Relationship Between Structural Characters and Synchronizability of Scale-free Networks

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Using Memory Tabu Search (MTS) algorithm, we investigate the relationship between structural characters and synchronizability of scale-free networks by maximizing and minimizing the ratio Q of the eigenvalues of the coupling matrix by edge-intercrossing procedures. The numerical results indicate that clustering coefficient C, maximal betweenness B_{max} are two important factors to scale-free network synchronizability, and assortative coefficient r and average distance D are the secondary ones. Moreover, the average degree \langle k \rangle affects the relationship between above structural characters and synchronizability of scale-free networks, and the minimal Q decreases when \langle k \rangle increases.

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Based on nonlinear dynamics, synchronization in coupled dynamical systems has been studied for many years. It is observed in a variety of fields [1, 2, 3, 4, 5]. In particular, synchronization in networks of coupled chaotic systems has received a great deal of attention over the past two decades [6, 7, 8, 9, 10, 11]. However, most of these works have been concentrated on networks with regular topological structures such as chains, grids, lattices, and fully connected graphs [12, 13, 14, 15]. Recent empirical studies have demonstrated that many real-world networks have two common statistical characteristics: small-world effect [16] and scale-free property [17], which cannot be treated as regular or random networks. Recently, an increasing number of studies have been devoted to investigating synchronization phenomena in complex networks with small-world and scale-free topologies [18, 19, 20, 21].

One of the goals in studying network synchronization is to understand how the network topology affects the synchronizability. The network synchronizability can be measured well by the eigenratio Q of the largest eigenvalue and the smallest nonzero eigenvalue [22, 23, 24, 25]; thus, our work is to understand the relationship between network structure and its eigenvalues. Since there are several topological characters of scale-free networks, what is the most important factor by which the synchronizability of the system is mainly determined?

In this brief report, we studied the relationship between structural characters and synchronizability of scale-free networks. Some detailed comparisons among various networks have been done, indicating the network synchronizability will be stronger with smaller heterogeneity, which can be measured by the variance of degree distribution or betweenness distribution [22, 26, 27], but the strict and clear conclusions have not been achieved because that previous studies are of both varying average distances and degree variances. Another extensively studied one is average distance D. Some works indicated the average distance D is one of the key factors to network synchronizability [28]. However, we have not achieved the consistent conclusion [19, 21, 25]. Some researchers considered that the randomicity is the more intrinsic factors leading to better synchronizability [29], which means that the intrinsic reason making small-world and scale-free networks having better synchronizability than regular ones is their random structures. Recently, several researches examine the effect of clustering coefficient on the synchronization by using Kuramoto model [30] or master stability function [31, 32]. Other researchers focus on the role played by maximal betweenness B_{max}, they found the network synchronizability will be better with smaller B_{max} [33]. Zhao et. al [27] enhanced the synchronizability by structural perturbations, they found that maximal betweenness plays a main role in network synchronization [34]. Zhou et. al [26] studied the average distance D to synchronizability by crossed double cycle. However, a network contain several statistical characteristics, such as degree distribution P(k), average distance D, clustering coefficient C, maximal betweenness B_{max} and so on. In the previous works, if one wants to show clearly how a structural character affects the network synchronizability, such as average distance D, he would investigate the network synchronizability with different D while keeping other structural characters constant approximately. However, this method neglect the influence made by the initial network structural characters. In fact, the network functions, such as the synchronizability, are affected by these characteristics simultaneously. Therefore, we should investigate these structural factors holistically. In order to find the real factors affect network synchronization and eliminate the influence made by the initial networks, we maximize and minimize the eigenratio Q by MTS algorithm [35] from the same initial network. The structural characters which change dramatically from maximal Q to minimal Q are the key factors influence network synchronizability, while the ones change little is not.
We investigate the synchronizability of a class of continuous-time dynamical networks with scale-free topology. Based on the synchronization criterion, we maximize and minimize the ratio $Q$ of the eigenvalues of the coupling matrix by edge-intercrossing procedures, which provides a way for observing the correlation between the synchronizability and those characteristics by keeping the degree distribution unchanged.

We start by considering a network of $N$ linearly coupled identical oscillators. The equation of motion reads

$$\dot{x}^i = F(x^i) + \sigma \sum_{j=1}^{N} G_{ij} H(x^j), \ i = 1, \ldots, N, \ (1)$$

where $x = F(x)$ govern the local dynamics of the vector field $x$ in each node, $H(x)$ is a linear vectorial function, $\sigma$ is the coupling strength, and $G$ is a coupling matrix.

Stability of the synchronous state $x^i(t) = x^j(t)$, $i = 1, \ldots, N$ can be accounted for by diagonalizing the linear stability equation, yielding $N$ blocks of the form $\zeta_i = JF(x)\zeta_i - \sigma \lambda_i H(\zeta_i)$, where $J$ is the Jacobian operator. Replacing $\sigma \lambda_i$ by $\nu$ in the equation, the master stability function (MSF) \cite{22} fully accounts for linear stability of the synchronization manifold. For a large class of oscillatory systems, the MSF is negative in a finite parameter interval $I_{st} \equiv (\nu_1 \leq \nu \leq \nu_2)$ \cite{22}. When the whole set of eigenvalues (multiplied by $\sigma$) enters the interval $I_{st}$, the stability condition is satisfied. This is accomplished when $\sigma \lambda_2 > \nu_2$ and $\sigma \lambda_N < \nu_2$ simultaneously. As $\nu_2$ and $\nu_1$ depend on the specific choice of $F(x)$ and $H(x)$, the key quantity for assessing the synchronizability of a network is the eigenratio

$$Q = \lambda_N / \lambda_2, \ (2)$$

which only depends on the topology of the network. The small $\lambda_N / \lambda_2$ is, the more packed the eigenvalues of $G$ are, leading to an enhanced $\sigma$ interval for which stability is obtained\cite{23}. In this paper, we will not address a particular dynamical system, but concentrate on how the network topology affects eigenratio $Q$.

The processes of heuristic algorithm, named MTS, is as follows.

**Step 1.** Generate an initial matrix $G_0$ of the extensional BA network \cite{33,35,36} with $N$ nodes and $E$ edges. Set the optimal network coupling matrix $G_* = G_0$ and the optimal network of tabu table $G_k = G_0$, and the time step $k = 0$. Compute the ratio $Q$ of $G_k^*$.  

**Step 2.** If a prescribed termination condition is satisfied, stop; Otherwise intercrossing a pairs of edges chosen randomly based on the network remains connected, denote by $G$.  

**Step 3.** If the ratio $Q$ of $G$, denoted by $Q_G$, satisfying $Q_G < Q_G^{k+1}, Q_G^{k+1} = Q_G$, else if $Q_G \leq Q_G^{k+1}, G_{k+1} := G$. When $Q_G > Q_G^{k+1}$, if $G$ does not satisfy the tabu condition, $|Q_G - Q_G^{k+1}| / R_G > \delta$ (where $\delta$ is a random number between 0.5 and 0.75), $G_{k+1} = G$, else $G_{k+1} = G$. Go to Step 2.

Since the MTS algorithm is heuristic, it can only find the approximate optimal solution. Thus, the termination condition of Step 2 should confirm by the experimentation solution.

The numerical results are experimented on extensional BA model for different network scales. The statistical properties of the optimal networks show similar trends. After many numerical experiments, we set the termination condition for maximizing $Q$ as 8000 time steps and the one for minimizing $Q$ as 3000 time steps, which can obtain the stability value using MTS algorithm.

We start from a network of size $N = 100, 200, 300, 400, 500$ and the average degree $\langle k \rangle = 6$ and then perform the optimization precesses. At each time step, we record the structural properties, such as $D, C, r$ and average node betweenness, when the objective function $Q$ is reduced. Let $D_{min}, C_{min}, r_{min}$ and $B_{min}$ denote the stable value when $Q$ reaches its minimal value $Q_{min}$, and $D_{max}, C_{max}, r_{max}$ and $B_{max}$ denote the stable value when $Q$ reaches its maximal value $Q_{max}$. Define the relative diversity function of the structural character $x$ as follows

$$f(x) = \frac{|x_{max} - x_{min}|}{x_{min}} \times 100, \ (3)$$

which can denote the difference of structural character $x$ to $Q_{max}$ and $Q_{min}$. The larger $f(x)$ is, the structural character $x$ change dramatically when the network leave
far from its optimal synchronizability state, which means \( r \) is more relevant to network synchronizability.

Figure 3 (b), (c) show \( C \) and \( r \) decrease to a stable value when maximizing and minimizing \( Q \), and \( C_{\text{max}} \) and \( r_{\text{max}} \) are both smaller than \( C_{\text{min}} \) and \( r_{\text{min}} \). The difference between \( C_{\text{min}} \) and \( C_{\text{max}} \), \( r_{\text{min}} \) and \( r_{\text{max}} \) means that the two structural characters are relevant to synchronizability of scale-free networks. Figure 3 (d) gives the change trend of average node betweenness when maximizing and minimizing \( Q \), which is consistent with \( D \). Figure 4 (a)-(d) demonstrate the stable value of \( D \), \( C \), \( r \) and \( B_{\text{max}} \) when \( N = 100, 200, 300, 400, 500 \). From Fig. 2 one can obtain that when \( N = 500 \), \( f(D) = 1.27 \), \( f(C) = 20.97 \) and \( f(r) = 1.27 \) and \( f(B_{\text{max}}) = 19.04 \). Moreover, one can see that the relative diversity of \( C \), \( r \) and \( B_{\text{max}} \) become large, while the one of \( D \) remain constant, which indicates that the influence produced by the structural characters \( C \), \( r \) and \( B_{\text{max}} \) to synchronizability of scale-free networks would become great when \( N \) become large. Furthermore, we investigate the relationship between average degree \( \langle k \rangle \) and \( f(x) \). Figure 3 demonstrates the \( Q_{\text{min}} \) obtained by MTS algorithm to different \( k \) when \( N = 500 \). The inset gives the functions of the structural characters obtained by different \( \langle k \rangle \). From the inset, one can see that if \( \langle k \rangle \) increase, the function of \( D \), \( r \) and \( B_{\text{max}} \) increases while the function of \( C \) decreases, which means that the influence of the structural characters to network synchronizability is affected by \( \langle k \rangle \). When \( \langle k \rangle \) increases, \( B_{\text{max}}, D \) and \( r \) become more relevant to synchronizability of scale-free networks, while \( C \) become less relevant.

In summary, using the MTS optimal algorithm, we maximized and minimized the network synchronizability by changing the connection pattern between different pairs of nodes while keeping the degree variance unchanged. Starting from extensional BA networks, we found the relationship between structural characters and synchronizability of scale-free networks. The numerical results indicate that \( D \), \( C \), \( r \) and \( B_{\text{max}} \) influence network synchronizability simultaneously. Especially, \( C \) and \( B_{\text{max}} \) are the two most important structural characters which affect synchronizability of scale-free networks, assortative coefficient \( r \) is the secondary character and \( D \) is the last one. Furthermore, the relationship is affected by the average degree \( \langle k \rangle \), and the maximal synchronizability of scale-free networks increases when \( \langle k \rangle \) increases.

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