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AN EFFECTIVE CONTRACTION ESTIMATE IN THE STABLE SUBSPACES OF PHASE POINTS IN HARD BALL SYSTEMS

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Abstract. In this paper we prove the following result, useful and often needed in the study of the ergodic properties of hard ball systems: In any hard ball system, for any phase point \( x_0 \in M \setminus \partial M \) with a non-singular forward orbit \( S^{(0, \infty)}x_0 \) and with infinitely many consecutive, connected collision graphs on \( S^{(0, \infty)}x_0 \), and for any number \( L > 0 \) one can always find a time \( t > 0 \) and a non-zero tangent vector \( (\delta q_0, \delta v_0) \in E^s(x_0) \) with \( \frac{|| (\delta q_t, \delta v_t) ||}{|| (\delta q_0, \delta v_0) ||} < L^{-1} \), where \( (\delta q_t, \delta v_t) = DS^t(\delta q_0, \delta v_0) \in E^s(x_t), x_t = S^tx_0 \). Of course, the Multiplicative Ergodic Theorem of Oseledets provides a much stronger conclusion, but at the expense of an unspecified zero-measured exceptional set of phase points, and this is not sufficient in the sophisticated studies the ergodic properties of such flows. Here the exceptional set of phase points is a dynamically characterized set, so that it suffices for the proofs showing how global ergodicity follows from the local one.

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§1. Introduction

Ever so often it happens that, while studying the ergodic properties of semi-dispersive billiard flows, one needs the following result:

**Theorem.** For any phase point \( x_0 \in M \setminus \partial M \) with a non-singular forward orbit \( S^{(0,\infty)}x_0 \) and with infinitely many consecutive, connected collision graphs on \( S^{(0,\infty)}x_0 \), and for any number \( L > 0 \) one can always find a time \( t > 0 \) and a non-zero tangent vector \( (\delta q_0, \delta v_0) \in E^s(x_0) \) with

\[
\frac{||(\delta q_t, \delta v_t)||}{||(\delta q_0, \delta v_0)||} < L^{-1},
\]

where \( (\delta q_t, \delta v_t) = DS^t(\delta q_0, \delta v_0) \in E^s(x_t), x_t = S^tx_0 \).

The last time when the above result was badly needed, at least in my own research, was a certain step in the conditional proof of the Boltzmann-Sinai Ergodic Hypothesis in [Sim(2006)].

We note that, according to Oseledets’ Multiplicative Ergodic Theorem and the full hyperbolicity of every hard ball system [Sim(2002)], for almost every phase point \( x_0 \) (with respect to the natural invariant measure, that is, the Liouville measure) every vector \( w \in E^s(x_0) \) contracts exponentially fast as \( t \to \infty \). The point in the present result is that it holds true for even more generic points \( x_0 \): for all points apart from a so called slim subset of the phase space, see Theorem 5.1 in [Sim(1992)-I]. What is more, even if we have a smooth, codimension-one, exceptional manifold \( J \) (capable of separating two distinct, open ergodic components of the flow) possessing a normal vector field \( n(x) = (z(x), w(x)) \) with \( Q(n(x)) = \langle z(x), w(x) \rangle < 0 \) \( (x \in J, \text{see the beginning of §3 in [Sim(2006)]}) \), almost every phase point \( x_0 \) of the hypersurface \( J \) will satisfy the hypotheses of the above theorem. This is indeed so, since the proof of Theorem 6.1 of [Sim(1992)-I] works without any essential change not only for singular phase points, but also for the points of the considered exceptional manifold \( J \). The only important ingredient of that proof is the transversality of the spaces \( E^s(x) \) to \( J \) for all \( x \in J \). According to that result, typical phase points \( x \in J \) (with respect to the hypersurface measure of \( J \)) indeed enjoy the above property of having infinitely many consecutive, connected collision graphs on their forward orbit \( S^{(0,\infty)}x_0 \).

We also note that, by slightly perturbing the shrinking vector \( (\delta q_0, \delta v_0) \in E^s(x_0) \) inside the stable space \( E^s(x_0) \), we can achieve that this vector be transversal to the manifold \( J \). This is another point where we use the fact that the stable spaces \( E^s(x) \) \( (x \in J) \) are transversal to \( J \), an immediate corollary of the negativity assumption on the infinitesimal Lyapunov form \( Q(n(x)) \).

It is worth noting here that the main phenomenon that makes the assertion of the theorem substantial (and the proof non-trivial) is the possibility of very long
free flights (i.e. orbit segments without collision), or at least the possibility of long segments on $S^{(0,\infty)}x_0$ in which all occurring collisions have very small relative velocities for the colliding balls. This phenomenon “flattens out” the stable manifold $\gamma^s(x_t)$ of $x_t = S^t x_0$ and, as a result, has the potential to make the contraction coefficient only slightly smaller than 1. The main part of the proof of the theorem is to show that the above phenomenon actually does not occur, at least when $S^{(0,\infty)}x_0$ has infinitely many consecutive, connected collision graphs (Corollary 3.12). It is just the proof of this corollary, more precisely, the proof of Proposition 3.9, that utilizes the assumption on the infinitely many connected collision graphs.

Finally, we note that the unstable version of the above theorem, claiming the arbitrarily big expansions (as $t \to \infty$) of the forward images of vectors $w \in E^u(x_0)$ is obviously true: It is easy to see that any tangent vector $w = (\delta q, \delta v) \in T_{x_0}M$ with $\langle \delta q, \delta v \rangle > 0$ expands at least linearly in time as $t \to \infty$, even without collisions, see Proposition 3.5 below.

§2. Prerequisites

Consider the $\nu$-dimensional ($\nu \geq 2$), standard, flat torus $\mathbb{T}^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu$ as the vessel containing $N$ ($\geq 2$) hard balls (spheres) $B_1, \ldots, B_N$ with positive masses $m_1, \ldots, m_N$ and (just for simplicity) common radius $r > 0$. We always assume that the radius $r > 0$ is not too big, so that even the interior of the arising configuration space $Q$ (or, equivalently, the phase space) is connected. Denote the center of the ball $B_i$ by $q_i \in \mathbb{T}^\nu$, and let $v_i = \dot{q}_i$ be the velocity of the $i$-th particle. We investigate the uniform motion of the balls $B_1, \ldots, B_N$ inside the container $\mathbb{T}^\nu$ with half a unit of total kinetic energy: $E = \frac{1}{2} \sum_{i=1}^{N} m_i ||v_i||^2 = \frac{1}{2}$. We assume that the collisions between balls are perfectly elastic. Since — beside the kinetic energy $E$ — the total momentum $I = \sum_{i=1}^{N} m_i v_i \in \mathbb{R}^\nu$ is also a trivial first integral of the motion, we make the standard reduction $I = 0$. Due to the apparent translation invariance of the arising dynamical system, we factorize the configuration space with respect to uniform spatial translations as follows: $(q_1, \ldots, q_N) \sim (q_1 + a, \ldots, q_N + a)$ for all translation vectors $a \in \mathbb{T}^\nu$. The configuration space $Q$ of the arising flow is then the factor torus $\left( (\mathbb{T}^\nu)^N / \sim \right) \cong \mathbb{T}^\nu^{(N-1)}$ minus the cylinders

$$C_{i,j} = \left\{ (q_1, \ldots, q_N) \in \mathbb{T}^\nu^{(N-1)} : \text{dist}(q_i, q_j) < 2r \right\}$$

$(1 \leq i < j \leq N)$ corresponding to the forbidden overlap between the $i$-th and $j$-th spheres. Then it is easy to see that the compound configuration point

$$q = (q_1, \ldots, q_N) \in Q = \mathbb{T}^\nu^{(N-1)} \setminus \bigcup_{1 \leq i < j \leq N} C_{i,j}$$
moves in $\mathcal{Q}$ uniformly with unit speed and bounces back from the boundaries $\partial C_{i,j}$ of the cylinders $C_{i,j}$ according to the classical law of geometric optics: the angle of reflection equals the angle of incidence. More precisely: the post-collision velocity $v^+$ can be obtained from the pre-collision velocity $v^-$ by the orthogonal reflection across the tangent hyperplane of the boundary $\partial \mathcal{Q}$ at the point of collision. Here we must emphasize that the phrase “orthogonal” should be understood with respect to the natural Riemannian metric (the kinetic energy) $||dq||^2 = \sum_{i=1}^{N} m_i ||dq_i||^2$ in the configuration space $\mathcal{Q}$. For the normalized Liouville measure $\mu$ of the arising flow $\{S^t\}$ we obviously have $d\mu = \text{const} \cdot dq \cdot dv$, where $dq$ is the Riemannian volume in $\mathcal{Q}$ induced by the above metric, and $dv$ is the surface measure (determined by the restriction of the Riemannian metric above) on the unit sphere of compound velocities

$$\mathbb{S}^{\nu(N-1)-1} = \left\{ (v_1, \ldots, v_N) \in (\mathbb{R}^\nu)^N : \sum_{i=1}^{N} m_i v_i = 0 \text{ and } \sum_{i=1}^{N} m_i ||v_i||^2 = 1 \right\}. $$

The phase space $\mathcal{M}$ of the flow $\{S^t\}$ is the unit tangent bundle $\mathcal{Q} \times \mathbb{S}^{d-1}$ of the configuration space $\mathcal{Q}$. (We will always use the shorthand notation $d = \nu(N-1)$ for the dimension of the billiard table $\mathcal{Q}$.) We must, however, note here that at the boundary $\partial \mathcal{Q}$ of $\mathcal{Q}$ one has to glue together the pre-collision and post-collision velocities in order to form the phase space $\mathcal{M}$, so $\mathcal{M}$ is equal to the unit tangent bundle $\mathcal{Q} \times \mathbb{S}^{d-1}$ modulo this identification.

A bit more detailed definition of hard ball systems with arbitrary masses, as well as their role in the family of cylindric billiards, can be found in §4 of [S-Sz(2000)] and in §1 of [S-Sz(1999)]. We denote the arising flow by $(\mathcal{M}, \{S^t\}_{t \in \mathbb{R}}, \mu)$.

**Collision graphs.** Let $S^{[a,b]}_x$ be a nonsingular, finite trajectory segment with the collisions $\sigma_1, \ldots, \sigma_n$ listed in time order. (Each $\sigma_k$ is an unordered pair $i,j$ of different labels $i,j \in \{1,2,\ldots,N\}$.) The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1,2,\ldots,N\}$ and set of edges $\mathcal{E} = \{\sigma_1, \ldots, \sigma_n\}$ is called the collision graph of the orbit segment $S^{[a,b]}_x$. For a given positive number $C$, the collision graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of the orbit segment $S^{[a,b]}_x$ will be called $C$-rich if $\mathcal{G}$ contains at least $C$ connected, consecutive (i. e. following one after the other in time, according to the time-ordering given by the trajectory segment $S^{[a,b]}_x$) subgraphs.

**No accumulation (of collisions) in finite time.** By the results of Vaserstein [V(1979)], Galperin [G(1981)] and Burago-Ferleger-Kononenko [B-F-K(1998)], in any semi-dispersive billiard flow there can only be finitely many collisions in finite time intervals, see Theorem 1 in [B-F-K(1998)]. Thus, the dynamics is well defined as long as the trajectory does not hit more than one boundary components at the same time.
Finally, for any phase point \( x \in M \setminus \partial M \) with a non-singular forward orbit \( S^{(0, \infty)}x \) (and with at least one collision, hence infinitely many collisions on it) we define the stable subspace \( E^s(x) \subset T_xM \) of \( x \) as

\[
E^s(x) = \{ (\delta q, \delta v) \in T_xM \mid \delta v = -B(x)[\delta q], \langle \delta q, \delta v \rangle < 0 \} \cup \{(0, 0)\},
\]

where the symmetric, non-negative operator \( B(x) \) (acting on the tangent space of \( Q \) at the footpoint \( q \), where \( x = (q, v) \)) is defined by the continued fraction expansion introduced by Sinai in [Sin(1979)], see also [Ch(1982)] or (2.4) in [K-S-Sz(1990)-I]. It is a well known fact that \( E^s(x) \) is the tangent space of the local stable manifold \( \gamma^s(x) \), if the latter object exists.

For any phase point \( x \in M \setminus \partial M \) with a non-singular backward orbit \( S^{(-\infty, 0)}x \) (and with at least one collision on it) the unstable space \( E^u(x) \) of \( x \) is defined as \( E^s(-x) \), where \( -x = (q, -v) \) for \( x = (q, v) \).

§3. Expansion and Contraction Rate Estimates

Proof of Theorem

We would like to get a useful lower estimate for the expansion of a tangent vector \( (\delta q_0, \delta v_0) \in T_xM \) with positive infinitesimal Lyapunov function \( Q(\delta q_0, \delta v_0) = \langle \delta q_0, \delta v_0 \rangle \). The expression \( \langle \delta q_0, \delta v_0 \rangle \) is the scalar product in \( \mathbb{R}^d \) defined via the mass (or kinetic energy) metric, see §2. It is also called the infinitesimal Lyapunov function associated with the tangent vector \( (\delta q_0, \delta v_0) \), see [K-B(1994)], or part A.4 of the Appendix in [Ch(1994)], or §7 of [Sim(2003)]. For a detailed exposition of the relationship between the quadratic form \( Q(\cdot, \cdot) \), the relevant symplectic geometry of the Hamiltonian system and the dynamics, please also see [L-W(1995)].

Note. The original idea of using infinitesimal Lyapunov exponents to measure the expansion rate of codimension-one submanifolds in the phase space of semi-dispersive billiards came from N. Chernov back in the late '80s. These ideas have been explored in detail and further developed by him and myself in recent personal communications, so that we obtained at least linear (but uniform!) expansion rates for such submanifolds with negative infinitesimal Lyapunov forms for their normal vector. These results are presented in our recent joint paper [Ch-Sim(2006)]. Also, closely related to the above said, the following ideas (to estimate the expansion rates of tangent vectors from below) are derived from the thoughts being published in [Ch-Sim(2006)].

Denote by \( (\delta q_t, \delta v_t) = (DS^t)(\delta q_0, \delta v_0) \) the image of the tangent vector \( (\delta q_0, \delta v_0) \) under the linearization \( DS^t \) of the map \( S^t \), \( t \geq 0 \). (We assume that the base phase
point $x_0$ — for which $(\delta q_0, \delta v_0) \in T_{x_0}M$ — has a non-singular forward orbit. The time-evolution $(\delta q_{t_1}, \delta v_{t_1}) \mapsto (\delta q_{t_2}, \delta v_{t_2})$ $(0 \leq t_1 < t_2)$ on a collision free segment $S^{[t_1,t_2]}x_0$ is described by the equations

\begin{align}
\delta v_{t_2} &= \delta v_{t_1}, \\
\delta q_{t_2} &= \delta q_{t_1} + (t_2 - t_1)\delta v_{t_1}.
\end{align}

(3.1)

Correspondingly, the change $Q(\delta q_{t_1}, \delta v_{t_1}) \mapsto Q(\delta q_{t_2}, \delta v_{t_2})$ in the infinitesimal Lyapunov function $Q(.)$ on the collision free orbit segment $S^{[t_1,t_2]}x_0$ is

\begin{align}
Q(\delta q_{t_2}, \delta v_{t_2}) &= Q(\delta q_{t_1}, \delta v_{t_1}) + (t_2 - t_1)||\delta v_{t_1}||^2,
\end{align}

(3.2)

thus $Q(.)$ steadily increases between collisions.

The passage $(\delta q_t^-, \delta v_t^-) \mapsto (\delta q_t^+, \delta v_t^+)$ through a reflection (i. e. when $x_t = S^t x_0 \in \partial M$) is given by Lemma 2 of [Sin(1979)] or formula (2) in §3 of [S-Ch(1987)];

\begin{align}
\delta q_t^+ &= R\delta q_t^-,
\delta v_t^+ &= R\delta v_t^- + 2 \cos \phi RV^*KV \delta q_t^-,
\end{align}

(3.3)

where the operator $R : TQ \rightarrow TQ$ is the orthogonal reflection (with respect to the mass metric) across the tangent hyperplane $T_{q_t}Q$ of the boundary $\partial Q$ at the configuration component $q_t$ of $x_t = (q_t, v_t^\pm)$, $V : (v_t^-)^\perp \rightarrow T_{q_t}Q$ is the $v_t^-$-projection of the orthocomplement hyperplane $(v_t^-)^\perp$ onto $T_{q_t}Q$, $V^* : T_{q_t}Q \rightarrow (v_t^-)^\perp$ is the adjoint of $V$ (i. e. $\nu(q_t)$-parallel projection of $T_{q_t}Q$ onto $(v_t^-)^\perp$, where $\nu(q_t)$ is the inner normal vector of $\partial Q$ at $q_t \in \partial Q$), $K : T_{q_t}Q \rightarrow T_{q_t}Q$ is the second fundamental form of the boundary $\partial Q$ at $q_t$ (with respect to the field $\nu(q_t)$ of inner unit normal vectors of $\partial Q$) and, finally, $\cos \phi = \langle \nu(q_t), v_t^\pm \rangle > 0$ is the cosine of the angle $\phi$ ($0 \leq \phi < \pi/2$) subtended by $v_t^\pm$ and $\nu(q_t)$. Regarding formulas (3.3), please see the last displayed formula in §1 of [S-Ch(1987)] or (iii)–(i) in Proposition 2.3 of [K-S-Sz(1990)-I]. The instantaneous change in the infinitesimal Lyapunov function $Q(\delta q_t, \delta v_t)$ caused by the reflection at time $t > 0$ is easily derived from (3.3):

\begin{align}
Q(\delta q_t^+, \delta v_t^+) &= Q(\delta q_t^-, \delta v_t^-) + 2 \cos \phi(V\delta q_t^-, KV\delta q_t^-)
\geq Q(\delta q_t^-, \delta v_t^-).
\end{align}

(3.4)

In the last inequality we used the fact that the operator $K$ is positive semi-definite, i. e. the billiard is semi-dispersive.

We are primarily interested in getting useful lower estimates for the expansion rate $||\delta q_t||/||\delta q_0||$. The needed result is
Proposition 3.5. Use all the notations above, and assume that
\[\langle \delta q_0, \delta v_0 \rangle / \|\delta q_0\|^2 \geq c_0 > 0.\]
We claim that \(\|\delta q_t\|/\|\delta q_0\| \geq 1 + c_0 t\) for all \(t \geq 0\).

Proof. Clearly, the function \(\|\delta q_t\|\) of \(t\) is continuous for all \(t \geq 0\) and continuously differentiable between collisions. According to (3.1), \(d/dt \delta q_t = \delta v_t\), so
\[
\frac{d}{dt} \|\delta q_t\|^2 = 2\langle \delta q_t, \delta v_t \rangle.
\]
Observe that not only the positive valued function \(Q(\delta q_t, \delta v_t) = \langle \delta q_t, \delta v_t \rangle\) is nondecreasing in \(t\) by (3.2) and (3.4), but the quantity \(\langle \delta q_t, \delta v_t \rangle / \|\delta q_t\|\) is nondecreasing in \(t\), as well. The reason is that \(\langle \delta q_t, \delta v_t \rangle / \|\delta q_t\| = \|\delta v_t\| \cos \alpha_t\) (\(\alpha_t\) being the acute angle subtended by \(\delta q_t\) and \(\delta v_t\)), and between collisions the quantity \(\|\delta v_t\|\) is unchanged, while the acute angle \(\alpha_t\) decreases, according to the time-evolution equations (3.1). Finally, we should keep in mind that at a collision the norm \(\|\delta q_t\|\) does not change, while \(\langle \delta q_t, \delta v_t \rangle\) cannot decrease, see (3.4). Thus we obtain the inequalities
\[
\langle \delta q_t, \delta v_t \rangle / \|\delta q_t\| \geq \langle \delta q_0, \delta v_0 \rangle / \|\delta q_0\| \geq c_0 \|\delta q_0\|,
\]
so
\[
\frac{d}{dt} \|\delta q_t\|^2 = 2\|\delta q_t\| \frac{d}{dt} \|\delta q_t\| = 2\langle \delta q_t, \delta v_t \rangle \geq 2c_0 \|\delta q_0\| \cdot \|\delta q_t\|
\]
by (3.6). This means that \(\frac{d}{dt} \|\delta q_t\| \geq c_0 \|\delta q_0\|\), so \(\|\delta q_t\| \geq \|\delta q_0\|(1 + c_0 t)\), proving the proposition. \(\square\)

Next we need an effective lower estimation \(c_0\) for the curvature \(\langle \delta q_0, \delta v_0 \rangle / \|\delta q_0\|^2\) of the trajectory bundle:

Lemma 3.7. Assume that the perturbation \((\delta q_0^-, \delta v_0^-) \in T_{x_0} M\) (as in Proposition 3.5) is being performed at time zero right before a collision, say, \(\sigma_0 = (1, 2)\) taking place at that time. Select the tangent vector \((\delta q^-_0, \delta v^-_0)\) in such a specific way that \(\delta q^-_0 = (m_2 w, -m_1 w, 0, 0, \ldots, 0)\) with a nonzero vector \(w \in \mathbb{R}^\nu\), \(\langle w, v^-_1 - v^-_2 \rangle = 0\). This scalar product equation is exactly the condition that guarantees that \(\delta q^-_0\) be orthogonal to the velocity component \(v^- = (v^-_1, v^-_2, \ldots, v^-_N)\) of \(x_0 = (q, v^-)\). The next, though crucial requirement is that \(w\) should be selected from the two-dimensional plane spanned by \(v^-_1 - v^-_2\) and \(q_1 - q_2\) (with \(||q_1 - q_2|| = 2r\) in \(\mathbb{R}^\nu\). The purpose of this condition is to avoid the unwanted phenomenon of “astigmatism” in our billiard system, discovered first by Bunimovich and Rehacek in [B-R(1997)] and [B-R(1998)]. Later on the phenomenon of astigmatism gathered further prominence.
in the paper [B-Ch-Sz-T(2002)] as the main driving mechanism behind the wild
non-differentiability of the singularity manifolds (at their boundaries) in hard ball
systems in dimensions bigger than 2. Finally, the last requirement is that the
velocity component \( \delta v_0^- \) (right before the collision (1,2)) is chosen in such a way
that the tangent vector \((\delta q_0^-, \delta v_0^-)\) belongs to the unstable space \(E^u(x_0)\) of \(x_0\). This
can be done, indeed, by taking \(\delta v_0^- = B^u(x_0)[\delta q_0^-]\), where \(B^u(x_0)\) the curvature
operator of the unstable manifold of \(x_0\) at \(x_0\), right before the collision (1,2) taking
place at time zero. Note that the argument \(\delta q_0^-\) of \(B^u(x_0)\) clearly belongs to the
positive subspace of the non-negative operator \(B^u(x_0)\).

We claim that

\[
\frac{\langle \delta q_0^+, \delta v_0^+ \rangle}{||\delta q_0||^2} \geq \frac{||v_1 - v_2||}{r \cos \phi_0} \geq \frac{||v_1 - v_2||}{r}
\]

for the post-collision tangent vector \((\delta q_0^+, \delta v_0^+)\), where \(\phi_0\) is the acute angle subtended by \(v_1^+ - v_2^+\) and the outer normal vector of the sphere \(\{ y \in \mathbb{R}^r \mid ||y|| = 2r \}\) at the point \(y = q_1 - q_2\). Note that in (3.8) there is no need to use + or − in \(||\delta q_0||^2\)
or \(||v_1 - v_2||\), for \(||\delta q_0|| = ||\delta q_0^+||, ||v_1^- - v_2^-|| = ||v_1^+ - v_2^+||\).

**Proof.** The proof of the equation in (3.8) is a simple, elementary geometric
argument in the plane spanned by \(v_1^- - v_2^-\) and \(q_1 - q_2\), so we omit it. We only note
that the outgoing relative velocity \(v_1^+ - v_2^+\) is obtained from the pre-collision relative
velocity \(v_1^- - v_2^-\) by reflecting the latter one across the tangent hyperplane of the
sphere \(\{ y \in \mathbb{R}^r \mid ||y|| = 2r \}\) at the point \(y = q_1 - q_2\). It is a useful advice, though,
to prove the first inequality of (3.8) in the case \(\delta v_0^- = 0\) first (this is the elementary
gometry exercise), then observe that this inequality can only be further improved
when we replace \(\delta v_0^- = 0\) with \(\delta v_0^- = B^u(x_0)[\delta q_0^-]\). \(\square\)

The previous lemma shows that, in order to get useful lower estimates for the
“curvature” \(\langle dq, dv \rangle/||dq||^2\) of the trajectory bundle, it is necessary (and sufficient)
to find collisions \(\sigma = (i, j)\) on the orbit of a given point \(x_0 \in M\) with a “relatively
big” value of \(||v_i - v_j||\). Finding such collisions will be based upon the following result:

**Proposition 3.9.** Consider orbit segments \(S^{[0,T]}x_0\) of \(N\)-ball systems with masses
\(m_1, m_2, \ldots, m_N\) in \(\mathbb{T}^r\) (or in \(\mathbb{R}^r\)) with collision sequences \(\Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)\) corresponding to connected collision graphs. (Now the kinetic energy is not necessarily
normalized, and the total momentum \(\sum_{i=1}^N m_i v_i\) may be different from zero.) We
claim that there exists a positive-valued function \(f(a;m_1, m_2, \ldots, m_N)\) \((a > 0, f\) is independent of the orbit segments \(S^{[0,T]}x_0\) with the following two properties:

1. If \(||v_i(t_l) - v_j(t_l)|| \leq a\) for all collisions \(\sigma_l = (i, j)\) \((1 \leq l \leq n, t_l\) is the time
of \(\sigma_l\) of some trajectory segment \(S^{[0,T]}x_0\) with a symbolic collision sequence \(\Sigma =\)
For \((\sigma_1, \sigma_2, \ldots, \sigma_n)\) corresponding to a connected collision graph, then the norm \(||v_i(t) - v_j(t)||\) of any relative velocity at any time \(t \in \mathbb{R}\) is at most \(f(a; m_1, \ldots, m_N)\); 

\[\lim_{a \to 0} f(a; m_1, \ldots, m_N) = 0\] for any \((m_1, \ldots, m_N)\).

**Proof.** We begin with

**Lemma 3.10.** Consider an \(N\)-ball system with masses \(m_1, \ldots, m_N\) (an \((m_1, \ldots, m_N)\)-system, for short) in \(\mathbb{T}^\nu\) (or in \(\mathbb{R}^\nu\)). Assume that the inequalities \(||v_i(0) - v_j(0)|| \leq a\) hold true \((1 \leq i < j \leq N)\) for all relative velocities at time zero. We claim that

\[(3.11) \quad ||v_i(t) - v_j(t)|| \leq 2a\sqrt{\frac{M}{m}}\]

for any pair \((i, j)\) and any time \(t \in \mathbb{R}\), where \(M = \sum_{i=1}^{N} m_i\) and 

\[m = \min \{m_i \mid 1 \leq i \leq N\}.\]

**Proof.** The assumed inequalities directly imply that \(||v'_i(0)|| \leq a\) \((1 \leq i \leq N)\) for the velocities \(v'_i(0)\) measured at time zero in the baricentric reference system. Therefore, for the total kinetic energy \(E_0\) (measured in the baricentric system) we get the upper estimation \(E_0 \leq \frac{1}{2}Ma^2\), and this inequality remains true at any time \(t\). This means that all the inequalities \(||v'_i(t)||^2 \leq \frac{M}{m_i}a^2\) hold true for the baricentric velocities \(v'_i(t)\) at any time \(t\), so

\[||v'_i(t) - v'_j(t)|| \leq a\sqrt{M\left(m_i^{-1/2} + m_j^{-1/2}\right)} \leq 2a\sqrt{\frac{M}{m}},\]

thus the inequalities 

\[||v_i(t) - v_j(t)|| \leq 2a\sqrt{\frac{M}{m}}\]

hold true, as well. \(\Box\)

**Proof of the proposition by induction on the number \(N\).**

For \(N = 1\) we can take \(f(a; m_1) = 0\), and for \(N = 2\) the function \(f(a; m_1, m_2) = a\) is obviously a good choice for \(f\). Let \(N \geq 3\), and assume that the orbit segment \(S^{[0,T]}x_0\) of an \((m_1, \ldots, m_N)\)-system fulfills the conditions of the proposition. Let \(\sigma_k = (i, j)\) be the collision in the symbolic sequence \(\Sigma_n = (\sigma_1, \ldots, \sigma_n)\) of \(S^{[0,T]}x_0\) with the property that the collision graph of \(\Sigma_k = (\sigma_1, \ldots, \sigma_k)\) is connected, while
Indeed, select a number non-zero tangent vector \((\delta q, \delta v)\) for which \(||v_i(0)| - v_j(t)|| \leq G(m_1,\ldots,m_N)\) at time zero. The normalization \(\sum_{i=1}^N m_i v_i = 0, \frac{1}{2} \sum_{i=1}^N m_i||v_i||^2 = \frac{1}{2}\). We claim that there exists a threshold \(G = G(m_1,\ldots,m_N) > 0\) (depending only on \(N, m_1,\ldots,m_N\)) with the following property:

In any orbit segment \(S^{[0,T]}x_0\) of the \((m_1,\ldots,m_N)\)-system with the standard normalizations and with a connected collision graph, one can always find a collision \(\sigma = (i,j)\), taking place at time \(t\), so that \(||v_i(t) - v_j(t)||| \geq G(m_1,\ldots,m_N)\).

**Proof.** Indeed, we choose \(G = G(m_1,\ldots,m_N) > 0\) so small that \(f(G; m_1,\ldots,m_N) < M^{-1/2}\). Assume, contrary to 3.12, that the norm of any relative velocity \(v_i - v_j\) of any collision of \(S^{[0,T]}x_0\) is less than the above selected value of \(G\). By the proposition, we have the inequalities \(||v_i(0) - v_j(0)||| \leq f(G; m_1,\ldots,m_N)\) at time zero. The normalization \(\sum_{i=1}^N m_i v_i(0) = 0\), with a simple convexity argument, implies that \(||v_i(0)||| \leq f(G; m_1,\ldots,m_N)\) for all \(i, 1 \leq i \leq N\), so the total kinetic energy is at most \(\frac{1}{2} M [f(G; m_1,\ldots,m_N)]^2 < \frac{1}{2}\), a contradiction. \(\square\)

**Corollary 3.13.** For any phase point \(x_0\) with a non-singular backward trajectory \(S^{(-\infty,0)}x_0\) and with infinitely many connected collision graphs on \(S^{(-\infty,0)}x_0\), and for any number \(L > 0\) one can always find a time \(-t < 0\) and a non-zero tangent vector \((\delta q_t, \delta v_t)\) in \(E^u(x_0)\) \((x_0, t = S^{-t}x_0)\) with \(||\delta q_t|| / ||\delta q_0|| > L\), where \((\delta q, \delta v) = D S^t(\delta q_0, \delta v_0) \in E^u(x_0)\).

**Proof.** Indeed, select a number \(t > 0\) so big that \(1 + \frac{1}{2} G(m_1,\ldots,m_N) > L\) and \(-t\) is the time of a collision (on the orbit of \(x_0\)) with the relative velocity \(v_i^-(t) - v_j^-(t)\), for which \(||v_i^-(t) - v_j^-(t)||| \geq G(m_1,\ldots,m_N)\). By Lemma 3.7 we can choose a
non-zero tangent vector \((\delta q_0^-, \delta v_0^-) \in E^u(x_{-t})\) right before the collision at time \(-t\) in such a way that the lower estimate

\[
\frac{\langle \delta q_0^+^-, \delta v_0^+^+ \rangle}{||\delta q_0^+||^2} \geq \frac{1}{r} G(m_1, \ldots, m_N)
\]

holds true for the “curvature” \(\langle \delta q_0^+^-, \delta v_0^+^+ \rangle/||\delta q_0^+||^2\) associated with the post-collision tangent vector \((\delta q_0^+, \delta v_0^+)\). According to Proposition 3.5, we have the lower estimate

\[
\frac{||\delta q_t||}{||\delta q_0||} \geq 1 + \frac{t}{r} G(m_1, \ldots, m_N) > L
\]

for the \(\delta q\)-expansion rate between \((\delta q_0^-, \delta v_0^-)\) and \((\delta q_t, \delta v_t) = DS^t(\delta q_0^-, \delta v_0^-)\). \(\square\)

We remind the reader that, according to the main result of [B-F-K(1998)], there exists a number \(\epsilon_0 = \epsilon_0(m_1, \ldots, m_N; r; \nu) > 0\) and a large threshold \(N_0 = N_0(m_1, \ldots, m_N; r; \nu) \in \mathbb{N}\) such that in the \((m_1, \ldots, m_N; r; \nu)\)-billiard flow amongst any \(N_0\) consecutive collisions one can always find two neighboring ones separated in time by at least \(\epsilon_0\). Thus, for a phase point \(x_{-t}\) at least \(\epsilon_0/2\)-away from collisions, the norms \(||\delta q_0||\) and \(\sqrt{||\delta q_0||^2 + ||\delta v_0||^2}\) are equivalent for all vectors \((\delta q_0, \delta v_0) \in E^u(x_{-t})\), hence we immediately get

**Corollary 3.14.** For any phase point \(x_0 \in M \setminus \partial M\) with a non-singular backward trajectory \(S^{(-\infty,0)}x_0\) and with infinitely many consecutive, connected collision graphs on \(S^{(-\infty,0)}x_0\), and for any number \(L > 0\) one can always find a time \(-t < 0\) and a non-zero tangent vector \((\delta q_0, \delta v_0) \in E^u(x_{-t})\) \((x_{-t} = S^{-t}x_0)\) with

\[
\frac{||\langle \delta q_t, \delta v_t \rangle||}{||\langle \delta q_0, \delta v_0 \rangle||} > L,
\]

where \((\delta q_t, \delta v_t) = DS^t(\delta q_0, \delta v_0) \in E^u(x_0)\). \(\square\)

The time-reversal dual of the previous result is immediately obtained by replacing the phase point \(x_0 = (q_0, v_0)\) with \(-x_0 = (q_0, -v_0)\), the backward orbit with the forward orbit, and the unstable vectors with the stable ones. We formulate this dual as our

**Theorem.** For any phase point \(x_0 \in M \setminus \partial M\) with a non-singular forward orbit \(S^{(0,\infty)}x_0\) and with infinitely many consecutive, connected collision graphs on \(S^{(0,\infty)}x_0\), and for any number \(L > 0\) one can always find a time \(t > 0\) and a non-zero tangent vector \((\delta q_0, \delta v_0) \in E^s(x_0)\) with

\[
\frac{||\langle \delta q_t, \delta v_t \rangle||}{||\langle \delta q_0, \delta v_0 \rangle||} < L^{-1},
\]

where \((\delta q_t, \delta v_t) = DS^t(\delta q_0, \delta v_0) \in E^s(x_t), x_t = S^tx_0\). \(\square\)
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