COMPLETE INTERSECTION HYPERKÄHLER FOURFOLDS
WITH RESPECT TO EQUIVARIANT VECTOR BUNDLES
OVER RATIONAL HOMOGENEOUS VARIETIES OF PICARD NUMBER ONE

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Abstract. We classify fourfolds with trivial canonical bundle which are zero loci of general global sections of completely reducible equivariant vector bundles over exceptional homogeneous varieties of Picard number one. By computing their Hodge numbers, we see that there exist no hyperkähler fourfolds among them. This implies that a hyperkähler fourfold represented as the zero locus of a general global section of a completely reducible equivariant vector bundle over a rational homogeneous variety of Picard number one is one of the two cases described by Beauville–Donagi and Debarre–Voisin.

1. Introduction

The holonomy group of the Levi-Civita connection on a Riemannian manifold is a Lie group given by parallel transports along loops based at a fixed point, and is a global invariant which measures constant tensors on the manifold. Riemannian manifolds with special holonomy groups are interesting in various aspects, and they include Calabi–Yau, hyperkähler, and quaternionic Kähler manifolds. Recall that a d-dimensional Riemannian manifold \((M, g)\) is called Calabi–Yau, hyperkähler, or quaternionic Kähler if its holonomy group \(\text{Hol}(M, g)\) is \(\text{SU}(n)\), \(\text{Sp}(m)\), or \(\text{Sp}(m) \cdot \text{Sp}(1)\), respectively, where \(d = 2n\), or \(4m\). However, it is extremely hard to construct Riemannian manifolds with a specific holonomy; see, for example, [Bes08] for details. For instance, there are no known examples of compact quaternionic Kähler manifolds that are not symmetric spaces. It is also a difficult problem to provide explicit examples of compact hyperkähler manifolds.

Known examples of compact hyperkähler manifolds are K3 surfaces, the Hilbert schemes of \(n\) points on K3 surfaces, and generalized Kummer varieties of abelian surfaces described by Beauville [Ben83]. We also have two explicit descriptions for compact hyperkähler manifolds of complex dimension 4 due to Beauville–Donagi [BD85] and Debarre–Voisin [DV10] even though these are deformation-equivalent to the Hilbert square of a K3 surface of genus 8 and 12, respectively. Beauville and Donagi proved that the variety of complex lines on a smooth cubic hypersurface in the complex projective space \(\mathbb{P}^5\) is hyperkähler, and it can be described as the zero locus of a general global section of the third symmetric power \(S^3\mathcal{U}^*\) of the dual universal subbundle of the complex Grassmannian \(\text{Gr}(2, 6)\). Debarre and Voisin gave another example of compact hyperkähler fourfolds as the zero locus of a general global section of the third exterior power \(\bigwedge^3\mathcal{U}^*\) of the dual universal subbundle of the Grassmannian \(\text{Gr}(6, 10)\). We might expect to find another example of compact hyperkähler fourfolds arising as the zero loci of general global sections of equivariant vector bundles over rational homogeneous varieties \(G/P\). Here, \(G\) is a complex semisimple Lie group and \(P \subset G\) is a maximal parabolic subgroup, that is, \(P\) is a maximal proper parabolic subgroup with respect to inclusion. We note that \(G/P\) is a rational homogeneous variety of Picard number one if and only if \(P\) is maximal. However, the actual results are far from this expectation.

Recently, Benedetti [Ben16a] proved that if \(Z\) is a hyperkähler fourfold which is the zero locus of a general global section of completely reducible, globally generated, equivariant vector bundles over a Grassmannian, or an isotropic (that is, orthogonal or symplectic) Grassmannian, then \(Z\) is either of Beauville–Donagi type or of Debarre–Voisin type. This provides a classification of hyperkähler fourfolds which are zero loci of general global sections of completely reducible equivariant vector bundles over rational homogeneous varieties of the classical Lie groups, that is, of type \(A, B, C, D\). Being motivated by the above works, we classify hyperkähler

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fourfolds appearing in rational homogeneous varieties of the exceptional Lie groups, that is, of type $E, F, G$. See Table 3 for the Dynkin diagrams of type $E, F, G$. More precisely, we obtain the following result.

**Theorem 1.1.** Let $G/P$ be a rational homogeneous variety of Picard number one, where $G$ is a complex simple Lie group of exceptional type and $P \subset G$ is a maximal parabolic subgroup. For a completely reducible, globally generated, equivariant vector bundle $F$ over $G/P$, if $Z \subset G/P$ is a fourfold which is the zero locus of a general global section of $F$, then $Z$ is not hyperkähler.

Together with Benedetti’s result [Ben18a], this theorem immediately implies the following.

**Corollary 1.2.** Let $F$ be a completely reducible, globally generated, equivariant vector bundle over a rational homogeneous variety $G/P$ of Picard number one. If $Z \subset G/P$ is a hyperkähler fourfold which is the zero locus of a general global section of $F$, then $Z$ is either of Beauville–Donagi type or of Debarre–Voisin type, that is, $(G/P, F)$ is either $(\text{Gr}(2, 6), S^2(U^*)$ or $(\text{Gr}(6, 10), \wedge^3 U^*)$ up to natural identifications of $\text{Gr}(k, n)$ with $\text{Gr}(n - k, n)$.

In order to prove Theorem 1.1, we give a classification of the pairs $(G/P, F)$ such that the zero locus of a general global section of $F$ is a fourfold with trivial canonical bundle when $G$ is a simple Lie group of exceptional type. A similar analysis has been parallely obtained in [Ben18a], with the usage of Macaulay2. For Grassmannians of classical Lie type, Benedetti classified such $d$-folds for $2 \leq d \leq 4$ in [Ben18a]. In [IM19], Inoue, Ito and Miura also gave a similar classification and geometric descriptions of Calabi–Yau 3-folds for Grassmannians.

**Theorem 1.3.** Let $G/P$ be a rational homogeneous variety of Picard number one, and let $F$ be a completely reducible, globally generated, equivariant vector bundle over $G/P$. If $G$ is a simple Lie group of exceptional type and the zero locus $Z$ of a general global section of $F$ is a fourfold with trivial canonical bundle, then the only possible pairs $(G/P, F)$ are listed in Table 1 up to natural identifications.

| No. | $G/P$ | $\dim(G/P)$ | $\nu(G/P)$ | $F$ | $h^{0,2}$ | $h^{1,1}$ | $h^{1,3}$ |
|-----|-------|-------------|-------------|-----|--------|--------|--------|
| 1   | $E_6/P_1$ | 16 | 12 | $O(1)^{\oplus 12}$ | 0 | 1 | 102 | Proposition 3.8 |
| 2   | $E_6/P_2$ | 21 | 11 | $\mathcal{E}_{\omega_7}^{\oplus 2} \oplus O(1)^{\oplus 5}$ | 0 | 1 | 87 | Proposition 3.9 |
| 2'  | $E_6/P_2$ | 21 | 11 | $\mathcal{E}_{\omega_1} \oplus \mathcal{E}_{\omega_6} \oplus O(1)^{\oplus 5}$ | 0 | 1 | 87 | |
| 2'' | $E_6/P_2$ | 21 | 11 | $\mathcal{E}_{\omega_6}^{\oplus 2} \oplus O(1)^{\oplus 5}$ | 0 | 1 | 87 | |
| 3   | $E_6/P_3$ | 25 | 9  | $\mathcal{E}_{\omega_7}^{\oplus 3} \oplus \mathcal{E}_{\omega_6}^{\oplus 3}$ | 0 | 1 | 48 | Proposition 3.11 |
| 4   | $E_6/P_3$ | 25 | 9  | $\mathcal{E}_{\omega_6}^{\oplus 2} \oplus O(1)$ | 0 | 1 | 72 | |
| 5   | $E_7/P_1$ | 33 | 17 | $\mathcal{E}_{\omega_7}^{\oplus 2} \oplus O(1)^{\oplus 5}$ | 0 | 1 | 87 | Proposition 3.12 |
| 6   | $F_4/P_1$ | 15 | 8  | $\mathcal{E}_{\omega_4} \oplus O(1)^{\oplus 5}$ | 0 | 1 | 86 | Proposition 3.13 |
| 1'  | $F_4/P_4$ | 15 | 11 | $O(1)^{\oplus 11}$ | 0 | 1 | 102 | |
| 7   | $F_4/P_4$ | 15 | 11 | $\mathcal{E}_{\omega_4} \oplus O(1)^{\oplus 4}$ | 0 | 1 | 87 | |
| 8   | $G_2/P_1$ | 5  | 5  | $O(5)$ | 0 | 1 | 356 | |
| 9   | $G_2/P_2$ | 5  | 3  | $O(3)$ | 0 | 1 | 258 | |

**Table 1.** Calabi–Yau 4-folds in exceptional homogeneous varieties of Picard number one.

In Table 1, $\nu(G/P)$ means the Fano index of a rational homogeneous variety $G/P$, and $\mathcal{E}_\lambda$ is the irreducible equivariant vector bundle associated to a $P$-dominant weight $\lambda$ (see Section 2.2 for details).

For the maximal parabolic subgroup $P_4$ associated to a simple root $\alpha_4$ of $F_4$, the rational homogeneous variety $F_4/P_4$ is a general hyperplane section of the Cayley plane $E_6/P_1 \subset \mathbb{P}^{26}$ (see [LM03, Section 6.3]). It follows from that the 27-dimensional fundamental $E_6$-module $V_{E_6}(\omega_1)$ may be regarded as an $F_4$-module using the embedding of $F_4$ in $E_6$ and it decomposes as $V_{F_4}(\omega_4) \oplus V_{F_4}(0)$ (see [Car05, Proposition 13.32]).
Hence, the Calabi–Yau fourfold in No. 1’ is the same as No. 1 in Table 1. The outer automorphism of $E_6$ induces a bundle isomorphism between $\mathcal{E}_{x_1}$ and $\mathcal{E}_{x_6}$; hence the Calabi–Yau fourfolds in No. 2’, 2’’ are the same as No. 2 in Table 1. Similarly, we can identify $E_6/P_3$ with $E_6/P_6$ (respectively, $E_6/P_3$ with $E_6/P_5$) by the projective equivalences induced from the outer automorphism of $E_6$.

As a direct consequence of the classification process, we also get the classification of all possible pairs $(G/P, \mathcal{F})$ such that the zero locus $Z$ of a general global section of $\mathcal{F}$ is a threefold with trivial canonical bundle.

**Proposition 1.4.** Let $G/P$ be a rational homogeneous variety of Picard number one, and let $\mathcal{F}$ be a completely reducible, globally generated, equivariant vector bundle over $G/P$. If $G$ is a simple Lie group of exceptional type and the zero locus $Z$ of a general global section of $\mathcal{F}$ is a threefold with trivial canonical bundle, then the only possible pairs $(G/P, \mathcal{F})$ are listed in Table 2 up to natural identifications.

| No. | $G/P$  | $\text{dim}(G/P)$ | $\iota(G/P)$ | $\mathcal{F}$ | $h^{0,1}$, $h^{0,2}$ | $h^{0,0}$, $h^{0,3}$, $h^{1,1}$ | $h^{1,2}$ | Euler characteristic $\chi$ |
|-----|--------|------------------|--------------|----------------|----------------------|-------------------------------|---------|-------------------|
| 1   | $E_6/P_3$ | 25               | 9            | $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_6}^{\oplus 4}$ | 0                    | 1                            | 31               | $-60$             |
| 2   | $G_2/P_1$ | 5                | 5            | $\mathcal{O}(1) \oplus O(4)$ | 0                    | 1                            | 89               | $-176$            |
| 3   | $G_2/P_1$ | 5                | 5            | $\mathcal{O}(2) \oplus O(3)$ | 0                    | 1                            | 73               | $-144$            |
| 4   | $G_2/P_1$ | 5                | 5            | $\mathcal{E}_{x_2} \oplus O(1)$ | 0                    | 1                            | 50               | $-98$             |
| 5   | $G_2/P_2$ | 5                | 3            | $\mathcal{O}(1) \oplus O(2)$ | 0                    | 1                            | 61               | $-120$            |
| 6   | $G_2/P_2$ | 5                | 3            | $\mathcal{E}_{x_3} \oplus O(1)$ | 0                    | 1                            | 50               | $-98$             |

Table 2. Calabi–Yau 3-folds in exceptional homogeneous varieties of Picard number one.

We notice that the result already obtained in [IMOU] with the aid of a computer program Mathematica and the Hodge numbers for No. 1, No. 4, No. 5, No. 6 were computed in that paper. As $G_2/P_1$ is isomorphic to the hyperquadric $Q^5 \subset \mathbb{P}^6$, the threefolds given in No. 2 and No. 3 are complete intersection Calabi–Yau threefolds in $\mathbb{P}^6$, and their Hodge numbers are well-known (for example, see [GHL89]).

The paper is organized as follows. In Section 2, we recall well-known properties of hyperkähler manifolds and equivariant vector bundles over rational homogeneous varieties. In Section 3 we list up all possible pairs $(G/P, \mathcal{F})$ such that the zero locus $Z$ of a general global section of $\mathcal{F}$ is a fourfold with trivial canonical bundle. In Section 4, the Hodge numbers $h^{2,0}(Z)$ and $h^{1,3}(Z)$ of $Z$ are computed to prove Theorem 1.1 and Theorem 1.3. Throughout this paper, we work over the complex number field $\mathbb{C}$.

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2. **Hyperkähler manifolds and equivariant vector bundles**

2.1. **Hyperkähler manifolds and irreducible holomorphic symplectic manifolds.** In this section, we recall notion of holomorphic symplectic forms and holomorphic symplectic manifolds from [Bes08] and their properties. Let $\mathbb{H}$ be the associative algebra of quaternions with the imaginary units $i, j, k$. The symplectic group $\text{Sp}(m)$ is defined as the group of $m \times m$ matrices $A$ over $\mathbb{H}$ satisfying $A^\dagger A = I$, where $x \mapsto x^\dagger$ is the conjugation given by $x = x_0 + x_1i - x_2j - x_3k$ for $x = x_0 + x_1i + x_2j + x_3k \in \mathbb{H}$.

**Definition 2.1.** A $4m$-dimensional Riemannian manifold $(M, g)$ is called hyperkähler if its holonomy group $\text{Hol}(M, g)$ is isomorphic to the symplectic group $\text{Sp}(m)$, that is, $\text{Hol}(M, g) \cong \text{Sp}(m)$. 
A hyperkähler manifold \((M, g)\) is naturally equipped with three (almost) complex structures \(J_1, J_2, J_3\) such that \(J_1 \circ J_2 = J_3\) and \(\nabla J_\ell = 0\) for \(\ell = 1, 2, 3\), where \(\nabla\) is the Levi-Civita connection of \(g\), i.e., \(g\) is a Kähler metric with respect to each of these complex structures. From the corresponding Kähler forms \(\omega_1, \omega_2, \omega_3\), we get a complex 2-form \(\omega_2 + \sqrt{-1}\omega_3\) with respect to \(J_1\) which makes \((M, J_1)\) a holomorphic symplectic manifold.

**Definition 2.2.** Let \((X, J)\) be a complex manifold of (complex) dimension \(2m\). A closed holomorphic 2-form \(\omega\) is called holomorphic symplectic if the \(m\)th wedge product \(\omega^m\) is nonzero at every point. An irreducible holomorphic symplectic manifold is a simply-connected compact complex Kähler manifold \((X, J)\) such that the space of global holomorphic 2-forms \(H^0(X, \Omega^2_X)\) is generated by a holomorphic symplectic 2-form.

**Remark 2.3.** Note that \(\omega^m\) is a nonvanishing holomorphic global section of the canonical line bundle \(K_X = \bigwedge^{2m} T^* X\). Thus, holomorphic symplectic manifolds have trivial canonical bundle, and admit Ricci-flat metrics by Yau’s theorem [Yau78].

**Definition 2.4.** A compact Kähler manifold of complex dimension \(n\) is called to be Calabi–Yau if its canonical bundle is trivial, or equivalently, it admits a (Ricci-flat) Kähler metric with holonomy group contained in \(SU(n)\).

We can relate these differential geometric concepts to algebraic geometric objects such as Dolbeault cohomology and sheaf cohomology due to Beauville.

**Proposition 2.5 ([Bea83 Proposition 4]).** If \((M, J, g)\) is a compact connected hyperkähler manifold, then \(M\) is simply-connected and admits a unique holomorphic symplectic structure \(\omega\) (up to scaling factor). Conversely, if \((M, J, \omega)\) is an irreducible holomorphic symplectic manifold, then it admits a hyperkähler metric.

Consequently, we use the names ‘irreducible holomorphic symplectic manifolds’ and ‘compact hyperkähler manifolds’ interchangeably.

**Corollary 2.6.** Let \((M, J)\) be a compact connected Kähler fourfold with trivial canonical bundle. Then \(M\) carries a hyperkähler metric \(g\) if and only if \(h^2(M, \mathcal{O}_M) = 1\) for the structure sheaf \(\mathcal{O}_M\) of \(M\).

**Proof.** Suppose that \(M\) admits a hyperkähler metric. Then \(M\) is an irreducible holomorphic symplectic manifold by Proposition 2.5, hence \(h^0(M, \Omega^2_M) = \dim H^0(M, \Omega^2_M) = 1\). Since the Dolbeault isomorphism theorem says that \(h^{2,0}(M) = h^0(M, \Omega^2_M)\) and \(h^{0,2}(M) = h^2(M, \mathcal{O}_M)\), we have \(h^2(M, \mathcal{O}_M) = 1\) from the conjugation \(h^{2,0}(M) = h^{0,2}(M)\).

Conversely, if \(M\) is connected and \(h^2(M, \mathcal{O}_M) = 1\), then the Euler–Poincaré characteristic \(\chi(\mathcal{O}_M)\) is equal to 3 so that \(M\) is simply-connected by [Ben18b Prop. 3.16] using the Beauville–Bogomolov decomposition theorem [Bea83 Théorème 1] for compact Kähler manifolds with trivial canonical bundle. As \(M\) is a compact simply-connected Kähler fourfold with trivial canonical bundle, the Bogomolov decomposition theorem in [Bog74] implies that \(M\) is either a product of two K3 surfaces, or a Calabi–Yau fourfold with \(h^0(M, \Omega^2_M) = 0\), or an irreducible holomorphic symplectic fourfold. From the assumption \(h^0(M, \Omega^2_M) = h^2(M, \mathcal{O}_M) = 1\), the manifold \(M\) is irreducible holomorphic symplectic and \(H^0(M, \Omega^2_M)\) is generated by a holomorphic symplectic 2-form. Thus, the result follows from Proposition 2.5.

**Proposition 2.7.** Let \(Z\) be a fourfold with trivial canonical bundle which is a general complete intersection with respect to a direct sum of line bundles over a smooth Fano variety \(X\). Then \(Z\) is not hyperkähler but Calabi–Yau.

**Proof.** Suppose that \(X\) has dimension \(n\) and Fano index \(\iota\). By the assumption, \(Z\) is the zero locus of a general global section of \(\mathcal{F} = \mathcal{O}_X(a_j)^{\oplus (n-4)}\) with \(\sum_{j=1}^{n-4} a_j = \iota\) and \(a_j > 0\). It is immediately checked that \(H^p(X, \bigwedge^q \mathcal{F}^*)\) are nontrivial only for \(p = n, q = n-4\) by the Kodaira vanishing theorem. Using the Serre duality, \(H^p(X, \bigwedge^{n-4} \mathcal{F}^*) = H^0(X, \mathcal{O}_X)^* = C\). Then, from the Koszul complex

\[
0 \to \bigwedge^{n-4} \mathcal{F}^* \to \bigwedge^{n-5} \mathcal{F}^* \to \cdots \to \bigwedge^2 \mathcal{F}^* \to \mathcal{F}^* \to \mathcal{O}_X \to \mathcal{O}_Z \to 0,
\]

we have \(h^4(Z, \mathcal{O}_Z) = 1\) and \(h^2(Z, \mathcal{O}_Z) = 0\) which means that \(Z\) is not hyperkähler by Corollary 2.6.

In general, complete intersections of dimension greater than 2 are never irreducible holomorphic symplectic, which is one reason why it is extremely hard to construct irreducible holomorphic symplectic manifolds.
2.2. Equivariant vector bundles over homogeneous varieties. Let $G$ be a simply-connected complex semisimple Lie group and $P \subset G$ be a parabolic subgroup. We will follow the notations in [FH91] for the basics on the representation theory of a Lie algebra. For an integral dominant weight $\lambda$ with respect to $P$, we have an irreducible representation $V_P(\lambda)$ of $P$ with the highest weight $\lambda$, and denote by $\mathcal{E}_\lambda$ the corresponding irreducible equivariant vector bundle $G \times P V_P(\lambda)^*$ over $G/P$:

$$\mathcal{E}_\lambda := G \times P V_P(\lambda)^* = (G \times V_P(\lambda)^*)/P,$$

where the equivalence relation is given by $(g, v) \sim (gp, p^{-1}v)$ for $p \in P$.

**Theorem 2.8** (Borel–Weil–Bott theorem [Bott57]). Let $G$ be a simply-connected complex semisimple Lie group and $P \subset G$ be a parabolic subgroup. Let $\rho$ denote the sum of fundamental weights of $G$. For an integral dominant weight $\lambda$ with respect to $P$, the following holds.

- If a weight $\lambda + \rho$ is singular, that is, it is orthogonal to some (positive) root of $G$, then the cohomology groups $H^i(G/P, \mathcal{E}_\lambda)$ vanish for all $i$.
- Otherwise, $\lambda + \rho$ is regular, that is, it lies in the interior of some Weyl chamber, then

$$H^{\ell(w)}(G/P, \mathcal{E}_\lambda) = V_G(w(\lambda + \rho) - \rho)^*$$

and any other cohomology vanishes. Here, $w \in W$ is a unique element of the Weyl group of $G$ such that $w(\lambda + \rho)$ is regular dominant, and $\ell(w)$ means the length of $w \in W$, that is, $\ell(w)$ is the minimum integer such that $w$ can be expressed as a product of $\ell(w)$ simple reflections.

Unfortunately, since $P$ is not reductive in general, its representations are difficult to study. Nevertheless, it is often possible to get results about an equivariant vector bundle from irreducible bundles by considering a filtration by $P$-submodules. If $V$ is a $P$-module, it has a filtration $0 \subset V_0 \subset \cdots \subset V_\ell \subset V = V\ell$ such that $V_i/V_{i+1}$ is an irreducible $P$-module. Moreover, we can consider $V$ as an $L$-module by the restriction for the reductive part $L$ of a Levi decomposition of $P = LU$, and define a new $P$-module structure by extending trivially to the unipotent radical $U$. The associated graded module $\text{gr}(V)$ is a direct sum of irreducible $P$-modules. By a series of extensions, $V$ can be reconstructed from the summands of $\text{gr}(V)$.

**Example 2.9** (Beauville–Donagi [BDS5]). Let $Z$ be the zero locus of a general global section of the third symmetric power $S^3U^*$ of the dual universal subbundle of the Grassmannian $\text{Gr}(2, 6)$. Recall that the dimension of $\text{Gr}(2, 6)$ is 8 and $K_{\text{Gr}(2, 6)} = \mathcal{O}(-6)$. Since the rank of $S^3U^*$ is 4 and $\det(S^3U^*) \cong \mathcal{O}(6)$ (see Proposition 3.2), $Z$ is a fourfold with trivial canonical bundle by the adjunction formula. Then we have the Koszul complex associated to a general section $s$ of the equivariant vector bundle $\mathcal{F} = S^3U^*$:

$$0 \to \wedge^1 F^* \to \wedge^3 F^* \to \wedge^2 F^* \to F^* \to \mathcal{O}_{\text{Gr}(2, 6)} \to \mathcal{O}_Z \to 0.$$ 

Using the Littlewood–Richardson rule (see [Weyl03] Section 2.3 for details), we obtain

$$0 \to \mathcal{O}(-6) \to S^4U(-3) \to S^4U(-1) \oplus \mathcal{O}(-3) \to S^3U \to \mathcal{O}_{\text{Gr}(2, 6)} \to \mathcal{O}_Z \to 0.$$ 

By the Borel–Weil–Bott theorem, $H^i(\text{Gr}(2, 6), \mathcal{O}(-6)) = H^i(\text{Gr}(2, 6), S^3U(-1)) = \mathbb{C}$ and the other non-trivial cohomologies vanish. For example, $S^3U(-3) = E_{3\varpi_1 - 6\pi_2}$ and the weight

$$3\varpi_1 - 6\varpi_2 + \rho = 4\varpi_1 - 5\varpi_2 + \varpi_3 + \varpi_4 + \varpi_5$$

is singular, where $\varpi_1, \ldots, \varpi_5$ are the fundamental weights of $\text{SL}(6, \mathbb{C})$. As a straightforward application of the Borel–Weil–Bott theorem, we see $H^i(\text{Gr}(2, 6), S^3U(-3)) = 0$ for all $i$. Therefore, $H^0(Z, \mathcal{O}_Z^2) \cong H^2(Z, \mathcal{O}_Z) = \mathbb{C}$ and $Z$ is a hyperkähler fourfold by Corollary 2.6.

**Remark 2.10.** Similarly, we can prove that the zero locus of a general global section of the third exterior power $\bigwedge^3 U^*$ of the dual universal subbundle of the Grassmannian $\text{Gr}(6, 10)$ is also a hyperkähler fourfold (see [DV10] Remark 2.6).

3. Classification of fourfolds with trivial canonical bundle

Let $G/P$ be an exceptional homogeneous variety of Picard number one, and $\mathcal{F}$ be a completely reducible, globally generated, equivariant vector bundle over $G/P$. Here, we say that a vector bundle $\mathcal{F}$ is completely reducible if it can be expressed as a direct sum of irreducible vector bundles. In this section, we list up all possible pairs $(G/P, \mathcal{F})$ such that the zero locus $Z$ of a general global section of $\mathcal{F}$ is a fourfold with trivial canonical bundle based on the method done by Kühle [Kuehle95] and Benedetti [Ben18a].
In what follows, we fix an ordering on the simple roots as in Table 3, our conventions agree with that in [FH91], which is called the Bourbaki ordering for the simple roots. Moreover, for a parabolic subgroup $P$, a weight $\lambda$ is called $P$-dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all positive roots $\alpha$ such that $g_\alpha \subset \mathfrak{t}$, where $g_\alpha$ is the root space corresponding to $\alpha$ and $\mathfrak{t}$ is the Lie algebra of the reductive part $L$ in a Levi decomposition of $P$.

**Definition 3.1.** Let $P_k \subset G$ be the $k$th maximal parabolic subgroup and $O(1)$ be the ample generator of the Picard group of $G/P_k$ giving the minimal embedding in the projective space $\mathbb{P}(V_G(\omega_k))$, that is, $O(1) = \mathcal{E}_{\omega_k}$. Here, $\omega_k$ is the $k$th fundamental weight. The $\text{dex}$ of a vector bundle $\mathcal{F}$ over $G/P_k$ is an integer $\text{dex}(\mathcal{F})$ defined by $\det(\mathcal{F}) = O(1)^{\text{dex}(\mathcal{F})}$. Similarly, the $\text{dex}$ of an integral $P$-dominant weight $\lambda$ is defined as $\text{dex}(\lambda) = \text{dex}(\mathcal{E}_\lambda)$.

The following proposition is essentially due to Benedetti in [Ben18a, Sections 3 and 4] but we provide its proof for reader’s convenience.

**Proposition 3.2.** Let $G$ be a semisimple Lie group of rank $r$, and $P_k$ be the $k$th maximal parabolic subgroup.

1. (A type) For a Grassmannian $\text{SL}(r + 1, \mathbb{C})/P_k = \text{Gr}(k, r + 1)$, and for $\lambda = \sum_{i=1}^r \lambda_i \omega_i$, we have
   
   $$\text{dex}(\lambda) = \left( \sum_{j=1}^k \sum_{i=j}^r \lambda_i \right) - \frac{\sum_{j=k+1}^r \sum_{i=j}^r \lambda_i}{r + 1 - k} \text{rank}(\mathcal{E}_\lambda).$$

2. (C type) For a symplectic Grassmannian $\text{Sp}(2r, \mathbb{C})/P_k = \text{IGr}(k, 2r)$, and for $\lambda = \sum_{i=1}^r \lambda_i \omega_i$, we have
   
   $$\text{dex}(\lambda) = \sum_{j=1}^k \sum_{i=j}^r \lambda_i \text{rank}(\mathcal{E}_\lambda).$$

3. (B, D types) Two homogeneous varieties $\text{SO}(2r, \mathbb{C})/P_r$ and $\text{SO}(2r - 1, \mathbb{C})/P_{r-1}$ are isomorphic, called a spinor variety $S_r$ of dimension $\frac{r(r-1)}{2}$. For a spinor variety $S_r = \text{SO}(2r, \mathbb{C})/P_r$, and for $\lambda = \sum_{i=1}^r \lambda_i \omega_i$, we have
   
   $$\text{dex}(\lambda) = 2 \cdot \frac{\sum_{j=1}^r (\sum_{i=j}^{r-2} \lambda_i + \lambda_{r-1} + \lambda_r)}{r} \text{rank}(\mathcal{E}_\lambda).$$

**Proof.** (1) The first formula is the same as in [Kie95, Lemma 3.4] and [Ben18a, Lemma 3.8]. Indeed, it follows from the facts that $\text{dex}(\lambda) \omega_k$ is the sum of all weights of the irreducible $P$-module with highest weight $\lambda$ and the set of weights is invariant under the action of the Weyl group of $P$.

(2) This is explained in [Ben18a, Remark 3.9 and Section 4.1]. Since the semisimple part of $P_k$ is isomorphic to $\text{SL}(k, \mathbb{C}) \times \text{Sp}(2r - 2k, \mathbb{C})$, its Weyl group $W$ is isomorphic to $\mathfrak{S}_k \times (\mathfrak{S}_{r-k} \ltimes \mathbb{Z}_2^{r-k})$, where $\mathfrak{S}_n$ stands for the symmetric group on $n$ letters. Since the sum of weights given by the signed symmetric group $\mathfrak{S}_{r-k} \ltimes \mathbb{Z}_2^{r-k}$ is equal to zero, we get the result.

(3) Let $\text{OGr}(r, 2r)$ be the variety parametrizing isotropic $r$-subspaces in $2r$-dimensional vector space equipped with a nondegenerate symmetric bilinear form. Then $\text{OGr}(r, 2r)$ has two connected components and these

| $\Phi$ | Dynkin diagram | $\Phi$ | Dynkin diagram |
|-------|----------------|-------|----------------|
| $E_6$ | ![Dynkin Diagram](image) | $E_7$ | ![Dynkin Diagram](image) |
| $E_8$ | ![Dynkin Diagram](image) | $F_4$ | ![Dynkin Diagram](image) |
| $G_2$ | ![Dynkin Diagram](image) |

Table 3. Dynkin diagrams of exceptional Lie groups.
two components are in fact indistinguishable as embedded varieties (for example, see [FH91] Section 23.3). The spinor variety $\mathcal{S}_r$ is one of two components of $\text{OGr}(r, 2r)$, and has the natural embedding $\mathcal{S}_r \subset \text{Gr}(r, 2r) \subset \mathbb{P}(\Lambda^r \mathbb{C}^{2r})$. However, the restriction of the Pl"ucker line bundle over $\text{Gr}(r, 2r)$ to $\mathcal{S}_r$ is divisible by two, and the square root of the Pl"ucker line bundle gives an embedding into the projectivization of a spin representation of $\text{SO}(2r, \mathbb{C})$. Because the Weyl group of $P_r$ is isomorphic to $\mathcal{S}_r$, a similar argument holds as in (1), but we have to multiply by 2 to get the correct formula. □

Let $Z = Z_\mathcal{F}$ be the zero locus of a general global section of $\mathcal{F}$ over a rational homogeneous variety $G/P$ of Picard number one. If $Z$ has the properties that $\dim Z = 4$ and $K_Z = O_Z$, by the adjunction formula for $Z \subset G/P$, we have

$$\text{rank}(\mathcal{F}) = \dim(G/P) - 4 \quad \text{and} \quad \text{dex}(\mathcal{F}) = \nu(G/P),$$

where $\nu(G/P)$ means the Fano index of a rational homogeneous variety $G/P$. Note that all rational homogeneous varieties $G/P$ are Fano varieties (for example, see Sections II.4.2–II.4.4 of [Jan03]).

**Lemma 3.3.** Let $G/P$ be a homogeneous variety of Picard number one. Let $\mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{F}_i$ be the direct sum of irreducible equivariant vector bundles over $G/P$. If the inequalities

$$\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} > \frac{\nu(G/P)}{\dim(G/P) - 4}$$

hold for all $i$, then there is no fourfold $Z$ such that it is the zero locus of a general global section of $\mathcal{F}$ and has trivial canonical bundle.

**Proof.** Assume that there is a fourfold $Z$ satisfying the given conditions. Since $\mathcal{F}$ is the direct sum of $\mathcal{F}_i$’s, we have $\text{rank}(\mathcal{F}) = \sum_i \text{rank}(\mathcal{F}_i)$ and $\text{dex}(\mathcal{F}) = \sum_i \text{dex}(\mathcal{F}_i)$. Then

$$\frac{\text{dex}(\mathcal{F})}{\text{rank}(\mathcal{F})} = \frac{\sum_i \text{dex}(\mathcal{F}_i)}{\sum_i \text{rank}(\mathcal{F}_i)} > \frac{\nu(G/P)}{\dim(G/P) - 4}$$

by the assumption. However, it contradicts to the conditions in (3.1). □

From the same argument used in the proof of Lemma 3.3 we have the following consequence directly.

**Proposition 3.4.** Let $G/P$ be a homogeneous variety of Picard number one. Let $\mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{F}_i$ be the direct sum of line bundles over $G/P$. Suppose that $\dim(G/P) = \nu(G/P) + 4$ and $\text{dex}(\mathcal{F}_i) \geq 1$ for all irreducible summands $\mathcal{F}_i$ of $\mathcal{F}$. If $Z \subset G/P$ is a fourfold with trivial canonical bundle which is the zero locus of a general global section of $\mathcal{F}$, then $Z$ is a general linear section of $G/P$, that is, $\mathcal{F}$ is isomorphic to $O(1)^{\oplus(\dim(G/P) - 4)}$.

Now, we will give a classification of possible pairs $(G/P, \mathcal{F})$ such that the zero locus $Z$ of a general global section of $\mathcal{F}$ is a fourfold with trivial canonical bundle when $G/P$ is an exceptional homogeneous variety of Picard number one. This is done by a case-by-case analysis using the ratio $\text{dex}/\text{rank}$ of irreducible equivariant vector bundles over $G/P$.

**Proposition 3.5.** Let $G/P$ be an exceptional homogeneous variety of Picard number one. Let $\mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{F}_i$ be the direct sum of irreducible equivariant vector bundles over $G/P$. Let $Z \subset G/P$ be the zero locus of a general global section of $\mathcal{F}$. If $Z$ is a fourfold with trivial canonical bundle, then the only possible cases for $G/P$ are $E_6/P_1, E_6/P_2, E_6/P_3, E_7/P_1, F_4/P_1, F_4/P_2, G_2/P_1$, and $G_2/P_2$ up to natural identifications.

**Proof.** It is well-known that dimensions and Fano indices of rational homogeneous varieties $G/P$ are computed from the root information (for example, see [Sno89]).

(1) Suppose that $G$ is of type $E_6$. Note that the dual Cayley plane $E_6/P_6 \subset \mathbb{P}^{28}$ is projectively equivalent to $E_6/P_4 \subset \mathbb{P}^{28}$ because the highest weight $E_6$-module $V_{E_6}(\varpi_6)$ is dual to $V_{E_6}(\varpi_5)$. Similarly, $E_6/P_3$ is projectively equivalent to $E_6/P_5$. Accordingly, it is enough to consider $P = P_4$.

The rational homogeneous variety $E_6/P_4 \subset \mathbb{P}^{2924}$ has dimension 29 and Fano index 7. Since the semisimple part of $P_4$ is isomorphic to $\text{SL}(3, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(3, \mathbb{C})$, we can easily compute $\text{dex}(\mathcal{E}_A)$ by Proposition 3.2(1). For example, $\text{dex}(\mathcal{E}_{\varpi_3}) = \frac{2}{3} \times 3 = 2$. Since $\mathcal{E}_{\varpi_2 + \varpi_3} = \mathcal{E}_{\varpi_2} \otimes \mathcal{E}_{\varpi_3}$, $\text{dex}(\mathcal{E}_{\varpi_2 + \varpi_3}) = \text{dex}(\mathcal{E}_{\varpi_2}) \cdot \text{rank}(\mathcal{E}_{\varpi_3}) + \text{dex}(\mathcal{E}_{\varpi_3}) \cdot \text{rank}(\mathcal{E}_{\varpi_2}) = 1 \times 3 + 2 \times 2 = 7$; see Table 4.
Table 4. The irreducible equivariant vector bundles over $E_6/P_4$ with $\text{dex} \leq 7$.

$$
\begin{array}{cccccccc}
\lambda & \varpi_1 & \varpi_2 & \varpi_3 & \varpi_5 & \varpi_6 & \varpi_1 + \varpi_2 & \varpi_2 + \varpi_3 \\
\hline
\text{rank}(E_{\lambda}) & 3 & 2 & 3 & 3 & 6 & 6 & \\
\text{dex}(E_{\lambda}) & 1 & 1 & 2 & 2 & 1 & 5 & 7 \\
\end{array}
$$

In general, for two vector bundles $F_1$ and $F_2$, the following holds

$$
\frac{\text{dex}(F_1 \otimes F_2)}{\text{rank}(F_1 \otimes F_2)} = \frac{\text{dex}(F_1)}{\text{rank}(F_1)} + \frac{\text{dex}(F_2)}{\text{rank}(F_2)}.
$$

Thus, we deduce that $\frac{\text{dex}(F_i)}{\text{rank}(F_i)} \geq \frac{1}{3}$ for any irreducible equivariant vector bundle $F_i$ over $E_6/P_4$. Since

$$
\frac{\iota(E_6/P_4)}{\dim(E_6/P_4) - 4} = \frac{7}{25} < \frac{1}{3},
$$

we cannot get a direct sum of irreducible equivariant vector bundles of which a general global section gives a fourfold with trivial canonical bundle by Lemma 3.3.

(2) Suppose that $G$ is of type $E_7$.

(a) The rational homogeneous variety $E_7/P_2 \subset \mathbb{P}^{911}$ has dimension 42 and Fano index 14. Note that irreducible equivariant vector bundles with rank $\leq 42 - 4 = 38$ are given in Table 5.

$$
\begin{array}{ccccccccccc}
\lambda & \varpi_1 & \varpi_2 & \varpi_3 & \varpi_4 & \varpi_5 & \varpi_6 & \varpi_7 & 2\varpi_1 & 2\varpi_7 \\
\hline
\text{rank}(E_{\lambda}) & 7 & 21 & 35 & 35 & 21 & 7 & 28 & 28 & \\
\text{dex}(E_{\lambda}) & 4 & 24 & 60 & 45 & 18 & 3 & 32 & 24 & \\
\end{array}
$$

Table 5. The irreducible equivariant vector bundles over $E_7/P_2$ with rank $\leq 38$.

Since

$$
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} \geq \frac{1}{3} \geq \frac{14}{38} = \frac{\iota(E_7/P_2)}{\dim(E_7/P_2) - 4} \quad \text{for all } i,
$$

there is no $Z$ with trivial canonical bundle by Lemma 3.3.

(b) The rational homogeneous variety $E_7/P_3 \subset \mathbb{P}^{8644}$ has dimension 47 and Fano index 11. Since the semisimple part of $P_3$ is isomorphic to $\text{SL}(2, \mathbb{C}) \times \text{SL}(6, \mathbb{C})$, by Proposition 3.2 we know

$$
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} \geq \frac{1}{3} \geq \frac{11}{43} = \frac{\iota(E_7/P_3)}{\dim(E_7/P_3) - 4} \quad \text{for all } i.
$$

(c) The rational homogeneous variety $E_7/P_4 \subset \mathbb{P}^{365749}$ has dimension 53 and Fano index 8. Since the semisimple part of $P_4$ is isomorphic to $\text{SL}(3, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times \text{SL}(4, \mathbb{C})$, by Proposition 3.2 we know

$$
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} \geq \frac{1}{3} \geq \frac{8}{49} = \frac{\iota(E_7/P_4)}{\dim(E_7/P_4) - 4} \quad \text{for all } i.
$$

(d) The rational homogeneous variety $E_7/P_5 \subset \mathbb{P}^{27663}$ has dimension 50 and Fano index 10. Since the semisimple part of $P_5$ is isomorphic to $\text{SL}(5, \mathbb{C}) \times \text{SL}(3, \mathbb{C})$, by Proposition 3.2 we know

$$
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} \geq \frac{1}{3} \geq \frac{10}{46} = \frac{\iota(E_7/P_5)}{\dim(E_7/P_5) - 4} \quad \text{for all } i.
$$

(e) The rational homogeneous variety $E_7/P_6 \subset \mathbb{P}^{1538}$ has dimension 42 and Fano index 13. From the direct computations (cf. Proposition 3.8),

$$
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} \geq \frac{1}{2} > \frac{13}{38} = \frac{\iota(E_7/P_6)}{\dim(E_7/P_6) - 4} \quad \text{for all } i.
(f) The rational homogeneous variety \(E_7/P_4 \subset \mathbb{P}^{55}\) has dimension 27 and Fano index 18. The semisimple part of the maximal parabolic subgroup \(P_4 \subset E_7\) is isomorphic to \(E_8\). Since the fundamental representation \(V_{E_8}(\varpi_1) \cong \mathbb{C}^{27}\) is a nontrivial irreducible representation of \(E_8\) with minimal dimension, an irreducible representation of \(E_8\) having dimension \(\leq 27 - 4 = 23\) is only the trivial representation. Consequently, \(\mathcal{F}\) is isomorphic to a direct sum of line bundles so that

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} = \text{dex}(\mathcal{F}_i) \geq 1 > \frac{18}{23} = \frac{\iota(E_7/P_4)}{\dim(E_7/P_4) - 4} \quad \text{for all } i.
\]

(3) Suppose that \(G\) is of type \(E_8\).

(a) The rational homogeneous variety \(E_8/P_1 \subset \mathbb{P}^{3874}\) has dimension 78 and Fano index 23. Note that irreducible equivariant vector bundles with rank \(\leq 78 - 4 = 74\) are \(\mathcal{E}_{\varpi_2}, \mathcal{E}_{\varpi_3}\) and \(\mathcal{E}_{\varpi_8}\). From the direct computations,

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} \geq \frac{1}{2} > \frac{23}{74} = \frac{\iota(E_8/P_1)}{\dim(E_8/P_1) - 4} \quad \text{for all } i.
\]

(b) The rational homogeneous variety \(E_8/P_2 \subset \mathbb{P}^{147249}\) has dimension 92 and Fano index 17. From the direct computations,

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} \geq \frac{3}{8} > \frac{17}{88} = \frac{\iota(E_8/P_2)}{\dim(E_8/P_2) - 4} \quad \text{for all } i.
\]

(c) The rational homogeneous variety \(E_8/P_3 \subset \mathbb{P}^{6695999}\) has dimension 98 and Fano index 13. Since the semisimple part of \(P_3\) is isomorphic to \(\text{SL}(2, \mathbb{C}) \times \text{SL}(7, \mathbb{C})\), by Proposition 3.2 we know

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} \geq \frac{2}{7} > \frac{13}{94} = \frac{\iota(E_8/P_3)}{\dim(E_8/P_3) - 4} \quad \text{for all } i.
\]

(d) The rational homogeneous variety \(E_8/P_4 \subset \mathbb{P}^{6899079263}\) has dimension 106 and Fano index 9. Since the semisimple part of \(P_4\) is isomorphic to \(\text{SL}(3, \mathbb{C}) \times \text{SL}(5, \mathbb{C})\), by Proposition 3.2 we know

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} \geq \frac{1}{5} > \frac{9}{102} = \frac{\iota(E_8/P_4)}{\dim(E_8/P_4) - 4} \quad \text{for all } i.
\]

(e) The rational homogeneous variety \(E_8/P_5 \subset \mathbb{P}^{146325269}\) has dimension 104 and Fano index 11. Since the semisimple part of \(P_5\) is isomorphic to \(\text{SL}(5, \mathbb{C}) \times \text{SL}(4, \mathbb{C})\), by Proposition 3.2 we know

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} \geq \frac{1}{4} > \frac{11}{100} = \frac{\iota(E_8/P_5)}{\dim(E_8/P_5) - 4} \quad \text{for all } i.
\]

(f) The rational homogeneous variety \(E_8/P_6 \subset \mathbb{P}^{2450239}\) has dimension 97 and Fano index 14. Since the semisimple part of \(P_6\) is isomorphic to \(\text{Spin}(10, \mathbb{C}) \times \text{SL}(3, \mathbb{C})\), from Proposition 3.2 and the direct computations,

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} \geq \frac{1}{3} > \frac{14}{93} = \frac{\iota(E_8/P_6)}{\dim(E_8/P_6) - 4} \quad \text{for all } i.
\]

(g) The rational homogeneous variety \(E_8/P_7 \subset \mathbb{P}^{30379}\) has dimension 83 and Fano index 19. From the direct computations,

\[
\frac{\text{dex}(\mathcal{F}_i)}{\text{rank}(\mathcal{F}_i)} > \frac{19}{79} = \frac{\iota(E_8/P_7)}{\dim(E_8/P_7) - 4} \quad \text{for all } i.
\]

(h) The rational homogeneous variety \(E_8/P_8 \subset \mathbb{P}^{247}\) has dimension 57 and Fano index 29. The semisimple part of the maximal parabolic subgroup \(P_8 \subset E_8\) is isomorphic to \(E_7\). Since the fundamental representation \(V_{E_7}(\varpi_1) \cong \mathbb{C}^{56}\) is a nontrivial irreducible representation of \(E_7\) with minimal dimension, an irreducible representation of \(E_7\) having dimension \(\leq 57 - 4 = 53\) is only the trivial representation. Consequently, \(\mathcal{F}\) is isomorphic to a direct sum of line bundles.

(4) Suppose that \(G\) is of type \(F_4\).
(a) The rational homogeneous variety \( F_4/P_2 \subset \mathbb{P}^{1273} \) has dimension 20 and Fano index 5. Since the semisimple part of \( P_2 \) is isomorphic to \( \text{SL}(2, \mathbb{C}) \times \text{SL}(3, \mathbb{C}) \), by Proposition 3.2, we know

\[
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} > \frac{5}{16} = \frac{\dim(F_i/P_2)}{\dim(F_i/ P_2) - 4} \quad \text{for all } i.
\]

(b) The rational homogeneous variety \( F_4/P_3 \subset \mathbb{P}^{272} \) has dimension 20 and Fano index 7. Since the semisimple part of \( P_3 \) is isomorphic to \( \text{SL}(3, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \), by Proposition 3.2, we know

\[
\frac{\text{dex}(F_i)}{\text{rank}(F_i)} > \frac{7}{16} = \frac{\dim(F_i/P_3)}{\dim(F_i/ P_3) - 4} \quad \text{for all } i.
\]

Therefore, the result follows. \( \square \)

The classification in Table 1 is a direct consequence of the following propositions.

**Proposition 3.6.** Let \( F \) be a completely reducible, globally generated, equivariant vector bundle over \( G_2/P \), and \( Z \) the zero locus of a general global section of \( F \).

1. If \( Z \subset G_2/P_1 \) is a fourfold with trivial canonical bundle, then \( F \cong \mathcal{O}(5) \).
2. If \( Z \subset G_2/P_2 \) is a fourfold with trivial canonical bundle, then \( F \cong \mathcal{O}(3) \).

**Proof.** Since the rational homogeneous variety \( G_2/P_1 \subset \mathbb{P}(V_{G_2}(\mathbb{C})) = \mathbb{P}^6 \) has dimension 5 and Fano index 5, \( F \) is a line bundle \( \mathcal{O}(5) \) of degree 5. Similarly, the rational homogeneous variety \( G_2/P_2 \subset \mathbb{P}(V_{G_2}(\mathbb{C})) = \mathbb{P}^{13} \) has dimension 5 and Fano index 3. Hence, in this case \( F \) is a line bundle \( \mathcal{O}(3) \) of degree 3.

**Remark 3.7.** It is well-known that the rational homogeneous variety \( G_2/P_1 \) is isomorphic to a 5-dimensional quadric \( Q^5 \) in \( \mathbb{P}^6 \) (e.g., [FH91, p. 391]). Thus, the fourfold \( Z \subset G_2/P_1 \) in Proposition 3.6(1) is the complete intersection of a smooth quadric hypersurface and a general quintic hypersurface in \( \mathbb{P}^6 \).

Moreover, since the second fundamental representation \( V_{G_2}(\mathbb{C}) \) is the adjoint representation of \( G_2/P_2 \), the adjoint variety of \( G_2 \). Thus, the fourfold \( Z \subset G_2/P_2 \) in Proposition 3.6(2) is the complete intersection of the adjoint variety of \( G_2 \) and a general cubic hypersurface in \( \mathbb{P}^{13} \).

The argument in Proposition 3.6 implies that a general global section of \( \mathcal{O}(1) \oplus \mathcal{O}(4) \) or \( \mathcal{O}(2) \oplus \mathcal{O}(3) \) over \( G_2/P_1 \) gives Calabi–Yau threefolds. As the equivariant bundle \( \mathcal{E}_{\mathbb{C}^3} \) over \( G_2/P_1 \) has rank 2 and dex 3, we have (hence the zero locus of a general global section of \( \mathcal{E}_{\mathbb{C}^3} \) is a Calabi–Yau threefold. Similarly, a general global section of \( \mathcal{O}(1) \oplus \mathcal{O}(2) \) or \( \mathcal{E}_{\mathbb{C}^3} \otimes \mathcal{O}(1) \) over \( G_2/P_2 \) gives a Calabi–Yau threefold. See Table 2.

**Proposition 3.8.** Let \( F \) be a completely reducible, globally generated, equivariant vector bundle over the Cayley plane \( E_6/P_1 \). If \( Z \subset E_6/P_1 \) is a fourfold with trivial canonical bundle which is the zero locus of a general global section of \( F \), then \( F \) is isomorphic to \( \mathcal{O}(1)^{\oplus 12} \).

**Proof.** The (complex) Cayley plane \( E_6/P_1 \) has dimension 16 and Fano index 12. Recall that the semisimple part of the maximal parabolic subgroup \( P_1 \subset E_6 \) is isomorphic to \( \text{Spin}(10, \mathbb{C}) \). Since irreducible representations of \( \text{Spin}(10, \mathbb{C}) \) having dimension \( \leq 16 - 4 = 12 \) are only the trivial and standard representation, it suffices to consider the line bundles \( \mathcal{O}(k) \) (\( k \geq 1 \)) and the equivariant vector bundle \( \mathcal{E}_{\mathbb{C}^6} \) of rank 10. However, since a general global section of \( \mathcal{E}_{\mathbb{C}^6} \) vanishes nowhere as explained in [PM15, Section 1.2], we obtain the result by Proposition 3.4. \( \square \)

**Proposition 3.9.** Let \( F \) be a completely reducible, globally generated, equivariant vector bundle over \( E_6/P_2 \). If \( Z \subset E_6/P_2 \) is a fourfold with trivial canonical bundle which is the zero locus of a general global section of \( F \), then \( F \) is isomorphic to either \( \mathcal{E}_{\mathbb{C}^2}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \) or \( \mathcal{E}_{\mathbb{C}^1} \oplus \mathcal{E}_{\mathbb{C}^6} \oplus \mathcal{O}(1)^{\oplus 5} \) or \( \mathcal{E}_{\mathbb{C}^2}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \).

**Proof.** The rational homogeneous variety \( E_6/P_2 \subset \mathbb{P}^{277} \) has dimension 21 and Fano index 11. Note that the semisimple part of the maximal parabolic subgroup \( P_2 \subset E_6 \) is isomorphic to \( \text{SL}(6, \mathbb{C}) \). Using the formula for the dex in Proposition 3.2, we obtain the dex of irreducible equivariant vector bundles with rank \( \leq 21 - 4 = 17 \) as in Table 6.
Table 6. The irreducible equivariant vector bundles over \( E_6/P_2 \) with rank \( \leq 17 \).

| \( \lambda \) | \( \varpi_1 \) | \( \varpi_3 \) | \( \varpi_5 \) | \( \varpi_6 \) |
|---|---|---|---|---|
| \( \text{rank}(E_\lambda) \) | 6 | 15 | 15 | 6 |
| \( \text{dex}(E_\lambda) \) | 3 | 15 | 15 | 3 |

Table 7. The irreducible equivariant bundles over \( E_6/P_3 \) with rank \( \leq 21 \) and \( \text{dex} \leq 9 \).

| \( \lambda \) | \( \varpi_1 \) | \( \varpi_3 \) | \( \varpi_5 \) | \( \varpi_6 \) |
|---|---|---|---|---|
| \( \text{rank}(E_\lambda) \) | 2 | 3 | 4 | 5 | 10 | 5 |
| \( \text{dex}(E_\lambda) \) | 1 | 3 | 6 | 3 | 8 | 2 |

Considering the conditions in (3.11), \( \mathcal{F} \) is isomorphic to either \( \mathcal{E}_{\varpi_1}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \) or \( \mathcal{E}_{\varpi_6} \oplus \mathcal{O}(1)^{\oplus 5} \) or \( \mathcal{E}_{\varpi_6}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \).

Remark 3.10. Since the outer automorphism of \( E_6 \) corresponding to the symmetry of the Dynkin diagram induces a bundle isomorphism between \( \mathcal{E}_{\varpi_1} \) and \( \mathcal{E}_{\varpi_6} \), the zero locus \( Z \subset E_6/P_2 \) of a general global section of \( \mathcal{E}_{\varpi_1}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \) in Proposition 3.10 is projectively equivalent to the zero locus of a general global section of \( \mathcal{E}_{\varpi_6} \oplus \mathcal{O}(1)^{\oplus 5} \) or \( \mathcal{E}_{\varpi_6}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \).

Proposition 3.11. Let \( \mathcal{F} \) be a completely reducible, globally generated, equivariant vector bundle over \( E_6/P_3 \). If \( Z \subset E_6/P_3 \) is a fourfold with trivial canonical bundle which is the zero locus of a general global section of \( \mathcal{F} \), then \( \mathcal{F} \) is isomorphic to either \( \mathcal{E}_{\varpi_1}^{\oplus 3} \oplus \mathcal{E}_{\varpi_6}^{\oplus 3} \) or \( \mathcal{E}_{\varpi_6}^{\oplus 4} \oplus \mathcal{O}(1) \) from the conditions in (3.1).

Proof. The rational homogeneous variety \( E_6/P_3 \) has dimension 25 and Fano index 9. Note that the rational homogeneous variety \( E_6/P_3 \subset \mathbb{P}^{50} \) is projectively equivalent to \( E_6/P_3 \subset \mathbb{P}^{350} \) because the highest weight \( E_6 \)-module \( V_{E_6}(\varpi_3) \) is dual to \( V_{E_6}(\varpi_3) \).

Since the semisimple part of the maximal parabolic subgroup \( P_3 \subset E_6 \) is isomorphic to \( \text{SL}(2, \mathbb{C}) \times \text{SL}(5, \mathbb{C}) \), by Proposition 3.2 irreducible equivariant vector bundles with rank \( \leq 25 - 4 = 21 \) and \( \text{dex} \leq 9 \) are given in Table 7.

| \( \lambda \) | \( \varpi_1 \) | \( 2\varpi_1 \) | \( 3\varpi_1 \) | \( \varpi_2 \) | \( \varpi_5 \) | \( \varpi_6 \) |
|---|---|---|---|---|---|---|
| \( \text{rank}(E_\lambda) \) | 2 | 3 | 4 | 5 | 10 | 5 |
| \( \text{dex}(E_\lambda) \) | 1 | 3 | 6 | 3 | 8 | 2 |

Except for \( \lambda = \varpi_6 \), we have

\[
\frac{\text{dex}(E_\lambda)}{\text{rank}(E_\lambda)} > \frac{9}{21} = \frac{t(E_6/P_3)}{\dim(E_6/P_3) - 4}
\]

so that \( \mathcal{F} \) must have \( \mathcal{E}_{\varpi_6} \) as direct summands. Repeating this process, \( \mathcal{F} \) is isomorphic to either \( \mathcal{E}_{\varpi_1}^{\oplus 3} \oplus \mathcal{E}_{\varpi_6}^{\oplus 3} \) or \( \mathcal{E}_{\varpi_6}^{\oplus 4} \oplus \mathcal{O}(1) \) from the conditions in (3.1).

The argument in Proposition 3.11 implies that a general global section of \( \mathcal{E}_{\varpi_1} \oplus \mathcal{E}_{\varpi_6}^{\oplus 4} \) over \( E_6/P_3 \) gives a Calabi–Yau threefold. See Table 2.

Proposition 3.12. Let \( \mathcal{F} \) be a completely reducible, globally generated, equivariant vector bundle over \( E_7/P_1 \). If \( Z \subset E_7/P_1 \) is a fourfold with trivial canonical bundle which is the zero locus of a general global section of \( \mathcal{F} \), then \( \mathcal{F} \) is isomorphic to \( \mathcal{E}_{\varpi_7}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \).

Proof. As the fundamental \( E_7 \)-module \( V_{E_7}(\varpi_1) \) is the adjoint representation \( \mathfrak{e}_7 \), the rational homogeneous variety \( E_7/P_1 \) is the adjoint variety of \( E_7 \). The \( E_7 \)-adjoint variety \( E_7/P_1 \subset \mathbb{P}^{132} \) has dimension 33 and Fano index 17. Since the semisimple part of the maximal parabolic subgroup \( P_1 \subset E_7 \) is isomorphic to \( \text{Spin}(12, \mathbb{C}) \), and irreducible representations of \( \text{Spin}(12, \mathbb{C}) \) having dimension \( \leq 33 - 4 = 29 \) are only the trivial and standard representation, it suffices to consider the line bundles \( \mathcal{O}(k) \) \( (k \geq 1) \) and the equivariant vector bundle \( \mathcal{E}_{\varpi_7} \) of rank 12.

From the direct computation using the action of the Weyl group, we know that the sum of all weights of the irreducible \( P_1 \)-module with highest weight \( \varpi_7 \) is equal to \( 6\varpi_1 \). Therefore, \( \text{dex}(\varpi_7) = 6 \) and \( \mathcal{F} \) is isomorphic to \( \mathcal{E}_{\varpi_7}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 5} \) from the conditions in (3.1).
Proposition 3.13. Let $\mathcal{F}$ be a completely reducible, globally generated, equivariant vector bundle over $F_4/P_1$. If $Z \subset F_4/P_1$ is a fourfold with trivial canonical bundle which is the zero locus of a general global section of $\mathcal{F}$, then $\mathcal{F}$ is isomorphic to $\mathcal{E}_{\varpi_4} \oplus \mathcal{O}(1)^{\oplus 5}$.

Proof. The rational homogeneous variety $F_4/P_1 \subset \mathbb{P}^{25}$ has dimension 15 and Fano index 8. Since the semisimple part of the maximal parabolic subgroup $P_1 \subset F_4$ is isomorphic to Sp(6, $\mathbb{C}$), and irreducible representations of Sp(6, $\mathbb{C}$) having dimension $\leq 15 - 4 = 11$ are only the trivial and standard representation, it suffices to consider the line bundles $\mathcal{O}(k)$ ($k \geq 1$) and the equivariant vector bundle $\mathcal{E}_{\varpi_4}$ of rank 6.

From the direct computation using the action of the Weyl group, we know that the sum of all weights of the irreducible $P_1$-module with highest weight $\varpi_4$ is equal to $3\varpi_1$. Therefore, $\text{dex}(\varpi_4) = 3$ and $\mathcal{F}$ is isomorphic to $\mathcal{E}_{\varpi_4} \oplus \mathcal{O}(1)^{\oplus 5}$ from the conditions in (3.1). □

Proposition 3.14. Let $\mathcal{F}$ be a completely reducible, globally generated, equivariant vector bundle over $F_4/P_4$. If $Z \subset F_4/P_4$ is a fourfold with trivial canonical bundle which is the zero locus of a general global section of $\mathcal{F}$, then $\mathcal{F}$ is isomorphic to either $\mathcal{O}(1)^{\oplus 11}$ or $\mathcal{E}_{\varpi_1} \oplus \mathcal{O}(1)^{\oplus 4}$.

Proof. The rational homogeneous variety $F_4/P_4 \subset \mathbb{P}^{25}$ has dimension 15 and Fano index 11. Since the semisimple part of the maximal parabolic subgroup $P_4 \subset F_4$ is isomorphic to Spin(7, $\mathbb{C}$), and irreducible representations of Spin(7, $\mathbb{C}$) having dimension $\leq 15 - 4 = 11$ are only the trivial, standard and spin representation, it suffices to consider the line bundles $\mathcal{O}(k)$ ($k \geq 1$), the equivariant vector bundles $\mathcal{E}_{\varpi_1}$ of rank 7 and $\mathcal{E}_{\varpi_3}$ of rank 8.

From the direct computation, the sum of all weights of the irreducible $P_4$-module with highest weight $\varpi_1$ is equal to $7\varpi_1$. Therefore, $\text{dex}(\varpi_1) = 7$ and $\mathcal{F}$ is isomorphic to $\mathcal{E}_{\varpi_1} \oplus \mathcal{O}(1)^{\oplus 4}$ from the conditions in (3.1). □

Remark 3.15. Since the 27-dimensional fundamental $E_6$-module $V_{E_6}(\varpi_1)$ may be regarded as an $F_4$-module using the embedding of $F_4$ in $E_6$ and it decomposes as $V_{F_4}(\varpi_4) \oplus V_{F_4}(0)$ (see [Car05, Proposition 13.32]),
the rational homogeneous variety $F_4/P_4$ is a general hyperplane section of the Cayley plane $E_6/P_3 \subset \mathbb{P}^{26}$ (see [LM03 Section 6.3]). Thus, the zero locus $Z \subset F_4/P_4$ of a general global section of $O(1)^{\oplus 11}$ in Proposition 3.14 is the same as the zero locus of a general global section of $O(1)^{\oplus 12}$ over $E_6/P_3$ in Proposition 3.8.

4. Computations of Hodge numbers

The Borel–Weil–Bott theorem (Theorem 2.8) is a powerful tool for computing cohomologies of equivariant vector bundles over homogeneous varieties. Using the Koszul complex, the conormal sequence and the exact sequences derived from this, the Hodge numbers of Calabi–Yau fourfolds and threefolds in Table 1 and Table 2 can be calculated by the similar way as in [Ku95].

All examples have $h^{0,0}(Z) = h^0(O_Z) = 1$ and $h^{0,1}(Z) = h^1(O_Z) = 0$ which mean that they are irreducible and Calabi–Yau in the strict sense.

4.1. Computations of $h^{0,q}(Z)$.

Let $G/P$ be a rational homogeneous variety of Picard number one, and $F$ be a completely reducible, globally generated, equivariant vector bundle over $G/P$ of rank $n - 4$, where $n = \dim(G/P)$. Using a similar argument as in Example 2.9, the Borel–Weil–Bott theorem and the Koszul complex

$$0 \to \bigwedge^{n-4} F^* \to \bigwedge^{n-5} F^* \to \cdots \to \bigwedge^2 F^* \to F^* \to O_G/P \to O_Z \to 0$$

allow us to calculate the Hodge numbers $h^{0,q}(Z) = \dim H^q(Z, O_Z)$ for the zero locus $Z$ of a general global section of $F$.

Proposition 4.1. Let $G$ be a simple Lie group of exceptional type. Let $G/P$ be a rational homogeneous variety of Picard number one, and $F$ be a completely reducible, globally generated, equivariant vector bundle over $G/P$ of rank $n - 4$. If the zero locus $Z$ of a general global section of $F$ is a fourfold with trivial canonical bundle, then $h^{0,0}(Z) = h^{0,4}(Z) = h^{0,8}(Z) = 1$ and $h^{0,q}(Z) = 0$ for $q = 1, 2, 3$.

Proof. First, if $Z$ is of No. 1, No. 8, No. 9 in the Table 1, then $h^{0,2}(Z) = 0$ by Proposition 2.7 and $h^{0,4}(Z) = h^{0,9}(Z) = 1$ from the straightforward computations. Using the same argument as Example 2.9, we compute the Hodge numbers $h^{0,0}(Z), h^{0,2}(Z), h^{0,4}(Z)$ of the other Calabi–Yau fourfolds in Table 1:

No. 2. $F = E^{\oplus 2} \oplus O(1)^{\oplus 5}$ over $E_6/P_2$
No. 3. $F = E^{\oplus 3} \oplus E^{\oplus 4}$ over $E_6/P_3$
No. 4. $F = E^{\oplus 3} \oplus O(1)$ over $E_7/P_3$
No. 5. $F = E^{\oplus 7} \oplus O(1)^{\oplus 5}$ over $E_7/P_1$
No. 6. $F = E^{\oplus 4} \oplus O(1)^{\oplus 5}$ over $F_4/P_1$
No. 7. $F = E^{\oplus 4} \oplus O(1)^{\oplus 4}$ over $F_4/P_3$

For instance, let $Z$ be the zero locus of a general global section of the equivariant vector bundle $F = E_{\pi_1} \oplus O(1)^{\oplus 4}$ over $F_4/P_4$. As we have already seen in Proposition 3.14, the reductive part of the maximal parabolic subgroup $P_3 \subset P_4$ is isomorphic to Spin($7, \mathbb{C}$) $\times \mathbb{C}^*$. But weights of $P_4$-modules have to be computed inside the weight lattice of $F_4$. Using representations of Spin($10, \mathbb{C}$) and adjusting the first Chern class, the wedge powers of the irreducible equivariant bundle $E_{\pi_1}$ are described as follows.

$$\bigwedge^2 E_{\pi_1} = E_{\pi_2}, \quad \bigwedge^3 E_{\pi_1} = E_{\pi_3}, \quad \bigwedge^5 E_{\pi_1} = E_{\pi_2+3\pi_4} = E_{\pi_2}(3), \quad \bigwedge^6 E_{\pi_1} = E_{\pi_1+5\pi_4} = E_{\pi_1}(5), \quad \bigwedge^7 E_{\pi_1} = E_{\pi_4} = O(7).$$

We have the Koszul complex associated to a general global section of the vector bundle $F$:

$$0 \to \bigwedge^{11} F^* \to \bigwedge^{10} F^* \to \cdots \to \bigwedge^2 F^* \to F^* \to O_{F_4/P_4} \to O_Z \to 0.$$
Then each irreducible equivariant bundle in $\bigwedge^p F^*$ is acyclic for $1 \leq p \leq 10$. For example, we can check that $E_{2\omega_3}(-7)$ is acyclic by the Borel–Weil–Bott theorem because the weights $2\omega_3 - 7\omega_4 + \rho = \omega_1 + \omega_2 + 3\omega_3 - 6\omega_4$ is singular. Indeed, $(s_3 s_1 s_2 s_3 s_4)(\omega_1 + \omega_2 + 3\omega_3 - 6\omega_4)$ is orthogonal to the simple root $\alpha_2$. On the other hand, by the Serre duality, we get

$$H^{15}(F_4/P_4, \bigwedge^1 F^*) = H^{15}(F_4/P_4, O(-11)) \cong H^0(F_4/P_4, O)^* \cong \mathbb{C},$$

which implies $H^4(Z, O_Z) \cong \mathbb{C}$. Consequently, we conclude $h^0(Z, O_Z) = h^4(Z, O_Z) = 1$ and $h^q(Z, O_Z) = 0$ for $q = 1, 2, 3$.

We consider one more example. Let $Z$ be the zero locus of a general global section of the equivariant vector bundle $F = E_{\omega_3}^{\oplus 4} \oplus O(1)$ over $E_6/P_3$. Using the computer program SageMath [SAGE], we provide weights appearing in each wedge power of the bundle $F^*$ in Table 8. Each vector $(\lambda_1, \ldots, \lambda_6)$ represents the weight $\lambda_1\omega_1 + \cdots + \lambda_6\omega_6$. For example, the third wedge power of $F^*$ is

$$\bigwedge^3 F^* = E_{\omega_3}(-3)^{\oplus 10} \oplus E_{2\omega_3 + \omega_2 - \omega_1}(-3)^{\oplus 20} \oplus E_{2\omega_2}(3)^{\oplus 5} \oplus E_{\omega_2}(3)^{\oplus 2}$$

and this decomposition corresponds to vectors $(0, 0, -3, 1, 0, 0), (0, 0, -2, 0, 1, 0), (0, 1, -3, 1, 0, 0), (0, 2, -3, 0, 0, 0), (0, 3, -3, 0, 0, 0)$. Using SageMath, we compute the weights appearing in $\bigwedge^p F^*$ for all $1 \leq p \leq 20$ and check that all of them are singular. Accordingly, by the Borel–Weil–Bott theorem, each irreducible equivariant bundle in $\bigwedge^p F^*$ is acyclic for $1 \leq p \leq 20$. Finally, we obtain

$$H^{25}(E_6/P_3, \bigwedge^{21} F^*) = H^{25}(E_6/P_3, O(-9)) \cong H^0(E_6/P_3, O)^* \cong \mathbb{C},$$

which implies $H^4(Z, O_Z) \cong \mathbb{C}$. This proves the claim for this case.

For the remaining cases, again using the computer program SageMath [SAGE], we compute the exterior powers of bundles and check whether they are acyclic or not.

| $p$ | weights appearing in $\bigwedge^p F^*$ |
|-----|-------------------------------------|
| 1   | $(0, 0, -1, 0, 0, 0), (0, 1, -1, 0, 0, 0)$ |
| 2   | $(0, 0, -2, 1, 0, 0), (0, 1, -2, 0, 0, 0)$ |
| 3   | $(0, 0, -3, 0, 0, 0), (0, 1, -3, 1, 0, 0)$ |
| 4   | $(0, 0, -4, 0, 0, 0)$ |
| 5   | $(0, 0, -5, 0, 0, 0)$ |
| 6   | $(0, 0, -6, 0, 0, 0)$ |
| 7   | $(0, 0, -7, 0, 0, 0)$ |
| 8   | $(0, 0, -8, 0, 0, 0)$ |
| 9   | $(0, 0, -9, 0, 0, 0)$ |
| 10  | $(0, 0, -10, 0, 0, 0)$ |

Table 8. Weights appearing in $\bigwedge^p F^*$ for $F = E_{\omega_3}^{\oplus 4} \oplus O(1)$ over $E_6/P_3$ for $1 \leq p \leq 10$. 

Proof of Theorem 1.1. From Proposition 4.1 and Corollary 2.6, each fourfold with trivial canonical bundle classified in Theorem 1.3 for exceptional homogeneous varieties of Picard number one is not hyperkähler. □

Proof of Corollary 1.2. Theorem 1.1 of [Ben15a] says that if $Z$ is a hyperkähler fourfold which is the zero locus of a general global section of completely reducible, globally generated, equivariant vector bundles over a Grassmannian, orthogonal or symplectic Grassmannian, then $Z$ is either of Beauville–Donagi type or of Debarre–Voisin type. Hence, by Theorem 1.1 we get the conclusion. □

4.2. Computations of $h^{1,q}(Z)$. First, recall the geometric meaning of the Hodge number $h^{1,3}(Z) = h^{3,1}(Z)$ of a Calabi–Yau fourfold $Z$. Since $Z$ has trivial canonical bundle, $H^1(Z, \Omega^3_Z) = H^1(Z, K_Z \otimes T_Z) = H^1(Z, T_Z)$. By Kodaira–Spencer deformation theory (cf. [Kod05]), $h^{1,3}(Z)$ is equal to the dimension of deformation parameter space of $Z$ because $Z$ is unobstructed from $H^2(Z, T_Z) = 0$.

If $G/P$ is not a Hermitian symmetric space of compact type, then the tangent bundle $T_{G/P}$ of $G/P$ is reducible. Nevertheless, using a filtration by $P$-submodules, the Borel–Weil–Bott theorem can be applied to compute cohomology groups of reducible equivariant vector bundles. Any $P$-module $V$ can be decomposed into a direct sum of irreducible $L$-modules $V = W_0 \oplus W_1 \oplus \cdots \oplus W_t$, where $L$ is the reductive part of $P$. Since the unipotent radical $U$ is a normal subgroup of $P$ and $U$ acts on $V$ in triangular fashion, these irreducible $L$-modules can be arranged such that $UW_i \subseteq W_j$ with $j > i$. Then $V$ has a filtration $0 \subseteq V_1 \subseteq \cdots \subseteq V_t \subseteq V_0 = V$ such that $V_i/V_{i+1} \cong W_i$.

Let $E_i = G \times_P V_i$ and $F_i = G \times_P W_i$ be the equivariant vector bundles associated with $V_i$ and $W_i$ over $G/P$, respectively. Then we get a short exact sequence of vector bundles $0 \to E_{i+1} \to E_i \to F_i \to 0$.

Clearly, when $W_i = V_P(\lambda_i)^*$ for a $P$-dominant weight $\lambda_i$, we have $F_i = Z(\lambda_i)$.

Definition 4.2. Let $E = G \times_P V$ be a (reducible) equivariant vector bundle over $G/P$. For an irreducible decomposition $V = W_0 \oplus W_1 \oplus \cdots \oplus W_t$ as $L$-modules, we denote a $P$-dominant weight $\lambda_i$ such that $W_i = V_P(\lambda_i)^*$. We define the set

$$\text{RegInd}(E) := \{\text{index}(\lambda_i + \rho) : \lambda_i + \rho \text{ is regular for } 0 \leq i \leq t\},$$

where index($\lambda$) is the minimum among the lengths $\ell(w)$ of the Weyl group elements $w$ making $w(\lambda)$ regular dominant.

Proposition 4.3 ([Gr06, Section 3]). Let $E$ be a (reducible) equivariant vector bundle over $G/P$. If $q \notin \text{RegInd}(E)$, then $H^q(G/P, E) = 0$.

Proof. The short exact sequence of vector bundles $0 \to E_{i+1} \to E_i \to F_i \to 0$ leads to the following long exact sequences

$$\cdots \to H^q(G/P, E_i) \to H^q(G/P, E_{i-1}) \to H^q(G/P, F_{i-1}) \to \cdots$$

$$\cdots \to H^q(G/P, E_{i-1}) \to H^q(G/P, E_{i-2}) \to H^q(G/P, F_{i-2}) \to \cdots$$

$$\cdots$$

$$\cdots \to H^q(G/P, E_1) \to H^q(G/P, E) \to H^q(G/P, F_0) \to \cdots$$

By Theorem 2.8 if $q \notin \text{RegInd}(E)$, then we have

$$H^q(G/P, E_i) = H^q(G/P, F_i) = 0, H^q(G/P, F_{i-1}) = 0, \ldots, H^q(G/P, F_0) = 0.$$ 

Hence we get $H^q(G/P, E) = 0$ from the above exact sequences. □

From now on, we only consider a rational homogeneous variety $G/P_k$ of Picard number one, that is, $P_k$ is the maximal parabolic subgroup associated to a simple root $\alpha_k$ of $G$. Let $p_k$ be the maximal parabolic subalgebra of $\mathfrak{g}$ associated to the simple root $\alpha_k$. Given an integer $\ell$, $-m \leq \ell \leq m$, $\Phi_\ell$ denotes the set of all roots $\alpha = \sum_{q=1}^r c_q \alpha_q$ with the $k$th coefficient $c_k = \ell$. Define

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\ell = \bigoplus_{\alpha \in \Phi_\ell} \mathfrak{g}_\alpha, \quad \ell \neq 0,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Then we have a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-m} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ of depth $m$ associated with the simple root $\alpha_k$. Using the natural identification

$$T_0(G/P_k) = \mathfrak{g}/p_k \cong \mathfrak{g}_{-m} \oplus \cdots \oplus \mathfrak{g}_{-1},$$
we describe the tangent bundle $T_{G/P_2}$ and the cotangent bundle $\Omega_{G/P_2}$ via irreducible equivariant vector bundles. Here, $o = eP \in G/P_k$ where $e$ is the identity element of $G$.

**Example 4.4** (No. 9 in Table 1). Let $Z$ be the zero locus of a general global section of the line bundle $\mathcal{O}(3)$ over $G_2/P_2$. Since $\Phi_1 = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}$ and $\Phi_2 = \{3\alpha_1 + 2\alpha_2\}$ for the root system of $G_2$, (4.2) $T_o(G_2/P_2) \cong g_{-2} \oplus g_{-1} = V_{P_2}(-3\alpha_1 - 2\alpha_2) \oplus V_{P_2}(-\alpha_2)$ has a filtration

$$0 \subset V_{P_2}(-\varpi_2) \subset V_{P_2}(-\varpi_2) \oplus V_{P_2}(3\varpi_1 - 2\varpi_2).$$

Moreover, we get a short exact sequence

(4.3) $0 \to \mathcal{E}_{-\varpi_2} \to \Omega_{G_2/P_2} \to \mathcal{E}_{3\varpi_1 - 2\varpi_2} \to 0$

from $\Omega_{G_2/P_2} = G_2 \times_{P_2} T_o(G_2/P_2)^*$. By Proposition 4.3 $H^q(G_2/P_2, \Omega_{G_2/P_2}) = 0$ for $q \neq 1$ because $\text{RegInd}(\Omega_{G_2/P_2}) = \{1\}$.

Moreover, tensoring the Koszul complex

(4.4) $0 \to \mathcal{O}(-3) \to \Omega_{G_2/P_2} \to \mathcal{O}_Z \to 0$

with $\Omega_{G_2/P_2}$, we obtain the exact sequence

$$0 \to \Omega_{G_2/P_2}(-3) \to \Omega_{G_2/P_2} \to \Omega_{G_2/P_2}|_Z \to 0.$$

Using $H^q(G_2/P_2, \Omega_{G_2/P_2}(-3)) = 0$ for $q \neq 5$ from $\text{RegInd}(\Omega_{G_2/P_2}(-3)) = \{5\}$, we see

(4.5) $H^1(Z, \Omega_{G_2/P_2}|_Z) \cong H^1(G_2/P_2, \Omega_{G_2/P_2}) \cong \mathbb{C}$,

(4.6) $H^4(Z, \Omega_{G_2/P_2}|_Z) \cong H^4(G_2/P_2, \Omega_{G_2/P_2}(-3)) \cong V_{G_2}(\varpi_2)^* \cong \mathbb{C}^{14}$,

and the other cohomologies vanish. Thus the conormal sequence $0 \to \mathcal{O}(-3)|_Z \to \Omega_{G_2/P_2}|_Z \to \Omega_Z \to 0$ leads to the following exact sequences

(4.7) $0 \to H^0(Z, \Omega_Z) \to H^1(Z, \mathcal{O}(-3)|_Z) \to H^1(Z, \Omega_{G_2/P_2}|_Z) \to H^1(Z, \Omega_Z) \to H^2(Z, \mathcal{O}(-3)|_Z) \to 0$;

(4.8) $0 \to H^2(Z, \Omega_Z) \to H^3(Z, \mathcal{O}(-3)|_Z) \to 0$;

(4.9) $0 \to H^3(Z, \Omega_Z) \to H^4(Z, \mathcal{O}(-3)|_Z) \to H^4(Z, \Omega_{G_2/P_2}|_Z) \to H^4(Z, \Omega_Z) \to H^5(Z, \mathcal{O}(-3)|_Z) \to 0$.

Because $H^4(Z, \Omega_Z) = H^1(Z) \cong H^1(Z, \mathcal{O}_Z) = 0$, we conclude that $H^5(Z, \mathcal{O}(-3)|_Z) = 0$ and the sequence (4.9) becomes

$$0 \longrightarrow H^3(Z, \Omega_Z) \longrightarrow H^4(Z, \mathcal{O}(-3)|_Z) \longrightarrow H^4(Z, \Omega_{G_2/P_2}|_Z) \longrightarrow H^4(Z, \Omega_Z) \longrightarrow H^5(Z, \mathcal{O}(-3)|_Z) \longrightarrow 0.$$

Accordingly, we have

(4.10) $H^3(Z, \Omega_Z) \cong H^4(Z, \mathcal{O}(-3)|_Z)/H^4(Z, \Omega_{G_2/P_2}|_Z)$.

Similarly, tensoring the Koszul complex (4.3) with $\mathcal{O}(-3)$ we obtain

$$0 \to \mathcal{O}(-6) \to \mathcal{O}(-3) \to \mathcal{O}(-3)|_Z \to 0.$$

The straightforward applications of the Borel–Weil–Bott theorem say that

$$H^5(G_2/P_2, \mathcal{O}(3)) = V_{G_2}(0)^* \cong \mathbb{C},$$

$$H^5(G_2/P_2, \mathcal{O}(-6)) = V_{G_2}(3\varpi_2)^* \cong \mathbb{C}^{273}.$$

Since $\text{RegInd}(\mathcal{O}(-3)) = \text{RegInd}(\mathcal{O}(-6)) = \{5\}$, we obtain

$$0 \longrightarrow H^4(Z, \mathcal{O}(-3)|_Z) \longrightarrow H^5(G_2/P_2, \mathcal{O}(-6)) \longrightarrow H^5(G_2/P_2, \mathcal{O}(-3)) \longrightarrow H^5(Z, \mathcal{O}(-3)|_Z) \longrightarrow 0.$$

Hence $H^4(Z, \mathcal{O}(-3)|_Z) \cong H^5(G_2/P_2, \mathcal{O}(-6))/H^5(G_2/P_2, \mathcal{O}(-3)) \cong \mathbb{C}^{272}$, and moreover, $H^4(Z, \mathcal{O}(-3)|_Z) = 0$ for $q \neq 4$. Therefore, by (4.5) the exact sequences (4.7) and (4.8) provide

$$H^1(Z, \Omega_Z) \cong H^1(Z, \Omega_{G_2/P_2}|_Z) \cong \mathbb{C}; \quad H^2(Z, \Omega_Z) \cong H^3(Z, \mathcal{O}(-3)|_Z) = 0.$$
Accordingly, we have $h^{1,1}(Z) = 1$ and $h^{1,2}(Z) = 0$. By considering (4.10), we get $h^{1,3}(Z) = \dim H^3(Z, \Omega_Z) = \dim H^4(Z, \mathcal{O}(-3)|_Z) - \dim H^4(Z, \Omega_{G_2/P_i}|_Z) = 272 - 14 = 258$.

**Example 4.5** (No. 8 in Table IV). Let $Z$ be the zero locus of a general global section of the line bundle $\mathcal{F} = \mathcal{O}(5)$ over $G_2/P_i$. The Lie algebra of $G_2$ has a gradation of depth 3 associated with the simple root $\alpha_1$. Since $\Phi_1 = \{\alpha_1, \alpha_2 + \alpha_2\}, \Phi_2 = \{2\alpha_1 + \alpha_2\}, \Phi_3 = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, we have the decomposition

$$T_\nu(G_2/P_i) \cong \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = V_{P_i}(−3\alpha_1 − \alpha_2) \oplus V_{P_i}(−2\alpha_1 − \alpha_2) \oplus V_{P_i}(−\alpha_2)$$

$$= V_{P_i}(−3\varpi_1 + \varpi_2) \oplus V_{P_i}(−\varpi_1) \oplus V_{P_i}(−2\varpi_1 + \varpi_2)$$

and this gives a filtration

$$0 \subset \mathfrak{g}_{-3} \subset \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = T_\nu(G_2/P_i).$$

Putting $E := G_2 \times P_i \left( \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \right)^*$, we get two short exact sequences

$$0 \rightarrow E \rightarrow \mathfrak{g}_{-31}, \mathfrak{g}_{-1} \rightarrow 0,$$

$$0 \rightarrow E \rightarrow \Omega_{G_2/P_i} \rightarrow \mathfrak{g}_{-21}, \mathfrak{g}_{-1} \rightarrow 0.$$

from $\Omega_{G_2/P_i} = G_2 \times P_i T_\nu(G_2/P_i)^*$. We notice that the weights $−\varpi_1$ and $−3\varpi_1 + \varpi_2$ are singular while $−2\varpi_1 + \varpi_2$ is regular with index $(−2\varpi_1 + \varpi_2) = 1$. By Proposition 4.13, $H^q(G_2/P_i, \Omega_{G_2/P_i}) = 0$ for $q \neq 1$ because $\text{RegInd}(\Omega_{G_2/P_i}) = \{1\}$.

Tensoring the Koszul complex

$$0 \rightarrow \mathcal{O}(−5) \rightarrow \mathcal{O}_{G_2/P_i} \rightarrow \mathcal{O}_Z \rightarrow 0$$

with $\Omega_{G_2/P_i}$, we obtain the exact sequence

$$0 \rightarrow \Omega_{G_2/P_i}(−5) \rightarrow \Omega_{G_2/P_i} \rightarrow \Omega_{G_2/P_i}|_Z \rightarrow 0.$$

Using $H^q(G_2/P_i, \Omega_{G_2/P_i}(−5)) = 0$ for $q \neq 5$ from $\text{RegInd}(\Omega_{G_2/P_i}(−5)) = \{5\}$, we see

$$H^1(Z, \Omega_{G_2/P_i}|_Z) = H^1(G_2/P_i, \Omega_{G_2/P_i}) \cong \mathbb{C},$$

$$H^4(Z, \Omega_{G_2/P_i}|_Z) = H^5(G_2/P_i, \Omega_{G_2/P_i}(−5)) = H^5(G_2/P_i, E(−5)) \cong \mathbb{C}^{21}$$

and the other cohomologies vanish:

$$H^q(Z, \Omega_{G_2/P_i}|_Z) = 0 \quad \text{for } q = 0, 2, 3.$$

Here, we explain how to get $H^5(G_2/P_i, E(−5)) \cong \mathbb{C}^{21}$ more precisely. Tensoring the first exact sequence in (4.11) with $\mathcal{F}^* = \mathcal{O}(−5)$, we obtain the exact sequence $0 \rightarrow \mathcal{E}_{−25}, (−8) \rightarrow E(−5) \rightarrow \mathcal{O}(−6) \rightarrow 0$. This induces

$$0 \rightarrow H^5(G_2/P_i, \mathcal{E}_{−25}, (−8)) \rightarrow H^5(G_2/P_i, E(−5)) \rightarrow H^5(G_2/P_i, \mathcal{O}(−6)) \rightarrow 0,$$

and we know $H^5(G_2/P_i, \mathcal{E}_{−25}, (−8)) = V_{G_2}(\varpi_1)^* \cong \mathbb{C}^7, H^5(G_2/P_i, \mathcal{O}(−6)) = V_{G_2}(\varpi_2)^* \cong \mathbb{C}^{14}$. This provides $H^5(G_2/P_i, E(−5)) \cong \mathbb{C}^{21}$.

Now the conormal sequence $0 \rightarrow \mathcal{O}(−5)|_Z \rightarrow \Omega_{G_2/P_i}|_Z \rightarrow \mathcal{O}_Z \rightarrow 0$ and (4.15) lead to the following long exact sequences

$$0 \rightarrow H^0(Z, \Omega_Z) \rightarrow H^1(Z, \mathcal{O}−5)|_Z) \rightarrow H^1(Z, \Omega_{G_2/P_i}|_Z) \rightarrow H^1(Z, \Omega_Z) \rightarrow H^2(Z, \mathcal{O}−5)|_Z) \rightarrow 0;$$

$$0 \rightarrow H^2(Z, \Omega_Z) \rightarrow H^3(Z, \mathcal{O}−5)|_Z) \rightarrow H^3(Z, \Omega_{G_2/P_i}|_Z) \rightarrow H^3(Z, \Omega_Z) \rightarrow H^4(Z, \mathcal{O}−5)|_Z) \rightarrow 0.$$

From $H^4(Z, \Omega_Z) = 0$, we get $H^5(Z, \mathcal{O}−5)|_Z) = 0$ and $H^3(Z, \Omega_Z) \cong H^4(Z, \mathcal{O}−5)|_Z) / H^4(Z, \Omega_{G_2/P_i}|_Z)$. Tensoring the Koszul complex (4.12) with $\mathcal{O}−5)\), we have the exact sequence

$$0 \rightarrow \mathcal{O}(−10) \rightarrow \mathcal{O}−5) \rightarrow \mathcal{O}−5)|_Z \rightarrow 0.$$

Since $\text{RegInd}(\mathcal{O}−10)) = \text{RegInd}(\mathcal{O}−5)) = \{5\}$, the computations $H^5(G_2/P_i, \mathcal{O}−10)) = V_{G_2}(5\varpi_1)^* \cong \mathbb{C}^{378}$ and $H^5(G_2/P_i, \mathcal{O}−5)) \cong \mathbb{C}^{377}$ imply

$$H^4(Z, \mathcal{O}−5)|_Z) \cong H^5(G_2/P_i, \mathcal{O}−10) / H^5(G_2/P_i, \mathcal{O}−5)) \cong \mathbb{C}^{377};$$
(4.20) \[ H^q(Z, O(5)|_Z) = 0 \quad \text{for } q \neq 4. \]

By (4.13) and (4.20), the exact sequences (4.10) and (4.17) provide:

\[ H^1(Z, \Omega_Z) \cong H^1(Z, \Omega_{G_2/P_2}|_Z) \cong \mathbb{C}; \quad H^2(Z, \Omega_Z) \cong H^3(Z, O(5)|_Z) = 0. \]

Moreover, the exact sequence (4.18) becomes:

\[ 0 \to H^3(Z, \Omega_Z) \to H^4(Z, O(5)|_Z) \to H^4(Z, \Omega_{G_2/P_2}|_Z) \to H^5(Z, \Omega_Z) \to H^5(Z, O(5)|_Z) \to 0 \]

| \[ \mathbb{C}^{377} \] | \[ \mathbb{C}^{21} \] |
|---|---|
| 0 | 0 |

Here, the isomorphisms come from (4.14) and (4.19). Accordingly, we get:

\[ h^{1,1}(Z) = \dim H^1(Z, \Omega_Z) = 1, \]

\[ h^{1,2}(Z) = \dim H^2(Z, \Omega_Z) = 0, \]

\[ h^{1,3}(Z) = \dim H^3(Z, \Omega_Z) = \dim H^4(Z, O(-5)|_Z) - \dim H^4(Z, \Omega_{G_2/P_2}|_Z) = 377 - 21 = 356. \]

In the same way as above, we obtain the Hodge number \( h^{1,3}(Z) \) of the other Calabi–Yau fourfolds as in Table 1.

**Proposition 4.6.** Let \( G \) be a simple Lie group of exceptional type. Let \( G/P \) be a rational homogeneous variety of Picard number one, and \( F \) be a completely reducible, globally generated, equivariant vector bundle over \( G/P \). If the zero locus \( Z \) of a general global section of \( F \) is a fourfold with trivial canonical bundle, then \( h^{1,0}(Z) = h^{1,2}(Z) = 0, h^{1,1}(Z) = 1, \) and \( h^{1,3}(Z) \) are given as in Table 1.

**Remark 4.7.** By the Hodge decomposition theorem and Proposition 4.3, the result \( h^{1,1}(Z) = 1 \) yields \( H^2(Z, \mathbb{C}) = H^{1,1}(Z) \cong \mathbb{C} \); hence \( Z \) has Picard number one and \( H^2(Z, \mathbb{C}) \) is spanned by the restriction of a Kähler form on \( G/P \).

In the same way as Section 3.4 we classify Calabi–Yau threefolds which are zero loci of general global sections of completely reducible equivariant vector bundles over exceptional homogeneous varieties of Picard number one. The Hodge numbers and the Euler–Poincaré characteristic \( \chi \) of such Calabi–Yau 3-folds are listed in Table 2.

### 4.3. Computations of \( h^{2,2}(Z) \)

In this section, we introduce a way to compute the remaining Hodge number \( h^{2,2}(Z) \). In order to compute the Hodge number \( h^{2,2}(Z) \), we may employ the exact sequence

\[ 0 \to S^2 F^*|_Z \to (F^* \otimes \Omega_{G/P})|_Z \to \Omega^2_{G/P}|_Z \to \Omega^2_Z \to 0 \]

obtained from the second exterior power of the conormal sequence; see, for example, [FM21, Section 3.9.1]. After determining the irreducible decomposition of each of \( S^2 F^* \otimes \wedge^i F^* \), \( (F^* \otimes \Omega_{G/P}) \otimes \wedge^i F^* \), and \( \Omega^2_{G/P} \otimes \wedge^i F^* \), several and rather long applications of the Borel–Weil–Bott theorem provide the Hodge number \( h^{2,2}(Z) \). We examine the above method in the following example. The interesting reader can use the same techniques for the remaining examples in Table 1.

**Example 4.8 (No. 9 in Table 1).** Let \( Z \) be the zero locus of a general global section of the line bundle \( F = O(3) \) over \( G_2/P_2 \). In this case, the sequence (4.21) becomes

\[ 0 \to O(-6)|_Z \to \Omega^2_{G_2/P_2}(-3)|_Z \to \Omega^2_{G_2/P_2}|_Z \to \Omega^2_Z \to 0. \]

We compute \( H^q(Z, O(-6)|_Z) \), \( H^q(Z, \Omega_{G_2/P_2}(-3)|_Z) \), and \( H^q(Z, \Omega^2_{G_2/P_2}|_Z) \) by considering the Koszul complex (4.3).

**Step 1.** We first consider \( H^q(Z, O(-6)|_Z) \). To compute the cohomology, tensoring the Koszul complex (4.4) with \( O(-6) \) yields

\[ 0 \to O(-9) \to O(-6) \to O(-6)|_Z \to 0. \]

Since \( \text{RegInd}(O(-6)) = \text{RegInd}(O(-9)) = \{5\} \), using the Borel–Weil–Bott theorem, we obtain:

\[ H^q(G_2/P_2, O(-6)) \cong \mathbb{C}^{273}; \quad H^q(G_2/P_2, O(-6)) = 0 \quad \text{for } q \neq 5, \]

\[ H^5(G_2/P_2, O(-9)) \cong \mathbb{C}^{3542}; \quad H^q(G_2/P_2, O(-9)) = 0 \quad \text{for } q \neq 5. \]
Accordingly, we obtain
\begin{align*}
H^4(Z, \mathcal{O}(-6)|_Z) &\cong H^5(G_2/P_2, \mathcal{O}(-9))/H^5(G_2/P_2, \mathcal{O}(-6)) \cong \mathbb{C}^{3269}, \\
H^q(Z, \mathcal{O}(-6)|_Z) &= 0 \quad \text{for } q \neq 4.
\end{align*}

**Step 2.** We consider $H^q(Z, \Omega_{G_2/P_2}(-3)|_Z)$. Tensoring the Koszul complex (4.3) with $\Omega_{G_2/P_2}(-3)$, we obtain
\begin{equation}
0 \to \Omega_{G_2/P_2}(-6) \to \Omega_{G_2/P_2}(-3) \to \Omega_{G_2/P_2}(-3)|_Z \to 0.
\end{equation}
Tensoring the exact sequence (4.3) with $\mathcal{O}(-3)$, we get
\begin{equation}
0 \to \mathcal{E}_{-3\pi_2} \to \Omega_{G_2/P_2}(-3) \to \mathcal{E}_{3\pi_1-5\pi_2} \to 0.
\end{equation}
Since the weight $3\pi_1 - 5\pi_2$ is singular and $\text{RegInd}(\mathcal{E}_{-4\pi_2}) = \{5\}$, applying the Borel–Weil–Bott theorem, all cohomologies $H^q(G_2/P_2, \mathcal{E}_{3\pi_1-5\pi_2})$ vanish and $H^5(G_2/P_2, \mathcal{E}_{-4\pi_2}) \cong \mathbb{C}^{14}$. Accordingly,
\begin{align*}
H^5(G_2/P_2, \Omega_{G_2/P_2}(-3)) &\cong H^5(G_2/P_2, \mathcal{E}_{-4\pi_2}) \cong \mathbb{C}^{14}; \\
H^8(G_2/P_2, \Omega_{G_2/P_2}(-3)) &= 0 \quad \text{for } q \neq 5.
\end{align*}
Similarly, tensoring the exact sequence (4.3) with $\mathcal{O}(-6)$, we get
\begin{equation}
0 \to \mathcal{E}_{-7\pi_2} \to \Omega_{G_2/P_2}(-6) \to \mathcal{E}_{3\pi_1-8\pi_2} \to 0.
\end{equation}
By applying the Borel–Weil–Bott theorem again, we obtain
\begin{align*}
H^4(G_2/P_2, \Omega_{G_2/P_2}(-6)) &\cong H^5(G_2/P_2, \mathcal{E}_{-7\pi_2}) \oplus H^5(G_2/P_2, \mathcal{E}_{3\pi_1-8\pi_2}) \cong \mathbb{C}^{748} \oplus \mathbb{C}^{1547} \cong \mathbb{C}^{2295}; \\
H^8(G_2/P_2, \Omega_{G_2/P_2}(-6)) &= 0 \quad \text{for } q \neq 5.
\end{align*}
Using the computations (4.25) and (4.26), from the exact sequence (4.24) we have
\begin{equation}
0 \to H^4(Z, \Omega_{G_2/P_2}(-3)|_Z) \to H^5(G_2/P_2, \Omega_{G_2/P_2}(-6)) \to H^5(G_2/P_2, \Omega_{G_2/P_2}(-3)) \to 0
\end{equation}
\[\cong \mathbb{C}^{2295} \oplus \mathbb{C}^{14} \quad \text{for } q \neq 4.
\]
Accordingly, we obtain
\begin{equation}
H^4(Z, \Omega_{G_2/P_2}(-3)|_Z) \cong \mathbb{C}^{2281}; \quad H^8(Z, \Omega_{G_2/P_2}(-3)|_Z) = 0 \quad \text{for } q \neq 4.
\end{equation}

**Step 3.** We consider $H^q(Z, \Omega^2_{G_2/P_2}|_Z)$. Tensoring the Koszul complex (4.3) with $\Omega^2_{G_2/P_2}$, we obtain
\begin{equation}
0 \to \Omega^2_{G_2/P_2}(-3) \to \Omega^2_{G_2/P_2} \to \Omega^2_{G_2/P_2}|_Z \to 0.
\end{equation}
Using $\Omega^2_{G_2/P_2} = \wedge^2 \Omega_{G_2/P_2} = G_2 \times P_2 \wedge^2 T_o(G_2/P_2)^*$ and the description of $T_o(G_2/P_2)$ in (4.2), we get a short exact sequence
\begin{equation}
0 \to \mathcal{E}_{3\pi_1-3\pi_2} \to \Omega^2_{G_2/P_2} \to \mathcal{E}_{-\pi_2} \oplus \mathcal{E}_{4\pi_1-3\pi_2} \to 0.
\end{equation}
By applying the Borel–Weil–Bott theorem on vector bundles $\mathcal{E}_{3\pi_1-3\pi_2}$, $\mathcal{E}_{-\pi_2}$, and $\mathcal{E}_{4\pi_1-3\pi_2}$, we obtain
\begin{align*}
H^q(G_2/P_2, \mathcal{E}_{3\pi_1-3\pi_2}) &= 0 \quad \text{for all } q, \\
H^q(G_2/P_2, \mathcal{E}_{-\pi_2} \oplus \mathcal{E}_{4\pi_1-3\pi_2}) &\cong \mathbb{C}; \\
H^q(G_2/P_2, \mathcal{E}_{-\pi_2} \oplus \mathcal{E}_{4\pi_1-3\pi_2}) &= 0 \quad \text{for } q \neq 2.
\end{align*}
Accordingly, we have
\begin{equation}
H^2(G_2/P_2, \Omega^2_{G_2/P_2}) \cong \mathbb{C}; \quad H^q(G_2/P_2, \Omega^2_{G_2/P_2}) = 0 \quad \text{for } q \neq 2.
\end{equation}
On the other hand, tensoring the exact sequence (4.29) with $\mathcal{O}(-3)$, we obtain
\begin{equation}
0 \to \mathcal{E}_{3\pi_1-6\pi_2} \to \Omega^2_{G_2/P_2}(-3) \to \mathcal{E}_{-4\pi_2} \oplus \mathcal{E}_{4\pi_1-6\pi_2} \to 0.
\end{equation}
Using the Borel–Weil–Bott theorem, we get
\begin{align*}
H^5(G_2/P_2, \mathcal{E}_{3\pi_1-6\pi_2}) &\cong \mathbb{C}^{77}; \quad H^9(G_2/P_2, \mathcal{E}_{3\pi_1-6\pi_2}) = 0 \quad \text{for } q \neq 5, \\
H^5(G_2/P_2, \mathcal{E}_{-4\pi_2} \oplus \mathcal{E}_{4\pi_1-6\pi_2}) &\cong \mathbb{C}^{14}; \quad H^9(G_2/P_2, \mathcal{E}_{-4\pi_2} \oplus \mathcal{E}_{4\pi_1-6\pi_2}) = 0 \quad \text{for } q \neq 5.
\end{align*}
Therefore, we have
\begin{align*}
H^5(G_2/P_2, \Omega^2_{G_2/P_2}(-3)) &= \mathbb{C}^{91}; \quad H^8(G_2/P_2, \Omega^2_{G_2/P_2}(-3)) = 0 \quad \text{for } q \neq 5.
\end{align*}
Using the computations (4.30) and (4.31), from the exact sequence (4.28) we obtain

\begin{equation}
H^q(Z, \Omega^2_{G_2/P_2}|_Z) \cong \begin{cases} 
\mathbb{C} & \text{if } q = 2, \\
\mathbb{C}^{91} & \text{if } q = 4, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

From the sequence (4.22), we obtain two short exact sequences

\begin{align}
0 & \to \mathcal{O}(-6)|_Z \to \Omega_{G_2/P_2}(-3)|_Z \to K \to 0, \\
0 & \to K \to \Omega^2_{G_2/P_2}|_Z \to \Omega^2_Z \to 0.
\end{align}

We note that \( h^{1,2}(Z) = h^{0,2}(Z) = 0 \) holds by Propositions 4.1 and 4.6. Using the Hodge symmetry \( h^{p,q}(Z) = h^{q,p}(Z) \) and the Serre duality \( h^{p,q}(Z) = h^{4-p,4-q}(Z) \), we obtain

\[ h^{2,3}(Z) = h^{3,2}(Z) = h^{1,2}(Z) = 0, \quad h^{2,4}(Z) = h^{4,2}(Z) = h^{0,2}(Z) = 0. \]

Accordingly, \( H^3(Z, \Omega^2_Z) = H^4(Z, \Omega^2_Z) = 0 \) and the sequence (4.31) provides

\begin{equation}
H^4(Z, K) \cong H^4(Z, \Omega^2_{G_2/P_2}|_Z) \cong \mathbb{C}^{91}.
\end{equation}

Here, the last isomorphism comes from (4.32). On the other hand, because \( H^q(Z, \mathcal{O}(-6)|_Z) = H^q(Z, \Omega_{G_2/P_2}(-3)|_Z) = 0 \) for \( q \neq 4 \) by (4.28) and (4.21), the sequence (4.33) provides \( H^q(Z, K) = 0 \) for \( q \neq 3, 4 \) and

\[ 0 \to H^3(Z, K) \to H^4(Z, \mathcal{O}(-6)|_Z) \to H^4(Z, \Omega_{G_2/P_2}(-3)|_Z) \to H^4(Z, K) \to 0. \]

\[ \mathbb{C}^{3269} \to \mathbb{C}^{2281} \to \mathbb{C}^{1079}. \]

Here, the isomorphisms come from (4.28), (4.21), and (4.35), respectively. Therefore, we obtain

\[ H^3(Z, K) \cong \mathbb{C}^{1079}. \]

From the sequence (4.31), we have

\[ 0 \to H^2(Z, \Omega^2_{G_2/P_2}|_Z) \to H^2(Z, \Omega^2_Z) \to H^3(Z, K) \to 0. \]

\[ \mathbb{C} \to \mathbb{C}^{1079}. \]

Therefore, we conclude \( h^{2,2}(Z) = 1079 + 1 = 1080. \)

We obtain the Hodge diamond as follows:

\[
\begin{array}{cccccccc}
& & & & & & & 1 \\
& & & & & h^{0,0} & & \\
& & & & h^{1,0} & & h^{0,1} & 0 0 \\
& & & h^{2,0} & & h^{1,1} & h^{0,2} & 0 1 0 \\
& & h^{3,0} & & h^{2,1} & & h^{1,2} & h^{0,3} & 0 0 0 0 \\
h^{4,0} & h^{3,1} & h^{2,2} & h^{1,3} & h^{0,4} & 1 258 1080 258 1 \\
h^{4,1} & h^{3,2} & h^{2,3} & h^{1,4} & 0 0 0 0 \\
h^{4,2} & h^{3,3} & h^{2,4} & 0 1 0 \\
h^{4,3} & h^{3,4} & 0 0 \\
h^{4,4} & 1
\end{array}
\]

The Euler–Poincaré characteristic \( \chi(Z) \) is

\[ \chi(Z) = 1 + 1 + (1 + 258 + 1080 + 258 + 1) + 1 + 1 = 1602. \]

**Remark 4.9.** In [GHL14], the Hodge numbers are computed for complete intersection Calabi–Yau fourfolds. On the other hand, as is mentioned in Remark 5.7, the fourfold \( Z \) given in No. 8 is a complete intersection Calabi–Yau fourfold in \( \mathbb{P}^9 \). Following the computational result in [GHL14] (indeed, \( Z \) corresponds to ‘MATRIX NUMBER : 2’ in the file), we obtain \( h^{2,2}(Z) = 1472 \), and moreover, the Euler–Poincaré characteristic \( \chi(Z) \) is 2190.
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