Advances on String Theory
in Curved Space Times

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String Quantum Gravity is motivated and introduced.
Advances in the study of the classical and quantum string dynamics in curved
spacetime are reported:
1 - New Classes of Exact Multistring Solutions in curved Spacetimes.
2 - Mass Spectrum of Strings in Curved Spacetimes.
3 - The Effect of a Cosmological Constant and of Spatial Curvature on
   Classical and Quantum Strings.
4 - Classical Splitting of Fundamental Strings.
5 - The General String Evolution in Constant Curvature Spacetimes.
6 - The Conformal Invariance Effects.

In 1987, we started a programme [1] to study the string dynamics in
curved spacetime and its associated physical phenomena. This study
revealed new insights and new physical phenomena with respect to string
propagation in flat spacetime (and with respect to quantum fields in curved
spacetime). The results are relevant both for fundamental (quantum) strings
and for cosmic strings, which behave essentially in a classical way. Approximative
and exact solving methods have been developed. Classical and
quantum string dynamics have been investigated in black hole spacetimes,
cosmological backgrounds, cosmic string spacetime, gravitational wave
backgrounds, supergravity backgrounds (which are necessary for fermionic
strings), and near spacetime singularities. Physical phenomena like classical
string instability and non oscillatory motion in time, quantum particle
transmutation, string scattering, string stretching, have been found. For the
results (1987-1994) see for example our Chalonge Erice Lectures 1989-1994

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and references therein. See (http://www.obspm.fr/chalonge), and “String Theory in Curved Space Times”, Ed. N. Sanchez, WSPC, (1998).

1 Multistring Solutions: A new feature for strings in curved spacetime

The discovery of the multistring property \([2, 3]\) in the propagation of strings in curved spacetimes is the consequence of several developments:

(i) The classical string equations of motion plus the string constraints were shown to be exactly integrable in D-dimensional De Sitter spacetime and equivalent to a sinh-Gordon model with a Hamiltonian unbounded from below \([4]\). Generalization of this result including the Cosh-Gordon and Liouville equations, for strings and multistrings in constant curvature spacetimes have been given in ref \([13]\).

(ii) Exact string solutions were systematically found by soliton methods using the linear system associated to the problem (the so-called dressing method in soliton theory) \([2]\). In particular, exact circular string solutions were found in terms of elementary \([3]\) and elliptic functions \([5]\).

(iii) All these solutions describe one string, several strings or even an infinite number of different and independent strings. A single world-sheet simultaneously describes many different strings. This new feature appears as a consequence of the coupling of the strings with the spacetime geometry. Here, interaction among the strings (like splitting and merging) is neglected, the only interaction is with the curved background. Different types of behaviour appear in the multistring solutions. For some of them the energy and proper size are bounded ("stable strings") while for many others the energy and proper size blow up for large radius of the universe ("unstable strings").

In all these works, strings are test objects propagating on the given fixed backgrounds. The string energy momentum tensor was computed and the string equation of state derived from the string dynamics in cosmological and black hole spacetimes. Strings obey the perfect fluid relation

\[
p = (\gamma - 1)\rho
\]
with three different behaviours:
(i) *Unstable* for large R, with negative pressure;
(ii) *Dual* for small R, with positive pressure, (as radiation);
(iii) *Stable* for large R, with vanishing pressure, (as cold matter).

We find the *back reaction effect* of these strings on the spacetime. This is achieved by considering *selfconsistently* the strings as matter sources for the Einstein equations, as well as for the complete effective string equations, for cosmological spacetimes at the classical level. The selfconsistent solution of the Einstein-Friedman equations for string dominated universes exhibits the realistic matter dominated behaviour for large times and the radiation dominated behaviour for early times. That is, the *standard cosmological evolution* is well generated by strings. It must be noticed that there is no satisfactory derivation of inflation in the context of the effective string equations. *De Sitter universe does not* emerge as solution of the effective string equations. The effective string action (whatever be the dilaton, its potential and the central charge term) is not the appropriate framework in which to address the question of string driven inflation.

More recently, new classes of exact *multistring* solutions were found. The multistring solutions were classified and their physical properties described.

In Anti de Sitter spacetime, the solutions describe an *infinity* number of infinitely long stationary strings of equal energy but different pressures. In De Sitter spacetime, outside the horizon, they describe infinitely many *dynamical* strings, infalling non radially, scattering at the horizon and going back to spatial infinity in different directions. For special values of the constant of motion, there are families of solutions with selected *finite* numbers of different and independent strings. The strings appear *distributed* in *packets*, the number of strings in each packet, the number of “turns” or “festoons” in each string is precisely determined and solely dictated by the dynamics, exactly solved in terms of elliptic functions.

In Black hole spacetimes, (without cosmological constant) no multi-string solutions are found. In the Schwarzschild black hole, inside the horizon, the string infalls, with *indefinitely* growing size and energy, into the r=0 singularity and the string motion stops there.

In the (2+1)- black hole anti-de Sitter background, the string stops at r=0 with *finite* length; the reason being that the point r=0 is not a strong curvature singularity in the (2+1)-black hole anti-de Sitter spacetime. Outside the horizon, in this spacetime, the multistring solution describes
infinitely many, infinitely long open strings.

2 The String Mass Spectrum in the presence of Cosmological Constant

The string mass spectrum in the presence of a cosmological constant (for both de Sitter and Anti de Sitter spacetimes) was found [9], [11]. New features as compared to the string spectrum in flat spacetime appear, as a fine structure effect (splitting of levels) at all the states beyond the graviton, (in both de Sitter (dS) and Anti de Sitter (AdS) spacetimes), and the absence of a critical Hagedorn temperature in AdS spacetime (the partition function for a gas of strings in AdS spacetime is well defined for all temperature).

The presence of a cosmological constant reduces (although do not totally removes) the degeneracy of states as compared with flat Minkowski spacetime. In AdS spacetime, the density of states $\rho(m)$ grows like $\exp[(\Lambda m)^{1/2}]$, (while, as is known, in Minkowski spacetime, $\rho(m)$ grows like the Exponential of the mass $m$). The high mass spectrum changes drastically with respect to flat Minkowski spacetime. The level spacing grows with the eigenvalue of the number operator, $N$, in AdS spacetime, while is approximatively constant (although smaller than in Minkowski spacetime and slightly decreasing) in dS spacetime. There is an infinite number of states with arbitrarily high mass in AdS space time in dS there is a finite number of oscillating states only.

The string mass has been expressed in terms of the Casimir operator $C = L_{\mu\nu} L^{\mu\nu}$ of the O(3,1) De Sitter group [O(2,2) group in Anti deSitter [11]. See also section 3 below.

3 Spatial Curvature Effects

The effects of the spatial curvature on the classical and quantum string dynamics is studied in ref.[10]. The general solution of the circular string motion in static Robertson-Walker spacetimes with closed or open sections has been found [10]. This is given closely and completely in terms of elliptic functions. The back reaction effect of these strings on the spacetime is found : the self-consistent solution to the Einstein equations is a spatially closed ($K>0$) spacetime with a selected value of the curvature index $K$, $K = (G/\alpha'')^{1/2}$ (the scale factor is normalized to unity). No self-consistent solutions with
K<0 exist. We semiclassically quantize the circular strings and find the mass \( m \) in each case. For \( K>0 \), the very massive strings, oscillating on the full hypersphere, have

\[
m^2 \sim KN^2, (N \geq N_0)
\]

independent of \( \alpha' \) and the level spacing grows with \( n \), while the strings oscillating on one hemisphere (without crossing the equator) have

\[
m^2 \sim \alpha' N
\]

and a finite number of states \( N \sim 1/(K\alpha') \). For \( K<0 \), there are infinitely many strings states with masses

\[
m \log m \sim N,
\]

that is the level spacing grows slower than \( N \).

The stationary string solutions as well as the generic string fluctuations around the center of mass are also found and analyzed in closed form.

4 Classical Splitting

We find exact solutions of the string equations of motion and constraints describing the classical splitting of a string into two [12]. For the same Cauchy data, the strings which split have smaller action that the string without splitting. This phenomenon is already present in flat space-time. The splitting process takes place in real (lorentzian signature spacetime).

The solutions in which the string splits are perfectly natural within the classical theory of strings. There is no need of extra interactions, (nor extra terms in the action to produce splitting). The difference with the non-splitting solutions is on the boundary conditions.

The mass, energy and momentum carried out by the strings are computed. We show that the splitting solution describes a natural decay process of one string of mass \( M \) into two strings with a smaller total mass and some kinetic energy. The standard non-splitting solution is contained as a particular case.

We also described the splitting of a closed string in the background of a singular gravitational plane wave, and showed how the presence of the strong gravitational field increases (and amplifies by an overall factor) the negative difference between the action of the splitting and non-splitting solutions.
5 General String Evolution in Constant Curvature Space-Times

In ref. [13], we have found that the fundamental quadratic form of the classical string propagation in (2+1)-dimensional constant curvature spacetimes, solves the sinh-Gordon equation, the cosh-Gordon equation, or the Liouville equation. In both de Sitter and anti-de Sitter spacetimes, (as well as in the 2+1 black hole anti-de Sitter spacetime), all three equations must be included to cover the generic string dynamics. This is particularly enlightening since generic properties of the string evolution can be thus directly extracted from the properties of these three equations and their associated Hamiltonians or potentials, irrespective of any solution.

These results complete and generalize our previous results on this topic (until now, only the sinh-Gordon sector in de Sitter spacetime was known). We also construct new classes of multistring solutions, in terms of elliptic functions, to all three equations in both de Sitter and anti de Sitter spacetimes, which generalize our previous ones.

These results can be straightforwardly generalized to constant curvature spacetimes of arbitrary dimension, by replacing the sinh-Gordon equation, the cosh-Gordon equation, and the Liouville equation by their higher dimensional generalizations.

Our results indicate the existence of various kinds of dualities relating the different sectors and their solutions in de Sitter and anti-de Sitter spacetimes: in the sinh-Gordon sector of de Sitter spacetime, small strings are dual (that is, under $S \rightarrow 1/S$, $S$ being the proper string size, they are mapped) to large strings. And, similarly, in the sinh-Gordon sector of anti-de Sitter spacetime. Furthermore, in the cosh-Gordon sector, small (large) strings in de Sitter spacetime are dual to large (small) strings in the anti-de Sitter spacetime.

6 Conformal Invariance Effects

Classical and quantum strings in the conformally invariant background corresponding to the $SL(2R)$ WZWN model has been studied in ref [14]. This background is locally anti-de Sitter spacetime with non-vanishing torsion. Conformal invariance is expressed as the torsion being parallelizing; and the precise effect of the conformal invariance on the dynamics of both circular and generic classical strings has been extracted [14].

In particular, the conformal invariance gives rise to a repulsive interaction of the string with the background which precisely cancels the dominant attrac-
tive term arising from gravity. We perform both semi-classical and canonical string quantization, in order to see the effect of the conformal invariance of the background on the string mass spectrum. Both approaches yield that the high-mass states are governed by

\[ m \sim HN(N \in N_0, N \text{“large”}), \]

where \( m \) is the string mass and \( H \) is the Hubble constant. It follows that the level spacing grows proportionally to \( N \):

\[ \frac{d(m^2\alpha'')}{dN} \sim N, \]

while the string entropy goes like

\[ S \sim \sqrt{m}. \]

Moreover, it follows that there is no Hagedom temperature, so that the partition function is well defined at any positive temperature. All results are compared with the analogue results in anti-de Sitter spacetime, which is a nonconformal invariant background. It appears that conformal invariance simplifies the mathematics of the problem but the physics remains mainly unchanged. Differences between conformal and non-conformal backgrounds only appear in the intermediate region of the string mass spectrum, but these differences are minor. For low and high masses, the string mass spectra in conformal and non-conformal backgrounds are identical. Interestingly enough, conformal invariance fixes the value of the spacetime curvature to be \(-69/(26\alpha')\).

It has been known for some time that the SL(2,R) WZWN model reduces to Liouville theory. In ref [15] we give a direct and physical derivation of this result based on the classical string equations of motion and the proper string size. This allows us to extract precisely the physical effects of the metric and antisymmetric tensor, respectively, on the exact string dynamics in the SL(2,R) background. The general solution to the proper string size has been also found [15]. We show that the antisymmetric tensor (corresponding to conformal invariance) generally gives rise to repulsion, and it precisely cancels the dominant attractive term arising from the metric. Both the sinh-Gordon and the cosh-Gordon sectors of the string dynamics in non-conformally invariant AdS spacetime reduce here to the Liouville equation (with different signs of the potential), while the original Liouville sector reduces to the free wave
equation.
Only the very large classical string size is affected by the torsion. Medium
and small size string behaviors are unchanged.
We also find illustrative classes of string solutions in the SL(2,R) background:
dynamical closed as well as stationary open spiralling strings, for which
the effect of torsion is somewhat like the effect of rotation in the metric.
Similarly, the string solutions in the 2+1 BH-AdS background with torsion
and angular momentum are fully analyzed [15].

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