Lagrangian description of cosmic fluids: mapping dark energy into unified dark energy

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We investigate the appropriateness of the use of different Lagrangians to describe various components of the cosmic energy budget, discussing the degeneracies between them, in the absence of non-minimal couplings to gravity or other fields, and clarifying some misconceptions in the literature. We then show that models with the same on-shell Lagrangian may have different proper energy densities and use this result to map dark energy models into unified dark energy models in which dark matter and dark energy are described by the same perfect fluid. We determine the correspondence between their equation of state parameters and sound speeds, briefly discussing the linear sound speed problem of unified dark energy models as well as a possible way out associated to the non-linear dynamics.

I. INTRODUCTION

The detection of a Higgs-like particle \(^{1, 2}\) reinforces the idea that scalar fields play a fundamental role in physics. In cosmology scalar fields are central to the primordial inflation paradigm \(^{3, 4}\) and potential candidates to explain the current accelerated expansion of the universe \(^{5, 12}\) or even cold dark matter \(^{13, 13}\) (see also \(^{16, 18}\) for recent reviews). More generally, scalar fields have also been proposed in the literature to unify primordial inflation and dark energy \(^{19}\) or to account for the entire dark sector (dark energy and dark matter) \(^{21, 29}\) (see also \(^{30, 32}\) for a unified description of primordial inflation, dark energy and dark matter).

It is well known that a minimally coupled scalar field in General Relativity admits a perfect fluid description \(^{33}\). Perfect fluids often provide a sufficiently general framework to model the source of the gravitational field. In particular, at cosmological scales (with homogeneity and isotropy being assumed) it is common to model the energy content of the Universe as a collection of perfect isentropic and irrotational fluids or, equivalently (under certain conditions, which we will explore in the present paper), as a collection of purely kinetic scalar fields \(^{34–36}\).

A number of action functionals, corresponding to at least three different on-shell Lagrangians \((\mathcal{L}_{\text{on-shell}} = -\rho, \; p \; \text{or} \; T)\), where \(\rho\), \(p\) and \(T\) represent, respectively, the proper density, the proper pressure and the trace of the energy-momentum tensor of the fluid) have been shown to define the dynamics of a perfect fluid \(^{37–43}\). Although some of these models may be used to describe the same physics in the context of General Relativity, in general this degeneracy is broken in the presence of a nonminimal coupling to gravity \(^{44–50}\) or to the other fields \(^{51, 56}\). For example, in \(f(R, \mathcal{L}_m)\) theories the Lagrangian of the matter fields appears explicitly in the equation of motion as a consequence of the nonminimal coupling between the geometry and the matter fields. Therefore, in these theories the identification of the correct form of the on-shell Lagrangian is essential in order to extract meaningful predictions \(^{46, 57–62}\).

Here, we shall explore the degeneracies between the energy-momentum tensor of a perfect fluid and the corresponding on-shell Lagrangian and using them to establish a correspondence between dark energy and unified dark energy models, clarifying some misconceptions in the literature. The outline of this paper is as follows. In Sec. \(\text{II}\) we start by considering four different models for a perfect fluid, discussing the degeneracies between them, in the absence of non-minimal couplings to gravity or other fields, and the appropriateness of the use of the corresponding Lagrangians to describe different components of the cosmic energy budget. In Sec. \(\text{III}\) we define a mapping between dark energy models described by purely kinetic Lagrangians and unified dark energy models. We also characterize the correspondence between their equation of state and sound speed parameters, briefly discussing the linear sound speed problem of unified dark energy models and a possible way out associated to the non-linear dynamics. Finally, we conclude in Sec. \(\text{IV}\).

Throughout this paper we use units such that \(c = k_B = 1\), where \(c\) is the value of the speed of light in vacuum and \(k_B\) is the Boltzmann constant. We also adopt the metric signature \((-\,+,+,+\,+)\). The Einstein summation convention will be used whenever a Greek index variable appears twice in a single term, once in an upper (super-script) and once in a lower (subscript) position.
II. PERFECT FLUID LAGRANGIAN DESCRIPTIONS

Consider a fluid characterized by the following intensive variables, defined in the local comoving inertial frame: the proper particle number density \( n \), energy density \( \rho \), isotropic pressure \( p \) and entropy per particle \( s \) [37]. Also, assume that there are no creation or annihilation processes, so that the particle number is conserved (or equivalently \( n \propto V^{-1} \), where \( V \) is the physical volume). In this case, the local form of the first law of thermodynamics may be written as

\[
d \left( \frac{F}{n} \right) = -p d \left( \frac{1}{n} \right) + T ds . \tag{1}
\]

In the case of an isentropic flow, the entropy per particle is conserved and, consequently, Eq. (1) simplifies to

\[
d \left( \frac{F}{n} \right) = -p d \left( \frac{1}{n} \right) . \tag{2}
\]

Defining an equation of state \( \rho = \rho (n) \) and solving Eq. (2) with respect to \( \rho \) leads to

\[
p(n) = \mu n - \rho (n) , \tag{3}
\]

where \( \mu = d\rho / dn \) is the chemical potential. On the other hand, if \( p = p(n) \) is given then Eq. (2) implies that

\[
\rho(n) = C n + n \int^{n} \frac{p(n')}{n'^{2}} dn' , \tag{4}
\]

where \( C \) is an integration constant.

A. Model I

The derivation of the equations of motion of a perfect fluid from an action functional has been studied by several authors [37–42]. Here we shall consider a model described by the action (see, e.g. [42])

\[
S = \int d^{4}x \sqrt{-g} \mathcal{L}(g_{\alpha\beta}, j^{\alpha}, \phi) , \tag{5}
\]

where

\[
\mathcal{L} = F (|j|) + j^{\alpha} \nabla_{\alpha} \phi , \tag{6}
\]

\( g = \det (g_{\alpha\beta}) \), \( g_{\alpha\beta} \) are the components of the metric tensor, \( j^{\alpha} \) are the components of a timelike vector field \( j \), \( \phi \) is a scalar field, \( F \) is a function of \( |j| \) and

\[
|j| = \sqrt{-j^{\alpha} j_{\alpha}} . \tag{7}
\]

Varying the action with respect to \( j^{\alpha} \) and \( \phi \) one obtains the following equations of motion

\[
\frac{\delta S}{\delta j^{\alpha}} = - \frac{1}{|j|} \frac{dF}{d|j|} j_{\alpha} + \nabla_{\alpha} \phi = 0 , \tag{8}
\]

\[
\frac{\delta S}{\delta \phi} = \nabla_{\alpha} j^{\alpha} = 0 . \tag{9}
\]

The energy-momentum tensor is given by

\[
T^{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L})}{\delta g_{\alpha\beta}} = 2 \frac{\delta \mathcal{L}}{\delta g_{\alpha\beta}} + \mathcal{L} g^{\alpha\beta} . \tag{10}
\]

Substituting the Lagrangian defined in Eq. (5) into Eq. (10) and using Eq. (8), one obtains

\[
T^{\alpha\beta} = - \frac{dF}{d|j|} j^{\alpha} j^{\beta} + \left( F - |j| \frac{dF}{d|j|} \right) g^{\alpha\beta} . \tag{11}
\]

Once the following identifications are made

\[
n = |j| , \tag{12}
\]

\[
\rho (n) = - F , \tag{13}
\]

\[
p (n) = F - n \frac{dF}{dn} , \tag{14}
\]

\[
u^{\alpha} = \frac{j^{\alpha}}{n} , \tag{15}
\]

the energy-momentum tensor may be written in a perfect fluid form

\[
T^{\alpha\beta} = (\rho + p) u^{\alpha} u^{\beta} + pg^{\alpha\beta} , \tag{16}
\]

where \( \rho \) and \( p \) are the proper density and pressure, and \( u^{\alpha} \) are the components of the 4-velocity (satisfying \( u^{\alpha} u_{\alpha} = -1 \)). With the identifications made above Eq. (8) now defines the 4-velocity of the fluid

\[
u^{\alpha} = - \frac{\nabla^{\alpha} \phi}{\mu} , \tag{17}
\]

associated to an irrotational flow (meaning that the spatial components of \( u^{\alpha} \) are curl-free in the local comoving inertial frame) while Eq. (9) represents the particle number conservation equation. Note that the condition \( u^{\alpha} u_{\alpha} = -1 \) implies that

\[
\mu^{2} = 2 X , \tag{18}
\]

where

\[
X \equiv - \frac{1}{2} \nabla^{\alpha} \phi \nabla_{\alpha} \phi > 0 . \tag{19}
\]

On the other hand, Eq. (8) may be obtained from Eqs. (13) and (14), thus implying that the Lagrangian given in Eq. (6) describes an isentropic flow satisfying

\[
\nabla_{\alpha} (sj^{\alpha}) = 0 . \tag{20}
\]

Since the entropy per particle \( s \) is not a dynamical variable of our model, Eq. (11) is, in this case, equivalent to the particle number conservation equation given in Eq. (4).

B. Model II

Using Eqs. (12), (13), (14), (15), (17), it is possible to show that the on-shell Lagrangian, defined off-shell in Eq. (6), is equal to

\[
\mathcal{L}_{\text{on-shell}} = - \rho + n \frac{d\rho}{dn} = p . \tag{21}
\]
If $\mu(n)$ is a strictly monotonic function of $n$ (such that there is a one-to-one relation between $\mu$ and $n$) Eq. 9 may be written as

$$p(\mu) = \mu n - \rho,$$  
(22)

where $p(\mu)$ is the Legendre transform of $\rho(n)$. The conjugate variables are related through

$$n = \frac{dp}{d\mu}, \quad \mu = \frac{d\rho}{dn}. \tag{23}$$

Taking into account that $\mu = \pm \sqrt{2X}$ and assuming $\mu > 0$ one finally obtains

$$n(X) = \frac{dX}{d\mu} p, X = \sqrt{2X} p X.$$  
(24)

where a comma denotes a partial derivative (e.g., $p_X \equiv dp/dX$). In combination with Eq. (22) this implies that the pure k-essence Lagrangian $L(X) = p(X)$ may be used to describe an irrotational perfect fluid with conserved particle number and constant entropy per particle 

$$\rho$$  
(22a)

The equation of motion of the scalar field

$$\nabla_a (L X \nabla^a \phi) = 0,$$  
(25)

provides the equivalent in the scalar field theory of the particle number conservation, given by Eq. (2). Interestingly, the identifications $L = p$, $u_\alpha = -\nabla_\alpha \phi/\sqrt{2X}$, in combination with $\rho = 2X L, X - L$ are also required in order that the energy momentum-tensor

$$T^{\alpha\beta} = L X \nabla^\alpha \phi \nabla^\beta \phi + \mathcal{L} g^{\alpha\beta},$$  
(26)

associated to an arbitrary scalar field Lagrangian $\mathcal{L}(\phi, X)$ may be written in a perfect fluid form.

C. Model III

The transformation

$$\mathcal{L} \to \mathcal{L} - \nabla_\alpha (\phi j^\alpha),$$  
(27)

leaves the action in Eq. (5) unchanged up to surface terms. This implies that the equations of motion given in Eqs. (3) and (4) are insensitive to this transformation. The resulting off-shell Lagrangian is given by

$$\mathcal{L} = F(n) + j^\alpha \nabla_\alpha \phi - \nabla_\alpha (\phi j^\alpha) = F(n) - \phi \nabla_\alpha j^\alpha.$$  
(28)

Varying the matter action with respect to the metric components one obtains

$$\delta S = \int d^4 x \frac{\delta (-\sqrt{-g} \mathcal{L})}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} = \frac{1}{2} \int d^4 x \sqrt{-g} T^{\alpha\beta} \delta g_{\alpha\beta},$$  
(29)

where

$$\delta (-\sqrt{-g} \mathcal{L}) = \sqrt{-g} \delta \mathcal{L} + \mathcal{L} \delta \sqrt{-g} = \sqrt{-g} \delta \mathcal{L} + \frac{\mathcal{L}}{2} \sqrt{-g} g^{\alpha\beta} \delta g_{\alpha\beta},$$  
(30)

with

$$\delta \mathcal{L} = -\frac{1}{2} dF \frac{j^\alpha j^\beta}{|j|} \delta g_{\alpha\beta} - \delta \phi (\nabla_\nu j^\nu),$$  
(31)

and

$$\delta \phi (\nabla_\nu j^\nu) = \delta \phi \left( \frac{\partial_\nu (\sqrt{-g} j^\nu)}{\sqrt{-g}} \right) = \frac{1}{2} g^{\alpha\beta} \delta g_{\alpha\beta} \nabla_\nu (\phi j^\nu) + \frac{1}{2} \nabla_\nu (\phi j^\nu) g^{\alpha\beta} \delta g_{\alpha\beta}. \tag{32}$$

Discarding the last term in Eq. (32) — this term gives rise to a vanishing surface term in Eq. (29) ($\delta g_{\alpha\beta} = 0$ on the boundary) — and using Eqs. (8) and (9) it is simple to show that the energy-momentum tensor associated to the transformed Lagrangian defined in (28) is still given by Eq. (11). However, in this case the on-shell Lagrangian is equal to

$$\mathcal{L}_{on-shell} = F = -\rho.$$  
(33)

Using this result, in combination with Eq. (11), it is possible to write the on-shell Lagrangian as

$$\mathcal{L}_{on-shell} = -C n - n \int p(n') \frac{dn'}{n^2},$$  
(34)

(see also [43] for an alternative derivation of this result).

D. Model IV

In many situations of interest a fluid (not necessarily a perfect one) may be simply described as a collection of many identical point particles undergoing quasi-instantaneous scattering from time to time [44, 45]. Hence, before discussing the Lagrangian of the fluid as a whole, let us start by considering the action of a single point particle with mass $m$

$$S = -\int d\tau m,$$  
(35)

and energy-momentum tensor

$$T^{\ast\alpha\beta} = \frac{1}{\sqrt{-g}} \int d\tau m u^\mu u^\beta \delta^4 (x^\mu - \xi^\mu(\tau)),$$  
(36)

where the * indicates that the quantity refers to a single particle, $\xi^\mu(\tau)$ represents the particle worldline and $u^\alpha$ are the components of the particle 4-velocity. If one
considers its trace $T^* = T^{\alpha \beta} g_{\alpha \beta}$ and integrates over the whole of space-time, we obtain

$$\int d^4x \sqrt{-g} T^* = - \int d^4x d\tau m \delta^4 (x^\mu - \xi^\mu (\tau)) = - \int d\tau m ,$$

which can be immediately identified as the action for a single massive particle, and therefore implies that the corresponding Lagrangian is simply given by

$$\mathcal{L}_{\text{on-shell}}^* = T^* .$$

If a fluid can be modelled as a collection of point particles then its on-shell Lagrangian at each point will be the average value of the single-particle Lagrangian over a small macroscopic volume around that point

$$\langle \mathcal{L}_{\text{on-shell}}^* \rangle = \frac{\int d^4x \sqrt{-g} \mathcal{L}_{\text{on-shell}}^*}{\int d^4x \sqrt{-g}} = \frac{\int d^4x \sqrt{-g} T^*}{\int d^4x \sqrt{-g}} = \langle T^* \rangle ,$$

where $\langle T^* \rangle = T$ is now the trace of the energy-momentum of the perfect fluid. This provides a further possibility for the on-shell Lagrangian of a perfect fluid:

$$\mathcal{L}_{\text{on-shell}} = T = -\rho + 3p ,$$

where $p = \rho (v^2)/3 = \rho T$, $\sqrt{\langle v^2 \rangle}$ is the root-mean-square velocity of the particles and $T$ is the temperature.

### E. Which Lagrangian?

We have shown that models I, II, III and IV, characterized by different matter Lagrangians, may be used to describe the dynamics of a perfect fluid. If the matter Lagrangian only couples minimally to gravity, then these models may even be used to describe the same physics. However, this degeneracy is generally broken in the presence of nonminimal coupling either to gravity \[46\] [54] or to other fields \[51\] [52], which makes the consideration of the appropriate Lagrangian a crucial one \[61\] [62]. Models I, II and III, described in the previous section, implied both the conservation of particle number and entropy — the same is true for more general models which consider additional degrees of freedom, such as a variable entropy per particle \[42\]. However, both the entropy and the particle number are in general not conserved in a fluid described as a collection of point particles. Hence, model IV has degrees of freedom that are not accounted for by models I, II and III. In model IV the pressure depends both on the temperature $T$ (or, equivalently, the root-mean-square velocity of the particles) and on the energy density $\rho$, with $p = \rho T$ where $T$ is the temperature, while in models I, II and III $p$ is a function of the number density alone. Still, in model IV the equation of state parameter $w = p/\rho$ must be in the interval $[0, 1/3]$, which while appropriate to describe a significant fraction of the energy content of the Universe, such as cold dark matter, baryons, photons and neutrinos, cannot be used to describe dark energy. On the other hand, models I, II and III are specially suited for dark energy, both because they allow for values of $w \sim -1$ but also because the requirement that $X > 0$ can only be met if the spatial variations of the scalar field $\phi$ are sufficiently small. In the following we shall use model II to describe both dark energy and unified dark energy. However, one should bear in mind that any successful unified dark energy model must account for the observed large scale structure of the Universe, and, therefore, a scalar field description of unified dark energy in terms of a perfect fluid will inevitably break down on non-linear scales \[64\].

### III. Mapping Dark Energy into Unified Dark Energy

The main feature of most unified dark energy (UDE) models is that of mimicking dark energy and dark matter with a single underlying perfect fluid or scalar field (see \[65\] for a discussion of the single fluid hypothesis). To construct a model with these properties we shall consider the Lagrangian

$$\mathcal{L}_{\text{ude}} = \mathcal{L}_{\text{de}} + \mathcal{L}_{\text{cdm}} .$$

Here, we shall assume that $\mathcal{L}_{\text{de}} \equiv \mathcal{L}_{\text{de}} (X)$ is an arbitrary pure kinetic dark energy Lagrangian and that the ratio between $\mathcal{L}_{\text{cdm}}$ and $\mathcal{L}_{\text{de}}$ vanishes on-shell (or is extremely small, so that the contribution of $\mathcal{L}_{\text{cdm}}$ to the total pressure can be neglected). Therefore, the unified dark energy Lagrangian $\mathcal{L}_{\text{ude}}$ describes a fluid with proper pressure

$$p_{\text{ude}} = \mathcal{L}_{\text{ude(on-shell)}} = \mathcal{L}_{\text{de(on-shell)}} = p_{\text{de}}$$

and energy density

$$\rho_{\text{ude}} = \rho_{\text{de}} + \rho_{\text{cdm}} ,$$

where $\rho_{\text{de}} = 2X \mathcal{L}_{\text{de,X}} - \mathcal{L}_{\text{de}}$. The new Lagrangian may be regarded as a unified dark energy model provided that $w_{\text{de}} = p_{\text{de}}/\rho_{\text{de}} \sim -1$.

One possible choice for $\mathcal{L}_{\text{cdm}}$ would be to consider

$$\mathcal{L}_{\text{cdm}} = \lambda (X - V (\phi)) ,$$

where $\lambda$ is a Lagrange multiplier and $V (\phi) > 0$ is a function of $\phi$ \[13\] [27]. This choice ensures that the constraint $X = V (\phi)$ is always satisfied on-shell, thus implying that $\mathcal{L}_{\text{cdm(on-shell)}} = 0$ or, equivalently, that $p_{\text{ude}} = \mathcal{L}_{\text{ude(on-shell)}} = \mathcal{L}_{\text{de(on-shell)}} = p_{\text{de}}$. On the other hand, the density of the UDE fluid is given by Eq. \[43\] with

$$\rho_{\text{cdm}} = \lambda (X + V (\phi)) = 2\lambda X .$$
however, the Lagrange multiplier $\lambda$ is also a dynamical field whose evolution is such as to ensure that the energy-momentum tensor of the UDE fluid, subject to the constraint $X = V(\phi)$, is covariantly conserved. Hence, in this case $\rho_{\text{ude}}(X, \lambda)$ thus implying that the UDE fluid would not in general be isentropic.

An alternative would be to consider a class of purely kinetic Lagrangians given by \[ L(X) = AX^\gamma, \] where $A$ and $\gamma$ are positive real constants. These models describe an isentropic perfect fluid with pressure $p = L(X)$ and energy density

\[
\rho = 2XL_X - L = (2\gamma - 1)AX^\gamma,
\]

with the equation of state parameter

\[
w \equiv \frac{p}{\rho} = \frac{1}{2\gamma - 1},
\]

being a constant. In the $\gamma \to \infty$ limit $w \to 0$. Hence, this fluid mimics pressureless dust in this limit. Thus another possible choice for $L_{\text{cdm}}$ would be

\[
L_{\text{cdm}}(X) = \lim_{\gamma \to \infty} A(\gamma)X^\gamma.
\]

The function $A(\gamma)$ is chosen in such a way that $p_{\text{cdm}}$ vanishes at every space time point in this limit but

\[
p_{\text{cdm}} = \lim_{\gamma \to \infty} (2\gamma - 1)A(\gamma)X^\gamma,
\]

is essentially unrestricted.

In a Friedmann-Lemaître-Robertson-Walker (FLRW) homogeneous and isotropic universe

\[
\rho_{\text{cdm}} = \rho_{\text{cdm}0}(1 + z)^3,
\]

where $z = 1/a - 1$ is the redshift, $a$ is the scale factor (normalized to unity at the present time), and $\rho_{\text{cdm}}(z = 0) = \rho_{\text{cdm}0}$. In this context, the equation of state parameter of the UDE fluid,

\[
w_{\text{ude}} = \frac{\rho_{\text{ude}}}{\rho_{\text{ude}} + \rho_{\text{cdm}}} = \frac{w_{\text{de}}}{1 + \rho_{\text{cdm}}/\rho_{\text{de}}},
\]

may be written as

\[
w_{\text{ude}}(z) = \frac{w_{\text{de}}(z)}{1 + \rho_{\text{cdm}0}(1 + z)^3/\rho_{\text{de}}(z)},
\]

where $w_{\text{de}}$ is the equation of state parameter of the original DE fluid. Since this model is defined by a purely kinetic Lagrangian the sound speed defined by \[ c_{s(ude)}^2(\rho_{\text{ude}}, z) = \frac{p_{\text{ude},X}}{p_{\text{ude},X} + 2Xp_{\text{ude},XX}} \]

coincides with the adiabatic sound speed given by

\[
c_{s(ude)}^2 = \frac{p_{\text{ude},z}}{\rho_{\text{ude},z}} = \left( 1 + \frac{3\rho_{\text{cdm}0}(1 + z)^2}{\rho_{\text{de},z}} \right) c_{s(de)}^2,
\]

where $c_{s(de)}^2 = p_{\text{de},X}/\rho_{\text{de},X} = p_{\text{de},z}/\rho_{\text{de},z}$ is the sound speed of the original DE fluid.

### A. Restrictions on isentropic UDE models

Let us assume the following parameterization of the equation of state of the original dark energy fluid \[ w_{\text{de}}(z) = w_0 + \Delta w \frac{z}{1 + z}, \]

where $w_0 \equiv w_{\text{de}}(z = 0)$, $w_\infty \equiv w_{\text{de}}(z = \infty)$ and $\Delta w \equiv w_\infty - w_0$. It is possible to show that this parameterization of $w(z)$ admits a purely kinetic Lagrangian formulation \[ \gamma. \] The energy density of the corresponding UDE fluid is equal to

\[
\rho_{\text{ude}} = \rho_{\text{ude}0} \left[ (1 + z)^{3(1 + w_\infty)} e^{3\Delta w/(1 + z)} + F (1 + z)^3 \right],
\]

and the sound speed squared is

\[
c_{s(ude)}^2 = \frac{\Delta w + 3w_0(1 + w_\infty) - 3\Delta w}{3(1 + w_0 + F e^{-3\Delta w})}.
\]

If one assumes that the original fluid is a dark energy fluid with $w_0$ sufficiently close to $-1$ one finds

\[
c_{s(ude)}^2 = \frac{w_\infty + 1}{3F} c^{3(w_\infty + 1)}.
\]

In order for the transformed fluid to play a unified dark energy role $F \sim \Omega_{\text{cdm}0}/\Omega_{\text{de}0} \sim 3/7$, where $\Omega_{\text{cdm}0}$ and $\Omega_{\text{de}0}$ are the fractional dark matter and dark energy densities inferred from the observations. This in turn implies that $c_{s(ude)}^2 \approx (w_\infty + 1)c^{3(w_\infty + 1)}$. Therefore, large sound speeds at recent times would be unavoidable, unless $|w_\infty + 1| \ll 1$. One can estimate how small this value has to be in order to be consistent with the standard growth of perturbation on linear scales by imposing that $c_{s(ude)}^2 \lesssim 10^{-3}$ \[ 71. \]

Hence the variation of $w$ is limited to $|1 + w_\infty| \lesssim 10^{-6}$, meaning that the original fluid has to follow very closely the behaviour of a cosmological constant. More generally, Eq. \[ 58 \] implies that large sound speeds at low redshifts can only be avoided if both $|w_\infty + 1|$ and $|w_0 + 1|$ are extremely small. Such stringent constraints regarding a non-null sound speed are typical for UDE models as far as linear perturbation theory is concerned \[ 71, 72. \] However, it has been shown that the clustering on non-linear scales can have a potential impact on the large scale evolution of the Universe, specially in UDE scenarios \[ 73, 73. \] Taking into account non-linear effects may render these models (ruled out in a linear analysis) consistent with cosmological observations \[ 74, 74. \]
IV. CONCLUSIONS

In this paper we have investigated the degeneracies between the energy-momentum tensor and the on-shell Lagrangian of a perfect fluid, explicitly showing that one does not univocally determine the other. We have discussed the appropriateness of four different Lagrangians to describe the dynamics of different components of the cosmic energy budget, distinguishing those that may be essentially modelled as a collection of point particles, such as baryons, photons or neutrinos, from those that do not, such as dark energy. This distinction is particularly relevant if a non-minimal coupling exists with the gravitational field or other matter fields, in which case the knowledge of the on-shell Lagrangian is essential to compute the corresponding dynamics. This point has been overlooked in the literature, where it is often wrongly assumed that there is a freedom of choice of the on-shell Lagrangian, even when describing standard model particles.

We have also explored the fact that models with the same on-shell Lagrangian may have different proper energy densities. We have used this result to establish a map between dark energy models described by purely kinetic Lagrangians and unified dark energy models, characterizing the correspondence between their equation of state and sound speed parameters. We have also briefly discussed the linear sound speed problem of unified dark energy models as well as a possible way out associated to their non-linear dynamics.

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