Higher-Order threshold effects to inclusive processes in QCD

V. Ravindran

Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad, India.

ABSTRACT

We present threshold enhanced QCD corrections to inclusive processes such as Deep inelastic scattering, Drell-Yan process and Higgs productions through gluon fusion and bottom quark annihilation processes using the resummed cross sections. The resummed cross sections are derived using renormalisation group invariance and mass factorisation theorem that these hard scattering cross sections satisfy and Sudakov resummation of QCD amplitudes. We show how these higher order threshold QCD corrections improve the theoretical predictions for the Higgs production through gluon fusion at hadron colliders.
Perturbative Quantum Chromodynamics (pQCD) provides a framework to successfully compute various observables involving hadrons at high energies. Wealth of precise data from hadronic experiments such as deep inelastic scattering (DIS) experiments at HERA complimented with the theoretical advances that lead to enormous success in computing the relevant observables very precisely has given us a better understanding of the structure of hadrons and also the strong interaction dynamics in terms of their constituents such as quarks and gluons at wide range of energies. With these successes we can now predict most of the important observables with less theoretical uncertainty for the physics studies at present collider Tevatron in Fermi-Lab as well as at the upcoming Large Hadron Collider (LHC) in CERN [1].

The Drell-Yan (DY) production of di-leptons is one of the important processes at hadronic colliders. It will serve not only as a luminosity monitor but also provide vital information on physics beyond Standard Model (SM). The other process which is equally important is Higgs boson production at such colliders because it will establish the Standard Model as well as beyond SM Higgs [2, 3]. In pQCD, the DY production of di-leptons and Higgs boson production are known up to next to next to leading order (NNLO) level in QCD [4–16]. Beyond NLO, the Higgs production cross sections are known only in the large top quark mass limit. Apart from these fixed order results, the resummation programs for the threshold corrections to both DY and Higgs productions have also been very successful [17, 18] (see also [19]). For next to next to leading logarithmic (NNLL) resummation, see [20, 21]. Due to several important results at three loop level that are available in recent times [22–27], the resummation up to $N^{3}LL$ has also become reality [28–31]. These results in fixed order as well as resummed calculations unravel the interesting structures in the perturbative results (for example: [31–34]).

In the QCD improved parton model, the infra-red safe observables, say, hadronic cross sections can be expandable in terms of perturbatively calculable partonic cross sections appropriately convoluted with non-perturbative operator matrix elements known as parton distribution functions (PDF). This is possible due to the factorisation property that certain hard scattering cross sections satisfy. The partonic cross sections are calculable in powers of the strong coupling constant $g_s$ within the framework of perturbative QCD because the coupling constant becomes small at high energies. On the other hand we only know the perturbative scale evolution of the parton distribution functions which otherwise are known/extracted from experiments.

The fixed order QCD predictions have limitations in applicability due to the presence of various logarithms of kinematical origins. These logarithms become large in some kinematical regions which otherwise can be probed by the experiments. In such regions, the applicability of fixed order perturbative results becomes questionable due to the missing higher order corrections that are hard to compute with the present day techniques. The alternate approach to probe these regions is to resum these logarithms in a closed form. Such an approach of resuming a class of large logarithms supplemented with fixed order results can almost cover the entire kinematic region of the phase space. In addition, these threshold corrections are further enhanced when the the flux of the incoming partons become large in those regions. In the case of Higgs production through gluon fusion, the gluon flux at small partonic energies becomes large improving the role
of threshold corrections. Here, we consider the inclusive cross sections of hadronic cross reactions such as deep inelastic scattering, DY, Higgs production through gluon fusion and bottom quark annihilation and study the effects of soft gluons that originate in the threshold region of the phase space. In these processes, large logarithms are generated when the gluons that are emitted from the incoming/outgoing partons become soft.

In [31], we extracted the soft distribution functions of Drell-Yan and Higgs production cross sections in perturbative QCD and showed that they are maximally non-abelien. That is, we found that the soft distribution function of Higgs production can be got entirely from the DY process by a simple multiplication of the colour factor $C_A/C_F$. In this article we extend the applicability of this method to the other important process, namely Higgs production through bottom quark annihilation. Using the soft distribution functions extracted from DY, and the form factor of the Yukawa coupling of Higgs to bottom quarks, we can now predict soft plus virtual part of the Higgs production through bottom quark annihilation beyond NNLO with the same accuracy that DY process and the gluon fusion to Higgs are known [28, 31]. This generalises our earlier approach to include any infrared safe inclusive cross section. In [31] we determined the threshold exponents $D_I^P$ upto three loop level for DY and Higgs productions using our resummed soft distribution functions. In this present work we provide all order proof which establishes the relation between soft distribution functions and the threshold resummation exponents and demonstrate the usefulness of this approach to derive higher order threshold enhanced corrections for any infrared safe inclusive cross sections. The results on DIS in this article provide a consistency check on various approaches (see [28, 31]) in the literature to study the soft part of the cross sections because some of the constants that go into our study were extracted from the deep inelastic scattering coefficient functions [25]. In the following, we systematically formulate a framework to resum the dominant soft gluon contributions in $z$ space to the inclusive cross sections. Here the variable $z$ is the appropriate scaling variable that enters the cross sections. To achieve this, we use renormalisation group(RG) invariance, mass factorisation and Sudakov resummation of QCD amplitudes as the guiding principles. Using the resummed results in $z$ space, we predict soft plus virtual part of the dominant partonic cross sections beyond $N^2LO$. The soft plus virtual corrections are also called threshold corrections. At the end, we show how these results affect the total cross section and improve the scale uncertainty for the Higgs production through gluon fusion.

Since we are only interested in the effect of soft gluons, the infra-red safe observable can be obtained by adding soft part of the cross sections with the virtual contributions and performing mass factorisation using appropriate counter terms. We call this infra-red safe combination a "soft plus virtual"(sv) part of the cross section. The soft plus virtual part of the cross section $(\Delta_{sv}^P(z, q^2, \mu_R^2, \mu_F^2))$ after mass factorisation is found to be

$$\Delta_{sv}^P(z, q^2, \mu_R^2, \mu_F^2) = C \exp \left( \Psi_p^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon) \right) \bigg|_{\varepsilon=0}$$

where $\Psi_p^I(z, q^2, \mu_R^2, \mu_F^2, \varepsilon)$ is a finite distribution. The subscript $P = S$ for Drell-Yan(DY) and Higgs productions and $P = SJ$ for deep inelastic scattering. The symbol $S$ stands for "soft" and $SJ$ stands for "soft plus jet". For DY and DIS, $I = q$(quark/anti-quark) and for Higgs production through
gluon fusion, $I = g$(gluon) and for bottom quark annihilation to Higgs boson, $I = b$(bottom quark). Here $\Psi_I^f(z,q^2,\mu_R^2,\mu_F^2,\varepsilon)$ is computed in $4 + \varepsilon$ dimensions.

$$\Psi_I^f(z,q^2,\mu_R^2,\mu_F^2,\varepsilon) = \left( \ln \left( Z^f(\hat{a}_s,\mu_R^2,\mu_F^2,\varepsilon) \right)^2 + \ln |\hat{F}^I(\hat{a}_s,\mu_F^2,\varepsilon)|^2 \right) \delta(1-z) + 2 \Phi_I^f(\hat{a}_s,q^2,\mu_F^2,z,\varepsilon) - 2 m C \ln \Gamma_I(\hat{a}_s,\mu_F^2,z,\varepsilon), \quad I = q, g, b \quad (2)$$

In the above $m = 1$ for DY and Higgs productions and $m = 1/2$ for DIS. The symbol "$\mathcal{C}$" means convolution. For example, $C$ acting on an exponential of a function $f(z)$ has the following expansion:

$$Ce^{f(z)} = \delta(1-z) + \frac{1}{1!} f(z) + \frac{1}{2!} f(z) \otimes f(z) + \frac{1}{3!} f(z) \otimes f(z) \otimes f(z) + \cdots \quad (3)$$

In the rest of the paper, the function $f(z)$ is a distribution of the kind $\delta(1-z)$ and $\mathcal{D}_i$, where

$$\mathcal{D}_i = \left[ \frac{\ln^i(1-z)}{(1-z)} \right]_+ \quad i = 0, 1, \cdots \quad (4)$$

and the symbol $\otimes$ means the Mellin convolution. Since we are only interested in the soft plus virtual part of the cross sections, we drop all the regular functions that result from various convolutions. $\hat{F}^I(\hat{a}_s,\mu_F^2)\mu_F^2$ are the form factors that enter in the Drell-Yan($I = g$) production, Higgs($I = g, b$) production and DIS($I = q/\bar{q}$) cross sections. In the form factors, $Q^2 = -q^2$. For the DY, $q^2 = M_{Z'}^2$ is the invariant mass of the final state di-lepton pair and for the Higgs production, $q^2 = m_H^2$, where $m_H$ is the mass of the Higgs boson. For DIS, $q^2 = q_{	au/Z}^2$ is the virtuality of the photon or $Z$. The variable $z$ in DIS is Björken scaling variable. In DY and Higgs production, $z$ is the ratio of $q^2$ over $\hat{s}$, where $\hat{s}$ is the center of mass of the partonic system. The functions $\Phi_I^f(\hat{a}_s,q^2,\mu_F^2,z)$ are called the soft distribution functions. The unrenormsised(bare) strong coupling constant $\hat{a}_s$ is defined as

$$\hat{a}_s = \frac{\hat{g}_s^2}{16\pi^2} \quad (5)$$

where $\hat{g}_s$ is the strong coupling constant which is dimensionless in $n = 4 + \varepsilon$, with $n$ being the number of space time dimensions. The scale $\mu$ comes from the dimensional regularisation in order to make the bare coupling constant $\hat{g}_s$ dimensionless in $n$ dimensions.

The bare coupling constant $\hat{a}_s$ is related to renormalised one by the following relation:

$$S_\varepsilon \hat{a}_s = Z(\mu_R^2\mu_R^2) a_s(\mu_R^2) \left( \frac{\mu^2}{\mu_R^2} \right)^{\frac{\varepsilon}{2}} \quad (6)$$

The scale $\mu_R$ is the renormalisation scale at which the renormalised strong coupling constant $a_s(\mu_R^2)$ is defined. The factorisation scale $\mu_F$ is due to mass factorisation.

$$S_\varepsilon = \exp \left\{ \frac{\varepsilon}{2} [\gamma_E - \ln 4\pi] \right\} \quad (7)$$
is the spherical factor characteristic of $n$-dimensional regularisation.

$$Z(\mu^2_R) = 1 + a_s(\mu^2_R) \frac{2\beta_0}{\varepsilon} + a_s^2(\mu^2_R) \left( \frac{4\beta_0^2}{\varepsilon^2} + \frac{\beta_1}{\varepsilon} \right) + a_s^3(\mu^2_R) \left( \frac{8\beta_0^3}{\varepsilon^3} + \frac{14\beta_0\beta_1}{3\varepsilon^2} + \frac{2\beta_2}{3\varepsilon} \right)$$

(8)

The renormalisation constant $Z(\mu^2_R)$ relates the bare coupling constant $\hat{a}_s$ to the renormalised one $a_s(\mu^2_R)$ through the eqn. (6).

The coefficients $\beta_0, \beta_1$ and $\beta_2$ are

$$\begin{align*}
\beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F n_f \\
\beta_1 &= \frac{34}{3} C_A^2 - 4 T_F n_f C_F - \frac{20}{3} T_F n_f C_A \\
\beta_2 &= \frac{2857}{54} C_A^3 - \frac{1415}{27} C_A^2 T_F n_f + \frac{158}{27} C_A T_F^2 n_f^2 \\
&\quad + \frac{44}{9} C_F T_F^2 n_f^2 - \frac{205}{9} C_F C_A T_F n_f + 2 C_F^2 T_F n_f
\end{align*}$$

(9)

where the color factors for $SU(N)$ QCD are given by

$$C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_F = \frac{1}{2}$$

(10)

and $n_f$ is the number of active flavours. In the case of the Higgs production, the number of active flavours is five because the top degree of freedom is integrated out in the large $m_{top}$ limit.

The factors $Z^l(\hat{a}_s, \mu^2_R, \mu^2, \varepsilon)$ are the overall operator renormalisation constants. For the vector current $Z^l(\hat{a}_s, \mu^2_R, \mu^2) = 1$, but the gluon operator gets overall renormalisation [35] given by

$$Z^g(\hat{a}_s, \mu^2_R, \mu^2, \varepsilon) = 1 + \hat{a}_s \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon} \left[ \frac{2\beta_0}{\varepsilon} \right] + \hat{a}_s^2 \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon} \left[ \frac{2\beta_1}{\varepsilon} \right]$$

$$+ \hat{a}_s^3 \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon^3} \left[ \frac{1}{\varepsilon^2} \left( -2\beta_0\beta_1 \right) + \frac{2\beta_2}{\varepsilon} \right]$$

$$+ \hat{a}_s^4 \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon^4} \left[ \frac{1}{\varepsilon^3} \left( \frac{8}{3} \beta_0^2 \beta_1 \right) + \frac{1}{\varepsilon^2} \left( -\frac{16}{3} \beta_0 \beta_2 \right) + \frac{1}{\varepsilon} \left( 2\beta_3 \right) \right]$$

(11)

and for the bottom quark coupled to Higgs, we have

$$Z^b(\hat{a}_s, \mu^2_R, \mu^2, \varepsilon) = 1 + \hat{a}_s \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon} \left[ \frac{1}{\varepsilon} \left( 2\gamma_0 \right) \right] + \hat{a}_s^2 \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon} \left[ \frac{1}{\varepsilon^2} \left( 2 \gamma_0^2 - 2 \beta_0 \gamma_0 \right) \right]$$

$$+ \frac{1}{\varepsilon} \left( \gamma_0 \right) + \hat{a}_s^3 \left( \frac{\mu^2_R}{\mu^2} \right) \frac{\varepsilon}{S_\varepsilon^3} \left[ \frac{1}{\varepsilon^3} \left( \frac{4}{3} \gamma_0^3 - 4 \beta_0 \gamma_0^2 + \frac{8}{3} \beta_0^2 \gamma_0 \right) \right]$$
The quantities $\gamma_i^0$ can be obtained from the quark mass anomalous dimensions [36]

$$\gamma_0^0 = 3C_F$$

$$\gamma_1^0 = \frac{3}{2} C_F^2 + \frac{97}{6} C_F C_A - \frac{10}{3} C_F T_F n_f$$

$$\gamma_2^0 = \frac{129}{2} C_F^2 - \frac{129}{4} C_F^2 C_A + \frac{11413}{108} C_F C_A^2 + \left( -46 + 48 \zeta_3 \right) C_F^2 T_F n_f$$

$$(13)$$

The bare form factors $\tilde{F}^I(\bar{\alpha}_s, Q^2, \mu^2, \varepsilon)$ (without overall renormalisation) of both fermionic and gluonic operators satisfy the following differential equation that follows from the gauge as well as renormalisation group invariances [37–40]. In dimensional regularisation,

$$Q^2 \frac{d}{d Q^2} \ln \tilde{F}^I(\bar{\alpha}_s, Q^2, \mu^2, \varepsilon) = \frac{1}{2} \left[ K^I \left( \bar{\alpha}_s, \frac{\mu_R^2}{\mu^2}, \varepsilon \right) + G^I \left( \bar{\alpha}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon \right) \right]$$

$$(14)$$

where $K^I$ contains all the poles in $\varepsilon$. On the other hand, $G^I$ collects rest of the terms that are finite as $\varepsilon$ becomes zero. Renormalisation group invariance (RG) of the $\tilde{F}^I(\bar{\alpha}_s, Q^2, \mu^2, \varepsilon)$ leads

$$\frac{\mu_R^2}{d \mu_R^2} K^I \left( \bar{\alpha}_s, \frac{\mu_R^2}{\mu^2}, \varepsilon \right) = - A^I(\alpha_s^2(\mu_R))$$

$$\frac{\mu_R^2}{d \mu_R^2} G^I \left( \bar{\alpha}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon \right) = A^I(\alpha_s^2(\mu_R))$$

$$(15)$$

The quantities $A^I$ are the standard cusp anomalous dimensions and they are expanded in powers of renormalised strong coupling constant $\alpha_s(\mu_R^2)$ as

$$A^I(\mu_R^2) = \sum_{i=1}^{\infty} a_i^I(\mu_R^2) A_i$$

$$(16)$$
The total derivative is given by
\[ \mu^2 \frac{df}{d\mu^2} = \mu^2 \frac{\partial}{\partial \mu^2} + \frac{d\alpha_s(\mu_R^2)}{d\mu_R^2} \frac{\partial}{\partial \alpha_s(\mu_R^2)} \]  

The RG equation for \( G^I \) can also be solved and the solution is found to be
\[ G^I \left( \frac{d\alpha_s}{d\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon \right) = G^I (\alpha_s(Q^2), 1, \varepsilon) + \int_{\mu^2}^{Q^2} \frac{d\lambda^2}{\lambda^2} A^I (\alpha_s(\lambda^2 \mu_R^2)) \]  

The finite function \( G^I (\alpha_s(Q^2), 1, \varepsilon) \) can also be expanded in powers of \( \alpha_s(Q^2) \) as
\[ G^I (\alpha_s(Q^2), 1, \varepsilon) = \sum_{i=1}^{\infty} a_i^I (Q^2) G_i^I (\varepsilon) \]  

The solution to the eqn. (14) can be obtained as a series expansion in the bare coupling constant in dimensional regularisation. The formal solution up to four loop level can be found in [24, 31].

The coefficients \( G_i^I (\varepsilon) \) can be found for both \( I = q \) and \( I = g \) in [25] to the required accuracy in \( \varepsilon \). We have extended this to the form factor corresponding to Yukawa interaction of Higgs boson to the bottom quarks. We have presented the logarithm of this form factor, \( \ln F_b \) in the appendix A. All these form factors satisfy
\[ G_1^I (\varepsilon) = 2 \left( B_1^I - \delta_{I,g} b_0 - \delta_{I,b} \gamma_0^b \right) + f_1^I + \sum_{k=1}^{\infty} \varepsilon^k g_1^I \]  
\[ G_2^I (\varepsilon) = 2 \left( B_2^I - 2 \delta_{I,g} b_1 - \delta_{I,b} \gamma_1^b \right) + f_2^I - 2 \beta_0 g_1^{I,1} + \sum_{k=1}^{\infty} \varepsilon^k g_2^I \]  
\[ G_3^I (\varepsilon) = 2 \left( B_3^I - 3 \delta_{I,g} b_2 - \delta_{I,b} \gamma_2^b \right) + f_3^I - 2 \beta_1 g_1^{I,1} - 2 \beta_0 \left( g_2^{I,1} + 2 \beta_0 g_1^{I,2} \right) + \sum_{k=1}^{\infty} \varepsilon^k g_3^I \]  
\[ G_4^I (\varepsilon) = 2 \left( B_4^I - 4 \delta_{I,g} b_3 - \delta_{I,b} \gamma_3^b \right) + f_4^I - 2 \beta_2 g_1^{I,1} - 2 \beta_1 \left( g_2^{I,1} + 4 \beta_0 g_1^{I,2} \right) \]
\[ -2 \beta_0 \left( g_3^{I,1} + 2 \beta_0 g_2^{I,2} + 4 \beta_0^2 g_1^{I,3} \right) + \sum_{k=1}^{\infty} \varepsilon^k g_4^I \]  

Notice that the single poles of the form factors contain the combination
\[ 2 \left( B_i^I - \delta_{I,g} i \beta_{i-1} - \delta_{I,b} \gamma_{i-1}^b \right) + f_i^I \]
at every order in \( \alpha_s \). The terms proportional to \(-2(\delta_{I,g} i \beta_{i-1} + \delta_{I,b} \gamma_{i-1}^b)\) come from the large momentum region of the loop integrals that are giving ultraviolet divergences. The poles containing
them will go away when the form factors undergo overall operator UV renormalisation through the renormalisation constants $Z^I$ which satisfy RG equations

$$
\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^g(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) = \sum_{i=1}^{\infty} a^g_i(\mu_R^2) \left( i \beta_{i-1} \right)
$$

$$
\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^b(\hat{a}_s, \mu_R^2, \mu^2, \varepsilon) = \sum_{i=1}^{\infty} a^b_i(\mu_R^2) \gamma^b_{i-1}
$$

(21)

where $\varepsilon \to 0$ is set. The constants $i \beta_{i-1}$ and $\gamma^b_{i-1}$ are anomalous dimensions of the renormalised form factors $F^g$ and $F^b$ respectively. The poles of the form factors containing terms proportional to $g^{l,k}_{i-1}$ multiplied by $\beta_i$ are due to coupling constant renormalisation. It is now straightforward to apply this procedure to form factor of many operators in QCD that are of interest. After the overall operator renormalisation through $Z^I$ and coupling constant renormalisation through $Z$, the remaining poles will contain only $B^I_i$ and $f^I_i$ in addition to the standard cusp anomalous dimensions $A^I_i$. The constants $B^I_i$ are known up to order $a^I_3$ thanks to the recent computation of three loop anomalous dimensions/splitting functions [22, 23]. They are found to be flavour independent, that is $B^I_i = B^b_i$. The constants $f^I_i$ are analogous to the cusp anomalous dimensions $A^I_i$ that enter the form factors. The cusp anomalous dimensions are flavour independent, $A^I_i = A^B_i$. It was first noticed in [41] that the single pole (in $\varepsilon$) of the logarithm of form factors up to two loop level ($a^I_2$) can be predicted due the presence of constants $f^I_i$ because these $f^I_i$ are found to be maximally non-abelian obeying the relation

$$
f^q_i = f^b_i = C_F \frac{C_A}{C_F} f^g_i
$$

(22)

similar to $A^I_i$. In [25], this relation has been found to hold even at the three loop level. From the experience up to three loop level, we can now predict all the poles of the form factor at every order in $\hat{a}_s$ from these constants $A^I_i, B^I_i, f^I_i$, their anomalous dimensions and the finite parts of lower order (in $\hat{a}_s$) contributions to the form factor. The last equality in eqn. (20) has been expanded again in terms of $f^I_{4i}$ and $g^{l,k}_{i-1}$ with $i, k = 1, 2, 3$ so that the single pole can again be predicted. One is not sure whether this structure will go through beyond three loop until an explicit calculation is available.

The collinear singularities that arise due to massless partons are removed in $\overline{MS}$ scheme using the mass factorisation kernel $\Gamma(z, \mu_F^2, \varepsilon)$ as shown in the eqn. (2). We have suppressed the dependence on $\hat{a}_s$ and $\mu^2$ in $\Gamma$. The kernel $\Gamma(z, \mu_F^2, \varepsilon)$ satisfies the following renormalisation group equation:

$$
\mu_F^2 \frac{d}{d\mu_F^2} \Gamma(z, \mu_F^2, \varepsilon) = \frac{1}{2} P(z, \mu_F^2) \otimes \Gamma(z, \mu_F^2, \varepsilon)
$$

(23)

The $P(z, \mu_F^2)$ are well known Altarelli-Parisi splitting functions (matrix valued) known up to three loop level [22, 23]:

$$
P(z, \mu_F^2) = \sum_{i=1}^{\infty} a^I_i(\mu_F^2) P^{(i-1)}(z)
$$

(24)
The diagonal terms of splitting functions $P^{(i)}(z)$ have the following structure

$$P^{(i)}_{II}(z) = 2 \left[ B_{i+1}^I \delta(1-z) + A_{i+1}^I \right] + P_{\text{reg},II}^{(i)}(z)$$

(25)

where $P_{\text{reg},II}^{(i)}$ are regular when the argument takes the kinematic limit (here $z \to 1$). The RG equation of the kernel can be solved in dimensional regularisation in powers of strong coupling constant. Since we are interested only in the soft plus virtual part of the cross section, only the diagonal parts of the kernels contribute. In the $\overline{MS}$ scheme, the kernel contains only poles in $\varepsilon$. The kernel can be expanded in powers of bare coupling $\hat{a}_s$ as

$$\Gamma(z,\mu_F^2,\varepsilon) = \delta(1-z) + \sum_{i=1}^{\infty} \hat{a}_s^I \left( \frac{\mu_F^2}{\mu^2} \right)^i \delta^I \Gamma^{(i)}(z,\varepsilon)$$

(26)

The constants $\Gamma^{(i)}(z,\varepsilon)$ expanded in negative powers of $\varepsilon$ up to four loop level can be found in [31]. The $\Gamma_{II}(\hat{a}_s,\mu_F^2,\mu^2,\varepsilon)$ in the eqn. (20) is the diagonal element of $\Gamma(z,\mu_F^2,\varepsilon)$.

The fact that $\Delta^{\alpha}_{I,P}$ are finite in the limit $\varepsilon \to 0$ implies that the soft distribution functions have pole structure in $\varepsilon$ similar to that of $\hat{F}^I$ and $\Gamma_{II}$. Hence it is natural to expect that the soft distribution functions also satisfy Sudakov type differential equation that the form factors $\hat{F}^I$ satisfy (see eqn. (14)):

$$q^2 \frac{d}{dq^2} \Phi_p^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) = \frac{1}{2} \left[ K_p^I(\hat{a}_s, \mu_F^2, \mu^2, z, \varepsilon) + G_p^I(\hat{a}_s, q^2, \mu_F^2, \mu^2, z, \varepsilon) \right]$$

(27)

where again the constants $K_p^I$ contain all the singular terms and $G_p^I$ are finite functions of $\varepsilon$. Also, $\Phi_p^I(\hat{a}_s, q^2, \mu^2, z)$ satisfy the renormalisation group equation:

$$\mu_R^2 \frac{d}{d\mu_R^2} \Phi_p^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) = 0$$

(28)

This renormalisation group invariance leads to

$$\mu_R^2 \frac{d}{d\mu_R^2} K_p^I(\hat{a}_s, \mu_F^2, \mu^2, z, \varepsilon) = - \overline{A}^I(a_s(\mu_F^2)) \delta(1-z)$$

$$\mu_R^2 \frac{d}{d\mu_R^2} G_p^I(\hat{a}_s, q^2, \mu_F^2, \mu^2, z, \varepsilon) = \overline{A}^I(a_s(\mu_F^2)) \delta(1-z)$$

(29)

If $\Phi_p^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon)$ have to contain the right poles to cancel the poles coming from $\hat{F}^I, Z^I$ and $\Gamma_{II}$ in order to make $\Delta^{\alpha}_{I,P}$ finite, then the constants $\overline{A}^I$ have to satisfy

$$\overline{A}^I = - A^I$$

(30)
Using the above relation, the solution to RG equation for \( \overline{G}_p^I \left( \hat{a}_s, \frac{q^2}{\mu_R^2}, \mu^2, z, \varepsilon \right) \) is found to be

\[
\overline{G}_p^I \left( \hat{a}_s, \frac{q^2}{\mu_R^2}, \mu^2, z, \varepsilon \right) = \overline{G}_p^I \left( a_s(\mu_R^2), \frac{q^2}{\mu_R^2}, z, \varepsilon \right) = \overline{G}_p^I \left( a_s(q^2), 1, z, \varepsilon \right) - \delta(1-z) \int_{\mu_R^2}^{\frac{q^2}{\mu_R^2}} \frac{d\lambda^2}{\lambda^2} A^I \left( a_s(\lambda^2 \mu_R^2) \right) \tag{31}
\]

With these solutions, it is straightforward to solve the Sudakov differential equation:

\[
\Phi_p^I(\hat{a}_s, q^2, \mu^2, z, \varepsilon) = \Phi_p^I(\hat{a}_s, q^2(1-z)^{2m}, \mu^2, \varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{q^2(1-z)^{2m}}{\mu^2} \right)^{\frac{i\varepsilon}{2}} S_{\varepsilon}^{i} \left( \frac{i m \varepsilon}{1-z} \right) \hat{\phi}_p^{I,(i)}(\varepsilon) \tag{32}
\]

where

\[
\hat{\phi}_p^{I,(i)}(\varepsilon) = \frac{1}{i\varepsilon} \left[ K_p^{I,(i)}(\varepsilon) + \overline{G}_p^{I,(i)}(\varepsilon) \right] \tag{33}
\]

The above solution depends on \( m \) so that \( \Psi_p^I \) or equivalently \( \Delta^{\gamma_\mu}_{I,J} \) is finite as \( \varepsilon \to 0 \)(see eqn.(2)). The constants \( K_p^{I,(i)}(\varepsilon) \) are independent of \( P \) and are determined by expanding \( K_p^I \) in powers of bare coupling constant \( \hat{a}_s \) as

\[
K_p^I \left( \hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \varepsilon \right) = \delta(1-z) \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{\mu_R^2}{\mu^2} \right)^{\frac{i\varepsilon}{2}} S_{\varepsilon}^{i} K_p^{I,(i)}(\varepsilon) \tag{34}
\]

and solving RG equation for \( K_p^I \left( \hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \varepsilon \right) \). We obtain

\[
K_p^{I,(1)}(\varepsilon) = \frac{1}{\varepsilon} \left( 2A_1^I \right)
\]

\[
K_p^{I,(2)}(\varepsilon) = \frac{1}{\varepsilon^2} \left( -2\beta_0 A_1^I + \frac{1}{\varepsilon} \left( A_2^I \right) \right)
\]

\[
K_p^{I,(3)}(\varepsilon) = \frac{1}{\varepsilon^3} \left( \frac{8}{3} \beta_0^2 A_1^I \right) + \frac{1}{\varepsilon^2} \left( -\frac{2}{3} \beta_1 A_1^I - \frac{8}{3} \beta_0 A_2^I \right) + \frac{1}{\varepsilon} \left( \frac{2}{3} A_3^I \right)
\]

\[
K_p^{I,(4)}(\varepsilon) = \frac{1}{\varepsilon^4} \left( -4\beta_0^3 A_1^I \right) + \frac{1}{\varepsilon^3} \left( \frac{8}{3} \beta_0 \beta_1 A_1^I + 6\beta_0^2 A_2^I \right)
\]

\[
+ \frac{1}{\varepsilon^2} \left( -\frac{1}{3} \beta_2 A_1^I - \beta_1 A_2^I - 3\beta_0 A_3^I \right) + \frac{1}{\varepsilon} \left( \frac{1}{2} A_4^I \right) \tag{35}
\]

9
The constants $\bar{G}_{P}^{I, (i)}(\varepsilon)$ are related to the finite function $\bar{G}_{P}^{j}(\alpha_{s}(q^{2}), 1, z, \varepsilon)$ through the distributions $\delta(1-z)$ and $\mathcal{D}_{j}$. Defining $\bar{G}_{P}^{I}(\varepsilon)$ through

$$\sum_{i=1}^{\infty} a_{i}^{I} \left(\frac{q^{2}(1-z)^{2m}}{\mu^{2}}\right)^{i/2} \delta_{\mathcal{F}} \bar{G}_{P}^{I, (i)}(\varepsilon) = \sum_{i=1}^{\infty} a_{i}^{I} \left(\frac{q^{2}(1-z)^{2m}}{\mu^{2}}\right) \bar{G}_{P, i}(\varepsilon)$$

we find

$$\bar{G}_{P}^{I, (1)}(\varepsilon) = \bar{G}_{P, 1}(\varepsilon)$$

$$\bar{G}_{P}^{I, (2)}(\varepsilon) = \frac{1}{\varepsilon} \left(-2 \beta_{0} \bar{G}_{P, 1}(\varepsilon) + \bar{G}_{P, 2}(\varepsilon) \right)$$

$$\bar{G}_{P}^{I, (3)}(\varepsilon) = \frac{1}{\varepsilon^{2}} \left(4 \beta_{0}^{2} \bar{G}_{P, 1}(\varepsilon) + \frac{1}{\varepsilon} \left(-\beta_{1} \bar{G}_{P, 1}(\varepsilon) - 4 \beta_{0} \bar{G}_{P, 2}(\varepsilon) \right) + \bar{G}_{P, 3}(\varepsilon) \right)$$

$$\bar{G}_{P}^{I, (4)}(\varepsilon) = \frac{1}{\varepsilon^{3}} \left(-8 \beta_{0}^{3} \bar{G}_{P, 1}(\varepsilon) + \frac{16}{3} \beta_{0} \beta_{1} \bar{G}_{P, 1}(\varepsilon) + 12 \beta_{0}^{2} \bar{G}_{P, 2}(\varepsilon) \right)$$

$$+ \frac{1}{\varepsilon^{2}} \left(-2 \beta_{2} \bar{G}_{P, 1}(\varepsilon) + 2 \beta_{1} \bar{G}_{P, 2}(\varepsilon) - 6 \beta_{0} \bar{G}_{P, 3}(\varepsilon) \right) + \bar{G}_{P, 4}(\varepsilon)$$

The $z$ independent constants $\bar{G}_{P, i}(\varepsilon)$ are obtained by demanding the finiteness of $\Delta_{I, P}^{\varepsilon}$ given in eqn. (1). Without setting $\varepsilon = 0$ in eqn. (1), we expand $\Delta_{I, P}^{\varepsilon}$ as

$$\Delta_{I, P}^{\varepsilon}(z, q^{2}, \mu_{R}^{2}, \mu_{F}^{2}, \varepsilon) = \sum_{i=0}^{\infty} a_{i}^{I}(\mu_{R}^{2}) \Delta_{I, P}^{\varepsilon, (i)}(z, q^{2}, \mu_{R}^{2}, \mu_{F}^{2}, \varepsilon)$$

Now, using the above expansion and eqn. (20), we determine these constants by comparing the pole as well as non-pole terms of the form factors, mass factorisation kernels and coefficient functions $\Delta_{I, P}^{\varepsilon, (i-1)}$ expanded in powers of $\varepsilon$ to the desired accuracy. The factor $m$ appearing in the solution turns out to be $1/2$ for the DY and Higgs productions(DIS). This choice is governed by the number of incoming partons that are responsible for the soft emissions. Since $\bar{G}_{P}^{I}(\varepsilon)$s in the form factors are found to satisfy a specific structure in terms of $B^{I}, f^{I}, \beta_{i}$ and $g^{I}$ as given in eqn. (20), we find that the constants $\bar{G}_{P, i}(\varepsilon)$ also satisfy similar looking expansion containing these constants.
\[
\bar{g}_{P,4}^I(\varepsilon) = -\left(f_4^I + B_4^I \delta_{P,SD}\right) - 2\beta_2 \bar{g}_{P,1}^{I,(1)} - 2\beta_1 \left(\bar{g}_{P,2}^{I,(1)} + 4\beta_0^I \bar{g}_{P,1}^{I,(2)}\right) - 2\beta_0 \left(\bar{g}_{P,3}^{I,(1)} + 2\beta_0 \bar{g}_{P,2}^{I,(2)} + 4\beta_0^2 \bar{g}_{P,1}^{I,(3)}\right) + \sum_{k=1}^{\infty} \varepsilon^k \bar{g}_{P,4}^{I,(k)}
\]

(39)

The above structure indicates that all the poles including the single pole of the soft distribution function can be predicted from that of the form factors, renormalisation constants and the mass factorisation kernels. This is possible because we have now better understanding [41] of the structure of even the single pole terms of the form factors. Notice that the terms proportional \(\varepsilon\) in the above expansion are due to the coupling constant renormalisation. The coefficients of single poles are proportional to the cusp anomalous dimension \(A^I\) and the combination \(-(f^I + B^I \delta_{P,SD})\) where the constants \(f^I\) and \(B^I\) are process independent. The \(\varepsilon\) dependent terms in \(\bar{g}_{P,1}^I(\varepsilon)\) can be obtained from the fixed order computations of cross sections and the finite parts of the form factors. At the moment, we know \(\bar{g}_{P,1}^I(\varepsilon)\) to all orders in \(\varepsilon\), \(\bar{g}_{P,2}^I(\varepsilon)\) to order \(\varepsilon\) and \(\bar{g}_{P,3}^I(\varepsilon)\) to order \(\varepsilon^0\). The lowest order term \(\bar{g}_{P,1}^I(\varepsilon)\) is known to all orders in \(\varepsilon\) because it is straightforward to compute the fixed order soft contribution. On the other hand, it is technically hard to determine \(\varepsilon\) dependent parts of soft cross sections beyond the lowest order \(\alpha_s\). We find,

\[
\begin{align*}
\bar{g}_{S,1}^{I,(1)} &= C_I \left(-3\zeta_2\right) \\
\bar{g}_{S,1}^{I,(2)} &= C_I \left(\frac{7}{3}\zeta_3\right) \\
\bar{g}_{S,1}^{I,(3)} &= C_I \left(-\frac{3}{16}\zeta_2\right) \\
\bar{g}_{S,2}^{I,(1)} &= C_I C_A \left(\frac{2428}{81} - \frac{469}{9}\zeta_2 + 4\zeta_2^2 - \frac{176}{3}\zeta_3\right) \\
&\quad + C_I m_f \left(-\frac{328}{81} + \frac{70}{9}\zeta_2 + \frac{32}{3}\zeta_3\right)
\end{align*}
\]

(40)

where \(C_I = C_F\) for \(I = q, b\) and \(C_I = C_A\) for \(I = g\). Interestingly these constants \(\bar{g}_{S,i}^I(\varepsilon)\) turn out to be maximally non-abelien. That is, they satisfy

\[
\bar{g}_{S,i}^q(\varepsilon) = \bar{g}_{S,i}^b(\varepsilon) = \frac{C_F}{C_A} \bar{g}_{S,i}^g(\varepsilon)
\]

(41)

Similarly, for the DIS, the constants \(\bar{g}_{S,i}^q\) are found to be

\[
\begin{align*}
\bar{g}_{S,i,1}^{q,(1)} &= C_F \left(\frac{7}{2} - 3\zeta_2\right) \\
\bar{g}_{S,i,1}^{q,(2)} &= C_F \left(-\frac{7}{2} + \frac{9}{8}\zeta_2 + \frac{7}{3}\zeta_3\right)
\end{align*}
\]
The threshold corrections dominate when the partonic scaling variable $z$ approaches its kinematic limit which is 1. They manifest in terms of the distributions $\delta(1-z)$ and $D_j$. Since the hadronic cross sections are expressed in terms of convolutions of partonic cross sections and the parton distribution functions, it is more convenient to study the threshold enhanced corrections in Mellin $N$ space where $z \to 1$ corresponds to large $N$. In Mellin $N$ space, all the convolutions become ordinary products and the $\delta(1-z)$ distribution becomes a constant and the distributions $D_j$ become functions of logarithm of $N$. The threshold resummation in Mellin $N$ space has been a successful approach thanks to several important works given in [17, 18, 42, 43]. We show in the following how the soft distribution function $\Phi_p^I(\hat{a}_s, q^2, \mu^2, \lambda^2, z, \varepsilon)$ captures all the features of the $N$ space resummation approach. The exponents of the $z$ space resummed cross sections get contributions from both form factor as well as the soft distribution functions. The form factor contributes to $\delta(1-z)$ part and the soft distributions functions contribute to $\delta(1-z)$ as well as to the distributions $D_j$. Using

$$
\frac{1}{1-z} \left[(1-z)^2m\right]^i = \frac{1}{\tilde{m} \varepsilon} \delta(1-z) + \left(\frac{1}{1-z} \left[(1-z)^2m\right]^i\right),
$$

we can express the soft distribution function (for any $m$) as

$$
\Phi_p^I(\hat{a}_s, q^2, \mu^2, \lambda^2, z, \varepsilon) = \left(\frac{m}{1-z}\left\{\int \frac{d\lambda^2}{\lambda^2} A_I(a_s(\lambda^2)) + \bar{G}_{p}^I(a_s(\lambda^2), \varepsilon)\right\}\right) + \\
+ \delta(1-z) \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2}{\mu^2}\right)^{i\varepsilon} S_{\varepsilon} \bar{G}_{p}^{I,(i)}(\varepsilon) + \\
+ \left(\frac{m}{1-z}\right) \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{\mu^2}{\mu^2}\right)^{i\varepsilon} S_{\varepsilon} \bar{K}^{I,(i)}(\varepsilon)
$$

where

$$
\bar{G}_{p}^I(a_s(q^2(1-z)^2m), \varepsilon) = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{q^2(1-z)^2m}{\mu^2}\right)^{i\varepsilon} S_{\varepsilon} \bar{G}_{p}^{I,(i)}(\varepsilon)
$$

(45)
For $m = 1$, that is, for DY and Higgs production, we can easily identify $G^I_s(a_s(q^2(1-z)^2), \varepsilon)$ with the threshold exponent $D^I(a_s(q^2(1-z)^2))$:

$$D^I(a_s(q^2(1-z)^2)) = \sum_{i=1}^{\infty} a_i^s(q^2(1-z)^2) D^I_i$$

$$= 2 \bar{G}^I_s(a_s(q^2(1-z)^2), \varepsilon) \bigg|_{\varepsilon=0}$$

Comparing the above result with the eqn.(36), we find

$$D^I_i = 2 \bar{G}^I_s(a_s(q^2(1-z)^2), \varepsilon) \bigg|_{\varepsilon=0} \quad (46)$$

For $m = 1/2$, that is, for DIS, we identify $G^I_{SJ}(a_s(q^2(1-z)), \varepsilon)$ with the threshold exponent $B^I_{DIS}(a_s(q^2(1-z)))$:

$$B^I_{DIS}(a_s(q^2(1-z))) = \sum_{i=1}^{\infty} a_i^s(q^2(1-z)) B^I_{DIS,i}$$

$$= \bar{G}^I_{SJ}(a_s(q^2(1-z)), \varepsilon) \bigg|_{\varepsilon=0} \quad (48)$$

Comparing the above result with the eqn.(36), we find

$$B^I_{DIS,i} = \bar{G}^I_{SJ,i}(\varepsilon = 0) \quad (49)$$

It is evident from the above derivation that the allowed value $m = 1/2$ for DIS does not allow an exponent similar to $D^I(a_i(q^2(1-z)^2))$ that appears in DY and Higgs productions. It is also straightforward to show that the so called conspicuous relation found in [26] follows directly from eqns.(39,47,49):

$$\frac{1}{2} D^I_i - B^I_{DIS,i} = B^I_i + \sum_{j=0}^{i-2} C^{(\beta)}_j \delta G^I_j \quad (50)$$

where the constants $C^{(\beta)}_j$ and $\delta G^I_j$ can be obtained from the eqns.(39,40,42).

Few important remarks about the eqn.(44) are in order. The third line in the eqn.(44) cancels against $D_0$ part of the mass factorisation kernel. The second line contains the right poles in $\varepsilon$ to cancel those coming from the form factor as well as the mass factorisation kernel. In addition, it contains the terms that are finite as $\varepsilon$ becomes zero through the constants $\bar{G}^I_{SJ,i}(\varepsilon)$. These finite constants contribute to soft part of the cross section at higher orders. Hence, adding the eqn.(44) with the renormalised form factor and the mass factorisation kernels, performing the coupling constant renormalisation and then finally taking Mellin moment, we reproduce the resummed result given in [17, 18, 42, 43] when $\varepsilon \to 0$.

In [31], it was observed that the soft distribution functions $\Phi^{I}(\hat{a}_s,q^2,\mu^2,z,\varepsilon)$ of DY and Higgs production are maximally non-abelien and hence the entire soft contribution of Higgs production
can be predicted from that of DY and vice versa. In fact the $\varepsilon$ dependent parts of $\overline{G}_{S,i}^q(\varepsilon)$ extracted from DY are needed to predict the soft part of the Higgs production. In addition, using the resummed result given in eqn. (1), and the available exponents $g_i^I(\varepsilon)$, one can also predict part of the higher order soft plus virtual contributions to the cross sections. The available exponents are

$$g_1^I, \overline{G}_1^I, \quad \text{for} \quad j = \text{all}$$

$$g_2^I, \overline{G}_2^I, \quad \text{for} \quad j = 0, 1$$

$$g_3^I, \overline{G}_3^I, \quad \text{for} \quad j = 0$$

in addition to the known $\beta_i (i = 0, 1, 2, 3)$, the constants in the splitting functions $A_i, B_i (i = 1, 2, 3)$, the maximally non-abelien constants $f_i (i = 1, 2, 3)$ and the anomalous dimensions $\chi_i^f (i = 0, 1, 2, 3)$. For $I = q, g$, the constants $g_2^{q,j}$ and $g_2^{g,j}$ are known for $j = 2, 3$ also (see [24]).

Using the resummed expression given in eqn. (1) and the known exponents, we present here the results for $\Delta_{i,P}^{sv}(i)$ for DY, Higgs production and DIS. As we have already mentioned in the beginning, for DIS, it is just a consistency check because the three loop form factors were derived from the DIS coefficient functions. The Drell-Yan coefficients $\Delta_{q,S}^{sv}(i)$ and Higgs coefficients $\Delta_{g,S}^{sv}(i)$ and $\Delta_{b,S}^{sv}(i)$ are known up to NNLO ($i = 0, 1, 2$) from the explicit computations (see [4]- [16]). For $N^3LO$ for $I = q, g$, a partial result $\Delta_{I,S}^{sv}(3)$, i.e., a result without $\delta(1-z)$ part is available from the work of [28]. Using the universal behavior of the soft distribution function for $P = S$, in the reference [31], the entire $\Delta_{I,S}^{sv}(i)$ for $I = q, g$ up to NNLO as well as partial $N^3LO$ for Drell-Yan and Higgs through gluon fusion, that is the coefficients $\Delta_{I,S}^{sv}(3)$ for $I = q, g$ were reproduced. This was achieved using the resummation formula given in eqn. (1). In this article, we predict for the first time the $N^3LO$ contribution $\Delta_{b,S}^{sv}(3)$ to Higgs production through bottom quark annihilation with the same accuracy that the coefficients $\Delta_{I,S}^{sv}(3)$ for $I = q, g$ are known. In addition, we extend this approach to $N^4LO$ order where we can predict partial soft plus virtual contribution coming from all $D_j$ except $j = 0, 1$ for Drell-Yan $N^4LO$ coefficient $\Delta_{q,S}^{sv}(4)$, gluon fusion to Higgs $N^4LO$ coefficient $\Delta_{g,S}^{sv}(4)$ and bottom quark annihilation to Higgs boson $N^4LO$ coefficient $\Delta_{b,S}^{sv}(4)$. Like $N^3LO \Delta_{I,S}^{sv}(3)$, here also we can not predict $\delta(1-z)$ part. These results are presented in the Appendix B for $\mu_R^2 = \mu_F^2 = q^2$. Using our resummed formula and the DIS exponents $\overline{G}_{S,j}^q(\varepsilon)$, we have reproduced the partial $N^4LO$ and $N^3LO$ coefficients $\Delta_{q,S}^{sv}(4), \Delta_{g,S}^{sv}(3)$ for DIS given in [26,45] and the lower order results (see [44] for $NNLO$). The computation of these soft plus virtual contributions to DY, Higgs productions and DIS involves convolutions of distribution functions $D_j$ with $j = 0, ..., 7$. We have computed them using the following integral representation given in [46].

$$D_1 \otimes D_j = \lim_{\varepsilon \to 0} \lim_{a,b \to 1} \left( \frac{\partial}{\partial a} \right)^i \left( \frac{\partial}{\partial b} \right)^j \frac{1}{\varepsilon^{i+j}} \int_0^1 dx \int_0^1 dy \ (1-x)^a \varepsilon^{-1} (1-y)^b \varepsilon^{-1}$$

$^1$ after fixing the typos in eqn. (4.19) of [45] and in eqns. (5.6,5.7) and eqn. (5.9) of [26].
\[
\delta(z - xy) - \delta(z - x) - \delta(z - y) + \delta(1 - z)
\]

The above integral reduces to a set of Hypergeometric functions and Euler Gamma functions which can be expanded around \(\varepsilon = 0\) to desired accuracy. We are interested only in those terms which are proportional to the distributions \(\delta(1 - z)\) and \(\mathcal{D}_i\) for our analysis. For this purpose, we have derived a formula to compute the convolutions of distributions for any arbitrary \(i, j\) using the above integral representation:

\[
\mathcal{D}_i \otimes \mathcal{D}_j = \lim_{\varepsilon \to 0} \lim_{a,b \to 1} \left( \frac{\partial}{\partial a} \right)^i \left( \frac{\partial}{\partial b} \right)^j \frac{1}{\varepsilon^{i+j+1}} \left( \frac{a+b}{ab} \right) \left( \frac{\Gamma(1+a\varepsilon)\Gamma(1+b\varepsilon)}{\Gamma(1+(a+b)\varepsilon)} - 1 \right)
\]

\[
\times \left[ \frac{1}{(a+b)\varepsilon} \delta(1-z) + \sum_{k=0}^{\infty} \frac{(a+b)\varepsilon}{k!} \mathcal{D}_k \right] + \sum_{k=1}^{\infty} \frac{(a+b)\varepsilon}{k!} \mathcal{D}_k
\]

where \(R(a,b,z,\varepsilon)\) is the remaining regular function. It is now straightforward to obtain \(\Delta_{S,S}^{2\varepsilon,(i)}\) for \(i = 1, ..., 4\) for both Higgs \((I = g, b)\) and DY \((I = q)\) productions.

The impact of partial soft plus virtual parts of \(N^3LO\) and \(N^4LO\) contributions to Higgs production through gluon fusion at LHC is presented in figure (1). The Higgs production cross section is given by

\[
\sigma^H(S, m_H^2) = \frac{\pi G_B^2}{8(N^2 - 1)} \sum_{a,b=q,\gamma,s} \int_1^1 dy \Phi_{ab}(y, \mu_F^2) \Delta_{ab}^H \left(\frac{x}{y}, m_H^2, \mu_F^2, \mu_R^2\right)
\]

where \(x = m_H^2/S, N = 3\) and the factor \(G_B\) can be found from [35]. The flux \(\Phi_{ab}(y, \mu_F^2)\) is given by

\[
\Phi_{ab}(y, \mu_F^2) = \int_1^1 \frac{dw}{w} f_a(w, \mu_F^2) f_b\left(\frac{y}{w}, \mu_F^2\right)
\]

where \(f_a(w, \mu_F^2)\) is the parton distribution function. The partonic cross section \(\Delta_{ab}^H\) contains both soft plus virtual as well as hard contributions:

\[
\Delta_{ab}^H(z, m_H^2, \mu_R^2, \mu_F^2) = \Delta_{S,S}^{2\varepsilon}(z, m_H^2, \mu_R^2, \mu_F^2) + \Delta_{ab}^{H,hard}(z, m_H^2, \mu_R^2, \mu_F^2)
\]

We choose the center of mass energy to be \(\sqrt{S} = 14 \text{ TeV}\) for LHC. The standard model parameters that enter our computation are the Fermi constant \(G_F = 4541.68\), \(Z\) boson mass \(M_Z = 91.1876\ \text{GeV}\), top quark mass \(m_t = 173.4\ \text{GeV}\). The strong coupling constant \(\alpha_s(\mu_R^2)\) is evolved using 4-loop RG equations depending on the order in which the cross section is evaluated. We choose \(\alpha_s^{LO}(M_Z) = 0.130\), \(\alpha_s^{NLO}(M_Z) = 0.119\), \(\alpha_s^{NNLO}(M_Z) = 0.115\) and \(\alpha_s^{N^2LO}(M_Z) = 0.114\) for \(i > 2\). We use MRST 2001 LO for leading order, MRST2001 NLO for next to leading order and
Figure 1: Total cross section for Higgs production through gluon fusion at LHC, and its scale dependence for the Higgs boson of mass $m_H = 150$ GeV with $\mu = m_R, \mu_0 = \mu_H$. The abbreviation "pSV" means partial soft plus virtual.

MRST 2002 NNLO for $N^iLO$ with $i > 1$ [47, 48]. In the first plot of fig.(1), we have shown the cross sections in pb for various values of Higgs mass. For LO,NLO and NNLO we used the exact results which contain both soft plus virtual as well as regular hard contributions. For $N^iLO$ ($i=3,4$), we use only soft plus virtual results extracted from the resummed formula. Here we have set $\mu_F = \mu_R = m_H$. We find that the inclusion of $N^iLO(i = 3, 4)$ does not change the cross section much confirming the reliability of the perturbative approach. In the second plot of fig.(1), we have plotted the scale variation of the cross section using the ratio:

$$R \left( \frac{\mu_R^2}{m_H^2} \right) = \frac{\sigma^H (S, m_H^2, \mu_R^2, \mu_F^2 = m_H^2)}{\sigma^H (S, m_H^2, \mu_R^2 = \mu_F^2 = m_H^2)}$$

(56)

It is clear from the second plot of fig.(1) that the inclusion of $N^iLO(i = 3, 4)$ soft plus virtual contributions reduces the scale ambiguity further.

To summarise, we have systematically studied the soft plus virtual correction to inclusive processes such as DIS, DY, Higgs productions through gluon fusion and bottom quark annihilation. Using renormalisation group invariance and Sudakov resummation of scattering amplitudes and the factorisation property of these hard scattering cross sections, the resummation of these corrections has been achieved. We have also shown how these resummed distributions are related to resummation exponents that appear in Mellin space. Using our resummed results we predict par-
tial soft plus virtual cross sections at $N^3LO$ and $N^4LO$. We have also shown the phenomenological consequences of such results for Higgs production through gluon fusion at LHC.

Acknowledgments: The author would like to thank Prakash Mathews for discussion.

A Appendix

In this appendix, we present the logarithm of the form factor $\ln \hat{f}^b$ for the Yukawa interaction of Higgs boson with bottom quarks upto three loop level expanded in powers of $\epsilon$ to the desired accuracy for our computation.

$$\ln \hat{f}^b(\hat{a}_s, Q^2, \mu^2, \epsilon) = \hat{a}_s \left( \frac{Q^2}{\mu^2} \right)^{\frac{\epsilon}{2}} S_{\epsilon} C_F \left[ \frac{1}{\epsilon^2} \left( -8 \right) + \epsilon \left( 2 - \frac{7}{3} \zeta_3 \right) + \epsilon^2 \left( -2 + \frac{1}{4} \zeta_2 + \frac{47}{80} \zeta_2^2 \right) \right]$$

$$+ \left( -2 + \zeta_2 \right) \right]$$

$$+ \hat{a}_s^2 \left( \frac{Q^2}{\mu^2} \right)^{\frac{\epsilon}{3}} S_{\epsilon}^3 \left[ \frac{1}{\epsilon^3} \left( n_f C_F \left( -\frac{8}{3} \right) + C_F C_A \left( \frac{44}{3} \right) \right) \right]$$

$$+ \frac{1}{\epsilon^2} \left( n_f C_F \left( \frac{20}{9} \right) + C_F C_A \left( -\frac{134}{9} + 4 \zeta_2 \right) \right) + \frac{1}{\epsilon} \left( n_f C_F \left( -\frac{92}{27} - \frac{2}{3} \zeta_2 \right) \right)$$

$$C_F C_A \left( \frac{440}{27} + \frac{11}{3} \zeta_2 - 26 \zeta_3 \right) + C_F^2 \left( -12 \zeta_2 + 24 \zeta_3 \right) \right) + n_f C_F \left( \frac{416}{81} \right)$$

$$+ C_F C_A \left( \frac{5}{9} \zeta_2 - \frac{26}{9} \zeta_3 \right) + C_F C_A \left( -\frac{1655}{81} - \frac{103}{18} \zeta_2 + \frac{44}{5} \zeta_2^2 + \frac{305}{9} \zeta_3 \right)$$

$$+ C_F^2 \left( 4 + 16 \zeta_2 - \frac{44}{5} \zeta_2^2 - 30 \zeta_3 \right)$$

$$+ \hat{a}_s^3 \left( \frac{Q^2}{\mu^2} \right)^{\frac{3\epsilon}{4}} S_{\epsilon}^4 \left[ \frac{1}{\epsilon^4} \left( n_f C_F C_A \left( \frac{1408}{81} \right) + n_f^2 C_F \left( -\frac{128}{81} \right) \right) \right]$$

$$+ C_F C_A^2 \left( -\frac{3872}{81} \right) + \frac{1}{\epsilon^3} \left( n_f C_F C_A \left( -\frac{8528}{243} + \frac{128}{27} \zeta_2 \right) + n_f C_F^2 \left( -\frac{16}{9} \right) \right)$$

$$n_f^2 C_F \left( \frac{640}{243} \right) + C_F C_A^2 \left( \frac{26032}{243} - \frac{704}{27} \zeta_2 \right) \right) + \frac{1}{\epsilon^2} \left( n_f C_F C_A \left( \frac{13640}{243} \right) \right)$$

$$+ \frac{1264}{81} \zeta_2 - \frac{1024}{27} \zeta_3 \right) + n_f C_F^2 \left( \frac{220}{27} - \frac{64}{3} \zeta_2 + \frac{320}{9} \zeta_3 \right)$$

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\[ + n_f^2 C_F \left( -\frac{128}{27} - \frac{16}{9} \zeta_2 \right) + C_F C_A^{2} \left( -\frac{38828}{243} - \frac{2212}{81} \zeta_2 - \frac{352}{45} \zeta_2^2 \right) \]

\[ + \frac{6688}{27} \zeta_3 \right) + C_F^2 C_A \left( \frac{352}{3} \zeta_2 - \frac{704}{3} \zeta_3 \right) \right) \right) + \frac{1}{\varepsilon} \left( n_f C_F C_A \left( -\frac{180610}{2187} \right) \right) \]

\[ - \frac{10066}{243} \zeta_2 + \frac{88}{5} \zeta_2^2 + \frac{10280}{81} \zeta_3 \right) + n_f C_F \left( -\frac{1171}{81} + \frac{454}{9} \zeta_2 - \frac{496}{45} \zeta_2^2 \right) \]

\[ - \frac{3104}{27} \zeta_3 \right) + n_f^2 C_F \left( \frac{19232}{2187} + \frac{80}{27} \zeta_2 - \frac{272}{81} \zeta_3 \right) + C_F C_A^{2} \left( \frac{385325}{2187} \right) \]

\[ + \frac{176}{9} \zeta_2 \zeta_3 + \frac{31966}{243} \zeta_2 - \frac{1604}{15} \zeta_2^2 - \frac{17084}{27} \zeta_3 + \frac{272}{3} \zeta_5 \right) + C_F^2 C_A \left( -12 \right) \]

\[ + \frac{32}{3} \zeta_2 \zeta_3 - \frac{2932}{9} \zeta_2 + \frac{3832}{45} \zeta_2^2 + \frac{5648}{9} \zeta_3 + 80 \zeta_5 \right) + C_F \left( -\frac{100}{3} \right) \]

\[ - \frac{64}{3} \zeta_2 \zeta_3 + 12 \zeta_2 + \frac{192}{5} \zeta_2^2 + \frac{136}{3} \zeta_3 - 160 \zeta_5 \right) \right) \right) \right] \]

\[ (1) \]

B Appendix

In this appendix we present soft plus virtual parts of Drell-Yan production of di-leptons and Higgs productions through bottom quark annihilation and gluon fusion.

\[ \Delta_{q,S}^{sv,(0)} = \delta(1-z) \]  \hspace{1cm} (2)

\[ \Delta_{q,S}^{sv,(1)} = C_F \left( \delta(1-z) \left[ -16 + 8 \zeta_2 \right] + D_1 \left[ 16 \right] \right) \]  \hspace{1cm} (3)

\[ \Delta_{q,S}^{sv,(2)} = \delta(1-z) \left[ n_f C_F \left( \frac{127}{6} - \frac{112}{9} \zeta_2 + 8 \zeta_3 \right) + C_F C_A \left( -\frac{1535}{12} + \frac{592}{9} \zeta_2 - \frac{12}{5} \zeta_2^2 + 28 \zeta_3 \right) \right] \]

\[ + C_F^2 \left( \frac{511}{4} - 70 \zeta_2 + \frac{8}{5} \zeta_2^2 - 60 \zeta_3 \right) \right] + D_0 \left[ n_f C_F \left( \frac{224}{27} - \frac{32}{3} \zeta_2 \right) + C_F C_A \left( -\frac{1616}{27} \right) \right] \]

\[ + \frac{176}{3} \zeta_2 + 56 \zeta_3 \right) + C_F^2 \left( 256 \zeta_3 \right) \right] + D_1 \left[ n_f C_F \left( -\frac{160}{9} \right) + C_F C_A \left( \frac{1072}{9} \right) - 32 \zeta_2 \right] \]

\[ + C_F^2 \left( -256 - 128 \zeta_2 \right) \right] + D_2 \left[ n_f C_F \left( \frac{32}{3} \right) + C_F C_A \left( -\frac{176}{3} \right) \right] + D_3 C_F^2 \left[ 128 \right] \]  \hspace{1cm} (4)
\[ \Delta_{q,S}^{(3)} = D_0 \left[ n_f C_F C_A \left( \frac{125252}{729} - \frac{29392}{81} \zeta_2 + \frac{736}{15} \zeta_2^2 - \frac{2480}{9} \zeta_3 \right) + n_f C_F^2 \left( -6 + \frac{1952}{27} \zeta_2 - \frac{1472}{15} \zeta_2^2 - \frac{5728}{9} \zeta_3 \right) + n_f C_F^2 \left( - \frac{3712}{729} + \frac{640}{27} \zeta_2 + \frac{320}{27} \zeta_3 \right) + C_F C_A^2 \left( - \frac{594058}{729} - \frac{352}{3} \zeta_2 \zeta_3 + \frac{98224}{81} \zeta_2 - \frac{2992}{15} \zeta_2^2 + \frac{40144}{27} \zeta_3 - 384 \zeta_5 \right) + C_F^2 C_A \left( \frac{25856}{27} - \frac{1472 \zeta_2 \zeta_3}{27} - \frac{12416}{27} \zeta_2 + \frac{1408}{3} \zeta_2^2 + \frac{26240}{9} \zeta_3 \right) + C_F^3 \left( -6144 \zeta_2 \zeta_3 - 4096 \zeta_3 + 12288 \zeta_5 \right) \right] \\
+ D_1 \left[ n_f C_F C_A \left( - \frac{32816}{81} + 384 \zeta_2 \right) + n_f C_F^2 \left( \frac{4288}{9} + \frac{2048}{9} \zeta_2 + 1280 \zeta_3 \right) + n_f C_F^2 \left( \frac{1600}{81} - \frac{256}{9} \zeta_2 \right) + C_F C_A^2 \left( \frac{124024}{81} - \frac{12032}{9} \zeta_2 + \frac{704}{5} \zeta_2^2 - 704 \zeta_3 \right) + C_F^2 C_A \left( - \frac{35572}{9} - \frac{11648}{9} \zeta_2 + \frac{3648}{5} \zeta_2^2 - 5184 \zeta_3 \right) + C_F^3 \left( 2044 + 2976 \zeta_2 - \frac{14208}{5} \zeta_2^2 - 960 \zeta_3 \right) \right] + D_2 \left[ n_f C_F C_A \left( \frac{9248}{27} - \frac{128}{3} \zeta_2 \right) + n_f C_F^2 \left( \frac{544}{9} - \frac{2048}{3} \zeta_2 \right) + n_f C_F^2 \left( - \frac{640}{27} + \frac{28480}{27} + \frac{704}{3} \zeta_2 \right) + C_F C_A^2 \left( - \frac{4480}{9} + \frac{11264}{3} \zeta_2 + 1344 \zeta_3 \right) + C_F^3 \left( 10240 \zeta_3 \right) \right] + D_3 \left[ n_f C_F C_A \left( - \frac{2816}{27} \right) + n_f C_F^2 \left( - \frac{2560}{9} \right) + n_f C_F^2 \left( \frac{256}{27} \right) + C_F C_A^2 \left( \frac{7744}{27} \right) + C_F^2 C_A \left( \frac{17152}{9} - 512 \zeta_2 \right) + C_F^3 \left( - 2048 - 3072 \zeta_2 \right) \right] + D_4 \left[ n_f C_F^2 \left( \frac{1280}{9} \right) + D_4 C_F^2 C_A \left( - \frac{7040}{9} \right) \right] + D_5 C_F^3 \left[ 512 \right] \tag{5} \]

\[ \Delta_{q,S}^{(4)} = D_2 \left[ n_f C_F C_A^2 \left( \frac{82216}{9} - \frac{64000}{9} \zeta_2 + \frac{1408}{5} \zeta_2^2 - 1408 \zeta_3 \right) + n_f C_F^2 C_A \left( \frac{337120}{243} \right) \right] \]
\[-31232\zeta_2 + \frac{13184}{3}\zeta_2^2 - \frac{334976}{9}\zeta_3 + n_f C_F^3 \left( \frac{2072}{3} + \frac{14144}{3}\zeta_2 - \frac{84224}{15}\zeta_2^2 \right) \]
\[-109184\zeta_2 + n_f^2 C_F C_A \left( \frac{131576}{27} + \frac{9728}{9}\zeta_2 \right) + n_f^2 C_F^2 \left( -\frac{29744}{243} + \frac{60416}{27}\zeta_2 \right) \]
\[+ \frac{28160}{9}\zeta_2 + n_f^3 C_F \left( \frac{3200}{81} - \frac{512}{9}\zeta_2 \right) + n_f C_F C_A^2 \left( -\frac{1651520}{81} + \frac{138880}{9}\zeta_2 \right) \]
\[-\frac{7744}{5}\zeta_2^2 + 7744\zeta_3 + C_F^2 C_A^2 \left( \frac{1426292}{243} - \frac{5504\zeta_2\zeta_3}{27} - \frac{337216}{15}\zeta_2^2 \right) \]
\[+ \frac{359296}{3}\zeta_3 - 9216\zeta_5 + C_F^3 C_A \left( \frac{139396}{9} - 93696\zeta_2\zeta_3 - \frac{87008}{3}\zeta_2 + \frac{437888}{15}\zeta_2^2 \right) \]
\[+ \frac{632128}{3}\zeta_3 + C_F^4 \left( -409600\zeta_2\zeta_3 - 163840\zeta_3 + 688128\zeta_5 \right) \]
\[+ D_3 \left[ n_f C_F C_A^2 \left( -\frac{117184}{27} + \frac{5632}{9}\zeta_2 \right) + n_f C_F^2 C_A \left( -\frac{2143808}{243} + \frac{50176}{3}\zeta_2 \right) \right] \]
\[+ \frac{7168}{9}\zeta_3 + n_f C_F^2 \left( \frac{44224}{9} + \frac{88064}{9}\zeta_2 + \frac{306176}{9}\zeta_3 \right) + n_f^3 C_F C_A \left( \frac{18304}{27} - \frac{512}{9}\zeta_2 \right) \]
\[+ n_f^2 C_F \left( \frac{124288}{243} - \frac{4096}{3}\zeta_2 \right) + n_f^3 C_F \left( -\frac{2560}{81} \right) + C_F C_A^3 \left( \frac{698368}{81} - \frac{15488}{9}\zeta_2 \right) \]
\[+ C_F^2 C_A^2 \left( \frac{7699456}{243} - \frac{52736\zeta_2}{5} + \frac{13824}{5}\zeta_2^2 - \frac{140800}{9}\zeta_3 \right) + C_F^3 C_A \left( -\frac{421792}{9} - \frac{536576}{9}\zeta_2 + \frac{100864}{5}\zeta_2^2 - \frac{1499648}{9}\zeta_3 \right) + C_F^4 \left( 16352 + 56576\zeta_2 - \frac{195584}{5}\zeta_2^2 - 7680\zeta_3 \right) \]
\[+ D_4 \left[ n_f C_F C_A^2 \left( \frac{7744}{9} \right) + n_f C_F^2 C_A \left( \frac{175360}{27} - \frac{2560}{3}\zeta_2 \right) \right] \]
\[+ n_f C_F \left( -\frac{14080}{27} - \frac{87040}{9}\zeta_2 \right) + n_f^2 C_F C_A \left( -\frac{1408}{9} \right) + n_f^2 C_F^2 \left( -\frac{12800}{27} \right) \]
\[+ n_f^3 C_F \left( \frac{256}{27} \right) + C_F C_A^3 \left( -\frac{42592}{27} \right) + C_F^2 C_A^2 \left( -\frac{536960}{27} + \frac{14080}{3}\zeta_2 \right) \]
\[+ C_F^3 C_A \left( \frac{79360}{27} + \frac{478720}{9}\zeta_2 + 8960\zeta_3 \right) + C_F^4 \left( \frac{286720}{3}\zeta_3 \right) \]
\[ \delta_1 \left[ \delta(1-z) + C_F \left( -4 + 8\xi_2 \right) \right] \]

\[ \delta_2 \left[ C_F \left( -16 + 8\xi_3 \right) \right] \]

\[ \delta_3 \left[ C_F \left( -32 + 8\xi_3 \right) \right] \]

\[ \delta_4 \left[ C_F \left( -64 + 8\xi_3 \right) \right] \]
\[
\begin{align*}
\Delta_{b,S}^{\text{sv}(4)} &= D_2 \left[ C_F^2 C_A^3 \left( -\frac{1651520}{81} + 7744\xi_3 + \frac{138880}{9}\xi_2 - \frac{7744}{5}\xi_2^2 \right) + C_F^2 C_A^2 \left( -\frac{6588656}{243} - 9216\xi_5 + \frac{365632}{3}\xi_3 + \frac{2785408}{27}\xi_2 - 5504\xi_2\xi_3 - \frac{337216}{15}\xi_2^2 \right) + C_F^2 C_A \left( \frac{43264}{9} - \frac{680512}{9}\xi_3 + \frac{52736}{3}\xi_2 - 93696\xi_2\xi_3 + \frac{437888}{15}\xi_2^3 \right) + C_F^4 \left( 688128\xi_5 - 40960\xi_3 - 409600\xi_2\xi_3 \right) + n_f C_F C_A^2 \left( \frac{82216}{9} - \frac{1408\xi_3}{9} - \frac{64000}{9}\xi_2 + \frac{1408}{5}\xi_2^2 \right) \\
&\quad + n_f^2 C_F C_A \left( \frac{2004352}{243} - \frac{338432}{9}\xi_3 - \frac{32640\xi_2}{3} + \frac{13184}{3}\xi_2^2 \right) + n_f C_F^3 \left( 2272 - \frac{109184}{3}\xi_3 - \frac{11264}{3}\xi_2 - \frac{84224}{15}\xi_2^2 \right) + n_f^2 C_F C_A \left( -\frac{31576}{27} + \frac{9728}{9}\xi_2 \right) \\
&\quad + n_f^2 C_F^2 \left( -\frac{151424}{243} + \frac{28160}{9}\xi_3 + \frac{62720}{27}\xi_2 \right) + n_f^3 C_F \left( \frac{3200}{81} - \frac{512}{9}\xi_2 \right) \right].
\end{align*}
\]
\[
\Delta_{g,S}^{\text{iv}(0)} = \delta(1 - z) \\
\Delta_{g,S}^{\text{iv}(1)} = C_A \left( \delta(1 - z) \left[ 8\xi_2 \right] + \mathcal{D}_1 \left[ 16 \right] \right) \\
\Delta_{g,S}^{\text{iv}(2)} = \delta(1 - z) \left[ n_f C_F \left( - \frac{67}{3} + 16\xi_3 \right) + n_f C_A \left( - \frac{80}{3} - \frac{80}{9} \xi_2 - \frac{8}{3} \xi_3 \right) + C_A^2 \left( 93 + \frac{536}{9} \xi_2 \right) \right]
\]
\[
\Delta_{g,s}^{w,3} = D_0 \left[ n_f C_A \left( \frac{224}{27} - \frac{32}{5} \xi_2 \right) + C^2_A \left( -\frac{1616}{27} + \frac{176}{3} \xi_2 + 312 \xi_3 \right) \right] + D_1 \left[ n_f C_A \left( -\frac{160}{9} \right) + C^2_A \left( \frac{1072}{9} - 160 \xi_2 \right) \right] + D_2 \left[ n_f C_A \left( \frac{32}{3} \right) + C^2_A \left( -\frac{176}{3} \right) \right] + D_3 C^2_A \left[ 128 \right]
\]

\[
\Delta_{g,s}^{w,4} = D_2 \left[ n_f C_F C_A \left( 6624 - 2176 \xi_2 - \frac{1536 \xi_2^2}{5} - 3968 \xi_3 \right) + n_f C_F^2 C_A \left( -16 \right) \right] + n_f C_A^3 \left( \frac{4591096}{243} - \frac{1186688}{27} \xi_2 - \frac{9472}{15} \xi_2^2 - 71808 \xi_3 \right) + n_f^2 C_F C_A \left( -\frac{5600}{9} \right) + n_f^2 C_A^2 \left( \frac{1280}{3} \xi_3 \right) + n_f C_A \left( \frac{3200}{81} \right)
\]
\[ \begin{align*} &- \frac{512}{9} \zeta_2 + C_A^4 \left( - \frac{13631360}{243} - 508800 \zeta_2 \zeta_3 + \frac{4104704}{27} \zeta_2 + \frac{15488}{3} \zeta_2^2 \right) \\
&+ \frac{3259328}{9} \zeta_2 + 678912 \zeta_5 \right) + \mathcal{D}_3 \left[ n_f C_F C_A^2 \left( - \frac{50368}{9} + 4096 \zeta_3 \right) \right. \\
&+ n_f C_A^3 \left( - \frac{4338368}{243} + \frac{248320}{9} \zeta_2 + \frac{94208}{3} \zeta_3 \right) + n_f C_A \left( - \frac{2560}{81} \right) + C_A^4 \left( \frac{13802368}{243} - \frac{1107584}{9} \zeta_2 \right) \\
&+ \frac{80896}{5} \zeta_2^2 - \frac{585728}{3} \zeta_3 \right) \right] + \mathcal{D}_4 \left[ n_f C_F C_A^2 \left( \frac{1280}{3} \right) + n_f C_A^3 \left( \frac{26048}{3} - \frac{94720}{9} \zeta_2 \right) \right. \\
&+ n_f C_A^2 \left( - \frac{17024}{27} \right) + n_f C_A \left( \frac{256}{27} \right) + C_A^4 \left( - \frac{838112}{27} + \frac{520960}{9} \zeta_2 + \frac{313600}{3} \zeta_3 \right) \right] \\
&+ \mathcal{D}_5 \left[ n_f C_A^3 \left( - \frac{91136}{27} \right) + n_f C_A^2 \left( \frac{4096}{27} \right) + C_A^4 \left( \frac{432640}{27} - 23552 \zeta_2 \right) \right. \\
&+ \mathcal{D}_6 \left[ n_f C_A^3 \left( \frac{7168}{9} \right) + C_A^4 \left( - \frac{39424}{9} \right) \right] + \mathcal{D}_7 C_A^4 \left[ \frac{4096}{3} \right] \end{align*} \] (16)

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