ON DRINFELD REALIZATION OF QUANTUM AFFINE ALGEBRAS

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Abstract. We provide a direct proof of the Drinfeld realization for the quantum affine algebras.

1. Introduction

In 1987 Drinfeld [Dr2] gave an extremely important realization of quantum affine algebras [Dr1][Jb]. This new realization has led to numerous applications such as the vertex representations [FJ][J]. The proof of this realization was not in print until Beck’s braid group interpretation for the untwisted types [B]. Some of lower rank cases were also studied in [D][S]. All these work started from the quantum group towards the quantum loop realization, and were based on Lusztig’s theory of braid group action on the quantum enveloping algebras [L]. However, Drinfeld did give the exact isomorphism between two definitions of quantum affine algebras in [Dr2]. In this paper we give another proof directly from the Drinfeld isomorphism. Our proof is self-contained and elementary and works from the opposite direction from the quantum loop algebras towards the quantum groups.

In doing this, we discovered that there are rich structures held by the \( q \)-loop algebra realization. We directly deform the argument used by Kac [K] to identify the affine Lie algebras and the Kac-Moody algebra defined by generators and relations. Here we must admit that the \( q \)-arguments are much more complicated than the classical analog, where we have the root space structure available. It is nontrivial to properly deform usual brackets by \( q \)-brackets. As we have shown here in many cases there are strong indications for us to follow. The key are the following identities:

\[
[a, [b, c]_u]_v = [[a, b]_x, c]_{uv/x} + x [a, c]_{x/2v/x}, \quad x \neq 0
\]

\[
[[a, b]_u, c]_v = [a, [b, c]_x]_{uv/x} + x [a, c]_{v/x, u/x}, \quad x \neq 0
\]

where one needs to choose an appropriate \( x \) to apply.

We also discuss the Drinfeld realization for the twisted quantum affine algebras using the same approach. The results are used to construct the intertwining operators between level one modules of twisted quantum affine algebras in [JK].

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2. Quantum affine algebras

Let $A = (a_{ij})$ ($i, j \in I = \{0, 1, \cdots, n\}$) be a generalized Cartan matrix of affine types [K]. Let $\mathfrak h$ be a vector space over $\mathbb C(q^{1/2})$ with a basis $\{h_0, h_1, \cdots, h_n, d\}$ and define the linear functionals $\alpha_i \in \mathfrak h^*$ ($i \in I$) by

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{i,0} \quad \text{for } j \in I. \tag{2.1}$$

Then the triple $(\mathfrak h, \Pi = \{\alpha_i \mid i \in I\}, \Pi' = \{h_i \mid i \in I\})$ is the realization of the matrix $A$. The Kac-Moody Lie algebra of type $\mathfrak g$ associated with the matrix $A$ is called the affine Kac-Moody algebra of type $A$ (cf. [K]). The algebra is generated as a Lie algebra by $e_i, f_i, h_i$ ($i \in I$) and $d$ subject to the usual relations. The elements of $\Pi$ (resp. $\Pi'$) are called the simple roots (resp. simple coroots) of $\mathfrak g$.

The standard nondegenerate symmetric bilinear form $(\cdot | \cdot)$ on $\mathfrak h^*$ satisfies

$$\langle \alpha_i | \alpha_i \rangle = d_i a_{ij}, \quad \langle \delta | \alpha_i \rangle = \langle \delta | \delta \rangle = 0 \quad \text{for all } i, j \in I. \tag{2.4}$$

where $d_i = (\alpha_i | \alpha_i)/2$ are rational numbers given as follows:

- $(ADE)^{(1)}$: $d_i = 1$
- $C_n^{(1)}$: $d_0 = 1, d_i = 1/2, d_n = 1$
- $B_n^{(1)}$: $d_0 = d_i = 1, d_n = 1/2$
- $G_2^{(1)}$: $d_0 = d_1 = 1, d_2 = 1/3$
- $F_4^{(2)}$: $d_0 = d_1 = d_2 = 1, d_3 = d_4 = 1/2$

and $q_i = q^{d_i}$ for $i \in I$. The quantum affine Lie algebra $U_q(A)$ is the associative algebra with 1 over $\mathbb C(q^{1/2})$ generated by the elements $e_i, f_i$ ($i \in I$) and $q^h$ ($h \in P'$) with the following defining relations:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P',$$

$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P'(i \in I),$$

$$e_i f_j - f_j e_i = \delta_{ij} t_i - t_i^{-1}, \quad \text{where } t_i = q^{h_i} \text{ and } i, j \in I,$$

$$\sum_{m+k=1-a_{ij}} (-1)^m e_i^{(m)} f_j f_i^{(n)} = 0, \quad \sum_{m+n=1-a_{ij}} (-1)^m f_i^{(m)} f_j e_i^{(n)} = 0 \quad \text{for } i \neq j, \tag{2.5}$$

where $e_i^{(k)} = e_i^k/[k]!$, $f_i^{(k)} = f_i^k/[k]!$, $[m]! = \prod_{k=1}^m [k]$, and $[k]_i = \frac{q_i^k - q_i^{-k}}{q - q^{-1}}$.

Let $\Omega$ be the anti-algebra involution of $U_q(\hat{\mathfrak g})$ over $\mathbb C$ given by

$$\Omega(e_i) = f_i, \quad \Omega(f_i) = e_i, \quad \Omega(q^h) = q^{-h}, \quad \Omega(q) = q^{-1}.$$

Now we give the Drinfeld realization for the untwisted types.
Let $U$ be the associative algebra with 1 over $\mathbb{C}(q^{1/2})$ generated by the elements $x^\pm_i(k), a_i(l), K_i^{\pm 1}, \gamma^\pm 1/2, q^{k/d} \ (i = 1, 2, \ldots, n, k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\})$ with the following defining relations:

\[
[\gamma^{\pm 1/2}, u] = 0 \quad \text{for all } u \in U,
\]
\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,
\]
\[
[a_i(k), a_j(l)] = \delta_{k+l,0} \frac{[a_{ij}]}{k} \gamma^k - \gamma^{-k},
\]
\[
[a_i(k), K_j^{\pm 1}] = [q^{\pm d}, K_j^{\pm 1}] = 0,
\]
\[
q^d x_i^\pm(k) q^{-d} = q^k x_i^\pm(k), \quad q^d a_i(l) q^{-d} = q^l a_i(l),
\]
\[
K_i x_j^\pm(k) K_i^{-1} = q^{\pm (\alpha_i | \alpha_j)} x_j^\pm(k),
\]
\[
[a_i(k), x_j^\pm(l)] = \frac{1}{k} [a_{ij}] \gamma^{\pm |j|/2} x_j^\pm(k + l),
\]
\[
x_i^+(k + 1) x_j^+(l) - q^{\pm (\alpha_i | \alpha_j)} x_j^+(l) x_i^+(k + 1) = q^{\pm (\alpha_i | \alpha_j)} x_j^+(k) x_i^+(l + 1) - x_j^+(l + 1) x_i^+(k),
\]
\[
[x_i^+(k), x_j^-(l)] = \sum_{m=0}^{\infty} s_{ij} \left( \gamma^{k} \psi_i(k + l) - \gamma^{l-k} \psi_i(k + l) \right),
\]

where $\psi_i(m)$ and $\varphi_i(-m)$ ($m \in \mathbb{Z}_{\geq 0}$) are defined by

\[
\sum_{m=0}^{\infty} \psi_i(m) z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_i(k) z^{-k} \right),
\]
\[
\sum_{m=0}^{\infty} \varphi_i(-m) z^{m} = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_i(-k) z^{k} \right),
\]
\[
Sym \prod_{i} \sum_{m=1-a_{ij}}^{m=0} (-1)^s \binom{m}{s} x_i^+(l_1) \cdots x_i^+(l_s) \times x_j^+(l_{s+1}) \cdots x_j^+(l_m) = 0, \quad \text{for } i \neq j.
\]

**Lemma 2.1.** Let $I_0 = \{1, 2, \ldots, n\}$ be the index set for the simple roots of a finite dimensional simple Lie algebra $\mathfrak{g}_0$. Then for each $i \in I_0$, there exists a sequence of indices $i = i_1, i_2, \ldots, i_{h-1}$ such that

\[
(\alpha_{i_1} | \alpha_{i_2}) = \epsilon_{1i},
\]
\[
(\alpha_{i_1} + \alpha_{i_2} | \alpha_{i_3}) = \epsilon_{i_2},
\]
\[
(\alpha_{i_1} + \cdots + \alpha_{i_{h-2}} | \alpha_{i_{h-1}}) = \epsilon_{i_{h-2}},
\]

where $h$ is the Coxeter number of the Lie algebra $\mathfrak{g}_0$, and $\epsilon_i \in \mathbb{Q}_{\geq 0}$.
If $A$ is an antimorphism, then $A([b_1, \cdots, b_n]_{v_1 \cdots v_{n-1}}) = [A(b_n), \cdots, A(b_1)]_{v_{n-1} \cdots v_1}$.
But if $B$ is an antimorphism such that $B(v_i) = v_i^{-1}$, then
\[
A([b_1, \cdots, b_n]_{v_1 \cdots v_{n-1}}) = [A(b_n), \cdots, A(b_1)]'_{v_{n-1} \cdots v_1}^{-1} = v_1^{-1} \cdots v_{n-1}^{-1} [B(b_1), \cdots, B(b_n)]_{v_1 \cdots v_{n-1}}
\]

The following identities follow from the definition.
\[
[a, bc]_v = [ab]_v c + x [a, ac]_v, \quad x \neq 0
\]
\[
[ac, b]_v = a[bc]_v + x [a, c]_v b, \quad x \neq 0
\]
\[
[a, [b, c]]_v = [[a, b], c]_v x + x [b, [a, c]]_v, \quad x \neq 0
\]
\[
[a, [b, c]]_v = [a, [b, c]]_v x + x [a, [c, b]]_v, \quad x \neq 0
\]

In particular, we have
\[
[a, b_1, \cdots, b_n]_{v_1, \cdots, v_{n-1}} = \sum_i [b_1, \cdots, b_i, b_{i+1}, \cdots, b_n]_{v_1, v_{i+1}, \cdots, v_{n-1}}.
\]
\[
(a, b, c)_{uv} = [a, b, c]_{uv} = a^2 b - (u + v)aba + uvba^2.
\]
The Serre relation for the case of $A_{ij} = -1$ can be written as:
\[
(x_i^+ (m), x_i^+ (m), x_i^+ (n))_{q_i q_{i-1}} = 0.
\]

**Theorem 2.2.** ([Dr]) Fix an $e$-sequence $i_1, i_2, \cdots, i_{h-1}$, and let $\theta = \sum_{j=1}^{h-1} \alpha_{i_j}$ be the maximal root of the finite dimensional simple Lie algebra $\mathfrak{g}$. Then there is a $\mathbb{C}(q^{1/s})$-algebra isomorphism $\Psi : U_q(\mathfrak{g}) \rightarrow \mathcal{U}$ defined by
\[
e_i \mapsto x_i^+(0), \quad f_i \mapsto x_i^-(0), \quad t_i \mapsto K_i \quad \text{for } i = 1, \cdots, n,
\]
\[
e_0 \mapsto [x_{i_{h-1}}^-(0), \cdots, x_{i_2}^-(0), x_{i_1}^-(1)]_{q_1 \cdots q_{h-2}^{-\epsilon} \gamma K_0^{-1}},
\]
\[
f_0 \mapsto a(-q)^{-\epsilon} \gamma K_0 [x_{i_{h-1}}^+(0), \cdots, x_{i_2}^+(0), x_{i_1}^+(1)]_{q_1 \cdots q_{h-2} \gamma K_0^{-1}},
\]
\[
t_0 \mapsto \gamma K_0^{-1}, \quad q^d \mapsto q^d,
\]
where $K_0 = K_{i_1} \cdots K_{i_{h-1}}$, $h$ is the Coxeter number, $\epsilon = \sum_{i=1}^{h-2} \epsilon_i$, and $s = 1, 2, 3$, the quotient of long roots by short roots. The constant $a$ is 1 for simply types $A_n, D_n$, $a = [2]_1$ for $C_{n}^{(1)}$, and $a = [2]^{1-\delta_{1,1}}$ for $B_{n}^{(1)}$.

**Proof.** Let $E_i, F_i, K_i, D$ be the images of $e_i, f_i, k_i, d$ in the algebra $\mathcal{U}$. We divide the proof into several steps.

Step 1. The elements $E_i, F_i, K_i, D$ satisfy the defining relations of $U_q(\mathfrak{g})$ given in (2.5). Clearly the defining relations of $\mathcal{U}$ imply that $E_i, F_i, K_i, i \neq 0$ generate a subalgebra isomorphic to $U_q(\mathfrak{g})$. Thus we are left with relation involving $i = 0$.

For $i \neq 0$ we have
\[
[E_0, F_i] = \left[ x_{i_{h-1}}^-(0), \cdots, x_{i_2}^-(0), x_{i_1}^-(1) \right]_{q_1 \cdots q_{h-2}^{-\epsilon} \gamma K_0^{-1}, x_{i_1}^-(0)} = - \left[ x_{i_1}^-(0), x_{i_{h-1}}^-(0), \cdots, x_{i_2}^-(0), x_{i_1}^-(1) \right]_{q_1 \cdots q_{h-2}^{-\epsilon} \gamma K_0^{-1}}
\]
We claim that \([x_i^- (0), x_j^- (0), \cdots , x_{i+1}^- (1)]_{q^{-1} q^{-1} q} = 0\) by the Serre relations. In fact this is seen by looking at rank 3 cases. We show the argument by working out all cases of \(A_3^{(1)}\). The first two use only one Serre relation, while the third one uses two Serre relations.

\[
\begin{align*}
[x_1^- (0), x_2^- (0), x_3^- (1)]_{q^{-1} q^{-1} q} &= [x_2^- (0), [x_3^- (0), x_2^- (0), x_1^- (1)]_{q^{-1} q^{-1} q}] = 0 \\
[x_2^- (0), x_3^- (0), x_1^- (1)]_{q^{-1} q^{-1} q} &= [x_3^- (0), [x_2^- (0), x_1^- (1)]_{q^{-1} q^{-1} q}] = 0
\end{align*}
\]

Using commutation relations it follows that

\[
\begin{align*}
[x_1^+ (0), x_2^+ (0), x_3^+ (0), \cdots , x_{i+1}^- (1)]_{q^{-1} \cdots q^{-1} q} &= 0 \\
[x_1^- (0), x_2^- (0), \cdots , x_{i-1}^- (1), x_{i+1}^- (1)]_{q^{-1} \cdots q^{-1} q} &= 0 \\
[x_1^- (0), x_2^- (0), \cdots , x_{i-1}^- (1), x_{i+1}^- (1)]_{q^{-1} \cdots q^{-1} q} &= 0
\end{align*}
\]
Writing $e_0 = e_0 \gamma^{-1} K_\theta$, we have

$$e_1 e_0^2 - (q + q^{-1}) e_0 e_1 e_0 + e_0^2 e_1$$

$$= (x_1^+(0) e_0^2, x_0^+(0)) \gamma^2 K_\theta^{-2}$$

$$= q^{-1} [e_0, x_1^+(0)]_{q^2} \gamma^2 K_\theta^{-2}$$

$$= q^{-1} \left[ e_0, [x_n^-(0), \ldots, x_2^-(0), -\gamma^{-1/2} K_1 a_1^+(1)]_{q^{-1}}, \gamma^2 K_\theta^{-2} \right]_{q^2}$$

$$= q^{-1} \left[ [x_n^-(0), \ldots, x_1^-(1)]_{q^{-1}}, [x_n^-(0), \ldots, x_2^-(1)]_{q^{-1}}, K_1 \gamma^2 K_\theta^{-2} \right]$$

$$= 0$$

where we used our earlier result: $[e_0, f_i] = 0$ for $i \neq 0$ and another identity $[e_0, x_2(1)] = 0$ by a similar argument as in (2.14).

Finally we check the relations $[e_i, f_i] = \frac{t_i - t_i^{-1}}{q - q^{-1}}$. Again it suffices to see the case of $i = 0$. We want to give two cases to show the argument.

First we consider the case of $A_n^{(1)}$:

$$\left[ [x_n^-(0), \ldots, x_1^-(1)]_{q^{-1}}, [x_n^+, \ldots, x_1^-(1)]_{q^{-1}} \right]$$

$$= \left[ [e_0, x_1^+(0), \ldots, x_n^-(0)]_{q^{-1}}, [e_0, x_1^+(0), \ldots, x_1^+(1)]_{q^{-1}} \right]$$

$$= \left[ [x_{n-1}^-(0), \ldots, x_1^-(1)]_{q^{-1}}, K_n, x_{n-1}^+(0), \ldots, x_1^+(1) \right]_{q^{-1}}, K_1^{-1} [x_{n-1}^-(0), \ldots, x_2^-(0)]_{q^{-1}}, \gamma$$

$$= (-q^{-1}) \left[ e_0 (n-1), f_0 (n-1), + q^{-1} K_1^{-1} \gamma [x_{n-1}^-(0), \ldots, x_3^+(0), x_0^+(0), x_{n-1}^-(0), \ldots, x_2^-(0)]_{q^{-1}}, \gamma \right]$$

$$= (-q)^{-n} \gamma K_\theta^{-1} K_n - \gamma^{-1} K_\theta$$

$$= (-q)^{-n} \gamma K_\theta^{-1} - \gamma^{-1} K_\theta$$

where we have reasoned as follows: the simplest Serre relations $[e_0, e_i] = 0$ for $i \geq 2$; an induction on rank $n$ as well as another induction on the second bracket in line two. The elements $e_0 (n-1), f_0 (n-1)$ refer to the corresponding ones for $A_n^{(1)}$.

The computations in other cases are similar. I just give $C_2^{(1)}$ to show some flavor.

We write

$$\tilde{e}_0 = [x_0^-(0), x_1^-(1)]_{q^{-1}}, \tilde{f}_0 = [x_0^+(0), x_1^+(1)]_{q^{-1}}$$
The algebra $U'$ is generated by $E_i, F_i, K_i^{\pm 1}, D^{\pm 1}$. Write $U' = \langle E_i, F_i, K_i^{\pm 1}, D^{\pm 1} \rangle$. The Cartan subalgebra is clearly generated by $K_i^{\pm 1}$ and $D^{\pm 1}$. Rewriting (2.13) we have

$$x_{i_1}(1) = a[E_{i_1}, E_{i_2}, \ldots, E_{i_{n-1}}, E_0]_{q^{11}, \ldots, q^{n-2}}$$

$$x_{i_1}^+(1) = b[F_{i_1}, F_{i_2}, \ldots, F_{i_{n-1}}, F_0]_{q^{11}, \ldots, q^{n-2}}$$

where $a, b$ are constants. It then follows from Drinfeld relations that

$$a_{i_1}(1) = K_{i_1}^{-1} \gamma^{-1/2} [x_{i_1}^+(0), x_{i_1}(1)]$$

$$a_{i_1}(-1) = K_{i_1} \gamma^{-1/2} [x_{i_1}^+(-1), x_{i_1}^-(0)]$$

which implies that $a_{i_1}(n) \in U'$, subsequently $x_{i_1}^+(n) \in U'$. Then we follow the $\epsilon$-sequence and get that $x_{i_1}^+(n) \in U'$. Thus $U' = U$.

Step 3. We now have an algebra epimorphism $\Phi$: $U_q(\mathfrak{g}) \to U$. It is clear that $Ker \Phi = 1$ when $q \to 1$ using Gabber-Kac [ GK]. Since quantization does not change the multiplicity we obtain that $\Phi$ is an automorphism.

Table 2.1. $\epsilon$-Sequences for simple Lie algebras

| $\mathfrak{g}$ | $\epsilon$-Sequence, $\epsilon = \sum \epsilon_i$ | $\epsilon$ |
|----------------|-----------------------------------------------|-----------|
| $A_n$          | $\gamma \rightarrow \cdots \rightarrow \alpha_n$ | $-n+1$    |
| $B_n$          | $\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow 0 \rightarrow \alpha_n \rightarrow \cdots \rightarrow \alpha_2$ | $-2n+4$   |
| $C_n$          | $\alpha_1 \rightarrow \gamma^{-1/2} \rightarrow \alpha_n \rightarrow \cdots \rightarrow \alpha_2$ | $-n+1$    |
| $D_n$          | $\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow 0 \rightarrow \alpha_n \rightarrow \cdots \rightarrow \alpha_2$ | $-2n+4$   |
| $E_6$          | $\alpha_1 \rightarrow \cdots \rightarrow \alpha_6 \rightarrow 0 \rightarrow \alpha_3 \rightarrow \alpha_2 \rightarrow \alpha_4 \rightarrow \alpha_3 \rightarrow \alpha_6$ | $-10$     |
| $E_7$          | $12345673245347321$, $\epsilon_i = -1$ | $-16$     |
| $E_8$          | $12345678543265843567458654321$, $\epsilon_i = -1$ | $-16$     |
| $F_4$          | $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ | $-7$      |
| $G_2$          | $\alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_2 \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_2$ | $2$       |

Remark. If $\mathfrak{g}$ is simply-laced, then $\epsilon = -h + 2$. The sequences are by no means unique, though $\epsilon$ is independent from the choice of the sequences. For example, we also have for $E_6$:

$$\alpha_1 \rightarrow \cdots \rightarrow \alpha_6 \rightarrow \alpha_3 \rightarrow \alpha_2 \rightarrow \alpha_4 \rightarrow \alpha_6 \rightarrow \alpha_3$$
We now derive the Drinfeld realizations for the twisted types. Let $X_N^{(1)}$ be a simply laced affine Cartan matrix. Let $\alpha_i'$ be the basis of the simple roots. The standard invariant bilinear form is normalized as

$$(\alpha_i' | \alpha_i') = 2r, \quad i = 0, 1, \cdots, N$$

Let $L_q(X_N^{(1)})$ be the quantum affine algebra associated with $X_N^{(1)}$ realized in the Drinfeld quantum loop form. We denote the corresponding generators be putting an extra prime to distinguish. Clearly the diagram automorphism $\sigma$ acts on the quantum affine algebra. We will use a different indexing for the type $A_{2n}^{(2)}$ from [K]. The action of $\sigma$ is given as follows:

- $A_N: \sigma(i) = N - i$
- $D_N: \sigma(i) = i, 1 \leq i \leq N - 2; \sigma(N - 1) = N$
- $E_6: \sigma(i) = 6 - i, 1 \leq i \leq 5; \sigma(6) = 6$
- $D_4: \sigma(1, 2, 3, 4) = (3, 2, 4, 1)$

We construct a special invariant subalgebra $U_\sigma$ generated by:

$$a_i(l) = \frac{1}{|d_i|\sqrt{r}} \sum_{s=0}^{r-1} a_{\sigma^s(i)}(l) \omega^{-ls}, \quad K_i = \prod_{s=0}^{r-1} K'_{\sigma^s(i)}$$

$$x_i^\pm(k) = \frac{1}{|d_i|\sqrt{r}} \sum_{s=0}^{r-1} x_{\sigma^s(i)}(k) \omega^{-ks}$$

where $\omega$ be a primitive $r$th root, and where $(d_0, \cdots, d_n) = (1, \cdots, 1, 2), (1, 2, \cdots, 2, 1), (2, 1, \cdots, 1, 1/2), (1, 1, 1, 2, 2)$ and $(1, 1, 3)$, for $A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_2^{(2)}, E_6^{(2)}$ and $D_4^{(2)}$ respectively. We denote $[k]_j = q^{\frac{k}{2}}_q - q^{-\frac{k}{2}}_q$ if $j$ belongs to the $\sigma$-orbit of $i$, then $[k]_j$ is defined for all $j = 1, \cdots, N$ though we use only $[k]_i$ for $i \in \{0\} \cup \Gamma_\sigma = \{0, 1, \cdots, n\}$. Easy and long calculation will lead to the following relations presented in the Drinfeld realization of twisted quantum affine algebras.

**Theorem 3.1.** The algebra $U_\sigma$ is the associative algebra with 1 over $\mathbb{C}(q^{1/2})$ generated by the elements $x_i^\pm(k), a_i(l), K_i^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d} (i = 1, 2, \cdots, N, k \in \mathbb{Z}, l \in$
\[ Z \setminus \{0\} \] with the following defining relations:

\[ x_{\sigma(i)}^\pm(k) = \omega^k x_i^\pm(k), \quad a_{\sigma(i)}(l) = \omega^l a_i(l), \]

\[ [\gamma^\pm 1/2, u] = 0 \quad \text{for all } u \in \mathbf{U}, \]

\[ [a_i(k), a_j(l)] = \delta_{k+l,0} \sum_{s=0}^{r-1} \frac{[k(\alpha_i^s(\alpha_j^s))]/rd_i]}{k} \omega^{ks} \gamma^k - \gamma^{-k} q_j - q_j^{-1}, \]

\[ [a_i(k), K_j^{\pm 1}] = [q^{\pm d}, K_j^{\pm 1}] = 0, \]

\[ q^d x_i^\pm(k)q^{-d} = q^k x_i^\pm(k), \quad q^d a_i(l)q^{-d} = q^l a_i(l), \]

\[ K_i x_j^\pm(k)K_i^{-1} = q^{\pm(\alpha_i,\alpha_j)} x_j^\pm(k), \]

\[ [a_i(k), x_j^\pm(l)] = \pm r^{-1} \sum_{s=0}^{r-1} \frac{[k(\alpha_i^s(\alpha_j^s))]/rd_i]}{k} \omega^{ks} \gamma^k [-\frac{1}{2} x_j^\pm(k) + l], \]

\[ \prod_s (z - \omega^s q^{\pm(\alpha_i^s(\alpha_j^s))}/r) x_i^\pm(w) x_j^\pm(z) = \prod_s (z q^{\pm(\alpha_i^s(\alpha_j^s))}/r - \omega^s w) x_i^\pm(w)x_j^\pm(z) \]

where \( \psi_i(m) \) and \( \varphi_i(-m) (m \in \mathbb{Z}_{\geq 0}) \) are defined by

\[ \sum_{m=0}^{\infty} \psi_i(m) z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{k=0}^{\infty} a_i(k) z^{-k} \right), \]

\[ \sum_{m=0}^{\infty} \varphi_i(-m) z^m = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{k=0}^{\infty} a_i(-k) z^k \right), \]

\[ \text{Sym}_{z_1, z_2} \prod_{i=1}^{2} (-1)^s \left[ \begin{array}{c} 2 \\ s \end{array} \right] x_i^\pm(z_1) \cdots x_i^\pm(z_s) x_j^\pm(w) x_j^\pm(z_{s+1}). \]

\[ \cdots x_i^\pm(z_2) = 0, \quad \text{for } A_{ij} = -1, \sigma(i) \neq j, \]

\[ \text{Sym}_{z_1, z_2, z_3} \left[ (q^{3r/4} z_1 - q^{-r/4} z_2 + q^{3r/4} z_3) x_i^\pm(z_1) x_i^\pm(z_2) x_j^\pm(z_3) \right] = 0, \]

\[ \text{for } A_{i, \sigma(i)} = -1 \]

where Sym means the symmetrization over \( z_i \), \( P_{ij}^\pm(z, w) \) and \( d_{ij} \) are defined as follows:

If \( \sigma(i) = i \), then \( P_{ij}^\pm(z, w) = 1 \) and \( d_{ij} = r \).

If \( A_{i, \sigma(i)} = 0 \) and \( \sigma(j) = j \), then \( P_{ij}^\pm(z, w) = z^r q^{2r} - w^r \) and \( d_{ij} = r \).

If \( A_{i, \sigma(i)} = 0 \) and \( \sigma(j) \neq j \), then \( P_{ij}^\pm(z, w) = 1 \) and \( d_{ij} = 1/2 \).

If \( A_{i, \sigma(i)} = -1 \), then \( P_{ij}^\pm(z, w) = z q^{r/2} + w \) and \( d_{ij} = r/4 \).
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the maximal root of the finite dimensional simple Lie algebra

The proof is similar to that of the untwisted case.

Proof. A basis of type Poincare-Birkhoff-Witt for the quantum affine algebra of

I. Damiani, A basis of type Poincare-Birkhoff-Witt for the quantum affine algebra of \(\mathfrak{sl}(2)\), Jour. Alg. 161 (1993), 291-310.

V. G. Drinfeld, Quantum groups, Proc. of Int’l. Cong. Math., Berkeley, Amer. Math. Soc. 1 (1987), 798-820.

V. G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988), 212-216.

I. Frenkel and N. Jing, Vertex representations of quantum affine algebras, Proc. Nat’l. Acad. Sci. USA 85 (1988), 9373-9377.

O. Gabber and V. G. Kac, On defining relations of certain infinite dimensional Lie algebras, Bull. Amer. Math. Soc. 5 (1981), 185-189.

M. Jimbo, A \(q\)-difference analog of \(U(\mathfrak{g})\) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 63-69.

N. Jing, Twisted vertex representations of quantum affine algebras, Invent. Math. 102 (1990), 663-690.

N. Jing and K. Misra, Vertex operators for twisted quantum affine algebras, preprint (1996).

V. Kac, Infinite dimensional Lie algebras (3rd edition), Cambridge Univ. Press, 1990.

G. Lusztig, Introduction to quantum groups, Birkhauser, Boston, 1993.

S. Lewandorskii, Y. Soibelman and V. Stukopin, Quantum Weyl group and universal quantum \(R\)-matrix for affine Lie algebra \(A_1^{(1)}\), Lett. Math. Phys. (1993).

Table 3.1. \(\epsilon\)-Sequences for simple Lie algebras

| \(\mathfrak{g}^{(r)}\) | \(\epsilon\)-Sequence | \(\epsilon\) | \(a\) |
|------------------|------------------|----------|----------|
| \(A_{2n-1}^{(2)}\) | \(\alpha_1 \frac{1}{2} \cdots \frac{1}{2} \alpha_{n-1} \frac{3}{2} \alpha_n \rightarrow \frac{1}{2} \rightarrow \alpha_2 \) | -2n+2 | -2 |
| \(D_{n+1}^{(2)}\) | \(\alpha_n \frac{3}{2} \rightarrow \cdots \rightarrow \alpha_1 \) | -2n+2 | \((-2)^{n+1}\) |
| \(A_{2n}^{(2)}\) | \(\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha_{n-1} \rightarrow \cdots \rightarrow \alpha_2 \rightarrow \alpha_1 \) | -2n+3 | \(-2^{2n-2}\) |
| \(D_{3}^{(2)}\) | \(\alpha_3 \rightarrow \frac{3}{2} \rightarrow \alpha_2 \rightarrow \alpha_1 \) | -4 | 3 |
| \(E_6^{(2)}\) | \(\alpha_1 \rightarrow \cdots \rightarrow \alpha_6 \rightarrow \alpha_5 \rightarrow \alpha_4 \rightarrow \alpha_3 \rightarrow \alpha_2 \) | -10 | |

Theorem 3.2. (Dr2) Fix an \(\epsilon\)-sequence \(i_1, i_2, \cdots, i_h\), and let \(\theta = \sum_{j=1}^{h-1} \alpha_{i_j}\) be the maximal root of the finite dimensional simple Lie algebra \(\mathfrak{g}\). Then there is a \(\mathbb{C}(q^{1/r})\)-algebra isomorphism \(\Psi : U_q(\mathfrak{g}^{(r)}) \rightarrow U\) defined by

\[
\begin{align*}
\epsilon_0 & \mapsto [x_{i_{h-1}}^+(0), \cdots, x_{i_1}^+(0), x_{i_1}^+(1)]_{q^1 \cdots q^{h-2}} \gamma K_{\theta}^{-1}, \\
 f_0 & \mapsto \epsilon(-q)^{-\gamma^{-1}} K_{\theta} [x_{i_{h-1}}^+(0), \cdots, x_{i_1}^+(0), x_{i_1}^+(-1)]_{q^1 \cdots q^{h-2}} \\
t_0 & \mapsto \gamma K_{\theta}^{-1}, \quad q^d \mapsto q^d,
\end{align*}
\]

where \(p_i = 1\) for \(\sigma(i) \neq i\), \(p_i = i\) otherwise, \(K_{\theta} = K_{i_1} \cdots K_{i_{h-1}}\), \(h\) is the Coxeter number, \(\epsilon = \sum_{i=1}^{h-1} \epsilon_i\), and \(d\) is a constant given by Table 3.1.

Proof. The proof is similar to that of the untwisted case.

References

[Be] J. Beck, Braid group action and quantum affine algebras, Commun. Math. Phys. 165 (1994), 555-568.

[Da] I. Damiani, A basis of type Poincare-Birkhoff-Witt for the quantum affine algebra of \(\mathfrak{sl}(2)\), Jour. Alg. 161 (1993), 291-310.

[Dr1] V. G. Drinfeld, Quantum groups, Proc. of Int’l. Cong. Math., Berkeley, Amer. Math. Soc. 1 (1987), 798-820.

[Dr2] V. G. Drinfeld, A new realization of Yangians and quantized affine algebras, Soviet Math. Dokl. 36 (1988), 212-216.

[FJ] I. Frenkel and N. Jing, Vertex representations of quantum affine algebras, Proc. Nat’l. Acad. Sci. USA 85 (1988), 9373-9377.

[GK] O. Gabber and V. G. Kac, On defining relations of certain infinite dimensional Lie algebras, Bull. Amer. Math. Soc. 5 (1981), 185-189.

[Jb] M. Jimbo, A \(q\)-difference analog of \(U(\mathfrak{g})\) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 63-69.

[J] N. Jing, Twisted vertex representations of quantum affine algebras, Invent. Math. 102 (1990), 663-690.

[JM] N. Jing and K. Misra, Vertex operators for twisted quantum affine algebras, preprint (1996).

[K] V. Kac, Infinite dimensional Lie algebras (3rd edition), Cambridge Univ. Press, 1990.

[L] G. Lusztig, Introduction to quantum groups, Birkhauser, Boston, 1993.

[LSS] S. Lewandorskii, Y. Soibelman and V. Stukopin, Quantum Weyl group and universal quantum \(R\)-matrix for affine Lie algebra \(A_1^{(1)}\), Lett. Math. Phys. (1993).