Self-dual supergravity and twistor theory

Martin Wolf

Theoretical Physics Group, The Blackett Laboratory, Imperial College London,
Prince Consort Road, London SW7 2AZ, UK

E-mail: m.wolf@imperial.ac.uk

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Abstract

By generalizing and extending some of the earlier results derived by Manin and Merkulov, a twistor description is given of four-dimensional $N$-extended (gauged) self-dual supergravity with and without cosmological constant. Starting from the category of $(4|4N)$-dimensional complex superconformal supermanifolds, the categories of $(4|2N)$-dimensional complex quaternionic, quaternionic Kähler and hyper-Kähler right-chiral supermanifolds are introduced and discussed. We then present a detailed twistor description of these types of supermanifolds. In particular, we construct supertwistor spaces associated with complex quaternionic right-chiral supermanifolds, and explain what additional supertwistor data allow for giving those supermanifolds a hyper-Kähler structure. In this way, we obtain a supersymmetric generalization of Penrose’s nonlinear graviton construction. We furthermore give an alternative formulation in terms of a supersymmetric extension of LeBrun’s Einstein bundle. This allows us to include the cases with nonvanishing cosmological constant. We also discuss the bundle of local supertwistors and address certain implications thereof. Finally, we comment on a real version of the theory related to Euclidean signature.

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1. Introduction and results

Since the discovery of twistor string theories by Witten [1] and Berkovits [2] about three years ago, many advancements in our understanding of the properties of (supersymmetric) Yang–Mills theory have been made. Despite the fact that these string theories describe supersymmetric Yang–Mills theory coupled to conformal supergravity [3], they provide an elegant way of describing some of the remarkable features exhibited by the scattering amplitudes of the gauge theory (see, e.g., [4, 5] and references therein). Surely, the appearance of conformal supergravity is awkward since it is inextricably mixed in with the gauge theory
beyond tree level in perturbation theory. This makes it impossible to solely compute gauge
theory scattering amplitudes beyond tree level by performing a string theory calculation. In
addition, one wishes to describe Einstein supergravity rather than conformal supergravity as
the latter is believed not to be a suitable candidate for describing nature due to its lack of
unitarity. In view of that, Abou-Zeid et al [6] proposed new twistor string theories which
indeed seem to yield supersymmetric Yang–Mills theory coupled to Einstein supergravity.
Among the already mentioned aspects, a variety of other related issues have been investigated
and are still being explored [7–19] (for recent reviews, see also [20]).

Despite the success, a consistent twistor string formulation of gravity remains an open
question. In order to find such a formulation, first it is certainly necessary to better understand
the twistor description of Einstein supergravity theories. Before trying to attempt to solve this
task in full generality, one may first consider a simplification of the theory by restricting the
focus to the much simpler theory of self-dual supergravity. In view of that, recall from the
early work by Penrose [21] that it is possible to associate with any complex-Riemannian four-
dimensional manifold $M$ (complex spacetime), which is equipped with a conformal structure
and has self-dual Weyl curvature, a complex three-dimensional twistor space $P$ which is
defined to be the space of maximal isotropic (totally null) complex submanifolds of $M$. All
the information about the conformal structure of $M$ is encoded in the complex structure of
the twistor space $P$. Some additional data on $P$ then allow for the construction of self-dual
metrics and conformal structures on $M$. For explicit constructions, see [22–37], for instance.
Moreover, hidden symmetries and hierarchies of self-dual gravity have been studied by the
authors of [38–42]. Note also that one may return to the realm of Riemannian geometry
by restricting the objects under consideration to the fixed-point set of an anti-holomorphic
involution.

Self-dual supergravity theories on four-dimensional spacetime have first appeared in the
works [43–47] and have subsequently been discussed, e.g., by the authors of [48, 49] within
the harmonic superspace framework (see also Galperin et al [50] and references therein). The
purpose of this paper is to give the twistor description of $N$-extended self-dual supergravity
with and without cosmological constant. In particular, we shall generalize and extend the
earlier results by Manin [51] and Merkulov [52–55]. For the majority of the time, we work
in the context of complex supermanifolds but at the end we also discuss a real version of
the theory. In the following section, starting from the category of complex superconformal
supermanifolds of dimension $(4|4N)$, the categories of

(i) complex quaternionic right-chiral (hereafter RC) supermanifolds,
(ii) complex quaternionic Kähler RC supermanifolds and
(iii) complex hyper-Kähler RC supermanifolds

are introduced and discussed. In this section, special attention is paid to the construction of
the connections and their properties under superconformal rescalings. In section 3, we first
discuss the twistor theory of complex quaternionic RC supermanifolds. We shall establish a
double fibration of the form

\[ \mathcal{P} \xrightarrow{\pi_2} \mathcal{M} \xleftarrow{\pi_1} \mathcal{P} \]

where $\mathcal{M}$ is a complex quaternionic RC supermanifold subject to additional restrictions and
$\mathcal{P}$ its associated supertwistor space. The supermanifold $\mathcal{F}$ is a certain $\mathbb{P}^1$-bundle over

\footnote{Note that Merkulov [54] has given a twistor description of minimal Einstein supergravity.}
and termed correspondence space. In this way, $\mathcal{M}$ is viewed as the space of complex submanifolds of $\mathcal{P}$ which are biholomorphically equivalent to the complex projective line $\mathbb{P}^1$ and have normal sheaf described by

$$0 \longrightarrow \Pi\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^n \longrightarrow \mathcal{N}_{\mathbb{P}^1,\mathcal{P}} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \longrightarrow 0.$$ 

Here, $\Pi$ is the Graßmann parity changing functor and $\mathcal{O}_{\mathbb{P}^1}(1)$ is the sheaf of sections of the dual tautological $(c_1 = 1)$ bundle over $\mathbb{P}^1$.

Having established this correspondence, we focus on the twistor description of complex hyper-Kähler RC supermanifolds—the case of interest in view of studying self-dual supergravity with zero cosmological constant. In particular, we give the supersymmetric analogue of Penrose’s nonlinear graviton construction [21], i.e. we shall show that in this case the supertwistor space is holomorphically fibred over the Riemann sphere $\mathcal{P} \rightarrow \mathbb{P}^1$ and equipped with a certain relative symplectic structure. Furthermore, we present an equivalent formulation of the Penrose construction in terms of a supersymmetric generalization of LeBrun’s Einstein bundle [56]. This construction allows for including a cosmological constant to the self-dual supergravity equations. In particular, the Einstein bundle is defined over the supertwistor space and as we shall see, its nonvanishing sections are in one-to-one correspondence with solutions to the self-dual supergravity equations. In particular, the Einstein bundle is defined of LeBrun’s Einstein bundle [56]. This construction allows for including a cosmological constant to the self-dual supergravity equations. In particular, the Einstein bundle is defined over the supertwistor space and as we shall see, its nonvanishing sections are in one-to-one correspondence with solutions to the self-dual supergravity equations with cosmological constant. Requiring its sections to be integrable amounts to putting the cosmological constant to zero. As in the purely bosonic situation, this bundle can explicitly be described in terms of certain intrinsic holomorphic data on the supertwistor space. Besides this, also in section 3, we introduce the bundle of local supertwistors over $\mathcal{M}$ and discuss certain implications thereof. For instance, we shall show that it can be reinterpreted in terms of a certain jet bundle over the supertwistor space by means of the Penrose–Ward transform. All these considerations are first generic in the sense of keeping arbitrary the number $N$ of allowed supersymmetries. However, like in the flat situation (see, e.g., Witten [1]), the $N = 4$ case is special and deserves a separate treatment.

Finally, in section 4 we discuss a real version of the theory, that is, we introduce certain real structures (anti-holomorphic involutions) on all the manifolds appearing in the above double fibration such that the underlying (ordinary) manifold of $\mathcal{M}$ is of Euclidean signature. A particular feature of Euclidean signature is that the number of allowed supersymmetries is restricted to be even.

1.1. Remarks

Some general remarks are in order. A complex supermanifold of dimension $m|n$ is meant to be a ringed space $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$, where $\mathcal{M}$ is a topological space and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of supercommutative $(\mathbb{Z}_2$-graded) rings on $\mathcal{M}$ such that, if we let $\mathcal{N}$ be the ideal subsheaf in $\mathcal{O}_{\mathcal{M}}$ of all nilpotent elements, the following is fulfilled:

(i) $\mathcal{M}_{\text{red}} := (\mathcal{M}, \mathcal{O}_{\text{red}} := \mathcal{O}_{\mathcal{M}}/\mathcal{N})$ is a complex manifold of (complex) dimension $m$ and

(ii) for any point $x \in \mathcal{M}$ there is neighbourhood $\mathcal{U} \ni x$ such that $\mathcal{O}_{\mathcal{M}}|\mathcal{U} \cong \mathcal{O}_{\text{red}}(\Lambda^* \mathcal{E})|\mathcal{U}$, where $\mathcal{E} := \mathcal{N}/\mathcal{N}^2$ is a rank-$n$ locally free sheaf of $\mathcal{O}_{\text{red}}$-modules on $\mathcal{M}$ and $\Lambda^*$ denotes the exterior algebra. We call $\mathcal{E}$ the characteristic sheaf of the complex supermanifold $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ and $\mathcal{O}_{\mathcal{M}}$ its structure sheaf. See, e.g., Manin [51] for more details. Such supermanifolds are said to be locally split. In this work, we will assume that the Graßmann odd directions have trivial topology, that is, we work in the category of globally split supermanifolds $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ with $\mathcal{O}_{\mathcal{M}} \cong \mathcal{O}_{\text{red}}(\Lambda^* \mathcal{E})$. For the sake of brevity, we shall be referring to them as split supermanifolds, in the following. The structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a split supermanifold admits the $\mathbb{Z}$-grading $\mathcal{O}_{\mathcal{M}} \cong \bigoplus_{p \geq 0} \mathcal{O}_{\mathcal{M}}^p$, where $\mathcal{O}_{\mathcal{M}}^0 \cong \mathcal{O}_{\text{red}}(\Lambda^0 \mathcal{E})$. Moreover, the assumption of
being split implies that there will always exist an atlas \([\mathcal{U}_a, \{\varphi_{ab}, \partial_{ab}\}]\) on \((\mathcal{M}, \mathcal{O}_\mathcal{M})\) such that, if we let \((z_a) = (z^1_a, \ldots, z^m_a)\) be Graßmann even coordinates and \((\eta_a) = (\eta^1_a, \ldots, \eta^n_a)\) be Graßmann odd coordinates on the patch \(\mathcal{U}_a \subset \mathcal{M}\), the transition functions on nonempty intersections \(\mathcal{U}_a \cap \mathcal{U}_b\) are of the form \(z_a = \varphi_{ab}(z_b)\) and \(\eta_a = (\partial^j_{ab}(z_b)\eta^i_b)\) for \(i, j = 1, \ldots, n\). We will frequently be working with such atlases without particularly referring to them.

An important example of a split supermanifold is the complex projective superspace \(\mathbb{P}^m|n\) given by

\[\mathbb{P}^m|n = (\mathbb{P}^m, \mathcal{O}_{\text{red}}(\wedge^1\mathcal{O}_\mathbb{P}^m(-1) \otimes \mathbb{C}^n)),\]

where \(\mathcal{O}_{\mathbb{P}^m}(-1)\) is the sheaf of sections of the tautological \((c_1 = -1)\) line bundle over the complex projective space \(\mathbb{P}^m\). The reason for the appearance of \(\mathcal{O}_{\mathbb{P}^m}(-1)\) is as follows. If we let \((z^0, \ldots, z^m, \eta^1, \ldots, \eta^n)\) be homogeneous coordinates\(^2\) on \(\mathbb{P}^m|n\), a holomorphic function \(f\) on \(\mathbb{P}^m|n\) has the expansion

\[f = \sum f_{i_1 \ldots i_r}(z^0, \ldots, z^m)\eta^{i_1} \ldots \eta^{i_r}.\]

Surely, for \(f\) to be well defined the total homogeneity of \(f\) must be zero. Hence, \(f_{i_1 \ldots i_r}\) must be of homogeneity \(-r\). This explains the above form of the structure sheaf of the complex projective superspace. Of particular interest in the context of (flat) twistor theory is \(\mathbb{P}^{3|2N}\), which is the super-twistor space associated with the superconformal compactification of the chiral complex superspace \(\mathbb{C}^{4|2N}\).

Moreover, two things are worth mentioning. First, any given complex supermanifold is actually a deformation of a split supermanifold [57], that is, to any complex supermanifold \((\mathcal{M}, \mathcal{O}_\mathcal{M})\) there is associated a complex analytic one-parameter family of complex supermanifolds \((\mathcal{M}_t, \mathcal{O}_{\mathcal{M}_t})\), for \(t \in \mathbb{C}\), such that \(\mathcal{O}_{\mathcal{M}_{t=0}} \cong \mathcal{O}_{\text{red}}(\wedge^1\mathcal{E})\) and \(\mathcal{O}_{\mathcal{M}_{t=1}} \cong \mathcal{O}_\mathcal{M}\), where \(\mathcal{E}\) is the characteristic sheaf of \((\mathcal{M}, \mathcal{O}_\mathcal{M})\). Second, smooth supermanifolds are always split due to Batchelor’s theorem [58]. The latter result follows because of the existence of a smooth partition of unity in the category of smooth supermanifolds. Since we are eventually interested in self-dual supergravity on a four-dimensional Riemannian manifold, this explains why we may restrict our discussion to split supermanifolds as stated above.

Furthermore, when there is no confusion with the underlying topological space, we denote the supermanifold \((\mathcal{M}, \mathcal{O}_\mathcal{M})\) simply by \(\mathcal{M}\). Finally, we point out that (holomorphic) vector bundles \(\mathcal{E}\) of rank \(r\) over some complex supermanifold \((\mathcal{M}, \mathcal{O}_\mathcal{M})\) are meant to be locally free sheaves of \(\mathcal{O}_{\mathcal{M}}\)-modules, that is, they are locally of the form \(\mathcal{O}_{\mathcal{M}} \otimes \mathbb{C}^r \oplus \Pi \mathcal{O}_{\mathcal{M}} \otimes \mathbb{C}^r\). Hence, the notions ‘vector bundle’ and ‘locally free sheaf’ are used interchangeably. This will also allow us to simplify notation. In addition, the dual of any locally free sheaf \(\mathcal{E}\) on \(\mathcal{M}\) is denoted by

\[\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\mathcal{E}, \mathcal{O}_{\mathcal{M}}).\]

For line bundles \(\mathcal{L}\), we instead write \(\mathcal{L}^{-1}\). If there is no confusion, the dimensionality (respectively, the rank) of ordinary manifolds (respectively, of ordinary vector bundles) will often be abbreviated by \(m|0 \equiv m\) (respectively, by \(r|0 \equiv r\)).

2. Self-dual supergravity

2.1. Superconformal structures

Remember that a (holomorphic) conformal structure on an ordinary four-dimensional complex spin manifold \(M\) can be introduced in two equivalent ways. The first definition states that a

\[\begin{align*}
\text{Recall that they are subject to the identification } (z^0, \ldots, z^m, \eta^1, \ldots, \eta^n) \sim (tz^0, \ldots, tz^m, t\eta^1, \ldots, t\eta^n), \text{ where } t \in \mathbb{C}\backslash\{0\}.
\end{align*}\]
conformal structure is an equivalence class $[g]$, the conformal class, of holomorphic metrics $g$ on $M$, where two given metrics $g$ and $g'$ are called equivalent if $g' = \gamma^2 g$ for some nowhere vanishing holomorphic function $\gamma$. Putting it differently, a conformal structure is a line subbundle $L$ in $\Omega^1 M \otimes \Omega^1 M$. The second definition assumes a factorization of the holomorphic tangent bundle $TM$ of $M$ as a tensor product of two rank-2 holomorphic vector bundles $S$ and $\tilde{S}$, that is, $TM \cong S \otimes \tilde{S}$. This isomorphism in turn gives (canonically) the line subbundle $\Lambda^2 S^\vee \otimes \Lambda^2 \tilde{S}^\vee$ in $\Omega^1 M \otimes \Omega^1 M$ which, in fact, can be identified with $L$.

Next one needs to extend the notion of a conformal structure to supermanifolds. We shall see that the generalization of the latter of the two approaches given above seems to be the appropriate one for our present purposes (see section 2.1.3 for some remarks regarding standard supergravities). Our subsequent discussion closely follows that given by Manin [51] and Merkulov [52], respectively.

2.1.1. Superconformal supermanifolds. To give the definition of a superconformal structure, let $(\mathcal{M}, \mathcal{O}_\mathcal{M})$ be a $(4|4N)$-dimensional complex supermanifold, where $N$ is a nonnegative integer.

**Definition 2.1.** A superconformal structure on $\mathcal{M}$ is a pair of integrable rank-0 $|2N$ distributions $T_l \mathcal{M}$ and $T_r \mathcal{M}$ which obey the following conditions:

(i) their sum in the holomorphic tangent bundle $T \mathcal{M}$ is direct,
(ii) there exist two rank-2 $|0$ locally free sheaves $\mathcal{I}$ and $\mathcal{J}$ and one rank-$0|N$ locally free sheaf $\mathcal{E}$ such that $T_l \mathcal{M} \cong \mathcal{I} \otimes \mathcal{E}$ and $T_r \mathcal{M} \cong \mathcal{E} \otimes \mathcal{J}$;
(iii) the Frobenius form $\Phi : T_l \mathcal{M} \otimes T_r \mathcal{M} \to T^0 \mathcal{M} := T \mathcal{M} / (T_l \mathcal{M} \oplus T_r \mathcal{M})$,

\[(X \otimes Y) \mapsto [X, Y] \mod (T_l \mathcal{M} \oplus T_r \mathcal{M})\]

coincides with the natural map $\mathcal{I} \otimes \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{J} \to \mathcal{I} \otimes \mathcal{J}$ and gives an isomorphism $T^0 \mathcal{M} \cong \mathcal{I} \otimes \mathcal{J}$. Here, $[\cdot, \cdot]$ denotes the graded Lie bracket.

From this definition it follows that $T_{l,r} \mathcal{M}$ define two foliations on $\mathcal{M}$. Let us denote the resulting quotients by $\mathcal{M}_{l,r}$. Furthermore, $\mathcal{M}_l$ and $\mathcal{M}_r$ are supermanifolds which are both of dimension $(4|2N)$. In addition, their structure sheaves $\mathcal{O}_{\mathcal{M}_{l,r}}$ are those subsheaves of $\mathcal{O}_\mathcal{M}$ which are annihilated by vector fields from $T_{l,r} \mathcal{M}$. By virtue of the inclusions $\mathcal{O}_{\mathcal{M}_{l,r}} \subset \mathcal{O}_\mathcal{M}$, we find the following double fibration:

\[
\begin{array}{ccc}
\mathcal{M} & \xymatrix{\ar[dr]_{\pi_l} & } & \mathcal{M} \\
\mathcal{M}_l & & \mathcal{M}_r
\end{array}
\] (2.1)

According to Manin [51], we shall call $\mathcal{M}_l$ and $\mathcal{M}_r$ left- and right-chiral supermanifolds, respectively. Moreover, we have

\[
0 \to T_{l,r} \mathcal{M} \to T \mathcal{M} \xymatrix{\ar[r]^-{\pi^*_r} & } T \mathcal{M}_{l,r} \to 0,
\] (2.2)

which is induced by the double fibration (2.1). Putting it differently, $T_r \mathcal{M}$ (respectively, $T_l \mathcal{M}$) is the relative tangent sheaf of the fibration $\pi_r : \mathcal{M} \to \mathcal{M}_r$ (respectively, of $\pi_l : \mathcal{M} \to \mathcal{M}_l$). Note that a superconformal structure is, by no means, just given by a conformal class of supermetrics.

\[\text{3 Do not confuse $\mathcal{E}$ with the characteristic sheaf of $(\mathcal{M}, \mathcal{O}_\mathcal{M})$.} \]
2.1.2. Some properties and an example. First of all, it should be noted that on the underlying four-dimensional manifold \( \mathcal{M}_{\text{red}} \), we naturally have the rank-2 holomorphic vector bundles \( \mathcal{S}_{\text{red}} \) and \( \mathcal{T}_{\text{red}} \). Part (iii) of definition 2.1 then guarantees a factorization of the holomorphic tangent bundle \( T\mathcal{M}_{\text{red}} \) of \( \mathcal{M}_{\text{red}} \) as \( T\mathcal{M}_{\text{red}} \cong \mathcal{S}_{\text{red}} \otimes \mathcal{T}_{\text{red}} \). Hence, \( \mathcal{M}_{\text{red}} \) comes naturally equipped with a conformal structure.

Furthermore, definition 2.1 implies that the holomorphic tangent bundle \( T\mathcal{M} \) of \( \mathcal{M} \) fits into the following short exact sequence:

\[
0 \longrightarrow T_{l}\mathcal{M} \oplus T_{r}\mathcal{M} \overset{\phi_{0,1}}{\longrightarrow} T\mathcal{M} \overset{\Phi_{1}}{\longrightarrow} T_{0}\mathcal{M} \longrightarrow 0,
\]

where \( i_{l,r} \) are the natural inclusion mappings and \( \Phi \) is the contracted Frobenius form which is invertible by assumption. Recall that \( T_{0}\mathcal{M} \cong \mathcal{F} \otimes \mathcal{F} \). Consider now the subsheaves \( \pi_{l,r}^{*}T\mathcal{M}_{l,r} \subset T\mathcal{M} \). It can be shown (cf Manin [51] and Merkulov [52]) that their structure is also described by a short exact sequence similar to that given above, i.e.

\[
0 \longrightarrow T_{l,r}\mathcal{M} \longrightarrow \pi_{l,r}^{*}T\mathcal{M}_{l,r} \longrightarrow T_{0}\mathcal{M} \longrightarrow 0.
\]

This implies that also the underlying manifolds \( \mathcal{M}_{l,r} \) of \( \mathcal{M}_{l,r} \) are naturally equipped with conformal structures in the usual sense.

The prime example of the above construction is the flag supermanifold

\[
\mathcal{M} = F_{20,2}|(\mathbb{C}^{4|N}) = (S^{2}|N) \subset S^{2} \subset \mathbb{C}^{4|N}.
\]

In this case, the double fibration (2.1) takes the following form:

\[
\begin{array}{ccccc}
\pi_{l} & \downarrow & \phi_{l} & \downarrow & \pi_{r} \\
\mathcal{T}_{l} & \rightarrow & \mathcal{T}_{\mathcal{M}_{l}} & \rightarrow & \mathcal{T}_{r} \\
\end{array}
\]

\[
\mathcal{M}_{l} = F_{20}|(\mathbb{C}^{4|N}) \quad \mathcal{M}_{r} = F_{20}(\mathbb{C}^{4|N}).
\]

In addition, there are four natural sheaves \( \mathcal{F}^{20} \subset \mathcal{F}^{2|N} \) and \( \mathcal{F}^{20} \subset \mathcal{F}^{2|N} \) on \( \mathcal{M} \), where \( \mathcal{F}^{20} \) and \( \mathcal{F}^{2|N} \) are the two tautological sheaves while the other two are defined by two short exact sequences

\[
0 \longrightarrow \mathcal{F}^{20} \longrightarrow \theta_{\mathcal{M}} \otimes \mathbb{C}^{4|N} \longrightarrow (\mathcal{F}^{2|N})^{\vee} \longrightarrow 0,
\]

\[
0 \longrightarrow \mathcal{F}^{2|N} \longrightarrow \theta_{\mathcal{M}} \otimes \mathbb{C}^{4|N} \longrightarrow (\mathcal{F}^{2|N})^{\vee} \longrightarrow 0.
\]

A short calculation shows that these two sequences together with (2.4) imply

\[
\mathcal{F} \cong (\mathcal{F}^{20})^{\vee}, \quad \mathcal{F} \cong (\mathcal{F}^{20})^{\vee} \quad \text{and} \quad \mathcal{E} \cong \mathcal{F}^{2|N} / \mathcal{F}^{20}.
\]

In addition, one may also verify that points (i) and (iii) of definition 2.1 are satisfied. For more details, see Manin [51]. The three flag supermanifolds \( F_{20,2}|(\mathbb{C}^{4|N}) \), \( F_{20}|(\mathbb{C}^{4|N}) \) and \( F_{20}(\mathbb{C}^{4|N}) \) play an important role in the twistor description of supersymmetric Yang–Mills theories, as they represent the superconformal compactifications of the flat complex superspaces \( \mathbb{C}^{4|N} \) and \( \mathbb{C}^{4|2N}_{l,r} \) (for recent reviews, see [20]).

2.1.3. Remarks. In this work, we shall only be concerned with \( \mathcal{N} \)-extended self-dual supergravities. These theories are conveniently formulated on right-chiral supermanifolds, as has been discussed in, e.g., [47–49]. By a slight abuse of notation, we denote \( \mathcal{M} \), simply by \( \mathcal{M} \). Henceforth, we shall be working with (complex) right-chiral (hereafter RC) supermanifolds of dimension \( (4|2N) \). Furthermore, for various technical reasons but also for reasons related to the real structures discussed in section 4, we restrict the number \( \mathcal{N} \) of allowed supersymmetries to be even. Note that for the complex situation (or the case of split signature), this restriction on \( \mathcal{N} \) is not necessary.
Furthermore, let us emphasize that for making contact with standard full non-self-dual supergravities with $\mathcal{N} > 2$ (this includes both Einstein and conformal supergravities), a superconformal structure should be defined slightly differently since due to certain torsion constraints (see, e.g., Howe [59]), there are no (anti-)chiral superfields. This means that one does not have the double fibration (2.1). Also for $\mathcal{N} > 2$, one should relax the condition of integrability of $T_r\mathcal{M}$ if one wishes to talk about self-dual supergravity since in that case only $T_r\mathcal{M}$ should be considered to be integrable. For our discussions given below, these issues can be left aside. They certainly deserve further studies in view of complementing the self-dual theory to the full theory.

### 2.2. Geometry of right-chiral supermanifolds

This section is devoted to some geometric aspects of RC supermanifolds. In particular, we discuss their structure group, introduce scales and vielbeins and talk about connections, torsion and curvature.

#### 2.2.1. Structure group

Let $\mathcal{M}$ be an RC supermanifold. From (2.4) we know that the holomorphic tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is described by a sequence of the form

$$0 \to \mathcal{E} \otimes \tilde{\mathcal{J}} \to T\mathcal{M} \to \mathcal{J} \otimes \tilde{\mathcal{J}} \to 0,$$

(2.9)

where $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are both of rank $2|0$ and $\mathcal{E}$ is of rank $0|\mathcal{N}$, respectively. By a slight abuse of notation, we are again using the same symbols $\mathcal{J}$, $\tilde{\mathcal{J}}$ and $\mathcal{E}$. Next we note that $T\mathcal{M}$ is given by the tensor product $\mathcal{H} \otimes \tilde{\mathcal{J}}$, where $\mathcal{H} \to \mathcal{M}$ is a rank-$2|\mathcal{N}$ holomorphic vector bundle over $\mathcal{M}$ described by

$$0 \to \mathcal{E} \to \mathcal{H} \to \mathcal{J} \to 0.$$

(2.10)

Hence, the structure group of $T\mathcal{M}$ is as follows: the supergroup $GL(2|\mathcal{N}, \mathbb{C})$ acts on the left on $\mathcal{H}$ and $GL(2|0, \mathbb{C})$ acts on the right on $\tilde{\mathcal{J}}$ by inverses. The resulting induced action on the tangent bundle $T\mathcal{M}$ yields a subsupergroup of $GL(4|2\mathcal{N}, \mathbb{C})$. Let us denote it by $G$ and its Lie superalgebra by $\mathfrak{g}$. In addition, there is a $|4-\mathcal{N}|$-fold cover of $G \subset GL(4|2\mathcal{N}, \mathbb{C})$,

$$1 \to \mathbb{Z}_{|4-\mathcal{N}|} \to S(GL(2|\mathcal{N}, \mathbb{C}) \times GL(2|0, \mathbb{C})) \to G \to 1,$$

(2.11)

where $S(GL(2|\mathcal{N}, \mathbb{C}) \times GL(2|0, \mathbb{C}))$ is the subsupergroup of $SL(4|\mathcal{N}, \mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix}
A_1 & 0 & A_2 \\
0 & b & 0 \\
A_3 & 0 & A_4
\end{pmatrix}, \quad \text{with} \quad \begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix} \in GL(2|\mathcal{N}, \mathbb{C})$$

(2.12)

and $b \in GL(2|0, \mathbb{C})$. Here, $A_i$, $s$, for $i = 1, \ldots, 4$, are the defining blocks of a supermatrix in standard format, that is, the matrices $A_{1,4}$ are Graßmann even while $A_{2,3}$ are Graßmann odd (see, e.g., Manin [51] for more details).

#### 2.2.2. Scales and vielbeins

Let us start by defining what shall be understood by a scale on an RC supermanifold $\mathcal{M}$ (cf also [51, 52, 55, 60]). Consider the sequences (2.9) and (2.10). They immediately give the following natural isomorphisms of Berezinian sheaves:

$$Ber T\mathcal{M} \cong (Ber \mathcal{H})^2 \otimes (Ber \mathcal{E})^2 \otimes (Ber \tilde{\mathcal{J}})^{2-\mathcal{N}}$$

$$\cong (Ber \mathcal{H})^2 \otimes (Ber \tilde{\mathcal{J}})^{2-\mathcal{N}}.$$

(2.13)
Hence, the Berezinian sheaf $\text{Ber}(\mathcal{M}) := \text{Ber} \Omega^1 \mathcal{M}$ of $\mathcal{M}$ is

$$\text{Ber}(\mathcal{M}) \cong (\text{Ber} \omega^\vee)^2 \otimes (\text{Ber} \tilde{T})^{2-N}. \quad (2.14)$$

In addition, we have $\text{Ber} \tilde{T} \cong \Lambda^2 \tilde{T}$ since $\tilde{T}$ is of purely even rank $2|0$. Moreover, without loss of generality, one may always make the identification

$$\text{Ber} \mathcal{M} \cong \text{Ber} \tilde{T}. \quad (2.15)$$

In this respect, recall that $N$ is assumed to be even. In fact, since the tangent bundle can be factorized as $T \mathcal{M} \cong \mathcal{M} \otimes \tilde{T}$, one can locally choose to have such an isomorphism.

Then we may give the following definition:

**Definition 2.2.** A scale on an RC supermanifold $\mathcal{M}$ is a choice of a particular nonvanishing volume form $\vartheta \in H^0(\mathcal{M}, \text{Ber} \tilde{T})$ on the vector bundle $\tilde{T}$. A superconformal rescaling is a change of scale.

Therefore, together with the identification $\text{Ber} \omega^\vee \cong \text{Ber} \tilde{T}$, a section of $\text{Ber} \tilde{T}$ gives a section of $\text{Ber}(\mathcal{M})$ on $\mathcal{M}$. We shall denote a generic section of $\text{Ber}(\mathcal{M})$ by $\text{Vol}$, in the following.

It is well known that a particular choice of a coordinate system on any supermanifold determines the corresponding trivialization of the (co)tangent bundle and hence of the Berezinian bundle. Let now $\mathcal{U}$ be an open subset of $\mathcal{M}$. On $\mathcal{U}$ we may introduce $(x^0, \eta^{0\mu})$ as local coordinates, where $\mu, \nu, \ldots = 1, 2, \hat{\mu}, \hat{\nu}, \ldots = 1, 2$ and $m, n, \ldots = 1, \ldots, N$. The entire set of coordinates is denoted by $x^M$, where $M = (\mu \nu, m\bar{\mu})$ is an Einstein index. We shall also make use of the notation $M = M\bar{\mu}$, where $M = (\mu, m)$. Then $\partial/\partial x^M$ (respectively, $dx^M$) are basis sections of the tangent bundle $T \mathcal{M}$ (respectively, of the cotangent bundle $\Omega^1 \mathcal{M}$) of $\mathcal{M}$. We may associate with the set $(\partial/\partial x^M)$ (respectively, with $\{dx^M\}$) a basis section of $\text{Ber}(\mathcal{M})$ which we denote by $D^{-1}(\partial/\partial x^M)$ (respectively, by $D(dx^M)$). An arbitrary (local) section of $\text{Ber}(\mathcal{M})$ then takes the following form:

$$\text{Vol} = \phi D^{-1} \left( \frac{\partial}{\partial x^M} \right) = \phi D(dx^M) := \phi dx^A d^2\eta, \quad (2.16)$$

where $\phi$ is a nonvanishing function on $\mathcal{U} \subset \mathcal{M}$. In the last step in the above equation, we have introduced a more conventional notation for the volume form.

Next we introduce (local) frame fields $E_A$, which generate the tangent bundle $T \mathcal{M}$, by setting

$$\text{Vol} = D^{-1}(E_A), \quad \text{with} \quad E_A := E^M_A \frac{\partial}{\partial x^M}. \quad (2.17)$$

Obviously, frame fields are unique up to $\mathcal{S}G$-transformations of the form $E_A \mapsto C_A^B E_B$, where $C = (C_A^B)$ is an $\mathcal{S}G$-valued function on $\mathcal{U}$ with $\mathcal{S}G$ being the subsupergroup of $G$ described by

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow S\mathcal{L}(2|\mathcal{N}, \mathbb{C}) \times S\mathcal{L}(2|0, \mathbb{C}) \longrightarrow \mathcal{S}G \longrightarrow 1, \quad (2.18)$$

as $\mathcal{N}$ is assumed to be even. Putting it differently, a choice of scale on $\mathcal{M}$ reduces the structure group from $G$ to $\mathcal{S}G$. By comparing (2.16) with (2.17), we see that the function $\phi$ is given by the superdeterminant of $E^M_A$, i.e. $\phi = \text{Ber}(E^M_A)$. This particular frame is also called the structure frame. In addition, (local) coframe fields $E^A$ are given by

$$E^A := dx^M E^A_M, \quad \text{with} \quad E_A \cdot E^B = \delta^B_A. \quad (2.19)$$

---

4 Later on, we additionally require that the resulting volume form $\text{Vol} \in H^0(\mathcal{M}, \text{Ber}(\mathcal{M}))$ obeys $\rho(\text{Vol}) = \text{Vol}$, where $\rho$ is a real structure on $\mathcal{M}$. See section 4 for more details.
They generate the cotangent bundle $\Omega^1 M$ of $M$. Here, $E^M_A$ and $E^A_M$ are called vielbein matrices which obey

$$E^M_A E^M_B = \delta^B_A \quad \text{and} \quad E^A_M E^N_A = \delta^N_M.$$  \hfill (2.20)

The structure group or structure frame indices $A, B, \ldots$ look explicitly as $A = (\alpha, \hat{\beta}, i, \hat{i})$, where $\alpha, \beta, \ldots = 1, 2, \hat{\alpha}, \hat{\beta}, \ldots = \hat{1}, \hat{2}$ and $i, j, \ldots = 1, \ldots, \hat{N}$, respectively. Again, we shall write $A = A\hat{\alpha}$ with $A = (\alpha, i)$.

Recall that by virtue of (2.14) and (2.15), a section of $\text{Ber} \tilde{T}$ gives a section of $\text{Ber}(M)$. If we rescale this section by some nonvanishing function $\gamma$, the volume form $\text{Vol}$ changes as $\text{Vol} \mapsto \hat{\text{Vol}} = \gamma^{4-N} \text{Vol}$. Up to $SG$-transformations (which can always be reabsorbed in the definition of the vielbein), the frame and coframe fields change accordingly as $E_A \mapsto \hat{E}_A = \gamma^{-\kappa} E_A$ and $E^A \mapsto \hat{E}^A = \gamma^\kappa E^A$, where

$$\kappa := \frac{4 - \hat{N}}{4 - 2N}. \hfill (2.21)$$

### 2.2.3. Connection

Generally speaking, an affine connection $\nabla$ on $M$ is a Graßmann even mapping on the tangent bundle $T M$,

$$\nabla : T.M \rightarrow T.M \otimes \Omega^1 M,$$  \hfill (2.22)

which satisfies the Leibniz formula

$$\nabla(f X) = df \otimes X + f \nabla X,$$  \hfill (2.23)

where $f$ is a local holomorphic function and $X$ a local section of $T M$. Setting $\nabla_A := E^A_J \nabla$, we may write equation (2.23) explicitly as

$$\nabla_A(f X) = (E_A f) X + (-)^{p/p_A} f \nabla_A X. \hfill (2.24)$$

Here, $p \in \mathbb{Z}_2$ denotes the Graßmann parity. Since $T M \cong M \otimes \tilde{T}$, we have the decomposition

$$\nabla = \nabla_{\mathcal{M}} \otimes \text{id}_{\tilde{T}} + \text{id}_{\mathcal{M}} \otimes \nabla_{\tilde{T}},$$  \hfill (2.25)

where

$$\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega^1 \mathcal{M} \quad \text{and} \quad \nabla_{\tilde{T}} : \tilde{T} \rightarrow \tilde{T} \otimes \Omega^1 M$$  \hfill (2.26)

are the two connections on $\mathcal{M}$ and $\tilde{T}$, respectively. Locally, the connection $\nabla$ is given in terms of a $g$-valued connection 1-form $\Omega = (\Omega_A^B) = (E^C \Omega_{CA}^B)$ which is defined by

$$\nabla E_A = \Omega_A^B E_B,$$  \hfill (2.27)

with

$$\Omega_A^B = \Omega_{A\hat{\alpha}}^{B\hat{\beta}} = \Omega_A^B \delta^\hat{\beta}_{A\hat{\alpha}} + \delta_A^B \Omega_{\hat{\alpha}}^{\hat{\beta}}, \hfill (2.28)$$

by virtue of (2.25). Therefore, equations (2.27) read explicitly as

$$\nabla E_A = \nabla E_{A\hat{\alpha}} = \Omega_A^B E_{B\hat{\alpha}} + \Omega_{\hat{\alpha}}^{\hat{\beta}} E_{A\hat{\beta}}.$$  

In the following, we shall not make any notational distinction between the three connections $\nabla$, $\nabla_{\mathcal{M}}$ and $\nabla_{\tilde{T}}$ and simply denote them commonly by $\nabla$. It will be clear from the context which of those is actually being considered.
2.2.4. Torsion. If we set

\[ [E_A, E_B] = f_{AB}^C E_C, \] (2.29)

where \( f_{AB}^C \) are the structure functions, the components of the torsion \( T = T^A E_A = \frac{1}{2} E_B \wedge E_A T_{AB}^C E_C \) of \( \nabla \), which is defined by

\[ T^A = -\nabla E^A = -dE^A + E_B \wedge \Omega_B^A, \] (2.30)

are given by

\[ T_{AB}^C = \Omega_{AB}^C - (-)^{p_A p_B} \Omega_{BA}^C - f_{AB}^C. \] (2.31)

Note that if we consider the space of differential 2-forms on \( \mathcal{M} \), we have

\[ \Lambda^2 \Omega^1 \mathcal{M} \cong \Lambda^2 (\mathcal{H}^{\nu} \otimes \mathcal{F}^{\nu}) \cong (\Lambda^2 \mathcal{H}^{\nu} \otimes \mathcal{F}^{\nu}) \oplus \Lambda^2 (\mathcal{H}^{\nu} \otimes \Lambda^2 \mathcal{F}^{\nu}), \] (2.32)

where \( \mathcal{H}^{\nu} \) denotes the \( \nu \)th (graded) symmetric power of the bundles in question. Therefore, \( T \) can be decomposed as

\[ T = T^- + T^+, \] (2.33)

with

\[ T^- \in H^0(\mathcal{M}, \Lambda^2 \mathcal{H}^{\nu} \otimes \mathcal{F}^{\nu} \otimes T \mathcal{M}), \]
\[ T^+ \in H^0(\mathcal{M}, \mathcal{H}^{\nu} \otimes \Lambda^2 \mathcal{F}^{\nu} \otimes T \mathcal{M}). \] (2.34)

In the structure frame, \( T^\pm \) look in components as

\[ T^- : T_{A\dot{a}B\dot{b} \dot{c}} C_{\dot{c} \dot{c}} \quad \text{and} \quad T^+ : T_{A\dot{a}B\dot{b}} C_{\dot{c} \dot{c}}, \] (2.35)

where parentheses denote normalized symmetrization while square brackets denote normalized antisymmetrization, respectively.

A tensor is called **totally trace free** if all possible supertraces with respect to upper and lower indices vanish. Then we have the following proposition:

**Proposition 2.1.** On any RC supermanifold \( \mathcal{M} \) with fixed scale, the totally trace-free parts of \( T^- \) and of \( T^+ \) are independent of the choice of connection, i.e. they are invariants of \( \mathcal{M} \).

**Proof.** Here, we are following ideas of [60] but adopted to the supersymmetric setting. Recall that \( T \mathcal{M} \cong \mathcal{H} \otimes \mathcal{F} \). Let \( \mu^A \) be a section of \( \mathcal{H} \) and \( \lambda^\dot{a} \) be a section of \( \mathcal{F} \), respectively. For fixed scale, a general change of a given connection \( \nabla_{A\dot{a}} = E_{A\dot{a}} \nabla \) to another one \( \tilde{\nabla}_{A\dot{a}} = E_{A\dot{a}} \tilde{\nabla} \) is given in terms of the contorsion tensors \( \Theta_{A\dot{a}B\dot{b}}^C \) and \( \Theta_{A\dot{a}\dot{b}}^C \) by

\[ (\tilde{\nabla}_{A\dot{a}} - \nabla_{A\dot{a}}) \mu^B = (-)^{p_A p_C} p_B \mu^C \Theta_{A\dot{a}C} B^B, \]
\[ (\tilde{\nabla}_{A\dot{a}} - \nabla_{A\dot{a}}) \lambda^\dot{b} = \lambda^\dot{b} \Theta_{A\dot{a}\dot{b}}. \]

Hence, for a section \( u^{A\dot{a}} \) of \( T \mathcal{M} \) this implies

\[ (\tilde{\nabla}_{A\dot{a}} - \nabla_{A\dot{a}}) u^{B\dot{b}} = (-)^{p_A p_C} p_B C^C \Theta_{A\dot{a}C} B^B, \]

with

\[ \Theta_{A\dot{a}B\dot{b}} C^C = \Theta_{A\dot{a}B\dot{c}} \delta^C_{\dot{c}} + \delta_B^C \Theta_{A\dot{a}\dot{b}}^C. \]

Note that \( p_A = p_A \). From

\[ [\tilde{\nabla}_{A\dot{a}}, \tilde{\nabla}_{B\dot{b}}] f = -T_{A\dot{a}B\dot{b}} C^C \tilde{\nabla}_{C\dot{c}} f, \]
where \( f \) is a local section of \( \mathcal{O}_M \), and from similar expressions for unhatted quantities, we thus obtain
\[
\hat{T}_{\mathcal{A}_a\mathcal{B}_b}^{\mathcal{C}_c} = T_{\mathcal{A}_a\mathcal{B}_b}^{\mathcal{C}_c} + 2\Theta_{\mathcal{A}_a\mathcal{B}_b}^{\mathcal{C}_c} \delta_{\mathcal{D}_d}^{\mathcal{E}_e} - 2\Theta_{\mathcal{A}_a\mathcal{D}_d}^{\mathcal{E}_e} \delta_{\mathcal{B}_b}^{\mathcal{C}_c}.
\]

Therefore, without loss of generality, we can always work with a connection \( \nabla \) on \( \mathcal{M} \) whose torsion tensors \( T^- \) and \( T^+ \) are totally trace free, since given any two connections on the bundles \( \mathcal{H} \) and \( \mathcal{F} \) it is always possible to find contorsion tensors such that the resulting connection induced on the tangent bundle \( T_M \) will be totally trace free. Indeed, we have the following proposition:

**Proposition 2.2.** On any RC supermanifold \( \mathcal{M} \) with fixed scale \( \bar{\varepsilon} \in H^0(\mathcal{M}, \text{Ber} \hat{T}) \) there always exits a connection such that

(i) the torsion tensors \( T^- \) and \( T^+ \) are totally trace free and

(ii) in addition we have that
\[
\nabla \bar{\varepsilon} = 0 = \nabla \bar{\varepsilon}.
\]

where \( \varepsilon \in H^0(\mathcal{M}, \text{Ber} \hat{T}) \) is determined by \( \bar{\varepsilon} \) via the isomorphism (2.15).

Furthermore, for \( N \neq 4 \), this connection is unique.

**Proof.** Existence is clear from our above discussion. It remains to prove uniqueness for \( N \neq 4 \). First of all, one notices that given two connections \( \nabla \) and \( \bar{\nabla} \) whose torsion tensors \( T^+ \) and \( \bar{T}^+ \) are totally trace free, then their contorsion tensors \( \Theta \) and \( \bar{\Theta} \) (obtained from an arbitrary connection one has started with) can only differ by the following terms:
\[
\hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b}^{\mathcal{C}_c} := \bar{\Theta}{\mathcal{A}_a}{\mathcal{B}_b}^{\mathcal{C}_c} - \Theta{\mathcal{A}_a}{\mathcal{B}_b}^{\mathcal{C}_c} = X_{[\mathcal{A}_a\mathcal{B}_b]}^{\mathcal{C}_c} + Y_{[\mathcal{A}_a\mathcal{B}_b]}^{\mathcal{C}_c},
\]

\[
\hat{\Theta}{\mathcal{A}_a}^{\mathcal{B}_b} := \bar{\Theta}{\mathcal{A}_a}^{\mathcal{B}_b} - \Theta{\mathcal{A}_a}^{\mathcal{B}_b} = -Y_{[\mathcal{A}_a]}^{\mathcal{B}_b} - X_{[\mathcal{A}_a]}^{\mathcal{B}_b},
\]

where \( X_{\mathcal{A}_a} \) and \( Y_{\mathcal{A}_a} \) are arbitrary differential 1-forms on \( \mathcal{M} \). This can be seen upon inspecting equations (2.36). Next one picks a volume form \( \varepsilon \in H^0(\mathcal{M}, \text{Ber} \hat{T}) \) and hence a volume form \( \varepsilon \in H^0(\mathcal{M}, \text{Ber} \hat{T}) \). In a structure frame, they are of the form (recall that \( \mathcal{M} \) is split)
\[
\varepsilon_{\mathcal{A}_a}^{\mathcal{B}_b} = \varepsilon_{\mathcal{B}_b}^{\mathcal{A}_a} = -\varepsilon^{\mathcal{A}_a\mathcal{B}_b} = \bar{\varepsilon}^{\bar{\mathcal{A}}_{\bar{a}} \bar{\mathcal{B}}_{\bar{b}}},
\]

where the \( \varepsilon \)-tensors are totally antisymmetric with \( \varepsilon_{12} = \varepsilon_{\hat{1}\hat{2}} = -\varepsilon_{\bar{1}\bar{2}} = -1 \). Since \( N \) is assumed to be even, we find
\[
\tilde{\nabla}_{\mathcal{A}_a} \varepsilon_{\mathcal{B}_b} = \nabla_{\mathcal{A}_a} \varepsilon_{\mathcal{B}_b} = -\hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b}^{\mathcal{C}_c} \varepsilon_{\mathcal{C}_c} + \hat{\nabla}_{\mathcal{A}_a} \varepsilon_{\mathcal{B}_b} = \nabla_{\mathcal{A}_a} \varepsilon_{\mathcal{B}_b} - \hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b}^{\mathcal{C}_c} \varepsilon_{\mathcal{C}_c}.
\]

These equations in turn imply that
\[
(-)^{\mathcal{B}} \hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b} = \hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b} - \hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b} = 0 = \hat{\Theta}{\mathcal{A}_a}{\mathcal{B}_b},
\]

since both, \( \nabla \) and \( \tilde{\nabla} \), are assumed to annihilate \( \varepsilon \) and \( \bar{\varepsilon} \), respectively. It is then a rather straightforward exercise to verify that \( X_{\mathcal{A}_a} \) and \( Y_{\mathcal{A}_a} \) must vanish for \( N \neq 4 \). Hence, \( \tilde{\nabla} = \nabla \) and the proof is completed.

Henceforth, we shall be working with a connection on \( \mathcal{M} \) which has totally trace-free torsion tensors \( T^+ \). Note that if \( T^+ \) is taken to be totally trace free, it must vanish identically.
This is seen as follows. One first notices that $T_{A[\dot{a}B\dot{b}]}C\dot{\gamma} = \epsilon_{\dot{a}\dot{b}} T_{A\dot{a}B\dot{b}}C\dot{\gamma}$ as the rank of $\mathcal{F}$ is 2|0. Since $T^{-}$ is totally trace free, it follows from

$$T_{A[\dot{a}B\dot{b}]}C\dot{\gamma} = \epsilon_{\dot{a}\dot{b}} T_{A\dot{a}B\dot{b}}C\dot{\gamma} = 0$$

that $T_{A\dot{a}B\dot{b}}C\dot{\gamma} = 0$. Altogether, the torsion tensor takes the form

$$T = T^{-}. \quad (2.37)$$

**Definition 2.3.** An RC supermanifold $\mathcal{M}$ is said to be complex quaternionic if it is equipped with a torsion-free connection which annihilates both volume forms $\varepsilon$ and $\tilde{\varepsilon}$.

For our later discussions, we need to know how a connection behaves under changes of scale.

**Proposition 2.3.** Suppose we are given an RC supermanifold $\mathcal{M}$ which is equipped with a connection $\nabla$ that obeys conditions (i) of (ii) given in proposition 2.2. Suppose further that $N \neq 4$. Under a rescaling of the form $\tilde{\varepsilon} \mapsto \gamma \varepsilon$, where $\gamma$ is a nonvanishing holomorphic function, the change of connection to the new one $\tilde{\nabla}$ is given by the following contorsion tensors:

$$\Theta_{A\dot{a}B\dot{c}} C = (-)^{\rho}\varepsilon B \gamma_{B\delta A} C - \frac{\kappa}{2} \gamma A \delta B C, \quad \Theta_{A\dot{a}\dot{b}} C \dot{\gamma} = \gamma_{A\delta A} \dot{\gamma} - \frac{\kappa}{2} \gamma A \delta A \dot{\gamma}.$$

Here, $\gamma_{A\dot{a} :} = E_{A\dot{a}} \log \gamma$ and the constant $\kappa$ has been introduced in (2.21). This implies that the new connection $\tilde{\nabla}_{A\dot{a}} = E_{A\dot{a}} - \tilde{\nabla}$, with $E_{A\dot{a}} = \gamma^{-1} E_{A\dot{a}}$, acts as follows:

$$\nabla_{A\dot{a}} \mu B \mapsto \tilde{\nabla}_{A\dot{a}} \mu B = \gamma^{-1} \nabla_{A\dot{a}} \mu B + \delta_{A\dot{a}} B \gamma C_{A B},$$

$$\nabla_{A\dot{a}} \hat{\lambda} \hat{B} \mapsto \tilde{\nabla}_{A\dot{a}} \hat{\lambda} \hat{B} = \gamma^{-1} \nabla_{A\dot{a}} \hat{\lambda} \hat{B} + \delta_{A\dot{a}} \hat{B} \gamma \hat{A} \hat{B},$$

$$\nabla_{A\dot{a}} \mu B \hat{B} \mapsto \tilde{\nabla}_{A\dot{a}} \mu B \hat{B} = \gamma^{-1} \nabla_{A\dot{a}} \mu B - \mu \gamma \gamma B \hat{B},$$

$$\nabla_{A\dot{a}} \lambda \hat{B} \hat{B} \mapsto \tilde{\nabla}_{A\dot{a}} \lambda \hat{B} \hat{B} = \gamma^{-1} \nabla_{A\dot{a}} \lambda \hat{B} \hat{B} - \gamma_{A B} \lambda \hat{B} \hat{B},$$

where $\mu^{A}$ and $\lambda^{\dot{A}}$ are sections of the vector bundles $\mathcal{M}$ and $\mathcal{F}$, respectively, together with $\mu^{A} = \gamma^{1/2} \mu$ and $\lambda^{\dot{A}} = \gamma^{1/2} \lambda$. 

**Proof.** The first thing one notices is that the components of the volume forms $\epsilon_{\dot{a}\dot{b}} i_{\dot{a}i_{\dot{b}}}$ and $\tilde{\epsilon}_{\dot{a}\dot{b}} i_{\dot{a}i_{\dot{b}}}$ scale as

$$\epsilon_{\dot{a}\dot{b}} i_{\dot{a}i_{\dot{b}}} \mapsto \tilde{\epsilon}_{\dot{a}\dot{b}} i_{\dot{a}i_{\dot{b}}} = \gamma^{1/2} \epsilon_{\dot{a}\dot{b}} i_{\dot{a}i_{\dot{b}}} \quad \text{and} \quad \tilde{\epsilon}_{\dot{a}\dot{b}} \mapsto \tilde{\tilde{\epsilon}}_{\dot{a}\dot{b}} = \gamma^{-1/2} \tilde{\epsilon}_{\dot{a}\dot{b}}.$$

Hence, the conditions

$$\tilde{\nabla}_{A\dot{a}} \tilde{\epsilon}_{\dot{a}i_{\dot{a}}i_{\dot{b}}} = 0 \quad \text{yield} \quad (-)^{\rho} \Theta_{A\dot{a}B\dot{c}} C = \Theta_{A\dot{a}B\dot{c}} C\dot{\gamma} = \Theta_{A\dot{a}B\dot{c}} C\dot{\gamma} = -\frac{\kappa}{4} \gamma A \delta B C\dot{\gamma}.$$

Furthermore, by the requirement that the parts $\tilde{T}^{\pm}$ of $\tilde{T}$ are totally trace free, we find that (see also the proof of proposition 2.2)

$$\Theta_{A\dot{a}B\dot{c}} C\dot{\gamma} = -\kappa \gamma A \delta B C\dot{\gamma}.$$

Combining these results, we arrive after some algebra at equations (2.38). Finally, equations (2.39) follow upon application of $\tilde{\nabla}_{A\dot{a}}$ on the appropriate sections. \[\square\]
2.2.5. Levi-Civita connection. In the class of affine connections on $\mathcal{M}$ which have totally trace-free torsion tensors $T^\sharp$ there exists a supersymmetric analogue of the Levi-Civita connection.

**Proposition 2.4.** Let $\mathcal{M}$ be an RC supermanifold. For a given pair of non-degenerate sections $e \in H^0(\mathcal{M}, \Lambda^2 \mathcal{H}^\vee)$ and $\tilde{\epsilon} \in H^0(\mathcal{M}, \Lambda^2 \tilde{\mathcal{S}}^\vee)$ there always exists a unique torsion-free connection $D$ on $\mathcal{M}$ such that

$$De = 0 = D\tilde{\epsilon}.$$  

In addition, there is a unique scale (up to multiplicative constants) for which this connection coincides with that given by proposition 2.2.

**Proof.** One first notices that

$$g := e \otimes \tilde{\epsilon} \in H^0(\mathcal{M}, \Lambda^2 \mathcal{H}^\vee \otimes \Lambda^2 \tilde{\mathcal{S}}^\vee) \subset H^0(\mathcal{M}, \Omega^1 \mathcal{M} \otimes \Omega^1 \mathcal{M})$$

can be regarded as a holomorphic metric on $\mathcal{M}$ which, in fact, reduces to an ordinary holomorphic metric on $\mathcal{M}_{red}$. Since $D(e \otimes \tilde{\epsilon}) = (De) \otimes \tilde{\epsilon} + e \otimes (D\tilde{\epsilon})$, we have further that $Dg = 0$. Together with the condition of vanishing torsion, the proof reduces to that one familiar from ordinary Riemannian geometry (modulo changes of signs due to the $\mathbb{Z}_2$-grading).

Next one realizes that

$$\text{Ber} \; \mathcal{H} \cong (\text{Ber} \; \Lambda^2 \mathcal{H})^{1/(1-N)},$$

which can be deduced from the definition of the Berezinian sheaf by using splitting principle arguments, for instance. Hence,

$$\text{Ber} \; \mathcal{H}^\vee \cong (\text{Ber} \; \Lambda^2 \mathcal{H}^\vee)^{1/(1-N)} \cong \text{Ber} \; \tilde{\mathcal{S}}^\vee.$$  

Thus, there exists a unique scale (up to multiplicative constants) where $D$ annihilates both $e \in H^0(\mathcal{M}, \Lambda^2 \mathcal{H}^\vee)$ and $\tilde{\epsilon} \in H^0(\mathcal{M}, \text{Ber} \; \mathcal{H}^\vee)$. Hence, by the uniqueness (for $N \neq 4$) shown in proposition 2.2, $D$ coincides with $\nabla$. 

Hence, $\mathcal{M}$ equipped with that type of connection is a complex quaternionic RC supermanifold. In full analogy with ordinary Riemannian geometry, we shall refer to this connection as the Levi-Civita connection.

2.2.6. $N = 4$ case. As shown in proposition 2.2, there is no unique connection $\nabla$ for $N = 4$ which is solely determined by the requirements of having totally trace-free torsion and simultaneously annihilating both volume forms on $\mathcal{H}$ and $\tilde{\mathcal{S}}$. To jump ahead of our story a bit, working with such a connection would result in a dependence of the supertwistor space $\mathcal{P}$ associated with an RC complex quaternionic supermanifold $\mathcal{M}$ on the chosen scale on the latter. Of course, the definition of $\mathcal{P}$ should only depend on the (super)conformal class of $\mathcal{M}$, that is, it should be independent of the particular scale.

Nevertheless, as seen above, the Levi-Civita connection $D$ will always exist no matter what the chosen value of $N$ is. Moreover, if $N = 4$, it is possible to compute the change of the Levi-Civita connection under superconformal rescalings since the usual torsion obstructions disappear.

**Proposition 2.5.** Let $\mathcal{M}$ be a (4|8)-dimensional RC supermanifold equipped with the Levi-Civita connection. Under a rescaling of the form $\tilde{\epsilon} \mapsto \gamma \tilde{\epsilon}$, where $\gamma$ is a nonvanishing

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5 Since $e$ and $\tilde{\epsilon}$ are assumed to be non-degenerate, their corresponding matrix representations are of full rank and hence as matrices they are invertible.
holomorphic function, the change of the Levi-Civita connection $D$ to the new one $\hat{D}$ is given by the following contorsion tensors:

$$\Theta_{A\hat{A} B} C = -\gamma_{A\hat{A}} \delta_B C \quad \text{and} \quad \Theta_{A\hat{A} \hat{B}} \hat{\gamma} = \gamma_{A\hat{A}} \delta_{\hat{B}} \hat{\gamma}. \quad (2.40)$$

Here, $\gamma_{A\hat{A}} := E_{A\hat{A}} \log \gamma$, as before. This implies that the new connection $\hat{D}_{A\hat{A}} = E_{A\hat{A}} \hat{D} = E_{A\hat{A}} \hat{D}$ acts as follows:

$$
\begin{align*}
D_{A\hat{A}} \mu_B &\mapsto \hat{D}_{A\hat{A}} \mu_B = D_{A\hat{A}} \mu_B - \gamma_{A\hat{A}} \mu_B, \\
D_{A\hat{A}} \lambda^\hat{B} &\mapsto \hat{D}_{A\hat{A}} \lambda^\hat{B} = D_{A\hat{A}} \lambda^\hat{B} + \gamma_{A\hat{A}} \lambda^\hat{B}, \\
D_{A\hat{A}} \mu_B &\mapsto \hat{D}_{A\hat{A}} \mu_B = D_{A\hat{A}} \mu_B + \gamma_{A\hat{A}} \mu_B, \\
D_{A\hat{A}} \lambda^\hat{B} &\mapsto \hat{D}_{A\hat{A}} \lambda^\hat{B} = D_{A\hat{A}} \lambda^\hat{B} - \gamma_{A\hat{A}} \lambda^\hat{B}.
\end{align*}
\quad (2.41)
$$

where $\mu^A$ and $\lambda^\hat{B}$ are sections of the vector bundles $\mathcal{H}$ and $\mathcal{T}$, respectively, together with $\mu^{\hat{A}} = \mu^A$ and $\lambda^\hat{B} = \lambda^\hat{B}$ and similarly for their duals.

\textbf{Proof.} Under a change of scale $\hat{e} \mapsto \hat{e} = \gamma \hat{e}$, the symplectic 2-form $e \in H^0(\mathcal{M}, \text{Ber}\Lambda^2 \mathcal{H}^{\vee})$ behaves as

$$e \mapsto \hat{e} = \gamma^{-1} \hat{\gamma} e = \gamma^{-1} e.$$  

This simply follows from the isomorphisms $\text{Ber}\mathcal{H}^{\vee} \cong (\text{Ber}\Lambda^2 \mathcal{H}^{\vee})^{1/(1-N)} \cong \text{Ber}\mathcal{T}^{\vee}$. Hence, $g = e \otimes \hat{e} \mapsto \hat{g} = g$. Note that $\kappa$ (see equation (2.21)) vanishes identically for $N = 4$. In addition, we have

$$\hat{D} \hat{e} = 0 = \hat{D} \hat{e}.$$

Hence, the induced contorsion tensor $\Theta_{A\hat{A} B} C \hat{\gamma}$ must be zero, i.e. $\hat{D} = D$ upon action on $T\mathcal{M}$ (this result was already expected by equations (2.14) and (2.15) for $N = 4$). It is then rather straightforward to verify that the above conditions imply equations (2.40).

\hfill \Box

2.2.7. \textbf{Curvature.} Given any connection $\nabla$ on $\mathcal{M}$, the associated curvature 2-form

$$R = (R_A^B) = \left(\frac{1}{2} e^D \wedge e^C R_{CDA}^B\right), \quad (2.42)$$

which takes values in $\mathfrak{g}$, is defined by

$$R_A^B = d\Omega_A^B + \Omega_A^C \wedge \Omega_C^B. \quad (2.43)$$

The components of the curvature read explicitly as

$$R_{ABC}^D = E_A \Omega_{BC}^D + (-)^{p_A p_B} E_B \Omega_{AC}^D + (-)^{p_A (p_B + p_C + p_D)} \Omega_{BC}^E \Omega_{ED}^F \Omega_{AB}^D - (-)^{p_A (p_C + p_D)} \Omega_{AC}^E \Omega_{DE}^F \Omega_{BD}^D. \quad (2.44)$$

In addition, torsion and curvature are combined into the standard formula

$$[\nabla_A, \nabla_B] u^D = (-)^{p_A (p_B + p_D)} u^A R_{ABC}^D - T_{ABC}^D \nabla_C u^D. \quad (2.45)$$

Here, $u^A$ is some tangent vector on $\mathcal{M}$. This equation might concisely be rewritten as

$$[\nabla_A, \nabla_B] u^C = R_{AB}^C u_C - \text{id}_\mathcal{F} \circ \text{id}_\mathcal{F} \otimes R_{AB}^C \quad (2.46)$$

where $R_{AB} = R_{ABC}^D M_D^C + R_{A\hat{A} B} \hat{\gamma} M_{\hat{A} \hat{B}} \hat{\gamma}$ together with the generators $M_A^B$ and $M_{\hat{A} \hat{B}} \hat{\gamma}$ of the Lie superalgebra $\mathfrak{g}$.

Note that because of the factorization $T\mathcal{M} \cong \mathcal{H} \otimes \mathcal{T}$, we have

$$R = R_{\mathcal{H}} \otimes \text{id}_\mathcal{F} + \text{id}_\mathcal{H} \otimes R_{\mathcal{T}}. \quad (2.47)$$
Here, $R_{\mathcal{F}}$ can be viewed as a section of $\Lambda^2\Omega^1_\mathcal{F} \otimes \text{End}\mathcal{F}$ while $R_{\mathcal{F}'}$ as a section of $\Lambda^2\Omega^1_\mathcal{F} \otimes \text{End}\mathcal{F}$. In the structure frame, the decomposition of $R$ looks as

$$R^\mathcal{F}_{\Lambda} = R_{\Lambda\delta} \delta^\Lambda + R_{\Lambda B} \delta^\Lambda B + \delta_{\Lambda B} \delta A B,$$  \hspace{1cm} (2.48)

Furthermore, recalling equation (2.32), we have further decompositions of $R$ (respectively, of $R_{\mathcal{F}'}$ and $R_{\mathcal{F}''}$) into $R^\mathcal{F}$ (respectively, into $R_{\mathcal{F}'}^\mathcal{F}$ and $R_{\mathcal{F}''}^\mathcal{F}$).

**Proposition 2.6.** Let $\mathcal{M}$ be a complex quaternionic RC supermanifold. In the structure frame, the curvature parts $R^\mathcal{F}_{\mathcal{F}'}$ and $R^\mathcal{F}_{\mathcal{F}''}$ of $R^\mathcal{F}$ are of the following form:

$$R^\mathcal{F}_{\mathcal{F}'} := R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda + R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda B + \delta_{\Lambda\Lambda\Lambda\Lambda} \delta A B,$$

$$R^\mathcal{F}_{\mathcal{F}''} := R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda + R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda B + \delta_{\Lambda\Lambda\Lambda\Lambda} \delta A B.$$

where $R^\mathcal{F}_{\Lambda\Lambda\Lambda\Lambda} := R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda$ and

$$C_{\Lambda\Lambda\Lambda\Lambda} := C_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda.$$

In addition, the Ricci tensor $\text{Ric}_{\Lambda\Lambda\Lambda\Lambda} := (-)^{p+q} R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda$ is given by

$$\text{Ric}_{\Lambda\Lambda\Lambda\Lambda} = -(2 - N) R_{\Lambda\Lambda\Lambda\Lambda} + (6 - N) \Lambda_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda,$$  \hspace{1cm} (2.50)

where $\text{Ric}_{\Lambda\Lambda\Lambda\Lambda} := (-)^{p+q} \text{Ric}_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda$.

**Proof.** The proof is based on Bianchi identities and certain index symmetries of the curvature tensor. However, the calculations are rather technical and lengthy, and therefore postponed to the appendix.

In the following, we shall refer to the quantity $\Lambda_{\Lambda\Lambda\Lambda\Lambda}$ as the **cosmological constant**.

### 2.3. Self-dual supergravity equations

#### 2.3.1. Self-duality

Let $\mathcal{M}$ be a complex quaternionic RC supermanifold which is equipped with the Levi-Civita connection. It is called **self-dual Einstein** if $C_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda = 0$ and simultaneously $R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda = 0$.

**Definition 2.4.** A complex quaternionic RC supermanifold is said to be complex quaternionic Kähler if it is equipped with the Levi-Civita connection and is also self-dual Einstein.

If, in addition, the cosmological constant $\Lambda_{\Lambda\Lambda\Lambda\Lambda}$ vanishes as well, we call $\mathcal{M}$ **self-dual**. In the latter case, the curvature $R$ is of the form $R = R^\mathcal{F}_{\mathcal{F}'} \otimes \text{id} \mathcal{F}$, i.e.

$$[D_{\Lambda\Lambda\Lambda\Lambda} R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda = \delta_{\Lambda\Lambda\Lambda\Lambda} \delta A B,$$  \hspace{1cm} (2.51)

where $R_{\Lambda\Lambda\Lambda\Lambda}$ is of the form $R_{\Lambda\Lambda\Lambda\Lambda} = C_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda M_D^C$. Furthermore, the connection $D$ has components

$$D_{\Lambda\Lambda\Lambda\Lambda} = E_{\Lambda\Lambda\Lambda\Lambda} M_D^\Lambda \delta_{\Lambda\Lambda\Lambda\Lambda} + \Omega_{\Lambda\Lambda\Lambda\Lambda} M_D^C.$$

(2.52)

Obviously, this says that $D$ on $\mathcal{F}$ of $T_{\mathcal{M}} \cong \mathcal{M} \otimes \mathcal{F}$ is flat. It should be noted that the superfield components of $R_{\Lambda\Lambda\Lambda\Lambda}$ are not independent of each other because of the Bianchi identities

$$D_{\Lambda\Lambda\Lambda\Lambda} R_{\Lambda\Lambda\Lambda\Lambda} \delta^\Lambda = 0.$$  \hspace{1cm} (2.53)
The field equations of self-dual supergravity with vanishing cosmological constant then follow from these identities together with \((2.51)\). Their explicit form can be found in Siegel \([47]\). We may summarize by giving the following definition:

**Definition 2.5.** A complex quaternionic Kähler RC supermanifold is called a complex hyper-Kähler RC supermanifold if the Levi-Civita connection on \(\tilde{\mathcal{F}}\) is flat.

In addition, proposition 2.6 shows that if \(\mathcal{M}\) is self-dual, it is also Ricci-flat. Altogether, a complex hyper-Kähler RC supermanifold \(\mathcal{M}\) is Ricci-flat and has trivial Berezinian sheaf \(\text{Ber} (\mathcal{M})\), i.e., it is a Calabi–Yau supermanifold. In this respect, it is worth mentioning that contrary to ordinary complex manifolds, complex supermanifolds with trivial Berezinian sheaf do not automatically admit Ricci-flat metrics (see, e.g., \([10]\)). We shall refer to this latter type of supermanifolds as formal Calabi–Yau supermanifolds. Furthermore, for an earlier account of hyper-Kähler supermanifolds of dimension \((4k|2k+2)\), though in a slightly different setting, see Merkulov \([55]\). See also Lindström et al \([14]\).

2.3.2. Second Plebanski equation. By analysing the constraint equations \((2.51)\) in a noncovariant gauge called the light-cone gauge, Siegel \([47]\) achieved reducing them to a single equation on a superfield \(\Theta\), which in fact is the supersymmetrized analogue of Plebanski’s second equation \([61]\). In particular, in this gauge the vielbeins turn out to be

\[
E_A^{\dot{\alpha} \dot{M}} = \delta_A^{\dot{M}} \delta_{\dot{\alpha} \dot{\beta}} + \frac{1}{2} (-)^{\rho \sigma} \delta_A^{\dot{N}} \delta_B^{\dot{O}} \partial_{\dot{O}} (\partial_{\dot{N}} \partial_{\dot{\alpha}} (\Theta)) \omega^{BC} \delta_C^{\dot{M}} \delta_{\dot{\beta}},
\]

\[
E_A^{\dot{\alpha} \bar{\dot{M}}} = \delta_A^{\bar{\dot{M}}} \delta_{\dot{\alpha} \dot{\beta}},
\]

where \((\partial_{\dot{M}}) = (\partial_{\dot{m}}, \partial_{\dot{\alpha}})\) with \(\partial_{\dot{\alpha}} := \partial / \partial x^{\dot{\alpha}}\) and \(\partial_{\dot{m}} := \partial / \partial \eta^{\dot{m}}\) and \(\omega^{AB} := (\epsilon^{\dot{\alpha} \dot{\beta}}, \partial^{ij})\). By a slight abuse of notation, we shall write \(\partial_{\dot{A}} \equiv \partial_A^{\dot{M}} \partial_{\dot{M}}\tilde{\Theta}\) in the following. Furthermore, the components of the connection 1-form in this gauge are given by

\[
\Omega_{\dot{A} \dot{B}}^C = -(-)^{\rho \sigma} \frac{1}{2} \partial_{\dot{A}} \partial_{\dot{B}} \partial_{\dot{C}} (\Theta) \omega^{DC},
\]

\[
\Omega_{\dot{A} \bar{\dot{B}}}^C = 0.
\]

The equation \(\Theta\) is being subject to is then

\[
\epsilon^{\dot{\alpha} \dot{\beta}} \partial_{\dot{A}} \partial_{\dot{B}} \partial_{\dot{C}} (\Theta) + \frac{1}{2} (-)^{\rho \sigma} (\partial_{\dot{A}} \partial_{\dot{C}} (\Theta)) \omega^{CD} (\partial_{\dot{D}} \partial_{\dot{B}} (\Theta)) = 0.
\]

In summary, the field equations of self-dual supergravity in light-cone gauge are equivalent to \((2.56)\).

2.3.3. Another formulation. The subject of this paragraph is to give another (equivalent) formulation of the self-dual supergravity equations with vanishing cosmological constant. The following proposition generalizes results of Mason and Newman \([62]\) to the supersymmetric situation.

**Proposition 2.7.** Let \(\mathcal{M}\) be a complex quaternionic RC supermanifold which is equipped with the Levi-Civita connection. Suppose further we are given vector fields \(V_{\dot{A}}\) on \(\mathcal{M}\) which obey

\[
[V_{\dot{A} \dot{\alpha}}, V_{\bar{\dot{B}}}] = 0.
\]

Then \((2.57)\) is an equivalent formulation of the self-dual supergravity equations, i.e., given vector fields \(V_{\dot{A}}\) on \(\mathcal{M}\) satisfying \((2.57)\), it is always possible to find frame fields \(E_A\) such that the self-dual supergravity equations with zero cosmological constant are satisfied thus making \(\mathcal{M}\) into a complex hyper-Kähler RC supermanifold. Conversely, given a complex hyper-Kähler RC supermanifold \(\mathcal{M}\), then there will always exist vector fields \(V_{\dot{A}}\) on \(\mathcal{M}\) which satisfy \((2.57)\).
Proof. In fact, it is not too difficult to see that (2.57) implies the self-dual supergravity equations with \( \Lambda_{AB} = 0 \). Indeed, by Frobenius’ theorem (see, e.g., Manin [51] for the case of supermanifolds) we may choose coordinates such that \( V^A_1 \) becomes coordinate derivatives, i.e.,

\[
V^A_1 = \partial_A. 
\]

In addition, by choosing a gauge such that \( V^A_1 \) take the form

\[
V^A_1 = \partial_A + \frac{1}{2}(-)^{p_A} (\partial_A \partial_B \Theta) \omega^C \partial_C, 
\]

where \( \Theta \) is some to-be-determined superfield, all equations (2.57) but one are identically satisfied. In particular, only \( [V^A_1, V^B_1] = 0 \) gives a nontrivial condition on \( \Theta \). In fact, this equation reduces to (2.56). Therefore, taking the vielbeins and the components of the connection 1-form as in (2.54) and (2.55), respectively, we arrive at the desired result.

Conversely, given some complex hyper-Kähler RC supermanifold \( \mathcal{M} \), the only nonvanishing components of the connection 1-form are \( \Omega^A_{\hat{a}\hat{b}} \). By virtue of the vanishing of the torsion, equations (2.31) imply

\[
f^C_{AB} = \Omega^C_{AB} - (-)^{p_A p_B} \Omega^C_{BA}. 
\]

Since \( \Omega^C_{AB} = \Omega^C_{\hat{a}\hat{b} \hat{c}} \delta^\hat{c}_C \), we find

\[
f^C_{A(\hat{a}\hat{b})} \delta^\hat{c}_C = \delta^\hat{c}_C \Omega^C_{A(\hat{a}\hat{b})}. 
\]

By the discussion given in the last paragraph of section 2, we know that there exists a gauge in which \( \Omega^C_{(\hat{a}\hat{b})} \) vanishes. Therefore, there will always exist vector fields \( V^a_\alpha \) obeying (2.57). This concludes the proof. □

3. Twistor theory

Above we have introduced and discussed complex quaternionic Kähler and hyper-Kähler RC supermanifolds by starting from complex quaternionic RC supermanifolds. In this section, we shall be concerned with their twistorial description. We first construct the supertwistor space, denoted by \( \mathcal{P} \), of a complex quaternionic RC supermanifold \( \mathcal{M} \). However, as in the purely bosonic situation, we shall see that this will only work if one makes certain additional assumptions about the properties of \( \mathcal{M} \). Having presented this construction, we then show which additional structures on \( \mathcal{P} \) are needed to render \( \mathcal{M} \) into a complex hyper-Kähler RC supermanifold. We further give an alternative formulation and eventually conclude this section by introducing the bundle of local supertwistors.

3.1. Supertwistor space \((\mathcal{N} \neq 4)\)

3.1.1. Conic structures. In order to proceed in finding an appropriate twistor description, so-called conic structures appear to be an adequate tool. Let us therefore recall their definition.

**Definition 3.1** (Manin [51]). Let \( \mathcal{M} \) be a complex supermanifold with holomorphic tangent bundle \( T\mathcal{M} \). A \((p|q)\)-conic structure on \( \mathcal{M} \) is a closed subsupermanifold \( \mathcal{F} \) in the relative Grassmannian \( G_{\mathcal{M}}(p|q; T\mathcal{M}) \),

\[
G_{\mathcal{M}}(p|q; T\mathcal{M}) := \{ \text{rank } p|q \text{ local direct summands of } T\mathcal{M} \},
\]

such that the projection \( \pi : \mathcal{F} \to \mathcal{M} \) is a submersion.

Putting it differently, at any point \( x \in \mathcal{M} \) such an \( \mathcal{F} \) determines a set of \((p|q)\)-dimensional tangent spaces in the fibre of \( T\mathcal{M} \) over \( x \) corresponding to the points \( \pi^{-1}(x) \subset G_{\mathcal{M}}(p|q; T_x\mathcal{M}) \).
3.1.2. \( \beta \)-plane bundle. Having given this definition, we may now introduce a canonical conic structure on a complex quaternionic RC supermanifold \( \mathcal{M} \). Recall again from (2.9) that the tangent bundle \( T\mathcal{M} \) of \( \mathcal{M} \) is of the form
\[
0 \rightarrow \mathcal{E} \otimes \tilde{T} \rightarrow T\mathcal{M} \rightarrow \mathcal{F} \otimes \tilde{T} \rightarrow 0,
\]
where \( \mathcal{F} \) and \( \tilde{T} \) are both of rank \( 2|0 \) and \( \mathcal{E} \) is of rank \( 0|\mathcal{N} \).

Now let \( \mathcal{F} \) be the relative projective line bundle \( P_{\mathcal{M}}(\tilde{T}) \) on \( \mathcal{M} \). Then the above sequence induces a canonical \( (2|\mathcal{N}) \)-conic structure on \( \mathcal{M} \), that is, an embedding \( \mathcal{F} \hookrightarrow G_{\mathcal{M}}(2|\mathcal{N}; T\mathcal{M}) \). In local coordinates, it is given by
\[
\mathcal{F} \rightarrow G_{\mathcal{M}}(2|\mathcal{N}; T\mathcal{M}),
\]
\[
[\lambda_{\alpha}] \mapsto \mathcal{F} := \{ \lambda^{\hat{\alpha}} E_{A\hat{\alpha}} \}.
\]

Here, \([\lambda_{\alpha}]\) are homogeneous fibre coordinates of \( \pi_{1} : \mathcal{F} \rightarrow \mathcal{M} \) and \( E_{A\hat{\alpha}} \)'s are the frame fields on \( \mathcal{M} \). This construction leads naturally to the following definition:

**Definition 3.2.** A \( \beta \)-surface \( \Sigma \) in a complex quaternionic RC supermanifold \( \mathcal{M} \) is a complex sub(super)manifold of dimension \( (2|\mathcal{N}) \) with the property that at each point \( x \in \Sigma \), the tangent space \( T_{x} \Sigma \) is spanned by vectors of the form \( (3.1) \), where \( \lambda_{\alpha} \) is fixed up to rescalings.

This in particular means that the components of a tangent vector on \( \Sigma \) are always of the form \( \mu^{\hat{\lambda}} \lambda^{\alpha} \), where \( \mu^{\hat{\lambda}} \) is arbitrary.

It is worth noting that on \( \mathcal{M}_{\text{red}} \) this notion of \( \beta \)-surfaces reduces to the standard one (see, e.g., [63, 64]). Next we introduce the notion of right-flatness.

**Definition 3.3.** A complex quaternionic RC supermanifold \( \mathcal{M} \) is said to be right-flat, if the \( R_{A\beta B\hat{\alpha} \delta} \)-components of the curvature tensor vanish.

Clearly, for \( \mathcal{N} = 0 \) this reduces to the standard definition of the vanishing of the anti-self-dual part of the Weyl tensor. Now we are in the position to prove the following proposition:

**Proposition 3.1.** Let \( \mathcal{M} \) be a complex quaternionic RC supermanifold. For any given point of \( \mathcal{F} = P_{\mathcal{M}}(\tilde{T}) \) there exists a corresponding \( \beta \)-surface \( \Sigma \) in \( \mathcal{M} \) if and only if \( \mathcal{M} \) is right-flat.

Putting it differently, the distribution defined by (3.1) is integrable, i.e. closed under the graded Lie bracket, if and only if \( \mathcal{M} \) is right-flat.

**Proof.** It is not too difficult to show that the graded Lie bracket of two vector fields \( E_{A} := \lambda^{\hat{\alpha}} E_{A\hat{\alpha}} \) and \( E_{B} := \lambda^{\hat{\beta}} E_{B\hat{\beta}} \) is given by
\[
[E_{A}, E_{B}] = 2\left( \lambda^{\hat{\alpha}} \nabla_{A\hat{\alpha}} \lambda^{\hat{\beta}} \right) E_{B\hat{\beta}} - \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} T_{A\beta B\hat{\alpha}} \epsilon^{\gamma} E_{C\gamma} - \lambda^{\hat{\alpha}} \Omega_{A\beta B} \epsilon^{\gamma} E_{C\gamma}.
\]

However, by virtue of the vanishing of the torsion, the second term on the right-hand side of this equation vanishes identically. Hence, the distribution generated by \( E_{A} \) is integrable if and only if
\[
\lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} \nabla_{A\hat{\alpha}} \lambda_{\hat{\beta}} = 0.
\]

Since the integrability condition of this equation is equivalent to the vanishing of the curvature components \( R_{A\beta B\hat{\alpha} \delta} \), we arrive at the desired result.

A remark is in order: if \( \mathcal{M} \) was not complex quaternionic but only equipped with a connection whose torsion is totally trace free (cf our discussion given in section 2.2.4), then the above requirement of the integrability of the distribution \( \mathcal{D} \) would enforce the vanishing of \( T_{A\beta B\hat{\alpha}} \epsilon^{\gamma} \) as the first term appearing in the equation for \([E_{A}, E_{B}]\) is proportional to the trace.
Hence, the whole torsion tensor $T_{\alpha\beta\delta}^{\gamma}$ would be zero (see equation (2.37)). By virtue of equation (3.2), $M$ would then become a right-flat complex quaternionic RC supermanifold.

Note that $\lambda_{\delta}$ can be normalized such that equation (3.2) becomes

$$\lambda^\delta \nabla_{\lambda^\delta} \lambda^\beta = 0,$$

i.e. $\lambda_{\delta}$ is covariantly constant (i.e. $\lambda_{\delta}$ is an auto-parallel cotangent spinor) on $\Sigma \hookrightarrow M$. In addition, we point out that this equation is scale invariant. This follows from the transformation laws (2.39) of the connection under rescalings. For $N = 4$, this equation is not scale invariant as $\nabla$ is not unique. Therefore, its solutions may depend on the chosen scale on $M$. We shall address this issue in more detail later on.

Following the terminology of Mason and Woodhouse [63], we shall call $F$ the $\beta$-plane bundle. We also refer to $F$ as the correspondence space.

3.1.3. Supertwistor space. Note that $\beta$-surfaces $\Sigma$ lift into $F$ and in addition also foliate $\mathcal{F}$. The lift $\tilde{\Sigma}$ of $\Sigma$ is a section of $F|_{\Sigma} \hookrightarrow \Sigma$ satisfying equations (3.2). The tangent vector fields on $\tilde{\Sigma}$ are then given by

$$\tilde{E}_A = E_A + \lambda^\delta \lambda^\gamma \Omega_{\lambda^\delta \lambda^\gamma} \frac{\partial}{\partial \lambda^\beta}.$$ (3.4)

Therefore, we canonically obtain an integrable rank-2$|N$ distribution $\mathcal{D}_F \subset T\mathcal{F}$ on the correspondence space generated by $\tilde{E}_A$, i.e. $\mathcal{D}_F = \langle \tilde{E}_A \rangle$. We shall refer to $\mathcal{D}_F$ as the twistor distribution. After quotienting $\mathcal{F}$ by the twistor distribution, we end up with the following double fibration:

$$\pi_2 \quad \pi_1$$

Here, $\mathcal{P}$ is a $(3|N)$-dimensional complex supermanifold which we call the supertwistor space of $M$. Note that this construction is well defined if we additionally assume that $M$ is civilized, that is, $\mathcal{P}$ is assumed to have the same topology as the supertwistor space associated with any convex region in flat superspace $C^{4|2N}$. Otherwise, one may end up with non-Hausdorff spaces, see e.g. Ward and Wells [64] and Mason and Woodhouse [63] for a discussion in the purely bosonic situation. Moreover, without this convexity assumption, the Penrose transform, which relates certain cohomology groups on $\mathcal{P}$ to solutions to certain partial differential equations on $M$, will not be an isomorphism (see also section 3.2).

By virtue of this double fibration, we have a geometric correspondence between the two supermanifolds $\mathcal{M}$ and $\mathcal{P}$. In particular, any point $x \in \mathcal{M}$ is associated with the set $\pi_2(\pi_1^{-1}(x))$ in $\mathcal{P}$ consisting of all $\beta$-surfaces being incident with $x$. Conversely, any point $z$ in supertwistor space $\mathcal{P}$ corresponds to a $\beta$-surface $\pi_1(\pi_2^{-1}(z))$ in $\mathcal{M}$. As $\mathcal{F} \rightarrow M$ is a $\mathbb{P}^1$-bundle over $M$, the submanifolds $\pi_2(\pi_1^{-1}(x))$ are biholomorphically equivalent to $\mathbb{P}^1$ and are parametrized by $x \in M$.

We may now state the following basic result:

**Theorem 3.1.** There is a one-to-one correspondence between

(i) civilized right-flat complex quaternionic RC supermanifolds $\mathcal{M}$ of dimension $(4|2N)$ and
(ii) \((3|N)\)-dimensional complex supermanifolds \(\mathcal{P}\) each containing a family of rational curves biholomorphically equivalent to \(\mathbb{P}^1\) and with normal bundle \(\mathcal{N}_{\mathbb{P}^1|\mathcal{P}}\) inside \(\mathcal{P}\) described by

\[
0 \longrightarrow \Pi \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{C}^N \longrightarrow \mathcal{N}_{\mathbb{P}^1|\mathcal{P}} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{C}^2 \longrightarrow 0,
\]

where \(\mathcal{O}_{\mathbb{P}^1}(1)\) is the dual tautological \((c_1 = 1)\) bundle on \(\mathbb{P}^1\) and \(\Pi\) is the Graßmann parity changing functor.

**Proof.** Let us first show \((i) \rightarrow (ii)\): in fact, we have already seen that for any complex quaternionic RC supermanifold \(\mathcal{M}\) with the above properties, there always exists an associated \((3|N)\)-dimensional complex supermanifold \(\mathcal{P}\) containing holomorphically embedded projective lines \(\pi_2(\pi_1^{-1}(x)) \cong \mathbb{P}^1\) for \(x \in \mathcal{M}\). It remains to verify that each of it has a normal bundle \(\mathcal{N}_{\mathbb{P}^1|\mathcal{P}}\) of the above type. To show this, we note that \(\mathcal{N}_{\mathbb{P}^1|\mathcal{P}}\) is described by the exact sequence

\[
0 \longrightarrow \mathbb{D}_\mathbb{P} \longrightarrow \pi_1^*T\mathcal{M} \longrightarrow \pi_2^*\mathcal{N}_{\mathbb{P}^1|\mathcal{P}} \longrightarrow 0,
\]

where \(\mathbb{D}_\mathbb{P}\) is the twistor distribution. Clearly, the distribution \(\mathbb{D}_\mathbb{P}\) is described by

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{C}^2 \longrightarrow \mathbb{D}_\mathbb{P} \longrightarrow \Pi \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{C}^N \longrightarrow 0
\]

when restricted to the fibres \(\pi_2^{-1}(x)\) of \(\mathcal{F} \to \mathcal{M}\). Furthermore, \(\pi_1^*T\mathcal{M}\) trivial is trivial when restricted to \(\pi_1^{-1}(x)\). Therefore, the maps of the above sequence are explicitly given by

\[
0 \longrightarrow \mathbb{D}_\mathbb{P} \longrightarrow \pi_1^*T\mathcal{M} \longrightarrow \pi_2^*\mathcal{N}_{\mathbb{P}^1|\mathcal{P}} \longrightarrow 0,
\]

\[
\mu^A \mapsto \mu^A\lambda^\delta, \quad \mu^A \mapsto u^{A\lambda}\delta,
\]

which completes the proof of the direction \((i) \rightarrow (ii)\).

To show the reverse direction \((ii) \rightarrow (i)\), one simply applies a supersymmetric version of Kodaira’s theorem of deformation theory [65]. First, one notices that the obstruction group \(H^1(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1|\mathcal{P}})\) vanishes which follows from the sequence (3.6) and its induced long exact cohomology sequence:

\[
0 \longrightarrow H^0(\mathbb{P}^1, \Pi \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{C}^N) \longrightarrow H^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1|\mathcal{P}}) \longrightarrow
\]

\[
\longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{C}^2) \longrightarrow H^1(\mathbb{P}^1, \Pi \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{C}^N) \longrightarrow
\]

\[
H^1(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1|\mathcal{P}}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{C}^2) \longrightarrow 0.
\]

Then there exists a dimension \(\dim H^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1|\mathcal{P}}) = 4|0 + 0|2N = 4|2N\) parameter family \(\mathcal{M}\) of deformations of \(\mathbb{P}^1\) inside \(\mathbb{P}\).

If we let \(\mathcal{F} := \{(z, \pi_2(\pi_1^{-1}(x)))| z \in \pi_2(\pi_1^{-1}(x)), z \in \mathcal{P}, x \in \mathcal{M}\} \subset \mathcal{P} \times \mathcal{M}\), then \(\mathcal{F}\) is a fibration over \(\mathcal{M}\). The typical fibres of \(\mathcal{F} \to \mathcal{M}\) are complex projective lines \(\mathbb{P}^1\). Hence, we obtain a double fibration

\[
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathcal{M} \\
\| & \| & \| \\
\mathcal{P} & \longrightarrow & \mathcal{M}
\end{array}
\]

where the fibres of \(\mathcal{F} \to \mathcal{P}\) are \((2|N)\)-dimensional complex subsupermanifolds of \(\mathcal{M}\).

Let \(T\mathcal{F}/\mathcal{P}\) be the relative tangent sheaf on \(\mathcal{F}\) given by

\[
0 \longrightarrow T\mathcal{F}/\mathcal{P} \longrightarrow T\mathcal{F} \longrightarrow \pi_2^*T\mathcal{P} \longrightarrow 0.
\]

Then (see above) we define a vector bundle \(\mathcal{N}\) on \(\mathcal{F}\) by

\[
0 \longrightarrow T\mathcal{F}/\mathcal{P} \longrightarrow \pi_1^*T\mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0.
\]

(3.8)
Clearly, the rank of $\mathcal{N}$ is $2|\mathcal{N}|$, and furthermore the restriction of $\mathcal{N}$ to the fibre $\pi_1^{-1}(x)$ of $\mathcal{F} \to \mathcal{M}$ for $x \in \mathcal{M}$ is isomorphic to the pull-back of the normal bundle of the curve $\pi_2(\pi_1^{-1}(x)) \to \mathcal{P}$. Hence, $\mathcal{N}$ may be identified with $\pi_2^* \mathcal{N}_{\mathcal{P}/\mathcal{P}}$ and moreover the relative tangent sheaf $T\mathcal{F}/\mathcal{P}$ with the twistor distribution $\mathcal{D}_\mathcal{F}$.

In addition, the bundle $\pi_1 : \mathcal{F} \to \mathcal{M}$ is of the form $P_{\mathcal{H}}(\mathcal{F}')$ for some rank $2|\mathcal{N}|$ vector bundle $\mathcal{F}'$ (determined below) over $\mathcal{M}$. Then we denote by $\mathcal{O}_{\mathcal{F}}(-1)$ the tautological $(c_1 = -1)$ bundle on $\mathcal{F}$. It then follows from the above that the direct images$^6$ $\pi_1^*(\Omega^1 \mathcal{F}/\mathcal{P} \otimes \mathcal{O}_{\mathcal{F}}(-2))$ and $\pi_1^*(\Omega^1 \mathcal{F}/\mathcal{P} \otimes \mathcal{O}_{\mathcal{F}}(-2))$ vanish. Therefore, we find that

$$\pi_1^*(T\mathcal{F}/\mathcal{P}) = 0 = \pi_1^*(T\mathcal{F}/\mathcal{P})$$

upon application of Serre duality$^7$.

Applying the direct image functor to the sequence (3.8), we obtain

$$0 \longrightarrow \pi_1^*(T\mathcal{F}/\mathcal{P}) \longrightarrow \pi_1^*(\pi^* T\mathcal{M}) \longrightarrow \pi_1^* \mathcal{N} \longrightarrow \pi_1^*(T\mathcal{F}/\mathcal{P}),$$

and hence

$$T\mathcal{M} \cong \pi_1^* \mathcal{N} \cong \pi_1^*(\pi_2^* \mathcal{N}_{\mathcal{P}/\mathcal{P}}).$$

Thus, the sequence (3.6) yields

$$0 \longrightarrow \pi_1^*(\pi_2^* (\Pi_\mathcal{P} \mathcal{O}_\mathcal{P} (1) \otimes \mathcal{N}^2)) \longrightarrow T\mathcal{M} \longrightarrow \pi_1^*(\Pi_\mathcal{P} \mathcal{O}_\mathcal{P} (1) \otimes \mathcal{N}^2)) \longrightarrow 0$$

since the first direct image sheaf $\pi_1^*(\pi_2^* (\Pi_\mathcal{P} \mathcal{O}_\mathcal{P} (1) \otimes \mathcal{N}^2))$ vanishes (see above). Above, we have introduced

$$\mathcal{F} := \pi_1^*(\Pi_\mathcal{P} \mathcal{O}_\mathcal{P} (1) \otimes \mathcal{C}^2)$$

$$\mathcal{F}' := \pi_1^*(\Pi_\mathcal{P} \mathcal{O}_\mathcal{P} (1))$$

$$\mathcal{E} := \pi_1^*(\Pi_\mathcal{P} \mathcal{O}_\mathcal{P} (1) \otimes \mathcal{N}^2).$$

Note that by construction, the bundle $\mathcal{F} \cong P_{\mathcal{H}}(\mathcal{F}')$ is an integrable $(2|\mathcal{N})$-conic structure on $\mathcal{M}$.

**3.1.4. Gindikin’s 2-forms and self-dual supergravity with $\Lambda_{AB} = 0$.** In this and the subsequent paragraph, we shall determine the structure on the supertwistor space $\mathcal{P}$ corresponding to a hyper-Kähler structure on $\mathcal{M}$. In view of that, recall that there always exists a scale where the Levi-Civita connection $\nabla$ coincides with the connection $\nabla$.

Let $E^{\hat{A}\hat{B}}$ be the coframe fields on some complex quaternionic Kähler supermanifold $\mathcal{M}$. On the correspondence space $\mathcal{F}$ of the double fibration (3.5), we may introduce a differential 2-form $\Sigma(\lambda)$ by setting

$$\Sigma(\lambda) := E^{\hat{B}\hat{D}} \wedge E^{\hat{A}\hat{B}} e_{\hat{A}\hat{B}} \lambda_{\hat{D}} \lambda_{\hat{B}},$$

where $e \in H^0(\mathcal{M}, \Lambda^2 \mathcal{F}')$ is assumed to be non-degenerate and to obey $\nabla e = 0$. Furthermore, let $d_{\hat{B}}$ be the exterior derivative on $\mathcal{F}$ holding $\lambda_{\hat{D}}$ constant.

---

$^6$ Given a mapping $\pi : \mathcal{M} \to \mathcal{N}$ of two complex supermanifolds $\mathcal{M}$ and $\mathcal{N}$, the $q$th direct image sheaf $\pi^*_q \mathcal{E}$ of a locally free sheaf $\mathcal{E}$ over $\mathcal{M}$ is defined by the presheaf $\mathcal{F} \supset \mathcal{U} \mapsto H^q(\pi^* \mathcal{U}, \mathcal{E})$ with the obvious restriction maps. The zeroth direct image sheaf $\pi^0 \mathcal{E}$ is usually denoted by $\pi_* \mathcal{E}$.

$^7$ Recall that Serre duality asserts that for any locally free sheaf $\mathcal{E}$ on a compact complex manifold $M$ of dimension $d$ we have an isomorphism $H^d(M, \mathcal{E}) \cong H^{d-q}(M, \mathcal{E}^* \otimes \mathcal{O}_M)$, where $\mathcal{E}^*$ is the canonical sheaf on $M$. 

Proposition 3.2. There is a one-to-one correspondence between gauge equivalence classes of solutions to the self-dual supergravity equations (2.51) with vanishing cosmological constant on $\mathcal{M}$ and equivalence classes of (global) $\mathfrak{d}_h$-closed non-degenerate differential 2-forms $\Sigma(\lambda)$ of the form (3.9) on the correspondence space $\mathcal{F}$.

Proof. First, let us define a differential 2-form $\Sigma^{AB}(\lambda)$ by setting

$$\Sigma^{AB}(\lambda) := \lambda_\beta\lambda_\delta E^{A\beta}\wedge E^{B\delta}. $$

It then follows that $d_\beta \Sigma^{AB}$ is given by

$$d_\beta \Sigma^{AB} = -2\lambda_\delta\lambda_\beta E^{A\delta}\wedge dE^{B\beta},$$

where $d$ is the exterior derivative on $\mathcal{M}$. Assuming the vanishing of the torsion and upon substituting equations (2.30) into this equation, we see that the connection 1-form on $\mathcal{M}$ will be of the form $\Omega_A^B = \Omega_{\alpha A}^B$ if and only if

$$d_\beta \Sigma^{AB} = -2\Sigma^{AC} \wedge \Omega_{\beta C}^B. $$

Therefore,

$$d_\beta \Sigma = d_\beta (\Sigma^{AB} e_{BA}) = 0,$$

since $d e_{AB} - 2\Omega_{(A|C} e_{C|B)} = 0$. \hfill $\Box$

The differential 2-form $\Sigma(\lambda)$ satisfying the properties stated in the immediately preceding proposition is a supersymmetric extension of the Gindikin 2-form [66] (see also [61]). Note that the twistor distribution $\mathcal{D} = (\tilde{E}_A)$ annihilates $\Sigma(\lambda)$, i.e. $\Sigma(\lambda)$ descends to $\mathcal{P}$ (also $d \Sigma(\lambda)$ is annihilated by $\mathcal{D}$).

3.1.5. Supertwistor space for complex hyper-Kähler RC supermanifolds. The question which now arises is how the Gindikin 2-form can be obtained from certain data given on the supertwistor space $\mathcal{P}$. In the following, we generalize the results known from the purely bosonic situation (see Penrose [21] and also Alekseevsky and Graev [67]).

Let us assume that the supertwistor space $\mathcal{P}$ is a holomorphic fibre bundle $\pi : \mathcal{P} \to \mathbb{P}^1$ over the Riemann sphere $\mathbb{P}^1$. Later on, in section 3.2.4 we shall see that this condition arises quite naturally. Furthermore, let us consider the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1 \hookrightarrow \mathcal{P}$ together with its pull-back $\pi^* \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{P}$.

In addition, let $\Omega^1 \mathcal{P}/\mathbb{P}^1$ be the sheaf of relative differential 1-forms on $\mathcal{P}$ described by

$$0 \longrightarrow \pi^* \Omega^1 \mathbb{P}^1 \longrightarrow \Omega^1 \mathcal{P} \longrightarrow \Omega^1 \mathcal{P}/\mathbb{P}^1 \longrightarrow 0. \quad (3.10)$$

According to Alekseevsky and Graev [67], we give the following definition (already adopted in our situation):

Definition 3.4. A section $\omega \in H^0(\mathcal{P}, \Lambda^2(\Omega^1 \mathcal{P}/\mathbb{P}^1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(2))$ is called a holomorphic relative symplectic structure of type $\mathcal{O}_{\mathbb{P}^1}(2)$ on $\mathcal{P}$ if it is closed and non-degenerate on the fibres. The integer $\deg(\mathcal{O}_{\mathbb{P}^1}(2)) = c_1(\mathcal{O}_{\mathbb{P}^1}(2)) = 2$ is called the weight of $\omega$.\footnote{Recall that $\mathcal{O}_{\mathbb{P}^1}(n) := \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes n}$.}

Then Gindikin’s 2-form $\Sigma(\lambda)$ on $\mathcal{F}$ can be obtained by pulling back the relative symplectic structure $\omega$ on $\mathcal{P}$ to the correspondence space $\mathcal{F}$ (and by dividing it by a constant section of $\pi^* \mathcal{O}_{\mathbb{P}^1}(2)$).

\footnote{Note that for $N = 0$ a relative differential 2-form is automatically relatively closed as in this case the fibres of $\pi$ are two dimensional.}
Altogether, we may now summarize all the findings from above by stating the following theorem:

**Theorem 3.2.** There is a one-to-one correspondence between civilized RC supermanifolds \( \mathcal{M} \) of dimension \((4|2N)\) which are equipped with a hyper-Kähler structure and complex supermanifolds \( \mathcal{P} \) of dimension \((3|N)\) such that

(i) \( \mathcal{P} \) is a holomorphic fibre bundle \( \pi : \mathcal{P} \to \mathbb{P}^1 \) over \( \mathbb{P}^1 \),

(ii) \( \mathcal{P} \) is equipped with a \((4|2N)\)-parameter family of sections of \( \pi \), each with normal bundle given by (3.6) and

(iii) there exists a holomorphic relative symplectic structure \( \omega \) of weight 2 on \( \mathcal{P} \).

### 3.2. Equivalent formulation (\( N \neq 4 \))

The purpose of this section is to provide an alternative formulation of our above considerations. In this way, we will also be able to describe the case with nonzero cosmological constant. Here, we are generalizing some of the results of Ward [24], LeBrun [56, 68], Bailey and Eastwood [60], and Merkulov [53, 54].

Let \( \mathcal{M} \) be a civilized right-flat complex quaternionic RC supermanifold with connection \( \nabla \). Let further \( \mathcal{P} \) be its associated supertwistor space. There exist several natural vector bundles on \( \mathcal{P} \) which encode information about the supermanifold \( \mathcal{M} \) and about \( \mathcal{P} \) itself, respectively. In the following, we shall be using the notation

\[
\mathcal{E}[m] := \mathcal{E} \otimes (\text{Ber} \tilde{S})^{-m} \cong \mathcal{E} \otimes (\Lambda^2 \tilde{\mathcal{F}})^{-m}
\]  

for any locally free sheaf \( \mathcal{E} \) on \( \mathcal{M} \).

#### 3.2.1. Universal line bundle.

Let us begin by recalling equation (3.3). In fact, this equation implies the existence of a natural holomorphic line bundle \( \mathcal{L} \to \mathcal{P} \) over \( \mathcal{P} \). Following LeBrun’s terminology [56], we shall refer to \( \mathcal{L} \) as the universal line bundle. It is defined as follows. Let \( \mathcal{O}_F(-1) \) be again the tautological bundle on \( F \). Furthermore, denote by \( \Omega^1 F / \mathcal{P} \) the sheaf of relative differential 1-forms on \( F \) described by the sequence

\[
0 \longrightarrow \pi_1^* \Omega^1 \mathcal{P} \longrightarrow \Omega^1 F \longrightarrow \Omega^1 F / \mathcal{P} \longrightarrow 0.
\]  

Then we may define the composition

\[
\nabla_{T_F / \mathcal{P}} : \mathcal{O}_F(-1) \overset{\pi_1^*}{\longrightarrow} \mathcal{O}_F(-1) \otimes \pi_1^* \Omega^1 \mathcal{M} \overset{\text{id} \otimes \text{res}}{\longrightarrow} \mathcal{O}_F(-1) \otimes \Omega^1 F / \mathcal{P},
\]  

where res denotes the restriction of differential 1-forms on \( F \) onto the fibres of the projection \( \pi_2 : F \to \mathcal{M} \). The universal line bundle \( \mathcal{L} \) is then defined by the zeroth direct image

\[
\mathcal{L} := \pi_2_* (\ker \nabla_{T_F / \mathcal{P}}).
\]  

Hence, the fibre of \( \mathcal{L} \) over a point \( z \in \mathcal{P} \) is the space of solutions to \( \lambda^\beta \nabla_{A_\beta} \lambda_\beta = 0 \) on the \( \beta \)-surface \( \pi_1(\pi^{-1}_1(z)) \). Note that \( \mathcal{L} \) restricted to \( \pi_2(\pi_1^{-1}(x)) \leftrightarrow \mathcal{P} \), for \( x \in \mathcal{M} \), can be identified with \( \mathcal{O}_\mathbb{P}^1(-1) \).
3.2.2. Jacobi bundle. The second bundle over the supertwistor space we are interested in is the so-called Jacobi bundle (see also [54, 56]). Let us denote it by $\mathcal{J}$. It is defined to be the solution space of the supertwistor equation

$$\lambda^\dagger \left( \nabla_{\lambda^A} \omega^B + \delta^B_A \pi^\dagger \right) = 0,$$

(3.15)
on the $\beta$-surface $\pi_1(\pi_2^{-1}(z))$. Here, $\lambda^\dagger$ is non-zero and obeys (3.3) and $\pi^\dagger$ is arbitrary. Note that (3.15) does not depend on the chosen scale on $\mathcal{M}$. Note further that the rank of $\mathcal{J}$ is 3$|\mathcal{N}$. Then we have the following result:

**Proposition 3.3.** There is a natural isomorphism $T \mathcal{P} \cong \mathcal{J} \otimes \mathcal{L}^{-1}$.

**Proof.** Let $z$ be a point in $\mathcal{P}$ and $\Sigma := \pi_1(\pi_2^{-1}(z))$ the associated $\beta$-surface in $\mathcal{M}$, and let $\lambda^\dagger$ be a section of $\mathcal{J}$.

It is always possible to have a one-parameter foliation of $\Sigma$ since

$$\lambda^\dagger \nabla_{\lambda^A} (\mu^B \lambda^\dagger) = \lambda^\dagger \nabla_{\lambda^A} \mu^B = 0,$$

where $\mu^A \lambda^\dagger$ is any tangent vector field to $\Sigma$. Let now $J = J^A \lambda^\dagger E_{A\dot{d}}$ be the associated Jacobi field on $\Sigma$. Then any tangent vector $(\omega^A, \pi^\dagger)$ at $z \in \mathcal{P}$ can be represented by Jacobi fields on $\Sigma$,

$$\omega^A = J^A \lambda^\dagger \quad \text{and} \quad \pi^\dagger = J^{\dot{\beta}} \nabla_{\dot{\beta}} \lambda^\dagger,$$

subject to the constraint

$$L_J X \text{mod} T \Sigma = [J, X] \text{mod} T \Sigma = 0 \quad \text{for any} \quad X \in T \Sigma.$$

Note that the above equations are unaffected by changes of the form $J^A \lambda^\dagger \mapsto J^A \lambda^\dagger + J^A \lambda^\dagger$, where $J^A \lambda^\dagger$ is also a Jacobi field which is in addition tangent to $\Sigma$. Therefore, tangent vectors at $z \in \mathcal{P}$ are actually represented by equivalence classes of Jacobi fields, where two Jacobi fields are said to be equivalent if their difference lies in $T \Sigma$.

Explicitly, the constraint $[J, X] \in T \Sigma$ reads

$$\lambda^\dagger \nabla_{\dot{\beta}} J^\beta \dot{\beta} - \delta^\beta_A J^\gamma \nabla_{\dot{\gamma}} \lambda^\dagger = 0.$$

(3.16)

Using this expression, one may straightforwardly check that $\omega^A$ obeys the supertwistor equation (3.15). Hence, the mapping $J \otimes \lambda \mapsto \omega$ defines a morphism $T \mathcal{P} \otimes \mathcal{L} \to \mathcal{J}$. Since the solution space of (3.15) is of the right dimensionality, we have thus constructed an isomorphism.

3.2.3. Einstein bundle. The last vector bundle we are about to define is the Einstein bundle. Originally, it was introduced by LeBrun [56] in the context of the ambitwistor space (the space of complex null-geodesics of some given complex four-dimensional manifold) and its relation to the (full) Einstein equations. He showed that nonvanishing sections of this bundle are in one-to-one correspondence with Einstein metrics in the given conformal class. Unfortunately, the Einstein bundle on ambitwistor space and its generalization to superambitwistor space in the context of $\mathcal{N} = 1$ supergravity (cf Merkulov [54]) seem only to be definable in terms of their inverse images on the associated correspondence space, that is, so far it lacks a description in terms of the intrinsic structure of the (super)ambitwistor space. As we shall see in a moment, this will not be the case if the (super)manifold under consideration is (super)conformally right-flat. It is this additional condition that allows for giving an explicit description of this bundle in terms of natural holomorphic sheaves on the (super)twistor space. As we shall see, this bundle will also yield a reinterpretation of the results given in theorem 3.2. Our subsequent discussion is a generalization of the ideas of [53, 54, 56, 60].
Next we introduce a second-order differential operator, $\Delta$, on the correspondence space $\mathcal{F}$ which is given in the structure frame by
\[
\Delta_{AB} := \lambda^A \lambda^B (\nabla_{[A} \nabla_{B]} + R_{A\hat{B}\hat{D}}),
\]
where $\lambda$ obeys (3.3) and $R_{A\hat{B}\hat{D}}$ is the $R^+$-part of the curvature as discussed in proposition 2.6. It then follows that $\Delta$ is independent of the choice of scale if it acts on sections of $\pi^{-1}_1 O(\mathcal{F})[1]$. This can be seen as follows: let $\psi$ be a section of $\mathcal{O}(\mathcal{F})[k]$. If one performs a change of scale according to $\tilde{\epsilon} \mapsto \gamma \tilde{\epsilon}$, the connection changes as follows:
\[
\tilde{\nabla}_{\tilde{\lambda}} \tilde{\psi} = \gamma^{-2} (\nabla_{\lambda} \psi - k \gamma \lambda \psi),
\]
where $\kappa$ was defined in (2.21) and $\gamma_{\lambda\hat{A}} := E_{\lambda \hat{A}} \log \gamma$, as before. Therefore, if one chooses $k = -1$ one arrives after a few lines of algebra at
\[
\tilde{\nabla}_{\tilde{\lambda}} \tilde{\nabla}_{\tilde{B}} \tilde{\psi} = \gamma^{-2} (\nabla_{[A\lambda} \nabla_{B]B} \psi + \nabla_{[A\lambda} \nabla_{B]} \dot{\psi} - \gamma_{[A\lambda} \gamma_{B]} \dot{\psi} \). \tag{3.19}
\]
In a similar manner, one may verify that
\[
\tilde{\nabla}_{\tilde{\lambda}} \tilde{\nabla}_{\tilde{B}} \tilde{\psi} = \gamma^{-2} (R_{\tilde{A} \tilde{B} \tilde{C}} - \nabla_{[A\lambda} \nabla_{B]} \dot{\psi} - \gamma_{[A\lambda} \gamma_{B]} \dot{\psi} \). \tag{3.20}
\]
Combining these two expressions, one arrives at the desired result.

As $\Delta$ acts on the fibres of $\pi_2 : \mathcal{F} \to \mathcal{P}$, we can define the Einstein bundle $\mathcal{E}$ on $\mathcal{P}$ by the following resolution:
\[
0 \to \pi_2^{-1} \mathcal{E} \to \pi_1^{-1} \mathcal{O}(\mathcal{F})[-1] \overset{\Delta}{\to} \pi_1^{-1} (\mathcal{O}(\mathcal{H})[-3]) \otimes \mathcal{O}(\mathcal{F}) \to 0, \tag{3.21}
\]
where $\mathcal{O}(\mathcal{F})$ is the second tensor power of the dual of the tautological bundle $\mathcal{O}(\mathcal{F})(-1)$ on the correspondence space $\mathcal{F}$.

We are now in the position to relate the four bundles $T \mathcal{P}$, $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{E}$ among themselves by virtue of the following proposition:

**Proposition 3.4.** There is a natural isomorphism of sheaves $\mathcal{E} \cong \Omega^1 \mathcal{P} \otimes \mathcal{L}^{-2} \cong \mathcal{F} \otimes \mathcal{L}^{-1}$.

**Proof.** The second isomorphism is the one proven in proposition 3.3. So it remains to verify the first one. Recall again that the fibre of $\mathcal{E}$ over some point $z \in \mathcal{P}$ is the space of solutions of the supertwistor equation on $\pi_2^{-1}(z)$ for $\omega^A$ being of homogeneous degree 1 in $\lambda$. The fibre of the Einstein bundle $\mathcal{E}$ over $z \in \mathcal{P}$ coincides with the kernel of $\Delta$ on the same subsupermanifold $\pi_2^{-1}(z) \hookrightarrow \mathcal{F}$.

Consider now the scalar
\[
Q := (2 - N) \omega^A \lambda^B \nabla_{[A\lambda} \psi - \tilde{\varphi} (-)^{\hat{B} \hat{C}} \lambda^B \nabla_{[A\lambda} \omega^B, \nonumber
\]
where $\varphi$ is a section of $\pi_1^{-1} \mathcal{O}(\mathcal{F})[-1]$ and $\omega^A$ a solution to the supertwistor equation (3.15). Clearly, $Q$ is of homogeneous degree 2 in $\lambda$ and as one may check it is independent of the choice of scale. In showing the latter statement, one needs the relation
\[
\lambda^B \nabla_{[A\lambda} \omega^B = \frac{1}{2 - N} \delta^B_A (-)^{\hat{E} \hat{F}} \lambda^E \nabla_{[C \lambda} \omega^C, \nonumber
\]
which follows from the supertwistor equation (3.15). In addition, upon using the very same equation (3.15) together with $\Delta_{AB} \psi = 0$, one finds that
\[
\lambda^B \nabla_{[A\lambda} Q = 0. \nonumber
\]
Hence, the quantity $Q$ corresponds to a point in the fibre of $\mathcal{L}^{-2}$ over the point $z \in \mathcal{P}$. Altogether, $Q$ provides a non-degenerate $\mathcal{L}^{-2}$-valued pairing of the fibres of tangent bundle $T \mathcal{P}$ and the Einstein bundle $\mathcal{E}$, thus establishing the claimed isomorphism. \(\square\)

This shows, as indicated earlier, that the Einstein bundle is fully determined in terms of the intrinsic structure of the supertwistor space.
3.2.4. Hyper-Kähler structures. The next step is to verify the following statement:

**Proposition 3.5.** There is a natural one-to-one correspondence between scales on a civilized right-flat complex quaternionic RC supermanifold \( M \) in which the \( R^+_{\bar{\gamma}} \)-part of the curvature vanishes and nonvanishing sections of the Einstein bundle \( \mathcal{E} \) over the associated supertwistor space \( \mathcal{P} \).

Putting it differently, nonvanishing sections of the Einstein bundle are in one-to-one correspondence with (equivalence classes of) solutions to the self-dual supergravity equations with nonzero cosmological constant.

**Proof.** By our convexity assumption (recall that \( M \) is assumed to be civilized; putting it differently, there is a Stein covering of \( M \)), we have

\[
H^q(\pi^{-1}_{-1/2}(z), \mathcal{C}) \cong 0 \text{ for } z \in \mathcal{P} \text{ and } q \geq 1.
\]

Let \( \mathcal{U} \subset M \) be an open subset, and set \( \mathcal{U}' := \pi^{-1}_{1}(\mathcal{U}) \subset \mathcal{F} \) and \( \mathcal{U}'' := \pi_{2}(\mathcal{U}') \subset \mathcal{P} \). Therefore, we have an isomorphism\(^{10}\)

\[
H^r(\mathcal{U}'', \mathcal{E}) \cong H^r(\mathcal{U}', \pi_{-1/2}^{-1}\mathcal{E}).
\]

Hence, in order to compute \( H^r(\mathcal{U}'', \mathcal{E}) \) we need to compute \( H^r(\mathcal{U}', \pi_{-1/2}^{-1}\mathcal{E}) \). However, the latter cohomology groups can be computed from the exact resolution (3.21) upon applying the direct image functor

\[
0 \longrightarrow \pi_{1*}(\mathcal{R}^0) \longrightarrow \pi_{1*}(\mathcal{R}^1) \longrightarrow \pi_{1*}(\pi_{-1/2}^{-1}\mathcal{E})
\]

where we have abbreviated

\[
\mathcal{R}^0 := \pi_{1*}(\mathcal{O}_M[-1]) \quad \text{and} \quad \mathcal{R}^1 := \pi_{1*}(\mathcal{H}^1\otimes \tilde{\mathcal{S}}\otimes \mathcal{O}_F(2))
\]

In addition, there is a spectral sequence converging to

\[
H^{p+q}(\mathcal{U}', \pi_{-1/2}^{-1}\mathcal{E})
\]

with

\[
E_1^{p,q} \cong H^0(\mathcal{U}', \pi_{1/2}^{p}\mathcal{R}^q).
\]

Note that the sheaves in resolution have vanishing higher direct images while the zeroth images are given by

\[
\pi_{1*}(\mathcal{R}^0) \cong \mathcal{O}_M[-1],
\]

\[
\pi_{1*}(\mathcal{R}^1) \cong (\mathcal{H}^1\otimes \tilde{\mathcal{S}}\otimes \mathcal{O}_F(2))[-1].
\]

Therefore, the cohomology group \( H^0(\mathcal{U}'', \mathcal{E}) \cong H^0(\mathcal{U}', \pi_{-1/2}^{-1}\mathcal{E}) \) is isomorphic to the kernel of a second-order differential operator which by virtue of our above discussion turns out to be the solution space of

\[
\nabla_{(A\dot{a})} \nabla_{B\dot{b}} + R_{A\dot{a}B\dot{b}} \psi = 0,
\]

where \( \psi \) is a nonvanishing section of \( \mathcal{O}_M[-1] \). As discussed above, this equation is independent of the choice of scale on \( M \). Since \( \psi \) is a nonvanishing section of \( \mathcal{O}_M[-1] \), we may always work in the scale where \( \psi = 1 \). Thus, the above equation implies that \( R_{A\dot{a}B\dot{b}} \) must vanish and the proof is completed. \( \square \)

Let now \( \tau \) be the section of \( \mathcal{E} \) corresponding to the scale where \( R^+_{\bar{\gamma}} \) vanishes. Obviously, it defines a \((2|N)\)-dimensional distribution on the supertwistor space \( \mathcal{P} \) given by the kernel of \( \tau \). One also says that \( \tau \) is a non-degenerate holomorphic contact form determining a holomorphic contact structure (a distribution of Grassmann even codimension 1) on \( \mathcal{P} \). Thus,

\(^{10}\) Note that this in fact holds true for any locally free sheaf on the supertwistor space.
non-degeneracy of \( \tau \) ensures a nonvanishing cosmological constant. In addition, let \( \nabla \) be the connection on \( \mathcal{M} \) defined by this chosen scale. In the remainder, we shall show that degenerate contact structures on \( \mathcal{P} \) are in one-to-one correspondence with (equivalence classes of) solutions to the self-dual supergravity equations with zero cosmological constant.

**Proposition 3.6.** The \((4|2N)\)-dimensional distribution on \( \mathcal{P} \) defined by \( \pi_2^* \tau \) coincides with the \((4|2N)\)-dimensional distribution defined by \( \pi_1^* \nabla \).

**Proof.** Recall that \( \mathcal{E} \cong \Omega^1 \mathcal{P} \otimes \mathcal{L}^{-2} \). Then we note that the pairing \( \Omega^1 \mathcal{P} \otimes \mathcal{L}^{-2} \times T \mathcal{P} \rightarrow \mathcal{L}^{-2} \) is given by
\[
(2 - N)\omega^A \alpha \nabla_{A\lambda} \varphi - \varphi(-)^{p+1} \lambda \beta \nabla_{A\lambda} \omega^A,
\]
as follows by the discussion given in the proof of proposition 3.4. Here, \( \varphi \) represents \( \tau \) on \( \mathcal{P} \) and \( \omega^A \) corresponds to a tangent vector on \( \mathcal{P} \). In the scale defined by \( \tau \), we have \( \varphi = 1 \) (cf. the proof of the immediately preceding proposition). By virtue of the supertwistor equation (3.15), we conclude that the distribution on the correspondence space \( \mathcal{F} \) defined by the vanishing of this pairing is given by
\[
\lambda \lambda \nabla_{A\lambda} \omega = 0.
\]
Equation (3.16) in turn implies that a solution to this equation must correspond to a Jacobi field \( J^{A\lambda} \) which satisfies
\[
J^{A\lambda} \lambda \beta \nabla_{A\lambda} \lambda \beta = 0.
\]
Therefore, we have a correspondence between subspaces of the fibre of \( T \mathcal{P} \) over a point \( z \in \mathcal{P} \) which are annihilated by the differential 1-form \( \tau \) and Jacobi fields on the \( \beta \)-surface \( \Sigma = \pi_1(\pi_2^{-1}(z)) \leftrightarrow \mathcal{M} \) which are annihilated by the differential 1-form \( E^{A\lambda} \lambda \beta \nabla_{A\lambda} \lambda \beta \).

In fact, this form is the push-forward to \( \mathcal{M} \) of the differential 1-form on \( \mathcal{F} \) defining the distribution given by \( \pi_1^* \nabla \).

Then we have the following result:

**Proposition 3.7.** Let \( \tau \) be the section of the Einstein bundle \( \mathcal{E} \rightarrow \mathcal{P} \) corresponding to the scale in which \( R^G \) vanishes. In this scale, the cosmological constant will vanish if and only if the distribution on \( \mathcal{P} \) defined by \( \tau \) is integrable. Hence, the Ricci tensor is zero.

This means that nonvanishing integrable sections of the Einstein bundle (that is, degenerate contact structures) are in one-to-one correspondence with (equivalence classes of) solutions to the self-dual supergravity equations with zero cosmological constant.

**Proof.** Obviously, showing integrability of the distribution on \( \mathcal{P} \) defined by \( \tau \) is equivalent to showing the integrability of the distribution on \( \mathcal{F} \) defined by the pull-pack \( \pi_2^* \tau \). Proposition 3.6 implies that the distribution defined by \( \pi_2^* \tau \) will be integrable if and only if \( \mathcal{F} \) is projectively flat in the scale defined by \( \tau \) (see also comment after proof of proposition 3.1 leading to equation (3.3)). We conclude from proposition 2.6 that in addition to \( R_{A\beta\beta} \) also \( \Lambda_{AB} \) must vanish. Hence, the Ricci tensor is zero.

Recall that \( \tau \) defines a \((2|N)\)-dimensional distribution on \( \mathcal{P} \). If this distribution is integrable, it gives a foliation of \( \mathcal{P} \) by \((2|N)\)-dimensional subsupermanifolds. In fact, it yields a holomorhpic fibration
\[
\mathcal{P} \rightarrow \mathbb{P}^1
\]
of the supertwistor space over the Riemann sphere (see also Penrose [21] for the purely bosonic situation). Remember that this fibration was one of the assumptions made in theorem 3.2. Therefore, we may conclude that if the distribution \( \tau \) is integrable the supertwistor space \( \mathcal{P} \) is equipped with a relative symplectic structure as stated in point (iii) of theorem 3.2.
3.2.5. Summary. Let us summarize all the correspondences derived above in the following table:

| Supertwistor spaces $\mathcal{P}$ | Civilized right-flat complex quaternionic RC supermanifolds, i.e. $C_{\dot{A}\dot{B}\dot{C}} = 0$ |
|-----------------------------------|------------------------------------------------------------------------------------------------|
| Supertwistor spaces $\mathcal{P}$ with non-degenerate holomorphic contact structures | Civilized right-flat complex quaternionic RC supermanifolds which are self-dual Einstein, i.e. $C_{\dot{A}\dot{B}\dot{C}} = 0$ and $R_{\dot{A}\dot{B}\dot{D}} = 0$ |
| Supertwistor spaces $\mathcal{P}$ with degenerate holomorphic contact structures | Civilized right-flat complex quaternionic RC supermanifolds which are self-dual, i.e. $C_{\dot{A}\dot{B}\dot{C}} = 0$, $R_{\dot{A}\dot{B}\dot{D}} = 0$ and $\Lambda_{\dot{A}\dot{B}} = 0$ |

We remind the reader that the curvature components can be found in proposition 2.6.

3.3. Bundle of local supertwistors ($N \neq 4$)

This subsection is devoted to the bundle of local supertwistors and its implications on the supermanifolds under consideration. Here, we give a generalization of methods developed by Penrose [69], LeBrun [56], and Bailey and Eastwood [60]. So, let $\mathcal{M}$ be a civilized right-flat complex quaternionic RC supermanifold with connection $\nabla$, in the following.

3.3.1. Bundle of local supertwistors. Let us start by recalling the jet sequence (for a proof, see e.g., Manin [51])

$$0 \rightarrow \Omega^1 \mathcal{M} \otimes \mathcal{E} \rightarrow \text{Jet}^1 \mathcal{E} \rightarrow \mathcal{E} \rightarrow 0,$$  \hspace{1cm} (3.23)

where $\mathcal{E}$ is some locally free sheaf on $\mathcal{M}$ and Jet$^1 \mathcal{E}$ is the sheaf of first-order jets of $\mathcal{E}$. Recall further the factorization of the tangent bundle of $\mathcal{M}$ as $T\mathcal{M} \cong \mathcal{H} \otimes \tilde{\mathcal{S}}$. Choose now $\mathcal{E}$ to be $\mathcal{H}$. Hence, the above sequence becomes

$$0 \rightarrow (\mathcal{H} \otimes \mathcal{H}^\vee)_0 \otimes \tilde{T}^i \rightarrow \text{Jet}^1 \mathcal{H} \rightarrow \mathcal{H} \rightarrow 0.$$  \hspace{1cm} (3.24)

Since $(\mathcal{H} \otimes \mathcal{H}^\vee)_0 \otimes \tilde{T}^i$, where $(\mathcal{H} \otimes \mathcal{H}^\vee)_0$ means the trace-free part of $\mathcal{H} \otimes \mathcal{H}^\vee$, is a subbundle of $\Omega^1 \mathcal{M} \otimes \mathcal{H}$, i.e.

$$0 \rightarrow (\mathcal{H} \otimes \mathcal{H}^\vee)_0 \otimes \tilde{T}^i \rightarrow \Omega^1 \mathcal{M} \otimes \mathcal{H},$$  \hspace{1cm} (3.25)

we may define a rank-$4|N$ bundle, denoted by $\tilde{T}$, over $\mathcal{M}$ by the following sequence:

$$0 \rightarrow (\mathcal{H} \otimes \mathcal{H}^\vee)_0 \otimes \tilde{T}^i \rightarrow \text{Jet}^1 \mathcal{H} \rightarrow \tilde{T} \rightarrow 0.$$  \hspace{1cm} (3.26)

We shall call $\tilde{T}$ the bundle of local supertwistors. The reason for naming it like this will become clear in due course of our subsequent discussion.

As a first result, we obtain from (3.24) and (3.26) a natural isomorphism:

$$\text{Ber} \tilde{T} \cong \text{Ber} \mathcal{H} \otimes (\text{Ber} \tilde{T})^{-1}.$$  \hspace{1cm} (3.27)

Hence, by virtue of (2.15) we may conclude that

$$\text{Ber} \tilde{T} \cong \mathcal{O}_{\mathcal{M}}.$$  \hspace{1cm} (3.28)

Furthermore, in a structure frame, $\tilde{T}$ may be described by natural fibre coordinates of the form $(\omega^A, \pi_\dot{A})$. Under a change of scale $\bar{\varepsilon} \mapsto \gamma \varepsilon$, these coordinates behave as

$$\omega^A \mapsto \bar{\omega}^A = \gamma^{1/2} \omega^A \quad \text{and} \quad \pi_\dot{A} \mapsto \bar{\pi}_\dot{A} = \gamma^{-1/2} (\pi_\dot{A} - \omega^A \gamma_{A\dot{A}}),$$  \hspace{1cm} (3.29)

which is an immediate consequence of the transformation laws (2.39). Remember that the constant $\kappa$ appearing above was introduced in (2.21) and $\gamma_{A\dot{A}}$ was defined to be
\[ \gamma_{A\dot{a}} = E_{A\dot{a}} \log \gamma. \] Altogether, these considerations imply that there is a canonical exact sequence
\[ 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{H} \rightarrow 0. \] (3.30)

3.3.2. Local supertwistor connection. In the class of affine connections on the bundle \( \mathcal{F} \), there exists a distinguished one referred to as the local supertwistor connection, in the following. This is an immediate consequence of the scaling behaviour (3.29), as we shall see now. Let us mention in passing that this particular connection will be unique and independent of the choice of scale on \( \mathcal{M} \).

Let us recall the supertwistor equation (3.15), which we repeat for the reader’s convenience at this stage
\[ \lambda^\dot{a}\left( \nabla_A \omega^B + \delta_A^B \pi_\dot{a} \right) = 0. \] (3.31)
Recall further from the proof of proposition 3.3 that tangent vectors at \( z \in \mathcal{P} \) can be represented by \( \omega^A = J^{A\dot{a}} \lambda_{\dot{a}} \) and \( \pi_\dot{a} = J^{B\dot{b}} \nabla_{\dot{b}} \lambda_{\dot{a}} \), where \( J = J^{A\dot{a}} E_{A\dot{a}} \) is a Jacobi field on the \( \beta \)-surface \( \pi_\dot{a} \{ \overline{\pi}_\dot{a}^{-1}(z) \} \rightarrow \mathcal{M} \). In the very same proof, we have argued that this \( \omega^A \) satisfies the supertwistor equation. Similarly, one may show that
\[ \lambda^\dot{a}\left( \nabla_A \pi_\beta + (-)^{p_a}(R_{AB \dot{a} \dot{b}} - \Lambda_{AB} \epsilon_{\dot{a}\dot{b}}) \omega^B \right) = 0. \] (3.32)

Here, we have made use of the curvature decompositions (2.49). Furthermore, the scaling behaviour (3.29) is exactly of the same form as that of \( \omega^A = J^{A\dot{a}} \lambda_{\dot{a}} \) and \( \pi_\dot{a} = J^{B\dot{b}} \nabla_{\dot{b}} \lambda_{\dot{a}} \), respectively. That is why we have denoted the fibre coordinates of the bundle \( \mathcal{F} \) by the same letters.

Altogether, equations (3.31) and (3.32) can be reinterpreted as an \( SL(4|N) \)-connection \( \mathcal{D} \) on \( \mathcal{F} \)—the local supertwistor connection:
\[ \lambda^\dot{a}\mathcal{D}_{\dot{a}} \left( \omega^B, \pi_\beta \right) = \left( \lambda^\dot{a}\left( \nabla_A \omega^B + \delta_A^B \pi_\dot{a} \right) \right). \] (3.33)

Thus, flat sections with respect to this connection correspond to solutions of the supertwistor equation. After all, this justifies the name local supertwistor bundle.

Let us make the following abbreviations: \( \mathcal{D}_A := \lambda^\dot{a}\mathcal{D}_{\dot{a}}, \nabla_A := \lambda^\dot{a}\nabla_\dot{a} \) and \( 'Z := (\omega^A, \pi_\dot{a}) \). Then we may rewrite equations (3.33) concisely as
\[ \mathcal{D}_A Z = \nabla_A Z + A_A Z, \] (3.34)
where
\[ A_A := \left( \begin{array}{cc} 0 & \delta_A^B \lambda^\dot{b} \\ (-)^{p_a} \lambda^\dot{a}(R_{AB \dot{a} \dot{b}} - \Lambda_{AB} \epsilon_{\dot{a}\dot{b}}) & 0 \end{array} \right) \] (3.35)
is the \( sl(4|N) \)-valued gauge potential. The local supertwistor connection \( \mathcal{D} \) is torsion free, since \( \nabla \) is torsion free. Furthermore, the \( \mathcal{F}^- \)-part of the curvature 2-form \( \mathcal{F} = \mathcal{D}^2 = \mathcal{F}^- + \mathcal{F}^\tau \) of \( \mathcal{D} \) (here, we are using the notation of proposition 2.6) is given in a structure frame by
\[ \mathcal{F}_{AB} = \lambda^\dot{a}\lambda^\dot{b}\mathcal{D}_{\dot{a}B\dot{b}} \]
\[ = \left[ \mathcal{D}_A, \mathcal{D}_B \right] \] (3.36)
\[ = R_{AB} + \nabla_A A_B - (-)^{p_a} p_b \nabla_B A_A + [A_A, A_B], \]
where \( R_{AB} = [\nabla_A, \nabla_B] = \lambda^\dot{a}\lambda^\dot{b}[\nabla_{\dot{a}}, \nabla_{\dot{b}}] \). The next one verifies that \( \nabla_A A_B \) actually vanishes, which is due to equations (3.3) and due to Bianchi identities of the curvature of \( \nabla \). Upon explicitly computing the commutator \( [A_A, A_B] \) and comparing it with \( R_{AB} \) thereby
Proposition 3.8. The $F^\ast$-part of the curvature $F$ of the local supertwistor connection $D$ on the local supertwistor bundle $\mathcal{F}$ over a civilized right-flat complex quaternionic RC supermanifold $\mathcal{M}$ is zero. Hence, the curvature $F$ is self-dual, that is, $F = F^\ast$.

Putting it differently, the connection $D$ is flat on any $\beta$-surface $\pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathcal{M}$ for all $z \in \mathcal{P}$. This is going to be important in the paragraph subsequent to the following one, where we will show that the bundle of first-order jets of the dual universal line bundle $\mathcal{L}$ over the supertwistor space, i.e. Jet$^1\mathcal{L}^{-1} \rightarrow \mathcal{P}$, corresponds to the dual of the bundle of local supertwisters $\mathcal{F}^\ast \rightarrow \mathcal{M}$ by means of the Penrose–Ward transform.

3.3.3. Penrose–Ward transform. In this paragraph, we briefly discuss the general form of the Penrose–Ward transform which relates certain holomorphic vector bundles over the supertwistor space $\mathcal{P}$ to holomorphic vector bundles over $\mathcal{M}$ and vice versa. However, we merely quote the result. A detailed proof goes along the lines presented by Manin [51] and can be done in the supersymmetric setting without difficulties.

Suppose we are given a locally free sheaf $\mathcal{E}_\mathcal{P}$ on $\mathcal{P}$. Suppose further that $\mathcal{E}_\mathcal{P}$ is free when restricted to any submanifold $\pi_1(\pi_2^{-1}(x)) \hookrightarrow \mathcal{P}$ for all $x \in \mathcal{M}$. In addition, let $\Omega^1\mathcal{F}/\mathcal{P}$ be the sheaf of relative differential 1-forms on $\mathcal{F}$ as given by the sequence (3.12). Furthermore, let

$$\mathcal{D}_{\mathcal{F}/\mathcal{P}} : \pi_2^\ast\mathcal{E}_\mathcal{P} \rightarrow \pi_1^\ast\mathcal{E}_\mathcal{P} \otimes \pi_1^\ast\Omega^1\mathcal{M} \xrightarrow{\text{id} \otimes \text{res}} \pi_2^\ast\mathcal{E}_\mathcal{P} \otimes \Omega^1\mathcal{F}/\mathcal{P}$$

(3.37)

be the relative connection on the pull-back $\pi_2^\ast\mathcal{E}_\mathcal{P}$ of $\mathcal{E}_\mathcal{P}$ to the correspondence space $\mathcal{F}$.

In order for the below theorem to work, one needs

$$\Omega^1\mathcal{M} \cong \pi_1^\ast\Omega^1\mathcal{F}/\mathcal{P},$$

(3.38)

since only then $\mathcal{D}_{\mathcal{F}/\mathcal{P}}$ gives rise to a connection $\mathcal{D} := \pi_1^\ast(\mathcal{D}_{\mathcal{F}/\mathcal{P}})$ on $\mathcal{E}_\mathcal{M} = \pi_1^\ast(\pi_2^\ast\mathcal{E}_\mathcal{P})$. One may check that this isomorphism indeed follows from the sequence (3.7) after dualizing and upon applying the direct image functor. In showing this, one uses the fact that the direct images $\pi_1^\ast(\pi_2^\ast\mathcal{N}_\mathcal{P})$ and $\pi_1^\ast(\pi_2^\ast\mathcal{N}_\mathcal{F})$ vanish due to Serre duality. Since the fibres of $\pi_1 : \mathcal{F} \rightarrow \mathcal{M}$ are compact and connected and those of $\pi_2 : \mathcal{F} \rightarrow \mathcal{P}$ are connected and simply connected (recall that $\mathcal{M}$ is assumed to be civilized), we have the following theorem:

Theorem 3.3. Let $\mathcal{M}$ be a civilized right-flat complex quaternionic RC supermanifold and $\mathcal{P}$ its associated supertwistor space. Then there is a natural one-to-one correspondence between

(i) the category of locally free sheaves $\mathcal{E}_\mathcal{P}$ on $\mathcal{P}$ which are free on any submanifold $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathcal{P}$ for all $x \in \mathcal{M}$ and

(ii) the category of pairs $(\mathcal{E}_\mathcal{M}, \mathcal{D})$, where $\mathcal{E}_\mathcal{M}$ is a locally free sheaf on $\mathcal{M}$ given by $\mathcal{E}_\mathcal{M} = \pi_1(\pi_2^\ast\mathcal{E}_\mathcal{P})$ and $\mathcal{D}$ is the push-forward of the relative connection on $\mathcal{F}$, i.e. $\mathcal{D} := \pi_1(\mathcal{D}_{\mathcal{F}/\mathcal{P}})$ which is flat on any $\beta$-surface $\pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathcal{M}$ for all $z \in \mathcal{P}$.

Note that flatness on any $\beta$-surface is equivalent to saying that the curvature of $\mathcal{D}$ is self-dual. The above correspondence is called Penrose–Ward transform.
3.3.4. **Penrose–Ward transform of \( \mathcal{T} \).** Consider the bundle of local supertwistors \( \mathcal{T} \) as defined in section 3.3.1. As we have shown in section 3.3.2, the local supertwistor connection is self-dual, i.e. flat on any \( \beta \)-surface in \( \mathcal{M} \). So one naturally asks for the Penrose–Ward transform of \( \mathcal{T} \). The answer gives the following proposition:

**Proposition 3.9.** The Penrose–Ward transform takes the bundle of local supertwistors \( \mathcal{T} \) over \( \mathcal{M} \) to the dual of the sheaf of first-order jets \( \text{Jet}^1 \mathcal{L}^{-1} \) of the dual universal line bundle \( \mathcal{L} \) over \( \mathcal{P} \).

**Proof.** As a first check, note that the restriction of \( \mathcal{L} \) to any \( \pi_2^{-1}(x) \) is \( \mathcal{O}_{\mathcal{P}}(-1) \). Hence, the dual restricts to \( \mathcal{O}_{\mathcal{P}}(1) \). Hence, one may check that the dual of the sheaf of first-order jets \( \text{Jet}^1 \mathcal{L}^{-1} \) of \( \mathcal{L}^{-1} \) is free when restricted to \( \pi_2^{-1}(x) \) as a consequence of the Euler sequence. So, \( (\text{Jet}^1 \mathcal{L}^{-1})^\vee \) satisfies point (i) of theorem 3.3.

In the following, we are using a supersymmetric generalization of an argument by LeBrun [68]. Let \( m_t : \mathcal{L} \rightarrow \mathcal{L} \) (zero section deleted), where \( t \in \mathbb{C}\{0\} \), denote the scalar multiplication map. Furthermore, let \( m_{t^*} : T \mathcal{L} \rightarrow T \mathcal{L} \) be its Jacobian. According to LeBrun [68], one has an isomorphism

\[
\text{Jet}^1 \mathcal{L}^{-1} \cong (\mathcal{L} \otimes (T \mathcal{L}/[m_{t^*}]))^\vee.
\]

Thus, we are about to verify that

\[
\mathcal{T} \cong \pi_1^*(\mathcal{T} \otimes (T \mathcal{L}/[m_{t^*}])).
\]

To do this, we first recall that a point \( \ell \) of \( \mathcal{L} \) is a pair \( (\pi_1^{-1}(z), \lambda_\alpha) \), where \( z \in \mathcal{P} \) and \( \lambda_\alpha \) is an auto-parallel tangent spinor, i.e. it satisfies equation (3.33). Therefore, a tangent vector at \( \ell \in \mathcal{L} \) can be represented by Jacobi fields as introduced and discussed in the proof of proposition 3.3. In particular, we may write

\[
(\omega^A, \pi_\alpha) = (J^A_\beta \lambda_\beta^\dagger, J^B_\beta \nabla_\beta \lambda_\alpha),
\]

for the tangent vector at \( \ell \in \mathcal{L} \). From our discussion given in section 3.3.2, we know that such \( (\omega^A, \pi_\alpha) \) satisfy

\[
\lambda_\alpha (\nabla_{A\alpha} \omega^B + \delta_A^B \pi_\alpha) = 0,
\]

\[
\lambda^\dagger_\alpha (\nabla_{A\alpha} \pi_B + (-)^{p_B} (R_{AB\beta\hat{\delta}} - \Lambda_{AB\epsilon\hat{\delta}}) \omega^B) = 0,
\]

i.e. they are annihilated by the local supertwistor connection (3.33). Since the transformation \( m_t : \lambda_\alpha \mapsto t \lambda_\alpha \) induces

\[
(\omega^A, \pi_\alpha) \mapsto (t \omega^A, t \pi_\alpha),
\]

we conclude that the Penrose–Ward transform takes \( \mathcal{T} \) to \( \mathcal{L} \otimes (T \mathcal{L}/[m_{t^*}]) \), that is, to \( (\text{Jet}^1 \mathcal{L}^{-1})^\vee \).

\( \square \)

This leads us to the following interesting result:

**Proposition 3.10.** There are natural isomorphisms of Berezinian sheaves:

(i) \( \text{Ber} \mathcal{J} \cong \mathcal{L}^{-1} \),

(ii) \( \text{Ber}(\mathcal{P}) := \text{Ber} \Omega^1 \mathcal{P} \cong \mathcal{L}^{4-N} \).

\( \text{11} \) Recall that the Euler sequence is given by \( 0 \rightarrow \mathcal{O}_{\mathcal{P}} \rightarrow \mathcal{O}_{\mathcal{P}}^p(1) \otimes \mathbb{C}^{n+1} \rightarrow T \mathcal{P}^n \rightarrow 0. \)
**Proof.** The starting point is the jet sequence (3.23). This sequence in particular implies that
\[ 0 \longrightarrow \Omega^1 \mathcal{P} \otimes \mathcal{L}^{-1} \longrightarrow \text{Jet}^1 \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \longrightarrow 0, \]
i.e.
\[ 0 \longrightarrow \mathcal{J}^\vee \longrightarrow \text{Jet}^1 \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \longrightarrow 0, \]
by virtue of proposition 3.3. Therefore, we obtain a natural isomorphism of Berezinian sheaves
\[ \text{Ber} \mathcal{J} \otimes \mathcal{L} \cong \text{Ber}(\text{Jet}^1 \mathcal{L}^{-1})^\vee. \]
Since
\[ \pi_2^*(\text{Jet}^1 \mathcal{L}^{-1})^\vee \cong \pi_1^* \mathcal{T} \]
and due to equation (3.28), we may conclude that
\[ \text{Ber}(\text{Jet}^1 \mathcal{L}^{-1})^\vee \cong \mathcal{O}_{\mathcal{P}}, \]
which, in fact, proves point (i).

To verify point (ii), we merely apply proposition 3.3 again. Indeed, from \( T \mathcal{P} \cong \mathcal{J} \otimes \mathcal{L}^{-1} \) we find that
\[ \text{Ber}(\mathcal{P}) = \text{Ber} \mathcal{J}^\vee \otimes \mathcal{L}^{3-N} = (\text{Ber} \mathcal{J} \otimes \mathcal{L})^\vee \otimes \mathcal{L}^{4-N} \cong \mathcal{L}^{4-N}. \]
This completes the proof. \( \square \)

### 3.4. Supertwistor space (\( N = 4 \))

Let us now discuss the \( N = 4 \) case. However, we can be rather brief on this, as the discussion is very similar to that given above. Furthermore, for the sake of illustration we only discuss the hyper-Kähler case.

#### 3.4.1. Conic structure and \( \beta \)-plane bundle

Let \( \mathcal{M} \) be a \((4|8)\)-dimensional RC supermanifold equipped with the Levi-Civita connection. Recall again the sequence (2.9). It is equivalent to
\[ 0 \longrightarrow \mathcal{E}[1] \otimes \mathcal{F}[-1] \longrightarrow T \mathcal{M} \longrightarrow \mathcal{P}[1] \otimes \mathcal{F}[-1] \longrightarrow 0. \tag{3.39} \]
The reason for making this particular choice will become clear momentarily.

Now let \( \mathcal{F} \) be the relative projective line bundle \( P_{\mathcal{M}}(\mathcal{F}^\vee)[1] \) on \( \mathcal{M} \). As before, the tangent bundle sequence induces a canonical \((2|4)\)-conic structure on \( \mathcal{M} \), which in local coordinates is given by
\[ \mathcal{F} \rightarrow G_{\mathcal{M}}(2|4; T \mathcal{M}), \]
\[ [\lambda_\alpha] \mapsto \mathcal{D} := (\lambda_\delta \mathcal{E}_{\delta \lambda}). \tag{3.40} \]
By a similar reasoning as given in proposition 3.1, this distribution will be integrable if and only if \( \mathcal{M} \) is right-flat. As before, \( \mathcal{F} \) will be called the \( \beta \)-plane bundle in this case. In addition, equation (3.3) is then substituted by
\[ \lambda_\delta D_{\delta \lambda} \lambda_\beta = 0, \tag{3.41} \]
i.e. \( \lambda_\delta \) is auto-parallel with respect to the Levi-Civita connection on the \( \beta \)-plane \( \Sigma \hookrightarrow \mathcal{M} \). Furthermore, as directly follows from the transformation laws given in proposition 2.5, this equation is scale invariant since \( \lambda_\delta \) is chosen to be a section of \( \mathcal{F}^\vee[1] \). This explains, why we have twisted \( \mathcal{F} \) by \( \mathcal{O}_{\mathcal{M}}(k) \), with \( k = -1 \) in (3.39).

\[ 12 \text{ Note that } \text{Ber} \mathcal{L} \cong \mathcal{L}. \]
\[ 13 \text{ Recall that } \pi_2^*(\text{Jet}^1 \mathcal{L}^{-1})^\vee \text{ is free when restricted to } \pi_1^{-1}(x) \text{ for all } x \in \mathcal{M}. \]
3.4.2. Supertwistor space. As before, we obtain the following double fibration:

\[
\begin{array}{c}
\mathcal{P} \\
\pi_2 \\
\pi_1 \\
\mathcal{M}
\end{array}
\] (3.42)

Here, \( \mathcal{P} \) is the \((3|4)\)-dimensional supertwistor space of \( \mathcal{M} \). Again, we need to assume that \( \mathcal{M} \) is civilized.

As already indicated, we shall now directly jump to the hyper-Kähler case. The following then gives the inverse construction.

**Theorem 3.4.** There is a one-to-one correspondence between civilized complex hyper-Kähler supermanifolds \( \mathcal{M} \) of dimension \((4|8)\) and \((3|4)\)-dimensional complex supermanifolds \( \mathcal{P} \) such that

(i) \( \mathcal{P} \) is a holomorphic fibre bundle \( \pi : \mathcal{P} \to \mathbb{P}^1 \),

(ii) \( \mathcal{P} \) is equipped with a \((4|8)\)-parameter family of sections of \( \pi \), each with normal bundle \( \mathcal{N}_{\mathcal{P}|\mathcal{P}_1} \) described by

\[
0 \to \Pi \mathcal{O}_{\mathcal{P}^1}(1) \otimes \mathbb{C}^4 \to \mathcal{N}_{\mathcal{P}|\mathcal{P}_1} \to \mathcal{O}_{\mathcal{P}^1}(1) \otimes \mathbb{C}^2 \to 0,
\]

and

(iii) there exists a holomorphic relative symplectic structure \( \omega \) of weight 2 on \( \mathcal{P} \).

In proving this result, one basically follows the argumentation given in section 3.1. The only modification is the replacement of \( \tilde{\mathcal{T}} \) by \( \tilde{\mathcal{T}}[-1] = \tilde{\mathcal{T}} \otimes \text{Ber} \tilde{\mathcal{T}} \). In this respect, we also point out that triviality of the bundle \( \tilde{\mathcal{T}}[-1] \) certainly implies triviality of \( \tilde{\mathcal{T}} \).

3.4.3. Remark. It is obvious how to define the universal line bundle, the Jacobi bundle and the bundle of local supertwistors in the context of the \( N = 4 \) supertwistor space. Proposition 3.3 can be modified accordingly. Point (ii) of proposition 3.10 is then substituted by the fact that the Berezinian sheaf \( \text{Ber} (\mathcal{P}) \) is globally trivial, i.e. \( \text{Ber} (\mathcal{P}) \cong \mathcal{O}_{\mathcal{P}} \). Hence, the \( N = 4 \) supertwistor space is a formal Calabi-Yau supermanifold.

4. Real structures

So far, we have been discussing only complex supermanifolds. The subject of this section is to comment on a real version of theory.

4.1. Almost quaternionic supermanifolds

Let us first present an overview about real structures on complex supermanifolds.

**Definition 4.1 (Manin [51]).** A real structure on a complex supermanifold \((\mathcal{M}, \mathcal{O}_{\mathcal{M}})\) of type \((\epsilon_1, \epsilon_2, \epsilon_3)\), where \( \epsilon_i = \pm 1 \) for \( i = 1, 2, 3 \), is an even \( \mathbb{R} \)-linear mapping \( \rho : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}} \) such that

\[
\rho(\alpha f) = \bar{\alpha} \rho(f),
\]

\[
\rho(\rho(f)) = (\epsilon_1)^{\rho_{\rho}} f,
\]

\[
\rho(fg) = \epsilon_3 (\epsilon_2)^{\rho_{\rho}} \rho(g) \rho(f),
\]

where \( f \) and \( g \) are local holomorphic functions on \( \mathcal{M} \) and \( \alpha \in \mathbb{C} \). The bar means complex conjugation. Furthermore, \( \rho(f(\cdot)) = f(\rho(\cdot)) \).
If $\mathcal{E}$ is a holomorphic vector bundle over $\mathcal{M}$, then a prolongation $\hat{\rho}$ of type $\eta = \pm 1$ of a given real structure $\rho : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ is an even $\mathbb{R}$-linear mapping $\hat{\rho} : \mathcal{E} \to \mathcal{E}$ such that

$$\hat{\rho}(\hat{\rho}(\sigma)) = \eta(\epsilon_1)^{P\rho} \rho(\sigma), \quad \hat{\rho}(f \sigma) = \epsilon_3(\epsilon_2)^{P\rho} \rho(f) \hat{\rho}(\sigma), \quad \hat{\rho}(\sigma f) = \epsilon_3(\epsilon_2)^{P\rho} \rho(f) \hat{\rho}(\sigma),$$

where $\sigma$ is a local section of $\mathcal{E}$ and $f$ is a local holomorphic function on $\mathcal{M}$. If $\eta = +1$ then the prolongation is called real while for $\eta = -1$ quaternionic.

Having recalled the definition of real structures and their extensions to vector bundles, we may now give the following definition:

**Definition 4.2.** A $(4|2N)$-dimensional RC supermanifold $\mathcal{M}$ is called an almost quaternionic RC supermanifold if there is a real structure $\rho$ on $\mathcal{M}$ of type $(-1, 1, 1)$ which leaves $\mathcal{E} \otimes \tilde{\mathcal{F}}$ invariant and which induces two quaternionic prolongations $\hat{\rho}_1 : \mathcal{F} \to \mathcal{F}$ and $\hat{\rho}_2 : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$, respectively. In addition, it is also assumed that $\rho$ has a (real) $(4|2N)$-dimensional supermanifold $\mathcal{M}_\rho$ of $\rho$-stable points in $\mathcal{M}$.

### 4.2. Structure group on $\mathcal{M}_\rho$

In section 2.2.1, we have discussed the form of the structure group $G$ of $T\mathcal{M}$. If $\mathcal{M}$ is equipped with an almost quaternionic structure, $G$ may be reduced to the real form $G_\rho$ on $\mathcal{M}_\rho$ which is described by

$$1 \longrightarrow \mathbb{Z}_{4-2N} \longrightarrow S(GL(1|\frac{1}{2}N, \mathbb{H}) \times GL(1|0, \mathbb{H})) \longrightarrow G_\rho \longrightarrow 1,$$

where $G_\rho \subset GL(4|2N, \mathbb{R})$. This makes it clear why it is necessary to have an even number $N$ of supersymmetries as otherwise one cannot endow an RC supermanifold with an almost quaternionic structure.

Furthermore, a scale is defined in this case as follows:

**Definition 4.3.** A scale on an almost quaternionic RC supermanifold $\mathcal{M}$ is a choice of a particular nonvanishing volume form $\tilde{\varepsilon} \in H^0(\mathcal{M}, \text{Ber}(\tilde{\mathcal{F}}))$ on the vector bundle $\tilde{\mathcal{F}}$ such that the corresponding volume form $\text{Vol} \in H^0(\mathcal{M}, \text{Ber}(\mathcal{M}))$ obeys $\rho(\text{Vol}) = \text{Vol}$.

Clearly, a choice of scale reduces the structure group $G_\rho$ on $\mathcal{M}_\rho$ further down to $SG_\rho$, which in fact is given by

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2|N) \times SU(2|0) \longrightarrow SG_\rho \longrightarrow 1,$$

as follows from (2.18).

Now one can basically repeat the analysis given in sections 2.2 and 2 starting from almost quaternionic RC supermanifolds. One eventually arrives at the notions of quaternionic, quaternionic Kähler and hyper-Kähler structures, that is, in definitions 2.3, 2.4 and 2.5 one simply needs to remove the word ‘complex’.

### 4.3. Supertwistor space

It remains to clarify the additional structure on the supertwistor space $\mathcal{P}$ needed in order to be associated with an RC supermanifold equipped with a real structure in the above sense.

On first notices that by starting from $\mathcal{M}$, the real structure $\rho$ on $\mathcal{M}$ naturally induces real structures on $\mathcal{F}$ and $\tilde{\mathcal{F}}$, respectively, which are, of course, of the same type as $\rho$, that is, $(-1, 1, 1)$. For instance, since $\rho$ is assumed to have a quaternionic prolongation

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14 See Salamon [70] for the purely bosonic situation.
\[ \hat{\rho}_2 : \hat{\mathcal{F}} \to \hat{\mathcal{F}}, \text{the induced real structure acts on the fibres of } \pi_1 : \hat{\mathcal{F}} \to \mathcal{M} \text{ as the antipodal map } (\lambda_1, \lambda_2) \mapsto (-\lambda_2, \lambda_1). \] Since \( \mathcal{P} \) foliates \( \mathcal{F} \), one obtains the induced real structure on \( \mathcal{P} \). The following theorem also clarifies the reverse direction:

**Theorem 4.1.** There is a one-to-one correspondence between

(i) civilized right-flat quaternionic RC supermanifolds \( \mathcal{M} \) of (complex) dimension \( (4|2N) \)

and

(ii) \( (3|N) \)-dimensional complex supermanifolds \( \mathcal{P} \) each containing a family of

holomorphically embedded projective lines \( \mathbb{P}^1 \) each having normal bundle \( N_{\mathbb{P}^1} |_{\mathcal{P}} \) inside \( \mathcal{P} \) described by (3.6) and in addition \( \mathcal{P} \) has a real structure of type \( (-1, 1, 1) \) which is compatible with the above data and which acts on the projective lines \( \mathbb{P}^1 \) as the antipodal map.

**Proof.** In fact, almost everything has been proven (cf also theorem 3.1). It remains to show that by going from (ii) \( \to \) (i) the antipodal map on \( \mathcal{P} \) indeed gives the correct real structure on \( \mathcal{M} \).

To see that the induced real structure \( \rho \) on \( \mathcal{M} \) yields two quaternionic prolongations \( \hat{\rho}_1 : \hat{\mathcal{F}} \to \hat{\mathcal{F}} \) and \( \hat{\rho}_2 : \hat{\mathcal{F}} \to \hat{\mathcal{F}} \), respectively, we apply the arguments of Hitchin et al [30].

In particular, consider \( \hat{\mathcal{F}} = \pi_1(\pi^*_2(\mathcal{O}_{\mathbb{P}^1}(1))) \). Then the prolongation \( \hat{\rho}_2 \), induced by the antipodal map on \( \mathcal{P} \), is given by

\[ \hat{\rho}_2 = \tilde{\alpha}_2 \lambda^1 \bar{\lambda}^2 - \bar{\tilde{\alpha}} \lambda^2 \bar{\lambda}^1. \]

Analogously, the antipodal map induces a quaternionic prolongation \( \hat{\rho}_1 \) on the bundle \( \mathcal{S} = \pi_1(\pi^*_2(\mathcal{O}_{\mathbb{P}^1}(1))) \).

\[ \hat{\rho}_1 = \tilde{\alpha}_2 \lambda^1 \bar{\lambda}^2 - \bar{\tilde{\alpha}} \lambda^2 \bar{\lambda}^1. \]

In a similar fashion, one may make the appropriate changes in theorem 3.2.

Finally, we have the following fact:

**Proposition 4.1.** Let \( \mathcal{M} \) be a civilized right-flat quaternionic RC supermanifold and \( \mathcal{P} \) its associated supertwistor space. Then there is a natural diffeomorphism \( \mathcal{P} \cong \mathcal{M} \rho \). Hence, we obtain a nonholomorphic fibration

\[ \mathcal{P} \to \mathcal{M}_\rho \]

of the supertwistor space over \( \mathcal{M}_\rho \subset \mathcal{M} \). Typical fibres of this fibration are 2-spheres \( S^2 \).

**4.4. Remark**

In the purely bosonic setting and for Euclidean signature, the twistor space has an alternative definition which is equivalent to the definition in terms of the projectivization of the right-chiral spin bundle. Let \( M \) be an oriented Riemannian four-manifold. The twistor space \( P \) of \( M \) can equivalently be defined as the associated bundle [72]

\[ P \cong P(M, SO(4)) \times_{SO(4)} (SO(4)/U(2)) \]

with

\[ P \to M. \]

Typical fibres of this bundle are 2-spheres \( S^2 \cong SO(4)/U(2) \) which parametrize almost complex structures on the fibre \( T_xM \) of \( TM \) over \( x \in M \). Recall that an almost complex structure \( J \) is an endomorphism of the tangent bundle that squares to minus the identity, i.e. \( J^2 = -1 \). Note that while a manifold \( M \) admits in general almost no complex structure, its twistor space \( P \) can always be equipped with an almost complex structure \( J \) (Atiyah et al [72]). Furthermore, \( J \) is integrable if and only if the Weyl tensor of \( M \) is self-dual.
[71, 72]. Then $P$ is a complex three-manifold with an antiholomorphic involution $\rho$ which maps $J$ to $-J$ and the fibres of the bundle (4.4) over $x \in M$ are $\rho$-invariant projective lines $\mathbb{P}^1$, each of which has normal bundle $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2$ in the complex manifold $P$. Here and in the following we make no notational distinction between real structures appearing on different (super)manifolds.

In the supersymmetric setting, the situation is slightly different. Let us consider $\mathcal{M}_\rho$ from above. The tangent spaces $T_x \mathcal{M}_\rho$ for $x \in \mathcal{M}_\rho$ are isomorphic to $\mathbb{R}^{2|2N}$. So most complex structures are parametrized by the supercoset space $\text{OSp}(4|2N)/U(2|N)$, which is a supermanifold of (real) dimension $2 + N(N + 1)|4N$, and whose even part is $\text{OSp}(4|2N)/U(2|N)$.

Thus, the supertwistor space $\mathcal{P} \to \mathcal{M}_\rho$, as viewed in proposition 4.1, cannot be reinterpreted as a space which does describe all possible almost complex structures on $\mathcal{M}_\rho$. Nevertheless, one can view $\mathcal{P}$ as a space describing a certain class of almost complex structures on $\mathcal{M}_\rho$. Remember that the complexified tangent bundle $T \mathcal{M}_\rho \otimes \mathbb{C}$ can be factorized as $T \mathcal{M}_\rho \otimes \mathbb{C} \cong \mathcal{H} \otimes J$. In particular, these complex structures, being compatible with this tangent bundle structure, are again parametrized by 2-spheres $S^2 \cong \text{SO}(4)/U(2)$ and are given in a structure frame by (cf Wolf [12])

$$J_{\hat{a}\hat{b}} = -i\lambda_\hat{a} \lambda_\hat{b} \lambda_{\hat{c}} \delta_{\hat{a}\hat{c}} + \lambda_\hat{b} \lambda_{\hat{c}} \delta_{\hat{a}\hat{c}},$$

(4.6)

where $\lambda_\hat{a}$ are homogeneous coordinates on $\mathbb{P}^1 (\cong S^2)$ and $i(x, y) := \langle \hat{x}, \hat{y} \rangle$ (see also the preceding paragraph). Now one may introduce an almost complex structure $J$ on $\mathcal{P} \hookrightarrow \mathcal{M}_\rho$ by setting $\mathcal{J}_z = J_z \oplus J_z$ for $z \in \mathcal{P}$. Here, $J_z$ is given in terms of (4.6) and $J_z$ in terms of the standard almost complex structure on $S^2$, respectively. In fact, following the arguments of Atiyah et al [72], this description of $J$ does not depend on the choice of local coordinates. Hence, $\mathcal{J}_z$ can be defined for all $z \in \mathcal{P}$ and thus $\mathcal{P}$ is equipped with a natural almost complex structure.

Next one can show that this almost complex structure is integrable if and only if $\mathcal{M}$ is right-flat and furthermore that the fibres of $\mathcal{P} \to \mathcal{M}_\rho$ are $\rho$-invariant projective lines $\mathbb{P}^1$ each having normal bundle $\mathcal{N}_{\mathbb{P}^1,\mathcal{P}}$ inside $\mathcal{P}$ described by

$$0 \longrightarrow \Pi \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^N \longrightarrow \mathcal{N}_{\mathbb{P}^1,\mathcal{P}} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \longrightarrow 0.$$  

(4.7)

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15 For more details, see e.g. Wolf [12].

16 Recall that if $G$ is a Lie supergroup and $H$ a closed Lie subsupergroup (i.e. $H_{\text{red}}$ is closed in $G_{\text{red}}$) then $G/H := (G_{\text{red}}/H_{\text{red}}, \phi_{G/H})$, where $\phi_{G/H}(\mathcal{U}) := \{ f \in \mathcal{O}(\pi^{-1}(\mathcal{U}))) \phi \stackrel{\text{red}}{=} \mathcal{U} f \}$ with $\mathcal{U} \subset H_{\text{red}}, \pi : G_{\text{red}} \to G_{\text{red}}/H_{\text{red}}$ and $\phi : G \times H \to G$ is the canonical projections and $\phi = (\phi, \phi) : G \times H \to G$ is the right action of $H$ on $G$. See Kostant [73] for more details. Hence, $(G/H)_{\text{red}} \cong G_{\text{red}}/H_{\text{red}}$. 
Appendix. A proof of proposition 2.6

Subject of this appendix is to give a proof of proposition 2.6. First, we show the second relation of equations (2.49). The proof of the third one follows similar lines as for the second one. So we omit it at this point and leave it to the reader. Eventually, we prove the first relation.

The curvature components \( R_{ABC}^D \) can be decomposed into irreducible \(^{17}\) pieces as
\[
R_{ABC}^D = C_{ABC}^D + D_{ABC}^D + E_{AB}^D \delta_C^D
\]
\[\text{+}(N - 2)(-)^{p_A(p_A + p_B)}E_{C[A|\delta_B]}^D - 2(-)^{p_C(p_A + p_B)}\Lambda_{C[A|\delta_B]}^D, \quad (A.1)\]
where \( C_{ABC}^D \) and \( \Lambda_{AB} \) obey the properties stated in proposition 2.6 and \( D_{ABC}^D \) is totally trace free.

Upon substituting
\[\text{=1}, \quad \text{=2}, \quad \text{=3}\]
However, the second line vanishes identically as it represents a Bianchi identity for the curvature of the bundle \( \mathcal{M} \rightarrow \mathcal{M} \). Therefore, we end up with

\[
R_{A\beta B\gamma C}^\delta = -2(-)^{pC(pA + pB)}R_{C[A\delta]}\delta_{B]}^\delta. \tag{A.9}
\]

Finally, we note that the form (2.50) of the Ricci tensor can straightforwardly be obtained by substituting equations (2.49) into its definition and by explicitly performing the appropriate index traces. This remark concludes the proof of proposition 2.6.

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