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Abstract—In this article, we update the reference [14] in two aspects. First, we note that in order for the control law (12) in [14] to be equivalent to the control law (3) in [14], we need to assume that the samplings for all subsystems must be synchronous, i.e., we need to assume that \( T_i = T \) for all \( i = 1, \ldots, N \). Second, we extend our results from periodic sampling to aperiodic sampling.

Index Terms—Sampled-data control, multi-agent systems, leader-following consensus.

I. INTRODUCTION

O

VER the years, cooperative control of multi-agent systems has attracted extensive attention from the control community. This is due to its wide range of applications in various engineering areas such as coordination of mobile robots, formation of unmanned vehicles, and synchronization of multiple spacecraft systems. The objective of cooperative control is to design a control law using only the information of the neighboring agents to achieve a collective behavior in the overall multi-agent system. Such a control law is called a distributed law. A fundamental cooperative control problem is called consensus. Depending on whether there is a leader system, the problem can be classified into two types: leaderless consensus and leader-following consensus. The leaderless consensus problem aims to make the states of a group of agents converge to a same trajectory, while the leader-following consensus problem further requires the states of a group of follower systems asymptotically track a prescribed trajectory produced by a so-called leader system. So far, both consensus problems have been widely studied. For example, the leaderless consensus problem has been studied in [15], [17], [19], [22], the leader-following consensus problem has been studied in [9], [11], [16], and both problems have been studied in [12], [20].

It is noted that most existing results on continuous-time multi-agent systems assume that the information is transmitted continuously and the control laws are also in the continuous-time form. However, many advanced communication networks only permit digital information transmission, and more and more practical controllers are implemented in digital platforms. Hence, it is more practical to take into account both digital information transmission and digital control laws. The sampled-data control approach has been a most commonly used method for implementing a continuous-time control law in a digital platform [11], [2], [4], and recently, this approach has also been used to address the consensus problem. For example, in [25], [26], the sampled-data leaderless consensus problem (SDLCP) for single-integrator multi-agent systems was studied for the static network case and the switching network case, respectively. The SDLCP was further studied for single-integrator multi-agent systems in [23] and double-integrator multi-agent systems in [5], where the communication networks are assumed to be switched and jointly connected. In [27], a control protocol depending on the sampled position data was proposed to solve the SDLCP for double-integrator multi-agent systems. Reference [7] further studied the SDLCP for double-integrator multi-agent systems based on the impulsive control strategy. In [29], [31], the sampled-data leaderless mean square consensus problem was studied for the general linear multi-agent systems with packet losses. Reference [30] studied the SDLCP for general linear multi-agent systems with switching topologies using the input delay method. In [21], the sampled-data leader-following consensus problem (SDLFPC) was studied for a class of multi-agent systems by using the direct discretization method, where the follower systems had the single-integrator dynamics and the leader system had the double-integrator dynamics. In [18], two weighted consensus tracking protocols via computing the network centrality were proposed to solve the SDLFPC for double-integrator multi-agent systems. Reference [24] studied the bounded SDLFPC for double-integrator multi-agent systems, and the tracking errors were guaranteed to be ultimately bounded. In [3], a delay-dependent stability criterion was derived to solve the SDLFPC for general linear multi-agent systems. However, the solvability conditions of the problem...
The dynamics of the leader system is described as follows: composed of \( N \) agent systems as special cases. Second, our approach applies as all the sampling intervals are smaller than this upper bound.

Next, we consider the following control law

\[ u_i(t) = K \sum_{j=0}^{N} \tilde{a}_{ij}(t)(x_j(t) - x_i(t)), \quad \forall t \in [t_s, t_{s+1}) \]  

where \( i = 1, \ldots, N, t_0 = 0, t_{s+1} = t_s + T_s, s \in \mathbb{N}, T_s \in \mathbb{T}, \mathbb{T} \) with \( \mathbb{T} \leq \mathbb{T} \) being two positive real numbers, and \( K \) is a constant matrix with proper dimension. Remark 2.1: The control law \( \{3\} \) is called a distributed sampled-data state feedback control law, since agent \( i \) can only make use of the sampled states of its neighbors and itself for feedback control. In fact, the control law \( \{3\} \) is motivated by sampling the continuous-time control laws used in \([16, 20]\). Other similar sampled-data control laws can also be found in the recent survey paper \([6]\).

We describe the sampled-data leader-following consensus problem as follows:

Problem 2.1: Given the multi-agent system composed of \([1\) and \([2]\\) and a switching digraph \( \bar{G}(t) \), design a control law of the form \([3]\\) with appropriate sampling intervals \( T_s, s \in \mathbb{N} \), such that, for any initial conditions \( x_i(0) \), \( \lim_{t \to \infty} (x_i(t) - x_0(t)) = 0 \) for \( i = 1, \ldots, N \).

To solve Problem 2.1 we introduce the following assumption.

Assumption 2.1: The pair \((A, B)\) is stabilizable.

Remark 2.2: Assumption 2.1 is a standard assumption for the consensus problem of general linear multi-agent systems, which has also been used in \([3, 16, 22, 39]\) etc.

III. A TECHNICAL LEMA

In this section, we will establish a technical lemma as follows.

Lemma 3.1: Suppose \( W(t) : [0, \infty) \to [0, \infty) \) is continuous, and there exists a sequence \( \{t_s : s \in \mathbb{N}, t_s \in [0, \infty)\} \) satisfying \( t_{s+1} - t_s \geq h \) for all \( s \in \mathbb{N} \) and some positive real number \( h \) such that \( W(t) \) is differentiable on each interval \([t_s, t_{s+1})\) and

\[ \dot{W}(t) \leq -\beta_1 W(t) + \beta_2 W(t_s), \quad \forall t \in [t_s, t_{s+1}) \]  

where \( \beta_1 \) and \( \beta_2 \) are two positive real numbers with \( \beta_2 < \beta_1 \). Then

\[ \lim_{t \to \infty} W(t) = 0. \]  

Proof: First, if \( W(t_s) = 0 \) for some \( s \in \mathbb{N} \), then, by \([2]\) and the fact that \( W(t) \geq 0 \) for all \( t \geq 0 \), we have \( W(t) = 0 \) for all \( t \geq t_s \). Thus \([5]\) holds.
Second, consider the case where \( W(t_s) \neq 0 \) for all \( s \in \mathbb{N} \).
For any \( t \in [t_s, t_{s+1}) \), solving (4) gives
\[
W(t) \leq e^{-\beta_1(t-t_s)}W(t_s) + \int_{t_s}^{t} e^{-\beta_1(t-\tau)} \beta_2 W(t_s) d\tau
\]
\[
= (e^{-\beta_1(t-t_s)} + \beta_2 e^{-\beta_1 t} \int_{t_s}^{t} e^{\beta_1 \tau} d\tau) W(t_s)
\]
\[
= (1 - \frac{\beta_2}{\beta_1}) e^{-\beta_1(t-t_s)} + \frac{\beta_2}{\beta_1} W(t_s).
\]
Thus,
\[
\lim_{t \to t_{s+1}^-} W(t) \leq \lim_{t \to t_{s+1}^-} \left( (1 - \frac{\beta_2}{\beta_1}) e^{-\beta_1(t-t_s)} + \frac{\beta_2}{\beta_1} W(t_s) \right)
\]
\[
= \left( (1 - \frac{\beta_2}{\beta_1}) e^{-\beta_1(t_{s+1}+t_s)} + \frac{\beta_2}{\beta_1} W(t_s) \right).
\]
Let
\[
\rho_s = (1 - \frac{\beta_2}{\beta_1}) e^{-\beta_1(t_{s+1}+t_s)} + \frac{\beta_2}{\beta_1}, \quad s \in \mathbb{N}
\]
\[
\rho = (1 - \frac{\beta_2}{\beta_1}) e^{-\beta_1 h} + \frac{\beta_2}{\beta_1}.
\]
Since \( t_{s+1} - t_s \geq h \) for all \( s \in \mathbb{N} \) and \( 0 < \beta_2 < \beta_1 \), we obtain
\[
\rho_s \leq \rho = e^{-\beta_1 h} + \frac{\beta_2}{\beta_1} (1 - e^{-\beta_1 h}) < e^{-\beta_1 h} + 1 - e^{-\beta_1 h} = 1.
\]
Since \( W(t) \) is continuous, using (7) and (8) gives
\[
W(t_{s+1}) = \lim_{t \to t_{s+1}^-} W(t) \leq \rho_s W(t_s) \leq \rho W(t_s).
\]
Therefore, \( W(t_s) \) converges to zero as \( s \) tends to infinity, which implies \( \lim_{s \to \infty} W(t) = 0 \) since \( W(t) \) is continuous over \([0, \infty)\).
\[\square\]

IV. STATIC NETWORK CASE

In this section, we will first consider the leader-following consensus problem for the multi-agent system composed of (11) and (12) under static networks by a distributed sampled-data state feedback control law.

To solve our problem, we need one more assumption on the communication graph as follows:

**Assumption 4.1:** Every node \( i = 1, \ldots, N \) is reachable from node 0 in the digraph \( \mathcal{G} \).

**Remark 4.1:** Assumption 4.1 allows the communication graph to be directed, and contains the undirected graph as a special case. Under this assumption, by Lemma 4 of (11), \( H \) is an M-matrix. Then, by Theorem 2.5.3 of (10), there exists a positive definite diagonal matrix \( D = \text{diag}(d_1, \ldots, d_N) \) such that \( DH + H^T D \) is positive definite.

For the static network case, the control law (3) can be simplified as follows:
\[
u_i(t) = K \sum_{j=0}^{N} \bar{a}_{ij} (x_j(t_s) - x_i(t_s)), \quad \forall t \in [t_s, t_{s+1})
\]
\[\text{where } i = 1, \ldots, N, \ t_0 = 0, \ t_{s+1} = t_s + T, \ s \in \mathbb{N}, \ \text{and} \ T \in [T, \bar{T}].
\]
For \( i = 0, 1, \ldots, N \), let
\[
\ddot{x}_i(t) = x_i(t) - x_0(t) \quad \ddot{x}_i(t) = \dddot{x}_i(t) = 0, \quad \forall t \in [t_s, t_{s+1})
\]
\[\text{Then, according to (1), (2), (10) and (11), for } i = 1, \ldots, N, \ \text{we have}
\]
\[
\dot{x}_i(t) = \dddot{x}_i(t) - \dot{x}_0(t)
\]
\[
=A x_i(t) + BK \sum_{j=0}^{N} \bar{a}_{ij} (x_j(t_s) - x_i(t_s)) - A x_0(t)
\]
\[
=Ax_i(t) + BK \sum_{j=0}^{N} \bar{a}_{ij} (x_j(t_s) - x_i(t_s))
\]
\[\text{where } \mu_1 \text{ and } \mu_2 \text{ are any positive real numbers.}
\]

Before giving our main result, we introduce some notation.

Let \( x = (x_1, \ldots, x_N) \) and \( \ddot{x} = (\dddot{x}_1, \ldots, \dddot{x}_N) \). Then we further put (12) into the following compact form:
\[
\dot{x}(t) = (I_N \otimes A) \ddot{x}(t) - (H \otimes BK) \dddot{x}(t)
\]
\[
= (I_N \otimes A - (H \otimes BK)) \dddot{x}(t)
\]
\[
\text{with } \ddot{x}(t) \text{ and } \dddot{x}(t) \text{ for all } t \in [t_s, t_{s+1}).
\]

Then we give the following result.

**Theorem 4.1:** Under Assumptions 2.1 and 4.1 let \( 0 < T \leq \bar{T} < \frac{1}{\lambda_M} \). Then the distributed sampled-data state feedback control law (13) with \( K = \alpha_1 B^T P \) and \( T_s \in [T, \bar{T}] \) for all \( s \in \mathbb{N} \) solves the leader-following consensus problem for the multi-agent system composed of (11) and (12).

**Proof:** First, note that, if \( \ddot{x}(t_s) = 0 \), then, according to (13), \( \ddot{x}(t) = 0 \) for all \( t \geq t_s \). Thus the problem is obviously solved.
Second, consider the case $\bar{x}(t_s) \neq 0$. Let

$$V(\bar{x}) = \bar{x}^T (D \otimes P) \bar{x}. \quad (16)$$

Then we have

$$\lambda_m \| \bar{x} \|^2 \leq V(\bar{x}) \leq \lambda_M \| \bar{x} \|^2. \quad (17)$$

Note that $K = \alpha_1 B^TP$. Then, along the trajectory of the closed-loop system \textsuperscript{(13)}, for any $t \in [t_s, t_{s+1})$, we have

$$\dot{V}(\bar{x}) = 2\bar{x}^T(t)(D \otimes P) \bar{x}(t) \leq 2\bar{x}^T(t)(D \otimes P)((I_N \otimes A - (H \otimes BK))\bar{x}(t) - (H \otimes BK)\bar{x}(t)) \leq \bar{x}^T(t)(D \otimes (PA + AT)P)\bar{x}(t) - \alpha_1(DH + H^TD) \otimes PBB^TP)\bar{x}(t) \quad (18)$$

$$\leq -\mu_1 D(PBB^TP)\bar{x}(t) + 2\alpha_1 \| \bar{x}(t) \|^2 \| DH \| \| PBB^TP \| \| \bar{x}(t) \| \leq -\mu_2 \bar{x}^T(t)(D \otimes A)x(t) + 2\alpha_2 \| \bar{x}(t) \|^2 \| \bar{x}(t) - \bar{x}(t) \| \leq -d_{m\mu_2} \| \bar{x}(t) \|^2 + \frac{d_{m\mu_2}}{2} \| \bar{x}(t) \|^2 + \frac{\alpha_3}{d_\mu_2} \| \bar{x}(t) - \bar{x}(t) \|^2 \leq -\frac{d_{m\mu_2}}{2} \| \bar{x}(t) \|^2 + \frac{\alpha_3}{d_\mu_2} \| \bar{x}(t) - \bar{x}(t) \|^2. \quad (19)$$

Based on \textsuperscript{(13)} and \textsuperscript{(17)}, for any $t \in [t_s, t_{s+1})$, we have

$$\| \dot{\bar{x}}(t) \| \leq \| (I_N \otimes A)\bar{x}(t) - (H \otimes BK)\bar{x}(t) \| \leq \| A \| \| \bar{x}(t) \| + \alpha_1 \| H \otimes BB^TP \| \| \bar{x}(t) \| \leq \frac{\| A \| \sqrt{V(\bar{x}(t))}}{\lambda_m} \| \sqrt{\frac{\| \bar{x}(t) \|}{\lambda_m}} \| + \frac{\alpha_1 \| H \otimes BB^TP \|}{\sqrt{\lambda_m}} \| \sqrt{\frac{\| \bar{x}(t) \|}{\lambda_m}} \| \leq \frac{\| A \| + \alpha_1 \| H \otimes BB^TP \|}{\sqrt{\lambda_m}} \| \sqrt{\frac{\| \bar{x}(t) \|}{\lambda_m}} \| \leq \frac{\| A \| + \alpha_1 \| H \otimes BB^TP \|}{\sqrt{\lambda_m}} \sqrt{V_M(t)} \quad (20)$$

where $V_M(t) = \max_{\tau \in [t_{s+1}, t]} V(\bar{x}(\tau))$ for any $t \in [t_s, t_{s+1})$. Note that $t_{s+1} - t_s = T_s \leq \bar{T}$ for any $s \in \mathbb{N}$. Then, for any $t \in [t_s, t_{s+1})$, we have

$$\| \bar{x}(t) - \bar{x}(t_s) \| \leq \int_{t_s}^{t} \| \dot{\bar{x}}(\tau) \| d\tau \leq \int_{t_s}^{t} \| A \| + \alpha_1 \| H \otimes BB^TP \| \sqrt{V_M(t)} d\tau \leq \frac{\| A \| + \alpha_1 \| H \otimes BB^TP \|}{\sqrt{\lambda_m}} \sqrt{V_M(t)} (t - t_s) \quad (21)$$

which further implies, for any $t \in [t_s, t_{s+1})$,

$$\| \bar{x}(t) - \bar{x}(t_s) \|^2 \leq \frac{\| A \|^2 + \alpha_1^2 \| H \otimes BB^TP \|^2}{\lambda_m} \bar{T} V_M(t) \quad (22)$$

According to \textsuperscript{(17)}, \textsuperscript{(18)} and \textsuperscript{(21)}, for any $t \in [t_s, t_{s+1})$, we have

$$\dot{V}(\bar{x}(t)) \leq -\frac{d_{m\mu_2}}{2 \lambda_m} \| \bar{x}(t) \|^2 + \frac{\alpha_3}{d_\mu_2} \| \bar{x}(t) - \bar{x}(t) \|^2 \quad (23)$$

If \textsuperscript{(23)} is not true, then there exists a time instant $t' \in [t_s, t_{s+1})$ such that $V(\bar{x}(t')) > V(\bar{x}(t_s))$. Note that, according to \textsuperscript{(13)},

$$\dot{V}(\bar{x}(t_s)) \leq -\frac{d_{m\mu_2}}{2} \| \bar{x}(t_s) \|^2 < 0, \quad \forall \bar{x}(t_s) \neq 0 \quad (24)$$

which implies that $V(\bar{x}(t))$ will decrease in a short time starting from $t_s$. Therefore, there exists another time instant $t'' \in [t_s, t']$ such that

$$V(\bar{x}(t'')) = V(\bar{x}(t_s))$$

$$\dot{V}(\bar{x}(t'')) > 0$$

$$V(\bar{x}(t)) \leq V(\bar{x}(t'')), \forall t \in [t_s, t''] \quad (25)$$

Note that $\bar{T} < \sqrt{\frac{\lambda_m}{\alpha_2}}$. Then, according to \textsuperscript{(22)} and the third inequality of \textsuperscript{(25)}, we have

$$\dot{V}(\bar{x}(t'')) \leq -c_1 V(\bar{x}(t'')) + c_2 \bar{T}^2 V(\bar{x}(t'')) < 0 \quad (26)$$

which contradicts the second inequality of \textsuperscript{(25)}. Thus we conclude that \textsuperscript{(23)} is true. Then, from \textsuperscript{(22)}, for any $t \in [t_s, t_{s+1})$,

$$\dot{V}(\bar{x}(t)) \leq -c_1 V(\bar{x}(t)) + c_2 \bar{T}^2 V(\bar{x}(t_s)) \quad (27)$$

Since $t_{s+1} - t_s = T_s \geq \bar{T}$ for all $s \in \mathbb{N}$, and $c_2 \bar{T}^2 < c_1$, by Lemma 3.1, we have the limit $\lim_{t \to \infty} V(\bar{x}(t)) = 0$, which implies $\lim_{t \to \infty} \| \bar{x}(t) \| = 0$.

Thus the proof is complete. \hfill \Box

**Remark 4.2:** In fact, it is possible to design a control law and an upper bound independent of the specific connection information of the graph. Since the number of all graphs with a finite number of nodes is finite, we can calculate all possible $H$ and hence $D$ off-line. For this purpose, let $\mathcal{J} = \{1, \ldots, N_0\}$, where $N_0$ is the total number of all connected graphs with the number of the nodes equal to $N + 1$. Then, all the parameters
defined in (15) can also be calculated off-line as follows:
\[ d_m = \min_{j \in \mathcal{J}} \{\lambda_{\min}(D_j)\}, \quad \lambda_m = \min_{j \in \mathcal{J}} \{\lambda_{\min}(D_j \otimes P)\} \]
\[ d_M = \max_{j \in \mathcal{J}} \{\lambda_{\max}(D_j)\}, \quad \lambda_M = \max_{j \in \mathcal{J}} \{\lambda_{\max}(D_j \otimes P)\} \]
\[ \lambda_1 = \min_{j \in \mathcal{J}} \{\lambda_{\min}(D_j H_j + H_j^T D_j)\}, \quad \alpha_1 = \frac{\mu_1 d_M}{\lambda_1} \]
\[ \alpha_2 = \max_{j \in \mathcal{J}} \{2\alpha_1 \|D_j H_j\| \|P B B T P\|\}, \quad \alpha_3 = \frac{\alpha_2^2}{2 d_m \mu_2} \]
\[ \alpha_4 = \max_{j \in \mathcal{J}} \left\{ \left(\|A\| + \alpha_1 \|H_j \otimes B B T P\|\right)^2 / \lambda_m \right\} \]
\[ c_1 = \frac{d_m \mu_2}{2 \lambda M}, \quad c_2 = \alpha_3 \alpha_4. \]

With the parameters given by (28), the control law (3) applies to all connected graphs with the number of the nodes equal to \( N + 1 \). Nevertheless, it should be noted that the parameters defined in (28) are more conservative than those defined in (15).

V. SWITCHING NETWORK CASE

In this section, we will further consider the leader-following consensus problem for the multi-agent system composed of (1) and (2) under switching networks by a distributed sampled-data state feedback control law (3) with \( K = \alpha_1 B^T P \) and \( T_s \in [T, T] \) for all \( s \in \mathbb{N} \) solves the leader-following consensus problem for the multi-agent system composed of (1) and (2).

**Theorem 5.1:** Under Assumptions 4.1 and 5.1 let \( 0 < T \leq \hat{T} < \frac{1}{2c_1} \). Then the distributed sampled-data state feedback control law (3) with \( K = \alpha_1 B^T P \) and \( T_s \in [T, T] \) for all \( s \in \mathbb{N} \) solves the leader-following consensus problem for the multi-agent system composed of (1) and (2).

**Proof:** The proof is similar to the proof of Theorem 4.1. Choose the same function \( V(\bar{x}) = \bar{x}^T (D \otimes P) \bar{x} \) as in (16). Note that, under the switching digraph \( \mathcal{G}_{\sigma(t)} \), \( V(\bar{x}(t)) \) is still continuous. However, the time derivative of \( V(\bar{x}(t)) \) is discontinuous not only at the sampling time instants but also at the switching time instants. Nevertheless, with \( \lambda_1, \alpha_2 \) and \( \alpha_4 \) being defined in (30), the time derivative of \( V(\bar{x}(t)) \) satisfies

\[ \dot{V}(\bar{x}) = \bar{x}^T (D \otimes (PA + AP^T)) \bar{x} \\
- \alpha_1 (DH_{\sigma(t)} + H_{\sigma(t)}^T D) \otimes P B B T P \bar{x}(t) \\
- 2\alpha_1 \bar{x}^T (D H_{\sigma(t)} \otimes P B B T P) \bar{x}(t) \\
\leq \bar{x}^T (D \otimes (PA + AP^T)) \bar{x}(t) \\
+ 2\alpha_1 \|\bar{x}(t)\| \|D H_{\sigma(t)}\| \|P B B T P\| \|\bar{x}(t)\| \\
\leq - \mu_2 \bar{x}^T (D \otimes I_n) \bar{x}(t) + \alpha_2 \|\bar{x}(t)\| \|\bar{x}(t)\| \\
\leq \frac{d_m \mu_2}{2} \|\bar{x}(t)\|^2 + \alpha_3 \|\bar{x}(t)\|^2 \|\bar{x}(t)\| - \bar{x}(t) \]

for all \( t \in [t_s, t_{s+1}) \). The remaining part of the proof is the same as that in the proof of Theorem 4.1.

**Remark 5.2:** The upper bound for the sampling intervals given in Theorems 4.1 and 5.1 may be conservative. In practice, even if the sampling intervals are greater than the given upper bound, the problem may still be solved by the proposed control law.

**Remark 5.3:** References [23] and [5] also studied the sampled-data consensus problem, where the communication graph condition is weaker than Assumption 5.1 and the time delay issue was considered in [23]. Nevertheless, there are at least four main differences or novelties between the results in this paper and the results in [23] and [5]. First, references [23] and [5] considered the sampled-data leaderless consensus problem, whereas we consider the sampled-data leader-following consensus problem. Second, references [23] and [5] considered single integrator systems and double integrator systems, respectively, whereas we consider a class of general linear multi-agent systems, which contains single integrator systems and double integrator systems as special cases. Third, in [23] and [5], the problem was transformed into the asymptotic stability problem of a discrete-time system, whereas we develop a new technical lemma to analyze the stability of the piecewise-continuous closed-loop system directly. Finally, we give an explicit upper bound for the sampling intervals that guarantees the stability and performance of the closed-loop system as long as all the sampling intervals are smaller than this upper bound.
VI. AN EXAMPLE

In this section, we consider a linear multi-agent system with the leader system as follows:

\[
\dot{x}_0 = \begin{bmatrix} -0.38 & 0.72 \\ -0.68 & 0.42 \end{bmatrix} x_0
\]

(32)

and the four follower systems as follows:

\[
\dot{x}_i = \begin{bmatrix} -0.38 & 0.72 \\ -0.68 & 0.42 \end{bmatrix} x_i + \begin{bmatrix} 0.26 \\ 0.31 \end{bmatrix} u_i
\]

(33)

for \( i = 1, 2, 3, 4 \). Clearly, Assumption 2.1 is satisfied.

A. Static Network Case

Consider the static communication graph \( \tilde{G} \) in Figure 1 where node 0 is associated with the leader system, and the other nodes are associated with the follower systems. It is easy to see that Assumption 4.1 is satisfied and

\[
H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\]

Choose \( D = I_4 \). Then it is easy to check that \( DH + HTD \) is positive definite. Choose \( \mu_1 = \mu_2 = 1 \). Then solving (14) for all \( \sigma \) gives

\[
P = \begin{bmatrix} 7.2138 & -3.6897 \\ -3.6897 & 6.3388 \end{bmatrix}
\]

Following the procedures described in Section IV, we obtain \( \bar{T} = 0.0186 \) and \( K = \begin{bmatrix} 0.8874 & 1.2195 \end{bmatrix} \). We further choose \( \overline{T} = 0.001 \). Then, by Theorem 4.1, the distributed sampled-data state feedback control law (3) with \( K = \begin{bmatrix} 0.8874 & 1.2195 \end{bmatrix} \) and \( T_s \in [0.001, 0.0186] \) for all \( s \in \mathbb{N} \) asymptotically, and thus the tracking errors of all agents approach zero asymptotically. Therefore, the leader-following consensus is achieved satisfactorily.

B. Switching Network Case

Consider the switching communication graph \( \tilde{G}_{\sigma(t)} \), where

\[
\sigma(t) = \begin{cases} 
1, & \text{if } lT_0 \leq t < (l + \frac{2}{3})T_0 \\
2, & \text{if } (l + \frac{2}{3})T_0 \leq t < (l + 1)T_0 
\end{cases}
\]

for \( l = 0, 1, 2, \ldots \) and \( T_0 = 1 \), and the two communication graphs are described in Figure 4. It is easy to obtain

\[
H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix}
\]

Choose \( D = I_4 \). Then it is easy to check that \( DH_1 + H_1^TD \) and \( DH_2 + H_2^TD \) are both positive definite. Thus Assumption 5.1 is also satisfied.
the same initial states as those for the static network case. The trajectories and tracking errors of all agents under the switching communication graph $\mathcal{G}_{\sigma(t)}$ are shown in Figure 5 and Figure 6 respectively. As expected, the trajectories of all follower systems approach the trajectory of the leader system asymptotically, and thus the tracking errors of all agents approach zero asymptotically. Therefore, the leader-following consensus is achieved satisfactorily.

VII. CONCLUSION

In this paper, we have studied the sampled-data leader-following consensus problem for a class of general linear multi-agent systems. Both the static network case and the switching network case have been studied. It has been shown that the problem can be solved by the proposed distributed sampled-data control law if all the sampling intervals are smaller than an explicitly given threshold.

It would be interesting to further consider the sampled-data leader-following consensus problem for linear multi-agent systems with time delay, parameter uncertainties, and to weaken the condition on communication topologies. The results of this paper and some existing results in [8], [23], [28] may shed some light on this future work.

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