Quark horizontal flavor symmetry and two-Higgs doublet in $(7+1)$-dimensional extended spin space

R. Romero and J. Besprosvany
Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, Ciudad de México 01000, México

An extended spin-space model in $7 + 1$ dimensions is presented that describes the standard-model electroweak quark sector. Up to four generations of massless and massive quarks and two-Higgs doublets derive from the associated representation space, in addition to the W- and Z-vector bosons. Other mass operators are obtained that put restrictions on additional non-Higgs scalars and their vacuum expectation value. After symmetry breaking, the scalar components give rise to a hierarchy effect vertically (within doublets) associated to the Higgs fields, and horizontally (within generations) associated to the non-Higgs elements.

I. INTRODUCTION

The Standard Model (SM) of particle physics is very successful in describing the fundamental particles and their interactions, but it is also phenomenological, requiring a large amount of experimental input such as fermion masses, coupling constants and mixing angles. The recent discovery of a Higgs-like particle at the LHC further validates the SM, but the absence of positive results in new particles searches [1, 2] also severely constrains favored beyond the SM proposals, such as supersymmetry (SUSY)[3–6].

Two elements frequently used in beyond-the-SM model building, that are within reach of experimental verification at the Large Hadron Collider (LHC) in upcoming years, are the extension of the Higgs sector, as in two-Higgs doublet (2HD) models[7], and the inclusion of horizontal (flavor) symmetries[8–11]. The latter can be global or gauged, and the flavor group can be either discrete or a continuous Lie group. In trying to explain the origin of the mass hierarchies and mixings of quarks and leptons, it has been proposed that the flavor symmetry might be broken by Froggatt-Nielsen scalars, or flavons[12], which are scalars transforming non-trivially under the flavor group that couple to fermions and acquire vacuum expectation values. Different choices for the flavor group, as well as the breaking scale and mechanism, lead to different phenomenological outcomes.

One common feature of models incorporating the above elements is that the additional Higgs structure and/or the flavor symmetry are proposed \textit{ab initio}, rather than derived. To mention some recent examples, in Ref. [13] a 2HD model is proposed where the two Higgses serve as the flavon and the flavor breaking scale is just the electroweak one. In Ref. [14] a 2HD and additional scalar singlets are used in conjunction with a gauged U(1) horizontal symmetry to explain the SM deviations in B-meson decays found by the LHCb run, while in Ref. [15] an extension of a 2HD model is considered in the context of the Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) axion model[16, 17].

In looking for model-building alternatives, and motivated by the original Kaluza-Klein idea of gauge-interactions and gravity unification through the addition of more spacetime dimensions, it has been proposed[18–20] to enlarge instead the dimensions of the abstract vector space in which spin-1/2 objects reside, commonly referred to as spin space, but keeping Lorentz symmetry in the standard four-dimensional spacetime. Hence, the extra dimensions are only relevant in spin space, while they are \textit{frozen} in spacetime. The motivation for doing so is that, regarding the classification of physical fields by their transformation properties under the Poincaré group, the spin-1/2 representation is the fundamental one, since all tensor representations can be obtained from the former, but not the other way around. In another interpretation, one proceeds by checking additional discrete spaces, described in terms of a Clifford algebra and independently of the configuration-space description, that allow for the inclusion of the relevant states and operators[18]. In this respect, a $(7+1)$-dimensional discrete space is the minimum space that allows for the description of different flavor quarks and the operators that classify them, since lower dimensional spaces have been studied and found not big enough for that purpose[18, 20–22].

Properties and results of the extended spin model, within the context of relativistic quantum mechanics, have been studied before in various dimensions[20, 22–24], and recently, a formal translation was presented between fields and Lagrangians in the extended spin space and the conventional formulation, and in particular for the SM. Thus, a direct interpretation of the model in terms of a field theory is now available. Given that the extended space is constructed...
in terms of ket-bras constructed from elementary doublets, a matrix-vector space results, which is suitably endowed with a metric, and is describable in terms of a Clifford algebra. Also, maintaining the 4-d Lorentz symmetry results in additional operators in the same matrix space that are associated to gauge and flavor symmetries; as these commute with the former, the Coleman-Mandula theorem is satisfied. The latter makes it possible for symmetry operators, both gauge and Lorentz, to be elements of the same matrix space, along with the physical fields they classify.

In this paper, we derive a model in the \((7 + 1)\)-d extended spin space, where we obtain up to four massless and massive quark generations, two Higgs doublets, and both horizontal and vertical mass relations. There are also mass operators that put restrictions on additional non-Higgs scalars and their vacuum expectation value. We show they constrain the quark mass spectrum; in particular, they generate linearly vertical and horizontal hierarchies. Thus, they play a similar role to flavons, with differences. All these features are obtained from the model under standard physical assumptions. The dimension of the space is chosen because it is the minimal one required for massive quarks, since the previous lower \((5 + 1)\)-d space was studied, and found not big enough to accommodate left- and right-handed quark fields with the SM quantum numbers. The paper is organized as follows: In section II, general properties of the model are reviewed, including operators, symmetry transformations, and field and Lagrangian representation. Section III presents the \((7 + 1)\)-d model, giving the classification of operators and states, and obtaining massless and massive quarks, and Higgs-like particles, and non-Higgs scalars, along with hierarchies, both horizontal and vertical. In Section IV, we give concluding remarks.

II. EXTENDED SPIN MODEL

Let us begin by providing a brief introduction to the features of the extended spin model.

A. General Properties

Let us consider an \(N\)-dimensional Clifford algebra \(\mathbb{C}_N\), with \(N\) even, generated by a set of \(N\) gamma matrices of dimension \(2^{N/2} \times 2^{N/2}\) obeying the defining property of the algebra:

\[ \gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2g_{\alpha\beta}, \]

(1)

where \(g_{\alpha\beta}\) is the metric tensor with signature \((+,−,...,−)\) and \(\alpha,\beta = 0,1,\ldots,3,5,\ldots,N\). The gamma matrices are taken with the standard Hermiticity properties:

\[ \gamma_0^\dagger = \gamma_0, \]
\[ \gamma_i^\dagger = -\gamma_i, \quad i = 1,\ldots,N. \]

(2)

The \(N\) gamma matrices and all their linearly independent products form a set of \(2^N\) elements, which constitutes a basis for the vector space of complex \(2^{N/2} \times 2^{N/2}\) matrices. The spin Lorentz generators and finite Lorentz transformations acting on spinors have standard expressions in the four dimensional Clifford algebra \(\mathbb{C}_4\), namely

\[ \sigma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu] \text{ with } \mu,\nu = 0,\ldots,3, \]

(3)

\[ S(\Lambda) = e^{-\frac{i}{2} \sigma_{\mu\nu} \omega^{\mu\nu}}. \]

(4)

As a result, the \((3 + 1)\)-d gamma matrices transform under \(S(\Lambda)\) as vectors

\[ S(\Lambda)\gamma^\mu S(\Lambda)^{-1} = (\Lambda^{-1})^\mu_{\nu} \gamma^\nu, \]

(5)

\(\mu,\nu = 0,\ldots,3\), while the remaining \(N - 4\) gamma matrices \(\gamma_a, a = 4,\ldots,N - 1\), and their products commute with \(\sigma_{\mu\nu}\), e.g. \([\gamma_6, \sigma_{01}] = i\gamma_0 \gamma_0 \gamma_1 - i\gamma_0 \gamma_1 \gamma_6 = 0\), so they are indeed Lorentz scalars

\[ S(\Lambda)\gamma^a S(\Lambda)^{-1} = \gamma^a. \]

(6)
These scalars can be identified with generators of continuous symmetries, either gauge or global, and they automatically satisfy the Coleman-Mandula theorem since they commute with the Lorentz generators. This latter feature allows for the inclusion of spacetime and gauge groups generators in the same space, and provides an alternative to a graded Lie algebra for evading the Coleman-Mandula theorem, as is done in SUSY. Furthermore, the states on which the operators act are also elements of the space, so we have a matrix space with the dimensionality restricting the type and number of allowed scalar symmetries. The scalars generate the unitary-group combination

$$S_{N-4} = \frac{1}{2}(I + \tilde{\gamma}_5)U \left(2^{(N-4)/2}\right) \oplus \frac{1}{2}(I - \tilde{\gamma}_5)U \left(2^{(N-4)/2}\right),$$

where $I$ stands for the $N$-d identity matrix and $\tilde{\gamma}_5$ is the 4-d chirality matrix, defined as

$$\tilde{\gamma}_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3.$$ 

Given that $U \left(2^{(N-4)/2}\right)$ possesses $2^{N-4}$ generators, the number of elements in $S_{N-4}$ is $2^{N-3}$.

### B. Operators and symmetry transformations

Physical fields are associated with elements of $\mathcal{C}_N$, classified by operators from $\mathcal{C}_4 \otimes S_{N-4}$, so in general, a field $\psi$ has the structure

$$(\text{elements of } 3+1 \text{ space }) \times \left(\text{combination of products of elements of } S_{N-4}\right).$$

Symmetry transformations act on fields as

$$\Psi(x) \rightarrow U\Psi(x)U^\dagger,$$

and commutators are used for the evaluation of operators

$$[\mathcal{O}, \Psi(x)] = \lambda\Psi(x).$$
Figure 2. Representation of states in extended spin space, classified according to their Lorentz transformation properties: fermion (F), vector (V), scalar (S), and anti-symmetric tensor (A). Antifermions (F̅) correspond to the Hermitian conjugate, and the blocks for V, S and A also contain antiparticle solutions.

This last equation is to be interpreted as an eigenvalue equation, that is, ψ is an eigenstate of O with eigenvalue λ. The inner product of two fields is defined accordingly to a matrix space

\[ \langle \phi | \Psi \rangle = \text{tr} (\phi^\dagger \Psi) \, . \] (12)

Vector and scalar fields transform respectively under Eq. (4) as the matrices \( \gamma^\mu, \mu = 0, \ldots, 3 \) and \( \gamma^a, a = 5, \ldots, N \), Eqs. (5) and (6).

For the fermion representation, a projector operator \( \mathcal{P} \), obtained from elements of \( S \), is used such that it acts on both the Poincaré generators \( J_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \) and the \( S_{N-4} \) symmetry operator space

\[ J'_{\mu\nu} = \mathcal{P} J_{\mu\nu} = \mathcal{P} \left[ i (x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right], \] (13)

\[ S' = \mathcal{P} S. \]

A fermion field is then required to include \( 1 - \mathcal{P} \) in the general structure of Eq. (9), so that Poincaré and gauge generators act trivially on its rhs when evaluating commutators as in Eq. (11), since \( (1 - \mathcal{P}) \mathcal{P} = 0 \). This leads to fermions transforming in the fundamental representation of both the Poincaré and gauge groups, that is, for those groups instead of Eq. (10) we have

\[ \Psi \rightarrow U \Psi, \] (14)

where \( \Psi \) stands for a fermion field. Flavor operators, on the contrary, act non-trivially from the right-hand side and give zero from the left, as will be shown in the next section. This is heuristically understood by thinking of \( \Psi \) as the matrix \( \Psi \sim |\psi\rangle \langle a| \), with the ket \( |\psi\rangle \) carrying Lorentz and gauge group information, while the bra \( \langle a| \) carries information about the flavor.

Fig. 1 presents schematically symmetry operators in the matrix space, and Fig. 2 shows the corresponding Lorentz states: scalars, vectors, fermions, and anti-symmetric tensors, arranged in the same matrix space.

C. Lagrangian formulation

Interactive Lagrangians can be given in terms of vector, scalar and fermion fields conforming to the general structure of Eq. (13). These fields are:
Vector field

\[ A^a_\mu(x) \gamma_0 \gamma_\mu I_a, \]  
\hspace{1cm} (15)

where \( \gamma_0 \gamma_\mu \in C_4 \) and \( I_a \in S'_{n-4} \) is a generator of a given unitary group.

Scalar field

\[ \phi^a(x) \gamma_0 \Gamma^S_a, \]  
\hspace{1cm} (16)

with \( \Gamma^S_a \) an element of \( S \).

Fermion field

\[ \psi^a_\alpha(x) L^a P_F \Gamma^F_a, \]  
\hspace{1cm} (17)

where \( \Gamma^F_a \) is an element of \( S \), and \( L^a \) represents a spin polarization component, e.g. \( L^1 = (\gamma_1 + i\gamma_2) \). The operator \( P_F \) is a projection operator of the type in Eq. (13) such that

\[ P_F^{-} \gamma^\mu = \gamma^\mu P_F^{+}, \]  
\hspace{1cm} (18)

with \( P_F^{\pm} = 1 - P_F \). Vector and scalar fields transform under gauge transformations as in Eq. (10), while fermions transform as in Eq. (14), with \( U \) given by

\[ U = \exp \left[-i I_a \alpha_a(x) \right]. \]  
\hspace{1cm} (19)

For example, a gauge-invariant fermion-vector Lagrangian is given by

\[ \frac{1}{N_f} \text{tr} \Psi^\dagger \left\{ \left[ i\partial_\mu - g A^a_\mu(x) I_a \right] \gamma^0 \gamma^\mu - m \gamma^0 \right\} \Psi, \]  
\hspace{1cm} (20)

where \( \Psi \) is a fermion field as in Eq. (17), \( g \) is the coupling constant, and \( N_f \) contains the normalization.

III. 7+1-DIMENSIONAL MODEL

We now present a 7 + 1 dimensional model of the electroweak quark sector in extended space.

A. Generators and operators

The (7 + 1)-d Clifford algebra is generated by the eight \( 16 \times 16 \) matrices

\[ \gamma_0, \gamma_1, \ldots, \gamma_8. \]  
\hspace{1cm} (21)

The first four matrices form the Lorentz generators \( \sigma_{\mu\nu} \) of Eq. (3), and the set of unitary scalars consists of 32 elements that generate the group combination \( S = P_+ U(4) \oplus P_- U(4) \), with \( P_\pm = \frac{1}{2} (1 \pm \gamma_5) \) and \( \gamma_5 \) the chirality matrix within 4-d, defined in Eq. (5). The 32 elements of \( S \) are: four matrices \( \gamma_a, a = 5, \ldots, 8 \), six pairs \( \gamma_{ab} = \gamma_a \gamma_b, \ldots, \gamma_{ab} = \gamma_a \gamma_b, \ldots, \gamma_{ab} = \gamma_a \gamma_b \).
$a < b$, four triplets $\gamma_{abc} \equiv \gamma_a \gamma_b \gamma_c$, and one quadruplet $\gamma_5 \gamma_6 \gamma_7 \gamma_8$. To these fifteen matrices we add the same matrices multiplied by $\hat{\gamma}_5$, plus $\hat{\gamma}_5$ itself and the identity to complete the 32 scalars.

The maximal number of elements in $S$ that commute with each other is eight, and these elements define the eight-dimensional Cartan subalgebra $\mathfrak{h}$ of $S$, for which we choose the basis

$$1, \hat{\gamma}_5, \gamma_5 \gamma_6, \gamma_7 \gamma_8, \gamma_5 \gamma_6 \gamma_7 \gamma_8, \gamma_5 \gamma_6 \gamma_5, \gamma_7 \gamma_5 \gamma_5, \gamma_5 \gamma_6 \gamma_7 \gamma_5.$$ (22)

This basis is used to build operators that classify the particles according to their quantum numbers, seeking a configuration as close as possible to the SM. One degree of freedom in $\mathfrak{h}$ is assigned to the projector $\mathcal{P}$, which characterizes fermions, as in Eq. (13). Two other are the $SU(2)_L \times U(1)_Y$ gauge-group generators that can be chosen diagonal, namely, the third component of isospin I$_3$ and the hypercharge $Y$. Of the remaining five, three are associated to flavor, to be described below, and the rest are just the identity and the chirality.

To obtain the diagonal operators let us recast the basis in Eq. (22) in terms of the projection operators

$$
\begin{align*}
P_{R1} &= \frac{1}{4}(1 + \gamma_5)(1 + i\gamma_5 \gamma_6)(1 + i\gamma_7 \gamma_8), \\
P_{R2} &= \frac{1}{4}(1 + \gamma_5)(1 + i\gamma_5 \gamma_6)(1 - i\gamma_7 \gamma_8), \\
P_{R3} &= \frac{1}{4}(1 + \gamma_5)(1 - i\gamma_5 \gamma_6)(1 + i\gamma_7 \gamma_8), \\
P_{R4} &= \frac{1}{4}(1 + \gamma_5)(1 - i\gamma_5 \gamma_6)(1 - i\gamma_7 \gamma_8), \\
P_{L1} &= \frac{1}{3}(1 - \gamma_5)(1 + i\gamma_5 \gamma_6)(1 + i\gamma_7 \gamma_8), \\
P_{L2} &= \frac{1}{3}(1 - \gamma_5)(1 + i\gamma_5 \gamma_6)(1 - i\gamma_7 \gamma_8), \\
P_{L3} &= \frac{1}{3}(1 - \gamma_5)(1 - i\gamma_5 \gamma_6)(1 + i\gamma_7 \gamma_8), \\
P_{L4} &= \frac{1}{3}(1 - \gamma_5)(1 - i\gamma_5 \gamma_6)(1 - i\gamma_7 \gamma_8),
\end{align*}
$$ (23)

which are schematically shown in the matrix space in Fig. 3. Each one of these projectors has two degenerate non-zero eigenvalues, set to one, and all scalar diagonal operators can be written as their linear combination. For the right-handed quarks description, we need two different hypercharge quantum numbers, namely 4/3 and $-2/3$, respectively, for up- and down-type quarks, so the right-handed part of the hypercharge operator must be a linear combination of at least two of the first four operators in Eq. (24), with the weights given by the two eigenvalues. If the operators $P_{R1}$ and $P_{R2}$ make room for flavor and $1 - \mathcal{P}$, necessary for the description of different fermion families, this leaves $P_{R3}$ and $P_{R4}$ as the only candidates, and we assign

$$\frac{1}{2}(1 + \gamma_5)Y = \frac{4}{3}P_{R3} - \frac{2}{3}P_{R4},$$ (24)

for the right-handed part of the hypercharge operator. (Alternatively, fixing $P_{R3}$ and $P_{R4}$ with Eq. (24), leaves $P_{R1}$ and $P_{R4}$ for $1 - \mathcal{P}$ and flavor association.) The choice of the two operators in Eq. (24) is not unique, with different matrix operator representations of and states, leading to the same physics.

The left-handed part of the hypercharge operator is symmetrically given as

$$\frac{1}{2}(1 - \gamma_5)Y = \frac{1}{3}(P_{L3} + P_{L4}),$$ (25)

where both weights equal 1/3 as such is the hypercharge eigenvalue for both up- and down-type left-handed quarks. Thus, the complete operator is

$$Y = \frac{1}{3}(4P_{R3} - 2P_{R4} + P_{L3} + P_{L4}),$$

$$= \frac{1}{6}(1 - i\gamma_5 \gamma_6)\left(1 + i\frac{3}{2}(1 + \gamma_5)\gamma_7 \gamma_8\right).$$ (26)

Tracing $Y$, operator $\mathcal{P}$ in Eq. (13) must be given by a linear combination of the same operators appearing in the hypercharge expression, Eq. (26), multiplied by the same, positive weight $\omega$. This leads to

$$\mathcal{P} = \omega(P_{R3} + P_{R4} + P_{L3} + P_{L4}),$$ (27)
The rest of the SU(2) left-handed projectors used in it follows from the choice of the projector \( P \). It is also possible to start by initially determining the \( B \) symmetry for fermions, which we associate with baryon number, so from now on \( P \) will be referred to as the baryon number operator \( B \), with \( \lambda_B = 1/3 \). In terms of the gamma matrices, the operator is

\[
B = \frac{1}{6}(1 - i\gamma_5\gamma_6) = \frac{1}{3}(P_{R3} + P_{R4} + P_{L3} + P_{L4}).
\] (30)

It is also possible to start by initially determining the \( B \) operator, requiring the spectrum to contain fermions with standard transformation properties with \( \lambda_B = 1/3 \). In terms of the gamma matrices, the operator is

\[
B = \frac{1}{6}(1 - i\gamma_5\gamma_6) = \frac{1}{3}(P_{R3} + P_{R4} + P_{L3} + P_{L4}).
\] (30)

The rest of the SU(2)_L generators are obtained from the non-diagonal elements of \( \mathcal{S} \), by requiring that they fulfill the group Lie algebra \([I_k, I_l] = i\varepsilon_{klm}I_m\), and that they also commute with \( Y \). They are given by

\[
I_3 = \frac{1}{2}(P_{L3} - P_{L4}) = \frac{i}{8}(1 - \gamma_5)(1 - i\gamma_5\gamma_6)\gamma_7\gamma_8.
\] (31)
Finally, the charge operator $Q$ is given by the Gell-Mann–Nishijima relation

$$Q = I_3 + \frac{Y}{2}. \quad (33)$$

### B. Flavor operators

From the elements of $S$ it is possible to obtain four additional groups that commute with $\text{SU}(2)_L \otimes \text{U}(1)_Y$, two $\text{SU}(2)$s and two $\text{U}(1)$s. These groups give zero eigenvalue when acting on non-fermionic states. Thus, they are suitably interpreted as flavor groups, denoted by $\text{SU}(2)_f$, $\text{SU}(2)_{\hat{f}}$, $\text{U}(1)_f$, and $\text{U}(1)_{\hat{f}}$. Their generators are

$$f_1 = \frac{i}{8} (1 + \gamma_5) \left( 1 + i\gamma_5 \gamma_6 \right) \gamma^7,$$

$$f_2 = \frac{i}{8} (1 + \gamma_5) \left( 1 + i\gamma_5 \gamma_6 \right) \gamma^8,$$

$$f_3 = \frac{i}{8} (1 + \gamma_5) \left( 1 + i\gamma_5 \gamma_6 \right) \gamma^7 \gamma^8, \quad (34)$$

$$\hat{f}_1 = \frac{i}{8} (1 - \gamma_5) \left( 1 + i\gamma_5 \gamma_6 \right) \gamma^7,$$

$$\hat{f}_2 = \frac{i}{8} (1 - \gamma_5) \left( 1 + i\gamma_5 \gamma_6 \right) \gamma^8,$$

$$\hat{f}_3 = \frac{i}{8} (1 - \gamma_5) \left( 1 + i\gamma_5 \gamma_6 \right) \gamma^7 \gamma^8, \quad (35)$$

respectively for by $\text{SU}(2)_f$ and $\text{SU}(2)_{\hat{f}}$, and

$$f_0 = i\gamma^5 \gamma^6 \tilde{\gamma}_5, \quad (36)$$

$$\hat{f}_0 = i\gamma^5 \gamma^6, \quad (37)$$

for $\text{U}(1)_f$, and $\text{U}(1)_{\hat{f}}$. In terms of the projectors in Eq. (23) the decomposition of the diagonal generators $f_3$ and $\hat{f}_3$ is

$$f_3 = \frac{1}{2} (P_{R1} - P_{R2}), \quad (38)$$

$$\hat{f}_3 = \frac{1}{2} (P_{L1} - P_{L2}). \quad (39)$$

The matrix space is schematically represented in Fig.4, with the operators that classify the states along the diagonal, and the states, to be described below, appearing as off-diagonal matrix elements.
Table I. Hamiltonian mass operators of the form $S\gamma$, with $S \in S$. Operators $M_1$ to $M_4$ form a maximal commuting set, and so do $M_5$ to $M_8$. The two sets are connected by a transformation involving $\gamma_5$.

| $M_1 = \gamma^0$ | $M_5 = i\gamma_5\gamma^0$ |
|-------------------|-----------------------------|
| $M_2 = i\gamma^5\gamma^8\gamma^0$ | $M_6 = \gamma^5\gamma^8\gamma_5\gamma^0$ |
| $M_3 = \gamma^5\gamma^8\gamma^0$ | $M_7 = \gamma^5\gamma^8\gamma_5\gamma^0$ |
| $M_4 = \gamma^5\gamma^8\gamma^0$ | $M_8 = i\gamma^5\gamma^8\gamma_5\gamma^0$ |

We obtain operators that transform as Lorentz scalars from elements of $S$ multiplied by $\gamma_0$ [30]. Within such a set, of the form $S\gamma_0$, with $S \in S$, in a Hamiltonian description, we concentrate on mass operators which commute with the charge operator. There are eight of these operators, shown in Table I, with a maximal commuting set, among them, given by either $M_1$ to $M_4$ or $M_5$ to $M_8$.

We look among the set for those with the Higgs-field quantum numbers: SU(2)$_L$ doublets, and hypercharge $\pm 1$, when classified by the relevant SU(2)$_L \otimes$ U(1)$_Y$ operators following Eq. (11), and find that there are at most two such doublets. This can be seen in Fig. 4, where the chiral projections of the diagonal operators $B$, $I_3$ and $Y$, according to Eqs. (20), (30), (31), and Fig. 3, are grouped together in the sets $Q_R = \frac{1}{2} (1 + \gamma_5) (B, I_3, Y)$ and $Q_L = \frac{1}{2} (1 - \gamma_5) (B, I_3, Y)$. The off-diagonal matrix elements represent the states, and following the matrix product rule, they are multiplied from the left by diagonal operators in the same row, and from the right by diagonal operators in the same column, producing the eigenvalues corresponding to the weights in Eqs. (20), (30), and (31).

Let us consider the matrix elements in rows 13, 14, and columns 7, 8 in Fig. 4, labeled $\varphi^\dagger_{L,1,2}$, and belonging to the left-chiral projection of the matrix space. Their classification stems from action of the left by $Q_L$, and from the right by $Q_R$; hence Eq. (11) gives the eigenvalues $(1/3, 1/2, 1/3) - (1/3, 0, -2/3) = (0, 1/2, 1)$, for baryon number, isospin and hypercharge, respectively. Thus, they have the quantum numbers of an SU(2)$_L$ upper doublet-component charged scalar field. Now the matrix elements in rows 5, 6, and columns 15, 16, labeled $\varphi^\dagger_{R,1,2}$, belong to the right-chiral projection of the matrix space; they are classified from the left by $Q_R$, and from the right by $Q_L$, giving the same quantum numbers as the previous ones because $(1/3, 0, 4/3) - (1/3, -1/2, 1/3) = (0, 1/2, 1)$. Therefore, a SM Higgs field (non-chirally projected) must be given by a combination of them. With the eight real degrees of freedom of the two blocks we obtain 4 complex elements, from which we can have two linearly independent, scalar charged fields: $\phi^+_{1,2}$. An analogous construction produces two neutral Higgs fields, denoted by $\phi^0_1$ and $\phi^0_2$, and connected to the charged ones by the ladder operators of SU(2)$_L$. The scalar Higgs doublets are given in Table II.

Only some combinations of mass operators correspond to the neutral Higgs fields, so that the latter can be given in terms of the former, e.g.,

$$\phi^0_1 = \frac{1}{8} (M_1 - M_2) + \frac{i}{8} (M_7 - M_8),$$

$$\phi^0_2 = \frac{1}{8} (M_3 + M_4) - \frac{i}{8} (M_5 + M_6).$$

There are also mass-operator terms that do not correspond to any combination of them. Thus, after electroweak symmetry breaking when only the neutral Higgs fields survive, we consider the following linear combinations of the neutral Higgs fields and their Hermitian conjugates, and of the Hamiltonian mass operators, taken from the first set in Table I.
The massless-quark spectrum consists of states with the structure of Eq. (41), and with the corresponding quantum numbers, when classified by baryon number, isospin, and hypercharge. The dimensionality of the matrix space gives rise to four generations of massless quarks \[ Q_{R} = \frac{1}{2} (1 + 7\gamma) (B, I_3, Y) \] and \[ Q_{L} = \frac{1}{2} (1 - 7\gamma) (B, I_3, Y) \]. Following matrix multiplication rules, operators act from the left on states in the same row, and from the right on states in the same column.

\[
M_1 = a_1 \left( \phi_1^0 + \phi_1^0 \right) + b_1 \left( \phi_2^0 + \phi_2^0 \right),
\]

\[
M_2 = \frac{a_2}{4} (M_1 + M_2) + \frac{b_2}{4} (M_3 - M_4),
\]

with \( a_{1,2} \) and \( b_{1,2} \) real parameters, and \( M_2 \) cannot be given in terms of the neutral Higgses (the 1/4 factor is just for normalization convenience). In Ref. 29 a Lagrangian for the Higgs-field doublets was given for a reduced model with only two quark flavors. As for the \( M_2 \) operator in Eq. (41), it is a non-Higgs-like scalar that also classifies the massive quark states, fulfilling the same function as a flavon field [12, 13]. In fact, \( M_2 \) produces a horizontal mass hierarchy, as is shown in Subsection III.E.

D. Massless quarks

The massless-fermion spectrum consists of states with the structure of Eq. (41), and with the corresponding quantum numbers, when classified by baryon number, isospin, and hypercharge. The dimensionality of the matrix space gives rise to four generations of massless quarks. In Fig. 4, the elements labeled \( U_{i}^{L} \), \( i = 1, \ldots, 4 \) in rows 13 and 14 represent all possible degrees of freedom with the quantum numbers (1/3, 1/2, 1/3) for the latter operators, respectively, and the corresponding flavor. Thus, they are interpreted as left-handed quarks of the up-type. There are 4 degrees of freedom for each \( U_{i}^{L} \) accounting for the two possible spin directions (up/down) and chirality (left/right-handed), so in this respect, even though they are represented by 16 \( \times \) 16 matrices, fermions in the extended spin space carry the same degrees of freedom as a massless 4-d Weyl spinor. This latter feature is a consequence of maintaining Lorentz symmetry in 4-d.

Analogous results hold for right-handed up-type quarks, shown in rows 5 and 6 in Fig. 4, and for down-type quarks, both left-handed and right-handed, shown respectively in rows 15 and 16, and rows 7 and 8 in Fig. 4. Thus, there are in total eight left-handed quarks arranged in four SU(2)_L doublets, with elements connected vertically by the ladder operators \( \frac{1}{\sqrt{2}} (I_1 \pm iI_2) \), and eight right-handed SU(2)_L singlets. The quark states are given in Tables III and IV, together with their quantum numbers.
Table III. Massless left-handed quark weak isospin doublets. Gauge and Lorentz operators act from the left and trivially from the right, while the reverse is true for flavor operators. To obtain the $-1/2$ polarization, the replacement must be made $(\gamma^0 + \gamma^5) \rightarrow (\gamma^1 - i\gamma^2)$, for $Q_{L,1}^1$, $Q_{L,1}^2$, and $(\gamma^0 - \gamma^5) \rightarrow (\gamma^1 + i\gamma^2)$, for $Q_{L,3}^1$, $Q_{L,4}^1$.

| Baryon number 1/3, hypercharge 1/3, and polarization 1/2 (operator $\frac{1}{\sqrt{2}}B\gamma^i\gamma^5$), left-handed quark doublets | $I_3$ | $Q$ | $f_3$ | $\bar{f}_3$ | $F$ |
|---|---|---|---|---|---|
| $Q_{L,1}^1 = \left( \begin{array}{c} U_{L,1}^1 \\ D_{L,1}^1 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( \gamma^1 + i\gamma^2 \right) \left( \gamma^0 + \gamma^5 \right) \\ \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( 1 - i\gamma^1 \gamma^5 \right) \left( \gamma^0 + \gamma^5 \right) \end{array} \right)$ | $1/2$ | $2/3$ | $-1/2$ | $0$ | $3/2$ |
| $Q_{L,2}^1 = \left( \begin{array}{c} U_{L,2}^1 \\ D_{L,2}^1 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( 1 + i\gamma^1 \gamma^5 \right) \left( \gamma^0 + \gamma^5 \right) \\ \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( 1 - i\gamma^1 \gamma^5 \right) \left( \gamma^0 + \gamma^5 \right) \end{array} \right)$ | $1/2$ | $2/3$ | $1/2$ | $0$ | $-1/2$ |
| $Q_{L,3}^1 = \left( \begin{array}{c} U_{L,3}^1 \\ D_{L,3}^1 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( \gamma^1 + i\gamma^2 \right) \left( \gamma^0 + \gamma^5 \right) \\ \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( 1 - i\gamma^1 \gamma^5 \right) \left( \gamma^0 + \gamma^5 \right) \end{array} \right)$ | $1/2$ | $2/3$ | $0$ | $-1/2$ | $1$ |
| $Q_{L,4}^1 = \left( \begin{array}{c} U_{L,4}^1 \\ D_{L,4}^1 \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( 1 + i\gamma^1 \gamma^5 \right) \left( \gamma^0 + \gamma^5 \right) \\ \frac{1}{\sqrt{2}} (1 - \gamma_5) \left( \gamma^0 - i\gamma^5 \right) \left( 1 - i\gamma^1 \gamma^5 \right) \left( \gamma^0 + \gamma^5 \right) \end{array} \right)$ | $1/2$ | $2/3$ | $0$ | $1/2$ | $0$ |

Table IV. Massless right-handed quark weak isospin singlets. Gauge and Lorentz operators act from the left and trivially from the right, while the reverse is true for flavor operators. To obtain the $-1/2$ polarization, the replacement must be made $(\gamma^0 + \gamma^5) \rightarrow (\gamma^1 - i\gamma^2)$, for $U_{R,1}^1$, $U_{R,2}^1$, $D_{R,1}^1$, $D_{R,2}^1$, and $(\gamma^0 - \gamma^5) \rightarrow (\gamma^1 + i\gamma^2)$, for $U_{R,3}^1$, $U_{R,4}^1$, $D_{R,3}^1$, $D_{R,4}^1$.

The flavor operators provide further symmetries: members of the first two families with the same charge are horizontal $SU(2)_F$ doublets when classified with the ladder operators $\frac{1}{\sqrt{2}} (\bar{f}_1 \pm i f_2)$ and $\frac{1}{\sqrt{2}} (f_1 - i f_2)$ of SU(2)$_F$, e.g.

\[
U_{L,1}^1 \frac{1}{\sqrt{2}} (f_1 - i f_2) = \frac{1}{\sqrt{2}} U_{L,2}^1, \quad U_{L,2}^1 \frac{1}{\sqrt{2}} (f_1 - i f_2) = 0, \quad U_{L,2}^1 \frac{1}{\sqrt{2}} (f_1 + i f_2) = -\frac{1}{\sqrt{2}} U_{L,1}^1, \quad U_{L,1}^1 \frac{1}{\sqrt{2}} (f_1 + i f_2) = 0,
\]

(42)

where, in accord with the discussion below [13], flavor operators act non-trivially only from the rhs. This horizontal flavor symmetry also applies to right-handed quarks.

There is also the possibility of projecting out a generation. As an example, let us consider the operator

\[
F = -\frac{1}{4} \left( \bar{f}_0 - 4 \bar{f}_3 - 8 f_3 \right),
\]

(43)

which yields $[F, U_{L,i}^1] = \lambda_i U_{L,i}^1$, $[F, D_{L,i}^1] = \lambda_i D_{L,i}^1$, $[F, U_{R,i}^1] = \lambda_i U_{R,i}^1$, $[F, D_{R,i}^1] = \lambda_i D_{R,i}^1$, $i = 1, \ldots, 4$, with $\lambda_1 = 3/2$, $\lambda_2 = -1/2$, $\lambda_3 = 1$, and $\lambda_4 = 0$. Therefore, it provides a horizontal (family) $U(1)$ symmetry for the quark states, with the zero value for $\lambda_4$ interpreted as projecting the fourth quark family out of flavor space, so effectively, we get three massless generations. This operator commutes with $B$, $Q$ and $M_1$, respectively in Eqs. (30), (33), and (11), but not with $M_2$ in the latter. Its complete characterization requires further investigation, which is left for future work and at present it will be no longer considered.
E. Mass hierarchy effects

Let us consider the general mass operator

$$\Omega \equiv M_1 + M_2,$$

(44)

with $M_1$ and $M_2$ given in Eq. (41). This operator can be diagonalized with the massless quark states to obtain massive ones, e.g., the combinations

$$U_M = \frac{1}{\sqrt{2}} \left( U_{1L,1}^1 + U_{1R,1}^1 \right),$$
$$D_M = \frac{1}{\sqrt{2}} \left( D_{1L,1}^1 - D_{1R,1}^1 \right),$$

(45)

have the quantum numbers of massive quarks of the up and down-type, respectively. Considering only the $M_1$ part in Eq. (44), that is setting $a_2 = b_2 = 0$, we obtain the eigenvalues

$$\frac{a_1 + b_1}{2}, \quad \frac{a_1 - b_1}{2},$$

(46)

respectively for $U_M$ and $D_M$ in Eq. (45), which constitute a vertical mass hierarchy depending on the order of the parameters involved, with the identifications

$$a_1 = m_u + m_d,$$
$$b_1 = m_u - m_d,$$

(47)

where $m_u$ and $m_d$ are current quark masses of the up and down type, respectively, for a given family. However, the structure of the states in Eq. (45) does not allow them to be eigenstates of the $M_2$ operator in Eq. (44), responsible for the horizontal mass hierarchy effect as we next show. In order for them to be also eigenstates of $M_2$, and therefore diagonalize $\Omega$, another generation of massless quarks, in a left and right combination, must be added, and the relative phases fixed, so that the states in Eq. (45) change to

$$U_{M,1}^1 = \frac{1}{2} \left( -U_{1L,1}^1 - U_{1R,1}^1 + U_{1L,3}^1 + U_{1R,3}^1 \right),$$
$$D_{M,1}^1 = \frac{1}{2} \left( -D_{1L,1}^1 + D_{1R,1}^1 + D_{1L,3}^1 - D_{1R,3}^1 \right),$$

(48)

The four generations of massive quarks is shown in Table V, and the vertical hierarchy effect in Eq. (46) remains valid for $U_{M,i}^1$ and $D_{M,i}^1$ quarks, $i = 1, \ldots, 4$.

Turning now to the $M_2$ part in Eq. (44), with $a_1 = b_1 = 0$, and considering the same states in Eq. (48), together with the second family from Table V

$$U_{M,2}^1 = \frac{1}{2} \left( U_{1L,2}^1 - U_{1R,2}^1 - U_{1L,4}^1 + U_{1R,4}^1 \right),$$
$$D_{M,2}^1 = \frac{1}{2} \left( D_{1L,2}^1 + D_{1R,2}^1 - D_{1L,4}^1 - D_{1R,4}^1 \right),$$

(49)

we obtain the eigenvalues

$$\frac{a_2 - b_2}{2}, \quad \frac{a_2 + b_2}{2},$$

(50)

with the first value corresponding to the generation in Eq. (48), and the second to the one in Eq. (49). Thus, we see that the operator $M_2$ produces a horizontal (family) mass hierarchy. The massive quark states are shown in Table V, with the eigenvalues from operators in Eqs. (45) and (44).
the model may provide new information in this regard. This makes this a promising research program, and because they are shared with dynamical symmetry-breaking models, bosons, and phenomenological implications for the CKM matrix and mixing angles. The features presented thus far work analogously to flavons, and also limit physics beyond the SM, as they are restricted by the (7+1)-dimensional hierarchy effects. The horizontal effect is implemented through non-Higgs fields/operators for mass generation, which its main features: producing up to four quark generations, two Higgs doublets, and both horizontal and vertical mass postulated; e. g., in some models of dynamical symmetry breaking and compositeness [32–34], four-quark generations but the model’s features could serve as a basis for further phenomenological studies, since they are derived and not serves as an alternative, or even a complement, to SUSY: this is a comparison fostered by the fact that both approaches gauge and spacetime, in a single matrix space. It also provides an alternative unification scheme that does not require a graded Lie algebra, as in SUSY. Thus, an interesting research direction is to investigate to what extent the model serves as an alternative, or even a complement, to SUSY; this is a comparison fostered by the fact that both approaches can use Clifford algebras for their representations.

The model suggests directions to research further, such as a complete characterization of leptons, gauge-vector bosons, and phenomenological implications for the CKM matrix and mixing angles. The features presented thus far make this a promising research program, and because they are shared with dynamical symmetry-breaking models, the model may provide new information in this regard.

IV. CONCLUDING REMARKS

In this paper, we presented an electroweak quark model derived from an extended (7+1)-dimensional spin space, with its main features: producing up to four quark generations, two Higgs doublets, and both horizontal and vertical mass hierarchy effects. The horizontal effect is implemented through non-Higgs fields/operators for mass generation, which work analogously to flavons, and also limit physics beyond the SM, as they are restricted by the (7+1)-dimensional space.

The mass relations obtained here are not enough to accurately reproduce the known quark masses on their own, but the model’s features could serve as a basis for further phenomenological studies, since they are derived and not postulated; e. g., in some models of dynamical symmetry breaking and compositeness [32–34], four-quark generations and two Higgs doublets are proposed at the outset, whereas in the present case they arise naturally from a minimum of physical assumptions. This is also true for the compositeness aspect, which is naturally embedded in the present model by its very construction.

As a SM extension, the extended spin-space model provides a unified description of fields and symmetries, both gauge and spacetime, in a single matrix space. It also provides an alternative unification scheme that does not require a graded Lie algebra, as in SUSY. Thus, an interesting research direction is to investigate to what extent the model serves as an alternative, or even a complement, to SUSY; this is a comparison fostered by the fact that both approaches can use Clifford algebras for their representations.

The model suggests directions to research further, such as a complete characterization of leptons, gauge-vector bosons, and phenomenological implications for the CKM matrix and mixing angles. The features presented thus far make this a promising research program, and because they are shared with dynamical symmetry-breaking models, the model may provide new information in this regard.

ACKNOWLEDGMENTS

The authors acknowledge support from DGAPA-UNAM, project IN112916.

\[ Q \mathcal{M}_1 \mathcal{M}_2 \Omega \]

| Baryon number 1/3 and polarization 1/2 | \( Q \mathcal{M}_1 \mathcal{M}_2 \Omega \) |
|--------------------------------------|----------------------------------|
| \( U^U_{M,1} = \frac{1}{2} (U^U_{1,1} - U^U_{2,1} + U^U_{3,1} + U^U_{4,1}) \) | \( \frac{2}{3} \) \( \frac{a_1 + b_1}{2} \) \( \frac{a_2}{2} \) \( \frac{a_1 + b_2 + a_2}{4} \) |
| \( U^U_{M,2} = \frac{1}{2} (U^U_{1,2} - U^U_{2,2} + U^U_{3,2} + U^U_{4,2}) \) | \( \frac{2}{3} \) \( \frac{a_1}{2} \) \( \frac{a_2 + b_2}{2} \) \( \frac{a_1 + b_1 + a_2 + b_2}{4} \) |
| \( U^U_{M,3} = \frac{1}{2} (U^U_{1,3} + U^U_{2,3} + U^U_{3,3} + U^U_{4,3}) \) | \( \frac{2}{3} \) \( \frac{a_1 + b_1}{2} \) \( \frac{a_2 - b_2}{2} \) \( \frac{a_1 + b_2 - a_2 - b_2}{4} \) |
| \( U^U_{M,4} = \frac{1}{2} (U^U_{1,4} + U^U_{2,4} + U^U_{3,4} - U^U_{4,4}) \) | \( \frac{2}{3} \) \( \frac{a_1 - b_1}{2} \) \( \frac{a_2 + b_2}{2} \) \( \frac{a_1 - b_2 - a_2 + b_2}{4} \) |
| \( D^D_{M,1} = \frac{1}{2} (D^D_{1,1} + D^D_{2,1} + D^D_{3,1} - D^D_{4,1}) \) | \( \frac{1}{3} \) \( \frac{a_1 + b_1}{2} \) \( \frac{a_2 + b_2}{2} \) \( \frac{a_1 - b_2 + a_2}{4} \) |
| \( D^D_{M,2} = \frac{1}{2} (D^D_{1,2} - D^D_{2,2} + D^D_{3,2} + D^D_{4,2}) \) | \( \frac{1}{3} \) \( \frac{a_1 - b_1}{2} \) \( \frac{a_2 - b_2}{2} \) \( \frac{a_1 - b_2 - a_2 + b_2}{4} \) |
| \( D^D_{M,3} = \frac{1}{2} (D^D_{1,3} + D^D_{2,3} + D^D_{3,3} - D^D_{4,3}) \) | \( \frac{1}{3} \) \( \frac{a_1 + b_1}{2} \) \( \frac{a_2}{2} \) \( \frac{a_1 + b_2 + a_2}{4} \) |
| \( D^D_{M,4} = \frac{1}{2} (D^D_{1,4} + D^D_{2,4} - D^D_{3,4} + D^D_{4,4}) \) | \( \frac{1}{3} \) \( \frac{a_1 - b_1}{2} \) \( \frac{a_2}{2} \) \( \frac{a_1 - b_2 - a_2}{4} \) |

Table V. Massive quark states

The authors acknowledge support from DGAPA-UNAM, project IN112916.

[1] N. Craig, in Beyond the Standard Model after the first run of the LHC Arcetri, Florence, Italy, May 20-July 13, 2013 (2013) arXiv:1309.0528 [hep-ph]
[2] S. Pokorski, Ann. Phys. 528, 84 (2016)
[3] M. Aaboud et al. (ATLAS), Eur. Phys. J. C 76, 392 (2016) arXiv:1605.03814 [hep-ex]
[4] G. Aad et al. (ATLAS), Phys. Rev. D 94, 032003 (2016) arXiv:1605.09318 [hep-ex]
[5] V. Khachatryan et al. (CMS), Phys. Lett. B 758, 152 (2016) arXiv:1602.06581 [hep-ex]
[6] V. Khachatryan et al. (CMS), Eur. Phys. J. C 76, 460 (2016)
[7] G. Branco, P. Ferreira, L. Lavoura, M. Rebelo, M. Sher, and J. P. Silva, Phys. Rept. 516, 1 (2012) arXiv:1106.0034 [hep-ph]
[8] W. A. Ponce, L. A. Wills, and A. Zepeda, Z. Phys. C 73, 711 (1997) arXiv:hep-ph/9507467 [hep-ph]
[9] H. Fritzsch and Z.-Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000), arXiv:hep-ph/9912358 [hep-ph].
[10] G. C. Ross, “Flavor physics for the millennium: Tasi 2000,” (World Scientific, 2001) Chap. Models of fermion masses, pp. 775–826.
[11] K. S. Babu, in Proceedings of Theoretical Advanced Study Institute in Elementary Particle Physics on The dawn of the LHC era (TASI), (2010) pp. 49–123, arXiv:0910.2948 [hep-ph].
[12] C. Froggatt and H. Nielsen, Nucl. Phys. B 147, 277 (1979).
[13] M. Bauer, M. Carena, and K. Gemmler, JHEP 2015, 1 (2015), arXiv:1506.01719 [hep-ph].
[14] A. Crivellin, G. D’Ambrosio, and J. Heeck, Phys. Rev. D 91, 075006 (2015), arXiv:1503.03477 [hep-ph].
[15] D. Espriu, F. Mescia, and A. Renau, Phys. Rev. D 92, 095013 (2015), arXiv:1503.02953 [hep-ph].
[16] M. Dine, W. Fischler, and M. Srednicki, Phys. Lett. B 104, 199 (1981).
[17] A. R. Zhitnitsky, Sov. J. Nucl. Phys. 31, 260 (1980), [Yad. Fiz. 31, 497 (1980)].
[18] J. Besprosvany and R. Romero, Symmetries in nature. Proceedings, Symposium in Memoriam Marcos Moshinsky, Cuernavaca, Mexico, August 7-14, 2010, AIP Conf. Proc. 1323, 16 (2010).
[19] J. Besprosvany and R. Romero, Nucl. Part. Phys. Proc. 267, 199 (2015), arXiv:1512.05395 [hep-th].
[20] J. Besprosvany, Int. J. Theor. Phys. 39, 2797 (2000), arXiv:hep-th/0203114 [hep-th].
[21] J. Besprosvany, Nucl. Phys. Proc. Suppl. 101, 323 (2001).
[22] J. Besprosvany, Int. J. Mod. Phys. A 20, 77 (2005), arXiv:hep-th/0206156.
[23] J. Besprosvany, Phys. Lett. B 578, 181 (2004), arXiv:hep-th/0203122 [hep-th].
[24] J. Besprosvany, Mod. Phys. Lett. A 18, 1877 (2003), arXiv:hep-ph/0305211 [hep-ph].
[25] S. R. Coleman and J. Mandula, Phys. Rev. 159, 1251 (1967).
[26] J. Snygg, Clifford algebra: a computational tool for physicists (Oxford University Press, 1997).
[27] I. Benn and R. Tucker, An Introduction to Spinors and Geometry With Applications in Physics (Adam Hilger, 1987).
[28] Following standard practice, the label 4 is omitted.
[29] J. Besprosvany and R. Romero, Int. J. Mod. Phys. A 29, 1450144 (2014), arXiv:1408.4066 [hep-th].
[30] This property depends also on the operator structure, and not only on multiplication by $\gamma_0$, so the latter is a necessary but not sufficient condition.
[31] The space dimension is not big enough to accommodate color SU(3) of QCD, but this can be done in 9+1 dimensions, see Ref. [23].
[32] S. Bar-Shalom, S. Nandi, and A. Soni, Phys. Rev. D 84, 053009 (2011), arXiv:1105.6095 [hep-ph].
[33] P. Hung and C. Xiong, Nucl. Phys. B 848, 288 (2011), arXiv:1012.4479 [hep-ph].
[34] H. S. Fukano and K. Tuominen, arXiv e-print (2011), arXiv:1102.1254 [hep-ph].