Coupling Constants and Brane Tensions from Anomaly Cancellation in M-theory

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Abstract

The theory of eleven dimensional supergravity on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ with super Yang-Mills theory on the boundaries is reconsidered. We analyse the general solution of the modified Bianchi identity for the four-form field strength using the equations of motion for the three-form and find that the four-form field strength has a unique value on the boundaries of $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$. Considering the local supersymmetry in the “downstairs” approach this leads to a relation between the eleven dimensional supergravity coupling constants in the “upstairs” and “downstairs” approaches. Moreover, it is shown using flux quantization that the brane tensions only have their standard form in the “downstairs” units. We consider the gauge variation of the classical theory and find that it cannot be gauge invariant, contrary to a recent claim. Finally we consider anomaly cancellation in the “downstairs” and “upstairs” approaches and obtain the values of $\lambda^6/\kappa^4$ and the two- and five-brane tensions.

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1 Introduction and summary

In several recent papers\cite{1,2,3,4,5,6,7} the eleven dimensional supergravity theory on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ with super Yang-Mills theory on the boundaries has been discussed. Three important issues have come up:

1) With the convention that $\frac{1}{\kappa^2} \int_{M_{11}} d^{11}x \frac{1}{2} \sqrt{-G} R$ is the Einstein-Hilbert term in the action in the “upstairs” approach and $\frac{1}{\kappa^2} \int_{M_D^{11}} d^{11}x \frac{1}{2} \sqrt{-G} R$ is the Einstein-Hilbert term in the action in the “downstairs” approach (see next section for definition of $M_{11}$ and $M_D^{11}$), it was argued in \cite{2} that $\bar{\kappa}^2 = 2\kappa^2$ and that $\bar{\kappa}$ was the eleven dimensional supergravity coupling constant. In \cite{5} it was argued that $\bar{\kappa}^2 = 2\kappa^2$, but that $\kappa$ was the eleven dimensional supergravity coupling constant. Finally, in \cite{6} it was argued that $\bar{\kappa} = \kappa$ so that both $\kappa$ and $\bar{\kappa}$ was the eleven dimensional supergravity coupling constant. It is crucial which of the coupling constants that is the right one, we have for instance the relation \cite{10} $2\kappa_{11}^2 = (2\pi)^8 (\alpha')^2$ here with the name $\kappa_{11}$ for the eleven dimensional supergravity coupling constant. In this paper we prove the relation $\bar{\kappa}^2 = 2\kappa^2$ and that $\kappa$ is the eleven dimensional supergravity coupling constant.

2) In \cite{7} it was conjectured that there exists a consistent classical theory of eleven dimensional supergravity with super Yang-Mills theory on the boundaries. This was based on the general solution to the Bianchi identity where an arbitrary parameter is introduced. For certain values of this parameter the theory was shown to be gauge invariant. This is completely contrary to the original claim of Ho\v{r}ava and Witten in \cite{2} that a consistent theory necessarily is a quantum theory, since the classical theory is not gauge invariant. In a quantum theory, the gauge variation of the classical theory can then cancel with the gauge anomaly in the effective action. In this paper we show that the gauge variation of the classical theory cannot be zero, so quantization of the theory is necessary.

3) Starting with a general solution of the modified Bianchi identity, it was claimed in \cite{6} that it was necessary to use gauge, gravitational and mixed anomaly cancellation and to include two- and five-brane quantization plus a half-integral quantization of $G^W/2\pi \kappa^4$ (see later for details) in order to determine the value of $\lambda^6/\kappa^4$. It was further claimed that one could not determine $\lambda^6/\kappa^4$ in the “downstairs” approach. In this paper we show that it is possible to determine $\lambda^6/\kappa^4$ by working with gauge anomaly cancellation in the “downstairs” approach alone.

We start in section 2 by considering the general solution to the modified Bianchi identity where we have an arbitrary parameter called $\beta$. By the equations of motion for the three-form $C$ it is shown that the four-form field strength $K$ surprisingly has a value on the boundaries of $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$ that is independent of $\beta$. In section 3 we consider the local supersymmetry of the theory in the “downstairs” approach and use this to find the value of the four-form $K$ on the boundaries of $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$. Comparing the two values of $K$, we find that $\bar{\kappa}^2 = 2\kappa^2$. In section 4 we find that the classical theory cannot be gauge invariant so that a quantized theory with anomalies is necessary. In section 5 we consider the gauge anomaly cancellation in the “downstairs” approach.
and find $\lambda^6/\kappa^4 = (4\pi)^5$ and $\lambda^6/\bar{\kappa}^4 = 256\pi^5$. In section 6 we use this to prove that $\kappa$ must be the eleven dimensional supergravity coupling constant by use of the flux quantization rule. In sections 7 and 8 we consider gauge, gravitational and mixed anomaly cancellations, in section 7 it is in the “downstairs” approach and in section 8 it is in the “upstairs” approach. The two methods give the same results.

An important conclusion to draw from this paper, is that one can derive all the results without using the “upstairs” method at all. The modification of the Bianchi identity is not necessary, since from section 3 we see that we can calculate the four-form field strength $K$ on the boundaries by working entirely in the “downstairs” approach. This is contrary to the “upstairs” approach where one has to use the four-form field strength in the “downstairs” approach, in order to use the flux quantization rule. So the “downstairs” approach seems to have all the advantages: It is more natural conceptually, there is no need for the modification of the Bianchi identity with the arbitrary parameter in the solution, and the anomaly cancellation is easier to work out.

2 Analysis of the modified Bianchi identity

The “downstairs” eleven dimensional space-time is $M_D^{11} = \mathbb{R}^{10} \times S^1/\mathbb{Z}_2 = \mathbb{R}^{10} \times [0, \pi \sqrt{\alpha'}]$ and the “upstairs” eleven dimensional space-time is $M_U^{11} = \mathbb{R}^{10} \times S^1$ with $x^{11}$ equivalent to $x^{11} + 2\pi \sqrt{\alpha'}$. $M^{10}$ is the boundary at $x^{11} = 0$ and $M'_{10}$ is the boundary at $x^{11} = \pi \sqrt{\alpha'}$.

The bosonic terms in the “upstairs” eleven dimensional supergravity theory is

$$S_{SUGRA} = \frac{1}{\kappa^2} \int_{M_U^{11}} d^{11}x \sqrt{-G} \left( \frac{1}{2} R - \frac{1}{48} K_{MNY} K^{MNY} \right) + S_{CKK}$$

with

$$S_{CKK} = -\frac{\sqrt{2}}{\bar{\kappa}^2} \int_{M_U^{11}} C \wedge K \wedge K$$

where $C$ is the three-form and $K = 6dC$ is the four-form field strength. We use uppercase greek letters for the 11 dimensional indices and lowercase greek letters for the 10 dimensional indices. The eleven dimensional metric $G_{MN}$ and the ten dimensional metric is $g_{\mu\nu}$. We choose $\Gamma_{11} = \Gamma_0 \Gamma_1 \cdots \Gamma_9$ where the bars on the indices indicate flat indices. In the “upstairs” approach we introduce an orbifold transformation acting as $x^{11} \to \bar{x}^{11} = -x^{11}$ and $x^{11} \to \bar{x}^{11} = x^{11}$. If we demand that the lagrangian is invariant under this transformation we get that $C_{\mu\nu\rho}(\bar{x}) = -C_{\mu\nu\rho}(x)$, $C_{11\mu\nu}(\bar{x}) = C_{11\mu\nu}(x)$, $K_{\mu\nu\rho}(\bar{x}) = -K_{\mu\nu\rho}(x)$ and $K_{11\mu\nu}(\bar{x}) = K_{11\mu\nu}(x)$. We combine this action with super Yang-Mills theory on $M^{10}$ and $M'_{10}$. On $M^{10}$ the bosonic part of the super

\footnote{We use the eleven dimensional supergravity action with the notation from [3], except for the indices and the renaming of the four-form field strength.}

\footnote{We use the metric signature $-++\cdots+$.}
Yang-Mills action is

\[ S_{\text{SYM}} = -\frac{1}{\lambda^2} \int_{M^{10}} d^{10}x \sqrt{-g} \frac{1}{4} \text{tr}(F_{\mu \nu} F^{\mu \nu}) \]

where \( F = DA + A^2 \) is the gauge field strength and \( A \) is the gauge field connection. As Ho\'rava and Witten pointed out in [2], local supersymmetry in the “upstairs” approach requires a modification of the Bianchi-identity \( dK = 0 \) to

\[ dK = \frac{1}{\sqrt{2} \lambda^2} \delta(x^{11}) dx^{11} \wedge I_4 \]

where \( I_4 = -\text{tr}(F^2) \). This identity has the general solution [4, 6]

\[ K = 6dC - \frac{\beta}{\sqrt{2} \lambda^2} \delta(x^{11}) dx^{11} \wedge I_3 + \frac{1}{2\sqrt{2}} (1 - \beta) \frac{\kappa^2}{\lambda^2} \epsilon(x^{11}) I_4 \]

where \( \beta \) is an arbitrary parameter and where \( I_3 = -\text{tr}(DA + \frac{2}{3} A^3) \) so that \( dI_3 = I_4 \). In [6] Lu added the condition that \( C_{\mu \nu \xi} = 0 \) on \( M^{10} \), but as we shall see, this is not consistent with the equations of motion.

The equations of motion for the three-form \( C \) in eleven dimensional supergravity is

\[ \partial_{\mu} (\sqrt{-G} K^{MN\Xi \Upsilon}) + \frac{\sqrt{2}}{72} \epsilon^{M_1 M_2 ... M_7 N \Xi \Upsilon} \partial_{\mu} (K_{M_1 M_2 M_3 M_4} C_{M_5 M_6 M_7}) = \frac{\sqrt{2}}{3456} \epsilon^{M_1 M_2 ... M_8 N \Xi \Upsilon} K_{M_1 M_2 M_3 M_4} K_{M_5 M_6 M_7 M_8} \]

where we again only consider the bosonic terms of the action. We see that the only term proportional to \( \partial_1 K_{11 \mu \nu \xi} \) is the term \( \sqrt{-G} \partial_1 K_{11 \mu \nu \xi} \), so if \( K_{11 \mu \nu \xi} \) has a term proportional to \( \delta(x^{11}) \) then \( \partial_1 K_{11 \mu \nu \xi} \) has a term proportional to \( \partial_1 \delta(x^{11}) \). But it is not possible for any of the other terms in the equations of motion to be proportional to \( \partial_1 \delta(x^{11}) \), so we conclude that \( K_{11 \mu \nu \xi} \) cannot have a term proportional to \( \delta(x^{11}) \). Since \( K_{11 \mu \nu \xi}(\bar{x}) = K_{11 \mu \nu \xi}(x) \) we cannot have a term proportional to \( \epsilon(x^{11}) \) in \( K_{11 \mu \nu \xi} \) either. This means that \( K_{11 \mu \nu \xi} \) is well-defined at \( x^{11} = 0 \), so we must have that

\[ dC_{11 \mu \nu \xi} = \frac{\beta}{6\sqrt{2} \lambda^2} \delta(x^{11}) (I_3)_{\mu \nu \xi} + U_{\mu \nu \xi} \]

where \( U_{\mu \nu \xi} \) is a 10 dimensional 3-form that is well-defined at \( x^{11} = 0 \). With the definition \( B_{\mu \nu} \equiv C_{11 \mu \nu} \) we have

\[ dC_{11 \mu \nu \xi} = \partial_{11} C_{\mu \nu \xi} - dB_{\mu \nu \xi} \]

but if we set \( dB_{\mu \nu \xi} = c \delta(x^{11}) (I_3)_{\mu \nu \xi} + U'_{\mu \nu \xi} \), where \( U'_{\mu \nu \xi} \) is a 10 dimensional 3-form without any terms proportional to \( \delta(x^{11}) \), we get \( d(dB)_{\mu \nu \xi \upsilon} = c \delta(x^{11}) (I_4)_{\mu \nu \xi \upsilon} + dU'_{\mu \nu \xi \upsilon} \) but since \( d(dB)_{\mu \nu \xi \upsilon} = 0 \) we must have \( c = 0 \) since \( dU'_{\mu \nu \xi \upsilon} \) cannot cancel a term.
proportional to $\delta(x^{11})$. Since $B_{\mu\nu}(\tilde{x}) = B_{\mu\nu}(x)$ there cannot be terms in $B_{\mu\nu}$ proportional to $\epsilon(x^{11})$, so $B_{\mu\nu}$ must be well-defined at $x^{11} = 0$. So since $\partial_{11} C_{\mu\nu\xi} = \frac{\beta \kappa^2}{6\sqrt{2} \lambda^2} \delta(x^{11}) (I_3)_{\mu\nu\xi} + U_{\mu\nu\xi} + dB_{\mu\nu\xi}$ we get

$$C = \frac{\beta}{12\sqrt{2} \lambda^2} \delta(x^{11}) I_3 + C'$$

where $C'$ is an 11 dimensional 3-form that is well-defined at $x^{11} = 0$. From this we obtain

$$dC = \frac{\beta \kappa^2}{6\sqrt{2} \lambda^2} \delta(x^{11}) dx^{11} \wedge I_3 + \frac{\beta}{12\sqrt{2} \lambda^2} \delta(x^{11}) I_4 + dC'$$

and

$$K = \frac{1}{2\sqrt{2} \lambda^2} \delta(x^{11}) I_4 + 6dC'$$

So the surprising result is that $K$ does not have any dependence on the arbitrary parameter $\beta$, contrary to what was found in [6, 7].

A membrane must experience the same field strength $K$ in the bulk in the two approaches. This means that we can find $K$ on $M^{10}$ in the “downstairs” approach by taking the limiting value of $K$ for $x^{11} \to 0^+$ in the “upstairs” approach[2, 5, 6]. This gives

$$K|_{M^{10}} = \frac{1}{2\sqrt{2} \lambda^2} I_4$$

### 3 Local supersymmetry in the “downstairs” approach

In [2] it was shown that under a local supersymmetry transformation, the $\psi \eta F^2$ terms in the variation of the super Yang-Mills action on $M^{10}$ is:

$$\Delta = \frac{1}{96\lambda^2} \int_{M^{10}} d^{10}x \sqrt{-g} \psi_\mu \Gamma^{\nu\xi\nu\rho} \eta \psi_\Theta \Gamma^{\nu\xi\nu\rho} \psi_\Theta K_{\nu\xi\nu\rho}$$

in the notation of [2] (except for the indices). To cancel this variation in the “upstairs” approach, one modifies the Bianchi identity for $K$ as described in the previous section. In the “downstairs” approach, we must instead consider total derivatives with respect to the eleventh coordinate in the supersymmetry variation of the lagrangian. We consider the following term in the eleven dimensional supergravity action[8, 2]:

$$-\frac{1}{\kappa^2} \int_{M^{11}_D} d^{11}x \sqrt{-G} \frac{\sqrt{2}}{192} \psi_M \Gamma^{MN\Xi\Theta\Psi} \psi_\Theta K_{\Xi\Theta\Psi}$$

\[4\] This is the variation of the modified super Yang-Mills action derived in [8] starting from the globally supersymmetric super Yang-Mills action.
in the notation of [2] (except for the indices and the renaming of the four-form $K$). Under a supersymmetry transformation we have that $\delta \psi_M = \partial_M \eta + \cdots$ where $\cdots$ represents the terms without derivatives of $\eta$. Ignoring the $\cdots$ terms we get the variation

$$-\frac{1}{\kappa^2} \int_{M_D^{11}} d^{11}x \sqrt{-G} \frac{\sqrt{2}}{96} \bar{\psi}_M \Gamma^{MNP\Theta} \partial_{\Theta} \eta K_{N\Xi P}$$

so the contribution from this containing a total eleventh derivative is

$$-\frac{1}{\kappa^2} \int_{M_D^{11}} d^{11}x \partial_{11} \left( \sqrt{-G} \frac{\sqrt{2}}{96} \bar{\psi}_\mu \Gamma^{\mu \nu \xi \upsilon} \eta K_{\nu \xi \upsilon} \right)$$

Ignoring the contribution from $M_{10}'$ this becomes

$$\frac{1}{\kappa^2} \int_{M_1^{10}} d^{10}x \sqrt{-g} \frac{\sqrt{2}}{96} \bar{\psi}_\mu \Gamma^{\mu \nu \xi \upsilon} \eta K_{\nu \xi \upsilon}$$

where we used that $\Gamma^{11} \eta = \sqrt{-g} \sqrt{-G} \eta$ on $M_1^{10}$. In order to cancel the $\Delta$ term, we must require

$$K\big|_{M_1^{10}} = -\frac{1}{\sqrt{2} \lambda^2} \text{tr}(F^2) = \frac{1}{\sqrt{2} \lambda^2} I_4$$

This is completely consistent with [4] from the “upstairs” approach provided that we have the relation

$$\bar{\kappa}^2 = 2 \kappa^2$$

So this relation can be seen as a consequence of demanding local supersymmetry in both the “upstairs” and the “downstairs” approach.

### 4 The gauge variation of the classical theory

In the following we calculate the gauge variation of the classical theory. In the “downstairs” approach we always have $K = 6dC$ so we have that

$$C\big|_{M_1^{10}} = \frac{1}{6\sqrt{2} \lambda^2} I_3$$

up to an irrelevant exact form. If we now make the gauge variation $\delta A = [D, v] = dv + [A, v]$ we have that $\delta K\big|_{M_1^{10}} = 0$ and

$$\delta C\big|_{M_1^{10}} = \frac{1}{6\sqrt{2} \lambda^2} dI_1^2$$

where $I_1^2 = -\text{tr}(vdA)$. The only possible non-gauge-invariant term in the combined supergravity and super Yang-Mills action is the $S_{CKK}$ term, and we have

$$\delta S_{CKK} = -\frac{\sqrt{2}}{\kappa^4} \int_{M_1^{11}} \delta C \wedge K \wedge K = -\frac{1}{6\lambda^2} \int_{M_1^{11}} d(I_1^2 \wedge K \wedge K)$$

$$= \frac{1}{6\lambda^2} \int_{M_1^{11}} I_1^2 \wedge K \wedge K = \frac{1}{12 \lambda^6} \int_{M_1^{11}} I_1^2 \wedge (I_4)^2$$
So the classical theory cannot be gauge-invariant, contrary to the claim in [7]. As explained in [2], we must then consider the combined eleven dimensional supergravity and super Yang-Mills theory as an effective low-energy theory for a quantum theory, so that the quantum anomalies in the effective action can cancel the gauge variation of the classical theory.

5 The gauge anomaly in the “downstairs” approach

In order to cancel the gauge variation of the classical theory, we must use the gauge group $E_8$ [2]. For ten dimensional Majorana-Weyl spinors, we have the 12-form (see [5]; the spinors have positive chirality under $\Gamma_{11}$)

$$I_{12} = \frac{1}{384(2\pi)^5} (I_4)^3$$

with the descent equations

$$I_{12} = dI_{11}, \quad \delta I_{11} = dI_{10}$$

so that the gauge anomaly in the effective action is

$$\delta W = \int_{M^{10}} I_{10} = -\frac{1}{384(2\pi)^5} \int_{M^{10}} I_2^1 \wedge (I_4)^2$$

Since we want $\delta S_{KK} + \delta W = 0$ we find

$$\frac{\lambda^6}{\kappa^4} = (4\pi)^5, \quad \frac{\lambda^6}{\kappa^4} = 256\pi^5$$

This is the same result as in [1]. It is important to note that we obtained this by using only the gauge anomaly in the “downstairs” approach. This was deemed impossible in [1].

6 Proof that $\kappa$ is the eleven dimensional supergravity coupling constant

We have proved that $\tilde{\kappa}^2 = 2\kappa^2$. To prove that $\kappa$ is the eleven dimensional supergravity coupling constant we start by assuming that $\tilde{\kappa}$ is the eleven dimensional supergravity coupling constant and show that this leads to an inconsistency. From [10] we have the quantization rule for the two-brane tension

$$(T_2)^3 = \frac{(2\pi)^2}{2\kappa^2 m}, \quad m \in \mathbb{Z}$$
From [11] we know that $G^W/2\pi$ should have a half integral period, where in our notation $G^W = \sqrt{2T_2 K}$, so since

$$\sqrt{2T_2 K}|_{M^{10}} = T_2 \frac{\kappa^2}{\lambda^2} \frac{1}{4\pi} \hat{I}_4 = \frac{1}{(2m)^{\frac{3}{2}}} \frac{1}{16\pi^2} \hat{I}_4$$

this means that we have the flux quantization rule

$$\frac{1}{(2m)^{\frac{3}{2}}} \in \mathbb{Z}$$

but that is impossible. If instead we assume that $\kappa$ is the eleven dimensional supergravity coupling constant, we can again write the two-brane tension quantization rule [10]

$$(T_2)^3 = \frac{(2\pi)^2}{2\kappa^2 m}, \quad m \in \mathbb{Z}$$

giving

$$\sqrt{2T_2 K}|_{M^{10}} = T_2 \frac{\kappa^2}{\lambda^2} \frac{1}{2\pi} \hat{I}_4 = \frac{1}{m^{\frac{3}{2}}} \frac{1}{16\pi^2} \hat{I}_4$$

so that the flux quantization rule [11] implies

$$\frac{1}{m^{\frac{3}{2}}} \in \mathbb{Z}$$

This is fulfilled if and only if $m = 1$ so that

$$(T_2)^3 = \frac{(2\pi)^2}{2\kappa^2}$$

This is the standard form for the two-brane tension (see for example [10]).

7 **Gauge, gravitational and mixed anomalies in the “downstairs” approach**

To extend our anomaly analysis to include gravitational and mixed anomalies, we must replace $I_4$ with $\hat{I}_4 = \frac{1}{2} \text{tr}(R^2) - \text{tr}(F^2)$ in (1) and (5). This was pointed out in [2] based on the knowledge of the structure of ten dimensional anomalies. Here $R = d\omega + \omega^2$ is the curvature two-form and $\omega$ is the spin connection. With the local Lorentz variation $\delta \omega = [D, \Theta]$, we can write

$$\hat{I}_3 = \frac{1}{2} \text{tr}(\omega d\omega + \frac{2}{3} \omega^3) - \text{tr}(\text{Ad}A + \frac{2}{3} A^3), \quad \hat{I}_2 = \frac{1}{2} \text{tr}(\Theta d\omega) - \text{tr}(\nu dA)$$

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5 In sections 7 and 8 we extend our anomaly analysis to include gravitational and mixed anomalies. This has the consequence that $I_4 = -\text{tr}(F^2)$ is replaced by $\hat{I}_4 = \frac{1}{2} \text{tr}(R^2) - \text{tr}(F^2)$ in (1) and (5). In the following we use these modified expressions for the four-form field strength on $M^{10}$.

6 See also [1, 3, 5].

7 Assuming $m$ is positive.
so that we have the descent equations

\[ d\hat{I}_3 = \hat{I}_4, \quad \delta\hat{I}_3 = d\hat{I}_2 \]

Replacing \( I \) with \( \hat{I} \) we have

\[ \delta K \big|_{M^{10}} = 0, \quad \delta C \big|_{M^{10}} = \frac{1}{6\sqrt{2} \nu^2} \hat{I}_2 \]

and

\[ \delta S_{C \bar{K} K} = \frac{1}{12} \kappa^4 \int_{M^{10}} \hat{I}_2 \wedge (\hat{I}_4)^2 \]

The anomalous 12-form with gauge, gravitational and mixed anomalies is (see \[5\])

\[ \hat{I}_{12} = -\frac{1}{96(2\pi)^5} \hat{I}_4 \wedge (\frac{1}{4}(\hat{I}_4)^2 - X_8) \]

where \( X_8 = -\frac{1}{8} \text{tr}(R^4) + \frac{1}{12}(\text{tr}(R^2))^2 \), with the descent equations

\[ \hat{I}_{12} = d\hat{I}_{11}, \quad \delta\hat{I}_{11} = d\hat{I}_{10} \]

so that the anomaly takes the form

\[ \delta W = \int_{M^{10}} \hat{I}_{10} = -\frac{1}{96(2\pi)^5} \int_{M^{10}} \hat{I}_2 \wedge (\frac{1}{4}(\hat{I}_4)^2 - X_8) \]

This only partly cancels with \( \delta S_{C \bar{K} K} \), so we have to introduce the five-brane term in the classical action. In “downstairs” units, it is \[3\]

\[ S_5 = \frac{\sqrt{2}}{8(2\pi)^3 \kappa^2 T_5} \int_{M^{11}} C \wedge X_8 \]

This term has the variation

\[ \delta S_5 = \frac{\sqrt{2}}{8(2\pi)^3 \kappa^2 T_5} \frac{1}{6\sqrt{2} \nu^2} \int_{M_{11}} d(\hat{I}_2 \wedge X_8) = -\frac{1}{48(2\pi)^3 \nu^2 T_5} \int_{M^{10}} \hat{I}_2 \wedge X_8 \]

So we obtain \( \delta W + \delta S_{C \bar{K} K} + \delta S_5 = 0 \) if and only if

\[ \frac{\lambda^6}{\kappa^4} = (4\pi)^5 \text{ and } (T_5)^3 = \frac{2\pi}{(2\nu^2)^2} \]

We see that \( T_5 \) has the standard form (see for example \[10\]).

8
8 Gauge, gravitational and mixed anomalies in the “upstairs” approach

In this section the anomaly cancellation in the “upstairs” approach is considered. The purpose is to check whether the anomalies cancel for the same values of $\lambda^6/\kappa^4$ and $T_5$ as in the “downstairs” method.

From (2) and (3) we have

$$C = \frac{\beta}{12\sqrt{2}} \frac{\tilde{\kappa}^2}{\lambda^2} \epsilon(x^{11}) \hat{I}_3 + C'$$

$$K = \frac{1}{2\sqrt{2}} \frac{\tilde{\kappa}^2}{\lambda^2} \epsilon(x^{11}) \hat{I}_4 + 6dC'$$

The variation of $C$ is

$$\delta C = \frac{\beta}{12\sqrt{2}} \frac{\tilde{\kappa}^2}{\lambda^2} \epsilon(x^{11}) \hat{I}_1 2 + \delta C'$$

Since we want $\delta K = 0$, we require $d(\delta C') = 0$. Using

$$K \wedge K = \frac{1}{8} \frac{\tilde{\kappa}^4}{\lambda^4} \hat{I}_4 (\hat{x}^{11})^2 + 36dC' \wedge dC' + \frac{6}{\sqrt{2}} \frac{\tilde{\kappa}^2}{\lambda^2} \epsilon(x^{11}) \hat{I}_4 \wedge dC'$$

we find

$$\int_{M_{U}^{11}} \delta C' \wedge K \wedge K = \frac{6}{\sqrt{2}} \frac{\tilde{\kappa}^2}{\lambda^2} \int_{M_{U}^{11}} \epsilon(x^{11}) \hat{I}_4 \wedge dC' \wedge \delta C' = 0$$

where we used that $C'_{\mu\nu\xi} = 0$ on $M^{10}$ since $C'_{\mu\nu\xi}$ is odd under the orbifold transformation. So

$$\delta S_{CKK} = \frac{\sqrt{2}}{\tilde{\kappa}^2} \int_{M_{U}^{11}} (\delta C - \delta C') \wedge K \wedge K = \frac{\beta}{12} \frac{1}{\lambda^2} \int_{M_{U}^{11}} \epsilon(x^{11}) d\hat{I}_2 \wedge K \wedge K$$

$$= \frac{\beta}{6} \frac{\tilde{\kappa}^4}{\lambda^6} \int_{M_{U}^{11}} \epsilon(x^{11})(d^{11} \hat{I}_2 \wedge K + \epsilon(x^{11}) \hat{I}_2 \wedge dK \wedge K)$$

$$= \frac{\beta}{6} \frac{\tilde{\kappa}^4}{16 \lambda^6} \int_{M_{U}^{11}} \epsilon(x^{11})^2 d^{11} \hat{I}_2 \wedge (\hat{I}_4)^2 = \frac{\beta}{48} \frac{\tilde{\kappa}^4}{\lambda^6} \int_{M_{U}^{11}} \delta(x^{11}) d^{11} \hat{I}_2 \wedge (\hat{I}_4)^2$$

$$= \frac{\beta}{48} \frac{\tilde{\kappa}^4}{\lambda^6} \int_{M_{10}} \hat{I}_2 \wedge (\hat{I}_4)^2$$

The five-brane term in “upstairs” units is

$$S_5 = \frac{\sqrt{2}}{8(2\pi)^4 \tilde{\kappa}^2 T_5} \int_{M_{U}^{11}} C \wedge X_8$$
Hence
\[
\delta S_5 = \frac{\sqrt{2}}{8(2\pi)^3 \kappa^2 T_5} \frac{\beta}{12 \sqrt{2} \lambda^2} \int_{M^1}^{} \epsilon(x^{11}) d\hat{I}_2^1 \wedge X_8 = \frac{\beta}{96(2\pi)^3 \lambda^2 T_5} \int_{M^1}^{} \epsilon(x^{11}) d\hat{I}_2^1 \wedge X_8
\]
\[= -\frac{\beta}{48(2\pi)^3 \lambda^2 T_5} \int_{M^1}^{} \delta(x^{11}) dx^{11} \wedge \hat{I}_2^1 \wedge X_8 = -\frac{\beta}{48(2\pi)^3 \lambda^2 T_5} \int_{M^{10}}^{} \hat{I}_2^1 \wedge X_8
\]
So the anomaly cancellation \(\delta W + \delta S_{CKK} + \delta S_5 = 0\) implies
\[
\frac{\lambda^6}{\kappa^4} = \beta(4\pi)^5 \quad \text{and} \quad (T_5)^3 = \beta^2 \frac{2\pi}{(2\kappa^2)^2}
\]
We have the quantization rules for the two- and five-brane tensions, \[9, 10\]
\[2\kappa^2 T_2 T_5 = 2\pi n, \quad n \in \mathbb{Z} \quad \text{and} \quad (T_2)^3 = \frac{(2\pi)^2}{2\kappa^2 m}, \quad m \in \mathbb{Z}
\]
So using this, we find
\[\beta^2 = mn^3
\]
Since
\[
\frac{\sqrt{2}}{2\pi} T_2 K|_{M^{10}} = \frac{1}{(\beta m)^3} \frac{1}{16\pi^2} \hat{I}_4 = \frac{1}{\sqrt{mn}} \frac{1}{16\pi^2} \hat{I}_4
\]
we have the flux quantization rule, \[11\]
\[\frac{1}{\sqrt{mn}} \in \mathbb{Z}
\]
so that the unique solution is\[8\]
\[\beta = m = n = 1
\]
which gives
\[
\frac{\lambda^6}{\kappa^4} = (4\pi)^5, \quad (T_2)^3 = \frac{(2\pi)^2}{2\kappa^2} \quad \text{and} \quad (T_5)^3 = \frac{2\pi}{(2\kappa^2)^2}
\]
So we get exactly the same solution as for the “downstairs” method.

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\[8\] Assuming \(m\) and \(n\) are positive.
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