Spinors and the AdS/CFT correspondence

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Abstract

We consider a free massive spinor field in Euclidean Anti-de Sitter space. The usual Dirac action in bulk is supplemented by a certain boundary term. The boundary conditions of the field are parametrized by a spinor on the boundary, subject to a projection. We calculate the dependence of the partition function on this boundary spinor. The result agrees with the generating functional of the correlation functions of a quasi-primary spinor operator, of a certain scaling dimension, in a free conformal field theory on the boundary.
1. Introduction

The relationship between $d$-dimensional conformal field theories and field theories in $(d + 1)$-dimensional Anti-de Sitter space has been the focus of studies from various viewpoints over the years. Conformal field theories provide a field theoretical interpretation for the so-called “singleton” representations [1,2] of the symmetry group $SO(d,2)$ [3], which correspond to gauge degrees of freedom everywhere except in the boundary [4]. A further development of this correspondence was that the singletons of AdS-spaces were related to brane solutions of supergravity via the indentification of the various brane world-volume fields with members of the corresponding (super)-singleton multiplets [5,6,7,8]. The AdS/CFT correspondence was exploited in [9] in connection with the physics of non-extremal black holes. In parallel, it has been conjectured that there is a relation between the large-$N$ limit of certain superconformal gauge theories (realized by branes) and the supergravity limit of $M$-theory or string theory [10].

A more precise AdS/CFT relationship was suggested in [11,12,13] as follows: $AdS_{d+1}$ has a boundary $M_d$ at spatial infinity. This means that the action functional $S[\phi]$ of a field theory on $AdS_{d+1}$ must be supplemented by some boundary conditions on the field $\phi$. (For notational simplicity we consider only the case of a single field. The generalization to several fields is straightforward.) To specify a boundary condition, we first choose a function $x^0$ on $AdS_{d+1}$ with a simple zero on the boundary. The boundary condition then amounts to

$$\lim_{x^0 \to 0} (x^0)^\Delta \phi = \phi_0,$$

for some (finite) field $\phi_0$ defined on the boundary $M_d$. The value of the constant $\Delta$ is fixed by the requirement that the classical equations of motion allow this behaviour of $\phi$. We can now calculate the partition function

$$Z_{AdS}[\phi_0] = \int_{\phi_0} \mathcal{D}\phi \exp(-S[\phi]) ,$$

where the subscript on the integral indicates that we should only integrate over field configurations that fulfil the boundary condition (1.1).

We can obtain a finite metric $g$ on $M_d$ by multiplying the metric $G$ on $AdS_{d+1}$ by $(x^0)^2$ and restricting to $M_d$. The freedom to choose the function $x^0$, subject to the restriction

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1 The case of $AdS_3$ is quite well understood [14]. Additional recent work concerning various aspects of the AdS/CFT correspondence can be found in [15,16,17,18,19,20,21].
that it has a simple zero on \( M_d \), means that only the conformal structure of \( M_d \) is uniquely determined. The \( O(d+1,1) \) isometry group of \( AdS_{d+1} \) acts as the conformal group on \( M_d \). We can therefore equate the partition function \( Z_{AdS}[\phi_0] \) with the generating functional

\[
Z_{CFT}[\phi_0] = \langle \exp \int_{M_d} d^d x \sqrt{g} O \phi_0 \rangle ,
\]

of a quasi-primary operator \( O \) in some conformal field theory on \( M_d \). Such an operator is characterized by its quantum numbers under an \( O(d) \times SO(1,1) \) subgroup of the conformal group, i.e. by its Lorentz group representation and scaling dimension [22]. Invariance of the \( \int d^d x \sqrt{g} O \phi_0 \) coupling implies that \( O \) and \( \phi_0 \) transform in conjugate Lorentz representations and that the sum of their scaling dimensions is \( d \). The boundary behaviour (1.1) and the fact that the metric on \( AdS_{d+1} \) has a double pole on \( M_d \) then determines the scaling dimension of \( O \) to be equal to \( \Delta \).

Some examples of this AdS/CFT correspondence were given in [13], where, in particular, the cases of a free massive scalar field and of a free \( U(1) \) gauge theory on \( AdS_{d+1} \) were considered. The exact partition function (1.2) is then given by the tree-level contribution, i.e. by the exponential of the action evaluated for a field configuration that solves the classical equations of motion with boundary conditions given by (1.1). For such a field configuration, the action can in fact be rewritten as a total derivative and thus reduces to a boundary term. The result is that the partition function (1.2) indeed equals the generating functional (1.3) of the correlation functions of a free quasi-primary operator of a certain scaling dimension.

Fermionic fields are obviously of equal importance to scalar and gauge fields. Hence, it will be the purpose of this paper to perform an analysis for the free Dirac spinor field \( \psi \) and its conjugate \( \bar{\psi} \) of mass \( m \) on \( AdS_{d+1} \). It turns out that the usual Dirac action in bulk must be supplemented by a certain boundary term. Also, one would expect that the boundary conditions are determined by a spinor field \( \psi_0 \) and its conjugate \( \bar{\psi}_0 \) on \( M_d \). For \( d \) even, \( \psi_0 \) would be a Dirac spinor, whereas for \( d \) odd, it would be a pair of Dirac spinors. Actually, it will turn out that half of \( \psi_0 \) (and of \( \bar{\psi}_0 \)) will have to be put to zero, so we are left with a chiral spinor or a single Dirac spinor for \( d \) even or odd respectively. Which half is retained is determined by the sign of the mass \( m \). In this way we find that the partition function indeed reproduces the correlation functions of a free quasi-primary spinor operator \( O \) of scaling dimension \( \Delta = \frac{d}{2} + |m| \).

For some of these issues there is an overlap with a recent paper [23], which appeared while we were completing this work.
2. The action

Our starting point is the usual free spinor action on $AdS_{d+1}$

$$S_0 = \int_{AdS} d^{d+1}x \sqrt{G} \bar{\psi} (\slashed{D} - m) \psi ,$$  \hspace{1cm} (2.1)

from which follow the Dirac equations of motion

$$ (\slashed{D} - m) \psi = 0 ,$$

$$ \bar{\psi} (\slashed{\partial} - m) = 0 .$$  \hspace{1cm} (2.2)

Before we determine the boundary conditions to be imposed at infinity, we will discuss a difficulty with the action (2.1). Since this is a free field theory, the exact partition function is given by the tree-level contribution. Up to a constant, that arises from the Gaussian path integral over the square integrable fluctuations of the fields, this equals the exponential of the action functional evaluated for a field configuration that solves the classical equations of motion and obeys the boundary conditions. However, for a spinor field, the action (2.1) vanishes for any field configuration that satisfies the equations of motion (2.2). The same is true for the total derivative term

$$\int_{AdS} d^{d+1}x \sqrt{G} \left( \bar{\psi} \slashed{\partial} \psi + \bar{\psi} \slashed{\partial} \psi \right) ,$$  \hspace{1cm} (2.3)

a multiple of which could be added to the action without changing the equations of motion. It would therefore seem that the partition function is actually independent of the boundary conditions and would not reproduce any conformal field theory correlation functions.

To avoid this conclusion, we propose that the action (2.1) be supplemented by a multiple of

$$S_1 = \lim_{\epsilon \to 0} \int_{M^\epsilon_d} d^dx \sqrt{G_\epsilon} \bar{\psi} \psi ,$$  \hspace{1cm} (2.4)

where $M^\epsilon_d$ is a closed $d$-dimensional submanifold of $AdS_{d+1}$, which approaches the boundary manifold $M_d$ of $AdS_{d+1}$ as $\epsilon$ goes to zero. The metric $G_\epsilon$ on $M^\epsilon_d$ appearing in the expression (2.4) is the one induced from the metric $G$ of $AdS_{d+1}$. While perhaps slightly unfamiliar, the addition of this term to the action preserves the crucial properties of the original action (2.1): it is invariant under the $O(d+1,1)$ isometry group of $AdS_{d+1}$, since this group maps $M_d$ to itself. The equations of motion in the bulk of $AdS_{d+1}$ are still given by (2.2). While the term (2.4) clearly depends on the boundary conditions on the fields, it does not affect the path integral over square integrable quantum fluctuations. In the present context, the relative coefficient between the terms (2.1) and (2.4) is not determined, and we will only assume that it is non-zero. In other theories, gauge invariance or supersymmetry, for example, will impose restrictions on the coefficients of various boundary terms.
3. The computation

In this section, we will show that the theory on $AdS_{d+1}$ described above is indeed equivalent to the free conformal field theory on $M_d$ of a quasi-primary spinor operator $O$. We choose coordinates $x^\mu = (x^0, x^i) = (x^0, x^i)$, $i = 1, \ldots, d$, such that Euclidean $AdS_{d+1}$ is represented by the domain $x^0 > 0$ and the metric $ds^2$ is given by

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = (x^0)^{-2}(dx^0 dx^0 + g_{ij}dx^i dx^j) = (x^0)^{-2}(dx^0 dx^0 + dx \cdot dx) .$$  \hspace{1cm} (3.1)

The boundary $M_d$ is defined by the hypersurface $x^0 = 0$ plus a single point at $x^0 = \infty$. The metric $d\tilde{s}^2$ on $M_d$ is obtained by multiplying $ds^2$ by $(x^0)^2$ and restricting to $M_d$ so that

$$d\tilde{s}^2 = g_{ij}dx^i dx^j = dx \cdot dx .$$  \hspace{1cm} (3.2)

To couple a spinor field to this background, we need to choose a local Lorentz frame, i.e. a vielbein $e^a_\mu$, $a = 0, \ldots, d$ such that $G_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. A convenient choice is

$$e^a_\mu = (x^0)^{-1}\delta^a_\mu ,$$  \hspace{1cm} (3.3)

for which the corresponding spin connection $\omega^a_{\mu b}$ has

$$\omega^0_i = -\omega^0_i = (x^0)^{-1}\delta^i_0$$  \hspace{1cm} (3.4)

and all other components vanishing. With these choices, the operator $\mathcal{D}$ is given by

$$\mathcal{D} = e^a_\mu \Gamma^a (\partial_\mu + \frac{1}{2}\omega^b_{\mu c} \Sigma^c_{bc}) = x^0 \Gamma_0 \partial_0 + x^0 \Gamma \cdot \nabla - \frac{d}{2} \Gamma^0 ,$$  \hspace{1cm} (3.5)

where $\Gamma^a = (\Gamma^0, \Gamma^i) = (\Gamma^0, \Gamma^i)$ fulfill $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ and $\partial_\mu = (\partial_0, \partial_i)$ is $(\partial_0, \nabla)$.

Next, we will construct solutions to the equations of motion (2.2) with various boundary behaviours. This can be accomplished by first constructing a solution with singular behaviour at a single point on the boundary $M_d$ (most conveniently the point at $x^0 = \infty$), and then applying an element of the $O(d + 1, 1)$ isometry group of $AdS_{d+1}$ to move the singularity to an arbitrary point on the boundary. (This transformation has to be accompanied by a compensating local Lorentz transformation to preserve the gauge choice (3.3).) In this way one constructs the field configurations

$$\psi(x^0, x) = \int d^d x' (x^0 \Gamma^0 + (x-x') \cdot \Gamma) \left((x^0)^2 + |x-x'|^2\right)^{-\frac{d+1}{2} + m \Gamma^0} (x^0)^{\frac{d}{2} - m \Gamma^0} \psi_0(x') ,$$

$$\bar{\psi}(x^0, x) = \int d^d x'' \bar{\psi}_0(x'') (x^0)^{\frac{d}{2} + m \Gamma^0} \left((x^0)^2 + |x-x''|^2\right)^{-\frac{d+1}{2} - m \Gamma^0} (x^0 \Gamma^0 + (x-x'') \cdot \Gamma) ,$$  \hspace{1cm} (3.6)
which solve (2.2) for arbitrary finite spinors \( \psi_0(x') \) and \( \bar{\psi}_0(x'') \).

To determine the boundary behaviour of these solutions, it is convenient to decompose \( \psi_0(x') \) and \( \bar{\psi}_0(x'') \) as

\[
\psi_0(x') = \psi_+(x') + \psi_-(x') , \\
\bar{\psi}_0(x'') = \bar{\psi}_+(x'') + \bar{\psi}_-(x''),
\]

where \( \Gamma^0 \psi_\pm(x') = \pm \psi_\pm(x') \) and \( \bar{\psi}_+ \bar{\psi}_+(x'') \Gamma^0 = \pm \bar{\psi}_\pm(x'') \). Without loss of generality we take the mass \( m \) to be positive. One then finds that

\[
\lim_{\epsilon \to 0} (x^0)^{-\frac{d}{2} + m} \psi(x^0, x) = -c \psi_-(x) + \int d^d x' |x - x'|^{-d - 1 + 2m} (x - x') \cdot \Gamma \psi_+(x') , \\
\lim_{\epsilon \to 0} (x^0)^{-\frac{d}{2} + m} \bar{\psi}(x^0, x) = c \bar{\psi}_+(x) + \int d^d x'' \bar{\psi}_-(x'') (x - x'') \cdot \Gamma |x - x''|^{-d - 1 + 2m},
\]

where the constant \( c = \pi^{d/2} \Gamma(m + \frac{1}{2})/\Gamma(m + \frac{d+1}{2}) \). For the right-hand side of (3.8) to be square integrable, with respect to the measure \( d^d x \) on \( M_d \), we will furthermore have to impose the conditions

\[
\psi_+(x') = 0 , \\
\bar{\psi}_-(x'') = 0 .
\]

(A similar condition has appeared in [24] for the case of a Rarita-Schwinger field.) Before (3.4) are imposed, \( \psi_0(x') \) and \( \bar{\psi}_0(x') \) both transform as a Dirac spinor or two Dirac spinors when \( d \) is even or odd, respectively. The two irreducible terms of this representation are distinguished by their \( \Gamma^0 \) eigenvalues. (When \( d \) is even, \( \Gamma^0 \) is simply the chirality operator on \( M_d \).) The conditions (3.9) thus mean that we are left with a chiral spinor when \( d \) is even and a single Dirac spinor when \( d \) is odd.

As discussed in the previous section, the bulk action (2.1) vanishes for such a configuration, so the entire contribution to the partition function comes from the boundary term (2.4). To calculate this term, we take the hypersurface \( M^\epsilon_d \) to be given by \( x^0 = \epsilon = \text{constant} \). The induced metric on \( M^\epsilon_d \) is then \( ds^2_\epsilon = \epsilon^{-2} d\mathbf{x} \cdot d\mathbf{x} \), with determinant \( G_\epsilon = \epsilon^{-2d} \). In this way we obtain

\[
Z_{AdS}[\psi_-, \bar{\psi}_+] = \exp(-S_1) = \exp \left( - \int d^d x'' \int d^d x' \bar{\psi}_+ (x'') \Omega(x'', x') \psi_-(x') \right), \tag{3.10}
\]

where

\[
\Omega(x'', x') = \lim_{\epsilon \to 0} \int d^d x \epsilon^{2m+1} \left( \epsilon^2 + |x - x''|^2 \right)^{-\frac{d+1}{2} - m} \left( \epsilon^2 + |x - x'|^2 \right)^{-\frac{d+1}{2} - m} (x'' - x') \cdot \Gamma, \tag{3.11}
\]

\[
= c |x'' - x'|^{-d - 2m} (x'' - x') \cdot \Gamma .
\]
4. Comparison to conformal field theory

We will now compare these results to what would be expected from a conformal field theory on $M_d$. The expression (3.10) can be interpreted as the generating functional $Z_{\text{CFT}}[\psi_-, \bar{\psi}_+]$ of correlation functions of a free-field operator $O^\alpha$ and its conjugate $\bar{O}^\alpha$ on $M_d$ with the two-point function

$$\langle \bar{O}^\beta(x'')O^\alpha(x') \rangle = \Omega^{\alpha\beta}(x'',x') \ ,$$

where we have written out the spinor indices $\alpha$ and $\beta$ explicitly.

The expression (4.1) is in fact the correct two-point function for a quasi-primary spinor operator of scaling dimension

$$\Delta = \frac{d}{2} + m \ .$$

Indeed, for an operator $O^\alpha$ transforming in some representation of the $O(d)$ Lorentz group and with scaling dimension $\Delta$, conformal invariance fixes the two-point function, up to normalization, to be [25]

$$\langle \bar{O}^\beta(x'')O^\alpha(x') \rangle = |x'' - x'|^{-2\Delta} D^{\alpha\beta}(R(x'',x')) \ ,$$

where $D^{\alpha\beta}(R(x'',x'))$ denotes the representation matrix in the appropriate representation of the $O(d)$ element

$$R^i_j(x'',x') = \delta^i_j - 2|x'' - x'|^{-2}(x'' - x')^i(x'' - x')_j \ .$$

We note that $\det(R^i_j) = -1$. For the spinor representation, we have

$$D^{\alpha\beta}(R(x'',x')) = -|x'' - x'|^{-1}(x'' - x') \cdot (\Gamma^0)^{\alpha\beta} \ ,$$

which can be checked by verifying the invariance of $(\Gamma^i)^{\alpha\beta}$ under the rotation (4.4): $R^i_j D^{-1} \Gamma^j D = \Gamma^i$. In the subspace of definite $\Gamma^0$ eigenvalue we are working on, we may set $\Gamma^0 = -1$ in (4.3). Then (4.3), with $\Delta = \frac{d}{2} + m$, agrees with (4.1).

We believe that techniques similar to those used in the present paper are appropriate for the spin-3/2 Rarita–Schwinger field on AdS. Having understood the AdS/CFT correspondence for free fields, a natural continuation would be to consider interacting theories. It would be interesting to see how the results of this paper, in particular the need for an extra boundary term in the action, generalize in that context.
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