Twisted conjugacy classes in nilpotent groups

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Abstract

Let $N$ be a finitely generated nilpotent group. Algorithm is constructed such, that for every automorphism $\phi \in Aut(N)$ defines the Reidemeister number $R(\phi)$. It is proved that any free nilpotent group of rank $r = 2$ or $3$ and class $c \geq 4r$, or rank $r \geq 4$ and class $c \geq 2r$, belongs to the class $R_\infty$.

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1 Introduction

Let $\phi : G \to G$ be an automorphism of a group $G$. One says that the elements $g, f \in G$ are $\phi$-twisted conjugated, denoted by $g \sim_\phi f$, if and only if there exists $x \in G$ such that $g = (x\phi)^{-1}fx$, or equivalently $(x\phi)g = fx$. A class of equivalence $[g]_\phi$ is called the Reidemeister class (or the $\phi$-conjugacy class of $\phi$). The number $R(\phi)$ of Reidemeister classes is called the Reidemeister number of $\phi$.

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There are different origins of interest in twisted conjugacy relations. In classical Nielsen-Reidemeister fixed point theory, $R(\varphi)$ plays a crucial role in estimating the Nielsen number $N(f)$ of a selfmap $f : C \to C$, where $C$ is a compact connected manifold. In such setting $\varphi$ appears as the homomorphism induced by $f$ on the fundamental group $\pi_1(C)$. The Nielsen number $N(f)$ is a homotopy invariant which provides a lower bound of size of the set $Fix f(C)$ of fixed $f$-points. The Selberg theory (see [16]), and Algebraic Geometry (see [9]) present other sources for the twisted conjugacy relations.

One of the central problems in the field concerns obtaining a twisted analogue of the classical Burnside-Frobenius theorem, that to show the coincidence of the Reidemeister number $R(\varphi)$ and the number of fixed points of the induced homeomorphism of an appropriate dual object. The authors of the paper [6] emphasize that one step in this process is to describe the class of groups $G$, such that $R(\varphi) = \infty$ for any automorphism $\varphi : G \to G$. In a number of papers ([19], [5], [6]) this class of groups is denoted $R_\infty$. Namely, a group $G$ has property $R_\infty$ (or is an $R_\infty$ group) if all of its automorphisms $\varphi$ have $R(\varphi) = \infty$.

It was shown by various authors that the following groups belong to the class $R_\infty$:

- non-elementary Gromov hyperbolic groups [11, 2];
- Baumslag-Soliter groups $B(m, n)$ except for $B(1, 1)$ [4];
- generalized Baumslag-Soliter groups, that is finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [13, 17];
- lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if and only if $2|n$ or $3|n$ [8];
- the solvable generalization $\Gamma$ of $BS(1, n)$ given by the short sequence $1 \to \mathbb{Z}[1/n] \to \Gamma \to \mathbb{Z}^k \to 1$ [18];
- relatively hyperbolic groups [3];
- the Grigorchuk group and Gupta-Sidki group [6].

For the immediate consequences of $R_\infty$ property for topological fixed point theory see, e.g. [17].

In the present paper we treat with finitely generated nilpotent groups. On the first glance nilpotent groups are too far to possess the $R_\infty$ property. Indeed, E.G. Kukina [12] noted that the Reidemeister spectrum

$$Spec_R(G) = \{ R(\varphi) | \varphi \in Aut(G) \}$$

of any free abelian group $G = \mathbb{Z}^k$, $k \geq 2$, coincides with $\mathbb{N} \cup \{ \infty \}$, i.e. is full.

She also calculated the Reidemeister spectra of the free nilpotent groups $N_{rc}$ of rank $r$ and class $c$ in cases $r = 2, 3$ and $c = 2$. Namely,

$$Spec_R(N_{zz}) = \{ 2k | k \in \mathbb{N} \} \cup \{ \infty \},$$
For any automorphism $\varphi$, twisted conjugacy classes in nilpotent groups are studied by F.K. Indukaev [10].

Any pair of elements $c_1, c_2 \in C$ are $\varphi$-conjugated in $G$ if and only if

$$c_1^{-1}c_2 \in L(C, \varphi).$$

**Lemma 2.1.**

**Proof.** Let $c_1 \sim c_2 \Rightarrow \exists x \in G | (x\varphi)c_1 = c_2x \Rightarrow x\varphi = c_1^{-1}c_2x \Rightarrow c_1^{-1}c_2 \in L(C, \varphi)$.

In opposite, let $c_1^{-1}c_2 \in L(C, \varphi) \Rightarrow \exists x \in G | x\varphi = c_1^{-1}c_2x \Rightarrow (x\varphi)c_1 = c_2x \Rightarrow c_1 \sim c_2$.

**Corollary 2.2.** The set of all $\varphi$-conjugacy classes of the elements $c \in C$ coincides with the set of all cosets $C \text{ w.r.t. } L(C, \varphi)$. In particular, $R(\varphi) \geq [C : L(C, \varphi)]$.

**Corollary 2.3.** In the case of a finitely generated abelian group $G = C$ there is an effective procedure for calculating the Reidemeister number $R(\varphi)$ of any automorphism $\varphi : G \rightarrow G$. Moreover, such procedure finds in a finite case a set of representatives of all $\varphi$-conjugacy classes.

**Proof.** Let $G = gp(a_1, ..., a_p)$, and $a_i\varphi = c_ia_i$, $i = 1, ..., p$. Then $L(C, \varphi) = gp(c_1, ..., c_p)$. A standard procedure easily defines either $[C : L(C, \varphi)] = \infty$, or not.

In a finite case we just need to calculate the index $[C : L(C, \varphi)]$ and to find any transversal set of $C \text{ w.r.t. } L(C, \varphi)$. It is well known that such effective procedure exists in any finitely generated effective presented abelian group $G$.

Now let $G$ be any group, and $\varphi : G \rightarrow G$ be an automorphism of $G$. Let $C$ be a central $\varphi$-admissible subgroup of $G$. Denote by $\eta : G \rightarrow \tilde{G} = G/C$ the standard epimorphism, and by $\tilde{\varphi} : \tilde{G} \rightarrow \tilde{G}$ the induced by $\varphi$ automorphism. Let $[\tilde{g}]_{\tilde{\varphi}}$ be any $\tilde{\varphi}$-conjugacy class in $\tilde{G}$, and $g \in G$ be any pre image of $\tilde{g}$ in $G$. Define an automorphism $\varphi_g : G \rightarrow G$ by $\varphi_g = \varphi \circ \sigma_g$, where $\sigma_g \in InnG, \sigma_g : h \mapsto g^{-1}hg$ for all $h \in G$. Define as before a subgroup $L(C, \varphi_g)$ of $G$. Then we have
Lemma 2.4. The full pre image in $G$ under the standard epimorphism $\eta : G \rightarrow \bar{G}$ of any $\bar{\varphi}$-equivalence class $[\bar{g}]_{\bar{\varphi}}$ is a disjoint union of $s = [C : L(C, \varphi_g)]$ $\varphi$-conjugacy classes

$$[gc_1]_{\varphi} \cup \ldots \cup [gc_s]_{\varphi},$$

where $\{c_1, \ldots, c_s\}$ is any transversal set of $C$ w.r.t. $L(C, \varphi_g)$.

Proof. Let we show first that all $\varphi$-conjugacy classes in (3) are different. Suppose that $gc_i \sim \varphi gc_j$ for some $i \neq j$. Then

$$\exists x \in G | (x\varphi)gc_i = gc_jx \Rightarrow (x\varphi) = c_i^{-1}c_jx = c_i^{-1}c_j \in L(C, \varphi_g),$$

which contradicts to our assumption.

Now we are to prove that every element $gc, c \in C$, is $\varphi$-conjugated to some of $gc_i, i = 1, \ldots, c_s$. Namely, if $c = c_ic'$, where $c' \in L(C, \varphi_g)$ then $gc \sim \varphi gc_i$. To prove it suppose that for $x \in G$ one has $x\varphi_g = c'x$. Then

$$x\varphi = g(x\varphi_g)g^{-1} = gxg^{-1}c' \Rightarrow (x\varphi)gc_i = gc_c'x \Rightarrow gc \sim \varphi gc_i.$$  

At last suppose that any element $f \in G$ belongs to the full pre image of $[\bar{g}]_{\bar{\varphi}}$. So,

$$f \sim \bar{\varphi} \bar{g} \Rightarrow \exists x \in G, c \in C | (x\varphi)g = fx \Rightarrow (x\varphi)gc = gc \Rightarrow f \sim \varphi gc_i,$$

where $c^{-1} = c'gc'$, $c' \in L(C, \varphi_g)$.

The following result was obtained in [15] even in more general form.

Lemma 2.5. Let $N$ be a finitely generated nilpotent group of class $k$, and $\varphi : N \rightarrow N$ be any automorphism. Then there is an effective procedure which gives a finite generating set of a subgroup

$$\text{Fix}_\varphi(N) = \{ x \in N | x\varphi = x \}.$$  

Proof. Obviously, such procedure exists in abelian case $k = 1$.

Let $C$ be a central $\varphi$-admissible subgroup of $N$, one can take the last non-trivial member $C = \gamma_{k-1}N$ of the lower central series of $N$. Let $\bar{\varphi} : N/C \rightarrow N/C$ be the induced by $\varphi$ automorphism. By induction on $k$ we have a generating set of

$$\text{Fix}_{\bar{\varphi}}(N/C) = gp(\bar{g}_1, \ldots, \bar{g}_l).$$

Then we can effectively obtain a generating set $\{g_1, \ldots, g_l, g_{l+1}, \ldots, g_p\}$ of the full pre image of $\text{Fix}_{\bar{\varphi}}(N/C)$ in $N$. Namely, we take some pre images $g_1, \ldots, g_l$ of the elements $\bar{g}_1, \ldots, \bar{g}_l$, respectively, and add a generating set $\{g_{l+1}, \ldots, g_p\}$ of $C$. See for details [11]. So,

$$g_i\varphi = c_ig_i, \ i = 1, \ldots, p.$$  

Introduce a homomorphism of the full pre image of $\text{Fix}_\varphi(N/C)$ in $N$ to $C$, uniquely defined by the map

$$\mu : g_i \mapsto c_i, \quad i = 1, \ldots, p. \quad (10)$$

Since the derived subgroup belongs to $\ker \mu$ we actually have a homomorphism of abelian groups. Then we can effectively find a generating set of

$$\text{Fix}_\varphi(N) = \ker \mu. \quad (11)$$

**Theorem 1.** Let $N$ be a finitely generated nilpotent group of class $k$. Then there is an effective procedure to calculate for any automorphism $\varphi : N \to N$ the Reidemeister number $R(\varphi)$.

Proof. By induction on $k$, starting with Corollary 2.3 we can assume that the statement is true for any finitely generated nilpotent group of class $\leq k - 1$.

Let $\varphi : N \to N$ be any automorphism of $N$. Consider some $\varphi$-admissible central series

$$N = C_1 > C_2 > \ldots > C_k > 1. \quad (12)$$

For example, one can take the lower central series in $N$.

By induction we have the number $R(\varphi_{k-1})$. Obviously, $R(\varphi_{k-1}) = \infty$ implies $R(\varphi_k) = R(\varphi) = \infty$.

Suppose, $R(\varphi_{k-1}) = r < \infty$. Let

$$[\bar{g}_1]_{\varphi_{k-1}}, \ldots, [\bar{g}_r]_{\varphi_{k-1}} \quad (13)$$

be the set of all $\varphi_{k-1}$ -conjugacy classes in $N_{k-1} = N/C_k$. As before $\bar{f}$ means the image of any element $f \in N$ under the standard epimorphism $N \to N_{k-1}$.

To apply Lemma 2.4 we need in subgroups $L(C_k, \varphi_{g_i}), \quad i = 1, \ldots, r$. Let $\psi : N \to N$ be any automorphism with $\psi(C_k) = C_k$, and $\psi_{k-1} : N_{k-1} \to N_{k-1}$ be the automorphism induced by $\psi$.

Firstly we derive by Lemma 2.4 a generating set $\{f_1, \ldots, f_q\}$ of the full pre image $H_\psi$ of $\text{Fix}_{\psi_{k-1}}(N_{k-1})$ in $N$. Then we compute

$$f_j \psi = c_j f_j, \quad c_j \in C_k, \quad j = 1, \ldots, q, \quad (14)$$

and conclude that

$$L(C_k, \psi) = gp(c_1, \ldots, c_q). \quad (15)$$

We repeat such process for every $\psi = \varphi_{g_i}, \quad i = 1, \ldots, r$. Every time we apply Lemma 2.4 to find a presentation of the full pre image $H_{\varphi_{g_i}}$ of $[\bar{g}_i]_{\varphi_{k-1}}$ in $N$ as a disjoint union of $\varphi$-conjugacy classes in $N$. The union of them is the set of all $\varphi$-conjugacy classes in $N$.

Theorem is proved.
3 Most of the free nilpotent groups are in $R_\infty$

Let $N$ be a finitely generated torsion free nilpotent group. Every quotient $A_i = \zeta_{i+1}N/\zeta_iN$, $i = 0, 1, \ldots, k - 1$, of the upper central series

$$\zeta_0N = 1 < \zeta_1N < \ldots < \zeta_kN = N$$

(16)

is a free abelian group of a finite rank. Any automorphism $\varphi : N \to N$ induces the automorphisms $\varphi_i : N_i \to N_i$, where $N_i = N/\zeta_iN$, $i = 1, \ldots, k - 1$, as well as the automorphisms $\bar{\varphi}_i : A_i \to A_i$, $i = 0, 1, \ldots, k - 1$.

**Lemma 3.1.** Let $A$ be a free abelian group of a finite rank $r$. Let $\varphi : A \to A$ be any automorphism of $A$ such that

$$Fix_{\varphi}A \neq 1.$$  

(17)

Then $R(\varphi) = \infty$.

Proof. It was shown in Lemma 2.1 that $R(\varphi) = [A : L(A, \varphi)]$, where in our case

$$L(A, \varphi) = im(\varphi - id).$$  

(18)

The subgroup $L(A, \varphi)$ is a free abelian group of rank $s = \text{range}(\varphi - id)$. If $Fix_{\varphi}(A) \neq 1$ we have $s < r$ and so $[A : L(A, \varphi)] = \infty$.

**Corollary 3.2.** Let $N$ be a finitely generated torsion free nilpotent group of class $k$, and $\varphi : N \to N$ be any automorphism of $N$. Suppose that $Fix_{\bar{\varphi}_i}(A_i) \neq 1$ for some $i = 0, 1, \ldots, k - 1$. Then $R(\varphi) = \infty$.

Proof. We can assume that $i$ is maximal with property $Fix_{\bar{\varphi}_i}(A_i) \neq 1$. Then $Fix_{\varphi_i+1} = 1$. Let in the denotions of Lemma 2.1 $G = N_i$ and $C = A_i$. Then by the statement of Lemma 3.1 $R(\bar{\varphi}_i) = \infty$, so $R(\varphi_i) = \infty$ too.

E. Formanek [7] classified all pairs $r, c$ for which the free nilpotent group $N_{rc}$ of rank $r \geq 2$ and class $c \geq 2$ has nontrivial elements fixed by all automorphisms. Note that V. Bludov (see [1], background to Problem N1 by A. Myasnikov) gave the first examples of such elements for $r = 2$, $c = 4k$, $k \geq 2$.

**Theorem of Formanek.** Let $N_{rc}$ be a free nilpotent group of rank $r$ and class $c$. Then there are nontrivial elements of $N_{rc}$ which are fixed by all automorphisms of $N_{rc}$ if and only if

(a) $r = 2$ or $r = 3$, and $c = 2kr$, $k \geq 2$.

(b) $r \geq 4$ and $c = 2kr$, $k \geq 1$.

This remarkable result allows to define place of most free nilpotent groups of finite rank in the class $R_\infty$.

**Theorem 2.** Let $N_{rc}$ be a free nilpotent group of rank $r \geq 2$ and class $c \geq 2$. Suppose that for $r = 2$ or $r = 3$ we have $c \geq 4r$ and for $r \geq 4$ we have $c \geq 2r$. Then $N_{rc} \in R_\infty$. 

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Proof. Denote $N = N_{rc}$. Let $\varphi : N \to N$ be any automorphism. It follows from the Formanek’s theorem that there is a maximal $k$ such that the induced automorphism $\varphi_j : N_j \to N_j$ has $Fix_{\varphi_j}(N_j) \neq 1$. Then $Fix_{\varphi_j}(A_j) \neq 1$, and so by Lemma 2.1 $R(\varphi_j) = \infty$. By Corollary 3.2 we have $R(\varphi) = \infty$, which gives the conclusion of the theorem.

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