We show how the prices of options can be determined with the help of double-fractional differential equation in such a way that their admixture to a portfolio of stocks provides a more reliable hedge against dramatic price drops than the use of options whose prices were fixed by the Black-Scholes formula.

PACS numbers: 89.65.Gh, 05.40.Fb
Keywords: Double-fractional diffusion; Lévy option pricing; Risk redistribution

I. INTRODUCTION

In 1995, the Nobel price in Economics was awarded to Merton and Scholes for the so-called Black-Scholes (BS) model first published in 1973 [1]. It was hailed as a milestone in derivative trading, and led to the development of elaborate hedging strategies which promise a safe growth of appropriately composed portfolios of financial assets. Unfortunately, however, the usefulness of the formula was based on a simplifying assumption, that fluctuations of assets follow Gaussian distributions. Only this made possible to set up a mixture of assets and options that has a chance of growing like a safe investment. In fact, this formula was initially quite successful to a number of professional speculators. Later, however, it led to a disaster, and was one of the major sources the ongoing financial crisis [2–4]. The reason for this is that rare events such as drastic falls of stock markets are much more frequent than would be expected from Gaussian distributions. From time to time, there can occur catastrophic outlier events which should not have happened in hundreds of years. Such events have been named ”black-swan” events, after a popular book by N. N. Taleb [5]. The existence of such events is a severe obstacle to all hedging attempts. Normally, one considers price changes as the result of random steps of a given finite size and demonstrates that these build up to Gaussian random walks. This is a consequence of the central limit theorem (CLT). But in general, some of the steps can be extremely large, and the combined random walks are of the so-called Lévy type. These possess power-like tails which may not even have finite variance. They are encountered in many rare events in nature, such as earthquakes, monster waves in the ocean, and giant drops in financial markets. Apparently, we need new option price formulas that incorporate the possibility of encountering large drops of prices, and are able to compensate them by a corresponding rise in the price of the derivative.

Many sophisticated models that go beyond Black-Scholes have been introduced in recent decades: among them e. g. models based on Lévy distributions [6], truncated Lévy distributions [7], Multifractal volatility [8], jump processes [9], and many other approaches. We would like to focus on models that are based on so-called stable distributions. These have found applications in many scientific fields such as multifractal thermodynamics [10], quantum field theory [11], evolutionary systems [12], complex dynamical systems [13], etc. These models usually exhibit self-similarity and power-law behavior in some particular time interval. We focus also on temporal scaling and show that models based on double-fractional diffusion (i.e., self-similar scaling in both spatial and temporal variables) can successfully simulate situations, when instant fluctuations cause high short-term volatility. In this case we can redistribute the risk for short-term options and long-term options and the volatility of the model can remain unchanged. This approach has a good possibility for further applications in more complex option pricing scenarios. By fitting the data of S&P 500 options, we show that for most of the time the optimal value of temporal-scaling is very close to classical diffusion based on Lévy flight, but for some particular days, e.g., after sudden drops, the risk redistribution can be accounted for by time-fractional diffusion process.
The paper is divided as follows: in Section II are briefly introduced stable distributions, in Section III we discuss various definitions of fractional derivatives and their relation to stable distributions of the Lévy type. Section IV revises previous models of log-Lévy option pricing. In Section V we introduce double-fractional diffusion equations, and discuss their properties. We also show different ways of representing the solutions. Section VI is dedicated to application of double-fractional diffusion to option pricing and the last section is devoted to conclusions and perspectives.

II. STABLE DISTRIBUTIONS

Stable distributions (also called Lévy distributions) constitute an important class of probability distributions. They are form-invariant under convolution, implying that the sum of two random variables governed by Lévy distributions follow themselves such a distribution. Gnedenko and Kolmogorov [14] showed that such distributions are limiting distributions of infinite sums of independent random variables with arbitrary distribution, and this is a content of generalized central limit theorem. Moreover, it is possible to express their probability density function by Fourier integrals. In probability theory, the logarithms of the Fourier transforms are referred to characteristic functions. In theoretical physics, these are the well-known Hamiltonian operators.

Stable distributions (also called Lévy distributions) constitute an important class of probability distributions. They are form-invariant under convolution, implying that the sum of two random variables governed by Lévy distributions follow themselves such a distribution. Gnedenko and Kolmogorov [14] showed that such distributions are limiting distributions of infinite sums of independent random variables with arbitrary distribution, and this is a content of generalized central limit theorem. Moreover, it is possible to express their probability density function by Fourier integrals. In probability theory, the logarithms of the Fourier transforms are referred to characteristic functions. In theoretical physics, these are the well-known Hamiltonian operators. The theoretical physics, these are the well-known Hamiltonian operators. The theoretical physics, these are the well-known Hamiltonian operators.

The only exception is the case of $\beta = 1$, so for $\beta = 1$, resp. $\beta = 2$, the Lévy distribution $L_{\alpha,\beta}(x)$ exhibits power-like Lévy tails that behave like $x^{-(\alpha+1)}$ for positive $x$ (see Ref. [15]), so

$$L_{\alpha,\beta}(x) \sim \alpha C_{\alpha}(1 + \beta) x^{-(\alpha+1)} \quad \text{for } x \to +\infty$$

with a similar behavior for negative $x \to -\infty$. The only exception is the case of $\beta = -1$, where for $\alpha > 1$ exhibits the distribution the following behavior

$$L_{\alpha,-1}(x) \sim \frac{1}{2(\alpha-1)} \left( \frac{x}{\alpha C_{\alpha}} \right)^{\alpha-2} \exp \left[ -(\alpha - 1) \left( \frac{x}{\alpha C_{\alpha}} \right)^{-\alpha/(\alpha - 1)} \right] \quad \text{for } x \to +\infty$$

with some constant $c_{\alpha}$. The proof can be found in [16]. A similar exponential decay holds for the negative tail and $\beta = 1$. For $\alpha \leq 1$, the support is bounded to the interval $[bar{x}, \infty)$ for $\beta = 1$, resp. $(-\infty, \bar{x})$ for $\beta = -1$. For $\alpha < 2$, the distribution has infinite moments $\langle |x|^{l} \rangle$ for $l \geq 1$.

The two-sided Laplace transform of $L_{\alpha,\beta}$ does not exist, unless $\beta = 1$, so for $\Re(\lambda) > 0$ (see Ref. [16]) the logarithm of Laplace image is equal to

$$\ln\left( e^{-\lambda x} \right) = -\lambda \bar{x} - \lambda^{\alpha} \sigma^{\alpha} \sec \frac{\pi \alpha}{2}.$$  

From symmetry argument we can deduce that the expectation value $\langle e^{i\tau x} \rangle$ for $\tau \geq 0$ exists only when $\beta = -1$. Sometimes, it is advantageous to use an alternative representation of the stable Hamiltonian, which is

$$H_{\alpha,\beta;\bar{x},c}(p) \equiv \ln\left( e^{ipx} \right) = i\bar{x}p - c|p|^\alpha e^{i\text{sign}(p) \theta} \frac{\sigma}{2},$$

where $c$ and $\theta$ are uniquely determined by the parameters $\alpha$, $\beta$ and $\sigma$. Accessible values of the parameter $\theta$ satisfy condition $|\theta| \leq \min(\alpha, 2 - \alpha)$. The corresponding area in $(\alpha, \theta)$-plane is called Feller-Takayasu diamond, and for certain values we obtain extremely asymmetric stable distributions (e.g., for $\alpha > 1$, the value $\theta = -\alpha + 2$ corresponds to the case $\beta = 1$ and $\theta = \alpha - 2$ corresponds to the case $\beta = -1$). Further properties of stable distributions, their representations and relations between them can be found in Refs. [15] [17] [18].
III. FRACTIONAL CALCULUS

Fractional calculus generalizes the classical integral and differential calculus to non-integer integration and differentiation. The motivation was originally to find such an operator, for which the second power of the operator would equal to the first derivative. Subsequently, the goal was to generalize the integral and differential calculus for all real values. This section presents the basic overview of some possible definitions.

Let us begin with the fractional integration, which generalizes the well-known Cauchy formula

\[ \int_{x_0}^{x} \int_{x_0}^{x_1} \ldots \int_{x_0}^{x_{n-1}} f(x_n)dx_n \ldots dx_1 = \frac{1}{(n-1)!} \int_{x_0}^{x} (x - y)^{n-1} f(y)dy \]

in a natural way from integer values \( n \) to non-integer values \( \nu \) by defining

\[ x_0 \mathcal{T}_x^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x - y)^{\nu-1} f(y)dy. \]  \hspace{1cm} (6)

The fractional integral is a linear operator that obeys the semigroup property

\[ x_0 \mathcal{T}_x^{\nu_1} \circ x_0 \mathcal{T}_x^{\nu_2} = x_0 \mathcal{T}_x^{\nu_1+\nu_2}. \]  \hspace{1cm} (7)

It is easy to show that \( \frac{d}{dx} (x_0 \mathcal{T}_x^{\nu+1}) = x_0 \mathcal{T}_x^{\nu} \), which is the baseline for the definition of fractional derivative.

A. Riemann-Liouville derivative

The relation between fractional integrals and derivatives suggests the introduction the Riemann-Liouville (RL) fractional derivative

\[ x_0 \mathcal{D}_x^{\nu} f(x) := \frac{d^{[\nu]}}{d^{[\nu]}x} \left( x_0 \mathcal{T}_x^{[\nu]-\nu}[f] (x) \right), \]  \hspace{1cm} (8)

where \([\nu]\) denotes the associated ceiling function, i.e., the lowest integer exceeding \( \nu \). Just as ordinary derivatives, these derivatives are linear operators, but contrary to them, they do not satisfy a composition law like (7), i.e.,

\[ x_0 \mathcal{D}_x^{\nu_1} \circ x_0 \mathcal{D}_x^{\nu_2} \neq x_0 \mathcal{D}_x^{\nu_2} \circ x_0 \mathcal{D}_x^{\nu_1}. \]  \hspace{1cm} (9)

In fact, they share only a few properties of ordinary derivative operators. On the other hand, some particular choices of derivatives, given by special values of \( x_0 \), recover some of the typical properties of ordinary derivatives. One example of such fractional derivative can be inferred from the derivative of integer power functions \( x^n \). For arbitrary integers \( m, n \) is the \( m \)-th derivative of \( x^n \) equal to

\[ \frac{d^m}{dx^m} x^n = \frac{n!}{(n-m)!} x^{n-m}. \]  \hspace{1cm} (10)

By performing a fractional integration, it is easy to show that the fractional derivative of a monomial has the same form as in Eq. (10), when \( x_0 = 0 \). This operator will be denoted \( \mathcal{D}_x^{\nu} \equiv 0 \mathcal{D}_x^{\nu} \). Moreover, this property is valid for arbitrary powers and arbitrary derivatives. As an example, we obtain for \( n = 0 \) that

\[ \mathcal{D}_x^{\nu} 1 = \frac{x^{-\nu}}{\Gamma(1-\nu)}. \]  \hspace{1cm} (11)

Expression (11) is equal to zero only for \( \nu \in \mathbb{N} \), because of poles in the Gamma function.

Sometimes, it is advantageous to discuss the formalism at the level of the Laplace images. If we perform the Laplace transform of Def. [3], we get

\[ \mathcal{L} \left[ \mathcal{D}_x^{\nu} f(x); s \right] = \left[ \hat{\mathcal{D}_x^{\nu}} f \right](s) = s^\alpha F(s) - \sum_{k=0}^{[\nu]} s^k \left[ \mathcal{D}_x^{\nu-k-1} f(x) \right] \Big|_{x=0}, \]  \hspace{1cm} (12)

where \([x]\) represents floor function, i.e., the largest integer not exceeding \( x \). More details can be found in Ref. [19].
B. Caputo fractional derivative

One has to notice that the Riemann-Liouville definition of fractional derivatives has some objectionable properties, among them non-zero derivative of constant function and unnatural initial conditions. Mainly the latter issue limits the application of the above derivative in physical and other real problems, because the fractional initial conditions have no physical meaning. These reasons compel us to introduce a different kind of fractional derivative, namely the Caputo (C) fractional derivative, which patches some of the unwanted properties of RL derivatives. It is defined as follows

\[
x_0 \, ^\ast D_x^\nu f(x) := \frac{1}{\Gamma([\nu] - \nu)} \int_{x_0}^x \frac{f[y]}{(x-y)^{\nu+1-[\nu]}} \, dy.
\]  

(13)

The Caputo derivative is more restrictive regarding its domain, because the function \( f \) must have at least \([\nu]\) derivatives. On the other hand, it recovers the desired properties, because \( ^\ast D_x^\nu 1 = 0 \), and the Laplace transform is equal to

\[
L[ ^\ast D_x^\nu f(x); s] \equiv \left[ ^\ast D_x^\nu f \right](s) = s^\nu F(s) - \sum_{k=0}^{[\nu]} s^{\nu-k-1} f^{(k)}(0).
\]  

(14)

Now, natural initial conditions are recovered. In case of the Caputo derivative, we can solve the fractional differential equation

\[
^\ast D_x^\nu f(x) = \lambda f(x)
\]  

(15)

by introduction of the Mittag-Leffler function

\[
E_{\nu,\zeta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\nu n + \zeta)}.
\]  

(16)

It is easy to see that the function \( f(x) = E_{\nu,1}(\lambda x^\nu) \) solves Eq. (15). Eventually, Riemann-Liouville and Caputo derivatives can be connected via the relation (to be found e.g. in Ref. [20])

\[
^\ast D_x^\nu f(x) = D_x^\nu f(x) - \sum_{k=0}^{[\nu]} \frac{x^k}{k!} f^{(k)}(0).
\]  

(17)

C. Riesz-Feller fractional derivative

Equation (15) is a generalization of another well-known property of derivatives, which holds for an exponential functions

\[
\frac{d^m}{dx^m} \exp(\lambda x) = \lambda^m \exp(\lambda x).
\]  

(18)

In the case of the Caputo derivative, this formula is generalized in a way that exponential function is replaced by the Mittag-Leffler function. On the other hand, this property can be preserved, when we send the lower bound to minus infinity; the operator

\[
D_x^\nu f(x) := \lim_{x_0 \to -\infty} x_0 \, ^\ast D_x^\nu f(x)
\]  

(19)

obeys Eq. (18) even for non-natural derivatives. Such operator has to be defined on a different space, because the fractional operator of this type is convergent for functions that decay faster than \( |x|^{-\nu} \) for \( x \to -\infty \). This operator is called Riesz-Feller derivative and is denoted by \( D_x^\nu \). Interestingly, we get the same operator, when we use the Caputo derivative approach. In case of Riesz-Feller derivative it is advantageous to transform into Fourier image, because it has the following form

\[
F[ D_x^\nu f(x); p] \equiv \hat{D_x^\nu f}(p) = \int_{\mathbb{R}} dx e^{ipx} \int_{-\infty}^x dy (x-y)^{-\nu-1} f(y) = (-ip)^\nu \hat{f}(p).
\]  

(20)
By direct differentiation it is easy to show that the option this price fulfills the celebrated Black-Scholes equation even more sophisticated option pricing rules, some examples are provided by Refs. [22, 25, 26].

The evolution of an underlying asset is modeled as a log-normal (or geometric Brownian) process

\[
C \equiv \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC - rS \frac{\partial C}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2},
\]

with the boundary condition \(C_{T} = [S_T - K]^+\).

This option pricing model is the most used worldwide, with many applications such as estimation of implied volatility [23]. On the other hand, a considerable amount of effort has been made during last two decades on the development of more advanced evolution models of the underlying assets. As an example, the fractional Brownian motion [24] is an elegant model. Furthermore, the complexity of financial markets motivated many authors to introduce even more sophisticated option pricing rules, some examples are provided by Refs. [22, 25, 26].

We adopt the approach introduced in Ref. [6] and assume that the price evolution is driven by the log-Lévy model

\[
dS(t) = rS(t)dt + \sigma S(t)dL_{\alpha,\beta}(t).
\]
Contrary to geometric Brownian process, the prices of log-Lévy stable process do not possess all moments $⟨S^α⟩$. What more, it is not possible to use Esscher transform. The only exception is the case, when $β = -1$, or similarly $θ = α - 2$, because then there exists the two-sided Laplace transform, which is expressed in Eq. (15). As a result of exponential decay of positive tail of PDF, all moments exist and are finite. Such model can better describe a dramatic price drops on the market, which is more frequent than envisaged by Black-Scholes theory [11]. The price process becomes the following time dependence

$$S_t = S_0 \exp \left[ (r + μ) t + σ L_{α, -1}(t) \right], \quad \text{(30)}$$

where $μ = σ^2 \sec \frac{πα}{2}$. The corresponding option price, which is given by (26), is equal to

$$C(S_t, t) = e^{-rτ} \int_0^T dt \left[ S_t e^{rτ} - K \right]^+ \int dσ e^{-σ^2}\left[ e^{(rα + σ^2)H_{α, -1}(σ)} \right], \quad \text{(31)}$$

where $τ = T - t$. By introduction of a new variable $z = \ln S_t$ and change of integration variable to $y = x + rτ + z$, we obtain

$$C(z, τ) = e^{-rτ} \int_0^T dy \left[ e^{y} - K \right]^+ \int dσ \frac{e^{ipx}}{2π} e^{rτ(rα + μσ) - μσ^2} \left[ e^{iσ^2} \right] = \int_0^T dy \left[ e^{y} - K \right]^+ \hat{g}_α(z, τ|y, 0), \quad \text{(32)}$$

where $\hat{g}_α(z, τ|y, 0)$ is the Green function (sometimes also called fundamental solution). By further transformations $ξ = z - y + τ(r + μ)$ and $g_α(ξ, τ) = e^{τ} \hat{g}_α(z, τ|y, 0)$ we obtain the equation for $g_α$ in the form

$$\frac{∂g_α(ξ, τ)}{∂τ} = -μ [α - 2 D_ξ^α \hat{g}_α] (ξ, τ), \quad \text{(33)}$$

together with the initial condition $g_α(ξ, 0) = δ(ξ)$. This equation is a fractional Black-Scholes equation for log-prices, and for $α = 2$, we recover classical diffusion equation. In the Fourier image, the equation has the form of a fractional-diffusion equation

$$\frac{∂\hat{g}_α(p, τ)}{∂τ} = H_{α, -1}(p)\hat{g}_α(p, τ). \quad \text{(34)}$$

**V. DOUBLE-FRACTIONAL DIFFUSION**

In some recent works, other models that involve fractional time derivatives have been studied [27–29]. It has been realized that complex time scaling introduces more general classes of solutions that exhibit interesting phenomena and enables one to price options more realistically. The original motivation was provided by fractional Brownian motion, originally introduced in [24], and later applied to option pricing in Refs. [26, 30]. The question at stake is, which particular fractional derivative is the best when generalizing the Black-Scholes equation, driven by a Green function $g(ξ, τ)$, obtained as a solution of a double-fractional diffusion equation

$$(K^2D_ξ^α g(ξ, τ) + μ[α - 2 D_ξ^α]) g(ξ, τ) = 0, \quad \text{(35)}$$

where we have considered two type of the temporal derivatives denoted by the parameter $K$, namely $^C D^α = ^C D^α$ (Caputo derivative), or $^RF D^α = D^α$ (Riesz-Feller derivative). The parameter $α$ is the degree of the spatial derivative and corresponds to the stability parameter. The parameter $γ$ is the degree of temporal fractional derivative (corresponding to parameter $ν$ in definitions of fractional derivatives). It is called the diffusion speed parameter, because, as discussed below, it influences the speed and type of diffusion behavior. In the following sections, we compare two classes of double-fractional diffusion equations, particularly equations with Riesz-Feller time derivatives, resp. Caputo time derivatives. Both of these equations are examples of a wide class of two-variable pseudo-differential equations, which is usually represented via the Laplace-Fourier (LF) image (which means the Fourier image in spatial variable and the Laplace image in the temporal variable) as

$$a(s)\hat{g}(p, s) - a_0(s)\hat{g}_0(p) = b(p)\hat{g}(p, s) \quad \text{(36)}$$

In our case, i.e. when Eq. (36) represents a LF image of the double-fractional diffusion equation, then $b_α(p) = H_{α, -1}(p)$, $a_α(s) = s^α$ and $a_0(s)$ is determined by the type of derivative, for Caputo derivative $a_0^C(s) = s^{α - 1}$, and for Riesz-Feller derivative is $a_0^RF(s) = 1$. We shall note that the equation requires one initial condition $\hat{g}_0(p)$, which is the Fourier
transform of \( g_0(\xi) \equiv g(\xi, \tau = 0) \). In the following, we consider that \( g_0(\xi) = \delta(\xi) \), which leads to \( \hat{g}_0(p) \equiv 1 \). This is valid for the case, when \( \gamma < 1 \) (sometimes called slow diffusion). In the following section we focus on that case, because there exists a representation which results into a nice interpretation of double fractional diffusion. On the other hand, when we consider \( 1 < \gamma \leq 2 \) (fast diffusion), we can also obtain the same form of equation as in Eq. (36), we have to only add the second initial condition of the particular form, namely

\[
\frac{\partial g}{\partial \tau}(\xi, \tau = 0) \equiv 0.
\]  

(37)

The question at stake is, if the solutions of Eq. (36) can be interpreted as probability distributions, i.e. if they are positive. In Ref. [31] is shown that for \( \gamma < 1 \) are solutions positive for all \( \alpha \in (0, 2] \), while in case \( \gamma > 1 \), the parameters have to obey the condition \( 0 < \gamma < \alpha \leq 2 \).

### A. Composition rule of Green function for \( \gamma < 1 \)

In case when \( \gamma < 1 \), we can derive a special form of the Green function, which gives a nice interpretation of double fractional Green function as a composition of space-fractional Green functions weighted by smearing kernel equal to time-fractional Green function. We begin with Eq. (36). The solution of this equation can be in Fourier-Laplace image expressed as

\[
\hat{g}(p, s) = \frac{a_0(s)\hat{g}_0(p)}{a(s) - b(p)}.
\]  

(38)

Under the assumption that \( \Re(a(s) + b(p)) > 0 \), we can apply Schwinger’s formula and get

\[
\hat{g}(p, s) = \int_0^\infty \! dl\, a_0(s)e^{-ls} \hat{g}_0(p)e^{lb(p)} = \int_0^\infty \! dl\, \hat{g}_1(s, l)\hat{g}_2(l, p).
\]  

(39)

Therefore, the original \((\xi, \tau)\) -dependence is given by

\[
g(\xi, \tau) = \int_0^\infty \! dl\, g_1(\tau, l)g_2(l, \xi).
\]  

(40)

Because \( \int_\mathbb{R} g(\xi, t) d\xi = 1 \), we obtain that \( \int_0^\infty g_1(\tau, l) dl = 1 \). One possible interpretation of the variable \( l \) is that it represents the generic time-parameter of the system and the function \( g_1(\tau, l) \) represents the smearing kernel, so that the resultant solution is obtained by integration over all solutions for different time-parameters with the weight factor given by the smearing kernel.

After plugging in for \( b(p) \) and \( g_0(p) \), we obtain the solution in the form

\[
\hat{g}(p, s) = \int_0^\infty \! dl\, a^0(s)e^{-la(s)}e^{lH_{a, -1}(p)} = \int_0^\infty \! dl\, \hat{g}_1(s, l)\hat{g}_0(l, p),
\]  

(41)

so the solution is given by superposition of space-fractional-diffusion Green functions for different times \( l \). The smearing kernel obeys the differential equation

\[
\frac{d}{dl}\hat{g}_1(s, l) = -a(s)\hat{g}_1(s, l)
\]  

(42)

with an initial condition \( \hat{g}_1(s, 0) = a_0(s) \).

When we assume that the time-derivative operator is equal the Riesz fractional derivative \( D_\tau^\gamma \), this results into expected property that for \( \tau < 0 \) is \( g_1(\tau, l) \equiv 0 \) [32]. We obtain that the Laplace image is equal to [33]:

\[
\frac{d}{dl}\hat{g}_1^{RF}(s, l) = -s^\gamma \hat{g}_1^{RF}(s, l)
\]  

(43)

with the initial condition \( \hat{g}_1^{RF}(s, 0) = 1 \), leading to the solution \( \hat{g}_1^{RF}(s, l) = e^{-l\gamma} \), which nothing else than the Laplace transform of the fully asymmetric stable distribution with stability parameter \( \gamma \), asymmetric parameter equal to 1 and the support contained in \( \mathbb{R}_+^\alpha \). The function \( \hat{g}_1^{RF}(t, l) \) is not normalized, so according to [32], we have

\[
\int_0^\infty \! dl\, \hat{g}_1^{RF}(\tau, l) = \int_0^\infty \! dl\, \int_\mathbb{R} \frac{e^{-ip\tau}}{2\pi}e^{-ip\mu(l)p^\gamma} = \int_\mathbb{R} \frac{e^{-ip\tau}}{2\pi} \frac{1}{\mu(l)\gamma} = \frac{\tau^{\gamma-1}}{\mu(\gamma)}.
\]  

(44)
Thus, we end with
\[ g^{RF}(\xi, \tau) = \int_0^\infty dl \left( \frac{\Gamma(\gamma)}{\tau^{\gamma-1}} \right) \frac{1}{\Gamma(\gamma)} L_{\gamma,1} \left( \frac{\tau}{\Gamma(\gamma)} \right) g_\alpha(\xi, l) \] (45)

where \( L_{\gamma,1} \) is the \( \gamma \)-stable Lévy asymmetric distribution.

In case of Caputo fractional derivative \( ^aD_\tau^\gamma \), the Laplace transform of \( \hat{g}_{1}^{C}(s, l) \) is equal to \( \hat{g}_{1}(s, l) = s^{\gamma-1}e^{-ls^{\gamma}} \). According to Ref. [34], the inverse Laplace transform is equal to
\[ g_{1}^{C}(\tau, l) = \frac{1}{\tau^{\gamma}} M_{\gamma,1} \left( \frac{l}{\tau^{\gamma}} \right) \] (46)

where \( M_{\nu}(z) \) is the M function of Wright type, which is defined by the infinite series
\[ M_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\nu n + (1 - \nu))} . \] (47)

Interestingly, the connection of \( M \)-function to asymmetric Lévy-distributions is provided by the relation
\[ \frac{1}{c^{1/\nu}} L_{\nu,1} \left( \frac{x}{c^{1/\nu}} \right) = \frac{c^{\nu}}{x^{\nu+1}} M_{\nu} \left( \frac{c}{x^{\nu}} \right) , \] (48)

which is valid for \( \nu \in (0, 1) \), \( c > 0 \) and \( x > 0 \). Hence, we can rewrite the Green function \( g^{C}(\xi, t) \) as
\[ g^{C}(\xi, \tau) = \int_0^\infty dl \, \frac{1}{\tau^{\gamma}} M_{\gamma} \left( \frac{l}{\tau^{\gamma}} \right) g_\alpha(\xi, l) = \int_0^\infty dl \left( \frac{\tau}{l^{\gamma}} \right) \frac{1}{\Gamma(\gamma)} L_{\gamma,1} \left( \frac{\tau}{\Gamma(\gamma)} \right) g_\alpha(\xi, l) . \] (49)

We compare properties of both smearing kernels in Appendix A. Now let us turn attention to an alternative representation of the Green function, which is more computationally tractable and includes \( \gamma > 1 \).

### B. Mellin-Barnes representation of double-fractional-diffusion Green function

It is possible to introduce an alternative, computationally effective representation of double-fractional diffusion equation which is based on Mellin-Barnes [35] transformation. This representation is also valid for the case \( \gamma > 1 \). Let us again begin with equation (36). It has the solution equal to
\[ \hat{g}(p, s) = \frac{s^{\gamma-\kappa}}{s^{\gamma} - \mathcal{H}_{\alpha, \beta}(p)} , \] (50)

where \( \kappa \) depends on the type of the derivative. For Riesz derivative it is \( \kappa = \gamma \), for Caputo derivative we have \( \kappa = 1 \). We shall note that because of computational reasons, we assume here only solutions for \( \xi > 0 \). The solution for negative values can then be easily obtained from the relation \( g_{\alpha, \theta, \gamma, \kappa}(\xi) = g_{\alpha, -\theta, \gamma, \kappa}(\xi) \). Therefore, we formally leave \( \theta \) undetermined, even if we assume only extremal cases, for which \( \beta = -1 \).

According to [19, 36], the inverse Laplace transform of Eq. (50) is equal to the Mittag-Leffler function
\[ \hat{g}(p, \tau) = \tau^{\kappa-1} E_{\gamma, \kappa} \left( \mathcal{H}_{\alpha, \theta}(p) \tau^{\gamma} \right) . \] (51)

The Mittag-Leffler function can be represented through the Mellin integral transform which is defined as
\[ \mathcal{M}[g(x); s] = \int_0^\infty g(x)x^{s-1}dx \] (52)
and the inverse transform is defined as (for some \( c \) given by the Mellin transform theorem [37])
\[ g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}[g](s)x^{-s}ds . \] (53)
FIG. 1. Comparison of Green functions for ordinary derivative ($\gamma = 1$), Riesz and Caputo derivative for $\gamma = 0.9$ (slow diffusion) and $\gamma = 1.1$ (fast diffusion) for $\alpha = 2$ and $\alpha = 1.6$. The Caputo Green function highlights the peak of the distribution, while Riesz-Feller Green function has slower decay in tails of the distribution. Note that for $\gamma > 1$, the green function exhibits wave-like behavior with two peaks receding in time.

This representation can provide the Green function in more tractable form which can be better exploited in numerical computations. The Mittag-Leffler function can be expressed as a complex integral

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s')\Gamma(1-s')}{\Gamma(b-as')} (-z)^{-s'} ds'$$

(54)

where $0 < \Re(c) < 1$. Plugging into the equation (51), we get that

$$\hat{g}(p,\tau) = \frac{\tau^{\kappa-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s')\Gamma(1-s')}{\Gamma(\kappa - \gamma s')} \left[ -\mu |p|^\alpha \exp \left( -\frac{i\pi \theta \text{sign}(p)}{2} \right) \right]^{-s'} ds'.$$

(55)

Transforming the variable $p$ back to variable $\xi$, we obtain

$$g(\xi,\tau) = \frac{\tau^{\kappa-1}}{2\pi i|\xi|} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s')\Gamma(1-s')}{\Gamma(\kappa - \gamma s')\Gamma(s'\alpha)} \left[ -\mu \tau^\gamma \xi^\alpha \right]^{-s'} ds'.$$

(56)

After change of variables $\alpha s' = s$ and taking into account the normalization [14], which can be in both cases written as $\tau^{\kappa-1}\Gamma(\kappa)$, we finally arrive at the normalized Green function

$$g^{DF}(\xi,\tau) = \frac{\Gamma(\kappa)}{2\alpha\pi i\xi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma \left( \frac{s}{\alpha} \right) \Gamma \left( 1 - \frac{s}{\alpha} \right) \Gamma(1-s)}{\Gamma \left( \kappa - \frac{\gamma}{\alpha} s \right) \Gamma \left( \frac{(\alpha-\theta)s}{2\alpha} \right) \Gamma \left( 1 - \frac{(\alpha-\theta)s}{2\alpha} \right)} \left[ \frac{\xi}{(-\mu\tau^\gamma)^{1/\alpha}} \right]^s ds.$$

(57)

From Eq. [57] it is apparent that $g(\xi,\tau) = \frac{1}{\tau^{\gamma/\alpha}} g \left( \frac{\xi}{\tau^{\gamma/\alpha}}, 1 \right)$, so the Green function has the expected scaling with the temporal scaling exponent equal to $\frac{\gamma}{\alpha}$. The ratio $\Omega = \frac{\gamma}{\alpha}$ is called the diffusion scaling exponent. Differences between Riesz-Feller and Caputo Green functions for various values of $\alpha$ and $\gamma$ are displayed in Fig. 1.
VI. DOUBLE-FRACTIONAL OPTION PRICING MODEL

The solution of option-pricing equation driven by double-fractional diffusion under interest rate $r$ and dividend yield $q$ can be obtained by integrating over all scenarios $[S_T - K]^+$, so it reads

$$C_{(\alpha,\gamma,\kappa)}(S_t, K, \tau) = e^{-r\tau} \int_R dy \left[ S_t e^{(r-q+\mu)y} - K \right]^+ g^{DF}(y, \tau) =$$

$$= e^{-r\tau} \int_R dy \left[ S_t e^{(r-q+\mu)y} - K \right]^+ \frac{\Gamma(k)}{2\alpha\pi i y} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(1 - \frac{k}{2}\right)\Gamma(1 - s)}{\Gamma\left(k - \frac{k}{2} - s\right)\Gamma\left(\frac{(\alpha-\theta)s}{2\alpha}\right)\Gamma\left(1 - \frac{(\alpha-\theta)s}{2\alpha}\right)} \left[ \frac{y}{(-\mu\tau)^{1/\alpha}} \right]^s ds. \quad (58)$$

In Fig. 2 are displayed Green functions for different pairs $(\alpha, \gamma)$ and semi-log plots for better tail-behavior illustration. There are also shown option prices derived from Green functions. The corresponding put price can be obtained through put-call parity relation, which reads:

$$P_{(\alpha,\gamma,\kappa)}(S_t, K, \tau) = C_{(\alpha,\gamma,\kappa)}(S_t, K, \tau) - S_t e^{-q\tau} + Ke^{-r\tau}. \quad (59)$$

We notice a few properties of option prices driven by double-fractional diffusion. Indeed, we recover classical BS model for $\alpha = 2$ and $\gamma = 1$. When $\alpha < 2$, the underlying return distribution is skewed and we have polynomial decay for the negative tail of the distribution. This results into an adjustment of the option price. Larger probability of fall of the underlying asset price results into the increase of option price for $K < F_{[38]}$, and options, for which $K > F$, become cheaper (both puts and calls). Similarly, the parameter $\gamma$ plays analogous role in temporal risk redistribution. For $\gamma < 1$, options with short expiration period become more expensive, while options with long expiration period become slightly cheaper. This behavior can be observed in situations, when we face to some kind of unexpected or sudden change of regime, as e.g., black day on the market, bankrupt of some company trading on the market, natural disaster, etc. Indeed, for options with long expiration is more important the long-term equilibrium volatility. Nevertheless, for options with short expiration are such jumps and short-term uncertainty the most important factor for price estimation. On the other hand, for $\gamma > 1$, the diffusion is faster than in case of space-fractional diffusion, and the options with long maturity time. We should note that the change of parameters $\alpha$ or $\gamma$ does not change all option prices in the same direction; there are always options which become cheaper and more expensive. This essential role is played by the parameter $\sigma$, which is the volatility of the system.
TABLE I. Estimated mean values and standard deviations of model parameters ($\alpha$, $\gamma$, $\sigma$) and mean aggregated error (MAE) for three considered models, i.e., Black-Scholes model, Lévy stable model with ordinary time derivative and Double-fractional model. The statistics is done for all options and separately for call options and put options. Apparently, the mean value of $\gamma$ is very close to one for all options. But when call options and put options are analyzed separately, the $\gamma$-value is larger than one. The mean square error of Double-fractional model exhibits significant improvement with respect to the BS model, but compared with Lévy stable model, it shows only a little improvement. This is caused by the fact, that the assumption of constant parameters does not fully describe the complex behavior of option markets. The situation improves if we calibrate the model for call and put options separately. In Fig. 3 is shown that on some particular days, the improvement of aggregated error is more significant.

A. Model calibration for S&P 500 options traded in November 2008

We calibrate our model on the data of S&P 500 options that were traded during November 2008. The choice of this period is mainly because of financial crisis, which brings about phenomena that are potentially interesting. We follow the methodology of Carr and Wu [6] and try to find such triplet ($\alpha$, $\gamma$, $\sigma$) that minimizes the aggregated option price error (AE) of all out-of-the-money options, so

$$(\alpha_O, \gamma_O, \sigma_O) = \arg \min_{(\alpha, \gamma, \sigma)} \sum_{\tau \in T, K \in K} |O_{\alpha, \gamma, \kappa} - O_{\text{market}}|.$$ (60)

We make the optimization for each trading day. We have chosen the out-of-the-money options, because in-the-money option prices are more determined by the boundary conditions in option pricing formula, rather than by particular underlying diffusion model [6]. The statistics of calibrated parameters is listed in Tab. I for Black-scholes model, Lévy stable model and Double-fractional model. Because of the fact that $\gamma$ was close to 1, the choice of derivative type did not have a large impact on the solutions and the results are practically the same, therefore the results are presented only for Caputo derivative, i.e., $\kappa = 1$. The parameter $\alpha$ is close to 1.5 in all cases, only for call options is fluctuating around 1.6. The analysis shows that the double-fractional model brings more improvement, when is fitted separately for call and put options. This could be connected to the discussion about the general validity of put-call parity and market efficiency [39]. Fig. 3 shows estimated parameters for every day and also the ratio between aggregated errors of Lévy stable model and Double-fractional model. Particularly for put options we can observe a noticeable improvement. Besides, there is apparent another interesting phenomenon connected with the decrease of $\alpha$ parameter. In such situation decreases also the $\gamma$ parameter, even to values smaller than one, which could be interpreted as the risk redistribution from long-term options to short term options. The parameter $\Omega$, which is the ratio between parameters $\gamma$ and $\alpha$, expresses the temporal scaling parameter of the system. This parameter produces more stable behavior, pointing to the simultaneous changes in both parameters $\gamma$ and $\alpha$. The differences between option prices in case of Double-fractional and Black-Scholes model are presented in Fig. 4. One can observe that the price differences are relatively small, even in some cases are Double-fractional prices below prices estimated
FIG. 3. Estimated values of stability parameter $\alpha$, diffusion speed parameter $\gamma$, scaling exponent $\Omega$ and the ratio of aggregated errors between Lévy model and Double-fractional model for each particular day, done for all options, resp. for calls and puts separately. We can observe that in case of calls and puts is the benefit of the Double-fractional model more significant. In case when we observe a drop in $\alpha$ parameter, we can simultaneously observe a drop in values of $\gamma$, which points to the risk redistribution from long-term to short-term options. This phenomenon could be an indicator of market regime change. The parameter $\Omega$ measures the ratio between $\gamma$ and $\alpha$ and corresponds to the temporal scaling exponent, so $g(\xi, \tau) \sim \tau^\Omega$. We shall note that for BS model $\Omega = \frac{1}{2}$, which corresponds to Brownian motion. The graphics shows that $\Omega$ exhibits more stable behavior, especially when estimated on both put and call options simultaneously.

by the Black-Scholes model. This is caused by the fact that the Double-fractional Green function redistributes the probability of particular scenarios. In cases, when the option execution is less probable than in case of Black-Scholes model, we obtain cheaper option price. Nevertheless, with increasing maturity time are the Double-fractional prices more expensive, but the difference is still not dramatic.

VII. CONCLUSIONS AND PERSPECTIVES

A novel method of option pricing was proposed on the basis of fully asymmetric Double-fractional diffusion, and a special example was calibrated for S&P 500 options traded in November 2008. The presence of power-law behavior in prices has been discussed before in case of cotton prices [40] and options [6, 9]. There have been various alternative attempts to reduce the risks in portfolios. Examples are provided by models based on regime switching [8], stochastic volatility [25], jump processes [9], etc. Alternatively, it is possible to redistribute the risk by a temporally-fractional derivative, which, similarly to spatial asymmetry, redistributes the risk to the either short-period options or long-term options. Such model enables one to treat differently the short-term, instant, risk coming from contemporary fluctuations, jump corrections, etc., and the long-term risk, which is determined by the slow volatility of the system.

While the long-term average behavior of markets tends to be similar in systems driven by ordinary, first time-derivative diffusion (i.e., diffusion speed parameter equal to one), an adjustment of diffusion speed parameter $\gamma$ to values $\gamma \neq 1$ can often describe the system better. Such fact coheres with the complex nature of financial markets and particularly option trading. Further investigations of the topic and its interrelation to other models will be the subject of future research. Connection with regime switching models or interpretation of $\gamma$ parameter as a “regime-switching” time-dependent parameter and the conjunction with other models are questions of high importance and can possibly reveal a further potential of models based on double-fractional diffusion.
FIG. 4. Estimated call and put prices of S&P 500 calibrated by Double-fractional (DF) model and Black-Scholes (BS) model as functions of strike price $K$ for several maturity times $\tau$. The market spot price is $S = 1000$. The model parameters are listed in Tab. I. One can observe that the price differences are relatively small, increasing with increasing $\tau$, which is caused by different scaling exponents $\Omega$. In some cases is the price of DF option even cheaper than in case of BS.

ACKNOWLEDGEMENTS

Authors want to thank to Petr Jizba and Václav Zatloukal for valuable discussions. J. K. acknowledges support from GAČR, Grant no. P402/12/J077.

[1] F. Black and M. S. Scholes, Journal of Political Economy 81, 637 (1973).
[2] V. V. Acharya and M. Richardson, Critical Review 21, 195 (2009).
[3] O. Merrouche and E. Nier, What Caused the Global Financial Crisis?: Evidence on the Drivers of Financial Imbalances 1999–2007, Tech. Rep. (International Monetary Fund, 2010).
[4] D. Colander, H. Fllmer, A. Haas, M. Goldberg, K. Juselius, A. Kirman, T. Lux, and B. Sloth, The Financial Crisis and the Systemic Failure of Academic Economics, WorkingPaper (Department of Economics, University of Copenhagen, 2009).
[5] N. Taleb, The Black Swan: The Impact of the Highly Improbable Fragility (Random House Publishing Group, 2010).
[6] P. Carr and L. Wu, Journal of Finance 58, 753 (2003).
[7] H. Kleinert, Physica A: Statistical Mechanics and its Applications 312, 217 (2002), [http://arxiv.org/abs/cond-mat/0202311].
[8] L. Calvet and A. Fisher, Multifractal Volatility: Theory, Forecasting, and Pricing, Academic Press Advanced Finance (Elsevier Science, 2008).
[9] P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC Financial Mathematics Series (Taylor & Francis, 2003).
[10] P. Jizba and T. Arimitsu, Annals of Physics 312, 17 (2004).
[11] H. Kleinert, Foundations of Physics, 1 (2013), [http://klnrt.de/409].
[12] C.-Y. Lee and X. Yao, Evolutionary Computation, IEEE Transactions on 8, 1 (2004).
[13] D. Turcotte, J. Rundle, H. Frauenfelder, and N. Sciences, Self-organized Complexity in the Physical, Biological, and Social Sciences, Arthur M. Sackler Colloquia of the National Academy of Sciences (National Academy of Sciences, 2002).
[14] B. Gnedenko and A. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, edited by K. L. Chung (Adison-Wesley, 1968).
[15] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets, 4th edn (World Scientific, 2009) [http://klnrt.de/b5].
[16] V. Zolotarev, One-dimensional Stable Distributions, Translations of mathematical monographs (American Mathematical Society, 1986).
[17] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge Studies in Advanced Mathematics (Cambridge University Press, 1999).
[18] G. Samoradnitsky and S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Stochastic Modeling Series (Taylor & Francis, 1994).
[19] I. Podlubny, Fractional Differential Equations, Volume 198: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their ... (Mathematics in Science and Engineering), 1st ed. (Academic Press, 1998).
[20] T. Abdeljawad, Comput. Math. Appl. 62, 1602 (2011).
[21] S. Samko, A. Kilbas, and O. Marichev, Fractional Integrals and Derivatives: Theory and Applications (Gordon and Breach Science Publishers, 1993).
[22] H. Gerber, E. Shiu, et al., Option Pricing by Esscher Transforms, Cahier / Institut de sciences actuarielles (HEC Ecole des hautes études commerciales, 1993).
FIG. 5. Dependence of $g^{RF}(\tau,l)$ (purple line) and $g^C(\tau,l)$ (blue line) to the variable $l$ for $\tau = 1$.

[23] R. N. Mantegna and H. E. Stanley, An introduction to econophysics: correlations and complexity in finance, Vol. 9 (Cambridge university press Cambridge, 2000).
[24] B. B. Mandelbrot and J. W. Van Ness, SIAM review 10, 422 (1968).
[25] S. L. Heston, The Review of Financial Studies 6, 327 (1993).
[26] C. Necula, Option Pricing in a Fractional Brownian Motion Environment, Advances in Economic and Financial Research - DOFIN Working Paper Series 2 (Bucharest University of Economics, Center for Advanced Research in Finance and Banking - CARFIB, 2008).
[27] G. Jumarie, Computers & Mathematics with Applications 59, 1142 (2010).
[28] L. Song and W. Wang, Abstract and Applied Analysis 2013 (2013).
[29] W. Wyss, Fractional Calculus & Applied Analysis 3 (2000).
[30] Y. Hu, B. Øksendal, and A. Sulem, Infinite Dimensional Analysis, Quantum Probability and Related Topics 06, 519 (2003).
[31] G. Pagnini, Fractional Calculus and Applied Analysis 16, 436 (2013).
[32] H. Kleinert and V. Zatloukal, Phys. Rev. E 88, 052106 (2013).
[33] H. Zhang and F. Liu, Numerical Mathematics, A Journal of Chinese Universities 16, 181 (2007).
[34] R. Gorenflo, Y. Luchko, and F. Mainardi, Fractional Calculus and Applied Analysis 2, 383 (1999).
[35] R. Paris and D. Kaminski, Asymptotics and Mellin-Barnes Integrals, Encyclopedia of Mathematics and its Applications (Cambridge University Press, 2001).
[36] K. Diethelm, The analysis of fractional differential equations an application-oriented exposition using differential operators of Caputo type (Springer-Verlag, Heidelberg New York, 2010).
[37] P. Flajolet, X. Gourdon, P. Dumas, D. T. D. Knuth, N. G. D. Bruijn, and H. Mellin, Theoretical Computer Science 144, 3 (1995).
[38] $F$ is the forward price, i.e., $F = S_t \exp((r - q)\tau)$.
[39] M. Brunetti and C. Torricelli, International Review of Financial Analysis 14, 508 (2005).
[40] B. B. Mandelbrot, The Journal of Business 36 (1963).
[41] A. Skorohod, Select. Transl. Math. Statist. and Probability 1, 157 (1961).

Appendix A: Comparison of Smearing Kernels for Riesz-Feller and Caputo Derivatives

In this appendix, we compare Green function of Double-fractional diffusion in case when time derivative is equal to Riesz-Feller derivative and Caputo derivative and $\gamma < 1$. The Green function is given by

$$g_{\alpha,\gamma}(\xi,\tau) = \int_0^\infty dl \psi_M(\tau,l) \frac{1}{\Gamma(\gamma)} L_{\gamma,1} \left( \frac{\tau l^{1/\gamma}}{\Gamma(\gamma)} \right) g_\alpha(\xi,l),$$  \hspace{1cm} (A1)

where $\psi_M$ is different according to used derivative, so

$$\psi_M(l, \tau) = \begin{cases} \frac{\Gamma(\gamma)}{\tau^{\gamma - 1}} & \text{for Riesz derivative,} \\
\frac{\Gamma(\gamma)}{\tau^{\gamma}} & \text{for Caputo derivative.} \end{cases}$$ \hspace{1cm} (A2)
Indeed, it is interesting to see, what happens with the smearing kernel

\[ g_{1,\gamma}(l, \tau) = \psi_{M}(\tau, l) \frac{1}{l^{1/\gamma}} L_{\gamma,1} \left( \frac{\tau}{l^{1/\gamma}} \right) \]  

(A3)

for small and large values. Firstly, when \( l \to 0 \), then the argument of \( \gamma \)-stable distribution distribution goes to infinity. According to Ref. [41], the asymptotic expansion gives us

\[ \frac{1}{l^{1/\gamma}} L_{\gamma,1} \left( \frac{\tau}{l^{1/\gamma}} \right) \sim \frac{\Gamma(\gamma + 1) \sin(\pi \gamma)}{\cos \left( \frac{\pi \gamma}{2} \right)} \frac{l}{\tau^{\gamma + 1}} \]  

for \( l \to 0 \),  

(A4)

which gives us

\[ g_{1}^{RF}(\tau, l) \sim \frac{l}{\tau^{2\gamma}} \frac{\Gamma(\gamma) \Gamma(\gamma + 1) \sin(\pi \gamma)}{\cos \left( \frac{\pi \gamma}{2} \right)} \]  

for \( l \to 0 \).  

(A5)

For Caputo case we get non-zero value of smearing kernel for \( l = 0 \). Particularly, it is equal to

\[ g_{1}^{C}(\tau, 0) = \left( \frac{1}{\tau^{\gamma}} \right) \frac{\Gamma(\gamma) \sin(\pi \gamma)}{\cos \left( \frac{\pi \gamma}{2} \right)} \]  

(A6)

On the other hand, when \( l \to \infty \), then it is necessary to use the Taylor expansion of \( L_{\gamma,1}(x) \), and again from [41] we obtain

\[ L_{\gamma,1}(x) \sim A_{\gamma} x^{-1 - \frac{\lambda_{\gamma}}{\gamma}} \exp \left( -B_{\gamma} x^{-\lambda_{\gamma}} \right) \]  

for \( x \to 0^{+} \),  

(A7)

where \( \lambda_{\gamma} = \frac{2}{\gamma - 1} \) and \( A_{\gamma} \) resp. \( B_{\gamma} \) are \( \gamma \)-dependent constants. The asymptotic behavior can be therefore described as

\[ g_{1}^{RF}(\tau, l) \sim C^{RF}(\tau) A_{\gamma} l^{\frac{1}{\gamma - 1}} \exp \left( -B_{\gamma} D(\tau) l^{\frac{1}{\gamma - 1}} \right) \]  

for \( l \to +\infty \)  

(A8)

and for Caputo case as

\[ g_{1}^{C}(\tau, l) \sim C^{C}(\tau) A_{\gamma} l^{\frac{1}{\gamma - 1}} \exp \left( -B_{\gamma} D(\tau) l^{\frac{1}{\gamma - 1}} \right) \]  

for \( l \to 0 \).  

(A9)

The \( \tau \)-dependent constants \( C^{RF}(\tau) \), resp. \( C^{C}(\tau) \) can be determined from previous expressions. Therefore, in both cases we become exponential decay in \( l \). The graphs of both smearing kernels are depicted in Fig. [5].