Sparse Hanson-Wright inequalities for subgaussian quadratic forms

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Abstract

In this paper, we provide a proof for the Hanson-Wright inequalities for sparse quadratic forms in subgaussian random variables. This provides useful concentration inequalities for sparse subgaussian random vectors in two ways. Let \( X = (X_1, \ldots, X_m) \in \mathbb{R}^m \) be a random vector with independent subgaussian components, and \( \xi = (\xi_1, \ldots, \xi_m) \in \{0,1\}^m \) be independent Bernoulli random variables. We prove the large deviation bound for a sparse quadratic form of \( (X \circ \xi)^T A (X \circ \xi) \), where \( A \in \mathbb{R}^{m \times m} \) is an \( m \times m \) matrix, and random vector \( X \circ \xi \) denotes the Hadamard product of an isotropic subgaussian random vector \( X \in \mathbb{R}^m \) and a random vector \( \xi \in \{0,1\}^m \) such that \( (X \circ \xi)_i = X_i \xi_i \), where \( \xi_1, \ldots, \xi_m \) are independent Bernoulli random variables. The second type of sparsity in a quadratic form comes from the setting where we randomly sample the elements of an anisotropic subgaussian vector \( Y = HX \) where \( H \in \mathbb{R}^{m \times m} \) is an \( m \times m \) symmetric matrix; we study the large deviation bound on the \( \ell_2 \)-norm \( \|D_x Y\|_2 \) from its expected value, where for a given vector \( x \in \mathbb{R}^m \), \( D_x = \text{diag}(x) \) denotes the diagonal matrix whose main diagonal entries are the entries of \( x \). This form arises naturally from the context of covariance estimation.

Keywords: Hanson-Wright inequality; Subgaussian concentration; Sparse quadratic forms.

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1 Introduction

In this paper, we explore the concentration of measure results for quadratic forms involving a sparse subgaussian random vector $X \in \mathbb{R}^m$. Sparsity can naturally come from the fact that the high dimensional vector $X \in \mathbb{R}^m$ is sparse, for example, when the elements of $X$ are missing at random, or when we intentionally sparsify the vector $X$ to speed up computation. The purpose of the paper is to prove the Hanson-Wright type of large deviation bounds for sparse quadratic forms in Theorems 1.1 and 1.2.

Sparsity comes in two forms. In Theorem 1.1, we randomly sparsify the subgaussian vector $X$ involved in the quadratic form $X^T A X$, where $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ is a random vector with independent subgaussian components, and $\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m$ consists of independent Bernoulli random variables. In particular, we consider $(X \circ \xi)^T A (X \circ \xi)$, where $X \circ \xi \in \mathbb{R}^m$ denotes the Hadamard product of random vectors $X$ and $\xi$ such that $(X \circ \xi)_i = X_i \xi_i$ and $A$ is an $m \times m$ matrix. The second type of sparsity comes into play when we sample the elements of an anisotropic subgaussian random vector $Y = D_0 X$ where $X \in \mathbb{R}^m$ is as defined in Theorem 1.1 and $D_0 \in \mathbb{R}^{m \times m}$ is an $m \times m$ symmetric matrix.

The bound in Theorem 1.2 allows the second type of sparsity in a quadratic form in the following sense. Suppose $A_0$ is an $m \times m$ symmetric positive semidefinite matrix and $A_{0}^{1/2}$ is the unique square root of $A_0$. Suppose we randomly sample the rows or columns of $A_{0}^{1/2}$ to construct a quadratic form as follows,

$$X^T A_{0}^{1/2} A_{0}^{1/2} X \rightarrow X^T A_{0}^{1/2} D_\xi A_{0}^{1/2} X.$$  \hfill (1)

We state in Theorem 1.2, where we replace $A_{0}^{1/2}$ with $D_0$, a symmetric $m \times m$ matrix, the large deviation bound for the sparse quadratic form on the right hand side of (1). These questions arise naturally in the context of covariance estimation problems, where we naturally take $A_0$ and $D_0$ as symmetric positive (semi)definite matrices.

The following definitions correspond to Definitions 5.7 and 5.13 in [15]. For a random variable $X$, the sub-gaussian (or $\psi_2$) norm of $X$ denoted by $\|X\|_{\psi_2}$ is defined to be [15]:

$$\|X\|_{\psi_2} = \sup_{p \geq 1} p^{-1/2} (\mathbb{E} |X|^p)^{1/p}$$

which is the smallest $K_2$ which satisfies $(\mathbb{E} |X|^p)^{1/p} \leq K_2 \sqrt{p} \forall p \geq 1$;

if $\mathbb{E}[X] = 0$, then $\mathbb{E} \exp(tX) \leq \exp(\|X\|_{\psi_2}^2) \text{ for all } t \in \mathbb{R}.$

For a symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{m \times m}$, we let $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ denote the largest and the smallest eigenvalue of $A$ respectively. Moreover, we order the $m$ eigenvalues algebraically and denote them by

$$\lambda_{\text{min}}(A) = \lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_m(A) = \lambda_{\text{max}}(A).$$
For a matrix $A$, the operator norm $\|A\|_2$ is defined to be $\sqrt{\lambda_{\text{max}}(A^T A)}$. Let $\|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2}$. Let $\text{diag}(A)$ be the diagonal of $A$. Let $\text{offd}(A)$ be the off-diagonal of $A$. For matrix $A$, $r(A)$ denotes the effective rank $\text{tr}(A)/\|A\|_2$. For two numbers $a, b$, $a \lor b := \max(a, b)$.

In particular, we prove:

**Theorem 1.1.** Let $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ be a random vector with independent components $X_i$ which satisfy $\mathbb{E}X_i = 0$ and $\|X_i\|_{\psi_2} \leq K$. Let $\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m$ be a random vector independent of $X$, with independent Bernoulli random variables $\xi_i$ such that $\mathbb{E}(\xi_i) = p_i$. Let $A = (a_{ij})$ be an $m \times m$ matrix. Then, for every $t > 0$,

$$
\mathbb{P} \left( |(X \circ \xi)^T A(X \circ \xi) - \mathbb{E}(X \circ \xi)^T A(X \circ \xi)| > t \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{K^4 \left( \sum_{k=1}^m p_k a_{kk}^2 + \sum_{i \neq j} a_{ij}^2 p_i p_j \right)}, \frac{t}{K^2 \|A\|_2} \right) \right)
$$

(2)

where $X \circ \xi$ denotes the Hadamard product of random vectors $X$ and $\xi$ such that $(X \circ \xi)_i = X_i \xi_i$.

Let $\xi$ be as defined in Theorem 1.1. We now randomly sample entries of a correlated subgaussian random vector $Y = D_0 X$ and study the large deviation bound on the norm of $\|D_0 Y\|_2^2$ from its expected value in Theorem 1.2, where for a given $x \in \mathbb{R}^m$, $D_x = \text{diag}(x)$ denotes the diagonal matrix whose main diagonal entries are the elements of $x$. And we write $D_x := \text{diag}(x)$ interchangeably. Partition a symmetric matrix $D_0 \in \mathbb{R}^{m \times m}$ according to its columns as $D_0 = [d_1, d_2, \ldots, d_m]$. Denote by

$$
A_0 := D_0^2 = \sum_{i=1}^m d_i d_i^T = (a_{ij}) \succeq 0.
$$

(3)

The bounds in Theorem 1.1 and Theorem 1.2 reduce to essentially the same type.

**Theorem 1.2.** Let $D_\xi$ be a diagonal matrix with elements from the random vector $\xi \in \{0, 1\}^m$, where $\mathbb{E} \xi_j = p_j$, for $0 \leq p_j \leq 1$. Let $X$ be as defined in Theorem 1.1, independent of $\xi$. Let $A_0 = (a_{ij}) = D_0^2$. Let $Y = D_0 X$. Then, for every $t > 0$,

$$
\mathbb{P} \left( |Y^T D_\xi Y - \mathbb{E}Y^T D_\xi Y| > t \right) =: \mathbb{P}(\|S\| > t) \leq 2 \exp \left( -c_2 \min \left( \frac{t^2}{K^4 \left( \sum_{i=1}^m p_i a_{ii}^2 + \sum_{i \neq j} a_{ij}^2 p_i p_j \right)}, \frac{t}{K^2 \|A_0\|_2} \right) \right)
$$

where $c_2, C$ are some absolute constants.

Theorem 1.3 shows a concentration of measure bound on a quadratic form with Bernoulli random variables where an explicit dependency on $p_i$, for all $i$, is shown. The setting here is different from Theorem 2.1 as we deal with a quadratic form which involves non-centered Bernoulli random variables. Theorem 1.3 is crucial in proving Theorem 1.2.

**Theorem 1.3.** Let $\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m$ be a random vector with independent Bernoulli random variables $\xi_i$ such that $\xi_i = 1$ with probability $p_i$ and 0 otherwise. Let $A = (a_{ij})$ be an
$m \times m$ matrix. Then, for every $0 \leq \lambda \leq \frac{1}{104 \max(||A||_1,||A||_\infty)}$, 

$$
\mathbb{E} \exp \left( \lambda \sum_{i,j} a_{ij} \xi_i \xi_j \right) \leq \exp \left( \lambda \left( \sum_{i=1}^{m} a_{ii} p_i + \sum_{i\neq j} a_{ij} p_i p_j \right) \right) \times 

\exp \left( \frac{1}{3} \lambda \sum_{j \neq i} |a_{ij}| \sigma_i^2 \sigma_j^2 \right) \times 

\exp \left( C_5 \lambda \left( \frac{1}{2} \sum_{i=1}^{m} |a_{ii}| p_i + \sum_{j \neq i} |a_{ij}| p_j p_j \right) \right)
$$

where $\sigma_i^2 = p_i(1 - p_i)$ and $C_5 \leq 0.04$.

The proof of Theorem 1.3 is deferred to Section 5.

Before we leave this section, we also introduce the following notation. For a random variable $X$, the sub-exponential (or $\psi_1$) norm of $X$ denoted by $\|X\|_{\psi_1}$ is defined to be the smallest $K_2$ which satisfies

$$(\mathbb{E} |X|^p)^{1/p} \leq K_2 p \forall p \geq 1; \text{ in other words}$$

$$\|X\|_{\psi_1} = \sup_{p \geq 1} p^{-1} (\mathbb{E} |X|^p)^{1/p}.$$  

Throughout this paper $C_0, C, C_1, c, c_1, \ldots$ denote positive absolute constants whose value may change from line to line. For a vector $X \in \mathbb{R}^m$, let $X_{\Lambda}$ denote $(X_i)_{i \in \Lambda}$ for a set $\Lambda \subseteq [m]$.

We use the following properties of the Hadamard product [8],

$$A \circ x x^T = D x A D x$$

and 

$$\text{tr}(D A \xi A^T) = \xi^T (A \circ A) \xi$$

from which a simple consequence is 

$$\text{tr}(D A \xi A) = \xi^T (A \circ I) \xi = \xi^T \text{diag}(A) \xi.$$ 

We use the following bounds throughout our paper. For any $x \in \mathbb{R}$,

$$e^x \leq 1 + x + \frac{1}{2} x^2 e^{x^2}. \quad (4)$$

We need the following result which follows from Proposition 3.4 in [13].

**Lemma 1.4.** Let $A = (a_{ij})$ be an $m \times m$ matrix. Let $a_{\infty} := \max_i |a_{ii}|$. Let $\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m$ be a random vector with independent Bernoulli random variables $\xi_i$ such that $\xi_i = 1$ with probability $p_i$ and 0 otherwise. Then for $|\lambda| \leq \frac{1}{4a_{\infty}}$, 

$$\mathbb{E} \exp \left( \lambda \sum_{i=1}^{m} a_{ii} (\xi_i - p_i) \right) \leq \exp \left( \frac{1}{2} \lambda^2 e^{p_i a_{\infty}} \sum_{i=1}^{m} a_{ii}^2 \sigma_i^2 \right)$$

where $\sigma_i^2 = p_i(1 - p_i)$.

We need to state Lemma 1.5, which provides an estimate of the moment generating function for the centered sub-exponential random variable $Z_k := X_k^2 - \mathbb{E}X_k^2$ for $X_k$ as defined in Theorem 1.1.
Lemma 1.5. Let $X \in \mathbb{R}$ be a sub-gaussian random variable which satisfies $\mathbb{E}X = 0$ and $\|X\|_{\psi_2} \leq K$. Let $|\tau| \leq \frac{1}{23.5eK^2}$. Denote by $C_0 := 38.94$. Then
\[
\mathbb{E}\left(\exp(\tau(X^2 - \mathbb{E}X^2))\right) \leq 1 + 38.94\tau^2K^4 \leq \exp(C_0\tau^2K^4).
\]
The proof follows essentially that of Lemma 5.15 in [15]; we provide here explicit constants.

The rest of the paper is organized as follows. In Section 2, we compare our results with those in the literature. We then prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. We prove Theorem 1.3 in Section 5. We leave certain calculations in Appendix A for the purpose of self-containment, namely, the proof of Lemmas 1.5 and 3.2.

2 Consequences and related work

In this section, we first compare with the following form of the Hanson-Wright inequality as recently derived in [12], as well as an even more closely related result in [11]. Such concentration of measure bounds were originally proved by [7, 16]. The bound as stated in Theorem 2.1 is proved in [12].

Theorem 2.1. [12] Let $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ be a random vector with independent components $X_i$ which satisfy $\mathbb{E}X_i = 0$ and $\|X_i\|_{\psi_2} \leq K$. Let $A$ be an $m \times m$ matrix. Then, for every $t > 0$,
\[
P\left(\|X^TAX - \mathbb{E}X^TAX\| > t\right) \leq 2 \exp\left(-c\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)\right).
\]

When $X$ is a vector whose coordinates are $\pm 1$ Bernoulli random variables, the following Lemma in the same spirit as in Theorem 1.1 is shown in [11].

Lemma 2.2. ([11]) Let $J$ be a random subset of $[m]$ of size $k < m$ uniformly chosen among all such subsets. Denote by $R_J = \sum_{j \in J} e_j e_j^T$ the coordinate projection on the set $J$. Let $Y = (\varepsilon_1, \ldots, \varepsilon_m)$ be vector whose coordinates are $\pm 1$ Bernoulli Random variables. Then for any $m \times m$ matrix $A$ and any $t > 0$
\[
P\left(\|Y^T R_JAR_J Y - \mathbb{E}Y^T R_JAR_J Y\| > t\right) \leq 2 \exp\left(-c\min\left(\frac{t^2}{k\|A\|_2^2}, \frac{t}{\|A\|_2}\right)\right).
\]

Other related results include [10, 9, 5, 2, 6, 1]. We refer to [12] for a survey of these and other related results.
Clearly, the large deviation bounds in Theorems 1.1 and 1.2 are determined by the following quantity

\[ M := \sum_{i=1}^{m} p_i a_{ii}^2 + \sum_{i \neq j} a_{ij}^2 p_i p_j. \]

We now state some consequences of Theorems 1.1 and 1.2 in Corollaries 2.3 and 2.4. Lemma 2.2 and Corollaries 2.3 and 2.4 show essentially a large deviation bound at roughly the same order given that

\[ p \| \text{diag}(A) \|^2_F + p^2 \| \text{offd}(A) \|^2_F \leq p m \| A \|^2_2, \]

while \( k \| A \|^2_2 = \frac{k}{m} m \| A \|^2_2 \).

The following Corollary 2.3 follows from Theorem 1.1 immediately.

**Corollary 2.3.** Let \( X, \xi \) be as defined in Theorem 1.1. Let \( p_1 = p_2 = \ldots = p_m = p \). Let \( A = (a_{ij}) \) be an \( m \times m \) matrix. Then, for every \( t > 0 \),

\[
\mathbb{P} \left( \left| X^T D_\xi A D_\xi X - \mathbb{E} X^T D_\xi A D_\xi X \right| > t \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{K^4 \left( p \| \text{diag}(A) \|^2_F + p^2 \| \text{offd}(A) \|^2_F \right)}, \frac{t}{K^2 \| A \|^2_2} \right) \right).
\]

**Corollary 2.4.** Let \( D_0, A_0, X, \xi, Y \) be as defined in Theorem 1.2. Let \( p_1 = p_2 = \ldots = p_m = p \). Then, for every \( t > 0 \),

\[
\mathbb{P} \left( \left| Y^T D_\xi Y - \mathbb{E} Y^T D_\xi Y \right| > t \right) = \mathbb{P} \left( \left| D_\xi D_0 X \right|_2 - \mathbb{E} \left| D_\xi D_0 X \right|_2 \right) \leq 2 \exp \left( -c \min \left( \frac{t^2}{K^4 (p \| \text{diag}(A_0) \|^2_F + p^2 \| \text{offd}(A_0) \|^2_F)}, \frac{t}{K^2 \| A_0 \|^2_2} \right) \right).
\]

**Corollary 2.5.** Suppose all conditions in Corollary 2.3 hold. Let \( A \in \mathbb{R}^{m \times m} \) be positive semidefinite. Suppose \( EX_i^2 = 1 \) and

\[
\log m \| A \|^2_2 = o(\text{ptr}(A)) \tag{5}
\]

Then with probability at least \( 1 - 4/m^4 \),

\[ X^T D_\xi A D_\xi X \leq \text{ptr}(A)(1 + o(1)). \]

**Proof.** Define

\[ S = \sum_{i,j} a_{ij} (X_i \xi_i X_j \xi_j - \mathbb{E} X_i \xi_i X_j \xi_j). \]

Thus \( \mathbb{E} S = \sum_i a_{ii} \mathbb{E} X_i^2 \mathbb{E} \xi_i = \text{ptr}(A) \). We have under conditions of Theorem 1.1, with probability at least \( 1 - 4/m^4 \), for some absolute constant \( C \),

\[
|S| := \left| X^T D_\xi A D_\xi X - \text{ptr}(A) \right| \leq C K^2 \log^{1/2} m \left( \log^{1/2} m \| A \|^2_2 + \sqrt{p \| \text{diag}(A) \|^2_F + p \| \text{offd}(A) \|^2_F} \right) =: t.
\]
where under condition (5), the deviation term is of a small order of the expected value \( \text{ptr}(A) \); that is,

\[
t \asymp \log m \|A\|_2 + \log^{1/2} m (\sqrt{p} \|\text{diag}(A)\|_F + p \|A\|_F) =: I + II = o(\text{ptr}(A)).
\]

To see this, notice that (5) immediately implies that the first term in \( t \) is of \( o(\text{ptr}(A)) \). Now in order for the second and third term to be of \( o(\text{ptr}(A)) \), we need that

\[
\sqrt{p} \|A\|_F \log^{1/2} m \ll \text{ptr}(A) \quad \text{and hence} \quad p \gg \log m \|A\|_F^2 / \text{tr}(A)^2
\]

which is satisfied by (5) given that \( \|A\|_{\text{tr}(A)} \geq \|A\|_F^2 / \|A\|_2 \), which in turn is due to \( \|A\|_F^2 \leq \text{tr}(A) \|A\|_2 \). □

**Corollary 2.6.** Suppose that (5) and all conditions in Corollary 2.4 hold. Assume \( EX_i^2 = 1 \). Then with probability at least \( 1 - \frac{4}{m^4} \), \( |X^T D_0 D_\xi D_0 X| = \text{ptr}(A_0)(1 + o(1)) \).

**Proof.** First by independence of \( X \) and \( \xi \), we have for \( EX_i^2 = 1 \),

\[
EX_i^T A_\xi X = E \sum_{k=1}^m X_k^T A_{\xi,k} = E(X_k^2) E(A_{\xi,k}) = E X_k^2 E A_{\xi,k} = \sum_{k=1}^m E X_k^2 E \sum_{\ell=1}^m \xi_{\ell,k}^2 d_{k\ell} = \sum_{\ell=1}^m p_{\ell,k} \sum_{k=1}^m d_{k\ell} = \sum_{\ell=1}^m p_{\ell} a_{\ell}.
\]

We have by Corollary 2.4, with probability at least \( 1 - \frac{4}{m^2} \),

\[
|X^T D_0 D_\xi D_0 X| \leq \sum_{i=1}^m a_{ii} p_i + CK^2 \log^{1/2} m \left( \log^{1/2} m \|A_0\|_2 + \sqrt{M} \right)
\leq p \|D_0\|_F^2 + CK^2 \log^{1/2} m \left( \log^{1/2} m \|A_0\|_2 + \sqrt{p} \|\text{diag}(A_0)\|_F + p \|\text{offd}(A_0)\|_F \right).
\]

for some absolute constants \( C \), where \( \sqrt{M} \leq \sqrt{p} \|\text{diag}(A_0)\|_F + p \|\text{offd}(A_0)\|_F \). The rest of the proof for Corollary 2.6 follows from that of Corollary 2.5. □

### 2.1 Implications when \( p_1, \ldots, p_m \) are not the same

We first need the following sharp statements about eigenvalues of a Hadamard product. See for example Theorem 5.3.4 [8].

**Theorem 2.7.** Let \( A, B \in \mathbb{R}^{m \times m} \) be positive semidefinite. Let \( a_{\infty} := \max_{i=1}^m a_{ii} \) and \( b_{\infty} := \max_{i=1}^m b_{ii} \). Any eigenvalue of \( \lambda(A \circ B) \) satisfies

\[
\lambda_{\min}(A) \lambda_{\min}(B) \leq (\min_{i=1}^m a_{ii}) \lambda_{\min}(B) \leq \lambda(A \circ B) \leq a_{\infty} \lambda_{\max}(B) \leq \lambda_{\max}(A) \lambda_{\max}(B).
\]
Corollary 2.8. Suppose all conditions in Theorem 1.2 hold. Suppose $EX_i^2 = 1$. Let $p = (p_1, \ldots, p_m)$. Let $|p|_1 := \sum_{i=1}^{m} p_i$ and $\|p\|_2^2 = \sum_{i=1}^{m} p_i^2$. Then with probability at least $1 - 4/m^4$,

$$|X^TD_0D_0X| \leq \sum_{i=1}^{m} p_i \|d_i\|_2^2 + CK^2 \log^{1/2} m \|D_0\|_2 \left(\log^{1/2} m \|D_0\|_2 + 2(\max_i \|d_i\|_2) |p|_2^{1/2}\right).$$

Proof. Recall $A_0 = (a_{ij}) = D_0^2 \succeq 0$. Let $a_\infty := \max_{i=1}^{m} a_{ii} = \max_i \|d_i\|_2$. Thus we have $a_\infty \leq \|D_0\|_2^2$. Denote by $p = (p_1, \ldots, p_m)$. We have by Theorem 2.7,

$$M = \sum_{i=1}^{m} p_i a_{ii}^2 + \sum_{i \neq j} a_{ij}^2 p_i p_j \leq \sum_{i=1}^{m} p_i a_{ii}^2 + p^T(A_0 \circ A_0)p \leq a_\infty^2 |p|_1 + \lambda_{\max}(A_0 \circ A_0) \|p\|_2^2 \leq a_\infty^2 \|A_0\|_2 \|p\|_2^2 \leq 2a_\infty \|A_0\|_2 |p|_1.$$

where $\|p\|_2^2 \leq |p|_1$. The corollary thus follows immediately from Theorem 1.2. □

Remark 2.9. Assume that $p_i \geq \log m \over m$ and hence $|p|_1 \geq \log m$. Then we have $\|p\|_2 \leq |p|_2^{1/2} \leq |p|_1$. Notice that the second term starts to dominate when $|p|_1^{1/2} \gg \log m$ while the total deviation remains to be a small order of the mean $\sum_{i=1}^{m} p_i \|d_i\|_2^2$ so long as

$$|p|_1 \gg \frac{\max_k \|d_k\|_2^2 \|D_0\|_2^2}{\min_k \|d_k\|_2^4}.$$

3 Proof of Theorem 1.1

The structure of our proof follows that of Theorem 1.1 by [12]. The problem reduces to estimating the diagonal and the off-diagonal sums.

Part I: Diagonal Sum. Define

$$S_0 := \sum_{k=1}^{m} a_{kk} \xi_k X_k^2 - \mathbb{E} \sum_{k=1}^{m} a_{kk} \xi_k X_k^2 \quad \text{where} \quad \mathbb{E} \sum_{k=1}^{m} a_{kk} \xi_k X_k^2 = \sum_{k=1}^{m} a_{kk} p_k \mathbb{E} X_k^2.$$

Lemma 3.1. Let $X$ and $\xi$ be defined as in Theorem 1.1. Let $A$ be an $m \times m$ matrix. Then, for every $t > 0$,

$$\mathbb{P} \left( \left| \sum_{k=1}^{m} a_{kk} \xi_k X_k^2 - \sum_{k=1}^{m} a_{kk} p_k \mathbb{E} X_k^2 \right| > t \right) \leq \mathbb{P}(S_0 > t) + \mathbb{P}(S_0 < -t) \leq 2 \exp \left( -\frac{1}{4e} \min \left( \frac{t^2}{3K^4 \sum_{k=1}^{m} a_{kk}^2 p_k}, \frac{t}{K^2 \max_k |a_{kk}|} \right) \right).$$
We prove Lemma 3.1 after we state Lemma 3.2. For the general case where $X_k$ are mean-zero independent sub-gaussian random variables with $\|X_k\|_{\psi_2} \leq K$, we first state the following bound on the moment generating function of $X_k^2$.

**Lemma 3.2.** Suppose that $|\lambda| < 1/(4eK^2 \max_k |a_{kk}|)$. Then for all $k$, we have for all $a_{kk} \in \mathbb{R}$

$$
\mathbb{E} \exp \left( \lambda a_{kk} X_k^2 \right) - 1 \leq \lambda a_{kk} \mathbb{E} X_k^2 + 16\lambda^2 a_{kk}^2 K^4. \quad (7)
$$

**Proof of Lemma 3.1.** We first state some simple fact: $\max_{k=1}^m \mathbb{E} X_i^2 \leq K^2$. By independence of $X_1, \ldots, X_k$ and $\xi_1, \ldots, \xi_k$, we bound the moment generating function of $S_0$ as follows: for $|\lambda| \leq \frac{1}{4eK^2 \max_k |a_{kk}|}$, we have

$$
\mathbb{E} \exp (\lambda S_0) = \mathbb{E} \exp (\lambda \sum_{k=1}^m \xi_k X_k^2 a_{kk} - \lambda \sum_{k=1}^m p_k a_{kk} \mathbb{E} X_k^2)
$$

$$
= \prod_{k=1}^m \left( \frac{\mathbb{E} \exp (\lambda a_{kk} \xi_k X_k^2)}{\exp (\lambda p_k a_{kk} \mathbb{E} X_k^2)} \right)
= \prod_{k=1}^m \frac{\mathbb{E} \exp (\lambda p_k a_{kk} \mathbb{E} X_k^2)}{\exp (\lambda p_k a_{kk} \mathbb{E} X_k^2)} \exp (\lambda p_k a_{kk} \mathbb{E} X_k^2)
$$

$$
\leq \prod_{k=1}^m \frac{1 + p_k (\lambda a_{kk} \mathbb{E} X_k^2 + 16\lambda^2 a_{kk}^2 K^4)}{\exp (\lambda p_k a_{kk} \mathbb{E} X_k^2)}
$$

$$
\leq \prod_{k=1}^m \frac{\exp (\lambda p_k a_{kk} \mathbb{E} X_k^2 + 16\lambda^2 p_k a_{kk}^2 K^4)}{\exp (\lambda p_k a_{kk} \mathbb{E} X_k^2)} = \exp \left( 16\lambda^2 K^4 \sum_{k=1}^m p_k a_{kk}^2 \right)
$$

where we used (7) for the first inequality and the fact that $1 + x \leq e^x$ for the second inequality. Hence for $0 < \lambda \leq \frac{1}{4eK^2 \max_k |a_{kk}|}$, we have

$$
\mathbb{P} (S_0 > t) \leq \frac{\mathbb{E} \exp (\lambda S_0)}{e^{\lambda t}} \leq \exp \left( -\lambda t + 16\lambda^2 K^4 \sum_{k=1}^m p_k a_{kk}^2 \right)
$$

for which the optimal choice of $\lambda$ is

$$
\lambda = \min \left( \frac{t}{32K^4 \sum_{k=1}^m p_k a_{kk}^2}, \frac{1}{4eK^2 \max_k |a_{kk}|} \right)
$$

Thus we have

$$
\mathbb{P} \left( \sum_{k=1}^m a_{kk} \xi_k X_k^2 - \sum_{k=1}^m a_{kk} p_k \mathbb{E} X_k^2 > t \right)
$$

$$
\leq \exp \left( -\frac{1}{4e} \min \left( \frac{t^2}{3K^4 \sum_{k=1}^m p_k a_{kk}^2}, \frac{t}{K^2 \max_k |a_{kk}|} \right) \right).
$$

We note that these constants have not been optimized. Repeating the arguments for $-A$ instead of $A$, we obtain for every $t > 0$, and for $S_0' := \sum_{k=1}^m (-a_{kk}) \xi_k X_k^2 - \sum_{k=1}^m (-a_{kk}) p_k \mathbb{E} X_k^2$

$$
\mathbb{P} \left( \sum_{k=1}^m a_{kk} \xi_k X_k^2 - \sum_{k=1}^m a_{kk} p_k \mathbb{E} X_k^2 < -t \right) = \mathbb{P} (S_0' > t)
$$

$$
\leq \exp \left( -\frac{1}{4e} \min \left( \frac{t^2}{3K^4 \sum_{k=1}^m p_k a_{kk}^2}, \frac{t}{K^2 \max_k |a_{kk}|} \right) \right).
$$
The lemma thus holds. □

**Part II: Off-diagonal Sum.** We now focus on bounding the off-diagonal part of the sum:

$$S_{\text{offd}} := \sum_{i \neq j}^m a_{ij} X_i X_j \xi_i \xi_j$$

where by independence of $X$ and $\xi$, $\mathbb{E} S_{\text{offd}} = \sum_{i \neq j}^m a_{ij} \mathbb{E} X_i \mathbb{E} X_j \mathbb{E} \xi_i \mathbb{E} \xi_j = 0$.

We will show that the following large deviation inequality holds for all $t > 0$,

$$\mathbb{P} (|S_{\text{offd}}| > t) \leq 2 \exp \left( -c \min \left( \frac{t^2}{K^4 \sum_{i \neq j} a_{ij}^2 p_i p_j}, \frac{t}{K^2 \|A\|_2} \right) \right) \quad (8)$$

First we prove a bound on the moment generating function for the off-diagonal sum $S_{\text{offd}}$. We assume without loss of generality that $K = 1$ by replacing $X$ with $X/K$. Let $C_4$ be a constant to be specified. It holds that for all $|\lambda| \leq \frac{1}{2\sqrt{C_4\|A\|_2}}$,

$$\mathbb{E} \exp(\lambda S_{\text{offd}}) \leq \exp \left( 1.44 C_4 \lambda^2 \sum_{i \neq j} a_{ij}^2 p_i p_j \right). \quad (9)$$

Thus we have for $0 \leq \lambda \leq \frac{1}{2\sqrt{C_4\|A\|_2}}$ and $t > 0$,

$$\mathbb{P} (S_{\text{offd}} > t) \leq \frac{\mathbb{E} \exp(\lambda S_{\text{offd}})}{e^{\lambda t}} \leq \exp \left( -\lambda t + 1.44 C_4 \lambda^2 \sum_{i \neq j} p_i p_j a_{ij}^2 \right).$$

Optimizing over $\lambda$, we conclude that

$$\mathbb{P} (S_{\text{offd}} > t) \leq \exp \left( -c \min \left( \frac{t^2}{\sum_{i \neq j} a_{ij}^2 p_i p_j}, \frac{t}{\|A\|_2} \right) \right) =: q_1 \quad (10)$$

Repeating the arguments for $-A$ instead of $A$, we obtain for $S' := \sum_{i \neq j}^m (-a_{ij}) X_i X_j \xi_i \xi_j = -S_{\text{offd}}$, $0 \leq \lambda \leq \frac{1}{2\sqrt{C_4\|A\|_2}}$ and $t > 0$,

$$\mathbb{P} (S' > t) \leq \frac{\mathbb{E} \exp(-\lambda S')} {e^{\lambda t}} = \frac{\mathbb{E} \exp(-\lambda S_{\text{offd}})} {e^{\lambda t}} \leq \exp \left( -\lambda t + 1.44 C_4 \lambda^2 \sum_{i \neq j} p_i p_j a_{ij}^2 \right) \leq q_1$$

by (9) and (10). Thus we have

$$\mathbb{P} (|S_{\text{offd}}| > t) = \mathbb{P} (S_{\text{offd}} > t) + \mathbb{P} (S_{\text{offd}} < -t) = \mathbb{P} (S_{\text{offd}} > t) + \mathbb{P} (S' > t) = 2q_1.$$ 

Thus (8) holds for all $t > 0$. The theorem is thus proved by summing up the bad events for diagonal sum and the non-diagonal sum while adjusting the constant $c$ in (2).

The proof of (9) follows essentially from the decoupling and reduction arguments in [12] and thus omitted from the main body of the paper. For completeness, we include the full proof in Appendix C. See for example [4, 3] for comprehensive discussions on modern decoupling methods. □
4 Proof of Theorem 1.2

Let $X, \xi, D_0$ and $D_\xi$ be defined as in Theorem 1.2. We assume without loss of generality that $K = 1$ by replacing $X$ with $X/K$. Denote by $\xi = (\xi_1, \ldots, \xi_m) \in \{0, 1\}^m$ a random vector with independent Bernoulli random variables $\xi_i$ such that $\xi_i = 1$ with probability $p_i$ and 0 otherwise.

We will bound the diagonal and the off-diagonal sums separately. Let $D_0 = [d_1, d_2, \ldots, d_m]$ be a symmetric matrix. Recall that we need to estimate

$$q := \mathbb{P} \left( \left| X^T A_\xi X - \mathbb{E} X^T A_\xi X \right| > t \right)$$

where $A_\xi = D_0 D_\xi D_0 =: (\tilde{a}_{ij})$.

We first separate the diagonal sum from the off-diagonal sum as follows:

$$\left| X^T A_\xi X - \mathbb{E} X^T A_\xi X \right| \leq \left| \sum_{i \neq j} X_i X_j A_{\xi,ij} \right| + \left| \sum_{k=1}^m X_k^2 A_{\xi, kk} - \mathbb{E} (X_k^2) \mathbb{E} (A_{\xi, kk}) \right|$$

where $S_{\text{offd}}$ and $S_{\text{diag}}$ denote the following random variables:

$$S_{\text{offd}} := \sum_{i \neq j} X_i X_j A_{\xi,ij} = \sum_{i \neq j} X_i X_j \tilde{a}_{ij}$$

and

$$S_{\text{diag}} := \sum_{k=1}^m X_k^2 A_{\xi, kk} - \mathbb{E} (X_k^2) \mathbb{E} (A_{\xi, kk}).$$

To prove Lemma 4.5, we need the following bounds on moment generating functions for the diagonal sum in $S_{\text{diag}}$ in Lemma 4.1 and the off-diagonal sum $S_{\text{offd}}$ in Lemma 4.4. Let $A_0 = D_0^2 = (a_{ij}) \succeq 0$. The constants in the expression for $N$ (and $M$) are not being optimized:

$$N = 82 \sum_{i=1}^m a_{ii}^2 p_i + 108 \sum_{i \neq j} a_{ij}^2 p_i p_j,$$

(11)

Lemma 4.1. For all $|\lambda| \leq \frac{1}{128\|A_0\|_2}$,

$$\mathbb{E} \exp(\lambda S_{\text{diag}}) \leq \exp \left( \lambda^2 N \right) \quad \text{and} \quad \mathbb{E} \exp(-\lambda S_{\text{diag}}) \leq \exp \left( \lambda^2 N \right).$$

To prove Lemma 4.1, first we write $S_{\text{diag}} = S_0 + S_\ast$ where

$$S_0 := \sum_{k=1}^m (X_k^2 - \mathbb{E} (X_k^2)) A_{\xi, kk} = \sum_{k=1}^m (X_k^2 - \mathbb{E} (X_k^2)) \sum_{\ell=1}^m d_{k\ell}^2 \xi_\ell,$$

(12)

$$S_\ast := \sum_{k=1}^m \mathbb{E} (X_k^2) A_{\xi, kk} - \mathbb{E} (X_k^2) \mathbb{E} (A_{\xi, kk}) = \sum_{k=1}^m \mathbb{E} (X_k^2) \sum_{\ell=1}^m d_{k\ell}^2 (\xi_\ell - \mathbb{E} (\xi_\ell))$$

(13)
where recall

\[ A_\xi = D_0 D_\xi D_0 , \quad \text{where } D_0 = [d_1, \ldots, d_m]. \]  

(14)

We now state the following bounds on the moment generating functions of \( S_0 \) and \( S_* \) in Lemmas 4.2 and 4.3 respectively. The estimate on the moment generating function stated in Lemma 4.1 then follows immediately from the Cauchy-Schwartz inequality, in view of Lemmas 4.2 and 4.3.

**Lemma 4.2.** Let \( a_{ii} = \|d_i\|_2^2 \) for \( d_i \) as defined in (14). Let \( a_{i\infty} = \max_i \|d_i\|_2^2 \). Then for \( |\lambda| < \frac{1}{4a_{i\infty}}, \)

\[ \mathbb{E} \exp(\lambda S_*) \leq \exp \left( \frac{1}{2} \lambda^2 e^{\lambda a_{i\infty}} \sum_{i=1}^{m} a_{ii}^2 \sigma_i^2 \right) \quad \text{where} \quad \mathbb{E}(X_k^2) \leq \|X_k\|_{\psi_2} = 1. \]

**Proof.** We have by independence of \( X \) and \( \xi \) and by definition of \( S_* \) in (13)

\[ S_* = \sum_{k=1}^{m} \mathbb{E}(X_k^2) \sum_{i=1}^{m} d_{ki}^2 (\xi_i - p_i) = \sum_{i=1}^{m} \left( \sum_{k=1}^{m} \mathbb{E}(X_k^2) d_{ki}^2 \right) (\xi_i - p_i) =: \sum_{i=1}^{m} a_{ii}' (\xi_i - p_i). \]

where by assumption, we have \( \mathbb{E}(X_k^2) \leq \|X_k\|_{\psi_2} \leq K = 1 \) and hence

\[ 0 \leq a_{ii}' := \sum_{k=1}^{m} \mathbb{E}(X_k^2) d_{ki}^2 \leq a_{ii} \quad \text{and thus} \quad \max_i a_{ii}' \leq a_{i\infty}. \]

The bound on the mgf of \( S_* \) follows from Lemma 1.4. For \( |\lambda| < \frac{1}{4a_{i\infty}}, \) we have

\[ \mathbb{E} \exp(\lambda S_*) = \mathbb{E} \exp(\lambda \sum_{i=1}^{m} a_{ii}' (\xi_i - p_i)) \leq \exp \left( \frac{1}{2} \lambda^2 e^{\lambda a_{i\infty}} \sum_{i=1}^{m} (a_{ii}')^2 \sigma_i^2 \right) \]

\[ \leq \exp \left( \frac{1}{2} \lambda^2 e^{\lambda a_{i\infty}} \sum_{i=1}^{m} a_{ii}^2 \sigma_i^2 \right). \]

\[ \square \]

**Lemma 4.3.** Denote by \( a_{ij} = \langle d_i, d_j \rangle \) for all \( i \neq j \) and \( a_{ii} = \|d_i\|_2^2 \) for \( d_i \) as defined in (14). Denote by \( a_{i\infty} := \max_i a_{ii} \). Let \( C_0 = 38.94 \). Then for \( |\lambda| \leq \frac{1}{64\|A_0\|_2} \leq \frac{1}{64a_{i\infty}}, \)

\[ \mathbb{E} \exp(\lambda S_0) \leq \exp \left( \lambda^2 (40 \sum_{j=1}^{m} p_j a_{jj}^2 + 54 \sum_{i \neq j} p_i p_j a_{ij}^2) \right). \]

(15)

**Lemma 4.4.** Let \( A_0 = (a_{ij}) = D_0^2 \). For all \( |\lambda| \leq \frac{1}{68\|C_0\|_2} \) for some constant \( C \)

\[ \mathbb{E} \exp(\lambda S_{offd}) \leq \mathbb{E} \exp(\lambda^2 C_2 \xi^T (A_0 \circ A_0) \xi) \leq \exp \left( \lambda^2 M \right) \]

\[ \mathbb{E} \exp(-\lambda S_{offd}) \leq \exp \left( \lambda^2 M \right) \]

where \( C_2 = 32C^2 \) and \( M = 11C^2 (3 \sum_{i=1}^{m} p_i a_{ii}^2 + 4 \sum_{i \neq j} a_{ij}^2 p_i p_j) \).
We defer the proof of Lemma 4.4 to Section 4.2 and Lemma 4.3 to Section 4.1. We are now ready to state the large deviation inequalities for the diagonal sum $S_{\text{diag}}$, followed by that for the off-diagonal sum $S_{\text{offd}}$.

**Lemma 4.5.** Let $A_0 = (a_{ij}) = D_0^2$. For all $t > 0$ and $N$ as defined in (11),
\[
P(\|S_{\text{diag}}\| > t/2) \leq 2 \exp \left( -\frac{1}{16} \min \left( \frac{t^2}{N}, \frac{t}{32 \|A_0\|_2} \right) \right).
\]

For the off-diagonal sum, we now state the following large deviation bound as in Lemma 4.6.

**Lemma 4.6.** Suppose all conditions in Lemma 4.4 hold. For all $t > 0$, and some large enough absolute constant $C$,
\[
P(\|S_{\text{offd}}\| > t/2) \leq 2 \exp \left( -\frac{1}{16} \min \left( \frac{t^2}{M}, \frac{t}{15C \|A_0\|_2} \right) \right)
\]
where $M = 11C^2 \left( 3 \sum_{i=1}^{m} p_i a_{ii}^2 + 4 \sum_{i \neq j} a_{ij}^2 p_i p_j \right)$.

The theorem is thus proved by summing up the two bad events:
\[
q = P(\|S_{\text{diag}} + S_{\text{offd}}\| > t) \leq P(\|S_{\text{offd}}\| > t/2) + P(\|S_{\text{diag}}\| > t/2)
\]
while adjusting the constant $c$ in (2).

It remains to prove Lemmas 4.1, 4.5 and 4.6.

**Proof of Lemma 4.1.** Suppose that $|\lambda| \leq \frac{1}{128 \|A_0\|_2}$. By Lemmas 4.3 and 4.2,
\[
E^{1/2} \exp(2\lambda S_*) \leq \exp \left( \lambda^2 e^{2|\lambda|a_{\infty}} \sum_{j=1}^{m} \sigma_j^2 a_{jj}^2 \right)
\]
\[
E^{1/2} \exp(2\lambda S_0) \leq \exp \left( 80\lambda^2 \sum_{j=1}^{m} p_j a_{jj}^2 \right) \exp \left( 108\lambda^2 \sum_{i \neq j} a_{ij}^2 p_i p_j \right).
\]

Now we have by the Cauchy-Schwartz inequality,
\[
E \exp(\lambda S_{\text{diag}}) = E \exp(\lambda(S_0 + S_*)) \leq E^{1/2} \exp(2\lambda S_0) E^{1/2} \exp(2\lambda S_*)
\]
\[
\leq \exp \left( 82\lambda^2 \sum_{j=1}^{m} \sigma_j^2 a_{jj}^2 \right) \exp \left( 108\lambda^2 \sum_{i \neq j} a_{ij}^2 p_i p_j \right).
\]

\[\square\]

**Proof of Lemma 4.5.** Lemma 4.5 follows from Lemma 4.1 immediately. Let $E_X$ and $E_\xi$ denote the expectation with respect to random variables in vectors $X$ and $\xi$ respectively.
First, by the Markov’s inequality, we have for $0 < \lambda \leq \frac{1}{128\|A_0\|_2}$

$$\mathbb{P}(S_{\text{diag}} > t/2) = \mathbb{P}(\lambda S_{\text{diag}} > \lambda t/2) = \mathbb{P}(\exp(\lambda S_{\text{diag}}) > \exp(\lambda t/2)) \leq \frac{\mathbb{E}\exp(\lambda S_{\text{diag}})}{e^{\lambda t/2}} \leq \exp(-\lambda t/2 + N\lambda^2)$$

Optimizing over $\lambda$, for which the optimal choice of $\lambda$ is $\lambda = \frac{t}{4N}$. Thus, we have for $t > 0$,

$$\mathbb{P}(S_{\text{diag}} > t/2) \leq \exp \left( -\min \left( \frac{t^2}{16N}, \frac{t}{4 * 128 \|A_0\|_2} \right) \right) \leq \exp \left( -\frac{1}{16} \min \left( \frac{t^2}{N}, \frac{t}{32 \|A_0\|_2} \right) \right) =: q_d$$

Repeating the argument for $-A_\xi$ instead of $A_\xi$, we now consider

$$S'_{\text{diag}} := \sum_{k=1}^{m} (X^2_k(-A_{\xi,kk}) + \mathbb{E}(X^2_k)\mathbb{E}(A_{\xi,kk}) = -S_{\text{diag}}.$$ 

By Lemma 4.1, we have for all $|\lambda| \leq \frac{1}{128\|A_0\|_2}$

$$\mathbb{E}\exp(\lambda S'_{\text{diag}}) = \mathbb{E}\exp(-\lambda S_{\text{diag}}) \leq \exp(\lambda^2 N).$$

Thus, we have for $t > 0$ and $0 < \lambda \leq \frac{1}{128\|A_0\|_2}$

$$\mathbb{P}(S'_{\text{diag}} > t/2) \leq \frac{\mathbb{E}\exp(\lambda S'_{\text{diag}})}{e^{\lambda t/2}} \leq \exp(-\lambda t/2 + N\lambda^2) \leq q_d$$

The lemma is thus proved, given that for $t > 0$

$$\mathbb{P}(S_{\text{diag}} < -t/2) = \mathbb{P}(S'_{\text{diag}} > t/2) \leq q_d$$

$$\mathbb{P}(|S_{\text{diag}}| > t/2) = \mathbb{P}(S_{\text{diag}} > t/2) + \mathbb{P}(S_{\text{diag}} < -t/2) \leq 2q_d.$$

Proof of Lemma 4.6. Lemma 4.6 follows immediately from Lemma 4.4. We have for $0 < \lambda \leq \frac{1}{58\|A_0\|_2}$ and $S := S_{\text{offd}}$,

$$\mathbb{P}(S > t/2) = \mathbb{P}(\exp(\lambda S) > \exp(\lambda t/2)) \leq \frac{\mathbb{E}\exp(\lambda S)}{e^{\lambda t/2}} \leq \exp(-\lambda t/2 + M\lambda^2)$$

for which the optimal choice of $\lambda$ is $\lambda = \frac{t}{4M}$. Thus we have for $t > 0$,

$$\mathbb{P}(S > t/2) \leq \exp(-\lambda t/2 + M\lambda^2) \leq \exp \left( -\frac{1}{16} \min \left( \frac{t^2}{M}, \frac{t}{15C \|A_0\|_2} \right) \right) =: q_{\text{offd}}$$

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Similarly, we have for \( \lambda, t > 0 \),
\[
\mathbb{P}(S < -t/2) = \mathbb{P}(-S > t/2) = \mathbb{P}(\exp(\lambda(-S)) > \exp(\lambda t/2)) \\
\leq \frac{\mathbb{E}\exp(\lambda(-S))}{e^{\lambda t/2}} \leq \exp(-\lambda t/2 + M\lambda^2) \leq q_{\text{offd}}.
\]
The lemma is thus proved using the union bound. \( \square \)

The Theorem is thus proved. \( \square \)

The plan is to first bound the moment generating function for the \( S_0 \) in the diagonal sum in Section 4.1. We then bound the moment generating function for the off-diagonal sum as stated in Lemma 4.4 in Section 4.2.

### 4.1 Proof of Lemma 4.3

Recall \( A_\xi = D_0 D_\xi D_0 = (\tilde{a}_{ij}) = (d_i^T D_\xi d_j) \). Then for \( \tilde{a}_{kk} = d_k^T D_\xi d_k = \sum_{i=1}^m d_{ki}^2 \xi_i \)
\[
S_0 := \sum_{k=1}^m (X_k^2 - \mathbb{E}X_k^2) A_{\xi, kk} = \sum_{k=1}^m (X_k^2 - \mathbb{E}X_k^2) \tilde{a}_{kk}
\]
To estimate the moment generating function of \( S_0 \), we first consider \( \xi \) as being fixed and thus treat \( \tilde{a}_{ij} \) as fixed coefficients. The bound on the moment generating function of \( S_0 \) as in (12) will involve the following symmetric matrices \( A_1 \) and \( A_2 \) which we now define:
\[
A_1 := D_0 \circ D_0 = [d_1 \circ d_1, \ldots, d_m \circ d_m],
\]
\[
A_2 = (a_{ij}^{\prime\prime}) = A_1^2 = (d_1 \circ d_1, \ldots, d_m \circ d_m) (d_1 \circ d_1, \ldots, d_m \circ d_m)^T \quad (16)
\]
\[
= \sum_{k=1}^m (d_k d_k^T) \circ (d_k d_k^T) = \sum_{k=1}^m (d_k \circ d_k) (d_k \circ d_k)^T \geq 0.
\]
Thus we have both \( A_0, A_2 \) being positive semidefinite, while in general \( A_1 \) is not positive semidefinite unless \( D_0 \succeq 0 \) by the Schur Product Theorem. See Theorem 5.2.1 [8].

#### Lemma 4.7
Suppose all conditions in Lemma 4.3 hold. Let \( C_0 = 38.94 \). Then for \( |\lambda| \leq \frac{1}{64\|A_0\|_2} \leq \frac{1}{64a_{\infty}} \),
\[
\mathbb{E}\exp(\lambda S_0) \leq \mathbb{E}\exp(C_0\lambda^2 \xi^T A_2 \xi) \leq \mathbb{E}\exp(C_0\lambda^2 \|\text{diag}(A_\xi)\|_F^2).
\]

\( \text{Proof.} \) We first compute the moment generating function for \( S_0 \) when \( \xi \) is fixed. Conditioned on \( \xi, \tilde{a}_{kk}, \forall k \) are considered as fixed coefficients. Indeed, for \( |\lambda| \leq \frac{1}{64a_{\infty}} \), by independence of \( X_i \)
\[
\mathbb{E}(\exp(\lambda S_0)|\xi) = \mathbb{E}_X \exp \left( \lambda \sum_{k=1}^m \tilde{a}_{kk} (X_k^2 - \mathbb{E}X_k^2) \right) = \prod_{k=1}^m \mathbb{E}_X \exp(\lambda \tilde{a}_{kk} (X_k^2 - \mathbb{E}X_k^2))
\]
\[
\leq \prod_{k=1}^m \exp \left( 38.94\lambda^2 \tilde{a}_{kk}^2 \right) = \exp \left( C_0\lambda^2 \sum_{k=1}^m \tilde{a}_{kk}^2 \right)
\]
where the inequality follows from Lemma 1.5 with $\tau := \lambda \tilde{a}_{kk}$ in view of (18):

$$\forall k, \forall \xi, \ |\lambda \tilde{a}_{kk}| \leq \frac{1}{64} \leq \frac{1}{23.5\epsilon} \quad \text{where} \quad |\tilde{a}_{kk}| \leq \langle d_k, d_k \rangle = a_{kk} \leq a_\infty \quad \tag{18}$$

Now

$$\sum_{k=1}^{m} \tilde{a}_{kk}^2 = \sum_{k=1}^{m} (d_k^T D \xi d_k)^2 = \sum_{k=1}^{m} \text{tr}(d_k^T D \xi d_k d_k^T D \xi d_k)$$

$$= \sum_{k=1}^{m} \text{tr}(D \xi d_k d_k^T D \xi d_k d_k^T) = \sum_{k=1}^{m} \xi^T ((d_k d_k^T) \circ d_k d_k^T) \xi =: \xi^T A_2 \xi$$

where $A_2 = (a_{ij}''') = (D_0 \circ D_0)^2$ is as defined in (16). Thus

$$E_X \exp(\lambda S_0) \leq \exp(C_0 \lambda^2 \xi^T A_2 \xi) = \exp \left( C_0 \lambda^2 \| \text{diag}(A_\xi) \|^2_F \right) \quad \tag{19}$$

and (17) is thus proved by taking expectation on both sides of (19) with respect to random variables in vector $\xi$. \□

To prove (15) in the Lemma statement, notice that for all $\xi \in \{0, 1\}^m$,

$$\sum_{k=1}^{m} \tilde{a}_{kk}^2 = \| \text{diag}(A_\xi) \|^2_F \leq \| A_\xi \|^2_F .$$

Thus we have for $|\lambda| \leq \frac{1}{64\|A_0\|_2}$,

$$E \exp(\lambda S_0) \leq E \exp(C_0 \lambda^2 \| \text{diag}(A_\xi) \|^2_F) \leq E \exp(C_0 \lambda^2 \| A_\xi \|^2_F) = E \exp(C_0 \lambda^2 \xi^T (A_0 \circ A_0) \xi)$$

where $A_0 = (a_{ij})$. Finally, we invoke Corollary 4.8 to finish the proof of Lemma 4.3.

**Corollary 4.8.** Let $A_0, \xi$ be as defined in Theorem 1.2. Then for $|\lambda| \leq \frac{1}{64\|A_0\|_2}$ and $C_0 \leq 38.94$

$$E \exp \left( C_0 \lambda^2 \sum_{i,j} \tilde{a}_{ij}^2 \xi_i \xi_j \right) \leq \exp(\lambda^2 N)$$

where $N = \left( 40 \sum_{j=1}^{m} p_j a_{jj}^2 + 54 \sum_{i\neq j} p_i p_j a_{ij}^2 \right)$. The proof of Corollary 4.8 follows exactly that of Corollary 4.11 in view of Theorem 1.3 and is thus omitted. The Lemma is thus proved. \□

**Remark 4.9.** An alternative bound can be stated as follows: for $\lambda \leq \frac{1}{64a_\infty}$,

$$E \exp(\lambda S_0) \leq \exp \left( 41 \lambda^2 \sum_{j=1}^{m} \sigma_j^2 a_{jj}^2 + 52 \lambda^2 \| A_1 p \|^2_2 \right)$$

where $p = [p_1, \ldots, p_m]$ and $\sigma_j^2 = p_j (1 - p_j)$. The proof follows from a direct analysis based on the quadratic form $\xi^T A_2 \xi$ on the RHS of (17), which is omitted from the present paper. This bound may lead to a slight improvement upon the final bound in (15). We do not pursue this improvement here because the bound in (15) is sufficient for us to obtain the final large deviation bound as stated in Theorem 1.2.
4.2 Proof of Lemma 4.4

Let $E_X$ and $E_\xi$ denote the expectation with respect to random variables in vectors $X$ and $\xi$ respectively. Recall

$$S_{\text{offd}} = \sum_{i \neq j} X_i X_j (A_{\xi,ij}) =: \sum_{i \neq j} \tilde{a}_{ij} X_i X_j$$

where $\tilde{a}_{ij} = d_i^T D_\xi d_j = \sum_{k=1}^m d_{ik} \xi_k d_{jk}$

To estimate the moment generating function of $S_{\text{offd}}$, we first consider $\xi$ as being fixed and thus treat $\tilde{a}_{ij}$ as fixed coefficients. Lemma 4.10 reduces the original problem of estimating the moment generating function of $S_{\text{offd}}$ to the new problem of estimating the moment generating function of $S := \xi^T (A_0 \circ A_0) \xi$, which involves a new quadratic form with independent non-centered random variables $\xi_1, \ldots, \xi_m \in \{0, 1\}$ and the symmetric matrix $(A_0 \circ A_0)$ as shown in (20). Lemma 4.10 follows from the proof of Theorem 1 [12] directly. We omit the proof in this paper.

**Lemma 4.10.** Consider $\xi \in \{0, 1\}^m$ as being fixed and denote by $A_\xi = D_0 D_\xi D_0$ and $A_0 = D_0^2 = (a_{ij})$. Then, for some constant $C$ and $|\lambda| \leq \frac{1}{12C \|A_0\|_2}$ and $C_2 = 32C^2$,

$$E_X \exp(\lambda S_{\text{offd}}) \leq \exp \left( C_2 \lambda^2 \|A_\xi\|_F^2 \right) = \exp \left( C_2 \lambda^2 \xi^T (A_0 \circ A_0) \xi \right). \tag{20}$$

Note that $\|D_\xi D_0\|_2 \leq \|D_0\|_2$ and hence by symmetry

$$\|A_\xi\|_2^2 = \|D_0 D_\xi D_0\|_2 \leq \|D_\xi\|_2 \|D_0\|_2^2 \leq \|D_\xi\|_2 \|D_0\|_2 = \|A_0\|_2$$

$$\|A_\xi\|_F^2 = \text{tr}(A_0 D_\xi A_0 D_\xi) = \xi^T (A_0 \circ A_0) \xi.$$

In order to estimate the moment generating function $S_{\text{offd}}$, we now take expectation with respect to $\xi$ on both sides of (20). Thus we have for $|\lambda| \leq \frac{1}{12C \|A_0\|_2}$

$$E_\xi \sum E_X \exp(\lambda S_{\text{offd}}) \leq E \exp \left( C_2 \lambda^2 \|A_\xi\|_F^2 \right) = E \exp \left( C_2 \lambda^2 \xi^T (A_0 \circ A_0) \xi \right). \tag{21}$$

**Corollary 4.11.** Then for $|\lambda| \leq \frac{1}{58C \|A_0\|_2}$ and $t := C_2 \lambda^2$, where $C_2 = 32C^2$ and $C$ is a large enough absolute constant,

$$E \exp \left( t \sum_{i,j} a_{ij}^2 \xi_i \xi_j \right) \leq \exp \left( \lambda^2 M \right)$$

where $M := C^2 \left( 33 \sum_{i=1}^m a_{ii}^2 p_i + 44 \sum_{i \neq j} a_{ij}^2 p_i p_j \right)$.

Combining Lemma 4.10, (21) and Corollary 4.11, we have for $|\lambda| \leq \frac{1}{58C \|A_0\|_2}$

$$E(\lambda S_{\text{offd}}) \leq E \exp \left( C_2 \lambda^2 \|A_\xi\|_F^2 \right) = E \exp \left( t \sum_{i,j} a_{ij}^2 \xi_i \xi_j \right) \leq \exp(\lambda^2 M).$$

Lemma 4.4 thus holds. □
Corollary 4.11 follows from Theorem 1.3 immediately, which is derived in the current paper for estimating the moment generating function of $S' := \xi^T A \xi$ where $A$ is an arbitrary matrix and $\xi$ is a Bernoulli random vector with independent elements as defined in Theorem 1.2.

Proof of Corollary 4.11. Clearly for the choices of $t$ and $\lambda$,

$$t = C_2 \lambda^2 \leq \frac{32 C^2}{58^2 C^2 \|A_0\|^2_2} \leq \frac{1}{104 \|A_0\|^2_2} \leq \frac{1}{104 \|A_0 \circ A_0\|_1 \sqrt{\|A_0 \circ A_0\|_\infty}},$$

where we use the fact that for symmetric $A_0$,

$$\|A_0 \circ A_0\|_1 = \|A_0 \circ A_0\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{m} a_{ij}^2 = \max_{1 \leq i \leq m} \|A_0 e_i\|^2_2 \leq \|A_0\|^2_2.$$

Thus we can apply Theorem 1.3 with $B := (A_0 \circ A_0)$ to obtain for $0 < t \leq \frac{1}{104 \|A_1 \circ A_1\|_\infty}$,

$$\mathbb{E} \exp \left( t \sum_{i,j} a_{ij}^2 \xi_i \xi_j \right) \leq \exp \left( 1.02 t \sum_{j=1}^{m} p_j a_{jj}^2 \right) \exp \left( 1.373 t \sum_{i \neq j} a_{ij}^2 p_i p_j \right) \leq \exp \left( C^2 \lambda^2 \left( 33 \sum_{j=1}^{m} p_j a_{jj}^2 + 44 \sum_{i \neq j} p_i p_j a_{ij}^2 \right) \right).$$

\[ \square \]

5 Proof of Theorem 1.3

We first state the following Theorem 5.1 from a note by Vershynin [14]; we state its consequence as follows.

**Theorem 5.1.** Let $A$ be an $m \times m$ matrix. Let $X = (X_1, \ldots, X_m)$ be a random vector with independent mean zero coefficients. Then, for every convex function $F$,

$$\mathbb{E} F\left( \sum_{i \neq j} a_{ij} X_i X_j \right) \leq \mathbb{E} F\left( 4 \sum_{i \neq j} a_{ij} X_i X'_j \right), \quad (22)$$

where $X'$ is an independent copy of $X$.

Let $Z_i := \xi_i - p_i$. Denote by $\sigma_i^2 = p_i(1 - p_i)$. For all $Z_i$, we have $|Z_i| \leq 1$, $\mathbb{E} Z_i = 0$ and

$$\mathbb{E} Z_i^2 = (1 - p_i)^2 p_i + p_i^2 (1 - p_i) = p_i (1 - p_i) = \sigma_i^2, \quad (23)$$

$$\mathbb{E} |Z_i| = (1 - p_i) p_i + p_i (1 - p_i) = 2p_i (1 - p_i) = 2 \sigma_i^2. \quad (24)$$

Proof of Theorem 1.3. Let $Z_i = \xi_i - p_i$. Denote by $\bar{a}_i := \sum_{j \neq i} (a_{ij} + a_{ji}) p_j + a_{ii}$. We express the quadratic form as follows:

$$\sum_{i=1}^{m} a_{ii} (\xi_i - p_i) + \sum_{i \neq j} a_{ij} (\xi_i \xi_j - p_i p_j) = \sum_{i \neq j} a_{ij} Z_i Z_j + \sum_{j=1}^{m} Z_j \bar{a}_i =: S_1 + S_2.$$
We first state the following bounds on the moment generating functions of $S_1$ and $S_2$ in (27) and (28). The estimate on the moment generating function for $\sum_{i,j} a_{ij} \xi_i \xi_j$ then follows immediately from the Cauchy-Schwartz inequality in view of (27) and (28).

**Bounding the moment generating function for $S_1$.** In order to bound the moment generating function for $S_1$, we start by a decoupling step following Theorem 5.1. Let $Z'$ be an independent copy of $Z$.

**Decoupling.** Now consider random variable $S'_1 := \sum_{i \neq j} a_{ij} Z_i Z_j'$, we have $\mathbb{E} \exp (2 \lambda S_1) \leq \mathbb{E} \exp (8 \lambda S'_1) =: f$

by (22). Thus we have by independence of $Z_i$,

$$f := \mathbb{E}_{Z'} \mathbb{E}_Z \exp \left( 8 \lambda \sum_{i=1}^m Z_i \sum_{j \neq i} a_{ij} Z'_j \right) = \mathbb{E}_{Z'} \prod_{i=1}^m \mathbb{E} (\exp (8 \lambda Z_i a_i)) . \quad (25)$$

First consider $Z'$ being fixed. Let us define

$$t_i := 8 \lambda \bar{a}_i \quad \text{where} \quad \bar{a}_i := \sum_{j \neq i} a_{ij} Z'_j.$$

Hence for all $0 \leq \lambda \leq \frac{1}{104 \| A \|_\infty}$ and $C_4 := \frac{4}{13} e^{1/13}$, and any given fixed $Z'$ by (4)

$$\mathbb{E} \exp (8 \lambda \bar{a}_i Z_i) := \mathbb{E} \exp (t_i Z_i) \leq 1 + \frac{1}{2} t_i^2 \mathbb{E} Z_i^2 e^{t_i} \leq \exp \left( \frac{1}{2} t_i^2 \mathbb{E} Z_i^2 e^{t_i} \right) \leq \exp \left( \frac{4}{13} e^{1/13} \lambda | \bar{a}_i | \sigma_i^2 \right) =: \exp \left( C_4 \lambda | \bar{a}_i | \sigma_i^2 \right) \quad (26)$$

where $Z_i, \forall i$ satisfies: $|Z_i| \leq 1, \mathbb{E} Z_i = 0$ and $\mathbb{E} Z_i^2 = \sigma_i^2$,

$$|t_i| = |8 \lambda \bar{a}_i| \leq 8 \lambda \sum_{j \neq i} |a_{ij}| |Z'_j| \leq 8 \lambda \| A \|_\infty \leq \frac{1}{13} \quad \text{and} \quad \frac{1}{2} t_i^2 \leq \frac{4}{13} \lambda | \bar{a}_i | .$$

Denote by $| \bar{a}_j | := \sum_{i \neq j} |a_{ij}| | \sigma_i^2 |$. Thus by (25) and (26)

$$f \leq \mathbb{E}_{Z'} \prod_{i=1}^m \exp \left( C_4 \lambda | \bar{a}_i | \sigma_i^2 \right) \leq \mathbb{E}_{Z'} \exp \left( C_4 \lambda \sum_{i=1}^m | \bar{a}_i | \sum_{j \neq i} |a_{ij}| |Z'_j| \right)$$

$$= \prod_{j=1}^m \mathbb{E} \exp \left( C_4 \lambda |Z'_j| \sum_{i \neq j} |a_{ij}| | \sigma_i^2 | \right) =: \prod_{j=1}^m \mathbb{E} \exp \left( C_4 \lambda | \bar{a}_j | |Z'_j| \right) .$$

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where we have by the elementary approximation (4) and $\tilde{t}_j := C_4 \lambda |\tilde{a}_j|$

$$\mathbb{E} \exp \left( C_4 \lambda |\tilde{a}_j| \right) = \mathbb{E} \exp \left( \tilde{t}_j \right) \leq 1 + \mathbb{E} \left( \tilde{t}_j \right) + \frac{1}{2} (\tilde{t}_j)^2 \mathbb{E} (Z_j)^2 e^{\tilde{t}_j}$$

$$\leq \exp \left( 2 \tilde{t}_j \sigma_j^2 + \frac{1}{2} (\tilde{t}_j)^2 \sigma_j^2 e^{0.0008} \right) \leq \exp \left( 2.0005 \tilde{t}_j \sigma_j^2 \right)$$

$$\leq \exp \left( 2.0005 C_4 \lambda |\tilde{a}_j| \sigma_j^2 \right) \leq \exp \left( \frac{2}{3} \lambda \sigma_j^2 \sum_{i \neq j} |a_{ij}| \sigma_i^2 \right)$$

where $\mathbb{E} (Z_j)^2 = \sigma_i^2$ and $\mathbb{E} |Z_j| = 2 \sigma_i^2$ following (23) and (24), and for $0 < \lambda \leq \frac{1}{104 \max(\|A\|_1,\|A\|_\infty)}$,

$$\tilde{t}_j := C_4 \lambda |\tilde{a}_j| = \frac{4}{13} e^{1/13} \lambda \sum_{i \neq j} |a_{ij}| \sigma_i^2 \leq \frac{4}{13} e^{1/13} \frac{1}{4} \sum_{i \neq j} |a_{ij}| \leq \frac{1}{13} e^{1/13} \frac{1}{104} < 0.0008.$$

Thus for every $0 \leq \lambda \leq \frac{1}{104 \max(\|A\|_1,\|A\|_\infty)}$,

$$\mathbb{E} \exp (\lambda 2 S_1) \leq \exp \left( \frac{2}{3} \lambda \sum_{i \neq j} |a_{ij}| \sigma_i^2 \sigma_j^2 \right). \quad (27)$$

**Bounding the moment generating function for $S_2$.** Recall

$$S_2 := \sum_{i=1}^m Z_i \left( \sum_{j \neq i} (a_{ij} + a_{ji}) p_j + a_{ii} \right) =: \sum_{i=1}^m Z_i \tilde{a}_i.$$ 

Let $a_\infty := \max_i |\tilde{a}_i| \leq \|A\|_\infty + \|A\|_1$. Thus we have by Lemma 1.4

$$g := \mathbb{E} \exp (2 \lambda S_2) = \mathbb{E} \exp \left( 2 \lambda \sum_{i=1}^m Z_i \tilde{a}_i \right)$$

$$\leq \exp \left( 2 \lambda^2 e^{2 \lambda |a_\infty|} \sum_{i=1}^m \tilde{a}_i^2 \sigma_i^2 \right) \leq \exp \left( C_5 |\lambda| \sum_{i=1}^m |\tilde{a}_i| p_i \right)$$

where $e^{2 \lambda |a_\infty|} \lambda |\tilde{a}_i| \leq \frac{1}{26} e^{1/26} =: C_5 \leq 0.04$ given that for all $|\lambda| \leq \frac{1}{52 (\|A\|_\infty + \|A\|_1)}$

$$2 \lambda |\tilde{a}_i| \leq \frac{2 (\|A\|_\infty + \|A\|_1)}{52 (\|A\|_\infty + \|A\|_1)} \leq \frac{1}{26} \text{ for all } i.$$ 

Thus we have for $0 < \lambda \leq \frac{1}{52 (\|A\|_\infty + \|A\|_1)},$

$$\mathbb{E} \exp (\lambda 2 S_2) \leq \exp \left( 0.02 \lambda \sum_{i=1}^m p_i |\tilde{a}_i| \right). \quad (28)$$
Hence by the Cauchy-Schwartz inequality, for all $0 < \lambda \leq \frac{1}{104(\|A\|_{\infty} \vee \|A\|_1)}$,

$$
\mathbb{E} \exp \left( \lambda \left( \sum_{i=1}^{m} a_{ii} (\xi_i - p_i) + \sum_{i \neq j} a_{ij} (\xi_i \xi_j - p_i p_j) \right) \right)
\leq \mathbb{E} \exp (\lambda (S_1 + S_2)) \leq \mathbb{E}^{1/2} \exp(2\lambda S_1) \mathbb{E}^{1/2} \exp(2\lambda S_2)
$$

The theorem is thus proved by multiplying $\exp \left( \lambda \left( \sum_{i=1}^{m} a_{ii} p_i + \sum_{i \neq j} a_{ij} p_i p_j \right) \right)$ on both sides of the above inequality. \(\square\)

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## A Proof of Lemma 1.5

Let $Z := X^2 - \mathbb{E}X^2$ and $Y := Z/\|Z\|_{\psi_1}$. Then $Y$ and $Z$ are both centered sub-exponential random variables with $\|Y\|_{\psi_1} = 1$ and

$$
\|Z\|_{\psi_1} = \|X^2 - \mathbb{E}X^2\|_{\psi_1} \leq 2 \|X^2\|_{\psi_1} \leq 4 \|X\|_{\psi_2}^2 \leq 4K^2
$$

which follows from the triangle inequality and Lemma 5.14 of [15].

Now set $t := \tau \|X^2 - \mathbb{E}X^2\|_{\psi_1}$, where for $|\tau| \leq \frac{1}{23.5eK^2}$,

$$
e |t| = e |\tau| \|X^2 - \mathbb{E}X^2\|_{\psi_1} \leq \frac{4K^2}{23.5K^2} < \frac{8}{47}
$$

and $2(e |t|)^3 \leq (e |t|)^2$. By Lemma 5.15 of [15], we have for all $k$,

$$
\mathbb{E} \exp(tY) = 1 + t\mathbb{E}Y + \sum_{p=2}^{\infty} \frac{t^p \mathbb{E}Y^p}{p!} \leq 1 + \sum_{p=2}^{\infty} \frac{|t|^p \mathbb{E}|Y|^p}{p!}
\leq 1 + \sum_{p=2}^{\infty} \frac{|t|^p e^p}{p!} \sqrt{2\pi p}
$$
Thus
\[
\mathbb{E} \exp(tY) \leq 1 + \frac{(te)^2}{2\sqrt{\pi}} + \frac{1}{\sqrt{6\pi}} \sum_{p=3}^{\infty} (e|t|)^p
\]
\[
\leq 1 + \frac{|t|^2 e^2}{2\sqrt{\pi}} + \frac{1}{\sqrt{6\pi}} \left(1 - e|t|\right) \leq 1 + \frac{|t|^2 e^2}{2\sqrt{\pi}} + \frac{8(e|t|)^2}{39\sqrt{6\pi}}
\]
\[
< 1 + e^2 \tau^2 \|X^2 - \mathbb{E}X^2\|_{\psi_1}^2 \left(\frac{1}{2\sqrt{\pi}} + \frac{8}{39\sqrt{6\pi}}\right)
\]
\[
\leq 1 + 38.94 |\tau|^2 K^4
\]
where we used the following form of Stirling’s approximation for all \(p \geq 2\),
\[
\frac{1}{p!} \leq \frac{e^p}{p^p \sqrt{2\pi p}} \leq \frac{e^p}{p^p 2\sqrt{\pi}}.
\]
The lemma is thus proved given that
\[
\mathbb{E} \exp \tau(X^2 - \mathbb{E}X^2) = \mathbb{E} \exp \left(\tau \|X^2 - \mathbb{E}X^2\|_{\psi_1} Y\right) = \mathbb{E} \exp(tY).
\]
\[
\square
\]

B  Proof of Lemma 3.2

Note that the following holds by Lemma 5.14 [15],
\[
\|X_i\|_{\psi_2}^2 \leq \|X_i^2\|_{\psi_1} \leq 2 \|X_i\|_{\psi_2}^2 = 2K^2.
\]
For all \(k\), let \(Y_k := \frac{X_k^2}{\|X_k^2\|_{\psi_1}}\). By definition, \(Y_k\) is a sub-exponential random variable with \(\|Y_k\|_{\psi_1} = 1\). We now set \(\tau_k := \lambda a_{kk} \|X_k^2\|_{\psi_1}\). Following the proof of Lemma 1.5, we first use the Taylor expansions to obtain for all \(k\),
\[
\mathbb{E} \exp(t_k Y_k) := \mathbb{E} \exp \left(\lambda a_{kk} X_k^2\right) = 1 + t_k \mathbb{E} Y_k + \sum_{p=2}^{\infty} \frac{t_k^p \mathbb{E} Y_k^p}{p!}
\]
\[
\leq 1 + t_k \mathbb{E} Y_k + \sum_{p=2}^{\infty} \frac{|t_k|^p \mathbb{E} |Y_k|^p}{p!}
\]
\[
\leq 1 + \lambda a_{kk} \mathbb{E} X_k^2 + \frac{|t_k|^2 e^2}{2\sqrt{\pi}} + \frac{1}{\sqrt{6\pi}} \sum_{p=3}^{\infty} (e |t_k|)^p
\]
\[
\leq 1 + \lambda a_{kk} \mathbb{E} X_k^2 + \frac{|t_k|^2 e^2}{2\sqrt{\pi}} + \frac{1}{\sqrt{6\pi}} 2(e |t_k|)^3
\]
\[
< 1 + \lambda a_{kk} \mathbb{E} X_k^2 + e^2 (\lambda a_{kk} \|X_k^2\|_{\psi_1})^2 \left(\frac{1}{2\sqrt{\pi}} + \frac{1}{\sqrt{6\pi}}\right) \leq 1 + \lambda a_{kk} \mathbb{E} X_k^2 + 16 |\lambda a_{kk}|^2 K^4.
\]
where
\[ e |t_k| \leq \frac{|a_{kk}| \|X_k^2\|_{\psi_1}}{4K^2 \max_k |a_{kk}|} \leq \frac{1}{2} \]
and \(2(e |t_k|)^3 \leq (e |t_k|)^2\). The lemma is thus proved. □

C Proof of (9)

The proof structure follows from the proof of Theorem 2.1 [12]. Recall \(S := \sum_{i \neq j} a_{ij} X_i X_j \xi_i \xi_j\).

We start with a decoupling step.

**Step 1. Decoupling.** Let \(\delta = (\delta_1, \ldots, \delta_m) \in \{0, 1\}^m\) be a random vector with independent Bernoulli random variables with \(E \delta_i = 1/2\), which is independent of \(X\) and \(\xi\). Let \(X_{\Lambda_\delta}\) denote \((X_i)_{i \in \Lambda_\delta}\) for a set \(\Lambda_\delta := \{i \in [m] : \delta_i = 1\}\). Let \(E_X, E_\xi\) and \(E_\delta\) denote the expectation with respect to random variables in \(X\), \(\xi\) and \(\delta\) respectively. Now consider random variable

\[ S_\delta := \sum_{i,j} \delta_i (1 - \delta_j) a_{ij} X_i X_j \xi_i \xi_j \quad \text{and hence} \quad S = 4E_\delta S_\delta. \]

By Jensen’s inequality, for all \(\lambda \in \mathbb{R}\),

\[ E \exp(\lambda S) = E_\xi E_X \exp(E_\delta 4\lambda S_\delta) \leq E_\xi E_X E_\delta \exp(4\lambda S_\delta). \tag{29} \]

where the last step holds because \(e^{ax}\) is convex on \(\mathbb{R}\) for any \(a \in \mathbb{R}\).

Consider \(\Lambda_\delta := \{i \in [m] : \delta_i = 1\}\). Denote by \(f(\xi, \delta, X_{\Lambda_\delta})\) the conditional moment generating function of random variable \(4S_\delta\):

\[ f(\xi, \delta, X_{\Lambda_\delta}) := E (\exp(4\lambda S_\delta)|\xi, \delta, X_{\Lambda_\delta}). \]

Conditioned upon \(X_{\Lambda_\delta}\) for a fixed realization of \(\xi\) and \(\delta\), we rewrite \(S_\delta\)

\[ S_\delta := \sum_{i \in \Lambda_\delta, j \in \Lambda_\delta^c} a_{ij} X_i X_j \xi_i \xi_j = \sum_{j \in \Lambda_\delta^c} X_j \left( \sum_{i \in \Lambda_\delta} \xi_j \sum_{i \in \Lambda_\delta} a_{ij} X_i \xi_i \right) \]

as a linear combination of mean-zero subgaussian random variables \(X_j, j \in \Lambda_\delta^c\), with fixed coefficients. Thus the conditional distribution of \(S_\delta\) is sub-gaussian with \(\psi_2\) norm being upper bounded by the \(\ell_2\) norm of the coefficient vector \((\xi_j \sum_{i \in \Lambda_\delta} a_{ij} X_i \xi_i)_{j \in \Lambda_\delta^c}\) [15](cf. Lemma 5.9).

Thus, conditioned upon \(\xi, \delta\) and \(X_{\Lambda_\delta}\),

\[ ||S_\delta||_{\psi_2} \leq C_0 \sigma_{\delta, \xi} \quad \text{where} \quad \sigma_{\delta, \xi}^2 = \sum_{j \in \Lambda_\delta^c} \xi_j \left( \sum_{i \in \Lambda_\delta} a_{ij} X_i \xi_i \right)^2. \tag{30} \]
Thus we have for some large absolute $C > 0$

$$f(\xi, \delta, X_\Lambda) = \mathbb{E}(\exp(4\lambda S_\delta)||\xi, \delta, X_\Lambda) \leq \exp(C\lambda^2 \|S_\delta\|_2^2) \leq \exp(C'\lambda^2 \sigma_{\delta,\xi}^2).$$

(31)

Taking the expectations of both sides with respect to $X_\Lambda$ and $\xi$, we obtain

$$\mathbb{E}_\xi \mathbb{E}_{X_\Lambda} f(\xi, \delta, X_\Lambda) = \mathbb{E}_\xi \mathbb{E}_{X_\Lambda} \mathbb{E}_\xi(\exp(4\lambda S_\delta)||\xi, \delta, X_\Lambda) \leq \mathbb{E}_\xi \mathbb{E}_{X_\Lambda} \exp(C'\lambda^2 \sigma_{\delta,\xi}^2) =: \tilde{f}_{\delta}$$

(32)

**Step 2. Reduction to normal random variables.** Let $\delta, \xi$ and $X_\Lambda$ be a fixed realization of the random vectors defined as above. Let $g = (g_1, \ldots, g_n)$, where $g_i$ i.i.d. $\sim N(0, 1)$. Let $\mathbb{E}_g$ denote the expectation with respect to random variables in $g$. Consider random variable

$$Z := \sum_{j \in \Lambda_\delta} g_j \left( \xi_j \sum_{i \in \Lambda_\delta} a_{ij} X_i \xi_i \right)$$

By the rotation invariance of normal distribution, for a fixed realization of random vectors $\xi, \delta, X$, the conditional distribution of $Z$ follows $N(0, \sigma_{\delta,\xi}^2)$ for $\sigma_{\delta,\xi}^2$ as defined in (30). Thus we obtain the conditional moment generating function for $Z$ denoted by

$$\mathbb{E}_g(\exp(tZ)) := \mathbb{E}(\exp(tZ)||\xi, \delta, X_\Lambda) = \exp(t^2 \sigma_{\delta,\xi}^2/2).$$

Choose $t = C_1 \lambda$ where $C_1 = \sqrt{2C'}$, we have

$$\mathbb{E}_g(\exp(C_1 \lambda Z)) = \exp(C'\lambda^2 \sigma_{\delta,\xi}^2)$$

which matches the RHS of (31).

Hence for a fixed realization of $\delta$, we can calculate $\tilde{f}_{\delta}$ using $Z$ as follows:

$$\tilde{f}_{\delta} := \mathbb{E}_\xi \mathbb{E}_X \exp(C'\lambda^2 \sigma_{\delta,\xi}^2) = \mathbb{E}_\xi \mathbb{E}_X \mathbb{E}_g(\exp(C_1 \lambda Z)) = \mathbb{E}(\exp(C_1 \lambda Z)||\delta).$$

(33)

Conditioned on $\delta$, $\xi$ and $g$, we can re-express $Z$:

$$Z = \sum_{i \in \Lambda_\delta} X_i \left( \xi_i \sum_{j \in \Lambda_\delta} a_{ij} g_j \xi_j \right)$$

as a linear combination of subgaussian random variables $X_i, i \in \Lambda_\delta$ with fixed coefficients, which immediately imply that

$$\mathbb{E}(\exp(C_1 \lambda Z)||\delta, \xi, g) \leq \exp \left( C_3 \lambda^2 \sum_{i \in \Lambda_\delta} \xi_i \left( \sum_{j \in \Lambda_\delta} a_{ij} g_j \xi_j \right)^2 \right).$$

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Let $P_\delta$ denote the coordinate projection of $\mathbb{R}^m$ onto $\mathbb{R}^{\Lambda_\delta}$. Then conditioned on $\delta$, we have by definition of $\tilde{f}_\delta$ as in (33) and the bounds on the conditional moment generating function of $Z$ immediately above,

$$
\tilde{f}_\delta = \mathbb{E}(\exp(C_1 \lambda Z) | \delta) = \mathbb{E}_{\xi, \eta} \mathbb{E}(\exp(C_1 \lambda Z) | \delta, \xi, \eta)
$$

$$
\leq \mathbb{E} \left[ \exp \left( C_3 \lambda^2 \sum_{i \in \Lambda_\delta} \xi_i \left( \sum_{j \in \Lambda_\delta} a_{ij} \eta_j \xi_j \right) \right) | \delta \right]
$$

$$
= \mathbb{E} \left[ \exp \left( C_3 \lambda^2 \| D_\xi P_\delta A(I - P_\delta) D_\xi g \|_2^2 \right) | \delta \right]
$$

$$
= \mathbb{E} \left[ \exp \left( C_3 \lambda^2 \| A_{\delta, \xi} g \|_2^2 \right) | \delta \right]
$$

(34)

where we denote by $A_{\delta, \xi} := D_\xi P_\delta A(I - P_\delta) D_\xi$. We will integrate $g$ out followed by $\xi$ in the next two steps.

**Step 3. Integrating out the normal random variables.** Conditioned upon $\delta$ and $\xi$, and by the rotation invariance of $g$, the random variables $\| A_{\delta, \xi} g \|_2^2$ follows the same distribution as $\sum_i s_i^2 g_i^2$ where $s_i$ denote the singular values of $A_{\delta, \xi}$, with

$$
\max_i s_i = \sqrt{\lambda_{\text{max}}(A_{\delta, \xi}^T A_{\delta, \xi})} =: \| A_{\delta, \xi} \|_2 \leq \| A \|_2, \quad \text{and}
$$

$$
\sum_i s_i^2 = \| A_{\delta, \xi} \|_F^2 = \text{tr}(A_{\delta, \xi}^T A_{\delta, \xi}) = \text{tr}(D_\xi P_\delta A(I - P_\delta) D_\xi A^T P_\delta D_\xi)
$$

$$
= \text{tr}(D_\xi P_\delta A(I - P_\delta) D_\xi A^T) = \sum_{i \in \Lambda_\delta} \xi_i \sum_{j \in \Lambda_\delta} a_{ij}^2 \xi_j
$$

(35)

First we note that $g_i^2, \forall i$ follow the $\chi^2$ distribution with one degree of freedom, and $\mathbb{E} \exp(t g^2) = \frac{1}{\sqrt{1 - 2t}} \leq e^{2t}$ for $t < 1/4$. Thus we have for a fixed realization of $\delta, \xi$, and for all $|\lambda| \leq \frac{1}{2 \sqrt{C_3 \| A \|_2}}$,

$$
\mathbb{E} \left[ \exp(C_3 \lambda^2 s_i^2 g_i^2) | \delta, \xi \right] = \frac{1}{\sqrt{1 - 2C_3 \lambda^2 s_i^2}} \leq \exp(2C_3 \lambda^2 s_i^2).
$$

Hence for any fixed $\delta$ and $\xi$, and for $C_4 = 2C_3$ and $|\lambda| \leq \frac{1}{2 \sqrt{C_4 \| A \|_2}}$, we have by independence of $g_1, g_2, \ldots$,

$$
\mathbb{E} \left[ \exp \left( C_3 \lambda^2 \| A_{\delta, \xi} g \|_2^2 \right) | \delta, \xi \right] = \mathbb{E} \left[ \exp \left( C_3 \lambda^2 \sum_i s_i^2 g_i^2 \right) | \delta, \xi \right]
$$

$$
= \prod_i \mathbb{E} \left[ \exp(C_3 \lambda^2 s_i^2 g_i^2) | \delta, \xi \right]
$$

$$
\leq \prod_i \exp(2C_3 \lambda^2 s_i^2).
$$

(36)
Thus we have by (34), (35) and (36)

\[
\tilde{f}_\delta \leq \mathbb{E}_\xi \mathbb{E} \left[ \exp \left( C_3 \lambda^2 \| A_\delta, \xi g \|_2^2 \right) | \delta, \xi \right] \leq \mathbb{E} \left[ \exp(2C_3\lambda^2 \sum_i s_i^2) | \delta \right]
\]

\[
= \mathbb{E} \left[ \exp \left( C_4 \lambda^2 \sum_{i \in \Lambda}_\delta \xi_i \sum_{j \in \Lambda_{\delta}^c} a_{ij}^2 \xi_j \right) | \delta \right].
\]

(37)

The key observation here is we are dealing with a quadratic form on the RHS of (37) which is already decoupled thanks to the decoupling Step 1.

**Step 4. Integrating out the Bernoulli random variables.** For any given realization of \( \delta \), we now need to bound the moment generating function for the decoupled quadratic form on the RHS of (37), which is the content of Lemma C.1 where we take \( t = C_4 \lambda^2 \) and conclude that for all \( \delta \) and for all \( |\lambda| \leq \frac{1}{2\sqrt{C_4 \| A \|_2^2}} \),

\[
\tilde{f}_\delta \leq \exp \left( 1.44 C_4 \lambda^2 \sum_{i \neq j} a_{ij}^2 p_i p_j \right).
\]

**Lemma C.1.** Let \( 0 < \tau \leq \frac{1}{4\| A \|_2^2} \). For any fixed realization of \( \delta \), we have

\[
\mathbb{E} \left[ \exp \left( \tau \sum_{i \in \Lambda}_\delta \xi_i \sum_{j \in \Lambda_{\delta}^c} a_{ij}^2 \xi_j \right) | \delta \right] \leq \exp \left( 1.44 \tau \sum_{i \neq j} a_{ij}^2 p_i p_j \right).
\]

**Proof.** As mentioned, we are dealing with a quadratic form which is already decoupled. Thus we integrate out \( \xi_i \) for all \( i \in \Lambda_\delta \) followed by those in \( \Lambda_{\delta}^c \). Recall for any realization of \( \delta \) and \( 0 < \tau \leq \frac{1}{4\| A \|_2^2} \) we have by independence of \( \xi_1, \xi_2, \ldots \),

\[
f_\delta := \mathbb{E} \left[ \exp \left( \tau \sum_{i \in \Lambda}_\delta \xi_i \sum_{j \in \Lambda_{\delta}^c} a_{ij}^2 \xi_j \right) | \delta \right]
\]

\[
= \mathbb{E}_{\xi_{\Lambda_\delta}} \mathbb{E} \left[ \exp \left( \tau \sum_{i \in \Lambda}_\delta \xi_i \sum_{j \in \Lambda_{\delta}^c} a_{ij}^2 \xi_j \right) | \xi_{\Lambda_\delta} \delta \right]
\]

\[
= \mathbb{E}_{\xi_{\Lambda_\delta}} \prod_{i \in \Lambda_\delta} \mathbb{E} \left[ \exp \left( \tau \xi_i \sum_{j \in \Lambda_{\delta}^c} a_{ij}^2 \xi_j \right) | \xi_{\Lambda_{\delta}^c} \delta \right].
\]

(38)

We will use the following approximation twice in our proof:

\[
e^x - 1 \leq 1.2x \quad \text{which holds for} \quad 0 \leq x \leq 0.35.
\]

(39)
First notice that for all realizations of $\delta$ and $\xi$, we have for $0 < \tau \leq \frac{1}{4\|A\|_2^2}$

$$0 \leq \tau \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \leq \tau \sum_{j} a^2_{ij} \leq \tau \|A\|_2^2 \leq 1/4$$

given that the maximum row $\ell_2$ norm of $A$ is bounded by the operator norm of matrix $A^T$: $\|A^T\|_2 = \|A\|_2 = \sqrt{\lambda_{\text{max}}(A^T A)}$. Hence, we have for $|\lambda| \leq \frac{1}{2\sqrt{4\|A\|_2}}$, (39) and the fact that $1 + x \leq e^x$,

$$\mathbb{E} \left[ \exp \left( \tau \xi_i \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \right) | \xi_{\Lambda^c_\delta}, \delta \right] = p_i \exp \left( \tau \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \right) + (1 - p_i)$$

$$\leq p_i \left( 1.2\tau \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \right) + 1 \leq \exp \left( 1.2\tau p_i \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \right). \quad (40)$$

Thus we have by independence of $\xi_1, \xi_2, \ldots$, (38) and (40)

$$f_\delta \leq \mathbb{E}_{\xi_{\Lambda^c_\delta}} \prod_{i \in \Lambda_\delta} \exp \left( 1.2\tau p_i \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \right) = \mathbb{E}_{\xi_{\Lambda^c_\delta}} \exp \left( \sum_{i \in \Lambda_\delta} 1.2\tau p_i \sum_{j \in \Lambda^c_\delta} a^2_{ij} \xi_j \right)$$

$$= \mathbb{E}_{\xi_{\Lambda^c_\delta}} \exp \left( 1.2\tau \sum_{j \in \Lambda^c_\delta} \xi_j \sum_{i \in \Lambda_\delta} a^2_{ij} p_i \right) = \prod_{j \in \Lambda^c_\delta} \mathbb{E}_{\xi_j} \exp \left( 1.2\tau \xi_j \sum_{i \in \Lambda_\delta} a^2_{ij} p_i \right)$$

$$= \prod_{j \in \Lambda^c_\delta} p_j \exp \left( 1.2\tau \sum_{i \in \Lambda_\delta} a^2_{ij} p_i \right) + (1 - p_j) \quad (41)$$

where for all $\delta$ and $0 < \tau \leq \frac{1}{4\|A\|_2^2}$, we have by the approximation in (39)

$$\exp \left( 1.2\tau \sum_{i \in \Lambda_\delta} a^2_{ij} p_i \right) - 1 \leq 1.44\tau \sum_{i \in \Lambda_\delta} a^2_{ij} p_i \quad (42)$$

given that the column $\ell_2$ norm of $A$ is bounded by the operator norm of $A$, and thus

$$1.2\tau \sum_{i \in \Lambda_\delta} a^2_{ij} p_i \leq 1.2\tau \sum_{i=1}^m a^2_{ij} p_i \leq 1.2 \sum_{i=1}^m a^2_{ij} / (4 \|A\|_2^2) \leq 0.3.$$
Now by (41), (42) and the fact that $x + 1 \leq e^x$

$$f_\delta \leq \prod_{j \in \Lambda_\delta^c} p_j \left[ \exp \left( 1.2\tau \sum_{i \in \Lambda_\delta} a_{ij}^2 p_i \right) - 1 \right] + 1$$

$$\leq \prod_{j \in \Lambda_\delta^c} p_j \left( 1.44\tau \sum_{i \in \Lambda_\delta} a_{ij}^2 p_i \right) + 1 \leq \prod_{j \in \Lambda_\delta^c} \exp \left( 1.44\tau p_j \sum_{i \in \Lambda_\delta} a_{ij}^2 p_i \right)$$

$$= \exp \left( \sum_{j \in \Lambda_\delta^c} 1.44\tau p_j \sum_{i \in \Lambda_\delta} a_{ij}^2 p_i \right) \leq \exp \left( 1.44\tau \sum_{i \neq j} a_{ij}^2 p_i p_j \right)$$

The lemma thus holds. □

Step 5. Putting things together.

By Jensen’s inequality (29), definition of $f(\xi, \delta, X_{\Lambda_\delta})$ in (31) and (32), we have for all $|\lambda| \leq \frac{1}{2\sqrt{C_4\|A\|_2}}$

$$\mathbb{E} \exp(\lambda S) \leq \mathbb{E}_\delta \mathbb{E}_\xi \mathbb{E}_{X_{\Lambda_\delta}} \exp(4\lambda S_\delta)$$

$$= \mathbb{E}_\delta \mathbb{E}_\xi \mathbb{E}_{X_{\Lambda_\delta}} \mathbb{E}(\exp(4\lambda S_\delta)|\xi, \delta, X_{\Lambda_\delta})$$

$$= \mathbb{E}_\delta \mathbb{E}_\xi \mathbb{E}_{X_{\Lambda_\delta}} f(\xi, \delta, X_{\Lambda_\delta})$$

$$\leq \mathbb{E}_\delta \tilde{f}_\delta \leq \exp \left( 1.44C_4\lambda^2 \sum_{i \neq j} a_{ij}^2 p_i p_j \right)$$

Thus (9) holds. □

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