Projection-operator optimization of controls of dynamic objects

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Abstract. The principles of application of projection-operator methods of mathematical programming for calculating optimal controls based on “immersion of control problems” in problems of finite-dimensional mathematical programming are formulated. The results obtained illustrate the technique of qualitative study of the stability of systems with optimizing feedback based on the contraction mapping principle of functional analysis. The considered technique allows one to synthesize locally optimal systems with nonlinear control objects. In this case, nonlinear models of objects are taken into account in the right-hand sides of linear manifolds, and the projection operator forms nonlinear optimal controls with feedback taking into account the operators of nonlinearities of the object.

1. Introduction
The article deals with the history, examples and development trends of mathematical operator methods and programming methods. The history of the issue goes back to the results of Corresponding Member USSR Academy of Sciences A.A. Lyapunov [1], devoted to logical schemes of programs. The development of approaches to solving this problem is known. In particular, in [2] the principles of hierarchy were used, in [3, 4] - an operator programming method for solving mathematical problems. In this paper, one of the variants of the projection-operator approach to the synthesis of controls is considered, which is constructive in connection with the quasianalytic form of specifying control operators.

At present, three main groups of synthesis methods are used to synthesize optimal controls for dynamic objects:

- the maximum principle of L.S. Pontryagin [5-8];
- method of dynamic programming by R. Bellman [9-11];
- numerical methods of mathematical programming [11 - 13].

The applied numerical methods of mathematical programming complement the classical dynamic programming method and the maximum principle. However, traditional numerical methods of mathematical programming do not allow for a qualitative analysis of the stability of synthesized systems of local or interval optimal control.

Further, the principles of reduction (“immersion”) of the problems of calculating optimal controls of linear or linearized dynamic objects to the problems of operator mathematical programming are described, which are solved on the basis of orthogonal or non-orthogonal projectors:

1. The principle of immersion of mathematical models of control systems, for example, difference operators of objects, into linear manifolds of mathematical programming constraints.
2. "The principle of operator solutions" for a countable number of control problems based on operator-projection methods of mathematical programming in real time.

3. "The principle of creating new models of systems" based on a programmable environment of linear manifold for the analysis of stability (convergence) of systems.

2. Projection operators of conditional finite-dimensional minimization of linear and quadratic functionals

Solving control synthesis problems based on projection-operator methods of mathematical programming for the conditional minimization of linear or quadratic functionals makes it possible to synthesize control systems in finite-dimensional spaces. These methods provide in general analytical solutions to conditional minimization problems for linear or quadratic functionals on affine-ellipsoidal sets with regularization based on nonsmooth operators, as well as the principle of “boundary solutions" [12–14].

**Theorem 1** [12, 13]. Let a pair of conditional optimization problems have the following formulation: to calculate the control vectors based on the solutions to the finite-dimensional optimization problems

\[
\begin{align*}
    x_* &= \arg\min \{ \varphi(x) : c^T x \mid x \in D = D^0 \cap D^1 \}, \\
    x^* &= \arg\max \{ \varphi(x) : c^T x \mid x \in D = D^0 \cap D^1 \},
\end{align*}
\]

where the admissible sets are defined by the equalities

\[
D^0 = \{ x \mid Ax = b, A \in \mathbb{R}^{m \times n}, \text{rang} A = m \}, D^1 = \{ x \mid x^T Rx \leq r^2, R = R^T > 0 \} \subseteq \mathbb{R}^n.
\]

In order for the pair of problems (1.a) to have analytical solutions, it is necessary and sufficient that the analytical projection-operator solutions of the problems of minimization or maximization of a linear functional at the intersection of a linear manifold and an ellipsoid have the form

\[
\begin{align*}
    x_* &= 0,5 \lambda^0_+ R^{-1} (-P_1 c + 2l_2 P_2 b), \\
    x^* &= 0,5 \lambda^0_- R^{-1} (-P_1 c + 2l_2 P_2 b).
\end{align*}
\]

where scalar parameters are defined by equalities

\[
\lambda_+ = \pm 0,25 c^T P_1 R^{-1} P_1 c (r^2 - 4b^T (P_2^+)^T R^{-1} P_2^+ b)^{-1}.
\]

Constraint compatibility condition: \( D = D^0 \cap D^1 \neq \emptyset \) given by the inequalities

\[
r^2 - 4b^T (P_2^+)^T R^{-1} P_2^+ b > 0,
\]

where \( P_1, P_2 \) — nonorthogonal ("skew") projectors onto a linear manifold and its complement:

\[
P_1 = E_n - A^T (AR^{-1}A^T)^{-1} AR^{-1}, \quad P_2 = A^T (AR^{-1}A^T)^{-1} AR^{-1}.
\]

Thus, the operator of minimization of linear functionals at the intersection of a linear manifold and an ellipsoid in the Euclidean vector space is presented in a form similar to (1.a), (1.b) using non-orthogonal projectors.

The operators for solving the problems of conditional minimization of quadratic functionals at the intersection of a programmable linear manifold in which the object model is "immersed" and the ellipsoid (ball) of inequality constraints also given by the projection operators of conditional minimization of the Euclidean norm, which supplement (1.a), (1.b).

**Theorem 2** [13, 14]. The projection optimization operator, consistent with the problem of calculating optimal controls, determine the solution to the problem of conditional minimization of the Euclidean norm

\[
\begin{align*}
    x_* &= P^* b + P^0 C \left( \frac{2}{\rho} \right)^{1/2} \in \mathbb{R}^n, \\
    x^* &= P^* b + P^0 C \left( \frac{2}{\rho} \right)^{1/2} \in \mathbb{R}^n.
\end{align*}
\]
where the orthogonal projectors \( P^+ = A^T (A A^T)^{-1} \) and \( P^0 = E_n - P^+ A \), parameters (2.b) are equal

\[
\eta = \left( \frac{1}{\rho} \right)^{\frac{1}{2}}, \quad \theta_o = 0.5(\theta_o | - 1 + 1) \in [0,1].
\]

The projection operator (2.b) allows calculating the controls for stabilizing the programmed motion of the control object in the form of a vector using orthogonal projectors. The conditional maximization operator for a problem of type (2.a) is defined in [13, 14].

3. "Immersion" of problems of locally optimal control in problems of operator mathematical programming

The problems of calculating optimal controls in real time for discrete objects described by linear (linearized) difference operators are represented by a countable number of mathematical programming problems. These problems define a countable family of numerical vectors of the type (1.a), (1.b) or (2.a), (2.b), which provide solutions to optimization problems.

The "immersion" of the computation of controls in the "programmable environment" (linear manifold) of the projection optimization operator determines the tasks: compute

\[
\mathcal{A} \mathbf{z}_k = \begin{bmatrix} A \quad 0 \\ \frac{1}{Q^2} \end{bmatrix} \mathbf{y}_k = \begin{bmatrix} b_{x_k} \\ \frac{1}{Q^2} \end{bmatrix} = \begin{bmatrix} H x_k \\ \frac{1}{Q^2} \end{bmatrix} \mathbf{d}_k = \mathbf{b}_z k; \quad \mathcal{A} \mathbf{z}_k \mathbf{y}_k = \begin{bmatrix} A \quad 0 \\ -E_n \end{bmatrix} \mathbf{y}_k \mathbf{d}_k = \mathbf{b}_z k.
\]

From (3) it follows that the linear manifold \( \mathcal{A} \mathbf{z}_k = \mathbf{b}_z k \) defines the model of the object in the form of a difference operator \( x_{k+1} = H x_k + F u_k \), "included" in the linear manifold so that an equivalent specification of the model takes place: \( \mathcal{A} \mathbf{z}_k = \mathbf{b}_z k \).

The solution to problem (3) is calculated using a projection operator of the type (2.b).

**Theorem 3.** The projection operator for the solution of the problem of conditional nonclassical minimization of the norm, defined by relations of type (3), by virtue of the generalization of Theorem 2, has the following form:

\[
\mathbf{x}_* = \mathbf{P}^+ \mathbf{b} + (1 - 2\theta_o) \mathbf{P}^0 \mathbf{C} \mathbf{y}_o, \quad \mathbf{P} = A^T (A A^T)^{-1}, \quad \mathbf{P}^0 = E_n - \mathbf{P}^+ A,
\]

\[
\theta_o = 0.5(\theta_o | - 1 + 1) \in [0,1], \quad \theta_o = 0.5(1 - \eta^{-1}),
\]

\[
\eta = \sigma^{-1} = \left( \frac{1}{\rho} \right)^{\frac{1}{2}}, \quad \alpha = r^2 - b^T (A A^T)^{-1} b, \quad \rho = C^T \mathbf{P}^0 \mathbf{C}.
\]

Further, we consider a technique for analyzing the stability of control systems with projection optimization operators, which makes it possible to formulate requirements for stability to the parameters of a locally optimal control system using methods of functional analysis [13 - 15].

4. Methods for the analysis of stability (convergence) of optimal control systems with difference operators of objects

The method for analyzing sufficient stability conditions for control systems described by difference operators for discrete systems with local minimization of the quadratic quality functional is presented as the following result.

**Theorem 4.** Let the difference operator of a discrete nonlinear dynamic control system have the form
and the properties of the control object and the system are defined below:

1). \( x_{k0} \in D \), where \( D \subset \mathbb{R}^n \) - domain of attraction of a dynamical system.

2). The object is controllable in the sense of R. Kalman and asymptotically stable, i.e.

\[
\| H \|_{2} < 1.
\]

3). Equation (4) satisfies the following properties of nonlinearity operators in phase coordinates:

\( \Phi(x_k) : \mathbb{R}^n \rightarrow \mathbb{R}^n \), and controls \( \Phi(u_k) : \mathbb{R}^m \rightarrow \mathbb{R}^m \), which are bounded in some balls of the phase space of states and satisfy the Lipschitz condition with constants, respectively equal to \( L_x \) and \( L_u \).

4). The phase coordinates and controls satisfy the previously introduced linear manifold and inequality constraint, which is approximated by a ball in the Euclidean space

\[
D_Q = \{ x_{k} \in (s_x(x_k + 1) \& |u_k| \}^T \| x_k \|_2 \leq r_L^2 \}.
\]

Then, to ensure the stability of the control system, it is sufficient to satisfy two related inequality constraints:

on the compression parameter \( s \in \mathbb{R} \) of the difference operator (5):

\[
s = \| H \|_{2} x_{k} + \| F_{l} \|_{2} \| L_{u} \|_2 (\beta_1 L_x + \beta_2 \| p_0 \|_2) < 1,
\]

(6.a)

on the parameter of the static operator feedback locally optimal control systems (5):

\[
| y | < (1 - \| H \|_{2} L_x) \times (\| F_{l} \|_{2} \| L_{u} \|_2) \times (\beta L_x + \| p_0 \|_2 L_p \beta^2)^{-1}.
\]

(6.b)

In inequalities (6.a) and (6.b), the following notation is used for the parameters of the locally optimal system

\[
\| p_0 \|_2 = 1, \beta_1 = L_x \| p^+ \|_2 \| H \|_2, \beta_2 = 2 r L_u L_p \beta_1.
\]

As a result, it was proved that the nonlinear difference operator of the locally optimal control system (5) has a stable equilibrium position \( x_* \).

Thus, the principles of application of projection-operator methods of mathematical programming for calculating optimal controls based on "immersion of control problems" in problems of finite-dimensional mathematical programming are formulated. The results obtained illustrate the technique of qualitative research of stability of systems with optimizing feedback based on the principle of contraction maps of functional analysis [15-17].

As follows from relations (2) - (4), the considered technique allows one to synthesize locally (or intervally) optimal systems with nonlinear control objects. In this case, nonlinear models of objects are taken into account in the right-hand sides of linear manifolds, and the projection operator forms nonlinear optimal controls with feedback taking into account the operators of nonlinearities of the object.

**Conclusion**

At the beginning of the article, a brief analysis of the stages of development of software for information management systems was carried out, reflecting the gradual transition from traditional programming in algorithmic languages to the use of fundamental methods for solving complex problems. A similar trend is realized in the methods of synthesis of optimal controls for dynamic objects. It can be noted that at various stages, various classes of operator methods were used for the mathematical support of control systems.

At present, resources for the development of mathematical support for information management systems have been created in the fundamental achievements of modern science. The use of these achievements can provide ample opportunities for the development of fundamental information management technologies. In addition, it is possible to give an answer to the question of the equivalence
of feedbacks obtained on the basis of numerical and operator optimization methods. It can be noted that the commonality of numerical and operator methods consists in the presence of a common "mathematically programmable environment" in the form of a linear manifold, the universality of which can be effectively used in the synthesis of complex functional laws of feedback control.

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