PALEY GRAPHS AND SÁRKÖZY'S THEOREM IN FUNCTION FIELDS

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Abstract. Sárközy’s theorem states that dense sets of integers must contain two elements whose difference is a $k^{th}$ power. Following the polynomial method breakthrough of Croot, Lev, and Pach [3], Green proved a strong quantitative version of this result for $\mathbb{F}_q[T]$. In this paper we provide a lower bound for Sárközy’s theorem in function fields by adapting Ruzsa’s construction [16] for the analogous problem in $\mathbb{Z}$. We construct a set $A$ of polynomials of degree $< n$ such that $A$ does not contain a $k^{th}$ power difference with $|A| = q^{n-n/2k}$.

Additionally, we prove a handful of results concerning the independence number of generalized Paley Graphs, including a generalization of a claim of Ruzsa, which helps with understanding the limit of the method.

1. Introduction

In 1978 Sárközy proved that if $A \subset \mathbb{Z}$ is a dense set of integers then it contains a $k^{th}$-power difference for any $k \geq 2$ [17]. For a set $A \subset \{1, \ldots, N\}$ that avoids $k^{th}$-power differences the best quantitative upper bound, due to Pintz, Steiger, and Szemerédi [14], takes the form

$$|A| \leq N^{1-o(1)}$$

and the best lower bound, due to Ruzsa [16] (improved in some cases by Lewko [9]), is of the form $N^{1-c}$ where $c \leq \frac{1-\delta}{k}$ for some small fixed $\delta > 0$. Following recent advances in the polynomial method [3, 4, 18], Green [6] gave a strong quantitative result for the function field analogue of Sárközy’s theorem. Let $P_{q,n}$ denote the space of polynomials in the variable $T$ over $\mathbb{F}_q$ of degree $< n$. For $k \geq 2$, Green proved that any subset $A \subset P_{q,n}$ that does not contain two distinct polynomials $u(T), v(T)$ such that $u(T) - v(T) = b(T)^k$ for some $b \in \mathbb{F}_q[T]$ has size at most

$$|A| \leq q^{n(1-c_k)}$$

where

$$c_k = \frac{1}{2k^2 D_q(k)^2 \log q}$$

and $D_q(k)$ denotes the sum of digits of $k$ in base $q$. In this paper, by adapting Ruzsa’s construction in $\mathbb{Z}$, we prove the following lower bound for the function field setting:

**Theorem 1.** Let $k \geq 2$, and suppose that $\gcd(k, q-1) > 1$. For $n \equiv 0 \pmod{2k}$ there exists a set $A \subset P_{q,n}$ of size

$$|A| = q^{n(1-\frac{k}{q})}$$

that does not contain a $k^{th}$ power difference.

The construction results in a bound for all $n$, since $P_{q,n} \subset P_{q,n+1}$. In section 2 we prove a handful of results concerning the independence number of generalized Paley graphs, including a generalization of a claim of Ruzsa. We prove a basic lower bound for the independence number of a product, which is ingredient in the proof of Theorem 1 and we also prove upper bounds to help understand the limits of the method. For $k \geq 3$, Theorem 1 could potentially be improved by finding larger independent sets in products of generalized Paley graphs. In subsection 3.2 we give a case with an improved bound, and discuss the limit of the method. When $k = 2$, we conjecture that the lower bound achieved, $q^{2n}$, is optimal.

Remarks. For $\gcd(k, q-1) = 1$, every element of $\mathbb{F}_q$ is a $k^{th}$ power, and it is not always possible to obtain a lower bound as strong as Theorem 1. For $k = q^r$, a pigeonhole argument achieves the upper bound

$$|A| \leq q^{n-n/q^r}.$$
For any $k$, there are at $q^{1+\lfloor \frac{n-1}{k} \rfloor}$ distinct $k^{th}$ powers with degree $\leq n - 1$, and so the greedy construction yields a set $A$ with no $k^{th}$ power differences of size

$$|A| = q^{n-1-\lfloor \frac{n-1}{k} \rfloor},$$

and hence in the case of $k = q^r$, this upper bound is tight within a factor of $q$.

1.1. **Graph Theoretic Notation.** We will use the following graph theoretic notation throughout the paper: For a directed graph $G = (V,E)$, let $\alpha(G)$ denote the size of the largest independent set, that is, the largest set such that no two vertices are connected by a directed edge

$$\alpha(G) = \max_{W \subseteq V} \{|W| : \forall x,y \in W, (x,y) \notin E\}.$$  

We let $\omega(G)$ denote the clique number, which equals $\alpha(G)$ where $\overline{G}$ is the complement graph. Here the complement graph is $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{(x,y) | (x,y) \notin V^2, x \neq y, (x,y) \notin E\}$.

The Strong Graph Product: Given two graphs $G = (V,E)$, $G = (V',E')$, the strong graph product of $G$ and $G'$ is the graph $G \boxtimes G'$ with vertex set $V \times V'$, and edge set defined by connecting $(v,v')$, $(u,u') \in V \times V'$ if $(v,u) \in E$ and $(v',u') \in E'$, or $(v,u) \in E$ and $v' = u'$, or $v = u$ and $(v',u') \in E'$. The strong graph product as defined applies to both directed and undirected graphs.

**Shannon Capacity:** For a graph $G$ and $n \geq 2$, we let $G^\otimes n$ denote the $n$-fold strong graph product of $G$ with itself. The Shannon Capacity of an undirected graph $G$ is defined to be

$$\Theta(G) = \limsup_{n \rightarrow \infty} \left( \alpha(G^\otimes n) \right)^\frac{1}{n}.$$  

**Lovász Theta Function:** For an undirected graph $G$, the Lovász Theta Function, $\vartheta(G)$, is a minimization over orthonormal representations of $G$, see [11] for a precise definition. In particular, for undirected $G, H$, $\vartheta$ satisfies

$$(1.2) \quad \Theta(G) \leq \vartheta(G),$$

$$(1.3) \quad \vartheta(G \boxtimes H) = \vartheta(G) \vartheta(H),$$

and if $G$ is vertex transitive, then

$$(1.4) \quad \vartheta(G) \vartheta(\overline{G}) = |G|.$$  

2. **Generalized Paley Graphs**

Generalized Paley Graphs were introduced by Cohen [2] and reintroduced by Lim and Praeger [10]. We expand their definition slightly to allow for the vertex set to be a ring rather than just a field, as this will be relevant later (see Theorem 9).

**Definition 3.** For a finite commutative ring $R$, let $\text{Paley}_k(R)$ be the (possibly directed) graph with vertex set $V = R$, and edge set

$$E = \{(x,y) : x - y = z^k \text{ for some } z \in R\}.$$  

With this notation, for $q \equiv 1 \pmod{4}$, $\text{Paley}_2(\mathbb{F}_q)$ is the usual Paley graph. The graph $\text{Paley}_k(\mathbb{F}_q)$, where two elements are connected if they differ by a $k^{th}$ power, is undirected if and only if $\frac{q-1}{\gcd(q-1,k)}$ is even, since in this case $-1$ is a $k^{th}$ power. If $\gcd(k,q-1) = 1$, then every element of $\mathbb{F}_q$ is a $k^{th}$ power, and so $\text{Paley}_k(\mathbb{F}_q)$ is complete graph on $q$ vertices, and if $\gcd(k,q-1) = d < k$, then replacing $k$ with $d$ does not change the graph. The assumption $q \equiv 1 \pmod{2k}$ is often used as it assures both that the graph is undirected and that $k^{th}$ powers are indeed relevant.

Ruzsa [16] gave a lower bound for Sárközy’s theorem in $\mathbb{Z}$ based on a maximization involving the quantity

$$\alpha \left( \text{Paley}_k(\mathbb{Z}/m\mathbb{Z}) \right)$$

for squarefree $m$. In Section 3 we give a lower bound for the function field case in terms of the independence number of a product of this graph. Our results are most naturally stated in terms of the independence number of products of generalized Paley graphs, and so we define
(2.1) \[ r_{k,n}(R) = \begin{cases} \alpha(Paley_k(R)) & \text{when } n = 1 \\
\alpha(Paley_k(R)^{\otimes n}) & \text{when } n \geq 2 \end{cases} \]

For products of rings, these graphs can be factored. For composite \(m\), we have the following lemma:

**Lemma 4.** Let \(n, m > 1\) be relatively prime. Then

\[ \operatorname{Paley}_k(\mathbb{Z}/mn\mathbb{Z}) = \operatorname{Paley}_k(\mathbb{Z}/m\mathbb{Z}) \boxtimes \operatorname{Paley}_k(\mathbb{Z}/n\mathbb{Z}). \]

**Proof.** This follows from the Chinese Remainder Theorem and the fact that an element is a \(k^n\)-power in \(\mathbb{Z}/mn\mathbb{Z}\) if and only if it maps to a \(k^n\)-power in both \(\mathbb{Z}/m\mathbb{Z}\) and \(\mathbb{Z}/n\mathbb{Z}\). \(\square\)

### 2.1. The Independence Number

For \(R = \mathbb{F}_q\), the clique number of the generalized Payley graph is a well studied quantity. See Yip’s Masters Thesis [19, 20] for a discussion of lower and upper bounds. In particular, when the graph is undirected, the best lower bound is due to Cohen [2]

\[ \frac{p}{(p-1)\log d} \left( \frac{1}{2} \log q - 2\log\log q \right) - 1 \leq \omega(Paley_k(\mathbb{F}_q)) = \alpha(Paley_k(\mathbb{F}_q)). \]

When \(k = 2\), the graph \(\operatorname{Paley}_k(\mathbb{F}_q)\) is self-complementary, and so \(\omega(Paley_2(\mathbb{F}_q)) = \alpha(Paley_2(\mathbb{F}_q))\) but for \(k > 2\), \(\operatorname{Paley}_k(\mathbb{F}_q)\) is isomorphic to a subgraph of \(\operatorname{Paley}_k(\mathbb{F}_q)\), and hence

\[ \omega(Paley_k(\mathbb{F}_q)) = \alpha(Paley_k(\mathbb{F}_q)) \leq \alpha(Paley_k(\mathbb{F}_q)), \]

that is the lower bounds for the clique number imply lower bounds for the independence number. In some unique cases, such as when \(q = p^s\) and \(k|\frac{q-1}{p-1}\) for some \(r\)'s, there are significantly stronger lower bounds for the size of the clique in \(\operatorname{Paley}_k(\mathbb{F}_q)\), see [1, 19] for more details. The upper bound for the clique number is \(\sqrt{q}\), and for \(k = 2\) this was recently improved in [8].

The case for the independence number is different however, as in some cases it grows above \(\sqrt{q}\) for larger \(k\). In the first non-trivial case, when \(k = 3\) and \(q = 7\), the Paley Graph is precisely the 7-cycle, and we have an independent set of size \(3 = \varpi_{645}^\prime\). Indeed, if \(q = 1 + 2k\), then the only \(k^n\)-powers in \(\mathbb{F}_q\) are \(-1, 0, 1\), and hence \(\operatorname{Paley}_k(\mathbb{F}_q)\) is isomorphic to \(C_{2k+1}\), the \((2k+1)\)-cycle. This contains an independent set of size \(k\), and so there are infinitely many graphs satisfying

\[ q^{1 - \frac{\log(2)}{\log(2k+1)}} \leq \alpha(\operatorname{Paley}_k(\mathbb{F}_q)). \]

The following theorem gives a basic upper bound for the independence number of these graphs.

**Theorem 5.** Suppose that \(-1\) is a \(k^n\)-power in \(\mathbb{F}_q\). Then we have that

\[ \vartheta(\operatorname{Paley}_k(\mathbb{F}_q)) \leq q^{1 - \frac{k}{k+1}} \]

where \(\vartheta\) is the Lovász Theta Function.

In particular, by [1, 2] and the definition of \(\Theta\) as a limsup, it follows that

\[ r_{k,2}(\mathbb{F}_q) \leq q^{2(1 - \frac{1}{k+1})}. \]

To prove this theorem, we need to prove the following fact about the complement graph:

**Lemma 6.** We have that

\[ \alpha(\overline{\operatorname{Paley}_k(\mathbb{F}_q)}) \geq q, \]

and hence when the graph is undirected

\[ \vartheta(\overline{\operatorname{Paley}_k(\mathbb{F}_q)}) \geq q^{\frac{k}{k+1}}. \]

**Proof.** If \(\gcd(k, q-1) = 1\), then \(\overline{\operatorname{Paley}_k(\mathbb{F}_q)}\) is the totally isolated graph, the complement of the complete graph, and so the result holds trivially. Assume that \(\gcd(k, q-1) > 1\), and let \(\beta \in \mathbb{F}_q\) be a cyclic generator of \(\mathbb{F}_q^\ast\). Consider the set

\[ A = \{ (x, \beta x, \beta^2 x, \ldots, \beta^{k-1} x) | x \in \mathbb{F}_q \}. \]
Let $x, y \in \mathbb{F}_q$ be two elements with $x - y \neq 0$, and write $(x - y) = \beta^a$ for some $a$. Then $(x - y)^{\beta^j}$ will be a $k^{th}$
power for the value of $j \in \{0, 1, \ldots, k-1\}$ satisfying $j \equiv -a \pmod{k}$. This proves that $A$ is an independent
set. Equation (2.7) then follows from (2.5) since the Lovász Theta Function upper bounds the size of the
largest independent set.

\[ \vartheta(Paley_k(F_q)) \leq q, \]

Equation (2.5) then follows from this and the lower bound in Lemma 6.

The following proposition gives a lower bound for the independence number of a product, which will be
used in the next section in the proof of Theorem 1.

**Proposition 7.** For $\gcd(k,q-1) > 1$ we have that $r_{k,2}(F_q) \geq q$. In particular, when $k = 2$ and $q$ is odd,
$r_{2,2}(\mathbb{F}_q) = q$.

**Proof.** Since $\gcd(k,q-1) > 1$, there exists $\beta \in \mathbb{F}_q$ that is not a $k^{th}$ power. Then $A = \{(x, \beta x) \mid x \in \mathbb{F}_q\}$ is an
independent set in $\text{Paley}_k(F_q) \boxtimes \text{Paley}_k(F_q)$, since for any distinct $x, y \in \mathbb{F}_q$, only one of $x - y$ and $\beta(x - y)$
can be a $k^{th}$ power. The final statement follows from the upper bound (2.6).

Proposition 2.7 and Theorem 5 together imply that

\[ q^{\frac{1}{k}} \leq \Theta(\text{Paley}_k(F_q)) \leq q^{1 - \frac{1}{k}}, \]

and

\[ q^{\frac{1}{k}} \leq \Theta(\text{Paley}_k(F_q)) \leq q^{\frac{1}{k}}. \]

For $k \geq 3$, for prime fields, we believe that neither of these inequalities are sharp.

**Conjecture 8.** For $k \geq 3$, there exists $a_k, b_k > 0$ such that for any prime $p \equiv 1 \pmod{k}$.

\[ p^{\frac{1}{k} + a_k} \leq \Theta(\text{Paley}_k(F_p)) \leq p^{1 - \frac{1}{k} - b_k}. \]

### 2.2. Composite m.

We conclude this section with a generalization of a result of Ruzsa from 10. For $k = 2$, the following Theorem was proven by Ruzsa, but the proof was not published.

**Theorem 9.** Let $m > 1$ be squarefree, let $k = d2^s$ where $d$ is odd, and suppose that each prime dividing $m$
is of the form $p \equiv 1 \pmod{2^{s+1}}$. Then if $A \subset \mathbb{Z}/m\mathbb{Z}$ does not contain two elements whose difference is a $k^{th}$ power we have

\[ |A| < m^{1 - \frac{1}{k}}. \]

**Proof.** We will prove that

\[ \vartheta(\text{Paley}_k(\mathbb{Z}/m\mathbb{Z})) < m^{1 - \frac{1}{k}}, \]

which implies the result since $\alpha(G) \leq \vartheta(G)$ for any $G$. Let $m = p_1 \cdots p_r$. Then by Lemma 1

\[ \text{Paley}_k(\mathbb{Z}/m\mathbb{Z}) = \text{Paley}_k(\mathbb{Z}/p_1\mathbb{Z}) \boxtimes \cdots \boxtimes \text{Paley}_k(\mathbb{Z}/p_r\mathbb{Z}). \]

The condition $p_i \equiv 1 \pmod{2^{s+1}}$ for each $p_i|m$ guarantees that $\frac{p_i - 1}{\gcd(p_i-1,k)}$ will be even, and hence that $-1$ is a $k^{th}$-power in $\mathbb{Z}/p_i\mathbb{Z}$, and so these graphs are undirected. The multiplicative property of the Lovász Theta Function 11 Lemma 2] implies that

\[ \vartheta(\text{Paley}_k(\mathbb{Z}/m\mathbb{Z})) \leq \prod_{i=1}^r \vartheta(\text{Paley}_k(\mathbb{Z}/p_i\mathbb{Z})), \]

and hence by equation (2.5)

\[ \vartheta(\text{Paley}_k(\mathbb{Z}/m\mathbb{Z})) \leq \prod_{i=1}^r p_i^{1 - \frac{1}{k}} = m^{1 - \frac{1}{k}}. \]

The inequality can be made strict since the left hand side is an integer, but the right hand side is not.
When $k$ is odd, then the bound holds for any odd squarefree integer $m$. For $k = 3$, Matolcsi and Ruzsa recently gave the superior upper bound $O_n(m^{\frac{3}{2}+\epsilon})$ for squarefree $m$ \cite{13} using methods from \cite{12}. In the case where $m$ is a product of primes that are not necessarily of the form $p \equiv 1 \pmod{2^{s+1}}$, it seems likely that the same upper bound holds, however for the $k = 2$ proving this seems more complex, see \cite{3} for more details.

Should such a result hold for all $m$, then Ruzsa’s lower bound construction in [1, $N$] for sets without a $k^{th}$ power difference cannot yield a set of size greater than $N^{1-1/k^2}$.

3. The Lower Bound Construction

To prove Theorem 1, we prove a general lower bound in terms of $r_{k,2}(\mathbb{F}_q)$, and then apply Proposition \[. We conclude this section with a discussion of the limits of this method.

3.1. Main Result.

**Theorem 10.** Suppose that $n \equiv 0 (2k)$ and let $F \in \mathbb{F}_q[T]$ be a polynomial of degree $k$. Then there exists a set $A \subset P_{q,n}$ of size

\[ |A| \geq (r_{k,1}(\mathbb{F}_q))^{\frac{1}{2}} T^{n(1-\frac{1}{k})} \]

that does not contain $p, p'$ such that $p' - p = F(u)$ for some $u \in \mathbb{F}_q[T]$. If $F(T) = b_k T^k$ for $b_k \neq 0$, then we have the improved bound

\[ |A| \geq (r_{k,2}(\mathbb{F}_q))^{\frac{1}{2}} T^{n(1-\frac{1}{k})}. \]

**Proof.** Suppose that $n \equiv 0 (2k)$, and let $S \subset \mathbb{F}_q$ be a maximal independent set in Paley$_k(\mathbb{F}_q)$ that contains 0, and let $b_k u^k$ be the first coefficient of $F(u)$. Let $A \subset P_{q,n}$ be the set of polynomials of the form

\[ c_0 + c_1 T + c_2 T^2 + \cdots + c_{n-2} T^{n-2} + c_{n-1} T^{n-1} \]

where

\[
\begin{cases} 
  c_i \in \mathbb{F}_q & \text{when } i \not\equiv 0 (k) \\
  c_i \in b_k S & \text{when } i \equiv 0 (k)
\end{cases}
\]

The first non-zero coefficient of the difference of two elements in $A$ will either be a power that is not divisible by $k$, or will equal $b_k (s-s') T^j k$ for some $j$ where $s, s' \in S$. For $u = c_j T^j + \cdots + c_0$, the first coefficient of $F(u)$ will equal $b_k (c_j T^j) k$, but since $s-s'$ is never a $k^{th}$ power by definition of $S$, this can never be of the form $b_k (s-s') T^j k$ for any $u \in P_{q,n}$. This proves \cite{3} since $|S| = r_{k,1}(\mathbb{F}_q)$. In the case where $F(u) = u^k$, we can improve the bound by making use of both the first and last coefficient. Let $U \subset \mathbb{F}_q \times \mathbb{F}_q$ be an independent set in Paley$_k(\mathbb{F}_q) \otimes$ Paley$_k(\mathbb{F}_q)$, and consider the set $A$ of polynomials of the form

\[ c_0 + c_1 T + c_2 T^2 + \cdots + c_{n-2} T^{n-2} + c_{n-1} T^{n-1} \]

where

\[
\begin{cases} 
  c_i \in \mathbb{F}_q & \text{when } i \not\equiv 0 (k) \\
  (c_i, c_{n-k-i}) \in U & \text{when } i \equiv 0 (k), \ i < \frac{n}{2}
\end{cases}
\]

Note that since $n \equiv 0 \pmod{2k}$, $T^{n-k}$ is the $k^{th}$-power with the largest degree in $P_{q,n}$. Suppose that $u$ is a difference of two elements of $A$, and write $u = \sum_{i=1}^{n-1} a_i T^i$ for coefficients $a_i$, some of which may equal 0. Let $j$ be the index of the non-zero coefficient $a_j$ whose degree is farthest from the middle, that is, let $j$ be such that $a_j \neq 0$, and $|j - \frac{n-k}{2}|$ is maximal. In the event of a tie between 2 non-zero coefficients, take $j > \frac{n-k}{2}$. If $j \not\equiv 0 \pmod{k}$, then $u$ cannot be a $k^{th}$-power. If $k|j$, consider the pair of coefficients $(a_j, a_{n-k-j})$, and assume without loss of generality that $j > \frac{n-k}{2}$. By definition of $j$, we have that

\[
  a_i = \begin{cases} 
    0 & \text{if } i > j \\
    0 & \text{if } i < n - k - j
  \end{cases}
\]

and so $a_j$ and $a_{n-k-j}$ must both be $k^{th}$-powers for $u$ to be a $k^{th}$-power. Note that 0 is a $k^{th}$-power, and $a_{n-k-j}$ could possibly equal 0. Since $a_j \neq 0$, and since $(a_j, a_{n-k-j}) \in U - U$, by definition of $U$ at least one
of \( a_j, a_{n-k-j} \) is not a \( k^{th} \)-power. This implies that \( u \) is not \( k^{th} \)-power, and hence \( A \) contains no \( k^{th} \)-power differences. Since \( |U| = r_{k,2}(\mathbb{F}_q) \), we have that
\[
|A| \geq (r_{k,2}(\mathbb{F}_q))^\frac{q}{m} q^n(1 - \frac{1}{2^k}).
\]
The result follows for \( F(T) = b_k T^k \) for \( b_k \neq 0 \) by multiplying the elements of \( U \) by \( b_k \) in the construction. \( \square \)

Equation (2.2), Cohen’s clique number lower bound, yields a nontrivial bound in \([3, 1]\) for sets avoiding a general polynomial \( F \). Theorem \( \ref{main} \) follows immediately from equation (2.2) and Proposition \( \ref{green} \).

**Remark 11.** Theorem \( \ref{uzma} \) is similar to Ruzsa’s lower bound in the integers. Let \( n = \log_m N \). Ruzsa proved that there exists \( A \subset [1, x] \) that contains no two elements whose difference is a \( k^{th} \) power satisfying
\[
|A| \geq \frac{1}{m} (r_{k,1}(\mathbb{Z}/m\mathbb{Z}))^\frac{1}{2^{k-1}} N(1 - \frac{1}{k}).
\]
(The quantitative result was improved by Lewko \([9]\) based on the fact that \( \log r_{k,1}(\mathbb{Z}/m\mathbb{Z})/\log m \) is larger for \( m = 205 \) than for \( m = 65 \). See also \([21]\) for an improved bound for a related problem). The bound for \( k^{th} \) powers in Theorem \( \ref{main} \) is similar, but utilizes the fact that there is no “overflow” when taking powers of a polynomial in \( \mathbb{F}_q \) so both the first and last coefficients play a role instead of only the last coefficient. This results in a better bound for function fields since we always have
\[
(r_{k,2}(\mathbb{F}_q))^\frac{q}{m} \geq r_k(\mathbb{F}_q).
\]

### 3.2. Limits of the Method

Theorem \( \ref{main} \) implies that one must use a different method to improve the lower bound in Theorem \( \ref{uzma} \) beyond
\[
|A| = q^n(1 - \frac{1}{2^k}).
\]
One can ask if the lower bound in proposition \( \ref{green} \) is optimal. This turns out not to be true in general for \( k \geq 3 \), as described in the comments preceding equation (2.3). When \( p = 2k + 1 \) is a prime, \( \text{Paley}_k(\mathbb{F}_p) \) is isomorphic to \( C_{2k+1}, \) the \( (2k+1) \)-cycle, and the largest independent set in \( C_{2k+1} \) has size \( k^2 + \lceil \frac{k}{2} \rceil \). [7, Theorem 7.1]. Hence, in this case
\[
r_{k,2}(\mathbb{F}_p) = k^2 + \left\lfloor \frac{k}{2} \right\rfloor.
\]
One can easily verify that in this case this construction results in a lower bound better than Theorem \( \ref{main} \) but weaker than the best possible from Equation (3.3). The following example helps illustrate the size of the gap between the upper and lower bounds with an explicit case.

**Example 12.** Consider the specific case of \( k = 3 \) and \( q = 7 \). The most precise upper bound obtained by Green’s method, where we calculate the value of the minimum instead of using a Chernoff bound, is
\[
2 \cdot \left( \min_{0 < t < 1} \frac{1 - t^2}{(1 - t)^{\frac{1}{6}}} \right)^n = 2 \cdot (6.903 \ldots)^n.
\]

Theorem \( \ref{uzma} \) gives the lower bound of \( 7^{\frac{n}{2}} = (5.061 \ldots)^n \). Since \( \text{Paley}_3(\mathbb{F}_7) \) is precisely \( C_7 \), the 7-cycle, using the fact that
\[
\alpha(C_7 \otimes C_7) = 10,
\]
Theorem \( \ref{uzma} \) yields the improved lower bound
\[
\left( 10^\frac{1}{2} 7^{\frac{1}{2}} \right)^n = (5.371 \ldots)^n,
\]
which is the limit of the method in this case. There is still a considerable gap between the upper bound and the best possible lower bound this method can produce.

Given the gap between upper and lower bounds, we may ask which is closer to the truth? We believe that in the Function Field setting the lower bound is close to the truth, and conjecture that it is exact when \( k = 2 \) (See [15] Section 1.4 for speculation on the integer setting).

**Conjecture 13.** Let \( k \geq 2 \), and suppose that \( \gcd(k, q - 1) > 1 \). For \( n \equiv 0 \mod (2k) \), any set \( A \subset P_{q,n} \) that does not contain a \( k^{th} \) power difference has size at most
\[
|A| \leq q^n(1 - \frac{1}{2^k}).
\]
In particular, for \( k = 2 \) and \( q \) odd, we conjecture that Theorem \( \ref{main} \) is tight.
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REFERENCES

[1] I. Broere, D. Döman, and J. N. Ridley. The clique numbers and chromatic numbers of certain Paley graphs. *Quaestiones Math.*, 11(1):91–93, 1988.

[2] Stephen D. Cohen. Clique numbers of Paley graphs. *Quaestiones Math.*, 11(2):225–231, 1988.

[3] Ernie Croot, Vsevolod F. Lev, and Péter Pál Pach. Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small. *Ann. of Math. (2)*, 185(1):331–337, 2017.

[4] Jordan S. Ellenberg and Dion Gijswijt. On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression. *Ann. of Math. (2)*, 185(1):339–343, 2017.

[5] Kevin Ford and Mikhail R. Gabdullin. Sets whose differences avoid squares modulo $m$. *Proc. Amer. Math. Soc.*, 149(9):3669–3682, 2021.

[6] Ben Green. Sárközy’s theorem in function fields. *Q. J. Math.*, 68(1):237–242, 2017.

[7] R. S. Hales. Numerical invariants and the strong product of graphs. *J. Combinatorial Theory Ser. B*, 15:146–155, 1973.

[8] Brandon Hanson and Giorgis Petridis. Refined estimates concerning sumsets contained in the roots of unity. *Proc. Lond. Math. Soc. (3)*, 122(3):353–358, 2021.

[9] Mark Lewko. An improved lower bound related to the Furstenberg-Sárközy theorem. *Electron. J. Combin.*, 22(1):Paper 1.32, 6, 2015.

[10] Tian Khoon Lim and Cheryl E. Praeger. On generalized Paley graphs and their automorphism groups. *Michigan Math. J.*, 58(1):293–308, 2009.

[11] László Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, 25(1):1–7, 1979.

[12] Máté Matolcsi and Imre Z. Ruzsa. Difference sets and positive exponential sums I. General properties. *J. Fourier Anal. Appl.*, 20(1):17–41, 2014.

[13] Máté Matolcsi and Z. Ruzsa. Difference Sets and Positive Exponential Sums. II: Cubic Residues in Cyclic Groups. *Tr. Mat. Inst. Steklova*, 314(Analiticheskaya i Kombinatornaya Teoriya Chisel):145–151, 2021.

[14] János Pintz, W. L. Steiger, and Endre Szemerédi. On sets of natural numbers whose difference set contains no squares. *J. London Math. Soc. (2)*, 37(2):219–231, 1988.

[15] Alex Rice. A maximal extension of the best-known bounds for the Furstenberg-Sárközy theorem. *Acta Arith.*, 187(1):1–41, 2019.

[16] I. Z. Ruzsa. Difference sets without squares. *Period. Math. Hungar.*, 15(3):205–209, 1984.

[17] A. Sárközy. On difference sets of sequences of integers. III. *Acta Math. Acad. Sci. Hungar.*, 31(3-4):355–386, 1978.

[18] Terence Tao. A symmetric formulation of the croot-lev-pach-ellenberg-gijswijt capset bound, 2016. URL:https://terrytao.wordpress.com/2016/05/18/a-symmetric-formulation-of-the-croot-lev-pach-ellenberg-gijswijt-capset-bound/.

[19] Chi Hoi Yip. On the clique number of paley graphs and generalized paley graphs. 2021.

[20] Chi Hoi Yip. On the clique number of Paley graphs of prime power order. *Finite Fields Appl.*, 77:Paper No. 101930, 16, 2022.

[21] Khalid Younis. Lower bounds in the polynomial szemerédi theorem. *ArXiv e-prints*, 2019. URL:https://arxiv.org/abs/1908.06058/.

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