Violation of the Spin Statistics Theorem and the Bose-Einstein Condensation of Particles with Half Integer Spin

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(Dated: July 4, 2014)

We consider the Bose condensation of particles with spin 1/2. The condensation is driven by an external magnetic field. Our work is motivated by ideas of quantum critical deconfinement and bosonic spinons in spin liquid states. We show that both the nature of the novel Bose condensate and the excitation spectrum are fundamentally different from that in the usual integer spin case. We predict two massive (“Higgs”) excitations and two massless Goldstone excitations. One of the Goldstone excitations has a linear excitation spectrum and another has quadratic spectrum. This implies that the Bose condensate does not support superfluidity, the Landau criterion is essentially violated. We formulate a “smoking gun” criterion for searches of the novel Bose condensation.

PACS numbers: 64.70.Tg, 75.40.Gh, 75.10.Jm

The phenomenon of Bose-Einstein condensation [1, 2] is of enormous importance in physics, from superfluidity of liquid 4He [3–5] to cold atoms [6]. The famous spin-statistics theorem [7] claims that particles with integer spin obey Bose statistics and particles with half integer spin obey Fermi statistics. Therefore only particles with integer spin can Bose condensate [8]. The spin statistic relation is based on Lorentz invariance, so the relation is a relativistic effect which is absolutely important even at extremely low energy. If the number of particles is conserved the stability of Bose condensate is not related to interaction between bosons. The repulsive interaction is only responsible for a linear in momentum spectrum of Goldstone excitations [5]. The linear spectrum guarantees superfluidity of the Bose condensate [4]. If the number of particles is not conserved, like in the case of magnetic field induced Bose condensation of magnons in a gapped quantum antiferromagnet [9, 10], then the stability of the condensate is only due to repulsion between particles. The excitation spectrum above the condensate is still linear in momentum. The linearity is a generic property of a conventional Bose condensation.

The idea of bosonic quasiparticles with spin 1/2, spinons, is a new paradigm in condensed matter physics. Bosonic spinons have been proposed as quasiparticles in some kinds of spin liquids [11]. Bosonic spinons also appear in the deconfined quantum criticality (DQC) scenario. The scenario has been suggested [12, 13] to explain properties of a magnetic quantum phase transition between a Neel antiferromagnet (AF) and a valence-bond solid (VBS) in a two-dimensional system of S=1/2 spins. According to this scenario the system fractionalize into independent (deconfined) spinons. Bosonic spinons do not contradict the spin statistics theorem since they are quasiparticles in a solid and hence the Lorentz invariance is explicitly violated. Both the spin liquid and the DQC are essentially strong coupling nonperturbative phenomena, therefore there is no simple theoretical model where the phenomena can be studied analytically. There are several spin-lattice models which, according to numerical studies, manifest the spin liquid behaviour or DQC. The J-Q spin-lattice model had been specially designed [14] to study DQC numerically. The model is assessable by quantum Monte Carlo (QMC) numerical method which allows the study of very large lattices. To the best of our knowledge this model enables the analysis of the largest lattices among all existing DQC and spin liquid lattice models. Therefore, we refer to numerical results on this model.

The QMC simulations of the J-Q model [14, 15] support the DQC scenario. The scenario violates the long-held Landau rule according to which an order-order transition breaking unrelated symmetries should be of first order. Some studies claim a generic first-order Neel-valence-bond-solid transition [16]. A weak discontinuity (first order) certainly cannot be ruled out based on numerical data. However, the most important issue is not the order of the transition, the most important issue is deconfinement of spinons. This issue was directly addressed in Ref. [17] using the Wilson ratio method (the ratio of uniform magnetic susceptibility over specific heat). The ratio was calculated numerically (QMC) within the strongly interacting J-Q model. The calculation has been performed in terms of original (bare) lattice spins and does not refer to any quasiparticles. On the other hand the Wilson ratio can be expressed in terms of quasiparticles, importantly the ratio is proportional to the quasiparticle spin squared. Comparison of the QMC results with the quasiparticle expression has demonstrated the following points. (i) The spin of the bosonic quasiparticle is really S=1/2. This is a major conclusion. Besides that the analysis [17] has demonstrated two more points. (ii) The spinons interact even weaker than it is predicted in the DQC scenario [12, 13]. (iii) There is a logarithmic correction to the Wilson ratio, the correction is due to large spinon occupation numbers in a weak magnetic field, $n_k \gg 1$, and hence, the correction is a precursor to spinon Bose condensation.
In the present work we consider Bose condensation of particles with $S=1/2$. The condensation is driven by an external magnetic field. The present study is motivated by models of spin liquids [11], the DQC scenario [12, 13] and by numerical results of Ref. [17] for the J-Q model. However, results of the present work are decoupled from the motivations. The results are generic and can be applied to bosonic spinons in solids and to particles in vacuum which for some reason do not respect the spin-statistics theorem (the case of fundamental violation of Lorentz invariance).

Following Refs. [12, 13] we use the CP$^1$ representation to describe spinons, so the mathematical object of the theory, $z$, is an SU(2) spinor. The effective Lagrangian of spinons is

$$\mathcal{L} = \{\partial_t z^\dagger + i S z^\dagger[\vec{\sigma} \cdot \vec{B}]\} \{\partial_t z - i S[\vec{\sigma} \cdot \vec{B}]z\} - (\nabla z^\dagger)(\nabla z) - m^2 z^\dagger z - \frac{\alpha}{2}(z^\dagger z)^2. \quad (1)$$

We set the spinon speed equal to unity, $\vec{B} = 2S \mu_B \vec{B}$, $\vec{B}$ is the external uniform static magnetic field, $S = 1/2$ is spin of the spinon, and $\mu_B$ is the Bohr magneton; $\vec{\sigma}$ are the usual Pauli matrices, $m$ is mass of the spinon. The interaction is repulsive, $\alpha > 0$ and first we assume that $m^2 \geq 0$. Unlike Refs. [12, 13] the Lagrangian (1) does not contain a dynamic gauge field. This is directly motivated by the QMC analysis [17] which does not show a contribution of the gauge field. We already pointed out that the same QMC analysis indicates a logarithmic correction to magnetic susceptibility. The correction can be due to a slightly nonpolynomial interaction of spinons. Here we disregard this correction and consider a simple quartic interaction $\frac{\alpha}{2}(z^\dagger z)^2$. While the DQC motivation comes from two-dimensional (2D) systems (spatial dimensions), the Lagrangian (1) can be considered both in two and three dimensions.

If magnetic field is smaller than mass, $B < m$, then the classical part of the field $z$ is zero and the quantum part obeys the following Euler-Lagrange equation

$$\ddot{z} - \nabla^2 z - 2i(\vec{\sigma} \cdot \vec{B})\dot{z} + [m^2 - B^2]z = 0 \quad (2)$$

We ignore the interaction here having in mind that it is reabsorbed in the quantum renormalization of mass. The plane wave solution is $z(t, \vec{x}) \rightarrow u_\lambda e^{ik_\lambda \vec{x}} - i \omega_{k_\lambda} t$, where the spinor $u_\lambda$ follows from $[\vec{\sigma} \cdot \vec{B}]u_\lambda = \lambda Bu_\lambda$, $\lambda = \pm 1$. Hence

$$\omega_{k_\lambda} = \Omega_k + \lambda B. \quad (3)$$

Here $\Omega_k = \sqrt{m^2 + k^2}$. The standard canonical quantization gives

$$z = \sum_{k_\lambda} \sqrt{\frac{1}{2\Omega_k}} u_\lambda \hat{a}_{k_\lambda} e^{i \omega_{k_\lambda} t - ik_\lambda \vec{x}} + \hat{b}^\dagger_{k_\lambda} e^{-i \omega_{k_\lambda} t + ik_\lambda \vec{x}}. \quad (4)$$

where, $\hat{a}_{k_\lambda}$ and $\hat{b}_{k_\lambda}$ are bosonic creation and annihilation operators. Importantly, $z$ is a usual spinor, $z \neq z^\dagger$, and therefore one must introduce two types of bosons, $a$ and $b$, or spinons and antispinons. This implies that at a given momentum there are four degrees of freedom, $(a, b) \times (\lambda = \pm 1)$ [12, 13].

Obviously if $B > m$ the dispersion (3) becomes negative and this implies Bose condensation at $k = 0$. Energy density corresponding to (1) is $E = \frac{1}{2} \dot{z}^2 + \nabla z \nabla z + [m^2 - B^2]z^2 + \frac{\alpha}{2}(z^\dagger z)^2$. Minimization of the energy gives the spinor condensate and the classical energy [18]

$$z_0^\dagger z_0 = \frac{(B^2 - m^2)}{\alpha}, \quad E_0 = -\frac{(B^2 - m^2)^2}{\alpha}. \quad (5)$$

This expression has the following implications. First of all, $z_0^\dagger z_0$ is a real number that effectively counts together the number of spinons and antispinons. However, the condensate $z_0^\dagger z_0$ gives no information as to the relative contributions of spinons and antispinons. The second implication, is that the ground state energy has no dependence on how the spin polarization vector $\vec{\zeta}_0 = z_0^\dagger \vec{\sigma} z_0 = (\sin \theta_0 \cos \varphi_0, \sin \theta_0 \sin \varphi_0, \cos \theta_0)$ is directed relative to the magnetic field ($\vec{B} || \vec{z}$), see Fig.1. The spinor condensate is

$$z_0 = A_0 e^{-i \gamma_0} \left(\cos \theta_0 / 2 \right), \quad (6)$$

$A_0 = \sqrt{(B^2 - m^2)/\alpha}$. Note, that the induced magnetization is still directed along $\vec{B}$, $\vec{M} = -\frac{\partial E_0}{\partial B} = 4 \mu_B (B^2 - m^2) \vec{B}$.

The degeneracy with respect to direction of $\vec{\zeta}_0$ implies that there are two Goldstone gapless excitations in the system corresponding to variation of angles $\theta_0$ and $\varphi_0$ in Fig.1. We already pointed out that the total number of modes is four. So, in the spinor Bose condensate phase we expect two gapless Goldstone modes and two gapped Higgs modes. Note that the situation is very different from Bose condensation of magnons in a quantum antiferromagnet (condensation of a vector field). The staggered magnetization is always orthogonal to the external magnetic field. This results in one gapless Goldstone mode and two gapped Higgs modes in the case of magnon condensation, see e.g. Refs. [19, 20].

![FIG. 1: (Color online. Orientation of the spinor condensate vector $\vec{\zeta}_0$ with respect to magnetic field.)](image-url)
To find explicitly the excitation modes in the Bose condensate phase we represent

$$z = z_0 + \delta z .$$  \hspace{1cm} (7)

The Euler-Lagrange equation for $\delta z$ reads

$$\delta \ddot{z} - \nabla^2 \delta z - 2i(\vec{\sigma} \cdot \vec{B})\delta \dot{z} + \alpha(\delta z^\dagger z_0)z_0 + \alpha(\delta z^\dagger \delta z)z_0 = 0$$  \hspace{1cm} (8)

A small mathematical complication is that this equation contains both $\delta z$ and $\delta z^\dagger$. Therefore, the solution is of the following form,

$$\delta z = z_+ e^{i\omega t - ikr} + z_- e^{-i\omega t + ikr} .$$  \hspace{1cm} (9)

After some algebra, presented in the supplementary material, we find the following modes

$$\omega_{1k} = \sqrt{B^2 + k^2 - B}$$  \hspace{1cm} (10)
$$\omega_{2k} = \sqrt{3B^2 - m^2 + k^2 - \sqrt{(3B^2 - m^2)^2 + 4B^2 k^2}}$$
$$\omega_{3k} = \sqrt{B^2 + k^2 + B}$$
$$\omega_{4k} = \sqrt{3B^2 - m^2 + k^2 + \sqrt{(3B^2 - m^2)^2 + 4B^2 k^2}} .$$

The first two modes are gapless (Goldstone), $\omega_{k=0} = 0$, and the second two modes are gapped (Higgs). Very surprisingly the first Goldstone mode has quadratic dispersion at small $k$, $\omega_{1k} \approx k^2/2B$, while the second Goldstone mode has the usual linear dispersion, $\omega_{2k} \approx ck$, $c = \sqrt{(B^2 - m^2)/m^2}$.

In order to gain an intuitive/geometric understanding of the excitation modes, it is instructive to look at the variation of the spin expectation vector in these modes, $\zeta = z^\dagger \delta z = \zeta_0 + \delta \vec{\zeta}$ where $\delta \vec{\zeta} \approx z_0^\dagger \delta z + \delta z^\dagger \sigma z_0$. Calculations presented in the supplementary material shows that for the Goldstone modes at small momenta $k \to 0$, the variations are

$$\delta \vec{\zeta}_1 \propto \begin{pmatrix} \cos \theta_0 \cos(\psi + \varphi_0) \\ \cos \theta_0 \sin(\psi + \varphi_0) \\ -\sin \theta_0 \cos \psi \end{pmatrix}$$
$$\delta \vec{\zeta}_2 \propto \begin{pmatrix} -\sin \theta_0 \sin \varphi_0 \\ \sin \theta_0 \cos \varphi_0 \\ 0 \end{pmatrix} \sin \psi .$$  \hspace{1cm} (11)

Here $\psi$ is the plane wave phase, $\psi = \omega t - \vec{k} \cdot \vec{r}$. For the Goldstone mode with quadratic dispersion, the variation $\delta \vec{\zeta}_1$ is shown in the panel (a) of Fig. 2. It represents an anticlockwise rotation of vector $\vec{\zeta}$ around the vacuum polarization vector $\vec{\zeta}_0$. The spin polarization variation $\delta \vec{\zeta}_2$ in the Goldstone mode with linear dispersion is shown in the panel (b) Fig. 2. The mode represents linear oscillations of vector $\vec{\zeta}$ around the vacuum polarization vector $\vec{\zeta}_0$ in the direction perpendicular the meridian.

Both Goldstone modes satisfy the orthogonality condition $\delta \vec{\zeta}_1 \cdot \vec{\zeta}_0 = \delta \vec{\zeta}_2 \cdot \vec{\zeta}_0 = 0$. The spin oscillations are in the plane orthogonal to the spin condensate vector. Polarizations of the Goldstone modes are intuitively quite natural, a circular polarization for the quadratic dispersion mode, similar to that in a ferromagnet and a linear polarization in the linear dispersion mode, similar to that in an antiferromagnet.

Variations of the spin expectation vectors in Higgs modes are (see supplementary material).

$$\delta \vec{\zeta}_{3,4} \propto \begin{pmatrix} -\cos \theta_0 \cos(\psi - \varphi_0) \\ \cos \theta_0 \sin(\psi - \varphi_0) \\ \sin \theta_0 \cos \psi \end{pmatrix}$$
$$\delta \vec{\zeta}_{4} \propto \begin{pmatrix} -\sin \theta_0 \sin \varphi_0 \\ \sin \theta_0 \cos \varphi_0 \\ 0 \end{pmatrix} \sin \psi + b_4 \zeta_0 \cos \psi .$$  \hspace{1cm} (12)

Where $b_4$, presented in the supplementary material, is a non-zero constant $b_4 \to const \neq 0$ at small momenta $k \to 0$. The third mode represents a clockwise rotation of vector $\vec{\zeta}$ around the vacuum polarization vector $\vec{\zeta}_0$, see panel (c) in Fig. 2. Finally the fourth mode represents elliptic oscillations in the plane perpendicular to the meridian, see panel (d) in Fig. 2. The mode has both a longitudinal component (the component parallel to $\vec{\zeta}_0$) and a transverse component.

Instead of the external magnetic field a Quantum Phase Transition (QPT) can be driven by some external parameter $g$. In the case of magnon condensation in the spin dimerized compound TiCuCl$_3$ such parameter is pressure [21–23]. In this case the mass squared term in the magnon Lagrangian changes sign at the transition [20]. Now we consider a similar QPT for spinons described by the Lagrangian (1), $m^2 \propto g - g_c$. The formula $m^2 \propto g - g_c$ assumes a continuous QPT, but in the end the continuity is not crucial, one can consider a weak discontinuous (first order) QPT too. There are two following crucial points which generally are independent of the kind of QPT, (i) the nature of quasiparticles is the same (spinons) on both sides of the QPT, (ii) the mass squared is negative, $m^2 = -|m^2|$, in the ordered phase. Now we consider zero magnetic field. Minimization of the energy gives the following spinor condensate in the ordered phase

$$z_0^\dagger z_0 = \frac{|m^2|}{\alpha} .$$  \hspace{1cm} (13)

The Euler-Lagrange equation for the variation $\delta z$, $\delta \ddot{z} - \nabla^2 \delta z + \alpha(\delta z^\dagger z_0)z_0 + \alpha(\delta z^\dagger \delta z)z_0 = 0$, can be easily solved using (9). This gives three Goldstone excitations and one gapped Higgs excitation

$$\omega_1 = \omega_2 = \omega_3 = k$$
$$\omega_4 = \sqrt{2|m^2| + k^2} .$$

It is instructive to compare QPTs of spinons (condensation of the spinor field, $S=1/2$) and QPTs of magnons
FIG. 2: Oscillations of the spin polarization vector $\vec{\zeta}$ with respect to the condensate vector $\vec{\zeta}_0$ in different excitation modes. The solid black arrow represents an arbitrary $\vec{\zeta}_0$. (a) Goldstone mode with quadratic dispersion. Anticlockwise rotation, the orange circle shows the path traced by the oscillations. (b) Goldstone mode with linear dispersion. Linear oscillations in the direction perpendicular to the meridian. (c) Higgs mode [3]. Clockwise rotation, the orange circle shows the path traced by the oscillations. (d) Higgs mode [4]. Elliptic oscillations in the plane perpendicular the meridian.

Here $\zeta(x)$ is Riemann’s zeta function. The unconventional specific heat is a smoking gun for the unconventional Bose condensate. The Mermin-Wagner theorem [24] is certainly valid for spinon condensates. This implies that there is no true long range order at $T \neq 0$ in 2D case. However, as usual the exponentially large correlation length does not influence the power of temperature in the specific heat (14).

When applying the present analysis to the AF $\rightarrow$ VBS transition in a quantum magnet one has to remember that the deconfined description is valid only within a vicinity of the quantum critical point. There is no doubt that deep inside the VBS phase the quasiparticles are usual triplons and that deep inside the AF phase the quasiparticles are usual transverse magnons. This implies that there is a spinon confinement length $\xi$ which depends on the detuning from the quantum critical point. It is possible that the gauge field [12, 13], while does not show up as a dynamic variable in (1), still acts as a constraint contributing to formation of the confinement length. The deconfined description, and in particular Eq.(14), is valid if the spinon thermal wavelength is smaller than the confinement scale, $\lambda T \sim 1/\sqrt{BT} \ll \xi$.

In conclusion, we have considered Bose condensation of particles with spin $S=1/2$. The Bose statistics for the half integer spin particles implies violation of the spin statistics theorem. While we believe that our results are generic, specifically we consider the Bose condensation of spinons driven by a magnetic field. The most surprising are the gapless Goldstone excitations in the spinon condensate. There are two such excitations, one has a linear in momentum dispersion and another has a quadratic dispersion. The quadratic dispersion implies that the condensate does not support superfluidity. This also implies an unconventional temperature behaviour of the specific heat. This is a smoking gun criterion/behaviour for iden-
tification of the unconventional Bose condensation. Observation of such behaviour in an experiment or in a numerical simulation would be an unambiguous indication for the Bose condensation of spinons.

We thank A. W. Sandvik for discussions.

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[18] We treat the energy in a simple mean-field approximation. There is no doubt that quantum fluctuations/corrections significantly modify Eqs.(5) and some other Eqs., especially in the 2D case. However, the goal of the present work is to find properties of quasiparticles/excitations. These properties are dictated by symmetries and hence they are not influenced by the dynamic approximations made. This point is valid for a usual quantum antiferromagnet [20] (condensation of a vector field) and it is equally valid for condensation of spinons.
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Supplementary Material

Part A

In this section we show how the Euler-Lagrange equation (8) generates the Bose condensate excitation modes (10).

It is convenient to define the following time independent spinors

\[
\begin{align*}
    u &= \begin{pmatrix} \cos \frac{\theta_0}{2} \\ \sin \frac{\theta_0}{2} e^{i\phi_0} \end{pmatrix}, \\
    \bar{u} &= \begin{pmatrix} -\sin \frac{\theta_0}{2} \\ \cos \frac{\theta_0}{2} e^{i\phi_0} \end{pmatrix}.
\end{align*}
\]

\[\text{(A.1)}\]

Hence the Bose condensate field (6) is \( z_0 = A_0 u \). Substitution of ansatz (9) in the Euler-Lagrange equation (8) results in the following algebraic equation for the “amplitude” spinors \( z_+ \) and \( z_- \)

\[
\begin{align*}
    (-\omega^2 + k^2)z_+ + 2\omega(\vec{\sigma} \cdot \vec{B})z_+ + (B^2 - m^2)(z_+^\dagger u + u^\dagger z_+)u &= 0 \\
    (-\omega^2 + k^2)z_- - 2\omega(\vec{\sigma} \cdot \vec{B})z_- + (B^2 - m^2)(z_+^\dagger u + u^\dagger z_-)u &= 0.
\end{align*}
\]

\[\text{(A.2)}\]

Projecting each equation (A.2), onto both \( u \) and \( \bar{u} \), and using the following definitions:

\[
\begin{align*}
    u^\dagger z_+ &= a_1 & a_1 + a_2^* &= a_+ \\
    u^\dagger z_- &= a_2 & a_1 - a_2^* &= a_- \\
    \bar{u}^\dagger z_+ &= b_1 & b_1 + b_2^* &= b_+ \\
    \bar{u}^\dagger z_- &= b_2 & b_1 - b_2^* &= b_- \text{.}
\end{align*}
\]

\[\text{(A.3)}\]
we get the following matrix equation for the amplitudes $a_+, a_-, b_+, b_-$,

$$
\begin{pmatrix}
  k^2 - \omega^2 + 2(B^2 - m^2) & 2\omega B \cos \theta_0 & 0 & -2\omega B \sin \theta_0 \\
  2\omega B \cos \theta_0 & k^2 - \omega^2 & -2\omega B \sin \theta_0 & 0 \\
  0 & -2\omega B \sin \theta_0 & k^2 - \omega^2 & -2\omega B \cos \theta_0 \\
 -2\omega B \sin \theta_0 & 0 & -2\omega B \cos \theta_0 & k^2 - \omega^2 \\
\end{pmatrix}
\begin{pmatrix}
  a_+ \\
  a_- \\
  b_+ \\
  b_- \\
\end{pmatrix} = 0.
$$

(A.4)

Frequencies of normal modes (10) immediately follow from this matrix equation. Amplitude vectors $(a_+, a_-, b_+, b_-)$, corresponding to the normal modes are of the following form

$$
|1\rangle \propto \begin{pmatrix}
  0 \\
  \sin \theta_0 \\
  1 \\
  \cos \theta_0 \\
\end{pmatrix};
|2\rangle \propto \begin{pmatrix}
  b_2 \\
  -\cos \theta_0 \\
  0 \\
  \sin \theta_0 \\
\end{pmatrix};
|3\rangle \propto \begin{pmatrix}
  0 \\
  \sin \theta_0 \\
  -1 \\
  \cos \theta_0 \\
\end{pmatrix};
|4\rangle \propto \begin{pmatrix}
  b_4 \\
  -\cos \theta_0 \\
  0 \\
  \sin \theta_0 \\
\end{pmatrix}
$$

$$
b_2 = \frac{\sqrt{(3B^2 - m^2)^2 + 4B^2k^2 - (3B^2 - m^2)}}{2B\omega_2},
$$

$$
b_4 = -\frac{\sqrt{(3B^2 - m^2)^2 + 4B^2k^2 - (3B^2 - m^2)}}{2B\omega_4}
$$

At small momenta, $k \to 0$, the coefficients behave as $b_2 \to 0$, $b_4 \to const \neq 0$. Note that there is no naive orthogonality of “eigenvectors”, in particular $\langle 2|4 \rangle \neq 0$. A similar nonorthogonality is typical for classical (non-quantum) coupled oscillators in magnetic field. In spite of the nonorthogonality, after an appropriate canonical transformation the Hamiltonian is transformed to the sum of independent Hamiltonians of the normal modes.
Part B

In this section we calculate spin vibrations \((11),(12)\) corresponding to different excitation modes.

Using Eqs. (A.1) and (A.3) we find

\[
\begin{align*}
  z^+ & \propto \left( \begin{bmatrix}
    (a_+ + a_-) \cos \frac{\theta_0}{2} - (b_+ + b_-) \sin \frac{\theta_0}{2} \\
    (a_+ + a_-) \sin \frac{\theta_0}{2} + (b_+ + b_-) \cos \frac{\theta_0}{2}
  \end{bmatrix}
  e^{i\phi_0} \right), \\
  z^- & \propto \left( \begin{bmatrix}
    (a_+ - a_-) \cos \frac{\theta_0}{2} - (b_+ - b_-) \sin \frac{\theta_0}{2} \\
    (a_+ - a_-) \sin \frac{\theta_0}{2} + (b_+ - b_-) \cos \frac{\theta_0}{2}
  \end{bmatrix}
  e^{i\phi_0} \right),
\end{align*}
\]

(B.1)

From here, using

\[
\delta \vec{\zeta} = \delta z^\dagger \vec{\sigma} z_0 + z_0^\dagger \vec{\sigma} \delta z,
\]

(B.2)

together with Eqs. (9) and (A.5) we find explicit formulas for the spin vibration vectors \(\delta \vec{\zeta}\) presented in Eqs. (11) and (12).