Amenability of semigroups and common multiples in $\ell_1^+$

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Abstract. In this note, we prove that a semigroup $S$ is left amenable if and only if every two nonzero elements of $\ell_1^+(S)$ have a common nonzero right multiple, where $\ell_1^+(S)$ is the positive part of the Banach algebra $\ell_1(S)$, or equivalently the semiring of finite measures on $S$. This characterization of amenability is new even for groups.

1. Introduction

A cancellative semigroup $S$ can be embedded into a group as soon as it satisfies the Ore condition. This sufficient condition comes in two versions: the right Ore condition states that for every $a, b \in S$, it must be the case that their principal ideals intersect,

$$aS \cap bS \neq \emptyset,$$

and likewise for the left Ore condition. Similarly, a ring $R$ without zero divisors can be embedded into a division ring as soon as for every $a, b \in R$, we have

$$aR \cap bR \neq \{0\},$$

and this fact generalizes straightforwardly to semirings [7]. We use the following terminology.

Definition 1. A semiring $R$ has common right multiples (CRMs) if for every nonzero $a, b \in R$, there are $x, y \in R$ such that

$$ax = by \neq 0.$$

For example, every commutative semiring without zero divisors has CRMs, since trivially $ab = ba$.

Whether group rings and semirings have CRMs has been of interest for a while, and similarly the case of semigroup rings and semirings has been studied. Recalling the start of the art first requires a brief digression on amenability. We recall Day’s definition and
characterization of amenability for semigroups, which unlike in the case of groups comes in a left and an analogously defined right version, which generally are not equivalent.

**Definition 2** (Day [2]). A semigroup $S$ is left amenable if the following equivalent conditions hold:

(i) $S$ has a left invariant mean, i.e. there is a linear map $\Lambda : \ell^\infty(S) \to \mathbb{R}$ satisfying:

$\triangleright$ Normalization: $\Lambda(1) = 1$.

$\triangleright$ Positivity: For $f \in \ell^\infty(S)$ with $f \geq 0$, we have $\Lambda(f) \geq 0$.

$\triangleright$ Invariance: For every $f \in \ell^\infty(S)$ and $a \in S$,

$$\Lambda(f) = \Lambda(x \mapsto f(ax))$$

(ii) For every finite $F \subseteq S$ and $\varepsilon > 0$, there is $\varphi \in \ell_+^1(S)$ with $\|\varphi\|_1 = 1$ such that

$$\|a\varphi - \varphi\|_1 < \varepsilon \quad \forall a \in F.$$

Although condition (ii) reads a little different than the original condition in [2, Lemma 1], it is easily seen to be equivalent. We also refer to [5] for a more recent account.

We can now summarize what is known about monoid rings and CRMs.

$\triangleright$ A classical result of Tamari states that the monoid ring $K[M]$ of a cancellative and left amenable monoid $M$ over any field $K$ has CRMs [8, 4].

$\triangleright$ Recently, Kielak has shown conversely that if $G$ is a group, and if $K[G]$ has CRMs but no zero divisors\(^1\), then $G$ is amenable [1].

Other variants of the CRM problem have also been studied, in particular for the monoid semirings $\mathbb{N}[S]$, which trivially do not contain zero divisors.

$\triangleright$ It was shown by Donnelly that if $\mathbb{N}[S]$ has CRMs for a cancellative monoid $S$, then $S$ is left amenable [3].

$\triangleright$ Very recently, Guba derived a simple but surprisingly powerful necessary criterion for CRMs in group semirings [6]. Using this, he showed that if $G$ is the metabelian group on two generators, then $\mathbb{N}[G]$ does not have CRMs although $G$ is amenable (since it is solvable). He also proved that if $M$ is the positive monoid of Thompson’s group $F$, then $\mathbb{N}[M]$ does not have CRMs. This excludes the possibility of proving amenability of $F$ through CRMs.

In this note, we prove that a semigroup $S$ is left amenable if and only if the associated semiring $\ell_+^1(S)$ has CRMs. We state this result avoiding the CRM jargon in order to make it more accessible.

**Theorem 3.** Let $S$ be a semigroup. Then the following are equivalent:

\(^1\)Recall Kaplansky’s conjecture stating that this is automatically the case as soon as $G$ is torsion-free, which is known to be true for many classes of groups.
(i) $S$ is left amenable.
(ii) For every nonzero $A, B \in \ell^1_+(S)$, there are nonzero $X, Y \in \ell^1_+(S)$ such that $AX = BY$.
(iii) For every nonzero $A, B \in \mathbb{N}[S]$, there are nonzero $X, Y \in \ell^1_+(S)$ such that $AX = BY$.

This result is new already for groups. We present the proof in the next section. As an immediate consequence, we obtain a strengthening of Donnelly’s result which drops the cancellativity assumption (and does not require the existence of a unit).

**Corollary 4.** Let $S$ be any semigroup for which $\mathbb{N}[S]$ has CRMs. Then $S$ is left amenable.

**Remark 5.** Especially in light of the density of finitely supported positive linear combinations in $\ell^1_+$, one may think that the existence of CRMs in $\ell^1_+(S)$ has a very similar flavour to the existence of CRMs in $\mathbb{R}_+[S]$ or in $\mathbb{N}[S]$. However, the simplicity of our proof for $\ell^1_+(S)$ suggests that the problem is generally much simpler in this case than for $\mathbb{R}_+[S]$ or $\mathbb{N}[S]$, for which combinatorial considerations play an essential role.

In particular, our result implies that if $G$ is the free metabelian group on two generators $a$ and $b$, then $\ell^1_+(G)$ has CRMs, while Guba’s argument mentioned above shows that $\mathbb{R}_+[G]$ does not. In fact, his argument shows that already for $A = 1 + a$ and $B = 1 + b$, the equation $AX = BY$ has no nontrivial solution with $X, Y \in \mathbb{R}_+[G]$; while our result is that a nontrivial solution must exist in $X, Y \in \ell^1(G)$, which is then necessarily infinitely supported.

Many open questions remain, among the most interesting of which is perhaps the problem of precisely characterizing the groups $G$ for which $\mathbb{N}[G]$ has CRMs. While this is trivially the case if $G$ is abelian, clearly false if $G$ is free, and false by Guba’s now result for $G$ the metabelian group on two generators, many intermediate cases remain open, starting with nilpotent groups. In fact, the CRM problem was originally communicated to us by Terence Tao in the following form.

**Problem 6.** Let $H$ be the discrete Heisenberg group. Then does $\mathbb{N}[H]$ have CRMs?

In this case, we have considered the problem for various particular choices of $A$ and $B$, partly with the help of a computer search implementing the relevant conditions on $X$ and $Y$ in terms of a linear program over $\mathbb{Q}$. This has resulted in explicit solutions for $X$ and $Y$ in all instances, but without the appearance of any detectable pattern that would allow for a general construction of $X$ and $Y$ in terms of $A$ and $B$.

2. Proof of Theorem 3

For the implication (i) $\Rightarrow$ (ii), we assume that $S$ is left amenable and will use a suitable infinite-dimensional linear programming duality argument to prove that $AX = BY$ has a nontrivial solution.

We start with a few general remarks for later use and for setting up notation.
For $s \in S$, we write $\delta_s \in \ell^1(S)$ for the associated indicator function.

In terms of the canonical pairing implementing the duality $\ell^\infty(S) = \ell^1(S)^*$, we have that for any $f \in L^\infty(S)$ and $X \in \ell^1(S)$,

$$\langle f, X \rangle = \left\langle f, \sum_s X_s \delta_s \right\rangle = \sum_s X_s \langle f, \delta_s \rangle = \sum_s X_s f(s),$$

where the second equation holds by linearity and continuity of the pairing. In particular, we have $f = (s \mapsto \langle f, \delta_s \rangle)$.

The augmentation homomorphism

$$\epsilon : \ell^1(S) \longrightarrow \mathbb{R}_+, \quad \sum_s X_s \delta_s \longmapsto \sum_s X_s$$

coinsides with the $\ell^1$-norm $\| - \|_1$ on the positive part $\ell^1_+(S)$.

If $\Lambda$ is a left invariant mean on $\ell^\infty(S)$, then its continuity and left invariance imply that

$$\Lambda(s \mapsto \langle f, X \delta_s \rangle) = \epsilon(X) \Lambda(f) \quad (2.1)$$

for all $X \in \ell^1(S)$.

It is a standard fact that the set of probability measures

$$\mathcal{P}(S) := \{X \in \ell^1_+(S) \mid \|X\|_1 = 1\}$$

is compact in the weak-* topology.

Now to find a nontrivial solution for $AX = BY$ for given nonzero $A, B \in \ell^1_+(S)$, we may as well assume that $A, B \in \mathcal{P}(S)$ by rescaling, and then look for a solution with $X, Y \in \mathcal{P}(S)$. Thus our goal is to prove that the set

$$\mathcal{C} := \{AX - BY : X, Y \in \mathcal{P}(S)\} \subseteq \ell^1(S)$$

contains $0$. Since $\mathcal{C}$ is by definition the linear and continuous image of the compact convex set

$$\mathcal{P}(S) \times \mathcal{P}(S) \subseteq \ell^1(S) \oplus \ell^1(S),$$

we know that $\mathcal{C}$ is convex and weak-* compact, and in particular closed.

Thus if $0 \not\in \mathcal{C}$, then by the Hahn-Banach separation theorem we can find $f \in \ell^\infty(S)$ such that

$$\langle f, AX - BY \rangle \geq 1 \quad \forall X, Y \in \mathcal{P}(S).$$

In particular, for all $s \in S$, we have

$$\langle f, A\delta_s \rangle \geq 1 + \langle f, B\delta_s \rangle.$$

Now considering both sides as functions of $s$ and applying $\Lambda$ yields

$$\Lambda(s \mapsto \langle f, A\delta_s \rangle) \geq 1 + \Lambda(s \mapsto \langle f, B\delta_s \rangle).$$
By (2.1) and $\epsilon(A) = \epsilon(B) = 1$, this reduces to
$$\Lambda(f) \geq 1 + \Lambda(f),$$
a contradiction.

The implication (ii) $\Rightarrow$ (iii) is trivial.

For the implication (iii) $\Rightarrow$ (i), we will prove amenability by establishing condition (ii) in Definition 2. So let finite $F \subseteq S$ and be given, and choose nonzero $k \in \mathbb{N}$. The assumption easily generalizes inductively to the statement that any finite set of nonzero elements has a nonzero CRM. Applying this to the formal sums $\sum_{j=1}^{k} m^j \in \mathbb{N}[S]$ for $m \in F$ shows that there is a family $(\psi_m)_{m \in F}$ of probability measures $\psi_m \in \mathcal{P}(S)$ such that the probability measure
$$\varphi := \frac{1}{k} \sum_{j=1}^{k} m^j \psi_m$$
is independent of $m \in F$. Then we get, again for $m \in F$,
$$m\varphi - \varphi = \frac{m^{k+1} - m}{k} \psi_m,$$
which implies the desired $\|m\varphi - \varphi\|_1 \leq 2k^{-1}$ by $\|\psi_m\|_1 = 1$.

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