On upper and lower completely contra e-irresolute fuzzy multifunction

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Abstract
In this paper the concepts of upper and lower fuzzy completely contra e-irresolute fuzzy multifunctions are introduced. Also the concepts of the upper and lower completely weakly e-irresolute fuzzy multifunctions are being discussed. Some characterizations of these classes and some basic interesting properties of such fuzzy multifunctions are obtained and the mutual relationship and with other existing fuzzy multifunctions are also discussed.

Keywords
Upper and lower fuzzy completely contra e-irresolute fuzzy multifunctions, upper and lower completely weakly e-irresolute fuzzy multifunctions.

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1. Introduction
In the year 1985, Papageorgiu\(^7\) introduced the fuzzy multifunction as a function from an ordinary topological space X to a fuzzy topological space Y. Further a group of researchers are engaged themselves and have studied different types of fuzzy multifunctions. The upper and lower inverses of a fuzzy multifunction are discussed and defined Papageorgiou. Mukherjee and Malakar\(^6\) have studied fuzzy multifunctions with q-coincidence with the definition of upper inverse that were discussed and defined in\(^5\) and with the definition of lower inverse which is defined by J. E. Joseph and M. H. Kwack in \(^4\). On the other hand, J. Dontchev\(^3\) introduced and discussed a lot about contracontinuous functions. Joseph and Kwack\(^4\) introduced another form of contra continuous functions. In recent years, several authors have studied some new forms of contra-continuity for functions and multifunctions. Also, Seenivasan, V and Kamala K, have defined and studied about Fuzzy e-continuity and fuzzy e-open sets\(^9\) and studied about Fuzzy Ce-I(ec, eo) functions and fuzzy completely Ce-I(rc, eo) functions via fuzzy e-open sets\(^10\). In this paper we introduce fuzzy upper (lower) completely contra e-irresolute multifunctions and fuzzy upper (lower) completely weakly contra e-irresolute multifunctions and give some characterizations and properties of such notions are discussed. Throughout this paper let us use the abbreviations as fuzzy topological space as fts, fuzzy multifunction as fmf, Regular open set as Ro, regular closed as Rc, fuzzy open set as FOS, fuzzy closed set as FCS, fuzzy e-closed set as fe-cs, fuzzy e-open set as fe-os, e-irresolute multifunction as e-irmf etc.

2. Preliminaries
In this section, we recall some definitions and basic results which will be used. Through this paper, by \((X, \tau)\) or simply by \(X\) we will mean a topological space in the classical sense, and \((Y, \sigma)\) or simply \(Y\) will stand for a fuzzy topological space as defined by Chang\(^2\). Fuzzy sets in \(Y\) will be denoted by \(\lambda, \mu, \rho, \eta, \gamma\) etc., and although subsets of \(X\) will be denoted by \(A, B, M, U, V, W\) etc. A mapping \(F : X \rightarrow Y\) is called...
a fuzzy multifunction if for each, $x \in X$, $F(x)$ is a fuzzy set in $Y$. For a fuzzy multifunction $F : X \rightarrow Y$, the upper inverse $F^+(\mu)$ and lower inverse $F^-(\mu)$ of a fuzzy set $\mu$ in $Y$ are defined as follows: $F^+(\mu) = \{ x \in X : F(x) \leq \mu \}$ and $F^-(\mu) = \{ x \in X : F(x) \geq \mu \}$.

**Definition 2.1.** [vi] A fuzzy set $\mu$ of a fte $X$ is called fe-os if $\mu \leq c(\mu) \vee \mu$, and Fe-os if $\mu \geq c(\mu) \wedge \mu$. The intersection of all fe-cs’s containing $\mu$ is called fe-closure of $\mu$ and is denoted by $fe-cl \mu$. The union of all fe-os’s contained in $\mu$ is called fuzzy e-interior of $\mu$ and is denoted by $fe-int \mu$.

**Definition 2.2.** [vii] intcl $F$ is fuzzy CC $+$ completely contra e-irresolute (briefly, fuzzy CC $+$) if for every $FCS \ F$ of $Y$, $F \in FCS \ F$, and for every $fe-cs \ G \subset \ F$, $F \in FCS \ G$. There exists an $Rc$ set $R$ such that $\forall \ x \in X$, $F(x) \in X \rightarrow R$.

**Theorem 3.3.** For every $FCS \ F$ of $Y$, $F \in FCS \ F$ for every $fe-cs \ G \subset \ F$, $F \in FCS \ G$. There exists an $Rc$ set $R$ such that $\forall \ x \in X$, $F(x) \in X \rightarrow R$.

**Example 3.6.** Let $X = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ b \}, \{ a, b \} \}$ and $Y = \{ 0, 1 \}$, $\sigma = \{ 0, 1, \mu, \lambda, \delta \}$ where $\mu(\gamma) = 0.4$, $\lambda(\nu) = 0.1$, $\delta(\gamma) = 0.7$, $\nu(\gamma) = 0.3$, $\gamma(\nu) = 0.5$. Consider the fmf $F : (X, \tau) \rightarrow (Y, \sigma)$ defined as $F(a) = v$, $F(b) = \gamma$, $F(c) = \delta$. Then $F^+(\mu) = \{ a, b \}$, $F^+(1 - \lambda) = X$, $F^+(1 - \delta) = \{ a \}$ and $F^-(1 - \mu) = \{ b, c \}$, $F^-(1 - \lambda) = X$, $F^-(1 - \delta) = \phi$ which is open but not $Ro$ in $(X, \tau)$. Then $F$ is fuzzy upper contra continuous but not fuzzy $CC^+$ e-irresolute (fuzzy $CC_L$ e-irresolute) multifunction.

**Theorem 3.4.** For a fmf $F : (X, \tau) \rightarrow (Y, \sigma)$ the following statements are equivalent:

(i) $F$ is fuzzy $CC^+$ e-irresolute

(ii) For each fe-cs $\mu$ and $x \in X$ such that $F(x) \leq \mu$, there exists an $Ro$ set $V$ containing $x$ such that if $y \in V$, then $F(y) \leq \mu$.

(iii) $F^+(\mu)$ is Ro in $X$ for any fe-cs $\mu$ in $Y$.

(iv) $F^+(\rho)$ is Re in $X$ for any fe-os $\rho$ in $Y$.

(v) For every $FOS \mu$ of $Y$, $F^+(fe-int(\mu))$ is Re in $X$.

(vi) For every $FCS \eta$ of $Y$, $F^+(fe-cl(\eta))$ is Ro in $X$.

(vii) $intcl F^+(\mu) = F^+(fe-cl(\mu))$, for every fuzzy set $\mu$ in $Y$.

**Proof:** (i) $\Leftrightarrow$ (ii) obvious.

(iii) $\Leftrightarrow$ (iv) Let $\mu$ be any fe-cs in $Y$ and $x \in F^+(\mu)$. By (iii), there exists a Ro set $M$ containing $x$ such that $M \supset F^+(\mu)$. Thus, $x \in intcl F^+(\mu)$ and hence $F^+(\mu)$ is an Ro set in $X$.

(iv) $\Leftrightarrow$ (v) Let $\rho$ be any fe-cs in $Y$ and $x \in F^+(\rho)$. By (iii), $F^+(\rho)$ is a Ro set in $X$. Take $V = F^+(\rho)$. Then $X \supset F^+(\rho)$. Hence, $F$ is fuzzy $CC^+$ e-irresolute.

(v) $\Leftrightarrow$ (vi) Let $\mu$ be a FCS of $Y$. Since $fe-int(\mu)$ is fuzzy open, then by (iv), $F^+(fe-int(\mu))$ is a Ro set in $X$. Converse is obvious.

(vi) $\Leftrightarrow$ (vii) Let $\eta$ be a FCS of $Y$. Then $fe-int(1 - \eta)$ is a FCS of $Y$. By (vi), $F^+(fe-int(1 - \eta))$ is a Ro set in $X$. This implies, $F^+(fe-int(1 - \eta)) = F^+(1 - fe-cl(\eta)) = 1 - F^+(fe-cl(\eta))$. Then $F^+(fe-cl(\eta))$ is Ro in $X$. Converse is obvious.

(vii) $\Rightarrow$ (vii) Let $\mu$ be any fuzzy set in $Y$. Since $fe-cl(\mu)$ is fuzzy closed in $Y$, then by (vi), $F^+(fe-cl(\mu))$ is Ro in $X$. Therefore, we obtain $intcl F^+(\mu) = intcl F^+(fe-cl(\mu))$. Since $F^+(fe-cl(\mu))$ is Ro in $X$ and hence $intcl F^+(\mu) = F^+(fe-cl(\mu))$.

(vii) $\Leftrightarrow$ (vi): obvious.

**Remark 3.5.** Every fuzzy $CC^+$ e-irresolute (fuzzy $CC_L$ e-irresolute) multifunction is fuzzy upper contra continuous. Converse is not true.
Remark 3.7. Every fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) multifunction is fuzzy upper(lower) almost continuous. Converse is not true.

Example 3.8. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$ and $Y = [0, 1]$, $\sigma = \{0, 1, \mu, \lambda, \eta\}$ where $\mu(y) = 0.4, \lambda(y) = 0.1, \eta(y) = 0.5, \xi(y) = 0.7, \gamma(y) = 0.6$. Consider the fnf $F : (X, \tau) \rightarrow (Y, \sigma)$ defined as $F(a) = \mu, F(b) = \lambda, F(c) = \eta$. Then $F^+((1 - \mu) = \{a, c\}, F^+((1 - \lambda) = X, F^+((1 - \eta) = \{a\}$ which is closed but not Ro in $(X, \tau)$. Then $F$ is fuzzy upper almost continuous but not fuzzy $CC_l^U$ e-irrmf.

Example 3.9. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}, \{a, b\}\}$ and $Y = [0, 1]$, $\sigma = \{0, 1, \mu, \rho, \eta\}$ where $\mu(y) = 0.4, \rho(y) = 0.2, \eta(y) = 0.5$. Consider the fnf $F : (X, \tau) \rightarrow (Y, \sigma)$ defined as $F(a) = \mu, F(b) = \rho, F(c) = \eta$. Then $F^+((1 - \mu) = \{a\}, F^+((1 - \rho) = \{a, c\}, F^+((1 - \eta) = \phi$ which is closed but not Ro in $(X, \tau)$. Then $F$ is fuzzy upper almost continuous but not fuzzy $CC_l^U$ e-irresolute multifunction.

Theorem 3.10. Let $\{Y_i : i \in I\}$ be a family of product spaces. If a function $F : X \rightarrow \prod_{i \in I} Y_i$ is fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute), then $P_i \circ F : X \rightarrow Y_i$ is fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) for each $i \in I$ where $P_i$ is the projection of $\prod_{i \in I} Y_i$ onto $Y_i$.

Proof : Let $\delta_i$ be any fe-os in $Y_i$. Since $P_i$ is a fuzzy continuous and fuzzy open set, it is a fnf. Now $P_i : \prod_{i \in I} Y_i \rightarrow Y_i, P_i^+ (\delta_i)$ is a fuzzy e-open in $\prod_{i \in I} Y_i$. Therefore, $P_i$ is a fuzzy e-irresolute function. Now $(P_i \circ F)^+ (\delta_i) = F^+ (P_i^+ (\delta_i)) = F^+ (\prod_{i \in I} Y_i \times \delta_i)$ since $F$ is fuzzy $CC_l^U$ e-irresolute. Hence $F^+ (P_i^+ (\delta_i))$ is a Ro set, since $P_i^+ (\delta_i)$ is a fuzzy e-open set. Hence $P_i \circ F$ is fuzzy $CC_l^U$ e-irresolute.

Theorem 3.11. If the function $F : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$, defined by $F(x_i) = \prod_{i \in I} F_i(x_i)$, is fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) multifunction, then $F_i : X_i \rightarrow Y_i$ is fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) multifunction for each $i \in I$.

Proof : Let $u_i$ be any fe-cs of $Y_i$, then $\prod_{i \in I} Y_i \times u_i$ is fuzzy e-closed in $\prod_{i \in I} Y_i$. Since $F$ is fuzzy $CC_l^U$ e-irrmf, then $F^+ (\prod_{i \in I} Y_i \times u_i) = \prod_{i \in I} X_i \times F_i^+ (u_i)$ is Ro in $\prod_{i \in I} X_i$ and hence $F_i^+ (u_i)$ is Ro in $X_i$. This implies, $F_i$ is fuzzy $CC_l^U$ e-irrmf.

Theorem 3.12. For a fnf $F : X \rightarrow Y$, if $\text{clint}(F^+ (\eta)) \leq F^+ (f e-K_0)$ for every fuzzy set $\eta$ in $Y$, then $F$ is fuzzy $CC_l^U$ e-irresolute.

Proof : Suppose that $\text{clint}(F^+ (\eta)) \leq F^+ (f e-K_0)$, for every fuzzy set $\eta$ in $Y$. By definition, $F^+ (f e-K_0) = F^-(\eta)$. This implies that, $\text{clint}(F^+ (\eta)) = F^-(\eta)$ and $F^-(\eta)$ is Re in $X$. Thus, by theorem 3.3, $F$ is fuzzy $CC_l^U$ e-irresolute.

Theorem 3.13. For a fnf $F : X \rightarrow Y$, if $\text{clint}(F^+ (\eta)) \leq F^+ (f e-K_0)$ for every fuzzy set $\eta$ in $Y$, then $F$ is fuzzy $CC_l$ e-irresolute.

Proof : It is similar to that of theorem 3.12.

Theorem 3.14. Let $\{V_i : i \in I\}$ be a Ro cover of $X$ and a fnf $F : X \rightarrow Y$ is a fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) iff $F|V_i : V_i \rightarrow Y$ is fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) for each $i \in I$.

Proof : Suppose that $F$ is fuzzy $CC_l^U$ e-irrmf. Let $x \in X$ and $x \in V_i$ for each $i \in I$. Let $\lambda$ be a fe-cs of $Y$ containing $F(V_i(x))$. Since $F$ is fuzzy $CC_l^U$ e-irrmf and $F(x) = F(V_i(x))$, there exists an Ro set $U$ containing $x$ such that $U \subset F^+ (\lambda)$. Take $W = U \cap V_i$. Then $W$ is a Ro set $V_i$ containing $x$ and $F|W(V_i) = F(W) \leq \lambda$. This implies that $W \subset F^+ (\lambda)$. Thus $F|V_i$ is fuzzy $CC_l^U$ e-irresolute.

Conversely, let $x \in X$ and $\lambda$ be fe-cs in $Y$ with $x \in F^+ (\lambda)$. Since $\{V_i : i \in I\}$ is a Ro cover for $X$, then $x \in V_i$. Since $F(V_i)$ is fuzzy $CC_l^U$ e-irresolute and $F(x) = F(V_i(x))$, there exists a Ro set $W$ such that $F|W(V_i) \leq \lambda$. Then we have, $W$ is Ro in $X$ and $W \subset F^+ (\lambda)$. Therefore, $F$ is fuzzy $CC_l^U$ e-irresolute.

Theorem 3.15. For a fnf $F : X \rightarrow Y$, if the fuzzy graph multifunction $G_F : X \times X \rightarrow Y$ is fuzzy $CC_l^U$ e-irresolute, then $F$ is fuzzy $CC_l^U$ e-irresolute.

Proof : Suppose that fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ is fuzzy $CC_l^U$ e-irresolute and $x \in X$. Let $\eta$ be fe-cs in $Y$ with $(x, \eta) \leq \eta$. Then $G_F(x) \leq x \times \eta$. Since the graph function $G_F$ is fuzzy $CC_l^U$ e-irresolute, there exists an Ro set $M$ containing $x$ such that $G_F(M) \leq x \times \eta$. For any $m_0 \in M$ and $y \in Y$, we have $F(M_0)(y) = G_F(m_0)(m_0, y) \leq (x \times \eta)(m_0, y) = \eta(y)$. Then we have $F(M_0)(y) = \eta(y)$ for all $y \in Y$. Thus, $F(M_0) \leq \eta$ for any $m_0 \in M$. Hence, $F$ is fuzzy $CC_l^U$ e-irresolute.

Theorem 3.16. For a fnf $F : X \rightarrow Y$, if the fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ is fuzzy $CC_l$ e-irresolute, then $F$ is fuzzy $CC_l$ e-irresolute.

Proof : Suppose that fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ is fuzzy $CC_l$ e-irresolute and $x \in X$. Let $\eta$ be fe-cs in $Y$ such that $F(x) \eta$. Then there exists $y \in Y$ such that $(F(x))(y) + \eta(y) > 1$. Then we have $G_F(x)(x, y) + (x \times \eta)(x, y) > 1$ which implies $G_F(x)(x) \eta$. Since fuzzy graph function $G_F$ is fuzzy $CC_l$ e-irresolute, there exists an Ro set $M$ such that $x \in M$ and $G_F(m_0)(X \times \eta)$ for all $m_0 \in M$. Suppose that there exists a point $m_0$ in $M$ such that $F(m_0) \eta$. Then for all $y \in Y$, $(F(m_0))(y) + \eta(y) \leq 1$ we have $G_F(m_0)(x, y) \leq F(m_0)(y)$ and $(x \times \eta)(x, y) \leq \eta(y)$. Thus, $G_F(m_0)(x, y) + (X \times \eta)(x, y) \leq 1$. Thus, $G_F(m_0)(X \times \eta)$, for any $n_0$ in $M$ which is a contradiction. Hence $F$ is fuzzy $CC_l$ e-irresolute.

Theorem 3.17. If $F : (X, \tau) \rightarrow (Y, \sigma)$ is a fuzzy $CC_l^U$ e-irresolute (fuzzy $CC_l$ e-irresolute) injective fnf and $F(x)$ be fuzzy $e-T_2$ space for every $x \in X$, then $X$ is Urysohn space.
Proof: Let $x_1$ and $x_2$ be any two distinct points in $X$. Since $F$ is injective, $F(x_1) \neq F(x_2)$ in $Y$. Since $F$ is fuzzy $e$-$T_2$, there exists fe-$os \eta$ and $\rho$ in $Y$ such that $F(x_1) \in \eta$ and $F(x_2) \in \rho$ and $\eta \cap \rho = 0$. This implies that $fe-cl(\eta)$ and $fe-cl(\rho)$ are fe-$cs$ in $Y$. Then, since $F$ is fuzzy $CC^U$ e-$irresolute$, there exists a Ro set $V$ and $W$ in $X$ containing $x_1$ and $x_2$ respectively, such that $F(V) \leq fe-cl(\eta)$ and $F(W) \leq fe-cl(\rho)$. This implies that $V \subseteq F^+(fe-cl(\eta))$ and $W \subseteq F^+(fe-cl(\rho))$. Thus, we have $F^+(fe-cl(\eta))$ and $F^+(fe-cl(\rho))$ are disjoint and hence $cl(V) \cap cl(W) = \emptyset$, and by definition, $X$ is Urysohn.

Theorem 3.18. Let $F : X \to Y$ be a fuzzy $CC^U$ e-$irresolute surjective multifunction and $F(x)$ is fuzzy e-closed for each $x \in X$. If $X$ is nearly compact, then $Y$ is fuzzy e-closed.

Proof: Let $\{u_\alpha : \alpha \in \Omega\}$ be any cover of $F(x)$ by fe-$cs$ of $Y$. Since $F(x)$ is fuzzy e-closed for any $x \in X$, there exists a finite subset $\Delta$ of $\Omega$ such that $F(x) \subseteq \bigcup_{\alpha \in \Delta} fe-cl(u_\alpha)$. Take $\lambda = \bigvee_{\alpha \in \Delta} fe-cl(u_\alpha)$. Since $F$ is fuzzy $CC^U$ e-$irresolute$, there exists a Ro set $A_\lambda$ of $X$ containing such that $F(A_\lambda) \subseteq \lambda$. Then $\{A_\lambda\}, x \in X$ is a Ro cover of $X$. Since $X$ is nearly compact, there exists $n, i = 1, \ldots, n$ in $X$ such that $X = \bigcup_{i=1}^n A_{\lambda_i}$ we have $Y = F(X) = F(\bigcup_{i=1}^n A_{\lambda_i}) = \bigcup_{i=1}^n F(A_{\lambda_i}) \leq \bigvee_{i=1}^n \lambda_i = \bigvee_{\alpha \in \Delta} fe-cl(u_\alpha)$. Thus, $Y$ is fuzzy e-closed.

Theorem 3.19. If $F : X \to Y$ is a fuzzy $CC^U$ e-$irresolute injection and $Y$ is fuzzy e-normal then $X$ is strongly normal.

Proof: Let $V$ and $W$ be a disjoint nonempty closed sets of $X$. Since $F$ is injective, $F(V)$ and $F(W)$ are disjoint FCSs. Since $Y$ is fuzzy e-normal, there exists fe-$os \mu$ and $\lambda$ such that $F(V) \leq \mu$ and $F(W) \leq \lambda$ and $\mu \cap \lambda = 0$. This implies that $fe-cl(\mu)$ and $fe-cl(\lambda)$ are fe-$cs$ in $Y$. Then, since $F$ is fuzzy $CC^U$ e-$irresolute$, $F^+(fe-cl(\mu))$ and $F^+(fe-cl(\lambda))$ are Ro sets. Then $V \subseteq F^+(fe-cl(\mu))$ and $W \subseteq F^+(fe-cl(\lambda))$, we have $F^+(fe-cl(\mu))$ and $F^+(fe-cl(\lambda))$ are disjoint, and by definition, $X$ is strongly normal.

4. Fuzzy $CC^U$ e-$irresolute and Fuzzy $CC^U_L$ e-$irresolute Multifunctions

Definition 4.1. A fnmf $F : X \to Y$ is called fuzzy lower completely weakly contra e-$irresolute$ (briefly, fuzzy $CC^U_L$ e-$irresolute$) multifunction if for any fe-$cs$ $\mu$ in $Y$ with $x \in F^+(\mu)$ (i.e.) $F(x) \cap \mu$, there exists an open set $V$ in $X$ containing $x$ such that $V \subseteq F^+(\mu)$.

Definition 4.2. A fnmf $F : X \to Y$ is called fuzzy upper completely weakly contra e-$irresolute$ (briefly, fuzzy $CC^U$ e-$irresolute$) multifunction if for any fe-$cs$ $\mu$ in $Y$ with $x \in F^+(\mu)$, there exists an open set $V$ in $X$ containing $x$ such that $V \subseteq F^+(\mu)$.

Remark 4.3. Every fuzzy $CC^U$ e-$irresolute$ (fuzzy $CC^U_L$ e-$irresolute$) multifunction is fuzzy $CC^U$ e-$irresolute$ (fuzzy $CC^U_L$ e-$irresolute$) multifunction.

Example 4.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ and $Y = \{0, 1\}$, $\sigma = \{0, 1, \mu, \lambda, \eta\}$ where $\mu(0) = 0.4, \lambda(0) = 0.2, \eta(0) = 0.8, \zeta(0) = 0.5, \gamma(0) = 0.6$. Consider the fnmf $F : (X, \tau) \to (Y, \sigma)$ is defined as $F(a) = \zeta, F(b) = \eta, F(c) = \gamma$. Then $F^+(1 - \mu) = \{a, c\}, F^+(1 - \lambda) = X, F^+(1 - \eta) = \{a\}$ which is open but not Ro in $(X, \tau)$. Then, $F$ is fuzzy $CC^U$ e-$irresolute$ but not fuzzy $CC^U_L$ e-$irresolute$.

Example 4.5. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $Y = \{0, 1\}$, $\sigma = \{0, 1, \zeta, \rho\}$ where $\zeta(0) = 0.1, \rho(0) = 0.6, \eta(0) = 0.4$. Consider the fnmf $F : (X, \tau) \to (Y, \sigma)$ is defined as $F(a) = \eta, F(b) = \rho, F(c) = \zeta$. Then $F^+(1 - \zeta) = \{a, b\}$, $F^+(1 - \eta) = \{b\}, F^+(1 - \rho) = \emptyset$ which is open but not Ro in $(X, \tau)$. Then $F$ is fuzzy $CC^U_L$ e-$irresolute$ but not fuzzy $CC^U$ e-$irresolute$.

Theorem 4.6. For a fnmf $F : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent:

(i). $F$ is fuzzy $CC^U$ e-$irresolute$

(ii). For each fe-$cs$ $\mu$ and $x \in X$ such that $F(x) \subseteq \mu$, there exists an open set $V$ containing $x$ such that if $y \in V$, then $F(y) \subseteq \mu$.

(iii). $F^+(\mu)$ is open in $X$ for any fe-$cs$ $\mu$ in $Y$.

(iv). $F^+(\rho)$ is closed in $X$ for any fe-$os$ $\rho$ in $Y$.

(v). For every FOS $\mu$ of $Y$, $F^+(fe-int(\mu))$ is closed in $X$.

(vi). For every FCS $\eta$ of $Y$, $F^+(fe-cl(\eta))$ is open in $X$.

Proof: (i) $\Rightarrow$ (ii): obvious.

(iii) $\Rightarrow$ (i): Let $\mu$ be any fe-$cs$ in $Y$ and $x \in F^+(\mu)$. By (i), there exists an open set $V$ containing $x$ such that $V \subseteq F^+(\mu)$. Thus, $x \in intF^+(\mu)$ and hence $F^+(\mu)$ is an open set in $X$.

(iv) $\Rightarrow$ (ii): Let $\rho$ be any fe-$os$ in $Y$ and $x \in F^+(\rho)$. By (iii), $F^+(\rho)$ is an open set in $X$. Take $V = F^+(\rho)$. Then $V \subseteq F^+(\rho)$.

(v) $\Rightarrow$ (iii): Let $\mu$ be a Fe-$cs$ in $Y$. Then $1_Y - \mu$ is a Fe-$os$ in $Y$. By (iv), $F^+(1_Y - \mu)$ is a closed set in $X$. Since $F^+(1_Y - \rho) = 1_X - F^+(\rho)$, then $F^+(\rho)$ is a closed set in $X$.

(vi) $\Rightarrow$ (v): Let $\mu$ be a Fe-$os$ in $Y$. Since $fe-int(\mu)$ is fuzzy e-open, then by (iv), $F^-(fe-int(\mu))$ is a closed set in $X$. Converse is obvious.

(iii) $\Rightarrow$ (vi): Let $\eta$ be a FCS of $Y$. Since $fe-cl(\eta)$ is fe-$cs$ of $Y$, then by (3), $F^+(fe-cl(\eta))$ is a open set in $X$. Converse is obvious.

(v) $\Rightarrow$ (vii): Let $\eta$ be a FCS of $Y$. Then $1_Y - \eta$ is a FOS of $Y$. Since $fe-int(1_Y - \eta)$ is fe-$os$ of $Y$. By (5), $F^-(fe-int(1_Y - \eta))$ is a closed set in $X$. This implies, $F^-(fe-int(1_Y - \eta)) = F^-(1_Y - fe-cl(\eta)) = 1_X - F^+(fe-cl(\eta))$. Then $F^+(fe-cl(\eta))$ is open in $X$. The converse is obvious.

Theorem 4.7. If $F : X \to Y$ is an upper almost continuous multifunction where $X$ and $Y$ are topological spaces and $G : Y \to Z$ is a fuzzy $CC^U$ e-$irresolute$ multifunction where $Z$ is a fuzzy topological space, then $G \circ F : X \to Z$ is fuzzy $CC^U$ e-$irresolute$.
Proof: Let \( x \in X \) and \( \rho \) be a fuzzy \( e \)-closed set in \( Z \) we have \((G \circ F)^+(\rho) = F^+(G^+(\rho))\). Since \( G \) is fuzzy \( CC^U \) \( e \)-irresolute, \( G^+(\rho) \) is \( Ro \) in \( Y \). Since \( F \) is upper almost continuous, \( F^+(G^+(\rho)) = (G \circ F)^+(\rho) \) is open in \( X \). Thus, \( G \circ F \) is fuzzy \( CC^U \) \( e \)-irresolute.

Theorem 4.8. If \( F_i : X \to Y \) for \( i = 1, 2, \ldots, n \), are fuzzy \( CC^U \) \( e \)-irmf, then \( \sqcup_{i=1}^n F_i \) is a fuzzy \( CC^U \) \( e \)-irmf.

Proof: Let \( \eta \) be a fe-cs of \( Y \) and \( F_i : X \to Y \) for \( i = 1, 2, \ldots, n \), are fuzzy \( CC^U \) \( e \)-irmf. Let \( x \in (\sqcup_{i=1}^n F_i)^+(\eta) \). Then, \( \sqcup_{i=1}^n F_i(x) \leq \eta \). Since \( F_i, i = 1, 2, \ldots, n \), are fuzzy \( CC^U \) \( e \)-irmf's, then there exists an open set \( V_i \) containing \( x \) such that \( F_i(x_0) \leq \eta \) for every \( x_0 \in V_i \). Let \( V = \bigcup_{i=1}^n V_i \). Then \( V \subseteq (\sqcup_{i=1}^n F_i)^+(\eta) \). Thus \((\sqcup_{i=1}^n F_i)^+(\eta)\) is open in \( X \) and hence \((\sqcup_{i=1}^n F_i)^+(\eta)\) is fuzzy \( CC^U \) \( e \)-irmf.

Theorem 4.9. If \( F_i : X \to Y \) for \( i = 1, 2, \ldots, n \), is fuzzy \( CC_L \) \( e \)-irmf, then \( \sqcup_{i=1}^n F_i \) is a fuzzy \( CC_L \) \( e \)-irmf.

Proof: Let \( \eta \) be a fe-cs of \( Y \) and \( F_i : X \to Y \) for \( i = 1, 2, \ldots, n \), are fuzzy \( CC_L \) \( e \)-irresolute. Let \( x \in (\sqcup_{i=1}^n F_i)^-(\eta) \). Then, \( \sqcup_{i=1}^n F_i(x) \geq \eta \). Then, \( \sqcup_{i=1}^n F_i(x) \geq \eta \). Since \( F_i, i = 1, 2, \ldots, n \), are fuzzy \( CC_L \) \( e \)-irmf's, then there exists an open set \( V_i \) containing \( x \) such that \( F_i(x_0) \geq \eta \) for every \( x_0 \in V_i \). Let \( V = \bigcup_{i=1}^n V_i \). Then \( V \subseteq (\sqcup_{i=1}^n F_i)^-(\eta) \). Thus \((\sqcup_{i=1}^n F_i)^-(\eta)\) is open in \( X \) and hence \((\sqcup_{i=1}^n F_i)^-(\eta)\) is fuzzy \( CC_L \) \( e \)-irmf.

5. Conclusion

Thus in this paper the concepts of upper and lower fuzzy completely contra \( e \)-irresolute fuzzy multifunctions were introduced. Also the concepts of the upper and lower completely weakly \( e \)-irresolute fuzzy multifunctions were being discussed. Some characteristics of these classes and some basic interesting properties of such fuzzy multifunctions were obtained and the mutual relationship with other existing fuzzy multifunctions were also discussed.

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