Fractional Graph Coloring for Functional Compression with Side Information

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Abstract—We describe a rational approach to reduce the computational and communication complexities of lossless point-to-point compression for computation with side information. The traditional method relies on building a characteristic graph with vertices representing the source symbols and edges that assign a source symbol to a collection of independent sets to be distinguished for the exact recovery of a function. Our approach uses fractional coloring for a b-fold coloring of characteristic graphs to provide a linear programming relaxation to the traditional coding method and achieves coding at a fine-grained granularity. We derive the fundamental lower bound for compression, given by the fractional characteristic graph entropy, through generalizing the notion of Körner’s graph entropy. We demonstrate the coding gains of fractional coloring over traditional coloring via a computation example. We conjecture that the integrality gap between fractional coloring and traditional coloring approaches the smallest b that attains the fractional chromatic number to losslessly represent the independent sets for a given characteristic graph, up to a linear scaling which is a function of the fractional chromatic number.

I. INTRODUCTION

We consider the problem of point-to-point compression for computing a function with decoder side information. Traditionally, this problem is referred to as Körner’s graph coloring problem [1]. This problem stems from source coding (compression), which has been the subject of extensive study in information theory dating back to the seminal work of Shannon [2], and its many extensions.

A. Coding for Compression

In the traditional compression approach, for a point to point compression of a source variable $X_1$ drawn from distribution $P_{X_1}$, the source coding theorem, as demonstrated by Shannon [2], states that in the limit, as the length of a stream of independent and identically-distributed (i.i.d.) random variable data tends to infinity, the best rate of compression of $X_1$ (quantified in the average number of bits per symbol) is the Shannon entropy of the source, $H(X_1) = \mathbb{E}[-\log P_{X_1}(X_1)]$.

A natural extension of the source coding theorem for networked settings is the problem of distributed compression. The problem of distributed lossless compression dates back to the seminal work of Slepian and Wolf [3], instantiated by random binning of the typical source sequences [4]. For concreteness, consider two random variables $X_1$ and $X_2$, jointly distributed according to $P_{X_1,X_2}$. Given two sequences $X_1^n = (X_{11}, X_{12}, \ldots, X_{1n})$ and $X_2^n = (X_{21}, X_{22}, \ldots, X_{2n})$ drawn i.i.d. from $P_{X_1,X_2}$, Slepian-Wolf Theorem gives a theoretical bound for the lossless coding rate of distributed coding [3]: To recover a joint source $(X_1^n, X_2^n)$ drawn from $P_{X_1,X_2}$ at a receiver that has access to side information $X_2^n$, it is both necessary and sufficient to encode the source $X_1^n$ up to the rate $H(X_1|X_2)$ [3]. Slepian-Wolf problem is a special case of the general distributed function compression problem.

Practical schemes for Slepian-Wolf compression have been proposed by several authors, including [5]–[8]. The generalization of the distributed compression scheme of Slepian-Wolf to trees and to networks beyond depth one uses random linear coding, as shown by Ho et al. in [8]. Distributed communication has also been considered in [8], and by Ahlswede et al. [9] and Yeung in [10] for multicasting under general network settings via random linear network coding.

B. Coding for Functional Compression

Distributed compression of source variables for the purpose of computing a deterministic function across a network, is referred to as distributed functional compression. To that end, since the pioneering work of Slepian-Wolf [3], different techniques have been explored, e.g., computation with decoder side information and functional distortion criterion in Wyner-Ziv settings [11], compression for multiple descriptions of functions [12], special functions such as addition [13], and multiplication with side information [14]. Function-oriented recovery, or compression of $f(X_1, X_2)$ is better understood through the lens of characteristic graph-entropy $H_{G_{X_1}}(X_1)$ [1], which quantifies the minimum number of bits required to represent a function of random variables. This notion was initially devised by Körner [1] for point-to-point compression.

The zero-error side information problem and the rate regions for the functional compression problem have been investigated by Witsenhausen [15], along with the formal introduction of the characteristic graph of $X_1, X_2$ and $f$, by Orlitsky-Roche [16] when one source is fully available at the receiver via conditional graph entropy $H_{G_{X_1}}(X_1|X_2)$, and for restricted and unrestricted inputs by Alon-Orlitsky [17]. The problem of distributed lossless compression has been studied by Doshi et al. [18], and also extended to tree networks (Feizi-Médard [19] and Doshi et al. [20]) via generalizing distributed compression (Slepian-Wolf [3]) to distributed functional compression.

The requirement for structured coding for computing. For distributed compression of a general function, trimming of (independently encoded) Slepian–Wolf partitions may not be feasible. In other instances, good computation codes may only achieve marginal gains in computing capacity over separation-based codes (see e.g., [21] and [19]) at the expense of a significant computation burden on the encoders and decoders. There also exist approaches to compression of graphical data.
in sparse scenarios, e.g., [22]–[24], which may not capture computations of general functions. Hence, designing efficient function-oriented codebooks requires a different vision to alleviate the redundancy of data-oriented encoding.

The need for constructive techniques to compression. For existing graph entropy-based approaches, we refer the reader to [1], [16]–[18], [25]–[27] and the references therein. The proofs of functional compression are by means of coloring characteristics graphs of functions but provide no natural constructive approach to instantiate functional compression.

Our contributions in this paper are summarized as follows:

• In Sect. II, we provide a relaxation to the traditional graph coloring approach (NP-hard) for functional compression. This is possible through fractional coloring, by generalizing traditional graph coloring that assigns one color per vertex. An $a:b$ fractional graph coloring assigns $b$ colors out of a total of $a$ available colors to each vertex of a graph such that adjacent vertices have disjoint colors.

• We introduce the concept of the fractional chromatic entropy and characterize it in Prop 1 exploiting Han’s theorem [28], which states that the average entropy decreases monotonically in the size of the subset. In Prop. 2, we derive the analytical expression for the fractional graph entropy, $H_{G_{X}}^f (X_1 | X_2)$, that gives the achievable rate for functional compression with side information.

• Sect. III presents our main results providing a rate bound on functional compression using fractional coloring. Exploiting several properties of fractional chromatic number, we state that $H_{G_{X}}^f (X_1 | X_2)$ lower bounds $H_{G_{X}} (X_1 | X_2)$ (Lemma 1). This new notion provides a refinement in coloring such that on average less colors are spent and the communication complexity is reduced.

• Sect. IV provides several bounds on the integrality gap between fractional and traditional colorings. To contrast the potential rate savings of our approach over traditional coloring, we show that the integrality gap is a monotonically increasing function of the source sequence length $n$ (Lemma 2), and approximate the integrality gap, and observe that (Prop. 3) it scales linearly with $b$, given a valid $b$-fold coloring. Hence, our approach yields a reduced communication complexity by a factor of $b$ (up to a linear scaling), in the number of communication or exchanged bits [29], [30], versus the traditional approach.

II. PROBLEM SETUP

We consider the problem of lossless distributed functional compression with side information introduced in [16] (via generalizing [11] using a characteristic graph approach). The encoder has source $X_1$. The decoder has source $X_2$, which is not accessible at the encoder side. Given two statistically dependent i.i.d. finite-alphabet random sequences $X_1^n$ and $X_2^n$, our goal is to give a theoretical bound for the lossless coding rate to encode $X_1$ for computing a function $f(X_1, X_2)$ to achieve arbitrarily small error probability for long sequences. Both the encoder and the decoder knows the function. In [16] and its extensions, this problem, for the zero-error setting, has been tackled using a traditional coloring approach and it has been proven that a zero-error compression up to a rate $H_{G_{X}} (X_1 | X_2)$ is possible when $X_2$ is available at the receiver.

A. Traditional Coloring of Characteristic Graphs

Let $G_{X}$ be the characteristic graph the encoder builds for computing the function $f(X_1, X_2)$, determined as function of $X_1, X_2$, and $f$. The characteristic graph is denoted by $G_{X} = (V_{G_{X}}, E_{G_{X}})$, where $V_{G_{X}} = X_1$ and an edge $(x_1, x_2) \in E_{G_{X}}$, if and only if there exists a $x_3 \in X_2$ such that $p(x_1, x_2, x_3) > 0$ and $f(x_1, x_2) \neq f(x_1, x'_2)$. We assign different codes (colors) to connected vertices, which corresponds to a graph coloring. Vertices that are not connected to each other can be assigned to the same or different colors. In this paper, we only consider vertex colorings. A valid coloring of a graph $G_{X}$ is such that each vertex of $G_{X}$ is assigned a color such that adjacent vertices receive disjoint colors.

To better motivate our approach and demonstrate the achievable description lengths, we illustrate the relevance of characteristic graph in compression via the following example.

Example 1. A characteristic graph and its entropy. Random variables $X_1$ and $X_2$ are over the alphabet $\mathcal{X} = \{-2, -1, 0, 1, 2\}$. The joint distribution has ordered entries:

$$P_{X_1, X_2} = \begin{bmatrix}
0.1 & 0.1 & 0 & 0 & 0 \\
0.1 & 0 & 0 & 0 & 0.1 \\
0 & 0.1 & 0.1 & 0 & 0 \\
0 & 0.1 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0.1 & 0.1 \\
\end{bmatrix},$$

where $X_1$ is uniformly distributed, and $f(X_1, X_2) = X_1 + X_2$ such that $G_{X_1}$ denotes the characteristic graph the encoder builds, where $V_{G_{X_1}} = X_1$ and $E_{G_{X_1}} = \{(-2, -1), (-2, 0), (0, 1), (1, 2), (2, -1)\}$. A valid coloring of $G_{X_1}$, denoted by $c_{G_{X_1}} (X_1)$ and shown in Fig. 1 (Left), has a distribution $P(c_1) = P(c_2) = 0.4$ and $P(c_3) = 0.2$ over $\{c_1, c_2, c_3\}$. This yields an entropy $H(c_{G_{X_1}} (X_1)) \approx 1.52$.

Next, we encode a source sequence with length two, $X_1^2 = (X_{11}, X_{12})$, which can take 25 values $\{(-2, -2), (-2, -1), (-2, 0), \ldots, (2, 2)\}$. To construct the characteristic graph for $X_1^2$, i.e., the second power graph $G_{X_1^2}$, we connect two vertices if at least one of the coordinates are connected in $G_{X_1}$. It is possible to color $G_{X_1^2}$ using 8 colors. The entropy of this coloring satisfies $\frac{1}{2} H(c_{G_{X_1^2}}) \approx 1.48 < H(c_{G_{X_1}}) \approx 1.52 < H(X_1) \approx 2.32$.

The chromatic number $\chi(G_{X_1})$ of a graph $G_{X_1}$ is the minimum number of colors needed to color the vertices in such a way that no two adjacent vertices have the same color.
Definition 1. (Chromatic entropy [17].) The chromatic entropy of a graph $G_{X_1}$ is defined as
\[
H_{X_1}^X(X_1|X_2) = \min_{c_{G_{X_1}}} \mathcal{H}^X(c_{G_{X_1}}(X_1)) ,
\]
where $\mathcal{H}^X(c_{G_{X_1}}(X_1)) = \{H(c_{G_{X_1}}(X_1)) : c_{G_{X_1}}(X_1) \text{ is a valid coloring of } G_{X_1} \}$ is the set of chromatic entropies over the set of valid colorings of $G_{X_1}$.

Let $G_{X_1}^n = (V_{X_1}^n, E_{X_1}^n)$ be the $n$-th power of a graph $G_{X_1}$ such that $V_{X_1}^n = X_1^n$ and $\{x_1^n, x_2^n\} \in E_{X_1}^n$, where $x_1^n = (x_1^{1_1}, x_1^{1_2}, \ldots, x_1^{1_n})$ and similarly for $x_2^n$, when there exists at least one coordinate $i \in \{1, 2, \ldots, n\}$ such that $(x_1^{1_i}, x_2^{1_i}) \in E_{X_1}$. We denote a coloring of $G_{X_1}^n$ by $c_{G_{X_1}^n}(X_1).$ Körner showed in [1] that, in the limit of large $n$, the chromatic entropy and the graph entropy are related as
\[
H_{G_{X_1}}(X_1|X_2) = \lim_{n \to \infty} \frac{1}{n} \min_{c_{G_{X_1}^n}} H(c_{G_{X_1}^n}(X_1)|X_2). \quad (3)
\]
The entropy of $c_{G_{X_1}^n}(X_1)$ characterizes the minimal description length needed to reconstruct with fidelity $f(X_1, X_2)$ [1]. The case of the identity function yields a complete graph.

B. Fractional Coloring of Characteristic Graphs

Fractional graph coloring is a natural extension of traditional coloring, where each vertex is assigned a set of colors and the adjacent vertices have disjoint sets. Traditional graph coloring problems may not be amenable to a linear programming approach. Solving (3) is equivalent to determining a coloring random variable which minimizes the entropy. However, finding the minimum entropy coloring of $G_{X_1}$ is an NP-hard problem [32]. To solve the coloring problem losslessly in polynomial time, we exploit the fractional coloring relaxation.

Definition 2 (Scheinerman and Ullman [31]). A valid $b$-fold coloring is an assignment of sets of size $b$ to vertices such that adjacent vertices receive disjoint sets of colors. A valid $a:b$ coloring is a valid $b$-fold coloring out of available colors. The notation $\chi_a(G)$ represents the $b$-fold chromatic number of graph $G$ that is the least such that an $a:b$ coloring exists.

The chromatic number $\chi(G)$ is subadditive, i.e., $\chi_{a+b}(G) \leq \chi_a(G) + \chi_b(G)$. If $g : \mathbb{Z}^+ \to \mathbb{R}$ is subadditive and $g(b) \geq 0$ for all $b$, then from the sub-additivity lemma [31], the limit $\lim_{b \to \infty} \frac{g(b)}{b}$ exists and is equal to the infimum of $\frac{g(b)}{b}$ ($b \in \mathbb{Z}^+$).

Fig. 2. The second power graph $G_{X_1}$ for Example 1, where $a = 8$ is the minimum number of colors for which an $a:1$ coloring is possible.

Fig. 3. A fractional coloring of $G_{X_1}$ for Example 1, where $a = 13$ is the minimum number of colors for which an $a:2$ coloring exists.

Definition 3. The fractional chromatic number is defined as
\[
\chi_f(G) := \lim_{b \to \infty} \frac{\chi_1(G)}{b} = \inf_b \frac{\chi_b(G)}{b} , \quad (4)
\]
where the existence of this limit follows from the sub-additivity of $b$-fold colorings, and the sub-additivity lemma [31].

From a probabilistic perspective, $\chi_f(G)$ represents the smallest $k$ for which there is a distribution over the independent sets (an independent set is a set of vertices in a graph, no two of which are adjacent) of $G$ such that for each vertex $v$, given an independent set $I$ drawn from the distribution, $\mathbb{P}(v \in I) \geq \frac{1}{k}$. Let $\mathcal{I}(G)$ be the set of all independent sets of $G$, and $\mathcal{I}(G, x)$ be the set of all those independent sets which include vertex $x$, and $x_I \in \mathbb{R}^+$ for each independent set $I$. Then, the fractional chromatic number $\chi_f(G)$ can be obtained as a solution of the following linear program [31]:
\[
\chi_f(G) = \min_{x \in \mathcal{I}(G)} \left\{ \sum_{I \in \mathcal{I}(G)} x_I : \sum_{I \in \mathcal{I}(G, x)} x_I \geq 1, x_I \geq 0 \right\}. \quad (5)
\]
This relaxation transforms traditional coloring, which is an integer programming problem (NP-complete), into a fractional coloring problem. We illustrate fractional coloring for Example 1 in Fig. 1 (right). A $5:2$ coloring achieves the optimal solution of (5), where $|\mathcal{I}(G)| = 10$, with 5 of those sets have cardinality 2, and 5 sets have size 1, and $|\mathcal{I}(G, x)| = 3$ such that $x_I = 0.5$ for sets of size 2 and $x_I = 0$ for the sets with size 1. Hence, $\chi_f(G_{X_1}) = 5$ and from (4) $\chi_f(G_{X_1}) = 2.5$.

Example 1 indicates that assigning colors to sufficiently large power graphs, we can compress $X_1$ more. A $5:2$ coloring yields $H_{G_{X_1}}^X(X_1|X_2) = 1.16$, providing a saving of 0.36 bits over $H_{G_{X_1}}^X(X_1|X_2) = 1.52$. We sketch the traditional coloring for $G_{X_1}^n$ in Fig. 2. The distribution of colors satisfies $P(c_k) = 0.16$, $k \in \{1, \ldots, 5\}$ and $P(c_6) = 0.08$, $k \in \{6, 7\}$ and $P(c_8) = 0.04$. Hence, $H_{G_{X_1}}^X(X_1|X_2) = 1.44$. For a $13:2$ coloring (Fig. 3), the colors satisfy $P(c_k) = 0.08$, $k \in \{1, \ldots, 12\}$ and $P(c_{13}) = 0.04$. Hence, $H_{G_{X_1}^{13}}^X(X_1|X_2) = 0.92$, i.e., 0.52 bits of savings from $H_{G_{X_1}}^X(X_1|X_2)$. Due to space constraints, we next focus on the theoretical notions and the achievable gains pertaining to fractional coloring. We explain an example code construction in [33] to demonstrate the practical savings of the proposed model.
III. Fractional Chromatic Entropy

We next formalize the notion of fractional chromatic entropy of a set of valid fractional colorings via extending Defn. 1.

Definition 4. (Fractional chromatic entropy.) $c^f_{G_{X_1}}(X_1)$ is a valid $a:b$ fractional coloring of $G_{X_1}$ if it assigns $b$ colors out of a total of $a$ available colors to each $X_1$, such that adjacent vertices have disjoint colors. We define $H^{f^f}(c^f_{G_{X_1}}(X_1)) = \{H(c^f_{G_{X_1}}(X_1)), c^f_{G_{X_1}}(X_1)\}$ is a valid $a:b$ coloring of $G_{X_1}$ to be the collection of fractional chromatic entropies over the set of all valid $a:b$ colorings of $G_{X_1}$, given $X_2$.

Proposition 1. The fractional chromatic entropy of a characteristic graph $G_{X_1}$, denoted by $H^{f^f}_{G_{X_1}}(X_1|X_2)$, is given as

$$H^{f^f}_{G_{X_1}}(X_1|X_2) = \inf_{\mathbf{b}} \left[ \frac{1}{b} \min_{c^f_{G_{X_1}}} \mathcal{H}^{f^f}(c^f_{G_{X_1}}(X_1)) \right].$$

Proof. Following from combining the definition in (6) and (2), we obtain (6). W

Proposition 2. The fractional graph entropy is given as

$$H^{f^f}_{G_{X_1}}(X_1|X_2) = \lim_{n \to \infty} \frac{1}{n} \inf_{\mathbf{b}} \left[ \frac{1}{b} \min_{c^f_{G_{X_1}}} \mathcal{H}^{f^f}(c^f_{G_{X_1}}(X_1)) \right].$$

Using (6) and (10) we can derive the fractional graph entropy, a natural generalization of (3). Prop. 2 is derived from Prop. 1 and (10), and we skip its proof.

IV. Coding Gains of Fractional Coloring

We denote the integrality gap (IG), i.e., ratio of the solutions of the fractional coloring versus the fractional coloring problems for encoding $G_{X_1}$, by $IG_n$. It is given as

$$IG_n = \frac{H_{G_{X_1}}^X(X_1|X_2)}{H_{G_{X_1}}^X(X_1|X_2)}, \quad n \geq 1.$$ 

From Lemma 1, it is immediate that $IG_n \geq 1$ for $n \geq 1$.
Proposition 3. \( b_k \) decreases with a higher rate versus \( a_k \). We infer that \( 1 \leq a_1/b_1 \leq a_2/b_2 \leq \ldots \leq a_n/b_n \). Hence, we can show that
\[
\frac{a_1}{b_1} - \frac{a_1 + a_2}{b_1 + b_2} - \ldots \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = IG_n.
\]
Hence, \( IG_n \) is an increasing function of \( n \).

In Example 1, \( IG_1 = 1/2 = 0.5 \), \( IG_2 = 1/2 + 1/2 = 1.0 \), and \( IG_n \) is higher for \( n > 2 \) (Lemma 2). The gain is due to cross-coding across graphs (Prop. 1). For the identity function, the graph \( G_{X_1} \) is complete, \( b^* = 1 \), and fractional coloring does not have savings. Significant gains are possible for sparse graphs.

For a valid \( a:b \) coloring of \( G_{X_1} \), let \( b^*_G \) be the smallest \( b \in [S] \) that achieves (6) for \( G_{X_1} \), and \( c_{G_{X_1}}(X_1) \) and \( c_{G_{X_1}}^f(X_1) \) be the valid colorings with distributions \( q = (q_1, \ldots, q_X(G_{X_1})) \) and \( r = (r_1, \ldots, r_X(G_{X_1})) \) that minimize the respective entropies of the colorings. Similarly, for \( n > 1 \), let \( b^*_G \) be the smallest \( b \in [S] \) for \( G_{X_1} \) satisfying (4), \( c_{G_{X_1}}(X_1) \) and \( c_{G_{X_1}}^f(X_1) \) be the valid colorings with \( q^n = (q_1, \ldots, q_X(G_{X_1})), r^n = (r_1, \ldots, r_X(G_{X_1})) \).

Proposition 3. The fractional coloring scheme attains
\[
IG_n \geq b^*_G \chi H(q^n) + \Delta_G^n,
\]
where
\[
\Delta_G^n = \sum_{j \in J_G} \left[ h \left( \frac{1}{b_{G_{X_1}}} \right) + b_{G_{X_1}}^{c_{G_{X_1}}(X_1)} - \log(m_G(j)(b_{G_{X_1}}^{c_{G_{X_1}}(X_1)} - 1)) \right], \quad j \in J_G = \{1, \ldots, \chi(G_{X_1}(s_1))\}.
\]

Proof. We first focus on \( n = 1 \). Then,
\[
IG_1 = b^*_G H(q^{X_{1}}(G_{X_1})) = b_{G_{X_1}} H(q) = b_{G_{X_1}} H(q).
\]
Using the grouping property of entropy [34, Ch. 2.],
\[
H(r) = H(q) + \sum_{j \in J_G} q_j h \left( \frac{r_j}{q_j} : \sum_{i \in j} r_{i} = q_j \right).
\]
(15)

Next we analyze the RHS of (15) for a given \( j \), assuming that there exist \( m_G(j) \) vertices in \( G_{X_1} \), in color class \( j \). Then it holds that in the \( b \)-fold coloring scheme \( q_j \) accumulates the probabilities of \( m_G(j) \times b \) vertices in \( G_{X_1}(s_1) \). The marginal distribution of colors in each \( G_{X_1} : i \in S \) is identical. Hence,
\[
\frac{r_j}{q_j} = \frac{1}{b} \frac{1}{r_{j}} \frac{1}{m_G(j)b} \leq H \left( \frac{1}{b} \frac{1}{m_G(j)b} \right) = \frac{1}{b} \left( \frac{1}{m_G(j)b} \right) + \frac{1}{b} \log(m_G(j)(b - 1))
\]
where the colors of \( G_{X_1} \) and \( G_{X_1} \) are the same, putting \( b \) of the mass of \( q_j \) in \( G_{X_1}(s_1) \), and leaving \( b - 1 \) of the mass to the remaining \( b - 1 \) graph replicas \( \{G_{X_1} : i \in S, i \neq j\} \) with \( m_G(j)(b - 1) \) vertices. The RHS holds when the colors are uniformly split among \( m_G(j)(b - 1) \) vertices. Hence,
\[
IG_1 \geq b^*_G \chi H(q) \geq \frac{b^*_G H(q)}{H(q) + \Delta_G^n}.
\]

For \( n > 1 \), employing the grouping property we can obtain the final result. We note that the count of class \( j \) vertices \( m_G(j) \) is less than \( m_G(j)^n \) for \( j \in J_G = \{1, \ldots, \chi(G_{X_1})\} \).

From Prop. 3, for any given \( G_{X_1} \), if not a complete graph, fractional coloring for \( n > 1 \) offers compression savings.

The following corollary implies theoretically that the achievable \( IG_n \) is lower bounded by \( b^*_G \) (up to a scaling), for a valid \( b^*_G \)-fold coloring that is determined by the function to be computed and its \( n \)-th power graph \( G_{X_1}^n \).

Corollary 2. Under the assumption that the colorings of \( G_{X_1} \) are uniform, the fractional coloring scheme attains
\[
IG_n \geq b^*_G \chi \left( \frac{1}{b} \log \chi H(f(G_{X_1})) / \log \chi b^*_G \right)
\]
(16)

Proof. Let \( b^*_G \) and \( b^*_G \) be the smallest \( b \) values such that (4) holds for \( G_{X_1} \) and \( G_{X_1}^n \), respectively. Then, provided that the colorings of \( G_{X_1} \) are uniform, we can simplify the expressions for \( \chi(H(q)) \) and \( H(r) \) and obtain
\[
IG_n \geq \frac{1}{b} \log \chi H(f(G_{X_1})) / \log \chi b^*_G \chi(n)
\]
(17)

or
\[
IG_n \geq b^*_G \chi(n) / \log \chi b^*_G \chi(n)
\]
(18)

where the lower bound (16) follows from using (13), and (17) is due to (14). Employing the relations \( \chi(G_{X_1}) = \chi_{b^*_G}(G_{X_1}) \), (14) yields \( \chi(f(G_{X_1})) = \chi_{b^*_G}(G_{X_1}) \), and \( \chi(b^*_G(G_{X_1})) \geq \chi_{b^*_G}(G_{X_1}) \), we obtain \( b^*_G \) \( \chi(n) \), which yields (18).

Discussion and future directions. Fractional coloring exploits the possibility of cross-coding between graphs and provides coding gains (at infinite and finite source sequence lengths). This approach provides a reduced communication complexity versus traditional coloring via decreasing the number of bits to send roughly from \( \log \chi \) to \( \log \chi f \). Lower bounding \( IG_n \) for different classes of functions and source distributions is of primary importance. Quantifying and upper bounding the IG in the limit of large \( n \), i.e., the ratio of the bits required by fractional compression, \( H(f(X_1 | X_2), H_{X_1}(X_1 | X_2)) \), is left as future work.

While fractional coloring is a less combinatorial problem than traditional coloring and accepts a linear programming solution (solvable in polynomial time), finding an independent set is strongly NP-hard [36]. Hence, the relaxation is in the class of NP-hard problems. This issue can be alleviated via using fractional edge coloring (versus fractional vertex coloring) for which a polynomial-time solution exists [31]. The characteristic graph approach is concerned with functional compression of source sequences when the adjacency matrix is a \( (0,1) \)-matrix. A possible generalization includes edge-weighted graphs to capture the distortion in reconstruction.

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