THE MEAN-FIELD LIMIT OF THE LIEB-LINIGER MODEL

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ABSTRACT. We start from the well-known Lieb-Liniger (LL) model, which describes \( N \) bosons interacting pairwise on the line via the \( \delta \)-potential, with initial data of the form \( \phi_0^\otimes N \) corresponding to all of the particles being in the same initial state. In the mean-field regime where the interaction term of the Hamiltonian has been scaled by a factor of \( \frac{1}{N} \), we show that for each \( k \in \mathbb{N} \), the \( k \)-particle reduced density matrix of the system converges in trace norm to the density matrix \( \langle \phi_0^\otimes k | \phi_0^\otimes k \rangle \), as \( N \to \infty \), where \( \phi \) is the unique solution to the one-dimensional cubic NLS with initial datum \( \phi_0 \in H^2(\mathbb{R}) \). Our proof is inspired by the method of Pickl [28, 29, 30] and Knowles and Pickl [17] and is based on an energy-type estimate for a time-dependent functional which gives a weighted count of the number of particles in the system at time \( t \) which are not in the state \( \phi(t) \). To overcome difficulties stemming from the singularity of the \( \delta \)-potential, we introduce a new short-range approximation argument that exploits the Hölder continuity of the \( N \)-body wave function in a single particle variable. In contrast to the previous work of Ammari and Breteaux [3], our simple proof makes no use of second quantization and provides an explicit rate of convergence to the mean-field limit.

1. Introduction

1.1. Background. We consider the Lieb-Liniger (LL) model, which describes a finite number of bosons in one dimension with two-body interactions governed by the \( \delta \) potential. Formally, the Hamiltonian for \( N \) bosons is given by

\[
\sum_{i=1}^{N} -\Delta_i + c \sum_{1 \leq i < j \leq N} \delta(X_i - X_j),
\]

where \( -\Delta_i \) denotes the Laplacian in the \( i \)-th particle variable \( x_i \in \mathbb{R} \), \( \delta(X_i - X_j) \) denotes multiplication by the distribution \( \delta(x_i - x_j) \), and \( c \in \mathbb{R} \) is the coupling constant determining the strength of the interaction and whether it is repulsive (\( c > 0 \)) or attractive (\( c < 0 \)). The LL model is named for Lieb and Liniger, who showed in the seminal works [20, 19] that when considered on a finite interval \([0, L]\) with periodic boundary conditions, the model is exactly solvable by Bethe ansatz\(^1\). While it was originally introduced as a toy quantum many-body system, the LL model has since attracted interest from both the physics community [24, 27, 8, 14, 21, 25, 7] and the mathematics community [22, 35] in modeling quasi-one-dimensional dilute Bose gases which have been realized in laboratory settings [6, 33, 38, 9].

In applications, the number of particles \( N \) is large, ranging upwards from \( N \approx 10^3 \) in the case of very dilute Bose-Einstein condensates. For large \( N \), it is computationally expensive to extract useful information about the time evolution of the system directly from its wave function. Thus, one seeks to find an evolution equation, for which one can more efficiently extract information, that provides an effective description of the \( N \)-body system for large values of \( N \).

Accordingly, the goal of this article is to rigorously obtain an effective description of the dynamics of the LL model in the limit as the number of particles tends to infinity. To obtain nontrivial dynamics in the limit, we consider the mean-field scaling regime, where the coupling constant \( c \) in (1.1) is taken to be equal to \( \kappa/N \), for some \( \kappa \in \mathbb{R} \setminus \{0\} \), so that the Hamiltonian becomes

\[
H_N = \sum_{i=1}^{N} -\Delta_i + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \delta(X_i - X_j).
\]

\(^1\)Bethe ansatz refers to a method in the study of exactly solvable models originally introduced by Hans Bethe to find exact eigenvalues and eigenvectors of the antiferromagnetic Heisenberg spin chain [4]. For more on this technique and its applications, we refer the reader to the monograph [11].
Note that the mean-field scaling is such that the free and interacting components of the Hamiltonian $H_N$ are of the same order in $N$. By means of quadratic forms (see Section 3), the expression (1.22) can be realized as a self-adjoint operator on the Hilbert space $L^2_{sym}(\mathbb{R}^N)$ consisting of wave functions $\Phi_N \in L^2(\mathbb{R}^N)$ satisfying
\begin{equation}
\Phi_N(x_{\pi(1)}, \ldots, x_{\pi(N)}) = \Phi_N(x_1, \ldots, x_N) \text{ almost everywhere, } \forall \pi \in S_N.
\end{equation}

By Stone’s theorem, the corresponding Schrödinger problem
\begin{equation}
\begin{cases}
i \partial_t \Phi_N = H_N \Phi_N \\
\Phi_N(0) = \Phi_{N,0} \in L^2_{sym}(\mathbb{R}^N)
\end{cases}
\end{equation}
has a unique global solution $\Phi_N(t) = e^{-iH_N t} \Phi_{N,0}$. Of particular interest are factorized initial data $\Phi_{N,0} = \phi_0 \otimes N$, for $\phi_0 \in L^2(\mathbb{R})$ satisfying $\|\phi_0\|_{L^2(\mathbb{R})} = 1$, which correspond to a system where the $N$ particles are all in the same initial state $\phi_0$. By rescaling spacetime, it suffices to consider the case $\kappa \in \{\pm 1\}$.

In general, factorization of the wave function $\Phi_N$ is not preserved under the time evolution due to the interaction between particles. However, it is reasonable to expect from the factor of $\frac{1}{N}$ in the potential term in (1.22) that the total potential experienced by each particle is approximately described by an effective \textit{mean-field} potential in the limit as $N \to \infty$. Formally, we may expect that
\begin{equation}
\Phi_N \approx \phi \otimes N \quad \text{as } N \to \infty,
\end{equation}
for some $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, in some sense to be made precise momentarily.

To find an equation satisfied by $\phi$ and to give rigorous meaning to the approximation (1.5), we argue as follows. Let $\Phi_N$ be the solution to the Schrödinger equation (1.3), and consider the \textit{density matrix}
\begin{equation}
\Psi_N := |\Phi_N \rangle \langle \Phi_N|
\end{equation}
associated to $\Phi_N$. This density matrix is the rank-one projection onto the state $\Phi_N$ with integral kernel
\begin{equation}
\Psi_N(t, \mathbf{x}; \mathbf{x}') = \Phi_N(t, \mathbf{x}) \Phi(t, \mathbf{x}'), \quad \forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^N, \ t \in \mathbb{R}.
\end{equation}

For $k \in \{1, \ldots, N\}$, we define the \textit{k-particle reduced density matrix} $\gamma_N^{(k)}$ associated to $\Phi_N$ by
\begin{equation}
\gamma_N^{(k)} := T_{k+1, \ldots, N} \Psi_N,
\end{equation}
where $T_{k+1, \ldots, N}$ denotes the partial trace over the coordinates $(x_{k+1}, \ldots, x_N)$. By conservation of mass for (1.4) (i.e. $\|\Phi_N(t)\|_{L^2(\mathbb{R}^N)} = \|\Phi_{N,0}\|_{L^2(\mathbb{R}^N)} = 1$), it follows that $T_{1, \ldots, k}(\gamma_N^{(k)}(t)) = 1$ for every $N \in \mathbb{N}$, $k \in \{1, \ldots, N\}$, and $t \in \mathbb{R}$. Using equation (1.3), one can show that $\{\gamma_N^{(k)}\}_{k=1}^N$ solve the coupled system of equations known as the \textit{Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy (BBGKY) hierarchy}:
\begin{equation}
\begin{split}
&i \partial_t \gamma_N^{(k)} = \left[ -\Delta_X, \gamma_N^{(k)} \right] + \frac{\kappa}{N} \sum_{1 \leq \ell < j \leq k} \left[ \delta(X_\ell - X_j), \gamma_N^{(k)} \right] \\
&\quad + \frac{N-k}{N} \sum_{j=1}^k T_{k+1, \ldots, N-1} \left[ \delta(X_j - X_{k+1}), \gamma_{N-1}^{(k+1)} \right],
\end{split}
\end{equation}
where we have introduced the notation $\Delta_X := \sum_{i=1}^k \Delta_i$ and $\left[ \cdot, \cdot \right]$ denotes the usual commutator bracket. As $N \to \infty$, the sequence $\{\gamma_N^{(k)}\}_{k \in N}$, where by convention $\gamma_N^{(k)} := 0$ for $k > N$, formally converges to a solution $\{\gamma_k\}_{k \in N}$ of the \textit{Gross-Pitaevskii (GP) hierarchy}:
\begin{equation}
\begin{split}
&i \partial_t \gamma^{(k)} = \left[ -\Delta_X, \gamma^{(k)} \right] + \kappa \sum_{j=1}^k T_{k+1, \ldots, N-1} \left[ \delta(X_j - X_{k+1}), \gamma^{(k+1)} \right].
\end{split}
\end{equation}
If there is some function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, such that the GP solution takes the form $\gamma^{(k)} = |\phi \otimes \mathbf{k}\rangle \langle \phi \otimes \mathbf{k}|$ for every $k \in \mathbb{N}$, it is an easy computation from (1.10) that $\phi$ solves the \textit{one-dimensional (1D) cubic nonlinear Schrödinger (NLS) equation}
\begin{equation}
(i \partial_t + \Delta) \phi = \kappa |\phi|^2 \phi, \quad \phi(0) = \phi_0.
\end{equation}
Thus, we formally refer to the 1D cubic NLS as the \textit{mean-field limit} of the LL model. It is quite interesting that just as the LL model is exactly solvable by Bethe ansatz, as we commented above, the 1D cubic NLS is exactly solvable by the inverse scattering transform \cite{39,10}. We do not investigate the relationship between N-body exact solvability and limiting exact solvability in this article; but we refer the interested reader to our recent joint work \cite{23} for a more extensive discussion of this connection.

Establishing the validity of the mean-field approximation to the Schrödinger problem (1.4) consists of showing convergence of the \(k\)-particle reduced density matrices \(\gamma_N^{(k)}\) to \(|\phi^{\otimes k}\rangle \langle \phi^{\otimes k}|\), as \(N \to \infty\), in trace norm:

\[
\forall k \in \mathbb{N}, \quad \lim_{N \to \infty} \text{Tr}_{1,\ldots,k} \left| \gamma_N^{(k)} - |\phi^{\otimes k}\rangle \langle \phi^{\otimes k}| \right| = 0.
\]

We refer to (1.12) as \textit{convergence to the mean-field limit} or, following terminology in the kinetic theory literature, as \textit{propagation of chaos}.

1.2. Prior results. The mathematical investigation of the validity of the mean-field approximation for the LL model was initiated by Adami, Bardos, Golse, and Teta \cite{1}. The authors proceed by the so-called BBGKY method, which was pioneered by Spohn \cite{36} for the study of quantum many-body systems. Namely, Adami et al. show that for each \(k \in \mathbb{N}\) fixed, the sequence \(\{\gamma_N^{(k)}\}_{N \in \mathbb{N}}\) has a limit point \(\gamma^{(k)}\) with respect to a topology weaker than trace norm. Then they show that the sequence \(\{\gamma^{(k)}\}_{k \in \mathbb{N}}\) is a solution to the GP hierarchy (1.10) with initial datum \(\{\phi^{\otimes k}\}_{k \in \mathbb{N}}\) in a certain class akin to the Sobolev space \(H^1\). In order to conclude their proof, they need to show that there can only be one such solution (i.e. prove uniqueness for the GP hierarchy in the class under consideration), from which propagation of chaos (1.12) follows. However, they could not prove this uniqueness, and to our knowledge, their argument has yet to be completed. We remark that the BBGKY approach does not yield a rate of convergence in (1.12) as \(N \to \infty\) and \(|t| \to \infty\).

Several years later, Ammari and Breteaux \cite{3} revisited the mean-field approximation to the LL model from the perspective of quantum field theory. Inspired by the approach of Rodnianski and Schlein \cite{34}, which in turn builds on earlier ideas of Hepp \cite{13} and Ginibre and Velo \cite{12}, the authors use the framework of second quantization and reformulate the problem of mean-field limit for the Hamiltonian (1.2) in terms of the semiclassical limit for a related Hamiltonian on the Fock space. Through a very technical argument involving abstract non-autonomous Schrödinger equations, they construct a time-dependent quadratic Hamiltonian which provides a semiclassical approximation for the evolution of coherent states. Borrowing an argument from \cite{34}, they are able to show the convergence (1.12) from their approximation result for coherent states.

We note that the authors do not provide a quantitative rate for the convergence (1.12) in terms of \(N \) and \(t\).

1.3. Overview of main results and their proof. Having introduced the LL model and the problem of establishing the mean-field approximation and having reviewed prior work on this problem, we are now prepared to state our main results.

Theorem 1.1 (Main result). Let \(\kappa \in \{\pm 1\}\), and let \(\phi_0 \in H^2(\mathbb{R})\) with \(|\phi_0|_{L^2(\mathbb{R})} = 1\). Then there exists an absolute constant \(C > 0\) such that for every \(N \in \mathbb{N}\) and \(k \in \{1, \ldots, N\}\),

\[
\text{Tr}_{1,\ldots,k} \left| \gamma_N^{(k)}(t) - |\phi(t)^{\otimes k}\rangle \langle \phi(t)^{\otimes k}| \right| \leq C \sqrt{|t|} k \left( \frac{||\phi_0||_{H^2(\mathbb{R})}^2}{N^{1/3}} + \frac{||\phi_0||_{H^2(\mathbb{R})}^2}{N^{1/2}} \right)^{1/2} e^{C ||\phi_0||_{H^2(\mathbb{R})}^2 |t|}, \quad \forall t \in \mathbb{R},
\]

where \(\gamma_N^{(k)}\) is the \(k\)-particle reduced density matrix defined in (1.8) and \(\phi\) is the unique solution to the cubic NLS (1.11) in \(C^0_0(\mathbb{R}; H^2_0(\mathbb{R}))\).\footnote{It is textbook that the 1D cubic NLS is globally well-posed in the class \(C^0_0(\mathbb{R}; H^2_0(\mathbb{R}))\) of functions which are continuous in time values in \(H^2(\mathbb{R})\). For instance, see \cite{5} and \cite{37}.}

Our Theorem 1.1 establishes the convergence to the mean-field limit (1.12) for the LL model with an explicit rate of convergence which holds for arbitrary lengths of time in both the repulsive and attractive settings. The \(H^2\) regularity assumption on the initial datum \(\phi_0\) is consistent with the assumption of Ammari
and Breteaux [3]. Additionally, an examination of the argument in Section 4 and Section 5 shows that if we replace the Hamiltonian $H_N$ in (1.2) with the “regularized Hamiltonian”

$$H_{N, \sigma} := \sum_{i=1}^{N} -\Delta_i + \frac{\kappa}{N}\sum_{1 \leq i < j \leq N} V_N(X_i - X_j), \quad \kappa \in \{\pm 1\},$$

(1.14)

where $V$ is a short-range potential satisfying certain regularity conditions and $V_N := N^\sigma V(N^\sigma \cdot)$, for some fixed $\sigma \in (0, \infty)$, then for any $T > 0$ fixed,

$$\forall k \in \mathbb{N}, \quad \lim_{N \to \infty} \sup_{0 \leq t \leq T} \text{Tr} \left| \gamma^{(k)}_{N, \sigma}(t) - |\phi(t)^\otimes k\rangle \langle \phi(t)^\otimes k| \right| = 0,$$

(1.15)

where $\gamma^{(k)}_{N, \sigma}$ is the $k$-particle reduced density matrix associated to the Schrödinger problem obtained by replacing $H_N$ in (1.4) with $H_{N, \sigma}$. One can extract a rate of convergence for (1.15) which tends to the rate (1.13) as $\sigma \to \infty$.

We now comment on the proof of Theorem 1.1 and highlight the major difficulties and differences from existing work. Inspired by the method of Pickl [28, 29, 30] and the refinement of this method developed by Knowles and Pickl [17] for derivation of the Hartree equation in the mean-field limit, our argument is based on an energy-type estimate for a functional $\beta_N$ of the solution $\Phi_N$ to equation (1.4), which gives a weighted count of the number of “bad particles” in the system at time $t$ which are not in the state $\phi(t)$, where $\phi$ solves the cubic NLS (1.11). The functional $\beta_N$ takes the form

$$\beta_N(t) := \left\langle \Phi_N(t) | n_N(t) \Phi_N(t) \right\rangle_{L^2(\mathbb{R}^N)} = \sum_{k=0}^{N} \sqrt{\frac{k}{N}} \langle \Phi_N(t) | P_k(t) \Phi_N(t) \rangle_{L^2(\mathbb{R}^N)}, \quad \forall t \in \mathbb{R},$$

(1.16)

where $\Phi_N$ is the solution to (1.4) and $P_k(t)$ is the projector mapping a wave function onto the subspace of $L^2_{\text{sym}}(\mathbb{R}^N)$ of functions corresponding to $k$ of the particles not being in the state $\phi(t)$. See (1.4) and more generally Section 4.1 for the precise definition and properties of these projectors. The main estimate for $\beta_N$ is given by Proposition 1.2 below. To state the proposition, we first introduce some notation. Let $E_N^\Phi$ denote the energy per particle of the $N$-body system (1.4), which is defined by

$$E_N^\Phi(t) := \frac{1}{N} \langle \Phi_N(t) | H_N \Phi_N(t) \rangle_{L^2(\mathbb{R}^N)} = ||\nabla 1 \Phi_N(t)||^2_{L^2(\mathbb{R}^N)} + \frac{\kappa(N-1)}{2N} ||\text{tr}_{1=2} \Phi_N(t)||^2_{L^2(\mathbb{R}^{N-1})},$$

(1.17)

where the ultimate equality follows from integration by parts and the symmetry (1.3). Let $E^\phi$ denote the cubic NLS energy, which is defined by

$$E^\phi(t) := ||\nabla \phi(t)||^2_{L^2(\mathbb{R})} + \frac{\kappa}{2} ||\phi(t)||^4_{L^4(\mathbb{R})}.$$

(1.18)

Above, we have used the notation $\text{tr}_{i=j}$ to denote the trace to the hyperplane $\{x_N \in \mathbb{R}^N : x_i = x_j\}$. Note that both $E_N^\Phi$ and $E^\phi$ are independent of time by conservation of energy for equations (1.4) and (1.11):

$$E_N^\Phi(t) = ||\nabla 1 \Phi_N,0||^2_{L^2(\mathbb{R}^N)} + \frac{\kappa(N-1)}{2N} ||\text{tr}_{1=2} \Phi_N,0||^2_{L^2(\mathbb{R}^{N-1})},$$

(1.19)

$$E^\phi(t) = ||\nabla \phi,0||^2_{L^2(\mathbb{R})} + \frac{\kappa}{2} ||\phi,0||^4_{L^4(\mathbb{R})}.$$  

(1.20)

**Proposition 1.2** (Evolution of $\beta_N$). Let $\kappa \in \{\pm 1\}$. Then there exists an absolute constant $C > 0$, such that for every $N \in \mathbb{N}$, there exists a continuous function $\mathfrak{A}_N : [0, \infty) \to [0, \infty)$ such that

$$\beta_N(t) \leq \mathfrak{A}_N(|t|) e^{C||\phi,0||^2_{H^1(\mathbb{R})}|t|}, \quad \forall t \in \mathbb{R},$$

(1.21)

where $\mathfrak{A}_N$ satisfies the bound

$$\mathfrak{A}_N(t) \leq \beta_N(0) + C|t| \left( \frac{||\phi,0||_{H^1(\mathbb{R})}}{N^{1/3}} + \frac{||\phi,0||_{H^2(\mathbb{R})}}{N^{1/2}} + (E_N^\Phi - E^\phi)||\phi,0||_{H^1(\mathbb{R})}^2 \right), \quad \forall t \in \mathbb{R}.$$

(1.22)

A function $\phi : \mathbb{R} \times \mathbb{R}^d$ satisfies the Hartree equation if $(i\partial_t + \Delta) \phi = (V * |\phi|^2)\phi$, where $V$ is a chosen locally integrable function. The cubic NLS (1.11) may be viewed as the special case of the Hartree equation with $V = \delta$. 

\[ \text{[Footnote]} \]
Remark 1.3. An examination of the argument in Section 5 for obtaining Theorem 1.1 from Proposition 1.2 shows that we have propagation of chaos for any sequence of initial wave functions $\Phi_{N,0} \in L_{sym}^2(\mathbb{R}^N)$ such that

$$\lim_{N \to \infty} \beta_N(0) = 0 \quad \text{and} \quad \lim_{N \to \infty} E_{N}^\Phi - E^\phi = 0. \tag{1.23}$$

To prove Proposition 1.2 we proceed by a Gronwall-type argument. Differentiating $\beta_N$ with respect to time and performing some simplifications, we find that we need to estimate the following three terms:

$$\text{Term}_1 := \left\langle \Phi_N | p_1 p_2 \left[ V_1^\phi, \hat{\nu}_N \right] q_1 q_2 \Phi_N \right\rangle_{L_{\infty}^2(\mathbb{R}^N)}, \tag{1.24}$$

$$\text{Term}_2 := \left\langle \Phi_N | q_1 q_2 \left[ (N - 1)V_{12} - NV_2^\phi, \hat{\nu}_N \right] q_1 q_2 \Phi_N \right\rangle_{L_{\infty}^2(\mathbb{R}^N)}, \tag{1.25}$$

$$\text{Term}_3 := \left\langle \Phi_N | p_1 p_2 [(N - 1)V_{12}, \hat{\nu}_N] q_1 q_2 \Phi_N \right\rangle_{L_{\infty}^2(\mathbb{R}^N)}, \tag{1.26}$$

where we have used the notation $V_{12} := \delta(X_1 - X_2)$ and $V_2^\phi := |\phi(X_j)|^2$ and we remind the reader that $[\cdot, \cdot]$ denotes the commutator. $V_{12}(q_1 q_2 \Phi_N)$ and $V_{12}^{\phi} = \langle \hat{\nu}_N \rangle_{L_{\infty}^2(\mathbb{R}^N)}$, similarly for the other terms, should be interpreted as elements of $H^{-1}(\mathbb{R}^N)$ and the inner product as a duality pairing. Here, $p_j$ is the rank-one projector $|\phi \rangle \langle \phi|$ acting in the $x_j$-variable, and $q_j = 1_N - p_j$, where $1_N$ is the identity operator on $L^2(\mathbb{R}^N)$ (see Section 4.4 for more details). As Term3 is the most difficult case in the analysis and where the existing arguments in the literature break down, we focus on it.

By expanding the commutator in the definition of Term3 and using Lemma 4.7 to shift the projectors $P_k$ in the definition of $\hat{\nu}_N$ (see Definition 4.4), we reduce to bounding the expression

$$\left\langle \Phi_N | p_1 p_2 V_{12} q_1 q_2 \hat{\nu}_N \Phi_N \right\rangle_{L_{\infty}^2(\mathbb{R}^N)}, \tag{1.27}$$

where $\hat{\nu}_N = \sum_{k=1}^N \nu_N(k) P_k$ is a time-dependent operator on $L_{sym}^2(\mathbb{R}^N)$ such that the coefficients satisfy $\nu_N(k) \lesssim n_N^{-1}(k)$. See (4.68) for the precise definition of $\nu_N$ and $\hat{\nu}_N$. To obtain an acceptable bound for our Gronwall argument, we need to produce an operator $\hat{\nu}_N^2$, so that

$$\hat{\nu}_N^2 \lesssim \hat{\nu}_N. \tag{1.28}$$

In [17], Knowles and Pickl had to contend with an expression similar to Term3 but with a much more regular potential $V$, which satisfies certain integrability assumptions of the form $V \in L^{p_0} + L^\infty$. In order to simplify the comparison, we assume that $V \in L^{p_0}$. To deal with their analogue of (1.27), they split the potential into its “regular” and “singular” parts by making an $N$-dependent decomposition of the form

$$V_{reg} := V_1(|V| \leq N^\sigma), \quad V_{sing} := V_1(|V| > N^\sigma), \tag{1.29}$$

where $1_A$ denotes the indicator function for the set $\{A\}$ and $\sigma \in (0, 1)$ is a parameter to be optimized at the end. For the singular part, they express the potential as the divergence of a vector field,

$$V = \nabla \cdot \xi \tag{1.30}$$

and integrate by parts. Crucially, their integrability assumption implies that $\xi \in L^2(\mathbb{R}^N)$ with $L^2$ norm $O(N^{-\delta})$, for some $\delta > 0$, which is necessary to close their estimate. For the regular part, the important idea is to exploit the symmetry (1.3) of the wave function, since the operator norm of $p_1 p_2 V_{12} q_1 q_2$ is much smaller on the bosonic subspace $L_{sym}^2(\mathbb{R}^N)$ than on the full space $L^2(\mathbb{R}^N)$. As the argument is a bit involved, we only comment that it importantly requires $V_{reg}$ to be integrable.

For $V = \delta(x)$, Knowles and Pickl’s argument described above breaks down. While we have the identity

$$\delta(x) = \frac{1}{2} \nabla \text{sgn}(x), \tag{1.31}$$

the signum function is only in $L^\infty$, not in $L^2$ as their singular-part argument requires. Additionally, since $\delta$ is only a distribution, we cannot assign meaning to $\delta^2$ in the regular part of their argument. In fact, the regular part of their argument is formally vacuous for the $\delta$ potential.
To overcome the difficulties stemming from the lack of integrability of the $\delta$ potential, we introduce a new short-range approximation argument as follows. We make an $N$-dependent mollification of the potential by setting
\begin{equation}
V_{\sigma}(x) := N^{\sigma} \tilde{V}(N^{\sigma} x), \quad \forall x \in \mathbb{R},
\end{equation}
where $\sigma \in (0, 1)$, $0 \leq \tilde{V} \leq 1$, $\tilde{V} \in C_c^\infty(\mathbb{R})$ is even, and $\int_\mathbb{R} dx \tilde{V}(x) = 1$. By the triangle inequality, we have
\begin{equation}
\left| \langle \Phi_N | p_1 p_2 V_{12} q_1 q_2 \tilde{V}^2 \Phi_N \rangle \right|_{L^2_\nu(\mathbb{R}^N)} \leq \left| \langle \Phi_N | p_1 p_2 (V_{12} - V_{\sigma,12}) q_1 q_2 \tilde{V}^2 \Phi_N \rangle \right|_{L^2_\nu(\mathbb{R}^N)} + \left| \langle \Phi_N | p_1 p_2 V_{\sigma,12} q_1 q_2 \tilde{V}^2 \Phi_N \rangle \right|_{L^2_\nu(\mathbb{R}^N)}.
\end{equation}
Combining the scaling relation
\begin{equation}
\int_\mathbb{R} dx |x|^{1/2} V_{\sigma}(x) \sim N^{-\sigma/2}
\end{equation}
with fact that the wave function $\Phi_N$ is $\frac{1}{2}$ Hölder-continuous in a single particle variable by conservation of mass and energy together with Sobolev embedding (see Lemma 2.3), we can estimate
\begin{equation}
\left| \langle \Phi_N | p_1 p_2 (V_{12} - V_{\sigma,12}) q_1 q_2 \tilde{V} \Phi_N \rangle \right|_{L^2_\nu(\mathbb{R}^N)} \lesssim N^{-\sigma} + \|\phi\|_{C^1(\mathbb{R})}^2 \|\phi\|_{H^1(\mathbb{R})}^2 \beta_N + \|\phi\|_{C^1(\mathbb{R})}^2 \left\| \nabla_1 q_1 \Phi_N \right\|_{L^2(\mathbb{R}^N)}.
\end{equation}
Note that by the Sobolev embedding $H^1(\mathbb{R}) \subset C^{1/2}(\mathbb{R})$ together with conservation of mass and energy for the cubic NLS (1.11), we have that $\|\phi\|_{L^\infty(\mathbb{R}; C^{1/2}(\mathbb{R}))} \lesssim \|\phi\|_{H^1(\mathbb{R})}$. We can estimate the second term in the right-hand side of (1.33) by proceeding similarly as to the aforementioned Knowles-Pickl argument for the regular part $V_{\text{reg}}$ of the potential. While $\|V_{\sigma}\|_{L^2(\mathbb{R})} \sim N^{\sigma/2}$, we are able to extract sufficient decay in $N$ from other factors to absorb this growth in $N$, provided we appropriately choose $\sigma$.

To close the proof of Proposition 1.2, we need to control the auxiliary quantity $\|\nabla_1 q_1 \Phi_N\|_{L^2(\mathbb{R}^N)}$ in terms of $\beta_N$ and other quantities which tend to zero as $N \to \infty$. The desired control is given by Proposition 4.10. Our argument exploits the conservation of mass and energy together with the identity (1.31) and integration by parts (cf. [17] Lemma 4.6). Crucially, $\text{sgn} \in L^\infty$ so that the multiplication operator $\text{sgn}(X_1 - X_2)$ is bounded on $L^2(\mathbb{R}^N)$.

Strictly speaking, we do not work in Section 4 directly with the wave function $\Phi_N$ and with the functional $\beta_N$ but rather with an approximation obtained by replacing the Hamiltonian $H_N$ in the Schrödinger problem (1.4) with the mollified Hamiltonian
\begin{equation}
H_{N,\varepsilon} := \sum_{i=1}^N -\Delta_i + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} V_\varepsilon(X_i - X_j), \quad \kappa \in \{\pm 1\},
\end{equation}
where $V_\varepsilon := \varepsilon^{-1} \tilde{V}(-/\varepsilon)$, for $\varepsilon > 0$ and $\tilde{V}$ as above. This step is purely technical to deal with issues of operator domains involved in differentiating the functional $\beta_N$ and to avoid awkward notation involving distributions. Since $H_{N,\varepsilon} \to H_N$, as $\varepsilon \to 0^+$, in norm-resolvent sense (see Section 3.3), we are able to obtain Proposition 1.2 from an analogous estimate for the mollified version of $\beta_N$ (see 4.19 and Proposition 4.10).

1.4. Organization of the paper. We now comment on the organization of the paper. Section 2 is devoted to basic notation and preliminary facts from functional analysis used extensively in the article. We begin the section with an index (see Table 1) of the frequently used notation in the article. Section 2.1 introduces the spaces of functions and distributions used in the body of the paper, and Section 2.2 contains some basic estimates for the traces of Sobolev functions, which we use in Section 3 and Section 4.

Section 3 gives the rigorous construction of the self-adjoint operator $H_N$ corresponding to the expression (1.2). The main result is Proposition 3.4. As the construction proceeds by means of quadratic forms, we first review such forms in Section 3.1 and then prove Proposition 3.4 in Section 3.2. We close the section by establishing a short-range approximation to $H_N$ in Section 3.3 which is used in Section 4. While most of the results of Section 2 seem to be folklore in the math physics community and have appeared in other
forms elsewhere in the literature (for instance, see [3 Proposition 3.3] for a presentation in terms of the Fock space formalism), we believe that our presentation is new.

In Section 4, we prove Proposition 1.2 which is the main estimate for the functional $\beta_N$ and the main ingredient for the proof of Theorem 1.1. As this section constitutes the bulk of the paper, we have divided it into several subsections corresponding to the steps in the proof of Proposition 1.2. In Section 4.1, we introduce the time-dependent projectors which underlie the definition of the functional $\beta_N$. In Section 4.2, we approximate the functional $\beta_N$ with a functional $\beta_{N,\varepsilon}$ obtained by regularizing the Hamiltonian $H_N$ (see (4.19)) and prove a preliminary estimate for $\beta_{N,\varepsilon}$, which is Proposition 4.9. In Section 4.3, we prove Proposition 4.10 which gives an estimate in terms of $\beta_{N,\varepsilon}$ for an auxiliary quantity appearing in Proposition 4.9. In Section 4.4, we send the regularization parameter $\varepsilon$ to zero and obtain Proposition 1.2 from Proposition 4.9 and Proposition 4.10.

Lastly, in Section 5, we show how to obtain Theorem 1.1 from Proposition 1.2. As the arguments used in this step are by now well-known, we only sketch the details.

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We include here a table of the notation frequently used in the article with an explanation for the notation and/or a reference to where the definition is given.

| Symbol | Definition |
|--------|------------|
| $A \lesssim B$, $A \sim B$ | There are absolute constants $C_1, C_2 > 0$ such that $A \leq C_1 B$ or $C_2 B \leq A \leq C_1 B$ |
| $x_k$, $x_{i+k}$ | $(x_1, \ldots, x_k), (x_i, \ldots, x_{i+k})$, where $x_j \in \mathbb{R}$ for $j \in \{1, \ldots, k\}$ or $j \in \{i, \ldots, i+k\}$ |
| $d x_k$, $d x_{i+k}$ | $dx_1 \cdots dx_k, dx_i \cdots dx_{i+k}$ |
| $N$, $N_0$ | natural numbers, natural numbers inclusive of zero |
| $S_n$ | symmetric group on $N$ elements |
| $C_c^\infty(\mathbb{R}^N)$ | smooth, compactly supported functions on $\mathbb{R}^N$ |
| $S(\mathbb{R}^N)$ | Schwartz space on $\mathbb{R}^N$ |
| $S'(\mathbb{R}^N)$ | tempered distributions on $\mathbb{R}^N$ |
| $L^p(\mathbb{R}^N)$, $\| \cdot \|_{L^p}$ | standard $p$-integrable function space: see (2.2) |
| $H^s(\mathbb{R}^N)$, $\| \cdot \|_{H^s}$ | standard $L^2$-based Sobolev function space: see (2.4) |
| $C^\gamma(\mathbb{R}^N)$, $\| \cdot \|_{C^\gamma}$ | standard Hölder-continuous function space: see (2.7) |
| sym | subscript which denotes functions symmetric under permutation of coordinates |
| $\langle \cdot | \cdot \rangle$ | $L^2(\mathbb{R}^N)$ inner product with physicist’s convention: $\langle f | g \rangle := \int_{\mathbb{R}^N} dx_N f(x_N)g(x_N)$ |
| $\langle \cdot | \rangle$ | duality pairing |
| $\langle \cdot | \cdot \rangle$ | Dirac’s bra-ket notation: see footnote [2] |
| $A^{(k)}_{i_1 \cdots i_k}$ | subscript denotes that the operator on $L^2(\mathbb{R}^N)$ acts in the variables $(x_{i_1}, \ldots, x_{i_k})$ |
| $\phi^{\otimes k}$ | $k$-fold tensor product of $\phi$ with itself realized as $\phi^{\otimes k}(x_k) = \prod_{i=1}^k \phi(x_i), x_k \in \mathbb{R}^k$ |
| $\text{Tr}_{1, \ldots, N}$ | trace on $L^2(\mathbb{R}^N)$ |
| $\text{Tr}_{k+1, \ldots, N}$ | partial trace on $L^2(\mathbb{R}^N)$ over $x_{k+1}, \ldots, x_N$ coordinates |
| $1$, $1_N$ | identity operator on $L^2(\mathbb{R})$ and on $L^2(\mathbb{R}^N)$ |
| $\Phi_N$, $\Phi_N^*$ | solution to Schrödinger problem (1.4) and to regularized problem (3.19) |
| $\phi$ | solution to cubic NLS (1.11) |
| $H_N$, $H_{N,x}$ | LL Hamiltonian and regularized LL Hamiltonian: see (1.2) and (3.19) |
| $p(t), q(t)$ | rank-one projector $| \phi(t) \rangle \langle \phi(t) |$ and $1 - | \phi(t) \rangle \langle \phi(t) |$: see (4.1) |
| $p_j$, $q_j$ | projectors $1 \otimes 1 \cdots 1 \otimes p \otimes 1^{N-j}$, $1 \otimes 1 \cdots 1 \otimes q \otimes 1^{N-j}$: see (4.2) |
| $P_k$ | projector onto subspace of $k$ particles not in the state $\phi(t)$: see (4.3) |
| $\hat{f}$, $\hat{f}^{-1}$ | operator $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by $\hat{f} := \sum_{k=0}^N f(k) P_k$, for $f : Z \to \mathbb{C}$: see (4.6) |
| $n_N, m_N$ | functions $Z \to \mathbb{C}$ and operators $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$: see Definition 4.4 |
| $\mu, \nu$, $\hat{\mu}, \hat{\nu}$ | time-dependent functionals of solution $\phi$ to (1.3) and $\Phi_N$ to (1.4): see Definition 4.4 |
| $\tau_n$ | shift operator on $\mathbb{C}^Z$: see (4.11) |
| $\text{tr}_{i=j}$ | trace of a function to hyperplane $\{x_N \in \mathbb{R}^N : x_i = x_j\}$: see (2.8) |
| $\Delta_k$ | Laplacian on $\mathbb{R}^k$: $\Delta_k := \sum_{i=1}^k \Delta_i$ |
| $[\cdot, \cdot]$ | commutator bracket: $[A, B] := AB - BA$ |

Table 1: Notation
2.1. Function spaces. Fix $N \in \mathbb{N}$. We denote the Schwartz space on $\mathbb{R}^N$ by $\mathcal{S}(\mathbb{R}^N)$ and the dual space of tempered distributions on $\mathbb{R}^N$ by $\mathcal{S}'(\mathbb{R}^N)$. The subspace of $\mathcal{S}(\mathbb{R}^N)$ consisting of functions with compact support is denoted by $C_c^\infty(\mathbb{R}^N)$. Given a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^N)$ and a tempered distribution $\Upsilon \in \mathcal{S}'(\mathbb{R}^N)$, we denote their duality pairing by
\begin{equation}
\langle \Phi, \Upsilon \rangle_{\mathcal{S}(\mathbb{R}^N) - \mathcal{S}'(\mathbb{R}^N)} := \Upsilon(\Phi).
\end{equation}
For $1 \leq p \leq \infty$, we define $L^p(\mathbb{R}^N)$ to be the usual Banach space of equivalence classes of measurable functions $\Phi : \mathbb{R}^N \to \mathbb{C}$ with respect to the norm
\begin{equation}
\|\Phi\|_{L^p(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\Phi(x)|^p \, dx \right)^{1/p}
\end{equation}
with obvious modification when $p = \infty$. We denote the inner product on $L^2(\mathbb{R}^N)$ by
\begin{equation}
\langle \Phi, \Psi \rangle_{L^2(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \overline{\Phi(x)} \Psi(x) \, dx.
\end{equation}
Note that we use the physicist’s convention that the inner product is complex linear in the second entry.

For $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^N)$ to be the completion of the space $\mathcal{S}(\mathbb{R}^N)$ with respect to the norm
\begin{equation}
\|\Phi\|_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\mathcal{F}(\Phi)(\xi)|^2 \, d\xi \right)^{1/2},
\end{equation}
where $\mathcal{F}$ denotes the Fourier transform defined via the convention
\begin{equation}
\mathcal{F}(\Phi)(\xi) := \int_{\mathbb{R}^N} \Phi(x) e^{-i\xi \cdot x} \, dx.
\end{equation}
We can anti-isomorphically identify $H^{-s}(\mathbb{R}^N)$ with the dual of $(H^s(\mathbb{R}^N))^*$ by
\begin{equation}
\langle \Phi, \Upsilon \rangle_{H^s(\mathbb{R}^N) - H^{-s}(\mathbb{R}^N)} := \langle \langle \nabla \rangle^{-s} \Upsilon \rangle \langle \nabla \rangle^s \Phi \rangle_{L^2(\mathbb{R}^N)},
\end{equation}
where $\langle \cdot \rangle := (1 + |x|^2)^{1/2}$ is the Japanese bracket and $\langle \nabla \rangle$ is the Fourier multiplier with symbol $\langle \xi \rangle$. For $\gamma \in (0, 1)$, we denote the Hölder norm on $\mathbb{R}^N$ of exponent $\gamma$ by
\begin{equation}
\|\Phi\|_{C^\gamma(\mathbb{R}^N)} := \sup_{x, y \in \mathbb{R}^N \atop x \neq y} \frac{|\Phi(x) - \Phi(y)|}{|x - y|^\gamma}, \quad \|\Phi\|_{C^\gamma(\mathbb{R}^N)} := \|\Phi\|_{L^\infty(\mathbb{R}^N)} + \|\Phi\|_{C^\gamma(\mathbb{R}^N)}.
\end{equation}

Remark 2.1. In the sequel, we generally omit the underlying domain for norms (e.g. we write $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_{L^p(\mathbb{R}^N)}$), as the domain will be clear from context. Similarly, we omit the underlying domain for the inner product $\langle \cdot, \cdot \rangle$ and for the duality pairing $\langle \cdot, \cdot \rangle$.

2.2. Some trace estimates. In this subsection, we establish some basic estimates pertaining to the trace of a Sobolev function. We use these trace estimates for the rigorous construction of the LL Hamiltonian (recall expression (1.2) in Section 3 and in the proof of Proposition 1.2 in Section 4).

For a Schwartz function $\Phi \in \mathcal{S}(\mathbb{R}^N)$ and indices $1 \leq i < j \leq N$, we let $\Phi_{i=j}$ denote the restriction of $\Phi$ to the hyperplane $\{x \in \mathbb{R}^N : x_i = x_j\}$. We recall from elementary functional analysis that for any $s > 1/2$, there is a unique bounded linear map
\begin{equation}
\text{tr}_{i=j} : H^s(\mathbb{R}^N) \to H^{s-\frac{1}{2}}(\mathbb{R}^{N-1}), \quad \|\text{tr}_{i=j}\Phi\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})} \lesssim \|\Phi\|_{H^s(\mathbb{R}^N)}
\end{equation}
with the property that for any $\Phi \in \mathcal{S}(\mathbb{R}^N)$,
\begin{equation}
\text{tr}_{i=j}(\Phi) = \Phi_{i=j}.
\end{equation}

For the next lemma, we first recall the elementary distributional identity
\begin{equation}
\delta(x) = \frac{1}{2} \nabla \text{sgn}(x), \quad \forall x \in \mathbb{R}.
\end{equation}
Lemma 2.2 (\(H^1\) Trace estimate). Let \(N \in \mathbb{N}\). For any \(1 \leq i < j \leq N\),
\begin{equation}
\left| \langle \text{tr}_{i=j} \Phi, \text{tr}_{i=j} \Psi \rangle_{L^2(\mathbb{R}^{N-1})} \right| \leq \frac{1}{2} \left( \| \nabla_i \Phi \|_{L^2(\mathbb{R}^N)} \| \Psi \|_{L^2(\mathbb{R}^N)} + \| \Phi \|_{L^2(\mathbb{R}^N)} \| \nabla_i \Psi \|_{L^2(\mathbb{R}^N)} \right), \quad \forall \Phi, \Psi \in H^1(\mathbb{R}^N).
\end{equation}
Consequently, if \(\Phi \in H^1(\mathbb{R}^N)\), then we can define \(\delta(X_i - X_j) \Phi := \Phi \delta(X_i - X_j) \in H^{-1}(\mathbb{R}^N)\) by
\begin{equation}
\langle \Psi, \delta(X_i - X_j) \Phi \rangle_{H^{-1}(\mathbb{R}^N)} := \langle \text{tr}_{i=j} \Psi, \text{tr}_{i=j} \Phi \rangle_{L^2(\mathbb{R}^{N-1})-L^2(\mathbb{R}^{N-1})},
\end{equation}
and
\begin{equation}
\| \delta(X_i - X_j) \Phi \|_{H^{-1}(\mathbb{R}^N)} \leq \| \Phi \|_{H^1(\mathbb{R}^N)}.
\end{equation}

Proof. By considerations of symmetry, it suffices to consider \((i, j) = (1, 2)\). Let \(\Phi, \Psi \in S(\mathbb{R}^N)\). Then by definition of the product distribution \(\delta(X_1 - X_2) \Phi \in S'(\mathbb{R}^N)\), we have that
\begin{equation}
\langle \Psi, \delta(X_1 - X_2) \Phi \rangle_{S'-S} = \langle \Psi_{1=2}, \Phi_{1=2} \rangle_{L^2\rightarrow L^2}.
\end{equation}
Substituting the distributional identity \(\int S^0\) into the left-hand side of the preceding equality and applying the definition of the distributional derivative together with the product rule, we obtain that
\begin{equation}
\langle \Psi_{1=2}, \Phi_{1=2} \rangle_{L^2\rightarrow L^2} = -\frac{1}{2} \left( \langle \nabla_1 \Psi, \text{sgn}(X_1 - X_2) \Psi \rangle_{L^2\rightarrow L^2} + \langle \Psi, \text{sgn}(X_1 - X_2) \nabla_1 \Phi \rangle_{L^2\rightarrow L^2} \right).
\end{equation}
Taking absolute values of both sides, applying the triangle inequality, followed by Cauchy-Schwarz, we obtain that
\begin{equation}
\left| \langle \Psi_{1=2}, \Phi_{1=2} \rangle_{L^2\rightarrow L^2} \right| \leq \frac{1}{2} \left( \| \nabla_1 \Psi \|_{L^2} \| \Phi \|_{L^2} + \| \Psi \|_{L^2} \| \nabla_1 \Phi \|_{L^2} \right).
\end{equation}
The conclusion \((2.11)\) then follows from density of \(S(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)\) and the continuity of the map \(\text{tr}_{1=2} : H^1(\mathbb{R}^N) \rightarrow H^{1/2}(\mathbb{R}^{N-1})\).

Next, given \(\Phi \in H^1(\mathbb{R}^N)\), we define the linear functional \(\delta(X_1 - X_2) \Phi\) on \(H^1(\mathbb{R}^N)\) by extending the definition of the product distribution for \(\Phi \in S(\mathbb{R}^N)\). Then by Cauchy-Schwarz and the estimate \((2.11)\),
\begin{equation}
\sup_{\| \Psi \|_{H^1} = 1} \left| \langle \Psi, \delta(X_1 - X_2) \Phi \rangle_{H^1(\mathbb{R}^N)} \right| = \sup_{\| \Psi \|_{H^1} = 1} \left| \langle \text{tr}_{1=2} \Psi, \text{tr}_{1=2} \Phi \rangle_{L^2\rightarrow L^2} \right|
\leq \sup_{\| \Psi \|_{H^1} = 1} \frac{1}{2} \left( \| \nabla_1 \Psi \|_{L^2} \| \Phi \|_{L^2} + \| \Psi \|_{L^2} \| \nabla_1 \Phi \|_{L^2} \right)
\leq \| \Phi \|_{H^1},
\end{equation}
which by duality, implies the desired conclusion. \(\square\)

We also record here a partial Hölder continuity result for functions in \(H^1(\mathbb{R}^N)\) used in Section 4.

Lemma 2.3 (Partial Hölder continuity). Let \(N \in \mathbb{N}\). For any \(i \in \{1, \ldots, N\}\), we have the estimate
\begin{equation}
\| \Phi \|_{L^2(X_{i=1}^{N-1} \mathbb{R}^{N-1}; C_i^{1/2}(\mathbb{R}))} \leq \| \nabla_i \Phi \|_{L^2(\mathbb{R}^N)}, \quad \forall \Phi \in S(\mathbb{R}^N),
\end{equation}
Consequently, every element of \(H^1(\mathbb{R}^N)\) has a modification belonging to \(L^2(X_{i=1}^{N-1} \mathbb{R}^{N-1}; C_i^{1/2}(\mathbb{R}))\).

Proof. By considerations of symmetry, it suffices to consider \(i = 1\). Let \(\Phi \in S(\mathbb{R}^N)\), and fix \(x_{2,N} \in \mathbb{R}^{N-1}\). Define the function
\begin{equation}
\phi_{x_{2,N}} : \mathbb{R} \rightarrow \mathbb{C}, \quad \phi_{x_{2,N}}(x) := \Phi(x, x_{2,N}), \quad \forall x \in \mathbb{R}.
\end{equation}
Applying the fundamental theorem of calculus to \(\phi_{x_{2,N}}\) followed by Cauchy-Schwarz, we obtain that
\begin{equation}
|\phi_{x_{2,N}}(x) - \phi_{x_{2,N}}(y)| \leq |x - y|^{1/2} \| \nabla \phi_{x_{2,N}} \|_{L^2(\mathbb{R})}, \quad \forall x, y \in \mathbb{R},
\end{equation}
which implies that \(\| \phi_{x_{2,N}} \|_{C_i^{1/2}(\mathbb{R})} \leq \| \nabla \phi_{x_{2,N}} \|_{L^2(\mathbb{R})}\). Therefore, we see from the Fubini-Tonelli theorem that
\begin{equation}
\int_{\mathbb{R}^{N-1}} d_{x_{2,N}} \| \phi_{x_{2,N}} \|_{C_i^{1/2}(\mathbb{R})} \leq \int_{\mathbb{R}^{N-1}} d_{x_{2,N}} \| \nabla \phi_{x_{2,N}} \|_{L^2(\mathbb{R})} = \| \nabla_1 \Phi \|_{L^2(\mathbb{R}^N)}^2.
\end{equation}
The conclusion of the proof then follows from the density of \(S(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)\). \(\square\)
3. Construction of the Hamiltonian $H_N$

In this section, we give the rigorous construction of the Hamiltonian $H_N$, which we recall from (1.2) corresponds to the expression

$$
\sum_{i=1}^{N} -\Delta_i + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j), \quad \kappa \in \{\pm 1\}.
$$

(3.1)

The construction requires some care due to the presence of the $\delta$ pair potential. The main ingredients in the construction are the KLMN theorem, which we recall in Proposition 3.3 below, and the trace estimate of Lemma 2.2. Before proceeding to the construction, we need to introduce some terminology from the theory of unbounded operators on Hilbert spaces. Our presentation follows that of Reed and Simon [32, 31]. In what follows, $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ is a separable complex Hilbert space.

3.1. Quadratic forms. We begin with the definition of and basic facts about quadratic forms.

**Definition 3.1** (Quadratic form). A quadratic form is a sesquilinear map $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$, where $\mathcal{Q}(q)$ is a dense subset of $\mathcal{H}$ called the form domain. If $q(\varphi, \psi) = q(\psi, \varphi)$ for all $\varphi, \psi \in \mathcal{Q}(q)$, then we say that $q$ is symmetric. If $q(\varphi, \varphi) \geq 0$ for every $\varphi \in \mathcal{H}$, then we say that $q$ is positive, and if there exists a constant $M > 0$ such that $q(\varphi, \varphi) \geq -M\|\varphi\|_{\mathcal{H}}^2$, then we say that $q$ is semibounded.

**Definition 3.2** (Closed quadratic forms). Let $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$ be a semibounded quadratic form with constant $M > 0$ such that

$$
q(\psi, \psi) \geq -M\|\psi\|_{\mathcal{H}}^2, \quad \forall \psi \in \mathcal{Q}(q).
$$

We say that $q$ is closed if $\mathcal{Q}(q)$ is complete under the norm

$$
\|\psi\|_q := \sqrt{q(\psi, \psi) + (M + 1)\|\psi\|_{\mathcal{H}}^2}, \quad \forall \psi \in \mathcal{Q}(q).
$$

(3.3)

If $q$ is closed and $D \subset \mathcal{Q}(q)$ is dense in $\mathcal{Q}(q)$ with respect to the norm $\| \cdot \|_q$, then we call $D$ a form core for $q$.

Let $A$ be a self-adjoint operator on $\mathcal{H}$. We define a subset of $\mathcal{H}$ by

$$
\mathcal{Q}(A) := \{\psi \in \mathcal{H} : \|A^{1/2}\psi\|_{\mathcal{H}} < \infty\}.
$$

We can then define the quadratic form $q$ associated to $A$ by setting $\mathcal{Q}(q) := \mathcal{Q}(A)$ and

$$
q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}, \quad q(\varphi, \psi) := \left\langle |A|^{1/2}U^*\varphi, |A|^{1/2}\psi \right\rangle_{\mathcal{H}}, \quad \forall \varphi, \psi \in \mathcal{H},
$$

(3.5)

where $A = U|A|$ is the polar decomposition for $A$ (see [32, Theorem VIII.32]). In the sequel, we agree to write $\left\langle \varphi |A^{1/2}\psi \right\rangle_{\mathcal{H}}$ for the quadratic form associated to $A$, even though $\psi \in \mathcal{Q}(A)$ may not belong to $\text{Dom}(A)$. We hope this abuse of notation causes no confusion for the reader.

We now are prepared to state the KLMN theorem.

**Proposition 3.3** (KLMN theorem, [31, Theorem X.17]). Let $A$ be a positive self-adjoint operator on $\mathcal{H}$ with domain $D(A)$. Suppose that $\beta : \mathcal{Q}(A) \times \mathcal{Q}(A) \to \mathbb{C}$ is a symmetric quadratic form such that there exist constants $a < 1$ and $b \in \mathbb{R}$ so that

$$
|\beta(\psi, \psi)| \leq a \left\langle |A|^{1/2}\psi \right\rangle_{\mathcal{H}} + b \left\langle |A|^{1/2}\psi \right\rangle_{\mathcal{H}}, \quad \forall \psi \in D(A).
$$

Then there exists a unique self-adjoint operator $C$ on $\mathcal{H}$ with $\mathcal{Q}(C) = \mathcal{Q}(A)$ and

$$
\left\langle \varphi |C^{1/2}\psi \right\rangle_{\mathcal{H}} = \left\langle \varphi |A^{1/2}\psi \right\rangle_{\mathcal{H}} + \beta(\varphi, \psi), \quad \forall \varphi, \psi \in \mathcal{Q}(C).
$$

Moreover, $C$ is bounded below by $-b$, and any domain of essential self-adjointness for $A$ is a form core for $C$.

If the quadratic form $q$ is semibounded, then it is in fact symmetric.
3.2. Existence of $H_N$. We now use Proposition 3.3 and Lemma 2.2 to realize $H_N$ as a self-adjoint operator on $L^2_{sym}(\mathbb{R}^N)$. Let $\Delta_N := \sum_{i=1}^{N} \Delta_i$ denote the Laplacian on $\mathbb{R}^N$. It is easy to check that $-\Delta_N$ is a positive, self-adjoint operator on $L^2_{sym}(\mathbb{R}^N)$ and that $\mathcal{Q}(-\Delta_N) = H^1_{sym}(\mathbb{R}^N)$. We then have the following proposition.

**Proposition 3.4 (Existence of $H_N$).** Let $N \in \mathbb{N}$, and let $\kappa \in \{\pm 1\}$. Then there exists a unique self-adjoint operator $H_N$ on $L^2_{sym}(\mathbb{R}^N)$ with form domain $\mathcal{Q}(H_N) = H^1_{sym}(\mathbb{R}^N)$ and such that

\begin{equation}
\langle \Phi | H_N \Psi \rangle_{L^2(\mathbb{R}^N)} = \langle \Phi | -\Delta_N \Psi \rangle_{L^2(\mathbb{R}^N)} + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \langle \text{tr}_{i=j} \Phi | \text{tr}_{i=j} \Psi \rangle_{L^2(\mathbb{R}^N)},
\end{equation}

\forall \Phi, \Psi \in H^1_{sym}(\mathbb{R}^N).

Moreover, $H_N$ is bounded from below by 0, if $\kappa = 1$, and $-\frac{N-1}{2}$, if $\kappa = -1$, and any domain of essential self-adjointness for $\Delta_N$ is a form core for $H_N$.

**Proof.** We want to use Proposition 3.3. To this end, we let

\begin{equation}
A := -\Delta_N : H^2_{sym}(\mathbb{R}^N) \rightarrow L^2_{sym}(\mathbb{R}^N),
\end{equation}

and we define the quadratic form

\begin{equation}
\beta : \mathcal{Q}(A) \times \mathcal{Q}(A) \rightarrow \mathbb{C}, \quad \beta(\Phi, \Psi) := \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \langle \text{tr}_{i=j} \Phi | \text{tr}_{i=j} \Psi \rangle,
\end{equation}

which is evidently symmetric. Using the symmetry of $\Phi, \Psi$ under exchange of particle labels, we see that

\begin{equation}
\beta(\Phi, \Psi) = \frac{\kappa(N-1)}{2} \langle \text{tr}_{1=2} \Phi | \text{tr}_{1=2} \Psi \rangle.
\end{equation}

By Lemma 2.2 and Young’s inequality for products, we have that

\begin{equation}
|\langle \text{tr}_{1=2} \Phi | \text{tr}_{1=2} \Psi \rangle| \leq \|\nabla_1 \Phi\|_{L^2} \|\Phi\|_{L^2} \leq \frac{1}{2} (\|\nabla_1 \Phi\|_{L^2}^2 + \|\Phi\|_{L^2}^2).
\end{equation}

Since by another application of the symmetry of $\Phi$,

\begin{equation}
\langle \Phi | -\Delta_N \Phi \rangle = N \|\nabla_1 \Phi\|_{L^2}^2,
\end{equation}

we obtain that

\begin{equation}
|\beta(\Phi, \Phi)| \leq \frac{1}{2} \langle \Phi | -\Delta_N \Phi \rangle + \frac{(N-1)}{2} \langle \Phi | \Phi \rangle.
\end{equation}

The desired conclusion then follows from application of Proposition 3.3. \qed

**Remark 3.5.** An examination of the proof of the KLMN theorem in [31] shows that the domain of $H_N$ consists of all $\Phi \in H^1_{sym}(\mathbb{R}^N)$ such that the distribution

\begin{equation}
\left( -\Delta_N + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} \delta(X_i - X_j) \right) \Phi \in H^{-1}(\mathbb{R}^N)
\end{equation}

may be (uniquely) identified with an element $\Psi \in L^2_{sym}(\mathbb{R}^N)$, which we denote by $H_N \Phi$. With a little more work, one can show that $\text{Dom}(H_N)$ consists of all functions

\begin{equation}
\Phi \in H^1_{sym}(\mathbb{R}^N) \cap H^2_{sym}(\mathbb{R}^N \setminus \bigcup_{1 \leq i < j \leq N} \{x_N \in \mathbb{R}^N : x_i = x_j\})
\end{equation}

such that

\begin{equation}
\lim_{x_i - x_j \to 0^+} (\nabla_i - \nabla_j) \Phi - \lim_{x_i - x_j \to 0^-} (\nabla_i - \nabla_j) \Phi = \frac{\kappa}{2N} \text{tr}_{i=j} \Phi.
\end{equation}

Note for $1 \leq i < j \leq N$ and almost every $(x_{1,i-1}, x_{i+1,j-1}, x_{j+1,N}) \in \mathbb{R}^{N-2}$ fixed, $\nabla_i \Phi$ and $\nabla_j \Phi$ are continuous away from the hyperplane $\{x_N \in \mathbb{R}^N : x_i = x_j\}$ by Sobolev embedding.
3.3. Approximation of $H_N$. We close this section with some approximation results obtained from mollifying the $\delta$ pair potential in the expression (1.2) for $H_N$. These approximation results are used extensively in Section 4.

More precisely, let $\tilde{V} \in C_c^\infty(\mathbb{R})$ be an even function such that $0 \leq \tilde{V} \leq 1$, $\int_\mathbb{R} d\tilde{V}(x) = 1$, and

$$\tilde{V}(x) = \begin{cases} 
1, & |x| \leq \frac{1}{2} \\
0, & |x| > \frac{1}{2}.
\end{cases}$$

For $\varepsilon > 0$, set $V_\varepsilon(x) := \varepsilon^{-1}V(x/\varepsilon)$. It is straightforward to check that the operator

$$H_{N,\varepsilon} := -\Delta_N + \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} V_\varepsilon(X_i - X_j), \quad \kappa \in \{\pm 1\}$$

is self-adjoint on $H^2_{sym}(\mathbb{R}^N)$. So by Stone’s theorem, $H_{N,\varepsilon}$ generates a strongly continuous one-parameter unitary group $\{e^{itH_{N,\varepsilon}}\}_{t \in \mathbb{R}}$. We set $\Phi_{\varepsilon} := e^{-itH_{N,\varepsilon}}\Phi_{N,0}$, where $\Phi_{N,0}$ is the same initial datum as in the Cauchy problem (1.4), so that $\Phi_{\varepsilon}$ is the unique global solution in $C^0_t(\mathbb{R}; L^2_{\mathcal{N}}(\mathbb{R}^N))$ to the Schrödinger equation

$$\begin{cases} 
i^{it}\Phi_{\varepsilon} = H_{N,\varepsilon}\Phi_{\varepsilon}, \\
\Phi_{\varepsilon}(0) = \Phi_{N,0} \end{cases}.$$ 

Given that $V_\varepsilon \to \delta$ in distribution, as $\varepsilon \to 0$, we expect that $H_{N,\varepsilon} \to H_N$ in some sense. The sense in which this convergence holds is that of norm-resolvent convergence.

**Definition 3.6 (Norm-resolvent convergence).** Let $\{A_n\}_{n=1}^\infty$ be a sequence of self-adjoint operators on $\mathcal{H}$. Then we say that $A_n$ converges to $A$ in norm-resolvent sense if $R_\lambda(A_n) \to R_\lambda(A)$ in norm, for every $\lambda$ with $\text{Im} \lambda \neq 0$, where $R_\lambda$ denotes the resolvent.

**Lemma 3.7.** Fix $N \in \mathbb{N}$. We have that $H_{N,\varepsilon} \to H_N$ in norm-resolvent sense, as $\varepsilon \to 0^+$. Consequently, $e^{iH_{N,\varepsilon}} \to e^{iH_N}$ strongly, as $\varepsilon \to 0^+$, uniformly on compact intervals of time.

**Proof.** Fix $\kappa \in \{\pm 1\}$. The second assertion regarding convergence of unitary groups follows from [15, Chapter 9, Theorem 2.16], so we focus on the first assertion. To show that $H_{N,\varepsilon} \to H_N$ in norm-resolvent sense, it suffices by [31, Theorem VII.25] to show that

$$\lim_{\varepsilon \to 0^+} \|H_{N,\varepsilon} - H_N\|_{H^1 \to H^{-1}} = 0,$$

where $\| \cdot \|_{H^1 \to H^{-1}}$ denotes the operator norm for maps $H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$. To see that (3.21) holds, we observe that for any $\Phi \in H^1(\mathbb{R}^N)$,

$$\begin{align*}
(H_N - H_{N,\varepsilon})\Phi &= \frac{\kappa}{N} \sum_{1 \leq i < j \leq N} (\delta(X_i - X_j) - \varepsilon(X_i - X_j))\Phi \in H^{-1}(\mathbb{R}^N).
\end{align*}$$

Since $H^{-1}(\mathbb{R}^N)$ is isomorphic to $(H^1(\mathbb{R}^N))^*$ and by considerations of symmetry, it suffices to estimate

$$\begin{align*}
&\left| \langle \Psi, (\delta(X_1 - X_2) - \varepsilon(X_1 - X_2))\Phi \rangle \right|_{H^1 \to H^{-1}} \\
&= \left| \int_{\mathbb{R}^{2N}} d\mathbf{x}_{2,N}(\text{tr}_1 = 2) \Psi(\mathbf{x}_{2,N})\Phi(\mathbf{x}_{2,N}) - \int_{\mathbb{R}^{2N}} d\mathbf{x}_{2,N} V_\varepsilon(x_1 - x_2)\Psi(\mathbf{x}_{2,N})\Phi(\mathbf{x}_{2,N}) \right|,
\end{align*}$$

for every $\Psi \in H^1(\mathbb{R}^N)$ with $\|\Psi\|_{H^1} \leq 1$. By the density of $\mathcal{S}(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, we may assume without loss of generality that $\Phi, \Psi$ are Schwartz. By Fubini-Tonelli, we see that

$$\int_{\mathbb{R}^N} d\mathbf{x}_{2,N} V_\varepsilon(x_1 - x_2)\Psi(\mathbf{x}_{2,N})\Phi(\mathbf{x}_{2,N}) = \int_{\mathbb{R}^{N-1}} d\mathbf{x}_{2,N} \int_{\mathbb{R}} dx_1 V_\varepsilon(x_1 - x_2)\Psi(\mathbf{x}_{2,N})\Phi(\mathbf{x}_{2,N}),$$

and since $\int_{\mathbb{R}} dx_1 V_\varepsilon(x) = 1$, it follows from translation invariance of Lebesgue measure that

$$\begin{align*}
&\int_{\mathbb{R}^{N-1}} d\mathbf{x}_{2,N}(\text{tr}_1 = 2) \Psi(\mathbf{x}_{2,N})\Phi(\mathbf{x}_{2,N}) \\
&= \int_{\mathbb{R}^{N-1}} d\mathbf{x}_{2,N} \int_{\mathbb{R}} dx_1 V_\varepsilon(x_1 - x_2)\Psi(x_2, \mathbf{x}_{2,N})\Phi(x_2, \mathbf{x}_{2,N}).
\end{align*}$$
Using the algebra property of Hölder norms followed by the dilation invariance of Lebesgue measure, we see that
\[
\left| \int_{\mathbb{R}} dx_1 V_\varepsilon(x_1 - x_2) \left( \Psi(x_2, \tilde{x}_2, N) \Phi(x_2, \tilde{x}_2, N) - \Psi(x_1, \tilde{x}_2, N) \Phi(x_1, \tilde{x}_2, N) \right) \right| \lesssim \varepsilon^{1/2} \| \Psi(\cdot, \tilde{x}_2, N) \|_{C^{1/2}} \| \Phi(\cdot, \tilde{x}_2, N) \|_{C^{1/2}}.
\]
(3.25)

Integrating both sides of the preceding inequality over \( \mathbb{R}^{N-1} \) with respect to \( \tilde{x}_2, N \) then applying Cauchy-Schwarz, we obtain that
\[
\int_{\mathbb{R}^{N-1}} d\tilde{x}_2, N \left| \int_{\mathbb{R}} dx_1 V_\varepsilon(x_1 - x_2) \left( \Psi(x_2, \tilde{x}_2, N) \Phi(x_2, \tilde{x}_2, N) - \Psi(x_1, \tilde{x}_2, N) \Phi(x_1, \tilde{x}_2, N) \right) \right| \lesssim \varepsilon^{1/2} \| \Psi \|_{L^2_{\tilde{x}_2, N} C^{1/2}_x} \| \Phi \|_{L^2_{\tilde{x}_2, N} C^{1/2}_x}.
\]
(3.26)
where the ultimate inequality follows from Lemma 2.3 and the assumption that \( \| \Psi \|_{H^1} \leq 1 \). We therefore conclude that
\[
\left| \langle \Psi, \delta(X_1 - X_2) - V_\varepsilon(X_1 - X_2) \rangle \right|_{H^{1\rightarrow H^{-1}}} \lesssim \varepsilon^{1/2} \| \Phi \|_{H^1},
\]
(3.27)
which implies that \( \| \delta(X_1 - X_2) - V_\varepsilon(X_1 - X_2) \|_{H^{1\rightarrow H^{-1}}} \lesssim \varepsilon^{1/2} \). It then follows from symmetry that
\[
\limsup_{\varepsilon \rightarrow 0^+} \| H_{N, \varepsilon} - H_N \|_{H^{1\rightarrow H^{-1}}} \lesssim \limsup_{\varepsilon \rightarrow 0^+} N\varepsilon^{1/2} = 0,
\]
which completes the proof of the lemma.

We remark that one can also prove the desired norm-resolvent convergence by modifying the argument from [2 Subsubsection I.3.2].

4. Control of \( \beta_N \)

4.1. Projectors. As the goal of Section 4 is to prove Proposition 1.2, we first define the projectors underlying the definition of the functional \( \beta_N \) in the statement of the proposition. Recall that \( \phi \in C^0_0(\mathbb{R}; \mathcal{H}_2(\mathbb{R})) \) is the unique solution to the cubic NLS (1.11) with initial datum \( \phi_0 \in \mathcal{H}_2(\mathbb{R}) \). We define the projectors
\[
p(t) := |\phi(t)| \langle \phi(t) \rangle, \quad q(t) := 1 - p(t), \quad \forall t \in \mathbb{R},
\]
(4.1)
where \( 1 \) denotes the identity operator on \( L^2(\mathbb{R}) \). For \( N \in \mathbb{N} \) and \( j \in \{1, \ldots, N\} \), we define
\[
p_j := 1^{\otimes j} \otimes p \otimes 1^{\otimes N-j}, \quad q_j := 1_N - p_j = 1^{\otimes j} \otimes q \otimes 1^{\otimes N-j},
\]
(4.2)
where \( 1_N = 1^{\otimes N} \) denotes the identity operator on \( L^2(\mathbb{R}^N) \). Since \( 1 = p + q \), it follows that
\[
1_N = (p_1 + q_1) \cdots (p_N + q_N),
\]
(4.3)
and therefore
\[
1_N = \sum_{k=0}^N P_k, \quad P_k := \sum_{\|\alpha_N\| = k} \prod_{j=1}^N p_j^{1-\alpha_j} q_j^{\alpha_j}.
\]
We define \( P_k \) to be the zero operator on \( L^2(\mathbb{R}^N) \) for \( k \in \mathbb{Z} \setminus \{0, \ldots, N\} \). Important properties of the operators \( P_k \) are the following:

(i) \( P_k \) is an orthogonal projector on \( L^2(\mathbb{R}^N) \);
(ii) \( P_k(\mathcal{L}^2_{\text{sym}}(\mathbb{R}^N)) \subset \mathcal{L}^2_{\text{sym}}(\mathbb{R}^N) \);
(iii) \( P_k P_l = \delta_{kl} P_k \), where \( \delta_{kl} \) is the Kronecker delta function;
(iv) \( p_j, q_j \) commute with \( P_k \), for any \( j \in \{1, \ldots, N\} \) and \( k \in \mathbb{Z} \).

Remark 4.1. Since the function \( \phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) underlying the definition of the projectors \( p_j, q_j \) is time-dependent, the projector \( P_k \) is also time-dependent (i.e. \( P_k(t) \) is a projector on \( \mathcal{L}^2_{\text{sym}}(\mathbb{R}^N) \) for each \( t \in \mathbb{R} \)). For convenience, we do not emphasize the dependence on time with our notation in the sequel.
Remark 4.2. In the sequel, we frequently use without comment the elementary fact that \(p_j, q_j\) are self-adjoint and that we have the operator norm identities

\[
\|p_j\|_{L^2_\mathbb{N}(\mathbb{R}^N) \to L^2_\mathbb{N}(\mathbb{R}^N)} = \|q_j\|_{L^2_\mathbb{N}(\mathbb{R}^N) \to L^2_\mathbb{N}(\mathbb{R}^N)} = 1.
\]

Given a function \(f : \mathbb{Z} \to \mathbb{C}\), we define the operator

\[
\hat{f} := \sum_{k \in \mathbb{Z}} f(k)P_k = \sum_{k=0}^N f(k)P_k.
\]

The reader may check that for \(f, g : \mathbb{Z} \to \mathbb{C}\), we have that \(\hat{f}g = \hat{f}g\). Furthermore, since \(p_j, q_j, P_k\) commute, it follows that \(\hat{f}\) commutes with \(p_j, q_j, P_k\). Additionally, if \(f, g\) are such that \(f \geq g\). Then \(\hat{f} \geq \hat{g}\). Indeed, since \(P_k\) is an orthogonal projector,

\[
\langle \Phi| (\hat{f} - g)\Phi \rangle = \sum_{k=0}^N \langle P_k\Phi|(\hat{f} - g)P_k\Phi \rangle \geq 0, \quad \forall \Phi \in L^2(\mathbb{R}^N).
\]

If \(f \geq 0\), then we agree to abuse notation by writing

\[
f^{-1}(k) := \frac{1}{f(k)}1_{>0}(k) \quad \text{and} \quad \hat{f}^{-1} := \sum_{k \in \mathbb{Z}} f^{-1}(k)P_k
\]

with the convention that \(0 \cdot \infty = 0\).

Remark 4.3. Since each \(P_k\) is time-dependent, as commented in Remark 4.1, the operator \(\hat{f}\) is also time-dependent. Out of convenience, we do not emphasize the dependence on time with our notation in the sequel.

Definition 4.4. Given \(N \in \mathbb{N}\), we define the functions \(m_N, n_N : \mathbb{Z} \to [0, \infty)\) by

\[
m_N(k) := \frac{k}{N}1_{\geq 0}(k) \quad \text{and} \quad n_N(k) := \sqrt{\frac{k}{N}}1_{\geq 0}(k), \quad \forall k \in \mathbb{Z}.
\]

Letting \(\Phi_N\) denote the solution to the Schrödinger equation (1.4) and with the notation introduced in (4.6), we define the time-dependent quantities

\[
\alpha_N(t) := \langle \Phi_N(t)|\hat{m}_N(t)\Phi_N(t)\rangle_{L^2(\mathbb{R}^N)} \quad \text{and} \quad \beta_N(t) := \langle \Phi_N(t)|\hat{n}_N(t)\Phi_N(t)\rangle_{L^2(\mathbb{R}^N)}, \quad \forall t \in \mathbb{R}.
\]

Remark 4.5. Since \(\sum_{k=0}^N P_k = 1_N\), we have that

\[
\frac{1}{N} \sum_{j=1}^N q_j = \frac{1}{N} \sum_{k \in \mathbb{Z}} \sum_{j=1}^N q_j P_k.
\]

By unpacking the definition of \(P_k\) in (4.4), the reader can check that \(\sum_{j=1}^N q_j P_k = kP_k\), which implies that

\[
\frac{1}{N} \sum_{j=1}^N q_j = \sum_{k \in \mathbb{Z}} \frac{k}{N}P_k = \hat{m}_N.
\]

It then follows from the symmetry of the wave function \(\Phi_N\) under exchange of particle labels that

\[
\alpha_N(t) = \langle \Phi_N(t)|\hat{m}_N(t)\Phi_N(t)\rangle = \frac{1}{N} \sum_{i=1}^N \langle \Phi_N(t)|q_i(t)\Phi_N(t)\rangle = \langle \Phi_N(t)|\hat{q_1}(t)\Phi_N(t)\rangle, \quad \forall t \in \mathbb{R}.
\]

We now record two technical lemmas from \[17\] pertaining to the operator \(\hat{m}_N\), which we frequently use in Section 4.

Lemma 4.6 ([17] Lemma 3.9). For any function \(f : \mathbb{Z} \to [0, \infty)\), the following hold:

(i)

\[
\|\hat{f}^{1/2}q_1\Phi_N\|_{L^2_\mathbb{N}}^2 = \left\langle \Phi_N|\hat{f}q_1\Phi_N\right\rangle_{L^2_\mathbb{N}} = \left\langle \Phi_N|\hat{m}_N\Phi_N\right\rangle_{L^2_\mathbb{N}},
\]
(ii)

\begin{equation}
\| \hat{f} q_1 q_2 \Phi_N \|^2_{L^2_{\text{sym}}(N)} = \left\langle \Phi_N, \hat{f} q_1 q_2 \Phi_N \right\rangle_{L^2_{\text{sym}}(N)} \leq \frac{N}{N-1} \left\langle \Phi_N, \hat{f} m_N^{-1} \Phi_N \right\rangle_{L^2_{\text{sym}}(N)}.
\end{equation}

Given \( n \in \mathbb{N} \), we define the shift operator

\begin{equation}
\tau_n: \mathbb{C}^Z \rightarrow \mathbb{C}^Z, \quad (\tau_n f)(k) := f(k+n), \quad \forall k \in \mathbb{Z}, \ f \in \mathbb{C}^Z.
\end{equation}

**Lemma 4.7** ([17, Lemma 3.10]). Let \( r \in \mathbb{N} \), and let \( A^{(r)} \) be a linear operator on \( L^2_{\text{sym}}(\mathbb{R}^r) \). For \( i \in \{1,2\} \), let \( Q_i \) be a projector of the form

\begin{equation}
Q_i = \#_1 \cdots \#_r,
\end{equation}

where each \( \# \) stands for either \( p \) or \( q \). Define the linear operator \( A_{1 \ldots r}^{(r)} := A^{(r)} \otimes 1^{N-r} \). Then for any function \( f : \mathbb{R} \rightarrow \mathbb{C} \), we have that

\begin{equation}
Q_1 A_{1 \ldots r}^{(r)} \hat{f} Q_2 = Q_1 (\tau_n f) A_{1 \ldots r}^{(r)} Q_2,
\end{equation}

where \( n := n_2 - n_1 \) and \( n_i \) is the number of factors \( q \) in \( Q_i \), for \( i \in \{1,2\} \).

4.2. **Evolution of** \( \beta_{N,\varepsilon} \). In this subsection, we would like to control the evolution of the quantity \( \beta_N \) introduced in Definition 4.3, thereby proving Proposition 4.2. As commented in Section 3.3 of the introduction, rather than work directly with \( \beta_N \), we work with the approximation \( \beta_{N,\varepsilon} \) defined in (4.19) below, which is obtained by replacing the \( N \)-body Hamiltonian \( H_N \) constructed in Proposition 3.4 with the mollified Hamiltonian \( H_{N,\varepsilon} \) from Section 3.3. The motivation is to justify some computations involving questions of operator domains and to avoid awkward notation involving distributions.

Similarly to \( \alpha_N \) and \( \beta_N \), we define the time-dependent quantities \( \alpha_{N,\varepsilon} \) and \( \beta_{N,\varepsilon} \) by

\begin{equation}
\alpha_{N,\varepsilon}(t) := \left\langle \Phi_N^\varepsilon(t) | \hat{f}(t) \Phi_N^\varepsilon(t) \right\rangle \quad \text{and} \quad \beta_{N,\varepsilon}(t) := \left\langle \Phi_N^\varepsilon(t) | m_N(t) \Phi_N^\varepsilon(t) \right\rangle, \quad \forall t \in \mathbb{R},
\end{equation}

where \( \Phi_N^\varepsilon \) is the solution to the regularized Schrödinger equation (3.20). As a corollary of Lemma 3.7, we obtain that \( \alpha_{N,\varepsilon} \rightarrow \alpha_N \) and \( \beta_{N,\varepsilon} \rightarrow \beta_N \) uniformly on compact intervals on time. This result is a consequence of the following more general lemma.

**Lemma 4.8.** Let \( T > 0 \), and let \( f : \mathbb{R} \rightarrow \mathbb{C} \) be bounded. For \( N \in \mathbb{N} \) and \( \varepsilon > 0 \), define the functions \( \vartheta_{N,\varepsilon} : \mathbb{R} \rightarrow \mathbb{C} \) by

\begin{equation}
\vartheta_{N,\varepsilon}(t) := \left\langle \Phi_N^\varepsilon(t) | \hat{f}(t) \Phi_N^\varepsilon(t) \right\rangle \quad \text{and} \quad \vartheta_{N,\varepsilon}(t) := \left\langle \Phi_N^\varepsilon(t) | m_N(t) \Phi_N^\varepsilon(t) \right\rangle, \quad \forall t \in \mathbb{R}.
\end{equation}

Then for \( N \) fixed,

\begin{equation}
\lim_{\varepsilon \rightarrow 0^+} \sup_{|t| \leq T} | \vartheta_{N,\varepsilon}(t) - \vartheta(t) | = 0.
\end{equation}

**Proof.** First, observe from the definition (4.5) for \( \hat{f} \) that for any \( \Psi_N \in L^2(\mathbb{R}^N) \),

\begin{equation}
\| \hat{f} \Psi_N \|^2_{L^2_{\text{sym}}(N)} = \sum_{k=0}^{N} \| f(k) P_k \Psi_N \|^2_{L^2_{\text{sym}}(N)} \leq \| f \|_{L^2} \sum_{k=0}^{N} \| P_k \Psi_N \|^2_{L^2_{\text{sym}}(N)} = \| f \|_{L^2} \| \Psi_N \|^2_{L^2_{\text{sym}}(N)},
\end{equation}

which implies that \( \| \hat{f} \|^2_{L^2_{\text{sym}}(N)} \leq \| f \|_{L^2} \). Now by definition, \( \Phi_N = e^{-itH_N} \Phi_{N,0} \) and \( \Phi_N^\varepsilon = e^{-itH_{N,\varepsilon}} \Phi_{N,0} \), so that by the triangle inequality and Cauchy-Schwarz,

\begin{align}
| \vartheta_{N}(t) - \vartheta_{N,\varepsilon}(t) | &= \left| \left\langle e^{-itH_N} \Phi_{N,0}, \hat{f}(t) e^{-itH_{N,\varepsilon}} \Phi_{N,0} \right\rangle - \left\langle e^{-itH_{N,\varepsilon}} \Phi_{N,0}, \hat{f}(t) e^{-itH_N} \Phi_{N,0} \right\rangle \right|
\leq ( \| e^{-itH_N} \Phi_{N,0} \|_{L^2} + \| e^{-itH_{N,\varepsilon}} \Phi_{N,0} \|_{L^2} ) \| \hat{f}(t) (e^{-itH_N} - e^{-itH_{N,\varepsilon}}) \Phi_{N,0} \|_{L^2}
\leq 2 \| f \|_{L^2} \| (e^{-itH_N} - e^{-itH_{N,\varepsilon}}) \Phi_{N,0} \|_{L^2},
\end{align}

where the ultimate inequality follows from the operator norm bound for \( \hat{f} \), unitarity of \( e^{-itH_N} \) and \( e^{-itH_{N,\varepsilon}} \), and \( \| \Phi_{N,0} \|_{L^2} = 1 \). The desired conclusion is then immediate from Lemma 3.7.

The goal of this subsection is to prove the following proposition. The reader will recall that \( \phi \) is the solution to the cubic NLS (1.11).
Proposition 4.9. Let $\kappa \in \{ \pm 1 \}$. Then we have the estimate
\begin{equation}
\dot{\hat{\beta}}_{N,\varepsilon}(t) \lesssim \frac{\| \phi(t) \|^2_{L^\infty(\mathbb{R})}}{N} + \frac{1}{N^\sigma} + \frac{\| \phi(t) \|^2_{L^1(\mathbb{R})}}{N^{(1-\sigma)/2}} + \frac{\| \phi(t) \|^2_{L^\infty(\mathbb{R})}}{N^{\delta/2}} + N^{\frac{2(\sigma-1)+\delta}{2}} + \varepsilon^{1/2} \| \phi(t) \|^2_{C^{1/2}(\mathbb{R})},
\end{equation}
for every $t \in \mathbb{R}$, uniformly in $(\varepsilon, \sigma, \delta) \in (0,1)^3$ and $N \in \mathbb{N}$.

Proof. By time-reversal symmetry, it is enough to consider $t \geq 0$. Using that
\begin{equation}
\phi \in C^1_c(\mathbb{R}; H^2(\mathbb{R})) \cap C^1_c(\mathbb{R}; L^2(\mathbb{R})) \quad \text{and} \quad \Phi_{N,0} \in H^2(\mathbb{R}^N) = \text{Dom}(H_{N,\varepsilon})
\end{equation}

for the subspace of the Lieb-Liniger model, Subsection 3.3.2, pg. 113], we see that $\beta_{N,\varepsilon}$ is differentiable with respect to $t$ and its derivative $\dot{\beta}_{N,\varepsilon}$ is given by
\begin{equation}
\dot{\beta}_{N,\varepsilon} = \frac{i \kappa}{2} \left[ \Phi_{\varepsilon}^{\infty} \left[ \frac{1}{N} \sum_{1 \leq i < j \leq N} V_{\varepsilon,ij} - \sum_{i=1}^{N} V_{\varepsilon}^{\phi}(X_i) \right] \Phi_{\varepsilon}^{\infty} \right],
\end{equation}
where we have introduced the notation
\begin{equation}
V_{\varepsilon,ij} := V_{\varepsilon}(X_i - X_j) \quad \text{and} \quad V_{\varepsilon}^{\phi} := |\phi(X_i)|^2.
\end{equation}

Using the symmetry of $\Phi_{\varepsilon}^{\infty}$ and $\widehat{n}_{\varepsilon}$ with respect to exchange of particle labels and the decomposition $1_N = (p_1 + q_1)(p_2 + q_2)$, then examining which terms cancel, we see that
\begin{equation}
\dot{\beta}_{N,\varepsilon} = \frac{i \kappa}{2} \left[ (N-1)V_{\varepsilon,12} - NV_{1}^{\phi} - NV_{2}^{\phi} \right] \Phi_{\varepsilon}^{\infty}
\end{equation}

\begin{equation}
= \text{Term}_1 + \text{Term}_2 + \text{Term}_3,
\end{equation}
where
\begin{equation}
\text{Term}_1 := 2 \text{ Re} \left\{ \frac{i \kappa}{2} \left[ (N-1)V_{\varepsilon,12} - NV_{1}^{\phi} - NV_{2}^{\phi} \right] q_1 p_2 \Phi_{\varepsilon}^{\infty} \right\},
\end{equation}
\begin{equation}
\text{Term}_2 := 2 \text{ Re} \left\{ \frac{i \kappa}{2} \left[ (N-1)V_{\varepsilon,12} - NV_{1}^{\phi} - NV_{2}^{\phi} \right] q_1 q_2 \Phi_{\varepsilon}^{\infty} \right\},
\end{equation}
\begin{equation}
\text{Term}_3 := \text{ Re} \left\{ \frac{i \kappa}{2} \left[ (N-1)V_{\varepsilon,12} - NV_{1}^{\phi} - NV_{2}^{\phi} \right] q_1 q_2 \Phi_{\varepsilon}^{\infty} \right\}.
\end{equation}

We proceed to estimate Term$_1$, Term$_2$, and Term$_3$ individually. In the sequel, we drop the subscript $N$, as the number of particles is fixed. For convenience, we also introduce the notation
\begin{equation}
V_{\varepsilon}^{\phi}(x) := (V_{\varepsilon} \ast |\phi|^2)(x) \quad \text{and} \quad V_{\varepsilon,ij}^{\phi} := (V_{\varepsilon} \ast |\phi|^2)(X_j), \quad \forall j \in \{ 1, \ldots, N \}.
\end{equation}
Note that by Young's inequality and $\| V_{\varepsilon} \|_{L^1} = 1$, we have the operator norm estimate
\begin{equation}
\| V_{\varepsilon,ij}^{\phi} \|_{L^2_{\varepsilon} - L^2_{\varepsilon}} \leq \| \phi \|^2_{L^\infty}, \quad \forall \varepsilon > 0, \ j \in \{ 1, \ldots, N \}.
\end{equation}

**Estimate for Term$_1$:** We first observe that since $q_1$ commutes with $V_{\varepsilon,ij}^{\phi}$ and $p_1, q_1$ are orthogonal,
\begin{equation}
\left\langle \Phi_{\varepsilon}^{\infty} p_1 p_2 \left[ NV_{\varepsilon,12}^{\phi} \widehat{n} \right] q_1 p_2 \Phi_{\varepsilon}^{\infty} \right\rangle_{L^2_{\varepsilon}} = \left\langle \Phi_{\varepsilon}^{\infty} \underbrace{q_1 q_1 q_2 q_2}_{=0} \left[ NV_{\varepsilon,12}^{\phi} \widehat{n} \right] p_2 \Phi_{\varepsilon}^{\infty} \right\rangle_{L^2_{\varepsilon}} = 0.
\end{equation}

Since $p_2 V_{\varepsilon,12} p_2 = V_{\varepsilon,12}^{\phi} p_2$, it follows that
\begin{equation}
|\text{Term}_1| \lesssim \left| \left\langle \Phi_{\varepsilon}^{\infty} p_1 p_2 \left[ (N-1)V_{\varepsilon,12}^{\phi} \widehat{n} \right] q_1 p_2 \Phi_{\varepsilon}^{\infty} \right\rangle_{L^2_{\varepsilon}} \right| = \left| \left\langle \Phi_{\varepsilon}^{\infty} p_1 p_2 \left[ (N-1)V_{\varepsilon,12}^{\phi} \widehat{n} \right] (\widehat{n} - \langle 1 \rangle) q_1 p_2 \Phi_{\varepsilon}^{\infty} \right\rangle_{L^2_{\varepsilon}} \right|.
\end{equation}
where the ultimate equality follows from an application of Lemma 4.7. Define the function
\begin{equation}
\mu : \mathbb{Z} \to \mathbb{R}, \quad \mu(k) := \frac{N(n(k) - (\tau - 1)n(k))}{k - 1} \quad \forall k \in \mathbb{Z},
\end{equation}
and observe that
\begin{equation}
\mu(k) = \frac{\sqrt{N}}{\sqrt{k + 1} > 0} \quad 1 \geq 0(k) \leq n^{-1}(k), \quad \forall k \in \mathbb{Z}.
\end{equation}

So by the triangle inequality,
\begin{equation}
|\text{Term}_1| \lesssim \frac{1}{N} \left| \left\langle \Phi^\varepsilon \right| p_1 p_2 V_{e_1}^\varepsilon \tilde{q} q_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{2N}} + \left| \left\langle \Phi^\varepsilon \right| p_1 p_2 (V_{e_1}^\varepsilon - V_{e_1}^\varepsilon) \tilde{q} q_1 p_2 \Phi^\varepsilon \right\rangle_{L^2_{2N}} \right|
\end{equation}
where the ultimate inequality follows from Cauchy-Schwarz and
\begin{equation}
\left| \langle \Phi^\varepsilon \rangle \right| \lesssim \frac{1}{N} \left| \left| \left| \langle \Phi^\varepsilon \rangle \right| \right|_{L^2_{2N}} + \left| \left| \left| \langle \Phi^\varepsilon \rangle \right| \right|_{L^2_{2N}} \right|
\end{equation}
where the ultimate inequality follows from Cauchy-Schwarz and \( \langle \Phi^\varepsilon \rangle_{L^2_{2N}} = 1 \). By translation invariance of Lebesgue measure and \( \int_\mathbb{R} dy V_{e_1} y = 1 \), for any \( x \in \mathbb{R} \),
\begin{equation}
\left| \langle \Phi^\varepsilon \rangle \right| \lesssim \frac{1}{N} \left| \left| \left| \left| \langle \Phi^\varepsilon \rangle \right| \right|_{L^2_{2N}} + \left| \left| \left| \langle \Phi^\varepsilon \rangle \right| \right|_{L^2_{2N}} \right| \right|
\end{equation}
where the ultimate inequality follows from dilatation invariance of Lebesgue measure and the algebra property of \( C^1_{x/2} \). Hence,
\begin{equation}
\left| \langle \Phi^\varepsilon \rangle \right| \lesssim \frac{1}{N} \left| \left| \left| \left| \langle \Phi^\varepsilon \rangle \right| \right|_{L^2_{2N}} + \left| \left| \left| \langle \Phi^\varepsilon \rangle \right| \right|_{L^2_{2N}} \right| \right|
\end{equation}
where the ultimate inequality follows from the bound (4.37) for \( \mu \) and an application of Lemma 4.6(i) together with recalling that \( \tilde{n}^2 = \tilde{m} \). Thus, we conclude that
\begin{equation}
|\text{Term}_1| \lesssim \frac{\| \Phi^\varepsilon \|_{L^\infty_{2N}}}{N} + \varepsilon^{1/2} \| \Phi^\varepsilon \|_{C^1_{x/2}}^2.
\end{equation}

**Estimate for** **Term** 2: Arguing similarly as in (4.34), we see that
\begin{equation}
\left\langle \Phi^\varepsilon \right| q_1 p_2 \left[ V_{1}^\varepsilon \tilde{n}, \tilde{n} \right] q_1 q_2 \Phi^\varepsilon \right\rangle_{L^2_{2N}} = 0.
\end{equation}

Therefore,
\begin{equation}
2 |\text{Term}_2| = \left| \left\langle \Phi^\varepsilon \right| q_1 p_2 \left[ (N - 1) V_{e_1, 12} - N V_{2}^\varepsilon \tilde{n} \right] q_1 q_2 \Phi^\varepsilon \right\rangle_{L^2_{2N}}
\end{equation}
where to obtain the penultimate equality have used Lemma 4.7 and introduced the notation \( \mu \) from (4.36) and to obtain the ultimate equality we have used the triangle inequality.

We first consider Term 2. By Cauchy-Schwarz together with the estimate (4.33),
\begin{equation}
\text{Term}_{2,2} \lesssim \| q_1 \Phi^\varepsilon \|_{L^2_{2N}} \| p_2 V_{2}^\varepsilon \tilde{q} q_1 q_2 \Phi^\varepsilon \|_{L^2_{2N}} \lesssim \| q_1 \Phi^\varepsilon \|_{L^2_{2N}} \| \phi \|_{L^\infty_{x}}^{2} \| \tilde{q} q_1 q_2 \Phi^\varepsilon \|_{L^2_{2N}}.
\end{equation}
By Remark 4.5 and Lemma 4.6[iii] respectively, together with the $\mu$ bound (4.37), we have that
\[(4.46) \quad \|q_1\Phi\|_{L^2_{\mathbb{Z}^N}} \leq \sqrt{\alpha_\varepsilon} \leq \sqrt{\beta_\varepsilon} \quad \text{and} \quad \|\hat{\mu}q_2\Phi\|_{L^2_{\mathbb{Z}^N}} \leq \sqrt{\beta_\varepsilon}.
\]
Therefore,
\[(4.47) \quad \text{Term}_{2,2} \lesssim \|\phi\|_{L^2_{\mathbb{Z}^N}} \beta_\varepsilon.
\]

We now consider $\text{Term}_{2,1}$. It follows from the distributional identity (2.10) and the fact that $\delta * V_\varepsilon = V_\varepsilon$ that
\[(4.48) \quad V_\varepsilon = \frac{1}{2}(\nabla \text{sgn} * V_\varepsilon) = \frac{1}{2}\nabla (\text{sgn} * V_\varepsilon).
\]
We introduce the notation $X_{\varepsilon,12} := \frac{1}{2}(\text{sgn} * V_\varepsilon)(X_1 - X_2)$. By Young’s inequality, $\|V_\varepsilon\|_{L^1} = \|\text{sgn}\|_{L^\infty} = 1$, so that
\[(4.49) \quad \|X_{\varepsilon,12}\|_{L^2_{\mathbb{Z}^N}} \rightarrow L^2_{\mathbb{Z}^N} \leq \frac{1}{2}.
\]
Therefore, we find from integrating by parts and applying the product rule and triangle inequality that
\[(4.50) \quad \text{Term}_{2,1} \leq \left| \langle \nabla_1 q_1 p_2 \Phi | X_{\varepsilon,12} \hat{\mu}q_1 q_2 \Phi \rangle \right|_{L^2_{\mathbb{Z}^N}} + \left| \langle \Phi | q_1 p_2 X_{\varepsilon,12} \nabla_1 \hat{\mu}q_1 q_2 \Phi \rangle \right|_{L^2_{\mathbb{Z}^N}} =: \text{Term}_{2,1,1} + \text{Term}_{2,1,2}.
\]
By Cauchy-Schwarz and the estimate (4.49),
\[(4.51) \quad \text{Term}_{2,1,1} \leq \|\nabla_1 q_1 p_2 \Phi\|_{L^2_{\mathbb{Z}^N}} \|\hat{\mu}q_1 q_2 \Phi\|_{L^2_{\mathbb{Z}^N}},
\]
so by application of the second estimate of (4.46) and $\|p_2\|_{L^2_{\mathbb{Z}^N}} \rightarrow L^2_{\mathbb{Z}^N} = 1$,
\[(4.52) \quad \text{Term}_{2,1,1} \lesssim \|\nabla_1 q_1 \Phi\|_{L^2_{\mathbb{Z}^N}} \sqrt{\beta_\varepsilon}.
\]

Next, we write $1 = p_1 + q_1$ and use the triangle inequality to obtain
\[(4.53) \quad \text{Term}_{2,1,2} \leq \left| \langle p_2 q_1 \Phi | X_{\varepsilon,12} p_1 \nabla_1 \hat{\mu} q_1 q_2 \Phi \rangle \right|_{L^2_{\mathbb{Z}^N}} + \left| \langle p_2 q_1 \Phi | X_{\varepsilon,12} q_1 \nabla_1 \hat{\mu} q_1 q_2 \Phi \rangle \right|_{L^2_{\mathbb{Z}^N}}.
\]

By Lemma 4.7 we have the operator identity
\[(4.54) \quad p_1 \nabla_1 \hat{\mu} q_1 = p_1 (\tau_1 \mu) \nabla_1 q_1.
\]

Hence,
\[(4.55) \quad \left| \langle p_2 q_1 \Phi | X_{\varepsilon,12} p_1 \nabla_1 \hat{\mu} q_1 q_2 \Phi \rangle \right|_{L^2_{\mathbb{Z}^N}} \leq \|X_{\varepsilon,12} p_2 q_1 \Phi\|_{L^2_{\mathbb{Z}^N}} \|p_1 (\tau_1 \mu) \nabla_1 q_1 q_2 \Phi\|_{L^2_{\mathbb{Z}^N}}
\]
\[\leq \|q_1 \Phi\|_{L^2_{\mathbb{Z}^N}} \|p_1 (\tau_1 \mu) \nabla_1 q_1 q_2 \Phi\|_{L^2_{\mathbb{Z}^N}}.
\]

By Remark 4.5, $\|q_1 \Phi\|_{L^2_{\mathbb{Z}^N}} \leq \sqrt{\beta_\varepsilon}$. Now using the $\mu$ bound (4.37), we have that
\[(4.56) \quad \langle \tau_1 \mu \rangle(k) \lesssim n^{-1}(k + 1) \lesssim n^{-1}(k), \quad \forall k \in \mathbb{Z}.
\]

Combining this estimate with the symmetry of $\Phi$ under permutation of particle labels, we find that
\[(4.57) \quad \|p_1 (\tau_1 \mu) \nabla_1 q_1 q_2 \Phi\|_{L^2_{\mathbb{Z}^N}} \lesssim \sqrt{\langle \nabla_1 q_1 \Phi | \nabla q_1 q_2 \Phi \rangle \rangle}_{L^2_{\mathbb{Z}^N}}
\]
\[= \sqrt{\frac{1}{N - 1} \sum_{i=2}^{N} \langle \nabla_1 q_1 \Phi | q_i \hat{n}^{-2} \nabla_1 q_1 \Phi \rangle} \rangle_{L^2_{\mathbb{Z}^N}}.
\]

Since the projector $q_1$ commutes with $\hat{n}^{-2}$ and $\hat{n}^{-2} \geq 0$, we have that
\[(4.58) \quad \langle \nabla_1 q_1 \Phi | q_i \hat{n}^{-2} \nabla_1 q_1 \Phi \rangle \rangle_{L^2_{\mathbb{Z}^N}} = \langle q_1 \nabla_1 q_1 \Phi | \hat{n}^{-2} q_i \nabla_1 q_1 \Phi \rangle \rangle_{L^2_{\mathbb{Z}^N}} \geq 0,
\]
so that by Remark 4.5 and the identity \( n^2 = m \),
\[
\sqrt{\frac{1}{N-1} \sum_{i=2}^{N} \langle \nabla q_1 \Phi^e \mid q_i \tilde{n}^{-2} \nabla q_1 \Phi^e \rangle_{L^2_{n}}} \lesssim \sqrt{\frac{1}{N} \sum_{i=1}^{N} \langle \nabla q_1 \Phi^e \mid q_i \tilde{n}^{-2} \nabla q_1 \Phi^e \rangle_{L^2_{n}}} = \sqrt{\langle \nabla q_1 \Phi^e \mid \tilde{n}^{-2} \nabla q_1 \Phi^e \rangle_{L^2_{n}}} = \| \nabla q_1 \Phi^e \|_{L^2_{n}}.
\]
(4.59)

After a little bookkeeping, we find that
\[
\left| \langle p_2 q_1 \Phi^e \mid X_{\epsilon,12} p_1 \nabla_1 \mu q_2 \Phi^e \rangle_{L^2_{n}} \right| \lesssim \sqrt{\beta \| \nabla q_1 \Phi^e \|_{L^2_{n}}}.
\]
(4.60)

Again by Lemma 4.7 we have the operator identity
\[
q_1 \nabla_1 \mu q_1 = q_1 \mu \nabla_1 q_1,
\]
and proceeding similarly as immediately above, we find that
\[
\left| \langle p_2 q_1 \Phi^e \mid X_{\epsilon,12} q_1 \nabla_1 \mu q_2 \Phi^e \rangle_{L^2_{n}} \right| \lesssim \sqrt{\beta \| \nabla q_1 \Phi^e \|_{L^2_{n}}},
\]
(4.62)

and therefore
\[
\text{Term}_{2,1,2} \lesssim \| \nabla q_1 \Phi^e \|_{L^2_{n}} \sqrt{\beta \epsilon}.
\]
Together the estimate (4.52) for Term_{2,1,1}, we obtain that
\[
\text{Term}_{2,1} \lesssim \| \nabla q_1 \Phi^e \|_{L^2_{n}} \sqrt{\beta \epsilon}.
\]
(4.64)

Collecting the estimates (4.61) for Term_{2,1} and (4.47) for Term_{2,2}, we conclude that
\[
\text{Term}_2 \lesssim \| \phi \|_{L^2_{\beta \epsilon}}^2 + \| \nabla q_1 \Phi^e \|_{L^2_{n}} \sqrt{\beta \epsilon}.
\]
(4.65)

**Estimate for Term_3:** We now consider Term_3, which is the most difficult portion of the analysis. We first note that by arguing similarly as in (4.31), we see that
\[
p_1 p_2 \left[ V^\phi_1, \tilde{n} \right] q_1 q_2 = 0 = p_1 p_2 \left[ V^\phi_2, \tilde{n} \right] q_1 q_2,
\]
where the reader will recall the notation \( V^\phi_j \) introduced in (4.27). Therefore,
\[
|\text{Term}_3| \lesssim \left| \langle \Phi^e \mid p_1 p_2 ((N-1)V_{\epsilon,12}, \tilde{n}) q_1 q_2 \Phi^e \rangle_{L^2_{n}} \right|
\]
(4.67)

where the ultimate equality follows from unpacking the commutator and applying Lemma 4.7 Analogously to the function \( \mu \) defined in (4.36), we define the function
\[
\nu : \mathbb{Z} \to \mathbb{R}, \quad \nu(k) := N(n(k)) - (\tau - n)(k), \quad \forall k \in \mathbb{Z}.
\]
(4.68)

It is a straightforward computation from the definition of \( n \) in Definition 4.1 that
\[
\nu(k) = \frac{2\sqrt{N}}{\sqrt{k + 1} \geq 2(k) \sqrt{k - 2} \geq 0(k)}, \quad \forall k \in \mathbb{Z},
\]
(4.69)

which implies that
\[
\nu(k) \lesssim n^{-1}(k), \quad \forall k \in \mathbb{Z}.
\]
(4.70)

We now introduce an approximation of the pair potential \( V_\epsilon \) as follows. Define \( V_\sigma(x) := N^\sigma \tilde{V}(N^\sigma x) \), where \( \sigma \in (0,1) \) is a parameter to be specified momentarily and \( \tilde{V} \) is as in Section 3.3. We convolve \( V_\epsilon \) with \( V_\sigma \) to define
\[
V_{\epsilon,\sigma} := V_\epsilon * V_\sigma \quad \text{and} \quad V_{\epsilon,\sigma,ij} := V_{\epsilon,\sigma}(X_i - X_j), \quad \forall 1 \leq i < j \leq N.
\]
(4.71)
By the triangle inequality,
\begin{equation}
\langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,12} \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\mathbb{Z}^N}} \leq \left| \langle \Phi^\varepsilon | p_1 p_2 (V_{\varepsilon,12} - V_{\varepsilon,\sigma,12}) \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\mathbb{Z}^N}} \right| + \left| \langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,\sigma,12} \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\mathbb{Z}^N}} \right| =: \text{Term}_{3,1} + \text{Term}_{3,2}.
\end{equation}

Observe that by moving $p_1 p_2$ over to the first entry of the inner product, writing out the convolution implicit in $V_{\varepsilon,\sigma,12}$, and using the Fubini-Tonelli theorem, we have that
\begin{equation}
\langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,\sigma,12} \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\mathbb{Z}^N}} = \int_{\mathbb{R}} dy \nu_q \int_{\mathbb{R}^2} dx_{1,2} V_\varepsilon (x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} dx_{3,N} (\langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle) (x_1, x_2, x_{3,N})
\end{equation}

\begin{equation}
= \int_{\mathbb{R}} dy \nu_q \int_{\mathbb{R}^2} dx_{1,2} V_\varepsilon (x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} dx_{3,N} \left( \left( \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle \right) (x_1, x_2, x_{3,N}) - \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle (x_1, x_2 + y, x_{3,N}) \right)
\end{equation}
\begin{equation}
+ \int_{\mathbb{R}} dy \nu_q \int_{\mathbb{R}^2} dx_{1,2} V_\varepsilon (x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} dx_{3,N} \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle (x_1, x_2 + y, x_{3,N}).
\end{equation}

By translation invariance of Lebesgue measure applied in the $x_2$-coordinate, we have that for any $y \in \mathbb{R}$,
\begin{equation}
\int_{\mathbb{R}^2} dx_{1,2} V_\varepsilon (x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} dx_{3,N} \left( \left( \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle \right) (x_1, x_2 + y, x_{3,N}) \right)
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^2} dx_{1,2} V_\varepsilon (x_1 - x_2) \int_{\mathbb{R}^{N-2}} dx_{3,N} \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle (x_1, x_2, x_{3,N})
\end{equation}
\begin{equation}
= \langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,12} \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\mathbb{Z}^N}},
\end{equation}

where the ultimate equality follows from using the Fubini-Tonelli theorem and the self-adjointness of $p_1 p_2$. Since $\int_{\mathbb{R}} dy = 1$, we conclude that
\begin{equation}
\int_{\mathbb{R}} dy \nu_q \int_{\mathbb{R}^2} dx_{1,2} V_\varepsilon (x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} dx_{3,N} \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle (x_1, x_2 + y, x_{3,N})
\end{equation}
\begin{equation}
= \langle \Phi^\varepsilon | p_1 p_2 V_{\varepsilon,12} \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle_{L^2_{\mathbb{Z}^N}}.
\end{equation}

Next, we have by definition of the Hölder norm in the $x_2$-coordinate that
\begin{equation}
\sup_{x_2 \in \mathbb{R}} \left| \left( \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle \right) (x_1, x_2, x_{3,N}) - \left( \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle \right) (x_1, x_2 + y, x_{3,N}) \right|
\end{equation}
\begin{equation}
\leq \| \left( \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle \right) (x_1, \cdot, x_{3,N}) \|_{C^{1/2}_{x_2}} |y|^{1/2}
\end{equation}
\begin{equation}
\leq \| \langle p_1 p_2 \Phi^\varepsilon | \tilde{\nu} q_1 q_2 \Phi^\varepsilon \rangle (x_1, \cdot, x_{3,N}) \|_{C^{1/2}_{x_2}} \| \tilde{\nu} q_1 q_2 \Phi^\varepsilon \|_{C^{1/2}_{x_2}} |y|^{1/2},
\end{equation}

for every $y \in \mathbb{R}$ and almost every $(x_1, x_{3,N}) \in \mathbb{R}^{N-1}$, where the ultimate inequality follows from the fact $C^{1/2}$ is an algebra. So by the Fubini-Tonelli theorem, followed by using the translation and dilation
invariance of Lebesgue measure and then Cauchy-Schwarz, we find that

\[
\int_{\mathbb{R}} dy \|V_\sigma(y)\int_{\mathbb{R}^2} d\varepsilon_{1,2} V_\varepsilon(x_1 - x_2 - y) \int_{\mathbb{R}^{N-2}} d\varepsilon_{3, N} \left| \left( (p_1 p_2 \Phi^\varepsilon)(\widehat{\theta q_2 \Phi^\varepsilon})(x_1, x_2, \varepsilon_{3, N}) - (p_1 p_2 \Phi^\varepsilon)(\widehat{\theta q_2 \Phi^\varepsilon})(x_1, x_2 + y, \varepsilon_{3, N}) \right) \right|
\]

\[
\leq \int_{\mathbb{R}^{N-1}} dx_1 d\varepsilon_{3, N} \left( \| (p_1 p_2 \Phi^\varepsilon)(x_1, \cdot, \varepsilon_{3, N}) \|_{C^1_{x_2}} \| (\widehat{\theta q_2 \Phi^\varepsilon})(x_1, \cdot, \varepsilon_{3, N}) \|_{C^1_{x_2}} \right.
\]

\[
\times \left( \int_{\mathbb{R}} dy |y|^{1/2} V_\sigma(y) \int_{\mathbb{R}} dx_2 V_\varepsilon(x_1 - x_2 - y) \right)
\]

\[
\leq N^{-\sigma/2} \| p_1 p_2 \Phi^\varepsilon \|_{L^2_{x_2, N} C^1_{x_1}} \| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N} C^1_{x_1}}.
\]

where in the ultimate inequality we use the symmetry of \( \Phi_\varepsilon \) to swap \( x_1 \) and \( x_2 \) in order to ease the burden of notation. By Fubini-Tonelli, Cauchy-Schwarz, and the normalization \( \| \phi \|_{L^2_\varepsilon} = 1 \), we have the estimate

\[
\| p_1 p_2 \Phi^\varepsilon \|_{L^2_{x_2, N} C^1_{x_1}} \leq \| \phi \|_{C^1_{x_2}} \| p_2 \Phi^\varepsilon \|_{L^2_{x_2, N}} \leq \| \phi \|_{C^1_{x_2}} \|
\]

where the ultimate inequality follows from the normalization \( \| \Phi^\varepsilon \|_{L^2_{x_2, N}} = 1 \). By Lemma 2.3 and the \( H^{1/2+} \subset L^\infty \) Sobolev embedding,

\[
\| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N} C^1_{x_1}} \leq \| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N} H^1_{x_1}} \leq \| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} + \| \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}},
\]

where the ultimate inequality follows from splitting the \( H^1_{x_1} \) norm and Fubini-Tonelli. Using the \( \nu \) estimate (4.70), Lemma 4.6 and the identity \( \tilde{m} = \tilde{n} \), we see that

\[
\| \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} \leq \sqrt{\langle \Theta^\varepsilon \tilde{m}^2 \Theta^\varepsilon \rangle_{L^2_{x_2, N}}} = \sqrt{\langle \Theta^\varepsilon \tilde{m}^2 \Theta^\varepsilon \rangle_{L^2_{x_2, N}}} = \sqrt{\alpha} \leq \sqrt{\beta^\varepsilon}.
\]

Next, inserting the decomposition \( \nabla_1 = p_1 \nabla_1 + q_1 \nabla_1 \) and applying the triangle inequality,

\[
\| \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} \leq \| p_1 \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} + \| q_1 \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}}.
\]

Since \( p_1 \nabla_1 = -\langle \phi \rangle \cdot \nabla \phi \),

\[
\| p_1 \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} \leq \| \nabla \phi \|_{L^2_{x_2, N}} \| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} \leq \| \nabla \phi \|_{L^2_{x_2, N}} \sqrt{\beta^\varepsilon},
\]

where the ultimate inequality follows from the estimate (4.80). By Lemma 4.7 followed by using the \( \nu \) estimate (4.70),

\[
\| q_1 \nabla_1 \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N}} = \| q_1 \nabla_1 \widehat{\theta q_1 \Phi^\varepsilon} \|_{L^2_{x_2, N}} \leq \sqrt{\langle \nabla_1 \phi^\varepsilon q_1^2 \nabla_1 \phi^\varepsilon \rangle_{L^2_{x_2, N}}} + \| \nabla_1 \phi^\varepsilon \|_{L^2_{x_2, N}},
\]

and arguing as for the estimate (4.59), we find that the right-hand side is \( \leq \| \nabla_1 q_1 \Phi^\varepsilon \|_{L^2_{x_2, N}} \). Therefore,

\[
\| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N} C^1_{x_1}} \leq (\| \nabla \phi \|_{L^2_{x_2, N}} \sqrt{\beta^\varepsilon} + \| \nabla_1 q_1 \Phi^\varepsilon \|_{L^2_{x_2, N}} \leq \| \phi \|_{H^1_{x_1}} \sqrt{\beta^\varepsilon} + \| \nabla_1 q_1 \Phi^\varepsilon \|_{L^2_{x_2, N}}.
\]

Collecting the estimates (4.78) and (4.84), we see that

\[
N^{-\sigma/2} \| p_1 p_2 \Phi^\varepsilon \|_{L^2_{x_2, N} C^1_{x_1}} \| \widehat{\theta q_2 \Phi^\varepsilon} \|_{L^2_{x_2, N} C^1_{x_1}} \leq N^{-\sigma/2} \| \phi \|_{C^1_{x_2}} \| \phi \|_{H^1_{x_1}} \sqrt{\beta^\varepsilon} + \| \nabla_1 q_1 \Phi^\varepsilon \|_{L^2_{x_2, N}}
\]

\[
\leq N^{-\sigma} + \| \phi \|_{C^1_{x_2}} \| \phi \|_{H^1_{x_1}} \sqrt{\beta^\varepsilon} + \| \nabla_1 q_1 \Phi^\varepsilon \|_{L^2_{x_2, N}}^2,
\]

where the ultimate line follows from Young’s inequality for products.

After a little bookkeeping, we conclude that

\[
| \text{Term}_3.1 | \leq N^{-\sigma} + \| \phi \|_{C^1_{x_2}}^2 \| \phi \|_{H^1_{x_1}}^2 \sqrt{\beta^\varepsilon} + \| \nabla_1 q_1 \Phi^\varepsilon \|_{L^2_{x_2, N}}^2,
\]

leaving us with \( \text{Term}_3.2 \).
For $\text{Term}_{3,2}$, we borrow an idea from [17] and introduce a partition of unity as follows. Let $\chi^{(1)}, \chi^{(2)} : \mathbb{Z} \to [0, \infty)$ be the two functions respectively defined by
\begin{equation}
\chi^{(1)}(k) := 1 - \chi^{(1)}(k) = 1_{\geq N^{1-\delta}}(k), \quad \forall k \in \mathbb{Z}.
\end{equation}
where $\delta \in (0, 1)$ will be optimized at the end. Trivially, we have that $\chi^{(j)} \in \{0, 1\}^\mathbb{Z}$, so that $(\chi^{(j)}(k))^2 = \chi^{(j)}(k)$, and $\chi^{(1)}(k) + \chi^{(2)}(k) = 1$. We insert this decomposition into the expression for $\text{Term}_{3,2}$ and use the triangle inequality to obtain
\begin{equation}
|\text{Term}_{3,2}| \leq \left| \left\langle \Phi^{\varepsilon} \left| p_1 p_2 V_{\varepsilon, \sigma, 12} \widehat{\chi}^{(1)}(k) q_2 \Phi^{\varepsilon} \right| \right\rangle_{L_2^{\mathbb{Z}}} \right| + \left| \left\langle \Phi^{\varepsilon} \left| p_1 p_2 V_{\varepsilon, \sigma, 12} \widehat{\chi}^{(2)}(k) q_2 \Phi^{\varepsilon} \right| \right\rangle_{L_2^{\mathbb{Z}}} \right|.
\end{equation}
We consider $\text{Term}_{3,2,1}$ and $\text{Term}_{3,2,2}$ separately.

For $\text{Term}_{3,2,1}$, we want to use the fact that the operator norm of $p_1 p_2 V_{\varepsilon, \sigma, 12} q_2$ is much smaller on the bosonic subspace $L_2^{\text{sym}}(\mathbb{R}^N)$ than on the full space $L^2(\mathbb{R}^N)$. Accordingly, we symmetrize the expression $p_2 V_{\varepsilon, \sigma, 12} q_2$ to write
\begin{equation}
\text{Term}_{3,2,1} = \frac{1}{N - 1} \left| \left\langle \Phi^{\varepsilon} \left| p_1 p_i V_{\varepsilon, \sigma, 12} q_i \chi^{(1)}(k) \widehat{q} q_1 \Phi^{\varepsilon} \right| \right\rangle_{L_2^{\mathbb{Z}}} \right|
\end{equation}

where the ultimate line follows from Cauchy-Schwarz. We claim that $\|\widehat{q} q_1 \Phi^{\varepsilon}\|_{L_2^{\mathbb{Z}}} \lesssim 1$. Indeed, by the $\nu$ bound (2.40) and Lemma 4.17
\begin{equation}
|\widehat{q} q_1 \Phi^{\varepsilon}|_{L_2^{\mathbb{Z}}} = \sqrt{\left\langle \Phi^{\varepsilon} | \widehat{q}^2 q_1 \Phi^{\varepsilon} \right\rangle_{L_2^{\mathbb{Z}}}} \lesssim \sqrt{\left\langle \Phi^{\varepsilon} | \widehat{n}^2 \Phi^{\varepsilon} \right\rangle_{L_2^{\mathbb{Z}}} } = 1,
\end{equation}

since $\widehat{n}^2 = \widehat{m}$ and $\|\Phi^{\varepsilon}\|_{L_2^{\mathbb{Z}}} = 1$. Now expanding the $L_2^{\mathbb{Z}}$ norm and using that $\chi^{(1)}(k)^2 = \chi^{(1)}(k)$, we see that
\begin{equation}
\left\| \sum_{i=2}^N \chi^{(1)}(k) q_i q_1 V_{\varepsilon, \sigma, 12} p_1 p_i \Phi^{\varepsilon} \right\|_{L_2^{\mathbb{Z}}} = \sqrt{\sum_{i, j=2}^N \left\langle \Phi^{\varepsilon} \left| p_1 p_i V_{\varepsilon, \sigma, 12} q_i q_1 \chi^{(1)}(k) q_j V_{\varepsilon, \sigma, 12} p_j p_i \Phi^{\varepsilon} \right| \right\rangle_{L_2^{\mathbb{Z}}}}
\end{equation}

where the ultimate inequality follows from the embedding $\ell^{1/2} \subset \ell^1$. Therefore,
\begin{equation}
\text{Term}_{3,2,1} \lesssim \frac{1}{N - 1} \left( \sqrt{B} + \sqrt{A} \right).
\end{equation}

We first consider $B$, which is the easy term. Since $\|q_1 q_i \chi^{(1)}(k) q_1 q_i \|_{L_2^{\mathbb{Z}}} \rightarrow L_2^{\mathbb{Z}} \leq 1$,
\begin{equation}
B \leq \sum_{i=2}^N \left\| V_{\varepsilon, \sigma, 12} p_i \Phi^{\varepsilon} \right\|_{L_2^{\mathbb{Z}}}^2 = \sum_{i=2}^N \left\langle \Phi^{\varepsilon} \left| p_1 p_i V_{\varepsilon, \sigma, 12} q_i q_1 V_{\varepsilon, \sigma, 12} p_i p_1 \Phi^{\varepsilon} \right| \right\rangle_{L_2^{\mathbb{Z}}}.
\end{equation}
Then applying Lemma 4.7 in order to obtain

\begin{align}
\|\phi\|^2(V_{\epsilon,\sigma}^2 * |\phi|^2)_{L_2} & \leq \|\phi\|^2(V_{\epsilon,\sigma}^2)_{L_2} \leq \|\phi\|^4_{L_2^4}.
\end{align}

It then follows from \(\|\Phi^\epsilon\|_{L_2^p} = 1\) that

\begin{align}
B & \leq (N - 1)N^\sigma \|\phi\|^4_{L_2^4}.
\end{align}

We proceed to consider \(A\). We first make a further decomposition of \(A\) by using that \((\chi(1))^2 = \chi(1))\) and then applying Lemma 4.7 in order to obtain

\begin{align}
A = & \sum_{2 \leq i \neq j \leq N} \langle \Phi^\epsilon | p_1 p_i V_{\epsilon,\sigma,1} q_1 q_i \chi(1) \chi(1) q_j q_1 V_{\epsilon,\sigma,1} p_j p_1 \Phi^\epsilon \rangle_{L_2^p} \\
= & \sum_{2 \leq i \neq j \leq N} \langle \Phi^\epsilon | p_1 p_i (\tilde{\chi(1)}) V_{\epsilon,\sigma,1} q_1 V_{\epsilon,\sigma,1} q_j q_1 V_{\epsilon,\sigma,1} p_j p_1 \Phi^\epsilon \rangle_{L_2^p} \\
= & \sum_{2 \leq i \neq j \leq N} \langle \Phi^\epsilon | p_1 p_i (\tilde{\chi(1)}) V_{\epsilon,\sigma,1} q_1 V_{\epsilon,\sigma,1} q_j q_1 V_{\epsilon,\sigma,1} p_j p_1 \Phi^\epsilon \rangle_{L_2^p} \\
= & A_1 - A_2
\end{align}

where the ultimate equality follows from writing \(q_1 = 1 - p_1\).

For \(A_1\), we have by the triangle inequality and self-adjointness of \((\tilde{\chi(1)}) q_j\) that

\begin{align}
|A_1| \leq & \sum_{2 \leq i \neq j \leq N} \left| \langle (\tilde{\chi(1)}) q_j \Phi^\epsilon | p_1 p_i V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1} q_j q_1 V_{\epsilon,\sigma,1} p_j p_1 \Phi^\epsilon \rangle_{L_2^p} \right|.
\end{align}

Using that \(V_{\epsilon,\sigma} \geq 0\) and commutativity of point-wise multiplication operators, we can write

\begin{align}
V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1} = (V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1})^{1/2} (V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1})^{1/2}
\end{align}

and then use Cauchy-Schwarz to obtain

\begin{align}
\left| \langle (\tilde{\chi(1)}) q_j \Phi^\epsilon | p_1 p_i V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1} p_j p_1 \Phi^\epsilon \rangle_{L_2^p} \right| \leq \|(V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1})^{1/2} p_1 p_i (\tilde{\chi(1)}) q_j \Phi^\epsilon \|_{L_2^p}
\end{align}

\times \|(V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1})^{1/2} p_j p_1 (\tilde{\chi(1)}) q_j \Phi^\epsilon \|_{L_2^p}.

From Young’s inequality for products and the symmetry of \(\Phi^\epsilon\) under permutation of particle labels, we then find that

\begin{align}
|A_1| \leq & \sum_{2 \leq i \neq j \leq N} \left| \langle \Phi^\epsilon | (\tilde{\chi(1)}) q_j p_1 p_i V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1} p_j p_1 q_j (\tilde{\chi(1)}) \Phi^\epsilon \rangle_{L_2^p} \right|.
\end{align}

Next, by computation of its integral kernel, we see that

\begin{align}
p_i V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1} p_i = p_i (V_{\epsilon,\sigma} * |\phi|^2)_1 V_{\epsilon,\sigma,1},
\end{align}

and

\begin{align}
(p_i (V_{\epsilon,\sigma} * |\phi|^2)_1 V_{\epsilon,\sigma,1} p_1) = p_i (V_{\epsilon,\sigma} * (|\phi|^2 (V_{\epsilon,\sigma} * |\phi|^2)))_1.
\end{align}
By Young’s inequality with \(\|V_{\epsilon,\sigma}\|_{L^1} = 1\), followed by Hölder’s inequality, and then another application of Young’s, we have that
\[
\|(V_{\epsilon,\sigma} \ast (|\phi|^2V_{\epsilon,\sigma} \ast |\phi|^2))\|_{L^\infty_x} \leq \|\phi\|_{L^2_x}^2 \|V_{\epsilon,\sigma} \ast |\phi|^2\|_{L^\infty_x} \leq \|\phi\|_{L^\infty_x}^4,
\]
which implies that
\[
\|p_1 p_1 V_{\epsilon,\sigma,1} V_{\epsilon,\sigma,1} p_1 p_1\|_{L^2_{\mathbb{Z}N} \rightarrow L^2_{\mathbb{Z}N}} \leq \|\phi\|_{L^\infty_x}^4.
\]
Applying this last estimate to the right-hand side of (4.101) and the symmetry of \(\Phi^\epsilon\), we obtain that
\[
|A_1| \leq \|\phi\|_{L^\infty_x}^4 \sum_{2 \leq i \neq j \leq N} \|(\tau_2 \chi^{(1)}) q_j \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}}^2 \leq N^2 \|\phi\|_{L^\infty_x}^4 \|(\tau_2 \chi^{(1)}) q_1 \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}}^2 \leq N^2 \|\phi\|_{L^\infty_x}^4 \|(\tau_2 \chi^{(1)}) \hat{n} \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}}^2,
\]
where the ultimate inequality follows by application of Lemma 4.6(i) to the factor \(\|(\tau_2 \chi^{(1)}) q_1 \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}}\). In order to estimate the last expression, we claim that
\[
(\tau_2 \chi^{(1)})(k) n(k) \leq N^{-\delta/2}, \quad \forall k \in \{0, \ldots, N\}.
\]
Indeed, recalling from (4.87) that \(\chi^{(1)} = 1_{\leq N-\delta}\), where \(\delta \in (0, 1)\), we see that
\[
(\tau_2 \chi^{(1)})(k) n(k) = 1_{\leq N-\delta}(k+2)1_{\geq 0}(k) \sqrt{\frac{(k+2) - 2}{N}} \leq 1_{\leq N-\delta}(k) \sqrt{\frac{N^{1-\delta} - 2}{N}},
\]
from which the claim follows. Applying this estimate to the right-hand side of (4.106) leads to the conclusion
\[
|A_1| \lesssim N^2 - \delta \|\phi\|_{L^\infty_x}^4.
\]
Now using the identity
\[
p_1 V_{\epsilon,\sigma,1} p_1 V_{\epsilon,\sigma,1} p_1 = p_1 (V_{\epsilon,\sigma} \ast |\phi|^2)_i (V_{\epsilon,\sigma} \ast |\phi|^2)_j,
\]
which follows from examination of the integral kernel, and arguing similarly as for \(A_1\), we find that
\[
|A_2| \leq \|V_{\epsilon,\sigma} \ast |\phi|^2\|_{L^\infty_x}^2 \sum_{2 \leq i \neq j \leq N} \|q_j (\tau_2 \chi^{(1)}) \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}} \|q_i (\tau_2 \chi^{(1)}) \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}} \lesssim N^2 - \delta \|\phi\|_{L^\infty_x}^4.
\]
Thus, we conclude from (4.109) and (4.111) that
\[
|A| \lesssim N^2 - \delta \|\phi\|_{L^\infty_x}^4.
\]
To conclude the estimate for Term_{3.2.1} defined in (4.88) above, we insert the estimate (4.96) for \(B\) and the estimate (4.112) for \(A\) into the right-hand side of (4.92), obtaining
\[
\text{Term}_{3.2.1} \lesssim \frac{1}{N-1} \left( \sqrt{(N-1) N^\sigma \|\phi\|_{L^2}^4} + \sqrt{N^2 - \delta \|\phi\|_{L^\infty}^4} \right) \lesssim \frac{\|\phi\|_{L^2}^4}{N^{(1-\sigma)/2}} + \frac{\|\phi\|_{L^\infty}^4}{N^{\delta/2}}.
\]
It remains for us to estimate Term_{3.2.2}, which we recall from (4.88) is defined by
\[
\text{Term}_{3.2.2} = \left| \left< \Phi^\epsilon p_1 p_2 V_{\epsilon,\sigma,12} \hat{\nu} \chi^{(2)} q_1 q_2 \Phi^\epsilon \right|_{L^2_{\mathbb{Z}N}} \right|.
\]
Writing \(\hat{\nu} = \hat{\nu}^{1/2} \hat{\nu}^{1/2}\) and using the same symmetrization trick as above, we find that
\[
\text{Term}_{3.2.2} = \frac{1}{N-1} \left| \left< \Phi^\epsilon \sum_{i=2}^N p_1 p_i V_{\epsilon,\sigma,1i} q_i \hat{\nu} \chi^{(2)} q_1 q_2 \Phi^\epsilon \right|_{L^2_{\mathbb{Z}N}} \right|
\leq \frac{1}{N-1} \|\hat{\nu}^{1/2} q_1 \Phi^\epsilon\|_{L^2_{\mathbb{Z}N}} \sum_{i,j=2}^N \left| \left< \Phi^\epsilon p_1 p_i V_{\epsilon,\sigma,1i} q_i \hat{\nu} \chi^{(2)} q_1 q_j V_{\epsilon,\sigma,1j} p_j p_1 \Phi^\epsilon \right|_{L^2_{\mathbb{Z}N}} \right|.
\]
where the ultimate inequality follows by Cauchy-Schwarz and expanding the $L^2_{\mathbb{Z}_N}$ norm of the second factor. By the $\nu$ estimate (4.70) together with Lemma 4.14 (4.115)

\[
\|\hat{\varphi}^{1/2} q_1 \Phi^\varphi\|_{L^2_{\mathbb{Z}_N}} = \sqrt{\langle \Phi^\varphi | \hat{\varphi} q_1 \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}}} \lesssim \sqrt{\langle \Phi^\varphi | \hat{n}^{-1} q_1 \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}}} \lesssim \sqrt{\beta_\varphi}.
\]

Thus, splitting the sum $\sum_{i,j} = \sum_{i} + \sum_{i \neq j}$ in the second factor of (4.115) and applying the embedding $\ell^{1/2} \subset \ell^1$, we obtain that

\[
\text{Term}_{3,2,2} \leq \frac{\sqrt{\beta_\varphi}}{N-1} \left( \sqrt{A} + \sqrt{B} \right),
\]

where

\[
B := \sum_{i=2}^{N} \langle \Phi^\varphi | p_1 p_i V_{\varphi,\sigma,1} q_i \chi(2) \bar{\varphi} V_{\varphi,\sigma,1} p_i \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}},
\]

\[
A := \sum_{2 \leq i \neq j \leq N} \langle \Phi^\varphi | p_1 p_i V_{\varphi,\sigma,1} q_i \chi(2) \bar{\varphi} q_j V_{\varphi,\sigma,1} p_j p_i \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}}.
\]

Note that in contrast to the inequality (4.92) for Term_{3,2,1}, we have a factor of $\sqrt{\beta_\varphi}$ in the right-hand side of inequality (4.117).

We first dispense with the easy case $B$. We recall from (4.87) that $\chi(2) = 1_{N^{1-\delta}}$, which together with the $\nu$ bound (4.70) implies the estimate

\[
\chi(2)(k) \nu(k) \lesssim 1_{N^{1-\delta}}(k) n^{-1}(k) = 1_{N^{1-\delta}}(k) \sqrt{\frac{N}{k}} < N^{\delta/2}, \quad \forall k \in \mathbb{Z}.
\]

Therefore, we have the $L^2_{\mathbb{Z}_N}$ operator norm estimate

\[
\|q_1 q_i \chi(2) \bar{\varphi}\|_{L^2_{\mathbb{Z}_N} \to L^2_{\mathbb{Z}_N}} \lesssim N^{\delta/2}, \quad \forall i \in \{1, \ldots, N\},
\]

which implies that

\[
B \lesssim N^{\delta/2} N^{-1} \sum_{i=2}^{N} \|V_{\varphi,\sigma,1} p_i \Phi^\varphi\|_{L^2_{\mathbb{Z}_N}}^2 = (N-1) N^{\delta/2} \|V_{\varphi,\sigma,1} p_1 p_2 \Phi^\varphi\|_{L^2_{\mathbb{Z}_N}}^2,
\]

where the ultimate identity follows from the symmetry of $\Phi^\varphi$. Since by Cauchy-Schwarz and Young's inequality,

\[
\|p_1 p_2 V_{\varphi,\sigma,1} p_1 p_2\|_{L^2_{\mathbb{Z}_N}} = 2 \|\phi\|^2_{L^2_{\mathbb{Z}_N}} \leq N^{\sigma},
\]

where we also use $\|V_{\varphi,\sigma}\|_{L^2_{\mathbb{Z}_N}} \lesssim N^{\sigma}$, we conclude that

\[
B \lesssim N^{1+\delta+\delta} \|\phi\|_{L^2_{\mathbb{Z}_N}}^4.
\]

For the hard case $A$, we again use Lemma 4.7 as in (4.97) to write $A = A_1 + A_2$, where

\[
A_1 := \sum_{2 \leq i \neq j \leq N} \langle \Phi^\varphi | p_1 p_i q_j (\tau_2 \chi(2))^{1/2} V_{\varphi,\sigma,1} V_{\varphi,\sigma,1} (\tau_2 \chi(2))^{1/2} q_i q_j p_1 \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}},
\]

\[
A_2 := \sum_{2 \leq i \neq j \leq N} \langle \Phi^\varphi | p_1 p_i q_j (\tau_2 \chi(2))^{1/2} V_{\varphi,\sigma,1} p_1 V_{\varphi,\sigma,1} (\tau_2 \chi(2))^{1/2} (\tau_2 \nu)^{1/2} q_i q_j p_1 \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}}.
\]

For $A_1$, we use that $V_{\varphi,\sigma} \geq 0$ to apply Cauchy-Schwarz and exploit the symmetry of $\Phi^\varphi$ under exchange of particle labels in order to obtain

\[
|A_1| \leq \sum_{2 \leq i \neq j \leq N} \langle \Phi^\varphi | q_j (\tau_2 \chi(2))^{1/2} p_1 p_i V_{\varphi,\sigma,1} V_{\varphi,\sigma,1} p_1 (\tau_2 \chi(2))^{1/2} q_i q_j p_1 \Phi^\varphi \rangle_{L^2_{\mathbb{Z}_N}}.
\]
Using the $L^2_{\mathbb{Z}_N}$ operator norm estimate (4.105), we conclude that

\begin{equation}
|A_1| \lesssim \|\phi\|_{L^\infty}^2 \sum_{2 \leq i \neq j \leq N} \left\langle \left(\tau_2 \chi^{(2)} \right) \left(\tau_2 \nu \right)^{1/2} q_1 \Phi^e \right\rangle_{L^2_{\mathbb{Z}_N}} \lesssim N^2 \|\phi\|_{L^\infty}^4 \left\langle \Phi^e | \vec{n} \Phi^e \right\rangle_{L^2_{\mathbb{Z}_N}} = N^2 \|\phi\|_{L^\infty}^4 \beta_\epsilon,
\end{equation}

where the penultimate inequality follows from the $\nu$ estimate (4.70) together with Lemma 4.11 and the ultimate equality is by definition of $\beta_\epsilon$ (recall (4.19)). Next, using the operator identity (4.110) and arguing similarly as for $A_2$ in the case of $\chi^{(1)}$, we also obtain the estimate

\begin{equation}
|A_2| \lesssim N^2 \|\phi\|_{L^\infty}^4 \beta_\epsilon,
\end{equation}

leading us to conclude that

\begin{equation}
|A| \lesssim N^2 \|\phi\|_{L^\infty}^4 \beta_\epsilon.
\end{equation}

Inserting the estimates (4.124) for $B$ and (4.130) for $A$ into the right-hand side of (4.117), we find from the normalization $\|\phi\|_{L^2} = 1$ and Young’s inequality for products that

\begin{equation}
\text{Term}_{3,2,2} \lesssim \sqrt{\frac{\sigma}{N-1}} \left( N \|\phi\|_{L^\infty} \sqrt{\tau} \beta_\epsilon + N^{\frac{1+\alpha}{2}} \|\phi\|_{L^1}^2 \right) \lesssim \|\phi\|_{L^\infty}^2 \beta_\epsilon + N^{\frac{2(\alpha-1)+\delta}{2}}.
\end{equation}

Collecting the estimates (4.113) for Term$_{3,2,1}$ and (4.131) for Term$_{3,2,2}$, we find that

\begin{equation}
|\text{Term}_{3,2}| \lesssim N^{-\sigma} \|\phi\|_{C_{L_x}^{1/2}}^2 + N^{-\frac{5}{2}} \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^2 \beta_\epsilon + N^{\frac{2(\alpha-1)+\delta}{2}}.
\end{equation}

Now inserting the estimates (4.86) for Term$_{3,1}$ and (4.132) for Term$_{3,2}$ into the right-hand side of (4.72), we conclude that

\begin{equation}
|\text{Term}_3| \lesssim N^{-\sigma} + \|\phi\|_{C_{L_x}^{1/2}}^2 \|\phi\|_{H^1}^2 \beta_\epsilon + \|\phi\|_{C_{L_x}^{1/2}}^2 \|\nabla_1 q_1 \Phi^e\|_{L^2_{\mathbb{Z}_N}}^2
\end{equation}

\begin{equation}
+ N^{-\frac{1}{2}} \|\phi\|_{L^2_x}^2 + N^{-\frac{5}{2}} \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^2 \beta_\epsilon + N^{\frac{2(\alpha-1)+\delta}{2}}.
\end{equation}

where the ultimate line follows from the trivial $C_{L_x}^{1/2} \subset L^\infty$ embedding and the fact $\|\phi\|_{H^1} \geq 1$.

We are now prepared to conclude the proof of the proposition. After a bookkeeping of the estimates (4.12) for Term$_1$, (4.15) for Term$_2$, and (4.133) for Term$_3$, we find that

\begin{equation}
\beta_\epsilon \lesssim \frac{\|\phi\|_{L^\infty}^2}{N} + \varepsilon^{1/2} \|\phi\|_{C_{L_x}^{1/2}}^2 + \|\phi\|_{L^\infty}^2 \beta_\epsilon + \|\nabla_1 q_1 \Phi^e\|_{L^2_{\mathbb{Z}_N}} \beta_\epsilon + \frac{1}{N^\sigma} + \|\phi\|_{C_{L_x}^{1/2}}^2 \beta_\epsilon
\end{equation}

\begin{equation}
+ \frac{\|\phi\|_{L^2_{L_x}}^2 \|\nabla_1 q_1 \Phi^e\|_{L^2_{\mathbb{Z}_N}}^2}{N(1-\sigma)/2} + \frac{\|\phi\|_{L^\infty}^2}{N^{\delta/2}} + N^{\frac{2(\alpha-1)+\delta}{2}}.
\end{equation}

The desired conclusion now follows from Young’s inequality for products, $\|\phi\|_{L^2} = 1$, and some algebra. □

4.3. Control of $\|\nabla_1 q_1 \Phi_N\|_{L^2}\). Before we can pass to the limit $\varepsilon \to 0^+$ to remove the regularization of the LL Hamiltonian, we need to control the auxiliary quantity $\|\nabla_1 q_1 \Phi^e_N\|_{L^2_{\mathbb{Z}_N}}$ appearing in the right-hand side of (4.134). To this end, we first introduce the energy per particle of the solution $\Phi^e$ to equation (3.20):

\begin{equation}
E_N^e := \frac{1}{N} \left\langle \Phi^e_N | H_{N,\epsilon} \Phi^e_N \right\rangle_{L^2_{\mathbb{Z}_N}(\mathbb{R}^N)} = \|\nabla_1 \Phi_N^e\|_{L^2_{\mathbb{Z}_N}(\mathbb{R}^N)}^2 + \frac{\kappa(N-1)}{2N} \left\langle \Phi^e_N, |V_{\epsilon,12} \Phi^e_N\right\rangle_{L^2_{\mathbb{Z}_N}(\mathbb{R}^N)},
\end{equation}

where the ultimate equality follows from conservation of energy, unpacking the definition (3.19) of $H_{N,\epsilon}$, and exploiting the symmetry of $\Phi^e$. We recall from (1.20) that the energy of the solution $\phi$ to the cubic NLS (1.11) is given by

\begin{equation}
E^\phi = \|\nabla \phi\|_{L^2(\mathbb{R}^3)}^2 + \frac{\kappa}{2} \|\phi\|_{L^2(\mathbb{R}^3)}^4 = \|\nabla \phi_0\|_{L^2(\mathbb{R})}^2 + \frac{\kappa}{2} \|\phi_0\|_{L^4(\mathbb{R})}^4.
\end{equation}
The reader will remember that \( \kappa \in \{ \pm 1 \} \) denotes the sign of the interaction (i.e. repulsive or attractive). The goal of this subsection is to prove the following proposition, which controls \( \| \nabla_1 q_1 \Phi_N \|^2_{L^2} \) in terms of \( \beta_\varepsilon \), \( N \), and \( (E_{N}^{\Phi_\varepsilon} - E^\phi) \).

**Proposition 4.10** (Control of \( \| \nabla_1 q_1 \Phi_N \|^2_{L^2} \)). Let \( \kappa \in \{ \pm 1 \} \). Then we have the estimate

\[
\| \nabla_1 q_1(t) \Phi_N(t) \|^2_{L^2(R^N)} \lesssim E_{N}^{\Phi_\varepsilon} - E^\phi + \varepsilon^{1/2} \| \phi(t) \|^2_{C^1(L^2)} + \| \phi(t) \|_{H^2(R)} \beta_\varepsilon(t) + \frac{\| \phi(t) \|_{H^2(R)}}{\sqrt{N}},
\]

for every \( t \in \mathbb{R} \), uniformly in \( \varepsilon > 0 \) and \( N \in \mathbb{N} \).

**Proof.** As before, we drop the subscript \( N \), as the number of particles is fixed throughout the proof. We introduce two parameters \( \kappa_1 \in (0, 1) \) and \( \kappa_2 > 0 \), the precise values of which we shall specify momentarily. Using the decomposition \( 1 = p_1 p_2 + (1 - p_1 p_2) \) and the normalizations \( \| \Phi_\varepsilon \|^2_{L^2} = 1 = \| \phi \|_{L^2} \), together with some algebraic manipulation of the quantities \( (4.135) \) and \( (4.136) \), we arrive at the identity

\[
(1 - \kappa_1) \| \nabla_1 (1 - p_1 p_2) \Phi_\varepsilon \|^2_{L^2} = E_{N}^{\Phi_\varepsilon} - E^\phi + \sum_{i=1}^{6} \text{Term}_i,
\]

where

\[
\text{Term}_1 := -\| \nabla_1 p_1 p_2 \Phi_\varepsilon \|^2_{L^2} + \| \nabla \phi \|^2_{L^2},
\]

\[
\text{Term}_2 := -\kappa_2 \langle \Phi_\varepsilon | p_1 p_2 \Phi_\varepsilon \rangle_{L^2} + \kappa_2,
\]

\[
\text{Term}_3 := -\frac{\kappa(N - 1)}{2N} \langle \Phi_\varepsilon | p_1 p_2 V_{\varepsilon,12} p_1 p_2 \Phi_\varepsilon \rangle_{L^2} + \frac{\kappa_2}{2} \| \phi \|^4_{L^2},
\]

\[
\text{Term}_4 := -2 \Re \{ \langle \nabla_1 (1 - p_1 p_2) \Phi_\varepsilon | \nabla_1 p_1 p_2 \Phi_\varepsilon \rangle_{L^2} \},
\]

\[
\text{Term}_5 := -\frac{\kappa(N - 1)}{N} \Re \{ \langle \Phi_\varepsilon | (1 - p_1 p_2) V_{\varepsilon,12} p_1 p_2 \Phi_\varepsilon \rangle_{L^2} \},
\]

\[
\text{Term}_6 := -\frac{\kappa(N - 1)}{2N} \| V_{\varepsilon,12} (1 - p_1 p_2) \Phi_\varepsilon \|^2_{L^2} - \kappa_1 \| \nabla_1 (1 - p_1 p_2) \Phi_\varepsilon \|^2_{L^2} - \kappa_2 \| (1 - p_1 p_2) \Phi_\varepsilon \|^2_{L^2}.
\]

We keep the term \( E_{N}^{\Phi_\varepsilon} - E^\phi \). We want to obtain upper bounds for the moduli of Term1, . . . , Term5, and we want to show that Term6 \( \leq 0 \) provided that we appropriately choose \( \kappa_1, \kappa_2 \) depending on \( \kappa \).

**Estimate for Term1:** Since \( \nabla_1 p_1 = (| \nabla \phi \rangle \langle \phi |)_1 \), it follows from 1 = \( \| \Phi_\varepsilon \|^2_{L^2} \) that

\[
\text{Term}_1 = \| \nabla \phi \|^2_{L^2} \left( 1 - \langle \Phi_\varepsilon | p_1 p_2 \Phi_\varepsilon \rangle_{L^2} \right) = \langle \Phi_\varepsilon | (1 - p_1 p_2) \Phi_\varepsilon \rangle_{L^2}.
\]

Since \( 1 - p_1 p_2 = q_1 p_2 + q_2 p_1 + q_1 q_2 \), it follows from Remark 3.3 and the triangle inequality that

\[
\langle \Phi_\varepsilon | (1 - p_1 p_2) \Phi_\varepsilon \rangle_{L^2} \leq 3 \alpha_\varepsilon \lesssim \beta_\varepsilon,
\]

leading us to conclude that

\[
\text{Term}_1 \lesssim \| \nabla \phi \|^2_{L^2} \beta_\varepsilon.
\]

**Estimate for Term2:** Using the identity \( \kappa_2 \| \Phi_\varepsilon \|^2_{L^2} = \kappa_2 \) and the estimate \( (4.146) \), we find that

\[
\text{Term}_2 = \kappa_2 \langle \Phi_\varepsilon | (1 - p_1 p_2) \Phi_\varepsilon \rangle_{L^2} \lesssim \kappa_2 \beta_\varepsilon.
\]

**Estimate for Term3:** First, observe that

\[
p_1 p_2 V_{\varepsilon,12} p_1 p_2 = \| \phi \|^4_{L^2} p_1 p_2 \quad \text{and} \quad p_1 p_2 V_{\varepsilon,12} p_1 p_2 = \| \phi \|^2_{L^2} \| \phi \|^2_{L^2} p_1 p_2.
\]

So by the triangle inequality,

\[
\text{Term}_3 \leq \frac{1}{2} \left| \langle \Phi_\varepsilon | p_1 p_2 (V_{\varepsilon,12} - V_{\varepsilon,12}) p_1 p_2 \Phi_\varepsilon \rangle_{L^2} \right| + \frac{\| \phi \|^4_{L^2}}{2} - \frac{(N - 1)}{N} \langle \Phi_\varepsilon | p_1 p_2 \Phi_\varepsilon \rangle_{L^2} + 1.
\]
Since \( \| \Phi^\varepsilon \|^2_{L^{2N}_2} = 1 \), the second term in the right-hand side equals

\[
\frac{\| \phi \|^4}{2} \left| \frac{1}{N} \langle \Phi^\varepsilon | p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} + \langle \Phi^\varepsilon | (1 - p_1 p_2) \Phi^\varepsilon \rangle_{L^{2N}_2} \right| \lesssim \| \phi \|_{L^4}^4 \left( \frac{1}{N} + \beta \varepsilon \right),
\]

where the ultimate inequality follows from the triangle inequality, \( \langle \Phi^\varepsilon | p_1 p_2 \Phi^\varepsilon \rangle \leq \| \Phi^\varepsilon \|^2_{L^{2N}_2} = 1 \), and the estimate (4.140). Again using that \( \| \Phi^\varepsilon \|^2_{L^{2N}_2} = 1 \), we see that the first term in the right-hand side of (4.150) is bounded by

\[
\frac{1}{2} \| \phi \|^2 \left( (V_\varepsilon * |\phi|^2) - |\phi|^2 \right) \|L^2 \| \lesssim \| \phi \|^2_{C^{1/2}_x} \varepsilon^{1/2},
\]

which follows from the estimate (4.40) and \( \| \phi \|_{L^2} = 1 \). Therefore,

\[
\text{Term} 3 \lesssim \varepsilon^{1/2} \| \phi \|^2_{C^{1/2}_x} + \| \phi \|^4_{L^4} \left( \frac{1}{N} + \beta \varepsilon \right).
\]

**Estimate for Term4:** By using the decomposition \( 1 - p_1 p_2 = q_1 p_2 + q_2 p_1 + q_1 q_2 \), the triangle inequality, and the fact that \( [q_2, \nabla_1] = 0 = q_2 p_2 \), we see that

\[
|\text{Term} 4| \lesssim \left| \langle \nabla_1 q_1 p_2 \Phi^\varepsilon | \nabla_1 p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} + \langle \nabla_1 q_2 p_1 \Phi^\varepsilon | \nabla_1 p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} + \langle \nabla_1 q_1 q_2 \Phi^\varepsilon | \nabla_1 p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} \right|
\]

\[
= \left| \langle q_1 \Phi^\varepsilon | (-\Delta_1) p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} \right|
\]

\[
= \left| \langle \nabla_1^{-1/2} q_1 \Phi^\varepsilon | \nabla_1^{1/2} (-\Delta_1) p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} \right|
\]

\[
(4.154)
\]

where the penultimate equality follows from integration by parts and the ultimate equality from writing \( 1 = \nabla_1^{-1/2} \nabla_1^{1/2} \). The reader will recall the definitions of \( n \) and \( \n \) from Definition 4.4. By Cauchy-Schwarz and \( q_1^2 = q_1 \),

\[
\left| \langle \nabla_1^{-1/2} q_1 \Phi^\varepsilon | \nabla_1^{1/2} (-\Delta_1) p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} \right| \leq \| \nabla_1^{-1/2} q_1 \Phi^\varepsilon \|_{L^{2N}_2} \| q_1 \nabla_1^{1/2} (-\Delta_1) p_1 p_2 \Phi^\varepsilon \|_{L^{2N}_2}
\]

\[
\leq \sqrt{\beta} \| q_1 \nabla_1^{1/2} (-\Delta_1) p_1 p_2 \Phi^\varepsilon \|_{L^{2N}_2},
\]

where the ultimate line follows from applying Lemma 4.4(3) to the first factor in the right-hand side of the first line. By Lemma 4.7, we have the operator identity

\[
\langle \nabla_1^{-1/2} q_1 \Phi^\varepsilon | (-\Delta_1) p_1 p_2 \Phi^\varepsilon \rangle_{L^{2N}_2} = q_1 (-\Delta_1) (\tau_1 n) \frac{1}{2} p_1 \n \n
\]

\[
(4.156)
\]

So writing \( q_1 = 1 - p_1 \) and using the triangle inequality together with the operator norm estimates

\[
\| (-\Delta_1) p_1 \|_{L^{2N}_2 \to L^2_{2N}} \leq \| \Delta \phi \|_{L^2} \quad \text{and} \quad \| p_1 (-\Delta_1) p_1 \|_{L^{2N}_2 \to L^2_{2N}} \leq \| \nabla \phi \|^2_{L^2}
\]

we find that

\[
\| q_1 \nabla_1^{-1/2} (-\Delta_1) p_1 p_2 \Phi^\varepsilon \|_{L^{2N}_2} \leq \| (-\Delta_1) p_1 (\tau_1 n) \n \|_{L^{2N}_2} + \| p_1 (-\Delta_1) p_1 (\tau_1 n) \|_{L^{2N}_2} \| \Phi^\varepsilon \|_{L^{2N}_2}
\]

\[
(4.157)
\]

\[
\leq \left( \| \Delta \phi \|_{L^2} + \| \nabla \phi \|^2_{L^2} \right) \| (\tau_1 n) \|^{1/2} \| \Phi^\varepsilon \|_{L^{2N}_2},
\]

where we eliminate \( p_2 \) using \( \| p_2 \|_{L^{2N}_2 \to L^{2N}_2} = 1 \). Using the embedding \( \ell^{1/2} \subset \ell^1 \), we see that

\[
(\tau_1 n)(k) = \sqrt{\frac{k + 1}{N}} I_{\geq 0}(k + 1) \leq \sqrt{\frac{k}{N}} I_{\geq 0}(k) + \frac{1}{\sqrt{N}} = n(k) + \frac{1}{\sqrt{N}}, \quad \forall k \in \mathbb{Z}.
\]

\[\text{This is the only place in this work where the } H^2 \text{ regularity assumption is strictly needed.} \]
By another application of $\ell^{1/2} \subset \ell^1$ together with $\|\Phi^e\|_{L^2_{xN}} = 1$, 

$$\| (\tau_1 n)^{1/2} \Phi^e \|_{L^2_{xN}} \leq \sqrt{\beta_e} + N^{-1/4}. \quad (4.160)$$

Using Young's inequality for products and interpolation of $H^s$ spaces with $\|\phi\|_{L^2_x} = 1$, we obtain that 

$$|\text{Term}_1| \lesssim \left( \|\Delta \phi\|_{L^2_x} + \|\nabla \phi\|_{L^2_x} \right) \sqrt{\beta_e} \left( \sqrt{\beta_e} + N^{-1/4} \right) \lesssim \|\phi\|_{H^2_x} \left( \beta_e + N^{-1/2} \right). \quad (4.161)$$

**Estimate for Term$_5$:** Using the decomposition $1 - p_1 p_2 = p_1 q_2 + p_2 q_1 + q_1 q_2$ together with the triangle inequality and the symmetry of $\Phi^e$ under exchange of particle labels, we have that 

$$|\text{Term}_5| \lesssim \left| \langle \Phi^e | p_1 p_2 V_{\epsilon,12} q_1 p_2 \Phi^e \rangle_{L^2_{xN}} + \langle \Phi^e | p_1 p_2 V_{\epsilon,12} q_2 p_1 \Phi^e \rangle_{L^2_{xN}} + \langle \Phi^e | p_1 p_2 V_{\epsilon,12} q_1 q_2 \Phi^e \rangle_{L^2_{xN}} \right| \quad (4.162)$$

$$=: \text{Term}_{5,1} + \text{Term}_{5,2},$$

For Term$_{5,1}$, we note from an examination of its integral kernel that 

$$p_1 p_2 V_{\epsilon,12} q_2 = p_1 p_2 V_{\epsilon,1}^\phi q_1,$$

where we use the notation $V_{\epsilon,1}^\phi$ introduced in (4.32). Now writing $1 = \tilde{n}^{-1/2} \tilde{n}^{1/2}$, we find that 

$$\text{Term}_{5,1} = \left| \langle \Phi^e | p_1 p_2 V_{\epsilon,1}^\phi \tilde{n}^{-1/2} \tilde{n}^{1/2} q_1 \Phi^e \rangle_{L^2_{xN}} \right|$$

$$= \left| \langle \Phi^e | p_1 p_2 (\tau_1 n)^{1/2} V_{\epsilon,1}^\phi \tilde{n}^{-1/2} q_1 \Phi^e \rangle_{L^2_{xN}} \right|$$

$$\leq \|p_1 p_2 (\tau_1 n)^{1/2} \Phi^e\|_{L^2_{xN}} \|V_{\epsilon,1}^\phi \tilde{n}^{-1/2} q_1 \Phi^e\|_{L^2_{xN}}, \quad (4.164)$$

where the penultimate line follows from an application of Lemma 4.47 and the ultimate line follows from Cauchy-Schwarz. Applying the operator norm identity $\|p_j\|_{L^2-\rightarrow L^2} = 1$ together with the estimate (4.160) to the first factor in (4.164), we obtain that 

$$|\text{Term}_{5,1}| \lesssim \left( \sqrt{\beta_e} + N^{-1/4} \right) \|V_{\epsilon,1}^\phi \tilde{n}^{-1/2} q_1 \Phi^e\|_{L^2_{xN}}. \quad (4.165)$$

Now since $\|V_{\epsilon,1}^\phi\|_{L^2_{xN} \rightarrow L^2_{xN}} \leq \|\phi\|_{L^\infty_x}$, we find that 

$$\|V_{\epsilon,1}^\phi \tilde{n}^{-1/2} q_1 \Phi^e\|_{L^2_{xN}} \leq \|\phi\|_{L^\infty_x} \|\tilde{n}^{-1/2} q_1 \Phi^e\|_{L^2_{xN}} \leq \|\phi\|_{L^\infty_x} \sqrt{\beta_e}. \quad (4.166)$$

where the ultimate equality follows from Lemma 4.6(i) and the trivial fact that $\tilde{n}^2 = \hat{n}$. Using the embedding $\ell^{1/2} \subset \ell^1$, we conclude that 

$$|\text{Term}_{5,1}| \lesssim \|\phi\|_{L^\infty_x} \sqrt{\beta_e} \left( \sqrt{\beta_e} + N^{-1/4} \right) \lesssim \|\phi\|_{L^\infty_x}^0 \left( \beta_e + N^{-1/2} \right). \quad (4.167)$$

For Term$_{5,2}$, we use, as in the proof of Proposition 4.9, the distributional identity (2.10) to write $V_{\epsilon,12} = (\nabla_1 X_{\epsilon,12})$, where $X_{\epsilon,12} := \frac{1}{2} (V_{\epsilon} \ast \text{sgn})(X_1 - X_2)$. Thus, 

$$\text{Term}_{5,2} = \left| \langle \Phi^e | p_1 p_2 (\nabla_1 X_{\epsilon,12}) q_1 q_2 \Phi^e \rangle_{L^2_{xN}} \right|$$

$$= \left| \langle \Phi^e | p_1 p_2 (\nabla_1 X_{\epsilon,12}) \tilde{n} \tilde{n}^{-1} q_1 q_2 \Phi^e \rangle_{L^2_{xN}} \right|$$

$$= \left| \langle \Phi^e | p_1 p_2 (\tau_2 n) (\nabla_1 X_{\epsilon,12}) \tilde{n}^{-1} q_1 q_2 \Phi^e \rangle_{L^2_{xN}} \right|$$

$$= \left| \langle \tau_2 n p_1 p_2 \Phi^e | (\nabla_1 X_{\epsilon,12}) \tilde{n}^{-1} q_1 q_2 \Phi^e \rangle_{L^2_{xN}} \right|, \quad (4.168)$$
where the penultimate line follows from an application of Lemma [4.7]. Now integrating by parts and then applying the product rule and triangle inequality, we obtain that
\[
\left| \left\langle (\tilde{\tau}_n) p_1 p_2 \Phi^\varepsilon \left| \nabla_1 X_{\varepsilon,12} \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon \right\rangle \right| \leq \left| \left\langle \nabla_1 (\tilde{\tau}_n) p_1 p_2 \Phi^\varepsilon \left| X_{\varepsilon,12} \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon \right\rangle \right| + \left| \left\langle (\tilde{\tau}_n) p_1 p_2 \Phi^\varepsilon \left| X_{\varepsilon,12} \nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon \right\rangle \right| \leq \text{Term}_{5,2,1} + \text{Term}_{5,2,2}.
\]

(4.169)

We first dispense with the easy case Term_{5,2,1}. By Cauchy-Schwarz and using the operator norm estimates
\[
\|\nabla_1 p_1\|_{L^2_\mathbb{Z}^N} \to L^2_\mathbb{Z}^N \leq \|\nabla \phi\|_{L^2_\mathbb{Z}^N} \quad \text{and} \quad \|X_{\varepsilon,12}\|_{L^2_\mathbb{Z}^N} \to L^2_\mathbb{Z}^N \leq \frac{1}{2},
\]
we obtain that
\[
\text{Term}_{5,2,1} \leq \|\nabla \phi\|_{L^2_\mathbb{Z}^N} \|(\tilde{\tau}_n) \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \|\hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N}.
\]

(4.170)

By arguing similarly as for the estimates [4.159] and (4.160), we find that
\[
\|(\tilde{\tau}_n) \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \leq \sqrt{\beta \varepsilon} + \frac{1}{\sqrt{N}},
\]
and by applying Lemma [4.4(ii)] we have that
\[
\|\hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \leq \sqrt{\beta \varepsilon}.
\]

Thus, we conclude that
\[
\text{Term}_{5,2,1} \lesssim \|\nabla \phi\|_{L^2_\mathbb{Z}^N} \left(\beta \varepsilon + \frac{1}{N}\right).
\]

(4.174)

For the hard case Term_{5,2,2}, we first use Cauchy-Schwarz and (4.170) to obtain
\[
\text{Term}_{5,2,2} \leq \|(\tilde{\tau}_n) p_1 p_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \|
abla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \lesssim \left(\sqrt{\beta \varepsilon} + N^{-1/2}\right) \|
abla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N},
\]

(4.175)

where the second line follows from applying the estimate (4.172) to the first factor in the right-hand side of the first line. For the remaining factor \(\|
abla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N}\), we write \(1 = p_1 + q_1\) and use the triangle inequality to obtain
\[
\|\nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \leq \|p_1 \nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} + \|q_1 \nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N}.
\]

(4.176)

Since \(\|p_1 \nabla_1\|_{L^2_\mathbb{Z}^N} \to L^2_\mathbb{Z}^N \leq \|\nabla \phi\|_{L^2_\mathbb{Z}^N}\), it follows that
\[
\|p_1 \nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \leq \|\nabla \phi\|_{L^2_\mathbb{Z}^N} \|\hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \lesssim \|\nabla \phi\|_{L^2_\mathbb{Z}^N} \sqrt{\beta \varepsilon},
\]

(4.177)

where the ultimate inequality follows from applying Lemma [4.4(ii)] and \(\hat{n}^2 = \hat{n}\). Next, observe that by Lemma [4.7], \(q_1 \nabla_1 \hat{n}^{-1} = q_1 \hat{n}^{-1} \nabla_1 q_1\), which implies that
\[
\|q_1 \nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} \leq \|\nabla_1 q_1 q_2 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N} = \sqrt{\langle \nabla_1 q_1 \Phi^\varepsilon | q_2 \hat{n}^{-2} \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_\mathbb{Z}^N}},
\]

(4.178)

where the ultimate equality follows from the fact that \(q_2\) commutes with \(\hat{n}^{-2} \nabla_1 q_1\) and \(q_2^2 = q_2\). By the symmetry of \(\Phi^\varepsilon\) with respect to permutation of particle labels and the operator identity
\[
\frac{1}{N-1} \sum_{i=2}^{N} q_i \hat{n}^{-2} \leq \left(\frac{N}{N-1}\right) \hat{n} \hat{n}^{-2} \lesssim 1,
\]

(4.179)

which follows from Remark [4.5], we see that
\[
\langle \nabla_1 q_1 \Phi^\varepsilon | q_2 \hat{n}^{-2} \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_\mathbb{Z}^N} = \frac{1}{N-1} \sum_{i=2}^{N} \langle \nabla_1 q_1 \Phi^\varepsilon | q_i \hat{n}^{-2} \nabla_1 q_1 \Phi^\varepsilon \rangle_{L^2_\mathbb{Z}^N} \lesssim \|\nabla_1 q_1 \Phi^\varepsilon\|_{L^2_\mathbb{Z}^N}^2.
\]

(4.180)
Hence,
\begin{equation}
\|q_1 \nabla_1 \hat{n}^{-1} q_1 q_2 \Phi^\varepsilon\|_{L^2_{\omega, N}} \lesssim \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}.
\end{equation}

We therefore conclude from another application of Young’s inequality that
\begin{equation}
\text{Term}_{5,2,2} \lesssim \| \nabla \phi\|_{L^2_{\omega}} (\beta_\varepsilon + N^{-1}) + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}.
\end{equation}

Collecting the estimate (4.174) for Term_{5,2,1} and the estimate (4.182) for Term_{5,2,2}, we find that
\begin{equation}
\text{Term}_{5,2} \lesssim \| \nabla \phi\|_{L^2_{\omega}} (\beta_\varepsilon + N^{-1}) + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}.
\end{equation}

Together with the estimate (4.167) for Term_{5,1}, we conclude that
\begin{equation}
|\text{Term}_5| \lesssim \| \phi\|_{L^{\infty}_\omega}^2 \left(\beta_\varepsilon + N^{-1/2}\right) + \| \nabla \phi\|_{L^2_{\omega}} (\beta_\varepsilon + N^{-1}) + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}.
\end{equation}

**Estimate for Term_{6}:** We want to show that Term_{6} \leq 0. We assume here that \( \kappa = -1 \); otherwise, it is trivial that Term_{6} \leq 0 and we can take \( \kappa_2 = 0 \). By the same argument used to prove Lemma 2.2
\begin{equation}
\|V_{\varepsilon,12}^{1/2}(1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 \leq \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}} \| (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}},
\end{equation}

and by Young’s inequality for products,
\begin{equation}
\frac{(N - 1)}{2N} \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}} \| (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}} \leq \kappa_1 \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 + \frac{(N - 1)^2}{4N^2 \kappa_1} \| (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2.
\end{equation}

We choose \( \kappa_2 > 1/(2 \kappa_1) \). Then,
\begin{align}
\text{Term}_6 &= \frac{(N - 1)}{2N} \|V_{\varepsilon,12}^{1/2}(1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 - \kappa_1 \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 - \kappa_2 \| (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 \\
&\leq \left(\frac{(N - 1)^2}{4N^2 \kappa_1} - \kappa_2\right) \| (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2
\end{align}

(4.187)

as desired.

Having estimated the terms Term_{1}, \ldots, Term_{6}, we can now complete the proof of the proposition. Combining estimate (4.147) for Term_{1}, (4.148) for Term_{2}, (4.153) for Term_{3}, (4.161) for Term_{4}, and (4.184) for Term_{5}, we see that there exists an absolute constant \( C > 0 \) such that
\begin{equation}
(1 - \kappa_1) \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 \leq \left(E^{\Phi^\varepsilon} - E^{\Phi}\right) + C \epsilon^{1/2} \| \phi\|_{C^{1/2}_2} + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}
\end{equation}

\begin{equation}
+ C \left(\| \phi\|_{L^\infty_{\omega}} + \| \phi\|_{H^2_{\omega}}\right) N^{-1/2} + \left(\| \nabla \phi\|_{L^2_{\omega}} + \| \phi\|_{C^1_{\omega}}\right) N^{-1}
\end{equation}

\begin{equation}
+ C \beta_\varepsilon \left(\| \nabla \phi\|_{L^2_{\omega}} + \kappa_2 (1 - \kappa_1)\| \phi\|_{H^2_{\omega}} + \| \phi\|_{L^\infty_{\omega}} + \| \phi\|_{C^1_{\omega}}\right).
\end{equation}

Note that by using Sobolev embedding, the interpolation property of \( H^s \) norms, and the normalization \( \| \phi\|_{L^2_{\omega}} = 1 \), we can simplify the right-hand side of (4.188) to
\begin{equation}
(1 - \kappa_1) \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 \leq \left(E^{\Phi^\varepsilon} - E^{\Phi}\right) + C \| \phi\|_{H^2_{\omega}} \left(N^{-1/2} + \beta_\varepsilon\right)
\end{equation}

\begin{equation}
+ C \left(\epsilon^{1/2} \| \phi\|_{C^{1/2}_{\omega}} + \left(\sqrt{\beta_\varepsilon} + N^{-1/2}\right) \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}\right),
\end{equation}

for some larger absolute constant \( C > 0 \). To close the proof of the lemma, we want to obtain a lower bound for the left-hand side of (4.189) in terms \( \| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}}^2 \). To this end, we note that
\begin{equation}
1 - p_1 p_2 = p_1 + q_1 - p_1 q_2 = q_1
\end{equation}

so that by the triangle inequality and the fact that \( q_2 \) commutes with \( \nabla_1 \),
\begin{equation}
\| \nabla_1 q_1 \Phi^\varepsilon\|_{L^2_{\omega, N}} \leq \| \nabla_1 (1 - p_1 p_2) \Phi^\varepsilon\|_{L^2_{\omega, N}} + \| \nabla_1 p_1 q_2 \Phi^\varepsilon\|_{L^2_{\omega, N}}.
\end{equation}
Since $\|\nabla_1 p_1\|_{L^2_{x,\epsilon}} \leq \|\nabla \phi\|_{L^2_{x}}$, it follows that
\begin{equation}
\|\nabla_1 p_1 q_2 \Phi^\epsilon\|_{L^2_{x,\epsilon}} \leq \|\nabla \phi\|_{L^2_{x}} \|q_2 \Phi^\epsilon\|_{L^2_{x,\epsilon}} \leq \|\nabla \phi\|_{L^2_{x}} \sqrt{\beta_\epsilon},
\end{equation}
where the ultimate inequality follows from Remark 4.5 and $\alpha_\epsilon \leq \beta_\epsilon$. Therefore,
\begin{equation}
\|\nabla_1 (1 - p_1 p_2) \Phi^\epsilon\|_{L^2_{x,\epsilon}}^2 \geq \left( \|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x,\epsilon}} - \|\nabla \phi\|_{L^2_{x}} \sqrt{\beta_\epsilon} \right)^2 \geq \frac{3\|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x,\epsilon}}^2}{4} - 15 \|\nabla \phi\|_{L^2_{x}}^2 \beta_\epsilon,
\end{equation}
where the ultimate inequality follows from application of Young's inequality for products. Inserting the preceding lower bound into the inequality (4.189) and rearranging, we find that
\begin{equation}
\frac{3}{4} \|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x,\epsilon}}^2 \leq \frac{E^{\Phi^\epsilon} - E^{\Phi}}{1 - \kappa_1} + \frac{C}{1 - \kappa_1} \left( \epsilon^{1/2} \|\phi\|_{C^{1/2}}^2 + \left( \sqrt{\beta_\epsilon} + N^{-1/2} \right) \|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x}} \right) + \frac{C \|\phi\|_{H^2}}{1 - \kappa_1} \left( \beta_\epsilon^{-1/2} + \beta_\epsilon \right) + 15 \|\nabla \phi\|_{L^2_{x}}^2 \beta_\epsilon.
\end{equation}
By Young's inequality for products,
\begin{equation}
\frac{C}{1 - \kappa_1} \|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x,\epsilon}} \left( \sqrt{\beta_\epsilon} + N^{-1/2} \right) \leq \frac{4C^2}{(1 - \kappa_1)^2} \left( \beta_\epsilon^{-1} + \frac{1}{N} \right) + \frac{1}{4} \|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x,\epsilon}}^2,
\end{equation}
The desired conclusion now follows after some algebra. \qed

### 4.4. Proof of Proposition 1.2

We now use the results of the previous subsections to pass to the limit $\epsilon \to 0^+$ and obtain an inequality for $\beta_N$, thereby proving Proposition 1.2.

**Proof of Proposition 1.2** Applying Proposition 4.10 to factors $\|\nabla_1 q_1 \Phi^\epsilon\|_{L^2_{x,\epsilon}}$ appearing in the right-hand side of the inequality given by Proposition 4.9 and using the majorization $\|\phi\|_{H^1_{x}}^2 \leq \|\phi\|_{H^2_{x}}$ together with a bit of algebra, we obtain the point-wise estimate
\begin{equation}
\hat{\beta}_{N,\epsilon} \lesssim \frac{\|\phi\|_{L^\infty_{x}}^2}{N} + \epsilon^{1/2} \|\phi\|_{C^{1/2}}^2 + \frac{\left( 1 + \|\phi\|_{C^{1/2}}^2 \right) \|\phi\|_{H^2_{x}}}{\sqrt{N}} + \frac{1}{N^{1/2}} \|\phi\|_{L^1_{x}}^4 + \frac{\|\phi\|_{L^\infty_{x}}^2}{N^{2/3}} + \frac{N^{2(\sigma-1)/2}}{N^{1/2}} \|\phi\|_{C^{1/2}}^2 \left( E^\Phi_{N} - E^\Phi + \epsilon^{1/2} \|\phi\|_{C^{1/2}}^2 \right).
\end{equation}

We now optimize the choice of $\delta, \sigma \in (0, 1)$. We choose $\delta, \sigma \in (0, 1)$ such that
\begin{equation}
1 - \sigma = \delta \quad \text{and} \quad \sigma = \frac{1 - \sigma}{2},
\end{equation}
which, after some algebra, implies that $(\delta, \sigma) = (2/3, 1/3)$. Inserting this choice of $(\delta, \sigma)$ into the right-hand side of inequality (4.196) and using Sobolev embedding together with the interpolation property of the $H^s$ norm, we obtain
\begin{equation}
\hat{\beta}_{N,\epsilon} \lesssim \frac{\|\phi\|_{H^2_{x}}^2}{\sqrt{N}} + \frac{\|\phi\|_{H^1_{x}}^2}{N^{1/3}} + \|\phi\|_{H^2_{x}} \beta_{N,\epsilon} + \left( 1 + \|\phi\|_{C^{1/2}}^2 \right) \left( E^\Phi_{N} - E^\Phi + \epsilon^{1/2} \|\phi\|_{C^{1/2}}^2 \right).
\end{equation}

Integrating both sides of the preceding inequality over the interval $[0, t]$ and applying the fundamental theorem of calculus, we obtain that
\begin{equation}
\beta_{N,\epsilon}(t) \leq \beta_{N,\epsilon}(0) + C \int_0^t ds \|\phi(s)\|_{H^2_{x}}^2 \beta_{N,\epsilon}(s) ds + C \int_0^t ds \left( \frac{\|\phi(s)\|_{H^2_{x}}^2}{\sqrt{N}} + \frac{\|\phi(s)\|_{H^1_{x}}^2}{N^{1/3}} + \left( 1 + \|\phi(s)\|_{C^{1/2}}^2 \right) \left( E^\Phi_{N} - E^\Phi + \epsilon^{1/2} \|\phi(s)\|_{C^{1/2}}^2 \right) \right),
\end{equation}
where $C > 0$ is an absolute constant. So applying the Gronwall-Bellman inequality, specifically [26, Theorem 1.3.1], we find that
\begin{equation}
\beta_{N,\epsilon}(t) \leq \mathcal{A}_{N,\epsilon}(t) \exp \left( C \int_0^t ds \|\phi(s)\|_{H^2_{x}}^2 \right), \quad \forall t \geq 0,
\end{equation}
where \( \mathfrak{A}_{N, \varepsilon} : [0, \infty) \rightarrow [0, \infty) \) is the function defined by

\[
\mathfrak{A}_{N, \varepsilon}(t) := \beta_{N, \varepsilon}(0) + C \int_0^t ds \left( \frac{\|\phi(s)\|_{L^2}^2}{\sqrt{N}} + \frac{\|\phi(s)\|_{H^1}^2}{N^{1/3}} + (1 + \|\phi(s)\|_{C^{1/2}}^2) \left( E_N^{\Phi^\varepsilon} - E_{\Phi} + \varepsilon^{1/2}\|\phi(s)\|_{C^{1/2}}^2 \right) \right),
\]

for every \( t \geq 0 \).

We now send \( \varepsilon \to 0^+ \). By Lemma 4.8 we have that \( \beta_{\varepsilon, N}(t) \to \beta_N(t) \), as \( \varepsilon \to 0^+ \), uniformly on compact intervals of time. Recalling the definition of the energy per particle \( E_N^{\Phi^\varepsilon} \) and the cubic NLS energy \( E_{\Phi} \) from (4.135) and (4.136), respectively, we see that

\[
E_N^{\Phi^\varepsilon} - E_{\Phi} = \|\nabla \Phi_{N, 0}\|_{L^2(\mathbb{R}^N)}^2 + \frac{\kappa(N-1)}{2N} \langle \Phi_{N, 0} | V_{\varepsilon, 12} \Phi_{N, 0} \rangle_{L^2(\mathbb{R}^N)} - \|\nabla \phi_0\|_{L^2(\mathbb{R})}^2 - \|\phi_0\|_{L^4(\mathbb{R})}^4.
\]

It follows from the proof of Lemma 3.7 that \( V_{\varepsilon, 12} \Phi_{N, 0} \to V_{12} \Phi_{N, 0} \) in \( H^{-1}(\mathbb{R}^N) \) as \( \varepsilon \to 0^+ \). Therefore,

\[
\lim_{\varepsilon \to 0^+} E_N^{\Phi^\varepsilon} - E_{\Phi} = E_N^{\Phi} - E_{\Phi},
\]

where \( E_N^{\Phi} \) is the energy per particle of the solution \( \Phi_N \) to equation (1.3) introduced in (1.17), so that

\[
\mathfrak{A}_{N, \varepsilon}(t) \to \beta_N(0) + C \int_0^t ds \left( \frac{\|\phi(s)\|_{L^2}^2}{\sqrt{N}} + \frac{\|\phi(s)\|_{H^1}^2}{N^{1/3}} + \left( E_N^{\Phi} - E_{\Phi} \right) \left( 1 + \|\phi(s)\|_{C^{1/2}}^2 \right) \right) =: \mathfrak{A}_N(t),
\]

as \( \varepsilon \to 0^+ \), locally uniformly. Using the higher conservation laws of the 1D cubic NLS (see [10, Chapter 1])\footnote{In fact, Koch and Tataru [18] have shown that there exist conserved quantities for the 1D cubic NLS corresponding to the \( H^s \) norm, for any \( s > -\frac{1}{2} \). See also the work [10] of Killip, Visan, and Zhang for a similar result for the case \(-\frac{1}{2} < s < 0 \).} we have that the \( H^k \) norms of \( \phi \) are bounded (up to an absolute constant) by \( \|\phi_0\|_{H^k} \), for any \( k \in \mathbb{N}_0 \). Thus, there exists an absolute constant \( C' \geq C \) such that

\[
\mathfrak{A}_N(t) \leq \beta_N(0) + C' t \left( \frac{\|\phi_0\|_{H^2}^2}{\sqrt{N}} + \frac{\|\phi_0\|_{H^1}^2}{N^{1/3}} + \|\phi_0\|_{H^1}^2 \left( E_N^{\Phi} - E_{\Phi} \right) \right), \quad \forall t \geq 0.
\]

Now taking the limit as \( \varepsilon \to 0^+ \) of the inequality (4.200) and using Lemma 4.8 once more, we obtain that

\[
\beta_N(t) \leq \mathfrak{A}_N(t) \exp \left( C' \|\phi_0\|_{H^2}^2 t \right), \quad \forall t \geq 0.
\]

Comparison with the statement of Proposition 1.2 completes the proof of the proposition. \( \Box \)

5. PROOF OF THEOREM 1.1

In this last section, we show how Proposition 1.2 implies Theorem 1.1. As the implication is well-known, we only sketch the details. We first recall two technical lemmas from [17].

**Lemma 5.1** ([17, Lemma 2.1]). Let \( k \in \mathbb{N} \), and let \( \{\gamma^{(j)}\}_{j=1}^k \) be a sequence of nonnegative, trace-class operators on \( L^2_{sym}(\mathbb{R}^j) \), for \( j \in \{1, \ldots, k\} \), with unit trace and such that

\[
\text{Tr}_{j+1} \gamma^{(j+1)} = \gamma^{(j)}, \quad \forall j \in \{1, \ldots, k - 1\}.
\]

Let \( \varphi \in L^2(\mathbb{R}) \) satisfy \( \|\varphi\|_{L^2} = 1 \). Then

\[
1 - \left< \varphi \otimes \gamma^{(k)} \varphi \otimes k \right> \leq k \left( 1 - \left< \varphi \gamma^{(1)} \varphi \right> \right).
\]

**Lemma 5.2** ([17, Lemma 2.3]). Let \( k \in \mathbb{N} \), and let \( \gamma^{(k)} \) be a nonnegative self-adjoint trace-class operator on \( L^2_{sym}(\mathbb{R}^k) \) with unit trace (i.e. a density matrix). Let \( \varphi \in L^2(\mathbb{R}) \) with \( \|\varphi\|_{L^2} = 1 \). Then

\[
1 - \left< \varphi \otimes \gamma^{(k)} \varphi \otimes k \right> \leq \text{Tr}_{1, \ldots, k} \left| \gamma^{(k)} - |\varphi \otimes \gamma^{(k)} \varphi \otimes k | \right| \leq \sqrt{8 \left( 1 - \left< \varphi \otimes \gamma^{(k)} \varphi \otimes k \right> \right)}.
\]
Proof of Theorem 1.1. For $k \in \{1, \ldots, N\}$, let $\gamma^{(k)}_N = \text{Tr}_{1,\ldots,k} (|\Phi_N\rangle \langle \Phi_N|)$ denote the $k$-particle reduced density matrix of the $N$-body system, where $\Phi_N$ is the solution to the Schrödinger equation (1.4). Let $\phi$ be the solution to the 1D cubic NLS (1.11). It is straightforward from the definition of partial trace that

$$
\langle \phi | \gamma^{(1)}_N | \phi \rangle_{L^2_{\mathbb{R}}} = \langle \Phi_N | ((|\phi\rangle \langle \phi|) \otimes 1^\otimes (N-1)) \Phi_N \rangle_{L^2_{\mathbb{R}}} = \langle \Phi_N | q_1 \Phi_N \rangle_{L^2_{\mathbb{R}}},
$$

which implies by Remark 4.5 that

$$
1 - \langle \phi | \gamma^{(1)}_N | \phi \rangle_{L^2_{\mathbb{R}}} = \langle \Phi_N | q_1 \Phi_N \rangle_{L^2_{\mathbb{R}}} = \alpha_N.
$$

Since $\alpha_N \leq \beta_N$, Proposition 1.2 implies that there is an absolute constant $C > 0$ such that

$$
1 - \langle \phi(t) | \gamma^{(1)}(t) | \phi(t) \rangle \leq (\beta_N(0) + C |t| \left( \frac{\|\phi_0\|_{L^1}^2}{N^{1/3}} + \frac{\|\phi_0\|_{L^2}^2}{N^{1/2}} + \frac{\|\phi_0\|_{L^4}^4}{N} \right) \right),
$$

for every $t \in \mathbb{R}$. Since $\Phi_{N,0} = \phi_0^{\otimes N}$, we see from unpacking Definition 1.1 for $\beta_N$ that

$$
\beta_N(0) = \langle \phi_0^{\otimes N} | n_N(0) \phi_0^{\otimes N} \rangle = \sum_{k=0}^{N} \sqrt{k} \langle \phi_0^{\otimes N} | P_k(0) \phi_0^{\otimes N} \rangle,
$$

where the reader will recall the definition of the projector $P_k$ from (1.1). For $k \in \{1, \ldots, N\}$, the terms in the definition of $P_k(0)$ contain a projector $q_j(0) = (1 - |\phi_0\rangle \langle \phi_0|)_j$, for some $j \in \{1, \ldots, N\}$, which is orthogonal to the state $\phi_0^{\otimes N}$. Thus,

$$
P_k(0) \phi_0^{\otimes N} = 0, \quad \forall k \in \{1, \ldots, N\},
$$

which together with the identity (5.7) implies that $\beta_N(0) = 0$. Additionally, using the normalization $\|\phi_0\|_{L^2} = 1$ and Fubini-Tonelli, we have that

$$
E_N^\phi - E^\phi = \|\nabla_1 \phi_0^{\otimes N}\|_{L^2(\mathbb{R}^N)}^2 + \frac{(N-1)\kappa}{2N} \|\phi_0 \otimes \phi_0^{(N-2)}\|_{L^2(\mathbb{R}^{N-1})}^2 - \|\nabla \phi_0\|_{L^2(\mathbb{R}^{N-1})}^2 - \frac{\kappa}{2} \|\phi_0\|_{L^4(\mathbb{R})}^4.
$$

Now by application of Lemma 5.1, Lemma 5.2, and the $\dot{H}^{1/4}_x \subset L^4_x$ Sobolev embedding, the inequality (5.6) implies that there is an absolute constant $C' \geq C$, such that for any $k \in \mathbb{N}$ fixed,

$$
\text{Tr}_{1,\ldots,k} | \gamma^{(k)}_N (t) - |\phi(t)\rangle \langle \phi(t)\rangle^{\otimes k} | \leq \left( 8kC' |t| \left( \frac{\|\phi_0\|_{L^1}^2}{N^{1/3}} + \frac{\|\phi_0\|_{L^2}^2}{N^{1/2}} + \frac{\|\phi_0\|_{L^4}^4}{N} \right) \right)^{1/2}, \quad \forall t \in \mathbb{R}.
$$

Thus, the proof of Theorem 1.1 is complete. \hfill $\square$

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