On the K-theory of $\mathbb{Z}/p^n$ – announcement

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April 7, 2022

Quillen introduced higher algebraic K-theory in [27] and computed the K-groups $K_*(\mathbb{F}_q)$ in [26]. Except in low degrees, the computation of the K-groups of closely related rings, for example $\mathbb{Z}/4$, has remained out of reach. In this paper, we announce new methods for computations of K-groups of such rings and outline new results. A full account will be given in [3].

We are interested in rings of the form $\mathbb{O}_K/\varpi^n$ where $K$ is a finite extension of $\mathbb{Q}_p$ of degree $d$, $\mathbb{O}_K$ is its ring of integers, and $\varpi^n$ is the $n$th power of a uniformizer $\varpi$. In particular, $p \in (\varpi^n)$ where $e$ is the degree of ramification of $K$ over $\mathbb{Q}_p$. When $n = 1$, $\mathbb{O}_K/\varpi^n$ is the residue field $k = \mathbb{F}_q$ of $\mathbb{O}_K$, where $q = p^f$ for some $f$, called the residual degree of the extension.

The problem of computing the K-groups of such rings, and of finite rings in general, was raised by Swan in the Battelle proceedings [13, Prob. 20].

1 History

For any field $k$, $K_0(k) \cong \mathbb{Z}$ and $K_1(k) \cong k^\times$. Quillen showed in [26] that if $\mathbb{F}_q$ is the finite field with $q = p^f$ elements, then for $r \geq 1$,

$$K_r(\mathbb{F}_q) \cong \begin{cases} 0 & \text{if } r \text{ is even} \\ \mathbb{Z}/(q^i - 1) & \text{if } r = 2i - 1. \end{cases}$$

Note in particular that there is no $p$-torsion in the K-groups of $\mathbb{F}_q$.

For each prime $\ell$ and ring $R$, $K(R, \mathbb{Z}_\ell)$ denotes the $\ell$-completion of the K-theory spectrum of $R$. In the main case of interest to us, namely when $R = \mathbb{O}_K/\varpi^n$, $K_r(R)$ is finitely generated torsion for $r > 0$ and $K_r(R, \mathbb{Z}_\ell)$ is the subgroup of $\ell$-primary torsion in $K_r(R)$.

Gabber’s rigidity theorem [12] implies that if $R$ is a commutative ring which is henselian with respect to an ideal $I$ and if $\ell$ is invertible in $R$, then

$$K(R, \mathbb{Z}_\ell) \cong K(R/I, \mathbb{Z}_\ell).$$

Examples of such henselian pairs are the rings of integers $\mathbb{O}_K$ as above with the ideal $(\varpi)$ or the quotients $\mathbb{O}/\varpi^n$, again with the ideal $(\varpi)$. It follows that for $\ell \neq p$ we have

$$K_*(\mathbb{O}_K; \mathbb{Z}_\ell) \cong K_*(\mathbb{O}/\varpi^n; \mathbb{Z}_\ell) \cong K_*(\mathbb{F}_q; \mathbb{Z}_\ell)$$

so that these $\ell$-adic K-groups are all determined by Quillen’s computation.

The situation of the $p$-adic K-theory of $\mathbb{O}_K$ or $\mathbb{O}_K/\varpi^n$ is very different. A result of Dundas–Goodwillie–McCarthy [11] implies that $K(\mathbb{O}/\varpi^n; \mathbb{Z}_p) \cong \tau_{2p} TC(\mathbb{O}/\varpi^n; \mathbb{Z}_p)$, while work of Hesselholt–Madsen [17] and of Panin [25] implies that $K(\mathbb{O}_K; \mathbb{Z}_p) \cong \tau_{p} TC(\mathbb{O}_K; \mathbb{Z}_p)$. Here, $TC(\mathbb{O}_K; \mathbb{Z}_p)$ and $TC(\mathbb{O}_K/\varpi^n; \mathbb{Z}_p)$ denote the $p$-adic topological cyclic homology spectra of $\mathbb{O}_K$ and $\mathbb{O}_K/\varpi^n$, respectively. This theory is built from topological Hochschild homology and is closely connected to $p$-adic cohomology theories thanks to the work of [6]. These results make the $p$-adic K-groups amenable to calculation using so-called trace methods.

Hesselholt and Madsen determine the structure of $TC(\mathbb{O}_K; \mathbb{Z}_p) \cong K_*(\mathbb{O}_K; \mathbb{Z}_p)$ in [18] and thereby verify the Quillen–Lichtenbaum conjecture for $\mathbb{O}_K$. This conjecture now follows in general from the proof of the Bloch–Kato conjecture due to Rost and Voevodsky; see for example [14], although the $p$-adic ranks of the groups $K_*(\mathbb{O}_K; \mathbb{Z}_p)$ had previously been computed by Wagoner [31].

The Hesselholt–Madsen approach uses logarithmic de Rham–Witt forms and TR, i.e., the classical approach to trace method computations. These have recently been revisited by Liu–Wang [21] who describe $K_*(\mathbb{O}_K; \mathbb{F}_p)$, the K-groups with mod $p$ coefficients, using new cyclotomic techniques from [6, 24].

The result is that

$$K_r(\mathbb{O}_K; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } r = 0, \\ H^1_{\mathbb{Q}_p}(\text{Spec } K, \mathbb{Z}_p(i)) & \text{if } r = 2i - 1, \text{ and} \\ H^2_{\mathbb{Q}_p}(\text{Spec } K, \mathbb{Z}_p(i)) & \text{if } r = 2i - 2, \end{cases}$$

where $\mathbb{Z}_p(i)$ is the $i$th Tate twist. These cohomology
2. New results

As $K(\mathbb{O}/\mathbb{w}^k; \mathbb{Z}_p) \simeq \tau_\geq 0 \mathrm{TC}(\mathbb{O}/\mathbb{w}^k; \mathbb{Z}_p)$ by [11, 18], it is enough to determine TC of these rings. To do so, we use the filtration on TC constructed by Bhattacharya–Morrow–Scholze in [6]. If $R$ is a quasicyclic ring, there is a complete decreasing filtration $F_{\mathrm{syn}}^{-i} \mathrm{TC}(R; \mathbb{Z}_p)$ with associated graded pieces

$$F_{\mathrm{syn}}^{-i} \mathrm{TC}(R; \mathbb{Z}_p) \simeq \mathbb{Z}_p(i)(R)[2i],$$

where $\mathbb{Z}_p(i)(R)$ is the weight $i$ syntomic cohomology of $R$ introduced in [6]. The syntomic complexes provide a $p$-adic analogue of the motivic filtration on $K$-theory.

As shown in [4], the weight $i$ syntomic cohomology $\mathbb{Z}_p(i)(R)$ is concentrated in $[0, i + 1]$, independent of $R$; this means that $H^r(\mathbb{Z}_p(i)(R)) = 0$ for $r \notin [0, i + 1]$. In the special case of $\mathbb{O}_K$ or $\mathbb{O}_K/\mathbb{w}^n$, an argument using the $\mathbb{w}$-adic associated graded implies that in fact the weight $i$ syntomic cohomology is in $[0, 2i]$; moreover, for $i \geq 1$, $H^0(\mathbb{Z}_p(i)(\mathbb{O}_K/\mathbb{w}^n)) = 0$ so the complex has cohomology concentrated in degrees $1$ and $2$.

One checks that $H^2(\mathbb{Z}_p(1)(\mathbb{O}_K/\mathbb{w}^n)) = 0$, so the spectral sequence associated to the syntomic filtration on $\mathrm{TC}$ collapses at the $E_2$-page for $\mathbb{O}/\mathbb{w}^n$ (or the $E_2$-page in the reindexing in [6, Thm. 1.12]). Hence,

$$\mathrm{TC}_{2i-1}(\mathbb{O}_K/\mathbb{w}^n; \mathbb{Z}_p) \cong H^1(\mathbb{Z}_p(i)(\mathbb{O}_K/\mathbb{w}^n))$$

for $i \geq 1$ and

$$\mathrm{TC}_{2i-2}(\mathbb{O}_K/\mathbb{w}^n; \mathbb{Z}_p) \cong H^2(\mathbb{Z}_p(i)(\mathbb{O}_K/\mathbb{w}^n))$$

for $i \geq 2$. Thus, it makes sense to speak of the syntomic weights of the $K$-groups of $\mathbb{O}_K/\mathbb{w}^n$.

**Theorem 2.1.** For $i \geq 1$, if the residue field of $\mathbb{O}_K$ has $q = p^f$ elements, then there is an explicit cochain complex

$$\left( \mathbb{Z}_p^{f(n-1)} \xrightarrow{\text{syn}_0} \mathbb{Z}_p^{f(n-1)} \xrightarrow{\text{syn}_1} \mathbb{Z}_p^{f(n-1)} \right)$$

quasi-isomorphic to $\mathbb{Z}_p(i)(\mathbb{O}_K/\mathbb{w}^n)$. The terms are free $\mathbb{Z}_p$-modules of the given ranks in cohomological degrees $0$, $1$, and $2$.

The proof of the existence of this explicit cochain complex model of the syntomic complex will be discussed in Sections 4 and 5.

The groups $K_v(\mathbb{O}_K/\mathbb{w}^n)$ are torsion for $* > 0$. In particular, the complex above is exact rationally. Thus, to find the cohomology of $\mathbb{Z}_p(i)(\mathbb{O}_K/\mathbb{w}^n)$, and hence the $p$-adic $K$-groups of $\mathbb{O}_K/\mathbb{w}^n$, it is enough to compute the matrices $\text{syn}_0$, $\text{syn}_1$ and their elementary divisors.
**Theorem 2.2.** The matrices \( \text{syn}_0 \) and \( \text{syn}_1 \) are effectively computable. Specifically, they can be determined with enough \( p \)-adic precision to guarantee computability of the effective divisors.

We have implemented our algorithm in SAGE [30] in the case where \( f = 1 \), i.e., when the residue field is \( \mathbb{F}_p \). Future work will include an implementation for general \( f \).

**Corollary 2.3.** There is an algorithm to determine the structure of \( K_r((O_K/\omega^n)/I) \) for any \( K \), \( n \), and \( r \).

Along the way, we extend the result of Angeltveit on the quotients of the orders from the unramified case to any \( O_K/\omega^n \).

**Corollary 2.4.** For any \( O_K/\omega^n \),

\[ \frac{\#K_{2i-1}(O_K/\omega^n; \mathbb{Z}_p)}{\#K_{2i-2}(O_K/\omega^n; \mathbb{Z}_p)} = q^{i(n-1)}, \]

where \( q = p^f \) is the order of the residue field of \( O_K \).

This corollary is especially powerful thanks to the following theorem.

**Theorem 2.5** (Even vanishing theorem). If

\[ i \geq \frac{p^2}{(p-1)^2} (p^{\frac{n}{2}} - 1), \]

then \( H^2(\mathbb{Z}_p(i)(O_K/\omega^n)) = 0 \) and hence

\[ K_{2i-2}(O_K/\omega^n) = 0 \]

if additionally \( i \geq 2 \).

**Corollary 2.6.** If

\[ i \geq \frac{p^2}{(p-1)^2} (p^{\frac{n}{2}} - 1), \]

then \( \#K_{2i-1}(O_K/\omega^n) = q^{i(n-1)} \cdot (q^i - 1) \).

**Corollary 2.7.** There is an algorithm to compute the orders of all of the \( K \)-groups of \( O/\omega^n \).

Indeed, Theorem 2.5 and Corollary 2.6 reduce the problem to the computation of the cohomology of the syntomic complexes \( \mathbb{Z}_p(i)(O_K/\omega^n) \) for finitely many \( i \); those satisfying

\[ i < \frac{p^2}{(p-1)^2} (p^{\frac{n}{2}} - 1). \]

This number grows rather quickly, but improvements are possible and will be described in our forthcoming work [3].

## 3 Computations

We present here four example calculations.

### 3.1 \( \mathbb{Z}/4 \)

The even vanishing theorem holds in syntomic weights \( i \geq 12 \). In fact, machine computations show in this case that \( K_{2i-2}(\mathbb{Z}/4) = 0 \) for all \( i \geq 3 \), while \( K_2(\mathbb{Z}/4) \cong \mathbb{Z}/2 \). Corollary 2.4 together with Quillen’s calculation implies that

\[ \#K_3(\mathbb{Z}/4) = 8 \cdot (2^2 - 1) \] and \( \#K_{2i-1}(\mathbb{Z}/4) = 2^i \cdot (2^2 - 1) \)

for \( i \geq 3 \). This gives the complete calculation of the orders of all \( K \)-groups of \( \mathbb{Z}/4 \).

The precise structure of the decomposition of \( p \)-primary part of the \( K \)-groups into cyclic groups remains unknown to us. Figure 1 displays a table of the output of our machine computations giving the groups in syntomic weights \( i \leq 16 \).

### 3.2 Chain rings of order 8

A chain ring is a commutative ring whose ideals are totally ordered with respect to inclusion. Examples include valuation rings or quotients of valuation rings. Every finite chain ring is of the form \( O_K/\omega^n \) for some \( 1 \leq n < \infty \). There are four chain rings of order 8, namely \( \mathbb{Z}/8 \), \( \mathbb{Z}[2^{1/2}]/2^{1/2} \) (so \( n = 3 \) in our notation), \( F_2[z]/z^3 \), and \( F_8 \); see [9]. The 2-adic \( K \)-groups \( K_1(F_2[z]; \mathbb{Z}_2) \) vanish for \( n \geq 1 \). Figure 2 displays the low-degree 2-adic \( K \)-groups of the other three chain rings of order 8.

### 3.3 Quotients of degree 2 totally ramified 2-adic fields

The lmfdb [22] provides tables of \( p \)-adic fields based on work of Jones–Roberts [19]. There are 6 totally ramified degree 2 extensions of \( \mathbb{Q}_2 \). In Figure 3, we give low-degree \( p \)-adic \( K \)-groups of the quotients of these fields.

### 3.4 \( \mathbb{Z}/9 \)

The even vanishing theorem holds in syntomic weights \( i \geq 18 \). Figure 4 displays a table of the output of our machine computations in syntomic weights \( i \leq 18 \). In particular, \( K_4(\mathbb{Z}/9) \cong \mathbb{Z}/3 \) and all other positive even \( K \)-groups vanish. In odd degrees,

\[ \#K_5(\mathbb{Z}/9) = 81 \cdot (3^3 - 1) \] and \( \#K_{2i-1}(\mathbb{Z}/9) = 3^i \cdot (3^i - 1) \)

for \( i \geq 1 \), \( i \neq 3 \). This gives the complete calculation of the orders of all \( K \)-groups of \( \mathbb{Z}/9 \).
4. Prismatic cohomology over $\delta$-rings

Our proofs are motivated by previous work of Krause–Nikolaus [20] and the approach of Liu–Wang [21]. There are two main new ideas: the notion of prismatic cohomology relative to a $\delta$-ring and the systematic use of the filtration on the syntomic complexes induced by the $\varpi$-adic filtration on $O_K/\varpi^n$. Similar filtrations have also been used by Angeltveit [2] and Brun [8] in the topological context.

Let $A^0 = W(F_q)[z]$ be the $\delta$-ring with $\delta(z) = 0$ and hence $\varphi(z) = z^p$. If $E(z)$ is an Eisenstein polynomial for $O_K$, then the pair $(A^0, (E(z)))$ is a prism. Bhatt and Scholze show that $\Delta_{(O_K/\varpi^n)/A^0}$ is discrete and admits a description as a prismatic envelope $A^0((\varpi/n)^\wedge)$ in the sense of [7, Prop. 3.13]; the prismatic envelope is an explicit pushout in $(p, E(z))$-complete $\delta$-rings over $A^0$.

The main idea is to determine the syntomic complexes $Z_{p(i)}(O/\varpi^n)$ by descent along the map $\Delta_{O/\varpi^n} \to \Delta_{(O_K/\varpi^n)/A^0}$ from absolute prismatic cohomology to relative prismatic cohomology. To make sense of this, we introduce prismatic cohomology rel-
ative to a $\delta$-ring. Let us outline the definition.

Given an arbitrary derived $p$-complete $\delta$-ring $A$ and a derived $p$-complete $A$-algebra $R$, let $X = \text{Spf } R$ and let $(X/A)_\Delta$ be the opposite of the category of commutative diagrams

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
R \\
\downarrow \\
B/J,
\end{array}
\]

where $(B, J)$ is a bounded prism and $A \to B$ is a map of $\delta$-rings.

By definition, $\Delta_{R/A} = R\Gamma((X/A)_\Delta, \mathcal{O}_\Delta)$, where $\mathcal{O}_\Delta$ is the prismatic structure sheaf, which sends a commutative diagram as above to $B$. Warning: this site-theoretic definition should be derived in general, but gives the correct answer under additional assumptions on $R$, in particular in the case of $R = O_K/\omega^n$ over the multivariable Breuil–Kisin prisms appearing in this paper.

**Example 4.1.** If $A = \mathbb{Z}_p$ is the initial (derived $p$-complete) $\delta$-ring, then $\Delta_{R/\mathbb{Z}_p}$ recovers absolute prismatic cohomology as introduced in [6, 7] and studied further in [5]. More generally, this is true if $A$ is replaced by the ring of $p$-typical Witt vectors of any perfect $\mathbb{F}_p$-algebra.

**Example 4.2.** If $(A, I)$ is a prism and $R$ is an $A/I$-algebra, then $\Delta_{R/A}$ agrees with derived relative prismatic cohomology as studied in [7].

Now, consider the augmented cosimplicial diagram $A^\bullet$ where $A^{-1} = W(F_q)$, $A^0 = W(F_q)[[z]]$, and $A^s = W(F_q)[z_0, \ldots, z_s]$. This is a completed descent complex for $W(F_q) \to W(F_q)[z]$.

In the cosimplicial diagram

\[
\begin{array}{c}
W(F_q) \\
\downarrow \\
A^0 \\
\downarrow \\
A^1 \\
\downarrow \\
\cdots
\end{array}
\]

the arrows are all $\delta$-ring maps and the entire diagram admits a map to $O_K$ sending each generator $z_j$ to $\omega$. As a result, for any $O_K$-algebra $R$, there is an induced augmented cosimplicial diagram in prismatic cohomology of $R$ relative to the $\delta$-rings $A^\bullet$.

**Theorem 4.3.** The augmented cosimplicial diagram

\[
\begin{array}{c}
\Delta_R \\
\downarrow \\
\Delta_{R/A^0} \\
\downarrow \\
\Delta_{R/A^1} \\
\downarrow \\
\cdots
\end{array}
\]

is a limit diagram for $R = O_K/\omega^n$.

Thus, the absolute prismatic cohomology of an $O_K$-algebra, such as $O_K/\omega^n$, can be computed by descent using the cosimplicial diagram above.

This does not make sense when speaking of prismatic cohomology as defined in [7] because there is no compatible way to equip the entire cosimplicial diagram with the structure of a cosimplicial prism. For example, if $E(z)$ is an Eisenstein polynomial making $A^0 = W(F_q)[[z]]$ into a prism, both $E(z_0)$ and $E(z_1)$ are distinguished elements in $A^1 = W(F_q)[z_0, z_1]$ making it into a prism in two different ways.

**Proposition 4.4.** For any $s \geq 0$, the relative prismatic cohomology $\hat{\Delta}(O_K/\omega^n)/A^s$ is discrete and isomorphic to a prismatic envelope

\[
A^s \cdot \left\{ \frac{z_0^n - z_1^n}{E(z_0)}, \frac{z_1^n - z_0^n}{E(z_0)}, \ldots, \frac{z_s^n - z_0^n}{E(z_0)} \right\}.
\]

The proposition follows immediately from Example 4.2. Note that while prismatic cohomology relative to $\delta$-rings is functorial in arbitrary maps of $\delta$-rings, the presentation of a given term $\hat{\Delta}_{R/A}$ as a prismatic envelope depends on the choice of a prism structure $J$ on $A^s$ making $R$ into an $A^s/J$-algebra. In the theorem above, we choose to make $A^s$ into a prism with respect to the ideal $(E(z_0))$.

It follows that the cosimplicial diagram appearing in Theorem 4.3 gives a resolution of $\hat{\Delta}_{O_K/\omega^n}$ as the limit of a cosimplicial diagram of discrete $\delta$-rings.

To give the main idea of the rest of the argument, we illustrate it here for prismatic cohomology instead of the syntomic complexes. The absolute prismatic cohomology of a quasisyntomic ring $R$ admits a Nygaard filtration $\hat{\Delta}^{\bullet}R_K$: Nygaard completion of prismatic cohomology is written $\hat{\Delta}_R$.

**Proposition 4.5.** The Nygaard-complete absolute prismatic cohomology groups $H^r(\hat{\Delta}_{O_K/\omega^n})$ vanish for $r \neq 0, 1$.

The proposition can be proved by computing directly with a Nygaard-complete, Frobenius-twisted variant of the cosimplicial diagram in Theorem 4.3 using the prismatic envelopes of Proposition 4.4. Alternatively, one can argue as follows: the $\omega$-adic filtration on $O_K/\omega^n$ induces a filtration on $\hat{\Delta}_{O_K/\omega^n}$ whose completion agrees with $\hat{\Delta}_{O_K/\omega^n}$, and whose associated graded is the same as that of the corresponding filtration on $\hat{\Delta}_{F_q[[z]]/\omega^n}$. This associated graded can be described using crystalline cohomology and vanishes away from cohomological degrees 0, 1. Thus, by dévissage and completeness, the same vanishing holds for $\hat{\Delta}_{O_K/\omega^n}$.

It follows from the proposition that the cochain complex $A^0 \to A^1 \to A^2 \to \cdots$ associated to the cosimplicial abelian group $\hat{\Delta}(O_K/\omega^n)/A^s$ is exact in degrees $\geq 2$. This reduces the computation of $\hat{\Delta}(O_K/\omega^n)$
that, for a small computation involving prismatic envelopes of $\mathcal{O}_K/\wp^m$ relative to $A^0$, $A^1$, and $A^2$.

However, we are interested not in the absolute prismatic cohomology of $\mathcal{O}_K/\wp^n$ but rather in its syntomic cohomology. Relative syntomic cohomology is defined in the setting of prismatic cohomology relative to a Frobenius ring. We first have to explain the Nygaard filtration and the Breuil–Kisin twist, following [7, 5].

The Frobenius twist $\varphi_R$ is defined to be $\varphi_R \mathcal{A}$, the base-change of $\Delta_R$ along the Frobenius map on $A$. The Frobenius twist admits a map $\Delta_R \rightarrow \Delta_R$ and the Nygaard filtration $N^{\geq 1} \Delta_R$ is a filtration which is taken by this map to the $I$-adic filtration on $\Delta_R$. If $\Delta_R$ is discrete (as in our examples of interest) then the Nygaard filtration is simply the preimage of the $I$-adic filtration.

Given a prism $(A, I)$, let $I_n$ be the invertible $A$-module $I^\varphi(I) \cdots \varphi^{-1}(I)$. If $(A, I)$ is transversal, meaning that $A/I$ is a torsion-free, then the canonical map $I_n/I_{n+1} \rightarrow I_n/I_{n+1}$ is divisible by $p$ and the induced map $I_n/I_{n+1} \rightarrow I_n/I_{n+1}$ is surjective. The Breuil–Kisin filtration is defined to be

$$A(1) = \lim \left( \cdots \rightarrow I_3/I_3 \rightarrow I_2/I_2 \rightarrow I/I^2 \right).$$

This is an invertible $A$-module. For a general $A$-module $M$, let $M(1) = M \otimes_A A(1)$.

The relative syntomic cohomology of $R$ over a Frobenius ring $A$ is

$$Z_p(i)(R/A) = \text{fib} \left( N^{\geq 1} \Delta_R \rightarrow \Delta_R \right),$$

where $\varphi$ is a Frobenius which exists on $N^{\geq 1} \Delta_R$. Note that in [6], the syntomic complexes are defined using Nygaard complete prismatic cohomology; however, the two definitions agree by [6, Lem. 7.22] or [4, Cor. 5.31].

It follows along the lines of Theorem 4.3 that, for each $i \geq 0$, the cosimplicial diagram

$$Z_p(i)(R/A^0) \rightarrow Z_p(i)(R/A^1) \rightarrow \cdots$$

is equivalent to $Z_p(i)(R)$ when $R = \mathcal{O}_K/\wp^k$.

The fact that the Nygaard-complete prismatic cohomology $\Delta_{\mathcal{O}_K/\wp^n}$ is concentrated in cohomological degrees 0, 1, 2 implies that $Z_p(i)(\mathcal{O}_K/\wp^n)$ is concentrated in cohomological degrees 0, 1, 2. In fact, it is not hard to show that, for $i \geq 1$, each relative syntomic complex $Z_p(i)(\mathcal{O}_K/\wp^n)/A^*$ is concentrated in cohomological degree 1. Thus, the spectral sequence associated to the limit diagram

$$Z_p(i)(\mathcal{O}_K/\wp^n) \simeq \lim_{\Delta} Z_p(i)(\mathcal{O}_K/\wp^n)/A^*$$

implies that $Z_p(i)(\mathcal{O}_K/\wp^n)$ is concentrated in cohomological degrees 1, 2 for $i \geq 1$.

By the same spectral sequence, to determine $Z_p(i)(\mathcal{O}_K/\wp^n)$, and hence $K_{2n-2}(\mathcal{O}_K/\wp^n; Z_p)$ and $K_{2n-1}(\mathcal{O}_K/\wp^n; Z_p)$, it is enough to compute the cohomology of the complex

$$H^1(Z_p(i)(R/A^0)) \rightarrow \ker \left( H^1(Z_p(i)(R/A^1)) \rightarrow H^1(Z_p(i)(R/A^2)) \right)$$

where $R = \mathcal{O}_K/\wp^n$. In the next section, we explain how to use the $\wp$-adic filtration to reduce this to a finite problem.

5 The syntomic matrices

In the cosimplicial diagram $A^*$, each term is a filtered $\Delta$-ring, where in $A^\bullet = W(k)[z_1, \ldots, z_s]$ the weight of $z_j$ is 1. A filtered $\Delta$-ring is a $R^\bullet$-ring such that $\delta(R^\geq 1) \subseteq R^\geq 2$. Since each $A^\bullet$ is a filtered $\Delta$-ring, all resulting invariants, such as prismatic or syntomic cohomology complexes admit induced filtrations, which we will write for instance as $F^* Z_p(i)(\mathcal{O}_K/\wp^n)/A^*$. Theorem 5.1. For $b \geq i \geq 1$, the natural maps

$$F^{[i, b]} Z_p(i)(\mathcal{O}_K/\wp^n) \rightarrow Z_p(i)(\mathcal{O}_K/\wp^n) \rightarrow F^{[i, b]} Z_p(i)(\mathcal{O}_K/\wp^n)$$

are equivalences.

The right-hand arrow is easy to handle because $F^{=0} Z_p(i)(\mathcal{O}_K/\wp^n) \simeq Z_p(i)(\mathcal{O}_K/\wp^n)$ for $i > 0$. For the left-hand arrow, we argue by an explicit study of the interaction between the $\mathcal{F}$-filtration and the Nygaard filtration on each $\mathcal{F}(\mathcal{O}_K/\wp^n)/A^*$. The entire problem has now been reduced to a finite computation. Set $R = \mathcal{O}_K/\wp^n$ and consider the commutative diagram

$$\begin{array}{c}
\mathcal{F}^{[1, b]} N^{\geq 1} \Delta_R(1) \rightarrow \mathcal{F}^{[1, b]} N^{\geq 1} \Delta_R(1) \\
\mathcal{F}^{[1, b]} \Delta_R(1) \rightarrow \mathcal{F}^{[1, b]} \Delta_R(1) \\
\mathcal{F}^{[1, b]} \Delta_R(1) \rightarrow \mathcal{F}^{[1, b]} \Delta_R(1) \\
\mathcal{F}^{[1, b]} \Delta_R(1) \rightarrow \mathcal{F}^{[1, b]} \Delta_R(1) \\
\end{array}$$

All four terms are finitely generated free $Z_p$-modules. The vertical fibers are $Z_p(i)(R/A^0)$ and $Z_p(i)(R/A^1)$, respectively.
Our approach to the computation avoids the more traditional approach of computing either $\TR(O_K/\mathbb{Z}^n)^{F=1}$ or computing $\TC(O_K/\mathbb{Z}^n)$ as the fiber of $\TC^-(O_K/\mathbb{Z}^n) \xrightarrow{\can-\mathbb{Z}} TP(O_K/\mathbb{Z}^n)$. It would nevertheless be very interesting to understand $TP(O_K/\mathbb{Z}^n)$.

Since the complexes $\mathcal{J}^{1,0}\mathbb{N}^{\geq 1}\Delta_R\{i\}$ and $\mathcal{J}^{1,0}\Delta_R\{i\}$ are torsion for $i \geq 1$ by another use of the $\mathbb{w}$-adic filtration, one can replace

$$
\ker \left( \mathcal{J}^{1,0}\mathbb{N}^{\geq 1}\Delta_R^{(1)}\{i\} \rightarrow \mathcal{J}^{1,0}\mathbb{N}^{\geq 1}\Delta_R^{(1)}\{i\} \right)
$$

with the saturation of the image of the top horizontal map, where by saturation we mean the sub-$\mathbb{Z}_p$-module consisting of elements $x$ such that $p^N x$ is in the image for some $N$, and similarly for $\ker \left( \mathcal{J}^{1,0}\Delta_R^{(1)}\{i\} \rightarrow \mathcal{J}^{1,0}\Delta_R^{(1)}\{i\} \right)$. Write $S^0$ and $S^1$ for the saturations. The resulting commutative square

$$
\begin{array}{ccc}
\mathcal{J}^{1,0}\mathbb{N}^{\geq 1}\Delta_R^{(1)}\{i\} & \rightarrow & S^0 \\
\downarrow & & \downarrow \\
\mathcal{J}^{1,0}\Delta_R^{(1)}\{i\} & \rightarrow & S^1
\end{array}
$$

consists of free $\mathbb{Z}_p$-modules of rank $bf$ and the total cohomology computes $\mathcal{J}^{1,0}\mathbb{Z}_p\{i\}(R)$ and hence $\mathbb{Z}_p\{i\}(O_K/\mathbb{Z}^n)$ for $i \geq 1$.

To conclude, we use explicit polynomial presentations of the relevant prismatic envelopes as well as Breuil–Kisin orientations to give explicit bases of all four terms and to compute the maps between them. Taking $b = in - 1$, the result is the matrices $\text{syn}_0$ and $\text{syn}_i$ and the complex appearing in Theorem 2.1.

**Acknowledgements.** We are very grateful to Bhargav Bhatt, Lars Hesselholt, Akhil Mathew, Noah Riggenbach, Peter Scholze, and Chuck Weibel for comments on a draft of this announcement.

The first author was supported by NSF grants DMS-2102010 and DMS-2120005 and a Simons Fellowship; he would like to thank Universität Münster for its hospitality during a visit in 2020. The second and third author were funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 427320536 – SFB 1442, as well as under Germany’s Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics–Geometry–Structure. They would also like to thank the Mittag–Leffler Institute for its hospitality while working on this project.

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Figure 3: The 2-adic K-groups in syntomic weights $i = 1, 2, 3, 4$ for the totally ramified degree 2 extensions of $\mathbb{Z}_2$. The \texttt{mfdb} [22] label is given in the top left corner together with an Eisenstein polynomial. The data gives the exponents of the elementary divisors in each degree: for example, the entry 1, 3 in the \( K_3 \) row of the \( \mathcal{O}_K/\omega^3 \) column means that \( K_3(\mathcal{O}_K/\omega^3; \mathbb{Z}_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8. \)
Figure 4: The $3$-adic $K$-groups of $\mathbb{Z}/9$ for syntomic weights $1 \leq i \leq 18$. The contribution of $K_{36}(\mathbb{Z}/9; \mathbb{Z}_3) = 0$ is a (null) group from weight 19.