Indices of collections of equivariant 1-forms and characteristic numbers

W. Ebeling and S. M. Gusein-Zade *

Abstract

If two $G$-manifolds are $G$-cobordant then characteristic numbers corresponding to the fixed point sets (submanifolds) of subgroups of $G$ and to normal bundles to these sets coincide. We construct two analogues of these characteristic numbers for singular complex $G$-varieties where $G$ is a finite group. They are defined as sums of certain indices of collections of 1-forms (with values in the spaces of the irreducible representations of subgroups). These indices are generalizations of the GSV-index (for isolated complete intersection singularities) and the Euler obstruction respectively.

Introduction

Two closed manifolds are (co)bordant if and only if all their Stiefel–Whitney characteristic numbers coincide. Unitary ($U$-) manifolds, that is manifolds with complex structures on the stable normal bundles, are generalizations of complex analytic manifolds adapted to (co)bordism theory. Two closed $U$-manifolds are (co)bordant if and only if all their Chern characteristic numbers coincide. (For the facts from cobordism theory see, e.g., [10].) Analogues of Chern numbers for (compact) singular complex analytic varieties were considered in [3] and [4]. In [3] they were defined for compact complex analytic varieties with isolated complete intersection singularities (ICIS). In [4] (other) characteristic numbers were defined for arbitrary compact analytic varieties. (For non-singular varieties both sets of characteristic numbers coincide with the usual Chern numbers.) A survey of these and adjacent results can be found in [5].

*Partially supported by DFG (Mercator fellowship, Eb 102/8–1), RFBR–13-01-00755, and NSh–5138.2014.1. Keywords: singularities, finite group action, equivariant 1-forms, indices, characteristic numbers. AMS 2010 Math. Subject Classification: 14B05, 58E40, 58A10, 57R20.
Each characteristic number from any of the sets is the sum of indices of singular (or special) points of (generic) collections of 1-forms defined in \[3\] and \[4\]. The indices from \[3\] are generalizations of the GSV-index of a 1-form defined initially for vector fields in \[5\] and \[8\]. The indices from \[4\] are generalizations of the Euler obstruction defined in \[7\]. Chern characteristic numbers of (closed) complex analytic manifolds cannot be arbitrary: they satisfy some divisibility conditions. In \[2\] it was shown that the characteristic numbers of singular analytic varieties from \[4\] can have arbitrary sets of values.

Let us consider varieties (manifolds) with actions of a finite group \(G\). Let \(V\) be a \(U\)-manifold (for example, a complex analytic manifold of (complex) dimension \(d\), that is with real dimension \(2d\)) with a \(G\)-action respecting the \(U\)-structure. This means that the trivial extension of the \(G\)-action on the tangent bundle \(TV\) to its stabilization preserves the complex structure on it. For a subgroup \(H \subset G\), let \(V^H\) be the set of fixed points of the subgroup \(H\). The manifold \(V^H\) is a \(U\)-manifold, i.e. its stable tangent bundle has a natural complex structure. The normal bundle of \(V^H\) has the structure of a complex vector bundle and the group \(H\) acts on it. Let \(r(\langle H \rangle)\) be the set of the isomorphism classes of the complex irreducible representations of the group \(H\) and let \(r^+\langle H \rangle\) be the set of non-trivial ones. For a component \(N\) of \(V^H\), the normal bundle of \(V\) is the direct sum of vector bundles of the form \(\alpha \otimes \nu_{\alpha}\) where \(\alpha\) runs over all the irreducible non-trivial representations of \(H\) and \(\nu_{\alpha}\) are complex vector bundles (with trivial \(H\)-actions). Moreover

\[
\sum_{\alpha \in r^+(H)} \text{rk} \alpha \cdot \text{rk} \nu_{\alpha} = (\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} V^H)/2.
\]

For a subgroup \(H \subset G\) and for a collection \(n = \{n_{\alpha}\}_{\alpha \in r(H)}\) of integers, let \(V^n\) be the union of the components of \(V^H\) for which the rank of the corresponding vector bundle \(\nu_{\alpha}\) is equal to \(n_{\alpha}\). Let \(V^1\) be compact of real dimension \(2d\), let \(n = \{n_{\alpha}\}_{\alpha \in r(H)}\) be a collection of integers, let \(n_1 := d - \sum_{\alpha \in r(H)} n_{\alpha} \text{rk} \alpha\), and let \(\nu_1\) be the stable tangent bundle to \(V^n\). In the sequel, the collection \(n\) will contain \(n_1\) as well.

For a collection of integers \(k = \{k_{\alpha,i}\}_{\alpha, i} \in \mathbb{N}\) with \(\sum_{\alpha, i} k_{\alpha, i} = n_1\), let \(c_{H, n_1, k}(V, G) = \langle \prod_{\alpha, i} c_{k_{\alpha, i}}(\nu_{\alpha}^*), [V^n] \rangle\) be the corresponding characteristic number ([\(V^n\)] \(\in H_{2n_1}(V^n)\) is the fundamental class of the manifold \(V^n\)). One can assume that \(k_{\alpha, i} \leq n_\alpha\) for all \((\alpha, i)\). These characteristic numbers are invariants of the bordism classes of \(U\)-manifolds with \(G\)-actions, i.e. if two \(G\)-\(U\)-manifolds \((V_1, G)\) and \((V_2, G)\) are (co)bordant, then, for any \(H \subset G\), \(n\) and \(k\), the corresponding characteristic numbers \(c_{H, n_1, k}(V_1, G)\) and \(c_{H, n_1, k}(V_2, G)\) coincide. One can see that these numbers depend only on the conjugacy class of the subgroup \(H\) (modulo the corresponding identification of the sets of representations of conjugate subgroups).
In this paper we construct, for compact complex singular $G$-varieties, two analogues of these characteristic numbers defined as sums of certain indices of collections of 1-forms (with values in the spaces of the irreducible representations of $H$). These indices are generalizations of the GSV-indices and the Chern obstructions defined in [3] and in [4] respectively.

1 Characteristic numbers in terms of indices of collections of 1-forms

Let us give a convenient description of the characteristic numbers defined above for complex analytic manifolds. Let $V$ be a compact complex analytic manifold of (complex) dimension $d$. We keep the notations of the Introduction. For each pair $(\alpha, i)$, let $\omega^{\alpha,i}_j$, $j = 1, \ldots, n_{\alpha} - k_{\alpha,i} + 1$, be generic (continuous) sections of the bundle $\nu^\alpha$ (i.e. linear forms on $\nu^\alpha$). The characteristic number $c_{H,2;k}(V,G)$ can be computed as the algebraic (i.e. counted with signs) number of points of $V^n$ where for each pair $(\alpha, i)$ the 1-forms $\omega^{\alpha,i}_1, \ldots, \omega^{\alpha,i}_{n_{\alpha} - k_{\alpha,i} + 1}$ are linearly dependent.

For arbitrary sections $\omega^{\alpha,i}_j$, $j = 1, \ldots, n_{\alpha} - k_{\alpha,i} + 1$, a point $x \in V^n$ will be called singular if for each pair $(\alpha, i)$ the 1-forms $\omega^{\alpha,i}_1, \ldots, \omega^{\alpha,i}_{n_{\alpha} - k_{\alpha,i} + 1}$ are linearly dependent at this point. Assume that all singular points of a collection $\{\omega^{\alpha,i}_j\}$ are isolated. In this case the characteristic number $c_{H,2;k}(V,G)$ is the sum of certain indices of the collection $\{\omega^{\alpha,i}_j\}$ at the singular points. Let us describe these indices as degrees of certain maps, as intersection numbers, and, in the case when all the 1-forms $\omega^{\alpha,i}_j$ are complex analytic, as dimensions of certain algebras.

For some simplicity, assume first that the group $G$ is abelian and therefore all the representations $\alpha \in r(H)$ are one-dimensional (i.e. they are elements of the group $H^\ast = \text{Hom}(H, \mathbb{C}^\ast)$ of characters of $H$). The linear functions $\omega^{\alpha,i}_j$ on $\nu^\alpha$ can be regarded as linear functions on $T_x V$ for $x \in V^n$ assuming that $\omega^{\alpha,i}_j|_{\beta \otimes \nu^\beta} = 0$ for $\beta \neq \alpha$. The form $\omega^{\alpha,i}_j$ on $T_x V$ satisfies the condition

$$\omega^{\alpha,i}_j(gu) = \alpha(g)\omega^{\alpha,i}_j(u) \quad (1)$$

for $g \in H$, $u \in T_x V$. The forms $\omega^{\alpha,i}_j$ can be extended to 1-forms on $V$ satisfying (1). If the group $G$ is not assumed to be abelian and thus among the irreducible representations $\alpha$ of the subgroup $H$ one may have those of dimension higher than 1, the form $\omega^{\alpha,i}_j$ should be considered as a 1-form on $V$ with values in the space $E_\alpha$ of the representation $\alpha$ satisfying the same condition (1) (where now $\alpha(g)$ is not a number, but the corresponding operator on $E_\alpha$). The latter simply means that the form $\omega^{\alpha,i}_j$ is $H$-equivariant.
For natural numbers \( p \) and \( q \) with \( p \geq q \), let \( M_{p,q} \) be the space of \( p \times q \) matrices with complex entries, and let \( D_{p,q} \) be the subspace of \( M_{p,q} \) consisting of matrices of rank less than \( q \). The complement \( W_{p,q} = M_{p,q} \setminus D_{p,q} \) is the Stiefel manifold of \( q \)-frames (collections of \( q \) linearly independent vectors) in \( \mathbb{C}^p \). The subset \( D_{p,q} \) is an irreducible complex analytic subvariety of \( M_{p,q} \) of codimension \( p-q+1 \). Therefore \( W_{p,q} \) is \( 2(p-q) \)-connected, \( H_{2p-2q+1}(W_{p,q};\mathbb{Z}) \cong \mathbb{Z} \), and one has a natural choice of a generator of this homology group: the boundary of a small ball in a smooth complex analytic slice transversal to \( D_{p,q} \) at a non-singular point.

For collections \( \underline{n} = \{n_\alpha\} \) and \( \underline{k} = \{k_{\alpha,i}\} \) with \( \sum_{\alpha,i} k_{\alpha,i} = n_1 \), \( k_{\alpha,i} \leq n_\alpha \) for all \((\alpha,i)\), let \( M_{\underline{n},\underline{k}} := \prod_{\alpha,i} M_{n_\alpha,n_{\alpha}-k_{\alpha,i}+1} \), \( D_{\underline{n},\underline{k}} := \prod_{\alpha,i} D_{n_\alpha,n_\alpha-k_{\alpha,i}+1} \subset M_{\underline{n},\underline{k}} \). The subset \( D_{\underline{n},\underline{k}} \) is an irreducible subvariety of \( M_{\underline{n},\underline{k}} \) of codimension \( \sum_{\alpha,i} k_{\alpha,i} = n_1 \), the complement \( W_{\underline{n},\underline{k}} = M_{\underline{n},\underline{k}} \setminus D_{\underline{n},\underline{k}} \) is \( (2n_1-2) \)-connected, \( H_{2n_1-1}(W_{\underline{n},\underline{k}};\mathbb{Z}) \cong \mathbb{Z} \), and one has a natural choice of a generator of this homology group. This choice defines the degree (an integer) of a map from an oriented manifold of dimension \( 2n_1 - 1 \) to \( W_{\underline{n},\underline{k}} \).

Assume that the subgroup \( H \) acts on

\[
\mathbb{C}^d = \bigoplus_{\alpha \in r(H)} n_\alpha E_\alpha = \bigoplus_{\alpha \in r(H)} \bigoplus_{s=1}^{n_\alpha} E_\alpha^{(s)}
\]

by the representation \( \sum_{\alpha \in r(H)} n_\alpha \alpha \left( \sum n_\alpha \text{rk } \alpha = d \right) \). Here \( E_\alpha^{(s)} \), \( s = 1, \ldots, n_\alpha \) are copies of the space \( E_\alpha \) of the representation \( \alpha \). (This means that as a germ of an \( H \)-set \( (\mathbb{C}^d,0) \) is isomorphic to \((V,x)\) for \( x \in V^*\).) The corresponding components of a vector from \( \mathbb{C}^d \) will be denoted by \( u_s^\alpha \) with \( 1 \leq s \leq n_\alpha \), \( u_s^\alpha \in E_\alpha \). The vector with the components \( u_s^\alpha \) will be denoted by \( [u_s^\alpha] \). For a collection \( \underline{k} = \{k_{\alpha,i}\} \) with \( \sum_{\alpha,i} k_{\alpha,i} = n_1 \), \( k_{\alpha,i} \leq \ell_\alpha \) for all \((\alpha,i)\), let \( \omega_\alpha^{i,j} \), \( j = 1, \ldots, n_\alpha - k_{\alpha,i} + 1 \), be \( H \)-equivariant (i.e. satisfying the condition \((\Pi)\)) \( 1 \)-forms on a neighbourhood of the origin in \( \mathbb{C}^d \) with values in the spaces \( E_\alpha \). According to Schur’s lemma, at a point \( p \in n_1 E_1 \subset \mathbb{C}^d \) (1 is the trivial representation of \( H \) and \( E_1 \) is its space: the complex line), the 1-form \( \omega_\alpha^{i,j} \) vanishes on \( \bigoplus_{\beta \neq \alpha} n_\beta E_\beta \) and on each copy \( E_\alpha^{(s)} \) \( (s = 1, \ldots, n_\alpha) \) it is the multiplication by a (complex) number (depending on \( p \)). Thus let \( \omega_\alpha^{i,j}(u_s^\alpha) = \sum_{s} \psi_{j,s}(p) u_s^\alpha \). Let \( \Psi \) be the map from \( E_1^{n_1} = (\mathbb{C}^n)^H \) to \( M_{\underline{n},\underline{k}} \) which sends a point \( p \in E_1^{n_1} \) to the collection
of \( n_\alpha \times (n_\alpha - k_{\alpha,i} + 1) \) matrices

\[
\left\{ \begin{pmatrix}
\psi_{1,1}^{\alpha,i}(p) & \cdots & \psi_{n_\alpha-k_{\alpha,i}+1,1}^{\alpha,i}(p) \\
\vdots & \ddots & \vdots \\
\psi_{1,n_\alpha}^{\alpha,i}(p) & \cdots & \psi_{n_\alpha-k_{\alpha,i}+1,n_\alpha}^{\alpha,i}(p)
\end{pmatrix} \right\},
\]

whose columns consist of the components \( \psi_{j,s}^{\alpha,i}(p) \) of the 1-forms \( \omega_j^{\alpha,i} \). Assume that the collection of the forms has no singular points on \( V \) outside of the origin (in a neighbourhood of it). This means that \( \Psi^{-1}(\mathcal{D}^{n_k}_{n_\alpha}) = \{ 0 \} \).

If \( \Psi^{-1}(\mathcal{D}^{n_k}_{n_\alpha}) = \{ 0 \} \), the origin in \( E_1^{n_\alpha} \subset \mathbb{C}^d \) is an isolated singular point of the collection \( \{ \omega_j^{\alpha,i} \} \). In this case let us define the index \( \text{ind} \, \mathcal{H}_{n_k}^{n_\alpha} \{ \omega_j^{\alpha,i} \} \) of the singular point 0 of the collection \( \{ \omega_j^{\alpha,i} \} \) as the degree of the map \( \Psi|_{S^{2n_\alpha-1}} : \mathcal{S}^{2n_\alpha-1} \to W_{n_k} \) or, what is the same, as the intersection number of the image \( \Psi(E_1^{n_\alpha}) \) with \( \mathcal{D}^{n_k}_{n_\alpha} \) at the origin.

The origin is a non-degenerate singular point of the collection \( \{ \omega_j^{\alpha,i} \} \) if the map \( \Psi \) is transversal to the variety \( \mathcal{D}^{n_k}_{n_\alpha} \) at a non-singular point of it. The index of a non-degenerate singular point is equal to \( \pm 1 \). If all the forms \( \omega_j^{\alpha,i} \) are complex analytic, the index is equal to +1.

The following statement is a reformulation of the definition of the Chern classes of a vector bundle as obstructions to the existence of several linearly independent sections of the bundle. Let \( V \) be a complex analytic manifold of dimension \( d \) with a \( G \)-action, let \( H \) be a subgroup of \( G \), let \( V^H \) be the set of fixed points of the subgroup \( H \) and, for a collection \( \{ n_\alpha \} \), let \( V^{\mathcal{F}} \) be the union of the corresponding components of \( V^H \). For \( \mathcal{F} = \{ k_{\alpha,i} \} \), let \( \{ \omega_j^{\alpha,i} \} \), \( j = 1, \ldots, n_\alpha - k_{\alpha,i} + 1 \), be a collection of equivariant 1-forms on \( V \) with values in the spaces \( E_j^{n_\alpha} \) with only isolated singular points on \( V^{\mathcal{F}} \). Then the sum of the indices \( \text{ind} \, \mathcal{H}_{n_k}^{n_\alpha} \{ \omega_j^{\alpha,i} \} \) of these points is equal to the characteristic number \( c_{\mathcal{H},n_k}^{n_\alpha}(V,G) \).

Assume that in the situation described above all the 1-forms \( \omega_j^{\alpha,i} \) on \( (\mathbb{C}^d, 0) \) are complex analytic. One has the following statement.

**Proposition 1** The index \( \text{ind} \, \mathcal{H}_{n_k}^{n_\alpha} \{ \omega_j^{\alpha,i} \} \) is equal to the dimension of the factor-algebra of the algebra \( \mathcal{O}_{\mathbb{C}^d,0} \) of germs of holomorphic functions on \( (\mathbb{C}^d, 0) \) by the ideal generated by the maximal (i.e. of size \( (n_\alpha - k_{\alpha,i} + 1) \times (n_\alpha - k_{\alpha,i} + 1) \)) minors of the matrices \( (\psi_{j,s}^{\alpha,i}(p)) \).

The proof repeats the one in [3].
2 Equivariant GSV-indices and characteristic numbers

Let

$$\mathbb{C}^N = \bigoplus_{\alpha \in r(H)} m_{\alpha} E_{\alpha} = \bigoplus_{\alpha \in r(H)} \bigoplus_{s=1}^{m_{\alpha}} E_{\alpha}^{(s)}$$

be the complex vector space with the representation \( \sum_{\alpha \in r(H)} m_{\alpha} \alpha \) of a finite group \( H \) (\( N = \sum m_{\alpha} \dim E_{\alpha} \)). Let \( f_{\alpha,\ell} \) with \( \alpha \in r(H), \ell = 1, \ldots, \ell_{\alpha} \), be \( H \)-equivariant germs of holomorphic functions (maps) \( (\mathbb{C}^N,0) \rightarrow (E_{\alpha},0) \) (i.e. \( f_{\alpha,\ell}(gx) = \alpha(g)f_{\alpha,\ell}(x) \) for \( x \in \mathbb{C}^N, g \in H \)) such that \( V, 0) = \{ f_{\alpha,\ell} = 0 \mid \alpha \in r(H), \ell = 1, \ldots, \ell_{\alpha} \} \) is an isolated complete intersection singularity in \( (\mathbb{C}^N,0) \) of codimension \( \sum_{\alpha} \ell_{\alpha} \dim E_{\alpha} \).

The differentials \( df_{\alpha,\ell} \) of the functions \( f_{\alpha,\ell} \) (with values in \( E_{\alpha} \)) at a point \( p \in V \cap m_{1} E_{1} \) is an \( H \)-equivariant linear map from \( T_{V,0} \) to \( E_{\alpha} \). According to Schur’s lemma \( df_{\alpha,\ell} \) vanishes on \( \bigoplus_{\beta \neq \alpha} m_{\beta} E_{\beta} \) and is the multiplication by a complex number (depending on the point \( p \)) on each copy \( E_{\alpha}^{(s)}, s = 1, \ldots, m_{\alpha} \). Let us denote this number by \( \partial_{\alpha} f_{\alpha,\ell}(p) \). Thus one has

$$df_{\alpha,\ell}|_{p}([u_{\beta}^{s}]) = \sum_{s} \partial_{\alpha} f_{\alpha,\ell}(p) u_{\alpha}^{s}.$$

For a collection \( \{ k_{\alpha,i} \} \) with \( \sum_{\alpha,i} k_{\alpha,i} = m_{1} - \ell_{1}, k_{\alpha}^{*} \leq n_{\alpha} := m_{\alpha} - \ell_{\alpha} \) for all \( (\alpha,i) \), let \( \omega_{j,i}^{\alpha} \), \( j = 1, \ldots, n_{\alpha} - k_{\alpha,i} + 1 \), be \( H \)-equivariant 1-forms on a neighbourhood of the origin in \( (\mathbb{C}^N,0) \) with values in the spaces \( E_{\alpha} \). For \( p \in E_{1}^{m_{1}} \cap V \), let \( \omega_{j,i}^{\alpha,\beta}|_{p} = \sum_{s} \psi_{j,i}^{\alpha,s}(p) u_{\beta}^{s} \). Let us define a map \( \Psi : E_{1}^{m_{1}} \cap V \rightarrow M_{m_{\alpha}k_{\alpha,i}} \) by

$$\Psi(p) = \left\{ \begin{array}{cccc}
\partial_{\alpha} f_{\alpha,1}(p) & \cdots & \partial_{\alpha} f_{\alpha,\ell_{\alpha}}(p) & \psi_{1,1}^{\alpha,i}(p) & \cdots & \psi_{n_{\alpha} - k_{\alpha,i} + 1,1}^{\alpha,i}(p) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\partial_{\alpha} f_{\alpha,1}(p) & \cdots & \partial_{\alpha} f_{\alpha,\ell_{\alpha}}(p) & \psi_{1,m_{\alpha}}^{\alpha,i}(p) & \cdots & \psi_{n_{\alpha} - k_{\alpha,i} + 1,m_{\alpha}}^{\alpha,i}(p)
\end{array} \right\},$$

where the right hand side of the equation is the collection of \( m_{\alpha} \times (m_{\alpha} - k_{\alpha,i} + 1) \) matrices \( A_{\alpha,i} \) whose first \( \ell_{\alpha} \) columns consist of the components of the differentials \( df_{\alpha,\ell_{\alpha}} \) of the functions \( f_{\alpha,\ell_{\alpha}} \) and the last \( n_{\alpha} - k_{\alpha,i} + 1 \) columns consist of the components \( \psi_{j,i}^{\alpha,s}(p) \) of the 1-forms \( \omega_{j,i}^{\alpha} \). Assume that the collection of the forms has no singular points on \( V \) outside of the origin (in a neighbourhood of it). This means that \( \Psi^{-1}(D_{m_{\alpha}k_{\alpha,i}}) = \{ 0 \} \).
Definition: The equivariant GSV-index $\text{ind}^{H,n}_{V,0}(\omega_j^{\alpha,i})$ of the collection $\{\omega_j^{\alpha,i}\}$ on the ICIS $(V,0)$ is the degree of the map $\Psi|_{E_1^{m_1} \cap V \cap S_2^{N-1}} : E_1^{m_1} \cap V \cap S_2^{N-1} \to W_{n,k}$ (or, what is the same, the intersection number of the image $\Psi(E_1^{m_1} \cap V)$ with $D_{n,k}$ at the origin).

Being an intersection number, the equivariant GSV-index satisfies the law of conservation of number. This means that if $\tilde{V} = V_{\lambda}$ is an $(H$-invariant) deformation of the ICIS $V$ and a collection $\{\tilde{\omega}_{j}^{\alpha,i}\} = \{\omega_{j,\lambda}^{\alpha,i}\}$ is a deformation of the collection $\{\omega_{j}^{\alpha,i}\}$ with isolated singular points on $V_{\lambda}$, then, for $\lambda$ small enough, one has

$$\text{ind}^{H,n}_{V,0}(\omega_{j}^{\alpha,i}) = \sum_{Q} \text{ind}^{H,n}_{\tilde{V},Q}(\tilde{\omega}_{j}^{\alpha,i}),$$

where the sum on the right hand side runs over all singular points $Q$ of the collection $\{\tilde{\omega}_{j}^{\alpha,i}\}$ on $\tilde{V}$ in a neighbourhood of the origin (including the singular points of $\tilde{V}$ itself).

This implies that, for a compact $H$-variety $V$ with only isolated $(H$-invariant) complete intersection singularities and for a collection of $H$-equivariant forms $\{\omega_{j}^{\alpha,i}\}$ on it with only isolated singular points, the sum

$$\sum_{Q} \text{ind}^{H,n}_{V,0}(\omega_{j}^{\alpha,i})$$

of the equivariant GSV-indices of the collection $\{\omega_{j}^{\alpha,i}\}$ over all its singular points $Q$ does not depend on the collection and is an invariant of $V$. It can be considered as an analogue of the corresponding Chern number. Moreover, this Chern number is the same for varieties with only isolated $(H$-invariant) complete intersection singularities from a, say, one-parameter family of them. In particular, if this family includes (as the generic member) a smooth variety, the defined Chern number coincides with the usual one.

Assume that all the forms $\omega_{j}^{\alpha,i}$, $\alpha \in r(H)$, $i = 1, \ldots, s_{\alpha}$, $j = 1, \ldots, n_{\alpha} - k_{\alpha,i} + 1$, are complex analytic (in a neighbourhood of the origin in $\mathbb{C}^N$). Let $I_{V,\{\omega_{j}^{\alpha,i}\}}$ be the ideal of the ring $\mathcal{O}_{\mathbb{C}^{m_1,0}}$ generated by the germs $f_{1,1}\mid_{\mathbb{C}^{m_1}}$, $\ldots$, $f_{1,\delta_1}\mid_{\mathbb{C}^{m_1}}$ and by the maximal (i.e. of size $(m_{\alpha} - k_{\alpha,i} + 1) \times (m_{\alpha} - k_{\alpha,i} + 1)$) minors of the matrices $A_{\alpha,i}$ for all $(\alpha,i)$.

**Theorem 1** One has

$$\text{ind}^{H,n}_{V,0}(\omega_{j}^{\alpha,i}) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{m_1,0}}/I_{V,\{\omega_{j}^{\alpha,i}\}}.$$

The proof is essentially the same as in [3].
3 Equivariant Chern obstructions and characteristic numbers

Let the group $H$ act (by a representation) on the affine space $\mathbb{C}^N$ and let $(X,0) \subset (\mathbb{C}^N,0)$ be an $H$-invariant (reduced) germ of a complex analytic variety of (pure) dimension $n$. Let $\pi : \hat{X} \to X$ be the Nash transformation of the variety $X \subset B_\varepsilon$ ($B_\varepsilon = B_\varepsilon^2\mathbb{C}^N$ is the ball of a sufficiently small radius $\varepsilon$ around the origin in $\mathbb{C}^N$) defined as follows. Let $X_{\text{reg}}$ be the set of smooth points of $X$ and let $G(n,N)$ be the Grassmann manifold of $n$-dimensional vector subspaces of $\mathbb{C}^N$. The action of the group $H$ on $\mathbb{C}^N$ defines an action on $G(n,N)$ and on the space of the tautological bundle over it as well. There is a natural map $\sigma : X_{\text{reg}} \to B_\varepsilon \times G(n,N)$ which sends a point $x \in X_{\text{reg}}$ to $(x,T_xX_{\text{reg}})$. The Nash transform $\hat{X}$ of the variety $X$ is the closure of the image $\text{Im} \sigma$ of the map $\sigma$ in $B_\varepsilon \times G(n,N)$, $\pi$ is the natural projection. The Nash bundle $\hat{T}$ over $\hat{X}$ is a vector bundle of rank $n$ which is the pullback of the tautological bundle on the Grassmann manifold $G(n,N)$. There is a natural lifting of the Nash transformation to a bundle map from the Nash bundle $\hat{T}$ to the restriction of the tangent bundle $T\mathbb{C}^N$ of $\mathbb{C}^N$ to $X$. This is an isomorphism of $\hat{T}$ and $TX_{\text{reg}} \subset T\mathbb{C}^N$ over the non-singular part $X_{\text{reg}}$ of $X$.

For a collection of integers $\underline{n} = \{n_\alpha\}$, let $X_{\underline{n}}$ be the closure in $X$ of the set of points $x \in X_{\text{reg}} \cap X^H$ such that the representation of the group $H$ on $T_xX_{\text{reg}}$ is $\bigoplus \alpha n_\alpha$ and let $\hat{X}_{\underline{n}}$ be the closure of the same set in $\hat{X}$. The restriction of the Nash bundle $\hat{T}$ to $\hat{X}_{\underline{n}}$ is the direct sum of the subbundles $\hat{T}_\alpha$ ($\alpha \in r(H)$) subject to the splitting of the representation of $H$ on $\hat{T}$ into the summands corresponding to different irreducible representations of $H$. Let the (complex) vector bundle $\hat{v}_\alpha^*$ of rank $n_\alpha$ over $\hat{X}_{\underline{n}}$ be defined by $\hat{v}_\alpha^* := \text{Hom}_H(\hat{T}, E_\alpha) = \text{Hom}_H(\hat{T}_\alpha, E_\alpha)$.

Let $\underline{k} = \{k_{\alpha,i}\}$, $\alpha \in r(H)$, $i = 1, \ldots, n_\alpha - k_{\alpha,i} + 1$, be a collection of positive integers with $\sum_{\alpha,i} k_{\alpha,i} = n_1$ and let $\{\omega_{\alpha,i}^j\}$, $j = 1, \ldots, n_\alpha - k_{\alpha,i} + 1$, be a collection of $H$-equivariant 1-forms on $(\mathbb{C}^N,0)$ with values in the spaces $E_\alpha$ of the representations $\omega_{\alpha,i}^j(gu) = \alpha(g)\omega_{\alpha,i}^j(u)$ for $g \in H$.

**Remark 1** One can see that all the constructions below are determined by the restrictions of the forms $\omega_{\alpha,i}^j$ to the regular part of the variety $X$.

Let $\varepsilon > 0$ be small enough so that there is a representative $X$ of the germ $(X,0)$ and representatives $\omega_{\alpha,i}^j$ of the germs of 1-forms inside the ball $B_\varepsilon \subset \mathbb{C}^N$. 

8
**Definition:** A point $P \in X^\infty$ is called a *special* point of the collection $\{\omega_j^{\alpha,i}\}$ of 1-forms on the variety $X$ if there exists a sequence $\{P_m\}$ of points from $X^\infty \cap X_{\text{reg}}$ converging to $P$ such that the sequence $T_{P_m}X_{\text{reg}}$ of the tangent spaces at the points $P_m$ has an ($H$-invariant) limit $L$ as $m \to \infty$ (in the Grassmann manifold of $n$-dimensional vector subspaces of $\mathbb{C}^N$) and the restrictions of the 1-forms $\omega_1^{\alpha,i}, \ldots, \omega_{n_\alpha-k_{\alpha,i}+1}^{\alpha,i}$ to the subspace $L \subset T_P\mathbb{C}^N$ are linearly dependent for each pair $(\alpha, i)$.

**Definition:** The collection $\{\omega_j^{\alpha,i}\}$ of 1-forms has an *isolated special point* on the germ $(X, 0)$ if it has no special points on $X$ in a punctured neighbourhood of the origin.

**Remark 2** On a smooth variety the notions “special point” and “singular point” coincide. (On a singular variety these notions are different: see [4, Remark 1.2].) A singular point of a collection of 1-forms $\{\omega_j^{\alpha,i}\}$ on a variety can be non-degenerate only if it is a smooth point of the variety and therefore it is a special point as well.

Let

$$L^{\infty,\hat{k}} = \prod_{\alpha,i} \left( \prod_{j=1}^{n_\alpha-k_{\alpha,i}+1} \text{Hom}_H(\mathbb{C}^N, E_{\alpha}) \right)$$

be the space of collections of $E_{\alpha}$-valued linear functions on $\mathbb{C}^N$ (i.e. of $H$-equivariant 1-forms with constant coefficients).

**Proposition 2** There exists an open and dense subset $U \subset L^{\infty,\hat{k}}$ such that each collection $\{\lambda_j^{\alpha,i}\} \in U$ has only isolated special points on $X^\infty$ and, moreover, all these points belong to the smooth part $X^\infty \cap X_{\text{reg}}$ of the variety $X^\infty$ and are non-degenerate.

**Proof.** Let $Y \subset X^\infty \times L^{\infty,\hat{k}}$ be the closure of the set of pairs $(x, \{\lambda_j^{\alpha,i}\})$ where $x \in X^\infty \cap X_{\text{reg}}$ and the restrictions of the linear functions $\lambda_1^{\alpha,i}, \ldots, \lambda_{n_\alpha-k_{\alpha,i}+1}^{\alpha,i}$ to the tangent space $T_xX_{\text{reg}}$ are linearly dependent for each pair $(\alpha, i)$. Let $\text{pr}_2 : Y \to L^{\infty,\hat{k}}$ be the projection to the second factor. One has codim$Y = \sum_{\alpha,i} k_{\alpha,i} = n_\infty$ and therefore dim$Y = \dim L^{\infty,\hat{k}}$.

Moreover, $Y \setminus ((X^\infty \cap X_{\text{reg}}) \times L^{\infty,\hat{k}})$ is a proper subvariety of $Y$ and therefore its dimension is strictly smaller than dim$L^{\infty,\hat{k}}$. A generic point $\Lambda = \{\lambda_j^{\alpha,i}\}$ of the space $L^{\infty,\hat{k}}$ is a regular value of the map $\text{pr}_2$ which means that it has only finitely many preimages, all of them belong to $(X^\infty \cap X_{\text{reg}}) \times L^{\infty,\hat{k}}$ and the map $\text{pr}_2$ is non-degenerate at them. Therefore $\Lambda \times (X^\infty \cap X_{\text{reg}})$ intersects $Y$ transversally. This implies the statement. \[\square\]
Corollary 1 Let \( \{ \omega_{j}^{\alpha,i} \} \) be a collection of 1-forms on \( X \) with an isolated special point at the origin. Then there exists a deformation \( \{ \tilde{\omega}_{j}^{\alpha,i} \} \) of the collection \( \{ \omega_{j}^{\alpha,i} \} \) whose special points lie in \( X^2 \cap X_{\text{reg}} \) and are non-degenerate. Moreover, as such a deformation one can use \( \{ \omega_{j}^{\alpha,i} + \lambda_{j}^{\alpha,i} \} \) with a generic collection \( \{ \lambda_{j}^{\alpha,i} \} \in \mathcal{L}^{\mathbb{A}} \times L \) small enough.

Let

\[
\hat{\mathbb{T}}_{a}^{\mathbb{A}} = \bigoplus_{a,i,j=1}^{n_{a}-k_{a,i}+1} \tilde{\nu}_{a,i,j}^{*},
\]

where \( \tilde{\nu}_{a,i,j}^{*} \) are copies of the vector bundle \( \tilde{\nu}_{a,i}^{*} \) numbered by \( i \) and \( j \). Let \( \hat{\mathbb{T}}_{\mathbb{A}} \subset \mathbb{T}_{\mathbb{A}} \) be the set of pairs \( (x, \{ \eta_{j}^{\alpha,i} \}), x \in \hat{X}^2, \eta_{j}^{\alpha,i} \in \tilde{\nu}_{a,i,j}^{*} \), such that \( \eta_{1}^{\alpha,i}, \ldots, \eta_{n_{a}-k_{a,i}+1}^{\alpha,i} \) are linearly dependent for each pair \( (\alpha,i) \).

The collection \( \{ \omega_{j}^{\alpha,i} \} \) defines a section \( \tilde{\omega} \) of the bundle \( \hat{\mathbb{T}}_{a}^{\mathbb{A}} \). The image of this section does not intersect \( \hat{\mathbb{T}}_{\mathbb{A}} \) outside of the preimage \( \pi^{-1}(0) \subset \hat{X}^2 \) of the origin. The map \( \hat{\mathbf{T}}_{a}^{\mathbb{A}} \setminus \hat{\mathbb{T}}_{\mathbb{A}} \rightarrow \hat{X}^2 \) is a fibre bundle. The fibre \( W_{x} = \hat{\mathbf{T}}_{a}^{\mathbb{A}} \setminus \hat{\mathbb{T}}_{\mathbb{A}} \) of it is \( (2n_{1} - 2) \)-connected, its homology group \( H_{2n_{1}-1}(W_{x};\mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \) and has a natural generator. (This follows from the fact that \( W_{x} \) is homeomorphic to \( M_{a}^{\mathbb{A}} \setminus D_{a}^{\mathbb{A}} \) from Section 1.) The latter fact implies that the fibre bundle \( \hat{\mathbf{T}}_{a}^{\mathbb{A}} \setminus \hat{\mathbb{T}}_{\mathbb{A}} \rightarrow \hat{X}^2 \) is homotopically simple in dimension \( 2n_{1} - 1 \), i.e. the fundamental group \( \pi_{1}(\hat{X}^2) \) of the base acts trivially on the homotopy group \( \pi_{2n_{1}-1}(W_{x}) \) of the fibre, the last one being isomorphic to the homology group \( H_{2n_{1}-1}(W_{x};\mathbb{Z}) \); see, e.g., [9].

Definition: The local Chern obstruction (index) \( \text{Ch}_{X_{0}}^{H_{a}^{\mathbb{A}}}(\omega_{j}^{\alpha,i}) \) of the collections of germs of 1-forms \( \{ \omega_{j}^{\alpha,i} \} \) on \( (X,0) \) at the origin is the (primary, and in fact the only) obstruction to extend the section \( \tilde{\omega} \) of the fibre bundle \( \hat{\mathbf{T}}_{a}^{\mathbb{A}} \setminus \hat{\mathbb{T}}_{\mathbb{A}} \rightarrow \hat{X}^2 \) from the preimage of a neighbourhood of the sphere \( S_{\varepsilon} = \partial B_{\varepsilon} \rightarrow \hat{X}^2 \), more precisely its value (as an element of \( H^{2n_{1}}(\pi^{-1}(X^2 \cap B_{\varepsilon}), \pi^{-1}(X^2 \cap S_{\varepsilon});\mathbb{Z}) \)) on the fundamental class of the pair \( (\pi^{-1}(X^2 \cap B_{\varepsilon}), \pi^{-1}(X^2 \cap S_{\varepsilon})) \).

The definition of the local Chern obstruction \( \text{Ch}_{X_{0}}^{H_{a}^{\mathbb{A}}}(\omega_{j}^{\alpha,i}) \) can be reformulated in the following way. The collection of 1-forms \( \{ \omega_{j}^{\alpha,i} \} \) defines also a section \( \tilde{\omega} \) of the trivial bundle \( X^2 \times \mathcal{L}^{\mathbb{A}} \rightarrow X^2 \), namely \( \tilde{\omega} : X^2 \rightarrow X^2 \times \mathcal{L}^{\mathbb{A}} \). Let \( \mathcal{D}_{a}^{\mathbb{A}} \subset X^2 \times \mathcal{L}^{\mathbb{A}} \) be the closure of the set of pairs \( (x, \{ \lambda_{j}^{\alpha,i} \}) \) such that \( x \in X^2 \cap X_{\text{reg}} \) and the restrictions of the linear functions \( \lambda_{1}^{\alpha,i}, \ldots, \lambda_{n_{a}-k_{a,i}+1}^{\alpha,i} \) to \( T_{x}X_{\text{reg}} \subset \mathbb{C}^{N} \) are linearly dependent for each pair \( (\alpha,i) \). For \( x \in X^2 \cap S_{\varepsilon} \) (\( \varepsilon \) small enough), \( \tilde{\omega} \notin \mathcal{D}_{a}^{\mathbb{A}} \) and the local Chern obstruction is the value on the fundamental class of the pair \( (X^2 \cap B_{\varepsilon}, X^2 \cap S_{\varepsilon}) \) of the first obstruction to extend the map \( \tilde{\omega} : X^2 \cap S_{\varepsilon} \rightarrow (X^2 \times \mathcal{L}^{\mathbb{A}}) \setminus \mathcal{D}_{a}^{\mathbb{A}} \) to a map \( X^2 \cap B_{\varepsilon} \rightarrow (X^2 \times \mathcal{L}^{\mathbb{A}}) \setminus \mathcal{D}_{a}^{\mathbb{A}} \). (This follows, e.g. from the fact
that, due to Corollary 1, the collection \( \{ \omega_{j;i}^{\alpha} \} \) can be deformed in such a way that all the special points of the deformed collection lie in the regular part of \( X^2 \) and thus the corresponding obstructions on \( X^2 \) and on \( \hat{X}^2 \) coincide.) The map \( \hat{\omega} \) is also defined on \( \mathbb{C}^N \) (a section of the trivial bundle \( \mathbb{C}^N \times L^2 \)).

For \( x \in S_\varepsilon \) (\( \varepsilon \) small enough), \( \hat{\omega} \notin D^{\varepsilon, k} \subset X^2 \times L^{2, k} \subset \mathbb{C}^N \times L^{2, k} \) and the local Chern obstruction is the value on the fundamental class of the pair \( (B_\varepsilon, S_\varepsilon) \) of the first obstruction to extend the map \( \hat{\omega} : S_\varepsilon \to (\mathbb{C}^N \times L^{2, k}) \setminus D^{\varepsilon, k} \). This means that it is equal to the intersection number \( \langle \hat{\omega}(\mathbb{C}^N) \circ D^{\varepsilon, k} \rangle_0 \) at the origin in \( \mathbb{C}^N \times L^k \).

Being a (primary) obstruction, the local Chern obstruction satisfies the law of conservation of number, i.e. if a collection of 1-forms \( \{ \tilde{\omega}_{j;i}^{\alpha} \} \) is a deformation of the collection \( \{ \omega_{j;i}^{\alpha} \} \) with only isolated special points on \( X \), then

\[
\text{Ch}^{H,n,k}_{X,0} \{ \omega_{j;i}^{\alpha} \} = \sum_Q \text{Ch}^{H,n,k}_{X,Q} \{ \tilde{\omega}_{j;i}^{\alpha} \},
\]

where the sum on the right hand side is over all special points \( Q \) of the collection \( \{ \tilde{\omega}_{j;i}^{\alpha} \} \) on \( X \) in a neighbourhood of the origin.

Along with Corollary 1 this implies the following statements.

**Proposition 3** The local Chern obstruction \( \text{Ch}^{H,n,k}_{X,0} \{ \omega_{j;i}^{\alpha} \} \) of a collection \( \{ \omega_{j;i}^{\alpha} \} \) of germs of 1-forms is equal to the algebraic (i.e. counted with signs) number of special points on \( X \) of a generic deformation of the collection.

If all the 1-forms \( \omega_{j;i}^{\alpha} \) and their generic deformations are holomorphic, the local Chern obstructions of the deformed collection at its special points are equal to 1 and therefore the local Chern obstruction \( \text{Ch}^{H,n,k}_{X,0} \{ \omega_{j;i}^{\alpha} \} \) is equal to the number of special points of the deformation.

**Proposition 4** If a collection \( \{ \omega_{j;i}^{\alpha} \} \) (\( \alpha \in r(H), i = 1, \ldots, s_\alpha, j = 1, \ldots, n_\alpha - k_{\alpha,i} + 1 \)) of 1-forms on a compact (say, projective) \( G \)-variety \( X \) has only isolated special points on \( X^H \), then the sum of the local Chern obstructions of the collection \( \{ \omega_{j;i}^{\alpha} \} \) at these points does not depend on the collection and therefore is an invariant of the variety.

One can consider this sum as the corresponding Chern number of the \( G \)-variety \( X \).

Let \( (X, 0) \) be an isolated \( H \)-invariant complete intersection singularity (see Section 2). As it was described there, a collection of germs of 1-forms \( \{ \omega_{j;i}^{\alpha} \} \) on \( (X, 0) \) with an isolated special point at the origin has an index \( \text{ind}^{H,n,k}_{X,0} \{ \omega_{j;i}^{\alpha} \} \) which is an analogue of the GSV-index of a 1-form. The fact that both the
Chern obstruction and the index satisfy the law of conservation of number and they coincide on a smooth manifold yields the following statement.

**Proposition 5** For a collection \( \{ \omega_j^{\alpha,i} \} \) of germs of 1-forms on an isolated \( H \)-invariant complete intersection singularity \((X,0)\) the difference

\[
\text{ind}_{X,0}^{H,\underline{n},k} \{ \omega_j^{\alpha,i} \} - \text{Ch}_{X,0}^{H,\underline{n},k} \{ \omega_j^{\alpha,i} \}
\]

does not depend on the collection and therefore is an invariant of the germ of the variety.

Since, by Proposition 2 \( \text{Ch}_{X,0}^{H,\underline{n},k} \{ \omega_j^{\alpha,i} \} = 0 \) for a generic collection \( \{ \omega_j^{\alpha,i} \} \) of linear functions on \( \mathbb{C}^N \), one has the following statement.

**Corollary 2** One has

\[
\text{Ch}_{X,0}^{H,\underline{n},k} \{ \omega_j^{\alpha,i} \} = \text{ind}_{X,0}^{H,\underline{n},k} \{ \omega_j^{\alpha,i} \} - \text{ind}_{X,0}^{H,\underline{n},k} \{ \nu_j^{\alpha,i} \}
\]

for a generic collection \( \{ \nu_j^{\alpha,i} \} \) of \( H \)-equivariant linear functions on \( \mathbb{C}^N \).

**Remark 3** Instead of working with \( E_{\alpha} \)-valued 1-forms on a compact variety \( X \), one can also consider 1-forms with values ”in local systems of coefficients”, i.e. in \((H-)\)vector bundles of the form \( E_{\alpha} \otimes L_{\alpha} \) where \( L_{\alpha} \) is a (usual) line bundle over \( X \). The local indices (both GSV- and Chern ones) of a collection of 1-forms are defined in the same way and the sums of these indices give invariants of a compact \( G \)-variety \( X \) with the bundles \( L_{\alpha} \) which can be regarded as analogues of the Chern numbers \( \langle \prod_{\alpha,i} c_{k_{\alpha,i}}(\nu_{\alpha}^* \otimes L_{\alpha}), [X] \rangle \) (see the Introduction).

**Remark 4** The definitions and constructions of this section work for \( G \) being a compact Lie group as well.

**References**

[1] J.-P. Brasselet, J. Seade, T. Suwa. Vector fields on singular varieties. Lecture Notes in Mathematics, 1987. Springer-Verlag, Berlin, 2009.

[2] A.Yu. Buryak. Existence of a singular projective variety with an arbitrary set of characteristic numbers. Math. Res. Lett. 17 (2010), no. 3, 395–400.

[3] W. Ebeling, S.M. Gusein-Zade. Indices of vector fields or 1-forms and characteristic numbers. Bull. London Math. Soc. 37 (2005), no.5, 747–754.
[4] W. Ebeling, S.M. Gusein-Zade. Chern obstructions for collections of 1-forms on singular varieties. In: Singularity theory, World Sci. Publ., Hackensack, NJ, 2007, 557–564.

[5] W. Ebeling, S.M. Gusein-Zade. Indices of vector fields and 1-forms on singular varieties. In: Global aspects of complex geometry, F. Catanese, H. Esnault, A. Huckleberry, K. Hulek, T. Peternell (Eds.), Springer, 2006, 129–169.

[6] X. Gómez-Mont, J. Seade, A. Verjovsky. The index of a holomorphic flow with an isolated singularity. Math. Ann. 291 (1991), 737–751.

[7] R. MacPherson. Chern classes for singular varieties. Annals of Math. 100 (1974), 423–432.

[8] J.A. Seade, T. Suwa. A residue formula for the index of a holomorphic flow. Math. Ann. 304 (1996), no.4, 621–634.

[9] N. Steenrod: The topology of fibre bundles. Princeton Math. Series, Vol.14, Princeton University Press, Princeton, N. J., 1951.

[10] R.E. Stong. Notes on cobordism theory. Mathematical notes, Princeton University Press, Princeton, N.J., 1968.

Leibniz Universität Hannover, Institut für Algebraische Geometrie, Postfach 6009, D-30060 Hannover, Germany E-mail: ebeling@math.uni-hannover.de

Moscow State University, Faculty of Mechanics and Mathematics, Moscow, GSP-1, 119991, Russia E-mail: sabir@mccme.ru