Threshold cascade dynamics on signed random networks

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Abstract

Relationships between individuals in a social network, genes in biological systems, and spins in magnetic systems often reflect a mixture of positive (friendly) and negative (antagonistic) interactions. However, most studies of complex networks have focused on networks consisting of solely positive interactions. Here, we study threshold cascades on signed networks composed of both positive and negative connections, focusing on when a pair of nodes connected by a negative link can only be activated exclusively to each other. We found that the negative interactions not only suppress global cascades, but also induce the heterogeneity in activation patterns manifesting from single-node to network levels. Our results suggest that negative interactions may be an important source of the variability in cascading dynamics.

Keywords: Signed networks, Threshold cascade, Negative links, Heterogeneous activation patterns

1. Introduction

Modeling how the cascades of activations occur in threshold-based dynamics is fundamental for understanding collective behaviours in social and biological complex systems [1, 2, 3]. In order to model the cascading phenomena triggered by a tiny perturbation, a threshold cascade model was proposed [2, 3]. This model was originally motivated by the behavioral and emotional contagions in a society where individuals are encouraged to follow what their connected neighbors are doing. In addition, threshold cascades driven by integrate-and-fire mechanisms are associated with the avalanches of neural activations [3, 4], the spread of economic crisis [5], and cascading failures in infrastructure networks [6, 7]. The key mechanism in this model is that nodes are activated when the fraction of activated neighbors exceeds their threshold assigned a priori. In this model, cascades with an extensive size, called global cascades, can occur from an extremely small fraction of seeds because the cascades of activations propagate along connected neighbors [8, 9].

In threshold cascade models on networks, links act as channels for cascade propagation, such that the influence or stimulus arriving from each neighbor contributes positively to reaching the threshold [3]. Although traditional cascade modeling, which consists exclusively of positive links [3, 9], renders the model simple and tractable, it overlooks the negative interactions in the cascade dynamics. Adversarial interactions are common and essential elements of many networked systems [10, 11]. “Dislike” relationships in social networks [12, 13, 14], inhibitory signals in genetic regulation [15], synaptic inhibition in neural networks [16], antagonistic competitions between nations [17, 18], and antiferromagnetic bonds in magnetic systems [19] are typical examples of adversarial relationships, to name a few. Not only negative links are widespread in real-world systems, but also they play a qualitatively different role in dynamical processes than positive links [16, 19].

Networks with both types of interactions can be better modeled as “signed networks” where links are either positive or negative [18, 20]. The concept of signed networks has long been proposed in psychology and sociology, through social balance theory [21, 22], and structural balance theory [23, 24]. In addition, from the perspective of statistical physics, coexistence of positive and negative interactions has important implications as a source of geometric frustration and dynamic heterogeneity [25, 26]. As such, the studies on signed networks have received due attention from statistical physics and network science communities [25, 26, 27]. However, studies on the impact of negative interactions on threshold cascade dynamics are still lacking.

In this work, we study the dynamics of a threshold cascade model on signed random networks or “signed” cascade, to be short. In our “signed” cascade model, nodes’ activation is completely blocked if there exist active adversarial neighbors. That is, no pair of nodes connected by a negative link can be activated at the same time. It models, in an idealized way, the following real-world scenarios: In the case of “distrust” or “dislike” relationships denoted by negative links in a signed social network, someone would never agree and follow with their negative-linked person’s opinion or behavior, regardless of what their friends are.
active positive neighbors out of total positive neighbors. The following two conditions are fulfilled: i) the fraction of active neighbors of node $i$ connected by positive links is larger than threshold $\theta = 0.4$ and there are no active neighbors connected by negative links, node $i$ becomes active. Note that there can be various scenarios of cascades activations depending on the sequence of activations.

In doing. In such cases, a pair of nodes connected by a negative link cannot become active at the same time. Theoretically speaking, our model implements the cascade dynamics where negative links co-operate with positive links in functionally-multiplicative manner rather than additive manner.

There are some related previous studies in statistical physics literature on signed networks such as random threshold networks [4], percolation of antagonistic multiplex networks [37], opinion models on evolving signed networks [35], epidemic spreading on signed networks [39, 40], and threshold model with anticonformity [41, 42]. Information diffusion and linear threshold models in signed networks have also received much attention and how the information diffusion dynamics in signed networks depends on the diffusion path and structural balance was studied [43, 44, 46, 47]. However, in contrast to our model most previous studies have considered the effect of negative links additively [4, 41, 43, 40] or focused on structural properties rather than dynamical consequences [37]. In this study, we implemented cascading dynamics in a way that maximizes the multiplicative coupling effect of negative links. We confirmed that the negative links can significantly reduce the size of the global cascades. We also found that negative interactions can produce the heterogeneity in the activation patterns at various scales of cascading dynamics.

2. Signed cascade model

We propose a model of threshold cascading dynamics on a signed network by explicitly implementing the role of negative links preventing the activation of a connected neighbor. Each node can be one of two states, active or inactive in multiplicative manner. In signed networks, each link can be positive or negative. Neighbors connected by positive (negative) links are referred to as positive (negative) neighbors for short. A node becomes active when the following two conditions are fulfilled: i) the fraction of active positive neighbors out of total positive neighbors exceeds the prescribed threshold $\theta$ as in the ordinary threshold model [3] and ii) there are no active negative neighbors. The rule clearly shows different roles of the positive and negative connections in the “signed” cascades. While positive links spread activations to a connected neighbor, negative links prevents neighbors from activation. Logical “AND” requirement of both conditions reflects the multiplicative coupling of positive and negative interactions. According to the model if there exists even one active negative neighbor, the activation of the node is completely blocked. We impose the strongest role of negative links in order to demonstrate the effect of the signed networks in a simple and dramatic way.

Let us describe the procedures for numerical simulations of threshold cascading on a signed network. Initially all nodes are inactive except for a small fraction $\rho_0$ of the seed nodes that are active at the beginning. The signed cascade proceeds as follows. i) At each step, we select a node, say $i$, at random. ii-a) For inactive node $i$, the state of node $i$ becomes active when the ratio of its active positive neighbors exceeds threshold $\theta$ and there is no active negative neighbor. For instance, suppose that there are $r$ active positive neighbors out of $k_p$ positive neighbors for node $i$. Then node $i$ becomes active when $r/k_p > \theta$ and there is no active negative neighbor. ii-b) If an active node including a seed is selected, nothing happens. It means that active nodes maintain their active state permanently. iii) The procedures repeat until the dynamics of activations reaches a steady state meaning that there exists no node that can be newly activated. An example of the signed cascade process is depicted in Fig. 1. We perform random sequential updates, so that we choose at random a single node in the network and update its state at every step. Note that if there is no negative link, our model reduces to the original Watts threshold model [3]. Contrary to the original Watts threshold model, cascading dynamics on signed networks is no longer deterministic because of the role of negative interactions. Specifically, the set of active nodes in a steady state can be diverse even when the cascading dynamics starts from identical seeds on the same network structure. The final configuration is stochastically realized among many possible configurations that satisfy the conditions of both activation and inactivation as illustrated in Fig. 1. The variability of the final configuration depending on the sequence of activations produces more heterogeneous and richer dynamics than that of the traditional threshold model.

3. Results

3.1. Suppression of Global Cascades

The primary effect of negative links is the suppression of cascading dynamics due to the local suppression by antagonistic connections. We examine how negative links globally suppress cascading dynamics on a signed network focusing on the global cascades. We measured the size of
global cascades as the fraction $\rho$ of active nodes at the steady state. We use Erdős-Rényi (ER) graphs with network size $N = 10^5$, seed fraction $\rho_0 = 10^{-3}$, and threshold $\theta = 0.2$, averaged over $10^5$ different realizations. As control parameters, we vary the average degrees for positive links $z_p$ and for negative links $z_n$. The global cascade size $\rho$ as a function of $z_p$ and $z_n$ is numerically calculated [Fig. 2(a)]. We found that the size $\rho$ of global cascades monotonically decreases by adding more negative links as shown in the vertical gradient in Fig. 2(a) owing to the suppression effect of negative links.

Let us examine the behaviour of $\rho$ as a function of $z_p$ with various $z_n = 0 \ (\circ)$, $z_n = 1 \ (\triangle)$, $z_n = 2 \ (\triangledown)$, $z_n = 3 \ (\vartriangle)$, $z_n = 4 \ (\diamond)$. In Fig. 2(b), the global cascade can occur only for a region with an intermediate range of $z_p$. When $z_p$ is less than $z_p^{(1)}$, it is the percolation threshold of random graphs [48], there is no global cascade in the limit $\rho_0 \to 0$ because the components cannot span a finite fraction of a network in the thermodynamic limit. Therefore, the effect of negative links cannot be significant when $z_p < z_p^{(1)}$. In addition, as the positive links become too dense meaning that $z_p$ is larger than the second threshold $z_p^{(2)}$, i.e., $z_p^{(2)} \approx 4.3$ for $z_n = 0$, nodes that exceed their threshold become too rare, so that the global cascade can hardly occur [8]. As a result, the size $\rho$ of global cascades drops abruptly to zero regardless of $z_n$ for large $z_p > z_p^{(2)}$. When the value of average degree $z_p$ is between $z_p^{(1)} < z_p < z_p^{(2)}$, the global cascade can occur. In this regime, negative links play an important role in preventing the activation of nodes connected to active nodes, so that the size of global cascades $\rho$ noticeably decreases as $z_n$ increases. Note that the second threshold $z_p^{(2)}$ is $\theta$-dependent.

Figure 2(c) shows the decrease of $\rho$ as $z_n$ increases for various $z_p = 1 \ (\circ)$, $z_p = 2 \ (\triangle)$, $z_p = 3 \ (\vartriangle)$, $z_p = 4 \ (\vartriangle)$, $z_p = 5 \ (\diamond)$. We again confirm the monotonic decrease of $\rho$ with an increasing number of negative links. To quantitatively measure this effect, we measured the difference $\Delta \rho$ of global cascades as a function of $z_p$ for various $z_n = 0, 1, 2, 3, 4$. (f) The duration $T$ of global cascades as a function of $z_n$ for various $z_p = 1, 2, 3, 4, 5$.
When the cascades of activation start from seed nodes, the cascade size increases rapidly in the early time and finally reaches $\rho$ in the steady state. The duration of the global cascade, $T$, is the time from when the cascading first starts to when it falls into a steady state. We computed the duration $T$ for different sets of $z_n$ and $z_p$. We found that the peaks of the duration $T$ are located near the transition points where global cascades emerge or disappear. A similar phenomenon which is related to the critical slowing down was observed in the previous studies of threshold models [15, 49].

3.2. Theory: mean-field approximation

We next develop a mean-field-type analytic approximation in order to estimate $\rho$ for given degree distribution and seed fraction. When there is no negative link, the global cascade size can be analytically calculated by using the generating function method [3, 11, 50]. For signed networks, however, the effect of negative links must be included. It is important to note that the aforementioned ‘blocking’ effect is applied only in one direction and the directionality is determined dynamically according to the order in which the state of nodes is determined. Such a direction cannot be exactly determined based on structural information alone. Therefore we apply the following simple approximation. In random networks, a zero-th order estimation for the probability that one node in a pair connected by a negative link will be determined for its state first is uniform as $1/n$. Then, we can assume that there are $k_n/2$ negative links that effectively act on a node having $k_n$ number of negative neighbors.

Using the above approximation, we derived the self-consistency equations for the probability $q_p$ ($q_n$) that a node reached along a randomly chosen positive (negative) link is active. The probabilities $q_p$ and $q_n$ satisfy

$$ q_p = \rho_0 + (1 - \theta - \rho_0 z_n) \sum_{k_p,k_n} \frac{k_p P(k_p,k_n)}{\langle k_p \rangle} (1) $$
$$ \times \sum_{r=0}^{k_p-1} \binom{k_p-1}{r} q_p^r (1 - q_p)^{k_p-1-r} F \left( \frac{r}{k_p} \right) (1 - q_n)^{k_n/2} $$
$$ q_n = \rho_0 + (1 - \theta - \rho_0 z_n) \sum_{k_p,k_n} \frac{k_n P(k_p,k_n)}{\langle k_n \rangle} (2) $$
$$ \times \sum_{r=0}^{k_p} \binom{k_p}{r} q_p^r (1 - q_p)^{k_p-r} F \left( \frac{r}{k_p} \right) (1 - q_n)^{(k_n-1)/2} $$

where $P(k_p,k_n)$ is the degree distribution of the network and $F(r/k_p)$ is the activation function. $F(r/k_p)$ is 1 for $r/k_p > \theta$ where $r$ is the number of active positive neighbors and $F(r) = 0$ otherwise. The term $\rho_0 z_n$ approximately corresponds to the probability that the fraction of nodes that are blocked by seed nodes, for a small $\rho_0$. Once obtaining $q_p$ and $q_n$ by simultaneously solving Eqs. (1) and (2), we can estimate the size $\rho$ of global cascades as

$$ \rho = \rho_0 + (1 - \rho_0 - \rho_0 z_n) \sum_{k_p,k_n} P(k_p,k_n) $$
$$ \times \sum_{r=0}^{k_p} \binom{k_p}{r} q_p^r (1 - q_p)^{k_p-r} F \left( \frac{r}{k_p} \right) (1 - q_n)^{k_n/2} $$

We found that the size of global cascades predicted by the theory shows good agreement with the numerical results as shown in Fig. 3. Figure 3(a) shows the size $\rho$ of global cascades as a function of $z_p$ with various $z_n = 0$ (○), $z_n = 1$ (△), $z_n = 2$ (▲), $z_n = 3$ (■), $z_n = 4$ (□). The theory implies that the condition for global cascades becomes [3, 11]

$$ \sum_{k_p=\infty}^{k_p} \frac{k_p (k_p - 1)}{z_p} P(k_p) F \left( \frac{1}{k_p} \right) > 1 - \rho_0 - \rho_0 z_n $$

where $F(1/k)$ is 1 if $1/k > \theta$ and 0 otherwise. Therefore the theory predicts that the first transition $z_p^{(1)}$ from no cascading to global cascades is located near $z_n^{(1)} = 1/(1 - \rho_0 - \rho_0 z_n)$, which is less sensitive to $z_n$ for a small $\rho_0$. In
addition, the second threshold $z_n^{(2)}$ from the global cascade to no cascade phases decreases with increasing $z_n$, because the number of nodes that are initially inactive increases with increasing $z_n$ by the term $\rho_0 z_n$. Figure 3(b) shows the behaviour of $\rho$ as a function of the threshold $\theta$ with $z_p = 3$ and various $z_n = 0 (~\circ~)$, $z_n = 1 (~\bigtriangleup~)$, $z_n = 2 (~\bigtriangledown~)$, $z_n = 3 (~\Diamond~)$, $z_n = 4 (~\bigcirc~)$. The theory successfully predicts the cascade condition for the threshold from Eq. (3) and the suppression of the global cascades with increasing $z_n$.

3.3. Heterogeneity in Cascades of Activations

Negative links in signed networks not only suppress global cascades but also render the patterns of activations more heterogeneous. When there is no negative link ($z_n = 0$), the final configuration of activations is uniquely defined for a given set of seeds on a given network, meaning that nodes are either always activated or never activated in different trials of cascading dynamics. When $z_n > 0$, however, the final configuration resulting from the cascades is not uniquely determined even though the dynamics starts from the same network structure as exemplified in Fig. 3. In other words, each node in the signed networks responds differently to the dynamics depending on the sequence of the cascades of activations. To quantify such heterogeneity, we define $m_i$ as the probability that node $i$ is active over different realizations on the same network configuration. We measured the probability distribution of $m_i$ for a given network structure. When $z_n = 0$, $P(m_i)$ is simply given by the two peaks at $m_i = 0$ and $m_i = 1$ [see Fig. 3(a)]. However, when $z_n > 0$ most nodes are active only occasionally meaning that $0 < m_i < 1$. As shown in Fig. 3(b), sporadically active nodes are widely distributed in $m_i$, accompanied by the two peaks at $m_i = 0$ and 1. In addition, the latter peak at $m_i = 1$ is substantially weaker than that of $z_n = 0$ case.

The response of nodes to cascading dynamics can fall into three classes: always active ($m_i = 1$), never active ($m_i = 0$), and occasionally active ($0 < m_i < 1$). We measured the fraction of nodes that are always active $h_1$, never active $h_0$, and occasionally active $h_m$ as a function of $z_n$, for a fixed signed ER network of $z_p = 3$ with an ensemble of random initial seed sets [Fig. 3(c)]. The sum of the three fractions satisfies the sum rule $h_0 + h_m + h_1 = 1$. With increasing $z_n$, the fraction $h_1$ of the always active nodes decreases rapidly and becomes almost negligible when $z_n \gtrsim 2$. When the density of negative links increases, most nodes in a network belong to occasionally active state $h_m$ because of the negative links. The fraction $h_0$ of nodes that are never activated maintains a low fraction for all range of $z_n$, but increases steadily as $z_n$ increases. The observation that the $h_m$-state nodes constitute a dominant fraction indicates the prevalence of system-wide heterogeneity of dynamics driven by the negative links.

Finally, in order to assess quantitatively the heterogeneity for different realizations of activation patterns, we computed the average overlap between different ‘samples’ (activation configurations) defined by [27, 54]

$$Q = \left\langle \sum_{i=1}^{N} s_i^\alpha s_i^\beta \right\rangle_{\alpha\beta},$$

where $\alpha$ and $\beta$ represent the indices of samples. Here a sample refers to the particular outcome of signed cascade dynamics on the same network structure but with random initial seed sets. $\langle \cdot \rangle_{\alpha\beta}$ represents the sample average. $s_i^\alpha$ represents the state of node $i$ at the steady state in sample $\alpha$, i.e., $s_i^\alpha = 1$ and 0 respectively represents that node $i$ is active and inactive in sample $\alpha$. (Note that in the spin-glass literature [27, 54] the same formula for the overlap is used but with the Ising spin variables $s_i = \pm 1$.)

The overlap would become maximized as $Q = \rho$ when the activation patterns for each node are completely identical for different realizations. On the other hand, if the activation patterns are completely uncorrelated for different realizations, the overlap $Q$ would become $Q = \rho^2$. As shown in Fig. 5, $Q$ is maximized at $z_n = 0$ exhibiting $Q \approx \rho$ and decreases steadily as $z_n$ increases. When $z_n > 0$, $Q$ is less than $\rho$ but greater than $\rho^2$, meaning that the configurations are correlated, but partially. Moreover, as $z_n$ increases, the different activation patterns become more and more uncorrelated. This can be inferred as the result that $Q$ approaches $\rho^2$ with increasing $z_n$. In addition, we can see in the inset of Fig. 5 that the overlap $Q$ remains a constant value regardless of the increase of sys-
Figure 5: The overlap $Q$ averaged over $10^3$ different realizations of activation patterns for ER networks with $N = 10^5$, $p_0 = 10^{-3}$, and $z_p = 3$. For comparison, $\rho$ and $\rho^2$ are also shown. Inset shows the overlap $Q$ with $(z_p, z_n) = (3, 3)$ for different network size $N$ from $2 \times 10^4$ to $10^5$.

The heterogeneity of the activation patterns suggests that many different activation patterns can come from the same network structure. This is reminiscent of infinitely many ground states in disordered spin systems [27]. It also implies that the final fate of each node cannot be fully predicted from the structure of networks.

3.4. On signed scale-free networks

In order to check the effect of negative links on signed heterogeneous networks, we study the signed cascade model on signed scale-free (SF) networks. For building signed SF networks, we use static SF network model [51]. Initially, there are $N$ isolated nodes, i.e., $N = 10^5$ in our study. By following rules, we constructed signed static SF networks. Each node $i$ has its inherent weight $\omega_i$ given by $\omega_i = i^{-\mu}/\sum_{j=1}^{N} j^{-\mu}$, where $\mu$ is a constant, $0 < \mu < 1$, which determines the degree exponent. We choose a pair of nodes, say $i$ and $j$ independently following the probability $\omega_i$ and $\omega_j$, and connect them unless they are already connected, for both positive and negative links. We repeat the procedures until the mean degree reaches $z_p$ for positive links and $z_n$ for negative links. The degree distribution of resulting networks is asymptotically scale-free with the tail decaying as $\sim (k_p + k_n)^{-\gamma}$ with the degree exponent $\gamma = (\mu + 1)/\mu$. Note that the degree distribution for positive or negative links respectively also has the same power-law tail and the positive degree $k_p$ and negative degree $k_n$ of a given node are correlated in this signed static SF networks.

We perform numerical simulations of signed cascade model on the signed static SF networks with the degree exponent $\gamma = 2.5$. Figure 6(a) shows the size $\rho$ of global cascades as a function of $z_p$ with various $z_n = 0$ (○), $z_n = 1$ (∆), $z_n = 2$ (▲), $z_n = 3$ (∇), $z_n = 4$ (○) with $\theta = 0.15$. Global cascade can occur under the condition $z_p^{(1)} < z_p < z_p^{(2)}$ as the signed ER networks. In this regime, we confirm the suppression of the global cascades with increasing $z_n$ in signed scale-free networks. Figure 6(b) shows the behaviour of $\rho$ as a function of threshold $\theta$ with $z_p = 3$ and various $z_n = 0$ (○), $z_n = 1$ (∆), $z_n = 2$ (▲), $z_n = 3$ (∇), $z_n = 4$ (○). We found that $\rho$ abruptly decreases at the critical threshold. Comparing with the cases on signed ER networks [Fig. 3(b)], the global cascade disappears at a lower value of $\theta$. To summarize, we observe qualitatively similar features on signed SF networks to those on signed ER networks: we confirm that the negative links suppress the size $\rho$ of global cascade in scale-free networks while the critical threshold is insensitive to $z_n$.

3.5. On signed real-world networks

We test the effect of negative links on signed “real-world” networks from the data of online social media. By using the real-world signed network data, we can not only study the model on more realistic setting, but also address the effect of higher-order structural features such as structural balance and degree correlations. We constructed signed networks from the Epinions and Slashdot datasets obtained from Stanford Network Analysis Project (SNAP) [52, 53]. Epinions is an online review web site where consumers can have signed relationships one another.
reduce the global cascade size in empirical networks by comparing \( \rho \) when the negative links are removed.

Second, we explore the effect of the structural balance using the real-world signed networks. In order to generate the signed network structures with different levels of structural balance, we take the empirical signed network data and perform the “sign-shuffling” while preserving the overall network connections intact to the original data. To be specific, we select a pair of links (irrespective of signs) in the network uniformly at random and exchange their signs. The insets of Figs. 7(c,d) show that the fraction \( B \) of balanced triads, a measure of the structural balance level, decreases monotonically with the number of edge-shufflings per link \( \sigma \). We measure the size \( \rho \) of global cascade as we perform the edge-shuffling. Figures 7(c,d) compare the case of \( \sigma = 0 \) (original unshuffled network), \( \sigma = 0.1 \), and \( \sigma = 0.9 \) for the real-world signed networks, showing that the size of global cascade decreases with \( \sigma \).

Finally, we measure the size \( \rho \) of the global cascade in “randomly-coupled” signed surrogates of empirical networks to evaluate the impact of the correlation between positive-link and negative-link degrees of a given node. We created the randomized surrogates that removes the degree correlation between the positive and negative links in the original network. Figures 7(e,f) shows \( \rho \) of the original network and randomized networks as a function of \( \theta \) for Epinions and Slashdot networks. The size of global cascade is greater in randomized networks without degree correlation than in original network as shown in Fig. 7(e,f).

4. Conclusion

To conclude, in this paper we proposed and studied a threshold cascade model on signed networks with both positive and negative interactions. In our model, we impose the role of negative links by the rule that a node’s activation is completely blocked if there is at least one active neighbor connected by a negative link. We found that the negative links not only suppress the global cascades but also produce the heterogeneity in the activation patterns. As the number of negative links increases, the activation patterns of threshold model become increasingly uncorrelated. The mean-field-type approximation theory developed for the effects of negative links successfully accounts for the observed results. We also address the impact of network structural balance and degree correlation between positive and negative links to the global cascade size on a signed network using real-world networks and their randomized surrogates. Our results imply that the inhibitory interactions which are widespread in many real-world systems may play not only to suppress the activations but also to render variability in dynamical patterns.

The model studied in this study is a toy model illustrating the essential functional role of negative links in threshold cascade dynamics, offering many rooms for extensions over its limitations. For example, from the model perspective, a rather relaxed or probabilistic form of suppressive...
function of negative links can be considered; a mixture of the currently-implemented multiplicative suppression and the additive suppression as considered in Ref. [20] is also worth the investigation. From signed-network structural perspective, more comprehensive study on the effects of higher-order organization of signed interactions such as social balance at the triad level and beyond needs systematic examination, which is by no means a trivial problem. From theoretical perspective, further understanding of similarities to and differences from spin-glass phenomenology might be of interest. As a final outlook, we anticipate the functional relevance and dynamic importance of negative links showcased in this paper to apply not only to the simple social dynamic processes considered in this study but also to many other different dynamical processes on signed networks, including brain, offering an ample arena for future works.

5. Acknowledgments

This work was supported in part by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MSIT) (No. 2020R1A2C2083669 (K-IG) and No. 2020R1I1A3068803 (BM)). SL was supported by Basic Science Research Program through the Korea government (MSIT) (No. 2020R1A2C2003669 (BM)). K-IG and No. 2061R1A6A3A11932833. K-IG would also like to thank the APCTP for its hospitality during the completion of this work.

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