Abstract. Constants of motion in Mechanics are usually inferred from groups of symmetry transformations of the given system, as, for example, a Lagrangian function that is time-invariant implies the conservation of energy. Here we wish to show that useful properties of a mechanical system can sometimes be deduced from a family of Noether-like transformations that are not inspired by any symmetry whatsoever. The sample system we concentrate on is the Lagrangian interpretation of Poincaré’s half plane of hyperbolic geometry, and the properties we will derive in a new way are the shape and the time parameterization of its geodesics.

Dedicated to our thesis advisors Giuseppe Da Prato and Aldo Bressan

1. INTRODUCTION. In the first half of the 19th century Nikolai Lobachevsky and János Bolyai introduced the hyperbolic geometry as an axiomatic theory, but, within a few decades, concrete models were worked out first by Eugenio Beltrami, and then by Felix Klein, in particular some called after Poincaré, see [1,4]. Here we are interested in the model that is called Poincaré’s half plane, which can be described like this: in the ambient set $D = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ we will call a “straight line” any half-circle with center on the $x$ axis and any vertical half-line. As in Figure 1, given a “straight line” $r$ (the thicker half-circle) and a point $P$, there are infinitely many “straight lines” that contain $P$ and do not intersect $r$.

Poincaré’s half plane can be formulated in the language of differential geometry, because the “straight lines” coincide with the geodesics with respect to the metric $(\frac{1}{0} \frac{0}{1})/y^2$. Then there is the point of view of variational mechanics, on which we will concentrate. The “straight lines” are the variational stationary points, or natural motions, for the action defined by the following Lagrangian function

$$L(t, q, \dot{q}) = \frac{\dot{q}_1^2 + \dot{q}_2^2}{2q_2^2}, \quad t \in \mathbb{R}, \quad q \in D, \quad \dot{q} \in \mathbb{R}^2,$$

where we switch to the notation that is familiar in mechanics: $(x, y) = q = (q_1, q_2)$. We were attracted to this Lagrangian formulation because we were in search of applications of our theory of “nonlocal constants of motion.” In so doing, we happened to find an unusual derivation of the well-known fact that the nonconstant motions for $L$ are indeed half-circles and vertical half-lines. The calculations are simple enough to be presented to a general audience, while the specialist may find it surprising that useful constants of motion can be found without appealing to Noether symmetries.

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[Monthly 130]
2. NONLOCAL CONSTANTS OF MOTION. By natural motions of a Lagrangian $L$ we mean the solutions to the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(t, q(t), \dot{q}(t)) - \frac{\partial L}{\partial q}(t, q(t), \dot{q}(t)) = 0,$$

which, for our Lagrangian (1) in Poincaré’s half-plane, become

$$\ddot{q}_1 - \frac{2}{q_2} \dot{q}_1 \dot{q}_2 = 0, \quad \ddot{q}_2 + \frac{1}{q_2} (\dot{q}_1^2 - \dot{q}_2^2) = 0.$$

This Lagrangian system has the following functions $E, p$ as immediate first integrals

$$\frac{\dot{q}_1^2 + \dot{q}_2^2}{2q_2^2} = E, \quad \frac{\dot{q}_1}{q_2} = p.$$

The first is energy, which is a property of any time-independent Lagrangian $L(t, q, \dot{q}) = \mathcal{L}(q, \dot{q})$, through the formula

$$E = \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) \cdot \dot{q} - \mathcal{L}(q, \dot{q})$$

(where the central dot is the scalar product in $\mathbb{R}^n$). The function $p = \dot{q}_1/q_2^2$ (the first component of momentum), is constant because of the lack of $q_1$ in the Lagrangian. Indeed, from the Euler-Lagrange equations (2)

$$\frac{\partial L}{\partial q_1}(t, q(t), \dot{q}(t)) \equiv 0 \implies \frac{\partial L}{\partial \dot{q}_1}(t, q(t), \dot{q}(t)) = \text{constant}. \quad (6)$$

Clearly, $E > 0$ for nonconstant geodesics.

In the spirit of Noether’s theorem, energy conservation is a consequence of the invariance of the Lagrangian by time-translations, and the other first integral comes from invariance under $q_1$-translations.

Dilations leave the Lagrangian invariant too, yielding one further independent first integral. In a previous paper [2, subsection 7.2.2], we made full use of the first integral. Here, we make a point to avoid it.

What we do is to consider $q_2$-translations. This may seem ludicrous at first sight, because $q_2$-translations do not leave $L$ invariant. Nevertheless, it makes perfect sense within our framework that generates nonlocal constants of motion, which can be stated simply enough.

**Theorem 1.** Let $L(t, q, \dot{q})$ be a smooth scalar valued Lagrangian function, $t \in \mathbb{R}$, $q, \dot{q} \in \mathbb{R}^n$. Let $q(t)$ be a solution to the Euler-Lagrange equation and let $q_{\lambda}(t), \lambda \in \mathbb{R}$, be a smooth family of perturbed motions, such that $q_0(t) \equiv q(t)$. Then the following
function of \( t \) is constant

\[
\frac{\partial L}{\partial \dot{q}}(t, q(t), \dot{q}(t)) \cdot \frac{\partial q_\lambda}{\partial \lambda}(t) \bigg|_{\lambda=0} - \int_{t_0}^t \frac{\partial}{\partial \lambda} L(s, q_\lambda(s), \dot{q}_\lambda(s)) \bigg|_{\lambda=0} ds .
\]

(7)

The proof is straightforward: we just take the derivative of the function in (7) and use the Euler-Lagrange equation and reverse the order of a double derivative.

This expression (7) will be called the constant of motion associated to the family \( q_\lambda(t) \). For a random family, we may expect the constant of motion to be trivial or inconsequential. In general it is nonlocal, which means that its value at a time \( t \) depends not only on the current state \( (t, q(t), \dot{q}(t)) \) at time \( t \), but also on the whole history between an arbitrary \( t_0 \) and \( t \).

In the original spirit of Noether’s theorem we focused our attention on the families \( q_\lambda(t) \) which make the integrand in (7) vanish whenever \( L \) enjoys an invariance property. For instance, the \( q_1 \)-translation family \( q_\lambda(t) = (q_1(t) + \lambda, q_2(t)) \), if plugged into formula (7) yields the first integral \( p \) in (4) for the geodesics of Poincaré’s half-plane.

As we will show in Section 3, with \( q_2 \)-translations the integral of formula (7) does not disappear, but, in spite of that, we will put the nonlocal constant of motion to good use, as it allows for a nonstandard separation of variables.

Theorem 1 is part of a line of research that we started in 2014. The interested reader may find references and a survey of applications in [3].

3. INTEGRATION OF THE GEODESICS. Consider the Lagrangian (1). As announced, let us apply Theorem 1 with the \( q_2 \)-translation family \( q_\lambda(t) = (q_1(t), q_2(t) + \lambda) \). Then

\[
\frac{\partial q_\lambda}{\partial \lambda}(t) \bigg|_{\lambda=0} = (0, 1), \quad \frac{\partial}{\partial \lambda} L(t, q_\lambda(t), \dot{q}_\lambda(t)) \bigg|_{\lambda=0} = -\frac{\dot{q}_1(t)^2 + \dot{q}_2(t)^2}{q_2(t)^3},
\]

and the nonlocal constant of motion becomes

\[
\frac{\dot{q}_2(t)}{q_2(t)^2} + \int_{t_0}^t \frac{\dot{q}_1(s)^2 + \dot{q}_2(s)^2}{q_2(s)^3} ds = -\frac{d}{dt} \frac{1}{q_2(t)} + 2E \int_{t_0}^t \frac{1}{q_2(s)} ds
\]

(8)

where we have used the energy conservation, i.e., the fact that the \( E \) in (4) is constant along the motions. Since (8) is constant along any solution to the Euler-Lagrange equation, its time derivative vanishes:

\[
-\frac{d^2}{dt^2} \frac{1}{q_2(t)} + 2E \frac{1}{q_2(t)} = 0.
\]

(9)

This linear equation in the variable \( 1/q_2 \) is integrated at once.

If \( E = 0 \) we already know that \( (q_1, q_2) \) is constant. If \( E > 0 \) the generic solution of (9) is

\[
q_2(t) = \left( c_1 e^{t\sqrt{2E}} + c_2 e^{-t\sqrt{2E}} \right)^{-1}.
\]

(10)

Since \( q_2(0) > 0 \) it must be that \( c_1 + c_2 > 0 \). Then eliminate \( \dot{q}_1 \) from the conservation laws in (4), apply formula (10), and set \( t = 0 \). This way we obtain a necessary condition for \( c_1, c_2 \) to give a geodesic:

\[
8c_1c_2E = p^2.
\]

(11)

If \( p = 0 \) the geodesic is contained in a vertical half-line because the conservation law \( \dot{q}_1 = pq_2^2 \) implies that \( q_1(t) \) is constant; also, formula (11) gives that either \( c_1 = 0 \)
or \( c_2 = 0 \). The geodesic then becomes fully explicit as \( q_2(t) = e^{\pm t \sqrt{2E}/c} \) with the constant \( c = 1/q_2(0) > 0 \). In either case \( t \) ranges in the whole of \( \mathbb{R} \). We can check that the original Euler-Lagrange equations (3) are verified too.

In the nondegenerate case \( p \neq 0 \), both \( c_1 > 0 \) and \( c_2 > 0 \), \( q_2(t) > 0 \) is defined for all \( t \in \mathbb{R} \), and we can deduce \( q_1 \) using \( \dot{q}_1 = pq_2^2 \) up to an integration constant \( c_3 \):

\[
q_1(t) = c_3 - \frac{p}{2c_1 \sqrt{2E}(c_2 + c_1 e^{2t \sqrt{2E}})},
\]

also defined for all \( t \in \mathbb{R} \). Let us write \( x = q_1(t) \), \( y = q_2(t) \) and eliminate the exponentials from formulas (10) and (12). We obtain the relation

\[
\frac{8c_1c_2E}{p^2} \left( x - c_3 + \frac{p}{2c_1c_2 \sqrt{8E}} \right)^2 + y^2 = \frac{1}{4c_1c_2},
\]

which is the equation of an ellipse. If we finally impose the necessary condition (11), the ellipse becomes a circle

\[
(x - c)^2 + y^2 = \frac{2E}{p^2}, \quad c = c_3 - \frac{1}{2 \sqrt{c_1c_2}},
\]

with center on the \( x \) axis and radius \( \sqrt{2E}/|p| \).

We have proved that the geodesics of the Poincaré half-plane are either half-lines or half-circles using the two conservation laws for \( E \) and \( p \) plus the nonlocal constant of motion (8).

Even better, the relation (11) is not only necessary, but also sufficient for the explicit formulas (12) and (10) to provide geodesics of the Poincaré half-plane, as can be easily verified by direct replacement into the Euler-Lagrange equations (3). The problem of finding the geodesics is fully solved.

On Poincaré’s half-plane every couple of points are connected by a geodesic. Figure 1 shows that given a geodesic \( r \) and a point \( P \) outside, there exist infinitely many geodesics through \( P \) that do not cross \( r \).

Remark 1. Equation (9), once it is known, can also be checked with an algebraic manipulation of the two first integrals \( E, p \) in (4) and some time derivatives, without referring to Theorem 1.

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