Quantum Multi Prover Interactive Proofs with Communicating Provers

Extended Abstract

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Abstract

Multi Prover Interactive Proof systems (MIPs) were first presented in a cryptographic context, but ever since they were used in various fields. Understanding the power of MIPs in the quantum context raises many open problems, as there are several interesting models to consider. For example, one can study the question when the provers share entanglement or not, and the communication between the verifier and the provers is quantum or classical. While there are several partial results on the subject, so far no one presented an efficient scheme for recognizing NEXP (or NP with logarithmic communication), except for [KM03], in the case there is no entanglement (and of course no communication between the provers).

We introduce another variant of Quantum MIP, where the provers do not share entanglement, the communication between the verifier and the provers is quantum, but the provers are unlimited in the classical communication between them. At first, this model may seem very weak, as provers who exchange information seem to be equivalent in power to a simple prover. This in fact is not the case—we show that any language in NEXP can be recognized in this model efficiently, with just two provers and two rounds of communication, with a constant completeness-soundness gap.

The main idea is not to bound the information the provers exchange with each other, as in the classical case, but rather to prove that any “cheating” strategy employed by the provers has constant probability to diminish the entanglement between the verifier and the provers by a constant amount. Detecting such reduction gives us the soundness proof. Similar ideas and techniques may help help with other models of Quantum MIP, including the dual question, of non communicating provers with unlimited entanglement.

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1 Introduction

Multi Prover Interactive Proofs (MIPs) have been studied extensively in the classical setting, and provide an exact characterization of $\text{NEXP}$ [BFL92]. Extending MIPs to the quantum setting poses many important open problems, and may give us more intuition regarding the power of entanglement. There are several possible generalizations for quantum multi-prover schemes, which differ in the power of the verifier (which can be quantum or classical), and in the relation between the provers (for example, how much entanglement they have). The first results for this problem were given by Kobayashi and Matsumoto [KM03]. They proved that as long as the provers share a bounded (polynomial) amount of entanglement, the set of languages which can be recognized is contained in $\text{NEXP}$ (even if the verifier is quantum).

We do not understand the power of the model when the verifier is classical and the provers share (limited or unlimited) entanglement. In particular, Cleve et al. [CHTW04] provide examples where the proof is valid if the provers share no entanglement, but is no longer sound when they do. Preda [Pre] showed that if the provers are not limited to quantum entanglement, but instead have an unlimited amount of nonlocal boxes [PR97], then the set of recognizable languages is contained in $\text{EXP}$.

There are also some positive results when the provers are quantum. Cleve et al. [CGJ07] provide a proof system for $\text{NP}$ when the verifier is classical and the provers share an unlimited amount of entanglement. The proof scheme provides a constant gap, but the communication is linear. Kempe et al. [KKMTV07] give a quantum protocol for recognizing languages in $\text{NP}$ by a quantum verifier with logarithmic communication, when the provers share unlimited entanglement. However, when $x \not\in L$ the probability that the verifier will discover this is $1 - O(1/n)$, which means that it is necessary to repeat the protocol a polynomial number of times to get constant soundness. Ito et al. [IKPSY07] use this result, and give a 3 prover proof system for $\text{NEXP}$ which is resistant to entanglement with soundness of just $1 - 2^{-\text{poly}}$.

1.1 Our Results

An important assumption underlying the work on multi prover schemes is that the provers are not allowed to pass information between themselves. The results of [KW00, Pre] could lead us to believe that a proof system with a quantum verifier and two provers who can pass classical information between them is limited to $\text{EXP}$. Surprisingly, this is not the case (assuming $\text{EXP} \neq \text{NEXP}$). We show that:

**Theorem 1.1.** Let $V$ be a polynomial time verifier that can exchange quantum messages with two computationally unbounded provers. The provers share no entanglement, but can freely communicate classically between them. Then for any $L \in \text{NEXP}$ there is a two round protocol for the verifier and provers such that for any string $x$

- (completeness) If $x \in L$ then there are two prover strategies such that $V$ will accept $x$ with probability 1.
- (soundness) If $x \not\in L$ then for any two prover strategies the probability that $V$ will accept $x$ is at most $c$ for some constant $0 < c < 1$. 


The communication between the verifier and the provers is is polynomial in the length of the input.

We note that augmenting the provers in our model with unlimited entanglement gives something which is contained in EXP [KW00] (as this is equivalent to quantum communication and thus to a single quantum prover). Bounding the verifier to be classical, would limit us to languages in PSPACE [Sha90] (as in this scenario is equivalent to a single prover and a classical verifier), so both conditions are necessary. This problem is in a way dual to the scenario where the provers do not have any means of communication but instead have unlimited entanglement, where much less is known.

Quantum MIPs are thought to be a model of computation which may give us better understanding of entanglement, and its powers. Surprisingly, our result, which is stated in a model with no entanglement between the provers, is based on following the entanglement between the provers and the verifier. Each message the verifier sends is a superposition of two classical queries. Measuring the message would ruin the superposition, and will be caught by the verifier. However, a strategy which does not measure it “enough” does not extract enough useful classical information, and prevents the provers from coordinating answers via the classical channel. Most of the paper follows the amount of entanglement between the verifier and the provers during the protocol, making sure that either the provers do not extract enough information to answer with very high probability (we note that from an information-theoretic point of view they extract many bits of information–so we use tailored bounds), or they have some chance of getting caught.

1.2 Related Work

It is interesting to view the results of this paper in light of the complexity class QMA(2), defined by Kobayashi, Mastumoto and Yamakami [KMY01]. Intuitively, this is the class of languages which can be recognized by a quantum verifier with two unentangled bounded pieces of quantum evidence. While there is no classical analog for this problem (having two classical witnesses is still NP), there is evidence that QMA(2) strictly contains QMA [LCV07]. Blier and Tapp [BT07] showed that a verifier can recognize an NP complete language with soundness $1 - O(1/n^6)$. A constant soundness completeness gap in their results would imply our own. We note however, that Aaronson et al. [ABDFS08] give evidence towards $\text{QMA}(2) \subseteq \text{PSPACE}$, and therefore we do not expect that this is the case.

The idea of using Private Information Retrieval [CGKS95, KdW03, KdW04] schemes (PIRs) has been suggested by Cleve et al. [CGJ07]. Our protocol is in a sense a cheat sensitive PIR where the verifier can check whether the prover has tried to learn information. A similar quantum PIR scheme has been independently presented by [GLM07] in a different context. It is important to note that information disturbance tradeoffs proposed by such quantum PIR schemes are by themselves insufficient to prove the soundness of our multi-prover protocol, since the leakage of even a small amount of information might enable the provers to succeed in cheating the verifier.

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1 Equivalently we can state our result for NP, bounding the communication to be logarithmic.
2 Preliminaries

We assume the reader is familiar with quantum computation (see [NC00] for example).

Let \( L \in \text{NEXP} \). By standard PCP machinery, we can assume that given \( x \) the verifier has an implicit efficient access to an exponentially long 3-SAT formula \( \Phi \), such that if \( x \in L \) then \( \Phi \) is satisfiable, and otherwise any assignment can satisfy at most a \( 1 - \gamma \) proportion of the clauses for some constant \( \gamma > 0 \). We can also assume that each variable appears exactly 5 times, and each clause contains three different variables. Let \( C \) denote the set of clauses and \( V \) the set of variables. If a variable \( v \in V \) appears in a clause \( c \in C \) we write \( v \in c \). Let \( M = |C| \) denote the number of clauses and \( N = |V| \) the number of variables. Let \( T \) be a truth assignment for \( \Phi \). For a variable \( x \in V \), let \( T(x) \) denote the value \( T \) assigns \( x \). For a clause \( y \in C \), if \( y \) contains the variables \( v_y^1, v_y^2, v_y^3 \), let \( T(y) = T(v_y^1), T(v_y^2), T(v_y^3) \).

Alice (Bob) has a private Hilbert space \( H^m_A (H^m_B) \), with some finite arbitrarily large dimension \( d \) (we assume without loss of generality that the dimensions are identical). The messages between Alice (Bob) and the verifier will be sent by passing a state which is in a Hilbert space \( H^m_A (H^m_B) \). For convenience, we partition the private Hilbert space of the verifier into three parts, \( H^v_A = H^v_A \otimes H^v_B \otimes H^v_B \). The Hilbert spaces \( H^v_A, H^v_B \) will be used with messages sent to different provers, but they are private spaces that belong to the verifier. We let the verifier send and receive classical messages from Alice.

3 Algorithm

Let \( \pi \) be a probability distribution which chooses two clauses \( y, \tilde{y} \) uniformly at random from \( C \), and two variables \( x, \tilde{x} \) uniformly at random from \( V \), with the constraint that \( x \) appears in \( y \) (\( \tilde{x} \) does not necessarily appear in \( \tilde{y} \)).

Protocol for verifier

1. Sample \( \pi \) to get \( y, \tilde{y}, x, \tilde{x} \). Generate the states on \( O(\log(N)) \) qubits
   \[
   \frac{1}{\sqrt{2}} (|yy\rangle + |\tilde{y}\tilde{y}\rangle) \otimes |000\rangle \in H^v_A \otimes H^m_A
   \]
   \[
   \frac{1}{\sqrt{2}} (|xx\rangle + |\tilde{x}\tilde{x}\rangle) \otimes |0\rangle \in H^v_B \otimes H^m_B
   \]
   Send Alice (Bob) the message space \( H^m_A (H^m_B) \), which consists of the last \( m + 3 \) \((n + 1)\) qubits.

2. Let \( T \) be a satisfying assignment for \( \Phi \) (if one exists). Alice should apply the unitary which takes \( |c\rangle \otimes |000\rangle \rightarrow |c\rangle \otimes |T(c)\rangle \) for any clause \( c \in C \), and Bob should apply the unitary which takes

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This can be done by using a larger space \( H^m_A \), with the verifier measuring the part of the space which should be used for the classical message. Thus, this does not change the model, and is only done for clarity.
\[ |v\rangle \otimes |0\rangle \rightarrow |v\rangle |T(v)\rangle \] for \( v \in V \). In fact the provers apply any Local Operations and Classical Communication protocol they want among themselves. Finally, Alice (Bob) returns the verifier the message space \( H^m_A (H^m_B) \).

3. Send Alice the classical values \( y, \tilde{y}, x, \tilde{x} \). Alice returns 8 bits: \( T(y), T(\tilde{y}), T(x), T(\tilde{x}) \). If Alice returned quantum values, the verifier measures them according to the standard basis.

4. Verify that the clause \( y \) is satisfied, and that \( T(x) \) matches \( T(y) \). Perform the SWAP test [BCWW01] between the state in \( H^v_A \otimes H^m_A \) and \( \frac{1}{\sqrt{2}} (|yy\rangle \otimes |T(y)\rangle + |\tilde{y}\tilde{y}\rangle \otimes |T(\tilde{y})\rangle) \) and between the state in \( H^v_B \otimes H^m_B \) and \( \frac{1}{\sqrt{2}} (|xx\rangle \otimes |T(x)\rangle + |\tilde{x}\tilde{x}\rangle \otimes |T(\tilde{x})\rangle) \). Accept if all tests passed.

Note that the verifier does not generate any entanglement between the provers. This means that it is possible to repeat the protocol in order to reduce the error probability.

Completeness: With a common satisfying assignment the provers can apply the required quantum transformation, and all the tests will be passed with probability 1.

4 Soundness of the Protocol

**Intuition** To simplify the analysis, we modify the protocol. First, we purify the verifier. This will enable us to talk about the probability of a set of queries given measurements by the provers. The second modification will be to strengthen the provers, allowing them to perform any joint separable measurement instead of Local Operations and Classical Communication (LOCC), which will enable us to write the state after their actions. We prove that the provers have a constant failure probability for any result \( k \) of the separable measurement they make. We begin by finding an estimate for the probability that the verifier measures \((y, \tilde{y}, x, \tilde{x})\) as a function of the provers’ result \( k \). Next, we show that if \( k \) is more probable given a clause \( y_1 \) then given another result \( y_2 \), and the verifier measured \((y_1, y_2, x, \tilde{x})\) for any \( x, \tilde{x} \), then there is constant probability that Alice fails the SWAP test (because such a measurement operator diminishes the entanglement between \( H^v_A \) and \( H^m_A \)).

We then show that either the measurement has a constant probability to diminish the entanglement, or after it there is still a large set of clauses (and variables which appear in them) which are all “almost uniformly” probable. The set will be large enough that no assignment will satisfy all of it. This means that if the provers succeed with very high (but constant) probability, they must succeed on a large portion of this “uniform” set, and thus they must succeed on a very large number of clauses and variables. This will give a strategy for the classical protocol which has success probability greater than \( 1 - \gamma/3 \), which is a contradiction.

**The Modified Protocol** As stated above, the first modification is to purify the sampling of \( \pi \), postponing it until after the provers act on the information. It uses \( H^v_{\text{aux}} \) with \( \text{dim}(H^v_{\text{aux}}) = M^2 N^2 \). The verifier generates

\[
\psi_{\pi} = \sum_{y, \tilde{y} \in C} \sum_{x, \tilde{x} \in V} \sum_{x, \tilde{x} \in V} |y\tilde{y}, x\tilde{x}\rangle \otimes \frac{1}{\sqrt{2}} (|yy\rangle + |\tilde{y}\tilde{y}\rangle) \otimes |000\rangle \otimes \frac{1}{\sqrt{2}} (|xx\rangle + |\tilde{x}\tilde{x}\rangle) \otimes |0\rangle \in H^v_{\text{aux}} \otimes H^v_A \otimes H^m_A \otimes H^v_B \otimes H^m_B
\]
As before, the verifier sends Alice (Bob) the Hilbert space $H_m^A$ ($H_m^B$). After Alice and Bob act on the message spaces they get and return $H_m^A$, $H_m^B$, the verifier measures $R_v^{aux}$ to get $y, \tilde{y}, x, \tilde{x}$ and sends them to Alice as in Protocol 1. This modification does not change the cheating power of the provers (they cannot tell what protocol is being used).

The second modification is to replace the LOCC done by the provers in the first stage with a single joint separable measurement. [BDF+98, BNS97] proved that this is strictly stronger than LOCC. In particular they showed how to transform any LOCC protocol into such a measurement. As the provers are not entangled, we can assume that their private spaces are initialized with the state $|0 \ldots 0\rangle$. Letting $\rho = |\psi_\pi\rangle\langle\psi_\pi|$, the provers’ operation now becomes applying a measurement with operators

$$(I_{M^2N^2} \otimes I_M \otimes A_k \otimes I_N \otimes B_k)\bigl((I_{M^2N^2} \otimes I_M \otimes A_k \otimes I_N \otimes B_k\bigr)$$

where $I_p$ is the $p \times p$ identity matrix, $A_k$ is an $8Md \times 8Md$ matrix, $B_k$ is a $2Nd \times 2Nd$ matrix and

$$\sum_k (A_k \otimes B_k)^\dagger (A_k \otimes B_k) = I_{16NMd^2}$$

The Hilbert spaces $H_m^A$, $H_m^B$ are then returned to the verifier.

We now calculate the probability that the verifier measured values $r = (y, \tilde{y}, x, \tilde{x})$, conditioned on the fact that the measurement result was $k$. Denote $A_k(y) = \text{tr}(A_k(|y\rangle\langle y| \otimes I)A_k)$, where we are tracing over the private qubits of the prover and the qubits which define the assignment, and similarly $B_k(x) = \text{tr}(B_k(|x\rangle\langle x| \otimes I)x_k)$. In Appendix A we prove that for $y \neq \tilde{y}, x \neq \tilde{x}$

$$\Pr(y, \tilde{y}, x, \tilde{x}|k) = \frac{(A(y) + A(\tilde{y}))(B(x) + B(\tilde{x}))}{\sum_{c,\tilde{c} \in C, v, \tilde{v} \in V} \Pr(c, v, \tilde{c}, \tilde{v}|k)}$$

(1)

where if $y = \tilde{y}$ the numerator changes to $4A(y)(B(x) + B(\tilde{x}))$, and similarly for $x, \tilde{x}$.

We give some intuition for Equation (1). The numerator is the product of two factors, because when the verifier measures before the provers (which is physically equivalent) the provers are unentangled, and therefore the probability of $k$ is just the $\text{tr}(A_kpA_k^\dagger) \cdot \text{tr}(B_kpB_k^\dagger)$. Alice’s factor is composed of two terms, because tracing out the verifier Alice just gets a mixed state of $\frac{1}{2}|y\rangle\langle y| + \frac{1}{2}|\tilde{y}\rangle\langle \tilde{y}|$.

Omitting the subindex $k$, and denoting $W_{A_k} = \sum_i A_k(i) = \text{tr}(A_k), W_{B_k} = \sum_i B_k(i) = \text{tr}(B_k), \tilde{W} = \sum_{c,\tilde{c} \in C} A_k(c)B_k(c)$ We show the following bound in In Appendix A by bounding the denominator

$$\Pr(y, \tilde{y}, x, \tilde{x}|k) \geq \frac{A(y)B(x) + A(\tilde{y})B(\tilde{x})}{2MNW + 22MW_AW_B}$$

(2)

### 4.1 Auxiliary Lemmas

We show that if $A_k$ is too skewed, then for certain values of $y, \tilde{y}$, Alice has a good chance of failing the SWAP test. Formally:

**Lemma 4.1.** Assume $A(y) \geq pA(\tilde{y}), p > 1$. Then for any assignment $T$, the probability that the verifier will catch Alice cheating in the SWAP test is at least $\frac{1}{2} - \frac{1}{1+p}$.
The proof is found in Appendix B as it is somewhat technical. The intuition is that the super-operator which acts on the state diminishes the entanglement between $H^A$ and $H^n_A$. Therefore, this is true for any assignment Alice will send in the second round of the protocol.

If the condition of lemma 4.1 holds, we say that the measurement $p$-damaged the state. An analogous lemma holds for Bob. The following lemma is trivial:

**Lemma 4.2.** If there exists a set $D \subset Y \times \tilde{Y} \times X \times \tilde{X}$ such that

1. For any $d = (y, \tilde{y}, x, \tilde{x}) \in D$ we have $x \in y$, and either $A$ or $B$ $p$-damage $d$ for some constant $p$.
2. $\sum_{d \in D} \Pr(d|k) > \epsilon_D$ for some constant $\epsilon_D$

Then at least one of the provers gets caught in the SWAP test with probability $\epsilon_D \left( \frac{1}{2} - \frac{\sqrt{p}}{1+p} \right)$.

In this case we say that $D$ is an $(\epsilon_D, p)$ bad set.

### 4.2 Large $NM\tilde{W}$

**Theorem 4.3.** If $NM\tilde{W} \geq 100MW_AW_B$ then at least one of the provers fails the SWAP test with probability $\frac{1}{6.96 \times 10^{-7}} = \min \{ \frac{1}{6.96 \times 10^{-7}}, \frac{1}{4.2 \times 10^{-7}} \}$.

The proof is by contradiction. We prove Lemma 4.4 which states that if $A$ and $B$ do not have a certain property then the provers have a constant probability of getting caught. We then prove that if $A$ and $B$ do have that property than either a second property holds or the provers get caught, with some probability. The second property implies $NM\tilde{W} < 100MW_AW_B$, which is a contradiction. Remember $\tilde{W} = \sum_{c \in C, v \in c} A(c)B(v)$. For $c \in C$, let $u(c) = \sum_{v \in c} A(c)B(v)$, and for $S \subset C$, $U(S) = \sum_{c \in S} u(c)$. Let

$$S_i = \left\{ c : \frac{\tilde{W}}{2^{i+1}} < u(c) \leq \frac{\tilde{W}}{2^i} \right\}$$

**Lemma 4.4.** If there exists an index $j$ such that $\sum_{i=0}^{j-1} U(S_i) > \tilde{W}/100$ and $\sum_{i=j+1}^{\infty} U(S_i) > \tilde{W}/100$, then the provers get caught with constant probability $\frac{1}{6.96 \times 10^{-7}}$, generated from a $(\frac{1}{4.8 \times 10^{-7}}, \sqrt{2})$ bad set.

The proof is found in Appendix C. It follows by constructing a bad set, such that the clauses (and variables) in $\bigcup_{i=0}^{j-1} S_i$ stand for $y, x$, and the clauses (and variables) in $\bigcup_{i=j+1}^{\infty} S_i$ stand for $\tilde{y}, \tilde{x}$, where we use the fact that if $u(c_1) > 2u(c_2)$ for some two clauses, then either Alice damages the state because $A(c_1) > \sqrt{2}A(c_2)$, or Bob $\sqrt{2}$ damages the state, or both of them do. We note that if $u(c_1) > 2u(c_2), u(c_1) > 2u(c_3)$ it may still be the case that $A(c_2) > A(c_1)$, and for each $v_3 \in c_3$ and each $v_1 \in c_1 B(v_3) > B(v_1)$. However, taking out such tuples only diminishes the size of the bad set $D$ by a factor of 36 (a factor of 9 comes from choosing one of the variables in the clause, and a factor of 4 comes from choosing the clause).
If the condition of Lemma 4.4 does not hold, then there must be an index \( j \) such that \( U(S_j) + U(S_{j+1}) > 0.98\bar{W} \). Define \( F = S_j \cup S_{j+1} \). Remembering that \( W_A = \sum_{c \in G} A(c) \), we partition the clauses in \( F \):

\[
T_i = \{ c \in F : \frac{W_A}{2^{i+1}} < A(c) \leq \frac{W_A}{2^i} \}
\]

**Lemma 4.5.** If there exists an index \( j \) such that \( \sum_{i=j}^{j+1} |T_i| > |F|/100 \), and \( \sum_{i=j}^{\infty} |T_i| > |F|/100 \), then the first prover gets caught with constant probability \( \frac{1}{4 \cdot 2^{10^\gamma}} \), generated from a \((\frac{1}{4 \cdot 2^{10^\gamma}}, 2)\) bad set.

The proof appears in Appendix C. It is very similar to the one of Lemma 4.4, but much simpler.

As before, if the condition of Lemmas 4.4 and 4.5 do not hold, then

\[
\exists i : |T_i| + |T_{i+1}| > 0.98|F| \geq 0.98^2 M > 0.96M 
\]

Let \( G = T_i \cup T_{i+1} \). As \( G \subset F \), and as \( \forall c_1, c_2 \in F : u(c_1) < 4u(c_2) \) we have

\[
U(G) > 0.25 \cdot 0.98U(F) > 0.25 \cdot 0.98^2 \bar{W} 
\]

(3)

Note \( \sum_{c \in G} \sum_{v \in c} B(v) \leq 5W_B \), as each variable appears 5 times. Also, since \( \forall c \in G : A(c) > W_A/2^{i+1} \)

\[
0.96M \frac{W_A}{2^{i+1}} < \frac{W_A}{2^{i+1}} |G| < \sum_{c \in G} A(c) < W_A 
\]

(4)

Putting this together, we get

\[
0.25 \cdot 0.98^2 \bar{W} \overset{3}{\leq} U(G) = \sum_{c \in G} u(c) = \sum_{c \in G} \sum_{v \in c} A[c]B[v] \leq \sum_{c \in G} \sum_{v \in c} \frac{W_A}{2^{i+1}} B[v] = 
\]

\[
\frac{4W_A}{2^{i+1}} \sum_{c \in G} \sum_{v \in c} B[v] \leq \frac{20W_A W_B}{2^{i+1}} \overset{4}{\leq} \frac{20W_A W_B}{0.96M} \leq \frac{20W_A W_B}{0.96N} 
\]

This is a contradiction to \( MN\bar{W} \geq 100MW_AW_B \). This proves Theorem 4.3

### 4.3 Small \( N\bar{W} \)

In this subsection we handle those values of \( k \) for which the premise of Theorem 4.3 does not hold, i.e., \( N\bar{W} < 100MW_AW_B \). Define \( S_i = \{ c \in C : \frac{W_A}{2^{i+1}} \leq A(c) < \frac{W_A}{2^i} \} \). For a set \( S \subset C \), let \( W(S) = \sum_{c \in S} A(c) \).

**Lemma 4.6.** If \( N\bar{W} < 100MW_AW_B \) and there exists an index \( i \) such that

\[
\sum_{j=0}^{i-1} W(S_j) > \gamma 10^{-4} W_A \wedge \sum_{j=i+1}^{\infty} |S_j| > \gamma 10^{-4} M 
\]

(5)

then Alice is caught cheating with probability \( \frac{\gamma^2}{2 \cdot 6 \cdot 10^{12^\gamma}} \), generated from a \((\frac{\gamma^2}{7.4 \cdot 10^{12^\gamma}}, 2)\) bad set.
Let Charlie and Diana be two classical provers who are faced with a classical verifier. The verifier sends Charlie a random clause \(c\), and Diana a random variable \(v\) which appears in \(c\). Charlie should answer with the values that some satisfying assignment gives the variables in \(c\), and Diana should answer with the value that same assignment gives the variable \(v\). If the original formula is \(\gamma\)-distant from being satisfiable, then the success probability of Charlie and Diana is bounded by \(1 - \frac{2}{\sqrt{\gamma}}\).

We prove a reduction from the quantum case to the classical one. First, a simple lemma.

**Lemma 4.9.** If \(|u| |v| \leq 1/2\) and \(|u| = |v| = |w| = 1\) then \(\langle u | v \rangle > 1 - \epsilon \Rightarrow \langle v | w \rangle < 1/2 + \frac{\sqrt{\epsilon}}{2} - \frac{3}{4} \).

The proof follows from Taylor’s approximation. A specific case: if \(\epsilon < 0.01\) then the final term is less than 0.99. Finally, let \(\text{FailProb}(y, \tilde{y}, x, \tilde{x}, k)\) denote the probability that the provers failed to convince the verifier, given the measurement results \((y, \tilde{y}, x, \tilde{x})\) and \(k\).

**Lemma 4.10.** If there exists an index \(k\), matrices \(A_k, B_k\) and a set of clauses \(R \subset C\) such that

1. \(|R| \geq (1 - \epsilon_1)M\)
2. \(\forall y \in R : \forall x \in y : |\{(\tilde{y}, \tilde{x}) \in C \times V : \text{FailProb}(y, \tilde{y}, x, \tilde{x}, k) > \epsilon_3\}| < \epsilon_2 MN\)
3. \(\epsilon_3 < 1/200\)
then there is a classical strategy for Charlie and Diana which gives them a success probability of at least

\[(1 - \epsilon_1)(1 - \epsilon_2)(1 - \epsilon_3)(1 - 200\epsilon_3)^2.\]

Proof. Charlie gets as an input a clause \(y\) from the verifier. He chooses a random \(\tilde{y}\), and simulates Alice, conjugating by \(A_k\). Then he finds the closest possible legal classical description to the state, by choosing \(T(y), T(\tilde{y})\) to maximize the fidelity. Similarly, Diane simulates Bob with her input \(x\).

The classical verifier chooses independently, and therefore with probability at least \(1 - \epsilon_1\) he chooses a clause from \(R\). With probability greater than \(1 - \epsilon_2\) the provers choose a pair \(\tilde{y}, \tilde{x}\) for which Alice and Bob have good success probability. If this is the case, Alice and Bob’s success probability is at least \(1 - \epsilon_3\). Since Alice passes the SWAP test with probability \(1 - \epsilon_3\), the state she sends in the first step must pass the SWAP test with the classical description she sent in the second step with probability \(1 - \epsilon_3\). If the latter is not the closest possible classical description, then her probability of failing the SWAP test, using \(\delta = 0.99\) to ensure that there are no closer alternatives, is at least \((1 - \delta^2)/2 > (1 - 0.99^2)/2 > 1/200\). Thus, the probability that this occurs is bounded by \(200\epsilon_3\). So with probability at least \(1 - 200\epsilon_3\) Alice and Charlie send the same assignment; given that, there is only an \(\epsilon_3\) chance of failure for Charlie. Finally, the same argument as before applies to Diane (simulating Bob), which contributes another factor of \(1 - 200\epsilon_3\) (the other factors have already been counted for both provers).

Lemma 4.11. If the failure probability of Alice and Bob given result \(k\) is less than \(3/1002\), then there exists a set \(R\) with the properties stated in Lemma 4.10 with \(\epsilon_1 = 0.003\gamma, \epsilon_2 = \gamma 10^{-3}\) and \(\epsilon_3 = \gamma 10^{-4}\).

Proof. Since the failure probability is less than \(3/1002\), we must have, by Theorem 4.3, that \(NMW < 100MW_AW_B\). By Lemmas 4.6, 4.7 and 4.8 we have a set \(H\) such that \(|H| \geq (1 - 0.002\gamma)M\), and

\[\forall y \in H : \forall x \in y : A(y) > W_A/(5M) \land B(x) > W_B/(5N)\]

Using (2), this means that for any tuple \((y, \tilde{y}, x, \tilde{x}) \in H\)

\[\Pr(y, \tilde{y}, x, \tilde{x}|k) \geq \frac{A(y)B(x)}{222MW_AW_B} \geq \frac{W_AW_B}{25MN \cdot 222MW_AW_B} = \frac{1}{5.55 \cdot 10^4 NM^2}\]

Denote \(L(y, x) = \{(\tilde{y}, \tilde{x}) : \text{FailProb}(y, \tilde{y}, x, \tilde{x}, k) > 10^{-4}\gamma\}\), and \(H_{\text{fail}} = \{y \in H : \exists x \in y : |L(y, x)| > 10^{-3}\gamma NM\}\). For any clause \(y \in H_{\text{fail}}\), let \(\text{fail}(y) \in y\) denote the variable in \(y\) for which \(L(y, x)\) is maximal. We bound Alice and Bob’s failure probability from below, to get an upper bound on \(|H_{\text{fail}}|\)

\[\Pr(\text{Provers fail to cheat}) \geq \sum_{y \in H_{\text{fail}}} \sum_{(\tilde{y}, \tilde{x}) \in L(y, x)} \text{FailProb}(y, \tilde{y}, x, \tilde{x}, k) \Pr(y, \tilde{y}, x, \tilde{x}|k)\]

\[\geq \sum_{y \in H_{\text{fail}}} \gamma 10^{-4} \Pr(y, \tilde{y}, \text{fail}(y), \tilde{x} : k)\]

\[\geq \sum_{y \in H_{\text{fail}}} \gamma 10^{-4} |L(y, \text{fail}(y))| \Pr(y, \tilde{y}, \text{fail}(y), \tilde{x} : k)\]

\[\geq \sum_{y \in H_{\text{fail}}} \gamma^2 NMW_AW_B \geq \frac{\gamma^2 |H_{\text{fail}}|}{25NM \cdot 10^7 \cdot 222MW_AW_B} = \frac{\gamma^2 |H_{\text{fail}}|}{M \cdot 5.55 \cdot 10^4}\]
Where the last inequality comes from taking a tuple in $H$. As $\Pr(\text{The provers fail}) < \gamma$, we have

$$|H_{\text{fail}}| < \frac{\gamma^3}{5.55 \cdot 10^{13}} \cdot \frac{M \cdot 5.55 \cdot 10^{10}}{\gamma^2} = 10^{-3}\gamma M$$

Taking $R = H \setminus H_{\text{fail}}$, we get $|R| \geq (1 - 0.002\gamma)M - |H_{\text{fail}}| \geq (1 - 0.003\gamma)M$ as required.

\[\square\]

**Proof of theorem 1.1** Assume $\Phi$ is not satisfiable, and assume by contradiction that the provers had some strategy which would work with success probability $\geq 1 - \frac{\gamma^3}{5.55 \cdot 10^{13}}$. Then there has to be a measurement result $k$ such that the success probability given $k$ is at least $1 - \frac{\gamma^3}{5.55 \cdot 10^{13}}$. However, according to the previous lemma, either the provers are caught with probability greater than $\frac{\gamma^3}{5.55 \cdot 10^{13}}$ (which contradicts our assumption on the success probability), or there exists a set $R$ as in the premises of that lemma. However, this would imply that there is a strategy in the classical protocol with success probability $> (1 - 0.003\gamma)(1 - \gamma 10^{-3})(1 - \gamma 10^{-4})(1 - 200\gamma 10^{-4})^2 > 1 - \gamma/3$, which is a contradiction.

\[\square\]

### 5 Conclusions and Open Problems

We have shown that NEXP can be recognized in a quantum MIP protocol, even if the provers have unlimited classical communication between them. Our protocol achieves perfect completeness and a constant gap. It only sends $O(\log(N))$ qubits, and thus can also be used for NP-complete languages with a polylogarithmic communication. Some interesting questions still remain open:

- What is the correct upper bound on the power of this proof system? Note that if the provers were allowed to make any joint separable measurement it would be exactly NEXP. Does adding provers or communication rounds help? What happens if there is just one quantum round?

- Is there a parallel repetition lemma for protocols when the provers are allowed to communicate with each other? The original proof of [Raz95] does not apply here.

- What happens in the dual problem, when the provers are allowed to share entanglement but are not allowed to communicate? Does our protocol still work, with a different proof?

- Does our result hold when the provers have a bounded amount of entanglement in addition to their communication channel?

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A Calculating Probabilities

Let \( r = (y, \tilde{y}, x, \tilde{x}) \). We wish to estimate \( \Pr(r|k) \). Bayes’ rule gives

\[
\Pr(r|k) = \frac{\Pr(k|r)\Pr(r)}{\Pr(k)} = \frac{\Pr(k|r)\Pr(r)}{\sum_s \Pr(k|s)\Pr(s)}
\]

where \( s \) denotes any legal tuple \( s = (c, \tilde{c}, v, \tilde{v}) \) with \( c, \tilde{c}, v, \tilde{v} \in V \) and \( v \in c \). As the prior distribution for all legal tuples is identical, we are only interested in calculating \( \Pr(k|s) \) for any legal tuple \( s = (c, \tilde{c}, v, \tilde{v}) \).

In the protocol we presented, the provers first apply their measurement and get \( k \), and then the verifier measures to get \( s \). However, it is physically equivalent to assume the verifier measured first. As the states sent to the provers are unentangled after tracing out the verifier, we have that

\[
\Pr(k|s) = \text{tr}((I \otimes A_k)\rho_A(I \otimes A_k)\dagger) \cdot \text{tr}((I \otimes B_k)\rho_B(I \otimes B_k)\dagger)
\]

Where \( \rho_A \) is the state in \( H_A^v \otimes H_A^M \), \( \rho_B \) is the state in \( H_B^v \otimes H_B^M \), and the identity is applied on the verifier’s side.

When considering states in \( H_A^v \otimes H_A^m \otimes H_A^p \), we stick to the convention that the first \( m \) qubits define the verifier’s private space, then next \( m + 3 \) describe the message qubits, and the last \( d \) define Alice’s private space. We can now calculate

\[
A_k(y) = \text{tr}(A_k(|y\rangle\langle y| \otimes I)A_k) = \sum_{j=1}^{8Md} \sum_{h=8d(y-1)+1}^{8Dy} A_k[j, h]A_k[j, h] = \sum_{j=1}^{8Md} \sum_{h=8d(y-1)+1}^{8Dy} |A_k[j, h]|^2
\]

\[
B_k(x) = \text{tr}(B_k(|x\rangle\langle x| \otimes I)B_k) = \sum_{j=1}^{2Nd} \sum_{h=2d(x-1)+1}^{2Dx} B_k[j, h]B_k[j, h] = \sum_{j=1}^{2Nd} \sum_{h=2d(x-1)+1}^{2Dx} |B_k[j, h]|^2
\]

We now assume that the \( x, \tilde{x} \) being traced out, and only look at the probabilities for \( y, \tilde{y} \), generated from \( \text{tr}((I \otimes A_k)\rho_A(I \otimes A_k)\dagger) \). As \( A_k(y) \) is just the trace out of the private data and the qubits which fit the assignment, then \( A_k(y) = \text{tr}((I \otimes A_k)\rho_A(I \otimes A_k)\dagger) \). We are analyzing the following expression:

\[
\text{tr}(I_M \otimes A_{8Md}\rho_A(I_M \otimes A_{8Md})\dagger)
\]

Up to normalization, \( \rho_A \) is a matrix which contains exactly four 1s, arranged: \( (a, a), (a, b), (b, a), (b, b) \). However, as we shall soon see, either \( a = b \) (in which case we have a single cell with a 4 in it) or else \( |a - b| \geq 8Md \) and thus, by the previous paragraph, we can ignore the off-diagonal entries. In both cases we can restrict our attention to the diagonal entries.

Thus the structure of the \( \rho_A \) matrix is:

\[
\rho = \frac{1}{\sqrt{2}} (|yy\rangle + |\tilde{y}\tilde{y}\rangle) \otimes |000\rangle\langle 000| + (|yy\rangle + |\tilde{y}\tilde{y}\rangle) \frac{1}{\sqrt{2}} \otimes |0_d\rangle\langle 0_d| \in H_A^v \otimes H_M^A \otimes H_p^A
\]
Note that the term 0\_d refers to element in a space of dimension d, as opposed to 000, an element in a space of dimension 2\^3. If y = \tilde{y} then obviously there is only one nonzero cell in the final matrix, on the diagonal. Otherwise, since |yy\rangle is located in the cell My + y = (M + 1)y, and \tilde{y} \neq y, they are differentiated (after tensoring) by at least (M + 1) \cdot 8 \cdot d > 8Md, as required.

Let A_k(i) = \sum_{j=1}^{8Md} \sum_{h=8d(i-1)+1}^{8Di} A_k[j, h]A_k[j, h] = \sum_{j=1}^{8Md} \sum_{h=8d(i-1)+1}^{8Di} |A_k[j, h]|^2. The probability that the verifier measures y, \tilde{y} in the modified protocol given k is

\[
P(y, \tilde{y}|k) = \frac{P(k|y, \tilde{y})P(y, \tilde{y})}{P(k)} = \frac{P(k|y, \tilde{y})P(y, \tilde{y})}{\sum_{z, \tilde{z}} P(k|z, \tilde{z})P(z, \tilde{z})} = \frac{\text{tr}(A_k\rho_{y, \tilde{y}}A_k^\dagger)}{\sum_{z, \tilde{z}} \text{tr}(A_k\rho_{z, \tilde{z}}A_k^\dagger)}
\]

(equal unless y = \tilde{y})

\[
\geq \frac{A_k(y) + A_k(\tilde{y})}{\sum_{z\neq \tilde{z}} (A_k(z) + A_k(\tilde{z})) + \sum_{z} 4A_k(z)} = \frac{A_k(y) + A_k(\tilde{y})}{\sum_{z\neq \tilde{z}} (A_k(z) + A_k(\tilde{z})) + \sum_{z} 2A_k(z)} = \frac{A_k(y) + A_k(\tilde{y})}{2MW_A + 2W_A}
\]

where W_A = \sum_z A_k(z). Note that if y = \tilde{y} we use 4A_k(y) instead of A_k(y) + A_k(\tilde{y}).

A.1 Bounding the Denominator

Let W_A = \sum_i A_k(i), W_B = \sum_i B_k(i), \tilde{W} = \Sigma_{c\in C, v\in C} A_k(c)B_k(v). We want to bound the denominator in

\[
\text{Pr}(y, \tilde{y}, x, \tilde{x}|k) = \frac{(A(y) + A(\tilde{y}))(B(x) + B(\tilde{x}))}{\sum_{c, \tilde{c}\in C, v\in C, \tilde{v}\in V} \text{Pr}(c, \tilde{c}, v, \tilde{v}|k)}
\]

Note that if c = \tilde{c}, then tr((I \otimes A_k)\rho_A(I \otimes A_k)^\dagger) = 4A_k(c). However, when c \neq \tilde{c}, we account this twice (because any of them can be considered first in the sum). Thus, the denominator becomes

\[
\sum_{c, \tilde{c}} \sum_{v\in C, \tilde{v}} (A_k(c) + A_k(\tilde{c}))(B_k(v) + B_k(\tilde{v})) + 2 \left( \sum_{c=\tilde{c}, v, \tilde{v}} + \sum_{c, \tilde{c}, v=\tilde{v}} \right) + 4 \sum_{c=\tilde{c}, v=\tilde{v}}
\]

where all the sums are on (A_k(c) + A_k(\tilde{c}))(B_k(v) + B_k(\tilde{v})), and factors of two and four come from c = \tilde{c}, and v = \tilde{v}. We begin by bounding the first two sums (which will contribute most of the weight). We omit the subindex k.
$$\sum_{c,\tilde{c}} \sum_{v \in c, \tilde{v}} (A(c) + A(\tilde{c}))(B(v) + B(\tilde{v})) = \sum_{c,\tilde{c}} \sum_{v \in c, \tilde{v}} A(c)B(v) + A(c)B(\tilde{v}) + A(\tilde{c})B(v) + A(\tilde{c})B(\tilde{v})$$

We now look at each of the four terms separately:

$$\sum_{c,\tilde{c}} \sum_{v \in c, \tilde{v}} A(c)B(v) = MN \sum_{v \in c, \tilde{v}} A(c)B(v)$$

$$\sum_{c,\tilde{c}} \sum_{v \in c, \tilde{v}} A(c)B(\tilde{v}) = 3M \sum_{c \in C, \tilde{v} \in V} A(c)B(\tilde{v})$$

$$\sum_{c,\tilde{c}} \sum_{v \in c, \tilde{v}} A(\tilde{c})B(v) = 5NW_AW_B$$

And $$\sum_{c,\tilde{c}} \sum_{v \in c, \tilde{v}} A(\tilde{c})B(\tilde{v}) = 3MW_AW_B$$. We used the fact that $$\Phi$$ is $$3-SAT$$, and that each variable appears exactly 5 times.

We return to bounding the sums in (6). By fixing $$c$$, we get that if $$c = \tilde{c}$$ the second sum is bounded, relative to the first, by a factor of $$2/M$$. Fixing $$\tilde{v}$$, we can bound the third sum by a factor of $$2/N$$. Fixing both, the fourth sum is bounded by a factor of $$4/(MN)$$. We get an overall bound for the denominator of:

$$(MN\tilde{W} + 3MW_AW_B + 5NW_AW_B + 3MW_AW_B)(1 + 2/M + 2/N + 4/(MN))$$

Since $$M$$ and $$N$$ are arbitrarily large, and $$M \geq N$$, we deduce our bound:

$$2(MN\tilde{W} + 11MW_AW_B)$$

which finally gives

$$\Pr(y, \tilde{y}, x, \tilde{x} | k) \geq \frac{A(y)B(x) + A(\tilde{y})B(\tilde{x})}{2MN\tilde{W} + 22MW_AW_B}$$

B Proof of Lemma 4.1

**Lemma 4.1** Assume $$A(y) \geq pA(\tilde{y})$$, $$p > 1$$. Then for any assignment $$T$$, the probability that the verifier will catch Alice cheating in the SWAP test is at least $$\frac{1}{2} - \frac{\delta^2}{1+p}$$.

**Proof.** Let $$\sigma = tr_{H_A} \left[ \frac{(I \otimes A_k)(I \otimes A_k)^T}{tr(I \otimes A_k)(I \otimes A_k)^T} \right]$$, and $$|\psi\rangle = 1/\sqrt{2}(|yy\rangle|T(y)\rangle + |\tilde{y}y\rangle|T(\tilde{y})\rangle)$$. Taking $$\delta = \sqrt{\langle \psi | \sigma | \psi \rangle}$$ the fidelity between $$|\psi\rangle$$ and $$\rho$$, the SWAP test has probability at least $$\frac{1-\delta^2}{2}$$ to distinguish between them [BCWW01].
To calculate $\sigma$, we utilize the result in Appendix A. Since $\rho$ consists of four elements in a rectangle $((8M + 8)y, (8M + 8)y), ((8M + 8)y, (8M + 8)y) + 8), ((8M + 8)y, (8M + 8)y) + 8), (8M + 8), (8M + 8)y)$, differentiated by a distance of at least $8M$, the nondiagonal elements do not contribute to the trace.

For a given assignment $T(y) \in \{0, 1\}^3$ and $T(y^\dagger) \in \{0, 1\}^3$, let $|\psi\rangle = 1/\sqrt{2}(|yyT(y)\rangle + |y\tilde{y}T(y^\dagger)\rangle)$. The fidelity between the pure state $\psi$ and the quantum state is $\sqrt{\langle \psi | \sigma | \psi \rangle}$.

$|\psi\rangle$ is an equal superposition of two base vectors, one corresponding to the base state $|yyT(y)\rangle$ and the other to $|y\tilde{y}T(y^\dagger)\rangle$. Thus the multiplication is effectively the sum of four elements arranged in a rectangle (multiplied by $1/2$). To calculate each of these four elements, we turn to Appendix A. Since, in the tensor product $I \otimes A_k$, any cell whose two coordinates differ by at least $8M$ is zero, we can simplify and get:

$$
\sigma[yyT(y), yyT(y)] = \text{tr}(A|yT(y)\rangle\langle yT(y)|A)
$$

We write the elements, sticking to the convention that the first $m$ qubits describe the verifier’s private space, the next $m$ fit the clause in the message space and the last three fit the value of the assignment:

$$
\sigma(8My + 8y + a, 8My + 8y + a) = \sum_i \sum_j |A[8dy + da + i, 8dy + da + j]|^2
$$

$$
\sigma((8M + 8)y + b, (8M + 8)y + b) = \sum_i \sum_j |A[8dy + db + i, 8dy + db + j]|^2
$$

$$
\sigma((8M + 8)y + a, (8M + 8)y + b) = \sum_i \sum_j A[8dy + da + i, 8dy + da + j]A^\dagger[8dy + db + i, 8dy + db + j]
$$

$$
\sigma((8M + 8)y + b, (8M + 8)y + a) = \sum_i \sum_j A[8dy + db + i, 8dy + db + j]A^\dagger[8dy + da + i, 8dy + da + j]
$$

Note that as $AA^\dagger$ is a measurement operator, we have that $A(y) \leq 1$, so $A(y^\dagger) \leq 1/p$. Now calculating, reindexing by $s = 8My + 8y + T(y)$ and $t = 8My + 8y + T(y^\dagger)$, and folding the sum into the expression, we get:

$$
\frac{|\sigma[s, s]|^2 + |\sigma[t, t]|^2 + |\sigma[s, t]|^2 + \sigma[s, s]|\sigma[s, t]|}{2(A(y) + A(y^\dagger))} \leq \frac{A(y) + A(y^\dagger) + 2\sqrt{A(y)A(y^\dagger)}}{2(A(y) + A(y^\dagger))}
$$

$$
= \frac{1}{2} + \frac{\sqrt{A(y)A(y^\dagger)}}{A(y) + A(y^\dagger)} < 1
$$

The last inequality follows from the AM-GM inequality. More precisely, since the ratio is at least $p$, the extreme value is achieved when it is exactly $p$, which (when substituting) gives what we need. When $p \geq \sqrt{2}$, this gives $1/2 + 2^{1/4}/(1 + \sqrt{2}) \leq 0.993$, as required. When $p \geq 2$, we get $1/2 - \sqrt{2}/3 \leq 0.975$.  \( \square \)
C Proofs for Lemmas 4.4 and 4.5

Remember that for \( c \in C \), \( u(c) = \sum_{v \in c} A(c) B(v) \), and for \( S \subset C \), \( U(S) = \sum_{c \in S} u(c) \). We also defined
\[
S_i = \left\{ c : \frac{W}{2^{i+1}} < u(c) \leq \frac{W}{2^i} \right\}
\]

**Lemma 4.4.** If there exists an index \( j \) such that \( \sum_{i=0}^{j-1} U(S_i) > \frac{W}{100} \) and \( \sum_{i=j}^{\infty} U(S_i) > \frac{W}{100} \), then the provers get caught with constant probability \( \frac{1}{6.96 \cdot 10^{-9}} \), generated from a \((1.48 \cdot 10^7, \sqrt{2})\) bad set.

**Proof.** We construct such a bad set \( D \). For any clause \( c \) in variables \( v_1, v_2, v_3 \), let \( \text{vmax}(c) \) denote the variable \( v_i \in c \) such that \( B[v_i] = \max\{B[v_j] : j = 1, 2, 3\} \), and define \( \text{vmin}(c) \) analogously. Let \( S_{\text{up}} = \cup_{i=0}^{j-1} S_i \), and \( S_{\text{down}} = \cup_{i=j}^{\infty} S_i \). As \( U(S_{\text{down}}) > \frac{W}{100} \), and \( S_{\text{down}} \) consists of “light” clauses, we must have \( |S_{\text{down}}| > M/100 \). Partition \( S_{\text{down}} \) arbitrarily into two sets \( S_l \) and \( S_r \), such that \( |S_l|, |S_r| \geq M/200 \). The idea is that each clause in \( S_{\text{up}} \) will contribute \( |S_l| \cdot |S_r| \) elements to \( D \).

For each \( c_{\text{up}} \in S_{\text{up}}, c_l \in S_l, c_r \in S_r \), we have \( u(c_{\text{up}}) > 2u(c_l), u(c_{\text{up}}) > 2u(c_r) \). Taking the maximal element in the sum for \( c_{\text{up}} \), and the minimal element for \( c_l, c_r \), we get:
\[
A(c_{\text{up}}) B(\text{vmax}(c_{\text{up}})) > 2A(c_l) B(\text{vmin}(c_l))
\]
\[
A(c_{\text{up}}) B(\text{vmax}(c_{\text{up}})) > 2A(c_r) B(\text{vmin}(c_r))
\]

Assume WLOG that \( A(c_l) < A(c_r) \). Then
\[
A(c_{\text{up}}) B(\text{vmax}(c_{\text{up}})) > 2A(c_l) B(\text{vmin}(c_r))
\]

So in the tuple \( (c_{\text{up}}, c_l, \text{vmax}(c_{\text{up}}), \text{vmin}(c_r)) \) at least one of the provers damages the state by at least \( \sqrt{2} \). We add this tuple to \( D \). Note that we have added a (distinct) element to \( D \) for each of the \( |S_l| \cdot |S_r| \) choices of \( c_l, c_r \), as desired. Let \( D_{\text{up}} \) denote the elements contributed to \( D \) by \( c_{\text{up}} \).

The next step is to prove that \( D \) has constant probability. Note that under the conditions of the lemma, we have
\[
\sum_{y, \tilde{y} \in C, x \in y, \tilde{x} \in V} \Pr(y, \tilde{y}, x, \tilde{x}|k) < 22MW_AW_B + 2NM\tilde{W} < 4NM\tilde{W}
\]
\[ \sum_{(y, \tilde{y}, x, \tilde{x}) \in D} \Pr((y, \tilde{y}, x, \tilde{x}) : k) = \sum_{c_{up}} \sum_{(y, \tilde{y}, x, \tilde{x}) \in D(c_{up})} \Pr((c_{up}, \tilde{y}, \max(c_{up}), \tilde{x}) : k) \]
\[ \geq \frac{1}{4NMW} \sum_{c_{up}} \sum_{(y, \tilde{y}, x, \tilde{x}) \in D(c_{up})} A(c_{up})B(\max(c_{up})) \]
\[ \geq \frac{1}{4NMW} |S_l| \cdot |S_r| \sum_{c_{up}} A(c_{up})B(\max(c_{up})) \]
\[ \geq \frac{1}{4NMW} |S_l| \cdot |S_r| \sum_{c_{up}} u(c_{up})/3 \]
\[ \geq \frac{1}{4NMW} |S_l| \cdot |S_r| \frac{W}{300} \]
\[ \geq \frac{1}{4NMW} \cdot \frac{M}{200} \cdot \frac{M}{200} \cdot \frac{W}{300} \]
\[ = \frac{1}{4.8 \cdot 10^7} \]

(\text{because } M > N) \]

Remember that \( F = S_j \cup S_{j+1} \) and \( T_i = \{ c \in F : \frac{W_A}{2^i} < A(c) \leq \frac{W_A}{2^{i+1}} \} \). We wish to prove \textbf{Lemma 4.5}. If there exists an index \( j \) such that \( \sum_{i=0}^{j-1} |T_i| > |F|/100 \), and \( \sum_{i=j+1}^{\infty} |T_i| > |F|/100 \), then the first prover gets caught with constant probability \( \frac{1}{4.2 \cdot 10^7} \), generated from a \( (1.2 \cdot 10^6, 2) \) bad set. \[ \square \]

\textbf{Proof}. Let \( T_{up} = \cup_{i=0}^{j-1} T_i \), \( T_{down} = \cup_{i=j+1}^{\infty} T_i \). Note that any clause from \( T_{up} \) at least 2-damages any clause in \( T_{down} \). Take \( D = \cup_{c_{up} \in T_{up}} \{ c_{up} \} \times T_{down} \times \{ \max(c_{up}) \} \times V \). Note that \( |T_{down}| > 0.98M/100 > M/200 \), and as \( T_{up} \subset F \), we have \( U(T_{up}) \geq \frac{0.98W}{400} \geq \frac{W}{500} \), and thus
\[ \Pr(D) \geq \frac{1}{4NMW} |T_{down}| N \frac{W}{1500} \geq \frac{1}{1.2 \cdot 10^6} \]
\[ \square \]

\section*{D Proofs for Lemmas 4.6, 4.7 and 4.8}

The proofs in this appendix are very similar and very easy. We recall some definitions, then state the lemmas. Define \( S_i = \{ c \in C : \frac{W_A}{2^i} \leq A(c) < \frac{W_A}{2^{i+1}} \} \). For a set \( S \subset C \), let \( W(S) = \sum_{c \in S} A(c) \).

\textbf{Lemma 4.6}. If \( NM\hat{W} < 100MW_AW_B \) and there exists an index \( i \) such that
\[ \sum_{j=0}^{i-1} W(S_j) > \gamma 10^{-4} W_A \wedge \sum_{j=i+1}^{\infty} |S_j| > \gamma 10^{-4} M \]

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then Alice is caught cheating with probability $\frac{\gamma^2}{2 \cdot 6 \cdot 10^{-12}}$, generated from a $(\frac{\gamma^2}{1 \cdot 4 \cdot 10^{-12}}, 2)$ bad set.

**Proof.** Let $S_{up} = \bigcup_{j=0}^{i-1} S_j$, $S_{down} = \bigcup_{j=i+1}^{\infty} S_j$. Let $D = \bigcup_{c \in S_{up}} \bigcup_{v \in c} \{c\} \times S_{down} \times \{v\} \times V$. Every $(y, y, x, \bar{x}) \in D$ is 2-damaged by Alice. On the other hand,

$$\Pr(y, y, x, \bar{x} \mid k) \geq \frac{\gamma}{22 M W_A W_B + 2 N M W} \geq \frac{\gamma}{222 M W_A W_B} \geq \frac{\gamma^2 W_A}{222 \cdot 10^8 W_A} \geq \frac{\gamma^2}{7.4 \cdot 10^{19}}$$

Summing this over $D$ gives

$$\Pr(D) \geq \sum_{y \in S_{up}} \sum_{x \in y} \sum_{\bar{x} \in V} \frac{\gamma}{222 M W_A W_B} \geq \sum_{y \in S_{up}} 3 \cdot 10^{-4} \gamma M W_B \frac{\gamma}{222 M W_A W_B} \geq \frac{3 \gamma^2 W_A}{222 \cdot 10^8 W_A} \geq \frac{\gamma^2}{7.4 \cdot 10^{19}}$$

**Lemma 4.6** If $N MW < 100 M W A W B$ and the second condition of Lemma 4.6 does not hold, then there exists an index $i$ such that for $F = S_i \cup S_{i+1}$ we have $|F| \geq (1 - 0.0002 \gamma) M \wedge W(F) \geq (1 - 0.0002 \gamma) W_A \wedge \forall c \in F : A(c) \geq \frac{W_A}{10^4}$.

**Proof.** Choose $t$ to be the smallest index for which the first half of the condition does hold, i.e., $\sum_{j=0}^{i-1} W(S_j) > \gamma 10^{-4} W_A$. Then the second half of the condition cannot hold, i.e.

$$\sum_{j=0}^{\infty} |S_j| \leq \gamma 10^{-4} M$$

Take $i = t - 1$ (note that $t \neq 0$ because otherwise the first half of the condition does not hold). So:

$$|S_i| + |S_{i+1}| = M - \sum_{j=0}^{i-1} |S_j| - \sum_{j=i+2}^{\infty} |S_j| \geq M - \sum_{j=0}^{i-1} |S_j| - \gamma 10^{-4} M \geq M - \gamma 10^{-4} M - \gamma 10^{-4} M$$

where the last inequality follows since the total weight $\sum_{j=0}^{i-1} W(S_j) < \gamma 10^{-4} W_A$, but each clause in the $S_j$s contributes at least $2^{-i} W_A$ to $W(S_j)$ while each clause outside of the $S_j$s contributes at most $2^{-i+1} W_A$. A similar argument now applies to the weight $W(S_i) + W(S_{i+1})$. Finally, for each $c \in F$ we have

$$A(c) \geq \frac{W(F)}{4|F|} \geq \frac{(1 - 0.0002 \gamma) W_A}{4 M} \geq \frac{W_A}{5 M}$$

Remember $S_i$: $T_i = \{ v \in V : \frac{W_a}{2^{|v|}} \leq B(v) < \frac{W_a}{2^{|v|}} \}$. We now prove

**Lemma 4.8** Either Bob gets caught cheating with probability $\gamma^2 \frac{3 \cdot 9 \cdot 10^{12}}{1 \cdot 1 \cdot 10^{-12}}$ which is generated from a $(\frac{\gamma^2}{1 \cdot 1 \cdot 10^{-12}}, 2)$ bad set, or else there exists an index $i$ such that for $G = T_i \cup T_{i+1}$ we have $|G| > (1 - 0.0002 \gamma) N$, $\sum_{v \in G} B(v) \geq (1 - 0.0002 \gamma) W_B$ and for each $v \in G$, $B(v) \geq \frac{W_B}{5 N}$. 

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Proof. If no such index exists then there is a separating index $i$ such that letting $T_{up} = \bigcup_{j=0}^{i-1} S_j$, $T_{down} = \bigcup_{j=i+1}^{\infty} S_j$, we have $\sum_{v \in T_{up}} B(v) > 10^{-4}\gamma W_B$, $|T_{down}| > 10^{-4}\gamma N$. Let $D = \bigcup_{v \in T_{up}} \bigcup_{c : v \in c} \{c\} \times C \times \{v\} \times T_{down}$.

$$\Pr(D : k) \geq \sum_{(y,x,\tilde{y},\tilde{x})} \frac{A(y)B(x)}{222 MW_A W_B} \geq \sum_{x \in T_{up}} \frac{\gamma NW_AB(x)}{222 MW_A W_B} \geq \frac{\gamma^2 NW_B}{2.22 \cdot 10^{10} MW_A} \geq \frac{\gamma^2 M}{1.1 \cdot 10^{11} M} = \frac{\gamma^2}{1.1 \cdot 10^{11}}$$

where we used the fact that each variable appears in the formula 5 times. \qed