Limit theorems for discrete-time quantum walks on trees

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Abstract. We consider a discrete-time quantum walk $W_t$ given by the Grover transformation on the Cayley tree. We reduce $W_t$ to a quantum walk $X_t$ on a half line with a wall at the origin. This paper presents two types of limit theorems for $X_t$. The first one is $X_t$ as $t \to \infty$, which corresponds to a localization in the case of an initial qubit state. The second one is $X_t/t$ as $t \to \infty$, whose limit density is given by the Konno density function [1–4]. The density appears in various situations of discrete-time cases. The corresponding similar limit theorem was proved in [5] for a continuous-time case on the Cayley tree.

1 Introduction

Let $G$ be a group generated by $\kappa \geq 2$ free involutions. The generating set is given by $\Sigma = \{ \epsilon_1, \epsilon_2, \ldots, \epsilon_\kappa \}$ with a relation $\epsilon_i^2 = e$, where $e$ is the identity. The Cayley tree $T_\kappa$ with the root $e$ is an infinite homogeneous $\kappa$-regular tree. The vertex set of $T_\kappa$ is defined by the all possible reduced words in $G$ such that

$$V(T_\kappa) = \{ \epsilon_{i_n} \epsilon_{i_{n-1}} \cdots \epsilon_{i_1} : \epsilon_{i_j} \neq \epsilon_{i'_j} \; j = 1, 2, \ldots, n \; (n \geq 1) \} \cup \{ e \}.$$ 

Vertices $g$ and $h$ are connected if and only if $gh^{-1} \in \Sigma$.

The state of a particle is described by a direct product of two Hilbert spaces $\mathcal{H}_P \otimes \mathcal{H}_C$, where $\mathcal{H}_P$ is generated by an orthonormal basis $\{|g\}; \; g \in V(T_\kappa)\}$ and $\mathcal{H}_C$ is associated with an orthonormal basis $\{|\epsilon_j\}; \; \epsilon_j \in \Sigma\}$. The unitary time evolution $U$ is expressed as $U = S \cdot C$, where shift operator $S$ and coin operator $C$ act on a state $|\Psi\rangle \in \mathcal{H}_P \otimes \mathcal{H}_C$ in the following: if $|g, \epsilon\rangle$ is a base of $\mathcal{H}_P \otimes \mathcal{H}_C$, then

$$C|g, \epsilon\rangle = \sum_{\tau \in S} (-\delta_{\epsilon \tau} + 2/\kappa)|g, \tau\rangle,$$

$$S|g, \epsilon\rangle = |\epsilon g, \epsilon\rangle.$$
Thus the one step unitary transition can be written as

\[ U|g, \epsilon\rangle = \sum_{\tau \in \Sigma} (-\delta_{e\tau} + 2/\kappa)|\tau g, \tau\rangle. \]

This paper is organized as follows. In Sect. 2, we reduce the quantum walk on \( \mathbb{T}_\kappa \) to a walk on \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). Section 3 presents two types of the limit theorems. Section 4 is denoted to summary and discussions. Appendices A and B give proofs Theorems 1 and 2, respectively.

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## 2 Reduction to half line

Throughout this paper, we will consider the quantum walk starting from the root \( e \) with the two cases of the initial qubit: for \( \kappa \geq 3 \), Case (A) Uniform initial qubit: \( \varphi^U_0 = \tau [1/\sqrt{\kappa}, \ldots, 1/\sqrt{\kappa}] \), Case (B) Weighted uniform initial qubit: \( \varphi^{WU}_0 = \tau [1/\sqrt{\kappa}, \omega_{2\kappa}/\sqrt{\kappa}, \ldots, \omega_{\kappa^{-1}}/\sqrt{\kappa}] \) with \( \omega_{\kappa} = e^{2\pi i/\kappa} \).

Let us devise the set \( V(\mathbb{T}_\kappa) \times \Sigma \) into a disjoint union of \( A_j^{(\pm)}(x), (j = 0, 1, \ldots, \kappa - 1, \ x \in \mathbb{Z}_+) \) with

\[
A_j^{(+)}(x) = \begin{cases} 
\{(e, \epsilon_j)\} & : x = 0, \\
\{(g, \epsilon) \in V(\mathbb{T}_\kappa) \times \Sigma : |\epsilon g| = x + 1, \text{the first letter of } g = \epsilon_j\} & : x \geq 1,
\end{cases}
\]

\[
A_j^{(-)}(x) = \{(g, \epsilon) \in V(\mathbb{T}_\kappa) \times \Sigma : |\epsilon g| = x - 1, \text{the first letter of } g = \epsilon_j\} : x \geq 1,
\]

where \( |g| \) means the length of the reduced word \( g \). To induce a reduction to a half line, we use the following lemma.

**Lemma 1** Let \( \alpha_t(g, \epsilon) \in \mathbb{C} \) be probability amplitude at \( (g, \epsilon) \) at time \( t \), where \( \mathbb{C} \) is the set of complex numbers.

1. Case (A) (the initial qubit \( \varphi^U_0 \)):
   If \( (g, \epsilon), (g', \epsilon') \in V(\mathbb{T}_\kappa) \otimes \Sigma \) with \( |\epsilon g| = |\epsilon' g'| \), then \( \alpha_t(g, \epsilon) = \alpha_t(g', \epsilon') \).

2. Case (B) (the initial qubit \( \varphi^{WU}_0 \)):
   If \( (g, \epsilon) \in A_j^{(\pm)}(x) \) and \( (g', \epsilon') \in A_j^{(\pm)}(x) \), then \( \alpha_t(g', \epsilon') = \omega_{\kappa}^{-i} \alpha_t(g, \epsilon) \).

**Proof.** For part (1), from the symmetry of \( \mathbb{T}_\kappa \) and the property of the Grover coin, we can show that for any \( (g, \epsilon), (g', \epsilon') \in A_j^{(\pm)}(x) \), \( \alpha_t(g, \epsilon) = \alpha_t(g', \epsilon') \) by induction on time step \( t \), (see a more detailed proof in [6, 7], for example). Then when the initial qubit is \( \varphi^U_0 \), we see that if \( (g, \epsilon), (g', \epsilon') \in V(\mathbb{T}_\kappa) \otimes \Sigma \) with \( |\epsilon g| = |\epsilon' g'| \), then \( \alpha_t(g, \epsilon) = \alpha_t(g', \epsilon') \). For part (2), let \( P \) be a permutation on \( \mathcal{H}_p \otimes \mathcal{H}_\kappa \) such that for a basis \( |\epsilon_j, \cdots, \epsilon_{j_i}, \epsilon_{k}\rangle \in \mathcal{H}_p \otimes \mathcal{H}_\kappa \), \( P|\epsilon_j, \cdots, \epsilon_{j_i}, \epsilon_k\rangle = |\epsilon_j, \cdots, \epsilon_{j_i+1}, \epsilon_{k+1}\rangle \), where \( x \oplus y = \text{mod } [x + y, \kappa] \) and \( \text{mod}[a, b] \) is the remainder of \( a/b \). We should note that if \( (g, \epsilon) \in A_j^{(\pm)}(x) \) and \( P|g, \epsilon\rangle = |g', \epsilon'\rangle \), then...
(g', e') ∈ A(r) j+1 \(x \ (r ∈ \{+, −\}) \). The group generated by \(P\) is an automorphism group of \(κ\)-colored \(T_κ\) with color set \(Σ\), i.e., \(PSP^{−1} = S\), (see [8] for a detail). Then from the symmetry of the Grover coin, we have \(PUP^{−1} = U\). Remark that the initial state \(\mid e, ϕ^{WU}_0 \rangle \) is the eigenvector of \(P\) with its eigenvalue \(e^{−iωκ}\). Let the total state at time \(t\) be \(\mid Ψ_t \rangle \equiv Ut\mid e, ϕ^{WU}_0 \rangle \). Therefore we have \(P\mid Ψ_t \rangle = e^{−iωκ} \mid Ψ_t \rangle \).

When the initial qubit is \(ϕ_U^0\) or \(ϕ^{WU}_0\), we can consider the time evolution under the subspace \(H' ⊂ H_P \otimes HC\) generated by the following new basis: for the initial qubit \(ϕ_U^0\),

\[
|x\rangle_{out} = \frac{1}{\sqrt{κ(κ − 1)x}} \sum_{(g, ε) \mid |g| = x + 1} |g, ε\rangle, \quad (x ≥ 0),
\]

\[
|x\rangle_{in} = \frac{1}{\sqrt{κ(κ − 1)x−1}} \sum_{(g, ε) \mid |g| = x − 1} |g, ε\rangle, \quad (x ≥ 1),
\]

and, for the initial qubit \(ϕ^{WU}_0\),

\[
|x\rangle_{out} = \frac{1}{\sqrt{κ(κ − 1)x}} \sum_{j=0}^{κ−1} \omega_κ^{−j} \sum_{(ε, g) \in A_j^{(+)}(x)} |g, ε\rangle, \quad (x ≥ 0),
\]

\[
|x\rangle_{in} = \frac{1}{\sqrt{κ(κ − 1)x−1}} \sum_{j=0}^{κ−1} \omega_κ^{−j} \sum_{(ε, g) \in A_j^{(−)}(x)} |g, ε\rangle, \quad (x ≥ 1).
\]

Therefore the one-step unitary transition defined by Eq. (1.1) on the space \(H'\) is described as follows. If \(ϕ_0 \in \{ϕ_U^0, ϕ^{WU}_0\}\) be the initial qubit, then

\[
(2.2) \quad U|x\rangle_{in} = −(1 − 2/κ)|x − 1\rangle_{out} + 2\sqrt{κ − 1/κ}|x + 1\rangle_{in} \quad : x ≥ 1,
\]

\[
(2.3) \quad U|x\rangle_{out} = \begin{cases} |1\rangle_{in} & : x = 0, \ ϕ_0 = ϕ_U^0, \\ −|1\rangle_{in} & : x = 0, \ ϕ_0 = ϕ^{WU}_0, \\ (1 − 2/κ)|x + 1\rangle_{in} + 2\sqrt{κ − 1/κ}|x − 1\rangle_{out} & : x ≥ 1, \ ϕ_0 ∈ \{ϕ_U^0, ϕ^{WU}_0\}. \end{cases}
\]

Now we will show that the reduced quantum walk under a subspace \(H'\) with the time evolution given by Eqs. (2.2) and (2.3) is equivalent to a special case of quantum walk with a reflection wall at the origin on \(Z_+\) introduced by Oka et al. [9] in the following. At first we give the definition of the quantum walk with the wall. The space is described as \(\tilde{H}_P \otimes \tilde{HC}\), where \(\tilde{H}_P\) is associated with an orthonormal basis \(\{|x\rangle : x ∈ Z\}\) and \(\tilde{HC}\) is generated by an orthonormal basis \(\{|R\rangle, |L\rangle\}\). The time evolution \(\tilde{U} = \tilde{S} \cdot \tilde{C}\) on \(Z\) with the initial state \(Φ_0 = |0, L\rangle\) is given by

(1) Coin operation: \(\tilde{C}|x, A\rangle = |x\rangle \otimes H(x)|A\rangle \ (A = R, L)\) with

\[
(2.4) \quad H(x) = (1 − δ_0(x))H_κ + e^{iγ}δ_0(x)σ,
\]

where \(γ\) is a real number, \(δ_0(x)\) is the delta measure at the origin,

\[
H_κ = \begin{bmatrix} 2\sqrt{κ − 1/κ} & −(1 − 2/κ) \\ 1 − 2/κ & 2\sqrt{κ − 1/κ} \end{bmatrix},
\]

and \(\tilde{S}\) is a special case of quantum walk with the wall.
and
\[ \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

(2) Shift operation:
\[ \tilde{S}|x, A\rangle = \begin{cases} |x + 1, R\rangle : (A = R), \\ |x - 1, L\rangle : (A = L). \end{cases} \]

Define a Hilbert space \( \tilde{\mathcal{H}}' \) as a subspace of \( \tilde{\mathcal{H}}_P \otimes \tilde{\mathcal{H}}_C \) generated by a basis set
\[ \{|0, L\rangle, |1, R\rangle, |1, L\rangle, |2, R\rangle, |2, L\rangle, \ldots\}. \]

Note that \( \tilde{U}^t|0, L\rangle \in \tilde{\mathcal{H}}' \) for any \( t \geq 0 \). We should remark that the time evolution \( U \) on \( \mathcal{H}' \) given by Eqs.(2.2) and (2.3) is equivalent to the time evolution \( \tilde{U} \) on \( \tilde{\mathcal{H}}' \) with the following one-to-one-correspondence:
\[ |x\rangle_{\text{out}} \leftrightarrow |x, L\rangle, \ |x\rangle_{\text{in}} \leftrightarrow |x, R\rangle. \]

Furthermore the case of \( \gamma = 0 \) (resp. \( \gamma = \pi \)) in Eq.(2.4) corresponds to the initial qubit \( \varphi_{0}^{U} \) (resp. \( \varphi_{0}^{WU} \)). Let \( W_t \) be the quantum walk on \( T_\kappa \) at time \( t \) and \( X_t \) be the quantum walk with the wall at time \( t \). By definition, so we have \( P(|W_t| = x) = P(X_t = x) \).

3 Limit theorems

In this section, we will show that a localization occurs in the case of the initial qubit \( \varphi_{0}^{WU} \). The definition of the localization considered here is that there exists a vertex \( v \in V(T_\kappa) \) such that \( \limsup_{t \to \infty} P(W_t = v) > 0 \). Figure 1 (resp. Fig. 2) depicts the distribution of \( W_t \) on \( T_3 \) at time \( t = 10 \) with the initial qubit \( \varphi_{0}^{U} \) (resp. \( \varphi_{0}^{WU} \)). We can see that if \( |g| = |h| \), then the finding probability at \( g \) is equal to one at \( h \) as we have shown in Lemma 1. Furthermore we can see a high probability at the origin with the initial qubit \( \varphi_{0}^{WU} \). Figure 3 (resp. Fig. 4) shows the distribution of \( X_t \) on \( \mathbb{Z}_+ \) at time 500 with the initial qubit \( \varphi_{0}^{U} \) (resp. \( \varphi_{0}^{WU} \)). The solid lines in Figs. 3 and 4 represent the quantum walk, and dotted lines in Figs. 3 and 4 represent the classical random walk.

From now on, we present the limit theorems corresponding to a localization for \( X_t \) and a weak convergence theorem for the rescaled \( X_t/t \). The first theorem describes the localization for Case (B) suggested by Figs. 2 and 4.

**Theorem 1** Let \( P_{x}^{(E)}(x) = \lim_{t \to \infty} P(X_{2t} = x) \) and \( P_{x}^{(O)}(x) = \lim_{t \to \infty} P(X_{2t+1} = x) \) for \( x \in \mathbb{Z}_+ \).

(1) Case (A) \( (\gamma = 0, \ i.e., \ \varphi_{0}^{U} \ \text{case}) \) : for \( x \in \mathbb{Z}_+ \),
\[ P_{x}^{(E)}(x) = P_{x}^{(O)}(x) = 0 \ (x \geq 0). \]
(2) Case (B) ($\gamma = \pi$, i.e., $\varphi^{WU}_0$ case): for $x \in \mathbb{Z}_+$,

\[
P^*_E(x) = \begin{cases} 
(\frac{\kappa-2}{\kappa-1})^2 \left\{ \delta_0(x) + (1 - \delta_0(x)) \kappa \left(\frac{1}{\kappa-1}\right)^x \right\} & ; \ x \text{ = even}, \\
0 & ; \ x \text{ = odd}.
\end{cases}
\]

\[
P^*_O(x) = \begin{cases} 
\kappa \left(\frac{\kappa-2}{\kappa-1}\right)^2 \left(\frac{1}{\kappa-1}\right)^x & ; \ x \text{ = odd}, \\
0 & ; \ x \text{ = even}.
\end{cases}
\]

The proof can be seen in Appendix A. Remark that for Case (A), $C_\kappa(A) \equiv \sum_{x \in \mathbb{Z}_+} P^*_E(x) = \sum_{x \in \mathbb{Z}_+} P^*_O(x) = 0$, and for Case (B),

\[
C_\kappa(B) \equiv \sum_{x \in \mathbb{Z}_+} P^*_E(x) = \sum_{x \in \mathbb{Z}_+} P^*_O(x) = \frac{\kappa-2}{\kappa-1} < 1.
\]

That is, $\{P^*_E(x) : x \in \mathbb{Z}_+\}$ and $\{P^*_O(x) : x \in \mathbb{Z}_+\}$ are not probability distributions for both cases. The following weak convergence theorem explains the vanishing values $1 - C_\kappa(A) = 1$ and $1 - C_\kappa(B) = 1/(\kappa - 1)(> 0)$.

**Theorem 2** As $t \to \infty$,

\[
X_t/t \Rightarrow Y,
\]

where “$\Rightarrow$” means the weak convergence. The limit measure is defined by

\[
\rho_\kappa(x) = \begin{cases} 
 f_\kappa(x) & ; \text{Case (A)}, \\
C_\kappa(B)\delta_0(x) + (1 - C_\kappa(B))f_\kappa(x) & ; \text{Case (B)},
\end{cases}
\]

(3.5)
where,

\[ f_\kappa(x) = (\kappa - 2) \frac{x^2 I_{[0,a_\kappa]}(x)}{\pi(1 - x^2)\sqrt{a_\kappa^2 - x^2}}, \quad a_\kappa = 2\sqrt{\kappa - 1/\kappa}, \]

and \( I_A(x) \) is the indicator function of a set \( A \).

As for the proof, see Appendix B. Note that the coefficient \( \delta_0(x) \) in Eq. (3.5) for Case (B), i.e., \( C_\kappa(B) \), corresponds to the localization. Furthermore \( f_\kappa(x) \) in Eq. (3.6) is described as the so-called Konno density function \( \mu_K \) [1,2] with a weight function \( \kappa x^2 \), that is, \( f_\kappa(x) = \kappa x^2 \mu_K(x,a_\kappa)I_{[0,\infty)}(x) \), where

\[ \mu_K(x,a) = \frac{\sqrt{1-a^2}}{\pi(1-x^2)\sqrt{a^2-x^2}}I_{(-|a|,|a|)}(x). \]

The Konno density function appears in discrete-time quantum walks on \( \mathbb{Z} \) [1–3] and on \( \mathbb{Z}^2 \) [10] as the limit density function for a suitable scaling, where \( \mathbb{Z} \) is the set of integers.

4 Summary and discussions

We reduced a discrete-time quantum walk \( W_t \) on the Cayley tree to a walk \( X_t \) on \( \mathbb{Z}_+ \). We have obtained two types of limit theorems for \( X_t \). The first one corresponds to a localization of \( X_t \). The second one is a weak convergence theorem for \( X_t \), where the limit density can be described by the Konno density function \( \mu_K \) [1–4]. To clarify a relation between the previous works of \([11–13]\) and our result seems to be challenging.

We can also reduce quantum walks on distance regular graphs such as the Hamming graph, the Johnson graph, etc., to a half line in a similar fashion. So the study on limit theorems for quantum walks on these graphs would be one of the future interesting problems.

Finally we give an interesting relation between our discrete-time quantum walk on \( \mathbb{T}_\kappa \) and the continuous-time quantum walk on \( \mathbb{T}_\kappa \) studied by [5] with respect to the weak convergence. The total Hilbert space of the continuous-time quantum walk on \( \mathbb{T}_\kappa \) is associated with an orthonormal basis \( \{|g\} : g \in V(\mathbb{T}_\kappa)\} \). The state \( \Psi_t^{(c)} \) at time \( t \) with the initial state \(|e\rangle \) is given by \( \Psi_t^{(c)} \equiv U^t|e\rangle \) with \( U_t = e^{itA_\kappa}/\sqrt{\pi} \), where \( A_\kappa \) is the adjacency matrix of \( \mathbb{T}_\kappa \), i.e., \( (A_\kappa)_{g,h} = I_{((g,h);gh-v\Sigma)}(g,h) \). Here \( I_X(x,y) = 1 \), if \((x,y) \in X, = 0, \) if \((x,y) \notin X \). Let \(|x\rangle = |e\rangle \quad (x = 0), = 1/\sqrt{\kappa(\kappa-1)}x^{-1}\sum_{g:|g|=x}|g\rangle \quad (x \geq 1) \). Then we can reduce the continuous-time quantum walk on \( \mathbb{T}_\kappa \) to a walk on the subspace \( \mathcal{H}^{(c)} \) generated by \( \{|x\rangle : x \in \mathbb{Z}_+\} \) as in the discrete-time case. Assume that \( \alpha_t(x) \) denotes the amplitude at time \( t \) at position \( x \) of the reduced walk on \( \mathbb{Z}_+ \). By a quantum probabilistic approach [14,15], the following limit theorem was shown in [5]:

\[ \lim_{\kappa \to \infty} \alpha_t(x) = (x+1)i^2 \frac{J_{x+1}(2t)}{t}, \]

where \( J_x(n) \) denotes the Bessel function of the first kind of order \( n \). Let \( X_t^{(c)} \) be a continuous-time quantum walk starting from the origin defined by

\[ P(X_t^{(c)} = x) = (x+1)^2 \frac{J_{x+1}^2(2t)}{t^2}. \]
Furthermore, the following weak limit theorem was proved in [5]: as $t \to \infty$,

$$X_t^{(c)}/t \Rightarrow Y^{(c)},$$

where $Y^{(c)}$ has the density

$$\rho^{(c)}(x) = x^2 \mu_A(x; 2) I_{[0, \infty)}(x),$$

with $\mu_A(x; a) = I_{[-|a|, |a|]}(x)/(\pi \sqrt{a^2 - x^2})$. As shown in [16], $\mu_A(x; a)$ is the rescaled limit density function for a continuous-time quantum walk on $\mathbb{Z}$. On the other hand, Theorem 2 gives a similar result:

$$f_\kappa(x) = \kappa x^2 \mu_K(x; a_\kappa) I_{[0, \infty)}(x),$$

where the Konno density function $\mu_K$ is the rescaled limit density function for a discrete-time quantum walk on $\mathbb{Z}$.

**Appendix A: Proof of Theorem 1**

Let $\Psi_t(x)$ be the coin state at time $t$ and position $x \in \mathbb{Z}_+$ of the quantum walk with the reflection wall at the origin. Let $\tilde{\Psi}(x; z)$ denote a generating function for $\Psi_t(x)$ such that $\tilde{\Psi}(x; z) = \sum_{t\geq 0} \Psi_t(x) z^t$. From the result of [9], we can obtain an explicit expression for $\tilde{\Psi}(x; z) = T[\tilde{\Psi}^{(L)}(x; z), \tilde{\Psi}^{(R)}(x; z)]$ in the following:

\begin{align}
\tilde{\Psi}^{(L)}(x; z) &= \begin{cases} 
  m_\kappa(\lambda(z))^{x-1}(a_\kappa z - \lambda(z))^{\nu(z)}/z^{2x-1} & ; x \geq 1, \\
  m_\kappa(z - a_\kappa \lambda(z))^{\nu(z)}/z^{2x-1} & ; x = 0,
\end{cases} \\
\tilde{\Psi}^{(R)}(x; z) &= \begin{cases} 
  z(\lambda(z))^{x-1}/z^{2x-1} & ; x \geq 1, \\
  0 & ; x = 0,
\end{cases}
\end{align}

with $a_\kappa = 2\sqrt{\kappa - 1}/\kappa$, $m_\kappa = \kappa/(\kappa - 2)$ (Case A), $= -\kappa/(\kappa - 2)$ (Case B), and

\begin{align*}
\lambda(z) &= z^2 + 1 - \sqrt{z^4 + 2(1 - 2a_\kappa^2)z^2 + 1}/2a_\kappa z, \\
\nu(z) &= 2 - m_\kappa + m_\kappa z^2 - m_\kappa \sqrt{z^4 + 2(1 - 2a_\kappa^2)z^2 + 1}/2(1 - m_\kappa).
\end{align*}

For $r_0 \in (0, 1)$, we get

$$\Psi_t(x) = \frac{1}{2\pi i} \int_{|z|=r_0} \tilde{\Psi}(x; z) \frac{dz}{z^{t+1}}.$$

Remark that $||\tilde{\Psi}(x; z)||^2 < 1$. Then $\int_{|z|=r_1} \tilde{\Psi}(x; z)/z^{t+1}dz \to 0$ with $r_1 > 1$ as $t \to \infty$. So we have

$$\Psi_t(x) \to -\left(\text{Res}(\tilde{\Psi}(x; z), 1) + \text{Res}(\tilde{\Psi}(x; z), -1)(-1)^{t+1}\right) \ (t \to \infty).$$
The above equation gives
\[
\lim_{t \to \infty} \Psi_t^{(L)}(x) = \begin{cases} 
(1 + (-1)^{x+t}) m_\kappa \nu(1) \frac{\kappa - 2}{\sqrt{\kappa - 1}} \left( \frac{1}{\sqrt{\kappa - 1}} \right)^{-1} & ; x \geq 1, \\
(1 + (-1)^{x+t}) m_\kappa \nu(1) (1/2 - 1/\kappa) & ; x = 0,
\end{cases}
\]
and
\[
\lim_{t \to \infty} \Psi_t^{(R)}(x) = \begin{cases} 
(1 + (-1)^{x+t}) \nu(1) \left( \frac{1}{\sqrt{\kappa - 1}} \right)^{-1} & ; x \geq 1, \\
0 & ; x = 0.
\end{cases}
\]
The desired conclusion follows from \( \nu(1) = 0 \) (Case (A)), = \( \kappa - 2)/(\kappa - 1) \) (Case (B)).

Appendix B: Proof of Theorem 2

Let \( \tilde{\Psi}(k; z) = T[\tilde{\Psi}^{(L)}(k; z), \tilde{\Psi}^{(R)}(k; z)] = \sum_x \tilde{\Psi}(x; z)e^{ikx} \). When \(|\lambda(z)| < 1\), Eqs. (4.7) and (4.8) imply
\[
\tilde{\Psi}^{(L)}(k; z) = \frac{\phi_0^{(L)}(k, z)}{(z + 1)(z - 1)} + \frac{\phi_1^{(L)}(k, z)}{z(z + 1)(z - 1)(z - e^{i\theta(k)})(z - e^{-i\theta(k)})},
\]
\[
\tilde{\Psi}^{(R)}(k; z) = \frac{\phi_0^{(R)}(k, z)}{(z + 1)(z - 1)(z - e^{i\theta(k)})(z - e^{-i\theta(k)})} + \frac{\phi_1^{(R)}(k, z)}{z(z + 1)(z - 1)(z - e^{i\theta(k)})(z - e^{-i\theta(k)})},
\]
where \( \phi_0^{(L)}(k, z), \phi_1^{(L)}(k, z), \phi_0^{(R)}(k, z), \phi_1^{(R)}(k, z) \) are some regular functions on \( \mathbb{C} \), and \( \cos \theta(k) = a_\kappa \cos k \). Since \( ||\tilde{\Psi}(k; z)||^2 < \infty \) \(|z| < 1\), we can rewrite \( \tilde{\Psi}(k; z) \) as \( \tilde{\Psi}(k; z) = \sum_{t \geq 0} \tilde{\Psi}_t(k)z^t \) with \( \tilde{\Psi}_t(k) = \sum_x \tilde{\Psi}_t(x)e^{ikx} \). For \( r_0 \in (0, 1) \), we have
\[
\tilde{\Psi}_t(k) = \frac{1}{2\pi i} \int_{|z| = r_0} \tilde{\Psi}(k, z) \frac{dz}{z^{t+1}}.
\]
Then for \(|z| > 1\), \( ||\tilde{\Psi}(k, z)||^2 < \infty \) implies \( \int_{|z| = r_1} \tilde{\Psi}(k, z) \frac{dz}{z^{t+1}} \to 0 \) \((t \to \infty)\) with \( r_1 > 1 \). So
\[
-\tilde{\Psi}_t(k) \to \psi_1(k) + \psi_{-1}(k)(-1)^{t+1} + \psi_1(k)e^{-i\theta(k)}(t \to \infty),
\]
where \( \psi_\pm(k) = \text{Res}(\tilde{\Psi}(k, z); \pm 1) \) and \( \psi_\pm(k) = \text{Res}(\tilde{\Psi}(k, z); e^{\pm i\theta(k)}) \). The definition of \( \tilde{\Psi}_t(k) \) gives
\[
E[e^{i\xi X_t}] = \int_0^{2\pi} \langle \tilde{\Psi}_t(k), \tilde{\Psi}_t(k + \xi) \rangle \frac{dk}{2\pi}.
\]
Hence
\[
\int_0^{2\pi} (||\psi_1(k)||^2 + ||\psi_{-1}(k)||^2) \frac{dk}{2\pi} = \frac{(\kappa - 2)/(\kappa - 1)}{2\pi} = 0.
\]
Note that the right-hand side of Eq. (4.10) is nothing but $C_x(B)$. Combining Eqs. (4.9), (4.10) and (4.11) with the Riemann-Lebesgue lemma, we have

$$\lim_{t \to \infty} E \left[ e^{i\xi X_t/t} \right] = \frac{\kappa - 2}{\kappa - 1} + \int_0^{2\pi} e^{-i\xi h(k)} p(k) \frac{dk}{2\pi} + \int_0^{2\pi} e^{i\xi h(k)} q(k) \frac{dk}{2\pi},$$

where $h(k) = d\theta(k)/dk$, $p(k) = ||\psi_+(k)||^2$, $q(k) = ||\psi_-(k)||^2$. An explicit expression for $h(k)$ is

$$h(k) = a_\kappa \sin k/\sqrt{1 - a_\kappa^2 \cos^2 k}.$$ 

Then $h'(k) \equiv dh(k)/dk = a_\kappa (1 - a_\kappa^2) \cos k/(1 - a_\kappa^4 \cos^2 k)^{1/2}$. Put $h_+(k) = I_{[0,\pi/2)\cup[3\pi/2,2\pi]}(k) h(k)$ and $h_-(k) = I_{(\pi/2,3\pi/2)}(k) h(k)$. If $x = h_+(k)$ with $|x| = a_\kappa$, then the solutions $k_\pm(x)$ are given by

$$\cos(k_\pm(x)) = \pm \frac{1}{a_\kappa} \sqrt{\frac{a_\kappa^2 - x^2}{1 - x^2}}, \quad \sin(k_\pm(x)) = \frac{\sqrt{1 - a_\kappa^2}}{a_\kappa} \frac{x}{\sqrt{1 - x^2}}.$$

Therefore we obtain

$$h'(k_\pm(x)) = \pm \frac{(1 - x^2) \sqrt{a_\kappa^2 - x^2}}{\sqrt{1 - a_\kappa^2}},$$

$$p(k_\pm(x)) = (1 + \text{sgn}(x)) \frac{(k - 2)^2}{4\kappa(k - 1)} \frac{x^2}{a_\kappa^2 - x^2},$$

$$q(k_\pm(x)) = (1 + \text{sgn}(x)) \frac{(k - 2)^2}{4\kappa(k - 1)} \frac{x^2}{a_\kappa^2 - x^2},$$

where $\text{sgn}(x) = 1$ ($x > 0$), $= 0$ ($x = 0$), $= -1$ ($x < 0$). Then by putting $h(k) = x$, the second and third terms of right-hand side of Eq. (4.12) can be expressed as

$$\int_0^{2\pi} \left( e^{-i\xi h(k)} p(k) + e^{i\xi h(k)} q(k) \right) \frac{dk}{2\pi} = \int_0^\infty e^{i\xi x} w(x) \mu_K(x; a_\kappa) dx,$$

where $\mu_K(x; a)$ is the Konno density function and weight function $w(x)$ is given by

$$w(x) = \begin{cases} \frac{\kappa x^2}{\kappa - 1}^2 & \text{Case (A)}, \\ \frac{\kappa x^2}{\kappa - 1}^2 & \text{Case (B)}. \end{cases}$$

Thus we obtain the desired conclusion.

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