Topological quantum mechanics and the first Chern class

Yishi Duan\textsuperscript{1}, Libin Fu\textsuperscript{2,1,*} and Hong Zhang

\textsuperscript{1}Physics Department, Lanzhou University, Lanzhou, Gansu, 730000, China
\textsuperscript{2}Institute of Applied Physics and Computational Mathematics
P.O.Box 8009(26), Beijing 100088, China

(June 18, 1999)

Abstract

Topological properties of quantum system is directly associated with the wave function. Based on the decomposition theory of gauge potential, a new comprehension of topological quantum mechanics is discussed. One shows that a topological invariant, the first Chern class, is inherent in the Schrödinger system, which is only associated with the Hopf index and Brouwer degree of the wave function. This relationship between the first Chern class and the wave function is the topological source of many topological effects in quantum system.

\*Corresponding author. E-mail: lbfu@263.net
I. INTRODUCTION

Topology now becomes absolutely necessary in physics\(^1\)–\(^3\). The $\phi$-mapping theory and the gauge potential decomposition theory\(^4\)–\(^7\) are found to significant in exhibiting the topological structure of physics system and have been used to study topological current of magnetic monopole\(^4,8\), topological string theory\(^6\), topological structure of Gauss-Bonnet-Chern theorem\(^9\), topological structure of the SU(2) Chern density\(^10\), topological characteristics of dislocations and disclinations continuum\(^11,12\), topological structure of the defects of space-time in early universe as well as its topological bifurcation\(^13,14\).

Topological properties of quantum system should be directly associated with the wave function. Recently, using the $\phi$-mapping theory, the topological structure of the London equation in superconductor has been studied\(^15\). It is showed that the topological structure of London equation is characterized by topological index of wave function.

In this paper, based on $\phi$-mapping theory and gauge potential decomposition theory, we reveal the inner relation between the topological property of Schrödinger system and the intrinsic properties of its wave function. For the first time, we point out that a topological invariant, the first Chern class, is inherent in the Schrödinger system, which is only associated with the wave function, without using any particular models or hypotheses. One can find that this relationship between the first Chern class and the wave function is the topological source of the inner structure of London equations in superconductor\(^15\).

II. DECOMPOSITION THEORY OF $U(1)$ GAUGE POTENTIAL AND THE FIRST CHERN CLASS

Considering a complex line bundle $\mathbb{R}^3 \times \mathbb{C}$, as one knows, the $U(1)$ gauge potential $A = A_i dx^i$ is a connection on this bundle. A section of the line bundle gives a complex valued functions $\psi$, and the covariant derivative on the line bundle is defined as

$$D\psi = d\psi - iA\psi,$$
and its complex conjugate

\[ D\psi^* = d\psi^* + iA\psi^* . \]

From these formula above, one can obtain

\[ A = \frac{i}{2} \frac{d\psi^* \psi - d\psi \psi^*}{\psi \psi^*} + \frac{i}{2} \frac{D\psi^* \psi - D\psi \psi^*}{\psi \psi^*}. \] (1)

The main feature of the decomposition theory of the gauge potential is that the gauge potential \( A \) can be generally decomposed as

\[ A = a + b, \] (2)

where \( a \) is required to satisfy the gauge transformation rule and \( b \) satisfies the vector covariant transformation, i.e.,

\[ a' = a + d\alpha \] (3)

and

\[ b' = b \] (4)

under \( U(1) \) transformation \( \psi' = e^{i\alpha} \psi \). One can show that the gauge potential \( A \) are rigorously satisfies the gauge transformation

\[ A' = A + d\alpha. \]

Comparing (3) with (4), we can obtain a decomposition expression of \( U(1) \) gauge potential by defining

\[ a = -\frac{i}{2} \frac{d\psi^* \psi - d\psi \psi^*}{\psi \psi^*}, \] (5)

and

\[ b = \frac{i}{2} \frac{D\psi^* \psi - D\psi \psi^*}{\psi \psi^*}. \] (6)

One can easily prove that this decomposition satisfies the transformation rules (3) and (4).
We know that the complex valued function $\psi$ can be denoted as

$$\psi = \phi^1 + i\phi^2,$$

in which $\phi^1$ and $\phi^2$ are real valued function and can be regarded as two components of a two-dimensional vector field $\phi = (\phi^1, \phi^2)$ on $\mathbb{R}^3$. The unit vector field is defined as

$$n^a = \frac{\phi^a}{||\phi||}, \quad ||\phi|| = (\phi^a\phi^a)^{1/2}, \quad a = 1, 2,$$

(7)

satisfying

$$n^a n^a = 1.$$

From (8) and (7), it can be seen that

$$a = -\epsilon_{ab} dn^a n^b.$$

We know that the characteristic class is the fundamental topological property, and it is independent of the gauge potential\cite{1, 16}. So, to discuss the Chern class, we can take $A$ as

$$A = -\epsilon_{ab} dn^a n^b.$$

One can regard it as a special gauge. Then the field strength (the curvature) $F$ can be expressed as

$$F = dA = \epsilon_{ab} dn^a \wedge dn^b.$$

(9)

Using (7) and

$$dn^a = \frac{d\phi^a}{||\phi||} - \frac{\phi^a d(||\phi||)}{||\phi||^2},$$

$$\frac{\partial}{\partial \phi^a} \ln ||\phi|| = \frac{\phi^a}{||\phi||^2},$$

$F$ changes into

$$F = \epsilon_{ab} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^b} \ln ||\phi|| d\phi^c \wedge d\phi^b.$$

By making use of the Laplacian relation in $\phi$ space:

$$\frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^a} \ln ||\phi|| = 2\pi \delta^2(\phi),$$

4
we obtain
\[ F = 2\pi \delta^2(\phi) \epsilon^{ab} d\phi^a \wedge d\phi^b, \] (10)

One finds that \( F \) does not vanish only at the zero points of \( \phi \), i.e.
\[ \phi(x) = 0. \] (11)

The solution of Eqs. (11) are generally expressed as
\[ x = x_i(v), \quad i = 1, \ldots, m, \] (12)
which represent \( m \) zero lines \( L_i \) \((i = 1, \ldots, m)\) with \( v \) as intrinsic coordinates. There exists
a two-dimensional manifold \( \Sigma \) which normally intersects \( L_i \) at the point \( x_i \) and \( u = (u_1, u_2) \) are the intrinsic coordinates on \( \Sigma \). In the \( \delta \)-function theory\(^ {17,18} \), one can prove that
\[ \delta^2(\phi) = \sum_{i=1}^{m} \beta_i \eta_i \int_{L_i} \frac{\delta^3(x - x_i(v))}{D(\phi/u)_\Sigma} dv, \] (13)
where
\[ D(\phi/u)_\Sigma = \frac{1}{2} \epsilon^{ij} \epsilon_{ab} \frac{\partial \phi^a}{\partial u^i} \frac{\partial \phi^b}{\partial u^j}. \] (14)

The positive \( \beta_i \) is the Hopf index of \( \phi \)-mapping and \( \beta_i \) is the Brouwer degree\(^ {19} \):
\[ \eta_i = \text{sgn} D(\phi/u)_\Sigma = \pm 1 \]

The meaning of the Hopf index \( \beta_i \) is that the vector field function \( \phi \) covers the corresponding
region \( \beta_i \) times while \( x \) covers the region neighborhood of zero \( z_i \) once.

The integration of \( F \) on \( \Sigma \)
\[ C_1 = \frac{1}{2\pi} \int_{\Sigma} F \] (15)
is the first Chern class, an important topological invariant of the line bundle. From (11) and (13), we can obtain
\[ C_1 = \sum_{i=1}^{m} \beta_i \eta_i. \] (16)

From this result, we see that the first Chern class is the sum of the index of zero points, and
labeled by the Hopf index and Brouwer degree, or the Winding number of \( \phi \).
III. TOPOLOGICAL INVARIANT IN QUANTUM MECHANICS

The topological property of quantum system should be directly associated with the intrinsic properties of the wave function. Considering a Schrödinger system:

\[ i\hbar \frac{\partial}{\partial t} \psi = H\psi, \]

where \( \psi \) is the wave function. The current density of this system is given by

\[ j(\mathbf{r},t) = -\frac{i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*). \]

For the purpose to study the topological property, we consider this system with a fixed time. Hence the wave function \( \psi \) can be regarded as a section of a complex line bundle over \( \mathbb{R}^3 \) and can be denoted as

\[ \psi(\mathbf{r}) = \phi^1(\mathbf{r}) + i\phi^2(\mathbf{r}). \]  

(17)

One defines a physical quantity \( V \) as

\[ V = -\frac{i}{2} \frac{(\psi^* \nabla \psi - \psi \nabla \psi^*)}{\psi \psi^*}. \]

(18)

Under \( U(1) \) transformation \( \psi' = e^{i\alpha} \psi \), one can see that \( V \) satisfies the gauge transformation

\[ V' = V + \nabla \alpha. \]

So, one should notice here that \( V \) is a composed \( U(1) \) gauge potential. One can prove that

\[ V = -\epsilon_{ab} \nabla n^a n^b = -\epsilon_{ab} \partial_i n^a n^b \vec{e}_i, \]

(19)

where \( \vec{e}_i \) (\( i = 1, 2, 3 \)) denote (\( \hat{x}, \hat{y}, \hat{z} \)), and

\[ \nabla \times V = \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b \vec{e}_i = 2\pi J^i \vec{e}_i, \]

with

\[ J^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b. \]

(20)
which is a topological current. From the discussion in above section, we have

\[ \nabla \times \mathbf{V} = 2\pi \delta^2(\phi) \tilde{D}(\frac{\phi}{x}), \]  

(21)

where

\[ \tilde{D}(\frac{\phi}{x}) = \epsilon^{ijk} \epsilon_{ab} \partial_j \phi \partial_j \phi \bar{e}_i. \]

And then

\[ \nabla \times \mathbf{V} = 2\pi \sum_{i=1}^{m} \beta_i \eta_i \int_{L_i} \delta^3(\bar{r} - \bar{r}_i(v)) \frac{\tilde{D}(\frac{\phi}{x})}{D(\frac{\phi}{u})} d\bar{v}. \]

(22)

From\(^7\), it is easy to see that

\[ \frac{\tilde{D}(\frac{\phi}{x})}{D(\frac{\phi}{u})} |_{\bar{r}_i(v)} = \frac{d\bar{r}_i}{dv}, \]

then the current (20) is turned to

\[ \mathbf{J} = \frac{1}{2\pi} \nabla \times \mathbf{V} = \sum_{i=1}^{m} \beta_i \eta_i \int_{L_i} \delta^3(\bar{r} - \bar{r}_i) d\bar{r}_i. \]

(23)

It is obvious to see that the formula (23) represents a current of \( m \) isolated vortices with the \( i \)-th vortex carries charge \( 2\pi \beta_i \eta_i \). And, one can prove that

\[ Q = \int_{\Sigma} \mathbf{J} \cdot d\bar{\sigma} = \sum_{i=1}^{m} \beta_i \eta_i. \]

(24)

Comparing this result with (15) and (16), we see that the total topological charge of this system is equal to the first Chern number. So, one finds that the Schrödinger system inherits a topological invariant, the first Chern class, which is only associated with the intrinsic properties of wave function, without using any particular models or hypotheses. One can find that this relationship between the first Chern class and the wave function is the source of many topological effects in quantum system.

As an example, let us now consider a homogeneous superconductor in a magnetic field which is weak compared with the critical field \( B_{c2} \) at which the superconductivity is lost. The relation between the superconductor current \( \mathbf{j}_s \) and the condensate wave function \( \psi \) is

\[ \mathbf{j}_s = \frac{e\hbar}{\mu} |\psi|^2 \mathbf{V} - \frac{2e^2}{\mu c} |\psi|^2 \mathbf{A}. \]

(25)
Let the body we studied be in a state of thermodynamic equilibrium, so that there is no normal current and \( j = j_s \). We shall also use the general Maxwell’s equations:

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} j, \tag{26}
\]

\[
\nabla \cdot \mathbf{B} = 0. \tag{27}
\]

To put them in the appropriate form, we first rewrite the relation (25) between the superconductivity current density and \( \mathbf{V} \) through (26):

\[
\mathbf{A} + \lambda^2 \nabla \times \mathbf{B} = \frac{\phi_0}{2\pi} \mathbf{V}. \tag{28}
\]

For the London approximation corresponds to the assumption that \( \lambda \) is constant, taking the curl of both sides of (28) and noting that \( \nabla \times \mathbf{A} = \mathbf{B} \) and (27), we have

\[
\mathbf{B} - \lambda^2 \nabla^2 \mathbf{B} = \frac{\phi_0}{2\pi} \nabla \times \mathbf{V}. \tag{29}
\]

Comparing this expression with (23), one obtain the topological structure of London equation:

\[
\mathbf{B} - \lambda^2 \nabla^2 \mathbf{B} = \phi_0 \sum_{i=1}^{m} \beta_i \eta_i \int_{L_i} d\mathbf{r} \delta^3(\mathbf{r} - \mathbf{r}_i). \tag{30}
\]

It is obvious to see that the equation (30) represents \( m \) isolated vortices of which the \( i \)th vortex carries flux \( \beta_i \eta_i \phi_0 \). On can conclude that includes vortex-antivortex pair\(^{20,21} \) (\( \beta_1 = \beta_2 \), \( \eta_1 = 1 \) and \( \eta_2 = -1 \)), vortex rings (\( L_i \) is a ring), multicharged vortices\(^{22} \).

**IV. DISCUSSION**

Based on the \( \phi \)-mapping theory, using gauge potential decomposition method, we reveal that the first Chern class is inherent in Schrödinger system, which is only associated with the intrinsic properties of the wave function. In fact, this topological property does not only relate to the Schrödinger system, but also relates to the non-linear Schrödinger system as well as the condensate Schrödinger system. We must point out that this relationship
naturally exists in (2+1)-dimensional quantum system and is associated with the discussion of topological structure of quantum Hall effect. We believe that this intrinsic topological property is a fundamental property of quantum system and is the source of many topological effects in quantum system.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China.

1S. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic, London, 1983).

2G. Morandi, *The Role of Topology in Classical and Quantum Physics*, Lecture Note in Physics Vol. M7 (Springer-Verlag, Berlin, 1992).

3A. P. Balachandran, Found. Phys. **24**(4), 455 (1994).

4Y. S. Duan and M. L. Ge, Sci. Sinica **11**, 1072 (1979).

5Y. S. Duan, SLAC-PUB-3301/84.

6Y. S. Duan and J. C. Liu, *Proceedings of Johns Hopkins Workshop 11* (World Scientific, Singapore, 1988).

7Y. S. Duan and X. H. Meng, J. Math. Phys. **34**(3), 4463 (1993).

8G. H. Yang and Y. S. Duan, Inter. J. Theor. Phys. **37** (1998) 2435.

9Y. S. Duan and X. H. Meng, J. Math. Phys. **34** (1993) 1149; Y. S. Duan, S. Li and G. H. Yang, Nucl. Phys. B **514** (1998) 705.

10Y. S. Duan and L. B. Fu, J. Math. Phys. **39** (1998) 4343.

11Y. S. Duan and S. L. Zhang, Int. J. Eng. Sci. **28**, 689 (1990); **29**, 153 (1991); **29**, 1593 (1991); **30**, 153 (1992).

12Y. S. Duan and X. H. Meng, Int. J. Engng. Sci. **31**, 1173 (1993).

13Y. S. Duan, S. L. Zhang, and S. S. Feng, J. Math. Phys. **35** (1994) 4463; Y. S. Duan, G. H. Yang, and Y. Jiang, Int. J. Mod. Phys. A **58** (1997) 513.

14Y. S. Duan, G. H. Yang, and Y. Jiang, Gen. Rel. Grav. **29**, 715 (1997).
15Y. S. Duan, H. Zhang and S. Li, Phys. Rev. B 58 (1998) 125.

16M. W. Hirsch, Differential Topology (Springer verlag. New York 1976).

17J. A. Schouten, Tensor Analysis for Physicists (Clarendon, Oxford, 1951).

18G. Y. Yang, Ph.D. thesis, Lanzhou University, China (1997).

19H. Hopf, Math. Ann. 96 209 (1929).

20B. A. Dubrovin, A. T. Fomenko and S. P. NoviKov, Modern Geometry and Applications (Springer-Verlag, New York, 1984).

21V. L. Berezinsky, Zh. Éksp. Tero. Fiz. 59, 907 (1970) [Sov. Phys. JEPT 32, 493 (1971)].

22I. Aranson and V. Steinberg, Phys. Rev. B 53, 75 (1996).

23Y. S. Duan and S. Li, Phys. Lett. A 246, 172 (1998).