FUNCTORIAL RESOLUTION EXCEPT FOR TOROIDAL LOCUS.
TOROIDAL COMPACTIFICATION

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Abstract. Let $X$ be any variety in characteristic zero. Let $V \subset X$ be an open subset that has toroidal singularities. We show the existence of a canonical desingularization of $X$ except for $V$. It is a morphism $f : Y \to X$, which does not modify the subset $V$ and transforms $X$ into a toroidal embedding $Y$, with singularities extending those on $V$. Moreover, the exceptional divisor has simple normal crossings on $Y$.

The theorem naturally generalizes the Hironaka canonical desingularization. It does not modify the nonsingular locus $V$ and transforms $X$ into a nonsingular variety $Y$.

The proof uses, in particular, the canonical desingularization of logarithmic varieties recently proved by Abramovich-Temkin-Wlodarczyk. It also relies on the established here canonical functorial desingularization of locally toric varieties with an unmodified open toroidal subset. As an application, we show the existence of a toroidal equisingular compactification of toroidal varieties. All the results here can be linked to a simple functorial combinatorial desingularization algorithm developed in this paper.

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1. Introduction

A question of good compactifications of algebraic varieties is of fundamental nature. It was studied in different contexts, by Nagata [Nag], Satake [Sat60], Baily-Borel [BW66], Mumford [AMRT75], Sumihiro [Sum] and many others. In a smooth case or the case of isolated singularities it is certainly possible, using Nagata imbedding and Hironaka desingularization, to compactify the variety $X \subset \overline{X}$, such that $\overline{X}$ is smooth outside of the singularities on $X$, and the complement $\overline{X} \setminus X$ is a simple normal crossing divisor. ([Hir64],[BM97],[Vil89],[Wlo05],[Kol07]) If $X$ admits some mild non-isolated singularities, we still would like to have a good compactification with the boundary divisor having simple intersections.

The problem is directly related to the existence of good or partial resolutions of singularities. The question of good desingularization was studied in the context of log resolution, in particular, by Szabó [Sza94], Kollár in [Kol08], and Bierstone-Milman [BM12a]. On the other hand in the papers of Gonzalez-Perez-Teissier [PGT02], Tevelev [Tev07], Kollár [Kol16], different versions of good combinatorial partial desingularization were introduced.

In this paper we consider a problem of a good partial resolution, which does not modify a given open subset with a certain type of singularities, and no new type of singularities is introduced. Moreover the boundary divisor have some simple intersections. This type of partial resolution can be used, in particular for constructing good compactifications.

The problem generalizes the Hironaka desingularization theorem which does not modify the smooth locus of the scheme and the exceptional locus is an SNC divisor.

A particular question of the existence of partial desingularization except for normal crossing (NC) locus was posed by Kollár in [Kol08].

Some results in this direction were proven by Bierstone-Milman in [BM12a]. They show that such a (partial) resolution except for SNC locus exists for any reduced and reducible scheme of finite type over a field of characteristic zero. Moreover, the SNC locus on the resolved (reducible) variety is the closure of the SNC locus on the given variety.

They also observed in the example of the “pinch point” or “Whitney umbrella”, that the partial resolution except for NC locus, should allow some more general singularities.

In their paper(s) [BM12a], and [BM12b] (jointly with Lairez) they give a complete list of possible singularities in a low dimension and a low codimension- “more general pinch points” which need to be allowed to resolve the schemes except for NC locus.

The present paper addresses the problem of good resolution in a more general situation where the unmodified set is defined by a strict toroidal embedding. This, in particular excludes the case of NC locus, which cannot be resolved. One shall mention, that Bierstone and Milman in their approach use their singularity invariant and alter some steps of their proof of Hironaka desingularization to prevent modification of the NC and SNC locus. Unfortunately, in general, the Bierstone-Milman invariant gives a very little geometric information, as it is specifically designed for the inductive structure of the resolution process.

An alternative resolution tool giving a more precise, more geometric and more efficient control over the resolution process was introduced by Mumford and others
They consider the language of toroidal embeddings, which allows to translate the resolution problems into the language of simpler combinatorial objects - conical complexes. The method was initially introduced in [KKMSD73] to solve the problem of semistable reduction of a dominant morphism to a curve, and proved successful for solving many fundamental problems in birational geometry, weak semistable reduction [AK00], weak factorization theorem[Wo03],[AKMW02],[Wo00], and many others. The language of toroidal embeddings was also used by Mumford and his collaborators in [AMRT75], to construct toroidal compactifications of locally symmetric varieties. Toroidal compactifications have many nice properties and posses simple, easy to describe singularities. They found many applications in the theory of Shimura Varieties.

One of the technical novelties of the paper is a simple functorial combinatorial desingularization which can potentially lead to many new applications using combinatorial methods (Theorem 4.6.1). In fact, all the results here can be linked to versions of this algorithm.

The main disadvantage of the toroidal resolution method is that it can be applied to very special toroidal singularities defined by the binomial equations. In practice, it means that to apply the method in general situation one needs to transform singularities to toroidal ones first.

The structures on toroidal embeddings defined by the divisor or, equivalently, monomials were further generalized in the language of Fontaine-Illusie logarithmic schemes founded in the papers of Kato [Kat89b]. This gives a more general viewpoint, where toroidal embeddings are called logarithmically smooth as they form a class of objects similar to the smooth varieties in the category of reduced schemes of finite type over the field. Similarly to smooth case the logarithmically smooth varieties have a relatively simple structure of the completions of local rings. They are generated by free parameters and an algebraically independent monomial part which forms a monoid. Moreover, similar to the smooth case, the module of the logarithmic differentials is free of the rank equal to the dimension of the ring. The latter is the direct sum of the free parameters part and a free monomial part corresponding to the groupification of the monoid.

In the recent papers by Abramovich-Temkin-Włodarczyk [ATW16], [ATW17] the authors prove the canonical desingularization of logarithmic varieties. The resolution is functorial with respect to arbitrary logarithmically smooth morphisms. The resulting resolved object is, as dictated by the strong functoriality properties, a quasi-log smooth variety as in [ATW17], or a toroidal orbifold with a locally toric coarse moduli space, as in [ATW16]. The result is thus a counterpart of the Hironaka desingularization in the logarithmic category. The functoriality properties imply that the log-smooth (toroidal) locus is unmodified in the process. Quasi log smooth varieties are log smooth in, so-called, Kummer étale topology (which allows to extract roots from the monomials). They are locally toric varieties which are finite toric quotients of log smooth varieties (toroidal embeddings). In particular, their logarithmic structure is defined by a smooth open subset which is the complement of a certain locally toric divisor.

In the paper, we give a proof of the canonical (partial) desingularizations of varieties with unmodified an open toroidal subset:

We obtain several results in this direction.
We prove a canonical desingularization of locally binomial varieties, and toroidal embeddings Theorems 7.18.1, 6.5.1. Any locally binomial variety or a toroidal embedding over a field of any characteristic can be resolved canonically by a projective morphism from a smooth variety such that the exceptional divisor is SNC.

**Theorem 1.0.1.** Let $X$ be any étale locally binomial variety or a toroidal embedding over a field $K$ of any characteristic. There exists a canonical resolution of singularities i.e. a birational projective $f : Y \to X$ such that

1. $Y$ is smooth over $K$.
2. $f$ is an isomorphism over the open set of the nonsingular points.
3. $f$ is a simple normal crossing divisor on $Y$.
4. If $(X, D_X)$ is toroidal then $(Y, D_Y)$ is strictly toroidal with $D_Y$ having SNC. Moreover the birational morphism $f : (Y, D_Y) \to (X, D_X)$ is toroidal.
5. $f$ is a composition of the normalization and the normalized blow-ups of the locally monomial filtered centers $\{ J_n \}_{n \in \mathbb{N}}$ defined locally by valuations.
6. $f$ commutes with smooth morphisms and field extensions, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.

Recall that the problem of a functorial canonical resolution of locally binomial or locally toric varieties was open in positive characteristic. The existence of non-canonical resolution of locally toric varieties preserving the nonsingular locus was proven over any algebraically closed fields in [Wlo03, Theorem 8.3.2].

The Hironaka approach of the embedded resolution to this problem works well only in the case of toroidal embeddings but fails in the locally binomial or the locally toric situation. We show in Example 9.0.1 that so-called maximal contact which constitutes a basis of the Hironaka method, does not exist even in the locally binomial situation. On the other hand, a version of the "combinatorial maximal contact" considered for "combinatorial blow-ups", was constructed by Bierstone-Milman in the case of toroidal embeddings or locally binomial varieties defined by the divisors ([BM06], see also [Blab],[Blaa],[BS]). Using the Hironaka method they prove the case of embedded desingularization of toroidal embeddings, or locally binomial varieties defined by the divisors.

By modifying the algorithm we show the canonical partial desingularization except for toroidal subset for locally binomial varieties, toroidal embeddings, logarithmic varieties, and for any varieties with a Weil divisor: Theorems 7.18.1, 6.6.1, 2.1.25, and 2.2.11.

**Theorem 1.0.2.** Let $X$ be a variety over a field $K$ of characteristic zero, and $V \subset X$ be its open subset which is a strict toroidal embedding $(V, D_V)$, defined by the divisor $D_V$. Assume that the divisor $D_V$ on $V$ defining the toroidal structure on $V$ has locally ordered components. Assume furthermore that one of the following holds

1. $X$ is a logarithmic variety and $\text{char}(K) = 0$.
2. $X$ is any variety with a Weil divisor $D$ and $\text{char}(K) = 0$.
3. $X$ is a locally toric variety with a locally toric divisor $D$.
4. $(X, D)$ is a toroidal embedding.

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1. Definition 6.3.11
We assume that the divisor $D_V$ is the restriction of the divisor $D$ on $X$, or, as in the case (1), defines the restricted logarithmic structure. In the second case we assume that the strict toroidal embedding $(V, D_V)$ is extendable. Then there exists a canonical resolution of singularities of $X$ except for $V$ i.e. a birational projective morphism $f : Y \to X$ such that

1. $f$ is an isomorphism over the open set $V$.
2. The variety $(Y, D_Y)$ is a strict toroidal embedding, where $D_Y := \overline{D_{V,Y}}$ is the closure of the divisor $D_V$ in $Y$. Moreover, $(Y, D_Y)$ is the saturation of the toroidal subset $(V, D_V)$ in $Y$. (In particular, $(Y, D_Y)$ has the same singularities as $(V, D_V)$.)
3. The exceptional divisor $E_{\text{exc}} \subseteq E_{V,Y}$ has simple normal crossings (SNC) with $D_Y$.
4. $f$ is a composition of a sequence of the blow-ups at functorial centers.
5. $f$ commutes with field extensions and smooth morphisms respecting the structure on $X$, the subset $V$, and the order of the components of $D_V$.
6. In particular, if $G$ is an algebraic group acting on $(X, \mathcal{M})$ and preserving the logarithmic structure $\mathcal{M}$ and the subset $V$, and the components of $D_V$ on (a $G$-stable) $V$ then the action of $G$ on $X$ lifts to $Y$, and $f : Y \to X$ is $G$-equivariant.

In other words any such variety containing an open subset which is a strict toroidal embedding can be modified canonically by a birational projective modification into a toroidal embedding. In the process the open subset is untouched and the resulting variety is a toroidal embedding with the singularities identical as on the open subset. This means that the (irreducible) equisingular strata of the resolved toroidal embedding extend the strata on the open unmodified set. Moreover, the exceptional boundary divisor has relatively simple normal crossings. By a relative simple normal crossing divisor we mean here a divisor whose components are locally described by a part of the coordinate system of free parameters. (see Definition 2.1.15)

The desingularization theorems are proven in the following order. First, we prove the canonical desingularization for locally binomial varieties and partial desingularization for locally binomial varieties with locally toric Weil divisors over a field of any characteristic (Theorems 7.17.1, 7.18.1). Then, using the canonical desingularization of logarithmic varieties combined with the desingularization of locally binomial varieties with an unmodified open toroidal subset we show a canonical desingularization of logarithmic varieties except for an open toroidal subset. (Theorem 2.1.25) To further generalize the result for arbitrary varieties with Weil divisors we canonically extend the logarithmic structure from the toroidal subset to the whole variety. (Lemma 2.2.9).

This can be done for arbitrary varieties with Weil divisor possessing an open extendable toroidal subset. The extendable toroidal embeddings satisfy certain conditions on restrictions of local Cartier divisors, like for example, toroidal varieties

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2Definition 2.1.22
3Definition 2.1.6
4Definition 2.1.15
5Note that our definition of toroidal embeddings differs slightly from the definition of Abramovich-Denef-Karu [ADR13, Section 2.2]. Both definitions agree over a perfect field.
with quotient singularities, toroidal varieties with a single closed stratum or with extendable local Cartier divisors). (Definition 2.2.2)

As a consequence, we show the existence of a functorial partial resolution except for open extendable toroidal subsets for arbitrary varieties with a given Weil divisor. (Theorem 2.2.11).

A particular version of this result shows that any variety with a locally toric singularity at a given point can be modified in such a way that a neighborhood of the point will remain unchanged, and the resulting variety, when equipped with a divisorial structure will become a toroidal embedding with a unique closed stratum passing through the point (Theorem 2.2.17).

Using the desingularization Theorem 2.2.11 we prove the existence of equisingular toroidal compactification of extendable toroidal embeddings (Theorem 2.2.14).

The canonical desingularization of logarithmic varieties from [ATW16], [ATW17] reduces the problem to the quasi-toroidal embeddings. The quasi-toroidal embeddings are, in particular, locally toric, with some locally toric divisors. This defines a natural (non-smooth) stratification induced by a given divisor and the singularity type. The resulting variety is not a toroidal embedding but it is a stratified toroidal variety. To deal with it we use the theory of stratified toroidal varieties. The theory was developed as a tool in the proof of the Weak factorization theorem [Wlo03]. It associates with the variety a semicomplex and allows us to run certain (sufficiently functorial) algorithms. In particular, only very special centers of the modifications (star subdivisions) can be used. In section 7 we give a crash course on the theory of stratified toroidal varieties, recalling and reproving a few most basic results used in the proof.

To resolve locally toric singularities with an unmodified toroidal subset we develop a simple fast and efficient functorial desingularization combinatorial algorithm, and its relative version. It can be applied to conical complexes and more general semicomplexes. The functoriality properties are critical for gluing the algorithm on the more general objects.

The method gives a functorial resolution of locally toric or locally binomial varieties with stratification over a field in any characteristic and its relative version with an unmodified toroidal subset (Theorems 7.17.1, 7.18.1).

The functorial desingularization algorithm of locally toric (or locally binomial) varieties over a field of any characteristic is much simpler, more efficient, and more geometric than Hironaka resolution in characteristic zero. Combined with the logarithmic desingularization of [ATW16] ([ATW17]) it gives also a more efficient algorithm of canonical desingularization of arbitrary varieties in characteristic zero. Moreover, the method gives a very good control over resolved singularities and allows to avoid undesired modifications.

Another and perhaps most straightforward application of the combinatorial algorithm is for the toroidal embeddings. In this case, we obtain a very simple and efficient method of functorial resolution, and its relative version. (Theorems 4.6.1, 6.4.1, 6.5.1, and 6.6.1).

Recall that an embedded functorial desingularization of (not necessarily normal) toroidal embeddings over a perfect field was proven, as it was mentioned earlier, by Bierstone-Milman in [BM06] (see also [Blab],[Blaa],[BS]). They extended resolution methods developed in characteristic zero. Similar results were shown by Nizioł,
She was using a combinatorial interpretation of the simplified Hironaka algorithm in [Niz06, Theorem 5.10]. The non-embedded resolution in characteristic 0, preserving a simple normal crossing locus was also proven by Illusie-Temkin [IT14, Theorem 3.3.16], Gillam-Molcho [GM15, Theorem 9.4.5]. Another simple combinatorial method was provided in [ACMW17, Theorem 4.4.2] by Abramovich-Chen-Marcus-Wise. Their method is, however nonfunctorial and modifies the normal crossings locus.

The combinatorial algorithm in this paper is based upon the ideas developed in [KKMSD73], and is closely related to the method considered in [ACMW17]. It is fully functorial and preserves the normal crossings locus. The method combines a version of barycentric subdivision with the lattice reduction algorithm of [KKMSD73, Theorem 11*] with respect to a certain natural order introduced in section 4.5. Unlike other functorial desingularizations ours does not depend upon the toroidal structure and is controlled by simple geometric invariants without additional bulk. When forgetting about the divisors defining the toroidal (or log smooth) structure we are left with locally toric varieties, which reduces the language to the previous situation of stratified toroidal varieties (Theorems 7.17.1) without changing the algorithm. This also explains why the algorithm works in a locally toric case in positive characteristic. On the other hand, the presentation of the algorithm in the paper from toroidal embeddings to locally binomial varieties, illustrates the main feature of the theory of stratified toroidal varieties which studies the combinatorial modifications independent of locally toric coordinates.

The paper is organized as follows. In Chapter 2 we formulate and prove the main theorems using the desingularization theorems from Chapter 7. In Chapter 3 we introduce the basic definitions and results on toroidal embeddings, and conical complexes. Chapter 4 is entirely devoted to proof for canonical desingularization of conical complexes.

In Chapter 5, we introduce the language of relative conical complexes, and prove the relative version of the canonical desingularization of the conical complexes. The algorithm in the relative version is nearly identical, and the introduced notions are perfectly analogous to the standard nonrelative situation. However, the language of relative complexes is somewhat more involved and thus perhaps less intuitive than the language of complexes. That is why we deal separately with the nonrelative and relative cases, although the first one is the particular case of the second, with the trivial relative structure.

Chapter 6 contains a proof of the functorial desingularization of toroidal embeddings, and its relative version with unmodified open subset. The results are quite immediate consequences of the canonical desingularization of complexes in Chapters 4, 5.

Finally, Chapter 7 contains a proof of the functorial desingularization of locally toric varieties over the fields and its relative version. The main tool in the proof is a theory of the stratified toroidal varieties. One constructs a stratification defined by the singularity type and by a given divisor (in the relative situation). These data define the associated conical semicomplex, which is, roughly, a collection of cones defined up to automorphisms groups associated with strata and some rather sparse face relation defined by the strata.
The functoriality of the algorithm developed in Chapters 4, 5 for complexes and the canonicity of the centers allow us to run it on semicomplexes to give rise to the desingularization of stratified toroidal varieties.

2. Main results

2.1. Desingularization of logarithmic varieties except for log smooth locus.

2.1.1. Logarithmic varieties. The logarithmic structures are used in this paper only in the formulation of Theorem 2.1.25, and in the Extension Lemma 2.2.9, in Chapter 2.

Recall that a logarithmic structure on a scheme $X$ of finite type is given by a sheaf of monoids $\mathcal{M}_X$ under multiplication, such that $j^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*$, where $\mathcal{O}_X^*$ is the sheaf of monoids of the invertible regular functions. A logarithimic structure is called coherent if for any point $x \in X$ there is a certain étale neighborhood $i_U : U \to X$, and a map of monoids $P \to \Gamma(U, i_U^*(\mathcal{M}_X))$, called chart, where $P$, is a finitely generated monoid. The map of the monoids induces a map of sheaves of monoids $\alpha : P^a \to i_U^*(\mathcal{M}_X)$, where $P^a$ is a locally constant sheaf defined by $\mathcal{P}_{\text{gp}}$. Moreover, we assume that $i_U^*(\mathcal{M}_X)$ is isomorphic to the push out of $P^a \hookrightarrow \alpha^{-1}(\mathcal{O}_U^*) \to \mathcal{O}_U^*$.[Kat89b, Sections 1.1, 1.2] [Kat89a, Sections 1.2]

A monoid $P$ is fine if $P$ is finitely generated and has no zero divisors, so injects in its finitely generated groupification $P^{\text{gp}}$. It is fine and saturated if it is fine and for any $a \in P^{\text{gp}}$, such that $a^n \in P$ we have that $a \in P$. In particular, the monoids defined by the intersections of rationally generated cones with lattices is an example of fine and saturated monoids.

A coherent logarithmic structure is called fine and saturated (fs) if any point admits a chart $P \to \Gamma(U, i_U^*(\mathcal{M}_X))$, with $P$ being fine and saturated.[Kat89b, Sections 1.1, 1.2]

Any scheme, with a coherent logarithmic structure can be made canonically into a variety or a scheme with a fine and saturated structure by the natural canonical procedure, called saturation.[Kat89a, Proposition 1.2.9], [Ogu16, Proposition 2.1.5]. It is étale locally described by

$$X^{\text{sat}} := X \times_{\text{Spec } \mathbb{Z}[P]} \text{Spec } \mathbb{Z}[P^{\text{sat}}],$$

with

$$P^{\text{sat}} := \{a \in P^{\text{gp}} \mid a^n \in i(P)\},$$

where $i : P \to P^{\text{gp}}$ is the natural map defined by the groupification. [Kat89a, Proposition 1.2.9] [Mau00, Page 1-2].

A logarithmic structure, which is a sheaf of monoids $\mathcal{M}_X$, is usually defined in the étale topology (for functoriality properties). Note that, when working over nonclosed fields or $\mathbb{Z}$ one needs to pass to étale neighborhoods to have nice properties of local rings and the corresponding monoids.

A coherent or fs logarithmic structure will be called strict if the charts are defined in the Zariski topology.
2.1.2. **Stratifications on schemes with logarithmic structures.** Let $X$ be a scheme. A collection $S$ of disjoint locally closed subsets of $X$, called *strata* will be called a *stratification* if the closure of a stratum is a union of strata, and $X$ contains an open dense stratum. This defines a natural order on the strata induced by generalization:

$s \leq s'$ iff $\overline{s} \subseteq \overline{s'}$.

The closures of strata in the stratification will be called the *closed strata*. The collection of the corresponding closed strata will be denoted by $\overline{S}$.

In this paper, we shall consider the resolution of logarithmic schemes in two different general situations:

By a *logarithmic variety* (respectively a *strict logarithmic variety*) we mean a variety equipped with a fine and saturated logarithmic structure $\mathcal{M}$ (respectively a strict fs logarithmic structure). Such a variety possesses a natural stratification defined by the rank of the associated monoids $\mathcal{M}/\mathcal{O}^*$ [IT14], [AT17].

On the other hand, we consider varieties with the logarithmic structure defined by an open subset $U \subset X$. The logarithmic structure $\mathcal{M}$ on $(X,U)$ is defined (in étale topology) as

$$\mathcal{M} = (\mathcal{O}_X)_{\text{ét}} \cap j_*(\mathcal{O}_U^*)_{\text{ét}},$$

where $j : U \to X$ is the open immersion. The strict logarithmic structure is defined in the Zariski topology as

$$\mathcal{M} = (\mathcal{O}_X) \cap j_*(\mathcal{O}_U^*).$$

If the open subset $U$ is the complement of a Weil divisor $D := X \setminus U$, then we shall often use the divisor $D$ to describe the logarithmic structure $\mathcal{M}_D$ associated with $U$ identifying the logarithmic structure on $(X,D)$ with $(X,U)$.

2.1.3. **Divisorial stratification.** In general, the logarithmic structure on $(X,D)$ is not coherent and does not have the stratification given by the rank. However, one can consider a divisorial stratification $S_D$ instead. Write $D = \bigcup_{i \in I} D_i$ with irreducible Weil components $D_i$. The *closed strata* of $\overline{S}_D$ are defined by the irreducible components of the intersections $\bigcap_{i \in I} D_i$ of $D_i$. The strata of $S_D$ are the components of

$$\bigcap_{i \in I} D_i \setminus \bigcup_{i \in (J \setminus I)} D_i.$$

2.1.4. **Logarithmic structures on toroidal embeddings.** Any toric variety $X \supset T$ over a base field $K$ admits a natural stratification by the orbits. It coincides with both stratifications: divisorial, defined by $D = X \setminus U$, and the one given by the rank of the rank of the logarithmic structure.

Toroidal and strict toroidal embeddings were introduced in [KKMSD73] by Mumford and others (initially over an algebraically closed field). They are modeled by toric varieties.

We review this theory over nonclosed fields in Section 3.4. Strict toroidal and toroidal embeddings are defined by an open subset $U \subset X$, and are locally (respectively locally in étale topology) étale isomorphic to toric varieties $(X_\sigma, T)$ with open torus $T$ corresponding to open subset $U$.

By definition, the sheaf of monoids $\mathcal{M}$ is generated at any point $x \in X$ by the monoid of the effective Cartier divisors $\text{Cart}_x^+(X,D)$ supported on $D$ and defined...
in a neighborhood (respectively étale neighborhood of $x$), so we can write

$$\mathcal{M}_x = \text{Cart}^+_x(X, D) \cdot \mathcal{O}^*_{X,x} \subset \mathcal{O}_X.$$  

The toric stratification by the orbits on a toric model $(X_\sigma, T)$ induces locally (respectively étale locally) the canonical stratification $S$ on strict toroidal and toroidal embeddings. (see Section 3.5). It coincides with the divisorial stratification in the case of strict toroidal embeddings.

This stratification is equisingular, which means that the completion of local rings at geometric $\overline{K}$-points on a stratum are isomorphic, where $\overline{K}$ is the closure of a base field $K$.

Each stratum $s$ on a strict toroidal embedding defines an open subset, called the star of $s$, defined as

$$\text{Star}(s, S) := \bigcup_{s' \geq s} s',$$

which is an open neighborhood of $s$.

One can associate with a stratum $s$ the monoid $\text{Cart}^+(s, S)$ of the effective Cartier divisors on $\text{Star}(s, S)$ supported on $\text{Star}(s, S) \cap D$. This monoid is isomorphic to $\text{Cart}^+_x(X, D)$ for any $x \in s$, with the natural isomorphism

$$\text{Cart}^+(s, S) \to \text{Cart}^+_x(X, D)$$

given by the restriction.

The monoid $P := \text{Cart}^+(s, S)$ generates the logarithmic structure $\mathcal{M}_{\text{Star}(s, S)}$ on $\text{Star}(s, S)$:

$$\mathcal{M}_{\text{Star}(s, S)} = \text{Cart}^+(s, S) \cdot \mathcal{O}^*_{\text{Star}(s, S)} \subset \mathcal{O}_{\text{Star}(s, S)}.$$  

For any point $x \in \text{Star}(s, S)$ there is a natural local injective chart $P \to \mathcal{M}(U)$, where $U \subset \text{Star}(s, S)$ is an open neighborhood of $x$. Moreover the stratum $s$ on $U$ is defined as vanishing locus of $P \setminus \{1\}$.

As it was observed in [KKMSD73], there exists a conical complex $\Sigma$ associated with a strict toroidal embedding $(X, D)$, with faces $\sigma \in \Sigma$ in bijective correspondence with strata $s = s(\sigma) \in S$ (see Section 3.5, and Theorem 3.8.11).

2.1.5. Saturated toroidal subsets.

**Definition 2.1.6.** By a toroidal subset of a scheme with logarithmic structure $(X, \mathcal{M})$ we mean an open subset $V \subset X$, with a Weil divisor $D_V$ such that $(V, D_V)$ is a strict toroidal embedding, which defines the restricted log structure $\mathcal{M}_{|V}$. By the toroidal saturation or simply saturation of an open toroidal subset $(V, D_V)$ of $(X, \mathcal{M})$ we mean the maximal open toroidal subset $(V^0, D^0)$ of $(X, \mathcal{M})$ which contains $(V, D_V)$, such that all the strata on $(V^0, D^0)$ intersect the subset $V$ (so extend the strata in $(V, D_V)$).

A toroidal subset $(V, D_V)$ of $(X, \mathcal{M})$ is called saturated if it is equal to its saturation.

The above definitions apply to logarithmic varieties, and varieties with Weil divisor.

**Example 2.1.7.** In the case of the strict toroidal embedding any open subset is toroidal. The saturated subsets are just those which are the unions of some stars.
**Definition 2.1.8.** The largest saturated toroidal subset of \((X, \mathcal{M})\) is called the toroidal locus (or the log smooth locus). It is the set of all points of \(X\) where \((X, \mathcal{M})\) is a strict toroidal embedding. We denote it by \((X, \mathcal{M})^\text{tor}\).

**Remark 2.1.9.** Observe that since the strata on toroidal embeddings are equisingular, any toroidal subset \(V \subset X\) has the same singularities as its toroidal saturation \(V^0\). So the saturated subsets, in particular, represent the sets of all the points with certain given types of toroidal singularities (including information on the divisor). For example, the smallest nonempty saturated toroidal subset on \(X\) is the set of all nonsingular points \((X^0)^{ns}\) of the set \(X^0\) of the points where the logarithmic structure is trivial \(\mathcal{M} = \mathcal{O}^*\) (resp. \(X^0 := X \setminus D\)). The largest saturated toroidal subset on \(X\) is its toroidal locus \(X^\text{tor}\). There are finitely many saturated toroidal subsets on \(X\), as these are exactly the unions of the stars of strata on \(X^\text{tor}\).

**Lemma 2.1.10.** Let \((X, D)\) be a toroidal embedding embedding (which is not necessarily strict). Let \((V, D_V)\) be its toroidal subset intersecting all the strata of \((X, D)\). Then \((X, D)\) is strict toroidal and it is the toroidal saturation of \((V, D_V)\).

**Proof.** Let \(x\) be any point in a certain stratum \(s\) on \(X\). Consider a Weil divisor \(E\) consisting of the components of \(D\), passing through \(x \in E\). Let \(\pi : (U, D_U) \to (X, D)\) be an \(\text{étale}\) neighborhood of \(x\) which is strict toroidal. This map being \(\text{étale}\) preserves the strata (of the rank stratification), so that the inverse image of the strata on \((X, D)\) are the strata on \((U, D_U)\). Let \(\overline{x} \in \overline{s}\) be a point over \(x \in s\), where \(\overline{\pi}\) is the stratification over \(s\).

It defines an \(\text{étale}\) map between the relevant strict toroidal embeddings \(\pi_U : (U_V, D_{U_V}) \to (V, D_V)\), where \(U_V := \pi^{-1}(V)\). By Lemma 3.8.2, there exists a bijective correspondence between the components of the Weil divisor of \(E\) passing through a generic point of \(s\) and the components of \(D_U\) passing through a generic point of \(\overline{s}\).

But the components of the divisor \(E\) (respectively \(D_U\)) through \(x\) and \(s\) (respectively \(\overline{x}\) and \(\overline{s}\)) are the same. This follows from the construction of the divisorial stratification on \(\text{étale}\) neighborhood \(U\) and the stratification on \(X\) with strata which are locally the images of strata on \(\text{étale}\) neighborhoods.

Consequently, the natural surjection between the irreducible components of Weil divisor \(D_U\) through \(\overline{x}\) and \(E\) through \(x\) defined by \(\pi : \text{Spec}(\mathcal{O}_{\overline{x}, U}) \to \text{Spec}(\mathcal{O}_{x, X})\) is a bijection. Thus the inverse image of an irreducible divisorial component is locally an irreducible divisorial component.

Consequently, each such a component of \(E\) itself is a toroidal embedding since it is strict toroidal in \(\text{étale}\) topology. (Its inverse image under \(\text{étale}\) morphism is a strict toroidal embedding). So, in particular, each component of \(E\) is normal. This implies, by Lemma 3.8.3, that \((X, D)\) is strictly toroidal.

**Corollary 2.1.11.** Let \((X, D)\) be a toroidal embedding embedding (which is not necessarily strict). Let \((V, D_V)\) be its saturated toroidal subset. Then for any \(\text{étale}\) morphism \(\phi : (U, D_U) \to (X, D)\) the inverse image \(\phi^{-1}(V)\) is a saturated toroidal subset of \((U, D_U)\).

**Proof.** Let \(V' := \phi^{-1}(V)^{\text{sat}}\) be the saturation of \(\phi^{-1}(V)^{\text{sat}}\). Then its image \((\phi(V'), D \cap \phi(V'))\) is a toroidal embedding such that all the strata on \(\phi(V')\) intersect \(V\). Then, by Lemma 2.1.10, \(\phi(V')\) is a strict toroidal embedding, and thus it is
contained in the saturation $V^{\text{sat}} = V$. This implies that $\phi(V') = V$, and $V' = \phi^{-1}(V)$.

\[\blacklozenge\]

2.1.12. **Relative SNC divisors on toroidal varieties.** In the classical Hironaka desingularization the exceptional locus has SNC (simple normal crossings). In the relative desingularization the exceptional divisor has similar properties.

**Definition 2.1.13.** Let $(X, D)$ be a toroidal embedding. By a *free coordinate system* on $(X, D)$ at a point $x$ we mean a set of parameters $u_1, \ldots, u_k$ on an open subscheme $X$ such that there is an étale neighborhood $U^{\alpha} \to U$ of $x$, and a monoid $P$, with an injective chart

$$P \to \mathcal{M}(U^{\alpha}) = \mathcal{O}(U^{\alpha}) \cap \mathcal{O}^*(U^{\alpha} \smallsetminus D_{U^{\alpha}}),$$

and an étale morphism $U^{\alpha} \to \text{Spec} \, K[x_1, \ldots, x_k, P]$ defined by $x_i \mapsto u_i$, and $P \to \mathcal{O}(U^{\alpha})$, where $x_1, \ldots, x_k$ are free variables independent of $P$, and the ideal $\mathcal{I}_s$ of the stratum $s$ through $\mathcal{I} \sigma$ over $x$ is generated by $P \setminus \{1\}$.

**Remark 2.1.14.** Equivalently $u_1, \ldots, u_k$ is a free coordinate system if their restrictions to the stratum $s$ define a local parameter system on a smooth subvariety $s$.

**Definition 2.1.15.** Let $(X, D)$ be a toroidal embedding and $E$ be a Weil divisor. We say that $E$ has *normal crossings* (NC) (resp. *simple normal crossings* (SNC)) on $(X, D)$ if it is étale locally (resp. locally) defined by a part of a free coordinate system. Equivalently we shall call $E$ a *relative NC divisor* (respectively a *relative SNC divisor*) on $(X, D)$.

**Remark 2.1.16.** It follows from Lemma 2.1.18, that if $E$ has SNC on a strict toroidal $(X, D)$ then then there is a part of a free coordinate system $u_1, \ldots, u_k$ locally on an open $U$, and an injective chart $P \to \mathcal{M}(U)$, with étale morphism $U \to \text{Spec} \, K[x_1, \ldots, x_k, P]$, defined by $x_i \mapsto u_i$, and $P \to \mathcal{O}(U^{\alpha})$.

One can rephrase Definition 2.1.15

**Lemma 2.1.17.** A divisor $E$ has *normal crossings* (NC) on $(X, D)$ iff $(X, D \cup E)$ is a toroidal embedding and any point $x \in X$ admits an étale neighborhood $\alpha : U \to X$ of $x$ and étale morphism

$$\phi : U \to X_{\sigma} \times X_{\tau} = X_{\sigma} \times \mathbb{A}^n,$$

where $X_{\tau} = \mathbb{A}^n$ such that, $\alpha^{-1}(D)|_U = \phi^{-1}(D_1 \times X_{\tau})$, and $\alpha^{-1}(E)|_U = \phi^{-1}(X_{\sigma} \times E_1)$, with a toric divisor $D_1$ on $X_{\sigma}$, and $E_1$ is an SNC toric divisor on $X_{\tau} = \mathbb{A}^n$. \[\blacklozenge\]

**Lemma 2.1.18.** With the preceding notation and the assumptions. The divisor $E$ has *simple normal crossings* (SNC) on $(X, D)$ if it has NC and its components are normal. If, additionally, $(X, D)$ is a strict toroidal embedding then $(X, D \cup E)$ is such. Moreover, any point $x \in X$ admits a neighborhood $U$ and an étale morphism

$$\phi : U \to X_{\sigma} \times X_{\tau} = X_{\sigma} \times \mathbb{A}^n,$$

where $X_{\tau} = \mathbb{A}^n$ such that, $D_1|_U = \phi^{-1}(D_1 \times X_{\tau})$, and $E_1|_U = \phi^{-1}(X_{\sigma} \times E_1)$, with a toric divisor $D_1$ on $X_{\sigma}$, and $E_1$ is an SNC toric divisor on $X_{\tau} = \mathbb{A}^n$.
Proof. If \((X, D)\) is a toroidal embedding and the components of \(E\) are locally defined by free parameters on a toroidal embedding, then, by definition, they are toroidal embeddings and hence normal. Conversely, if \(E\) has NC on \((X, D)\), and its components are normal then, by definition, the components of \(E\) are étale locally defined by local parameters. Denote by \(E'\) the inverse image of \(E\) on an étale neighborhood. Since the components of \(E\) are normal on \(X\) then, by Lemma 3.8.2, the irreducible components of \(E'\) on étale neighborhood of \(X\) are locally the inverse images of the components on \(X\). By definition, the components of \(E'\) are defined by a part of a free coordinate system in the étale neighborhood. Thus, by the second part of Lemma 3.8.2, these components descend to Cartier divisors defined locally by a part of free coordinate system on \((X, D)\) describing components of \(E\). Thus \(E\) has SNC on \((X, D)\).

Now, if \((X, D)\) is strict toroidal, then all the components of \(D \cup E\) are normal, and since \((X, D \cup E)\) is toroidal, it is also strict toroidal by Lemma 3.8.3. Thus there is locally an étale morphism to \(X_\sigma \times X_\tau\).

Remark 2.1.19. If \(E\) is relative NC (respectively SNC) on \((X, D)\) then its restriction \(E \cap (X \setminus D)\) to the logarithmically trivial locus \((X \setminus D)\) is an NC (resp. SNC) divisor in the usual sense on a smooth subset \((X \setminus D)\).

Lemma 2.1.20. Let \((X^0, D_{X^0})\) be a strict toroidal embedding, and \(E\) has NC with \(D_{X^0}\). Set \(X := X^0 \setminus E\), \(D_X := D_{X^0}\). Then \((X^0, D_{X^0})\) is the saturation of \((X, D_X)\).

Proof. We verify this property on a strict toroidal étale neighborhood, where it reduces to the obvious fact for toric varieties that \(X_\sigma \times \mathbb{A}^n\) is the saturation of \(X_\sigma \times (\mathbb{A}^n \setminus E)\). Then, by Lemma 2.1.10, \((X, D_X)\) is strict toroidal, and is the toroidal saturation of \((X^0, D_{X^0})\).

2.1.21. Divisors with locally ordered components.

Definition 2.1.22. Let \((X, D)\) be a strict toroidal embedding. We say that a Weil divisor \(D\) on \(X\) has locally ordered components if there is given a partial order on the set of components, which is total for any subset of the components passing through a common point.

Remark 2.1.23. The condition of ordering components for the canonical desingularization except toroidal subset is unavoidable given Example 5.13.1!

2.1.24. Desingularization of logarithmic varieties except for log smooth locus.

Theorem 2.1.25. Let \((X, M)\) be a logarithmic variety over a field \(K\) of characteristic zero. Let \(V\) be an open toroidal subset of \(X\). Assume that the divisor \(D_V\) on \(V\) defining the smooth logarithmic (toroidal) structure on \(V\) has locally ordered components.

There exists a canonical resolution of singularities of \((X, M)\) except for \(V\) i.e. a birational projective morphism \(f : Y \to X\) such that

1. \(f\) is an isomorphism over the open set \(V\).

---

6Definition 2.1.6
7Definition 2.1.22
(2) The variety \((Y, D_Y)\) is a strict toroidal embedding, where \(D_Y := \overline{D_{V,Y}}\) is the closure of the divisor \(D_Y\) in \(Y\). Moreover, \((Y, D_Y)\) is the saturation\(^8\) of the toroidal subset \((V, D_V)\) in \(Y\). (In particular, \((Y, D_Y)\) has the same singularities as \((V, D_V)\).)

(3) The complement \(E_{V,Y} := Y \smallsetminus V\) of \(V\) in \(Y\) is a divisor with simple normal crossings (SNC) with \(D_Y\) \(^9\). So is the exceptional divisor \(E_{exc} \subseteq E_{V,Y}\).

(4) In particular, if \(V\) is smooth and \(D_Y\) is an SNC divisor on \(V\) then \(Y\) is smooth, and \(D_Y \cup E_{V,Y}\) is an SNC divisor.

(5) \(f\) is a composition of a sequence of the blow-ups at functorial centers.

(6) \(f\) commutes with field extensions and smooth morphisms respecting the logarithmic structure, the subset \(V\), and the order of the components of \(D_Y\).

(7) In particular, if \(G\) is an algebraic group acting on \((X, M)\) and preserving the logarithmic structure \(M\) and the subset \(V\), and the components of \(D_Y\) on \((a G\text{-stable})\) \(V\) then the action of \(G\) on \(X\) lifts to \(Y\), and \(f : Y \to X\) is \(G\)-equivariant.

**Proof.** Let \(X\) be a logarithmic variety. We can assume that the complement \(X \smallsetminus V\) is the support of a Cartier divisor \(E_X\). To this end consider the blow-up of the ideal of the complement \(X \smallsetminus V\), and let \(E_X\) be the exceptional divisor. We can also assume that the logarithmic structure \(M\) is not trivial anywhere outside of \(V\), by replacing \(M\) with the saturation of the log-structure generated by \(M\) and \(E_X\) (The push-out \(M \leftarrow \mathcal{O}^* \to M_{E_X}\)). Then, in particular, the components of \(E_X\) are the closed irreducible strata of \(M\). This is done as a preliminary step to ensure the condition (3) of the Theorem. The open subset \(V\) is saturated on the newly created logarithmic variety \((X, M)\). Then the open subset \(V_0 := V \smallsetminus D_V\) coincides with the logarithmically trivial locus \(X^{tr}\) of \(X\).

By [ATW16], and [ATW17] we can canonically desingularize \((X, M)\) that is transform birationally \(X\) to a *quasi- toroidal embedding* \(X\) (modifying its logarithmic structure on the way). The process is functorial for logarithmically smooth morphisms, and thus it preserves a toroidal subset \(V\).

The quasi-toroidal embedding \(X\) is, in particular, étale locally toric, with the locally toric divisor

\[
E_{V_0,X} := X \smallsetminus V_0 = \overline{D_{V,X}} \cup E_{V,X},
\]

where \(\overline{D_{V,X}}\) is the closure of \(D_V \subset V\) in \(X\), and \(E_{V,X} := X \smallsetminus V\).

By Theorem 7.18.1, applied to the open saturated toroidal subset \((V, D_V) \subset (X, E_{V_0,X})\), the variety \(X\) can be transformed into a strict toroidal embedding \((Y, D_Y)\), where \(D_Y\) is the closure of \(D_V\), and the complement divisor \(E_{V,Y} := Y \smallsetminus \overline{V}\) of \(V\) in \(Y\) is an SNC divisor on \((Y, D_Y)\). The conditions (1)-(7) follow from Theorem 7.18.1.

**Remark 2.1.26.** Consider a divisor \(D\) with the ”pinch point” singularity defined by \(y^2 = x^2 z\) in \(\mathbb{A}^3\). The natural logarithmic structure on \(\mathbb{A}^3\) defined by the complement of \(D\) is not fine and saturated (not even coherent). Its restriction to \(V := \mathbb{A}^3 \setminus \{0\}\) defines a toroidal embedding with NC singularities. The ”pinch point” singularity however, cannot be resolved without modifying the NC locus \(V\).

\(^8\)Definition 2.1.6
\(^9\)Definition 2.1.15
The open subscheme $V$ is not a toroidal subset of $(\mathbb{A}^3, D)$ since it is not a strict toroidal embedding. (see also [BM12a]).

### 2.2. Desingularization of varieties except for log smooth locus.

#### 2.2.1. Extendable toroidal embeddings.

**Definition 2.2.2.** Let $(X, D)$ be a strict toroidal embedding and $S$ be the induced stratification. For any stratum $s \in S$ consider the monoid of the effective Cartier divisors $\text{Cart}(s, S)^+$ on an open neighborhood

$$U_s := \text{Star}(s, S) = \bigcup_{s' \geq s} s'$$

supported on $D_s := D \cap U_s$. Then $(X, D)$ will be called extendable if there exists a Cartier system $\Phi = \Phi(X)$, that is, a collection $\Phi = \{\Phi_s\}_{s \in S}$ of finite nonempty subsets $\Phi_s \subset \text{Cart}(s, S)^+$ of effective Cartier divisors on $U_s$ for $s \in S$, such that

1. $\Phi_s \subset \text{Cart}(s, S)^+$ generates $\text{Cart}(s, S)^+$ (as monoid).
2. If $s \leq s'$ then $U_s \subset U_{s'}$, and the induced restriction of the Cartier divisors defines a surjective map $\Phi_s \to \Phi_{s'}$ of the sets.

**Example 2.2.3.** The SNC divisors on smooth varieties are extendable, with the Cartier systems $\Phi_s$, defined by the components of the Weil divisors through $s$.

**Lemma 2.2.4.** Strict toroidal embeddings with quotient singularities are extendable. Moreover the Cartier systems can be chosen functorial for smooth morphisms dominating on the strata.

**Proof.** Define the Cartier system $\Phi = \Phi(X) := \{\Phi_s\}$, with $\Phi_s := \text{Cart}(s, U_s)^+_{\leq n}$ to be the set all the effective Cartier divisors on $U_s \subset X$ which are linear combinations of their Weil components with the coefficients $\leq n$.

This defines a Cartier system for sufficiently large $n$. Indeed, for any irreducible Weil divisor $E$ on $U$, let $n_E$ denote the smallest integer for which $n_E E$ is Cartier. Let $n$ be the integer $\geq n_E$, where $E$ ranges over all the components of $D$ and such that each $\Phi_s = \text{Cart}(s, S)^+_{\leq n}$ generates $\text{Cart}(s, S)^+$ for each stratum $s$.

If $s \leq s'$ then any effective Cartier divisor $E_{s'} = \sum m_E E$ in $\Phi_{s'}$, defined on $U_{s'}$ extends to an effective Cartier divisor

$$E_s = \sum m_E E + \sum s_E' E'$$

in $\text{Cart}(s, S)^+$ by an elementary fact for the cones (See also Example 2.2.7), where $E'$ are the Weil component in $U_s \setminus U_{s'}$. Then for any such $E'$ any multiple $k \cdot (n_E' E')$ is a Cartier divisor and can be subtracted from the effective Cartier divisor $E_s$. Consequently, the coefficient $s_{E'}$ can be adjusted to a new (nonnegative) coefficient $s_{E'}'$ with $s_{E'}' < n_{E'} \leq n$ in the presentation of the modified Cartier divisor

$$E_s' = \sum m_{dE} + \sum s_{E'} E' \in \Phi_s.$$ 

This shows the surjectivity of the map $\Phi_s \to \Phi_{s'}$. ♦

**Example 2.2.5.** If $(X, D_X)$ is a strict toroidal embedding for which any effective Cartier divisor on any subset $U_s \subset X$ supported on $D_X \cap U_s$ extends to an effective Cartier divisor on $X$ supported on $D_X$, then $(X, D_X)$ is extendable. We can simply set $\Psi_s$ to be the set of the extensions of generators of $\text{Cart}(s, S)^+$ to $X$ and put $\Psi = \bigcup_{s \in S} \Psi_s$, then we set $\Phi_s$ to be the set of restrictions of $\Psi$ to $U_s$. 


The conditions for strict extendable toroidal embeddings can be translated into the language of the associated conical complexes. (Section 3.8.10)

**Lemma 2.2.6.** A strict toroidal embedding \((X, D)\) is extendable if the associated complex \(\Sigma\) satisfies the condition: There exists a functor \(L\) associating with each face \(\sigma\) a finite set of integral linear functions \(L_\sigma\) from \(\sigma \cap N_\sigma\) to \(\mathbb{Z}_{\leq 0}\), such that

1. \(L_\sigma\) generates the monoid \(\sigma^\vee \cap M_\sigma\) of the integral linear functions which are nonnegative on \(\sigma \cap N_\sigma\).
2. The restriction defined by a face inclusion \(i_{\tau \sigma}: \tau \to \sigma\) yields the surjective map of the sets \(L_\sigma \to L_\tau\).

We shall call such a complex **extendable**.

**Example 2.2.7.** If \(\Sigma\) is a complex with a single maximal face \(\sigma\) then it is extendable. (See also Example 2.2.5). In particular, any strict toroidal embedding \((X, D)\) with a single minimal (closed) stratum is extendable.

**Proof.** Follows from a well-known fact, that any nonnegative integral linear function on a face \(\tau\) of \(\sigma\) extends to a nonnegative integral linear function on \(\sigma\). (See for instance [Ful93]). So one can take \(L_\sigma\) to be the set of extensions of the generators of all \(\tau^\vee \cap M_\tau\), where \(\tau\) ranges over all faces of \(\sigma\). Then we simply put \(L_\tau\) to be the set of the restrictions of the elements in \(L_\sigma\) to \(\tau\).

\[\blacksquare\]

2.2.8. **Extension of log smooth logarithmic structures.**

**Lemma 2.2.9.** Let \((X, D)\) be a variety with a Weil divisor. Let \((V, D_V)\) be an extendable toroidal subset of a variety \((X, D)\) with a Cartier system \(\Phi\). Then there is a canonical projective birational modification \(\tilde{X}\) of \(X\) which is an isomorphism on \(V\), and a canonical extension \(M\) of the logarithmic structure on \((V, D_V)\) such that \((\tilde{X}, M)\) is a logarithmic variety. Moreover, the logarithmic structure \(M\) on \(\tilde{X}\) is functorial for smooth morphisms respecting \(V\) and \(D\) and a Cartier system \(\Phi\).

**Proof.** Let \(D_X := D_V\) be the closure of the Weil divisor \(D_V\). Let \(S\) be the toroidal stratification on the toroidal subset \((V, D_V)\). For any stratum \(s \in S\) on \(V\) consider the monoid \(\Phi_s \subset \text{Cart}^+(s, S)\) of effective Cartier divisors on \(V_s = \text{Star}(s, S) \subset V\).

The closures of the divisors in \(\Phi_s\) can be thought as Weil divisors on \(X\).

For any point \(x \in X\) let \(\overline{s_{1x}}, \ldots, \overline{s_{nx}}\) be the minimal sets (with respect to inclusion) which are the closures of the strata of \(S\) through \(x\). Let \(n_x\) be their number. Observe that \(n_x = 1\) when \(x \in V\), as any such point belongs to a unique minimal stratum of \(S\). Set \(N := \max\{n_x \mid x \in X\}\) and denote by \(I_{s_{1x}}\ldots + I_{s_{nx}}\) the ideals of the subschemes \(\overline{s_{1x}}\ldots \overline{s_{nx}}\) and consider the blow-ups at \(I_{s_{1x}} \ldots + I_{s_{nx}}\) cosupported on the disjoint (by maximality of \(N\)) subsets \(\overline{s_{1x}} \cap \ldots \cap \overline{s_{nx}}\), with \(n_x = N\). In particular, as long as \(N > 1\) the centers of the blow-ups are disjoint.

\[\blacksquare\]
from $V$. After each such a blow-up the strict transforms of $\overline{s_1x}, \ldots, \overline{s_nx}$ will have no common intersection, and the number $N$ drops. By continuing this process we arrive at the situation with $N = 1$ so that each point belongs to a unique minimal closure of a single stratum.

The process does not affect the open subset $V$. Moreover, for any two closures of strata $\overline{s_1}$ and $\overline{s_2}$ the intersection $\overline{s_1} \cap \overline{s_2}$ is the union of the closures of the strata, as any point $x \in \overline{s_1} \cap \overline{s_2}$ belong to a unique minimal $\overline{s} \subset \overline{s}_i$, for $i = 1, 2$.

Now for any $s \in S$ consider the corresponding locally closed subset $t = t(s) \in T$ defined as

$$t := \overline{s} \smallsetminus \left( \bigcup_{s' \subseteq s \smallsetminus \overline{s}} \overline{s'} \right)$$

Then $t \cap V = s$, $\overline{t} = \overline{s}$, and all subsets $t$ are disjoint. Thus $T$ is a stratification. Moreover, the order $\leq$ on $S$ defines the order on $T$: $t(s) \leq t(s')$ if $s \subset s'$. So we can write

$$t := \overline{t} \smallsetminus \left( \bigcup_{s' < t} \overline{s'} \right).$$

For any $t = t(s) \in T$, let

$$U_t := \text{Star}(t, T) = \bigcup_{t \leq t'} t'$$

be an open subset of $X$. Then $U_t \cap V = V_s$.

Denote by $\Phi_t$ with $t = t(s) \in T$ the set of closures in $U_t$ of divisors in $\Phi_s$ in $V_s$, which are Weil divisors on $U_t$.

If $t_1 < t_2$, then $U_{t_1} \supset U_{t_2}$, and, by the assumption on $\Phi$, there is a natural surjective map of the sets of divisors on $U_{t_2} : \Phi_{t_1 \mid U_{t_2}} \rightarrow \Phi_{t_2}$.

Consider the ideal sheaf

$$I_t := \prod_{D \in \Phi_t} \mathcal{O}_X(-D)$$

on the open subset $U_t$. We define a modification $\overline{X} \rightarrow X$ to be the blow-up of the ideal sheaf $I_t$ on $U_t$. Note that the blow-up of $I_t$ on $U_t$ is equivalent to the composition of the blow-ups at $\mathcal{O}(-D)$, with $D \in \Phi_t$. If $t \leq t'$ then, by condition (2) of Definition 2.2.2, the set of of the restrictions $\mathcal{O}(-D \mid U_{t'})$, where $D \in \Phi_t$, coincides with the set of the $\mathcal{O}(-D')$ with $D' \in \Phi_{t'}$.

Thus the blow ups of the ideals $I_t$ on $U_t$ glue to define a unique transformation $\sigma : \overline{X} \rightarrow X$ which is an isomorphism on $V$.

Moreover, any divisor $\sigma^{-1}(D)$, where $D \in \Phi_t$ is Cartier on $\overline{U}_t$. For any such $\overline{U}_t$ consider the monoid $P(\overline{U}_t)$ of the effective Cartier divisors on $\overline{U}_t$ generated by the inverse image

$$\sigma^{-1}(\Phi_t) := \{ \sigma^{-1}(D), D \in \Phi_t \}.$$ 

This induces the canonical coherent logarithmic structure

$$\mathcal{M}_{\overline{U}_t} = P(\overline{U}_t) \cdot \mathcal{O}_{\overline{U}_t} \subset \mathcal{O}_{\overline{U}_t}$$

on each $\overline{U}_t$ which glues to the coherent logarithmic structure $\mathcal{M}_{\overline{X}} \subset \mathcal{O}_{\overline{X}}$ on $\overline{X}$.

By condition (1) of Definition 2.2.2, the restriction $\sigma^{-1}(\Phi_t) \rightarrow \Phi_s$ of Cartier divisors defines a surjective map to the set of generators $\Phi_s$ of the monoid Cart$(V_s, D_V \cap$
of the Cartier divisors on $U_s$ supported on $D_V \cap U_s$. Thus the logarithmic structure $\mathcal{M}_X$ extends the toroidal logarithmic structure on $V$ induced by $D_V$.

Finally the saturation functor $\text{sat}$ transforms functorially the logarithmic variety $(\tilde{X}, \mathcal{M}_X)$ with a coherent logarithmic structure into the variety $(\tilde{X}^{\text{sat}}, \mathcal{M}_{\tilde{X}}^{\text{sat}})$ with a fine and saturated logarithmic structure.

2.2.10. Desingularization of varieties except for a toroidal subset.

**Theorem 2.2.11.** Let $X$ be a variety over a field $K$ of characteristic zero, and $D_X$ be any Weil divisor on $X$. Let $(V, D_V)$ be an open extendable toroidal subset of $(X, D_X)^{11}$. Assume that $D_V$ has locally ordered components $^{12}$. There exists a canonical resolution of singularities of $(X, D_X)$ except for the toroidal subset $V$, i.e. a projective birational morphism $f : Y \to X$ such that

1. $f$ is an isomorphism over the open set $V$.
2. The variety $(Y, D_Y)$ is a strict toroidal embedding, where $D_Y := \overline{D_V \cap Y}$ is the closure of the divisor $D_V$ in $Y$.
3. $(Y, D_Y)$ is the saturation of $(V, D_V)^{13}$. In particular, $(Y, D_Y)$ is an extendable toroidal embedding.
4. The complement $E_{V,Y} := Y \setminus V$ is a divisor which has simple normal crossings with $D_Y^{14}$. So is the exceptional divisor $E_{\text{exc}} \subseteq E_{V,Y}$.
5. In particular, if $V$ is smooth and $D \cap V$ is an SNC divisor on a smooth subset $V \subseteq X$ then $Y$ is smooth and $D_Y \cup E_{V,Y}$ is an SNC divisor.
6. $f$ commutes with field extensions and smooth morphisms respecting the saturated toroidal subset, the order of the components $D_V$, and the Cartier system on $(V, D_V)$.
7. If $G$ is an algebraic group acting on $(X, D)$ and preserving the open subset $V$, and the components of the divisor $D_V$ on $V$ then the action of $G$ on $X$ lifts to $Y$, and $f : Y \to X$ is $G$-equivariant.

**Proof.** By Lemma 2.2.9, we extend canonically the logarithmic structure on $(V, D_V)$ to the structure $\mathcal{M}_{\tilde{X}}$ of logarithmic variety on a certain birational modification $\tilde{X}$ of $X$ preserving $V$. To finish the proof we apply Theorem 2.1.25 to $(\tilde{X}, \mathcal{M}_{\tilde{X}})$.

**Remark 2.2.12.** The theorem extends the nonembedded Hironaka desingularization. We consider the zero divisor $D = 0$ on a variety $X$, and the nonsingular locus $V = X^{\text{ns}}$. Then $V = (X, 0)^{\text{tor}}$ is the toroidal locus of the variety $(X, 0)$ with a single nonsingular stratum $V$ on $V$. Let $Y \to X$ be the desingularization except for $V$. Since $Y$ is the saturation of $V$ it is nonsingular with one big smooth stratum $Y$ which is an extension of the unique smooth stratum $V$ on $(V, 0)$. Consequently, the resolution $Y \to X$ except for $V$ is simply a canonical Hironaka desingularization which keeps nonsingular locus singular locus $V = X^{\text{ns}}$ untouched. Moreover, the exceptional locus $E = Y \setminus X^{\text{ns}}$ is a simple normal crossing divisor.

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$^{11}$Definitions 2.1.6, 2.2.2
$^{12}$Definition 2.1.22
$^{13}$Definition 2.1.6
$^{14}$Definition 2.1.15
The above theorem, in very particular, is closely related to a variant of the Bierstone-Milman desingularization theorem except for SNC locus. ([BM12a], Theorem 3.4) Consider a reducible divisor $D$ on a smooth variety $X$. Let $V = (X, D)^{snc}$ will be the locus of point in $X$ where $(X, D)$ is SNC. By Theorem 2.2.11, there exists a resolution $(Y, D_Y)$ of $(X, D)$ except for $V$, which is an isomorphism on $V$ and such that $D_Y \cup E_{V,Y}$ is SNC.

2.2.13. Toroidal compactifications of toroidal embeddings. As an immediate corollary from the above we obtain the following result.

**Theorem 2.2.14.** Let $(X, D)$ be a extendable toroidal embedding over a field of characteristic zero\(^{15}\). There exists a toroidal compactification $(\overline{X}, \overline{D})$ of $(X, D)$ such that

1. $(\overline{X}, \overline{D})$ is a complete strict toroidal embedding, where $\overline{D}$ is the closure of $D$ in $\overline{X}$.
2. $(\overline{X}, \overline{D})$ is the saturation of $(X, D)$ in $(\overline{X}, \overline{D})^{16}$. In particular it is extendable (and admits the same singularities as $(X, D)$).
3. The complement $E := \overline{X} \setminus X$ of $X$ is a divisor, which has SNC with $\overline{D}$.\(^{17}\)
4. If $X$ is quasi-projective then $\overline{X}$ is projective.
5. Moreover, if an algebraic group $G$ acts on $(X, D)$ preserving the components then there exists a $G$-equivariant compactification $(\overline{X}, \overline{D})$ satisfying the above properties.

**Proof.** We consider the Nagata ([Nag]) or a projective completion $X_0$ of $X$, and let $D_0$ be the closure of $D$ in $X_0$. To finish the proof we apply Theorem 2.2.11 to the toroidal subset $(X, D) \subset (X_0, D_0)$. If $G$ acts on $X$ we use the Sumihiro compactification theorem ([Sum]) to construct $X_0$.

The above theorem is a natural extension of a much simpler fact of the compactification of toric varieties due to Sumihiro,([Sum]) while controlling singularities. We give here a strenghtening of the Sumihiro result

**Theorem 2.2.15.** Let $X$ be a (normal) toric variety containing a torus $T$, and let $D_X := X \setminus T$ be a toric divisor. The toric variety $(X, D_X)$, admits an equivariant toric compactification $\overline{X}$ having the same singularities as $X$, and such that $E := \overline{X} \setminus X$ is a (toric) divisor having SNC with the closure $\overline{D}_X \subset \overline{X}$ of $D_X$.

**Proof.** We find a Sumihiro compactification $X_0 \supset X$ first. Note that since the set $X$ is $T$-stable it is the union of the orbits (strata) and thus it is saturated in $X_0$. Then we apply a $T$-equivariant toroidal desingularization from Theorem 6.6.1 except for $X$, to the toric compactification $X_0$, and $D = X_0 \setminus T$.

2.2.16. Desingularization except for locally toric singularities.

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\(^{15}\)Definitions 2.1.6, 2.2.2

\(^{16}\)Definition 2.1.6

\(^{17}\)Definition 2.1.15
Theorem 2.2.17. Let $X$ be a variety over a field of characteristic zero. Let $x \in X$ be a point where $X$ has a locally toric singularity. There exists a projective birational transformation $Y \to X$, which is an isomorphism over a certain neighborhood $U$ of $x \in X$, and a Weil divisor $D_Y$ on $Y$ such that $(Y, D_Y)$ is a strict toroidal embedding which has a single closed stratum, and all its strata pass through $x \in X$.

Proof. Consider the structure of a toroidal embedding on a neighborhood $(U, D_U)$ of $x$. We can assume (by shrinking $U$ if necessary) that there exists a unique closed stratum on $U$ which passes through $x$. This implies that $(U, D_U)$ is extendable, by Example 2.2.7. Let $D_X = D_U$ be the closure of $D_U$, and $(V, D_V) \subset (X, D_X)$ be the saturation. It suffices to apply Theorem 2.2.11 to $(V, D_V) \subset (X, D_X)$.

3. Toroidal embeddings

3.1. Toric varieties. Let $K$ be a field and $T = \text{Spec}(K[[x_1, x_1^{-1}], \ldots, x_n, x_n^{-1}])$ be an $n$-dimensional torus over $K$. Denote by

$$M := \text{Hom}_{alg.gr.}(T, K^*) = \langle x_1, x_1^{-1}, \ldots, x_n, x_n^{-1} \rangle = \{x^\alpha | \alpha \in \mathbb{Z}^n\}$$

the lattice of the group homomorphisms to $K^*$, i.e. characters of $T$. Then the dual lattice

$$N = \text{Hom}_{alg.gr.}(K^*, T) \simeq M^\vee$$

can be identified with the lattice of 1-parameter subgroups of $T$, that is the homomorphisms

$$K^* \to T, \quad t \mapsto (t^{a_1}, \ldots, t^{a_n}).$$

The vector space $M^\mathbb{Q} := M \otimes \mathbb{Q}$ is dual to $N^\mathbb{Q} := N \otimes \mathbb{Q}$. Denote by $(v, w)$ the relevant pairing for $v \in N^\mathbb{Q}, w \in M^\mathbb{Q}$.

By a cone in this paper we mean a convex set

$$\sigma = Q_{\geq 0} \cdot v_1 + \ldots + Q_{\geq 0} \cdot v_k \subseteq N^\mathbb{Q}$$

in $N^\mathbb{Q}$, for some vectors $v_i \in N^\mathbb{Q}$. A cone is strictly convex if it contains no line. For any cone $\sigma \subset N^\mathbb{Q}$ we denote by $\sigma^\vee$ the dual cone,

$$\sigma^\vee := \{m \in M^\mathbb{Q} | (v, m) \geq 0 \text{ for any } v \in \sigma\} \subseteq M^\mathbb{Q},$$

and by $(\sigma^\vee)_{\text{integ}} := \sigma^\vee \cap M$ the monoid of the integral vectors in $\sigma^\vee$.

This associates with a cone $\sigma$ a (normal) affine toric variety

$$X_\sigma := \text{Spec}(K[\langle (\sigma^\vee)_{\text{integ}} \rangle]) \supseteq T.$$

Definition 3.1.1. (see [Dan], [Oda88]). By a fan $\Sigma$ in $N^\mathbb{Q}$ we mean a finite collection of finitely generated strictly convex cones $\sigma$ in $N^\mathbb{Q} \supset N$ such that

- any face of a cone in $\Sigma$ belongs to $\Sigma$,
- any two cones of $\Sigma$ intersect in a common face.

By the support of the fan we mean the union of all its faces, $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

If $\sigma$ is a face of $\sigma'$ we shall write $\sigma \leq \sigma'$.
To a fan $\Sigma$ there is associated a toric variety $X_\Sigma \supset T$, obtained by gluing $X_\sigma$, where $\sigma \in \Sigma$.

It is a normal variety on which a torus $T$ acts effectively with an open dense orbit (see [KKMSD73], [Dan], [Oda88], [Ful93]).

3.2. The orbit stratification of toric varieties. To each cone $\sigma \in \Sigma$ corresponds an open affine invariant subset $X_\sigma$ and its unique closed orbit $O_\sigma$.

The orbits form a locally closed smooth stratification, and $\tau \preceq \sigma$ if and only if $O_\sigma \subset \bar{O}_\tau$ (or in our notation $O_\sigma \leq O_\tau$).

Definition 3.2.1. Let $\Sigma$ be a fan and $\tau \in \Sigma$. The star of the cone $\tau$ is defined as follows:

$$\text{Star}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \tau \preceq \sigma\},$$

The orbits in the closure $\overline{O}_\tau$ of the orbit $O_\tau$ on the toric variety $X_\Sigma$ correspond to the cones of $\text{Star}(\tau, \Sigma)$.

Denote by $N_\tau^\Sigma$ the subspace of $N_\tau^\Sigma$ spanned by the cone $\tau$. It defines the lattice $N_\tau := N \cap N_\tau^\Sigma$. The dual space to $N_\tau^\Sigma$ is isomorphic to $M_\tau^\Sigma := M_\tau^\Sigma / \tau^\perp$, where $\tau^\perp := \{v \in M_\tau^\Sigma \mid (v, w) = 0 \mid w \in \tau\}$, and the dual cone to $\tau^\perp := (\tau, N_\tau^\Sigma)$ is isomorphic to $\tau^\vee := \tau^\vee / \tau^\perp$ in $M_\tau^\Sigma = M_\tau^\Sigma / \tau^\perp$.

The subtorus $T_\tau$ corresponding to the sublattice $N_\tau := N \cap N_\tau^\Sigma$ is isomorphic to the stabilizer of the points in $O_\tau$.

The quotient torus $\overline{T}_\tau := T / T_\tau$ corresponds to the quotient space $\overline{N}_\tau^\Sigma := N_\tau^\Sigma / N_\tau^\Sigma$. It acts transitively on the big orbit $O_\tau \subset \overline{O}_\tau$ making $\overline{O}_\tau$ into a toric variety.

Denote by $\pi_\tau : N_\tau^\Sigma \to \overline{N}_\tau^\Sigma$ the projection map. The toric subvariety $\overline{O}_\tau \supset O_\tau$ corresponds to the quotient fan

$$\text{Star}(\tau, \Sigma)/\tau := \{\pi_\tau(\sigma) \mid \sigma \in \text{Star}(\tau, \Sigma)\},$$

(see [KKMSD73], [Dan], [Oda88], [Ful93]).

If $\sigma \in \text{Star}(\tau, \Sigma)$ then

$$\pi_\tau(\sigma)^\vee = \sigma^\vee \cap \tau^\perp.$$

The natural embedding $\overline{O}_\tau \cap X_\sigma \to X_\sigma$ is given by the projection

$$\sigma^\vee \cap M \to \sigma^\vee \cap \tau^\perp \cap M,$$

such that $m \mapsto m$ if $m \in \sigma^\vee \cap \tau^\perp \cap M$, and $m \mapsto 0$ otherwise. This defines an epimorphism:

$$K[(\sigma^\vee)^{\text{integ}}] \to K[(\pi_\tau(\sigma)^\vee)^{\text{integ}}],$$

identifying the closure of the orbit $\overline{O}_\tau$ with a toric variety. In particular, we have that

Lemma 3.2.2. The closures of the toric orbits are normal (toric) varieties.

3.3. Birational morphisms of toric varieties.

Definition 3.3.1. (see [KKMSD73], [Oda88], [Dan], [Ful93]). A birational toric morphism of toric varieties $X_\Sigma \to X_\Delta$ is a morphism identical on $T \subset X_\Sigma, X_\Delta$.

Definition 3.3.2. (see [KKMSD73], [Oda88], [Dan], [Ful93]). A subdivision of a fan $\Sigma$ is a fan $\Delta$ such that $|\Delta| = |\Sigma|$ and any cone $\sigma \in \Sigma$ is the union of cones $\delta \in \Delta$. 
Theorem 3.3.3. (see [KKMSD73], [Oda88], [Dan], [Ful93]) There exists a bijective correspondence between proper toric birational morphisms $X_\Sigma \to X_\Delta$ and the subdivisions $\Sigma$ of the fan $\Delta$.

The theorem was originally stated over algebraically closed field but remains valid without this assumption with unchanged proof.

3.4. Toroidal embeddings. Toroidal embeddings were introduced in [KKMSD73] over an algebraically closed field. The following definition over an arbitrary non-closed field is essentially due to Mumford. It is equivalent to the definition of Kato ([Kat89b]), who considered toroidal embeddings in a more general context of logarithmic geometry (and refer to them as logarithmically smooth varieties). It is also equivalent to another Mumford’s definition over a base field which is algebraically closed.

Definition 3.4.1. A strict toroidal embedding (respectively toroidal embedding) is a variety $X$ with an open subset $U$ such that any point $x \in X$ admits an open neighborhood $V \subset X$ (respectively an étale neighborhood $f : V \to X$), and an étale morphism $\phi : (V,U_V) \to (X_\sigma,T)$, where $U_\sigma = U \cap V$ (respectively $U_\sigma = f^{-1}(U)$), and $\phi^{-1}(T) = U_\sigma$. Such a morphism is called an étale chart. (In the sequel, and prequel we often represent a toroidal embedding $(X,U)$ as $(X,D)$ for the reduced divisor $D = X \setminus U$.)

The definition and Lemma 3.2.2 imply immediately that

Lemma 3.4.2. The irreducible components of the divisor $D = X \setminus U$ on a strict toroidal embedding are strict toroidal embeddings. In particular, they are normal.

Proof. By Lemma 6.2.6, the components of a toric divisor are normal. On the other hand, the inverse images of the components of a toric divisor under étale morphism, are also normal and thus, in particular, locally irreducible. Hence the irreducible components of $D$ are locally the inverse images of irreducible components of a toric divisor, and thus are normal. Hence they are étale equivalent to the components of the toric divisor, which are, by Lemma 6.2.6, isomorphic to toric varieties. ♣

Remark 3.4.3. The irreducible components of the divisor $D$ on toroidal embeddings are not necessarily normal, as those components may admit self-intersections.

Remark 3.4.4. (1) Equivalently, one rephrases Definition 3.4.1 of toroidal embeddings in the language of the completion of local rings at $K$-points: A variety $X$ over an algebraically closed field $K = \overline{K}$ with an open subset $U$ is a toroidal embedding if for any point $x \in X$ there is an isomorphism $\hat{\phi} : \widehat{O}_{X,x} \to \widehat{O}(X_\sigma)_y$, where $X_\sigma$ is a toric variety containing a torus $T$ and corresponding to the cone $\sigma$ of the maximal dimension and $y \in O_\sigma$ is a closed point in the orbit $O_\sigma$, and $\hat{\phi}$ takes the ideal of $X \setminus U$ to that of $X_\sigma \setminus T$. The above definition is also due to Mumford [KKMSD73].

(2) A toroidal embedding is strict (or without self-intersections) if, additionally, the irreducible components of the divisor $D = X \setminus U$ are normal (so they do not have self-intersections). (see [KKMSD73], and Lemma
This characterization of strict toroidal embeddings is due to Mumford [KKMSD73]. Later it was proven over any field not necessarily algebraically closed, in particular, in [Den] (see also Lemma 3.8.3).

The theory of toric, and toroidal varieties was initially considered over algebraically closed fields, and mostly in the language of the completions of the local rings. As was observed by Kato in [Kat89b], most of the results can be extended to the case of nonclosed fields in a more convenient language of logarithmic geometry which uses charts in the Zariski or étale topology and does not require assumption on the algebraically closed base field.

3.5. **Conical complexes.** The notion of the conical complex associated with a strict toroidal embedding is a natural extension of the fan associated with a toric variety.

The following definition of the conical complex is equivalent to the original one from [KKMSD73]. We use this formalism since in the later sections we are going to consider a variation of this notion in a more general setting of semicomplexes. (Definition 7.7.2)

**Definition 3.5.1.** By a conical complex Σ we mean a finite partially ordered set of finitely generated strictly convex cones σ of maximal dimension in $N^Q_σ \supset N_σ$ such that

1. For any $τ \preceq σ$ there is a linear injective map $i_{τ,σ} : τ \to σ$ such that $i_{τ,σ}(τ)$ is a face of σ, with the lattice $i_{τ,σ}(N_τ)$ saturated in $N_σ$. Moreover, each face of σ can be presented in such a form.
2. If $τ \preceq σ \preceq δ$ then $i_{τ,δ} = i_{σ,δ}i_{τ,σ}$.

The definition implies that the intersection of two cones is the union of common faces.

3.6. **Support of a complex.** By the support of a complex Σ we mean the topological space

$$|Σ| := \bigsqcup_{σ ∈ Σ} σ/ \sim$$

where ~ is the equivalence relation generated by the inclusions $i_{τ,σ} : τ \to σ$.

There is an inclusion $ϕ_σ : σ \to |Σ|$, onto the closed subset $|σ| \subset |Σ|$ homeomorphic to σ.

Denote by $\text{int}(σ)$ the relative interior of the cone σ. This means the interior of σ in $N^Q_σ$. There is an inclusion $\text{int}(σ) \to |Σ|$ onto a locally subset $|\text{int}(σ)|$ which allows to write the support of $|Σ|$ as the disjoint union

$$|Σ| := \bigcup_{σ ∈ Σ} \text{int}(σ)$$

In general, by the support of any subset $Σ_0$ of a complex Σ is defined as

$$|Σ_0| := \bigcup_{σ ∈ Σ_0} \text{int}(σ)$$
3.7. Mumford’s definition of complexes. Using the above we see that the conical complexes define topological spaces that are covered by the closed cones. This allows to define conical complexes as topological spaces with a local cone structure.

**Definition 3.7.1.** ([KKMSD73], [Pay06]) A conical complex $\Sigma$ is a topological space $|\Sigma|$ together with a finite collection $\Sigma$ of closed subsets $|\sigma| \in \Sigma$ of $|\Sigma|$ such that

1. For each $|\sigma|$ there is a finitely generated lattice $M_\sigma$ of continuous functions on $|\sigma|$, and the dual lattice $N_\sigma = Hom(M, \mathbb{Z})$ in the vector space $N_\sigma^Q = N_\sigma \otimes \mathbb{Q}$.
2. The natural map $\phi_\sigma : |\sigma| \to N_\sigma^Q$ given by $x \mapsto (u \mapsto u(x))$ maps $\sigma$ homeomorphically onto a rational convex cone $\sigma := \phi_\sigma(|\sigma|)$.
3. The inverse image under $\phi_\sigma$ of each face of $\sigma$ is some $|\tau| \in \Sigma$, with $|\tau| \subset |\sigma|$ and $M_\tau = \{u|_\tau : u \in M_\sigma\}$.
4. The topological space $|\Sigma|$ is the disjoint union of the relative interiors of the $|\sigma| \in \Sigma$.

Identifying $\sigma$ with $|\sigma|$ we obtain the natural maps $i_{\tau \sigma} := \phi_\sigma \phi_\tau^{-1} : \tau \to \sigma$ satisfying the conditions from Definition 3.5.1.

3.8. Conical complexes associated with strict toroidal embeddings.

3.8.1. Mumford’s lemma. The following useful results are essentially due to Mumford.

**Lemma 3.8.2.** ([KKMSD73], [Den]) Let $f : (X, D_X) \to (Y, D_Y)$ be an étale morphism of normal varieties, mapping a point $x \in X$ to a point $y = f(x) \in Y$, and such that

1. The components of $D_Y$ are normal.
2. $f^{-1}(D_Y) = D_X$.

Then $f$ defines a bijective correspondence between the components of $D_X$ through $x$ and the components of $D_Y$ through $y$. The above correspondence extends to a bijective correspondence between the effective Cartier divisors on $\text{Spec}(\mathcal{O}_{X,x})$ supported on $D_X$ and the effective Cartier divisors on $\text{Spec}(\mathcal{O}_{Y,y})$ supported on $D_Y$.

**Proof.** The correspondence between Weil or Cartier divisors can be deduced by adapting the arguments of the proof of [KKMSD73, Lemma 1, p.60], or more precisely the proof of [Den, Lemma 2.3].

First, the bijective correspondence between the irreducible components of Weil divisors through $x$ and $y$ follows from the fact that the inverse image of any irreducible component is normal so is locally an irreducible component. In other words, any component of $D_Y$ through $y$ induces a unique component of $D_X$ through $x$, which is locally its inverse image.

Moreover, since the morphism $\text{Spec}(\mathcal{O}_{X,x}) \to \text{Spec}(\mathcal{O}_{Y,y})$ is flat and even étale the schematic pull-back of a divisor is a divisor. Thus there is a bijective correspondence between the Weil divisors of $\text{Spec}(\mathcal{O}_{X,x})$ supported on $D_X$ and those of $\text{Spec}(\mathcal{O}_{Y,y})$ supported on $D_Y$.

We need to show that the image $E_Y$ of an effective Cartier divisor $E_X$ on $\text{Spec}(\mathcal{O}_{X,x})$ supported on $D_X$ is Cartier on $\text{Spec}(\mathcal{O}_{Y,y})$. We will show that $I_{E_Y}$ is principal at a point $y$. 
By the assumption, $\mathcal{I}_{E_X} = \mathcal{I}_{E_Y} \cdot \mathcal{O}_X$ is principal so $\tilde{\mathcal{I}}_{E_X} = \mathcal{I}_{E_Y} \cdot \hat{\mathcal{O}}_{X,x}$ is principal in $\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{Y,y} \otimes_{k(y)} k(x)$ as well. This, in turn, implies that the ideal of the initial forms in the graded ring

$$\text{in}(\mathcal{I}_{E_X}) = \text{in}(\tilde{\mathcal{I}}_{E_X}) \subset \mathcal{O}_X/m_x \oplus \ldots \oplus m_x^n/m_x^{n+1} \oplus \ldots$$

is principal and generated by any homogenous form of the lowest degree. Such a generating initial form can be chosen as in($f$), where $f \in \mathcal{I}_{E_Y}$. Then for any $g \in \mathcal{I}_{Y,y}$, with \text{in}($g$) $\in$ \text{in}(\mathcal{I}_{Y,y}) one can write as \text{in}($g$) $= H \cdot \text{in}(f)$, with $H \in \text{in}(\mathcal{O}_{X,x}) = \text{in}(\mathcal{O}_{Y,y}) \otimes_{k(y)} k(x)$. This implies that all the coefficients of $H$ are in $k(y)$, and $H \in \text{in}(\mathcal{O}_{Y,y})$. So in$_y(\mathcal{I}_{E_Y}) \subset \text{in}(\mathcal{O}_{Y,y})$ is principal and generated by \text{in}($f$). Thus $\mathcal{I}_{E_Y} \cdot \hat{\mathcal{O}}_{Y,y}$ is principal in $\hat{\mathcal{O}}_{Y,y}$ and generated by $f \in \mathcal{I}_{E_Y}$. Now by faithful flatness of the completion of local ring the ideal $\mathcal{I}_{E_Y} = f \cdot \mathcal{O}_{Y,y}$ is principal in $\mathcal{O}_{Y,y}$.

\[\square\]

The following result shows that the Mumford condition on the normality of the components can be used to detect strict toroidal embeddings.

**Corollary 3.8.3.** [KKMSD73, page 195 footnote]

Let $(Y, D)$ be a toroidal embedding. Then $(Y, D)$ is a strict toroidal embedding iff the divisor $D$ has normal components.

**Proof.** The ”only if” part follows from Lemma 3.4.2. We need to prove ” if ” part of the corollary. The question is local. Consider an étale strict toroidal neighborhood $(U, D_U) \rightarrow (Y, D_Y)$ of $y \in Y$ with a point $x \in X$ over $y$. Let $s_x$ be the stratum through $x \in X$, and $s_y$ be the stratum through $y \in Y$.

One can assume that there is an étale morphism

$$\phi : (U, D_U) \rightarrow (X_\sigma \times T, D_\sigma \times T)$$

with closed orbit $O_\sigma \times T$, where $\sigma$ is a cone of maximal dimension in $N_\sigma$. Note that the induced map

$$s_x \rightarrow O_\sigma \times T \simeq T$$

is an étale morphism of smooth subvarieties.

Moreover, any morphism

$$\phi : (U, D_U) \rightarrow (X_\sigma \times T, D_\sigma \times T)$$

which is étale at $x$, is defined by a smooth morphism $(U, D_U) \rightarrow (X_\sigma, D_\sigma)$, and a morphism $U \rightarrow T$, such that its restriction

$$s_x \rightarrow O_\sigma \times T \simeq T$$

to the stratum $s_x$ is étale. In such a case the completion of a local ring at $x$ is given by

$$\hat{\mathcal{O}}_{x,X} \simeq \hat{\mathcal{O}}_{t,X,x} \otimes_{K(t)} K(x) = K(x)[[u_1, \ldots, u_k, (\mathcal{O}_V^{\text{integ}})]]$$

where $(\mathcal{O}_V^{\text{integ}})$ generate $\mathbb{Z}_{s_x}$, and $u_1, \ldots, u_k$ determine the étale morphism $s_x \rightarrow T$.

By the above, given an étale morphism $\phi$, a morphism $\psi : (U, D_U) \rightarrow (X_\sigma, D_\sigma)$ is étale if the pull-backs in $\psi^*(\mathcal{O}_V^{\text{integ}})$ differ from $\phi^*(\mathcal{O}_V^{\text{integ}})$ by units, and the restriction morphism $s_x \rightarrow T$ is étale, that is given by a set of local parameters $v_1, \ldots, v_k$. 
We need to show that there is such a morphism for a certain Zariski neighborhood $V$ of $y$. First, the existence of the étale morphism $s_y \to T$ is clear as $s_y$ is smooth étale isomorphic to $s_x$. The morphism $s_y \to T$ is given by some parameters and extends to a morphism $\psi_V : V \to T$ from a certain neighborhood $V$ of $y$.

On the other hand, the morphism $(U, D_U) \to (X_\sigma, D_\sigma)$ is given by the pull-backs of the $T$-invariant generators of $K[X_\sigma] = K[(\sigma^\vee)^\text{integ}]$. By the previous lemma, the Cartier divisors on $U$, which are also pull-backs of the principal toric divisors defined by characters in $M_\sigma$ on $X_\sigma$ descend to Cartier divisors in a neighborhood of $y \in Y$. This means that such descent exists up to units, defining the smooth morphism $\phi_V : (V, D_V) \to (X_\sigma, D_\sigma)$. The pull-backs of these functions define an étale morphism $\psi : (U, D_U) \to (X_\sigma, D_\sigma)$, which factors through a morphism

$$\phi_U \times \psi_V : (V, D_V) \to (X_\sigma \times T, D_\sigma \times T).$$

It follows that

$$\tilde{O}_{Y,Y} \simeq \tilde{O}_{Y,X_\sigma} \otimes_{K(t)} K(y) = K(y)[[v_1, \ldots, v_k, (\sigma^\vee)^\text{integ}]],$$

and the morphism $\phi_U \times \psi_V$ is étale. This shows that $(Y, D)$ is a strict toroidal embedding.

3.8.4. The monoids of Cartier divisors associated with strata. Let $(X, D)$ be a strict toroidal embedding with the stratification $S = S_D$. Following [KKMSD73] we associate with a stratum $s \in S$ the canonical monoids (commutative semigroups) and groups (lattices):

1. $M_s^+ = \text{Cart}(s, S)^+$ is the monoid of the Cartier divisors on $\text{Star}(s, S) = \bigcup_{s \leq s'} s'$ supported on $D \cap \text{Star}(s, S)$

2. $M_s = \text{Cart}(s, S)$ is the free abelian group of the Cartier divisors on $\text{Star}(s, S)$ supported on $D \cap \text{Star}(s, S)$. So $M_s$ is a lattice which is the groupification of $M_s^+$

3. $N_s := \text{Hom}(M_s, \mathbb{Z})$ is the dual lattice with the vector space $N_s^Q := N_s \otimes \mathbb{Z}Q$.

4. $\sigma_s = \{v \in N_s^Q \mid F(v) \geq 0, \quad F \in M_s^+\}$ is the associated strictly convex cone of maximal dimension in $N_s^Q$.

Lemma 3.8.5. [KKMSD73] Let $Y \subset \text{Star}(s, S)$ be an open subset intersecting $s \in S$. Let $\phi : (Y, U) \to (X_\sigma, T)$ be an étale map mapping $s$ into a closed orbit $O_\sigma \subset X_\sigma$. Then the group $\text{Cart}(Y, U)$ (resp. $\text{Cart}(Y, U)^+$) of Cartier divisors (resp. effective Cartier divisor) supported on $Y \setminus U$ is the pull-back of the group of toric Cartier divisors supported on $X_\sigma \setminus T$.

In particular,

$$\text{Cart}(Y, U) \simeq M_s \simeq M_{\sigma} = M_{\sigma}/(\sigma^\perp)^\text{integ},$$

$$\text{Cart}(Y, U)^+ \simeq M_s^+ \simeq (\sigma^\vee)^\text{integ} = (\sigma^\perp)^\text{integ}/(\sigma^\perp)^\text{integ},$$

where $\sigma := (\sigma, N_\sigma^Q)$ is the associated cone of maximal dimension in $N_\sigma^Q \subset N^Q$.

Proof. The Lemma is a consequence of Lemma 3.8.2. ♣
Corollary 3.8.6. With the preceding notation, the cone $\sigma = (\sigma, N^\sigma_\sigma)$ (defined by the chart) is dual to $M^\sigma_s$. In particular, the cone $\sigma$ with the lattice $N^\sigma_\sigma$ is independent of chart and uniquely defined for the stratum $s \in S : \sigma_s = \sigma$.

Corollary 3.8.7. [KKMSD73] (see also [Oda88], [Ful93]) With the preceding notation, if $s \leq s'$ then $\text{Star}(s, S) \supset \text{Star}(s', S)$, is an open subset.

Let $Y \subseteq \text{Star}(s, S)$ be an open subset with étale chart $\phi : (Y, U) \to (X_\sigma, T)$ and $Y' : = Y \cap \text{Star}(s', S)$ be its open subset. The restriction $\phi|_{Y'}$ of $\phi$ defines an étale morphism

$$\phi|_{Y'} : (Y', U) \to (X_{\sigma'}, T)$$

into the open subset $X_{\sigma'} \subset X_\sigma$, with $\sigma' \leq \sigma$ corresponding to $s'$.

There is a natural surjection

$$\text{Cart}^+(Y, U) \cong M^+_s \cong (\sigma^\vee)^{\text{integ}} \quad \longrightarrow \quad \text{Cart}^+(Y', U) \cong M_{s'} \cong (\sigma'^\vee)^{\text{integ}}$$

induced by the restriction of the Cartier divisors. Its dual map corresponds to the face inclusion $\sigma' \hookrightarrow \sigma$.

Proof. This translates into a a well known fact of toric varieties and cones [KKMSD73], [Oda88], [Ful93]. If $\sigma' \leq \sigma$, then the open immersion $X_{\sigma'} \hookrightarrow X_\sigma$ correspond to the localization $\mathcal{O}(X_{\sigma'}) = \mathcal{O}(X_\sigma)_m$ by a monomial $m \in (\sigma'^\vee)^{\text{integ}} \subset \mathcal{O}(X_\sigma)$. Hence

$$(\sigma'^\vee)^{\text{integ}} = (\sigma^\vee)^{\text{integ}} + \mathbb{Z} \cdot m = (\sigma^\vee)^{\text{integ}} + ((\sigma')^\perp)^{\text{integ}}.$$ 

Consequently, there is a surjection

$$(\sigma'^\vee)^{\text{integ}} \cong (\sigma^\vee)^{\text{integ}} / (\sigma'^\perp)^{\text{integ}} \cong (\sigma^\vee)^{\text{integ}} / (\sigma'^\perp)^{\text{integ}} = (\sigma'^\vee)^{\text{integ}},$$

and the dual injective map of cones and lattices: $i_{\sigma' \sigma} : \sigma' \to \sigma$, where $i_{\sigma' \sigma}(\sigma')$ is described as the face $\{m\}^\perp \subset \sigma$.

3.8.8. Toroidal embeddings and logarithmic smoothness.

Theorem 3.8.9. (Kato-Mumford) , [Kat89b] Let $(X, D)$ be a strict toroidal embedding, and $s$ be the stratum through $x \in X$. Then

\begin{enumerate}
\item The stratum $s$ is locally the intersection of the components of the divisor $D$
\item $\mathcal{O}_{x,X} \cong K(x)[[u_1, \ldots, u_k, M^+_s]]$, where $u_1, \ldots, u_k$ generate $\mathcal{O}_{x,X}/\mathcal{I}_s \cong \mathcal{O}_{x,s}$.
\item $\dim(s) + \text{rank}(M_s) = \dim(X)$.
\end{enumerate}

Proof. Let $\phi : (X, U) \to (X_\sigma, T)$ be an étale map mapping a certain $x \in s$ to a point $t$ in the closed orbit $O_\sigma \subset X_\sigma$.

The statements are valid for the toric varieties. Moreover, by corollary 3.8.7, we have $M^+_s \cong (\sigma^\vee)^{\text{integ}}$, so that

$$\mathcal{O}_{x,X} \cong \mathcal{O}_{t,X_\sigma} \otimes_{K(t)} K(x) = K(x)[[u_1, \ldots, u_k, (\sigma^\vee)^{\text{integ}}]],$$

where $(\sigma^\vee)^{\text{integ}} = (\sigma^\vee)^{\text{integ}} / (\sigma^\perp)^{\text{integ}}$ is isomorphic to the semigroup $M^+_s$. 

\hfill \Box
3.8.10. Conical complexes associated with strict toroidal embeddings.

**Theorem 3.8.11.** ([KKMSD73]) A strict toroidal embedding \((X, U)\) determines a unique associated conical complex \(\Sigma\). Moreover, there is a bijective correspondence between the strata on a toroidal embedding \(X\) and the cones of the complex.

\[ \tau \rightarrow s_\tau. \]

Furthermore \(\tau \leq \sigma\) iff \(s_\sigma \preceq s_\tau\) (i.e., \(s_\sigma\) is contained in the closure \(\overline{s_\tau}\) of \(s_\tau\)).

**Proof.** The theorem was initially proven over algebraically closed field but it extends to non-closed field using the results above: Lemma 3.8.5, Corollaries 3.8.6, 3.8.7. The conical complex \(\Sigma := \{\sigma_s : s \in S\} \) is obtained glueing of \(\sigma_s\) along the natural inclusion maps \(\sigma_s \hookrightarrow \sigma_{s'}\) for \(s \leq s'\) as in Corollary 3.8.7. The verification is straightforward. The gluing is essentially the same as for the cones in fans corresponding to toric varieties \(X_\sigma\).

**Definition 3.8.12.** By a piecewise linear functions \(f : |\Sigma| \rightarrow \mathbb{Z}\) we mean a function \(f\), such that the restriction \(f|_{\sigma_s \cap N_{\sigma}}\) is defined by the functional \(m \in (\sigma^\vee)^{\text{integ}}\).

Another consequence of Corollary 3.8.7 is the following result.

**Corollary 3.8.13.** ([KKMSD73]) The piecewise linear functions \(|\Sigma| \rightarrow \mathbb{Z}\) are in bijective correspondence with Cartier divisors on \(X\) supported on \(D = X \setminus U\).

**Proof.** The Cartier divisors on Star(\(s, S\)) supported on \(D\) correspond to the integral linear functions defined by \(m_\sigma \in M_\sigma\) on \(\sigma = \sigma_s\). Moreover, by Corollary 3.8.7, if \(s \leq s'\) and \(s'\) corresponds to a face \(\sigma'\) of \(\sigma\) then \(m_{\sigma'|\sigma}\) is the restriction \(m_{\sigma'|}\).

3.8.14. Saturated subsets. Recall that a subset of a strict toroidal embedding is called saturated if it is the union of strata. (see also Definition 2.1.6.)

Let \(\Sigma\) be the conical complex associated with a strict toroidal variety \(X\). Similarly with any subset \(\Sigma_0\) of \(\Sigma\) one can associate the constructible saturated subset \(X(\Sigma_0) := \bigcup_{\tau \in \Sigma_0} s_\tau\).

Then it follows immediately that

**Lemma 3.8.15.** \(X(\Sigma_0)\) is open (closed under generization) iff \(\Sigma_0\) is a subcomplex.

In particular, with a cone \(\sigma\) one can associate the open subset \(X(\sigma) = \bigcup_{\tau \preceq \sigma} s_\tau = \text{Star}(s, S)\),

where \(s\) is the stratum corresponding to \(\sigma\).

**Lemma 3.8.16.** The following are equivalent for \(\Sigma_0 \subset \Sigma\):

1. \(\Sigma_0\) is a subcomplex
2. \(X(\Sigma_0) \subset X\) is open
3. \(|\Sigma_0|\) is a closed subset of \(|\Sigma|\).

**Proof.** The property is local and can be verified for the the closed cover \(|\sigma|\) of \(|\Sigma|\).
3.9. Maps of conical complexes.

**Definition 3.9.1.** ([KKMSD73]) A map of conical complexes $f : \Sigma \rightarrow \Sigma'$, is a function which assigns to a cone $\sigma \in \Sigma$ a unique cone $\sigma' \in \Sigma'$, together with the linear map $f_{\sigma,\sigma'} : (\sigma, N^Q_\sigma) \rightarrow (\sigma', N^Q_{\sigma'})$, such that,

1. $f_{\sigma,\sigma'}(N_\sigma) \subseteq (N_{\sigma'})$.
2. $f_{\sigma,\sigma'}(\text{int}(\sigma)) \subseteq \text{int}(\sigma')$.
3. If $\tau \preceq \sigma$ then $\tau' \preceq \sigma'$ and $f_{\sigma,\sigma'}|_{\tau,\sigma} = i_{\tau',\sigma'}f_{\tau,\tau'}$.

Note that the map $f$ induces a unique continuous map of topological spaces $[f] : [\Sigma] \rightarrow [\Sigma']$. Equivalently

**Definition 3.9.2.** ([KKMSD73], [Pay06]) A map of conical complexes $f : \Sigma \rightarrow \Sigma'$ is a continuous map of topological spaces $[\Sigma] \rightarrow [\Sigma']$ such that, for each cone $\sigma \in \Sigma$ there is some $\sigma' \in \Sigma'$ with $f(\sigma) \subseteq \sigma'$ and $f^*M_{\sigma'} \subseteq M_{\sigma}$.

**Definition 3.9.3.** A subdivision of a complex $\Sigma$ is a map $\Delta \rightarrow \Sigma$ such that $|f|$ is a homeomorphism identifying $|\Delta| = |\Sigma|$, so that any cone $\sigma \in \Sigma$ is a union of cones $\delta \in \Delta$. A subdivision $\Delta$ of $\Sigma$ which is regular is called desingularization of $\Sigma$.

A map $f : \Sigma \rightarrow \Sigma'$ will be called a local isomorphism if each $f_{\sigma,\sigma'}$ is an isomorphism.

If $f$ is bijective and is a local isomorphism then $f$ is called an isomorphism.

If $f$ is injective and is a local isomorphism then $\Sigma$ is isomorphic to a subcomplex of $\Sigma'$.

A map $f : \Sigma \rightarrow \Sigma'$ is called a regular local projection if for each $\sigma \in \Sigma$ there is a decomposition $\sigma = \sigma' \times \tau$, where $\tau$ is regular and $f_{\sigma,\sigma'} : \sigma = \sigma' \times \tau \rightarrow \sigma'$ is the projection on the first component.

**Lemma 3.9.4.** Let $f : \Delta \rightarrow \Sigma$, be a subdivision, and $|f| : |\Delta| \rightarrow |\Sigma|$ be the induced homeomorphism of the topological spaces. Then

$$\Delta^\sigma := \{ \tau \in \Delta : |\tau| \subseteq |f|^\circ(|\sigma|) \}$$

defines a fan in $N^Q_\sigma$, which is a subdivision of the cone $\sigma$.

3.10. Toroidal morphisms of toroidal embeddings. The following definitions are equivalent to the definitions of log smooth morphisms in characteristic zero.

**Definition 3.10.1.** A morphism of strict toroidal embeddings $f : (X, U) \rightarrow (Y, V)$ is strictly toroidal if there exists the induced map of open neighborhoods $f' : (X', U') \rightarrow (Y', V')$, and a commutative diagram of

$$(X', U') \rightarrow (X'_{\sigma'}, T') \downarrow f' \downarrow (Y', V') \rightarrow (X_{\sigma}, T)$$

with horizontal morphisms étale and a vertical toric map $(X_{\sigma'}, T') \rightarrow (X_{\sigma}, T)$.

**Definition 3.10.2.** A morphism of toroidal embeddings $f : (X, U) \rightarrow (Y, V)$ is toroidal if there exists the induced map of open étale neighborhoods $f' : (X', U') \rightarrow (Y', V')$, and a commutative diagram as above.
3.11. **Canonical birational toroidal maps.** The following definition is equivalent to [KKMSD73, Definition 3 p.87, Definition 1 p.73] in view of Theorem 3.11.2.

**Definition 3.11.1.** A birational morphism of strict toroidal embeddings $f : (Y, U) \to (X, U)$ will be called *canonical toroidal* if for any $x \in s \subseteq X$ there exists an open neighborhood $U_x$ of $x$, an étale morphism $U_x \to X$ and a fan $\Delta^s$ mapping to $\sigma = \sigma_s$ and the fiber square of morphisms of varieties

\[
U_x \times_{\mathcal{X}_{\sigma_s}} \mathcal{X}_s \cong \left(f^{-1}(U_x), f^{-1}(U_x) \cap U \right) \to \left(X_{\Delta^s}, T\right)
\]

\[
\downarrow f
\]

\[
\left(U_x, U_x \cap U\right) \to \left(X_{\sigma_s}, T\right)
\]

Here $f^{-1}(U_x) := U_x \times_X Y$.

We will prove later that such a local description of canonical toroidal morphisms does not depend upon the choice of étale charts. This fact can be described nicely using the following Hironaka condition:

For any geometric $\overline{K}$-points $x, y$ which are in the same stratum every isomorphism $\alpha : \mathcal{X}_{\overline{K}} \to \mathcal{X}_{\overline{y}}$ preserving stratification can be lifted to an isomorphism $\alpha' : \mathcal{X} \times_X \mathcal{X}_{\overline{K}} \to \mathcal{Y} \times_X \mathcal{X}_{\overline{y}}$ preserving stratification.

(Here $\overline{K}$ is the algebraic closure of $K$, $\mathcal{X}_{\overline{K}} := \mathcal{X} \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$, and $\mathcal{X}_{\overline{y}} := \text{Spec}(\mathcal{O}_{X, y})$).

The Hironaka condition is extremely important when considering stratified toroidal varieties. As we can see the condition is satisfied for the canonical birational maps of strict toroidal embeddings. (Lemma 7.14.6).

**Theorem 3.11.2.** ([KKMSD73, Theorem 6 p.90]) Let $(X, U)$ be a strict toroidal embedding, and $\Sigma$ be the associated semicomplex. Then there is a bijective correspondence between the subdivisions of $\Sigma$ and the canonical proper birational toroidal maps.

*Proof.* Again, the theorem was originally proven over an algebraically closed field but it can be extended to arbitrary fields (with our definition). (It also can be further generalized to the case of the stratified toroidal varieties in Theorem 7.14.4).

If $Y \to X$ is any morphism of strict toroidal embeddings then the strata are mapped into strata. Moreover, if $t \in T$ maps to $s \in S$ then its face $t' \leq t$ maps to $s' \leq s$. These and other properties of Definition 3.9.1 can be verified locally in the charts where they follow from the properties of the toric maps. Moreover, we get the map of cones between the cones associated with the strata. Because of Lemma 3.8.5, and Corollaries 3.8.6, 3.8.6, the maps between the cones are independent of charts. The restrictions of $f$ to the open stars Star$(s, Sy) \to \text{Star}(t, S_X)$, induce the maps $M_{s,X}^+ \to M_{t,Y}^+$ and the dual maps of cones $\sigma_s \to \sigma_t$ commuting with face inclusions (by Lemma 3.8.7) and defining the maps of the conical complexes $f : \Sigma_Y \to \Sigma_X$.

Moreover, if $Y \to X$ is a canonical birational toroidal morphism then the correspondence between cones defined by the charts shows that the set $\Delta^s := \{ \tau \in \Delta_Y : |\tau| \subseteq |f|^{-1}(|\sigma|)\}$ is a fan inside of $\sigma$. Since the map is proper by using valuative criterion of properness we see that $|\Delta^s| = \sigma$, so $f$ is, in fact a subdivision.
Conversely, a subdivision $\Delta$ of the conical complex $\Sigma$ defines locally for any chart $U \to X_\sigma$ the variety $\tilde{U} := U \times_{X_\sigma} X_{\Delta^\sigma}$ over $U$. The subset $\tilde{U}$ is the union of the canonical open subsets $\tilde{U} = \bigcup_{\tau \in \Delta} \tilde{U}_\tau$, where

$$\tilde{U}_\tau := U \times_{X_\sigma} X_\tau = \text{Spec} \mathcal{O}(U)((\tau^\vee)^{\text{integ}}) = \text{Spec} \left( \sum_{D \in (\tau^\vee)^{\text{integ}}} \mathcal{O}(U)(-D) \right)$$

(see also [KKMSD73], page 74) Note that the Cartier divisors in $(\tau^\vee)^{\text{integ}} \subset M_\sigma$ are naturally contained in $M_\sigma \subset K(X)/\mathcal{O}_X^\times$, where $K(X)$ is a constant sheaf of the rational functions on $X$, so $\sum_{D \in (\tau^\vee)^{\text{integ}}} \mathcal{O}(U)(-D)$ is a subsheaf of $K(X)$ over $U$.

Consequently, the subsets $\tilde{U}_\tau$ are independent of charts, and the canonically determined morphisms $\tilde{U}_\tau \to U$ are birational. This allows to represent $Y$ canonically by gluing the open subsets $\tilde{U}_\tau$ over $U$ along the subsets corresponding to their faces so that

$$Y = \bigcup_{\sigma \in \Sigma} \bigcup_{\tau \in \Delta^\sigma} \text{Spec} \left( \sum_{D \in (\tau^\vee)^{\text{integ}}} \mathcal{O}(U_\sigma)(-D) \right)$$

The natural projection $Y \to X$ is proper and separated as it is locally represented by the morphism $\tilde{U}_\tau := U \times_{X_\sigma} X_\tau \to U$.

As a corollary from the proof we get:

**Corollary 3.11.3.** The strictly toroidal maps determine the maps of the associated conical complexes.

3.11.4. **Smooth maps.**

**Definition 3.11.5.** A morphism of strictly toroidal varieties $f : (Y, D_Y) \to (X, D_X)$ will be called smooth (respectively étale) if it is a smooth (resp. étale) and $f^{-1}(D_X) = D_Y$.

**Lemma 3.11.6.** If $f : (Y, D_Y) \to (X, D_X)$ is a strictly toroidal morphism which is smooth then the corresponding map of the associated conical complexes $\Sigma_X \to \Sigma_Y$ is a local isomorphism.

**Proof.** Consider a local chart $\alpha : U \to X_\sigma$, and the induced smooth morphism

$$\beta := \alpha \circ f : V := f^{-1}(U) \to X_\sigma$$

preserving strata. Then one can locally find a coordinate system $u_1, \ldots, u_s$ on the stratum $\beta^{-1}(O_\sigma) \subset V$ defining an étale chart $V \to X_\sigma \times \mathbb{A}^s$. We can additional assume that $u_1, \ldots, u_s$ do not vanish on $V$, and they induce an étale chart of the form $V \to X_\sigma \times T^s$, by shrinking $V$ if necessary. Here $T^s \subseteq \mathbb{A}^s$ is an $s$-dimensional torus. Thus the morphism $f$ in the étale charts $\alpha, \beta$ is represented by the toric map $X_\sigma \times T^s \to X_\sigma$, defining an isomorphism of the associated cones.

4. **Functorial desingularization of complexes**

4.1. **Regular and singular subcomplexes.** In the sequel, we shall say that cones of a conical complex are disjoint if their intersection is the zero cone.
**Definition 4.1.1.** We say that any nonzero integral vector \( v \in N_\sigma \) is **primitive** if it generates the monoid \( \mathbb{Q}_{\geq 0} \cdot v \cap N_\tau \).

Any strongly convex, finitely generated cone can be written uniquely as
\[
\sigma = \langle v_1, \ldots, v_k \rangle := \mathbb{Q}_{\geq 0} \cdot v_1 + \ldots + \mathbb{Q}_{\geq 0} \cdot v_k,
\]
such that \( v_i \) are primitive vectors, and \( k \) is minimal. We shall call vectors \( v_i \) **vertices** of \( \sigma \).

**Definition 4.1.2.** ([KKMSD73]) We say that a cone \( \sigma \) in \( N^Q \) is **regular** or **nonsingular** if it is generated by a part of a basis of the lattice \( e_1, \ldots, e_k \in N \), written \( \sigma = \langle e_1, \ldots, e_k \rangle \). If the cone is not regular it will be called **singular**. A complex \( \Sigma \) is **regular** or **nonsingular** if all cones \( \sigma \in \Sigma \) are regular.

**Lemma 4.1.3.** ([KKMSD73]) Let \((X, U)\) be a strict toroidal embedding, and \( \Sigma \) be the associated semicomplex. Then a cone \( \sigma \in \Sigma \) is regular iff the corresponding open subset \( X(\sigma) \) is nonsingular. In particular, \( \Sigma \) is regular if \( X = X(\Sigma) \) is nonsingular.

**Definition 4.1.4.** A cone \( \sigma \) is called **irreducible singular** or simply **irreducible** if it cannot be written as \( \sigma = \tau \times \sigma_1 \), with \( \sigma_1 \) being regular.

Any singular cone \( \sigma \) contains a unique maximal irreducible singular face denoted by \( \text{sing}(\sigma) \), so we can write
\[
\sigma = \text{sing}(\sigma) \times \text{reg}(\sigma),
\]
where \( \text{reg}(\sigma) \) is the maximal regular face of \( \sigma \) disjoint with \( \text{sing}(\sigma) \). This follows from a simple observation that an irreducible face of \( \tau \times \text{reg}(\sigma) \) is contained in \( \tau \).

Denote by \( \text{sing}(\Sigma) \) the subset of all irreducible singular faces of \( \Sigma \), and let \( \text{Sing}(\Sigma) \) denote the minimal subcomplex of \( \Sigma \) containing \( \text{sing}(\Sigma) \). As we will see later the subset \( \text{sing}(\Sigma) \) describes the maximal components of the singular set on \( X \).

On the other hand, let \( \text{Reg}(\Sigma) \) the set of all the regular cones in \( \Sigma \). Then \( \text{Reg}(\Sigma) \) corresponds to the open subset of nonsingular points on \( X \). We see immediately from the definition that

**Lemma 4.1.5.** The restriction of a regular local projection \( f : \Sigma \rightarrow \Sigma' \) is a local isomorphism
\[
\text{Sing}(f) : \text{Sing}(\Sigma) \rightarrow \text{Sing}(\Sigma')
\]
on the subcomplexes.

**Proof.** By definition 3.9.3, \( \sigma \simeq (f(\sigma)) \times \tau \), where \( \tau \) is regular. So if \( \sigma \in \text{sing}(\Sigma) \) then \( f(\sigma) \simeq (\sigma) \). In particular, the cone \( f(\sigma) \) is irreducible, and thus it is in \( \text{Sing}(\Sigma') \). Consequently \( f \) defines isomorphisms between the corresponding cones in \( \text{sing}(\Sigma) \) and in \( \text{sing}(\Sigma') \), as well as between their faces in \( \text{Sing}(\Sigma) \) and in \( \text{Sing}(\Sigma') \).

**4.1.6. Toric divisors.**

**Definition 4.1.7.** Any \( T \)-stable divisor on a toric variety \((X, T)\) will be called a **toric divisor**.

**Remark 4.1.8.** The maximal toric divisor on on a toric variety \((X, T)\) id given by \( D := X \times T \).

**Remark 4.1.9.** Note that any toric divisor \( D \) on a smooth toric variety \( X \), induces the structure of a strict toroidal embedding \((X, D)\).
Lemma 4.1.10. Let $D = \bigcup D_i$ be a toric divisor on $X_\sigma$ with the components $D_i$ such that the closed orbit $O_\sigma$ is the intersection of all $D_i$. Then $(X_\sigma, D)$ is a strict toroidal embedding if and only if

$$D = D_{\sigma} := X_\sigma \setminus T$$

(So $D$ contains all the irreducible toric divisors on $X_\sigma$ and is the maximal toric divisor on $X_\sigma$.)

Proof. Let $x \in O_\sigma$ be a closed point. By Theorem 3.8.9, we get that the group of the Cartier divisors supported on $D_{\sigma}$ on $X_\sigma$ is given by

$$\text{Cart}(X_\sigma, D_{\sigma}) = \frac{M}{(\sigma^\perp)^\text{integ}} = M_\sigma.$$  

Moreover,

$$\text{rank}(\text{Cart}(X_\sigma, D_{\sigma})) + \text{dim}(O_\sigma) = \text{dim}(X)$$

Since $(X, D)$ is strict toroidal and $O_\sigma$ is a stratum defined by $D$ we have the same equalities for our $D$ at the point $x$. Hence

$$\text{rank}(\text{Cart}(X_\sigma, D_{\sigma})) = \text{rank}(\text{Cart}(X_\sigma, D)),$$

as $D$ and $D_{\sigma}$. The group $\text{Cart}(X_\sigma, D_{\sigma})$ can be interpreted as a group of integral functionals on $N_\sigma \subset N$, so

$$\text{rank}(\text{Cart}(D_{\sigma})) = \text{rank}(M_\sigma) = \text{dim}(\sigma).$$

If $D \neq D_{\sigma}$, say there exists a component $E_i \in D_{\sigma} \setminus D$ corresponding to the one dimensional faces (rays) $\rho_i$ of $\sigma$. Then $\text{Cart}(X_\sigma, D_{\sigma})$ is a proper subgroup of $\text{Cart}(X_\sigma, D)$ which corresponds to a subgroup of the integral functionals vanishing on $\rho_i$. This implies that $\text{rank}(\text{Cart}(s_\sigma, D)) < \text{rank}(\text{Cart}(s_\sigma, D_{\sigma}))$, which is a contradiction. So $D = D_{\sigma}$ on $X_\sigma$. ♦

Lemma 4.1.11. Let $D$ be a toric divisor on $X_\sigma$. Then $(X_\sigma, D)$ is a strict toroidal embedding if and only if $\sigma = \tau \times \sigma_1$, where $\sigma_1$ is regular, and $D = D_\tau \times X_{\sigma_1}$.

Proof. Let $O_\sigma$ be the generic orbit in the intersection of the divisor components $D_i$ of $D$ corresponding to the rays $\rho_i$. Passing to $X_\tau$ we see, by Lemma 4.1.10, that $\rho_i$ generate $\tau = \langle \rho_1, \ldots, \rho_k \rangle$, and $D_{X_\tau} = D_\tau$.

Now let $x \in O_\sigma$. If $(X_\sigma, D)$ is a strict toroidal embedding at $x$ then $s_\tau = O_\tau = \bigcap D_i$ is the unique smooth toroidal stratum through $x$. Then, by the characterization of the group of Cartier divisors around a stratum in Lemma 3.8.5, we have that $\text{Cart}(X_\sigma, D) \simeq \text{Cart}(X_\tau, D_\tau)$ is the subgroup of $\text{Cart}(X_\sigma, D_{\sigma})$ consisting of the toric Cartier divisors on $X_\sigma$ supported on $D$. In other words, the natural map defined by the restriction of Cartier divisors

$$\text{Cart}(X_\sigma, D) \to \text{Cart}(X_\tau, D_\tau)$$

is an isomorphism.

But this map is the restriction to the subgroup $\text{Cart}(X_\sigma, D) \subset \text{Cart}(X_\sigma, D_{\sigma})$, of the another natural surjection map

$$\text{Cart}(X_\sigma, D_{\sigma}) \to \text{Cart}(X_\tau, D_\tau).$$

Consequently, any nonnegative integral functional $F$ on $\tau$ extends uniquely to an integral functional $\tilde{F}$ on $\sigma \supset \tau$, such that $\tilde{F}_\rho = 0$ for all one dimensional rays $\rho$.
in $\sigma \setminus \tau$. In particular, those rays form a face $\sigma_1$. We have the exact sequence of the maps of monoids

$$0 \to (\tau^\vee)^\text{integ} \to (\sigma^\vee)^\text{integ} \to ((\sigma_1)^\vee)^\text{integ} \to 0,$$

where $(\sigma^\vee)^\text{integ} \to ((\sigma_1)^\vee)^\text{integ}$ is defined by the restrictions, and the exact sequence of the corresponding lattices:

$$0 \to M_\tau \to M_\sigma \to M_{\sigma_1} \to 0.$$

Both exact sequences split as we have the natural restriction map $(\sigma^\vee)^\text{integ} \to ((\tau)^\vee)^\text{integ}$, and $M_{\sigma_1} \to M_\tau$. So

$$M_{\sigma} \cong M_\tau \times M_{\sigma_1},$$

Dualizing

$$\sigma \cong \tau \times \sigma_1.$$

Moreover, the orbit $O_\tau$ on $X_\sigma = X_\tau \times \sigma_1$ corresponds to $T_{\sigma_1} \times O_\tau$, and its closure is equal to $O_\tau = O_\tau \times X_{\sigma_1}$, and since it is a smooth stratum we conclude that $X_{\sigma_1}$ is smooth, and $\sigma_1$ is regular.

\[\Box\]

4.2. The determinants of simplicial cones.

Definition 4.2.1. A cone $\sigma$ is simplicial if it is generated over $\mathbb{Q}$ by linearly independent primitive vectors $v_1, \ldots, v_k$, written $\sigma = \langle v_1, \ldots, v_k \rangle$.

To control the singularities of the simplicial cones $\sigma = \langle v_1, \ldots, v_k \rangle$ one introduces the multiplicity or determinant $\text{det}(\sigma)$ of the cone $\sigma$ to be the absolute value of $\text{det}(v_1, \ldots, v_k)$, where the determinant is computed with respect to any basis of the lattice $N_\sigma$. Set $\text{Vert}(\sigma) := \{v_1, \ldots, v_k\}$ for the set of vertices of $\sigma$.

Let $N_{\text{Vert}(\sigma)} := \bigoplus \mathbb{Z}v_i \subset N_\sigma$ be the sublattice generated by $v_i$.

Lemma 4.2.2. Let $\sigma = \langle v_1, \ldots, v_k \rangle$ be a simplicial cone. Then

1. The order of the quotient group $\frac{N_\sigma}{N_{\text{Vert}(\sigma)}}$ is equal to $n := \text{det}(\sigma)$, and the cosets in the quotient group $\frac{N_\sigma}{N_{\text{Vert}(\sigma)}}$ have representatives which are integral vectors of $N_{\sigma}$ of the form $\sum a_i v_i$, where $0 \leq a_i < 1$, $a_i \in \frac{1}{n} \cdot \mathbb{Z}_{\geq 0}$.
2. $\sigma$ is regular iff $\text{det}(\sigma) = 1$.

Proof. It is a well known fact.

1. By a triangular linear modification (which does not change determinant) one can transform $\text{Vert}(\sigma)$ into a set $\{n_1 e_1, \ldots, n_k e_k\}$ where $\{e_1, \ldots, e_k\}$ is a basis of $N_\sigma$ so that

$$\text{det}(\sigma) = \text{det}(n_1 e_1, \ldots, n_k e_k) = n_1 \cdots n_k = [N_\sigma : N_{\text{Vert}(\sigma)}].$$

If $\sum a_i v_i \in N_\sigma$ and $n = [N_\sigma : N_{\text{Vert}(\sigma)}]$ then $n \sum a_i v_i \in N_{\text{Vert}(\sigma)}$, so $na_i \in \mathbb{Z}$.

2. The condition $\text{det}(\sigma) = 1$ means that the set $\text{Vert}(\sigma)$ is a basis of $N_{\sigma}$.

\[\Box\]
4.2.3. Minimal vectors.

**Definition 4.2.4.** By a minimal point or a minimal vector of a cone $\sigma$ we shall mean an integral vector $v \in \sigma$, which is not a vertex, and which cannot be decomposed as $v = x + y$, where $x, y \in \sigma^{\text{integ}} \setminus \{0\}$.

**Definition 4.2.5.** More generally a small vector $v$ is a nonzero integral vector of a cone $\sigma = \langle v_1, \ldots, v_k \rangle$ which cannot be written as $v = v_i + y$, where $y \in \sigma^{\text{integ}} \setminus \{0\}$.

By definition any minimal vector is small. Moreover,

**Lemma 4.2.6.** A small vector of a simplicial cone $\sigma = \langle v_1, \ldots, v_k \rangle$ is a nonzero integral vector of the form $v = a_1 v_1 + \cdots + a_k v_k \in \mathbb{N} \sigma$ with $0 \leq a_i < 1$.

Now, let $v \in \sigma$ be a small vector, which is not minimal. Then there is a decomposition $v = x + y$, where both $x$ and $y$ are small vectors again. By continuing this decomposition process we show the following:

**Lemma 4.2.7.**

1. Any small vector in a cone is a nonnegative integral combination of the minimal vectors.
2. Any integral vector of a cone is a nonnegative integral combination of minimal vectors and vertices.

**Lemma 4.2.8.** All the small and the minimal points in $\sigma$ are necessarily in $\text{sing}(\sigma)$, and conversely, $\text{sing}(\sigma)$ is the smallest face of $\sigma$ containing all the minimal or small points.

**Proof.** Any nonzero vector of $\sigma = \text{sing}(\sigma) + \text{reg}(\sigma)$ which is not in $\text{sing}(\sigma)$ is the sum of a nonzero vector in $\text{reg}(\sigma)$ and a vector in $\text{sing}(\sigma)$, and thus cannot be minimal. So the minimal vectors are in $\text{sing}(\sigma)$, and thus their combinations are as well.

**Example 4.2.9.** (D. Abramovich) Let $\sigma = \mathbb{Q}_2^3$, with $N^\mathbb{Q}_\sigma = \mathbb{Q}^3$, and the lattice

$$N_\sigma = \{(a_1, a_2, a_3) : a_i \in \mathbb{Z}, a_1 + a_2 + a_3 \in 2\mathbb{Z}\} \subset N^\mathbb{Q}_\sigma$$

Then $\sigma$ is generated by $\{(1, 1, 0), (1, 0, 1), (0, 1, 1), (2, 0, 0), (0, 2, 0), (0, 0, 2)\}$, with the vertices $\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}$. Thus $\sigma$ contains no minimal points in its relative interior.

Note that the semigroup (or monoid) of a simplicial cone $\tau$ containing no minimal vectors is necessarily generated by $v_i$, and thus $\tau$ is regular. Summarizing

**Lemma 4.2.10.**

1. A simplicial cone $\sigma$ is regular if it contains no minimal points.
2. If $\det(\sigma) = n > 1$ then there exists a minimal or a small point of the form $\sigma, v = a_1 v_1 + \ldots + a_k v_k \in N_\sigma$ with $0 \leq a_i < 1$, where $a_i \in \frac{1}{n} \cdot N$. 

♠
4.3. Star subdivisions.

**Definition 4.3.1.** Let $\Sigma$ be a conical complex and $\tau \in \Sigma$. The **star** of the cone $\tau$, the **closed star**, and the **link** of $\tau$ are defined as follows:

\[
\text{Star}(\tau, \Sigma) := \{ \sigma \in \Sigma \mid \tau \preceq \sigma \}, \\
\overline{\text{Star}}(\tau, \Sigma) := \text{Star}(\tau, \Sigma) \\
\text{Link}(\tau, \Sigma) = \overline{\text{Star}}(\tau, \Sigma) \setminus \text{Star}(\tau, \Sigma)
\]

For any $\sigma \in \text{Link}(\tau, \Sigma)$, there is a unique minimal face of $\text{Star}(\tau, \Sigma)$, which contains $\sigma$. We denote it, by the abuse of notations by $\tau + \sigma$.

**Definition 4.3.2.** Let $\Sigma$ be a conical complex and $v$ be a primitive vector in the relative interior of $\tau \in \Sigma$. Then the **star subdivision** $v \cdot \Sigma$ of $\Sigma$ at $v$ is defined to be

\[
v \cdot \Sigma = (\Sigma \setminus \text{Star}(\tau, \Sigma)) \cup \{ \langle \sigma_v \mid \sigma \in \text{Link}(\tau, \Sigma) \}\}
\]

where $\sigma_v := \langle v \rangle + \sigma \subset \tau + \sigma$,

with $i_{\sigma, \sigma_v}$ is the restriction of the unique map $i_{\sigma, \tau + \sigma}$. The vector $v$ will be called the **center** of the star subdivision.

One can extend the definition of the center for the purpose of functoriality:

**Definition 4.3.3.** Let $\Sigma$ be a complex and $V = \{v_1, \ldots, v_k\}$ be a set of the primitive vectors $v_i \in \text{int}(\tau_i)$ in the relative interiors of the cones $\tau_i \in \Sigma$ for $i = 1, \ldots, k$ defining disjoint stars $\text{Star}(\tau_i, \Sigma)$.

The **star subdivision** $V \cdot \Sigma$ of $\Sigma$ at $V$ is defined to be

\[
V \cdot \Sigma = v_1 \cdot \ldots \cdot v_k \cdot \Sigma = (\Sigma \setminus \bigcup_{i=1,\ldots,k} \text{Star}(\tau_i, \Sigma)) \cup \bigcup_{\sigma \in \text{Link}(\tau_i, \Sigma)} \langle v_i \rangle + \sigma.
\]

The set of vectors $V$ is the **center** of the star subdivision.

**Remark 4.3.4.** Let $V = \{v_1, \ldots, v_k\}$ be the center of the star subdivision of $\Sigma$. Then $V \cdot \Sigma = v_1 \cdot \ldots \cdot v_k \cdot \Sigma$ is independent of the order of the vectors $v_i$ in $V$.

**Definition 4.3.5.** A **multiple star subdivision** of $\Sigma$ is a subdivision obtained as a sequence of the star subdivisions at the consecutive centers $V_1, \ldots, V_k$. A regular subdivision is called a **desingularization**.

The star subdivisions at minimal points allow to resolve singularities of simplicial faces.

**Lemma 4.3.6.** ([KKMSD73]) Let $v \in \text{int}(\tau)$ be a minimal point of $\tau \in \Sigma$. Then for any cone $\sigma \in \text{Star}(\tau, \Sigma)$ the resulting cones in $v \cdot \sigma$ in the the star subdivision $v \cdot \Sigma$ of the complex $\Sigma$ have smaller determinants than $\det(\sigma)$.

**Proof.** Let $\tau = \langle v_1, \ldots, v_k \rangle$, and $\sigma = \langle v_1, \ldots, v_s \rangle$, and write $v = a_1 v_1 + \ldots + a_k v_k$ with $0 \leq a_i < 1$. Then for the cone $\sigma_i = \langle v_1, \ldots, \hat{v}_i, \ldots, v_s \rangle$ we have

\[
\det(\sigma_i) = |\det(v, v_1, \ldots, \hat{v}_i, \ldots, v_s)| = a_i \det(v_1, \ldots, v_s) = a_i \det(\sigma).
\]
4.4. **Barycenters and irreducible barycentric subdivisions.** In our desingularization algorithm, one considers the canonical centers of the star subdivisions in the relative interiors of the irreducible cones. There are several ways of doing this. One could associate with any cone $\sigma = \langle v_1, \ldots, v_k \rangle$ the canonical center $v_1 + \ldots + v_k$. However, for the applications in Section 7, we need another choice of the centers in the desingularization of semicomplexes.

**Definition 4.4.1.** By a minimal internal vector of $\sigma$ we mean an integral vector $v \in \text{int}(\sigma)$ which cannot be represented as the sum of two nonzero integral vectors in $\sigma$, such that at least one of them is in the relative interior $\text{int}(\sigma)$. Then the sum of all its minimal internal vectors $v_\sigma$ will be called the canonical barycenter of $\sigma$.

**Definition 4.4.2.** Let $\Sigma$ be a conical complex. By the canonical irreducible barycentric subdivision of $\Sigma$, we mean a sequence of the star subdivisions at the sets of all barycenters of all the irreducible faces in $\Sigma \setminus \Omega$ of the same dimension in order of decreasing dimension.

**Lemma 4.4.3.** If $\Delta$ is a canonical irreducible barycentric subdivision of $\Sigma$ then $\Delta$ is simplicial.

**Proof.** If $\delta$ is a face of $\Delta$ then all its new rays (or vertices) are linearly independent of the other rays. So $\delta$ has a unique maximal face $\tau$ which is in $\Sigma$, and this face is regular, as it contains no irreducible face and thus $\text{sing}(\tau) = \{0\}$. Consequently, $\text{Vert}(\delta) \setminus \text{Vert}(\tau)$ are linearly independent from $\text{Vert}(\tau)$, and $\text{Vert}(\tau)$ are linearly independent.

4.5. **Marked complexes.** The procedure described in Lemma 4.3.6 allows to resolve singularities of simplicial complexes by applying the star subdivisions at the minimal points. Unfortunately, the choice of such minimal points is highly non-canonical. To eliminate choices, we introduce here the concept of marking.

**Definition 4.5.1.** A marking on a complex $\Sigma$ is a partially ordered subset $V$ of the set of all vertices $\text{Vert}(\Sigma)$ of $\Sigma$ such that the following conditions are satisfied.

1. For any cone $\sigma$ in $\Sigma$ the set $V(\sigma) := V \cap \text{Vert}(\sigma)$ is linearly independent of the remaining vertices in $\text{Vert}(\sigma) \setminus V$. It means $\sum v_i \in V(\sigma) c_i v_i = 0$ implies that $c_i = 0$ for each $v_i \in V(\sigma)$.
2. The order on $V$ is total on each subset $V(\sigma)$

A complex with a marking will be called marked. We say that a face $\tau$ of a complex $\Sigma$, is completely marked if $V(\sigma) = \text{Vert}(\sigma)$. A subcomplex $\Sigma_0$ is completely marked if all its faces are completely marked. A face is unmarked if $V(\sigma) = \emptyset$. The set of all unmarked faces of $\Sigma$ forms the maximal unmarked subcomplex $U(\Sigma)$.

**Definition 4.5.2.** A complex with marking will be called regularly marked if all the unmarked faces are regular.

**Remark 4.5.3.** The empty set $V = \emptyset$ defines the trivial marking on a complex.

4.5.4. **Natural order on marked complexes.** Let $\Sigma$ be a complex with a marking $V \subset \text{Vert}(\Sigma)$. One can introduce the natural partial order on vectors in $|\Sigma|$. Each
such vector \( v \) is in the relative interior of a single cone \( \sigma \in \Sigma \). Then it can be written as

\[
v = \sum_{v_i \in V(\sigma)} c_i v_i + \sum_{v_i \in \text{Vert}(\sigma) \setminus V} c_i v_i
\]

By Condition (1) of Definition 4.5.1, we shall associate with it uniquely the linear combination

\[
\pi(v) := \sum_{v_i \in V(\sigma)} c_i v_i
\]

Then we say that \( \pi(v) > \pi(v') \) if in the difference

\[
\pi(v) - \pi(v') = \sum (c_i - c'_i)v_i
\]

for any vertex \( v_i \) with negative coefficient there exists at least one vertex \( v_j > v_i \) with a positive coefficient.

Then we set \( v > v' \) if \( \pi(v) > \pi(v') \).

This defines a partial order which is total for the points in any completely marked cone. The order in such a face is lexicographic, when the coefficients form a sequence determined by the descending order on the vertices.

The following simple observation is critical for the algorithm.

**Lemma 4.5.5.** In any regularly marked face there exists a unique small vector which is minimal with respect to the marking order.

**Proof.** Note that any regularly marked face \( \sigma = (v_1, \ldots, v_k) \) is simplicial, and its maximal unmarked face \( \sigma_0 \) is regular and spanned by \( v_i \in \text{Vert}(\sigma) \setminus V \).

For \( j = 0, 1 \) we can write two minimal vectors \( w_1, w_2 \) of \( \sigma \) as

\[
w_j = \sum_{v_i \in V \cap \text{Vert}(\sigma)} a_{ji} v_i + \sum_{v_i \in \text{Vert}(\sigma_0)} b_{ji} v_i.
\]

Since both vectors are minimal for the marking order, the coefficients \( a_{ji} \) are the same for \( j = 0, 1 \). Since the vectors \( w_j \) are small and the cone \( \sigma_0 \) is regular we have that \( 0 \leq b_{ji} < 1 \). The difference

\[
w_0 - w_1 = \sum_{v_i \in \text{Vert}(\sigma_0)} (b_{0i} - b_{1i}) v_i
\]

is thus an integral vector with the coefficients satisfying inequalities

\[-1 < (b_{0i} - b_{1i}) < 1\]

So \( (b_{0i} - b_{1i}) = 0 \), since the cone \( \sigma_0 \) is regular, and the coefficients \( (b_{0i} - b_{1i}) \) are integral. \( \Box \)

4.5.6. **Marking defined by star subdivisions.** Marking naturally occurs when applying star subdivisions of complexes.

**Lemma 4.5.7.** Let \( \Sigma \) be a complex with marking \( V \subset \text{Vert}(\Sigma) \). Let \( \Sigma' \) be obtained by a sequence of star subdivisions of a complex \( \Sigma \) at the consecutive centers \( V_1, \ldots, V_k \). (see Definition 4.3.3). Then there exists a natural marking \( V' := V \cup V_1 \cup \ldots \cup V_k \), which extends the order on \( V \setminus (V_1 \cup \ldots \cup V_k) \) and such that

1. All the vectors in \( V_1 \cup \ldots \cup V_k \) are greater than vertices in \( V \setminus (V_1 \cup \ldots \cup V_k) \)
2. \( v_i < v_j \) if \( i < j \), and \( v_i \in V_i \), \( v_j \in V_j \) are in a common face of \( \Sigma \).
Proof. Note that all the new vertices of \( \Sigma' \) are the centers of the star subdivisions. On the other hand, each center of a star subdivision is linearly independent from other vertices in the newly constructed faces. Moreover, this property is preserved under the consecutive star subdivisions. Thus, for any face \( \sigma' \) of \( \Sigma' \), the new vertices (the centers of the star subdivisions) are linearly independent and marked. So Condition (1) of Definition 4.5.1 is satisfied.

Any cone has vertices that are either some vertices of a cone in \( \Sigma \) or are elements of some \( V_i \). By the construction, each cone may have at most one vertex in any set \( V_i \). This implies Condition (2) of Definition 4.5.1.

\[ \blacksquare \]

The following is the auxiliary result used in the proof of Theorem 4.6.1

**Lemma 4.5.8.** Let \( \Sigma \) be a regularly marked complex. There exists a canonical multiple star desingularization \( \bigstar V_1 \cdots \bigstar V_k \cdot \Sigma \) of \( \Sigma \) such that

1. The centers \( V_i \) are sets of points which lie in \( |\text{sing}(\Sigma)| \), and no regular faces are affected.
2. The centers of the consecutive star subdivisions are sets of minimal points in the interior of singular irreducible faces.

The algorithm is functorial for regular local projections and local isomorphisms of complexes, in the sense that the centers transform functorially with the trivial subdivisions removed.

Proof. By definition, a regularly marked complex is simplicial. By Lemma 4.3.6, any star subdivision at minimal points improves singularities. To control this process we shall consider here a polynomial invariant

\[ P_{\text{det}}(\Sigma)(t) := \sum a_i t^i \]

whose coefficients \( a_i \) of \( t^i \) are the numbers of the maximal faces with determinant equal to \( i \). We can order the polynomials lexicographically. Equivalently:

\[ P_{\text{det}} > P'_{\text{det}} \quad \text{if} \quad \lim_{t \to \infty} (P_{\text{det}} - P'_{\text{det}}) > 0. \]

Then, by Lemma 4.3.6, after any star subdivision \( \Sigma' \) of \( \Sigma \) at minimal points the polynomial invariant drops \( P_{\text{det}}(\Sigma') < P_{\text{det}}(\Sigma) \). We choose canonically the center of the star subdivision to be the set of the integral vectors of \( |\Sigma| \) which are minimal for the canonical partial order defined by the marking. Then, by Lemma 4.5.5,

1. Any face \( \sigma \) of the complex contains at most one point \( P_\sigma \in \text{int}(\sigma) \) which is in the center of a star subdivision.
2. The point \( P_\sigma \) is a minimal vector of a cone \( \sigma \) (in the sense of Definition 4.2.4).
3. The choice of the center point \( P_\sigma \) is independent of the other faces of the complex.

The procedure of the star subdivisions at such centers clearly terminates since the set of the polynomials with nonnegative integral coefficients has a dcc property so any descending sequence stabilizes.

\[ \blacksquare \]
4.6. Canonical desingularization of conical complexes.

**Theorem 4.6.1.** There exists a canonical desingularization $\Sigma'$ of a conical complex $\Sigma$, that is a sequence of star subdivisions $(\Sigma_i)_{i=0}^n$ of $\Sigma$ such that $\Sigma_0 = \Sigma$ and $\Sigma_n = \Sigma'$ is regular. Moreover,

1. All the centers are in $|\text{sing}(\Sigma)|$, where $\text{sing}(\Sigma)$ is the subset of all the irreducible singular faces of $\Sigma$.

2. The centers of the consecutive star subdivisions are either sets of minimal points, or sets of the canonical barycenters of irreducible faces.

3. The subdivision does not affect the set $\text{Reg}(\Sigma)$ of all the regular cones.

4. The algorithm is functorial for regular local projections and local isomorphisms of complexes, in the sense that the centers transform functorially with the trivial subdivisions removed. In the case of surjective local isomorphisms, the centers of the star subdivisions transform functorially.

**Proof.** We take the trivial (the empty) marking on $\Sigma$. Consider the canonical irreducible barycentric subdivision $B(\Sigma)$ of $\Sigma$ (as in Definition 4.4.2). By Lemma 4.5.7, we create some marking on the set of new vertices, defined by the dimensions of the subdivided cones. The subdivision will not affect any regular cones. It induces a simplicial subdivision of $\Sigma$, with the induced marking $V$ which is regularly marked, as all untouched faces must be regular. Then the resulting regularly marked complex can be resolved by Lemma 4.5.8.

Commutativity with regular local projections follows immediately from Lemmas 4.1.5, and 4.5.8.

5. Desingularization of relative complexes

In this Chapter we extend the combinatorial desingularization algorithm to a relative situation.

5.1. Relative complexes.

**Definition 5.1.1.** By a relative conical complex $(\Sigma, \Omega)$ we mean a pair of a conical complex $\Sigma$ and its subcomplex $\Omega \subseteq \Sigma$.

We say that a cone $\sigma \in \Sigma$ is balanced if it contains a unique maximal face $\omega$ which is in $\Omega$. In such a case we shall call $(\sigma, \omega)$ a pair in $(\Sigma, \Omega)$, and write $(\sigma, \omega) \in (\Sigma, \Omega)$. The relative complex $(\Sigma, \Omega)$ is balanced if any $\sigma \in \Sigma$ is balanced.

Let $(\sigma, \omega) \in (\Sigma, \Omega)$, where $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s \rangle \in \Sigma$ be a (balanced) cone with the maximal face $\omega = \langle w_1, \ldots, w_s \rangle \in \Omega$. We say that a pair $(\sigma, \omega)$ is regular if $\text{sing}(\sigma) \preceq \omega$. We say that $(\sigma, \omega)$ is simplicial if $v_1, \ldots, v_k$ are linearly independent from $w_1, \ldots, w_s$. Equivalently we say that $\sigma$ is relatively regular (respectively relatively simplicial) if $\sigma$ is a balanced cone with a maximal face $\omega \in \Omega$, and $(\sigma, \omega)$ is regular (resp. simplicial). If $\sigma$ is not relatively regular (In particular, if it is not balanced or simplicial) it will be called relatively singular.

The relative complex $(\Sigma, \Omega)$ is balanced (respectively regular or simplicial) if any cone $\sigma \in \Sigma$ is is balanced (resp. relatively regular or relatively simplicial).

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19 Definition 4.1.4
20 Definition 3.9.3
5.2. **Relative irreducibility.** As in the previous case of $\Omega = \{0\}$ we can introduce a relative version of irreducibility. Let $(\Sigma, \Omega)$ be a relative complex. We say that $\sigma \in \Sigma$ is relatively irreducible if it contains no proper face $\tau$ which contains both $\text{sing}(\sigma)$ and $\sigma \cap |\Omega|$. In particular, any balanced relatively irreducible face $\sigma$ which is not in $\Omega$ is singular since $\text{sing}(\sigma) \neq \{0\}$.

Any cone $\sigma \in \Sigma$ contains a unique maximal relatively irreducible face denoted by $\text{sing}_\Omega(\sigma)$. Equivalently, $\text{sing}_\Omega(\sigma)$ is a unique minimal face containing $\text{sing}(\sigma)$ and $\sigma \cap |\Omega|$. By definition $\text{sing}(\sigma) \subseteq \text{sing}_\Omega(\sigma)$, and thus as in Definition 4.1.4, we can write $\sigma = \text{sing}_\Omega(\sigma) \times \text{reg}_\Omega(\sigma)$, where $\text{reg}_\Omega(\sigma)$ is the maximal regular face of $\sigma$ disjoint from $\text{sing}_\Omega(\sigma)$ (intersecting it at the zero cone).

**Lemma 5.2.1.** $\text{sing}_\Omega(\sigma) \in \Omega$ iff $\sigma$ is relatively regular.

**Proof.** If $\omega := \text{sing}_\Omega(\sigma) \in \Omega$ then $\omega = \sigma \cap |\Omega|$ is a unique maximal face of $\sigma$ in $\Omega$. Moreover, $\text{sing}(\sigma) \subseteq \omega$.

Conversely, suppose that $\sigma$ is relatively regular. Then it is balanced and $\text{sing}(\sigma) \subseteq \omega$, where $\omega$ is a unique maximal face of $\sigma$ in $\Omega$. Then $\text{sing}_\Omega(\sigma) = \omega \in \Omega$. ♣

5.3. **Singular faces.** Denote by $\text{sing}(\Sigma, \Omega)$ the subset of all relatively irreducible faces of $\Sigma$, and let $\text{Sing}(\Sigma, \Omega)$ be its the smallest subcomplex of $\Sigma$ containing $\text{sing}(\Sigma, \Omega)$. Denote by $\text{Reg}(\Sigma, \Omega)$ the set of all the relatively regular cones in $\Sigma$.

**Lemma 5.3.1.**

1. $\text{sing}(\Sigma, \Omega)$ contains $\Omega$.
2. $\text{Reg}(\Sigma, \Omega)$ is a subcomplex of $\Sigma$ containing $\Omega$.
3. $\text{sing}(\Sigma, \Omega) \cap \text{Reg}(\Sigma, \Omega) = \Omega$.

**Proof.** (1) follows from definition.

(2) if $\sigma$ is relatively regular with maximal cone $\omega \preceq \sigma$ which is in $\Omega$, and $\tau$ is its face then $\omega \cap \tau \in \Omega$ is maximal in $\tau$. Moreover, $\text{sing}(\tau) \subseteq \text{sing}(\sigma) \subseteq \omega$, hence $\text{sing}(\tau) \subseteq \omega \cap \tau$. So $\text{Reg}(\Sigma, \Omega)$ is a subcomplex of $\Sigma$.

(3) If $\sigma$ is relatively regular and irreducible then $\sigma = \text{sing}_\Omega(\sigma) = \omega$. ♣

**Remark 5.3.2.** These results are interpreted geometrically in Lemma 7.5.8 in the toric situation. In the toroidal case, they have the same meaning. The set $\text{sing}(\Sigma, \Omega)$ describes the strata of the singular locus defined by both: the singularity type and the divisor $D = D_\Omega$. On the other hand, the open subset $X_{\text{Reg}(\Sigma, \Omega)}$ is the saturation of the open set $(X_\Omega, D_\Omega)$ defined by $\Omega$ in $(X_\sigma, D)$. (See Lemmas 5.4.6, and 7.5.8 (in the toric situation)).

5.4. **Regular relative complexes and SNC divisors.**

**Definition 5.4.1.** Let $(X, D_X)$ be a strict toroidal embedding. By a toroidal divisor on $X$ we mean a divisor $D \subset D_X$. Usually $(X, D)$ is not a strict toroidal embedding. By the closed strata of $D = \bigcup_{i \in I} D_i$ with irreducible Weil components $D_i$ we mean the irreducible components of the intersections $\bigcap_{i \in J} D_i$ of $D_i$. Denote by $S_D$ the induced stratification with strata the components of $\bigcap_{i \in I} D_i \setminus \bigcup_{i \in (J \setminus I)} D_i$. 
and by $\overline{S}_D$ the set of the induced closed strata.

Note that the strata of $S_D$ are the unions of those on the toroidal embedding $(X, D_X)$ (defined by $D_X$). Thus $S_D$ is coarser than the canonical stratification on $(X, D_X)$. Moreover, the strata in $S_D$ are not necessarily smooth.

**Lemma 5.4.2.** Let $(X, D_X)$ be a strict toroidal embedding with the associated complex $\Sigma$. There exists a bijective correspondence between the saturated open toroidal subsets $V = X(\Omega)$ of $(X, D_X)$ and the subcomplexes $\Omega \subset \Sigma$. (see notation in Section 3.8.14).

*Proof.* The Lemma is a rephrasing of Lemma 3.8.15.

---

**Definition 5.4.3.** A subcomplex $\Omega \subset \Sigma$ will be called **saturated** if any cone of $\sigma$ with rays (one dimensional faces) in $\Omega$ is in $\Omega$.

Immediately from the definition we obtain the following characterization of saturated subcomplexes:

**Lemma 5.4.4.** If $(\Sigma, \Omega)$ is balanced then the subcomplex $\Omega \subset \Sigma$ is saturated.

The saturated subcomplexes have the following geometric description:

**Lemma 5.4.5.** The subcomplex $\Omega \subset \Sigma$ is saturated in $\Sigma$ iff the complement

$$X \setminus X(\Omega) = X(\Sigma \setminus \Omega)$$

is a divisor.

*Proof.* Assume $E := X \setminus X(\Omega)$ is a divisor. If the vertices of $\sigma = \langle v_1, \ldots, v_k \rangle \in \Sigma$ are in $\Omega$ then the closure of the corresponding stratum $s_{\sigma}$ is the intersection of the irreducible components $D_i$ corresponding to $v_i$. But the stratum $s_{\sigma}$ is not in $E$, since the only divisors which contain it are $D_i$, and they are all intersecting $X(\Omega)$ and thus are not contained in $E$. So $s_{\sigma} \subset X \setminus E = X(\Omega)$, and $\sigma \in \Omega$. We use the property that any closed stratum is the intersection of the irreducible divisors containing it.

Conversely, if $\Omega$ is saturated, then any stratum $s$ in $E = X \setminus X(\Omega)$, corresponds to a cone $\sigma \in \Sigma \setminus \Omega$. Since $\Omega$ is saturated, at least one of the vertices of $\sigma$ is not in $\Omega$ and corresponds to a divisor $D \subset E$. Thus $s$ is contained in an irreducible Weil divisor $D \subset E$. This implies that $E$ is a divisor.

---

**Lemma 5.4.6.** Let $(X, D_X)$ be a strict toroidal embedding, with the associated complex $\Sigma$. There is a bijective correspondence between the toroidal divisors $D \subset D_X$ and the saturated subcomplexes $\Omega$ of $\Sigma$. Given a toroidal divisor $D$ on $X$, there is a unique saturated subcomplex $\Omega \subset \Sigma$ which is defined by the set of vertices of $\Sigma$ corresponding to the components of $D$. Conversely, $D = \overline{D}_\Omega$, where

$$D_\Omega := X(\Omega) \setminus U = X(\Omega) \cap D_X.$$

---

**Lemma 5.4.7.** Let $(X, D_X)$ be a strict toroidal embedding with the associated complex $\Sigma$. Let $\Omega$ be a saturated subcomplex in $\Sigma$, and let $D = \overline{D}_\Omega \subseteq D_X$. Then
(1) \((X, D)\) is a strict toroidal embedding if \((\Sigma, \Omega)\) is a regular relative complex. In such a case \((X, D)\) is the saturation of \((X(\Omega), D_{\Omega}) \subset (X, D)\), and \(E := D_X \setminus D = X \setminus X(\Omega)\) is a relative SNC divisor on \((X, D)\).

(2) Conversely, if \((X, D)\) is a strict toroidal embedding, where \(D \subseteq D_X\) then \(D = \overline{D_{\Omega}}\), where \((\Sigma, \Omega)\) is regular relative complex.

(3) The toroidal locus \((X, D)_{\text{tor}}\) of \((X, D)\) is defined by the saturated subcomplex \(\text{Reg}(\Sigma, \Omega)\) of \(\Sigma\). It is the toroidal saturation of \((X(\Omega), D_{\Omega})\) in \((X, D)\).

Proof. (1) Using a chart we can reduce the situation to the toric case. Let \((\sigma, \omega) \in (\Sigma, \Omega)\) be a regular pair. Then \(X_\sigma = X_\omega \times X_\tau = X_\omega \times \mathbb{A}^k\). By construction, the divisor \(D \cap X_\sigma = D_\omega \times \mathbb{A}^k\), corresponds to the vertices of \(\omega = \Omega \cap \sigma\). The complement \(E \cap X_\sigma = X_\omega \times D_\tau\) corresponds to the remaining vertices of \(\sigma\) and has SNC with \(D_\omega \times \mathbb{A}^k\). Moreover the set \((X(\Omega), D_{\Omega})\) corresponds in a local chart to \((X_\omega \times T, D_\omega \times T)\) and \((X_\sigma, D_\omega \times \mathbb{A}^k)\) is its saturation.

(2) Conversely, assume that \((X, D)\) is a strict toroidal embedding. Then, by Lemma 4.1.11, for any \(\sigma \in \Sigma\), we have that \(\sigma = \omega \times \tau\), where \(\tau\) is regular, and \(D \cap X_\sigma = D_\omega \times X_\tau\), whence \((\Sigma, \Omega)\) is regular.

(3) Follows from (2).

5.5. Subdivisions of relative complexes.

Definition 5.5.1. A map \(f : (\Sigma, \Omega) \to (\Sigma', \Omega')\) is a map of complexes \(\Sigma \to \Sigma'\) which induces a map of the subcomplexes \(\Omega \to \Omega'\).

A subdivision of a relative complex \((\Sigma, \Omega)\) is a subdivision \(\Delta \to \Sigma\) which is identical on \(\Omega \subset \Delta\), and thus defines a relative complex \((\Delta, \Omega)\). A subdivision \((\Delta, \Omega)\) of \((\Sigma, \Omega)\), which is a regular relative complex is called desingularization of \((\Sigma, \Omega)\).

A map \(f : (\Sigma, \Omega) \to (\Sigma', \Omega')\) is a local isomorphism if each \(f_{\sigma, \sigma'}\) is an isomorphism mapping \(\Omega \cap \sigma\) to \(\Omega' \cap \sigma'\).

A map \(f : (\Sigma, \Omega) \to (\Sigma', \Omega')\) is an isomorphism if \(f\) is a bijection of the sets and a local isomorphism.

A map \(f : (\Sigma, \Omega) \to (\Sigma', \Omega')\) is called a regular local projection if for each \(\sigma \in \Sigma\) there is an isomorphism \(\sigma \simeq \sigma' \times \tau\), where \(\tau\) is regular, which takes \(\Omega \cap \sigma\) to \((\Omega' \cap \sigma') \times \tau\), and such that

\[f_{\sigma, \sigma'} : \sigma \simeq \sigma' \times \tau \to \sigma'\]

is the projection on the first component.

Lemma 5.5.2. A regular local projection \(f : (\Sigma, \Omega) \to (\Sigma', \Omega')\) induces a local isomorphism of the relative subcomplexes:

\[\text{Sing}(f) : \text{Sing}(\Sigma, \Omega) \to \text{Sing}(\Sigma', \Omega').\]
5.6. Minimal vectors in relative complexes.

**Definition 5.6.1.** Let $\sigma \in \Sigma$ be a cone with a unique maximal face $\omega \in \Omega$.

A minimal vector (respectively small vector) of the pair $(\sigma, \omega)$ is a minimal (respectively small) vector of the cone $\sigma$ which is not in $\omega$.

**Lemma 5.6.2.** Let $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s \rangle \in \Sigma$ be a relatively simplicial cone with a unique maximal face $\omega = \langle w_1, \ldots, w_s \rangle \in \Omega$. Then a minimal vector of $(\sigma, \omega)$ can be written down in the form $\sum c_i v_i + \sum d_j w_j$, where $0 \leq c_i < 1$, and at least one $c_i \neq 0$.

**Lemma 5.6.3.** Let $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s \rangle \in \Sigma$ be a relatively simplicial cone with a unique maximal face $\omega = \langle w_1, \ldots, w_s \rangle \in \Omega$. Then $(\sigma, \omega)$ is regular if there is no minimal vector of $(\sigma, \omega)$.

**Proof.** If $(\sigma, \omega)$ is regular. Then $\sigma = \omega \times \text{reg}_\Omega(\sigma)$, and all the minimal vectors of $\sigma$ are in $\omega$. Conversely, suppose all the minimal vectors of $\sigma$ are in $\omega$. Then, by Lemma 4.2.7(2), any integral vector $v \in \sigma$ can be written as a nonnegative combination of the vertices $v_1, \ldots, v_k$, and vectors in $\omega^{\text{integ}}$. Since $v_1, \ldots, v_k$ are linearly independent from vectors in $\omega$ we infer that $N_\sigma$ is generated by $v_1, \ldots, v_k$, and a basis of $N_\omega$. Hence $v_1, \ldots, v_k$ is a part of a basis of $N_\sigma$, and generate a regular face $\text{reg}_\Omega(\sigma)$ of $\sigma$, whence $\sigma^{\text{integ}} = \omega^{\text{integ}} \times \text{reg}_\Omega(\sigma)^{\text{integ}}$.

5.7. Star subdivisions. Let $(\Sigma, \Omega)$ be a relative simplicial complex.

**Definition 5.7.1.** Let $(\Sigma, \Omega)$ be a relative conical complex and $v$ be a primitive vector in the relative interior of $\tau \in \Sigma \setminus \Omega$. Then the star subdivision $v \cdot (\Sigma, \Omega)$ of $(\Sigma, \Omega)$ at $v$ is defined to be

$$v \cdot (\Sigma, \Omega) := (v \cdot \Sigma, \Omega)$$

Analogously if $V = \{v_1, \ldots, v_k\}$ is a set of the primitive vectors $v_i$ in the relative interior of the cones $\tau_i \in \Sigma \setminus \Omega$ for $i = 1, \ldots, k$ defining the disjoint stars $\text{Star}(\tau_i, \Sigma)$ then the star subdivision $V \cdot (\Sigma, \Omega)$ of $(\Sigma, \Omega)$ at $V$ is defined to be

$$V \cdot (\Sigma, \Omega) := (V \cdot \Sigma, \Omega)$$

A multiple star subdivision of $(\Sigma, \Omega)$ is a subdivision obtained as a sequence of star subdivisions at the consecutive centers $V_1, \ldots, V_k$. A subdivision of $(\Sigma, \Omega)$ which is a regular relative complex is called desingularization.

5.8. Determinants of relative subdivision. We introduce the measure for the singularity of the relative simplicial cones:

If $(\sigma, \omega)$ is simplicial with $\omega = \langle w_1, \ldots, w_s \rangle$, and $\sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s \rangle$ then we put

$$\det(\sigma, \omega) := N_\sigma/(N_{\text{Vert}(\sigma)} + N_\omega) = \det(\overline{v}_1, \ldots, \overline{v}_k),$$

where $\overline{v}_1, \ldots, \overline{v}_k$ are the images of $v_1, \ldots, v_k$ are in $N_\sigma/N_\omega$. The following lemma is a consequence of the definition:

**Lemma 5.8.1.** If $(\sigma, \omega)$ is simplicial then $\det(\sigma, \omega) = 1$ iff $(\sigma, \omega)$ is regular.
Proof: If \( \det(\sigma, \omega) = 1 \) then \( v_1, \ldots, v_k \) generate that lattice \( N_\sigma/N_\omega \). So \( \sigma \simeq \langle v_1, \ldots, v_k \rangle \times \omega \) is relatively regular.

\[ \hat{\bullet} \]

5.9. **The invariant of relative cones.** We introduce a somewhat richer invariant measuring a progress under the star subdivisions at minimal vectors. Set

\[ \mu(\sigma, \omega) := (\dim(\omega), \det(\sigma, \omega)) \in \mathbb{N}^2. \]

We order the set of values \( \mathbb{N}^2 \) of \( \mu(\sigma, \omega) \) lexicographically.

The following extends Lemma 4.3.6([IKKMSD73]):

**Lemma 5.9.1.** Assume that \( (\Sigma, \Omega) \) is simplicial and let \( (\tau, \omega_r) \in (\Sigma, \Omega) \) Let \( v \in \text{int}(\tau) \) be a minimal point of \( (\tau, \omega) \). Then for any pair \( (\sigma, \omega) \), where \( \sigma \in \text{Star}(\tau, \Sigma) \) the resulting pairs \( (\sigma', \omega') \) in \( v \cdot (\sigma, \omega) \subseteq v \cdot \Sigma \) of the relative complex \( (\Sigma, \Omega) \) have a strictly smaller invariant \( \mu(\sigma', \omega') \) than \( \mu(\sigma, \omega) \).

**Proof.** Let \( \omega_r = \langle w_1, \ldots, w_r \rangle \), \( \tau = \langle v_1, \ldots, v_l, w_1, \ldots, w_l \rangle \), \( \sigma = \langle v_1, \ldots, v_s, w_1, \ldots, w_k \rangle \), and write

\[ v = a_1 v_1 + \ldots + a_l v_l + c_1 w_1 + \ldots + c_r w_r \]

with \( 0 \leq a_i < 1 \).

There are two types of the faces of maximal dimension of \( v \cdot \sigma \) denoted by \( \sigma_i \), and \( \delta_j \), where

\[ \sigma_i := \langle v, v_1, \ldots, \bar{v}, \ldots, v_s, w_1, \ldots, w_k \rangle, \]

for \( i \leq l \) contain \( \omega \), and

\[ \delta_j := \langle v_1, \ldots, v_r \rangle + \omega_j, \]

containing \( \omega_j \in \Omega \), which is a facet (codimension one face) of \( \omega \).

Then \( (\delta_j, \omega_j) \in (v \cdot \Sigma, \Omega) \) and \( \mu(\delta_j, \omega_j) < \mu(\sigma, \omega) \), since \( \dim(\omega_j) < \dim(\omega) \). On the other hand, for each cone \( \sigma_i \) we have

\[ \det(\sigma_i, \omega) = | \det(\bar{\nu}_1, \ldots, \bar{\nu}_l, \ldots, \bar{\nu}_k) | = a_i | \det(\bar{\nu}_1, \ldots, \bar{\nu}_k) | = a_i \det(\sigma, \omega) < \det(\sigma, \omega), \]

where \( \bar{\nu} \) are the images of \( v_i \) in \( N_\sigma/N_\omega \). This implies that \( \mu(\delta_j, \omega_j) < \mu(\sigma, \omega) \). \( \hat{\bullet} \)

5.10. **Barycentric subdivision.**

**Definition 5.10.1.** Let \( (\Sigma, \Omega) \) be a relative conical complex. Let \( \sigma \in \Sigma \) be a relatively irreducible cone. Denote by \( w_\sigma \) the sum of all vertices of \( \Omega \cap \sigma \), and by \( z_\sigma \) the sum of all the minimal internal vectors in \( \sigma \). Then we shall call the sum \( v_\sigma := w_\sigma + z_\sigma \in \text{int}(\sigma) \) the **canonical barycenter** of \( \sigma \).

**Definition 5.10.2.** Let \( (\Sigma, \Omega) \) be a relative conical complex. By the **canonical irreducible barycentric subdivision** of \( (\Sigma, \Omega) \) we mean the sequence of the star subdivisions at the sets of all barycenters \( v_\sigma \) of all the irreducible faces in \( \Sigma \setminus \Omega \) of the same dimension in order of decreasing dimension.

**Lemma 5.10.3.** If \( (\Delta, \Omega) \) is the canonical irreducible barycentric subdivision of \( (\Sigma, \Omega) \) then \( (\Delta, \Omega) \) is simplicial. Moreover, all faces of \( \Sigma \) which are not subdivided are relatively regular.
Proof. Let \( \delta \) be a face of \( \Delta \). Then, by definition of the star subdivision, all its new rays (vertices) are linearly independent of the other "old" rays, and these old vertices form a face which is in \( \Sigma \). Hence \( \delta \) has a unique maximal face \( \sigma \in \Sigma \). Then \( \sigma \) and its irreducible face \( \omega := \text{sing}_\Omega(\sigma) \) are intact, and thus \( \omega \) is in \( \Omega \), since otherwise it would have been decomposed. This implies, by Lemma 5.2.1, that \( \sigma \) is relatively regular, and \((\sigma, \omega)\) is regular. Moreover, the rays defined by the vertices \( \text{Vert}(\delta) \setminus \text{Vert}(\sigma) \) are not in \( \Omega \), so \( \omega \) is a unique maximal face of \( \delta \) which is in \( \Omega \), so \( \delta \) is balanced. Since \( \text{Vert}(\delta) \setminus \text{Vert}(\sigma) \) are linearly independent from \( \text{Vert}(\sigma) \), and \( \text{Vert}(\sigma) \setminus \text{Vert}(\omega) \) are linearly independent from \( \text{Vert}(\omega) \) we conclude that \( \text{Vert}(\delta) \setminus \text{Vert}(\omega) \) are linearly independent from \( \text{Vert}(\omega) \).

5.11. Marked relative complexes.

Definition 5.11.1. A marking on a relative complex \( (\Sigma, \Omega) \) is a partially ordered subset \( V \) of the set of all vertices \( \text{Vert}(\Sigma) \) of \( \Sigma \) such that the following conditions are satisfied.

1. \( \text{Vert}(\Omega) \subseteq V \)
2. Set \( V(\sigma) := V \cap \text{Vert}(\sigma) \) for any \( \sigma \in \Sigma \). The set \( V(\sigma) \setminus \Omega \) is linearly independent of the remaining vertices in \( (\text{Vert}(\sigma) \setminus V(\sigma)) \cup \Omega \).
3. For any \( \sigma \in \Sigma \) the order on \( V(\sigma) \) is total on each subset \( V(\sigma) \).

A face \( \sigma \in \Sigma \) is unmarked if \( V(\sigma) = \emptyset \). A face \( \sigma \in \Sigma \) with \((\sigma, \omega) \in (\Sigma, \Omega)\) is relatively unmarked if \((\sigma, \omega)\) is relatively simplicial, and all the vertices in \( \text{Vert}(\sigma) \setminus \text{Vert}(\omega) \) are unmarked. The set of all unmarked faces of \( (\Sigma, \Omega) \) forms the maximal unmarked subcomplex \( U(\Sigma, \Omega) \). Note that \(|U(\Sigma, \Omega)| \cap |\Omega| = \{0\} \). A marked relative complex \( (\Sigma, \Omega) \) is regularly marked if it is relatively simplicial and all the relatively unmarked faces are relatively regular. In particular all the unmarked faces in \( \Sigma \) are regular.

One can extend Lemma 4.5.7 to the relative case.

Lemma 5.11.2. Let \( (\Sigma, \Omega) \) be a relative complex with marking \( V \subset \text{Vert}(\Sigma) \). Let \( (\Sigma', \Omega) \) be obtained by a sequence of star subdivisions of a complex \( (\Sigma, \Omega) \) at the consecutive centers \( V_1, \ldots, V_k \). (disjoint from \( |\Omega| \)). Then there exists a natural marking \( V' := V \cup V_1 \cup \ldots \cup V_k \), which extends the order on \( V \setminus (V_1 \cup \ldots \cup V_k) \supseteq \text{Vert}(\Omega) \) and such that

1. All the vectors in \( V_1 \cup \ldots \cup V_k \) are greater than vertices in \( V \setminus (V_1 \cup \ldots \cup V_k) \)
2. \( v_i < v_j \) if \( i < j \), and \( v_i \in V_1 \), \( v_j \in V_j \), and \( v_i, v_j \) are in a face of \( \Sigma \).

Proof. The reasoning is identical as in the proof of Lemma 4.5.7.

Lemma 5.11.3. Given any marked relative complex \( (\Sigma, \Omega) \), one can define the canonical partial order \( \leq \) on the set of all the vectors in \( |\Sigma| \).

Proof. Let \( \sigma = \langle v_1, \ldots, v_k, w_1, \ldots, w_s \rangle \) be a face of \( \sigma \), with \( V(\sigma) = \{w_1, \ldots, w_s\} \). We order the set of vertices \( V(\sigma) = \{w_1, \ldots, w_s\} \) according to the order on \( V \), which is total on each face of \( \Sigma \). For any \( v \in \sigma \) we write

\[
v = \sum c_i v_i + \sum d_j w_j.
\]
Then we define the lexicographic order on the the coefficients $d_j$ in the presentation
\[
\pi(v) := \sum d_j w_j.
\]
Since $v_1, \ldots, v_k$ are linearly independent of $w_1, \ldots, w_s$ the vector $\pi(v)$ is unique. Although the presentation of $\pi(v)$ is, in general, not unique there is a unique smallest presentation of $\pi(v)$ with respect to the lexicographic order determined by the total order on $w_1, \ldots, w_s$. We put $v > v'$ if $\pi(v) > \pi(v')$.

**Lemma 5.11.4.** In any regularly marked face, there exists a unique small vector which is minimal for the marking order.

**Proof.** Note that in any regularly marked face there exists a unique maximal unmarked face which is regular. Then we apply the same argument as in the proof of Lemma 4.5.5.

**Lemma 5.11.5.** Let $(\Sigma, \Omega)$ be a regularly marked simplicial relative complex. There exists a canonical multiple star desingularization $V_1 \cdot \ldots \cdot V_k \cdot \Sigma$ of $\Sigma$ such that

1. The centers lie in $|\text{sing}_\Omega(\Sigma)|$, and no faces in $\text{Reg}_\Omega(\Sigma)$ are affected.
2. The centers of the consecutive star subdivisions $V_i$ are the sets of minimal vectors (definition 5.6.2) in the interior of nonempty relatively irreducible faces of $\Sigma$ which are not in $\Omega$.
3. The algorithm is functorial for regular local projections and local isomorphisms of complexes, preserving the order.

**Proof.** The proof is nearly identical to the proof of Lemma 5.11.5. By the assumption, $(\Sigma, \Omega)$ is regularly marked, and, in particular, it is relatively simplicial. We consider the polynomial invariant associated with any subdivision $(\Delta, \Omega)$ of the relative complex $(\Sigma, \Omega)$:
\[
P_{\mu}(\Delta, \Omega) = \sum c_{ij} x^i y^j,
\]
where $c_{ij}$ is the number of the maximal faces $\sigma$ of $\Delta$ with the invariant $\mu(\sigma, \omega) = (i, j)$ from Lemma 5.9.1. We consider the lexicographic order on the monomials $x^i y^j$. This extends to the lexicographic order on the polynomials.

We apply the star subdivisions at the sets of the minimal points for the partial order defined by the marking. By Lemma 4.5.5, these sets of minimal vectors are uniquely determined, and there is at most one in each face.

Note also that, by Lemma 5.11.2, the intermediate subdivisions of $(\Sigma, \Omega)$ are marked.

By Lemma 5.9.1, after each star subdivision the invariant $P_{\mu}(\Delta, \Omega)$ drops. So the procedure terminates. It terminates when there is no minimal vector in $(\Sigma, \Omega)$. By Lemma 5.6.3, the relative complex $(\Delta, \Omega)$ without minimal vectors is relatively regular.

5.12. Canonical desingularization of relative conical complexes.

**Theorem 5.12.1.** Let $(\Sigma, \Omega)$ be a relative conical complex. Assume that there is a partial order on the set of vertices $\text{Vert}(\Omega)$, which is total on each set $\text{Vert}(\omega)$, with $\omega \in \Omega$. 
There exists a canonical desingularization \((\Delta, \Omega)\) of a relative conical complex \((\Sigma, \Omega)\), that is a sequence of star subdivisions \((\Sigma_i, \Omega)\) such that \((\Sigma_0, \Omega) = (\Sigma, \Omega)\) and \((\Sigma_n, \Omega) = (\Delta, \Omega)\) is regular, that is \(\text{sing}(\Delta, \Omega) = \Omega\).  

Moreover,

(1) All the centers of the star subdivisions are in \(|\text{sing}(\Sigma, \Omega)|\). The centers are in the relative interiors of relatively irreducible faces \(\sigma\) of the intermediate subdivisions.

(2) The centers of the consecutive star subdivisions are either the barycenters of the relatively irreducible faces or the minimal vectors of the relatively simplicial irreducible cones.

(3) The subdivision does not affect the set \(\text{Reg}(\Sigma, \Omega)\).

(4) The algorithm is functorial for regular local projections and local isomorphisms of complexes preserving the order on \(\Omega\), in the sense that the centers transform functorially with the trivial subdivisions removed.

Proof. The proof is a slight modification of the proof of Theorem 4.6.1. The order on \(V := \text{Vert}(\Omega)\) defines a marking on \((\Sigma, \Omega)\).

Consider the canonical relative barycentric subdivision \((\Delta, \Omega)\) of \((\Sigma, \Omega)\) as in Lemma 4.4.3. This induces, by Lemmas 5.10.3, 5.11.2, a relatively simplicial complex \((\Delta, \Omega)\), with the induced marking \(V\) such that all unchanged cones are relatively regular. Consequently, the relative complex \((\Delta, \Omega)\) is regularly marked and can be canonically resolved by Lemma 5.11.5.

5.13. Finite group actions and obstructions to an equivariant relative desingularization.

Example 5.13.1. Consider the monoid \(\sigma = \sigma \cap N_\sigma\) generated by \((1,1,0), (1,0,1), (0,1,1), (2,0,0), (0,2,0), (0,0,2)\) as in the Abramovich Example 4.2.9. Let \(G = \mathbb{Z}_2\) acts on \(N_\sigma\) by permuting first two coordinates. Let

\[ \omega = \langle (2,0,0), (0,2,0) \rangle. \]

be the face defining the submonoid \(\omega \cap N_\sigma\) generated by \((2,0,0), (1,1,0), (0,2,0)\).

Suppose that there is a \(G\)-equivariant desingularization \((\Delta, \omega)\) of \((\sigma, \omega)\). Then there is a unique 3-dimensional relatively regular simplicial cone \(\sigma_0 \in \Delta\) containing the face \(\omega\). But such a cone is \(G\)-stable. So it can be written as

\[ \sigma_0 = \langle (2,0,0), (0,2,0), w \rangle, \]

where \(w\) is \(G\)-invariant and has a form

\[ w = (a,a,2b), \]

where \(a,b \in \mathbb{Z}_{>0}\). But then the pair \((\sigma_0, \omega)\) is not regular.

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Thus to have an equivariant desingularization one has to restrict permutations of the vertices of \( \Omega \). To this end we introduce an order of the vertices of \( \Omega \), and add the condition that the automorphisms should respect this order.

6. **Functorial desingularization of strict toroidal and toroidal embeddings**

6.1. **Canonical stratification defined by an NC divisor.** Let \( E \) be a divisor that has SNC on a toroidal embedding \( (X, D) \) (as in Definition 2.1.15). Then \( E \) induces the divisorial stratification \( S_E \). Its closed strata are defined by the intersections of the divisorial components of \( E \). This stratification is preserved by étale morphisms. This allows to extend the construction of the stratification to NC divisors by taking locally the images of the strata on étale neighborhoods.

Let \( \phi : (U, E_U) \to (X, E_X) \) be an étale neighborhood with \( V = \phi(U) \) for which the inverse image \( \phi^{-1}(E_X) = E_U \) of the NC divisor \( E_X \) is an SNC divisor \( E_U \). We define the strata of \( E_X \cap V \) to be the images \( \phi(s) \) of \( s \in S_{E_U} \). This is well defined, and is independent of the choice of étale neighborhoods.

Let \( \phi' : (U', E'_{U'}) \to (X, E_X) \) be another étale neighborhood of \( x \). Let \( x \in s \) be the image of points \( y \in U \), and \( y' \in U' \), so we can write \( x = \phi(y) = \phi'(y') \). Then the points \( y, y' \) define a point \( y'' \) in the component \( U'' \) of the étale morphism of fiber product

\[
U \times_X U' \to X.
\]

The strata \( s_y \) and \( s_{y'} \) through \( y \), and \( y' \) are the images of the stratum \( s_{y''} \) on \( U'' \) under the relevant projection. Hence the images of \( s_y \) and \( s_{y'} \) and \( s_{y''} \) coincide in a neighborhood of \( x \) defining the same stratum.

Moreover, the closure of the stratum is the union of strata. If \( s \) is a stratum through \( x \), and assume that \( x \in \bar{s}_1 \). Then for a point \( y \) over \( x \), there is a stratum \( s_Y \) containing \( y \) and dominating \( s \). Moreover, since \( \phi \) is open and thus closed under genericization there is a stratum \( s_{Y'} \) dominating \( s_Y \), and such that \( s_{Y'} \) contains \( y \). So the stratum \( s_Y \) intersects \( \bar{s}_{Y'} \), and thus \( s_{Y'} \subset \bar{s}_{Y'} \). Consequently, the image \( \phi(s_Y) \subset \phi(s_{Y'}) \subset \bar{s}_1 \). But \( \phi(s_{Y'}) \subset s_Y \) is open and dense. So locally we have

\[
s \subset \phi(s_{Y'}) \subset \phi(s_{Y'}) \subset \bar{s}_1.
\]

6.2. **Transforming a relative NC divisor into a relative SNC divisor.**

**Definition 6.2.1.** We say that a locally closed subscheme \( C \) on a toroidal embedding \( (X, D) \) is relatively nonsingular if its ideal is locally generated by a set of free coordinates on \( (X, D) \) (see Definition 2.1.13).

**Lemma 6.2.2.** Let \( E \) be a relative NC divisor on a toroidal embedding \( (X, D) \). Then all the locally closed strata of \( E \) are relatively nonsingular.

**Proof.** Let \( s \) be a stratum of \( E \), and \( x \in s \). We can assume that \( s \) is minimal and thus closed and irreducible in a certain Zariski neighborhood \( V \) of point \( x \). Consider an étale neighborhood \( \pi : U \to V \subseteq X \) of \( x \), such that \( \pi^{-1}(E) \) is a relatively SNC divisor on \( U \). Let \( \pi \in U \) be a point over \( x \). Then, by shrinking \( U \), if necessary, we can assume that the inverse image \( s_U := \pi^{-1}(s) \) is irreducible, contains \( s \), and thus is a minimal stratum on \( U \). Consider the ideal \( I_s \subset O_{\pi,X} \). By definition of a relative NC divisor, \( s_U \) is relatively nonsingular on \( (U, \pi^{-1}(D)) \) and the induced
ideal $I_{s,u,x} = I_s \cdot \mathcal{O}_{x,U} \subset \mathcal{O}_{x,U}$ is generated by free parameters $(u_1, \ldots, u_k)$ at $x$. Since $\mathcal{O}_{x,s} \rightarrow \mathcal{O}_{x,x}$ is a flat morphism there is a natural isomorphism

$$\widehat{\mathcal{O}}_{x,U} = \mathcal{O}_{x,X} \otimes_{K(x)} K(\mathcal{O})$$

Consider a set of generators of $I_{s,u,x} \subset \mathcal{O}_{x,X}$, and choose a subset $v_1, \ldots, v_k \in \mathcal{O}_{x,X}$ of $k$ functions with linearly independent linear parts at $x$. Then $(v_1, \ldots, v_k)$ generates $I_{s,u,x} = (u_1, \ldots, u_k) = (v_1, \ldots, v_k)$ since it generates

$$(I_{s,u,x} + m_{x,s}^2 I_{s,u}) / m_{x,s}^2 = ((I_{x,s} + m_{x,s}^2) / m_{x,s}^2) \otimes_{K(x)} K(\mathcal{O})$$

so by Nakayama, it generates $I_{s,u,x}$. Since $\mathcal{O}_{x,s} \rightarrow \mathcal{O}_{x,x}$ is faithfully flat it generates $I_s$ as well.

**Definition 6.2.3.** Let $E$ be an NC divisor on a toroidal embedding $(X, D)$. We say that a relatively nonsingular locally closed subscheme $C$ on $(X, D)$ has SNC with $E$, if there is an étale neighborhood $(U, D_U)$ such that the center and the components of $E$ are described by a part of a free coordinate system on $(U, D_U)$.

**Remark 6.2.4.** Let $E$ be an SNC divisor on $(X, D)$ and a $C$ has SNC on $(X, D)$. Then, by Lemma 2.1.18, the subscheme $C$, and $E$ are described by a part of a free coordinate system locally on Zariski open neighborhoods $(U, D_U)$.

**Lemma 6.2.5.** Let $E$ be an an NC divisor on a toroidal embedding $(X, D)$. Let $C$ be a closed relatively nonsingular center having SNC with $E$. Consider the blow-up $X' \rightarrow X$ of $C$. Denote by $E_C$ is the exceptional divisor and by $E'$, and $D'$ the strict transforms of $E$ and $D$. Then the divisor $E_C$ has SNC on $(X', D')$.

**Proof.** Assume that $(X, D)$ is a strict toroidal embedding and $E$ has SNC on $(X, D)$. Then there is a system of free parameters $u_1, \ldots, u_r$ on a neighborhood $U$ of $X$ such that the center $C$ is described by the ideal $I_C = (u_1, \ldots, u_k)$, and $E$ by the equation $u_1 \cdot u_2 \cdot \ldots \cdot u_{j_1} = 0$. Then for any $x \in U$ we have

$$\widehat{\mathcal{O}}_{x,U} = K(x)[[u_1, \ldots, u_r, P_0]]$$

The blow-up of $C$ will transform the center $C$ into the exceptional divisor $E_C$ defined locally, without loss of generality, by $x := u'_1 := u_1$, with $u'_i = u_i / u_1$, for $i = 1, \ldots, k$, and $u'_j = u_j$ for $j = k + 1, \ldots, r$. The local rings and their completions will be transformed accordingly

$$\widehat{\mathcal{O}}_{x', U'} = K(y)[[u'_1, \ldots, u'_r, P_0]]$$

where $u'_1, \ldots, u'_r$ are free parameters on $X$. The divisor $E' \cup E$ is, up to multiplicities described as $\sigma^*(u'_1 \cdot \ldots \cdot u'_j) = u'_j' \cdot u'_j' \cdot (u'_1')^d$ so has SNC on $X$.

The proof for NC divisor $E$ is the same, except we need to consider the effect of the blow-up on an étale neighborhood. Note that $E_C$ is locally (on $X'$) defined by a parameter and thus it is SNC on $(X', D')$.

**Proposition 6.2.6.** Let $E$ be a relative NC divisor on a toroidal embedding $(X, D)$, where $D'$ is the closure of $D \setminus E$ on $X'$. Then there exists a canonical sequence of blow-ups of strata of $E$ transforming the divisor $E$ into a relative SNC divisor on the resulting toroidal embedding $(X', D')$. The procedure is functorial for smooth morphisms.
Proof. Consider the blow-ups of all the strata of $E$ of the minimal dimension $r$. By minimality, the strata are disjoint closed, and relatively nonsingular. The blow-up will create a new NC divisor of the form $E^1 \cup E_1$, where $E_1$ is the exceptional divisor, and $E^1$ is the strict transform of $E$. The divisor $E_1$ is relatively SNC, and both $E^1$ and $E^1 \cup E_1$ are a relatively NC divisor, with $E^1$ having no strata in dimension $\leq r$. (All such strata were blown up). Then we blow-up the strata of $E^1$ of the minimal dimension $r + 1$. They have SNC with $E_1$. We create $E^2 \cup E_2$, where $E_2$ is the union of the exceptional divisors and the strict transforms of $E_1$, and $E^2$ are the strict transforms of $E^1$. As before $E_2$ is a relatively SNC, and $E^2$ and $E^2 \cup E_2$ are a relatively NC divisors, with $E^2$ having no strata in dimension $\leq r + 1$. We continue the process until we eliminate all the components in $E^k$. Functoriality follows from the construction of the centers.

\[\square\]

6.3. Blow-ups of toroidal valuations.

6.3.1. Toric valuations. We interpret the star subdivisions in the desingularization theorem 4.6.1 as the blow-ups at some functorial centers associated with the sets of valuations.

Each vector $v \in N$ defines a linear integral function on $M$ which determines a discrete valuation $\text{val}(v)$ on $X_{\Sigma}$. For any regular function $f = \sum_{w \in M} a_w x^w \in K[T]$ we set

$$\text{val}(v)(f) := \min\{(v, w) : a_w \neq 0\}.$$ 

Thus $N$ can be perceived as the lattice of all $T$-invariant integral valuations of the function field of $X_{\Sigma}$.

An integral vector $v \in |\Sigma|$, and the valuation $\text{val}(v)$ define the coherent ideal sheaves on $X_{\Sigma}$:

$$I_{\text{val}(v), a} := \{f \in \mathcal{O}_{X_{\Sigma}} \mid \text{val}(v)(f) \geq a\}$$

for all natural $a \in \mathbb{N}$.

6.3.2. Toric valuations. Let $\Sigma$ be a fan, and $X_{\Sigma}$ be the corresponding toric variety.

Recall that any nonnegative function $F : |\Sigma| \to \mathbb{Q}$ which is convex and piecewise linear on each cone $\sigma \in \Sigma$ defines a collection monoids

$$I_{\sigma} = \{m \in \sigma^\vee \cap M \mid m|_{\sigma} \geq F|_{\sigma}\}.$$ 

These monoids generate ideals $I_{F, \sigma}$ of $K[X_{\sigma}]$ which glue to form a coherent ideal sheaf of ideals $I_F$.

Conversely, a coherent $T$-stable ideal sheaf $\mathcal{I}$ is determined by the monoids

$$I_{\sigma} = \{m \in (\sigma^\vee)^{\text{integ}} \cap \mathcal{I}_{X_{\sigma}} \subset \sigma^\vee \}$$

which define a piecewise linear function

$$F_\mathcal{I} = \min\{m(w) : m \in I_{\sigma}\}.$$ 

If $F$ is integral, i.e $F(\sigma^{\text{integ}}) \subset \mathbb{N} \cup \{0\}$ for any $\sigma \in \Sigma$ then $F_\mathcal{I} = F$.

Lemma 6.3.3. Any primitive vector $v$ in $|\Sigma|$ defines a piecewise linear function $F_{v, a} : |\Sigma| \to \mathbb{Q}$, such that for any vector $w \in \sigma \subset |\Sigma|$, we have

$$F_{v, a}(w) := \min\{L(w) : L \in \sigma^\vee, \ L(v) = a\}.$$ 

Moreover,
(1) $F_{v,a}$ is a piecewise linear function which is convex on each face.
(2) It is integral if $a \in \mathbb{N}$ is sufficiently divisible.
(3) $F_{v,a} = I_{\text{val}(v),a}$, and for sufficiently divisible $a \in \mathbb{N}$, $F_{I_{\text{val}(v),a}} = F_{v,a}$.
(4) The star subdivision at $v$ creates an exceptional $Q$-Cartier divisor $D$ on the birational modification $X_{v,\Sigma}$ corresponding to $v$ such that the multiple $E = aD$ is a Cartier divisor.

Proof. It is a well known fact. To construct $F_{v,a}$ in more explicit terms, we consider the star subdivision $\langle v \rangle \cdot \Sigma$ of $\Sigma$, and define $F_{v,a}$ on all vertices of 1-dimensional rays of $\langle v \rangle \cdot \Sigma$. We put $F_{v,a}(u) = 0$ for $u \in \text{Vert}(\Sigma) \setminus \{v\}$, and $F_{v,a}(v) = a$. This canonically extends to all faces of $\langle v \rangle \cdot \Sigma$ by linearity. If $a$ is sufficiently divisible then $F_{v,a}$ is integral function.

The function $F_{v,a}$ defines monoids $I_{\text{val}(v),a,\sigma} = \{m \in \sigma^\vee \text{integ} \mid m(v) \geq 0\}$ which give rise to the sheaf $I_{\text{val}(v),a}$.

Finally the integral function $F_{v,a}$ on the star subdivision $\langle v \rangle \cdot \Sigma$, is linear on each cone. Thus it defines a Cartier divisor $E$. For any vertex $w$ in $\Sigma$ let $D_w$ be the corresponding divisor. Then the Cartier divisor $E$ can be written as

$$E = \sum_{w \in \text{Vert}(\Sigma)} F_{v,a}(w)D_w = mD_v.$$  

6.3.4. Toroidal valuations. One can easily extend the result to strict toroidal embeddings

Lemma 6.3.5. Let $(X,D)$ be a strict toroidal embedding, and $\Sigma$ be the associated conical complex. Any vector $v$ in $|\Sigma|$ defines a piecewise linear function $F_{v,a} : |\Sigma_X| \to \mathbb{Q}$, as above which is a convex on each face and piecewise linear function. It is integral if $a$ is sufficiently divisible. For such $a$, there is a unique coherent ideal $I_{F_{v,a}}$ and a toroidal valuation $\text{val}(v)$ on $X$, such that $I_{F_{v,a}} = I_{\text{val}(v),X,a}$.

Proof. To define the valuation $\text{val}(v)$ on $X$ we consider the canonical birational transformation $Y \to X$ associated with the star subdivision $\langle v \rangle \cdot \Sigma$. One can describe the valuation $\text{val}_Y(v)$ by using étale charts. By the previous lemma it follows that $\text{val}_Y(v)$ is the valuation of the exceptional divisor $D$ of $Y \to X$ as in the toric case. If $a$ is sufficiently divisible, the multiple $aD$ as in Lemma 6.3.3, is a Cartier divisor on $Y$. It is locally a pull-back of a toric Cartier divisor. Since $Y \to X$ is birational $\text{val}_Y(v)$ determines a unique valuation on $\text{val}_X(v)$ on $X$, with $K(X) = K(Y)$.

6.3.6. Blow-ups at toroidal valuations. Denote by $\text{bl}_J(X) \to X$ the blow-up of any coherent sheaf of ideals $J$.

Definition 6.3.7. Let $X$ be a toric variety (respectively strict toroidal embedding), with the associated fan (respectively the associated complex) $\Sigma$. Let $v \in |\Sigma|$ be an integral vector.
By the blow-up \( \text{bl}_{\text{val}(v)}(X) \) of \( X \) at a toric (respectively toroidal) valuation \( \text{val}(v) \) we mean the normalization of

\[
\text{Proj}(\mathcal{O} \oplus \mathcal{I}_{\text{val}(v),1} \oplus \mathcal{I}_{\text{val}(v),2} \oplus \ldots).
\]

**Lemma 6.3.8.** \( (5.2.8 \ [\text{Wlo05}]) \) Let \( X_\Sigma \) be a toric variety (resp. strict toroidal embedding) associated with a fan (resp. with a conical complex) \( \Sigma \) and \( v \in |\Sigma| \) be an integral vector. Then \( \text{bl}_{\text{val}(v)}(X_\Sigma) \) is the toric variety (toroidal embedding) associated with the star subdivision \( \langle v \rangle \cdot \Sigma \) of \( \Sigma \). Moreover, for any sufficiently divisible integer \( d \), \( \text{bl}_{\text{val}(v)}(X_\Sigma) \) is the normalization of the blow-up of \( \mathcal{I}_{\text{val}(v),d} \). The valuation \( \nu \) is induced by an irreducible \( \mathbb{Q} \)-Cartier divisor on the variety \( \text{bl}_{\text{val}(v)}(X) \).

**Proof.** By [KKMSD73], Ch. II, Th. 10, the normalized blow-up of \( \mathcal{I}_{\text{val}(v),d} \) corresponds to the minimal subdivision \( \Sigma' \) of \( \Sigma \) such that \( F_{\mathcal{I}_{\text{val}(v),D}} = F_{\nu,d} \) is linear on each cone in \( \Sigma' \). On the other hand, if \( d \) is sufficiently divisible then \( \mathcal{I}_{\text{val}(v),nd} = \mathcal{I}_{\text{val}(v),d} \) for any \( n \in \mathbb{N} \). See also Lemma 5.2.8 [Wlo03].

\[ \blacksquare \]

With any irreducible Weil divisor \( D \), which is Cartier at its generic point \( \nu = \nu_D \) we associate a valuation \( \text{val}_D \), such that for any \( f \) which is regular at \( \nu \), \( \text{val}_D(f) = n \), if we have the equality of ideals \( (f) = \mathcal{I}_{\text{val}(D),n} \) in the local ring \( \mathcal{O}_{X,\nu} \).

\[
\pi : Y = \text{bl}_\nu(X) \to X
\]

be the associated blow-up with the exceptional Weil, \( \mathbb{Q} \)-Cartier divisor \( D \).

**Corollary 6.3.9.** Let \( \nu \) be a toric (respectively toroidal) valuation on a toric variety (resp. strict toroidal embedding) \( X \), and

\[
\pi : Y = \text{bl}_\nu(X) \to X
\]

be the associated blow-up with the exceptional Weil, \( \mathbb{Q} \)-Cartier divisor \( D \).

Then for the ideals \( \mathcal{I}_{D,n} := \mathcal{I}_{\text{val}(D),n} = \{ f \in \mathcal{O}_X \mid \nu_D(f) \geq a \} \) on \( Y \) we have

\[
\pi_* (\mathcal{I}_{D,n}) = \{ f \in \pi_* \mathcal{O}_Y = \mathcal{O}_X \mid \nu_D(f) \geq a \} = \mathcal{I}_{\nu,a}.
\]

Thus the valuation \( \nu \) is induced by an irreducible exceptional Weil (\( \mathbb{Q} \)-Cartier) divisor \( D \) on the variety \( Y = \text{bl}_\nu(X) \).

6.3.10. **Filtered centers.** The centers of the blow-ups of the valuations \( \nu \) are represented by the sets of the ideals \( \mathcal{I}_{\nu,a} \). It is possible to associate with the center one of these ideals for a sufficiently divisible \( a \), but this correspondence won’t be functorial.

**Definition 6.3.11.** By a filtered center of a blow-up we mean a filtration \( \{ \mathcal{I}_n \}_{n \in \mathbb{N}} \) of ideals \( \mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \ldots \) on a variety \( X \) defining the graded Rees algebra

\[
\mathcal{O} \oplus \mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \ldots.
\]

By the blow-up of the filtered center \( \{ \mathcal{I}_n \}_{n \in \mathbb{N}} \) we mean

\[
\text{Proj}(\mathcal{O} \oplus \mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \ldots).
\]
6.4. Canonical desingularization of strict toroidal embeddings.

Theorem 6.4.1. For any strict toroidal embedding \((X, D)\) over a field \(K\) there exists a canonical resolution of singularities i.e., a birational projective morphism \(f : (Y, D_Y) \to (X, D_X)\) such that

1. \((Y, D_Y)\) is a nonsingular strict toroidal embedding. In particular, \(Y\) is nonsingular, and \(D_Y\) is an SNC divisor.
2. \(f\) is an isomorphism over the open set of nonsingular points. In particular, it is an isomorphism over \(U := X \setminus D\).
3. The inverse image \(f^{-1}(\text{Sing}(X)) \subseteq Y \setminus U = D_Y\) of the singular locus \(\text{Sing}(X)\) is a simple normal crossing divisor on \(Y\).
4. \(f\) is a composition of the normalized blow-ups of the locally monomial filtered centers \(\{J_n\}_{n \in \mathbb{N}}\) defined by the sets of valuations.
5. \(f\) commutes with smooth morphisms and field extensions, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.

Proof. Consider the conical complex \(\Sigma\) associated with \((X, D)\). Its combinatorial desingularization \(\Delta\) from Theorem 4.6.1 determines a canonical desingularization \((Y, D_Y)\) of \((X, D_X)\), with \(D_Y\) being SNC. Note that any toroidal divisor on a smooth strict toroidal embedding is SNC.

6.5. Canonical desingularization of toroidal embeddings.

Theorem 6.5.1. Let \((X, D)\) be a toroidal embedding over a field of any characteristic.

There exists a canonical desingularization \((Y, D_Y)\) of \((X, D)\), that is a projective birational toroidal morphism \(f : (Y, D_Y) \to (X, D)\) such that

1. \(Y\) is a smooth variety, and \(D_Y\) is an SNC -divisor (respectively NC- divisor).
2. If \(\text{Sing}(X)\) is the set of the singular points on \(X\) then \(f^{-1}(\text{Sing}(X))\) is an SNC divisor.
3. \(f\) is an isomorphism for all points where \(X\) is smooth, and \(D_X\) is and an SNC divisor (respectively NC divisor).
4. \(f\) is a composition of the normalized blow-ups of the locally monomial filtered centers \(\{J_n\}_{n \in \mathbb{N}}\) defined by the sets of valuations.
5. \(f\) commutes with smooth morphisms.
6. \(f\) commutes with the field extensions

Proof. Let \(X_0\) be an étale cover of \(X\) consisting of strict toroidal embeddings, and let \(\pi_i : X_1 := X_0 \times_X X_0, i = 1, 2\) be two natural projections. Consider the functorial resolution of \(X_0\) from Theorem 6.4.1. By functoriality, the centers lift to \(\pi_1^*(I_{j,n}) = \pi_2^*(I_{j,n})\), and thus by the flat descent the ideals \(I_{j,n}\) on \(X_0\) generate the ideals \(I_{j,n,X}\) on \(X\). The blow-ups at \(I_{j,n,X}\) determine the functorial birational modification \(Y\) of \(X\), which is smooth in étale topology, and hence it is smooth. The inverse image of the \(f^{-1}(\text{Sing}(X))\) of the singular locus \(\text{Sing}(X)\) is SNC in étale topology, and thus it is NC in Zariski topology. By Proposition 6.2.6, the
exceptional locus $D_Y$ can be further modified functorially, if needed, to an SNC divisor $D'_Y$ by a sequence of blow-ups at smooth strata.

6.6. Canonical partial desingularization of toroidal and strict toroidal embeddings.

**Theorem 6.6.1.** Let $(X, D)$ be a toroidal embedding over a field of any characteristic with $U = X \setminus D$. Let $V \subset X$ be an open saturated toroidal subset of $X$. Denote by $D_V = V \cap D$ the induced divisor on $V$, and assume that $D_V$ has locally ordered components.

There exists a canonical resolution of singularities of $(X, D)$ except for $V$, i.e. a projective birational toroidal morphism $f : Y \to X$ such that

(1) $f$ is an isomorphism over the open set $V$.
(2) The variety $(Y, D_Y)$ is a strict toroidal embedding, where $D_Y := \overline{D_V}$ is the closure of the divisor $D_V$ in $Y$.
(3) The variety $(Y, D_Y)$ is the saturation of $(V, D_V)$.
(4) The complement $E_{V,Y} := Y \setminus V$ is a divisor which has simple normal crossings with $D_Y$, and so does the exceptional divisor $E_{exc} \subset E_{V,Y}$. In particular, $(Y, D_Y)$ is a strict toroidal embedding.
(5) $f$ commutes with field extensions and smooth morphisms preserving the order of the components $D_V$, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.

**Proof.** Assume first that $(X, D)$ is a strict toroidal embedding with the associated complex $\Sigma$. Thus the open saturated subset $V$ has a form $V = X(\Omega)$, where $(\Sigma, \Omega)$ is a relative complex. Consider its canonical relative desingularization $(\Delta, \Omega)$ from Theorem 5.12.1. This, by Lemma 5.4.7, defines the associated strict toroidal variety $(Y, U)$, such that $(Y, \overline{D_V})$ is also a strict toroidal embedding, and the complement $E_{V,Y} := Y \setminus V$ is a divisor which has SNC with $\overline{D_V}$.

Now if $(X, D)$ is a toroidal embedding, then we repeat the reasoning from the proof of Theorem 6.6.1. We obtain a toroidal embedding $(Y, \overline{D_V})$, which is strictly toroidal in the étale charts. Since $(Y, \overline{D_V})$ contains an open subset $(V, D_V)$ intersecting all strata and which is strictly toroidal then $(Y, \overline{D_V})$ is strictly toroidal by Lemma 2.1.10. Consequently, $E_{V,Y} := X \setminus V$ has NC on $(Y, \overline{D_V})$. To obtain the condition that $E_{V,Y}$ has SNC with $\overline{D_V}$ it suffices to apply the additional blow-ups from Proposition 6.2.6. The latter process commutes with smooth morphisms and field extensions. The previous steps commute with field extensions and smooth morphisms preserving the order of the components $D_V$.

7. Canonical desingularization of locally toric varieties

7.1. Locally toric and locally binomial varieties.
Definition 7.1.1. A variety $X$ will be called locally toric (respectively étale locally toric) if for any point $x \in X$ there is an étale morphism called a chart $\varphi_x : U_x \to X_\sigma$ from an open neighborhood (respectively an étale neighborhood) of $x$ to a toric variety $X_\sigma$. Similarly, $X$ will be called locally binomial (respectively étale locally binomial) if it locally (respectively étale locally) admits an étale morphism to a variety $Z \subset \mathbb{A}^n = \text{Spec}(K[x_1, \ldots, x_n])$ with the (reduced) ideal $I_Z = (F_1, \ldots, F_s)$ defined by the binomials $F_i = x^{\alpha_i} - x^{\beta_i}$, with $\alpha_i, \beta_i \in \mathbb{Z}^k$.

Consider the homomorphism of tori $\phi : T^k \to T^s$, where $T^k = \text{Spec}(K[x_1, x_1^{-1} \ldots, x_n, x_n^{-1}]) \subset \mathbb{A}^n$, $T^s = \text{Spec}(K[x_1, x_1^{-1} \ldots, x_s, x_s^{-1}]) \subset \mathbb{A}^s$ given by $\phi(x) = (x^{\alpha_1 - \beta_1}, \ldots, x^{\alpha_s - \beta_s})$. Its kernel defines a variety $T_Z := \ker(\phi) = T^k \cap Z$, which is an open subvariety of $Z$. It is also a closed subtorus of $T^k$. Thus $Z \supset T_Z$ is a variety with a torus action with a big open orbit on which torus $T_Z$ acts transitively. It can be described as $Z = \text{Spec}(K[G_1, \ldots, G_s])$, where $G_i = y^\gamma_i$ are Laurent monomials in $K[T_Z] = K[y_1, y_1^{-1}, \ldots, y_r, y_r^{-1}]$. The normalization transforms $Z$ into a normal toric variety.

Consequently, the normalization of étale locally binomial variety transforms it into an étale locally toric variety.

7.2. Stratified toric varieties. Locally toric varieties can be equipped with various natural stratifications making them into a more general class of stratified toroidal varieties.

7.2.1. Stratified toric varieties.

Definition 7.2.2. [Wlo03, Definition 3.1.3] A stratified toric variety is a toric variety $(X, T)$ over a field $K$ with a $T$-stable equisingular stratification, such that for any stratum $s$ and two geometric $K$-points $x, x' \to s$ the completions of local rings $\widehat{O}_{X, x}$ and $\widehat{O}_{X, x'}$ are isomorphic over $\text{Spec} K$, where $\overline{K}$ is the algebraic closure of $K$, and

$$X_{\overline{K}} := X \times_{\text{Spec} K} \text{Spec} \overline{K}$$

The important difference between toric varieties and stratified toric varieties is that the latter come with stratifications which may be coarser than the one given by orbits. As a consequence, the combinatorial object associated with a stratified toric variety, called a semifan, consists of those faces of the fan of the toric variety corresponding to strata. The faces which do not correspond to the strata are ignored (see Definition 7.2.7).

7.2.3. Demushkin’s theorem and equisingularity. Recall that $\text{sing}(\sigma)$ is a unique maximal irreducible singular face of $\sigma$. (Definition 4.1.4)

The equisingularity condition can be well understood by applying the following criterion:
Theorem 7.2.4. ([Dem82], [Wlo03, Theorem 2.5.1]) Let $\sigma$ and $\tau$ be two cones of maximal dimension in isomorphic lattices $N_\sigma \simeq N_\tau$. Set $\tilde{X}_\sigma^K := \text{Spec}(K[[\sigma^\vee]\text{integ}])$, $\tilde{X}_\tau^K := \text{Spec}(K[[\tau^\vee]\text{integ}])$ for any field $K$. Then the following conditions are equivalent:

1. $\sigma \simeq \tau$.
2. $\tilde{X}_\sigma^K \simeq \tilde{X}_\tau^K$.
3. $\tilde{X}_\sigma^K \simeq \tilde{X}_\tau^K$.
4. $\text{sing}(\sigma) \simeq \text{sing}(\tau)$.

Remark 7.2.5. Theorem 7.2.4 was originally proven over algebraically closed field. But we have obvious implications $1 \Rightarrow 2 \Rightarrow 3$.

7.2.6. Semifans. For any subset $\Sigma_0$ of a fan $\Sigma$ by the closure of $\Sigma_0$ we shall mean the smallest subcomplex $\overline{\Sigma_0}$ of $\Sigma$ containing $\Sigma_0$.

Definition 7.2.7. [Wlo03, Definition 3.1.5] An embedded semifan is a subset $\Omega \subset \Sigma$ of a fan $\Sigma$ in a $NQ$ such that for every $\sigma \in \Sigma$ there is a unique maximal face $\omega(\sigma) \in \Sigma$ satisfying

1. $\omega(\sigma) \preceq \sigma$ and any other $\omega \in \Omega$ with $\omega \preceq \sigma$ is a face of $\omega(\sigma)$,
2. $\sigma = \omega(\sigma) \oplus r(\sigma)$ for some regular cone $r(\sigma) \in \Sigma$.

An embedded fan is an embedded semifan $\Omega \subset \Sigma$, where $\Omega$ is a subfan of $\Sigma$.

A semifan in a lattice $N$ is a set $\Omega$ of cones in $NQ$ such that the set $\Sigma = \bigcup \Omega$ of all faces of the cones of $\Omega$ is a fan in $NQ$ and $\Omega \subset \Sigma$ is an embedded semifan.

By the support of the semifan $\Omega$ we mean the union of all its faces, $|\Omega| = \bigcup_{\sigma \in \Omega} \sigma$.

As in Definition 4.1.4, we can consider the unique decomposition $\sigma = \text{sing}(\sigma) \oplus \text{reg}(\sigma)$ in the context of fans.

Example 7.2.8. [Wlo03, Example 3.1.2] (see Definition 4.1.4) Consider any fan $\Sigma$, and the subset $\text{sing}(\Sigma) := \{\text{sing}(\sigma) \mid \sigma \in \Sigma\}$.

Then $\text{sing}(\Sigma) \subset \Sigma$ is an embedded semifan.

Proposition 7.2.9 (Proposition 3.1.7). [Wlo03] Let $\Sigma$ be a fan with a lattice $N$, and let $X$ denote the associated toric variety. There is a canonical bijective correspondence between the toric equisingular stratifications of $X$ and the embedded semifans $\Omega \subset \Sigma$:

1. If $S$ is a toric stratification of $X$, then the corresponding embedded semifan $\Omega \subset \Sigma$ consists of all those cones $\omega \in \Sigma$ that describe the big orbit of some stratum $s \in S$.
2. If $\Omega \subset \Sigma$ is an embedded semifan, then the strata of the associated toric stratification $S_\Omega$ of $X$ arise from the cones of $\Omega$ via $\omega \mapsto \text{strat}(\omega) := \bigcup_{\omega(\sigma) = \omega} O_\sigma$.

Proof. (1) Let $S$ be a $T$-invariant equisingular stratification on $X$. Since the strata of $S$ are $T$-invariant and disjoint, each orbit $O_\sigma$ belongs to a unique stratum $s$. Let $\omega \in \Omega$ describe the big open orbit of $s$. Write $s = \text{strat}(\omega)$. If the orbit $O_\sigma$ is
contained in \( s \), then it is contained in the closure of \( O_\omega \). Hence \( \omega = \omega(\tau) \) is a face of \( \tau \). If any other \( \omega' \in \Omega \) is a face of \( \tau \) with \( O_\tau \subset s \) then \( O_\tau \) is in the closure of \( O_{\omega'} = \text{strat}(\omega') \). Consequently, the stratum \( s = \text{strat}(\omega) \) is contained in \( \text{strat}(\omega') \). This implies that \( \omega' \) is a face of \( \omega \) and shows Condition (1) of Definition 7.2.7.

Moreover, by Definition 7.2.2, there exists an isomorphism of the completions of the local rings \( \mathcal{O}_{X_\tau, x} \) and \( \mathcal{O}_{X_\omega, y} \) of two points \( x \in O_\tau \) and \( y \in O_\omega \). By Theorem 7.2.4, and since \( \text{sing}(\tau) \supseteq \text{sing}(\omega) \), we infer that \( \text{sing}(\tau) = \text{sing}(\omega) \). Hence \( \omega = \text{sing}(\omega) \oplus \text{reg}(\omega) \) and \( \tau = \text{sing}(\tau) \oplus \text{reg}(\tau) = \omega \oplus \tau' \), where \( r(\tau) = r(\omega) \oplus \tau' \). This proves Condition (2) of Definition 7.2.7.

(2) By definition, the strata \( \text{strat}(\omega) \) in \( S_\Omega \) are the unions of orbits that correspond to the collections of cones:

\[
\{ \sigma \in \Sigma \mid \omega(\sigma) = \omega \} = \text{Star}(\omega, \Sigma) \setminus \bigcup_{\omega' \in \Omega} \text{Star}(\omega', \Sigma)
\]

Under this correspondence

\[
\text{strat}(\omega) = \overline{O_\omega} = \{ O_\tau : \tau \in \text{Star}(\omega, \Sigma) \}
\]

In particular,

\[
\text{strat}(\omega) = \overline{\text{strat}(\omega)} \setminus \bigcup_{\omega' \in \Omega} \overline{O_{\omega'}}
\]

are locally closed, and

\[
\text{strat}(\omega) = \bigcup_{\omega' \in \Omega} \overline{O_{\omega'}}.
\]

Moreover, by definition, \( \text{strat}(\omega) \) are disjoint, and each orbit \( O_\tau \) is contained in a unique stratum \( \text{strat}(\omega) \), where \( \omega = \omega(\sigma) \). Thus the sets \( \text{strat}(\omega) \), where \( \omega \in \Omega \) form a stratification on \( X \). By Section 3.2, each \( \text{strat}(\omega) \subset \overline{O_\omega} \) is a toric variety with a fan

\[
\Sigma' := \{ \tau/\omega \mid \omega(\tau) = \omega \}
\]

in \( (N^Q)' := N^Q / \text{span}(\omega) \). Since \( \text{sing}(\tau) = \text{sing}(\omega) \), then \( \tau/\omega \) is regular and the strata \( \text{strat}(\omega) \) are smooth. Moreover, by Theorem 7.2.4, the points in \( \text{strat}(\omega) \) have isomorphic local rings (and their completions).

**Corollary 7.2.10.** The strata on a stratified toric variety are smooth.

### 7.3. Isomorphisms of embedded semifans over partially ordered sets.

**Definition 7.3.1.** By an automorphism of an embedded semifan \( \Omega \subset \Sigma \) in \( N^Q \) we mean an automorphism \( \alpha : N^Q \to N^Q \) of the vector space \( N^Q \), preserving the lattice \( N \), which defines an automorphism of the fan \( \Sigma \) such that \( \alpha(\omega) = \omega \) for any \( \omega \in \Omega \). The group of the automorphisms will be denoted by \( \text{Aut}(\Sigma, \Omega) \).

**Remark 7.3.2.** The automorphisms in \( \text{Aut}(\Sigma, \Omega) \) of the embedded semifan \( \Omega \subset \Sigma \) induce the automorphisms of the toric variety \( X_\Sigma \) preserving the strata in \( S_\Omega \) (and the torus \( T \subset X_\Sigma \)).

The cones in \( \Omega \) are in the natural bijective correspondence with the strata of \( S_\Omega \), and this bijection respects the natural partial order. The other cones of \( \Sigma \) have no geometric meaning and are dependent of a particular torus action. When comparing two embedded semifans associated with the same stratification \( S \), it is convenient to consider the natural bijection \( \Omega \to S_\Omega \) of the partially ordered sets.
Definition 7.3.3. Let \( S \) be a partially ordered set. By an embedded semifan over \( S \) we mean a semifan \( \Omega \subset \Sigma \) with an injective map \( j_{\Omega,S} : \Omega \to S \) of the partially ordered sets respecting the order.

Definition 7.3.4. Let \( \Omega_i \subset \Sigma_i \) in \( N_i^Q \supset N_i \) for \( i = 1, 2 \), be two embedded semifans over \( S \). By the isomorphism \( (\Sigma_1, \Omega_1) \to (\Sigma_2, \Omega_2) \) of embedded semifans over \( S \) we mean a vector space isomorphism \( i_{N_1, N_2} : N_1^Q \to N_2^Q \) preserving the lattice structures which defines an isomorphism of the fans \( j_{\Sigma_1, \Sigma_2} : \Sigma_1 \to \Sigma_2 \) and induces a bijection \( j_{\Omega_1, \Omega_2} : \Omega_1 \to \Omega_2 \) of the sets of isomorphic cones, commuting with the maps to \( S \):

\[
j_{\Omega_2,S} \circ j_{\Omega_1,\Omega_2} = j_{\Omega_1,S}
\]

Remark 7.3.5. Any embedded semifan \( \Omega \subset \Sigma \) is defined over the stratification \( S_\Omega \) with the natural map \( j_{\Omega,S} : \Omega \to S_\Omega \) identifying the cones of \( \Omega \) with the strata of \( S_\Omega \). This interpretation is useful for glueing properties when comparing different charts.

Any embedded semifan \( \Omega \subset \Sigma \) can be considered naturally as an embedded semifan over \( S := \Omega \) with the identical map \( \Omega \to \Omega = S \). Then the isomorphisms of \( \Omega \subset \Sigma \) into itself over \( S \) are simply the automorphisms in \( \text{Aut}(\Sigma, \Omega) \).

Analogously one can define the stratified toric varieties over \( S \) and their isomorphisms.

Definition 7.3.6. Let \( S \) be a partially ordered set. By a stratified toric variety over \( S \) we mean a stratified toric variety \( X \) with a stratification \( T \), and an injective map \( j_{T,S} : T \to S \) of the partially ordered sets respecting the order.

By the isomorphism \( (X_1, T_1) \to (X_2, T_2) \) of stratified toric varieties over \( S \) we mean an isomorphism of toric varieties preserving stratifications which defines a bijection \( j_{T_1, T_2} : T_1 \to T_2 \) of the sets of strata, commuting with the maps to \( S \):

\[
j_{T_2,S} \circ j_{T_1, T_2} = j_{T_1,S}
\]

7.3.7. Generalized Demushkin’s theorem. Recall that for any cone \( \tau \) by \( \Sigma \) we mean its closure, that is, the fan consisting of the faces of \( \tau \).

Theorem 7.3.8. [Wlo03, Theorem 4.6.1] Let \( \overline{K} \) be an algebraically closed field. Let \( S \) be a partially ordered set.

Let \( \sigma \) and \( \tau \) be two cones of maximal dimension in isomorphic lattices \( N_\sigma \cong N_\tau \). Let \( S_\sigma \), and \( S_\tau \) denote the equisingular stratifications on \( X_\sigma \), and \( X_\tau \), and \( \Omega_\sigma \subset \overline{\sigma} \), \( \Omega_\tau \subset \overline{\tau} \) be the corresponding embedded semifans with the natural bijections \( \Omega_\sigma \to S_\sigma \) and \( \Omega_\tau \to S_\tau \), commuting with given injective maps \( S_\sigma \to S \), and \( S_\tau \to S \).

Then the following conditions are equivalent:

1. \( (\overline{\sigma}, \Omega_\sigma) \) and \( (\overline{\tau}, \Omega_\tau) \) are isomorphic over \( S \).
2. \( (\overline{X_\sigma}, \overline{S_\sigma}) \) and \( (\overline{X_\tau}, \overline{S_\tau}) \) are isomorphic over \( S \) (defining the correspondence between the strata in \( \overline{S_\sigma} \) and \( \overline{S_\tau} \)).

Proof. We sketch the idea of the proof. It is very similar to the proof of Theorem 7.2.4. For the details see [Wlo03].

The isomorphism \( (\overline{X_\sigma}, \overline{S_\sigma}) \to (\overline{X_\tau}, \overline{S_\tau}) \) over \( S \) defines two different toric actions on the same variety \( (\overline{X_\sigma}, \overline{S_\sigma}) := (\overline{X_\sigma}, \overline{S_\sigma}) \) by the relevant tori \( T_\tau \) and \( T_\sigma \). The action
of these tori determine uniquely (up to constants) the semi-invariant parameters generating the dual cones $\sigma^\vee$ and $\tau^\vee$.

The problem translates into the fact that the tori $T_\tau$ and $T_\sigma$ are maximal and conjugate in the proalgebraic group $\mathrm{Aut}((\hat{X}_x, \hat{S}_x))$ of all the automorphisms preserving the stratification, that is $T_\sigma = \phi^{-1}T_\tau \phi$, where $\phi \in \mathrm{Aut}((\hat{X}_x, \hat{S}_x))$. (In fact there is a natural map to the linear group of the tangent space, and its kernel is a unitary proalgebraic group (see Section 7.10.)

The conjugation $\phi$ defines the desired automorphism of $(\hat{X}_x, \hat{S}_x)$ which translates into isomorphism $\psi : (\hat{X}_\sigma, \hat{S}_\sigma) \rightarrow (\hat{X}_\tau, \hat{S}_\tau)$, respecting tori actions. Thus it defines the isomorphism of the cones

$$\sigma^\vee\integ \simeq \psi^*\left((\sigma^\vee\integ\right) = (\tau^\vee\integ$$

and their relevant faces, that is the embedded semifans $(\Sigma, \Omega_\sigma)$ and $(\tau, \Omega_\sigma)$. 

7.4. Stratification on toric varieties by the singularity type. Let $X_\Sigma$ be a toric variety of dimension $n$ associated with $\Sigma$. For any closed point $x \in X_\Sigma$ with the residue field $K(x) \subset K$ we define the singularity type $\text{sing}(x)$ to be the isomorphism class of the cones $(\text{sing}(\sigma), N^\mathbb{Q}_{\text{sing}(\sigma)})$, where $\sigma$ is a cone of dimension $n$ such that $\hat{X}_x = \hat{X}_x^{K(x)}$.

By the Demushkin Theorem 7.2.4, $\text{sing}(x)$ is independent of the toric structure on $X = X_\Sigma$. If $x \in O_\sigma \subset X_\Sigma$, then $\text{sing}(x)$ is the isomorphism class of $\text{sing}(\sigma)$. We shall write it as $\text{sing}(x) = \text{sing}(\sigma)$.

Lemma 7.4.1. [Wlo03, Lemma 4.1.7] The canonical stratification $\text{Sing}(X_\Sigma)$ on $X_\Sigma$ defined by the singularity type $\text{sing}$ on a toric variety $X_\Sigma$ corresponds to the embedded semifan $\text{sing}(\Sigma) \subset \Sigma$ from Example 7.2.8.

That is, for a stratum $s \in \text{Sing}(X_\Sigma)$ through $x \in X_\Sigma$ we have

$$s \cap U = \{ x \in U \mid \text{sing}(x) = \text{sing}(y) \},$$

for some open neighborhood $U$ of $x$.

Proof. Consider the stratification corresponding to the embedded semifan $\text{Sing}(\Sigma) \subset \Sigma$. Let $x \in s = \text{strat}(\sigma)$ where $\sigma$ is an irreducible face in $\text{Sing}(\Sigma)$. Then $x \in O_\sigma$, with $\tau$ satisfying $\text{sing}(\tau) = \sigma$ in the open set $X_\tau$. We can write

$$\tau = \text{sing}(\tau) \oplus \text{reg}(\tau),$$

so the singularity type $\text{sing}$ (defined up to isomorphism of cones) is the same for all points of the stratum $\text{strat}(\sigma)$ and equal to $\text{sing}(\sigma)$. It differs from the points in $X_\tau \setminus s$ which have the singularity types of the irreducible cones of smaller dimension which are the proper faces of $\text{sing}(\sigma)$. Thus any stratum $\text{strat}(\sigma)$, where $\sigma \in \text{Sing}(\Sigma)$ can be characterized as the set of points with locally constant singularity type equal $\text{sing}(\sigma)$.

7.5. Toric varieties with toric divisors. One can adapt the terminology of relative complexes to the situation of fans. (See Sections 5.1, 5.2, 5.4)

Definition 7.5.1. By a relative fan we mean a pair $(\Sigma, \Omega)$ where $\Omega \subseteq \Sigma$ are fans. A relative fan is regular if any cone $\sigma \in \Sigma$ contains a unique maximal face $\omega \in \Omega$ such that $\sigma = \omega \oplus \tau$ for a regular cone $\tau$. A relative fan $(\Sigma, \Omega)$ will be called saturated if any face of $\Sigma$ with vertices (one dimensional rays) in $\Omega$, is in fact in $\Omega$.
Equivalently we say that a subfan $\Omega \subset \Sigma$ is saturated. For any $\sigma \in \Sigma$, by $\text{sing}_\Omega(\sigma)$ we mean the smallest face containing $\text{sing}(\sigma)$, and the faces of $\sigma$ which are in $\Omega$. Then $\sigma = \text{sing}_\Omega(\sigma) \oplus \tau$, where $\tau$ is a regular cone. Moreover, $\text{sing}_\Omega(\sigma)$ is a unique minimal face of $\sigma$ which gives such decomposition.

**Example 7.5.2.** Consider a relative fan $(\Sigma, \Omega)$. Let

$$\text{sing}(\Sigma, \Omega) := \{ \text{sing}_\Omega(\sigma) | \sigma \in \Sigma \}.$$  

Then $\text{sing}(\Sigma, \Omega) \subset \Sigma$ is an embedded semifan. Moreover, $\text{sing}(\Sigma, \Omega) \supseteq \Omega$.

For any toric variety $X_\Delta \supset T$ associated with the fan $\Delta$ denote by

$$D_\Delta := X_\Delta \setminus T$$

the toric divisor of the complement of the big torus $T$. It is the maximal toric divisor on $X_\Delta$ (see Definition 4.1.7). The following lemma rephrases Lemma 5.4.6 in the context of fans and toric varieties.

**Lemma 7.5.3.** There is a bijective correspondence between the toric divisors $D$ on a toric variety $X_\Sigma$ and the saturated subfans $\Omega$ of $\Sigma$. Given a toric divisor $D$ on $X_\Sigma$, there is a unique saturated subfan $\Omega \subset \Sigma$ which is defined by the set of vertices of $\Sigma$ corresponding to the components of $D$. Conversely, $D = \overline{D_\Omega}$, where $D_\Omega = X_\Omega \setminus T$ is a toric divisor on $X_\Omega$.

**Proof.** The proof is identical to the proof of Lemma 5.4.6. 

7.5.4. Embedded fans. There is a relation between the notions of embedded fans (Definition 7.2.7) and relative fans as both represent properties of stratifications. The following lemma is a consequence of definition.

**Lemma 7.5.5.** A relative fan $(\Sigma, \Omega)$ is an embedded fan iff it is a regular relative fan. Moreover, if $(\Sigma, \Omega)$ is an embedded fan then $\Omega \subset \Sigma$ is a saturated subfan, and $\text{Reg}(\Sigma, \Omega)) = \Omega$.

**Corollary 7.5.6.** Let $X_\Sigma$ be a toric variety associated with the fan $\Sigma$. Let $\Omega$ be a saturated subfan of $\Sigma$, corresponding to a toric divisor $D = \overline{D_\Omega}$. Then

1. $(X, D)$ is a strict toroidal embedding if $(\Sigma, \Omega)$ is a regular relative fan (or embedded fan). Then $(X, D)$ is the saturation of $(X_\Omega, D_\Omega) \subset (X, D)$, and $E := D_\Sigma \setminus D = X \setminus X_\Omega$ is a relative SNC divisor on $(X, D)$.
2. Conversely, if $(X, D)$ is a strict toroidal embedding, where $D \subset D_\Sigma$ then $D = \overline{D_\Omega}$, where $(\Sigma, \Omega)$ is regular relative fan.
3. In general, the toroidal locus $(X, D)^\text{tor}$ of $(X, D)$ is defined by the saturated subfan $\text{Reg}(\Sigma, \Omega)$ of $\Sigma$. It is the toroidal saturation of $(X_\Omega, D_\Omega)$ in $(X, D)$.

**Proof.** The proof is identical to the proof of the analogous statement for complexes in Lemma 5.4.7.
7.5.7. Canonical stratification on toric varieties with toric divisors. Let \((X, D)\) be a pair of a toric variety \(X = X_\Sigma\) and a toric divisor \(D\).

Let \(x \in X\), \(x \in O_\sigma\), with \(\Omega_\sigma := \sigma \cap \Omega\). We define the singularity type at \(x\) on \((X, D)\) to be

\[\text{sing}_D(x) := (\text{sing}(x), n_D(x)),\]

where \(n_D(x)\) is the number of the components of \(D\) through \(x \in X\), and by \(\text{Sing}_D(X)\) the corresponding stratification.

Recall that the divisor \(D\) determines the divisorial stratification \(S_D\) with closed strata being the intersection components of \(D\).

The vertices of \(\Sigma\) associated with components of \(D\) determine the saturated subfan \(\Omega \subset \Sigma\).

**Lemma 7.5.8.** Let \(X_\Sigma\) be a toric variety and \(D = \overline{D_\Omega}\) be a toric divisor on \(X_\Sigma\) corresponding to a saturated subfan \(\Omega \subset \Sigma\). Then the stratification \(\text{Sing}_\Omega(X_\Sigma)\) on \(X_\Sigma\) defined by the embedded semifan \(\text{sing}(\Sigma, \Omega) \subset \Sigma\) from Example 7.5.2, coincides with the stratification \(\text{Sing}_D(X)\), defined by the singularity type \(\text{sing}_D(x)\). Moreover,

1. The strata of \(\text{Sing}_\Omega(X_\Sigma)\) on \(X_\Sigma\) correspond to the cones in \(\text{sing}(\Sigma, \Omega)\). In particular, \(\text{Sing}_\Omega(X_\Sigma)\) contains the strata which are extensions of the orbits in \(X_\Omega\), and \(\text{sing}(\Sigma, \Omega)\) contains \(\Omega\).

2. The stratification \(S_D\) is coarser than \(\text{Sing}_\Omega(X_\Sigma)\).

3. The saturation of \((X_\Omega, D_\Omega)\) in \((X_\Sigma, D_\Omega)\) is an open subset \(X_{\text{Reg}(\Sigma, \Omega)}\) corresponding to the subfan \(\text{Reg}(\Sigma, \Omega)\) of \(\Sigma\) containing \(\Omega\). Moreover, it defines the toroidal locus

\[(X_\Sigma, D)^{\text{tor}} = X_{\text{Reg}(\Sigma, \Omega)}.;\]

4. The restriction of the stratification \(\text{Sing}_\Omega(X_\Sigma) = \text{Sing}_D(X)\) to \(V = (X_\Sigma, D)^{\text{tor}}\) coincides with the divisorial stratification \(S_D|_V\).

5. In particular, if \((X, D)\) is a strict toroidal embedding then the stratifications \(S_D\) and \(\text{Sing}_D(X)\) coincide.

**Proof.** Consider the embedded semifan \(\text{sing}(\Sigma, \Omega) \subset \Sigma\). Let \(\sigma \in \text{sing}(\Sigma, \Omega)\) be a relatively irreducible cone of \(\Sigma\), with its subfan \(\Omega_\sigma = \Omega \cap \sigma\). If \(x \in \text{strat}(\sigma)\), then \(x \in O_\tau\) with \(\text{sing}_\Omega(\tau) = \sigma\). This implies that \(\text{sing}_D(x) = (\text{sing}(\sigma), n_D)\), where \(n_D\) is the number of vertices in the subfan \(\Omega_\sigma = \Omega \cap \tau\). If \(y \in X_\tau\), and \(y \notin \text{strat}(\sigma)\), then \(y \in O_{\tau'}\), where \(\tau' \leq \tau\) with \(\text{sing}_\Omega(\tau') < \sigma\). By definition \(\text{sing}_\Sigma(\tau)\) is the smallest face containing \(\text{sing}(\tau)\) and \(\Omega_{\tau'}\). So either \(\text{sing}(\tau') \subset \text{sing}(\sigma)\) or \(\Omega_{\tau'} \subset \Omega_\sigma\). In any case

\[(\dim(\sigma_y), n_D(y)) < (\dim(\sigma_x), n_D(x))\]

with respect to the lexicographic order. This shows that the stratifications \(\text{Sing}_\Omega(X_\Sigma)\) and \(\text{Sing}_D(X)\) coincide.

1. Follows from Definition.

2. By Lemma 7.5.3, the closed strata of the stratification \(S_D\) are the intersections of the irreducible divisorial components. These components correspond to the vertices of \(\Omega\). Since \(\Omega\) is saturated their intersections correspond to the cones in \(\Omega\). Thus the cones in the subfan \(\Omega\) correspond to the strata and the closed strata in \(S_D\). In this correspondence a cone \(\omega \in \Omega\) corresponds to the closed stratum \(\text{strat}(\omega) = \overline{\Omega_\omega}\). The fact that \(\text{sing}(\Sigma)\) contains \(\Omega\) means that the \(S_D\) is coarser than \(\text{Sing}_D\).
(3) Follows from Lemma 7.5.6(3).
(4) Follows from (1) and (3). In this case we also have
\[ \text{sing}((\text{Reg}(\Sigma, \Omega), \Omega)) = \Omega. \]
(5) Follows from (4).

7.6. Stratified toroidal varieties.

**Definition 7.6.1.** [Wlo03, Definition 4.1.6] By a *stratified toroidal variety* we mean a stratified variety \((X, S)\) such that for any \(x \in s, s \in S\) there is an étale map called a *chart* \(\varphi : (U, S \cap U) \to (X_{\sigma}, S_{\sigma})\) from an open neighborhood to a stratified toric variety \((X_{\sigma}, S_{\sigma})\) such that all strata in \(U\) are the preimages of strata in \(S_{\sigma}\).

**Remark 7.6.2.** The stratified toric varieties \((X_{\sigma}, S_{\sigma})\) in the above definition correspond to the embedded semifans \(\sigma \supseteq \Omega_{\sigma}\), where \(\sigma\) is the fan consisting of all the faces of \(\sigma\), and \(\Omega_{\sigma}\) describes the generic orbits of the strata in \(S_{\sigma}\). Moreover, the definition implies the existence of a natural injective map from \(\Omega_{\sigma} \to S_{\sigma} \to S\), associating with faces the corresponding strata. So the induced semifans can be considered as semifans over the stratification \(S\). This observation is important and helps to understand how gluing works.

Toroidal embeddings with the natural stratification are particular example of a stratified toroidal variety.

7.6.3. Canonical stratification on locally toric varieties. Using Demushkin’s theorem, we are in position to define a combinatorial invariant on a locally toric variety.

We associate with any closed point \(x \in X\), with residue field \(K(x)\) the invariant \(\text{sing}(x)\) to be the isomorphism class of the cones \(\text{sing}(\sigma)\), where \(\sigma\) are the cones of the maximal dimension, where
\[
\hat{X}_x \simeq \hat{X}_{K(x)}^{\text{sing}(x)}.
\]
The invariant \(\text{sing}(x) = \text{sing}(\sigma)\) is well defined by Theorem 7.2.4.

**Lemma 7.6.4.** [Wlo03, Lemma 4.2.1] Let \(X\) be a locally toric variety. There is a locally closed stratification \(\text{Sing}(X)\) with smooth irreducible strata \(s \in \text{Sing}(X)\), such that for any point \(y \in X\), and the stratum \(s\) through \(y\), there is an open neighborhood \(U\) such that
\[
s \cap U = \{ x \in U \mid \text{sing}(x) = \text{sing}(y) \}.
\]
Moreover, the stratification \(\text{Sing}(X)\) is locally defined via charts by the inverse images of strata on \((X_{\sigma}, \text{Sing}(X_{\sigma}))\) associated with the embedded semifan \((\sigma, \text{sing}(\sigma))\).

**Proof.** Consider an étale chart \(U \to X_{\sigma}\). By Lemma 7.4.1, the stratification \(\text{Sing}(X_{\sigma})\) on \(X_{\sigma}\) is defined by the singularity type \(\text{sing}(x)\). So the stratification \(\text{Sing}(X)\) is induced locally by the stratification \(\text{Sing}(X_{\sigma})\). The strata of \(\text{Sing}(X)\) are locally the inverse images of strata in \(\text{Sing}(X_{\sigma})\).
7.6.5. Canonical stratification on locally toric varieties with divisors.

**Definition 7.6.6.** Let $X$ be locally toric variety. A divisor $D$ on $X$ will be called a locally toric divisor if there exists locally a chart $U \to X_\sigma$, where $D \cap U$ is the inverse image of a toric divisor on $X_\sigma$.

The above extends to the relative case. Let $X$ be a locally toric variety, and $D$ be a locally toric divisor on $X$. For any closed point $x \in X$ set as before (see Section 7.5.7)

$$\text{sing}_D(x) := (\text{sing}(x), n_D(x)).$$

**Theorem 7.6.7.** Let $X$ be locally toric variety with a locally toric divisor $D$. There is a locally closed stratification $\text{Sing}_D(X)$ defined by the invariant $\text{sing}_D(x)$ such that $(X, \text{Sing}_D(X))$ is a stratified toroidal variety. Moreover, the divisorial stratification $S_D$ is coarser than $\text{Sing}_D(X)$. If $(X, D)$ is a strict toroidal embedding then the divisorial stratification $S_D$ coincides with the stratification $\text{Sing}_D(X)$.

**Proof.** The proof uses Theorem 7.5.8. The rest is identical as for the proof of Theorem 7.6.4.

7.7. Conical semicomplexes.

7.7.1. **Semicones.** Next we generalize the notion of cone. The stratified toroidal varieties are locally described by the charts to affine stratified toric varieties $(X_\sigma, S_\sigma)$. The induces semifan consists of the cone $\sigma$ and some of their faces -those corresponding to the strata in $S_\sigma$. We will call such semifans semicones. In analogy to usual cones, we denote the semicones by small Greek letters $\sigma, \tau$, etc.:

We also shall consider semicones over partially ordered set $S$, i.e., a semifan $\sigma$ over $S$ consisting of some faces of $\sigma$ with an injective map of partially ordered sets $\sigma \to S$.

Again by the previous Remark 7.6.2, the semicones arising from the charts are naturally defined over the stratification $S$.

**Definition 7.7.2.** [Wlo03, Definition 4.3] Let $N$ be a lattice in the vector space $N^\mathbb{Q}$. A **semicone** in $N^\mathbb{Q}$ is a semifan $\sigma$ in $N^\mathbb{Q}$ such that the support $|\sigma|$ of $\sigma$ occurs as an element of $\sigma$.

This implies that the semifan $\sigma$ consists of the cone $|\sigma|$ and some of their faces. It defines an embedded semifan $(\overline{\sigma}, \sigma)$, where $\overline{\sigma}$ is the the fan consisting of the faces of the cone $|\sigma|$. Moreover, one can write the semicone $\sigma$ as the collection of the faces

$$\sigma = \{|\tau| \mid \tau \leq \sigma\}.$$

The cone $|\sigma|$ corresponds to the minimal stratum $\text{strat}(|\sigma|) = O_{|\sigma|}$. By abuse of notation we will use the notation $O_\sigma = \text{strat}(\sigma)$ in the future.

The **dimension** of a semicone is the dimension of its support. Moreover, for an injection $i: N^\mathbb{Q} \to (N')^\mathbb{Q}$ of vector spaces, the **image** $i(\sigma)$ of a semicone $\sigma$ in $N^\mathbb{Q}$ is the semicone consisting of the images of all the elements of $\sigma$.

Note that every cone becomes a semicone by replacing it with the set of all its faces. Moreover, every semifan is a union of maximal semicones. Generalizing this observation, we build up in the next sections the **semicomplexes** associated with stratified toroidal varieties from semicones.
We will rephrase the Demushkin Theorem 7.3.8 in this language.

**Corollary 7.7.3.** Let $S$ be a partially ordered set. Let $\sigma_1$ and $\sigma_2$ be two semicones over $S$ with isomorphic lattices $N_1 \simeq N_2$. Let $S_{\sigma_1}$ and $S_{\sigma_2}$ denote the equisingular stratifications on $X_{\sigma_1}$ and $X_{\sigma_2}$ with injective maps $S_{\sigma_1} \to S$, and $S_{\sigma_2} \to S$. For $i = 1, 2$ denote by $(\hat{X}_{\sigma_i}^S) = \text{Spec}(\mathcal{O}(\hat{X}_{\sigma_i,x_i}^S))$, where $x_i \in \hat{O}_{\sigma_i}$, is a closed point.

Then the following conditions are equivalent:

1. The semicones $(\sigma_1, N_{\sigma_1})$ and $(\sigma_2, N_{\sigma_2})$ are isomorphic over $S$.
2. $(\hat{X}_{\sigma_1,x_1}^S, \hat{S}_{\sigma_1,x_1}^S)$ are isomorphic over $(\hat{X}_{\sigma_2,x_2}^S, \hat{S}_{\sigma_2,x_2}^S)$ over $S$.

**Proof.** The theorem is a rephrasing of the Theorem 7.3.8. Let $\delta_1 = |\sigma_1| \times \tau_1$, and $\delta_2 = |\sigma_2| \times \tau_2$ be two cones of maximal dimension, in $N_1^Q$, where $\tau_i$ are regular.

Then $(\hat{X}_{\sigma_1,x_1}^S, \hat{S}_{\sigma_1,x_1}^S) \simeq (\hat{X}_{\sigma_2,x_2}^S, \hat{S}_{\sigma_2,x_2}^S)$, and their isomorphism implies the isomorphism of the embedded semifans $(\hat{S}_1, \sigma_1) \to (\hat{S}_2, \sigma_2)$, which in turn defines the isomorphism of the semicones $(\sigma_1, N_{\sigma_1}) \to (\sigma_2, N_{\sigma_2})$.

---

**7.7.4. Conical semicomplexes.** Let $\sigma$ be as semicone over $S$. Denote by $\text{Aut}(\sigma)$ the group of automorphism of the semicone $\sigma = (\sigma, N_{\sigma})$ (see Definition 7.3.1), and let $\text{Aut}_S(\sigma)$ the groups of automorphisms of the semicone $\sigma$ over $S$. In particular, we have $\text{Aut}(\sigma) = \text{Aut}_S(\sigma)$ is the group of automorphisms $\sigma \to \sigma$ defined over $S = \sigma$.

**Definition 7.7.5.** [Wlo03, Definition 4.3.1]

Let $\Sigma$ be a finite collection of semicones $\sigma$ in $N_{\sigma}^Q \supset N_{\sigma}$ with $\dim(\sigma) = \dim(N_{\sigma})$. Moreover, suppose that there is a partial ordering “$\leq$” on $\Sigma$. We associate with each $\sigma \in \Sigma$ the group of automorphisms $\text{Aut}(\sigma)$.

We call $\Sigma$ a **semicomplex** if for any pair $\tau \leq \sigma$ in $\Sigma$ there is an associated linear injection $i^\sigma_\tau : N_{\sigma}^Q \to N_{\sigma}^Q$ such that $i^\sigma_\tau(\tau) \subseteq \sigma$. In particular, $i^\sigma(|\tau|) \in \sigma$ is a face of the cone $|\sigma|$, $i^\sigma(|\tau|) \subseteq N_{\sigma}$ is a saturated sublattice and

1. $i^\sigma_\rho \circ i^\sigma_\varrho = i^\sigma_\varrho \alpha_\rho$, where $\alpha_\rho \in \text{Aut}(\rho)$.
2. $i^\sigma_\tau(|\rho|) = i^\sigma_\tau(|\tau|)$ implies $\varrho = \tau$.
3. $\sigma = \bigcup_{\tau \leq \sigma} i^\sigma_\tau(\tau) = \{i^\sigma_\tau(|\tau|) \mid \tau \leq \sigma\}$.

**Remark 7.7.6.** [Wlo03] The condition (1) is equivalent to the following condition:

$$i^\sigma_\rho \circ i^\sigma_\varrho(\varrho) = i^\sigma_\varrho(\varrho).$$

Indeed there is an induced isomorphism $\overline{i}^\sigma_\rho : \rho \to i^\sigma_\rho(\varrho)$, with the inverse $(\overline{i}^\sigma_\rho)^{-1} : i^\sigma_\rho(\varrho) \to \rho$ which defines the automorphism $\alpha_\rho := (\overline{i}^\sigma_\rho)^{-1}(i^\sigma_\rho \circ i^\sigma_\rho)$.

The gluing of the faces and the subdivisions of the semicomplex are defined up to the automorphisms in $\text{Aut}(\sigma)$.

**Definition 7.7.7.** We say that the semicomplex $\Sigma$ is defined over a partially ordered set $S$ if there is an injective map $\Sigma \to S$ respecting the order.

**Remark 7.7.8.**

1. Any semicomplex $\Sigma$ can be considered as a semicomplex over itself with the identical map $\text{Id} : \Sigma \to \Sigma$.
2. If a semicomplex $\Sigma$ is defined over $S$ then the semicones $\sigma \in \Sigma$ are defined over $S$ with natural injective map map $i_\sigma : \sigma \to \Sigma \to S$. 

(3) As we see later the semicomplexes associated with toroidal stratification $S$ are naturally defined over $S$.

**Definition 7.7.9.** [Wlo03] By an isomorphism of the semicomplexes $\Sigma \to \Sigma'$ over $S$ we mean a bijection of the sets $\Sigma \to \Sigma'$ commuting with the maps to $S$ and defining the isomorphisms of the corresponding semicones $\phi_{\sigma,\sigma'} : \sigma \to \sigma'$ over $S$.

**Lemma 7.7.10.** If $\phi : \Sigma \to \Sigma'$ is an isomorphism over $S$ then for any $\tau \leq \sigma$ there is a unique $\tau' \leq \sigma'$. Moreover, $\phi_{\sigma,\sigma'}i^\tau = i^\tau,\phi_{\tau,\tau'}\alpha_\tau$, for some $\alpha_\tau \in \text{Aut}(\tau)$.

**Proof.** Note that the semicones $\phi_{\sigma,\sigma'}i^\tau(\tau)$ and $i^\tau,\phi_{\tau,\tau'}(\tau)$ are equal as both are equal to the subset $i^\tau,\tau'$ of $\sigma'$ corresponding to the same subset of $S$. So they define the same semifan and two different isomorphisms $\tau \to i^\tau,\tau' \to \sigma'$ over $S$ which are different by the automorphism of $\tau$ over $S$ so an element $\alpha_\tau \in \text{Aut}(\tau)$.

\[\heartsuit\]

As a special case of the above notion, we recover the notion of a (conical) complex introduced by Kempf, Knudsen, Mumford and Saint-Donat in [KKMSD73]:

**Definition 7.7.11.** [Wlo03, Definition 4.3.2] A conical complex is a semicomplex $\Sigma$ such that every $\sigma \in \Sigma$ is a fan consisting of all the faces of its support cone $|\sigma|$.  

**Remark 7.7.12.** If $\Sigma$ is a complex, then the the semicones can be identified with the cones without loss of information. The groups $\text{Aut}(\sigma)$ are trivial since the automorphisms preserve the vertices of $\sigma$ and are identical on each cone $|\sigma|$. In such a case, we obtain the definition of the conical complex. (Definition 3.5.1).

### 7.8. Associated semicomplexes.

**Definition 7.8.1.** [Wlo03, Definition 4.8.1] Let $(X,S)$ be a stratified toroidal variety. We say that a semicomplex $\Sigma$ over $S$ is associated with $(X,S)$ if there is a bijection $i : \Sigma \to S$ of ordered sets with the following properties:

- Let $\sigma \in \Sigma$ map to $s = \text{strat}_X(\sigma) \in S$. Then any $x \in s$ admits an open neighborhood $U_\sigma \subset X$ and $\varphi_\sigma : U_\sigma \to X_\sigma$ of stratified varieties such that $s \cap U_\sigma$ equals $\varphi_\sigma^{-1}(O_\sigma) = \varphi_\sigma^{-1}(\text{strat}(\sigma))$ and the intersections $s' \cap U_\sigma, s' \in S$, are precisely the inverse images of the relevant strata of $X_\sigma$ (defined by the bijection $i$):

$$s' \cap U_\sigma = \varphi_\sigma^{-1}(\text{strat}(\tau)),$$

where $\tau \in \sigma$ corresponds to $s' = i(\tau)$. Recall that $O_\sigma \subset X_\sigma$ is a stratum corresponding to the maximal cone $|\sigma| \in \sigma$. (See the formula after Definition 7.7.2)

We call the smooth morphisms $U_\sigma \to X_\sigma$ from the above definition charts. A collection of charts satisfying the conditions from the above definition is called an atlas.

**Lemma 7.8.2.** [Wlo03, Lemma 4.8.2] For any stratified toroidal variety $(X,S)$ there exists a unique (up to an isomorphism over $S$) associated semicomplex $\Sigma$ over $S$. Moreover, $(X,S)$ is a toroidal embedding iff $\Sigma$ is a complex.

**Proof.** For any stratum $s$ consider the associated local étale chart $U \to X_\sigma$ which defines the corresponding semicone $\sigma = \sigma_s$ over $S$, i.e. semifan over $S$ consisting of the faces of $\sigma_s$ corresponding to the strata in $\text{Star}(s,S)$. This correspondence defines a natural map $\sigma \to S$ between faces $\tau$ in $\sigma$ the strata $s, \tau \in S$. 

The semicone is defined uniquely up to an isomorphism over $S$. If one chart associates to a stratum a semicone $\sigma_s^1$ over $S$, and another associates $\sigma_s^2$ over $S$, then by Demushkin’s theorem (7.7.3) $\sigma_s^1$ and $\sigma_s^2$ are isomorphic over $S$. This gives us the correspondence between strata and semicones and defines a bijection $\Sigma \to S$.

Let $s \leq s'$. Consider the semicones $\sigma_s$ and $\sigma_{s'}$ defined by the charts. Then there is a face $\tau_s$ of $\sigma_{s'}$ corresponding to the stratum $s$ with a face inclusion $i : \tau_s \to \sigma_{s'}$ over $S$. Composing it with Demushkin’s isomorphism $\alpha_s : \sigma_s \to \tau_s$ over $S$ produces a face inclusion $i_{ss'} : \sigma_s \to \sigma_{s'}$ over $S$.

The conditions (1) (2) (3) are satisfied. They exactly mean that the face inclusions are defined over $S$. (See also the equivalent conditions in Remark 7.7.8(2).

By construction and definition the semicomplex $\Sigma$ is determined uniquely up to isomorphism over $S$.

For any two such semicomplexes $\Sigma$ and $\Sigma'$ defined over $S$ there is a natural bijection $\Sigma \to \Sigma'$. Moreover, the corresponding cones $\sigma_s$ and $\sigma_{s'}$ are defined over $S$ and thus related by the Demushkin isomorphism over $S$.

If a stratified toroidal variety is a toroidal embedding then a semicone $\sigma \in \Sigma$ consists all the faces of the cone $|\sigma|$. One can replace semicones with cones without loss of data. The associated semicomplex is equivalent to a usual conical complex in that case.

Conversely, if the associated semicomplex of the stratified toroidal variety is a complex, then locally, the strata are defined by toric orbits, so are induced by the intersecting of the divisors (codimension one strata) defining the structure of a toroidal embedding.

**Example 7.8.3.** Let $(X, S)$ be a stratified toric variety associated with the embedded semifan $(\Sigma, \Omega)$ over $S = S_{\Omega}$. Then the associated semicomplex is given by

$$\Omega_{red} := \{(\omega, N^Q_\omega) \mid \omega \in \Omega\}$$

with the natural face inclusions. It is a semicomplex defined over the stratification $S$.

### 7.9. Relative semicomplexes and locally toric varieties with a divisor.

One can extend the definition of saturated subsets and toroidal divisors on strict toroidal embeddings as in Definition 5.4.1 to the case of the stratified toroidal varieties.

**Definition 7.9.1.** Let $(X, S)$ be a stratified toroidal variety. We say that a subset $V$ of $X$ is *saturated* in $(X, S)$ if it is the union of strata of $S$. A divisor $D$ on $X$ will be called *toroidal* if it saturated.

The strata whose closures are defined by the intersecting components of $D$ will be called *divisorial strata*.

**Lemma 7.9.2.** Let $X$ be a locally toric variety, and $D$ be a locally toric divisor. Then $D$ is a toroidal divisor on the induced stratified toroidal variety $(X, \text{Sing}_D(X))$.

**Proof.** It is a rephrasing of the statement in Theorem 7.6.7 that $S_D$ is coarser than $\text{Sing}_D(X)$. ♣

We extend Definition 5.4.3, and Lemma 5.4.6, to the situation of stratified toroidal varieties.
Definition 7.9.3. By a relative semicomplex we mean a pair \((\Sigma, \Omega)\), of a semicomplex \(\Sigma\) and a subcomplex \(\Omega \subset \Sigma\). A complex \(\Omega \subset \Sigma\) will be called saturated if any cone of \(\sigma \in \Sigma\) with vertices (one dimensional faces) in \(\Omega\) is in \(\Omega\).

We extend Theorem 7.6.7:

**Theorem 7.9.4.** Let \(X\) be locally toric variety with a locally toric divisor \(D\). Let \(V := (X, D)_{\text{tor}}\) be its toroidal locus. Consider a locally closed toroidal stratification \(\text{Sing}_D(X)\), and let \(\Sigma\) be the associated semicomplex.

Then

1. \(\text{Sing}_D(X)_{|V}\) is the divisorial stratification associated with the strict toroidal embedding \((V, D \cap V)\). It corresponds to a subcomplex \(\Omega\) of \(\Sigma\).
2. \(\Omega\) is a maximal subcomplex of \(\Sigma\). In particular, it is saturated in \(\Sigma\).
3. The toroidal locus of \((X, D)\) is described as 
   \[V = (X, D)_{\text{tor}} = X(\Omega) := \bigcup_{\omega \in \Omega} \text{strat}(\omega),\]
4. The vertices of \(\Omega\) correspond to the irreducible components of \(D\).
5. \(D\) is a toroidal divisor on \((X, \text{Sing}_D(X))\).
6. \(\text{codim}(X \setminus V) \geq 2\)
7. \(D = D_{X(\Omega)}\).

**Proof.** (1) By Theorem 7.6.7, the divisorial stratification on \(V\) and the stratification \(\text{Sing}_D(X)_{|V}\) defined by the singularity type coincide on a toroidal embedding \((V, D \cap V)\). They both correspond to a subcomplex \(\Omega\) of \(\Sigma\).

(2) Let \(\Omega' \subset \Sigma\) be a subcomplex. Then, by the definition, \(X(\Omega')\) is the union of locally closed strata which is closed under generization, so open. The stratification \(\text{Sing}_D(X)\) on \(X(\Omega')\) corresponds to a complex \(\Omega'\). So locally it corresponds to a toric variety \(X_\sigma \supset T\) with the orbit stratification corresponding to a toric divisor \(D_\sigma = X_\sigma \setminus T\), and thus it is toroidal. Consequently \(X(\Omega') \subseteq V\), and \(\Omega' \subseteq \Omega\).

(3) Follows from (2).

(4) Any irreducible component of \(D\) defines a toroidal embedding around its generic point. So it intersects \(V\), and defines a vertex of \(\Omega\). On the other hand, any vertex of \(\Omega\) corresponds to an irreducible component of \(D_{|V}\).

(5) Follows from the fact that \(S_D(X)\) is coarser than \(\text{Sing}_D(X)\).

(6) Follows from (4).

(7) follows from (6). 

7.10. **Inverse systems of affine algebraic groups.** We shall give the groups of the automorphisms of the completions of the local rings the structure of proalgebraic groups. All the proalgebraic groups here are considered over an algebraically closed field \(K = \overline{K}\).

**Definition 7.10.1.** [Wlo03, Definition 4.4.1] By an affine proalgebraic group we mean an affine group scheme that is the limit of an inverse system \((G_i)_{i \in \mathbb{N}}\) of affine algebraic groups and algebraic group homomorphisms \(\varphi_{ij} : G_i \to G_j\), for \(i \geq j\).

**Lemma 7.10.2.** [Wlo03, Lemma 4.4.2] Consider the natural morphism \(\varphi_i : G \to G_i\). Then \(H_i := \varphi_i(G)\) is an algebraic subgroup of \(G_i\), all induced morphisms
$H_j \to H_i$ for $i \leq j$ are epimorphisms and $G = \lim_{\leftarrow} G_i = \lim_{\leftarrow} H_i$. In particular $K[H_i] \subset K[H_{i+1}]$ and $K[G] = \bigcup K[H_i]$.

Lemma 7.10.3. [Wlo03, Lemma 4.4.3] The set $G^K$ of $K$-rational points of $G$ is an abstract group which is the inverse limit $G^K = \lim_{\leftarrow} G_i^K$ in the category of abstract groups.

By abuse of notation, we shall identify $G$ with $G^K$.

Example 7.10.4. [Wlo03, Example 4.5.4] Let $\varphi_n : \text{Aut}(\hat{X}, S) \to \text{Aut}(X^{(n)}_x, S)$ denote the natural morphisms. For $n = 1$ we get the differential mapping:

$$d = \varphi_1 : \text{Aut}(\hat{X}, S) \to \text{Aut}(X^{(1)}_x, S) \subset \text{GL(Tan}_{X,x}).$$

Definition 7.10.5. [Wlo03, Definition 4.9.1] We shall call a proalgebraic group $G$ connected if it is a connected affine scheme. For any proalgebraic group $G = \lim_{\leftarrow} G_i$, denote by $G^0 = \lim_{\leftarrow} G^0_i$.

7.11. Oriented semicomplexes. Let $\sigma$ be a semicone in $N^\sigma_Q$. Denote by $\text{Aut}(\sigma)$ the group of automorphisms of the semicone $\sigma$. Consider the natural inclusion $\phi : \text{Aut}(\hat{X}, S) \to \text{Aut}(X^{(1)}_x, S) \subset \text{GL(Tan}_{X,x})$.

Definition 7.11.1. [Wlo03, Definition 4.11.1] By an oriented semicomplex we mean a semicomplex $\Sigma$ together with the associated groups $\text{Aut}(\sigma)^0$ such that for any $\sigma \leq \tau \leq \gamma$ there is $\alpha_\sigma \in \text{Aut}(\sigma)^0$ for which $\iota_\gamma \iota_\tau = \iota_\sigma \alpha_\sigma$.

We will not use the notion of oriented semicomplexes in this paper. It was introduced for [Wlo03, Definition 4.10.1] to apply certain nonfunctorial algorithms. Since our algorithm of the combinatorial desingularization in Section is functorial for arbitrary automorphisms of semicomplexes our reasoning will not require this notion. However we will use the subgroups $\text{Aut}(\sigma)^0$.

7.12. Subdivisions of semicomplexes. If $\Sigma$ is a fan and $\sigma$ be its face. Then, by definition, for any subdivision $\Delta$ of $\Sigma$ and restriction

$$\Delta|\sigma := \{\delta \in \Delta \mid \delta \subset \sigma\}$$

of $\Delta$ to the cone $\sigma$ is a subdivision of the fan $\overrightarrow{\sigma}$.

We shall use this notation in the context of semicomplexes. Recall that we can associate with any semicone $\sigma$ a fan $\overrightarrow{\sigma}$ consisting of all faces of the cone $|\sigma|$.

Definition 7.12.1. [Wlo03, Definition 4.11.4] A subdivision of a semicomplex (respectively an oriented semicomplex) $\Sigma$ is a collection $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ of fans $\Delta^\sigma$ in $N^\sigma_Q$ where $\sigma \in \Sigma$ such that

1. For any $\sigma \in \Sigma$, $\Delta^\sigma$ is a subdivision of the fan $\overrightarrow{\sigma}$ which is $\text{Aut}(\sigma)$-invariant (resp. $\text{Aut}(\sigma)^0$-invariant).
2. For any $\tau \leq \sigma$, $\Delta^\sigma|\tau = \iota_\sigma^\tau(\Delta^\tau)$.

Remark 7.12.2. (1) By abuse of terminology, we shall understand by a subdivision of a semicone $\sigma$ simply a subdivision of the relevant fan $\overrightarrow{\sigma}$.
(2) By definition, the vectors in faces $\sigma$ of a semicomplex (respectively an oriented semicomplex $\Sigma$) are defined up to automorphisms from $\text{Aut}(\sigma)$ (respectively from $\text{Aut}(\sigma)^0$). Consequently, the faces of subdivisions $\Delta^\sigma$ do not have any geometric meaning as they are defined up to automorphisms from $\text{Aut}(\sigma)^0$. However, subdivisions $\Delta^\sigma$ describe the birational modifications locally.

(3) The condition that $\Delta^\sigma$ are $\text{Aut}(\sigma)$-invariant subdivisions is replaced for canonical subdivisions with a somewhat stronger condition of similar nature which says that the induced morphism $\tilde{X}_{\Delta^\sigma} := X_{\Delta^\sigma} \times_{X_{\sigma}} \tilde{X}_{\sigma} \to \tilde{X}_{\sigma}$ is $\text{Aut}(\tilde{X}_{\sigma})$-equivariant.

7.13. Toroidal modifications. We will ignore the notion of orientation in the sequel, which makes the reasoning easier for most of the part.

**Definition 7.13.1.** [Wlo03, Definition 4.12.1] Let $(X, S)$ be a stratified toroidal variety. We say that $Y$ is a toroidal modification of $(X, S)$ if

1. There is given a proper morphism $f : Y \to X$ such that for any $x \in s = \text{strat}_X(\sigma)$ there exists a chart $x \in U_\sigma \to X_\sigma$, a subdivision $\Delta^\sigma$ of $\sigma$, and a fiber square

$$
\begin{array}{ccc}
U_\sigma & \xrightarrow{\varphi_\sigma} & X_\sigma \\
\uparrow f & & \uparrow \\
U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} & \simeq & f^{-1}(U_\sigma) \xrightarrow{\tilde{\varphi}} X_{\Delta^\sigma}
\end{array}
$$

2. (Hironaka’s condition) For any geometric point $\mathfrak{m} : \text{Spec}(K) \to s \subset X$ in a stratum $s$, every automorphism $\alpha$ of $\tilde{X}_\mathfrak{m}$ preserving strata can be lifted to an automorphism $\alpha'$ of $Y \times_X \tilde{X}_\mathfrak{m}$. Here

$$\tilde{X}_\mathfrak{m} = (X \times_{\text{Spec}(K)} (\text{Spec}(K)))_\mathfrak{m}.$$  

7.14. Canonical subdivisions of semicomplexes. We will rewrite the Hironaka condition in the above definition in a more convenient form

**Lemma 7.14.1.** [Wlo03, Lemma 4.13.1] Let $(X, S)$ be a stratified toroidal variety of dimension $n$ with associated semicomplex $\Sigma$ and let $f : Y \to (X, S)$ be a toroidal modification. Let $\mathfrak{m} : \text{Spec}(K) \to \{x\} \subset X$ be a geometric point in the stratum $\text{strat}_X(\sigma) \in S$, $\varphi_\sigma : U \to X_\sigma$ a chart of a neighborhood $U$ of $x$, and $\Delta^\sigma$ a subdivision of $\sigma$ for which there is a fiber square

$$
\begin{array}{ccc}
U & \xrightarrow{\varphi_\sigma} & X_\sigma \\
\uparrow f & & \uparrow \\
f^{-1}(U) & \xrightarrow{\tilde{\varphi}} & X_{\Delta^\sigma}
\end{array}
$$

where the horizontal morphisms are smooth. Set

$$\text{reg}(\sigma) := \langle e_1, \ldots, e_{n - \dim(N_\sigma)} \rangle = \langle e_1, \ldots, e_{\dim(\text{strat}_X(\sigma))} \rangle.$$  

Then
(1) there is a fiber square of étale extensions

\[
\begin{array}{ccc}
U & \xrightarrow{\tilde{\phi}_\sigma} & X_{\tilde{\sigma}} \\
\uparrow f & & \uparrow \\
f^{-1}(U) & \xrightarrow{\tilde{\phi}_\Delta} & X_{\Delta^\sigma}
\end{array}
\]

where the horizontal morphisms are étale and where

\[\tilde{\sigma} := \sigma \times \text{reg}(\sigma) \quad \text{and} \quad \tilde{\Delta}^\sigma := \Delta^\sigma \times \text{reg}(\sigma).\]

(2) \(X_{\tilde{\sigma}}\) is a stratified toric variety with the strata described by the embedded semifan \(\sigma \subset \tilde{\sigma}\). Moreover, the strata on \(U\) are exactly the inverse images of strata of \(X_{\tilde{\sigma}}\).

(3) There is a fiber square of isomorphisms

\[
\begin{array}{ccc}
\hat{X}_K & \xrightarrow{\hat{\phi}_\sigma} & \tilde{X}_\sigma \\
\uparrow \hat{f} & & \uparrow \\
Y \times_X \hat{X}_K & \xrightarrow{\hat{\phi}_\Delta} & \tilde{X}_{\Delta^\sigma}
\end{array}
\]

where

\[
\tilde{X}_\sigma := \hat{X}_K^\sigma \quad \text{and} \quad \tilde{X}_{\Delta^\sigma} := X_{\Delta^\sigma} \times_{X_\sigma} \tilde{X}_\sigma = X_{\Delta^\sigma}^K \times_{X_\sigma^K} \tilde{X}_\sigma.
\]

(4) \(\tilde{X}_\sigma\) is a stratified toroidal scheme with the strata described by the embedded semifan \(\sigma \subset \sigma\). The isomorphism \(\hat{\phi}_\sigma\) preserves strata.

(5) The morphism \(\hat{f}_x : Y \times_X \hat{X}_K^\sigma \to \hat{X}_K^\sigma\) is \(\text{Aut}(\hat{X}_K^\sigma, S)\)-equivariant.

(6) The morphism \(\tilde{X}_{\Delta^\sigma} \to \tilde{X}_\sigma\) is \(\text{Aut}(\tilde{X}_\sigma)\)-equivariant.

\[\blacksquare\]

Proof. The Lemma is a reinterpretation of the Hironaka condition and follows directly from Definition 7.13.1. For more details see [Wlo03].

\[\blacksquare\]

We shall assign to the faces of an semicomplex \(\Sigma\) the collection of connected proalgebraic groups over \(K\):

\[G_\sigma := \text{Aut}(\tilde{X}_\sigma), \quad G_\sigma^0 := \text{Aut}(\tilde{X}_\sigma)^0.\]

Definition 7.14.2. [Wlo03, Definition 4.13.3] A subdivision \(\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}\) of a semicomplex \(\Sigma\) is called canonical if for any \(\sigma \in \Sigma\), \(G_\sigma\) acts on \(\tilde{X}_{\Delta^\sigma}\) (as an abstract group) and the morphism \(\tilde{X}_{\Delta^\sigma} \to \tilde{X}_\sigma\) is \(G_\sigma\)-equivariant.

Lemma 7.14.3. ([Dem82], [Wlo03, Lemma 7.3.2]) Let \(\sigma\) be a semicone. Then \(\text{Aut}(\tilde{X}_\sigma)^0 \subset \text{Aut}(\tilde{X}_\sigma)\) is a normal subgroup and there is a natural surjection \(\text{Aut}(\sigma) \to \text{Aut}(\tilde{X}_\sigma)/\text{Aut}(\tilde{X}_\sigma)^0\).

Theorem 7.14.4. ([Wlo03, Theorem 4.14.1]) Let \((X, S)\) be stratified toroidal variety with the associated semicomplex \(\Sigma\). There exists a bijective correspondence between the toroidal modifications \(Y\) of \((X, S)\) and the canonical subdivisions \(\Delta\) of \(\Sigma\).
(1) If $\Delta$ is a canonical subdivision of $\Sigma$ then the toroidal modification associated with it is defined locally by

$$U_\sigma \rightarrow X_\sigma$$

$$U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} \simeq f^{-1}(U_\sigma) \rightarrow X_{\Delta^\sigma}$$

(2) If $Y^1 \rightarrow X$, $Y^2 \rightarrow X$ are toroidal modifications associated with canonical subdivisions $\Delta_1$ and $\Delta_2$ of $\Sigma$ then the natural birational map $Y^1 \rightarrow Y^2$ is a morphism iff $\Delta_1$ is a subdivision of $\Delta_2$.

Proof. Sketch of the proof. Let $\Delta$ be a canonical subdivision of $\Sigma$. The variety $Y$ is obtained by gluing the pieces $V = U \times_{X_\sigma} X_{\Delta^\sigma}$, defined by the toric charts $\phi : U \rightarrow X_\sigma$ which are birational to $U \subset X$. We need to show that the gluing is independent of the charts. Suppose that there are two different charts $\phi_1, \phi_2$ inducing birational varieties $V_1$ and $V_2$ over $X$ which do not glue over a certain point $x \in s$, where $s$ is a stratum corresponding to the cone $\sigma$. If we consider the graph $V$ of $V_1 \rightarrow V_2$, then at least one of the birational morphisms $V \rightarrow V_1$ or $V \rightarrow V_2$ is not an isomorphism over $x$, so contracts a curve.

Observe that by shrinking and restricting charts we can reduce the situation to two smooth charts of the form $\phi_1, \phi_2 : U \rightarrow X_\sigma$. Then we can further assume that the charts are étale replacing $\phi_i$ with $\tilde{\phi}_i : U \rightarrow X_\sigma = \tilde{X}_\sigma$ with both charts taking $x$ to the closed orbit $O_\sigma$.

Passing to the algebraic closure $\overline{K}$, and to the local rings at a geometric point $\overline{x}$ over $x$ we see that the spaces $\tilde{V}^i_{\overline{K}} := U^i_{\overline{K}} \times_{X^i_{\overline{K}}} X^i_{\Delta^\sigma}$ does not glue over $\overline{x}$. In other words, the induced spaces $\tilde{V}^i_{\overline{K}} := \tilde{X}^i_{\overline{K}} \times_{\tilde{X}^i_{\overline{K}}} \tilde{X}^i_{\Delta^\sigma}$ are not isomorphic, for two different induced isomorphisms $\tilde{\phi}_{\overline{x}} : \tilde{X}^i_{\overline{K}} \rightarrow \tilde{X}_\sigma$ since the map $\tilde{V}^i_{\overline{K}} \rightarrow \tilde{V}^j_{\overline{K}}$ is induced from the map between $V_i \times_X \text{Spec}(O_{X,x})$ by the faithful flat change of base $\text{Spec}((O_X)_{\overline{x}}) \rightarrow \text{Spec}(O_{X,x})$.

On the other hand, by the assumption, the isomorphisms $\tilde{\phi}_{\overline{x}}$ differ by an automorphism in $\text{Aut}(\tilde{X}_\sigma)$, and this automorphism lifts to an automorphism of $\tilde{X}_{\Delta^\sigma}$ inducing the isomorphism between $\tilde{V}^i_{\overline{K}} = U^i_{\overline{K}} \times_{X^i_{\overline{K}}} X^i_{\Delta^\sigma}$, which is a contradiction.

The converse follows from the Definition and Lemma 7.14.1. By definition, for any $\sigma$ there is a subdivision $\Delta^\sigma$ of $\sigma$ such that $\tilde{X}_{\Delta^\sigma} \rightarrow \tilde{X}_\sigma$ is $G_\sigma$-equivariant.

One needs to show that the subdivisions $\Delta^\sigma$ of $\sigma$ are defined uniquely. To this end we use diagram (3) from Lemma 7.14.1. The isomorphisms $\tilde{\phi}_{\overline{x}}$ differ by element of $G_\sigma$. On the other hand, the induced morphisms $\tilde{X}_{\Delta^\sigma} \rightarrow \tilde{X}_\sigma$ are $G_\sigma$-equivariant. Thus we conclude that there is an isomorphism $\tilde{X}_{\Delta^\sigma} \rightarrow \tilde{X}_\delta$ over $\tilde{X}_\sigma$. It is a torus equivariant and takes affine torus subschemes $\tilde{X}_\delta \subset \tilde{X}_{\Delta^\sigma}$ to $\tilde{X}_{\delta'} \subset \tilde{X}_{\Delta^\sigma'}$, and defines an isomorphism of the cones of semisimply functions $(\tilde{\delta})^{\text{integ}} \rightarrow ((\tilde{\delta'})^{\text{integ}}$ and $\delta \simeq \delta'$.

(2) This part follows from the analogous properties of toric varieties. We can assume that both morphisms are locally described in the same étale chart by the toric morphisms. For the details, see [Wlo03].

♣
7.14.5. Canonical birational modifications of strict toroidal embeddings. The theorem below shows that the canonical birational modifications of strict toroidal embeddings can be considered as a particular case of the canonical morphisms of stratified toroidal varieties (see also Theorem 3.11.2). In particular, we have

**Lemma 7.14.6.** [Wlo03, Lemma 6.3.1(2)]. Let \((X, S)\) be a strict toroidal embedding with the associated complex \(\Sigma\). Then for any subdivision \(\Delta^\sigma\) of \(\sigma\) the induced morphism \(\tilde{X}_{\Delta^\sigma} \to \tilde{X}_\sigma\) is \(G_\sigma\)-equivariant. Consequently, all subdivisions of \(\Sigma\) are canonical.

**Proof.** Let \(\delta \in \Delta^\sigma\). Since \(X_\sigma\) is a toroidal embedding (and \(\tilde{X}_\sigma\) is the completion of its local ring) each automorphism \(g\) from \(G_\sigma\) preserves the toric irreducible divisors on \(\tilde{X}_\sigma\) defined by the rays of \(\sigma\), hence it multiplies the generating monomials in \((\sigma^\vee)^{\text{integ}}\) by invertible functions. This implies that the action \(g\) on \(\tilde{X}_\sigma\) lifts to an automorphism \(g'\) of \(\tilde{X}_\Delta = \tilde{X}_\sigma \times \tilde{X}_{\tilde{\delta}}\) which also multiplies monomials by suitable invertible functions. Therefore \(g\) lifts to \(\tilde{X}_\delta\) and to to the scheme \(\tilde{X}_{\Delta^\sigma} = \bigcup_{\delta \in \Delta^\sigma} \tilde{X}_\delta\).

7.15. Use of minimal vectors. The following observations are critical. They allow to run certain desingularization combinatorial algorithms on stratified toroidal varieties.

By abuse of terminology a vector \(v \in \sigma\) will be called \(G_\sigma\)-invariant (respectively \(G_\sigma^0\)-invariant) if the corresponding valuation \(\text{val}(v)\) on \(\tilde{X}_\sigma\) is \(G_\sigma^0\)-invariant (respectively \(G_\sigma^0\)-invariant).

There are not too many \(G_\sigma\)-invariant vectors, but quite a few \(G_\sigma^0\)-invariant ones.

**Lemma 7.15.1.** [Wlo03, Lemma 5.3.15] Let \(\sigma\) be a semicone and \(\tilde{X}_{\Delta^\sigma} \to \tilde{X}_\sigma\) be a \(G_\sigma^0\)-equivariant birational morphism induced by a toric morphism \(X_{\Delta^\sigma} \to X_\sigma\) associated with subdivision \(\Delta^\sigma\) of \(\sigma\).

(1) [Wlo03, Lemma 5.3.15(3,4)] Let \(\delta\) be an irreducible face of \(\Delta^\sigma\). Then all minimal internal points of \(\delta\) are \(G_\sigma^0\)-invariant.

(2) [Wlo03, Lemma 5.3.15(5)] If \(v\) is a vector in the ray (one dimensional face) of the semicone \(\sigma\) then \(v\) is \(G_\sigma^0\)-invariant.

(3) [Wlo03, Lemma 6.2.1(1)] The set of the \(G_\sigma^0\)-invariant vectors in \(\sigma\) is convex.

**Corollary 7.15.2.** With the above notation

(1) Let \(\delta \in \Delta^\sigma\) be an irreducible face. Then the sum of the minimal internal vectors of \(\delta\) are \(G_\sigma^0\)-invariant. In particular, the canonical barycenter of \(\delta\) is \(G_\sigma^0\)-invariant.

(2) Let \(\delta \in \Delta^\sigma\). Then the minimal vectors of \(\delta\) are \(G_\sigma^0\)-invariant.

**Proof.** A minimal vector of \(\delta\) is in the relative interior of an irreducible face \(\delta'\) of \(\delta\). It is a minimal internal vector of \(\delta'\) and, hence it is \(G_\sigma^0\)-invariant, by Lemma 7.15.1.

**Corollary 7.15.3.** Let \(\sigma\) be a semicone containing a complex \(\Omega_\sigma \subset \sigma\) and \(\tilde{X}_{\Delta^\sigma} \to \tilde{X}_\sigma\) be a \(G_\sigma^0\)-equivariant birational morphism induced by a toric morphism \(X_{\Delta^\sigma} \to X_\sigma\) associated with subdivision \((\Delta^\sigma, \Omega_\sigma)\) of \((\bar{\Sigma}, \Omega_\sigma)\).

(1) All the integral vectors in \(|\Omega_\sigma|\) are \(G_\sigma^0\)-invariant.
Let $\delta \in \Delta^a$ be a relatively irreducible face of $(\Delta^a, \Omega_\sigma)$. Then the canonical barycenter of $(\delta, \Omega_\delta)$ is $G^0_\sigma$-invariant.

(3) Let $(\delta, \omega) \in (\Delta^a, \Omega_\sigma)$ be a simplicial pair then the the minimal vectors of $(\delta, \omega)$ are $G^0_\sigma$-invariant.

Proof. (1) The vertices of $\Omega_\sigma$ are $G^0_\sigma$-invariant by Lemma 7.15.1(2) so, by Lemma 7.15.3, their nonnegative linear combinations are also $G^0_\sigma$-invariant.

(2) The canonical barycenter is the sum of the minimal internal vectors of sing$(\delta)$ and the vertices in $\Omega_\delta := \Omega \cap \delta$.

(3) The minimal vectors of the pair $(\delta, \omega)$ are the minimal vectors of $\delta$. Hence they are $G^0_\sigma$-invariant by Lemma 7.15.1(3).

Remark 7.15.4. Both corollaries state that the centers used in the desingularization algorithm of the cones (or relative cones) are $G^0_\sigma$-invariant.

7.16. Locally toric valuations. Let $X$ be an algebraic variety and $\nu$ be a valuation of the field $K(X)$ of rational functions. By the valuative criterion of separatedness and properness the valuation ring of $\nu$ dominates the local ring of a uniquely determined point (in general nonclosed) $c_\nu$ on a complete variety $X$. (If $X$ is not complete such a point may not exist). We call the closure of $c_\nu$ the center of the valuation $\nu$ and denote it by $Z(\nu)$ or $Z(\nu, X)$. For any $x \in Z(\nu)$ and $a \in \mathbb{Z}_{\geq 0}$ let

$$I_{\nu, a, x} := \{ f \in O_{X,x} \mid \nu(f) \geq a \}$$

be an ideal in $O_{X,x}$. For a fixed $a$ these ideals define a coherent sheaf of ideals $I_{\nu,a}$ supported at $Z(\nu)$.

The following is a well-known fact from the theory of toric varieties.

Lemma 7.16.1. Let $\Sigma$ be a fan, and $X$ be the associated toric variety. Let $v$ be an integral vector in the support of the fan $\Sigma$. Then the toric valuation $\text{val}(v)$ on $X$ is centered on $O_{\sigma}$, where $\sigma$ is the cone whose relative interior contains $v$.

Proof. If $v \in \text{int}(\sigma)$ then $\text{val}(v)(m) = (v, m)$ > 0 for any $m \in (\sigma^0)^{\text{integ}} \setminus (\sigma^\vee)^{\text{integ}}$. But the set $(\sigma^\vee)^{\text{integ}} \setminus (\sigma^\vee)^{\text{integ}}$ generates the ideal $I_{\sigma}$ of $O_{\sigma}$.

Definition 7.16.2. Let $X$ be a locally toric variety. We say that the valuation $\nu$ is locally toric if for any $x \in X$ there exists a neighborhood $U$ of $x$, and an étale morphism $\phi : U \to X_{\sigma}$ and a vector $v \in \sigma \cap N_{\sigma}$ such that $I_{\nu,a} = \phi^{-1}(I_{\text{val}(v), a})$ for any $a \in \mathbb{N}$.

Smooth morphisms to toric varieties (charts) allow to define valuations locally:

Lemma 7.16.3. [Wlo03] Let $X_{\Sigma}$ be a toric variety (associated with a fan $\Sigma$ and $f : U \to X_{\Sigma}$ be a smooth morphism. Let $v \in \text{int}(\sigma)$, where $\sigma \in \Sigma$, be an integral vector. Assume that the inverse image of $O_{\sigma}$ is irreducible. Then there exists a unique valuation $\mu$ on $U$ such that $I_{\mu,a} = f^{-1}(I_{\text{val}(v), a}) : O_U$.

Proof. We consider the completion $\hat{O}_{\Sigma, X_{\Sigma}}$ of the local ring at a geometric $\overline{K}$-point $\pi$ over a closed point $x \in f^{-1}(O_{\sigma})$. The smooth morphism $f$ defines étale morphism $\overline{f} : U \to X_{\overline{K}} \times A^n_{\overline{K}}$ in a sufficiently small neighborhood $U$. 

We have the induced isomorphism $\hat{f}^*: \hat{O}_{\hat{f}(\mathcal{I}),X}^{\mathbb{P}^1} \to \hat{O}_{\mathcal{I},X}^{\mathbb{P}^1}$. The vector $v$ defines a valuation on $\hat{O}_{\hat{f}(\mathcal{I}),X}^{\mathbb{P}^1}$ and on $\hat{O}_{\mathcal{I},X}^{\mathbb{P}^1}$. Its restriction to $\mathcal{O}_{x,X}$ defines a valuation on $U$. The verification of the condition $\mathcal{I}_{\mu,a} = f^{-1}(\mathcal{I}_{\text{val}(v),a}) \cdot \mathcal{O}_U$ and independence of $f$ and $x$ is straightforward.

7.16.4. Filtered centers. As before one can represent the center of the blow-ups of a locally toric valuation $\nu$ by the filtered sequence of ideals $\mathcal{I}_{\nu,a}$.

Definition 7.16.5. [Wlo03, Definition 5.2.6] (see also Definition 6.3.11) By the blow-up $\text{bl}_{\nu}(X)$ of $X$ at a locally toric valuation $\nu$ we mean the normalization of the blow-up at the filtered center $\{\mathcal{I}_{\nu,k}\}$:

$$\text{Proj}(\mathcal{O} \oplus \mathcal{I}_{\nu,1} \oplus \mathcal{I}_{\nu,2} \oplus \ldots)$$

Proposition 7.16.6. [Wlo03, Proposition 5.2.9] For any locally toric valuation $\nu$ on $X$ there exists an integer $d$ such that

- $\text{bl}_{\nu}(X) = \text{bl}_{\mathcal{I}_{\nu,d}}(X)$.
- If $Y := \text{bl}_{\nu}(X) \to X$ is the blow-up of $\nu$, then the exceptional divisor $D$ is irreducible and $\mathbb{Q}$-Cartier on $X$. Moreover, $\nu = \mathcal{O}_D$ on $Y$.

Proof. The proof is identical as the proof of Lemma 6.3.8. Locally the blow-ups are induced by blow-ups of toric valuations of the form $\text{bl}_{\text{val}(v)}(X)$. By quasi-compactness of $X$ and using Lemma 6.3.3, we can find the same sufficiently divisible $d$ for the open cover with toric charts.

Also we have

Corollary 7.16.7. Let $\nu$ be a locally toric valuation on a locally toric variety $X$, and $\pi: Y := \text{bl}_{\nu}(X) = \text{bl}_{\mathcal{I}_{\nu,d}}(X) \to X$ be the associated normalized blow-up with the exceptional Weil, $\mathbb{Q}$-Cartier divisor $D$.

Then for the ideals $\mathcal{I}_{aD} := \mathcal{I}_{\text{val}(v),a} = \{f \in \mathcal{O}_Y \mid \nu_D(f) \geq a\}$ we have

$$\pi^*(\mathcal{I}_{aD}) = \mathcal{I}_{\nu,a}.$$

Thus the valuation $\nu$ is induced by an irreducible exceptional Weil ($\mathbb{Q}$-Cartier) divisor on the variety $Y = \text{bl}_{\text{val}(v)}(X)$.

Proof. The problem reduces locally to the toric situation, where we use Corollary 6.3.9.

7.17. Desingularization theorems.

Theorem 7.17.1. For any étale locally binomial (or locally toric) variety $X$ over a field $K$ of any characteristic there exists a canonical resolution of singularities i.e. a birational projective $f: Y \to X$ such that

1. $Y$ is smooth over $K$.
2. $f$ is an isomorphism over the open set of the nonsingular points.

27 Definition 7.1.1
(3) The inverse image $f^{-1}(\text{Sing}(X))$ of the singular locus $\text{Sing}(X)$ is a simple normal crossing divisor on $Y$.

(4) $f$ is a composition of the normalization and the normalized blow-ups of the locally monomial filtered centers $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ defined locally by valuations.

(5) $f$ commutes with smooth morphisms and field extensions, in the sense that the centers are transformed functorially, and the trivial blow-ups are omitted.

(6) Moreover, if $D$ is an étale locally toric divisor $D$ on an étale locally toric $X$, then there is functorial desingularization of $X$ as in Lemma 7.15.3.

Proof. First, we prove the theorem in the case of Zariski locally binomial variety $X$. By normalizing $X$ we can further assume that it is locally toric.

Consider the canonical stratification $\text{Sing}(X)$ on $X$ as in Theorem 7.6.4. (In the case of a locally toric divisor $D$ on $X$, as in (5), we consider the stratification $\text{Sing}_D(X)$ as in Theorem 7.6.7, instead of $\text{Sing}(X)$.) Then the pair $(X, \text{Sing}(X))$ is a stratified toroidal variety.

Let $\Sigma$ be the associated conical semicomplex. Consider the complex $\Sigma$ consisting of the disjoint union of the fans $\sigma$ associated with the semicones $\sigma \in \Sigma$.

Let

$$\Sigma := V_0 \cdot V_1 \cdot \ldots \cdot V_i \cdot \ldots$$

be the canonical desingularization of $\Sigma$ as in Lemma 4.6.1. It is obtained by a sequence of the star subdivisions at sets $V_i$ of the points which are either minimal vectors or the barycenters in singular irreducible faces. Denote by

$$\Sigma_i := V_0 \cdot V_1 \cdot \ldots \cdot V_i$$

the intermediate subdivisions.

The subdivision $\Sigma$ defines for any face $\sigma \in \Sigma$ the canonical desingularization

$$\Delta^\sigma = V_k^\sigma \cdot \ldots \cdot V_1^\sigma \cdot \sigma,$$

where $V_i^\sigma = V_i \cap |\sigma|$, and the intermediate subdivisions:

$$\Delta_i^\sigma = V_k^\sigma \cdot \ldots \cdot V_i^\sigma \cdot \sigma$$

By Lemmas 7.15.1(2), 7.15.2(1), 7.15.3(2), all the points in the sets $V_i^\sigma$ define the $G_\sigma^0$- invariant valuations on $\tilde{X}_\sigma$. The action of $G_\sigma^0$ on $\tilde{X}_\sigma$ lifts to $\tilde{X}_{\Delta^\sigma}$, and $\tilde{X}_{\Delta^\sigma} \rightarrow \tilde{X}_\sigma$ is $G_\sigma^0$-equivariant. Also, by the canonicity of the algorithm, the action of $\text{Aut}(\sigma)$ on $\sigma$ lifts to $\Delta^\sigma$, so that the subdivisions $\Delta^\sigma \rightarrow \sigma$ are $\text{Aut}(\sigma)$ invariant. By Lemma 7.14.3, $G_\sigma$ is generated by $G_\sigma^0$ and $\text{Aut}(\sigma)$. This implies that the action of $G_\sigma$ on $\tilde{X}_\sigma$ lifts to each scheme $\tilde{X}_{\Delta^\sigma}$. On other hand, the functoriality of the algorithm implies that $(\Delta_i^\sigma)|_\pi = \Delta_i^\tau$ for $\tau \leq \sigma$. These data define canonical subdivisions $\Delta_i = \{\Delta_i^\sigma \mid \sigma \in \Sigma\}$ of $\Sigma$ (see Definition 7.12.1). By Theorem 7.13.1, there exists a unique toroidal modification $X_i$ of $X$ locally defined by the relevant diagram.

The canonical subdivisions $\{\Delta_i^\sigma\}$ define the intermediate varieties $X_i$. The morphisms $\pi_i : X_i \rightarrow X_{i-1}$ are locally described by the star subdivisions $\Delta_i^\sigma = V_i^\sigma \cdot \Delta_{i-1}^\sigma$ of fans of $\Delta_{i-1}^\sigma$. Thus, by Corollary 7.16.7, the exceptional divisors $D_{ij}$ of each $\pi_i$
define locally toric valuations, corresponding to vectors \( v_{ij} \in V_i^\sigma = \{ v_{i1}, \ldots, v_{ik} \} \). This implies that the sets of the vectors \( V_i^\sigma \) correspond to the sets of locally toric valuations, and each morphism \( X_i \rightarrow X_{i-1} \) is a composition of the blow-ups at valuations with disjoint centers.

In particular, the corresponding filtered ideals

\[
\mathcal{I}_{i,n} := \prod_{v \in V_i^\sigma} \pi_* (\mathcal{I}_{n D_i}),
\]

don \( X_{i-1} \), defining the blow-ups are locally described, by Lemma 6.3.9, as the pull-backs of the monomial ideals associated with \( V_i^\sigma \) on \( X_{\Delta^\sigma} \):

\[
\mathcal{I}_{V_i,n} := \prod_{v \in V_i^\sigma} \mathcal{I}_{\text{val}(v_{ij}, n)}
\]

Then, by Corollary 7.16.7, the morphism \( X_i \rightarrow X_{i-1} \) is the blow-up of the filtered ideal center \( \mathcal{I}_{i,n} \).

The resulting variety \( Y = X_k \) defined locally by \( \{ \Delta^\sigma \mid \sigma \in \Sigma \} \) is regular. Since the desingularizations \( \Delta^\sigma \) do not affect regular cones of \( \sigma \) the nonsingular points remain unaffected. The morphism \( Y \rightarrow X \) is the composition of the blow-ups of the functorial filtered centers \( \mathcal{I}_{i,n} \). The inverse image \( f^{-1}(\text{Sing}(X)) \) of the singular locus \( \text{Sing}(X) \) is defined locally by a toric divisor on a nonsingular toric variety \( X_{\Delta^\sigma} \) so it is SNC.

The algorithm commutes with smooth morphisms and field extensions. The pull-backs of the charts can be used to describe the algorithm locally.

For the condition (5), the strict transforms of locally toric divisors, and the exceptional components are both local pull-backs of toric divisors on smooth toric varieties and thus they have SNC.

Now assume that \( X \) is étale locally binomial variety. Consider its normalization. Since the normalization commutes with étale maps we obtain an étale locally toric variety. By the previous case there exist compatible desingularizations on étale locally toric cover. These compatible desingularizations descend to the desingularization \( Y \) of \( X \). Moreover, they define an SNC exceptional divisor \( E \) on a strict toroidal cover which descends to an NC divisor on \( Y \). We apply Proposition 6.2.6 to further transform it to an SNC divisor. We use identical arguments as in the proof of Theorem 6.5.1.

7.18. **Desingularization of étale locally toric varieties except for a toroidal subset.**

**Theorem 7.18.1.** Let \( X \) be an étale locally toric variety over a field \( K \) with a locally toric Weil divisor \( D \). Assume that \( D \) has locally ordered components. Let \( (V, D_V) \subseteq (X, D) \) be an open saturated toroidal subset, where \( D_V := D \cap V \) in \( (X, D) \).

There exists a canonical resolution of singularities of \((X, D)\) except for \( V \) i.e. a birational projective toroidal map \( f : Y \rightarrow X \) such that

1. \( f \) is an isomorphism over the open set \( V \).

---

30 Definition 2.1.22
31 Definitions 7.1.1, 7.6.6
32 Definition 2.1.6
(2) The variety \((Y, D_Y)\) is a strict toroidal embedding, where \(D_Y := \overline{D_Y}\) is the closure of the divisor \(D_Y\) in \(Y\).

(3) The variety \((Y, D_Y)\) is the saturation of \((V, D_V)\) in \((Y, D_Y)\). 33

(4) The complement \(E_{V,Y} := Y \setminus V\) is a divisor which has simple normal crossings on \((Y, D_Y)\), and so does the exceptional divisor \(E_{exc} \subset E_{V,Y}\).

(5) If \(V = (X, D)^{tor}\) is the toroidal locus of \((X, D)\) then \(E_{exc} = E_{V,Y} = Y \setminus V\).

(6) \(f\) is obtained by a sequence of blow-ups at the canonical filtered centers 34.

(7) \(f\) commutes with smooth morphisms and field extensions respecting the subset \(V\), and the order of the components \(D\).

Proof. **Case 1.** Assume that \(X\) is a locally toric variety with a locally toric divisor \(D\) and \(V = (X, D)^{tor}\) is the toroidal locus of \((X, D)\).

By Theorem 7.9.4, the complement \(X \setminus V\) is of codimension 2. Also, the locally toric divisor \(D\) on the variety \(X\) defines the natural stratification \(S = \text{Sing}_D(X)\), and the associated semicomplex \(\Sigma\). Moreover, the divisor \(D\) defines a saturated subcomplex \(\Omega \subset \Sigma\).

Consider a chart \(\phi : U \to X_\sigma\), where \(\sigma \in \Sigma\) is a semicone. The intersection of \(\sigma\) with \(\Omega\) defines a unique saturated subcomplex \(\Omega_\sigma\) of \(\sigma\). It corresponds to the divisor \(\overline{D_{\Omega_\sigma}}\) on \(X_\sigma\), such that \(D_U = \phi^{-1}(\overline{D_{\Omega_\sigma}})\). Here \(\overline{D_{\Omega_\sigma}}\) is the closure of \(D_{\Omega_\sigma} \subset X_{\Omega_\sigma}\) on \(X_\sigma\) (see Lemma 7.5.3, and Theorem 7.9.4(7)).

Denote the embedded fan \(\text{Reg}(\sigma, \Omega_\sigma)\) by \((\Omega^0_\sigma, \Omega_\sigma)\). Then, by Lemma 7.5.6(3), the open subset \(X_{\Omega_\sigma}^0\) is the toroidal saturation of \((X_{\Omega_\sigma}, D_{\Omega_\sigma})\) in \(X_\sigma\), and \(X_{\Omega_\sigma}^0 = (X_\sigma, \overline{D_{\Omega_\sigma}})^{tor}\) is the toroidal locus.

Thus the toroidal locus \((U, D \cap U)^{tor} = V \cap U\) is defined locally as \(V \cap U = \phi^{-1}(X_{\text{Reg}(\sigma, \Omega_\sigma)})\).

As before consider the complex \(\widehat{\Sigma}\) consisting of the disjoint union of the fans \(\sigma\), of all the faces of the cones \(|\sigma|\), where \(\sigma \in \Sigma\), and its subcomplex \(\overline{\Omega}\) defined by \(\Omega_\sigma\) where \(\sigma \in \Sigma\).

Let \(\overline{\Sigma} := V_k \cdot \ldots \cdot V_1 \cdot \overline{\Sigma}\) be the canonical desingularization of the relative complex \((\overline{\Sigma}, \overline{\Omega})\) from Theorem 5.12.1.

In the process we use centers which are either barycenters, so the sums of the minimal internal vectors in faces \(\sigma\) and the vertices in \(\sigma \cap \Omega\) or the minimal generators in \(\sigma\). Such centers are \(G^0_\sigma\)-invariant by Lemma 7.15.3. Since the algorithm is functorial they are also \(\text{Aut}(\sigma)\)-invariant. By the same argument as before \(\overline{X_{\Delta^\sigma}} \to \overline{X_\sigma}\) is \(G^0_\sigma\)-equivariant.

Moreover, locally for any semicone \(\sigma \in \Sigma\) we obtain a subdivision complex \((\Delta^\sigma, \Omega_\sigma)\) which is relatively regular. This defines a collection of the subdivisions \((\Delta^\sigma, \Omega_\sigma)\) which determines the subdivision \(\Delta = \{\Delta^\sigma\}\) and, induces, by Theorem 7.13.1, a unique transformation \(Y\) of \((X, S)\).

Repeating the reasoning from the previous proof we see that the modification is given by a sequence of the blow-ups \(\pi_i : X_i \to X_{i-1}\) at invariant valuations.

33Definition 2.1.6
34Definition 6.3.11
The desingularization $\Delta^\sigma$ does not affect relatively regular cones with respect to $\Omega_\sigma$. Thus, in particular, the cones in $\Omega_\sigma^0$ remain unaffected. Consequently, the points in $V$ are not modified in the process. Since $(\Delta^\sigma, \Omega_\sigma)$ is a regular relative fan (an embedded fan) we get by Proposition 7.5.6, that:

1. $(X_{\Delta^\sigma}, D_{\Omega_\sigma})$ is a strict toroidal embedding locally corresponding to $(Y, D_Y)$.
2. The exceptional divisor $E_\sigma \subseteq X_{\Delta^\sigma} \setminus X_{\Omega^0}$ of $X_{\Delta^\sigma} \to X$ has SNC. So $E_{\text{exc}}$ has SNC on $(Y, D_Y)$.
3. $(X_{\Delta^\sigma}, D_{\Omega_\sigma})$ is the saturation of $(X_{\Omega^0}, D_{\Omega^0})$ and of $(X_{\Omega^0}, D_{\Omega^0})$ so $(Y, D_Y)$ is the saturation of $(V, D_V)$.

By definition, $Y \setminus V$ coincides with the exceptional divisor $E_{\text{exc}}$. Otherwise $Y$ would contain an unmodified toroidal open subset of $X$ strictly bigger than $V$. This contradicts the assumption that $V$ is the toroidal locus of $X$.

**Case 2.** Assume that the variety $(X, D)$ is Zariski locally toric with a locally toric divisor, and the open subset $V$ is an arbitrary saturated toroidal subset.

Let $V_0 := (X, D)_{\text{tor}} \subset V$ be the toroidal locus of $(X, D)$. By definition, $V$ is a toroidal subset which is saturated in $V_0$. By Case 1, there is a desingularization except for $V_0$. Now $(Y, D_Y)$ is a strict toroidal embedding and $E := Y \setminus V_0$ is a divisor having SNC on $(Y, D_Y)$. So $(Y, D_Y \cup E)$ is a strict toroidal embedding. Since $V$ is saturated in $V_0$ and $V_0$ is saturated in $Y$ it follows that $(V, D_V)$ is an open saturated toroidal subset of $(Y, D_Y \cup E)$. It suffices to use Theorem 6.6.1 and apply the partial desingularization of toroidal embeddings $(V, D_V) \subset (Y, D_Y \cup E)$.

**Case 3.** Assume that the variety $(X, D)$ is étale locally toric with an étale locally toric divisor, and the open subset $V$ is an arbitrary saturated toroidal subset of $(X, D)$. We repeat the reasoning from the proof of Theorem 6.4.1.

We consider an étale covering $(X^0, D^0)$ of $(X, D)$ consisting of locally toric varieties with locally toric divisor. Note that, by Lemma 2.1.11, the inverse image $V^0$ of $V$ is a saturated toroidal subset of $(X^0, D^0)$. By Step 2, there is a desingularization at functorial centers on an $X^0$. By functoriality, the centers descend (by the flat descent) to the centers on $X$. We obtain a toroidal embedding $(Y, D_Y \cup E_{V,Y})$ containing a toroidal subset $(V, D_V)$. Since $(Y, D_Y)$ contains an open toroidal subset $(V, D_V)$ intersecting all strata then $(Y, D_Y)$ is a strictly toroidal embedding, by Lemma 2.1.10. Consequently, $E_{V,Y} := Y \setminus V$ has NC on $(Y, D_Y)$. To obtain the condition that $E_{V,Y}$ has SNC with $D_V$ it suffices to apply Proposition 6.2.6. The process commutes with field extensions and smooth morphisms preserving the order of the components $D_V$.

8. **Comparison of the desingularization algorithm for toroidal embeddings and locally toric varieties**

When we forget the toroidal structure, a strict toroidal embedding is a locally toric variety. The desingularization algorithm is identical in both cases. If $\Sigma$ is the complex associated with the toroidal structures then $\text{sing}(\Sigma)$ defines the semicomplex associated with the canonical singular stratification $S = \text{Sing}(X)$ on $X$.

The same algorithm in both situations is induced by the decompositions of the faces of $\text{sing}(\Sigma)$. In the case of toroidal embedding the decomposition of $\text{sing}(\Sigma)$
simply extends uniquely to decomposition of $\Sigma$ defining the relevant toroidal modification. In the locally toric situation the decomposition of $\text{sing}(\Sigma)$ induces the modification directly via the charts.

A point $x$ on a toroidal embedding belongs to an open neighborhood associated with a face $\sigma \in \Sigma$. The modification of $\sigma$ is induced by the subdivision of its singular part $\text{sing}(\sigma)$. The very same point on the same variety with a locally toric structure with the singular stratification will be associated via a chart with a face $\text{sing}(\sigma)$ of the semicomplex $\text{sing}(\Sigma)$, and the transformation will be again defined by the same subdivision of $\text{sing}(\sigma) \in \text{sing}(\Sigma)$. So both algorithms agree locally and globally.

9. **Counterexample to the Hironaka desingularization of locally binomial varieties in positive characteristic**

Let $x \in X$ be a point on a smooth variety over $K$. Let $I$ be an ideal sheaf on $X$ of maximal order $p$. Denote by $\text{Sing}_p(I)$ the locus of the points of order $p$. Recall that a **maximal contact** is a local parameter $u$ such that its zero locus $H = V(u)$ contains $\text{Sing}_p(I)$, and this property will be preserved after any sequence of blow-ups of smooth centers contained in $\text{Sing}_p(I)$. The following example shows that there is no maximal contact for locally binomial varieties over a field of positive characteristic. Thus the Hironaka characteristic zero approach fails already in this case.

**Example 9.0.1.** Let $X \subset \mathbb{A}^4 = \text{Spec}(K[x,y,z,w])$ be the hypersurface over a field of characteristic $p$, defined by a single equation $x^p - y^p z$. It admits the action of the group of automorphisms $\mathbb{A}^1 = \text{Spec}(K[t])$:

\[
x \mapsto x + ty, \quad y \mapsto y, \quad z \mapsto z + t^p.
\]

The locus $\text{Sing}_p(X)$ of the points of maximal order $p$ is given by the smooth subvariety $Z = V(x,y)$. Suppose that there is a maximal contact $u \in \mathcal{O}_X$ such that $V(u) \supset \text{Sing}_p(X) = V(x,y)$ and that the property will be preserved after blow-ups at smooth centers contained in $\text{Sing}_p(X)$. Thus $u$ is a local parameter which can be written in the form

\[
u = a(x,z,w)x + b(x,y,z,w)y.
\]

Using automorphisms above we can always assume that $b(0,0,0,0) \neq 0$. Applying the blow-up at $x = y = z = w = 0$ in the chart of $w$:

\[
(x, y, z, w) \mapsto (xw, yw, zw, w)
\]

we transform $x^p - y^p z$, into

\[
u^p(x^p - y^p zw).
\]

Moreover, the equation of the new maximal contact

\[
u' = 1/w(a(xw, yw, zw, w)wx + b(xw, yw, zw, w)wy) = \\
a(xw, yw, zw, w)y + b(xw, yw, zw, w)y
\]

will maintain its form with the condition $b(0,0,0,0) \neq 0$ in the corresponding chart of $w$.

After $p$ such blow-ups (and factoring the exceptional divisors) we obtain the form

\[
x^p - y^p zw^p.
\]
The order $p$ locus $\text{Sing}_{p}(X)$ has now two components $V(x, y)$ and $V(x, w)$. The intersection $V(x, w) \cap V(u)$ is equal to $V(x, y, w)$ in a neighborhood with $b(x, y, z, w) \neq 0$. Thus the component $V(x, w)$ is not contained in $V(u)$ which contradicts to the assumption on maximal contact.

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