UNIVERSAL TROPICAL STRUCTURES FOR CURVES IN EXPLODED MANIFOLDS

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Abstract. For any stable curve $f$ in an exploded manifold, this paper constructs a family of curves $\hat{f}$ with universal tropical structure which contains $f$. Such a family has the property that any other family of curves containing $f$ is locally a small modification of a family which factors through $\hat{f}$. As such, families of curves with universal tropical structure play an important role in the analysis of the moduli stack of curves and the construction of Gromov-Witten invariants on exploded manifolds.

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1. INTRODUCTION

This paper is one of a series of papers defining Gromov-Witten invariants of exploded manifolds and in the process proving new gluing theorems for Gromov-Witten invariants of symplectic manifolds. For an introduction to exploded manifolds, see [6]. For an outline of the relationship between exploded manifolds and log schemes, and an explanation of how exploded Gromov-Witten invariants are related to log Gromov-Witten invariants, see [7].

In the category of smooth manifolds, any map $h$ which is close to a given map $h_0$ may be described as $h_0$ followed by exponentiation of some vector field. The same is not true in the category of exploded manifolds. Indeed, if $h_0$ and $h$ are nearby maps of exploded manifolds, then $h$ may be written as $h_0$ followed by exponentiation of some vector field only if $h_0$ and $h$ have the same tropical part.

Similarly, in the category of smooth manifolds, any smooth map defined on a submanifold extends to a map defined on a neighborhood, but the same property does not hold for exploded manifolds. Again the obstruction can be seen by analyzing the tropical structure. For example $(T^1)^2$ contains the exploded submanifold $\{\hat{z}_1\hat{z}_2 = 1t^1\}$. The coordinate function $\hat{z}_1$ gives an isomorphism of this sub exploded manifold to $T^1_{[0,1]}$. If this isomorphism extended to a map $h : (T^1)^2 \to T^1_{[0,1]}$ defined on an open neighborhood of $\{\hat{z}_1\hat{z}_2 = 1t^1\}$ then the tropical part of $h$ would

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be an integral affine map \([0, \infty)^2 \rightarrow [0, 1]\), which would be equal to \(x_1\) restricted to \(\{x_1 + x_2 = 1\} \subset [0, \infty)^2\). No such map exists.

This paper is concerned with dealing with the above two related problems for the moduli stack of stable curves in an exploded manifold. In particular, for any stable curve \(f\) in an exploded manifold, we shall construct a family of curves \(\hat{f}\) with universal tropical structure. This family \(\hat{f}\) has the property any stable curve \(f'\) which is close to \(f\) may be written as the composition of a nearly holomorphic map to the domain of a curve in \(\hat{f}\), followed by \(\hat{f}\), followed by exponentiation of some small vector field. Moreover, any family of curves \(f'\) close to \(f\) may be written as the composition of a nearly fiberwise holomorphic map to the domain of \(\hat{f}\) followed by \(\hat{f}\) followed by exponentiation of some small vector field.

In the case that \(f\) is a curve in a smooth manifold \(M\), \([f]\) is a stable curve in \(M\), which may have nodes. In this case, \([\hat{f}]\) is a family of curves in \(M\) which includes all possible smoothings of the nodes of \([f]\). In the general case when the target of \(f\) is an exploded manifold \(B\), the construction of \(\hat{f}\) is more involved—In \([3]\), an inelegant such construction with the simplifying assumption that \(B\) is basic plays a pivotal role. The condition of a family of curves having universal tropical structure is related to the ‘basic’ condition on curves which Gross and Siebert require for defining log Gromov-Witten invariants in \([2]\), and the similar condition of a minimal log structure used by Abramovich and Chen in \([1]\), and Kim in \([3]\).

In \([8]\), the existence of families of curves with universal tropical structure is used to construct the moduli stack \(\mathcal{M}(pt)\) of stable holomorphic curves mapping to a point, and the forgetful map \(\mathcal{M}^{st}(B) \rightarrow \mathcal{M}(pt)\) from the moduli stack of stable curves in \(B\) to \(\mathcal{M}(pt)\). More crucially, they are also used in \([8]\) to construct ‘core families’ \(\hat{f}\) with an automorphism group \(G\) with the property that any family of curves close enough to \(\hat{f}\) may be written uniquely as a fiberwise holomorphic map to the domain of \(\hat{f}/G\) followed by \(\hat{f}\) followed by exponentiation of a vectorfield. This enables the moduli stack of curves close to \(f\) to be studied by studying modifications of \(\hat{f}\) by exponentiating vectorfields. The properties of the \(\bar{\partial}\) equation on the moduli stack of curves close to \(f\) then fall readily to the analysis in \([5]\).

2. Notation and definitions

The definitions in \([6]\) shall be essential for understanding this paper. We shall be studying the moduli stack \(\mathcal{M}^{st}(B)\) of stable curves in a family \(\hat{B} \rightarrow G\) of exploded manifolds. We shall use the notation

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \hat{B} \\
\downarrow & & \downarrow \\
* & \xrightarrow{} & G
\end{array}
\]

for a \(C^{\infty, 1}\) curve \(f\) in \(\hat{B}\). We shall call \(f\) stable if both \(f\) and the smooth part \([f]\) of \(f\) do not have an infinite automorphism group. For a family \(\hat{f}\) of curves, we shall use the notation

\[
\begin{array}{ccc}
\mathbb{C}(\hat{f}) & \xrightarrow{f} & \hat{B} \\
\downarrow & & \downarrow \\
\mathbb{F}(\hat{f}) & \xrightarrow{} & G
\end{array}
\]
2.1. Tropical structure.
Recall from section 4 of [6], that the tropical structure \((\mathbb{B}_T, \mathcal{P})\) of an exploded manifold \(\mathbb{B}\) associates to every point \(p \in \mathbb{B}\) a polytope \(\mathcal{P}(p)\) so that \(p\) is in the interior strata of a coordinate chart with tropical part \(\mathcal{P}(p)\).

For every homotopy class of continuous path \(\gamma : [0, 1] \rightarrow [\mathbb{B}]\) so that if \(t_1 \geq t_0\), then \(t_1\) is in the closure of the strata containing \(t_0\), there is a kind of parallel transport of \(\mathcal{P}(\gamma(0))\) along \(\gamma\) which gives an integral affine map

\[
\mathcal{P}(\gamma) : \mathcal{P}(\gamma(0)) \rightarrow \mathcal{P}(\gamma(1))
\]

which identifies \(\mathcal{P}(\gamma(0))\) with some face of \(\mathcal{P}(\gamma(1))\).

The above information may be formalized by introducing a category \(\mathbb{B}_T\) with objects the points of \([\mathbb{B}]\) and morphisms homotopy classes of continuous paths \(\gamma\) obeying the conditions above, then defining a functor \(\mathcal{P}\) from \(\mathbb{B}_T\) to the category of integral affine polytopes as above.

The construction of this tropical structure is functorial, so given a map \(h : \mathbb{A} \rightarrow \mathbb{B}\)

there is an induced map of tropical structures

\[
(h_T, \mathcal{P}_h) : (\mathbb{A}_T, \mathcal{P}) \rightarrow (\mathbb{B}_T, \mathcal{P})
\]

consisting of a functor

\[
h_T : \mathbb{A}_T \rightarrow \mathbb{B}_T
\]

and a natural transformation

\[
\mathcal{P}_h : \mathcal{P} \rightarrow \mathcal{P} \circ h_T
\]

The functor \(h_T\) is really just the map induced from \([h] : [\mathbb{A}] \rightarrow [\mathbb{B}]. The natural transformation \(\mathcal{P}_h\) restricted to a point \(x \in [\mathbb{A}]\) is the integral affine map

\[
\mathcal{P}_h : \mathcal{P}(x) \rightarrow \mathcal{P}(h(x))
\]

which is the tropical part of \(h\) restricted to coordinate charts containing \(x\) and \(h(x)\) in their interior strata.

Throughout this paper, any arrow

\[
(\mathbb{A}_T, \mathcal{P}) \rightarrow (\mathbb{B}_T, \mathcal{P})
\]

shall always mean such a functor and natural transformation.

2.2. Extensions of the tropical structure of a curve.
The tropical structure of a family of curves \(\hat{f}\) in \(\hat{B}\) consists of the diagram

\[
\begin{array}{ccc}
(C(\hat{f}))_T, \mathcal{P} & \xrightarrow{(fr, \mathcal{P})} & (\hat{B}_T, \mathcal{P}) \\
& \downarrow & \\
(F(\hat{f})_T, \mathcal{P}) & \rightarrow & (G_T, \mathcal{P})
\end{array}
\]

In the case of a single curve \(f\), \(F(f)\) is a single point \(\ast\), and \(\mathcal{P}\) has image the polytope which is a single point, so we shall simplify the notation for the tropical
structure of a curve $f$ to be

$$
\begin{array}{c}
(C_T, P) \\
\downarrow \\
(*, *) \\
\end{array}
\xrightarrow{(f_T, P)}
\begin{array}{c}
(B_T, P) \\
\downarrow \\
(G_T, P) \\
\end{array}
$$

If a family of curves $\hat{f}$ contains $f$, then we get a commutative diagram which factorizes $(f_T, P_f)$

$$
\begin{array}{c}
(C_T, P) \\
\downarrow \\
(*, *) \\
\end{array}
\xrightarrow{(f_T, P_f)}
\begin{array}{c}
(C(\hat{f})_T, P) \\
\downarrow \\
(F(\hat{f})_T, P) \\
\end{array}
\xrightarrow{(\hat{B}_T, P)}
\begin{array}{c}
(G_T, P) \\
\downarrow \\
(G_T, P) \\
\end{array}
$$

Restricting $P$ to the subcategory $C_T$ of $C(\hat{f})_T$ gives a functor $P'$, and natural transformation $P'f : P' \rightarrow P \circ f_T$. Restricting $(F(\hat{f})_T, P)$ to the image of $*$ gives a functor $P'$ with a map $(*, P') \rightarrow (G_T, P)$. The commutative diagram

$$
\begin{array}{c}
(C_T, P') \\
\downarrow \\
(*, P') \\
\end{array}
\xrightarrow{(\hat{B}_T, P)}
\begin{array}{c}
(B_T, P) \\
\downarrow \\
(G_T, P) \\
\end{array}
$$

is an example of an extended tropical structure $P'$ on $f_T$, which we shall define more precisely below in Definition 2.1.

In Lemma 2.2 below, we shall show that there is a notion of the pull back of an extended tropical structure. The left hand square in the following diagram coming from the inclusion of $f$ in $\hat{f}$ is an example of a pullback diagram of extended tropical structures.

$$
\begin{array}{c}
(C_T, P) \\
\downarrow \\
(*, *) \\
\end{array}
\xrightarrow{(f_T, P_f)}
\begin{array}{c}
(C_T, P') \\
\downarrow \\
(*, P') \\
\end{array}
\xrightarrow{(\hat{B}_T, P)}
\begin{array}{c}
(B_T, P) \\
\downarrow \\
(G_T, P) \\
\end{array}
$$

We can regard any extended tropical structure which pulls back to give the tropical structure of $f$ to be an extension of the tropical structure of $f$. This notion of an extension of tropical structures is analogous to Gross and Siebert’s notion of a ghost sheaf of a given type in [2]. We shall construct a universal extension of the tropical structure below in Theorem 3.1 so that any extension of the tropical structure of $f$ may be uniquely expressed as the pullback of this universal extension. We shall say that a family $\hat{f}$ has universal tropical structure when its tropical structure restricted to any individual curve is the universal extension of the tropical structure of that curve.
Definition 2.1. An extended tropical structure on $f_T$ is a pair of functors $\mathcal{P}'$ from $\mathcal{C}_T$ and $\ast$ to the category of integral affine polytopes and a commutative diagram

$$
\begin{array}{ccc}
(C_T, \mathcal{P}') & \xrightarrow{(f_T, \mathcal{P}'f)} & (\hat{B}_T, \mathcal{P}) \\
\downarrow (\pi_T, \mathcal{P}'\pi) & & \downarrow \\
(\ast, \mathcal{P}') & \xrightarrow{} & (G_T, \mathcal{P})
\end{array}
$$

so that

1. given any path $\gamma$ along which parallel transport is defined, $\mathcal{P}'(\gamma) : \mathcal{P}'(\gamma(0)) \to \mathcal{P}'(\gamma(1))$ is always an isomorphism onto a face of $\mathcal{P}'(\gamma(1))$.

2. The natural transformation $\mathcal{P}'\pi : \mathcal{P}' \to \mathcal{P}' \circ \pi_T$ gives for each point $x \in [C]$ an integral affine map

$$
\mathcal{P}'\pi : \mathcal{P}'(x) \to \mathcal{P}'(\ast)
$$

This integral affine map is:

(a) an isomorphism in the case that $x$ is a point in a smooth component of $C$,

(b) or a pullback of the map $[0, \infty)^2 \to [0, \infty)$ given by $(a, b) \mapsto a + b$ in the case that $x$ is a node of $[C]$,

(c) or the pullback of the map $[0, \infty) \to \{0\}$ in the case that $x$ is a marked point of $[C]$. (In other words, in the case that the inverse image of $x$ in $C$ is isomorphic to $T_{(0, \infty)}$.)

So long as $C$ is not $T$, given any family $\hat{f}$ of curves containing $f$, the tropical structure of $\hat{f}$ restricted to $[C] \subset [C(\hat{f})]$ gives an extended tropical structure on $f_T$ in the above sense. Throughout this paper, we shall be ignoring the case that the domain of $f$ is $T$, because the moduli stack of curves with domain $T$ is very easily analyzed.

Just as we can pull back families of curves, we may pull back extended tropical structures as follows:

Lemma 2.2. Given an extended tropical structure on $f_T$

$$
\begin{array}{ccc}
(C_T, \mathcal{P}') & \xrightarrow{(f_T, \mathcal{P}'f)} & (\hat{B}_T, \mathcal{P}) \\
\downarrow (\pi_T, \mathcal{P}'\pi) & & \downarrow \\
(\ast, \mathcal{P}') & \xrightarrow{} & (G_T, \mathcal{P})
\end{array}
$$

and any integral affine map $Q \to \mathcal{P}'(\ast)$
there is a unique pullback extended tropical structure on $f_T$

\[
\begin{align*}
(C_T, Q) & \xrightarrow{(f_T, Qf)} (\hat{B}_T, P) \\
\downarrow^{(\pi_T, Q\pi)} & \downarrow \\
(\ast, Q) & \longrightarrow (G_T, P)
\end{align*}
\]

with a commutative diagram

\[
\begin{align*}
\begin{array}{ccc}
(C_T, Q) & \xrightarrow{(f_T, Qf)} & (\hat{B}_T, P) \\
\downarrow^{(\pi_T, Q\pi)} & & \downarrow \\
(\ast, Q) & \longrightarrow & (G_T, P)
\end{array}
\end{align*}
\]

so that

(1) \[ Q(\ast) = Q \]

and the map \[ Q(\ast) \longrightarrow P'(\ast) \]

is the given map $Q \longrightarrow P'(\ast)$.

(2) For all $x \in [C]$, the following diagram is a pull back diagram:

\[
\begin{align*}
Q(x) & \xrightarrow{\eta} P'(x) \\
\downarrow & \downarrow \\
Q(\ast) & \longrightarrow P'(\ast)
\end{align*}
\]

Proof:

For notational simplicity, let $P := P'(\ast)$.

Define $Q(x)$ and $\eta : Q(x) \longrightarrow P'(x)$ via the pullback diagram

\[
\begin{align*}
Q(x) & \xrightarrow{\eta} P'(x) \\
\downarrow & \downarrow \\
Q & \longrightarrow P
\end{align*}
\]

Given any path $\gamma$ from $x_1$ to $x_2$ so that $P'(\gamma)$ is defined, the commutative diagram

\[
\begin{align*}
\begin{array}{ccc}
Q(x_1) & \xrightarrow{\eta} P'(x_1) & \xrightarrow{P'(\gamma)} P'(x_2) \\
\downarrow & & \downarrow \\
Q & \longrightarrow & P
\end{array}
\end{align*}
\]

induces a map $Q(x_1) \longrightarrow Q(x_2)$ because $Q(x_2)$ is defined as a pullback. Define $Q(\gamma)$ to be this map. Such a $Q$ is the unique functor from $C_T$ to the category of integral affine polytopes so that $\eta$ is a natural transformation.
We now have a commutative diagram

\[(C_T, Q) \xrightarrow{(f_T, P') \circ \eta)} (\hat{B}_T, P) \]
\[\downarrow \quad \downarrow \]
\[(\ast, Q) \xrightarrow{} (G_T, P) \]

As $P'(\gamma)$ is always an isomorphism onto a face, and $Q(\gamma)$ is obtained from $P'(\gamma)$ via a base change, $Q(\gamma)$ is also always an isomorphism onto a face. Similarly, condition 2 of Definition 2.1 is preserved under base change, so it is also satisfied by $Q$.

It follows that our commutative diagram above is the unique extended tropical structure on $f_T$ which satisfies the conditions above to be a pullback. □

A pullback diagram

\[(C_T, Q) \xrightarrow{} (C_T', P') \xrightarrow{} (\hat{B}_T, P) \]
\[\downarrow \quad \downarrow \]
\[(\ast, Q) \xrightarrow{} (\ast, P') \xrightarrow{} (G_T, P) \]

from Lemma 2.2 is the correct notion of a map of extended tropical structures from $Q$ to $P'$.

**Definition 2.3.** Say that an extended tropical structure $P'$ on $f_T$ is an extension of the tropical structure of $f$ if the tropical structure of $f$ considered as an extended tropical structure on $f_T$ is a pullback of $P'$.

### 3. Construction of a universal extension of tropical structure

**Theorem 3.1.** If $f$ is a curve in $\hat{B}$ so that the domain $C$ of $f$ is not $T$, then there exists a universal extension of the tropical structure of $f$,

\[(C_T, P_u) \xrightarrow{(f_T, P_u f)} (\hat{B}_T, P) \]
\[\downarrow \quad \downarrow \]
\[(\ast, P_u) \xrightarrow{} (G_T, P) \]

with the property that given any other extension

\[(C_T, P') \xrightarrow{(f_T, P' f)} (\hat{B}_T, P) \]
\[\downarrow \quad \downarrow \]
\[(\ast, P') \xrightarrow{} (G_T, P) \]

of the tropical structure of $f$, there exists a unique map

$P'(*) \longrightarrow P_u(*)$
which pulls back \( P_u \) to \( P' \).

\[
\begin{align*}
(C_T, P') &\xrightarrow{(f_T, P'f)} (C_T, P) \\
* &\xrightarrow{(f_T, P_u) \circ f_T} (B_T, P) \\
(C_T, P) &\xrightarrow{(f_T, P_u)} (G_T, P)
\end{align*}
\]

Proof:

We shall use \( P \) to refer to the polytope which is \( P' \), and \( P_u \) to refer to \( P_u \).

The structure of the proof shall be as follows: We shall
1. Define a functor \( Q \) from \( C_T \) to the category of integral affine polytopes.
2. Prove that every extension \( P' \) of the tropical structure of \( f \) may be written uniquely as a pullback of \( Q \).
3. Find obstructions to constructing a natural transformation \( f \circ Q \) from \( Q \) to \( P \).
4. Construct \( P_u \) from \( Q \) by solving integral affine equations corresponding to the obstructions to defining \( Q \).

Choose a point \( p_i \) on each smooth component of \( C \). We shall use the polytopes \( P(f(p_i)) \). For each node \( e \) of \( C \) (corresponding to an internal edge of \( C \)), define

\[
Q_e := [0, \infty) \\
\prod_{i} P(f(p_i)) \times \prod_{e' \neq e} I_{e'}
\]

For a path \( \gamma \) joining \( x \) to one side of the node \( e \), set \( Q(\gamma) \) to be the map induced by inclusion of \( I_e := [0, \infty) \) as \( \{0\} \times [0, \infty) \subset [0, \infty)^2 \). (Let \( Q(\gamma) \) be the identity.
on all other factors.) For a path $\gamma'$ joining $x'$ to the other side of the node $e$, let $Q(\gamma')$ be the inclusion of $I_e$ as $[0, \infty) \times \{0\} \subset [0, \infty)^2$ and the identity on all other factors of $Q(x') = Q$.

A marked point $y$ of $[C]$ is a point in $[C]$ with inverse image in $C$ isomorphic to $T^1(0, \infty)$, so $y$ corresponds to an external edge of $C$. For any marked point $y$ of $[C]$, define $Q(y) = Q \times [0, \infty)$. For a path $\gamma$ joining a point $x$ in a smooth component of $C$ to $y$, define $Q(\gamma) : Q(x) \to Q(y)$ to be the inclusion of $Q$ as the face $Q \times \{0\} \subset Q \times [0, \infty)$.

This completes the definition of a functor $Q$ from $C_T$ to the category of integral affine polytopes. Define $Q(*)$ to be $Q$. There is a map

$$(C_T, Q) \to (\ast, Q)$$

given by

- the identity map
  $$Q(x) = Q \to Q(*) = Q$$
  when $x$ is in a smooth component of $C$,
- the projection
  $$Q(y) := Q \times [0, \infty) \to Q$$
  when $y$ is a marked point of $[C]$,
- and the map
  $$[0, \infty)^2 \to I_e$$
  $$(a, b) \mapsto a + b$$
  on the components of $Q(e)$ and $Q(*)$ corresponding to the node $e$, and the identity map on all other factors of $Q(e)$ and $Q(*)$.

We have not yet defined a natural transformation $Qf$, only the map $Qf : Q(x) \to P(f(x))$ for $x$ in a smooth component of $C$. Nevertheless, the following claim may be thought of as saying that we may write any extension $P'$ of the tropical structure of $f$ uniquely as a pullback of $Q$.

**Claim 3.2.** Given any extension $P'$ of the tropical structure of $f$, there exists a unique natural transformation

$$\eta_0 : P' \to Q$$

so that

1. the following diagram commutes

$$
\begin{array}{ccc}
(C_T, P') & \xrightarrow{\text{id}, \eta_0} & (C_T, Q) \\
\downarrow & & \downarrow \\
(\ast, P') & \xrightarrow{} & (\ast, Q)
\end{array}
$$

2. for all $x$ in a smooth component of $C$,

$$P' f : P'(x) \to P(f(x))$$

factorizes as

$$P' f = Qf \circ \eta_0$$
(3) and for all points \( x \) in \([C]\), the following is a pullback diagram:

\[
\begin{array}{ccc}
P' & \xrightarrow{\eta_0} & Q(x) \\
\downarrow & & \downarrow \\
P & \xrightarrow{} & Q
\end{array}
\]

To prove Claim 3.2, note that condition 2 of Definition 2.1 tells us that at points \( x \) in the smooth part of \( C \), the map \( P'(x) \rightarrow P \) must be an isomorphism. The commutativity of the diagram in item 1 above and the fact that \( Q'(x) \rightarrow Q \) is an isomorphism tells us that \( P'(x) \rightarrow Q'(x) \) must be determined by the map \( P \rightarrow Q \).

On the other hand, item 2 above tells us that the map \( P'(p_i) = P \rightarrow Q = Q(p_i) \) followed by projection to \( P(f(p_i)) \) must be equal to \( P'f \).

Condition 2 of Definition 2.1 and the definition of a pullback in Lemma 2.2 imply that there is a pullback diagram:

\[
\begin{array}{ccc}
P' & \xrightarrow{\eta_0} & Q \\
\downarrow & & \downarrow \\
P & \xrightarrow{} & Q
\end{array}
\]

Similarly, for a marked point \( y \) of \([C]\),

\( \eta_0 : P'(y) \rightarrow Q(y) \)

is uniquely determined by condition 2 of Definition 2.1 and item 3 above. Given any path \( \gamma \) joining \( x \) to \( y \), the diagram required to show that \( \eta_0 \) is a natural transformation commutes:

\[
\begin{array}{ccc}
P' & \xrightarrow{\eta_0} & Q \\
\downarrow & & \downarrow \\
P & \xrightarrow{} & Q
\end{array}
\]

because in each case the inclusion corresponding to \( \gamma \) is the pullback of the inclusion \( 0 \rightarrow [0, \infty) \).
It follows that $\eta_0 : P' \to Q$ as defined is a natural transformation, and is the unique natural transformation satisfying the above conditions. This completes the proof of Claim 3.2.

So far, we have only defined maps $Qf : Q(x) \to P(f(x))$ for $x$ in smooth components. We shall now attempt to define $Qf$ as a natural transformation, and discover two types of obstructions along the way.

For $Qf$ to be a natural transformation, we need for any path $\gamma$ joining $x_1$ to $x_2$ within a given smooth component, the following diagram to commute:

$$
\begin{array}{ccc}
Q & \xrightarrow{Qf} & P(f(x)) \\
\downarrow{\text{id} = Q(\gamma)} & & \downarrow{P(f_T(\gamma))} \\
Q & \xrightarrow{Qf} & P(f(x_2))
\end{array}
$$

If $x_1 = x_2$, then this is not possible unless $Qf$ has image contained in the subset of $P(f(x_1))$ where $P(f_T(\gamma))$ is the identity.

Define $P_1(f(x)) \subset P(f(x))$ to be the sub polytope of $P(f(x))$ which is fixed by $P(f_T(\gamma))$ for all such $\gamma$ from $x$ to itself. Define $Q_1$ to be the subpolytope of $Q$ which is the product of $P_1(f(p_i))$ for all $i$ with $I_e$ for each node $e$.

$$
Q_1 := \prod_i P_1(f(p_i)) \times \prod_e I_e
$$

As $Pf = Qf \circ \eta$ restricted to $x$, and $P'(\gamma)$ is the identity for any loop $\gamma$, the map $P \to Q$ from Claim 3.2 must have image inside $Q_1$.

Now define $Qf : Q(y) \to P(f(y))$ for a marked point $y$ of $[C]$. Choose a path $\gamma$ to $y$ from a point $x$ in the smooth component of $[C]$ with closure containing $y$. For $Qf$ to be a natural transformation, we want the following diagram to commute.

$$
\begin{array}{ccc}
Q(y) & \xrightarrow{Qf} & P(f(y)) \\
\uparrow{Q(\gamma)} & & \uparrow{P(f_T(\gamma))} \\
Q(x) & \xrightarrow{Qf} & P(f(x))
\end{array}
$$

As $Q(y) = Q \times [0, \infty)$ and $Q(\gamma)$ has image $Q \times \{0\}$, this specifies $Q(f)$ on the face $Q \times \{0\} \subset Q \times [0, \infty)$. Claim 3.2 gives a unique inclusion $\eta_0 : P(y) \to Q(y)$ as some fiber of the projection $Q \times [0, \infty) \to Q$. As the image of $P(y)$ is complementary to $Q \times \{0\}$, we may define a unique map $Qf : Q(y) \to P(f(y))$ so that diagram (3) commutes and so that $Pf = Qf \circ \eta_0$. Note that $Qf$ as defined depends on the choice of $\gamma$, but restricted to the inverse image of $Q_1 \subset Q$, it does not depend on the choice of $\gamma$. 

The map $\eta_0 : P'(y) \longrightarrow Q(y)$ pulls back $Qf$ to $P'f$ because the following diagram commutes.

\[
\begin{array}{ccc}
P(y) & \xrightarrow{\eta} & Q(y) \\
\downarrow & & \downarrow \phi \qquad \downarrow \psi \\
P'(y) & \xrightarrow{\eta_0} & P(f(y))
\end{array}
\]

In particular, the bottom left hand triangle commutes because of the uniqueness property of $\eta_0$, the top loop commutes by the definition of $Qf$, and the outer loop commutes because $P$ is a pullback of $P'$. It follows that the bottom right hand triangle commutes on the image of $P(y)$. On the other hand, the bottom right hand triangle commutes on the complimentary face of $P'(y)$ because of condition \ref{claim:pullback} from Claim \ref{claim:pullback} and the commutativity of diagram \ref{diagram:commute}.

Let us now try to define $Qf : Q(e) \longrightarrow P(f(e))$ similarly for a node $e$ of $[C]$. Given a path $\gamma$ joining $p_i$ to the node $e$, if $Qf$ is to be a natural transformation, there must be a commutative diagram

\[
\begin{array}{ccc}
Q(e) & \xrightarrow{Qf} & P(f(e)) \\
\downarrow Q(\gamma) & & \downarrow \phi(f(\gamma)) \\
Q(p_i) & \xrightarrow{Qf} & P(f(p_i))
\end{array}
\]

Claim \ref{claim:pullback} gives us an inclusion of $P(e)$ as a fiber of $Q(e) \longrightarrow Q$. As the pullback of $Qf$ under this inclusion is required to be $Pf$, we now have our prospective map $Qf$ specified on the face which is the image of $Q(\gamma)$ and the complementary subspace $P(e) \subset Q(e)$. If we embed $P(f(e))$ in $\mathbb{R}^n$, this specifies a unique integral affine map

\[A_\gamma : Q(e) \longrightarrow \mathbb{R}^n \supset P(f(e))\]

so that the following diagram commutes

\[
\begin{array}{ccc}
P(e) & \xrightarrow{\eta_0} & Q(e) \\
\downarrow Q(\gamma) & & \downarrow A_\gamma \\
Q(x) & \xrightarrow{Qf} & P(f(x))
\end{array}
\]

Consider the following diagram.

\[
\begin{array}{ccc}
P(e) & \xrightarrow{\eta_0} & Q(e) \\
\downarrow & & \downarrow A_\gamma \\
P'(e) & \xrightarrow{\eta_0} & P(f(e))
\end{array}
\]

The top, lower left and outer loops of the above diagram commute, so the bottom righthand triangle commutes when $P'(e)$ is restricted to the image of $P(e)$. On the
other hand, consider the diagram

\[
\begin{array}{c}
\mathcal{P}'(e) \\
\downarrow \eta_0 \quad \downarrow \quad \eta \quad \downarrow \quad \eta \quad \downarrow \quad \eta_0 \\
\mathcal{Q}(e) \quad \mathcal{Q}(e) \quad \mathcal{Q}(e) \quad \mathcal{Q}(e) \\
\mathcal{P}'(x) \quad \mathcal{P}(x) \quad \mathcal{P}(x) \quad \mathcal{P}(x) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathbb{R}^n \quad \mathcal{P}(f(e)) \quad \mathcal{P}(f(x)) \quad \mathcal{P}(f(x)) \\
\mathcal{P}(x) \quad \mathcal{P}(x) \quad \mathcal{P}(x) \quad \mathcal{P}(x) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathcal{Q}'f \quad \mathcal{Q}'f \quad \mathcal{Q}'f \quad \mathcal{Q}'f \\
\end{array}
\]

The outer loop, two inner squares and bottom triangle commute, therefore the top triangle commutes when \( \mathcal{P}'(e) \) is restricted to the image of \( \mathcal{P}'(x) \). Therefore both of the above diagrams commute, because the questionable triangle commutes when \( \mathcal{P}'(e) \) is restricted to the complimentary images of \( \mathcal{P}(e) \) and \( \mathcal{P}(x) \). In particular, this implies that the pullback of \( A_\gamma \) to \( \mathcal{P}'(e) \) must equal \( \mathcal{P}'f \).

Given another path \( \gamma' \) joining a marked point \( p_j \) to \( e \), we get an analogous map

\[
A_{\gamma'} : \mathcal{Q}(e) \to \mathbb{R}^n \supset \mathcal{P}(f(e))
\]

As the diagram

\[
\begin{array}{c}
\mathcal{Q}(e) \\
\downarrow \eta_0 \\
\mathcal{P}'(e) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathbb{R}^n \\
\downarrow \eta \quad \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{P}(f(e)) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{Q}'f \\
\downarrow \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\]

commutes, \( \eta_0(\mathcal{P}'(e)) \) must be in the subset of \( \mathcal{Q}(e) \) on which \( A_\gamma = A_{\gamma'} \). This subset of \( \mathcal{Q}(e) \) is the inverse image of the subset of \( Q = \mathcal{Q}(p_j) \) on which the diagram

\[
\begin{array}{c}
\mathcal{Q}(p_j) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{P}(f(p_j)) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{P}(f(e)) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{Q}(\gamma') \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{P}(f(\gamma')) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(e) \\
\end{array}
\]

commutes.

Define \( P_u \) as the subpolytope of \( Q_1 \subset Q \) on which the above diagram commutes for all such pairs, \( \gamma \) and \( \gamma' \).

Define \( \mathcal{P}_u \) as the pull back of \( \mathcal{Q} \) under the inclusion \( P_u \to Q \). In other words define \( \mathcal{P}_u(x) \) via the pullback diagram

\[
\begin{array}{c}
\mathcal{P}_u(x) \\
\downarrow \\
P_u \\
\end{array}
\quad
\begin{array}{c}
\mathcal{Q}(x) \\
\downarrow \\
\mathcal{Q} \\
\end{array}
\quad
\begin{array}{c}
P_u \\
\downarrow \\
Q \\
\end{array}
\]

and define \( \mathcal{P}_u(\gamma) \) to be the map induced by the diagram

\[
\begin{array}{c}
\mathcal{P}_u(x_1) \\
\downarrow \\
P_u \\
\end{array}
\quad
\begin{array}{c}
\mathcal{Q}(x_1) \\
\downarrow \quad \eta_0 \\
\mathcal{Q}(x_2) \\
\end{array}
\quad
\begin{array}{c}
\mathcal{Q}(x_2) \\
\downarrow \\
\mathcal{Q} \\
\end{array}
\]


For points $x$ in $[C]$ which are not nodes, we may define $P_u f$ via the composition

$$P_u(x) \xrightarrow{P_u f} Q(x) \xrightarrow{Q f} P(f(x))$$

To define $P_u f$ at a node $e$ of $[C]$, consider the commutative diagram

$$\begin{array}{ccc}
P_u(p_j) & \xrightarrow{P_u f} & P(f(p_j)) \\
\downarrow{P_u(\gamma')} & & \downarrow{P(f(\gamma'))} \\
P_u(e) & \xrightarrow{A_\gamma} & R^n \supset P(f(e)) \\
\downarrow{P_u(\gamma)} & & \downarrow{P(f(\gamma))} \\
P_u(p_i) & \xrightarrow{P_u f} & P(f(p_i))
\end{array}$$

As $P_u(e)$ is the convex hull of the image of $P_u(\gamma)$ and $P_u(\gamma')$, and all polytopes above are convex, the composition

$$P_u(e) \xrightarrow{Q f} R^n \supset P(f(e))$$

has image inside $P(f(e))$, so we may define $P_u f$ using the commutative diagram

$$\begin{array}{ccc}
P_u(e) & \xrightarrow{Q f} & R^n \supset P(f(e))
\end{array}$$

As discussed above, $P_u f$ thus defined obeys the commutativity requirements to be a natural transformation. This completes the construction of our universal extension of the tropical structure of $f$

$$\left(\mathcal{C}_T, P_u\right) \xrightarrow{(f_T, P_u f)} \left(\mathcal{B}_T, P\right)$$

As the map $P \rightarrow Q$ has image in $P_u$, we get a unique natural transformation $\eta : \mathcal{P}' \rightarrow P_u$ so that the following factorizes the pullback diagram for $\eta_0$ from Claim 3.2

$$\begin{array}{ccc}
\mathcal{P}'(x) & \xrightarrow{\eta} & P_u(x) \\
\downarrow & & \downarrow \\
P_u & \xrightarrow{\eta_0} & Q
\end{array}$$

As $A_\gamma \circ \eta_0 = \mathcal{P}' f$ and $Q f \circ \eta_0 = \mathcal{P}' f$, we get that

$$P_u f \circ \eta = \mathcal{P}'$$

Claim 3.2 and Lemma 2.2 then imply that this map $P \rightarrow P_u$ is the required unique map which pulls back our universal extension $P_u$ of the tropical structure of $f$ to the given extension $\mathcal{P}'$ of the tropical structure of $f$. □
Remark 3.3. For each point \( x \) in a smooth strata of of \([ C ]\) there is a map
\[ A_x : P_u \to \mathcal{P}(f(x)) \]
which is equal to \( P_u f : P_u(x) \to \mathcal{P}(f(x)) \) when \( P_u \) is identified with \( P_u(x) \). This map \( A_x \) should be thought of as parametrizing the potential tropical positions of the image of \( x \). For each node \( e \) of \([ C ]\), there is a map \( \rho_e : P_u \to I_e := [0, \infty) \) which should be thought of as parametrizing the possible lengths of the edge of the tropical curve corresponding to \( e \), so \( P_u(e) \) is defined as a fiber product
\[
\begin{array}{rcl}
P_u(e) & \to & [0, \infty)^2 \\
\downarrow & & \downarrow \rho_e \\
P_u & \to & [0, \infty)
\end{array}
\]
Our construction of \( P_u \) implies that if there are \( n \) nodes \( \{e_1, \ldots, e_n\} \), and if \( \{x_1, \ldots, x_m\} \) is any finite collection of points in smooth stata of \([ C ]\) so that each strata contains at least one of our \( x_i \), then the map
\[
(e_1, \ldots, e_n, A_{x_1}, \ldots, A_{x_m}) : P_u \to \prod_{j=1}^n I_{e_j} \times \prod_{i=1}^m \mathcal{P}(f(x_i))
\]
is an injective map which represents \( P_u \) as a subpolytope of the target which corresponds to the solution of some integral affine equations. Roughly speaking, these integral affine equations are the consistency conditions which must be satisfied for tropical curves with the data given by \( e_i \) and \( A_{x_i} \) to exist.

If \( f \) is a curve in a smooth manifold, then \( \mathcal{P}(f(x_i)) \) is always a point, and
\[
(e_1, \ldots, e_n) : P_u \to [0, \infty)^n
\]
is an integral affine isomorphism.

4. Families of curves with universal tropical structure

Definition 4.1. Say that a family of curves \( \hat{f} \) containing \( f \) has universal tropical structure at \( f \) if the restriction of the tropical structure of \( \hat{f} \) to \( C_T \) is the universal extension of the tropical structure of \( f \).

Say that a family of curves has universal tropical structure if it has universal tropical structure at every curve which it contains.

Lemma 4.2. Given a curve \( f \) in \( \hat{B} \) with domain not equal to \( T \), there exists a family of curves
\[
\begin{array}{c}
\mathbf{C}(\hat{f}) \xrightarrow{f} \hat{B} \\
\downarrow \\
\mathbf{F}(\hat{f}) \xrightarrow{f} \hat{G}
\end{array}
\]
containing \( f \) which has universal tropical structure at \( f \) so that the smooth part of the strata of \( \mathbf{F}(\hat{f}) \) containing \( f \) consists of a single point.

Proof:

The base of our family of curves \( \mathbf{F}(\hat{f}) \) shall be an open subset of \( T^m_{P_u} \). There is a distinguished point in \( P_u \) corresponding to the map of the tropical structure of \( f \) to \( P_u \). Choose some point \( * \in T^m_{P_u} \) with tropical part this distinguished point.

Choose a finite collection of coordinate charts \( U_i \) on the domain \( C \) of \( f \) so that \( U_i \) with its complex structure is either
(1) An open subset of $T^1_{[0,1]}$ in the form of
\[ 1 > |\tilde{z}| > rt^l \]
(2) an open subset of $T^1_{[0,\infty)}$ in the form
\[ 1 > |\tilde{z}| \]
(3) or an open subset of $\mathbb{C}$, so that
- any non empty intersection of two different charts involves at least one chart of type 2,
- and each chart of type 3 which intersects a chart of type 1 does so either on the subset
\[ 1 > |\tilde{z}| > \frac{1}{2} \]
or on the subset
\[ 2rt^l > |\tilde{z}| > rt^l \]
but does not intersect both of these subsets.

We shall extend these coordinate charts $U_i$ to coordinate charts $\tilde{U}_i$ on a family $\hat{f}$ of curves as follows:
In the case that $U_i$ is a chart of type 1, proceed as follows: Denote by $V$ the open subset of $T^2_{[0,\infty)^2}$ in the form of
\[ \{(\tilde{z}_1,\tilde{z}_2) : |\tilde{z}_1| < 1, |\tilde{z}_2| < 1, |\tilde{z}_1\tilde{z}_2| < \frac{1}{4}\} \subset T^2_{[0,\infty)^2} \]
There is a submersion
\[ V \longrightarrow T^1_{[0,\infty)} \]
in the form of $(\tilde{z}_1,\tilde{z}_2) \mapsto \tilde{z}_1\tilde{z}_2$.
Suppose that $U_i$ is the coordinate chart containing the node $e$. Remark 3.3 implies that for each node $e$ of $[C]$ there is a pullback diagram
\[
\begin{array}{ccc}
\mathcal{P}(e) & \longrightarrow & [0,\infty)^2 \\
\downarrow & & \downarrow \rho_e \\
\mathcal{P}_u & \longrightarrow & [0,\infty)
\end{array}
\]
There is a monomial map
\[ q_i : T^m_{\mathcal{P}_u} \longrightarrow T^1_{[0,\infty)} \]
with tropical part equal to $\rho_e$ so that the fiber of $V$ over the image of $*$ is isomorphic to $U_i$ (with its complex structure). Define $F(\hat{f}) \subset T^m_{\mathcal{P}_u}$ to be the subset where $|q_i| > \frac{1}{4}$ for all such maps $q_i$. Then define $\tilde{U}_i$ via the pullback diagram
\[
\begin{array}{ccc}
\tilde{U}_i & \longrightarrow & V \\
\downarrow & & \downarrow \\
F(\hat{f}) & \longrightarrow & T^1_{[0,\infty)}
\end{array}
\]
and give $\tilde{U}_i \longrightarrow F(\hat{f})$ the fiberwise complex structure pulled back from $V \longrightarrow T^1_{[0,\infty)}$. Note that $U_i$ is the fiber of $\tilde{U}_i$ over $* \in F(\hat{f})$.

If $U_i$ is any other coordinate chart (of the form 2 or 3), then define $\tilde{U}_i$ as $U_i \times F(\hat{f})$ with the obvious projection to $F(\hat{f})$ and the fiberwise complex structure from $U_i$. 
Similarly, if $U_I$ indicates the intersection of $U_i$ for all $i \in I$, and $|I| > 1$, then define $\bar{U}_I := U_I \times F(\hat{f})$ with the obvious projection to $F(\hat{f})$.

To define a family of curves $C(\hat{f}) \rightarrow F(\hat{f})$ using the system of coordinate charts $\bar{U}_I$, we still need to define transition maps. If $\bar{U}_I$ is not a chart of type $\mathbb{P}$, then there is an obvious induced map

$$\tilde{\psi}_{I,I} : \bar{U}_I \rightarrow \bar{U}_I \times F(\hat{f})$$

which is a fiberwise holomorphic isomorphism onto its image. If on the other hand $U_i$ is a chart of type $\mathbb{P}$ and $I$ contains $i$, and $\psi_{I,i}$ has image within $1 > |\tilde{z}| > \frac{1}{2}$, then define $\tilde{\psi}_{I,i}$ as the unique map

$$\tilde{\psi}_{I,i} : \bar{U}_I \rightarrow \bar{U}_i$$

so that the following two diagrams commute

$$\begin{array}{ccc}
\bar{U}_I & \xrightarrow{\psi_{I,i}} & \bar{U}_i \\
\downarrow & & \downarrow \\
\bar{U}_I \times F(\hat{f}) & \rightarrow & \bar{U}_i \times F(\hat{f})
\end{array}$$

$$\begin{array}{ccc}
\bar{U}_I & \xrightarrow{\psi_{I,i}} & \bar{U}_i \\
\downarrow & & \downarrow \\
U_i & \xrightarrow{\psi_{I,i}} & \bar{U}_i
\end{array}$$

Similarly, if $\psi_{I,i}$ has image with $2r^t > \tilde{z} > r^t$, then define $\tilde{\psi}_{I,i}$ to be the unique map so that the following diagram and the first of the above diagrams commute.

$$\begin{array}{ccc}
\bar{U}_I & \xrightarrow{\psi_{I,i}} & \bar{U}_i \\
\downarrow & & \downarrow \\
\bar{U}_i & \xrightarrow{\psi_{I,i}} & \bar{U}_i \\
\downarrow & & \downarrow \\
U_i & \xrightarrow{\psi_{I,i}} & \bar{U}_i \\
\downarrow & & \downarrow \\
U_i & \xrightarrow{\psi_{I,i}} & \bar{U}_i
\end{array}$$

As in each case, these coordinates of $V$ are (fiberwise) holomorphic functions on $\bar{U}_i$, these transition maps are fiberwise holomorphic. $\tilde{\psi}_{I,i}$ is also an isomorphism onto its image.

With these definitions, compatibility of the coordinate maps $\tilde{\psi}_{I,i}$ follows from compatibility of the coordinate maps $\psi_{I,i}$. As the map $\prod_{i \in I} \psi_{I,i}$ is proper, it follows that $\prod_{i \in I} \tilde{\psi}_{I,i}$ is proper too. Therefore $\bar{U}_I$ with these transition maps define a coordinate system on some family of curves $C(\hat{f}) \rightarrow F(\hat{f})$. As the fiber of $\bar{U}_i$ over $* \in F(\hat{f})$ is always $U_i$ and the restriction of $\tilde{\psi}_{I,i}$ to this fiber is $\psi_{I,i}$, the fiber of $C(\hat{f})$ over $* \in F(\hat{f})$ is isomorphic to $C$.

**Remark 4.3.** In the proof of Lemma 4.8 below, we shall use that $\{\bar{U}_I\}$ is an equivariant set of coordinate charts on $C(\hat{f}) \rightarrow F(\hat{f})$ in a sense defined precisely in [3]. In particular, the projections to $T_{P_\mu}$ and the maps $\tilde{\psi}_{I,i}$ are compatible with multiplication of exploded coordinates in the source and target by elements of $C^\ast t^\mathbb{P}$.

Now we shall define a map $\hat{f} : C(\hat{f}) \rightarrow \hat{B}$ so that

1. $\hat{f}$ restricted to $C \subset C(\hat{f})$ is equal to $f$
(2) and so that the tropical structure of $\hat{f}$ restricted to $C_T$ is equal to

$$(C_T, \mathcal{P}_u) \xrightarrow{P_u f} (\hat{B}_T, \mathcal{P})$$

We have constructed $\hat{f}$ so that the tropical structure of $C(\hat{f}) \to F(\hat{f})$ restricted to $C_T$ is equal to the left hand side of the above diagram. The above condition (2) on the tropical structure of $\hat{f}$ translates to requiring that restricted to any coordinate chart on $C(\hat{f})$ containing a point $x \in [C]$ in its interior strata, the tropical part of $\hat{f}$ in this coordinate chart is equal to $P_u f$. The fact that $P_u f$ is a natural transformation implies that condition (2) does not depend on the point $x \in [C]$ chosen, and that the restriction of a map satisfying condition (2) on one coordinate chart to another coordinate chart will still satisfy condition (2).

Choose a map $T_m \xrightarrow{P_u} G$ with the correct tropical part which sends $* \in F(\hat{f})$ to the point in $G$ whose fiber in $\hat{B}$ contains the image of $f$.

As $P_u$ extends the tropical structure of $f$, the tropical condition (2) is compatible with the condition (1) that $\hat{f}$ restricted to $C$ is equal to $f$. Therefore, around any point in $C$, there locally exists a map obeying the above two conditions and projecting to the given map $T_m \xrightarrow{P_u} G$. Any two such maps differ by exponentiation of a section of the pullback of $T_{vert} B$ which vanishes on $C \subset C(\hat{f})$, and exponentiation of such a vector field always preserves these conditions (1) and (2). Therefore, we may extend a map satisfying conditions (1) and (2) on some set of coordinate charts to the next coordinate chart by using a cutoff function to interpolate between the old map and a new map on the next coordinate chart.

In this way, we may construct a map

$$(C(\hat{f}), \mathcal{P}_u) \xrightarrow{\mathcal{P}_u f} (\hat{B}, \mathcal{P})$$

extending $f$ with tropical structure which restricts to $C_T$ to be the universal extension of the tropical structure of $f$.

\[\square\]

**Lemma 4.4.** Suppose that $\hat{f}$ is a family of curves with universal tropical structure at some curve $f$. Then $\hat{f}$ also has universal tropical structure at any curve $f'$ in a strata of $F(\hat{f})$ with closure containing $f$.

**Proof:**

Use the notation $\mathcal{P}'$ for the tropical structure of $\hat{f}$ restricted to either $f'$ or $f$, and $\mathcal{P}_u$ for the universal extension of the tropical structure of $f'$ or $f$. Use the notation

$$P' = \mathcal{P}'(F(f'))$$
$$P'_u = \mathcal{P}_u(F(f'))$$
$$P_u = \mathcal{P}_u(F(f)) = \mathcal{P}'(F(f))$$

The defining property of $\mathcal{P}_u$ gives us a map

$$P' \to P'_u$$

which pulls back $\mathcal{P}_u$ to $\mathcal{P}'$. It suffices to prove that this map is an isomorphism.

The curve $f'$ is in a strata of $F(f)$ with closure containing $f$ if and only if there exists a path $\gamma$ joining $f'$ to $f$ in $[F(\hat{f})]$ along which parallel transport of tropical
structure is defined. Parallel transport along $\gamma$ gives an inclusion of $P'$ as a face of $P = P_u$. Similarly parallel transport along a lift of this path joining $x'$ to $x$ gives an inclusion of $P'(x')$ as the face of $P'(x) = P_u(x)$ which is the inverse image of $P' \subseteq P_u$.

We shall be using the notation established at the start of the proof of Theorem 3.1 to define the polytope $Q$. The path $\gamma$ may be lifted to a path $\gamma_i$ in $[C(f')]$ along which parallel transport is defined which joins some point $p_i'$ in $[C(f')]$ to $p_i$ in $[C(f)]$. Parallel transport along $\gamma$, $\gamma_i$ and $f_T\gamma_i$ fits into the following commutative diagram.

$$\begin{array}{ccc}
P(f'(p_i)) & \xrightarrow{P(f')\gamma_i} & P(f(p_i)) \\
\downarrow_{Pf} & & \downarrow_{Pf} \\
P'(p_i') & \xrightarrow{P(\gamma_i)} & P_u(p_i) \\
\downarrow & & \downarrow \\
P' & \xrightarrow{P(\gamma)} & P_u
\end{array}$$

(5)

Define the polytope $Q'$ to be the product of $P(f(p_i))$ for each $i$ with $I_e := [0, \infty)$ for each node $e$ of $[C(f')]$.

$$Q' := \prod_i P(f(p_i)) \times \prod_e I_e$$

Note the similarity of $Q'$ to the polytope $Q$ defined in the proof of Theorem 3.1 on page 8. Every node of $[C(f')]$ may be followed over the path $\gamma$ to identify with a node of $[C(f)]$. This identifies the nodes of $[C(f')]$ with a subset of the nodes of $[C(f)]$. There is therefore an inclusion of $Q'$ as a face of $Q$ induced by the maps $P(f_T\gamma_i)$, the identity maps on the $I_e$ factors corresponding to identified nodes, and the 0 map on $I_e$ factors of $Q'$ corresponding to nodes of $[C(f)]$ which do not come from nodes of $[C(f')]$.

There is then a commutative diagram

$$\begin{array}{ccc}
Q' & \longrightarrow & Q \\
\uparrow & & \uparrow \\
P' & \longrightarrow & P_u
\end{array}$$

(6)

where the map $P' \longrightarrow Q'$ is defined analogously to the map $P \longrightarrow Q$ from Claim 3.2 in the proof of Theorem 3.1.

We may choose our sections $\gamma_i$ so that if $p_i'$ and $p_j'$ are in the same smooth component of $C(\hat{f})$, then $p_i' = p_j'$. Restrict to the sub polytope of $Q'$ defined by requiring coordinates on $Q'$ corresponding to such $P(f(p_i'))$ and $P(f(p_j'))$ to be equal, and then follow the same construction as in the proof of Theorem 3.1. This constructs $P'_u$ as a subpolytope of $Q'$. The map $P' \longrightarrow Q'$ has image in $P'_u \subseteq Q'$.

As the lower and righthand arrows in the above commutative diagram (6) are inclusions of sub polytopes with images defined by the solutions of integral affine equations, it follows that the map $P' \longrightarrow P'_u \subseteq Q'$ is an inclusion of $P'$ as a sub polytope of $P'_u$ with image defined by the solution of integral affine equations. To prove that $P' \longrightarrow P'_u$ is an isomorphism, it therefore suffices to show that this map is surjective, or equivalently, show that the image of $P'_u \subseteq Q'$ in $Q$ under the top arrow of the above diagram is contained in the image of $P'$ in $P_u \subseteq Q$.

Note that in the case that $f'$ is in the same strata as $f$, the construction of $P'_u$ and $P_u$ uses isomorphic polytopes $Q$ and $Q'$, and isomorphic relations to define the sub polytopes $P_u$ and $P'_u$. In this case, $P'_u$, $P_u$, and $P'$ have the same dimension,
so the inclusion $P' \rightarrow P''$ must be an isomorphism because it is an inclusion as a subpolytope defined by integral affine equations.

Without losing generality, we may now reduce to the case that $P'$ is one dimension smaller than $P_u$. As any face of $P_u$ of smaller dimension may be reached by taking a sequence of subfaces of codimension 1, the general case follows from repeating the argument that follows. Keep in mind what we need to do is prove that the image of $P''$ in $Q$ is contained in the image of $P'$ in $Q$.

Because $P'$ is one dimension smaller than $P_u$, $f'$ is in a different strata from $f$, so the image of the curve $f$ in $P_u \subset Q$ can not be in $Q' \subset Q$, because for that to be the case, either some $p_i$ would need to have image not in the interior of $\mathcal{P}(f(p_i))$, or some edge of $\mathcal{C}(f)$ would need to have zero length. Therefore, $Q' \cap P_u$ is at least one dimension smaller than $P_u$. As the image of $P'$ in $Q$ is one dimension smaller than $P_u$ and contained in $Q' \cap P_u$, the image of $P'$ in $Q$ must actually be equal to $Q' \cap P_u$. (This uses that all these polytopes are the intersection of $Q$ with an affine subspace.)

It remains to show that the image of $P''$ in $Q$ is contained in the image of $P' = Q' \cap P_u$, so we must show that the image of $P''$ in $Q$ is contained in $P_u \subset Q$. To achieve this, we shall show that $P'' \subset Q$ satisfies the integral affine equations used to define $P_u \subset Q$.

Given a loop $\gamma_0$ in a smooth component of $\mathcal{C}(f)$ starting and ending at $p_i$, there is a map

$$\mathcal{P}(f \gamma_0) : \mathcal{P}(f(p_i)) \rightarrow \mathcal{P}(f(p_i))$$

The polytope $P_u$ is contained in the subset of $Q$ with coordinates fixed by this map $\mathcal{P}(f \gamma_0)$.

There exists a loop $\gamma'_0$ in a smooth component of $\mathcal{C}(f')$ starting and ending at $p'_i$ so that $\gamma'_0$ followed by $\gamma_i$ is homotopic to $\gamma_i$ followed by $\gamma_0$. Therefore

$$\mathcal{P}(f' \gamma'_0) \circ \mathcal{P}(f \gamma_0) = \mathcal{P}(f' \gamma'_0) \circ \mathcal{P}(f \gamma_0)$$

As $P'' \subset Q'$ with coordinates fixed by $\mathcal{P}(f' \gamma'_0)$, and the inclusion $Q' \rightarrow Q$ on the relevant factor is $\mathcal{P}(f' \gamma'_0)$, it follows that the image of $P''$ in $Q$ also has coordinates fixed by $\mathcal{P}(f' \gamma'_0)$. In other words, $P''$ has image inside the polytope $Q_1 \subset Q$ from the proof of Theorem 3.1.

The remaining equations defining $P_u \subset Q_1 \subset Q$ come from pairs of paths $h_1$ and $h_2$ joining $p_i$ and $p_j$ to some node $e$ of $[\mathcal{C}(f)]$. (These paths were referred to as $\gamma$ and $\gamma'$ in the proof of Theorem 3.1, but there are already enough $\gamma$’s floating around in this proof.)

We may lift $\gamma$ to a path $\gamma_e$ in $[\mathcal{C}(f)]$ which ends at $e$ and along which parallel transport is possible. Let $e' \in [\mathcal{C}(f')]$ be the starting point of $\gamma_e$. There are paths $h'_1$ and $h'_2$ in $[\mathcal{C}(f')]$ joining $p'_i$ and $p'_j$ to $e'$ so that $h'_1$ followed by $\gamma_e$ is homotopic to $\gamma_i$ followed by $h_1$, and $h'_2$ followed by $\gamma_e$ is homotopic to $\gamma_j$ followed by $h_2$. Parallel transport along the image of these paths under $[\hat{f}]$ gives the following commutative diagram:
There is a unique map $P_u' \rightarrow I_e = [0, \infty)$ so that $P_u'(e') \rightarrow P_u$ is obtained by the pullback diagram

$$
\begin{array}{c}
\mathcal{P}(f'(p_i')) \rightarrow \mathcal{P}(f(p_i)) \\
\mathcal{P}(f'(e')) \rightarrow \mathcal{P}(f(e)) \\
\mathcal{P}(f'(p_j')) \rightarrow \mathcal{P}(f(p_j))
\end{array}
$$

(7)

If $e'$ is not a node of $\lceil C(f') \rceil$, then this map has image 0 in $[0, \infty)$ and $P_u'(e')$ is isomorphic to $P_u$. As $Q(e)$ is constructed similarly as a pullback of $[0, \infty)^2 \rightarrow [0, \infty)$ over the corresponding coordinate of $Q$, we get a canonical map $P_u'(e') \rightarrow Q(e)$ so that the following diagram commutes

$$
\begin{array}{c}
P_u'(p_i') = P_u' \rightarrow Q = Q(p_i) \\
P_u'(h_i') \rightarrow Q(h_i) \\
P_u'(e') \rightarrow Q(e) \\
P_u'(h_i') \rightarrow Q(h_i) \\
P_u'(p_j') = P_u' \rightarrow Q = Q(p_j)
\end{array}
$$

(8)

Define $A_h$ analogously to the definition of $A_\gamma$ given in diagram (4) on page 12.

Consider the following diagram.

$$
\begin{array}{c}
\mathcal{P}(e) \xrightarrow{\eta} Q(e) \xrightarrow{A_{h_1}} \mathbb{R}^n \supset \mathcal{P}(f(e)) \\
\mathcal{Q}(h_1) \xrightarrow{\mathcal{Q}(f(h_1))} \mathcal{P}(f(x))
\end{array}
$$

(9)
Diagram (1) implies that the outer loop commutes, the naturality of $P'u'f'$ implies that the left hand square commutes, the middle square commutes because of diagram (8), the right hand square commutes because of diagram (9), and the top loop commutes because the definition of the map $P'u' \subset Q' \rightarrow Q$ involves the map $P'(f'r\gamma_i)$ on the relevant factor projected onto by $P'_u(e')$ and $Qf$. This implies that the bottom loop commutes when $P'_u(e')$ is restricted to the image of $P'_u(p'i')$. The bottom loop also commutes when $P'_u(e')$ is restricted to the complementary image of $P(e')$, therefore the bottom loop of the above diagram commutes.

Repeating the above argument for $h_2$ gives that the following diagram commutes

$$
\begin{array}{ccc}
P'_u(e') & \xrightarrow{P'u'f'} & P(f'(e')) \\
\downarrow & & \downarrow \\
Q(e) & \xrightarrow{P'(f'r\gamma_e)} & P(f(e)) \subset \mathbb{R}^n
\end{array}
$$

It follows that the image of $P'_u(e')$ in $Q_e$ is in the subset where $A_{h_1} = A_{h_2}$. As $P_u$ is the subset of $Q_1$ over which $A_{h_1} = A_{h_2}$ for all such pairs of paths, it follows that the image of $P'_u$ in $Q$ is contained inside $P_u$. As noted above, this implies that the map $P' \rightarrow P'_u$ is an isomorphism, so $\hat{f}$ has universal tropical structure at $f'$.

**Lemma 4.5.** Let $\hat{f}$ be a family of curves with universal tropical structure at some curve $f$. Given any other family of curves $\hat{h}$ containing $f$, by restricting to a neighborhood of $f$ in $\hat{h}$ there exists a map

$$
\begin{array}{ccc}
C(\hat{h}) & \xrightarrow{\Phi} & C(\hat{f}) \\
\downarrow & & \downarrow \\
F(\hat{h}) & \rightarrow & F(\hat{f})
\end{array}
$$

which is the identity map on $C(f)$ within $C(\hat{h})$ and $C(\hat{f})$, and so that in a metric on $\mathcal{B}$, the distance between the maps $\hat{h}$ and $f \circ \Phi$ is bounded.

**Proof:** The bound on the distance between the maps $\hat{h}$ and $\hat{f} \circ \Phi$ ensures that they have the same tropical part. Such a map $\Phi$ induces a map of extensions of tropical structure of $f$, which must therefore coincide with the unique map $\eta$ from the definition of a universal extension of tropical structures. In this proof, we shall use this unique map of tropical structures to construct $\Phi$.

In particular, given any point $x$ in $[C(f)]$, we get a specification of what the tropical part of $\Phi$ should be restricted to coordinate charts on $C(\hat{h})$ and $C(\hat{f})$ containing $x$. The naturality of the map of extended tropical structures implies the following: given a map defined on one coordinate chart with the correct tropical part, the restriction of this map to another coordinate chart will again have the correct tropical part.

Choose a map $F(\hat{h}) \rightarrow F(\hat{f})$ with the correct tropical part and sending $f$ to $f$. (If necessary, restrict to a neighborhood of $f$ in $F(\hat{h})$ so that such a map exists.) Now construct a $\Phi$ so that:

- The following diagram commutes

$$
\begin{array}{ccc}
C(\hat{h}) & \xrightarrow{\Phi} & C(\hat{f}) \\
\downarrow & & \downarrow \\
F(\hat{h}) & \xrightarrow{} & F(\hat{f})
\end{array}
$$
\textbullet{} Φ is the identity restricted to $C(f)$

\textbullet{} The tropical part of Φ in a coordinate chart containing $x \in C(f)$ is the same as the map coming from the unique map of extended tropical structures.

Around $x$ in $C(f)$, there locally exists a map satisfying the above properties. Any two such maps differ by the flow of a vertical vector field which vanishes on $C(f)$, and the flow of any such vertical vector field preserves the above properties.

If Φ is defined on some collection of coordinate charts on $C(\hat{h})$ intersecting $C(f)$, we can extend the domain of definition of Φ to include the next coordinate chart by using a cutoff function to interpolate between the previously defined Φ and a map defined on the new coordinate chart. This constructs such a Φ on a neighborhood of $C(f)$.

Actually, Lemma 4.5 also holds for any family $\hat{h}$ of curves containing a curve $f'$ which is a refinement of $f$ or which is $f$ with some extra external edges or bubble components. This is used in [5] to define an evaluation map to Deligne-Mumford space, and is also useful for proving that Gromov-Witten invariants of exploded manifolds are preserved by the operation of refinement.

**Lemma 4.6.** Let $\hat{f}$ be a family of curves with universal tropical structure at $f$. Let $\hat{h}$ be a family of curves containing a curve $f'$ with a degree 1 holomorphic map $\phi : C(f') \rightarrow C(f)$ so that $f' = f \circ \phi$. Then by restricting to a neighborhood of $f'$ in $\hat{h}$ there exists an extension of $\phi$ to a map

$$
\begin{array}{c c c c}
C(\hat{h}) & \xrightarrow{\phi} & C(\hat{f}) & \\
\downarrow & & \downarrow & \\
F(\hat{h}) & \rightarrow & F(\hat{f})
\end{array}
$$

so that in a metric on $\hat{B}$, the distance between the maps $\hat{h}$ and $\hat{f} \circ \Phi$ is bounded.

**Proof:**

We shall first construct the tropical part of Φ. Then the existence of a map Φ satisfying the required properties will follow as in the proof of Lemma 4.5.

Denote the tropical structure of $\hat{h}$ restricted to $C(f')_T$ by $P_0$. To construct the tropical part of Φ, we need to construct a map from $P_0$ to the universal extension of the tropical structure of $f$. To achieve this, it suffices to construct an extension $P_1$ of the tropical structure of $f$ with a map from $P_0$.

Denote by $P$ the polytope $P_0(\mathcal{F}(f'))$. We shall construct $P_1$ so that $P_1(\mathcal{F}(f)) = P$ too.

We shall define $P_1$ and the natural transformation $\eta$ separately on the inverse image of each point in $C(f)$, and then show these constructions may be glued together.

For any point $x$ in a smooth strata of $C(f)$, set $P_1(x)$ equal to $P$. For any point $y$ in $C(f')_T$ sent to $x$ by $\phi_T$, define $\eta : P_0(y) \rightarrow P_1(x)$.
to be the projection to $P$ coming from the tropical structure of the projection $C(\hat{h}) \rightarrow \mathbf{F}(\hat{h})$. For any smooth strata $C$ of $\mathbf{C}(f)$, the fact that $\phi$ is holomorphic and degree 1 implies that there exists a unique smooth component $C_0$ of $\phi^{-1}(C) \subset \mathbf{C}(f')$ on which $\phi$ is injective. The image $\phi(C_0) \subset C$ is then dense. The inverse image of $x \in C$ is either a single point in $C_0$ or a connected and closed union of strata of $\mathbf{C}(f')$ attached to $C_0$ which map constantly into $x$. If $y$ is a node or marked point of $[\mathbf{C}(f')]$, the fact that the strata corresponding to $y$ maps to $x$ implies that the map $\mathbf{P}_0 f' : \mathbf{P}_0(y) \rightarrow \mathbf{P}(f(x))$ factors through projection to $P$ followed by a map $P \rightarrow \mathbf{P}(f(x))$. The same holds for all points $y$ in the inverse image of $x$ and the inverse image of $x$ is connected, therefore the maps $P \rightarrow \mathbf{P}(f(x))$ must be the same for each $y$ in the inverse image of $x$. We may therefore define

$$\mathbf{P}_1 f : \mathbf{P}_1(x) \rightarrow \mathbf{P}(f(x))$$

so that for any path $\gamma$ in $\phi^{-1}(x)$ along which parallel transport is defined, the diagram

$$\begin{array}{ccc}
\mathbf{P}_0(\gamma(0)) & \xrightarrow{\mathbf{P}_0(\gamma)} & \mathbf{P}_0(\gamma(1)) \\
\downarrow{\eta} & \phantom{\uparrow{\phi f'}} & \phantom{\downarrow{\phi f' \mathbf{P} f}} \\
P = \mathbf{P}_1(x) & \xrightarrow{\mathbf{P}_1 f} & \mathbf{P}(f(x))
\end{array}$$

commutes. For $x$ a point in a smooth component of $\mathbf{C}(f)$, we have now constructed $\mathbf{P}_1$ for the point $x$, and constructed $\eta$ on $\phi^{-1}(x)$.

Suppose that $e$ is a node or puncture of $[\mathbf{C}(f)]$. We shall now construct $\mathbf{P}_1(e)$ and $\eta$ on $\phi^{-1}(e)$. Choose an integral affine identification of the edge of $\mathbf{C}(f)$ corresponding to $e$ as a sub interval of $[0, \infty)$ with closure containing 0. The inverse image of $e$ in $\mathbf{C}(f)$ is a connected, closed union of strata. There is a finite collection of strata $e_1, \ldots, e_n$ of $\phi^{-1}(e)$ with non-constant image in $[0, \infty)$. The fact that $\phi$ is a degree 1 map implies that the image of these strata do not intersect, and each maps isomorphically to a subinterval of $[0, \infty)$. Order the $e_i$ so that $e_i$ has image in $[0, \infty)$ before the image of $e_{i+1}$. In the case that $e$ is a node, these strata correspond to nodes $e_1, \ldots, e_n$. In the case that $e$ is a puncture, $e_n$ is a puncture and the other $e_i$ are nodes. According to Definition 2.1 part 2 corresponding to each node $e_i$, there is a pullback diagram

$$\begin{array}{ccc}
\mathbf{P}_0(e_i) & \xrightarrow{[0, \infty)^2} & [0, \infty)^2 \\
\downarrow{\rho_i} & & \downarrow{a + b} \\
P & \xrightarrow{\mathbf{P}} & [0, \infty)
\end{array}$$

If $e$ is a node, define $\mathbf{P}_1(e)$ via the pullback diagram

$$\begin{array}{ccc}
\mathbf{P}_1(e) & \xrightarrow{[0, \infty)^2} & [0, \infty)^2 \\
\downarrow{\text{sum}} & & \downarrow{a + b} \\
P & \xrightarrow{\sum_i \rho_i} & [0, \infty)
\end{array}$$

Let $p \in P$ be the point corresponding to the tropical part of the inclusion $\mathbf{F}(f') \rightarrow \mathbf{F}(\hat{h})$. The length of the edge in $\mathbf{C}(f')$ corresponding to $e_i$ is $\rho_i(p)$, so the length of the edge corresponding to $e$ in $\mathbf{C}(f)$ is equal to $\sum_i \rho_i(p)$. In particular, we may choose an isomorphism of $\mathbf{P}(e)$ with the fiber of $\mathbf{P}_1(e) \rightarrow P$ over $p$. 

If on the other hand, $e$ is an external edge, define $\mathcal{P}_1(e) := P \times [0, \infty)$. In this case, the fiber of $\mathcal{P}_1(e)$ over $p \in P$ is uniquely isomorphic to $\mathcal{P}(e)$.

In either case, for any $y$ in the inverse image of $e$, there exists a unique integral affine map

$$\eta : \mathcal{P}_0(y) \to \mathcal{P}_1(e)$$

so that the following diagram commutes.

Given any path $\gamma$ in the inverse image of $e$ joining $y'$ to $y$, the following diagram commutes

Therefore, the uniqueness of $\eta$ implies that the composition of the arrows in the middle row is $\eta : \mathcal{P}_0(y') \to \mathcal{P}_1(e)$. Therefore, $\eta$ defines a natural transformation from $\mathcal{P}_0$ restricted to $\phi_T^{-1}(e)$ to the constant functor with image $\mathcal{P}_1(e)$.

So far, for every object $x$ in $C_T(f)$ we have the following commuting diagram of functors from $\phi_T^{-1}(x)$

In fact, the $\mathcal{P}_1 \circ \phi_T$ in the above diagram may be regarded as a pushout in the following sense:

**Claim 4.7.** Given any constant functor $\mathcal{P}'$ from $\phi_T^{-1}(x)$ to a polytope $P'$ and a commuting diagram of functors on $\phi_T^{-1}(x)$,

there exists a unique map

$$\mathcal{P}_1(x) \to P'$$
corresponding to a natural transformation $\mathcal{P}_1 \circ \Phi_T \rightarrow \mathcal{P}'$ so that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\mathcal{P}_\phi} & \mathcal{P} \circ \phi_T \\
\downarrow \eta & & \downarrow \exists \\
\mathcal{P}_0 \circ \phi_T & \rightarrow & \mathcal{P}'
\end{array}
\]

To prove Claim 4.7, consider the diagram

\[
\begin{array}{ccc}
\mathcal{P}(y) & \xrightarrow{} & \mathcal{P}(x) \\
\downarrow & & \downarrow \\
\mathcal{P}_0(y) & \xrightarrow{} & \mathcal{P}_1(x)
\end{array}
\]

for any object $y$ in $\phi_T^{-1}(x)$. The associated diagram featuring the affine spaces generated by the affine polytopes is a pushout diagram of integral linear spaces. This may be seen by considering the following three cases: If the dimensions of $\mathcal{P}(y)$ and $\mathcal{P}(x)$ are equal, then the map $\mathcal{P}_0(y) \rightarrow \mathcal{P}_1(x)$ generates an isomorphism of integral affine spaces. If $y$ is a node or edge and $x$ is a point in a smooth component, then $\mathcal{P}(y) \rightarrow \mathcal{P}_0(y)$ is injective, and the integral affine map generated by $\mathcal{P}_0(y) \rightarrow \mathcal{P}_1(x)$ is the quotient map which sends the span of the image of $\mathcal{P}(y)$ to a single point. If $y$ is a point in a smooth component, and $x$ is a node or edge, then the integral affine space spanned by $\mathcal{P}_0(y)$ is injective, and the integral affine space generated by $\mathcal{P}_0(y) \rightarrow \mathcal{P}_1(x)$ is the span of the complementary images of $\mathcal{P}_0(y)$ and $\mathcal{P}(x)$. In each of the above three cases, the diagram of integral affine spaces generated by the above diagram is a pushout diagram.

Therefore, if we embed $\mathbb{P}'$ into $\mathbb{R}^n$, there exists a unique map $\mathcal{P}_1(x) \rightarrow \mathbb{R}^n$ so that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{P}_1(x) & \xleftarrow{} & \mathcal{P}(x) \\
\eta & \swarrow & \downarrow \\
\mathcal{P}_0(y) & \rightarrow & \mathbb{R}^n
\end{array}
\]

The uniqueness of this map and the fact that $\phi_T^{-1}(x)$ is connected implies that the same map is obtained using any $y$ in the inverse image of $x$. As the union of the images of $\eta$ is all of $\mathcal{P}_1(x)$, this map $\mathcal{P}_1(x) \rightarrow \mathbb{R}^n$ must have image inside $\mathbb{P}'$. This constructs the unique map $\mathcal{P}_1(x) \rightarrow \mathbb{P}'$ and completes the proof of Claim 4.7.

We may now use Claim 4.7 to complete the description of the functor $\mathcal{P}_1$ and verify that $\eta$ is indeed a natural transformation. Let $\gamma$ be a path in $\mathcal{C}(f')_T$ joining $y$ to $y'$ so that $\phi_T \gamma$ is non constant and joins $x$ to $x'$. Then the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{P}(y) & \xrightarrow{\mathcal{P}_\phi} & \mathcal{P}(x) \\
\downarrow \mathcal{P}(\gamma) & & \downarrow \mathcal{P}(\phi_T \gamma) \\
\mathcal{P}_0(y') & \xrightarrow{\mathcal{P}_\phi} & \mathcal{P}(x') \\
\downarrow \mathcal{P}_0(\gamma) & & \downarrow \mathcal{P}_1(\gamma) \\
\mathcal{P}_0(y') & \xrightarrow{\eta} & \mathcal{P}_1(x')
\end{array}
\]
As $\phi_T\gamma$ is non constant, $y$ must be equal to $\phi^{-1}(x)$. Consider $P_1(x')$ as a constant functor from $\phi_T^{-1}(x) = y$ and use the outer loop in the above commutative diagram to apply Claim 4.7. There therefore exists a unique map

$$P_1(x) \longrightarrow P_1(x')$$

so that the following diagram commutes

\[
\begin{array}{ccc}
P(y) & \xrightarrow{P} & P(x) \\
\downarrow & & \downarrow \phi_T \\
P_0(y) & \xrightarrow{\eta} & P_1(x) \\
\downarrow & & \downarrow P_1(x') \\
\end{array}
\]

Define $P_1(\phi_T\gamma)$ to be this map $P_1(x) \longrightarrow P_1(x')$. As every other map in the above diagram commutes with projection to $P$, Claim 4.7 implies that $P_1(\phi_T\gamma)$ also commutes with projection to $P$.

$$P_1(x) \xrightarrow{P_1(\phi_T\gamma)} P_1(x')$$

The fact that $\phi_T\gamma$ is non constant implies that $P_0(y) = P = P_1(x)$. If $P_1(x') = P$ too, then the above commutative diagram implies that $P_1(\phi_T\gamma)$ is the identity. Otherwise, the map $P_1(\phi_T\gamma) : P_1(x) \longrightarrow P_1(x')$ is the inclusion of $P$ as the face of $P_1(x')$ which contains the image of $P(\phi_T\gamma)$ in $P(x') \subset P_1(x')$. For non constant paths $\gamma$ which are not in the image of $\Phi$, we may similarly define $P_1(\gamma)$ to be the inclusion of $P_1(\gamma(0)) = P$ as the face of $P_1(\gamma(1))$ which contains the image of $P(\gamma)$. This defines $P_1$ as a functor.

We have already observed that $\eta$ restricted to the inverse image of any object in $C(f)_T$ is a natural transformation. The bottom left hand square of diagram (10) then implies that $\eta$ is a natural transformation.

To describe $P_1$ as an extension of the tropical structure of $f$ we must also describe a natural transformation $P_1f$ from $P_1$ to $P \circ f_T$. Consider the commutative diagram of functors from $C(f')_T$ to the category of integral affine polytopes:

\[
\begin{array}{ccc}
P & \xrightarrow{P} & P \circ \phi_T \\
\downarrow & & \downarrow P_f \circ \phi_T \\
P_0 & \xrightarrow{P_0f'} & P \circ f_T \circ \phi_T \\
\end{array}
\]

Claim 4.7 then gives us for every object $x$ in $C(f)_T$ a unique map

$$P_1f : P_1(x) \longrightarrow P(f(x))$$

so that for any $y$ in $\phi_T^{-1}(x)$, the following diagram commutes

\[
\begin{array}{ccc}
P_1(x) & \xleftarrow{\eta} & P(x) \\
\uparrow & & \uparrow P_f \\
P_0(y) & \xrightarrow{P_0f'} & P(f(x)) \\
\end{array}
\]
We already know that every map apart from $P_1 f$ in the above diagram comes from a natural transformation. Let $\gamma$ be a path from $y$ to $y'$ and $\phi_T(\gamma)$ be a path from $x$ to $x'$, and consider the diagram

\[ \begin{array}{ccc}
P_1(x) & \xrightarrow{\phi_T(\gamma)} & P_1(x') \\
p & \downarrow & p \\
P_0(y) & \xrightarrow{P_0 f'} & P(f(x)) \\
\end{array} \quad \begin{array}{ccc}
P_1(x) & \xrightarrow{\phi_T(\gamma)} & P_1(x') \\
p & \downarrow & p \\
P_0(y) & \xrightarrow{P_0 f'} & P(f(x')) \\
\end{array} \]

The commutativity of the squares corresponding to maps which we know are natural transformations imply that square which should commute if $P_1 f$ is a natural transformation commutes when $P_1(x)$ is restricted to the image of either $P_0(x)$ or $P_0(y)$. As these two sub polytopes span $P_1(x)$, it follows that the square required for $P_1 f$ to be a natural transformation commutes. As any path in $C(f)_T$ may be written as a composition of paths in the image of $\phi_T$ or their inverses, it follows that $P_1 f$ is a natural transformation.

The diagram (11) above therefore corresponds to a commutative diagram of natural transformations. It follows that $P_1$ is an extension of the tropical structure of $f$, and $P_1 f \circ \eta = P_0 f'$.

We have now constructed the required commutative diagram.

\[ \begin{array}{ccc}
(C(f')_T, P_0) & \xrightarrow{(f'_T, P_0 f')} & (f_T, P_0 f') \\
\downarrow_{(\phi_T, \eta)} & & \downarrow_{(\phi_T, \eta)} \\
(C_T, P_1) & \xrightarrow{(f_T, P_1 f)} & (B_T, P) \\
\end{array} \]

As $P_1$ must be a pullback of the universal extension of the tropical structure of $f$, we therefore get a commutative diagram

\[ \begin{array}{ccc}
(C(f')_T, P_0) & \xrightarrow{(f'_T, P_0 f')} & (f_T, P_0 f') \\
\downarrow & & \downarrow \\
(C_T, P_1) & \xrightarrow{(f_T, P_1 f)} & (B_T, P) \\
\downarrow & & \downarrow \\
(C_T, P_u) & \xrightarrow{(f_T, P_u f)} & (B_T, P) \\
\end{array} \]

This gives the tropical part of our required map $\Phi$ from a neighborhood of $C(f')$ in $C(\hat{h})$ to $C(\hat{f})$. We may now construct $\Phi$ as in the proof of Lemma 4.5.

The following theorem is used in [8] to construct a concrete local model form the moduli stack of stable curves in $B$.

**Theorem 4.8.** For any stable curve $f$ in $\hat{B}$ with domain not equal to $T$, there exists a family of curves $\hat{f}$ containing $f$ with universal tropical structure so that

1. there is a group $G$ of automorphisms of $\hat{f}$ which acts freely and transitively on the set of maps of $f$ into $\hat{f}$.
2. There is only one strata $F_0$ of $F(\hat{f})$ which contains the image of a map $f \rightarrow \hat{f}$, and the smooth part of this strata, $\lceil F_0 \rceil$ is a single point.
3. The action of $G$ on $\lceil C(\hat{f}) \rceil$ restricted to the inverse image of $[F_0]$ is effective, so $G$ may be regarded as a subgroup of the group of automorphisms of $[f]$.

□
Proof:
To start with, Lemma 4.2 constructs a family of curves \( \hat{f}_0 \) which contains \( f \) and which has universal tropical structure at \( f \). By restricting \( \hat{f}_0 \) to a subfamily if necessary, we may assume that \( \mathbf{F}(\hat{f}_0) \) is an open subset of \( T^m_{\hat{P}_n} \) and the strata containing \( f \) corresponds to the interior \( P^n_u \) of \( P_n \). Lemma 4.4 implies that \( \hat{f}_0 \) has universal tropical structure. It follows that any map of \( f \) into \( \hat{f}_0 \) must have image inside \( T^m_{\hat{P}_n} \subset T^m_{P_n} \). In what follows, we shall first construct a group \( G \) which acts transitively and freely on the set of maps of \( f \) into \( \hat{f}_0 \), then verify that \( G \) is a subgroup of the automorphisms of \([f]\), then construct a \( G \) action on \( C(\hat{f}_0) \), then modify \( \hat{f}_0 \) to be \( G \)-invariant.

Lemma 4.5 gives that we may extend each inclusion of \( f \) into \( \hat{f}_0 \) to a map \( \Phi : C(\hat{f}_0) \rightarrow C(\hat{f}_0) \) so that in any metric on \( \hat{B} \), the distance between \( \hat{f}_0 \) and \( \hat{f}_0 \circ \Phi \) is bounded. In particular, this implies that restricted to the inverse image of \( T^m_{\hat{P}_n} \) within \( C(\hat{f}_0) \), \( \hat{f}_0 \circ \Phi = \hat{f}_0 \). The tropical part of \( \Phi \) is uniquely determined by the universal property of \( P_n \), therefore \( \Phi \) is uniquely determined restricted to the inverse image of \( T^m_{\hat{P}_n} \), because the smooth part of \( T^m_{\hat{P}_n} \) is a point.

To summarize, any two inclusions of \( f \) in \( \hat{f}_0 \) are exchanged by a unique automorphism of \( \hat{f}_0 \) restricted to the inverse image of \( T^m_{\hat{P}_n} \). Let \( G \) be the group of automorphisms of \( \hat{f}_0 \) restricted to \( T^m_{\hat{P}_n} \). So far, we have that \( G \) acts freely and transitively on the set of maps of \( f \) into \( \hat{f}_0 \).

The smooth part of the inverse image of \( T^m_{\hat{P}_n} \) in \( C(\hat{f}_0) \) is equal to the smooth part of \( C(f) \). There is therefore an action of \( G \) as automorphisms of \([f]\). Remark 3.3 implies that there is an injective map with domain \( T^m_{\hat{P}_n} \) which evaluates \( \hat{f}_0 \) at a collection of marked point sections, and records how the smooth strata of \( C(\hat{f}) \) over \( T^m_{\hat{P}_n} \) are glued together at nodes. Any two inclusions of \( f \) into \( \hat{f} \) with the same smooth part must therefore have the same image in \( T^m_{\hat{P}_n} \), as they will have the same labeling of marked points and nodes. Any element of \( G \) which acts trivially on \( \{C(f)\} \) is therefore an automorphism of \( f \) itself. The only automorphism of \( f \) which acts trivially on \( \{C(f)\} \) is the identity. As we have that \( G \) acts freely on the set of maps of \( f \) into \( \hat{f}_0 \), it follows that the action of \( G \) on \( \{C(f)\} \) is effective, so we may regard \( G \) as a subgroup of the automorphisms of \([f]\).

For curves in a smooth manifold, \( G \) is equal to the automorphism group of \([f]\), but in general, \( G \) is a subgroup of the group of automorphisms of \([f]\).

We shall now extend the action of \( G \) to \( C(\hat{f}_0) \). In the construction of \( C(\hat{f}_0) \) from the proof of Lemma 4.2, we started with a collection of coordinate charts \( U_i \) on \( C(f) \) which we then extended to coordinate charts \( \hat{U}_i \) on \( C(\hat{f}) \). We may make the further assumption that the action of \( G \) preserves the smooth part of these coordinate charts, so the smooth part of \( U_i \) is always sent to the smooth part of some coordinate chart \( U_j \). (One way to achieve this is by choosing a \( G \) invariant metric on \( \{C(f)\} \) in the conformal class determined by the complex structure, and choosing the coordinate charts of type 4 and 2 to have image in \( \{C(f)\} \) the set of points within some given distance of the given node or marked point.)

The smooth part of \( \hat{U}_i \) restricted to the inverse image of \( T^m_{\hat{P}_n} \) is equal to the smooth part of \( U_i \). Therefore, restricted to \( T^m_{\hat{P}_n} \), \( G \) sends \( \hat{U}_i \) isomorphically into some \( \hat{U}_j \). These maps are equivariant in the sense defined in 4. In other words these maps send smooth monomials to smooth monomials. As noted in Remark 4.3.
\{\tilde{U}_i\} is an equivariant collection of coordinate charts on \( C(\hat{f}_0) \), therefore we may uniquely modify \( \Phi \) outside of the inverse image of \( T_{P_m}^* \) to make it an equivariant map. In practice, this means that if \( U_i \) is a chart of type 1 or 2 we may make \( \Phi : \tilde{U}_i \longrightarrow \hat{U}_j \) be the unique monomial map which restricts to be the correct map on the inverse image of \( T_{P_m}^* \). If \( U_i \) is a chart of type 3 then \( \tilde{U}_i \) is \( U_i \) times \( F(f) \), and we may uniquely modify \( \Phi : \tilde{U}_i \longrightarrow \hat{U}_j \) to be the product of a map \( U_i \longrightarrow U_j \) with a monomial automorphism of \( F(f) \) so that \( \Phi \) is the correct map restricted to the inverse image of \( T_{P_m}^* \). The definition of the fiberwise complex structure on \( \hat{U}_i \) is such that these equivariant maps are automatically fiberwise complex.

\( \Phi \) with this extra condition that it is equivariant in the given coordinate charts is unique. This uniqueness implies that such equivariant \( \Phi \)'s extend the action of \( G \) to an action as a group of automorphisms of \( C(\hat{f}_0) \).

By an averaging procedure, we may modify the map \( \hat{f}_0 \) outside the inverse image of \( T_{P_m}^* \) to a map \( \hat{f} \) which is preserved by this action of \( G \). In particular, let \( U \) be a \( G \)-invariant open subset of \( C(\hat{f}_0) \) so that

- each connected component of \( U \) is simply connected, and intersects the inverse image of \( T_{P_m}^* \) in a connected subset,
- and \( \hat{f}_0(U) \) is contained in a coordinate chart \( V = \mathbb{R}^n \times T_{P_m}^* \) on \( \hat{b} \) so that \( T_{\text{vert}} \hat{b} \) restricted to \( V \) corresponds to the span of some fixed subset of the standard basis vectors \( \{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_j} \} \).

Then \( \hat{f}_0 \circ \Phi \) on \( U \) is equal to \( f \) followed by the exponentiation of some section \( v_{\Phi} \) of \( f_0 T_{\text{vert}} \hat{b} \) using the (locally defined) connection which preserves the standard basis vectors on \( V \). The section \( v_{\Phi} \) is unique when we impose the condition that it vanishes on the inverse image of \( T_{P_m}^* \). The map \( \hat{f}_0 \) followed by exponentiation of the section which is the average of \( v_{\Phi} \) for all \( \Phi \) is a \( G \)-invariant map \( U \longrightarrow V \) which agrees with \( \hat{f}_0 \) on the inverse image of \( T_{P_m}^* \). Using a cut off function to interpolate between \( \hat{f}_0 \) and this new map, we obtain a new map which is \( G \)-invariant on a compactly contained open subset \( U_0 \) of \( U \).

In fact, \( v_{\Phi} \) will be 0 on the connected components of the subsets of \( U \) where \( \hat{f}_0 \) is already \( G \)-invariant which intersect the inverse image of \( T_{P_m}^* \). Therefore, this new \( G \)-invariant map will agree with \( \hat{f}_0 \) on these subsets, and therefore remain \( G \)-invariant there. We may repeat the argument some finite number of times for new subsets \( U_0 \) until we obtain a map \( \hat{f} \) which is \( G \)-invariant on a \( G \)-invariant neighborhood of the inverse image of \( T_{P_m}^* \) in \( C(\hat{f}_0) \).

Restricting \( \hat{f} \) to this \( G \)-invariant neighborhood gives the required family with automorphism group \( G \). As \( \hat{f} \) agrees with \( \hat{f}_0 \) on the inverse image of \( T_{P_m}^* \), \( G \) acts freely and transitively on the set of maps \( f \longrightarrow \hat{f} \), and the tropical structure of \( \hat{f} \) restricted to \( C_T(f) \) is the universal extension of the tropical structure of \( f \).

\[ \Box \]

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