Some new similarity measures for hesitant fuzzy sets and their applications in multiple attribute decision making

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Abstract. Similarity measure is a very important topic in fuzzy set theory. Torra (2010) proposed the notion of hesitant fuzzy set (HFS), which is a generalization of the notion of Zadeh' fuzzy set. In this paper, some new similarity measures for HFSs are developed. Based on the proposed similarity measures, a method of multiple attribute decision making under hesitant fuzzy environment is also introduced. Additionally, a numerical example is given to illustrate the application of the proposed similarity measures of HFSs to decision-making.

Keywords: fuzzy set; hesitant fuzzy set; distance measure; similarity measure; multiple attribute decision making

1 Introduction

Ever since the notion of fuzzy set was given by Zadeh [38], many new theories and approaches dealing with uncertainty and imprecision have been introduced. Some of them, such as intuitionistic fuzzy set (IFS) [1], interval-valued fuzzy set (IVFS) [2], vague set [9], and type-2 fuzzy set (T2FS) [39], are extensions of ordinary fuzzy set theory. After that, many researchers have studied this topic and obtained a lot of meaningful results in cluster analysis, multi-criteria decision, aggregation and grey relational analysis.

As is well known, the similarity measure is a very important concept, for it provides the degree of similarity between two fuzzy objects. Since Zadeh [40] introduced the similarity relation concept, similarity measures of fuzzy sets have been widely studied from different aspects and applied in various fields, such as decision-making, cluster analysis, machine learning, market prediction, approximate reasoning, image processing, and pattern recognition. Fan and Xie [8] as well as Liu [18] gave the axiom definition and

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studied some properties of similarity measures between fuzzy sets. Pappis and Karacapilidis [21] investigated three similarity measures of fuzzy sets based on intersection and union operations, the maximum difference and the differences as well as the sum of membership grades. In [26], Wang proposed two new similarity measures between fuzzy sets and between elements. Turksen and Zhong [25] applied similarity measures of fuzzy sets for an approximate analogical reasoning. In a multimedia database query, Candan et al. [3] applied similarity measures to develop query processing with different fuzzy semantics. Moreover, lots of similarity measures for IFSs, IVFSs, vague sets and T2FSs have also been widely developed in the literatures [6, 7, 9–17, 32, 36, 41, 42].

Recently, to deal with hesitant and incongruous problems, Torra and Narukawa [23, 24] introduced the concept of hesitant fuzzy set (HFS), which is also an extension of the classic fuzzy set, for it permits the membership degree of an element to a set to be represented as several possible values between 0 and 1. After the pioneering work of Torra, the HFS has received much attention from many authors and has been used in decision-making and clustering analysis [5, 22, 27, 31, 34, 37, 43]. For example, Chen [5] systematically investigated the correlation coefficients of HFSs and applied them to clustering analysis, Xia and Xu [28] studied the aggregation operators of hesitant fuzzy sets and applied them to decision making. Xu and Xia [30] gave the axiom definitions of distance and similarity measures between HFSs, they also presented some distance measures for HFSs and obtained some similarity measures corresponding to the distances of HFSs. However, their axiom definitions of distance and similarity measures only satisfy three properties, respectively. The more reasonable definitions of distance and similarity measures, in general, should have four properties like the notions of fuzzy sets [8–18], IFSs [12–14, 19], IVFSs [41] and T2FSs [13]. Therefore, in this paper we modify the axiom definitions of distance and similarity measures for HFSs and propose some new distance and similarity measures between HFSs.

The rest of this paper is organized as follows. In Section 2, we review the notions of HFS and give the modified axiom definitions of distance and similarity measures for HFSs. In Section 3, we present some new geometric distance and similarity measures between HFSs base on geometric distance model and set-theory approach. We apply the proposed similarity measures of HFSs to hesitant fuzzy decision-making in Section 4. We make the conclusions in Section 5.

2 Preliminaries

In this section, we briefly recall the necessary definitions and notations of HFS and modify the axiom definitions of distance and similarity measures between HFSs, which were first given by Xu and Xia [30].

HFSs are very useful in dealing with the situations where people have hesitation in providing their preferences over objects in a decision-making process. The definition of HFS was first introduced by Torra and Narukawa [23, 24] as follows.

**Definition 2.1** Let X be a reference set, an HFS on X is in terms of a function that when applied to X
returns a subset of \([0, 1]\), which can be represented as \(H = \{\frac{h_H(x)}{x} | x \in X\}\), where \(h_H(x)\) is a set of some values in \([0, 1]\), denoting the possible membership degrees of the element \(x \in X\) to the set \(H\).

For convenience, Xu and Xia \cite{30} called \(h_H(x)\) an hesitant fuzzy element (HFE) with respect to \(x\) of \(H\). It is worth noting that the number of values of different HFEs may be different, let \(n(h_H(x))\) be the number of values of \(h_H(x)\). We arrange the values of \(h_H(x)\) in decreasing order, and let \(h^{(i)}_H(x)\) be the \(i\)th smallest value of \(h_H(x)\).

Distance and similarity measures are the fundamental and important issues of theory of sets. For HFSs, the axiom definitions of distance and similarity measures were first addressed by Xu and Xia \cite{30}.

**Definition 2.2** Let \(A\) and \(B\) be two HFSs on \(X = \{x_1, x_2, \ldots, x_m\}\). Then the distance measure between \(A\) and \(B\) is defined as \(d(A, B)\), which satisfies the following properties:
1. \(0 \leq d(A, B) \leq 1\);
2. \(d(A, B) = 0 \iff A = B\);
3. \(d(A, B) = d(B, A)\).

**Definition 2.3** Let \(A\) and \(B\) be two HFSs on \(X = \{x_1, x_2, \ldots, x_m\}\). Then the similarity measure between \(A\) and \(B\) is defined as \(s(A, B)\), which satisfies the following properties:
1. \(0 \leq s(A, B) \leq 1\);
2. \(s(A, B) = 1 \iff A = B\);
3. \(s(A, B) = s(B, A)\).

In many cases, however, \(n(h_A(x)) \neq n(h_B(x))\). To operate correctly, it is requested that two HFEs have the same length when they are compared. Thus we should extend the shorter one such that their length is the same. For this, Xu and Xia \cite{30} give the following regulation:

If \(n(h_A(x)) > n(h_B(x))\), then \(h_B(x)\) should be extended by adding the minimum value in it until it has the same length with \(h_A(x)\); If \(n(h_A(x)) < n(h_B(x))\), then \(h_A(x)\) should be extended by adding the minimum value in it until it has the same length with \(h_B(x)\). For instance, let \(h_A(x) = \{0.5, 0.4\}, h_B(x) = \{0.7, 0.4, 0.2\}\). Clearly, \(n(h_A(x)) < n(h_B(x))\), so we should extend \(h_A(x)\) to \(h_A(x) = \{0.5, 0.4, 0.4\}\).

In fact, we can extend the shorter HFE by adding any value in it until it has the same length with the longer one according to the decision makers’ preferences and actual situations. In this paper, we assume that the decision makers all adopt the above regulation.

Based on the above regulation, we define the following comparison laws.

**Definition 2.4** Let \(A\) and \(B\) be two HFSs on \(X\), and \(n_x = \max\{n(h_A(x)), n(h_B(x))\}\) for all \(x \in X\). Then
1. \(h_A(x)\) is said to be inferior to \(h_B(x)\), denoted by \(h_A(x) \preceq h_B(x)\), if \(h^{(i)}_A(x) \leq h^{(i)}_B(x)\) for all \(i = 1, 2, \ldots, n_x\). Especially, if \(n_x = n(h_A(x)) = n(h_B(x))\) and \(h^{(i)}_A(x) \leq h^{(i)}_B(x)\) for all \(i = 1, 2, \ldots, n_x\), then \(h_A(x)\) is said to be less than \(h_B(x)\), denoted by \(h_A(x) \leq h_B(x)\).
(2) \( h_A(x) \) is said to be equal to \( h_B(x) \) if \( h_A^{(i)}(x) = h_B^{(i)}(x) \) for all \( i = 1, 2, \ldots, n_x \), denoted by \( h_A(x) = h_B(x) \).

(3) HFS \( A \) is said to be an hesitant fuzzy quasi subset of HFS \( B \), denoted by \( A \subseteq B \), if \( h_A(x) \preceq h_B(x) \) for all \( x \in X \). Especially, if \( h_A(x) \leq h_B(x) \) for all \( x \in X \), then \( A \) is called an hesitant fuzzy subset of \( B \), denoted by \( A \subseteq B \).

(4) HFS \( A \) is said to be equal to HFS \( B \), denoted by \( A = B \), if \( h_A(x) = h_B(x) \) for all \( x \in X \).

**Proposition 2.5** Let \( A \) and \( B \) be two HFSs on \( X \). If \( h_A(x) = h_B(x) \), then \( n(h_A(x)) = n(h_B(x)) \).

**Proof.** The proof is easily obtained from Definition 2.4.

Based on Definition 2.4, we modify the axiom definitions of the distance and similarity measures as follows:

**Definition 2.6** Let \( A \) and \( B \) be two HFSs on \( X = \{x_1, x_2, \ldots, x_m\} \), and \( d_{\text{max}} = \max\{d(A, B)\} \). Then the distance measure between \( A \) and \( B \) is defined as \( d(A, B) \), which satisfies the following properties:

- \( (D1) \ 0 \leq d(A, B) \leq d_{\text{max}} \);
- \( (D2) \ d(A, B) = 0 \iff A = B \);
- \( (D3) \ d(A, B) = d(B, A) \);
- \( (D4) \ Let C be an HFS, if A \subseteq B \subseteq C, then d(A, B) \leq d(A, C) \) and \( d(B, C) \leq d(A, C) \).

If \( (D1') \) replaces \( (D1) \), then \( d(A, B) \) is called a normalized distance measure, where

- \( (D1') \ 0 \leq d(A, B) \leq 1 \).

**Definition 2.7** Let \( A \) and \( B \) be two HFSs on \( X = \{x_1, x_2, \ldots, x_m\} \). Then the similarity measure between \( A \) and \( B \) is defined as \( s(A, B) \), which satisfies the following properties:

- \( (P1) \ 0 \leq s(A, B) \leq 1 \);
- \( (P2) \ s(A, B) = 1 \iff A = B \);
- \( (P3) \ s(A, B) = s(B, A) \);
- \( (P4) \ Let C be an HFS, if A \subseteq B \subseteq C, then s(A, C) \leq (A, B) \) and \( s(A, C) \leq s(B, C) \).

### 3 Some new similarity measures for hesitant fuzzy sets

Let \( A \) and \( B \) be two HFSs on \( X = \{x_1, x_2, \ldots, x_m\} \). In this section, we introduce some new distance and similarity measures between hesitant fuzzy sets.

#### 3.1 Similarity measures based on geometric distance model

Xu and Xia [30] introduced a lot of geometric distance models between hesitant fuzzy sets \( A \) and \( B \). Some of them are given as follows:
(1) Hesitant normalized Hamming distance:

\[ d_1(A, B) = \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} | h^{(j)}_{A}(x_i) - h^{(j)}_{B}(x_i) | \right) \]  

(2) Hesitant normalized Euclidean distance:

\[ d_2(A, B) = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} | h^{(j)}_{A}(x_i) - h^{(j)}_{B}(x_i) |^2 \right)} \] 

(3) Generalized hesitant normalized distance:

\[ d_3(A, B) = \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} | h^{(j)}_{A}(x_i) - h^{(j)}_{B}(x_i) |^p \right) \right]^{1/p}, \quad p > 0. \]  

Clearly, if \( p = 1 \), then Eq. (3) is reduced to Eq. (1).

From Eq. (1), we know that

\[ d_i = \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} | h^{(j)}_{A}(x_i) - h^{(j)}_{B}(x_i) | \]

indicates the distance between the \( i \)th HFE of \( A \) and \( B \), and \( d_i(A, B) \) indicates the mean of distances between all elements of \( A \) and \( B \). From the point of view, we define another generalized normalized distance of \( A \) and \( B \) as:

\[ d_4(A, B) = \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} | h^{(j)}_{A}(x_i) - h^{(j)}_{B}(x_i) |^p \right) \right]^{1/p}, \quad p > 0, \]  

which we call type-2 generalized hesitant normalized distance. It is clear that Eq. (4) is different from Eq. (3). But if \( p = 1 \), then Eq. (4) is also reduced to Eq. (1). If \( p = 2 \), then Eq. (4) becomes type-2 hesitant normalized Euclidean distance:

\[ d_5(A, B) = \frac{1}{m} \sum_{i=1}^{m} \sqrt{\frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} | h^{(j)}_{A}(x_i) - h^{(j)}_{B}(x_i) |^2}. \]  

Then it is natural to ask “Is the defined distance \( d_4(A, B) \) reasonable?” We answer this question in Theorem 3.1.

**Theorem 3.1** \( d_4(A, B) \) is a normalized distance measure between HFSs \( A \) and \( B \).
Proof. It is easy to see that $d_4(A, B)$ satisfies the properties (D1') – (D3). We therefore only prove (D4). Let $A \subseteq B \subseteq C$, then $h_A(x_i) \leq h_B(x_i) \leq h_C(x_i)$ for each $x_i \in X$. It follows that

$$
|h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \leq |h_A^{\sigma(j)}(x_i) - h_C^{\sigma(j)}(x_i)|^p,
$$

$$
|h_B^{\sigma(j)}(x_i) - h_C^{\sigma(j)}(x_i)|^p \leq |h_B^{\sigma(j)}(x_i) - h_C^{\sigma(j)}(x_i)|^p,
$$

$$
\Rightarrow \frac{1}{n_{xi}} \sum_{j=1}^{n_{xi}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \leq \frac{1}{n_{xi}} \sum_{j=1}^{n_{xi}} |h_A^{\sigma(j)}(x_i) - h_C^{\sigma(j)}(x_i)|^p,
$$

$$
\frac{1}{n_{xi}} \sum_{j=1}^{n_{xi}} |h_B^{\sigma(j)}(x_i) - h_C^{\sigma(j)}(x_i)|^p \leq \frac{1}{n_{xi}} \sum_{j=1}^{n_{xi}} |h_B^{\sigma(j)}(x_i) - h_C^{\sigma(j)}(x_i)|^p,
$$

$$
\Rightarrow d_4(A, B) \leq d_4(A, C), d_4(B, C) \leq d_4(A, C).
$$

Thus the property (D4) is obtained. □

Based on Eq. (4), we further define type-2 generalized hesitant distances as follows:

$$
d_5(A, B) = \sum_{i=1}^{m} \left( \frac{1}{n_{xi}} \sum_{j=1}^{n_{xi}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p}, p > 0.
$$

(6)

$$
d_6(A, B) = \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n_{xi}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p}, p > 0.
$$

(7)

$$
d_7(A, B) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n_{xi}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p}, p > 0.
$$

(8)

Theorem 3.2 $d_i(A, B)(i = 5, 6, 7)$ is a distance measure between HFSs A and B, and satisfies the following properties:

1. $0 \leq d_5(A, B) \leq m$;
2. $0 \leq d_6(A, B) \leq \frac{1}{m} \sum_{i=1}^{m} (n_{xi})^{1/p}$;
3. $0 \leq d_7(A, B) \leq \sum_{i=1}^{m} (n_{xi})^{1/p}$.

Proof. The proof of (D2) – (D4) is similar to Theorem 5.1. We only prove (1) – (3). Let $h_A^{\sigma(j)}(x_i) = 1$ and $h_B^{\sigma(j)}(x_i) = 0$ for all $x_i \in X$ and $j = 1, 2, \cdots, n_{xi}$, then $d_5(A, B) = m$, $d_6(A, B) = \frac{1}{m} \sum_{i=1}^{m} (n_{xi})^{1/p}$ and $d_7(A, B) = \sum_{i=1}^{m} (n_{xi})^{1/p}$. □

The $L_p$ metric is very important and has been used to measure the distance of fuzzy sets and IFSs [10]. If we apply the $L_p$ metric to the distance measure between HFSs, then a hesitant $L_p$ distance is given as

$$
d_8(A, B) = \frac{1}{m} \sum_{i=1}^{m} \left( \sum_{j=1}^{n_{xi}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p}, p \geq 1.
$$

(9)
Clearly, if \( p \geq 1 \), then the type-2 generalized hesitant distance \( d_6(A, B) \) becomes the hesitant \( L_p \) distance \( d_8(A, B) \).

However, there is an interesting result: if \( p \to \infty \), then the hesitant \( L_p \) distance \( d_8(A, B) \) is reduced to hesitant normalized Hamming-Hausdorff distance

\[
d_9(A, B) = \frac{1}{m} \sum_{i=1}^{m} \max_j |h^\sigma_A(x_i) - h^\sigma_B(x_i)|,
\]

which is defined by Xu and Xia \([30]\).

To prove the above result, the following lemma is needed.

**Lemma 3.3** Let \( a_i \in \mathbb{R} \) and \( a_i \geq 0, i = 1, 2, \cdots, k \). Then

\[
\lim_{p \to \infty} (a_1^p + a_2^p + \cdots + a_k^p)^{1/p} = \max_i \{a_i\}, \quad p \geq 1.
\]

**Proof.** It is obvious whenever (i) \( a_i = 0 (i = 1, 2, \cdots, k) \), or (ii) \( a_1 = a_2 = \cdots = a_k \), because \( \lim_{p \to \infty} k^{1/p} = 1 \).

If \( a_i \neq a_j, i \neq j, i, j = 1, 2, \cdots, k \), then the following show that

\[
\lim_{p \to \infty} (a_1^p + a_2^p + \cdots + a_k^p)^{1/p} = \max_i \{a_i\}.
\]

Without loss of generality, we suppose that \( a_1 \geq a_2 \geq \cdots \geq a_k \), and let \( y = (a_1^p + a_2^p + \cdots + a_k^p)^{1/p} \). Then

\[
\lim_{p \to \infty} ln y = \lim_{p \to \infty} \frac{a_1^p + a_2^p + \cdots + a_k^p}{p}.
\]

Using L’Hospital’s rule, we have

\[
\lim_{p \to \infty} ln y = \lim_{p \to \infty} \frac{a_1^p \ln a_1 + a_2^p \ln a_2 + \cdots + a_k^p \ln a_k}{a_1^p + a_2^p + \cdots + a_k^p} = \lim_{p \to \infty} \frac{(\ln a_1 + (a_2/a_1)^p \ln a_2 + \cdots + (a_k/a_1)^p \ln a_k)}{1 + (a_2/a_1)^p + \cdots + (a_k/a_1)^p} = \ln a_1.
\]

Therefore,

\[
\lim_{p \to \infty} y = \lim_{p \to \infty} (a_1^p + a_2^p + \cdots + a_k^p)^{1/p} = a_1 = \max_i \{a_i\}. \quad \Box
\]

**Theorem 3.4** \( \lim_{p \to \infty} d_8(A, B) = \frac{1}{m} \sum_{i=1}^{m} \max_j |h^\sigma_A(x_i) - h^\sigma_B(x_i)|. \)

**Proof.** It can be obtained directly by Lemma \([3.3]\) \( \Box \)

In many practical problems, however, the weight of the element \( x_i \in X \) should be taken into account. Especially for multiple attribute decision making problems, the considered attributes usually are of different importance. Thus we need to consider the weight of the element so that we get the following weighted
distance between HFSs. Assume that \( w_i(i = 1, 2, \ldots, m) \) is the weight of the element \( x_i \in X, w_i \in [0, 1] \) and \( \sum_{i=1}^{m} w_i = 1 \), then we obtain a type-2-generalized hesitant weighted distance

\[
d_{10}(A, B) = \sum_{i=1}^{m} w_i \left( \frac{1}{n_x} \sum_{j=1}^{n_x} [h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)]^p \right)^{1/p}, \quad p > 0. \tag{11}
\]

and a hesitant \( L_p \) weighted distance

\[
d_{11}(A, B) = \sum_{i=1}^{m} w_i \left( \frac{1}{n_x} \sum_{j=1}^{n_x} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p}, \quad p \geq 1. \tag{12}
\]

Obviously, if each element has the same importance, that is, \( w_i = 1/m, (i = 1, 2, \ldots, n) \), then the Eq.s (11) and (12) are reduced to Eq.s (4) and (9), respectively.

It is seen that all the above distance measures are discrete, if both the universe of discourse and the weight of element are continuous, then we get the continuous distances. Let the weight of \( x \in X = [a, b] \) be \( w(x) \) with \( w(x) \in [0, 1] \) and \( \int_a^b w(x)dx = 1 \), we define a type-2 continuous hesitant weighted Euclidean distance and type-2 generalized continuous hesitant weighted distance as follows, respectively:

\[
d_{12}(A, B) = \int_a^b w(x) \left[ \frac{1}{n_x} \sum_{j=1}^{n_x} [h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x)]^2 \right]^{1/2} \, dx \tag{13}
\]

\[
d_{13}(A, B) = \int_a^b w(x) \left[ \frac{1}{n_x} \sum_{j=1}^{n_x} |h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x)|^p \right]^{1/p} \, dx, \quad p > 0. \tag{14}
\]

Especially, if \( w(x) = 1/(b - a) \) for all \( x \in [a, b] \), then the type-2 continuous hesitant weighted Euclidean distance is reduced to a type-2 continuous hesitant normalized Euclidean distance

\[
d_{14}(A, B) = \frac{1}{(b - a)} \int_a^b \left[ \frac{1}{n_x} \sum_{j=1}^{n_x} [h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x)]^2 \right]^{1/2} \, dx \tag{15}
\]

and the type-2 generalized continuous hesitant weighted distance is reduced to a type-2 generalized continuous hesitant normalized distance

\[
d_{15}(A, B) = \frac{1}{(b - a)} \int_a^b \left[ \frac{1}{n_x} \sum_{j=1}^{n_x} |h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x)|^p \right]^{1/p} \, dx, \quad p > 0. \tag{16}
\]

Based on \( L_p \) metric, we define a continuous hesitant weighted \( L_p \) distance

\[
d_{16}(A, B) = \int_a^b w(x) \left[ \sum_{j=1}^{n_x} [h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x)]^p \right]^{1/p} \, dx, \quad p \geq 1. \tag{17}
\]
Especially, if \( w(x) = 1/(b - a) \) for all \( x \in [a, b] \), then the continuous hesitant weighted \( L_p \) distance is reduced to a continuous hesitant average \( L_p \) distance

\[
d_{17}(A, B) = \frac{1}{(b - a)} \int_a^b \left[ \sum_{j=1}^{n_h} |h_A^{(j)}(x) - h_B^{(j)}(x)|^p \right]^{1/p} dx, \quad p \geq 1. \tag{18}
\]

Motivated by the ordered weighted idea [33], similar to literature [30], we can get the hesitant ordered weighted distances corresponding to aforementioned distances.

As is well known, an exponential operation is very useful in dealing with the similarity relation [40], classical Shannon entropy [20] and in cluster analysis [35]. We therefore adopted the exponential operation to a distance of HFSs and get a new distance measure between HFSs. Let \( d(A, B) \) be a distance between HFSs \( A \) and \( B \) and \( d_{\text{max}} = \max(d(A, B)) \), then we define an exponential-type distance measure:

\[
d_{18}(A, B) = \frac{1 - \exp(-d(A, B))}{1 - \exp(-d_{\text{max}})} \tag{19}
\]

we give the following lemma to prove Eq. (19) is a reasonable distance measure.

**Lemma 3.5** Let \( f(x) = \frac{1 - \exp(-x)}{1 - \exp(-m)} \), \( x \in [0, m] \), then \( f_{\text{min}}(x) = f(0) = 0 \) and \( f_{\text{max}}(x) = f(m) = 1 \).

**Proof.** Since \( f'(x) = \frac{\exp(-x)}{1 - \exp(-m)} > 0, x \in [0, m] \), then \( f(x) \) is increasing in \( [0, m] \). □

**Theorem 3.6** Let \( d(A, B) \) be a distance between HFSs \( A \) and \( B \), and \( d_{\text{max}} = \max(d(A, B)) \). Then \( d_{18}(A, B) \) is a normalized distance measure of HFSs \( A \) and \( B \).

**Proof.** \( (D1') - (D3) \) is easily obtained. We only prove \( (D4) \). Since \( d(A, B) \) is a distance measure between HFSs \( A \) and \( B \), then \( d(A, B) \leq d(A, C) \) and \( d(B, C) \leq d(A, C) \) for \( A \subseteq B \subseteq C \). By Lemma [3.5] we have \( d_{18}(A, B) \leq d_{18}(A, C) \) and \( d_{18}(B, C) \leq d_{18}(A, C) \) for \( A \subseteq B \subseteq C \). Thus the property \( (D4) \) is obtained. □

From Theorem 3.6 we know that \( d_{18}(A, B) \) is a normalized distance of \( d(A, B) \), that is to say, we can use Eq. (19) to generate a normalized distance of \( d(A, B) \).

It is well known that the similarity measure and distance measure are dual concepts. Hence we may use a distance measure to define a similarity measure.

**Theorem 3.7** Let \( A \) and \( B \) be HFSs. Let \( f \) be a monotone decreasing function, \( d \) a distance measure and \( d_{\text{max}} \) the maximal distance. We define

\[
s_0(A, B) = \frac{f(d(A, B)) - f(d_{\text{max}})}{f(0) - f(d_{\text{max}})}, \tag{20}
\]

then \( s_0(A, B) \) is a similarity measure between HFSs \( A \) and \( B \).
Proof. (1) Since \( f \) be a monotone decreasing function and \( 0 \leq d(A, B) \leq d_{\text{max}} \), then \( f(d_{\text{max}}) \leq f(d(A, B)) \leq f(0) \). This implies
\[
0 \leq \frac{f(d(A, B)) - f(d_{\text{max}})}{f(0) - f(d_{\text{max}})} \leq 1.
\]
(2) \( d(A, B) = 0 \Leftrightarrow A = B \) implies \( s_0(A, B) = 1 \Leftrightarrow A = B \).
(3) \( d(A, B) = d(B, A) \) implies \( s_0(A, B) = s_0(B, A) \).
(4) Let \( C \) be an HFS, and \( A \subseteq B \subseteq C \), then \( d(A, B) \leq d(A, C) \) and \( d(B, C) \leq d(A, C) \). Since \( f \) be a monotone decreasing function, then \( f(d(A, C)) \leq f(d(A, B)) \) and \( f(d(A, C)) \leq f(d(B, C)) \). These imply \( s_0(A, C) \leq s_0(A, B) \) and \( s_0(A, C) \leq s_0(B, C) \). \( \Box \)

By Theorem 3.7 if we choose \( f(x) = 1 - x \) (or \( e^{-x} \) or \( \frac{1}{1+x} \)), then the corresponding similarity measures between \( A \) and \( B \) can be obtained. For example, let \( f(x) = 1 - x \), then \( s_0(A, B) = 1 - \frac{d(A, B)}{d_{\text{max}}} \). Based on Eq.s (3), (4) and (7), we obtain the similarity measures corresponding to the distance measures as follows, respectively:
\[
s_1(A, B) = 1 - \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} |l_A^{\sigma(j)}(x_i) - l_B^{\sigma(j)}(x_i)|^p \right) \right]^{1/p}, \tag{21}
\]
\[
s_2(A, B) = 1 - \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} |l_A^{\sigma(j)}(x_i) - l_B^{\sigma(j)}(x_i)|^p \right) \right]^{1/p}, \tag{22}
\]
\[
s_3(A, B) = 1 - \left[ \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} |l_A^{\sigma(j)}(x_i) - l_B^{\sigma(j)}(x_i)|^p \right) \right]^{1/p}. \tag{23}
\]
where \( p > 0 \).

If we take the weight of each element \( x \in X \) into account, then we define the weighted similarity measures as:
\[
s_4(A, B) = 1 - \left[ \sum_{i=1}^{m} w_i \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} |l_A^{\sigma(j)}(x_i) - l_B^{\sigma(j)}(x_i)|^p \right) \right]^{1/p}, \tag{24}
\]
\[
s_5(A, B) = 1 - \left[ \sum_{i=1}^{m} w_i \left( \frac{1}{n_{x_i}} \sum_{j=1}^{n_{x_i}} |l_A^{\sigma(j)}(x_i) - l_B^{\sigma(j)}(x_i)|^p \right) \right]^{1/p}, \tag{25}
\]
\[
s_6(A, B) = 1 - \left[ \frac{m}{\sum_{i=1}^{m} (n_{x_i})^{1/p}} \sum_{i=1}^{m} w_i \left( \sum_{j=1}^{n_{x_i}} |l_A^{\sigma(j)}(x_i) - l_B^{\sigma(j)}(x_i)|^p \right) \right]^{1/p}. \tag{26}
\]
where \( p > 0, w_i(i = 1, 2, \cdots, m) \) with \( w_i \in [0, 1] \) and \( \sum_{i=1}^{m} w_i = 1 \).

Especially, if each element has the same importance, that is, \( w_i = 1/m, (i = 1, 2, \cdots, n) \), then the Eq.s (24), (25) and (26) are reduced to Eq.s (21), (22) and (23), respectively.

Let the universe of discourse \( X = [a, b] \), the weight of element \( x \in X \) be \( w(x) \) with \( w(x) \in [0, 1] \) and \( \int_{a}^{b} w(x)dx = 1 \), then we define the continuous similarity measures based on Eq.s (24)-(25) as follow, respectively:

\[
s_{7}(A, B) = 1 - \int_{a}^{b} w(x) \left( \frac{1}{n_x} \sum_{j=1}^{n_x} h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x) \right)^{1/p} dx
\]

\[
s_{8}(A, B) = 1 - \frac{m}{\sum_{i=1}^{m} (n_{x_i})^{1/p}} \int_{a}^{b} w(x) \left( \sum_{j=1}^{n_{x_i}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p} dx.
\]

where \( p > 0 \).

Specially, if \( w(x) = 1/(b-a) \) for all \( x \in [a, b] \), then Eq.s (27)-(28) becomes respectively

\[
s_{9}(A, B) = 1 - \frac{1}{b-a} \int_{a}^{b} \left( \frac{1}{n_x} \sum_{j=1}^{n_x} h_A^{\sigma(j)}(x) - h_B^{\sigma(j)}(x) \right)^{1/p} dx,
\]

\[
s_{10}(A, B) = 1 - \frac{m}{(b-a) \sum_{i=1}^{m} (n_{x_i})^{1/p}} \int_{a}^{b} \left( \sum_{j=1}^{n_{x_i}} |h_A^{\sigma(j)}(x_i) - h_B^{\sigma(j)}(x_i)|^p \right)^{1/p} dx.
\]

where \( p > 0 \).

It can be verified that \( s_i(A, B)(i = 4, 6, \cdots, 10) \) also have the properties (P1)-(P4).

### 3.2 similarity measures based on the set-theoretic approach

The set-theoretic approach is used usually to similarity measures for fuzzy sets [21] and intuitionistic fuzzy sets [32]. Thus we also define a similarity measure between two hesitant fuzzy sets \( A \) and \( B \) from the point of set-theoretic views as follows:

\[
s_{11}(A, B) = \frac{1}{m} \sum_{i=1}^{m} \frac{\sum_{j=1}^{n_{x_i}} \min_{j=1}^{n_{x_i}} \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)}{\sum_{j=1}^{n_{x_i}} \max_{j=1}^{n_{x_i}} \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)}
\]
Theorem 3.8 \( s_{11}(A, B) \) is a similarity measure of HFSs \( A \) and \( B \).

Proof. It is obvious that \( s_{11}(A, B) \) satisfies the properties (P1)-(P3). we only prove (P4). Let \( A \subseteq B \subseteq C \), then \( h_A(x_i) \leq h_B(x_i) \leq h_C(x_i) \) for each \( x_i \in X \). It follows that \( 0 < h_A^{\sigma(j)}(x_i) \leq h_B^{\sigma(j)}(x_i) \leq h_C^{\sigma(j)}(x_i) \) for all \( x_i \in X \) and \( j = 1, 2, \cdots, n_x \). Then we have

\[
\begin{align*}
\frac{\sum_{j=1}^{n_x} \min \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)}{\sum_{j=1}^{n_x} \max \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)} &= \frac{\sum_{j=1}^{n_x} h_A^{\sigma(j)}(x_i)}{\sum_{j=1}^{n_x} h_B^{\sigma(j)}(x_i)} \\
\frac{\sum_{j=1}^{n_x} h_A^{\sigma(j)}(x_i)}{\sum_{j=1}^{n_x} \max \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)} &= \frac{\sum_{j=1}^{n_x} \min \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)}{\sum_{j=1}^{n_x} h_B^{\sigma(j)}(x_i)} \leq \frac{1}{m} \sum_{i=1}^{m} \frac{\sum_{j=1}^{n_x} \min \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)}{\sum_{j=1}^{n_x} \max \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)} \\
\end{align*}
\]

Thus, \( s_{11}(A, C) \leq s_{11}(A, B) \). Similarly, we have \( s_{11}(A, C) \leq s_{11}(B, C) \). □

If we take the weight of each element \( x \in X \) into account, then we obtain

\[
s_{12}(A, B) = \sum_{i=1}^{m} w_i \frac{\sum_{j=1}^{n_x} \min \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)}{\sum_{j=1}^{n_x} \max \left( h_A^{\sigma(j)}(x_i), h_B^{\sigma(j)}(x_i) \right)} \tag{32}
\]

where \( w_i \in [0, 1] \) and \( \sum_{i=1}^{m} w_i = 1 \). Specially, if \( w_j = 1/m, (i = 1, 2, \cdots, m) \), then the Eq. \( 32 \) are reduced to the Eq. \( 31 \).

Let the weight of element \( x \in X = [a, b] \) be \( w(x) \) with \( w(x) \in [0, 1] \) and \( \int_{a}^{b} w(x)dx = 1 \), then we define the continuous similarity measures corresponding to Eq. \( 32 \) as follow:

\[
s_{13}(A, B) = \int_{a}^{b} w(x) \frac{\sum_{j=1}^{n_x} \min \left( h_A^{\sigma(j)}(x), h_B^{\sigma(j)}(x) \right)}{\sum_{j=1}^{n_x} \max \left( h_A^{\sigma(j)}(x), h_B^{\sigma(j)}(x) \right)} dx \tag{33}
\]
Especially, if \( w(x) = 1/(b - a) \) for all \( x \in [a, b] \), then Eq. (33) become

\[
s_{14}(A, B) = \frac{1}{b - a} \int_a^b \frac{\sum_{j=1}^{n_x} \min \left( h_A^{\sigma(j)}(x), h_B^{\sigma(j)}(x) \right)}{\sum_{j=1}^{n_x} \max \left( h_A^{\sigma(j)}(x), h_B^{\sigma(j)}(x) \right)} \, dx
\]

(34)

It is obvious that \( s_i(A, B)(i = 11, 12, \ldots, 14) \) also satisfies the properties (P1)-(P4).

4 An application in multiple attribute decision making

In this section, we apply the above proposed similarity measures to multiple attribute decision making under hesitant fuzzy environment.

For a multiple attribute decision making problem, let \( H = \{H_1, h_2, \ldots, h_p\} \) be a set of alternatives, \( X = \{x_1, x_2, \ldots, x_m\} \) a set of attributes and \( w = \{w_1, w_2, \ldots, w_m\}^T \) the weight vector of attributes, where \( w_i \in [0, 1] \) and \( \sum_{i=1}^{m} w_i = 1 \).

Now we define respectively the notions of positive ideal \( HFS \) and negative ideal \( HFS \) as follows:

\[
H^+ = \left\{ \frac{h_{H^+}(x_i)}{x_i} \ | x_i \in X \right\}
\]

(35)

and

\[
H^- = \left\{ \frac{h_{H^-}(x_i)}{x_i} \ | x_i \in X \right\}
\]

(36)

where

\[
h_{H^+}(x_i) = \left\{ h^{\sigma(k)}(x_i) \ | h^{\sigma(k)}(x_i) = \max_j \{h^{\sigma(k)}_H(x_i)\}, k = 1, 2, \ldots, n_{x_i} \right\},
\]

\[
h_{H^-}(x_i) = \left\{ h^{\sigma(k)}(x_i) \ | h^{\sigma(k)}(x_i) = \min_j \{h^{\sigma(k)}_H(x_i)\}, k = 1, 2, \ldots, n_{x_i} \right\}.
\]

Based on the aforementioned formulae of similarity measures between \( HFS \s \), we can calculate the degree of similarity of the positive ideal \( HFS \) \( H^+ \) and alternative \( H_i \), denoted by \( s(H^+, H_i) \), and the degree of similarity of the negative ideal \( HFS \) \( H^- \) and alternative \( H_i \), denoted by \( s(H^-, H_i) \), respectively.

Then we define the relative similarity measure \( s_i \) corresponding to the alternative \( H_i \) as follows:

\[
s_i = \frac{s(H^+, H_i)}{s(H^+, H_i) + s(H^-, H_i)}, \quad i = 1, 2, \ldots, m.
\]

(37)

Obviously, the bigger the value \( s_i \), the better the alternative \( H_i \).

To illustrate the proposed similarity measures of \( HFS \s \) and the above approach of decision making, we give an example adapted from Example 1 in [30] as follows:
Example 4.1

With the economic development of societies, energy is an essential factor. Therefore, the correct energy policy affects economic development and environment directly. Hence, the most appropriate energy policy selection is very important. Now we suppose that there are five energy projects as alternatives $H_i (i = 1, 2, 3, 4, 5)$ to be invested, and four attributes ($x_1$: technological; $x_2$: environmental; $x_3$: socio-political; $x_4$: economic) to be considered. The weight vector of the attributes is $w = (0.15, 0.3, 0.2, 0.35)^T$. Several decision makers are invited to evaluate the performances of the five alternatives. For an alternative under an attribute, though all of the decision makers provide their evaluated values, some of these values may be repeated. However, here we only consider all the possible values for an alternative under an attribute, that is to say these values repeated many times appear only once (Xu and Xia explained the reason in [30]). In this case, all possible evaluations for an alternative under the attributes can be regarded as an HFS. For convenience, we use an hesitant fuzzy decision matrix to express the results evaluated by the decision makers, which is given in Table 1.

|    | $x_1$ | $x_2$               | $x_3$               | $x_4$               |
|----|-------|---------------------|---------------------|---------------------|
| $H_1$ | [0.5,0.4,0.3] | [0.9,0.8,0.7,0.1] | [0.5,0.4,0.2] | [0.9,0.6,0.5,0.3] |
| $H_2$ | [0.5,0.3]     | [0.9,0.7,0.6,0.5,0.2] | [0.8,0.6,0.5,0.1] | [0.7,0.3,0.4] |
| $H_3$ | [0.7,0.6]     | [0.9,0.6]           | [0.7,0.5,0.3]     | [0.6,0.4]          |
| $H_4$ | [0.8,0.7,0.4,0.3] | [0.7,0.4,0.2]     | [0.8,0.1]         | [0.9,0.8,0.6]     |
| $H_5$ | [0.9,0.7,0.6,0.3,0.1] | [0.8,0.7,0.6,0.4] | [0.9,0.8,0.7]     | [0.9,0.7,0.6,0.3] |

If we use the formulae of similarity measure $s_i(A, B) (i = 4, 5, 6, 11)$ to calculate the degree of similarity between each alternative $H_i$ and the positive ideal alternative $H_i^+$ (or negative ideal alternative $H_i^-$), then we get the rankings of these alternatives by Eq. (37). The results are listed in Tables 2-5, respectively. We find that $H_5 > H_3$ and they are superior to others whichever formula of similarity measure is used. From Tables 2-4, it is seen that, similar to literature [30], the rankings are different except Table 3 when the different values of the parameter $p$ (which can be considered as the decision makers’ risk attitude) are given. Therefore, the proposed similarity measures can provide the decision makers more choices according to the decision makers’ risk attitudes and actual situations.
Table 2: Results obtained by the similarity measure $s_4(A, B)$.

|      | $H_1$  | $H_2$  | $H_3$  | $H_4$  | $H_5$  | Rankings                  |
|------|-------|-------|-------|-------|-------|---------------------------|
| $p = 1$ | 0.4719 | 0.47033 | 0.5111 | 0.47788 | 0.5547 | $H_5 > H_3 > H_4 > H_1 > H_2$ |
| $p = 2$ | 0.46814 | 0.48052 | 0.5138 | 0.46197 | 0.55475 | $H_5 > H_3 > H_2 > H_1 > H_4$ |
| $p = 6$ | 0.47238 | 0.48158 | 0.52557 | 0.4262 | 0.55783 | $H_5 > H_3 > H_2 > H_1 > H_4$ |
| $p = 10$ | 0.47854 | 0.47206 | 0.53101 | 0.40649 | 0.56777 | $H_5 > H_3 > H_1 > H_2 > H_4$ |

Table 3: Results obtained by the similarity measure $s_5(A, B)$.

|      | $H_1$  | $H_2$  | $H_3$  | $H_4$  | $H_5$  | Rankings                  |
|------|-------|-------|-------|-------|-------|---------------------------|
| $p = 1$ | 0.4719 | 0.47033 | 0.5111 | 0.47788 | 0.5547 | $H_5 > H_3 > H_4 > H_1 > H_2$ |
| $p = 2$ | 0.47016 | 0.46967 | 0.50993 | 0.48055 | 0.55334 | $H_5 > H_3 > H_4 > H_1 > H_2$ |
| $p = 6$ | 0.47058 | 0.45747 | 0.51003 | 0.48376 | 0.54219 | $H_5 > H_3 > H_4 > H_1 > H_2$ |
| $p = 10$ | 0.47124 | 0.4518 | 0.51049 | 0.48481 | 0.5389 | $H_5 > H_3 > H_4 > H_1 > H_2$ |

Table 4: Results obtained by the similarity measure $s_6(A, B)$.

|      | $H_1$  | $H_2$  | $H_3$  | $H_4$  | $H_5$  | Rankings                  |
|------|-------|-------|-------|-------|-------|---------------------------|
| $p = 1$ | 0.4728 | 0.4735 | 0.51883 | 0.4735 | 0.54951 | $H_5 > H_3 > H_2 > H_4 > H_1$ |
| $p = 2$ | 0.46962 | 0.48329 | 0.51937 | 0.45865 | 0.55016 | $H_5 > H_3 > H_2 > H_1 > H_4$ |
| $p = 6$ | 0.4976 | 0.49856 | 0.50208 | 0.4905 | 0.50783 | $H_5 > H_3 > H_2 > H_1 > H_4$ |
| $p = 10$ | 0.49985 | 0.49978 | 0.50015 | 0.49819 | 0.50167 | $H_5 > H_3 > H_1 > H_2 > H_4$ |

Table 5: Results obtained by the similarity measures based on the set-theoretic approach.

|      | $H_1$  | $H_2$  | $H_3$  | $H_4$  | $H_5$  | Rankings                  |
|------|-------|-------|-------|-------|-------|---------------------------|
| $s_{11}(A, B)$ | 0.49857 | 0.49975 | 0.57059 | 0.49975 | 0.6122 | $H_5 > H_3 > H_2 > H_4 > H_1$ |

5 Conclusion

In this paper, we have presented the modified axiom definitions of distance and similarity measure between HFSs and proposed a series of hesitant distance measures based on the Hamming distance, the Euclidean distance, $L_p$ metric and exponential operation. We have also investigated the relationship between distance measures and similarity measures, and according to their relationships, the corresponding similarity measures between HFSs have been obtained. Furthermore, we have also developed the similarity measures for HFSs based on set-theoretic approach and applied our similarity measures to hesitant
fuzzy decision-making. The experiment results have showed that the proposed similarity measures and approach of decision making for HFSs are reasonable and efficient.

Acknowledgments

We would like to thank the anonymous reviewers for their helpful comments and valuable suggestions.

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