How are Feynman graphs resummed by the Loop Vertex Expansion?

Vincent Rivasseau, Zhituo Wang
Laboratoire de Physique Théorique, CNRS UMR 8627,
Université Paris XI, F-91405 Orsay Cedex, France
E-mail: rivass@th.u-psud.fr, zhituo.wang@th.u-psud.fr
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Abstract

The purpose of this short letter is to clarify which set of pieces of Feynman graphs are resummed in a Loop Vertex Expansion, and to formulate a conjecture on the $\phi^4$ theory in non-integer dimension.

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1 Introduction

In quantum field theory (hereafter QFT) any connected (i.e. interesting) quantity is written as a sum of amplitudes for a certain category of connected graphs

\[ S = \sum_{G \text{ connected}} A_G \]  

but this formula is not a valid definition of \( S \) since usually

\[ \sum_{G \text{ connected}} |A_G| = \infty. \]

This phenomenon, known since [1], is basically due to the very large number of elements at order \( n \) in the species [2] of Feynman graphs. Accordingly the \textit{generating functional for the Feynman graphs species}, namely the series

\[ \sum_n \frac{\lambda^n a_n}{n!}, \]

where \( a_n \) is the number of Feynman graphs at order \( n \), has zero radius of convergence as power series in \( \lambda \). We call such a species a \textit{proliferating species}. In zero space-time dimension, quantum field theory reduces to this generating functional, hence to graphs counting. In higher dimensions quantum field theory is in fact a \textit{weighted} such species, that is Feynman graphs have to be pondered with weights, called Feynman amplitudes. For an introduction to the structure of Feynman graphs, see [3]. Nevertheless these Feynman amplitudes tend to behave as \( K^n \) at order \( n \) (at least in low dimensions). Hence the perturbation series eg for the \( \phi^4 \) Euclidean Bosonic quantum field theory tends to behave as \( \sum_n (-\lambda)^n K^n n! \) and it has been proved to have zero radius of convergence in one, two and three dimensions ([4, 5]). Nothing is yet known for sure in dimension 4 but there are strong reasons to expect also the renormalized Feynman series to diverge there as well (see [6] and references therein).
In contrast Cayley’s theorem, which states that the total number of labeled trees at order \( n \) is \( n^{n-2} \), implies that the species of trees is not proliferating. This fact can be related to the local existence theorems for flows in classical mechanics, since classical perturbation theory is indexed by trees [7]. These theorems have no quantum counterpart, but constructive theory can be seen as various recipes to replace the ordinary divergent Feynman graph expansions by convergent ones, indexed by trees rather than graphs [8]. It can therefore be considered a bridge between QFT and classical mechanics, since it repacks the loops which are the fundamental feature of QFT, and brings the expansion closer to those of classical mechanics. Historically constructive theory used cumbersome non canonical tools borrowed from lattice statistical mechanics, such as cluster expansions which did not respect the rotational invariance of the underlying theory [9, 6]. The Loop Vertex Expansion [10, 11] is a more canonical way to replace the ordinary perturbative divergent expansion by a convergent one, which in principle allows to compute quantities to arbitrary accuracy.

One of us (VR) was recently asked exactly which (pieces of) Feynman graphs are resummed by this expansion. The answer is contained in the initial papers, but perhaps not easy to extract. The purpose of this little note is therefore to explain more explicitly exactly which pieces of which Feynman graphs of different orders are combined together by the loop vertex expansion to create a convergent expansion. This reshuffling is fully explicitied up to third order for the simplest of all possible examples, namely the \( \phi^4 \) quantum field theory. Finally we also propose a conjecture, which, if true, would allow to define QFT in non-integer dimensions of space-time.
2 Relative Tree Weights in a Graph

A graph may contain many (spanning) forests, and a forest can be extended into many graphs with loops. So the relationship between graphs and their spanning forests is not trivial.

The forest formula which we use [13, 14] can be viewed as a tool to associate to any pair made of a graph $G$ and a spanning forest $\mathcal{F} \subset G$ a unique rational number or weight $w(G, \mathcal{F})$ between 0 and 1, called the relative weight of $\mathcal{F}$ in $G$.

The numbers $w(G, \mathcal{F})$ are multiplicative over disjoint unions. Hence it is enough to give the formula for $(G, \mathcal{F})$ only when $G$ is connected and $\mathcal{F} = \mathcal{T}$ is a spanning tree in $G$.

The definition of these weights is

**Definition 2.1.**

$$w(G, \mathcal{T}) = \int_0^1 \prod_{\ell \in \mathcal{T}} dw_\ell \prod_{\ell' \notin \mathcal{T}} x^\mathcal{T}_\ell(\{w\})$$

where $x^\mathcal{T}_\ell(\{w\})$ is the infimum over the $w_{\ell'}$ parameters over the lines $\ell'$ forming the unique path in $\mathcal{T}$ joining the ends of $\ell$.

**Lemma 2.1.** The relation

$$\sum_{\mathcal{F} \subset G} w(G, \mathcal{F}) = 1$$

holds for any connected graph $G$.

---

1 And also over vertex joints of graphs, just as in the universality theorem for the Tutte polynomial.

2 It is enough in fact to compute such weights for 1-particle irreducible and 1-vertex-irreducible graphs, then multiply them in the appropriate way for the general case.
Proof It is a simple consequence of the forest formula \[13, 14\] applied to the lines of the graph \(G\).

2.1 Examples

For a fixed spanning tree inside a graph, we call \textit{loop lines} the lines not in the tree.

Consider the graph \(G\) of Figure 1. There are 5 spanning trees inside this graph:

\[T_{12} = \{l_1, l_2\}, \ T_{13} = \{l_1, l_3\}, \ T_{14} = \{l_1, l_4\}, \ T_{23} = \{l_2, l_3\}, \ T_{24} = \{l_2, l_4\}.\]

For example, for the tree \(T_{12} = \{l_2, l_4\}\), the loop lines are \(l_1\) and \(l_3\).

To take into account the weakening factors \(x_{\ell}^T(\{w\})\) of (4) for each loop line \(\ell\), it is convenient to decompose the integration domain \([0, 1]^{\vert\mathcal{T}\vert}\) into \(|\mathcal{T}|!\) sectors corresponding to complete orderings of the \(w_{\ell}\) parameters for \(\ell \in \mathcal{T}\).

Let us compute in this way the relative weights of the five trees of \(G\). First consider the contribution of the tree \(T_{12}\). In this case the loop lines are
and $l_4$. For each of them we have a factor $\inf(w_1w_2)$. Hence

$$w(G, T_{12}) = \int_0^1 \int_0^1 dw_1 dw_2 [\inf(w_1, w_2)]^2$$

$$= 2 \int_0^1 dw_2 \int_0^{w_2} dw_1 w_1^2 = \frac{2}{12} = \frac{1}{6}.$$ 

Next we consider the spanning tree $T_{13}$. In this case the ”loop lines“ are $l_2$ which connects the vertices $v_1$ and $v_3$ and $l_4$ which connects $v_2$ and $v_3$. So we have:

$$w(G, T_{13}) = \int_{w_1 < w_3} dw_1 \int dw_3 \inf(w_1w_3)w_3$$

$$+ \int_{w_3 < w_1} dw_1 \int dw_3 \inf(w_1w_3)w_3$$

$$= \int_0^1 dw_3 \int_{w_1}^{w_3} dw_1 w_1 w_3 + \int_0^1 dw_1 \int_{w_3}^{w_2} dw_3 w_3^2 = \frac{1}{8} + \frac{1}{12} = \frac{5}{24}. $$

With the same method we find that

$$w(G, T_{14}) = w(G, T_{21}) = w(G, T_{23}) = \frac{5}{24},$$

and we have

$$\sum_{T \in G} w(G, T) = \frac{1}{6} + 4 \frac{5}{24} = 1. $$

Let us treat a second example. Consider the graph $G'$ of Fig. 2 which has 6 edges:

$$\{l_1, l_2, l_3, l_4, l_5, l_6\}. $$

To each edge $l_i$ we associate a factor $w_i$. There are 12 spanning trees:

$$\{l_1, l_2, l_3\}, \{l_1, l_2, l_4\}, \{l_1, l_3, l_4\}, \{l_2, l_3, l_4\}, \{l_1, l_2, l_5\}, \{l_1, l_2, l_6\},$$

$$\{l_3, l_4, l_5\}, \{l_3, l_4, l_6\}, \{l_1, l_5, l_4\}, \{l_1, l_6, l_4\}, \{l_3, l_5, l_2\}, \{l_3, l_6, l_2\}. $$

$$6$$
Let us compute the relative weight for each of these spanning trees in $G'$. First of all consider $T_{123} = \{l_1, l_2, l_3\}$. The other edges are drawn in dotted lines. See figure(3) As is easily seen the corresponding loop lines are $l_4, l_5$ and $l_6$. The weakening factor for $l_5$ and $l_6$ is $\inf(w_1, w_3)$ and the weakening factor for $l_4$ is $\inf(w_1, w_2, w_3)$. Therefore we have

$$w(G', T_{123}) = \int_{0<w_1<w_2<w_3<1} dw_1 dw_2 dw_3 \inf(w_1, w_3)^2 \inf(w_1, w_2, w_3)$$

+ other permutations of $w_1, w_2, w_3$

$$= \int_{w_1<w_2<w_3} dw_1 dw_2 dw_3 w_1^3 + \int_{w_2<w_3<w_1} dw_1 dw_2 dw_3 w_2^3 w_3$$

$$+ \int_{w_3<w_1<w_2} dw_1 dw_2 dw_3 w_3^3 + \int_{w_2<w_1<w_3} dw_1 dw_2 dw_3 w_1^2 w_2$$

$$+ \int_{w_3<w_2<w_1} dw_1 dw_2 dw_3 w_3^2 + \int_{w_1<w_3<w_2} dw_1 dw_2 dw_3 w_1^3.$$ 

We compute only two of the integrals explicitly as others are obtained by
changing the names of variables.

\[
\int_{w_1 < w_2 < w_3} dw_1 dw_2 dw_3 \ w_1^3 = \int_0^1 dw_3 \int_0^{w_3} dw_2 \int_0^{w_2} dw_1 \ w_1^3 = \frac{1}{120}. \quad (10)
\]

\[
\int_{w_1 < w_2 < w_3} dw_1 dw_2 dw_3 \ w_3^2 \ w_2 = \frac{1}{60}. \quad (11)
\]

So we have

\[w(G', \mathcal{T}_{123}) = \frac{1}{120} \times 4 + \frac{1}{60} \times 2 = \frac{1}{15}. \quad (12)\]

The relative weights in \( G' \) of the spanning trees \( \mathcal{T}_{124}, \mathcal{T}_{134} \) and \( \mathcal{T}_{234} \) are the same.

Now we consider the tree \( \{l_1, l_2, l_5\} \). (See figure 4). To the loop line \( l_3 \) is associated a weakening factor \( \inf(w_1, w_5) \). To the loop line \( l_4 \) is associated a weakening factor \( \inf(w_2, w_5) \). To the loop line \( l_6 \) is associated a weakening factor \( w_5 \). So we have

\[
w(G', \mathcal{T}_{125}) = \int_{w_1 < w_2 < w_5} dw_1 dw_2 dw_5 \inf(w_1, w_5) \inf(w_2, w_5) w_5 \quad (13)
\]

\[
+ \text{ other permutations of } w_1, w_2, w_5
\]

\[
= \int_{w_1 < w_2 < w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 + \int_{w_5 < w_1 < w_2} dw_1 dw_2 dw_5 w_5^3 \\
+ \int_{w_2 < w_5 < w_1} dw_1 dw_2 dw_5 w_5^2 w_2 + \int_{w_2 < w_1 < w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 \\
+ \int_{w_1 < w_5 < w_2} dw_1 dw_2 dw_5 w_5^2 w_1 + \int_{w_5 < w_2 < w_1} dw_1 dw_2 dw_5 w_5^3.
\]
We have

\[ \int_{w_1 < w_2 < w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 = \frac{1}{48}, \quad (14) \]

\[ \int_{w_5 < w_1 < w_2} dw_1 dw_2 dw_5 w_5^3 = \frac{1}{120}, \quad (15) \]

\[ \int_{w_2 < w_5 < w_1} dw_1 dw_2 dw_5 w_2 w_5^2 = \frac{1}{60}. \quad (16) \]

Similarly we get

\[ w(G', \mathcal{T}_{125}) = \frac{1}{120} \times 2 + \frac{1}{60} \times 2 + \frac{1}{48} \times 2 = \frac{11}{120}. \quad (17) \]

By the same method we find that this is also the relative weight of trees \( \mathcal{T}_{126}, \mathcal{T}_{345}, \mathcal{T}_{346}, \mathcal{T}_{125}, \mathcal{T}_{145}, \mathcal{T}_{146}, \mathcal{T}_{235} \) and \( \mathcal{T}_{236} \).

We can check again that

\[ \sum_{T \in G'} w(G', \mathcal{T}) = 4 \times \frac{1}{15} + 8 \times \frac{11}{120} = 1. \quad (18) \]

3 Resumming Feynman Graphs

3.1 Naive Repacking

Consider the expansion (1) of a connected quantity \( S \). The most naive way to reorder Feynman perturbation theory according to trees rather than graphs is to insert for each graph the relation (4)

\[ S = \sum_G A_G = \sum_G \sum_{T \subseteq G} w(G, T) A_G \quad (19) \]

and exchange the order of the sums over \( G \) and \( T \). Hence it writes

\[ S = \sum_T A_T, \quad A_T = \sum_{G \supseteq T} w(G, T) A_G. \quad (20) \]
This rearranges the Feynman expansion according to trees, but each tree has the same number of vertices as the initial graph. Hence it reshuffles the various terms of a given, fixed order of perturbation theory. Remark that if the initial graphs have say degree 4 at each vertex, only trees with degree less than or equal to 4 occur in the rearranged tree expansion.

For Fermionic theories this is typically sufficient and one has for small enough coupling

$$\sum_{\mathcal{T}} |A_{\mathcal{T}}| < \infty$$

because Fermionic graphs mostly compensate each other at a fixed order by Pauli’s principle; mathematically this is because these graphs form a determinant and the size of a determinant is much less than what its permutation expansion suggests. This is well known [15, 16, 17].

But this repacking fails for Bosonic theories, because the only compensations there occur between graphs of different orders. Hence if we were to perform this naive reshuffling, eg on the $\phi^4_0$ theory we would still have

$$\sum_{\mathcal{T}} |A_{\mathcal{T}}| = \infty.$$  \hfill (22)

4 The Loop Vertex Expansion

The loop vertex expansion overcomes this difficulty by exchanging the role of vertices and propagators before applying the forest formula. The corresponding regrouping is completely different and each tree resums an infinite number of pieces of the previous graphs. It relies on a technical tool (which physicists call the intermediate field representation) which decomposes any interaction of degree higher than three in terms of simpler three-body interactions. It is
particularly natural for 4-body interactions but can be generalized to higher interactions as well [18].

This quite universal and powerful trick is linked to various deep physical and mathematical tools, such as the color 1/N expansion and the Matthews-Salam and Hubbard-Stratonovich methods in physics and the Kaufmann bracket of a knot and many similar ideas in mathematics.

It is easy to describe the intermediate field method in terms of functional integrals, as it is a simple generalization of the formula

$$e^{-\lambda \phi^4/2} = \int e^{-\sigma^2/2} e^{i\sqrt{\lambda} \sigma \phi^2} d\sigma.$$  \hspace{1cm} (23)

In this section we introduce the graphical procedure equivalent to this formula.

In the case of a $\phi^4$ graph $G$ each vertex has exactly four half-lines hence there are exactly three ways to pair these half-lines into two pairs. Hence each fully labeled (vacuum) graph of order $n$ (with labels on vertices and half-lines), which has $2n$ lines can be decomposed exactly into $3^n$ labeled graphs $G'$ with degree 3 and two different types of lines

- the $2n$ old ordinary lines
- $n$ new dotted lines which indicate the pairing chosen at each vertex (see Figure 5).

![Figure 5: The extension and collapse for the order 1 graph](image)

Such graphs $G'$ are called the 3-body extensions of $G$ and we write $G' \text{ ext } G$ when $G'$ is an extension of $G$. Let us introduce for each such
extension $G'$ an amplitude $A_{G'} = 3^{-n}A_G$ so that

$$A_G = \sum_{G' \text{ ext } G} A_{G'}$$

(24)

when $G'$ is an extension of $G$.

Now the ordinary lines of any extension $G'$ of any $G$ must form cycles. These cycles are joined by dotted lines.

**Definition 4.1.** We define the collapse $\bar{G}'$ of such a $G'$ graph as the graph obtained by contracting each cycle to a "bold" vertex (see Figure 5). We write $\bar{G}'$ coll $G'$ if $\bar{G}'$ is the collapse of $G'$, and define the amplitude of the collapsed graph $\bar{G}'$ as equal to that of $G'$.

Remark that collapsed graphs, made of bold vertices and dotted lines, can have now arbitrary degree at each vertex. Remark also that several different extensions of a graph $G$ can have different collapsed graphs, see Figure 5.

Now the loop vertex expansion rewrites

$$S = \sum_G A_G = \sum_{G' \text{ ext } G} A_{G'} = \sum_{\bar{G}' \text{ coll } G'} \sum_{G' \text{ ext } G} A_{G'}.$$ 

(25)

Now we perform the tree repacking according to the graphs $\bar{G}'$ with the $n$ dotted lines and *not* with respect to $G$. This is a completely different repacking:

$$A_{\bar{G}'} = \sum_{\bar{T} \in \bar{G}'} w(\bar{G}', \bar{T})A_{\bar{G}'}.$$ 

(26)

so that

$$S = \sum_{G' \text{ ext } G} A_{\bar{G}'} = \sum_{\bar{T} \in \bar{G}'} A_{\bar{T}},$$

(27)

$$A_{\bar{T}} = \sum_{G' \supset \bar{T}} w(G', \bar{T})A_{G'}.$$ 

(28)
The "miracle" is that

**Theorem 4.1.** For $\lambda$ small

$$
\sum_{\mathcal{T}} |A_{\mathcal{T}}| < \infty \quad (29)
$$

the result being the Borel sum of the initial perturbative series [12].

The proof of the theorem will not be recalled here (see [10, 11, 12]) but it relies on the positivity property of the $x_{\mathcal{T}}(\{w\})$ symmetric matrix, and the representation of each $A_{\mathcal{T}}$ amplitude as an integral over a corresponding normalized Gaussian measure of a product of resolvents bounded by 1. This convergence would not be true if we had chosen naive $w(T, G)$ barycentric weights such as $1/5$ for each of the five trees of the graph in Figure 1.

This method is valid for any $\phi^4$ model in any dimension with cutoffs [11]. It is not limited to $\phi^4$ but works eg for any stable interaction at the cost of introducing more intermediate particles until three body elementary interactions are reached [18]. It also reproduces correctly the large $N$ behaviour of $\phi^4$ matrix models, which was the key property for which this expansion was found [10].

## 5 Examples

In this section we give the extension and collapse of the Feynman graphs for $Z$ and log $Z$ for the $\phi^4_0$ model up to order 3. We also recover the combinatorics of those graphs through the ordinary functional integral formula for the loop vertex expansion formula of [12].
The extension and collapse at order 1 was shown in Figure (6). In this case the tree structure is easy. We find only the trivial "empty" tree with one vertex and no edge and the "almost trivial" tree with two vertices and a single edge. The weight for these trees is 1.

\[
\begin{align*}
\text{extension} & \quad 3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \\
\text{collapse} & \quad + \quad +
\end{align*}
\]

Tree structure in loop vertex expansion:

\[\text{extension} \quad \text{collapse} \]

Figure 6: The extension and collapse for order 1 graph, with combinatoric weight shown below, and the list of corresponding trees.

At second order we find one disconnected Feynman graph and two connected ones. Only the connected ones survive in the expansion of \(\log Z\).

The corresponding graphs and tree structures are shown in Figure (7) and Figure (8). Using the loop vertex expansion formula we begin to see that graphs that come from different order of the expansion of \(\lambda\) are associated to the same trees by the loop vertex expansion. Indeed we recover contributions for the trivial and almost trivial trees of the previous figure. But we find also a new contribution belonging to a tree with two edges.

At order three the computation becomes a bit more involved but the process is clear. We could start from the ordinary Feynman graphs and get the graphs of loop vertex expansion by extension and collapse. This is shown in Figure (10). The number under each collapsed graph means the number of the corresponding graphs, as in the previous case. The tree structure is shown in Figure (12). In this figure the weight factor \(w\) means always \(w(G, \mathcal{T})\).
Figure 7: The extension and collapse for order 2 graph and the number of graphs.

We could also get the graphs and combinatorics by using directly the loop vertex expansion, namely we integrate the $\phi$ fields and consider only the Wick contractions of the $\sigma$ fields. This is shown in the appendix and Figure (11). In this process we expand both $\exp V$ and the vertex $V = \text{tr} \log(1 + 2i\sqrt{2} \lambda \sigma)$.

The interactions terms are then the loop vertices $V$ with various attached $\sigma$ fields. This is shown on the left hand of Figure (11). For example, the symbol 123 means we consider the $V^3/3!$ term in $\exp V$. We expand one of the $V$ to order $\lambda^{1/2}$, namely with one $\sigma$ field attached, one to the order $\lambda$, namely with 2 $\sigma$ fields attached and the third one to $\lambda^{3/2}$, namely with 3 $\sigma$ fields. Then we contract the sigma fields with respect to the Gaussian measure, obtaining
Figure 8: The connected graphs and the tree structure from the Loop vertex expansion.

Figure 9: The order 3 vacuum graph and the number of graphs.

all the contracted graphs. The total number of 123 graphs could be read directly from this Gaussian integration. To get the combinatoric factor of each graph we need to compute the relative weights of these graphs. This is shown in the following example:

**Example 5.1.** We consider the 123 case for example. This is shown more
explicitly in Figure 13. We use $a, b, \cdots, f$ to label the $\sigma$ fields attached to the vertices. After the Wick contractions we get three different graphs $A$, $B$ and $C$. The number of possibilities to get $A$ is $3$, the number to get $B$ is $2 \times 3 = 6$ and the number to get $C$ is also $6$. So the relative weight for graph $A$ is $3/(3+6+6) = 1/5$ and the relative weights for $B$ and $C$ are both $6/(3+6+6) = 2/5$. As we could read directly from the loop vertex formula that the total number of 123 contraction graphs is $960$, we get finally the combinatoric factor of graph $A$ to be $960 \times 1/5 = 192$, and the corresponding factors for graphs $B$ and $C$ are $960 \times 2/5 = 384$. This result agrees with the one coming from the Feynman graph computation.

From these examples we find that the structure of loop vertex expansion is totally different from that of Feynman graph calculus. At each order of
Figure 11: The graph structure and combinatorics from the loop vertex expansion at order 3. The symbols like 1122 means we have 4 loop vertices V, two of them have one $\sigma$ field each and two of them have two $\sigma$ fields each, as we could read directly from this figure.

the loop vertex expansion we combine terms in different orders of $\lambda$. 
6 Non-integer Dimension

Let us now consider, eg for $0 < D \leq 2$ the Feynman amplitudes for the $\phi^4_D$ theory. They are given by the following convergent parametric representation

$$A_{D,G} = \int_0^\infty d\alpha \frac{e^{-m^2 \sum_\ell \alpha_\ell}}{U_G^{D/2}}$$

(30)
where $m$ is the mass and $U_G$ is the Kirchoff-Symanzik polynomial for $G$

$$U_G = \sum_{T \in G} \prod_{\ell \notin T} \alpha_{\ell},$$  \hspace{1cm} (31)

All the previous decompositions working at the level of graphs, they are independent of the space-time dimension. We can therefore repack the series of Feynman amplitudes in non integer dimension to get the $D$ dimensional tree amplitude:

$$A_{D,T} = \sum_{G \supset T} w(\bar{T}, G) A_{D,G}$$ \hspace{1cm} (32)

We know that for $D = 0$ and $D = 1$ the loop vertex expansion is convergent. Therefore it is tempting to conjecture, for instance at least for $D$ real and $0 \leq D < 2$ (that is when no ultraviolet divergences require renormalization)

**Conjecture 6.1.** For $\lambda$ small

$$\sum_{\bar{T}} |A_{D,T}| < \infty$$ \hspace{1cm} (33)

the result being the Borel sum of the initial perturbative series.

Progress on this conjecture would be extremely interesting as it would allow to bridge quantum field theories in various dimensions of space time, and ie perhaps justify the Wilson-Fisher $4 - \epsilon$ expansion that allows good numerical approximate computations of critical indices in 3 dimensions.

We know however that when renormalization is needed, ie for $D \geq 2$, this approach has to be completed with the introduction of the correct counterterms. Presumably in this case the tree expansion should be adapted to select
optimal trees with respect to renormalization group scales. This is work in progress.

An other possible approach to quantum field theory in non integer dimension, also based on the forest formula but more radical, is proposed in [19].

7 Conclusion

The lessons we may draw from the Loop Vertex Expansion are

- Interactions should be decomposed into three body elementary interactions. The corresponding fields might be more fundamental than the initial ones.

- Tree formulas solve the constructive problem ie resum perturbation theory at the cost of loosing locality of the new vertices.

It may be also interesting to further understand why trees are so central both in the parametric formulas (30) for single Feynman amplitudes and in the non-perturbative treatment of the theory. The answer might imply a complete refoundation of quantum field theory around the notion of trees, rather than Feynman graphs or even functional integrals [19].

8 Appendix

In this Appendix we compute the weight of collapsed Feynman graphs using the Loop Vertex Expansion.
For the $\phi^4$ model we have:

$$
Z = \frac{1}{\sqrt{2\pi}} \int d\phi e^{-\frac{1}{2}\phi^2 - \lambda\phi^4} = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2 - \frac{1}{2}\log(1 + 2i\sqrt{2}\lambda\sigma)}. \tag{34}
$$

We define

$$
V = \frac{1}{2} \log(1 + 2i\sqrt{2}\lambda\sigma). \tag{35}
$$

In what follows we compute the vacuum graphs up to order 3 in $\lambda$. We expand $Z$ into powers of $V$:

$$
Z = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} \left[1 - V + \frac{1}{2!} V^2 - \frac{1}{3!} V^3 + \frac{1}{4!} V^4 - \frac{1}{5!} V^5 + \frac{1}{6!} V^6\right], \tag{36}
$$

and we have

$$
\log(1 + 2i\sqrt{2}\lambda\sigma) = 2\sqrt{2}\lambda i\sigma + 4\lambda\sigma^2 - \frac{16\sqrt{2}}{3}\lambda^{3/2}\sigma^3 - 16\lambda^2\sigma^4 + \frac{128\sqrt{2}}{5}\lambda^{5/2}\sigma^5 + \frac{256}{3}\lambda^3\sigma^6. \tag{37}
$$

The first term

$$
\frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} 1 = 1 \tag{38}
$$

is trivial.

The order $V$ terms give:

$$
-\frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} V = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} \left[-2\lambda\sigma^2 + 8\lambda^2\sigma^4 - \frac{128}{3}\lambda^3\sigma^6\right] = -2\lambda + 24\lambda^2 - 640\lambda^3. \tag{39}
$$

The $V^2$ terms give:

$$
\frac{1}{2!} \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} V^2 = \frac{1}{2!} \left(\frac{1}{2}\right)^2 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} \left[-8\lambda\sigma^2 + 16\lambda^2\sigma^4\right] - \frac{64}{9}\lambda^3\sigma^6 + \frac{128}{3}\lambda^2\sigma^4 - \frac{128 \times 8}{5}\lambda^3\sigma^6 - 128\lambda^3\sigma^6] = -\lambda + 22\lambda^2 - \frac{320}{3}\lambda^3 - 624\lambda^3. \tag{40}
$$
The \( V^3 \) terms give:

\[
- \frac{1}{3!} \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2} \sigma^2} V^3 = - \frac{1}{3!} \left( \frac{1}{2} \right)^3 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2} \sigma^2} [64\lambda^3 \sigma^6 - 96\lambda^2 \sigma^4 + 384\lambda^3 \sigma^6 + 512\lambda^3 \sigma^6] \\
= 6\lambda^2 - 300\lambda^3.
\]  

(41)

The \( V^4 \) terms give:

\[
\frac{1}{4!} \left( \frac{1}{2} \right)^4 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2} \sigma^2} [64\lambda^2 \sigma^4 - \frac{2048}{3} \lambda^3 \sigma^6 - 768\lambda^3 \sigma^6] \\
= \frac{1}{2} \lambda^2 - \frac{80}{3} \lambda^3 - 30\lambda^3.
\]

(42)

The \( V^5 \) terms give:

\[
- \frac{1}{5!} \left( \frac{1}{2} \right)^5 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2} \sigma^2} 1280\lambda^3 \sigma^6 = -5\lambda^3.
\]  

(43)

The \( V^6 \) term gives:

\[
- \frac{1}{6!} \left( \frac{1}{2} \right)^6 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2} \sigma^2} 512\lambda^3 \sigma^6 = -\frac{1}{6} \lambda^3.
\]

(44)

So up to 3rd order in \( \lambda \) we recover

\[
Z = -3\lambda + \frac{105}{2} \lambda^2 - \frac{10395}{6} \lambda^3 = -4!!\lambda + \frac{8!!}{2!!} \lambda^2 - \frac{12!!}{3!!} \lambda^3,
\]

(45)

which of course coincide with the number of ordinary Wick contractions derived by the regular \( \lambda \phi^4 \) Feynman expansion.

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