A comparison of formulations for the simple assembly line balancing problem

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Assembly line balancing is a well-known problem in operations research. It consists in finding an assignment of tasks to some arrangement of stations where the tasks are executed. Typical objective functions require to minimize the cycle time for a given number of stations or the number of stations for a given cycle time.

Common to all variants of assembly line balancing are the precedence constraints among the tasks. Different inequalities for modeling these constraints have been proposed in the literature and lead to different formulations for problems such as the simple assembly line balancing problem (SALBP). We theoretically compare these formulations in terms of the strength of their linear relaxations. Moreover, we propose two new families of inequalities and a stronger formulation of the station limits for the SALBP and show that one of the resulting formulations dominates all existing ones.

We evaluate the efficiency of these formulations experimentally and show that the proposed formulation produces equal or better results than existing formulations.
1 Introduction

Assembly lines are an efficient way of producing large quantities of standardized products (Boysen et al., 2008). The products are successively assembled on some arrangement of stations, most commonly a linear sequence. On each station workers execute the subset of tasks assigned to that station. Due to technical restrictions the tasks usually have to respect precedence constraints and therefore cannot be assigned arbitrarily to stations.

The station time is the total time needed to execute the tasks assigned to a station and the cycle time of an assignment is the maximum station time. In problems of type 1 the goal is to minimize the number of used stations for a given cycle time, while in problems of type 2 the goal is to minimize the cycle time for a given number of stations. Other problem types include the feasibility problem (type F) of finding some solution for a given number of stations and a given cycle time, and the problem of type E of maximizing the line efficiency, i.e., minimizing the product of the number of stations and the cycle time.

The most simple and well studied problem of this kind is the simple assembly line balancing problem (SALBP) (Salveson, 1955). In this problem each task has a fixed execution time and the stations are arranged in a linear order. The four variants mentioned above are known as SALBP-1, SALBP-2, SALBP-F, and SALBP-E. All these problem variants are NP-complete (Scholl and Becker, 2006).

In this paper we theoretically and experimentally compare several formulations for the SALBP-1 and SALBP-2, and propose two new families of valid inequalities for expressing the precedence constraints.

The remainder of this paper is organized as follows. In the next section we present a formal definition and a mixed-integer linear model of the SALBP-1 and the SALBP-2. Next, we discuss different models proposed in the literature. In Section 3 we theoretically compare these models and two new families of valid inequalities for expressing the precedence constraints. A computational study is presented in Section 5 and we conclude in Section 6.

2 Mathematical models for the SALBP

In this section we review mathematical models for SALBP-1 and SALBP-2 that have been proposed in the literature. Let \((N, \leq)\) be a partially ordered set of tasks. For tasks \(i, j \in N\), we write \(i < j\) if \(i \leq j\) and \(i \neq j\) and \(i \ll j\) if \(i\) immediately precedes \(j\). Each task \(i \in N\) has execution time \(t_i\). Let \(S\) be the set of stations. In the following we suppose that for the SALBP-1 an upper bound \(m\) on the number of stations is given (\(|N|\) is such an upper bound), and that \(S = [m]\). For the SALBP-2 the number of stations \(m\) is known. In this case we set \(S = \{m\}\).

There have been three types of models proposed in the literature. Bowman (1960) (in the revised formulation of White (1961)) proposed two formulations, one based on binary impulse variables, indicating the station to which a task is allocated, and another based on time variables, representing the starting time of the tasks. Scholl (1999) proposed a formulation using binary step variables, which indicate that a task is allocated to some station or a preceding station.

\[\text{We write } [m] = \{1, \ldots, m\}\]
Since the formulation using time variables has been found inferior by Pastor et al. (2007) we focus in the following on formulations using impulse and step variables.

2.1 Formulations using impulse variables

An allocation of tasks to stations can be formulated using variables $x_{si} \in \{0, 1\}$ indicating that task $i \in N$ has been allocated to station $s \in S$. Any feasible allocation has to satisfy the occurrence constraints

$$\sum_{s \in S} x_{si} = 1 \quad \forall i \in N,$$  

which make sure that every task is assigned to a station, the precedence constraints

$$x_{tj} \leq \sum_{s \in S|s \leq t} x_{si} \quad \forall i, j \in N, i \leq j, \forall t \in S,$$  

and the nondivisibility constraints

$$x_{si} \in \{0, 1\} \quad \forall i \in N, s \in S.$$  

These constraints have been first proposed by Bowman (1960). Their above form is due to White (1961).

For a given cycle time $c$ and an upper bound $m$ on the number of stations, the SALBP-1 can be formulated as

$$(\text{BW1-1}) \quad \text{minimize} \quad \sum_{s \in S} s y_s,$$  

$$\text{subject to} \quad \sum_{i \in N} t_i x_{si} \leq cy_s \quad \forall s \in S,$$  

Equations (1)–(3),  

$$y_s \in \{0, 1\} \quad \forall s \in S,$$  

For the SALBP-2 the cycle time $c$ is variable and the number of stations $m$ is fixed. It can be formulated as

$$(\text{BW1-2}) \quad \text{minimize} \quad c,$$  

$$\text{subject to} \quad \sum_{i \in N} t_i x_{si} \leq c \quad \forall s \in S,$$  

Equations (1)–(3),  

$$c \in \mathbb{R}.$$  

These formulations are due to Baybars (1986), which also presents a survey of different formulations. In the following two subsections we present alternative formulations for the station limits and the precedence constraints.
2.1.1 Station limits

For a given cycle time $c$ and a given number of stations $m$, Patterson and Albracht (1975) observed that a task $i \in N$ cannot be assigned to a station earlier than

$$E_i(c) = \left\lceil \sum_{j \in N|j \leq i} t_j/c \right\rceil$$

or to a station later than

$$L_i(c, m) = m + 1 - \left\lceil \sum_{j \in N|j \leq i} t_j/c \right\rceil.$$ 

Let $A_i(c, m) = [E_i(c), L_i(c, m)]$ be the set of admissible stations for a task $i \in N$. Then we can add to the formulation of the SALBP-1 the constraints

$$x_{si} = 0 \quad \forall i \in N, \forall s \not\in A_i(c, \overline{m}),$$

where $\overline{m}$ is an upper bound on the number of stations. Similarly, if $\overline{c}$ is an upper bound on the cycle time $c$ we can add the constraints

$$x_{si} = 0 \quad \forall i \in N, \forall s \not\in A_i(\overline{c}, m)$$

to the formulation of the SALBP-2.

These inequalities have been first proposed by Patterson and Albracht (1975). They can be strengthened as proposed by Pastor and Ferrer (2009). For the SALBP-1, by taking into account currently used stations, we obtain

$$x_{Li(c,s),i} \leq y_s \quad \forall s \in S, \forall i \in N.$$ 

These inequalities are valid, since if a station $s$ is unused, we can assume that all later stations are also unused and therefore the latest possible station is now $L_i(c, s - 1) = L_i(c, s) - 1$.

For the SALBP-2 we can model the cycle time explicitly to obtain better bounds on the stations. Let $\underline{c}$ be a lower bound on the cycle time and let $C = [\underline{c}, \overline{c}]$ be the set of admissible cycle times. Then we can represent the cycle time by

$$c = \sum_{t \in C} r_t,$$

$$\sum_{t \in C} r_t = 1,$$

$$r_t \in \{0, 1\} \quad t \in C.$$ 

This allows to add the following valid inequalities to the formulation of the SALBP-2:

$$x_{ei} \leq 1 - \sum_{t \in C|e < E_i(t)} r_t \quad \forall i \in N, \forall s \not\in A_i(\overline{c}),$$

$$x_{li} \leq 1 - \sum_{t \in C|L_i(t, m) < l} r_t \quad \forall i \in N, \forall s \not\in A_i(\overline{c}, m).$$ 

These inequalities are easily seen to be valid. Suppose, for example, that $\sum_{t \in C|e < E_i(t)} r_t = 1$ in equation (20). Then station $e$ comes before the earliest possible station for the current cycle time, and therefore $x_{ei} = 0$. Similarly, if $\sum_{t \in C|L_i(t, m) < l} r_t = 1$ in equation (21) station $l$ is after the latest possible station for the current cycle time, and therefore $x_{li} = 0$. 

4
2.1.2 Precedence constraints

Patterson and Albracht (1975) have proposed to formulate the precedence constraints as

\[ \sum_{s \in S} sx_{si} \leq \sum_{s \in S} sx_{sj} \quad \forall i, j \in N, i < j. \quad (22) \]

Thangavelu and Shetty (1971) observe that an alternative formulation of the precedence constraints is given by

\[ \sum_{s \in S} (m - s + 1)(x_{si} - x_{sj}) \geq 0 \quad \forall i, j \in N, i < j. \quad (23) \]

2.2 Formulations using step variables

An alternative formulation can be obtained using step variables \( x_{si} \in \{0, 1\} \) indicating that task \( i \in N \) has been allocated to a station \( t \in S \) such that \( t \leq s \). When using step variables, we obtain occurrence constraints

\[ x_{si} = 1 \quad \forall i \in N, \quad (24) \]

and precedence constraints

\[ x_{si} \geq x_{sj} \quad \forall i, j \in N, i < j. \quad (25) \]

Additionally, the step variables have to satisfy the continuity constraints

\[ x_{si} \leq x_{s+1,i} \quad \forall i \in N, s \in S, s < m. \quad (26) \]

To simplify the formulation, we assume that \( x_{0i} = 0 \) for all \( i \in N \). Then, the SALBP-1 can be formulated as

\[(SC1-1) \quad \text{minimize} \quad \sum_{s \in S} sy_s, \quad (27)\]

subject to

\[ \sum_{i \in N} t_i(x_{si} - x_{s-1,i}) \leq cy_s \quad \forall s \in S, \quad (28)\]

Equations (24)–(26),

\[ y_s \in \{0, 1\} \quad \forall s \in S. \quad (30)\]

and the SALBP-2 can be formulated as

\[(SC1-2) \quad \text{minimize} \quad c, \quad (31)\]

subject to

\[ \sum_{i \in N} t_i(x_{si} - x_{s-1,i}) \leq c \quad \forall s \in S, \quad (32)\]

Equations (24)–(26),

\[ c \in \mathbb{R}. \quad (34)\]
2.2.1 Station limits

The station limits proposed for the formulations using impulse variables can also be applied to the formulation using step variables. Using the above notation we can add to the formulation of SALBP-1

\[
\begin{align*}
\bar{x}_{E_i(c)} - 1, i & = 0 \quad \forall i \in N \quad (35) \\
\bar{x}_{L_i(c, m)} & = 1 \quad \forall i \in N \quad (36)
\end{align*}
\]

and to the formulation of SALBP-2

\[
\begin{align*}
\bar{x}_{E_i(c)} - 1, i & = 0 \quad \forall i \in N \quad (37) \\
\bar{x}_{L_i(c, m)} & = 1 \quad \forall i \in N \quad (38)
\end{align*}
\]

Similarly, the strengthened station limits in the step model are

\[
\begin{align*}
\bar{x}_{E_i(c)} - 1, i & = 1 - y_s \quad \forall i \in N \quad (39)
\end{align*}
\]

for SALBP-1 and

\[
\begin{align*}
\bar{x}_{E_i(c)} & \leq 1 - \sum_{t \in C | E_i(t) < E_i(c)} r_t \quad \forall i \in N, E_i(c) \leq e < E_i(c), \quad (40) \\
\bar{x}_{E_i(c)} & \geq \sum_{t \in C | E_i(c, m) < l} r_t \quad \forall i \in N, L_i(c, m) < l \leq L_i(c, m) \quad (41)
\end{align*}
\]

together with equations (17)–(19) for the SALBP-2.

3 New inequalities for the SALBP

3.1 Station limits

We can strengthen the station limits proposed by Pastor and Ferrer (2009) as follows. In inequality (16), when a task cannot be assigned to station \(L_i(c, s)\) it also cannot be assigned to any later station. This justifies

\[
\sum_{u \in S | u \geq L_i(c, s)} x_{ui} \leq y_s \quad \forall s \in S, i \in N. \quad (42)
\]

Similarly, in equations (20) and (21) when a task cannot be assigned earlier than station \(e\) this holds also for stations preceding \(e\), and when a task cannot be assigned later than station \(l\), this holds also for stations following \(l\). Therefore we can strengthen these inequalities to

\[
\sum_{u \in S | u \leq e} x_{ui} \leq 1 - \sum_{t \in C | E_i(t) < E_i(c)} r_t \quad \forall i \in N, E_i(c) \leq e < E_i(c), \quad (43)
\]

and

\[
\sum_{u \in S | u \geq l} x_{ui} \leq 1 - \sum_{t \in C | L_i(c, m) < l} r_t \quad \forall i \in N, L_i(c, m) < l \leq L_i(c, m). \quad (44)
\]
3.2 Precedence constraints

We propose two new formulations of the precedence constraints:

\[ x_{si} + x_{tj} \leq 1 \quad \forall i, j \in N, i \preceq j, t < s \]  
\[ \sum_{s \in S \mid s \geq k} x_{si} \leq \sum_{s \in S \mid s \geq k} x_{sj} \quad \forall i, j \in N, i \preceq j, k \in S. \]  

Inequalities (45) are valid, since for \( i \preceq j \) task \( i \in N \) cannot be executed on a later station than task \( j \in N \).

**Proposition 1** Inequalities (46) are valid for BW1-1 and BW1-2.

**Proof.** If \( \sum_{s \in S \mid s \geq k} x_{si} = 0 \) the inequality is trivially satisfied. Otherwise, task \( i \) is executed on station \( k \) or later. But since \( i \preceq j \) task \( j \) has to be executed also on station \( k \) or a later station, i.e., \( \sum_{s \in S \mid s \geq k} x_{sj} = 1 \).

**Proposition 2** Inequalities (46) imply inequalities (22).

**Proof.** Summing over all \( k \in S \) yields, for all \( i, j \in N, i \preceq j \),

\[ \sum_{k \in S} \sum_{s \in S \mid s \geq k} x_{si} \leq \sum_{k \in S} \sum_{s \in S \mid s \geq k} x_{sj} \iff \sum_{s \in S} sx_{si} \leq \sum_{s \in S} sx_{sj} \]

**Proposition 3** Inequalities (23) are equivalent to inequalities (22).

**Proof.**

\[ \sum_{s \in S} (m - s + 1)(x_{si} - x_{sj}) \geq 0 \]

\[ \iff \sum_{s \in S} (m - s + 1)x_{si} \geq \sum_{s \in S} (m - s + 1)x_{sj} \]

\[ \iff (m + 1) \sum_{s \in S} x_{si} - \sum_{s \in S} sx_{si} \geq (m + 1) \sum_{s \in S} x_{sj} - \sum_{s \in S} sx_{sj}, \]

which, in the presence of equalities (1), are equivalent to

\[ \sum_{s \in S} sx_{si} \leq \sum_{s \in S} sx_{sj}. \]

**Proposition 4** Inequalities (46) imply inequalities (2).
**Proof.** From (46), in the presence of equality (1), and considering that all \(x_{sj} \geq 0\), we obtain, for all \(i, j \in N, i \prec j\), and for all \(k \in S\),

\[
1 - \sum_{s \in S|s < k} x_{si} \leq \sum_{s \in S|s < k-1} x_{sj} + \sum_{s \in S|s \geq k} x_{sj},
\]

and again in the presence of equality (1)

\[
1 - \sum_{s \in S|s < k} x_{si} \leq 1 - x_{kj},
\]

\[
\Leftrightarrow 1 - \sum_{s \in S|s \leq k-1} x_{si} \leq 1 - x_{kj},
\]

\[
\Leftrightarrow x_{kj} \leq \sum_{s \in S|s < k-1} x_{sj}.
\]

Therefore, for all \(t = k-1, k \in S\)

\[
x_{ij} \leq \sum_{s \in S|s \leq t} x_{si},
\]

We are missing the last inequality

\[
x_{mj} \leq \sum_{s \in S} x_{si},
\]

but this is trivially obtained in the presence of equality (1), since \(\sum_{s \in S} x_{si} = 1\). ■

**Proposition 5** Inequalities (2) imply inequalities (45).

**Proof.** In the presence of equality (1) we can rewrite inequalities (2), for all \(i, j \in N, i \prec j, t \in S\), as

\[
x_{ij} \leq 1 - \sum_{s \in S|s > t} x_{si}
\]

\[
\Leftrightarrow x_{ij} + \sum_{s \in S|s > t} x_{si} \leq 1.
\]

Therefore, for all \(i, j \in N, i \prec j, t < s\),

\[
x_{si} + x_{ij} \leq \sum_{s \in S|s > t} x_{si} + x_{ij} \leq 1.
\]

■

**Proposition 6** Inequalities (46) dominate inequalities (22).

**Proof.** Let \(N = \{a, b\}\) and \(a \prec b\) and suppose \(m = 3\). Then, \(x_{1a} = x_{2a} = 1/2, x_{1b} = 3/4\), and \(x_{3b} = 1/4\) satisfies (22), but not (46). ■
Table 1: Different formulations of the SALBP-1.

| Precedence constraints | Station limits |
|------------------------|----------------|
| Name | Equations | Res. | Name | Equations | Res. |
| PA | (22) | on² | 1 | - | - |
| BW | (2) | on²m | 2 | (14) | - |
| TS | (23) | on² | 3 | (14),(16) | m\(n\) |
| RC | (45) | on² | 4 | (14),(42) | m\(n\) |
| RC' | (46) | on²m² | - | - | - |

Base model

| Suffix | Equations | Var. | Res. |
|--------|-----------|------|------|
| -1 | 1,3,4,5,7 | m(n + 1) | m + n |

**Proposition 7** Inequalities (46) dominate inequalities (2).

**Proof.** Let \( N = \{a, b\} \) and \( a < b \) and suppose \( m = 3 \). Then, \( x_{1a} = x_{1b} = x_{2b} = x_{3a} = 1/2 \) satisfies (2), but not (46). ■

**Proposition 8** Inequalities (22) and (2) are incomparable.

**Proof.** The example given in the proof of Proposition 6 satisfies (22), but not (2). The example from Proposition 7 satisfies (2), but not (22). ■

**Proposition 9** Inequalities (22) and (45) are incomparable.

**Proof.** The example from Proposition 6 satisfies (22), but not (45). The example from Proposition 7 satisfies (45), but not (22). ■

In summary, inequalities (46) dominate, to the best of our knowledge, all inequalities proposed in the literature for formulating the precedence constraints. Tables 1 and 2 give a summary of models with different precedence constraints and station limits for the SALBP-1 and the SALBP-2 and their number of variables and restrictions. We obtain a different model for each combination of the precedence constraints, the stations limits and the base model. Observe that equations (14) and (15) do not increase the number of restrictions, but reduce the number of variables. In both tables \( o \) denotes the order strength of the instance. The order strength of an instance is at most \( \binom{n}{2} \).

Figure 1 shows the relationships between these models.

## 4 Comparison of formulations using impulse and step variables

In this section we demonstrate that RC implies SC under the variable conversion

\[
\text{x}_{sl} = \sum_{u \in S | u \leq s} x_{su}.
\]
Table 2: Different formulations of the SALBP-2.

| Precedence constraints | Station limits |
|------------------------|----------------|
| Name | Equations | Res. | Name | Equations | Res. |
| PA (22) | $on^2$ | 1 | - | - |
| BW (2) | $on^2m$ | 2 | (15) | - |
| TS (23) | $on^2$ | 3 | (15),(17),(21) | mn |
| RC (45) | $on^2m$ | 4 | (15),(17),(19),(43),(44) | mn |
| RC’ (46) | $on^2m^2$ | | | |

Base model

| Suffix | Equations | Var. | Res. |
|--------|-----------|------|------|
| -2 | 1,3,8,9,11 | nm + 1 | m + n |

Figure 1: Relationships between SALBP formulations. The relationships are due to different precedence constraints and are valid for SALBP-1 as well as SALBP-2 formulations with the same station limits.
Using this conversion, since \( x_{si} \geq 0 \) for all \( s \in S \) and \( i \in N \), the continuity constraints of SC1 are satisfied because

\[
\bar{x}_{si} = \sum_{u \in S, u \leq s} x_{ui} \leq \sum_{u \in S, u \leq s+1} x_{ui} = \bar{x}_{s+1,i},
\]

and by equality (1) we obtain

\[
\bar{x}_{si} = \sum_{u \in S, u \leq |S|} x_{ui} = \sum_{u \in S} x_{ui} = 1,
\]

i.e., the occurrence constraints hold. We can write the precedence constraints (25) as

\[
\sum_{u \in S, u \leq s} x_{ui} \geq \sum_{u \in S, u \leq s} x_{u j} \iff 1 - \sum_{u \in S, u > s} x_{ui} \geq 1 - \sum_{u \in S, u > s} x_{u j}
\]

\[
\sum_{u \in S, u > s} x_{ui} \leq \sum_{u \in S, u > s} x_{u j},
\]

which shows that the precedence constraints in both formulations are equivalent, and we can write equation (28) as

\[
\sum_{i \in N} \left( \sum_{u \in S, u \leq s} x_{ui} - \sum_{u \in S, u \leq s-1} x_{ui} \right) \leq cy_s
\]

\[
\iff \sum_{i \in N} l_i x_{si} \leq cy_s,
\]

which shows that the cycle time constraints in both formulations are equivalent.

The implication also applies when inequalities for station limits are present. By equation (14) we have \( x_{si} = 0 \) for \( s < E_i(c) \) and for \( s > L_i(c, m) \) and therefore equation (35)

\[
\bar{x}_{L_i(c, m)} = \sum_{s \leq L_i(c, m)} x_{si} = \sum_{s \in S} x_{si} = 1
\]

and equation (36)

\[
\bar{x}_{E_i(c) - 1,i} = \sum_{s < E_i(c)} x_{si} = 0.
\]

hold. By equation (42) we have

\[
\bar{x}_{L_i(c, s) - 1,i} = \sum_{s \in S, s < L_i(c, s)} x_{si} = 1 - \sum_{s \in S, s \geq L_i(c, s)} x_{si} \geq 1 - y_s
\]

and by equations (43) and (44) we obtain

\[
\bar{x}_{ei} = \sum_{s \in S, s \leq e} x_{si} \leq 1 - \sum_{t \in C, e < E_i(t)} r_t \quad \forall i \in N, E_i(\bar{\tau}) \leq e < E_i(\bar{\mu})
\]

and

\[
\bar{x}_{l - 1,i} = \sum_{s \in S, s \leq l - 1} x_{si} = 1 - \sum_{s \in S, s \geq l} x_{si} \leq \sum_{t \in C, l \geq L_i(t, m), l < l} r_t \quad \forall i \in N, L_i(\bar{\nu}, m) < l \leq L_i(\bar{\tau}, m).
\]

This shows that the dynamic station limits in RC imply the dynamic station limits in SC.
5 Computational experiments

We have empirically evaluated the performance of the formulations of the SALBP-1 and the SALBP-2 presented in Tables 1 and 2. We limited the comparison to the best known formulations PA and BW from the literature and the new formulation RC for the precedence constraints combined with all four sets of equations for the station limits, for a total of 12 SALBP-1 and SALBP-2 formulations.

The formulations have been tested on 269 instances of the SALBP-1 and 302 instances of the SALBP-2. These instances are available online (ALB, 2011). Details about them can be found in Scholl (1993). Currently the optimal value is known for all except one of the SALBP-1 instances and 14 of the SALBP-2 instances. In the evaluations below solutions for instances without a known optimum were never considered optimal.

All experiments were performed on a PC with an Intel Core i7 CPU running at 2.8 GHz and 12 GB of main memory. We used the solver CPLEX 12.3.0 with standard options, except for a MIP optimality gap of $10^{-5}$, running in deterministic mode with two threads for a maximum time of 600 seconds. All computation times reported are in seconds of real time. Following Pastor and Ferrer (2009) we use in our experiments lower bounds on the number of stations $m = LM1 := \lceil \sum_{i \in N} t_i / c \rceil$ and on the cycle time $c = LC1 := \max \{ \max_{i \in N} t_i, \lceil \sum_{i \in N} t_i / m \rceil \}$ for the SALBP-1 and SALBP-2 respectively (Scholl and Becker, 2006) and an upper bound on the number of stations of $m = \min \{ 2m, |N| \}$ and on the cycle time of $\bar{c} = 2c$.

Tables 3 and 4 show the results for the formulations of the SALBP-1 and the SALBP-2, respectively. For each tested model, the first column presents the relative deviation of the LP bound from the optimal value in percent. The next three columns present the number of instances for which the branch-and-cut solver of CPLEX found a provably optimal solution, the average solution time for these instances, and the average number of nodes explored in the branch-and-cut tree. The next two columns present the number of instances for which an optimal solution was found, but could not be proven to be optimal and the average number of nodes explored for these instances. The following two pairs of columns present the same information for the instances which terminated with a feasible, but not optimal solution, and for the instances for which no feasible solution was found. The last column presents the number of instances where the formulation obtained the best value found over all 12 formulations of the same problem type.

In our evaluation, when comparing solution values of different formulations we inform the $p$-values obtained by a conservative, non-parametric paired sign test, where half of the ties were assigned to each sample (Dixon and Mood, 1946).

In both problem types, the relative deviations of the LP relaxation from the best known value improve with stronger station limits. The new station limits produce the best bounds, but improve only slightly over the station limits of Pastor and Ferrer (2009). For the SALBP-1 they lead to more provably optimal and more best solutions, but not for the SALBP-2. In both problem types the solution value does not improve significantly compared to other forms of station limits (for $p = 0.05$). The relative deviations are the same under different formulations of the precedence constraints. Therefore, the influence of the precedence constraints on the results, discussed below, is due to different relaxations and cuts of subproblems during the execution of the branch-and-cut algorithm.

Concerning the SALBP-1, formulations BWn and RCn obtain better results than the corre-
Table 3: Comparison of formulations of the SALBP-1. Time and number of nodes are averages over the instances of each category.

| Model | Dev. | Optimal | Unproven | Sub-optimal | Feasible | Infeasible | Best |
|-------|------|---------|----------|-------------|----------|------------|------|
|       |      | Proven  |          |             | Feasible | Infeasible |      |
| PA1   | 5,408 | 158     | 32.5     | 25027       | 24       | 667091     | 47   | 372095 | 40 | 24038 | 205 |
| PA2   | 5,171 | 175     | 24.8     | 38739       | 22       | 495261     | 35   | 454219 | 37 | 60031 | 218 |
| PA3   | 4,349 | 184     | 34.5     | 41886       | 18       | 993955     | 31   | 507752 | 36 | 4963  | 215 |
| PA4   | 4,348 | 186     | 35.2     | 43814       | 18       | 1020934    | 33   | 561443 | 32 | 13527 | 223 |
| BW1   | 5,408 | 174     | 52.5     | 66862       | 20       | 166021     | 75   | 109242 | 0  | -    | 221 |
| BW2   | 5,171 | 178     | 28.0     | 17152       | 20       | 280644     | 71   | 101728 | 0  | -    | 230 |
| BW3   | 4,349 | 189     | 32.8     | 34982       | 16       | 195908     | 64   | 108796 | 0  | -    | 232 |
| BW4   | 4,348 | 190     | 22.5     | 36286       | 21       | 443164     | 58   | 137030 | 0  | -    | 247 |
| RC1   | 5,408 | 184     | 37.7     | 32217       | 16       | 397677     | 69   | 94612  | 0  | -    | 234 |
| RC2   | 5,171 | 188     | 28.0     | 30878       | 18       | 217022     | 63   | 80017  | 0  | -    | 242 |
| RC3   | 4,349 | 189     | 30.5     | 30624       | 15       | 326869     | 65   | 168141 | 0  | -    | 242 |
| RC4   | 4,348 | 197     | 35.6     | 22313       | 17       | 284820     | 55   | 204217 | 0  | -    | 260 |

Table 4: Comparison of formulations of the SALBP-2. Time and number of nodes are averages over the instances of each category.

| Model | Dev. | Optimal | Unproven | Sub-optimal | Feasible | Infeasible | Best |
|-------|------|---------|----------|-------------|----------|------------|------|
|       |      | Proven  |          |             | Feasible | Infeasible |      |
| PA1   | 2,100| 164     | 70.4     | 77770       | 17       | 258272     | 121  | 437489 | 0  | -    | 199 |
| PA2   | 2,100| 170     | 55.3     | 56314       | 12       | 549199     | 120  | 330365 | 0  | -    | 196 |
| PA3   | 0.692| 187     | 57.4     | 55120       | 9        | 657220     | 86   | 381573 | 21 | 1037  | 206 |
| PA4   | 0.691| 184     | 28.3     | 45916       | 11       | 1036212    | 93   | 529207 | 14 | 1151  | 206 |
| BW1   | 2,100| 188     | 49.3     | 59523       | 17       | 430656     | 97   | 254536 | 0  | -    | 219 |
| BW2   | 2,100| 192     | 50.1     | 79907       | 11       | 347230     | 99   | 334954 | 0  | -    | 220 |
| BW3   | 0.692| 186     | 31.3     | 49803       | 10       | 714271     | 97   | 359153 | 9  | 106   | 210 |
| BW4   | 0.691| 188     | 24.9     | 47618       | 10       | 334468     | 97   | 321281 | 7  | 85    | 213 |
| RC1   | 2,100| 189     | 34.1     | 54955       | 14       | 399572     | 99   | 424456 | 0  | -    | 234 |
| RC2   | 2,100| 186     | 32.4     | 62847       | 15       | 224409     | 101  | 451419 | 0  | -    | 232 |
| RC3   | 0.692| 200     | 41.9     | 86384       | 10       | 441196     | 92   | 308320 | 0  | -    | 225 |
| RC4   | 0.691| 196     | 47.0     | 65991       | 8        | 402442     | 98   | 349380 | 0  | -    | 224 |
sponding formulation PA\(n\) (with \(p \leq 0.04\) for BW\(n\) and \(p \leq 0.02\) for RC\(n\), \(1 \leq n \leq 4\)). They are able to prove and find more optimal solutions, always find a feasible solution, and more often find the best result. The computation times are comparable, except the higher execution time of BW1. Formulations RC\(n\) prove and find more optimal solutions and more best values than BW\(n\), but the difference in solution value is not statistically significant (for \(p = 0.05\)). From a practical point of view formulation RC4 could be considered best, since it provides the largest number of best solutions in a time comparable to the other formulations.

Concerning the SALBP-2, the formulations BW\(n\) find and prove more optimal solutions than formulations PA\(n\), but the improvement in solution value is not statistically significant. The reason is that BW\(n\) obtains worse solutions on the 28 instances proposed by Scholl (1993), for which both formulations find only feasible solutions. Formulations RC\(n\) find and prove more optimal solutions than BW\(n\) and PA\(n\), except for RC2, and always find a feasible solution. They obtain better solutions values than PA\(n\) (with \(p \leq 0.001\)). The improvement over BW\(n\) is marginally significant (\(p \leq 0.04\) for \(1 \leq n \leq 3\), and \(p = 0.06\) for \(n = 4\)). Formulations RC\(n\) also find most best values. The computation times of formulations RC\(n\) and BW\(n\) are less than those of PA\(n\), except for RC4. Computation times of RC\(n\) and BW\(n\) are similar, but a comparison of the computation times on the 142 instances that all formulations could prove optimal shows that RC\(n\) tends to increase computation time. The difference in solution value among formulations RC\(n\) is not statistically significant and we can observe a trade-off between proving more solutions optimal and finding better solution values, where formulation RC4 tends to find and prove less optimal solutions. From a practical viewpoint one may prefer formulation RC1 which find the largest number of best solutions in a short time. Formulations BW2, RC1, and RC2 were able to prove that the previously open instance Wee-Mag 23 has optimal solution value 65.

In a computational study Pastor et al. (2004) found no significant difference between formulations PA and BW, but observe that formulation BW leads to shorter solving times. Our results do confirm that only for the SALBP-2. The different results may come from the difference between the used solvers (CPLEX 8.0 and CPLEX 12.3). Pastor and Ferrer (2009) find that the dynamic station limits (PA3) increase the number of provably optimal solutions over formulation PA2, which is corroborated by our findings. The solution values, on the other hand, do not decrease significantly, and the dynamic station limits tend make it more difficult to find feasible solutions in the SALBP-2.

6 Conclusions

We have proposed two new formulations of the precedence constraints for the SALBP, and shown that one of them dominates, to the best of our knowledge, the precedence constraints proposed in the literature. They are applicable to other assembly line balancing problems with precedence constraints. We have further given a variable conversion under which the model for the SALBP using impulse variables implies the model using step variables when using the newly proposed precedence constraints and the strengthened station limits. Comparing the new and existing models, we have provided a complete classification of the relationships between the models currently existing in the literature.
An experimental evaluation corroborates the theoretical comparison. The newly proposed precedence constraints can improve upon the constraints of Patterson and Albracht (1975) and Bowman (1960), finding and proving more solutions optimal, and finding more best values. A conservative statistical test shows that the improvement of the solution value is significant, with one exception for SALBP-1, where the newly proposed constraints and those of Bowman (1960) have to be considered practically equivalent.

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