name frames of subspaces) and [11], and have quickly turned into an industry (see www.fusionframes.org). Recent developments include applications to sensor networks [12], filter bank fusion frames [13], applications to coding theory [1, 6], compressed sensing [2], construction methods [3, 4, 7–9], sparsity for fusion frames [10], and frame potentials and fusion frames [18]. Until now, most of the work on fusion frames has centered on developing their basic properties and on constructing fusion frames with specific properties. We now know that there are very few tight fusion frames without weights. For example, in [9] the authors classify all triples \((K, L, N)\) so that there exists a tight fusion frame \(\{W_i\}_{i=1}^{K}\) with \(\dim W_i = L\), for all \(i = 1, 2, \ldots, K\) in \(\mathcal{H}_N\).

A major stumbling block for the application of fusion frame theory is that in practice, we generally do not get to construct the fusion frame, but instead it is thrust upon us by the application. In a majority of fusion frame applications, such as in sensor network data processing, each sensor spans a fixed subspace \(W_i\) of \(\mathcal{H}\) generated by the spatial reversal and the translates of the sensor’s impulse response function [16, 17]. There is no opportunity then for subspace transformation, manipulation and/or selection. As a result, the fusion frame operator \(S_W\) is always non-sparse with an extremely high probability. The lack of sparsity of \(S_W\) is a significant hinderance in computing \(S_W^{-1}\) and its inverse, which is necessary to apply the theory. So the central issue in the effective application of fusion frames is to have sparsity for the fusion frame operator—preferably for it to be a diagonal operator.

We have long suspected that there has to be a way to ensure that \(S_W\) is no more than a block diagonal operator with each block having the dimension of the corresponding subspace. It turns out that a notion of non-orthogonal fusion frames achieves that and this is the central theme of this paper.

2 Non-orthogonal Fusion Frames

Nonorthogonal fusion frames are a modification of fusion frames [5, 11] with a sequence of non-orthogonal projections operators. A non-orthogonal projection onto \(W\) is a linear mapping \(P_W\) from \(\mathcal{H}\) onto \(W\) which satisfies \(P_W^2 = P_W\). An important property is that the adjoint \(P_W^*\) is also a non-orthogonal projection from \(\mathcal{H}\) onto \(\mathcal{N}(P_W)^\perp\) with \(W^\perp\) being the null space (of \(P^*\)). Here \(\mathcal{N}(P_W) = \{f \in \mathcal{H} : P_W f = 0\}\). Also observe that we must have that \(\mathcal{N}(P_W) \cap W = \{0\}\), i.e., \(\mathcal{N}(P_W) \oplus W = \mathcal{H}\).

Definition 2.1 Let \(I\) be a countable index set. Let \(\{W_i\}_{i \in I}\) be a family of closed subspaces in \(\mathcal{H}\), and let \(\{v_i\}_{i \in I}\) be a family of positive weighting scalers. Denote by \(P_i\) a non-orthogonal projection onto \(W_i\). Then \(\{P_i, v_i\}_{i \in I}\) is a non-orthogonal fusion frame of \(\mathcal{H} = \text{span}(\sum_{i \in I} W_i)\) if there are constants \(0 < C \leq D < \infty\) such that

\[
\forall f \in \mathcal{H}, \quad C\|f\|^2 \leq \sum_{i \in I} v_i^2 \|P_i f\|^2 \leq D\|f\|^2. \tag{1}
\]
Remarks (1) Throughout this paper we will use $\pi$ for an orthogonal projection. It is obvious that if $\mathcal{P}_i$ is an orthogonal projection $\pi_i$, then our notion of a non-orthogonal fusion frame becomes the standard fusion frame.

(2) In general, let $(\sum \oplus W_i)_{l^2} \equiv \{(f_i) | f_i \in W_i \text{ and } \|f_i\| \in l^2(I)\}$. Define the analysis operator

$$T_W : H \rightarrow \left( \sum_{i \in I} \oplus W_i \right)_{l^2}$$

by

$$T_W f = \{v_i \mathcal{P}_i f\}_{i \in I}, \quad \text{for all } f \in H.$$ 

Then

$$T_W^* f = \sum_{i \in I} v_i \mathcal{P}_i^* (f_i), \quad \text{for all } f = \{f_i\}_{i \in I} \in \left( \sum_{i \in I} \oplus W_i \right)_{\ell^2}.$$ 

The new (non-orthogonal) fusion frame operator $S_W : H \rightarrow H$ becomes

$$S_W \equiv T_W^* T_W = \sum_{i \in I} v_i^2 \mathcal{P}_i^* \mathcal{P}_i.$$ 

We compare this to the standard fusion frame operator

$$S_W \equiv T_W^* T_W = \sum_{i \in I} v_i^2 \pi_i.$$ 

It is also true that the non-orthogonal fusion frame condition (1) is equivalent to that

$$C \text{Id} \leq S_W \leq D \text{Id}.$$ 

(3) If the standard (orthogonal) fusion frame (OGFF) condition [5, 11] holds, there will be no loss of information with non-orthogonal projection operators. Instead, there are infinitely many flexibilities now available which is highly beneficial to the sparsity of $S_W$ as we demonstrate next.

(4) Oftentimes, subspaces $\{W_i\}$ are given a priori by applications. Subspace manipulation does not exist nor is allowed in those applications. As a result, the fusion frame operator $S_W$ given by the orthogonal projections are fixed, and are non-sparse with probability nearly 1. For instance, in $\mathbb{R}^3$, let $W_1 = \{z = 0\}$, and $W_2 = \{x + y + z = 0\}$ be two planes. $S_W$ by the OGFF definition [5, 11] $S_W = \pi_1 + \pi_2$ gives rise to a full matrix with no zero entry.

If on the other hand, if we take $\mathcal{P}_1 = \pi_1$, but let $\mathcal{P}_2$ be the non-orthogonal projection with $\mathcal{N}(\mathcal{P}_2) = \{z = 0\} \cap \{y = 0\}$ so that

$$\mathcal{P}_2 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
then $P_2^*P_2$ is a $2 \times 2$ block matrix
\[
P_2^*P_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 3 & 1 \\
0 & 1 & 2
\end{pmatrix}.
\]

Also, the corresponding non-orthogonal fusion frame operator $S_W$ has the standard matrix representation
\[
S_W = \pi_1 + P_2^*P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 3 & 1 \\
0 & 1 & 2
\end{pmatrix},
\]
which is now a relatively sparse representation—already much better than that of the orthogonal projections.

**Diagonal $S_W$** One can achieve more in this example with non-orthogonal projections. Say, if we take $N(P_1) = \text{span}\{e_2 + e_3\}$. Then $N(P_1) \cap W_1 = 0$, and
\[
P_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then
\[
P_1^*P_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix}.
\]

Consequently,
\[
S_W = P_1^*P_1 + P_2^*P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{pmatrix},
\]
which yields a diagonal non-orthogonal fusion frame operator $S_W$. This situation is highly beneficial to all fusion frame applications.

(5) Suppose that fusion frames are used in sensor network applications. Each subspace $W_i$ represents a sensor. The measurement of each sensor is a typical frame expansion $\{\langle f, w_n \rangle\}$ [16]. Therefore, not only the subspaces $\{W_i\}$ are fixed by the sensors in the network, but also the sensor measurements are given a priori. So diagonalizing $S_W$ through subspace transformations and/or rotations are not permitted.

(6) Relating to the notion of nonorthogonal fusion frames, there is the notion of g-frames [19, 20]. Actually, g-frames are more general classes of “operator frames”. Though nonorthogonal fusion frames are a class of g-frames with projection operators, the study of this (nonorthogonal) projection class has never been carried out, and the restriction to (nonorthogonal) projection operators also makes the analysis less flexible than that of the general operator frames. Yet, it is this class of projection operators that actually find realistic applications in sensor array or distributed
system data fusion. Because projection operators really have the physical interpretation, namely, signals measured by sensors are really projections of the original signal/function onto the subspace $W$ spanned by the sensor. Linear measurements of a signal by sensors and/or linear devices are typically modeled by an orthogonal projection operator [16]. Sensors and/or linear devices can also function in a nonorthogonal way, the principle of which and the computational advantage of the non-orthogonal FF are discussed in detail in Sect. 6.

Our work here has also led to a synthesis of positive and self-adjoint operator $T$ by projections $P_{ij}$ in the form of $\sum_{ij} v_i^2 P_{ij}^* P_{ij}$. These ideas will be developed in a later article.

3 Main Problem Statements

We list here some of the problems needed to be resolved in the topic of non-orthogonal fusion frames. In this article, we provide solutions to several of these problems.

**Problem 3.1** (Main Problem) Given subspaces $\{W_i\}_{i \in I}$ of $H_N$ which span $H_N$, does there exist a family of non-orthogonal projections $\{P_i\}_{i \in I}$ with $P_i$ mapping onto $W_i$ so that

$$\sum_{i \in I} P_i^* P_i = \lambda I?$$

Alternatively,

$$\sum_{i \in I} P_i^* P_i = D, \quad D \text{ a diagonal operator?}$$

**Conjecture 3.2** We believe that Problem 3.1 has a negative answer with strict diagonal right hand sides. But, sparsity to certain degree is always achievable.

Since non-orthogonal projections onto a given subspace are no longer unique, the following problem is very natural, and likely to have a positive answer.

**Problem 3.3** Given subspaces $\{W_i\}_{i \in I}$ of $H_N$ which span $H_N$, do there exist multiple non-orthogonal projections $\{P_i\}_{i \in I}$ and $\{Q_i\}_{i \in I}$ and weights $\{v_i\}_{i \in I}$ and $\{w_i\}_{i \in I}$ with $P_i, Q_i$ mapping onto $W_i$ so that

$$\sum_{i \in I} (v_i^2 P_i^* P_i + w_i^2 Q_i^* Q_i) = \lambda I?$$

Or perhaps some number of projections—which should not be too large.

**Remark 3.4** Problem 3.3 has a positive answer with $v_i = 1$ for every $i \in I$ if the subspaces all have dimension $\geq \frac{N}{2}$. We show this in Proposition 8.2.
Since for every subspace $W$, either $W$ or $W^\perp$ (or alternatively, $(I - P)W$ for any projection $P$ onto $W$) has dimension $\geq \frac{N}{2}$, it would be interesting to solve the next problem.

**Problem 3.5** Given subspaces $\{W_i\}_{i \in I}$, weights $\{v_i\}_{i \in I}$, and projections $\{P_i\}_{i \in I}$ onto the $W_i$ satisfying

$$\sum_{i \in I} v_i^2 P_i^* P_i = \lambda I,$$

does there exist projections $\{Q_i\}_{i \in I}$ onto $\{W_i^\perp\}_{i \in I}$ (or onto $\{(I - P_i)H_N\}_{i \in I}$) and weights $\{w_i\}_{i \in I}$ so that

$$\sum_{i \in I} w_i^2 Q_i^* Q_i = \mu I?$$

### 4 Block Diagonal Characterization of the Non-orthogonal Projection $P$

We first show that every subspace $W$ with $\dim W = k$ has a projection $P_W$ onto it for which the matrix of $P_W^* P_W$ is a $k \times k$ block matrix.

**Proposition 4.1** Let $H_N$ be an $N$-dimensional Hilbert space with standard orthonormal basis $\{e_i\}_{i=1}^N$, and let $W$ be a $k$-dimensional subspace of $H_N$. Then there is a subset $K \subset \{1, 2, \ldots, N\}$ with $|K| = k$ and a projection $P_W$ onto $W$ so that the matrix of $P_W^* P_W$ with respect to the orthonormal basis $\{e_i\}_{i=1}^N$ has non-zero entries only on the entries of $K \times K$.

**Proof** Given an orthonormal basis $\{x_i\}_{i=1}^k$ for $W$, if we row reduce it, we will find a set $K \subset \{1, 2, \ldots, N\}$ with $|K| = k$ so that the restriction of the operator $\pi_K : W \rightarrow V_K = \text{span}\{e_i\}_{i \in K}$ is invertible on $V_K$. Define a mapping

$$P_W = (\pi_K|_V)^{-1}\pi_K.$$

Then $P_W$ is a projection onto $W$. Also, $P_We_j = 0$ if $j \notin K$ implies that for $j \notin K$ we have for all $i$:

$$\langle P_W^* P_We_i, e_j \rangle = \langle P_We_i, P_We_j \rangle = \langle P_We_i, 0 \rangle = 0.$$

So the only non-zero entries in the matrix of $P_W^* P_W$ are the entries from $K \times K$. □

An alternative argument of the proof goes as follows. Since $\dim(W) = k$, one can always find a set $K' \subset \{1, 2, \ldots, N\}$ with $|K'| = N - k$ such that $W' = \text{span}\{e_j\}_{j \in K'}$ complements $W$, i.e., $W' \cap W = \{0\}$ and $W + W' = H_N$. Now, set the null space of $P_W$ to be $N(P_W) = W'$. Then $P_We_j = 0$ for all $j \in K'$. The rest follows by the last 3 lines of the previous proof. Note, there are consequently $N - k$ columns of zeros in the matrix of $P_W$ with respect to the orthonormal basis $\{e_j\}$.

In fact, more can be said about the sparsity of $P_W$. 

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Proposition 4.2 Let $\mathcal{H}$ be Hilbert space with orthonormal basis $\{e_i\}_{i=1}^N$. Then for every subspace $W \subseteq \mathcal{H}$, there exists a projection $P_W$ such that the matrix of $P_W$ is triangular with respect to $\{e_i\}_{i=1}^N$.

Proof Choose $K \subseteq \{1, \ldots, N\}$, $|K| = k = \dim(W)$, $K = \{i_1, i_2, \ldots, i_k\}$ and $V = \text{span}\{e_j\}_{j=1}^k$, so that the orthogonal projection $\pi_V$ onto $V$ is a bijection between $V$ and $W$. We know there exists $x_1 \in W$ so that $\pi_V x_1 = e_{i_1}$. Let $W_1 = \{w \in W : \langle w, x_1 \rangle = 0\}$ and $A_1 = \{w \in W : \pi_V w \in \text{span}\{e_{i_1}, e_{i_2}\}\}$. Then $A_1 \cap W_1$ is a one dimensional subspace of $W$, so choose $x_2$ in this subspace. Repeat inductively so that $\pi_V x_j \in \text{span}\{e_{i_1}, \ldots, e_{i_j}\}$. Now define $U : V \to W$ so that $U \pi_V x_j = x_j$, and define $P_W = U \pi_V$. Then $P_W x_j = U \pi_V x_j = x_j$, so $P_W^2 = P_W$. Also, for $i \in K$ we have $e_i = \sum_{j=1}^i b_{ij} \pi_V x_j$. Therefore, for $i, \ell \in K$ with $\ell > i$ we have

$$\langle P_W e_i, e_\ell \rangle = \sum_{j=1}^i b_{ij} \langle \pi_V x_j, e_\ell \rangle = 0.$$ 

Also, if $i \notin K$ then $P_W (e_i) = 0$ so for all $\ell$,

$$\langle P_W e_i, e_\ell \rangle = 0. \quad \square$$

Remark Since $P_W^* P_W$ is self adjoint, it is triangular if and only if it is diagonal. Consequently, the triangular nature of $P_W$ may only result in $K \times K$ block diagonal nature in $P_W^* P_W$. In Sect. 5, we will provide a characterization of when $P_W^* P_W$ can always be diagonal.

But first, let us examine an immediate consequence of the non-orthogonal fusion frame applied to conventional frames. The evaluation of dual frames (to any conventional frames) becomes effortless. There is a corresponding Parseval fusion associated with any given conventional frame.

Example (The case of conventional frames) Let $\{w_i\}_{i=1}^M$ be a conventional finite frame of $\mathcal{H}_N$. The following is immediate.

Proposition 4.3 Let $\{w_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ and let $W_i = \text{sp}\{w_i\}$. Then there exists projections $P_i$ onto $W_i$ and weights $v_i$ so that $\sum_{i=1}^M v_i^2 P_i^* P_i = I$. That is, $\{P_i, v_i\}_{i=1}^M$ is a non-orthogonal Parseval fusion frame.

Proof Let $\{e_j\}_{j=1}^N$ be the standard orthonormal basis for $\mathcal{H}_N$. By permuting the frame vectors, we may assume that $w_{jj} \neq 0$ for all $1 \leq j \leq N$. For $1 \leq i \leq N$, select $P_i$ such that

$$N(P_i) = \text{span}\{e_j\}_{j \neq i}. \quad (2)$$

Write $\tilde{w}_i = \frac{1}{w_{ii}} w_i$. Then
\[
P_{i} = \begin{pmatrix}
0 & \cdots & 0 & w_{i1} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & w_{ii-1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & w_{ii+1} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & w_{iN} & 0 & \cdots & 0 
\end{pmatrix}, \quad 1 \leq i \leq N.
\] (3)

As a result, for \(1 \leq i \leq N\),

\[
(P_{i}^* P_{i})_{mn} = \begin{cases}
r_i = \|w_i\|^2, & m = n = i, \\
0, & \text{otherwise.}
\end{cases}
\]

For all \(N + 1 \leq i \leq M\), chose any projection \(P_i\) so that the matrix of \(P_i^* P_i\) has only one nonzero entry \(r_i\), and this entry is on the diagonal. For all \(k = 1, 2, \ldots, N\), let \(I_k = \{i : P_i^* P_i(k, k) \neq 0\}\). If we let \(v_i^2 = (\sum_{i \in I_k} r_i)^{-1}\) for each \(i \in I_k\), then \(\sum_{i=1}^{M} v_i^2 P_i^* P_i = I\).

With such selections of non-orthogonal projections \(P_i\) and the associated weights \(v_i\), we have constructed a (non-orthogonal) Parseval fusion frame \(\{P_i, v_i^2\}\).

5 Diagonal Characterization of \(P_W^* P_W\)

In a simplified version of Problem 3.1, we consider in this section the conditions for which one individual \(P_W^* P_W\) can be diagonal.

**Proposition 5.1** Fix an orthonormal basis \(\{e_i\}_{i=1}^{N}\) for \(\mathcal{H}\). Let \(W \subseteq \mathcal{H}\) be a \(k\)-dimensional subspace. The following are equivalent:

(i) There exists a projection \(P_W\) such that the matrix of \(P_W^* P_W\) is diagonal with respect to \(\{e_i\}_{i=1}^{N}\).

(ii) There exists a subset \(K \subseteq \{1, 2, \ldots, N\}, |K| = k\) such that there exists an orthonormal basis \(\{x_i\}_{i \in K}\) for \(W\) such that \(x_i(j) = \delta_{ij}\) for \(i, j \in K\).

(iii) There exists a projection \(Q_W\) such that \(\{Q_W e_i\}_{i \in K}\) is an orthogonal basis for \(W\) and \(Q_W e_i(j) = \delta_{ij}\) for \(i, j \in K\).

(iv) There exists a projection \(P_W\) such that \(\{P_W e_i\}_{i \in K}\) is an orthogonal basis for \(W\).

(v) There exists a subset \(K \subseteq \{1, 2, \ldots, N\}, |K| = k\) and there exists an orthogonal basis \(\{x_i\}_{i \in K}\) for \(W\) such that \(\pi_K x_i\) is an orthonormal basis for \(\text{span}\{e_i\}_{i \in K}\).

Moreover, in all of the above cases, the diagonal elements of \(P_W^* P_W\) are \(\|x_i\|^2\) for cases (ii) and (v); \(\|Q_W e_i\|^2\) in case (iii), and \(\|P_W e_i\|^2\) for case (iv).

**Proof** (iii) \(\Rightarrow\) (iv) \(\Rightarrow\) (i) is clear.

We first show (i) \(\Rightarrow\) (ii). We know that \(\langle P_W e_i, P_W e_j \rangle = 0\) for \(i \neq j\), so \(\{P_W e_i\}_{i=1}^{N}\) is an orthogonal set. But \(\dim(W) = k\) so there exists a \(K \subseteq \{1, \ldots, N\}\) such
that $P_W e_i = 0$ for $i \notin K$ and $\{P_W e_i\}_{i \in K}$ is an orthogonal basis for $W$. Let $V = \text{span}\{e_i\}_{i \in K}$. Observe that for $x, y \in V$

$$\langle x, y \rangle = 0 \text{ if and only if } \langle P_W x, P_W y \rangle = 0. \quad (4)$$

Now write

$$P_W e_i = \pi_V P_W e_i + (I - \pi_V) P_W e_i$$

to see that

$$P_W P_W e_i = P_W \pi_V P_W e_i + P_W (I - \pi_V) P_W e_i.$$

But $P_W (I - \pi_V) P_W e_i = 0$, since $P_W e_i = 0$ for $i \notin K$. Therefore, since $P_W$ is a projection (i.e., $P_W^2 = P_W$) we have that

$$P_W e_i = P_W \pi_V P_W e_i.$$

Hence, (4) now implies that $\{\pi_V P_W e_i\}_{i \in K}$ is an orthogonal basis for $V$. Now observe that $\pi_V$ is a bijection between $V$ and $W$ so we can choose $\{x_i\}_{i \in K}$ so that $\pi_V x_i = e_i$.

(ii) $\Rightarrow$ (iii). By (ii) we know that there is an orthogonal basis $\{x_i\}_{i \in K}$ for $W$ with the desired properties, so we just need to show that there is a projection $Q_W$ such that $Q_W e_i = x_i$ for $i \in K$. Define $U : \mathcal{H} \to \mathcal{H}$ by

$$U e_j = \begin{cases} 0, & \text{if } j \notin K, \\ x_j, & \text{if } j \in K. \end{cases}$$

We now claim that $Q_W = U \pi_V$ satisfies (iii). Clearly, $Q_W e_i = x_i$, so we just need to check that $Q_W$ is in fact a projection. If $j \notin K$ then clearly $Q_W^2 e_j = 0$. If $j \in K$, then $Q_W^2 e_j = Q_W x_j = U \pi_K x_j = U e_j = x_j$ so $Q_W$ is a projection.

(ii) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i): Define,

$$P_W e_j = 0 \text{ if } j \notin K, \quad P_W \pi_K (x_j) = x_j.$$

It is immediate that $P_W$ is a projection. Also, $P_W$ is an orthogonal operator when restricted to $\text{span}\{e_i\}_{i \in K}$. Hence, if $i, j \in K, i \neq j$ we have

$$\langle P_W^* P_W e_i, e_j \rangle = \langle P_W e_i, P_W e_j \rangle = 0.$$

On the other hand, if $j \notin K$ then $P_W e_j = 0$ and so

$$\langle P_W^* P_W e_i, e_j \rangle = \langle P_W e_i, P_W e_j \rangle = \langle P_W e_i, 0 \rangle = 0.$$

So we have (i).

The "moreover" part of the theorem is obvious from the proofs. \qed

We now check that for large dimensional subspaces $W$ of $\mathcal{H}$, there is a fundamental restriction for finding a projection $P_W$ onto $W$ so that $P_W^* P_W$ is diagonal with respect to $\{e_i\}$. 

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Proposition 5.2 Let $\mathcal{H}$ be an $N$ dimensional Hilbert space and $W$ a $k$-dimensional subspace. If $k > \frac{N}{2}$ and there is a projection $P_W$ such that the matrix of $P^*_W P_W$ is diagonal with respect to $\{e_i\}_{i=1}^N$ then there are at least $2k - n$ $e_i$'s in $W$.

Proof By Proposition 5.1 part (ii) we can find a $K \subseteq \{1, \ldots, N\}$ such that $|K| = k$ and an orthogonal basis $\{x_i\}_{i \in K}$ for $W$ which satisfies $x_i(j) = \delta_{ij}$ for $i, j \in K$. Therefore, we know that $\langle \pi_V x_i, \pi_V x_j \rangle = 0$ for $i, j \in K$ which means that $\langle (I - \pi_V) x_i, (I - \pi_V) x_j \rangle = 0$ for $i, j \in K$ (since we know $\langle x_i, x_j \rangle = 0$). Therefore $\{(I - \pi_V) x_i\}_{i \in K}$ is an orthogonal set inside an $N - k$ dimensional space, which means there is a $J \subseteq K$ such that $|J| \geq 2k - N$ and $(I - \pi_V) x_j = 0$ for every $j \in J$. Then for each $j \in J$ we actually have $x_j = e_j$. □

6 The Implementation of $P_W$ via Pseudoframes for Subspaces

Pseudoframes for subspaces (PFFS) [14] are an extension of frames for subspaces $W$ where both frame-like sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ are not necessarily in $W$, yet for every $f \in W$

$$f = \sum_n \langle f, x_n \rangle \tilde{x}_n.$$  

Furthermore, the frame-like condition holds for all vectors in $W$. Namely, there are constants $0 < A \leq B < \infty$ such that for all $f \in W$

$$A\|f\|^2 \leq \sum_n |\langle f, x_n \rangle|^2 \leq B\|f\|^2.$$  

Bringing in a projection operator $P$ onto $W$, and a PFFS gives rise to

$$P g = \sum_n \langle P g, x_n \rangle \tilde{x}_n, \tag{5}$$

for every $g \in \mathcal{H}$.

6.1 PFFS as Non-orthogonal Projections

We recall the property of $P$-consistent PFFS with an assumption that the sequence $\{x_n\}$ is Bessel in $\mathcal{H}$. Using the same terminology of Aldroubi and Unser [21], we say a PFFS is $P$-consistent [15] if $U P = U$, where $U : \mathcal{H} \to l^2$ is the analysis operator functioning as the measuring device in the form $U f = \{\langle f, x_n \rangle\}$ for all $f \in \mathcal{H}$. The $P$-consistent principle is to say that the direct measurement of a function $f$ equals the measurement of a projection (approximation) $P f$ of $f$ onto (in) $W$. This clearly depends on the direction of the projection. We also recall that a $P$-consistent PFFS expansion is precisely a non-orthogonal projection operator, and the direction of the projection can be arbitrarily adjusted by steering the $sp\{x_n\}$ [14, 15].

It is known that a PFFS is a $P$-consistent PFFS if and only if the direction of the projection (or the null space of $P$) $N(P) = sp\{x_n\}^\perp$ [15], and it is always achievable.
Consequently, one can always have $\mathcal{N}(\mathcal{P}) = \overline{\mathcal{P}}[x_n]$, and the range of $\mathcal{P}^*$, $\mathcal{R}(\mathcal{P}^*) = \mathcal{N}(\mathcal{P})^\perp = \overline{\mathcal{P}}[x_n]$. The resulting non-orthogonal projection is given by

$$
\mathcal{P} g = \sum_n \langle g, \mathcal{P}^* x_n \rangle \tilde{x}_n = \sum_n \langle g, x_n \rangle \tilde{x}_n,
$$

for every $g \in \mathcal{H}$.

If we denote by $\mathcal{P}_{W, \mathcal{N}(\mathcal{P})}$ the projection operator with the first index $W$ as the range, and the second index $\mathcal{N}(\mathcal{P})$ as the “direction” of the projection, then PFFS always produces a projection onto $W$ along the direction $\overline{\mathcal{P}}[x_n]$, namely $\mathcal{P}_{W, \mathcal{N}(\mathcal{P})}$.

From another point of view, if $\mathcal{N}(\mathcal{P})$ is given, one can always select $\{x_n\}$ so that $\overline{\mathcal{P}}[x_n] = \mathcal{N}(\mathcal{P})^\perp$, and thereby construct a non-orthogonal projection via PFFS. More importantly, the selection of $\{x_n\}$ for a given $W$ is made easy by the following proposition.

**Theorem 6.1** [14] Let $\{x_n\}$ be a Bessel sequence with respect to $W$, and let $\{\tilde{x}_n\}$ be a Bessel sequence in $\mathcal{H}$. The following are equivalent:

1. $\{x_n, \tilde{x}_n\}$ is a PFFS for $W$.
2. If $\pi_W$ is the orthogonal projection of $\mathcal{H}$ onto $W$, both of the following hold:
   a. $\{\pi_W x_n\}$ is a frame for $W$ with a dual frame $\{\pi_W \tilde{x}_n\}$.
   b. For all $f \in W$ we have $\sum_n \langle f, \pi_W x_n \rangle (I - \pi_W) \tilde{x}_n = 0$.
3. There is a frame $\{w_n\}$ of $W$ with a dual frame $\{\tilde{w}_n\} \subseteq W$, a sequence $\{z_n\}$ in $(I - \pi_W)\mathcal{H}$ and a sequence $\{y_n\} \in \mathcal{U}(\{w_n\})$ and a unitary operator $T : \overline{\mathcal{P}}[y_n] \rightarrow (I - \pi_W)\mathcal{H}$ so that

$$
x_n = w_n + z_n
$$

and

$$
\tilde{x}_n = \tilde{w}_n + T(y_n).
$$

Here $\mathcal{U}$ is defined as follows. If $\{w_n\}$ is a frame for $W$, then

$$
\mathcal{U}(\{w_n\}) \equiv \left\{\text{Bessel}(y_n) : \sum_n \langle f, x_n \rangle y_n = 0, \text{ for all } f \in W\right\}.
$$

Therefore, if we construct $\{x_n\}$ so that $\overline{\mathcal{P}}[x_n] = \mathcal{N}(\mathcal{P})^\perp$, it turns out the choice of $\{x_n\}$ is fairly easy—adding to a frame $\{w_n\}$ of $W$ orthogonal components $\{z_n\} \subseteq W^\perp$ so that $\overline{\mathcal{P}}[x_n] = \mathcal{N}(\mathcal{P})^\perp$.

6.2 Implementation of $\mathcal{P}_W$ Is Not Free, but Feasible Through PFFS

The implementation of non-orthogonal projections $\mathcal{P}_W$ certainly does not come free, but it is practically feasible. However, as we analyze shortly later that the diagonalization of the orthogonal $S_{W}$ through subspace transformation is not (in sensor network data collection applications).

In sensory data collection applications, each sensor is spanned by a sensory frame $\{w_n\}$ given by the elementary transformation (often simple shifts) of the spatial reversal of the sensor’s impulse response function [16].
The measurement of each sensor is thus given by \{⟨f, w_n⟩\} a priori by the physics of the sensor. Any post processing/fusion operation would have to make use of such a priori measurements. Implementation through PFFS makes use exactly the measurement \{⟨f, w_n⟩\}, together with needed auxiliary (but achievable) measurements. Recall, for all \(f \in \mathcal{H}\)

\[
P_W f = \sum_n ⟨f, x_n⟩ \tilde{x}_n = \sum_n (⟨f, w_n + z_n⟩) \tilde{x}_n = \sum_n ((⟨f, w_n⟩ + ⟨f, z_n⟩) \tilde{x}_n. \tag{6}
\]

Consequently, in order to control the direction of the data collection, the sensor need to be bundled with an auxiliary sensor with spanning components \{z_n\} ⊆ \mathcal{W}^⊥.

For instance, if the sensor has a low-pass characteristic, the auxiliary sensor would need to have the high-pass or band-pass nature, having certain complementary information. Depending on the degree of controllability of the direction of the projection, the span of the auxiliary sensor may need to be little, or maybe very broad.

Another simpler way of implementation is to split an original sensor into the primary part, represented by \(\mathcal{W}\), and the secondary part, contained in \(\mathcal{W}^⊥\). As long as the measurement in \(\mathcal{W}\) and in the secondary part are separable (which can be designed so at the sensor manufacturing stage), the direction of the projection can be controlled easily by the principle stated in (6).

To summarize, one needs to add a controllable measurement in the primary sensor’s complement subspace via \{(f, z_n)\}. The implementation of non-orthogonal projections \(P_W\) will be achieved together with the a-priori sensor measurement \{(f, w_n)\}. An example will be given shortly after the next subsection with discussions why subspace transformation/rotation is not feasible is sensor network data fusion applications.

We note that in special cases, if a signal \(f\) is within the primary sensory subspace spanned by \(\mathcal{P}\{w_n\}\), then \(⟨f, x_n⟩ = ⟨f, w_n⟩\). No additional complementary information is needed for any direction of projections.

### Subspace Transformation is Not Feasible

We now turn to the explanation why subspace transformation in achieving diagonalization of \(S_W\) is not practically feasible in sensor array data fusion applications.

Diagonalization of \(S_W\) involves a unitary operator \(T\) such that

\[
D = T^H \left( \sum_i v_i^2 \pi_{\mathcal{W}_i} \right) T = \sum_i v_i^2 T^H (\tilde{X}_i^H X_i) T,
\]

where \(X_i\) and \(\tilde{X}_i\) are frame matrices with columns being the frame elements \(\{w_{ij}\}_j\) and its dual \(\{\tilde{w}_{ij}\}_j\).

On the one hand, it seems that a transformation in the form of \(F = X_i T\) would have diagonalized \(S_W\). On the other hand, the new frame system \(F = X_i T\) would have to “measure” functions \(f\) through \(X_i T(f)\). But this requires that \(T\) acts on \(f\) prior to the sensor \(X_i\) acts on \(f\), which is simply not practically feasible in (at least) sensor array data collection applications.

This is why non-orthogonal fusion frames is a more natural tool to achieve the sparsity of the fusion frame operator.
An Example of the PFFS Implementation In an extreme case study, let us go back to the example of conventional frames \( \{ w_i \}_{i=1}^M \) in \( \mathcal{H}_N \) and Proposition 4.3. We shall show how the sequence \( \{ z_i \} \subseteq W_i^\perp \) are constructed, and what additional measurements (in addition to the sensor measurements \( \langle f, w_i \rangle \)) we need to diagonalize \( S_W \) in this example.

Since \( W_i = \mathcal{sp}\{ w_i \}_{i=i} = \mathcal{sp}\{ w_i \}^\perp \), the canonical dual frame of \( w_i \) in \( W_i \) is \( w_i/\| w_i \|^2 \). Hence elements of the PFFS sequence \( \{ x_i \} \) is given by
\[
x_i = \frac{w_i}{\| w_i \|^2} + z_i,
\]
where, by the analysis of the PFFS implementation of \( P_i \), \( z_i \in W_i^\perp = \mathcal{sp}\{ w_i \}^\perp \).

That \( z_i \in \mathcal{sp}\{ w_i \}_{i=i} \) suggests that the vector \( z_i = (z_{ik})_k \) must be in the co-dimension 1 subspace,
\[
w_i_1 z_i + w_i 2 z_2 + \cdots + w_i N z_i N = 0. \quad (7)
\]

By the construction equation (2) in the proof of Proposition 4.3, for \( 1 \leq i \leq N \), we have selected
\[
\mathcal{N}(P_i) = \mathcal{sp}\{ x_i \}_{i=i} = \mathcal{sp}\{ e_j \}_{j\neq i}. \quad (8)
\]
For \( N + 1 \leq i \leq M \), let us take \( j_i \in \{ 1, \ldots, N \} \) be such that \( |w_{i,j_i}| \geq |w_{ij}| > 0 \) for all \( 1 \leq j \leq N \) (repetition of \( j_i \) is allowed), and define
\[
\mathcal{N}(P_i) = \mathcal{sp}\{ x_i \}_{i=i} = \mathcal{sp}\{ e_{ji} \}_{ji\neq i}. \quad (9)
\]
Then the choice of (8) and (9), together with (7), imply that the solutions to the sequence \( \{ z_i \} \) are given by
\[
z_{ik} = \begin{cases} -\frac{w_{ik}}{\| w_i \|^2}, & k \neq i, \\ \sum_{j \neq k} \frac{w_{ij}^2}{w_{ij}^\perp}, & k = i, \end{cases}, \quad i = 1, \ldots, N,
\]
and
\[
z_{ik} = \begin{cases} -\frac{w_{ik}}{\| w_i \|^2}, & k \neq j_i, \\ \sum_{j \neq j_i} \frac{w_{ij}^2}{w_{ij}^\perp}, & k = j_i, \end{cases}, \quad i = N + 1, \ldots, M.
\]

That is, we need to bundle such an auxiliary sequence \( \{ z_i \} \) with the primary (sensor) frame \( \{ w_i \} \) so that
\[
\forall f \in \mathcal{H}_N, \quad P_i f = \langle f, x_i \rangle w_i = \left( \langle f, \frac{w_i}{\| w_i \|^2} \rangle + \langle f, z_i \rangle \right) w_i
\]
has the null space \( \mathcal{N}(P_i) \) as chosen, and \( P_i^* P_i \) has only one non-zero component in the diagonal.

We see, again, that the implementation of the non-orthogonal projection is not free, but feasible in sensor array applications.
One major point of the non-orthogonal fusion frame is that it allows us to design sensor and auxiliary sensor bundles so that the direction of the projection is controllable at the signal processing stage upon the signal is “sensed”. Such a “hardware” bundle makes the new fusion frame operator $S_W$ sparse or diagonal, which in turn enables a seamless fusion process (involving $S_W^{-1}$).

**Remark** This is an extreme case study, one of the most difficult scenarios. Because each $W_i \equiv \mathfrak{sp}(w_i)$ is 1-dimensional. The sequence $\{z_i\} \subseteq W_i^\perp$ needs to be chosen from the relatively large $N - 1$ dimensional subspace.

In general, if $\text{dim}(W_i^\perp) \ll N$, the auxiliary sequence $\{z_i\}$ is easier to obtain, and easier to control.

For instance, in an opposite extreme where $\text{dim}(W_i^\perp) = 1$, then $z_i$ is a 1-dimensional vector. The projection $P_i$ as seen in the PFFS expansion is parameterized by a set of scalars $\{\alpha_n\}$ in a form

$$\forall f \in \mathcal{H}_N, \quad P_i f = \sum_n \langle f, w_n + \alpha_n z_i \rangle \tilde{w}_n,$$

where $\{w_n\}$ and $\{\tilde{w}_n\}$ are a pair of frame and dual frame of $W_i$. In such a case, the direction of the projection $\mathcal{N}(P_i) = (\mathfrak{sp}(w_n + \alpha_n z_i))^\perp$ is completely determined by the scalars $\{\alpha_n\}$, which is not only easy, but also entirely maneuverable in the post processing stage (given the sensor measurements of $\{\langle f, w_n \rangle\}$ and the auxiliary measurement $\langle f, z_i \rangle$). All possible directions are achievable.

### 7 Eigen-Properties of $P_W$

We will compare non-orthogonal projections to orthogonal projections. The first two propositions are included for clarity and handy references.

**Proposition 7.1** Let $W$ be a $k$-dimensional subspace of $\mathcal{H}_N$ and $P_W$ be a (non-orthogonal) projection onto $W$. Let:

1. $\{x_i\}_{i=1}^k$ be an orthonormal basis for $W$.
2. $\{y_j\}_{i=k+1}^N$ be an orthonormal basis for the $N - k$-dimensional space $V = (I - P_W)\mathcal{H}_N$.

Then $\{x_i, y_j\}_{i=1}^k, j=k+1$ are the eigenvectors of $P_W$ with eigenvalues “1” for $x_i$ and eigenvalues “0” for $y_j$.

In particular, $P_W$ is an orthogonal projection if and only if $V = W^\perp$.

**Proof** Since $P_W$ is a projection, we have

$$P_W x_i = x_i,$$

i.e. $x_i$ is an eigenvector for $P_W$ with eigenvalue “1”. Also,
\[ P_W y_i = 0. \]

So \( y_i \) is an eigenvector for \( P_W \) with eigenvalue “0”. \( \square \)

The converse of the above proposition is also true.

**Proposition 7.2** Given \( W, V \) subspaces of \( \mathcal{H}_N \) with \( W \cap V = \{0\} \), and

\[
\dim W = k, \quad \text{and} \quad \dim V = N - k.
\]

Choose orthonormal bases \( \{x_i\}_{i=1}^k \) and \( \{y_i\}_{i=k+1}^N \) for \( W \) and \( V \) respectively. Given \( x \in \mathcal{H}_N \), there are unique scalars \( \{a_i\}_{i=1}^N \) so that

\[
x = \sum_{i=1}^k a_i x_i + \sum_{i=k+1}^N a_i y_i.
\]

Define

\[
P_W(x) = \sum_{i=1}^k a_i x_i.
\]

Then \( P_W \) is a projection on \( \mathcal{H}_N \) (and hence, \( P_W \) has eigenvectors \( \{x_i, y_j\}_{i=1, j=k+1}^N \) and eigenvalues “1” for \( x_i \) and “0” for \( y_j \)).

**Proof** We compute:

\[
P_W(P_W x) = P_W \left( \sum_{i=1}^k a_i x_i \right) = \sum_{i=1}^k a_i x_i = P_W(x) \quad \text{(by definition)}. \quad \square
\]

The above tells us what we can get out of non-orthogonal projections if we are projecting along a subset of the basis. To keep the notation simple, we will project along \( \text{span} \{e_i\}_{i=k+1}^N \). But this clearly works exactly the same for any \( K \subset \{1, 2, \ldots, N\} \) with \( |K| = k \).

**Corollary 7.3** In \( \mathcal{H}_N \), let \( K = \{1, 2, \ldots, k\} \). Choose an orthogonal set of vectors \( \{y_i\}_{i=1}^k \) in \((I - P_K)\mathcal{H}_N = P_K^c \mathcal{H}_N \) and for each \( i = 1, 2, \ldots, k \) let

\[
x_i = \frac{1}{\|e_i + y_i\|} e_i + \frac{1}{\|e_i + y_i\|} y_i.
\]

(Note that if \( N - k < k \), then \( 2k - N \) of the \( y_i \) will be zero.) Let \( W = \text{span} \{x_i\}_{i=1}^k \). Define \( P_W : \mathcal{H}_N \rightarrow W \) by

\[
P_W e_i = \|e_i + y_i\| x_i, \quad \text{if } i \in K,
\]

and

\[
P_W e_i = 0, \quad \text{if } i \in K^c.
\]
Then $P_W$ is a (non-orthogonal) projection having eigenvectors $\{x_i\}_{i=1}^k$ with eigenvalues “1” and eigenvectors $\{e_n\}_{n=N-k}^N$ with eigenvalues “0” for $n \in K^c$.

Moreover, $P_W^*P_W$ is a diagonal matrix with eigenvectors $\{e_n\}_{n=1}^N$ and eigenvalues “0” for $i = k + 1, k + 2, \ldots, N$ and eigenvalues $\|e_i + y_i\|^2 = 1 + \|y_i\|^2$ for $i = 1, 2, \ldots, k$.

Finally, if $Q$ is the orthogonal projection of $\mathcal{H}_N$ onto the same span $W$, then $Q$ has eigenvectors $\{x_i\}_{i=1}^k$ with eigenvalues “1” and eigenvectors $\{z_i\}_{i=k+1}^N$ an orthonormal basis for $W^\perp$ with eigenvalues “0”.

**Proof** For $x_i \in W$,

$$P_W(x_i) = P_W\left(\frac{1}{\|e_i + y_i\|} e_i\right) + P_W\left(\frac{1}{\|e_i + y_i\|} y_i\right)$$

$$= \frac{1}{\|e_i + y_i\|} P_W e_i + 0$$

$$= \|e_i + y_i\| \left(\frac{1}{\|e_i + y_i\|}\right) x_i$$

$$= x_i.$$

So $P_W$ is a projection.

For the moreover part, if $j$ is not in $K$ then

$$\langle P_W^*P_W e_i, e_j \rangle = \langle P_W e_i, P_W e_j \rangle = \langle P_W e_i, 0 \rangle = 0.$$

If $i \neq j \in K$ then

$$\langle P_W^*P_W e_i, e_j \rangle = \langle P_W e_i, P_W e_j \rangle$$

$$= \langle \|e_i + y_i\| x_i, \|e_j + y_j\| x_j \rangle$$

$$= \|e_i + y_i\| \|e_j + y_j\| \langle x_i, x_j \rangle = 0.$$

And if $i = j \in K$ then

$$\langle P_W^*P_W e_i, e_i \rangle = \|P_W e_i\|^2 = \|e_i + y_i\|^2.$$

The finally part is clear. $\square$

**Remark** It is worth understanding intuitively why $P_W^*P_W$ has all of its non-zero eigenvalues $\geq 1$. This is happening because by forcing ourselves to project along a set of $e_j$’s, we see that $P_W$ must project a set of vectors of the form $e_i$ to vectors of the form $e_i + y_i$ where $y_i \perp e_i$, and hence

$$\|P_W e_i\|^2 = \|e_i\|^2 + \|y_i\|^2 \geq \|e_i\|^2 = 1.$$

Now we can see what diagonal entries we can get when $P_W^*P_W$ is a diagonal matrix.
Corollary 7.4 Fix $1 \leq k \leq N$ and choose $K \subset \{1, 2, \ldots, N\}$ with $|K| = k$.

(1) If $\dim k \leq N/2$ and any numbers $\{a_n\}_{n \in K}$ are given with $a_n \geq 1$, there is a subspace $W$ of $\mathcal{H}_N$ and a (non-orthogonal) projection $P_W$ onto $W$ so that the eigenvectors of $P_W^* P_W$ are $\{e_n\}_{n=1}^N$ with respective eigenvalues $\{a_n\}_{n \in K}$ and “0” if $n \notin K$. That is, $P_W^* P_W$ is a diagonal matrix with non-zero diagonal entries $\{a_n\}_{n \in K}$.

(2) If $k > N/2$, there is a $K_1 \subset K$ with $|K_1| = N - k$ and if $\{a_n\}_{n \in K_1}$ are given with $a_n \geq 1$, then there is a subspace $W$ of $\mathcal{H}_N$ and a (non-orthogonal) projection $P_W$ onto $W$ so that $P_W^* P_W$ is a diagonal matrix with diagonal entries “0” if $n \notin K$, diagonal entries “1” if $n \in K \setminus K_1$, and diagonal entries $a_n$ if $n \in K_1$.

Proof (1) Since $\dim W \leq N/2$, we have that

$$N - \frac{N}{2} = \frac{N}{2} \geq \dim W.$$

Hence, there is an orthogonal set of vectors $\{y_n\}_{n \in K}$ satisfying:

(a) $y_n \in P_{K^c} \mathcal{H}_N$.
(b) $\|y_n\|^2 = a_n - 1$.

By Corollary 7.3, there exists a subspace $W$ with

$$W = \text{span}\left\{\frac{1}{\|e_n + y_n\|} (e_n + y_n) : n \in K\right\},$$

and a projection $P_W$ so that $P_W^* P_W$ has eigenvectors $\{e_n\}_{n=1}^N$ and non-zero eigenvalues only for $n \in K$ which are of the form:

$$\|e_n + y_n\|^2 = \|e_n\|^2 + \|y_n\|^2 = 1 + (a_n - 1) = a_n.$$

(2) We just do as in (1) except now, we can only find $N - k$ orthogonal vectors $y_n$ in $P_{K^c} \mathcal{H}_N$. So we pair these $y_n$’s with $N - k$ of the $e_n$’s in $K$ and put $e_n \in W$ for the rest of the $n \in K$.

\[\square\]

8 Tight and Multiple Fusion Frames

Nonorthogonal fusion frames bring in some quite unique properties that the orthogonal fusion frames do not have. For instance, we can now easily construct examples of tight fusion frames for non-orthogonal projections which do not exist in orthogonal fusion frames. In fact, we may have quite spectacular examples where tight fusion frames can be constructed via one (proper) subspace.

One immediate observation is that we may have multiple non-orthogonal projections onto one given subspace, now that (non-orthogonal) projections are no longer unique. We show that by applying multiple projections onto one and each subspace, tight nonorthogonal fusion frames exist.
Remark 8.1 First, let us observe that there is an obvious restriction on the number of projections we need. That is, if $W$ has dimension $k$ in $\mathcal{H}_N$, then each projection onto $W$ has at most $k$ non-zero eigenvalues (and $P^*_W P_W$ also has the same). So if we want $\sum_{i=1}^{L} P^*_i P_i = \lambda I$, then

$$L \geq \left\lfloor \frac{N}{k} \right\rfloor + 1.$$ 

In the next proposition we will see that this works if $k$ divides $N$. However, it can be shown that if $k$ does not divide $N$ then this result fails in general.

Proposition 8.2 Let $W \subseteq \mathcal{H}$ be a subspace of dimension $k \geq 1$.

1. If $k \geq \frac{N}{2}$, there are non-orthogonal projections $\{P_i\}_{i=1}^{2}$ onto $W$ so that

$$P^*_1 P_1 + P^*_2 P_2 = 2I.$$

2. If $N = kL$, there are non-orthogonal projections $\{P_i\}_{i=1}^{L}$ onto $W$ so that

$$\sum_{i=1}^{L} P^*_i P_i = LI.$$

Before we prove the proposition, we give some simple examples to show how the proof will work.

Example 8.3 There is a subspace $W$ in $\mathcal{H}_3$ with dim $W = 2$ and two (non-orthogonal) projections $P_W$ and $Q_W$ giving a 2-tight fusion frame for $\mathcal{H}_3$. We also know [9] that there is no tight fusion frame for $\mathcal{H}_3$ made from orthogonal projections and two, 2-dimensional subspaces. Moreover, the example above is “unique” in that the only way to produce a 2-tight (non-orthogonal) fusion frame out of projections $P$ with $P^*P$ diagonal is to produce the above example up to applying a unitary operator.

To do this, we consider the 2-dimensional subspace of $\mathcal{H}_3$ given by:

$$W_1 = \text{span}\left\{ (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}.$$ 

Now, by our Corollary 7.3, if we project onto $W$ along $e_3$ with $P_W$, then $P^*_W P_W$ will have eigenvectors $\{e_n\}_{n=1}^{3}$ with respective eigenvalues $\{1, 2, 0\}$ and if we project onto the subspace $W$ along $e_2$ with $Q_W$, then $Q^*_W Q_W$ has eigenvectors $\{e_n\}_{n=1}^{3}$ with respective eigenvalues $\{1, 0, 2\}$. So

$$P^*_W P_W + Q^*_W Q_W = 2I.$$

This example is unique since if we pick any two subspaces $W_1, W_2$ of $\mathcal{H}_3$ with dim $W_i = 2$ and choose any projections $P_{W_1}, P_{W_2}$ to get diagonal operators $P^*_i P_i$, then each projection must have a unit vector in its span and be projecting along another unit vector. Hence, all you can get for eigenvalues is $\{1, a_1, 0\}$ and $\{1, 0, a_2\}$ and we know that $a_1, a_2 \geq 1$ and these two sets of 3 eigenvalues must be arranged.
so their respective sums are all the same. Checking cases we can easily see that this only happens if the eigenvalues are lined up as they are and hence $a_1 = a_2 = 2$ and backtracking through the corollary, we get exactly our example back.

We give one more example which illustrates how the general case will go.

**Example 8.4** In $\mathcal{H}_{2N}$ let

$$W = \text{span}\{e_i + e_{N+i} : 1 \leq i \leq N\}.$$  

Then $\dim W = N$ (i.e. half the dimension of the space). We will construct two projections $P_1, P_2$ so that

$$P_1^*P_1 + P_2^*P_2 = 2I.$$  

To do this let $P_1$ be the projection onto $W$ along $\{e_i : 1 \leq i \leq N - k\}$ and $P_2$ the projection onto $\{e_i : i = N - k + 1, N - k + 2, \ldots, N\}$. Then by Corollary 7.3 we have $P_1^*P_1$ has eigenvectors $\{e_i\}^{N-k}_{i=1}$ with non-zero eigenvalues 2 for each and $\{e_i\}^{N}_{i=1}$ for $P_2^*P_2$, i.e. $P_1^*P_1 + P_2^*P_2 = 2I$.

For the proof of the proposition we need a simple lemma.

**Lemma 8.5** Let $\{W_i\}_{i \in I}$ be subspaces of a Hilbert space $\mathcal{H}$. Assume there exists a unitary operator $U$ on $\mathcal{H}$ and projections $\{P_{W_i}\}_{i \in I}$ onto the spaces $UW_i$ so that

$$\sum_{i \in I} P_{W_i}^*P_{W_i} = \lambda I.$$  

Then $\{U^*P_{W_i}U\}_{i \in I}$ is a family of projections onto $\{W_i\}_{i \in I}$ satisfying $\sum_{i \in I} U^*P_{W_i}^*P_{W_i}U = \lambda I$.

**Proof** Since $U$ is unitary, $U^*$ is a unitary operator taking $UW_i$ onto $W_i$. Also,

$$U^*P_{W_i}U(U^*P_{W_i}U) = U^*P_{W_i}(UU^*P_{W_i}U) = U^*P_{W_i}P_{W_i}U = U^*P_{W_i}U.$$  

That is, $U^*P_{W_i}U$ is a projection onto $W_i$. Finally,

$$\sum_{i \in I} U^*P_{W_i}^*P_{W_i}U = U^*(\sum_{i \in I} P_{W_i}^*P_{W_i}) = U^*\lambda I U = U^*U\lambda I = \lambda I.$$  

□

**Proof of Proposition 8.2** Let $\dim W = k$. We have to look at the two cases.

**Case 1:** We have $k \geq N \over 2$.

By Lemma 8.5, we may assume that our fixed subspace $W$ is

$$W = \text{span}\{e_i + e_{k+i}^{N-k}_{i=1} \cup e_i^{k}_{i=N-k+1}\}.$$  

By Corollary 7.3, if we project with $P_1$ onto $W$ along $\{e_i : i = 1, 2, \ldots, N - k\}$ we have that $P_1^*P_1$ has eigenvectors $\{e_i\}^{N-k}_{i=1}$ with respective eigenvalues “1” for $\{e_i\}^{k}_{i=N-k+1}$, “2” for $i = 1, 2, \ldots, N - k$, and “0” otherwise. Let $P_2$ be the projection along $\{e_i : i = k + 1, k + 2, \ldots, N\}$. Then by Corollary 7.3 $P_2^*P_2$ has eigenvectors...
\( \{e_i\}_{i=1}^{N} \) with eigenvalues “1” for \( \{e_i\}_{i=N-k+1}^{k} \), “2” for \( i = k + 1, k + 2, \ldots, N \) and “0” otherwise. Hence,

\[
P_1^*P_1 + P_2^*P_2 = 2I.
\]

**Remark** It is worthwhile to note an important property of the projections we constructed. By Corollary 7.3, we have that

\[
P_1e_i = \begin{cases} 
0, & \text{if } i = k + 1, k + 2, \ldots, N, \\
e_i, & \text{if } i = \lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{2} \rfloor + 1, \ldots, k
\end{cases}
\]

and

\[
P_2e_i = \begin{cases} 
0, & \text{if } i = 1, 2, \ldots, N - k, \\
e_i, & \text{if } i = \lfloor \frac{N}{2} \rfloor, \lfloor \frac{N}{2} \rfloor + 1, \ldots, k
\end{cases}
\]

This property carries over to our original subspace \( W \) since we can let

\[
V = \text{span}\left\{ e_i : i = \left\lfloor \frac{N}{2} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor + 1, \ldots, k \right\}.
\]

Now we need to see that this works when we use Lemma 8.5. But this is really immediate. Our original subspace is now \( U^*W \) and our projections are \( U^*P_1U, U^*P_2U \), and so

\[
U^*P_1U U^*P_2U = U^*P_1P_2U.
\]

**Case 2:** We have \( N = kL \).

By Lemma 8.5, we may assume that our subspace \( W \) is:

\[
W = \text{span}\{e_i, e_{k+i}, e_{2k+i}, \ldots, e_{(L-1)k+i} : i = 1, 2, \ldots, k\}.
\]

For \( j = 1, 2, \ldots, L \), let \( K_j = \{i = (j-1)k+1, (j-1)k+2, \ldots, jk\} \) and let \( P_j \) be the projection onto \( W \) along \( \{e_i\}_{i \in K_j} \). Then by Corollary 7.3, \( P_j^*P_j \) has eigenvectors \( \{e_i\}_{i=1}^{N} \) with eigenvalues “0” for \( i \in K_j^c \) and eigenvalues “L” for \( i \in K_j \). Since

\[
\bigcup_{j=1}^{L} K_j = \{1, 2, \ldots, N\},
\]

and the sets \( \{K_j\}_{j=1}^{L} \) are disjoint, it follows that

\[
\sum_{i=1}^{L} P_i^*P_i = LI.
\]

For fixed \( W \) with \( \dim(W) = k \), but \( k \) does not divide \( N \), the same conclusion as in Proposition 8.2 may not hold. The following existence conclusion still applies.
Proposition 8.6 If $N = kL + M$ and $1 \leq M < k$, there exists a subspace $W$ of dimension $k$ and non-orthogonal projections $\{P_i\}_{i=1}^{L+1}$ onto $W$ so that $\sum_{i=1}^{L} P_i^* P_i$ has $\{e_i\}_{i=1}^{N}$ as eigenvectors with eigenvalues “$L + 1$” for $\{e_i\}_{i=1}^{N-M}$ and eigenvalues “$L$” for $\{e_i\}_{i=N-M+1}^{N}$.

Proof We may define the subspace $W$ by

$$W = \text{span}\left[ \left\{ \sum_{j=0}^{L} e_{jk+i} : i = 1, 2, \ldots, M \right\} \right.$$

$$\cup \left\{ \sum_{j=0}^{L-1} e_{jk+i} : j = M + 1, M + 2, \ldots, k \right\} \bigg] .$$

For $j = 1, 2, \ldots, L$, let $K_j = \{i = (j-1)k + 1, (j-1)k + 2, \ldots, jk\}$, and let $K_L = \{N - kL + 1, N - kL + 2, \ldots, N\}$. Then the result follows by projecting onto $W$ along the sets $\{K_j\}_{j=1}^{k}$. □

Concluding Remarks Non-orthogonal fusion frames are clearly natural extensions of orthogonal fusion frames previously introduced [5, 11]. With non-orthogonal fusion frames, not only can we always make the (new) fusion frame operator $\mathcal{F}$ sparse, but also sometimes enable $\mathcal{F}$ to become diagonal or tight. In sensor network data fusion applications, non-orthogonal fusion frames is seen as a flexible tool to resolve the non-sparse nature of the orthogonal fusion frames operator since sensor subspaces and their relationships are given a priori by the sensor physics and the deployment of sensors. Sparsity considerations through non-orthogonal fusion frames seems to be the only effective approach in such applications. The implementation of the non-orthogonal projections through pseudoframes for subspaces are also discussed in detail.

It is also seen that the flexibility of non-orthogonal fusion frames brings in rather unique and a broad range of properties to the notion of fusion frames. Our on-going subsequent work includes multi-fusion frame constructions with diagonal or tight $\mathcal{F}$, complete tight fusion frame constructions based on one (proper) subspace, classification of positive and self-adjoint operators by projections, and non-orthogonal fusion frames analysis for a given set of subspaces (such as in sensor networks) so that $\mathcal{F}$ is either diagonal or tight. This last task is an ultimate goal.

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