Sample-based and Feature-based Federated Learning via Mini-batch SSCA

Chenceng Ye, Ying Cui

Abstract—Due to the resource consumption for transmitting massive data and the concern for exposing sensitive data, it is impossible or undesirable to upload clients’ local databases to a central server. Thus, federated learning has become a hot research area in enabling the collaborative training of machine learning models among multiple clients that hold sensitive local data. Nevertheless, unconstrained federated optimization has been studied mainly using stochastic gradient descent (SGD), which may converge slowly, and constrained federated optimization, which is more challenging, has not been investigated so far. This paper investigates sample-based and feature-based federated optimization, respectively, and considers both the unconstrained problem and the constrained problem for each of them. We propose federated learning algorithms using stochastic successive convex approximation (SSCA) and mini-batch techniques. We show that the proposed algorithms can preserve data privacy through the model aggregation mechanism, and their security can be enhanced via additional privacy mechanisms. We also show that the proposed algorithms converge to Karush-Kuhn-Tucker (KKT) points of the respective federated optimization problems. Besides, we customize the proposed algorithms to application examples and show that all updates have closed-form expressions. Finally, numerical experiments demonstrate the inherent advantages of the proposed algorithms in convergence speeds, communication costs, and model specifications.

Index Terms—Federated learning, non-convex optimization, stochastic optimization, stochastic successive convex approximation.

I. INTRODUCTION

Machine learning with distributed databases has been a hot research area [2]. The amount of data at each client can be large, and hence the data uploading to a central server may be constrained by energy and bandwidth limitations. Besides, local data may contain highly sensitive information, e.g., travel records, health information, and web browsing history, and thus a client may be unwilling to share it. Therefore, it is impossible or undesirable to upload distributed databases to a central server. Recent years have witnessed the growing interest in federated learning, where data is maintained locally and model parameters or gradients are exchanged during the collaborative training of the server and clients [3]. [4]. Federated learning can protect data privacy for privacy-sensitive applications and can also improve communication efficiency.

Model aggregation (or model averaging), cryptographic methods, and differential privacy are three main privacy mechanisms in federated learning. They provide different privacy guarantees. Specifically, model aggregation protects data privacy by sharing model updates, e.g., gradients and some intermediate parameters, instead of raw data [5]. Communicating model updates generally reveals much less information than communicating raw data. Cryptographic methods and differential privacy can further enhance the privacy of federated learning. Cryptographic methods, such as secret sharing [6] and homomorphic encryption [7], preserve privacy by encrypting the model updates before sharing, at the cost of communication and computation efficiency reduction. As a large computation and communication overhead is usually induced, the resulting federated learning systems are inefficient. Differential privacy [8] preserves privacy by adding random noise to updated parameters at the cost of model performance decline.

Existing works on federated learning [5], [9]–[13] investigate only unconstrained optimization problems mainly using mini-batch stochastic gradient descent (SGD). Depending on whether data is distributed over the sample space or feature space, federated learning can be typically classified into sample-based (horizontal) federated learning and feature-based (vertical) federated learning [14]. In sample-based federated learning, which is more widely studied, the datasets of different clients have the same feature space but no (or little) intersection on the sample space [5], [9]–[14]. The most commonly used sample-based federated learning algorithm is the Federated Averaging (FedAvg) algorithm [5]. In FedAvg, the global model is iteratively updated at the server by aggregating and averaging the clients’ locally computed models. Specifically, at one communication round, each client downloads the latest global model parameters and conducts one or multiple SGD updates to refine its local model. Multiple local SGD updates can reduce the required number of model averaging steps, thereby saving the communication cost. However, it may diverge in the presence of heterogeneous local datasets across clients. Some relevant follow-up works [9], [10] address the convergence issue by carefully designing the number of local SGD updates for each client. Besides, some other works aim at improving the communication efficiency by selecting only some clients to participate in each model averaging step [11] or modifying local SGD updates [12], [13].

On the contrary, in feature-based federated learning, the datasets of different clients share the same sample space but differ in the feature space. Feature-based federated learning is more challenging, as a client cannot obtain the gradient of a loss function relying purely on its local data. Compared with the sample-based federated learning algorithms, the existing feature-based federated learning algorithms [15]–[18] impose additional restrictions on the structure of a loss function to
preserve privacy and require the exchange of intermediate parameters among clients to calculate the gradients for updating local model parameters. Besides, most existing feature-based federated learning algorithms are designed for only two clients [15]–[17]. Reference [18] is the only work for an arbitrary number of clients.

SGD has long been used for obtaining stationary points of unconstrained stochastic optimization problems [19] or Karush–Kuhn–Tucker (KKT) points of stochastic optimization problems with deterministic convex constraints [20]. Recently, stochastic successive convex approximation (SSCA) is proposed to obtain KKT points of stochastic optimization problems with deterministic convex constraints [21] and with general stochastic nonconvex constraints (including deterministic nonconvex constraints as a special case) [22], [23]. Apparently, SSCA applies to more types of constraints. It has also been shown in [21] that SSCA empirically achieves a higher convergence speed than SGD, as SGD utilizes only first-order information of the objective function. Notice that the existing SSCA algorithms [21]–[23] use only one sample at each iteration and may converge slowly when applied to machine learning problems with large datasets. Some recent works [24]–[27] have combined the SSCA algorithm in [21] and mini-batch techniques to solve unconstrained or convex constrained machine learning problems. However, it is still unknown whether mini-batch techniques can be combined with the SSCA algorithms in [22], [23] for solving machine learning problems with general stochastic nonconvex constraints. Besides, SSCA has not been used for solving federated optimization problems so far.

In summary, there are several interesting questions: 1) whether SSCA and mini-batch techniques can apply to a broader range of federated optimization problems, including unconstrained federated optimization and constrained federated optimization, 2) whether mini-batch SSCA can converge faster than mini-batch SGD, and 3) whether mini-batch SSCA can preserve privacy in federated learning. In this paper, we would like to address the above questions. We investigate sample-based and feature-based federated optimization, respectively. For each of them, we consider both the unconstrained problem and the constrained problem. We focus on designing mini-batch SSCA-based federated learning algorithms that preserve data privacy through the model aggregation mechanism. To the best of our knowledge, this is the first work that applies SSCA to solve federated optimization and also the first work that investigates federated optimization with general stochastic nonconvex constraints. The main contributions of this paper are summarized as follows.

- First, we investigate unconstrained and constrained sample-based federated optimization, respectively. For each problem, we propose a privacy-preserving algorithm to obtain a KKT point using mini-batch SSCA and analyze its convergence and security. The proposed algorithm for unconstrained sample-based federated optimization empirically converges faster (i.e., achieves a lower communication cost) than the SGD-based ones [5], [12]–[14] and achieves the same order of computational complexity as the SGD-based ones [5], [12]–[14].
- Then, we investigate unconstrained and constrained feature-based federated optimization, respectively. For each problem, we propose a privacy-preserving algorithm to obtain a KKT point using mini-batch SSCA and analyze its convergence and security. The proposed algorithm for unconstrained feature-based federated optimization also has a faster convergence speed empirically than the SGD-based ones [15]–[18].
- Next, we consider two application examples in classification, and customize the proposed SSCA-based algorithms to them. We show that all updates at each iteration have closed-form expressions, and the updates in SSCA algorithms for unconstrained optimization are as computationally efficiently as those in the SGD-based algorithms. We also characterize the relationship between the two formulations.
- Finally, numerical experiments demonstrate that the proposed SSCA-based algorithms for unconstrained federated optimization converge faster (i.e., yield lower communication costs) than the existing SGD-based ones [5], [12]–[18]. Numerical experiments also show that the proposed algorithms for constrained federated optimization can more flexibly specify a training model.

II. SYSTEM SETTING

Consider $N$ data samples, each of which has $K$ features. For all $n \in \mathcal{N} \triangleq \{1, \ldots, N\}$, the $K$ features of the $n$-th sample are represented by a $K$-dimensional vector $x_n \in \mathbb{R}^K$.

Consider a central server connected with $I$ local clients, each of which maintains a local dataset. We assume that the server and clients are honest-but-curious, i.e., they will follow the predetermined algorithms but will attempt to infer private information using all data received throughout the protocol execution [28]. Depending on whether data is distributed over the sample space or feature space, federated learning can be typically classified into sample-based (horizontal) federated learning and feature-based (vertical) federated learning [15], as shown in Fig. 1.

In sample-based federated learning, the clients have the same feature space but differ in the sample space. Specifically, partition $\mathcal{N}$ into $K$ disjoint subsets, denoted by $\mathcal{N}_i$, $i \in \mathcal{I} \triangleq \{1, \ldots, I\}$, where $\mathcal{N}_i \triangleq |\mathcal{N}_i|$ denotes the cardinality of the $i$-th subset and $\sum_{i \in \mathcal{I}} |\mathcal{N}_i| = N$. For all $i \in \mathcal{I}$, the $i$-th client maintains a local dataset containing $\mathcal{N}_i$ samples, i.e., $x_n, n \in \mathcal{N}_i$. For example, two companies with similar business in different cities may have different user groups (from their respective regions) but the same type of data, e.g., users’ occupations, ages, incomes, deposits, etc, as illustrated in Fig. 1(a).

The server and $I$ clients collaboratively train a model from the local datasets stored on the $I$ clients under the condition that each client cannot expose its local raw data to the others. This training process is referred to as sample-based (horizontal) federated learning [15]. The underlying optimization, termed...
expose its local raw data to the others. This training process is to train a model under the condition that each client cannot observe the model parameters of other clients. Therefore, we impose additional restrictions on the structure of $f_{f,0}(\omega, x_n)$. It is worth noting that in existing work on privacy-preserving feature-based federated optimization, the cost function is either a special case of $F_{f,0}(\omega)$ (e.g., $I = 2$) [13], [16], [31] and the cost function is the mean square error function [13] or cross-entropy function [16], [31], or shares the same form as $F_{f,0}(\omega)$ [18].

In Section III and Section V, we investigate sample-based federated learning and feature-based federated learning, respectively. To guarantee the convergence of the proposed SCA-based federated learning algorithms, we assume that $f_{s,0}(\omega, x_n)$ and $f_{f,0}(\omega, x_n)$ both satisfy the following assumption in the rest of the paper:

**Assumption 1 (Assumption on $f(\omega, x)$):** [21], [22] For any given $x$, each $f(\omega, x)$ is continuously differentiable, and its gradient is Lipschitz continuous.

Note that Assumption 1 is also necessary for the convergence of SCA [21]–[23] and SGD [13].

### III. Sample-based Federated Learning

In this section, we investigate two types of sample-based federated optimization, namely, unconstrained sample-based federated optimization and constrained sample-based federated optimization. For each optimization problem, we propose a privacy-preserving federated learning algorithm using mini-batch SCA.

#### A. Sample-based Federated Learning for Unconstrained Optimization

In this part, we consider the following unconstrained sample-based federated optimization problem:

**Problem 1 (Unconstrained Sample-based Federated Optimization):**

$$\min_{\omega} F_{s,0}(\omega)$$

where $F_{s,0}(\omega)$ is given by (7).

Problem 1 (whose objective function has a large number of terms) is usually transformed to an equivalent stochastic optimization problem and solved using stochastic optimization algorithms. The SGD-based algorithms in [5], [12]–[14], proposed to obtain a KKT point of Problem 1 may have unsatisfactory convergence speeds and high communication costs, as SGD utilizes only the first-order information of an objective function. In the following, we propose a privacy-preserving sample-based federated learning algorithm, i.e., Algorithm 1 to obtain a KKT point of Problem 1 using mini-batch SCA. It has been shown in [21] that SCA empirically achieves a higher convergence speed than SGD. Later in Section VI, we shall numerically show that the proposed SCA-based algorithm converges faster than the SGD-based algorithms in [5], [12]–[14].
1) Algorithm Description: The main idea of Algorithm 1 is to solve a sequence of successively refined convex problems, each of which is obtained by approximating \( F_{s,0}(\omega) \) with a convex function based on its structure and the samples in a randomly selected mini-batch by the server. Specifically, at iteration \( t \), we choose:

\[
F_{s,t}(\omega) = (1 - \rho(t)) F_{s,t-1}(\omega) + \rho(t) \left( \sum_{i \in I} N_{i} \sum_{n \in N_{i}} \widehat{f}_{s,0}(\omega, \omega_{i}^{t}, x_{n}) \right) \quad (3)
\]

with \( \widehat{F}_{s,0}(\omega) = 0 \) as an approximation function of \( F_{s,0}(\omega) \), where \( \rho(t) \) is a stepsize satisfying:

\[
\rho(t) > 0, \quad \lim_{t \to \infty} \rho(t) = 0, \quad \sum_{t=1}^{\infty} \rho(t) = \infty, \quad (4)
\]

\( N_{i}^{t} \subseteq N_{i} \) is a randomly selected mini-batch by client \( i \) at iteration \( t \), \( B \leq N_{i} \) is the batch size, and \( f_{s,0}(\omega, \omega_{i}^{t}, x_{n}) \) is a convex approximation of \( f_{s,0}(\omega, x_{n}) \) around \( \omega_{i}^{t} \) satisfying the following assumptions. A common example of \( f_{s,0} \) will be given later.

Assumption 2 (Assumptions on \( f(\omega, \omega^{t}, x) \) for Approximating \( f(\omega, x) \) Around \( \omega^{t} \)): [22] 1) \( \nabla f(\omega, x) = \nabla \hat{f}(\omega, x) \); 2) \( \hat{f}(\omega, \omega^{t}, x) \) is strongly convex in \( \omega \); 3) \( \hat{f}(\omega, \omega^{t}, x) \) is Lipschitz continuous in both \( \omega \) and \( \omega^{t} \); 4) \( f(\omega, \omega^{t}, x) \), its derivative, and its second order derivative w.r.t. \( \omega \) are uniformly bounded.

Assumption 3 is necessary for the convergence of SCA [23]. Note that for all \( i \in I \) and any mini-batch \( N_{i}^{t} \subseteq N_{i} \) with batch size \( B \leq N_{i} \), \( \sum_{n \in N_{i}^{t}} \hat{f}_{s,0}(\omega, \omega^{t}, x_{n}) \) can be written as \( \sum_{n \in N_{i}^{t}} F_{s,0}(\omega, \omega^{t}, x_{n}) \) with \( F_{s,0} : \mathbb{R}^{D_{0}+d} \to \mathbb{R} \) and \( \mathbf{q}_{s,0} : \mathbb{R}^{BK+d} \to \mathbb{R}^{D_{0}} \). Assume that the expressions of \( f_{s,0}, p_{s,0} \), and \( \mathbf{q}_{s,0} \) are known to the server and \( N \) clients. Each client \( i \in I \) computes \( \mathbf{q}_{s,0}(\omega_{i}^{t}, (x_{n})_{n \in N_{i}^{t}}) \) and sends it to the server. Then, the server solves the following convex approximate problem to obtain \( \omega_{s}^{t} \).

Problem 2 (Convex Approximate Problem of Problem 1):

\[
\omega_{s}^{t} = \arg \min_{\omega} F_{s,t}(\omega)
\]

Problem 2 is convex and can be solved with conventional convex optimization techniques. Given \( \omega_{s}^{t} \), the server updates \( \omega_{s}^{t+1} \) according to:

\[
\omega_{s}^{t+1} = (1 - \gamma(t)) \omega_{s}^{t} + \gamma(t) \omega_{s}^{t}, \quad t = 1, 2, \ldots \]

where \( \gamma(t) \) is a stepsize satisfying:

\[
\gamma(t) = 0, \quad \lim_{t \to \infty} \gamma(t) = 0, \quad \sum_{t=1}^{\infty} \gamma(t) = \infty, \quad \sum_{t=1}^{\infty} (\gamma(t))^{2} < \infty, \quad \lim_{t \to \infty} \frac{\gamma(t)}{\rho(t)} = 0. \quad (6)
\]

The detailed procedure is summarized in Algorithm 1 and the convergence of Algorithm 1 is summarized below.

Theorem 1 (Convergence of Algorithm 1): Suppose that \( f_{s,0} \) satisfies Assumption 1 \( f_{s,0} \) satisfies Assumption 2 and the sequence \( \{\omega_{s}^{t}\} \) generated by Algorithm 1 is bounded. Then, every limit point of \( \{\omega_{s}^{t}\} \) is a KKT point of Problem 1 almost surely.

**Proof:** Please refer to Appendix A. ■

2) Security Analysis: For all \( i \in I \) and any mini-batch \( N_{i}^{t} \subseteq N_{i} \) with batch size \( B \leq N_{i} \), the system of equations w.r.t. \( \mathbf{z} \in \mathbb{R}^{BK} \), i.e., \( \mathbf{q}_{s,0}(\omega^{t}, (x_{n})_{n \in N_{i}^{t}}) = \mathbf{q}_{s,0}(\omega^{t}, (x_{n})_{n \in N_{i}^{t}}) \), has an infinite (or a sufficiently large) number of solutions, then raw data \( x_{n} \), \( n \in N_{i}^{t} \) cannot be extracted from \( \mathbf{q}_{s,0}(\omega^{t}, (x_{n})_{n \in N_{i}^{t}}) \) in Step 4 of Algorithm 1 and hence Algorithm 1 can preserve data privacy. Otherwise, extra privacy mechanisms can be applied to preserve data privacy. For example, if \( \omega_{s}^{t} \) is linear in \( \mathbf{q}_{s,0}(\omega_{s}^{t}, (x_{n})_{n \in N_{i}^{t}}) \), \( i \in I \), then homomorphic encryption can be applied; if \( \omega_{s}^{t} \) is a polynomial of \( x_{n} \) and \( \omega_{s}^{t} \), then secret sharing can be applied.

3) Algorithm Example: Finally, we provide an example of \( f_{s,0} \) which satisfies Assumption 2 and yields an analytical solution of Problem 1

\[
\widetilde{f}_{s,0}(\omega, \omega_{i}^{t}, x_{n}) = \left( \nabla f_{s,0}(\omega_{i}^{t}, x_{n}) \right)^{T} (\omega - \omega_{i}^{t}) + \tau \| \omega - \omega_{i}^{t} \|_{2}^{2}, \quad (7)
\]

where \( \tau > 0 \) can be any constant, and the term \( \| \omega - \omega_{i}^{t} \|_{2}^{2} \) is used to ensure strong convexity. Obviously, \( \widetilde{f}_{s,0} \), given by (7), satisfies Assumption 2. Substituting (7) into (3), \( \widetilde{F}_{s,0}(\omega) \) can be rewritten as:

\[
\tilde{F}_{s,0}(\omega) = \left( \tilde{F}_{s,0,1}^{(t)} \right)^{T} \omega + \tau \| \omega \|_{2}^{2}, \quad (8)
\]

where \( \tilde{F}_{s,0,1}^{(t)} \in \mathbb{R}^{d} \) is given by:

\[
\tilde{F}_{s,0,1}^{(t)} = (1 - \rho(t)) \tilde{F}_{s,0,1}^{(t-1)}.
\]

Algorithm 1 Mini-batch SCA for Problem 1

1: initialize: choose any \( \omega_{s}^{0} \) at the server.
2: for \( t = 1, 2, \ldots, T - 1 \) do
3: the server sends \( \omega_{s}^{t} \) to all clients.
4: for all \( i \in I \), client \( i \) randomly selects a mini-batch \( N_{i}^{t} \subseteq N_{i} \), computes \( \mathbf{q}_{s,0}(\omega_{s}^{t}, (x_{n})_{n \in N_{i}^{t}}) \) and sends it to the server.
5: the server obtains \( \omega_{s}^{t} \) by solving Problem 2 and updates \( \omega_{s}^{t+1} \) according to (5).
6: end for
7: Output: \( \omega_{s}^{T} \)
is an unconstrained convex quadratic programming w.r.t. \( \omega \). By the first-order optimality condition, it has the following analytical solution:

\[
\hat{\omega}_s^{(t)} = \frac{1}{2 \tau} \hat{f}_{s,0}^{(t)},
\]

The computational complexity for calculating \( \hat{\omega}_s^{(t)} \) is \( O(d) \).

For \( \tilde{f}_{s,0} \) given by (7), \( \sum_{n \in \mathcal{N}_t} \nabla f_{s,0}(\omega_s^{(t)}, x_n) \) can be viewed as \( q_{s,0}(\omega_s^{(t)}, x_n) \). If for all \( i \in \mathcal{I} \) and \( \mathcal{N}_t' \subseteq \mathcal{N}_t \) with batch size \( B \), the system of equations w.r.t. \( (z_n)_{n=1}^B \) where \( z_n \in \mathbb{R}^N \), \( n = 1, \ldots, B \), i.e., \( \sum_{n=1}^B \nabla f_{s,0}(\omega_s^{(t)}, z_n) = \sum_{n \in \mathcal{N}_t'} \nabla f_{s,0}(\omega_s^{(t)}, x_n), \) has an infinite (or a sufficiently large) number of solutions, then Algorithm 1 with \( \tilde{f}_{s,0} \) given by (7) can preserve data privacy. Otherwise, homomorphic encryption can be applied to preserve data privacy, since \( \hat{\omega}_s^{(t)} \) is linear in \( \nabla f_{s,0}(\omega_s^{(t)}, x_n) \).

Remark 1 (Comparison Between Algorithm 1 and Sample-based Federated Learning Algorithms via SGD): Notice that under the sample-based federated learning algorithms via SGD [5], [12]–[14], each client conducts one SGD update, and the server averages the locally computed models. Thus, under Algorithm 1 with \( \tilde{f}_{s,0} \) given by (7) and the sample-based federated learning algorithm via SGD [5], [12]–[14], the computational complexities at each client have the same order, and the computational complexities at the server have the same order (which is \( O(d) \)).

B. Sample-based Federated Learning for Constrained Optimization

In this part, we consider the following constrained sample-based federated optimization problem:

\textbf{Problem 3 (Constrained Sample-based Federated Optimization):}

\[
\begin{align*}
\min_{\omega} & \quad F_{s,0}(\omega) \\
\text{s.t.} & \quad F_{s,m}(\omega) \leq 0, \quad m = 1, 2, \ldots, M,
\end{align*}
\]

where \( F_{s,0}(\omega) \) is given by (1), and

\[
F_{s,m}(\omega) \triangleq \frac{1}{N} \sum_{n \in \mathcal{N}} f_{s,m}(\omega, x_n), \quad m = 1, 2, \ldots, M.
\]

To be general, \( F_{s,m}(\omega), m = 0, \ldots, M \) are not assumed to be convex in \( \omega \). Notice that federated optimization with nonconvex constraints has not been investigated so far. It is quite challenging, as the stochastic nature of a constraint function may cause infeasibility at each iteration of an ordinary stochastic iterative method [23]. In the following, we propose a privacy-preserving sample-based federated learning algorithm, i.e., Algorithm 2 to obtain a KKT point of Problem 3 by combining the exact penalty method [23] for SCCA in our previous work [23] and mini-batch techniques.

1) Algorithm Description: First, we transform Problem 3 to the following stochastic optimization problem whose objective function is the weighted sum of the original objective and the penalty for violating the original constraints.

\textbf{Problem 4 (Transformed Problem of Problem 3):}

\[
\begin{align*}
\min_{\omega, s} & \quad F_{s,0}(\omega) + c \sum_{m=1}^M s_m \\
\text{s.t.} & \quad F_{s,m}(\omega) \leq s_m, \quad m = 1, 2, \ldots, M, \\
& \quad s_m \geq 0, \quad m = 1, 2, \ldots, M,
\end{align*}
\]

where \( s \triangleq (s_m)_{m=1}^M \) are slack variables, and \( c > 0 \) is a penalty parameter that trades off the original objective function and the slack penalty term.

At iteration \( t \), we choose \( \tilde{f}_{s,0}^{(t)}(\omega) \) given by (7) as an approximation function of \( F_{s,0}(\omega) \) and choose:

\[
\bar{F}_{s,m}^{(t)}(\omega) = (1 - \rho^{(t)}) \bar{F}_{s,m}^{(t-1)}(\omega) + \rho^{(t)} \sum_{i \in \mathcal{I}} \frac{N_i}{B N} \sum_{n \in \mathcal{N}_t'} \tilde{f}_{s,m}(\omega, x_n), \quad m = 1, \ldots, M
\]

with \( \bar{F}_{s,m}^{(0)}(\omega) = 0 \) as an approximation function of \( F_{s,m}(\omega) \), for all \( m = 1, \ldots, M \), where \( \rho^{(t)} \) is a stepsize satisfying (4). \( N_i^{(t)} \) is the randomly selected mini-batch by client \( i \) at iteration \( t \), and \( f_{s,m}(\omega, x_n) \) is a convex approximation of \( f_{s,m}(\omega, x_n) \) around \( \omega_s^{(t)} \) satisfying \( f_{s,m}(\omega, x) = f_{s,m}(\omega, x) \) and Assumption 2 for all \( m = 1, \ldots, M \). A common example of \( f_{s,m} \), \( m = 0, \ldots, M \) will be given later.

Note that for all \( i \in \mathcal{I} \) and any mini-batch \( N_i' \subseteq \mathcal{N}_t \) with batch size \( B \leq N_i \), \( \sum_{n \in \mathcal{N}_t'} f_{s,m}(\omega, x_n) = p_{s,m}(q_{s,m}(\omega, x_n) \in \mathcal{N}_t')), m = 0, \ldots, M \) with \( p_{s,m} : \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( q_{s,m} : \mathbb{R}^{\mathcal{I}} \rightarrow \mathbb{R}^{\mathcal{I}} \). Assume that the expressions of \( f_{s,m}, p_{s,m} \) and \( q_{s,m} \) are known to the server and \( N \) clients. Each client \( i \in \mathcal{I} \) computes \( \tilde{f}_{s,m}(\omega, x_n) \), m = 0, \ldots, M and send them to the server. Then, the server solves the following approximate problem to obtain \( \omega_s^{(t)} \):

\textbf{Problem 5 (Convex Approximate Problem of Problem 4):}

\[
\begin{align*}
(\bar{\omega}_s^{(t)}, \bar{s}^{(t)}) & \triangleq \arg \min_{\omega, s} \bar{F}_{s,0}(\omega) + c \sum_{m=1}^M s_m \\
\text{s.t.} & \quad \bar{F}_{s,m}(\omega) \leq s_m, \quad m = 1, 2, \ldots, M, \\
& \quad s_m \geq 0, \quad m = 1, 2, \ldots, M.
\end{align*}
\]
Algorithm 2 Mini-batch SSCA for Problem 3

1: initialize: choose any \( \omega_s^1 \) and \( c > 0 \) at the server.
2: for \( t = 1, 2, \ldots, T - 1 \) do
3: the server sends \( \omega_s^t \) to all clients.
4: for all \( i \in I \), client \( i \) randomly selects a mini-batch \( \mathcal{N}_i \subseteq \mathcal{N}_s \), computes \( q_{s,m}(\omega_s^t, (X_n)_{n \in \mathcal{N}_i}) \), \( m = 0, 1, \ldots, M \), and sends them to the server.
5: the server obtains \( (\omega_s^{t+1}, s^{t+1}) \) by solving Problem 5 and updates \( \omega_s^{t+1} \) according to 5.
6: end for
7: Output: \( \omega_s^T \)

the server updates \( \omega_s^{t+1} \) according to 5. The detailed procedure is summarized in Algorithm 2 and the convergence of Algorithm 2 is summarized below. Consider a sequence \( \{c_j\} \). For all \( j \), let \( (\omega_s^{*,j}, s^{*,j}) \) denote a limit point of \( \{(\omega_s^{t}, s^{t})\} \) generated by Algorithm 2 with \( c = c_j \).

**Theorem 2 (Convergence of Algorithm 2):** Suppose that \( f_{s,m}, \ m = 0, \ldots, M \) satisfy Assumption 1. \( f_{s,0} \) satisfies Assumption 2 and \( f_{s,m} \) satisfies \( f_{s,m}(\omega, x, \mathbf{w}) = f_{s,m}(\omega, x) \) and Assumption 3 for all \( m = 1, \ldots, M \), the constraint set of Problem 4 is compact, and the sequence \( \{c_j\} \) satisfies \( 0 < c_j < c_{j+1} \) and \( \lim_{j \to \infty} c_j = \infty \). Then, the following statements hold.

i) For all \( j \), if \( s^n_{*,j} = 0 \), then \( \omega_s^{*,j} \) is a KKT point of Problem 3 almost surely; ii) A limit point \( (\omega_s^{*,*}, s^{*,*}) \), denoted by \( \{(\omega_s^{*,t}, s^{*,t})\} \), satisfies that \( s^{*,*} = 0 \), and \( \omega_s^{*,*} \) is a KKT point of Problem 3 almost surely.

**Proof:** Please refer to Appendix B.

In practice, we can choose a sequence \( \{c_j\} \) which satisfies that \( 0 < c_j < c_{j+1} \), \( \lim_{j \to \infty} c_j = \infty \), and \( c_1 \) is large and repeat Algorithm 2 with \( c = c_j \) until \( \|s^n_{*,*}\|_2 \) is sufficiently small.

2) Security Analysis: If for all \( i \in I \) and any mini-batch \( \mathcal{N}_i \), batch size \( B \leq \mathcal{N}_s \), the system of equations w.r.t. \( z \in \mathbb{R}^{BK} \), i.e., \( q_{s,m}(\omega, z) = q_{s,m}(\omega, (X_n)_{n \in \mathcal{N}_i}) \), \( m = 0, \ldots, M \), has an infinite (or a sufficiently large) number of solutions, then raw data \( x_n \), \( n \in \mathcal{N}_i \) cannot be extracted from \( q_{s,m}(\omega_{s,m}^{t}, (X_n)_{n \in \mathcal{N}_i}) \), \( m = 0, \ldots, M \) in Step 4 of Algorithm 2 and hence Algorithm 2 can preserve data privacy. Otherwise, extra privacy mechanisms need to be exploited. Note that federated learning with constrained optimization has not been studied so far, let alone the privacy mechanisms for it.

3) Algorithm Example: We provide an example of \( f_{s,m}, m = 0, \ldots, M \) satisfying Assumption 2 and \( f_{s,0} \) satisfying \( f_{s,m}(\omega, x) = f_{s,m}(\omega, x) \) and Assumption 3 for all \( m = 1, \ldots, M \). Specifically, we can choose \( f_{s,0} \) given by 7 and choose \( f_{s,m} \) as follows:

\[
\tilde{f}_{s,m}(\omega^{t}, x_n) = f_{s,m}(\omega^{t}, x_n) + \left( \nabla f_{s,m}(\omega_s^{t}, x_n) \right)^T (\omega - \omega_s^{t}) + \tau \|\omega - \omega_s^{t}\|_2^2, \quad m = 1, \ldots, M,
\]

where \( \tau > 0 \) can be any constant. Obviously, \( f_{s,0} \) given by 7 satisfies Assumption 1 and \( f_{s,m} \) given by 11 satisfies \( f_{s,m}(\omega, x) = f_{s,m}(\omega, x) \) and Assumption 3 for all \( m = 1, \ldots, M \). Recall that with \( f_{s,0} \) given in 7, \( F_{s,0}(\omega) \) is given in 8. In addition, for all \( m = 1, \ldots, M \), substituting 11 into 10, \( F_{s,m}(\omega) \) can be rewritten as:

\[
\tilde{F}_{s,m}(\omega) = \tilde{f}_{s,m}(\omega) = f_{s,m}(\omega) + (\tilde{f}_{s,m}(\omega))^{T} (\omega + \tau \omega_s^{t})_2^2, \quad m = 1, \ldots, M,
\]

where \( \tilde{f}_{s,m} \) and \( \tilde{f}_{s,m}^{*,1} \in \mathbb{R}^d \) are given by:

\[
f_{s,m}(\omega) = (1 - \rho(t)) f_{s,m}^{*,1} + \rho(t) \sum_{i \in I} \sum_{n \in \mathcal{N}_i} (f_{s,m}(\omega_s^{t}, x_n) - \omega) + \tau \|\omega - \omega_s^{t}\|_2^2, \quad m = 1, \ldots, M,
\]

where \( \tilde{f}_{s,m} \) and \( \tilde{f}_{s,m}^{*,1} \) are given by:

\[
f_{s,m}^{*,1} = (1 - \rho(t)) f_{s,m}^{*,1 - 1} + \rho(t) \sum_{i \in I} \sum_{n \in \mathcal{N}_i} (f_{s,m}(\omega_s^{t}, x_n) - \omega_s^{t}) + \tau \|\omega_s^{t} - \omega_s^{t}\|_2^2, \quad m = 1, \ldots, M,
\]

with \( f_{s,m}^{*,1} = 0 \) and \( \tilde{f}_{s,m}^{*,1} = 0 \). Apparently, Problem 5 with \( f_{s,0} \) given by 7 and \( f_{s,m}, m = 1, \ldots, M \) given by 11 is a convex quadratically constrained quadratic programming and can be solved using an interior point method.

For \( f_{s,m} \) given by 11, \( \sum_{n \in \mathcal{N}_i} \nabla f_{s,m}(\omega_s^{t}, x_n) \) can be viewed as \( q_{s,m}(\omega_s^{t}, (X_n)_{n \in \mathcal{N}_i}) \). If for all \( i \in I \) and \( \mathcal{N}_i \), batch size \( B \), the system of equations w.r.t. \( (z_n)_{n=1}^{B} \), i.e., \( \sum_{m=1}^{B} f_{s,m}(\omega_s^{t}, z_n) = z_n \), \( m = 1, \ldots, B \), then Algorithm 2 with \( f_{s,0} \) given by 7 and \( f_{s,m}, m = 1, \ldots, M \) given by 11 can preserve data privacy.

IV. FEATURE-BASED FEDERATED LEARNING

In this section, we investigate two types of sample-based federated optimization, namely, unconstrained feature-based federated optimization and constrained feature-based federated optimization. For each optimization problem, we propose a privacy-preserving federated learning algorithm using mini-batch SSCA.

A. Feature-based Federated Learning for Unconstrained Optimization

In this part, we consider the following unconstrained feature-based federated optimization problem:

**Problem 6 (Unconstrained Feature-based Federated Optimization):**

\[
\min_{\omega} F_{f,0}(\omega)
\]

where \( F_{f,0}(\omega) \) is given by 4.
Algorithm 3 Mini-batch SSCA for Problem 6
1: initialize: choose any $\omega_0^T$ at the server.
2: for $t = 1, 2, \ldots, T - 1$ do
3: the server randomly selects a mini-batch with the index set denoted by $N^{(t)} \subseteq N$, and sends $N^{(t)}$ and $(\omega_0^{(t)}, \omega_{i}^{(t)})$ to client $i$ for all $i \in I$.
4: for all $i \in I$, client $i$ computes $h_{0,i}(\omega_{i}^{(t)}, x_{n,i})$, $n \in N^{(t)}$ and sends them to the other clients.
5: for all $i \in I$, client $i$ computes $q_{f,0,i}(\omega_0^{(t)}, \omega_{i}^{(t)}, (x_{n,i}, h_{0,j}(\omega_{j}^{(t)}, x_{n,j}))_{j \in I})$ and sends it to the server.
6: the client with the highest computation speed (or any client) computes $q_{f,0,0}(\omega_0^{(t)}, (h_{0,i}(\omega_{i}^{(t)}, x_{n,i}))_{i \in I})$ and sends it to the server.
7: the server obtains $\omega_f^T$ by solving Problem 7 and updates $\omega_{f}^{(t+1)}$ according to (5).
8: end for
9: Output: $\omega_f^T$

Similarly, Problem 3 can be solved using stochastic optimization algorithms. The SGD-based algorithms in [15–18], proposed to obtain a KKT point of Problem 6 for $I = 2$ [15–17] and $I > 2$ [18], may have unsatisfactory convergence speeds and high communication costs. Similarly to Algorithm 1 in the following, we propose a privacy-preserving feature-based federated learning algorithm, i.e., Algorithm 3, to obtain a KKT point of Problem 6 using mini-batch SSCA. Later in Section VI, we shall numerically show that the proposed SSCA-based algorithm converges faster than the SGD-based algorithms in [15–18].

1) Algorithm Description: At iteration $t$, we choose

$$F_f^{(t)}(\omega) = (1 - \rho^{(t)}) F_f^{(t-1)}(\omega) + \rho^{(t)} \frac{1}{B} \sum_{n \in N^{(t)}} f_{f,0}(\omega, \omega_{f}^{(t)}, x_{n})$$

(13)

with $F_f^{(0)}(\omega) = 0$ as an approximation function of $F_{f,0}(\omega)$, where $\rho^{(t)}$ is a stepsize satisfying (4). $N^{(t)} \subseteq N$ is a randomly selected mini-batch by the server at iteration $t$. $B \leq N$ is the batch size, and $f_{f,0}(\omega, \omega_{f}^{(t)}, x_{n})$ is a convex approximation of $f_{f,0}(\omega, x_{n})$ around $\omega_{f}^{(t)}$ satisfying Assumption 2. A common example of $f_{f,0}$ will be given later.

Note that for any mini-batch $N^{(t)} \subseteq N$ with batch size $B \leq N$, $\sum_{n \in N^{(t)}} f_{f,0}(\omega, \omega_{f}^{(t)}, x_{n})$ can be written as (14), as shown at the top of this page, with $f_{f,0} : \mathbb{R}^{En+d} \to \mathbb{R}$ and $q_{f,0,i} : \mathbb{R}^{B(K+1)+d+di} \to \mathbb{R}^{En}, i \in I$, Assume that the expressions of $f_{f,0}, f_{f,0,i}, h_{0,i}$, and $h_{0,j}$, $i \in I$ are known to the server and $N$ clients. Each client $i \in I$ computes $h_{0,i}(\omega_0^{(t)}, x_{n,i})$, $n \in N^{(t)}$ and sends them to the other clients. Based on $h_{0,i}(\omega_0^{(t)}, x_{n,i})$, $i \in I$, $n \in N^{(t)}$, each client $i \in I$ computes $q_{f,0,i}(\omega_0^{(t)}, \omega_{i}^{(t)}, (x_{n,i}, h_{0,j}(\omega_{j}^{(t)}, x_{n,j}))_{j \in I})$ and sends it to the server. Moreover, the client with the highest computation speed (or any client) computes $q_{f,0,0}(\omega_0^{(t)}, (h_{0,i}(\omega_{i}^{(t)}, x_{n,i}))_{i \in I})$ and sends it to the server. Then, the server solves the following convex approximate problem to obtain $\omega_f^T$.

Problem 7 (Convex Approximate Problem of Problem 6):

$$\omega_f^T \triangleq \arg \min_{\omega} \left\{ F_f^{(t)}(\omega) \right\}$$

Problem 7 is convex and can be readily solved. Given $\omega_f^T$, the server updates $\omega_f^T$ according to (5). The detailed procedure is summarized in Algorithm 3 and the convergence of Algorithm 3 is summarized below.

Theorem 3 (Convergence of Algorithm 3): Suppose that $f_{f,0}$ satisfies Assumption 1 $f_{f,0}$ satisfies Assumption 2 and the sequence $\{\omega_f^T\}$ generated by Algorithm 3 is bounded almost surely. Then, every limit point of $\{\omega_f^T\}$ is a KKT point of Problem 6 almost surely.

Proof: Please refer to Appendix A.
Algorithm 3 can preserve data privacy if the two assumptions mentioned above are not satisfied, extra privacy mechanisms are required. For instance, if $\omega_f^{(t)}$ is linear in $q_f,0,0$ and $q_f,0,1$, $i \in I$, then homomorphic encryption \[7\] can be applied.

3) Algorithm Example: We provide an example of $\bar{f}_{f,0}$ which satisfies Assumption 2 and yields an analytical solution of Problem 6

$$\bar{f}_{f,0}(\omega, \omega_f^{(t)}, x_n) = f_{f,0}(\omega_f^{(t)}, x_n) + \nabla f_{f,0}(\omega_f^{(t)}, x_n)^T (\omega - \omega_f^{(t)})$$

$$+ \tau \|\omega - \omega_f^{(t)}\|_2^2, \quad n \in N(t),$$

where $\tau > 0$ can be any constant. Obviously, $\bar{f}_{f,0}$ given by (15) satisfies Assumption 2. By the chain rule, we have:

$$\nabla_{\omega_n} \bar{f}_{f,0}(\omega_f^{(t)}, x_n) = \nabla_{\omega_n} g_0(\omega_0^{(t)}, h_0,i,\omega_i^{(t)}, x_{n,i})_{i \in I}, \quad n \in N(t),$$

$$\nabla_{\omega_n} \bar{f}_{f,0}(\omega_f^{(t)}, x_n) = \frac{\partial g_0(\omega_0^{(t)}, h_0,i,\omega_i^{(t)}, x_{n,i})}{\partial h_0,i,\omega_i^{(t)}, x_{n,i}} \times \nabla_{\omega_n} h_0,i,\omega_i^{(t)}, x_{n,i}, \quad n \in N(t), i \in I.$$ (16)

Substituting (15) into (13), $F_{f,0}(\omega)$ can be rewritten as:

$$F_{f,0}(\omega) = \left(\frac{1}{\rho_{f,0,1}}\right)^T \omega + \tau \|\omega\|_2^2,$$ (18)

where $\rho_{f,0,1} \in \mathbb{R}^d$ is given by:

$$R_{f,0,1} = (1 - \rho_{f,0,1})R_{f,0,1} + \rho_{f,0,1} \sum_{n \in N(t)} (\nabla_{f,0}(\omega_f^{(t)}, x_n) - 2\tau \omega_f^{(t)}).$$

with $R_{f,0,1} = 0$. Similarly to Problem 2 with $\bar{f}_{f,0}$ given by (7), Problem 3 with $\bar{f}_{f,0}$ given by (15) is a unconstrained quadratic programming w.r.t. $\omega$ and hence has the following analytical solution:

$$\omega_f^{(t)} = \frac{1}{2\tau} R_{f,0,1}.$$ (19)

The computational complexity for calculating $\omega_f^{(t)}$ in (19) is $O(d)$.

For $\bar{f}_{f,0}$ given by (15), $\sum_{n \in N(t)} \nabla_{\omega_n} f_{f,0}(\omega_f^{(t)}, x_n)$ can be viewed as $q_{f,0,0}(\omega_0^{(t)}, h_0,i,\omega_i^{(t)}, x_{n,i})_{n \in N(t), i \in I}$, and $\sum_{n \in N(t)} \nabla_{\omega_n} f_{f,0}(\omega_f^{(t)}, x_n)$ can be viewed as $q_{f,0,i}(\omega_0^{(t)}, \omega_i^{(t)}, x_{n,i}, h_0,j,\omega_j^{(t)}, x_{n,j})_{n \in N(t), j \in I}$.

4) Suppose 1) for any mini-batch $N' \subseteq N$ with batch size $B$, the system of equations w.r.t. $(x_n)_{n=1,...,B}$, where $x_n \in \mathbb{R}^K$, $n = 1, ..., B$, i.e.,

$$\sum_{n=1}^B \nabla f_{f,0}(\omega_f^{(t)}, x_n) = \sum_{n \in N(t)} \nabla f_{f,0}(\omega_f^{(t)}, x_n),$$

has an infinite (or a sufficiently large) number of solutions; 2) for all $i \in I$ and any mini-batch $N' \subseteq N$ with batch size $B$, the system of equations w.r.t. $(\theta, x_n)_{n \in N(t)}$, where $\theta \in \mathbb{R}^d$, and $x_n \in \mathbb{R}^K$, i.e., $h_0,i(\theta, x_n) = h_0,i(\omega_f^{(t)}, x_n)$, $n \in N(t)$, has an infinite (or a sufficiently large) number of solutions. In that case, the existing feature-based federated learning algorithms via SGD [3, 12–14] can preserve data privacy.

5) The condition for Algorithm 3 with $f_{f,0}$ given by (15) to preserve data privacy without extra privacy mechanisms is the same as that for the existing feature-based federated learning algorithm via SGD [5, 12–14], as illustrated in Footnote 4.
Algorithm 4 Mini-batch SSCA for Problem 8

1: initialize: choose any \( \omega_0^f \), \( c > 0 \) at the server.
2: for \( t = 1, 2, \ldots, T - 1 \) do
3: the server randomly selects a mini-batch with the index set denoted by \( \mathcal{N}(t) \subset \mathcal{N} \), and sends \( \mathcal{N}(t) \) and \( (\omega_0^f(t), \omega_1^f(t)) \) to client \( i \) for all \( i \in \mathcal{I} \).
4: for all \( i \in \mathcal{I} \), client \( i \) computes \( h_{m,i}(\omega_i^f(t), x_{n,i}) \), \( n \in \mathcal{N}(t) \), \( m = 0, 1, \ldots, M \) and sends them to the other clients.
5: for all \( i \in \mathcal{I} \), client \( i \) computes \( q_{f,m,i}(\omega_0^f(t), \omega_1^f(t), (x_{n,i}, h_{m,j}(\omega_j^f(t), x_{n,j}))_{n \in \mathcal{N}(t), j \in \mathcal{I}}) \), \( m = 0, 1, \ldots, M \) and sends them to the server.
6: the client with the highest computation speed (or any client) computes \( q_{f,m,0}(\omega_0^f(t), (h_{m,i}(\omega_i^f(t), x_{n,i}))_{n \in \mathcal{N}(t), i \in \mathcal{I}}) \), \( m = 0, 1, \ldots, M \) and sends them to the server.
7: the server obtains \( (\omega_0^f(t+1), s(t)) \) by solving Problem 10 and updates \( \omega_i^f(t+1) \) according to (5).
8: end for
9: Output: \( \omega_T^f \)

s.t. \( F_{f,m}(\omega) \leq s_m, \quad m = 1, 2, \ldots, M, \quad s_m \geq 0, \quad m = 1, 2, \ldots, M. \)

At iteration \( t \), we choose \( \hat{F}_{f,m}(\omega) \) given in (20) as an approximation function of \( F_{f,m}(\omega) \) and choose:

\[
\hat{F}_{f,m}(\omega) = (1 - \rho(t)) \hat{F}_{f,m}(-1) + \rho(t) \frac{1}{|\mathcal{N}(t)|} \sum_{n \in \mathcal{N}(t)} F_{f,m}(\omega, \omega_f, x_n), \quad m = 1, \ldots, M. \tag{20}
\]

with \( \hat{F}_{f,m}(\omega) = 0 \) as an approximation function of \( F_{f,m}(\omega) \), for all \( m = 1, \ldots, M, \) where \( \rho(t) \) is a stepsize satisfying (4). \( \mathcal{N}(t) \) is a randomly selected mini-batch by the server at iteration \( t \), and \( f_{f,m}(\omega, \omega_f, x_n) \) is a convex approximation of \( f_{f,m}(\omega, x_n) \) around \( \omega \) satisfying \( f_{s,m}(\omega, x) = f_{s,m}(\omega, x) \) and Assumption 2 for all \( m = 1, \ldots, M \). A common example of \( f_{f,m}, m = 1, \ldots, M \) will be given later.

Note that for any mini-batch \( \mathcal{N}' \subseteq \mathcal{N} \) with batch size \( B \leq N \), \( \sum_{n \in \mathcal{N}'} f_{f,m}(\omega, \omega_f, x_n) \) can be written as (21), as shown at the top of the next page, with \( p_{f,m} : \mathbb{R}^{1 | \mathcal{E}| + d} \to \mathbb{R} \) and \( q_{f,m,i} : \mathbb{R}^{B(K_i + 1) + d_0 + d_i} \to \mathbb{R}^{|\mathcal{E}|_i}, i \in \mathcal{I} \). Assume that the expressions of \( f_{f,m}, p_{f,m}, q_{f,m,i}, h_{m,i}, m = 0, \ldots, M, i \in \mathcal{I} \) are known to the server and \( N \) clients. Each client \( i \in \mathcal{I} \) computes \( h_{m,i}(\omega_i^f(t), x_{n,i}), n \in \mathcal{N}(t), m = 0, \ldots, M \) and sends them to the other clients. Based on \( h_{m,i}(\omega_i^f(t), x_{n,i}), m = 0, \ldots, M, n \in \mathcal{N}(t), i \in \mathcal{I} \), each client \( i \in \mathcal{I} \) computes \( q_{f,m,i}(\omega_0^f(t), \omega_1^f(t), (x_{n,i}, h_{m,j}(\omega_j^f(t), x_{n,j}))_{n \in \mathcal{N}(t), j \in \mathcal{I}}), m = 0, \ldots, M \) and sends them to the server. Moreover, the client with the highest computation speed (or any client) computes \( q_{f,m,0}(\omega_0^f(t), (h_{m,i}(\omega_i^f(t), x_{n,i}))_{n \in \mathcal{N}(t), i \in \mathcal{I}}), m = 0, \ldots, M \) and sends them to the server. Then, the server solves the following convex approximate problem to obtain \( \omega_0^f(t) \).

Problem 10 (Convex Approximate Problem of Problem 9):

\[
(\omega_0^f(t), s(t)) = \arg \min_{\omega, s} F_{f,m}(\omega) + c \sum_{m=1}^{M} s_m
\]

s.t. \( F_{f,m}(\omega) \leq s_m, \quad m = 1, 2, \ldots, M, \quad s_m \geq 0, \quad m = 1, 2, \ldots, M. \)

Problem 10 is convex and can be readily solved. Given \( \omega_0^f(t) \), the server updates \( \omega_0^f(t) \) according to (5). The detailed procedure is summarized in Algorithm 4 and the convergence of Algorithm 4 is summarized below. Consider a sequence \( \{c_j\} \).

Theorem 4 (Convergence of Algorithm 4): Suppose that \( f_{f,m} \) satisfies Assumption 1 for all \( m = 0, \ldots, M \), \( f_{f,0} \) satisfies Assumptions 2 and 4 for all \( m = 1, \ldots, M \), the constraint set of Problem 3 is compact, and the sequence \( \{c_j\} \) satisfies \( 0 < c_j < c_{j+1} \) and \( \lim_{j \to \infty} c_j = \infty \). Then, the following statements hold. i) For all \( j \), if \( s_{\omega,\infty}^* = 0 \), then \( \omega_{\omega,\infty}^* \) is a KKT point of Problem 8 almost surely; ii) A limit point of \( \{s_{\omega,\infty}^*, \omega_{\omega,\infty}^*\} \), denoted by \( \{s_{\omega,\infty}^*, \omega_{\omega,\infty}^*\} \), satisfies that \( s_{\omega,\infty}^* = 0 \), and \( \omega_{\omega,\infty}^* \) is a KKT point of Problem 8 almost surely.

Proof: Please refer to Appendix B.
addition, for all $f$ that with $\bar{x}$ by (15) satisfies Assumption 2, and where $\tau > (15)$ and choose above are not satisfied, extra privacy mechanisms need to be preserve data privacy. If the two assumptions mentioned above are not satisfied, extra privacy mechanisms need to be investigated.

3) Algorithm Example: We provide an example of $\bar{f}$, $m = 0, \ldots, M$ with $\bar{f}, \bar{f}_0$ satisfying Assumption 2 and $\bar{f}, \bar{f}_0$ satisfying $\bar{f}, m^\tau, \omega, x = \bar{f}, m^\tau, \omega, x$ and Assumption 2 for all $m = 1, \ldots, M$. Specifically, we can choose $\bar{f}, \bar{f}_0$ given by [15] and choose $\bar{f}, m, m = 1, \ldots, M$ as follows:

$$\bar{f}, m^\tau, \omega, \omega, x = \bar{f}, m^\tau, \omega, x + (\nabla \bar{f}, m^\tau, \omega, x) T (\omega - \omega^\tau) + \tau ||\omega - \omega^\tau||^2_2, \quad m = 1, \ldots, M, \quad (22)$$

where $\tau > 0$ can be any constant. Obviously, $\bar{f}, \bar{f}_0$ given by [15] satisfies Assumption 2 and $\bar{f}, \bar{f}_0$ given by [22] satisfies $\bar{f}, m^\tau, \omega, x = \bar{f}, m^\tau, \omega, x$ and Assumption 2 for all $m = 1, \ldots, M$. Note that $\nabla \bar{f}, m^\tau, \omega, x$ can be computed according to the chain rule, similarly to [16] and [17]. Recall that with $\bar{f}, \bar{f}_0$ given in [15], $\bar{f}, m^\tau, \omega, x$ is given in [18]. In addition, for all $m = 1, \ldots, M$, substituting [22] into [20], $\bar{f}, \bar{f}_0$ can be rewritten as:

$$\bar{f}, m^\tau, \omega, \omega, x = \bar{f}, m^\tau, \omega, x + (\nabla \bar{f}, m^\tau, \omega, x) T (\omega - \omega^\tau) + \tau ||\omega - \omega^\tau||^2_2, \quad (23)$$

where $\bar{f}, m^\tau, \omega, x = 0$ and $\bar{f}, m^\tau, \omega, x = 0$. Analogously to Problem 5 with $\bar{f}, m^\tau, \omega, x$ given by [7] and $\bar{f}, m^\tau, \omega, x$ given by [11]. Problem 10 with $\bar{f}, m^\tau, \omega, x$ given by [15] and $\bar{f}, m^\tau, \omega, x$ given by [22] is a convex quadratically constrained quadratic programming and can be solved using an interior point method.

For $\bar{f}, m^\tau, \omega, x$ given by [22],

$$\left(\sum_{n \in \mathcal{N}(\bar{\omega})} \bar{f}, m^\tau, \omega, x, \sum_{n \in \mathcal{N}(\bar{\omega})} \nabla \bar{f}, m^\tau, \omega, x\right) \in \mathcal{E}$$

can be viewed as $\bar{f}, m^\tau, \omega, x, \bar{f}, m^\tau, \omega, x$ can be viewed as $\bar{f}, m, \bar{f}, m, \omega, x, \bar{f}, m, \omega, x$ and $\bar{f}, m, \bar{f}, m, \omega, x, \bar{f}, m, \omega, x$ can be viewed as $\bar{f}, m^\tau, \omega, x, \bar{f}, m^\tau, \omega, x$ and $\bar{f}, m^\tau, \omega, x, \bar{f}, m^\tau, \omega, x$ can be viewed as $\bar{f}, m, \bar{f}, m, \omega, x, \bar{f}, m, \omega, x$.

V. APPLICATION EXAMPLES

In this section, we customize the proposed algorithmic frameworks to some applications and provide detailed solutions for the specific problems. Define $\mathcal{K} \triangleq \{1, \ldots, K\}$, $\mathcal{J} \triangleq \{1, \ldots, J\}$, and $\mathcal{L} \triangleq \{1, \ldots, L\}$. Consider an $L$-class classification problem with a dataset of $N$ samples $(x_n, y_n) \in \mathcal{N}$, $y_n \triangleq (y_n, i) \in \mathcal{L}$ with $x_n, k \in \mathbb{R}$ and $y_n, l \in \{0, 1\}$. Consider a three-layer neural network, including an input layer composed of $K$ cells, a hidden layer composed of $J$ cells, and an output layer composed of $L$ cells. We use the swish activation function $S(z) = z/(1 + \exp(-z))$ [33] for the hidden layer and the softmax activation function for the output layer. We consider the cross entropy loss function. Thus, the resulting cost functions for sample-based and feature-based federated learning are given by:

$$F(\omega) \triangleq \frac{1}{N} \sum_{n \in \mathcal{N}} \sum_{L \in \mathcal{L}} \log (Q(\omega, x_n)),$$

with $\omega \triangleq (\omega_1, \ldots, \omega_J) \in \mathcal{K}$ and $y_n \triangleq (y_n, i) \in \mathcal{L}$

and

$$Q(l, \omega, x_n) \triangleq \exp(\sum_{j \in \mathcal{J}} \omega_{l, j, k} S(\sum_{k \in \mathcal{K}} \omega_{l, j, k} x_n)),$$

$$h = 1, \ldots, L.$$  

(26)

A. Unconstrained Federated Optimization

For $a = s, f$, one unconstrained federated optimization formulation for the $L$-class classification problem is to minimize
the weighted sum of the cost function $F(\omega)$ in (25) together with the $\ell_2$-norm regularization term $\|\omega\|_2^2$:
\[
\min_{\omega} \quad F_{a,0}(\omega) = F(\omega) + \lambda \|\omega\|_2^2 \tag{27}
\]
where $\lambda > 0$ is the regularization parameter that trades off the cost and model sparsity. Obviously, $F(\omega) + \lambda \|\omega\|_2^2$ satisfies the additional restrictions on the structure of $F_{f,0}(\omega)$. We can apply Algorithm 4 with $\tilde{f}_{a,0}(\omega, (\omega^{(f)}_t, x_n))$ given by (7) to solve the problem in (27) for $a = s$ and apply Algorithm 3 with $\tilde{f}_{f,0}(\omega, (\omega^{(f)}_t, x_n))$ given by (15) to solve the problem in (22) for $a = f$. Theorem 1 and Theorem 3 guarantee the convergences of Algorithm 4 and Algorithm 3 respectively, as Assumption 1 and Assumption 2 are satisfied.

First, we present the details of Step 4 in Algorithm 1 and the details of Steps 4-6 in Algorithm 3. In Step 4 of Algorithm 1, each client $i$ computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server. Here, $S(z) = \frac{1}{1 + \exp(-z)}(1 + \frac{1}{1 + \exp(-z)})$. In Steps 4-6 of Algorithm 3, each client $i$ computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server. The client with the highest computation speed (or any client) computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server.

Next, we present the details of Step 5 in Algorithm 1 and the details of Step 7 in Algorithm 3. For $a = s, f$, the server solves the following convex approximte problem:
\[
\min_{\omega} \quad F_{a,0}(\omega) = F_{a}(\omega) + 2\lambda(\beta(\omega)^T)^T \omega \tag{28}
\]
where $F_{a}(\omega)$ is given by
\[
F_{a}(\omega) = \sum_{j \in J} \sum_{k \in K} B_{a,j,k}^{(t)} \omega_{a,j,k} + \sum_{l \in L} \sum_{j \in J} C_{a,l,j}^{(t)} \omega_{a,l,j} + \tau \|\omega\|_2^2 \tag{29}
\]
and $\beta(\omega)$ is updated according to:
\[
\beta(t) = (1 - \rho(t))\beta(t-1) + \rho(t) \omega_a^{(t)}, \quad a = s, f,
\]
\[
B_{a,j,k}^{(t+1)} = (1 - \rho(t))B_{a,j,k}^{(t-1)} + \rho(t) (\tilde{B}_{a,j,k}^{(t)} - 2\tau \omega_{a,1,j,k}), \quad a = s, f, \quad j \in J, \quad k \in K, \tag{30}
\]
\[
C_{a,l,j}^{(t+1)} = (1 - \rho(t))C_{a,l,j}^{(t-1)} + \rho(t) (\tilde{C}_{a,l,j}^{(t)} - 2\tau \omega_{a,2,l,j}), \quad a = s, f, \quad l \in L, \quad j \in J, \tag{31}
\]
respectively, with $\beta(0) = 0$ and $B_{a,j,k}^{(0)} = C_{a,l,j}^{(0)} = 0$. Here, $\tilde{B}_{a,j,k}$ and $\tilde{C}_{a,l,j}$ are given by (33) and (34), as shown at the top of the next page. By (9) for $a = s$ and (19) for $a = f$, the closed-form solutions of the problem in (28) for $a = s, f$ are given by:
\[
\omega_{a,1,j,k}^{(t)} = -\frac{1}{2\tau} (B_{a,j,k}^{(t)} + 2\lambda \beta_{a,1,j,k}^{(t)}), \quad j \in J, \quad k \in K, \tag{35}
\]
\[
\omega_{a,2,l,j}^{(t)} = -\frac{1}{2\tau} (C_{a,l,j}^{(t)} + 2\lambda \beta_{a,2,l,j}^{(t)}), \quad l \in L, \quad j \in J. \tag{36}
\]
Thus, in Step 5 in Algorithm 1 and Step 7 in Algorithm 3 the server only needs to compute $\omega$ according to (35) and (36).

B. Constrained Federated Optimization

For $a = s, f$, one constrained federated optimization formulation for the $L$-class classification problem is to minimize the $\ell_2$-norm of the network parameters $\|\omega\|_2^2$ under a constraint on the cost function $F(\omega)$ in (25):
\[
\min_{\omega} \quad F_{a,0}(\omega) = \frac{1}{2}\|\omega\|_2^2 \tag{37}
\]
s.t. $F_{a,1}(\omega) = F(\omega) - U \leq 0,$
where $U$ represents the limit on the cost. We can apply Algorithm 2 with $\tilde{f}_{a,0}(\omega, (\omega^{(f)}_t, x_n))$ given by (7) and $f_{s,m}(\omega, (\omega^{(f)}_t, x_n))$ given by (11) to solve the problem in (37) for $a = s$ and apply Algorithm 4 with $\tilde{f}_{f,0}(\omega, (\omega^{(f)}_t, x_n))$ given by (15) and $f_{f,m}(\omega, (\omega^{(f)}_t, x_n))$ given by (22) to solve the problem in (37) for $a = f$. The convergences of Algorithm 2 and Algorithm 4 are guaranteed by Theorem 2 and Theorem 3 respectively, as Assumption 1 and Assumption 2 are satisfied.

First, we present the details of Step 4 in Algorithm 2 and the details of Steps 4-6 in Algorithm 3. In Step 4 of Algorithm 2, each client $i$ computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server. The client with the highest computation speed (or any client) computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server. In Steps 4-6 of Algorithm 3, each client $i$ computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server. The client with the highest computation speed (or any client) computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server. The client with the highest computation speed (or any client) computes $\sum_{n \in N^{(i)}} \sum_{l \in L} (Q_l(\omega^{(f)}_t, x_n) - y_{n,l})S(\sum_{k' = 1}^{K} \omega_{a,1,j,k'}^{(f)} x_{n,k'})$, $j \in J$, $l \in L$, and sends them to the server.
where \( \bar{U}_a(t) \) is given by (29) with \( B_{a,j,k}^{(t)} \), \( C_{a,l,j}^{(t)} \), and \( A_{a}^{(t)} \) updated according to (30), (31), and

\[
A_{a}^{(t)} = (1 - \rho^{(t)}) A_{a}^{(t-1)} + \rho^{(t)} \left[ \sum_{j \in J} \sum_{k \in K} B_{a,j,k}^{(t)} \omega_{a,1, j, k} - \sum_{l \in L} \sum_{j \in J} C_{a,l,j}^{(t)} \right],
\]

respectively, with \( A_{a}^{(0)} = 0 \) and \( \bar{U}_a(t) \) given by (32), as shown at the top of the next page. The KKT conditions, the closed-form solutions of the problem in (38) for \( a = s, f \) are given as follows.

**Lemma 1 (Optimal Solution of Problem in (38)):**

\[
\bar{U}_{a,1,j,k}^{(t)} = -\frac{\nu B_{a,j,k}^{(t)}}{2(1 + \nu \tau)}, \quad j \in J, \ k \in K,
\]

\[
\bar{U}_{a,2,l,j}^{(t)} = -\frac{\nu C_{a,l,j}^{(t)}}{2(1 + \nu \tau)}, \quad l \in L, \ j \in J,
\]

where

\[
\nu = \begin{cases} \frac{b}{c} & \text{if } \sqrt{b + 4\tau(U - A_{a}^{(t)})} - 1 \geq 0, b + 4\tau(U - A_{a}^{(t)}) > 0 \\ \frac{b}{c} & \text{if } \sqrt{b + 4\tau(U - A_{a}^{(t)})} - 1 < 0, b + 4\tau(U - A_{a}^{(t)}) \leq 0 \end{cases}
\]

\[
b = \sum_{j \in J} \sum_{k \in K} (B_{a,j,k}^{(t)})^2 + \sum_{l \in L} \sum_{j \in J} (C_{a,l,j}^{(t)})^2.
\]

Here, \( [x]_+^a \triangleq \min \{ \max \{x, 0\}, c \} \).

**Proof:** Please refer to Appendix C.

Thus, in Step 5 of Algorithm 2 and Step 7 of Algorithm 4 the server only needs to compute \( \omega \) according to (40) and (41).

**C. Comparisons of Two Formulations**

Both the unconstrained federated optimization formulation in (27) and constrained federated optimization formulation in (37) allow tradeoffs between the cost and model sparsity. The equivalence between the two formulations are summarized in the following theorem.

**Theorem 5 (Equivalence between Problems in (27) and (37)):** i) If \( \omega^* \) is a locally optimal solution of the problem in (27) with \( \lambda > 0 \), then there exists \( U \geq 0 \) such that \( \omega^* \) is a locally optimal solution of the problem in (37). ii) If \( \omega^1 \) is a locally optimal solution of the problem in (37) with \( U > 0 \), which is regular and satisfies the KKT conditions together with a corresponding Lagrange multiplier \( \xi > 0 \), then there exists \( \lambda > 0 \) such that \( \omega^1 \) is a stationary point of the problem in (27). Furthermore, if \( \lambda \) and \( \omega^1 \) satisfy \( \nabla^2 F(\omega^1) + \lambda I \succeq 0 \), then \( \omega^1 \) is a locally optimal solution.

**Proof:** Please refer to Appendix D.

By the above theorem, we know that the problem in (27) and the problem in (37) have the same locally optimal solution for certain \( \lambda \) and \( U \) under some conditions. Besides, we can tradeoff between the training accuracy and model sparsity of each formulation. It is evident that with constrained federated optimization formulation, one can set an explicit constraint on the training cost to control the test accuracy effectively.

**VI. NUMERICAL RESULTS**

In this section, we numerically compare the proposed algorithms with the federated learning algorithms via SGD [5], [9]–[18], using the application examples in Section V. We carry our experiments on Mnist dataset. For the training model, we set \( N = 60000 \), \( I = 10 \), \( K = 784 \), \( J = 128 \), and \( L = 10 \). For the proposed algorithms, we choose \( T = 100 \), \( \tau = 0.1 \), \( c = 10^5 \), \( \rho^{(t)} = a_1/t_1^{\alpha} \) and \( \gamma^{(t)} = a_2/t_1^{\alpha + 0.05} \) with \( a_1 = 0.4, 0.6, 0.9 \), \( a_2 = 0.4, 0.9, 0.9 \), and \( \alpha = 0.4, 0.3, 0.3 \) for batch sizes \( B = 1, 10, 100 \) in sample-based federated learning and \( B = 10, 100, 1000 \) in feature-based federated learning. For the federated learning algorithms via SGD [5], [12]–[18], the learning rate is set as \( r = \bar{a}/t^\alpha \), where \( \bar{a} \) and \( \alpha \) are selected using a grid search method. Moreover, let \( E \) denote the number of local SGD updates for the sample-based federated learning algorithms via SGD. Note that all the results are given by averaging over 100 runs.

Fig. 3 illustrates the training cost and test accuracy versus the iteration index in sample-based federated learning. From Fig. 4 we can see that the proposed algorithms with larger batch sizes converge faster. Fig. 2(a) and Fig. 2(c) show that for unconstrained federated optimization, Algorithm 1 converges faster than the SGD-based algorithm with \( E = 1 \) and the same batch size. Also, Algorithm 1 with \( B = 10 \) (100) converges faster than the SGD-based algorithm with \( B = 5 \) (50) and \( E = 2 \) at the same computation cost (for calculating the gradients) at clients.

Fig. 5 illustrates the training cost and test accuracy versus the iteration index in feature-based federated learning. Similarly, from Fig. 5 we can see that the proposed algorithms with...
strained sample-based and feature-based federated optimization problems using SSCA and mini-batch techniques. By comparing Fig. 2 and Fig. 3, we can observe that for unconstrained federated optimization, Algorithm 3 converges faster than the SGD-based algorithm at the same batch size. Fig. 4 shows the tradeoff curve between the model sparsity and training cost of each proposed algorithm. From Fig. 4(b), we see that with constrained sample-based federated optimization, one can set an explicit constraint on the training cost to control the test accuracy effectively.

**VII. CONCLUSIONS**

In this paper, we investigated sample-based and feature-based federated optimization, respectively, and considered both the unconstrained problem and constrained problem for each of them. We proposed federated learning algorithms that converge to KKT points of the respective federated optimization problems using SSCA and mini-batch techniques. We showed that data privacy could be preserved through the model aggregation mechanism and further enhanced via additional privacy mechanisms. Numerical experiments demonstrate that the proposed SSCA-based algorithms for unconstrained sample-based and feature-based federated optimization problems converge faster than the existing SGD-based algorithms, and the proposed SSCA-based algorithms for constrained sample-based and feature-based federated optimization problems obtain models that strictly satisfy non-convex constraints. To the best of our knowledge, this is the first work that provides an SSCA framework for federated optimization and highlights the value of constrained federated optimization. This paper opens up several directions for future research. An important direction is to design advanced SSCA-based federated learning algorithms that allow multiple local updates to reduce communication costs further. Another interesting direction is to design more privacy mechanisms for SSCA-based federated learning algorithms.

**APPENDIX A: PROOFS OF THEOREM 1 AND THEOREM 3**

The proofs of Theorem 1 and Theorem 3 are identical. In the following proof, we omit the subscript $s, f$ for notation simplicity. First, we introduce the following preliminary results.
Lemma 2: Let \( \{\omega(t)\} \) be the sequence generated by Algorithm 1 (Algorithm 3). Then, we have:

\[
\begin{align*}
\lim_{t \to \infty} \|\nabla F_0(\omega(t)) - \nabla F_0(\omega(t))\| &= 0, \\
\lim_{t \to \infty} |F_0(\omega) - G_0(\omega, \omega(t))| &= 0, \quad \forall \omega \in \mathbb{R}^d, \\
\end{align*}
\]

almost surely, where \( G_0(\omega, \omega(t)) \) is assumed to be bounded. Condition (b) of [35, Lemma 1] comes from the stepsize rules in (44) and (46). Condition (c)-(d) of [35, Lemma 1] come from the Lipschitz property of \( F_0(\omega) \) from Assumption 1 and the stepsize rules in (46).

**Proof:** From Assumption 2.3, \( F_0(\omega) \) is uniformly strongly convex, and thus

\[
\begin{align*}
\nabla^T \tilde{F}_0(\omega) \tilde{F}_0(\omega) - \tilde{F}_0(\omega) - \tilde{F}_0(\omega) &
\leq -\mu \|\omega(t) - \omega(t)\| + \tilde{F}_0(\omega) - \tilde{F}_0(\omega), \\
&\leq -\mu \|\omega(t) - \omega(t)\|, \\
\end{align*}
\]

where the last inequality follows from the optimality of \( \omega(t) \). Suppose \( \nabla F_0(\omega) \) is Lipschitz continuous with constant \( L > 0 \), we have:

\[
\begin{align*}
F_0(\omega(t+1)) - F_0(\omega(t)) \\
&\leq \omega(t+1) - \omega(t) \nabla F_0(\omega(t)) + \tilde{F}_0(\omega) - \tilde{F}_0(\omega) \\
&\leq \chi > 0 \text{ with a positive probability}. \\
\end{align*}
\]

Then, we show by contradiction that \( \lim \inf_{t \to \infty} \|\omega(t) - \omega(t)\| = 0 \) almost surely. Suppose \( \lim \inf_{t \to \infty} \|\omega(t) - \omega(t)\| \geq \chi > 0 \) with a positive probability. Then we can find a realization such that \( \|\omega(t) - \omega(t)\| \geq \chi > 0 \) for all \( t \). We focus next on such a realization. By \( \|\omega(t) - \omega(t)\| \geq \chi > 0 \) and Lemma 4 we have:

\[
\begin{align*}
F_0(\omega(t+1)) - F_0(\omega(t)) &
\leq -\chi(t) \left( \mu - \frac{\tilde{L}}{2} \chi(t) - \frac{1}{\chi} \|\nabla F_0(\omega(t)) - \nabla \tilde{F}_0(\omega(t))\| \right) \\
&\times \|\omega(t) - \omega(t)\|^2. \\
\end{align*}
\]

Since \( \lim_{t \to \infty} \|\nabla \tilde{F}_0(\omega(t)) - \nabla F_0(\omega(t))\| = 0, \lim_{t \to \infty} \gamma(t)
\]
and $\mu > 0$, there exists a $t_0$ sufficiently large such that
\[
\mu \frac{\hat{L}}{2} (t) - \frac{1}{\chi} \left\| \nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t)) \right\| \geq \bar{\mu}, \ \forall t \geq t_0,
\]
for some $\bar{\mu} \in (0, \mu)$. Therefore, it follows from (55), (55) and $\|\omega(t) - \omega(t)\| \geq \chi$ for all $t \geq t_0$ that
\[
F_0(\omega(t)) - F_0(\omega(n)) \leq -\bar{\mu} \chi^2 \sum_{n=t_0}^{(t)} \gamma(t),
\]
which, in view of $\sum_{n=t_0}^{\infty} \gamma(t) = \infty$, contradicts the boundedness of $\{F_0(\omega(t))\}$. Therefore, it must be $\lim \inf_{t \to \infty} \|\omega(t) - \omega(t)\| = 0$ almost surely.

Next, we show by contradiction that $\lim \sup_{t \to \infty} \|\omega(t) - \omega(t)\| = 0$ almost surely. Suppose $\lim \sup_{t \to \infty} \|\omega(t) - \omega(t)\| > 0$ with a positive probability. We focus next on a realization along with $\lim \sup_{t \to \infty} \|\omega(t) - \omega(t)\| > 0$, $\lim \inf_{t \to \infty} \|\omega(t) - \omega(t)\| = 0$, and $\lim_{t \to \infty} e(t_1, t_2) = 0$, where $e(t_1, t_2)$ is defined in Lemma 4. It follows from $\lim \sup_{t \to \infty} \|\omega(t) - \omega(t)\| > 0$ and $\lim \inf_{t \to \infty} \|\omega(t) - \omega(t)\| = 0$ that there exists a $\delta > 0$ such that $\|\Delta \omega(t)\| \geq 2\delta$ (with $\Delta \omega(t) = \omega(t) - \omega$) for infinitely many $t$ and also $\|\Delta \omega(t)\| \leq \delta$ for infinitely many $t$. Therefore, one can always find an infinite set of indices, say $T$, having the following properties: for any $t \in T$, we have
\[
\|\Delta \omega(t)\| \leq \delta,
\]
and there exists an integer $i_t > t$ such that
\[
\|\Delta \omega(i_t)\| \geq 2\delta, \quad \delta \leq \|\Delta \omega(n)\| \leq 2\delta, \quad t < n < i_t.
\]
Thus, for all $t \in T$, we have:
\[
\delta \leq \|\Delta \omega(i_t)\| - \|\Delta \omega(t)\| \leq \|\Delta \omega(i_t) - \Delta \omega(t)\|
\leq \|\omega(i_t) - \omega(t)\| + \|\omega(i_t) - \omega(t)\|
\leq (1 + \hat{L})\|\omega(i_t) - \omega(t)\| + e(i_t, t)
\leq (1 + \hat{L}) \sum_{n=t}^{i_t-1} \gamma(n) + e(i_t, t)
\leq 2\delta(1 + \hat{L}) \sum_{n=t}^{i_t-1} \gamma(n) + e(i_t, t),
\]
where (a) is due to Lemma 3 and (b) is due to $\delta_t$ and $\delta_t$. By $\delta_t$ and $\lim_{t \to \infty} e(i, t) = 0$, we have:
\[
\lim \inf_{T \ni t \to \infty} \sum_{n=t}^{i_t-1} \gamma(n) \geq \delta_t \geq \frac{1}{2(1 + \hat{L})} > 0.
\]
Proceeding as in (59), for all $t \in T$, we also have:
\[
\|\Delta \omega(t+1)\| - \|\Delta \omega(t)\| \leq \|\Delta \omega(t+1) - \Delta \omega(t)\|
\leq (1 + \hat{L})\|\Delta \omega(t)\| + c(t, t + 1),
\]
which leads to
\[
(1 + (1 + \hat{L})\gamma(t)) \|\Delta \omega(t)\| + c(t, t + 1) \geq \|\Delta \omega(t+1)\| \geq \delta,
\]
where the second inequality follows from (58). It follows from (62) and $\lim_{t \to \infty} c(t, t + 1) = 0$ that there exists a $\delta_2 > 0$ such that for a sufficiently large $t \in T$,
\[
\|\Delta \omega(t)\| \geq \delta - c(t, t + 1) \geq \delta_2 > 0.
\]
Here we assume w.l.o.g. that (63) holds for all $t \in T$ (in fact one can always restrict $\{\omega(t)\} \in T$ to a proper subsequence). We show now that (63) is in contradiction with the convergence of $\{F_0(\omega(t))\}$. By Lemma 4 for all $t \in T$, we have:
\[
F_0(\omega(t+1)) - F_0(\omega(t)) \leq -\gamma(t) \left(\mu - \frac{\hat{L}}{2} \gamma(t)\right) \|\omega(t) - \omega(t)\|^2
\]
\[
+ \gamma(t) \delta \left\| \nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t)) \right\|^2,
\]
and for $t < n < i_t$,
\[
F_0(\omega(n+1)) - F_0(\omega(n)) \leq -\gamma(n) \left(\mu - \frac{\hat{L}}{2} \gamma(n)\right) \|\omega(n) - \omega(n)\|^2
\]
\[
- \gamma(n) \delta \left\| \nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t)) \right\|^2,
\]
where the second inequality follows from (58). Adding (64) and (65) over $n = t + 1, \ldots, i_t - 1$ and, for $t \in T$ sufficiently large (so that $\mu - \frac{\hat{L}}{2} \gamma(t) - \delta^{-1} \left\| \nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t)) \right\| \geq \bar{\mu} > 0$ and $\|\nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t))\| < \bar{\mu} \delta_2 \delta^{-1}$), we have:
\[
F_0(\omega(i_t)) - F_0(\omega(t))
\leq -\gamma(n) \left(\mu - \frac{\hat{L}}{2} \gamma(n)\right) \|\omega(n) - \omega(n)\|^2 + \gamma(t) \delta \left\| \nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t)) \right\|^2
\leq -\mu \delta_2 \delta^{-1} \sum_{n=t}^{i_t-1} \gamma(n)
\leq -\mu \delta_2 \delta^{-1} \sum_{n=t}^{i_t-1} \gamma(n),
\]
where (a) follows from $\mu - \frac{\hat{L}}{2} \gamma(t) - \delta^{-1} \left\| \nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t)) \right\| \geq \bar{\mu} > 0$; (b) follows from (63); and (c) follows from $\|\nabla F_0(\omega(t)) - \nabla \hat{F}_0(t)(\omega(t))\| < \bar{\mu} \delta_2 \delta^{-1}$. Since $\{\omega(t)\}$ converges, it must be $\lim \inf_{T \ni t \to \infty} \sum_{n=t}^{i_t-1} \gamma(n) = 0$, which contradicts (66). Therefore, it must be $\lim \sup_{t \to \infty} \|\omega(t) - \omega(t)\| = 0$ almost surely.

Finally, we show that a limit point of the sequence $\{\omega(t)\}$ generated by Algorithm 1 (Algorithm 3), i.e., $\omega^*$, is a KKT
point of Problem 1 (Problem 6). It follows from first-order optimality condition for $\omega^{(1)}$ that
\[
(\omega - \omega^{(1)})^T \nabla F_0^{(1)}(\omega^{(1)}) \geq 0, \quad \forall \omega.
\] (67)
Taking the limit of (67) over the index set $T$, we have:
\[
\lim_{T \ni t \to \infty} (\omega - \omega^{(1)})^T \nabla F_0^{(1)}(\omega^{(1)}) = (\omega - \omega^*)^T \nabla F_0(\omega^*) \geq 0, \quad \forall \omega.
\] (68)
where the equality follows from $\lim_{t \to \infty} \|\omega^{(1)} - \omega(t)\| = 0$ (which is due to $\liminf_{t \to \infty} \|\omega^{(1)} - \omega(t)\| = 0$ and $\limsup_{t \to \infty} \|\omega^{(1)} - \omega(t)\| = 0$) and $\lim_{t \to \infty} \left| \nabla F_0(\omega^{(1)}) - \nabla F_0^{(1)}(\omega^{(1)}) \right| = 0$. This is the desired first-order optimality condition and $\omega^*$ is a KKT point of Problem 1 (Problem 6).

APPENDIX B: PROOFS OF THEOREM 2 AND THEOREM 4

The proofs of Theorem 2 and Theorem 4 are identical. In the following proof, we omit the subscript $s, f$ for notation simplicity. We first introduce the following preliminary results.

**Lemma 5:** Denote the constraint set of Problem 5 (Problem 3) by $\Omega$. Then, we have:
\[
\lim_{t \to \infty} \left| \frac{\partial^2 L}{\partial m \partial \omega}(\omega^{(1)}) - F_m(\omega^{(1)}) \right| = 0, \quad m = 1, \ldots, M,
\]
\[
\lim_{t \to \infty} \left| \nabla F_m(\omega(t)) - \nabla F_m(\omega(t)) \right| = 0, \quad m = 1, \ldots, M,
\]
\[
\lim_{t \to \infty} \left| \nabla F_m(\omega(t)) - G_m(\omega(t)) \right| = 0, \quad \forall \omega \in \Omega, m = 1, \ldots, M,
\]
almost surely, where $G_m(\omega, \omega^{(1)}) \triangleq \sum_{n \in N} g_m(\omega, \omega^{(1)}, x_n)$.

**Proof:** Lemma 5 is a consequence of Lemma 1. We just need to verify that all the technical conditions therein are satisfied. Specifically, Condition (a) of Lemma 1 is satisfied because $\Omega$ is assumed to be compact. Condition (b) of Lemma 1 comes from Assumption 2. Conditions (c)-(d) of Lemma 1 come from the stepsize rules in 4 and 6. Condition (e) of Lemma 1 comes from the Lipschitz property of $F(\omega)$ from Assumption 2 and the stepsize rule in 6.

For notation simplicity, we omit the index $a = s, f$ in the rest of the proof without loss of generality.

**Lemma 6:** Consider a subsequence $\{\omega^{(1)}_l\}_{l=1}^\infty$ generated by Algorithm 2 (Algorithm 3) with $c = c_j$ converging to a limit point $\omega_j^*$. There exist uniformly continuous functions $F_m(\omega)$, $m = 0, \ldots, M$ such that
\[
\lim_{t \to \infty} F_m^{(1)}(\omega_t) = F_m(\omega), \quad \forall \omega \in \Omega, m = 0, \ldots, M
\] almost surely. Moreover, we have:
\[
F_m^{(1)}(\omega_j^*) = F_m(\omega_j^*), \quad m = 1, \ldots, M,
\]
\[
\nabla F_m^{(1)}(\omega_j^*) = \nabla F_m(\omega_j^*), \quad m = 0, \ldots, M.
\] (71)

**Proof:** It readily follows from Assumption 2 that the families of functions $\{F_m^{(1)}(\omega)\}$ are equicontinuous. Moreover, they are bounded and defined over a compact set $\Omega$. Hence, the Arzela–Ascoli theorem implies that, by restricting to a subsequence, there exists uniformly continuous functions $\tilde{F}_m(\omega)$ such that (69) is satisfied. Finally, (70) and (71) follow immediately from (69) and Lemma 5.

By Assumption 1 Assumption 2 and Lemma 5, we can show $\lim_{t \to \infty} (\omega(t) - \omega(t)) = 0$. As the proof is similar to that in Appendix A, the details are omitted for conciseness. Next, we show that a limit point of the sequence $\{s^{(1)}_m(t)\}$ generated by Algorithm 2 (Algorithm 3) with $c = c_j$, i.e., $\{s^*_m(\omega_j^*)\}$, is a KKT point of Problem 4 (Problem 9). Consider the subsequence $\{\omega_j^{(1)}\}_{j=1}^\infty$ converging $\omega_j^*$. By $\lim_{t \to \infty} (\omega(t) - \omega(t)) = 0$ and $\lim_{t \to \infty} \omega(t) = \omega_j^*$, we have $\lim_{t \to \infty} (\omega(t) = \omega_j^*)$. Then, by $\lim_{t \to \infty} (\omega(t) = \omega_j^*)$, (69), and Problem 4 (Problem 9) we have:
\[
(\omega_j^*, s^*_j) = \arg \min \tilde{F}_0(\omega) + c \sum_{m=1}^M s_m
\] (72)

st. $\tilde{F}_m(\omega) \leq s_m, \quad m = 1, 2, \ldots, M.$

As $(\omega_j^*, s^*_j)$ satisfies the KKT conditions of the problem in (72) and (70) and (71) holds in Lemma 6, hold, $\{s^*_m(\omega_j^*)\}$ also satisfies together with the Lagrange multipliers $\lambda \triangleq \lambda_m m = 1, \ldots, M$ and $\mu \triangleq (\mu_m m = 1, \ldots, M),$ the KKT conditions of the KKT conditions of Problem 5 (Problem 9), i.e.,
\[
\nabla \omega \tilde{F}_0(\omega_j^*) + \sum_{m=1}^M \lambda_m \nabla \omega \tilde{F}_0(\omega_j^*) = 0,
\]
\[
F_m(\omega_j^*) \leq s^*_m, \quad m = 1, 2, \ldots, M,
\]
\[
\lambda_m(\tilde{F}_m(\omega_j^*) - s^*_m) = 0, \quad \mu_ms^*_m = 0, \quad m = 1, \ldots, M,
\]
\[
c - \lambda_m - \mu_m = 0, \quad m = 1, \ldots, M.
\]

In addition, it is obvious that (73) and $s^*_j = 0$ imply the KKT conditions of Problem 5 (Problem 9). Therefore, we can show the first statement.

Finally, it readily follows the proof of (73) that a limit point of $\{s^*_m(\omega_j^*)\}$, denoted by $\{s^*_m(\omega_j^*)\}$, satisfies that $s^*_m = 0$. Then it follows from the first statement that $\omega_j^*$ is a KKT point of Problem 5 (Problem 9) almost surely. Therefore, we can show the second statement.

APPENDIX C: PROOF OF LEMMA 1

As the problem in 35 is convex and the Slater’s condition holds, we can solve the problem in 35 by solving its dual problem. The Lagrange function of the problem in 35 is given by:
\[
\mathcal{L}(\omega, s, \nu, \mu) = \|\omega\|_2^2 + cs + \nu \left( \tilde{F}(\omega(t)) + A_0^t - U - s \right) + \mu(-s)
\]
\[
= \|\omega\|_2^2 + \nu \left( \tilde{F}(\omega(t)) + A_0^t - U \right) + (c - \nu - \mu)s,
\]
where $\nu$ and $\mu$ are the Lagrange multipliers. Thus, the Lagrange dual function is given by:
\[
g(\nu, \mu) = \inf_{\omega \geq 0} \mathcal{L}(\omega, s, \nu, \mu)
\]
\[
= \begin{cases}
\inf_{\omega \geq 0} \left( \|\omega\|_2^2 + \nu \left( \tilde{F}(\omega(t)) + A_0^t - U \right) \right), & c - \nu - \mu \geq 0, \\
-\infty, & c - \nu - \mu < 0.
\end{cases}
\]
As $L(\omega, s, \nu, \mu)$ is convex w.r.t. $\omega$, by taking its derivative and setting it to zero, we can obtain the optimal solution:

\[
\bar{\omega}^{(t)}_{a,1,j,k} = \frac{-\nu F_{a,j,k}^{(t)}}{2(1 + \nu \tau)}, \quad j \in \mathcal{J}, \ k \in \mathcal{K},
\]

\[
\bar{\omega}^{(t)}_{a,2,j,l} = \frac{-\nu C_{a,l,j}^{(t)}}{2(1 + \nu \tau)}, \quad l \in \mathcal{L}, \ j \in \mathcal{J},
\]

and the optimal value:

\[
h(\nu) = \nu \left( A^{(t)}_a - U - \frac{b \nu}{4(1 + \nu \tau)} \right),
\]

where $b$ is given in (42). Therefore, the dual problem of the problem in (38) is given by:

\[
\max_{\nu, \mu} \ h(\nu) \quad \text{s.t.} \quad c - \nu - \mu \geq 0, \quad \nu \geq 0, \quad \mu \geq 0,
\]

which is equivalent to the following problem:

\[
\nu^{*} = \arg \max_{\nu} \ h(\nu) \quad \text{s.t.} \quad 0 \leq \nu \leq c.
\]

As $h(\nu)$ is convex in $\nu$, and $h'(\nu) = b - (b + 4\nu(U - A^{(t)}_a))2(1 + \nu \tau)^2$, by the optimality conditions of problem in (72), we have:

\[
\nu^{*} = \begin{cases} 
\frac{b}{4} \left( \sqrt{\frac{b}{b + 4\nu(U - A^{(t)}_a)}} - 1 \right)^2, & b + 4\nu(U - A^{(t)}_a) > 0 \\
\frac{b}{c}, & b + 4\nu(U - A^{(t)}_a) \leq 0
\end{cases}, \quad \nu^{*} = \frac{b}{c}, \quad \nu^{*} = \frac{b}{4} \left( \sqrt{\frac{b}{b + 4\nu(U - A^{(t)}_a)}} - 1 \right)^2,
\]

which completes the proof.

**APPENDIX D: PROOF OF THEOREM 5**

As $\omega^{*}$ is a locally optimal solution of the problem in (27), there exists $\varepsilon > 0$ such that for all $\omega$ with $\|\omega - \omega^{*}\|_2 < \varepsilon$, we have:

\[
F(\omega^{*}) + \lambda \|\omega^{*}\|_2^2 \leq F(\omega) + \lambda \|\omega\|_2^2.
\]

Set $U = F(\omega^{*})$. Then, for all $\omega$ with $\|\omega - \omega^{*}\|_2 < \varepsilon$ and $F(\omega) \leq U$, we have:

\[
\|\omega\|_2^2 \leq \frac{1}{\lambda} (F(\omega) - F(\omega^{*})) + \|\omega^{*}\|_2^2 \leq \|\omega^{*}\|_2^2,
\]

where (a) is due to (75) and (b) is due to $F(\omega) \leq U = F(\omega^{*})$. Therefore, $\omega^{*}$ is a locally optimal solution of the problem in (27). The first statement holds.

As $\omega^{\dagger}$ is a locally optimal solution of the problem in (27), the following necessary KKT condition holds:

\[
\nabla \|\omega^{\dagger}\|_2^2 + \lambda \nabla F(\omega^{\dagger}) = 0.
\]

Set $\lambda = \frac{1}{\lambda}$. Then, we have $\nabla F(\omega^{\dagger}) + \lambda \nabla \|\omega^{\dagger}\|_2^2 = 0$. Therefore, $\omega^{\dagger}$ is a stationary point of the problem in (27). Moreover, if $\lambda$ and $\omega^{\dagger}$ satisfy $\nabla^2 F(\omega^{\dagger}) + \lambda I \geq 0$, i.e., the Hessian Matrix is semi-definite, then $\omega^{\dagger}$ is a locally optimal solution. The second statement holds.

**REFERENCES**

[1] C. Ye and Y. Cui, “Sample-based Federated Learning via Mini-batch SSSCA,” in *Proc. IEEE ICC*, 2021, pp. 1–6.

[2] M. Li, D. G. Andersen, A. J. Smola, and K. Yu, “Communication efficient distributed machine learning with the parameter server,” in *Advances in NIPS*, 2014, pp. 19–27.

[3] Q. Li, Z. Wen, and B. He, “Federated learning systems: Vision, hype and reality for data privacy and protection,” [Online]. Available: https://arxiv.org/abs/1907.09693

[4] T. Li, A. K. Sahu, A. Talwalkar, and V. Smith, “Federated learning: Challenges, methods, and future directions,” *IEEE Signal Process. Mag.*, vol. 37, no. 3, pp. 50–60, 2020.

[5] B. McMahan, E. Moore, D. Ramage, S. Hampson, and B. A. y Arcas, “Communication-efficient learning of deep networks from decentralized data,” in *AISTATS*, 2017, pp. 1273–1282.

[6] P. Mohassel and Y. Zhang, “SecureML: A system for scalable privacy-preserving machine learning,” in *Securit and Privacy*, 2017, pp. 19–38.

[7] L. T. Phong, Y. Aono, T. Hayashi, L. Wang, and S. Morais, “Privacy-preserving deep learning via additively homomorphic encryption,” *IEEE Trans. Inf. Forensics Security*, vol. 13, no. 5, pp. 1333–1345, 2018.

[8] S. Song, K. Chaudhuri, and A. D. Sarwate, “Stochastic gradient descent with differentially private updates,” in *Proc. IEEE GlobalSIP*, 2013, pp. 245–248.

[9] T. Li, A. K. Sahu, M. Zaheer, M. Sanjabi, A. Talwalkar, and V. Smith, “Federated optimization in heterogeneous networks,” 2020. [Online]. Available: https://arxiv.org/abs/1812.06127

[10] C. Xie, S. Koyeis, and P. Gupta, “Asynchronous federated optimization,” 2020. [Online]. Available: https://arxiv.org/abs/1903.03934

[11] T. Nishio and R. Yonetani, “Client selection for federated learning with heterogeneous resources in mobile edge,” in *Proc. IEEE ICC*, 2019, pp. 1–7.

[12] J. Konenčný, H. B. McMahan, D. Ramage, and P. Richtárik, “Federated optimization: Distributed machine learning for on-device intelligence,” 2016. [Online]. Available: https://arxiv.org/abs/1610.02527

[13] H. Yu, S. Yang, and S. Zhu, “Parallel restarted sgd with faster convergence and less communication: Demystifying why model averaging works for deep learning,” in *Proc. AAAI*, vol. 33, 2019, pp. 5693–5700.

[14] J. Konenčný, Z. Liu, T. Q. Quek, and H. V. Poor, “Scheduling policies for federated learning in wireless networks,” *IEEE Trans. Commun.*, vol. 68, no. 1, pp. 317–333, 2019.

[15] Q. Yang, Y. Liu, T. Chen, and Y. Tong, “Federated machine learning: Concept and applications,” *ACM Trans. Intell. Syst. Technol.*, vol. 10, no. 2, pp. 1–19, 2019.

[16] S. Hardy, W. Henecka, H. Ivey-Law, R. Nock, G. Patrini, G. Smith, and B. Thorne, “Private federated learning on vertically partitioned data via entity resolution and additively homomorphic encryption.” [Online]. Available: https://arxiv.org/abs/1911.05237

[17] S. Yang, B. Ren, X. Zhou, and L. Liu, “Parallel distributed logistic regression for vertical federated learning without third-party coordinator.” [Online]. Available: https://arxiv.org/abs/1911.09824

[18] T. Chen, X. Jin, Y. Sun, and W. Yin, “VFL: a method of vertical asynchronous federated learning.” [Online]. Available: https://arxiv.org/abs/2007.06081

[19] H. Robbins and S. Monro, “A stochastic approximation method,” *The Annals of Mathematical Statistics*, vol. 22, no. 3, pp. 400–407, 1951.

[20] L. Bottou, F. E. Curtis, and J. Nocedal, “Optimization methods for large-scale machine learning.” *SIAM Review*, vol. 60, no. 2, pp. 223–311, 2018.

[21] Y. Yang, G. Scutari, D. P. Palomar, and M. Pesavento, “A parallel decomposition method for nonconvex stochastic multi-agent optimization problems,” *IEEE Trans. Signal Process.*, vol. 64, no. 11, pp. 2949–2964, 2016.

[22] A. Liu, V. K. Lau, and B. Kanann, “Stochastic successive convex approximation for non-convex constrained stochastic optimization,” *IEEE Trans. Signal Process.*, vol. 67, no. 16, pp. 4189–4203, 2019.

[23] C. Ye and Y. Cui, “Stochastic successive convex approximation for general stochastic optimization problems,” *IEEE Wireless Commun. Lett.*, vol. 9, no. 6, pp. 755–759, 2019.
[24] P. Di Lorenzo and S. Scardapane, “Parallel and distributed training of neural networks via successive convex approximation,” in Proc. IEEE MLSP Workshop, 2016, pp. 1–6.
[25] S. Scardapane and P. Di Lorenzo, “Stochastic training of neural networks via successive convex approximations,” IEEE Trans. Neural Netw. Learn. Syst., vol. 29, no. 10, pp. 4947–4956, 2018.
[26] A. Koppel, A. Mokhtari, and A. Ribeiro, “Parallel stochastic successive convex approximation method for large-scale dictionary learning,” in Proc. IEEE ICASSP, 2018, pp. 2771–2775.
[27] Y. Yang, Y. Yuan, A. Chatzimichailidis, R. J. van Sloun, L. Lei, and S. Chatzinotas, “Proxsgd: Training structured neural networks under regularization and constraints,” in Proc. ICLR, 2019.
[28] S. Truex, N. Baracaldo, A. Anwar, T. Steinke, H. Ludwig, R. Zhang, and Y. Zhou, “A hybrid approach to privacy-preserving federated learning,” in Proc. 12th ACM Workshop on AISec, 2019, p. 1–11.
[29] H. Chen, K. Laine, and P. Rindal, “Fast private set intersection from homomorphic encryption,” in Proc. ACM CCS, 2017, pp. 1243–1255.
[30] B. Pinkas, T. Schneider, and M. Zohner, “Scalable private set intersection based on ot extension,” ACM Trans. Priv. Secur., vol. 21, no. 2, pp. 1–35, 2018.
[31] K. Yang, T. Fan, T. Chen, Y. Shi, and Q. Yang, “A quasi-newton method based vertical federated learning framework for logistic regression,” [Online]. Available: https://arxiv.org/abs/1912.00513
[32] D. P. Bertsekas, W. Hager, and O. Mangasarian, Nonlinear programming. Athena Scientific Belmont, MA, 1998.
[33] P. Ramachandran, B. Zoph, and Q. V. Le, “Searching for activation functions.” [Online]. Available: https://arxiv.org/abs/1710.05941
[34] S. Foucart and H. Rauhut, “A mathematical introduction to compressive sensing,” Bull. Am. Math, vol. 54, pp. 151–165, 2017.
[35] A. Ruszczynski, “Feasible direction methods for stochastic programming problems,” Mathematical Programming, vol. 19, pp. 220–229, 1980.
[36] N. Dunford and J. T. Schwartz, Linear operators part I: general theory. Interscience Publishers New York, 1958, vol. 243.
[37] A. H. Phan, H. D. Tuan, H. H. Kha, and D. T. Ngo, “Nonsmooth optimization for efficient beamforming in cognitive radio multicast transmission,” IEEE Trans. Signal Process., vol. 60, no. 6, pp. 2941–2951, 2012.