SCHRÖDINGER OPERATORS PERIODIC IN OCTANTS

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Abstract. We consider Schrödinger operators with periodic potentials in the positive quadrant for dim > 1 with Dirichlet boundary condition. We show that for any integer $N$ and any interval $I$ there exists a periodic potential such that the Schrödinger operator has $N$ eigenvalues counted with the multiplicity on this interval and there is no other spectrum on the interval. Furthermore, to the right and to the left of it there is an essential spectrum. Moreover, we prove similar results for Schrödinger operators for other domains. The proof is based on the inverse spectral theory for Hill operators on the real line.

1. Introduction and main results

1.1. Introduction. We consider Schrödinger operators $H$ on the domain $D$ given by

$$H = -\Delta^+_x - \Delta_y + V(x, y),$$

$$\quad (x, y) \in D = \mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}, \quad d_1 + d_2 = d \geq 2, \quad d_1, d_2 \geq 0.$$  \hfill (1.1)

Here the operator $\Delta^+_x$ is the Laplacian in the octant $\mathbb{R}^{d_1}_+$ with the Dirichlet boundary conditions on the boundary $\partial \mathbb{R}^{d_1}_+$ and the operator $\Delta_y$ is the Laplacian in the space $\mathbb{R}^{d_2}$. We assume that the potential $V$ belongs to $L^\infty_{\text{real}}(D)$ and is octant periodic, see Condition V.

In order to define octant periodic potentials we need additional definitions. Let $\omega = (\omega_j)_{j=1}^m$ be a sequence of $+$ or $-$ and the set all these sequences we denote by $\Omega_m$. For any $\omega \in \Omega_m$ we define the octants $\mathcal{R}_\omega \subset \mathbb{R}^m$ by

$$\mathcal{R}_\omega = \mathbb{R}_{\omega_1} \times \mathbb{R}_{\omega_2} \times \ldots \times \mathbb{R}_{\omega_m}, \quad \omega = (\omega_j)_{j=1}^m \in \Omega_m.$$  \hfill (1.2)

In particular, if $d = 3$ and $\omega = (+, +, +)$, then we have the positive octant $\mathcal{R}_\omega = \mathbb{R}^3_+$. Note that three axial planes ($x_1 = 0, x_2 = 0, x_3 = 0$) divide space $\mathbb{R}^3$ into eight octants, each with a coordinate signs from $(-, -, -)$ to $(+, +, +)$.

Definition V. A potential $V(z), z = (x, y) \in \mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}$ is called octant periodic if it has the decomposition

$$V(x, y) = \sum_{\omega \in \Omega_m} V_\omega(x, y) \chi_\omega(y),$$  \hfill (1.3)

where $\chi_\omega$ is the characteristic function of the octant $\mathcal{R}_\omega$ and the function $V_\omega(z), z = (x, y)$ is periodic in $\mathbb{R}^d$ and satisfies

$$V(z + p_je_j) = V(z), \quad \forall z \in \mathbb{R}^d, \quad j = 1, \ldots, d,$$  \hfill (1.4)

for some constants $p_j > 0, j = 1, 2, \ldots, d$, where and $e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots)$ is the standard basis in $\mathbb{R}^d$.

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For each \( \omega \) we define Schrödinger operators \( H_\omega \) with periodic potentials \( V_\omega \) on \( \mathbb{R}^d \) by
\[
H_\omega = -\Delta + V_\omega(z).
\]
(1.4)

It is well known that the spectrum of each operator \( H_\omega \) is absolutely continuous and is an union of an unbounded interval and a finite number of non-degenerated bounded intervals.

In the next theorem we show the existence of eigenvalues of \( H \) with some octant periodic potentials.

**Theorem 1.1.**

i) Let \( H = -\Delta_x - \Delta_y + V(x,y) \), where \( (x,y) \in D = \mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}_+ \), \( d_1, d_2 \geq 0 \), \( d_1 + d_2 = d \geq 2 \) and the potential \( V \in L^\infty(D) \) is octant periodic. Then
\[
\bigcup_{\omega \in \Omega_{d_2}} \sigma(H_\omega) \subseteq \sigma_{\text{ess}}(H).
\]
(1.5)

ii) Let \( I = (a,b) \subset \mathbb{R} \) be a finite open interval. Then for any integer \( N \geq 0 \) there exists an octant periodic potential \( V \in L^\infty(D) \) such that \( H = -\Delta_x - \Delta_y + V(z) \) on \( D \) has \( N \) eigenvalues counted with multiplicity on \( I \). Moreover, the interval \( I \) does not contain the essential spectrum, to the right and to the left of it there is a essential spectrum.

**Remark.**

1) The relation (1.5) is standard. Its proof follows from the Floquet theory for periodic operators.

2) In the simple case we have Schrödinger operators \( H = -\Delta_x^+ + V(x) \) on the quadrant \( \mathbb{R}^2_+ \), where the operator \( \Delta_x^+ \) is the Laplacian in the domain \( \mathbb{R}^2_+ \) with the Dirichlet boundary conditions on the boundary \( \partial \mathbb{R}^2_+ \). We assume that the potential \( V \) is octant periodic. This theorem describe the spectrum of \( H \) on the positive quadrant.

3) The proof of ii) is based on the inverse spectral theory from [K99].

Now we discuss an existence of eigenvalue of the operator \( H \) acting on \( \mathbb{R}^d \) below the essential spectrum.

**Theorem 1.2.** There exists a octant periodic real potential \( V \) such that the operator \( H = -\Delta_y + V(y) \) on \( \mathbb{R}^d \) has exactly one simple eigenvalue on some interval \( (-\infty, E) \) and there are no other spectrum on it.

**Remark.** There is a problem to show that there is only one eigenvalue below the continuous spectrum.

1.2. **Historical review.** Davies and Simon [DS78] study the Schrödinger operators with potential, which periodic in half-space. The model one dimensional case with periodic potentials on the half-line is considered in [K00], [K05]. In series of papers [HK11], [HK13], [HKSV15], Hempel and Kohlmann with co-authors consider the different type of dislocation problem in solid state physics. The surface density of states is investigated in [JL01], [KS01]. Scattering on the periodic boundary is considered in [F06], [JL03].

In the simple case \( d = 1 \) we need to describe the spectrum of two operators:

- In Section 2 we consider the Schrödinger operator \( h \) on the space \( L^2(\mathbb{R}_+) \) given by
\[
hy = -y'' + vy,
\]
with the boundary condition \( y(0) = 0 \). Here the potential \( v \) is real 1-periodic and \( v \in L^1(0,1) \). In our proof results from [K99], [K06] about the inverse problem for \( h \) from is crucial and is presented in Theorem 2.1.
• We also consider the so-called half-solid operator $T_\tau, \tau \in \mathbb{R}$ acting on $L^2(\mathbb{R})$ and given by

$$T_\tau = -\frac{d^2}{dx^2} + q_\tau(x), \quad q_\tau(x) = \begin{cases} 
\tau & \text{if } x < 0 \\
v(x) & \text{if } x > 0
\end{cases},$$

(1.6)

where the potential $v \in L^2_{\text{real}}(\mathbb{T})$. By the physical point of view $v$ is the potential of a crystal and the constant $\tau$ is the potential of a vacuum. Roughly speaking the potential $q_\tau$ is the octant periodic and has the form $q_\tau = V_+\chi_+ + V_-\chi_-, \text{ where } V_+ = v$ and $V_- = \tau$ and $\chi_\pm$ are the characteristic functions of the half-line $\mathbb{R}_\pm$. In our proof results from [K05] about spectral properties of $h$ from is crucial and is presented in Theorem 2.1. Moreover, we need additional results formulated in Lemmas 3.1 and 3.2.

2. Periodic Schrödinger operators on the half-line

2.1. Preliminary. We consider the Schrödinger operator $h$ acting on the space $L^2(\mathbb{R}_+)$ and given by

$$hy = -y'' + vy,$$

with the boundary condition $y(0) = 0$. Here the potential $v$ is real 1-periodic and satisfies $v \in L^1(0,1)$. The spectrum of $h$ consists of an absolutely continuous part $\sigma_{ac}(h)$ (the union of the bands $\sigma_n, n \geq 0$ separated by gaps $\gamma_n$) plus at most one eigenvalue in each non-empty gap $\gamma_n, \ n \in \mathbb{N}$, [KS12], [Z69]. Here the bands $\sigma_n$ and gaps $\gamma_n$ are given by (see Fig. 1)

$$\sigma_{ac}(h) = \bigcup_{n \geq 0} \sigma_n, \quad \sigma_n = [\lambda_n^+, \lambda_{n+1}^-], \quad \gamma_n = (\lambda_n^-, \lambda_n^+), \quad n \in \mathbb{N},$$

(2.1)

We also set $\gamma_0 = (-\infty, \lambda_0^+)$ and $\lambda_0^+ = 0$. Here $\lambda_n^\pm$ satisfy

$$\lambda_n^- < \lambda_1^- \leq \lambda_2^- \leq \ldots \leq \lambda_{n-1}^- < \lambda_n^+ \leq \lambda_{n+1}^- \leq \ldots,$$

$$\lambda_n^+ = (\pi n)^2 + O(1) \quad \text{as } n \to \infty.$$  

(2.2)

The bands $\sigma_n, n \geq 0$ satisfy (see e.g. [MS1] or [K97])

$$|\sigma_n| = \lambda_{n+1}^- - \lambda_n^+ \leq \pi^2(2n + 1), \quad \forall \ n \geq 0.$$  

(2.3)

The sequence (2.2) is the spectrum of the equation

$$-y'' + v(x)y = \lambda y, \quad \lambda \in \mathbb{C},$$

(2.4)

with the 2-periodic condition $y(x + 2) = y(x)$ ($x \in \mathbb{R}$). If a gap degenerates, $\gamma_n = \emptyset$ for some $n \geq 1$, then the corresponding bands $\sigma_n$ and $\sigma_{n+1}$ touch. This happens when $\lambda_n^- = \lambda_{n+1}^+$; this number is then a double eigenvalue of the 2-periodic problem (2.4). The lowest eigenvalue $\lambda_0^+ = 0$ is always simple and has a 1-periodic eigenfunction. Generally, the eigenfunctions corresponding to eigenvalues $\lambda_n^\pm$ are 1-periodic, those for $\lambda_n^\pm$, are 1-anti-periodic in the sense that $y(x + 1) = -y(x)$ ($x \in \mathbb{R}$).

Introduce the two canonical fundamental solutions $\vartheta(x, \lambda), \varphi(x, \lambda)$ of the equation (2.4), satisfying the initial conditions $\varphi'(0, \lambda) = \vartheta(0, \lambda) = 1$ and $\varphi(0, \lambda) = \vartheta'(0, \lambda) = 0$. Here and in the following $u'$ denotes the derivative w.r.t. the first variable. The Lyapunov function (Hill discriminant) of the periodic equation (2.4) is then defined by

$$\mathfrak{F}(\lambda) = \frac{1}{2}(\varphi'(1, \lambda) + \vartheta(1, \lambda)), \quad \lambda \in \mathbb{C}.$$  

Note that $\sigma_{ac}(h) = \{ \lambda \in \mathbb{R} : \mathfrak{F}(\lambda) \in [-1, 1] \}$, see [T158].
2.2. Riemann surface. The function $\mathfrak{F}(\lambda)$ is entire and is real on the real line. Introduce the function $\phi$, which is analytic in $\mathbb{C}_+$ and given by

$$\phi(\lambda) = (1 - \mathfrak{F}(\lambda))^2, \quad \lambda \in \mathbb{C}_+, \quad (2.5)$$

where the branch is defined by the condition $\phi(\lambda + i0) > 0$ for $\lambda \in \sigma_0 = [\lambda_0^+, \lambda_1^-]$. We also introduce the two-sheeted Riemann surface $\Lambda$ of $\phi(\lambda)$ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C} \setminus \sigma_{ac}(h)$ in the usual (crosswise) way, see e.g. [KS12].

We denote the $n$-th open gap on the first, physical sheet $\Lambda_1$ by $\gamma_n^{(1)}$ and its counterpart on the second, nonphysical sheet $\Lambda_2$ by $\gamma_n^{(2)}$, and set the "circle" gap $\gamma_n^*$ by

$$\gamma_n^* := \gamma_n^{(1)} \cup \gamma_n^{(2)}. \quad (2.6)$$

2.3. Floquet solutions. The Floquet solutions $\psi^\pm(x, \lambda), \lambda \in \Lambda$, of the equation (2.4) are given by

$$\psi^\pm(x, \lambda) = \vartheta(x, \lambda) + m^\pm(\lambda)\varphi(x, \lambda),$$

where

$$m^\pm = \frac{a \pm i\phi}{\varphi(1, \cdot)}, \quad a(\lambda) = \frac{1}{2}(\varphi'(1, \lambda) - \vartheta(1, \lambda)), \quad (2.7)$$

$$\varphi(1, \lambda)\varphi_+(\cdot, \lambda) \in L^2(\mathbb{R}_+) \quad \forall \lambda \in \Lambda_1 \subset \mathbb{C} \setminus \sigma_{ac}(h).$$

Note that in the trivial case $v = 0$, we have $\psi^\pm(x, \lambda) = e^{\pm ix\sqrt{\lambda}}$.

Introduce a function $b(\lambda) = -i\phi(\lambda)$. It is known that (see e.g. [K05])

$$b(\lambda) = (-1)^n \sqrt{\mathfrak{F}(\lambda)^2 - 1}, \quad \lambda \in \gamma_n, \quad (2.8)$$

where the branch $\sqrt{\mathfrak{F}(\lambda)^2 - 1} > 0$ as $\lambda \in \gamma_n^1 \subset \Lambda^1$ and $\sqrt{\mathfrak{F}(\lambda)^2 - 1} < 0$ as $\lambda \in \gamma_n^2 \subset \Lambda^2$.

Below we need the simple identities

$$a^2 + 1 - \mathfrak{F}(\lambda)^2 = 1 - \varphi'(1, \cdot)\vartheta(1, \cdot) = -\varphi(1, \cdot)\vartheta'(1, \cdot). \quad (2.9)$$

2.4. Eigenvalues and resonances. It is well known (see e.g. [KS12], [Z69]) that, for each $f \in C^\infty_0(\mathbb{R}_+), f \neq 0$, the function $g(\lambda) = ((h - \lambda)^{-1} f, f)$ has a meromorphic extension from the physical sheet $\Lambda_1$ to the whole Riemann surface $\Lambda$. By definition,

If $g$ has a pole at some $\lambda_0 \in \Lambda_1$ for some $f$, then $\lambda_0$ is an eigenvalue of $h$ and $\lambda_0 \in \bigcup_{n \geq 0} \gamma_n^{(1)}$.

If $g$ has a pole at some $\lambda_0 \in \Lambda_2$ for some $f$, then $\lambda_0$ is a resonance of $h$. In particular, if $\lambda_0 \in \gamma_n^{(2)}$ for some $n \geq 0$, then $\lambda_0$ is anti-bound state of $h$ and $\lambda_0 \in \bigcup_{n \geq 1} \gamma_n^{(2)}$.

If $g$ has asymptotics $g(\lambda) = \frac{1}{\sqrt{t}}(1 + O(t))$ as $t = \lambda - \lambda_0 \to 0$ for some $\lambda_0 \in \{\lambda_n^+, \lambda_n^-, \}, n \geq 1$, then $\lambda_0$ is a virtual state of $h$. 

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**Figure 1.** The cut domain $\mathbb{C} \setminus \cup S_n$ and the cuts (bands) $S_n = [E_n^+, E_n^-], n \geq 1$. 

- $\lambda^+_n$ and $\lambda^-_n$ represent the upper and lower cuts on the physical sheet $\Lambda_1$.
- $\lambda^+_n$ and $\lambda^-_n$ represent the upper and lower cuts on the nonphysical sheet $\Lambda_2$. 
- The branches $\lambda^+_n$ and $\lambda^-_n$ are defined by the condition $\phi(\lambda + i0) > 0$.

The function $\mathfrak{F}(\lambda)$ is entire and is real on the real line.
It is well known that for the case \( v \neq \text{const} \), see [KST2], [Z69], the function \( g \) has exactly one simple pole \( \mu_n^* \) on each "circle" gap \( \gamma_n^* \neq \emptyset, n \geq 1 \) and there are no others. Here \( \mu_n^* \in \gamma_n^* \) is a so-called state of \( h \) and its projection onto the complex plane coincides with the \( n \)-th eigenvalue, \( \mu_n \), of the Dirichlet boundary value problem

\[
-y_n'' + vy_n = \mu_n y_n, \quad y_n(0) = y_n(1) = 0, \quad x \in [0, 1], \quad n \geq 1.
\]

Moreover, if \( \gamma_n \neq \emptyset \), then exactly one of the following three cases holds,

1) \( \mu_n^* \in \gamma_n^{(1)} \) is an eigenvalue,
2) \( \mu_n^* \in \gamma_n^{(2)} \) is a resonance (it is a so-called anti-bound state),
3) \( \mu_n^* \in \{ \lambda_n^+, \lambda_n^- \} \) is a virtual state. Here the function \( g(\mu_n^* + z^2) \) has a pole at 0.

There are no other states of \( h \), so \( h \) has only eigenvalues, virtual states and anti-bound states. If there are exactly \( N \geq 1 \) non-degenerate gaps in the spectrum of \( \sigma_{ac}(h) \), then the operator \( h \) has exactly \( N \) states; the closed gaps \( \gamma_n = \emptyset \) and the semi infinite gap \( (-\infty, 0) \) do not contribute any states. In particular, if \( \gamma_n = \emptyset \) for all \( n \geq 1 \), then \( v = \text{const} \) (K98, K99) and thus \( h \) has no states. A more detailed description of the states of \( h \) is given in Theorem 2.1 below.

2.5. Inverse problem. We need the following results from the inverse spectral theory for the operator \( h \) on the half-line, in the form convenient for us. We define the real Hilbert spaces

\[
H_\alpha = \left\{ q \in L^2(0, 1) : \int_0^1 q(x)dx = 0, \quad q(0) \in L^2(0, 1) \right\}, \quad \alpha \geq 0,
\]

and let \( H = H_0 \). Introduce the real Hilbert spaces \( \ell^2_\alpha, \alpha \in \mathbb{R} \) of the sequences \((f_n)_n^\infty\) equipped with the norms

\[
\|f\|_{\ell^2_\alpha}^2 = \sum_{n=1}^\infty (2\pi n)^{2\alpha} f_n^2 < \infty,
\]

and let \( \ell^2 = \ell^2_0 \).

Defining the mapping \( \xi : H \to \ell^2 \oplus \ell^2 \) by

\[
v \mapsto \xi = (\xi_n)_n^\infty, \quad \xi_n = (\xi_{1n}, \xi_{2n}) \in \mathbb{R}^2
\]

where the components \( \xi_{1n}, \xi_{2n} \) are given by

\[
\xi_{1n} = \frac{\lambda_n^- + \lambda_n^+}{2} - \mu_n^2, \quad \xi_{2n} = \frac{|\gamma_n|^2}{4} - \xi_{1n}^2 \, \text{sign}_n, \quad \xi_n^2 = \frac{1}{4}|\gamma_n|^2, \quad (2.10)
\]

where

\[
\text{sign}_n = \begin{cases} +1 & \text{if } \mu_n^* \text{ is an eigenvalue}, \\ -1 & \text{if } \mu_n^* \text{ is a resonance}, \\ 0 & \text{if } \mu_n^* \text{ is a virtual state}, \end{cases} \quad n \geq 1. \quad (2.11)
\]

This mapping \( \xi \) is described by the following result from [K99], [K98], [K06]:

Theorem 2.1. The mapping \( v \to \xi \) acting from \( H \) to \( \ell^2 \oplus \ell^2 \) is a real analytic isomorphism between the real Hilbert spaces \( H \) and \( \ell^2 \oplus \ell^2 \) and satisfies

\[
\|v\| \leq 4\xi \|(1 + \|\xi\|^{\frac{4}{3}}), \quad \|\xi\| \leq \|v\|(1 + \|v\|)^{\frac{4}{3}}, \quad (2.12)
\]

where \( \|v\|^2 = \int_0^1 v^2(x)dx \) and \( \|\xi\|^2 = \frac{1}{4} \sum_{n \geq 1} |\gamma_n|^2 \). Moreover, if \( \alpha \in \mathbb{N} \), then \((|\gamma_n|)_n^\infty \in \ell^2_\alpha \) iff the derivative \( q(\alpha) \) belongs to \( H \).
Remark. 1) Thus we have that for any non-negative sequence \( \kappa = (\kappa_n)_{n=0}^{\infty} \in \ell^2 \), there are unique 2-periodic eigenvalues \( \lambda_n^\pm \) \( (n \in \mathbb{N}) \), for some \( p \in \mathcal{H} \), such that each \( \kappa_n = \lambda_n^+ - \lambda_n^- \), \( (n \in \mathbb{N}) \). Consequently, from the gap lengths \( (\gamma_n)_{n=0}^{\infty} \) one can uniquely recover the Riemann surface \( \Lambda \) as well as the points \( \lambda_n^- = \lambda_n^+ \) where \( \kappa_n = 0 \). Furthermore, for any additional sequence \( \tilde{\lambda}_n^0 \in \gamma_n^* \) \( (n \in \mathbb{N}) \), there is a unique potential \( p \in \mathcal{H} \) such that each state \( \lambda_n^0 \) of the corresponding operator coincides with \( \tilde{\lambda}_n^0 \) \( (n \in \mathbb{N}) \). The results of [K99] were extended in [K03] to periodic distributions \( v \) such that \( \rho \in \mathcal{H} \).

2) We can consider the Schrödinger operator \( \tilde{h} \) acting on the space \( L^2(\mathbb{R}_+) \) and given by
\[
hy = -y'' + vy,
\]
with the Neumann boundary condition \( y(0) = 0 \). The spectrum of \( \tilde{h} \) consists of an absolutely continuous part \( \sigma_{ac}(\tilde{h}) = \sigma_{ac}(h) \) \( (\text{the union of the bands } \sigma_n, n \geq 0 \text{ separated by gaps } \gamma_n, \text{ see (2.1))} \) plus at most one eigenvalue in each non-empty gap \( \gamma_n, n \in \mathbb{N} \). Here also we can consider the resonances for the Schrödinger operator \( \tilde{h} \), similar to the resonances for the Schrödinger operator \( h \). These eigenvalues and the resonances for \( \tilde{h} \) coincide with the eigenvalues \( \nu_n, n \geq 0 \) of the problem
\[
-y'' + vy = \lambda y, \quad y'(0) = y'(1) = 0. \tag{2.13}
\]
The eigenvalues \( \nu_n, n \geq 0 \) satisfy
\[
\nu_0 \leq \lambda^+_n, \quad \nu_n \in [\lambda^-_n, \lambda^+_n] \quad \forall n \geq 1. \tag{2.14}
\]
Here also we can consider the resonances for the Schrödinger operator \( \tilde{h} \), similar to the resonances for the Schrödinger operator \( h \).

3. One dimensional half-solid

In this section we consider the case of one-dimensional octant periodic potentials in the specific form given by (3.1). We consider the half-solid operator \( T_\tau, \tau \in \mathbb{R} \) acting on \( L^2(\mathbb{R}) \) and given by
\[
T_\tau = -\frac{d^2}{dx^2} + q_\tau(x), \quad q_\tau(x) = \begin{cases} 
\tau & \text{if } x < 0 \\
v(x) & \text{if } x > 0
\end{cases}, \tag{3.1}
\]
where the potential \( v \in L^2_{\text{real}}(\mathbb{T}) \). By the physical point of view \( v \) is the potential of a crystal and the constant \( \tau \) is the potential of a vacuum. In order to describe the spectrum of \( T_\tau \) we use some properties of the operator \( h = -\frac{d^2}{dx^2} + v \) on the half-line from Section 2.

We recall needed results about operators \( T_\tau \) from [K03]. We have the following simple results about the spectrum of \( \sigma(T_\tau) \) given by
\[
\sigma(T_\tau) = \sigma_{ac}(T_\tau) \cup \sigma_{disc}(T_\tau), \quad \sigma_{ac}(T_\tau) = \sigma_{ac}(h) \cup [\tau, \infty). \tag{3.2}
\]
Our goal is to study the eigenvalues in the gaps \( \gamma_n(T_\tau), n \geq 0 \), and to find how these eigenvalues depend on \( \tau \). We take any integer \( N \geq 1 \). We describe the basic properties of the one dimensional half-solid operator \( T_\tau \):

- If \( \tau \leq \lambda^-_1 \), then there is no any gap in the spectrum of \( T_\tau \) and we get
  \[
  \sigma_{ac}(T_\tau) = (\tau_0, \infty), \quad \text{where } \tau_0 = \min\{\lambda^+_0, \tau\}. \tag{3.3}
  \]
- If \( \tau \in \gamma_N \cup \sigma_N = (\lambda_N, \lambda^-_{N+1}] \), for some \( N \geq 1 \), then the spectrum of \( T_\tau \) has the form:
  \[
  \sigma(T_\tau) = \sigma_0 \cup \sigma_1 \cup \ldots \cup \sigma_{N-1} \cup \tilde{\sigma}_N, \quad \tilde{\sigma}_N = [\tau_N, \infty), \quad \tau_N = \min\{\lambda^+_N, \tau\}. \tag{3.4}
  \]
Thus, there are possible gaps in the spectrum $\sigma_{ac}(T_\tau)$ given by
\[
\gamma_j(T_\tau) = \gamma_j(h), \quad j = 0, 1, ..., N - 1, \quad \gamma_N(T_\tau) = (\lambda_N, \tau_N).
\] (3.5)
Moreover, in each open gap $\gamma_j(T_\tau) \neq \emptyset$, $j = 0, 1, ..., N$ there is at most one eigenvalue $\mu_j(\tau)$.

- We introduce the Weyl-type functions $\Psi_\pm$, which are solutions of the equation
\[
-y'' + q_\tau y = \lambda y
\] (3.6)
and satisfy
\[
\Psi_\pm(\cdot, \lambda) \in L^2(\mathbb{R}_\pm), \quad \forall \lambda \in \Lambda_\tau = \mathbb{C} \setminus \sigma_{ac}(T_\tau).
\]
They have the forms
\[
\Psi_+(x, \lambda) = \psi^+(x, \lambda), \quad x \geq 0;
\]
\[
\Psi_-(x, \lambda) = e^{\lambda \sqrt{\tau - x}}, \quad x \leq 0,
\]
for $\lambda < \tau, \lambda \in \Lambda_\tau$ and here $\psi^+$ given by (2.7). These functions $\Psi_\pm(x, \lambda)$ are analytic in the cut domain $\Lambda_\tau$ and are continuous up to the boundary.

- We define the Wronskians
\[
w(\lambda) = \{\Psi_-, \Psi_+\} = m_+(\lambda) - \sqrt{\tau - \lambda}, \quad \lambda \in \Lambda_\tau.
\] (3.7)
The function $w(\lambda)$ on the first sheet $\Lambda_\tau$ has finite number of zeros, which are simple and coincide with eigenvalues of the operator $T_\tau$.

- Using (2.7) we rewrite the Wronskian $w(\lambda)$ in the gap $\gamma_n \subset \Lambda_\tau, \lambda < \tau$ in the form
\[
w(\lambda) = m^+(\lambda) - \sqrt{\tau - \lambda} = \frac{a(\lambda) - b(\lambda)}{\varphi(1, \lambda)} - \sqrt{\tau - \lambda}, \quad \lambda < \tau, \quad \lambda \in \gamma_n \subset \Lambda_\tau,
\] (3.8)
where
\[
\sqrt{\tau - \lambda} > 0, \quad \text{if} \quad \lambda < \tau, \lambda \in \Lambda_\tau, \quad (-1)^n b(\lambda) = \sqrt{\Delta^2(\lambda) - 1} > 0, \quad \lambda \in \gamma_n \subset \Lambda_\tau, \quad (3.9)
\]

**Lemma 3.1.** Assume that the operator $hy = -y'' + vy, y(0) = 0$ on $L^2(\mathbb{R}_+)$ has an open gap $I = (\lambda^-, \lambda^+)$ in the continuous spectrum and an eigenvalue $\mu \in (\lambda^-, \lambda^+)$ for some $v \in L^2(\mathbb{T})$. Then for any constant $\tau$ large enough the operator $T_\tau$ defined by (3.1) has an eigenvalue $\mu_\tau \in I$ such that
\[
\mu_\tau - \mu = \frac{c(\mu)}{\sqrt{\tau}} + O(1) \quad \text{as} \quad \tau \to \infty,
\] (3.10)
where $c(\mu) = \frac{2b(\mu)}{\varphi(1, \mu)} \neq 0$ and here $\varphi(1, \mu) = \frac{\partial}{\partial \lambda} \varphi(1, \mu)$.

**Proof.** i) Due to (3.8) the eigenvalues of $T_\tau$ are zeros of the Wronskian
\[
w(\lambda) = m_+(\lambda) - \sqrt{\tau - \lambda} = \frac{a(\lambda) - b(\lambda)}{\varphi(1, \lambda)} - \sqrt{\tau - \lambda}
\]
on the first sheet $\Lambda_\tau$. Consider the two functions $m_+(\lambda) = \frac{a(\lambda) - b(\lambda)}{\varphi(1, \lambda)}$ and $\sqrt{\tau - \lambda}$ on the gap $(\lambda^-, \lambda^+)$, where $\tau >> \lambda_\pm$. The point $\mu \in I$ is an eigenvalue of the operator $h$. Then due to (2.9) we have $a^2(\mu) = b^2(\mu) \neq 0$. Then the function $m_+(\lambda)$ is a meromorphic in the disk around the centrum of the gap $\gamma_n$ and has the following asymptotics
\[
m_+(\lambda) = \frac{c(\mu)}{\lambda - \mu} + O(1) \quad \text{as} \quad \lambda \to \mu.
\] (3.11)
Thus the equation \( m_+(\lambda) = \sqrt{\tau - \lambda} \) has a unique solution \( \mu_+ \to \mu \) as \( \tau \to \infty \) given by (3.10), since
\[
\frac{c(\mu)}{\mu_+ - \mu} + O(1) = \sqrt{\tau} + O(\tau^{-\frac{1}{2}}).
\]

We now prove the main result of this section.

**Lemma 3.2.** i) Let integer \( N \geq 1, \alpha \geq 0 \) and let \( \gamma > 0 \). Then there exists a potential \( v \in \mathcal{A}_\alpha \) such that the first \( N \) gaps in the spectrum of the operator \( h \) on \( L^2(\mathbb{R}_+) \) are open and satisfy
\[
|\gamma_j| = \gamma, \quad \forall \ j = 1, 2, ..., N.
\]

Moreover, in addition for any points \( \lambda_j \in \gamma_j, j = 1, 2, ..., N \), then exists a periodic potential \( v \in \mathcal{A}_\alpha \) such that each \( \lambda_j = \mu_j, j = 1, 2, ..., N \) is an eigenvalue of the operator \( h \).

ii) Let in addition \( q \) be given by (3.1) and let \( \tau \) be large enough. Then each \( \gamma_j, j = 1, 2, ..., N \) is a gap in the spectrum of \( T_\tau \) and on each \( \gamma_j \) exists an eigenvalue \( \mu_j(\tau) \in \gamma_j \) such that
\[
\mu_j(\tau) - \mu_j = \frac{c(\mu_j)}{\tau^2} + O(1) \quad \text{as} \quad \tau \to \infty,
\]
where \( c(\mu) = \frac{2b(\mu)}{a(1, \mu)} \neq 0 \).

iii) Let \( \lambda_0^+ = 0 \) and let \( \nu_0 \leq \lambda_0^+ = 0 \) be the first Neumann eigenvalue (see (2.14)) of the problem (2.13). Then for any \( \alpha \geq 0 \) there exists a potential \( v \in \mathcal{A}_\alpha \) such that \( m_+(0) > 0 \) and the operator \( T_\tau \) has an eigenvalue \( E < 0 \) for each \( \tau \in (\nu_0, m_+(0)^2) \neq \emptyset \).

**Proof.** i) It follows from Theorem 2.1

ii) It follows from Lemma 3.1

iii) We define \( \rho := m_+(\lambda_0^+) = \frac{a(\lambda_0^+)}{\varphi(1, \lambda_0^+)} \) for some potential \( v \in \mathcal{A}_\alpha \). We recall a needed result from [K05] about the first eigenvalue:

if \( \rho < 0 \), then \( \# (T_\tau, \gamma_0(T_\tau)) = 0 \),

if \( \rho > 0 \), then
\[
\# (T_\tau, \gamma_0(T_\tau)) = \begin{cases} 0, & \text{if} \ \tau \leq \nu_0 \ or \ \tau \geq \rho^2, \\ 1, & \text{if} \ \nu_0 < \tau < \rho^2. \end{cases}
\]

Assume that \( \rho > 0 \) for some potential \( v \in \mathcal{A}_\alpha \). Then due to (3.14) the operator \( T_\tau \) has an eigenvalue \( E < \lambda_0^+ = 0 \) for each \( \tau \in (\nu_0, \rho^2) \).

We show that \( \rho > 0 \) for some potential \( v \in \mathcal{A}_\alpha \). Below we take \( v = p(x + t) \) for some \( p \in L^2(\mathbb{T}) \) and small \( t \). We assume that \( p \) satisfies

**Condition P.** 1) the function \( p, p'' \in L^2(\mathbb{T}); \)

2) \( p(1 - x) = p(x), \ \forall \ x \in [0, 1], \ i.e., \ the \ potential \ p \ is \ even \ on \ the \ interval \ [0, 1]; \)

3) \( p(x) > \delta > 0 \) for all \( x \in [0, \varepsilon] \) for some small constants \( \delta, \varepsilon > 0 \).

Recall that we put \( \lambda_0^+ = 0 \) and in this case we have \( \int_0^1 p(x) dx > 0 \) (see e.g., [K97]), and then item 3) in Condition P is possible since \( \int_0^1 p(x) dx > 0 \). We define the fundamental solutions \( \varphi(x, \lambda, t), \vartheta(x, \lambda, t) \) of the following equation with the shifted potential
\[
-y'' + p(x + t)y = \lambda y, \quad x \in \mathbb{C}, \quad t \in \mathbb{R};
\]
\[
\varphi_x(0, \lambda, t) = \vartheta(0, \lambda, t) = 1, \quad \varphi(0, \lambda, t) = \vartheta(0, \lambda, t) = 0.
\]

For the shifted potential \( v = p(\cdot + t) \) we define the Lyapunov functions \( \mathcal{F}(\lambda, t) = \frac{1}{2}(\varphi'(1, \lambda, t) + \vartheta(1, \lambda, t)) \). Note that we have \( \mathcal{F}(\lambda, t) = \mathcal{F}(\lambda, 0) \), i.e., the Lyapunov function \( \mathcal{F}(\lambda, t) \) for (3.15)
coincides with the Lyapunov function for the case \( t = 0 \) (see \([L87]\)). We also define the functions
\[
a(\lambda, t) = \frac{1}{2}(\varphi'(1, \lambda, t) - \vartheta(1, \lambda, t)), \quad m_+(\lambda, t) = \frac{a(\lambda, t) - \phi(\lambda)}{\varphi(1, \lambda, t)}.
\]

Let \( \dot{u} = \frac{\partial}{\partial t}u \). We have the equations
\[
\dot{a}(\lambda, t) = -\vartheta_x(1, \lambda, t) - (\lambda - p(t))\varphi(1, \lambda, t), \quad \forall (\lambda, t) \in \mathbb{C} \times \mathbb{R},
\]
see \([L87]\). Then the properties of \( p \) give
\[
\dot{a}(0, t) = -\vartheta'(1, 0, t) + p(t)\varphi(1, 0, t) > \delta\varphi(1, 0, t) > 0 \quad \forall t \in [0, \varepsilon],
\]
since \( \vartheta'(1, 0, t) \leq 0 \) (its first zero \( \nu_0 \leq 0 \)) and \( \varphi(1, 0, t) > 0 \) (its first zero \( \mu_1 > 0 \)) which yields
\[
a(0, t) = a(0, 0) + \int_0^t \dot{a}(0, \tau)d\tau = \int_0^t \dot{a}(0, \tau)d\tau > 0,
\]
since \( a(0, 0) = 0 \) for all even potentials (see Lemma 3.4. in \([K05]\)). This implies \( m_+(0, t) = \frac{a(0, t)}{\varphi(1, 0, t)} > 0 \) for the potential \( v(x) = p(x + t) \) all \( t \in [0, \varepsilon] \).

4. Model operators on \( \mathbb{R}^d_+ \) and \( \mathbb{R}^d \)

4.1. Specific periodic Schrödinger operators on the half-line. Consider the Schrödinger operator given by
\[
hf = -f'' + vf \quad \text{on} \quad L^2(\mathbb{R}_+), \quad f(0) = 0.
\]

Recall that the spectrum of \( h \) consists of an absolutely continuous part (which is a union of non-degenerate spectral bands \( \sigma_n = [\lambda_n^-, \lambda_n^+] \), \( n \geq 0 \)) plus at most one eigenvalue in each open gap \( \gamma_n = (\lambda_n^-, \lambda_n^+) \), \( n \geq 1 \) between bands \([KST12]\, [Z69]\) (see Fig.1) and the \( \lambda_n^\pm \) satisfy \((2.2)\).

Now we begin to construct a specific potential \( v \). Here we use results about the gap-lengths mapping from Lemma 3.2 i). Due to these results about the gap-lengths mapping, we take the potential \( v \in \mathcal{H}_\alpha \) for any fixed \( \alpha \geq 0 \) such that the first \( N \) gaps \( \gamma_1, ..., \gamma_N \) and other ones \( \gamma_{n+N}, n > N \) in the spectrum of \( h \) satisfy
\[
\gamma = |\gamma_1| = |\gamma_2| = |\gamma_3| = ....|\gamma_N| = \frac{\pi^24(N+1)^2}{\kappa}, \quad 0 < \kappa << 1,
\]
\[
\sum_{n>N} n^{2\alpha}|\gamma_n|^2 = Q < \infty.
\]

The value \( Q \) is not important in our consideration and thus we can take any \( \alpha \geq 0 \). Thus \((4.1)\) and the estimate \((2.3)\) give
\[
\lambda_n^- = \gamma(n-1) + A_n, \quad \lambda_n^+ = \gamma n + A_n, \quad A_n = \sum_{j=0}^n |\sigma_j|,
\]
\[
|A_n| \leq \pi^2\sum_{j=0}^n (2j+1) = \pi^2(n+1)^2 \leq \pi^2(N+1)^2 \leq \frac{\kappa}{4}\gamma.
\]

Due to Theorem 2.1 in each open gap \( \gamma_n \), \( n = 1, 2, ..., N \) we choose exactly one eigenvalue \( \mu_n^* \) by
\[
\mu_n^* = \lambda_n^- + \frac{\gamma}{4d} \in \gamma_n^1.
\]
Moreover, (4.2) gives
\[ \mu_n^* = \lambda_n + \frac{n}{4d} = \gamma(n - 1 + \frac{n}{4d}) + A_n, \quad \forall \ n = 1, \ldots, N. \] (4.4)

It is convenient to define the ”normalized” operator
\[ h_\gamma = \frac{1}{\gamma} h. \]

Then the spectrum of \( h_\gamma \) consists of an absolutely continuous part \( \sigma_{ac}(h_\gamma) = \bigcup_{n \geq 0} s_n \) plus at most one eigenvalue in each non-empty gap \( g_n, n \in \mathbb{N} \), but exactly one eigenvalue \( e_n = \frac{\mu_n}{\gamma} \) in each open gap \( g_n, n = 1, \ldots, N \). Here the spectrum of \( h_\gamma \) has the bands \( s_n = \frac{\alpha_n}{\gamma} \) and gaps \( g_n = \frac{\gamma_n}{\gamma} \). In particular, we have
\[ s_0 = \frac{\alpha_0}{\gamma}, \quad s_n = \frac{\alpha_n}{\gamma} = [n + \frac{A_n}{\gamma}, n + \frac{A_{n+1}}{\gamma}], \quad g_n = \frac{\gamma_n}{\gamma}, \quad n = 1, 2, \ldots, N, \]
where \( A_n \) is defined in (4.2). The first bands satisfy
\[ |s_n| = \frac{|\alpha_n|}{\gamma} \leq \frac{\gamma^2}{\gamma} (2n + 1) \leq \frac{\pi}{2(N+1)}, \quad \forall \ n = 0, \ldots, N. \] (4.5)

Thus these spectral bands \( s_n \) are very small and very close to the points \( n \) and satisfy
\[ \text{dist}\{s_n, n\} \leq \frac{A_{n+1}}{\gamma} \leq \frac{\pi}{4}, \quad n = 0, 1, \ldots, N. \] (4.6)

In this case due to (4.1) the gaps \( g_n \) satisfy
\[ |g_n| = 1, \quad n = 1, \ldots, N. \] (4.7)

In each gap \( g_n, n = 1, \ldots, N \), there exists exactly one eigenvalue \( e_n \) of \( h_\gamma \) such that
\[ e_n = \frac{\mu_n}{\gamma} = e_n^0 + \frac{A_n}{\gamma}, \quad e_n^0 = n - 1 + \frac{1}{4d} = n - 1 + e_1, \quad e_1 = \frac{1}{4d}, \quad \frac{A_n}{\gamma} \leq \frac{\pi}{4}. \] (4.8)

4.2. Schrödinger operators on \( \mathbb{R}^2_+ \). We consider Schrödinger operators \( H_+ = h_1 + h_2 \) on the quadrant \( \mathbb{R}^2_+ \). Here \( h_1 \) and \( h_2 \) are defined on the half-line and depend on one variable and given by
\[ h_j y = -y'' + v(x_j) y, \quad y(0) = 0, \quad j = 1, 2, \]
where the potential \( v \in L^2(\mathbb{T}) \). For large constant \( \gamma \) we define a new operator
\[ H_\gamma = \frac{1}{\gamma} H_+ = \frac{1}{\gamma} (h_1 + h_2). \]

We take the operator \( H_\gamma \), when the variables are separated. We show that \( H_\gamma \) has (first) bands which are very small and their positions are very close to the integer \( n \). The union of groups of bands close to the integer \( n \) is a cluster \( K_n \). Between the two neighbor clusters \( K_n \) and \( K_{n+1} \) there exists a big gap. On this gap there exist \( n \) eigenvalues.

- We define bands (i.e., the basic bands) \( S^0_{i,j} \) of the operator \( H_\gamma \) and their clusters \( K^0_n \) by
\[ S^0_{i,j} = s_i + s_j, \quad K^0_n = \bigcup_{i+j=n} S^0_{i,j}, \quad i, j, n \in \mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}, \] (4.9)
where we define \( A + B = \{z = x + y : x \in A, b \in B\} \) for sets \( A, B \). In particular, we have
\[ K^0_0 = S^0_{0,0}, \quad K^0_1 = S^0_{1,0}, \quad K^0_2 = S^0_{2,0} \cup S^0_{1,1}, \ldots, \] (4.10)
If $\gamma$ is large enough, then due to (4.13), (4.15) we estimate the position of bands $S^0_{i,j}$ and their lengths $|S^0_{i,j}|$ by
\[
\text{dist}\{S^0_{i,j}, i + j\} \leq \frac{\kappa}{2}, \quad |S^0_{i,j}| \leq \frac{\kappa}{(N + 1)}
\] (4.11)
for all $i, j = 0, 1, \ldots, N$. This yields the position of their clusters $K^0_n$ and their diameters by
\[
\text{dist}\{K^0_n, n\} \leq \frac{\kappa}{2}, \quad \text{diam} K^0_n \leq \frac{3}{2} \kappa
\] (4.12)
for all $n = 0, 1, \ldots, N$.

- We define the surface bands $S^1_{i,j}$ of the operator $H_\gamma$ and their clusters $K^1_n$, by
\[
S^1_{i,j} = e_i + s_j, \quad K^1_n = \bigcup_{i+j=n+1} S^1_{i,j}, \quad i, j, n \geq 0.
\] (4.13)
In particular, we have
\[
K^1_0 = S_{0,1}, \quad K^1_1 = S^1_{0,2} \cup S^1_{1,1}, \quad K^1_2 = S^1_{0,3} \cup S^1_{1,2} \cup S^1_{2,1}, \quad \ldots.
\] (4.14)
Using arguments similar to the case of the bands $S^0_{i,j}$ we determine the position of surface bands $S^1_{i,j}$ and their lengths $|S^1_{i,j}|$ by
\[
\text{dist}\{S^1_{i,j}, e_i^0 + j\} \leq \frac{\kappa}{2}, \quad |S^1_{i,j}| \leq \frac{\kappa}{2(N + 1)}
\] (4.15)
for all $i, j = 0, 1, \ldots, N$. This yields the position of their clusters $K^1_n$ and their diameters by
\[
\text{dist}\{K^1_n, e^{01}_n + n\} \leq \frac{\kappa}{2}, \quad \text{diam} K^1_n \leq \kappa
\] (4.16)
for all $n = 0, 1, \ldots, N$.
- The operator $H_\gamma$ has eigenvalues $E_{i,j}$ and their cluster $K^e_n$ given by
\[
E_{i,j} = e_i + e_j, \quad i, j \geq 1, \quad K^e_n = \{E = E_{i,j}, i + j = n + 1\}, \quad n \geq 1,
\] (4.17)
The cluster $K^e_n$ has $n$ eigenvalues of the operator $H_+$. In particular, we have
\[
K^1_1 = \{E_{1,1}\}, \quad K^2_2 = \{E_{1,2}, E_{2,1}\}, \quad K^e_3 = \{E_{1,3}, E_{2,2}, E_{3,1}\}, \quad \ldots.
\] (4.18)
The identity (4.8) gives
\[
E_{i,j} = e_i^0 + e_j^0 + \frac{A_i}{\gamma} + \frac{A_j}{\gamma}, \quad |E_{i,j} - e_i^0 - e_j^0| \leq \frac{\kappa}{2}
\] (4.19)
for all $i, j = 1, 2, \ldots, N$. This yields the position of their clusters $K^e_n$ and their diameters by
\[
\text{dist}\{K^e_n, 2e_1 + n - 1\} \leq \frac{\kappa}{2}, \quad \text{diam} K^e_n \leq \kappa
\] (4.20)
for all $n = 1, 2, \ldots, N$.
- Thus we can describe $\sigma_{ac}(H_+)$ and $\sigma_{disc}(H_+)$ by
\[
\sigma_{ac}(H_+) = \cup_{n \geq 0} (K^0_n \cup K^1_n), \quad \sigma_{disc}(H_+) = \cup_{n \geq 1} K^e_n.
\] (4.21)
Now combining all estimates (4.12)-(4.16) we deduce that between two sets $K^0_n \cup K^1_n$ and $K^0_{n+1} \cup K^1_{n+1}$ for each $n = 0, 1, \ldots, N$ there exists an interval $I_n$ given by
\[
I_n = [I^-_n, I^+_n] = [e^0_n + n + 4\kappa, n + 1 - 4\kappa],
\] (4.22)
such that $\text{dist}\{I_n, \sigma_{ac}(H_+)\} \geq 2\kappa$. 

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i.e., the distance between the interval $I_n$ and two sets $K^0_n \cup K^1_n$ and $K^0_{n+1} \cup K^1_{n+1}$ is greater than $2\pi$. Moreover, due to (4.18)-(4.20) the eigenvalue cluster $K^0_n$ satisfies:

$$K^0_n \subset I_n, \quad \forall n = 1, \ldots, N.$$  

(4.23)

4.3. Schrödinger operators on $\mathbb{R}^3_+$. We consider Schrödinger operators $H_+$ on the corner $\mathbb{R}^3_+$ given by

$$H_+ = -\Delta_+ + V_+, \quad V_+(x) = v(x_1) + v(x_2) + v(x_3), \quad x = (x_j)^3 \in \mathbb{R}^3_+$$

(4.24)

where the potential $v$ is 1-periodic and $v \in L^2(0, 1)$. We rewrite the operator $H$ in the form

$$H_+ = h_1 + h_2 + h_3, \quad h_j = h_0 + v(x_j)$$

(4.25)

Define the operator $H_\gamma$ by

$$H_\gamma = \frac{1}{\gamma}H = h_{1,\gamma} + h_{2,\gamma} + h_{3,\gamma}, \quad h_{j,\gamma} = \frac{h_j}{\gamma}$$

- We define bands $S^0_{i,j,k}$ of the operator $H_\gamma$ and their clusters $K^0_n, n = 0, 1, \ldots, N$ by

$$S^0_{i,j,k} = s_i + s_j + s_k, \quad i, j, k \in \mathbb{Z}_+, \quad K^0_n = \bigcup_{i+j+k=n} S^0_{i,j,k}, \quad n \in \mathbb{Z}_+,$$

(4.26)

and in particular,

$$K^0_0 = S^0_{0,0,0} = s_0 + s_0 + s_0, \quad K^0_1 = S^0_{0,0,1}, \quad K^0_2 = S^0_{0,0,2} \cup S^0_{0,1,1}, \ldots.$$  

Recall that we define $A + B$ for sets $A, B$ by $A + B = \{z = x + y : (x, y) \in A \times B\}$. Similar to 2dim case we deduce that

$$S^0_{i,j,k} \sim i + j + k, \quad K^0_n \sim n, \quad \forall \quad n = 1, 2, \ldots, N.$$  

(4.27)

In 3-dimensional case we have two types of the surface (guided) bands $S^1_{i,j,k}$ and $S^2_{i,j,k}$.

- The first type of surface (guided) bands. We define the surface (guided) bands $S^1_{i,j,k}$ of the operator $H_\gamma$ and their clusters $K^1_n, n = 1, \ldots, p$ by

$$S^1_{i,j,k} = s_i + s_j + e_k, \quad K^1_n = \bigcup_{i+j+k=n+1} S^1_{i,j,k}, \quad i, j, n \in \mathbb{Z}_+, \quad k \in \mathbb{N},$$

(4.28)

The position of surface bands $S^1_{i,j,k}$ and their clusters $K^1_n$ are given by

$$S^1_{i,j,k} \sim i + j + k - 1 + e_1 = n - 1 + e_1, \quad K^1_n \sim n - 1 + e_1.$$  

(4.29)

These clusters are separated by gaps $G_n, n = 1, 2, \ldots, p$. Thus we have

$$K^1_1 = S^1_{0,1}, \quad K^1_2 = S^1_{0,2}, \quad K^1_3 = S^1_{0,3} \cup S^1_{1,2}, \ldots, K^1_N = S^1_{0,N} \cup S^1_{1,N-1} \cup \ldots.$$  

(4.30)

- The second type of surface (guided) bands. We define the surface (guided) bands $S^2_{i,j,k}$ of the operator $H_\gamma$ and their clusters $K^2_n, n = 1, \ldots, p$ by

$$S^2_{i,j,k} = e_i + e_j + s_k, \quad K^2_n = \bigcup_{i+j+k=n+2} S^2_{i,j,k}, \quad i, j, k \in \mathbb{N}, \quad k, n \in \mathbb{Z}_+.$$  

(4.31)

The positions of the surface bands $S^2_{i,j,k}$ and the cluster $K^2_n$ are given by

$$S^2_{i,j} \sim i + j + k - 2 + 2e_1 = n - 2 + 2e_1, \quad K^2_n \sim n - 1 + 2e_1.$$  

(4.32)

These clusters are separated by gaps $G_n, n = 1, 2, \ldots, p$. Thus we have

$$K^1_1 = S^2_{0,1}, \quad K^2_2 = S^2_{0,2}, \quad K^1_3 = S^2_{0,3} \cup S^2_{1,2}, \ldots, K^2_N = S^2_{0,N} \cup S^2_{1,N-1} \cup \ldots.$$  

(4.33)
• Eigenvalues. The operator $H_\gamma$ has eigenvalues $E_{i,j,k}$ and their cluster $K^n_\gamma$ given by

$$E_{i,j,k} = e_i + e_j + e_k, \quad i,j,k \in \mathbb{N}, \quad K^n_\gamma = \{E = E_{i,j,k}, i+j+k = n+3\}, \quad n \in \mathbb{Z}_+. \quad (4.34)$$

The positions of eigenvalues $E_{i,j,k}, i+j+k = n+3$ and their cluster $K^n_\gamma$ are given by

$$E_{i,j,k} \sim i + j + k - 3 + 3e_1 = n + 3e_1, \quad K^n_\gamma \sim n + 3e_1. \quad (4.35)$$

The sets $\sigma_{ac}(H)$ and $\sigma_{disc}(H)$ are given by

$$\sigma_{ac}(H) = \bigcap_{n \geq 0} (K^n_\gamma), \quad \sigma_{disc}(H) = \bigcup_{n \geq 1} K^n_\gamma \quad (4.36)$$

Later on we repeat the proof for the case $d = 2$. \hfill \blacksquare

4.4. Specific 1dim half-solid potentials. Consider the operator $T_\tau f = -f'' + q_\tau f$ on $L^2(\mathbb{R})$, where the potential $q_\tau$ is given by (3.31). We take any fix integer $N \geq 1$ and numbers $\gamma, \tau > 0$ large enough. Due to Lemma 3.2 we obtain that there exists a periodic potential $v$ such that the first $N$ gaps $\gamma_j, j = 1, 2, ..., N$ in the spectrum of the operator $h$ are open. Moreover, there exists an eigenvalue $\mu_j(\tau)$ in each this gap $\gamma_j$ and they satisfy

$$\mu_j(\tau) \in \gamma_j \quad |\gamma_j| = \gamma, \quad \forall \ j = 1, 2, ..., N, \quad (4.37)$$

$$\sigma_{ac}(T_\tau) = \sigma_0 \cup \sigma_1 \cup ... \cup \sigma_{N-1} \cup \sigma_N, \quad \sigma_\gamma \subset [\lambda^+_{N}, \infty). \quad (4.38)$$

Here the bands $\sigma_0, \sigma_1, ..., \sigma_{N-1}$ and the set $\sigma_N$ are separated by gaps $\gamma_j, j = 1, 2, ..., N$ and each eigenvalue $\mu_n(\tau)$ satisfies (4.38).

Now we begin to construct a specific potential $v$. Here we use results about the gap-lengths mapping from [K99]. Due to Theorem 2.1 about the gap-lengths mapping, we take the potential $v \in L^2(\mathbb{T})$ such that the first $N$ gaps $\gamma_1, ..., \gamma_N$ in the spectrum of $h$ are open and these gaps satisfy

$$\gamma = |\gamma_1| = ... = |\gamma_N| = \frac{\pi^24(N+1)^2}{\kappa}, \quad 0 < \kappa << 1, \quad (4.39)$$

and in each big gap $\gamma_n, n = 1, 2, ..., N$ there exists exactly one eigenvalue $\mu_n(\tau)$.

Define the operator $T_{\tau,\gamma} = \frac{1}{\gamma} T_\tau$. From the properties of $T_\tau$ we deduce that the spectrum of $T_{\tau,\gamma}$ consists of an absolutely continuous part $\sigma_{ac}(T_{\tau,\gamma}) = \bigcup_{n \geq 0} s_n$ plus at most one eigenvalue in each non-empty gap $g_n, n \in \mathbb{N}$, where the bands $s_n$ and gaps $g_n$ are given by

$$s_0 = \frac{\sigma_0}{\gamma}, \quad s_n = \frac{\sigma_n}{\gamma}, \quad g_n = \frac{\gamma_n}{\gamma}, \quad n \in \mathbb{N}_0 = \{1, ..., N\},$$

and they satisfy (4.15)-(4.17). In each gap $g_n, n = 1, ..., N$, there exists exactly one eigenvalue $e_n$ given by

$$e_n = \frac{\mu_n}{\gamma} = e^0_n + \varepsilon_n, \quad e^0_n = n - 1 + \frac{1}{4d}, \quad |\varepsilon_n| \leq \frac{\kappa}{4}, \quad n \in \mathbb{N}_0, \quad (4.40)$$

since we take $\tau$ large enough. Thus roughly speaking the spectrum of the operators $T_\tau$ on $L^2(\mathbb{R})$ and $h$ (on $L^2(\mathbb{R}_+)$) is the same on the interval $[0, \lambda^+_{N}]$. They have the same bands $\sigma_0, ..., \sigma_N$ and the same gaps $\gamma_1, ..., \gamma_N$. Moreover, their eigenvalues in each gap $\gamma_n$ are very close, since we take $\tau$ large enough.
4.5. Model Schrödinger operators on $\mathbb{R}^2$. We consider Schrödinger operators $H$ on the plane $\mathbb{R}^2$ given by
\[ H = T_{r,1} + T_{r,2}, \] (4.41)
the proof for the case $\mathbb{R}^d, d \geq 3$ is similar. Here each $T_{r,j} = -\frac{d^2}{dx^2} + q_r(x_j)$ acts on $\mathbb{R}$ and the potential $q_{js}$ is determined in Subsection 4.4. The spectrum of $T_{r,j}$ and $h_j$ are similar on the interval $[0, \lambda_N^2]$. Then the spectrum of the sum $T_{r,1} + T_{r,2}$ is similar to the spectrum of $h_1 + h_2$ on the interval $[0, 2\lambda_N^2]$. The proof repeats the case $h_1 + h_2$.

4.6. Schrödinger operators on $\mathbb{R}_+ \times \mathbb{R}$. Consider the operator $H = h_1 + T_{r,2}$ on the half-plane $\mathbb{R}_+ \times \mathbb{R}$, where the operator $h_1 y = -y'' + v(x_1) y, y(0) = 0$ acts on the half-line and depends on one variable $x_1 > 0$; the operator $T_{r,2} = -\frac{d^2}{dx^2} + q_s(x_2)$ acts on $\mathbb{R}$ and the potential $q_{s}(x_2)$ is defined by (5.1) and the constant $\tau$ is large enough. The spectrum of $T_{r,2}$ and $h_1$ are similar on the interval $[0, \lambda_N^2]$ for $N, \tau$ large enough. The proof repeats the case $h_1 + h_2$.

5. Proof of main Theorems

Proof Theorem 1.1 i) We consider an operator $H_+ = -\Delta + V$ on $\mathbb{R}_+^2$, where $V$ is $\mathbb{Z}^2$-periodic, the proof for other cases is similar. Let $H = -\Delta + V$ on $\mathbb{R}^2$. Define functions $g_n \in C_0^\infty(\mathbb{R})$ and $G_n \in C_0^\infty(\mathbb{R}^2), n \geq 1$ by:
\[ g_n|_{w_n} = 1, \quad w_n = [4^n, 4^n + 1], \quad \text{supp} \; g_n = [4^n - 1, 4^n + n + 1], \quad \text{supp} \; G_n \subset \mathbb{R}^2_+. \] (5.1)

Let $\mathcal{T}_2 = \mathbb{R}^2/\mathbb{Z}^2$. For any $\lambda \in \sigma(H)$ there exists a function $\psi(x, k) = e^{ikx} u(x, k)$, which satisfies
\[ (-\Delta + V(x)) \psi(x, k) = \lambda \psi(x, k), \quad \forall x \in \mathbb{R}^2, \quad u(\cdot, k) \in L^2(\mathcal{T}_2), \quad \int_{\mathcal{T}_2} |u(x, k)|^2 dx = 1, \] (5.2)
for some $k \in \mathbb{R}^2$. Define the sequence $\psi_n(x, k) = \frac{1}{c_n} G_n(x) \psi(x, k)$, where $c_n > 0$ is given by
\[ c_n^2 = \int_{\mathbb{R}^2} |G_n(x) \psi(x, k)|^2 dx. \]

The function $u(x, k)$ is $\mathbb{Z}^2$ periodic, then due to (5.2) we obtain
\[ c_n^2 = \int_{\mathbb{R}^d} |G_n(x) \psi(x, k)|^2 dx = n^2 + O(n) \] (5.3)
as $n \to \infty$. Thus the sequence $\psi_n$ satisfies
1) $\|\psi_n(\cdot, k)\| = 1$ and $\Delta \psi_n \in L^2(\mathbb{R}^2)$, for all $n \in \mathbb{N}$,
2) $\psi_n \perp \psi_m$ for all $n \neq m$, and $\psi_n \rightharpoonup 0$ weakly as $n \to \infty$.

Thus $\lambda \in \sigma_{ess}(H_+)$, since standard arguments imply
\[ \| (H_+ - \lambda) \psi_n(\cdot, k) \| = \| (H - \lambda) \psi_n(\cdot, k) \| \to 0 \quad \text{as} \quad n \to \infty. \]

ii). Consider an operator $H_\varepsilon = H_+ + \varepsilon W$ on $\mathbb{R}_+^d$ for the case $d = 2$, the proof for the case $\varepsilon = x, d \geq 3$ is similar. Here the operator $H_+$ is defined in subsection 4.2. Recall that for for any $n \geq 1$ there exists a specific potential $v \in L^2(\mathbb{R})$ such that on the interval $I_n = (a_n, b_n)$ (defined by (4.22)) contains $n$ eigenvalues of the operator $H_+$. Moreover, the distance between the interval $I_n$ and two cluster spectral sets $K_0^0 \cup K_1^1$ and $K_{n+1}^0 \cup K_{n+1}^1$ is greater than $2\varepsilon$. 


We have $H_0 = H_+$, where the real coupling constant $\varepsilon$ is small enough and $W$ satisfies
\[ W \in L^\infty(\mathbb{R}_+^2), \quad \|W\|_{L^\infty(\mathbb{R}_+^2)} \leq 1, \quad 0 < \varepsilon \leq \varkappa^3. \] (5.4)
We define contours $c_n = \{ \lambda \in \mathbb{C} : \text{dist}\{\lambda, I_n\} = \varkappa \}$.
Due to (5.4) the operator $H_\varepsilon$ has $n$ eigenvalues inside the contours $c_n$, since we have
\[ P_n(\varepsilon) = -\frac{1}{2\pi i} \int_{c_n} R_\varepsilon(z) dz, \]
\[ R_\varepsilon(z) = R_+(z) - R_+(z)\varepsilon W R_\varepsilon(z), \] (5.5)
\[ \|R_\varepsilon(z) - R_+(z)\| \leq \varepsilon\|R_+(z)\| \leq \frac{\varepsilon}{\varkappa^2} \leq \varkappa \quad \forall \ z \in c_n. \]
This yields
\[ \|P_n(\varepsilon) - P_n(0)\| \leq \varkappa < 1. \] (5.6)
and then the operators $P_n(\varepsilon)$ and $P_n(0)$ have the same dimension.
In order to show that the intervals $[a_-, a]$ and $[b, b_+]$ contain the essential spectrum of $H$ for some $a_- < a$ and $b < b_+$
we use similar arguments.

Now we consider Schrödinger operators $H_\varepsilon = H_0 + \varepsilon W$ on the domain $D = \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$, $d_1 + d_2 = d \geq 2$, where the real coupling constant $\varepsilon$ is small enough and $W$ satisfies
\[ W \in L^\infty(D), \quad \|W\|_{L^\infty(D)} \leq 1. \] (5.7)
The operator $H_0 = H_{01} + H_{02}$, where $H_{01}, H_{02}$ are given by
\[ H_{01} = -\Delta_x + \sum_{j=1}^{d_1} v(x_j), \quad H_{02} = -\Delta_y + \sum_{j=1}^{d_2} v(y_j), \quad x = (x_j) \in \mathbb{R}_+^{d_1}, y = (y_j) \in \mathbb{R}^{d_2}. \]
The proof for this case is similar and is based on the □

**Proof Theorem 1.2.** We consider Schrödinger operators $H = T_{\tau,1} + T_{\tau,2}$ on the plane $\mathbb{R}^2$, the proof for the case $\mathbb{R}^d, d \geq 3$ is similar. Here each operator $T_{\tau, j}$ acts on $\mathbb{R}$ and given by
\[ T_{\tau, j} = -\frac{d^2}{dy_j^2} + q_\tau(y_j), \quad y = (y_1, y_2) \in \mathbb{R}^2, \]
where the potential $q_\tau$ is determined in (3.1). By Lemma 3.2 iii), each operator $T_{\tau, j}, j = 1, 2$ on $L^2(R)$ has an eigenvalue $E_0$ below the continuous spectrum $\sigma_{ac}(T_{\tau, j}) = [\tau_0, \infty)$ for some potential $v \in L^2(\mathbb{T})$ and some $\tau \in \mathbb{R}$, where $\tau_0 = \min\{\tau, \lambda_0^j\}$. Then the operators $H = T_{\tau,1} + T_{\tau,2}$ has the eigenvalue $E_0$ below the continuous spectrum $\sigma_{ac}(H) = [\tau_0 + E_0, \infty)$. Consider an operator $H_\varepsilon = H + \varepsilon W$, where $\varepsilon > 0$ is small enough and $W$ satisfies (1.2). It is well know that under small perturbation, the isolated eigenvalue is still eigenvalue. □

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