One-dimensional Multi-particle DLA – a PDE approach

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Abstract

In the present note we analyze the one-dimensional multi-particle diffusion limited aggregation (MDLA) model: the initial number of particles at each site \( x \in \mathbb{N} \) has Poisson distribution with mean \( \mu \), independently of all other sites. Particles perform independent continuous-time simple symmetric random walks until they come to the site neighbouring the sticky aggregate, which initially consists only of the origin. If a particle tries to jump on the aggregate, the size of the aggregate increases by one, i.e., its rightmost point moves to the right by one unit. All particles which are present at the site neighbouring the aggregate at the moment when the aggregate advances, are immediately deleted.

The \( d \)-dimensional MDLA model, \( d \geq 1 \), was introduced in 1980 in \cite{4}, and studied numerically in \cite{7}. The one dimensional model exhibits a phase transition for the rate of growth of the aggregate: in \cite{3} it was proven that if \( \mu < 1 \) then the size \( R(t) \) of the aggregate grows like \( \sqrt{t} \) and in \cite{6} it is proved that if \( \mu > 1 \) then \( R(t) \) grows linearly.

In this note we give heuristic predictions about the constant \( c(\mu) \) for which
\begin{itemize}
  \item \( R(t) \approx c(\mu)\sqrt{t} \) in the subcritical case \( \mu < 1 \),
  \item \( R(t) \approx c(1+\varepsilon)t \) in the barely supercritical case \( \mu = 1 + \varepsilon \) and
  \item \( R(t) \approx c(1)t^{2/3} \) in the critical case \( \mu = 1 \).
\end{itemize}

We compare our predictions with new computer simulation results of the 1-dimensional multi-particle DLA model.

Keywords: diffusion limited aggregation, partial differential equation, Stefan problem

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1 Introduction

In this work we focus on the multi-particle diffusion limited aggregation model - the predecessor of the paradigm diffusion-limited aggregation (DLA) model of Witten and Sander [8]. The MDLA was introduced in [4] with multiple particles moving and aggregating possibly at the same time, describing the growth of a sticky aggregate which interacts with a Poissonian cloud of particles. This inspired Witten and Sander to create their two dimensional model with random walks started from a distant point which become attached one by one to an aggregate that originally consists of one vertex, the origin.

Later the multi-particle DLA model was studied numerically in [7] and its one dimensional version was rigorously studied in [3] and in [6], establishing presence of the phase transition in the rate of growth. In the higher dimensional case and for high enough densities of particles the linear growth and a type of shape theorem were established in [5].

1.1 Definition of the model

In this note we study the multi-particle DLA model in the one-dimensional setting.

Denote the time parameter by $t \in [0, \infty)$. Denote by $R(t)$ the size of the sticky aggregate at time $t$. Initially, the length of the aggregate is $R(0) = 0$, and $R(t)$ is a non-decreasing function of $t$. For all $t \in [0, \infty)$ and $i \in \mathbb{N}, i > R(t)$ we denote by $N(i,t)$ the number of particles at time $t$ at position $i$.

Initially $N(i,0), i = 1, 2, \ldots$ are i.i.d. with $\text{Poi}(\mu)$ distribution, where $0 < \mu$ is the parameter of the model. The particles perform independent continuous time simple symmetric random walks with unit jump rate until they hit the sticky aggregate. As soon as a particle touches the aggregate, it stays put forever and we say that it “dies”.

At time $t = 0$ there is a sticky particle at the origin $i = 0$. If at time $t$ there is a particle at position $R(t) + 1$ which attempts to jump to the left, then it sticks to the aggregate and dies, moreover the size of the sticky aggregate increases by one, i.e. $R(t_+) = R(t) + 1$. We also assume that if $R(t_+) = R(t) + 1$, then all other particles at position $R(t) + 1$ at time $t$ are also swallowed by the sticky aggregate and die (but these surplus particle deaths do not contribute to the increase of the aggregate).

1.2 Previous results

One is interested in the asymptotic speed of growth of the size $R(t)$ of the aggregate when $t \to \infty$ and the dependence of this speed on the initial density $\mu$ of the particles.

The conjectured behaviour is that the model undergoes phase transition as $\mu$ varies:

$$R(t) \asymp t^\alpha,$$

where $\alpha = \begin{cases} 1/2 & \text{if } \mu < 1, \\ 2/3 & \text{if } \mu = 1, \\ 1 & \text{if } \mu > 1. \end{cases}$

We say that the model is subcritical if $\mu < 1$, supercritical if $\mu > 1$ and critical if $\mu = 1$. 

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In [3] it was proved that for any \( \mu \in \mathbb{R}_+ \) the aggregate grows at most linearly, i.e., we have \( \lim_{t \to \infty} R(t)/t < \infty \) (see [3, Theorem 1]), moreover for any \( \varepsilon > 0 \) there exists \( \eta(\mu, \varepsilon) \) such that \( \lim_{t \to \infty} \mathbb{P}(R(t)/\sqrt{t} > \eta) \geq 1 - \varepsilon \) (see [3, Theorem 3]). It was also shown that if \( \mu < 1 \) then \( \lim_{t \to \infty} R(t)/\sqrt{t} = \infty \) (see [3, Theorem 2]).

Recently, in [6] it was proved that for any \( \mu > 1 \) the aggregate grows linearly, i.e., we have \( \lim_{t \to \infty} R(t)/t = r \) for some \( r \in (0, +\infty) \).

Let us observe that our methods involve partial differential equations with moving boundaries (see Sections 2 and 3), called Stefan (free-boundary) equations which have been linked to particle systems similar to ours in [1] and more recently in [2].

The one dimensional “freezing” model introduced in [1] involves a configuration of particles that perform symmetric simple exclusion process and get attached to a growing sticky boundary. Note that in this model the density of particles has to satisfy \( \mu \in [0, 1) \), and the Stefan problem which arises as the hydrodynamic limit is essentially the same as the one we study in Section 3, see Remark 3.1.

The difference between the dynamics of the one dimensional “frictionless” growth model introduced in [2] and the MDLA model studied in this note is that in the frictionless model, the size of the aggregate is always equal to the number of particles swallowed by the aggregate, whereas in the MDLA model some particles are “lost” (note that this property of the MDLA model crucially enters our analysis of the \( \mu \geq 1 \) case, c.f. Section 4.2). The asymptotic behaviour of the frictionless model and the MDLA model should be the same in the subcritical case \( \mu < 1 \) (i.e., Claim 1.1 should also hold for the frictionless model) as the number of “lost” particles is negligible in the subcritical MDLA model, see Remark 3.2.

In the frictionless model, the size of the aggregate almost surely becomes infinite in finite time if \( \mu > 1 \), whereas it grows linearly in the MDLA model. In [2, Corollary 1.5(a)] it is shown that \( R(t)/t^{2/3} \) converges in law to a non-trivial proper probability distribution as \( t \to \infty \) in the frictionless model with critical density \( \mu = 1 \), whereas in the critical MDLA model we conjecture that \( R(t)/t^{2/3} \) converges to a deterministic constant, see Claim 1.3.

### 1.3 Statement of our predictions

In this short note we provide some non-rigorous predictions about the phase transition as well as the near-critical behaviour of the one-dimensional multi-particle DLA model.

Despite the fact that our arguments are not mathematically rigorous, we hope that the ideas presented in this note will serve as valuable intermediate steps on the way to the fully rigorous understanding of the near-critical and critical behaviour of the model. We elaborate on the conjecture \( R(t) \asymp t^{2/3} \) in the sense that we provide the exact constant for which \( R(t) \approx c \cdot t^{2/3} \) (see Claim 1.3) and give a (hopefully convincing) short argument explaining why we think that the model behaves this way.

In Section 5 we recount the results of our computer simulations and compare them with the non-rigorous predictions that we are about to present.

The following heuristic claim is an elaboration of [3, Open Problem 3].
Claim 1.1. If $\mu < 1$ then we have

$$\lim_{t \to \infty} \frac{R(t)}{\sqrt{t}} = r(\mu),$$

where $r(\mu) = r$ is the unique positive solution of the equation

$$\mu = \int_0^\infty e^{-x} \exp \left( -\frac{1}{2} \frac{x^2}{r^2} \right) dx. \quad (1.1)$$

In particular, when $\mu = 1 - \varepsilon$, then

$$\lim_{\varepsilon \to 0^+} \sqrt{\varepsilon} \cdot r(1 - \varepsilon) = 1. \quad (1.2)$$

The following heuristic claim is an elaboration of \cite{3} Open Problem 2].

Claim 1.2. If $\mu > 1$ then we have

$$\lim_{t \to \infty} \frac{R(t)}{t} = r(\mu),$$

where

$$\lim_{\varepsilon \to 0^+} \frac{r(1 + \varepsilon)}{\varepsilon} = \frac{1}{2}. \quad (1.3)$$

Our last heuristic claim elaborates on the conjecture $R(t) \propto t^{2/3}$ in the critical model.

Claim 1.3. If $\mu = 1$ then we have

$$\lim_{t \to \infty} \frac{R(t)}{t^{2/3}} = \frac{1}{2} \left( \frac{3}{2} \right)^{2/3}. \quad (1.4)$$

Let us now outline the structure of the rest of this note.

In Section 2 we introduce the system of differential equations that describe the time evolution of particle densities given the evolution of the aggregate.

In Section 3 we study the $\mu < 1$ case: we write down and solve the Stefan PDE problem which describes the evolution of the asymptotic shape of the particle density profile as well as that of the size of the aggregate in the $t \to \infty$ limit.

In Section 4 we study the $\mu \geq 1$ case. Here we again approximate the particle density profile with a PDE (Section 4.1) and also keep track of the number of particles that were swallowed by the aggregate without contributing to its growth (Section 4.2). We then link the number of such “lost” particles to the scaling of the particle density profile in order to determine the speed of aggregate growth in the barely supercritical case $\mu = 1 + \varepsilon$ in Section 4.3 and in the critical case $\mu = 1$ in Section 4.4.

In Section 5 we recount the results of our computer simulations of the discrete-time one-dimensional multi-particle DLA model.
2 Differential equations

Recall the definition of the model from Section 1.1. We define the $\sigma$-algebras

$\mathcal{F}_t = \sigma(R(s), 0 \leq s \leq t), \quad t \geq 0$.

**Claim 2.1.** Conditionally on $\mathcal{F}_t$,

$$(N(i, t))_{i=R(t)+1}^\infty \text{ are independent with } (\text{Poi}(\lambda(i, t)))_{i=R(t)+1}^\infty \text{ distribution},$$

(2.1)

where $\lambda(i, t), i > R(t)$ are $\mathcal{F}_t$-measurable random variables which can be determined in the following way. Initially we have $\lambda(i, 0) = \mu, i > 0$, and the rates $\lambda(i, t)$ satisfy the system of differential equations

$$\frac{d}{dt}\lambda(i, t) = \frac{1}{2}\lambda(i-1, t) + \frac{1}{2}\lambda(i+1, t), \quad i > R(t)$$

(2.2)

with the boundary condition

$$\lambda(i, t) = 0, \quad i \leq R(t).$$

(2.3)

**Remark 2.1.** The statement of Claim 2.1 was suggested to us by T. Kurtz (private communication). An alternative formula for $\lambda(i, t)$ can be derived by taking advantage of the time-reversibility of simple symmetric random walk. If $X_i(s), s \geq 0$ is a random walk starting from $X_0 = i$ which is independent of $\mathcal{F}_t$ then

$$\lambda(i, t) = \mu \cdot \mathbb{P}(X(s) > R(t-s) \text{ for all } 0 \leq s \leq t \mid \mathcal{F}_t),$$

see [6, Section 2] for a description of this approach.

Note that for all $t \geq 0$ we have

$$\lim_{i \to \infty} \lambda(i, t) = \mu.$$  

(2.4)

Also note that the aggregate grows at a rate

$$\lim_{dt \to 0^+} \frac{1}{dt}\mathbb{P}(R(t + dt) = R(t) + 1 \mid \mathcal{F}_t) = \frac{1}{2}\lambda(R(t) + 1, t),$$

(2.5)

because each of the $N(R(t) + 1, t)$ particles jump to the left at rate $\frac{1}{2}$.

3 Subcritical model, $\mu < 1$

In this section we give a heuristic proof of Claim 1.1. We derive the parabolic PDE with the Stefan-type moving boundary condition which describes the hydrodynamics of the evolution of the density profile of the particles as well as that of $R(t)$. 

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Let $1 \ll n$ denote a large number. We introduce the continuous space and time parameters $x \in \mathbb{R}_+$ and $s \in \mathbb{R}_+$ as well as the smooth function $r : \mathbb{R}_+ \to \mathbb{R}_+$ and the smooth function $\mu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. We introduce the scaling
\[ t = n \cdot s, \quad i = \lfloor \sqrt{n} \cdot x \rfloor, \quad r(s) \approx \frac{1}{\sqrt{n}} R(n \cdot s), \quad \mu(x, s) \approx \lambda(\lfloor \sqrt{n} \cdot x \rfloor, n \cdot s). \] (3.1)

It follows from (2.4) that we have
\[ \lim_{x \to \infty} \mu(x, s) = \mu. \] (3.2)

It follows from our assumptions (2.3) and (3.1) that we have
\[ \mu(x, s) = 0, \quad x \leq r(s). \] (3.3)

By (3.1) we can see that (2.2) is an approximation of the heat equation:
\[ x > r(s) \implies \partial_s \mu(x, s) = \frac{1}{2} \partial_{xx} \mu(x, s). \] (3.4)

Now we are going to derive the driving equation of $r(s)$:
\[ \frac{d}{ds} r(s) = \frac{1}{2} \partial_x \mu(r(s), s). \] (3.5)

According to (3.1), we assume $ds = \frac{dt}{n}$,
\[ \frac{d}{ds} r(s) \approx \frac{r(s + ds) - r(s)}{ds} \approx \frac{1}{\sqrt{n}} \sqrt{n} \cdot \mathbb{E}(R(t + dt) - R(t)) = \frac{1}{\sqrt{n}} \mathbb{P}(R(t + dt) = R(t) + 1) \] (3.6)
\[ \approx \frac{\sqrt{n}}{2} \lambda(R(t) + 1, t) \approx \frac{\sqrt{n}}{2} (\lambda(R(t) + 1, t) - \lambda(R(t), t)) \approx \frac{\sqrt{n}}{2} \left( \mu \left( \frac{r(s) + 1}{\sqrt{n}}, s \right) - \mu(r(s), s) \right) \approx \frac{1}{2} \partial_x \mu(r(s), s), \] (3.7)

thus we have shown (3.5).

Now we are going to solve (3.2), (3.3), (3.4), (3.5), (3.7) with initial conditions
\[ \mu(0, x) \equiv \mu, \quad r(0) = 0. \] (3.8)

Remark 3.1. Uniqueness of the weak solution of the heat equation (3.4) with the Stefan boundary condition (3.5) under some additional technical assumptions (which include the subcriticality assumption $\mu(0, x) < 1$) is proved in the Appendix of [1]. In this non-rigorous paper we assume without proof that the solution of (3.2), (3.3), (3.4), (3.5), (3.7) that we are about to construct is unique.
Since the choice of \( n \) in (3.1) was arbitrary, we look for a self-similar solution of form
\[
    r(s) = r \cdot \sqrt{s}, \quad \mu(s, x) = \mu \left( \frac{x}{\sqrt{s}} \right), \quad \mu : [r, \infty) \to [0, \mu].
\] (3.8)

The boundary conditions (3.3) and (3.2) are transformed into
\[
    \mu(r) = 0, \quad \lim_{x \to \infty} \mu(x) = \mu.
\] (3.9)

Under the assumption (3.8) the equation (3.5) is transformed into
\[
    \mu'(r) = r.
\] (3.10)

Under the assumption (3.8) the equation (3.4) is transformed into
\[
    x > r \implies -\mu'(x) \cdot x = \mu''(x),
\] (3.11)

thus we have \( \mu'(x) = c \exp(-\frac{1}{2}x^2) \) for some \( c > 0 \) and if we combine this with (3.9), we get
\[
    \mu(x) = \mu \cdot \frac{\int_r^\infty \exp(-\frac{1}{2}y^2) \, dy}{\int_r^\infty \exp(-\frac{1}{2}y^2) \, dy}.
\] (3.12)

We still have to choose \( r \) so that (3.10) is satisfied. We use (3.12) to find such \( r \):
\[
    \mu \cdot \frac{\exp(-\frac{1}{2}r^2)}{\int_r^\infty \exp(-\frac{1}{2}y^2) \, dy} = r \iff \mu = r \int_r^\infty \exp \left( \frac{y^2}{2} - \frac{x^2}{2} \right) \, dx \implies \mu = \int_0^\infty e^{-x} \exp \left( -\frac{1}{2}x^2 \right) \, dx.
\]

Now if we define
\[
    \mu(r) = \int_0^\infty e^{-x} \exp \left( -\frac{1}{2}x^2 \right) \, dx
\] (3.13)

then it is easy to see that \( \mu(r) \) is a strictly increasing, continuous function of \( r \) with \( \lim_{r \to 0} \mu(r) = 0 \) and \( \lim_{r \to \infty} \mu(r) = 1 \). Thus the equation \( \mu(r) = \mu \) has a unique solution for every \( \mu \in (0, 1) \), as we claimed above (1.1). This is the value of \( r \) for which (3.10) is satisfied.

We have constructed a solution to (3.2), (3.3), (3.4), (3.5), which also satisfies the initial condition (3.7) by (3.8) and (3.9). The first statement of Claim 1.1 now follows from (3.1) and (3.8) by choosing \( s = 1 \).

Now we prove (1.2). If \( 1 \ll r \), then \( 1 - \mu(r) \) can be approximated in the following way:
\[
    1 - \mu(r) = \int_0^\infty e^{-x} \left( 1 - \exp \left( -\frac{1}{2}x^2 \right) \right) \, dx \approx \int_0^\infty e^{-x} \frac{1}{2}x^2 \, dx \approx \frac{1}{r^2}.
\]

Thus if \( \varepsilon = 1 - \mu \) and \( \varepsilon \ll 1 \), then \( r(\mu) \approx \frac{1}{\sqrt{\varepsilon}} \), hence (1.2) holds.
Remark 3.2. Note that if $\mu(x,s)$ and $r(s)$ solve (3.4) and (3.5) with the boundary conditions (3.2) and (3.3), then the solution of these equations reflects the asymptotic conservation of mass in our subcritical system, as we now explain.

The mass of particles that died before time $s$ is

$$\int_0^\infty (\mu(x,0) - \mu(x,s)) \, dx = \int_{r(s)}^\infty (\mu - \mu(x,s)) \, dx + \mu \cdot r(s).$$

On the other hand, the mass of the particles in the aggregate is $r(s)$. It is straightforward to show that no mass is lost, i.e., that

$$\int_{r(s)}^\infty (\mu - \mu(x,s)) \, dx + \mu \cdot r(s) = r(s)$$

holds for all $s \geq 0$ (3.14) by checking that the time derivative of the l.h.s. of (3.14) is the same as the time derivative of the r.h.s. of (3.14).

4 Barely supercritical and supercritical model, $\mu \geq 1$

In Sections 4.1 and 4.2 we make some preliminary observations that will be useful in the barely supercritical case $\mu = 1 + \varepsilon$ (Section 4.3) and the critical case $\mu = 1$ (Section 4.4).

4.1 Shape of cloud

The following lemma states that if the function $R(t)$ grows at a low constant speed $r$ for a sufficiently long time interval $[t_0 - \Delta t, t_0]$, then we can identify the shape of the cloud of particles in front of the tip of the aggregate.

Lemma 4.1. If $t_0, r, \Delta t \in \mathbb{R}_+$ satisfy

$$0 < r \ll 1, \quad r^{-2} \ll \Delta t \leq t_0,$$

moreover we have

$$R(t_0) - R(t) \approx r \cdot (t_0 - t) \quad \text{for any } t \in [t_0 - \Delta t, t_0]$$

then we also have

$$\lambda(R(t_0) + i, t_0) \approx \mu \cdot (1 - e^{-2ri}).$$

Heuristic proof. We introduce the scaling parameter $\delta = 4r^2$.

We introduce the continuous space and time parameters $x \in \mathbb{R}_+$ and $s \in \mathbb{R}_+$. We define the function $u : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$u(x, s) = \lambda \left( \frac{1}{\sqrt{\delta}} x, \frac{1}{\delta} s \right).$$
By $\delta \ll 1$ we see that (2.2) approximates the heat equation under this scaling:

\[
\frac{1}{\sqrt{\delta}} x > R(t) \implies \partial_s u(x, s) = \frac{1}{2} \partial_x^2 u(x, s).
\]  

(4.5)

We let $t = \frac{1}{\delta} s$ and $i = \lfloor \frac{1}{\sqrt{\delta}} x \rfloor$ and we define the smooth function $\mu(x, s)$ by

\[
\mu(x, s) := \lambda(R(t) + i, t) \approx \lambda(R(t_0) - rt_0 + rt + i, t) = \lambda\left(R(t_0) - rt_0 + \frac{1}{\delta} x, \frac{1}{\delta} s\right) \overset{(*)}{=} \lambda\left(x_0 + \frac{r}{\sqrt{\delta}} s + x, \frac{1}{\delta} s\right) \overset{4.4}{=} u\left(\frac{r}{\sqrt{\delta}} s + x_0 + x, s\right),
\]

(4.6)

where in (*) we introduced the notation $x_0 = \sqrt{\delta} \cdot (R(t_0) - rt_0)$.

We obtain from (4.5) the following PDE for $\mu(x, s)$:

\[
x > 0 \implies \partial_s \mu(x, s) = \frac{1}{2} \partial_x^2 \mu(x, s) + \frac{1}{2} \partial_x \mu(x, s).
\]  

(4.7)

Note that by (2.3), (2.4) and (4.6) we obtain the boundary conditions

\[
\mu(0, s) = 0, \quad \lim_{x \to \infty} \mu(x, s) = \mu.
\]  

(4.8)

Note that (4.7) holds on a time interval of width $\delta \cdot \Delta t \gg 1$ by our assumptions (4.1) and $\delta = 4r^2$, thus we may assume that $\mu(x, \delta t_0)$ is the stationary solution $\mu(x)$ of the PDE (4.7) with boundary condition (4.8).

The solution of $0 = \frac{1}{2} \partial_x^2 \mu(x) + \frac{1}{2} \partial_x \mu(x)$ with the boundary conditions (4.8) is

\[
\mu(x) = \mu \cdot (1 - e^{-x}).
\]  

(4.9)

By our choice $\delta = 4r^2$ we have $i = \lfloor \frac{1}{2r} x \rfloor$, thus (4.3) follows from the definition (4.6) of $\mu(x, s)$ and (4.9).

\[\square\]

### 4.2 Lost particles

Assume that each particle has unit mass, thus for any $i > R(t)$ the average total mass of particles at site $i$ at time $t$ is $\lambda(i, t)$. We define the mass of the aggregate at time $t$ to be $R(t)$, thus the mass of a particle that tries to jump on the aggregate is added to the mass of the aggregate. The mass of a particle can only disappear if it is swallowed by the aggregate as it advances. Denote by $L(t)$ the number of particles that got lost by time $t$ for this reason.
Lemma 4.2. Assume that (4.3) holds at time $t$ for some $r \ll 1$. Then we have

$$L(t) \approx R(t) \cdot (\mu - 1) + \frac{\mu}{2r} \quad (4.10)$$

and

$$\frac{d}{dt} E(L(t)) \approx 2r^2. \quad (4.11)$$

Heuristic proof. Denote by $D(t)$ the number of particles that died (i.e., touched the sticky aggregate) by time $t$. First we derive (4.10):

$$L(t) = D(t) - R(t) \approx \sum_{i=0}^{\infty} (\lambda(i, 0) - \lambda(i, t)) - R(t) \approx R(t) \cdot \mu + \sum_{j=1}^{\infty} (\mu - \lambda(R(t) + j, t)) - R(t) \approx R(t) \cdot (\mu - 1) + \sum_{j=1}^{\infty} \mu \cdot e^{-2rj} \approx R(t) \cdot (\mu - 1) + \frac{\mu}{2r}.$$

Next we observe that $L(t+) = L(t) + k - 1$ if and only if $D(t+) = D(t) + k$ for some $k \geq 1$, since exactly one of the $k$ particles that die at time $t$ contributes to $R(t+) = R(t) + 1$. Denote by $\lambda = \lambda(R(t) + 1, t)$. Now we derive (4.11):

$$E(L(t + dt) - L(t)) = \sum_{k=2}^{\infty} (k - 1)P(D(t + dt) - D(t) = k) \approx \sum_{k=2}^{\infty} (k - 1) e^{-\lambda} \frac{\lambda^k}{k!} \frac{dt}{2} \sum_{k=0}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!} = \frac{dt}{2} \lambda^2 \approx \frac{dt}{2} \left(1 - e^{-2r}\right)^2 \approx 2r^2 dt,$$

where in (*) we used that $P(N(i, t) = k) \approx e^{-\lambda} \frac{\lambda^k}{k!}$ (see (2.1)) and each of these $k$ particles jump to the left at rate $\frac{1}{2}$.

4.3 Barely supercritical model, $\mu = 1 + \varepsilon$

In this section we give a heuristic proof of Claim 1.2. We assume that if $\mu = 1 + \varepsilon$ for some $\varepsilon \ll 1$ then we have

$$R(t) \approx r \cdot t, \quad r \ll 1. \quad (4.12)$$

We will use the results of Sections 4.1 and 4.2 to identify how $r$ in (4.12) depends on $\varepsilon$.

We assume that $r^{-2} \ll t$ so that the result Lemma 4.1 holds, so that we can infer

$$\lim_{t \to \infty} \frac{1}{t} L(t) \approx \frac{\mu}{2r} \quad \text{and} \quad \lim_{t \to \infty} \frac{R(t) \cdot \varepsilon}{t} \approx \frac{\mu}{2r} \cdot \varepsilon \quad (4.13)$$

Assuming convergence to stationarity as $t \to \infty$, (4.13) implies

$$\frac{d}{dt} E(L(t)) \approx r \cdot \varepsilon \quad (4.14)$$

Now we can combine (4.14) and (4.11) to obtain $r \approx \frac{1}{2} \varepsilon$, which gives (1.3).
### 4.4 Critical model, $\mu = 1$

In this section we give a heuristic proof of Claim 1.3. We assume that

$$R(t) \approx c \cdot t^\alpha, \quad \frac{1}{2} < \alpha < 1.$$  

We will use the results of Sections 4.1 and 4.2 to identify the values of $\alpha$ and $c$.

We want to apply Lemma 4.1. The speed of the aggregate at time $t \gg 1$ is

$$r = \frac{d}{dt} c t^\alpha = c \alpha t^{\alpha - 1}. \tag{4.15}$$

Note that $0 < \frac{1}{2} < \alpha$ is the right assumption if we want to find $\Delta t$ such that (4.1) and (4.2) hold at time $t$. From now on we assume that the conclusion (4.3) of Lemma 4.1 holds.

We have

$$L(t) \approx \frac{1}{2r} \approx \frac{1}{2c\alpha} t^{1-\alpha}, \tag{4.10}$$

which implies

$$\frac{d}{dt} \mathbb{E}(L(t)) \approx \frac{1 - \alpha}{2c\alpha} t^{-\alpha}. \tag{4.16}$$

On the other hand, we have

$$\frac{d}{dt} \mathbb{E}(L(t)) \approx \frac{1}{2r} \approx \frac{1}{2c\alpha} t^{1-\alpha}, \tag{4.11}$$

Comparing (4.10) and (4.11) we obtain $\alpha = 2/3$ and $c = \frac{1}{2} \left(\frac{3}{2}\right)^{2/3}$. The heuristic proof of Claim 1.3 is complete.

### 5 Simulations (discrete-time)

We performed computer simulations of the discrete-time version of the one-dimensional multi-particle DLA model. Let us briefly explain how to modify the definitions of Section 1.1 and the predictions of Section 1.3 in this setting.

The time parameter is now $T = 0, 1, 2, \ldots$. Denote by $R(T)$ the size of the sticky aggregate at time $T$. For all $T \in \mathbb{N}$ and $i \in \mathbb{N}$, $i > R(T)$ we denote by $N(i, T)$ the number of particles at time $T$ at position $i$. The particles perform independent discrete-time simple symmetric random walks until they hit the sticky aggregate. If at time $T$ there is a particle at position $R(T) + 1$ which attempts to jump to the left, then it sticks to the aggregate and dies, moreover the size of the sticky aggregate increases by one, i.e. $R(T + 1) = R(T) + 1$. If $R(T + 1) = R(T) + 1$, then all other particles that were either at position $R(T) + 1$ or at position $R(T) + 2$ at time $T$ and tried to jump to the left are also swallowed by the aggregate (while those particles that jumped to the right survive). In words: first the particles jump, and then the aggregate increases and swallows the particles.
The conjectured behaviour of the continuous and discrete-time models are the same in the subcritical setting $\mu < 1$, thus Claim 1.1 remains the same in the above-described discrete-time model: $\lim_{T \to \infty} R(T)/\sqrt{T} = r(\mu)$, where $r(\mu)$ is defined in (1.1).

However, in the critical and barely supercritical case, some of the constants need to be adjusted because of the difference between the discrete-and continuous-time model. Namely, in the discrete-time case, (4.11) has to be replaced by

$$E[L(T + 1) - L(T)] \approx \frac{5}{2} r^2,$$

and consequently equation (1.3) in Claim 1.2 is replaced by

$$\lim_{\varepsilon \to 0^+} \frac{r(1 + \varepsilon)}{\varepsilon} = \frac{2}{5},$$

where $\lim_{T \to \infty} \frac{R(T)}{T} = r(\mu)$ (5.1)

and equation (1.4) in Claim 1.3 is replaced by

$$\lim_{T \to \infty} \frac{R(T)}{T^{2/3}} = \sqrt{\frac{9}{40}} \approx 0.608.$$ (5.2)

Below we list our numerical findings for different values of the parameter $\mu$. For each value of $\mu$ we performed 100 independent experiments, where an experiment consists of running the simulation for $10^5$ time-steps.

- **Subcritical case:** $\mu = 0.4382$. By (1.1), we have $r(\mu) = \frac{1}{2}$, thus at time $T = 10^5$ we should see $R(10^5) \approx 158.1$ according to Claim 1.1. Our computer simulations produced the empirical mean $\overline{R}(10^5) = 154.6$, moreover the empirical standard deviation of the 100 independent experiments of $R(10^5)$ was $\sigma = 17.17$.

- **Critical case:** $\mu = 1$. By (5.2) we should see $R(10^5) \approx 1309.8$. The empirical mean was $\overline{R}(10^5) = 1150.42$ and the empirical standard deviation was $\sigma = 259.98$.

- **Supercritical case:** $\mu = 1.02$. Using the notation of (5.1), we have $\varepsilon = 0.02$. In this case we declared the initial values of the particle densities (c.f. (2.1)) to be $\lambda(0, z) = \mu \cdot \left(1 - e^{-\frac{1}{4}z}\right)$, because this is the conjectured stationary shape of particle densities, c.f. Lemma 4.1. By (5.1) we should see $R(10^5) \approx 800$. The empirical mean was $\overline{R}(10^5) = 762.87$ and the empirical standard deviation was $\sigma = 166.43$.

These simulation results are reassuring in the subcritical case, but in the (super)critical case they raise some doubts, especially because in these cases the empirical standard deviations are quite big.

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