WAVE FUNCTIONS OF THE TODA CHAIN WITH BOUNDARY INTERACTION

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Abstract

In this contribution, we give an integral representation of the wave functions of the quantum $N$-particle Toda chain with boundary interaction. In the case of the Toda chain with one-boundary interaction, we obtain the wave function by an integral transformation from the wave functions of the open Toda chain. The kernel of this transformation is given explicitly in terms of $\Gamma$-functions. The wave function of the Toda chain with two-boundary interaction is obtained from the previous wave functions by an integral transformation. In this case, the difference equation for the kernel of the integral transformation admits separation of variables. The separated difference equations coincide with the Baxter equation.

1 Introduction

Recently, some progress in the derivation of the eigenfunctions of the Hamiltonians of some integrable quantum chains with finite number of particles has been achieved [1]–[7]. It is connected with the development of the method of separation of variables [1] for quantum integrable models. The first steps in the elaboration of this method were taken by Gutzwiller [2], who has found a solution of the eigenvalue problem for $N = 2, 3, 4$-particle periodic Toda chain.

Using the $R$-matrix formalism, Sklyanin [3] proposed an algebraic formulation of the method of separation of variables applicable to a broader class of integrable quantum chains. The next important step was taken by Kharchev and Lebedev [4], who combined the analytic method of Gutzwiller and algebraic approach of Sklyanin. They obtained the eigenfunctions of the $N$-particle periodic Toda chain by some integral transformation of the eigenfunctions of an auxiliary problem, the open $(N - 1)$-particle Toda chain. It turned out that the kernel of this transformation admits separation of variables. The separated equations coincide with the Baxter equation. A solution of this equation has been found in [8] (see also [4]).

Later Kharchev and Lebedev [5] have found a remarkable recurrence relation between the eigenfunctions of the $N$-particle and $(N - 1)$-particle open Toda chains. Understanding these formulas from the viewpoint of the representation theory [6] made it possible to extend their approach to other integrable systems [6, 7].

In this paper, we apply this method to the derivation of the eigenfunctions of the commuting Hamiltonians of the $N$-particle quantum Toda chain with boundary interaction. We use the Sklyanin approach [9] to the boundary problems for the quantum integrable models. The $N$-particle eigenfunctions of the quantum Toda chain in which the first and last particles exponentially interact with the walls (the two-boundary interaction) is constructed by means of an integral transformation of the eigenfunctions for the Toda chain with one-boundary interaction (the auxiliary problem). These eigenfunctions, in turn, are constructed using the eigenfunctions of the $N$-particle open Toda chain. Such a complicated hierarchy allows one to separate the variables in the difference equation for the kernel of the mentioned integral transformation reducing it to a version of the Baxter equation. We note that, for the classical Toda chain with general boundary interaction, the separation of variables was performed by Kuznetsov [10].

2 Integrals of motion of the open Toda chain

To describe the integrals of motion of the quantum $N$-particle open Toda chain, we use the $L$-operators (one for each particle)

$$L_k(u) = \begin{pmatrix} u - p_k & e^{-q_k} \\ -e^{q_k} & 0 \end{pmatrix}, \quad k = 1, 2, \ldots, N.$$
where $N$ is the number of particles in the chain, $p_k$ and $q_k$ are the operators of momentum and position of the $k$-th particle, respectively. The monodromy matrix is defined as

$$T(u) := L_N(u) L_{N-1}(u) \cdots L_2(u) L_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (1)$$

The commutation relations for the matrix elements of $T(u)$ follow from the canonical commutation relations

$$[p_k, q_l] = -i\hbar \delta_{kl}$$

and can be written as

$$R(u-v) (T(u) \otimes \mathbf{1}) (\mathbf{1} \otimes T(v)) = (\mathbf{1} \otimes T(v)) (T(u) \otimes \mathbf{1}) R(u-v), \quad (2)$$

where $R(u)$ is the rational $R$-matrix:

$$R(u) = \begin{pmatrix} 1 + \frac{i\hbar}{u} & 0 & 0 & 0 \\ 0 & 1 & \frac{i\hbar}{u} & 0 \\ 0 & \frac{i\hbar}{u} & 1 & 0 \\ 0 & 0 & 0 & 1 + \frac{i\hbar}{u} \end{pmatrix}. \quad (3)$$

From (1) it follows that $A(u)$ is a polynomial of degree $N$ in $u$:

$$A(u) = \sum_{m=0}^{N} (-1)^m u^{N-m} H_m(p_1, q_1; p_2, q_2; \ldots; p_N, q_N) = u^N - H_1 u^{N-1} + H_2 u^{N-2} - \cdots + (-1)^N H_N.$$ 

In particular, relations (2) give

$$[A(u), A(v)] = 0,$$

and, therefore, $[H_m, H_k] = 0$, that is, $A(u)$ is a generating function for the commuting operators $H_m$. Since

$$H_1 = \sum_{k=1}^{N} p_k, \quad H_2 = \sum_{k,l} p_k p_l - \sum_{k=1}^{N-1} e^{q_k-q_{k+1}},$$

we get the Hamiltonian for the open Toda chain in the form

$$H = H_1^2/2 - H_2 = \sum_{k=1}^{N} \frac{p_k^2}{2} + \sum_{k=1}^{N-1} e^{q_k-q_{k+1}}.$$ 

Therefore, the operators $H_m$ are Hamiltonians for the open Toda chain.

### 3 Wave functions for the open Toda chain

Let a wave function $\psi(q_1, \ldots, q_N)$ for the open Toda chain be a common eigenfunction of the commuting Hamiltonians $H_m$:

$$H_m \psi(q_1, \ldots, q_N) = E_m \psi(q_1, \ldots, q_N).$$

Then

$$A(u) \psi_{\gamma_N}(q_1, \ldots, q_N) = \prod_{l=1}^{N} (u - \gamma_{NL}) \psi_{\gamma_N}(q_1, \ldots, q_N),$$

where $\gamma_N = (\gamma_{N1}, \gamma_{N2}, \ldots, \gamma_{NN})$ are the quantum numbers of the $N$-particle system, $E_m = e_m(\gamma_{N1}, \gamma_{N2}, \ldots, \gamma_{NN})$, and $e_m$ is the $m$-th elementary symmetric polynomial. For every set $\gamma_N$, the space of eigenfunctions is $N!$ dimensional. The physical eigenfunction $\psi_{\gamma_N}$ is fixed by the requirement that $\psi_{\gamma_N}$ rapidly decreases in the classically
forbidden region, that is, for \( q_k >> q_{k+1} \) for some \( k \). For \( q_1 << q_2 << \cdots << q_N \), \( \psi_{\gamma N} \) is a superposition of plane waves.

Recently, Kharchev and Lebedev [5] have found a recursive procedure of constructing the \( N \)-particle wave function \( \psi_{\gamma N}(q_1, q_2, \ldots, q_N) \) through the \( (N-1) \)-particle wave functions \( \psi_{\gamma_{N-1}}(q_1, q_2, \ldots, q_{N-1}) \). The recurrence relation is

\[
\psi_{\gamma N}(q_1, q_2, \ldots, q_N) = \int d\gamma_{N-1} \mu(\gamma_{N-1}) Q(\gamma_{N-1}|\gamma_N) \psi_{\gamma_{N-1}}(q_1, q_2, \ldots, q_{N-1}) e^{\frac{i\pi}{\hbar} \sum_{j=1}^{N-1} \gamma_{N,j} - \sum_{k=1}^{N} \gamma_{N-1,k} q_N},
\]

where integration is carried out with respect to \( \gamma_{N-1,k} \), \( k = 1, 2, \ldots, N-1 \), along any set of the lines parallel to the real axis and such that

\[
\min_k \text{Im} \gamma_{N-1,k} > \max_j \text{Im} \gamma_{N,j}, \quad j = 1, \ldots, N,
\]

\[
Q(\gamma_{N-1}|\gamma_N) = \prod_{k=1}^{N-1} \prod_{j=1}^{N} h^{N-1,k-\gamma_{N,j}} \Gamma \left( \frac{\gamma_{N-1,k} - \gamma_{N,j}}{i\hbar} \right), \quad \mu^{-1}(\gamma_{N-1}) = \prod_{k,l} \Gamma \left( \frac{\gamma_{N-1,k} - \gamma_{N-1,l}}{i\hbar} \right).
\]

In a similar way, the \( (N-1) \)-particle wave functions can be expressed through the \( (N-2) \)-particle wave functions, and so on. The wave function for the 1-particle open Toda chain is just a plane wave:

\[
\psi_{\gamma 11}(q_1) = e^{\frac{i\pi}{\hbar} \gamma 11 q_1}.
\]

In what follows, we use the notation \( \gamma := \gamma N \), \( \gamma_k := \gamma_{N,k} \). As shown in [5], the wave function \( \psi_{\gamma} \) satisfies the relations

\[
A(u) \psi_{\gamma} = \prod_{l=1}^{N} (u - \gamma_l) \psi_{\gamma},
\]

\[
B(u) \psi_{\gamma} = i^{N-1} \sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{u - \gamma_l}{\gamma_p - \gamma_l} \right) \psi_{\gamma + p},
\]

\[
C(u) \psi_{\gamma} = i^{-N-1} \sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{u - \gamma_l}{\gamma_p - \gamma_l} \right) \psi_{\gamma - p},
\]

where \( \psi_{\gamma \pm p} := \psi_{\gamma_1, \gamma_2, \ldots, \gamma_p \pm i\hbar, \ldots, \gamma_N} \).

In order to find the action of \( D(u) \) on \( \psi_{\gamma} \), we use the following property of the quantum determinant of \( T(u) \) for the Toda chain:

\[
D(u) A(u - i\hbar) - C(u) B(u - i\hbar) = 1.
\]

The result is

\[
D(u) \psi_{\gamma} = \sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{u - \gamma_l}{\gamma_p - \gamma_l} \right) \frac{1}{i\hbar} \left( \frac{1}{\prod_{l \neq p} (\gamma_p - \gamma_l + i\hbar)} - \frac{1}{\prod_{l \neq p} (\gamma_p - \gamma_l - i\hbar)} \right) \psi_{\gamma - p} \psi_{\gamma + p - q}.
\]

### 4 Integrals of motion of the Toda chain with boundary interaction

In this section, we give a sketch of the \( R \)-matrix formalism for the quantum Toda chain with boundary interaction proposed by Sklyanin [9]. This formalism is important for the construction of wave functions. The key object in this approach is the matrix

\[
U(u) := T(u) K^{(-)}(u - \frac{i\hbar}{2}) \hat{T}(-u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},
\]
where $T(u)$ is the monodromy matrix \(^{(1)}\) of the $N$-particle open Toda chain, and

\[
\hat{T}(-u) = \sigma_2 T^t(-u)\sigma_2 = (\sigma_2 L_1^t(-u)\sigma_2)(\sigma_2 L_2^t(-u)\sigma_2) \cdots (\sigma_2 L_N^t(-u)\sigma_2).
\]

Here, $\sigma_2$ is the Pauli matrix. The matrix $K(-)(u - i\hbar/2)$ is

\[
K(-)(u - i\hbar/2) = \begin{pmatrix}
\alpha_1 & u - \frac{i\hbar}{2} \\
-\beta_1(u - \frac{i\hbar}{2}) & \alpha_1
\end{pmatrix}, \quad (13)
\]

As shown in \(^{(3)}\), the matrix $U(u)$ satisfies the reflection equation

\[
R(u - v) (U(u) \otimes 1) R(u + v - i\hbar) (1 \otimes U(v)) = (1 \otimes U(v)) R(u + v - i\hbar) (U(u) \otimes 1) R(u - v), \quad (14)
\]

where $R(u)$ is given by \(^{(3)}\).

This equation implies $\mathcal{B}(u)\mathcal{B}(v) = \mathcal{B}(v)\mathcal{B}(u)$. Therefore, the expansion of $\mathcal{B}(u)$ in powers of $u$ gives commuting operators which, in fact, are the Hamiltonians of the one-boundary Toda chain

\[
\mathcal{B}(u) = (-1)^N(u - i\hbar/2)^2 (u^{2N} - u^{2N-2} H_1^B + u^{2N-4} H_2^B - \cdots + (-1)^N H_N^B), \quad (15)
\]

where

\[
H_k^B = \sum_{k=1}^N p_k^2 + 2 \sum_{k=1}^{N-1} e^{q_k - q_{k+1}} - 2\alpha_1 e^{-q_1} + \beta_1 e^{-2q_1}.
\]

Here the last two terms describe interaction of the first particle with the wall.

The Sklyanin’s transfer-matrix

\[
t(u) := \text{Tr} K^{(+)}(u + i\hbar/2) U(u), \quad (16)
\]

where

\[
K^{(+)}(u + i\hbar/2) = \begin{pmatrix}
\alpha_N & \beta_N(u + \frac{i\hbar}{2}) \\
-(u + \frac{i\hbar}{2}) & \alpha_N
\end{pmatrix},
\]

satisfies the commutation relation \(^{(9)}\)

\[
t(u)t(v) = t(v)t(u). \quad (17)
\]

Hence, $t(u)$ is a generating function for commuting operators which, in fact, are the Hamiltonians of the two-boundary Toda chain.

For simplicity, in what follows, we fix $\beta_1 = \beta_N = 0$ and use the notation $e^{\kappa_1} := -2\alpha_1$, $e^{-\kappa_N} := -2\alpha_N$. In this case, we have

\[
t(u) = (-1)^{N-1}(u^2 + k^2/4) (u^{2N} - u^{2N-2} H_1^{BB} + u^{2N-4} H_2^{BB} - \cdots + (-1)^N H_N^{BB}) + 2\alpha_1\alpha_N, \quad (18)
\]

where

\[
H_1^{BB} = \sum_{k=1}^N p_k^2 + 2 \sum_{k=1}^{N-1} e^{q_k - q_{k+1}} - 2\alpha_1 e^{-q_1} - 2\alpha_N e^{q_N}.
\]

In the case of the Toda chain, the matrix $U(u)$ has some additional symmetry (unitarity) \(^{(9)}\):

\[
\begin{pmatrix}
\mathcal{A}(-u) & \mathcal{B}(-u) \\
\mathcal{C}(-u) & \mathcal{D}(-u)
\end{pmatrix} = \frac{1}{2u - i\hbar} \begin{pmatrix}
-i\hbar\mathcal{A}(u) + 2u\mathcal{D}(u) & -(2u + i\hbar)\mathcal{B}(u) \\
-2u\mathcal{C}(u) - i\hbar\mathcal{A}(u) & 2u\mathcal{A}(u) - i\hbar\mathcal{D}(u)
\end{pmatrix}. \quad (19)
\]

In particular, this leads to

\[
\mathcal{A}(u) = \frac{1}{u} \left( (u - \frac{i\hbar}{2})\mathcal{D}(-u) + \frac{i\hbar}{2}\mathcal{D}(u) \right). \quad (20)
\]

Therefore, using this equality and \(^{(16)}\), we obtain

\[
t(u) = \alpha_N \left( \frac{u + \frac{i\hbar}{2}}{u} \right) \mathcal{D}(u) + \alpha_N \left( \frac{u - \frac{i\hbar}{2}}{u} \right) \mathcal{D}(-u) - \left( u + \frac{i\hbar}{2} \right) \mathcal{B}(u). \quad (21)
\]
Using (12), we obtain the following expressions for the matrix elements of $U(u)$ in terms of the matrix elements of the monodromy matrix $T(u)$ for the $N$-particle open Toda chain:

$$A(u) = \alpha_1 (A(u)D(-u) - B(u)C(-u)) - \left( u - \frac{i}{2} \right) A(u)C(-u),$$  \hfill (22)

$$B(u) = -\alpha_1 (A(u)B(-u) - B(u)A(-u)) + \left( u - \frac{i}{2} \right) A(u)A(-u),$$  \hfill (23)

$$C(u) = \alpha_1 (C(u)D(-u) - D(u)C(-u)) - \left( u - \frac{i}{2} \right) C(u)C(-u),$$  \hfill (24)

$$D(u) = \alpha_1 (D(u)A(-u) - C(u)B(-u)) + \left( u - \frac{i}{2} \right) C(u)A(-u).$$  \hfill (25)

We give some examples:

$N = 1$:

$$B(u) = -(u - i\hbar/2)(u^2 - (p_1^2 + e^{\gamma_1})),
\quad t(u) = (u^2 + \hbar^2/4) \left( u^2 - (p^2 + e^{\gamma_1 - \gamma_2} + e^{\gamma_1 - \gamma_2}) + 2\alpha \alpha' \right).$$

$N = 2$:

$$B(u) = (u - i\hbar/2) \left( u^4 - u^2 (p_1^2 + p_2^2 + 2e^{\gamma_1 - \gamma_2} + e^{\gamma_1 - \gamma_2}) + (p_1p_2 - e^{\gamma_1 - \gamma_2})^2 - \alpha_1 p_2^2 e^{-\gamma_1} - 2\alpha_1 e^{-\gamma_2} \right),
\quad t(u) = -(u^2 + \hbar^2/4) \left( u^4 - u^2 (p_1^2 + p_2^2 + 2e^{\gamma_1 - \gamma_2} + e^{\gamma_1 - \gamma_2} + e^{-\gamma_2}) + \cdots \right) + 2\alpha_1 \alpha_2.$$

5 Wave functions for the one-boundary Toda chain

We define the function $\Psi_{\lambda} \equiv \Psi_{\lambda_1, \ldots, \lambda_N}$ as

$$\Psi_{\lambda}(q_1, \ldots, q_N) = \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda)e^{-\frac{\hbar}{2} \Gamma(\gamma_1 - \gamma_N)} \psi_\gamma(q_1, \ldots, q_N),$$  \hfill (26)

where $e^{\gamma_1} = -2\alpha_1$ and

$$Q(\gamma|\lambda) = \prod_{k<l} \frac{\Gamma(\frac{\lambda_k - \gamma_k}{\hbar}) \Gamma(\frac{-\lambda_l - \gamma_k}{\hbar})}{\Gamma(\frac{-\lambda_k - \gamma_l}{\hbar}) \Gamma(\frac{\lambda_l - \gamma_k}{\hbar})} \prod_k \hbar \frac{\Gamma(N+1)\gamma_k}{\Gamma(\frac{\lambda_k - \gamma_k}{\hbar})},
\quad \mu^{-1}(\gamma) = \prod_{k<l} \Gamma\left( \frac{\gamma_k - \gamma_l}{\hbar} \right).$$  \hfill (27)

We show that this is a wave function for the quantum one-boundary Toda chain, and

$$B(u)\Psi_{\lambda}(q_1, \ldots, q_N) = (-1)^N (u - \frac{i\hbar}{2}) \prod_{l=1}^N (u^2 - \lambda_l^2) \Psi_{\lambda}(q_1, \ldots, q_N),$$  \hfill (28)

where the structure of the right-hand side corresponds to (15). The integration in (26) is carried out along any set of lines parallel to the real axis and such that

$$\max_k \text{Im} \gamma_k < -\min_j \text{Im} \lambda_j, \quad k = 1, 2, \ldots, N, \quad j = 1, \ldots, N.$$  \hfill (29)

First, we prove the absolute convergence in (26). For this, we use the inequalities

$$|\Gamma(x + iy)| \leq \Gamma(x)p_x(|y|)e^{-\frac{x|y|}{2}}, \quad x > 0,$$

where $p_x(|y|)$ is some polynomial in $|y|$ with degree linearly depending on $x$,

$$\frac{1}{|\Gamma(x + iy)|} \leq \frac{\left(1 + \frac{|y|}{x}\right) e^{-\frac{x|y|}{2}}}{\Gamma(x)}, \quad x > 0,$$
and also from Appendix A:
\[
\sum_{k,l=1}^{N} \left( |\tilde{\lambda}_k - \tilde{\gamma}_{N,l}| + |\tilde{\lambda}_k + \tilde{\gamma}_{N,l}| \right) + \sum_{r=1}^{N-1} \sum_{k,l} |\tilde{\gamma}_{r+1,k} - \tilde{\gamma}_{r,l}| - 2 \sum_{r=2}^{N} \sum_{k<l} |\tilde{\gamma}_{r,k} - \tilde{\gamma}_{r,l}| - \sum_{k<l} |\tilde{\gamma}_{N,k} + \tilde{\gamma}_{N,l}| \geq
\]
\[
\geq -2N \sum_{k=1}^{N} |\tilde{\lambda}_k| + 2 \sum_{k<l} \left( |\tilde{\lambda}_k - \tilde{\lambda}_l| + |\tilde{\lambda}_k + \tilde{\lambda}_l| \right) + \frac{2}{N} \sum_{r=1}^{N} \sum_{k=1}^{r} |\tilde{\gamma}_{r,k}|,
\]
which is valid for any set of real variables \( \tilde{\lambda}_k, k = 1, 2, \ldots, N; \tilde{\gamma}_{r,l}, l = 1, 2, \ldots, r, r = 1, 2, \ldots, N \). A proof of the last inequality is given in Appendix A. For our purposes, we fix \( \tilde{\lambda}_k \) (respectively, \( \tilde{\gamma}_{r,l} \)) to be equal to \( \text{Re} \lambda_k \) (respectively, \( \text{Re} \gamma_{r,l} \)).

Presenting \( \Psi \) as
\[
\Psi(\gamma_1, \ldots, q_N) = \int \prod_{r=1}^{N} \prod_{k=1}^{r} d\tilde{\gamma}_{r,k} F(\gamma_1, \gamma_2, \ldots, \gamma_N, \lambda; q_1, \ldots, q_N),
\]
we obtain the following inequality for the dependence of the integrand on \( \gamma_{r,k} \):
\[
|F(\gamma_1, \gamma_2, \ldots, \gamma_N, \lambda; q_1, \ldots, q_N)| \leq P(\{\tilde{\gamma}_{r,k}\}) \exp \left( -\frac{\pi}{B_N} \sum_{r=1}^{N} \sum_{k=1}^{r} |\tilde{\gamma}_{r,k}| \right),
\]
where \( P(\{\tilde{\gamma}_{r,k}\}) \) has polynomial dependence on the variables \( \tilde{\gamma}_{r,k} \) and certain dependence on the other variables. Estimate \( \int \) leads to absolute convergence of the integral on the right-hand side of \( \psi \). We would like to mention that integral \( \int \) does not depend on the values of the imaginary parts of \( \gamma_{r,k} \) (that is, lines of integration) provided the mentioned inequalities \( \psi \) and \( \int \) for them are satisfied. This follows from two facts. First, we do not encounter poles as we shift the integration contour. Second, due to estimate \( \int \), the integrand is vanishing at the infinities of the integration contours. This justifies the correctness of shifting of the integration contours which we use in what follows.

From the physical viewpoint, the function \( \Psi(q_1, \ldots, q_N) \) given by \( \psi \) has correct asymptotic behaviour rapidly decreasing in the classically forbidden region, that is, where \( q_k > q_{k+1} \) for some \( k \) or where \( q_1 << 0 \). In the region \( 0 << q_1 << q_2 << \cdots << q_N \), the function \( \Psi(q_1, \ldots, q_N) \) is a superposition of plane waves.

The formulas for the action of the matrix elements of \( \mathcal{U}(u) \) on \( \Psi \), in particular \( \psi \), are derived in Appendix B. Here we give some heuristic explanation of formulas \( \psi \),
\[
\mathcal{D}(\lambda_r) \Psi = \alpha_1 \Psi_{\lambda+r}, \quad \mathcal{D}(-\lambda_r) \Psi = \alpha_1 \Psi_{\lambda+r},
\]
which are proved in Appendix B. Let \( \Psi(q_1, \ldots, q_N) \) be an eigenfunction of \( \mathcal{B}(u) \) satisfying \( \psi \). Then the commutation relation
\[
(u^2 - (v - i\hbar)^2) \mathcal{D}(v) \mathcal{B}(u) - (u^2 - v^2) \mathcal{B}(u) \mathcal{D}(v) = i\hbar(u + v - i\hbar)\mathcal{D}(u) \mathcal{B}(v) + i\hbar(u - v)\mathcal{A}(u) \mathcal{B}(v),
\]
which follows from \( \psi \), gives
\[
\mathcal{B}(u) \mathcal{D}(\lambda_r) \Psi = (-1)^N (u - \frac{i\hbar}{2}) (u^2 - (\lambda_r - i\hbar)^2) \prod_{k=1}^{N} (u^2 - \lambda_k^2) \cdot \mathcal{D}(\lambda_r) \Psi
\]
at \( v = \lambda_r \), and, therefore, \( \mathcal{D}(\lambda_r) \Psi \) is an eigenfunction of \( \mathcal{B}(u) \) with \( \lambda_r \) replaced by \( (\lambda_r - i\hbar) \). Clearly, this argumentation is not sufficient to prove the relation \( \mathcal{D}(\lambda_r) \Psi = \alpha_1 \Psi_{\lambda+r} \).

### 6 Wave functions for the two-boundary Toda chain

Taking into account \( \psi \), it is useful to introduce
\[
\tilde{i}(u) := (-1)^{N-1} \frac{\mathcal{D}(u) - 2\alpha_1 \alpha_N}{u^2 + \left( \frac{\hbar}{2} \right)^2} = u^{2N} - u^{2N-2} H_1^{BB} + u^{2N-4} H_2^{BB} - \cdots + (-1)^N H_N^{BB}.
\]
Let \( \Phi_\rho(q) \) be a wave function for the two-boundary Toda chain:

\[
\hat{t}(u)\Phi_\rho(q) = \prod_{k=1}^{N} (u^2 - \rho_k^2)\Phi_\rho(q) =: \hat{t}(u|\rho)\Phi_\rho(q),
\]

where \( \rho = \{\rho_1, \rho_2, \ldots, \rho_N\} \) are the quantum numbers of the corresponding state. We look for \( \Phi_\rho(q) \) in the form

\[
\Phi_\rho(q) = \int d\lambda_1 \cdots d\lambda_N \, \tilde{\mu}(\lambda)C(\lambda|\rho)\Psi_\lambda(q),
\]

where

\[
\tilde{\mu}^{-1}(\lambda) = \prod_{i \neq j} \left( \Gamma\left( \frac{\lambda_i - \lambda_j}{\hbar} \right) \Gamma\left( -\frac{\lambda_i - \lambda_j}{\hbar} \right) \right) \prod_{i \leq j} \left( \Gamma\left( \frac{\lambda_i + \lambda_j}{\hbar} \right) \Gamma\left( -\frac{\lambda_i + \lambda_j}{\hbar} \right) \right),
\]

and the integration with respect to \( \{\lambda_k\} \) is carried out along arbitrary lines parallel to the real axis. Using (31) and

\[
\frac{\tilde{\mu}(\lambda^{+p})}{\tilde{\mu}(\lambda)} = \frac{(\lambda_p + i\hbar)p}{\lambda_p} \prod_{l \neq p} (\lambda_p + i\hbar)^2 - \lambda_l^2, \]

we obtain

\[
(-1)^{N-1} \hat{t}(u|\rho)\Phi_\rho(q) = \int d\lambda_1 \cdots d\lambda_N \, \Psi_\lambda(q) \left[ \alpha_1 \alpha_N \sum_{p=1}^{N} \left[ \frac{\tilde{\mu}(\lambda^{+p})C(\lambda^{+p}|\rho)}{(\lambda_p + i\hbar)(\lambda_p + \frac{i\hbar}{2})} \prod_{l \neq p} \frac{u^2 - \lambda_l^2}{(\lambda_p + i\hbar)^2 - \lambda_l^2} \right] \right. \]

\[
+ \left. \frac{\tilde{\mu}(\lambda^{-p})C(\lambda^{-p}|\rho)}{(\lambda_p - i\hbar)(\lambda_p - \frac{i\hbar}{2})} \prod_{l \neq p} \frac{u^2 - \lambda_l^2}{(\lambda_p - i\hbar)^2 - \lambda_l^2} \right] \right. \]

\[
= \int d\lambda_1 d\lambda_2 \cdots d\lambda_N \, \mu(\lambda)C(\lambda|\rho)\Psi_\lambda(q) \left[ \alpha_1 \alpha_N \sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{u^2 - \lambda_l^2}{(\lambda_p + i\hbar)^2 - \lambda_l^2} \right) \right. \]

\[
\times \left. \left[ \frac{1}{\lambda_p(\lambda_p + \frac{i\hbar}{2})} C(\lambda^{+p}|\rho) + \frac{1}{\lambda_p(\lambda_p - \frac{i\hbar}{2})} C(\lambda^{-p}|\rho) \right] \right. \]

\[
- \left. \left[ 2 \lambda_p(\lambda_p + \frac{i\hbar}{2}) \right] - \prod_{l=1}^{N}(\lambda_l^2 - u^2) \right].
\]

We set \( u = \lambda_p \). Then the previous relation is satisfied if

\[
(-1)^{N-1} \hat{t}(\lambda_p|\rho) = \frac{t(\lambda_p|\rho) - 2\alpha_1 \alpha_N}{\lambda_p^2 + \frac{i\hbar}{4}} =
\]

\[
= \alpha_1 \alpha_N \left[ \frac{1}{\lambda_p(\lambda_p + \frac{i\hbar}{2})} C(\lambda^{+p}|\rho) + \frac{1}{\lambda_p(\lambda_p - \frac{i\hbar}{2})} C(\lambda^{-p}|\rho) \right] - \frac{2}{\lambda_p^2 + \frac{i\hbar}{4}},
\]

where \( t(u|\rho) = (-1)^{N-1}(u^2 + \hbar^2/4)\prod_{k=1}^{N}(u^2 - \rho_k^2) + 2\alpha_1 \alpha_N \). This multidimensional difference equation admits separation of variables. Namely, we suppose the factorization property

\[
C(\lambda|\rho) = \prod_{p=1}^{N} c(\lambda_p|\rho).
\]

Then \( c(\lambda|\rho) \) satisfies the Baxter equation

\[
\frac{1}{\lambda(\lambda + \frac{i\hbar}{2})} c(\lambda + i\hbar|\rho) + \frac{1}{\lambda(\lambda - \frac{i\hbar}{2})} c(\lambda - i\hbar|\rho) = \frac{t(\lambda|\rho)c(\lambda|\rho)}{\alpha_1 \alpha_N(\lambda^2 + \frac{i\hbar}{4})}.
\]
or, equivalently,
\[
(\lambda - \frac{i\hbar}{2})c(\lambda + i\hbar|\rho) + (\lambda + \frac{i\hbar}{2})c(\lambda - i\hbar|\rho) = \frac{\lambda t(\lambda|\rho) c(\lambda|\rho)}{\alpha_1 \alpha_N}.
\]

Solutions of this equation can be constructed in terms of ratios of infinite-dimensional determinants as was done in the case of the periodic Toda chain [8 4]. We expect that, similarly to the case of the periodic Toda chain [9 4], the requirement of the analytical properties of \(c(\lambda|\rho)\) (which is important, in particular, for the convergence of integral (31)) restricts possible values of \(\rho\) to the discrete spectrum of the quantum two-boundary Toda chain.

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**Appendix A: A proof of some inequalities**

**Proposition.** For any two sets of real variables \(\tilde{\gamma}_{N-1} = \{\tilde{\gamma}_{N-1,1}, \tilde{\gamma}_{N-1,2}, \ldots, \tilde{\gamma}_{N-1,N-1}\}\) and \(\tilde{\gamma}_N = \{\tilde{\gamma}_{N,1}, \tilde{\gamma}_{N,2}, \ldots, \tilde{\gamma}_{N,N}\}\), the following inequality is valid:

\[
F(\tilde{\gamma}_{N-1}, \tilde{\gamma}_N) := \sum_{k=1}^{N} \sum_{l=1}^{N-1} |\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N-1,l}| - \sum_{k_1 < k_2}^{N} \sum_{l_1 < l_2}^{N-1} |\tilde{\gamma}_{N,k_1} - \tilde{\gamma}_{N,k_2}| - \sum_{l_1 < l_2}^{N-1} |\tilde{\gamma}_{N-1,l_1} - \tilde{\gamma}_{N-1,l_2}| \geq 0. \tag{32}
\]

**Proof.** Since \(F(\tilde{\gamma}_{N-1}, \tilde{\gamma}_N)\) is invariant with respect to permutations of elements in the sets \(\tilde{\gamma}_{N-1}\) and \(\tilde{\gamma}_N\), we can restrict ourselves to the case where \(\tilde{\gamma}_{N-1,1} \geq \tilde{\gamma}_{N-1,2} \geq \cdots \geq \tilde{\gamma}_{N-1,N-1}\) and \(\tilde{\gamma}_{N,1} \geq \tilde{\gamma}_{N,2} \geq \cdots \geq \tilde{\gamma}_{N,N}\). In this case, simple combinatorics shows that

\[
\sum_{k_1 < k_2}^{N} |\tilde{\gamma}_{N,k_1} - \tilde{\gamma}_{N,k_2}| + \sum_{l_1 < l_2}^{N-1} |\tilde{\gamma}_{N-1,l_1} - \tilde{\gamma}_{N-1,l_2}| = \sum_{k=1}^{N} \left( \sum_{l=1}^{k-1} (\tilde{\gamma}_{N-1,l} - \tilde{\gamma}_{N,k}) + \sum_{l=k}^{N-1} (\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N-1,l}) \right).
\]

In fact, both sides are equal to

\[
\sum_{k=1}^{N} (N + 1 - 2k)\tilde{\gamma}_{N,k} + \sum_{l=1}^{N-1} (N - 2k)\tilde{\gamma}_{N-1,l}.
\]

Since \(|a| - a \geq 0\) for any real \(a\), we have

\[
F(\tilde{\gamma}_{N-1}, \tilde{\gamma}_N) = \sum_{k=1}^{N} \left( \sum_{l=1}^{k-1} (|\tilde{\gamma}_{N-1,l} - \tilde{\gamma}_{N,k}| - (\tilde{\gamma}_{N-1,l} - \tilde{\gamma}_{N,k})) + \sum_{l=k}^{N-1} (|\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N-1,l}| - (\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N-1,l})) \right) \geq 0.
\]

We note that \(F(\tilde{\gamma}_{N-1}, \tilde{\gamma}_N) = 0\) if and only if

\(\tilde{\gamma}_{N,1} \geq \tilde{\gamma}_{N-1,1} \geq \tilde{\gamma}_{N,2} \geq \tilde{\gamma}_{N-1,2} \geq \cdots \geq \tilde{\gamma}_{N,N-1} \geq \tilde{\gamma}_{N-1,N-1} \geq \tilde{\gamma}_{N,N}\). \(\square\)

As a corollary, we obtain

\[
G(\tilde{\gamma}_{N-1}, \tilde{\gamma}_N) := F(\tilde{\gamma}_N, \{\tilde{\gamma}_{N-1,1}, \tilde{\gamma}_{N-1,2}, \ldots, \tilde{\gamma}_{N-1,N-1}, 0, 0\}) = \sum_{k=1}^{N} \sum_{l=1}^{N-1} |\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N-1,l}| -
\]

\[
- \sum_{k_1 < k_2}^{N} |\tilde{\gamma}_{N,k_1} - \tilde{\gamma}_{N,k_2}| - \sum_{l_1 < l_2}^{N-1} |\tilde{\gamma}_{N-1,l_1} - \tilde{\gamma}_{N-1,l_2}| + 2 \sum_{k=1}^{N} |\tilde{\gamma}_{N,k}| - 2 \sum_{l=1}^{N-1} |\tilde{\gamma}_{N-1,l}| \geq 0. \tag{33}
\]
For $N$ arbitrary sets $\tilde{\gamma}_s = \{\tilde{\gamma}_{s,1}, \tilde{\gamma}_{s,2}, \ldots, \tilde{\gamma}_{s,s}\}$, $s = 1, \ldots, N$, of real numbers, combining inequalities (32) and (33), we get

$$H_s := \sum_{r=1}^{N-1} F(\tilde{\gamma}_r, \tilde{\gamma}_{r+1}) + \sum_{r=s}^{N-1} G(\tilde{\gamma}_r, \tilde{\gamma}_{r+1}) \geq 0, \quad s = 1, \ldots, N.$$ Explicitly,

$$H_s := \sum_{r=1}^{N-1} \sum_{k=1}^{r} \sum_{l=1}^{r} |\tilde{\gamma}_{r+1,k} - \tilde{\gamma}_{r,l}| - \sum_{k<l}^{N} |\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N,l}| - 2 \sum_{r=2}^{N-1} \sum_{k<l}^{r} |\tilde{\gamma}_{r,k} - \tilde{\gamma}_{r,l}| +$$

$$+2 \sum_{k=1}^{N} |\tilde{\gamma}_{N,k}| - 2 \sum_{s=1}^{N} \sum_{l=1}^{s} |\tilde{\gamma}_{s,l}| \geq 0, \quad s = 1, \ldots, N.$$ Therefore, the inequality $\sum_{s=1}^{N} H_s/N \geq 0$ is equivalent to

$$\sum_{r=1}^{N-1} \sum_{k=1}^{r} \sum_{l=1}^{r} |\tilde{\gamma}_{r+1,k} - \tilde{\gamma}_{r,l}| - \sum_{k<l}^{N} |\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N,l}| - 2 \sum_{r=2}^{N-1} \sum_{k<l}^{r} |\tilde{\gamma}_{r,k} - \tilde{\gamma}_{r,l}| \geq$$

$$\geq -2 \sum_{k=1}^{N} |\tilde{\gamma}_{N,k}| + 2 \sum_{s=1}^{N} \sum_{l=1}^{s} |\tilde{\gamma}_{s,l}|.$$ (34)

We need one more inequality for real $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N$ and $\tilde{\gamma}_{N,1}, \tilde{\gamma}_{N,2}, \ldots, \tilde{\gamma}_{N,N}$. It is just inequality (33) for $G(\{\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N, -\tilde{\lambda}_1, -\tilde{\lambda}_2, \ldots, -\tilde{\lambda}_N, \{\tilde{\gamma}_{N,1}, \tilde{\gamma}_{N,2}, \ldots, \tilde{\gamma}_{N,N}, 0, 0, \ldots, 0\})$ with $N - 1$ zeros. It is

$$\sum_{k,l=1}^{N} (|\tilde{\lambda}_k - \tilde{\gamma}_{N,l}| + |\tilde{\lambda}_k + \tilde{\gamma}_{N,l}|) - \sum_{k<l}^{N} (|\tilde{\gamma}_{N,k} - \tilde{\gamma}_{N,l}| + |\tilde{\gamma}_{N,k} + \tilde{\gamma}_{N,l}|) \geq$$

$$\geq -2N \sum_{k=1}^{N} |\tilde{\lambda}_k| + 2 \sum_{k<l}^{N} (|\tilde{\lambda}_k - \tilde{\lambda}_l| + |\tilde{\lambda}_k + \tilde{\lambda}_l|) + 2 \sum_{k=1}^{N} |\tilde{\gamma}_{N,k}|.$$ (35)

Adding (34) and (35), we obtain the main inequality

$$\sum_{k,l=1}^{N} (|\tilde{\lambda}_k - \tilde{\gamma}_{N,l}| + |\tilde{\lambda}_k + \tilde{\gamma}_{N,l}|) + \sum_{r=1}^{N-1} \sum_{k,l}^{r} |\tilde{\gamma}_{r+1,k} - \tilde{\gamma}_{r,l}| - 2 \sum_{r=2}^{N} \sum_{k<l}^{r} |\tilde{\gamma}_{r,k} - \tilde{\gamma}_{r,l}| - \sum_{k<l}^{\tilde{\gamma}_{N,k} + \tilde{\gamma}_{N,l}} \geq$$

$$\geq -2N \sum_{k=1}^{N} |\tilde{\lambda}_k| + 2 \sum_{k<l}^{N} (|\tilde{\lambda}_k - \tilde{\lambda}_l| + |\tilde{\lambda}_k + \tilde{\lambda}_l|) + 2 \sum_{k=1}^{N} |\tilde{\gamma}_{N,k}|.$$ (36)

Appendix B: Formulas for action of the matrix elements of $U(u)$ on $\Psi_\lambda$

In this Appendix, we prove the following action formulas:

$$B(u)\Psi_\lambda = (-1)^N(u - \frac{i\hbar}{2}) \prod_{l=1}^{N} (u^2 - \lambda_l^2) \Psi_\lambda,$$ (37)

$$D(u)\Psi_\lambda = \alpha_1 \sum_{p=1}^{N} \left( \prod_{l \neq p} (u^2 - \lambda_l^2) \right) \left[ \left( \frac{u + \lambda_p}{2\lambda_p} \right) \left( \frac{u - i\hbar}{2\lambda_p} \right) \Psi_\lambda - \left( \frac{u - \lambda_p}{2\lambda_p} \right) \left( \frac{u + i\hbar}{2\lambda_p} \right) \Psi_\lambda^+ \right] +$$

$$+ \alpha_1 \left( \prod_{l=1}^{N} \frac{\lambda_l^2 - u^2}{\lambda_l^2 + (\frac{i\hbar}{2})^2} \right) \Psi_\lambda,$$ (38)
\[ \tilde{t}(u)\Psi_\lambda = (-1)^{N-1} \frac{\tilde{t}(u) - 2\alpha_1\alpha_N}{u^2 + \left(\frac{1}{4}\right)^2} \Psi_\lambda = \prod_{l=1}^{N} \left( u^2 - \lambda_l^2 \right) \Psi_\lambda + \]
\[ + (-1)^{N-1}\alpha_1\alpha_N \sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{u^2 - \lambda_l^2}{\lambda_p - \lambda_l^2} \right) \left[ \frac{1}{\lambda_p(\lambda_p - \frac{1}{2})} \Psi_{\lambda - p} + \frac{1}{\lambda_p(\lambda_p + \frac{1}{2})} \Psi_{\lambda + p} - \frac{2}{\lambda_p^2 + \left(\frac{1}{2}\right)^2} \Psi_\lambda \right]. \] (39)

In particular, formula (35) gives
\[ \mathcal{D}(\lambda_r)\Psi_\lambda = \alpha_1 \Psi_{\lambda - r}, \quad \mathcal{D}(-\lambda_r)\Psi_\lambda = \alpha_1 \Psi_{\lambda + r}, \quad \mathcal{D}(i\hbar/2)\Psi_\lambda = \alpha_1 \Psi_\lambda. \] (40)

The action of \( \mathcal{A}(u) \) and \( \mathcal{C}(u) \) on \( \Psi_\lambda \) can be derived using (20) and Sklyanin determinant [9] for \( U(u) \), respectively.

Before presenting a proof of the action formulas, we give useful relations for \( \mu(\gamma) \) and \( Q(\gamma|\lambda) \) from (27):

\[ \mu(\gamma^+q) = \prod_{l \neq p, q} \frac{\gamma_l - \gamma_q - i\hbar}{\gamma_l - \gamma_p}, \quad \mu(\gamma^-p) = \prod_{l \neq p} \frac{\gamma_l - \gamma_p - i\hbar}{\gamma_l - \gamma_p}, \]
\[ \mu(\gamma^{-p,q}) = \prod_{l \neq p, q} \frac{\gamma_p - \gamma_q - 2i\hbar}{\gamma_p - \gamma_l}, \quad \mu(\gamma^{+p,q}) = \prod_{l \neq p, q} \frac{\gamma_p - \gamma_q + 2i\hbar}{\gamma_p - \gamma_l}. \] (41)

\[ Q(\gamma^+q|\lambda) = \frac{i^{-N+1} \prod_{l \neq p, q} (\gamma_q + \gamma_l + i\hbar)}{\prod_{l \neq p} (\lambda_l^2 - (\gamma_q + i\hbar)^2)}, \quad Q(\gamma^-p|\lambda) = \frac{i^{-N+1} \prod_{l \neq p} (\lambda_l^2 - \gamma_p^2)}{\prod_{l \neq p} (\gamma_p + \gamma_l)}, \]
\[ Q(\gamma^{-p,q}|\lambda) = \prod_{l} \frac{\gamma_l^2 - \gamma_p^2}{\gamma_l^2 - \gamma_q^2} \prod_{l \neq p, q} \gamma_p + \gamma_q + i\hbar. \] (43)

Action formula for \( \mathcal{B}(u) \).

To prove formula (31), we use (23). Taking into account (7) and (8), we get
\[ (A(u)B(-u) - B(u)A(-u))\Psi_\lambda = 2i^{N-1}(u - \frac{i\hbar}{2}) \sum_{p=1}^{N} \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda) \times \]
\[ \times e^{-i\lambda_1^+(\gamma_1^+ + \cdots + \gamma_N^+) \kappa_1} \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{\gamma_l^2 - \gamma_p^2} \psi_{\gamma^+p}. \]

In every summand with fixed \( p \), we make the change of variable \( \gamma_p \to \gamma_p - i\hbar \). Then we use formulas (11) and (13) to transform \( \mu(\gamma) \) and \( Q(\gamma|\lambda) \) to the original forms. This leads to an additional factor. Finally, we shift the contour of integration with respect to \( \gamma_p \) to the original one. This possibility was explained in Section 5. Thus, we have
\[ -\alpha_1 \left( A(u)B(-u) - B(u)A(-u) \right)\Psi_\lambda = \]
\[ = 2\alpha_1i^{N-1}(u - \frac{i\hbar}{2})e^{-\kappa_1} \sum_{p=1}^{N} \int d\gamma_1 \cdots d\gamma_N \mu(\gamma^-p)Q(\gamma^-p|\lambda)e^{-i\lambda_1^+(\gamma_1^+ + \cdots + \gamma_N^+) \kappa_1} \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{\gamma_l^2 - \gamma_p^2} \psi_{\gamma^-p}, \]
\[ = (u - \frac{i\hbar}{2}) \sum_{p=1}^{N} \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda)e^{-i\lambda_1^+(\gamma_1^+ + \cdots + \gamma_N^+) \kappa_1} \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{\gamma_l^2 - \gamma_p^2} \prod_{l \neq p} (\gamma_l^2 - \gamma_p^2) \psi_p, \]
where we used \( e^{\kappa_1} = -2\alpha_1 \). Since
\[ (u - \frac{i\hbar}{2})A(u)A(-u)\Psi_\lambda = (u - \frac{i\hbar}{2}) \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda)e^{-i\lambda_1^+(\gamma_1^+ + \cdots + \gamma_N^+)} \prod_{l \neq p} (\gamma_l^2 - \gamma_p^2) \psi_p, \]
we get
\[ \mathcal{B}(u)\Psi_\lambda = (u - \frac{i\hbar}{2}) \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda)e^{-i\lambda_1^+(\gamma_1^+ + \cdots + \gamma_N^+)} \times \]

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Having in mind the expression (25) and the action formulas (7)–(10), we begin calculation. We have

$$\sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{\gamma_l^2 - \gamma_p^2} \right) \prod_{l=1}^{N} (\lambda_l^2 - \gamma_p^2) + \prod_{l=1}^{N} (\gamma_l^2 - u^2) = \prod_{l=1}^{N} (\lambda_l^2 - u^2).$$

(46)

It follows from the identity

$$\prod_{l=1}^{N} (b_l - x) - \prod_{l=1}^{N} (a_l - x) = \sum_{p=1}^{N} \left( \prod_{l \neq p} \frac{a_l - x}{a_l - a_p} \right) \prod_{l=1}^{N} (b_l - a_p)$$

(47)

for the variables $x, a_l, b_l, l = 1, 2, \ldots, n$, if one substitutes $x := u^2, a_l := \gamma_l^2, b_l := \lambda_l^2$. Identity (47) is just the Lagrange interpolation formula for the polynomial $\prod_{l=1}^{N} (b_l - x) - \prod_{l=1}^{N} (a_l - x)$ in $x$ of degree $N - 1$ reconstructed from its values at $N$ points $x = a_p, p = 1, 2, \ldots, N$. Using (45) and (46), we obtain (37).

Action formula for $D(u)$.

Our strategy is to derive (44) and then to reconstruct formula (35) using the Lagrange interpolation formula. Having in mind the expression (29) and the action formulas (7)–(10), we begin calculation. We have

$$C(u)B(-u)\psi_\gamma = -\sum_{p,q} \frac{(u - \gamma_p - i\hbar)}{(\gamma_p - \gamma - i\hbar)} \left( \frac{u + \gamma_q}{(\gamma_q - \gamma_p - i\hbar)} \right) \left( \prod_{l \neq p,q} \frac{\gamma_l^2 - u^2}{(\gamma_l - \gamma_p)(\gamma_l - \gamma_q)} \right) \psi_{\gamma+p,-q}$$

$$- \sum_{p} \left( \prod_{l \neq p} \frac{\gamma_l + u - \gamma_l}{\gamma_l - \gamma_p} \right) \left( \prod_{l \neq p} \frac{u - \gamma_l}{\gamma_l - \gamma + i\hbar} \right) \psi_\gamma,$$

$$D(u)A(-u)\psi_\gamma = -\sum_{p,q} \frac{(u + \gamma_p)}{(\gamma_p - \gamma - i\hbar)} \left( \frac{u + \gamma_q}{(\gamma_q - \gamma_p - i\hbar)} \right) \left( \prod_{l \neq p,q} \frac{\gamma_l^2 - u^2}{(\gamma_l - \gamma_p)(\gamma_l - \gamma_q)} \right) \psi_{\gamma+p,q} +$$

$$+ \sum_{p} \left( \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{\gamma_l - \gamma_p} \right) \frac{1}{\hbar} \left( \frac{u + \gamma_p}{\prod_{l \neq p}(\gamma_p - \gamma_l + i\hbar)} - \frac{-(u + \gamma_p)}{\prod_{l \neq p}(\gamma_p - \gamma_l - i\hbar)} \right) \psi_\gamma.$$ 

Therefore,

$$(D(u)A(-u) - C(u)B(-u))\psi_\gamma =$$

$$-2(u - \frac{i\hbar}{2}) \sum_{p,q} \frac{u + \gamma_q}{(\gamma_p - \gamma_q)(\gamma_p - \gamma - i\hbar)} \left( \prod_{l \neq p,q} \frac{\gamma_l^2 - u^2}{(\gamma_l - \gamma_p)(\gamma_l - \gamma_q)} \right) \cdot \psi_{\gamma+p,q} +$$

$$+ \sum_{p} \left( \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{\gamma_l - \gamma_p} \right) \left( \prod_{l \neq p} \frac{1}{\hbar} \left( \frac{u + \gamma_p}{\prod_{l \neq p}(\gamma_p - \gamma_l + i\hbar)} + \frac{1}{\hbar} \prod_{l \neq p}(\gamma_p - \gamma_l - i\hbar) \right) \right) \psi_\gamma.$$ 

After the action of $D(u)A(-u) - C(u)B(-u)$ on $\Psi$, the integrand becomes a linear combination of $\psi_\gamma$ with non-shifted indices $\gamma$ and $\psi_{\gamma+p,-q}$ with all possible $p, q, p \neq q$. Changing variables and shifting the integration contours as was described above for $B(u)$, we rewrite the result as

$$\int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda) \left( \sum_{p,q} R_{p,q}(u) \right) e^{-i(\gamma_1 + \cdots + \gamma_N)u_1} \psi_\gamma,$$
where
\[
R_{p,q}(u) = -2(u - \frac{ih}{2}) \frac{\mu(\gamma-p,q)}{\mu(\gamma)} \frac{Q(\gamma-p,q|\lambda)}{Q(\gamma|\lambda)} \frac{(u+\gamma_q+ih)}{\prod_{l \neq p,q} (\gamma_l - \gamma_l + ih)} \frac{\gamma_l^2 - u^2}{(\gamma_p - \gamma_q - 2ih)(\gamma_q - \gamma + ih)} = \]
\[
= 2 \left( u - \frac{ih}{2} \right) \frac{(u+\gamma_q+ih)}{\prod_{l \neq q} (\gamma_l - \gamma_q)(\gamma_q - \gamma - i\hbar)} \frac{\gamma_l^2 - u^2}{(\gamma_q - \gamma + h)(\gamma_q - \gamma + ih)} \frac{\gamma + \gamma_q + ih}{\prod_{l \neq q} (\gamma_l - \gamma_q)(\gamma_q - \gamma - i\hbar)} = \]
Thus, we have
\[
(D(u)A(-u) - C(u)B(-u))\Psi_{\lambda} = \]
\[
= \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda) \left( \sum_{p \neq q} R_{p,q}(u) + \sum_q (R_{q}^{(1)}(u) + R_{q}^{(2)}(u) + R_{q}^{(3)}(u)) \right) \exp \left( \frac{i(\gamma_1 + \cdots + \gamma_N+\gamma_{\mu})}{\hbar} \right) \psi_I, \]
where
\[
R_{q}^{(1)}(u) = \prod_{l \neq q} \left( \gamma_l^2 - u^2 \right) \frac{\gamma_l}{(\gamma_q - \gamma_l)(\gamma_q - \gamma + ih)}, \]
\[
R_{q}^{(2)}(u) = \frac{(u+\gamma_q)}{ih} \prod_{l \neq q} \left( \gamma_l^2 - u^2 \right) \frac{\gamma_l}{(\gamma_q - \gamma_l)(\gamma_q - \gamma - i\hbar)}, \]
\[
R_{q}^{(3)}(u) = \frac{(u+\gamma_q)}{ih} \prod_{l \neq q} \left( \gamma_l^2 - u^2 \right) \frac{\gamma_l}{(\gamma_q - \gamma_l)(\gamma_q - \gamma - i\hbar)}. \]

Since
\[
C(u)A(-u)\psi_I = -i^{-N-1} \sum_{p} (u + \gamma_p) \left( \prod_{l \neq p} \frac{\gamma_l^2 - u^2}{(\gamma_p - \gamma_l)} \right) \psi_I, \]
after appropriate shift of the integration contours and change of variables, the action of \((u - ih/2)C(u)A(-u)\) on \(\Psi_{\lambda}\) becomes
\[
(u - ih/2)C(u)A(-u)\Psi_{\lambda} = \alpha_1 \int d\gamma_1 \cdots d\gamma_N \mu(\gamma)Q(\gamma|\lambda) \left( \sum_q R_q(u) \right) \exp \left( \frac{i(\gamma_1 + \cdots + \gamma_N+\gamma_{\mu})}{\hbar} \right) \psi_I, \]
where
\[
R_q(u) = 2(u - \frac{ih}{2}) \frac{\mu(\gamma+q)}{\mu(\gamma)} \frac{Q(\gamma+q|\lambda)}{Q(\gamma|\lambda)} \frac{1}{\prod_{l \neq q} \gamma_l^2 - u^2} \cdot \frac{1}{\prod_{l \neq q} \gamma_l^2 - \gamma_q^2} \cdot \frac{1}{(\gamma_q + i\hbar)^2} = \]
\[
= -2(u - \frac{ih}{2})(u + \gamma_q + ih) \prod_{l \neq q} \frac{(u^2 - \gamma_l^2)(\gamma_q + \gamma + i\hbar)}{(\gamma_q - \gamma_l)} \prod_{l} \frac{1}{\lambda_l^2 - (\gamma_q + i\hbar)^2}. \]

We denote
\[
R_q^{(0)}(u) := R_q(u) + \sum_{p \neq q} R_{p,q}(u) = \]
\[
= -2(u - \frac{ih}{2})(u + \gamma_q + ih) \prod_{l \neq q} \frac{(u^2 - \gamma_l^2)(\gamma_q + \gamma + i\hbar)}{(\gamma_q - \gamma_l)} \prod_{l} \frac{1}{\lambda_l^2 - (\gamma_q + i\hbar)^2} \cdot \frac{\gamma_q^2 - \lambda_q^2}{(\gamma_q - \gamma_q + i\hbar)^2} + \]
\[
+ 2 \left( u - \frac{ih}{2} \right) \sum_{p \neq q} \frac{(u + \gamma_q + ih)}{(\gamma_q - \gamma_p)(\gamma_q - \gamma_p + i\hbar)} \left( \prod_{l \neq p,q} \frac{(\gamma_l^2 - u^2)}{(\gamma_q - \gamma_l)(\gamma_q - \gamma_l)} \frac{\gamma_q + \gamma + i\hbar}{\gamma_q + \gamma_p} \right) \cdot \prod_{l} \frac{\gamma_q^2 - \lambda_q^2}{(\gamma_q + i\hbar)^2 - \lambda_q^2}. \quad (48) \]

Now we prove the following relation:
\[
R_q^{(0)}(\lambda_r) = \frac{2(\lambda_r - ih/2)}{(\gamma_q - \lambda_r + i\hbar)} \prod_{l \neq q} \frac{\gamma_q^2 - \lambda_q^2}{(\gamma_q - \gamma_l)(\gamma_q - \gamma_l + i\hbar)}. \quad (49) \]
We start from the identity for the variables \( a_1, a_2, \ldots, a_N \) and \( b_1, b_2, \ldots, b_{N-1} \):

\[
\sum_{p=1}^{N} \prod_{i=1}^{N-1} \frac{(a_p - b_i)}{(a_p - a_i)} = 1.
\]

Separating the summand with \( p = q \), we rewrite the identity in the form

\[
1 + \sum_{p \neq q} \left( \prod_{i \neq p,q} \frac{1}{a_p - a_i} \right) \frac{1}{a_q - a_p} \prod_{l=1}^{N-1} (a_p - b_l) = \frac{\prod_{i \neq q}^{N-1} (a_q - a_i)}{\prod_{i \neq q} (a_q - b_i)}.
\]

Dividing this equality by its right-hand side, we get

\[
\frac{\prod_{i \neq q} (a_q - a_i)}{\prod_{i = 1}^{N-1} (a_q - b_i)} + \sum_{p \neq q} \left( \prod_{i \neq p,q} \frac{a_q - a_i}{a_p - a_i} \cdot \prod_{l=1}^{N-1} \frac{a_p - b_l}{a_q - b_l} \right) = 1.
\]

With \( a_q := (\gamma_q + \text{i}h)^2 \), \( a_l := \lambda_l^2 \) \((l \neq q)\), \( \{b_1, \ldots, b_{N-1}\} := \{\lambda_1^2, \ldots, \lambda_{r-1}^2, \lambda_{r+1}^2, \ldots, \lambda_N^2\} \), we have

\[
\frac{\prod_{i \neq q} (\gamma_q + \text{i}h)^2 - \gamma_l^2}{\prod_{i \neq q} (\gamma_q + \text{i}h)^2 - \lambda_q^2} + \sum_{p \neq q} \left( \prod_{i \neq p,q} \frac{(\gamma_q + \text{i}h)^2 - \gamma_l^2}{\gamma_l^2} \cdot \prod_{i \neq r} \frac{\gamma_l^2 - \lambda_q^2}{\gamma_q + \text{i}h)^2 - \lambda_q^2} \right) = 1.
\]

Multiplying the obtained identity by

\[
\frac{2(\lambda_r - \text{i}h/2)}{(\gamma_q - \lambda_r + \text{i}h)} \prod_{q \neq q} (\gamma_q - \gamma_l)(\gamma_q - \gamma_l + \text{i}h),
\]

after simple transformations, we obtain (49). Using this result, we get

\[
R_q^{(0)}(\lambda_r) + R_q^{(1)}(\lambda_r) + R_q^{(2)}(\lambda_r) = -\frac{1}{\text{i}h(\gamma_q - \lambda_r + \text{i}h)} \prod_{i \neq q} (\gamma_q^2 - \lambda_q^2).
\]

Consider the identity

\[
\sum_{m=1}^{2N} \prod_{s \neq m}^{2N} \frac{u - a_s}{a_m - a_s} = 1
\]

with \( \{a_1, \ldots, a_{2N}\} := \{\gamma_1, \ldots, \gamma_N, \gamma_1 + \text{i}h, \ldots, \gamma_N + \text{i}h\} \) and \( u = \lambda_r \). Explicitly, we have

\[
\sum_{q=1}^{N} \left\{ \frac{1}{\text{i}h} \cdot \prod_{q \neq i} (\gamma_q - \lambda_r) \prod_{l \neq q} (\gamma_q - \gamma_l) \cdot \frac{1}{\text{i}h} \cdot \prod_{q \neq i} (\gamma_q - \lambda_r) \prod_{l \neq q} (\gamma_q - \gamma_l + \text{i}h) \right\} = 1.
\]

Multiplying the both sides by \( \prod_i (\gamma_q + \lambda_r)/\prod_i (\gamma_q - \lambda_r + \text{i}h) \), we get

\[
\sum_{q=1}^{N} \left\{ \frac{(\gamma_q + \lambda_r)}{\text{i}h} \cdot \prod_{i \neq q} \frac{(\gamma_q^2 - \lambda_q^2)}{(\gamma_q - \gamma_l)(\gamma_q - \gamma_l - \text{i}h)} - \frac{1}{\text{i}h(\gamma_q - \lambda_r + \text{i}h)} \cdot \prod_{q \neq i} (\gamma_q - \lambda_r) \prod_{l \neq q} (\gamma_q - \gamma_l + \text{i}h) \right\} = \frac{\prod_i (\gamma_q + \lambda_r)}{\prod_i (\gamma_q - \lambda_r + \text{i}h)},
\]

which, in fact, coincides with

\[
\sum_{q=1}^{N} \left( R_q^{(3)}(\lambda_r) + R_q^{(0)}(\lambda_r) + R_q^{(1)}(\lambda_r) + R_q^{(2)}(\lambda_r) \right) = \frac{\prod_i (\gamma_q + \lambda_r)}{\prod_i (\gamma_q - \lambda_r + \text{i}h)}.
\]
Since
\[ Q(\gamma|\lambda) \prod_{i}(\gamma + \lambda_{i}) = Q(\gamma|\lambda^{-}), \]
we obtain \( D(\lambda_{r})\Psi_{\lambda} = \alpha_{1}\Psi_{\lambda^{-r}}. \) In complete analogy with the above derivation, we find \( D(-\lambda_{r})\Psi_{\lambda} = \alpha_{1}\Psi_{\lambda^{+r}}. \) From \( (25) \) and \( (10) \) at \( u = i\hbar/2, \) it follows that \( D(i\hbar/2) = \alpha_{1}. \) Thus, we proved \( (40). \)

From \( (7) \) (resp. \( (8), (9), (10) \)), it follows that the polynomial \( A(u) \) (resp. \( B(u), C(u), D(u) \)) has degree \( N \) (resp. \( (N - 1), (N - 1), (N - 2) \)) in \( u. \) Hence, using \( (24) \), we find that polynomial \( D(u) \) has degree \( 2N \) in \( u. \) We know the results \( (40) \) of the action of \( D(u) \) on \( \Psi_{\lambda} \) in \( 2N + 1 \) points. Applying the Lagrange interpolation formula, we obtain \( (39). \)

Action formula for \( t(u). \)

Now we calculate the action of \( t(u) \), given by \( (24), \) on \( \Psi_{\lambda}. \) Using formulas \( (37) \) and \( (38), \) we find
\[ t(u)\Psi_{\lambda} = \left[ u^{2} + \left( \frac{\hbar}{2} \right)^{2} \right] \left[ \alpha_{1}\alpha_{N} \prod_{p=1}^{N} \left( \frac{u^{2} - \lambda_{p}^{2}}{\lambda_{p}^{2} - \lambda_{p}^{2}} \right) \right] \left( \frac{1}{\lambda_{p}(\lambda_{p} - \lambda)} \right) \Psi_{\lambda^{-}} + \frac{1}{\lambda_{p}(\lambda_{p} + \lambda)} \Psi_{\lambda^{+}} + \right. \]
\[ + \left. \frac{2\alpha_{1}\alpha_{N}}{u^{2} + \left( \frac{\hbar}{2} \right)^{2}} \left( \prod_{l=1}^{N} \lambda_{l}^{2} - u^{2} \right) \Psi_{\lambda} - \prod_{l=1}^{N} \lambda_{l}^{2} - u^{2}) \right] \Psi_{\lambda}. \]

The identity
\[ \frac{1}{u^{2} + \left( \frac{\hbar}{2} \right)^{2}} \left[ \left( \prod_{l=1}^{N} \lambda_{l}^{2} - u^{2} \right) \right] - 1 = - \sum_{p=1}^{N} \frac{1}{\lambda_{p}^{2} + \left( \frac{\hbar}{2} \right)^{2}} \left( \prod_{l=1}^{N} \lambda_{l}^{2} - \lambda_{p}^{2} \right), \]
which can be obtained using the Lagrange interpolation formula, implies \( (39). \)

References

[1] E. Sklyanin, Separation of variable. New trends, Prog. Theoret. Phys. Suppl. (1995), 118, 35–60.
[2] M. Gutzwiller, The quantum mechanical Toda lattice II, Ann. of Phys., (1981), 133, 304–331.
[3] E. Sklyanin, The quantum Toda chain, Lect. Notes in Phys., (1985), 226, 196–233.
[4] S. Kharchev, D. Lebedev, Integral representation for the eigenfunctions of quantum periodic Toda chain, Lett.Math.Phys., (1999) 50, 53–77.
[5] S. Kharchev, D. Lebedev, Eigenfunctions of \( GL(N, R) \) Toda chain: The Mellin-Barnes representation, Pisma Zh.Eksp.Teor.Fiz. 71 (2000) 338–343; JETP Lett. 71 (2000) 235–238.
[6] A. Gerasimov, S. Kharchev, D. Lebedev, Representation Theory and the Quantum Inverse Scattering Method: The Open Toda Chain and the Hyperbolic Sutherland Model, Int. Math. Res. Not. 2004:17 (2004) 823-854.
[7] S. Kharchev, D. Lebedev, M. Semenov-Tian-Shansky, Unitary representations of \( U_{q}(sl(2, R)), \) the modular double, and the multiparticle q-deformed Toda chains, Commun. Math. Phys., (2002), 225, 573–609.
[8] V. Pasquier, M. Gaudin, The periodic Toda chain and a matrix generalization of the Bessel function recursion relations, J. Phys. A, (1992), 25, 5243–5252.
[9] E. Sklyanin, Boundary conditions for integrable quantum systems, J. Phys. A, (1988) 21, 2375–2389.
[10] V. Kuznetsov, Separation of variables for the \( D_{n} \) type periodic Toda lattice, J. Phys. A (1997) V.30, 2127-2138.