Study of the Wheeler Propagator *

C.G. Bollini$^1$ and M.C. Rocca$^{1,2}$

$^1$Departamento de Física, Fac. de Ciencias Exactas,
Universidad Nacional de La Plata.
C.C. 67 (1900) La Plata. Argentina.

$^2$Departamento de Matemáticas, Fac. de Ciencias Exactas
Universidad Nacional del Centro de la Pcia de Bs. As.
Pintos 390, C.P.7000, Tandil. Argentina.

August 1, 1997

Abstract

*This work was partially supported by Consejo Nacional de Investigaciones Científicas
and Comisión de Investigaciones Científicas de la Pcia. de Buenos Aires; Argentina.
We study the half advanced and half retarded Wheeler Green function and its relation to Feynman propagators. First for massless equation. Then, for Klein-Gordon equations with arbitrary mass parameters; real, imaginary or complex. In all cases the Wheeler propagator lacks an on-shell free propagation. The Wheeler function has support inside the light-cone (whatever the mass). The associated vacuum is symmetric with respect to annihilation and creation operators.

We show with some examples that perturbative unitarity holds, whatever the mass (real or complex). Some possible applications are discussed.

PACS: 10.14.14.80-j 14.80.Pb
1 Introduction

More than half a century ago, J. A. Wheeler and R. P. Feynman published a work [1] in which they represented electromagnetic interactions by means of a half advanced and half retarded potential. The charged medium was supposed to be a perfect absorber, so that no radiation could possibly escape the system.

We are going to call this kind of potential a “Wheeler function” (or propagator), although it had been used before by P. A. M. Dirac [2] when trying to avoid some run-away solutions. Later, in 1949, J. A. Wheeler and R. P. Feynman showed that, in spite of the fact that it contains an advanced part, the results do no contradict causality [3].

Of course, the success of QED and renormalization theory made soon unnecessary or not advisable, to follow that line of research (at least for electromagnetism).

One of the distinctive characteristics of the Green function used in references [1, 2, 3] is its lack of asymptotic free waves. This is the reason behind the choice of a “perfect absorber” for the medium through which the field propagates. As the quantization of free waves is associated to free particles,
the above mentioned feature of Wheeler functions imply that no free quantum of the field can ever be observed. Nevertheless, we are now habituated to the existence of confined particles. They do not manifest themselves as free entities.

We can give some examples (outside QCD) where such a behaviour can be present.

A Lorentz-invariant higher order equation can be decomposed into Klein-Gordon factors, but the corresponding mass parameters need not be real. For instance, the equation:

\[
\left( \Box^2 + m^4 \right) \varphi = \left( \Box + im^2 \right) \left( \Box - im^2 \right) \varphi = 0 \tag{1}
\]

gives rise to a pair of constituent fields [4] obeying:

\[
\left( \Box \pm im^2 \right) \varphi_{\pm} = 0 \tag{2}
\]

Any solution of eq.(2) blows-up asymptotically. We can say that the corresponding fields should be forbidden to appear asymptotically as free waves. Therefore, they should have a Wheeler function as propagator [5].

Equations similar to (1), or more general:
\begin{align*} 
(\Box^n \pm m^{2n}) \varphi = 0 
\tag{3}
\end{align*}

appear in a natural way in supersymmetric models for higher dimensional spaces [6].

Another example is provided by fields obeying Klein-Gordon equations with the wrong sign of the mass term. A careful analysis shows that the propagator should be a Wheeler function [7, 8]. Accordingly no tachyon can ever be observed as a free particle. They can only exist as “mediators” of interactions.

To define the propagators in a proper way, we have to solve the equations for the Green functions, with suitable boundary conditions.

For the wave equation:

\begin{align*} 
\Box \tilde{G}(x) = \delta(x) \tag{4}
\end{align*}

a Fourier transformation gives:

\begin{align*} 
G(p) = \left(\vec{p}^2 - p_0^2\right)^{-1} &\equiv (p_\mu p^\mu)^{-1} \equiv P^{-1} \tag{5}
\end{align*}

Of course, it is necessary to specify the nature of the singularity. Different
determinations imply different types of Green functions. For the classical solution of (4) it is natural to use the retarded function \( \tilde{G}_{rt} \). It corresponds to the propagation towards the future of the effect produced by the sources. This function can be obtained by means of a Fourier transform of (5) in which the \( p_0 \) integration is taken along a path from \(-\infty\) to \(+\infty\), leaving the poles to the right. In practice, we add to \( p_0 \) a small positive imaginary part:

\[
G_{rt}(p) = \left[ \vec{p}^2 - (p_0 + i0)^2 \right]^{-1} = \left( \vec{p}^2 - p_0^2 - i0 \text{sgn}p_0 \right)^{-1} = (P - i0 \text{sgn}p_0)^{-1}
\]

(6)

The advanced solution is the complex conjugate of (6):

\[
G_{ad}(p) = \left( \vec{p}^2 - p_0^2 + i0 \text{sgn}p_0 \right)^{-1} = (P + i0 \text{sgn}p_0)^{-1}
\]

(7)

For the Feynman propagator we have to add a small imaginary part to \( P \) (not just to \( p_0 \)):

\[
G_{\pm}(p) = (P \pm i0)^{-1}
\]

(8)

And, in the massive case:

\[
G_{\pm}(p) = \left( P + m^2 \pm i0 \right)^{-1}
\]

(9)
The Wheeler function is half advanced and half retarded. It is easy to see that it is also half Feynman and half its conjugate (we will not use any index for the Wheeler propagator):

\[
G(p) = \frac{1}{2}G_+(p) + \frac{1}{2}G_-(p) \tag{10}
\]

On the real axis, the Wheeler function coincides with Cauchy’s “principal value” Green function, which is known to be zero on the mass-shell (no free waves).

We can write:

\[
G_\pm(p) = G(p) \pm i\pi\delta\left(P + m^2\right) \tag{11}
\]

where:

\[
 i\pi\delta\left(P + m^2\right) = \frac{1}{2}G_+(p) - \frac{1}{2}G_-(p) \tag{12}
\]

Equation (11) is a decomposition of the Feynman function into two terms. The first one only contains virtual propagation. The second one is a Lorentz invariant solution of the homogeneous equation representing the free particle.

To perform convolution integrations in \(p\)-space, we will utilize the method
presented in reference [9]. Essentially, it consists in the use of the Bochner theorem for the reduction of the Fourier transform to a Hankel transform. The nucleus of this transformation is made to correspond to an arbitrary number of dimensions $\nu$, taken as a free parameter. In this way, starting with a given propagator in $p$-space, we get a function in $x$-space whose singularity at the origin depends analytically on $\nu$. It exists then a range of values (of $\nu$) such that the product of Green functions is allowed and well determined.

In $x$-space we define:

$$Q = r^2 - x_0^2 = x_\mu x^\mu$$

The Fourier transform of the massless Feynman function is:

$$\mathcal{F}\{(P - i0)^{-1}\}(x) = \frac{1}{(2\pi)^\frac{\nu}{2}} \int d^\nu p (P - i0)^{-1} e^{ipx}$$

By means of a “Wick rotation”, the $p_0$-integration can be made to run along the imaginary axis, without crossing any pole. Mathematically, we perform a dilatation:

$$p_0 = ap'_0 ; \quad x_0 = \frac{1}{a} x'_0 ; \quad p_0 x_0 = p'_0 x'_0$$  \hspace{1cm} (13)$$

A subsequent continuation to $a = i$ produces the transformation:
\[ P \Rightarrow \vec{p}^2 + p_0^2 = P' \]
\[ Q \Rightarrow \vec{x}^2 + x_0^2 = Q' \]

The new quadratic forms are Euclidean and Bochner theorem [10] tells that the Fourier transformation reduces to:

\[
\mathcal{F} \left\{ (P - i0)^{-1} \right\} (x) = \frac{i}{x^{\frac{\nu}{2}}} \int_0^\infty dy \frac{y^{\nu}}{y^{2}} \mathcal{J}_{\nu-1}(xy) \tag{14}
\]

where \( \mathcal{J}_\alpha \) is a Bessel function of the first kind and order \( \alpha \).

In eq.(13) we see that a Wick rotation in p-space \( (a \rightarrow i) \) implies an anti-Wick rotation in x-space \( (a^{-1} \rightarrow -i) \). We must then choose:

\[ x = (Q + i0)^{\frac{1}{2}} \]

Note that the imaginary unit in eq.(14) (r.h.s.), is due to the transformation \( dp_0 \rightarrow idp'_0 \).

From ref.[11] we take:

\[
\int_0^\infty dy \ y^\mu \mathcal{J}_\mu(ay) = 2^\mu a^{-\mu-1} \frac{\Gamma \left( \frac{1+\rho+\mu}{2} \right)}{\Gamma \left( \frac{1+\rho-\mu}{2} \right)}
\]

I.e.
\[ \mathcal{F} \left\{ (P - i0)^{-1} \right\} (x) = i2^{\frac{\nu}{2}} \Gamma \left( \frac{\nu}{2} - 1 \right) (Q + i0)^{1-\frac{\nu}{2}} \]  

More general, for a function \( f(P \pm i0) \), we obtain:

\[ \mathcal{F} \left\{ f(P \pm i0) \right\} (x) = \mp \frac{i}{x^{\frac{\nu}{2} - 1}} \int_0^\infty dy \ y^\frac{\nu}{2} \ f \left( y^2 \right) J_{\frac{\nu}{2} - 1}(xy) \]  

where \( x = (Q \mp i0)\frac{1}{2} \).

The right hand side of eq.(16) is a Hankel transform of the function \( f(y^2) \) [12].

## 2 Massless case

a) Fourier transforms

With the procedures described in section 1, we can obtain the Fourier transforms of general massless Feynman functions, defined as \( (P \pm i0)^\alpha \).

From (18) and ref.[11], we get ( compare with ref.[13] ):

\[ \mathcal{F} \left\{ (P \pm i0)^\alpha \right\} (x) = \mp i2^{2\alpha + \frac{\nu}{2}} \frac{\Gamma \left( \alpha + \frac{\nu}{2} \right)}{\Gamma(-\alpha)} (Q \mp i0)^{-\alpha - \frac{\nu}{2}} \]  

The exponent of \( Q \mp i0 \) can be deduced by dimensional considerations.
Furthermore, if we interchange the quadratic forms $P \leftrightarrow Q$ and write $\mathcal{F}^{-1}$ for $\mathcal{F}$, then eq.(17) is still valid.

We define the massless Wheeler propagator as:

$$P^\alpha = \frac{1}{2} (P + i0)^\alpha + \frac{1}{2} (P - i0)^\alpha$$  \hspace{1cm} (18)

(We will not use any index when a Wheeler function is meant)

The Fourier transform of (18) is (cf. eq.(17)):

$$\mathcal{F}\{P^\alpha\}(x) = i2^{2\alpha+\frac{\nu}{2}} \frac{\Gamma\left(\alpha + \frac{\nu}{2}\right)}{\Gamma\left(-\alpha\right)} \left[\frac{1}{2} (Q + i0)^{-\alpha-\frac{\nu}{2}} - \frac{1}{2} (Q - i0)^{-\alpha-\frac{\nu}{2}}\right]$$ \hspace{1cm} (19)

But we also have the relation (valid for any quadratic form [13])

$$(Q \pm i0)^\lambda = Q_+^\lambda + e^{\pm i\pi \lambda} Q_-^\lambda$$  \hspace{1cm} (20)

where

$$Q_+^\lambda = \begin{cases} Q^\lambda & Q > 0 \\ 0 & Q \leq 0 \end{cases}$$

$$Q_-^\lambda = \begin{cases} (-Q)^\lambda & Q < 0 \\ 0 & Q \geq 0 \end{cases}$$
So that we can write (19) in the form:

$$\mathcal{F}\{P^\alpha\}(x) = 2^{2\alpha + \frac{\nu}{2}} \frac{\Gamma\left(\alpha + \frac{\nu}{2}\right)}{\Gamma(-\alpha)} \sin\pi \left(\alpha + \frac{\nu}{2}\right) Q^{-\alpha - \frac{\nu}{2}}$$

(21)

Equation (21) shows another interesting property of Wheeler functions. They are real and have support inside the light-cone of the coordinates. Furthermore, for \(\alpha = -1\), the trigonometric function tends to zero for \(\nu \to 4\), but \(Q^{-1 - \frac{\nu}{2}}\) has a pole at \(\nu = 4\) with residue \(\delta(Q)\) [14]. Then:

$$\lim_{\nu \to 4} \mathcal{F}\{P^{-1}\}(x) = \delta(Q)$$

(22)

In four dimensions the massless Wheeler function is concentrated on the light cone. From eq.(21) we obtain:

$$\mathcal{F}^{-1}\{Q^\lambda\}(p) = -\frac{2^{2\lambda + \frac{\nu}{2}} \Gamma\left(\lambda + \frac{\nu}{2}\right)}{\sin\pi \lambda \Gamma(-\lambda)} \frac{\pi}{\sin \pi z}$$

$$= 2^{2\lambda + \frac{\nu}{2}} \Gamma\left(\lambda + 1\right) \Gamma\left(\lambda + \frac{\nu}{2}\right) P^{-\lambda - \frac{\nu}{2}}$$

(23)

where the relation

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

has been used.

From (17) we can also get:
\[ F^{-1} \{ Q^\lambda_+ \} (p) = -2^{2\lambda + \frac{\nu}{2}} \Gamma (\lambda + 1) \Gamma \left( \lambda + \frac{\nu}{2} \right) \times \]
\[ \left[ \cos \pi \lambda P^\lambda_+ - \frac{\nu}{2} \cos \pi \nu P^{-\lambda - \frac{\nu}{2}}_+ \right] \]  
(24)

From relation (12) we find:

\[ 2\pi i \delta(P) = (P - i0)^{-1} - (P + i0)^{-1} \]  
(25)

And

\[ F \{ \delta(P) \} (x) = \frac{2^{\nu - 2}}{2\pi} \Gamma \left( \frac{\nu}{2} - 1 \right) \left[ (Q + i0)^{1 - \frac{\nu}{2}} + (Q - i0)^{1 - \frac{\nu}{2}} \right] \]  
(26)

\[ F \{ \delta(P) \} (x) = \frac{2^{\nu - 2}}{2\pi} \Gamma \left( \frac{\nu}{2} - 1 \right) \left[ P^{1 - \frac{\nu}{2}}_+ - \cos \frac{\pi \nu}{2} Q^{1 - \frac{\nu}{2}}_- \right] \]  
(27)

Note that eq.(21) allows us to obtain the advanced and retarded components of the Wheeler functions in x-space.

\[ \tilde{G}_{rt}(x) = 2\Theta(t) F \{ P^\alpha \} (x) \quad \tilde{G}_{ad}(x) = 2\Theta(-t) F \{ P^\alpha \} (x) \]

where \( \Theta(t) \) is Heaviside's step function.

b) Convolutions
The well-known convolution theorem asserts that the Fourier transform of a convolution product is the product of the Fourier transform of each factor:

\[ f(p) * g(p) = c \mathcal{F}^{-1} \left\{ \mathcal{F} \{ f(p) \} (x) \mathcal{F} \{ g(p) \} (x) \right\} (p) \]

\[ c = (2\pi)^{\nu/2} \quad (28) \]

Formula (28) can be taken as a definition of the operation \(*\). The product of distributions inside the curly brackets, can be taken in a suitable range of \(\nu\), and analytically extended to other values.

For Feynman propagators (28) gives:

\[ (P - i0)^\alpha * (P - i0)^\alpha = ic2^{-\frac{\nu}{2}} \frac{\Gamma^2 \left( \alpha + \frac{\nu}{2} \right)}{\Gamma \left( -\alpha \right)} \times \]

\[ \frac{\Gamma \left( -2\alpha - \frac{\nu}{2} \right)}{\Gamma \left( 2\alpha + \nu \right)} (P - i0)^{2\alpha + \frac{\nu}{2}} \quad (29) \]

An analogous equation can be obtained by changing the sign of the imaginary unit in (29). The convolution of two Feynman functions of the same kind (+ or -) gives another Feynman function of the same kind.

For Wheeler propagators, eqs.(21), (23) and (28) give:

\[ P^\alpha * P^\alpha = 2^{-\frac{\nu}{2}} c \frac{\Gamma^2 \left( \alpha + \frac{\nu}{2} \right)}{\Gamma \left( -\alpha \right)} \times \frac{\Gamma \left( -2\alpha - \frac{\nu}{2} \right)}{\Gamma \left( 2\alpha + \nu \right)} \times \]

\[ t g \pi \left( \alpha + \frac{\nu}{2} \right) P^{2\alpha + \frac{\nu}{2}} \quad (30) \]
So that the convolution of two Wheeler functions gives another Wheeler function.

For the wave equation we choose $\alpha = -1$:

\[
(P - i0)^{-1} \cdot (P - i0)^{-1} = 2ia(\nu) (P - i0)^{\frac{\nu}{2} - 2} \tag{31}
\]

\[
(P + i0)^{-1} \cdot (P + i0)^{-1} = -2ia(\nu) (P + i0)^{\frac{\nu}{2} - 2} \tag{32}
\]

\[
P^{-1} \ast P^{-1} = a(\nu) \tan\pi \left(\frac{\nu}{2} - 1\right) P^{\frac{\nu}{2} - 2} \tag{33}
\]

where

\[
a(\nu) = c2^{-\frac{\nu}{2} - 1}\Gamma^2 \left(\frac{\nu}{2} - 1\right) \frac{\Gamma \left(2 - \frac{\nu}{2}\right)}{\Gamma (\nu - 2)}
\]

Eqs.(31) and (32) have a pole for $\nu \to 4$ (the usual ultraviolet divergence), while (33) is well determined in that limit.

The convolution of $\delta(P)$ function can be found with the help of eqs.(23), (24), (27) and (28):

\[
\pi^2 \delta(P) \ast \delta(P) = a(\nu) \tan\pi \left(\frac{\nu}{2} - 1\right) \left[P^{\frac{\nu}{2} - 2} - \cos\frac{\pi}{2} \nu P^{\frac{\nu}{2} - 2}\right] \tag{34}
\]

A comparison with eq.(31) shows that:

\[
\pi^2 \delta(P) \ast \delta(P) = \text{sgn}P \cdot P^{-1} \ast P^{-1} \tag{35}
\]
This relation implies:

\[(P - i0)^{-1} \ast (P + i0)^{-1} + \pi^2 \delta(P) \ast \delta(P) = 2\Theta(P) \ast P^{-1} \ast P^{-1}\]

The convolution of two Feynman propagators of different kinds has support outside the light-cone in p-space.

3 Massive case

A bradyon field obeys a normal Klein-Gordon equation. Its Feynman propagator is given by eq.(9). The Wheeler function is:

\[ (P + m^2)^{-1} = \frac{1}{2} (P + m^2 + i0)^{-1} + \frac{1}{2} (P + m^2 - i0)^{-1} \]  

(36)

The on-shell \(\delta\)-function, solution of the homogeneous equation is:

\[ \delta (P + m^2) = \frac{1}{2\pi i} \left[ (P + m^2 + i0)^{-1} - (P + m^2 - i0)^{-1} \right] \]  

(37)

\textbf{a) Fourier transforms}

To find the Fourier transform of the Feynman propagators we use eq.(16) and ref.[11] ( p. 687 - 6.566 - 2 ):

\[ \mathcal{F} \left\{ (P + m^2 \pm i0)^{-1} \right\} (x) = \mp im^{\frac{3}{2} - 1} (Q \mp i0)^{\frac{1}{2}(1 - \frac{3}{2})} K_{\frac{3}{2} - 1} \left[ m (Q \mp i0)^{\frac{3}{2}} \right] \]
where $\mathcal{K}_\alpha$ is a Bessel function of the third kind and order $\alpha$. (For the definition of different Bessel functions we follow ref.[11], p.951 - 8.40). Note also that with the more general formula of ref.[11] (p.687 - 6.565 - 4), we could find the Fourier transform of arbitrary powers of the Feynman propagator.

Using the relation (20), we can write:

$$\mathcal{F}\left\{\left(P + m^2 \pm i0\right)^{-1}\right\}(x) = \mp im^{\frac{\nu}{2} - 1} \left[ Q^{\frac{1}{2}(1-\frac{\nu}{2})}_+ \mathcal{K}_{\frac{\nu}{2} - 1} \left( mQ^\frac{1}{2}_+ \right) + \right.$$  
$$\left. \frac{i}{2} \pi Q^{\frac{1}{2}(1-\frac{\nu}{2})}_- \mathcal{H}_{1-\frac{\nu}{2}} \left( mQ^\frac{1}{2}_- \right) \right]$$  

(38)

where $\beta = 1$ for the upper sign and $\beta = 2$ for the lower sign.

It is now easy to find the transforms of (36) and (37):

$$\mathcal{F}\left\{\left(P + m^2\right)^{-1}\right\}(x) = \frac{\pi}{2} m^{\frac{\nu}{2} - 1} Q^{\frac{1}{2}(1-\frac{\nu}{2})}_- \mathcal{J}_{1-\frac{\nu}{2}} \left( mQ^\frac{1}{2}_- \right)$$  

(39)

$$\mathcal{F}\left\{\delta \left(P + m^2\right)\right\}(x) = -\frac{1}{\pi} m^{\frac{\nu}{2} - 1} \left[ Q^{\frac{1}{2}(1-\frac{\nu}{2})}_+ \mathcal{K}_{\frac{\nu}{2} - 1} \left( mQ^\frac{1}{2}_+ \right) - \right.$$  
$$\left. \frac{\pi}{2} Q^{\frac{1}{2}(1-\frac{\nu}{2})}_- \mathcal{N}_{1-\frac{\nu}{2}} \left( mQ^\frac{1}{2}_- \right) \right]$$  

(40)

Also for the massive case, the Wheeler function is zero outside the light-cone.

With some slight changes in notation, we can check that (38) and (40) coincide with the results of ref.[13].
b) Convolutions

We are going to follow the procedure already used in section 2.

For the convolution of a massive Feynman propagator with a massless one, eqs.(17), (38) and (28) give:

\[
\left( P + m^2 - i0 \right)^{-1} * (P - i0)^{-1} = -2^{\frac{\nu}{2} - 1} cm^{\nu - 1} \Gamma \left( \frac{\nu}{2} - 1 \right) \times \\
\mathcal{F}^{-1} \left\{ (Q + i0)^{\frac{\nu}{2}} \mathcal{K}_{\frac{\nu}{2} - 1} \left[ m (Q + i0)^{\frac{1}{2}} \right] \right\} (p)
\]

And with the help of eq.(16) and ref.[11] (p.643 - 6.576 - 3), we get:

\[
\left( P + m^2 - i0 \right)^{-1} * (P - i0)^{-1} = \pi \delta \left( P + m^2 \right)^{-1} \times \\
\frac{m^{\nu - 2}}{\Gamma \left( \frac{\nu}{2} \right)} \Gamma \left( \frac{\nu}{2} - 1 \right) \Gamma \left( 2 - \frac{\nu}{2} \right) \times \\
F \left( 1, 2 - \frac{\nu}{2}, \frac{\nu}{2}; - \frac{P - i0}{m^2} \right) \quad (41)
\]

where \( F(a, b, c; z) \) is Gauss hypergeometric function.

For two Wheeler propagators and two on-shell \( \delta \)-functions we can follow the same method. It is then not difficult to prove the relation (compare with (35)):

\[
\pi^2 \delta \left( P + m^2 \right) * \delta (P) = sgnP \cdot \left( P + m^2 \right)^{-1} * P^{-1}
\]

Or, more general:
\[ \pi^2 \delta \left( P + m_1^2 \right) \ast \delta \left( P + m_2^2 \right) = \text{sgn} P \cdot \left( P + m_1^2 \right)^{-1} \ast \left( P + m_2^2 \right)^{-1} \]  \hbox{(42)}

4 Tachyons

A tachyon field obeys a Klein-Gordon equation with the wrong sign of the “mass” term. The Green function is an inverse of \( P - \mu^2 \) (we use \( \mu^2 = -m^2 \) for the mass of the tachyon).

Although it is not easy to see what a Feynman propagator should be when the inverse of \( P - \mu^2 = p^2 - p_0^2 - \mu^2 \) has a pair of imaginary poles (if \( \bar{p}^2 < \mu^2 \)), we may nevertheless define:

\[ \left( P - \mu^2 \pm i0 \right)^{-1} = - \left( -P + \mu^2 \mp i0 \right)^{-1} \]

Or, introducing the “dual” quadratic form:

\[ +P = -P = p_0^2 - \bar{p}^2 \]

\[ \left( P - \mu^2 \pm i0 \right)^{-1} = - \left( +P + \mu^2 \mp i0 \right)^{-1} \]

In other words, instead of propagators with the wrong sign of the mass, we can say that we have propagators with the wrong sign of the metric.
To find the Fourier transform, we recall (see section 2) that we have an “i” for the Wick rotation of $p_0$. For the dual metric we have three Wick rotations. The three factors $d\vec{p}$ contribute with $i^3 = -i$. We also note that $+Q_+ = -Q_-$ and $+Q_- = Q_+$. So that (compare with (38)): $$F\left\{ \left( P - \mu^2 \pm i0 \right)^{-1} \right\}(x) = \pm i\mu^{\frac{\varepsilon}{2}} \left[ \frac{K_{\frac{\varepsilon}{2}-1} \left( \mu Q_+^{\frac{\varepsilon}{2}} \right)}{Q_-^{\frac{1}{2}}(\frac{\varepsilon}{2}-1)} \right]$$

$$= \pi \frac{\beta}{2} \frac{H_{\frac{\varepsilon}{2}-1} \left( \mu Q_+^{\frac{\varepsilon}{2}} \right)}{Q_+^{\frac{1}{2}}(\frac{\varepsilon}{2}-1)}$$

(43)

where $\beta = 2$ for the upper sign and $\beta = 1$ for the lower sign.

The real part of (43) is:

$$\frac{1}{2} F\left\{ \left( P - \mu^2 + i0 \right)^{-1} \right\}(x) + \frac{1}{2} F\left\{ \left( P - \mu^2 - i0 \right)^{-1} \right\}(x) =$$

$$ReF\left\{ \left( P - \mu^2 \pm i0 \right)^{-1} \right\}(x) = \frac{\pi}{2} \mu^{\frac{\varepsilon}{2}-1} Q_+^{\frac{1}{2}(1-\frac{\varepsilon}{2})} J_{\frac{\varepsilon}{2}-1} (\mu Q_+)$$

(44)

This real part has support outside the light-cone, while for bradyons, the real part of the Feynman propagator is zero for $x^\mu$ space-like (cf. eq.(39)).

We will now show that (44) is not the Wheeler propagator for the tachyon. To this aim, we go back to the original definition. Namely, a half retarded and half advanced propagator.
\[(P - \mu^2)^{-1} = \frac{1}{2} (P - \mu^2)^{-1}_{Ad} + \frac{1}{2} (P - \mu^2)^{-1}_{Rt} \quad (45)\]

The Fourier transform is:

\[
\mathcal{F} \left\{ (P - \mu^2)^{-1} \right\} (x) = \frac{1}{(2\pi)^{\nu/2}} \frac{1}{2} \int d^\nu p \frac{e^{ipx}}{(P - \mu^2)_{Ad}} + \frac{1}{(2\pi)^{\nu/2}} \frac{1}{2} \int d^\nu p \frac{e^{ipx}}{(P - \mu^2)_{Rt}} \quad (46)
\]

We will first evaluate the advanced part of (46):

\[
\mathcal{F} \left\{ (P - \mu^2)^{-1}_{Ad} \right\} (x) = \frac{1}{(2\pi)^{\nu/2}} \int d^{\nu-1} p e^{i\vec{p} \cdot \vec{r}} \int_{Ad} dp_0 \frac{e^{-ip_0x_0}}{p^2 - p_0^2 - \mu^2} \quad (47)
\]

Where the path of integration runs parallel to the real axis and below both poles of the integrand. For \(x_0 < 0\) the path can be closed on the upper half plane of \(p_0\). The contribution will be the residues at the poles:

\[
p_0 = \pm \omega = \pm \sqrt{\vec{p}^2 - \mu^2} \quad , \quad \text{if} \quad \vec{p}^2 \geq \mu^2
\]

\[
p_0 = \pm i\omega' = \pm i\sqrt{\mu^2 - \vec{p}^2} \quad , \quad \text{if} \quad \vec{p}^2 \leq \mu^2
\]

\[
\mathcal{F} \left\{ (P - \mu^2)^{-1}_{Ad} \right\} (x) = \frac{1}{(2\pi)^{\nu/2}} \int d^{\nu-1} p e^{i\vec{p} \cdot \vec{r}} 2\pi i \left[ \left( \frac{e^{i\omega x_0}}{2\omega} - \frac{e^{-i\omega x_0}}{2\omega} \right) \Theta \left( \vec{p}^2 - \mu^2 \right) \right.
\]

\[
+ \left( \frac{-e^{i\omega' x_0}}{2i\omega'} + \frac{e^{-i\omega' x_0}}{2i\omega'} \right) \Theta \left( \mu^2 - \vec{p}^2 \right) \right]
\]
\[
\frac{-2\pi}{(2\pi)^{\nu/2}} \int d^{\nu-1}p e^{i\vec{p}\cdot\vec{r}} \left[ \frac{\sin\omega x_0}{\omega} \Theta(\bar{p}^2 - \mu^2) + \frac{\sinh\omega' x_0}{\omega'} \Theta(\mu^2 - \bar{p}^2) \right]
\]

Note that we can write the brackets as (cf. eq. (20))

\[
\left[ \frac{\sin\omega x_0}{\omega} \Theta(\bar{p}^2 - \mu^2) + \frac{\sinh\omega' x_0}{\omega'} \Theta(\mu^2 - \bar{p}^2) \right] = \sin \left[ \frac{x_0 (\bar{p}^2 - \mu^2 + i0)^{\frac{1}{2}}}{(\bar{p}^2 - \mu^2 + i0)^{\frac{1}{2}}} \right] \frac{\sin x_0 \Omega}{\Omega}
\]

For the spatial Fourier transform (48), we use Bochner theorem (cf. eq.14))

\[
\mathcal{F} \left\{ \left( P - \mu^2 \right)^{-1} \right\}(x) = (2\pi)^{\frac{1}{2}} \frac{\Theta(-t)}{t^{\nu + 1}} \int_0^\infty dk k^{\nu - 1} \frac{\sin |t| \Omega}{\Omega} \mathcal{J}_{\nu - 1}(rk)
\]

Formula 6.737-5, p.761 of the table of ref.[11] gives \((b = i\mu + 0)\)

\[
\mathcal{F} \left\{ \left( P - \mu^2 \right)^{-1} \right\}(x) = \pi \mu^{\frac{\nu}{2} - 1} Q_{-}^{\frac{1}{2} - \frac{\nu}{2}} \mathcal{L}_{1 - \frac{\nu}{2}} \left( \mu Q_{-}^{\frac{1}{2}} \right) \quad (x_0 < 0)
\]

And of course:

\[
\mathcal{F} \left\{ \left( P - \mu^2 \right)^{-1} \right\}(x) = 0 \quad for \ x_0 \geq 0
\]

The retarded Fourier transform reproduces (48), with a change of sign, for \(x_0 > 0\) and is zero for \(x_0 \leq 0\). We then get, for the Wheeler Green function (46):
\[
\mathcal{F}\left\{(P - \mu^2)^{-1}\right\}(x) = \frac{\pi}{2^{1/2}} \mu^{\frac{1}{2} - 1} Q_\mu^{\frac{1}{2}}(1 - \xi) I_{1-\xi} \left(\mu Q_\mu^{\frac{1}{2}}\right)
\]

(51)

Again, the Wheeler propagator has support inside the light-cone. But, instead of a Bessel function of the first kind, we have now a Bessel function of the second kind.

It is clear that by taking into account eq.(28), we can evaluate convolutions of different Green functions. As it was done in section 3 for bradyons fields.

5 Fields with complex mass parameters

The decomposition in Klein-Gordon factors of a higher order equation, often leads to complex mass parameters. Equation (1) is an example. The constituent fields obey eq.(2). A simple higher order equation such as (3) presents the same behaviour. Of course for a real equation the masses come in complex conjugate pairs. We consider:

\[
\left(\Box - M^2\right) \phi = 0 \ , \ M = m + i\mu \quad (m > 0)
\]

(52)

This type of equation has been analyzed elsewhere (ref.[5]). The Green
functions are inverses of \( P + M^2 = \Omega^2 - p_0^2 \), where \( \Omega = (\bar{p}^2 + M^2)^{1/2} \). The two poles at \( p_0 \pm \Omega \), move when \( \bar{p}^2 \) varies from 0 to \( \infty \), on a line contained in a band of width \( \pm i\mu \), centered at the real axis.

The retarded Green function is obtained with a \( p_0 \)-integration that runs parallel to the real axis, with \( \text{Im} p_0 > |\mu| \). For the advanced solution, the integration runs below both poles (\( \text{Im} p_0 < -|\mu| \)):

\[
\mathcal{F}\left\{ (P + M^2)^{-1}_\text{Ad} \right\}(x) = \frac{1}{(2\pi)^{\nu/2}} \int d^{\nu-1}p \ e^{i\vec{p} \cdot \vec{x}} \int_{Ad} dp_0 \ \frac{e^{i\text{pot}}}{\Omega^2 - p_0^2} = \\
\frac{1}{(2\pi)^{\nu/2}} \int d^{\nu-1}p \ e^{i\vec{p} \cdot \vec{r}} \frac{2\pi i \Theta(-t)}{2\Omega} \left( e^{i\Omega t} - e^{-i\Omega t} \right) = \\
-2\pi \frac{\Theta(-t)}{(2\pi)^{\nu/2}} \int d^{\nu-1}p \ e^{i\vec{p} \cdot \vec{r}} \frac{\sin \Omega t}{\Omega} \\
\mathcal{F}\left\{ (P + M^2)^{-1}_\text{Ad} \right\}(x) = (2\pi)^{\frac{\nu}{2}} \Theta(-t) M^{\frac{\nu}{2} - 1} Q_{\frac{\nu}{2}}^{\frac{1}{2}}(1 - \nu) \int_0^\infty dk \ K_{\frac{\nu}{2} - 1} \frac{\sin \Omega t}{\Omega} J_{\frac{\nu}{2} - 1}(r k) 
\]

And according to ref.[11] (6.735 - 5):

\[
\mathcal{F}\left\{ (P + M^2)^{-1}_\text{Ad} \right\}(x) = \pi \Theta(-t) M^{\frac{\nu}{2} - 1} Q_{\frac{\nu}{2}} \left( M Q_{\frac{\nu}{2}}^{\frac{1}{2}} \right) 
\]

For the retarded Green function we get the same answer, with \( \Theta(+t) \) replacing \( \Theta(-t) \).

The Wheeler propagator is then:
\[ \mathcal{F} \left\{ \left( P + M^2 \right)^{-1} \right\} (x) = \frac{\pi}{2} M^{2\nu-1} Q_\nu^\ast (1-\nu) J_{\nu+\frac{1}{2}} \left( M Q_\nu^\ast \right) \tag{53} \]

Now we have the general result: The Wheeler function propagates inside the light-cone for any value of the mass, real (bradyons), imaginary (tachyons) or complex \((M = m + i\mu)\).

In the case of complex masses, a natural definition for the Feynman propagator is obtained by a \(p_0\)-integration along the real axis. It is not difficult to see that

\[ \mathcal{F} \left\{ \left( P + M^2 \right)^{-1} \right\}_F (x) = \sqrt{\frac{\pi}{2}} r^\frac{3-\nu}{2} \int_0^\infty dk \ k^\frac{\nu-1}{2} \left( \frac{\sin \Omega |t|}{\Omega} - i \ sgn \mu \ \cos \Omega |t| \right) J_{\nu+\frac{1}{2}} (rk) \tag{54} \]

The first term in the right hand side is the Wheeler function. The second term corresponds to a positive loop around the pole in the upper half-plane, and a negative loop around the pole in the lower half-plane.

The conjugate Feynman function (not the complex conjugate one), is obtained by changing the sign of the second term. I.e.: by changing the sign of both loops.

The term in \(\cos \Omega |t|\) can be read in ref.[11] (6.735 - 6). With these definitions the Wheeler function is half Feynman and half its conjugate.
Note that for a real mass, $\mu$ corresponds to the “small” negative imaginary part added to the mass in Feynman’s definition. Accordingly, for real mass we take $\text{sgn}\mu = -1$.

Equation (54) can be utilized to define:

$$\mathcal{F}\left\{\delta \left( P + M^2 \right) \right\}(x) = -\frac{\text{sgn}\mu}{\pi} \sqrt{\frac{r}{2}} \int_{0}^{\infty} dq \frac{1}{2} \frac{1}{\sqrt{\nu + 1}} \cos \frac{q}{\Omega} \left| t \right| J_{\nu+\frac{1}{2}}(rt)$$

(55)

$$\left( P + M^2 \right)_F^{-1} = \left( P + M^2 \right)^{-1} + i\pi\delta \left( P + M^2 \right)$$

(56)

The last formula is valid for any mass, real or complex.

6 Associated vacuum

It is well known that the perturbative solution to the quantum equation of motion leads to a Green function which is the vacuum expectation value of the chronological product of field operators (VEV). Furthermore, when the quanta are not allowed to have negative energies, the VEV turns out to be Feynman’s propagator.

However, when the energy-momentum vector is space-like the sign of its energy component is not Lorentz invariant. It is then natural to have symme-
try between positive and negative energies. It has been shown in references [7] and [8] that under this premise, the VEV is a Wheeler propagator.

To see clearly the origin of the difference between both types of propagators, we are going to compare the usual procedure with the symmetric one.

A quantized Klein-Gordon field can be written as:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} \left[ a(\vec{k}) e^{ik\cdot x} + a^+(\vec{k}) e^{-ik\cdot x} \right]$$  \hspace{1cm} (57)

where

$$[a(\vec{k}), a^+(\vec{k}')] = \delta(\vec{k} - \vec{k}') ; \quad \omega = \sqrt{\vec{k}^2 + m^2}$$

For simplicity, we are going to consider a single (discretized) degree of freedom.

The raising and lowering operators obey:

$$[a, a^+] = 1$$  \hspace{1cm} (58)

The energy operator is:

$$h = \frac{\omega}{2} (aa^+ + a^+a) = \omega a^+a + \frac{\omega}{2} = h_0 + \frac{\omega}{2}$$
Usually, the energy is redefined to be \( h_0 \). The vacuum then obeys:

\[
h_0 \mid 0 \rangle = 0
\]  

(59)

It is a consequence of (58) and (59) that:

\[
< 0 \mid aa^+ \mid 0 \rangle = 1 \quad , \quad < 0 \mid a^+a \mid 0 \rangle = 0
\]  

(60)

On the other hand, the symmetric vacuum is defined to be the state that has zero “true energy”:

\[
h \mid 0 \rangle = \frac{\omega}{2} \left( aa^+ + a^+a \right) \mid 0 \rangle = 0
\]  

(61)

Equations (58) and (61) imply:

\[
< 0 \mid aa^+ \mid 0 \rangle = \frac{1}{2} \quad , \quad < 0 \mid a^+a \mid 0 \rangle = -\frac{1}{2}
\]  

(62)

Let as assume, for the sake of the argument, that we define a “ceiling” state (as opossed to a ground state):

\[
a^+ \mid 0 \rangle = 0
\]  

(63)

Equations (58) and (63) give:
\[ <0|aa^+|0> = 0, \quad <0|a^+a|0> = -1 \] 

(64)

The usual normal case, eq.(60) leads to the Feynman propagator. The “inverted” case, eq.(64), leads to its complex conjugate. Then eq.(62), which is one half of (60) and one half of (64), leads to one half of the Feynman function and one half of its conjugate. This is the Wheeler propagator defined in section 1.

The space of states generated by successive applications of \( a \) and \( a^+ \) on the symmetric vacuum has an indefinite metric.

The scalar product can be defined by means of the holomorphic representation [14]. The functional space is formed by analytic functions \( f(z) \), with the scalar product:

\[
\langle f, g \rangle = \int dz \, d\bar{z} e^{-z\bar{z}} f(z) \, \overline{g(z)} 
\] 

(65)

Or, in polar coordinates:

\[
\langle f, g \rangle = \int_0^\infty d\rho \, \rho e^{-\rho^2} \int_0^{2\pi} d\phi \, f(z) \overline{g(z)} 
\] 

(66)

The raising and lowering operators are represented by:
\[ a^+ = z \quad , \quad a = \frac{d}{dz} \]  

(67)

The symmetric vacuum obeys:

\[
\left( \frac{d}{dz} z + z \frac{d}{dz} \right) f_0 = \left( 1 + 2z \frac{d}{dz} \right) f_0 = 0
\]

whose normalized solution is:

\[
f_0 = \left(2\pi^{3/2}\right)^{-1/2} z^{-1/2}
\]

The energy eigenfunctions are:

\[
f_n = \left[ 2\pi \Gamma \left( n + \frac{1}{2} \right) \right]^{-1/2} z^{-1/2} z^n
\]

(68)

7 Unitarity

In QFT, the equations of motions for the states of a system of interacting fields are formally solved by means of the evolution operator.

\[
U(t, t_0) |t_0 >= |t >
\]

The interactions between the quanta of the fields is supposed to take place in a limited region of space-time. The initial and final times can be taken to be \( t_0 \rightarrow -\infty \) and \( t \rightarrow +\infty \). Thus defining the S-operator:
We do not intend to discuss the possible problems of such a definition. Here we are only interested in its relation to Wheeler propagators.

Usually, the initial and final states are represented by free particles. However, when Wheeler fields are present, their quanta either mediate interactions between other particles, or they end up at an absorber. This circumstance had been pointed out by J.A. Wheeler and R.P. Feynman in references [1] and [3]. In consequence, the S-matrix not only contains the initial and final free particles, but it also contains the states of the absorbers. Through the latter we can determine the physical quantum numbers of the Wheeler virtual “asymptotic particles”. For these reasons, even if the initial and final states do not contain any Wheeler free particle, for the verification of perturbative unitarity it is necessary to take them into account.

We shall illustrate this point with some examples. Let us consider four scalar fields $\phi_s$ (s=1,...,4) obeying Klein-Gordon equations with mass parameters $m_s^2$ and the interaction $\Lambda = \lambda \phi_1 \phi_2 \phi_3 \phi_4$. They can be written as in eq.(57).
Unitarity implies:

\[ SS^+ = 1 \]

or, with \( S = 1 - T \):

\[ T + T^+ = TT^+ \]

We introduce the initial and final states and also a complete descomposition of the unit operator:

\[
< \alpha | T + T^+ | \beta > = \int d\sigma_\gamma < \alpha | T | \gamma > < \gamma | T^+ | \beta >
\]

For the perturbative development:

\[
T = \sum_n \lambda^n T_n
\]

\[
< \alpha | T_n + T_n^+ | \beta > = \sum_{s=1}^{n-1} \int d\sigma_\gamma < \alpha | T_{n-s} | \gamma > < \gamma | T^+_s | \beta >
\] (69)

In particular, \( T_0 = 0 \) and \( T_1 = \) pure imaginary.

For \( n = 2 \)

\[
< \alpha | T_2 + T_2^+ | \beta > = \int d\sigma_\gamma < \alpha | T_1 | \gamma > < \gamma | T^+_1 | \beta >
\] (70)

where we will take \( T_1 = i\phi_1\phi_2\phi_3\phi_4 \).

32
\( \phi_1 \) and \( \phi_2 \) are supposed to be normal fields whose states can be obtained from the usual vacuum.

\[
|\alpha> = a_2^+ a_1^+ |0>, \quad |\beta> = a_2^+ a_1^+ |0>
\]

On the other hand, for \( \phi_3 \) and \( \phi_4 \) we leave open the possibility of a choice between the usual vacuum or the symmetric one.

The left hand side of (70) comes from the second order loop formed with the convolution of a propagator for \( \phi_3 \) and another for \( \phi_4 \). When both fields are normal, we have the convolution of two Feynman propagators, where the real part is:

\[
\text{Re} \left[ \left( P + m_3^2 - i0 \right)^{-1} \ast \left( P + m_4^2 - i0 \right)^{-1} \right] = \left( P + m_3^2 \right)^{-1} \ast \left( P + m_4^2 \right) - \pi^2 \delta \left( P + m_3^2 \right) \ast \delta \left( P + m_4^2 \right)
\]

And according to (42), we have in the physical region (P negative):

\[
\text{Re} \left[ \left( P + m_3^2 - i0 \right)^{-1} \ast \left( P + m_4^2 - i0 \right)^{-1} \right] = 2 \left( P + m_3^2 \right)^{-1} \ast \left( P + m_4^2 \right)^{-1}
\]

\( (P < 0) \) (71)

Equation (71) implies that the left hand side of (70) for Feynman particles is twice the value corresponding to Wheeler particles.
The relation (70) is known to be valid for normal fields. So, there is no point in proving each. We are going to show where the relative factor 2 comes from.

The decomposition of unity for normal fields is:

\[ I = \int d\sigma \gamma |\gamma > < \gamma | = |0 > < 0| + \int d^{\nu-1}q; a^+(q) |0 > < 0|a(q) + \]
\[ \int d^{\nu-1}q_1 d^{\nu-1}q_2 \frac{1}{\sqrt{2}} a^+(\vec{q}_1) a+(\vec{q}_2) |0 > < 0|a(\vec{q}_1) a(\vec{q}_2) \frac{1}{\sqrt{2}} + .... \] (72)

Then, for the \( T_1 \) matrix we have:

\[ < \alpha | T_1 | \gamma > = \frac{1}{2\pi} \frac{1}{2^{(\nu-1)}} \frac{\delta (p - q_3 - q_4)}{4\sqrt{\omega_1 \omega_2 \omega_3 \omega_4}} (p = p_1 + p_2) \] (74)

And

\[ < \gamma | T_1 | \beta > = \frac{1}{2\pi} \frac{1}{2^{(\nu-1)}} \frac{\delta (q_3 + q_4 - p')}{{4}\sqrt{\omega'_1 \omega'_2 \omega'_3 \omega'_4}} (p' = p'_1 + p'_2) \] (75)
Multiplying together (74) and (75) and adding all possible $|\gamma><\gamma|$ (all $\vec{q}_3$ and $\vec{q}_4$), we get:

$$
\int d\sigma_\gamma <\alpha|T_1|\gamma><\gamma|T_1^+|\beta> = \frac{\delta(p-p')}{16 (2\pi)^{2\nu-4} \sqrt{\omega_1 \omega_2 \omega'_1 \omega'_2}} 
$$

$$
\int d\vec{q} \frac{\delta(p^0 - \omega_3 (\vec{q}) - \omega_4 (\vec{p} - \vec{q}))}{\omega_3 (\vec{q}) \omega_4 (\vec{p} - \vec{q})} 
$$

(76)

This result coincides with (70) (l.h.s.) when the $p^0$-convolution is carried out.

Suposse now that one of the fields, says $\phi_4$, has the Wheeler function as propagator. Instead of eq.(71) we have:

$$
Re \left[ \left( P + m_3^2 - i0 \right)^{-1} \ast \left( P + m_4^2 - i0 \right)^{-1} \right] = 

\left( P + m_3^2 \right)^{-1} \ast \left( P + m_4^2 \right)^{-1} 
$$

(77)

Half the value of (71).

To evaluate the matrix $< T_1 >$ for this case we note that the decomposition of unity for the states of $\phi_4$ (with an indefinite metric) is now:

$$
I = \int d\sigma_\gamma |\gamma><\gamma| = |0><0| + \int d^{\nu-1}q \sqrt{2}a^+ (\vec{q}) |0><0|a (\vec{q}) \sqrt{2} - 

\int d^{\nu-1}q \sqrt{2}a (\vec{q}) |0><0|a^+ (\vec{q}) \sqrt{2} + .... 
$$

(78)
The normalization factors come from the VEV quoted in section 6, eq. (62). It is not necessary to evaluate again the matrix element (73). Its last vacuum expectation value has now a factor 1/2 from eq. (62), and a factor $\sqrt{2}$ form the normalization in (78). When the matrix for $T_1$ and $T_1^+$ are multiplied together, we get an extra factor $(\sqrt{2}/2)^2 = 1/2$. As it should be for unitarity to hold.

When both fields $\phi_3$ and $\phi_4$, are of the Wheeler type, we get for the convolution of the respective Wheeler propagators the same result (77).

The matrix element of $T_1$ gains now two factors $\sqrt{2}/2$, i.e. a factor 1/2. When we multiply $<T_1><T_1^+>$ we get a factor $1/2 \cdot 1/2 = 1/4$. And we seem to be in trouble with unitarity. However, in this case a new matrix contributes to $<T_1>$. It is:

$$<0|a_1(p_1)\phi_1(x)a_1^+(q_1)\phi_2(x)a_2^+(q_2)|0> = <0|a_3^+(q_3)|0> <0|a_4^+(q_4)|0>$$

(79) is only possible when both, $\phi_3$ and $\phi_4$ are associated with symmetric vacua.

For the first matrix factor we have:

$$<0|a_1(p_1)\phi_1(x)a_1^+(q_1)\phi_2(x)a_2^+(q_2)|0>$$
\[
\delta (p_1 - q_1) \frac{e^{-iq_1 x}}{\sqrt{2\omega_1 (q_1)}} + \delta (p_1' - q_1') \frac{e^{-iq_1' x}}{\sqrt{2\omega_1 (q_1)}}
\] (80)

A similar matrix factor from \( < T^+_1 > \) gives:

\[
< 0|a_1 (\vec{q}_1) a_1 (\vec{q}_1') \phi_1 (y) a_1 (\vec{p}_1') |0 > = \\
\delta (p'_1 - q'_1) \frac{e^{iq_1 y}}{\sqrt{2\omega_1 (q_1)}} + \delta (p_1' - q_1) \frac{e^{iq_1'y}}{\sqrt{2\omega_1 (q_1')}}
\] (81)

When we multiply together (80) and (81), the crossed terms do not contribute \((\delta(p_1 - p'_1) = 0)\). The other two terms give equal contributions. A similar evaluation can be done for the second factor of (79) and the corresponding factor of \( < T^+ > \). For this reason we are going to keep only the first terms from (80) and (81) (multiplied with the appropriate constants):

\[
< \alpha | T_1 | \gamma > = \frac{2}{(2\pi)^{2(\nu - 1)}} \delta (p_1 - q_1) \frac{e^{-iq_1 x}}{\sqrt{2\omega_1 (q_1)}} \delta (p_2 - q_2) \times \\
\frac{e^{-iq_2 x}}{\sqrt{2\omega_2 (q_2)}} \frac{e^{iq_3 x}}{2\sqrt{2\omega_3 (q_3)}} \frac{e^{iq_4 x}}{2\sqrt{2\omega_4 (q_4)}}
\]

And after performing the x-integration,

\[
< \alpha | T_1 | \gamma > = \frac{(2\pi)^{\nu}}{2(2\pi)^{2(\nu - 1)}} \delta (-q'_1 - q'_2 + q_3 + q_4) \delta (p_1 - q_1) \delta (p_2 - q_2)
\]

Analogously:
\[ < \gamma | T_1^+ | \beta > = \frac{(2\pi)^\nu}{2(2\pi)^{2(\nu-1)}} \frac{\delta (q_1 + q_2 - q_3 - q_4) \delta (p_1' - q_1') \delta (p_2' - q_2')}{4\sqrt{\omega_1' \omega_2' \omega_3 \omega_4}} \]

The sum \[ \int d\sigma \gamma < \alpha | T_1 | \gamma > < \gamma | T_1^+ | \beta > \] corresponds to an integration on \( \vec{q}_1, \vec{q}_1', \vec{q}_2, \vec{q}_2' \). It is easy to see that after this operations we get one fourth of (76). Thus completing the proof of unitarity, for the proposed example.

Let us now consider the case in which \( \phi_3 \) and \( \phi_4 \) are fields obeying Klein-Gordon equations with complex mass parameters (See section 5).

The solution of eq.(52):

\[ (\Box - M^2) \phi = 0 \; , \; M = m + i\mu \; (m > 0) \]

can be written as:

\[ \phi(x) = \frac{1}{(2\pi)^{\frac{\nu-1}{2}}} \int d^{\nu-1}p \frac{e^{i\vec{p} \cdot \vec{x}}}{\sqrt{2\Omega}} \left[ a(\vec{p}) e^{-i\Omega t} + b(\vec{p}) e^{i\Omega t} \right] \] (82)

where

\[ \Omega = \left( \vec{p}^2 + M^2 \right)^{\frac{1}{2}} \] (83)

The quantization of \( \phi \) leads to the rules (ref.[5]):

\[ [a(\vec{p}), b(\vec{p}')] = \delta (\vec{p} - \vec{p}') \]
and to the adoption of the symmetric vacuum. From which we get:

\[< 0|a\left(p\right)b\left(p^{'}\right)|0> = -< 0|b\left(p\right)a\left(p^{'}\right)|0> = \frac{1}{2}\delta\left(p - p^{'}\right)\]  

(84)

Accordingly, the decomposition of unity for complex mass fields is (compare with (78)):

\[
\mathbf{I} = |0><0| + \int d^{n-1}q\sqrt{2b}\left(q\right)|0><0|a\left(q\right)|0 > \sqrt{2} - \\
\int d^{n-1}q\sqrt{2a}\left(q\right)|0><0|b\left(q\right)|0 > \sqrt{2} + ....
\]

The matrix \(\langle a|T_{1}\gamma >\) have the form given in eq.(73), but now, instead of \(a_{3}^{+}\) and \(a_{4}^{+}\), we have to write the four possible operators: \(a_{3}, a_{4}; a_{3}, b_{4}; b_{3}, a_{4}\) and \(b_{3}, b_{4}\).

When multiplied with \(\langle \gamma|T_{1}^{+}\beta >\) as in (76) they give similar contributions except for the signs of \(\omega_{3}\) and \(\omega_{4}\) in the arguments of the \(\delta\)-functions. Then, the \(\delta\)-function for each of the terms in the integral of (76) (r.h.s), should be:\(\delta(p_{0} - \Omega_{3} - \Omega_{4}), \delta(p_{0} - \Omega_{3} + \Omega_{4}), \delta(p_{0} + \Omega_{3} - \Omega_{4})\) and \(\delta(p_{0} + \Omega_{3} + \Omega_{4})\), respectively.

This is exactly one half of the convolution product of the Wheeler propagators for \(\phi_{3}\) and \(\phi_{4}\). The other half comes from the contribution of the matrix elements of the form (79).
Of course, for every field corresponding to a complex mass $M$, there is another field corresponding to the complex conjugate mass $M^*$. We have:

$$\phi^+(x) = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int d^{d-1}p \frac{e^{-i\vec{p} \cdot \vec{x}}}{\sqrt{2\Omega^*}} \left[ b^+ (\vec{p}) e^{-i\Omega^* t} + a^+ (\vec{p}) e^{i\Omega^* t} \right]$$

$$\left[ b^+ (\vec{p}), a^+ (\vec{p}') \right] = \delta \left( \vec{p} - \vec{p}' \right)$$

$$< 0 | b^+ (\vec{p}) a^+ (\vec{p}') | 0 > = - < 0 | a^+ (\vec{p}) b^+ (\vec{p}') | 0 > = \frac{1}{2} \delta \left( \vec{p} - \vec{p}' \right)$$

$$I = | 0 > < 0 | + \int d^{d-1}q \sqrt{2} a^+ (\vec{q}) | 0 > < 0 | b^+ (\vec{q}) | 0 > \sqrt{2} - \int d^{d-1}q \sqrt{2} b^+ (\vec{q}) | 0 > < 0 | a^+ (\vec{q}) | 0 > \sqrt{2} + ...$$

The T-matrix can be constructed with the four possible contributions: $\phi_3, \phi_4; \phi_3^+, \phi_4^+$; $\phi_3^+, \phi_4$ and $\phi_3^+, \phi_4^+$. Correspondingly, for each Wheeler propagator with mass $M$, there is another one with mass $M^*$. And of course, four possible convolution products. It is easy to see that the total convolution is real, as well as the total T-matrix.

The proof of unitarity for the chosen example, where the masses are complex numbers, contains as particular cases all fields with symmetric vacua. For bradyons $M=m$. For tachyons, the limit $M \rightarrow i\mu$, must be taken. 

40
Similarly in other cases a proof of unitarity for Feynman propagators, based on the decomposition (72), can be converted into a proof of unitarity for Wheeler propagators, by using the corresponding decomposition (78).

8 Discussion

We have shown that the Wheeler propagator has several interesting properties. In the first place we have the fact that it implies only virtual propagation. The on-shell $\delta$-function, solution of the free equation, is absent. No quantum of the field can be found in a free state. The function is always zero for space-like distances. The field propagation takes place inside the light-cone. This is true for bradyons, but is also true for fields that obey Klein-Gordon equations with the wrong sign of the mass term and even for complex mass fields. The convolution of two Wheeler functions gives another Wheeler function. In $p$-space, this convolution coincides for $P < 0$, with the convolution of the two on-shell $\delta$-functions. In spite of the fact that each of the latter only contains free propagation, while each of the former only contains virtual propagation. The usual vacuum state is annihilated by the descending operator $a$, and gives rise to the Feynman propagator.
The Wheeler Green function is associated to the symmetric vacuum. This vacuum is not annihilated by $a$, but rather by the “true energy” operator, a symmetric combination of annihilation and creation operators. The space of states generated by $a$ and $a^+$ has an indefinite metric. There are known methods to deal with this kind of space. In particular we can define and handle all scalar products by means of the “holomorphic representation” [14]. Due to the absence of asymptotic free waves, no Wheeler particle will appear in external legs of the Feynman diagrams. Only the propagator will appear explicitly, associated to internal lines. So the theoretical tools to deal with matrix elements in spaces with indefinite metric, will not, in actual facts be necessary for the evaluation of cross-sections. However, the decomposition of unity for spaces with indefinite metric, is needed for the proof of another important point. The inclusion of Wheeler fields and the correspondings Wheeler propagators do not produce any violation of unitarity, when only normal particles are found in external legs of Feynman diagrams.

To complete the theoretical framework for a rigorous mathematical analysis, it is perhaps convenient to notice that the propagators we have defined, are continuous linear functionals on the space of the entire analytic functions rapidly decreasing on the real axis. They are known in general as “Tempered
Ultradistributions” [15, 16, 17, 18]. The Fourier transformed space contains the usual distributions and also admits exponentially increasing functions (distributions of the exponential type) (see also ref. [19]).

We must also answer the important question. What are the possible uses of the Wheeler propagators?

In the first place we would like to stress the fact that the quanta of Wheeler fields can not appear as free particles. They can only be detected as virtual mediators of interactions. It is in the light of this observation that we must look for probable applications.

We will first take the case of a tachyon field. It is known that unitarity can not be achieved, provided we accept the implicit premise that only Feynman propagators are to be used, with the consequent presence of free tachyons. This work can also be considered to be a proof of the incompatibility of unitarity and Feynman propagator for tachyons. To this observation we add the fact that if the propagator is a Wheeler function, a tachyon can not propagate freely. Consequently we are led to the acceptance of the complete spectrum: \( \vec{p}^2 < \mu^2 \) and \( \vec{p}^2 \geq \mu^2 \). With the remark that the real exponentials one gets for \( \vec{p}^2 < \mu^2 \) are not eigenfunctions of the Hamiltonian (ref. [8, 20]). Furthermore, this procedure fits naturally into the treatment
for complex mass fields of section 5. To this case it could be related the Higgs problems. The scalar field of the standard model behaves as a tachyon field for low amplitudes. The fact that the Higgs has not yet been observed suggest the possibility that the corresponding propagator might be a Wheeler function [21]. It is easy to see that this assumption does not spoil any of the experimental confirmations of the model (On the contrary, it adds the non observation of the free Higgs).

Another possible application appear in higher order equations. Those equations appear for example in some supersymmetric models for higher dimensional spaces [6]. They can be decomposed into Klein-Gordon factors with general mass parameters. The corresponding fields have Wheeler functions as propagators. It is interesting that there are models of higher order equations, coupled to electromagnetism, which can be shown to be unitary, no matter how high the order is [22].

The acceptance of thachyons as Wheeler particles, might be of interest for the bosonic string theory. With the symmetric vacuum we can show that the Virasoro algebra turns out to be free of anomalies in spaces of arbitrary number of dimensions [23]. The excitations of the string are Wheeler functions in this case.
References

[1] J.A.Wheeler and R.P.Feynman: Rev. of Mod. Phys. 17, 157 (1945).

[2] P.A.M.Dirac: Proc. Roy. Soc. London A 167, 148 (1938).

[3] J.A.Wheeler and R.P.Feynman: Rev. of Mod. Phys. 21, 425 (1949).

[4] D.G.Barci, C.G.Bollini, L.E.Oxman and M.C.Rocca: Int. J. of Mod. Phys. A 9, 4169 (1994).

[5] C.G.Bollini and L.E.Oxman: Int. J. of Mod. Phys. A 7, 6845 (1992).

[6] C.G.Bollini and J.J.Giambiagi: Phys. Rev D 32, 3316 (1985).

[7] D.G.Barci, C.G.Bollini and M.C.Rocca: Il Nuovo Cimento 106 A, 603 (1993).

[8] D.G.Barci, C.G.Bollini and M.C.Rocca: Int. J. of Mod. Phys. A 9, 3497 (1994).

[9] C.G.Bollini and J.J.Giambiagi: Phys. Rev D 53, 5761 (1996).

[10] S.Bochner: “Lectures on Fourier Integrals”. Princeton University Press, N.Y., 235 (1939).
[11] I.S.Gradshteyn and I.M.Ryzhik: “Table of Integrals, Series and Products”. Academic Press (1980).

[12] “The Bateman Project: Table of Integrals Transforms”, Vol.2. McGraw-Hill, N.Y. (1954).

[13] I.M.Guelfand and G.E.Shilov: “Les Distributions”, Vol.1, Dunod, Paris (1972).

[14] L.D.Faddeev and A.A.Slavnov: “Gauge Fields. Introduction to Quantum Theory”. The Benjamin-Cummings Publishing Company, Inc.(1970).

[15] J.Sebastiao e Silva: Math. Ann. 136, 38, (1958).

[16] M.Hasumi: Tohoku Math. J. 13, 94 (1961).

[17] M.Morimoto: Proc. Japan Acad. Sci. 51, 83, (1978).
   M.Morimoto: Proc. Japan Acad. Sci. 51, 213, (1978)

[18] C.G.Bollini, L.E.Oxman and M.C.Rocca: J. of Math. Phys. 35, 4429 (1994).
[19] C.G.Bollini, O.Civitarese, A.L.De Paoli and M.C.Rocca: J. of Math. Phys. 37, 4235 (1996).

[20] C.G.Bollini and J.J.Giambiagi: “On Tachyon Quantization”. Prof. Jaime Tiomno Fernschrift. World Scientific Publishing Company (1992).

[21] C.G.Bollini and M.C.Rocca: “Is the Higgs a visible particle”. To be published in Il Nuovo Cimento A.

[22] C.G.Bollini, L.E.Oxman and M.C.Rocca: Int. J. of Mod. Phys A 12, 2915 (1997).

[23] C.G.Bollini and M.C.Rocca: “Vacuum State of the Quantum String without anomalies in any number of dimensions”. To be published in Il Nuovo Cimento A.