Remarks on Grothendieck’s standard conjectures

A. Beilinson
The University of Chicago

We show that Grothendieck’s standard conjectures (over a field of characteristic zero) follow from either of two other motivic conjectures, namely, that of existence of the motivic t-structure and (a weak version of) Suslin’s Lawson homology conjecture. I am grateful to H. Esnault, E. Friedlander, and B. Kahn for a stimulating exchange of letters.

§1. The motivic t-structure conjecture yields standard conjectures

1.1. Below $k$ is our base field, $\text{Var}_k$ is the category of smooth varieties over $k$, and $DM_k$ is the triangulated category of geometric motives with $\mathbb{Q}$-coefficients over $k$, so we have the motive functor $M: \text{Var}_k \to DM_k$. Recall that $DM_k$ is an idempotently complete triangulated rigid tensor $\mathbb{Q}$-category, and $M$ yields a fully faithful embedding of tensor $\mathbb{Q}$-categories $\text{CHM}_k \hookrightarrow DM_k$, were $\text{CHM}_k$ is the category of Chow motives with $\mathbb{Q}$-coefficients over $k$.

Denote by $\text{Vec}_\mathbb{Q}, \text{Vec}_\mathbb{Q}_\ell$ the categories of finite-dimensional $\mathbb{Q}$- and $\mathbb{Q}_\ell$-vector spaces. For $\ell$ prime to the characteristic of $k$ one has the $\ell$-adic realization functors $r_{Q_\ell}: DM_k \to D^b(\text{Vec}_{Q_\ell})$. For $k$ of characteristic 0, each embedding $\iota: k \to \mathbb{C}$ yields the Betti realization functor $r_{\iota}: DM_k \to D^b(\text{Vec}_\mathbb{Q})$. These are tensor triangulated functors; there are canonical identifications $r_{\iota} \otimes \mathbb{Q}_\ell \sim r_{Q_\ell}$.

Let $r$ be one of the realization functors. For a variety $X$, the vector spaces $H^a_r(X) := H^a r(M(X))$ are appropriate homology of $X$; they vanish unless $-2d_X \leq a \leq 0$ (here $d_X := \dim X$).

1.2. Let $\mu$ be a t-structure on $DM_k$. Denote by $DM^\leq 0_k, DM^\geq 0_k$ the positive and negative parts of $DM_k$, by $M_k$ its heart, and by $^\mu H: DM_k \to M_k$ the cohomology functor.

**Definition.** $\mu$ is said to be motivic if it is non-degenerate and compatible with $\otimes$ and $r$ (i.e., $\otimes$ and $r$ are t-exact).

**Conjecture** (cf. [A] Ch. 21). A motivic t-structure exists.

Assuming the conjecture, let us deduce some of its corollaries.

1.3. For a motivic $\mu$, let $^\mu r: M_k \to \text{Vec}_{Q_\ell}$ be the restriction of $r$ to $M_k$.

**Observation.** $M_k$ is a Tannakian $\mathbb{Q}$-category, and $^\mu r$ is a fiber functor.

**Proof.** $M_k$ is an abelian tensor $\mathbb{Q}$-category; the endomorphism ring of its unit object $Q(0)$ is $\mathbb{Q}$. It is rigid (for t-exactness of $\otimes$ implies that the duality is t-exact). Since $^\mu r$ is an exact tensor functor, we are done (see e.g. [Del2] 2.8).
Corollary. $\mu^r$ is faithful (hence conservative), and every object of $\mathcal{M}_k$ has finite length. The functor $r$ is conservative.

Proof. The first assertion is a part of the Tannakian story. It implies the second one, for our $t$-structure is non-degenerate and $\mu^r\mu^H = H^r$ (since $r$ is $t$-exact). □

Corollary. (i) Any object $P$ of $DM_k$ has only finitely many non-zero cohomology objects $\mu^H a P$ (i.e., $\mu$ is bounded).
(ii) $P$ lies in $DM_k^{\leq 0}$, resp. $DM_k^{>0}$, if and only if the complex $r(P)$ has trivial positive, resp. negative, cohomology.

Proof. (i) Since $r$ is $t$-exact, one has $H^r(P) = \mu^r\mu^H(P)$. The first assertion follows then from the conservativity of $\mu^r$. (ii) Since $\mu$ is non-degenerate, $P$ lies in $DM_k^{\leq 0}$, resp. $DM_k^{>0}$, if and only if it has trivial positive, resp. negative, cohomology $\mu^H P$. We are done by the conservativity of $\mu^r$. □

Remarks. (a) By (ii) above, a motivic $t$-structure is unique (for given $r$). (b) By (ii) above, the Tate motive $Q(\ell)$ follows then from the conservativity of $\mu^r$. Let $\ell$ be the Tate twist is $t$-exact.

1.4. Let $X$ be a smooth projective variety. Then $CH^n(X)_Q = \text{Hom}(M(X), Q(n)[2n])$, and the intersection product on $CH^*(X)_Q$ comes from the canonical coalgebra structure on $M(X)$ (the coproduct is the diagonal map $M(X) \to M(X \times X) = M(X) \otimes M(X)$) and the evident algebra structure on $Q(\cdot)[2\cdot]$. Thus the Chow ring acts on $M(X)(\cdot)[2\cdot]$; explicitly, the multiplication by $c \in CH(n)X_Q$ is the composition $\cap c$ of $M(X) \to M(X) \otimes M(X) \overset{id_{M(X)} \otimes c}{\longrightarrow} M(X)(n)[2n]$.

Thus, let $L \in CH^1(X)_Q = \text{Hom}(M(X), Q(1)[2])$ be the class of hyperplane section, so we have the morphisms $\cap L_i : M(X) \to M(X)(i)[2i], i \geq 0$.

Proposition. (i) (hard Lefschetz) For any $i \geq 0$ the morphism $\cap L_i : \mu^{-i-dX} M(X) \to \mu^{-i-dX} M(X)(i)$ is an isomorphism.
(ii) The object $M(X)$ is isomorphic to the direct sum of its cohomology objects: $M(X) \cong \oplus \mu^H a M(X)[-a]$.
(iii) (primitive decomposition) There is a unique collection of subobjects $\mu^p a(X) \subset \mu^H a M(X)$, $-2dX \leq a \leq -dX$, such that for every $b$ the maps $\cap L_i$ provide isomorphisms $\mu^b \oplus \mu^{b-2i}(-i) \overset{\sim}{\longrightarrow} \mu^b M(X)$.

Proof. Applying $\mu^r$ to the morphism in (i), we get an isomorphism from the usual hard Lefschetz theorem for $H^r$, and (i) follows since $\mu^r$ is conservative; (i) implies (ii) by [Del3]; (iii) follows from (i) by a usual linear algebra argument. □

Remarks. (a) $\mu^r$ sends the decomposition of (iii) to a similar decomposition of $\mu^r \mu^H M(X) = H^r(M(X))$, which is the usual primitive decomposition of the homology $H^r(X)$. (b) The decomposition in (ii) is usually non unique.

Corollary. The standard conjectures of Lefschetz and Künneth type\(^3\) are true.

Proof. Recall that $\text{Hom}(M(X), M(X)/(a)[2a]) = CH^{dX+a}(X \times X)_Q$, and for any $\lambda : M(X) \to M(X)/(a)[2a]$ the map $\mu^r(\lambda)$ is the action on $H^r(X)$ of the

\(^3\)See e.g. [A] Ch. 5.
corresponding algebraic correspondence. The K"unneth type assertion follows if we take for \( \lambda \) the projector to any of the components of the decomposition from (ii). To deduce the Lefschetz type assertion, consider \( \lambda : M(X) \to M(X)(-i)[-2i] \) whose only non-zero component with respect to the decomposition in (ii) is the inverse to the isomorphism from (i).

\[ \square \]

1.5. From now on we assume that \( k \) has characteristic 0. Then (see [A] Ch. 5) the standard conjecture of Lefschetz type implies all the standard conjectures.

**Proposition.** For \( X \) as in 1.4, the objects \( \mu H^a M(X) \) of \( \mathcal{M}_k \) are semi-simple.

**Proof (cf. [J]).** Due to the primitive decomposition (see 1.4(iii)) it suffices to check that the objects \( \mu P^a(X) \), \(-2d_X \leq a \leq -d_X\), are semi-simple, which means that any subobject \( Q \subset \mu P^a(X) \) admits a complement \( Q^\perp \).

Recall that the dual \( M(X)^* \) identifies naturally with \( M(X)(-d_X)[-2d_X] \). The corresponding canonical pairing \( M(X) \otimes M(X) \to \mathbb{Q}(d_X)[2d_X] \) yields a pairing \( (\cdot, \cdot) : \mu H^a M(X) \otimes \mu H^{a-2d_X} M(X) \to \mathbb{Q}(d_X) \). Consider the pairing \( (\cdot, \cdot)_L := (\cdot, \cap L^{-a-d_X}) : \mu P^a(X) \otimes \mu P^a(X) \to \mathbb{Q}(2a) \). Let \( Q^\perp \subset \mu P^a(X) \) be the orthogonal complement to \( Q \) for this pairing.

It remains to show that \( Q \oplus Q^\perp \to \mu P^a(X) \). It suffices to check this after applying a fiber functor \( \mu r_i \). Then \( (\cdot, \cdot)_L \) becomes the usual polarization pairing on primitive cycles, and \( \mu r_i(Q^\perp) \) is the orthogonal complement to \( \mu r_i(Q) \) with respect to the polarization. Since \( (\cdot, \cdot)_L \) is non-degenerate on \( \mu r_i(Q) \) (as on every Hodge substructure) by the Hodge index theorem, we are done.

\[ \square \]

**Corollary.** Each irreducible object of \( \mathcal{M}_k \) can be realized as a Tate twist of a direct summand of some \( \mu H^{-d_X} M(X) \) where \( X \) is projective and smooth.

**Proof.** Every irreducible object can be realized as a subquotient of a Tate twist of some \( \mu H^a M(Y) \), \( Y \in \text{Var}_k \). Writing \( \mu H^a M(Y) \) in terms of cohomology of smooth projective varieties [Del1], we see that it can be realized as a subquotient of a Tate twist of some \( \mu H^a M(Y) \) with \( Y \) projective and smooth. By Lefschetz, one can realize it as a subquotient of a Tate twist of \( \mu H^{-d_X} M(X) \) with \( X \) projective and smooth. We are done by the proposition.

\[ \square \]

1.6. The motivic t-structure can be characterized in purely geometric terms (without the reference to \( r \)). Namely, consider a filtration \( DM_{k(0)} \subset DM_{k(1)} \subset \ldots \) on \( DM_k \) where \( DM_{k(n)} \) is the thick subcategory of \( DM_k \) generated by all motives of type \( M(X)(a) \) with \( d_X \leq n \).

**Proposition.** The motivic t-structure is a unique t-structure compatible with the filtration \( DM_{k(n)} \) and such that the heart of the induced t-structure on any successive quotient \( DM_{k(n)}/DM_{k(n-1)} \) contains all objects \( M(X)[-n] \), \( X \) is projective smooth of dimension \( n \).

**Proof.** By 1.4(ii), 1.5, and the argument from [Del1], \( DM_{k(n)} \) is generated by all irreducible objects of \( \mathcal{M}_k \) that can be realized as a Tate twist of a direct summand of some \( \mu H^{-d_X} M(X) \) where \( X \) is projective and smooth of dimension \( \leq n \). This shows that \( \mu \) induces a t-structure on each \( DM_{k(n)} \), i.e., \( \mu \) is compatible with the filtration \( DM_{k(n)} \). In such a situation, the t-structure on \( DM_k = \bigcup DM_{k(n)} \) is uniquely determined by the t-structures induced on the successive quotients \( DM_{k(n)}/DM_{k(n-1)} \).
The heart of the t-structure on $DM_{k(n)}$ is $\mathcal{M}_k \cap DM_{k(n)}$, which is the Serre subcategory of $\mathcal{M}_k$ generated by irreducibles occurring as direct summands of some $\mu H^{−d_X} M(X)(a)$ where $X$ is projective and smooth of dimension $\leq n$. The irreducibles in the heart of $DM_{k(n)}/DM_{k(n−1)}$ are the images of those of them with $d_X = n$. For such an $X$ the image of $M(X)(a)[−n]$ in $DM_{k(n)}/DM_{k(n−1)}$ equals the image of $\mu H^{−n} M(X)(a)$ (since $\mu H^{−n} M(X)(a) \in DM_{k(n−1)}$). Since the t-structure on $DM_{k(n)}/DM_{k(n−1)}$ is bounded and its heart is Artinian, it is uniquely defined by the datum of irreducible objects in its heart.\footnote{Precisely, $(DM_{k(n)}/DM_{k(n−1)})^{≤0}$ is the left orthogonal complement to the set of objects $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell < −n$; $(DM_{k(n)}/DM_{k(n−1)})^{≥0}$ is the right orthogonal complement to the set of $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell > −n$.} q.e.d.

1.7. Proposition. (i) Objects of $\mathcal{M}_k$ carry a natural finite increasing filtration $W$, such that each morphism is strictly compatible with $W$. It is characterized by the next property: an irreducible object $P$ has weight $m$, i.e., has property $W_m P = P$, $W_{m−1} P = 0$, if and only if it occurs in some $\mu H^{i} M(X)(a)$ where $X$ is smooth projective and $m = i − 2a$.

Proof. (cf. [Del4] 3.8). It suffices to check that if irreducible objects $P$, $Q$ occur in, respectively, $\mu H^{i} M(X)(a)$ and $\mu H^{j} M(Y)(b)$, $X$ and $Y$ are smooth projective, then $\text{Hom}(P, Q[\ell]) = 0$ for $\ell > (i − 2a) − (j − 2b)$. By Lefschetz, we can assume that $i = −d_X$, $j = −d_Y$. By 1.5 and 1.4(ii), $\text{Hom}(P, Q[\ell])$ is a subquotient of $\text{Hom}(M(X)(a), M(Y)(b)[\ell + d_X − d_Y]) = \text{Hom}(M(X \times Y), Q(b−a+d_Y)[\ell+d_X+d_Y])$, which is 0 for $\ell > (2b + d_Y) − (2a + d_X)$ due to the next lemma:

Lemma. If $X$ is any smooth variety, then $\text{Hom}(M(X), Q(n)[\ell]) = 0$ for $\ell > n + \text{min\{d}_X, n\}$.\footnote{Precisely, $(DM_{k(n)}/DM_{k(n−1)})^{≤0}$ is the left orthogonal complement to the set of objects $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell < −n$; $(DM_{k(n)}/DM_{k(n−1)})^{≥0}$ is the right orthogonal complement to the set of $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell > −n$.}

Proof. $R\text{Hom}(M(X), Q(n))$ is Bloch’s complex of relative cycles (see Lecture 19 from [MVW]). Thus $\text{Hom}(M(X), Q(n)[\ell])$ is a subquotient of the group of codimension $n$ cycles on $X \times \mathbb{A}^{2n−\ell}$, which is 0 for $\ell > d_X + n$ or $\ell > 2n$.\footnote{Precisely, $(DM_{k(n)}/DM_{k(n−1)})^{≤0}$ is the left orthogonal complement to the set of objects $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell < −n$; $(DM_{k(n)}/DM_{k(n−1)})^{≥0}$ is the right orthogonal complement to the set of $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell > −n$.}

1.8. Suppose an irreducible $P \in \mathcal{M}_k$ is effective, i.e., occurs in some $\mu H^{i} M(Y)$. By the argument from [Del1], it occurs then in $\mu H^{i} M(X)$ with $X$ smooth and projective of dimension $\leq d_Y$. The level of $P$ is the smallest dimension of such an $Y$. For any effective $P \in \mathcal{M}_k$ its level is the maximal level of its irreducible subquotients.

Proposition. If $P$, $Q$ are effective of level $\leq \ell$, then $\text{Hom}(P, Q[\ell]) = 0$.

Proof. It suffice to check this when $P$, $Q$ are irreducible. Then $P$ occurs in some $\mu H^{−d_X} M(X)(a)$ where $X$ is smooth projective with $d_X \leq \ell$ and $0 \leq a \leq \ell − d_X$; same for $Q$. As in the proof in 7, one can realize $\text{Hom}(P, Q[\ell])$ as a subquotient of $\text{Hom}(M(X \times Y), Q(b−a+d_Y)[\ell+d_X+d_Y])$. Now use the lemma in 1.7.\footnote{Precisely, $(DM_{k(n)}/DM_{k(n−1)})^{≤0}$ is the left orthogonal complement to the set of objects $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell < −n$; $(DM_{k(n)}/DM_{k(n−1)})^{≥0}$ is the right orthogonal complement to the set of $M(X)(a)[\ell]$, $X$ is projective smooth of dimension $n$, $\ell > −n$.}

§2. Suslin’s Lawson homology conjecture yields standard conjectures

2.1. For a complex projective variety $X$ we have its Lawson homology groups $L_r H^{2r+1}(X, \mathbb{Z}) := \pi_r(C_r(X)^+)$; here $C_r(X)$ is the topological Chow monoid of effective $r$-cycles on $X$, and $C_r(X)^+$ is its group completion. They form the “homology” part of a Bloch-Ogus style cohomology theory for complex algebraic
varieties, see [Fr]. There is another cohomology theory with cohomology groups $H^r_C(X, \mathbb{Z}(n)) := H^r(X_{\text{Zar}}, \tau_{=n} R\pi_* \mathbb{Z}(n))$; here $\pi : X_{\text{cl}} \to X_{\text{Zar}}$ is the map from the classical topology of $X$ to the Zariski one, $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}$ is the constant sheaf on $X_{\text{cl}}, \tau_{\leq n}$ is the truncation. There is a natural morphism from the former cohomology theory to the latter one, and the Suslin conjecture asserts that this morphism is an isomorphism. More concretely, this means that for smooth projective $X$ the canonical map $L_r H_a(X, \mathbb{Z}) \to H_a(X, \mathbb{Z}(-r))$ is an isomorphism for $a \geq \dim X + r$.

Remark. The Suslin conjecture with finite coefficients ($\mathbb{Z}$ replaced by $\mathbb{Z}/\ell$) is known to be true: indeed, by [SV], $L_r H_a(X, \mathbb{Z}/\ell)$ equals the motivic homology with coefficients in $\mathbb{Z}/\ell(-r)$, so the assertion comes from the Milnor-Bloch-Kato conjecture established by Voevodsky, Rost,... Therefore the Suslin conjecture with $\mathbb{Z}$-coefficients amounts to the assertion that the groups $L_r H_a(X, \mathbb{Z})$ for $a \geq \dim X + r$ are finitely generated. And the Suslin conjecture with $\mathbb{Z}$-coefficients is equivalent to that with $\mathbb{Q}$-coefficients.

2.2. From now on all the (co)homology have $\mathbb{Q}$-coefficients, which are omitted, as well as the Tate twist, in the notation.

Proposition. The next conjectures are equivalent:

(i) For any smooth projective $X$ the maps $L_r H_a(X) \to H_a(X)$ are surjective for $a \geq \dim X + r$.

(ii) For $X$ as in (i) one can find a finite correspondence $f : X \to Y$ with $Y$ projective smooth and $\dim Y = \dim X - 1$ such that $f^* : H^i(Y) \to H^i(X)$ is surjective for $i < \dim X$.

(iii) For $X$ as in (i) and any $j \geq 0$ one can find a finite correspondence $f_j : X \to Y_j$ with $Y_j$ projective smooth of dimension $j$ such that $f_j^* : H^j(Y_j) \to H^j(X)$ is surjective.

(iv) The standard conjectures (for varieties over $\mathbb{C}$).\(^6\)

Proof. (iii)⇒(ii): Take $Y = \bigsqcup_{j < \dim X} Y_j$, $f = \sum_{j < \dim X} f_j$. (ii)⇒(iii): Take $(Y_{\dim X}, f_{\dim X}) = (X, \text{id}_X)$. For $j < \dim X$, find $(Y_j, f_j)$ using downward induction by $j$: namely, $(Y_j, f_j) = (Y, ff_{j+1})$ where $(Y, f)$ comes from (ii) for $X$ replaced by $Y_{j+1}$. For $j > \dim X$ the assertion is evident (say, take $Y_j = X \times \mathbb{P}^{j-\dim X}$).

(i)⇒(iii): We can assume that $r = \dim X - j \geq 1$. Recall that $C_r(X)$ is disjoint union of projective varieties, and we have a universal family of $r$-cycles on $X$ parametrized by $C_r(X)$. Viewed as a correspondence between $C_r(X)$ and $X$, it yields a map $H_a(C_r(X)) \to H_{a+2r}(X)$, and (i) says that this map is surjective for $a \geq j$. Replace $C_r(X)$ by its sufficiently large component so that surjectivity still holds. Let $Z$ be a resolution of singularities of the latter; the map $H_a(Z) \to H_{a+2r}(X)$ for $a \geq j$ is still surjective by a usual mixed Hodge theory argument. Let $Y$ be a generic iterated hyperplane section of dimension $j$, so $H_j(Y) \to H_j(Z)$ is surjective by weak Lefschetz, hence $H_j(Y) \to H_{j+2r}(X)$, or, replacing the homology by cohomology, $H^j(Y) \to H^j(X)$. By construction, this map is the action of a correspondence given by a cycle of dimension $\dim X$ on $X \times Y$. By [FV] 7.1, it can be replaced by a finite correspondence, and we are done.

(iv)⇒(i): Consider the “inverse Lefschetz” endomorphism $\Lambda : H^i(X) \to H^{i-2}(X)$.

\(^5\)The implication (iv) ⇒ (ii) was observed independently by S. Bloch and B. Kahn.

\(^6\)Hence over any field of characteristic 0.
The map $\Lambda^r : H^{i+2r}(X) \to H^i(X)$ is surjective for $i \leq \dim X - r$, i.e., the corresponding map on homology $H_{r-2i}(X) \to H_a(X)$ is surjective for $a \geq \dim X + r$. Realizing $\Lambda^r$ as an $X$-family of $r$-cycles on $X$ (by (iv) and [FV] 7.1), we factor the latter map through $L_i H_a(X) \to H_a(X)$, which yields (i).

2.3. It remains to prove that (iii) implies (iv). Recall that the standard conjectures reduce to the Lefschetz type conjecture.

For a smooth projective variety $X$, consider the next three conjectures $L(X), l(X), S(X)$ about the cohomology of $X$:
- $L(X)$ is the Lefschetz type standard conjecture for $X$;
- $l(X)$ is the next assertion: for every $i > 0$ one can find a correspondence on $X$ that yields an isomorphism $H^{\dim X+i}(X) \simeq H^{\dim X-i}(X)$;
- $S(X)$ is conjecture (iii) from 2.2 for our $X$.

Let $L(n)$ be the assertion that $L(X)$ is true for all $X$ of dimension $\leq n$; same for $l(n), S(n)$. We will show that $S(n)$ implies $L(n)$. This takes two steps:
(a) $S(n) & L(n-1)$ implies $l(n)$, and (b) $l(X)$ implies $L(X)$.

Proof of (a). By $S(n)$, we can find smooth projective $Y$ of dimension $n-i$ and a correspondence $f : X \to Y$ such that $f^* : H^{n-i}(Y) \to H^{n-i}(X)$ is surjective. Pick an ample line bundle on $Y$ and consider the corresponding primitive decomposition of $H^i(Y)$. By $L(n-1)$ the projectors $\pi_a$ on its components are given by algebraic correspondences. Denote by $\pi_+, \pi_-$ the sum of $\pi_a$'s such that $\pi_+ + \pi_- = 0$.

Set $f_{\pm} := \pi_{\pm} f$. Thus $f^* : H^{n-i}(Y) \to H^{n-i}(X)$ equals $f^*_+ + f^*_-$. Consider the maps $f_{\pm} : H^{n+i}(Y) \to H^{n-i}(X)$. Consider the maps $f_{\pm} : H^{n+i}(Y) \to H^{n-i}(X)$.

Lemma. For almost all non-zero rational numbers $a$ the restriction of the Poincaré bilinear form to the image of $af_+ + f_- : H^{n-i}(X) \to H^{n-i}(Y)$ is non-degenerate.

The lemma implies (a): Indeed, pick $a$ as above; set $f' := af_+ + f_-$. Then $f'^* : H^{n-i}(Y) \to H^{n-i}(X)$ is surjective (since such is $f^*$ and $a \neq 0$), hence its adjoint (with respect to the Poincaré pairings) $f'_* = af_+ + f_- : H^{n+i}(X) \to H^{n-i}(Y)$ is injective. The condition of the lemma implies then that $f'^* f'_* : H^{n+i}(X) \to H^{n-i}(X)$ is an isomorphism, q.e.d.

Proof of Lemma. Consider our cohomology groups with real coefficients. Our picture decomposes into the direct sum of $\mathbb{R}$-Hodge structure isotypical pieces. It suffices to prove the lemma for one such piece. Our Hodge structures look as $V \otimes H$, where $H$ is a fixed irreducible Hodge structure (rank 2 or rank 1) and $V$ is a real vector space (i.e., a Hodge structure of type $(0,0)$). If $H$ is a subspace of $H^{n-i}(Y)$, then the Poincaré pairing is the tensor product of a symmetric bilinear form $q$ on $V$ and a fixed polarization on $H$; if we live in $\pi_{\pm} H^{n-i}(Y)$, then $q$ is either positive or negative definite. Now the lemma follows from the next linear algebra assertion: Let $V_+, V_-$ be $\mathbb{R}$-vector spaces equipped with, respectively, positive and negative definite symmetric bilinear forms $q_+, q_-$.

\footnote{Which follows from the fact that $U$ can be decomposed into a direct sum of 1-dimensional subspaces orthogonal with respect to both bilinear forms $g^*_+(q_+)$ and $g^*_-(q_-)$.}
and \( g_\pm : U \to V_\pm \) be linear maps; then for almost all non-zero real \( a \) the form \( q_+ \oplus q_- \) is non-degenerate on the image of \( ag_+ \oplus g_- : U \to V_+ \oplus V_- \).

**Proof of (b).** Assuming \( l(X) \), we want to find for every \( i > 0 \) a correspondence \( c \) on \( X \) (here \( n := \dim X \)) whose action on \( H^n(X) \) is the inverse to Lefschetz \( H^{n+i}(X) \to H^{n-i}(X) \), all other components are 0. We do downward induction by \( i \). By the induction assumption, all the projectors \( p_j \) on \( H^{n+j}(X) \), \( |j| > i \), come from correspondences. By \( l(X) \), we can find a correspondence \( c' \) that provides an isomorphism \( H^{n+i}(X) \to H^{n-i}(X) \). Multiplying \( c' \) by the product of \( (1-p_j) \), \( j > i \), from the right and by the product of \( (1-p_j) \), \( j < -i \), from the left, we can assume that the isomorphism \( H^{n+i}(X) \to H^{n-i}(X) \) is the only non-zero component of the action of \( c' \) on \( H^n(X) \).

The composition \( A \) of \( c' \) with the \( i \)th power of Lefschetz acts as an automorphism on \( H^{n-i}(X) \), and all its other components are 0. Thus there is a polynomial \( f \) in \( \mathbb{Q}[t] \) such that \( f(A)A \) acts as identity on \( H^{n-i}(X) \). The promised \( c \) is \( f(A)c' \). □

**References**

[A] Y. André, *Une introduction aux motifs*, Panoramas et Synthèses, vol. 17, SMF, 2004.

[BV] A. Beilinson, V. Vologodsky, *A DG guide to Voevodsky’s motives*, GAFA **17** (2007), 1709–1877.

[Dég] F. Déglise, *Finite correspondences and transfers over a regular base*, Algebraic Cycles and Motives, vol. 1, London Mathematical Society Lecture Notes Series, vol. 344, Cambridge University Press, 2007, pp. 138–205.

[Del1] P. Deligne, *Théorie de Hodge II*, Publ. Math. IHES **40** (1972), 1–57.

[Del2] P. Deligne, *Catégories Tannakiennes*, The Grothendieck Festschrift, vol. 2, Progress in Mathematics, vol. 87, Birkhäuser, 1990, pp. 111–195.

[Del3] P. Deligne, *Décompositions dans la catégorie dérivée*, Motives, part 1, Proceedings of Symposia in Pure Mathematics, vol. 55, AMS, 1994, pp. 115–128.

[Del4] P. Deligne, *A quoi servent les motifs?*, Motives, part 1, Proceedings of Symposia in Pure Mathematics, vol. 55, AMS, 1994, pp. 143–161.

[F] E. Friedlander, *Bloch-Ogus properties for topological cycle theory*, Ann. Sci. ENS **33** (2000), 57–79.

[FV] E. Friedlander, V. Voevodsky, *Bivariant cycle cohomology*, Cycles, transfers, and motivic homology theories, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, pp. 138–187.

[J] U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Inv. Math. **107** (1992), 447–452.

[MVW] C. Mazza, V. Voevodsky, C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematical Monographs, vol. 2, AMS, 2006.

[SV] A. Suslin, V. Voevodsky, *Singular homology of abstract algebraic varieties*, Inv. Math. **123** (1996), 61–94.

[V] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, pp. 188–238.