Behavior of Cosmological Perturbations in the Brane-World Model

Hideo Kodama

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan

In this paper we present a gauge-invariant formalism for perturbations of the brane-world model developed by the author, A. Ishibashi and O. Seto recently, and analyze the behavior of cosmological perturbations in a spatially flat expanding universe realized as a boundary 3-brane in AdS^5 in terms of this formalism. For simplicity we restrict arguments to scalar perturbations. We show that the behavior of cosmological perturbations on superhorizon scales in the brane-world model is the same as that in the standard no-extradimension model, irrespective of the initial condition for bulk perturbations, in the late stage when the cosmic expansion rate H is smaller than the inverse of the bulk curvature scale ℓ. Further, we give rough estimates which indicate that in the early universe when H is much larger than 1/ℓ, perturbations in these two models behave quite differently, and the conservation of the Bardeen parameter does not hold for superhorizon perturbations in the brane-world model.

I. INTRODUCTION

Recently, motivated by the S^1/Z_2 reduction of the M-theory and the hierarchy problem in particle physics, a fascinating model was proposed by Randall and Sundrum, in which 4-dimensional Minkowski spacetime is realized as a boundary 3-brane in the 5-dimensional anti-de Sitter spacetime \[ \text{AdS}^5 \]. The most surprising feature of this model is that the 5-dimensional gravity is confined in the brane in the sense that the standard Newtonian law of gravity holds in the non-relativistic limit on scales much larger than the bulk curvature scale ℓ, and the massive Kaluza-Klein modes of gravitational waves decouple from the massless mode and their mode functions are suppressed near the brane.

After the paper by Randall and Sundrum, brane-world cosmology has been actively studied by many people, and it has been shown that any spatially homogeneous and isotropic universe can be embedded as 3-brane in AdS^5. Further, in order to see whether the brane-world model can provide a new framework for cosmology consistent with observations, formalisms to investigate behavior of perturbations in the brane-world model have been proposed by various people. In particular, the author developed with A. Ishibashi and O. Seto a complete gauge-invariant formalism for perturbations in the brane-world model in which the Einstein equations for perturbations are reduced to a single master equation with a boundary condition on a dynamical brane. In the present paper, we first briefly describe the main point of this formalism specialized to scalar perturbations, and then address the important issue whether the behavior of scalar perturbations in the brane-world model is the same as that in the conventional no-extradimension model.

II. GAUGE INVARIANT FORMALISM OF PERTURBATIONS

In this section we give the gauge-invariant formalism for scalar perturbations in the \((n+2)\)-dimensional brane-world model in which the bulk spacetime \((M, g_{MN})\) is empty and its unperturbed background is given by \(\text{AdS}^{n+2}\) with the negative cosmological constant \(\Lambda = -n(n+1)/(2\ell^2)\). The starting point is the Einstein equations for the bulk, \(\bar{G}_{MN} + \bar{\Lambda}g_{MN} = 0\), and Israel’s junction condition for \(\mathbb{Z}_2\) symmetry at the \(n\)-brane \(\Sigma\),

\[
\kappa^2 T^\mu_\nu = 2(K^\mu_\nu - K\delta^\mu_\nu),
\]

where \(\kappa^2\) is the gravitational constant for the bulk, \(T^\mu_\nu\) is the energy-momentum tensor of the brane, and \(K^\mu_\nu\) is the extrinsic curvature of the brane.
A. Unperturbed Background

We assume that the unperturbed configuration including the brane is invariant under the isometry group $G_n$ isomorphic to the symmetry group of the $n$-dimensional constant curvature space $\mathcal{K}^n$, and that the the bulk manifold $\mathcal{M}^{n+2}$ is written as $\mathcal{N}^2 \times \mathcal{K}^n$, where $\mathcal{N}^2$ is the 2-dimensional orbit space: $\mathcal{M}^{n+2} = \mathcal{N}^2 \times \mathcal{K}^n \ni (y^a, x^r) = (z^M)$. Under this decomposition the bulk background metric takes the form

$$ds^2 = g_{MN}dz^Mdz^N = g_{ab}(y)dy^ady^b + r^2(y)d\sigma_n^2,$$

where $d\sigma_n^2$ is the metric of $\mathcal{K}^n$ with a constant curvature $K$.

The unperturbed structure of the brane is described by the Robertson-Walker metric $ds^2 = -d\tau^2 + a(\tau)^2d\sigma_n^2$, and the energy-momentum tensor $T_{\tau\tau} = \rho, T_{\tau i} = 0, T^i_j = p\delta^i_j$. The embedding of this brane into the bulk background spacetime is represented by functions $y^a(\tau)$ such that $a(\tau) = r(y(\tau))$ and $dr^2 = -g_{ab}dy^ady^b$. The junction condition is expressed as

$$D_{\perp}r = -\tilde{\kappa}^2/2n \rho, \quad (n - 1)D_{\perp}r - K^r_r = \tilde{\kappa}^2/2p,$$

where $D$ is the covariant derivative with respect to the metric $g_{ab}(y)$ of $\mathcal{N}^2$ and $D_\perp = n^aD_a$ with $n^M$ being the unit normal to $\Sigma$.

B. Bulk Perturbation

Since the unperturbed background has $G_n$ symmetry, perturbations of the bulk geometry and those of the brane can be decomposed into scalar-type, vector-type and tensor-type components with respect to the transformation behavior under diffeomorphisms of $\mathcal{K}^n$, and components of different types do not couple in the Einstein equations [8]. Further, each component can be expanded in terms of the harmonic tensors of the same type on $\mathcal{K}^n$. In particular, for the scalar-type perturbation, the bulk metric perturbation $h_{MN} := \delta g_{MN}$ is expressed in terms of the scalar harmonics $S$ defined by $(\tilde{\Delta} + k^2)S = 0$, as

$$h_{ab} = f_{ab}S, \quad h_{ai} = -rf_a\hat{D}_iS, \quad h_{ij} = 2r^2(\tilde{\mathcal{R}}_{ij}S + H_T\hat{D}_i\hat{D}_jS),$$

where $\hat{D}$ denotes the covariant derivative with respect to the metric $d\sigma_n^2$. We can construct two gauge-invariant combinations from the expansion coefficients $f_{ab}, f_a, \tilde{\mathcal{R}}$ and $H_T$ as

$$F = \tilde{\mathcal{R}} + \frac{1}{r}D^a_{\perp}X_a, \quad F_{ab} = f_{ab} + D_aX_b + D_bX_a,$$

where $X_a = rf_a + r^2D_aH_T$.

From the Einstein equations for the bulk we can show that these gauge-invariants are expressed as

$$r^{n-2}F = 2n \left( \Box - \frac{2}{\ell^2} \right) \Omega, \quad r^{n-2}F_{ab} = D_aD_b\Omega - \left( \frac{n-1}{n} \Box - \frac{n-2}{n\ell^2} \right) \Omega g_{ab},$$

in terms of the master variable $\Omega$ satisfying the wave equation

$$\Box \Omega - \frac{n}{r}D_T \cdot D\Omega - \left( \frac{k^2 - nK}{r^2} - \frac{n-2}{\ell^2} \right) \Omega = 0,$$

where $\Box = D^aD_a$.

C. Brane Perturbation

The intrinsic metric perturbation $\delta g_{\mu\nu}$ of the brane can be expanded in terms of the harmonics as

$$\delta g_{\tau\tau} = -2\alpha S, \quad \delta g_{\tau i} = a\beta \hat{D}_iS, \quad \delta g_{ij} = 2a^2(\mathcal{R}S_{ij} + h_T\hat{D}_i\hat{D}_jS_{ij}),$$

where
and the perturbation of the brane position is determined by 
\[ n_M \delta z^M = Z_\perp S, \]
from which we can construct a gauge-invariant quantity 
\[ Y_\perp := Z_\perp - X_\perp. \]
The standard gauge-invariants for the intrinsic metric perturbation are expressed in terms of the bulk gauge-invariants and 
\[ Y_\perp. \]

The intrinsic matter perturbation of the brane is expanded as
\[ \delta T^\tau_\tau = -\delta \rho S, \quad \delta T^\tau_i = -a(\rho + p)(v - \beta) \hat{D}_i S, \quad \delta T^i_j = \delta p S_{ij} + \pi T \left( \frac{1}{k^2} \hat{D}_i \hat{D}_j S + \frac{1}{n} \delta^i_j S \right). \]

We can construct the following three gauge-invariants from these perturbation variables other than the anisotropic stress perturbation \( \pi T \), which is gauge invariant by itself:
\[ V := k(v - \dot{a} h_T), \quad \rho \Delta := \delta \rho - a \dot{\rho}(v - \beta), \quad \Gamma := \delta p - c_s^2 \delta \rho. \]

D. Junction Condition

By expressing the perturbation of the junction condition \([4]\) in terms of the gauge-invariant quantities defined so far, we obtain the following four equations:
\[ \frac{\kappa^2}{k^2 - nK} a^n \rho \Delta = -r D_\perp \left( \frac{\Omega}{r} \right) - 2a^{n-2} Y_\perp, \]
\[ \kappa^2 a^{n-1} (\rho + p)V = \frac{k}{a} \hat{D}_i \hat{D}_j S + \frac{2a^{n-1}}{n} \delta^i_j S, \]
\[ \frac{1}{a} (aV) = \frac{k}{a} \Psi + \frac{\kappa}{\rho + p} \Gamma + \frac{c_s^2 \rho \Delta}{\rho + p} - \frac{n - 1}{a} \frac{k^2 - nK}{ak} \pi T, \]
\[ 2\frac{k^2}{a^2} Y_\perp = \kappa^2 \pi T. \]

Among these, the first two give expressions for the intrinsic matter gauge-invariant variables \( \Delta \) and \( V \) in terms of the bulk variable \( \Omega \) and \( Y_\perp \). On the other hand, the rest give the boundary conditions on the latter, because the anisotropic stress perturbation \( \pi T \) and the entropy perturbation \( \Gamma \) are not dynamical and are expressed in terms of other dynamical variables when the matter model is specified. In particular, when the anisotropic stress perturbation vanishes, they give the following boundary condition for the bulk master variable:
\[ \left[ r D_\perp \left( \frac{\Omega}{r} \right) \right] + (2 + n c_s^2) \frac{\dot{a}}{a} \left[ r D_\perp \left( \frac{\Omega}{r} \right) \right] + \left\{ -n(1 + w)(2n - 2 + nw) \left( \frac{D_\perp r}{r} \right)^2 \right\} \left[ r D_\perp \left( \frac{\Omega}{r} \right) \right] - (n - 1)(1 + w) \frac{k^2}{a^2} \frac{D_\perp r}{r} \Omega = \kappa^2 a^{n-2} \Gamma. \]

III. BEHAVIOR OF COSMOLOGICAL PERTURBATIONS

In this section we examine the behavior of scalar perturbations of the brane using the formalism presented in the last section. We only consider the case \( n = 3 \) and \( \pi_T = 0 \).
A. Low Energy Region

If we put $\rho = \rho_0 + \rho_m$ where $\kappa^2 \rho_0 / 6 = 1 / t^2$ and $\kappa^2 = \kappa^2 / \ell$, in the low energy region for which $\rho_m / \rho_0 \ll 1$ and $k \ell / a \ll 1$, the evolution equations of the brane universe are approximately given by

$$H^2 := \left( \frac{\dot{a}}{a} \right)^2 \simeq \frac{\kappa^2}{3} \rho_m - \frac{K}{a^2}, \quad \dot{\rho} = -3(1 + w_m)\rho_m H,$$

where $w_m = p_m / \rho_m$, and the junction condition is expressed as

$$-(k^2 - 3K)a \Gamma \simeq (a^3 \rho \Delta + (2 + 3\kappa^2)H(a^3 \rho \Delta)'' + \left[ \frac{3}{2}(1 + w_m) \left( \frac{H^2}{a^2} + \frac{K}{a^2} \right) + c_s^2 \frac{k^2 - 3K}{a^2} \right](a^3 \rho \Delta).$$

These equations are the same as those for the standard model. Hence the density perturbation of the brane universe behaves in the same way as that in the standard model. However, this does not immediately imply that the same result holds for other quantities such as the curvature perturbation, because we do not have the corresponding to the time $t \sim 1 / \omega$ at which $w_m$ changes. For $K = 0$ (a spatially flat model) the general solution for $\Omega$ is given by

$$\Omega(r, t) = 2t^2 r \int d\omega [A(\omega)J_0(m \ell / r) + B(\omega)N_0(m \ell / r)] e^{-i \omega t},$$

where $\omega^2 = m^2 + k^2$. By inserting this expression into the junction condition, we obtain the following estimates for superhorizon perturbations with $k \ell / a H \ll 1$ in the low energy region:

$$\Phi \simeq \frac{2}{3} \frac{g}{m^2} d\omega B(\omega) e^{-i \omega t}, \quad Z \simeq \frac{5 + 3w_m}{3(1 + w_m)} \Phi,$$

$$\Phi + \Psi \simeq O(h^2) \Phi, \quad \frac{a^2}{2k^2} \kappa^2 \rho \Delta \simeq (1 + O(h^2)) \Phi,$$

where $h = HT$, $M$ is a constant of the order $1 / t$ and $Z := \Phi - aHV / k$ is the Bardeen parameter. This together with the previous argument on $\Delta$ implies that for superhorizon modes in the low energy region the cosmological perturbation of the brane universe behaves in the same way as that of the standard model with no extra-dimension as far as the growing mode is concerned. This confirms the same conclusion obtained by Koyama and Soda [5] by a cruder estimate in the Gaussian normal coordinates. Here note that $B(\omega)$ should have a $\delta$-function type peak at $\omega = 0$ in order for $\Phi$ to approach a non-vanishing constant in the $t \to \infty$ limit. Further $B(\omega)$ should also have peaks for the values of $\omega$ corresponding to the time $t \sim 1 / \omega$ at which $w_m$ changes.

In contrast, in the high energy region where $h \gg 1$, we obtain the estimate $Z = O(h^2) \Phi$. Since $Z$ and $\Phi$ become constants of the same order for superhorizon perturbations in the standard models, this implies that the behavior of perturbations in the brane-world model is quite different from that in the standard model in the high energy region.

[1] Randall, L. and Sundrum, R.: Phys. Rev. Lett. **83**, 3370 (1999); ibid., 4690 (1999).
[2] Mukohyama, S.: hep-th/0004065 (2000); hep-th/0006144 (2000).
[3] Maartens, R.: hep-th/0004166 (2000).
[4] Bruck, van de C., Dorca, M., Brandenberger, R. and Lukas, A.: hep-th/0005073 (2000).
[5] Koyama, K. and Soda, J.: gr-qc/0001033 (2000).
[6] Langlois, D., Maartens, R. and Wands, D.: hep-th/0006007 (2000).
[7] Kodama, H., Ishibashi, A. and Seto, O.: hep-th/0004160 (2000).
[8] Kodama, H. and Sasaki, M.: Prog. Theor. Phys. Suppl. **78**, 1–166 (1984).