NON-CRITICAL SUPERSTRINGS: A COMPARISON BETWEEN CONTINUUM AND DISCRETE APPROACHES

A. ZADRA
Instituto de Física, University of São Paulo
CP 20516, São Paulo, Brazil

E. ABDALLA†
CERN-TH, CH-1211 Geneva 23, Switzerland

Abstract

We review the relation between the matrix model and Liouville approaches to two-dimensional gravity as elaborated by Moore, Seiberg and Staudacher. Then, based on the supersymmetric Liouville formulation and the discrete eigenvalue model proposed by Alvarez-Gaumé, Itoyama, Mañes and Zadra, we extend the previous relation to the supersymmetric case. The minisuperspace approximation for the supersymmetric case is formulated, and the corresponding wave equation is found.
1. Introduction

It is generally accepted that the $m$-th multicritical point of the Hermitian 1-matrix model corresponds to the coupling of the Liouville theory to perturbations of the $(2, 2m - 1)$ minimal conformal model. However it took some time to understand this relation. A precise dictionary between Liouville and matrix model operators, correlation functions and loop amplitudes was proposed by Moore, Seiberg and Staudacher in Ref. [1]. They recognized how part of the difficulties were related to contact terms. After a careful examination of macroscopic loop amplitudes in two-dimensional gravity, which must satisfy the Wheeler-de Witt (WdW) equation, they managed to determine a new frame of scaling operators and couplings (called conformal frame) in the matrix theory, whose correlation functions and scaling dimensions were in perfect agreement with the Liouville predictions.

This successful program motivated us to generalize their computations for the supersymmetric case. We considered the amplitudes of the $N = 1$ super-Liouville theory coupled to $c < 3/2$ matter, which were calculated in Ref. [2]. As for the discrete counterpart, there is no actual generalized matrix model. So far the most successful description has been given by the discrete model introduced by Alvarez-Gaumé, Itoyama, Mañes and Zadra in Ref. [3] in terms of super-eigenvalue variables. Nevertheless we shall prove that, in spite of the missing interpretation in terms of matrices, this discrete theory also shows a complete agreement with continuum results.

We dedicate sect. 2 to a brief review of the purely bosonic case, emphasizing the main results from the continuum approach in subsect. 2.1 and describing in subsect. 2.2 the discrete theory and the procedure that determines the conformal frame of operators. In sect. 3 we present the supersymmetric version of the models: the super Liouville theory is summarized in subsect. 3.1; the super-eigenvalue model, its standard scaling operators and their correlation functions are reviewed in subsect. 3.2; then, in subsect. 3.3 we follow the strategies of the bosonic case to define a superconformal frame; their wave functions are studied in subsect. 3.4. We leave sect. 4 for the final comments and a brief conclusion.

2. Review of the purely bosonic case

2.1 Continuum approach

Let us take the David-Distler-Kawai (DDK) description of $c < 1$ conformal theories coupled to 2d gravity, where the partition function $Z$ on the Euclidian sphere is defined by the following integral over “matter” $(X)$ and Liouville $(\phi)$ fields,

$$Z = \int \frac{\mathcal{D}  \hat{g}  \mathcal{D}  \phi}{V_{SL(2, C)}} e^{-S},$$

$$S = \frac{1}{8\pi} \int d^2 w \sqrt{\hat{g}} \left[ \hat{g}^{ab} \partial_a \phi \partial_b  \phi + \hat{g}^{ab} \partial_a X \partial_b X - Q \hat{R} \phi + 2i \alpha_0 \hat{R} X + 8\pi \bar{\mu} e^{\alpha \phi} \right].$$

$\hat{g}_{ab}$ as in (2) is the fiducial metric fixed by the conformal gauge, $\hat{R}$ is the respective scalar curvature and $V_{SL(2, C)}$ is the volume of the residual $SL(2, C)$ symmetry (a
residual symmetry of the conformal gauge). The parameter $\alpha_0$ and the background charge $Q$ are related to the matter central charge $c$ by the relations

$$c = 1 - 12\alpha_0^2, \quad Q = 2\sqrt{2 + \alpha_0^2} = \sqrt{\frac{25 - c}{3}}. \quad (3)$$

The last term in Eq. (2), $\bar{\mu} \int d^2w \sqrt{\hat{g}} e^{\alpha \phi}$, measures the area of the spherical surface and is called the cosmological term. The coupling $\bar{\mu}$ is a bare cosmological constant and the parameter $\alpha$ is chosen in such a way that $e^{\alpha \phi}$ has conformal dimension one, $\Delta(e^{\alpha \phi}) = -\frac{1}{2}\alpha(\alpha + Q) = 1$. \quad (4)

The above equation has two solutions, namely $\alpha_\pm = -Q/2 \pm |\alpha_0|$. We choose $\alpha = \alpha_+ \geq -Q/2$, which exhibits the correct classical limit and also satisfies Seiberg’s energy bound\(^5\) ($E(\alpha) = \alpha + Q/2 \geq 0$).

As stated in the Introduction, we are interested in the scaling behaviour of correlation functions of vertex operators. In the continuum approach we take vertex operators “dressed” by gravity, $V_i = \int d^2z \sqrt{\hat{g}} \mathcal{O}_i e^{\beta_i \phi}$, where $\mathcal{O}_i$ is the matter field or vertex and the dressing factor $e^{\beta_i \phi}$ is determined by imposing that $\mathcal{O}_i e^{\beta_i \phi}$ has conformal dimension 1,

$$\Delta(\mathcal{O}_i e^{\beta_i \phi}) = h_i - \frac{1}{2} \beta_i(\beta_i + Q) = 1. \quad (5)$$

Above, $h_i$ is the conformal dimension of $\mathcal{O}_i$. In particular, for a tachyon $\mathcal{O}_i = e^{ik_iX}$, one finds $h_i = \frac{1}{2} k_i(k_i - 2\alpha_0)$ and therefore its dressing charge reads $\beta_i(k_i) = -Q/2 + |k_i - \alpha_0|$. Notice the degeneracy in this representation: the momenta $k_i$ and $2\alpha_0 - k_i$ imply the same conformal dimension and dressing. Therefore we may take either

$$T_k = \int d^2z \sqrt{\hat{g}} e^{ikX + \beta(k)\phi} \quad \text{or} \quad T_{2\alpha_0 - k} = \int d^2z \sqrt{\hat{g}} e^{(2\alpha_0 - k)X + \beta(k)\phi} \quad (6)$$

to represent the same dressed operator, choosing one or the other according to kinematic criteria. We call the above vertex ($T_k$) the tachyon, and $k$ is its momentum. In particular the cosmological term or area operator $T_0 = \int d^2z \sqrt{\hat{g}} e^{\alpha \phi}$ previously mentioned may be as well represented by $T_{2\alpha_0} = \int d^2z \sqrt{\hat{g}} e^{2\alpha_0 X + \alpha \phi}$.

An $N$-tachyon correlation function $\mathcal{A}_N \equiv \langle T_{k_1} \cdots T_{k_N} \rangle$ is calculated, using Coulomb gas techniques. We choose the fiducial metric $\hat{g}_{\alpha \beta} = \delta_{\alpha \beta}$. Integrating over the matter field $X$ we obtain the momentum-conservation law

$$\sum_{i=1}^N k_i = 2\alpha_0 \quad , \quad (7)$$

which must be satisfied by every non-vanishing correlation function.

The gravitational contribution is more difficult to deal with, due to the exponential self-interaction in the Liouville action. At this point we turn to the zero-mode integration procedure\(^6\): defining $\phi = \dot{\phi} + \phi_0$, where $\phi_0$ is the Liouville
zero mode, we can explicitly integrate over $\phi_0$ and use the free field propagator
\[ \langle e^{\beta \tilde{\phi}(z)} e^{\gamma \tilde{\phi}(w)} \rangle = (z-w)^{-\beta \gamma} \]
for the remaining field $\tilde{\phi}$.

After renormalizing the cosmological constant ($\bar{\mu} \to \mu$), we find the $N$-point function
\[ A_N = \frac{\partial^{N-3}}{\partial \mu^{N-3}} \mu^{N+s-3} \prod_{j=1}^{N} \Delta \left( \frac{1}{2} (\beta_j^2 - k_j^2) \right) \sim \mu^s , \]  
where $\Delta(x) = \frac{\Gamma(1-x)}{\Gamma(x)}$, and $s = -\frac{1}{\alpha_+} \left( Q + \sum_{i=1}^{N} \beta_i \right)$.

After writing Eq. (8) we analytically continue the results, originally derived for $s \in \mathbb{Z}$, to $s \in \mathbb{C}$. Moreover we constrain the kinematics choosing $N-1$ values of $k_i$ to be bigger than $\alpha_0$ and one smaller (for a thorough discussion on the subject see Refs. [7,10]). Notice that $A_N$ factorizes in terms of external leg contributions and its scaling behaviour with respect to the cosmological constant $\mu$, $A_N \sim \mu^s = \mu^{-Q/\alpha_+ - \sum_{i=1}^{N} \beta_i / \alpha_+}$, provides the scaling dimension of the dressed vertex operators
\[ T_k = \int d^2 z \ e^{ikX + \beta(k)\phi} \to \mu^{-\beta(k)/\alpha_+} . \]  

The scaling of the partition function can also be taken from Eq. (8) and defines the so-called string susceptibility $\gamma$ as
\[ Z \sim A_0 \sim \mu^{-Q/\alpha_+} = \mu^{2-\gamma} , \]
i.e. $\gamma = 2 + Q/\alpha_+$.

From the momentum conservation (7) we conclude that there is only one tachyonic operator, with momentum $k = 2\alpha_0$ and dressing charge $\beta = \alpha_+$ (i.e. the area operator), whose expectation value is non-vanishing,
\[ \langle T_{2\alpha_0} \rangle \sim \mu^{-Q/\alpha_+ - 1} . \]  

The remaining 1-tachyon functions are expected to vanish.

The conservation law (7) also implies that $T_{k_1}$ and $T_{2\alpha_0 - k_1}$ represent the same operator, and we conclude that 2-point functions are diagonal and scale as $\mu^{-Q/\alpha_+ + 2\beta(k)/\alpha_+}$.

In Ref. [1] the authors used the wave functions of dressed vertex operators to establish the correct Liouville/matrix model dictionary. The starting point was the WdW equation\(^1,5\):
\[ [H - \Delta_O] \Psi_l(O) = 0 \]  
where the wave function $\Psi_l(O)$ corresponds to the insertion of a dressed operator $O$ on a surface of boundary $l$; $\Delta_O = 1 - h$ is the conformal weight of $O$; $H$ is the Liouville Hamiltonian
\[ H = \frac{1}{2} (\phi' + 4\pi P)^2 + Q(\phi' + 4\pi P)' + \frac{\mu}{2\alpha_+^2} e^{\alpha_+ \phi} + \frac{Q^2}{8} . \]  

\[ (13a) \]
In the minisuperspace approximation one ignores the space dependence and drops derivative terms in $H$, which then reads

$$H \equiv \frac{1}{2} p^2 + \frac{\mu}{2 \alpha_+} e^{\alpha_+ \phi} + \frac{Q^2}{8}, \quad (13b)$$

with $p = -i \partial / \partial \phi$. Moreover, since the boundary length $l$ is measured by the contour integral $l = \oint d\xi \  e^{\frac{1}{2} \alpha_+ \phi}$, we can associate

$$\frac{\partial}{\partial \phi} \leftrightarrow \frac{\alpha_+}{2} \left( i \frac{\partial}{\partial l} \right), \quad \frac{\mu}{2 \alpha_+} e^{\alpha_+ \phi} \leftrightarrow \frac{\alpha_+^2}{8} \mu l^2 \quad (14)$$

in which case Eq. (12) becomes the Bessel differential equation

$$\left[- \left( i \frac{\partial}{\partial l} \right)^2 + \mu l^2 + \nu^2 \right] \Psi_l(\mathcal{O}) = 0 \quad , \quad (15)$$

where $\nu^2 = \frac{Q^2 - 8 \Delta}{\alpha_+} = \left( \frac{Q}{\alpha_+} + 2 \frac{\beta}{\alpha_+} \right)^2$.

Since the wave functions $\Psi_l(\mathcal{O})$ are expected to decay in the infrared (large $l$) limit, we take the modified Bessel functions of second kind $\Psi_l(\mathcal{O}) \propto K_\nu(\sqrt{\mu l})$ as appropriate solutions. This result will be taken as a guide to unravel some tangles in the continuum/discrete translation.

2.2 Discrete (matrix model) approach

The Hermitian 1-matrix model is defined by the partition function

$$Z = \int \mathcal{D} \phi \ e^{-N \text{tr} V(\phi)}, \quad \phi^\dagger = \phi \quad , \quad (16)$$

$$\mathcal{D} \phi = \prod_i \mathcal{D} \phi_i \prod_{i<j} \mathcal{D} (\Re \phi_{ij}) \mathcal{D} (\Im \phi_{ij}) \ , \quad V(\phi) = \sum_k g_k \phi^k$$

One can generate different critical regimes by tuning the coupling constants $g_k$. The potentials found by Kazakov in Ref. [8] exemplify this property. In the $m$-th critical regime the continuum limit of the model is defined by a double scaling limit, where the matrix size gets large ($N \rightarrow \infty$) and the constant $\Lambda$ approaches a critical value ($\Lambda \rightarrow \Lambda_c$), while the combination $N(\Lambda_c - \Lambda)^{1+1/2m}$ is kept constant. The resulting theory is described by the free energy

$$\mathcal{F} = -\frac{1}{\kappa^2} \partial t_n^{-2} u(t_n) \quad , \quad (17)$$

where $\kappa$ is the renormalized string coupling and $u$ is the specific heat satisfying the string equation

$$-t_0 = \sum_{n=1} u n t_n R_n[u] \quad ; \quad (18)$$
$R_n[u]$ are the Gel'fand-Dikii polynomials. The dependence on the renormalized couplings $t_n$ is ruled by the KdV flows

$$\frac{\partial}{\partial t_n} u = \frac{\partial}{\partial t} R_{n+1}[u] .$$

The exact $m$-th critical regime is defined by the limit

$$t_0 \to t \ ; \ t_m \to -1 \ ; \ t_n \to 0 , \ n \neq 0 , m ,$$

in which case the string equation becomes $mR_m[u] = t$. Within the planar or spherical approximation, given by the limit $\kappa \to 0$, the Gel'fand-Dikii polynomials tend to $R_n[u] = u^n/n$, and the string equation (18) becomes

$$\sum_{n \geq 0} t_n u^n = 0 ,$$

which in the $m$-th critical regime (20) implies $u^m = t$.

We associate a set of scaling operators $\sigma_n$ to the couplings $t_n$, i.e. $\sigma_n \leftrightarrow \frac{\partial}{\partial t_n}$. This frame of operators and couplings was called the KdV frame by the authors in Ref. [1]. Correlation functions are described by the KdV flows, with the following results in the planar approximation

$$\left\langle \prod_{i=1}^n \sigma_{a_i} \right\rangle = -\frac{1}{\kappa^2} \partial_t^{n-2} \frac{u^{a+1}}{a+1} , \ a = \sum_{i=1}^n a_i .$$

In the regime (20) we observe the scaling behaviour of the free energy as given by $F \sim t^{2+1/m} = t^{2-\gamma}$, which defines the susceptibility $\gamma_m = -1/m$. On the other hand the $n$-point functions scale as $t^{2+1/m+\sum(a_i/m-1)}$, which therefore defines the scaling dimension $d_n = n/m$ of the operator $\sigma_n$. Notice that neither any one of the 1- nor of the 2-point functions vanish, as opposed to the continuum results. Such differences obstruct a direct association between vertex operators $T_k$ and the KdV frame of scaling operators $\sigma_n$.

These difficulties can also be felt as one studies the wave functions of scaling operators. We start defining macroscopic loop operators $W(l)$ of finite length $l$, whose expectation value reads

$$\langle W(l) \rangle = \frac{1}{\kappa \sqrt{\pi l}} \int_{t_0}^\infty dy \ e^{-lu(y;t_n)} .$$

We suggest Ref. [8] for a review on the subject. Notice that we can actually integrate Eq. (23) by using the planar string equation (21), finding as a result

$$\langle W(l) \rangle = \frac{1}{\kappa \sqrt{\pi}} \sum_{n \geq 0} t_n u^{n+1/2} \psi_n(lu) ,$$
The functions $\psi_n(lu)$ display the dependence on the loop length $l$. They are proportional to the wave functions associated to the insertion of scaling operators, as follows

$$\Psi_l(\sigma_n) = \langle \sigma_n W(l) \rangle = \frac{\partial}{\partial t_n} \langle W(l) \rangle \propto \psi_n(lu)$$

i.e. Eq. (24) can be taken as an expansion of the loops in terms of wave functions. However we see that $\psi_n(lu)$ does not obey a Bessel equation, which reinforces the differences between the KdV basis and the vertex operators.

In Ref. [1] the authors resolved these discrepancies defining a new frame of scaling operators $\hat{\sigma}_n$ and the respective couplings $\hat{t}_n$, which they called the conformal field theory frame. They assumed that the differences between the observed correlation functions should arise from contact terms. On the other hand a change of contact terms is equivalent to an analytic redefinition of coupling constants. Thus one should look for the proper set of couplings.

In order to change coupling constants and yet preserve scaling properties we must examine their dimensions. From the way the loop length $l$ comes out in Eq. (23), we take the dimension of the specific heat, $[u] = [\text{length}]^{-1}$. Thus the dimension of the couplings follow from the string equation (21),

$$[t_n] = [\text{length}]^{n-m}$$

The “physical” cosmological constant is usually taken to be the one coupled to the area operator and should therefore have the dimension of inverse of area or $[\text{length}]^{-2}$. Since $[t] = [t_0] = [\text{length}]^{-m}$ we conclude that $t$ deserves the name of cosmological constant only when $m = 2$. In general $t_{m-2}$ is the coupling with the proper dimension of a cosmological constant. If one wishes to compare results with the Liouville approach one should select another regime, different from the one defined by (20), where scaling properties are measured with respect to some coupling as $t_{m-2}$.

We can determine the conformal frame of couplings according to the following criteria. Consider an analytical transformation relating the KdV frame $\{t_n\}$ to another frame of couplings $\{\hat{t}_n\}$, with the general form

$$t_n = C^a_n \hat{t}_i + C^{ij}_{n} \hat{t}_i \hat{t}_j + \cdots$$

In order to maintain scaling properties (the operators $\hat{\sigma}_n$ should be scaling fields as $\sigma_n$ already were), let us impose that the analytic transformation preserves the dimension of couplings, $[\hat{t}_n] = [t_n]$. In analogy to the limit (20) we shall define another $m$-th critical regime in terms of the new set of couplings,

$$\hat{t}_{m-2} \rightarrow \mu \quad ; \quad \hat{t}_m \rightarrow -1 \quad ;$$
$$\hat{t}_n \rightarrow 0 \quad , \quad n \neq m-2, m$$
where \( \mu \) is the “physical” cosmological constant. The solution of the string equation will become \( u^2 = \mu \) and correlation functions will scale as functions of \( \mu \). The operator \( \hat{\sigma}_{m-2} \) is expected to correspond to the area or cosmological term.

Our immediate concern is to have 1- and 2-point functions compatible with the Liouville predictions. In this case it is sufficient\(^1\) to consider lowest (linear)-order terms in the transformation (28) around the critical point (29). Thus it is convenient to shift the non-vanishing couplings as \( \hat{t}_{m-2} \to \mu + \hat{t}_{m-2}, \hat{t}_m \to -1 + \hat{t}_m \), so that the critical regime is achieved by the limit \( \hat{t}_n \to 0 \) and we can deal with the \( \hat{t}_n \)’s as perturbative couplings.

Considering these scaling restrictions and lowest-order approximations, the transformation (28) is reduced to the form

\[
\hat{t}_n = b_n \mu^{(m-n)/2} + \sum_{s=0}^{\infty} a_s^{(n+2s)} \mu^s \hat{t}_{n+2s},
\]

where \( b_n \) and \( a_s^{(n)} \) are dimensionless coefficients to be determined. However, the coefficients \( b_n \) are not independent: they are given by

\[
b_{m-2i} = \frac{1}{i} a_{i-1}^{(m-2)}; \quad b_n = 0, \quad n > m.
\]

From Eq. (30) we derive \( \frac{\partial}{\partial \hat{t}_n} = \sum_{s=0}^{[n/2]} a_s^{(n)} \frac{\partial}{\partial \hat{t}_{n-2s}} \), which gives the transformation of scaling operators

\[
\hat{\sigma}_n = [n/2] \sum_{s=0}^{[n/2]} a_s^{(n)} u^{2s} \sigma_{n-2s}.
\]

Therefore the coefficients \( a_s^{(n)} \) uniquely characterize a given basis. In order to find the conformal basis we can adopt two strategies, described below.

2.2.a. Minisuperspace approximation strategy

In this case we use the wave functions as a guide\(^1\) to determine the coefficients \( a_s^{(n)} \). From the Liouville description (see Eq. (15)) we expect to find Bessel functions. We take the expansion (24) for the 1-loop function, which holds in any regime, and substitute the transformation (30). Recalling that \( u^2 \to \mu \) we find

\[
\langle W(l) \rangle = \frac{1}{\kappa \sqrt{\pi}} u^{m+1/2} \sum_{s=0}^{[m/2]} b_{m-2s} \psi_{m-2s}(lu) + \frac{1}{\kappa \sqrt{\pi}} \sum_{n \geq 0} \hat{t}_n u^{n+1/2} \left( \sum_{s=0}^{[n/2]} a_s^{(n)} \psi_{n-2s}(lu) \right).
\]

Indeed we can obtain Bessel functions using

\[
K_{n+1/2}(x) = \sqrt{\frac{\pi}{2}} \sum_{s=0}^{[n/2]} a_s^{(n)} \psi_{n-2s}(x),
\]
with
\[ a_s^{(i)} = \frac{(-1)^s}{2^i s!} \frac{(2i - 2s)!}{s!(i - s)!(i - 2s)!} . \]  

These are the coefficients we were looking for. Using Eq. (31) we also find the corresponding expression for the remaining coefficients
\[ b_{m-2s} = \frac{a_{s-1}^{(m-2)} - a_s^{(m)}}{(m - 1/2)} , \quad b_m = -\frac{a_0^{(m)}}{(m - 1/2)} . \]  

Therefore the 1-loop function becomes an expansion in Bessel functions
\[ \langle W(l) \rangle = \frac{\sqrt{2}}{\kappa \pi} \frac{u^{m+1/2}}{(m - 1/2)} \left[ K_{m-3/2}(lu) - K_{m+1/2}(lu) \right] 
+ \frac{\sqrt{2}}{\kappa \pi} \sum_{n \geq 0} \hat{t}_n u^{n+1/2} K_{n+1/2}(lu) , \]  

from which we obtain the wave functions
\[ \Psi_l(\hat{\sigma}_n) = \langle \hat{\sigma}_n W(l) \rangle = \frac{\sqrt{2}}{\kappa \pi} u^{n+1/2} K_{n+1/2}(lu) , \]  

which obey the Bessel equation (15) for \( \nu = n + 1/2 \). Notice also that in the exact regime (29) we have
\[ \langle W(l) \rangle = \frac{\sqrt{2}}{\kappa \pi} \frac{u^{m+1/2}}{(m - 1/2)} \left[ K_{m-3/2}(lu) - K_{m+1/2}(lu) \right] 
= -\frac{2 \sqrt{2}}{l \kappa \pi} u^{m-1/2} K_{m-1/2}(lu) = -\frac{2}{l} \langle \hat{\sigma}_{m-1} W(l) \rangle , \]  

i.e. \( \langle \hat{\sigma}_{m-1} W(l) \rangle = -\frac{l}{2} \langle W(l) \rangle \), indicating\(^1\) that \( \hat{\sigma}_{m-1} \) is the boundary operator which measures the loop length \( l \).

We remark that the coefficients (35) of the conformal basis have an interesting interpretation in terms of orthogonal polynomials: indeed from Eqs. (25) and (34) we can write
\[ K_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}} \int_x^\infty \, dz \, P_n \left( \frac{z}{x} \right) e^{-z} , \]  

where \( P_n(x) = \sum_{s=0}^{[n/2]} a_s^{(n)} x^{n-2s} \) are the Legendre polynomials. We shall understand the appearance of this family of polynomials in the following subsection.
2.2.b Orthogonalization strategy

The coefficients \( a_s^{(n)} \) can also be determined by imposing that the 2-point functions \( \langle \hat{\sigma}_i \hat{\sigma}_j \rangle \) be diagonal. In fact, from the general transformation (32) and the 2-point functions of the KdV frame \( \langle \sigma_i \sigma_j \rangle \), which follow from Eq. (14), in the regime \( u^2 \to \mu \), we find

\[
\langle \hat{\sigma}_i \hat{\sigma}_j \rangle = -\frac{\mu (i+j+1)/2}{\kappa} g_{ij}(a),
\]

where

\[
g_{ij}(a) = \sum_{s=0}^{[i/2]} \sum_{r=0}^{[j/2]} \frac{a_s^{(i)} a_r^{(i)}}{(i + j - 2s - 2r + 1)}
\]

(42)

can be seen as a characteristic metric of the basis defined by the coefficients \( a_s^{(n)} \). Our aim is therefore to choose \( a_s^{(n)} \) so that \( g_{ij} \) becomes diagonal. Observe that \( g_{ij} \) satisfies

\[
[1 - (-1)^{i+j+1}] g_{ij}(a) = \int_{-1}^{+1} dx P_i(x) P_j(x), \quad P_i(x) = \sum_{s=0}^{[i/2]} a_s^{(i)} x^{i-2s},
\]

(43)

where \( P_i(x) = \sum_{s=0}^{[i/2]} a_s^{(i)} x^{i-2s} \) are characteristic polynomials of a given basis. Thus our problem has been translated into finding, among the characteristic polynomials, those which are orthogonal with respect to the internal product \( \langle P_i, P_j \rangle = \int_{-1}^{+1} dx P_i(x) P_j(x) \). The well-known solution is given by the Legendre polynomials, whose coefficients are precisely those defined in (35). Therefore the 2-point functions in the conformal frame read

\[
\langle \hat{\sigma}_i \hat{\sigma}_j \rangle = \begin{cases} 
-\frac{1}{\kappa^2} \mu^{i+1/2} \delta_{ij}, & \text{if } i + j \text{ even} \\
\text{analytical in } \mu, & \text{if } i + j \text{ odd}
\end{cases},
\]

(44)

i.e. they are diagonal up to analytical terms in the coupling constants, as expected from the Liouville approach.

Following any of the strategies we conclude that the conformal frame of operators \( \hat{\sigma}_n \) is related to the KdV frame \( \sigma_n \) by

\[
\hat{\sigma}_n = \sum_{s=0}^{[n/2]} \frac{(-1)^s}{2^n} \frac{(2n - 2s)!}{s!(n-s)!(n-2s)!} \mu^s \sigma_{n-2s},
\]

(45a)

which can also be inverted

\[
\sigma_n = \sum_{s=0}^{[n/2]} \sqrt{\pi n}! \frac{(2n - 4s + 1)}{2^n} \frac{(2n - 4s - 1)!}{s!(n-s+3/2)!} \mu^s \hat{\sigma}_{n-2s}.
\]

(45b)
We must also test the 1-point functions. From Eqs. (44) and (45b) we can calculate \( \frac{\partial}{\partial \mu} \langle \sigma_n \rangle = \langle \tilde{\sigma}_{m-2}\sigma_n \rangle \) which, after integration, gives

\[
\langle \sigma_{m-2+2s} \rangle = -\sqrt{\pi} \frac{\mu^{m+k-1/2}}{\kappa^2} \frac{(m-2+2k)!}{2^{m-2+2k}k!\Gamma(m+k+1/2)} .
\]  

Now we can take Eq. (45a) and easily calculate the 1-point functions \( \langle \hat{\sigma}_n \rangle \). It turns out that the only non-vanishing cases are

\[
\langle \hat{\sigma}_{m-2} \rangle = -\frac{4}{\kappa^2} \frac{\mu^{m-1/2}}{(2m-1)(2m-3)} , \quad \langle \hat{\sigma}_m \rangle = \frac{4}{\kappa^2} \frac{\mu^{m+1/2}}{(2m+1)(2m-1)} .
\]

This is also in agreement with the Liouville approach: \( \langle \hat{\sigma}_{m-2} \rangle \) is the average area, as in Eq. (11); on the other hand \( \langle \hat{\sigma}_m \rangle \) can be interpreted as a non-tachyonic part of the average energy. Since \( \langle \hat{\sigma}_{m-2} \rangle = \frac{\partial}{\partial \mu} F \), we can integrate Eq. (47a) and obtain the scaling behaviour of the free energy in this regime: \( F \sim \mu^{m+1/2} \).

Higher-order correlation functions would require the analysis of higher-order terms in the transformation (30). Nevertheless we know that, within the regime (29), the \( N \)-point functions must scale as

\[
\langle \prod_{i=1}^{N} \hat{\sigma}_{n_i} \rangle \sim \mu^{m+1/2+\sum_i \frac{(n_i-m)}{2} ,}
\]

which is the proper expression to be compared with Eq. (8).

As concerns the string equation one comment is in order. In the regime (29) the string equation (21) becomes

\[
\sum_n b_n \mu^{(m-n)/2} u^n = \frac{\mu^{m/2}}{(m-1/2)} \left[ P_{m-2} \left( \frac{u}{\sqrt{\mu}} \right) - P_m \left( \frac{u}{\sqrt{\mu}} \right) \right] = 0 .
\]

Recalling that Legendre polynomials satisfy \( P_n(1) = 1 \) we conclude that the solution \( u = \sqrt{\mu} \) has been consistently chosen.

### 2.3 Continuum/discrete dictionary

Suppose that the \( m \)-th critical Hermitian 1-matrix model corresponds to a \((p,q)\)-minimal model coupled to gravity. In order to determine \( p \) and \( q \) we compare scaling exponents in the \( t \) and \( \mu \) regimes.

In the continuum approach the partition function scales as \( Z \sim \mu^{-Q/\alpha_+} \) as stated in Eq. (10). On the other hand we learned that \( t \) should be the coupling of the minimal-weight operator, so that in a \( t \)-type regime we expect the partition function to scale as \( Z \sim t^{-Q/\beta_{\text{min}}} \), where \( \beta_{\text{min}} \) is the dressing charge of the minimum weight operator. These exponents can be written in terms of \( p \) and \( q \),

\[
-\frac{Q}{\beta_{\text{min}}} = \frac{2(p+q)}{(p+q) - 1} , \quad -\frac{Q}{\alpha_+} = \frac{1 + \frac{q}{p}}{p} .
\]
Both cases were also studied in the discrete \( m \)-th critical model, where the free energy scales as \( F \sim t^{2+1/m} \) and \( F \sim \mu^{m+1/2} \), in the \( t \) and \( \mu \) regime respectively. Identifying the scaling exponents from each approach we find two equations,

\[
\frac{2(p + q)}{(p + q) - 1} = 2 + \frac{1}{m}, \\
1 + \frac{q}{p} = m + \frac{1}{2},
\]

which uniquely determine \((p, q) = (2, 2m - 1)\).

Next we examine the scaling operators. In the \((p, q)\) model there are \( \frac{1}{2}(p - 1)(q - 1) \) primary operators, usually labelled \( O_{rr'}, 1 \leq r \leq q - 1 \) and \( 1 \leq r' \leq p - 1 \), with conformal weights \( \Delta_{rr'} = \frac{(rp - r'q)^2 - (q - p)^2}{4pq} \). With this notation the index \( \nu \) that characterizes the wave function \( \Psi_l(O_{rr'}) \) is given by

\[
\nu = \sqrt{Q^2 - 8\Delta_{rr'}} = \frac{r'q}{p} - r.
\]

For \((p, q) = (2, 2m - 1)\) we have \( r' = 1 \) and \( 1 \leq r \leq m - 1 \) (we can ignore the range \( m \leq r \leq 2(m - 1) \), which only doubles the spectrum of weights). Therefore \( \nu = m - r - 1/2 \). On the other hand the discrete model predicts \( \nu = n + 1/2 \). We conclude that \( r = m - 1 - n \), with \( n = 0, \ldots, m - 2 \), spans the set of primary operators and we can finally draw the identification

\[
\hat{\sigma}_n \leftrightarrow \int d^2 \xi e^{\beta_n \phi} O_{m-1-n,1}, \quad n = 0, \ldots, m - 2.
\]

Concerning the remaining scaling operators, we have two special cases: as indicated by Eq. (39), \( \hat{\sigma}_{m-1} \) corresponds to the boundary operator

\[
\hat{\sigma}_{m-1} \leftrightarrow \int d\xi e^{\alpha + \phi/2},
\]

while \( \hat{\sigma}_m \) is part of the energy operator, as suggested in Ref. [1]:

\[
\hat{\sigma}_m - \frac{(2m - 3)}{(2m + 1)} \mu \hat{\sigma}_{m-2} \leftrightarrow \text{energy}.
\]

For the operators \( \hat{\sigma}_n, n > m \), we verify that their conformal weights differ from the weights of \( \hat{\sigma}_n, n \leq m \), by integers, and thus correspond to secondary operators. The 1- and 2-point functions found in the discrete and continuum approaches are in perfect agreement. Higher-order correlators involve the issue of fusion rules and lay beyond the approximation assumed in Eq. (10). Nevertheless the scaling factors agree with the predictions of the continuum.

Now we are ready to generalize the previous computations for the non-critical superstring.
3. Supersymmetric non-critical strings

3.1 Super-Liouville approach to non-critical superstrings

The results in this section have been obtained in Ref. [2], using the formulation of Ref. [11] (see also the review [10]). The total supersymmetric action is given by

\[ S = S_{SL} + S_M + S_{gh}, \]

where

\[ S_{SL} = \int \frac{d^2 z d^2 \theta}{4\pi} E \left( \frac{1}{2} D_\alpha \Phi_{SL} D^\alpha \Phi_{SL} - \frac{Q}{2} Y \Phi_{SL} - 4i \theta \bar{\epsilon} e^{\alpha_+ \Phi_{SL}} \right) \]

is the super-Liouville action, representing the supergravity sector of the superstring in the superconformal gauge; \( \Phi_{SL} = \phi + \theta \psi + \bar{\theta} \bar{\psi} \) is the Liouville superfield; \( \pi \) is a bare cosmological constant; \( E \) is the superdeterminant of the super-zweibein and \( Y \) the curvature superfield\(^2,\)\(^10,\)\(^11\). Also,

\[ S_M = \int \frac{d^2 z d^2 \theta}{4\pi} E \left( \frac{1}{2} D_\alpha \Phi_M D^\alpha \Phi_M - i \alpha_0 Y \Phi_M \right) \]

corresponds to the \( c < 3/2 \) supermatter action in the Coulomb gas formulation, with matter superfield \( \Phi_M = X + \theta \zeta + \bar{\theta} \bar{\zeta} + \theta \bar{\theta} G \). The term \( S_{gh} \) stands for the ghost action whose explicit form we do not need here. The background charge \( \alpha_0 \) is related to the matter central charge:

\[ c = \frac{3}{2} (1 - 8 \alpha_0^2). \]

To compute \( Q \) and \( \alpha_\pm \) we have to impose respectively that the total central charge \( c_T \) vanishes and \( e^{\alpha_+ \Phi_{SL}} \) be a \( (1/2, 1/2) \) conformal operator. We need the bosonic piece of the super energy momentum tensor which is given in components by

\[ T_{SL} = -\frac{1}{2} : \partial \phi \partial \phi : - \frac{1}{2} : \psi \partial \psi : + \frac{Q}{2} \partial^2 \phi, \quad T_M = -\frac{1}{2} : \partial X \partial X : - \frac{1}{2} : \zeta \partial \zeta : - i \alpha_0 \partial^2 X \]

\[ T_{gh} = T_{bc} + T_{\beta \gamma} = : c \partial b : + : 2 (\partial \bar{c}) b : - \frac{3}{2} : (\partial \gamma) \beta : - \frac{1}{2} : \gamma \partial \beta : \]

where \( b, c (\beta, \gamma) \) are ghost fields. We use free propagators for all fields. The central charges for super-Liouville and ghosts are computed in the usual way, that is, \( c_{SL} = \frac{3}{2}(1 + 2 Q^2) \) and \( c_{gh} = -15 \). Imposing that \( c_{SL} + c + c_{gh} = 0 \) we deduce that \( Q = 2 \sqrt{\alpha_0^2 + 1} \). The conformal weight \( \Delta \) of an operator \( e^{\alpha \Phi} \) is \( \Delta(e^{\alpha \Phi}) = -\alpha (\alpha + Q) \).

Requiring that \( \Delta = 1/2 \) we get \( \alpha_\pm = -\frac{Q}{2} \pm \frac{1}{2} \sqrt{Q^2 - 4} = -\frac{Q}{2} \pm |\alpha_0|, \) with \( \alpha_+ \alpha_- = 1 \).

The particle content consists of a scalar (Neveu-Schwarz, NS) and a spinor (Ramond, R) vertex, both massless. The Neveu-Schwarz vertex is the supersymmetric extension of the tachyon,

\[ \psi_{NS}(k) = \int d^2 z d^2 \theta \ e^{i k \Phi_M + \beta \Phi_{SL}} \]

\[ = \int d^2 z e^{i k x + \beta \phi} \left[ (ik \zeta + \beta \psi)(i k \bar{\zeta} + \bar{\beta} \bar{\psi}) + \beta F + ik G \right]. \]
As before, we impose that $e^{ik\Phi_M+\beta\Phi_{SL}}$ be a $(1/2,1/2)$ conformal operator,

$$\Delta(e^{ik\Phi_M+\beta\Phi_{SL}}) = \frac{1}{2}k(k-2\alpha_0) - \frac{\beta}{2}(\beta + Q) = \frac{1}{2},$$  \hspace{1cm} (60)

which fixes the NS dressing charge; as in the bosonic case, $\beta(k) = -\frac{Q}{2} + |k - \alpha_0|$. The auxiliary fields $F$ and $G$ appear in a trivial way in the action. Their propagators consist of delta functions, $\langle F(z_i)F(z_j) \rangle = \langle G(z_i)G(z_j) \rangle \sim \delta^{(2)}(z_i - z_j)$. The contractions of other fields typically give $\langle e^{a\phi(z_i)}e^{b\phi(z_j)} \rangle \sim |z_i - z_j|^{-2ab}$. This problem will be circumvented by discarding such fields, i.e. we fix $F = G = 0$. This amounts to assuming $ab < 0^{12}$. After such simplification the NS vertex becomes

$$\psi_{NS}(k) = \int d^2zd^2\theta \ e^{ik\Phi_M+\beta\Phi_{SL}} = \int dz(ik\zeta + \beta\psi) \int d\bar{\zeta}(ik\bar{\zeta} + \bar{\beta}\bar{\psi}) e^{ikx+\beta\phi}. \hspace{1cm} (61)$$

In order to calculate correlation functions we must consider the residual $OSP(2,1)$ symmetry of the superconformal gauge. After choosing $\tilde{z}_1 = 0$, $\tilde{z}_2 = 1$, $\tilde{z}_3 = \infty$, $\tilde{\theta}_2 = \tilde{\theta}_3 = 0$ and renormalizing the cosmological constant $\mu \rightarrow \mu$, one finds

$$A_N = \frac{\partial^{N-3}}{\partial \mu^{N-3}} \mu^{N+s-3} \prod_{j=1}^{N} \Delta \left( \frac{1}{2} + \frac{1}{2}(\beta_j^2 - k_j^2) \right), \hspace{1cm} (62)$$

with $s = -(Q + \sum \beta_i)/\alpha_+$, i.e. the results are completely similar to the bosonic case (recall Eq. (8)).

For the Ramond vertex we have to consider other fermionic contributions. First we bosonize the fermions $\psi$ and $\zeta$ into a bosonic massless field $h$ in the usual way, that is

$$\psi \pm i\zeta = \sqrt{2}e^{\pm ih} \hspace{1cm} (63)$$

with the contraction $\langle h(z)h(w) \rangle = -\ln(z - w)$ and the superselection rule

$$\left\langle \sum_i e^{q_ih(z_i)} \right\rangle \neq 0 \hspace{1cm} \text{iff} \hspace{1cm} \sum_i q_i = 0. \hspace{1cm} (64)$$

Because of supersymmetry the Ramond vertex must also contain a ghost spinor field $\Sigma$ whose bosonized form $\Sigma_{\pm1/2} = e^{\pm\sigma/2}$ involves a massless bosonic field $\sigma$ with propagator $\langle \sigma(z)\sigma(w) \rangle = -\ln(z - w)$. The two solutions have different conformal dimensions, $\Delta(e^{-\sigma/2}) = 3/8$ and $\Delta(e^{\sigma/2}) = -5/8$. The requirement

$$\Delta \left(e^{\frac{1}{2}eh(z)+ikX(z)+\beta\phi(z)}\Sigma(z) \right) = \frac{1}{8} + \frac{1}{2}k(k-2\alpha_0) - \frac{1}{2}\beta(\beta + Q) + \Delta(\Sigma) = 1$$  \hspace{1cm} (65)

is enough to determine the proper choice. Indeed the Ramond vertex $V_R(k,\epsilon)$ should represent a massless particle. The on-shell condition $(k - \alpha_0)^2 - (\beta + Q/2)^2 = 0$,

13
equivalent to $E^2 - p^2 = 0$, implies that $\Delta(\Sigma) = 3/8$, which selects the solution $\Sigma = \Sigma_{-1/2} = e^{-\sigma/2}$. Therefore our spinor vertex can be written as

\[ V_R(k, \epsilon) = V_{-\frac{1}{2}}(k, \epsilon) = \int d^2z e^{-\frac{1}{2}\sigma(z) + \frac{1}{2}ch(z) + ikX(z) + \beta\phi(z)} . \quad (66) \]

From the Dirac equation we find for the dressing the expression $\beta(k, \epsilon) = -\frac{q}{2} + \epsilon(k - \alpha_0)$. There are further versions of these operators obtained by a certain procedure defined in Ref. [13], leading to the so-called pictures of vertices. We remark that the field $\sigma$ has background charge $-2$, which implies another superselection rule,

\[ \left\langle \sum_i e^{q_i\sigma(z_i)} \right\rangle \neq 0 \quad \text{iff} \quad \sum_i q_i = -2 . \quad (67) \]

Another useful picture of the NS vertex is the following

\[ \psi_{NS}^{(-1)}(k) = \int d^2z e^{-\sigma(z) + ikX(z) + \beta\phi(z)} . \quad (68) \]

The above vertex is BRST-invariant and can be “picture-changed” into $\psi_{NS} = [Q_{BRST}, \xi \psi_{NS}^{(-1)}]$, where $\xi$ is the ghost zero mode (see [13] for details). This is however not BRST-exact.

A mixed $N$-point correlator

\[ A_N^{(n,N-n)} = \left\langle \prod_{i=1}^{n} V_{-\frac{1}{2}}(k_i, \epsilon_i) \prod_{j=n+1}^{N} \psi_{NS}(k_j) \right\rangle \quad (69) \]

is obtained by integrating over matter ($X_0$) and Liouville ($\phi_0$) bosonic zero modes. The total momentum conservation law (7) still holds, supplemented by the rules (64) and (67). As in the bosonic case we find a factorizable result

\[ A_N^{(n,N-n)} \sim \mu^s \prod_{i=1}^{n} \Delta \left( \frac{1}{2} (\beta_i^2 - k_i^2) \right) \prod_{j=n+1}^{N} \Delta \left( \frac{1}{2} + \frac{1}{2} (\beta_j^2 - k_j^2) \right) , \quad (70) \]

where the exponent $s$, defined in (8), now includes spinor vertex momenta. Notice that the inclusion of the vertices $V_{-\frac{1}{2}}$ does not alter the scaling of the partition function and Eq. (10) still holds. The insertion of a NS or R vertex modifies the scaling behaviour by a factor $\mu^{-\beta/\alpha_+}$, $\beta$ being the corresponding dressing charge.

The bosonic case has taught us how important the analysis of 1- and 2-point functions are in the comparison between continuum and discrete results. So they must be in the supersymmetric theory. As in Eq. (11), momentum conservation
and the extra rules (64) and (67) imply that the “area” operator $\psi_{NS}(2\alpha_0)$ is the only vertex with non-vanishing expectation value, scaling as

$$\langle \psi_{NS}(2\alpha_0) \rangle \sim \mu^{-Q/\alpha} \cdot$$

(71)

In the NS sector we have an orthogonality property similar to the one valid before for the operators (6), namely

$$\langle \psi_{NS}(k)\psi_{NS}(2\alpha_0 - k) \rangle \sim \mu^{-2\alpha+2\beta(k)/\alpha} \cdot$$

(72)

Concerning the R sector, the superselection rule (64) implies that every 1-point function $\langle V_{\frac{1}{2}} \rangle$ must vanish. From Eq. (70) and the rules (64) and (67) we also conclude that $A_2^{(2,0)} = 0$. However, considering the picture-changed NS vertex (68) we can build a non-vanishing 2-spinor function, $\langle V_{\frac{1}{2}}(k,\epsilon)\psi_{NS}(-1)(0) \rangle \neq 0$. The operator $\psi_{NS}^{(-1)}(0) = \int d^2z e^{-\sigma+\alpha+\phi}$ acts as a screening required by the rule (67), its “engineering” dimension is $[\text{length}]^2$ and its coupling constant must tend to $\mu^2$ in the $\mu$-regime. The screening vertex $\mu^2\psi_{NS}^{(-1)}(0)$ thus defined contributes with a scaling factor $\mu^{-\alpha+\alpha+2} = \mu$. We therefore have the following screened 2-point function

$$\langle V_{\frac{1}{2}}(k,\epsilon)V_{\frac{1}{2}}(2\alpha_0 - k, -\epsilon)\mu^2\psi_{NS}^{(-1)}(0) \rangle \sim \mu^{-Q/\alpha+2\beta/\alpha+1} \cdot$$

(73)

We could interpret the above result in the following way: the spinor vertex $V_{\frac{1}{2}}(k,\epsilon)$ contributes with the scaling factor $\mu^{-\beta/\alpha}$, while its “conjugate” vertex in this representation is the composite operator $\mu^2\psi_{NS}^{(-1)}(0) \times V_{\frac{1}{2}}(2\alpha_0 - k, -\epsilon)$, which scales as $\mu^{-\beta/\alpha+1}$. This interpretation will be further supported by the discrete model results.

A delicate issue concerning the comparison with the matrix models results is the definition of the matter sector of the Ramond vertex, since the gravitational and matter fermions merge into $e^{\frac{1}{2}h}$, and an identification of matter contribution with the Kac table results turns out to be difficult. Recall that in general for a Ramond vertex $\frac{3}{8} + \frac{1}{8} + \frac{1}{2}k(k - \alpha_0) - \frac{1}{2}\beta(\beta + Q) = 1$, where the $3/8$ comes from the ghost contribution, and $1/8$ from $e^{\frac{1}{2}h}$. Therefore, since the latter is neither pure matter nor pure gravity, we have to disentangle their contributions and find the relation between the pure matter conformal dimension $\Delta_{Kac}(\Delta_K)$ and the dressing. Gravitational dressing gives a contribution $-\frac{1}{2}\beta(\beta + Q) + \frac{1}{16}$, as argued in Ref. [11]. (The Neveu-Schwarz field is expanded in half-integer components and displays no zero mode, while the Ramond field is expanded in integer modes: the zero mode contributes $1/16$). Put in another way, in order to maintain the form of the Virasoro algebra we must have, in the Ramond sector, shift $L_0$ into $L_0 + \frac{1}{16}$, where $1/16$ is the ground-state energy. Therefore we have, $\Delta_K = \Delta + \frac{1}{16}$, where $\Delta = \frac{1}{2}k(k - \alpha_0)$. 

15
3.2 Discrete super-eigenvalue model

In Ref. [3] a discrete model was proposed as a supersymmetric extension of the effective eigenvalue theory, which in its turn comes from the angular integration of the Hermitian 1-matrix model. It is defined by the supersymmetric partition function

$$Z_s = \int \prod_{n=1}^{N} d\lambda_n d\theta_n e^{-\frac{N}{\kappa} \sum_i V_s(\lambda_i, \theta_i) \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j)} ,$$

$$V_s(\lambda_i, \theta_i) = \sum_{k=0}^{\infty} (g_k \lambda_i^k + \xi_{k+1/2} \theta_i \lambda_i^k) ,$$

where $\lambda_i, \theta_i$ are commuting and anticommuting eigenvalue variables, respectively, and $g_k, \xi_{k+1/2}$ are the commuting and anticommuting coupling constants.

The planar superloop equations and the double scaling limit of this model were studied in Refs. [3,14]. Higher genera were also considered in Ref. [14] but it was only after Ref. [15] that a non-perturbative definition was given in terms of a supersymmetric KdV hierarchy.

To compare results with the continuum approach the planar approximation will suffice. The $m$-th critical double scaling limit is similar to the bosonic case. For even bosonic potentials the resulting theory is described by the supersymmetric free energy

$$F_s = -\frac{1}{2\kappa^2} \partial_t^2 [1 - \partial_t \tau_+ \tau_- \partial_t] u ,$$

where the functions $u$ and $\tau_\pm$, respectively bosonic and fermionic, satisfy the equations

$$t = u^m - \sum_{n \geq 0} t_n u^n , \quad \tau_\pm(u) = \sum_{n \geq 0} \tau_\pm^n u^n ,$$

in terms of the renormalized bosonic (fermionic) coupling constants $t_n (\tau_\pm^n)$. It is also convenient to define the functions $\rho_n[u] \equiv (1 - \partial_t \tau_+ \tau_- \partial_t)^{\frac{n}{2}}$, generalizing the monomials $R_n[u] = u^n / n$ of the bosonic theory, whose flows are given by

$$\frac{\partial}{\partial t_n} \rho_m[u] = \frac{\partial}{\partial t} \rho_{m+n}[u] , \quad \frac{\partial}{\partial \tau^n_\pm} \rho_m[u] = \frac{\partial}{\partial \tau_\pm^n} \rho_{m+n}[u] .$$

Using the string equation (23) we can also write the functions $\rho_n$ as

$$\rho_n[u] = (1 - D_+ D_-) \frac{u^n}{n} ,$$

where

$$D_\pm \equiv \sum_{i \geq 0} \tau_i^\pm \frac{\partial}{\partial t_i} .$$
In this case we can make explicit the quadratic dependence\(^3,15\) of fermionic couplings exhibited by the free energy:

\[
F_s = (1 - D_+ D_-) \left[ -\frac{1}{2\kappa^2} \partial t^{-2} u \right] = \frac{1}{2}(1 - D_+ D_-) F_{\text{bosonic}} \quad .
\] (79)

We associate scaling operators to the couplings, \(\sigma_n \leftrightarrow \frac{\partial}{\partial t_n}, \nu^\pm_n \leftrightarrow \frac{\partial}{\partial \tau^\pm_n}\) and their correlation functions follow easily

\[
\langle \prod_{i=1}^n \sigma_{a_i} \rangle = -\frac{1}{2\kappa^2} \partial_t^{n-2} \rho_{a+1} [u] , \quad a = \sum_i a_i , \quad \tag{80a}
\]

\[
\langle \nu^+_k \prod_{i=1}^n \sigma_{a_i} \rangle = -\frac{1}{2\kappa^2} \partial_t^{n-2} \frac{\partial}{\partial \tau^\pm} \rho_{k+a+1} [u] , \quad \tag{80b}
\]

\[
\langle \nu^+_k \nu^-_l \prod_{i=1}^n \sigma_{a_i} \rangle = -\frac{1}{2\kappa^2} \partial_t^{n-2} \frac{\partial}{\partial \tau^+} \frac{\partial}{\partial \tau^-} \rho_{k+l+a+1} [u] . \quad \tag{80c}
\]

In analogy to the bosonic case and considering the results of Ref. [15] we shall call this set of operators/couplings the super-KdV frame. Now we can compute correlation functions in the \(t\)-regime given by the limits \(t_n, \tau^\pm_n \to 0\). The free energy scales as in the bosonic model,

\[
F \sim t^{2+1/m} \Rightarrow \gamma = -1/m
\] (81)

while higher-order correlation functions scale as\(^3\)

\[
\langle \prod_{i=1}^n \sigma_{a_i} \rangle \sim t^{2+\frac{1}{m}+\sum \left(\frac{n}{m} - 1\right)} , \quad \tag{82}
\]

\[
\langle \nu^+_k \nu^-_l \prod_{i=1}^n \sigma_{a_i} \rangle \sim t^{2+\frac{1}{m}+\left(\frac{n}{m} - \frac{1}{2m} - 1\right)+\left(\frac{1}{m} + \frac{1}{2m} - 1\right)+\sum \left(\frac{n}{2m} - 1\right)} , \quad \tag{83}
\]

implying the scaling dimensions \(d_{\sigma_n} = \frac{n}{m}\) and \(d_{\nu^+_n} = \frac{n}{m} + \frac{1}{2m}\). Further correlators vanish.

We can foresee difficulties with the super-KdV basis similar to the bosonic case: indeed we have infinitely many non-vanishing 1-point functions, \(\langle \sigma_i \rangle \neq 0\), while the 2-point functions \(\langle \sigma_i \sigma_j \rangle\) and \(\langle \nu^+_i \nu^-_j \rangle\) are not diagonal. The results (71) to (73) show that we cannot associate the operators \((\sigma_i, \nu_i)\) to \((\psi_{NS}, V_{-\frac{1}{2}})\) in a simple way.

17
3.3 Superconformal frame of scaling operators

Let us begin to investigate the dimension of the coupling constants. As in the bosonic case the specific heat $u$ has dimension of inverse length, $[u] = [\text{length}]^{-1}$. From the string equation and the renormalization procedure used in Ref. [3] we verify that

$$[t_n] = [\text{length}]^{n-m}, \quad [\tau^\pm_n] = [\text{length}]^{n-m\pm\frac{1}{2}}. \quad (84)$$

Recalling the super-Liouville action (56), we observe that the “area” operator actually has the dimension of a length. In this case the “physical” cosmological constant $\mu$ has dimension of inverse length. We conclude that in general $t_{m-1}$ is the coupling with the proper dimension of a cosmological constant rather than $t$. The correct comparison with the continuum results requires that we replace the $t$-regime by a $\mu$-like scaling regime.

As outlined in the bosonic case, we take the string equation (21), consider an analytical transformation from the super-KdV basis $\{t_n, \tau^\pm_n\}$ to a generic one $\{\hat{t}_n, \hat{\tau}^\pm_n\}$, and replace the $t$-regime (20) by the following one:

$$\hat{t}_{m-1} \to \mu; \quad \hat{t}_m \to -1; \quad \hat{t}_n \to 0, \quad n \neq m-1, m; \quad \hat{\tau}^\pm_n \to 0, \quad \forall n, \quad (85)$$

with solution $u = \mu$ for every $m$-th critical model. Since we are going to concentrate on the properties of 1- and 2-point functions we must make an extra shift of the non-vanishing couplings, $\hat{t}_{m-1} \to \mu + \hat{t}_{m-1}$ and $\hat{t}_m \to -1 + \hat{t}_m$, and then treat $\hat{t}_n, \hat{\tau}^\pm_n$ as perturbative couplings. Imposing also that the dimensions are preserved, i.e. $[\hat{t}_n] = [t_n]$ and $[\hat{\tau}^\pm_n] = [\tau^\pm_n]$, the lowest-order transformations must have the form

$$t_n = B_n \mu^{m-n} + \sum_{s=0}^{\infty} A^{(n+s)}_s \mu^s \hat{\tau}^\pm_{n+s}, \quad (86)$$

with

$$B_{m-i} = \frac{1}{i} A^{(m-1)}_{i-1}; \quad B_n = 0, \quad n > m \quad (87)$$

and

$$\hat{\tau}^\pm_n = \sum_{s=0}^{\infty} A^{(n+s)}_s \mu^s \hat{\tau}^\pm_{n+s}. \quad (88)$$

The critical regime is therefore defined by the limit $\hat{t}_n, \hat{\tau}^\pm_n \to 0$. As we shall see, the choice of the same coefficients $A^{(n)}_s$ in both transformations (86) and (88) implies that $\langle \hat{\sigma}_i \hat{\sigma}_j \rangle$ and $\langle \hat{\nu}_i^+ \hat{\nu}_j^- \rangle$ are simultaneously diagonalized and also preserves supersymmetry.

From Eqs. (86) and (88) we obtain the new set of scaling operators,

$$\hat{\sigma}_n = \sum_{s=0}^{n} A^{(n)}_s \mu^s \sigma_{n-s}, \quad \hat{\tau}^\pm_n = \sum_{s=0}^{n} A^{(n)}_s \mu^s \tau^\pm_{n-s}. \quad (89)$$
In order to find the proper coefficients \( A^{(n)} \) we shall follow the orthogonalization procedure described in the bosonic model. From the transformation (89) and the general correlators given in (80) we find

\[
\langle \hat{\nu}^+_i \hat{\nu}^-_j \rangle = \langle \hat{\sigma}^+_i \hat{\sigma}^-_j \rangle = -\frac{1}{2\kappa^2} \mu^{i+j+1} g_{ij}(A) ,
\]

with the symmetric matrix

\[
g_{ij}(A) \equiv \sum_{s=0}^{i} \sum_{r=0}^{j} \frac{A^{(i)}_s A^{(j)}_r}{(i+j-s-r+1)} = \int_0^1 dx \left( \sum_{s=0}^{i} A^{(i)}_s x^{i-s} \right) \left( \sum_{r=0}^{j} A^{(j)}_r x^{j-r} \right)
\]

Once more we have reduced the issue of diagonalizing 2-point functions to a problem of orthogonal polynomials. Notice the differences with respect to the purely bosonic case: the integral goes from 0 to 1 now, and the polynomials have no definite parity. The solution is given in terms of the \( \pi \)-polynomials defined as

\[
\pi_i(x) = P_i(2x - 1) = \sum_{s=0}^{i} A^{(i)}_s x^{i-s} , \quad A^{(i)} = \frac{(-1)^s (2i-s)!}{s! [(i-s)!]^2} ,
\]

for which \( g_{ij} = \frac{1}{2i+1} \delta_{ij} \). The corresponding coefficients \( B_n \) are given by

\[
B_{m-s} = \frac{A^{(m-1)}_s - A^{(m)}_s}{2m} , \quad B_m = -\frac{A^{(m)}_0}{2m} .
\]

Also the choice (92) guarantees that the only non-vanishing 1-point functions are

\[
\langle \hat{\sigma}_{m-1} \rangle = -\frac{1}{2\kappa^2} \frac{\mu^{2m}}{2m(2m-1)} , \quad \langle \hat{\sigma}_m \rangle = \frac{1}{2\kappa^2} \frac{\mu^{2m+1}}{2m(2m+1)} ,
\]

the former corresponds to the average “area” to be compared with Eq. (71), while the latter is expected to represent part of the average energy, as in the bosonic case.

Therefore we propose the following definition of the superconformal basis of scaling operators

\[
\hat{\sigma}_n = \sum_{s=0}^{n} \frac{(-1)^s (2n-s)!}{s! [(n-s)!]^2} \mu^s \sigma_{n-s} , \quad \hat{\tau}_n^{\pm} = \sum_{s=0}^{n} \frac{(-1)^s (2n-s)!}{s! [(n-s)!]^2} \mu^s \tau_{n-s}^{\pm}
\]

which is our main result. We emphasize, though, that Eq. (95) corresponds to a lowest-order transformation. It can also be inverted, so that the super-KdV frame reads

\[
\sigma_n = [n!]^2 \sum_{s=0}^{n} \frac{(2n + 1 - 2s)}{s!(2n + 1 - s)!} \mu^s \sigma_{n-s} , \quad \tau_n^{\pm} = [n!]^2 \sum_{s=0}^{n} \frac{(2n + 1 - 2s)}{s!(2n + 1 - s)!} \mu^s \tau_{n-s}^{\pm} .
\]
We also notice that, within the approximation (86)-(88), we have
\[ \hat{D}_\pm = \sum_i \tau_i^\pm \frac{\partial}{\partial t_i} = \sum_i \tau_i^\pm \frac{\partial}{\partial t_i} = D_\pm , \] (97)
and the free energy can still be written as \( \mathcal{F}_s = \frac{1}{2}(1 - \hat{D}_+ \hat{D}_-) \mathcal{F}_{bosonic} \). In the new regime we verify that the free energy scales as
\[ \mathcal{F} \sim \mu^{m+1/2} , \] (98)
while general \( N \)-point functions scale as
\[ \left\langle \prod_i \hat{\sigma}_i \right\rangle \sim \mu^{m+1/2 + \sum_i \delta_i} , \] (99)
with the scaling exponents \( \delta_{\sigma_n} = n - m \) and \( \delta_{\nu_n^\pm} = n - m \mp \frac{1}{2} \), to be compared with the continuum values \(-\beta/\alpha_+\).

The \( m \)-th critical string equation becomes
\[ \sum_n B_n \mu^{m-n} u^n = \frac{u^m}{2m} \left[ \pi_{m-1} \left( \frac{u}{\mu} \right) - \pi_m \left( \frac{u}{\mu} \right) \right] = 0 , \] (100)
in agreement with the solution \( u \rightarrow \mu \) due to the general property \( \pi_n(1) = P_n(1) = 1 \) of the \( \pi \)-polynomials.

3.4 Super-Liouville/eigenvalue dictionary

As in the bosonic case, we suppose that the \( m \)-th critical supersymmetric model corresponds to some dressed \((p,q)\)-minimal superconformal theory; we determine the numbers \( p \) and \( q \) by analysing the scalings in two different regimes, namely the \( t \)- and \( \mu \)-averaged regimes.

In the continuum theory the partition function behaves as \( Z \sim t^{-Q/\beta_{\min}} \) and \( Z \sim \mu^{-Q/\alpha_+} \). For the supersymmetric \((p,q)\) model, we have \( \alpha_0 = \frac{p-q}{2\sqrt{pq}} \) and
\[ -\frac{Q}{\beta_{\min}} = \frac{2(p+q)}{(p+q) - 2} , \quad -\frac{Q}{\alpha_+} = 1 + \frac{q}{p} . \] (101)

In the discrete theory we have found \( \mathcal{F}_s \sim t^{2+1/m} \) and \( \mathcal{F}_s \sim \mu^{2m+1} \), thus implying the following pair of equations:
\[ \frac{2(p+q)}{(p+q) - 2} = 2 + \frac{1}{m} , \] (102)
\[ 1 + \frac{q}{p} = 2m + 1 \] ,
whose solution \( (p, q) = (2, 4m), m = 1, 2, \ldots \). The corresponding central charges are \( c = \frac{3}{2} \left[ 1 - \frac{(2m-1)^2}{m} \right] = 0, -\frac{21}{4}, \ldots \). The case \( m = 1, c = 0 \) corresponds to pure supergravity and is the only unitary model within this series.

Concerning the spectrum of primary operators, we have two sectors (NS and R): the total number of primaries is \( \frac{1}{2} ((p-1)(q-1)+1) \), which makes up \( 2m \) operators (\( m \) in each sector) in our case. The conformal weight \( \Delta_{r,r'} \) of such operators is given by the formula

\[
\Delta_{r,r'} = \frac{(rp - r'q)^2 - (q-p)^2}{8pq} + \frac{1 - (-1)^{r-r'}}{32},
\]

with \( 1 \leq r' \leq p-1 \) (i.e. \( r' = 1 \) in our case) and \( 1 \leq r \leq q-1 = 4m-1 \). When the difference \( r - r' \) is even, the operator belongs to the NS sector,

\[
\Delta_{2i+1,1}^{NS} = \frac{i(i - 2m + 1)}{4m}, \quad i = 0, 1, \ldots, 2m - 1,
\]

and for \( r - r' \) odd one has the R sector,

\[
\Delta_{2i+2,1}^{R} = \frac{(i + 1 - m)^2 - (m - 1/2)^2}{4m} + \frac{1}{16}, \quad i = 0, 1, \ldots, 2m - 2.
\]

In fact we can restrict the label \( i \) to the range \( i = 0, \ldots, m - 1 \), since the remaining values only duplicate the spectrum. From the weights \( \Delta_{r,r'} \) it is straightforward to calculate the corresponding momenta \( k_{r,r'} \) and dressing charges \( \beta_{r,r'} \). Then, by comparing the values of \( -\beta_{r,r'}/\alpha_+ \) with the exponents in Eq. (99) we can associate the bosonic scaling operators with the NS vertices,

\[
\hat{\sigma}_n \leftrightarrow \psi_{NS}(k_{2(m-n)-1,1}), \quad n = 0, \ldots, m - 1
\]

and the fermionic operators with the R vertices in the following subtle manner

\[
\hat{\nu}_n^+ \leftrightarrow V_{-\frac{1}{2}}(k_{2(m-n),1})
\]

\[
\hat{\nu}_n^- \leftrightarrow \mu^2 \psi_{NS}^{(-1)}(0) V_{-\frac{1}{2}}(2\alpha_0 - k_{2(m-n),1}), \quad n = 0, \ldots, m - 1.
\]

The special operator \( \hat{\sigma}_{m-1} \) corresponds to the “area” or cosmological term. We can also define an energy operator,

\[
\varepsilon_m = \hat{\sigma}_m - \frac{(2m-1)}{(2m+1)} \mu \hat{\sigma}_{m-1}
\]

which points out the role of the operator \( \hat{\sigma}_m \). The remaining operators correspond to secondary fields. Notice the absence of a boundary operator, as opposed to the bosonic case.
3.5 Wave functions and minisuperspace approximation

We shall determine the supersymmetric version of Eq. (15) obeyed by the wave functions of scaling operators in the superconformal frame. In this way we expect to circumvent some ambiguities, which usually arise in attempts\textsuperscript{16} to derive the super WdW equation in minisuperspace approximations from the continuum theory.

We start from the macroscopic superloop $W_{\pm}(l, \theta_{\pm})$ defined in Ref. [3], whose expectation value reads

$$\langle W_{\pm}(l, \theta_{\pm}) \rangle = \frac{1}{\kappa \sqrt{\pi l}} \left( -\partial_{l}^{-1} + \tau_{+} \tau_{-} \pm \theta_{\pm} \partial_{l}^{-1} \tau_{\mp} \partial_{l} \right) e^{-l u} \ .$$

Using the string equation and the definition (78b), this equation can be rewritten as

$$\langle W_{\pm}(l, \theta_{\pm}) \rangle = \left[ 1 - D_{+}D_{-} \mp \theta_{\pm} D_{\mp} \right] \frac{1}{\kappa \sqrt{\pi l}} \int_{t_{0}}^{\infty} dy \ e^{-l u(y)} \ ,$$

to be compared with Eq. (23). In fact we have the same integral of the bosonic model, implying an expansion in terms of the functions $\psi_{n}$ defined in Eq. (25).

Using the following identity between Bessel functions and $\pi$-polynomials,

$$e^{-x/2} K_{n+1/2} \left( \frac{x}{2} \right) = \sqrt{\frac{\pi}{x}} \int_{x}^{\infty} dz \pi_{n} \left( \frac{z}{x} \right) e^{-z} \ ,$$

we can calculate the integral in Eq. (109) in the superconformal frame, finding as a result the following expansion

$$\frac{1}{\kappa \sqrt{\pi l}} \int_{t_{0}}^{\infty} dy \ e^{-l u(y)} = \frac{1}{\kappa \pi} \frac{u^{m+1/2}}{2m} e^{-lu/2} \left[ K_{m-1/2} \left( \frac{lu}{2} \right) - K_{m+1/2} \left( \frac{lu}{2} \right) \right]$$

$$+ \frac{1}{\kappa \pi} \sum_{n=0}^{\infty} \hat{t}_{n} u^{n+1/2} e^{-lu/2} K_{n+1/2} \left( \frac{lu}{2} \right) \ .$$

We therefore obtain, in the regime $\hat{t}_{n}, \hat{t}_{n}^{\pm} \to 0$, the following wave functions

$$\Psi_{i}^{NS} = \langle \hat{\sigma}_{i} W_{\pm}(l, \theta_{\pm}) \rangle = \frac{1}{\kappa \pi} \mu^{i+1/2} e^{-l \mu/2} K_{i+1/2} \left( \frac{l \mu}{2} \right) \ ,$$

$$\Psi_{i \pm} = \langle \hat{\sigma}_{i}^{\pm} W_{\pm}(l, \theta_{\pm}) \rangle = \mp \theta_{\mp} \Psi_{i}^{NS} \ .$$

Both satisfy the wave equation

$$\left\{ - \left( l \frac{\partial}{\partial l} + \frac{l u}{2} \right)^{2} + \frac{\mu^{2} l^{2}}{4} + \left( i + \frac{1}{2} \right)^{2} \right\} \Psi_{i} = 0 \ ,$$

22
which we expect to be the supersymmetric WdW equation in the minisuperspace approximation. Indeed, after dropping the space dependence of the fields in the continuum theory, we have the super-Liouville Hamiltonian

$$H = \frac{1}{2} p^2 + \frac{\mu}{2 \alpha^2_+} \psi \psi e^{\frac{1}{2} \alpha^+ \phi} + \frac{\mu^2}{2 \alpha^2_+} e^{\alpha^+ \phi} + \frac{Q^2}{8},$$  \hspace{1cm} (114)$$

suggesting$^{17}$ the associations

$$p^2 + \frac{\mu}{\alpha^2_+} \psi \psi e^{\frac{1}{2} \alpha^+ \phi} \rightarrow - \left( L \frac{\partial}{\partial L} + \frac{\mu L}{2} \right)^2, \hspace{1cm} (115a)$$

$$\frac{\mu^2}{2 \alpha^2_+} e^{\alpha^+ \phi} \rightarrow \mu^2 L^2. \hspace{1cm} (115b)$$

Notice that the fermionic contribution is summarized by an $L$-dependent shift of the momentum $p$, which amounts to the extra exponential factor $e^{-\mu L/2}$ in comparison to the Bessel wave functions of the bosonic model. The understanding of this effect in terms of the continuum formulation is at present under investigation.

4. Conclusion

Even though a matrix formulation is still missing, we have found the superconformal background of couplings and operators of the discrete super-eigenvalue model, which can be completely translated into the super-Liouville language, and a precise correspondence to the dressed $(2,4m)$ minimal superconformal theory can be established. As a by-product we have derived the wave equation that should be equivalent to the proper minisuperspace approximation of the WdW equation that supersymmetric loop amplitudes are expected to obey.

References

[1] G. Moore, N. Seiberg and M. Staudacher, Nucl. Phys. B362 (1991) 665.
[2] E. Abdalla, M.C.B. Abdalla, D. Dalmazi and K. Harada, Phys. Rev. Lett. 68 (1992) 1641, Int. J. Mod. Phys. A7 (1992) 2437, IFT-Prep. 042/91; L. Alvarez-Gaumé and Ph. Zaugg, Phys. Lett. B273 (1991) 81; K. Aoki and E. D’Hoker, Mod. Phys. Lett. A7 (1992) 333.
[3] L. Alvarez-Gaumé, H. Itoyama, J.L. Mañez and A. Zadra, Int. J. Mod. Phys. A7 (1992) 5337.
[4] J. Distler and H. Kawai, Nucl. Phys. B321 (1988) 171; F. David, Mod. Phys. Lett. A3 (1988) 1651.
[5] N. Seiberg, Lecture at 1990 Yukawa Int. Sem. on Common Trends in Math. and Quantum Field Theory, and Cargèse Meeting on Random Surfaces, Quantum Gravity and Strings, 1990.
[6] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.
[7] P. di Francesco and D. Kutasov, Phys. Lett. B261B (1991) 385; Nucl. Phys. B375 (1992) 119.
[8] V.A. Kazakov, Mod. Phys. Lett. A4 (1989) 2125.
[9] D.J. Gross and A.A. Migdal, Nucl. Phys. B340 (1990) 333;
    M.R. Douglas and S.H. Shenker, Nucl. Phys. B335 (1990) 635;
    E. Brézin and V.A. Kazakov, Phys. Lett. B36B (1990) 144.
[10] E. Abdalla, M.C.B. Abdalla, D. Dalmazi and A. Zadra, 2D Gravity in Non-
    Critical Strings: continuum and discrete approaches, Lecture Notes in Physics,
    series M, Springer Verlag (to appear).
[11] J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. A5 (1990) 391.
[12] M. Green and N. Seiberg Nucl. Phys. B299 (1988) 559.
[13] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B241 (1986) 93.
[14] L. Alvarez-Gaumé, K. Becker, M. Becker, R. Emparán and J.L. Mañes, Int. J.
    Mod. Phys. A8 (1993) 2297.
[15] K. Becker and M. Becker, Mod. Phys. Lett. A8 (1993) 1205.
[16] P.D. D’Earth and D.I. Hughes, Nucl. Phys. B378 (1992) 381.
[17] E. Abdalla and A. Zadra, Trieste Summer School and Workshop in High Energy
    Physics and Cosmology, 1993, to appear