We investigate interacting dark energy models in the framework of fractal cosmology. We discuss a fractal FRW universe filled with the dark energy and dark matter which interact with each other. We obtain the equation for the relative density of dark matter and dark energy and the deceleration parameter. This model demonstrates new types of evolution, which are not common to cosmological models with this type of interaction.

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I. INTRODUCTION

In the present paper, we study interacting dark energy models in the framework of fractal cosmology proposed by Calcagni. In \cite{1,2} he proposed a model for a power-counting renormalizable field theory living in a fractal spacetime. The action in this model is Lorentz covariant and equipped with a Stieltjes measure.

For this model we calculate the relative density of dark matter and dark energy, the deceleration parameter and discuss their physical implications via the numerical integration of derived equations of motion.

The logic motivating Lebesgue–Stieltjes field theories is the following\cite{3}. At first, one notices that most theories of quantum gravity display universal properties of dimensional flow. Then, one can ask whether and how these properties are related to the problem of UV finiteness. For this purpose, it a geometry and field theory constructed\cite{4} where dimensional flow is an intrinsic (not indirect) property, and where one can check the UV finiteness explicitly. In this respect, fractional theories are independent from other models of quantum gravity and forcing two independent models to fit one another (say, fractional theories versus quantum Einstein gravity, or versus Hořava–Lifshitz) may cause misleading interpretations of otherwise constructive independent insights.

The fractal properties of quantum gravity theories in $D$ dimensions have been explored in several contexts. At first, renormalizability of perturbative gravity at and near two topological dimensions drew much interest into $D = 2 + \epsilon$ models, with the hope to understand the $D = 4$ case better\cite{5–12}.

Assuming that matter is minimally coupled with gravity, the total action is \cite{1,2}

$$S = S_g + S_m, \quad (1)$$

where $S_g$ is

$$S_g = \frac{M_p^2}{2} \int d\rho(x) \sqrt{-g} \left( R - 2\lambda - \omega \partial_\mu v \partial^\mu v \right), \quad (2)$$

and

$$S_m = \int d\rho \sqrt{-g} \mathcal{L}_m \quad (3)$$

is the matter action. Here $g$ is the determinant of the dimensionless metric $g_{\mu\nu}$, $M_p^{-2} = 8\pi G$ is reduced Planck mass, $R$ is Ricci scalar, $\lambda$ the bare cosmological constant, and the term proportional to $\omega$ has been added because $v$, like the other geometric field $g_{\mu\nu}$, is now dynamical. Note that $d\rho(x)$ is Lebesgue–Stieltjes measure generalizing the $D$-dimensional measure $d^D x$. The scaling dimension of $g$ is $[g] = -D\alpha \neq -D$, where $\alpha > 0$ is a positive parameter.
The derivation of the Einstein equations goes almost like in scalar-tensor models. Taking the variation of the action \( S \) with respect to the Friedmann-Robertson-Walker (FRW) metric \( g_{\mu \nu} \), one can obtain the Friedmann equations in a fractal universe as was found in \[1\] (D - 1) \( \frac{H^2}{2} + \dot{H} - \frac{\omega}{D - 1} \ddot{v}^2 = \frac{1}{M_p^2(D - 1)} \rho + \frac{\lambda}{D - 1} - \frac{k}{a^2} \), \( \Box v = -(D - 2) \left( H^2 + \dot{H} - \frac{\omega}{D - 1} \ddot{v}^2 \right) + \frac{2\lambda}{D - 1} = \frac{1}{M_p^2(D - 1)} [(D - 3) \rho + (D - 1) p] \). (4)

where \( H = \dot{a}/a \) is the Hubble parameter, \( \rho \) and \( p \) are the total energy density and pressure of the ideal fluid composing the Universe. The parameter \( k \) denotes the curvature of the Universe, where \( k = -1, 0, +1 \) for the close, flat and open Universe respectively. As is easy can see when \( v = 1 \), Eqs. (4) and (5) transform to the standard Friedmann equations in Einstein GR.

If \( \rho + p \neq 0 \), one can get a purely gravitational constraint (see \[2\] for detail):

\[ \dot{H} + (D - 1) H^2 + \frac{2k}{a^2} + \Box v + \dot{H} - \frac{\omega}{v} \ddot{v}^2 + \omega(v \Box v - \ddot{v}^2) = 0 \]. (6)

The continuity equation in fractal cosmology takes the form

\[ \dot{\rho} + \left[ (D - 1) H + \frac{\dot{v}}{v} \right] (\rho + p) = 0 \], (7)

When \( v = 1 \) and \( D = 4 \), we recover the standard Friedmann equations in four dimensions, eqs. (11) and (12) (no gravitational constraint):

\[ H^2 = \frac{1}{3M_p^2} \rho + \frac{\lambda}{3} - \frac{k}{a^2} \], (8)

\[ H^2 + \dot{H} = -\frac{1}{6M_p^2} (3p + \rho) + \frac{\lambda}{3} \]. (9)

On the other hand, for the measure weight

\[ v = t^{-\beta} \], (10)

where \( \beta \) is given by \( \beta \equiv D(1 - \alpha) \), the gravitational constraint is switched on. The UV regime, in fact, describes short scales at which inhomogeneities should play some role. If these are small, the modified Friedmann equations define a background for perturbations rather than a self-consistent dynamics.

Recently \[13\] the holographic, new agegraphic and ghost dark energy models in the framework of fractal cosmology was investigated. In the next section we consider the universe in which dark energy interacting with dark matter.

II. DARK SECTOR INTERACTION IN FRACTAL COSMOLOGY

In this section, we derive the first order differential equations that describe the evolution of interacting dark matter and dark energy in the framework of spatially flat \( k = 0 \) fractal cosmology.

A. The general equations

It is well known that interaction between the components in the Universe must be introduced in such a way that preserves the covariance of the energy-momentum tensor \( T_{\mu \nu} \) \( (\mu, \nu = 0) \), therefore \( T_{\mu \nu} = -T_{\nu \mu} \neq 0 \), where \( u_\nu \) is the 4-velocity. The conservation equations in that case take the form:

\[ u_\nu T_{\mu \nu} = -u_\nu T_{\mu \nu} = -Q \]. (11)
For four-dimensional space with FRW-metric in fractal case and natural parameterization of the function \( v = t^{-\beta} \), equations (7) transform to:

\[
\dot{\rho}_m + (3H - \beta t^{-1})\rho_m = Q, \quad \dot{\rho}_x + (1 + w_x) (3H - \beta t^{-1})\rho_x = -Q, \tag{12, 13}
\]

where \( \rho_m \) and \( \rho_x \) are densities of dark matter and dark energy respectively, \( w \) is the state parameter for dark energy.

It is convenient to use the relative energy densities of dark energy and dark matter in accordance with standard definitions:

\[
\Omega_m = \frac{\rho_m}{3M_p^2H^2}, \quad \Omega_x = \frac{\rho_x}{3M_p^2H^2}. \tag{14}
\]

The above equation can be written in terms of this density parameters as the following:

\[
\dot{\Omega}_m + (3H - \beta t^{-1})\Omega_m + 2\Omega_m \frac{\dot{H}}{H} = \frac{Q}{3M_p^2H^2}, \tag{15}
\]

\[
\dot{\Omega}_x + (1 + w_x) (3H - \beta t^{-1})\Omega_x + 2\Omega_x \frac{\dot{H}}{H} = -\frac{Q}{3M_p^2H^2}, \tag{16}
\]

where the dot denotes derivative with respect to cosmic time \( t \). The differential equation for the Hubble parameter has the form

\[
\dot{H} + H^2 = -\frac{\beta H}{2t} + \frac{\beta(\beta + 1)}{2t^2} + \frac{\omega\beta^2}{3t^2(\beta + 1)} = -\frac{1}{2}((1 + 3w)\Omega_x + \Omega_m)H^2. \tag{17}
\]

In order to obtain the Friedmann equation in terms of the relative densities it is necessary to enter fictitious density same way as \( \Omega_k = k/(a^2H^2) \). So, we introduce the fractal relative density:

\[
\Omega_f = \frac{\omega\dot{v}^2}{6H^2} - \frac{\dot{v}}{Hv}. \tag{18}
\]

Taking into account the ansatz \( v = t^{-\beta} \), we obtain the equation of motion for fractal relative density

\[
\Omega_f = \frac{\omega\beta^2}{6H^2(2\beta + 1)} + \frac{1}{Ht}. \tag{19}
\]

Thus, the Friedman equation can be re-written in a very elegant form

\[
\sum_{\alpha=k,f,x,m} \Omega_{\alpha} = 1. \tag{20}
\]

Note that in frames of this definition the values of the relative density \( \Omega_x \) or \( \Omega_m \) can exceed 1.

### B. Linear interaction of dark matter and dark energy

In view of the continuity equations, the interaction between dark matter and dark energy must be a function of the energy densities multiplied by a quantity with dimensions of inverse time. For the latter, in the context of cosmology, the obvious choice is the Hubble parameter \( H \). So, the interaction between dark components could be expressed phenomenologically in such forms as \( Q = Q(H\rho_x + \rho_m) \). Most generally in the form \( Q = Q(H\rho_x, H\rho_m) \) which leads to \( Q \approx \delta H\rho_x + \gamma H\rho_m \) as the first term in its power law expansion of more general expression. Next we consider the simplest form of interaction – linear combination of the densities of dark matter and dark energy:

\[
Q \equiv H(\delta \rho_x + \gamma \rho_m) \tag{21}
\]

In this case, the equations of motion take the form

\[
\dot{\Omega}_m + (3H - \beta t^{-1})\Omega_m + 2\Omega_m \frac{\dot{H}}{H} = H(\delta \Omega_x + \gamma \Omega_m), \tag{22}
\]

\[
\dot{\Omega}_x + (1 + w_x) (3H - \beta t^{-1})\Omega_x + 2\Omega_x \frac{\dot{H}}{H} = -H(\delta \Omega_x + \gamma \Omega_m),
\]

Since the equations explicitly depend on time, is not possible to find their analytical solution.
C. Analyzable case of dark matter and dark energy interaction

The analytical solution can be found only in the case when the Hubble parameter is inversely proportional to time, which is typical, for example, at the stage of nonrelativistic matter dominance. Suppose that at this stage the Hubble parameter has the form $H = \sigma t^{-1}$, then the equations take the following form

$$\begin{align*}
\dot{\Omega}_m &= \theta \dot{\Omega}_m + \sigma \dot{\Omega}_x, \\
\dot{\Omega}_x &= -\delta \gamma \dot{\Omega}_m + \nu \dot{\Omega}_x,
\end{align*}$$

(23)

where $\theta = 2 + \gamma \sigma + \beta - 3\sigma$, $\nu = 2 - (1 + w)(3\sigma - \beta) - \delta \sigma$, and the prime denotes derivative with respect to logarithm of cosmic time $t' \equiv \frac{d}{d \ln m}$. Note also that the parameter physical is meaningful if under condition $\sigma > 0$, because we do not consider the collapsing universe. In this regime of evolution of the Universe the system of equations is autonomous and can be exactly solved. Characteristic equation of the system (23) has the form

$$\tau^2 - (\theta + \nu)\tau + \delta^2 \gamma + \theta \nu = 0,$$

(24)

its roots of this equation are equal to:

$$\tau_{\pm} = \frac{\theta + \nu}{2} \left[ 1 \pm \sqrt{1 - 4 (\delta^2 \gamma + \theta \nu) / (\theta + \nu)^2} \right]$$

(25)

Let us consider possible types of solutions, and indicate the critical points corresponding to them. As one can see, this model contains many parameters, making it cumbersome to analyse. Note that due to this feature the system describes all possible types of critical points typical for coarse equilibrium states.

Recall that the values of $\beta$ in the IR and UV regimes are $\beta_{IR} = 0$ and $\beta_{UV} = 2$, respectively. The UV regime, in fact, describes short scales at which inhomogeneities should play some role. If these are small, the modified Friedmann equations define a background for perturbations rather than a self-consistent dynamics.

There are six types of critical points:

1. Stable node $\tau_{\pm} \in \mathbb{R}$, $\tau_{\pm} < 0$, $\tau_+ > \tau_- > 0$, $\theta + \nu < 0$, $4(\delta^2 \gamma + \theta \nu) < (\theta + \nu)^2, \delta^2 \gamma + \theta \nu > 0$.
2. Unstable node: $\tau_{\pm} \in \mathbb{R}$, $\tau_{\pm} > 0$, $\tau_+ > \tau_- > 0, \theta + \nu > 0$, $4(\delta^2 \gamma + \theta \nu) < (\theta + \nu)^2, \delta^2 \gamma + \theta \nu > 0$.
3. Saddle point: $\tau_{\pm} \in \mathbb{R}$, $\tau_+ \tau_- < 0$, $\delta^2 \gamma + \theta \nu < 0$.
4. Stable spiral point: $\tau_{\pm} \in \mathbb{C}$, $\tau_{\pm} = \tau_1 \pm i \tau_2$, $\tau_1, \tau_2 \in \mathbb{R}$ $\tau_1, \tau_2 > 0$, $\theta + \nu < 0, (\theta + \nu)^2 < 4(\delta^2 \gamma + \theta \nu)$.
5. Unstable spiral point: $\tau_{\pm} \in \mathbb{C}$, $\tau_{\pm} = \tau_1 \pm i \tau_2$, $\tau_1, \tau_2 \in \mathbb{R}$ $\tau_1, \tau_2 < 0$, $\theta + \nu > 0, (\theta + \nu)^2 < 4(\delta^2 \gamma + \theta \nu)$.
6. Elliptic fixed point $\tau_{\pm} \in \mathbb{S}$, $\tau_{\pm} = \pm i \tau$, $\tau \in \mathbb{R}$, $\theta = \nu$, $\delta^2 \gamma + \theta \nu > 0$.

This completes a variety of critical points in the system (23). Thus, there are only unstable equilibrium states in this dynamic system. This means that the universe can not last long in a state with the Hubble parameter $H = \sigma t^{-1}$, corresponding to slowdown expanding universe.

D. The numerical solution for interesting case

Dissimilar a linear system of ordinary differential equations (23), a non-linear system (23) allows for singular structures which are more intricate than that of the singular points, fixed lines or periodic orbits.

In this subsection we consider the case which seems to us particularly interesting, mainly because it corresponds to a very interesting regime of evolution - the transient acceleration.

Starobinsky [14] with co-authors, based on independent observational data, including the brightness curves for SNe Ia, cosmic microwave background temperature anisotropy and baryon acoustic oscillations (BAO), were able to show, that the acceleration of Universe expansion reached its maximum value and now decreases. In terms of the deceleration parameter it means that the latter reached its minimum value and started to increase. Thus the main result of the analysis is the following: SCM is not unique though the simplest explanation of the observational data, and the accelerated expansion of Universe presently dominated by dark energy is just a transient phenomenon.

Transient acceleration appears in many models of dark energy as the interaction between dark energy and dark matter as in interaction-free models. One of the key features of this model is that it contains only of the transient acceleration regime.
Another feature of this model (both in interacting and interacting-free cases) is the fact that even in the early stages of the evolution of the universe, its expansion was accelerated. So, this model has a finite period of accelerated expansion (inflation) in the early stages of the Universe evolution. After this transition, the Universe enters a stage of slow expansion. The duration of this regime depends on values of the parameters, it is significant that it is limited. Transition period ends by phantom mode expansion, which tends asymptotically to \( q \rightarrow \infty \).

The dependence of relative density of dark matter and dark energy is also differs significantly in most models. Since the early times of the evolution of the universe, dominated by the fractal energy density \( \Omega_f \), the following holds \( \Omega_f + \Omega_m + \Omega_x = 1 \). So, the contribution of the fractal component is significant only in the early stages of the evolution of the universe.

III. CONCLUSIONS

In present paper we consider a new model of interacting dark energy and dark matter with additional time dependent term within the framework of fractal cosmology. This model demonstrates new types of evolution, which are not common to cosmological models with this type of interaction. Within this model cosmological parameters depend on time in a strongly nonlinear manner. In particular, in some solutions deceleration parameter is a non-monotonic function of time, revolving between the stages of accelerated and decelerated expansion, thus demonstrating transient acceleration. The numerical solutions for some interesting cases are also shown.

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