On Karatsuba’s Problem Concerning the Divisor Function $\tau(n)$¹

M.A. Korolev

Abstract. We study an asymptotic behavior of the sum $\sum_{n \leq x} \frac{\tau(n)}{\tau(n + a)}$. Here $\tau(n)$ denotes the number of divisors of $n$ and $a \geq 1$ is a fixed integer.

1. Introduction

This paper deals with a problem stated by A.A. Karatsuba in November, 2004: to determine an asymptotic behavior of the following sum

$$S(x) = \sum_{n \leq x} \frac{\tau(n)}{\tau(n + 1)}$$

(Here $\tau(n)$ denotes the number of positive divisors of $n$). Since

$$\frac{1}{x} \sum_{n \leq x} \tau(n) \sim \ln x, \quad \frac{1}{x} \sum_{n \leq x} \frac{1}{\tau(n)} \sim \frac{c}{\sqrt{\ln x}}$$

for some $c > 0$, the below assumption seems reasonable:

$$\frac{1}{x} S(x) \sim \frac{K}{\sqrt{\ln x}} \cdot \ln x \sim K \sqrt{\ln x}, \quad K > 0.$$ 

The aim of this paper is to prove the following

Theorem. Let $a$ be a fixed integer, $a \geq 1$, and let

$$S_a(x) = \sum_{n \leq x} \frac{\tau(n)}{\tau(n + a)}.$$

Then

$$S_a(x) = K(a) x \sqrt{\ln x} + O(x \ln \ln x),$$

as $x \to +\infty$. The constant $K(a)$ has the form $K(a) = K \cdot \kappa(a)$, where

$$K = \frac{1}{\sqrt{\pi}} \prod_{p} \left( \frac{1}{\sqrt{p(p-1)}} + \sqrt{1 - \frac{1}{p} (p-1) \ln \frac{p}{p-1}} \right) = 0.757 \, 827 \, 651 \ldots,$$

$$\kappa(a) = \beta(a) \prod_{p|a} \frac{1 + \sum_{k=1}^{+\infty} e_a(p^k) p^{-k}}{1 + \beta(p) \sum_{k=1}^{+\infty} p^{-k} p^{-k} p^{-k} + 1},$$

$$e_a(n) = \frac{1}{\beta(a)} \sum_{d | (a,n)} \beta \left( \frac{an}{d^2} \right), \quad \beta(a) = \prod_{p|a} \frac{(p-1)^2}{p^2 - p + 1}.$$ ¹This research was supported by the Programme of the President of the Russian Federation ‘Young Candidates of the Russian Federation’ (grant no. MK-4052.2009.1).
For the below, we need the following notations:

- \( \varphi(q) \) denotes Euler function;
- \( \chi \) denotes Dirichlet’s character modulo \( q, q \geq 3 \);
- the symbols \( \sum_{\chi \mod q} \) and \( \sum_{\chi \neq \chi_0} \) denote the sums over all non-principal characters modulo \( q \);
- the symbols \( \sum_{\chi \mod q}^* \) and \( \sum_{\chi}^* \) denote the sums over all primitive characters modulo \( q \);
- \( s = \sigma + it, \sigma, t \) are real numbers;
- \( L(s, \chi) \) is Dirichlet’s \( L \)-function corresponding to the character \( \chi \);
- the symbol \( N(\sigma; T, \chi) \) means the number of zeros of \( L(s, \chi) \) in the rectangle \( \sigma < \text{Re} \, s \leq 1, |\text{Im} \, s| \leq T \);
- \( (a, b) \) denotes the great common divisor of \( a \) and \( b \);
- symbols \( \theta, \theta_1, \theta_2, \ldots \) denote complex numbers such that \( |\theta|, |\theta_1|, |\theta_2|, \ldots \leq 1 \), in general, different in different relations.

2. Auxiliary assertions

**Lemma 1.** Suppose \( S(t) \) is a smooth complex-valued function for \( t_0 \leq t \leq t_k \), \( t_0 < t_1 < \ldots < t_k \) and \( \min_{0 \leq j \leq k-1} (t_{j+1} - t_j) = \delta > 0 \). Then the following inequality holds:

\[
\sum_{j=1}^{k} |S(t_j)|^2 \leq \frac{1}{\delta} j_1 + 2 \sqrt{j_1 j_2},
\]

where

\[
j_1 = \int_{t_0}^{t_k} |S(t)|^2 dt, \quad j_2 = \int_{t_0}^{t_k} |S'(t)|^2 dt.
\]

For the proof, see [1, Chapter VII, §1].

**Lemma 2.** Suppose \( M, N, Q \) are integers. Then for any sequence of complex numbers \( a_n \) the following estimation holds:

\[
\sum_{q \leq Q} \sum_{\chi \mod q}^* \left| \sum_{n=\frac{M}{q}+1}^{M+N} a_n \chi(n) \right|^2 \ll (Q^2 + N) \sum_{n=\frac{M}{q}+1}^{M+N} |a_n|^2,
\]

where the constant in the symbol \( \ll \) is absolute.

For the proof, see [1, Chapter IX, answers to problems].

**Lemma 3.** Suppose \( q \geq 3, \chi \) is non-principle Dirichlet’s character modulo \( q \). Then for any \( s, Y \) such that \( \text{Re} \, s \geq \sigma_0 > 0 \) and \( Y \geq q(|t| + 1)/\pi \) the following equality holds:

\[
L(s, \chi) = \sum_{n \leq Y} \frac{\chi(n)}{n^s} + O(qY^{-\sigma}),
\]

where the constant in the symbol \( O \) is absolute.

For the proof, see [2, §26].
Lemma 4. Let $c$ be a sufficiently small positive absolute constant and let $q \geq 3$. Suppose $\chi$ is a complex character modulo $q$. Then the function $L(s, \chi)$ has no zeros in the domain

$$\Re s > 1 - \frac{c}{\ln q(|t| + 1)}, \quad -\infty < t < +\infty;$$

now if $\chi$ is a real non-principal Dirichlet’s character modulo $q$ then the function $L(s, \chi)$ has no zeros in the domain

$$\Re s > 1 - \frac{c}{\ln q(|t| + 1)}, \quad |t| > 0.$$

For the proof, see [1, Chapter IX, §2].

Lemma 5 (Siegel). For any $\varepsilon$, $0 < \varepsilon < \frac{1}{2}$, there exists $c = c(\varepsilon) > 0$ such that if $\chi$ is a real character modulo $q$ and $\beta$ is a real zero of $L(s, \chi)$, then

$$\beta < 1 - \frac{c}{q^\varepsilon}.$$

For the proof, see [1, Chapter IX, §2]. The constant $c = c(\varepsilon)$ is not effective. This means that it is impossible to find or estimate $c(\varepsilon)$ from a given $\varepsilon$. Therefore all statements (including main theorem of this paper) in which this lemma is essentially used are ineffective, too.

Lemma 6 (Montgomery). For any $Q \geq 3$, $T \geq 3$ the following estimation holds:

$$\sum_{q \leq Q} \sum_{\chi \bmod q}^* N(\sigma; T, \chi) \ll (Q^2T)^{\vartheta(\sigma)}(\ln QT)^{14},$$

where

$$\vartheta(\sigma) = \begin{cases} 3(1 - \sigma), & \text{if } \frac{1}{2} \leq \sigma \leq \frac{4}{5}, \\ 2(1 - \sigma), & \text{if } \frac{4}{5} \leq \sigma \leq 1, \end{cases}$$

and the constant in the symbol $\ll$ is absolute.

For the proof, see [3, Chapter 12].

Lemma 7. Suppose $q \geq 1$ is an integer, $\chi$ is a character modulo $q$ and $\gamma = \beta + i\gamma$ runs through all non-trivial zeros of $L(s, \chi)$. Then

$$\sum_{|\gamma| \leq T} \frac{1}{|\gamma| + 1} \ll (\ln qT)^2,$$

as $T \to +\infty$.

This estimation follows from asymptotic formula for $N(T, \chi)$ - the number of zeros of $L(s, \chi)$ in the rectangle $0 \leq \Re s \leq 1$, $|\Im s| \leq T$. 

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3. Basic assertions

**Lemma 8.** Let $\chi$ be a non-primitive character modulo $q = q_1 r$ induced by primitive character $\chi_1$ modulo $q_1$. Then

$$|L(s, \chi)| \leq \tau(r)|L(s, \chi_1)|$$

in the half-plane $\Re s \geq 0$.

**Proof.** Using the formula

$$L(s, \chi) = L(s, \chi_1) \prod_{p \mid q, p \nmid q_1} \left(1 - \chi_1(p) \frac{1}{p^s}\right),$$

we get

$$|L(s, \chi)| \leq |L(s, \chi_1)| \prod_{p \mid r} \left(1 + \frac{1}{p^s}\right) \leq |L(s, \chi_1)| \prod_{p \mid r} 2 \leq \tau(r)|L(s, \chi_1)|.$$

**Lemma 9.** Suppose $f(n)$ is non-negative multiplicative function such that $f(n) = O(n^\varepsilon)$, $0 < \varepsilon < \frac{1}{2}$, and the function

$$F(s) = \sum_{n=1}^{+\infty} \frac{f(n)}{n^s}$$

satisfies the identity

$$F(s) = \sqrt{\zeta(s)} \Phi(s)$$

for $\Re s > 1$ ($\sqrt{z} > 0$ for $z > 0$), where $\Phi(s)$ is regular in the half-plane $\Re s > \frac{1}{2}$ and obeys the estimate

$$|\Phi(\sigma + it)| \ll \max\left\{1, (\sigma - \frac{1}{2})^{-c}\right\}$$

for any $\sigma > \frac{1}{2}$ and some $c > 0$. Then

$$\sum_{n \leq x} f(n) = \frac{x}{\sqrt{\ln x}} \left(\frac{\Phi(1)}{\sqrt{\pi}} + O\left(\frac{1}{\ln x}\right)\right)$$

where the constant in the symbol $O$ depends on $f$ only.

**Proof.** Since $f(n) = O(n^\varepsilon)$, without loss of generality we may assume that $x$ has the form $N + \frac{1}{2}$ for some integer $N$. Suppose $T$ differs from the imaginary part of any zero of $\zeta(s)$ and $2 \leq T \leq x$. Then, by Perron’s formula we get

$$\sum_{n \leq x} f(n) = I + O\left(\frac{x}{T \ln x}\right)$$

where

$$I = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} \, ds, \quad b = 1 + \frac{1}{\ln x}. $$
Let $c_1 > 0$ be small positive constant such that $\zeta(s)$ has no zeros in the rectangle
with vertices $1 \pm iT$, $\alpha \pm iT$,
\[
\alpha = 1 - c_1 (\ln T)^{-\frac{2}{3}} (\ln \ln T)^{-\frac{1}{3}}.
\]
Then, by the identity $F(s) = \sqrt{\zeta(s)} \Phi(s)$ the function $F(s)$ continues analytically
to the domain $\alpha \leq \Re s \leq 1$, $|\Im s| \leq T$ with horizontal cut going straight
from the point $s = \alpha$ to the point $s = 1$. By Cauchy’s theorem,
\[
I = - \sum_{k=1}^{6} I_k,
\]
where the symbols $I_1, \ldots, I_4$ denote the integrals along the segments connecting
points $b+iT$, $\alpha+iT$, $\alpha$, $\alpha-iT$, $b-iT$ and the symbols $I_5, I_6$ denote the integrals
along the upper and lower edges of the cut respectively.

Since the bound
\[
\zeta(\sigma + it) = O\left(\ln^\frac{2}{3}(|t| + 2)\right)
\]
holds along the contour, we obtain:
\[
|I_1| + |I_4| \ll \int_\alpha^b \left( |F(\sigma + iT)| + |F(\sigma - iT)| \right) \frac{x^\sigma d\sigma}{\sqrt{\alpha^2 + T^2}} \ll \frac{x}{T} (\ln T)^\frac{1}{3},
\]
\[
|I_2| + |I_3| \ll \int_{-T}^T |F(\alpha + it)| \frac{x^\alpha dt}{\sqrt{\alpha^2 + t^2}} \ll x^\alpha \int_0^T \frac{\ln^\frac{1}{3}(t+2)}{t+2} dt \ll x^\alpha (\ln x)^\frac{4}{3}.
\]

Consider the function $u(s) = (s-1)\zeta(s)$, $u(1) = 1$. Since $u(s) \neq 0$ for $|s-1| \leq \frac{1}{3}$,
it follows that
\[
I_5 + I_6 = \frac{1}{2\pi i} \int_\alpha^1 \left( \sqrt{\zeta(\sigma + i0)} - \sqrt{\zeta(\sigma - i0)} \right) \frac{\Phi(\sigma) x^\sigma d\sigma}{\sigma} =
\]
\[
= \frac{1}{2\pi i} \int_\alpha^1 \left( \frac{1}{\sqrt{\sigma - 1 + i0}} - \frac{1}{\sqrt{\sigma - 1 - i0}} \right) \Phi(\sigma) \frac{x^\sigma d\sigma}{\sigma} =
\]
\[
= \frac{1}{2\pi i} \int_0^{1-\alpha} \left( \frac{1}{i\sqrt{v}} - \frac{1}{-i\sqrt{v}} \right) \frac{\Phi(1-v) \sqrt{u(1-v)}}{1-v} x^{1-v} dv =
\]
\[
= - \frac{x}{\pi} \int_0^{1-\alpha} \frac{x^{-v} \Phi(1-v) \sqrt{u(1-v)}}{\sqrt{v}} dv.
\]
The functions $\Phi(s)$ and $u(s)$ are bounded for $|s-1| \leq \frac{1}{3}$. This implies that
\[
\frac{\Phi(1-v) \sqrt{u(1-v)}}{1-v} = \Phi(1) + O(|v|)
\]
where the constant in the $O$-symbol depends on $\Phi$ only. Thus we obtain
\[
I_5 + I_6 = - \frac{x}{\pi} \int_0^{1-\alpha} \frac{x^{-v}}{\sqrt{v}} (\Phi(1) + O(v)) dv =
\]
\[
= - \frac{x}{\pi} \Phi(1) \left( \int_0^\infty \frac{x^{-v}}{\sqrt{v}} dv - \int_{1-\alpha}^{+\infty} \frac{x^{-v}}{\sqrt{v}} dv \right) + O\left(x \int_0^{+\infty} \sqrt{v x^{-v}} dv\right).
\]
Since
\[ (1 - \alpha) \ln x = c_1(\ln x)(\ln T)^{-\frac{2}{3}}(\ln \ln T)^{-\frac{1}{3}} \geq c_1 \left( \frac{\ln x}{\ln \ln x} \right)^{\frac{1}{3}} > 1, \]
we get
\[ \int_{1-\alpha}^{+\infty} \frac{x^{-v}}{\sqrt{v}} \, dv = \frac{1}{\sqrt{\ln x}} \int_{(1-\alpha)\ln x}^{+\infty} \frac{e^{-w}}{\sqrt{w}} \, dw < \frac{1}{\sqrt{\ln x}} \int_{(1-\alpha)\ln x}^{+\infty} e^{-w} \, dw = \frac{x^{\alpha-1}}{\sqrt{\ln x}}, \]
\[ I_5 + I_6 = -\frac{\Phi(1)}{\sqrt{\pi}} \frac{x}{\sqrt{\ln x}} + O \left( \frac{x^{\alpha-1}}{\ln \ln x} \right) + O \left( \frac{x}{(\ln x)^{3/2}} \right). \]
Using the above inequalities for \( I_1, \ldots, I_4 \) and substituting \( T = e^{\sqrt{\ln x}} \), we finally obtain
\[ I = \frac{\Phi(1)}{\sqrt{\pi}} \frac{x}{\sqrt{\ln x}} \left( 1 + O \left( \frac{1}{\ln x} \right) \right). \]
The proof is complete.

**Lemma 10.** Let \( m \) be an integer, \( m \geq 1 \). Then
\[ \sum_{q \leq x \atop (q,m)=1} \frac{1}{\varphi(q)} = C \beta(m) \left( \ln x + \gamma - \sum_{p} \frac{\ln p}{p^2 - p + 1} + \sum_{p|m} \frac{p^2 \ln p}{(p - 1)(p^2 - p + 1)} \right) + O \left( \frac{\ln^2 x}{x} \right) + O \left( \frac{\tau(m) \ln x}{x} \right), \]
where \( \gamma \) is Euler’s constant,
\[ C = \prod_p \left( 1 + \frac{1}{p(p-1)} \right) = \frac{\zeta(2) \zeta(3)}{\zeta(6)}, \]
and the constants in \( O \)’s are absolute.

**Proof.** Note that
\[ \frac{1}{\varphi(q)} = \frac{1}{q} \prod_{p|q} \left( 1 - \frac{1}{p} \right)^{-1} = \frac{1}{q} \prod_{p|q} \left( 1 + \frac{1}{p - 1} \right) = \frac{1}{q} \prod_{p|q} \left( 1 + \frac{1}{\varphi(p)} \right) = \frac{1}{q} \sum_{d|q} \mu^2(d) \varphi(d). \]
Let us use the prime sign for the summation over the numbers coprime to \( m \). Thus we get
\[ \sum_{q \leq x} \frac{1}{\varphi(q)} = \sum_{q \leq x} \frac{1}{q} \sum_{d|q} \mu^2(d) \varphi(d) = \sum_{d\leq x} \frac{\mu^2(d)}{\varphi(d)} \sum_{q\leq x \atop q\equiv 0(d\mod)q} \frac{1}{q} = \]
\[ = \sum_{d\leq x} \mu^2(d) \varphi(d) \sum_{k\leq x/d} \frac{1}{k} = \sum_{d\leq x} \mu^2(d) \varphi(d) \sum_{k\leq x/d} \frac{1}{k} = \]
\[ = \sum_{d\leq x} \mu^2(d) \varphi(d) \left( \sum_{k=1}^{x/d} \frac{\mu(\delta)}{\delta} \right) \frac{1}{k} = \sum_{d\leq x} \mu^2(d) \varphi(d) \sum_{k=1}^{x/d} \frac{\mu(\delta)}{\delta} \sum_{\delta|m \atop k\equiv0(\delta\mod)\delta} \frac{1}{k} = \]
\[ = \sum_{\delta|m} \mu(\delta) \sum_{d\leq x} \frac{\mu^2(d)}{d \varphi(d)} \sum_{r\leq x/d\delta} \frac{1}{r\delta} = \]
\[ = \sum_{\delta|m} \mu(\delta) \sum_{d\leq x} \frac{\mu^2(d)}{d \varphi(d)} \left( \ln \frac{x}{d\delta} + \gamma + O \left( \frac{d\delta}{x} \right) \right). \]
The contribution of the error term if the brackets to the initial sum does not exceed in order
\[
\sum \frac{1}{\delta} \sum'_{d \leq x} \frac{1}{d \varphi(d)} \frac{d\delta}{x} = \frac{1}{x} \left( \sum_{\delta|m} 1 \right) \sum_{d \leq x} \frac{1}{\varphi(d)} = O\left( \frac{\tau(m) \ln x}{x} \right).
\]
The contribution of all other terms has the form
\[
\sum_{\delta|m} \frac{\mu(\delta)}{\delta} \sum'_{d \leq x} \frac{\mu^2(d)}{d \varphi(d)} (\ln x + \gamma - \ln d - \ln \delta) =
\]
\[
= (\ln x + \gamma) \left( \sum_{\delta|m} \frac{\mu(\delta)}{\delta} \right) \sum'_{d \leq x} \frac{\mu^2(d)}{d \varphi(d)} - \left( \sum_{\delta|m} \frac{\mu(\delta)}{\delta} \right) \sum_{d \leq x} \frac{\mu^2(d) \ln d}{d \varphi(d)} -
\]
\[
- \left( \sum_{\delta|m} \frac{\mu(\delta) \ln \delta}{\delta} \right) \sum'_{d \leq x} \frac{\mu^2(d)}{d \varphi(d)}.
\]
Let us replace all the sums over \(d \leq x\) by infinite sums over \(d\) coprime to \(m\). Using the inequality
\[
\sum_{q \leq x} \frac{1}{\varphi(q)} \ll \ln x,
\]
we obtain:
\[
\sum'_{d \leq x} \frac{\mu^2(d)}{d \varphi(d)} = \sum_{d=1}^{+\infty} \frac{\mu^2(d)}{d \varphi(d)} + O\left( \frac{\ln x}{x} \right),
\]
\[
\sum'_{d \leq x} \frac{\mu^2(d) \ln d}{d \varphi(d)} = \sum_{d=1}^{+\infty} \frac{\mu^2(d) \ln d}{d \varphi(d)} + O\left( \frac{\ln^2 x}{x} \right).
\]
Since
\[
\sum_{d=1}^{+\infty} \frac{\mu^2(d) \ln d}{d \varphi(d)} = \prod_{p|m} \left( 1 + \frac{1}{p(p-1)} \right) = \prod_{p} \left( 1 + \frac{1}{p(p-1)} \right) \prod_{p|m} \frac{p(p-1)}{p^2 - p + 1} =
\]
\[
= C \prod_{p|m} \frac{p(p-1)}{p^2 - p + 1},
\]
we have
\[
\sum'_{n \leq x} \frac{1}{\varphi(q)} = C \frac{\varphi(m)}{m} (\ln x + \gamma) \prod_{p|m} \frac{p(p-1)}{p^2 - p + 1} - \frac{\varphi(m)}{m} \sum_{d=1}^{+\infty} \frac{\mu^2(d) \ln d}{d \varphi(d)} -
\]
\[
- C \prod_{p|m} \frac{p(p-1)}{p^2 - p + 1} \left( \sum_{\delta|m} \frac{\mu(\delta) \ln \delta}{\delta} \right).
\]
Note that non-zero terms in the sum over \(\delta|m\) correspond to squarefree divisors \(\delta\), so we have
\[
\ln \delta = \sum_{p|\delta} \ln p.
\]
Therefore,
\[
\sum_{\delta|m} \mu(\delta) \ln \delta \cdot \sum_{\delta|m} \frac{\mu(\delta)}{\delta} = \sum_{\delta|m} \ln p \cdot \sum_{\delta|m} \frac{\mu(\delta)}{\delta} =
\]
\[
= \sum_{p|m} \ln p \cdot \sum_{\delta|p} \frac{\mu(p)\mu(\delta)}{p\delta} = -\sum_{p|m} \ln p \cdot \sum_{\delta|p, (\delta, p)=1} \frac{\mu(\delta)}{\delta} =
\]
\[
= -\sum_{p|m} \ln p \cdot \prod_{r|m, r\neq p} \left(1 - \frac{1}{r}\right) = -\sum_{p|m} \frac{\prod_{p|m} (1 - \frac{1}{r})}{1 - \frac{1}{p}} =
\]
\[
= -\frac{\varphi(m)}{m} \sum_{p|m} \frac{\ln p}{p-1}.
\]

Next, by the same arguments we obtain
\[
\sum_{d=1}^{+\infty} \frac{\mu^2(d) \ln d}{d \varphi(d)} = \sum_{d=1}^{+\infty} \frac{\mu^2(d)}{d \varphi(d)} \sum_{p|d} \ln p = \sum_{p} \ln p \cdot \sum_{d=1}^{+\infty} \frac{\mu^2(d)}{d \varphi(d)} =
\]
\[
= \sum_{p} \ln p \cdot \sum_{d_1=1}^{+\infty} \frac{\mu^2(pd_1)}{pd_1 \varphi(pd_1)} = \sum_{p} \ln p \cdot \sum_{d_1=1}^{+\infty} \frac{\mu^2(d_1)}{d_1 \varphi(d_1)} =
\]
\[
= \sum_{p} \ln p \cdot \prod_{q|m} \left(1 + \frac{1}{q(q-1)}\right) =
\]
\[
= \sum_{p} \frac{\ln p}{p(p-1)} \cdot \prod_{q|m} \left(1 + \frac{1}{q(q-1)}\right) =
\]
\[
= \sum_{p} \frac{\ln p}{p(p-1)} \cdot \frac{p(p-1)}{p^2 - p + 1} \prod_{q|m} \left(1 + \frac{1}{q(q-1)}\right)^{-1} =
\]
\[
= \sum_{p} \frac{\ln p}{p^2 - p + 1} \prod_{q|m} q(q-1) =
\]
\[
= \sum_{p} \frac{\ln p}{q^2 - q + 1} \prod_{q|m} \frac{q(q-1)}{q^2 - q + 1} =
\]
\[
= \sum_{p} \frac{\ln p}{q^2 - q + 1} \left(\sum_{p} \frac{\ln p}{p^2 - p + 1} - \sum_{p|m} \frac{\ln p}{p^2 - p + 1}\right).
\]

Substituting these relations in the above formula for the initial sum and taking into account the relation
\[
\frac{\varphi(m)}{m} \prod_{p|m} \frac{p(p-1)}{p^2 - p + 1} = \prod_{p|m} \frac{(p-1)^2}{p^2 - p + 1} = \beta(m),
\]
we get
\[ \sum_{q \leq x} \frac{1}{\varphi(q)} = C \beta(m) \left( \ln x + \gamma - \sum_p \frac{\ln p}{p^2 - p + 1} + \sum_{p|a} \frac{\ln p}{p^2 - p + 1} + \sum_{p|a} \frac{\ln p}{p - 1} \right) + 
+ O \left( \frac{\ln^2 x}{x} \right) + O \left( \frac{\tau(a) \ln x}{x} \right). \]

**Corollary.** Under the same conditions,
\[ \sum_{q \leq x} \frac{1}{\varphi(q)} = C \beta(m) \ln x + O(\tau(m)) \]
where the constant in the symbol \( O \) is absolute.

**Lemma 11.** Let \( d \geq 2 \) be a fixed integer. Suppose \( \delta \) runs over an increasing sequence that contains 1 and all integers which are not divisible by prime numbers coprime to \( d \). Suppose also
\[ D_1(x) = \sum_{\delta \leq x} 1, \quad D_2(x) = \sum_{\delta > x} \frac{1}{\delta}. \]

Then
\[ D_1(x) \ll (\ln x)^s, \quad D_2(s) \ll \frac{(\ln x)^s}{x}, \]
where \( 1 \leq s \leq \tau(d) \).

**Proof.** Let \( p_1^{a_1} \ldots p_s^{a_s} \) be the unique decomposition of \( d \) into prime-powers. Then \( D_1(x) \) equals to the number of solutions of the inequality \( p_1^{\beta_1} \ldots p_s^{\beta_s} \leq x \) or
\[ \beta_1 \ln p_1 + \ldots + \beta_s \ln p_s \leq \ln x \]
with non-negative integers \( \beta_1, \ldots, \beta_s \). Since \( \ln p_1 \geq \ln 2, \ldots, \ln p_s \geq \ln 2 \), it follows that \( D_1(x) \) does not exceed the number of solutions of the inequality
\[ \beta_1 + \ldots + \beta_s \leq m, \quad m = \left[ \frac{\ln x}{\ln 2} \right], \]
that is \( D_1(x) \leq \binom{m+s}{s} \). Applying Stirling’s formula and Cauchy’s inequality we obtain
\[ \binom{m+s}{s} = \frac{1}{s!} (m+s) \ldots (m+1) \leq \frac{(m+s)^s}{s!} \leq \frac{2^{s-1}(m^s + s^s)}{(s/e)^s} = \]
\[ = \frac{1}{2} \left( \frac{2em}{s} \right)^s + (2e)^s \right) \leq (2e m)^s \leq \left( \frac{2e \ln x}{\ln 2} \right)^s < (8 \ln x)^s. \]

Further,
\[ D_2(x) = \sum_{k=0}^{+\infty} \sum_{2^k x < \delta \leq 2^{k+1} x} \frac{1}{\delta} \leq \sum_{k=0}^{+\infty} \frac{1}{2^k x} \sum_{\delta \leq 2^{k+1} x} 1 \leq \sum_{k=0}^{+\infty} \frac{D_1(2^{k+1} x)}{2^k x} < \]
\[ < \frac{1}{x} \sum_{k=0}^{+\infty} \left( \frac{8 \ln (2^{k+1} x)}{2^k} \right)^s \leq \frac{1}{x} \sum_{k=0}^{+\infty} \frac{(\ln x)^s + (k+1)^s}{2^k} \leq \frac{(\ln x)^s}{x}. \]
It remains to note that \( s \leq \tau(d) \).

**Lemma 12.** For any fixed \( a \geq 1 \)
\[
S_a(x) = C(\ln x)\beta(a)E_a(x) + \theta R_a(x) + O(x \ln \ln x),
\]
where the constant \( C \) is defined in lemma 10,
\[
E_a(x) = \sum_{n \leq x} \frac{e_a(x)}{\tau(n)}, \quad e_a(n) = \frac{1}{\beta(a)} \sum_{d|(a,n)} \beta\left(\frac{an}{d^2}\right),
\]
\[
R_a(x) = \sum_{d|a} \sum_{\delta \leq (\ln x)^3} R_{a,d,\delta}(x),
\]
\[
R_{a,d,\delta}(x) = \sum_{q \leq y/d \atop (a, \frac{q}{\delta})=1} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \left| \sum_{m \leq \frac{x}{d\delta}} \frac{\chi(m)}{\tau(m)} \right|^2,
\]
\( y = \sqrt{x}(\ln x)^{-A}, \ A > 0 \) is arbitrary fixed number and \( \delta \) runs through the sequence defined in lemma 11.

**Proof.** By definition of \( \tau(n) \),
\[
S_a(x) = \sum_{uv \leq x} \frac{1}{\tau(uv + a)} = \left( \sum_{u \leq \sqrt{x}} \sum_{v \leq \frac{x}{u}} + \sum_{\sqrt{x} < u \leq x} \sum_{v \leq \frac{x}{u}} \right) \frac{1}{\tau(uv + a)} =
\]
\[
= \left( \sum_{u \leq \sqrt{x}} \sum_{v \leq \frac{x}{u}} + \sum_{\sqrt{x} < u \leq x} \sum_{v \leq \frac{x}{u}} \right) \frac{1}{\tau(uv + a)} =
\]
\[
= 2 \sum_{u \leq \sqrt{x}} \sum_{v \leq \frac{x}{u}} \frac{1}{\tau(uv + a)} - \theta \frac{1}{2} x = 2 \sum_{q \leq \sqrt{x}} \sum_{1 \leq n \leq x + a \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} - \theta \frac{1}{2} x.
\]
Replacing the domain of \( n \) in the inner sum to the interval \( 1 \leq n \leq x \), we obtain
\[
S_a(x) = 2 \sum_{q \leq \sqrt{x}} \left( \sum_{1 \leq n \leq x \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} - \sum_{1 \leq n \leq a \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} + \sum_{x+n \leq x+a \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} \right) \frac{1}{\tau(n)} - \theta \frac{1}{2} x =
\]
\[
= 2 \sum_{q \leq \sqrt{x}} \sum_{1 \leq n \leq x \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} + 2 \theta_1 \sum_{q \leq \sqrt{x}} \frac{a}{q} \sum_{q \leq \sqrt{x}} \left( \frac{a}{q} + 1 \right) - \theta \frac{1}{2} x =
\]
\[
= 2 \sum_{q \leq \sqrt{x}} \sum_{1 \leq n \leq x \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} + \theta_2 x.
\]
Suppose \( A \) is an arbitrary positive number and let \( y = \sqrt{x}(\ln x)^{-A} \). Then the summands corresponding to the values \( y < q \leq \sqrt{x} \) do not exceed
\[
2 \sum_{y < q \leq \sqrt{x}} \sum_{n \equiv x \atop n \equiv a(\text{mod } q)} \frac{1}{\tau(n)} \leq \sum_{y < q \leq \sqrt{x}} \left( \frac{x}{q} + 1 \right) <
\]
\[
< x(\ln \sqrt{x} - \ln y + O(y^{-1})) + \sqrt{x} = A \ln \ln x + 2\sqrt{x}(\ln x)^A.
\]

Thus we get

\[ S_a(x) = V_a(x) + 2\theta A \ln \ln x, \quad V_a(x) = 2 \sum_{q \leq y} \sum_{n \equiv a (\text{mod } q)} \frac{1}{\tau(n)}. \]

Let us transform the last sum. Suppose \( d = (q, a) \). Then \( q = dq_1 \), \( a = da_1 \)
where \( (q_1, a_1) = 1 \). If \( n \equiv a (\text{mod } q) \) then for some \( k \geq 0 \) we obtain

\[ n = a + kq = d(a_1 + kq_1) = dm, \]

where \( m \equiv a_1 (\text{mod } q_1) \) and \( m \leq x_d = x/d \). Thus the inner sum takes the form

\[ \sum_{m \leq x_d \atop m \equiv a_1 (\text{mod } q_1)} \frac{1}{\tau(dm)}. \]

Since all possible values of \( d \) are among the divisors of \( a \), it follows that

\[ V_a(x) = 2 \sum_{d \mid a} \sum_{q \leq y \atop (q,a) = d} \sum_{m \leq x_d \atop m \equiv a_1 (\text{mod } q_1)} \frac{1}{\tau(dm)} = 2 \sum_{d \mid a} \sum_{q_1 \leq y_d \atop (q_1, a_1) = 1} \sum_{m \leq x_d \atop m \equiv a_1 (\text{mod } q_1)} \frac{1}{\tau(dm)}, \]

where \( y_d = y/d \).

Suppose \( d \) is an arbitrary fixed divisor of \( a \) and \( \delta \) runs through an increasing sequence that contains 1 and positive integers all whose prime divisors are among prime divisors of \( d \). (In particular, if \( d = p \) is prime then \( \delta \) take values 1, \( p, p^2, p^3, \ldots \)). Then for any integer \( n \) there exists a unique representation in the form \( \delta m \) where \( \delta \) belongs to the above sequence and \((m, d) = (m, \delta) = 1 \). Obviously for such \( n = \delta m \) we have \( \tau(dm) = \tau(d\delta m) = \tau(d\delta)\tau(m) \). Thus we obtain

\[ V_a(x) = 2 \sum_{d \mid a} \sum_\delta V_{a,d,\delta}(x) \]

where

\[ V_{a,d,\delta}(x) = \sum_{q_1 \leq y_d \atop (q_1, a_1) = 1} \sum_{m \leq x_d \atop \delta m \equiv a_1 (\text{mod } q_1)} \frac{1}{\tau(d\delta m)}. \]

Suppose that \((q_1, a_1) = 1 \) and the congruence \( \delta m \equiv a_1 (\text{mod } q_1) \) holds. If both the numbers \( \delta \) and \( q_1 \) have the same divisor \( \delta' > 1 \) then \( \delta' \) divides \( a_1 \). Therefore \((q_1, a_1) \geq \delta' > 1 \). This contradiction shows that \((q_1, \delta) = 1 \). Thus the solutions of the above congruence have the form \( m \equiv a_1 \delta*(\text{mod } q_1) \). This implies that

\[ V_{a,d,\delta}(x) = \sum_{q_1 \leq y_d \atop (q_1, a_1) = (q_1, \delta) = 1} \sum_{m \leq x_d \atop \delta m \equiv a_1 \delta*(\text{mod } q_1)} \frac{1}{\tau(d\delta m)}. \]
Let $\delta_0 = (\ln x)^3$. All the sums $V_{a,d,\delta}(x)$ corresponding to $\delta > \delta_0$ are estimated trivially:

$$V_{a,d,\delta}(x) \leq \sum_{q_1 \leq y_d} \sum_{m \leq \frac{x}{\delta}} \sum_{d a \delta \equiv 1 \pmod{q_1}} 1 \leq \sum_{q_1 \leq y_d} \left( \frac{x_d}{\delta q_1} + 1 \right) \leq \frac{x}{d \delta} \ln x + \frac{y}{d}$$

where the constants in the symbols $\ll$ are absolute. By lemma 11, the contribution of these terms to $V_a(x)$ do not exceed in order

$$\sum_{d|a} \sum_{\delta > \delta_0} \left( \frac{x}{d \delta} \ln x + \frac{y}{d} \right) \ll x \ln x \left( \sum_{d|a} \frac{1}{\delta} \right) \sum_{\delta > \delta_0} \frac{1}{\delta} + y \left( \sum_{d|a} \frac{1}{\delta} \right) \left( \sum_{\delta < \delta \leq x} \frac{1}{\delta} \right) \ll$$

$$\ll x \ln x \frac{(\ln \delta_0)^\nu}{\delta_0} + y (\ln x)^\nu \ll \frac{x (\ln \ln x)^\nu}{(\ln x)^2} \ll \frac{x}{\ln x},$$

where $\nu = \tau(a)$ and the constants in the symbols $\ll$ depend only on $a$. Thus,

$$V_a(x) = 2 \sum_{d|a} \sum_{\delta \leq \delta_0} V_{a,d,\delta}(x) + O\left( \frac{x}{\ln x} \right).$$

Next, for $1 \leq \delta \leq \delta_0$ we have

$$V_{a,d,\delta}(x) = \sum_{q_1 \leq y_d} \sum_{\chi \equiv a \mod{q_1}} \sum_{m \leq \frac{x}{\delta}} \left( \frac{1}{\varphi(q_1)} \chi(a_1 \delta \nu) \chi(m) \right) \frac{1}{\tau(d \delta m)} =$$

$$= \sum_{q_1 \leq y_d} \frac{1}{\varphi(q_1)} \sum_{\chi \equiv a \mod{q_1}} \chi(\delta) \chi(a_1) \sum_{m \leq \frac{x}{\delta}} \frac{\chi(m)}{\tau(d \delta m)} =$$

$$= \sum_{q_1 \leq y_d} \frac{1}{\varphi(q_1)} \sum_{m \leq \frac{x}{\delta}} \frac{1}{\tau(d \delta m)} +$$

$$+ \frac{1}{\tau(d \delta)} \sum_{q_1 \leq y_d} \frac{1}{\varphi(q_1)} \sum_{\chi \equiv a \mod{q_1}, \chi \neq \chi_0} \frac{\chi(\delta) \chi(a_1)}{\tau(m)} \sum_{m \leq \frac{x}{\delta}} \frac{\chi(m)}{\tau(m)}.$$

Therefore,

$$V_a(x) = W_a(x) + 2 \theta R_a(x) + O\left( \frac{x}{\ln x} \right)$$

where

$$W_a(x) = 2 \sum_{d|a} \sum_{\delta \leq \delta_0} \sum_{q_1 \leq y_d} \frac{1}{\varphi(q_1)} \sum_{m \leq \frac{x}{\delta}} \frac{1}{\tau(d \delta m)},$$

$$R_a(x) = \sum_{d|a} \sum_{\delta \leq \delta_0} R_{a,d,\delta}(x),$$

$$R_{a,d,\delta}(x) = \sum_{q \leq y_d} \frac{1}{\varphi(q_1)} \sum_{\chi \equiv a \mod{q}, \chi \neq \chi_0} \frac{1}{\tau(m)} \sum_{m \leq \frac{x}{\delta}} \frac{\chi(m)}{\tau(m)}.$$
Let us replace the domain of \( \delta \) in \( W_a(x) \) by the segment \( 1 \leq \delta \leq x_d \). Additional summands do not exceed

\[
2 \sum_{d|a} \sum_{\delta_0 < \delta \leq x_d} \sum_{q_1 < y_d} \frac{1}{\varphi(q_1)} \sum_{m \leq \frac{x_d}{\delta}} \frac{1}{2} \ll \sum_{d|a} \sum_{\delta_0 < \delta \leq x_d} \frac{x}{d \delta} \ln x \ll \\
\ll x \ln x \left( \sum_{d|a} \frac{1}{d} \right) \sum_{\delta > \delta_0} \frac{1}{\delta} \ll x \ln x \left( \ln \delta_0 \right)^{\nu} \ll \frac{x}{\ln x}.
\]

Thus,

\[
W_a(x) = 2 \sum_{d|a} \sum_{\delta \ q_1 \leq y_d \ (q_1, a_1 \delta) = 1} \frac{1}{\varphi(q_1)} \sum_{m \leq \frac{x_d}{\delta}} \frac{1}{\tau(d \delta m)} + O\left( \frac{x}{\ln x} \right).
\]

Now our purpose is to transform double sum over \( \delta \) and \( m \leq \frac{x_d}{\delta} \) into single inner sum. We put \( n = \delta m \). Since for any \( n \), \( 1 \leq n \leq x_d \), there exists a unique representation of the form \( n = \delta m \) where \( (m, d) = 1 \) and \( \delta \) belongs to the above sequence, the condition \( n = \delta m \) may be omitted. Next, the condition \( (m, d) = 1 \) in the inner sum in \( W_a(x) \) may be omitted. Indeed, the equality \( (q_1, a_1 \delta) = 1 \) in \( W_a(x) \) implies that the sum over \( \delta \) contains the terms that obey the condition \( (q_1, \delta) = 1 \). Then both the conditions \( (q_1, m) = 1 \) and \( (q_1, m) = 1 \) are equivalent to the single condition \( (q_1, n) = 1 \). Thus we get

\[
W_a(x) = \sum_{d|a} \sum_{\delta \ q_1 \leq y_d \ (q_1, a_1 \delta) = 1} \frac{1}{\varphi(q_1)} \sum_{n \leq x_d \ (q_1, n) = 1} \frac{1}{\tau(dn)} + O\left( \frac{x}{\ln x} \right) = \\
= 2 \sum_{d|a} \sum_{n \leq x_d} \frac{1}{\tau(dn)} \sum_{q_1 \leq y_d \ (q_1, na_1) = 1} \frac{1}{\varphi(q_1)} + O\left( \frac{x}{\ln x} \right).
\]

Applying the consequence of Lemma 9, we obtain

\[
W_a(x) = 2 \sum_{d|a} \sum_{n \leq x_d} \frac{1}{\tau(dn)} (C \beta(na_1) \ln y_d + O(\tau(na_1))) + O\left( \frac{x}{\ln x} \right).
\]

Since \( C \beta(na_1) \ln d + O(\tau(na_1)) \ll (\ln x) \), it follows that

\[
W_a(x) = \\
= 2 \sum_{d|a} \sum_{n \leq x_d} \frac{1}{\tau(dn)} (C \beta(na_1)(\frac{1}{2} \ln x - A \ln \ln x + O(\tau(n))) + O\left( \frac{x}{\ln x} \right) = \\
= C(\ln x) \sum_{d|a} \sum_{n \leq x_d} \beta(na_1) + r_a(x),
\]

\[\text{Page 13}\]
where
\[ r_a(x) \ll (\ln \ln x) \sum_{d \mid a} \sum_{n \leq x} \frac{1}{\tau(dn)} + \sum_{d \mid a} \sum_{n \leq x} \frac{\tau(n)}{\tau(dn)} + \frac{x}{\ln x} \ll \]
\[ \ll (\ln \ln x) \left( \sum_{d \mid a} 1 \right) \sum_{m \leq x} \frac{1}{\tau(m)} + \sum_{d \mid a} \frac{x}{d} + \frac{x}{\ln x} \ll \]
\[ \ll \frac{x \ln \ln x}{\sqrt{\ln x}} + x + \frac{x}{\ln x} \ll x. \]

Thus,
\[ W_a(x) = C(\ln x) \sum_{d \mid a} \sum_{n \leq x/d} \frac{\beta(na/d)}{\tau(nd)} + O(x). \]

Changing the order of summation, we deduce that
\[ W_a(x) = C(\ln x) \sum_{m \leq x} \frac{\beta(m)}{\tau(m)} \sum_{d \mid m} \frac{\beta(na/d)}{\tau(d)} + O(x) = \]
\[ = C(\ln x) \beta(a) \sum_{m \leq x} \frac{e_a(m)}{\tau(m)} + O(x), \]

where
\[ e_a(m) = \frac{1}{\beta(a)} \sum_{d \mid (a,m)} \beta(na/d^2) \]

This completes the proof of the lemma.

The following lemma is the main assertion of the paper.

**Lemma 13.** Let \( d \) be a fixed integer, \( d \geq 1 \). Then for any \( B > 0 \) there exists \( A = A(B) > 0 \) such that the estimation
\[ R = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \mod q} \left| \sum_{n \leq N} \frac{\chi(n)}{\tau(n)} \right| \ll x(\ln x)^{-B} \]
holds for any \( Q, N \) such that \( Q \leq \sqrt{x(\ln x)^{-A}}, N \leq x \).

**Remark.** This assertion holds true for \( A = \frac{16}{3} B + 4 \).

**Proof.** First we prove that if \( T \) do not coincide with an ordinate of a zero of \( L(s, \chi) \) then the following inequality holds:
\[ \left| \sum_{n \leq N} \frac{\chi(n)}{\tau(n)} \right| \ll \]
\[ \ll (\ln x)^{\alpha} \left( \frac{N}{T} \ln x + \frac{N}{T} \int_{\sigma_1}^{\sigma_2} \left( |L(\sigma + iT, \chi)|^2 + |L(\sigma + iT, \chi)|^2 \right) d\sigma + \right. \]
\[ + \sqrt{N} \int_0^T \left( |L(\sigma_1 + it, \chi)|^2 + |L(\sigma_1 + it, \chi)|^2 \right) \frac{dt}{t + 1} + \right. \]
\[ + \sum_{|\gamma| \leq T} \frac{1}{|\gamma| + 1} \int_{\sigma_1}^{\sigma_2} \left| L(\sigma + i\gamma, \chi) \right|^2 N^\delta d\sigma \). \]
Here \( \alpha = \frac{1}{24} \), \( \sigma_1 = \frac{1}{3} + (\ln x)^{-1} \), \( \sigma_2 = \sigma_1 + \frac{1}{2} \) and \( \varphi = \beta + i \gamma \) runs through all zeros of \( L(s, \chi) \) in the rectangle \( \sigma_1 < \beta \leq 1, \, |\gamma| \leq T \). Assume that \( \text{Re} \, s > 1 \). Then the generating function \( F(s; \chi, d) = F(s) \) of the sequence \( \chi(n)/\tau(n) \), \( (n, d) = 1 \), has the form

\[
F(s) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{\tau(n)} n^{-s} = \prod_{p|d} F_p(s), \quad F_p(s) = 1 + \sum_{k=1}^{\infty} \frac{\chi(p^k)}{k+1} p^{-ks}.
\]

Taking for brevity \( \kappa = \chi(p)p^{-s} \) we obtain for \( \sigma > 1 \):

\[
F_p(s) = 1 + \frac{1}{2} \kappa + \frac{1}{3} \kappa^2 + \frac{1}{4} \kappa^3 + \ldots = (1 - \kappa)^{-\frac{1}{2}} (1 - \kappa^2)^{-\frac{1}{24}} (1 - \kappa^3)^{-\frac{1}{24}} f_p(s)
\]

where the symbol \( f_p(s) \) denotes a convergent series of the form

\[
1 - \frac{49}{2880} \kappa^4 - \frac{49}{1440} \kappa^5 + \frac{1447}{362880} \kappa^6 - \frac{3383}{120960} \kappa^7 + \ldots.
\]

Therefore,

\[
F(s) = \prod_{p|d} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-\frac{1}{2}} \left( 1 - \frac{\chi^2(p)}{p^{2s}} \right)^{-\frac{1}{24}} \left( 1 - \frac{\chi^3(p)}{p^{3s}} \right)^{-\frac{1}{24}} f_p(s) = \\
= \left( L(s, \chi) \right)^{\frac{1}{2}} \left( L(2s, \chi^2) \right)^{-\frac{1}{24}} \left( L(3s, \chi^3) \right)^{-\frac{1}{24}} \times \\
\times \prod_{p|d} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-\frac{1}{2}} \left( 1 - \frac{\chi^2(p)}{p^{2s}} \right)^{-\frac{1}{24}} \left( 1 - \frac{\chi^3(p)}{p^{3s}} \right)^{-\frac{1}{24}} \prod_{p|d} f_p(s) = \\
= \left( \frac{L(s, \chi)}{L(2s, \chi^2)L(3s, \chi^3)} \right)^{\alpha} \Phi(s),
\]

where \( \alpha = \frac{1}{24} \) and the function \( \Phi(s) \) is regular in the half-plane \( \text{Re} \, s > \frac{1}{4} \).

Suppose \( T \gg 2 \) do not coincide with an ordinate of a zero of \( L(s, \chi) \). Taking \( N_1 = [N] + \frac{1}{2}, \, \sigma_2 = 1 + (\ln x)^{-1} \), by Perron’s formula we get

\[
\sum_{n \leq N, \, (n, d) = 1} \frac{\chi(n)}{\tau(n)} = \frac{1}{2\pi i} \int_{\sigma_2-iT}^{\sigma_2+iT} F(s) \frac{N^s}{s} ds + O \left( \frac{N}{T} \ln x \right) + O(1).
\]

Suppose \( \varphi = \beta + i \gamma \) runs through all the zeros of \( L(s, \chi) \) in the domain \( \sigma_2 < \beta \leq 1, \, |\gamma| \leq T \). Let \( \Gamma \) be a boundary of the rectangle with vertices \( \sigma_1 \pm iT, \, \sigma_2 \pm iT \) and with horizontal cuts going from the left side of the rectangle to each zero \( \varphi \). Applying Cauchy’s theorem we obtain

\[
\frac{1}{2\pi i} \int_{\sigma_2-iT}^{\sigma_2+iT} F(s) \frac{N^s}{s} ds = -(I_1 + I_2 + I_3 + \sum_{|\gamma| \leq T} I(\varphi))
\]

where the symbols \( I_1, I_2, I_3 \) denote the integrals

\[
\frac{1}{2\pi i} \int_{\sigma_2+iT}^{\sigma_1+iT} F(s) \frac{N^s}{s} ds, \quad \frac{1}{2\pi i} \int_{\sigma_1-iT}^{\sigma_2-iT} F(s) \frac{N^s}{s} ds, \quad \frac{1}{2\pi i} \text{v.p.} \int_{\sigma_1+iT}^{\sigma_1-iT} F(s) \frac{N^s}{s} ds
\]
respectively, and $I(\varphi)$ means the sum of the integrals over the upper and lower edges of the cut:

$$I(\varphi) = \frac{1}{2 \pi i} \left( \int_{\sigma_1+i(\gamma+0)}^{\beta+i(\gamma+0)} F(s) \frac{N^s}{s} ds + \int_{\beta+i(\gamma-0)}^{\sigma_1+i(\gamma-0)} \right).$$

Since the infinite product for $\Phi(s)$ converges absolutely in the half-plane $\Re s > \frac{1}{4}$, then $|\Phi(s)| = O(1)$ along the contour $\Gamma$. Moreover, for $\sigma \geq \sigma_1$ we obtain

$$|L(3s, \chi^3)|^{-1} = \left| \sum_{n=1}^{+\infty} \frac{\mu(n)\chi^3(n)}{n^{3s}} \right| \leq \sum_{n=1}^{+\infty} n^{-3/2} = O(1),$$

$$|L(2s, \chi^2)|^{-1} = \left| \sum_{n=1}^{+\infty} \frac{\mu(n)\chi^2(n)}{n^{2s}} \right| \leq \sum_{n=1}^{+\infty} n^{-2\sigma} \leq \sum_{n=1}^{+\infty} n^{2\sigma_1} \leq$$

$$\leq 1 + \int_{1}^{+\infty} u^{-2\sigma_1} du = \frac{2\sigma_1}{2\sigma_1 - 1} = \frac{1}{2} \ln x + 1 < \ln x.$$

Thus,

$$|F(s)| \ll (\ln x)^\alpha |L(s, \chi)|^{1/2}$$

along the contour $\Gamma$. Therefore,

$$I_1 \ll \int_{\sigma_1}^{\sigma_2} |F(\sigma + iT)| \frac{N^{\sigma} ds}{\sqrt{T^2 + \sigma^2}} \ll \frac{N^{\sigma_2}}{T} (\ln x)^\alpha \int_{\sigma_1}^{\sigma_2} |L(\sigma + iT, \chi)|^{1/2} d\sigma \ll$$

$$\ll \frac{N}{T} (\ln x)^\alpha \int_{\sigma_1}^{\sigma_2} |L(\sigma + iT, \chi)|^{1/2} d\sigma,$$

$$I_2 \ll \frac{N}{T} (\ln x)^\alpha \int_{\sigma_1}^{\sigma_2} |L(\sigma + iT, \chi)|^{1/2} d\sigma.$$

Next,

$$|I_3| = \frac{1}{2\pi} \text{v.p.} \int_{-T}^{T} F(\sigma_1 + it) N^{\sigma_1+it} \frac{1}{\sigma_1 + it} dt \ll N^{\sigma_1} \int_{-T}^{T} \frac{|F(\sigma_1 + it)| dt}{\sqrt{T^2 + \sigma_1^2}} \ll$$

$$\ll \sqrt{N} (\ln x)^\alpha \int_{-T}^{T} \frac{|L(\sigma_1 + it, \chi)|^{1/2} dt}{\sqrt{T^2 + \sigma_1^2}} \ll$$

$$\ll \sqrt{N} (\ln x)^\alpha \int_{0}^{T} \left( L(\sigma_1 + it, \chi) \right)^{1/2} dt + L(\sigma_1 + it, \chi) \frac{1}{t+1}.$$

Finally, each of the integrals $I(\varphi)$ obeys the inequality

$$|I(\varphi)| \ll \frac{(\ln x)^{\alpha+1}}{|\gamma| + 1} \int_{\sigma_1}^{\beta} |L(\sigma_1 + it, \chi)|^{1/2} N^\sigma d\sigma.$$

Summing the above estimates we obtain the required inequality for the sum of $\chi(n)/\tau(n)$, $n \leq N$, $(n, d) = 1$. Summing these inequalities over all non-principle characters $\chi \mod q$ and over $q \leq Q$, we get

$$R \ll (\ln x)^\alpha \sum_{j=1}^{4} R_j,$$
where

\[
R_1 = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \frac{N}{T} \ln x,
\]

\[
R_2 = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \frac{N}{T} \int_{\sigma_1}^{\sigma_2} \left( |L(\sigma + iT, \chi)|^{\frac{1}{2}} + |L(\sigma + iT, \chi_0)|^{\frac{1}{2}} \right) d\sigma,
\]

\[
R_3 = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \sqrt{N} \int_{1}^{T} \left( |L(\sigma_1 + it, \chi)|^{\frac{1}{2}} + |L(\sigma_1 + it, \chi_0)|^{\frac{1}{2}} \right) \frac{dt}{t+1},
\]

\[
R_4 = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \sum_{|\gamma| \leq T} \left( |N| L(\sigma + i\gamma, \chi)|^{\frac{1}{2}} \right) d\sigma.
\]

The values of \(T\) depends on \(q\) and \(\chi\) only: \(T = T(q, \chi)\). We choose \(T(q, \chi)\) in a following way. The points \(Q \cdot 2^{-k}, k = 0, 1, 2, \ldots\) split the segment \((1, Q]\) into the intervals of the type \((M, M_1]\), where \(M_1 \leq 2M, M_1 \leq Q\). Suppose the character \(\chi\) modulo \(q\) is induced by a primitive character \(\chi_1\) modulo \(q_1, q = q_1r\), where \(M < q_1 \leq M_1\). Then we put \(T = M(\ln x)^C\), where the constant \(C, 0 < C < A\), will be chosen later. Therefore, it follows that \(T(q, \chi) = T(q_1, \chi_1)\); in particular, \(T(q, \chi)\) do not depends on \(r\). Replacing (if it is necessary) the value \(T(q, \chi)\) by the value \(T(q, \chi) + h\) for some \(h, 0 < h \leq c(\ln x)^{-1}\), we may assume that \(T(q, \chi)\) does not coincide with an ordinate of a zero of \(L(s, \chi)\). In each case, we obviously have

\[
M(\ln x)^C \leq T(q, \chi) \leq M(\ln x)^C + c(\ln x)^{-1} \leq 2M(\ln x)^C.
\]

In the following, the sums over \(\chi \mod q, \chi \neq \chi_0\) are replaced by the sums over primitive characters \(\chi_1 \mod q_1\).

1°. Estimation of \(R_1\). Obviously we have

\[
R_1 \ll N(\ln x) \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \frac{1}{T(q, \chi)} \ll \]

\[
\ll N(\ln x) \sum_{q_1 \leq Q} \sum_{r \leq Q_{q_1}^{\chi_1 \mod q_1}} \frac{1}{\varphi(q_1 r)} \sum_{\chi_1 \mod q_1} \frac{1}{T(q_1, \chi_1)}.
\]

Using the symbol \(\sum_{M \leq q}^\prime\) for the summation over the points \(M = Q \cdot 2^{-k}\), by the inequality \(\varphi(q_1 r) \geq \varphi(q_1) \varphi(r)\) we obtain

\[
R_1 \ll N(\ln x) \sum_{M \leq Q} \sum_{M < q_1 < M_1} \frac{1}{\varphi(q_1)} \sum_{r \leq Q_{q_1}^{\chi_1 \mod q_1}} \frac{1}{\varphi(r)} \sum_{\chi_1 \mod q_1} \frac{1}{M(\ln x)^C} \ll
\]

\[
\ll \frac{N}{(\ln x)^C} \sum_{M \leq Q} \frac{1}{M} \sum_{M < q_1 < M_1} \frac{1}{\varphi(q_1)} \left( \sum_{\chi_1 \mod q_1} 1 \right) \sum_{r \leq Q_{q_1}^{\chi_1 \mod q_1}} \frac{1}{\varphi(r)} \ll
\]

\[
\ll \frac{N(\ln x)^2}{(\ln x)^C} \sum_{M \leq Q} \frac{1}{M} \left( \sum_{M < q_1 < M_1} 1 \right) \ll \frac{N(\ln x)^2}{(\ln x)^C} \sum_{M \leq Q} \frac{1}{M} \ll \frac{N(\ln x)^3}{(\ln x)^C}.
\]
2°. Estimation of $R_2$. Since $\chi$ and $\chi$ run through the same set of characters, we get

$$R_2 \ll N \int_{\sigma_1}^{\sigma_2} r_2(\sigma) d\sigma,$$

where

$$r_2(\sigma) = \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \mod q} \left| L(\sigma + iT, \chi ) \right| \frac{1}{T(q, \chi)}.$$

Using the same arguments as above, by lemma 7 we obtain ($T = T(q_1, \chi_1)$):

$$r_2(\sigma) \ll \sum_{q_1 \leq Q} \frac{1}{\varphi(q_1)} \sum_{r \leq \frac{Q}{q_1}} \sqrt{\tau(r)} \sum_{\chi_1 \mod q_1} \left| L(\sigma + iT, \chi_1) \right| \frac{1}{T(q_1, \chi_1)} \ll$$

$$\ll \sum_{M \leq Q} \frac{1}{\varphi(q_1)} \sum_{r \leq \frac{Q}{q_1}} \sqrt{\tau(r)} \sum_{\chi_1 \mod q_1} \left| L(\sigma + iT, \chi_1) \right| \frac{1}{M (\ln x)^C} \ll$$

$$\ll (\ln x)^{-C} \sum_{M \leq Q} \frac{1}{M} \sum_{M < q_1 \leq M_1} \frac{1}{\varphi(q_1)} \sum_{r \leq \frac{Q}{q_1}} \sqrt{\tau(r)} \sum_{\chi_1 \mod q_1} \left| L(\sigma + iT, \chi_1) \right| \frac{1}{M (\ln x)^C} \ll$$

By the inequalities

$$\sum_{r \leq X} \sqrt{\tau(r)} \leq \left( X \sum_{r \leq X} \tau(r) \right)^{\frac{1}{2}} \ll X \sqrt{\ln X}, \quad \frac{1}{\varphi(r)} \ll \frac{\ln \ln (r + 3)}{r},$$

we deduce that

$$\sum_{r \leq \frac{Q}{q_1}} \sqrt{\tau(r)} \ll (\ln Q) \sum_{r \leq Q} \frac{\sqrt{\tau(r)}}{r} \ll$$

$$\ll (\ln x) \left( \frac{1}{Q} \sum_{r \leq Q} \sqrt{\tau(r)} + \int_2^Q \left( \sum_{r \leq u} \sqrt{\tau(r)} \right) \frac{du}{u^2} \right) \ll (\ln x) \int_2^Q \frac{\sqrt{\ln u} \, du}{u} \ll$$

$$\ll (\ln x)^{\frac{3}{2}} \ln \ln x$$

and therefore

$$r_2(\sigma) \ll \frac{(\ln x)^{\frac{3}{2}} \ln x}{(\ln x)^C} \sum_{M \leq Q} \frac{1}{M} \sum_{M < q_1 \leq M_1} \frac{1}{\varphi(q_1)} \sum_{\chi_1 \mod q_1} \left| L(\sigma + iT, \chi_1) \right| \frac{1}{M (\ln x)^C} \ll$$

$$\ll \frac{(\ln x)^{\frac{3}{2}} (\ln x)^2}{(\ln x)^C} \sum_{M \leq Q} \frac{1}{M^2} r_2(\sigma, M),$$
where
\[ r_2(\sigma, M) = \sum_{M < q_1 \leq M_1} \sum_{\chi \mod q_1}^* |L(\sigma + iT, \chi_1)|^2. \]

Applying Hölder’s inequality, we obtain
\[ r_2^4(\sigma, M) \ll \left( \sum_{M < q_1 \leq M_1} \sum_{\chi \mod q_1}^* 1 \right)^3 \sum_{M < q_1 \leq M_1} \sum_{\chi \mod q_1}^* |L(\sigma + iT, \chi_1)|^2 \ll \]
\[ \ll M^6 \sum_{M < q_1 \leq M_1} \sum_{\chi \mod q_1}^* |L(\sigma + iT, \chi_1)|^2. \]

Let \( Y = 2M_1T(q_1, \chi_1) \); then for any \( q_1 \) such that \( M < q_1 \leq M_1 \), the following inequality holds: \( Y \geq q_1(T(q_1, \chi_1) + 1)/\pi \). By lemma 3, we get
\[ |L(\sigma + iT, \chi_1)|^2 \ll \left| \sum_{n \leq Y} \frac{\chi(n)}{n^{\sigma + iT}} \right|^2 + (q_1Y^{-\sigma})^2. \]

Since
\[ q_1Y^{-\sigma} \leq \frac{M_1}{\sqrt{Y}} = \frac{M_1}{\sqrt{2M_1T(q_1, \chi_1)}} < \sqrt{\frac{M_1}{T(q_1, \chi_1)}} < \]
\[ < \sqrt{\frac{2M_1}{M(\ln x)^C}} = \sqrt{2}(\ln x)^{-C/2} < 1, \]

for \( \sigma \geq \frac{1}{2} \), we get
\[ r_2^4(\sigma, M) \ll M^8 + M^6 \sum_{q_1 \leq M_1} \sum_{\chi \mod q_1}^* \left| \sum_{n \leq Y} \frac{\chi(n)}{n^{\sigma + iT}} \right|^2. \]

Setting \( Q = M_1, M = 0, N = Y \) in lemma 2, we obtain
\[ r_2^4(\sigma, M) \ll M^8 + M^6(M^2 + Y) \sum_{n \leq Y} \frac{1}{n^{2\sigma}} \ll M^8(\ln x)^{C+1} \]

and therefore
\[ r_2(\sigma) \ll \frac{(\ln x)^{\frac{3}{4}}(\ln \ln x)^{2}}{(\ln x)^C} \sum_{M \leq Q}^* \frac{1}{M^2} M^2(\ln x)^{\frac{1}{4}(C+1)} \ll \]
\[ \ll \frac{(\ln x)^{\frac{3}{4}}(\ln \ln x)^{2}}{(\ln x)^\frac{3}{4}C} \sum_{M \leq Q}^* 1 \ll \frac{(\ln x)^{\frac{11}{4}}(\ln \ln x)^{2}}{(\ln x)^\frac{3}{4}C}. \]

Thus,
\[ R_2 \ll N \frac{(\ln x)^{\frac{11}{4}}}{(\ln x)^\frac{3}{4}C}(\ln \ln x)^2. \]
3°. Estimation of $R_3$. By the same arguments we obtain

$$R_3 \ll \sqrt{N} \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \mod q, \chi \neq \chi_0} \int_0^{T(q, \chi)} |L(\sigma_1 + it, \chi)|^{\frac{1}{2}} \frac{dt}{t+1} \ll$$

$$\ll \sqrt{N} \sum_{q_1 \leq Q} \varphi(q_1) \sum_{r \leq \frac{Q}{q_1}} \sum_{\chi_1 \mod q_1} \int_0^{T(q_1, \chi_1)} |L(\sigma_1 + it, \chi)|^{\frac{1}{2}} \frac{dt}{t+1}.$$  

Since the terms in the inner sum over $\chi_1$ do not depend on $r$, we easily get

$$R_3 \ll \sqrt{N} (\ln x)^{\frac{3}{2}} (\ln \ln x) \sum_{q_1 \leq Q} \varphi(q_1) \sum_{\chi_1 \mod q_1} \int_0^{T(q_1, \chi_1)} |L(\sigma_1 + it, \chi)|^{\frac{1}{2}} \frac{dt}{t+1} \ll$$

$$\ll \sqrt{N} (\ln x)^{\frac{3}{2}} (\ln \ln x) \sum_{M \leq Q} M \sum_{q_1 \leq M} \frac{1}{\varphi(q_1)} \sum_{\chi_1 \mod q_1} \int_0^{T(q_1, \chi_1)} |L(\sigma_1 + it, \chi)|^{\frac{1}{2}} \frac{dt}{t+1}$$

$$\ll \sqrt{N} (\ln x)^{\frac{3}{2}} (\ln \ln x)^2 \sum_{M \leq Q} M \int_0^{T_1} \frac{r_3(t, M)}{t+1} dt,$$

where $T_1 = 2M(\ln x)^C$,

$$r_3(t, M) = \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \mod q_1} |L(\sigma_1 + it, \chi_1)|^{\frac{1}{2}}.$$  

From Hölder’s inequality and lemma 2 it follows that

$$r_3(t, M) \ll M^2 (\ln x)^{\frac{4}{4}(C+1)}.$$  

Thus we find

$$R_3 \ll \sqrt{N} (\ln x)^{\frac{3}{2}} (\ln \ln x)^2 \sum_{M \leq Q} \frac{1}{M} M^2 (\ln x)^{\frac{4}{4}(C+1)} \ll$$

$$\ll Q \sqrt{N} (\ln x)^{\frac{1}{4}(C+1)} (\ln \ln x)^2.$$  

4°. Estimation of $R_4$. For given character $\chi \mod q$ and a zero $\varrho = \beta + i\gamma$ of $L(s, \chi)$ we define the function $g_\chi(\varrho, \sigma)$ as follows:

$$g_\chi(\varrho, \sigma) = \begin{cases} 1, & \text{if } \beta > \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

It’s obvious that if $\chi_1 \mod q_1$ induces $\chi \mod q$ then $g_{\chi_1}(\varrho, \sigma)$ coincides with $g_\chi(\varrho, \sigma)$ for $\sigma \geq \frac{1}{2}$. Thus we have

$$R_4 \leq \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{\chi \mod q, \chi \neq \chi_0} \sum_{|\gamma| \leq T(q, \chi)} \frac{1}{|\gamma| + 1} \int_0^{1} g_\chi(\varrho, \sigma) N^\sigma |L(\sigma + i\gamma, \chi)|^{\frac{1}{2}} d\sigma \leq$$

$$\leq \int_0^{1} N^\sigma \sum_{q_1 \leq Q} \frac{1}{\varphi(q_1)} \sum_{r \leq \frac{Q}{q_1}} \frac{\sqrt{T(r)}}{\varphi(r)} \sum_{\chi_1 \mod q_1} \sum_{|\gamma| \leq T(q_1, \chi_1)} \frac{g_{\chi_1}(\varrho, \sigma)}{|\gamma| + 1} |L(\sigma + i\gamma, \chi_1)|^{\frac{1}{2}} d\sigma$$
\[ (\ln x)^2 \sum_{|\gamma| \leq T(q,\chi)} \sum_{\beta > 0.5} g_{\chi_1}(\beta, \sigma) |L(\sigma + i\gamma, \chi_1)|^{1/2} d\sigma \]

\[ (\ln x)^2 \sum_{M < Q} j(M), \]

where

\[ j(M) = \frac{1}{M} \int_{0.5}^{1} N^\sigma \sum_{|\gamma| \leq T(q,\chi)} \sum_{\beta > 0.5} g_{\chi_1}(\beta, \sigma) |L(\sigma + i\gamma, \chi_1)|^{1/2} d\sigma, \]

\[ r_4(\sigma, M) = \sum_{M < q < M_1} \sum_{\chi_1 \mod q_1} |\gamma| \leq T(q,\chi) \sum_{\beta > 0.5} g_{\chi_1}(\beta, \sigma) |L(\sigma + i\gamma, \chi_1)|^{1/2}. \]

In the below, we consider the cases of «small» and «large» \(M\) separately.

4a°. The case of «small» \(M\): \(1 \leq M \leq (\ln x)^A\). Obviously we have

\[ T(q_1, \chi_1) \leq 2M(\ln x)^C \leq 2(\ln x)^{A+C}. \]

If \(\rho = \beta + i\gamma\) is a zero of \(L(s, \chi)\) and \(|\gamma| \leq T(q_1, \chi_1)\) then lemma 4 implies that

\[ \beta \leq 1 - \frac{c_1}{\ln q_1(|\gamma| + 2)} \leq 1 - \frac{c}{\ln (2(\ln c)^{2A+C})} < 1 - \frac{1}{\sqrt{\ln x}}. \]

Further, if there exists a real zero \(\beta\) of \(L(s, \chi)\) then lemma 5 implies (with \(\varepsilon = (2A)^{-1}\)) that

\[ \beta \leq 1 - \frac{c}{q_1^\varepsilon} \leq 1 - \frac{c}{(\ln x)^{A\varepsilon}} = 1 - \frac{c}{\sqrt{\ln x}}. \]

for some \(c > 0\). Without loss of generality, we may assume that \(0 < c < 1\). Then it follows that for any zero \(\rho\) of the function \(L(s, \chi)\) under considering the inequality \(|\gamma| \leq T(q_1, \chi_1)\) implies that \(g_{\chi_1}(\rho, \sigma) = 0\) for any \(\sigma \geq 1 - \frac{1}{\sqrt{\ln x}}\).

Now, if \(\frac{1}{2} \leq \sigma \leq 1\) then from lemma 3 it follows that

\[ |L(\sigma + it, \chi_1)| \ll \sum_{n \leq q_1(|t| + 1)} \chi(n)n^{-\sigma-it} + q_1(|t| + 1))^{-\sigma} \]

\[ \ll (q_1(|t| + 1))^{1-\sigma} \ln q_1(|t| + 1). \]

By lemma 7, we easily get

\[ |L(\sigma + i\gamma, \chi_1)|^{1/2} \ll (q_1 T(q_1, \chi_1))^{1-\sigma} \ln \ln x \ll (M^2(\ln x)^C)^{1-\sigma} \ln \ln x < \]

\[ (\ln x)^{2A} \ln \ln x, \]

\[ r_4(\sigma, M) \ll (\ln x)^{\frac{3}{2}A} \ln \ln x \sum_{M < q < M_1} \sum_{\chi_1 \mod q_1} |\gamma| \leq T(q_1, \chi_1) \frac{1}{|\gamma| + 1} \ll \]

\[ (\ln x)^{\frac{3}{2}A} \ln \ln x \sum_{M < q < M_1} \sum_{\chi_1 \mod q_1} \ln^2 q_1 T(q_1, \chi_1) \ll (\ln x)^{\frac{3}{2}A} (\ln \ln x)^3 M^2 \ll \]

\[ M(\ln x)^{2A}. \]
Thus we finally obtain:

\[ j(M) \ll \frac{1}{M} \int_{0.5}^{1 - e^{\frac{c}{\ln x}}} N^\sigma M (\ln x)^A d\sigma \ll N^{1 - e^{\frac{c}{\ln x}}} (\ln x)^{2A} \ll xe^{-\frac{c}{2} \sqrt{\ln x}}. \]

4b°. The case of «large» \( M: (\ln x)^A < M \leq \sqrt{x} (\ln x)^{-A} \). Let us divide the domain of \( \gamma \) into segments \( U < |\gamma| \leq U_1 \) where

\[ U_1 \leq 2U, \quad U_1 \leq T(q_1, \chi_1) \leq 2M(\ln x)^C = T_1. \]

Thus we have

\[ r_4(\sigma, M) \ll \sum' \frac{1}{U} r_4(\sigma, M, U), \]

\[ r_4(\sigma, M, U) = \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \ong q_1} \sum_{U < |\gamma| \leq U_1}^{\beta > 0.5} g_{\chi_1}(q, \sigma) |L(\sigma + i\gamma, \chi_1)|^{\frac{1}{2}}. \]

Applying Hölder’s inequality, we find

\[ r_4'(\sigma, M, U) \ll \left( \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \ong q_1} \sum_{U < |\gamma| \leq U_1}^{\beta > 0.5} g_{\chi_1} \right)^{\frac{3}{2}} \times \]

\[ \times \left( \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \ong q_1} \sum_{U < |\gamma| \leq U_1}^{\beta > 0.5} |L(\sigma + i\gamma, \chi_1)| \right) = r_5 \cdot r_6^3, \]

where the notations \( r_5, r_6 \) are obvious.

First we get:

\[ r_5 \ll \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \ong q_1} \sum_{|\gamma| \leq U_1} |L(\sigma + i\gamma, \chi_1)|^2. \]

Let us divide the domain of \( \gamma \) into the segments

\[ 2n \leq \gamma < 2n + 1, \quad n = 0, \pm 1, \pm 2, \ldots, \]

and

\[ 2n + 1 \leq \gamma < 2(n + 1), \quad n = 0, \pm 1, \pm 2, \ldots. \]

Thus the sum \( r_5 \) splits into sums \( r_5' \) and \( r_5'' \). Further, let us sort all the ordinates \( \gamma \) in each segment \( 2n \leq \gamma < 2n + 1 \) in increasing order:

\[ 2n \leq \gamma^{(1)} \leq \gamma^{(2)} \leq \ldots \leq \gamma^{(s)} < 2n + 1. \]

It’s obvious that \( s = O(\ln q_1 (|n| + 1)) \). Now we place the terms of \( r_5' \) that corresponds to the first ordinates \( \gamma^{(1)} \) into sum \( r_5^{(1)} \). The terms corresponding to the second ordinates \( \gamma^{(2)} \) are placed into sum \( r_5^{(2)} \) and so on. Thus \( r_5' \) splits into \( s_0 = O(\ln x) \) sums \( r_5^{(s)} \), \( s = 1, 2, \ldots, s_0 \). The ordinates \( \gamma, \gamma' \) that corresponds to the neighbouring summands in the sum \( r_5^{(s)} \) satisfy the condition \( |\gamma - \gamma'| > 1. \)
Finally, we apply the same transformation to the sum \( r_5'' \). Thus the sum \( r_5 \) splits into \( \leq 2s_0 = O(\ln x) \) sums \( r \) of the following type:

\[
 r = \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \mod q_1} \sum' |L(\sigma + i\gamma, \chi_1)|^2
\]

where the prime sign means the summation over the «rarefied» ordinates \( \gamma \).

Taking \( Y = M_1 (T_1 + 1) \) we have \( Y \geq q_1 (|\gamma| + 1)/\pi \) for all \( q_1 \) and \( \gamma \) under considering. Then Lemma 3 implies that

\[
 r \ll \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \mod q_1} \sum' \left( \left| \sum_{n \leq Y} \frac{\chi_1(n)}{n^{\sigma + i\gamma}} \right|^2 + 1 \right) \ll \quad \ll M^2 U + \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \mod q_1} \sum' \left| \sum_{n \leq Y} \frac{\chi_1(n)}{n^{\sigma + i\gamma}} \right|^2.
\]

The application of Lemma 1 yields:

\[
 \sum' \left| \sum_{|\gamma| \leq U} \frac{\chi_1(n)}{n^{\sigma + i\gamma}} \right|^2 \leq j_1 + 2\sqrt{j_1 j_2},
\]

where

\[
 j_1 = \int_{-(U_1 + 1)}^{U_1 + 1} \left| \sum_{n \leq Y} \frac{\chi_1(n)}{n^{\sigma + it}} \right|^2 dt, \quad j_2 = \int_{-(U_1 + 1)}^{U_1 + 1} \left| \sum_{n \leq Y} \frac{\chi_1(n) \ln n}{n^{\sigma + it}} \right|^2 dt.
\]

Since \( U_1 + 1 \leq 2U + 1 \leq 3U, 2\sqrt{j_1 j_2} \leq j_1 + j_2 \), by Lemma 2 we obtain

\[
 r \ll M^2 U + \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \mod q_1} \left( j_1 + j_2 \right) \ll \quad \ll M^2 U + \int_{-3U}^{3U} \sum_{M < q_1 \leq M_1} \sum_{\chi_1 \mod q_1} \left( \left| \sum_{n \leq Y} \frac{\chi_1(n)}{n^{\sigma + it}} \right|^2 + \left| \sum_{n \leq Y} \frac{\chi_1(n) \ln n}{n^{\sigma + it}} \right|^2 \right) dt \ll \quad \ll M^2 U + \int_{-3U}^{3U} \left( M^2 + Y \right) \left( \sum_{n \leq Y} \frac{1}{n^{2\sigma}} + \sum_{n \leq Y} \frac{\ln^2 n}{n^{2\sigma}} \right) dt \ll \quad \ll M^2 U + \int_{-3U}^{3U} M^2 (\ln x)^{C + 3} dt \ll M^2 U (\ln x)^{C + 3}
\]

and therefore

\[
 r_4(\sigma, M, U) \ll M^2 U (\ln x)^{C + 4}.
\]

Further, using Lemma 6, we obtain

\[
 r_6 \ll \sum_{q_1 \leq M_1} \sum_{\chi_1 \mod q_1} \sum g_{\chi_1}(\sigma, \sigma) = \sum_{q_1 \leq M_1} \sum_{\chi_1 \mod q_1} N(\sigma; U_1, \chi_1) \ll \quad \ll (M^2 U)^{\vartheta(\sigma)} (\ln x)^{14}.
\]
Therefore,
\[ r_4^4(\sigma, M, U) \ll M^2 U (\ln x)^{C+4} (M^2 U)^{3\vartheta(\sigma)} (\ln x)^{42} \ll (M^2 U)^{1+3\vartheta(\sigma)} (\ln x)^{C+46}, \]
\[ r_4(\sigma, M, U) \ll (M^2 U)^{\frac{1}{4} + \frac{3}{4}\vartheta(\sigma)} (\ln x)^{\frac{1}{4}+\frac{23}{2}}, \]
\[ r_4(\sigma, M) \ll \sum_{U \leq T_1} U^{-1} \cdot (M^2 U)^{\frac{1}{4} + \frac{3}{4}\vartheta(\sigma)} (\ln x)^{\frac{1}{4}+\frac{23}{2}} \ll \]
\[ \ll (\ln x)^{\frac{1}{4}+\frac{23}{2}} \sum_{U \leq T_1} U^{\frac{3}{4}(\vartheta(\sigma)-1)} M^{1+\frac{3}{2}\vartheta(\sigma)}. \]

Since \( \vartheta(\sigma) \leq 1 \) for \( \frac{1}{2} \leq \sigma \leq 1 \), it follows that
\[ r_4(\sigma, M) \ll M^{\frac{1}{2}+\frac{3}{2}\vartheta(\sigma)} (\ln x)^{\frac{1}{4}+\frac{25}{2}} \]
and
\[ j(M) = M^{-1} \int_{\sigma_1}^{1} N^\sigma r_4(\sigma, M) d\sigma \ll (\ln x)^{\frac{1}{4}+\frac{25}{2}} \int_{0.5}^{1} \psi(\sigma) d\sigma, \]
where
\[ \psi(\sigma) = x^\sigma M^{u(\sigma)}, \quad u(\sigma) = -\frac{1}{2} + \frac{3}{2} \vartheta(\sigma). \]

Let us consider several cases.

i°. For \( \frac{1}{2} \leq \sigma \leq \frac{4}{5} \) we obviously have
\[ u(\sigma) = \frac{1}{4} + \frac{3(4-5\sigma)}{4(2-\sigma)} \geq \frac{1}{4}. \]

The condition \( N \leq \sqrt{x}(\ln x)^{-A} \) implies that
\[ \psi(\sigma) \leq x^\sigma (\sqrt{x}(\ln x)^{-A})^{u(\sigma)} \leq x^{\sigma + \frac{1}{2}u(\sigma)} (\ln x)^{-\frac{1}{4}A}. \]

Since
\[ \sigma + \frac{1}{2} u(\sigma) = 1 - \frac{(\sigma - \frac{1}{2})^2}{2 - \sigma} \leq 1, \]
we get
\[ \psi(\sigma) \leq x(\ln x)^{-\frac{1}{4}A}. \]

ii°. Suppose \( \frac{4}{5} \leq \sigma \leq \frac{6}{7} \). Then
\[ u(\sigma) = \frac{6 - 7\sigma}{2\sigma} \geq 0 \]
and
\[ \psi(\sigma) \leq x^\sigma (\sqrt{x}(\ln x)^{-A})^{u(\sigma)} \leq x^{\sigma + \frac{1}{2}u(\sigma)}. \]

From the inequality
\[ \frac{d}{d\sigma} \left( \sigma + \frac{1}{2} u(\sigma) \right) = 1 - \frac{3}{2\sigma^2} < 0 \]
it follows that the function $\sigma + \frac{1}{2} u(\sigma)$ is monotonically decreasing on the segment $\frac{4}{5} \leq \sigma \leq \frac{6}{7}$ and attains its maximum at a point $\sigma = \frac{4}{5}$:

$$\sigma + \frac{1}{2} u(\sigma) \leq \frac{4}{5} + \frac{1}{2} u\left(\frac{4}{5}\right) = \frac{37}{50}.$$  

Thus

$$\psi(\sigma) \leq x^{\frac{37}{50}} \leq x(\ln x)^{-\frac{1}{4} A}.$$

iii°. Suppose $\frac{6}{7} \leq \sigma \leq \frac{12}{13}$. Then $u(\sigma) \leq 0$ and therefore

$$\psi(\sigma) \leq x^\sigma \leq x^{\frac{12}{13}} \leq x(\ln x)^{-\frac{1}{4} A}.$$

iv°. Suppose that $\frac{12}{13} \leq \sigma \leq 1$. Then

$$u(\sigma) = -\frac{1}{4} - \frac{13\sigma - 12}{4\sigma} \leq -\frac{1}{4}.$$

Since $M > (\ln x)^A$, it implies that

$$\psi(\sigma) \leq x^\sigma M^{-\frac{1}{4}} \leq x(\ln x)^{-\frac{1}{4} A}.$$  

Thus the inequality

$$\psi(\sigma) \leq x(\ln x)^{-\frac{1}{4} A}$$

holds for any $\sigma$ such that $\frac{1}{2} \leq \sigma \leq 1$. Finally we get

$$j(M) \ll x(\ln x)^{-\frac{1}{4} A + \frac{1}{4} C + \frac{25}{2}}$$

$$R_4 \ll (\ln x)^{\frac{3}{2} (\ln \ln x)^2} \left( \sum'_{M \leq (\ln x)^C} x e^{-\frac{6}{2} \sqrt{\ln x}} + \sum'_{(\ln x)^C < M \leq Q} x(\ln x)^{-\frac{1}{4} A + \frac{1}{4} C + \frac{25}{2}} \right)$$

$$\ll x(\ln x)^{-\frac{1}{4} A + \frac{1}{4} C + 15} (\ln \ln x)^2.$$  

Summing the upper bounds for $R_j$, $1 \leq j \leq 4$, we obtain

$$R \ll (\ln x)^{\alpha} \left( N(\ln x)^{-C+3} + N(\ln x)^{-\frac{3}{2} C + \frac{11}{4}} (\ln \ln x)^2 + Q \sqrt{N(\ln x)^{\frac{3}{2} (C+1)} (\ln x)^2 + x(\ln x)^{-\frac{1}{4} A + \frac{1}{4} C + 15} (\ln \ln x)^2} \right) \ll$$

$$\ll (\ln x)^{\alpha} (\ln \ln x)^2 \left( x(\ln x)^{-C+3} + x(\ln x)^{-\frac{3}{2} C + \frac{11}{4}} + x(\ln x)^{-A+\frac{3}{4} (C+1)} + x(\ln x)^{-\frac{1}{4} A + \frac{1}{4} C + 15} (\ln \ln x)^2 \right) \ll$$

$$\ll x(\ln x)^{\alpha} (\ln \ln x)^2 \left( (\ln x)^{-\frac{3}{4} C + \frac{11}{4}} + (\ln x)^{-\frac{1}{4} A + \frac{1}{4} C + 15} \right).$$

Taking $C = \frac{1}{4} (A - 49)$, we get the required inequality:

$$R \ll x(\ln x)^{\alpha} (\ln \ln x)^2 (\ln x)^{-\frac{3}{16} A + \frac{101}{16}} \ll x(\ln x)^{-\left(\frac{3}{16} A - 12\right)}.$$  

If we put $B = \frac{3}{16} A - 12$, then $A = \frac{16}{3} B + 4$. This completes the proof of the lemma.
4. Main theorem

By Lemma 12, it remains to estimate $R_a(x)$ and find an asymptotic formula for $E_a(x)$. Setting $Q = xd^{-1} = \sqrt{x}d^{-1}(\ln x)^{-A}$, $N = x(d\delta)^{-1}$ in Lemma 13, we obtain

$$R_{a,d,\delta} \ll x(\ln x)^{-\left(\frac{3}{16}A - 12\right)}, \quad R_a(x) \ll x(\ln x)^{-\left(\frac{3}{16}A - 15\right)}.$$  

Taking $A = 85\frac{1}{3}$, we get the formula

$$S_a(x) = C(\ln x)\beta(a)E_a(x) + O(x\ln x),$$

where

$$E_a(x) = \sum_{n \leq x} \frac{e_a(n)}{\tau(n)}.$$  

In order to calculate the sum $E_a(x)$, let us prove that $e_a(n)$ is multiplicative function for any fixed $a \geq 1$. Indeed, suppose $(m,n) = 1$. Then there exists a unique decomposition $a = a_1a_2a'$ where the factors $a_1, a_2, a'$ are defined as follows. All prime divisors of $a_1$ and $a_2$ are among the sets of prime divisors of $m$ and $n$ respectively, and $(a', mn) = 1$. Then

$$(a_1, a_2) = (a_1, a') = (a_2, a') = (a_1, n) = (a_2, m) = 1.$$  

Further, if $d$ divides $(a, mn)$ then $d$ has the form $d_1d_2$ where $d_1|(a, m)$ and $d_2|(a, n)$. Thus,

$$e_a(mn) = \sum_{d|(a, mn)} \frac{\beta\left(\frac{an^2}{d^2}\right)}{\beta(a)} = \sum_{d_1|(a, m)} \sum_{d_2|(a, n)} \frac{\beta\left(\frac{a_1a_2a'mn}{d_1^2d_2^2}\right)}{\beta(a_1a_2a')} =$$

$$= \sum_{d_1|(a, m)} \sum_{d_2|(a, n)} \frac{\beta\left(\frac{a_1m}{d_1^2}\right)\cdot\frac{a_2n}{d_2^2}}{\beta(a_1a_2)\beta(a')}.$$  

Since the numbers $\frac{a_1m}{d_1^2}$ and $\frac{a_2n}{d_2^2}$ are integral and coprime, we have

$$e_a(mn) = \sum_{d_1|(a, m)} \frac{\beta\left(\frac{a_1m}{d_1^2}\right)}{\beta(a_1)} \sum_{d_2|(a, n)} \frac{\beta\left(\frac{a_2n}{d_2^2}\right)}{\beta(a_2)} =$$

$$= \sum_{d_1|(a, m)} \frac{\beta\left(\frac{a_1m}{d_1^2}\right)}{\beta(a_1)} \beta(a_2a') \sum_{d_2|(a, n)} \frac{\beta\left(\frac{a_2n}{d_2^2}\right)}{\beta(a_2)} \beta(a_1a') =$$

$$= \sum_{d_1|(a, m)} \frac{\beta\left(\frac{am}{d_1^2}\right)}{\beta(a)} \sum_{d_2|(a, n)} \frac{\beta\left(\frac{an}{d_2^2}\right)}{\beta(a)} = \sum_{d_1|(a, m)} \frac{\beta\left(\frac{am}{d_1^2}\right)}{\beta(a)} \sum_{d_2|(a, n)} \frac{\beta\left(\frac{an}{d_2^2}\right)}{\beta(a)} =$$

$$= e_a(m)e_a(n).
Suppose $\text{Re} \ s > 1$. By definition, put
\[ F_a(s) = \sum_{n=1}^{+\infty} \frac{e_a(n)}{\tau(n)} n^{-s}. \]

Then
\[ F_a(s) = \prod_p F_{a,p}(s), \quad F_{a,p}(s) = 1 + \sum_{k=1}^{+\infty} \frac{e_a(p^k)}{k+1} p^{-ks}. \]

Since
\[ e_a(n) = \frac{\beta(an)}{\beta(a)} = \beta(n) \]
for the case $(a, n) = 1$, we obtain
\[
F_a(s) = \prod_{p \nmid a} \left(1 + \sum_{k=1}^{+\infty} \frac{\beta(p^k)}{k+1} p^{-ks} \right) \prod_{p \mid a} F_{a,p}(s) = \\
= \prod_{p \nmid a} \left(1 + \beta(p) \sum_{k=1}^{+\infty} \frac{p^{-ks}}{k+1} \right) \prod_{p \mid a} F_{a,p}(s) = \\
= \prod_{p \nmid a} \left(1 - \beta(p) - \beta(p)p^s \ln(1-p^{-s}) \right) \prod_{p \mid a} F_{a,p}(s) = F(s) \psi_a(s),
\]

where
\[
F(s) = \prod_p \left(1 - \beta(p) - \beta(p)p^s \ln(1-p^{-s}) \right), \\
\psi_a(s) = \prod_{p \nmid a} \frac{1 + \sum_{k=1}^{+\infty} \frac{e_a(p^k)}{k+1} p^{-ks}}{1 - \beta(p) - \beta(p)p^s \ln(1-p^{-s})}.
\]

Further, $F(s) = \sqrt{\zeta(s)} \Phi(s)$, where
\[
\Phi(s) = \prod_p \Phi_p(s), \quad \Phi_p(s) = \left(1 - \frac{1}{p^s} \right) \frac{1}{2} \left(1 - \beta(p) - \beta(p)p^s \ln(1-p^{-s}) \right).
\]

Setting
\[
1 - \beta(p) - \beta(p)p^s \ln(1-p^{-s}) = 1 + \frac{1}{2p^s} + u(s), \quad \left(1 - \frac{1}{p^s} \right)^{\frac{1}{2}} = 1 - \frac{1}{2p^s} + v(s)
\]
and using the decomposition
\[
\beta(p) = 1 - \frac{p}{p^2 - p + 1} = 1 - \frac{p(p+1)}{p^3 + 1} = 1 - \frac{1}{p} \left(1 + \frac{1}{p} \right) \left(1 + \frac{1}{p^3} \right)^{-1} = \\
= 1 + \sum_{k=0}^{+\infty} (-1)^{k+1} \left(\frac{1}{p^{3k+1}} + \frac{1}{p^{3k+2}} \right),
\]

and
we find
\[ u(s) = \frac{1}{3p^{2s}} + \frac{1}{4p^{3s}} + \frac{1}{5p^{4s}} + \ldots - \\
- \left( \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^4} - \frac{1}{p^5} + \ldots \right) \left( \frac{1}{2p^s} + \frac{1}{3p^{2s}} + \frac{1}{4p^{3s}} + \ldots \right). \]

\[ v(s) = \sum_{k=2}^{\infty} 2^{-2k} \binom{2k}{k} \frac{p^{-ks}}{2k-1}. \]

Suppose now \( \sigma > \frac{1}{2} \). Then
\[ |u(s)| \leq \frac{1}{3p^{2\sigma}} + \frac{1}{4p^{3\sigma}} + \frac{1}{5p^{4\sigma}} + \ldots + \left( \frac{1}{p} + \frac{1}{p^2} \right) \left( \frac{1}{2p^\sigma} + \frac{1}{3p^{2\sigma}} + \frac{1}{4p^{3\sigma}} + \ldots \right) \leq \]
\[ \leq \frac{1}{3p^\sigma} \left( 1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} + \ldots \right) + \frac{3}{2p^\sigma} \left( \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} + \ldots \right) \leq \]
\[ \leq \left( \frac{1}{3p^{2\sigma}} + \frac{3}{4p^{\sigma+1}} \right) \frac{1}{1-p^{-\sigma}} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \left( \frac{1}{3p^{2\sigma}} + \frac{3}{4p^{\sigma+1}} \right), \]
and therefore
\[ |u(s)| \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \left( \frac{1}{3} + \frac{3}{4} \right) \max \left( \frac{1}{p^{2\sigma}}, \frac{1}{p^{\sigma+1}} \right) \leq \left\{ \begin{array}{ll}
3.7p^{-2\sigma}, & \text{if } \frac{1}{2} < \sigma \leq 1, \\
3.7p^{-(\sigma+1)}, & \text{if } \sigma \geq 1.
\end{array} \right. \]

Further,
\[ |v(s)| \leq \sum_{k=2}^{\infty} 2^{-2k} \binom{2k}{k} \frac{p^{-ks}}{2k-1} = 1 - \frac{1}{2p^\sigma} - \sqrt{1 - \frac{1}{p^\sigma}} = \]
\[ = \frac{(1 - \frac{1}{2p^\sigma} - \sqrt{1 - \frac{1}{p^\sigma}})}{\left( 1 - \frac{1}{2p^\sigma} + \sqrt{1 - \frac{1}{p^\sigma}} \right)} \leq \frac{1}{4p^{2\sigma}} < \frac{1}{4p^{2\sigma}} \]
for any \( \sigma > \frac{1}{2} \). In particular, in the case \( \sigma \geq 1 \) we get
\[ |v(s)| < \frac{1}{4p^{2\sigma}} \leq \frac{1}{4p^{\sigma+1}}. \]

Thus we obtain for \( \frac{1}{2} \leq \sigma \leq 1 \)
\[ |\Phi_p(s)| = \left| 1 - \frac{1}{4p^{2s}} + u(s) - v(s) - \frac{u(s)}{2p^s} - \frac{v(s)}{2p^s} - u(s)v(s) \right| \leq \]
\[ \leq 1 + \frac{1}{4p^{2\sigma}} + \frac{3.7}{p^{2\sigma}} + \frac{1}{4p^{2\sigma}} + \frac{1}{2p^{\sigma}} \left( \frac{3.7}{p^{2\sigma}} + \frac{1}{4p^{2\sigma}} \right) + \frac{3.7}{4p^{4\sigma}} < \\
< 1 + \frac{1}{2p^{2\sigma}} + \frac{3.7}{p^{2\sigma}} + \frac{3.95}{2\sqrt{2}p^{2\sigma}} + \frac{3.7}{8p^{2\sigma}} < 1 + \frac{7}{p^{2\sigma}} < \left( 1 - \frac{1}{p^{2\sigma}} \right)^7 \]

and therefore
\[ |\Phi(s)| < \prod_p \left( 1 - \frac{1}{p^{2\sigma}} \right)^7 \zeta(2\sigma) < \left( \frac{2\sigma}{2\sigma - 1} \right)^7 = (\sigma - \frac{1}{2})^{-7}. \]

Similarly, in the case \( \sigma \geq 1 \) we have
\[ |\Phi(s)| < \zeta(\sigma + 1) \leq \zeta(2) = O(1). \]

Thus we get for \( \text{Re} \ s > 1 \):
\[ F_a(s) = \sqrt{\zeta(s)} \Phi_a(s), \quad \Phi_a(s) = F(s)\psi_a(s) \]

where the function \( \Phi_a(s) \) is regular in the half-plane \( \text{Re} \ s > \frac{1}{2} \) and obeys the inequality
\[ |\Phi_a(s)| \ll \max \left( (\sigma - \frac{1}{2})^{-7}, 1 \right). \]

Now Lemma 8 implies that
\[ E_a(x) = \frac{x}{\sqrt{\ln x}} \left( \Phi_a(1) \sqrt{\pi} + O \left( \frac{1}{\ln x} \right) \right) \]

and therefore
\[ S_a(x) = K(a)x\sqrt{\ln x} + O(x \ln \ln x) \]

where
\[ K(a) = \frac{1}{\sqrt{\pi}} C\beta(a)\Phi_a(1). \]

Finally, considering the constant \( K(a) \), we get \( K(a) = K \cdot \kappa(a) \), where
\[ K = \frac{1}{\sqrt{\pi}} C\Phi(1) = \frac{C}{\sqrt{\pi}} \prod_p \sqrt{1 - \frac{1}{p}} \left( 1 - \beta(p) - \beta(p)p \ln(1 - p^{-1}) \right) = \\
= \frac{1}{\sqrt{\pi}} \prod_p \left( 1 + \frac{1}{p(p-1)} \right) \sqrt{1 - \frac{1}{p}} \frac{p(1 + (p-1)^2 \ln \frac{p}{p-1})}{p^2 - p + 1} = \\
= \frac{1}{\sqrt{\pi}} \prod_p \sqrt{1 - \frac{1}{p}} \frac{p^2 - p + 1}{p(p-1)} \frac{p(1 + (p-1)^2 \ln \frac{p}{p-1})}{p^2 - p + 1} = \\
= \frac{1}{\sqrt{\pi}} \prod_p \left( \frac{1}{\sqrt{p(p-1)}} + \sqrt{1 - \frac{1}{p} (p - 1) \ln \frac{p}{p-1}} \right), \\
\kappa(a) = \beta(a)\psi_a(1) = \beta(a) \prod_{p|a} \frac{1 + \sum_{k=1}^{+\infty} e_a(p^k) p^{-k}}{1 + \beta(p) \sum_{k=1}^{+\infty} p^{-k}}. \]
Theorem is completely proved.

Remark. The above arguments may be applied to the sums of the following type:

\[ C_a = \sum_{a < n \leq x} \frac{\tau(n)}{\tau(n - a)}, \quad S_{a,k}(x) = \sum_{n \leq x} \frac{\tau(n)}{\tau_k(n + a)}; \]

where \( a \geq 1 \) and \( \tau_k(n) \) denotes the number of solutions of the equation

\[ x_1 \ldots x_k = n \]

in the natural numbers \( x_1, \ldots, x_k \). One can obtain the formula

\[ S_{a,k}(x) = K(a, k) x (\ln x)^{\frac{1}{k}} + O(x \ln \ln x). \]

5. Calculation of the constants \( \kappa(a) \)

First we note that \( \kappa(a) \) is the multiplicative function of \( a \). Indeed, suppose \( a = a_1 a_2 \) where \( (a_1, a_2) = 1 \), and let \( p \) be a prime divisor of \( a_1 \). Then \( (p, a_2) = 1 \) and for any \( k \geq 1 \)

\[ e_a(p^k) = \frac{1}{\beta(a)} \sum_{d \mid (a,p^k)} \beta\left(\frac{dp^k}{d^2}\right) = \frac{1}{\beta(a_1) \beta(a_2)} \sum_{d \mid (a_1,p^k)} \beta\left(\frac{a_1 p^k}{d^2}\right) = \frac{1}{\beta(a_1)} \sum_{d \mid (a_1,p^k)} \beta\left(\frac{a_1 p^k}{d^2}\right) = e_{a_1}(p^k). \]

Now it follows that

\[ \kappa(a_1 a_2) = \beta(a_1 a_2) \prod_{p \mid a_1} \frac{1 + \sum_{k=1}^{+\infty} e_a(p^k) p^{-k}}{1 - \beta(p) - \beta(p)p \ln \frac{p}{p - 1}} \cdot \prod_{p \mid a_2} \frac{1 + \sum_{k=1}^{+\infty} e_a(p^k) p^{-k}}{1 - \beta(p) - \beta(p)p \ln \frac{p}{p - 1}} = \beta_{a_1} \prod_{p \mid a_1} \frac{1 + \sum_{k=1}^{+\infty} e_{a_1}(p^k) p^{-k}}{1 - \beta(p) - \beta(p)p \ln \frac{p}{p - 1}} \cdot \beta_{a_2} \prod_{p \mid a_2} \frac{1 + \sum_{k=1}^{+\infty} e_{a_2}(p^k) p^{-k}}{1 - \beta(p) - \beta(p)p \ln \frac{p}{p - 1}} = \kappa(a_1) \kappa(a_2). \]

Further, \( e_a(p^k) = \beta(p) \) for \( (a, p) = 1 \) and

\[ e_a(p^k) = \begin{cases} \min(k + 1, m + 1), & \text{if } k \neq m, \\ m + \frac{1}{\beta(p)}, & \text{if } k = m. \end{cases} \]

for the case \( a = p^m a_1, (a_1, p) = 1, m \geq 1 \). Now we calculate the values of \( \kappa(a) \) for the cases \( a = p, p^2, p^3, p^4 \) (\( p \) is prime).
1°. Suppose $a = p$; then

$$e_a(p) = 1 + \frac{1}{\beta(p)}, \quad e_a(p^k) = 2, \quad k = 2, 3, 4, \ldots.$$ 

Therefore,

$$1 + \sum_{k=1}^{\infty} e_a(p^k) p^{-k} = 1 + \frac{1}{2p} \left( 1 + \frac{1}{\beta(p)} \right) + 2 \sum_{k=2}^{\infty} \frac{p^{-k}}{k+1} =$$

$$= 2p \ln \frac{p}{p-1} - 1 - \frac{1}{2p} + \frac{1}{2p\beta(p)}.$$

Hence,

$$\kappa(p) = \frac{2p \ln \frac{p}{p-1} - 1 - \frac{1}{2p} + \frac{1}{2p\beta(p)}}{p \ln \frac{p}{p-1} - 1 + \frac{1}{\beta(p)}}.$$ 

In particular,

$$\kappa(2) = \frac{2 \ln 2 - \frac{1}{4}}{\ln 2 + 1} = 0.671 \, 113 \, 754 \ldots$$

$$\kappa(3) = \frac{2 \ln 3 - \frac{7}{3}}{\ln 3 + \frac{1}{3}} = 0.792 \, 206 \, 241 \ldots$$

$$\kappa(5) = \frac{2 \ln 5 - \frac{31}{16}}{\ln 5 + \frac{1}{16}} = 0.884 \, 098 \, 735 \ldots$$

$$\kappa(7) = \frac{2 \ln 7 - \frac{71}{504}}{\ln 7 + \frac{1}{504}} = 0.920 \, 297 \, 714 \ldots$$

$$\kappa(11) = \frac{2 \ln 11 - \frac{199}{2200}}{\ln 11 + \frac{1}{2200}} = 0.951 \, 150 \, 347 \ldots$$

$$\kappa(13) = \frac{2 \ln 13 - \frac{287}{1944}}{\ln 13 + \frac{1}{1944}} = 0.959 \, 100 \, 63 \ldots$$

$$\kappa(17) = \frac{2 \ln 17 - \frac{511}{8704}}{\ln 17 + \frac{1}{8704}} = 0.969 \, 157 \, 895 \ldots$$

$$\kappa(19) = \frac{2 \ln 19 - \frac{647}{12312}}{\ln 19 + \frac{1}{12312}} = 0.972 \, 537 \, 955 \ldots.$$

2°. Suppose $a = p^2$; then

$$1 + \sum_{k=1}^{\infty} \frac{e_a(p^k)}{k+1} \frac{1}{p^k} = 3p \ln \frac{p}{p-1} - 2 - \frac{1}{2p} - \frac{1}{3p^2} + \frac{1}{3p^2 \beta(p)},$$

$$\kappa(p^2) = \frac{3p \ln \frac{p}{p-1} - 2 - \frac{1}{2p} - \frac{1}{3p^2} + \frac{1}{3p^2 \beta(p)}}{p \ln \frac{p}{p-1} - 1 + \frac{1}{\beta(p)}}.$$
In particular,
\[
\kappa(2^2) = \frac{3 \ln 2 - \frac{25}{24}}{\ln 2 + 1} = 0.612926558 \ldots
\]
\[
\kappa(3^2) = \frac{3 \ln \frac{3}{2} - \frac{77}{108}}{\ln \frac{3}{2} + \frac{1}{4}} = 0.768053638 \ldots
\]

3°. Suppose \( a = p^3 \); then
\[
1 + \sum_{k=1}^{+\infty} \frac{e_a(p^k)}{k+1} p^{-k} = 4p \ln \frac{p}{p-1} - 3 - \frac{1}{p^2} - \frac{1}{3p^2} - \frac{1}{4p^3} + \frac{1}{4p^4 \beta(p)}
\]
\[
\kappa(p^3) = \frac{4p \ln \frac{p}{p-1} - 3 - \frac{1}{p^2} - \frac{1}{3p^2} - \frac{1}{4p^3} + \frac{1}{4p^4 \beta(p)}}{p \ln \frac{p}{p-1} - 1 + \frac{1}{\beta(p)}}
\]

In particular,
\[
\kappa(2^3) = \frac{4 \ln 2 - \frac{169}{96}}{\ln 2 + 1} = 0.597805121 \ldots
\]

4°. Suppose \( a = p^4 \); then
\[
1 + \sum_{k=1}^{+\infty} \frac{e_a(p^k)}{k+1} p^{-k} = 5p \ln \frac{p}{p-1} - 4 - \frac{3}{2p} - \frac{2}{3p^2} - \frac{1}{4p^3} - \frac{1}{5p^4} + \frac{1}{5p^5 \beta(p)}
\]
\[
\kappa(p^3) = \frac{5p \ln \frac{p}{p-1} - 4 - \frac{3}{2p} - \frac{2}{3p^2} - \frac{1}{4p^3} - \frac{1}{5p^4} + \frac{1}{5p^5 \beta(p)}}{p \ln \frac{p}{p-1} - 1 + \frac{1}{\beta(p)}}
\]

In particular,
\[
\kappa(2^4) = \frac{5 \ln 2 - \frac{2363}{960}}{\ln 2 + 1} = 0.59314251 \ldots
\]

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