A TWISTED $\bar{\partial}_f$-NEUMANN PROBLEM AND TOEPLITZ $n$-TUPLES FROM SINGULARITY THEORY

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ABSTRACT. A twisted $\bar{\partial}_f$-Neumann problem associated to a singularity $(\mathcal{O}_n, f)$ is established. By constructing the connection to the Koszul complex for toeplitz $n$-tuples $(f_1, \cdots, f_n)$ on Bergman spaces $B^0(D)$, we can solve this $\bar{\partial}_f$-Neumann problem. Moreover, the cohomology of the $L^2$ holomorphic Koszul complex $(B^*(D), \partial f \wedge)$ can be computed explicitly.

1. Introduction

Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ smooth boundary $\partial D$ and $f$ a holomorphic function on $D$ with only isolated critical points in $D$ and no critical points on $\partial D$. Under such assumption, we get two objects in the framework of analysis.

The first object is the toeplitz $n$-tuples with symbols $(f_1, f_2, \cdots, f_n)$ defined on the Bergman space on $D$, where the $f_i$'s are partial derivatives of $f$. One can study the $L^2$ holomorphic complex $(B^*(D), \partial f \wedge)$ given by

$$0 \to B^0(D) \xrightarrow{\partial f \wedge} B^1(D) \xrightarrow{\partial f \wedge} \cdots \xrightarrow{\partial f \wedge} B^n(D) \to 0.$$ 

Note that if without $L^2$ condition, this complex is an algebraic Koszul complex. If assuming $(f_1, \cdots, f_n)$ is regular, then the homology of the algebraic Koszul complex will only be nontrivial on the top term and is isomorphic to the Jacobian ring of $f$ on $D$. In the assumption of $L^2$ integrability, lack of noetherian ring structure make things complicated. This complex is an important example in Taylor’s multivariable spectral theory (ref. [Ta]) and has been studied a lot. The spectral picture, spectral mapping theorem and the index theory were all developed (ref. [EP]). The index of this complex is computed to be the dimension of $\text{Jac}(f)$ on $D$ (ref. [EP], Chapter 10). The fact that the cohomology is concentrated at the $n^{th}$ degree should be known (we were informed by M. Putinar [Pu] that this can be proved via the spectral

† Supported by NSFC(11271028), NSFC(11325101), and Doctoral Fund of Ministry of Education of China(20120001110060).
localization technique), but the direct proof seems not so easy. In this paper, we will reprove this result via the study of $\bar{\partial}_f$ operator.

On the other hand, we can define the twisted Cauchy-Riemann operator $\bar{\partial}_f := \bar{\partial} + \partial f \wedge$ on $D$, which only preserving the real grading of the differential forms, not the Hodge grading. This operator was used by physicists to study the topological field theory of Landau-Ginzburg model from the B side (ref. [Ce, CV]). In recent years, LG model has been found to be a very important part of 2-d topological field theory, mirror symmetry and categorification theory of open strings (ref. [FJR, CR, FJ, GMW, KKS]). Inspired by the physicists’ work, the second author proposed an approach ([Fa]) to study the singularity theory of $f$ by constructing the Hodge theory for the operator $\bar{\partial}_f$ and the twisted Laplacian $\Delta_f = \bar{\partial}_f \bar{\partial}_f^* + \bar{\partial}_f^* \bar{\partial}_f$. The aim is to construct the Saito’s Frobenius manifold structure (ref. [ST]) for singularities and eventually treat the quantization problem of LG model from the B side. Recently, a different method via the theory of polyvector fields was built by Li-Li-Saito [LLS] for studying the singularity and the related primitive forms, which however did not touch the Hodge structure. The paper [Fa] can only treat the marginal deformation of a general singularity, but not the universal deformation of a singularity. Hence to recover Saito’s Frobenius manifold structure from the analytical method, we must study some boundary value problem of $\bar{\partial}_f$ operator.

In this paper, we will study the $\bar{\partial}_f$-Neumann problem on $D$. This problem is related to the $L^2$ complex $(L^2(D), \bar{\partial}_f)$, whose cohomology group is denoted by $H^*_{{(2), \bar{\partial}_f}}$. As the first result, we can solve the $\bar{\partial}_f$-Neumann problem by proving the strong Hodge decomposition theorem as below.

**Theorem 1.1.** Let $D$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ smooth boundary $\partial D$ and $f$ a holomorphic function on $\bar{D}$ with only isolated critical points in $D$ and no critical points on $\partial D$. Then we have the decomposition

$$H^*(D) = \mathcal{H}^* \oplus \text{im} \bar{\partial}_f \oplus \text{im} \bar{\partial}_f^*,$$

(1.1)

and then the isomorphism

$$H^*_{{(2), \bar{\partial}_f}} \cong \mathcal{H}^* .$$

(1.2)

Furthermore, all the spaces $\mathcal{H}^*$ are of finite dimensional.

Theorem 1.1 is a direct conclusion of Theorem 3.7 and Corollary 3.8. To prove this theorem, we first show that $\Delta_f$ with $\bar{\partial}_f$-Neumann boundary condition is a self-adjoint operator, hence there exists a weak
Hodge decomposition. Usually, to prove the strong decomposition, we need a global a priori estimate for the Green operator which can naturally deduce the compactness by Rellich theorem. However, things are different in our $\bar{\partial}_f$-Neumann problem. The $\Delta_f$ operator always mix $(k,0)$-forms with other types of forms, thus the a priori estimate becomes complicated because there is no global estimate to control the Sobolev norms of holomorphic $k$ form in $\bar{\partial}$-Neumann problem. However, we have an indirect way to get around this problem. We can construct an isomorphism between the $L^2$ complex $(L^2(D), \bar{\partial}_f)$ and the $L^2$ holomorphic complex $(B^*(D), \partial f \wedge)$. By Taylor’s joint spectral theory, the cohomology of the later complex can be proved to be of finite dimension. Using the finite dimensionality and a theorem from functional analysis, we can prove the range of $\bar{\partial}_f$ and $\bar{\partial}_f^*$ are all closed. So this proves the strong Hodge decomposition, meanwhile we can prove that the spectrum of $\Delta_f$ has a gap at 0.

Conversely, by studying the complex $(L^2(D), \bar{\partial}_f)$ in $C^\infty$ category, we can calculate the cohomology of $(B^*(D), \partial f \wedge)$ as mentioned above. The second main result is as follows.

**Theorem 1.2.** If $D$ is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ smooth boundary $\partial D$ and $f$ is a holomorphic function on $D$ with isolated critical points in $D$ and no critical points on $\partial D$, then the dimension of the Koszul cohomology on Bergman spaces is concentrated at the $n^{th}$ degree and equal to the number of critical points, with multiplicities accounted, of $f$ in $D$.

**Corollary 1.3.** Under the assumption of Theorem 1.2, The cohomology groups

$$H^*_\partial(D), H^*_\bar{\partial}(\bar{D}), H^*_\partial(c,\bar{\partial}_f), H^*_\partial((2),\bar{\partial}_f), H^*_\partial(0,\bar{\partial}_f)$$

are all isomorphic to the space $\mathcal{H}^*$.

**Remark 1.4.** For arbitrary $n$-tuples $(f_1, f_2, \cdots, f_n)$ satisfying the condition that they have only finite common zeros and have no common zeros on $\partial D$, the proof in our article can be applied to the operator $\partial + (f_1 dz_1 + \cdots + f_n dz_n)\wedge$ and all our results still holds. In fact, throughout our article, we will not use the fact that the $f_i$’s are the partial derivatives of a single function.

**Notation 1.5.** We use the super-bracket

$$[A, B] := AB - (-1)^{\deg(A) \deg(B)} BA$$

in this paper.
2. $\bar{\partial}_f$-Neumann Problem

Let $h = \sum_i \frac{1}{2} dz^i \otimes d\bar{z}^i$ be the standard hermitian metric of $\mathbb{C}^n$ in the coordinate system $\{z_i, i = 1, \ldots, n\}$. Let $D$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and $f$ a holomorphic function on $\bar{D}$.

The study of pseudoconvexity is one of the central topic in the theory of functions of several complex variables. $D$ is called pseudoconvex if it can be exhausted by a continuous plurisubharmonic function. Every (geometrically) convex domain in $\mathbb{C}^n$ is pseudoconvex. If the boundary $\partial D$ is $C^2$, then this is equivalent to the Levi pseudoconvexity we will explain below.

Let $r$ be a $C^2$ function defined in a neighborhood of $p \in \partial D$ satisfying $r|_{\partial D} = 0$ and $\|dr\| = 1$ on $\partial D$. Then we can define a Levi form $L_p$ along the $n-1$ dimensional subspace $\{\xi \in T_p\mathbb{C}^n | \sum_{j=1}^{n} \frac{\partial r}{\partial z_j} \xi_j = 0\}$ by

$$L_p(\xi, \eta) = \sum_{i,j} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} \xi_i \eta_j.$$  \hspace{1cm} (2.1)

If the Levi form $L_p$ is semi-positive at all points $p \in \partial D$, then $D$ is said to be Levi pseudoconvex. If $L_p$ is positive at all points of $\partial D$, then $D$ is said to be strongly pseudoconvex. The balls in $\mathbb{C}^n$ are strongly pseudoconvex.

Denote by $A^p(D)$ (or $A^{p,q}(D)$) the space of smooth $p$ forms (or $(p,q)$-forms) on $D$ and $A(D) = \oplus_p A^p(D)$. Let $A^p(\bar{D})$ be the subspace of $A^p(D)$ whose elements can be extended smoothly to a small neighborhood of $\bar{D}$ and $A(\bar{D}) = \oplus_p A^p(\bar{D})$. $A^p_c(D)$ is a subspace of $A^p(\bar{D})$ whose elements have compact support disjoint from $\partial D$. Similarly, we have the definitions of $A^{p,q}(\bar{D})$ and $A^{p,q}_c(\bar{D})$.

For any form $\varphi \in A^{p,q}(\bar{D})$, we have the expression

$$\varphi = \sum_{I,J}' \varphi_{I,J} dz^I \wedge d\bar{z}^J,$$

where $\sum'$ means summation over strictly increasing multi-indices and $\varphi_{I,J}$'s are antisymmetric for arbitrary $I$ and $J$.

For any $(p,q)$-forms $\varphi = \sum_{I,J} \varphi_{I,J} dz^I \wedge d\bar{z}^J$ and $\psi = \sum_{I,J} \psi_{I,J} dz^I \wedge d\bar{z}^J$, we can define the $L^2$ inner product:

$$\langle \varphi, \psi \rangle = \sum_{I,J}' \langle \varphi_{I,J}, \psi_{I,J} \rangle = \sum_{I,J}' \int_D \varphi_{I,J} \overline{\psi_{I,J}} dV$$

where $dV$ denote the volume element on $D$ defined by $h$. 
Let \( \| \cdot \| \) be the corresponding \( L^2 \)-norm and \( L^2_{(p,q)}(D) \) be the \( L^2 \)-completion space of \( A^{p,q}(\bar{D}) \). Define \( L^2_k(D) = \oplus_{p+q=k} L^2_{(p,q)}(D) \) and \( L^2(D) = \oplus_k L^2_k(D) \). Furthermore, the Sobolev \( s \)-norms \( \| \cdot \|_s \) and the corresponding Sobolev spaces \( W^s_{(p,q)}(D), W^s_k(D), W^s(D) \) can be defined. For example, for non-negative integer \( s \), elements of \( W^s(D) \) has derivatives in \( L^2(D) \) up to \( s \) order and \( \| \varphi \|_s \) is the sum of \( L^2 \) norms of derivatives of \( \varphi \) up to \( s \) order. In particular, we have \( W^0(D) = L^2(D) \).

Now any differential operator \( T \) defined on \( A(\bar{D}) \) can be extended to a unbounded closed operator in \( L^2(D) \) by means of generalized derivatives. Remember that if \( T \) is a closed operator defined on \( \text{Dom}(T) \subset L^2(D) \) if and only if the following holds: if \( \varphi_i \in A^p(D) \cap L^2(D) \) and \( T(\varphi_i) \in L^2(D) \) are function sequences such that \( \varphi_i \to \varphi \) and \( T(\varphi_i) \to \psi \in L^2(D) \), then \( \varphi \) is in \( \text{Dom}(T) \) and \( T(\varphi) = \psi \).

Now the Cauchy-Riemann operator \( \bar{\partial} \) and the twisted operator operator \( \bar{\partial}_f = \bar{\partial} + \bar{\partial}f \wedge : A^k(\bar{D}) \to A^k(\bar{D}) \) can be extended to closed operators in \( L^2(D) \) such that

\[
\text{Dom}(\bar{\partial}) = \{ \varphi \in L^2(D) | \bar{\partial}\varphi \in L^2(D) \} \\
\text{Dom}(\bar{\partial}_f) = \{ \varphi \in L^2(D) | \bar{\partial}_f\varphi \in L^2(D) \}.
\]

Since \( f \) is bounded on \( D \), the multiplication operator \( \partial f \wedge \) has the domain \( \text{Dom}(\partial f \wedge) = L^2(D) \) and actually we have

\[
\text{Dom}(\bar{\partial}_f) = \text{Dom}(\bar{\partial}).
\]

Now we consider the adjoint of \( \bar{\partial}_f \) under the \( L^2 \) norm. By definition the Hilbert space adjoint \( \bar{\partial}_f^* \) of \( \bar{\partial}_f \) is defined on the domain \( \text{Dom}(\bar{\partial}_f^*) \) consisting of all \( \varphi \in L^2_k(D) \) such that \( |\langle \varphi, \bar{\partial}_f(\psi) \rangle| \leq c \|\psi\| \) for some positive constant \( c \) and for all \( \psi \in \text{Dom}(\bar{\partial}_f) \). As \( \bar{\partial}f \wedge \) is bounded, the above inequality is equivalent to \( |\langle \varphi, \bar{\partial}(\psi) \rangle| \leq c' \|\psi\| \) for some positive constant \( c' \). This means \( \text{Dom}(\bar{\partial}_f^*) = \text{Dom}(\bar{\partial}^*) \) and they have the same Neumann boundary conditions.

For \( \varphi, \psi \in A(\bar{D}) \), we have the integration by parts:

\[
\langle \bar{\partial}_f \varphi, \psi \rangle = \langle \varphi, \bar{\partial}_f \psi \rangle + \int_{\partial D} \langle \sigma(\bar{\partial}, dr) \varphi, \psi \rangle \quad (2.2)
\]
\[
\langle \partial_f \varphi, \psi \rangle = \langle \varphi, \partial_f \psi \rangle + \int_{\partial D} \langle \sigma(\partial, dr) \varphi, \psi \rangle. \quad (2.3)
\]

Here

\[
\partial_f = \partial + \bar{f}_j \partial_j,
\]
where \( \vartheta \) represents the formal adjoint of \( \bar{\partial} \) and \( \iota_{\partial_j} \) is the contraction operator with the vector \( \partial_j = \frac{\partial}{\partial z_j} \), and

\[
\sigma(\bar{\partial}, dr) = \bar{\partial} r \wedge = \frac{\partial r}{\partial z_j} d\bar{z}^j \wedge, \quad \sigma(\vartheta, dr) = -\frac{\partial r}{\partial z_j} \iota_{\partial_j}.
\]

Hence we have

\[
\text{Dom}(\bar{\partial}^* f) \cap A(\bar{\partial} D) = \{ \phi \in A(\bar{\partial} D) | \sigma(\vartheta, dr) \phi = 0 \text{ on } \partial D \}
\]

Denote by \( D^{p,q} = \{ \phi \in A(\bar{\partial} D) | \sigma(\vartheta, dr) \phi = 0 \text{ on } \partial D \} \) and \( D^k = \oplus_{p+q=k} D^{p,q} \).

**Definition 2.1.** Let \( \Delta_f = [\bar{\partial}_f, \bar{\partial}_f^*] = \bar{\partial}_f \vartheta + \vartheta \bar{\partial}_f^* \) be the operator from \( L^2(D) \) to \( L^2(D) \) with domain \( \text{Dom}(\Delta_f) = \{ \phi \in L^2(D) | \phi \in \text{Dom}(\bar{\partial}_f) \cap \text{Dom}(\bar{\partial}_f^*); \bar{\partial}_f \phi \in \text{Dom}(\bar{\partial}_f^*) \} \).

**Proposition 2.2.** \( \Delta_f \) is a linear, densely defined, closed self-adjoint operator.

**Proof.** The proof is the same as the proof of Proposition 1.3.8 in [FK]. \( \square \)

**Remark 2.3.** We can consider the formal Laplacian \( \hat{\Delta}_f = \bar{\partial}_f \vartheta + \vartheta \bar{\partial}_f^* + I \) defined on \( D^{p,q} \). This operator has a unique Friedrichs self-adjoint extension related to the quadratic form \( Q(\varphi, \phi) = (\bar{\partial}_f \varphi, \vartheta \bar{\partial}_f^* \psi) + (\vartheta \bar{\partial}_f \varphi, \vartheta \bar{\partial}_f^* \psi) + (\varphi, \psi) \). This extended self-adjoint operator is just \( \Delta_f + I \) and the equivalence relation is clear by the standard abstract theorem in functional analysis.

The self-adjointness of \( \Delta_f \) is due to the \( \bar{\partial} \)-Neumann boundary condition which is characterized by

\[
\text{Dom}(\Delta_f) \cap A(\bar{\partial} D) = \{ \phi \in A(\bar{\partial} D) | \sigma(\vartheta, dr) \phi = 0 \text{ on } \partial D \} \quad \text{(2.4)}
\]

Similar to the \( \bar{\partial} \)-Neumann problem, here we want to solve the equation \( \Delta_f \varphi = \eta \in L^2(D) \) under the \( \bar{\partial} \)-Neumann boundary condition. We call this as \( \bar{\partial}_f \)-Neumann problem.

Since \( \Delta_f \) is self-adjoint and \( \text{Im}(\bar{\partial}_f) \perp \text{Im}(\bar{\partial}_f^*) \), we get a weak Hodge decomposition

\[
L^2_k(D) = \mathcal{H}^k \oplus \text{Im}(\Delta_f) = \mathcal{H}^k \oplus \text{Im}(\bar{\partial}_f) \oplus \text{Im}(\bar{\partial}_f^*) \quad \text{(2.5)}
\]

where \( \mathcal{H}^k \) denote the kernel of \( \Delta_f \).

To solve the \( \bar{\partial}_f \)-Neumann problem, we need to prove that all the range in the above decomposition are closed. The \( \bar{\partial}_f \)-Neumann problem will display different nature compared to the \( \bar{\partial} \)-Neumann problem, in which \( f \) will play dominant role. This will be shown in next section.
3. $\bar{\partial}_f$-COMPLEXES, FINITE DIMENSIONALITY AND SPECTRAL GAP

In this section, we first discuss various $\bar{\partial}_f$ complexes defined on a bounded pseudoconvex domain. Then we will show that the $L^2 \bar{\partial}_f$-complex has finite dimensional cohomology groups and there exists a spectral gap between 0 and other spectra of $\Delta_f$. In the $\bar{\partial}$-Neumann problem, there is no estimate for $L^2$ integrable holomorphic $(p,0)$-forms, which is in the kernel of $\Delta_{\bar{\partial}}$, near the boundary. For this reason, we solve the $\bar{\partial}_f$-Neumann problem in a indirect way. We avoid to estimate directly the behavior of the operator $\Delta_f$, which twist the $(p,0)$-forms and other types of forms, instead, we will use a classical result in multivariable spectra theory about the complex $(B^* (D), \partial f \wedge)$ and some results in the theory of unbounded linear operators.

3.1. $\bar{\partial}_f$-complexes.

There are various $\bar{\partial}_f$-complexes which are defined by smoothness or boundary value conditions. At first, we have the $L^2 \bar{\partial}_f$-complex

$$L^2(D) : L^2_0(D) \xrightarrow{\bar{\partial}_f} L^2_1(D) \xrightarrow{\bar{\partial}_f} L^2_2(D) \xrightarrow{\bar{\partial}_f} \cdots \quad (3.1)$$

corresponding to $L^2$ integrable $p$-forms. The cohomology group is defined as

$$H^k_{(\bar{\partial}_f,\bar{\partial}_f)} = \frac{\{ \varphi \in \text{Dom}(\bar{\partial}_f) | \bar{\partial}_f \varphi = 0 \}}{\bar{\partial}_f (\text{Dom}(\bar{\partial}_f))}$$

In addition, there are $\bar{\partial}_f$-complexes $\mathcal{A}^*(D), \mathcal{A}^*(\bar{D}), \mathcal{A}_{\bar{\partial}}^*(D)$, which correspond to smooth $p$-form on $D$, on $\bar{D}$, and having compact support in $D$ respectively. We denote by $H^k_{\bar{\partial}_f}(D), H^k_{\bar{\partial}}(D), H^k_{(\bar{\partial}_f,\bar{\partial}_f)}(D)$ the corresponding cohomology groups.

Let $\mathcal{C}^{p,q} = \{ \varphi \in \mathcal{A}(\bar{D}) | \sigma(\bar{\partial}, dr) \varphi = 0 \text{ on } \partial D \}$ and $\mathcal{C}^k = \bigoplus_{p+q=k} \mathcal{C}^{p,q}$. We can take $\bar{\partial}_f$ as a closed operator in $H(D)$ at first and then consider the $\bar{\partial}_f$-Neumann problem, and in this case, we have $\mathcal{C}^k = \mathcal{A}^k(D) \cap \text{Dom}(\bar{\partial}_f^*)$.

**Lemma 3.1.** $\bar{\partial}_f \mathcal{C}^k \subset \mathcal{C}^{k+1}$.

**Proof.** If $\psi \in \mathcal{C}^k$, then it can be written as $\psi = \bar{\partial}_f r \wedge \alpha + r \beta$ for $\alpha \in \mathcal{A}^{k-1}(D), \beta \in \mathcal{A}^k(D)$. Then $\bar{\partial}_f \psi = \bar{\partial}_f r \wedge (\beta - \bar{\partial}_f \alpha) + r \bar{\partial}_f \beta$ which is in $\mathcal{C}^k$. \qed

This lemma shows that $(\mathcal{C}^*, \bar{\partial}_f)$ forms a complex and has cohomology $H^k_{\bar{\partial}_f}(\mathcal{C})$. 

As in [FK], we also have the Dirichlet or zero-boundary value cohomology

\[
H^k_{(0,\partial D)} = \left\{ \psi \in A^k(\tilde{D}) \mid \partial f \psi = 0, \psi|_{\partial D} = 0 \right\} / \left\{ \partial f \left( \psi \in A^{k-1}(\tilde{D}) \mid \psi|_{\partial D} = 0, \partial f \psi|_{\partial D} = 0 \right) \right\}.
\] (3.2)

**Proposition 3.2.** There exists isomorphism \( i : H^k_{(0,\partial f)} \cong H^k_{\partial f}(\mathcal{C}) \).

**Proof.** Suppose that \( \phi \in A^k(\tilde{D}), \phi|_{\partial D} = 0 \), and \( \phi = \partial f \psi \) with \( \psi \in \mathcal{C}^{k-1} \). Then \( \psi \) has the form \( \psi = \partial r \wedge \alpha + r \beta \). This can be rewritten as

\[
\psi = \partial (r \wedge \alpha) + r(-\partial \alpha + \beta).
\]

Let \( \psi_0 = r(-\partial \alpha + \beta) \). This gives \( \phi = \partial f \psi_0 \), which shows that \( i \) is a well-defined injective map. To prove the surjectivity, suppose \( \phi \in \mathcal{C}^k \) and \( \partial f \phi = 0 \). Then \( \phi \) also has the expression \( \phi = \partial (r \wedge \alpha) + r(-\partial \alpha + \beta) \). Hence \( \phi \) is cohomologous to \( r(-\partial \alpha + \beta) \), which vanishes on \( \partial D \). \( \square \)

We will discuss other relations between these cohomologies in the following sections. Above all, we want to discuss the relation between the \( L^2 \) complex \((L^2(D), \partial f)\) and the \( L^2 \) holomorphic Koszul complex \((B^*(D), \partial f \wedge)\).

### 3.2. Koszul complex, finite dimensionality and spectral gap

Let \( B^k(D) \) be the \( L^2 \) integrable holomorphic \( k \)-form on \( D \), i.e., \( B^0(D) \) is the Bergman space on \( D \) and \( B^k(D) \) can be viewed as direct products of \( B^0(D) \). The complex \((B^*(D), \partial f \wedge)\) is defined as

\[ 0 \to B^0(D) \xrightarrow{\partial f \wedge} B^1(D) \xrightarrow{\partial f \wedge} \cdots \xrightarrow{\partial f \wedge} B^n(D) \to 0, \]

whose cohomology are denoted by \( H^*_f(D) \).

In 1970, J. L. Taylor [Ta] developed a multivariable (joint) spectral theory. Given a Hilbert space \( X \) and a commuting \( n \)-tuples of bounded linear operators \( T = (T_1, \cdots, T_n) \) on \( X \), the joint spectra \( \sigma(T, X) \) is the set of all \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n \) such that \( K^*(T - \lambda, B(D)) \) is not acyclic. The essential joint spectra \( \sigma_e(T, X) \) is the set of all \( \lambda \) such that the cohomology of \( K^*(T - \lambda, B(D)) \) is not finite dimensional. The finite complex \( K^*(T - \lambda, B(D)) \) consists of the spaces

\[ K^p(T - \lambda, X) = X \otimes \mathbb{C} L^p(\mathbb{C}^n) \quad (0 \leq p \leq n) \]

and the coboundary operators

\[ d^p : K^p(T - \lambda, X) \to K^{p+1}(T - \lambda, X), \quad d^p(\varphi) = \tau \wedge \varphi \]

where \( \tau = (T_1 - \lambda_1) \otimes e_1 + (T_2 - \lambda_2) \otimes e_2 + \cdots (T_n - \lambda_n) \otimes e_n \) and \((e_1, e_2, \cdots, e_n)\) is the canonical basis of \( \mathbb{C}^n \).
The $L^2$ $\bar{\partial}f \wedge$-complex $(B^*(D), \bar{\partial}f \wedge)$ can be viewed as a model for Taylor’s joint spectral theory. The Bergman space is a Hilbert space and the toeplitz operators defined by multiplication by $f_i = \frac{\partial f}{\partial z_i}, 1 \leq i \leq n$, is a commuting $n$-tuples of bounded linear operators. The $dz_i$’s can be viewed as a basis of $\mathbb{C}^n$. Thus the associated Koszul complex is exactly $(B^*(D), \bar{\partial}f \wedge)$.

Under our assumption, $D$ is bounded and pseudoconvex. By Theorem 8.1.1 and corollary 8.1.2 of [EP], we have

$$\sigma(z_1, \cdots, z_n, B^0(D)) = D$$

and

$$\sigma_e(z_1, \cdots, z_n, B^0(D)) \subset \partial D.$$ 

Furthermore by Theorem 8.2.1 and Proposition 8.2.5 of [EP], we have

$$\sigma(f_1, \cdots, f_n, B^0(D)) = (f_1, \cdots, f_n)(D)$$

and

$$\sigma_e(f_1, \cdots, f_n, B^0(D)) = (f_1, \cdots, f_n)(\sigma_e(z_1, \cdots, z_n, B^0(D)))$$

$$\subset (f_1, \cdots, f_n)(\partial D).$$

Hence we have the simple conclusion:

**Proposition 3.3.** Assume that $f$ is holomorphic on $D$ and has no critical points on $\partial D$, then

$$0 \notin \sigma_e(f_1, \cdots, f_n, B^0(D)),$$

which says that the complex $(B^*(D), \bar{\partial}f \wedge)$ has at most finite dimensional cohomology group.

Now we turn to the discussion of the $L^2$ complex $(H^*(D), \bar{\partial}f)$. The key theorem in this section is as follows.

**Theorem 3.4.** There exists a quasisomorphism between the $L^2$ complex $(L^2(D), \bar{\partial}f)$ and the complex $(B^*(D), \bar{\partial}f \wedge)$. Moreover, their $p$th cohomology group vanishes for $n < p \leq 2n$.

We are working in $L^2$ integrable category, so we must be careful to control the norms. Before proving Theorem 3.4 we need the $L^2$ existence theorem for $\bar{\partial}$-Neumann problem.

*The assumption $n \geq 2$ doesn’t matter in our references. This is because when $n = 1$, the $\bar{\partial}$-Neumann condition is exactly the $0$-value Dirichlet condition for $(p, 1)$-forms and all the existence and regularity theorems clearly hold by standard elliptic estimate.
**Theorem 3.5** ([Sh]). Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ smooth boundary. If $\bar{\partial}\varphi = 0$ for some $\varphi \in L^2_{(p,q+1)}(D)$, then there exists $\psi \in L^2_{(p,q)}(D)$ such that $\varphi = \bar{\partial}\psi$ and $||\psi|| \leq c||\varphi||$. Here $0 \leq p \leq n$, $0 \leq q \leq n-1$ and $c$ is independent of the choice of $\varphi$.

We will also need the Banach’s closed range theorem as below.

**Theorem 3.6** ([Sh]). Let $T : X \to Y$ be a closed linear operator between two Hilbert spaces and $T'$ be the transpose of $T$. Then the following conditions are equivalent:

1. $T$ has closed range in $Y$.
2. $T'$ has closed range in $X$.
3. There exists a positive constant $c$, such that $||Tx|| \geq c ||x||$, $\forall x \in \text{Dom}(T) \cap \text{Ker}(T)'$
4. There exists a positive constant $c$, such that $||T'y|| \geq c ||y||$, $\forall y \in \text{Dom}(T') \cap \text{Ker}(T)'$

**Proof of Theorem 3.4.** Assume $\bar{\partial}\varphi = 0$ for some $\varphi \in L^2_p(D)$. To avoid too heavy notation, here and below we will use $a \lesssim b$ to denote 'there exists a constant $c>0$ such that $a \leq c \cdot b$'.

Firstly we assume $n < p \leq 2n$. Let

$$\varphi = \varphi^{n,p-n} + \varphi^{n-1,p-n+1} + \ldots + \varphi^{p-n,n}$$

Then we have $\bar{\partial} \varphi^{p-n,n} = 0$. By Theorem 3.5, there exists $\psi^{p-n,n-1}$ such that

$$\bar{\partial} \psi^{p-n,n-1} = \varphi^{p-n,n}$$

and

$$||\psi^{p-n,n-1}|| \lesssim ||\varphi^{p-n,n}||$$

Then

$$\partial f \wedge \varphi^{p-n,n} + \bar{\partial} \varphi^{p-n+1,n-1} = \bar{\partial} (\varphi^{p-n+1,n-1} - \partial f \wedge \psi^{p-n,n-1})$$

$$= 0$$

Again by Theorem 3.5, there exists $\psi^{p-n+1,n-2}$ such that

$$\bar{\partial} \psi^{p-n+1,n-2} = \varphi^{p-n+1,n-1} - \partial f \wedge \psi^{p-n,n-1}$$

and

$$||\psi^{p-n+1,n-2}|| \lesssim ||\varphi^{p-n+1,n-1} - \partial f \wedge \psi^{p-n,n-1}||$$

$$\lesssim ||\varphi^{p-n+1,n-1}|| + ||\psi^{p-n,n-1}||$$

$$\lesssim ||\varphi^{p-n+1,n-1}|| + ||\varphi^{p-n,n}||$$
Inductively, we have \( \psi^{p-n+k,n-1-k} \in L^2_{(p-n+k,n-1-k)}(D) \) such that

\[ \bar{\partial}\psi^{p-n+k,n-1-k} = \varphi^{p-n+k,n-k} - \partial f \wedge \psi^{p-n+k-1,n-k} \]

and

\[ ||\psi^{p-n+k,n-1-k}|| \lesssim \sum_{i=0}^{k} ||\varphi^{p-n+i,n-i}|| \]

Note that \( \psi^{p-n-1,n} = 0 \) here. Let \( \psi = \sum_{k=0}^{2n-p} \psi^{p-n+k,n-1-k} \), then \( \bar{\partial}_f \psi = \varphi \) and \( ||\psi|| \lesssim ||\varphi|| \). This means that the \( L^2 \) \( \bar{\partial}_f \)-complex is exact at \( p \)th degree and thus the \( p \)th cohomology vanish for \( n < p \leq 2n \). Moreover, by Theorem 3.6, \( \bar{\partial}_f \) has closed range at these degrees.

Then we consider the case \( 0 \leq p \leq n \). Let

\[ \varphi = \varphi^{0,0} + \varphi^{p-1,1} + \cdots + \varphi^{0,p} \]

Because \( \bar{\partial}_f \varphi = 0 \), we have

\[ \bar{\partial}_f \varphi^{p-k,k} + \partial f \wedge \varphi^{p-k-1,k+1} = 0, 0 \leq k \leq p \]

Similar to the discussion above, we have \( \psi^{p-1-k,k}, 0 \leq k \leq p-1 \) such that

\[ \bar{\partial}_f \psi^{p-k,k-1} = \varphi^{p-k,k} - \partial f \wedge \psi^{p-1-k,k}, 1 \leq k \leq p-1 \]

and

\[ \bar{\partial}(\varphi^{p,0} - \partial f \wedge \psi^{0,0}) = 0 \quad (3.3) \]

\[ \partial f \wedge (\varphi^{p,0} - \partial f \wedge \psi^{0,0}) = \partial f \wedge \varphi^{p,0} = 0 \quad (3.4) \]

Let \( \psi = \sum_{k=0}^{p-1} \psi^{p-1-k,k} \), then

\[ \varphi = \bar{\partial}_f \psi + (\varphi^{p,0} - \partial f \wedge \psi^{0,0}) \]

Similar to the discussion above, norm of \( \varphi^{p,0} - \partial f \wedge \psi^{p-1,0} \) can be controlled by that of \( \varphi \) and \( \psi \) and eventually by \( \varphi \). Thus by (3.3) and (3.4), \( \varphi^{p,0} - \partial f \wedge \psi^{p-1,0} \) is in \( B^p(D) \) and represents an element in \( H^2_{\bar{\partial}_f \wedge}(D) \).

We define a map between the two complexes at the level of \( L^2 \) cohomology by:

\[ u : H^p_{\bar{\partial}_f}(D) \to H^p_{\bar{\partial}_f \wedge}(D), [\varphi] \mapsto [\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}] \]

If \( [\varphi] = 0 \in H^p_{\bar{\partial}_f}(D) \), then we have \( \varphi = \bar{\partial}_f \eta \), together with

\[ \varphi = \bar{\partial}_f \psi + (\varphi^{p,0} - \partial f \wedge \psi^{p-1,0}) \]

we have

\[ \varphi^{p,0} - \partial f \wedge \psi^{p-1,0} = \bar{\partial}_f (\eta - \psi) \]

by counting degrees, we have

\[ \varphi^{p,0} - \partial f \wedge \psi^{p-1,0} = \partial f \wedge (\eta - \psi)^{p-1,0} \]
i.e. \([\phi^p,0 - \partial f \wedge \psi^p,0 - 1,0] = 0 \in H^p_{\partial f \wedge}(D)\) and thus \(u\) is well defined.

If \(\eta \in B^p(D)\) represent a cohomology class, we have \(\bar{\partial} f \eta = 0\) and \(u([\eta]) = [\eta]\) and \(u\) is surjective.

If \(u([\varphi]) = [\phi^p,0 - \partial f \wedge \psi^p,0 - 1,0] = 0 \in H^p_{\partial f \wedge}(D)\), then

\[
\varphi = \bar{\partial} f \psi + (\phi^p,0 - \partial f \wedge \psi^p,0) \\
= \bar{\partial} f \psi + \partial f \wedge \theta \\
= \bar{\partial} f (\psi + \theta)
\]

i.e. \([\varphi] = 0 \in H^p_{\partial f}(D)\), and \(u\) is injective. Thus \(u\) is an isomorphism of cohomology groups. \(\square\)

Now by the weak Hodge decomposition

\[L^2_k(D) = \mathcal{H}^k \oplus \overline{\text{Im}(\bar{\partial} f)} \oplus \overline{\text{Im}(\bar{\partial}^* f)}\]

we have

\[H^*_k((2), \bar{\partial} f)(D) = \text{Ker}(\bar{\partial} f)/\text{Im}(\bar{\partial} f) \cong \mathcal{H}^k \oplus \overline{\text{Im}(\bar{\partial} f)}/\text{Im}(\bar{\partial} f),\]

which is finite dimensional by Theorem 3.3. By Corollary IV.1.13 of [Go]: if a closed operator from a Banach space to another Banach space has finite cokernel, it must have closed range; we can conclude that \(\text{Im}(\bar{\partial} f)\) is closed. Now by Theorem 3.6 \(\text{Im}(\bar{\partial}^* f)\) is also closed. Thus we have

**Theorem 3.7** (strong Hodge decomposition).

\[L^2_k(D) = \mathcal{H}^k \oplus \text{Im}(\bar{\partial} f) \oplus \text{Im}(\bar{\partial}^* f) \quad (3.5)\]

and the isomorphism

\[H^*_k((2), \bar{\partial} f)(D) \cong \mathcal{H}^k \quad (3.6)\]

For any

\[\psi \in (\mathcal{H}^*)^\perp = \text{Ker}(\bar{\partial} f)^\perp + \text{Ker}(\bar{\partial}^* f)^\perp,\]

let \(\psi = \psi_1 + \psi_2 + \psi_3\) be the orthogonal decomposition of \(\psi\) into \(\text{Ker}(\bar{\partial} f)^\perp \cap \text{Ker}(\bar{\partial}^* f), \text{Ker}(\bar{\partial} f)^\perp \cap \text{Ker}(\bar{\partial}^* f)^\perp\) and \(\text{Ker}(\bar{\partial} f) \cap \text{Ker}(\bar{\partial}^* f)^\perp\).

By Theorem 3.6 and closeness of range of \(\bar{\partial} f\) and \(\bar{\partial}^* f\), we then have

\[
\langle \Delta f \psi, \psi \rangle = ||\bar{\partial} f(\psi)||^2 + ||\bar{\partial}^* f(\psi)||^2 \\
= ||\bar{\partial} f(\psi_1 + \psi_2)||^2 + ||\bar{\partial}^* f(\psi_2 + \psi_3)||^2 \\
\geq c||\psi_1 + \psi_2||^2 + c||\psi_2 + \psi_3||^2 \\
= c(||\psi_1||^2 + 2||\psi_2||^2 + ||\psi_3||^2) \\
\geq c||\psi||^2
\]
for some positive constant $c$. That’s, a spectral gap exists between 0 and other spectra of $\Delta_f$. By Proposition 3.6, the range of $\Delta_f$ is closed. So we obtain

**Corollary 3.8.** Let $D$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ smooth boundary. Let $f$ be a holomorphic function in $D$ without critical points on the boundary $\partial D$. Then the twisted Laplacian $\Delta_f$ has finite dimensional kernel and there exists a spectral gap between 0 and other spectra of $\Delta_f$. The complex $(L^2(D), \bar{\partial}f)$ has finite dimensional cohomology for $0 \leq p \leq n$ and zero cohomology for $n < p \leq 2n$.

Now Theorem 3.7 and Corollary 3.8 gives Theorem 1.1.

**Notation 3.9.** Similar to the equality $I = \Delta N + P$ for the $\bar{\partial}$-Neumann problem, let $P : L^2(D) \to \mathcal{H}^s(D)$ be the projection operator and $G : L^2(D) \to \text{Dom}(\Delta_f)$ be the Green operator, we have the decomposition

$$I = \Delta_f G + P.$$  

4. **Global regularity**

The operator $\Delta_f$ has the following expansion:

$$\Delta_f = \Delta_{\bar{\partial}} + \sum_{i,j} (f_{ij}(d\bar{z}_j \wedge dz_i) + \bar{f}_{ij}d\bar{z}_i \wedge (dz_j \wedge))^* + \sum_{i=1}^n |f_i|^2.$$  

Here $\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$ and the last two summands, denoted by $L_f$ and $|\nabla f|^2$ respectively, are of order 0. Hence $\Delta_f$ is an elliptic operator of second order and has the interior regularity estimate. In the following, we say that an operator $T$ is globally regular if and only if it preserves $\mathcal{A}(\bar{D})$. $T$ is exactly regular if it maps $W^s(D)$ continuously to itself for any non-negative integers. Exact regularity *a priori* means global regularity by Sobolev imbedding theorem.

To obtain a global estimate, we need sharper estimate about $\bar{\partial}$-Neumann operator on strongly pseudoconvex domains. We state the results needed in our proof in the following.

**Theorem 4.1.** For the $\bar{\partial}$-Neumann problem on a strongly pseudoconvex domain $D$ with smooth boundary, let $P : L^2(D) \to \mathcal{H}^s(D)$ be the projection operator and $N_{(p,q)}$ be the Neumann operator on $(p,q)$-forms. Then there exists positive constants $c_s$ depends only on $s$ such that the following global estimates hold:

1. $P$ is exactly regular, i.e. it maps $W^s(D)$ continuously to itself.
2. $\|N_{(0,0)}\psi\|_{s+\frac{1}{2}} \leq c_s \|\psi\|_s.$
\[ (3) \ |\tilde{\partial}^*N_{(p,q)}\psi|_{s+\frac{1}{2}} + |\tilde{\partial}N_{(p,q)}\psi|_{s+\frac{1}{2}} \leq c_s|\psi|_s \text{ for } q \geq 1. \\
(4) \ |N_{(p,q)}\psi|_{s+1} \leq c_s|\psi|_s \text{ for all } q \geq 1. \] 

**Proof.** All of them are classical results for \( \bar{\partial} \)-Neumann problem. For (1), see comments behind Corollary 5.2.7 and Theorem 6.2.2 in [Sh]. (3) and (4) is exactly Theorem 5.3.10 in [Sh]. For (2), by Equality 5.3.34 in [Sh], (3) and (4)

\[ |N_{(0,0)}\psi|_{s+\frac{1}{2}} \lesssim |N_{(0,1)}\bar{\partial}\psi|_s \lesssim |\bar{\partial}\psi|_{s-1} \lesssim |\psi|_s. \]

□

**Lemma 4.2.** Assume \( D \) is strongly pseudoconvex with smooth \( \partial D \) and \( f \) has no critical points on \( \partial D \). \( u \) is a function in \( \text{Dom}(\Delta_f) \). If

\[ \Delta u + |\nabla f|^2 u = g \]

for some function \( g \in W^s_0(D) \), then \( u \in W^s_0(D) \).

Notice that the 0\(^{th}\) cohomology \( H^0_{(\bar{\partial}f)} \) is obviously zero, so 0 is not in the spectrum of \( \Delta_f \). Therefore by Corollary 3.8, the spectrum of \( \Delta_f = \Delta + |\nabla f|^2 \) has a positive lower bound and \( \Delta + |\nabla f|^2 \) has a bounded inverse \( G_0 \) with \( ||G_0g|| \leq c||g|| \).

**Proof.** By definition, \( u \in L^2_0(D) \). Now assume \( u \in W^k_0(D) \) for some \( k \leq s - \frac{1}{2} \). Let \( u = Pu + u^\perp \) be the decomposition of \( u \) into a holomorphic function and the orthogonal part. Then there is

\[ \Delta u^\perp = g - |\nabla f|^2 u, \quad Pg = P(|\nabla f|^2 u). \]

By Theorem Lemma 4.1 we have

\[ u^\perp = N_{(0,0)}(g - |\nabla f|^2 u) \in W^{k+\frac{1}{2}}_0(D). \]

On the other hand, we have

\[ P(|\nabla f|^2 Pu) + P(|\nabla f|^2 u^\perp) = P(|\nabla f|^2 u) = Pg \in W^s_0(D). \]

Then it follows that

\[ P(|\nabla f|^2 u^\perp), P(|\nabla f|^2 Pu) \in W^{k+\frac{1}{2}}_0(D). \]
Now we have
\[ \|\nabla f|^2 P u \|_{k+\frac{1}{2}} = \| P(\nabla f|^2 P u) + \bar{\partial}^* N_{(0,1)} \bar{\partial}(\nabla f|^2 P u) \|_{k+\frac{1}{2}} \]
\[ \leq \| P(\nabla f|^2 P u) \|_{k+\frac{1}{2}} + \| \bar{\partial}^* N_{(0,1)} \bar{\partial}(\nabla f|^2 P u) \|_{k+\frac{1}{2}} \]
\[ \lesssim \| P(\nabla f|^2 P u) \|_{k+\frac{1}{2}} + \| \bar{\partial} \nabla f|^2 P u \|_k \]
\[ = \| P(\nabla f|^2 P u) \|_{k+\frac{1}{2}} + \| P u \bar{\partial} f|^2 P u \|_k \]
\[ \lesssim \| P(\nabla f|^2 P u) \|_{k+\frac{1}{2}} + \| u \|_k , \]
which gives
\[ |\nabla f|^2 P u \in W_0^{k+\frac{1}{2}}(D) . \]

Then there is
\[ |\nabla f|^2 u = |\nabla f|^2 P u + |\nabla f|^2 u^\perp \in W_0^{k+\frac{1}{2}}(D) . \]

Since $\Delta + |\nabla f|^2$ is elliptic in the interior of $D$ and $|\nabla f|^2$ is nonzero on the boundary $\partial D$, we can apply elliptic estimate in the interior and divide by $|\nabla f|^2$ near the boundary to conclude that $u \in W_0^{k+\frac{1}{2}}(D)$. Now by induction, $u \in W_0^s(D)$ holds. \qedhere

Remark 4.3. We can NOT expect that $u$ has higher regularity than $g$, which will lead to the compactness of $G_0$ by Rellich’s lemma. For example, when $|\nabla f|^2$ happens to be a positive constant $c$, which is the case when $f = \sum_{i=1}^n z^i$, $G_0$ will have a non-zero eigenvalue $\frac{1}{c}$ and the infinite dimensional Bergman space as the corresponding eigenspace. Thus $G_0$ can not be compact and that’s why we use an indirect way to prove the strong decomposition theorem.

**Proposition 4.4.** Assume $D$ is strongly pseudoconvex with smooth $\partial D$ and $f$ has no critical points on $\partial D$. Then the Green operator $G$ is exactly regular and $\mathcal{H}^* \subset A^*(D)$.

**Proof.** Assume $\Delta_f \varphi = \psi$ and $\psi \in W_p^*(D)$.

For $0 \leq p \leq n$, every $\varphi \in L^2_p(D)$ can be decomposed by types as $\varphi = \varphi^{p,0} + \varphi'$. We have
\[ \Delta \varphi' = \psi' - |\nabla f|^2 \varphi' - (L_f \varphi)' \in L^2_p(D) . \]
According to Theorem 4.4, $\varphi' \in W^1_p(D)$. As $L_f$ always transform $(p, q)$-forms into sum of $(p-1, q+1)$-forms and $(p+1, q-1)$-forms, $\varphi^{p,0}$ does not contribute to the $(p, 0)$ component of $L_f \varphi$. Therefore
\[ \Delta \varphi^{p,0} + |\nabla f|^2 \varphi^{p,0} = \psi^{p,0} - (L_f \varphi)^{p,0} \]
\[ = \psi^{p,0} - (L_f \varphi')^{p,0} \in W_0^1(D) . \]
Since the Neumann boundary condition for \((p, 0)\)-forms are the same as that for functions, so by Lemma \[4.2\] \(p^{0,0} \in W^1_p(D)\). Now by induction, exact regularity for \(G_p\) holds.

For \(n < p \leq 2n\), as no \((k, 0)\)-forms are involved, by Theorem \[4.1\] (4), the proof follows as the standard bootstrap argument.

Finally, let \(\psi = 0\), then Sobolev’s imbedding theorem guarantees that forms in \(H^*\) are all smooth up to the boundary. \(\square\)

5. Dimension computation

In this section, we will compute the dimension of the \(L^2\) cohomology group \(H^p_{((2),\bar{\partial}_f)}\) for \(0 \leq p \leq n\).

**Proposition 5.1.** The complex \((\mathbb{A}^*(\bar{D}), \bar{\partial}_f)\) is quasi-isomorphic to the \(L^2\) complex \((L^2(D), \bar{\partial}_f)\).

**Proof.** For any \(\phi \in \mathbb{A}^*(\bar{D})\), \(\bar{\partial}_f \phi = 0\), we can solve the \(\bar{\partial}_f\)-Neumann problem to obtain:

\[\phi = P\phi + \bar{\partial}_f \bar{\partial}^*_f G\phi.\]

By Proposition \[4.4\] \(P\phi \in \mathbb{A}^*(\bar{D})\) and \(\bar{\partial}^*_f G\phi \in \mathbb{A}^{*-1}(\bar{D})\). This gives an isomorphism

\[H^*(\mathbb{A}^*(\bar{D}), \bar{\partial}_f) \cong H^* \cong H^*_{((2),\bar{\partial}_f)}(D).\]

\(\square\)

If we ignore the \(L^2\) condition, we have the smooth complex \((\mathbb{A}^*(D), \bar{\partial}_f)\), whose cohomology can be computed by the spectral sequence as follows.

This complex can be viewed as a total complex of the double complex \((\mathbb{A}^{*,*}(D), \bar{\partial}, \partial f \wedge)\) with horizontal operator \(\bar{\partial}\) and vertical operator \(\partial f \wedge\). We consider the spectral sequence associated to the filtration

\[\mathcal{F}^k \mathbb{A}(D) = \bigoplus_{i \geq k} \mathbb{A}^{i,*}\]

for \(k \in \mathbb{Z}\). Since \(D\) is pseudo-convex, it is stein, therefore the first page is concentrated at the first column with \((k, 0)\) term given by holomorphic \(k\)-forms. Since the \(f_i\)'s have only finite common zeros, they form a regular sequence on \(D\). Therefore the cohomology of the holomorphic Koszul complex

\[0 \rightarrow \Omega^0(D) \xrightarrow{\partial f \wedge} \Omega^1(D) \xrightarrow{\partial f \wedge} \cdots \xrightarrow{\partial f \wedge} \Omega^n(D) \xrightarrow{\partial f \wedge} 0\]

which is \(E_2\), is concentrated at the top term \(\Omega^n(D) / \partial f \wedge \Omega^{n-1}(D) \cong \text{Jac}(f)\). Thus the spectral sequence degenerate at the \(E_2\)-stage and the cohomology of the smooth \(\bar{\partial}_f\) complex is concentrated at the middle.
Proposition 5.2.

\[ H^k_{\bar{\partial}_f}(D) = \begin{cases} \text{Jac}(f) & k = n \\ 0 & k \neq n. \end{cases} \]  

(5.1)

To construct the cohomology of the complex \((\mathcal{A}_c^*(D), \bar{\partial}_f)\) consisting of the forms with compact support, we want to use a homotopy introduced in [LLS].

Let \(\rho\) be a smooth function with compact support in \(D\) such that it equals to 1 in a neighborhood of \(\text{Crit}(f)\). Define the following operator

\[ V_f = \sum_{i=1}^{n} \frac{\bar{f}_i}{|\nabla f|^2} (dz_i \wedge)^*: \mathcal{A}^{*,*}(D \setminus \text{Crit}(f)) \to \mathcal{A}^{*,*}(D \setminus \text{Crit}(f)) \]

A direct calculation gives the following result.

Lemma 5.3.

\[ [df \wedge, V_f] = 1 \]  

(5.2)

and

\[ [\bar{\partial}, [\bar{\partial}, V_f]] = [df \wedge, [\bar{\partial}, V_f]] = [V_f, [\bar{\partial}, V_f]] = 0 \]  

(5.3)

Define two operators on \(D\):

\[ T_\rho = \rho + (\bar{\partial}\rho) V_f \frac{1}{1 + [\bar{\partial}, V_f]}; \quad R_\rho = (1 - \rho) V_f \frac{1}{1 + [\bar{\partial}, V_f]} \]  

(5.4)

Lemma 5.4.

\[ [\bar{\partial}_f, R_\rho] = 1 - T_\rho \quad \text{on} \quad \mathcal{A}^*(D) \]  

(5.5)

Proof. By Lemma 5.3

\[ [\bar{\partial}_f, R_\rho] = [\bar{\partial}_f, 1 - \rho] V_f \frac{1}{1 + [\bar{\partial}, V_f]} + (1 - \rho)[\bar{\partial}_f, V_f] \frac{1}{1 + [\bar{\partial}, V_f]} \]

\[ = (-\bar{\partial}\rho) V_f \frac{1}{1 + [\bar{\partial}, V_f]} + (1 - \rho)(1 + [\bar{\partial}, V_f]) \frac{1}{1 + [\bar{\partial}, V_f]} \]

\[ = 1 - T_\rho \quad \text{on} \quad \mathcal{A}^*(D) \]

□

The Lemma 5.4 built a homotopy from \((\mathcal{A}(D), \bar{\partial}_f)\) to \((\mathcal{A}_c(D), \bar{\partial}_f)\), hence we have

Proposition 5.5.

\[ H^*_\bar{\partial}_f(D) \cong H^*_c(D). \]  

(5.6)
Proof. Let \( i : \mathcal{A}_c^k(D) \to \mathcal{A}^k(D) \) be the inclusion. Since \( \bar{\partial}_f(\mathcal{A}_c(D)) \subset \bar{\partial}_f(\mathcal{A}(D)) \), we have the well-defined homomorphism \( i_* : H^k_{(c,\bar{\partial}_f)}(D) \to H^k_{\bar{\partial}_f}(D) \). Assume that \([i(b)] = 0 \in H^k_{\bar{\partial}_f} \), then there exists a \( c \in \mathcal{A}^{k-1}(D) \) such that \( b = \bar{\partial}_f c \). By Lemma 5.4, we have

\[
b = \bar{\partial}_f(T_\rho c + R_\rho \bar{\partial}_f c) = \bar{\partial}_f(T_\rho + R_\rho b),
\]

where \( T_\rho + R_\rho b \in \mathcal{A}_c(D) \). This shows that \([b]\) is the zero class in \( H^k_{(c,\bar{\partial}_f)}(D) \). Hence \( i_* \) is injective. On the other hand, if \( \bar{\partial}_f b = 0 \) for \( b \in \mathcal{A}^k(D) \), then \( b = T_\rho b + \bar{\partial}_f R_\rho b \), which shows that \( i_* \) is also surjective. \( \square \)

Let us check \( R_\rho \) more carefully. In a small neighborhood of \( \text{Crit}(f) \), \( R_\rho = 0 \). Outside such a neighborhood,

\[
R_\rho = (1 - \rho)V_f \sum_{k=1}^{n} (-1)^k [\bar{\partial}, V_f]^k
\]

Here \( V_f \) is of order 0 and

\[
[\bar{\partial}, V_f] = \sum_{i,j} \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i \wedge \frac{\bar{f}_j}{|\nabla f|^2} (d\bar{z}_j \wedge)^*
\]

\[
= \sum_{i,j} \frac{\partial}{\partial \bar{z}_i} (\frac{\bar{f}_j}{|\nabla f|^2}) d\bar{z}_i \wedge (d\bar{z}_j \wedge)^*
\]

is also of order 0. So \( R_\rho \) is actually smooth and bounded, and it defines a bounded operator from \( \mathcal{A}(\bar{D}) \) to itself. Now using the homotopy in Lemma 5.4, we can also have the following result.

**Proposition 5.6.**

\[
H^*(\mathcal{A}(\bar{D}), \bar{\partial}_f) \cong H^*_{\bar{\partial}_f}(D).
\]  \hspace{1cm} (5.7)

Combining the results of Proposition 5.1, 5.2, 5.5 and 5.6, we obtain Theorem 1.2.

**Remark 5.7.** When \( D \) is only pseudoconvex with smooth boundary, interior regularity of \( \Delta_f \) shows \( \mathcal{H}^* \subset \mathcal{A}^*(D) \) and \( G \) preserves \( \mathcal{A}^*(D) \). Thus similar argument like Proposition 5.1 can be applied to show \( H^*(L^2(D) \cap \mathcal{A}^*(D)) \cong \mathcal{H}^* \). Moreover, like Proposition 5.6, \( T_\rho \) and \( R_\rho \) can be used to give an isomorphism between \( H^k(L^2(D) \cap \mathcal{A}^*(D)) \) and \( H^k_{(c,\bar{\partial}_f)} \). So Theorem 1.2 still holds.
Likewise, $H^k_{(0,\bar{\partial}f)} \cong H^k_{(c,\bar{\partial}f)}$ by using $T_\rho$ and $R_\rho$. Thus by Proposition 3.2 $H^n_{\partial_{\bar{\partial}f}}(C)$ and $H^n_{\partial f}(\bar{D})$ are isomorphic when $D$ is strongly pseudoconvex. We can also obtain this result by proving the following duality theorem.

**Theorem 5.8.** We have the isomorphism

$$H^k_{\partial_{\bar{\partial}f}}(C) \cong (H^{2n-k}_{\partial f}(\bar{D}))^*$$

(5.8)

**Proof.** The idea is to construct a pairing

$$\varphi, \psi \mapsto \int_D \varphi \wedge \psi$$

for $\varphi \in H^{2n-k}_{\partial f}(\bar{D})$ and $\psi \in H^k_{\partial_{\bar{\partial}f}}(C)$. Almost the same proof as the Proposition 5.1.5 in [FK], except the arise of $\bar{\partial}f \wedge$, which gives a minus sign here, shows that this is indeed a pairing. To show it is non-degenerate, we only need Theorem 3.7 to take the role of the condition $Z(q)$ in Proposition 5.1.5 in [FK].

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