Gradient estimate and log-Harnack inequality for reflected SPDEs driven by multiplicative noises

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Abstract

In this paper, the gradient estimate and the log-Harnack inequality for the stochastic partial differential equation with reflection and driven by multiplicative noises are mainly investigated. To our purposes, the penalization method and the comparison principle for stochastic partial differential equations are adopted. We also apply the log-Harnack inequality to the study of the strong Feller property, uniqueness of invariant measures, the entropy-cost inequality, and some estimates of the transition density relative to its invariant measure.

Keywords: Gradient estimate, log-Harnack inequality, SPDEs with reflection, entropy-cost inequality

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1 Introduction

Consider the following reflected stochastic partial differential equation (SPDE) on the bounded interval $[0,1]$ driven by a space-time white noise:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t,x) + b(u(t,x)) \\
& \quad + \sigma(u(t,x))\dot{W}(t,x) + \eta(dtdx), \quad t > 0, \quad x \in (0,1), \\
& \quad u(t,0) = u(t,1) = 0, \quad t \geq 0, \\
& \quad u(0,x) = h(x), \quad x \in [0,1], \\
& \quad u(t,x) \geq 0, \quad t > 0, \quad x \in [0,1] \text{ a.s.,}
\end{align*}
\]

(1.1)

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where the initial value $h$ is a non-negative, continuous function defined on the interval $[0, 1]$ with $h(0) = h(1) = 0$, the coefficients $b$ and $\sigma$ are measurable real-valued functions defined on $\mathbb{R}$, $\{\dot{W}(t, x) : t \geq 0, x \in [0, 1]\}$ denotes the space-time white noise on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$, and $\eta$ denotes a positive random measure on $[0, \infty) \times [0, 1]$. In this paper, without loss of generalization, we consider the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by $\dot{W}(t, x)$, that is, $\mathcal{F}_t := \sigma(W(s, x) : s \in [0, t], x \in [0, 1]) \lor \mathcal{N}$, where $\mathcal{N}$ denotes the family of $\mathbb{P}$-null sets.

The problem (1.1) is one kind of random parabolic obstacle problems and is also regarded as an infinite-dimensional Skorokhod problem. It is initially proposed by D. Nualart and E. Pardoux [15] in 1992 for additive noises (i.e. $\sigma \equiv 1$) and then it is generalized to the case of multiplicative noises by C. Donati-Martin and E. Pardoux [9] in 1993. About nine years later, T. Funaki and S. Olla [10] in 2001 proved that the fluctuation of a $\nabla \phi$-interface model on a hard wall weakly converges to the stationary solution of the reflected SPDE driven by additive noises, which is known as a famous and interesting application of reflected SPDEs. In addition, various important properties, such as reversible measures [10, 29] and hitting properties [4], of the solution to reflected SPDE (1.1) driven by additive noises were studied.

However, for the general case of the reflected SPDE (1.1), the study of it becomes more difficult. In fact, even for the uniqueness of its solution is left open in [9] for very long time and at last it is successfully solved in [27] until 2009. After that, the strong Feller property and the large deviation principle of the Freidlin-Wentzell type relative to the solution were also investigated, see [27, 31] and references therein. For more information on the research of random obstacle problems, we also like to refer the readers to the monograph [30] authored by L. Zambotti in 2017 and references therein.

The main purpose of this paper is to investigate the log-Harnarck inequality, a weak type of the dimension-free Harnack inequality, with respect to the Markov semigroup generated by the solution of (1.1). The dimension-free Harnack inequality is initially introduced in [21] by F.-Y. Wang to study the log-Sobolev inequality of a diffusion process on Riemannian manifolds and then it becomes as a very powerful and effective tool to the study of various important properties of diffusion semigroups, such as, Li-Yau type heat kernel bound [2], hypercontractivity, ultracontractivity [18], strong Feller property, estimates on the heat kernels [22] and Varadhan type small time asymptotics [11, 31]. Generally speaking, the dimension-free Harnack inequality for the Markov semigroup $(P_t)_{t \geq 0}$ relative to a Markov process with values in a Banach space $E$ is formulated as

$$
\psi(P_t \Phi(x)) \leq (P_t(\psi(\Phi))(y)) \exp\{\Psi(t, x, y)\}, \ x, y \in E, 0 \leq \Phi \in B_b(E), t > 0, \ (1.2)
$$

where $\psi : [0, \infty) \to [0, \infty)$ is convex, $\Psi$ is a non-negative function defined on $[0, \infty) \times E \times E$ with $\Psi(t, x, x) = 0$, and $B_b(E)$ denotes the family of all measurable and bounded functions defined on $E$. In particular, for the special case $\psi(r) = r^p, p > 1$, (1.2) is reduced to

$$(P_t \Phi(x))^p \leq (P_t(\Phi^p)(y)) \exp\{\Psi(t, x, y)\}, \ x, y \in E, 0 \leq \Phi \in B_b(E), t > 0,$$
which is called the Harnack inequality with power $p$ and for $\psi(r) = e^r$, (1.2) is called the log-Harnack inequality, which is equivalent to

$$P_t \log \Phi(x) \leq \log P_t \Phi(y) + \Psi(t, x, y), \quad x, y \in E, 0 \leq \Phi \in B_0(E), t > 0.$$ 

Here, we would also like to refer the readers to the monograph [23] by F.-Y. Wang for more information of the dimension-free Harnack inequality.

Recently, such dimension-free Harnack inequalities have been actively studied and applied to the field of stochastic partial differential equations. For example, we refer the readers to [6, 12, 11, 22, 24, 31] for the study of the Harnack inequality with power and respectively to [19, 25, 26] for the study of the log-Harnack inequality. In particular, in [31], T.-S. Zheng studied the Harnack inequality with power relative to the solution of (1.1) driven by additive noises by the coupling method with the help of the Girsanov theorem and then applied it to the study of strong Feller property, hyperbound property and others.

However, most of the works listed above studied the SPDEs with additive (colored or white) noise and monotonic drifts. In this paper, we devote to the study of the dimension-free Harnack inequality relative to (1.1) with multiplicative noises and Lipschitz continuous coefficients. In this general case, it seems that the Harnack inequality with power is not available, see [19, 23] for more explanations. Hence, instead of the Harnack inequality with power, for this general case, we will investigate the log-Harnack inequality to the solution of (1.1) under the assumptions A1)-A3) below. The log-Harnack inequality relative to SPDEs driven by multiplicative noises is first studied in [19] for a special case of the coefficient $\sigma$ and then it is generalized in [26] to the general coefficient $\sigma$. All of these works deal with the SPDEs without reflection term $\eta$. It is valuable to point out that their results obtained in [19] and [26] can not be applied to reflected SPDEs (1.1), see Remark 4.1 for details. This is one of the main motivations of this paper. To investigate the log-Harnack inequality relative to the solution $u(t,x)$ of (1.1), the estimate of the Gradient estimate of the solution is studied by the approach of penalization and a comparison principle of SPDEs. We finally point out that SPDEs with two reflections, in particular, the existence and uniqueness of solutions, invariant measures and large deviation principles, are also actively studied, see for example, [16] for the case of additive noises and [28, 32, 33] for that of multiplicative noises. In fact, the method used in this paper can also be applied to the study of Harnack type inequalities for SPDEs with two reflections, which is now in progress [14].

Let us introduced some notations, which will be used throughout this paper. Let $H = L^2(0,1)$ and denote its scalar inner product and norm by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ respectively, that is

$$\langle h, \tilde{h} \rangle = \int_0^1 h(x)\tilde{h}(x)dx \quad \text{and} \quad |h| = \langle h, h \rangle^{\frac{1}{2}}, \quad h, \tilde{h} \in H.$$ 

Let $K_0 := \{ h \in H : h(x) \geq 0, x \in [0,1] \}$, the family of all non-negative functions $h \in H$. It is known that $K_0$ is the closed subset of $H$, and in particular it is Polish. Furthermore, we denote by $C_{0}^{k}(0,1)$ the class of all $k$-th continuously
differentiable functions \( h \) defined \([0, 1]\) with \( h(0) = h(1) = 0 \). For simplicity, we write \( C_0(0, 1) \) for \( C^0_0(0, 1) \).

Let us now formulate the definition of (1.1) according to the initial introduction in [9], see also [27] and [31].

**Definition 1.1.** Suppose the initial value \( h \in K_0 \cap C_0(0, 1) \). A pair \((u, \eta)\) is said to be a solution of the reflected SPDE (1.1) if the following are satisfied.

(i) The process \( \{u(t, x) : t \geq 0, x \in [0, 1]\} \) is non-negative and continuous on \([0, \infty) \times [0, 1]\). Moreover, \((u(t, \cdot))_{t \geq 0}\) is \( \mathcal{F}_t \)-adapted.

(ii) \( \eta \) is a positive random measure on \([0, \infty) \times [0, 1]\) such that
   (a) \( \eta([t \times (0, 1)) = 0, t \geq 0; \)
   (b) \( \int_0^t \int_0^1 x(1-x) \eta(dsdx) < \infty, t \geq 0; \)
   (c) \( \eta \) is \( \mathcal{F}_t \)-adapted, that is, for any Borel set \( B \subset [0, t] \times [0, 1] \), \( \eta(B) \) is \( \mathcal{F}_t \)-measurable.

(iii) For any \( \phi \in C^2_0(0, 1) \), the following stochastic integral equation is fulfilled.

\[
\langle u(t), \phi \rangle = \langle h, \phi \rangle + \frac{1}{2} \int_0^t \langle u(s), \phi'' \rangle ds + \int_0^t \langle b(u(s)), \phi \rangle ds
\]

\[
+ \int_0^t \int_0^1 \sigma(u(s, x)) \phi(x) W(dsdx) + \int_0^t \int_0^1 \phi(x) \eta(dsdx), \ t \geq 0 \ a.s.,
\]

where \( u(t) := u(t, \cdot) \) and the first term in second line is understood in the sense of Itô’s integral, see [20].

(iv) The support of \( \eta \) is a subset of \( \{(t, x) : u(t, x) = 0, t \geq 0, x \in [0, 1]\} \), that is,

\[
\int_0^\infty \int_0^1 u(t, x) \eta(dt dx) = 0 \ a.s.
\]

**Remark 1.1.** It is clear that (c) in (ii) is equivalent to the following:
the stochastic process \( \int_0^t \int_0^1 \phi(s, x) \eta(dsdx), t \geq 0 \) is \( \mathcal{F}_t \)-measurable for any measurable function \( \phi : [0, \infty) \times [0, 1] \rightarrow [0, \infty) \).

Based on the researches of [9] and [27], let us formulate the existence and uniqueness of the solution of (1.1) under the following assumptions on \( b \) and \( \sigma \).

**A1** \( b : \mathbb{R} \rightarrow \mathbb{R} \) is globally Lipschitz continuous, that is, there exists a constant \( L_b > 0 \) such that for any \( u, v \in \mathbb{R} \)

\[
|b(u) - b(v)| \leq L_b |u - v|.
\]

**A2** \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is globally Lipschitz continuous, that is, there exists a constant \( L_\sigma > 0 \) such that for any \( u, v \in \mathbb{R} \)

\[
|\sigma(u) - \sigma(v)| \leq L_\sigma |u - v|.
\]

To clarify the relations between our results and the Lipschitz constants of \( b \) and \( \sigma \) as below, we described the above assumptions separately.

Let us now formulate the result of the existence and uniqueness of the solution to the reflected SPDE (1.1).
Theorem 1.1. Suppose the assumptions A1) and A2) are satisfied. Then for each \( h \in K_0 \cap C_0(0, 1) \), the reflected SPDE (1.1) has a unique solution \((u, \eta)\) such that for any \( T > 0 \) and \( p \geq 1 \)

\[
E \left[ \sup_{(t, x) \in [0, T] \times [0, 1]} |u(t, x)|^p \right] < \infty.
\]

Remark 1.2. (i) We refer the readers to Theorem 3.1 [9] and Theorem 2.1 [27] for the proof of Theorem 1.1. For the case of multiplicative noises, the uniqueness of the solution is more difficult. In fact, although in 1993, C. Donati-Martin and E. Pardoux [9] succeeded in the construction a solution of (1.1) by the approach of penalization and showed that it is minimal, the uniqueness of the solution is left as an open problem. It is at last solved by T.-G. Xu and T.-S. Zhang [27] until 2009.

(ii) The Hölder continuity of \( u(t, x) \) is also studied in [5]. It is proved that \( u(t, x) \) is Hölder continuous with the index \( \left( \frac{1}{4} - \frac{1}{2} \right) \) on \([0, T] \times [0, 1]\), which is same as that of the solution to its penalized equation, see (4.1) below.

Throughout this paper, we will sometimes make use of the notations \( u(t) \), \( u(t, x; h) \) or \( u(t; h) \) instead of \( u(t, x) \) according to our purposes. For example, whenever we are going to emphasize its initial value \( h \) and regard the solution \( u(t, x) \) as an \( H \)-valued process, we will use the notation \( u(t; h) \).

This paper is organized as follows: In Section 2, the main results, including the gradient estimate and the log-Harnack inequality for the solution of (1.1), are formulated. In addition, some applications of the log-Harnack inequality are stated with brief proofs. From Section 3, we devote to the proofs of the main results. In Section 4 we give a brief proof of Proposition 2.1, which shows that the solution is continuous in its initial value. The proofs of the gradient estimate and the log-Harnack inequality relative to the solution are stated in Section 5 by the penalization method and a comparison theorem principle, and then as another application of the gradient estimate, the proof of Theorem 2.4 is described in last section.

2 Main Results

In this section, we first state the continuity of the solution \( u(t; h) \) on its initial data. After that, we formulate the main results of this paper including the gradient estimate and the log-Harnack inequality for the solution \( u(t; h) \) of (1.1) and finally state some applications of the log-Harnack inequality with their brief proofs.

Proposition 2.1. Suppose assumptions in Theorem 1.1 are satisfied and let the pair \((u(t; h_1), \eta_1)\) and respectively \((u(t; h_2), \eta_2)\) denote the unique solutions of (1.1) with initial data \( h_1 \) and \( h_2 \in K_0 \cap C_0(0, 1) \). Then for each \( p \geq 1 \) and \( T > 0 \), there exists a constant \( K = K(T, p) > 0 \) such that

\[
E \left[ \sup_{(t, x) \in [0, T] \times [0, 1]} |u(t, x; h_1) - u(t, x; h_2)|^p \right] \leq K |h_1 - h_2|_\infty^p,
\]

where \( |h|_\infty = \sup_{x \in [0, 1]} |h(x)| \) for \( h \in C_0(0, 1) \).
From now on, we will formulate the main theorems of this paper. Let us here introduce some notations. Let $B_b(H)$ (respectively $C_b(H)$) denote the class of all bounded and measurable real-valued functions (respectively bounded and continuous real-valued functions) defined on $H$. Similarly, we will use the notations $B_b(K_0)$ and $C_b(K_0)$. By considering the natural projection $\Pi$ from $H$ to $K_0$, we can identify $B_b(K_0)$ (respectively $C_b(K_0)$) with a subspace of $B_b(H)$ (respectively $C_b(H)$) by means of the injection: $B_b(K_0) \ni \Phi \mapsto \Phi(\Pi) \in B_b(H)$, see Section 2 \cite{29} for details.

Let us define $P_t$ by

$$P_t \Phi(h) = \mathbb{E}[\Phi(u(t; h))], \ t \geq 0, h \in B_b(K_0).$$

It is shown that $(P_t)_{t \geq 0}$ forms a Markov semigroup \cite{31}. To study the gradient estimate and the log-Harnack inequality relative to $P_t$, we will further suppose the following assumption A3).

A3) There exist two constants $0 < \kappa_1 < \kappa_2$, such that for $u \in \mathbb{R}$,

$$\kappa_1 \leq |\sigma(u)| \leq \kappa_2.$$ 

Throughout this paper, for any function $\Phi$ defined on $H$, we denote by $|\nabla \Phi|(h)$ the local Lipschitz constant of $\Phi$ at $h \in H$, i.e.,

$$|\nabla \Phi|(h) = \limsup_{|h - \tilde{h}| \to 0} \frac{|\Phi(\tilde{h}) - \Phi(h)|}{|h - \tilde{h}|}.$$ 

To state the main results, let us introduce the constant $M = M(L_b, L_\sigma) > 0$ and the function $\zeta(t), t \geq 0$ as below.

$$M = \max \left\{ 3, \frac{9L_\sigma^2}{\sqrt{\pi}}, \frac{8L_b^2}{L_\sigma^4}, \frac{144L_b^2}{L_\sigma^2 \sqrt{\pi}}, \frac{864L_b^2}{\sqrt{\pi}} \right\}, \quad (2.2)$$

and

$$\zeta(t) = t^{1/2} + \frac{9L_\sigma^4}{4} t + \frac{3L_b^2}{2} t^2 + \frac{18L_b^2 L_\sigma^2}{5 \sqrt{\pi}} t^{3/2}, \ t \geq 0. \quad (2.3)$$

It is important to see that the constant $M$ and the coefficients of $\zeta(t), t \geq 0$ depend only on $L_b$ and $L_\sigma$. The first main theorem is the gradient estimate of $P_t$, which plays a key role to the log-Harnack inequality.

**Theorem 2.2.** Suppose the assumptions A1)-A3) are satisfied. Then we have that for any $\Phi \in C_b^1(K_0)$,

$$|\nabla P_t \Phi|^2 \leq 2M \exp\{\zeta(t)\} P_t|\nabla \Phi|^2, \ t \geq 0. \quad (2.4)$$

According to the approaches used in \cite{19} and \cite{26}, the log-Harnack inequality of $P_t$ can be obtained based on Theorem 2.2.
Theorem 2.3. Under the assumptions of Theorem 2.2, the log-Harnack inequality holds for $P_t$, $t > 0$. More precisely, for any strictly positive $\Phi \in B_b(K_0)$,
\begin{equation}
\tag{2.5}
P_t \log \Phi(h_1) \leq \log P_t \Phi(h_2) + \frac{M|h_1 - h_2|^2}{\kappa_1^2 \int_0^t \exp\{-\zeta(s)\} ds}, \quad h_1, h_2 \in K_0, t > 0,
\end{equation}
where $\kappa_1$ is the constant in A3).

By Theorem 2.2, we can also obtain an estimate for $P_t \Phi^2 - (P_t \Phi)^2$ and the Lipschitz continuity of $P_t \Phi$ as below.

Theorem 2.4. Under the assumptions of Theorem 2.2, we have that
(i) For any $\Phi \in C^1_b(K_0)$,
\begin{equation}
\tag{2.6}
P_t \Phi^2 - (P_t \Phi)^2 \leq 2Mk_2^2 \int_0^t \exp\{\zeta(s)\} ds, \quad t \geq 0,
\end{equation}
where $\kappa_2$ is the constant in A3).
(ii) For any $\Phi \in B_b(K_0)$,
\begin{equation}
\tag{2.7}
|\nabla P_t \Phi|^2 \leq \frac{2M}{k_1^2 \int_0^t \exp\{-\zeta(s)\} ds} \{P_t \Phi^2 - (P_t \Phi)^2\}, \quad t > 0.
\end{equation}

In particular, $P_t \Phi$ is Lipschitz continuous for each $\Phi \in B_b(K_0)$ and $t > 0$.

From now on, let us formulate some applications of the log-Harnack inequality of $P_t$. Let us first study the entropy-cost inequality. To formulate it, let us introduce the definition of $L^2$-Wasserstein distance between two probability measures. For any two probability measures $\nu$ and $\mu$ on $K_0$, we denote by $W_2(\nu, \mu)$ the $L^2$-Wasserstein distance between them with respect to the cost function $(h_1, h_2) \to |h_1 - h_2|$, i.e.,
\begin{equation}
W_2(\nu, \mu) = \inf_{P \in \mathcal{C}(\nu, \mu)} \left\{ \int_{K_0 \times K_0} |h_1 - h_2|^2 P(dh_1, dh_2) \right\}^{\frac{1}{2}},
\end{equation}
where $\mathcal{C}(\nu, \mu)$ denotes the family of all couplings of $\nu$ and $\mu$. It is also called $L^2$-transportation cost between $\nu$ and $\mu$.

Under the assumptions A1)-A3), it follows from Theorem 2.1 [28] that the reflected SPDE (1.1) has an invariant probability measure $\pi$ on $K_0 \cap C_0(0, 1)$ and then we can regard $\pi$ as the invariant measure of $P_t$ on $K_0$.

As the application of Theorem 2.3, we have the following entropy-cost inequality for the adjoint operator of $P_t$ in $L^2(\pi)$.

Corollary 2.5. Let $\pi$ be one invariant measure of $P_t$ and let $P_t^*$ be the adjoint operator of $P_t$ in $L^2(\pi)$. Then for any non-negative $\Phi \in B_b(K_0)$ with $\pi(\Phi) = 1$, we have the following entropy-cost inequality:
\begin{equation}
\pi\left( (P_t^* \Phi) \log P_t^* \Phi \right) \leq \frac{MW_2(\Phi, \pi)^2}{\kappa_1^2 \int_0^t \exp\{-\zeta(s)\} ds}, \quad t > 0,
\end{equation}
where $\kappa_1$ is the constant in A3).
The relation between the log-Harnack inequality and the entropy-cost inequality is initially observed in Corollary 1.2 \cite{19} and then it is described for different frameworks \cite{25,26}. This corollary can be easily shown by analogy from Corollary 1.2 \cite{19}. Simply speaking, it is enough to apply (2.5) to $P^t_\pi \Phi$ in stead of $\Phi$ and then make use of the invariance of $\pi$. The details are omitted here.

In the end, let us apply the log-Harnack inequality to the study of the uniqueness of invariant measures of (1.1), and then some entropy inequalities relative to the transition density of the transition semigroup $P_t$ with respect to its invariant measure $\pi$.

**Corollary 2.6.** The Markov semigroup $(P_t)_{t \geq 0}$ is strong Feller, that is, for any $\Phi \in B_b(K_0)$,

$$\lim_{|h-h'| \to 0} P_t \Phi(h) = P_t \Phi(h), \ h \in K_0, \ t > 0,$$

and $P_t$ is irreducible, i.e., for any open set $\emptyset \neq A \subset K_0$, $P_t 1_A(h) > 0$ for all $h \in K_0$. In particular, the invariant measure of $P_t$ is unique and is fully supported on $K_0$, which will be still denoted by $\pi$ in the following.

Moreover, for each $t > 0$, $P_t$ is absolutely continuous with respect to its unique invariant measure $\pi$ with a strictly positive transition density $p_t(\hat{h}, h)$ and the following estimates relative to the transition density $p_t(\hat{h}, h)$ are satisfied.

(i) The log-Harnack inequality (2.5) is equivalent to the heat kernel entropy inequality

$$\int_{K_0} p_t(h_1, h) \log \frac{p_t(h_1, h)}{p_t(h_2, h)} \pi(\text{d}h) \leq \frac{M|h_1 - h_2|^2}{k^2 t^2} \int_0^t \exp\{-\zeta(s)\} \text{d}s, \ h_1, h_2 \in K_0, \ t > 0.$$

(ii) For any $h_1 \in K_0$,

$$\int_{K_0} p_t(h_1, h) \log p_t(h_1, h) \pi(\text{d}h) \leq -\log \int_{K_0} \exp \left\{ -\frac{M|h_1 - h|^2}{k^2 t^2} \int_0^t \exp\{-\zeta(s)\} \text{d}s \right\} \pi(\text{d}h), \ t > 0.$$

(iii) For any $h_1, h_2 \in K_0$,

$$\int_{K_0} p_t(h_1, h)p_t(h_2, h) \pi(h) \geq \exp \left\{ -\frac{M|h_1 - h_2|^2}{k^2 t^2} \int_0^t \exp\{-\zeta(s)\} \text{d}s \right\}, \ t > 0.$$

**Proof.** The above results are well-known as the applications of the log-Harnack inequality. Let us only state the proof of the irreducibility of $P_t$ and refer the readers to Theorem 1.4.2 (2) \cite{23} for the proof of (i) and for the others, to Theorem 1.4.1 \cite{23} for the abstract results and to \cite{19,25,26} for different frameworks. For irreducibility, it is enough to show that $P_t 1_A(h) > 0$, $t > 0$ with $A = A(\hat{h}) := \{h : |\hat{h} - h| < \gamma\}$ for any $\hat{h}$ and $\gamma > 0$. By the continuity of $u(t, x)$ on $[0, \infty) \times [0, 1]$, for each $\gamma > 0$, there exists $\hat{t} > 0$ such that $P_{\hat{t} 1_A(h)} = \mathbb{P}(u(\hat{t}; h) - h < \gamma) > \frac{1}{2}$. By contradiction, we assume there exists $h_0 \in K_0$ such that $P_{\hat{t} 1_A(h_0)} = 0$. Applying
to $1 + n1_A, n \in \mathbb{N}$, we have \( \log P_t(1 + n1_A)(h) = 0 \) and then for \( t < \tilde{t} \) and all \( n \in \mathbb{N} \),

\[
\frac{1}{2} \log(1 + n) \leq P_t \log(1 + n1_A(\tilde{h})) \leq \frac{M[\tilde{h} - h_0]^2}{\kappa_1^2 \int_0^t \exp\{-\zeta(s)\}ds},
\]

which is impossible, because of the boundedness of the last term. Therefore, for any \( t \leq \tilde{t} \), \( P_t \) is irreducible and then for all \( t > 0 \) by the Markov property of \( P_t \).

\[\square\]

Remark 2.1. The uniqueness of invariant measures of (1.1) is also established in [28] by using the coupling approach, which is initially introduced to the study of uniqueness of invariant measures of SPDEs by C. Mueller [13]. We would also like to refer the reader to [3] and references therein for another approaches. In this paper, the uniqueness of invariant measures is proved as the application of the log-Harnack inequality and some properties of the transition density are also studied.

3 Proof of Proposition 2.1

Let \( v(t, x) \) be a continuous function defined on \([0, \infty) \times [0, 1]\) such that \( v(0, x) = h(x) \in K_0 \cap C_0(0, 1) \) and \( v(t, 0) = v(t, 1) = 0, t \geq 0 \). Then by Theorem 4.1 [15] and its proof, we have the following lemma.

Lemma 3.1. (1) There exists a unique pair \((z, \eta)\) such that:

(i) \( z(t, x) \) is continuous on \([0, \infty) \times [0, 1]\) such that \( z(0, x) = 0, x \in [0, 1], z(t, 0) = z(t, 1) = 0 \) and \( z(t, x) + v(t, x) \geq 0 \).

(ii) \( \eta \) is a positive measure on \([0, \infty) \times [0, 1]\) with \( \eta([0, T] \times (\epsilon, 1 - \epsilon)) < \infty \) for all \( \epsilon \in (0, \frac{1}{2}) \) and each \( T > 0 \).

(iii) For any \( t \geq 0 \) and \( \phi \in C_0^2(0, 1) \),

\[
\langle z(t), \phi \rangle = \frac{1}{2} \int_0^t \langle z(s), \phi'' \rangle ds + \int_0^t \int_0^1 \phi(x)\eta(dxds).
\]

(iv) \( \int_0^\infty \int_0^1 (z(t, x) + v(t, x))\eta(dxdt) = 0 \).

(2) Let \( v_1(t, x) \) and \( v_2(t, x) \) be two functions which satisfy the same assumptions as those of \( v(t, x) \) in (1). If \((z_1, \eta_1)\) and \((z_2, \eta_2)\) are the unique pairs determined in (1) with respect to \( v_1(t, x) \) and respectively \( v_2(t, x) \), then for each fixed \( T > 0 \)

\[
|z_1 - z_2|_{T, \infty} \leq 2|v_1 - v_2|_{T, \infty}.
\]

Hereafter, \( |z|_{t, \infty} := \sup_{(s,x) \in [0,t] \times [0,1]} |z(s,x)|, t > 0 \) for a continuous function \( z(t, x) \) defined on \([0, \infty) \times [0, 1]\).

Using the above lemma, let us describe the proof of Proposition 2.1 briefly. Proof of Proposition 2.1: To show this proposition, it is enough to prove that there exists a \( p_0 > 1 \) such that for any \( p > p_0 \)

\[
\mathbb{E} \left[ |u(t; h_1) - u(t; h_2)|_{T, \infty}^p \right] \leq C|h_1 - h_2|^p_{\infty}
\]

(3.1)
holds for a constant $C = C(T, p) > 0$. In fact, if this is true, then (2.1) can be easily proved by Jensen’s inequality.

Let $v_i(t, x), i = 1, 2$ be defined by

$$v_i(t, x) = \int_0^1 p(t, x, y)h_i(y)dy + \int_0^t \int_0^1 p(t - s, x, y)b(u_i(s, y; h_i))dsdy$$

$$+ \int_0^t \int_0^1 p(t - s, x, y)\sigma(u_i(s, y; h_i))W(dsdy).$$

Hereafter, the function $p(t, x, y)$ denotes the fundamental solution of the linear part of (1.1). Then it is well-known that $v_i(t, x), i = 1, 2$ satisfy all of the assumptions in Lemma 3.1. Set

$$z_i(t, x) = u_i(t, x; h_i) - v_i(t, x), \ t \geq 0, x \in [0, 1], i = 1, 2.$$ 

From the proof of Theorem 2.1 [27], it follows that $(z_i, \eta_i)$ is the unique pair which satisfies all the conditions (i)-(iv) in Lemma 3.1 for $i = 1, 2$. Hence by Lemma 3.1, we have

$$|z_1 - z_2|_{T, \infty} \leq |v_1 - v_2|_{T, \infty}$$

and then

$$|u_1 - u_2|_{T, \infty} \leq 2|v_1 - v_2|_{T, \infty},$$

which in particular implies that

$$E[|u_1 - u_2|^p_{T, \infty}] \leq 2^pE[|v_1 - v_2|^p_{T, \infty}], \ p > 1. \quad (3.2)$$

From the definitions of $v_i, i = 1, 2$, it follows that for any $p > 1$

$$E\left[|v_1 - v_2|^p_{T, \infty}\right] \leq 3^{p-1}\sup_{(t, x) \in [0, \infty) \times [0, 1]} \left|\int_0^1 p(t, x, y)(h_1(y) - h_2(y))dy\right|^p$$

$$+ \left|\int_0^t \int_0^1 p(\cdot - s, \cdot, y)(b(u_1(s, y; h_1)) - b(u_2(s, y; h_2)))dsdy\right|^p_{T, \infty}$$

$$+ \left|\int_0^t \int_0^1 p(\cdot - s, \cdot, y)(\sigma(u_1(s, y; h_1)) - \sigma(u_2(s, y; h_2)))W(dsdy)\right|^p_{T, \infty}$$

$$= 3^{p-1}(I_1 + I_2(T) + I_3(T)).$$

By the property of $p(t, x, y)$, it is easy to know that

$$I_1 \leq |h_1 - h_2|^p_{\infty}, \ t \geq 0.$$
Using the ideas used in the proof of Theorem 2.1 [27], see also the proof of Theorem
3.1 [9], and by the assumptions A1)-A2), we have that for each \( p > 20 =: p_0 \), there
exists a constant \( C_1 = C_1(T, p) > 0 \) such that

\[
I_2(T) + I_3(T) \leq C_1(L_b^p + L_p^p) \int_0^T \mathbb{E}[|u_1(\cdot; h_1) - u_2(\cdot; h_2)|_{t, \infty}^p]dt.
\]

Combining the above estimates with (3.2), we have that there exists a constant
\( C_2 = C_2(T, p, L_b, L_\sigma) > 0 \) such that

\[
\mathbb{E}[|u_1 - u_2|_{T, \infty}^p] \leq C_2|h_1 - h_2|_{t, \infty}^p + C_2 \int_0^T \mathbb{E}[|u_1(\cdot; h_1) - u_2(\cdot; h_2)|_{t, \infty}^p]dt.
\]

Consequently, using Gronwall’s inequality, we can conclude the proof of (3.1), which
completes the proof.

\[\square\]

4 Proofs of Theorem 2.2 and Theorem 2.3

We will adopt the approach of penalization initially introduced in [9] for our pur-
pose. The main idea to prove Theorem 2.2 is enlightened by the paper [31] and the
comparison principle for SPDEs plays a very important role.

Let \( e_n(x) = \sqrt{2} \sin(n\pi x), x \in [0, 1] \). Then it is easy to see that

\[
d^2\frac{d}{dx^2} e_n(x) = -n^2\pi^2 e_n(x), \ n \in \mathbb{N}
\]

and \( \{e_n(x)\}_{n \in \mathbb{N}} \) forms a complete orthonormal basis of \( H \).

Set

\[
D(A) = \left\{ h \in H : \sum_{n=1}^{\infty} n^4 \langle h, e_n \rangle^2 < \infty \right\}
\]

and

\[
A_1 = -\frac{1}{2} \sum_{n=1}^{\infty} n^2\pi^2 \langle h, e_n \rangle e_n, \ h \in D(A).
\]

Then the operator \( A \) with its domain \( D(A) \) is the closure in \( H \) of \( \frac{d^2}{dx^2} \) on \( C_0^2(0, 1) \).

Remark that \( D(A) \) coincides with the Sobolev space \( H^2(0, 1) \cap H_0^1(0, 1) \).

Let

\[
B(u)(x) = b(u(x)), \ x \in [0, 1], u \in H,
\]

and

\[
\Sigma(u)[h](x) = \sigma(u(x))h(x), \ x \in [0, 1], u, h \in H.
\]

Then, A1) gives that \( B \) is a Lipschitz continuous mapping from \( H \) to \( H \). By A3),
we know that \( \Sigma : H \to \mathcal{L}(H) \), where \( \mathcal{L}(H) \) denotes the class of all bounded linear
operators on \( H \). However, we point out that A2) does not implies the Lipschitz
continuity of \( \Sigma \) on \( H \), which is different from \( B \). Define \( w_n(t) \) by

\[
w_n(t) = \int_0^t \int_0^1 e_n(x) W(dsdx), \ t \geq 0.
\]
It is easy to know that \( \{w_n(t), t \geq 0\}_{n \in \mathbb{N}} \) is a sequence of independent standard one-dimensional Brownian motions. Set
\[
W(t) = \sum_{n=1}^{\infty} w_n(t) e_n, \quad t \geq 0.
\]
Then, the stochastic process \( (W(t))_{t \geq 0} \) is a cylindrical Brownian motion on \( H \), and moreover we have that
\[
\int_0^t \int_0^1 \sigma(u(s, x)) W(dsdx) = \int_0^t \Sigma(u(s)) dW(s),
\]
where the stochastic integral in the right hand side is understood in the sense of Itô’s integral with respect to a cylindrical Brownian motion on \( H \), see [7, 8].

Let \( f: \mathbb{R} \to [0, \infty) \) be a non-increasing and Lipschitz continuous function such that
\[
f(u) \equiv 0, \quad u \geq 0 \quad \text{and} \quad f(u) > 0, \quad u < 0.
\]
For example, we can take \( f(u) = u^- := \max\{-u, 0\} \) or \( f(u) = \arctan((u \wedge 0)^2) \). In particular, \( f(u) = \arctan((u \wedge 0)^2) \) is twice continuously differentiable and bounded.

Let us consider the following penalized SPDE, which is used to construct the minimal solution of (1.1) in [9]. For each \( \epsilon > 0 \) and \( h \in H \), let us consider the SPDE
\[
\begin{aligned}
\frac{\partial u^\epsilon}{\partial t}(t, x) &= \frac{1}{2} \frac{\partial^2 u^\epsilon}{\partial x^2}(t, x) + b(u^\epsilon(t, x)) + \frac{1}{\epsilon} f(u^\epsilon(t, x)) \\
&\quad + \sigma(u^\epsilon(t, x)) W(t, x), \quad t > 0, \quad x \in (0, 1), \\
u^\epsilon(0, x) &= h(x), \quad x \in (0, 1).
\end{aligned}
\]
(4.1)

In the following to emphasize the coefficients \( b, f \) and \( \sigma \), we will sometimes denote this equation by SPDE\((b, f, \sigma)\). Under our assumptions, it is known that for each \( \epsilon > 0 \), (4.1) has a unique mild solution \( u^\epsilon(t, x; h) \), which forms a Markov process on \( H \).

Moreover, it is proved that \( u^\epsilon(t, x) \geq u^{\epsilon'}(t, x) \) a.s. for \( \epsilon < \epsilon' \) and for any \( p \geq 1 \),
\[
\lim_{\epsilon \downarrow 0} \mathbb{E} \left[ \left| u - u^\epsilon \right|^p_{T, \infty} \right] = 0, \quad T > 0,
\]
where \( u(t, x) \) denotes the solution of the reflected SPDE (1.1), see Theorem 4.1 [9] or Theorem 2.1 [27]. In particular, we have
\[
\lim_{\epsilon \downarrow 0} \sup_{t \in [0, T]} \mathbb{E} \left[ \left| u(t) - u^\epsilon(t) \right|^2 \right] = 0, \quad T > 0.
\]
(4.2)

Therefore, to prove Theorem 2.2 and Theorem 2.3, it is important to show similar results hold for \( P^\epsilon_t \Phi \), where
\[
P^\epsilon_t \Phi(h) = \mathbb{E} \left[ \Phi(u^\epsilon(t; h)) \right], \quad \Phi \in B_b(H).
\]
Introduce
\[ F(u)(x) = f(u(x)), \quad x \in [0, 1], u \in H. \]

Then it is easy to know that (4.1) can be written in its abstract form.
\[
\begin{cases}
  du'(t) = \left( Au'(t) + B(u'(t)) + \frac{1}{\epsilon} F(u'(t)) \right) dt + \Sigma(u'(t))dW(t), \\
  u'(0) = h.
\end{cases}
\] (4.3)

Let us now state the estimate of \(|\nabla P_{\epsilon} \Phi|^2\) for \(t \geq 0\) and \(\Phi \in C^1_b(H)\).

**Theorem 4.1.** Under the assumptions of A1)-A3), we have that for all \(\epsilon > 0\), the following holds.
\[
|\nabla P_{\epsilon} \Phi|^2 \leq 2M \exp\{\zeta(t)\} P_{\epsilon}^t|\nabla \Phi|^2, \quad \Phi \in C^1_b(H), t \geq 0,
\] (4.4)

where \(M > 0\) is the constant defined by (2.2) and \(\zeta(t), t \geq 0\) is the function defined by (2.3).

**Proof.** The proof will be divided into two steps.

**Step 1:** In this step, let us further assume that \(b\) and \(\sigma\) are differentiable such that
\[
\sup_{u \in \mathbb{R}} |b'(u)| \leq L_b \quad \text{and} \quad \sup_{u \in \mathbb{R}} |\sigma'(u)| \leq L_{\sigma}.
\] (4.5)

Combining the assumptions A1)-A3) with the assumptions on \(f\), we easily know that the operators \(B, F\) and \(\Sigma\) appeared in (4.3) satisfy all of the conditions in Theorem 5.4.1 (i) in [8]. Hence, we have that \(u'(\cdot; h)\) is continuously differentiable in its initial data \(h\). Let \(\nabla_k u'(t; h)\) denote the directional derivative of \(u'(t; h)\) along the direction \(k \in H\), that is,
\[
\nabla_k u'(t; h) = \lim_{\delta \downarrow 0} \frac{u'(t; h + \delta k) - u'(t; h)}{\delta}.
\]

We claim that the following estimate holds for all \(\epsilon > 0\) and \(h \in H\):
\[
\mathbb{E}[|\nabla_k u'(t; h)|^2] \leq M|k|^2 \exp\{\zeta(t)\}, \quad t \geq 0,
\] (4.6)

where the constant \(M > 0\) and the function \(\zeta(t), t \geq 0\) are same as those stated in this theorem.

By Theorem 5.4.1 (i) in [8], we also know that its directional derivative \(\nabla_k u'(t; h)\) along \(k \in H\) is the unique solution of the SPDE
\[
\begin{cases}
  d\nabla_k u'(t; h) = \left( A\nabla_k u'(t; h) + DB(u'(t); h) \cdot \nabla_k u'(t; h) \right) dt \\
  \quad + \frac{1}{\epsilon} DF(u'(t); h) \cdot \nabla_k u'(t; h) dt \\
  \quad + D\Sigma(u'(t); h) \cdot \nabla_k u'(t; h) dW(t), \quad t \geq 0, \\
  \nabla_k u'(0; h) = k,
\end{cases}
\] (4.7)
where $DB, DF$ and $D\Sigma$ denote the Fréchet derivatives of $B, F$ and $\Sigma$. Clearly, \eqref{4.7} can be rewritten as
\[
\begin{cases}
\frac{\partial}{\partial t} \nabla_k u^\epsilon(t, x; h) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \nabla_k u^\epsilon(t, x; h) + b'(u^\epsilon(t, x; h)) \nabla_k u^\epsilon(t, x; h) \\
+ \frac{1}{\varepsilon} f'(u^\epsilon(t, x; h)) \nabla_k u^\epsilon(t, x; h) + \sigma'(u^\epsilon(t, x; h)) \nabla_k u^\epsilon(t, x; h) \dot{W}(t, x), \quad t > 0, x \in (0, 1),
\end{cases}
\tag{4.8}
\]
\[
\nabla_k u^\epsilon(t, 0; h) = \nabla_k u^\epsilon(1, 1; h) = 0, \quad t \geq 0,
\]
\[
\nabla_k u^\epsilon(0, x; h) = k(x), \quad x \in [0, 1].
\]

Let us first assume $k$ is non-negative, i.e., $k \in K_0$. By the linearity of \eqref{4.8}, we know that if $k(x) \equiv 0, x \in [0, 1]$, then $\nabla_k u^\epsilon(t, x; h) \equiv 0$ is the unique solution of \eqref{4.8}. Hence, by using the comparison theorem of SPDEs, see Theorem 2.1 \cite{9} for example, we have the solution $\nabla_k u^\epsilon(t, x; h)$ of \eqref{4.8} is almost surely non-negative for any $k(x) \in K_0$, that is,
\[
\nabla_k u^\epsilon(t, x; h) \geq 0, \quad t \geq 0, x \in [0, 1] \ a.s.
\tag{4.9}
\]

On the other hand, by the non-increasing property of $f$, we have that
\[
f'(u) \leq 0, \quad u \in \mathbb{R}.
\]

Hence, by the non-negativity of the solution \eqref{4.8}, see \eqref{4.9}, and using the comparison theorem of SPDEs again, we have that
\[
0 \leq \nabla_k u^\epsilon(t, x; h) \leq v^\epsilon(t, x) \ a.s.,
\tag{4.10}
\]

where $v^\epsilon(t, x)$ denotes the unique solution of
\[
\begin{cases}
\frac{\partial v^\epsilon}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 v^\epsilon(t, x)}{\partial x^2} + b'(u^\epsilon(t, x; h)) v^\epsilon(t, x) \\
+ \sigma'(u^\epsilon(t, x; h)) v^\epsilon(t, x) \dot{W}(t, x), \quad t > 0, x \in (0, 1),
\end{cases}
\]
\[
v^\epsilon(t, 0; h) = v^\epsilon(1, 1; h), \quad t \geq 0,
\]
\[
v^\epsilon(0; h) = k(x), \quad x \in [0, 1].
\]

In the following, we use the mild solution of $v^\epsilon(t, x)$, that is,
\[
v^\epsilon(t, x) = \int_0^t p(t, x, y) k(y) dy + \int_0^t \int_0^1 p(t - s, x, y) b'(u^\epsilon(s, y; h)) v^\epsilon(s, y) dy ds dy
\tag{4.11}
\]
\[
+ \int_0^t \int_0^1 p(t - s, x, y) \sigma'(u^\epsilon(s, y; h)) v^\epsilon(s, y) W(dsy), \quad t \geq 0 \ a.s.
\]
\[
=: I_1(t) + I_2(t) + I_3(t).
\]

Firstly, the property of $p(t, x, y)$ gives easily that
\[
|I_1(t)| \leq e^{-\frac{c_1^2}{2}|k|}, \quad t \geq 0.
\]
Secondly, using Cauchy-Schwarz’s inequality and (4.5), we can easily obtain that

$$\mathbb{E}[|I_2(t)|^2] \leq L^2_\psi t \int_0^t e^{-\pi^2(t-s)} \mathbb{E}[|v^\epsilon(s)|^2]ds, \quad t \geq 0.$$  

Finally, Itô’s isometric property gives that

$$\mathbb{E}[|I_3(t)|^2] = \int_0^t \int_0^t \int_0^1 p^2(t - s, x, y) \mathbb{E}[|\sigma^\epsilon(u^\epsilon(s, y); h)v^\epsilon(s, y)|^2]dsdxdy$$

$$\leq L^2_\sigma \int_0^t \int_0^t \int_0^1 q^2(t - s, x - y) \mathbb{E}[|v^\epsilon(s, y)|^2]dxdudy$$

$$= L^2_\sigma \int_0^t \int_0^1 \frac{1}{2\sqrt{\pi}(t - s)} \mathbb{E}[|v^\epsilon(s, y)|^2]dsw$$

$$= \frac{L^2_\sigma}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t - s}} \mathbb{E}[|v^\epsilon(s)|^2]ds, \quad t \geq 0,$$  

where $q(t, x)$ denotes the density of the Gaussian distribution with mean 0 and variance $t$, the assumption (4.5) and $p(t, x, y) \leq q(t, x - y)$ have been used for the second line. Therefore, by the above estimates and (4.11), we obtain that

$$\mathbb{E}[|v^\epsilon(t)|^2] \leq 3|k|^2 + 3L^2_b t \int_0^t \mathbb{E}[|v^\epsilon(s)|^2]ds$$  

$$+ \frac{3L^2_\sigma}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t - s}} \mathbb{E}[|v^\epsilon(s)|^2]ds, \quad t \geq 0.$$  

Let $\psi^\epsilon(t) = \mathbb{E}[|v^\epsilon(t)|^2], \quad t \geq 0$. Using the relation (4.12), we have

$$\int_0^t \frac{1}{\sqrt{t - s}} \psi^\epsilon(s)ds \leq 6|k|^2t^{1/2} + 3L^2_b \int_0^t \int_0^s \frac{s}{\sqrt{t - s}} \psi^\epsilon(\theta)d\theta$$

$$+ \frac{3L^2_\sigma}{2\sqrt{\pi}} \int_0^t \int_0^s \frac{1}{\sqrt{t - s}\sqrt{s - \theta}} \psi^\epsilon(\theta)d\theta ds$$

$$\leq 6|k|^2t^{1/2} + \left\{ 6L^2_b t^{3/2} + \frac{3L^2_\sigma}{2\sqrt{\pi}} \right\} \int_0^t \psi^\epsilon(\theta)d\theta,$$  

where $\int_0^t \sqrt{t - s}ds = \pi$ has been used for the second inequality.

Inserting this estimate to the last part of (4.12), we see that

$$\psi^\epsilon(t) \leq \left\{ 3 + \frac{9L^2_\sigma t^{1/2}}{\sqrt{\pi}} \right\} |k|^2$$

$$+ \left\{ \frac{9L^4_\sigma}{4} + 3L^2_b t + \frac{9L^2_\sigma L^2_b t^{3/2}}{\sqrt{\pi}} \right\} \int_0^t \psi^\epsilon(s)ds$$

$$:= \alpha(t)|k|^2 + \beta(t) \int_0^t \psi^\epsilon(s)ds, \quad t \geq 0.$$
Since \( \alpha(t) \) and \( \beta(t) \) do not depend on \( \epsilon \). Owing to the generalized Gronwall inequality, see Lemma 4.2 below, we deduce that for all \( \epsilon > 0 \),

\[
\psi^\epsilon(t) \leq \alpha(t)|k|^2 + \beta(t) \int_0^t \alpha(s)|k|^2 \exp \left\{ \int_s^t \beta(\theta)d\theta \right\} ds, \quad t \geq 0.
\]

Since \( \alpha(t) \) is increasing in \( t \in [0, \infty) \) and \( \beta(t) \geq \frac{9L^4}{4} \) for all \( t \geq 0 \), we have

\[
\psi^\epsilon(t) \leq \alpha(t)|k|^2 + \alpha(t)\beta(t)|k|^2 \int_0^t \exp \left\{ \int_s^t \beta(\theta)d\theta \right\} ds
\]

\[
\leq \alpha(t)|k|^2 + \frac{4}{9L^4} \alpha(t)\beta(t)|k|^2 \int_0^t \beta(s) \exp \left\{ \int_s^t \beta(\theta)d\theta \right\} ds
\]

\[
= \alpha(t)|k|^2 + \frac{4}{9L^4} \alpha(t)\beta(t)|k|^2 \left\{ \exp \left\{ \int_0^t \beta(\theta)d\theta \right\} - 1 \right\}.
\]

Using the relation \( \beta(t) \geq \frac{9L^4}{4} \) for all \( t \geq 0 \) again, the above estimate gives that

\[
\psi^\epsilon(t) \leq \frac{4\alpha(t)\beta(t)}{9L^4}|k|^2 \exp \left\{ \int_0^t \beta(\theta)d\theta \right\}, \quad t \geq 0.
\]

(4.13)

By the definitions of \( \alpha(t) \) and \( \beta(t) \), it is easy to check that

\[
\frac{4\alpha(t)\beta(t)}{9L^4} \leq M \left\{ 1 + t^{1/2} + \frac{t^{3/2}}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} \right\} \leq M \exp\{t^2\}, \quad t \geq 0;
\]

recalling that \( M \) is the constant defined by (2.2).

Combining this estimate with (4.13), we obtain that

\[
\psi^\epsilon(t) \leq M|k|^2 \exp \left\{ t^{1/2} + \int_0^t \beta(s)ds \right\}, \quad t \geq 0.
\]

Therefore, by the definition of \( \beta(t) \) and recalling the definition of \( \zeta(t) \), see (2.3), we easily obtain the relation

\[
\mathbb{E}[|v^\epsilon(t)|^2] \leq M|k|^2 \exp\{\zeta(t)\}, \quad t \geq 0.
\]

As a consequence of the above estimate and the relation (4.10), we have for all \( h \in H \) and \( k \in K_0 \),

\[
\mathbb{E}[|\nabla_k u^\epsilon(t; h)|^2] \leq M|k|^2 \exp\{\zeta(t)\}, \quad t \geq 0.
\]

(4.14)

From now on, let us consider the general \( k \in H \). In this case, we have

\[
k(x) = k^+(x) - k^-(x), \quad x \in [0, 1],
\]

where \( k^+(x) = \max\{k(x), 0\} \) and \( k^-(x) = \max\{-k(x), 0\} \). On the other hand, we also have that

\[
\nabla_k u^\epsilon(t; h) = \nabla_{k^+} u^\epsilon(t; h) - \nabla_{k^-} u^\epsilon(t; h).
\]
Therefore, using (4.14), we have that
\[
\mathbb{E}[|\nabla_k u^\epsilon(t; h)|^2] \leq 2\{\mathbb{E}[|\nabla_{k^+} u^\epsilon(t; h)|^2] + \mathbb{E}[|\nabla_{k^-} u^\epsilon(t; h)|^2]\} \\
\leq 2M(|k^+|^2 + |k^-|^2) \exp\{\zeta(t)\} \\
= 2M|k|^2 \exp\{\zeta(t)\}, \ t \geq 0,
\]
which is the desired result (4.6).

**Step 2:** Let us formulate the proof of (4.4) under the assumptions of this theorem. To do that, let \( \phi \in C_0(\mathbb{R}) \) denote a symmetric and positive mollifier and set
\[
\phi_n(x) = n\phi(nx), \ n \in \mathbb{N}.
\]

Let us now construct approximating sequences \( b_n, f_n \) and \( \sigma_n, n \in \mathbb{N} \) for \( b, f \) and \( \sigma \) as below.
\[
b_n(u) = b \ast \phi_n(u), \ f_n(u) = f \ast \phi_n(u) \text{ and } \sigma_n(u) = \sigma \ast \phi_n(u), \ u \in \mathbb{R},
\]
where \( \ast \) denotes the convolution of two functions. From the properties of the mollifier, it is easy to know that \( b_n, f_n \) and \( \sigma_n \) are smooth, and as \( n \to \infty \), \( b_n, f_n \) and \( \sigma_n \) converge to \( b, f \) and \( \sigma \) respectively. In addition, by Rademacher’s Theorem, **A1** and **A2**), we have that
\[
\sup_{n \in \mathbb{N}, u \in \mathbb{R}} |b_n'(u)| \leq L_b \text{ and } \sup_{n \in \mathbb{N}, u \in \mathbb{R}} |\sigma_n'(u)| \leq L_\sigma. \quad (4.15)
\]
Moreover, by the assumptions of \( f \), we know that
\[
f_n'(u) \leq 0, \ u \in \mathbb{R}.
\]
Therefore, \( b_n, f_n \) and \( \sigma_n \) satisfy all of the assumptions used in **Step 1**. Let \( u^{\epsilon,n}(t; x; h) \) denote the solution of SPDE \( (b_n, f_n, \sigma_n) \) (4.13) and \( \nabla_k u^{\epsilon,n}(t; h) \) denote its directional derivative along \( k \in H \). By (4.14) and the estimate (4.16) established in **Step 1**, we have that for all \( \epsilon > 0, n \in \mathbb{N} \) and \( h \in H \), the following holds.
\[
\mathbb{E}[|\nabla_k u^{\epsilon,n}(t; h)|^2] \leq 2M|k|^2 \exp\{\zeta(t)\}, \ t \geq 0. \quad (4.16)
\]
Owing to (4.13) and the proof of (4.14), it is easy to know that \( M \) and \( \zeta(t), t \geq 0 \) are independent of \( \epsilon \) and \( n \) and are same as those in (4.16). Set
\[
P_t^{\epsilon,n} \Phi(h) = \mathbb{E}[\Phi(u^{\epsilon,n}(t; h))], \ \Phi \in B_b(H).
\]
By Cauchy-Schwarz’s inequality and (4.16), we have
\[
|\nabla_k P_t^{\epsilon,n} \Phi(h)|^2 = \mathbb{E}[(\nabla \Phi(u^{\epsilon,n}(t; h)), \nabla_k u^{\epsilon,n}(t; h))] \\
\leq P_t^{\epsilon,n}|\nabla_k \Phi|^2(h) \mathbb{E} \{ |\nabla_k u^{\epsilon,n}(t; h)|^2 \} \\
\leq 2M|k|^2 \exp\{\zeta(t)\} P_t^{\epsilon,n}|\nabla_k \Phi|^2(h).
\]
Noting now that
\[ |\nabla P_{\epsilon, n}^t \Phi(h)| = \sup \{ \nabla_k P_{\epsilon, n}^t(h) : k \in H \text{ with } |k| = 1 \}, \]
we easily obtain that
\[ |\nabla P_{\epsilon, n}^t \Phi|^2 \leq 2M \exp\{ \zeta(t) \} P_{\epsilon, n}^t|\nabla \Phi|^2, \Phi \in C^1_b(H). \tag{4.17} \]
On the other hand, we can easily show that
\[ \lim_{n \to \infty} \sup_{t \in [0,T]} E[|u^{\epsilon, n}(t; h) - u^\epsilon(t; h)|^2] = 0, \]
see, for example, Theorem 7.1 \[8\] or Theorem A.1 \[17\]. Consequently, we can obtain the desired result (4.4) by letting \( n \to \infty \) in (4.17).

Let us formulate the following generalized Gronwall inequality, which is used in the proof of Theorem 4.1 and state its brief proof for the reader’s convenience.

**Lemma 4.2.** Suppose functions \( \psi(t), \alpha(t), \beta(t) \) and \( \gamma(t) \) are non-negative and locally integrable on \([0, \infty)\). If the function \( \psi(t) \) satisfies the relation
\[ \psi(t) \leq \alpha(t) + \beta(t) \int_0^t \gamma(s)\psi(s)ds, \ t \geq 0, \tag{4.18} \]
then we have
\[ \psi(t) \leq \alpha(t) + \beta(t) \int_0^t \alpha(s)\gamma(s) \exp \left\{ \int_s^t \beta(\theta)\gamma(\theta)d\theta \right\} ds, \ t \geq 0. \]

**Proof.** The proof is very easy. In fact, let
\[ \phi(s) = \exp \left\{ - \int_0^s \beta(\theta)\gamma(\theta)d\theta \right\} \int_s^s \gamma(\theta)\psi(\theta)d\theta, \ s \geq 0. \]
Noting that
\[ \phi'(s) = \gamma(s) \exp \left\{ - \int_0^s \beta(\theta)\gamma(\theta)d\theta \right\} \left\{ \psi(s) - \beta(s) \int_0^s \gamma(\theta)\psi(\theta)d\theta \right\}, \]
by the relation (4.18), we easily know that
\[ \phi(t) \leq \int_0^t \alpha(s)\gamma(s) \exp \left\{ - \int_0^s \beta(\theta)\gamma(\theta)d\theta \right\} ds. \]
Hence, by the definition of \( \phi \) above, we have that
\[ \int_0^t \gamma(s)\psi(s)ds = \phi(t) \exp \left\{ \int_0^t \beta(\theta)\gamma(\theta)d\theta \right\} \leq \int_0^t \alpha(s)\gamma(s) \exp \left\{ \int_0^t \beta(\theta)\gamma(\theta)d\theta \right\} ds. \]
Consequently, inserting the above estimate into the right hand side of (4.18), we can easily obtain the desired relation. \( \square \)
Theorem 4.3. Suppose the assumptions A1-A3 are satisfied. Then, the log-Harnack inequality holds for $P_t^\epsilon$. More precisely, for any strictly positive $\Phi \in B_b(H)$,

$$P_t^\epsilon \log \Phi(h_1) \leq \log P_t^\epsilon \Phi(h_2) + \frac{M|h_1 - h_2|^2}{k_1^2 \int_0^t \exp\{-\zeta(s)\} ds}, \quad h_1, h_2 \in H, t > 0, \quad (4.19)$$

where $M$ and $\zeta(t), t \geq 0$ are same as those in Theorem 4.1.

Proof. As we saw in the proof of Theorem 4.1, without loss of generality, we assume $b, f$ and $\sigma$ are twice differentiable with bounded derivatives. Let us first claim that for each $\Phi \in C^2_b(H)$ and $\epsilon > 0$

$$P_s^\epsilon \log P_{t-s}^\epsilon \Phi(h) = \log P_t^\epsilon \Phi(h) - \frac{1}{2} \int_0^s P_\theta^\epsilon |\Sigma(h)\vartheta|D\log P_{t-\theta}^\epsilon \Phi(h)|^2 d\theta, \quad s \in [0, t]. \quad (4.20)$$

To prove this assertion, let $A_n$ be the Yosida approximation of $A$, that is, $A_n = nA_n(n - A)^{-1}$ and let $\Pi_n$ be the orthogonal projection from $H$ to $\text{span}\{e_1, e_2, \ldots, e_n\}, n \in \mathbb{N}$. Consider the following approximating stochastic partial differential equation for the penalized SPDE (4.3):

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{du_n^\epsilon(t)}{dt} = \left( A_n u_n^\epsilon(t) + B(u_n^\epsilon(t)) + \frac{1}{\epsilon} F(u_n^\epsilon(t)) \right) dt + \Sigma(u_n^\epsilon(t))\Pi_n dW(t), \\
u_n^\epsilon(0) = h.
\end{array} \right.
\end{align*} \quad (4.21)$$

It is known that for each $\epsilon > 0$ and $n \in \mathbb{N}$, this equation has a unique strong solution (see Section 7.4 [7]) and moreover,

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \mathbb{E} \left[ |u_n^\epsilon(t) - u^\epsilon(t)|^2 \right] = 0. \quad (4.22)$$

Moreover, the solution $(u_n^\epsilon(t))_{t \geq 0}$ of (4.21) can be represented by the stochastic equation

$$u_n^\epsilon(t) = h + \int_0^t \left( A_n u_n^\epsilon(s) + B(u_n^\epsilon(s)) + \frac{1}{\epsilon} F(u_n^\epsilon(s)) \right) ds + \int_0^t \Sigma(u_n^\epsilon(s))\Pi_n dW(s) \text{ a.s.}$$

Let us introduce the Markov semigroup

$$P_{n, t}^\epsilon \Phi(h) = \mathbb{E} [\Phi(u_n^\epsilon(t); h)], \quad \Phi \in B_b(H), t \geq 0.$$ 

Then, by Theorem 5.4.2 [8], for each $\Phi \in C_b^2(H)$, it is known that

$$P_{n, t}^\epsilon \Phi(h) \in C_b^{1, 2}([0, \infty) \times H).$$
Then, it is easy to know that $\alpha$ in Theorem 1.1 [19] and Proposition 3.2 [26]. Define $\alpha$ as the following.

Define the operator $\mathcal{L}$ on $C^2_c(H)$ by

$$
\mathcal{L}\Phi(h) = \left\langle A_n(h) + B(h) + \frac{1}{\epsilon} F(h), D\Phi(h) \right\rangle \\
+ \frac{1}{2} \text{Tr} \left[ D^2\Phi(h)(\Sigma(h)\Pi_n)(\Sigma(h)\Pi_n)^\ast \right], \ h \in H.
$$

Applying Itô’s formula (for example, see Theorem 4.32 [7]) to the solution $u_n^\epsilon(s)$ and log $P_{n,t-s}^\epsilon\Phi(h)$, $s \leq t$, we have that

$$
d\log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)) = \partial_s \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)) + \mathcal{L} \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)) \\
+ \left\langle D \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)), \Sigma(u_n^\epsilon(s))\Pi_n dW(s) \right\rangle \\
= -\frac{1}{2} \left\langle (\Sigma(u_n^\epsilon(s))\Pi_n)^\ast D \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)) \right\rangle^2 \\
+ \left\langle D \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)), \Sigma(u_n^\epsilon(s))\Pi_n dW(s) \right\rangle,
$$

where for the second equation, the relation

$$
\frac{\mathcal{L} \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s))}{\log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s))} = \frac{1}{2} \left\langle (\Sigma(u_n^\epsilon(s))\Pi_n)^\ast D \log P_{n,t-s}^\epsilon\Phi(u_n^\epsilon(s)) \right\rangle^2
$$

has been used. Noting the last term in (4.23) is a martingale, integrating of both sides of (4.23) from 0 to $s$ and then taking expectation, we have that

$$
\mathbb{E} \left[ \log P_{n,t-s}^\epsilon(u_n^\epsilon(s)) \right] \\
= \log P_{n,t}^\epsilon\Phi(h) - \frac{1}{2} \int_0^s \mathbb{E} \left[ \left\langle (\Sigma(u_n^\epsilon(\theta))\Pi_n)^\ast D \log P_{n,t-\theta}^\epsilon\Phi(u_n^\epsilon(\theta)) \right\rangle^2 \right] d\theta, \ s \in [0, t].
$$

Using the notation of the Markov semigroup $P_{n,t}^\epsilon$, we can rewrite the above equality as the following.

$$
P_{n,s}^\epsilon \log P_{n,t-s}^\epsilon(h) \\
= \log P_{n,t}^\epsilon\Phi(h) - \frac{1}{2} \int_0^t P_{n,\theta}^\epsilon(\Sigma(h)\Pi_n)^\ast D \log P_{n,t-\theta}^\epsilon\Phi(h) \left\rangle^2 d\theta, \ s \in [0, t].
$$

Recalling (4.22), we can conclude the proof of (4.20) by letting $n \to \infty$ in the above equation.

From now on, let us formulate the proof of (4.19) based on the main ideas used in Theorem 1.1 [19] and Proposition 3.2 [26]. Define $\alpha(s)$ by

$$
\alpha(s) = \frac{\int_0^s \exp\{-\zeta(\theta)\} d\theta}{\int_0^t \exp\{-\zeta(\theta)\} d\theta}, \ s \in [0, t].
$$

Then, it is easy to know that $\alpha(s) \in C^1([0, t])$ is increasing in $s$ and $\alpha(0) = 0, \alpha(t) = 1$. For any $h_1, h_2 \in H$, define $h(s) \in H$ by

$$
h(s) = h_1\alpha(s) + (1 - \alpha(s))h_2, \ s \in [0, t].
$$

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It is clear that \( h(0) = h_2 \) and \( h(t) = h_1 \). Using (4.20) and the relation
\[
P_t^e \log \Phi(h_1) = \log P_t^e \Phi(h_2) + \int_0^t \frac{d}{ds} \left\{ (P_s^e \log P_{t-s}^e \Phi)(h(s)) \right\} ds,
\]
we have that
\[
P_t^e \log \Phi(h_1) - \log P_t^e \Phi(h_2)
= -\frac{1}{2} \int_0^t P_s^e |\Sigma(s) D \log P_{t-s}^e \Phi|^2(h(s)) ds
+ \int_0^t \alpha'(s) \langle DP_s^e \log P_{t-s}^e \Phi(h(s)), h_1 - h_2 \rangle ds.
\]
Noting that the assumption \( 0 < \kappa_1 \leq |\sigma(u)| \), see the assumption \textbf{A3}), we have
\[
\kappa_1 \leq |\Sigma(h)| \text{ for all } h \in H.
\]
Therefore, Theorem 4.1, we obtain that
\[
P_t^e \log \Phi(h_1) - \log P_t^e \Phi(h_2)
\leq -\frac{\kappa_1^2}{4} \int_0^t \exp\{-\zeta(s)\} |\nabla P_s^e \log P_{t-s}^e \Phi|^2(h(s)) ds
+ \int_0^t \alpha'(s) \|h_1 - h_2\| |\nabla P_s^e \log P_{t-s}^e \Phi|(h(s)) ds
\leq \frac{M|h_1 - h_2|^2}{\kappa_1^2} \int_0^t \exp\{-\zeta(s)\}(\alpha'(s))^2 ds
= \frac{M|h_1 - h_2|^2}{\kappa_1^2} \int_0^t \exp\{-\zeta(s)\} ds,
\]
where the Young inequality \(|ab| \leq 2^{-1}\delta a^2 + (2\delta)^{-1}b^2, a, b \in \mathbb{R}, \delta > 0 \) and the definition of \( \alpha(s) \) have been used for the third inequality and the last equality respectively. Consequently, the proof of this theorem is completed.

Based on Theorem 4.1 and Theorem 4.3, we can easily describe the proofs of Theorem 2.2 and Theorem 2.3.

**Proof.** (Proofs of Theorem 2.2 and Theorem 2.3): Noting the relation (4.2) and letting \( \epsilon \downarrow 0 \) in (4.4), we can obtain (2.4) and then complete the proof of Theorem 2.2.

Let us turn to the proof of Theorem 2.3. By the relation (4.2) and letting \( \epsilon \downarrow 0 \) in (4.19), we can easily see that (2.5) holds for any Lipschitz continuous function \( \Phi \) and it can be extended for all \( \Phi \in B_0(K_0) \) by the monotone class theorem. Thus, the proof of Theorem 2.3 is completed.
Remark 4.1. Noting that $f$ is Lipschitz continuous with the Lipschitz constant 1, under the assumptions A1)-A3), we can easily testify that for any $h_1, h_2 \in H$ and $t > 0$, the following are fulfilled.

$$
\left| T_t \left( B(h_1) + \frac{1}{\epsilon} F(h_1) \right) - T_t \left( B(h_2) + \frac{1}{\epsilon} F(h_2) \right) \right|^2 \leq \left( L_b + \frac{1}{\epsilon} \right)^2 |h_1 - h_2|^2,
$$

$$
\| T_t \Sigma(h_1) - T_t \Sigma(h_1) \|_{HS}^2 \leq 2 L_a^2 |h_1 - h_2| \sum_{n=1}^{\infty} \exp\{-n^2 \pi^2 t\},
$$

where $T_t h(x) = \int_0^1 p(t, x, y) h(y) dy, h \in H$. Therefore, for each $\epsilon > 0$, according to Theorem 1.1 and Theorem 4.1 [26], we can easily obtain that

$$
|\nabla P_t^\epsilon \Phi|_2 \leq 6^{1+1/\epsilon} P_t^\epsilon |\nabla \Phi|_2, \Phi \in C^1_b(H) \quad (4.24)
$$

and the log-Harnack inequality holds for $P_t^\epsilon$, i.e., for all $0 \leq \Phi \in B_0(H)$,

$$
P_t^\epsilon \log \Phi(h_1) \leq \log P_t^\epsilon \Phi(h_2) + \frac{3 \log 6}{k_1 t_0(\epsilon) (1 - 6^{-t/t_0(\epsilon)})} |h_1 - h_2|^2, h_1, h_2 \in H, t > 0, \quad (4.25)
$$

where

$$
t_0(\epsilon) := \sup \left\{ t > 0 : \left( \frac{L_b + \frac{1}{\epsilon}}{\pi^2} \right) t (1 - \exp\{-\pi^2 t\}) + \frac{2 L_a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \exp\{-n^2 \pi^2 t\}}{n^2} \leq \frac{1}{6} \right\}.
$$

However, it is clear that $t_0(\epsilon) > 0$ converges to 0 as $\epsilon \to 0$. Consequently, we can not deduce our many results in Theorem 2.2 and Theorem 2.3 from (4.24) and (4.25).

5 Proof of Theorem 2.4

As in Section 4, let us first show the following results for the Markov semigroup $P_t^\epsilon$ relative to the solution of the penalized SPDE (4.1).

Theorem 5.1. Suppose the assumptions A1)-A3) are satisfied. Then, we have

(i) For any $\Phi \in C^1_b(H)$,

$$
P_t^\epsilon \Phi^2 - (P_t^\epsilon \Phi)^2 \leq 2 M k_1^2 P_t^\epsilon |\nabla \Phi|^2 \int_0^t \exp\{\zeta(s)\} ds, t > 0. \quad (5.1)
$$

(ii) For any $\Phi \in B_0(H)$,

$$
|\nabla P_t^\epsilon \Phi|^2 \leq \frac{2 M (P_t^\epsilon \Phi^2 - (P_t^\epsilon \Phi)^2)}{k_1^2 \int_0^t \exp\{-\zeta(s)\} ds}, t > 0. \quad (5.2)
$$

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Proof. The main idea to prove this theorem is similar to that used in Theorem 4.3. We assume that the coefficients $b, f$ and $\sigma$ are twice continuously differentiable with bounded derivatives and use the same notations introduced in the proof of Theorem 4.3. Considering the same approximating equation (4.21) as we did and then applying Itô’s formula, we have that

$$
d(P^\epsilon_n, \tau_{n,t-s})^2(u^\epsilon_n(s)) = \partial_s(P^\epsilon_n, \tau_{n,t-s})^2(u^\epsilon_n(s)) + \mathcal{L}(P^\epsilon_n, \tau_{n,t-s})^2(u^\epsilon_n(s))$$

$$+ \langle D(P^\epsilon_n, \tau_{n,t-s})^2(u^\epsilon_n(s)), \Sigma(u^\epsilon_n(s)) \Pi_n dW(s) \rangle$$

$$= |(\Sigma(u^\epsilon_n(s)) \Pi_n)^* D P^\epsilon_n, \tau_{n,t-s})^2(u^\epsilon_n(s))|^2$$

$$+ \langle D(P^\epsilon_n, \tau_{n,t-s})^2(u^\epsilon_n(s)), \Sigma(u^\epsilon_n(s)) \Pi_n dW(s) \rangle, \ s \in [0, t].$$

Then, integrating both sides of the above equation from 0 to $s \leq t$ and then taking expectation, we obtain that

$$P^\epsilon_n, \tau_{n,t-s}^2(h) = (P^\epsilon_n, \tau_t)^2(h) + \int_0^s P_n, \theta|\Sigma(\cdot) \Pi_n)^* D P^\epsilon_n, \tau_{n,t-s}^2(h)|^2(h) d\theta.$$  \hfill (5.3)

In particular, let $s = t$ in the above equation and then let $n \to \infty$, we have

$$P^\epsilon_t^2(h) = (P^\epsilon_t)^2(h) + \int_0^t P^\epsilon_s |\Sigma(\cdot)^* \nabla P^\epsilon_{t-s}^2(h)|^2(h) ds, \ t > 0.$$  \hfill (5.3)

Then we can easily complete our proofs. Let us first prove (i) by using this relation. In fact, since for all $u \in \mathbb{R}$, $|\sigma(u)| \leq \kappa_2$, from (5.3) and Theorem 4.1, it follows that

$$P^\epsilon_t^2(h) \leq (P^\epsilon_t)^2(h) + \kappa_2^2 \int_0^t P^\epsilon_s |\nabla P^\epsilon_{t-s}^2(h)|^2(h) ds$$

$$\leq (P^\epsilon_t)^2(h) + 2M \kappa_2^2 \int_0^t P^\epsilon_s P^\epsilon_{t-s}^2 h \exp \{\zeta(t-s)\} ds$$

$$= (P^\epsilon_t)^2(h) + 2M \kappa_2^2 \int_0^t \exp \{\zeta(s)\} ds, \ t > 0,$$

which completes the proof of (i).

Finally, let us briefly formulate the proof of (ii). By (4.4) and the assumption A3, we have that

$$P^\epsilon_s |\Sigma(\cdot)^* \nabla P^\epsilon_{t-s}^2(h)|^2(h) \geq \kappa_2^2 P^\epsilon_s |D P^\epsilon_{t-s}^2(h)|^2(h)$$

$$\geq \kappa_2^2 \frac{\exp \{-\zeta(s)\}}{2M} |\nabla P^\epsilon_t^2(h), \ s \in [0, t].$$

Inserting this into the right hand side of (5.3), we have

$$P^\epsilon_t^2(h) \geq (P^\epsilon_t)^2(h) + \frac{\kappa_2^2 a t \exp \{-\zeta(s)\}}{2M} |\nabla P^\epsilon_t^2(h),$$

which gives the desired result (5.2). \qed
In the end, let us conclude this paper with the proof of Theorem 2.4.

Proof. (Proof of Theorem 2.4): It is easy to show this theorem by using Theorem 5.1. In fact, as the proof of Theorem 2.3 by the monotone class theorem, it is enough to show (2.6) holds for any Lipschitz continuous function $\Phi$ on $K_0$. However, noting (4.2), they can be easily obtained by letting $\epsilon \downarrow 0$ in (5.1). On the other hand, (2.7) can be obtained by letting $\epsilon \downarrow 0$ in (5.2). Thus the proof of this theorem is completed.

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