ON A CERTAIN LOCAL IDENTITY FOR LAPI D–MAO’S
CONJECTURE AND FORMAL DEGREE CONJECTURE : EVEN
UNITARY GROUP CASE

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Abstract Lapid and Mao formulated a conjecture on an explicit formula of Whittaker–Fourier coefficients
of automorphic forms on quasi-split reductive groups and metaplectic groups as an analogue of the Ichino–
Ikeda conjecture. They also showed that this conjecture is reduced to a certain local identity in the case of
unitary groups. In this article, we study the even unitary-group case. Indeed, we prove this local identity
over $p$-adic fields. Further, we prove an equivalence between this local identity and a refined formal degree
conjecture over any local field of characteristic zero. As a consequence, we prove a refined formal degree
conjecture over $p$-adic fields and get an explicit formula of Whittaker–Fourier coefficients under certain
assumptions.

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ation theory of $p$-adic reductive groups

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1. Introduction

A relationship between periods of automorphic forms and special values of certain $L$-functions has been studied in several situations. For example, Gross and Prasad [14] conjectured a relationship between the nonvanishing of Gross–Prasad periods on special orthogonal groups and the nonvanishing of central values of certain $L$-functions. Moreover, Ichino and Ikeda [19] studied a refinement of the Gross–Prasad conjecture. Indeed, they conjectured an explicit formula between Gross–Prasad periods and central $L$-values. As an analogue of the Ichino–Ikeda conjecture, Lapid and Mao [30] formulated a similar conjecture on an explicit formula of Whittaker periods for quasi-split reductive groups and metaplectic groups, which is in the spirit of Sakellaridis and Venkatesh [47].

In this article, we consider Lapid and Mao’s conjecture in the case of a quasi-split skew-Hermitian even unitary group $U_{2n}$ associated to a quadratic extension of number fields $L/K$. Let us recall their conjecture in this case. For an automorphic form $\varphi$ on $U_{2n}(\mathbb{A}_K)$,
we consider the Whittaker–Fourier coefficient
\[ W^\psi_N(\varphi) := (\text{vol}(N(K)\backslash N(\mathbb{A}_K)))^{-1} \int_{N(K)\backslash N(\mathbb{A}_K)} \varphi(n)\psi_N(n)^{-1} \, dn. \]

Here, \( N \) is a maximal unipotent subgroup of \( U_{2n} \) and \( \psi_N \) is a nondegenerate character of \( N(\mathbb{A}_F) \). We also consider the \( U_{2n}(\mathbb{A}_K) \)-invariant pairing
\[ (\varphi_1, \varphi_2) := (\text{vol}(U_{2n}((K)\backslash U_{2n}(\mathbb{A}_K)))^{-1} \int_{U_{2n}((K)\backslash U_{2n}(\mathbb{A}_K))} \varphi_1(g)\varphi_2(g) \, dg \]

of two square-integrable automorphic forms \( \varphi_1, \varphi_2 \) on \( U_{2n}(\mathbb{A}_K) \). Then we would like to construct a Whittaker functional using the pairing \((-,-)\). Given a finite set of places \( S \) of \( F \), Lapid and Mao [30] defined a stable integral
\[ \int_{N(\mathbb{A}_S)}^{st} f(n) \, dn \]

for a suitable class of smooth function \( f \) on \( N(\mathbb{A}_K) \), with \( \mathbb{A}^S = \prod_{v \in S} K_v \). In particular, when \( S \) consists of finite places, there is a sufficiently large compact open subgroup \( N'_0 \) of \( N(\mathbb{A}_S) \) such that the previous integral is equal to
\[ \int_{N'_0(\mathbb{A}_S)} f(n) \, dn \]

for any compact open subgroup \( N'_0 \supset N_0 \). Let \( \sigma \) be an irreducible \( \psi_N \)-generic cuspidal automorphic representation of \( U_{2n}(\mathbb{A}_K) \). Lapid and Mao [30] proved that for \( \varphi \in \sigma \), \( \varphi^\vee \in \sigma^\vee \) and for a sufficiently large finite set \( S \) of places,
\[ I(\varphi, \varphi^\vee) := \int_{N(\mathbb{A}_S)}^{st} (\sigma(n)\varphi, \varphi^\vee)\psi_N^{-1}(n) \, dn \]

is defined, and \( I(\varphi, \varphi^\vee) \) satisfies
\[ I(\sigma(n)\varphi, \sigma^\vee(n)^{-1}\varphi^\vee) = \psi_N(nn')I(\varphi, \varphi'). \]

Then they formulated a conjecture on an explicit formula between these two Whittaker functionals following the Ichino–Ikeda conjecture.

Let \( \pi \) be the base change lift of \( \sigma \) to \( GL_{2n}(\mathbb{A}_L) \) constructed by Kim and Krishnamurthy [25]. It is an isobaric sum \( \pi_1 \boxtimes \cdots \boxtimes \pi_k \), where \( \pi_i \) is an irreducible cuspidal automorphic representation of \( GL_{n_i}(\mathbb{A}_F) \) with \( n_1 + \cdots + n_k = 2n \), such that \( L^S(s, \pi_i, \mathbb{A}^-) \) has a simple pole at \( s = 1 \). Here, \( L^S(s, \pi_i, \mathbb{A}^-) \) is the partial Asai \( L \)-function of \( \pi_i \) defined in [13]. Also, we have another Asai \( L \)-function \( L^S(s, \pi_i, \mathbb{A}^+) \) of \( \pi_i \), which satisfies \( L^S(s, \pi_i, \mathbb{A}^-) = L^S(s, \pi_i \otimes \mathbb{Y}, \mathbb{A}^+) \) where \( \mathbb{Y} \) is a character of \( \mathbb{A}_K^\times / L^S \) whose restriction to \( \mathbb{A}_K^\times \) is the quadratic character \( \eta_{L/K} \) corresponding to the quadratic extension \( L/K \).

**Conjecture 1** (Conjectures 1.2 and 5.1 in [30]). Let \( \sigma \) and \( \pi \) be as before. Then for any \( \varphi \in \sigma \) and \( \varphi^\vee \in \sigma^\vee \) and any sufficiently large finite set \( S \) of places, we have
\[ W^\psi_N(\varphi)W^\psi_N^{-1}(\varphi^\vee) = 2^{-k} \prod_{j=1}^{2n} \left( \frac{L^S(j, \sigma; \eta_{E/F})}{L^S(1, \mathbb{A}^+)} \right)^{-1} \int_{N(F_S)}^{st} (\sigma(u)\varphi, \varphi^\vee)\psi_N(u)^{-1} \, du. \]

Here, \( O_S \) is the ring of \( S \)-integers of \( K \).
In [32], for any place $v$ of $K$, Lapid and Mao defined the constant $c_{\pi_v}$ depending on $\pi_v$ (also on $\sigma_v$ and the choice of measures), which is given as a proportionality constant of two local Whittaker periods. Then they proved the following theorem:

**Theorem 1.1** (Theorem 5.5 in [32]). Keep the previous notation. Then for any $\varphi \in \sigma$ and $\varphi^\vee \in \sigma^\vee$ and any sufficiently large finite set $S$ of places, we have

$$W_{N^1}(\varphi)W_{N^1}^{-1}(\varphi^\vee) = 2^{1-k} \left( \prod_v c_{\pi_v}^{-1}(\varphi_v^\vee) \right) \frac{\prod_{j=1}^{2n} L^S(j, n_{L/K})}{L^S(1, \pi, \text{As}^+)} \left( \text{vol}(N(O_S)\backslash N(A_S^S)) \right)^{-1} \int_{N(F_S)}^{st} (\sigma(u)\varphi, \varphi') \psi_{N}(u)^{-1} du.$$

**Remark 1.1.** In [32, Theorem 5.5], Lapid and Mao assume two working assumptions. One, on the basic properties of the local zeta integral of Shimura type, is proved by Ben-Artzi and Soudry [6]; and the other, on the irreducibility of global descents, is proved by the author in his previous work [43]. Hence, Theorem 1.1 holds without any assumption.

Because of Theorem 1.1, in order to prove Conjecture 1 it suffices to show that $\prod_v c_{\pi_v} = 1$. Indeed, Lapid and Mao conjectured the following identity:

**Conjecture 2** (Conjecture 5.8 in [32]). Let $v$ be a place of $K$. Then

$$c_{\pi_v} = \omega_{\sigma_v}(-1).$$

In particular, this conjecture concludes that $\prod_v c_{\pi_v} = 1$. Note that when $L_v := L \otimes K_v$ is a quadratic extension, we have $\omega_{\sigma_v}(-1) = \omega_{\pi_v}(\tau)$ for $\tau \in L_v$, such that $c(\tau) = -\tau$ with $1 \neq \tau \in \text{Gal}(L_v/K_v)$. Then Conjecture 2 is equivalent to

$$c_{\pi_v} = \omega_{\pi_v}(\tau).$$

(1.0.1)

One of the local main theorems of this article is the proof of this conjecture in an inert case.

**Theorem 1.2.** Suppose that $v$ is a non-split finite place (i.e., $U_{2n}(K_v)$ is a quasi-split unitary group). Then Conjecture 2 holds.

A similar identity was proved by Lapid and Mao [33] in the case of metaplectic groups. In this article, following their argument we shall prove this theorem. Indeed, we give a rigorous proof of a formal argument from [32, Section 6]. Unlike the argument in [33], in the present case we will encounter a gap between integrals over $E$ and integrals over $F$. However, we can fill this gap by model transitions in [44]. Moreover, we need several preliminary results to complete a proof. Indeed, we shall study local base change lifts (Theorems A.4 and A.5) and extend the stability of certain oscillatory integrals in [28] to quasi-split groups (Theorems B.1 and B.2).

The local identity in Conjecture 2 is important not only for Lapid and Mao’s conjecture but also for the formal degree conjecture by Hiraga, Ichino and Ikeda [17, 18]. Using an observation by Gross and Reeder [15], Gan and Ichino [11, Section 14.5] formulated a refinement of this conjecture for classical groups. We call this refinement the refined formal degree conjecture. In [20], Ichino, Lapid and Mao showed that a similar local identity
to Conjecture 2 (proved in [33]) is equivalent to the refined formal degree conjecture for metaplectic groups. As a consequence of [33], they proved the refined formal degree conjecture in this case. In a similar argument to [20], we prove the equivalence between the refined formal degree conjecture for $U_{2n}$ and Theorem 1.2. As a consequence, we can prove the refined formal degree conjecture.

**Theorem 1.3.** Let $F$ be a non-Archimedean local field of characteristic zero and $E$ be a quadratic extension of $F$. Let $\pi$ be an irreducible representation of $GL_{2n}(E)$ of the form $\pi = \tau_1 \times \cdots \times \tau_k$, where $\tau_i$ are mutually inequivalent irreducible discrete series representations of $GL_{n_i}(E)$ such that $n = n_1 + \cdots + n_k$ and $L(s, \tau_i, As^+)$ has a pole at $s = 0$. Write the local descent of $\pi$ by $\sigma' = D_{\psi - 1}(\psi(1, c(\pi)))$, which is an irreducible generic discrete series representation of $G'$ (see Theorem A.5). Then we have

$$d_\psi = 2^k \lambda(E/F)^n \omega_{\sigma'}(-1) \gamma(1, c(\pi), As^-, \psi) d_{\sigma'}.$$ 

Here, $d_{\sigma'}$ is the formal degree (see Section 8) and $d_\psi$ is a certain measure on $U_{2n}(F)$ (see Section 2.4).

**Remark 1.2.** As we prove in Theorem A.5, any generic discrete series representation is given as in Theorem 1.3.

**Remark 1.3.** Recently, Beuzart-Plessis [7] proved the original formal degree conjecture from [17, 18] for any unitary groups using a different method.

On the other hand, we can show the refined formal degree conjecture for $U_{2n}$ over a real field using computations in [17]. In a similar argument as the non-Archimedean case, we can prove the equivalence between the refined formal degree conjecture and Conjecture 2. Hence, we obtain the following result:

**Theorem 1.4.** Suppose that $v$ is a real place and that $L_v \simeq \mathbb{C}$ – that is, $U_{2n}(K_v) \simeq U_{2n}(\mathbb{R})$. Then Conjecture 2 holds for any generic discrete series representation of $U_{2n}(\mathbb{R})$.

As a corollary of Theorems 1.2 and 1.4 and [32, Lemma 5.4], the following global formula holds:

**Corollary 1.1.** Let $\sigma$ be as in Conjecture 1. Suppose that $K$ is a totally real field and $L$ is an imaginary quadratic extension of $K$. Assume that

$$\begin{cases} 
v \text{ is inert in } L & \text{ if } v \mid 2 \\
\sigma_v \text{ is unramified} & \text{ if } v \text{ is a split finite place} \\
\sigma_v \text{ is discrete series} & \text{ if } v \text{ is a real place} \\
\end{cases}$$

for any place $v$ of $K$. Then the formula in Conjecture 1 holds for $\sigma$.

This article is organised as follows. In Section 2, we define basic notations. In Section 3, we formulate Lapid and Mao’s local conjecture and reduce it to the tempered case. In Sections 4 through 6, following the idea by Lapid and Mao [33], we rewrite the required identity as another local identity using local analysis and certain functional equations.
(See [33, Section 4] for the idea of the proof). Most parts are proved in a similar argument as in [33]. Hence, we give a proof only when there are nontrivial differences, and we give only statements if the proof is essentially the same as the corresponding results in [33]. In Section 7, we complete the proof of Theorem 1.2. In Section 8, we prove Theorems 1.3 and 1.4. In Appendix A, we determine the image of local base change lifts of generic discrete series representations of $U_{2n}$ using a similar argument as in [22, 35]. In Appendix B, we prove the stability of certain oscillatory integrals for quasi-split reductive algebraic groups, which is a generalisation of [28] to the quasi-split case.

2. Notation and Preliminaries

2.1. Groups, homomorphisms and group elements

- Let $F$ be a local field of characteristic zero and $E$ be a quadratic extension of $F$. Let $\epsilon$ be a nontrivial element of $\text{Gal}(E/F)$. Take $\tau \in E$ such that $\epsilon(\tau) = -\tau$.
- $\eta_{E/F}$ denotes the quadratic character of $F^\times$ corresponding to $E/F$. Fix a character $\Upsilon$ of $E^\times$ such that $\Upsilon|_{F^\times} = \eta_{E/F}$.
- $I_m$ is the identity matrix in $\text{GL}_m$, and $w_m$ is the $m \times m$ matrix with 1s on the nonprincipal diagonal and 0s elsewhere.
- For any group $Q$, $Z_Q$ is the centre of $Q$; $e$ is the identity element of $Q$. Denote the modulus function of $Q$ (i.e., the quotient of a right Haar measure by a left Haar measure) by $\delta_Q$.
- $\text{Mat}_m$ is the vector space of $m \times m$ matrices over $F$.
- For $x = (x_{ij}) \in \text{Mat}(E)$, $x^\epsilon$ denotes the matrix $(\epsilon(x_{ij}))$.
- $x \mapsto \dagger x$ is the transpose on $\text{Mat}_m$; $x \mapsto x^\vee$ is the twisted transpose map on $\text{Mat}_m$ given by $x^\vee = w_m^{-1} t x^\epsilon w_m$; $g \mapsto g^\ast$ is the outer automorphism of $\text{GL}_m$ given by $g^\ast = w_m^{-1}(t g^\epsilon)^{-1} w_m$.
- $\mathfrak{u}_n = \{x \in \text{Mat}_n(E): x^\vee = x\}$.
- $\mathcal{M} = \text{GL}_{2n}(E), \mathcal{M}' = \text{GL}_n(E)$.
- $G = U_{4n} = \left\{ g \in \text{GL}_{4n}(E): t g \begin{pmatrix} -w_{2n} & w_{2n} \\ -w_{2n} & w_{2n} \end{pmatrix} g = \begin{pmatrix} -w_{2n} & w_{2n} \\ -w_{2n} & w_{2n} \end{pmatrix} \right\}$, acting on the 4$n$-dimensional skew-Hermitian space with the standard basis $e_1, \ldots, e_{2n}, e_{-2n}, e_{-2n+1}, \ldots, e_{-1}$.
- $G' = U_{2n} = \left\{ g \in \text{GL}_{2n}(E): t g \begin{pmatrix} -w_n & w_n \\ -w_n & w_n \end{pmatrix} g = \begin{pmatrix} -w_n & w_n \\ -w_n & w_n \end{pmatrix} \right\}$.
- $G'$ is embedded as a subgroup of $G$ via $g \mapsto \eta(g) = \text{diag}(I_n, g, I_n)$, and thus it is the subgroup of $G$ consisting of elements fixing $e_1, \ldots, e_n$ and $e_{-n}, \ldots, e_{-1}$.
- $P = MU$ (resp., $P' = M'U'$) is the Siegel parabolic subgroup of $G$ (resp., $G'$), with its standard Levi decomposition.
- $\overline{P} = \dagger P$ is the opposite parabolic of $P$, with unipotent radical $\overline{U} = \dagger U$.
- The isomorphism $\varphi(g) = \text{diag}(g, g^\ast)$ identifies $\mathcal{M}$ with $M \subset G$. Similarly, for $\varphi': \mathcal{M}' \to M' \subset G'$.
- The embeddings $\eta_M(g) = \text{diag}(g, I_n)$ and $\eta_M'(g) = \text{diag}(I_n, g)$ identify $\mathcal{M}'$ with subgroups of $\mathcal{M}$. Set also $\eta_M = \varphi \circ \eta_M$ and $\eta_M' = \varphi \circ \eta_M' = \eta \circ \varphi'$.
- $K$ is the standard maximal compact subgroup of $G$ (in the $p$-adic case it consists of the matrices with integral entries).
$N$ is the standard maximal unipotent subgroup of $G$ consisting of upper unipotent matrices; $T$ is the maximal torus of $G$ consisting of diagonal matrices; $B = TN$ is the Borel subgroup of $G$.

For any subgroup $X$ of $G$, write $X = \eta^{-1}(X)$, $X_M = X \cap M$ and $X_M = \varrho^{-1}(X_M)$; similarly, $X_M' = X' \cap M'$ and $X_M' = (\varrho')^{-1}(X_M')$.

$\ell_M : \text{Mat}_n \to N_M$ is the group embedding given by $\ell_M(x) = \left( \begin{array}{cc} I_n & x \\ I_n & \end{array} \right)$ and $\ell_M = \varrho \circ \ell_M$.

$\ell : u_{2n} \to U$ is the group isomorphism given by $\ell(x) = \left( \begin{array}{cc} I_n & x \\ I_n & \end{array} \right)$.

$\xi_m = (0, \ldots, 0, 1) \in F^m$.

$P$ is the mirabolic subgroup of $M$ consisting of the elements $g$ such that $\xi_{2n}g = \xi_{2n}$.

Set $H_M = \text{GL}_{2n}(F)$.

$t = \text{diag}(1, -1, \ldots, 1, -1) \in M$.

$w'_0 = (-w_0, w_n) \in G'$ represents the longest Weyl element of $G'$.

$w_U = (-I_{2n}, I_{2n}) \in G$ represents the longest $M$-reduced Weyl element of $G'$.

$w'_U = (-I_n, I_n) \in G'$ represents the longest $M'$-reduced Weyl element of $G'$.

$w'_0 = w_{2n} \in M$ represents the longest Weyl element of $M$; $w_0^M = \varrho(w_0^M)$.

$w'_0 = w_{n} \in M'$ represents the longest Weyl element of $M$; $w_0^M = \varrho(w_0^M)$.

$w_{2n,n} = (I_n, I_n) \in M$, $w'_{2n,n} = (w_0^M, I_n) \in M$.

$\gamma = w_U \eta(w'_U)^{-1} = \left( \begin{array}{cc} I_n & I_n \\ -I_n & I_n \end{array} \right) \in G$.

$d = \text{diag}(1, -1, \ldots, (-1)^{n-1}) \in \text{Mat}_n$, $\epsilon_1 = (\hat{w} \gamma)^{-1} \varrho(\epsilon_3) w_U = \ell_M((-1)^n d)$, $\epsilon_2 = \ell_M(d)$, $\epsilon_3 = w_{2n,n} \epsilon_2$, $\epsilon_4 = \ell_M(-\frac{1}{2} \hat{d} w_0^M)$.

$V$ (resp. $V^\#$) is the unipotent radical of the standard parabolic subgroup of $G$ with Levi $\text{GL}_1(E)^n \times U_{2n}$ (resp., $\text{GL}_1(E)^{n-1} \times U_{2n+2}$). Thus, $N = \eta(N') \times V$, $V^\#$ is normal in $V$ and $V/V^\#$ is isomorphic to the Heisenberg group of dimension $2n + 1$ over $E$. Also, $V = V_M \ltimes V_U$ where $V_U = V \cap U = \{ \ell \left( \begin{array}{cc} x & y \\ x & x \end{array} \right) : x \in \text{Mat}_n(E), y \in u_n \}$.

$V_- = V_M \ltimes V_U$ (recall that $V_M^\# = V^\# \cap M$ by our convention).

$V_\gamma = V \cap \gamma^{-1}N \gamma = \eta(w'_U) V_M \eta(w'_U) \times \{ \ell \left( \begin{array}{cc} x & y \\ x & x \end{array} \right) : x \in \text{Mat}_n(E) \} \subset V_-$.

$V_+ \subset V$ is the image under $\ell_M$ of the space of $n \times n$ matrices over $E$ whose rows are zero except possibly for the last one. Thus, $V = V_+ \ltimes V_-$. For $c = \ell_M(x) \in V_+$, we denote by $c$ the last row of $x$.

$N^\# = V_- \rtimes \eta(N')$. It is the stabiliser in $N$ of the character $\psi_U$ defined in the next subsection.

$N_M^3 = (N_M^2)^n$.

$J$ is the subspace of $\text{Mat}_n$ consisting of the matrices whose first column is zero.

$\hat{R} = \left\{ \ell \left( \begin{array}{cc} I_n & x \\ x & I_n \end{array} \right) : x \in J, n \in N_M^\# \right\}$. 

2.2. Characters

Fix a nontrivial additive character $\psi_F$ of $F$ and define an additive character $\psi$ of $E$ by $\psi(x) = \psi_F \left( \frac{x + \xi(x)}{2} \right)$ for $x \in E$.

- $\psi_{N_M}(u) = \psi((u_1, 2 + \cdots + u_{2n-1}, 2n))$.
- $\psi_{N_M} \circ \tilde{Q} = \psi_{N_M}$.
- $\psi_{N_M'}(u) = \psi((u'_1, 2 + \cdots + u'_{n-1}, n))$.
- $\psi_{N_M'} \circ \tilde{Q}' = \psi_{N_M'}$.
- $\psi_{N_M'(nu)} = \psi_{N_M}(n)\psi\left(\frac{1}{2}u_n, n+1\right)^{-1}$, $n \in N_M'$, $u \in U'$.
- $\psi_N = \psi_{N_M}(n)$, $n \in N_M$, $u \in U$ (a degenerate character). Then $\psi_{N_M'}(u) = \psi_N(\gamma \eta(u)^{-1})$.
- $\psi_V(vu) = \psi_{N_M}(w_UuW_U^{-1})$, where we write an element of $V$ by $vu$ so that $u$ fixes $e_1, \ldots, e_n$, $v$ fixes $e_{2n+1}, e_{n+2}, \ldots, e_{2n-1}, n-1$. $\psi_{V}(vu) = \psi_{N_M}(v)^{-1}\psi_{U}(u)$, $v \in V_{M}', u \in U$. (Note that this is not a restriction of $\psi_V$ to $V_\omega$.)

- $\psi_{U}(v) = \psi\left(\frac{1}{2}(v_n, n+1 - v_{2n-1})\right)$.
- $\psi_{U}(\tilde{v}) = \psi(v_{2n+1, 1})$.
- $\gamma_M(q(g)) = \gamma(\det g)$, $g \in \mathbb{M}$, and $\gamma_M(q'(g')) = \gamma(\det g')$, $g' \in \mathbb{M}'$.

2.3. Other notation

- We use the notation $a \ll d$ to mean that $a \leq cb$, with $c > 0$ a constant depending on $d$.
- For any $g \in G$, define $v(g) \in \mathbb{R} > 0$ by $v(uQ(m)k) = |\det m|_F$ for any $u \in U$, $m \in \mathbb{M}$, $k \in K$. Let $v'(g) = v(\eta(g))$ for $g \in G'$.
- $\mathcal{CSR}_Q(Q)$ is the set of compact open subgroups of a topological group $Q$.
- For an $\ell$-group $Q$, let $C(Q)$ (resp., $S(Q)$) be the space of continuous (resp., Schwartz) functions on $Q$ respectively.
- When $F$ is $p$-adic, if $Q'$ is a closed subgroup of $Q$ and $\chi$ is a character of $Q'$, denote by $C(Q' \setminus Q, \chi)$ (resp., $C_{\text{sm}}(Q' \setminus Q, \chi)$; $C_{\text{c}}(Q' \setminus Q, \chi))$ the spaces of continuous (resp. $Q$-smooth; smooth and compactly supported modulo $Q'$) complex-valued left $(Q', \chi)$-equivariant functions on $Q$.
- For an $\ell$-group $Q$, write $\text{Irr}Q$ for the set of equivalence classes of irreducible representations of $Q$. If $Q$ is reductive, also write $\text{Irr}_{\text{sug}}Q$ and $\text{Irr}_{\text{temp}}Q$ for the subsets of irreducible unitary square-integrable (modulo centre) and tempered representations, respectively. Write $\text{Irr}_{\text{gen}}\mathbb{M}$ and $\text{Irr}_{\text{ut}}\mathbb{M}$ for the subset of irreducible generic representations of $\mathbb{M}$ and irreducible representations of unitary type (see below for the definition), respectively. For the set of irreducible generic representations of $G$, we use the notation $\text{Irr}_{\text{gen}}\psi_{N_M'}G'$ to emphasise the dependence on the character $\psi_{N_M'}$.
- For $\pi \in \text{Irr}_{\text{gen}}GL_m(F)$, we say that $\pi$ is of unitary type if it has a nontrivial $GL_m(F)$-invariant linear form.
- For $\pi \in \text{Irr}Q$, let $\pi^\vee$ be the contragredient of $\pi$.
- For $\pi \in \text{Irr}_{\text{gen}}\mathbb{M}$, $\mathbb{W}^{\psi_{N_M}}(\pi)$ denotes the (uniquely determined) Whittaker space of $\pi$ with respect to the character $\psi_{N_M}$. Similarly, we use the notation $\mathbb{W}^{\psi_{N_M}}$, $\mathbb{W}_{\text{gen}}^{\psi_{N_M}}$, $\mathbb{W}_{\text{gen}}^{\psi_{N_M}}$, $\mathbb{W}_{\text{gen}}^{\psi_{N_M}}$.
For $\pi \in \text{Irr} \, \text{gen} \, M$, let $\text{Ind}(\mathcal{W}^\psi_{NM}(\pi))$ be the space of smooth left $U$-invariant functions $W : G \to \mathbb{C}$ such that for all $g \in G$, the function $m \mapsto \delta_F(m)^{-\frac{1}{2}} W(mg)$ on $M$ belongs to $\mathcal{W}^\psi_{NM}(\pi)$. Similarily, define $\text{Ind}(\mathcal{W}^\psi_{NM}^{-1}(\pi))$.

- If a group $G_0$ acts on a vector space $W$ and $H_0$ is a subgroup of $G_0$, denote by $W_{H_0}$ the subspace of $H_0$-fixed points.

- We use the following bracket notation for iterated integrals: $\int \int (\int \int ...)$... implies that the inner integrals converge as a double integral, and after they are evaluated, the outer double integral is absolutely convergent.

### 2.4. Measures

The Lie algebra $\mathfrak{m}$ of $\text{Res}_E/F \text{GL}_m$ consists of the $m \times m$ matrices over $E$. Let $\mathfrak{m}_\mathbb{O}$ be the lattice of integral matrices in $\mathfrak{m}$. For any algebraic subgroup $Q$ of $\text{Res}_E/F \text{GL}_m$ defined over $F$, denote by $q \cap \mathfrak{m}_\mathbb{O}$ the Lie algebra of $Q$. The lattice $q \cap \mathfrak{m}_\mathbb{O}$ gives rise to a gauge form of $Q$ (determined up to multiplication by an element of $\mathbb{O}_E^*$), which we use (together with $\psi$) to define a Haar measure on $Q(F)$ by the recipe of Kneser [26]. For example, when $n = 1$, the measure on $N_M = \{(1, \frac{1}{2}) : x \in E\} \simeq E$ is the self-dual Haar measure $dx$ on $E$ with respect to $\psi$. It is written as follows, using the measure on $F$. Let $dx_i (i = 1, 2)$ and $dy_i (i = 1, 2)$ be self-dual Haar measures on $F$ with respect to $\psi$ — i.e. the measure on $F$ such that

$$\int_F \int_F f(x_1, y_1) \psi(x_1, y_1) \, dx_1 \, dy_1 = f(0)$$

for a smooth function $f$ on $F$, provided the integral converges. Define a Haar measure on $E$ by

$$dz_i = |\tau|^\frac{1}{2} \, dx_i \, dy_i \quad \text{with} \quad z_i = x_i + \tau y_i.$$  \hspace{1cm} (2.4.1)

Then for a smooth function $g$ on $E$, we have

$$\int_E \int_E g(z_1, z_2) \, \psi(z_1, z_2) \, dz_1 \, dz_2 = g(0),$$

provided the integral converges. Namely, $dz_i$ are self-dual Haar measures with respect to $\psi$. Further, note that for $f \in L^1(F)$ we have

$$\int_F \int_F f(x) \psi(ax, y) \, dx \, dy = |a|^{-1} \cdot f(0)$$

and

$$\int_F \left( \int_{F \setminus E} f(x) \psi_E(\tau xy) \, dx \right) \, dy = |\tau|^{-\frac{1}{2}} f(0),$$

$$\int_F \left( \int_{F \setminus E} f(x) \psi_E(\tau^{-1} xy) \, dx \right) \, dy = |\tau|^\frac{1}{2} f(0).$$  \hspace{1cm} (2.4.2)

### 2.5. Weil representation

Let $Y$ be a $2n$-dimensional space over $E$, equipped with a nondegenerate, skew-Hermitian form $(\cdot, \cdot)$. Assume that it has a maximal isotropic subspace of dimension $n$ over $E$. Denote its isometry group by $U(Y) \simeq U_{2n}$. We can view $Y$ as a symplectic space over $F$ with
the symplectic form \( \langle \cdot, \cdot \rangle = \text{Tr}_{E/F}(\cdot, \cdot) \). Then it is a \( 4n \)-dimensional vector space over \( F \), denoted by \( Y' \). Let \( \tilde{\text{Sp}}(Y') \) be the metaplectic cover of \( \text{Sp}(Y') \) with respect to the cocycle given in [39, p. 59] (see also [12, p. 455]). It is clear that \( \text{U}(Y) \subset \tilde{\text{Sp}}(Y') \), and we know that \( \tilde{\text{Sp}}(Y') \) splits over \( \text{U}(Y) \). Fix a character \( \Upsilon \) of \( E^\times \) such that \( \Upsilon|_{F^\times} = \eta_{E/F} \), and choose the splitting as in [13, p. 9] corresponding to \( \Upsilon \). Write

\[
Y = Y^+ + Y^-
\]
as a direct sum of two maximal isotropic subspaces of \( Y \), which are dual under \( \langle \cdot, \cdot \rangle \). When we consider \( Y^\pm \) as a subspace of \( Y' \), they are isotropic for \( \langle \cdot, \cdot \rangle \) and thus in duality under \( \langle \cdot, \cdot \rangle \). Consider the Weil representation \( \omega^\Upsilon_\psi \) of the group \( \mathcal{H}_Y \ltimes \tilde{\text{Sp}}(Y') \) on \( \Phi \in \mathcal{S}(E^n) \), where \( \mathcal{H}_Y := Y \oplus F \) is the Heisenberg group attached to \( Y \) with the multiplication

\[
(w_1, t_1) \cdot (w_2, t_2) = \left( w_1 + w_2, t_1 + t_2 + \frac{1}{2} \text{Tr}_{E/L}(w_1, w_2) \right).
\]

Explicitly, for any \( \Phi \in \mathcal{S}(Y^+) \) and \( X \in Y^+ \), the action of \( \mathcal{H}_Y \) is given by

\[
\omega^\Upsilon_\psi(a, 0) \Phi(X) = \Phi(X + a), \quad a \in Y^+,
\]

\[
\omega^\Upsilon_\psi(b, 0) \Phi(X) = \psi((X, b)) \Phi(X), \quad b \in Y^-,
\]

\[
\omega^\Upsilon_\psi(0, t) \Phi(X) = \psi(t) \Phi(X), \quad t \in F,
\]

while the action of \( \tilde{\text{Sp}}(Y') \) is (partially) given by

\[
\omega^\Upsilon_\psi \left( \begin{pmatrix} g & 0 \\ y & I \end{pmatrix}, \varepsilon \right) \Phi(X) = \varepsilon \gamma_\psi \left( N_{E/F}(\det g) \right) \gamma(\det g) |\det g|^{1/2} \Phi(X \cdot g),
\]

\[
\omega^\Upsilon_\psi \left( \begin{pmatrix} I & y \\ 0 & I \end{pmatrix}, \varepsilon \right) \Phi(X) = \varepsilon \psi \left( \frac{1}{2} (X, X \cdot y) \right) \Phi(X),
\]

where \( \gamma_\psi \) is the Weil factor. We now take \( Y = E^{2n} \) with the skew-Hermitian form

\[
((x_1, \ldots, x_{2n}), (y_1, \ldots, y_{2n})) = \sum_{i=1}^n x_i c(y_{2n+1-i}) - \sum_{i=1}^n y_i c(x_{2n+1-i})
\]

and the standard polarisation \( Y^+ = \{(y_1, \ldots, y_n, 0, \ldots, 0)\} \) and \( Y^- = \{(0, \ldots, 0, y_1, \ldots, y_n)\} \), with the standard basis \( e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1} \). For \( \Phi \in \mathcal{S}(E^n) \), define the Fourier transform

\[
\hat{\Phi}(X) = \int_{E^n} \Phi(X') \psi \left( (X, X' (-I_{2n} I_{n}) ) \right).
\]

Then, reapplied on \( \mathcal{S}(E^n) \), the Weil representation satisfies

\[
\omega^\Upsilon_\psi \left( \varphi'(g) \right) \Phi(X) = \alpha_\psi(\varphi'(g)) Y_{M'}(\varphi'(g)) \nu'(\varphi'(g))^{1/2} \Phi(X \cdot g),
\]

\[
\omega^\Upsilon_\psi \left( \begin{pmatrix} I & y \\ 0 & I \end{pmatrix} \right) \Phi(X) = \psi \left( \frac{1}{2} (X, X \cdot y) \right) \Phi(X),
\]

\[
\omega^\Upsilon_\psi (w'_{\Upsilon}) \Phi(X) = \alpha_\psi(w'_{\Upsilon}) \hat{\Phi}(X),
\]

where \( \alpha_\psi(\cdot) \) is a certain eighth root of unity, which satisfies \( \alpha_\psi(\cdot)^{-1} = \alpha_{\psi^{-1}}(\cdot) \).
Let $V_0 \subset V$ be the unipotent radical of the standard parabolic subgroup of $G$ with Levi $\text{GL}_1(E)^{n-1} \times U_{2n+2}$. Then the map

$$v \mapsto v_H := \left( (v_{n,n+j})_{j=1,\ldots,2n}, \frac{1}{2} \text{Tr}_{E/F}(v_{n,3n+1}) \right)$$

gives an isomorphism from $V/V_0$ to a Heisenberg group $H_Y$. Then we may regard $\omega_{Y}\psi$ as a representation of $\tilde{\text{Sp}}(Y) \ltimes V/V_0$. Further, extend $\omega_{Y}\psi$ to a representation $\omega_{\psiNM}$ of $V \ltimes G'$ by setting

$$\omega_{\psi}(vg)\Phi = \psi_v(v)\omega_{\psi}(v_H)(\omega_{\psi}(g)\Phi), \quad v \in V, g \in G'.$$

### 2.6. Stable integral

Many integrals studied in this article do not absolutely converge. In order to study such integrals, we shall use the notion of stable integrals as in [33]. For the convenience of the reader, we shall recall the notion of a stable integral. Suppose that $F$ is a non-Archimedean local field. Let $U_0$ be a unipotent subgroup over $F$ with a fixed Haar measure on $U_0$.

**Definition 2.1** (Definition 2.1 in [33]). Let $f$ be a smooth function on $U_0$. We say that $f$ has a stable integral over $U_0$ if there exists $U_1 \in \text{CSGR}(U_0)$ such that for any $U_2 \in \text{CSGR}(U_0)$ containing $U_1$, we have

$$\int_{U_2} f(u) \, du = \int_{U_1} f(u) \, du. \quad (2.6.1)$$

In this case, we write $\int_{U_0}^{\text{st}} f(u) \, du$ for the common value (2.6.1) and say that $\int_{U_0}^{\text{st}} f(u) \, du$ stabilises at $U_1$. In other words, $\int_{U_0}^{\text{st}} f(u) \, du$ is the limit of the net $(\int_{U_1} f(u) \, du)_{U_1 \in \text{CSGR}(U_0)}$ with respect to the discrete topology of $\mathbb{C}$.

We also define a uniformly stable integral (see [33, p. 10])

**Definition 2.2.** Given a family of functions $f_x \in C^\text{sm}(U_0)$, we say that the integral $\int_{U_0}^{\text{st}} f_x(u) \, du$ stabilises uniformly in $x$ if $U_1$ as in Definition 2.1 can be chosen independently of $x$. Similarly, if $x$ ranges over a topological space $X$, then we say that $\int_{U_0}^{\text{st}} f_x(u) \, du$ stabilises locally uniformly in $x$ if any $y \in X$ admits a neighbourhood on which $\int_{U_0}^{\text{st}} f_x(u) \, du$ stabilises uniformly.

### 3. Explicit local descents and a certain local identity

#### 3.1. Local Fourier–Jacobi transform

For any $f \in C(G)$ and $s \in \mathbb{C}$, define $f_s(g) = f(g)v(g)^s$, $g \in G$. Let $\pi \in \text{Irr}_{\text{genM}}$, with Whittaker model $W_{\psiNM}(\pi)$. Let $\text{Ind}(W_{\psiNM}(\pi))$ be the space of $G$-smooth left $U$-invariant functions $W: G \rightarrow \mathbb{C}$ such that for all $g \in G$, the function $\delta_{\rho}^{\frac{1}{2}}(m)W(mg)$ on $M$ belongs to $W_{\psiNM}(\pi)$. For any $s \in \mathbb{C}$ we have a representation $\text{Ind}(W_{\psiNM}(\pi), s)$ on the space
Ind($\mathbb{W}^{N_M}(\pi)$) given by $(I(s, g) W)_s(x) = W_s(xg)$, $x, g \in G$. It is equivalent to the induced representation of $\pi \otimes \psi^*$ from $P$ to $G$. The family $W_s$, $s \in \mathbb{C}$, is a holomorphic section of this family of induced representations.

For any $W \in C^{sm}(N \setminus B, \psi_N)$ and $\Phi \in \mathcal{S}(E^n)$, define a function on $G'$ by

$$A^{\psi, \gamma}(W, \Phi, g, s) := \int_{V_{\gamma^{-1}} V} W_s(\gamma v g) \omega_{N_M}^{-1}(vg) \Phi(\xi_n) \, dv, \quad g \in G',$$

where $\xi_n, \gamma$, $V$ and $V_{\gamma}$ are defined in Section 2.1. Its properties were studied in [32, Lemma 5.1]. In particular, it satisfies

$$A^{\psi, \gamma}(I(s, vx) W, \omega_{N_M}^{-1}(vx) \Phi, g, s) = A^{\psi, \gamma}(W, \Phi, gx, s) \quad (3.1.1)$$

for any $g, x \in G'$ and $v \in V$ (see [32, (5.2)]). Further, the integrand of the definition is compactly supported, and $A^{\psi, \gamma}$ gives rise to a $V \ltimes G$-intertwining map

$$A^{\psi, \gamma} : C^{sm}(N \setminus G, \psi_N) \otimes \mathcal{S}(E^n) \rightarrow C^{sm}(N' \setminus G', \psi_{N'})$$

where $V \ltimes G'$ acts via $V \ltimes \eta(G')$ by right translation on $C^{sm}(N' \setminus G', \psi_{N'})$.

In order to simplify the notation, we introduce the map $A^{\psi, \gamma}_{\#}$ as follows. Let $V_+ \subset V$ be the image under $\ell_M$ of the space of $n \times n$ matrices whose rows are zero except possibly for the last one. For $c = \ell_M(x) \in V_+$, denote by $\underline{c} \in E^n$ the last row of $x$. Then we have

$$A^{\psi, \gamma}(W(c), \Phi(\underline{c} + \underline{s}), g) = A^{\psi, \gamma}(W, \Phi, g), \quad c \in V_+, g \in G'.$$

This implies that $A^{\psi, \gamma}(W, \Phi, \cdot)$ factors through $W \otimes \Phi \mapsto \Phi * W$, where for any function $f \in C^{\infty}(G)$ we set

$$\Phi * f(g) = \int_{V_+} f(gc) \Phi(c) \, dc.$$

Then define the map

$$A^{\psi, \gamma}_{\#} : C^{sm}(N \setminus G, \psi_N) \rightarrow C^{sm}(N' \setminus G', \psi_{N})$$

by

$$A^{\psi, \gamma}_{\#}(W, \cdot) = A^{\psi, \gamma}(W, \Phi, \cdot) \quad (3.1.2)$$

for any $\Phi \in \mathcal{S}(E^n)$ such that $\Phi * W = W$. Note that $A^{\psi, \gamma}_{\#}$ is not invariant by $G'$ and $V$, but it has the following equivariance property. Set

$$\psi_{V_{\gamma}}(vu) = \psi_{N_M}(v)^{-1} \psi_U(u).$$

Recall that $\psi_U(\ell(x)) = \psi\left(\frac{1}{2}(x_{n,n+1} - x_{2n,1})\right)$.

**Lemma 3.1** (cf. Lemma 3.1 in [33]). For any $v \in V_-$, $p = g'(u)u \in P'$ where $g \in M'$ and $u \in U'$, we have

$$A^{\psi, \gamma}_{\#}(W(v \eta(p)), g) = v'(m) \frac{1}{2} \alpha_{M'}(m) \alpha_{N_M}^{-1}(m)^{-1} \psi_{V_{\gamma}}(v) A^{\psi, \gamma}_{\#}(W, gp).$$
Proof. This is proved in the same argument as the proof of [33, Lemma 3.1]. Indeed, for \( \Phi \in \mathcal{S}(E^n) \) with sufficiently small support and \( \int_{E^n} \Phi(x) \, dx = 1 \), we can show that by (2.5.2), (2.5.3), (2.5.6) and (2.5.7),

\[
\Phi' \ast W (\cdot vp) = \psi V_{\cdot} (v)\alpha_{\psi^{-1}}(m) v'(m)^{-\frac{1}{2}} \Upsilon_{M'}(m)^{-1} W (\cdot vp),
\]

where \( \Phi' = \alpha_{\psi^{-1}}^{-1}(vp) \Phi \). This identity shows that

\[
A^\psi,\Upsilon (W (\cdot v \eta(p)), \Phi', g) = \alpha_{\psi^{-1}}^{-1}(m) \Upsilon^{-1} \psi V_{\cdot} (v)^{-1} A^\psi,\Upsilon (W (\cdot v \eta(p)), g).
\]

Further, from the definition, we have

\[
A^\psi,\Upsilon (W (\cdot v \eta(p)), \Phi', g) = A^\psi,\Upsilon (W, \Phi, gp),
\]

and the lemma follows from these two identities.

As in [33, Lemma 3.2], we have the following result, which is proved as in the proof of [34, Lemma 4.5] (see also [32, Lemma 5.1]):

**Lemma 3.2.** For any \( K_0 \in \mathcal{CSGR}(G) \), there exists \( \Omega \in \mathcal{CSGR}(V_U) \) such that for any \( W \in C(N \setminus G, \psi_N)^{K_0} \), the support of \( W (\gamma \cdot) \psi^{-1} \) is contained in \( V_{\gamma \eta(w_U')} \Omega^{-1} \).

### 3.2. Explicit local descent

Define the intertwining operator

\[
M(\pi, s) = M(s) : \text{Ind} \left( \mathbb{W}^{\psi_{NM}}(\pi), s \right) \to \text{Ind} \left( \mathbb{W}^{\psi_{NM}}(\pi^\vee), -s \right)
\]

by (the analytic continuation of)

\[
M(s) W(g) = v(g)^s \int_U W_s(\varphi(t) w_U u g) \, du,
\]

where \( t = \text{diag}(1, -1, \ldots, 1, -1) \) is introduced in order to preserve the character \( \psi_{NM} \) and

\[
w_U = \begin{pmatrix} -1_{2n} & 1_{2n} \end{pmatrix}.
\]

Then we know that \( M(s) \) is holomorphic at \( s = \frac{1}{2} \) when \( \pi \in \text{Irr}_{\text{gen}} M \) is of unitary type by [32, Proposition 2.1]. By abuse of notation, we will also denote by \( M(\pi, s) \) the intertwining operator \( \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi), s) \to \text{Ind}(\mathbb{W}^{\psi_{NM}}(\pi^\vee), -s) \), defined in the same way.

For simplicity, denote \( M^*_s W := (M(s) W)_s \), so that

\[
M^*_s W = \int_U W_s(\varphi(t) w_U u \cdot) \, du
\]

for \( \text{Re}(s) \gg 1 \). Set \( M^* W := M^*_s W \).
Definition 3.1 (Explicit local descent). Suppose that \( \pi \in \text{Irr}_{\text{gen}} \mathbb{M} \) is of unitary type. Then denote by \( D_{\psi}^Y(\pi) \) the space of the Whittaker function on \( G' \) generated by

\[
A_{\psi,Y} \left( M \left( \frac{1}{2} \right) W, \Phi, \cdot, -\frac{1}{2} \right), \quad W \in \text{Ind}(\mathbb{W}_{\psi}^{\psi_{N,M}}(\pi)), \Phi \in \mathcal{S}(E^n).
\]

Note that from [32, Proposition 2.1] and the comment below [32, Lemma 5.1], \( D_{\psi}^Y(\pi) \) is not zero.

3.3. Good representations

Let \( \pi \in \text{Irr}_{\text{gen}} \mathbb{M} \) and \( \sigma' \in \text{Irr}_{\text{gen}, \psi^{-1}} G' \) with the Whittaker model \( \mathbb{W}_{\psi}^{-1}(\sigma) \). For any \( W' \in \mathbb{W}_{\psi}^{-1}(\sigma') \) and \( W \in \text{Ind}(\mathbb{W}_{\psi}^{\psi_{N,M}}(\pi)) \), define

\[
J(W', W, s) := \int_{N' \setminus G'} W'(g') A_{\#}^{\psi,Y}(W_s, g') \, dg'.
\]

By [6, Proposition 3.1], \( J \) converges for \( \text{Re}(s) \gg_{\pi, \sigma'} 1 \) and admits a meromorphic continuation in \( s \). Moreover, by [6, Proposition 4.1], for any \( s \in \mathbb{C} \) we can choose \( W \) and \( W' \) such that \( J(W', W, s) \neq 0 \).

Definition 3.2. Let \( \pi \in \text{Irr}_{\text{gen,ut}} \mathbb{M} \). Say \( \pi \) is good if it satisfies the following three conditions:

1. \( D_{\psi}^Y(\pi) \) and \( D_{\psi}^{-1}(\psi(\pi)) \) are irreducible. Write \( \sigma = D_{\psi}^Y(\pi) \) and \( \sigma' = D_{\psi}^{-1}(\psi(\pi)) \).
2. \( J(W', W, s) \) is holomorphic at \( s = \frac{1}{2} \) for any \( W' \in D_{\psi}^{-1}(\psi(\pi)) \) and \( W \in \text{Ind}(\mathbb{W}_{\psi}^{\psi_{N,M}}(\pi)) \).
3. There exists a nondegenerate \( G' \)-invariant bilinear form \( [\cdot, \cdot]_{\sigma'} \) on \( D_{\psi}^{-1}(\psi(\pi)) \times D_{\psi}^Y(\pi) \) such that

\[
J \left( W', W, \frac{1}{2} \right) = \left[ W', A_{\#}^{\psi,Y}(M^* W, \cdot) \right]_{\sigma'}.
\]

Good representations appear as a local component of certain automorphic representations as follows. Let \( K \) be a number field and \( L \) be a quadratic extension of \( K \). For an irreducible cuspidal automorphic representation \( \pi \) of \( GL_m(A_L) \), whose central character is trivial on \( A_K^{\times} \), say that \( \pi \) is of unitary type if

\[
\int_{A_K^{\times} \backslash GL_m(K) \backslash GL_m(A_K)} \varphi(h) \, dh \neq 0
\]

for some \( \varphi \in \pi \). Write \( \text{Cusp}_{\text{uni}} GL_m \) for the set of irreducible cuspidal automorphic representations of \( GL_m(A_L) \) of unitary type. Consider the set \( \text{U} \text{Cusp} \mathbb{M} \) of isobaric automorphic representations \( \pi_1 \boxplus \cdots \boxplus \pi_k \) of pairwise inequivalent \( \pi_i \in \text{Cusp}_{\text{uni}} GL_{n_i} \) for \( 1 \leq i \leq k \) and \( n_1 + \cdots + n_k = 2n \). Then we have the following crucial fact, which holds without any assumption as remarked in Remark 1.1:

Theorem 3.1 (Theorem 5.5 in [32]). If \( \pi \in \text{U} \text{Cusp} \mathbb{M} \), then its local components \( \pi_v \) are good.
The definition of good representations was introduced and discussed in [32, Section 5]. In particular, we know that if \( \pi \) is good, then there exists a constant \( c_\pi \) such that for any \( W' \in \mathbb{W}_N^1(\sigma') \) and \( (W')^\vee \in \mathbb{W}_N^1(\sigma) \), we have
\[
\int_{N'} \langle \sigma'(n) W', (W')^\vee \rangle_{\sigma'} \psi_{N'}(n) \, dn = c_\pi \, W'(e)(W')^\vee(e).
\]
(3.3.1)

More explicitly, for any \( W \in \text{Ind}(\mathbb{W}_{NM}^1(\pi)) \) and \( W^\vee \in \text{Ind}(\mathbb{W}_N^1(\pi)) \) we have
\[
\int_{N'} J \left( A^\psi_{\#, Y^{-1}}(M^* W^\vee, n), W, \frac{1}{2} \right) \psi_{N'}(n) \, dn = c_\pi A^\psi_{\#, Y^{-1}}(M^* W^\vee, e) A^\psi_{\#, Y}(M^* W, e).
\]
(3.3.2)

In the rest of this article will prove the following statement:

**Theorem 3.2.** For any unitarisable \( \pi \in \text{Irr}_{\text{gen,ut}\mathbb{M}} \) which is good, we have \( c_\pi = \omega_{\pi}(\tau) \).

**Remark 3.1.** As remarked in Section 1, from the definition, we have \( \omega_{\pi}(\tau) = \omega_{\sigma'}(-1) = \omega_{\sigma'}(-1) \).

The following special case will be used to prove the formal degree conjecture:

**Corollary 3.1.** Let \( \pi \) be an irreducible representation of \( \text{GL}_{2n}(E) \) of the form \( \pi = \tau_1 \times \cdots \times \tau_k \), where \( \tau_i \) are mutually inequivalent irreducible discrete series representations of \( \text{GL}_{n_i}(E) \) such that \( n = n_1 + \cdots + n_k \) and \( L(s, \tau_i, \text{As}) \) has a pole at \( s = 0 \). Then \( \pi \) is good and \( \sigma' = D^Y_{\psi^{-1}}(c(\pi)) \) is a discrete series. Moreover,
\[
\int_{N'} J \left( W, W', \frac{1}{2} \right) \psi_{N'}(n) \, dn = \omega_{\sigma'}(-1) \, W(e) A^\psi_{\#, Y}(M^* W', e)
\]
for any \( W \in \mathbb{W}_N^1(\sigma') \) and \( W' \in \text{Ind}(\mathbb{W}_{NM}^1(\pi)) \).

**Proof.** By Theorem A.5, we find that \( \sigma' \) is a discrete series representation. Moreover, by [20, Corollary A.6] (see also the proof of Theorem A.5), we see that \( \pi \) is good. Then the assertion follows from Theorem 3.2. \( \square \)

### 3.4. A reduction to the tempered case

In this section, we shall reduce Theorem 1.2 to the case of tempered good representations. For this reduction, we shall need some lemmas concerning analytic properties of local integrals. Using the same argument as the proof of [34, Lemma 4.6], we can prove the following lemma using [50, Proposition 2.1]:

**Lemma 3.3.** Suppose that \( \pi \) is an irreducible generic admissible representation of \( \mathbb{M} \) and \( \alpha \in \mathbb{R} \) with \( \alpha > e(\pi) \). Then for any \( W \in \text{Ind}(\mathbb{W}_{NM}^1(\pi)) \) and \( \Phi \in \mathcal{S}(E^n) \), there exists \( c > 0 \) such that for any \( t' = q'(t) \in T' \) with \( t = (t_1, \ldots, t_n) \in \mathbb{M}' \), \( k \in K' \) and \( s \in \mathbb{C} \), we have
\[
A^\psi_{\#}(W, \Phi, q'(t) k', s) = 0 \quad \text{unless} \quad |t_i| \leq c|t_{i+1}| \quad \text{for} \quad i = 1, \ldots, n \quad \text{with} \quad t_{n+1} = 1, \quad \text{in which case}
\]
\[
|A^\psi_{\#}(W, \Phi, q'(t) k', s)| \ll \delta_{B'}(t')^{\frac{1}{2}} |\text{det}(t)|^{\text{Re}(s)+\frac{1}{2}-\alpha}.
\]
Similar to the proof of [34, Lemma 4.12], the following lemma readily follows from Lemma 3.3.

**Lemma 3.4.** Suppose that \( \sigma_0 \in \text{Irr}_{\psi^{-1}, \text{gen}} G' \) is an essentially discrete series representation and \( \alpha = e(\pi) \). Then there exists \( \delta > 0 \) such that \( J(W', W, \Phi, s) \) is absolutely convergent locally uniformly (hence, holomorphic) for \( \text{Re}(s) \geq \alpha - \delta - \frac{1}{2} \), where \( W' \in \mathcal{W}_{N'}^{-1}(\sigma_0) \), \( W \in \text{Ind}(\mathcal{W}_{M'}^{N}(\pi)) \) and \( \Phi \in \mathcal{S}(E^n) \). Similarly, for any \( W \in \text{Ind}(\mathcal{W}_{M'}^{N}(\pi)) \), \( W^\vee \in \text{Ind}(\mathcal{W}_{M'}^{N}(\pi)) \), and \( \Phi, \Phi^\vee \in \mathcal{S}(E^n) \),

\[
J \left( A^{\psi^{-1}, \mathcal{Y}^{-1}} \left( M \left( \frac{1}{2} \right), \right. \Phi^\vee, \cdot, \left. - \frac{1}{2} \right), W, \Phi, s \right)
\]

converges absolutely and locally uniformly (and hence, is holomorphic) for \( \text{Re}(s) > 2\alpha - \frac{1}{2} \). In particular, if \( \pi \) is unitarizable, then \( J(W', W, \Phi, s) \) is holomorphic at \( s = \frac{1}{2} \) for any \( W' \in \mathcal{W}_{N'}^{-1}(\mathcal{D}_{\psi^{-1}}(\pi)) \), \( W \in \text{Ind}(\mathcal{W}_{M'}^{N}(\pi)) \) and \( \Phi \in \mathcal{S}(E^n) \).

When the base field \( F \) is a real field, a similar result can be proven as with [32, Lemma 4.11], using the same argument as that proof. For later use, we state them here.

**Lemma 3.5.** Suppose that \( \pi \) is an irreducible generic admissible representation of \( \mathfrak{M} \) and \( \alpha \in \mathbb{R} \) with \( \alpha > e(\pi) \). Then for any \( W \in \text{Ind}(\mathcal{W}_{M'}^{N}(\pi)) \) and \( \Phi \in \mathcal{S}(E^n) \), there exists a compact set \( D \subset \mathbb{C} \) and \( m \geq 1 \), and we have

\[
|A^{\psi, \mathcal{Y}}(W, \Phi, q'(t) k', s)| \ll \delta_{D'}(t')^{\frac{1}{2}} |\det(t)|^{\text{Re}(s) + \frac{1}{2} - \alpha (1 + |t_1 t_{i+1}^{-1}|)^{-m}}
\]

for any \( t' = q'(t) \in T' \), with \( t = (t_1, \ldots, t_n) \in \mathfrak{M}' \), \( k \in K' \) and \( s \in D \).

**Remark 3.2.** By the same argument as the proof of [32, Lemma 4.12] with Lemma 3.5, Lemma 3.4 holds over a real field.

Suppose that the base field \( F \) is a non-Archimedean local field. Then the aim in this section is to prove the following reduction:

**Proposition 3.1.** Suppose that (1.0.1) holds for any good \( \pi \in \text{Irr}_{\text{temp, ut}} \mathfrak{M} \) such that \( \mathcal{D}_{\psi^{-1}}(\pi) \) is tempered. Then Theorem 3.2 holds.

For a proof, recall the following classification of generic representations of unitary type by Matringe.

**Theorem 3.3** (Theorem 5.2 in [37]). The set \( \text{Irr}_{\text{gen, ut}} \mathfrak{M} \) consists of the irreducible representations of the form

\[
\pi = c(\sigma_1) \times \sigma_1^\vee \times \cdots \times c(\sigma_k) \times \sigma_k^\vee \times \delta_1 \times \cdots \times \delta_l,
\]

where \( \sigma_1, \ldots, \sigma_k \) are essentially square-integrable representations and \( \delta_1, \ldots, \delta_l \) are square-integrable representations of unitary type (i.e., \( L(0, \tau_i, A) = \infty \) for all \( i \)). Here, \( \times \) denotes the parabolic induction of \( \text{GL}_m \).
We prove the reduction following the argument of [33, Section 3F].

**Proof.** Let $\pi \in \text{Irr}_{\text{gen,ut}}M$ be unitarizable and good. Then by Theorem 3.3, we can write

$$\pi = c(\sigma_1)[a_1] \times \sigma_1^\vee[-a_1] \times \cdots \times c(\sigma_k)[a_k] \times \sigma_k^\vee[-a_k] \times \delta_1 \times \cdots \times \delta_l,$$

where $\sigma_i \in \text{Irr}_{\text{sq}}GL_{n_i}(E)$, $\delta_i \in \text{Irr}_{\text{sq}}GL_{m_i}(E)$ and $(a_1, \ldots, a_k) \in \mathbb{C}^k$ is in the domain

$$\mathcal{D} = \{(s_1, \ldots, s_k) \in \mathbb{C}^k : -\frac{1}{2} < \text{Res}_i < \frac{1}{2} \text{ for all } i\}.$$

Here, write $\sigma[s]$ for the twist of $\sigma \in \text{Irr}GL_m(E)$ by $|\det|^s$, with $s \in \mathbb{C}$. For $\underline{s} = (s_1, \ldots, s_k) \in \mathbb{C}^k$, consider the representation

$$\pi(\underline{s}) := c(\sigma_1)[s_1] \times \sigma_1^\vee[-s_1] \times \cdots \times c(\sigma_k)[s_k] \times \sigma_k^\vee[-s_k] \times \delta_1 \times \cdots \times \delta_l,$$

which is irreducible for $\underline{s} \in \mathcal{D}$ by [54, Theorem 9.7].

Take a quadratic extension of number fields $L/K$ such that for some finite place $v_0$, $K_{v_0} = F$ and $L_{v_0} = E$. Let $\rho \in \text{Irr}_{\text{sq},\text{gen}}U_m$ be the representation corresponding to $\delta_1 \times \cdots \times \delta_l$ under the correspondence established in Theorem 3.4, with $m = m_1 + \cdots + m_l$. Then set

$$\sigma(\underline{s}) := c(\sigma_1)[s_1] \times \sigma_1^\vee[-s_1] \times \cdots \times c(\sigma_k)[s_k] \times \sigma_k^\vee[-s_k] \times \rho.$$

By [27, Corollary 4.3], for a dense open subset of $\underline{s} \in i\mathbb{R}^k$, $\sigma(\underline{s}) \in \text{Irr}_{\text{temp}}U_{2n}$. Further, by [20, Corollary A.8], for a dense open subset of $\underline{s} \in i\mathbb{R}^k$, $\sigma(\underline{s})$ is a local component of an irreducible cuspidal automorphic representation of $U_{2n}(\mathbb{A}_K)$. From the proof of Theorem 3.4, its base change lift to $GL_{2n}(\mathbb{A}_L)$ by Kim and Krishnamurthy [25] gives a globalisation of $\pi(\underline{s})$, and $\mathcal{D}^{-1}_{\psi^{-1}}(\pi(\underline{s})) = \sigma(\underline{s})$. In particular, by [32, Theorem 5.5], $\pi(\underline{s})$ is good for such $\underline{s}$. Further, it is tempered, since $\underline{s} \in i\mathbb{R}^k$. Note that $\omega_{\pi(\underline{s})}(\tau) = \omega_{\tau_1}(\tau) \cdots \omega_{\tau_l}(\tau) = \omega_{\pi}(\tau)$. Hence, from my assumption, for $\underline{s} \in i\mathbb{R}^k$ given earlier, we have

$$c_{\pi(\underline{s})} = \omega_{\pi(\underline{s})}(\tau) = \omega_{\pi}(\tau). \quad (3.4.1)$$

On the other hand, by the same argument as [30, 3.6], [32, (5.2)] and [30, Proposition 2.11] imply that both sides of (3.3.2) are holomorphic functions of $\underline{s} \in \mathcal{D}$. This shows that $c_{\pi(\underline{s})}$ is a meromorphic function on $\underline{s} \in \mathcal{D}$, and thus $c_{\pi(\underline{s})}$ is a constant function of $\underline{s}$ by (3.4.1). In particular, we have

$$c_{\pi} = \omega_{\pi}(\tau). \quad \square$$

Until Section 7, my aim is to prove Theorem 3.2 – that is, the Main Identity

$$\int_{N'} \left( \int_{N' \setminus G'} A_{\#}^{\phi, Y}(W_s, g) A_{\#}^{\phi^{-1}, Y^{-1}}(M^s W^\wedge, gu) dg \right) \psi_{N'}(u) \, du \bigg|_{s = \frac{1}{2}} = \omega_{\pi}(\tau) A_{\#}^{\phi^{-1}, Y^{-1}}(M^s W^\wedge, e) A_{\#}^{\phi, Y}(M^s W, e) \quad \text{(MI)}$$

under the assumptions that $\pi \in \text{Irr}_{\text{temp,ut}}M$ is good and $\sigma' := \mathcal{D}^{-1}_{\psi^{-1}}(c(\pi))$ is tempered.
4. A bilinear form

From this section to Section 7, we suppose that the base field $F$ is a non-Archimedean local field of characteristic zero. Following [32], we shall study the main identity. A key ingredient of this section is the stability of the integral defining a Bessel function, which is given in Appendix B. Indeed, we apply Theorem B.2 to the function $A_{\#}^\psi(Y, v)$ and obtain the following proposition (see also Remark B.1):

**Proposition 4.1.** Let $K_0 \in CSGR(G)$. Then the integral

$$Y_{\psi, \psi}(W, t) := \int_{N'}^{st} A_{\#}^\psi(W, w_U^t w_0^M t n) \psi_{N'}(n)^{-1} dn, \quad t \in T',$$

stabilises uniformly for $W \in C(N \setminus G, \psi_{N'})^{K_0}$ and locally uniformly in $t \in T'$. In particular, $Y_{\psi, \psi}(W, t)$ is entire in $s \in \mathbb{C}$, and if $\pi \in \text{Irr}_{\text{gen}} \mathbb{M}$ and $W \in \text{Ind}(\mathbb{W}^N_M (\pi))$, then $Y_{\psi, \psi}(M^* W, t)$ is meromorphic in $s$. Both $Y_{\psi, \psi}(W, t)$ and $Y_{\psi, \psi}(M^* W, t)$ are locally constant in $t$, uniformly in $s \in \mathbb{C}$.

Finally, assuming $\pi \in \text{Irr}_{\text{meta, gen}} \mathbb{M}$ and that $\sigma' = D_{\psi, \psi}^{-1}(\psi(\pi))$ is irreducible, then for any $W^\wedge \in \text{Ind}(\mathbb{W}^N_M (\pi))$, we have

$$Y_{\psi, \psi}(M^* W^\wedge, t) = B_{\sigma'}(w_U^t w_0^M t) A_{\#}^\psi, Y^{-1}(M^* W^\wedge, e).$$

Theorem B.2 can be used in another way, as follows, which is proved by the same argument as [33, Lemma 5.4]:

**Lemma 4.1.** Let $W' \in C^\text{sm}(N \setminus G', \psi_{N'})^\wedge$ and $(W')^\wedge \in C^\text{sm}(N \setminus G', \psi_{N'}^{-1})$. Assume that the function $(t, n) \in T' \times N' \mapsto W'(w_0^t n)$ is compactly supported. (In particular, $W' \in C^\text{sm}(N \setminus G', \psi_{N'})$. ) Then the iterated integral

$$\int_{N'}^{st} \left( \int_{N' \setminus G'} W'(g)(W')^{\wedge}(gu) dg \right) \psi_{N'}(u) du$$

is well defined and is equal to

$$\int_{T'} \left( \int_{N'} \delta_B(t) W'(w_U^t w_0^M t n) \psi_{N'}(n)^{-1} dn \right) \left( \int_{N'}^{st} (W')^{\wedge}(w_U^t w_0^M t u) \psi_{N'}(u) du \right) dt.$$

4.1.

In order to apply Lemma 4.1 for $A_{\#}^\psi(W, s)$, make a special choice of $W$. Consider the $P$-invariant subspace $\text{Ind}(\mathbb{W}^N_M (\pi))$ of $\text{Ind}(\mathbb{W}^N_M (\pi))$ consisting of functions supported in the big cell $Pw_U P = Pw_U U$. Any element of $\text{Ind}(\mathbb{W}^N_M (\pi))$ is a linear combination of functions of the form

$$W(u' m w_U u') \delta_P(m) \frac{1}{2} W^M (m) \phi(u), \quad m \in M, u, u' \in U,$$

(4.1.1)

with $W^M \in \mathbb{W}^N_M (\pi)$ and $\phi \in C^\infty_c(U)$. Let $\eta_M$ be the embedding $\eta_M(g) = (g I_n)_{U}$ of $M'$ into $\mathbb{M}$. Also let $\eta_M = \varphi \circ \eta_M$, so that $\eta_M(g) = \left(\begin{array}{cc}g & I_{2n} \\ I_n & g^* \end{array}\right)$. 

Definition 4.1. Let \( \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))_{\#}^o \) be the linear subspace of \( \text{Ind}(\mathbb{W}^{\psi N_M}(\pi)) \) generated by \( W \)'s as in (4.1.1) that satisfies the additional property that the function \((t, n) \mapsto W^M(\eta_M(tw_0^M n))\) is compactly supported on \( T'_M \times N'_M \), or equivalently that the function \( W^M \circ \eta_M \) on \( M' \) is supported in the big cell \( B'_{M'} w_{0}^M N'_M \) and its support is compact modulo \( N'_M \).

Note that this space is nonzero by the proof of [33, Lemma 6.13]. Further, this space is invariant under \( \eta(T') \ltimes N \).

Lemma 4.2. For any \( W \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))_{\#}^o \), the function \( A_{\#}^{\psi, \gamma} (W, s) \) is compactly supported in \( t \in T' \) and \( n \in N' \) uniformly in \( s \in \mathbb{C} \).

Proof. The proof is identical to that of [33, Lemma 5.6]. \( \square \)

4.2.

For \( W \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))_{\#}^o \) and \( W^\gamma \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi^\gamma)) \), define

\[ B(W, W^\gamma, s) := \int_{N'} \int_{\text{N}' \backslash G'} A_{\#}^{\psi, \gamma} (W, g) A_{\#}^{\psi^{-1}, \gamma^{-1}} (W^\gamma, g u) \psi_N(u) \, du. \]

By Lemma 4.2, we can apply the argument in the proof of [33, Lemma 5.4]. Then we have

\[ \int_{T'} \int_{N'} \int_{\Omega} \delta_B'(t) A_{\#}^{\psi, \gamma} (W, s, tw_0^M t n) A_{\#}^{\psi^{-1}, \gamma^{-1}} (W^{-\gamma}, s, tw_0^M t n u) \psi_N(u) \, du \, dn \, dt \]

for any sufficiently large \( \Omega \in \mathcal{CSGR}(N') \). This implies that \( B(W, W^\gamma, s) \) is an entire function of \( s \), and from Proposition 4.1 we have

\[ B(W, W^\gamma, s) = \int_{T'} Y^{\psi, \gamma} (W, s, t) Y^{\psi^{-1}, \gamma^{-1}} (W^{-\gamma}, s, t) \delta_B'(t) \, dt \quad (4.2.1) \]

for any \( W \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))_{\#}^o \) and \( W^\gamma \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi^\gamma)) \).

Assume that \( \pi \in \text{Irr}_{\text{gen}, uM} \) and that \( \sigma = D_{\psi^{-1}}^\gamma (c(\pi)) \) is irreducible. Then for \( W \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))_{\#}^o \) and \( W^\gamma \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi^\gamma)) \),

the left-hand side of the Main Identity (MI) is \( B \left( W, M \left( \frac{1}{2} W^\gamma, \frac{1}{2} \right) \right) \quad (4.2.2) \)

5. Further analysis

Fix an element \( \varepsilon_1 \) of the form \( \ell_M(X) \), where \( X \in \text{Mat}_{n \times n} \) and the last row of \( X \) is \( -\xi_n \). Then we can check that

\[ \psi_{V_\gamma}(v^{-1} v \varepsilon_1) = \psi_{V_\gamma}(v) \psi(v_{n, 2n+1}). \]

For any \( W \in C^s m(N \backslash G, \psi_N) \), define

\[ A_\psi^W (W) := \int_{V_{\gamma} \backslash V_\gamma} W(\gamma v \varepsilon_1) \psi_{V_\gamma}(v^{-1} v \varepsilon_1)^{-1} \, dv. \quad (5.0.1) \]
Lemma 5.1. For any \( W \in C^\text{sm}(N \setminus G, \psi_N) \), the integrand in (5.0.1) is compactly supported on \( V_p \setminus V \), and we have \( A_{\pi, \gamma}^\psi(W, e) = A_{e}^\psi(W) \).

Proof. The proof is identical to the proof of [33, Lemma 6.1], because of Lemma 3.2 and (2.5.1)–(2.5.3).

Remark 5.1. The definition of \( A_{\pi, \gamma}^\psi(W, \cdot) \) depends on the choice of \( \gamma \), and its invariance also depends on \( \gamma \) by Lemma 3.1. In Lemma 5.1, \( A_{\pi, \gamma}^\psi(W, \cdot) \) is evaluated only at \( e \), and we see that this value is independent of the choice of \( \gamma \).

We now explicate \( A_{\pi, \gamma}^\psi(W_s, \cdot) \) on the big cell \( N'w_U', P' \). By Lemma 3.1, it is enough to consider the element \( w_U' \).

Lemma 5.2. Let \( \pi \in \text{Irr}_{\text{gen}} M \). Then for \( \text{Res} \gg_{\pi} 1 \) and \( W \in \text{Ind}(\mathbb{V}^{\psi N_M}_{\pi}(\pi)) \), we have

\[
A_{\pi, \gamma}^\psi(W_s, w_U') = \alpha_{\psi^{-1}}(w_U') \int_{V_U} W_s(w_U v) \psi_U(v)^{-1} dv = \alpha_{\psi^{-1}}(w_U') \int_{V_M^\pi \setminus V_{-}} W_s(w_U v) \psi_{V_+}(v)^{-1} dv. \quad (5.0.2)
\]

Proof. The proof is identical to that of [33, Lemma 6.3].

5.1.

We have the following convergence result, whose proof is identical to that of [33, Lemma 6.8]:

Lemma 5.3. Let \( \pi \in \text{Irr}_{\text{gen}} M \). Then for \( \text{Res} \gg_{\pi} 1 \), we have

\[
\int_{N_{\mathbb{V}^{\psi N_M}_{\pi}(\pi)}} \int_{U} |W_s(\varrho(n)w_U g)| \, du \, dn < \infty
\]

and

\[
\int_{N_{\mathbb{V}^{\psi N_M}_{\pi}(\pi)}} \int_{U} |W_s(\varrho(n)w_U g)| \, du \, dn < \infty
\]

for any \( W \in \text{Ind}(\mathbb{V}^{\psi N_M}_{\pi}(\pi)), \ g \in G \).

Because of this lemma, we can now explicate \( Y_{\pi, \gamma}^\psi(W_s, t) \) for \( \text{Res} \gg 1 \).

Lemma 5.4. Let \( \pi \in \text{Irr}_{\text{gen}} M \). For \( \text{Res} \gg_{\pi} 1 \) and any \( W \in \text{Ind}(\mathbb{V}^{\psi N_M}_{\pi}(\pi)), \ t \in T' \), we have the identity

\[
Y_{\pi, \gamma}^\psi(W_s, t) = v'(t)^{n-\frac{1}{2}} Y_{M'}(t)^{-1} \alpha_{\psi^{-1}}(w_U') \alpha_{\psi^{-1}}(w_{0}) \int_{V_M^\pi \setminus N^\#} W_s(w_U \eta(w_{0}' t) v) \psi_{N^\#}(v)^{-1} dv,
\]

where the right-hand side is absolutely convergent.
Proof. This lemma readily follows from Lemmas 3.1, 5.2 and 5.3, as in the proof of [33, Lemma 6.6].

5.2.

We now go back to the bilinear form $B$. The following lemma is proved by the same argument as the proof of [33, Lemma 6.10]:

Lemma 5.5. For $W \in \text{Ind}(\mathcal{W}^{\psi NM}(\pi))^{0}_{#}$, the integrand on the right-hand side of (5.1.1) is compactly supported in $t, v$ uniformly in $s$ (i.e., the support in $(t, v)$ is contained in a compact set which is independent of $s$). In particular, (5.1.1) holds for all $s \in \mathbb{C}$.

Lemma 5.6. Let $\pi \in \text{Irr}_{\text{gen}}M$. Then for $-\text{Re}(s) \gg 1$, we have

$$
B(W, W^{\vee}, s) = \int_U \int_{(N M')^\#} \int_{N M' \backslash M} W_s(\eta_M(g) w_U v) W_{-s}(\eta_M(g) w_U u) \psi_N^{#}(v)^{-1} \psi_U(u) \, dg \, dv \, du
$$

for any $W \in \text{Ind}(\mathcal{W}^{\psi NM}(\pi))^{0}_{#}$, $W^{\vee} \in \text{Ind}(\mathcal{W}^{\psi NM}(\pi^{\vee}))$, with the integral being absolutely convergent.

Proof. This is proved by a similar argument as the proof of [33, Lemma 6.11], except for some minor differences. For the sake of completeness, we repeat that argument in this setting.

Suppose that $-\text{Re}(s) \gg 1$. Then by (4.2.1), Lemmas 5.4 and 5.5 and the fact that

$$
\delta_{B'}(t) = \delta_{B'M'}(t) t^n, \quad t \in T',
$$

$B(W, W^{\vee}, s)$ is equal to

$$
\int_{T'_{M'}} \int_{\eta(N_{M'})^\times U} \int_{\eta(N_{M'})^\times U} W_s(\eta_M((w_0 M')^* t^{*}) w_U v_1) W_{-s}(\eta_M((w_0 M')^* t^{*}) w_U v_2) |\det t|^{3n-1} \psi_N^{#}(v_1)^{-1} \psi_N^{#}(v_2) \, dv_1 \, dv_2 \, dt,
$$

where the integral is absolutely convergent. By the change of variable, $B(W, W^{\vee}, s)$ is equal to

$$
\int_{T'_{M'}} \int_{N_{M'}^0} \int_{N_{M'}^0} \int_{U^0} W_s(\eta_M((w_0 M') t n)^* w_U w_1) W_{-s}(\eta_M((w_0 M') t n)^* w_U u) |\det t|^{3n-1} \psi_N^{#}(v_1)^{-1} \, dv_1 \, dn \, dt.
$$

Finally, by the Bruhat decomposition and the fact that

$$
\delta_{P}(\eta_M(g)) = |\det g|^{2n}, \quad \text{for any } g \in M',
$$

the lemma readily follows.
Define when convergent
\[
\{ W, W^\vee \} := \int_{N_M \setminus M'} W(\eta_M(g)) W^\vee(\eta_M(g)) \delta_P(\eta_M(g))^{-1} \det g^{1-n} \, dg,
\]
which converges for any \((W, W^\vee) \in \text{Ind}(W^{\psi N_M}(\pi)) \times \text{Ind}(W^{\psi N_M^{-1}}(\pi^\vee))\) when \(\pi\) is unitarizable by [29, Lemma 1.2]. Then for any \(W \in \text{Ind}(W^{\psi N_M}(\pi))^\circ_{\#}\), \(W^\vee \in \text{Ind}(W^{\psi N_M^{-1}}(\pi^\vee))\) and \(-\text{Re}(s) \gg 1\), we get
\[
B(W, W^\vee, s) = \int_U \int_{V_M \setminus N} \{ W_s(\cdot w_U v), W^\vee_s(\cdot w_U u) \} \psi_{N^\#}(v)^{-1} \psi_U(v) \, dv \, du. \quad (5.2.1)
\]

5.3.

By the same argument as the proof of [33, Lemma 6.13], we can prove the following lemma using Theorem B.3, Proposition 4.1 and Lemma 5.5 instead of [33, Theorem A.1, Corollary 5.3, Lemma 6.10]:

**Lemma 5.7.** Assume that \(\pi \in \text{Irr}_{\text{gen, ut}} M\) and that \(\sigma' = D^{T-1}_{\psi^{-1}}(c(\pi))\) is irreducible and tempered. Then the bilinear form \(B(W, M(\frac{1}{2}) W^\wedge, \frac{1}{2})\) is not identically zero on \(W \in \text{Ind}(W^{\psi N_M}(\pi))^\circ_{\#} \times \text{Ind}(W^{\psi N_M^{-1}}(c(\pi)))\).

By (4.2.2) and Lemmas 5.1 and 5.7, we have the following conclusion:

**Corollary 5.1.** Suppose that \(\pi \in \text{Irr}_{\text{ut, temp}} M\) is good and \(\sigma' = D^{T-1}_{\psi^{-1}}(c(\pi))\) is tempered. Then
\[
B \left( W, M \left( \frac{1}{2} \right) W^\wedge, \frac{1}{2} \right) = c_\pi A^\psi_e(M^* W) A^\psi_{e^{-1}}(M^* W^\wedge)
\]
for all \(W \in \text{Ind}(W^{\psi N_M}(\pi))^\circ_{\#}\) and \(W^\vee \in \text{Ind}(W^{\psi N_M^{-1}}(c(\pi)))\). Moreover, the linear form \(A^\psi_e(M^* W)\) does not vanish identically on \(\text{Ind}(W^{\psi N_M}(\pi))^\circ_{\#}\).

In other words, because of Proposition 3.1, Theorem 3.2 is reduced to the following statement:

**Proposition 5.1.** Assume that \(\pi \in \text{Irr}_{\text{ut, temp}} M\) is good and \(\sigma' = D^{T-1}_{\psi^{-1}}(c(\pi))\) is tempered. Then for any \(W \in \text{Ind}(W^{\psi N_M}(\pi))^\circ_{\#}\) and \(W^\wedge \in \text{Ind}(W^{\psi N_M^{-1}}(c(\pi)))\), we have
\[
B \left( W, M \left( \frac{1}{2} \right) W^\wedge, \frac{1}{2} \right) = \omega_\pi(\tau) A^\psi_e(M^* W) A^\psi_{e^{-1}}(M^* W^\wedge). \quad (5.3.1)
\]

6. Application of functional equations

Define \(B(W, W^\vee, s)\) to be the right-hand side of (5.2.1) whenever the integral defining \(\{\cdot, \cdot\}\) and the double integrals over \(V_M \setminus N\#\) and \(U\) are absolutely convergent. Clearly for \(g \in M'\), with \(|\det g| = 1\),
\[
\{ W(\cdot \eta_M(g)), W^\vee(\cdot \eta_M(g)) \} = \{ W, W^\vee \}.
\]
Then, as in [33, (7.1)], $B(W, W^\vee, s)$ is equal to
\[
\int_{N_M'} \int_U \int_U \{ W_s(-\eta_M(n)w_U v), W_s^\vee(-w_U u) \} \psi_U(v)^{-1} \psi_U(u) \psi_{N_M'}(n) \, dv \\ du \\ dn \\ dv \\ dn \\
= \int_{N_M'} \int_U \int_U \{ W_s(-w_U v), W_s^\vee(-\eta_M(n)w_U u) \} \psi_U(v)^{-1} \psi_U(u) \psi_{N_M'}(n)^{-1} \\ dv \\ du \\ dn \\ dv \\ dn \\
= \int_{V_M^\vee \setminus N^\#} \int_U \{ W_s(-w_U v_1), W_s^\vee(-w_U v_2) \} \psi_U(v_1)^{-1} \psi_{N^\#}(v_2) \\ dv_1 \\ dv_2.
\] (6.0.1)

By (5.2.1), for any $\pi \in \text{Irr}_{gcn} M$, $W \in \text{Ind}(W_{\psi N M}^\psi (\pi))$, $W^\vee \in \text{Ind}(W_{\psi N M}^{-1} (\pi(\psi)))$ and $-\text{Re}(s) \gg 1$, we have
\[
B(W, W^\vee, s) = B(W, W^\vee, s).
\] (6.0.2)

The following proposition is proved in a similar way as the argument in [29, Appendix B], practically word for word. Hence, we omit its proof.

**Proposition 6.1.** Let $\pi \in \text{Irr}_{\text{temp}} M$. Then

1. For $\text{Re}(s) \gg 1$, $B(W, W^\vee, s)$ is well defined for any $W \in \text{Ind}(W_{\psi N M}^\psi (\pi))$, $W^\vee \in \text{Ind}(W_{\psi N M}^{-1} (\pi^\vee))$.

2. For $-\text{Re}(s) \gg 1$, $B(W, W^\vee, s)$ is well defined for any $W \in \text{Ind}(W_{\psi N M}^\psi (\pi))$, $W^\vee \in \text{Ind}(W_{\psi N M}^{-1} (\pi^\vee))$.

3. For $-\text{Re}(s) \gg 1$, we have
\[
B(W, M(s) W^\wedge, s) = B(M(s) W, W^\wedge, -s)
\]
for any $W \in \text{Ind}(W_{\psi N M}^\psi (\pi))$, $W^\wedge \in \text{Ind}(W_{\psi N M}^{-1} (\pi^\vee))$.

Recall the definition of the space $W^\vee \in \text{Ind}(W_{\psi N M}^{-1} (\pi^\vee))$ in Section 4.1.

Combined with (6.0.2) and Proposition 6.1 we get the following identity:

**Corollary 6.1.** For $-\text{Re}(s) \gg 1$, we have
\[
B(W, M(s) W^\wedge, s) = B(M(s) W, W^\wedge, -s)
\]
for any $W \in \text{Ind}(W_{\psi N M}^\psi (\pi))$, $W^\wedge \in \text{Ind}(W_{\psi N M}^{-1} (\pi))$.

6.1. Set $\varepsilon_2 = \ell_M(e_{1,1} + J)$, where $e_{1,1}$ is the matrix in $\text{Mat}_n$ with 1 in the upper left corner and 0 elsewhere, $\varepsilon_3 = w_{2n, n}^\prime \varepsilon_2$ and $\varepsilon_4$ is an arbitrary element of $N_M$. As in [44, Section 9], define
\[
\Delta(t) := |t_1|^\varepsilon_4 \frac{1}{2} (q(t)), \quad t = \text{diag}(t_1, \ldots, t_{2n}) \in T_M.
\]

In particular, when $t \in N_M^\psi(t')$ with $t' \in T_M'$, we have $\Delta(t) = \delta_B^\varepsilon(q'(t'))$. 

Let \( T'' = \eta_{M}(T'_{M'}) \times Z_{M} \). For any \( W \in C^{\infty}(N \backslash G, \psi_{N}) \) and \( t \in T'' \), define

\[
E^{\psi}(W, t) := \Delta(t)^{-1} \int_{\eta_{M}(N_{M'}) \backslash N_{\#}} W(\varphi(t\varepsilon) w_{U} v) \psi_{N}(v)^{-1} dv = \Delta(t)^{-1} \psi_{\eta_{M}}(t \varepsilon \eta_{s} t^{-1}) \int_{\eta_{M}(N_{M'}) \backslash N_{\#}} W(\varphi(t\varepsilon) w_{U} v) \psi_{N}(v)^{-1} dv. \tag{6.1.1}
\]

### Definition 6.1.
Recall the definition of a certain subspace of \( \mathbb{W}^{1}_{NM}(\pi) \) in [33, Definition 7.5]. Let \( \mathbb{W}^{1}_{NM}(\pi)_{s} \) be the subspace of \( \mathbb{W}^{1}_{NM}(\pi) \) consisting of \( W \) such that

\[
W(-\varepsilon) \big|_{P^{s}} \in \mathcal{C}^{\infty}(N_{M'} \backslash P^{s}, \psi_{s}^{-1}) \quad \text{and} \quad W(-\varepsilon) \big|_{\eta_{s}(T'_{M'})} \in \mathcal{C}^{\infty}(\eta_{s}^{\vee}(T'_{M'}), K). 
\]

As remarked in [33, p. 745], this space is nonzero, and thus the following space is also nonzero:

### Definition 6.2.
Let \( \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))_{s} \) be the linear subspace of \( \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))^{s} \) spanned by the functions which vanish outside \( Pu_{U}N \) and on the big cell are given by

\[
W(u' m w u) = \delta_{P^{s}}(m) W^{s}(m) \phi(u), \quad m \in M, u, u' \in U,
\]

with \( \phi \in \mathcal{C}^{\infty}_{c}(U) \) and \( W^{s} \circ \phi \in \mathbb{W}^{1}_{NM}(\pi)_{s} \).

By the same argument as the proof of [33, Lemma 7.9], the following lemma is proved:

**Lemma 6.1.** Let \( \pi \in \text{Irr}_{\text{gen}}M. \) For \( \text{Re}(s) \gg 1 \), (7.1.1) defining \( E^{\psi}(W, t) \) converges for any \( W \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi)) \) and \( t \in S \) uniformly for \( (s, t) \) in a compact set. Hence, \( E^{\psi}(W, t) \) is holomorphic for \( \text{Re}(s) \gg 1 \) and continuous in \( t \). If \( W^{\wedge} \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))^{s} \), then \( E^{\psi^{-1}}(W_{s}^{\wedge}, t) \) is entire in \( s \) and locally constant in \( t \), uniformly in \( s \). If, moreover, \( W^{\wedge} \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))_{s}^{s} \), then \( E^{\psi^{-1}}(W^{\wedge}, t) \) is compactly supported in \( t \in S, \) uniformly in \( s \).

**Proposition 6.2.** Let \( \pi \in \text{Irr}_{\text{temp}}M. \) Then for \( \text{Re}(s) \gg 1 \), we have

\[
B(W, W^{\wedge}, s) = \int_{\eta_{M}(T'_{M'})} E^{\psi}(W_{s}, t) E^{\psi^{-1}}(W_{s}^{\wedge}, t) \frac{dt}{|\det t|} \tag{6.1.2}
\]

for any \( W \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi)) \), \( W^{\wedge} \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))^{s} \) where the integrand on the right-hand side is continuous and compactly supported.

**Proof.** This is proved by the same argument as the proof of [33, Proposition 7.10] using (6.0.1) and Lemma 6.1 instead of [33, (7.1) and Lemma 7.9], respectively.

Combining Proposition 6.2 with Corollary 6.1, we get the following:

**Proposition 6.3.** Let \( \pi \in \text{Irr}_{\text{temp}}M. \) Then for \( -\text{Re}(s) \gg 1 \) and any \( W \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))_{s}^{s} \), \( W^{\wedge} \in \text{Ind}(\mathbb{W}^{1}_{NM}(\pi))_{s}^{s} \), we have

\[
B(W, M(s) W^{\wedge}, s) = \int_{S} E^{\psi}(M^{\wedge}_{s} W, t) E^{\psi^{-1}}(W^{\wedge}, t) \frac{dt}{|\det t|},
\]

where the integrand is continuous and compactly supported. Here, for simplicity, we write \( S = \eta_{M}(T'_{M'}) \).
7. Proof of Proposition 5.1

7.1.

Let \( \mathfrak{d} = \text{diag}(1, -1, \ldots, (-1)^{n-1}) \in \text{Mat}_n \). Now fix

\[
\varepsilon_4 = \ell_M \left( -\frac{1}{2} \mathfrak{d} w_0^{M} \right) \in N_M.
\]

This element is denoted by \( \varepsilon' \) in the beginning of [44, Section 8], with the parameter \( \mathfrak{a} = -\frac{1}{2} \). Also fix \( \varepsilon_2 = \ell_M(\mathfrak{d}) \) and (correspondingly) \( \varepsilon_3 = w_{2n, n}^{\varepsilon_2} \). Then we have

\[
\psi_U(\bar{v}) = \psi_U((\varrho(\varepsilon_4 \varepsilon_3) w_U)^{-1} \bar{v} \varrho(\varepsilon_4 \varepsilon_3) w_U)^{-1} \psi(\bar{v}_{2n+1, 1}), \quad \bar{v} \in \bar{U}.
\]

Note that in [44, Section 4.1], the character \( \psi_U \) is denoted by \( \psi_{U, F} \). As in [33, 8.1], we can rewrite (for \( \Re(s) \gg 1 \))

\[
E^\psi(W_s, t) = \Delta(t)^{-1} \int_{N_M^*(N_M^*)' \setminus N_M^*} \int_{U} W_s(\varrho(t) \bar{v} \varrho(\varepsilon_4 \varepsilon_3) w_U) \psi_U(\bar{v}) \psi_{N_M^*}^{-1}(r) \, d\bar{v} \, dr
= \Delta(t)^{-1} \int_{\bar{R}} \int_{\bar{U}} W_s(\varrho(t) \bar{v} \varrho(\varepsilon_4 \varepsilon_3) w_U) \psi_U(\bar{v}) \psi_{\bar{R}}(r) \, d\bar{v} \, dr. \quad (7.1.1)
\]

Here, recall that \( N_M^0 = (N_M^*)' \) and \( \psi_{N_M^0}(m) = \psi_{N_M^*}(m^*) \), and define \( \psi_{\bar{R}}(r) = \psi_{N_M^*}(\varepsilon_3^{-1} r \varepsilon_3)^{-1} \) and

\[
\bar{R} = \varepsilon_3(\eta_M(N_M^*)' \ltimes \ell_M(J)) \varepsilon_3^{-1} = w_{2n, n}(\eta_M(N_M^*)' \ltimes \ell_M(J)) w_{2n, n}^{-1}
= \left\{ \begin{pmatrix} I_n & 0 \\ x & t_n \end{pmatrix} : x \in J, n \in N_M^* \right\} \subset tN_M \cap \mathcal{P}^*.
\]

We now quote a pertinent result from [44]. Let

\[
T_i := \{ \text{diag}(1, -1, z, 1, 2, \ldots, z) : z \in E^x \}.
\]

Then we have \( S = \prod_{i=1}^n T_i \). For any \( f \in C_c^\infty(S) \) and \( g \in C(S) \), write \( f \ast g(\cdot) = \int_S f(t) g(\cdot t) \, dt \).

**Theorem 7.1.** Let \( K_i \) be a compact open subgroup of \( E^x \) and let \( f_{K_i} \) be the characteristic function of \( K_i \). Regard \( f_{K_i} \) as a function on \( T_i \), and put \( f = f_{K_1} \otimes \cdots \otimes f_{K_n} \in \otimes_i C_c^\infty(T_i) = C_c^\infty(S) \).

For any \( W \in C^\infty(N \backslash G, \psi_N) \) which is left invariant under a compact open subgroup of \( Z_M \), the function \( f \ast E^\psi(W_s, t) \) extends to an entire function in \( s \) which is locally constant in \( t \), uniformly in \( s \). Moreover, if \( \pi \in \text{Irr}_{\text{ut, temp}} \mathbb{M} \), then
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\[ f \ast E_\psi^\psi(W_s, t)|_{s = \frac{1}{2}} = \begin{cases} 
\omega_\pi(t)^n A_\psi^\psi(M^* W) \int_{\mathcal{S}_F} f(t') dt' & \text{if } t_i \in F^s K_i \\
0 & \text{otherwise}
\end{cases} \]

for \( t = \text{diag}(1, \ldots, 1, t_1, \ldots, t_n) \). Here, \( \mathcal{S}_F = S \cap \text{GL}_n(F) \).

This theorem follows from a result in [44]. As in [44, Section 4.1], let \( Z \) be the unipotent subgroup of \( \mathbb{M} \) given by

\[ Z = \{ m \in \mathbb{M} : m_{i,i} = 1 \forall i, m_{i,j} = 0 \}
\]

if either \(( j > i \) and \( i + j > 2n \)) or \(( i > j \) and \( i + j \leq 2n + 1 \)),

and let \( \psi_{Z,F} \) be its character:

\[ \psi_{Z,F}(m) = \psi(m_{1,2} + \cdots + m_{n-1,n} - m_{n+2,n+1} - m_{n+3,n+2} - \cdots - m_{2n,2n-1}). \]

The group \( \varrho(Z) \) stabilises the character \( \psi_{\bar{U}} \). Let \( \mathcal{E} = \varrho(Z) \times \bar{U} \) and let \( Z^+ = Z \cap N_\mathbb{M} \), \( V_\Delta = Z \cap t N_\mathbb{M} \) and

\[ N_{\mathbb{M},\Delta} = \{ t \ell_M(X) : t X \in J, X_{i,j} = 0 \text{ if } i + j \geq n + 1 \}. \]

We have \( Z = Z^+ \cdot V_\Delta \) and \( \bar{R} = V_\Delta \cdot N_{\mathbb{M},\Delta} \).

The character \( \psi_{\bar{R}} \) is given by \( \psi_{\bar{R}}(v) = \psi_{V_\Delta,F}(v), \) \( v \in V_\Delta, \) \( n \in N_{\mathbb{M},\Delta}, \) where \( \psi_{V_\Delta,F}(v) \) is defined in [44, 4.1]. Then the expression in [44, (10.6)] evaluated at \( \varrho(\varepsilon_3)w_U \) is \( (f \Delta) \ast E_\psi^\psi(W_s, t) \), and its analytic continuation is given by the expression in [44, (10.5)], which amounts to a finite sum.

Meanwhile, from the definition of \( A_\psi^\psi \) in (5.0.1) we have

\[ A_\psi^\psi(W) = \int_{V_{M,M} \gamma V_{\mathcal{S}} \gamma^{-1}} W(v) \gamma \varepsilon v \tau_{V_{\mathcal{S}} \gamma^{-1}}((\gamma \varepsilon v \tau_{V_{\mathcal{S}} \gamma^{-1}}) \psi_{V_{\mathcal{S}} \gamma^{-1}}) \psi_{V_{\mathcal{S}} \gamma^{-1}}(v) dv. \]

We can integrate over the group \( \bar{U}^- \) in [44, (10.2)], consisting of elements \( \bar{u} \) such that \( \bar{u}_{2n+i,j} = \) whenever \( i \geq n \) and \( j \leq n \). Then the character \( \bar{u} \mapsto \psi_{V_{\mathcal{S}} \gamma^{-1}}(\gamma \varepsilon v \tau_{V_{\mathcal{S}} \gamma^{-1}}) \) on \( \bar{U}^- \) is the character denoted by \( \hat{\psi}_{\bar{U},\Delta} \) in the definition of \( \mathcal{T}' \) in [44, 10.2]. Thus, \( A_\psi^\psi(W) \) is \( \mathcal{T}'(W)(\gamma \varepsilon v) \) in the notation of [44, 10.2]. The second part amounts to the first statement of [44, Corollary 10.8] upon taking

\[ \varepsilon_1 = (\hat{\psi}(\gamma \varepsilon v))^{-1} \varrho(\varepsilon_3)w_U = \ell_M((-1)^n \mathfrak{d}), \]

where we write

\[ \hat{\psi} := \varrho(w_U') \varrho(w_{2n,n}'), \]

as in [44, 4.1]. This theorem is crucial step for my proof of Proposition 5.1, which is a unitary analogue of [33, Theorem 8.1] and proved in [31].
Corollary 7.1. Suppose that \( \pi \in \text{Irr}_{\text{ut, temp}} M \). Then for any \( W \in \text{Ind}(\mathbb{W}_N^\psi M(\pi))^{\#}_\circ \), \( W^\wedge \in \text{Ind}(\mathbb{W}_N^{-1} N_M(\pi))^{\#}_\circ \), we have

\[
B\left(W, M\left(\frac{1}{2}\right) W^\wedge, \frac{1}{2}\right) = \omega_{\pi}(\tau)^n A_e^\psi (M^* W) \int_{S_F} E^\psi_1 (W^\wedge, t) \frac{dt}{|\det t|},
\]

where, recall, \( S_F = S \cap \text{GL}_{4n}(F) \).

Proof. By Lemma 6.1, for any \( W^\wedge \in \text{Ind}(\mathbb{W}_N^{-1} N_M(\pi))^{\#}_\circ \) there exist \( K_i \in \text{CSGR}(E^\times) \) (\( 1 \leq i \leq n \)) such that \( E^\psi (W^\wedge, \cdot) \in C(S)K_0 \) for all \( s \), and \( E^\psi (W_s^\wedge, \cdot) \) is compactly supported on \( S \) uniformly in \( s \). Here, regard \( K_i \) as a subgroup of \( S \) and set \( K_0 = \prod_{i=1}^n K_i \).

Suppose \( f := f_{K_1} \otimes \cdots \otimes f_{K_n} \in C_\circ(S) \) and let \( f^\psi(t) := f(t^{-1}) \). By Proposition 6.3, for \( -\text{Re}(s) \gg 1 \) and \( W \in \text{Ind}(\mathbb{W}_N^\psi M(\pi))^{\#}_\circ \), we have

\[
B(W, M(s) W^\wedge, s) \int_S f(t) dt = \int_S E^\psi (M^* W, t) f^\psi \ast E^\psi (W^\wedge, t) \frac{dt}{|\det t|} = \int_S f \ast E^\psi (M^* W, t) E^\psi (W^\wedge, t) \frac{dt}{|\det t|}.
\]

Since \( B(W, W^\wedge, s) \) is an entire function of \( s \) for \( W \in \text{Ind}(\mathbb{W}_N^\psi M(\pi))^{\#}_\circ \), \( W^\wedge \in \text{Ind}(\mathbb{W}_N^{-1} N_M(\pi))^{\#}_\circ \), the first part of Theorem 7.1 implies that both sides of (7.1.3) are meromorphic functions and the identity holds whenever \( M(s) \) is holomorphic. Then by [32, Proposition 2.1], we can specialise \( s = \frac{1}{2} \). Using the second part of Theorem 7.1, we find that the right-hand side of (7.1.3) is equal to

\[
\omega_{\pi}(\tau)^n A_e^\psi (M^* W) \left( \int_{S_F} f(t) dt \right) \cdot \int_{S \cap K_0^\circ} E^\psi (W^\wedge, t) \frac{dt}{|\det t|}
\]

\[
= \omega_{\pi}(\tau)^n A_e^\psi (M^* W) \left( \int_{S_F} f(t) dt \right) \cdot \left( \int_{K_0 \cap \text{GL}_{4n}(F) \backslash K_0} dt \right) \int_{S_F} E^\psi (W^\wedge, t) \frac{dt}{|\det t|},
\]

where \( K_0' = \prod_{i=1}^n (F^\times K_i) \). The required formula readily follows, since

\[
\left( \int_{S_F} f(t) dt \right) \cdot \left( \int_{K_0 \cap \text{GL}_{4n}(F) \backslash K_0} dt \right) = \int_{S} f(t) dt.
\]

7.2.

In order to complete our proof of Proposition 5.1, it remains to compute the integral on the right-hand side of (7.1.2). Again, this is essentially done in [44], and it relies heavily on the fact that \( \pi \in \text{Irr}_{\text{ut}} M \).

Let

\[
\mathbb{W}_N^\psi M(\pi)_{\mathbb{C}} = \{ W \in \mathbb{W}_N^\psi M(\pi) : W|_{\mathcal{D}^\times} \in C_c^\infty (N_M^\psi \backslash \mathcal{D}^\times, \psi_{N_M^\psi}) \}
\]

and

\[
W|_{\eta_{M^\psi}^\psi (T_{M^\psi})^*} \in C_c^\infty (\mathcal{Z}^+ \backslash \eta_{M^\psi}^\psi (T_{M^\psi})^* \times \mathcal{Z}, \psi_{\mathcal{Z}}).
\]
Theorem 7.2. Let $\pi \in \text{Irr}_{\text{ut, temp}} M$. Then

1. \([46, \text{Corollary 4.1}]; \text{see also } [44, \text{Lemma 3.5}]\) The integral

$$\Psi^H_M(W) := \int_{(H_M \cap N_M) \setminus H_M \cap P} W(\tau^* p) \, dp$$

converges and defines a nonzero $H_M$-invariant functional on $\mathbb{W}_{NM}^\psi(\pi)$. Here we define $H_M = \text{GL}_2(F)$ and $\tau^* = \text{diag}(\tau, 1, \tau, 1, \ldots, \tau, 1) \in M$.

2. \([44, \text{Corollary 10.4}]\) Set

$$x(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

which is $b''(n)$ in the notation of [44]. Then for any $W \in \mathbb{W}_{NM}^\psi(\pi)_2$, we have

$$\int_{\mathbb{Z}^+ \setminus \mathbb{Z}} \int_{S_F} \Delta(t)^{-1} \det(t)^{-\frac{n+1}{2}} W(tr) \psi_Z, F(r)^{-1} \, dt \, dr$$

$$= \omega_\pi(\tau)^n |\tau|^{y(n)} \int_{\mathbb{Z}^+ \setminus \mathbb{Z}} \Psi^H_M(\pi(n \tau_o) W) \psi_Z(n)^{-1} \, dn,$$

where $\tau_o \in M$ is defined by $\varphi(\tau_o) = \hat{w}\varphi(\tau^*)^{-1} \hat{w}^{-1}$ and we define $\psi_Z(m) = \psi_Z, F(\tau_o^{-1} m \tau_o)$.

3. \([44, \text{Lemma 4.2}]\) The integral

$$L_W(g) := \int_{(P \cap H) \setminus H} \int_{(H_M \cap N_M) \setminus H_M \cap P} W(\varphi(\tau^* p) h g) \det p|^{-(n+\frac{1}{2})} \, dp$$

$$= \int_{H \cap \bar{U}} \Psi^H_M(\left( \delta_p^{-\frac{1}{2}} I \left( \frac{1}{2}, \bar{u} g \right) W \right) \circ \varphi) \, d\bar{u}$$

converges for any $W \in \text{Ind}(\mathbb{W}_{NM}^\psi(\pi), \frac{1}{2})$ and defines an intertwining map

$$\text{Ind}\left(\mathbb{W}_{NM}^\psi(\pi), \frac{1}{2}\right) \to C^{sm}(H \setminus G).$$

4. \([44, \text{Corollary 10.8, second statement}]\) Set

$$y(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ n - \frac{1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

which is $a(n) + n^2$ in the notation of [44]. Then we have

$$A^\psi_\varepsilon(M^* W) = \omega_\pi(\tau)^{y(n)} \left( \int_{N_M, A} \int_{H \cap \varepsilon \setminus \varepsilon} L_W(v \varphi(\tau_o \varepsilon u \varepsilon_3) w_U) \psi^{-1}_\varepsilon(v) \, dv \right) \, du,$$

where $\varepsilon$ we define

$$\psi_\varepsilon(\varphi(m) \bar{u}) = \psi_Z, F(\tau_o^{-1} m \tau_o) \psi_U^{-1}(\tau_o^{-1} \bar{u} \tau_o) = \psi_Z(m) \psi_U^{-1}(\tau_o^{-1} \bar{u} \tau_o).$$
Corollary 7.2. Let $\pi \in \text{Irr}_{\text{ut,temp}} M$. Then for any $W \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))_z^\circ$, we have

$$
\int_{S_F} E^{\psi}(W_{\frac{1}{2}}, t) \frac{dt}{|\det t|} = \omega_\pi(\tau)^{n+1} A^\psi_e(M^* W).
$$

(7.2.2)

Proof. We may assume without loss of generality that

$$
W_{\frac{1}{2}}(u', \rho(m) w_U u) = W^M(m) |\det m|^{\frac{1}{2}} \delta_P(\rho(m)) \frac{1}{2} \phi(u), \quad m \in M, \ u, u' \in U,
$$

(7.2.3)

with $W^M \in \mathbb{W}^{\psi N_M}(\pi)_z$ and $\phi \in C_c^\infty(U)$. Evaluate the left-hand side $I$ of (7.2.2) using the last expression in (7.1.1), where recall that the integrand is compactly supported by Lemma 6.1). Thus,

$$
I = I' \int_U \phi(u) \psi_U^{-1}(v) dv,
$$

where

$$
I' = \int_{S_F} \Delta(t)^{-1} |\det t|^{n-\frac{1}{2}} W^M(t \epsilon_4 r \epsilon_3) \psi_{\bar{R}}(r) dr dt.
$$

The integrand in $I'$ is compactly supported because $W^M \in \mathbb{W}^{\psi N_M}(\pi)_z$. Note that for $r \in V_\Delta = Z \cap \bar{R}$, $\psi_{\bar{R}}(r)^{-1} = \psi_Z F(r)$. Write

$$
I' = \int_{N_M} \left( \int_{V_\Delta} \int_{S_F} \Delta(t)^{-1} |\det t|^{n-\frac{1}{2}} W^M(t \epsilon_4 r \epsilon_3) \psi_{Z \cdot F}(r)^{-1} dt dr \right) du
$$

$$
= \int_{N_M} \left( \int_{Z^+ \cap Z} \int_{S_F} \Delta(t)^{-1} |\det t|^{n-\frac{1}{2}} W^M(t \epsilon_4 r \epsilon_3) \psi_{Z \cdot F}(r)^{-1} dt dr \right) du
$$

$$
= \int_{N_M} \left( \int_{Z^+ \cap Z} \int_{S_F} \Delta(t)^{-1} |\det t|^{n-\frac{1}{2}} W^M(t \epsilon_4 r \epsilon_3) \psi_{Z \cdot F}(r)^{-1} dt dr \right) du,
$$

since $\epsilon_4$ stabilises $\psi_{Z \cdot F}$. For the double integral in the brackets, apply Theorem 7.2, part 2, to $\pi(\epsilon_4 u \epsilon_3) W^M$ (which is applicable, since $W^M \in \mathbb{W}^{\psi N_M}(\pi)_z$). This yields

$$
I' = \omega_\pi(\tau)^n \left| \tau \right|^{x(n)} \int_{N_M,\Delta} \left( \int_{Z \cap H_M \cap Z} \mathfrak{P}^{H_M}(\pi(n \tau \epsilon_4 u \epsilon_3) W^M) \psi_Z(n)^{-1} du \right).
$$

Thus,

$$
I = \omega_\pi(\tau)^n \left| \tau \right|^{x(n)} \int_{N_M,\Delta} \left( \int_{Z \cap H_M \cap Z} \int_U \mathfrak{P}^{H_M}(\pi(n \tau \epsilon_4 u \epsilon_3) W^M)
$$

$$
\times \psi_Z(n)^{-1} \phi(v) \psi_U^{-1}(v) dv dn \right) du.
$$

From (7.2.3),

$$
\left( \delta_P^{-\frac{1}{2}} I \left( \frac{1}{2}, \rho(m) w_U u \right) W \right) \circ \rho = \phi(v) \delta_P^{\frac{1}{2}}(\rho(m)) |\det m|^{\frac{1}{2}} \pi(m) W^M
$$
for any \( v \in U \) and \( m \in \mathbb{M} \). Thus, We get

\[
I = \omega_\pi(\tau)^n |\tau|^{x(n)} \delta_P^{\frac{1}{2}}(\varphi(\tau_\circ))| \det \tau_\circ|^{-\frac{1}{2}} \times \\
\int_{N_{M,\Delta}} \left( \int_{\mathbb{Z} \cap H_M} \int_U \mathcal{Q}_{\mathbb{H}^M}(\delta_P^{\frac{1}{2}} I(\pi, \varphi(\tau_\circ u \varepsilon_3) w_U v) W) \psi_Z(n)^{-1} \psi_U(v)^{-1} dv \, dn \right) du.
\]

Since \( \varepsilon_3^{-1} N_{M,\Delta} \varepsilon_3 \subset N_{M}^\circ \), the group \( \varphi(\varepsilon_3^{-1} N_{M,\Delta} \varepsilon_3) \) stabilises the character \( \psi_U(w_U^{-1} \cdot w_U) \) on \( \tilde{U} \). Making a change of variable

\[
v \mapsto (\varphi(\tau_\circ u \varepsilon_3) w_U)^{-1} \varphi(\tau_\circ u \varepsilon_3) w_U
\]
on \( U \), we obtain

\[
I = \omega_\pi(\tau)^n |\tau|^{x(n)} \delta_P^{\frac{1}{2}}(\varphi(\tau_\circ))| \det \tau_\circ|^{-\frac{1}{2}} \times \\
\int_{N_{M,\Delta}} \left( \int_{\mathbb{Z} \cap H_M} \int_{H \cap \tilde{U}} \mathcal{Q}_{\mathbb{H}^M}(\delta_P^{\frac{1}{2}} I(\pi, x \varphi(\tau_\circ u \varepsilon_3) w_U) W) \psi_{\mathbb{E}}(n)^{-1} dx \right) du.
\]

From Theorem 7.2, part 3, we get

\[
I = \omega_\pi(\tau)^n |\tau|^{x(n)} \delta_P^{\frac{1}{2}}(\varphi(\tau_\circ))| \det \tau_\circ|^{-\frac{1}{2}} \int_{N_{M,\Delta}} \int_{\mathbb{Z} \cap H_M} L_W(\varphi(\tau_\circ u \varepsilon_3) w_U) \psi_{\mathbb{E}}(n)^{-1} \, dv \, du.
\]

From Theorem 7.2, part 4, this is equal to

\[
\omega_\pi(\tau)^{n+1} |\tau|^{x(n)-y(n)} \delta_P^{\frac{1}{2}}(\varphi(\tau_\circ))| \det \tau_\circ|^{-\frac{1}{2}} A^\psi_e(M^* W) = \omega_\pi(\tau)^{n+1} A^\psi_e(M^* W),
\]

using the fact that \( |\tau|^{x(n)-y(n)} \delta_P^{\frac{1}{2}}(\varphi(\tau_\circ))| \det \tau_\circ|^{-\frac{1}{2}} = 1 \) (see [44, Remark 10.1]).

Now to complete the proof of Proposition 5.1. From Corollaries 7.1 and 7.2, we have that (5.3.1) holds for any \( W \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))^\circ \) and \( W^\wedge \in \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))^\circ \), namely,

\[
B \left( W, M \left( \frac{1}{2} \right) W^\wedge, \frac{1}{2} \right) = \omega_\pi(\tau) A^\psi_e(M^* W) A^\psi_e^{-1}(M^* W^\wedge).
\]

On the other hand, as in [33, Section 8C], by a similar argument to the proof of [33, Lemma 6.13] with (7.1.1), we see that the linear map \( \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))^\circ \rightarrow C^\infty_c(T_{M}^\wedge) \) given by \( W^\wedge \mapsto E^\psi(W^\wedge, \cdot) \) is onto. Therefore, by Corollary 7.2, the linear form \( A^\psi_e^{-1}(M^* W^\wedge) \) does not vanish on \( \text{Ind}(\mathbb{W}^{\psi N_M}(\pi))^\circ \). From Corollary 5.1, we conclude that (5.3.1) holds for all \( W^\wedge \), which completes the proof of Proposition 5.1. 

\( \square \)
8. Refined formal degree conjecture and Conjecture 2

8.1. Non-Archimedean case

In this section, we show an equivalence between the refined formal degree conjecture and Theorem 1.2 for generic discrete series representations of $G' = U_{2n}$. In particular, we obtain a proof of the refined formal degree conjecture in the case of this article. Suppose that a base field $F$ is a non-Archimedean local field of characteristic zero.

Recall the refined formal degree conjecture by Gross and Reeder [15] and Gan and Ichino [11, Section 14.5]. For an irreducible discrete series representation of $G'$, the formal degree $d_\sigma$ for $\sigma$ is the measure on $G'$ satisfying

$$
\int_{G'} (\sigma(g)v_1, v'_1)(\sigma(g^{-1})v_2, v'_2) d_\sigma g = (v_1, v'_1)(v_2, v'_2)
$$

that holds for any $v_1, v'_1, v_2, v'_2$, where $(\cdot, \cdot)$ is a $G'$-invariant inner product on $V_\sigma$. On the other hand, denote the measure on $G'$ defined in Section 2.4 by $d_\psi$ to clarify the difference between these measures.

Hiraga, Ichino and Ikeda [17, 18] formulated a conjecture on a relationship between these measures $d_\sigma$ and $d_\psi$ in terms of absolute values of special values of adjoint gamma factors. Recently, Gan and Ichino [11] computed the sign of this special value for Steinberg representations of classical groups using an important observation by Gross $\gamma$-factors. Recently, Gan and Ichino [11] computed the sign of this special value for Steinberg representations of classical groups using an important observation by Gross and Reeder [15] on the sign. Then [11, Section 14.5] conjectured a refinement of [17, Conjecture 1.4]. My purpose in this section is to prove this refined version.

**Theorem 8.1.** Let $\pi$ be an irreducible representation of $GL_{2n}(E)$ of the form $\pi = \tau_1 \times \cdots \times \tau_k$, where $\tau_i$ are mutually inequivalent irreducible discrete series representations of $GL_{n_i}(E)$ such that $n = n_1 + \cdots + n_k$ and $L(s, \tau_i, As^\pm)$ has a pole at $s = 0$. Write $\sigma' = D_{\psi}^{-1}(c(\pi))$, which is an irreducible generic discrete series representation of $G'$ (see Theorem A.5). Then the refined formal degree conjecture holds; namely, we have

$$
d_\psi = 2^k \lambda(E/F, \psi)^n \omega_{\sigma'}(-1) \gamma(1, c(\pi), As^-, \psi) d_\sigma.
$$

Here, $\lambda(E/F, \psi)$ is the Langlands $\lambda$-function and the $\gamma$-factor is defined by the Langlands–Shahidi method [49].

We shall prove this theorem by a similar argument to the proof of the refined formal degree conjecture for metaplectic groups given by Ichino, Lapid and Mao [20].

Recall the following functional equation. Let $W \in \mathcal{W}_{\psi}^{-1}(\sigma')$ and $W' \in \text{Ind}(\mathcal{W}_{\psi}^{NM}(\pi))$. Let $\gamma(s, \pi, As^\pm, \psi)$ be Asai gamma factors defined by the Langlands–Shahidi method [49]. In [6], a functional equation was shown relating $J(W, M(s)W', -s)$ and $J(W, W', s)$, and local $\gamma$-factors were defined associated to $J(W, W', s)$. Further, in [45] we proved that this local $\gamma$-factor coincides with the $\gamma$-factors defined by the Langlands–Shahidi method [49]. We obtain the functional equation

$$
J(W, M(s)W', -s) = \lambda(E/F, \psi)^n \frac{\gamma(s + \frac{1}{2}, \sigma' \times (\pi \otimes \Upsilon^{-1}), \psi)}{\gamma(2s, c(\pi), As^+, \psi)} J(W, W', s),
$$

(8.1.2)
where the $\gamma$-factor $\gamma(s + \frac{1}{2}, \sigma' \times (\pi \otimes \Upsilon^{-1}), \psi)$ is defined by the Langlands–Shahidi method \cite{49}. Note that the factor $\lambda(E/F, \psi)$ comes from the normalisation of an intertwining operator (see, e.g., \cite{1, (3)}).

By Theorem A.5, we have

$$
\gamma \left( s + \frac{1}{2}, \sigma' \times (\pi \otimes \Upsilon^{-1}), \psi \right) = \gamma \left( s + \frac{1}{2}, \pi \otimes c(\pi), \psi \right).
$$

Moreover, we have

$$
\gamma \left( s + \frac{1}{2}, \pi \otimes c(\pi), \psi \right) = \gamma \left( s + \frac{1}{2}, c(\pi), As^+, \psi \right) \gamma \left( s + \frac{1}{2}, c(\pi), As^-, \psi \right).
$$

Hence, we get

$$
\lim_{s \to \frac{1}{2}} \frac{\gamma(s + \frac{1}{2}, \sigma' \times (\pi \otimes \Upsilon^{-1}), \psi)}{\gamma(2s, c(\pi), As^+, \psi)} = 2^k \gamma(1, c(\pi), As^-, \psi).
$$

(8.1.3)

Define a nondegenerate $G'$-invariant bilinear form $(\cdot, \cdot)_{\sigma'}$ on $\mathbb{H}^{\psi N^{-1}}(\sigma') \times \mathbb{H}^{\psi N'}((\sigma')^\vee)$ by

$$( W, W' )_{\sigma'} = \int_{N \back G} W(g) W'(g) d\psi(g),$$

which converges absolutely by \cite[Proposition 3.2]{10}. On the other hand, recall that Definition 3.2 defines a $G'$-invariant bilinear form $[\cdot, \cdot]$. Indeed, it is defined so that

$$J \left( W, W', \frac{1}{2} \right) = \left[ W, A^\psi_{\#} (M^* W', \cdot) \right]_{\sigma'},$$

for $W \in \mathbb{H}^{\psi N^{-1}}(\sigma')$ and $W' \in \text{Ind}(\mathbb{H}^{\psi N_M}(\pi))$. Then by (8.1.2), we obtain

$$( W, A^\psi_{\#} (M^* W', \cdot) )_{\sigma'} = J \left( W, M^* W', -\frac{1}{2} \right) = \lambda(E/F, \psi)^n \lim_{s \to \frac{1}{2}} \frac{\gamma(s + \frac{1}{2}, \sigma' \times (\pi \otimes \Upsilon^{-1}), \psi)}{\gamma(2s, c(\pi), As^+, \psi)} J \left( W, W', \frac{1}{2} \right) = \lambda(E/F, \psi)^n \lim_{s \to \frac{1}{2}} \frac{\gamma(s + \frac{1}{2}, \pi \otimes c(\pi), \psi)}{\gamma(2s, c(\pi), As^+, \psi)} [ W, A^\psi_{\#} (M^* W', \cdot) ].$$

Hence, by (8.1.3),

$$( W, A^\psi_{\#} (M^* W', \cdot) )_{\sigma'} = \lambda(E/F, \psi)^n 2^k \gamma(1, c(\pi), As^-, \psi) [ W, A^\psi_{\#} (M^* W', \cdot) ]_{\sigma'}.$$

(8.1.4)

Recall that the formal degree $d_{\sigma'}$ is defined so that

$$\int_{G'} [\sigma'(g) W_1, W'_1]_{\sigma'} [\sigma(g^{-1}) W_2, W'_2]_{\sigma'} d_{\sigma'} = [ W_1, W'_1 ]_{\sigma'} [ W_2, W'_2 ]$$

for $W_1, W_2 \in \mathbb{H}^{\psi N^{-1}}(\sigma')$ and $W'_1, W'_2 \in \mathbb{H}^{\psi N'}(\pi(\sigma))$. Assume that $W'_1 = A^\psi_{\#} (M^* W', \cdot)$. Then by the definition,
\[
\int_{G'} [\sigma(g) W_1, W'_1]_{\sigma'} [\sigma(g^{-1}) W_2, W'_2]_{\sigma'} d_\psi(g) \\
= \int_{G'} \int_{N' \setminus G'} W_1(xg) A^\psi_\#_{\#} (M^* W', x) [\sigma'(g^{-1}) W_2, W'_2]_{\sigma'} d_\psi(x) d_\psi(g). \quad (8.1.5)
\]

By the same argument as [20, pp. 1316–1317], we see that this double integral converges absolutely by Lemma 3.3. Then we can change the order of the integration, and changing the variable \( g \mapsto x^{-1} g \) gives

\[
\int_{N' \setminus G'} \int_{G'} W_1(g) A^\psi_\#_{\#} (M^* W', x) [\sigma'(g^{-1}) W_2, W'_2]_{\sigma'} d_\psi(x) d_\psi(g) \\
= \int_{N' \setminus G'} \int_{N' \setminus G'} \int_{N'} \psi_{N'}(u)^{-1} W_1(g) A^\psi_\#_{\#} (M^* W', x) \\
\times [\sigma'(g^{-1} u^{-1} x) W_2, W'_2]_{\sigma'} d_\psi(x) d_\psi(u) d_\psi(g) \\
= \int_{N' \setminus G'} \int_{N' \setminus G'} \int_{N'} \psi_{N'}(u) W_1(g) A^\psi_\#_{\#} (M^* W', x) \\
\times [\sigma'(ux) W_2, (\sigma')^{-1} (g) W'_2]_{\sigma'} d_\psi(x) d_\psi(u) d_\psi(g).
\]

By (3.3.1) and Corollary 3.1, this is equal to

\[
\omega_{\sigma'}(-1) \int_{N' \setminus G'} \int_{N' \setminus G'} W_1(g) A^\psi_\#_{\#} (M^* W', x) W_2(x) W'_2(g) d_\psi(x) d_\psi(g) \\
= \omega_{\sigma'}(-1) \cdot (W_1, W'_2)_{\sigma'} [W_2, W'_1]_{\sigma'}.
\]

Hence, we get

\[
\int_{G'} [\sigma(g) W_1, W'_1]_{\sigma'} [\sigma(g^{-1}) W_2, W'_2]_{\sigma'} d_\psi(g) = \omega_{\sigma'}(-1) \cdot (W_1, W'_2)_{\sigma'} [W_2, W'_1]_{\sigma'}. \quad (8.1.6)
\]

Then Theorem 8.1 follows from (8.1.4) and (8.1.6).

**8.1.1. General case.** Let \( U^-_{2n} \) be the even unitary group over \( F \) whose discriminant is different from that of \( U_{2n} \) and whose dimension is \( 2n \). For convenience, write \( U^+_{2n} = U_{2n} \).

In this section, we prove (8.1.1) for any discrete series representations of \( U^\pm_{2n} \), assuming local Langlands conjecture for these groups. Indeed, the local Langlands conjecture was established by Mok [42] for \( U^+_{2n} \) and Kaletha, Minguez, Shin and White [24] for \( U^-_{2n} \) with the stabilisation of the twisted trace formula established by Moeglin and Waldspurger [40, 41] – assuming the weighted fundamental lemma for quasi-split groups, which is proved in Chaudouard and Laumon [9] only in the split case.

Recall briefly the local Langlands conjecture in this case. Fix a splitting such that it gives Whittaker data \((B, \psi_{N'})\), and denote the \( L \)-group of \( U^\pm_{2n} \) by \( L U_{2n} = GL_{2n}(\mathbb{C}) \rtimes \text{Gal}(E/F) \) with the action \( \theta \in \text{Gal}(E/F) \) on \( GL_{2n}(\mathbb{C}) \) defined by \( \theta(g) = w_0 t^{-1} g^{-1} (w_0')^{-1} \).

Note that this action preserves the splitting. Also denote the connected component of \( L U_{2n} \) by \( \widehat{U}_{2n} \). The local Langlands conjecture for \( U^+_{2n} \) asserts that there exists a partition

\[
\text{Irr}_{\text{sqr}} U^+_{2n} = \bigsqcup_{\phi} \Pi_{\phi}
\]
Certain local identity and formal degree conjecture

into $L$-packets, where the disjoint union on the right-hand side runs over conjugacy classes of square integrable $L$-parameters $\phi : WD_F \to ^L U_{2n}$. Here, say that a continuous homomorphism $\phi : WD_F \to ^L U_{2n}$ is an $L$-parameter if $\phi$ is semisimple and $\phi|_{SL_2(C)}$ is algebraic, and that $\phi$ is square-integrable if the centraliser $S_\phi$ of the image of $\phi$ in $U_{2n}$ is finite, in which $S_\phi$ is an elementary abelian 2-group. Moreover, denoting by $\hat{S}_\phi$ the group of characters of $S_\phi$, there exists an injection

$$\Phi_\phi \to \hat{S}_\phi, \quad \sigma \mapsto \langle \cdot, \sigma \rangle,$$

whose image consists of the characters trivial on $\pm I_{2n}$ and which satisfies the suitable character identity. Furthermore, the Langlands conjecture for even unitary groups asserts that there exists a partition

$$\text{Irr}_{sqr}^+ U_{2n} \sqcup \text{Irr}_{sqr}^- U_{2n} = \bigsqcup /\Pi_1 \phi$$

indexed by equivalence classes of square-integrable $L$-parameters $\phi : WD_F \to ^L U_{2n}$ under the conjugation by $\hat{U}_{2n}$, and for each such $\phi$, there is a bijection

$$\Phi_\phi \to \hat{S}_\phi, \quad \sigma \mapsto \langle \cdot, \sigma \rangle$$

such that

$$\Pi_1^\pm : = \{ \sigma' \in \Pi_\phi : \langle -I_{2n}, \sigma' \rangle = \pm 1 \} = \Pi_\phi \cap \text{Irr}_{sqr} U_{2n}^\pm$$

and the endoscopic character relations hold.

**Corollary 8.1.** Assume that the local Langlands conjecture holds for $U_{2n}^\pm$. Then

$$d_\psi^G = |S_\phi| \lambda(E/F, \psi)^n \omega_{\sigma'}(-1)^\gamma(1, \sigma', \text{As}^-, \psi) d_{\sigma'}$$

(8.1.7)

holds for any square-integrable $L$-parameter $\phi : WD_F \to ^L U_{2n}$ and any $\sigma' \in \Pi_\phi$.

**Proof.** First, note that by the same argument as [20, p. 1325], we can reduce (8.1.7) in the case of $U_{2n}^-$ to the case of $U_{2n}^+$ using the endoscopic relations, (8.1.7) in the case of Steinberg representations in [17, 18], and the proof of [49, Corollary 9.10]. Moreover, in the case of $U_{2n}^+$, as in the proof of [20, Corollary 5.1], it can be shown using the character identity that all representations in $\Pi_\phi$ have the same formal degree. Hence, we can assume that $\langle \cdot, \sigma' \rangle$ is trivial, and thus it is generic by [42, Corollary 9.2.4] (see Remark 8.1). In this case, in a similar way as the proof of [42, Corollary 9.2.4] or [3, Proposition 8.3.2], we can find a quadratic extension of number fields $k'/k$, a place $v_0$ of $k$, an automorphic representation $\Pi$ of $\text{GL}_2n(A_{k'})$ and an irreducible globally generic cuspidal automorphic representation $\Sigma$ of $U_{2n}(A_{k'})$ such that

- $k_{v_0} = F, k' \otimes k_{v_0} = E$,
- $\Pi_{v_0}$ corresponds to the $L$-parameter $\iota \circ \phi$, where $\iota : ^L U_{2n}$ is the stable base change lift,
- $\Sigma$ weakly lifts to $\Pi$,
- $\Sigma_{v_0} \in \Pi_\phi$ and $\langle \cdot, \Sigma_{v_0} \rangle$ is trivial, i.e., $\Sigma_{v_0} = \sigma'$.

On the other hand, since the base change lift is strong by [25], we obtain

$$\Pi_{v_0} = \text{BC}(\Sigma_{v_0}).$$
Hence, by Theorem A.5, $BC(\sigma')$ should correspond to $\iota \circ \phi$. Then the corollary follows from Theorem 8.1.

**Remark 8.1.** Atobe [4, Theorem 3.1] gave a precise proof that $\sigma'$ is generic if $\langle \cdot, \sigma' \rangle$ is trivial, using expected desiderata on the local Langlands conjecture.

### 8.2. Archimedean case

In this section, we prove Conjecture 2 for generic discrete series representations of $U_{2n}(\mathbb{R})$ as a consequence of the formal degree conjecture.

**Lemma 8.1.** Let $\phi$ be a square-integrable $L$-parameter of $U_{2n}(\mathbb{R})$. Then we have

$$d_\psi = |S_\phi| \gamma(1, \sigma', As^{-}, \psi) d_{\sigma'},$$

for any $\sigma' \in \Pi_\phi$.

**Proof.** By [17, Proposition 2.1], we have

$$d_\psi = |S_\phi| |\gamma(1, \sigma', As^{-}, \psi)| d_{\sigma'}.$$

In this case, we can write $\phi = \oplus \chi_i$, with $\chi_i : WD_E \to \mathbb{C}^\times$. Then by a direct computation (or a similar computation to [11, Lemma 14.2]), we see that

$$\lambda(E/F, \psi)^n \omega_{\sigma'}(-1) \gamma(1, \sigma', As^{-}, \psi)$$

is a positive real number with the absolute value $|\gamma(1, \sigma', As^{-}, \psi)|$. Thus, the required identity holds.

**Theorem 8.2.** Conjecture 2 holds for any generic discrete series representations of $U_{2n}(\mathbb{R})$.

**Proof.** Suppose $\sigma' \in \text{Irr}_{\text{sqr}, \text{gen}} U_{2n}(\mathbb{R})$. Then $\pi := BC(\sigma') = \chi_1 \times \cdots \times \chi_{2n}$, where $\chi_i$ are mutually different unitary characters of $\mathbb{C}^\times$. Then we can find irreducible automorphic representations $\Pi_i$ of $A_{\mathbb{Q}(i)}$ such that $\Pi_i,\infty = \chi_i$. Set $\Pi = \Pi_1 \times \cdots \times \Pi_{2n}$. We see that $\pi$ is good, since $\Pi_\infty = BC(\sigma')$. In particular, there exists $c_\pi$ such that

$$\int_{N'} J \left( W, W', \frac{1}{2} \right) \psi_{N'}(n) dn = c_\pi W(e) A_\#^{\psi, \pi}(M^s W', e)$$

for any $W \in W_{N'}(\sigma')$ and $W' \in \text{Ind}(W_{N'M}(\pi))$. Then the same argument as in the proof of Theorem 8.1 gives

$$c_\pi = \omega_{\sigma'}(-1) \iff d_\psi = 2^k \lambda(E/F, \psi)^n \omega_{\sigma'}(-1) \gamma(1, c(\pi), As^{-}, \psi) d_{\sigma'},$$

where we have used the following in the real case:

1. The integral $J$ converges absolutely uniformly near $s = -\frac{1}{2}$ (cf. Remark 3.2).
2. Any Whittaker function $W \in W_{N'}(\sigma')$ is square-integrable over $N' \setminus G'$ (cf. [53, Theorem 15.3.4]).
Theorem A.4 for $U_{2n}(\mathbb{R})$ readily follows from the Langlands correspondence.

The absolute convergence of (8.1.5) can be proven by the same argument as [20, pp. 1316–1317], using [52, Theorem 7.2.1], [53, Theorem 15.2.4] and Lemma 3.5.

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**Appendix A. On the image of local base change lifts for generic representations**

In this appendix, using a similar argument as Jiang and Soudy [22] and Liu [35], we shall determine the image of local base change lifts for generic discrete series representations of even unitary groups $U_{2n}$ over a non-Archimedean local field $F$ of characteristic zero. Note that as in those two articles, we can write down Langlands parameters corresponding to these representations.

In this appendix, we will use the same notation as in the main body of the article. Further, for simplicity, say that a representation of $U_{2n}$ is generic if it is $\psi_{N}$-generic. For given representations $\tau_{i}$ of $GL_{k_{i}}(E)$, $i = 1, \ldots, a$, and $\rho$ of $U_{2m}$, denote by

$$\tau_{1} \times \cdots \times \tau_{a} \quad (\text{resp. } \tau_{1} \times \cdots \times \tau_{a} \times \rho)$$

the parabolic induction of $GL_{k_{1}+\cdots+k_{a}}$ (resp. $U_{2(k_{1}+\cdots+k_{a}+m)}$) for the parabolic subgroup with the Levi part $GL_{k_{1}}(E) \times \cdots \times GL_{k_{1}}(E)$ (resp. $GL_{k_{1}}(E) \times \cdots \times GL_{k_{1}}(E) \times U_{2m}$).

We would like to start my observation with the case of supercuspidal representations. Let $\Pi^{(sg)}(U_{2n})$ be the set of all equivalence classes of irreducible supercuspidal generic representations of $U_{2n}$. Let $\Pi^{(sg)}(GL_{2n}(E))$ be the set of all equivalence classes of irreducible tempered representations of $GL_{2n}(E)$ of the form

$$\tau_{1} \times \cdots \times \tau_{r},$$

where $\tau_{i}$ is an irreducible supercuspidal representation of $GL_{n_{i}}(E)$ such that $L(s, \tau_{i}, As^{+})$ has a pole at $s = 0$ and for $i \neq j$, $\tau_{i} \nleq \tau_{j}$. In [25], Kim and Krishnamurthy constructed local base change lifts of generic representations of $U_{2n}$ explicitly. Indeed, we have the following result:

**Theorem A.1** (Proposition 8.4 in [25]). There is a map $l$ from $\Pi^{(sg)}(U_{2n})$ to $\Pi^{(sg)}(GL_{2n}(E))$ which preserves local $\gamma$-factors with $GL$-twist, namely

$$\gamma^{Sh}(s, \pi \times \sigma, \psi) = \lambda(E/F, \psi)^{2nk} \gamma^{RS}(s, l(\pi) \times \sigma, \psi)$$

for any $\pi \in \Pi^{(sg)}(U_{2n})$ and any irreducible generic representation $\sigma$ of $GL_{k}(E)$ with $k \in \mathbb{N}$. Here, the $\gamma$-factor on the right-hand (resp. left-hand) side is defined by the Rankin–Selberg method [21] (resp. Langlands–Shahidi method [48, 49]).
Recently, we proved the following result:

**Theorem A.2** (Corollary 9.2 in [43]). The map \( l \) in Theorem A.1 is bijective and unique.

Theorem A.2 can be extended to the case of generic discrete series representations. Recall the construction of generic discrete series representations by Moeglin and Tadić [38]. To explain their construction, let us define some representations of general linear groups. Let \( \rho \) be an irreducible supercuspidal representation of \( \text{GL}_k(E) \). Then for integers \( l \geq m > 0 \) with the same parity, we write by \( D(l, m, \rho) \) the unique irreducible subrepresentation of

\[
\text{Ind}(|\det|^{(l-1)/2} \rho \otimes |\det|^{(l-1)/2-1} \rho \otimes \cdots \otimes |\det|^{-(m-1)/2} \rho)
\]

and by \( D(l, \rho) \) the unique subrepresentation of

\[
\text{Ind}(|\det|^{(l-1)/2} \rho \otimes |\det|^{(l-1)/2-1} \rho \otimes \cdots \otimes |\det|^{(l+1)/2-|l/2|} \rho).
\]

Then \( D(l, m, \rho) \) (resp. \( D(l, \rho) \)) is an essentially square integrable representation of \( \text{GL}_{k(l+m)}(E) \) (resp. \( \text{GL}_{l+|l/2|}(E) \)). Further, write

\[
\text{St}(\rho, l) = D(l, l, \rho).
\]

Next consider a parabolic induction

\[
D(l_1, m_1, \rho_1) \times \cdots \times D(l_r, m_r, \rho_r) \times D(l_{r+1}, \rho_{r+1}) \times \cdots \times D(l_t, \rho_t) \times \tau_0,
\]

where \( l_i > m_i > 0 \), \( \tau_0 \) is an irreducible supercuspidal representation of a smaller even unitary group and \( \rho_i \) is irreducible supercuspidal representations of \( \text{GL}_{k_i}(E) \) such that \( \rho_i \) is conjugate self-dual. Further, suppose that

\[
L(s, \rho_i, \text{As}^+) \text{ has a pole at } s = 0 \text{ if and only if } l_i \text{ is odd.}
\]

Since \( L(s, \rho_i, \text{As}^+) \) has a pole at \( s = 0 \) if and only if \( \rho_i \) is \( \text{GL}_{k_i}(F) \)-distinguished, according to Anandavardhanan, Kable and Tandon [2, Corollary 1.5], this condition is equivalent to the condition that

\[
\rho_i \text{ is } \text{GL}_{k_i}(F)\text{-distinguished if and only if } l_i \text{ is odd.}
\]

We know that \( \rho_i \) should be (\( \text{GL}_{k_i}(F), \eta_{E/F} \))-distinguished or \( \text{GL}_{k_i}(F) \)-distinguished by Kable [23, Theorem]. Moreover if \( \rho_i \) is (\( \text{GL}_{k_i}(F), \eta_{E/F} \))-distinguished, \( \rho_i \) is not \( \text{GL}_{k_i}(F) \)-distinguished, and vice versa [2, Corollary 1.6]. When \( \rho_i \) satisfies the condition (A.0.2), the unique generic constituent of the induced representation (A.0.1) is a discrete series representation.

When it is a representation of \( U_{2n} \), every discrete series representation is obtained in this way. Let \( \Pi^{(dg)}(U_{2n}) \) be the set of irreducible generic discrete series representations of \( U_{2n} \), namely the set of all irreducible representation obtained in this way.

Let \( \Pi^{(dg)}(\text{GL}_{2n}(E)) \) be the set of irreducible representations of \( \text{GL}_{2n}(E) \) of the form \( \pi = \tau_1 \times \cdots \times \tau_r \), where \( \tau_i \) is an irreducible discrete series representation of \( \text{GL}_{m_i}(E) \) such that \( L(s, \tau_i, \text{As}^+) \) has a pole at \( s = 0 \). We note that \( \tau_i \not\cong \tau_j \) if \( i \neq j \), because of the
irreducibility of $\pi$. Further, we know that $\tau_i$ is conjugate self-dual, in particular it is unitary. Then by [8] we can write

$$\tau_i = \text{St}(\rho_i, a_i),$$

where $a_i \in \mathbb{Z}_{\geq 0}$ and $\rho_i$ is an irreducible supercuspidal representation of $GL_{n_i}(E)$ such that $n_i = a_i m_i$. We have a necessary and sufficient condition for $\text{St}(\rho_i, a_i)$ to be $GL_{n_i}(F)$-distinguished.

**Theorem A.3** (Corollary 4.2 in [36]). Let $\rho$ be an irreducible supercuspidal representation of $GL_k(E)$ and $a \in \mathbb{N}$. Then the generalised Steinberg representation $\text{St}(\rho, a)$ of $GL_k(E)$ is $GL_k(F)$-distinguished if and only if $\rho$ is $(GL_k(F), \eta_{E/F}^{a-1})$-distinguished.

From this theorem, we see that $\rho_i$ is $\eta_{E/F}^{a_i-1}$-distinguished for any $i$. In particular, $\rho_i$ is conjugate self-dual. We prove the following generalisation of Theorems A.1 and A.2 to discrete series representations:

**Theorem A.4.** There is a bijective map $l$ from $\Pi^{(dg)}(U_{2n})$ to $\Pi^{(dg)}(GL_{2n}(E))$ satisfying the condition

$$\gamma^S(s, \pi \times \sigma, \psi) = \lambda(E/F, \psi)^{2nk} \gamma^R(s, l(\pi) \times \sigma, \psi)$$

for any $\pi \in \Pi^{(dg)}(U_{2n})$ and any irreducible generic representation $\sigma$ of $GL_k(E)$ with $k \in \mathbb{N}$. Further, this map is unique.

**Proof.** First, note that the uniqueness follows from a local converse theorem for generic representations of $GL_{2n}(E)$ by Henniart [16]. Following Kim and Krishnamurthy [25], define a map from $\Pi^{(dg)}(U_{2n})$ to $\Pi^{(dg)}(GL_{2n}(E))$. Let $\pi$ be an element of $\Pi^{(dg)}(U_{2n})$, and write it as (A.0.1). Then define $l(\pi)$ by

$$\text{St}(\rho_1, l_1) \times \text{St}(\rho_1, m_1) \times \cdots \times \text{St}(\rho_r, l_r) \times \text{St}(\rho_r, m_r) \times \text{St}(\rho_{r+1}, l_{r+1}) \times \cdots \times \text{St}(\rho_t, l_t) \times l(\tau_0),$$

which is irreducible by Bernstein and Zelevinsky [8]. Let us check that this representation is in $\Pi^{(dg)}(GL_{2n}(E))$. From Theorem A.1, $l(\tau_0)$ should be of the form $\Pi_1 \times \cdots \times \Pi_u$, where $\Pi_i$ are mutually distinct irreducible supercuspidal representations such that $L(s, \Pi_i, \text{As}^+)$ has a pole at $s = 0$. Thus, it suffices to check that $\text{St}(\rho_i, l_i)$ is $GL(F)$-distinguished. Indeed, $\text{St}(\rho_i, l_i)$ is $GL_{k_i}(F)$-distinguished if and only if $\rho_i$ is $(GL(F), \omega_{E/F}^{l_i-1})$-distinguished, by Theorem A.3. From (A.0.2), if $l_i$ is odd (resp. even), then $\rho_i$ is $GL_{k_i}(F)$-distinguished (resp. $(GL_{k_i}(F), \omega_{E/F}^{l_i-1})$-distinguished). Thus, $\text{St}(\rho_i, l_i)$ is $GL_{k_i}(F)$-distinguished.

From the definition of $\Pi^{(dg)}(GL_{2n}(E))$, the surjectivity of the map $l$ is clear. Further, from the local converse theorem [43, Theorem 9.4] for $U_{2n}$, its injectivity follows. 

Finally, we shall realise the map by local descent.

**Theorem A.5.** The map $\pi \mapsto D^Y_\psi(\pi)$ defines a bijection

$$D^Y_\psi : \Pi^{(dg)}(GL_{2n}(E)) \to \Pi^{(dg)}(U_{2n}).$$

Moreover, if $\pi \in \Pi^{(dg)}(GL_{2n}(E))$ and $\tilde{\pi} = D^Y_\psi(\pi \otimes \gamma)$, then

$$\gamma(s, \tilde{\pi} \times \tau, \psi) = \lambda(E/F)^{2nk} \gamma(s, \pi \times \tau, \psi)$$

for any irreducible generic representation $\tau$ of $GL_k(E)$. 


**Proof.** First, we shall prove that $D^\gamma\psi(\pi)$ is irreducible. From Theorem A.4, there is an irreducible discrete series representation $\sigma$ of $U_{2n}$ such that $l(\sigma) = \pi$, where $l$ is the map constructed in that theorem. Indeed, from its definition and the explicit description of local base change lift by Kim and Krishnamurthy [25], $l(\sigma) = \text{BC}(\sigma)$. Take number fields $L/K$ such that for some place $v_0$ of $K$, $L_{v_0} \simeq E$ and $K_{v_0} \simeq F$. If necessary, replacing $\psi$ by $\psi^a$ with some $a \in (F^\times)^2$, we can suppose that $\psi$ is a $v_0$-component of an additive character $\psi_{h_K}$ of $\mathbb{A}_K/K$. Then by [20, Corollary A.6], there is $\psi_{h_K}$-generic irreducible cuspidal automorphic representation $\Sigma$ of $U_{2n}(\mathbb{A}_K)$ such that $\Sigma_{v_0} = \sigma$. From the explicit construction of base change lifts, we have $\text{BC}(\Sigma)_{v_0} = \text{BC}(\sigma) = \pi$. Take a character $\eta$ of $\mathbb{A}_L$ such that its restriction to $\mathbb{A}_E$ is the quadratic character $\omega_{L/K}$ corresponding to $L/K$ and $\eta_{v_0} = \Upsilon$. Then by [32], $D^\eta\psi_{h_K}(\text{BC}(\Sigma))_{v_0} = D^\eta\psi(\pi)$ is irreducible.

Second, we shall prove $\text{BC}(D^\gamma\psi(\pi)) \in \Pi^{(d g)}(GL_{2n})$. From the preceding argument, we have a globalisation $\Pi := \text{BC}(\Sigma)$ of $\pi$. Further, from the unramified computation and strong multiplicity one theorem for $GL_{2n}$, we get $\text{BC}(D^\eta\psi_{h_K}(\Pi)) = \Pi \otimes \Upsilon^{-1}$ and thus $\text{BC}(D^\eta\psi_{h_K}(\Pi))_{v_0} = \pi \otimes \Upsilon^{-1}$. From the construction of base change lifts, we find that $\text{BC}(D^\eta\psi_{h_K}(\Pi))_{v_0} = \text{BC}(\text{BC}(D^\eta\psi_{h_K}(\Pi))_{v_0})$. From the definition of explicit local descent, we get $(D^\eta\psi_{h_K}(\Pi))_{v_0} = D^\gamma\psi(\pi)$. Therefore,

$$\text{BC}(D^\gamma\psi(\pi)) = \pi \otimes \Upsilon^{-1} \in \Pi^{(d g)}(GL_{2n}). \quad (A.0.3)$$

Third, we prove $D^\gamma\psi(\pi) \in \Pi^{(d g)}(G')$. From Theorem A.4 and the previous claim, there is $\sigma' \in \Pi^{(d g)}(G')$ such that $l(\sigma') = \text{BC}(D^\gamma\psi(\pi))$. The base change lift is strong, so that $\gamma(s, \text{BC}(D^\gamma\psi(\pi)) \times \tau, \psi) = \lambda(E/F)^n \gamma(s, D^\gamma\psi(\pi) \times \tau, \psi)$ for any irreducible generic representation $\tau$ of $GL_i$ with $1 \leq i \leq 2n$. Again by Theorem A.4, $\gamma(s, l(\sigma') \times \tau, \psi) = \lambda(E/F)^n \gamma(s, \sigma' \times \tau, \psi)$. Hence

$$\gamma(s, D^\gamma\psi(\pi) \times \tau, \psi) = \gamma(s, \sigma' \times \tau, \psi),$$

and the local converse theorem [43, Theorem 9.4] implies that

$$D^\gamma\psi(\pi) = \sigma' \in \Pi^{(d g)}(U_{2n}).$$

Further, note that if $D^\gamma\psi(\pi_1) = D^\gamma\psi(\pi_2)$, then $\pi_1 \simeq \pi_2$ by (A.0.3).

Finally, we shall prove $l(\pi) = D^\gamma\psi(\pi \otimes \Upsilon)$. Since the base change is strong, we have

$$\lambda(E/F)^{2nk} \gamma(s, \text{BC}(D^\gamma\psi(\pi)) \times \tau, \psi) = \gamma(s, D^\gamma\psi(\pi) \times \tau, \psi).$$

On the other hand, we have

$$\gamma(s, \text{BC}(D^\gamma\psi(\pi)) \times \tau, \psi) = \gamma(s, \pi \otimes \Upsilon^{-1} \times \tau, \psi)$$

by (A.0.3). Hence, we get

$$\lambda(E/F)^{2nk} \gamma(s, \pi \times \tau, \psi) = \gamma(s, D^\gamma\psi(\pi \otimes \Upsilon) \times \tau, \psi).$$

Then the assertion follows from the uniqueness in Theorem A.4. □
Appendix B. Stability of certain oscillatory integrals for quasi-split reductive groups and nonvanishing of Bessel functions

In this appendix, we shall prove the stability of certain oscillatory integrals for a quasi-split reductive group $G$ over a non-Archimedean local field $F$ of characteristic zero. This is a generalisation of [28] to quasi-split reductive groups.

B.1. Main results

Denote $G = G(F)$. Fix a Borel subgroup $B$ of $G$, and denote its unipotent radical by $N$. Let $\psi_N$ be a nondegenerate character of $N$. Consider the space $\Omega(N \backslash G, \psi_N)$ of smooth functions on $G$ such that $f(ng) = \psi_N(n)f(g)$ for all $n \in N, g \in G$. Denote the regular representation of $G$ on $\Omega(N \backslash G, \psi_N)$ by $R$. For any compact open subgroup $K$ of $G$, let $\Omega(N \backslash G, \psi_N)^K$ denote the subspace of right $K$-invariant functions. Let $w_0 \in G$ be a representative of the longest Weyl element $w_0$. Fix a Haar measure on $N$. For any compact open subgroup $N'$ of $N$ and $W \in \Omega(N \backslash G, \psi_N)$, let

$$R_{N', \psi_N} W := \frac{1}{\text{vol}(N')} \int_{N'} (R(n') W) \psi_N(n')^{-1} dn' \in \Omega(N \backslash G, \psi_N).$$

**Theorem B.1.** For any open subgroup $K$ of $G$, there exists an open compact subgroup $N'$ of $N$ such that for any $W \in \Omega(N \backslash G, \psi_N)^K$, $(R_{N', \psi_N} W)(\overline{w_0} \cdot \cdot)$ is compactly supported on $N$.

As a consequence, for any $W \in \Omega(N \backslash G, \psi_N)$ and a compact open subgroup $N'$ of $N$ such that $(R_{N', \psi_N} W)(g \cdot)$ is compactly supported on $N$, we can define the stable integral

$$\int_N W(\overline{w_0} n) \psi_N(n)^{-1} dn := \int_N (R_{N', \psi_N} W)(\overline{w_0} n) \psi_N(n)^{-1} dn.$$

More generally, we say that $g \in G$ is relevant (with respect to $\psi_N$) if $\psi_N(gng^{-1}) = \psi_N(n)$ for all $n \in N \cap g^{-1}Ng$. For any $g \in G_{\text{rel}}$ which is the set of relevant elements, the procedure in Theorem B.1 can be used to regularise the integral.

**Theorem B.2.** For any open subgroup $K$ of $G$ and $g \in G$, there exists an open compact subgroup $N'$ of $N$ such that for any $W \in \Omega(N \backslash G, \psi_N)^K, (R_{N', \psi_N} W)(g \cdot)$ is compactly supported on $g^{-1}Ng \cap N \backslash N$. In particular, when $g \in G_{\text{rel}}$, we can define

$$\int_{g^{-1}Ng \cap N \backslash N} W(gn) \psi_N(n)^{-1} dn := \int_{g^{-1}Ng \cap N \backslash N} (R_{N', \psi_N} W)(gn) \psi_N(n)^{-1} dn.$$

**Remark B.1.** Let $\pi$ be a $\psi_N$-generic irreducible admissible representation of $G$. Then by the uniqueness of Whittaker functionals, there is a function $\mathbb{B}_{\pi}^{\psi_N}$ on $G$ such that

$$\int_{g^{-1}Ng \cap N \backslash N} W(gn) \psi_N(n)^{-1} dn = \mathbb{B}_{\pi}^{\psi_N}(g) W(e)$$

for $g \in G_{\text{rel}}$ and $\mathbb{B}_{\pi}^{\psi_N}(g) = 0$ for $g \in G \setminus G_{\text{rel}}$. Then function $\mathbb{B}_{\pi}^{\psi_N}$ is called the Bessel function attached to $\pi$. 

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B.2. Notation

Suppose that \( G \) is a semisimple quasi-split group defined over \( F \) of rank \( r \), since the main results are easily reduced to the semisimple case. Write \( G = G(F) \). Let \( B \) be a Borel subgroup of \( G \), \( T \) be a maximal torus contained in \( B \), \( A \subset T \) be a maximal \( F \)-split torus of \( G \) and \( N \) be the unipotent radical of \( B \) so that \( B = TN \). Let \( \overline{B} = T\overline{N} \) be the opposite Borel subgroup with respect to \( T \). Let \( K \) be a hyperspecial maximal compact open subgroup of \( G \) in good position with respect to \( B \). Then we have the Iwasawa decomposition \( G = TNK \). Let \( X^*(Y) \) (resp. \( X_*(Y) \)) be the lattice of rational characters (resp. cocharacters) of a group \( Y \).

B.2.1. Relative roots and weights. Let \( \Phi \subset X^*(T) \) be the set of roots of \( T \) in \( g := \text{Lie}(G) \). Further, let \( \Phi_{\text{rel}} \subset X^*(A) \) be the set of roots of \( A \) in \( g \), \( \Phi_{\text{rel},+} \) be the subset of positive roots in \( \Phi_{\text{rel}} \) and \( \Delta_0 \) be the subset of simple roots with respect to \( (B, A) \) in \( \Phi_{\text{rel}} \). Similarly, let \( \Delta_0^\vee \subset \Phi_{\text{rel},+}^\vee \subset \Phi_{\text{rel}}^\vee \subset X_*(A) \) be the sets of (simple, or positive) co-roots. For \( \alpha \in X^*(A) \), denote by \( g_\alpha \) the \( \alpha \)-eigenspace in \( g \). Denote by \( \alpha \leftrightarrow \alpha^\vee \) the canonical bijection between \( \Phi_{\text{rel}} \) and \( \Phi_{\text{rel}}^\vee \) (resp. \( \Phi_{\text{rel},+} \leftrightarrow \Phi_{\text{rel},+}^\vee \)). Denote \( a_T := X_*(T) \otimes \mathbb{R} \).

For \( \alpha \in \Phi_{\text{rel}} \), denote by \( N_\alpha \) the unipotent subgroup whose Lie algebra is \( g_\alpha + g_{2\alpha} \). For \( \alpha \in \Delta_0 \), let \( P_\alpha \) be the standard parabolic subgroup with respect to \( \alpha \). Then its Levi subgroup \( M_\alpha \) has rank 1, as does the simply connected covering \( \tilde{M}_\alpha \). Then \( \tilde{M}_\alpha \) is \( SL_2(F_\alpha) \) or the quasi-split special unitary group \( SU(3)_\alpha \) with respect to a quadratic extension \( F_\alpha \) of \( F_\alpha \), where \( F_\alpha \) is a finite extension of \( F \). We say that \( \alpha \in \Delta_0 \) is of type (I) (resp. type (II)) if \( \tilde{M}_\alpha \) is isomorphic to \( SL_2(F_\alpha) \) (resp. \( SU(3)_\alpha \)).

Suppose that \( \alpha \in \Delta_0 \) is of type (II). Then \( SU(3)_\alpha \) can be realised by

\[
SU(3)_\alpha = \left\{ g \in SL(3, E_\alpha) : \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} g = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.
\]

Further, we can take the following group as a Borel subgroup of \( SU(3)_\alpha \):

\[
\tilde{B}_\alpha = T_\alpha \tilde{N}_\alpha,
\]

where

\[
T_\alpha = \left\{ t_\alpha(y) := \begin{pmatrix} y & \bar{y}/y & \bar{y}^{-1} \\ \bar{y}/y & \bar{y}^{-1} & \bar{y} / y \end{pmatrix} : y \in E_\alpha^* \right\}
\]

and

\[
\tilde{N}_\alpha = \left\{ x_\alpha(r, m) := \begin{pmatrix} 1 & r & m \\ 1 & -\bar{r} & \bar{r} m \end{pmatrix} : x, y \in E_\alpha \text{ s.t. } N_{E_\alpha/F_\alpha}(r) = -\text{Tr}_{E_\alpha/F_\alpha}(m) \right\}.
\]

Here, \( N_{E_\alpha/F_\alpha} \) (resp. \( \text{Tr}_{E_\alpha/F_\alpha} \)) is the norm (resp. trace) map from \( E_\alpha \) to \( F_\alpha \). Note that

\[
x_\alpha(r, m)^{-1} = x_\alpha(-r, \bar{m}).
\]
Hereafter, fix a covering map of $M_\alpha$ – namely, fix a map
\[ 1 \to \pi_1(M_\alpha) \to \tilde{M}_\alpha \xrightarrow{f_\alpha} M_\alpha \to 1, \]
so that $f_\alpha(\tilde{T}_\alpha) = T_\alpha := T \cap M_\alpha$ and $f_\alpha(\{x_\alpha(r, m) : r, m \in \mathcal{O}_{E_\alpha}\}) = N_\alpha \cap K$, where $\mathcal{O}_{E_\alpha}$ denotes the ring of integers of $E_\alpha$. Further, denote the image of $t_\alpha(y)$ and $x_\alpha(r, m)$ by the same symbol. Then $N_\alpha$ is the unipotent subgroup of $G$ generated by $\{x_\alpha(r, m) : r, m \in E_\alpha \}$ such that $N_{E_\alpha/F_\alpha}(r) = -\text{Tr}_{E_\alpha/F_\alpha}(m)$. For $\alpha \in \Delta_0$, write $N^\alpha$ for the unipotent radical of the parabolic subgroup $B \cup B_{\alpha}B$. Then we can write
\[ N = N_\alpha \times N^\alpha. \]
Further, for $r, m \in E_\alpha$ such that $N_{E_\alpha/F_\alpha}(r) = -\text{Tr}_{E_\alpha/F_\alpha}(m)$, define
\[ x_{-r\alpha}(r, m) := \left( \begin{array}{cc} 1 & -\bar{r} \\ -\bar{r} & 1 \\ m & r & 1 \end{array} \right). \]
Write the image of this element by the same symbol. From the definition, we can take the co-character $\alpha^\vee$ so that
\[ \alpha^\vee(x) = \left( \begin{array}{c} x \\ 1 \\ x^{-1} \end{array} \right), \quad x \in F^\times. \]
Further, extend $\alpha^\vee$ to $E_\alpha^\times \to T_\alpha$ by
\[ \alpha^\vee(x) = \left( \begin{array}{c} x \\ \bar{x}/x \\ \bar{x}^{-1} \end{array} \right), \quad x \in E_\alpha^\times. \]
More generally, for $\beta = w\alpha$ with $w \in W$ and $\alpha \in \Delta_0$ of type (II), define
\[ x_{\beta}(r, m) := wx_\alpha(r, m)w^{-1} \quad \text{and} \quad \beta^\vee := w \cdot (\alpha^\vee). \]
Suppose that $\alpha \in \Delta_0$ is of type (I). Then choose a parametrisation $n_\alpha : F_\alpha \to N_\alpha$ such that $n_\alpha(\mathcal{O}_{F_\alpha}) = N_\alpha \cap K$. Here, $\mathcal{O}_{F_\alpha}$ denotes the ring of integers of $F_\alpha$. Further, define
\[ \alpha^\vee(x) = \left( \begin{array}{c} x \\ x^{-1} \end{array} \right), \quad x \in F^\times, \]
and extend $\alpha^\vee$ to $F_\alpha^\times \to T_\alpha$ by
\[ \alpha^\vee(x) = \left( \begin{array}{c} x \\ x^{-1} \end{array} \right), \quad x \in F_\alpha^\times. \]
Similar to before, for $\beta = w\alpha$ with $w \in W$ and $\alpha \in \Delta_0$ of type (I), define
\[ n_\beta(x) := wx_\alpha(x)w^{-1} \quad \text{and} \quad \beta^\vee := w \cdot (\alpha^\vee). \]
On the other hand, let \( \lambda = \sum_{\alpha \in \Delta_0} r_\alpha \alpha \in a_0^* \), and define a character \( |\lambda| : A \to \mathbb{R}_+ \) by
\[
\lambda(t) = \prod_{\alpha \in \Delta_0} |\alpha(t)|^{r_\alpha}.
\]
For \( \alpha \in \Delta_0 \), we can consider \( \alpha \) as a character of \( \alpha^\vee(F_0^+) \) in the trivial way. Further, when \( \alpha \) is of type (II), we can extend a character \( |\alpha| \) to a character \( |\alpha_0| : T_{\alpha} \to \mathbb{R}_+ \), and naturally a character of \( T \). Indeed, define
\[
|\lambda_0| = \prod_{\alpha \in \Delta_0} |\alpha_0|^{r_\alpha}.
\]

By the Iwasawa decomposition, extend \( |\lambda_0| \) to a left \( N \)- and right \( K \)-invariant function on \( G \). For any compact subset \( C \subset G \), there exists a constant \( \kappa_C \) (depending also on \( \lambda \)) such that
\[
\kappa_C^{-1} |\lambda_0|(g) \leq |\lambda_0|(gh) \leq \kappa_C |\lambda_0|(g) \tag{B.0.1}
\]
for all \( g \in G \) and \( h \in C \).

Let \( H : T \to a_T \) be the Harish-Chandra map given by
\[
\exp(\chi, H(t)) = |\chi(t)|_F, \quad \chi \in X^*(T)_F.
\]
(Note that \( X^*(T) \otimes \mathbb{R} \simeq X^*(A) \otimes \mathbb{R} \).)

**B.2.2. Weyl group, Bruhat order and Bruhat decomposition.** Let \( W = N_G(A)/Z_G(A) = N_G(A)/T \) be the relative Weyl group of \( G \). Fix once and for all representatives \( \bar{w} \in K \). Note that a choice of \( \bar{w} \) is not unique in general. Then for any \( w_1, w_2 \in W \), we have \( w_1 w_2 = t w_1 \bar{w}_2 \), where \( t \in T \cap K \). As in [28], denote unspecified element on \( T \cap K \) by \( * \). Thus
\[
\bar{w}_1 \bar{w}_2 = \bar{*} w_1 \bar{w}_2,
\]
where \( * \) depends on \( w_1 \) and \( w_2 \) and also the choice of \( \bar{w}_1 \) and \( \bar{w}_2 \). We have the Bruhat decomposition
\[
G = \bigcup_{w \in W} B w B.
\]
Then for any \( g \in G \), we can write \( g = b \bar{w} n' \), where \( b \in B \), \( w \in W \) and \( n' \in N_{\bar{w}} := N \cap \bar{w}^{-1} N \bar{w} \). When we have \( g \in N t \bar{w} N \) with \( t \in T \) and \( w \in W \), we write \( w(g) = w \) and denote \( a(g) = t \).

Let
\[
C^*_+ = \left\{ \sum_{\alpha \in \Delta_0} c_\alpha \alpha : c_\alpha \in \mathbb{Z}_{\geq 0} \right\}.
\]
We write \( \leq \) for the Bruhat order on \( W \). If \( w_1 \leq w_2 \in W \), then for any dominant \( \lambda \in a_0^* \), we have
\[
w_1 \lambda - w_2 \lambda \in C^*_+.
\]
For any $w, w' \in W$, we have

$$BwB' B \subset \bigcup_{w'' \leq w'} Bw'' B$$

(cf. [28, (12)]).

Let $S(w)$ be the set of simple roots which appear in a reduced decomposition of $w$. (This set does not depend on the reduced decomposition.) It is the smallest set $S \subset \Delta_0$ such that $w$ belongs to the group generated by $\{s_\alpha : \alpha \in S\}$. Alternatively,

$$S(w) = \{ \alpha \in \Delta_0 : w\alpha^* \neq \alpha^* \}.$$ 

If $w_1 \leq w_2$, then $S(w_1) \subset S(w_2)$. Let $S^\circ(w) = S(ww_0)$. Thus $S^\circ(w) = \emptyset$ if and only if $w = w_0$, while $S^\circ(1) = \Delta_0$. Also, $S^\circ(w_1) \supset S^\circ(w_2)$ if $w_1 \leq w_2$.

For any $S \subset \Delta_0$, let $\Phi_{rel}(S)$ be the set of roots in $\Phi_{rel}$ which are linear combinations of roots in $S$. This is a root subsystem of $\Phi_{rel}$ which corresponds to the standard Levi subgroup determined by $S$. We have

$$\Phi_{rel}(S) = \{ \beta \in \Phi : \langle \alpha^*, \beta^\vee \rangle = 0 \text{ for all } \alpha \notin S \}.$$ 

Note that for any $w \in W$, $w_0S(w_0w) = -S^\circ(w)$ and hence

$$w_0 \Phi_{rel}(S(w_0w)) = \Phi_{rel}(S^\circ(w)).$$

For $\beta \in \Phi_{rel}$, denote by $s_\beta \in W$ the corresponding reflection.

**B.2.3. Spaces of Whittaker functions.** Let $\Omega(N \setminus G, \psi_N)$ denote the space of Whittaker functions on $G$. It is well known that for any normal open subgroup $K$ of $K$, we have

$$\sup |\alpha_0|(\text{supp}(W)) \ll_K 1$$

for all $W \in \Omega(N \setminus G, \psi_N)^K$ and $\alpha \in \Delta_0$ with an extension to $\alpha_0$ — that is, the image of $|\alpha_0|$ on $\text{supp} W$ is bounded above in terms of $K$ only. As in [28], following [5, Definition 5.1], we consider the space $\Omega^\circ(N \setminus G, \psi_N)$ consisting of those $W \in \Omega(N \setminus G, \psi_N)$ such that for any $w \in \mathbb{W}$ and $\alpha \in S^\circ(w)$, we have $\inf|\alpha_0|(\text{supp}_{BwB} W) > 0$. Here, $\alpha_0$ is an extension of $\alpha$ to $T$.

For $\alpha \in \Phi_{rel}$, denote by $\Phi_\alpha$ the subset of $\Phi$ consisting of roots whose restriction to $A$ is $\alpha$. Note that for any $\alpha \in \Phi_{rel}$, there is an extension $\beta \in \Phi$, but this is not unique in general.

For any $w \in W$ and $\epsilon > 0$, let

$$A^\epsilon(w) = \{ t \in T : |\beta^*(t)| \geq \epsilon \text{ for all } \beta \in \Phi_\alpha \text{ with } \alpha \notin S^\circ(w) \}.$$ 

Also, let

$$B^\epsilon(w) = \{ t \in T : |\beta^*(t)| \leq 1/\epsilon \text{ for all } \beta \in \Phi_\alpha \text{ with } \alpha \in S(w), |\beta^*(t)| \geq \epsilon \text{ for all } \beta \in \Phi_\alpha \text{ with } \alpha \notin S^\circ(w) \}.$$ 

Further, set $A^\epsilon_s(w) = A^\epsilon(w) \cap A$ and $B^\epsilon_s(w) = B^\epsilon(w) \cap A$. 

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*Certain local identity and formal degree conjecture*
We say that a set $C \subset G$ is bounded modulo $N$ if there exists a compact set $D \subset G$ such that $C \subset ND$. Denote by $\Omega^\#(N \backslash G, \psi_N)$ the subspace of those $W \in \Omega(N \backslash G, \psi_N)$ such that for any $w \in W$ and $\varepsilon > 0$, $\supp_{N_{A^\#}^w(wN)} W$ is bounded modulo $N$.

**B.2.4. The main technical statements.** Theorems B.1 and B.2 readily follow from the following technical results.

**Proposition B.1** (cf. Proposition 1 in [28] and Theorem 9.5 in [5]). We have $\Omega^\alpha(N \backslash G, \psi_N) \subset \Omega^\#(N \backslash G, \psi_N)$.

**Lemma B.1** (cf. Lemma 1 in [28]). If $W \in \Omega^\#(N \backslash G, \psi_N)$, then for any $w \in W$ and $\varepsilon > 0$, the function $(t,n) \mapsto W(t\bar{w}n)$, $(t,n) \in B^\varepsilon(w) \times N_w^-$, is compactly supported.

**Proposition B.2** (cf. Proposition 2 in [28] and Theorem 7.3 in [5]). For any normal open subgroup $K$ of $K$, there exists a compact open subgroup $N'$ of $N$ such that $R_{N',\psi_N} W \in \Omega^\alpha(N \backslash G, \psi_N)$ for any $W \in \Omega(N \backslash G, \psi_N)^K$.

**B.2.5. A lemma on Bruhat orders.** Suppose that $\alpha \in \Delta_0$ is of type (I). We can choose $n_{\pm \alpha}$ and $\overline{s}_\alpha$ and so that

$$n_\alpha(x)n_{-\alpha}(-x^{-1}) = \alpha^\vee(x)\overline{s}_\alpha n_\alpha(-x) \quad \text{ (B.0.2)}$$

and

$$\overline{s}_\alpha^{-1}n_\alpha(x)\overline{s}_\alpha = n_{-\alpha}(-x) = n_\alpha(-x^{-1})\alpha^\vee(x^{-1})\overline{s}_\alpha n_\alpha(-x^{-1}), \quad x \in F^\times_a \text{.} \quad \text{ (B.0.3)}$$

On the other hand, suppose that $\alpha$ is of type (II). Then we can take $\overline{s}_\alpha$ so that

$$x_\alpha(r,m)x_{-\alpha}(-rm^{-1}, \bar{m}^{-1}) = \alpha^\vee(m)\overline{s}_\alpha x_\alpha(-r\bar{m})/m, \bar{m} \quad \text{ (B.0.4)}$$

and

$$\overline{s}_\alpha x_\alpha(r,m)\overline{s}_\alpha^{-1} = x_{-\alpha}(-r,m) = x_\alpha(-rm^{-1}, m^{-1})\alpha^\vee(m^{-1})\overline{s}_\alpha x_\alpha(-r\bar{m}^{-1}, \bar{m}^{-1}) \quad \text{ (B.0.5)}$$

for any $r, m \in E_a$ such that $N_{E_a/F_a}(r) = -\Tr_{E_a/F_a}(m)$.

**Lemma B.2.** Let $w, w' \in W$ and $\alpha \in \Phi_{\text{rel},+}$. Assume that $w's_\alpha \leq w$ and $w\alpha \in \Phi_{\text{rel},+}$. Then $w\alpha \in \Phi(S(ww'^{-1}))$.

**Proof.** This is just a restatement of [28, Lemma 2] for $\Phi_{\text{rel}}$. \hfill $\Box$

**Lemma B.3.** Let $K$ be a compact open normal subgroup of $K$. Then for any $g \in G$ and $\alpha \in \Delta_0$, either

1. there exists $k \in K$ such that $w(g^{-1}k) = w(g)s_\alpha$ and $H(a(g^{-1}k)) = H(a(g))$, or
2. $w(g)\alpha \in \Phi_{\text{rel},-}$, $w(g^{-1}k) = w(g)$ and $H(a(g^{-1}k)) - H(a(g)) = x(w(w) \cdot (\alpha^\vee))$, with $x \ll K$.

**Proof.** We follow the proof of [28, Lemma 3]. Indeed, when $\alpha$ is of type (I), my lemma is proved in the same way as theirs. Now suppose that $\alpha$ is of type (II). Further, suppose that $g = \overline{s}n$, where $w = w(g) \in W$ and $n \in N_w^-$. When $w\alpha \in \Phi_{\text{rel},+}$, case (1) holds with
\( k = 1 \), since \( \overline{s_a}^{-1}n\overline{s_a} \in N \). Hence, we can assume that \( w\alpha \in \Phi_{rel-} \) and let \( w' = ws_a \), so that \( w'\alpha \in \Phi_{rel+} \).

Write \( n = n_1x_a(r, m) = x_a(r, m)n_2 \), where \( n_1, n_2 \in N^a \) and \( r, m \in E_a \), such that \( N_{E_a/F_a}(r) = -\text{Tr}_{E_a/F_a}(m) \). Suppose that \( |m| \) is so small that \( x_a(r, m) \in K \). Then \( k := \overline{s_a}^{-1}x_a(r, m)\overline{m} \in K \), and we have
\[
g\overline{s_a} = \overline{w}n_1x_a(r, m)\overline{s_a}k = \overline{w}n_1\overline{s_a} = *\overline{w}\overline{s_a}^{-1}n_1\overline{s_a} \in *\overline{w}N.
\]

Otherwise, write
\[
g\overline{s_a} = \overline{w}x_a(r, m)n_2\overline{s_a} = \overline{w}x_a(r, m)\overline{s_a}n_3,
\]
where \( n_3 = \overline{s_a}^{-1}n_2\overline{s_a} \in N \). From (B.0.5), we have
\[
g\overline{s_a} = *\overline{w}\overline{s_a}^{-1}x_a(r, m)\overline{s_a}n_3
\]
\[
= *\overline{w}x_a(-r\overline{m}^{-1}, m^{-1})\overline{\alpha}(\overline{m}^{-1})\overline{s_a}x_a(-r\overline{m}^{-1}, \overline{m}^{-1})n_3
\]
\[
= *(\overline{w}x_a(-r\overline{m}^{-1}, m^{-1})\overline{w}'\overline{\alpha}(\overline{m}^{-1})\overline{w}')\overline{x_a}(-r\overline{m}^{-1}, \overline{m}^{-1}).
\]

Since \( w'\alpha \in \Phi_{rel+} \), \( (\overline{w}x_a(-r\overline{m}^{-1}, m^{-1})\overline{w}') \in N \). Further, we know
\[
\overline{w}'\overline{\alpha}(\overline{m}^{-1})\overline{w}' = \overline{w}.(\overline{\alpha}(m)).
\]

Thus, we have
\[
H(a(g\overline{s_a})) = x(w(w) \cdot (\overline{\alpha})),
\]
with \( x \ll \text{K} 1 \).

For any subset \( S \subset \Delta_0 \) and \( X > 0 \), define
\[
\mathcal{C}(S)_{\geq -X} = \left\{ \sum_{\alpha \in S} c_\alpha \overline{\alpha} : c_\alpha \geq -X \quad \forall \alpha \in S \right\}.
\]

For \( w \in \text{W} \), fix its reduced decomposition by \( w = s_{a_k} \cdots s_{a_1} \), with \( \alpha_i \in \Delta_0 \). We shall use the following notational conventions, which implicitly depend on the chosen reduced decomposition (see the beginning of [28, 3.2]).

For any \( i = 1, \ldots, k \), let \( w_i = s_{a_i-1} \cdots s_{a_1} \) and \( iw = s_{a_k} \cdots s_{a_{i+1}} \), so that \( w = iw_{a_i}w_i \). Also write \( \beta_i = w_{a_i}^{-1}\alpha_i \), so that \( \{\beta_1, \ldots, \beta_k\} = \{\beta \in \Phi_{rel+} : w\beta \in \Phi_{rel-}\} \). Note that \( w\beta_i = -i_ww\alpha_i \in \Phi_{rel}(S(w)) \) for all \( \beta_i \).

We can also write \( N_i^{-} = i_NN_{\beta_i}N_i \), where \( N_i = N_{w_i}^{-} \) and \( iN = w_{a_i}^{-1}s_{a_i}^{-1}N_{i-w}^{-}s_{a_i}w_i = w_{i+1}^{-1}N_{i-w}^{-}w_{i+1} \). Note that \( \overline{N} = k \overline{N} = 1, N_{i+1} = N_{\beta_i} \times N_i \) and \( i_{-1}N = i_{-1} \times N_{\beta_i} \). Let \( n \in N_{w_i}^{-} \). For any \( i \), we can write uniquely
\[
n = \begin{cases} 
i_{n\beta_i}(x_i)n_i & \text{if } \alpha_i \text{ is of type (I)} \\
i_{n\beta_i}(r_i, m_i)n_i & \text{if } \alpha_i \text{ is of type (II)}, \end{cases}
\]
where \( i n < N \) \( n_i \in N_i \), \( x_i \in F_{\alpha_i} \) and \( r_i, m_i \in E_{\alpha_i} \) such that \( N_{E_{\alpha_i}/F_{\alpha_i}}(r_i) = -\text{Tr}_{E_{\alpha_i}/F_{\alpha_i}}(m_i) \).

We have

\[
n_{i+1} = \begin{cases} n_{\beta_i}(x_i)n_i & \text{if } \alpha_i \text{ is of type (I)} \\ n_{\beta_i}(r_i, m_i)n_i & \text{if } \alpha_i \text{ is of type (II)}. \end{cases}
\]  

(B.0.6)

\[\text{Lemma B.4.} \text{ Let } w = \iota ws_{\alpha_i} w_i \text{ be as before and let } g = \iota w n w_i, \text{ with } n \in N. \text{ Then } w(g) < w. \text{ Moreover, for any } K \text{ and any } n \in N, \text{ there exists } k \in K \text{ such that}
\]

(1) \( w(gk) = \iota w \tilde{w}_i \text{ for some } \tilde{w}_i \leq w_i \) (depending on \( g \)), and

(2) \( H(a(gk)) \in C(S^c(w(gk)))_{\leq X}, \text{ with } X \ll_K 1. \)

\[\text{Proof.} \text{ This lemma is proved similarly to [28, Lemma 4], using Lemma B.3 instead of [28, Lemma 3].} \]

\[\text{Lemma B.5.} \text{ Let } w \text{ be as before and suppose that } \alpha_i \text{ is of type (II). Let } g = \bar{w} n w_i^{-1} x_{\alpha_i}(r, m) w_i, \text{ with } n \in N \text{ and } r, m \in E_{\alpha_i}, \text{ such that } N_{E_{\alpha_i}/F_{\alpha_i}}(r) = -\text{Tr}_{E_{\alpha_i}/F_{\alpha_i}}(m). \text{ Then } w(g w_i^{-1} x_{-\alpha_i}(-r m^{-1}, m^{-1}) w_i) < w = w(g). \text{ Moreover, given } K, \text{ either } |m| \ll_K 1 \text{ or there exists } k \in K \text{ such that}
\]

(1) \( w(gk) < w = w(g) \),

(2) \( H(a(gk)) - v(m) w w_i^{-1} (\alpha_i^\vee) \in C(S^c(w(gk)))_{\leq X}, \text{ with } X \ll_K 1, \text{ and}
\]

(3) \( w w_i^{-1} \alpha_i \in \Phi_{rel}(S^c(w(gk))). \)

\[\text{Proof.} \text{ Let } k := w_i^{-1} x_{-\alpha_i}(-r m^{-1}, m^{-1}) w_i. \text{ If } |m| \text{ is sufficiently large with respect to } K, \text{ then } k \in K. \text{ By (B.0.4), we have}
\]

\[
gk = \bar{w} n w_i^{-1} \alpha_i^\vee(m) \bar{x}_{\alpha_i}(-r m/m, m) w_i,
\]

which can be written as

\[
(\bar{w} w_i^{-1} \alpha_i^\vee(m) w_i \bar{w}^{-1} \bar{w} n' w_i^{-1} \bar{x}_{\alpha_i}(-r m/m, m) w_i
\]

where \( n' = (w_i^{-1} \alpha_i^\vee(m) w_i)^{-1} n w_i^{-1} \alpha_i^\vee(m) w_i \). Changing \( n' \) for a conjugate of itself by an element of \( T \cap K \), if necessary, we can write this as

\[
*(\bar{w} w_i^{-1} \alpha_i^\vee(m) w_i \bar{w}^{-1}) \bar{w} n'' \bar{x}_{\alpha_i}(-r m/m, m) w_i
\]

where \( n'' = \bar{x}_{\alpha_i} n' \bar{x}_{\alpha_i} \bar{w}^{-1} \in N_{w}. \text{ Then we conclude that}
\]

\[
gk \in *(\bar{w} w_i^{-1} \alpha_i^\vee(m) w_i \bar{w}^{-1}) \bar{w} n \bar{w}.
\]

All but the last part directly follows from Lemma B.4. The last part follows from Lemma B.2 as in the proof of [28, Lemma 5].

\[\square \]

\[\text{Similarly, when } \alpha_i \text{ is of type (I), we have the following lemma, which is proved in the same way as [28, Lemma 5]:} \]
Lemma B.6. Let w be as before and suppose that \( \alpha_i \) is of type (I). Let \( g = \overline{w}n_{w_i}^{-1}n_{a_i}(x)w_i \), with \( n \in N \) and \( x \in F_{a_i} \).

Then \( \overline{w}n_{w_i}^{-1}x_{-a_i}(-x^{-1})w_i < w = \overline{w}(g) \). Moreover, given K, either \(|x| \ll_K 1\) or there exists \( k \in K \) such that

1. \( \overline{w}(gk) < w = \overline{w}(g) \),
2. \( H(\overline{a}(gk)) - v(m)\overline{w}n_{w_i}^{-1} \cdot (\alpha_i^\vee) \in C(S^\circ(\overline{w}(gk)))_{-X} \), with \( X \ll_K 1 \), and
3. \( \overline{w}n_{w_i}^{-1}\alpha_i \in \Phi_{rel}(S^\circ(\overline{w}(gk))) \).

Lemma B.7. Let \( w \in W \) and let \( \Phi_w = \{ \beta \in \Phi_{rel,+} : w^{-1}\beta < 0 \} \). Then for any \( n \in N \), we have \( H(\overline{w}n) = \sum_{\alpha \in \Phi_w} c_{\alpha} \alpha^\vee \), with \( c_{\alpha} \leq 0 \), for all \( \alpha \in \Phi_w \). Thus, \( |\alpha^*|(a(\overline{w}n)) \leq 1 \) for all \( \alpha \in \Delta_0 \), with equality if \( \alpha \not\in S(w) \). Moreover, the map \( n \mapsto H(\overline{w}n) \) from \( N^{-}_w \) to \( \alpha_T \) is proper.

Proof. This lemma is proved in the same way as [28, Lemma 6], using the relation

\[ H(\overline{w}n_{\alpha}(r, m)) = \min(0, -v(y))\alpha^\vee \]

for any \( \alpha \in \Delta_0 \) of type (I) (resp. type (II)), because of (B.0.3) (resp. (B.0.5)). \( \square \)

The following lemma is proved in the same way as [28, Lemma 7], which is a special case of Proposition B.1:

Lemma B.8. For any \( W \in \Omega^\circ(N \setminus G, \psi_N) \), \( w \in W \) and \( \varepsilon > 0 \), \( \text{supp}_{A^\varepsilon(w)w} W \) is compact.

B.3. Proof of Proposition B.1

We follow the argument in [28, 3.4] (see also the proof of [5, Theorem 9.5]).

We prove by induction on \( \ell(w) \) that for any \( w \in W \) and \( \varepsilon > 0 \),

\[ \text{for any } W \in \Omega^\circ(N \setminus G, \psi_N), \text{ supp}_{A^\varepsilon(w)wN} W \text{ is bounded modulo } N. \] \hfill (B.0.1)

For \( w = 1 \), this is a special case of Lemma B.8. Assume now that (B.0.1) holds for all \( w' \), with \( \ell(w') < \ell(w) \).

Let \( w = s_{a_k} \cdots s_{a_1} \) be a reduced decomposition of \( w \) and let \( \beta_1, \ldots, \beta_k, i w, w_i, N_i, N \) be as before. Let \( n \in N^{-}_w \) and write

\[ n = \begin{cases} \iota \nu \beta_i(x)n_i & \text{if } \alpha_i \text{ is of type (I)} \\ \iota \nu \beta_i(r, m)n_i & \text{if } \alpha_i \text{ is of type (II)} \end{cases} \]

To prove (B.0.1), we will show by induction on \( i \) that

\[ \text{for any } W \in \Omega^\circ(N \setminus G, \psi_N), \text{ supp}_{N^{-}_w N} W \text{ is bounded modulo } N. \] \hfill (B.0.2)

For \( i = k \) this follows from Lemma B.8, since \( k N = 1 \). For the induction step, assume that (B.0.2) holds for \( i \); we will show it for \( i - 1 \).

Let \( W \in \Omega^\circ(N \setminus G, \psi_N)^K \). Suppose that \( \alpha_i \) is of type (I), and assume that \( W(g) \neq 0 \), with \( g = t \overline{w}n_n \nu \beta_i(x), n \in i N \) and \( t \in A^\varepsilon(w) \). Then we can apply the same argument as the proof of [28, Proposition 1] by Lemma B.6, and thus (B.0.2) holds for \( i - 1 \). Now suppose
that \( \alpha_i \) is of type (II), and assume that \( W(g) \neq 0 \), with \( g = t \tilde{w}^{n}w_i^{-1}n_{a_i}(r, m)w_i \), \( n \in iN \) and \( t \in A^e(w) \).

By Lemma B.5, either \( |m| \ll_{W} 1 \) or there exists \( k \in K \) such that \( w' := w(gk) < w \), \( w\beta_i \in \Phi_{rel}(S^\circ(w')) \), and if \( t' = a(gk) \), then \( H(t') - H(t) - v(m)w_i^{-1} \cdot (\alpha_i^*) \in C(S^\circ(w'))_T \) with \( T \ll_{K} 1 \).

In the former case, we can use the induction hypothesis of (B.0.2) for the finitely many translations \( R(n') W \), where \( n' \) lies in a suitable compact subgroup of \( \tilde{w}_i^{-1}N_{a_i}\tilde{w}_i \) (depending only on \( W \)).

In the latter case,

\[ |\alpha^*(t')| = |\alpha^*(t)| \]

for all \( \alpha \notin S^\circ(w') \). Hence, \( t' \in A^e(w') \), since \( t \in A^e(w) \). Since \( w' < w \), we can apply the inductive assumption, namely (B.0.1) for \( w' \). Then we find that \( gk \) and therefore \( g \) is compactly supported modulo \( N \), which finishes the proof of Proposition B.1. \( \square \)

B.4. Proof of Lemma B.1
This is proved in the same way as [28, Lemma 1], using Lemma B.7 instead of [28, Lemma 6]. \( \square \)

B.5. Proof of Proposition B.2
Let \( U_0 = N \cap K \). Fix \( a \in A \) such that \( |\alpha(a)| > 1 \) for all \( \alpha \in \Delta_0 \) (and hence for all \( \alpha \in \Phi_{rel,+} \)). For any \( m \geq 1 \), define \( U_m = a^m U_0 a^{-m} \). Thus, \( U_1 \subset U_2 \subset \cdots \) and \( \bigcup_{m=1}^{\infty} U_m = N \). Set

\[ W_m := R_{U_m, \psi_N} W = \frac{1}{\text{vol}(U_m)} \int_{U_m} R(u) W \psi_N(u)^{-1} \, du \in \Omega(N \setminus G, \psi_N). \]

Clearly,

\[ W_m(gu) = \psi_N(u) W_m(g), \quad \text{for all } u \in U_m, g \in G. \quad (B.0.1) \]

Fix the reduced decomposition \( s_{\tilde{a}_1} \cdots s_{\tilde{a}_1} \) of \( w_0 \). Note that \( N_{w_0} = N \). Set

\[ \tilde{\beta}_i = \tilde{w}_i^{-1} \tilde{a}_i. \]

After renumbering if necessary, we can suppose that \( \tilde{a}_1, \ldots, \tilde{a}_s \) are of type (I) and \( \tilde{a}_{s+1}, \ldots, \tilde{a}_l \) are of type (II).

Lemma B.9 (cf. Lemma 8 in [28]). Let

\[ n = n_{\tilde{\beta}_i}(x_1) \cdots n_{\tilde{\beta}_m}(x_m) \cdot x_{m+1}(p_{m+1}, q_{m+1}) \cdots x_{p_1}(p_1, q_1). \]

Then \( u \in U_m \) if and only if \( |x_i| \leq |\tilde{\beta}_i(a^m)| \) and \( |q_j| \leq |\tilde{\beta}_i(a^{2m})| \) for all \( i, j \).

Proof. Clearly, we have

\[ a^m(N_{\tilde{\beta}_i} \cap K) a^{-m} = n_{\tilde{\beta}_i}(\tilde{\beta}_i(a^m)O) \]

for \( 1 \leq i \leq s \), and

\[ a^m(N_{\tilde{\beta}_i} \cap K) a^{-m} = \{ n_{\tilde{\beta}_i}(\tilde{\beta}_i(a^m)p, \tilde{\beta}_i(a^{2m})q) : p, q \in O_{E_{a_i}}, N_{E_{a_i}/E_{a_i}}(p) = -\text{Tr}_{E_{a_i}/R_{a_i}}(q) \} \]
for $s + 1 \leq i \leq l$. If $\tilde{\beta}_i(a^{2m})q \in \mathcal{O}_{E_{a_i}}$, then $\tilde{\beta}_i(a^m)p \in \mathcal{O}_{E_{a_i}}$, because of the relation $N_{E_{a_i}/F_{a_i}}(p) = -\text{Tr}_{E_{a_i}/F_{a_i}}(q)$. Then the ‘if’ direction follows.

We prove the ‘only if’ direction for $i$. For the induction step and the base of the induction, we can assume that $p_1 = q_1 = \cdots = p_{j-1} = q_{j-1} = 0$ – that is, $n \in \tilde{\mathcal{N}}$. In this case, observe that $x_{\beta_j} = \tilde{\beta}_j(a^{-m})x_{\beta_j}(a^{-2m})q_j = \bar{w_i}^{-1} \pi_{\beta_i}(\bar{w_i}^{-1} \bar{w_i}) \bar{w_i}$, where $\pi_{\beta_i} : N \to N_{a_i}$ is the canonical projection. Write $n = a^m n'a^{-m}$, where $n' \in \tilde{\mathcal{N}} \cap \mathbf{K}$. Then since $\pi_{\beta_i}$ is equivariant with respect to conjugation by $A$, we get

$$x_{\beta_j}(p_j, q_j) = \bar{w_i}^{-1} \pi_{\beta_i}(\bar{w_i} a^m n'a^{-m} \bar{w_i}) \bar{w_i} = a^m \bar{w_i}^{-1} \pi_{\beta_i}(\bar{w_i} n' \bar{w_i}^{-1}) \bar{w_i} a^{-m}.$$ 

or

$$x_{\beta_j}(\tilde{\beta}_i(a^{-m})p_j, \tilde{\beta}_i(a^{-2m})q_j) = \bar{w_i}^{-1} \pi_{\beta_i}(\bar{w_i} n' \bar{w_i}^{-1}) \bar{w_i} \in \mathbf{K}.$$ 

Then the required inequality follows.

**Lemma B.10** (cf. Lemma 9 in [28]). Let $n = i n x_{\beta_j}(p_i, q_i)$, with $i n \in \tilde{\mathcal{N}}$ and $p_i, q_i \in E_{a_i}$, such that $N_{E_{a_i}/F_{a_i}}(p_i) = -\text{Tr}_{E_{a_i}/F_{a_i}}(q_i)$. Assume that $x_{\beta_j}(p_i, q_i) \not\in U_m$. Then for any $n' \in U_m$, we have $n n' = i n x_{\beta_j}(p_i, q_i) n_i', n' \in \tilde{\mathcal{N}} \cap \mathbf{K}$, where $i n \in \tilde{\mathcal{N}}$ and $|q_i| = |q_i'|$.

**Proof.** Write $n' = i n' x_{\beta_j}(p_i', q_i') n_i'$, with $i n' \in \tilde{\mathcal{N}}$, $p_i, q_i \in E$, such that $N_{E/L}(p_i) = -\text{Tr}_{E/L}(q_i)$ and $n_i \in \tilde{\mathcal{N}}$. Then $n n' = i n x_{\beta_j}(p_i, q_i) n' x_{\beta_j}(p_i', q_i') n_i' = i n n' x_{\beta_j}(p_i + p_i', q_i + q_i' - p_i p_i') n_i'$, where $n' = x_{\beta_j}(p_i, q_i) n' x_{\beta_j}(p_i, q_i)$ and $n_i = x_{\beta_j}(p_i, q_i) n_i$. From the definition of $\mathcal{N}$, we can write $n_i = n_{\beta_i(i)}(x_{\beta_i(i)} \cdots n_{\beta_m(i)}(x_{\beta_m(i)} \cdots x_{\beta_{m(i)+1}}(p_{m(i)+1}, q_{m(i)+1}) \cdots x_{\beta_1}(p_1, q_1))$. Since $n_{j+1} = n_{j} x_{\beta_j}$ or $n_{j+1} = n_{j} x_{\beta_j}$, we see that $|q_i| = |\beta_i(a^{2m})|$ by Lemma B.9. Repeatedly using the lemma, we see that $n_i \in U_m$. Further, we see that $|q_i| = |\beta_i(a^{2m})|.

Since $x_{\beta_i}(p_i, q_i) \not\in U_m$, we have $|\beta_i(a^{2m})| < |q_i|$.

Then it is easy to see that $|p_i p_i'| < |q_i|$, and thus $|q_i + q_i' - p_i p_i'| = |q_i|$. The lemma follows.

Now let $w \in \mathbf{W}$ and set $N_w^+ = N \cap \hat{w}^{-1} N \hat{w}$. Fix a reduced decomposition $s_{a_k} \cdots s_{a_1}$ of $w$ and use the notation of Section B.2.5.
Corollary B.1. Let \( i = 1, \ldots, k \) and \( n = i n p_i(p_i, q_i) \), with \( iN \) and \( p_i, q_i \in E_{a_i} \), such that \( N_{E_{a_i}/F_{a_i}}(p_i) = -\text{Tr}_{E_{a_i}/F_{a_i}}(q_i) \). Assume that \( n_{p_i}(p_i, q_i) \notin U_m \). Then for any \( n' \in U_m \), we have \( nn' = \tilde{n}n_{p_i}(p_i, q_i)n_i \), where \( \tilde{n} \in N_{w}^+ \cdot iN \) \( n_i \in U_m \) and \( |q_i| = |\tilde{q}_i| > |\beta(a^{2m})| \).

For the convenience of the reader, we state a similar result in the case of type (I) by Lapid and Mao [28]:

Corollary B.2 (Corollary 1 in [28]). Let \( i = 1, \ldots, k \) and \( n = i n p_i(x_i) \) with \( iN \). Assume that \( n_{p_i}(x_i) \notin U_m \). Then for any \( n' \in U_m \), we have \( nn' = \tilde{n}n_{p_i}(p_i, q_i)n_i \), where \( \tilde{n} \in N_{w}^+ \cdot iN \) \( n_i \in U_m \) and \( |x_i| > |\beta(a^{2m})| \).

B.5.1. A special case.

Lemma B.11. There exists \( M \) such that for all \( W \in \Omega(N \setminus \Gamma, \psi_N) \), \( w \in W \) \( m \geq M \) and \( \alpha \in S(w) \), we have \( \inf|\alpha_0|(\text{supp}\_{Bw} W_m) = 0 \). Here, \( \alpha_0 \) is an extension of \( \alpha \) to \( T \).

Proof. Suppose that \( \alpha \in \Delta_0 \) is of type (I) (resp. type (II)). Then let \( \psi_{\alpha} : F_{\alpha} \rightarrow \mathbb{C} \) (resp. \( \psi_{\alpha} : E_{\alpha} \rightarrow \mathbb{C} \)) be the nontrivial character defined by

\[
\psi(n_{\alpha}(x)) \quad (\text{resp. } \psi \left( x_{\alpha} \left( x, -\frac{N_{E_{\alpha}/F_{\alpha}}(x)}{2} \right) \right))
\]

Denote by \( \text{cond}(\psi_{\alpha}) \) its conductor, namely the maximal fractional ideal of \( \mathcal{O}_{F_{\alpha}} \) or \( \mathcal{O}_{E_{\alpha}} \) on which \( \psi_{\alpha} \) is trivial. For any \( \alpha, \beta \in \Delta_0 \) of the same type, let \( \epsilon_{\alpha, \beta} \in F_{\alpha}^\times \) or \( E_{\alpha}^\times \) (depending on the type) such that \( \psi_{\beta} = \psi_{\alpha}(\epsilon_{\alpha, \beta}) \).

Suppose that \( \beta \in \Delta_0 \) is of type (II) and there exists \( \alpha \in \Phi_{\text{rel.}+, \Delta_0} \) such that \( \beta = w^{-1}\alpha \). Then we have

\[
\psi_{\beta}(x) W_m(t\overline{w}) = W_m \left( t\overline{w}x_{\beta} \left( x, -\frac{N_{E_{\alpha}/F_{\alpha}}(x)}{2} \right) \right) = W_m \left( x_{\alpha} \left( \ast_{\alpha_0}(t)x, -\frac{N_{E_{\alpha}/F_{\alpha}}(\ast_{\alpha_0}(t)x)}{2} \right) \overline{t} \right) = W_m(t\overline{w})
\]

for any \( t \in T \) and all \( x \in \beta(a^{m})\mathcal{O}_{E_{\alpha}} \). It follows that in fact \( W_m(s\overline{w}) = 0 \) for all \( s \in T \), provided that \( \beta(a^{m}) \notin \text{cond}(\psi_{\alpha}) \). If \( \beta \in \Delta_0 \) is of type (I) and there exists \( \alpha \in \Phi_{\text{rel.}+, \Delta_0} \) such that \( \beta = w^{-1}\alpha \), then my claim is provable in a similar way as before or as in the proof of [28, Lemma 10].

On the other hand, suppose that there does not exist \( \beta \in \Delta_0 \) such that \( \beta = w^{-1}\alpha \) for some \( \alpha \in \Phi_{\text{rel.}+, \Delta_0} \). Then using Steinberg [51, Lemma 89], in a similar way as [28, Lemma 10], we find that if \( W_m(s\overline{w}) \neq 0 \) and \( m \) is sufficiently large, then

\[
|\alpha_0|(s) = |\epsilon_{\alpha, w^{-1}\alpha}|
\]

which concludes the proof. \( \square \)
B.5.2. The general case. To prove Proposition B.2, we will show by induction on $\ell(w)$ that for any $w \in W$,

there exists $M$ depending on $K$ such that for any $W \in \Omega(N \setminus G, \psi_N)^K$, $m \geq M$ and $\alpha \in S^0(w)$, we have $\inf |\alpha_0|(\text{supp}_{B_w} W_m) > 0$. (B.0.2)

The case $w = 1$ follows from Lemma B.11. To carry out the induction step, assume that (B.0.2) holds for all $w' < w$. Fix a reduced decomposition of $w$ and use the notation of Section B.2.5. For any $m$ and $i = 1, \ldots, k$, let

$$B_w(i, m) = \{b \tilde{w} n : b \in B, n \in N_w^-, n_i \in U_m, n_{i+1} \notin U_m\}.$$ 

Consider the following auxiliary statement:

There exists $M$ depending on $K$ such that for any $W \in \Omega(N \setminus G, \psi_N)^K$, $m \geq M$ and $\alpha \in S^0(w)$, we have $\inf |\alpha_0|(\text{supp}_{B_w(i, m)} W_m) > 0$.

We will show this statement by induction on $i$. This will yield (B.0.2) for $w$. Indeed, we can take $M$ for which (B.0.3) holds for all $i$. Then for any $m \geq M$, we have

$$\inf |\alpha_0|(\text{supp}_{\cup_{i=1}^k B_w(i, m)} W_m) > 0.$$ (B.0.3)

On the other hand, the complement of $\cup_{i=1}^k B_w(i, m)$ in $B_w$ is $BwU_m$, and by (B.0.1) we have

$$\text{supp}_{BwU_m} W_m = (\text{supp}_{Bw} W_m) U_m.$$ 

Therefore, Lemma B.11 implies that

$$\inf |\alpha_0|(\text{supp}_{BwU_m} W_m) > 0$$

for all $\alpha \in S^0(w)$ as well.

It remains to prove (B.0.3). By (B.0.1), we can replace $B_w(i, m)$ in (B.0.3) by the set

$$B'_w(i, m) = \{b \tilde{w} n : b \in B, n \in N_w^-, n_i = 1, n_{i+1} \notin U_m\}.$$ 

Set $M$ (depending on $K$) such that (B.0.3) holds for all $j < i$ and (B.0.2) holds for all $w' < w$. Choose $M_1 \geq M$ depending only on $K$ such that

$$n_{-\beta_i}(\beta_i(a^{-M_1}) \mathcal{O}) \subset \bigcap_{n \in U_M} nKn^{-1}$$

when $\alpha_i$ is of type (I), and

$$\{n_{-\beta_i}(\tilde{\beta}_i(a^{-M_1}) p, \tilde{\beta}_i(a^{-2M_1}) q) : p, q \in \mathcal{O}_{E_{a_i}}, N_{E_{a_i}/E_{a_i}}(p) = -\text{Tr}_{E_{a_i}/F_{a_i}}(q)\} \subset \bigcap_{n \in U_M} nKn^{-1}$$

when $\alpha_i$ is of type (II) for $\beta_i = w_i^{-1} \alpha_i$. 

Assume that \( W \in \Omega(N \setminus G, \psi_N)^K \), \( g \in B'_w(i, m) \) and \( W_m(g) \neq 0 \), with \( m \geq M_1 \). Write \( g = t \omega n \). Since

\[
W_m(g) = \frac{1}{\text{vol}(U_m)} \int_{U_m} W_M(gn')\psi_N(n')^{-1} dn',
\]

there exists \( n' \in U_m \) such that \( W_M(gn') \neq 0 \). Let \( nn' = n_+ \tilde{n} \), where \( n_+ \in N_w^+ \) and \( \tilde{n} \in N_w^- \).

Write

\[
\tilde{n} = \begin{cases} 
1 \tilde{n}n_{\beta_i}(\tilde{x}_i)\tilde{n}_i & \text{if } \alpha_i \text{ is of type (I)}, \\
1 \tilde{n}n_{\beta_i}^{-}(\tilde{x}_i)\tilde{n}_i & \text{if } \alpha_i \text{ is of type (II)}. 
\end{cases}
\]

From the assumption on \( n \) and \( n' \), by Corollaries B.1 and B.2 we have \( |\tilde{x}_i| > |\beta_i(a)^m| \) and \( |\tilde{n}_i| > |\beta_i(a)^{2m}| \). In particular, \( \tilde{n}_{i+1} \notin U_m \) by (B.0.6).

Let \( j \leq i + 1 \) be the smallest index for which \( \tilde{n}_j \notin U_M \). If \( j < i \), then \( \bar{g} \in B_w(j - 1, M) \). Thus, we can apply our inductive assumption, and by the choice of \( M \) we have \( |\alpha_0(\bar{g})| \geq \delta_1 \) for all \( \alpha \in S^a(w) \), where \( \delta_1 > 0 \) depends only on \( W \). Hence, by (B.0.1) we also have \( |\alpha_0(g)| \geq \delta_2 \) for a suitable constant \( \delta_2 = \delta_2(m, W) > 0 \).

Assume that \( j = i + 1 \). Then \( \tilde{n}_i \in U_M \), and therefore \( W_M(g') \neq 0 \), where

\[
g' = \begin{cases} 
\tilde{t} \tilde{w}_i \tilde{n}_{\beta_i}(\tilde{x}_i) & \text{if } \alpha_i \text{ is of type (I)}, \\
\tilde{t} \tilde{w}_i \tilde{n}_{\beta_i}^{-}(\tilde{x}_i) \tilde{n}_i & \text{if } \alpha_i \text{ is of type (II)}. 
\end{cases}
\]

Here, note that \( g' \in N \tilde{g} \tilde{n}_i^{-1} \), since \( tw(n_+)(tw)^{-1} \). On the other hand, since \( \tilde{n}_{i+1} \notin U_M \), we get

\[
\begin{cases} 
x_i^{-1} \beta_i(a^{-m})O_{F_{\alpha_i}} & \text{if } \alpha_i \text{ is of type (I)}, \\
m_i^{-1} \beta_i(a^{-m})O_{F_{\alpha_i}} & \text{if } \alpha_i \text{ is of type (II)}, 
\end{cases}
\]

and thus \( W_M \) is right invariant by \( n_{-\beta_i}(-\tilde{x}_i^{-1}) \) (resp. \( n_{-\beta_i}(-rm^{-1}, \tilde{n}_i^{-1}) \)) if \( \alpha_i \) is of type (I) (resp. type (II)), since these belong to \( \cap n \in U_M nKn^{-1} \) by the choice of \( M_1 \). Hence, \( W_M(g'') \neq 0 \), where \( g'' := g' n_{-\beta_i}(-\tilde{x}_i^{-1}) \) (resp. \( g' n_{-\beta_i}(-rm^{-1}, \tilde{n}_i^{-1}) \)) if \( \alpha_i \) is of type (I) (resp. type (II)). By the first part of Lemmas B.5 and B.6, we have \( g'' = Bw'B \) with \( w' < w \). We conclude from the choice of \( M \) that \( |\alpha_0(g')| = |\alpha_0(g'')| \geq \delta_3 \) for all \( \alpha \in S^a(w') \) for a suitable constant \( \delta_3 = \delta_3(W, M) \), by my inductive assumption. In particular, this holds for all \( \alpha^2(w) \). Once again, by (B.0.1) we infer that \( |\alpha_0(g)| \geq \delta_4 \) for a suitable constant \( \delta_4 = \delta_4(W, m) \). This concludes the proof of Proposition B.2. \( \square \)

### B.6. Nonvanishing of Bessel functions

Keep the previous notations. Set \( B_0 = A \times N \) and \( G^0 = B_0 w_0 B_0 \).

**Theorem B.3.** For any tempered \( \pi \in \text{Irr}_{\text{gen}, \psi_N} G \), the function \( B_\pi \) is not identically zero on \( G^0 \).

**Proof.** By the same argument as the proof of [33, Theorem A.1], we can prove Theorem B.3 by substituting \( U_n \) for \( N_n \). \( \square \)
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