PAIR PRODUCTION MULTIPLEITIES IN ROTATION-POWERED PULSARS

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ABSTRACT

We discuss the creation of electron-positron cascades in the context of pulsar polar cap acceleration models and derive several useful analytic and semianalytic results for the spatial extent and energy response of the cascade. Instead of Monte Carlo simulations, we use an integrodifferential equation that describes the development of the cascade energy spectrum in one space dimension quite well, when it is compared to existing Monte Carlo models. We reduce this full equation to a single integral equation, from which we can derive useful results, such as the energy loss between successive generations of photons and the spectral index of the response. We find that a simple analytic formula represents the pair cascade multiplicity quite well, provided that the magnetic field is below $10^{12}$ G and that an only slightly more complex formula matches the numerically calculated cascade at all other field strengths. Using these results, we find that cascades triggered by $\gamma$-rays emitted through inverse Compton scattering of thermal photons from the neutron star’s surface, both resonant and nonresonant, are important for the dynamics of the polar cap region in many pulsars. In these objects, the expected multiplicity of pairs generated by a single input particle is lower than previously found in cascades initiated by curvature emission, frequently being on the order of 10 rather than ~1000 as usually quoted. Such pulsars also are expected to be less luminous in polar cap $\gamma$-rays than when curvature emission triggers the cascade, a topic that will be the subject of a subsequent paper.

Subject headings: acceleration of particles — gamma rays: theory — pulsars: general

1. INTRODUCTION

The polar cap pair production model of pulsar emission has remained the chief theory of the region thought to give rise to the radiative emission for the past 30 years. Particles accelerate from the surface of the neutron star, drawn by the induced fields of the rotating magnetosphere. As first discussed by Sturrock (1971), these particles emit $\gamma$-rays that pair produce in the high background magnetic field. These pairs then short out the magnetic field (Ruderman & Sutherland 1975) at a surface known as the pair formation front (PFF), preventing further acceleration.

The physics of this pair production process has been partially explored, with progress ranging from the initial generational models of Tademaru (1973), the PFF structure calculations of Arons & Scharlemann (1979) and Arons (1981), the detailed Monte Carlo simulations of Daugherty & Harding (1982), the pair creation and acceleration model of Harding & Muslimov (1998), and the more recent “full generational” model of Zhang & Harding (2000). Our paper describes an improved analytical and theoretical understanding of this pair production mechanism, not only reproducing the final $\gamma$-ray and pair spectra calculated by Daugherty & Harding (1982), but detailing the spatial extent of the pair production process itself, giving a more thorough understanding of the process.

These straightforward analytic approximations to the pair production process were the basis of the model of Hibschman & Arons (2001, hereafter Paper I). In this paper we exhibit the full shower theory underlying those approximations and find a variety of formulae that represent the numerical solutions of the cascade equations, thus allowing users of the cascade physics to apply the results easily to other problems. We give a full description of the pair plasma emergent from pulsar polar caps. We reserve discussion of the emergent $\gamma$-ray luminosities and spectra for a separate paper, as these are of direct observational relevance.

2. GAMMA-RAY OPACITY

The dominant opacity for pair production over pulsar polar caps is the $\gamma-B$ process, wherein a high-energy photon interacts with the background magnetic field to form an electron-positron pair. The competing process of $\gamma-\gamma$ pair production, wherein a high-energy photon interacts with a background X-ray to form a pair, is only competitive if the background X-ray flux is equivalent to a blackbody with temperature nearly $10^7$ K, well beyond the observed limits on stellar temperatures.

According to Erber (1966), the opacity for $\gamma-B$ pair production is

$$\alpha\theta(\epsilon, \psi) = 0.23 \frac{e B}{\zeta c} \sin \psi \exp \left(-\frac{8}{\chi a}\right),$$

where $B_q$ is the critical quantum magnetic field, $B_q = \epsilon/\zeta \lambda_c^2 = 4.41 \times 10^{-13}$ G, $\alpha = e^2/hc \approx 1/137$ is the fine structure constant, $\lambda_c = h/mc = 3.86 \times 10^{-11}$ cm is the reduced Compton wavelength, $B$ is the local magnetic field strength, $\psi$ is the pitch angle between the photon momentum and the local magnetic field, and

$$\chi \equiv \epsilon B_{q} / B_q \sin \psi,$$

where $\epsilon$ is the photon energy, in units of $mc^2$. For the remainder of this paper, all energies are quoted in terms of $mc^2$ for convenience.

This expression is accurate provided that $\epsilon \sin \psi > 1$, so that the created pair is in a high Landau level, and provided that $B$ is small compared to $B_q$: $B < B_q/3$ suffices. For pair production into low Landau levels, the full cross section
must be used (Daugherty & Harding 1983), and for magnetic fields equal to or higher than $B_\gamma$, higher order corrections become important, as discussed in Harding, Baring, & Gonthier (1997). Most pulsars, however, pair produce into high Landau levels for the most significant photons and have $B < B_\gamma/3$. As this work intends to describe the “typical” pulsar, we neglect these high-energy and high-field effects.

2.1. Optical Depth

Because the relativistic primary electron beam follows the magnetic field, the primary photons are beamed close to parallel to the local field. If the beam electrons have a Lorentz factor $\gamma$, the primary photons are beamed into a cone of opening angle $\psi \sim 1/\gamma$. Typical beam Lorentz factors are of order $10^3$ or higher, giving an initial pitch angle small compared to the pitch angles required for pair production. To a good approximation, then, we can treat the photons as if they are injected precisely parallel to the field lines.

As the photons propagate through the magnetosphere, the pitch angle between the photon momentum and the background magnetic field steadily increases. This pitch angle is by $\sin \psi = |\mathbf{k} \times \mathbf{B}(r)|$, where $\mathbf{k}$ is the photon momentum direction and $\mathbf{B}$ is the local magnetic field direction. If we assume $\psi$ small enough that $\sin \psi \approx \psi$, the change in $\psi$ along the photon path is then $d\psi/ds = (|\mathbf{k} \times (\mathbf{k} \cdot \mathbf{B})\mathbf{B}(r)|)$. Since $\psi$ is small, $\mathbf{k}$ remains close to $\mathbf{B}$, so we can substitute, yielding $d\psi/ds \approx |\mathbf{B} \times (\mathbf{B} \cdot \mathbf{B})\mathbf{B}(r)| = \rho(r) - 1$, where $\rho(r)$ is the local magnetic field radius of curvature. This is accurate to second order in $\psi$.

For a dipole field, the photon pitch angle depends on radius (through second order in the colatitude, $\theta$) as

$$\psi = \psi_0 \left(1 - \frac{r_c}{r}\right),$$

where $r_c$ is the radius of emission and $r$ is the current radius, both measured from the center of the dipole, and $\psi_0 \equiv r_c/\rho_0$, where $\rho_0 = (4/3)(R_*/\theta_0) f_s(r_c/R_*)^{3/2}$ is the field line radius of the magnetic colatitude of the field line in question at the surface of the star and $f_s \ll 1$ is a factor used to take crudely into account the possibility of non-dipolar components to the magnetic field at low altitude. Numerically this is $\psi_0 = 0.011(\theta_0/\theta_c) P^{1/2}(r/R_*)^{1/2} f_s$, with $\theta_c$ the colatitude of the last closed field line, $\theta_c = (R_s/R_*)^{1/2}$, and $P$ the period in seconds. If the dipole moment is displaced close to the surface of the star, general relativistic (GR) bending of the photon paths would change this, but for almost all other cases GR perturbations are negligible.

The magnetic field and radius of curvature may then be rewritten as

$$B = B_* \left(\frac{r}{r_c}\right)^{-3} = B_\gamma \left(1 - \frac{\psi}{\psi_0}\right)^3,$$

$$\rho = \rho_0 \left(\frac{r}{r_c}\right)^{1/2} = \rho_\gamma \left(1 - \frac{\psi}{\psi_0}\right)^{-1/2},$$

where $B_\gamma$ and $\rho_\gamma$ are, respectively, the magnetic field and radius of curvature at the emission point.

Using these relations to write the optical depth using $t \equiv \psi/\psi_\infty$ as the integration variable, letting $\sin \psi \approx \psi$, yields

$$\tau(t) = \frac{0.23}{\kappa_c} \frac{B_\gamma}{B_q} \psi_\infty^2 \int_0^1 \left(1 - t\right)^{5/2}$$

$$\times \exp \left[- \frac{Z_\infty}{t(1 - t)}\right] dt,$$

where

$$Z_\infty \equiv \frac{8}{3} \frac{B_q}{eB_\gamma \psi_\infty}$$

and $\epsilon$ is the photon energy in units of $mc^2$.

The exponential above has a maximum at $t = 1/2$. Physically, this is the point where the increase in opacity due to the increasing pitch angle balances the decrease in opacity due to the steadily declining magnetic field.

Provided that $Z_\infty$ is not too small, the integrand is a sharply peaked function of $t$ and can therefore be integrated using the steepest descents method, yielding

$$\tau(e) = \frac{0.23}{\kappa_c} \frac{B_\gamma \psi_\infty^2}{B_q} \left(\frac{3t}{2}\right)^{1/2} \sqrt{\frac{6\pi}{Z_\infty(e)}} \exp \left[- \frac{256}{27} Z_\infty(e)\right].$$

When compared to numerical calculations, this expression is accurate to better than 10% in the neighborhood of $\tau = 1$, for the range of magnetic fields and radii of curvature found in pulsars.

Photons with $\tau < 1$ will escape the magnetosphere, while those with $\tau > 1$ will be absorbed. If $\tau(e) > 1$, most of the absorption takes place on the leading edge of the exponential. Since the opacity is increasing exponentially in that regime, we can approximate the integral by expanding around the upper endpoint to find

$$\tau(e, t) \approx \Lambda_1 t^3 e^{-Z_\infty f(t)},$$

where

$$\Lambda_1 = 0.086 \frac{\kappa_c}{\kappa_c} \left(\frac{B_\gamma}{B_q}\right)^2 e\psi_\infty$$

and $f(t)$ has been defined to be $t^{-1}(1 - t)^{-3}$ for $t < \frac{1}{2}$ and $256/27$ for $t > \frac{1}{2}$. (This result was previously derived in Arons & Scharlemann 1979.) Since the opacity decreases rapidly after $t = \frac{1}{2}$, the optical depth saturates at that point.

2.2. Absorption Peak

Given this opacity, we can find where any given photon will be absorbed. As a result of the exponential dependence of the opacity, all photons are effectively absorbed at the maximum of $\exp(-\tau)$, or approximately at $\tau = 1$. Using equation (9) for $\tau$, we find that this peak occurs at

$$t_a(1 - t_a)^3 = \frac{8}{3} \frac{B_q}{eB_\gamma \ln \Lambda},$$

where $\ln \Lambda \equiv \ln [\Lambda_1(e)t^3]$. In the limit of small $t_a$, this becomes

$$\psi = \frac{8}{3} \frac{B_q}{eB_\gamma \ln \Lambda}.$$
Since the opacity saturates past $\psi_a = \psi_a / \Lambda$, the minimum photon energy required to pair produce is

$$\epsilon_a = \frac{32}{3} B_{\psi_a} \frac{1}{2B \psi_a \ln \Lambda} mc^2.$$  \hspace{1cm} (13)

This energy should be thought of as a critical scaling energy, not as the actual minimum energy that will pair produce, since the $(1 - t)$ term was neglected. The actual minimum energy that will pair produce is

$$\epsilon_{\text{min}} = \frac{64}{57} \epsilon_a.$$  \hspace{1cm} (14)

As long as $\epsilon \gtrsim 5 \epsilon_a$, the small-$t$ limit (eq. [12]) is appropriate, and we find that a photon will be absorbed after propagating

$$\Delta r = \frac{1}{4} \frac{\epsilon_a}{\epsilon} r_c.$$  \hspace{1cm} (15)

These expressions can be made tractable by treating $\ln \Lambda$ as a constant. Self-consistently evaluating $\ln \Lambda$ at $t = \frac{t}{\pi}$ then yields (Arons & Scharlemann 1979)

$$\ln \Lambda = 16.2 + \ln B_{t_2} - \frac{1}{2} \ln P.$$  \hspace{1cm} (16)

3. SYNCHROTRON EMISSION

A charged particle with pitch angle $\psi$ with respect to the magnetic field emits synchrotron radiation and rapidly spirals down to its lowest Landau level.

The basic rate of synchrotron emission is

$$\frac{dN}{dt} = \frac{\sqrt{3} \pi \epsilon B \sin \psi \int_{\infty}^{\infty} K_{5/3}(x) dx}{2 \lambda_c \epsilon_a} ,$$  \hspace{1cm} (17)

where $B_{\psi} = B/B_{\psi}$, $K_{5/3}$ is the modified Bessel function of order 5/3, and $\epsilon_a$ is the characteristic synchrotron energy,

$$\epsilon_a = \frac{3}{2} \epsilon B_{\psi} \gamma^2 (\sin \psi) mc^2.$$  \hspace{1cm} (18)

This corresponds to a total power loss of (Jackson 1975)

$$P_{\gamma} = \frac{2 \pi \epsilon^2 B_{\psi}}{3 \lambda_c} \gamma^2 (\sin^2 \psi) mc^2 s^{-1}.$$  \hspace{1cm} (19)

As a result of synchrotron radiation, the Lorentz factor of the particle decreases according to $\dot{\gamma} = -P_{\gamma}$. This energy loss occurs over a distance of $1.8 \times 10^{-6} \gamma$ cm, effectively instantaneous compared to stellar scales, where we have used equation (24) self-consistently to fix $\gamma$ and set $\sin \psi \approx \psi$. The final Lorentz factor may be found by noticing that the parallel component of the particle’s velocity is conserved, as can be seen by transforming into a comoving frame. If the initial particle has a Lorentz factor of $\gamma_i = (1 - \beta^2)^{-1/2}$ and is moving at an angle $\psi$ with respect to the magnetic field, then the final Lorentz factor is

$$\gamma_f = \frac{1}{\sqrt{1 - \beta^2 \cos^2 \psi}} \approx \frac{\gamma_i}{\sqrt{1 + \gamma^2 \psi^2}}.$$  \hspace{1cm} (20)

Integrating equation (17) over all time gives the number of photons generated during the decay. Using equation (19) to change the variable of integration from time to $\gamma$ gives

$$N_{\gamma}(\gamma_i, \psi) = \frac{3 \sqrt{3}}{8\pi} (\epsilon B_{\psi} \sin \psi)^{-1} \int_{\gamma_i}^{\infty} \frac{d\gamma}{\gamma} \epsilon_i(\gamma)^{-1} \int_{0}^{\infty} dx K_{5/3}(x) .$$  \hspace{1cm} (21)

Changing variables to $z = \epsilon / \epsilon_i(\gamma)$ and rearranging gives

$$N_{\gamma}(\gamma_i, \psi) = \frac{3}{4 \sqrt{2\pi}} (\epsilon B_{\psi} \sin \psi)^{-1/2} e^{-3/2} [F(z_f) - F(z_i)] ,$$  \hspace{1cm} (22)

where

$$F(t) = \frac{3}{2} \int_{t/\gamma}^{\infty} dz' t'^{1/2} \int_{\gamma}^{\infty} dx K_{5/3}(x) = \int_{\gamma}^{\infty} dx K_{5/3}(x)(x^{3/2} - t^{1/2}) .$$  \hspace{1cm} (23)

Equation (22) gives the total number of photons of energy $\epsilon$ emitted by a particle with initial Lorentz factor $\gamma_i$ and initial pitch angle $\psi$. Nearly all of these photons are emitted in a cone with opening angle $\psi$ because the Lorentz factor of the particle remains high, so the angle $\psi$ only changes once almost all of the particle energy has been radiated away.

In the case of a particle created by $\gamma-B$ pair production from a photon of initial energy $\epsilon_i$, $\gamma_i = \epsilon_i / 2$ and $\psi = \psi_d(\epsilon_i)$. Defining

$$a \equiv \gamma_i \psi_a = \frac{4}{3 \ln \Lambda} \frac{1}{B_{\psi}},$$  \hspace{1cm} (24)

we find $z_f = \ln \Lambda \epsilon_i / \epsilon_i$ and $z_f = (1 + a^2) \ln \Lambda / \epsilon_i$.

With these substitutions and counting the emission from both generated particles, the total number of synchrotron photons produced by one incident photon pair producing at $\psi = \psi_a$ is

$$N_{\gamma}(\epsilon_i) = \frac{3 \sqrt{3}}{8\pi} \sqrt{\ln \Lambda} (\epsilon_i)^{-3/2} \int [F(z_f) - F(z_i)] .$$  \hspace{1cm} (25)

We can understand the synchrotron spectrum by examining its asymptotic limits. Figure 1 shows the function $F(t)$ and its deviation from its zero-point value. The function is nearly constant for $t < 1$, with $F(0) - F(t) \approx e^{3/2}$, and exponentially decreases for $t > 1$. This divides the synchrotron response into three regimes. If $\epsilon < \epsilon_i(1 + a^2) \ln \Lambda$, both $z_i$ and $z_f$ are less than 1, and $N_{\gamma}(\epsilon_i) \approx \epsilon_i^{-3/2}$; this is the usual synchrotron spectral index. If $\epsilon_i(1 + a^2) \ln \Lambda < \epsilon < \epsilon_i / \ln \Lambda$, $z_i < 1$ but $z_f > 1$, and $N_{\gamma}(\epsilon_i) \approx \epsilon_i^{-3/2}$; this reflects the decline of the particle Lorentz factor. If $\epsilon > \epsilon_i / \ln \Lambda$, both $z_i$ and $z_f$ are greater than 1, and $N_{\gamma}(\epsilon_i)$ decays exponentially.

3.1. Integral Equation

Since particles are assumed to be beamed along the field lines, the specific number intensity at any point is

$$I_z(r, \psi, \epsilon) = \rho(r)Q(z, r, \epsilon)e^{-\sigma(r, r, \epsilon)},$$  \hspace{1cm} (26)

where $r_z$ is a function of $r$ and $\psi$ via equation (3), $Q_z$ is the volumetric photon emission rate, $\rho$ is the radius of curvature of the field line, and $\sigma(r, r, \epsilon)$ is the optical depth attained by a photon of energy $\epsilon$ propagating from $r_z$ to $r$. The radius of curvature enters since $\rho d\psi$ is the path length where the field points in direction $d\psi$.

The local volumetric rate of pair production events is then

$$Q_{pp}(r, \epsilon) = \int_{0}^{\infty} d\psi x_{pp}(r, \psi, \epsilon) I_z(r, \psi, \epsilon).$$  \hspace{1cm} (27)
Using the synchrotron response, equation (25), the synchrotron emission rate is

$$Q_{\gamma, \text{syn}}(r, \epsilon) = \int_0^\infty d\epsilon_i N_\epsilon(\epsilon_i) \int_0^\infty d\psi \propto \epsilon_i I(r, \psi, \epsilon_i) .$$

(28)

Using equation (26) for \( I_{\gamma} \), \( dt = \rho \, d\psi \), and the sharp-peaked nature of the absorption, we can exactly evaluate the \( \psi \) integral, yielding

$$Q_{\gamma, \text{syn}}(r, \epsilon) = \int_0^\infty d\epsilon_i N_\epsilon(\epsilon_i)(1 - e^{-\tau_{\text{max}}})Q_{\gamma}(\tilde{r}_e, \epsilon_i) ,$$

(29)

where \( \tilde{r}_e = r_e(r, \psi) \), \( \tilde{r}_e \) is the peak of the \( \psi \) integral and \( \tau_{\text{max}} \) is the optical depth attained by a photon of energy \( \epsilon \) propagating from the stellar surface to \( r \).

Since, by equation (15), a photon emitted at \( r_e \) travels at most 0.25\( r_e \) before pair producing, we can treat the cascade as if it occurred on the spot (OTS). Photons with energies well above \( \epsilon_{\text{min}} \) pair produce in a correspondingly shorter distance, so the photons near \( \epsilon_{\text{min}} \) set the spatial expanse of the cascade. This yields an integral equation for the synchrotron spectrum,

$$Q_{\gamma, \text{syn}}(\epsilon) = \int_0^\infty d\epsilon_i[1 - e^{-\tau_{\text{max}}(\epsilon_i)}]$$

$$\times \frac{1}{\epsilon_i} K\left(\frac{\epsilon}{\epsilon_i}\right)[Q_{\gamma, \text{src}}(\epsilon_i) + Q_{\gamma, \text{syn}}(\epsilon_i)] ,$$

(30)

where we have replaced \( \tau_{\text{max}} \) with \( \tau_{\infty} \) and where

$$K\left(\frac{\epsilon}{\epsilon_i}\right) = \frac{3\sqrt{3}}{8\pi} \sqrt{\ln \Lambda \left(\frac{\epsilon}{\epsilon_i}\right)^{-3/2}}$$

$$\times \left[ F\left(\ln \Lambda \frac{\epsilon}{\epsilon_i}\right) - F\left(\phi \ln \Lambda \frac{\epsilon}{\epsilon_i}\right) \right] ,$$

(31)

where \( \phi \equiv (1 + a^2) \).

This can be easily solved by matrix operator methods. If \( Q_{\gamma, \text{syn}} \) and \( Q_{\gamma, \text{src}} \) are represented by vectors \( \mathbf{Q}_{\text{syn}} \) and \( \mathbf{Q}_{\text{src}} \) and the integral operator above by a matrix \( \mathbf{K} \), the solution is simply \( \mathbf{Q}_{\text{syn}} = (1 - \mathbf{K})^{-1} \mathbf{Q}_{\text{src}} \),

This gives an excellent approximation to the final \( \gamma \)-ray spectrum and a close approximation to the final pair spectrum, as shown in Figures 2 and 3, respectively. The numerically calculated pair spectrum extends to lower energies than the OTS spectrum, as a result of the change in magnetic field over the course of the cascade.

3.2. Moment Equations

From the integral equation for the synchrotron photon distribution, we find several useful relations connecting the moments of the photon spectrum. Using equation (30) to

![Graph 2](image2.png)
calculate successive generations of synchrotron photons, we find

\[ Q^{(i+1)}_n(\varepsilon) = \int_{\varepsilon_{\text{min}}}^{\varepsilon} d\varepsilon' \frac{1}{\varepsilon_i} K_n(\varepsilon') Q^{(i)}_n(\varepsilon), \]

where \( Q^{(i)}_n \) is the \( i \)th-generation \( \gamma \)-ray spectrum and where we have replaced \( 1 - \exp(-\varepsilon_{\text{min}}) \) with a step function at \( \varepsilon_{\text{min}} \).

Multiplying by \( \varepsilon^n \) and integrating the above expression over all energies gives

\[ Q^{(i+1)}_n = R^{(i)}_n K_n, \]

where

\[ Q^{(i)}_n \equiv \int_{\varepsilon_{\text{min}}}^{\varepsilon} d\varepsilon \varepsilon^n Q^{(i)}_n(\varepsilon), \]

\[ R^{(i)}_n \equiv \int_{\varepsilon_{\text{min}}}^{\varepsilon} d\varepsilon \varepsilon^n Q^{(i)}_n(\varepsilon), \]

\[ K_n \equiv \int_{\varepsilon_{\text{min}}}^{\varepsilon} d\varepsilon \varepsilon^n N(\varepsilon). \]

Since \( K \) is known, the \( K_n \) are known. Given a known source function, \( R^{(0)}_n \) may be calculated. The moments of successive generations may then be calculated by assuming that \( R_n \approx Q_n \) so that \( Q^{(i)}_n = K_n Q^{(i)}_0 \). The limited number of generations truncates the sum of successive moments.

Integrating the kernel, we find that the first two moments are

\[ K_0 = \frac{15\sqrt{3}}{8} \ln (\phi^{1/2} - 1), \]

\[ K_1 = 1 - \phi^{-1/2}. \]

In general, the \( n \)th moment is

\[ K_n = \frac{9\sqrt{3}}{8\pi} \frac{2^n \ln \Lambda^{1-n}}{2n^2 + n - 1} (1 - \phi^{1/2}) \Gamma \left( \frac{n + 1}{2} \right) \Gamma \left( \frac{n + 11}{6} \right). \]

From \( K_0 \) and \( K_1 \), respectively, we can see that the total number of photons steadily increases while almost all of the energy remains in the photon spectrum, provided that \( \phi = (1 + a^2) \) is large. The fraction of energy left in the pair spectrum is \( 1 - K_1 = \phi^{-1/2} \), which becomes significant for values of \( \phi \) near 1, which occurs at magnetic fields greater than \( 3 \times 10^{-12} \) G.

3.3. Generations

A cascade of pair production then divides into several successive generations of photons converting into pairs that emit more particles. Following the work of Tademaru (1973) and later papers (Lu, Wei, & Song 1994; Wei, Song, & Lu 1997; Zhang & Harding 2000), we find, in our notation, that the characteristic synchrotron energies of successive generations are related via

\[ \varepsilon_{j+1} = \frac{\varepsilon_j}{\ln \Lambda}, \]

while the particles produced by generation \( j \) have final Lorentz factors

\[ \gamma_j = \frac{\varepsilon_j}{2\sqrt{1 + a^2}} \approx \frac{\varepsilon_j}{2a}. \]

where the approximate form assumes that \( a \) is well greater than 1, which is true provided that \( B < 3 \times 10^{-12} \) G. These correspond exactly to the expressions cited by Zhang & Harding (2000), given the equivalence \( \ln \Lambda = 4/3 \gamma \).

From this, it is clear that for each absorbed photon, a fraction of the energy,

\[ f_j = 1 - \frac{1}{\sqrt{1 + a^2}} = K_1, \]

is immediately reemitted in lower energy photons, with the remainder left in the electron-positron pair. As discussed by Zhang & Harding (2000), much of the energy in the pairs is later radiated at energies near \( 2e_0\gamma_j^2 \) via resonant inverse Compton scattering (ICS), but this process generally does not create further pairs.

These approximations effectively collapse the entire synchrotron emission of a created pair into a pulse of photons at the peak energy of the synchrotron spectrum. By using the moments of the photon spectra, we can derive similar approximations, but with wider applicability.

At each generation, the number of photons increases by a factor of \( K_0 \), while the total energy in the photons decreases by a factor of \( K_1 \), so the average energy of the \( j \)th generation is

\[ \bar{\varepsilon}_j = \bar{\varepsilon}_0 \left( \frac{K_0}{K_j} \right)^{-j}, \]

where \( \bar{\varepsilon}_0 \) is the average energy of the input spectrum. The total number of photons in the \( j \)th generation is

\[ N_j = N_0 K_0. \]
where \( N_0 \) is the initial number of photons. Provided that \( \varepsilon_j > \varepsilon_n \), most of the photons will convert into pairs. If we truncate the generations when the average energy \( \langle \varepsilon_j \rangle \) reaches \( \varepsilon_n \), we find a power-law relationship between the initial energy and the final pair multiplicity,

\[
M_{\text{tot}}(\varepsilon_0) \propto \left( \frac{\varepsilon_0}{\varepsilon_n} \right)^v,
\]

where the power-law exponent is given by

\[
v = \frac{\ln K_0}{\ln K_0 - \ln K_1}.
\]

The constant of variation depends on the behavior of the lowest energy photons near \( \varepsilon_{\text{min}} \), and is discussed in §5.2.

We can apply similar arguments to the spatial extent of the cascade. From equation (15), the photons of generation \( j \), with mean energy \( \langle \varepsilon_j \rangle \), must propagate a distance \( \Delta s = 0.25(\varepsilon_j / c) R_s \) before pair producing. Relating the multiplicity of generation \( j \), equation (44), to this propagation distance gives a power-law dependence between the multiplicity and the distance,

\[
M(s) \propto s^v,
\]

where the power-law exponent, \( v \), is identical to that of the energy-multiplicity relation, equation (46).

4. NUMERICAL METHOD

Since the OTS method cannot spatially resolve the pair cascade, we turn to a numerical calculation. Equation (29) provides the fundamental structure for the calculation. It gives the synchrotron spectrum injected into the magnetosphere at any point, as a function of the photons absorbed at that point.

To calculate the resultant spectrum, we simply evaluate the photon spectrum \( \sigma(\varepsilon) \) at logarithmically spaced points in \( \varepsilon \). The relation between the synchrotron photon output and the absorbed spectrum at a given radius is then a simple matrix multiplication by \( N(\sigma) \). As a result of the variation of magnetic field with altitude, this matrix will also depend on the altitude.

We then discretize the radial steps on a variable scale, with either purely logarithmic steps, in cases in which we are primarily concerned with the \( \gamma \)-ray output and the location of the PFF, or a combination of logarithmic steps, up to a polar cap radius, and linear steps beyond, to calculate a smooth pair spectrum. Since the final pair energies depend on the local magnetic field, the pair spectrum is more sensitive to the spatial variations in the field, requiring this scheme.

To begin the calculation, at each radial point we inject an initial spectrum. These spectra are either a \( \delta \)-function source at the lowest radius, for measuring the single-photon response, or the expected spectra emitted by a single beam particle as it passes through each radial bin, for a full physical cascade. In the second case, the spectrum is adjusted to prevent binning effects from affecting the total power injected.

Using the expression \( \tau_n \), equation (8), we find which photons are absorbed within the magnetosphere and which escape. The escaping photons are simply accumulated. The absorbed photons are assumed to be absorbed at \( \psi(\varepsilon) = \psi_n(\varepsilon) \), or at a radius of \( r' = r + \rho(r) \psi_n \). Each photon energy from each radial bin is absorbed at a different height; these final heights are then simply rebinned into the grid.

For each absorbed photon, equation (25) is used to determine the synchrotron spectrum injected at the absorption
point. This process is then iterated, to find the secondaries produced by those synchrotron photons, and so on, until the energy remaining in synchrotron photons is negligible.

These calculations are streamlined by precalculating the synchrotron response matrices, \( N(\varepsilon; r) \), and the total absorption, \( \tau_\infty(\varepsilon; r) \), at each altitude. In addition, the radial absorption matrix, \( r'(r; \varepsilon) \), is computed for each energy, reducing each step of the iteration to a series of matrix multiplications.

Figures 4 and 5 show, respectively, the final \( \gamma \)-ray and pair spectra for both this model and the Monte Carlo model of Daugherty & Harding (1982). The predicted \( \gamma \)-ray spectra match extremely well, while the pair spectra mismatch at low energies, as a result of the low resolution of the Monte Carlo method in that range, and at high energies, as a result of the approximation in this model of treating \( \ln \Lambda \) as a constant, rather than allowing it to vary with photon energy.

5. SINGLE-PHOTON RESULTS

To understand the cascade process itself, we first examine the response to a single photon injected into the magnetosphere at the surface of the star. Physically, the resultant cascade depends on the energy of the input photon, the local magnetic field strength, and the local radius of curvature of the field lines.

5.1. Gamma-Ray Spectra

From the discussion of the optical depth, we know that only photons emitted at radius \( r \) with energies less than \( \varepsilon_{\text{min}}(r) \) will escape the magnetosphere. As a result of the decline of the magnetic field with \( r \), \( \varepsilon_{\text{min}} \) steadily increases with altitude.

However, the cascade produced by a single injected high-energy photon remains localized. If there are multiple generations of pair production, the photon energy must be greater than \( \ln \Lambda \varepsilon_{\text{min}} \), from equation (40), which pair produces in \( \Delta r < r_a/4 \ln \Lambda \). Since \( \ln \Lambda \) is on the order of 20, this distance is short compared to \( r_a \), and the magnetic field remains effectively unchanged. Hence, we may treat the cascade process as if it occurred OTS, and the energy cutoff remains at \( \varepsilon_{\text{min}}(r_a) \).

The typical synchrotron response has a power-law exponent \( v = -3/2 \), extending from energy \( \varepsilon_i/\ln \Lambda \) to \( \varepsilon_i/(1 + a^2) \ln \Lambda \), and a power-law exponent of \( v = -3/2 \), the typical value for synchrotron radiation, for lower energies. After processing through multiple generations, this translates to an output spectrum with \( v = -3/2 \) up to \( \varepsilon_{\text{min}}/(1 + a^2) \ln \Lambda \) and \( v = -3/2 \) from there to \( \varepsilon_{\text{min}} \).

Figure 6 shows the \( \gamma \)-ray spectra produced by various input energies. This demonstrates the expected cutoff at \( \varepsilon_{\text{min}} \), as well as the transition from a power-law exponent of \( -3/2 \) to \( -3/2 \) at \( \varepsilon_{\text{min}}/(1 + a^2) \ln \Lambda \). The shape of the spectrum is different at low energies, as a result of the effects of the exponential tails of the opacity and synchrotron emission, but it quickly converges to the asymptotic form.

5.2. Pair Multiplicity

Since the total photon energy at each pair production event is roughly conserved, cascades in low- and mid-field pulsars effectively convert high-energy photons into photons with energy \( \varepsilon_{\text{min}} \). At the penultimate generation of an extended cascade, however, a significant fraction of the synchrotron photons have \( \varepsilon < \varepsilon_{\text{min}} \) and escape. From the

\[
M_{\text{tot}} = 1 + \frac{1}{\sqrt{\ln \Lambda}} \left( \frac{\varepsilon}{\varepsilon_{\text{min}}} \right)^v,
\]

with \( v \) as given in equation (46).

For pulsars with moderate and low fields, \( B \lesssim 3 \times 10^{12} \), the quantity \( a \) is large, a negligible fraction of the energy of each absorbed photon is left in the generated pair, and \( v \approx 1 \), yielding a simple linear relation between input photon energy and final pair multiplicity.

In the opposite regime, very high fields, \( a \) is small, so the generated pair retains almost all of the initial photon energy, and there is little or no cascade; high-energy photons simply transmute into high-\( \gamma \) particles.

In Figure 7 we plot the numerically calculated energy dependence of the total multiplicity along with the theoretical results. The simplest multiplicity formula, equation (48) with the exponent \( v \) set to 1, works well for magnetic fields below \( 10^{12} \) G but overestimates the multiplicity for higher fields, as a result of the increase in the amount of energy left in the particles in the higher field cascades. For all magnetic fields, the spectrum has three well-defined regions. If \( \varepsilon < \varepsilon_{\text{min}} \), the initial photon is below threshold, and no pairs are produced. If \( \varepsilon_{\text{min}} < \varepsilon < \ln \Lambda \varepsilon_{\text{min}} \), the initial photon will pair produce, but all of the secondary synchrotron photons have energies below \( \varepsilon_{\text{min}} \) and create no further pairs. Finally, for \( \varepsilon > \ln \Lambda \varepsilon_{\text{min}} \), both the initial photon and the secondary synchrotron photons may create pairs, producing a smoothly increasing multiplicity.

Figure 8 shows this change in the power-law exponent with magnetic field, along with the theoretical result, equa-
Injected photon energy ($\epsilon/\epsilon_a$)

Number of pairs

$10^{-1}$

$10^0$

$10^1$

$10^2$

$10^3$

$10^4$

$10^5$

$10^{-1}$

$10^0$

$10^1$

$10^2$

$10^3$

$10^4$

$10^5$

**Fig. 7.—** Final pair multiplicity produced by injecting a single photon of varying energy at the stellar surface. The topmost solid line shows the theoretical prediction for $B = 10^{11}$ G, while the other lines represent the numerical results for different magnetic field strengths: $1 \times 10^{11}$, $3 \times 10^{11}$, $5 \times 10^{11}$, $1 \times 10^{12}$, and $1 \times 10^{13}$ G, respectively. The decrease in the power-law slope of the response with increasing magnetic field is clearly visible.

From equation (47), we expect that the spatial development of the cascade should be a similar power law in space. To find a closer approximation, we use an argument akin to that for the final multiplicity.

Since the first several generations of a high-energy cascade occur in the high-energy regime where equation (15) applies, the cascade develops as if $\epsilon_a$ were the minimum energy, only deviating when the photon energy drops near $\epsilon_{min}$. Hence, a good approximation to the high-energy cascade is

$$M(\epsilon, s) = 1 + \frac{1}{\sqrt{\ln \Lambda}} \left[ \frac{\epsilon}{\epsilon_a} \frac{4(s - s_1)}{r_e} \right]^\nu,$$

where $s$ is the propagation distance from the emission point, $s_1$ is the point at which the initial photon pair produces, $s_1 = 0.25(\epsilon_a/\epsilon) r_e$, $\nu$ is given in equation (46), and the factor of $27/64$ follows from the truncation of the cascade at $\epsilon_{min}$ rather than $\epsilon_a$.

Examining the numerical results, shown in Figure 9, we found that, indeed, the pair production rate follows this rule very well. The total length of the cascade is approximately $0.11 R_\ast$, which is the pair production distance expected for a photon of the minimum energy capable of pair producing, $64\epsilon_a/27$, according to the $\psi_a$ model. The computed power law, shown in Figure 10, follows the predicted values almost exactly.

### 5.3. Pair Spectra

Figure 11 shows the variation in the output pair spectrum, for different values of the input energy. The pair spectra are characterized by a sharp rise near $\epsilon_a/\ln \Lambda$, followed by a $\nu = -3/2$ power law up to $\epsilon_a/\ln \Lambda$, where $\epsilon_0$ is the energy of the injected photon, followed by an exponential decline. The spike corresponding to the input photon is also visible.

**Fig. 8.—** Final pair multiplicity power-law exponent, $\nu$, where $M_\infty(\epsilon) \propto \epsilon^\nu$. The solid line shows the computed value, while the dotted line shows the theoretical prediction.

**Fig. 9.—** Cumulative pair production as a function of distance, for several values of the magnetic field $B$ and $\Phi_{op} = 10^{14}$ V. The topmost solid line shows the theoretical prediction for $B = 10^{11}$ G, while the other lines represent different magnetic field strengths: $1 \times 10^{11}$, $3 \times 10^{11}$, $1 \times 10^{12}$, $3 \times 10^{12}$, and $1 \times 10^{13}$ G, respectively.
Figure 10.—Variation of the spatial power-law exponent with $B$. The solid line is the computed power-law exponent and the dotted line the theoretical prediction.

Figure 12 shows the variation in the output pair spectrum, for a fixed ratio of input photon energy to $\epsilon_\text{p}$, $\epsilon_\text{i}/\epsilon_\text{p} = 10^4$, for different values of the magnetic field. Here we notice the similarity of the pair spectra at different field strengths. The lower limit of the pair spectra is at $\epsilon_\text{p}(1 + a^2)^{1/2}$, which is independent of $B$, provided that $a \gg 1$. The upper limit is a fixed multiple of this value and so remains constant as well.

Figure 11.—Final pair spectra produced by injecting a single photon at the stellar surface, evaluated at $B = 10^{12}$ G and $\Phi_\text{cap} = 10^{14}$ V, for various input photon energies. The input photon energies used are $10, 10^2, 10^3$, and $10^4$ times $\epsilon_\text{i}$ with higher input energies corresponding to higher amplitudes. For these parameters, $\epsilon_\text{i} = 1530\text{mc}^2$.

6. SEMINUMERICAL MODEL

The analytic models of Paper I did not take into account the variation of the power-law exponent, $\nu$, with magnetic field. This does not affect the calculation of either the curvature or the resonant ICS cascades, since those mechanisms operate primarily on the primary photons, but the decline of $\nu$ with $B$ limits the effectiveness of nonresonant ICS for magnetic fields above $10^{12}$ G.

Allowing for the change of the power law makes the analytic form impractical, but we can quantify the approximation by using a simple numerical model. If we still assume that all photons emitted by a given emission mechanism are emitted at the peak energy and use equation (49) for the spatial range of pair production, we can calculate the location of the PFF.

In practice, we find that this method yields results that are nearly indistinguishable from the full numerical method, in the regime where the full numerical method predicts a finite PFF height. In Figure 13, we show the calculated PFF heights from the analytic model and in Figure 14 those from the seminumerical model.

7. FULL CASCADE

The same numerical procedure used to calculate the expected response to one photon may be applied to a full model of a pulsar polar cap. We ignore any variation across the polar cap, concentrating on a typical field line instead.

Given the pair production physics discussed above, a polar cap model only requires knowledge of the electric potential accelerating particles off the surface of the star, the relevant emission mechanisms that inject high-energy photons into the magnetosphere, and a model for the temperature of the star.

As a model of the accelerating potential, we use the cubic approximation to the Muslimov & Tsygan (1992) GR accel-
eration derived in Paper I. This model is easily computable and sufficiently accurate for our purposes.

The emission mechanisms included are curvature radiation and ICS, both resonant and nonresonant. The spectrum of curvature radiation is well known (see Jackson 1975), while we use equation (A17) for nonresonant scattering, with spatial dilution terms included, and equation (A2) derived from Dermer (1990) for resonant scattering. See Appendix A for descriptions of the precise emissivities used.

The stellar temperature is the most uncertain quantity in these detailed models. Not only is the precise cooling rate for neutron stars still a topic of debate, but the polar cap is likely heated by beam particles reversed and accelerated back down onto the cap. To minimize the uncertainties, we simply assume a fixed stellar temperature of $10^6$ K.

From the emission mechanisms, we find the Lorentz factor of the beam as a function of altitude and may then inject the appropriate emission spectrum at each radial bin in our grid. To ensure energy conservation, we slightly adjust the peak amplitude of each injected spectrum to compensate for the effects of the discrete energy grid.

We run the model multiple times. First, given an assumed polar cap temperature, we run it as described above to find the PFF altitude, namely, the altitude at which $\kappa_\delta$ pairs have been produced per primary. At this point the pair plasma is sufficiently dense to short out the accelerating electric field.

We then recompute the Lorentz factor of the beam, with acceleration halting at the PFF, inject the new spectra, and follow the cascade to produce the final output $\gamma$-ray and pair spectra.

A typical output pair spectrum is shown in Figure 15, and a typical output $\gamma$-ray spectrum is shown in Figure 16. These show many of the characteristics of the single-photon response. The pair spectrum shows the rapid rise, $\nu = -3/2$ power law, and exponential tail, but it also shows a low-amplitude tail of high-energy particles, as a result of the effects of ICS. For most pulsars, the final pair and $\gamma$-ray spectra are similar to those in Figures 15 and 16.

In Figure 17 we plot the multiplicity as a function of pulsar period and cap potential for a fixed stellar temperature of $10^6$ K. Given that fixed temperature, most pulsars have multiplicities ranging from 0.1 to 100, while most of the millisecond pulsars have lower multiplicities. In Figure 18 we set the radius of curvature of the field lines equal to the stellar radius, in order to model roughly the effects of an offset dipole. Here most pulsars have multiplicities in the range of 10–1000, while the millisecond pulsars remain with significantly lower multiplicities. However, a parallel offset of the dipole moment would also increase the
surface field, effectively increasing the cap potential. Given the sharp gradient in multiplicities at low $P$, this could easily bring the multiplicities of the millisecond pulsars into parity with those of the other pulsars.

8. CONCLUSIONS

This paper has given several approximations and descrip-
tions of the pair creation process in pulsars, in the hopes that they will be useful for other researchers examining these objects. The development of the pair cascade in space is deceptively simple, a fact that has been used in Paper I to obtain several useful results about the regimes of dominance of the various emission mechanisms. A more detailed model, maintaining the variability of the power-law exponent, produces qualitatively identical results. Many pulsars have their PFF set by nonresonant ICS, with curvature dominating at high potential, and resonant ICS at high magnetic field.

In addition, as a result of the effects of nonresonant ICS, many pulsars seem to operate with comparatively low pair multiplicities. The total $\kappa$ is in the range of 1–10 more often than it is in the 1000s predicted by curvature models.

The OTS approximation to the pair cascade describes the spectrum of $\gamma$-rays and particles produced by the pair production process very well, although the OTS pair spectrum underestimates the number of low-energy pairs, as a result of the decline of the magnetic field with altitude.

The full cascade model may be used to predict the $\gamma$-ray and pair spectra of individual pulsars. Although the pair spectrum produced will be modified by resonant ICS, the high-energy $\gamma$-ray spectrum predicted should be observable. The $\gamma$-ray output from pulsars will be the subject of a subsequent paper.

We wish to acknowledge the assistance of Alice Harding, who provided the data for the comparisons of Figures 4 and 5.
EMISSION MECHANISMS

For convenience, we describe the forms of the various emission mechanisms used in the numerical calculations.

A1. CURVATURE EMISSION

For curvature emission, we use the standard formula,

\[ \frac{\partial^2 N_e}{\partial t \partial \epsilon_1} = \frac{\sqrt{3}}{3\pi} \frac{\alpha_{e}c}{\lambda_c} \frac{1}{\gamma^2} \int_{\epsilon_{1c}}^{\infty} dx K_{5/3}(x), \]

where \( \epsilon_c = (3/2)\lambda_c \gamma^{-3} \) and \( K_{5/3} \) is the modified Bessel function of order 5/3.

A2. RESONANT ICS

For resonant ICS, we use the result from Dermer (1990),

\[ \frac{\partial^2 N_e}{\partial t \partial \epsilon_1} = \frac{1}{2} \frac{\alpha_{e}c}{\lambda_c} \frac{T}{\gamma^3 B^2} \ln \frac{\gamma T}{\epsilon_B} \cdot \frac{H}{\epsilon_B} \left( \epsilon_1, \epsilon_B, 2\epsilon_B \right), \]

where \( H \) is the top-hat function, defined before.

A3. NONRESONANT ICS

Although the curvature emission and resonant ICS spectra are well represented in the literature, the nonresonant ICS spectrum is not, to the accuracy we desire. The treatment of Sturner (1995), for example, examines the energy loss rate, but not the spectrum.

The chief difference between the physical situations considered by Blumenthal & Gould (1970) and conditions in the pulsar magnetosphere is that the background X-rays in pulsars are not isotropic. They originate from the hot surface of the star itself and only occupy a range in \( \theta \) from \( \theta = 0 \) to at most \( \theta = \pi \). In terms of \( \mu = \cos \theta \), thermal photons are emitted with angles from 1 to \( \mu \), where

\[ \mu_c = \frac{a}{\sqrt{a^2 + z^2}}, \]

where \( a \) is the radius of the polar cap, typically \( a = \theta_c R_* \), and \( z \) is the height above the stellar surface. For simplicity, we only consider emission along the \( z \)-axis, where the incoming photons are symmetric in \( \phi \). The range of \( \mu \) populated with photons is then

\[ \Delta \mu = 1 - \mu_c = 1 - \frac{z}{\sqrt{a^2 + z^2}}. \]

For the isotropic case, \( \Delta \mu = 2 \). If \( \Delta \mu < 2 \), the simplest consequence is that there are fewer photons, by a factor of \( \Delta \mu/2 \), potentially scattering off of the particle. Furthermore, these photons are concentrated near \( \mu = 1 \), so the mean energy of the photons in the particle rest frame is reduced.

From Blumenthal & Gould (1970), the full nonresonant inverse Compton cross section is

\[ \frac{\partial \sigma(\epsilon, \mu)}{\partial \epsilon_1 \partial \mu_1} = \frac{1}{2} \frac{\alpha_{e}c}{\lambda_c} \frac{T}{\gamma^3 B^2} \ln \frac{\gamma T}{\epsilon_B} \cdot \frac{H}{\epsilon_B} \left( \epsilon_1, \epsilon_B, 2\epsilon_B \right) \left[ \frac{\epsilon_1 - \epsilon'}{1 + (\epsilon'/mc^2)(1 + \mu_1^2)} \right]. \]

The primed frame is the particle rest frame, the unprimed frame is fixed to the neutron star itself, and the subscript 1 indicates the scattered particles and angles. We have made the approximation that all of the incoming photons in the particle rest frame are beamed down the \( z \)-axis, so that the angle between the incoming and the scattered photon is simply equal to the latitude of the scattered photon, \( \Theta' = \pi - \theta' \). This approximation works very well for large Lorentz factors but is less accurate in the Thomson regime. However, it gives negligible error in the Klein-Nishina regime.

The rest-frame scattered number emissivity is then

\[ j'(\epsilon'_1, \mu'_1) = 2\pi n' \int d\epsilon' d\mu' \frac{\partial \sigma(\epsilon', \mu')}{\partial \epsilon'_1 \partial \mu'_1} I(\epsilon', \mu'). \]

Using the \( \delta \)-function in the cross section to perform the \( \epsilon' \) integral yields

\[ j'(\epsilon'_1, \mu'_1) = \frac{1}{2} \frac{n'r_0}{\epsilon'_1} \left( \frac{\epsilon'_0 + \epsilon'_1}{\epsilon'_0} + (1 - \mu'_1^2) \right) 2\pi \int d\mu' I(\epsilon'_0, \mu'), \]

where

\[ \epsilon'_0 = \frac{\epsilon'_1}{1 - (\epsilon'_1/mc^2)(1 + \mu'_1)}. \]
The integral over the source intensity is

$$2\pi \int_{-1}^{1} d\mu' I(\epsilon_0, \mu') = 2\pi \int_{-1}^{1} d\mu' I(\epsilon(\epsilon_0, \mu'), \mu(\epsilon_0, \mu)) \left[ \frac{\epsilon_0}{\epsilon(\epsilon_0, \mu')} \right]^2. \quad (A9)$$

Changing variables to $\epsilon$, using $\epsilon = \gamma \epsilon_0 (1 + \beta \mu')$, this becomes

$$2\pi \int_{-1}^{1} d\mu' I(\epsilon_0, \mu') = 2\pi \frac{\epsilon_0}{\gamma \beta} \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} d\epsilon \ e^{-\epsilon^2 I[\epsilon, \mu]} \ . \quad (A10)$$

The limits of the $\epsilon$ integration are set by which energies may be scattered such that they have energy $\epsilon_0$ in the rest frame. The minimum possible energy is $\epsilon_{\text{min}} = \epsilon_0 / (1 - \beta \mu)$, where the maximum is $\epsilon_{\text{max}} = \epsilon_0 / (1 - \beta) \approx 2\gamma \epsilon_0$, where the approximations assume that $\beta \approx 1$.

The source function we adopt is that of a blackbody, emitting at temperature $T$ in units of $mc^2$, into a range of solid angle $\delta \mu = 1 - \mu_c$. This corresponds to a specific intensity of

$$I(\epsilon, \mu) = \frac{c}{4\pi^3} \frac{\epsilon^2}{\lambda_c^2} \exp \left( \frac{\epsilon}{T} \right) - 1 \ H(\mu, \mu_c, 1), \quad (A11)$$

where $H(x, a, b) = 1$ for $a < x < b$ and 0 otherwise.

Combining all these components, we find

$$2\pi \int_{-1}^{1} d\mu' I(\epsilon_0, \mu') = \frac{c}{2\pi^2} \frac{\epsilon_0}{\lambda_c} \gamma \beta \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} \frac{1}{\exp (\epsilon / T) - 1} \quad (A12)$$

$$= \frac{c}{2\pi^2} \frac{\epsilon_0}{\lambda_c} T \ln \left[ \frac{1 - \exp \left( -2\gamma \epsilon_0 / T \right)}{1 - \exp \left( -\epsilon_0 / T \right) \Delta \mu} \right]. \quad (A13)$$

The numerator in the log is equivalent to 1 for all energies with significant emission. We then have

$$N(\epsilon_1) = 2\pi \frac{c}{n} \int d\epsilon' d\mu' \frac{1}{\gamma (1 + \beta \mu')} \delta \left[ \epsilon' - \frac{\epsilon_1}{\gamma (1 + \beta \mu')} \right] f(\epsilon_1, \mu'),$$

$$N(\epsilon_1) = \frac{c}{2\pi} \frac{\alpha^2 c}{\lambda_c} \frac{T}{\gamma^2 \beta} \int d\epsilon' d\mu' \left[ \frac{1}{1 + \mu'} \right] \left[ 1 - \frac{\epsilon_0}{\epsilon_1} \right] \left[ 1 - \frac{\epsilon_0}{\epsilon_1} \right] \ln \left[ 1 - \exp \left( -\epsilon_0 / (1 + \beta \mu')(1 - \epsilon_1) T \Delta \mu \right) \right]. \quad (A14)$$

where we have substituted $\gamma \beta c = \alpha^2 / \lambda_c$.

If we then assume that $\beta \approx 1$ so $\epsilon_1 / \epsilon_0 = 1 - \epsilon_1 / \gamma$ and evaluate the $\epsilon_1$ integral, we obtain

$$\frac{\partial^2 N_{\epsilon}}{\partial t \partial \epsilon_1} = \frac{c}{2\pi} \frac{\alpha^2 c}{\lambda_c} \frac{\epsilon_1 T}{\gamma^2 \beta (1 - \epsilon_1)} \left[ 1 - \frac{1}{1 - \epsilon_1} \right] \left[ 1 - \frac{1}{1 - \epsilon_1} - 1 + \mu_1^2 \right] \ln \left[ 1 - \exp \left( -\epsilon_1 / (1 + \beta \mu')(1 - \epsilon_1) T \Delta \mu \right) \right]. \quad (A15)$$

where we have defined $\bar{\epsilon}_1 = \epsilon_1 / \gamma$.

Changing variables to

$$y = \frac{\epsilon_1}{\gamma^2 (1 + \beta \mu')(1 - \epsilon_1) T \Delta \mu}, \quad (A15)$$

gives, after setting $\beta = 1,

$$N(\epsilon_1) = \frac{c}{2\pi} \frac{\alpha^2 c}{\lambda_c} \frac{T^2 \Delta \mu}{\gamma^2 (1 - \epsilon_1)} \int dy [1 + (\mu_1^2 - 1)(1 - \epsilon_1) + (1 - \epsilon_1)^2] \ln (1 - e^{-y}). \quad (A16)$$

Approximating this further by replacing $\mu_1^2$ with its average over the range, $\langle \mu_1^2 \rangle$, and performing the integral yields

$$N(\epsilon_1) = \frac{c}{2\pi} \frac{\alpha^2 c}{\lambda_c} \frac{T^2 \Delta \mu}{\gamma^2 (1 - \epsilon_1)} \left[ 1 + (\langle \mu_1^2 \rangle - 1)(1 - \epsilon_1) + (1 - \epsilon_1)^2 \right] \text{Li}_2 \left( e^{-y} \right)_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}}, \quad (A17)$$

where

$$\int dy \ln (1 - e^{-y}) = \text{Li}_2 \left( e^{-y} \right), \quad (A18)$$

$$y_{\text{min}} = \frac{\epsilon_1}{2\gamma^2 (1 - \epsilon_1) T \Delta \mu}, \quad (A19)$$

$$y_{\text{max}} = \frac{\epsilon_1}{(1 - \epsilon_1) T \Delta \mu}, \quad (A20)$$
where \( \text{Li}_2(x) \) is the dilogarithm function, defined as

\[
\text{Li}_2(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^2}.
\]  

(A21)

The dilogarithm \( \text{Li}_2(x) = x \) for small \( x \), and \( \text{Li}_2(1) = \pi^2/6. \)

The angle average, \( \langle \mu_i^2 \rangle \), clearly lies between 0 and 1. Its value is given by

\[
\langle \mu_i^2 \rangle = \int_{z_0}^{z_1} \frac{dz z^{-2}(-\ln[1-\exp(-\kappa/z)])}{z^{-2}(-\ln[1-\exp(-\kappa/z)])},
\]  

(A22)

where

\[
\kappa \equiv \frac{\bar{v}}{\gamma^2 \Delta \mu T (1 - \bar{v}_i)},
\]  

(A23)

\[
z \equiv 1 + \beta \mu_i',
\]  

(A24)

\[
z_0 = 1 - \beta \approx 0,
\]  

(A25)

\[
z_1 = 1 + \beta \approx 2.
\]  

(A26)

Adopting the approximate forms of \( z_0 \) and \( z_1 \) does not perceptibly change the value of the integral but reduces \( \langle \mu_i^2 \rangle \) to being a function only of the single parameter \( \kappa \). A good approximation to the value of \( \langle \mu_i^2 \rangle \) is

\[
\langle \mu_i^2 \rangle \approx 1 - 0.76 \exp \left[ \frac{-(\ln \kappa)^2}{10} \right].
\]  

(A27)

For the computation, we simply tabulate \( \langle \mu_i^2 \rangle \).

The power emitted by the scattering has two asymptotic limits. If \( \gamma T \Delta \mu < 1 \), then emitted power matches the Thomson value,

\[
P_n^{(T)} = \frac{4\pi^3}{135} \frac{\alpha_F^2 c}{\kappa_c} \gamma^2 \Delta \mu^3 T^4,
\]  

(A28)

while if \( \gamma T \Delta \mu > 1 \), the emitted power is approximately

\[
P_n^{(KN)} \approx \frac{\pi}{12} \frac{\alpha_F^2 c}{\kappa_c} T^2 \Delta \mu \ln 2 \gamma T \Delta \mu,
\]  

(A29)

which matches the Klein-Nishina limit of Blumenthal & Gould (1970) when \( \Delta \mu = 2 \). The transition between the two limits is purely a function of \( \gamma T \Delta \mu \). The emitted power is

\[
P_n = \frac{1}{2\pi} \frac{\alpha_F^2 c}{\kappa_c} T^2 \Delta \mu f(\gamma T \Delta \mu),
\]  

(A30)

where \( f \) is a numerically calculated function that has asymptotic limits

\[
f(x) \approx \begin{cases}
8\pi^4 x^2 / 135, & x \ll 1, \\
\pi^2 / 6 \ln 2x, & x \gg 1.
\end{cases}
\]  

(A31)

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