COMPARISON OF STEKLOV EIGENVALUES ON A
DOMAIN AND LAPLACIAN EIGENVALUES ON ITS
BOUNDARY IN RIEMANNIAN MANIFOLDS

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Abstract. We prove that in Riemannian manifolds the $k$-th Steklov
eigenvalue on a domain and the square root of the $k$-th Laplacian eigen-
value on its boundary can be mutually controlled in terms of the max-
imum principal curvature of the boundary under sectional curvature
conditions. As an application, we derive a Weyl-type upper bound for
Steklov eigenvalues. A Pohozaev-type identity for harmonic functions
on the domain and the min-max variational characterization of both
eigenvalues are important ingredients.

1. Introduction

Let $(M, g)$ be an $(n+1)$-dimensional Riemannian manifold and let $\Omega \subset M$
be a relatively compact domain with smooth boundary $\Sigma = \partial \Omega$. The Steklov
eigenvalue problem, introduced by V. A. Steklov in 1895 (see [14]), is
\begin{align}
\begin{cases}
\Delta_{\Omega} u = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \sigma u, & \text{on } \Sigma,
\end{cases}
\end{align}
where $\nu$ is the outward unit normal along $\Sigma$. Equivalently, the Steklov eigen-
values form the spectrum of the Dirichlet-to-Neumann map $\Lambda : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by
\begin{equation}
\Lambda f = \frac{\partial (Hf)}{\partial \nu}, \quad f \in C^\infty(\Sigma),
\end{equation}
where $Hf$ is the harmonic extension of $f$ to the interior of $\Omega$. The Dirichlet-
to-Neumann map $\Lambda$ is a first order elliptic pseudodifferential operator [17,
pp. 37–38] and its spectrum is nonnegative, discrete and unbounded:
\begin{equation}
0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty.
\end{equation}
There is an extensive literature concerning the Steklov eigenvalue problem.
We refer to the recent survey [6] and the references therein for an account
of this topic.

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fold, Weyl eigenvalue asymptotics.
On the other hand, better-known is the Laplacian eigenvalue problem. Let $\Delta_\Sigma$ denote the Laplace-Beltrami operator acting on smooth functions on the boundary. Then the Laplacian eigenvalue problem is
\[ -\Delta_\Sigma f = \lambda f, \quad \text{on } \Sigma, \] (4)
and it admits an increasing discrete sequence of non-negative eigenvalues
\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty. \] (5)
It is well known that the principal symbol of the Dirichlet-to-Neumann map $\Lambda$ is the square root of the principal symbol of the Laplacian $\Delta_\Sigma$. See e.g. [17, p. 38 and p. 453] and [16]. Consequently, we have
\[ \sigma_j \sim \sqrt{\lambda_j}, \quad \text{as } j \to \infty. \] (6)
Recently, Luigi Provenzano and Joachim Stubbe [15] confirmed this phenomenon explicitly for a $C^2$ domain $\Omega$ in Euclidean spaces. More precisely, they proved that $|\sigma_j - \sqrt{\lambda_j}|$ can be controlled in terms of the geometry of the domain. Our purpose in the present paper is to investigate the same problem for domains in Riemannian manifolds. Our main result can be stated as follows.

**Theorem 1.** Let $(M^{n+1}, g)$ be an $(n + 1)$-dimensional complete, simply connected Riemannian manifold. Denote by $K_M$ its sectional curvature. Let $\Omega \subset M^{n+1}$ be a bounded domain with boundary $\Sigma = \partial \Omega$ of class $C^2$. Denote by $\Pi$ the second fundamental form of $\Sigma$.

1. If $-a \leq K_M \leq 0$ and $\sqrt{a} \leq \Pi \leq \kappa_+$, then
\[ \lambda_j \leq \sigma_j^2 + n\kappa_+ \sigma_j, \quad \sigma_j \leq \frac{\kappa_+^2}{4} + \sqrt{\frac{\kappa_+^2}{4} + \lambda_j}, \quad j \in \mathbb{N}. \] (7)

In particular,
\[ |\sigma_j - \sqrt{\lambda_j}| \leq \max\{n/2, 1\}\kappa_. \] (8)

2. If $0 < K_M \leq a$ and $0 \leq \Pi \leq \kappa_+$, then
\[ \lambda_j \leq \sigma_j^2 + n\sqrt{a + \kappa_+^2} \sigma_j, \quad \sigma_j \leq \sqrt{a + \kappa_+^2} + \sqrt{\frac{a + \kappa_+^2}{4} + \lambda_j}, \quad j \in \mathbb{N}. \] (9)

Likewise,
\[ |\sigma_j - \sqrt{\lambda_j}| \leq \max\{n/2, 1\}\sqrt{a + \kappa_+^2}. \] (10)

**Remark 2.** Case (1) of Theorem 1 includes the result in Euclidean spaces due to [15], and the one in hyperbolic spaces; while Case (2) includes the spherical result, which degenerates to the Euclidean case as $a \to 0^+$. 

**Remark 3.** In hyperbolic case, e.g. $K_M = -1$, the condition $\Pi \geq 1$ is called “horo-convex”, which is a natural convexity. See e.g. [9] where this kind of convexity is essentially required. It is also worth mentioning that, in space forms it is very likely to prove results for even non-convex domains, just as
Here we present the results for convex (horo-convex) domains just for simplicity. In addition, we note that under the conditions in Theorem 1 the domain $\Omega$ has only one boundary component (see e.g. [1][7]).

Remark 4. There are other types of comparison between the Steklov eigenvalue $\sigma_j$ and the Laplacian eigenvalue $\lambda_j$, see e.g. [3][10][19][20]. Therefore any bound for $\lambda_j$ will imply a bound for $\sigma_j$. In particular, by the Weyl-type bound for $\lambda_j$ [2], we obtain the following:

**Corollary 5.** Notations as in Theorem 1. If $-a \leq K_M \leq 0$ and $\sqrt{a} \leq \Pi \leq \kappa_+$, then

$$\sigma_j \leq \kappa_+ + C_n \left( \frac{j}{|\Sigma|} \right)^{\frac{1}{n}}, \quad j \in \mathbb{N};$$  \hfill (11)

if $0 < K_M \leq a$ and $0 \leq \Pi \leq \kappa_+$, then

$$\sigma_j \leq \sqrt{a + \kappa_+^2} + C_n \left( \frac{j}{|\Sigma|} \right)^{\frac{1}{n}}, \quad j \in \mathbb{N},$$  \hfill (12)

where $C_n$ is a constant depending only on $n$.

It is easy to see that the upper bounds in Corollary 5 is compatible with the well-known Weyl asymptotic formula (see e.g. [6])

$$\sigma_j = 2\pi \left( \frac{j}{\omega_n |\Sigma|} \right)^{\frac{1}{n}} + O(1), \quad \text{as } j \to \infty,$$  \hfill (13)

where $\omega_n$ is the volume of the $n$-dimensional Euclidean unit ball. Note that the power $1/n$ in Corollary 5 is optimal.

The proof of Theorem 1 follows Provenzano and Stubbe’s work [15]. First we prove a Pohozaev-type identity for a harmonic function $u$ on $\Omega$ by integrating $\Delta u \cdot \langle F, \nabla u \rangle = 0$ over $\Omega$, where $F$ is any Lipschitz vector field on $\Omega$. Then we choose a suitable $F$ which is supported on a tubular neighbourhood of the boundary $\Sigma$, so as to relate the two boundary integrals $\int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma$ and $\int_{\Sigma} |\nabla \Sigma u|^2 d\sigma$. Finally the min-max characterization for both eigenvalues implies the required result.

The paper is built up as follows. In Section 2 we fix some notations, construct a potential function $\eta(x)$ on the tubular neighbourhood of the boundary in terms of the distance to the boundary and estimate the eigenvalues of its Hessian $\nabla^2 \eta$. Then in Section 3 we establish for general Lipschitz vector fields a Pohozaev-type identity and choose $F = \nabla \eta(x)$ to obtain the equivalence of $\int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma$ and $\int_{\Sigma} |\nabla \Sigma u|^2 d\sigma$. The final Section 4 contains the proofs of Theorem 1 and Corollary 5.

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2. Preliminaries

Let \( (M^{n+1}, g) \) be an \((n+1)\)-dimensional complete, simply connected Riemannian manifold with Levi-Civita connection \( \nabla \). The Riemannian curvature tensor \( R \) is given by

\[
R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z
\]

for any \( X, Y, Z \in \mathfrak{X}(M) \). Let \( p \in M \) and \( u, v \in T_p M \) linearly independent. Then the sectional curvature of a two-plane \( u \wedge v \) at \( p \) is defined by

\[
K_M(u \wedge v) = \frac{\langle R(u, v)u, v \rangle}{||u \wedge v||^2} = \frac{||R(u, v)u, v||^2 - \langle u, v \rangle^2}{||u||^2 ||v||^2}
\]

Assume \( \Omega \subset M \) is a domain with \( C^2 \) boundary \( \Sigma = \partial \Omega \). For any \( x \in \Sigma \), let \( \nu(x) \) be the outward unit normal to \( \Sigma \). Then the second fundamental form \( \Pi \) of \( \Sigma \) at \( x \) is defined by

\[
\Pi(X, Y) := \langle \nabla_X \nu, Y \rangle, \quad X, Y \in T_\Sigma.
\]

Denote by \( \kappa_1(x), \ldots, \kappa_n(x) \) the principal curvatures of \( \Sigma \) at \( x \). Then there exist \( \kappa_- \) and \( \kappa_+ \) in \( \mathbb{R} \) such that

\[
\kappa_- = \inf_{x \in \Sigma} \inf_{i=1,\ldots,n} \kappa_i(x), \quad \kappa_+ = \sup_{x \in \Sigma} \sup_{i=1,\ldots,n} \kappa_i(x),
\]

in which case we also write \( \kappa_- \leq \Pi \leq \kappa_+ \) for short.

For any \( x \in \bar{\Omega} \), set

\[
d_0(x) := \text{dist}(x, \Sigma).
\]

Then we define an \( h \)-tubular neighbourhood \( \omega_h \) of \( \Sigma \) as

\[
\omega_h := \{ x \in \Omega : d_0(x) < h \}.
\]

Since \( \Sigma \) is of class \( C^2 \), every point in \( \omega_h \) has a unique nearest point on \( \Sigma \), provided \( h > 0 \) is sufficiently small. Let \( \bar{h} \) be a real positive number to be chosen such that for any \( h \in (0, \bar{h}) \) any point in \( \omega_h \) has a unique nearest point on \( \Sigma \). In the following we always assume \( h \in (0, \bar{h}) \).

For the Hessian of the distance function \( d_0 \), we recall the following comparison result due to A. Kasue \[11, 12\] (See also \[18\] Theorem 1.2.2).

**Lemma 6.** For constants \( k, \theta \in \mathbb{R} \), let

\[
f(t) := \begin{cases} 
\cos \sqrt{k}t - \frac{\theta}{\sqrt{k}} \sin \sqrt{k}t, & \text{if } k > 0, \\
1 - \theta t, & \text{if } k = 0, \quad t \geq 0, \\
cosh \sqrt{-k}t - \frac{\theta}{\sqrt{-k}} \sinh \sqrt{-k}t, & \text{if } k < 0,
\end{cases}
\]

Let \( f^{-1}(0) \in (0, \infty) \) be the first zero point of \( f \) and \( h^+ \) be the supremum of the width of the tubular neighbourhood in which \( d_0 \) is smooth.
Define \( \rho \) as the eigenvalue of \( \nabla^2 d_0(X, X) \geq \frac{f'}{f}(d_0(x)). \)

**Lemma 8.** Notations as above.

**Remark 7.** The definition of \( \eta(x) \) in the case \( 0 < K_M \leq a \) is such that \( \nabla \eta(x) \) is a conformal vector field for a geodesic ball \( \Omega \) in spheres, which is inspired by [9].

Let \( \{\rho_i(x)\}_{i=1}^{n+1} \) be the eigenvalues of \( \nabla^2 \eta(x) \). Assume that \( \rho_1(x) \leq \rho_2(x) \leq \cdots \leq \rho_{n+1}(x) \). Then we can estimate these eigenvalues as follows.

**Lemma 8.** Notations as above.

1. If \( -a \leq K_M \leq 0 \) and \( \sqrt{a} \leq \kappa_- \leq \Pi \leq \kappa_+ \), then under a suitable basis the eigenvalues of \( \nabla^2 \eta(x) \) are

\[
0 \leq \rho_i(x) \leq 1, \ 1 \leq i \leq n; \ \rho_{n+1}(x) = 1.
\]

2. If \( 0 < K_M \leq a \) and \( 0 \leq \kappa_- \leq \Pi \leq \kappa_+ \), then under a suitable basis the eigenvalues of \( \nabla^2 \eta(x) \) are

\[
0 \leq \rho_i(x) \leq a \cos \sqrt{ad}(x), \ 1 \leq i \leq n; \ \rho_{n+1}(x) = a \cos \sqrt{ad}(x).
\]

**Proof.** (1) First we notice that for any \( X, Y \in T_xM \),

\[
\nabla^2 \eta(x)(X, Y) = \langle \nabla d, X \rangle \langle \nabla d, Y \rangle + d(x) \nabla^2 d(X, Y).
\]

Then there is an eigenvalue \( \rho_{n+1}(x) = 1 \) corresponding to the direction \( \nabla d \). Assume that \( \{E_i\}_{i=1}^n \) of unit length are the directions corresponding to \( \{\rho_i(x)\}_{i=1}^n \). Then for any \( E_i \):

\[
\rho_i(x) = \nabla^2 \eta(x)(E_i, E_i) \leq -d(x) \frac{-\kappa_+}{1 - \kappa_+ d_0(x)}.
\]
Note that by [4, Theorem 3.11] we can choose $\bar{h} = \kappa_+^{-1} > h$. Thus we have

$$\rho_i(x) \leq \frac{d(x)}{\kappa_+^{-1} - d_0(x)} = \frac{d(x)}{h - d_0(x)} \leq 1. \quad (28)$$

Similarly, we have

$$\rho_i = \nabla^2 \eta(E_i, E_i) \geq -d(x) \frac{\sqrt{a} \sinh \sqrt{ad} - \kappa_- \cosh \sqrt{ad}}{\cosh \sqrt{ad} - \frac{\kappa_-}{\sqrt{a}} \sinh \sqrt{ad}} \geq \sqrt{ad}(x) \geq 0,$$

if $a > 0$; if $a = 0$, it is easy to see that the conclusion also holds.

(2) In this case we notice that for any $X, Y \in T_xM$,

$$\nabla^2 \eta(X, Y) = a \cos \sqrt{ad}(x) \langle \nabla^2 \eta, X \rangle \langle \nabla^2 \eta, Y \rangle + \sqrt{a} \sin \sqrt{ad}(x) \nabla^2 d(X, Y).$$

Then there is an eigenvalue $\rho_{n+1}(x) = a \cos \sqrt{ad}(x)$ corresponding to the direction $\nabla d$. Assume that $\{E_i\}_{i=1}^n$ of unit length are the directions corresponding to $\{\rho_i\}_{i=1}^n$. Then for any $E_i$:

$$\rho_i = \nabla^2 \eta(E_i, E_i) \leq -\sqrt{a} \sin \sqrt{ad}(x) \frac{\sqrt{a} \sin \sqrt{ad} - \kappa_- \cos \sqrt{ad}}{\cos \sqrt{ad} - \frac{\kappa_-}{\sqrt{a}} \sin \sqrt{ad}}.$$ 

Note that by [4] Theorems 3.11 and 3.22 we can choose $\bar{h}$ such that $\tan(\sqrt{ah}) = \frac{\sqrt{a}}{\kappa_+}$. Therefore, we obtain:

$$\rho_i(x) \leq a \sin \sqrt{ad}(x) \frac{\frac{\sqrt{a}}{\kappa_+} \tan \sqrt{ad} + 1}{\frac{\sqrt{a}}{\kappa_+} - \tan \sqrt{ad}} = \frac{a \sin \sqrt{ad}(x)}{\tan \sqrt{a}(h - d_0(x))} \leq a \cos \sqrt{ad}(x). \quad (30)$$

Similarly, we have

$$\rho_i(x) \geq -\sqrt{a} \sin \sqrt{ad}(x) \frac{-\kappa_-}{1 - \kappa_- d_0(x)} \geq 0.$$

\[\square\]

3. Pohozaev identity and its consequences

In this section we aim at proving the equivalence of two integrals $\int_{\Sigma} (\frac{\partial u}{\partial \nu})^2 d\sigma$ and $\int_{\Sigma} |\nabla_{\nabla} u|^2 d\sigma$ for a harmonic function $u$ on $\Omega$. First we establish the following Pohozaev identity for $u$. The proof for it is similar to that in [15], except that here we need to take the covariant derivatives with respect to the connection $\nabla$. 

Lemma 9. Let $F \in \Gamma(T\Omega)$ be a Lipschitz vector field. Let $u \in H^2(\Omega)$ with $\Delta u = 0$ in $\Omega$. Then
\[ \int_{\Sigma} \frac{\partial u}{\partial \nu} (F, \nabla u) d\sigma - \frac{1}{2} \int_{\Sigma} |\nabla u|^2 (F, \nu) d\sigma + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \cdot \text{div} F \, dv - \int_{\Omega} \nabla (\nabla u) \cdot F \, dv = 0. \]

Here and in the sequel $H^k(\Omega)$ denotes the standard Sobolev space $W^{k,2}(\Omega)$. 

Proof. Since $u$ is harmonic, there holds $\Delta u \cdot (F, \nabla u) = 0$ in $\Omega$. Then we obtain
\[ 0 = \int_{\Omega} \Delta u \cdot (F, \nabla u) \, dv = \int_{\Sigma} \frac{\partial u}{\partial \nu} (F, \nabla u) d\sigma - \int_{\Omega} (\nabla u, \nabla (F, \nabla u)) \, dv 
\]
\[ = \int_{\Sigma} \frac{\partial u}{\partial \nu} (F, \nabla u) d\sigma - \int_{\Omega} \nabla F (\nabla u, \nabla u) \, dv - \int_{\Omega} \nabla^2 u (F, \nabla u) \, dv. \quad (31) \]

Now take $\{e_i\}_{i=1}^{n+1}$ as an orthonormal basis for $T\Omega$. Then
\[ \int_{\Omega} \nabla^2 u (F, \nabla u) \, dv = \int_{\Omega} u_{ij} F_i u_{ij} \, dv = \int_{\Omega} \left( (u_j F_i u_{ij}) - u_{j,i} u_{ij} - u_j F_i u_{ji} \right) \, dv 
\]
\[ = \int_{\Omega} |\nabla u|^2 (F, \nu) d\sigma - \int_{\Omega} |\nabla u|^2 \cdot \text{div} F \, dv - \int_{\Omega} \nabla^2 u (F, \nabla u) \, dv, \]

which implies
\[ \int_{\Omega} \nabla^2 u (F, \nabla u) \, dv = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 (F, \nu) d\sigma - \frac{1}{2} \int_{\Omega} |\nabla u|^2 \cdot \text{div} F \, dv. \quad (32) \]

Plugging (32) into (31), we complete the proof of the lemma. \hfill \Box

Now we choose
\[ F(x) := \begin{cases} 0, & \text{if } x \in \Omega \setminus \omega_h, \\ \nabla \eta, & \text{if } x \in \omega_h, \end{cases} \quad (33) \]

where we recall that
\[ \eta(x) := \begin{cases} \frac{1}{2} d(x)^2, & \text{if } K_M \leq 0, \\ 1 - \cos \sqrt{ad}(x), & \text{if } 0 < K_M \leq a. \end{cases} \quad (34) \]

Then $F$ is a Lipschitz vector field. If $K_M \leq 0$, we have $F(x) = h \cdot \nu(x)$ for $x \in \Sigma$, and then by Lemma \[\]
\[ 0 = h \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma - h \int_{\Sigma} |\nabla u|^2 d\sigma + \int_{\omega_h} (|\nabla u|^2 \Delta \eta - 2\nabla^2 \eta(\nabla u, \nabla u)) \, dv; \]
while if $0 < K_M \leq a$, we have $F(x) = \sqrt{a} \sin \sqrt{a} h \cdot \nu(x)$ for $x \in \Sigma$, and then again by Lemma 9

$$0 = \sqrt{a} \sin \sqrt{a} h \int_{\Sigma} \left( \frac{\partial u}{\partial v} \right)^2 d\sigma - \sqrt{a} \sin \sqrt{a} h \int_{\Sigma} |\nabla u|^2 d\sigma$$

$$+ \int_{\omega_h} (|\nabla u|^2 \Delta \eta - 2\nabla^2 \eta(\nabla u, \nabla u)) dv.$$

In both cases we need to estimate the last term in the expressions, which is the content of the following lemma.

**Lemma 10.** Let $\Omega$ be a bounded domain in $M^{n+1}$ of class $C^2$ and $u \in H^1(\Omega)$.

1. If $-a \leq K_M \leq 0$ and $\sqrt{a} \leq \kappa_- \leq \Pi \leq \kappa_+$, then

$$- \int_{\Omega} |\nabla u|^2 dv \leq \int_{\omega_h} (|\nabla u|^2 \Delta \eta - 2\nabla^2 \eta(\nabla u, \nabla u)) dv \leq n \int_{\Omega} |\nabla u|^2 dv. \quad (35)$$

2. If $0 < K_M \leq a$ and $0 \leq \kappa_- \leq \Pi \leq \kappa_+$, then

$$- a \int_{\Omega} |\nabla u|^2 dv \leq \int_{\omega_h} (|\nabla u|^2 \Delta \eta - 2\nabla^2 \eta(\nabla u, \nabla u)) dv \leq na \int_{\Omega} |\nabla u|^2 dv. \quad (36)$$

**Proof.** In fact we will first prove a pointwise inequality. Then integrating it yields the result. Let $x \in \omega_h$. Denote by $\xi_i(x)$, $i = 1, \ldots, n + 1$, the normalized eigenvectors of $\nabla^2 \eta(x)$ corresponding to the eigenvalues $\rho_i(x)$, $i = 1, \ldots, n + 1$. Then we can decompose $\nabla u(x)$ as

$$\nabla u(x) = \sum_{i=1}^{n+1} \alpha_i(x) \xi_i(x). \quad (37)$$

Then

$$Q := |\nabla u|^2 \Delta \eta - 2\nabla^2 \eta(\nabla u, \nabla u)$$

$$= |\nabla u|^2 \sum_{i=1}^{n+1} \rho_i(x) - 2 \sum_{i=1}^{n+1} \rho_i(x) \alpha_i(x)^2.$$

Assume $\nabla u(x) \neq 0$. We can normalize $\alpha_i(x)$ to get

$$\tilde{\alpha}_i(x) := \frac{\alpha_i(x)}{\sqrt{\sum_{i=1}^{n+1} \alpha_i(x)^2}} = \frac{\alpha_i(x)}{|\nabla u(x)|}. \quad (38)$$

Therefore we obtain

$$Q = \sum_{i=1}^{n+1} \rho_i(x)(1 - 2\tilde{\alpha}_i(x)^2)|\nabla u(x)|^2. \quad (39)$$

Then direct computation yields (recall $\rho_1(x) \leq \rho_2(x) \leq \cdots \leq \rho_{n+1}(x)$)

$$\sum_{i=1}^{n} \rho_i(x) - \rho_{n+1}(x) \leq \sum_{i=1}^{n+1} \rho_i(x)(1 - 2\tilde{\alpha}_i(x)^2) \leq \sum_{i=2}^{n+1} \rho_i(x) - \rho_1(x). \quad (40)$$
Now in Case (1), for a lower bound, we notice that
\[ \sum_{i=1}^{n+1} \rho_i(x)(1 - 2\tilde{\alpha}_i(x)^2) \geq -1; \]
while for an upper bound, we have
\[ \sum_{i=1}^{n+1} \rho_i(x)(1 - 2\tilde{\alpha}_i(x)^2) \leq n. \]
Consequently,
\[ -|\nabla u|^2 \leq Q \leq n|\nabla u|^2. \] (41)
Then by integrating the inequality we finish the proof of Case (1).

Case (2) can be handled similarly, with further using \( \cos \sqrt{a}d(x) \leq 1. \) So we complete the proof of the lemma. \( \square \)

In the following we only deal with the case \( -a \leq K_M \leq 0 \) and \( \sqrt{a} \leq \kappa_- \leq \Pi \leq \kappa_+ \), since the other case is analogous. The following proposition shows that the two integrals \( \int_{\Sigma} |\nabla_\Sigma u|^2d\sigma \) and \( \int_{\Sigma} (\frac{\partial u}{\partial \nu})^2d\sigma \) are equivalent.

**Proposition 11.** Assume \( -a \leq K_M \leq 0 \) and \( \sqrt{a} \leq \kappa_- \leq \Pi \leq \kappa_+ \). Let \( u \in H^2(\Omega) \) satisfy \( \Delta u = 0 \) in \( \Omega \) and normalized such that \( \int_{\Sigma} u^2d\sigma = 1. \) Then we have
\[ \int_{\Sigma} |\nabla_\Sigma u|^2d\sigma \leq \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma + n\kappa_+ \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma \right)^{\frac{1}{2}}, \] (42)
and
\[ \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma \right)^{\frac{1}{2}} \leq \kappa_+ + \sqrt{\frac{\kappa_+^2}{4} + \int_{\Sigma} |\nabla_\Sigma u|^2d\sigma}. \] (43)

**Proof.** For the first inequality, by Lemma 10 we have
\[ \int_{\Sigma} |\nabla_\Sigma u|^2d\sigma = \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma + \frac{1}{h} \left( \int_{\omega_h} \Delta \eta - 2\nabla^2 \eta(\nabla u, \nabla u) \right)d\nu \]
\[ \leq \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma + \frac{n}{h} \int_{\Omega} |\nabla u|^2d\nu \]
\[ = \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma + \frac{n}{h} \int_{\Sigma} u \frac{\partial u}{\partial \nu}d\sigma \]
\[ \leq \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma + \frac{n}{h} \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2d\sigma \right)^{\frac{1}{2}}, \]
where in the last step we have used the Cauchy-Schwarz inequality. Then letting \( h \to h = \kappa_+^{-1} \) we get the first inequality.
For the second one, we get

\[
\int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma = \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma - \frac{1}{h} \left( \int_{\omega_h} (|\nabla u|^2 \Delta \eta - 2\nabla^2 \eta (\nabla u, \nabla u)) dv \right)
\]

\[
\leq \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma + \frac{1}{h} \int_{\Omega} |\nabla u|^2 dv
\]

\[
= \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma + \frac{1}{h} \int_{\Sigma} u \frac{\partial u}{\partial \nu} d\sigma
\]

\[
\leq \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma + \frac{1}{h} \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma \right)^{\frac{1}{2}},
\]

where again the Cauchy-Schwarz inequality has been used. Solving \( \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma \right)^{\frac{1}{2}} \) from it we have

\[
\left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma \right)^{\frac{1}{2}} \leq \frac{1}{2h} + \sqrt{\frac{1}{4h^2} + \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma}.
\]

Likewise letting \( h \to \bar{h} = \kappa_{-1}^{-1} \) we get the second inequality.

\( \square \)

4. Proofs of main results

Proof of Theorem 1. The proof mainly utilizes Proposition 11 and the min-max variational characterizations of the eigenvalues of problems (1) and (4), i.e.

\[
\sigma_j = \inf_{V \subset H^1(\Omega), \text{dim} V = j + 1} \sup_{0 \neq u \in V, \int_{\Omega} |\nabla u|^2 dv = 1} \int_{\omega_h} |\nabla u|^2 dv,
\]

(45)

for all \( j \geq 0 \), and

\[
\lambda_j = \inf_{V \subset H^1(\Sigma), \text{dim} V = j + 1} \sup_{0 \neq u \in V, \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma = 1} \int_{\Sigma} |\nabla_{\Sigma} u|^2 d\sigma,
\]

(46)

for all \( j \geq 0 \). More precisely, take the case \(-a \leq K_M \leq 0 \) and \( \sqrt{a} \leq \Pi \leq \kappa_{-1} \) for example. Assume that \( \{u_k\}_{k=0}^\infty \subset H^1(\Omega) \) is the sequence of eigenfunctions of the Steklov problem (1) corresponding to the eigenvalues \( \{\sigma_k\}_{k=0}^\infty \). Moreover assume that \( \int_{\Sigma} u_k u_l d\sigma = \delta_{kl} \) for \( k, l \geq 0 \). Then for fixed \( j \geq 0 \), considering \( V = \text{span}\{u_0, u_1, \ldots, u_j\} \), by the min-max variational
characterization \cite{46}, we have

\begin{equation}
\lambda_j \leq \sup_{\sum_{k=0}^{j} c_k^2 = 1} \left( \int_{\Sigma} \left| \nabla_{\Sigma} \left( \sum_{k=0}^{j} c_k u_k \right) \right|^2 d\sigma \right) \leq \sup_{\sum_{k=0}^{j} c_k^2 = 1} \left( \int_{\Sigma} \left( \frac{\partial}{\partial \nu} \left( \sum_{k=0}^{j} c_k u_k \right) \right)^2 d\sigma + n\kappa_+ \left( \int_{\Sigma} \left( \frac{\partial}{\partial \nu} \left( \sum_{k=0}^{j} c_k u_k \right) \right)^2 d\sigma \right)^{\frac{j}{2}} \right) = \sigma_j^2 + n\kappa_+ \sigma_j.
\end{equation}

Here the second inequality is due to Proposition \ref{11}. The other inequality can be proved similarly. See also \cite{15} for more details. So we finish the proof. \hfill \Box

To prove Corollary \ref{5}, we recall the following Weyl-type estimate due to P. Buser \cite{2}. (See also \cite{13} and \cite{8}.)

**Theorem 12** (\cite{2}). Let \((\Sigma, g)\) be a compact Riemannian manifold without boundary of dimension \(n\) such that \(\text{Ric}_g(\Sigma) \geq -(n-1)\kappa^2, \kappa \geq 0\). Then

\begin{equation}
\lambda_j \leq \frac{(n-1)\kappa^2}{4} + c_n \left( \frac{j}{|\Sigma|} \right)^{\frac{2}{n}}, \tag{47}
\end{equation}

where \(c_n > 0\) depends only on \(n\).

Now we are ready to prove Corollary \ref{5}

**Proof of Corollary \ref{5}** By Gauss equation, we have for an orthonormal basis \(\{e_i\}_{i=1}^n \subset T\Sigma\),

\begin{equation}
R_{ijkl}^\Sigma = R_{ijkl} + \Pi_{ik}\Pi_{jl} - \Pi_{il}\Pi_{jk}. \tag{48}
\end{equation}

Then it is easy to see for both cases in Corollary \ref{5} \(\text{Ric}_g(\Sigma) \geq 0\). So \(\lambda_j \leq c_n \left( \frac{j}{|\Sigma|} \right)^{\frac{2}{n}}\). Then using Theorem \ref{11} in the case \(-a \leq K_M \leq 0\) and \(\sqrt{a} \leq \kappa_- \leq K \leq \kappa_+\), we get

\begin{equation}
\sigma_j \leq \kappa_+ + \sqrt{\lambda_j} \leq \kappa_+ + c_{n} \left( \frac{j}{|\Sigma|} \right)^{\frac{1}{n}}, \tag{49}
\end{equation}

which is as claimed. The other case can be dealt with similarly. \hfill \Box
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