JACOB’S LADDERS AND THE OSCILLATIONS OF THE FUNCTION $|\zeta(1/2 + it)|^2$ AROUND ITS MEAN-VALUE; LAW OF THE ALMOST EXACT EQUALITY OF CORRESPONDING AREAS

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ABSTRACT. The oscillations of the function $Z^2(t)$, $t \in [0, T]$ around the main part $\sigma(T)$ of its mean-value are studied in this paper. It is proved that an almost equality of the corresponding areas holds true. This result cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic.

1. INTRODUCTION

1.1. The Titchmarsh-Kober-Atkinson (TKA) formula

$$\int_0^\infty Z^2(t)e^{-2\delta t}dt = \frac{c - \ln(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^N c_n \delta^n + O(\delta^{N+1})$$

(see [17], p. 131) remained as an isolated result for the period of 56 years. We have discovered (see [5]) the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\mu[x(T)]t}dt = \int_0^T Z^2(t)dt$$

in which the essence of the TKA formula is encoded. Namely, we have shown in [5] that the following almost exact expression for the Hardy-Littlewood integral

$$\int_0^T Z^2(t)dt = \frac{\varphi(T)}{2} \ln \frac{\varphi(T)}{2} + (c - \ln(2\pi)) \frac{\varphi(T)}{2} + c_0 + O \left( \frac{\ln T}{T} \right)$$

takes place, where $\varphi(T)$ is the Jacob’s ladder, i.e. an arbitrary solution to the nonlinear integral equation (1.2).

Remark 1. Our formula (1.3) for the Hardy-Littlewood integral

$$\int_1^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \int_1^T Z^2(t)dt$$

(see [3], pp. 122, 151-156).
1.2. Let us remind that

(A) The Good’s Ω - theorem (see [2]) implies for the Balasubramanian formula

(1.5) \[ \int_0^T Z^2(t)dt \sim T \ln T + (2c - 1 - \ln 2\pi)T + R(T), \quad R(T) = O(T^{1/3+\epsilon}) \]

that

(1.6) \[ \limsup_{T \to \infty} |R(T)| = +\infty, \]

i.e. the error term in (1.5) is unbounded at \( T \to \infty \).

(B) In the case of our formula (1.3) the error term definitely tends to zero

(1.7) \[ \lim_{T \to \infty} r(T) = 0; \quad r(T) = O\left(\frac{\ln T}{T}\right), \]

i.e. our formula is almost exact (see [4]).

Remark 2. In this paper the geometric interpretation of (1.6) and (1.7) is obtained.

1.3. For the mean-value of the function \[ |\zeta\left(\frac{1}{2} + it\right)|^2 = Z^2(t), \]

where

\[ Z(t) = e^{\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \quad \vartheta(t) = -\frac{t^2}{2} \ln \pi + \text{Im} \ln \Gamma\left(\frac{1}{4} + \frac{t}{2}\right), \]

we obtain from (1.3)

(1.8) \[ \frac{1}{T} \int_0^T Z^2(t)dt = \frac{\varphi(T)}{2T} \ln \frac{\varphi(T)}{2} + (c - \ln 2\pi)\frac{\varphi(T)}{2T} + \frac{c_0}{T} + O\left(\frac{\ln T}{T}\right). \]

Let

(1.9) \[ \sigma(T) = \frac{\varphi(T)}{2T} \ln \frac{\varphi(T)}{2} + (c - \ln 2\pi)\frac{\varphi(T)}{2T} + \frac{c_0}{T} \]

denote the main part of the mean-value (1.8). In this paper the oscillation of the values of the function \( Z^2(t), \quad t \in [0, T] \) around the main part \( \sigma(T) \) of its mean-value are studied.

Remark 3. The main result of this paper is the following statement: the areas of the figures corresponding to the parts of the graph of the function \( Z^2(t), \quad t \in [0, T] \) given by inequalities \( Z^2(t) \geq \sigma(T) \) and \( Z^2(t) \leq \sigma(T) \), respectively, are almost exactly equal.

This paper is a continuation of the series [5]-[16].

2. Result

2.1. Let (see (1.9))

(2.1) \[ S^+(T) = \{ t : Z^2(t) \geq \sigma(T), \quad t \in [0, T] \}, \]

\[ S^-(T) = \{ t : Z^2(t) < \sigma(T), \quad t \in [0, T] \} \]

and

(2.2) \[ \Pi^+(T) = \{ (t, y) : \sigma(T) \leq y \leq Z^2(t), \quad t \in S^+(T) \}, \]

\[ \Pi^-(T) = \{ (t, y) : Z^2(t) \leq y \leq \sigma(t), \quad t \in S^-(T) \}, \]
i.e. $\Pi^+$ is the figure that corresponds to the parts of the graph of $y = Z^2(t)$, $t \in [0, T]$ lying above the segment $y = \sigma(T)$ and similarly $\Pi^-$ corresponds to the parts of the graph lying under that segment. Let $m\{\Pi^+(T)\}$, $m\{\Pi^-(T)\}$ denote measures of corresponding figures, i.e.

\[
m\{\Pi^+(T)\} = \int_{S^+(T)} \{Z^2(t) - \sigma(T)\} dt,
\]
\[
m\{\Pi^-(T)\} = \int_{S^-(T)} \{\sigma(T) - Z^2(t)\} dt.
\]

(2.3)

The following theorem holds true.

**Theorem.** First of all, we have the formula

\[
m\{\Pi^+(T)\} = m\{\Pi^-(T)\} + \mathcal{O}\left(\frac{\ln T}{T}\right)
\]

(see (1.3), (1.9), (2.1)-(2.3)). Next, the structure of the formula (2.4) is as follows: there are the functions $\eta_1(T)$, $\eta_2(T)$ that the following formulae hold true, and

\[
m\{\Pi^+(T)\} = \frac{1 + o(1)}{2\pi^2} \frac{T \ln^4 T}{\eta_1 - \eta_2} - \frac{\eta_2}{\eta_1 - \eta_2} \mathcal{O}\left(\frac{\ln T}{T}\right),
\]
\[
m\{\Pi^-(T)\} = \frac{1 + o(1)}{2\pi^2} \frac{T \ln^4 T}{\eta_1 - \eta_2} - \frac{\eta_1}{\eta_1 - \eta_2} \mathcal{O}\left(\frac{\ln T}{T}\right)
\]

hold true, and

\[
AT^{2/3} \ln^4 T < m\{\Pi^+(T)\}, m\{\Pi^-(T)\} < AT \ln T.
\]

In addition to (2.6): on the Lindelöf hypothesis

\[
m\{\Pi^+(T)\}, m\{\Pi^-(T)\} > A(\epsilon)T^{1-\epsilon},
\]

and on Riemann hypothesis

\[
m\{\Pi^+(T)\}, m\{\Pi^-(T)\} > T^{1 - \frac{\epsilon}{\ln T}}.
\]

**Corollary.** We have by (2.5), (2.6)

\[
\eta_1(T) - \eta_2(T) > A \ln^3 T.
\]

\[
\lim_{T \to \infty} \frac{m\{\Pi^+(T)\} - m\{\Pi^-(T)\}}{T} = 0
\]

follows then we have the almost exact equality of the areas $m\{\Pi^+(T)\}$ and $m\{\Pi^-(T)\}$.

**Remark 4.** Since from (2.4)

\[
\lim_{T \to \infty} \frac{m\{\Pi^+(T)\} - m\{\Pi^-(T)\}}{T} = 0
\]

(2.10)

follows then we have the almost exact equality of the areas $m\{\Pi^+(T)\}$ and $m\{\Pi^-(T)\}$.

2.2. In the case of the Balasubramanian formula (1.5) we have (comp. (1.3), (1.9))

\[
\sigma_1(T) = \ln T + 2\epsilon - 1 - \ln 2\pi.
\]

Let $S_1^+(T), S_1^-(T), \Pi_1^+(T), \Pi_1^-(T), m\{\Pi_1^+(T)\}, m\{\Pi_1^-(T)\}$ correspond to $\sigma_1(T)$ similarly to (2.1), (2.3). Then from (1.3) we obtain

\[
m\{\Pi_1^+(T)\} = m\{\Pi_1^-(T)\} + \mathcal{O}(T^{1/3+\epsilon}), \quad T \to \infty,
\]

and (see (1.6))

\[
\lim_{T \to \infty} \sup |m\{\Pi_1^+(T)\} - m\{\Pi_1^-(T)\}| = +\infty.
\]

**Remark 5.** The following holds true:
3. Proof of Theorem

3.1. We obtain from (1.3), (2.1)
\[
\int_{S^+(T)} (Z^2(t) - \sigma(T)) \, dt + \int_{S^-(T)} (Z^2(t) - \sigma(T)) \, dt = O \left( \frac{\ln T}{T} \right)
\]
and from (3.1) by (2.3) the formula
\[
(3.2) \quad m\{\Pi^+(T)\} - m\{\Pi^-(T)\} = O \left( \frac{\ln T}{T} \right)
\]
follows, i.e. (2.4).

3.2. Next, from the Ingham formula (see [4], p. 277, [17], p. 125)
\[
(3.3) \quad \int_0^T Z^4(t) \, dt = \frac{1}{2\pi^2} T \ln^4 T + O(T \ln^3 T)
\]
we obtain (see (1.3))
\[
(3.4) \quad \int_0^T \{Z^4(t) - \sigma^2(T)\} \, dt = \frac{1}{2\pi^2} T \ln^4 T - T \sigma^2(T) + O(T \ln^3 T).
\]
Since \((\varphi(T) \sim T)\)
\[
T \sigma^2(T) = O \left( \frac{\varphi^2(T)}{T} \ln^2 \frac{\varphi(T)}{2} \right) = O(T \ln^2 T),
\]
then from (3.3) the formula
\[
(3.5) \quad \int_0^T \{Z^4(t) - \sigma^2(T)\} \, dt = \frac{1 + o(1)}{2\pi^2} T \ln^4 T
\]
follows.

3.3. Since \(Z^4(t) - \sigma^2(T) = (Z^2 - \sigma)(Z^2 + \sigma)\) and \(Z^2(t) - \sigma(T)\) is always of the same sign on \(S^+(T)\) and on \(S^-(T)\), respectively, then from (3.5) we obtain (see (2.3))
\[
(3.6) \quad \eta_1(T)m\{\Pi^+(T)\} - \eta_2(T)m\{\Pi^-(T)\} = \frac{1 + o(1)}{2\pi^2} T \ln^4 T,
\]
where \(\eta_1 = \eta_1(T), \eta_2 = \eta_2(T)\) are the mean-values of \(Z^2(t) + \sigma(T)\) relatively to the values of the functions \(Z^2(t) - \sigma(T)\) and \(\sigma(T) - Z^2(t)\), respectively on the sets \(S^+(T)\) and \(S^-(T)\), respectively. It is clear that
\[
(3.7) \quad A \ln T < \eta_1(T), \eta_2(T) < AT^{1/3}
\]
(see (1.9); \(|Z(t)| < t^{1/6}\)). Next, \(\eta_1(T) \neq \eta_2(T)\) is also true. Since if \(\eta_1 = \eta_2\) then by (3.2), (3.6), (3.7) we would have the contradiction. Hence, from the simple system of linear equations (3.2), (3.6) we obtain (2.5).
3.4. Since
\[0 < \eta_1 - \eta_2 < \eta_1 < AT^{1/3}\]
(see (2.5), (3.7)) then we obtain from (2.5) the lower estimates in (2.6). Next we have (see (1.9), (2.3)
\[m\{\Pi^+(T)\}, m\{\Pi^-(T)\} < \int_0^T \{Z^2(t) - \sigma(T)\} dt < AT \ln T\]
i.e. the upper estimates in (2.6) hold true.

3.5. Following the Lindelöf and the Riemann conjectures the estimates
\[Z^2(t) < A(\epsilon)t^\delta, \quad t = \frac{A}{\ln \ln t}\]
take place correspondingly and then the conditional estimates (2.7) and (2.8) follow.

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