Genericity Results on Bloch Spectrum of Periodic Elliptic Operators and Applications to Homogenization

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Abstract

We consider a second-order elliptic operator in divergence form with periodic coefficients. The spectrum of such operators is completely described by Bloch eigenvalues. We study the structure and regularity properties of Bloch eigenvalues near the edge of a spectral gap. We show that a Bloch eigenvalue can be made simple at a point, and hence in a neighborhood of that point. Using this result, we prove that under small fiberwise perturbation of coefficients, a Bloch eigenvalue can be made simple for all parameter values. Further, under different regularity assumptions on the coefficients of the periodic operator, we obtain simplicity of a spectral edge under small perturbations of the coefficients. These spectral tools are used to establish Bloch wave homogenization at an internal edge in the presence of multiplicity.

Keywords: Bloch eigenvalues, Periodic Operators, Homogenization

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1. Introduction

The goal of the paper is to study the structure of spectral edges of a periodic second-order elliptic operator in divergence form. Namely, we consider the following unbounded operator in $L^2(\mathbb{R}^d)$.

$$
\mathcal{A}u := -\frac{\partial}{\partial y_k} \left( a_{kl}(y) \frac{\partial u}{\partial y_l} \right),
$$

where the Einstein summation convention is assumed. We make the following assumptions on the coefficients of the operator (1.1): The coefficients $a_{kl}(y)$ are measurable bounded functions defined on the set $Y = [0, 2\pi)^d$, which is a basic cell for a lattice of periods in the $d$-dimensional euclidean space $\mathbb{R}^d$. Further, $a_{kl}$ is $Y$-periodic. We will denote the space of measurable bounded periodic functions in $Y$ by $L_\infty^p(Y)$. Hence, $a_{kl} \in L_\infty^p(Y)$. In many instances, we will identify $Y$ with a torus $\mathbb{T}^d$ and the space $L_\infty^p(Y)$ with $L_\infty(\mathbb{T}^d)$, in the

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standard way. The matrix \( A = (a_{kl}) \) is real symmetric, i.e., \( a_{kl}(y) = a_{lk}(y) \). Further, the matrix \( A \) is coercive, i.e., there exists \( \alpha > 0 \) such that \( \forall \ v \in \mathbb{R}^d \) a.e. \( y \in \mathbb{R}^d \),

\[
\langle A(y)v, v \rangle \geq \alpha ||v||^2 .
\]

Let \( Y' = \left[ -\frac{1}{2}, \frac{1}{2} \right]^d \) be a basic cell for the dual lattice in \( \mathbb{R}^d \). Then, the spectrum of \( A \) can be studied by evaluating, for \( \eta \in Y' \), the spectrum of the shifted operator

\[
A(\eta) = e^{-i\eta.y} A e^{i\eta.y} = - \left( \frac{\partial}{\partial y_k} + i\eta_k \right) a_{kl}(y) \left( \frac{\partial}{\partial y_l} + i\eta_l \right). 
\]

This is an unbounded operator in \( L^2(\mathbb{R}^d) \). The operator \( A \) in \( L^2(\mathbb{R}^d) \) is unitarily equivalent to the fibered operator

\[
\int_{Y'} A(\eta) d\eta
\]

in the Bochner space \( L^2(Y', L^2(Y')) \). As a consequence of this fact, the spectrum of \( A \) is the union of the spectra of \( A(\eta) \) in \( L^2(Y) \) as \( \eta \) varies in \( Y' \). For a proof of this fact, see [33, p. 284]. By the usual theory of compact self-adjoint operators, there exists a sequence of increasing eigenvalues for \( A(\eta) \), denoted by \( (\lambda_n(\eta))_{n=1}^\infty \). The functions \( \eta \mapsto \lambda_n(\eta) \) are known as the Bloch eigenvalues of the operator \( A \). Let \( \sigma^-_n = \min_{\eta \in Y'} \lambda_n(\eta) \) and \( \sigma^+_n = \max_{\eta \in Y'} \lambda_n(\eta) \), then, the spectrum of the operator \( A \) is given by \( \sigma(A) = \bigcup_{n \in \mathbb{N}} [\sigma^-_n, \sigma^+_n] \). The spectrum of \( A \) is a union of closed intervals, which may overlap. The complement is a union of open intervals. The connected components of this union are known as spectral gaps. Let \( \lambda_0 \) be the upper endpoint of a spectral gap. Then, there exists a Bloch eigenvalue \( \lambda_m \) and a dual parameter value \( \eta_0 \) such that \( \lambda_0 = \lambda_m(\eta_0) \).

In the first part of the paper, we will study regularity of Bloch eigenvalues near \( \eta_0 \) and other points in the dual parameter space where the spectral edge is attained. Regularity properties of the Bloch eigenvalues in the parameter are important in applications in the theory of effective mass [7] and Bloch wave method in homogenization [14]. For periodic Schrödinger operators, Wilcox [46] proved that outside a set of measure zero in the dual parameter space, the Bloch eigenvalues are analytic and the Bloch eigenfunctions may be chosen to be analytic. However, this measure zero set might intersect the spectral edge, which would limit applicability of such a result.

Differentiability properties of parametrized eigenvalues is an active area of research, even at the level of matrices [3, 32]. Broadly speaking, regularity results for parametrized eigenvalues of self-adjoint operators are available in two cases: (i) for one parameter eigenvalue problems [33], [20]. (ii) for simple eigenvalues, regardless of the number of parameters. Multiple parameters are unavoidable in most applications of interest. Examples include propagation of singularities for hyperbolic systems of equations with multiple characteristics leading to novel phenomena such as conical refraction [26], [16], stability of hyperbolic initial-boundary-value problems [29] and Bloch waves for elasticity system [40]. Hence, an assumption of simplicity is useful in applications [4], [5], [6]. It can be shown that under
perturbations of some relevant parameter like domain shape, coefficients, potentials etc, a multiple eigenvalue can be made simple. In a well-known paper [2], Albert proves that, for a compact manifold $M$, the set of all smooth potentials $V \in C^\infty(M)$ for which the operator $-\Delta + V$ has only simple eigenvalues is a residual set in the space of all smooth admissible potentials. Similar results were proved by Uhlenbeck [44] using topological methods.

Hence, it is possible to conclude regularity of parametrized eigenvalues provided that they can be made simple in the manner of Albert [2]. Therefore, we are interested in the Bloch eigenvalues being separated, i.e., $\lambda_i(\eta) \neq \lambda_{i+1}(\eta)$ either locally in $\eta$ or globally for $\eta \in Y'$. As depicted in Figure 1, Bloch eigenvalues can be separated without there being a spectral gap between them. However, the genericity result of Albert is not applicable as it does not take into account second-order perturbations and the presence of the dual parameter $\eta$ in the Bloch eigenvalue problem. In this paper, we shall prove that the Bloch eigenvalues can be made simple locally in the parameter through a perturbation in the coefficients of the operator $A$. Further, by applying fiberwise perturbation on the fibered operator $\int_{Y'} A(\eta) d\eta$ operator, we can make sure that the corresponding eigenvalue of interest is simple for all parameter values.

The latter part of the paper is concerned with the theory of Bloch wave homogenization.
In homogenization, one studies the limits of solutions to equations with highly oscillatory coefficients, such as

$$-\nabla \cdot \left( A \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right) + \varepsilon^2 u^\varepsilon = f \quad \text{in } \mathbb{R}^d,$$

(1.2)

for $f \in L^2(\mathbb{R}^d)$ and $\varepsilon > 0$.

Suppose that $u^\varepsilon$ converges weakly in $H^1(\mathbb{R}^d)$ to $u^*$. Then, the theory of homogenization [43] characterizes the matrix $A^*$ such that the limit $u^*$ is a solution to the equation:

$$-\nabla \cdot (A^* \nabla u^*) + \varepsilon^2 u^* = f \quad \text{in } \mathbb{R}^d.$$

(1.3)

Bloch wave method of homogenization achieves this characterization through the properties of Bloch eigenvalues at spectral edges. In particular, the homogenized matrix $A^*$ is characterized by the Hessian of the lowest Bloch eigenvalue at $0 \in Y'$ [14]. Similarly, in the theory of internal edge homogenization [10], the homogenized coefficients are characterized by Bloch eigenvalues near the spectral edge. In particular, we require the following regularity conditions on the spectral edges:

(A) The spectral edge must be attained at finitely many points by a Bloch eigenvalue.

(B) The spectral edge must be simple, i.e., it is attained by a single Bloch eigenvalue.

(C) The spectral edge must be non-degenerate. If the Bloch eigenvalue $\lambda_m(\eta)$ attains the spectral edge, $\lambda_0$ at the points $\{\eta_j\}_{j=1}^l$, then the Bloch eigenvalue must satisfy

$$\left( \lambda_m(\eta) - \lambda_0 \right) = (\eta - \eta_j)^T B_j (\eta - \eta_j) + O(|\eta - \eta_j|^3),$$

(1.4)

for $\eta$ near $\eta_j$, $j = 1, 2, \ldots, l$, where $B_j$ are positive definite matrices.

While these features are readily available for the lowest Bloch eigenvalue corresponding to the divergence-type scalar elliptic operator, these properties may not be available for other spectral gaps of the same operator [23]. Also, these features are not available for elliptic systems of differential equations, like, the linear elasticity system for which the lowest Bloch eigenvalue has multiplicity 3.

It has been proved by Filonov and Kachkovskiy [18] that in two dimensions, the spectral gaps of the periodic elliptic differential operators are attained at finitely many points in $Y'$. The question of finiteness in higher dimensions is still open.

The lack of simplicity, which results in lack of analyticity for the Bloch eigenvalues, was handled in establishing homogenization of the elasticity operator by using the directional analyticity of the Bloch eigenvalues by Sivaji Ganesh and Vanninathan [40].

If the spectral edge is simple, it is possible to define the Hessian at points where the spectral edge is attained. The requirement of Hessian to be positive definite serves multiple purposes. In establishing homogenization limits, the strong convexity of the spectral edge
allows us to separate the spectral edge from the rest of the spectrum. Further, the homogenized limit equation is solvable on account of the ellipticity of the effective tensor. However, a spectral edge may not be non-degenerate, for example, Shterenberg [38] has constructed a magnetic Laplacian with a degenerate lowest Bloch eigenvalue. However, Parnovski and Shterenberg [30] have proved that a spectral edge can be made non-degenerate with a small perturbation of the potential with a larger period, for the two-dimensional Schrödinger operator.

The validity of hypotheses (A), (B), (C) is usually assumed in the literature [23]; for example, in establishing Green’s function asymptotics [24], [21], for internal edge homogenization [10] and to establish localization for random Schrödinger operators [45]. In particular, simplicity at a point is assumed in the study of diffractive geometric optics [5], [6] and homogenization of periodic systems [4]. Following Klopp and Ralston [22], we obtain simplicity of spectral edge under perturbation of coefficients, provided that the coefficients are $W^{1,\infty}$. However, for $L^\infty$ coefficients, we show that a spectral edge can be made simple under a small perturbation of the coefficients with the added assumption that the spectral edge is attained at only finitely many points. Thus, our results suggest a possible interplay between the validity of these assumptions and the regularity of the coefficients. Further, these spectral tools also allow us to achieve homogenization at an internal edge in the presence of multiplicity.

More details on the spectrum of elliptic periodic operators may be found in Reed and Simon [33]. Also, see the review by Kuchment for the state of the art on periodic differential operators [23]. A rigorous account of direct integral decomposition of operators, such as the one described above for periodic operators, may be found in [37] and [28].

1.1. Main Results

Let $Sym(d)$ denote the space of all symmetric matrices. Let

$$M^>_B = \{ A : \mathbb{R}^d \rightarrow Sym(d) : a_{ij} \in L^\infty(Y) \text{ and } A \text{ is coercive} \}.$$ 

$M^>_B$ is a space of $d(d+1)/2$-tuples of $L^\infty$ functions and we shall use the norm-topology on this space in our further discussion. We shall call a property generic if it holds on a set of the second category, i.e., it contains a countable intersection of open and dense sets. A Baire space is a topological space in which the countable intersection of dense open sets is dense. It is noteworthy that $M^>_B$ is an open subset of the space of all symmetric $L^\infty$ matrices which forms a complete metric space, and hence $M^>_B$ is a Baire Space.

We prove the following theorems.

**Theorem 1.1.** Let $\eta_0 \in Y^\ast$. The eigenvalues of the shifted operator $A(\eta_0)$ are generically simple with respect to the coefficients $A = (a_{ij})_{i,j=1}^d$ in $M^>_B$.

**Remark 1.2.** Theorem 1.1 is an extension of the theorem of Albert [2] which proves that the eigenvalues of $-\Delta + V$ are generically simple with respect to $V \in C^\infty(M)$ for compact manifold $M$. The potential $V$ is the quantity of interest for Schrödinger operator, $-\Delta + V$. For the applications that we have in mind, for example, the theory of homogenization, the
periodic matrix $A$ in the divergence type elliptic operator $-\nabla \cdot (A \nabla)$ is of physical importance. The spectrum of such operators is not discrete, and must be analyzed through Bloch eigenvalues, which introduces an extra parameter $\eta \in Y'$ to the problem. The determination of “real” perturbation for the shifted operator $A(\eta)$, which has complex coefficients, poses additional difficulties, when coupled with the lack of regularity of the coefficients which the applications demand.

**Theorem 1.3.** Given a Bloch eigenvalue $\lambda_n(\eta)$ of the periodic operator $A$ there exists a perturbation of $A$ such that the perturbed eigenvalue $\tilde{\lambda}_n(\eta)$ is simple for all $\eta \in Y'$.

A spectral edge is said to be simple if it is attained only by a single Bloch eigenvalue.

**Theorem 1.4.** Suppose that the coefficients of the matrix $A = (a_{ij})_{i,j=1}^d$ in the operator $A = -\nabla \cdot (A \nabla)$ belong to the class $W^{1,\infty}_Y(Y)$. Then, a spectral edge can be made simple by a small perturbation in the coefficients.

**Theorem 1.5.** Suppose that the coefficients of the matrix $A = (a_{ij})_{i,j=1}^d$ in the operator $A = -\nabla \cdot (A \nabla)$ belong to the class $L^{\infty}_Y(Y)$. Assume that its spectral edge $\lambda_0$ is achieved by the Bloch eigenvalue $\lambda_m(\eta)$ at finitely many points $\eta_1, \eta_2, \ldots, \eta_k$. There exists a matrix $B = (b_{ij})_{i,j=1}^d$ with $L^{\infty}_Y(Y)$-entries and $t_0 > 0$ such that for every $t \leq t_0$, a spectral edge is achieved by the Bloch eigenvalue $\lambda_m(\eta; A + tB)$ of the operator $A = -\nabla \cdot (A + tB) \nabla$ and the spectral edge is simple.

**Remark 1.6.**

1. While Theorem 1.3 achieves global simplicity for a Bloch eigenvalue, the perturbed operator is no longer a differential operator. Non-local operators are becoming increasingly important in the theory of effective mass [27] and homogenization [15], [13], particularly for non-uniformly bounded operators [12], [11].

2. Theorem 1.4 is an adaptation of the theorem of Klopp and Ralston [22] to divergence-type operators. Their proof relies heavily on the Hölder regularity results for weak solutions of divergence-type operators. In our proof, we require Hölder continuity of the solutions as well as their derivatives. Hence, we have to impose $W^{1,\infty}$ condition on the coefficients.

3. In Theorem 1.5, we drop the $W^{1,\infty}$ requirement on the coefficients under assumption of finiteness on the number of points at which the spectral edge is attained. This is essential for the applications that we have in mind, in the theory of homogenization, where only $L^{\infty}$ regularity is available on the coefficients.

We also prove the following theorem on internal edge homogenization. Let $m \in \mathbb{N}$. Suppose that $\lambda_0$ is the upper edge of an internal gap in the spectrum of operator $A$ (1.1), which is attained by the Bloch eigenvalues $\lambda_m(\eta)$ and $\lambda_{m+1}(\eta)$. After a perturbation such as in Theorem 1.5, this spectral edge will become simple. Suppose that the perturbed spectral edge is attained at a single point and the Hessian of the Bloch eigenvalue $\tilde{\lambda}_m(\eta)$ of
the perturbed operator is positive definite at this point. Similarly, assume that the Bloch eigenvalue \( \tilde{\lambda}_{m+1}(\eta) \) also has a unique minimum in a neighborhood of the old spectral edge and its Hessian at the minimum is positive definite. Then, the homogenized coefficients may be approximated using the Bloch eigenvalues of the perturbed operator as specified in the theorem below.

**Theorem 1.7.** Let \( \mathcal{A} \) be the operator in \( L^2(\mathbb{R}^d) \) defined by (1.1). Suppose that the entries of the matrix \( A \) belong to \( L_\infty^\#(Y) \). Let \( \lambda_0 \) be the upper edge of a gap in the spectrum of \( \mathcal{A} \). Let \( \varepsilon > 0 \) be small enough so that \( \lambda_0 - \varepsilon^2 \) remains in the spectral gap. Let \( \mathcal{A}' \) be defined as \( \mathcal{A}' = -\nabla \cdot (A(\frac{x}{\varepsilon}) \nabla) \) on \( L^2(\mathbb{R}^d) \).

Let \( \tilde{\mathcal{A}}(t) = \mathcal{A} + t\mathcal{B} \) be the perturbation of \( \mathcal{A} \) specified by theorem 1.5 such that the perturbed operator has a simple spectral edge at \( \tilde{\lambda}_0(t) \). Let \( \tilde{\mathcal{A}}'(t) = -\nabla \cdot (\tilde{\mathcal{A}}(\frac{x}{\varepsilon}) + t\tilde{\mathcal{B}}(\frac{x}{\varepsilon})) \nabla \). Choose \( t = O(\varepsilon^4) \). Under the assumptions stated above,

\[
||R(\varepsilon) - \tilde{R}^0(\varepsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\varepsilon),
\]

as \( \varepsilon \to 0 \), where

\[
R(\varepsilon) := (\mathcal{A}' - (\varepsilon^{-2}\lambda_0 - \varepsilon^2)I)^{-1}, \quad \text{and}
\]

\[
\tilde{R}^0(\varepsilon) := [\tilde{\psi}_m^\varepsilon]\left(\tilde{B}_0(t)\nabla^2 + \varepsilon^2I\right)^{-1}[(\tilde{\psi}_m^\varepsilon)^*] + [\tilde{\psi}_{m+1}^\varepsilon]\left(\tilde{B}_1(t)\nabla^2 + \varepsilon^2I\right)^{-1}[(\tilde{\psi}_{m+1}^\varepsilon)^*],
\]

where \( \tilde{\psi}_m^\varepsilon \) and \( \tilde{\psi}_{m+1}^\varepsilon \) are the Bloch eigenfunctions at the minimum of the Bloch eigenvalues \( \tilde{\lambda}_m(\eta) \) and \( \tilde{\lambda}_{m+1}(\eta) \) and \( \tilde{B}_0(t), \tilde{B}_1(t) \) are their respective Hessians.

**Remark 1.8.**

1. Theorem 1.7 allows the computation of the homogenized coefficients through the perturbation of original coefficients that make the corresponding spectral edge simple.

2. The homogenized coefficients require both the branches of the Bloch eigenvalues which crossed before the perturbation was applied.

The plan of this paper is as follows; in the next section 2, we prove the genericity result 1.1 stated above. In Section 3, we prove Theorem 1.3 and in subsequent sections 4 and 5, we prove Theorems 1.4 and 1.5 concerning simplicity of spectral edges. In the final section 6, we give a short introduction to internal edge homogenization and furnish applications of perturbation theory to Bloch wave homogenization by proving Theorem 1.7.
2. Local Simplicity of Bloch eigenvalues

Let \( P \) be the set defined by

\[
P := \{ A \in M_B^\geq : \text{the eigenvalues of } A(\eta_0) \text{ are simple} \}.
\]

We can write the set \( P \) as an intersection of countably many sets as follows: Let \( P_0 := M_B^\geq \), and

\[
P_n := \{ A \in M_B^\geq : \text{the first } n \text{ eigenvalues of } A(\eta_0) \text{ are simple} \}.
\]

Note that,

\[
P \subseteq \ldots \subseteq P_n \subseteq P_{n-1} \subseteq \ldots \subseteq P_1 \subseteq P_0,
\]

and \( P = \cap_{n=0}^{\infty} P_n \).

We shall require the following two lemmas.

Lemma 2.1. \( P_n \) is open in \( M_B^\geq \) for all \( n \in \mathbb{N} \cup \{0\} \).

Lemma 2.2. \( P_{n+1} \) is dense in \( P_n \), for all \( n \in \mathbb{N} \cup \{0\} \).

Proof. (of Theorem 1.1) We can write \( P \) as the countable intersection \( P = \bigcap_{n=0}^{\infty} P_n \), each of which is an open and dense set in \( M_B^\geq \). Hence, the simplicity of eigenvalues of \( A(\eta_0) \) is a generic property in \( M_B^\geq \).

The proofs of Lemmas 2.1 and 2.2 is the content of next subsections.

2.1. Eigenvalues are continuous functions of coefficients

In this subsection, we shall prove continuous dependence of the eigenvalues of the shifted operator \( A(\eta) \) on its coefficients. The main tool in this proof is Courant-Fischer min-max principle, which states that

\[
\lambda_m(\eta_0) = \min_{\dim F = m} \max_{v \in F} \frac{\int_Y A(\nabla + i\eta_0)v \cdot (\nabla + i\eta_0)v \, dx}{\int_Y v^2 \, dx},
\]

where \( F \) ranges over all subspaces of \( H^1_{\#}(Y) \) of dimension \( m \).

Proposition 2.3. Let \( A_1, A_2 \in M_B^\geq \) and let \( \eta \mapsto \lambda_n^1(\eta) \), \( \eta \mapsto \lambda_n^2(\eta) \) be the \( n \)-th Bloch eigenvalues of the operators \( A_1 \) and \( A_2 \) respectively. Then

\[
|\lambda_n^1(\eta_0) - \lambda_n^2(\eta_0)| \leq c_n(\eta_0)||A_1 - A_2||_{L^\infty},
\]

where \( c_n(\eta_0) \) is the \( n \)-th eigenvalue of the shifted Laplacian \(-(\nabla + i\eta_0)^2\) on \( Y \) with periodic boundary conditions.

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Proof. Let \( a_1(v) = \int_Y A_1(\nabla + i\eta_0)v.\nabla v \, dy \) and \( a_2(v) = \int_Y A_2(\nabla + i\eta_0)v.\nabla v \, dy \) be the quadratic forms that appear in the min-max principle.

\[
|a_1(v) - a_2(v)| = \left| \int_Y (A_1 - A_2)(\nabla + i\eta_0)v.\nabla v \, dy \right| \\
\leq d||A_1 - A_2||_{L^\infty} \int_Y |(\nabla + i\eta_0)v|^2 \, dy,
\]

Therefore,

\[
a_1(v) \leq a_2(v) + d||A_1 - A_2||_{L^\infty} \int_Y |(\nabla + i\eta_0)v|^2 \, dy.
\]

Now, divide both sides by \( \int_Y |v|^2 \, dy \), the \( L^2(\mathcal{Y}) \) inner product of \( v \) with itself and apply the appropriate min-max to obtain

\[
\lambda^1_m(\eta_0) \leq \lambda^2_m(\eta_0) + dc_m(\eta_0)||A_1 - A_2||_{L^\infty}.
\]

Notice that the constant \( c_m(\eta_0) \) is precisely the \( m^{th} \) eigenvalue of the shifted Laplacian operator \( -(\nabla + i\eta_0)^2 \) on \( \mathcal{Y} \) with periodic boundary conditions. By interchanging the role of \( A_1 \) and \( A_2 \), the inequality

\[
\lambda^2_m(\eta_0) \leq \lambda^1_m(\eta_0) + dc_m(\eta_0)||A_2 - A_1||_{L^\infty},
\]

is obtained, which completes the proof of this proposition.  

Remark 2.4. In [14], the Bloch eigenvalues have been proved to be Lipschitz continuous in \( \eta \in \mathcal{Y}'. \) Indeed, with a little more effort, one may prove that the Bloch eigenvalues are jointly continuous in \( \eta \in \mathcal{Y}' \) and the coefficients of the operator.

Proof of Lemma 2.7. Let \( A \in P_n \) and

\[
\delta = \min\{\lambda_{j+1}(\eta_0) - \lambda_j(\eta_0) : j = 1, 2, \ldots, n\}.
\]

Let \( c = \max_{1 \leq j \leq n} dc_j(\eta_0) \), where \( c_j(\eta_0) \) is the \( j^{th} \) eigenvalue of the shifted Laplacian \( -(\nabla + i\eta_0)^2 \) on \( \mathcal{Y} \) with periodic boundary conditions.

Let

\[
U = \left\{ A' \in M^\mathcal{Y}_B : ||A - A'||_{L^\infty} < \frac{\delta}{4c} \right\}.
\]

\( U \) is an open set in \( M^\mathcal{Y}_B \) containing \( A \). We shall show that \( U \) is a subset of \( P_n \). Let \( \{\lambda'_j(\eta), j = 1, 2, \ldots, n\} \) be the Bloch eigenvalues of operator \( A' \) associated to \( A' \in U \). For \( j = 1, 2, \ldots, n \), we have:

\[
|\lambda'_j(\eta_0) - \lambda_j(\eta_0)| \leq dc_j(\eta_0)||A - A'||_{L^\infty} \\
\leq dc_j(\eta_0)\frac{\delta}{4c} \leq \frac{\delta}{4},
\]
Hence,
\[
\delta \leq \lambda_{j+1}(\eta_0) - \lambda_j(\eta_0) \\
\leq |\lambda'_{j+1}(\eta_0) - \lambda_{j+1}(\eta_0)| + |\lambda'_j(\eta_0) - \lambda'_{j+1}(\eta_0)| + |\lambda'_j(\eta_0) - \lambda_j(\eta_0)| \\
\leq \frac{\delta}{4} + |\lambda'_j(\eta_0) - \lambda'_{j+1}(\eta_0)| + \frac{\delta}{4} \\
= \frac{\delta}{2} + \lambda'_{j+1}(\eta_0) - \lambda'_j(\eta_0).
\]

Therefore, \(\lambda'_{j+1}(\eta_0) - \lambda'_j(\eta_0) \geq \frac{\delta}{2} > 0\) for \(j = 1, 2, \ldots, n\). Therefore, the first \(n\) Bloch eigenvalues of \(A'\) are simple at \(\eta_0\), as required.

\[\square\]

2.2. Perturbation Theory

In this section, we show that a perturbation in the coefficients of the operator \(A\) gives rise to a corresponding holomorphic family of sectorial forms of type \((a)\). Further, the self-adjointness of the forms coupled with the compactness of the resolvent for the operator family ensures that it is a self-adjoint holomorphic family of type \((B)\). For definition of these notions, see Kato [20].

Let \(A \in M^>_B\) and \(B = (b_{kl})\) be a symmetric matrix with \(L^\infty(Y)\) entries. Then, for \(\sigma < \alpha/dC\), \(A + \sigma B\) belongs to \(M^>_B\), where \(\alpha\) is the coercivity constant for \(A\) and \(C := ||B||_{L^\infty}\) is an upper bound for \(B\). For a fixed \(\eta_0 \in Y'\) and for \(\sigma_0 := \frac{\alpha^2}{2d||B||_{L^\infty}}\), let us define the operator family
\[
A(\eta_0)(\tau) = -(\nabla + i\eta_0) \cdot (A + \tau B)(\nabla + i\eta_0), \quad \tau \in R,
\]
where \(R = \{z \in \mathbb{C} : |\text{Re}(z)| < \sigma_0, |\text{Im}(z)| < \sigma_0\}\). For these values of \(\tau\), \(A + \tau B\) has coercivity constant \(\alpha/2\). The holomorphic family of sesquilinear forms \(t(\tau)\) associated to operator \(A + \tau B\), with the \(\tau\)-independent domain \(\mathcal{D}(t(\tau)) = H^1_\gamma(Y)\), is defined as
\[
t(\tau)[u, v] := \int_Y (a_{kl}(y) + \tau b_{kl}(y)) \frac{\partial u}{\partial y_i} \frac{\partial \overline{v}}{\partial y_k} dy \\
+ i\eta_{0,l} \int_Y (a_{kl}(y) + \tau b_{kl}(y)) u \frac{\partial \overline{v}}{\partial y_k} dy \\
- i\eta_{0,k} \int_Y (a_{kl}(y) + \tau b_{kl}(y)) \overline{v} \frac{\partial u}{\partial y_l} dy \\
+ \eta_{0,l} \eta_{0,k} \int_Y (a_{kl}(y) + \tau b_{kl}(y)) u \overline{v} dy,
\]
where \(\eta := (\eta_{0,1}, \eta_{0,2}, \ldots, \eta_{0,d})\) and summation over repeated indices is assumed.

**Theorem 2.5.** \(t(\tau)\) is a holomorphic family of type \((a)\).
Proof. The quadratic form associated with $t(\tau)$ is as follows:

$$t(\tau)[u] := \int_Y (a_{kl}(y) + \tau b_{kl}(y)) \frac{\partial u}{\partial y_i} \frac{\partial \overline{u}}{\partial y_k} \, dy$$

$$+ i \eta_{0,l} \int_Y (a_{kl}(y) + \tau b_{kl}(y)) u \frac{\partial \overline{u}}{\partial y_k} \, dy$$

$$- i \eta_{0,k} \int_Y (a_{kl}(y) + \tau b_{kl}(y)) \overline{u} \frac{\partial u}{\partial y_l} \, dy$$

$$+ \eta_{0,k} \eta_{0,l} \int_Y (a_{kl}(y) + \tau b_{kl}(y)) \overline{u} \, dy.$$

$(i)$ $t(\tau)$ is sectorial.

Let us write $\tau = \rho + i\gamma$, then the quadratic form $t(\tau)$ can be written as the sum of its real and imaginary parts:

$$t(\tau) = \Re t(\tau)[u] + \Im t(\tau)[u]$$

where the real part is

$$\Re t(\tau)[u] := \int_Y (a_{kl}(y) + \rho b_{kl}(y)) \frac{\partial u}{\partial y_i} \frac{\partial \overline{u}}{\partial y_k} \, dy$$

$$+ i \eta_{0,l} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) u \frac{\partial \overline{u}}{\partial y_k} \, dy$$

$$- i \eta_{0,k} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) \overline{u} \frac{\partial u}{\partial y_l} \, dy$$

$$+ \eta_{0,k} \eta_{0,l} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) \overline{u} \, dy. \tag{2.1}$$

and the imaginary part is

$$\Im t(\tau)[u] := \int_Y (\gamma b_{kl}(y)) \frac{\partial u}{\partial y_i} \frac{\partial \overline{u}}{\partial y_k} \, dy$$

$$+ 2 \Im \left( \eta_{0,l} \int_Y (\gamma b_{kl}(y)) u \frac{\partial \overline{u}}{\partial y_k} \, dy \right)$$

$$+ \eta_{0,k} \eta_{0,l} \int_Y (\gamma b_{kl}(y)) \overline{u} \, dy. \tag{2.2}$$

The real part (2.1) of $t(\tau)[u]$ may also be written as
\[ \Re(t)[u] := \int_Y \left( a_{kl}(y) + \rho b_{kl}(y) \right) \frac{\partial u}{\partial y_l} \frac{\partial \overline{u}}{\partial y_k} dy + 2\Re \left( i\eta_{0,l} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) u \frac{\partial \overline{u}}{\partial y_k} dy \right) + \eta_{0,k}\eta_{0,l} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) u \overline{u} dy. \]  

(2.3)

The first term on the right of (2.3) is estimated from below as follows:

\[ \int_Y (a_{kl}(y) + \rho b_{kl}(y)) \frac{\partial u}{\partial y_l} \frac{\partial \overline{u}}{\partial y_k} dy \geq \alpha^2 \int_Y |\nabla u|^2 dy. \]  

(2.4)

The second term may be bounded from above as follows:

\[ \Re \left( i\eta_{0,l} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) u \frac{\partial \overline{u}}{\partial y_k} dy \right) \leq C_1 ||u||_{L^2_\sharp(Y)} ||\nabla u||_{L^2_\sharp(Y)} \leq C_1 C_* ||u||_{L^2_\sharp(Y)}^2 + \frac{\alpha}{4} ||\nabla u||_{L^2_\sharp(Y)}^2 \]  

(2.5)

where \( C_* = \frac{4C_1}{\alpha} \) and \( C_1, C_2 \) are generic constants.

The last term in (2.3) is estimated as

\[ \eta_{0,k}\eta_{0,l} \int_Y (a_{kl}(y) + \rho b_{kl}(y)) u \overline{u} dy \leq C_3 ||u||_{L^2_\sharp(Y)}^2 \]  

(2.6)

Finally, combining (2.4), (2.5) and (2.6), we obtain

\[ \Re(t)[u] \geq \frac{\alpha}{4} ||u||_{H^1_\sharp(Y)}^2 - C_4 ||u||_{L^2_\sharp(Y)}^2. \]  

(2.7)

Estimating the imaginary part (2.2) from above, we obtain

\[ \Im(t)[u] \leq C_5 ||\nabla u||_{L^2_\sharp(Y)}^2 + C_6 ||u||_{L^2_\sharp(Y)}^2. \]  

(2.8)

Now, choose a scalar \( C_7 \) so that \( C_5 = \alpha C_7^2 \).

The inequality (2.7) may be written as

\[ \Re(t)[u] + C_4 ||u||_{L^2_\sharp(Y)}^2 + \frac{C_6}{C^2_7} ||u||_{L^2_\sharp(Y)}^2 \geq \frac{\alpha}{4} ||u||_{H^1_\sharp(Y)}^2 + \frac{C_6}{C^2_7} ||u||_{L^2_\sharp(Y)}^2. \]  

(2.9)
Now, we define a new quadratic form \( \tilde{t}[u] := t[u] + (C_4 + \frac{C_6}{C_7})||u||^2_{L^2_\sharp(Y)} \), then inequality (2.9) becomes

\[
\Re\tilde{t}(\tau)[u] \geq \frac{\alpha}{4}||u||^2_{H^1_\sharp(Y)} + \frac{C_6}{C_7}||u||^2_{L^2_\sharp(Y)}.
\]

This may be further written as

\[
\Re\tilde{t}(\tau)[u] - \frac{\alpha}{4}||u||^2_{L^2_\sharp(Y)} \geq \frac{\alpha}{4}||\nabla u||^2_{L^2_\sharp(Y)} + \frac{C_6}{C_7}||u||^2_{L^2_\sharp(Y)}.
\]

On multiplying throughout by \( C_7 \), the inequality (2.11) becomes

\[
C_7 \left\{ \Re\tilde{t}(\tau)[u] - \frac{\alpha}{4}||u||^2_{L^2_\sharp(Y)} \right\} \geq C_5||\nabla u||^2_{L^2_\sharp(Y)} + C_6||u||^2_{L^2_\sharp(Y)}.
\]

Since \( \Re\tilde{t}[u] = \Re t[u] \), combining the inequalities (2.8) and (2.12), we obtain

\[
|\Re\tilde{t}(\tau)[u]| \leq C_7 \left\{ \Re\tilde{t}(\tau)[u] - \frac{\alpha}{4}||u||^2_{L^2_\sharp(Y)} \right\}.
\]

This proves that the form \( \tilde{t} \) is sectorial. However, the property of sectoriality is invariant under a shift. Therefore, \( t \) is sectorial, as well.

(iii) \( t(\tau) \) is closed.

This follows from the inequality (2.10). If \( u_n \xrightarrow{t-convergence} u \) then \( \Re t[u_n - u_m] \to 0 \) as \( n, m \to \infty \). By (2.10), \( (u_n) \) is a Cauchy sequence in \( H^1_\sharp(Y) \). By completeness, there is \( v \in H^1_\sharp(Y) \) to which the sequence converges. However, \( t \)-convergence implies \( L^2 \) convergence, and therefore, \( u = v \). Clearly, \( t[u_n - u] \to 0 \).

We have proved that \( t(\tau)[u] \) is sectorial and closed. It remains to prove that the form is holomorphic. This is easily done since \( t(\tau)[u] \) is linear in \( \tau \) for each fixed \( u \in H^1_\sharp(Y) \).

The first representation theorem of Kato ensures that there exists a unique m-sectorial operator with domain contained in \( H^1_\sharp(Y) \) associated with each \( t(\tau) \). A proof may be found in [20, p.322]. The family of such operators associated with a holomorphic family of sesquilinear forms of type (a) is called a holomorphic family of type (B). The aforementioned m-sectorial operator is given by

\[
A(\eta_0)(\tau) = -(\nabla + i\eta_0)(A + \tau B)(\nabla + i\eta_0).
\]

It follows from the symmetry of the matrix \( A + \tau B \) that the family \( A(\eta_0)(\tau) \) is a self-adjoint holomorphic family of type (B). Moreover, by the compact embedding of \( H^1_\sharp(Y) \) in \( L^2_\sharp(Y) \), the operator \( A(\eta_0)(\tau) + C^* I \) has compact resolvent for each \( \tau \in R \) for some appropriate constant \( C^* \), independent of \( \tau \in R \).
In the next section, we shall make use of the following theorem which asserts the existence of a sequence of eigenpairs associated with a self-adjoint holomorphic family of type (B), analytic in \( \tau \in (-\sigma_0, \sigma_0) \). The proof of this theorem dates back to Rellich, hence we shall call these eigenvalue branches as Rellich branches.

**Theorem 2.6. (Kato-Rellich)** Let \( A(\eta_0)(\tau) \) be a selfadjoint holomorphic family of type (B), defined for \( \tau \in R \) where \( R = \{ z \in \mathbb{C} : |\text{Re}(z)| < \sigma_0, |\text{Im}(z)| < \sigma_0 \} \) and \( \sigma_0 := \frac{\alpha}{2d|B|_{L^\infty}} \). Let \( A(\eta_0)(\tau) + C_*I \) have compact resolvent for some \( C_* \in \mathbb{R} \). Then, there exists a sequence of scalar-valued functions \( (\lambda_k(\tau; \eta_0))_{k=1}^\infty \) and \( L^2(Y) \)-valued functions \( (u_k(\tau; \eta_0))_{k=1}^\infty \) defined on \( I = (-\sigma_0, \sigma_0) \), such that

1. For each fixed \( \tau \in I \), the sequence \( (\lambda_k(\tau; \eta_0))_{k=1}^\infty \) represents all the eigenvalues of \( A(\eta_0)(\tau) \) counting multiplicities and the functions \( (u_k(\tau; \eta_0))_{k=1}^\infty \) represent the corresponding eigenvectors.

2. For each \( k \in \mathbb{N} \), the functions \( (\lambda_k(\tau; \eta_0))_{k=1}^\infty \) and \( (u_k(\tau; \eta_0))_{k=1}^\infty \) are analytic on \( I \) with values in \( \mathbb{R} \) and \( L^2(Y) \) respectively.

3. The sequence \( (u_k(\tau; \eta_0))_{k=1}^\infty \) is orthonormal in \( L^2(Y) \).

4. Suppose that the \( i^{th} \) eigenvalue of \( A(\eta_0)(\tau) \) at \( \tau = 0 \) has multiplicity \( p \), i.e.,

   \[
   \lambda_i(0; \eta_0) = \lambda_{i+1}(0; \eta_0) = \ldots = \lambda_{p+i}(0; \eta_0) = \lambda_{p+i-1}(0; \eta_0).
   \]

   Then, for each interval \( K \subset \mathbb{R} \) with \( \overline{K} \) containing the eigenvalue \( \lambda_i(0; \eta_0) \) and no other eigenvalue, \( \lambda_1(\tau; \eta_0), \lambda_{i+1}(\tau; \eta_0), \ldots, \lambda_{p+i-1}(\tau; \eta_0) \) are the only eigenvalues of \( A(\eta_0)(\tau) \), counting multiplicities, lying in the interval \( K \).

2.3. Eigenvalues are generically simple

Let \( A \in M^\infty_B \) and \( B \) be a symmetric matrix with \( L^\infty_B(Y) \)-entries. For \( |\tau| < \sigma_0 := \frac{\alpha}{2d|B|_{L^\infty}} \), \( A + \tau B \in M^\infty_B \). Consider the operator \( A(\eta_0) + \tau B(\eta_0) \) in \( L^\infty_B(Y) \). It was proved in the previous subsection that the operator family \( F(\tau) = A(\eta_0) + \tau B(\eta_0) \) is a holomorphic family of type (B) for \( |\tau| < \sigma_0 \). Further, by the perturbation theorem 2.6, an eigenvalue \( \lambda(\eta_0) \) of \( F(0) \) of multiplicity \( h \), splits into \( h \) analytic functions \( (\lambda_j(\tau; \eta_0))_{j=1}^h \). Further, the corresponding eigenfunctions \( (u_j(\tau; \eta_0))_{j=1}^h \) are also analytic. Suppose that the \( h \) eigenvalues and eigenvectors of \( F(\tau) \) have the following power series expansions at \( \tau = 0 \) for \( j = 1, 2, \ldots, h \):

\[
\lambda_j(\tau; \eta_0) = \lambda(\eta_0) + \tau a_j(\eta_0) + \tau^2 \beta_j(\tau, \eta_0)
\]

\[
u_j(\tau; \eta_0) = u_j(\eta_0) + \tau \nu_j(\eta_0) + \tau^2 w_j(\tau, \eta_0).
\]

(2.14)

The eigenpairs satisfy the following equation:

\[
(-\nabla + i\eta_0) \cdot (A + \tau B)(\nabla + i\eta_0) - \lambda_j(\tau, \eta_0) u_j(\tau, \eta_0) = 0.
\]

(2.15)
Differentiating \([2.15]\) with respect to \(\tau\) and setting \(\tau\) to 0, we obtain:

\[
- (\nabla + i\eta_0) \cdot A(\nabla + i\eta_0) u_j(\eta_0) - (\nabla + i\eta_0) \cdot B(\nabla + i\eta_0) u_j(\eta_0) \\
- \lambda(\eta_0) v_j(\eta_0) - a_j(\eta_0) u_j(\eta_0) = 0
\]  

(2.16)

Finally, multiply by \(u_k(\eta_0)\) and integrate over \(Y\) to conclude that

\[
\int_Y B(\nabla + i\eta_0) u_l(\eta_0) \cdot (\nabla - i\eta_0) \overline{u_m(\eta_0)} \, dy = a_l(\eta_0) \delta_{lm}.
\]  

(2.17)

The last equation is interpreted to mean that for any perturbation \(B\) with \(L^\infty(Y)\)-entries, we obtain a set of \(h\) unperturbed eigenfunctions, \(E = \{u_j(\eta_0)\}_{j=1}^h\) that define a diagonal matrix \(G_B(E)\) whose \((l,m)\)th entry is given by the right hand side of \((2.17)\).

In general, given a perturbation \(B\) and a basis for the unperturbed eigenspace \(N(\eta_0) : = \ker(\mathcal{A}(\eta_0) - \lambda(\eta_0)I)\), \(F = \{f_1, f_2, \ldots, f_h\}\), we can define a matrix \(G_B(F)\) whose \((l,m)\)th entry is given by

\[
\int_Y B(\nabla + i\eta_0) f_l \cdot (\nabla - i\eta_0) \overline{f_m} \, dy.
\]

As noted earlier, in the basis of unperturbed eigenfunctions \(E\), \(G_B(E)\) is a diagonal matrix,

\[G_B(E) = \text{diag}(a_1(\eta_0), a_2(\eta_0), \ldots, a_h(\eta_0)).\]

In the next proposition, we shall prove that the perturbation \(B\) may be chosen in such a way that \(a_k(\eta_0) \neq a_l(\eta_0)\) for some \(k, l \in \{1, 2, \ldots, h\}\).

**Proposition 2.7.** Let \(\lambda(\eta_0)\) be an eigenvalue of \(\mathcal{A}(\eta_0)\) of multiplicity \(h\). Let \(\{2.14\}\) represent the analytic branches of the eigenpairs corresponding to any perturbation \(B\). There exists a symmetric matrix \(B\) with \(L^\infty(Y)\)-entries such that at least two of the \((a_j(\eta_0))_{j=1}^h\) are distinct.

**Proof.** On choosing \(B\), we obtain a set of unperturbed eigenfunctions, \(E\), such that the matrix \(G_B(E)\) is a diagonal matrix, \(G_B(E) = \text{diag}(a_1(\eta_0), a_2(\eta_0), \ldots, a_h(\eta_0))\). If all the \((a_j(\eta_0))_{j=1}^h\) are equal, then \(G_B(E)\) will be a scalar matrix independent of the choice of the basis \(E\) for the eigenspace, \(\ker(\mathcal{A}(\eta_0) - \lambda(\eta_0)I)\). This is due to the fact that any two matrices, \(G_B(E)\) and \(G_B(F)\) are similar. However, if we can find a basis, \(F\) for the eigenspace and a matrix \(B\), corresponding to which, the matrix \(G_B(F)\) has a non-zero off-diagonal entry, then for that choice of \(B\), \(G_B(E)\) will not be a scalar matrix, and hence, \(a_k(\eta_0) \neq a_l(\eta_0)\) for some \(k, l \in \{1, 2, \ldots, h\}\).

Let \(F = \{f_1, f_2, \ldots, f_h\}\) be any basis of \(\ker(\mathcal{A}(\eta_0) - \lambda(\eta_0)I)\). Suppose that for some \(j \in \{1, 2, \ldots, d\}\),

\[
(\partial_j + i\eta_{0,j}) f_1(\partial_j - i\eta_{0,j}) f_2 \neq 0,
\]

(2.18)
where \( \eta_0 = (\eta_{0,1}, \eta_{0,2}, \ldots, \eta_{0,d}) \). Since, \( f_i \in H^1_s(Y), \ g := (\partial_j + i\eta_{0,j})f_i(\partial_j - i\eta_{0,j})\overline{f_2} \in L^1_s(Y) \). Hence, by Hahn-Banach Theorem, there is a continuous linear functional \( l \in (L^1_s(Y))^* \), such that \( l(g) = \|g\| \neq 0 \). However, by duality, there exists \( k \in L^\infty_s(Y) \), such that \( l(g) = f_Yk g = \|g\| \neq 0 \).

Now, either \( f_Y \text{Re}(k)g \neq 0 \) or \( f_Y \text{Im}(k)g \neq 0 \). Suppose, without loss of generality that \( f_Y \text{Re}(k)g \neq 0 \) and define
\[
B = \text{diag}(0, 0, \ldots, 0, \text{Re}(k), 0, \ldots, 0)
\]
with \( \text{Re}(k) \) in the \( j^{th} \) place, then
\[
(G_B(F))_{1,2} = \int_Y B(\nabla + i\eta_0)f_1 \cdot (\nabla - i\eta_0)\overline{f_2} \ dy
= \int_Y \text{Re}(k)(\partial_j + i\eta_{0,j})f_1(\partial_j - i\eta_{0,j})\overline{f_2} \ dy
= \int_Y \text{Re}(k)g \ dy \neq 0.
\]
Alternatively, if \( (\nabla + i\eta_0)f_1 \cdot (\nabla - i\eta_0)\overline{f_2} \equiv 0 \), then there exists \( j \in \{1, 2, \ldots, d\} \), such that
\[
|(\partial_j + i\eta_{0,j})f_1|^2 = |(\partial_j + i\eta_{0,j})f_2|^2 \neq 0.
\]
(2.19)
It is easy to see that if \( (2.18) \) and \( (2.19) \) do not hold, then \( f_1 \) and \( f_2 \) would be linearly dependent, which contradicts the fact that they are basis elements of \( N = \ker(A(\eta_0) - \lambda(\eta_0)I) \).

Since, \( f_i, g \in H^1_s(Y), \ g' := |(\partial_j + i\eta_{0,j})f_1|^2 = |(\partial_j + i\eta_{0,j})f_2|^2 \in L^1_s(Y) \). Hence, by Hahn-Banach Theorem, there is a continuous linear functional \( l' \in (L^1_s(Y))^* \), such that \( l'(g') = \|g'\| \neq 0 \). However, by duality, there exists \( k' \in L^\infty_s(Y) \), such that \( l'(g') = \int_Y k' g' = \|g'\| \neq 0 \).

Define
\[
B = \text{diag}(0, 0, \ldots, 0, k', 0, \ldots, 0)
\]
with \( k' \) in the \( j^{th} \) place, then in the new basis \( F' = \{f_1 + f_2, f_1 - f_2, f_3, \ldots, f_h\} \), the \( (1,2)^{th} \) entry of \( G_B(F') \) is given by
\[
\left\{ \int_Y B(\nabla + i\eta_0)(f_1 + f_2) \cdot (\nabla - i\eta_0)(\overline{f_1} - \overline{f_2}) \ dy \right\}
= \int_Y k' |(\partial_j + i\eta_{0,j})f_1|^2 - |(\partial_j + i\eta_{0,j})f_2|^2 \ dy \neq 0.
\]
Thus, either way, we have found a basis in which an off-diagonal entry of \( G_B(F) \) is non-zero. Hence, the matrix \( G_B(E) \) cannot be a scalar matrix. \( \square \)
Proof of Lemma 2.2. Suppose that $A \in P_{n}$. We want to find $A' \in P_{n+1}$ arbitrarily close to $A$. We shall construct $A'$ in the form $A' = A + \tau B$, where $B$ is a symmetric matrix with $L^{\infty}(Y)$-entries. By Lemma 2.2, we can choose $\tau$ and $B$ so that $A' \in P_{n}$. Hence, the first $n$ eigenvalues of the operator $-(\nabla + i\nu_{h}) \cdot (A + \tau B)(\nabla + i\nu_{h})$ are simple. Subsequently, we must choose $\tau$ such that $|\tau| < \sigma_{0} = \frac{\sigma}{2||B||_{L^{\infty}}} \quad \text{in order to apply the perturbation theorem 2.6}$

Now, suppose that the $(n + 1)^{th}$ eigenvalue of $A(\nu_{h})$ has multiplicity $h$. By perturbation theorem 2.6, the $h$ eigenvalue branches of the perturbed operator $A(\nu_{h}) + \tau B(\nu_{h})$ are given by the following power series at $\tau = 0$, for $j = 1, 2, \ldots, h$:

$$
\lambda_{j}(\tau; \nu_{h}) = \lambda(\nu_{h}) + \tau a_{j}(\nu_{h}) + \tau^{2} \beta_{j}(\tau; \nu_{h}).
$$

(2.20)

If there are $l, m \in \{1, 2, \ldots, h\}$ such that $a_{l}(\nu_{h}) \neq a_{m}(\nu_{h})$, then for small $\tau$, $\lambda_{i}(\tau; \nu_{h}) \neq \lambda_{m}(\tau; \nu_{h})$. Since two of the $h$ eigenvalue branches are distinct for small $\tau$, the multiplicity of the perturbed eigenvalue, which can only go down for small $\tau$, must be $h - 1$. An application of Proposition 2.7 gives us a matrix $B$ such that at least two of $(a_{j}(\nu_{h}))_{j=1}^{h}$ are distinct. Finally, after a finite number of such steps, we can reduce the multiplicity of the $(n + 1)^{th}$ eigenvalue to 1. Each perturbation must be chosen so that the total perturbation $\tau$ is as small as desired.

Remark 2.8. Theorem 1.1 proves that an eigenvalue $\lambda(\nu_{h})$ of the shifted operator $A(\nu_{h})$ can be made simple by a perturbation of the matrix $A \in M_{B}^{n}$. However, the Bloch eigenvalues are Lipschitz continuous functions of the parameter $\nu \in Y'$. Therefore, the perturbed eigenvalue $\lambda(\nu)$ will continue to remain simple in some neighbourhood of $\nu_{h}$. A proof of Lipschitz continuity of Bloch eigenvalues with respect to $\nu \in Y'$ may be found in [14].

The multiplicity of a Bloch eigenvalue $\nu \mapsto \lambda(\nu)$ can be reduced at a finite number of points in the dual parameter by application of the same perturbation. This will be the content of the next proposition. To this end, we require the following lemma.

Lemma 2.9. Let $k \in \mathbb{N}$. Let $X$ be a normed linear space and let $x_{1}, x_{2}, \ldots, x_{k}$ be non-zero elements of $X$. Then there is $x^{*} \in X^{*}$ such that $\langle x^{*}, x_{l} \rangle \neq 0 \forall l = 1, 2, \ldots, k$.

Proof. Consider the finite dimensional subspace $F$ of $X$ spanned by $x_{1}, x_{2}, \ldots, x_{k}$. Let $F^{*}_{l}$ denote the subspace of $F^{*}$ containing $x^{*} \in F^{*}$ such that $\langle x^{*}, x_{l} \rangle = 0$. Then, $F^{*} \neq \bigcup_{l=1}^{k} F^{*}_{l}$ since a vector space cannot be written as a finite union of proper subspaces. Hence, there exists $x^{*} \in F^{*}$ such that $x^{*} \notin \bigcup_{j=1}^{k} F^{*}_{l}$. Hence, $\langle x^{*}, x_{l} \rangle \neq 0$ for all $l = 1, 2, \ldots, k$. Finally, extend $x^{*}$ to $X^{*}$ using the Hahn Banach Theorem.

Proposition 2.10. Fix $m \in \mathbb{N}$. Let $S = \{\eta_{1}, \eta_{2}, \ldots, \eta_{k}\}$ be a finite collection of points in $Y'$. Then, there exists a matrix $B$ with $L^{\infty}(Y)$-entries and $t_{0} > 0$ such that for all $t \leq t_{0}$, the Bloch eigenvalue $\lambda_{i}(t, \eta_{i})$ of the operator $A + tB = -\nabla \cdot (A + tB)\nabla$ is simple for all $\eta_{i} \in S$, $1 \leq i \leq k$.

Proof. As a part of the proof of Lemma 2.2, we prove that for a given $m \in \mathbb{N}$ and $\nu_{h} \in Y'$, there exists $t_{0} > 0$ such that for all $t \leq t_{0}$, the Bloch eigenvalue $\lambda(t, \eta)$ of the perturbed operator $A + \tau B$ is simple at $\nu_{h}$. In the present proposition, we shall make a Bloch eigenvalue
\[ \lambda_m(\eta) \] of the operator \( A \) simple at a finite number of points in \( Y' \) through a perturbation in the coefficients.

As in the proof of Proposition 2.7, the perturbation at any \( \eta_i \in S \) gives rise to a self-adjoint holomorphic family of type \((B)\), analytic in \( \tau \in (-\sigma_0,\sigma_0) \), where \( \sigma_0 = \frac{C}{\|B\|_{L^\infty}} \). Suppose that the eigenvalue \( \lambda_m(\eta_i) \) of the operator \( A(\eta_i) \) has multiplicity \( h_i \). For the perturbed operator \(-\nabla \cdot (A + \tau B) \nabla\), the eigenvalue \( \lambda_m(\eta_i) \) splits into \( h_i \) branches. Suppose that the \( h_i \) eigenvalues and eigenvectors are given as follows. We have dropped the subscript \( m \) for convenience. For \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, h_i \):

\[
\lambda_j(\tau; \eta_i) = \lambda(\eta_i) + \tau a_j(\eta_i) + \tau^2 \beta_j(\tau, \eta_i) \\
u_j(\tau; \eta_i) = u_j(\eta_i) + \tau v_j(\eta_i) + \tau^2 w_j(\tau, \eta_i). 
\]

As before, the following equation holds true for \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, h_i \):

\[
\int_Y B(\nabla + i\eta_i)u_j(\eta_i) \cdot (\nabla + i\eta_i)u_k(\eta_i) \, dy = a_j(\eta_i)\delta_{jk}. \tag{2.21}
\]

The above equations define matrices that act on the unperturbed eigenspaces at each \( \eta_i \). The multiplicity would go down if we find \( B \) and bases for the unperturbed eigenspaces in which some off-diagonal entry, in particular, the \((1,2)\)-entry is non-zero. To achieve this, we proceed as before. For any choice of basis of the unperturbed eigenspace at \( \eta_i \), we find that either (2.18) or (2.19) holds. However, we cannot use this idea anymore, since, different \( \eta_i \) would have different matrices \( B \). To remedy this, we notice that, at each \( \eta_i = (\eta_i,1, \eta_i,2, \ldots, \eta_i,d) \), for a basis given by \( \{f_1, f_2, \ldots, f_{h_i}\} \) either

\[
\sum_{l=1}^d (\partial_l + i\eta_i) f_1(\partial_l - i\eta_i) \overline{f_2} \neq 0, \tag{2.22}
\]

or, if the above sum is zero, then in the modified basis \( \{f_1, f_1 + f_2, f_3, \ldots, f_{h_i}\} \),

\[
\sum_{l=1}^d (\partial_l + i\eta_i) f_1(\partial_l - i\eta_i)(f_1 + f_2) = \sum_{l=1}^d |(\partial_l + i\eta_i)f_1|^2 \neq 0, \tag{2.23}
\]

provided that \( f_1 \neq \exp(-i\eta_i \cdot y) \).

We can always choose \( f_1 \) to be a function different from \( \exp(-i\eta_i \cdot y) \) since at any of the \( \eta_i \), we have an eigenspace of dimension greater than 1. For each \( \eta_i \), call the non-zero sum among (2.22) and (2.23) as \( p_i \). Further, take either \( \Re(p_i) \) or \( \Im(p_i) \) depending on whichever is non-zero. This will make sure that we have a collection of only real functions.

By the above procedure, we have a set of \( k \) elements of \( L^1_\# (Y, \mathbb{R}) \), labelled as \( \{p_1, p_2, \ldots, p_k\} \). By Lemma 2.9, there is a common \( l \in (L^1_\# (Y))^* = L^\infty_\# (Y, \mathbb{R}) \) such that \( \int_Y lp_i \, dy \neq 0 \).

Define \( B = \text{diag}(l, l, \ldots, l) \), then either,

\[
\Re \int_Y B(\nabla + i\eta_i)f_1 \cdot (\nabla + i\eta_i)f_2 \, dy \neq 0, \tag{2.24}
\]

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or
\[ \Re \int_Y B(\nabla + i\eta_i) f_1 \cdot (\nabla - i\eta_i)(\bar{f}_1 + \bar{f}_2) \, dy \neq 0, \quad (2.25) \]
depending on \( \eta_i \).

At the end of this step, the multiplicity at each of the points \( \eta_i \) will reduce by 1. We repeat the procedure with the points among \{\eta_1, \eta_2, \ldots, \eta_k\} where the eigenvalue is still multiple. Finally, we require at most \( M \) steps to make the Bloch eigenvalue simple at each of these points, where \( M = \max_{1 \leq i \leq k} h_i \). \( \square \)

**Remark 2.11.**

- Bloch wave method may be thought of as belonging to the family of multiplier techniques in partial differential equations. In particular, exponential type multipliers, \( e^{r\phi} \), with real exponents, are used in obtaining Carleman estimates for elliptic operators [35].

- The perturbation formula (2.17) may be thought of as a variation of the Hellmann-Feynman theorem in the physics literature. The coefficients of the differential operator (1.1) are real functions, in as much as they are related to properties of materials. The presence of complex coefficients in the perturbation formula complicates the choice of the real perturbation \( B \).

- In the theory of homogenization, the coefficients of the second order divergence-type periodic elliptic operator are usually only measurable and bounded. By regularity theory [25], the eigenfunctions of the shifted operator \( A(\eta) \) are known to be Hölder continuous. However, derivatives of eigenfunctions, which may not be bounded, appear in the perturbation formula (2.17). Therefore, the perturbation \( B \) is chosen using the Hahn-Banach Theorem.

- Any operator of the form \(-\nabla \cdot A\nabla \) in \( L^2(\mathbb{R}^d) \) may be written in direct integral form, provided \( A \) is periodic. A satisfactory spectral theory for such operators is available for symmetric \( A \). However, non-self-adjoint operators are becoming increasingly important in physics [11]. For non-symmetric \( A \), the eigenvalues of the fibers \( A(\eta) \) may no longer be real and the eigenfunctions may not form a complete set. These difficulties were surmounted in proving the Bloch wave homogenization theorem for non-self-adjoint operators in [39]. Nevertheless, the generalized eigenfunctions form a complete set for a large class of elliptic operators of even order [11]. However, we are not aware of physical interpretations of complex Bloch-type eigenvalues.

- Most of the results of this paper would have similar analogues for internal edges of an elliptic system of equations, for example, the elasticity system. It would be interesting to consider these problems for the spectrum of non-elliptic operators such as the Maxwell operator.
3. Global Simplicity

In the previous section, we have proved that a given Bloch eigenvalue \( \lambda_m(\eta) \) of the operator \( \mathbf{A} \) can be made simple at a finite number of points in \( \mathcal{Y}' \) through a small perturbation in the coefficients. In this section, we shall perform perturbation on the operator \( \mathbf{A} \) in such a way that its spectrum still retains the fibered character, i.e., \( \sigma(\tilde{\mathbf{A}}) = \bigcup_{\eta \in \mathcal{Y}'} \sigma(\tilde{\mathbf{A}}(\eta)) \) and the \( m^{th} \) eigenvalue function \( \eta \mapsto \tilde{\lambda}_m(\eta) \) is simple for all \( \eta \in \mathcal{Y}' \). However, the perturbed operator \( \tilde{\mathbf{A}} \) is no longer a differential operator.

**Proof of Theorem 1.3.** The operator (1.1) has a direct integral decomposition \( \mathbf{A} = \int_{\eta \in \mathbb{T}^d} \mathbf{A}(\eta) \) where \( \mathbf{A}(\eta) = -(\nabla + i\eta) \cdot \mathbf{A} (\nabla + i\eta) \) is an unbounded operator in \( L^2_\sharp(\mathcal{Y}) \). We would like to point out that \( \mathcal{Y}' \) is understood to be a parametrization of the torus. Consider the \( m^{th} \) Bloch eigenvalue, \( \lambda_m(\eta) \) of \( \mathbf{A} \). By Lemma 2.2, at any point \( \eta_0 \in \mathcal{Y}' \), we can find a perturbation of the coefficients \( \mathbf{A}(\eta_0) = (a_{ij}) \) of \( \mathbf{A}(\eta_0) \) so that the perturbed eigenvalue \( \tilde{\lambda}_m(\eta_0) \) is simple. By Remark 2.8, there is a neighborhood of \( \eta_0 \), \( \mathcal{G}_{\eta_0} \) in which the perturbed eigenvalue \( \tilde{\lambda}_m(\eta) \) of the perturbed shifted operator \( \tilde{\mathbf{A}}(\eta) \) is simple. In this manner, for each \( \xi \in \mathbb{T}^d \), we obtain a perturbation \( B_\xi \) and a neighborhood, \( \mathcal{G}_\xi \) in which the eigenvalue of the perturbed operator \( \tilde{\mathbf{A}}(\eta) = -(\nabla + i\eta) \cdot (\mathbf{A} + B_\xi)(\nabla + i\eta) \) is simple. These neighborhoods form an open cover of the torus. The finite subcover has the property that in each of the neighborhoods \( \mathcal{G}_\xi \) in the subcover, the corresponding perturbation \( B_\xi \) causes the perturbed eigenvalue \( \tilde{\lambda}_m(\eta) \) to be simple in \( \mathcal{G}_\xi \).

Let \( \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k\} \) be the finite subcover of the torus obtained above. Define \( \mathcal{O}_1 = \mathcal{G}_1 \). Now, once \( \mathcal{O}_p \) has been defined, define \( \mathcal{O}_{p+1} = \mathcal{O}_p \setminus \bigcup_{j=1}^{p} \mathcal{G}_j \). Suppose that \( B_j \) is the perturbation corresponding to the set \( \mathcal{O}_j \).

Now, define the parametrized operator

\[
\tilde{\mathbf{A}}(\eta) = -(\nabla + i\eta) \cdot (\mathbf{A} + \sum_{j=1}^{k} B_j \chi_{\mathcal{O}_j})(\nabla + i\eta)
\]

which depends measurably on \( \eta \in \mathbb{T}^d \). Finally, define the direct integral \( \tilde{\mathbf{A}} = \int_{\eta \in \mathbb{T}^d} \tilde{\mathbf{A}}(\eta) \), where each of the fibers is a differential operator in \( L^2_\sharp(\mathcal{Y}) \). By Reed and Simon [33, p.284],

\[
\sigma(\tilde{\mathbf{A}}) = \bigcup_{\eta \in \mathcal{Y}'} \sigma(\tilde{\mathbf{A}}(\eta)).
\]

Hence, we may define an \( m^{th} \) eigenvalue function \( \eta \mapsto \tilde{\lambda}_m(\eta) \) with the property that

\[
|\lambda_m(\eta) - \tilde{\lambda}_m(\eta)| \leq C \max_{j=1}^{k} ||B_j||_{L^\infty},
\]

where \( \lambda_m(\eta) \) is the \( m^{th} \) Bloch eigenvalue of \( \mathbf{A} \).

**Remark 3.1.**
• Although the \( m^{th} \) eigenvalue of the perturbed operator is simple for all parameter values, \( \lambda_m(\eta) \) is only measurable in \( \eta \in Y' \). However, \( \tilde{\lambda}_m(\eta) \) is analytic in each \( O_i \subset T^d \).

• The perturbed operator \( \tilde{A} \) is no longer a differential operator, even though each fiber \( \tilde{A}(\eta) \) is a differential operator. In fact, we shall prove that \( \tilde{A} \) is a differential operator only if \( B_1 = B_2 = \ldots = B_k \).

Lemma 3.2. Let \( B \) be a symmetric matrix with \( L^\infty(Y) \)-entries. Define \( \mathcal{B}(\eta) = -(\nabla + i\eta) \cdot B(\nabla + i\eta) \). Let \( O \subset Y' \) be a proper subset of \( Y' \). Then, the direct integral defined by

\[
\mathcal{B} = \int_{\eta \in T^d} \mathcal{B}(\eta) \chi_O
\]

is not a differential operator.

Proof. A linear operator \( \mathcal{B} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d) \) is a differential operator if and only if \( \text{supp}(Pu) \subset \text{supp}(u) \) for all \( u \in \mathcal{D}(\mathbb{R}^d) \) \cite{31, 17, p. 236}. In order to show that \( \mathcal{B} \) is not a differential operator, we will show that it does not preserve support.

Given \( g \in \mathcal{D}(\mathbb{R}^d) \), we define its Zak transform as

\[
g_\sharp(y, \eta) = \sum_{p \in \mathbb{Z}^d} g(y + 2\pi p) e^{-i(y+2\pi p) \cdot \eta}.
\]

This is a function in \( L^2(Y', L^2_\sharp(Y)) \). The map from \( g \in \mathcal{D}(\mathbb{R}^d) \) to \( g_\sharp \) can be extended to a unitary isomorphism from \( L^2(\mathbb{R}^d) \) to \( L^2(Y', L^2_\sharp(Y)) \). We shall show that \( \mathcal{B}(g) \) is not compactly supported. \( \mathcal{B}(g) \) is a tempered distribution defined as:

\[
(\mathcal{B}(g), \phi) = \int_O \int_Y \mathcal{B}(\nabla + i\eta) g_\sharp(y, \eta) \cdot (\nabla - i\eta) \overline{\phi_\sharp}(y, \eta) \, dy \, d\eta
\]

(3.1)

We may define the Fourier transform of \( \mathcal{B}(g) \) in \( \mathcal{S}'(\mathbb{R}^d) \) as

\[
(\hat{\mathcal{B}(g)}, \phi) = (2\pi)^d (\mathcal{B}(g), \mathcal{F}^{-1}(\phi)),
\]

where \( \mathcal{F}^{-1}(\phi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \phi(\eta) e^{-i\eta \cdot y} \, d\eta \) is the inverse Fourier transform of \( \phi \). Since \( \phi \in \mathcal{S}(\mathbb{R}^d) \), there exists \( \psi \in \mathcal{S}(\mathbb{R}^d) \) such that \( \phi = \hat{\psi} \). Therefore,

\[
(\hat{\mathcal{B}(g)}, \phi) = (2\pi)^d (\mathcal{B}(g), \mathcal{F}^{-1}(\phi)) = (2\pi)^d (\mathcal{B}(g), \psi).
\]

By Poisson Summation Formula \cite{19, p. 171}, we conclude that

\[
\psi_\sharp(y, \eta) = \sum_{p \in \mathbb{Z}^d} \psi(y + 2\pi p) e^{-i(y+2\pi p) \cdot \eta}
\]

\[
= \frac{1}{(2\pi)^d} \sum_{q \in \mathbb{Z}^d} \hat{\psi}(\eta + q) e^{-iq \cdot y}
\]

\[
= \frac{1}{(2\pi)^d} \sum_{q \in \mathbb{Z}^d} \phi(\eta + q) e^{-iq \cdot y}.
\]

(3.2)
Now, suppose that $\phi \in S(\mathbb{R}^d)$ vanishes in $\bigcup_{q \in \mathbb{Z}^d} (\mathcal{O} + q)$, then $\psi$, as obtained in [3.2], vanishes in $\mathcal{O}$. Hence,

$$ (\hat{B(g)}, \phi) = (2\pi)^d (B(g), \psi) $$

$$ = (2\pi)^d \int_{\mathcal{Y}} \int_{\mathcal{O}} B(\nabla + i\eta) g_\sharp(y, \eta) \cdot (\nabla - i\eta) \overline{\psi}_\sharp(y, \eta) \, d\eta dy $$

$$ = 0. $$

Therefore, $\hat{B(g)}$ vanishes on the open set $\bigcup_{q \in \mathbb{Z}^d} (\mathcal{O} + q)$. By Schwartz-Paley-Wiener Theorem [36, p. 191], $\hat{B(g)}$ cannot be the Fourier transform of a compactly supported distribution, i.e., $B(g)$ is not compactly supported.

**Theorem 3.3.** Let $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_p\}$ be a partition of $\mathcal{Y}$ up to measure zero, i.e., $\mathcal{Y} \setminus \bigcup_{i=1}^p \mathcal{O}_i$ is a set of measure zero. Define $B : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ by $B(g) = \sum_{i=1}^p \int_{\eta \in \mathcal{O}_i} B_i(\eta)$ where $B_i(\eta) = -(\nabla + i\eta) \cdot B_i(\nabla + i\eta)$ where $B_i$ are matrices with $L^\infty(\mathbb{Y})$-entries for all $i \in \{1, 2, \ldots, p\}$, then $B$ is a differential operator if and only if $B_1 = B_2 = \ldots = B_p$.

**Proof.** If $B := B_1 = B_2 = \ldots = B_p$, then $B(g) = -\nabla \cdot B \nabla(g)$ which is a differential operator.

Conversely, without loss of generality, assume that $B_1 \neq B_2$ and suppose that $B$ is a differential operator. Then,

$$ B(g) = \int_{\eta \in \mathcal{Y}} B_1(\eta) + \int_{\eta \in \mathcal{O}_2} (B_2 - B_1)(\eta) + \int_{\eta \in \mathcal{O}_3} (B_3 - B_1)(\eta) $$

$$ + \ldots + \int_{\eta \in \mathcal{O}_p} (B_p - B_1)(\eta) $$

$$ B(g) - \int_{\eta \in \mathcal{Y}} B_1(\eta) = \sum_{j=2}^p \int_{\eta \in \mathcal{O}_j} (B_j - B_1)(\eta) $$

The left hand side of the above equation is a differential operator. We will show that the right hand side is not a differential operator to obtain a contradiction.

We proceed as in the previous lemma 3.2.

Given $g \in \mathcal{D}(\mathbb{R}^d)$, we define its Zak transform as

$$ g_\sharp(y, \eta) = \sum_{p \in \mathbb{Z}^d} g(y + 2\pi p) e^{-i(y + 2\pi p) \cdot \eta}. $$

Define $C : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ by

$$ (C(g), \phi) = \sum_{j=2}^p \int_{\mathcal{O}_j} \int_{\mathcal{Y}} (B_j - B_1)(\nabla + i\eta) g_\sharp(y, \eta) \cdot (\nabla - i\eta) \overline{\phi}_\sharp(y, \eta) \, dy d\eta \quad (3.3) $$

It is easy to see that $C(g) \in \mathcal{S}'(\mathbb{R}^d)$.

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Therefore, we may define its Fourier transform by

\[(\widehat{\mathcal{C}(g)})(\phi) = (2\pi)^d (\mathcal{C}(g), \mathcal{F}^{-1}(\phi)),\]

where \(\mathcal{F}^{-1}(\phi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \phi(\eta) e^{-i\eta \cdot \eta} \, d\eta\) is the inverse Fourier transform of \(\phi\). Since \(\phi \in \mathcal{S}(\mathbb{R}^d)\), there exists \(\psi \in \mathcal{S}(\mathbb{R}^d)\) such that \(\phi = \widehat{\psi}\). Therefore,

\[(\widehat{\mathcal{C}(g)})(\phi) = (2\pi)^d (\mathcal{C}(g), \mathcal{F}^{-1}(\phi)) = (2\pi)^d (\mathcal{C}(g), \psi).\]

By Poisson Summation Formula [19, p. 171], we conclude that

\[
\psi_\sharp(y, \eta) = \sum_{p \in \mathbb{Z}^d} \psi(y + 2\pi p) e^{-i(y + 2\pi p) \cdot \eta} = \frac{1}{(2\pi)^d} \sum_{q \in \mathbb{Z}^d} \widehat{\psi}(\eta + q) e^{-i\eta \cdot q} = \frac{1}{(2\pi)^d} \sum_{q \in \mathbb{Z}^d} \phi(\eta + q) e^{-i\eta \cdot q}. \tag{3.4}
\]

Now, suppose that \(\phi \in \mathcal{S}(\mathbb{R}^d)\) vanishes in \(\bigcup_{q \in \mathbb{Z}^d} (\bigcup_{j=2}^p \mathcal{O}_j + q)\), then \(\psi_\sharp\), as obtained in (3.4), vanishes in \(\bigcup_{j=2}^p \mathcal{O}_j\). Hence,

\[(\widehat{\mathcal{C}(g)})(\phi) = (2\pi)^d (\mathcal{C}(g), \psi) = (2\pi)^d \sum_{j=2}^p \int_Y \int_{\mathcal{O}_j} (B_j - B_1)(\nabla + i\eta) g_\sharp(y, \eta) \cdot (\nabla - i\eta) \widehat{\psi_\sharp}(y, \eta) \, dy \, d\eta = 0.\]

Therefore, \(\widehat{\mathcal{C}(g)}\) vanishes on the open set \(\bigcup_{q \in \mathbb{Z}^d} (\bigcup_{j=2}^p \mathcal{O}_j + q)\). By Schwartz-Paley-Wiener Theorem [36, p. 191], \(\widehat{\mathcal{C}(g)}\) cannot be the Fourier transform of a compactly supported distribution, i.e., \(\mathcal{C}(g)\) is not compactly supported. Therefore, \(\mathcal{C}\) is not a differential operator. 

4. Proof of Theorem 1.4

In this section, we prove that a spectral edge of a periodic elliptic differential operator can be made simple through a perturbation in the coefficients. The proof essentially follows Klopp and Ralston [22], with the straightforward modification that the coefficients must come from \(W_{1,\infty}^1(Y)\). This condition is required to ensure that the eigenfunctions and their derivatives are Hölder continuous functions. We produce the proof here for completeness.

Suppose that the coefficients of the operator (1.1), \(a_{ij} \in W_{1,\infty}^1(Y)\). Note that the Bloch eigenvalues which are defined for \(\eta \in Y'\) are Lipschitz continuous in \(\eta\) and may be extended as a periodic function over \(\mathbb{R}^d\). In the sequel, we shall treat the Bloch eigenvalues as functions on \(\mathbb{T}^d\), which is identified with \(Y'\) in a standard way. Also, we shall write \(\lambda_j(\eta, A)\) to specify that a Bloch eigenvalue corresponds to a particular matrix \(A\). We shall require the following lemma.

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Lemma 4.1. Let $\lambda_0$ correspond to the upper edge of a spectral gap of a periodic elliptic differential operator \((1.1)\) which is attained by the Bloch eigenvalue $\lambda_m$, then

(L1) there exist numbers $a, b \in \mathbb{R}$ such that $a < \lambda_m(\eta) < b$ for all $\eta \in Y'$.

(L2) there exists $M \in \mathbb{N}$ such that $M > m$ and $\lambda_M(\eta) > b$ for all $\eta \in Y'$.

(L3) Let $B$ be a symmetric matrix with $L^\infty(Y)$-entries. There is a finite cover of $Y'$, \(\{G_1, G_2, \ldots, G_p\}\) such that for each $G_i$, we have an orthonormal set

\[
\{\phi_m^{(i)}(\eta, A - tB), \phi_{m+1}^{(i)}(\eta, A - tB), \ldots, \phi_{R_i}^{(i)}(\eta, A - tB)\}
\]

(4.1)

of functions analytic for $\eta \in G_i$ and small $t$.

(L4) The linear subspace generated by the functions \(\phi_m^{(i)}\) contains the eigenspace corresponding to eigenvalues of $-\nabla \cdot (A - tB)\nabla$ between $a$ and $b$.

(L5) Further, these functions \(\phi_m^{(i)}\) may be chosen such that

\[
\langle \phi_r^{(i)}, \phi_s^{(i)} \rangle = 0.
\]

Proof.

Proof of (L1) As noted in Remark 2.8, the Bloch eigenvalues are Lipschitz continuous functions on a compact set $T^d$. Hence, the function $\eta \mapsto \lambda_m(\eta)$ is bounded.

Proof of (L2) By Weyl’s Law [33], the eigenvalues of the periodic Laplacian satisfy the following inequality, for some $s > 0$,

\[
\lambda_M(0, I) \geq \lambda_M^N \geq C_1 M^s,
\]

(4.3)

where $\lambda_M^N$ denotes the $M$th eigenvalue of the Neumann Laplacian on $Y$.

By Lipschitz continuity of Bloch eigenvalues in the dual parameter, we have

\[
|\lambda_M(\eta, I) - \lambda_M(0, I)| \leq C|\eta| \leq C_2.
\]

Therefore, for all $\eta \in Y'$,

\[
\lambda_M(\eta, I) \geq \lambda_M(0, I) - C_2.
\]

(4.4)

On combining (4.3) and (4.4), for all $\eta \in Y'$, we obtain

\[
\lambda_M(\eta, I) \geq C_1 M^s - C_2
\]

It follows from a usual argument involving min-max principle, that

\[
|\lambda(\eta, I)| \leq C_3 ||A^{-1}||_{L^\infty(Y)} |\lambda_M(\eta, A)|.
\]

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Therefore, for all $\eta \in Y'$,

$$\lambda_M(\eta, A) \geq \frac{1}{C_3 ||A^{-1}||_{L^\infty(W)}} |\lambda_M(\eta, I)| \geq \frac{C_1 M^s}{C_3 ||A^{-1}||_{L^\infty(W)}} - \frac{C_2}{C_3 ||A^{-1}||_{L^\infty(W)}}. \quad (4.5)$$

Finally to prove (L2), choose $M$ large enough so that

$$\frac{C_1 M^s}{C_3 ||A^{-1}||_{L^\infty(W)}} - \frac{C_2}{C_3 ||A^{-1}||_{L^\infty(W)}} > b.$$

**Proof of (L3) and (L3)** For each $\xi \in \mathbb{T}^d$, there is a circle $\Gamma_\xi$ in the complex plane containing the eigenvalues of $A(\xi)$ between $a$ and $b$. Let $B$ be a $d \times d$ symmetric matrix with $W^{1,\infty}(Y)$ entries. Observe that the operator $P_\xi$ defined by

$$P_\xi(\eta; A-tB) := -\frac{1}{2\pi i} \int_{\Gamma_\xi} (A(\eta; A-tB) - zI)^{-1} \, dz \quad (4.6)$$

is real analytic in a neighbourhood $R_\xi$ of $\xi$ and for small $t$, where

$$A(\eta; A-tB) := -(\nabla + i\eta) \cdot (A-tB)(\nabla + i\eta).$$

The operator $P_\xi$ is an orthogonal projection onto the eigenspace of $A(\eta; A-tB)$ corresponding to the eigenvalues between $a$ and $b$. The analyticity of the projection operator follows from the analyticity of the integrand, which is a consequence of $A(\eta; A-tB)$ being a holomorphic family of type (B). A proof of this fact is available in [39] for perturbation in $\eta$. For a perturbation in $t$, a proof is given in subsection 2.2. More details about holomorphic families of type (A) and type (B) may be found in [20].

Therefore, in a neighbourhood of $\eta = \xi, t = 0$, we obtain an orthonormal basis for the range of $P_\xi(\eta; A-tB)$ In this manner, we obtain an open cover of $\mathbb{T}^d$. By compactness of $\mathbb{T}^d$, the open cover has a finite subcover $\{G_1, G_2, \ldots, G_p\}$ with the following properties.

- For each $G_i$, we have an orthonormal set

$$\{\phi_m^{(i)}(\eta, A-tB), \ldots, \phi_R^{(i)}(\eta, A-tB)\}$$

whose elements are analytic for $\eta \in G_i$ and $|t| < \delta$.

- The linear subspace generated by

$$\{\phi_m^{(i)}(\eta, A-tB), \ldots, \phi_R^{(i)}(\eta, A-tB)\}$$

contains the eigenspace corresponding to eigenvalues of $A(\eta; A-tB)$ between $a$ and $b$. 

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Proof of (L5) Let \( \tilde{\phi}_i = \sum_{j=m} u_{ij} \phi_j \), then
\[
\langle \dot{\tilde{\phi}}_i, \tilde{\phi}_j \rangle = \sum_j u_{rj} \bar{u}_{sj} \dot{\phi}_i + \sum_{i,j} u_{ri} \bar{u}_{sj} \langle \dot{\phi}_i, \phi_j \rangle.
\]
If we set \( U \) to be the matrix with entries \( u_{ij} \) and \( A \) to be the matrix with entries \(-\langle \phi_i, \dot{\phi}_j \rangle\), (4.2) will hold if
\[
\dot{U} = UA.
\]
This is solved with the initial condition \( U(0) = I \). The matrix \( A \) is skew-symmetric, therefore, \( U(t) \) is unitary and analytic for \( \eta \in G \). Replace \( \phi_j \) with \( \tilde{\phi}_j \) to complete the proof of (L5).

Proof of Theorem 1.4.
Assume that the spectral edge is not attained by a single Bloch eigenvalue, for otherwise, there is nothing to prove. The desired properties for the perturbed eigenvalues will be established by using the min-max principle.

Consider the sesquilinear form
\[
a(\eta, t)(u, v) := \int_Y (A - tB)(\nabla + i\eta)u \cdot (\nabla - i\eta)v.
\]
For the functions constructed in (L3), \( \langle \phi_i^{(l)}, \phi_i^{(m)} \rangle = 0 \) for all \( l, m \). Thus,
\[
\frac{d}{dt} \left( a(\eta, t)(\phi_i^{(l)}(\eta, t), \phi_i^{(m)}(\eta, t)) \right)
= -\int_Y B(\nabla + i\eta)\phi_i^{(l)}(\eta, t) \cdot (\nabla - i\eta)\bar{\phi}_i^{(l)}(\eta, t).
\]

The eigenfunctions of \( A(\eta, A) \) with periodic boundary conditions, when multiplied by \( \exp(-i\eta \cdot y) \), become eigenfunctions of \( A := -\nabla \cdot A \nabla \) with \((\eta - Y)\)-periodic boundary conditions, i.e., there are \( \lambda \) and \( u \) such that \(-\nabla \cdot (A \nabla)u = \lambda u \), where \( u \) is \((\eta - Y)\)-periodic. Since \( u \) is a complex function, the regularity theorem [25, Chapter 3, Section 15], cannot be applied directly. However, since the operator is linear, we may write \( u = v + iw \) and express the eigenvalue equation for \( u \) as two equations for the real functions \( v \) and \( w \). In particular, \( v \) and \( w \) satisfy \(-\nabla \cdot (A \nabla)v = \lambda v \) and \(-\nabla \cdot (A \nabla)w = \lambda w \) in the interior of \( Y \). Hence, by the regularity theory for elliptic equations with \( W^{1,\infty} \) coefficients, \( v \) and \( w \) and their derivatives are H"older continuous in the interior of \( Y \). Further, the H"older estimates in the interior of \( Y \) are independent of \( \eta \in Y' \). Consequently, \( u \) and its derivatives are H"older continuous in the interior of \( Y \).

Choose \( \hat{\eta} \) and \( \phi_0 \) such that \( A(\hat{\eta}, A)\phi_0 = \lambda_0 \phi_0 \). Choose \( \phi_0 \neq \exp(-i\hat{\eta} \cdot y) \). This can be achieved because the multiplicity of the Bloch eigenvalue at \( \hat{\eta} \) is greater than one. Therefore, \((\nabla + i\hat{\eta})\phi_0 \neq 0 \). Consequently, there exist \( y_0 \) in the interior of \( Y \) and \( j \) with \( 1 \leq j \leq d \).
and $\theta > 0$ such that $\left| \left( \frac{\partial}{\partial x_j} + i \hat{\eta}_j \right) \phi_0(y_0) \right|^2 \geq \theta$. Since $\phi_0$ and its derivatives are H"older continuous in the interior of $Y$, there is a small $\epsilon_0 > 0$ such that for all $|y - y_0| < \epsilon_0$, $\left| \left( \frac{\partial}{\partial x_j} + i \hat{\eta}_j \right) \phi_0(y) \right|^2 > \frac{2\theta}{3}$.

Additionally, since $\phi^{(i)}_j$ obtained earlier in (L3) are linear combinations of eigenfunctions, by the H"older continuity of the eigenfunctions and their derivatives, we have the following inequality

$$\sum_{k=m}^{R_i} \left| \left( \frac{\partial}{\partial x_j} + i \eta_j \right) \phi^{(i)}_k(y, \eta, A) - \left( \frac{\partial}{\partial x_j} + i \eta_j \right) \phi^{(i)}_k(y_0, \eta, A) \right|^2 < \frac{\theta}{3}, \quad (4.10)$$

for $\eta \in G_i$ and $|y - y_0| < \epsilon_0$. Define the matrix $B = diag(b_1, b_2, \ldots, b_d)$ as a diagonal matrix all of whose diagonal entries are zero other than $b_j$ which is chosen as a function $b_j \in C_0^{\infty}(|y - y_0| < \epsilon_0)$ such that $b_j \geq 0$ and $\int_Y b_j = 1$. Extend $B$ periodically in $\mathbb{R}^d$.

There is an index $i$ such that $\hat{\eta} \in G_i$. Therefore, $\phi_0(\hat{\eta}) = \sum_{k=m}^{R_i} c_k \phi^{(i)}_k(\hat{\eta}, A)$. Define $\phi_0(t) = \sum_{k=m}^{R_i} c_k \phi^{(i)}_k(\hat{\eta}, A - tB)$. Then,

$$\frac{d}{dt} (a(\eta,t)(\phi_0(t), \phi_0(t))) \big|_{t=0} = - \int_Y b_j \left( \frac{\partial}{\partial y_j} + i \hat{\eta}_j \right) \phi_0(y) \left( \frac{\partial}{\partial y_j} - i \hat{\eta}_j \right) \phi_0(y) \, dy \leq - \frac{2\theta}{3}. \quad (4.11)$$

Hence,

$$a(\eta,t)(\phi_0(t), \phi_0(t)) \leq \lambda_0 - \frac{2\theta}{3} t + \beta(t)(t^2). \quad (4.12)$$

On each $\eta \in G_i$, we define the function

$$\phi^{(i)}_*(\eta, t) = \sum_{k=m}^{R_i} (\partial_j + i \eta_j) \phi^{(i)}_k(y_0, \eta, A) \phi^{(i)}_k(\eta, t). \quad (4.13)$$

For $\phi(\eta, t) = \sum_{k=m}^{R_i} a_k \phi^{(i)}_k(\eta, A - tB)$, $\phi(\eta, t)$ is perpendicular to $\phi^{(i)}_*(\eta, t)$ if and only if

$$\sum_{k=1}^{R_i} a_k (\partial_j + i \eta_j) \phi^{(i)}_k(y_0, \eta, A) = 0. \quad (4.14)$$

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For $\phi(\eta, t)$ satisfying (4.14) and $||\phi||_{L^2(Y)} = 1$, the following holds for $\eta \in G_i$,

$$
\frac{d}{dt} (a(\eta,t)(\phi(\eta,t), \phi(\eta,t))) |_{t=0} = -\int_Y b_j \left( \frac{\partial}{\partial y_j} + i\eta_j \right) \phi(\eta,0) \left( \frac{\partial}{\partial y_j} - i\eta_j \right) \overline{\phi}(\eta,0) \, dy
$$

$$
= -\int_{B_{\alpha}(y_0)} \left| \sum_{k=m}^{R_i} a_k \left( (\partial_j + i\eta_j) \phi_k^{(i)}(y, \eta, A) - (\partial_j + i\eta_j) \phi_k^{(i)}(y_0, \eta, A) \right) \right|^2 b_j \, dy
$$

$$
\geq -\frac{\theta}{3}.
$$

Therefore, the following holds true, uniformly for $\eta \in G_i$ and $||\phi||_{L^2(Y)} = 1$,

$$
a(\eta,t)(\phi(\eta,t), \phi(\eta,t)) \geq \lambda_0 - \frac{\theta}{3} + \gamma(t)(t^2).
$$

To find an upper bound for $\lambda(\hat{\eta},t)$, we apply the following variational characterization of the eigenvalues of $A(\eta,t)$ to (4.12). If $\phi_1, \phi_2, \ldots, \phi_{m-1}$ are the first $m - 1$ eigenfunctions corresponding to the self-adjoint operator $A(\eta,t)$, then the $m^{th}$ eigenvalue of $A(\eta,t)$ is given by the formula

$$
\lambda_m(\eta,t) = \min_{\phi \in \{\phi_1, \phi_2, \ldots, \phi_{m-1}\}, \|\phi\|_{L^2(Y)} = 1} a(\eta,t)(\phi, \phi).
$$

Therefore,

$$
\lambda_m(\hat{\eta},t) < \lambda_0 - \frac{7\theta}{12} t,
$$

(4.17)

for $t$, small enough. To find a lower bound for $\lambda_{m+1}(\eta,t)$, we apply another variational characterization for the eigenvalues to (4.16), viz.,

$$
\lambda_{m+1}(\eta,t) = \max_{\dim V = m} \min_{\phi \in V, \|\phi\|_{L^2(Y)} = 1} a(\eta,t)(\phi, \phi),
$$

(4.18)

where $V$ varies over $m$-dimensional subspaces of $H_1^2(Y)$.

For each fixed $\eta$ and $t$, take the $m$-dimensional subspace $V$ spanned by the first $m - 1$ eigenfunctions of $A(\eta,t)$ and $\phi^{(i)}$ as defined in (4.13), i.e.,

$$
V = \{\phi_1(\eta,t), \phi_2(\eta,t), \ldots, \phi_{m-1}(\eta,t), \phi^{(i)}(\eta,t)\}.
$$

Then, $\phi(\eta,t)$ satisfying the equation (4.14) is perpendicular to $V$ and allows us to conclude that

$$
\lambda_{m+1}(\eta,t) > \lambda_0 - \frac{5\theta}{12} t,
$$

(4.19)

for small $t$. The two estimates obtained above (4.17) and (4.19) together imply that the perturbed spectral edge is attained by a single Bloch eigenvalue. \hfill \Box
Remark 4.2. The proof of Theorem 1.4 depends crucially on the interior Hölder continuity of the Bloch eigenfunctions and their derivatives. This requires the coefficients of the elliptic operator to have $W^{1,\infty}(Y)$ entries. We attempt to reduce this regularity requirement to $L^\infty$ in the next section.

5. Proof of Theorem 1.5

The aim of this section is to achieve simplicity of spectral edge through perturbation of coefficients of the operator (1.1) when the spectral edge is attained at finitely many points. This is known to occur for operators on $R^2$.

The proof follows Parnovski and Shterenberg [30] and is divided into the following steps:

1. Let $A := -\nabla \cdot (A \nabla)$ be a periodic elliptic differential operator in divergence form such that it has a spectral gap $(\lambda_-, \lambda_+)$. Assume that $\lambda_+$ is achieved by the Bloch eigenvalue $\lambda_m(\eta)$ at finitely many points $\eta_1, \eta_2, \ldots, \eta_k$ in $Y'$.

2. By Proposition 2.10, there is a single perturbation $B$ of the coefficients so that the new operator $-\nabla \cdot (A + tB)\nabla$ has simple eigenvalues at the points $\eta_1, \eta_2, \ldots, \eta_k$.

3. However, the perturbation creates new points at which the new spectral edge has been attained. We shall prove that given $\delta > 0$, we can find perturbation parameter $t$ such that all the points at which the spectral edge is attained are within $\delta$-distance of the old spectral edge (Lemma 5.1).

4. We prove that these new spectral edges are not multiple.

We shall denote the operator $-\nabla \cdot (A + tB)\nabla$ as $A + tB$. Let $S_t$ denote the set of points at which the new spectral edge is attained, i.e.,

$$S_t := \{ \eta \in Y' : \text{The Bloch eigenvalue } \lambda_m(\eta; A + tB) \text{ attains the spectral edge at } \eta \}. \quad (5.1)$$

We shall require the following lemma.

Lemma 5.1. Let $k \in \mathbb{N}$. Let $A = -\nabla \cdot (A \nabla)$ be a periodic elliptic differential operator. Assume that the spectral edge $\lambda_+$ is attained by the Bloch eigenvalue $\lambda_m(\eta)$ at finitely many points $\eta_1, \eta_2, \ldots, \eta_k$ in $Y'$. Consider the perturbed operator $A + tB$. Given $\delta > 0$, $0 < \delta < 1$, there is a $t_0 > 0$ such that for every $t \leq t_0$

$$S_t \subset \bigcup_{i=1}^{k} B(\eta_i, \delta)$$

Proof. We prove this lemma by contradiction. Assume that there is $\delta$, $0 < \delta < 1$ and sequences $(t_n)_{n \in \mathbb{N}}$ and $\xi_n \in S_{t_n}$ such that $\forall \ 1 \leq l \leq k$,

$$|\xi_n - \eta_l| \geq \delta. \quad (5.2)$$

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We may choose $t_n$ so that $t_n \leq \frac{1}{n}$. Let $\lambda_+(A + tB)$ denote the spectral edge for the operator $A + tB$. The perturbed spectral edge satisfies the following inequality.

\[
|\lambda_+(A) - \lambda_+(A + t_n B)| \\
\leq |\min_{\eta \in \mathcal{Y}} \lambda_m(\eta; A) - \min_{\eta \in \mathcal{Y}} \lambda_m(\eta; A + t_n B)| \\
\leq | - \max_{\eta \in \mathcal{Y}} ( - \lambda_m(\eta; A)) + \max_{\eta \in \mathcal{Y}} ( - \lambda_m(\eta; A + t_n B))| \\
\leq \max_{\eta \in \mathcal{Y}} |\lambda_m(\eta; A) - \lambda_m(\eta; A + t_n B)| \\
\leq C t_n \leq \frac{C}{n}. \quad (5.3)
\]

Now, $\{\xi_n\}_{n>0}$ is a bounded sequence in $\mathcal{Y}$. There is a convergent subsequence, i.e., $\xi_n \to \hat{\xi}$.

We shall prove that

\[
\lambda_m(\xi_n; A + t_n B) \to \lambda_m(\hat{\xi}; A). \quad (5.4)
\]

Consider,

\[
\left| \int (A + t_n B)(\nabla + i\xi_n)u(\nabla - i\xi_n)\bar{u} - \int A(\nabla + i\hat{\xi})u(\nabla - i\hat{\xi})\bar{u} \right| \\
\leq \left| \int A(\nabla + i\xi_n)u(\nabla - i\xi_n)\bar{u} - \int A(\nabla + i\hat{\xi})u(\nabla - i\hat{\xi})\bar{u} \right| \\
+ \left| \int t_n B(\nabla + i\xi_n)u(\nabla - i\xi_n)\bar{u} \right|
\]

Divide throughout by $||u||^2_{L^2(\mathcal{Y})}$ and apply the min-max principle to obtain the following inequality.

\[
|\lambda_m(\xi_n; A + t_n B) - \lambda_m(\hat{\xi}; A)| \\
\leq |\lambda_m(\xi_n) - \lambda_m(\hat{\xi})| + t_n |\lambda_m(\xi_n; B)|. \quad (5.5)
\]

In order to establish (5.4), notice that the first part of (5.5) is taken care of by the Lipschitz continuity of $\lambda_m(\cdot)$ and in the second summand, $\lambda_m(\xi_n; B)$ is bounded independent of $n$.

The convergence (5.3) and (5.4) together imply that $\lambda_+(A) = \lambda_m(\hat{\xi}; A)$ and therefore, $\hat{\xi}$ is also a spectral edge. By (5.2), this contradicts the initial assumption that there are only $k$ points at which the spectral edge is attained.
Proof of Theorem 1.5. The spectral edge of the operator \( \mathcal{A} \) is attained at finitely many points \( \eta_1, \eta_2, \ldots, \eta_k \) in \( Y' \). Choose among \( \eta_1, \eta_2, \ldots, \eta_k \) the points where the Bloch eigenvalue \( \lambda_m(\eta) \) is not a simple eigenvalue. Now, apply Corollary 2.10 to these points, so that for the perturbed operator \( \mathcal{A} + t\mathcal{B} \), the corresponding Bloch eigenvalue becomes simple at these points.

There is a neighbourhood \( \mathcal{O}_i \) of each of the points \( (\eta_i)_{i=1}^k \) in which the Bloch eigenvalue is simple for a range of \( t \). Each of these neighbourhoods contain a ball, \( B(\eta_i, \delta_i) \) of radius \( \delta_i \) centered at \( \eta_i \). Let \( \delta = \min_{1 \leq i \leq k} \delta_i \), then by Lemma 5.1, there exists \( t_0 > 0 \) such that the spectral edge of the perturbed operator \( \mathcal{A} + t\mathcal{B} \) is contained in the union of the balls \( \bigcup_{i=1}^k B(\eta_i, \delta) \) for all \( t \leq t_0 \).

Hence, we have obtained a perturbation of the operator \( \mathcal{A} \) such that its spectral edge is simple.

6. Applications to the theory of homogenization

Birman and Suslina [9] have characterized homogenization as a spectral threshold effect. Their analysis focuses on finding operator error estimates of different orders. They have also extended the notion of homogenization to edges of spectral gaps other than the lowest one [10], [42].

6.1. Internal Edge Homogenization

In this subsection, we review the internal edge homogenization theorem of Birman and Suslina [10]. Consider the equation (1.1) corresponding to the operator \( \mathcal{A} \). We know that \( \inf \sigma(\mathcal{A}) = 0 \). The usual homogenization is interpreted as a spectral threshold effect at the lowest edge.

Consider an internal gap \( (\lambda_-, \lambda_+) \) in the spectrum of \( \mathcal{A} \), where \( \lambda_-, \lambda_+ \in \sigma(\mathcal{A}) \). Then, \( \lambda_- > 0 \). There exists \( m \in \mathbb{N} \), such that

\[
\lambda_- = \max_{\eta \in Y'} \lambda_{m-1}(\eta), \quad \lambda_+ = \min_{\eta \in Y'} \lambda_m(\eta).
\]

Birman and Suslina [10] make the following regularity assumptions on the lower end of the spectral edge. These correspond exactly to the properties of spectral edge that are required in order to define effective mass in theory of motion of electrons in solids [18].

(B1) \( \lambda_+ \) is attained by the \( m^{th} \) Bloch eigenvalue \( \lambda_m(\eta) \) at finitely many points \( \eta_j, j = 1, 2, \ldots, l \).

(B2) \( \lambda_m(\eta) \) is simple in a neighbourhood of \( \eta_j, j = 1, 2, \ldots, l \), therefore, \( \lambda_m(\eta) \) is analytic in \( \eta \) near \( \eta_j, j = 1, 2, \ldots, l \).
(B3) \( \lambda_m(\eta) \) is non-degenerate at \( \eta_j, j = 1, 2, \ldots, l \), i.e.,

\[
(\lambda_m(\eta) - \lambda_+) = (\eta - \eta_j)^T B_j (\eta - \eta_j) + O(|\eta - \eta_j|^3),
\]

for \( \eta \) near \( \eta_j, j = 1, 2, \ldots, l \), where \( B_j \) are positive definite matrices.

Under these assumptions, the internal edge homogenization theorem is proved.

**Theorem 6.1** [10]. Let \( \mathcal{A} \) be the operator on \( L^2(\mathbb{R}^d) \) defined by (1.1) and let \( (\lambda_-, \lambda_+) \) be a gap in the spectrum of \( \mathcal{A} \). Assume conditions (B1), (B2), (B3) and let \( \varepsilon > 0 \) be small enough so that \( \lambda_+ - \varepsilon^2 > \lambda_- \). Let \( \mathcal{A}' \) be defined as the unbounded operator \( \mathcal{A}' = -\nabla \cdot (A(\xi)\nabla) \) in \( L^2(\mathbb{R}^d) \). Then,

\[
||R(\varepsilon) - R^0(\varepsilon)||_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = O(\varepsilon), \tag{6.1}
\]

where

\[
R(\varepsilon) = (A' - (\varepsilon^{-2}\lambda_+ - \varepsilon^2)I)^{-1}
\]

and

\[
R^0(\varepsilon) := \sum_{j=1}^{l} [\psi_j^*] (B_j \nabla^2 + \varepsilon^2 I)^{-1} [(\psi_j)^*],
\]

where \( \psi_j(y, \eta_j) = \exp(i\eta \cdot \eta_j) \phi_j(y) \), where \( \phi_j \) is the eigenvector corresponding to the eigenvalue \( \lambda_+ = \lambda_m(\eta_j) \) of the operator \( \mathcal{A}(\eta) = - (\nabla + i\eta_j) \cdot A(\nabla + i\eta_j) \) for \( 1 \leq j \leq l \).

6.2. **Internal Edge Homogenization in the presence of multiplicity: Proof of Theorem 1.7**

In this section, we shall prove a theorem corresponding to internal edge homogenization of the operator \( \mathcal{A}' = -\nabla \cdot (A(\xi)\nabla) \) in \( L^2(\mathbb{R}^d) \) in the presence of multiplicity. We shall interpret the three assumptions [B1], [B2], [B3] that have been made on the spectral edge as hypotheses on the shape and structure of the spectral edge. Without knowledge of the shape and structure of the spectral edge, it is not possible to obtain any explicit homogenization result.

Starting with a spectral edge which is not simple, we shall appeal to Theorem 1.5 to modify the spectral edge so that it becomes simple. We shall make the following assumptions on the spectral edge. We assume the finiteness of the number of points at which the spectral edge is attained, however, since the contributions from different points are added up, we may as well assume that the spectral edge is attained uniquely at one point. Therefore, suppose that for the operator (1.1), a spectral gap exists. We shall label the upper edge of the spectral gap as \( \lambda_0 \), for convenience. There exists \( m \in \mathbb{N} \) such that \( \lambda_0 = \min_{\eta \in \mathcal{O'}} \lambda_m(\eta) \).

Suppose that the spectral edge is attained uniquely at the point \( \eta_0 \in Y' \). Also suppose that the eigenvalue \( \lambda_0 \) has multiplicity 2. Therefore, there exists a neighbourhood of \( \eta_0, \mathcal{O} \) on which the Bloch eigenvalue \( \lambda_m(\eta) \) is simple except at \( \eta_0 \). Now, a perturbation matrix \( B \) with \( L^\infty(Y) \) entries, as in Theorem 1.5, is applied to the coefficients of operator \( \mathcal{A} \), so that the new operator \( \mathcal{A}(t) = \mathcal{A} + tB \), has a simple spectral edge, \( \lambda_0(t) \). However, for small enough \( t \),
the perturbed Bloch eigenvalues \( \tilde{\lambda}_m(\eta, t) \) and \( \tilde{\lambda}_{m+1}(\eta, t) \) are simple in the neighbourhood \( \mathcal{O} \).

These properties follow from the analyticity of the projection operator \( \tilde{P}(\eta; A + tB) \), which is a consequence of the operator family \( \tilde{A}(t) \) being a holomorphic family of type \( (B) \).

For more details, see subsection 2.2.

For the perturbed spectral edge, we assume the following hypothesis

\[ \text{(C1) } \tilde{\lambda}_m(\eta; t) \text{ attains a unique minimum } \tilde{\lambda}_0(t) \text{ at a point } \eta_0(t) \in \mathcal{O} \text{ and is non-degenerate on } \mathcal{O}, \quad \text{i.e.,} \]

\[ \tilde{\lambda}_m(\eta; t) - \tilde{\lambda}_0(t) = (\eta - \eta_0(t))^T \tilde{B}_0(t)(\eta - \eta_0(t)) + O(|\eta - \eta_0(t)|^3), \quad (6.2) \]

for \( \eta \in \mathcal{O} \), where \( \tilde{B}_0(t) \) is positive definite, i.e., there is \( \alpha_0 > 0 \), independent of \( t \), such that \( \tilde{B}_0(t) > \alpha_0 I \). Further, the order in \( (6.2) \) is independent of \( t \).

\[ \text{(C2) } \tilde{\lambda}_{m+1}(\eta; t) \text{ attains a unique minimum } \tilde{\lambda}_1(t) \text{ at a point } \eta_1(t) \in \mathcal{O} \text{ and is non-degenerate on } \mathcal{O}, \quad \text{i.e.,} \]

\[ \tilde{\lambda}_{m+1}(\eta; t) - \tilde{\lambda}_1(t) = (\eta - \eta_1(t))^T \tilde{B}_1(t)(\eta - \eta_1(t)) + O(|\eta - \eta_1(t)|^3), \quad (6.3) \]

for \( \eta \in \mathcal{O} \), where \( \tilde{B}_1(t) \) is positive definite, i.e., there is \( \alpha_1 > 0 \), independent of \( t \), such that \( \tilde{B}_1(t) > \alpha_1 I \). Further, the order in \( (6.3) \) is independent of \( t \).

In essence, we are asking for the Bloch eigenvalues to have the following shapes before (Figure 2a) and after the perturbation (Figure 2b).

We will now set up some notation for the internal homogenization theorem that we are going to prove. For \( j = 0, 1 \), let

\[ \tilde{\psi}_{m+j}(y, \eta_j(t)) = \exp(iy \cdot \eta_j(t)) \tilde{\phi}_{m+j}(y; t), \]

where \( \tilde{\phi}_{m+j} \) is a normalized eigenvector corresponding to the eigenvalue \( \tilde{\lambda}_j(t) = \tilde{\lambda}_{m+j}(\eta_j(t)) \) of the operator

\[ \tilde{A}(\eta_j; t) = -(\nabla + i\eta_j) \cdot (A + tB)(\nabla + i\eta_j). \]

In what follows, we shall choose \( t = O(\epsilon^4) \). Define the following operators

\[ R(\epsilon) := (\mathcal{A}^\epsilon - (\epsilon^{-2}\lambda_0 - \epsilon^2 I))^{-1}, \quad (6.4) \]

and
\[ \tilde{R}^0(\epsilon) := \left[ \tilde{\psi}_m^\epsilon \right] \left( \tilde{B}_0(t) \nabla^2 + \epsilon^2 I \right)^{-1} \left[ (\tilde{\psi}_m^\epsilon)^* \right] + \left[ \tilde{\psi}_{m+1}^\epsilon \right] \left( \tilde{B}_1(t) \nabla^2 + \epsilon^2 I \right)^{-1} \left[ (\tilde{\psi}_{m+1}^\epsilon)^* \right]. \] (6.5)

We shall require the following two lemmas.

**Lemma 6.2.** Let
\[ \tilde{R}(\epsilon) := \left( \tilde{A}^\epsilon(t) - (\epsilon^{-2} \tilde{\lambda}_0(t) - \epsilon^2)I \right)^{-1}, \] (6.6)
where \( \tilde{A}^\epsilon(t) = -\nabla \cdot (A(\tilde{z}) + \epsilon t B(\tilde{z})) \nabla \) is an unbounded operator in \( L^2(\mathbb{R}^d) \), satisfying assumptions (C1) and (C2). Choose \( t = O(\epsilon^4) \). Then,
\[ ||R(\epsilon) - \tilde{R}(\epsilon)||_{L^2(\mathbb{R}^d)} = O(\epsilon), \] (6.7)
as \( \epsilon \to 0 \).

**Lemma 6.3.** With the same notation as in the previous lemma 6.2, it holds that
\[ ||\tilde{R}(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)} = O(\epsilon), \] (6.8)
as \( \epsilon \to 0 \).

The proofs of these lemma will be the content of the subsections 6.3 and 6.4. Now, we restate Theorem 1.7 with detailed hypotheses:

**Theorem 6.4.** Let \( A \) be the operator in \( L^2(\mathbb{R}^d) \) defined by (1.1). Suppose that the entries of the matrix \( A \) belong to \( L^\infty_2(Y) \). Let \( \lambda_0 \) be the upper edge of a gap in the spectrum of \( A \). Let \( \epsilon^2 > 0 \) be small enough so that \( \lambda_0 - \epsilon^2 \) remains in the spectral gap. Let \( \tilde{A}^\epsilon \) be defined as \( \tilde{A}^\epsilon = -\nabla \cdot (A(\tilde{z}) \nabla) \) on \( L^2(\mathbb{R}^d) \).

Let \( \tilde{A}(t) = A + \epsilon t B \) be the perturbation of \( A \) specified by theorem 1.5, such that the perturbed operator has a simple spectral edge at \( \tilde{\lambda}_0(t) \). Let \( \tilde{A}^\epsilon(t) = -\nabla \cdot (A(\tilde{z}) + \epsilon t B(\tilde{z})) \nabla \). Choose \( t = O(\epsilon^4) \). Assume conditions (C1), (C2) on the perturbed eigenvalues. Then,
\[ ||R(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)} = O(\epsilon), \] (6.9)
as \( \epsilon \to 0 \), where \( R(\epsilon) \) and \( \tilde{R}^0(\epsilon) \) are defined in (6.4) and (6.5), respectively.

**Proof of Theorem 1.7.** Observe that
\[ ||R(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)} \leq ||R(\epsilon) - \tilde{R}(\epsilon)||_{L^2(\mathbb{R}^d)} + ||\tilde{R}(\epsilon) - \tilde{R}^0(\epsilon)||_{L^2(\mathbb{R}^d)}. \] (6.10)
Applying Lemmas 6.2 and 6.3 to (6.10), we obtain (6.9).
6.3. Continuity of Resolvents

The aim of this section is to prove Lemma 6.2. We begin by introducing some notation. Define the two resolvents $S(\epsilon)$ and $\tilde{S}(\epsilon)$ by

$$S(\epsilon) = (A - (\lambda_0 - \epsilon^2 \varkappa^2)I)^{-1}$$  \hspace{1cm} (6.11)

and

$$\tilde{S}(\epsilon) = (\tilde{A}(t) - (\tilde{\lambda}_0(t) - \epsilon^2 \varkappa^2)I)^{-1}$$  \hspace{1cm} (6.12)

Define

$$h[u] := \int_{\mathbb{R}^d} A \nabla u \cdot \nabla u \, dy - \lambda_0 \int_{\mathbb{R}^d} |u|^2 \, dy.$$

Then, $h$ is a closed sectorial form with domain $H^1(\mathbb{R}^d)$.

Consider another form $p(t)$ with domain $H^1(\mathbb{R}^d)$ defined by

$$p(t)[u] := \int_{\mathbb{R}^d} t B \nabla u \cdot \nabla u \, dy - (\tilde{\lambda}_0 - \lambda_0) \int_{\mathbb{R}^d} |u|^2 \, dy.$$

To the sectorial forms $h$ and $p$, we shall apply the following theorem about continuity of resolvents which can be found in [20, p. 340].

**Theorem 6.5** [20]. Let $h$ be a densely defined, closed sectorial form bounded from below and let $p$ be a form relatively bounded with respect to $h$, so that $D(h) \subset D(p)$ and

$$|p[u]| \leq a||u||^2 + b h[u],$$  \hspace{1cm} (6.13)

where $0 \leq b < 1$, but $a$ may be positive, negative or zero. Then $h + p$ is sectorial and closed. Let $H, K$ be the operators associated with $h$ and $h + p$, respectively. If $\zeta$ is not in the spectrum of $H$ and

$$||(a + bH)R(\zeta, H)|| < 1,$$  \hspace{1cm} (6.14)

then $\zeta$ is not in the spectrum of $K$ and

$$||R(\zeta, K) - R(\zeta, H)|| \leq \frac{4||(a + bH)R(\zeta, H)||}{(1 - ||(a + bH)R(\zeta, H)||^2)}||R(\zeta, H)||.$$  \hspace{1cm} (6.15)

In order to apply the theorem, we must verify the hypotheses (6.13) and (6.14). We shall prove that $p$ is relatively bounded with respect to $h$, i.e., there exist $a, b \in \mathbb{R}$, such that:

$$|p[u]| \leq a||u||^2 + b h[u],$$
Observe that
\[ h[u] \geq \alpha \int_{\mathbb{R}^d} |\nabla u|^2 \, dy - \lambda_0 \int_{\mathbb{R}^d} |u|^2 \, dy, \]
and
\[ p[u] \leq t||B||_{L^\infty} \int_{\mathbb{R}^d} |\nabla u|^2 \, dy + (\lambda_0 - \lambda_0) \int_{\mathbb{R}^d} |u|^2 \, dy \]
\[ = t||B||_{L^\infty} \left\{ \int_{\mathbb{R}^d} \alpha |\nabla u|^2 \, dy - \lambda \int_{\mathbb{R}^d} |u|^2 \, dy \right\} \]
\[ + \left\{ \lambda_0 - \lambda_0 \right\} \int_{\mathbb{R}^d} |u|^2 \, dy \]
\[ = bh[u] + a||u||^2, \quad (6.16) \]
where \( a = \left\{ \lambda_0 - \lambda_0 + \frac{||B||_{L^\infty} \alpha}{\lambda} \right\} = c_1 t \) and \( b = \frac{||B||_{L^\infty}}{\alpha} = c_2 t \) for some constants \( c_1 \) and \( c_2 \).

Next, observe that for self-adjoint operator \( H \), the resolvent \( R(\zeta, H) \) is a normal operator. Therefore, \( ||R(\zeta, H)|| \leq \frac{1}{\text{dist}(\zeta, \sigma(H))} \). We have,
\[ ||(a + b\xi)R(\zeta, H)|| \leq ||aR(\zeta, H)|| + ||b\xi R(\zeta, H)|| \leq \frac{a}{\text{dist}(\zeta, \sigma(H))} + ||b(I - \zeta R(\zeta, H))|| \leq \frac{a}{\text{dist}(\zeta, \sigma(H))} + b||I|| + b||\zeta R(\zeta, H)|| \leq \frac{a}{\text{dist}(\zeta, \sigma(H))} + b + \frac{|\zeta|}{\text{dist}(\zeta, \sigma(H))}. \]

The operator corresponding to the sectorial form \( h \) is \( H := -\nabla \cdot A\nabla - \lambda_0 I \), therefore, \( 0 \in \sigma(H) \), so that, for \( \zeta = -\epsilon^2 x^2 \)
\[ ||(a + bH)R(\zeta, H)|| \leq \frac{a}{\epsilon^2 x^2} + 2b. \]

Notice that \( R(\zeta, H) = S(\epsilon) \) and \( R(\zeta, K) = \tilde{S}(\epsilon) \). Let us assume that \( t \) is small enough so that the above theorem can be applied to the resolvents in (6.11), (6.12). In particular, we have
\[ ||S(\epsilon) - \tilde{S}(\epsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = ||R(\zeta, H) - R(\zeta, K)|| \leq \frac{4(c_1 t + 2c_2 t \epsilon^2 x^2)}{(\epsilon^2 x^2 - c_1 t - 2c_2 t \epsilon^2 x^2)^2}. \]
Choose \( t \) so that \( t = O(\epsilon^4) \), then,

\[
||S(\epsilon) - \tilde{S}(\epsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \frac{4(1 + 2c_3\epsilon^2 \varepsilon^2)}{\varepsilon^2(1 - \epsilon^2 - 2c_3\epsilon^4 \varepsilon^2)^2}.
\]

Further, for \( \epsilon^2 < 1/2 \),

\[
||S(\epsilon) - \tilde{S}(\epsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \frac{16(1 + c_3 \varepsilon^2)}{\varepsilon^2(1 - c_3 \varepsilon^2)^2}.
\]

Proof of Lemma 6.2. Define the scaling transformation \( T_{\epsilon} \) by

\[
T_{\epsilon} : u(y) \mapsto \epsilon^{d/2} u(\epsilon y).
\]

These are unitary operators on \( L^2(\mathbb{R}^d) \). For the operators (6.4) and (6.6), it holds that

\[
R(\epsilon) = \epsilon^2 T_{\epsilon}^* S(\epsilon) T_{\epsilon}
\]

\[
\tilde{R}(\epsilon) = \epsilon^2 T_{\epsilon}^* \tilde{S}(\epsilon) T_{\epsilon}.
\]

Proving Lemma 6.2 is equivalent to proving that

\[
||S(\epsilon) - \tilde{S}(\epsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O\left(\frac{1}{\epsilon}\right).
\]

In (6.17), in fact, we prove

\[
||S(\epsilon) - \tilde{S}(\epsilon)||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(1).
\]

6.4. Internal Homogenization Result

The aim of this section is to prove Lemma 6.3. Let \( \left( \tilde{\lambda}_l(\eta; t) \right)_{l=1}^{\infty} \) and \( \left( \tilde{\phi}_l(y, \eta; t) \right)_{l=1}^{\infty} \) be the Bloch eigenvalues and the corresponding orthonormal Bloch eigenvectors for the operator \( \tilde{A}(t) \), defined in Theorem 6.4. Let, \( \tilde{\psi}_l(y, \eta; t) = e^{iy \cdot \eta} \tilde{\phi}_l(y, \eta; t) \). In the sequel, we shall suppress the dependence on \( t \) for notational convenience. The operator \( \tilde{A} \) may be decomposed in terms of the Bloch eigenvalues as in the Theorem below, a proof of which may be found in [3].

Theorem 6.6. Let \( g \in L^2(\mathbb{R}^d) \). Define \( m^{th} \) Bloch coefficient of \( g \) as follows:

\[
(\tilde{B}_l g)(\eta) = \int_{\mathbb{R}^d} \tilde{\psi}_l^*(y, \eta) g(y) \ dy, \ l \in \mathbb{N}, \eta \in Y'.
\]

Then, the following inverse formula holds.

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\begin{equation*}
g(y) = \sum_{l=1}^{\infty} \int_{Y'} (\tilde{B}_l g)(\eta)\psi_l(y,\eta) \, d\eta = \sum_{l=1}^{\infty} (\tilde{B}_l^*)(\tilde{B}_l g).
\end{equation*}

In particular, the following representation holds for the operator \( \tilde{A} \):

\begin{equation*}
\tilde{A} = \sum_{l \in \mathbb{N}} \tilde{B}_l^* \tilde{\lambda}_l \tilde{B}_l.
\end{equation*}

Also,

\begin{equation*}
R(\zeta, \tilde{A}) = \left( \tilde{A} - \zeta I \right)^{-1} = \sum_{l \in \mathbb{N}} \tilde{B}_l^*(\tilde{\lambda}_l - \zeta)^{-1} \tilde{B}_l.
\end{equation*}

(6.18)

Also, define the Fourier Transform

\begin{equation*}
(Fu)(\eta) = \int_{\mathbb{R}^d} e^{-iy \cdot \eta} u(y) \, dy.
\end{equation*}

**Proof of Lemma 6.3.** Define the operator

\begin{equation*}
\tilde{S}_0^{(\epsilon)} := [\tilde{\psi}_m] \left( \tilde{B}_0 \nabla^2 + \epsilon^2 \kappa^2 I \right)^{-1} [(\tilde{\psi}_0)^*] + [\tilde{\psi}_{m+1}] \left( \tilde{B}_1 \nabla^2 + \epsilon^2 \kappa^2 I \right)^{-1} [(\tilde{\psi}_{m+1})^*]
\end{equation*}

(6.19)

For the operators (6.5) and (6.6), it holds that

\begin{align*}
\tilde{R}^{(\epsilon)} &= \epsilon^2 T_\epsilon^* \tilde{S}(\epsilon) T_\epsilon \\
\tilde{R}^{0, (\epsilon)} &= \epsilon^2 T_\epsilon^* \tilde{S}_0^{(\epsilon)} T_\epsilon.
\end{align*}

Therefore, to prove Lemma 6.3, it is sufficient to prove that

\begin{equation*}
\| \tilde{S}(\epsilon) - \tilde{S}_0^{(\epsilon)} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = O \left( \frac{1}{\epsilon} \right).
\end{equation*}

(6.20)

We may assume that

\begin{align*}
2(\tilde{\lambda}_m(\eta) - \tilde{\lambda}_0) &\geq \tilde{B}_0(\eta - \eta_0)^2, \quad \eta \in \mathcal{O} \\
2(\tilde{\lambda}_{m+1}(\eta) - \tilde{\lambda}_1) &\geq \tilde{B}_1(\eta - \eta_1)^2, \quad \eta \in \mathcal{O},
\end{align*}

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by making $O$ smaller if required (see (C1) and (C2)). Let $\chi$ be the characteristic function of $O$, then the projections $F = \tilde{B}_m^* \chi \tilde{B}_m + \tilde{B}_{m+1}^* \chi \tilde{B}_{m+1}$ and $F^\perp = I - F$ commute with $\tilde{A}$.

Now, observe that

$$||\tilde{S}(\epsilon) - \tilde{S}_0^0(\epsilon)||_{L^2 \to L^2} = ||\tilde{S}(\epsilon)F^\perp + \tilde{S}(\epsilon)F - \tilde{S}_0^0(\epsilon)F^\perp||_{L^2 \to L^2} \leq ||\tilde{S}(\epsilon)F^\perp||_{L^2 \to L^2} + ||\tilde{S}(\epsilon)F - \tilde{S}_0^0(\epsilon)F||_{L^2 \to L^2} + ||\tilde{S}_0^0(\epsilon)F^\perp||_{L^2 \to L^2}$$

(6.21)

We shall prove that

$$||\tilde{S}(\epsilon)F^\perp||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(1)$$

(6.22)

and

$$||\tilde{S}_0^0(\epsilon)F^\perp||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(1),$$

(6.23)

so that, in order to prove (6.20), it is sufficient to prove that

$$||\tilde{S}(\epsilon)F - \tilde{S}_0^0(\epsilon)F||_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O \left( \frac{1}{\epsilon} \right).$$

(6.24)

Notice that we may expand $\tilde{S}(\epsilon)$ in Bloch waves as

$$\tilde{S}(\epsilon) = \sum_{l=1}^{\infty} \tilde{B}_l^* \left( \tilde{\lambda}_l - \tilde{\lambda}_0 + \epsilon^2 \chi^2 \right)^{-1} \tilde{B}_l.$$

We may write,

$$\tilde{S}(\epsilon) = \tilde{S}(\epsilon)F + \tilde{S}(\epsilon)F^\perp,$$

where

$$\tilde{S}(\epsilon)F = \tilde{B}_m^* \left( \tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2 \chi^2 \right)^{-1} \chi \tilde{B}_m + \tilde{B}_{m+1}^* \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2 \chi^2 \right)^{-1} \chi \tilde{B}_{m+1},$$

(6.25)

and

$$\tilde{S}(\epsilon)F^\perp = \sum_{l \neq m,m+1} \tilde{B}_l^* \left( \tilde{\lambda}_l - \tilde{\lambda}_0 + \epsilon^2 \chi^2 \right)^{-1} \tilde{B}_l + \tilde{B}_m^* \left( \tilde{\lambda}_m - \tilde{\lambda}_0 + \epsilon^2 \chi^2 \right)^{-1} (1 - \chi) \tilde{B}_m + \tilde{B}_{m+1}^* \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2 \chi^2 \right)^{-1} (1 - \chi) \tilde{B}_{m+1}.$$

(6.26)

To prove (6.22), notice that in the first term of (6.26), the sum does not include indices $m$ and $m + 1$, therefore, the Bloch eigenvalues $\tilde{\lambda}_l$ are bounded away from the spectral edge.
\( \hat{\lambda}_0 \), uniformly in \( \epsilon \) and hence, the expression \( \left( \hat{\lambda}_l - \hat{\lambda}_0 + \epsilon^2 \kappa^2 \right)^{-1} \) is bounded independent of \( \epsilon \), for \( l \neq m, m+1 \). Due to the non-degeneracy conditions assumed in \([\text{C1}]\) and \([\text{C2}]\), the Bloch eigenvalues \( \hat{\lambda}_m \) and \( \hat{\lambda}_{m+1} \) are bounded away from \( \hat{\lambda}_0 \) outside \( O \), independent of \( \epsilon \). This takes care of the last two terms in \( (6.26) \).

Similarly, we may write

\[
\tilde{S}^0(\epsilon) = \tilde{S}^0(\epsilon) F + \tilde{S}^0(\epsilon) F^\perp,
\]

where

\[
\tilde{S}^0(\epsilon) F = [\tilde{\psi}_m] \left( \tilde{B}_0 \nabla^2 + \epsilon^2 \kappa^2 I \right)^{-1} \chi([\tilde{\psi}_m]^*) + [\tilde{\psi}_{m+1}] \left( \tilde{B}_1 \nabla^2 + \epsilon^2 \kappa^2 I \right)^{-1} \chi([\tilde{\psi}_{m+1}]^*) \tag{6.27}
\]

and

\[
\tilde{S}^0(\epsilon) F^\perp = [\tilde{\psi}_m] \left( \tilde{B}_0 \nabla^2 + \epsilon^2 \kappa^2 I \right)^{-1} (1 - \chi) [\tilde{\psi}_m]^* + [\tilde{\psi}_{m+1}] \left( \tilde{B}_1 \nabla^2 + \epsilon^2 \kappa^2 I \right)^{-1} (1 - \chi) [\tilde{\psi}_{m+1}]^*. \tag{6.28}
\]

\( \tilde{S}^0(\epsilon) F^\perp \) may be further written as

\[
\tilde{S}^0(\epsilon) F^\perp = [\tilde{\psi}_m] F^* \left( \tilde{B}_0 (\eta - \eta_0)^2 + \epsilon^2 \kappa^2 I \right)^{-1} (1 - \chi) F([\tilde{\psi}_m]^*) + [\tilde{\psi}_{m+1}] F^* \left( \tilde{B}_1 (\eta - \eta_1)^2 + \epsilon^2 \kappa^2 I \right)^{-1} (1 - \chi) F([\tilde{\psi}_{m+1}]^*].
\]

The proof of \( (6.23) \) follows from the positive-definiteness of \( \tilde{B}_0 \) and \( \tilde{B}_1 \) assumed in \([\text{C1}]\) and \([\text{C2}]\), which makes the operator norm of the terms in \( (6.28) \) independent of \( \epsilon \). Now, it only remains to prove \( (6.24) \).

Write \( \tilde{S}(\epsilon) F = S_0 + S_1 \), where, for \( j = 0, 1 \),

\[
S_j := \tilde{B}_{m+j}^* \left( \hat{\lambda}_{m+j} - \hat{\lambda}_0 + \epsilon^2 \kappa^2 \right)^{-1} (\chi) \tilde{B}_{m+j} \\
= X_{m+j}^* \left( \hat{\lambda}_{m+j} - \hat{\lambda}_0 + \epsilon^2 \kappa^2 \right)^{-1} X_{m+j}, \tag{6.29}
\]

where

\[
(X_{m+j} u)(\eta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \chi \tilde{\psi}_{m+j}^*(y, \eta) u(y) \, dy, \quad j = 0, 1.
\]
Write $\tilde{S}^0(\epsilon) F = S^0_0 + S^0_1$, where, for $j = 0, 1$,

$$ S^0_j = [\tilde{\psi}_{m+j}] F^* \left( \tilde{B}_j(\eta - \eta_j)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} (\chi) \mathcal{F}[\tilde{\psi}_{m+j}]^* $$

$$ = (X^0_{m+j})^* \left( \tilde{B}_j(\eta - \eta_j)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} X^0_{m+j}, \quad (6.30) $$

where

$$ (X^0_{m+j} u)(\eta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \chi e^{-iy\eta} \tilde{\phi}_{m+j}(y, \tilde{\eta}_j) u(y) \, dy. $$

Observe that,

$$ ||\tilde{S}(\epsilon) F - \tilde{S}^0(\epsilon) F||_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq ||S_0 - S^0_0||_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} + ||S_1 - S^0_1||_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}. $$

Therefore, to prove (6.24), it remains to prove that for $j = 0, 1$,

$$ ||S_j - S^0_j||_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = O \left( \frac{1}{\epsilon} \right), \quad (6.31) $$

where $S_j, S^0_j$ are defined in (6.29), (6.30).

Consider,

$$ \epsilon ||S_0 - S^0_0|| = \epsilon ||X^*_m[\lambda_m - \tilde{\lambda}_0 + \epsilon^2 \mathcal{X}^2]^{-1} X_m - (X^0_m)^* \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} X^0_m|| $$

$$ \leq \epsilon ||X^*_m[\lambda_m - \tilde{\lambda}_0 + \epsilon^2 \mathcal{X}^2]^{-1} X_m - X^*_m \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} X_m|| + \epsilon ||X^*_m \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} X_m - (X^0_m)^* \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} X^0_m|| \quad (6.32) $$

The first of the two summands in (6.32) is estimated by using the following chain of inequalities.

$$ \epsilon |\lambda_m - \tilde{\lambda}_0 + \epsilon^2 \mathcal{X}^2|^{-1} - \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} | $$

$$ \leq c \epsilon |\eta - \eta_0|^3 |\lambda_m - \tilde{\lambda}_0 + \epsilon^2 \mathcal{X}^2|^{-1} \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} $$

$$ \leq \left( c|\eta - \eta_0|^2 \left( \tilde{B}_0(\eta - \eta_0)^2 \right)^{-1} \right) \left( 2\epsilon |\eta - \eta_0| \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} \right) \leq C_1 $$

$$ \leq c \epsilon |\eta - \eta_0|^3 |\lambda_m - \tilde{\lambda}_0 + \epsilon^2 \mathcal{X}^2|^{-1} \left( \tilde{B}_0(\eta - \eta_0)^2 + \epsilon^2 \mathcal{X}^2 I \right)^{-1} \quad (6.33) $$
The proof of the boundedness of the second term in (6.32) hinges on the analyticity of the Bloch eigenfunctions, and may be found in [10]. Finally, consider

\[
\epsilon ||S_1 - S_0|| = \\
\epsilon ||X_{m+1}^* \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2 \kappa^2 \right)^{-1} |X_{m+1} - (X_{m+1}^0)^* \left( \tilde{B}_1(\eta - \eta_0)^2 + \epsilon^2 \kappa^2 I \right)^{-1} X_{m+1}^0 || \\
\leq \epsilon ||X_{m+1}^* \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2 \kappa^2 \right)^{-1} |X_{m+1} - X_{m+1}^* \left( \tilde{B}_1(\eta - \eta_0)^2 + \epsilon^2 \kappa^2 I \right)^{-1} X_{m+1} || \\
+ \epsilon ||X_{m+1}^* \left( \tilde{B}_1(\eta - \eta_0)^2 + \epsilon^2 \kappa^2 I \right)^{-1} X_{m+1} \\
- (X_{m+1}^0)^* \left( \tilde{B}_1(\eta - \eta_0)^2 + \epsilon^2 \kappa^2 I \right)^{-1} X_{m+1}^0 ||
\]

(6.34)

The first of the two summands in (6.34) is estimated by using the following chain of inequalities.

\[
\epsilon | \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_0 + \epsilon^2 \kappa^2 \right)^{-1} - \left( \tilde{B}_1(\eta - \eta_1)^2 + \epsilon^2 \kappa^2 I \right)^{-1} | \\
\leq \epsilon | \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_1 + \epsilon^2 \kappa^2 \right)^{-1} - \left( \tilde{B}_1(\eta - \eta_1)^2 + \epsilon^2 \kappa^2 I \right)^{-1} | \\
\leq \epsilon |\eta - \eta_1|^2 \left( \tilde{\lambda}_{m+1} - \tilde{\lambda}_1 + \epsilon^2 \kappa^2 \right)^{-1} \left( \tilde{B}_1(\eta - \eta_1)^2 + \epsilon^2 \kappa^2 I \right)^{-1} \\
\leq \left( \epsilon |\eta - \eta_1|^2 \left( \tilde{B}_1(\eta - \eta_0)^2 \right)^{-1} \right) \left( 2\epsilon |\eta - \eta_1| \left( \tilde{B}_1(\eta - \eta_0)^2 + \epsilon^2 \kappa^2 I \right)^{-1} \right) \\
\leq C_2
\]

(6.35)

As before, the proof of the boundedness of the second term in (6.34) may be found in [10].

\[\square\]

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