Viscosity to entropy density ratio: violation of the lower bound at small temperatures

A. Jakovac

Department of Theoretical Physics, University of Wuppertal, Gaußstrasse 20, Wuppertal 42119, Germany

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Abstract

We show that in theories where the lowest energy excitations are not quasiparticles but they form a continuum, the shear viscosity to entropy density ratio goes to zero as the temperature goes to zero. In these theories therefore there is no lower bound for the shear viscosity to entropy density ratio, in contrast to the predictions coming from the AdS/CFT correspondence.

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1 Introduction

Experimental evidences from RHIC suggest that the plasma formed after heavy ion collisions is very close to a perfect fluid. The measured large value of $v_2$ [1, 2] indicates a streaming with very small viscosity [3, 4]: fits to the $v_2$ values suggest an upper bound $\eta/s < 0.16$ [5].

There is a tremendous effort in the recent literature to give an account for such small values of viscosity from the theory side. Perturbation theory a priori performs an expansion around the ideal gas limit, where $\eta = \infty$, and so the weak coupling expansion usually yields large $\eta/s$ ratios proportional to $1/g^4 \ln 1/g$ [6, 7, 8, 9]. Still there is hope to access experimentally acceptable ratios using higher order processes in Boltzmann equations [10, 11]. There exist analytic methods beyond weak coupling perturbation theory, like large $N_f$ expansion [12] or strong coupling expansion in pure Yang-Mills theories [13].

One can use Monte Carlo simulations to calculate energy-momentum tensor correlators, which is related to the viscosity via an integral equation. From this
one can extract the viscosity for pure Yang-Mills theories [14, 15, 16]. These results are, however, not fully reliable yet [17].

A method proved to be very useful in studying the viscosity to entropy ratio is based on the AdS/CFT correspondence: in $\mathcal{N} = 4$ super Yang-Mills theory at strong 't Hooft coupling $\lambda$ and at large $N_c$ one finds $\eta/s = 1/4\pi$ [18]. In lot of other gravity models we find the same result [19]. One can also calculate the $1/\lambda$ corrections [20], which turned out to be positive. It is very natural then to expect, that there is a lower bound to the viscosity to entropy density ratio in the conformal case. One can argue that this may be true for other models, too [21]: since the viscosity over entropy ratio parametrically is proportional to $E\tau$, where $E$ is the quasiparticle energy, $\tau$ is its lifetime, we expect that Heisenberg uncertainty relation constrains this ratio. The value of the lower bound is expected to be given by the conformal case, $1/4\pi$.

There are, however, caveats in the argumentation, in two aspects. One is the gravity side: in higher derivative gravity models the ratio can be smaller than $1/4\pi$. In the construction of Ref. [22] the lower bound in the unitary regime is $16/25 \times 1/4\pi$ [23]. There are other string theory constructions, where one can violate the $1/4\pi$ lower bound [24]. Recently there are several gravitational models, where the value of the lower bound is challenged further [25, 26]. This may mean, that in a generic higher derivative gravity model there is no lower bound at all.

The other caveat comes from the particle physics side: large number of particles can lead to a large Gibbs mixing entropy, while the viscosity is constant [27, 28, 29, 12], and so the $\eta/s$ ratio approaches zero. In [30] it is shown that the large number of species can be in the same quantum channel as excited states.

In quantum field theories the density of states generally contains a continuous part, which can be considered as the generalization of the large number of excited states. In these models under certain conditions we may expect the violation of the $\eta/s$ bound. This was observed in strongly coupled pure Yang-Mills theories [13]: when going to zero temperature the viscosity to entropy density ratio approached zero. From the details of the calculation one finds that this behavior is the consequence of that that in the spectrum there are no quasiparticles, only a continuum above a mass gap.

In this paper we show that this behavior is a common feature of a large class of relativistic quantum field theories, where the lowest lying excitations of the energy spectrum form a continuum instead of a discrete set (quasiparticles). As opposed to systems with quasiparticle excitations (Bose gases or liquids), these bosonic systems are non-Bose gases (liquids). In these models the $\eta/s$ ratio approaches zero when the temperature goes to zero. In fact the argumentation is based on kinematics, and therefore it is even more generic: we should find this behavior for all transport coefficients. This result may suggest that these theories have, if they have at all, a nonlocal dual gravity model.

Before the actual calculation we note that formally all relativistic theories at zero chemical potential are superfluids, since the spectral function must be zero below the light cone. It does not mean that the viscosity goes to zero at zero temperature: in fact also in superfluid $^4$He the shear viscosity is divergent.
As we will show the $T \to 0$ limit of $\eta/s$ can be very different for different models: it can be infinite (massive quasiparticles), finite (conformal case) or zero (non-Bose liquids).

## 2 General setup

To start we consider a self-adjoint operator $A(x)$, and study its correlator at low temperatures. We use the standard technique \cite{32} to write it with the help of matrix elements of $A(0)$. To this end we choose a basis in the Hilbert space and write at finite temperature

$$C_A(x) = \langle [A(x), A(0)] \rangle = \frac{1}{Z} \sum_{n,m} \left[ \text{Tr} \langle n | e^{-\beta H} A(x) | m \rangle \langle m | A(0) | n \rangle \right] -$$

$$\text{Tr} \langle m | e^{-\beta H} A(0) | n \rangle \langle n | A(x) | m \rangle \right]$$

where $Z = \sum_{n \in \mathcal{N}} \text{Tr} e^{-\beta H}$. For the states we assume that they are represented by the four-momentum of the state $p = (p_0, \mathbf{p})$, and other quantum numbers $Q$, so we have $|n⟩ = |p, Q⟩$. Note, that $p_0$ and $\mathbf{p}$ are not connected in general by a dispersion relation, not yet the possible values of $p_0$ form a discrete set.

To describe the generic situation we introduce the density of states of a given quantum channel $\rho_Q(p)$, with the definition

$$\sum_n \langle n | \ldots | n \rangle = \sum_Q \int \frac{d^4p}{(2\pi)^4} \rho_Q(p) \langle p, Q | \ldots | p, Q \rangle.$$  \hspace{1cm} (2)

We will also assume that the normalization of the states are done for unit volume – then all calculated quantities are densities. We emphasize here that the volume of the system is infinite, only the normalization volume of the states is fixed!

We can write $A(x) = e^{iP^n x^n} A(0) e^{-iP^n x^n}$ with the use of the generator of space-time translation, and so $\langle q, Q | A(x) | p, P \rangle = e^{-i(p-q)x} \langle q, Q | A(0) | p, P \rangle$. We also perform Fourier transformation, and find finally

$$C_A(k) = \frac{1}{Z} \sum_{Q, P} \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \rho_Q(q) \rho_P(p) e^{-\beta q_0} \times$$

$$\times \left( 1 - e^{-\beta q_0} \right) (2\pi)^4 \delta(k + q - p) \langle q, Q \langle 0 \rangle p, P \rangle.$$  \hspace{1cm} (3)

This form demonstrates also that $C(k) > 0 \forall k_0 > 0$, and $C(-k) = -C(k)$.

The transport coefficient corresponding to $A$, which is denoted here as $\eta_A$, comes from the Kubo formula \cite{32}:

$$\eta_A = \lim_{k_0 \to 0} \frac{C_A(k)}{k_0}.$$  \hspace{1cm} (4)
It reads with the current representation

\[ \eta_A = \frac{\beta}{Z} \sum_{Q,P} \int \frac{d^4q}{(2\pi)^4} \varrho_Q(q) \varrho_P(q) e^{-\beta q_0} |\langle q | A | q \rangle|^2 \]  

(5)

where we have omitted the zero argument of \( A \).

Now we go to low temperatures, \( \beta \to \infty \). The \( \exp(-\beta q_0) \) factor forces to keep only those quantum channels, which contain the lowest energy levels while the matrix element is not zero there. For simplicity we assume, that this is true for a single quantum channel, but this is not crucial for the later discussion. Therefore in the followings we will suppress the notion of the quantum channel, and write

\[ \eta_A = \frac{\beta}{Z} \int \frac{d^4q}{(2\pi)^4} \varrho^2(q) e^{-\beta q_0} |\langle q | A | q \rangle|^2. \]  

(6)

Analogous result could be found for the viscosity in strong coupling expansion at small temperatures [13].

We should work out the low temperature expression for the entropy, too. The free energy is the logarithm of \( Z \), which can be written, by explicitly separating the vacuum contribution:

\[ Z - 1 = \sum_{|n\rangle \neq |0\rangle} \langle n | e^{-\beta H} | n \rangle = \sum_{Q \neq |0\rangle} \int \frac{d^4q}{(2\pi)^4} \varrho_Q(q) e^{-\beta q_0}. \]  

(7)

At low temperatures the correction to 1 is small, and so we can approximate \( \ln(1 + \delta Z) = \delta Z \) to leading order. We again assume that the most important contributions come from a single quantum channel, and suppress the index \( Q \) in the followings. It could also happen, that the relevant quantum channel for entropy is different from the one for \( \eta_A \), but we will consider only those transport coefficients, where this is not the case. Transport coefficients coming from the commutator of the energy-momentum tensor (like the shear viscosity), are in this class, since all excitations necessarily contribute to the energy-momentum tensor. Then at low temperatures the free energy density and the entropy density read:

\[ f = -T \int \frac{d^4q}{(2\pi)^4} \varrho(q) e^{-\beta q_0}, \quad s = -\frac{\partial f}{\partial T}. \]  

(8)

At low temperatures, therefore, the ratio of \( \eta_A \) and the entropy density reads

\[ \frac{\eta_A}{s} = \frac{\beta \int \frac{d^4q}{(2\pi)^4} \varrho^2(q) e^{-\beta q_0} |\langle q | A | q \rangle|^2}{\frac{\partial}{\partial T} T \int \frac{d^4q}{(2\pi)^4} \varrho(q) e^{-\beta q_0}}, \]  

(9)

where we have taken into account that at low temperatures \( Z \approx 1 \).

In the following we make kinematical considerations to estimate the temperature dependence of the \( \eta/s \) ratio. There we should assume something about
the matrix element $|\langle q | A | q \rangle|^2$. Since we calculate transport coefficients, we may assume that $A$ is a current, and therefore its matrix element is zero, when there is no gradient of the corresponding density. Therefore we assume that

$$|\langle q | A | q \rangle|^2 = q^2 A. \tag{10}$$

The coefficient $A$ can be temperature dependent, but at small temperatures we may approximate it with its zero temperature limit. One should note that for the shear viscosity the current is a tensor, because we describe the current of the energy-momentum four vector. But it does not alter the statement that the matrix element remains to be proportional to $q^2$ where $q$ describes the change of the momentum in the perpendicular direction.

So we can write:

$$\frac{\eta_A}{s} = \frac{A\beta}{\frac{\partial}{\partial T}} \int \frac{d^4 q}{(2\pi)^4} \rho^2(q) q^2 e^{-\beta q_0} \tag{11}$$

We will consider two different types of systems: one with a quasiparticle behavior at the lowest energies, the other with a threshold behavior, and examine the $\eta_A/s$ ratio.

### 3 Quasiparticle case

Here we assume a Breit-Wigner-type distribution of the lowest energy eigenvalues:

$$\rho(q) = \frac{2\Gamma}{(q_0 - \varepsilon_q)^2 + \Gamma^2}, \tag{12}$$

where $\varepsilon_q$ is the dispersion relation of the quasiparticle, $\Gamma$ is the quasiparticle width. Here we will assume a relativistic dispersion relation $\varepsilon_q^2 = q^2 + m^2$. If $\Gamma \to 0$ this form approaches $2\pi \delta(q_0 - \varepsilon_q)$. We assume that we are in the “small width” regime, which means that always, where we can, we should send $\Gamma \to 0$.

This policy suggests that we should treat the product of two distribution function in the sense of the Fermi’s Golden Rule, i.e., we treat one of them as if it was a Dirac-delta:

$$\rho^2(q) \approx \rho(q_0 = \varepsilon_q) \rho(q) = \frac{2}{\Gamma} \rho(q). \tag{13}$$

Thereafter the integrals both in the numerator and in the denominator contain one spectral function, which can now be approximated by the delta function.

The free energy density reads

$$f = -T \int \frac{d^4 q}{(2\pi)^4} \rho(q) e^{-\beta q_0} = -T \frac{1}{2\pi^2} \int dq q^2 e^{-\beta \varepsilon_q} =$$

$$= -\frac{T m^3}{2\pi^2} \int dz \sqrt{z^2 - 1} e^{-\beta mz} = -\frac{T^4}{2\pi^2} (\beta m)^2 K_2(\beta m), \tag{14}$$
where $K_2$ is the modified Bessel function of the second kind. At zero mass (conformal case) the relevant limit is $\beta m \to 0$, then we find

$$f|_{\text{conf}} = \frac{T^4}{\pi^2}, \quad s|_{\text{conf}} = \frac{4T^3}{\pi^2}. \quad (15)$$

In the massive case at temperatures $T \ll m$ we find in the leading order:

$$f|_m = -T^4 \left( \frac{\beta m}{2\pi} \right)^{3/2} e^{-\beta m}, \quad s|_m = \frac{m^{5/2} T^{1/2}}{(2\pi)^{3/2}} e^{-\beta m}. \quad (16)$$

For the transport coefficient we find with help of (13)

$$\eta_A = A\beta \int \frac{d^4 q}{(2\pi)^4} \varrho^2(q) q^2 e^{-\beta q_0} \approx \frac{2A}{TT} \int \frac{d^4 q}{(2\pi)^4} \varrho(q) q^2 e^{-\beta q_0} = \frac{Am^5}{TT \pi^2} \int_1^\infty dz \frac{z(z^2 - 1)^{3/2}}{z^{3/2}} e^{-\beta mz} = \frac{3Am^3T}{\Gamma \pi^2} K_3(\beta m), \quad (17)$$

where $K_3$ is the modified Bessel function of the third kind. In the conformal case:

$$\eta_A|_{\text{conf}} = \frac{24AT^4}{\Gamma \pi^2}. \quad (18)$$

In the massive case, at temperatures $T \ll m$ we find in the leading order:

$$\eta_A|_m = \frac{6A (mT)^{5/2}}{\Gamma T (2\pi)^{3/2}} e^{-\beta m}. \quad (19)$$

Therefore the $\eta_A/s$ ratio reads in both cases

$$\frac{\eta_A}{s} = 6A \frac{T}{\Gamma}. \quad (20)$$

As we can see, the desired ratio is proportional to $T/\Gamma$. This is the manifestation of the qualitative argumentation of Ref. [21]; since $T$ is the kinetic energy and $1/\Gamma$ is the lifetime, therefore $\eta_A/s \sim E_{\text{kin}}T$. Therefore this ratio is constrained by the Heisenberg relation, and we expect to have a lower bound. In fact, in the conformal case $\Gamma$ must be proportional to the temperature (there is no other scale), and the ratio is constant. In the massive case, on the other hand, the width is exponentially small. This is because to form a width, the particle must scatter on thermal states, but their abundance is $\sim e^{-M/T}$, where $M$ is the energy of the lowest lying scattering state. Therefore

$$\left| \frac{\eta_A}{s} \right|_{\text{conformal}} \sim \text{const.}, \quad \left| \frac{\eta_A}{s} \right|_{\text{massive}} \sim T e^{M/T} \quad T \to \infty. \quad (21)$$

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4 Non-quasiparticle case

The other big class of theories is when the low energy spectrum cannot be described by a narrow quasiparticle peak, but instead there is a continuous density of state. In this case it is worth to stay in the four-dimensional formalism, and write the integral in 4D polar-coordinates. We should take into account that, since the energy spectrum has a lower bound (for stability) which can be chosen to be zero, the $q_0$ integral is restricted to positive values. Moreover the integrand is spatially symmetric. Then we can use:

$$\int \frac{d^4q}{(2\pi)^4} \Theta(q_0) \rightarrow \frac{1}{4\pi^3} \int_0^\infty dqq^3 \int_0^\infty d\eta \sinh^2 \eta. \quad (22)$$

The expression for the free energy can be rewritten as

$$f = -\frac{T}{4\pi^3} \int_0^\infty dq \int dq^3 \eta e^{-\beta q \cosh \eta} - \frac{T^2}{4\pi^3} \int_0^\infty dq^2 (q^2) \left( K_1(\beta q) \right). \quad (23)$$

For the transport coefficient we obtain

$$\eta_A = \frac{A\beta}{4\pi^3} \int_0^\infty dq \int dq^3 \eta^2 e^{-\beta q \cosh \eta} = \frac{3A}{4\pi^3} \int_0^\infty dq^5 \eta^2 (q^2) \left( K_2(\beta q) \right). \quad (24)$$

At $T \rightarrow 0$ limit we will use the zero temperature limit of the spectral density which is Lorentz-invariant. Since the modified Bessel functions all decrease as $K(x) \rightarrow (\pi/2x)^{1/2} e^{-x}$, the $\beta q$ argument in the above expressions enhances the lowest momentum part of the spectral function, i.e. the threshold region. Sufficiency close to the threshold the spectral density can be approximated by a power law: so we will assume that the spectral density has the form in the relevant regime as

$$\rho(q) = C \Theta(q - M)(q - M)^w, \quad (25)$$

where $M$ is the threshold value and $C$ is some constant.

If there is no mass gap, then the expressions for $f$ and $\eta_A$ can be exactly performed, yielding for the entropy density

$$s = \frac{C(5 + w)}{4\pi^3} T^{4+w} 2^{2+w} \Gamma \left( \frac{5 + w}{2} \right) \Gamma \left( \frac{3 + w}{2} \right), \quad (26)$$

and

$$\eta_A = \frac{3AC^2}{4\pi^3} T^{5+2w} 4^{1+w} \Gamma(1+w)\Gamma(3+w). \quad (27)$$

The transport coefficient to entropy ratio is therefore

$$\frac{\eta_A}{s} = \alpha w AC T^{1+w} \quad (28)$$
where \( \alpha_w = 3 2^w \frac{\Gamma(1 + w) \Gamma(3 + w)}{\Gamma((5 + w)/2) \Gamma((3 + w)/2)} \).

If \( M \neq 0 \) then at \( T \ll M \) we can use the asymptotic form for the modified Bessel functions, and obtain

\[
s = \sqrt{\frac{\pi}{2}} \frac{C \Gamma(w + 1)}{4\pi^3} T^{w+3/2} M^{5/2} e^{-\beta M},
\]

and

\[
\eta_A = \sqrt{\frac{\pi}{2}} \frac{3AC^2 \Gamma(2w + 1)}{4\pi^3} T^{5/2+2w} M^{5/2} e^{-\beta M}
\]

The transport coefficient over entropy ratio is therefore

\[
\frac{\eta_A}{s} = \bar{\alpha}_w AC T^{1+w},
\]

where \( \bar{\alpha}_w = 3\Gamma(1 + 2w)/\Gamma(1 + w) \).

So finally from both the gapped and gapless case we obtained the same result:

\[
\frac{\eta_A}{s} \sim T^{1+w} \xrightarrow{T \to 0} 0
\]

for any realistic threshold behavior \( w > -1 \).

We remark here that the result could be guessed by dimensional argumentation. The coefficient \( C \) must be dimensionfull: \( \nu \) is of dimension \( 1/E \), \( q, M \sim E \) therefore \( C \sim E^{-1-w} \). Since \( \eta_A \) contains \( C^2 \), the entropy density \( C \), their ratio is \( \sim C \). But the ratio is dimensionless, so a quantity with dimension \( E^{1+w} \) must appear. If \( M = 0 \) the only candidate is \( T \), so we must have \( T^{1+w} \). If \( M \neq 0 \), in principle there could be also some \( M \) factor, but it turns out from the concrete calculation, that the \( M \) factors drop out.

## 5 Conclusion

For conclusion we recall that we studied the low temperature behavior of the ratio of some transport coefficient \( \eta_A \) (where the notation is motivated by the shear viscosity) and the entropy density. The computation is based on exact formulae which are used near zero temperature. We studied two classes of models, both describing relativistic superfluids.

The first is the class of the Bose-liquids, where the lowest energy excitation is a stable particle at zero temperature which becomes a narrow width quasi-particle at small temperatures. To this class belong a lot of particle physics models, for example the \( \Phi^4 \) model, or the sigma model. There we have found that the \( \eta_A/s \) ratio is proportional to \( T/T \), in agreement with the qualitative expectations \[21\]. In the conformal limit this ratio is constant, in the massive case the ratio is diverging as \( T \to 0 \). In this class therefore there is a lower bound for the \( \eta_A/s \): in the case of viscosity it is probably \( 1/4\pi \) \[18\].

The second class of the studied models is the non-Bose-liquids: here the lowest lying excitations are form a continuous spectrum. The pure Yang-Mills theories may belong to this class, according to the strong coupling expansion
The result in these models is that $\eta_A/s \sim T^{1+w}$ where $w$ is the power of the power law appearing in the density of states when expanded around the threshold. In generic case this ratio goes to zero at zero temperature. Therefore in these class of models there is no lower bound for the shear viscosity entropy density ratio.

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