Müntz-type theorems on the half-line with weights

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Abstract
We consider the linear span $S$ of the functions $t^{a_k}$ (with some $a_k > 0$) in weighted $L^2$ spaces, with rather general weights. We give one necessary and one sufficient condition for $S$ to be dense. Some comparisons are also made between the new results and those that can be deduced from older ones in the literature.

1 Introduction
The first "if and only if" solution to a problem of S. N. Bernstein [4] was given by Ch. H. Müntz [21]:

Theorem A
Let $0 = \lambda_0 < \lambda_1 < \ldots$ be an increasing sequence of real numbers. The linear subspace span $\{t^{\lambda_k} : k = 0, 1, \ldots\}$ is dense in $C([0, 1])$, if and only if $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$.

This classical result was first proved in $L_2[0, 1]$ and then extended to $C[0, 1]$, as stated above. Also, it was stated only for increasing sequences $\lambda_k$. Subsequently, this theorem has had several different proofs and generalizations, and there are several surveys in this topic (see for instance the papers of J. Almira and A. Pinkus [11, 23]).

On $C[0, 1]$ and $L_p(0, 1)$, "full Müntz theorems", i.e. theorems with rather general exponents, were later proved by eg. P. Borwein, T. Erdélyi, W. B. Johnson and V. Operstein (17, 18, 12, 22). Versions of Müntz’s theorem on

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compact subsets of positive measure \([8], [9]\), and on countable compact sets \([2]\) were also proved. Further results can be found for instance in the monographs of P. Borwein, T. Erdélyi \([10]\), and B. N. Khabibullin \([16]\).

In this paper we are interested in Müntz-type theorems on \((0, \infty)\). Several papers were written in the ’40s on the completeness of the set \(\{t^{\lambda_k}e^{-t}\}\) in \(L_2(0, \infty)\) (see eg. \([14], [5], [6]\)). In particular, we will use some ideas of W. Fuchs. His theorem is the following:

**Theorem B**

Let \(a_k\) be positive numbers, such that

\[
a_{k+1} - a_k > c > 0 \quad (k = 1, 2, \ldots),
\]

and let \(\log \Psi(r) = 2 \sum_{a_k < r} \frac{1}{a_k} \) if \(r > a_1\), and \(\log \Psi(r) = \frac{2}{a_1}\) if \(r \leq a_1\). Then \(\{e^{-t^a_k}\}\) is complete in \(L_2(0, \infty)\), if and only if

\[
\int_1^\infty \frac{\Psi(r)}{r^2} = \infty
\]

A. F. Leontev \([18]\) and G. V. Badalyan \([3]\) proved similar theorems with more general weights (the weight being \(e^{-t}\) in the above theorem). In 1980, by the Hahn-Banach theorem technique, R. A. Zalik \([27]\) proved a Müntz type theorem on the half-line with weights \(|w| \leq c \exp(-|\log t|^a)\) \((a > 0)\). In 1996 Kroó and Szabados \([17]\) also had a related result on \((0, \infty)\).

In Theorem 1 and Theorem 2 below we will prove Müntz-type theorems on the half-line with more general weights, which generalize all the results mentioned above.

Closely related to our topic (by a \(\log t\) substitution) are the results on the whole real line for exponential systems. The basic paper in this respect was written by P. Malliavin \([20]\), and by this tool there are some nice generalizations of the above mentioned results, for instance by B. V. Vinnitskii, A. V. Shapovalovskii \([25]\), by G. T. Deng \([11]\), and by E. Zikkos \([28]\).

# 2 Definitions, Results

Let us begin with a rather general definition. Some specific examples are given subsequently.

**Definition 1** We say that a weight function \(w(t) = \nu(t)\mu(t)\) is admissible on \([0, \infty)\), if \(\nu(t)\) and \(\mu(t)\) are positive and continuous on \((0, \infty)\), \(w^2\) has finite moments, and there is a function \(\gamma\) on \([0, \infty)\), such that

\[
\gamma(t) = \sum_{k=0}^\infty c_k t^{\gamma_k},
\]

where \(c_k > 0\) for all \(k\), and \(0 \leq \gamma_0 < \gamma_1 < \gamma_2 < \ldots\), and there is a \(C_0 > 0\) such that \(\forall t > C_0\)

\[
\frac{1}{w^2(t)} \leq \gamma(t)
\]

(1)
and there is a $C > 1$, such that
\[ \int_0^\infty \gamma \left( \frac{t}{C} \right) w^2(t)dt < \infty. \]  
(2)
Furthermore we require that
\[ \lim_{t \to 0^+} \mu(t) \in (0, \infty), \]  
(3)
and there is an $a > 0$ such that
\[ \int_0^1 \left( \frac{t^a-1}{\nu(t)} \right)^2 < \infty. \]  
(4)

Here and in the followings $C, C_i$ and $c$ are absolute constants, and the value of them will not be the same at each occurrence.

Remark: If $\nu(t) \equiv 1$ (as in Theorem B) then we can choose $a = 1$. Also, it is easy to see that we can always assume $a \geq 1$.

Examples:

- $w(t) = t^\beta e^{-Dt^\alpha}$, where $\beta > -\frac{1}{2}$ and $\alpha > 0$ is admissible, namely it has finite moments, and $\gamma(t) = e^{3Dt^\alpha}$ serves the purpose. When $\beta = 0$ and $D = \alpha = 1$, we get back the original case of Fuchs (Theorem B). When $\beta > -\frac{1}{2}$, $D = \frac{1}{2}$ and $\alpha = 1$, then $w^2$ is a Laguerre weight. When $\beta = 0$, $D > 0$ and $\alpha > 1$, then $w$ is a Freud weight.

Let $w(t) = (4 + \sin t)t^\beta \prod_{k=1}^n e^{-D_k t^\alpha_k}$, and let us assume, that $\beta > -\frac{1}{2}$, and $0 \leq \alpha_1 < \alpha_2 \ldots < \alpha_n$, and $D_n > 0$. Then $w$ is admissible, and $e^{Dt^n}$ is a suitable choice for $\gamma(t)$, if $D$ is large enough. In particular, if $w(t) = t(4 + \sin t)e^{-t}$ then the second derivative of $-\log(\nu(t))$ takes some negative values on $(A, \infty)$ for any $A > 0$. This means that the results of [28] are not applicable in this case.

Definition 2 Let $w$ be a positive continuous weight function with $w^2$ having finite moments. Then define $\varphi(x)$ and $K(x)$ corresponding to $w(x)$ as
\[ \varphi(x) = \left( \int_{0}^{\infty} t^{2x} w^2(t)dt \right)^{\frac{1}{2x}}, \quad x > 0. \]  
(5)
Furthermore let us define another property of a weight function. The classical weight functions, and also our examples above, fulfil this "normality" condition, as we can see later.

Definition 3 Let us call a weight function $w^2$ with finite moments "normal", if the largest zero of the $n^{th}$ orthogonal polynomial $(x_{1,n})$ with respect to $w^2$, can be estimated as:

\[ x_{1,n} \leq e^{cn}, \]  
where $c = c(w)$ is a positive constant independent of $n$. 

3
Remark:

In the cases of Laguerre and Freud weights $x_{1,n} \leq cn^\lambda$, where $\lambda = \lambda(w)$ is a positive constant depending on the weight function, moreover the same estimation is valid for a more general classes of weights on the real axis (see [19] p. 313, Th. 11.1). As an application of the result of A. Markov ([24] p. 115, Th. 6.12.2), we can get a similar estimation for the examples above; for instance $w(x) = x^\gamma e^{-x^\alpha}$, there is a $\beta > 0$ such that with $W(x) = x^\beta e^{-x}$, the quotient $\frac{W}{w}$ is increasing on $(0, \infty)$; if $w(x) = x(1 + \sin x)e^{-x}$, then the corresponding $W$ can be $W(x) = x^2 e^{-\frac{x}{2}}$.

Definition 4 Let $\{a_k\}_{k=1}^{\infty}$ be positive numbers in increasing order. We define (as in [14] and Theorem B above)

$$m(r) = \begin{cases} \frac{1}{a_1}, & \text{if } 0 \leq r \leq a_1 \\ \sum_{a_k < r} \frac{1}{a_k}, & \text{if } r > a_1 \end{cases} \quad (6)$$

and let

$$\Psi(r) = e^{2m(r)}. \quad (7)$$

Let us also introduce the following notations:

Notation:

Let $w$ be a positive continuous weight function, and let us define the weighted $L^2$ space as $L^2_w(0, \infty) = \{ f | fw \in L^2(0, \infty) \}$, and $\| f \|_{L^2_w} = \| fw \|_{L^2(0, \infty)}$.

$S = \text{span}\{ t^{a_k} : k = 1, 2, \ldots \} \quad (8)$

with $0 < a_1 < a_2 < \ldots$.

We are now in position to state the main results of this note (the proofs will follow in the next Section).

Theorem 1 Let $w$ be an admissible and normal weight function on $[0, \infty)$. If there exists a monotone increasing function $f$ on $[0, \infty)$, such that for all $0 < x \leq r$

$$x \log \frac{\Psi(r)}{\varphi(x)} \leq f(r), \quad (9)$$

and

$$\int_1^\infty \frac{f(r)}{r^2} < \infty, \quad (10)$$

then $S$ is incomplete in $L^2_w(0, \infty)$.

This result is then nicely complemented by the following positive result.
Theorem 2 Let $w$ be positive and continuous on $(0, \infty)$, such that $w^2$ has finite moments. Let us suppose that there is a constant $d > 0$ such that
\[ a_{k+1} - a_k > d \]  
(11)
If there exists a monotone increasing function $h$ on $[0, \infty)$, for which
\[ C < \frac{h(r)}{h(r_1)} < D, \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{r_1} \leq 2 \]  
(12)
with some $0 < C, D$, and there are $\alpha, c, C > 0$, such that for all $0 < x \leq r$
\[ 0 < h(r) \leq C \frac{C_1}{\varphi_\alpha(x)} \Psi_\alpha(r), \]  
(13)
and
\[ \int_1^\infty \frac{h(r)}{r^2} = \infty, \]  
(14)
then $S$ is complete in $L^2_w(0, \infty)$.

Comparing the conditions of the above theorems we conclude the following:

Corollary:
If $w$ is admissible and normal on $(0, \infty)$, and there is a $d$ such that $a_{k+1} - a_k > d > 0$, and $f(r) = ch(r)$, where $h$ has the same properties as in Theorem 2, then $S$ is dense in $L^2_w(0, \infty)$ if and only if $\int_1^\infty \frac{h(r)}{r^2} = \infty$.

Remark:
(1) Let
\[ B_\alpha(r) = \inf_{x \in (0, r)} C_1 \frac{x}{\varphi_\alpha(x)}. \]
Then assuming (11) and (12), if there exists a $0 \leq h(r) \leq cB_\alpha(r)\Psi_\alpha(r)$, for which (14) is valid, then $S$ is dense in $L^2_w(0, \infty)$.

(2) Theorem 2 can be stated also in $L^p_w(0, \infty)$, with $1 \leq p < \infty$, and in $C_w(0, \infty)$ with the same proof. That is, let us define
\[ \varphi_p(x) = \left( \int_0^\infty t^{px} w(t) dt \right)^{\frac{1}{p}}, \quad x > 0, \quad 1 \leq p < \infty \]
and
\[ \varphi_c(x) = \left( \sup_{t>0} t^x w(t) dt \right)^{\frac{1}{x}}, \quad x > 0. \]
Using the standard notations $L^p_w(0, \infty) = \{ f : \|fw\|_{p,(0,\infty)} < \infty \}$, and $C_w(0,\infty) = \{ f \in C(0,\infty) : \lim_{t \to +\infty} f(t)w(t) = 0 \}$, we can formulate the following theorem:
Theorem 3 Let \( w \) be positive and continuous on \((0, \infty)\), and let us assume that \( t^xw(t) \in L^p_w(0, \infty) \) in the \( L^p_w \)-case, and that for all \( a > 0 \) \( \lim_{t \to \infty} t^xw(t) = 0 \) in the \( C_w \)-case. Furthermore let \( \{a_k\} \) be a sequence of positive numbers for which (11) is satisfied. If there is a monotone increasing function \( h \) on \((0, \infty)\) with the properties (12) and (14), and for which there are \( \alpha, C, c > 0 \), such that for all \( 0 < x \leq r \)

\[
0 < h(r) \leq C \frac{cx}{r^p(e(x)} \Psi^\alpha(r),
\]

then \( S \) is complete in \( L^p_w(0, \infty) \)/in \( C_w, (0, \infty) \).

(3) If \( B_\alpha(r) > B > 0 \) (\( \forall r \geq 1 \)), then \( h(r) = B \Psi^\alpha(r) \). This is the situation when \( w(t) = e^{-Dt^\alpha} \). Furthermore with suitable \( D \) and \( \alpha \inf_{x \in (0, r)} \frac{x}{\varphi^\alpha(x)} > B > 0 \). In this case

\[
K(x) = \int_0^\infty t^x e^{-2Dt^\alpha} dt = \frac{1}{\alpha(2D)^{\frac{2\alpha+1}{2\alpha}}} \Gamma\left(\frac{2x+1}{\alpha}\right)
\]

By Stirling’s formula (see eg. [15])

\[
\frac{x}{\varphi^\alpha(x)} = \left(\frac{\sqrt{\pi} \left(2x+1\right)^{\frac{2\alpha+1}{2\alpha}} e^{-\frac{2x+1}{2\alpha}} e^{\frac{2\pi+1}{2\alpha}}}{\alpha(2D)^{\frac{2\alpha+1}{2\alpha}}}\right)^{\frac{1}{x}} = (\ast),
\]

where \( J \) is the Binet function. For \( x > 0 \) we have \( 0 < J(x) < \frac{1}{12\pi} \). That is,

\[
(\ast) \geq \frac{2De\alpha x}{2x+1} \left(\frac{2De\alpha}{2x+1} \left(\frac{\alpha(2x+1)}{2\pi}\right)^\frac{1}{x} \frac{1}{e^{\frac{x^2}{12(2x+1)^2}}}\right)^{\frac{1}{x}} = b(D, \alpha, x)
\]

and \( b(D, \alpha, x) \) tends to \( De\alpha \) when \( x \) tends to infinity, and if

\[
C(\alpha, D) = \frac{\sqrt{2De\alpha}}{e^{\frac{\alpha}{12}(2x+1)^2}} > 1 \quad \text{then } \lim_{x \to 0^+} \frac{x}{\varphi^\alpha(x)} = \infty. \quad \text{(In the case of Fuchs,} \quad \alpha = D = 1, C(\alpha, D) > 1.)
\]

(4) For \( \alpha = D = 1 \), \( h(r) = f(r) = \Psi(r), r \geq 0 \). By the substitution \( t = Du^\alpha \) (without any further restrictions on the exponents \( a_k \) for \( \alpha \geq 1 \), and with the restriction \( a_k \neq \frac{1}{2}(\frac{1}{\alpha} - 1) \) for \( 0 < \alpha < 1 \)), after some obvious estimations one can deduce from the result of Fuchs (Theorem B), that \( \{e^\alpha e^{-Dt^\alpha}\} \) is dense if and only if \( \int_0^\infty \frac{\Psi^\alpha(x)}{\varphi^\alpha(x)} = \infty \). We get the same from Theorems 1 and 2. After the third remark we need to check the assumptions of Theorem 1. Now \( \varphi^\alpha(x) = \sqrt{K(x)} \leq (cx)^{\frac{1}{x}} \); and so

\[
\left(\frac{\varphi(x)}{\Psi(r)}\right)^x \leq \left(\frac{cx}{\Psi^\alpha(r)}\right)^{\frac{1}{x}}.
\]
Theorem 4  With the notations of Theorem 1

\[ f(r) = C + r \max \left\{ \frac{1}{2} K'(r), 2m(r) \right\} - \frac{1}{2} \log K(r) \quad (15) \]

is a good choice for \( f(r) \) with a suitable \( C \). That is, if \( w \) is admissible on \([0, \infty)\), and

\[ \int_{1}^{\infty} \frac{r \max \left\{ \frac{1}{2} K'(r), 2m(r) \right\} - \frac{1}{2} \log K(r)}{r^2} < \infty \quad (16) \]

then \( S \) is incomplete in \( L^2_w(0, \infty) \).

Remark:

If \( w(t) = e^{-Dt^\alpha} \), and \( \frac{1}{2} K'(r) > 2m(r) \) on a set \( H \), then on \( H \)

\[
\begin{align*}
f(r) &= r \frac{1}{2} K'(r) - \frac{1}{2} \log K(r) \\
&= r \left( \frac{\Gamma'}{\Gamma} \left( \frac{2r+1}{\alpha} \right) - \log(2D) \right) - \frac{1}{2} \log \left( \frac{1}{\alpha(2D)^{\frac{2r+1}{\alpha}}} \Gamma \left( \frac{2r+1}{\alpha} \right) \right) \\
&= r \log \frac{2r+1}{\alpha} - \frac{r}{2(2r+1)} - \frac{r}{\alpha} I \left( \frac{2r+1}{\alpha} \right) - \frac{r}{\alpha} \log(2D) \\
&\quad - \frac{1}{2} \log \frac{2\pi (2r+1)^{\frac{2r+1}{\alpha}}}{\alpha(2D)^{\frac{2r+1}{\alpha}}} e^{-\frac{2r+1}{\alpha} e^{I(2r+1)}} \\
&= \frac{r}{\alpha} - 2 - \frac{2 - \alpha}{4\alpha} \log \frac{2r+1}{\alpha} + O(1)
\end{align*}
\]

(In the last step we used that \( \frac{\Gamma'}{\Gamma} (z) = \log z - \frac{1}{2z} - I(z) \) where \( I(z) = \int_{0}^{\infty} \frac{t^z}{(t+1)^{z+1}} dt \) (see eg [26]).) That is, if \( H \) is large then the integral in (10) is divergent.

3 Proofs

For the proof of the first theorem, at first we need a lemma:

Lemma 1 Let \( a = m \) be a positive integer. If \( w^2 \) is a continuous, positive, normal weight function on \((0, \infty)\) with finite moments, then there is a function \( b(z) \) such that \( \frac{1}{b(z)} \) is regular on \( \Re z \geq -a \), and it fulfils the inequality on \( \Re z \geq -\frac{1}{2} : \)

\[ \sqrt{\frac{K(x+a)}{K(x)}} \leq |b(z)|, \]

where \( z = x + iy \).
Proof:
At first let \( x = n \) be also a positive integer. Then, using the Gaussian quadrature formula on the zeros of the \( N^{th} \) orthogonal polynomials \((x_{1,N} > \ldots > x_{k,N} > \ldots > x_{N,N})\) with respect to \( w^2 \), where \( N = n + m + 1 \), we get, that
\[
\frac{K(n+m)}{K(n)} = \int_0^{\infty} \frac{t^{2(n+m)}w^2}{t^{2n}u^2} = \frac{\sum_{k=1}^{N} \lambda_{k,N} x_{k,N}^{2(n+m)}}{\sum_{k=1}^{N} \lambda_{k,N} x_{k,N}^{2n}} \leq x_{1,N}^{2m},
\]
that is, by the condition of "normality"
\[
\sqrt{\frac{K(n+m)}{K(n)}} \leq e^{cN}.
\]
Now we can consider, that \( \frac{K(x+a)}{K(x)} \) is increasing on \( \Re z > -\frac{1}{2} \), namely
\[
\left( \frac{K(x+a)}{K(x)} \right)' = \frac{K(x+a)}{K(x)} \left( \frac{K'(x+a)}{K'(x)} - \frac{K'(x)}{K'(x+a)} \right),
\]
which is nonnegative, because \( \frac{K'}{K} \) is increasing. The last statement can be seen by the Cauchy-Schwarz inequality, that is the derivative of \( \frac{K'}{K} \) is nonnegative
\[
\left( 2 \int_0^{\infty} t^{2x} \log t |w^2(t)| dt \right)^2 \leq \int_0^{\infty} t^{2x} w^2(t) dt \int_0^{\infty} t^{2x} 4 \log^2 tw^2(t) dt.
\]
So with \( a = m \) and \( x > 0 \),
\[
\sqrt{\frac{K(x+a)}{K(x)}} \leq e^{ca(a+1+[x])} \leq e^{ca(a+2+x)} = C(a) |e^{caz}|.
\]
Remark:
(1) If \( \left( \frac{\varphi(2x)}{\varphi(x)} \right)^x \) does not grow too quickly, then one can choose \( b(z) = c(a)b_1(z) \), where \( b_1(z) \) is independent of \( a \), because
\[
\sqrt{\frac{K(x+a)}{K(x)}} \leq K^{\frac{1}{2}}(2a) \frac{K^{\frac{1}{2}}(2x)}{K^{\frac{1}{2}}(x)} = c(a) \left( \frac{\varphi(2x)}{\varphi(x)} \right)^x
\]
(2) Usually we can give \( b(z) \) in polynomial form, for instance if \( w(t) = e^{-Dt} \)
then
\[
\sqrt{\frac{K(x+a)}{K(x)}} = \frac{1}{(2D)^{\frac{1}{2}}} \sqrt{\frac{\pi}{\Gamma\left(\frac{x+1+2a}{2}\right)}} \leq c(2x + 1 + 2a)^{\frac{n}{2}},
\]
and so \( b(z) = c(z + 2a)^{n} \), where \( n > \frac{x}{2} \) is an integer.
Proof: of Theorem 1.
Let us extend \( f(r) \) to \( \Re \) as \( f(-r) = f(r) \). Let \( a \geq 1 \) be as in (4). Furthermore let \( a \) be an integer. Because \( \int_1^{\infty} \frac{f(r)}{r} < \infty \), the function
\[
p(z) = p(x + iy) = |p(r e^{it})| = \frac{2}{\pi} (x + a) \int_{-\infty}^{\infty} \frac{f(t)}{(x + a)^2 + (t - y)^2} dt
\]
(17)
is harmonic on $\Re z > -a$. Since $f(t)$ is increasing, and $x^2 + y^2 = r^2$
\[
p(z) \geq \frac{2}{\pi} f(r) \int_{|t| > r} \frac{x + a}{(x + a)^2 + (t - y)^2} \, dt
\]
\[
= f(r) \frac{2}{\pi} \left( \pi - \left( \arctan \frac{r - y}{x + a} + \arctan \frac{r + y}{x + a} \right) \right) > f(r).
\]
(In the last inequality we applied the height theorem of a triangle.) Let us choose $q$ so that $-p + iq$, and hence $g(z) = g_a(z) = e^{-p+iq}$, be regular on $\Re z > -a$. According to the assumptions of Theorem 1, for this $g(z) \neq 0$ on $\Re z \geq -a$ we have that
\[
|g(z)| \leq e^{-f(r)} \leq \left( \frac{\varphi(x)}{\Psi(r)} \right)^x \Re z \geq 0.
\]
We will show that in this case $S$ is not dense. For this let us define a regular function on the half plane $\Re z \geq 0$ by
\[
H(z) = \prod_{k=1}^{\infty} \frac{a_k - z}{a_k + z}.
\]
According to a Lemma of Fuchs ([14] L.5)
\[
|H(z)| \leq (C\Psi(r))^x \quad \text{on} \quad \Re z \geq 0.
\]
Let us replace the $a_k$'s in the definition of $H(z)$ by $a_k + a$, and let us denote the new function by $H^*(z)$. Now, with the help of $g$ and $H^*$ we can define a function $G(z) = G_a(z)$ which is regular on $\Re z \geq -a$:
\[
G(z) = \frac{g(z + a)H^*(z + a)}{b(z)C_1^{x+a}},
\]
where, according to Lemma 1, $\frac{1}{b(z)}$ is regular on $\Re z \geq -a$, and on $\Re z \geq -\frac{1}{2}$ we have
\[
\sqrt{\frac{K(x + a)}{K(x)}} \leq |b(z)|
\]
Because for an $a > 0$ $\frac{K(x+a)}{K(x)}$ is positive, and it tends to zero, when $x$ tends to $-\frac{1}{2}$, according to Lemma 1, we can suppose that $|b(z)| > \delta > 0$ on $\Re z \in [-a, -\frac{1}{2}]$.

Now, because $|H^*(z + a)| \leq (C\Psi(r))^{x+a} \quad (x \geq -a)$ (see (20)), we have that if $C_1$ is large enough, than according to (22)
\[
|G(z)| \leq (\varphi(x))^x \quad \text{on} \quad \Re z > -\frac{1}{2},
\]
and because $a > \frac{1}{2}$, on $\Re z \in [-a, -\frac{1}{2}]$:
\[
|G(z)| \leq \frac{(\varphi(x + a))^{x+a}}{|b(z)|} \leq \frac{1}{\delta} \max_{x \in [-a, -\frac{1}{2}]} \sqrt{K(x + a)} = M
\]
In the followings we will show that if there exists a function $G$ which is not identically zero, and is regular on $\Re z \geq -a$, and fulfils the equations $G(a_k) = 0$ ($k = 1, 2, \ldots$), and the inequalities (23) and (24) are valid, then $S$ is not complete.

For the purpose of showing this, we need to construct a function $0 \neq k(t) \in L^2_w(0, \infty)$ such that $$
\int_0^\infty t^a k(t) w^2(t) dt = 0 \quad (k = 1, 2, \ldots).$$
We give $k(t)$ by the inversion formula for the Mellin transform of $G(z)$: on $\Re z \geq -a$ let us define the function $u(t)$ by an integral along a line parallel with the imaginary axis
\begin{equation}
\nu(t)u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(z) \frac{\nu(t)w^2(t)}{(1+a+z)^2} t^{-z} dy \quad (25)
\end{equation}

It can be easily seen (by taking the integral round a rectangle $x_k \pm iL k = 1, 2, \ldots$, where $L \to \infty$) that the integral is independent of $x$. Let us choose $k(t) = \nu(Ct)u(Ct)$, $w^2(t)$, where $C$ is the same as in (2). Using that
\begin{equation}
G(z) = \frac{1}{(1+a+z)^2} \int_0^\infty \nu(t)u(t)t^a dt,
\end{equation}
we have that
\begin{equation}
\int_0^\infty t^a k(t) w^2(t) dt = \frac{1}{C_1^{a+k}} \int_0^\infty t^{a+k-1} \nu u(v) \nu(v) dv
= \frac{1}{C_1^{a+k}} \frac{G(a_k)}{(1+a+a_k)^2} = 0
\end{equation}

We have to show, that $k(t) \in L^2_w(0, \infty)$.
\begin{equation}
\|k\|^2_{2,w} = \int_0^\infty \frac{w^2(Ct)\nu^2(Ct)}{w^2(t)} dt = \int_0^\infty \frac{\nu^2(Ct)}{w^2(t)} dt + \int_0^\infty \frac{\nu^2(Ct)}{w^2(t)} dt = I + II,
\end{equation}
where $A = \max\{1, CC_0\}$.

According to (1), and by the positivity of the coefficients in $\gamma$,
\begin{equation}
II \leq \int_0^\infty \nu^2(Ct)u^2(Ct) \sum_{k=0}^\infty c_k t^\gamma_k dt \leq \sum_{k=0}^\infty c_k C^{\gamma_k+1} \int_A^\infty t^\gamma_k \nu^2(t) u^2(t) dt
= \sum_{\gamma_k < \frac{1}{2}} (\cdot) + \sum_{\gamma_k \geq \frac{1}{2}} (\cdot) = S_1 + S_2
\end{equation}

Using Parseval’s formula for the Mellin transform (see e.g. [15])
\begin{equation}
\int_0^\infty t^{2x+1} \nu^2(t) u^2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{G(z)}{(1+a+z)^2} \right|^2 dy
\end{equation}
\[
\leq c (\varphi(x))^{2x} \leq c \left( \varphi(x + \frac{1}{2}) \right)^{2x+1}, \quad (30)
\]

where the equality is valid on \( \Re z \geq -a \), the first inequality is on \( \Re z > -\frac{1}{2} \), and the last inequality is on \( \Re z \geq -\frac{1}{3} \) say, where we used again that \( \frac{K(x+a)}{K(x)} \) is increasing, that is

\[
0 < c \leq \frac{K\left(\frac{1}{3}\right)}{K\left(-\frac{1}{4}\right)} \leq \frac{K(x + \frac{1}{2})}{K(x)}
\]

Therefore, by (30)

\[
S_2 \leq c \sum_{k \geq \frac{1}{2}} \frac{c_k}{C^{\gamma k+1}} \int_0^\infty t^{\gamma_k} \nu^2(t) u^2(t) dt \leq c \sum_{k=0}^\infty \frac{c_k}{C^{\gamma k+1}} \left( \varphi\left(\frac{\gamma_k}{2}\right) \right)^{\gamma_k}
\]

\[
\leq c \sum_{k=0}^\infty c_k \int_0^\infty t^{\gamma_k} w^2(t) \leq c \sum_{k=0}^\infty c_k \int_0^\infty \left( \frac{t}{C} \right)^{\gamma_k} w^2(t)
\]

\[
= c \int_0^\infty \gamma \left( \frac{t}{C} \right) w^2(t) < \infty \quad (31)
\]

To estimate \( S_1 \) and \( I \), we will use that by (25), with \( x = -\frac{1}{3} \)

\[
\nu^2(t) u^2(t) \leq ct^{-\frac{4}{3}} \varphi \left( -\frac{1}{3} \right)^{-\frac{4}{3}} \int_{-\infty}^{\infty} \frac{1}{\left( \frac{2}{3} + a + iy \right)^2} dy = ct^{-\frac{4}{3}} \quad (32)
\]

That is

\[
t^{\gamma_k} \nu^2(t) u^2(t) \leq ct^{\beta_k}, \quad \text{where } \beta_k < -1,
\]

and therefore \( S_1 \) is bounded. Similarly, if instead of \( x = -\frac{1}{3} \) we use \( x = -a \) in (32), we obtain by (25) that \( \nu^2(t) u^2(t) \leq cM^2 t^{2a-2} \), and so by (4)

\[
I = \int_0^\infty \frac{\nu^2(t) u^2(t)}{\nu^2(t) u^2(t)} dt \leq c \int_0^\infty \frac{t^{2(a-1)}}{\nu^2(t)} < \infty \quad (33)
\]

This proves Theorem 1.

We now turn to the proof of Theorem 2. We will need a technical lemma. Following carefully the proof of Lemma 7 – Lemma 11 in [14], actually W. Fuchs proved the following:

**Lemma 2** If there is a nonnegative, monotone increasing function \( h \) on \((0, \infty)\), which fulfills (12), and

\[
\int_1^\infty \frac{h(r)}{r^2} = \infty, \quad (34)
\]
and if there is a function \( g \) regular on \( \mathbb{R} z \geq 0 \) such that there are \( C, c > 0, \alpha > 0 \)

\[
|g(z)| \leq C \left( \frac{c x}{h(r)} \right)^{\frac{\alpha}{x}},
\]

then

\[
g \equiv 0 \quad \text{on} \quad \mathbb{R} z \geq 0
\]

Remark:

In Lemma 2 \( C \) and \( c \) means that instead of a regular function \( g \) another regular function: \( bA^2g(z) \) can be considered (\( A, b \) are positive constants). It means that \( \Psi(r) \) can be replaced by a function \( \Psi_1(r) \) such that \( \frac{\Psi}{\Psi_1} \) lies between finite positive bounds, and \( \Psi_1(r) \) has a continuous derivative. Therefore in the followings we will assume that \( \Psi(r) \), that is \( m(r) \), is continuously differentiable, if it is necessary. Furthermore since \( m(r) \) is increasing, we will assume that the derivative of \( m \) is nonnegative. If it is necessary, we can assume the same on \( h \).

Proof: of Theorem 2.

From (13), and the previous lemma it follows that if a function \( g(z) \) is regular on \( \mathbb{R} z \geq 0 \), and it satisfies the inequality

\[
|g(z)| \leq \left( \frac{\varphi(x)}{\Psi(r)} \right)^x,
\]

then \( g \equiv 0 \). Namely, if \( r \geq x > 0 \), then (13) and (37) together gives (35), and by the definition of \( \varphi \) and \( \Psi \), \( \lim_{x \to 0^+} \left( \frac{\varphi(x)}{\Psi(r)} \right)^x = \|w\|_{2,(0,\infty)} \), so we can choose a constant \( C \), such that \( \left( \frac{\varphi(x)}{\Psi(r)} \right)^x \leq C \left( \frac{c x}{h(r)} \right)^{\frac{\alpha}{x}} \) on \( \mathbb{R} z \geq 0 \).

Now let us assume, by contradiction, that \( S \) is not dense in \( L^2_w \). In this case there is a function \( f \neq 0 \) in \( L^2_w \), such that the function

\[
G(z) = \int_0^\infty t^z f(t)w^2(t)dt
\]

defined on \( \mathbb{R} z \geq 0 \), satisfies the equalities

\[
G(a_k) = 0 \quad k = 1, 2, \ldots
\]

and we can estimate its modulus by

\[
|G(z)| \leq \|f\|_{2,w} (\varphi(x))^x
\]

Let us define now on \( \mathbb{R} z \geq 0 \)

\[
g(z) = \frac{G(z)}{H(z)C_{x+1}},
\]

where \( H \) is as in (19). By Lemma 4

\[
|H(z)| \geq (C_2\Psi(r))^x
\]
on \( \mathbb{C} \setminus \cup_{k=1}^{\infty} B \left( a_k, \frac{4}{3} \right) \), where \( B \left( a_k, \frac{4}{3} \right) \) are balls around \( a_k \) with radius depending on \( d \) (see (11)), and on the imaginary axis without exception. This implies that

\[
|g(z)| \leq \left( \frac{\varphi(x)}{\Psi(r)} \right)^x
\]

on \( \Re z \geq 0 \setminus \cup_{k=1}^{\infty} B \left( a_k, \frac{4}{3} \right) \) (and on the imaginary axis). But \( g \) is regular on \( \Re z \geq 0 \), so this inequality holds on the whole half-plane, and thus by Lemma 2 \( g \equiv 0 \), and hence \( G \equiv 0 \), a contradiction.

**Proof:** of Theorem 4.

Let us introduce the following notation on \( 0 \leq x \leq r \), where \( r \geq 0 \) is fixed:

\[
v_r(x) = 2x m(r) - \frac{1}{2} \log K(x)
\]

Because \( \frac{K'}{K} \) is increasing (see the first remark after the proof of Theorem 1), \( v_r(x) \) is concave on \([0, r]\). That is, we need to distinguish three cases: (a) \( v_r(x) \) is strictly decreasing on \((0, r]\), (b) \( v_r(x) \) has a maximum on \((0, r]\), (c) \( v_r(x) \) is strictly increasing on \([0, r]\).

In case (a) the first derivative of \( v_r(x) \) is negative on \([0, r]\), that is

\[
2m(r) < \frac{K'}{2K}(r) \quad 0 \leq x \leq r.
\]

Furthermore \( \frac{K'}{K} \) is increasing, and it means that

\[
2m(r) \leq \frac{K'}{2K}(0)
\]

Since the right-hand side is constant, and the left-hand side is increasing, there is an \( r_0 \), such that for all \( r > r_0 \) (45) must be wrong.

In case (b) there is an \( 0 < x_0 = x_0(r) \leq r \), where \( v'_r(x_0) = 0 \). That is, for all \( 0 \leq x \leq r \)

\[
v_r(x) \leq v_r(x_0) = \frac{x_0}{2} \left( \frac{K'}{K}(x_0) - \log K(x_0) \right) \leq \frac{r}{2} \left( \frac{K'}{K}(r) - \frac{\log K(r)}{r} \right)
\]

because \( \frac{r}{2} \left( \frac{K'}{K}(x) - \frac{\log K(x)}{x} \right) \) is increasing, since it’s derivative is \( \frac{1}{2} x \left( \frac{K'}{K} \right)'(x) \), which is nonnegative.

In case (c) \( 2m(r) > \frac{K'}{2K}(r) \) if \( 0 \leq x \leq r \). That is, \( 2m(r) > \frac{K'}{2K}(r) \). In this case \( v_r(x) \leq v_r(r) \), \( \frac{r}{2} \left( \frac{K'}{K}(r) - \frac{\log K(r)}{r} \right) \leq v_r(r) \), and \( v_r(r) \) itself is increasing, because using the remark after Lemma 1

\[
v_r(r)' = \left( 2m(r) - \frac{1}{2} \frac{K'}{K}(r) \right) + 2rm'(r) \geq 0
\]

That is, we can find a constant \( C \), such that \( v_r(x) \leq f(r) \) even in case (a), and \( f \) is increasing.
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