The Two-Loop Infrared Structure of Amplitudes with Mixed Gauge Groups

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The infrared structure of (multi-loop) scattering amplitudes is determined entirely by the identities of the external particles participating in the scattering. The two-loop infrared structure of pure QCD amplitudes has been known for some time. By computing the two-loop amplitudes for $f f \rightarrow X$ and $f f \rightarrow V_1 V_2$ scattering in an $SU(N) \times SU(M) \times U(1)$ gauge theory, I determine the anomalous dimensions which govern the infrared structure for any massless two-loop amplitude.

I. INTRODUCTION

The infrared structure of gauge theory amplitudes is governed by a set of anomalous dimensions. The anomalous dimensions at a particular loop-level can be computed directly or extracted from a small number of relatively simple amplitude calculations. Once determined, these anomalous dimensions allow one to predict, for any amplitude, no matter how complex, the complete infrared structure to the given loop level [1, 2]. In QCD, the anomalous dimensions are known completely, in both the massless and massive cases for one and two loop amplitudes, and their properties beyond the two-loop level are being actively studied [3–12]. Because of the many diagrams involved and the complexity of the resulting amplitudes, foreknowledge of the infrared structure is extremely valuable. This knowledge was an important guide for the groundbreaking calculations of two-loop parton scattering amplitudes [13–20].

Precision measurements in particle physics often involve the interaction of more than one gauge group. In particular, at hadron colliders, nominally electroweak processes always involve some interaction with QCD. Precision calculations of such processes, therefore, require the computation of higher-order corrections in mixed gauge groups [21].

In the current letter, I consider a theory with the following structure: There are three gauge interactions, obeying an $SU(N) \times SU(M) \times U(1)$ symmetry. Fermions occur in four different representations: $F_l$, which carry $U(1)$ charge $Q_l$ and are singlets under $SU(N)$ and $SU(M)$; $F_n$, which are in the fundamental representation of $SU(N)$, carry $U(1)$ charge $Q_n$ and are singlets under $SU(M)$; $F_m$, which are in the fundamental representation of $SU(M)$, carry $U(1)$ charge $Q_m$ and are singlets under $SU(N)$; and $F_b$, which are in the
fundamental representation of both $SU(N)$ and $SU(M)$ and carry $U(1)$ charge $Q_b$. Note that this is precisely the structure of the (unbroken) Standard Model, where the $SU(N)$ theory corresponds to QCD, the $SU(M)$ theory to the weak $SU(2)_L$ and the $U(1)$ to the hypercharge interaction. Under this identification, the $F_l$ multiplets correspond to the right-handed leptons, the $F_m$ multiplets to the left-handed leptons, the $F_n$ multiplets to the right-handed quarks and the $F_b$ multiplets to the left-handed quarks.

I will compute the two-loop amplitudes for $f_x f_x \rightarrow X$ (where $X$ is a massive vector boson, neutral under the $SU(N) \times SU(M) \times U(1)$ gauge symmetry) and $f_x f_x \rightarrow V_1 V_2$ for various combinations of fermions and gauge bosons. These calculations will give me redundant extractions of the anomalous dimensions for each particle type in the mixed gauge structure. As a cross-check, I can compare my results for the anomalous dimensions in a pure structure to the known results in the literature. All calculations are performed in the conventional dimensional regularization scheme [22].

II. THE INFRARED STRUCTURE OF QCD AMPLITUDES

The infrared structure of pure QCD interactions is well known. For a general $n$-parton scattering process, I label the set of external partons by $f = \{f_i\}_{i=1,\ldots,n}$. In the formulation of Refs. [2–4], a renormalized amplitude may be factorized into three functions: the jet function $J_f$, which describes the collinear dynamics of the external partons that participate in the collision; the soft function $S_f$, which describes soft exchanges between the external partons; and the hard-scattering function $|H_f\rangle$, which describes the short-distance scattering process,

$$\langle M_{\alpha_s}^{\mu^2, \epsilon}(p_i, \mu_i^2, \alpha_s(\mu^2), \epsilon) | H_f(p_i, \mu_i^2, \alpha_s(\mu^2)) \rangle = J_f(\alpha_s(\mu^2), \epsilon) S_f(p_i, \mu_i^2, \alpha_s(\mu^2), \epsilon) | H_f(p_i, \mu_i^2, \alpha_s(\mu^2)) \rangle.$$ (1)

The notation indicates that $|H_f\rangle$ is a vector and $S_f$ is a matrix in color space [1, 23, 24]. As with any factorization, there is considerable freedom to move terms about from one function to the others. It is convenient [3, 4] to define the jet and soft functions, $J_f$ and $S_f$, so that they contain all of the infrared poles but only contain infrared poles, while all infrared finite terms, including those at higher-order in $\epsilon$, are absorbed into $|H_f\rangle$.

A. The jet function in QCD

The jet function $J_f$ is found to be the product of individual jet functions $J_{f_i}$ for each of the external partons,

$$J_f(\alpha_s(\mu^2), \epsilon) = \prod_{i \in f} J_{f_i}(\alpha_s(\mu^2), \epsilon).$$ (2)
Each individual jet function is naturally defined in terms of the anomalous dimensions of the Sudakov form factor \[2\].

\[
\ln \mathcal{A}(\alpha_s(\mu^2), \varepsilon) = -\left(\frac{\alpha_s}{\pi}\right) \left[ \frac{1}{8\varepsilon^2} \gamma^{(1)}_{K_1} + \frac{1}{4\varepsilon} \gamma^{(1)}_{j_i}(\varepsilon) \right] + \left(\frac{\alpha_s}{\pi}\right)^2 \left\{ \beta_0 \frac{1}{8\varepsilon^2} \left[ \frac{3}{4\varepsilon^2} \gamma^{(1)}_{K_1} + \gamma^{(1)}_{j_i}(\varepsilon) \right] - \frac{1}{8} \left[ \frac{\gamma^{(2)}_{K_1}}{4\varepsilon^2} + \frac{\gamma^{(2)}_{j_i}(\varepsilon)}{\varepsilon} \right] \right\} + \ldots ,
\]

where

\[
\begin{align*}
\gamma^{(1)}_{K_1} &= 2C_i, \quad \gamma^{(2)}_{K_1} = C_iK_i = C_i \left[ C_A \left( \frac{67}{18} - \zeta_2 \right) - \frac{10}{9} T_f N_f \right], \quad C_q \equiv C_F, \quad C_g \equiv C_A, \\
\gamma^{(1)}_q &= \frac{3}{2} C_F + \frac{\varepsilon}{2} C_F (8 - \zeta_2), \quad \gamma^{(1)}_g = 2\beta_0 - \frac{\varepsilon}{2} C_A \zeta_2, \\
\gamma^{(2)}_q &= C_F^2 \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) + C_F C_A \left( \frac{2545}{432} + \frac{11}{12} \zeta_2 - \frac{13}{4} \zeta_3 \right) - C_F T_f N_f \left( \frac{209}{108} + \frac{1}{3} \zeta_2 \right), \\
\gamma^{(2)}_g &= 4\beta_1 + C_A \left( \frac{10}{27} - \frac{11}{12} \zeta_2 - \frac{1}{4} \zeta_3 \right) + C_A T_f N_f \left( \frac{13}{27} + \frac{1}{3} \zeta_2 \right) + \frac{1}{2} C_F T_f N_f, \\
\beta_0 &= \frac{11}{12} C_A - \frac{1}{3} T_f N_f, \quad \beta_1 = \frac{17}{24} C_F^2 - \frac{5}{12} C_A T_f N_f - \frac{1}{4} C_F T_f N_f.
\end{align*}
\]

Although \( \gamma_i \) and \( \gamma_{K_1} \) are defined through the Sudakov form factor, they can be extracted from fixed-order calculations \[25-31\]. \( \gamma_{K_1} \) is the cusp anomalous dimension and represents a pure pole term. The \( \gamma_i \) anomalous dimensions contain terms at higher order in \( \varepsilon \), but I only keep terms in the expansion that contribute poles to \( \ln (\mathcal{A}) \). \( \beta_0 \) and \( \beta_1 \) are the first two coefficients of the QCD \( \beta \)-function, \( C_F = (N_c^2 - 1)/(2N_c) \) denotes the Casimir operator of the fundamental representation of \( SU(N_c) \), while \( C_A = N_c \) denotes the Casimir of the adjoint representation. \( N_f \) is the number of quark flavors and \( T_f = 1/2 \) is the normalization of the QCD charge of the fundamental representation. \( \zeta_n = \sum_{k=1}^{\infty} 1/k^n \) represents the Riemann zeta-function of integer argument \( n \).

### B. The soft function in QCD

The soft function is determined entirely by the soft anomalous dimension matrix \( \Gamma_{S_f} \),

\[
S_f \left( p_i, \mu^2, \alpha_s(\mu^2), \varepsilon \right) = 1 + \frac{1}{2\varepsilon} \left( \frac{\alpha_s}{\pi} \right) \Gamma^{(1)}_{S_f} + \frac{1}{8\varepsilon^2} \left( \frac{\alpha_s}{\pi} \right)^2 \Gamma^{(1)}_{S_f} \times \Gamma^{(1)}_{S_f} - \frac{\beta_0}{4\varepsilon^2} \left( \frac{\alpha_s}{\pi} \right)^2 \Gamma^{(2)}_{S_f} + \frac{1}{4\varepsilon} \left( \frac{\alpha_s}{\pi} \right)^2 \Gamma^{(2)}_{S_f} + \ldots .
\]

In the color-space notation of Refs. \[1,23,24\], the soft anomalous dimension is given by \[3,4\]

\[
\Gamma^{(1)}_{S_f} = \frac{1}{2} \sum_{i \neq j} T_i \cdot T_j \ln \left( \frac{\mu^2}{-s_{ij}} \right), \quad \Gamma^{(2)}_{S_f} = \frac{K}{2} \Gamma^{(1)}_{S_f},
\]

where \( K = C_A \left( \frac{67}{18} - \zeta_2 \right) - 10 T_f N_f / 9 \) is the same constant that relates the one- and two-loop cusp anomalous dimensions. The \( T_i \) are the color generators in the representation of parton \( i \) (multiplied by \( -1 \) for incoming quarks and gluons and outgoing anti-quarks).
III. THE INFRARED STRUCTURE OF MIXED GAUGE GROUPS

When one includes additional gauge symmetries, the dominant effect on the infrared structure is a replication of the QCD structure, with appropriate changes accounting for the size of the gauge group and the Abelian character of the $U(1)$. There are, however, new terms that correspond to intrinsically mixed gauge interactions. It is these mixed terms I am interested in computing in this letter. In reference [21], some of the two-loop anomalous dimensions for QCD $\times$ QED amplitudes were determined, while the forms of others, particularly those involving external gauge bosons, were merely conjectured. The current calculation explicitly determines all of the two-loop mixed anomalous dimensions.

In a theory with the $SU(N) \times SU(M) \times U(1)$ symmetry described above, the jet function for an external parton of species $i$ is

$$\ln J_i (\alpha_N, \alpha_M, \alpha_U, \varepsilon) = -\left(\frac{\alpha_N}{\pi}\right) \left[ \frac{1}{8 \varepsilon^2} \gamma^{(100)}_{K_i} + \frac{1}{4 \varepsilon} \gamma^{(100)}_{J_i} (\varepsilon) \right]$$

$$+ \left(\frac{\alpha_N}{\pi}\right)^2 \left\{ \frac{1}{8 \varepsilon^2} \left[ \frac{3}{4 \varepsilon} \gamma^{(100)}_{K_i} + \gamma^{(100)}_{J_i} (\varepsilon) \right] - \frac{1}{8} \left[ \frac{\gamma^{(200)}_{K_i}}{4 \varepsilon^2} + \frac{\gamma^{(200)}_{J_i} (\varepsilon)}{\varepsilon} \right] \right\}$$

$$- \left(\frac{\alpha_M}{\pi}\right) \left[ \frac{1}{8 \varepsilon^2} \gamma^{(010)}_{K_i} + \frac{1}{4 \varepsilon} \gamma^{(010)}_{J_i} (\varepsilon) \right]$$

$$+ \left(\frac{\alpha_M}{\pi}\right)^2 \left\{ \frac{1}{8 \varepsilon^2} \left[ \frac{3}{4 \varepsilon} \gamma^{(010)}_{K_i} + \gamma^{(010)}_{J_i} (\varepsilon) \right] - \frac{1}{8} \left[ \frac{\gamma^{(020)}_{K_i}}{4 \varepsilon^2} + \frac{\gamma^{(020)}_{J_i} (\varepsilon)}{\varepsilon} \right] \right\}$$

$$- \left(\frac{\alpha_U}{\pi}\right) \left[ \frac{1}{8 \varepsilon^2} \gamma^{(001)}_{K_i} + \frac{1}{4 \varepsilon} \gamma^{(001)}_{J_i} (\varepsilon) \right]$$

$$+ \left(\frac{\alpha_U}{\pi}\right)^2 \left\{ \frac{1}{8 \varepsilon^2} \left[ \frac{3}{4 \varepsilon} \gamma^{(001)}_{K_i} + \gamma^{(001)}_{J_i} (\varepsilon) \right] - \frac{1}{8} \left[ \frac{\gamma^{(002)}_{K_i}}{4 \varepsilon^2} + \frac{\gamma^{(002)}_{J_i} (\varepsilon)}{\varepsilon} \right] \right\}$$

(7)

$$- \left(\frac{\alpha_N}{\pi}\right) \left(\frac{\alpha_M}{\pi}\right) \left[ \frac{1}{8 \varepsilon^2} \gamma^{(110)}_{K_i} + \frac{1}{4 \varepsilon} \gamma^{(110)}_{J_i} (\varepsilon) \right]$$

$$- \left(\frac{\alpha_N}{\pi}\right) \left(\frac{\alpha_U}{\pi}\right) \left[ \frac{1}{8 \varepsilon^2} \gamma^{(101)}_{K_i} + \frac{1}{4 \varepsilon} \gamma^{(101)}_{J_i} (\varepsilon) \right]$$

$$- \left(\frac{\alpha_M}{\pi}\right) \left(\frac{\alpha_U}{\pi}\right) \left[ \frac{1}{8 \varepsilon^2} \gamma^{(011)}_{K_i} + \frac{1}{4 \varepsilon} \gamma^{(011)}_{J_i} (\varepsilon) \right] + \ldots .$$

To deal with the multiplicity of gauge couplings, I have introduced some new notations. $\alpha_N$, $\alpha_M$, $\alpha_U$, are the renormalized gauge couplings of the $SU(N)$, $SU(M)$ and $U(1)$ symmetries respectively. Their $\beta$-function coefficients are indexed by the powers of the gauge couplings (in $N,M,U$ order) that multiply that coefficient. For example,

$$\beta^N (\alpha_N, \alpha_M, \alpha_U) = \mu^2 \frac{d}{d\mu^2} \left(\frac{\alpha_N}{\pi}\right)$$

$$= -\left(\frac{\alpha_N}{\pi}\right)^2 \beta^N_{200} (\alpha_N, \alpha_M, \alpha_U) - \left(\frac{\alpha_N}{\pi}\right)^3 \beta^N_{300} (\alpha_N, \alpha_M, \alpha_U) - \left(\frac{\alpha_N}{\pi}\right)^2 \left(\frac{\alpha_M}{\pi}\right) \beta^N_{210} (\alpha_N, \alpha_M, \alpha_U) - \left(\frac{\alpha_N}{\pi}\right) \left(\frac{\alpha_U}{\pi}\right) \beta^N_{201} (\alpha_N, \alpha_M, \alpha_U) + \ldots .$$

(8)
where
\[
\beta_{200}^N = \frac{11}{12} C_{4N} - \frac{1}{6} (N_{f} + C_{A_{4}} N_{f}), \quad \beta_{300}^N = \frac{17}{24} C_{4N} - \left( \frac{5}{24} C_{A_{4}} + \frac{1}{8} C_{F_{N}} \right) (N_{f} + C_{A_{4}} N_{f}), \\
\beta_{210}^N = -\frac{1}{16} C_{A_{4} N_{f}} C_{F_{M}}, \quad \beta_{201}^N = -\frac{1}{16} \left( \sum_{i=1}^{N_{f} N_{f}} Q_{f_{i}}^{2} + C_{A_{4}} \sum_{i=1}^{N_{f}} Q_{f_{i}}^{2} \right),
\]
(9)

Similarly, the cusp (\(\gamma_{k}\)) and \(\mathcal{G}\) anomalous dimensions are indexed by the powers of the gauge couplings that multiply their leading appearance in the jet functions. The explicit values of all of the anomalous dimensions that appear through two loops are given in Appendix A.

The soft anomalous dimension of a mixed gauge structure, like the log of the jet function, consists of the sum of the soft anomalous dimensions for each of the separate gauge interactions, plus possible terms that are due exclusively to the mixed interaction. The structure of such a mixed soft anomalous dimension would have to involve (at least) pairs of generators from each of the mixing gauge groups. The least complicated of such terms would be of the form
\[
\Gamma^{(110)}_{S_{f}} = \frac{\left(\alpha_{N}\right)}{2} \sum_{i \neq j} \left( T_{N_{i}} \cdot T_{N_{j}} \right) \left( T_{M_{i}} \cdot T_{M_{j}} \right) \ln \left( \frac{\mu^2}{-s_{ij}} \right)
\]
\[
\Gamma^{(101)}_{S_{f}} = \frac{\left(\alpha_{M}\right)}{2} \sum_{i \neq j} \left( T_{N_{i}} \cdot T_{N_{j}} \right) \sum_{Q_{i} Q_{j}} \ln \left( \frac{\mu^2}{-s_{ij}} \right)
\]
(10)

The resulting soft function is
\[
S_{r} = 1 + \left(\frac{\alpha_{N}}{\pi}\right) \frac{1}{2} \Gamma^{(100)}_{S_{f}} + \left(\frac{\alpha_{N}}{\pi}\right)^2 \left( \frac{1}{8 \epsilon^2} \Gamma^{(100)}_{S_{f}} \right) \times \Gamma^{(100)}_{S_{f}} - \frac{\beta^{200}}{4 \epsilon^2} \Gamma^{(200)}_{S_{f}} + \frac{1}{4 \epsilon} \Gamma^{(200)}_{S_{f}}
\]
\[
+ \left(\frac{\alpha_{M}}{\pi}\right) \frac{1}{2 \epsilon} \Gamma^{(010)}_{S_{f}} + \left(\frac{\alpha_{M}}{\pi}\right)^2 \left( \frac{1}{8 \epsilon^2} \Gamma^{(010)}_{S_{f}} \right) \times \Gamma^{(010)}_{S_{f}} - \frac{\beta^{020}}{4 \epsilon^2} \Gamma^{(020)}_{S_{f}} + \frac{1}{4 \epsilon} \Gamma^{(020)}_{S_{f}}
\]
\[
+ \left(\frac{\alpha_{U}}{\pi}\right) \frac{1}{2 \epsilon} \Gamma^{(001)}_{S_{f}} + \left(\frac{\alpha_{U}}{\pi}\right)^2 \left( \frac{1}{8 \epsilon^2} \Gamma^{(001)}_{S_{f}} \right) \times \Gamma^{(010)}_{S_{f}} - \frac{\beta^{002}}{4 \epsilon^2} \Gamma^{(002)}_{S_{f}} + \frac{1}{4 \epsilon} \Gamma^{(002)}_{S_{f}}
\]
(11)

Any new terms that might arise from mixing are parameterized by \(\Gamma^{(110)}_{S_{f}}, \Gamma^{(101)}_{S_{f}}, \text{ and } \Gamma^{(001)}_{S_{f}}\).

IV. EXTRACTING THE ANOMALOUS DIMENSIONS

I will extract the anomalous dimensions be performing a few, relatively simple, explicit calculations. The anomalous dimensions associated with the fermions can be extracted from a Sudakov-type calculation, \(\bar{f} \to f \to X\), where \(X\) is a massive vector boson that is uncharged under the \(SU(N) \times SU(M) \times U(1)\)
symmetry. In this case the infrared structure of the amplitude is uniquely associated with the $f_i$ fermions. Alternatively, one could extract the fermion anomalous dimensions from a set of calculations of the form $\bar{f}_i f_i \rightarrow \bar{f}_x f_x$. For instance, because $f_i$ carries only the $U(1)$ charge, the mixed infrared structure can again be uniquely associated with the $f_x$ fermions. This is the method used in Ref. [21], where the $SU(3) \times U(1)$ anomalous dimensions were determined from the mixed corrections to $\bar{q}q \rightarrow l^+ l^-$. I could extract the boson anomalous dimensions from another Sudakov-type calculation, that of “Higgs” production, $V_i V_i \rightarrow H$. The problem with this calculation is that the scalar must either carry quantum numbers of the vector boson, in which case it contributes to the infrared structure of the amplitude, or it must couple to the vectors through an effective interaction, for which one would need to determine the renormalization properties and Wilson coefficients. I will instead extract the gauge boson anomalous dimensions from calculations of the more complicated amplitudes, $\bar{f}_x f_x \rightarrow V_1 V_2$. The extraction of the boson anomalous dimensions from these amplitudes is made simpler by the fact that I have already determined the fermion anomalous dimensions from Sudakov-type amplitudes.

A. Extracting the fermion anomalous dimensions

The fermion anomalous dimensions are extracted from calculations of the Sudakov-type amplitudes $\bar{f}_x f_x \rightarrow X$. The Feynman diagrams (see Fig. 1) are essentially the same as for two-loop QCD corrections to Drell-Yan production. I generate the Feynman diagrams using QGRAF [32] and implement the Feynman rules and perform algebraic manipulations with FORM [33]. The resulting loop integrals are reduced to master integrals using the integration-by-parts (IBP) method [34] in combination with Laporta’s algorithm [35, 36] as implemented in the program REDUCE2 [37].

There are only four master integrals (see Fig. 2) that contribute to these processes and all can be evaluated in closed form by standard Feynman parameter integrals. The results of the reduction to master integrals and the values of the master integrals are inserted into the FORM program, and the amplitude is evaluated as a Laurent series in the dimensional regularization parameter $\epsilon$. After renormalization, the poles in $\epsilon$ are

![Sample diagrams of $\bar{f}_b f_b \rightarrow X$](image-url)
FIG. 2: Master Integrals for two-loop Sudakov-type amplitudes.

entirely infrared in origin. Most of the infrared terms can be readily associated with pure $SU(N)$, $SU(M)$ or $U(1)$ interactions, or with the overlap of two one-loop terms. Once these terms are accounted for, however, one obtains the two-loop mixed contribution to the fermion anomalous dimensions. I find that there are no mixed cusp anomalous dimensions for the fermions, nor is there a mixed soft anomalous dimension involving only fermions. There are, however, mixed $G$ anomalous dimensions. The results are collected in Appendix A.

B. Extracting the boson anomalous dimensions

The boson anomalous dimensions are extracted from two-loop, two-to-two fermion to di-boson scattering amplitudes. Sample diagrams are shown in Fig. 3. In addition to the four master integrals that contribute to two-loop Sudakov-type diagrams, there are six more that contribute to massless two-to-two scattering (see Fig. 4). In this case the infrared structure of the amplitudes involves the overlap of the infrared structure of the fermions and the two gauge bosons. The soft anomalous dimensions can be identified by their dependence on the logs of kinematic invariants. The gauge boson contributions to the jet functions must be determined by taking different combinations of the external gauge bosons and accounting for the contributions of the already-determined quark anomalous dimensions. As with the quarks, I find that there are no mixed cusp or soft anomalous dimensions at two loops, but that there are non-vanishing mixed $G$ anomalous dimensions.
V. CONCLUSION

I have computed the anomalous dimensions that govern the two-loop infrared structure of mixed gauge interactions. I have presented results for a general $SU(N) \times SU(M) \times U(1)$ gauge structure with fermions that lie in the fundamental representations of both non-Abelian gauge groups ($F_b$), the fundamental representation of one and the singlet representation of the other ($F_n$ and $F_m$), or are singlets under both non-Abelian gauge groups ($F_l$). All fermions are assumed to carry $U(1)$ charges. I note that this is the gauge structure and fermion content of the unbroken Standard Model. However, I have treated the fermions as vector-like, and therefore do not have the chiral structure of the Standard Model. Since the chiral anomaly and anomaly cancellation are ultraviolet issues, they should not affect the infrared structure at all. If one were to make the fermion multiplets chiral, so that $F_b$ and $F_m$ represent the left-handed quarks and leptons, respectively, while $F_n$ and $F_l$ represent the right-handed quarks and leptons, one would only need to weight factors of $N_{f_i}$ by a factor of $1/2$ to account for the chiral projector in the fermion trace. Since I have expressed the anomalous dimensions so that explicit factors of $N_{f_i}$ only appear in the coefficients of the $\beta$-functions, it is only there that one would need to make this change. The rest of the formulæ in Appendix A remain unchanged.

The connection of the current results to applications in QCD $\times$ QED is more direct. Here, I can identify the $SU(N)$ symmetry as QCD, and the $U(1)$ as QED and drop the $SU(M)$ interaction. In this case, I need only $F_n$ and $F_l$ vector-like representations of fermions. One can readily check that the mixed $G$ anomalous dimensions determined here agree with those determined for the quarks in Reference [21].

The results determined here are not surprising and were largely anticipated in Reference [21] by examining the structure of the QCD anomalous dimensions. The argument was that there can be no non-Abelian structure in the mixed terms because the generators of the different gauge groups commute with one another.
and two-loop amplitudes are not sufficiently complicated to allow both mixed interactions and non-Abelian structures of a single gauge group in the same term. Therefore, all factors of $C_A$ that appear in the two-loop QCD anomalous dimensions should be set to zero. Furthermore, it was postulated that all factors of $N_f$ that appear should be associated with coefficients of the $\beta$-functions. However, contributions to two-loop anomalous dimensions that might arise from corrections to one-loop terms would only involve leading coefficients of the $\beta$-functions. Because of the Ward identity, mixing first appears in the $\beta$-functions of gauge couplings at second order. Therefore, corrections that are proportional to leading coefficients of the $\beta$-functions should also be set to zero.

From this, one expects that there will be no mixed cusp or soft anomalous dimensions at two-loops. The factor $K$ which governs the two-loop corrections to both of these terms can be written as a linear combination of $C_A$ and the leading coefficient of the $\beta$-function. Thus, by this reasoning, the only mixed anomalous dimensions that one expects at two-loops are $G$ terms. If I assume that the mixed $G$ anomalous dimensions will have essentially the same form as those of QCD, the only terms that remain are proportional to $C_F^2$ or to $\beta_1$. The minimal possible change that is consistent with the mixed terms is to change each factor of $C_F$ to one of $\{C_F, C_{F_M}, Q^2_f\}$ and to change $\beta_1$ to the appropriate one of $\{\beta_{110}^N, \beta_{110}^N, \beta_{110}^M, \beta_{110}^M, \beta_{110}^U, \beta_{110}^U\}$.

It turns out that these simple transformations give exactly the correct result.

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### Appendix A: Infrared Anomalous Dimensions

#### 1. $\beta$-Functions

The $\beta$-function of the $SU(N)$ coupling is

\[
\beta^N(\alpha_N, \alpha_M, \alpha_U) = \mu^2 \frac{d}{d\mu^2} \left( \frac{\alpha_N}{\pi} \right) = -\left( \frac{\alpha_N}{\pi} \right)^2 \beta_{200}^N - \left( \frac{\alpha_N}{\pi} \right)^3 \beta_{300}^N - \left( \frac{\alpha_M}{\pi} \right)^2 \left( \frac{\alpha_N}{\pi} \right)^2 \beta_{210}^N - \left( \frac{\alpha_U}{\pi} \right)^2 \left( \frac{\alpha_N}{\pi} \right) \beta_{201}^N + \cdots, \tag{A1}
\]

where

\[
\beta_{200}^N = \frac{11}{12} C_A - \frac{1}{3} T_f (N_{f_n} + C_A N_{f_b}), \quad \beta_{300}^N = \frac{17}{24} C_A^2 - \left( \frac{5}{12} C_A + \frac{1}{4} C_F \right) T_f (N_{f_n} + C_A N_{f_b}),
\]

\[
\beta_{210}^N = -\frac{1}{8} C_{F_M} C_A T_f N_{f_b}, \quad \beta_{201}^{N} = -\frac{1}{16} \left( \sum_{i = 1}^{N_{f_n}} Q^2_{f_i} + C_A \sum_{i = 1}^{N_{f_b}} Q^2_{f_i} \right), \tag{A2}
\]
For the SU(M) coupling,

$$\beta^M_1 = \mu^2 \frac{d}{d\mu^2} \left( \frac{\alpha_M}{\pi} \right) = \frac{4}{\pi} \mu^2 \left[ \beta_{020}^M - \frac{1}{3} \beta_{030}^M - \frac{2}{3} \beta_{120}^M - \left( \frac{\alpha_M}{\pi} \right)^2 \beta_{021}^M + \cdots \right],$$

where

$$\beta_{020}^M = \frac{11}{12} C_{AM} - \frac{1}{3} T_f (N_{j_f} + N_{j_b}) \quad \beta_{030}^M = \frac{17}{24} C_{AM} - \left( \frac{5}{12} C_{AM} + \frac{1}{4} C_{FM} \right) T_f (N_{j_f} + N_{j_b}) \quad \beta_{120}^M = -\frac{1}{8} C_{FM} C_{AN} T_f N_{j_b},$$

while for the U(1),

$$\beta^U_1 = \mu^2 \frac{d}{d\mu^2} \left( \frac{\alpha_M}{\pi} \right) = \frac{4}{\pi} \mu^2 \left[ \beta_{020}^U - \frac{1}{3} \beta_{030}^U - \frac{2}{3} \beta_{120}^U - \left( \frac{\alpha_M}{\pi} \right)^2 \beta_{021}^U + \cdots \right],$$

where

$$\beta_{020}^U = -\frac{1}{3} \left( \sum_{i=1}^{N_{j_f}} Q_{j_i}^2 + C_{AM} \sum_{i=1}^{N_{j_f}} Q_{j_i}^2 + C_{AN} C_{AM} \sum_{i=1}^{N_{j_b}} Q_{j_i}^2 \right),$$

$$(A6) \quad \beta_{030}^U = -\frac{1}{4} \left( \sum_{i=1}^{N_{j_f}} Q_{j_i}^4 + C_{AM} \sum_{i=1}^{N_{j_f}} Q_{j_i}^4 + C_{AN} \sum_{i=1}^{N_{j_b}} Q_{j_i}^4 \right),$$

$$\beta_{120}^U = -\frac{1}{8} C_{AN} \left( \sum_{i=1}^{N_{j_f}} Q_{j_i}^2 + C_{AM} \sum_{i=1}^{N_{j_b}} Q_{j_i}^2 \right) \quad \beta_{021}^U = -\frac{1}{8} C_{AM} \left( \sum_{i=1}^{N_{j_f}} Q_{j_i}^2 + C_{AN} \sum_{i=1}^{N_{j_b}} Q_{j_i}^2 \right).$$

2. The Cusp Anomalous Dimensions

$$\gamma_{K_f}^{(100)} = \gamma_{K_f}^{(100)} = 2 C_{F_f} \quad \gamma_{K_{AN}}^{(100)} = 2 C_{AN} \quad \gamma_{K_{f_f}}^{(100)} = \gamma_{K_{f_l}}^{(100)} = \gamma_{K_{AM}}^{(100)} = \gamma_{K_{AU}}^{(100)} = 0\quad (A7)$$

$$\gamma_{K_{f_f}}^{(010)} = \gamma_{K_{f_l}}^{(010)} = 2 C_{F_f} \quad \gamma_{K_{AM}}^{(010)} = 2 C_{AM} \quad \gamma_{K_{f_f}}^{(010)} = \gamma_{K_{f_l}}^{(010)} = \gamma_{K_{AM}}^{(010)} = \gamma_{K_{AN}}^{(010)} = 0 \quad (A7)$$

$$\gamma_{K_{f_l}}^{(001)} = 2 Q_{j_i}^2 \quad \gamma_{K_{f}}^{(001)} = \gamma_{K_{AM}}^{(001)} = \gamma_{K_{AU}}^{(001)} = 0 \quad (A7)$$

$$\gamma_{K_{x}}^{(020)} = \gamma_{K_{x}}^{(020)} = K_{x}^{(020)} = \gamma_{K_{AM}}^{(020)} = \gamma_{K_{AU}}^{(020)} = \gamma_{K_{AN}}^{(020)} \quad (A7)$$

$$\gamma_{K_{x}}^{(002)} = \gamma_{K_{x}}^{(002)} = K_{x}^{(002)} = \gamma_{K_{AM}}^{(002)} = \gamma_{K_{AU}}^{(002)} \quad (A7)$$

$$\gamma_{K_{x}}^{(110)} = \gamma_{K_{x}}^{(101)} = \gamma_{K_{x}}^{(011)} = 0 \quad x \in \{ f_f, f_l, f_b, A_N, A_M, A_U \} \quad (A7)$$
3. The $\mathcal{G}$ Anomalous Dimensions

\[ \mathcal{G}_{f_n}^{(100)} = \mathcal{G}_{f_0}^{(100)} = \frac{3}{2} C_{F_n} + \frac{\varepsilon}{2} C_{F_n} (8 - \zeta_2) \]
\[ \mathcal{G}_{f_m}^{(100)} = \mathcal{G}_{f_0}^{(100)} = \frac{3}{2} C_{F_m} + \frac{\varepsilon}{2} C_{F_m} (8 - \zeta_2) \]
\[ \mathcal{G}_{A_N}^{(100)} = \mathcal{G}_{A_0}^{(100)} = 2 \beta_{200}^N - \frac{\varepsilon}{2} C_{A_N} \zeta_2 \]
\[ \mathcal{G}_{f_{m,l}}^{(100)} = \mathcal{G}_{A_{m,l}}^{(100)} = 0 \]

\[ \mathcal{G}_{f_n}^{(010)} = \mathcal{G}_{f_0}^{(010)} = \frac{3}{2} C_{F_n} + \frac{\varepsilon}{2} C_{F_n} (8 - \zeta_2) \]
\[ \mathcal{G}_{f_m}^{(010)} = \mathcal{G}_{f_0}^{(010)} = \frac{3}{2} C_{F_m} + \frac{\varepsilon}{2} C_{F_m} (8 - \zeta_2) \]
\[ \mathcal{G}_{A_N}^{(010)} = \mathcal{G}_{A_0}^{(010)} = 2 \beta_{200}^M - \frac{\varepsilon}{2} C_{A_M} \zeta_2 \]
\[ \mathcal{G}_{f_{m,l}}^{(010)} = \mathcal{G}_{A_{m,l}}^{(010)} = 0 \]

\[ \mathcal{G}_{f_n}^{(001)} = \frac{3}{2} Q_{f_n}^2 + \frac{\varepsilon}{2} Q_{f_n}^2 (8 - \zeta_2) \quad (x \in \{ l, m, n, b \}) \]
\[ \mathcal{G}_{A_n}^{(001)} = 2 \beta_{002}^U \quad \mathcal{G}_{A_{n,l}}^{(001)} = 0 \]

\[ \mathcal{G}_{f_n}^{(200)} = \mathcal{G}_{f_0}^{(200)} = C_{F_n} \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) + C_{F_n} \beta_{200}^N \left( \frac{209}{36} + \zeta_2 \right) + C_{F_n} C_{A_N} \left( \frac{41}{72} - \frac{13}{4} \zeta_3 \right) \]
\[ \mathcal{G}_{A_N}^{(200)} = 2 \beta_{200}^N + C_{A_N} \beta_{200}^N \left( \frac{19}{18} - \zeta_2 \right) + C_{A_N} \beta_{200}^M \left( \frac{177}{216} + \frac{1}{4} \zeta_3 \right) \]
\[ \mathcal{G}_{f_{n,l}}^{(200)} = \mathcal{G}_{A_{n,l}}^{(200)} = 0 \]

\[ \mathcal{G}_{f_n}^{(020)} = \mathcal{G}_{f_0}^{(020)} = C_{F_n} \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) + C_{F_n} \beta_{020}^M \left( \frac{209}{36} + \zeta_2 \right) + C_{F_n} C_{A_M} \left( \frac{41}{72} - \frac{13}{4} \zeta_3 \right) \]
\[ \mathcal{G}_{A_M}^{(020)} = 2 \beta_{020}^M + C_{A_M} \beta_{020}^M \left( \frac{19}{18} - \zeta_2 \right) + C_{A_M} \beta_{020}^M \left( \frac{177}{216} + \frac{1}{4} \zeta_3 \right) \]
\[ \mathcal{G}_{f_{n,l}}^{(020)} = \mathcal{G}_{A_{n,l}}^{(020)} = 0 \]

\[ \mathcal{G}_{f_n}^{(002)} = \mathcal{G}_{f_0}^{(002)} = Q_{f_n}^2 \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) + Q_{f_n}^2 \beta_{002}^M \left( \frac{209}{36} + \zeta_2 \right) \]
\[ \mathcal{G}_{A_n}^{(002)} = 2 \beta_{002}^U \quad \mathcal{G}_{A_{n,l}}^{(002)} = 0 \]

\[ \mathcal{G}_{f_n}^{(110)} = C_{F_n} \beta_{110} \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) \]
\[ \mathcal{G}_{A_N}^{(110)} = 2 \beta_{110}^N \quad \mathcal{G}_{A_M}^{(110)} = 2 \beta_{110}^M \quad \mathcal{G}_{f_{m,n,l}}^{(110)} = \mathcal{G}_{A_{m,n,l}}^{(110)} = 0 \]

\[ \mathcal{G}_{f_n}^{(011)} = C_{F_n} Q_{f_n}^2 \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) \]
\[ \mathcal{G}_{A_N}^{(011)} = 2 \beta_{011}^N \quad \mathcal{G}_{A_M}^{(011)} = 2 \beta_{011}^M \quad \mathcal{G}_{f_{m,n,l}}^{(011)} = \mathcal{G}_{A_{m,n,l}}^{(011)} = 0 \]

(A8)

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