Analytical Results for the Four-Loop RG Functions in the 2D Non-Linear $O(n)$ $\sigma$-Model on the Lattice

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Abstract—We recalculate four-loop renormalization group functions in 2-dimensional nonlinear $O(n)$ $\sigma$-model using coordinate-space method. The high accuracy of calculation allow us to find the analytical form of $\hat{\beta}$- and $\hat{\gamma}$-function (anomalous dimension).

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1. INTRODUCTION

Non-linear $\sigma$-models have been the objects of the intensive studies for many years. The particular case of these models, considered in this paper, is the 2-dimensional non-linear $O(n)$ $\sigma$-model. This model is known to be asymptotically free and can be applied, e.g. to the study of ferromagnetic systems. It can also serve as a toy model for the strong interactions in particle physics.

In calculations of physically interesting characteristics it is important to know the $\beta$-function and anomalous dimension $\gamma$. The know of them allows, in particular, to predict the correlation length $\xi$ and the spin susceptibility $\chi$. In the regime of weak coupling $\beta$- and $\gamma$-function can be evaluated as perturbative series in the coupling constant. In order to study the whole range of the coupling constant one has to appeal to the lattice simulations. Due to the asymptotical freedom this model is especially suitable for such study. For the precise comparision of Monte Carlo data with perturbative series in the lattice regularization are required. Such calculation to two-loops have been done analytically in [1] and then pushed forward to four-loops in [2] numerically and checked in [3]. At the same time the analogous results at the four-loop order in the continuum limit are known analytically [4].

The goal of this work is to find the analytical expressions for the renormalization group (RG) coefficients to the four-loop order in the lattice perturbation theory. In order to do this we use the methods proposed in the continuum field theory for the evaluation of the multiloop integrals. Diagrams on the lattice, as well as in the continuum limit, are related to each other algebraically. Such relations arise due to the integration by part method [5], which leads, in general, to the reduction of the number of independent integrals. However the realization of this algorithm in the lattice already at the three-loop level is quite difficult task.

In Section 2 we give the definitions and discuss the method. In Section 3 our results are presented and in Appendix A we give all integrals from [2] separately.

2. DEFINITIONS

The action of the non-linear $O(n)$ $\sigma$-model is usually written in the form

$$S = \frac{1}{2f_0} \int d^2 x (\partial_\mu q(x) \partial^\mu q(x)),$$

where $q(x)$ is an $n$-component real vector field of unit length and $f_0$ be the bare coupling constant. In the lattice formulation the derivatives are, as usually, understood as finite differences.

The perturbative expansions of the $\hat{\beta}$ - and $\hat{\gamma}$ -functions can be written as follows

$$\hat{\beta}(f) = -a \frac{d}{da} f_0 = -2\pi(n-2) \sum_{L=1} b^{(L)}(f_0) \frac{f_0}{2\pi}^{L+1},$$

$$\hat{\gamma}(f) = -a \frac{d}{da} \ln Z = 2\pi(n-1) \sum_{L=1} c^{(L)}(f_0) \frac{f_0}{2\pi}^{L+1},$$

where $a$ is the lattice spacing and $Z$ is the renormalization constant of the field. Prefactors $(n-2)$ and $(n-1)$ in the above formulae always factorize and we take them in front of the expressions.

Coefficients $b^{(L)}$ and $c^{(L)}$ can be computed using technique of Feynman diagrams. Generally, Feynman diagrams on the lattice are, generally, more difficult to evaluate than the ones in the continuum field theory. Therefore the analytical results in the lattice are

\[\hat{\gamma}^{(L)}\] and $c^{(L)}$ are defined slightly different than these in [2].
known only to two loops [1], while analogous quantities in the continuum theory are known to four loops [4]. The RG coefficients were computed on the lattice numerically to four loops [2], where they were expressed in terms 12 different integrals. The evaluation of this integrals has been repeated in [3] to somewhat better accuracy (about ~ 10^{-9}) and the wrong notation of [2] was clarified in [6].

It is known that between different Feynman diagrams there are many algebraic relations, which can be obtained by partial integration [5]. This explains the fact that a big amount of different integrals could be expressed as linear combinations of few constants (irrationalities) with rational coefficients. Moreover there were proposed some rules how to predict the constants that occur in higher loop calculations [7, 8]. The interesting question arises: which constants that occur in higher loop calculations [7, 8].

Given accuracy d, “detection threshold” ε and norm bound N, the PSLQ test allows to find out whether relation (4) exists or not (see details in [9]). This approach has been applied in several calculations (see e.g. [10]).

Thus we come the following basis elements
\[
\begin{align*}
\pi, \log 2, \\
\pi^2, \pi \log 2, \log^2 2, G, \\
\pi^3, \pi^2 \log 2, \pi \log^2 2, \log^3 2, G \pi, G \log 2, \zeta_3, Ls_3(\pi/2)
\end{align*}
\]

where \( \zeta_k = \zeta(k) \) is Riemann \( \zeta \)-function, \( G = 0.91596559417721901... \) is the Catalan constant and the constant \( Ls_2(\pi/3) = 1.014941606409653625... \) is defined through the so-called log-sine integral [13]

\[
Ls_k(\theta) = \int_0^{k-1} \left( 2 \sin \frac{\theta}{2} \right) d\theta.
\]

In Eqs. (5) and (6) the first, second and third lines correspond to weights 1, 2 and 3 respectively. The elements of higher weights would correspond to higher loop integrals and do not appear here.

3. RESULTS AND DISCUSSION

We applied the ideas explained above to the lattice integrals presented in [2]. The integrals were computed to accuracy better than 10^{-40} using the coordinate-space method proposed in [14]. The most problematic integrals \( V_5 \) and \( V_6 \) were computed even to higher accuracy. The analysis established that these integrals can be expressed within bases (5) and (6) plus one more constant, introduced below. From 28 elements of (5) and (6) only 5 do contribute. Namely, we were able to express all integrals evaluated numerically in [2, 3] in terms of the following six irrational constants

\[
\pi, \pi^2, \zeta_3, G, \frac{Ls_3(\pi/3)}{3}, \text{and } (2\pi)^3 K,
\]

where integral \( K \) is the same three-loop bubble as in [2, 3].

Among these integrals only for \( K \) we did not find a relation to the bases (5) and (6). Therefore we include it as an independent constant. However it is not excluded that \( (2\pi)^3 K \) can be rewritten as a linear combination of elements (5) and (6) and the possible rea-
son for our misreading is the lack of the accuracy for
the numerical value of this integral.

For the last constant $K$ we give numerical result accurate to $10^{-37}$
\[ (2\pi)^3 K = 23.7849506237378578142256363314563137344 \]

For the anomalous dimension (3) we have
\[ \zeta^{(1)} = 1, \]
\[ \zeta^{(2)} = \frac{1}{2}\pi, \]
\[ \zeta^{(3)} = \frac{n + 9}{24} \pi^2 - \frac{n - 2}{2}, \]
\[ \zeta^{(4)} = \frac{(n - 2)(127n - 121)}{24} \zeta_3 + \frac{(n - 2)(n + 1)}{16} (2\pi)^3 K - \frac{4n - 11}{24} \pi^3 \]
\[ + 5(n - 2) \frac{L_{s_2}(\pi/3)}{\sqrt{3}} - 10(n - 2) \pi G \]
\[ - \frac{3n^3 - 11n + 2}{6} \pi + (n - 2)(7n + 8) \frac{L_{s_2}(\pi/3)}{\sqrt{3}} \]
\[ + (2n - 4) G + \frac{13(n - 2)}{4} \pi - \frac{(n - 2)(10n - 21)}{2}. \]

In conclusion, we expressed RG functions within the lattice regularization in terms of six irrational constants given by (7). The algebraic structure of the above results suggests that there should exist a method of algebraic reduction of diagrams to a set of a few master integrals. As it is mentioned in the beginning of the paper such method exists in continuum field theory and is based on the integration by parts (5) in the momentum space. On the lattice however reduction algorithms are not so obvious. In the simplest case of vacuum one-loop bubble diagrams algebraic method was discussed in [15]. In more complicated cases only few investigations has been done in this directions (see e.g. [16]). The development of algebraic methods is desirable and they could be very useful tools for higher loop computations on the lattice.

A INTEGRALS

In this appendix we present separately our analytical results for the integrals that enter RG functions. They are given in [14] and [3] numerically. So our results for these integrals read

\[ (2\pi)^3 G_1 = \frac{1}{2} \zeta_3 + 1, \]
\[ (2\pi)^3 R = \frac{L_{s_2}(\pi/3)}{\sqrt{3}}, \]
\[ (2\pi)^3 J = -24 \zeta_2 \pi + 96 \zeta_3, \]
\[ (2\pi)^3 L_1 = -\frac{7}{2} \zeta_3 + 3 \zeta_2, \]
\[ (2\pi)^3 V_1 = \frac{7}{2} \zeta_3, \]
\[ (2\pi)^3 V_2 = \frac{14}{3} \zeta_3 - 4 \zeta_2 + 8 \frac{L_{s_2}(\pi/3)}{\sqrt{3}} - 4, \]
\[ (2\pi)^3 V_3 = \frac{56}{3} \zeta_3 - 16 \zeta_2 + 24 \frac{L_{s_2}(\pi/3)}{\sqrt{3}} - 16 + (2\pi)^3 K, \]
\[ (2\pi)^3 V_4 = -\frac{13}{24} \zeta_3, \]
\[ (2\pi)^3 V_5 = \frac{19}{2} \zeta_3 - 3 \pi \zeta_2 + 4 \zeta_3, \]
\[ (2\pi)^3 V_6 = \frac{14}{3} \zeta_3 - 8 \zeta_2 + \frac{1}{2} (2\pi)^3 K, \]
\[ (2\pi)^3 W_1 = -\frac{1}{2} \frac{L_{s_2}(\pi/3)}{\sqrt{3}}, \]
\[ (2\pi)^3 W_2 = \frac{1}{2} \zeta_3 + \frac{3}{2} \pi \frac{L_{s_2}(\pi/3)}{\sqrt{3}} - \frac{5}{2} \pi G \]
\[ + \frac{1}{2} \zeta_3 + \frac{11}{2} \frac{L_{s_2}(\pi/3)}{\sqrt{3}} + \frac{1}{2} G - \frac{1}{2}. \]
And according to [6]

\[
W_2 = \hat{W}_2 + \frac{85}{2304 \pi^3} \zeta_3.
\]

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REFERENCES

1. M. Falcioni and A. Treves, Nucl. Phys. B 265, 671 (1986); S. Caracciolo and A. Pelissetto, Nucl. Phys. B 420, 141 (1994).
2. S. Caracciolo and A. Pelissetto, Nucl. Phys. B 455, 619 (1995).
3. D. Shin, Nucl. Phys. B 546, 669 (1999).
4. S. Hikami and E. Brezin, J. Phys. A 11, 1141 (1978); S. Hikami, Phys. Lett. B 98, 208 (1981); Nucl. Phys. B 215, 555 (1983); W. Bernreuther and F. J. Wegner, Phys. Rev. Lett. 57, 1383 (1986); F. Wegner, Nucl. Phys. B 316, 663 (1989).
5. K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B 192, 159 (1981).
6. B. Alles, S. Caracciolo, A. Pelissetto, and M. Pepe, Nucl. Phys. B 562, 581 (1999).
7. D. J. Broadhurst, Eur. Phys. J. C 8, 311 (1999).
8. J. Fleischer and M. Yu. Kalmykov, Phys. Lett. B 470, 168 (1999).
9. H. R. P. Ferguson and D. H. Bailey, *RNR Technical Report*, RNR-91-032; H. R. P. Ferguson, D. H. Bailey, and S. Arno, *NASA Technical Report*, NAS-96-005.
10. B. A. Kniehl, A. V. Kotikov, A. I. Onishchenko, and O. L. Veretin, Phys. Rev. Lett. 97, 042001 (2006); B. A. Kniehl, A. V. Kotikov, and O. L. Veretin, Phys. Rev. Lett. 101, 193401 (2008); B. A. Kniehl, A. V. Kotikov, and O. L. Veretin, arXiv:0909.1431 [hep-ph].
11. A. I. Davydchev and M. Yu. Kalmykov, Nucl. Phys. B 605, 266 (2001).
12. M. Yu. Kalmykov and O. L. Veretin, Phys. Lett. B 483, 315 (2000); J. Fleischer and M. Yu. Kalmykov, Comput. Phys. Commun. 128, 531 (2000).
13. L. Lewin, *Polylogarithms and Associated Functions* (North-Holland, Amsterdam, 1981).
14. D. Shin, Nucl. Phys. B 525, 457 (1998).
15. S. Caracciolo, P. Menotti, and A. Pelissetto, Nucl. Phys. B 375, 195 (1992).
16. T. Becher and K. Melnikov, Phys. Rev. D 66, 074508 (2002).