RELATIVELY AMENABLE ACTIONS OF THOMPSON’S GROUPS

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Abstract
We investigate the notion of relatively amenable topological action and show that the action of Thompson’s group \( T \) on \( S^1 \) is relatively amenable with respect to Thompson’s group \( F \). We use this to conclude that \( F \) is exact if and only if \( T \) is exact. Moreover, we prove that the groupoid of germs of the action of \( T \) on \( S^1 \) is Borel amenable.

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1. Introduction

In [14], Spielberg showed that every Kirchberg (that is, simple, nuclear, purely infinite and separable) algebra which satisfies the universal coefficient theorem (UCT) admits a Hausdorff groupoid model and hence admits a Cartan subalgebra. Conversely, it was shown by Barlak and Li in [2] that any separable and nuclear \( C^* \)-algebra which has a Cartan subalgebra satisfies the UCT.

Given an étale non-Hausdorff groupoid \( G \), there are dynamical criteria which ensure that the essential \( C^* \)-algebra of \( G \) is a Kirchberg algebra. Since, in general, \( C^\text{ess}(G) \) does not admit any obvious Cartan subalgebra, it seems natural to look at such groupoids as potential sources of counterexamples to the UCT problem (which asks whether every separable nuclear \( C^* \)-algebra satisfies the UCT).

Let \( G(T,S^1) \) be the groupoid of germs of the action of Thompson’s group \( T \) on \( S^1 \). In [7], Kalantar and the author showed that the reduced \( C^* \)-algebra of \( G(T,S^1) \) is not simple, even though \( G(T,S^1) \) is minimal and effective. Moreover, as observed in [7], it follows from results of Kwaśniewski and Meyer [8] that \( C^\text{ess}(G(T,S^1)) \) is purely infinite and simple. In this paper, we show that \( G(T,S^1) \) is Borel amenable. Since, as observed by Renault in [13], the results on nuclearity of groupoid \( C^* \)-algebras from the work of Anantharaman-Delaroche and Renault [1] use only Borel amenability and hold in the non-Hausdorff setting as well, we conclude that \( C^\text{ess}(G(T,S^1)) \) is a
Kirchberg algebra. We leave open the question of whether $C^\ast_{\text{ess}}(G(T, S^I))$ admits a Cartan subalgebra (equivalently, whether it satisfies the UCT).

Let $\Gamma$ be a group acting on a locally compact Hausdorff space $X$ and on a set $K$. In [11], Ozawa studied the existence of nets of continuous approximately equivariant maps $\mu_i: X \rightarrow \text{Prob}(K)$. Clearly, the existence of such maps generalises both topological amenability (in the case $K = \Gamma$) and set-theoretical amenability (in the case that $X$ consists of a single point). If such a property holds in the case that $K$ is a set of left cosets $\Gamma/\Lambda$, we say that $X$ is $(\Gamma, \Lambda)$-amenable. We show that this property generalises the notion of relative co-amenability introduced by Caprace and Monod in [6] (in the more general setting of locally compact groups).

Consider Thompson’s groups $F \leq T$. We show that $S^I$ is $(T, F)$-amenable and use this fact to conclude that $F$ is exact if and only if $T$ is exact.

2. Relatively amenable actions

Given a set $Y$, we consider Prob$(Y) := \{\mu \in \ell^1(Y) : \mu \geq 0, \|\mu\|_1 = 1\}$ equipped with the pointwise convergence topology.

Given a group $\Gamma$ acting by homeomorphisms on a locally compact Hausdorff space $X$, we say that $X$ is a \emph{locally compact $\Gamma$-space}. Given $\Lambda \leq \Gamma$, we say that $X$ is $(\Gamma, \Lambda)$-amenable if there exists a net of continuous functions $\mu_i: X \rightarrow \text{Prob}(\Gamma/\Lambda)$ which is \emph{approximately invariant} in the sense that

$$\lim \sup_{i} \sup_{x \in K} \|s\mu_i(x) - \mu_i(sx)\|_1 = 0$$

for all $s \in \Gamma$ and $K \subset X$ compact. If $\Lambda = \{e\}$, this is the usual notion of (topologically) amenable action on a space $X$ [1, Example 2.2.14(2)]. If $\Lambda$ is co-amenable in $\Gamma$, then any $\Gamma$-space is $(\Gamma, \Lambda)$-amenable.

We will need the following result.

**Proposition 2.1** [4, Proposition 5.2.1]. Let $X$ be a compact $\Gamma$-space which is $(\Gamma, \Lambda)$-amenable for some $\Lambda \leq \Gamma$. If $\Lambda$ is exact, then $\Gamma$ is exact.

Let us now characterise $(\Gamma, \Lambda)$-amenability in the case of a discrete $\Gamma$-space.

**Proposition 2.2.** Let $S$ be a discrete $\Gamma$-space and $\Lambda \leq \Gamma$. The space $S$ is $(\Gamma, \Lambda)$-amenable if and only if there exists a unital positive $\Gamma$-equivariant linear map $\varphi: \ell^\infty(\Gamma/\Lambda) \rightarrow \ell^\infty(S)$.

**Proof.** We identify the space of bounded linear maps $\mathcal{L}(\ell^\infty(\Gamma/\Lambda), \ell^\infty(S))$ with $\ell^\infty(S, \ell^\infty(\Gamma/\Lambda)^*)$. Under this identification, a unital positive $\Gamma$-equivariant map $\varphi \in \mathcal{L}(\ell^\infty(\Gamma/\Lambda), \ell^\infty(S))$ corresponds to a map $\psi: S \rightarrow \ell^\infty(\Gamma/\Lambda)^*$ such that $\psi(s)$ is a state and $\psi(gs) = g(\psi(s))$ for every $s \in S$ and $g \in \Gamma$.

Suppose that $S$ is $(\Gamma, \Lambda)$-amenable and let $\mu_i: S \rightarrow \text{Prob}(\Gamma/\Lambda) \subset \ell^\infty(\Gamma/\Lambda)^*$ be a net of approximately invariant functions. By taking a subnet, we may assume that, for each $s \in S$, $\mu_i(s)$ converges in the weak-*$\ast$ topology to a state $\psi(s) \in \ell^\infty(\Gamma/\Lambda)^*$. Clearly, $\psi: S \rightarrow \ell^\infty(\Gamma/\Lambda)^*$ has the desired properties.
Conversely, suppose that there exists a map $\psi \in \ell^\infty(S, \ell^\infty(\Gamma/\Lambda)^*)$ which is unital, positive and $\Gamma$-equivariant. Since $\ell^1(\Gamma)$ is weak-* dense in $\ell^\infty(\Gamma)^*$, we can find a net $\mu_i : S \to \text{Prob}(\Gamma/\Lambda) \subset \ell^\infty(\Gamma/\Lambda)^*$ such that, for each $s \in S$, $\mu_i(s) \to \psi(s)$ in the weak-* topology. By $\Gamma$-equivariance of $\psi$, the net $g\mu_i(s) - \mu_i(gs)$ converges to zero weakly in $\ell^1(\Gamma/\Lambda)$ for each $g \in \Gamma$ and $s \in S$.

Given $\epsilon > 0$ and finite subsets $E \subset \Gamma$ and $F \subset S$, we claim that there is a function $\mu : S \to \text{Prob}(\Gamma/\Lambda)$ such that $\|g\mu(x) - \mu(gx)\|_1 < \epsilon$ for each $x \in F$ and $g \in E$. From the previous paragraph, it follows that 0 is in the weak closure of the convex set

\[
\bigoplus_{s \in F} \{g\mu(s) - \mu(gs) \mid \mu : S \to \text{Prob}(\Gamma/\Lambda) \subset \ell^\infty(\Gamma/\Lambda)^*\}.
\]

By the Hahn–Banach separation theorem, the claim follows. Thus, $S$ is $(\Gamma, \Lambda)$-amenable.

**Remark 2.3.** Given a group $\Gamma$ and subgroups $\Lambda_1, \Lambda_2 \leq \Gamma$, Proposition 2.2 implies that $\Gamma/\Lambda_2$ is $(\Gamma, \Lambda_1)$-amenable if and only if $\Lambda_1$ is co-amenable to $\Lambda_2$ relative to $\Gamma$ in the sense of [6, Section 7.C].

For completeness, we record the following permanence property. The proof follows the argument in [4, Proposition 5.2.1].

**Proposition 2.4.** Let $X$ be a locally compact $\Gamma$-space and $\Lambda_1 \leq \Lambda_2 \leq \Gamma$ be such that $X$ is $(\Gamma, \Lambda_2)$-amenable and $(\Lambda_2, \Lambda_1)$-amenable. Then $X$ is $(\Gamma, \Lambda_1)$-amenable.

**Proof.** Fix $E \subset \Gamma$ finite, $\epsilon > 0$ and $K \subset X$ compact. Take $\eta : X \to \text{Prob}(\Gamma/\Lambda_2)$ continuous such that $\sup_{s \in K} \|s\eta - \eta^{sx}\| < \epsilon/2$ for all $s \in E$. By arguing as in [4, Lemma 4.3.8], we may assume that there is $F \subset \Gamma/\Lambda_2$ finite such that $\text{supp} \eta^x \subset F$ for all $x \in X$.

Fix a cross-section $\sigma : \Gamma/\Lambda_2 \to \Gamma$. Let

\[
E^* := \{\sigma(sa)^{-1}s\sigma(a) : a \in F, s \in E\} \subset \Lambda_2
\]

and

\[
L := \bigcup_{a \in F} \sigma(a)^{-1}K.
\]

Take $\nu : X \to \text{Prob}(\Lambda_2/\Lambda_1) \subset \text{Prob}(\Gamma/\Lambda_1)$ continuous such that

\[
\max_{s \in E^*} \sup_{y \in L} \|s\nu(y) - \nu(sy)\|_1 < \epsilon/2.
\]

Let

\[
\mu : X \to \text{Prob}(\Gamma/\Lambda_1)
\]

\[
x \mapsto \sum_{a \in F} \eta^x(a)\sigma(a)y^{\sigma(a)^{-1}x}.
\]
Given \( s \in E \) and \( x \in K \),
\[
\begin{align*}
    s\mu(x) &= \sum_{a \in F} \eta^x(a) s\sigma(a) V^{\sigma(a)^{-1}x} \\
    &= \sum_{a \in \Gamma / \Lambda_2} \eta^x(a) s\sigma(sa)^{-1} s\sigma(a) V^{\sigma(a)^{-1}x} \\
    &\approx \frac{\epsilon}{2} \sum_{a \in \Gamma / \Lambda_2} \eta^x(a) s\sigma(sa) V^{\sigma(a)^{-1}sx} \\
    &\approx \frac{\epsilon}{2} \sum_{a \in \Gamma / \Lambda_2} \eta^x(sa) \sigma(a) V^{\sigma(a)^{-1}sx} \\
    &= \sum_{b \in \Gamma / \Lambda_2} \eta^x(b) \sigma(b) V^{\sigma(b)^{-1}sx} = \mu(sx). \quad \square
\end{align*}
\]

**Thompson’s groups.** Thompson’s group \( V \) consists of piecewise linear, right continuous bijections on \([0,1]\) which have finitely many points of nondifferentiability, all being dyadic rationals, and have a derivative which is an integer power of two at each point of differentiability.

Let \( W \) be the set of finite words in the alphabet \([0,1]\). Given \( w \in W \) with length \(|w|\), let \( C(w) := \{(x_n) \in [0,1]^\mathbb{N} : x_{\lfloor l \rfloor} = w\} \). Also let \( \psi : W \to [0,1] \) be the map given by \( \psi(w) := \sum_{n=1}^{|w|} x_n 2^{-n} \) for \( w \in W \). By identifying a set of the form \( C(w) \) with the half-open interval \( [\psi(w), \psi(w) + 2^{-|w|}] \), we can view \( V \) as the group of homeomorphisms of \([0,1]^\mathbb{N}\) consisting of elements \( g \) for which there exist two partitions \( \{C(w_1), \ldots, C(w_n)\} \) and \( \{C_z, \ldots, C_{z_n}\} \) of \([0,1]^\mathbb{N}\) such that \( g(w_i) = z_i x \) for every \( i \) and infinite binary sequence \( x \).

Let \( D := \{(x_n) \in [0,1]^\mathbb{N} : \text{there exists } k \in \mathbb{N} \text{ such that } x_l = 0 \text{ for all } l \geq k\} \). Notice that \( D \) is \( V \)-invariant. Given \( w \in W \), let \( w0^\infty \) be the element of \( D \) obtained by extending \( w \) with infinitely many 0’s.

**Theorem 2.5.** There is a sequence of continuous maps \( \mu_N : [0,1]^\mathbb{N} \to \operatorname{Prob}(D) \) such that
\[
\lim_{N \to \infty} \sup_{x \in [0,1]^\mathbb{N}} \|s\mu_N(x) - \mu_N(sx)\|_1 = 0 \tag{2.1}
\]
for every \( s \in V \).

**Proof.** Given \( N \in \mathbb{N} \), let \( \mu_N : [0,1]^\mathbb{N} \to \operatorname{Prob}(D) \) be defined by
\[
\mu_N(x) := \frac{1}{N} \sum_{j=1}^N \delta_{x_{\lfloor j \rfloor}0^\infty}.
\]
Clearly, for each \( d \in D \) and \( N \in \mathbb{N} \), the map \( x \mapsto \mu_N(x)(d) \) is continuous. We claim that \( (\mu_N) \) satisfies (2.1).

Fix \( s \in V \). There exist two partitions \( \{C(w_1), \ldots, C(w_n)\} \) and \( \{C_z, \ldots, C_{z_n}\} \) of \([0,1]^\mathbb{N}\) such that \( s(w_i) = z_i x \) for every \( i \) and infinite binary sequence \( x \).
Let \( k(s) := \max_i \{|w_i|, |z_i| - |w_i|\} \). Fix \( 1 \leq i \leq n \) and \( x \in \mathcal{C}(w_i) \). Let \( \alpha_i := |z_i| - |w_i| \). Given \( k > k(s) \),

\[
s(x|_{1:k}|0^\infty) = z_i x|_{|w_i|+1:k}|0^\infty = s(x)|_{1,|z_i|}|s(x)|_{|z_i|+1,k+|z_i|-|w_i|}|0^\infty = s(x)|_{1,k+\alpha_i}|0^\infty.
\]

For \( N > 2k(s) \),

\[
\|s\mu_N(x) - \mu_N(sx)\| = \frac{1}{N} \left\| \sum_{j=1}^N \delta_{s(x|_{1:j}|0^\infty)} - \delta_{s(x|_{1:j}|0^\infty)} \right\| \\
\leq \frac{1}{N} \sum_{j=k(s)+1}^{N-k(s)} \delta_{s(x|_{1:j}|0^\infty)} - \sum_{l=k(s)+1+\alpha_i}^{N-k(s)+\alpha_i} \delta_{s(x|_{1:l}|0^\infty)} + \frac{4k(s)}{N} \\
= \frac{4k(s)}{N}.
\]

Thompson’s group \( T \) is the subgroup of \( V \) consisting of elements which have at most one point of discontinuity. By identifying \([0,1)\) with \( S^1 \), the elements of \( T \) can be seen as homeomorphisms on \( S^1 \). Thompson’s group \( F \) is the subgroup of \( T \) which stabilises \( 1 \in S^1 \).

**Corollary 2.6.** The spaces \( \{0,1\}^\mathbb{N} \) and \( S^1 \) are \((T,F)\)-amenable.

**Proof.** Notice that \( T \) acts transitively on \( D \subset \{0,1\}^\mathbb{N} \). Since \( F \) is the stabiliser of \( 0^\infty \in D \), it follows immediately from Theorem 2.5 that \( \{0,1\}^\mathbb{N} \) is \((T,F)\)-amenable.

Let \( \varphi: S^1 \to \{0,1\}^\mathbb{N} \) be the map which, given \( \theta \in [0,1) \), sends \( e^{2\pi i \theta} \) to the binary expansion of \( \theta \). Clearly, \( \varphi \) is \( T \)-equivariant and Borel measurable. Since \( \{0,1\}^\mathbb{N} \) is \((T,F)\)-amenable, composition with \( \varphi \) gives rise to a sequence \( u_n: S^1 \to \text{Prob}(T/F) \) of approximately \( T \)-equivariant pointwise Borel maps (in the sense that for each \( d \in T/F \), the map \( x \mapsto u_n(x)(d) \) is Borel). It follows from [4, Proposition 5.2.1] (or [11, Proposition 11]) that \( S^1 \) is \((T,F)\)-amenable. \( \square \)

The next result follows immediately from Proposition 2.1 and Corollary 2.6.

**Corollary 2.7.** Thompson’s group \( F \) is exact if and only if Thompson’s group \( T \) is exact.

The next result has been recorded in [9, Section 3.2] as a consequence of hyperfiniteness of the equivalence relation of \( T \) on \( S^1 \). It also follows from the fact that stabilisers of amenable actions are amenable, Proposition 2.4 and Corollary 2.6.

**Corollary 2.8** [9]. The following conditions are equivalent:

(i) \( F \) is amenable;
(ii) \( T \curvearrowright \{0,1\}^\mathbb{N} \) is amenable;
(iii) \( T \curvearrowright S^1 \) is amenable.
3. Groupoids of germs

We say that a topological groupoid $G$ is \textit{étale} if its unit space $G^{(0)}$ is Hausdorff and the range and source maps $r, s: G \to G^{(0)}$ are local homeomorphisms. If $G$ is also second countable, then $G$ is said to be \textit{Borel amenable} [13, Definition 2.1] if there exists a sequence $(m_n)_{n \in \mathbb{N}}$, where each $m_n$ is a family $(m_n^g)_{g \in G^{(0)}}$ of probability measures on $r^{-1}(x)$ such that:

(i) for all $n \in \mathbb{N}$, $m_n$ is Borel in the sense that for all bounded Borel functions $f$ on $G$, $x \mapsto \sum_{g \in r^{-1}(x)} f(g)m_n^g(g)$ is Borel;

(ii) for all $g \in G$, we have $\sum_{h \in r^{-1}(r(g))} |m_n^h(g^{-1}h) - m_n^g(h)| \to 0$.

\textbf{Remark 3.1.} Let $G$ be a second countable \textit{étale} groupoid and $A \subset G^{(0)}$ a measurable subset which is invariant in the sense that $r^{-1}(A) = s^{-1}(A)$. In this case, $G_A := s^{-1}(A)$ is a subgroupoid of $G$. If $G$ is Borel amenable, then clearly $G_A$ is also Borel amenable.

Conversely, if $G_A$ and $G_{G^{(0)} \setminus A}$ are Borel amenable, then, since $G = G_A \cup G_{G^{(0)} \setminus A}$, also $G$ is Borel amenable.

Let $\Gamma$ be a group acting on a compact Hausdorff space $X$. Given $x \in X$, let $\Gamma^0_x := \{g \in \Gamma : g$ fixes pointwise a neighbourhood of $x\}$ be the open stabiliser at $x$. Consider the following equivalence relation on $\Gamma \times X$: $(g, x) \sim (h, y)$ if and only if $x = y$ and $g\Gamma^0_x = h\Gamma^0_y$. As a set, the \textit{groupoid of germs} of $\Gamma \rightrightarrows X$ is $G(\Gamma, X) := (\Gamma \times X)/\sim$. The topology on $G(\Gamma, X)$ is the one generated by sets of the form $[g, U] := \{[g, x] : x \in U\}$ for $U \subset X$ open and $g \in \Gamma$. Inversion in $G(\Gamma, X)$ is given by $[g, x]^{-1} = [g^{-1}, gx]$. Two elements $[h, y], [g, x] \in G(\Gamma, X)$ are multiplicable if and only if $y = gx$, in which case $[h, y][g, x] := [hg, x]$. With this structure, $G(\Gamma, X)$ is an \textit{étale} groupoid.

\textbf{Example 3.2.} Let $G_{[2]}$ be the Cuntz groupoid introduced in [12, Definition III.2.1]. Since Thompson’s group $T$ can be seen as a covering subgroup of the topological full group of $G_{[2]}$ [3, Example 3.3], it follows from [10, Proposition 4.10] that $G(T, \{0, 1\}^\mathbb{N}) \simeq G_{[2]}$. Hence, $G(T, \{0, 1\}^\mathbb{N})$ is Borel amenable by [12, Proposition III.2.5].

\textbf{Theorem 3.3.} The groupoid of germs of $T \rightrightarrows S^1$ is Borel amenable.

\textbf{Proof.} Let $X := \{e^{2\pi i \theta} : \theta \in \mathbb{Z}[1/2]\}$ and $Y := S^1 \setminus X$. Notice that $X$ is $T$-invariant. We will show that $G(T, S^1)_X$ and $G(T, S^1)_Y$ are Borel amenable. From Remark 3.1, it will follow that $G(T, S^1)$ is Borel amenable.

Let $\varphi: S^1 \to \{0, 1\}^\mathbb{N}$ be the $T$-equivariant map, which, given $\theta \in [0, 1)$, sends $e^{2\pi i \theta}$ to the binary expansion of $\theta$. Notice that $\varphi|_Y: Y \to \varphi(Y)$ is a homeomorphism. Furthermore, the map

$$\tilde{\varphi}: G(T, S^1)_Y \to G(T, \{0, 1\}^\mathbb{N})_{\varphi(Y)}$$

$$[g, y] \mapsto [g, \varphi(y)]$$

is an isomorphism of topological groupoids. Therefore, $G(T, S^1)_Y$ is Borel amenable by Remark 3.1 and Example 3.2.
Notice that $G(T, S^1)_X$ is a countable set. It follows from [5, Theorem 4.1] that the open stabiliser $T_0^1$ is equal to the commutator subgroup $[F, F]$ and $F/[F, F] \cong \mathbb{Z}^2$. Therefore, $G(T, S^1)_X$ is Borel isomorphic to the transitive discrete groupoid $X \times X \times \mathbb{Z}^2$, which, due to the amenability of the isotropy group, is Borel amenable.

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