Generalized Noether symmetry in $f(T)$ gravity

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Abstract

We consider modified teleparallel gravity ($f(T)$ gravity), as a framework to explain the present accelerated expansion of the universe. The matter component is assumed to be cold dark matter. To find the explicit form of the function $f$, we utilize generalized Noether theorem and use generalized vector fields as variational symmetries of the corresponding Lagrangian. We study the cosmological consequences of the obtained results.

1 Introduction

In teleparallel gravity [1], the gravitational interaction is described using torsion, instead of the curvature used in general relativity; and instead of the torsion-less Levi-Civita connection, curvature-less Weitzenböck [2] connection is employed. The gravitational action of this model is given by

$$S_T = \frac{1}{16\pi} \int |e| T d^4x,$$

where $|e| = \det (e^a_\mu)$, and the metric components are related to tetrad via

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu.$$

In [1], the torsion scalar $T$ is

$$T = \frac{1}{2} (K^{\mu\nu} + \delta^\mu_\sigma T^{\sigma\nu} - \delta^\nu_\sigma T^{\sigma\mu}) T_{\mu\nu},$$

where

$$T^{\sigma}_{\mu\nu} = e^a_\sigma \left( \partial_\mu e^a_\nu - \partial_\nu e^a_\mu \right),$$

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is torsion of the Weitzenböck connection, and

\[ K^{\mu \nu \sigma} = \frac{1}{2} (T^{\omega \mu \sigma} + T^{\sigma \mu \nu} - T^{\mu \nu \sigma}) , \quad (5) \]

is the contorsion tensor.

A candidate to describe the present accelerated expansion of our universe [3], is the modified theory of gravity [4]. In modified theories based on general relativity, the gravitational action \( S = \frac{1}{16\pi} \int R\sqrt{-g}d^4x \) is replaced by \( S = \frac{1}{16\pi} \int f(R)\sqrt{-g}d^4x \). Inspired by this model modified teleparallel gravity has been proposed to study the acceleration expansion of our universe [5]. This model is described

\[ S = \frac{1}{16\pi} \int d^4x |e| f(T) + S_m, \quad (6) \]

where \( S_m \) is matter action.

In some papers, the form of \( f(T) \) is suggested and then its cosmological consequences are investigated [6]. In some other papers, the Hubble parameter or the behavior of the effective energy density is specified and then using modified Friedmann equations, the form of \( f(T) \) is obtained [7].

Another way to determine a specific form for \( f(T) \), is to use symmetries of the problem. Noether symmetry provides us a mean to get some insights about the form of \( f(T) \) [8]. This method was used to determine the form of the potential in quintessence model, and to specify the modifications in modified theories of gravity [9]. In the aforementioned papers, the studies were restricted to Noether symmetries corresponding to vector fields whose coefficients were assumed to depend only on time and coordinates (in configuration space). In generalized symmetries, the coefficient functions of the vector fields contain first and higher order time derivatives of coordinates. In this situation, a generalization of Noether theorem can be realized. To see a discussion about this case see [10], where the generalized Noether symmetry and their corresponding generalized vector fields are studied. To have an insight of this subject, consider the motion of a particle under influence of a central force, characterized by the Lagrangian

\[ L = \frac{1}{2} \left[ \dot{r}^2 + r^2 \dot{\theta}^2 \right] + \frac{k}{r}. \quad (7) \]

In this problem, one can find the generalized vector fields [11]

\[ X_1 = r^2 \cos \theta \dot{\theta} \frac{\partial}{\partial r} + \left( \cos \theta \dot{r} - 2r \sin \theta \dot{\theta} \right) \frac{\partial}{\partial \theta} \]
\[ X_2 = r^2 \sin \theta \dot{\theta} \frac{\partial}{\partial r} + \left( \sin \theta \dot{r} + 2r \cos \theta \dot{\theta} \right) \frac{\partial}{\partial \theta}. \quad (8) \]

as variational symmetries of lagrangian related to Runge-Lenz vector [10].
In this manuscript, we consider modified teleparallel gravity ($f(T)$ model), as a framework to explain the present accelerated expansion of the universe and intend to use generalized Noether symmetry corresponding to generalized vector fields to find the explicit form of the function $f(T)$.

The structure of the manuscript is as follows: In the second section after some preliminaries, we introduce the Lagrangian formalism for modified teleparallel gravity. We assume that the dominant matter component is cold dark matter. Based on generalized Noether theorem, we introduce a generalized vector field and obtain a system of partial differential equations for the coefficients and the function $f$. In the third section, the system of differential equations is solved and based on explicit form derived for $f(T)$, and integrals of motion, some cosmological consequences of the obtained results are discussed.

We use units $\hbar = c = G = 1$ through the paper.

2 Generalized Noether symmetry

2.1 Preliminaries

We consider a spatially flat Friedmann-Robertson-Walker (FRW) space time in comoving coordinates

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2).$$  \hspace{1cm} (9)

$a$ is the scale factor and in terms of the Hubble parameter, $H = \frac{\dot{a}}{a}$, the scalar torsion \cite{21} is given by

$$T = -6H^2.$$  \hspace{1cm} (10)

The modified Friedmann equation is

$$H^2 = \frac{8\pi}{3} \left( \frac{\rho_m - \frac{f}{16\pi}}{2f,T} \right) = \frac{8\pi}{3} (\rho_m + \rho_T),$$  \hspace{1cm} (11)

where $f,T = \frac{df}{dT}$, and the effective dark energy density is

$$\rho_T = -\frac{1}{16\pi} (T + f) + \frac{Tf,T}{8\pi}.$$  \hspace{1cm} (12)

The Raychaudhury equation is given by

$$48H^2 f,T T - f,T (4\dot{H} + 12H^2) - f = 16\pi P_m,$$  \hspace{1cm} (13)

where $P_m$ is matter pressure. $\rho_m$ and $P_m$ satisfy The continuity equation

$$\rho_m + 3H(P_m + \rho_m) = 0.$$  \hspace{1cm} (14)
The equation of state parameter (EoS) of the universe can be written as

$$w = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}$$

$$= -1 + \frac{2Tf_{,T} - f}{T(2Tf_{,TT} + f_{,T})}. \quad (15)$$

We assume that the matter component is dominated by cold dark matter characterized by $P_m = 0$, leading to $\rho_m = \rho_{m0}a^{-3}$, where $\rho_{m0}$ is a constant. By adopting a suitable Lagrangian $L$

$$L = a^3(t)(f - f_{,T}T) - 6f_{,T}a(t)a^2(t) - 16\pi\rho_{m0}, \quad (16)$$

in the configuration space $q_i = \{a, T\}$, the Euler Lagrange equation $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$ results in $T = -6H^2$, when

$$f_{,T}T \neq 0. \quad (17)$$

The modified Raychoudhury equation (13) is deduced from

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \quad (18)$$

which is the same as (11) rewritten for cold dark matter.

### 2.2 Generalized symmetry

A generalized vector field is expressed as $X$

$$X = \epsilon(t, q^i, \dot{q}^i, ...) \frac{\partial}{\partial t} + \sum_{j=1}^{n} \phi^j(t, q^i, \dot{q}^i, ...) \frac{\partial}{\partial q^j}, \quad (20)$$

where $\epsilon$ and $\phi^j$ are smooth functions of $t$, $n$ canonical coordinates $q^i$, and their first and higher order time derivatives. $X$ is the generator of a variational symmetry of the Lagrangian if there there exists a continuous function $B$ such that

$$pr^1X(L) + L\frac{de}{dt} = \frac{dB}{dt}, \quad (21)$$

where the first prolongation of $X$ is

$$pr^1X = X + \sum_{j=1}^{n} \left( \phi^j - \dot{q}^j \dot{\epsilon} \right) \frac{\partial}{\partial q^j}. \quad (22)$$
Inspired by [8], and for the sake of simplicity, in the same manner as [10], we restrict ourselves to the case where the coefficients are linear in the velocities

\[ \begin{align*}
X &= \epsilon(a, T) \frac{\partial}{\partial t} + \left( \epsilon_1(a, T) + \alpha(a, T) \dot{a} + \beta(a, T) \dot{T} \right) \frac{\partial}{\partial \dot{a}} \\
&\quad + \left( \epsilon_2(a, T) + \lambda(a, T) \dot{a} + \gamma(a, T) \dot{T} \right) \frac{\partial}{\partial \dot{T}}.
\end{align*} \tag{23} \]

The Noether integral is

\[ \begin{align*}
P &= B - \epsilon \mathcal{L} - \left( \epsilon_1 + \alpha \dot{a} + \beta \dot{T} \right) \frac{\partial \mathcal{L}}{\partial \dot{a}} - \left( \epsilon_2 + \lambda \dot{a} + \gamma \dot{T} \right) \frac{\partial \mathcal{L}}{\partial \dot{T}} \\
&\quad + \epsilon \alpha \frac{\partial \mathcal{L}}{\partial \dot{a}} + \epsilon \beta \frac{\partial \mathcal{L}}{\partial \dot{T}}. \tag{24} \end{align*} \]

By putting (16) and (23) in (21), and equating expressions containing the same order of time derivatives of \( a \) and \( T \) in both sides, we get:

\[ B = p(a, T) - 6a \dot{f},T \alpha \dot{a}^2, \tag{25} \]

\[ \beta = 0, \tag{26} \]

where \( p(a, T) \) is a continuous function, and also a system of differential equations:

\[ \begin{align*}
\lambda f,_{TT} + \alpha,_{a} f,_{T} - \epsilon,_{a} f,_{T} &= 0, \\
3a^2 \alpha (f,_{TTT} - f) + Ta^3 \lambda f,_{TTT} + \epsilon,_{a} a^3 (f,_{TT} - f) + \rho_{m0} \epsilon,_{a} + p,_{a} &= 0, \\
\gamma f,_{TT} - f,_{T} \epsilon,_{T} + f,_{T} \alpha,_{T} - \alpha f,_{TTT} &= 0, \\
T a^3 \gamma f,_{TTT} + a^3 \epsilon,_{T} (T f,_{TT} - f) + \rho_{m0} \epsilon,_{T} + p,_{T} &= 0, \\
\epsilon_1 f,_{T} + a \epsilon_2 f,_{TT} + 2a f,_{T} \epsilon_1,_{a} &= 0, \\
3a^2 (f,_{TTT} - f) \epsilon_1 + a^3 f,_{TTT} \epsilon_2 &= 0, \\
12a f,_{T} \epsilon_1,_{T} &= 0. \tag{27} \end{align*} \]

The integral of motion is

\[ \begin{align*}
P &= p - 12\epsilon_1 a \dot{a} f,_{T} + a^3 \epsilon (f,_{TTT} - f) - 6a \dot{a}^2 (\alpha - \epsilon) f,_{T} + 16\pi \epsilon \rho_{m0}. \tag{28} \end{align*} \]

### 3 Solutions

\( \alpha, \beta, \lambda \) and \( \gamma \) do not involve in the three last equations in (27). These three equations are sufficient to determine the form of \( f(T) \) if either of \( \epsilon_1 \) and \( \epsilon_2 \) is non zero. In this situation one obtains a power law expression for \( f(T) = \mu T^n \), which using (19) leads to \( a(t) \propto t^{2n} \) whose the cosmological consequences are discussed in [8]. In this case by solving (27), an additional
Noether symmetry corresponding to the generalized vector field specified by

\[ \alpha = F(y), \quad \lambda = \frac{3T^2 F'(y) a^{3-n}}{n(n-1)}, \quad \gamma = F(y) - \frac{T a^3 F'(y)}{n-1}, \quad (29) \]

where \( y = T a^3 \) and \( F \) is an arbitrary continuous function, is attained.

For \( \epsilon_1 = \epsilon_2 = 0 \), obtaining an analytical solution for the system \((27)\) is very complicated, if not impossible. So to go further, one should examine specific cases. Here we consider solutions characterized by \( \epsilon = 0 \). By this simplification the following specific solutions for \( f(T) \) are derived (using the Maple 13 PDEtools package):\n
\[ f(T) = C_1 T + C_2 \]

\[ f(T) = C_1 \sqrt{-T} + C_2, \quad (30) \]

which are not acceptable because the first one is not consistent with \( f, T_T \neq 0 \) used in our procedure, and the second one when inserted in \((19)\) gives \( \rho_m = \frac{C_2}{16 \pi} \), which does not describe cold dark matter. The third specific solution is

\[ f(T) = \pm \sqrt{2 C_1 T + 2 C_2} \]

\[ \gamma = 0 \]

\[ p = \pm \frac{a^6 C_3}{\sqrt{2}} + C_4 \]

\[ \alpha = \frac{C_3 a^3 \sqrt{C_1 T + C_2}}{\sqrt{C_1 T + C_2}} \]

\[ \lambda = \frac{6 a^2 C_3 \sqrt{C_1 T + C_2}}{C_1}, \quad (31) \]

with the condition

\[ C_1 T + C_2 \geq 0, \quad (32) \]

implying that \( H \) is real. For this solution, the Noether integral is

\[ \mathcal{P} = p + 6a \dot{a}^2 \alpha f, T, \quad (33) \]

which can be rewritten as

\[ \pm \frac{C_2}{C_2 - 6 C_1 H^2} = \frac{d}{a^6}, \quad (34) \]

where the constant \( d \) is defined by \( d := \frac{(\mathcal{P} - C_4) \sqrt{2}}{C_3} \).

By using \((19)\) and after some computations we obtain \( d = \frac{\tilde{\rho}_m}{2 c_2} \), where \( \tilde{\rho}_m^0 = 16 \pi \rho_m^0 \), leading to

\[ \frac{c_2}{c_2 - 6 c_1 h^2} = \frac{1}{2 c_2 a^6}, \quad (35) \]
Dimensionless parameters $c_1$ and $c_2$ are defined through $C_2 = c_2 \hat{\rho}_{m0}$; $C_1 = \hat{\rho}_{m0} c_1$. We use $\tau = t \sqrt{\hat{\rho}_{m0}}$ instead of the cosmic time and dimensionless Hubble parameter $h = \frac{1}{a(\tau)} \frac{da(\tau)}{d\tau}$ is considered. (32) may be rewritten as

$$c_2 - 6c_1 h^2 \geq 0,$$  \hfill (36)

and (35) implies

$$\frac{c_2}{2c_1} - \frac{c_2^2}{c_1} a^6 \geq 0.$$  \hfill (37)

To see whether the phantom divide line ($w = -1$) crossing is allowed in this model, we must compute $\frac{dh}{d\tau}$. Using (35), after some calculations, we obtain

$$\frac{dh}{d\tau} = - \frac{c_2^2 a^6}{c_1}.$$  \hfill (38)

Therefore the transition from $\frac{dh}{d\tau} < 0$ to $\frac{dh}{d\tau} > 0$ and vice versa are not possible. In the following we consider $\frac{dh}{d\tau} < 0$ which corresponds to $c_1 > 0$. (36) leads to $c_2 > 0$. To study the cosmological consequences of this model we try to solve the differential equation (35), with the constraint \{\(c_1 > 0, c_2 > 0\)\}. We write (35) as

$$\left( \frac{da(\tau)}{d\tau} \right)^2 = \frac{1}{6} \left( \frac{c_2}{c_1} - 2 \frac{c_2^2 a^6(\tau)}{c_1} \right) a^2(\tau),$$  \hfill (39)

whose solution is given by

$$a^6(\tau) = \frac{24c_1 c_2 e^{\pm \sqrt{\frac{6c_2}{c_1} (C-\tau)}}}{\left( 12c_1 c_2^2 + e^{\pm \sqrt{\frac{6c_2}{c_1} (C-\tau)}} \right)^2},$$  \hfill (40)

Where $C$ is a constant. Using $h = \frac{1}{6a^6(\tau)} \frac{da(\tau)}{d\tau}$ one obtains

$$h = \left( \pm \sqrt{\frac{c_2}{6c_1}} \right) \left( \frac{-12c_1 c_2^2 + e^{\pm \sqrt{\frac{6c_2}{c_1} (C-\tau)}}}{12c_1 c_2^2 + e^{\pm \sqrt{\frac{6c_2}{c_1} (C-\tau)}}} \right).$$  \hfill (41)

As $h$ is a decreasing function of $\tau$, we expect that $h$ becomes negative after some time, dubbed as turnaround time (see fig(1) and fig(2)). This turnaround occurs at

$$\tau = C \mp \sqrt{\frac{c_1}{6c_2} \ln(12c_1 c_2^2)}.$$  \hfill (42)

As crossing the phantom divide line is not permitted, the Hubble parameter continues to decrease and reach at

$$h(\tau \rightarrow \infty) = -\sqrt{\frac{c_2}{6c_1}},$$  \hfill (43)
asymptotically.

To study the acceleration of the universe, we consider

\[
S := \frac{dh}{d\tau} + h^2, \tag{44}
\]

which has the same sign as \( \ddot{a} \). \( S = 0 \) occurs at the times \( \tau_1 \) and \( \tau_2 \) specified by

\[
\tau_1 = C \mp \sqrt{\frac{c_1}{6c_2}} \ln(167.14c_1^2) \\
\tau_2 = C \mp \sqrt{\frac{c_1}{6c_2}} \ln(0.859c_1^2). \tag{45}
\]

To elucidate the behavior of \( S \), we compute \( \frac{dS}{d\tau} \) at these points. The result is

\[
\frac{dS}{d\tau}(\tau_1) = \mp 0.35354 \left( \frac{c_2}{c_1} \right)^{\frac{3}{2}}, \\
\frac{dS}{d\tau}(\tau_2) = \pm 0.35285 \left( \frac{c_2}{c_1} \right)^{\frac{3}{2}}, \tag{46}
\]

which shows that in this model the universe has a positive acceleration (in the sense that \( \ddot{a} > 0 \)) for \( \tau < \min.\{\tau_1, \tau_2\} \) and \( \tau > \max.\{\tau_1, \tau_2\} \). For \( \min.\{\tau_1, \tau_2\} < \tau < \max.\{\tau_1, \tau_2\} \) we have \( \ddot{a} < 0 \).

So far, as we have not fixed the values of \( c_1, c_2, \) and \( C \), our discussions and results were general. To be more specific, we must put some physical conditions on these parameters. To do so, we consider the effective EoS parameter of dark sector, \( w_T = \frac{P_T}{\rho_T} \), where \( P_T \) and \( \rho_T \) are the effective pressure and energy density attributed to modified teleparallel gravity respectively [6]. We also consider the relative densities defined by \( \Omega_m = \frac{8\pi\rho_m}{3H^2} \), and \( \Omega_T = \frac{8\pi\rho_T}{3H^2} \).

The relative dark matter density can be written as

\[
\Omega_m = \frac{a^{-3}}{6h^2}. \tag{47}
\]

Using \( w = \Omega_T w_T \) [8], and \( \Omega_m + \Omega_T \approx 1 \), we obtain

\[
w_T = \frac{-4\frac{dh}{d\tau} - 6h^2}{6h^2 - a^{-3}}. \tag{48}
\]

Equations (47) and (48) provide us a tool to determine the parameters of the model. We take the present time to be at \( \tau = 0 \) and take [12]

\[
\Omega_m(\tau = 0) \approx 0.25 \\
w_T(\tau = 0) \approx -1, \tag{49}
\]

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as initial conditions (i.e. conditions respected by the model in a specific time, e.g. the present epoch). By solving (49), \(c_1\) and \(c_2\) are obtained in terms of \(C\) as

\[
\begin{align*}
    c_1 &= \frac{0.04572 W(-1.5951 |C|)^2}{C^2} \\
    c_2 &= \frac{0.7621 W(-1.5951 |C|)^4}{C^4},
\end{align*}
\]

or

\[
\begin{align*}
    c_1 &= \frac{0.04572 W(0.788 |C|)^2}{C^2} \\
    c_2 &= \frac{0.7621 W(0.788 |C|)^4}{C^4},
\end{align*}
\]

where \(W\) denotes Lambert \(W\) function. Our choices for \(C\) may be restricted by adopting further conditions. One of these conditions may be that the dark sector of the universe encounters a phase transition from \(w_T > -1\) to \(w_T < -1\) in the present era, implying

\[
\frac{dw_T}{d\tau}(\tau = 0) < 0.
\]

We must also take into account \(h(\tau = 0) > 0\) and \(S(\tau = 0) > 0\), corresponding to the fact that the present expansion of the universe has a positive acceleration.

As an illustration of our results, in fig. (1) (fig. (2)), \(S\) and \(h\) are depicted in terms of \(\tau\), for \(\{c_1 = 0.7435, c_2 = 201.5087, C = -0.23\}\) and for solutions with negative (positive) sign in (40). Fig. (2), as is related to a contacting universe at \(\tau = 0\), does not represent our actual universe. In fig. (1), we have \(\rho_{m0} \approx 0.0005 H^2(\tau = 0)\), where \(H(\tau = 0) = \frac{7^{1/2}}{\sqrt{10000}} Mpc^{-1}\) is the value of the Hubble parameter at the present era [12]. As we have not taken \(a(0) = 1\), \(\rho_{m0}\) must not be confused with the value of matter density at the present time.
Figure 1: $S$ (line) and $h$ (points) in terms of dimensionless time $\tau$, for $\{c_1 = 0.7435, c_2 = 201.5087, C = -0.23\}$ corresponding to solution with negative sign in [10].

Figure 2: $S$ (line) and $h$ (points) in terms of dimensionless time $\tau$, for $\{c_1 = 0.7435, c_2 = 201.5087, C = -0.23\}$ corresponding to solution with positive sign in [10].
Note that, the qualitative behaviors of $h$ and $S$ in these figures are consistent with was discussed after eq. (46), and do not depend on the specific values of the parameters. But as we have shown, astrophysical data, by fixing the initial conditions, put some quantitative constraints on these behaviors.

The explicit form of crossing the line $w_T = -1$, in the present era, is depicted in fig. (3).

Figure 3: $w_T$ in terms of dimensionless time $\tau$, for $\{c_1 = 0.7435, c_2 = 201.5087, C = -0.23\}$ corresponding to solution with negative sign in (40).

This figure shows that, by a suitable choice of the parameter $C$, the dark sector can exhibit a transition from $w_T > -1$ to $w_T < -1$ in the present era.

4 Conclusion

The modified teleparallel gravity ($f(T)$ model of gravity) is a framework to study the present accelerated expansion of the universe. To determine the form of $f(T)$, one can use the Noether symmetry. This method was vastly used in the literature to specify the form of modifications in modify theories of gravity as well as to determine the form of the scalar field potential in dark energy models. In [8], where the coefficients of vector fields are taken to be functions of coordinates, only one form for $f(T)$ is deduced: $f(T) = \mu T^n$, where $\mu$ and $n$ are two real numbers. The scale factor is $a(t) \propto t^{2n/3}$ and consequently the Hubble parameter is obtained as $H(t) = \frac{2n}{3a}$. Therefore the
EoS parameter of the universe is a constant $w = \frac{1}{n} - 1$, so $n > \frac{3}{2}$ results in an eternal acceleration for the expanding universe.

In this paper, this approach was extended to the generalized Noether symmetry where the coefficients of generalized vector fields contain the terms linear in velocities (time derivative of configuration space coordinates). In our study, we assumed that the matter component of the universe is dominated by cold dark matter. Beside the solution of [8], we obtained new explicit forms for $f(T)$ (see (31)). The novel aspects of these solutions are that the EoS parameter of the universe is no more a constant, and besides a turn around, the universe encounters successive acceleration deceleration phases in the era where the matter component is dominated by cold dark matter.

In our model the universe cannot cross the phantom divide line. Indeed, as was explained, we have always $\dot{H} < 0$, whence $w > -1$. Despite this, the EoS parameter of the dark sector, namely $w_T$, may be lower than or equal to $-1$, and $w_T > -1$ to $w_T < -1$ transition, as illustrated in fig. (3), may occur in our present era. This characteristic is absent in some previous frameworks where an exponential form, or other forms inspired by viable $f(R)$ gravities, for $f(T)$ were proposed [13]. As a summary, on the base of fundamental symmetry theories, generalizing the Noether symmetry provides us a mean to obtain new forms for $f(T)$, which may be physically more viable.

In our study the coefficients in vector fields were linear in the velocities (see eq.(23)). By adopting more generalized form of vector fields, it may be quite possible to obtain new $f(T)$ models which may be more consistent with astrophysical data. However, due to computational hurdles, finding analytical solutions in these cases is very complicated.

References

[1] A. Unzicker and T. Case, arXiv:physics/0503046 [physics.hist-ph]; C. Pellegrini and J. Plebanski, K. Dan. Vidensk. Selsk. Mat. Fys. Skr. 2, No. 2 (1962); K. Hayashi and T. Nakano, Prog. Theor. Phys. 38 (1967) 491; V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira, arXiv:gr-qc/0011087

[2] R. Weitzenböck, Invarianten Theorie, (Nordhoff, Groningen, 1923).

[3] S. Perlmutter et al, Nature (London) 391 (1998) 51; E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D 15 (2006) 1753.

[4] A. D. Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010) 3; T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010) 451; S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4 (2007) 115; S. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011); H. M. Sadjjadi, Phys.
R. Myrzakulov Eur. Phys. J. C 72 (2012) 1998; M. Jamil, D. Momeni, and R. Myrzakulov Eur. Phys. J. C 72 (2012) 2137; K. Atazadeh, and F. Darabi, Eur. Phys. J. C 72 (2012) 2016; B. Vakili and F. Khazaie, Class. Quantum Grav. 29 (2012) 035015.

[10] L. Fatibene, M. Ferraris, M. Francaviglia, and R. G. McLenagaghan, Jour. Mth. Phys. 43(2002)3147.

[11] L. Fatibene, M. Francaviglia, and S. Mercadante, arXiv:1001.2886 [gr-qc].

[12] E. Komatsu et al., Astrophys. J. Suppl. 192, 18 (2011), arXiv:1001.4538 [astro-ph.CO].

[13] K. Bamba., C. Geng, C. Lee, and L. Luo, JCAP 01, (2011) 021, arXiv:1011.0508 [astro-ph.CO]; K Bamba, C. Geng, and C. Lee, arXiv:1008.4036 [astro-ph.CO].