Virtual quarks, vacuum stability and scalar meson physics

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Results are reviewed, which provide relations between the response (and eventual instability) of the chiral QCD vacuum to an increase of the number of massless quarks in the theory and the observed violations of the large $N_c$ expansion in the scalar meson sector, by combining chiral perturbation theory expansions in $m_s$ with sum rule methods. An approach based on the construction of scalar form-factors was recently confirmed by an independent approach which uses the $\pi K$ scattering amplitudes.

I. INTRODUCTION

This talk collects some results obtained in ref. [1] and also ref. [2] as well as recent related work presented in ref. [3]. A relationship is established between some peculiar properties of the scalar mesons on the one hand, and, on the other hand a specific aspect of the chiral vacuum in QCD, namely the response of the vacuum to quantum fluctuations of the virtual massless quarks. First, what is peculiar about the scalar mesons? Other mesons can be classified into nonets and each nonet satisfies (to a reasonable approximation) the property of ideal mixing. Consider the example of the vector mesons: this means that the following flavour structure can be ascribed to each state,

$$\phi \sim \bar{s}s$$
$$K^{*+} \sim \bar{s}u$$
$$\omega, \rho^0 \sim \bar{u}u \pm \bar{d}d$$

This structure implies that the $\rho$ and $\omega$ are degenerate in mass and that the equal mass splitting formula applies,

$$m_\phi - m_{K^*} = m_{K^*} - m_\rho$$

These properties hold to first order in an expansion in the light quark masses and are well satisfied experimentally. Furthermore, according to this structure the OZI rule suppresses the decay of the $\phi$ meson into $\rho \pi$. These properties of ideal mixing and the OZI rule can be understood on the basis of QCD (at a qualitative level), by performing the large $N_c$ expansion (see e.g. [4]). When it comes to the light scalar mesons, none of the above properties seem to be satisfied. For instance, consider the $a_0(980)$ and $f_0(980)$. Since they are light and nearly degenerate one would be tempted to ascribe the flavour structure $\bar{u}u \pm \bar{d}d$ to them. However, the $f_0(980)$ appears to be much more strongly coupled to $K\bar{K}$ than to $\pi\pi$ which means either that the OZI rule is violated, or ideal mixing is violated or that these mesons are exotic (see e.g. ref. [5] for a review and a list of references): in all cases this implies a failure of the large $N_c$ expansion in QCD applied to the scalar mesons.

II. INCREASING THE NUMBER OF MASSLESS QUARKS IN QCD

Consider QCD in a chiral limit where the two lightest quark masses are set exactly to zero,

$$m_u = m_d = 0 .$$

The lagrangian is then invariant under a global $SU(2) \times SU(2)$ symmetry group but, as is well known, the vacuum is not implying spontaneous breakdown of this symmetry down the $SU(2)$ (isospin) symmetry group. As a result, order parameters have non-vanishing expectation values, for instance,

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq 0, \quad F_\pi \neq 0 .$$

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$F_\pi$ is the coupling of the goldstone bosons to the axial current, its order parameter status can be appreciated from the expression,

$$F_\pi^2 = -\frac{i}{6} \int d^4x \langle T\{ \bar{u}(x)\gamma^\mu(1 - \gamma^5)d(x)d\bar{u}(0)\gamma^\mu(1 + \gamma^5)u(0)\} \rangle .$$  \hspace{1cm} (5)

Let us now ask ourselves what are the roles of the quarks of other flavours, say the strange or the charmed quark, which are present in the theory on the size of the $\langle \bar{u}u \rangle$ condensate. It is easy to see that these quarks act via internal quark loops. Such graphs are suppressed perturbatively if the quarks are heavy. Suppose we now lower the mass of one of these quarks, say the strange quark, eventually down to zero mass. Still on the basis of the large $N_c$ expansion its effect would be suppressed. However, it is not difficult to see how this large $N_c$ prediction can fail here.

For this purpose, consider the Banks-Casher formula \[6\] which relates the value of the quark condensate to the density of small eigenvalues of the Dirac operator. We write the formula with QCD coupled to $N_F$ flavours of quarks, among which $N_F^0$ flavours have exactly zero mass and the remaining ones are assumed (for simplicity) to have the same mass $M$. Then, the $\langle \bar{u}u \rangle$ condensate is given by the following functional average,

$$\langle \bar{u}u \rangle_{N_F^0} = \frac{-\pi}{2\pi} \int d\mu[A]\rho_A(0)[\det(i\not{D}_A)]^{N_F^0} [\det(i\not{D}_A+iM)]^{N_F-N_F^0} e^{-S_{YM}(A)} .$$  \hspace{1cm} (6)

In this expression $\rho_A(\lambda)$ is the density of eigenvalues of the Dirac operator $i\not{D}_A$ for a given gauge field configuration, i.e.

$$\rho_A(\lambda) = \sum_n \delta(\lambda - \lambda_n)$$  \hspace{1cm} (7)

the eigenvalues $\lambda_n$ are real and occur in sign conjugate pairs. In the leading large $N_c$ approximation, each determinant is set to a constant and drops out of the formula, implying that $\langle \bar{u}u \rangle$ is independent of $N_F^0$. However, since

$$\det(i\not{D}_A) = \prod_n \lambda_n$$  \hspace{1cm} (8)

it is clear that the weight of the gluon configurations that generate many small eigenvalues are all the more suppressed by this determinant factor that $N_F^0$ is large. From the Banks-Casher formula \[6\] we expect the condensate to decrease if $N_F^0$ is increased. Eventually, if $N_F^0$ exceeds some critical value $N_F^{crit}$, then, chiral symmetry should be restored. Furthermore, if $N_F^0$ is not very small compared to $N_F^{crit}$ we expect a variation of $N_F^0$ to induce a strong variation of the condensate, in contradiction with the large $N_c$ expectation. In the next section, we show how this aspect of the vacuum is related to the physics of the scalar mesons using chiral perturbation theory (ChPT) together with sum rules.

### III. VARYING $N_F^0$ IN CHPT

In ChPT we can make use of the fact that in nature, the mass of the strange quark is sufficiently small (yet, not too small) and evaluate how the condensate $\langle \bar{u}u \rangle$ and also $F_\pi$ vary when we increase $N_F^0$ from $N_F^0 = 2$ to $N_F^0 = 3$ as an expansion in powers of the kaon mass. Explicitly, at one loop, one finds

$$\langle \bar{u}u \rangle_3 = \langle \bar{u}u \rangle_2 \left\{ 1 - \frac{m_sB_0}{F_\pi^2} \left[ 8L_4(\mu) - \frac{1}{32\pi^2} \log \frac{m_sB_0}{\mu^2} \right] + O(m_s^2) \right\}$$  \hspace{1cm} (9)

where the value of $N_F^0$ is indicated as a subscript, and

$$\langle \bar{u}u \rangle_3 = \langle \bar{u}u \rangle_2 \left\{ 1 - \frac{m_sB_0}{F_\pi^2} \left[ 32L_6(\mu) - \frac{1}{16\pi^2} \left( \frac{11}{9} \log \frac{m_sB_0}{\mu^2} + \frac{2}{9} \log \frac{4}{3} \right) \right] + O(m_s^2) \right\} .$$  \hspace{1cm} (10)

In these formulas, if $m_s$ is the physical strange quark mass, the product $m_sB_0$ can be expressed in terms of the physical pion and Kaon masses,

$$m_sB_0 = m_K^2 - \frac{1}{2}m_\pi^2 .$$  \hspace{1cm} (11)

At this order, the information on the variation with $N_F^0$ is contained in the two Gasser-Leutwyer \[7\] coupling-constants $L_4(\mu)$ and $L_6(\mu)$. The values of these couplings were not determined in ref. \[7\], it was only pointed out there that they
should be suppressed in the large $N_c$ expansion. How can one determine these coupling-constants? I will show two different approaches. The first approach is based on the consideration of scalar form factors i.e. the matrix elements of the scalar currents $\bar{u}u + \bar{d}d$ and $\bar{s}s$. For instance, between pion states

\[
\langle \pi^i(p) | \bar{u}u + \bar{d}d | \pi^j(p') \rangle = \delta^{ij} F_\pi(t) , \quad t = (p - p')^2
\]

Contrary to the case of the vector and axial-vector currents there are no physical sources of scalar currents available, so the matrix elements above are not directly measurable. However, it was argued in ref. [8] that these matrix elements can be constructed, using dispersion relation methods in an energy region where two-channel unitarity holds to a good approximation (i.e. $\sqrt{t} < 1.3$ GeV). In this energy region, the imaginary part of the form-factors are given in terms of the $\pi\pi$ and the $K\bar{K}$ scattering T-matrix elements and the form-factors themselves,

\[
\text{Im} \left( \frac{F^\pi}{F^K} \right) = \begin{pmatrix} T_{11}^* & T_{12}^* \\ T_{21}^* & T_{22}^* \end{pmatrix} \begin{pmatrix} \theta(t - 4m_{\pi}^2)\sqrt{1 - \frac{4m_{\pi}^2}{t}} & 0 \\ 0 & \theta(t - 4m_{K}^2)\sqrt{1 - \frac{4m_{K}^2}{t}} \end{pmatrix} \begin{pmatrix} F^\pi \\ F^K \end{pmatrix} .
\]

Using these relations, the dispersion relations for the form-factors take the form of a set of coupled Muskhelishvili-Omnès equations [9]. The construction is then based on the assumption that one can continue using the unitarity relations (13) in the high energy regions of the dispersion integrals (where they are no longer valid) without affecting the output at sufficiently low energies, and the related assumption to impose asymptotic conditions on the T-matrix which insure a unique solution of the the set of Muskhelishvili-Omnès integral equations, given initial value conditions $F^\pi(0), F^K(0)$. These values are given by ChPT at leading order. Since these assumptions are involved, it will be useful to perform some consistency checks and compare with other sources of information (further checks are provided by application of this method to the strangeness $S = 1$ scalar currents, see the recent work of ref. [10]). One must also note that the T-matrix elements $T_{ij}$ needed in this construction can be taken from experiment, but $T_{12}$ is also needed in an energy region below the $K\bar{K}$ threshold where it cannot be measured. Dispersion relation techniques allow one to extrapolate $T_{12}$ down to this region in a reliable way, this was redone recently in ref. [3] where references to earlier work can be found (in refs. [1] [8] this extrapolation was based on models). In the results shown below we have also made use of the recent Roy equation solutions of ref. [11] which constrain the $\pi\pi$ phase-shifts below 0.8 GeV.

FIG. 1. Absolute value of the pion form-factor of the $\bar{u}u + \bar{d}d$ current (divided by $B_0$ which makes it scale independent and dimensionless) from the dispersive construction described in the text.
Results for the pion form-factors based on this construction are shown in figs.1,2 which correspond to the $\bar{u}u + \bar{d}d$ and the $\bar{s}s$ current respectively. In the first case, one observes a wide but neat bump which one can associate with a $\sigma(500)$ resonance (the shape here is determined by the use of the Roy equation solutions) while the $f_0(980)$ resonance appears as a dip rather than a bump. The form-factor associated with $\bar{s}s$ has a very different shape displaying only a narrow peak corresponding to the $f_0(980)$.

![Figure 2: Absolute value of the pion form-factor of the $\bar{s}s$ current](image)

Let us now discuss how the informations we are looking for concerning the coupling constants $L_4$ and $L_6$ can be extracted from these form-factors. Consider, first, the derivatives at the origin of two pion form-factors. These derivatives can be expressed as an expansion using ChPT. For the $\bar{u}u + \bar{d}d$ form-factor, one has

$$
\dot{F}_\pi^u(0) = \frac{4}{F_\pi^2} \left\{ 2L_4(\mu) + L_5(\mu) - \frac{1}{64\pi^2}(L_\pi + \frac{1}{4}L_K + \frac{4}{3}) + O(m_s) \right\}
$$

(14)

with

$$
L_P = \log \frac{m_\rho^2}{\mu^2}.
$$

(15)

Using the known value of the coupling $L_5$ from ref. [7], $L_5(m_\rho) = (1.4 \pm 0.5) \times 10^{-3}$ we obtain the following central value for $L_4$,

$$
L_4(m_\rho) \simeq 0.20 \times 10^{-3}, \quad \text{implying} \quad \frac{(F_\pi^u)_3}{(F_\pi^u)_2} \simeq 1 - 0.12.
$$

(16)

Let us consider now the derivative of the $\bar{s}s$ form-factor. In this case, ChPT gives,

$$
\dot{F}_\pi^s(0) = \frac{8}{F_\pi^2} \left\{ L_4(\mu) - \frac{1}{256\pi^2}(L_K + 1) + O(m_s) \right\}.
$$

(17)

From this expression we obtain the following central value for $L_4$,

$$
L_4(m_\rho) \simeq 0.30 \times 10^{-3}, \quad \text{implying} \quad \frac{(F_\pi^s)_3}{(F_\pi^s)_2} \simeq 1 - 0.15.
$$

(18)
It is reassuring that these two determinations based on the two very different looking form-factors turn out to be extremely close, a further determination will be mentioned later. Also the result does correspond to an expected decrease of $F_\pi$ as one increases the number of chiral quarks. A discussion of the various sources of errors and the role of higher chiral orders can be found in ref. [1].

In order now to access the second coupling constant of interest, $L_6$, one may consider the following correlation function,

$$\Pi_6(t) = i \int d^4x e^{ipx} \langle T[\bar{u}u(x) + \bar{d}d(x)]\bar{s}s(0)\rangle, \quad t = p^2.\quad (19)$$

The information of interest is contained, as we will see, in the value of $\Pi_6(0)$ which is given by the spectral integral,

$$\Pi_6(0) = \frac{1}{\pi} \int_0^\infty dt \frac{Im \Pi_6(t)}{t}.\quad (20)$$

In order to compute this integral we observe, first, that in the chiral $m_u = m_d = 0$ limit (which is very close to the real world) the spectral function is constrained by a Weinberg-type superconvergence relation,

$$\int_0^\infty dt Im \Pi(t) = 0.\quad (21)$$

Next, inserting a complete set of states we can express the spectral function in the following form,

$$16\pi Im \Pi_6(t) = \theta(t - 4m_\pi^2)\sqrt{1 - \frac{4m_\pi^2}{t}} F_\pi^u(t)F_\pi^* (t)$$

$$+ \theta(t - 4m_K^2)\sqrt{1 - \frac{4m_K^2}{t}} F_K^u(t)F_K^* (t) + \rho_\pi(t) + \rho_{\pi K}(t) + ...\quad (22)$$

The important point here is that the contributions to the spectral function from intermediate states with four or more pseudo-scalars are completely negligible below 1 GeV$^2$ so that in this region the spectral function is given from the pion and Kaon scalar form-factors discussed above. Including only these contributions we obtain the spectral function shown in fig.3. While this spectral function is no longer quantitatively reliable above 1 GeV, it does change sign as one expects from the superconvergence relation[21].

![Graph](https://via.placeholder.com/150)

FIG. 3. Plot of $16\pi Im \Pi_6(E^2)$ (divided by $B_0^2$ to make it scale independent and dimensionless) including the $\pi\pi$ and $K\bar{K}$ contributions.
The spectral integral in eq. (21) is then computed by splitting the integral into two regions: below one GeV$^2$, which we expect to be the dominant region, we use the construction based on two-channel unitarity and in the higher energy range we use a simple Breit–Wigner parametrisation, using the known position of the resonances and fixing the normalisation from eq. (21). Having obtained $\Pi_6(0)$ allows us to determine the coupling-constant $L_6$ using the ChPT expansion expression,

$$\Pi_6(0) = 64 L_6(\mu) - \frac{1}{16 \pi^2} (2 L_K + \frac{4}{9} L_\eta + \frac{22}{9}) + O(m_s).$$

This yields the following central value for $L_6$,

$$L_6(m_\rho) \simeq 0.30 \times 10^{-3}$$

which implies the following behaviour of the quark condensate,

$$\frac{\langle \bar{u}u \rangle_3}{\langle \bar{u}u \rangle_2} \simeq 1 - 0.45.$$

One observes a rather large decrease of this order parameter. Errors and two-loop chiral corrections are discussed in ref. [1]. If we ignore the errors, what is the interpretation of the finding that the condensate decreases much faster than $F_\pi$? One can possibly argue that the condensate has a dimension of $[mass]^3$ and it could behave as $F_\pi^3$, this would be compatible with our results. Another possibility, is that increasing $N_C^0$ one reaches a phase of “weak” chiral symmetry breaking which has $\langle \bar{u}u \rangle = 0$ while $F_\pi \neq 0$. Existence of such a phase was proposed by Stern [12] (see, however, ref. [13]).

The results above on $L_4$ and $L_6$ have been derived using scalar form-factors. Since assumptions are involved in the construction of these objects, it is interesting that $L_4$ can be determined from a different method. We only sketch the idea here, details may be found in refs. [3]. The starting point is the pion-Kaon scattering amplitude. This amplitude was computed in ChPT at one-loop [14] and its expression involves seven coupling constants $L_i$, $i = 1...8 (i \neq 7)$. One then matches this expression with a dispersive representation of the amplitude in which one makes use of crossing symmetry and Regge asymptotic constraints. A number of sum rules are then obtained for the $L_i$’s one of which concerns $L_4$ and has the following form,

$$L_4(\mu) + \frac{1}{512 \pi^2} \left( -2 L_\pi + \frac{5}{4} R_{\pi K} + \frac{1}{4} R_{\eta K} + \frac{m_\pi^2}{2 (m_\pi^2 + m_K^2)} \log \frac{m_\pi^2}{m_\eta^2} - \frac{7}{2} \right) =$$

$$\frac{2 F_\pi^2}{3 (m_\pi^2 + m_K^2)} \left[ \sqrt{3} \int_{4 m^2}^{\infty} \frac{dt}{t^2} (1 + \frac{2 (m_\pi^2 + m_K^2)}{t}) \text{Im} g_0^0(t) \right.$$
on a construction of scalar form-factors are confirmed by sum rules based on the $\pi K$ amplitude. Such calculations can also be compared with lattice QCD formulations once such calculations with properly incorporated fermionic determinant become available. In both cases a decrease was found, in agreement with expectation from functional integral expressions. This aspect of the QCD vacuum was shown to be related to the properties of the scalar mesons. The fact that large $N_c$ expectations appear violated in this sector could possibly reflect the fact that the critical number of massless fermions that the chiral vacuum can sustain is not much larger than three, the number of light quarks available in nature.

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