Invertible Extractors and Wiretap Protocols

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Abstract

A wiretap protocol is a pair of randomized encoding and decoding functions such that knowledge of a bounded fraction of the encoding of a message reveals essentially no information about the message, while knowledge of the entire encoding reveals the message using the decoder. In this paper we study the notion of efficiently invertible extractors and show that a wiretap protocol can be constructed from such an extractor. We will then construct invertible extractors for symbol-fixing, affine, and general sources and apply them to create wiretap protocols with asymptotically optimal trade-offs between their rate (ratio of the length of the message versus its encoding) and resilience (ratio of the observed positions of the encoding and the length of the encoding). We will then apply our results to create wiretap protocols for challenging communication problems, such as active intruders who change portions of the encoding, network coding, and intruders observing arbitrary boolean functions of the encoding.

Keywords: Wiretap Channel, Extractors, Network Coding, Active Intrusion, Exposure Resilient Cryptography.

1 Introduction

Suppose that Alice wants to send a message to Bob through a communication channel, and that the message is partially observable by an intruder. This scenario arises in various practical situations. For instance, in a packet network, the sequence transmitted by Alice through the channel can be fragmented into small packets at the source and/or along the way and different packets might be routed through different paths in the network in which an intruder may have compromised some of the intermediate routers. An example that is similar in spirit is furnished by transmission of a piece of information from multiple senders to one receiver, across different delivery media, such as satellite, wireless, and/or wired networks. Due to limited resources, a potential intruder may be able to observe only a fraction of the lines of transmission, and hence only partially observe the message. As another example, one can consider secure storage of data on a distributed medium that is physically accessible in parts by an intruder, or a sensitive file on a hard drive that is erased from the file system but is only partially overwritten with new or random information, and hence, is partially exposed to a malicious party.

An obvious approach to solve this problem is to use a secret key to encrypt the information at the source. However, almost all practical cryptographic techniques are shown to be secure only under unproven hardness assumptions and the assumption that the intruder possesses bounded computational power. This might be undesirable in certain situations. In the problem we consider, we assume the intruder to be information theoretically limited, and our goal will be to employ this
limitation and construct a protocol that provides unconditional, information-theoretic security, even in the presence of a computationally unbounded adversary.

The problem described above was first formalized by Wyner [50] and subsequently by Ozarow and Wyner [36] as an information-theoretic problem. In its most basic setting, this problem is known as the wiretap II problem (the description given here follows from [36]). Consider a communication system with a source which outputs a sequence $X = (X_1, \ldots, X_m)$ in $F^m_2$ uniformly at random. A randomized algorithm, called the encoder, maps the output of the source to a binary string $Y \in F^n_2$. The output of the encoder is then sent through a noiseless channel (called the direct channel) and is eventually delivered to a decoder $D$ which maps $Y$ back to $X$. Along the way, an intruder arbitrarily picks a subset $S \subseteq [n]$ of size $t \leq n$, and is allowed to observe $Z := Y|_S$ (through a so-called wiretap channel), i.e., $Y$ on the coordinate positions corresponding to the set $S$. The goal is to make sure that the intruder learns as little as possible about $X$, regardless of the choice of $S$.

The security of the system is defined by the conditional entropy $\Delta := \min_{|S| = t} H(X|Z)$. When $\Delta = m$, the intruder obtains no information about the transmitted message and we have perfect privacy in the system. Moreover, when $\Delta \to m$ as $m \to \infty$, we call the system asymptotically perfectly private.

Remark 1. The assumption that $X$ is sampled from an i.i.d. and uniform random source should not be confused with the fact that Alice is transmitting one particular message to Bob that is fixed and known to her before the transmission. In this case, the randomness of $X$ in the model captures the a priori uncertainty about $X$ for the outside world, and in particular the intruder, but not the transmitter. As an intuitive example, suppose that a random key is agreed upon between Alice and a trusted third party, and now Alice wishes to securely send her particular key to Bob over a wiretapped channel. Or, assume that Alice wishes to send an audio stream to Bob that is encoded and compressed using a conventional audio encoding method. Furthermore, the particular choice of the distribution on $X$ as a uniformly random sequence will cause no loss of generality.

If the distribution of $X$ is publicly known to be non-uniform, the transmitter can use a suitable source-coding scheme to compress the source to its entropy prior to the transmission, and ensure that from the intruder’s point of view, $X$ is uniformly distributed. On the other hand, it is also easy to see that if a protocol achieves perfect privacy under uniform message distribution, it achieves perfect privacy under any other distribution as well.

1.1 Our Model

The model that we will be considering is motivated by the original wiretap channel problem but is more stringent in terms of its security requirements. In particular, instead of using Shannon entropy as a measure of uncertainty, we will rely on statistical indistinguishability which is a stronger measure that is more widely used in cryptography.

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\footnote{We will use the following notation: We denote the set $\{1, \ldots, n\}$ by $[n]$. For a vector $x = (x_1, \ldots, x_n)$ and a subset $S \subseteq [n]$, we denote by $x|_S$ the projection of $x$ onto the coordinate positions given by elements of $S$. If $X$ and $Y$ are random variables, then $X|_S(Y = y)$ is the random variable $X$ conditioned on the event $Y = y$. We use $U_S$ to denote the uniform distribution on the set $S$, $U_n$ as a short-hand for $U_{F^n_2}$, and $X \sim U_S$ to denote a random variable with distribution $U_S$. Two distributions $A$ and $B$ are called $\epsilon$-close, in symbols $A \sim_\epsilon B$, if their statistical distance is at most $\epsilon$. All the necessary background for this work is presented in detail in Appendix A.}

\footnote{Ozarow and Wyner also consider the case in which the decoder errs with negligible probability, but we are going to consider only error-free decoders.}
\textbf{Definition 2.} Let $\Sigma$ be a set of size $q$, $m$ and $n$ be positive integers, and $\epsilon, \gamma > 0$. A $(t, \epsilon, \gamma)_q$-resilient wiretap protocol of block length $n$ and message length $m$ is a pair of functions \( E: \Sigma^m \times \mathbb{F}_2^q \rightarrow \Sigma^n \) (the encoder) and \( D: \Sigma^n \rightarrow \Sigma^m \) (the decoder) that are computable in time polynomial in $m$, such that

\begin{itemize}
  \item[(a)] (Decodability) For all $x \in \Sigma^m$ and all $z \in \mathbb{F}_2^r$ we have $D(E(x, z)) = x$,
  \item[(b)] (Resiliency) Let $X \sim \mathcal{U}_{\Sigma^m}$, $R \sim \mathcal{U}_r$, and $Y = E(X, R)$. For a set $S \subseteq [n]$ and $w \in \Sigma^{|S|}$, let $X_{S,w}$ denote the distribution of $X$ conditioned on the event $Y|_S = w$. Define the set of \textit{bad observations} as $B_S := \{ w \in \Sigma^{|S|} \mid \text{dist}(X_{S,w}, \mathcal{U}_{\Sigma^{|S|}}) > \epsilon \}$. Then we require that for every $S \subseteq [n]$ of size at most $t$, $\Pr[Y|_S \in B_S] \leq \gamma$.
\end{itemize}

The \textit{encoding} of a vector $x \in \Sigma^k$ is accomplished by choosing a vector $Z \in \mathbb{F}_2^r$ uniformly at random, and calculating $E(x, Z)$. The quantities $R = m/n$, $\epsilon$, and $\gamma$ are called the rate, the error, and the \textit{leakage} of the protocol, respectively. By a slight abuse of notation, we call $\delta = t/n$ the \textit{(relative) resilience} of the protocol.

In our definition the imperfection of the protocol is captured by the two parameters $\epsilon$ and $\gamma$. When $\epsilon = \gamma = 0$, the above definition coincides with the original wiretap channel problem for the case of perfect privacy. When $\gamma = 0$, we will have a \textit{worst-case} guarantee, namely, that the intruder’s views of the message before and after his observation are statistically close, \textit{regardless} of the outcome of the observation. When $\gamma > 0$, a particular observation might potentially reveal to the intruder a lot of information about the message. However, a negligible $\gamma$ will ensure that such a bad event (or \textit{leakage}) happens only with negligible probability. All the constructions in this paper achieve zero leakage (i.e., $\gamma = 0$), except for the general result in subsection 5.3 for which a nonzero leakage is inevitable. The significance of zero-leakage protocols is that they assure \textit{adaptive} resiliency in the \textit{weak} sense introduced in [13] for exposure-resilient functions: if the intruder is given the encoded sequence as an oracle that he can adaptively query at up to $t$ coordinates and is afterwards presented with a challenge which is either the original message or an independent uniformly chosen random string, he will not be able to distinguish between the two cases.

In general, it is straightforward to verify that our model can be used to solve the original wiretap II problem, with $\Delta \geq m(1 - \epsilon - \gamma)$. This is proved in Appendix B.1. Hence, we will achieve asymptotically perfect privacy when $\epsilon + \gamma = o(1/m)$. For all the protocols that we present in this paper this quantity will be superpolynomially small.

\subsection{1.2 Related Notions in Cryptography}

There are several interrelated notions in the literature on Cryptography and Theoretical Computer Science that are also closely related to our definition of the wiretap protocol. These are \textit{resilient functions} (RF) and \textit{almost perfect resilient functions} (APRF), \textit{exposure-resilient functions} (ERF), and \textit{all-or-nothing transforms} (AONT) (cf. [9] [18] [39] [44] [17] [8] [27] and [11] for a comprehensive account of several important results in this area).

The notion of resilient functions was introduced in [3] (and also [49] as the \textit{bit-extraction problem}). A deterministic polynomial-time computable function $f: \mathbb{F}_2^q \rightarrow \mathbb{F}_2^n$ is called $t$-resilient if whenever any $t$ bits of the its input are arbitrarily chosen by an adversary and the rest of the bits are chosen uniformly at random, then the output distribution of the function is (close to) uniform. ERFs, introduced in [3], are similar to resilient functions except that the entire input is chosen...
uniformly at random, and the view of the adversary from the output remains (close to) uniform even after observing any $t$ input bits of his choice. ERFs and resilient functions are known to be useful in a scenario similar to the wiretap channel problem where the two parties aim to agree on any random string, for example a session key (Alice generates $x$ uniformly at random which she sends to Bob, and then they agree on the string $f(x)$). Here no control on the content of the message is required, and the only goal is that at the end of the protocol the two parties agree on any random string that is uniform even conditioned on the observations of the intruder. Hence, our definition of a wiretap protocol is more stringent than that of resilient functions, since it requires the existence and efficient computability of the encoding function $E$ that provides a control over the content of the message (see Remark 1).

Another closely related notion is that of all-or-nothing transforms, which was suggested in [39] for protection of block ciphers. A randomized polynomial-time computable function $f : \mathbb{F}_2^m \rightarrow \mathbb{F}_2^n$, $(m \leq n)$, is called a (statistical, non-adaptive, and secret-only) $t$-AONT with error $\epsilon$ if it is efficiently invertible and for every $S \subseteq [n]$ such that $|S| \leq t$, and all $x_1, x_2 \in \mathbb{F}_2^n$ we have that the two distributions $f(x_1)|_S$ and $f(x_2)|_S$ are $\epsilon$-close. An AONT with $\epsilon = 0$ is called perfect. It is easy to see that perfectly private wiretap protocols are equivalent to perfect adaptive AONTs. It was shown in [13] that such functions can not exist (with positive, constant rate) when the adversary is allowed to observe more than half of the encoded bits. A similar result was obtained in [9] for the case of perfect linear RFs.

As pointed out in [13], AONTs can be used in the original scenario of Ozarow and Wyner’s wiretap channel problem. However, the best known constructions of AONTs can achieve rate-resilience trade-offs that are far from the information theoretic optimum (see Figure 1). While an AONT requires indistinguishability of intruder’s view for every fixed pair $(x_1, x_2)$ of messages, the relaxed notion of average-case AONT requires the expected distance of $f(x_1)|_S$ and $f(x_2)|_S$ to be at most $\epsilon$ for a uniform random message pair. Hence, for a negligible $\epsilon$, the distance will be negligible for all but a negligible fraction of message pairs. Up to a loss in parameters, wiretap protocols are equivalent to average case AONTs:

**Lemma 3.** Let $(E, D)$ be an encoding/decoding pair for a $(t, \epsilon, \gamma)_2$-resilient wiretap protocol. Then $E$ is an average-case $t$-AONT with error at most $2(\epsilon + \gamma)$. Conversely, an average-case $t$-AONT with error $\eta^2$ can be used as a $(t, \eta, \eta)$-resilient wiretap encoder. \qed

A proof of the above lemma is given in Appendix B.2. Note that the converse direction does not guarantee zero leakage, hence, zero leakage wiretap protocols are in general stronger than average-case AONTs. An average-case to worst-case reduction for AONTs was shown in [8] which, combined with the above lemma, shows that any wiretap protocol can be used to construct an AONT.

A simple universal transformation was proposed in [8] to obtain an AONT from any ERF, by one-time padding the message with a random string obtained from the ERF. This construction can also yield a wiretap protocol with zero leakage. However, it has the drawback of significantly weakening the rate-resilience trade-off. Namely, even if an information theoretically optimal ERF is used in this reduction, the resulting wiretap protocol will only achieve half the optimal rate (see Figure 1).

Another concept that is close to our work is that of privacy amplification [2, 4, 33, 24]. Similar to resilient functions, the goal here is for Alice and Bob to agree on a common secret share, but the intruder is given more power. This is compensated by the fact that Alice and Bob have access
Figure 1: A comparison of the rate vs. resilience trade-offs achieved by the wiretap protocols for the binary alphabet (left) and larger alphabets (right, in this example of size 64). (1) Information theoretic bound, attained by Theorem 17; (2) The bound approached by 27; (3) Protocol based on best nonexplicit binary linear codes 20 48; (4) AONT construction of 8, assuming that the underlying ERF is optimal; (5) Random walk protocol of Corollary 8; (6) Protocol based on the best known explicit 47 and non-explicit 20 48 linear codes.

to a secure channel not available to the intruder over which they can transmit securely some bits.

The main focus of this paper is on asymptotic trade-offs between the rate $R$ and the resilience $δ$ of an asymptotically perfectly private wiretap protocol. Following 36, it is easy to see that in this case, an information-theoretic bound $R ≤ 1 − δ + o(1)$ must hold. Lower bounds for $R$ in terms of $δ$ have been studied by a number of researchers. For the case of perfect privacy ($ε = 0$), Ozarow and Wyner 36 give a construction of a wiretap protocol using linear error-correcting codes, and show that the existence of an $[n, k, d]_q$-code implies the existence of a perfectly private, $(d − 1, 0, 0)_q$-resilient wiretap protocol of message length $n − k$ and block length $n$. As a result, the Gilbert-Varshamov bound on linear codes 20 48 implies that asymptotically $R ≥ 1 − h_q(δ)$, where $h_q$ is the $q$-ary entropy function (defined in Appendix A). If $q ≥ 49$ is a square, the bound can be further improved to $R ≥ 1 − δ − 1/(\sqrt{q} − 1)$ using Goppa’s AG-codes 21 47. In these protocols, the encoder can be seen as an adaptively secure, perfect AONTs and the decoder is an adaptive perfect RF. Moving away from perfect to asymptotically perfect privacy, it was shown in 27 that for any $γ > 0$ there exist binary asymptotically perfectly private wiretap protocols with $R ≥ 1 − 2δ − γ$ and exponentially small error$^3$. This bound strictly improves the coding theoretic bound of Ozarow and Wyner for the binary alphabet.

$^3$Actually, what is proved in this paper is the existence of $t$-resilient functions which correspond to decoders in our wiretap setting; however, it can be shown that these functions also possess efficient encoders, so that it is possible to construct wiretap protocols from them.
1.3 Overview of our Results

In this paper we prove several lower bounds for the rate of asymptotically perfectly private wiretap protocols with negligible, i.e., superpolynomially small, error. Our main tool is the design of various types of invertible extractors, a concept defined in Section 2. Our first bound, described in Section 3 shows that if the alphabet size is \(d\), then there exists \(\alpha_d \in (0, 1)\) such that for every \(\eta > 0\) and every constant resilience \(\delta \in [0, 1)\), we have rate \(R \geq \max\{\alpha_d(1 - \delta), 1 - \delta/\alpha_d\} - \eta\) with exponentially small error. This is achieved by suitably modifying the symbol-fixing extractor of Kamp and Zuckerman \[23\]. Contrary to the coding theoretic construction of Ozarow and Wyner, for a fixed alphabet size our bound gives a positive rate for every constant resilience \(\delta \in [0, 1)\).

Even though the bound in Section 3 is superceded by our main result in Section 4, we have included it because of its simplicity and potential for practical use. Our second bound (Theorem 17) matches the information-theoretic upper bound of Ozarow and Wyner. Namely, for any prime power alphabet size \(q\), and any resilience \(\delta \in [0, 1)\), we construct a wiretap protocol with superpolynomially small error, zero leakage and rate \(\geq 1 - \delta - o(1)\). In fact, this bound holds in a more general setting in which the intruder is not only allowed to look at a \(\delta\)-fraction of the symbols of Alice’s message, but is also allowed to perform any linear preprocessing of Alice’s message before doing so. The power of this result stems largely from a black box transformation which makes certain seedless extractors invertible. More specifically, the results of this section are obtained by applying this transformation to certain affine extractors.

In sections 5.1 and 5.2 we will demonstrate several important applications of this fact in the context of network coding and wiretapped communication in the presence of noise and active intruders. In particular we provide, for the first time, an optimal solution to the wiretap problem in network coding \[7\] without imposing any restrictive assumptions. A plot of the bounds can be found in Figure 1.

The final application in Section 5.3 studies an all-powerful intruder who is only limited by the amount of information he can obtain from Alice’s encoded message, and not by the nature of the observations. By inverting seeded extractors with nearly-optimal output lengths, we will show that if Alice and Bob have access to a side channel over which Alice can publicly send a polylogarithmic number of bits to Bob, then their communication on the main channel can be made secure even if the intruder can access the values of any \(t\) Boolean functions of Alice’s encoded message.

2 Inverting Extractors

In this section we will introduce the notion of invertible extractors and its connection with wiretap protocols\[3\]. Later we will use this connection to construct wiretap protocols with good rate-resilience trade-offs.

**Definition 4.** Let \(\Sigma\) be a finite alphabet and \(f\) be a mapping from \(\Sigma^n\) to \(\Sigma^m\). For \(\gamma \geq 0\), a function \(A: \Sigma^m \times \mathbb{F}_2^\gamma \to \Sigma^n\) is called a \(\gamma\)-inverter for \(f\) if the following conditions hold:

\[\text{Another notion of invertible extractors was introduced in } \text{[12]} \text{ and used in } \text{[14]} \text{ for a different application (entropic security) that should not be confused with the one we use. Their notion applies to seeded extractors with long seeds that are efficiently invertible bijections for every fixed seed. Such extractors can be seen as a single-step walk on highly expanding graphs that mix in one step. This is in a way similar to the multiple-step random walk used in the seedless extractor of section } \text{[3]} \text{ that can be regarded as a single-step walk on the expander graph raised to a certain power.} \]
(a) (Inversion) Given \( x \in \Sigma^m \) such that \( f^{-1}(x) \) is nonempty, and for every \( z \in \mathbb{F}_2^r \), we have \( f(A(x, z)) = x \).

(b) (Uniformity) \( A(\mathcal{U}_{\Sigma^m}, \mathcal{U}_r) \sim_{\gamma} \mathcal{U}_{\Sigma^n} \).

A \( \gamma \)-inverter is called efficient if there is a randomized algorithm that runs in worst case polynomial time and, given \( x \in \Sigma^m \) and \( z \) as a random seed, computes \( A(x, z) \). We call a mapping \( \gamma \)-invertible if it has an efficient \( \gamma \)-inverter, and drop the prefix \( \gamma \) from the notation when it is zero.

Remark 5. If a function \( f \) maps the uniform distribution to a distribution that is \( \epsilon \)-close to uniform (as is the case for all extractors), then any randomized mapping that maps its input \( x \) to a distribution that is \( \gamma \)-close to the uniform distribution on \( f^{-1}(x) \) is easily seen to be an \( (\epsilon + \gamma) \)-inverter for \( f \). In some situations designing such a function might be easier than directly following the above definition.

The idea of random pre-image sampling was proposed in [13] for construction of adaptive AONTs from APRFs. However, they ignored the efficiency of the inversion, as their goal was to show the existence of (not necessarily efficient) information-theoretically optimal adaptive AONTs. Moreover, the strong notion of APRF and a perfectly uniform sampler is necessary for their construction of AONTs. As wiretap protocols are weaker than (worst-case) AONTs, they can be constructed from slightly imperfect inverters as shown by the following lemma who proof can be found in Appendix B.3.

Lemma 6. Let \( \Sigma \) be an alphabet of size \( q > 1 \) and \( f : \Sigma^n \to \Sigma^m \) be a \( (\gamma^2/2) \)-invertible \( q \)-ary \((k, \epsilon)\) symbol-fixing extractor. Then, \( f \) and its inverter can be seen as a decoder/encoder pair for an \((n - k, \epsilon + \gamma, \gamma)\_q\)-resilient wiretap protocol with block length \( n \) and message length \( m \). \( \square \)

3 A Wiretap Protocol Based on Random Walks

In this section we describe a wiretap protocol that achieves a rate \( R \) within a constant fraction of the information theoretically optimal value \( 1 - \delta \) (the constant depending on the alphabet size). We will use preliminaries from Appendix A.2.

To achieve our result, we will modify the symbol-fixing extractor of Kamp and Zuckerman [23] to make it efficiently invertible without affecting its extraction properties, and then apply Lemma 6 above to obtain the desired wiretap protocol. The extractor of [23] starts with a fixed vertex in a large expander graph and interprets the input as the description of a walk on the graph. Then it outputs the label of the vertex reached at the end of the walk. Notice that a direct approach to invert this function will amount to sampling a path of a particular length between a pair of vertices in the graph, uniformly among all the possibilities, which might be a difficult problem for good families of expander graphs. We work around this problem by choosing the starting point of the walk from the input sequence. In particular we show the following:

Theorem 7. Let \( G \) be a constructible \( d \)-regular graph with \( d^m \) vertices and second largest eigenvalue \( \lambda_G \geq 1/\sqrt{d} \). Then there exists an explicit invertible \((k, 2s/2)_d\) symbol-fixing extractor \( \text{SFExt} : [d]^n \to \)

\footnote{The idea of choosing the starting point of the walk from the input sequence has been used before in extractor constructions [52], but in the context of seeded extractors for general sources with high entropy.}
such that
\[
s := \begin{cases} 
  m \log d + k \log \lambda^2 \quad & \text{if } k \leq n - m, \\
  (n - k) \log d + (n - m) \log \lambda^2 \quad & \text{if } k > n - m.
\end{cases}
\]

Proof. (Sketch) We first describe the extractor and its inverse. Given an input \((v, w) \in [d]^m \times [d]^{n-m}\), the function \text{SFExt} interprets \(v\) as a vertex of \(G\) and \(w\) as the description of a walk starting from \(v\). The output is the index of the vertex reached at the end of the walk. The inverter \text{Inv} works as follows: Given \(x \in [d]^m\), \(x\) is interpreted as a vertex of \(G\). Then \text{Inv} picks \(W \in [d]^{n-m}\) uniformly at random. Let \(V\) be the vertex starting from which the walk described by \(W\) ends up in \(x\). The inverter outputs \((V, W)\). It is easy to verify that \text{Inv} satisfies the properties of a 0-inverter. The proof that \text{SFExt} is an extractor with the given parameters is similar to the original proof of Kamp and Zuckerman, but takes our modifications into account. We defer the details to Appendix B.4.

Combining this with Lemma 6 and setting up the the right asymptotic parameters, we obtain our protocol for the wiretap channel problem.

Corollary 8. Let \(\delta \in [0,1)\) and \(\gamma > 0\) be arbitrary constants, and suppose that there is a constructible family of \(d\)-regular expander graphs with spectral gap at least \(1 - \lambda\), for a constant \(\lambda < 1\). Moreover suppose that there exists a constant \(c \geq 1\) such that for every large enough \(N\) the family contains a graph with at least \(N\) and at most \(cN\) vertices. Then for every large enough \(n\) there is a \((\delta n, 2^{-\Omega(n)}, 0)\)-\(d\)-resilient wiretap protocol with block length \(n\) and rate \(R = \max\{\alpha(1-\delta), 1-\delta/\alpha\} -\gamma\), where \(\alpha := -\log d \lambda^2\).

Proof. (Sketch) For the case \(c = 1\) we use Lemma 6 with the extractor \text{SFExt} of Theorem 7 and its inverse. Every infinite family of graphs must satisfy \(\lambda \geq 2\sqrt{d-1}/d\) \cite{35}, and in particular we have \(\lambda \geq 1/\sqrt{d}\), as required by Theorem 7. We choose the parameters \(k := (1-\delta)n\) and \(m := n(\max\{\alpha(1-\delta), 1-\delta/\alpha\} -\gamma)\), which gives \(s = -\Omega(n)\), and hence, exponentially small error. The case \(c > 1\) is similar, but involves technicalities for dealing with lack of graphs of arbitrary size in the family. We will elaborate on this in Appendix B.5.

Using explicit constructions of Ramanujan graphs that achieve \(\lambda \leq 2\sqrt{d-1}/d\) when \(d - 1\) is a prime power \cite{31, 33, 37}, one can obtain \(\alpha \geq 1 - 2/\log d\), which can be made arbitrarily close to one (hence, making the protocol arbitrarily close to the optimal bound) by choosing a suitable alphabet size that does not depend on \(n\).

Remark 9. The invertible symbol-fixing extractor above can be used to construct an invertible bit-fixing extractor with exponentially small error by the partitioning method introduced in \cite{23}. Kamp and Zuckerman used their extractor as an APRF in the one-time pad construction of \cite{8} to obtain an adaptively secure statistical AONT. We note that using the invertible bit-fixing extractor as an APRF, there will be no need for the one-time pad technique and one can directly obtain an adaptive AONT (which is simply the corresponding inverter) with a smaller blow-up compared to \cite{23}.


4 Invertible Affine Extractors and Asymptotically Optimal Wiretap Protocols

In this section we will construct a black box transformation for making certain seedless extractors invertible. The method is described in detail for affine extractors, and leads to wiretap protocols with asymptotically optimal rate-resilience trade-offs.

A seeded extractor is called linear if it is a linear function for every fixed choice of the seed. Trevisan [46] gave a fascinating explicit construction of strong linear extractors based on pseudorandom generators from hard Boolean functions. His construction was later improved by Raz, Reingold and Vadhan [38]. Trevisan’s extractor (and its subsequent improvement) can be viewed as a careful puncturing of an $F_2$-linear error-correcting code, where the puncturing depends on the choice of the seed. This immediately implies the linearity of the extractor. In this work, we will use the following result.

**Theorem 10.** [38] There is an explicit strong linear seeded $(k, \epsilon)$-extractor $\text{Ext} : F_2^n \times F_2^d \to F_2^m$ with $d = O(\log^3(n/\epsilon))$ and $m = k - O(d)$.

**Remark 11.** We note that our arguments would identically work for any other linear seeded extractor as well, for instance those constructed in [45, 42]. However, the most crucial parameter in our application is the output length of the extractor, being closely related to the rate of the wiretap protocols we obtain. Among the constructions we are aware of, the result quoted in Theorem 10 is the best in this regard.

A recent result by Shaltiel [41] gives a general framework for transforming every seedless extractor (for a family of sources satisfying a certain closedness condition) with short output length to one with an almost optimal output length. The construction uses the imperfect seedless extractor to extract a small number of uniform random bits from the source, and will then use the resulting sequence as the seed for a seeded extractor to extract more random bits from the source. For a suitable choice of the seeded extractor, one can use this construction to extract almost all min-entropy of the source. The closedness condition needed for this result to work for a family $C$ of sources is that, letting $E(x, s)$ denote the seeded extractor with seed $s$, for every $X \in C$ and every fixed $s$ and $y$, the distribution $(X|E(X, s) = y)$ belongs to $C$. If $E$ is a linear function for every fixed $s$ (as Trevisan’s extractor is), the result will be available for affine sources (since we are imposing a linear constraint on an affine source, it remains an affine source). A more precise statement of Shaltiel’s main result is the following:

**Theorem 12.** [41] Let $C$ be a class of distributions on $F_2^n$ and $F : F_2^n \to F_2^t$ be an extractor for $C$ with error $\epsilon$. Let $E : F_2^n \times F_2^t \to F_2^m$ be a function for which $C$ satisfies the closedness condition above. Then for every $X \in C$, $E(X, F(X)) \sim_{\epsilon, 2^{-3}} E(X, U_t)$.

The explicit affine extractor that we will use for this presentation is the following result of Bourgain [5]:

**Theorem 13.** [5] For every constant $0 < \delta < 1$, there is an explicit affine extractor $\text{AExt} : F_2^n \to F_2^m$ for min-entropy $\delta n$ with output length $m = \Omega(n)$ and error at most $2^{-\Omega(m)}$.

Now, having these tools in hand, we are ready to describe our construction of invertible affine extractors with nearly optimal output length.
**Theorem 14.** For every constant \( \delta \in (0, 1) \) and every \( \alpha \in (0, 1) \), there is an explicit invertible affine extractor \( D: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m \) for min-entropy \( \delta n \) with output length \( m = \delta n - O(n^{\alpha}) \) and error at most \( O(2^{-n^{\alpha/3}}) \).

**Proof.** (Sketch) Let \( \epsilon := 2^{-n^{\alpha/3}} \) and \( t := O(\log^3(n/\epsilon)) = O(n^{\alpha}) \) be the seed length required by the extractor \( \text{Ext} \) in Theorem 10 for input length \( n \) and error \( \epsilon \), and further, let \( n' := n - t \). Set up \( \text{Ext} \) for input length \( n' \), min-entropy \( \delta n - t \), seed length \( t \) and error \( \epsilon \). Also set up Bourgain’s extractor \( \text{AExt} \) for input length \( n' \) and entropy rate \( \delta' \), for an arbitrary constant \( \delta' < \delta \). Then the function \( F \) will view the \( n \)-bit input sequence as a tuple \((s, x)\), \( s \in \mathbb{F}_2^t \) and \( x \in \mathbb{F}_2^{n'} \), and outputs \( \text{Ext}(x, s + \text{AExt}(x))|_q \).

First we show that this is an affine extractor. Suppose that \((S, X) \in \mathbb{F}_2^t \times \mathbb{F}_2^{n'}\) is a random variable sampled from an affine distribution with min-entropy \( \delta n \). The variable \( S \) can have an affine dependency on \( X \). Hence, for every fixed \( s \in \mathbb{F}_2^t \), the distribution of \( X \) conditioned on the event \( S = s \) is affine with min-entropy at least \( \delta n - t \), which is at least \( \delta' n' \) for large enough \( n \). Hence \( \text{AExt}(X) \) will be \( 2^{-\Omega(n)} \)-close to uniform by Theorem 13. This implies that \( \text{AExt}(X)|_q + S \) can extract \( t \) random bits from the affine source with error \( 2^{-\Omega(n)} \). Combining this with Theorem 12 noticing the fact that the class of affine extractors is closed with respect to linear seeded extractors, we conclude that \( D \) is an affine extractor with error at most \( \epsilon + 2^{-\Omega(n)} \cdot 2^{t+3} = O(2^{-n^{\alpha/3}}) \).

Now the inverter works as follows: Given \( y \in \mathbb{F}_2^m \), first it picks \( Z \in \mathbb{F}_2^t \) uniformly at random. The seeded extractor \( \text{Ext} \), given the seed \( Z \) is a linear function \( \text{Ext}_Z: \mathbb{F}_2^t \rightarrow \mathbb{F}_2^m \). Without loss of generality, assume that this function is surjective.\(^6\) Then the inverter picks \( X \in \mathbb{F}_2^{n'} \) uniformly at random from the affine subspace defined by the linear constraint \( \text{Ext}_Z(X) = y \), and outputs \((Z + \text{AExt}(X)|_q, X)\). It is easy to verify that the output is indeed a valid preimage of \( y \). To see the uniformity of the inverter, note that if \( y \) is chosen uniformly at random, the distribution of \((Z, X)\) will be uniform on \( \mathbb{F}_2^t \). Hence \((Z + \text{AExt}(X)|_q, X)\), which is the output of the inverter, will be uniform. \( \square \)

**Remark 15.** In the above construction we are using Bourgain’s extractor as a black box and hence, it can be replaced by an arbitrary affine extractor working for an arbitrary field size (however, depending on the particular affine extractor being used, we may need to adapt the parameters given above accordingly). In particular, over large fields one can use the affine extractor given by Gabizon and Raz \( [19] \) that works for sub-constant entropy rates as well\(^7\).

**Remark 16.** Bourgain presented his affine extractor for the most challenging underlying field, namely, the binary field. However, his argument can be adapted to work for larger fields as well \( [6] \). Hence, one can obtain invertible affine extractors for arbitrary finite fields.

An affine extractor is in particular, a symbol-fixing extractor. Hence the above theorem, combined with Lemma \( [6] \) gives us a wiretap protocol with almost optimal parameters:

**Theorem 17.** Let \( \delta \in [0, 1) \) and \( \alpha \in (0, 1/3) \) be constants. Then for a prime power \( q > 1 \) and every large enough \( n \) there is a \((\delta n, O(2^{-n^{\alpha}}), 0)q\)-resilient wiretap protocol with block length \( n \) and rate \( 1 - \delta - o(1) \). \( \square \)

\(^6\)Because the seeded extractor is strong and linear, for most choices of the seed it is a good extractor, and hence necessarily surjective. Hence if \( \text{Ext} \) is not surjective for some seed \( z \), one can replace it by a trivial surjective linear mapping without affecting its extraction properties.

\(^7\)Using this extractor, we can achieve the best parameters by combining it with the seeded extractor for affine extractors over large fields that is constructed in the same paper.
5 Further Applications

In this section we will sketch some important applications of our technique to more general wiretap problems.

5.1 Noisy Channels and Active Intruders

Suppose that Alice wants to transmit a particular sequence to Bob through a noisy channel. She can use various techniques from coding theory to encode her information and protect it against noise. Now what if there is an intruder who can partially observe the transmitted sequence and even manipulate it? Modification of the sequence by the intruder can be regarded in the same way as the channel noise; thus one gets security against active intrusion as a bonus by constructing a code that is resilient against noise and passive eavesdropping. There are two natural and modular approaches to construct such a code. A possible attempt would be to first encode the message using a good error-correcting code and then applying a wiretap encoder to protect the encoded sequence against the wiretapper. However, this will not necessarily keep the information protected against the channel noise, as the combination of the wiretap encoder and decoder does not have to be resistant to noise. Another attempt is to first use a wiretap encoder and then apply an error-correcting code on the resulting sequence. Here it is not necessarily the case that the information will be kept secure against intrusion anymore, as the wiretapper now gets to observe the bits from the channel-encoded sequence that may reveal information about the original sequence. However, the wiretap protocol given in Theorem 17 is constructed from an invertible affine extractor, and guarantees resiliency even if the intruder is allowed to observe arbitrary linear combinations of the transmitted sequence. Hence, we can use the second approach with our protocol and still ensure privacy against an active intruder, provided that the error-correcting code is linear. This immediately gives us the following result:

Theorem 18. Suppose that there is a $q$-ary linear error-correcting code with rate $r$ that is able to correct up to a $\tau$ fraction of errors (via unique or list decoding). Then for every constant $\delta \in [0, 1)$ and $\alpha \in (0, 1/3)$ and large enough $n$, there is a $(\delta n, O(2^{-n\alpha}), 0)$-resilient wiretap protocol with block length $n$ and rate $r - \delta - o(1)$ that can also correct up to a $\tau$ fraction of errors.

The same idea can be used to protect fountain codes, e.g., LT- [32] and Raptor Codes [43], against wiretappers without affecting the error correction capabilities of the code. Obviously this simple composition idea can be used for any type of channel so long as the inner code is linear, at the cost of reducing the total rate by almost $\delta$. Hence, if the inner code achieves the capacity of the direct channel (in the absence of the wiretapper), the composed code will achieve the capacity of the wiretapped channel, which is less than the original capacity by $\delta$ [10].

5.2 Network Coding

Our wiretap protocol from invertible affine extractors is also applicable in the more general setting of transmission over networks. A communication network can be modeled as a directed graph, in which nodes represent the network devices and information is transmitted along the edges. One particular node is identified as the source and $m$ nodes are identified as receivers. The main problem in network coding is to have the source reliably transmit information to the receivers at the highest possible rate, while allowing the intermediate nodes arbitrarily process the information along the
way. Suppose that the min-cut from the source to each receiver is \( n \). It was shown in [1] that the source can transmit information up to rate \( n \) to all receivers, and in [28, 25] that linear network coding is in fact sufficient to achieve this rate (See [51] for a comprehensive account of these and other relevant results).

Designing wiretap protocols for networks is an important question in network coding, which was first posed by Cai and Yeung [7]. In this problem, an intruder can choose a bounded number, say \( t \), of the edges and eavesdrop all the packets going through those edges. They designed a network code that could provide the optimal multicast rate of \( n - t \) with perfect privacy. However, this code requires an alphabet size of order \( (|E|) \), where \( E \) is the set of edges. Their result was later improved in [16] who showed that a random linear coding scheme can provide privacy with a much smaller alphabet size if one is willing to achieve a slightly sub-optimal rate. Namely, they obtain rate \( n - t(1 + \epsilon) \) with an alphabet of size roughly \( \Theta(|E|^{1/\epsilon}) \), and show that achieving the exact optimal rate is not possible with small alphabet size. El Rouayheb and Soljanin [15] suggested to use the original code of Ozarow and Wyner [36] as an outer code at the source and showed that a careful choice of the network code can provide optimal rate with perfect privacy. However, their code eventually needs an alphabet of size at least \( (\frac{1}{t+1}) + m \). Building upon this work, Silva and Kschischang [26] constructed an outer code that provides similar results while leaving the underlying network code unchanged. However, their protocol imposes the assumption that the packets in the network are transmitted in bundles of length at least \( n \), (hence, one needs to have an estimate of the min-cut size of the network), which bounds the intruder to observe the same set of links within each bundle, practically enlarging the field size from \( q \) to at least \( q^n \).

Using the protocol given in Theorem 17 as an outer-code in the source node, one can construct an asymptotically optimal wiretap protocol for networks that is completely unaware of the network and eliminates all the restrictions in the above results. Hence, extending our notion of \( (t, \epsilon, \gamma) \)-resilient wiretap protocols naturally to communication networks, we obtain the following:

**Theorem 19.** Let \( \delta \in [0, 1) \) and \( \alpha \in (0, 1/3) \) be constants, and consider a network that uses a linear coding scheme over a finite field \( \mathbb{F}_q \) for reliably transmitting information at rate \( n \), for \( n \) large enough\(^8\). Then the source and the receiver nodes can use an outer code of rate \( 1 - \delta - o(1) \) which is completely independent of the network, leaves the network code unchanged, and provides resilience \( \delta \) with error \( O(2^{-n^\alpha}) \) and zero leakage over a \( q \)-ary alphabet.

### 5.3 Arbitrary Processing

In this section we consider the erasure wiretap problem in its most general setting, which is still of practical importance. Suppose that the information emitted by the source goes through an arbitrary communication medium and is arbitrarily processed on the way to provide protection against noise, to obtain better throughput, or for other reasons. Now consider an intruder who is able to eavesdrop a bounded amount of information at various points of the channel. One can model this scenario in the same way as the original point-to-point wiretap channel problem, with the difference that instead of observing \( t \) arbitrarily chosen bits, the intruder now gets to choose an arbitrary circuit \( C \) with \( t \) output bits (which captures the accumulation of all the intermediate bits).
processing) and observes the output of the circuit when applied to the transmitted sequence. Obviously there is no way to guarantee resiliency in this setting, since the intruder can simply choose \( C \) to compute \( t \) output bits of the wiretap decoder. However, suppose that in addition there is an auxiliary communication channel between the source and the receiver (that we call the side channel) that is separated from the main channel, and hence, the information passed through the two channel do not blend together by intermediate processings.

We call this scenario the general wiretap problem, and extend our notion of \((t, \epsilon, \gamma)\)-resilient protocol to this problem, with the slight modification that now the output of the encoder (and the input of the decoder) is a pair of strings \((y_1, y_2) \in \mathbb{F}_2^n \times \mathbb{F}_2^d\), where \( y_1 \) \((y_2)\) is sent through the main \( \text{(side)} \) channel. Now we call \( n + d \) the block length and let the intruder choose an arbitrary pair of circuits \((C_1, C_2)\), one for each channel, that output a total of \( t \) bits and observe \((C_1(y_1), C_2(y_2))\).

The information-theoretic upper bounds for the achievable rates in the original wiretap problem obviously extend to the general wiretap problem as well. Below we show that for the general problem, secure transmission is possible at asymptotically optimal rates even if the intruder intercepts the entire communication passing through the side channel.

Similar as before, our idea is to use invertible extractors to construct general wiretap protocols, but this time we use invertible strong seeded extractors. Strong seeded extractors were used in [8] to construct ERFs, and this is exactly what we use as the decoder in our protocol. As the encoder we will use the corresponding inverter, which outputs a pair of strings, one for the extractor’s input which is sent through the main channel and another as the seed which is sent through the side channel. Hence we will obtain the following result, which is proved in Appendix B.6:

**Theorem 20.** Let \( \delta \in [0, 1) \) be a constant. Then for every \( \alpha, \epsilon > 0 \), there is a \((\delta n, \epsilon, 2^{-\alpha n} + \epsilon)\)-resilient wiretap protocol for the general wiretap channel problem that needs to send \( n \) bits through the main channel and \( d := O(\log^3(n/\epsilon)) \) bits through the side channel and achieves rate \( 1 - \delta - \alpha - O(d/(n + d)) \). The protocol is secure even when the entire communication through the side channel is observable by the intruder.

The above theorem uses the linear seeded extractor of Theorem 10 to achieve almost perfect resiliency and approaches the optimal rate using only \( O(\log^3 n) \) bits of side communication. It is possible to drop this amount down to \( O(\log n) \) bits by inverting existing constructions of seeded extractors that work with seeds of length \( O(\log(n/\epsilon)) \) and extract almost the whole min-entropy (e.g., [31] [52] [22]). However, we defer this improvement to subsequent work.

We observe that it is not possible to guarantee zero leakage for the general wiretap problem above. Specifically, suppose that \((C_1, C_2)\) are chosen in a way that they have a single preimage for a particular output \((w_1, w_2)\). With nonzero probability the observation of the intruder may turn out to be \((w_1, w_2)\), in which case the entire message is revealed. Nevertheless, it is possible to guarantee negligible leakage as the above theorem does. Moreover, when the general protocol above is used for the original wiretap II problem (where there is no intermediate processing involved), there is no need for a separate side channel and the entire encoding can be transmitted through a single channel. Contrary to Theorem 17 however, the general protocol will not guarantee zero leakage even for this special case.

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9 In fact this models a harder problem, as in our problem the circuit \( C \) is given by the communication scheme and not the intruder. Nevertheless, we solve the harder problem.
References

[1] R. Ahlswede, N. Cai, S-Y. R. Li, and R. W. Yeung. Network information flow. IEEE Trans. Information Theory, 46(4):1204–1216, 2000.

[2] C. Bennett, G. Brassard, and J. M. Robert. Privacy amplification by public discussion. SIAM J. Comp., 17:210–229, 1988.

[3] C.H. Bennett, G. Brassard, and J-M. Robert. How to reduce your enemy’s information. In Proc. of CRYPTO, volume 218 of LNCS, page 468–476, 1985.

[4] Charles H. Bennett, Gilles Brassard, Claude Crépeau, and Ueli Maurer. Generalized privacy amplification. IEEE Trans. Information Theory, 41(6):1915–1923, 1995.

[5] J. Bourgain. On the construction of affine extractors. Geom. and Func. Analysis, 17(1):33–57, 2007.

[6] J. Bourgain. Personal Communication, March 2008.

[7] N. Cai and R. W. Yeung. Secure network coding. In Proc. of ISIT, 2002.

[8] R. Canetti, Y. Dodis, S. Halevi, E. Kushilevitz, and A. Sahai. Exposure-resilient functions and all-or-nothing transforms. In Proc. of Eurocrypt, volume 1807 of LNCS, page 453–469, 2000.

[9] B. Chor, O. Goldreich, J. Håstad, J. Friedmann, S. Rudich, and R. Smolensky. The bit extraction problem or t-resilient functions. In Proc. of FOCS, page 396–407, 1985.

[10] I. Csiszár and J. Körner. Broadcast channels with confidential messages. IEEE Trans. Information Theory, 24(3):339–348, 1978.

[11] Y. Dodis. Exposure-Resilient Cryptography. PhD thesis, MIT, 2000.

[12] Y. Dodis. On extractors, error-correction and hiding all partial information. In Proc. of ITW, 2005.

[13] Y. Dodis, A. Sahai, and A. Smith. On perfect and adaptive security in exposure-resilient cryptography. In Proc. of Eurocrypt, volume 2045 of LNCS, page 301–324, 2001.

[14] Y. Dodis and A. Smith. Entropic security and the encryption of high-entropy messages. In Proc. of TCC, volume 3378 of LNCS, page 556–577, 2005.

[15] S. Y. El Rouayheb and E. Soljanin. On wiretap networks II. In Proc. of ISIT, page 24–29, 2007.

[16] J. Feldman, T. Malkin, R. Servedio, and C. Stein. On the capacity of secure network coding. In Proc. of Allerton Conference, 2004.

[17] K. Friedl and S-C. Tsai. Two results on the bit extraction problem. Discrete Applied Math., 99, 2000.

[18] J. Friedmann. On the bit extraction problem. In Proc. of FOCS, page 314–319, 1992.
[19] A. Gabizon and R. Raz. Deterministic extractors for affine sources over large fields. In Proc. of FOCS, page 407–418, 2005.

[20] E. N. Gilbert. A comparison of signaling alphabets. Bell Sys. Tech. J., 31:504–522, 1952.

[21] V. D. Goppa. Codes on algebraic curves. Sov. Math. Dokl., 24:170–172, 1981.

[22] V. Guruswami, C. Umans, and S. Vadhan. Unbalanced expanders and randomness extractors from Parvaresh-Vardy codes. In Proc. of CCC, 2007.

[23] J. Kamp and D. Zuckerman. Deterministic extractors for bit-fixing sources and exposure-resilient cryptography. SIAM J. Comp., 36:1231–1247, 2006.

[24] Robert Koenig and Ueli Maurer. Generalized strong extractors and deterministic privacy amplification. In Crypt. and Coding, volume 3796 of LNCS, pages 322–339, 2005.

[25] R. Kötter and M. Médard. An algebraic approach to network coding. IEEE/ACM Trans. Networking, 11(5):782–795, 2003.

[26] F. R. Kschischang and D. Silva. Security for wiretap networks via rank-metric codes. arXiv: cs.IT/0801.0061, 2007.

[27] K. Kurosawa, T. Johansson, and D. Stinson. Almost k-wise independent sample spaces and their cryptologic applications. J. Cryptology, 14(4):231–253, 2001.

[28] S-Y. R. Li, R. W. Yeung, and N. Cai. Linear network coding. IEEE Trans. Information Theory, 49(2):371–381, 2003.

[29] L. Lovász. Random walks on graphs: A survey. Combinatorics, Paul Erdős is Eighty, Vol. 2, János Bolyai Math. Society, page 353–398, 1996.

[30] C. J. Lu, O. Reingold, S. Vadhan, and A. Wigderson. Extractors: Optimal up to constant factors. In Proc. of STOC, pages 602–611, 2003.

[31] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. Combinatorica, 8:261–277, 1988.

[32] M. Luby. LT-codes. In Proc. of FOCS, page 271–280, 2002.

[33] Ueli Maurer and Stefan Wolf. Privacy amplification secure against active adversaries. In Proc. CRYPTO, volume 1294 of LNCS, pages 307–321, 1997.

[34] M. Morgenstern. Existence and explicit constructions of $q + 1$ regular ramanujan graphs for every prime power $q$. J. Comb. Theory B, 62:44–62, 1994.

[35] A. Nilli. On the second eigenvalue of a graph. Discrete Math., 91:207–210, 1991.

[36] L. H. Ozarow and A. D. Wyner. Wire-tap channel II. Bell Labs Tech. J., 63:2135–2157, 1984.

[37] A. K. Pizer. Ramanujan graphs and Hecke operators. Bulletin of the AMS, 23(1):127–137, 1990.
[38] R. Raz, O. Reingold, and S. Vadhan. Extracting all the randomness and reducing the error in Trevisan’s extractor. *JCSS*, 65(1):97–128, 2002.

[39] R. Rivest. All-or-nothing encryption and the package transform. In *Proc. of Int. Workshop on Fast Software Encryption*, volume 1267 of *LNCS*, page 210–218, 1997.

[40] R. Shaltiel. Recent developments in explicit constructions of extractors. *Bulletin of the EATCS*, 77:67–95, 2002.

[41] R. Shaltiel. How to get more mileage from randomness extractors. In *Proc. of CCC*, page 46–60, 2006.

[42] R. Shaltiel and C. Umans. Simple extractors for all min-entropies and a new pseudorandom generator. *JACM*, 52(2):172–216, 2005.

[43] A. Shokrollahi. Raptor codes. *IEEE Trans. Information Theory*, 52:2551–2567, 2006.

[44] D. Stinson. Resilient functions and large set of orthogonal arrays. *Congressus Numerantium*, 92:105–110, 1993.

[45] A. Ta-Shma, D. Zuckerman, and S. Safra. Extractors from Reed-Muller codes. *JCSS*, 72:786–812, 2006.

[46] L. Trevisan. Extractors and pseudorandom generators. *JACM*, 48(4):860–879, 2001.

[47] M. A. Tsfasman, S. G. Vlăduţ, and T. Zink. Modular curves, Shimura curves, and Goppa codes better than the Varshamov-Gilbert bound. *Math. Nachrichten*, 109:21–28, 1982.

[48] R. R. Varshamov. Estimate of the number of signals in error correcting codes. *Dokl. Akademii Nauk SSSR*, 117:739–741, 1957.

[49] U.V. Vazirani. Towards a strong communication complexity theory or generating quasi-random sequences from two communicating semi-random sources. *Combinatorica*, 7(4):375–392, 1987.

[50] A. D. Wyner. The wire-tap channel. *The Bell Sys. Tech. J.*, 54:1355–1387, 1975.

[51] R. W. Yeung, S-Y. R. Li, N. Cai, and Z. Zhang. *Network Coding Theory*. Found. and Trends in Communications and Inf. Theory. Now Publishers, 2005.

[52] D. Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6):103–128, 2007.

### A Preliminaries and Basic Facts

For a prime power $q$, we use $\mathbb{F}_q$ to denote the finite field with $q$ elements. We will occasionally use the notation $\mathbb{F}_2$ for the set $\{0, 1\}$, even if we do not need to use the field structure. For a positive integer $n$, define $[n]$ as the set $\{1, 2, \ldots, n\}$. For a vector $x = (x_1, x_2, \ldots, x_n)$ and a subset $S \subseteq [n]$, we denote by $x|_S$ the vector of length $|S|$ that is obtained from $x$ by removing all the coordinates $x_i$, $i \notin S$. For an integer $k > 0$, we will use the notation $\mathcal{U}_k$ for the uniform distribution on $\mathbb{F}_2^k$. 

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More generally, for a finite set $\Omega$, we will use $U_\Omega$ for the uniform distribution on $\Omega$. For a function $f$, we denote by $f^{-1}(x)$ the set of the preimages of $x$, i.e., the set $\{y : f(y) = x\}$.

We denote the probability measure defined by a distribution $X$ by $\Pr_X$, hence, $\Pr_X(x)$ and $\Pr_X[S]$ for $x \in \Omega$ and $S \subseteq \Omega$ denote the probability that $X$ assigns to an outcome $x$ and an event $S$, respectively. We will use $X|S$ to denote the conditional distribution of $X$ restricted to the set (event) $S$, and $X \sim X$ to denote that a random variable $X$ is distributed according to $X$.

**Definition 21.** The **support** of a distribution is the set of all the elements of the sample space to which it assigns nonzero probabilities. The **min-entropy** of a distribution $X$ with finite support $S$ is defined as

$$H_\infty(X) := \min_{x \in S} \{-\log \Pr_X(x)\},$$

where typically $\log(\cdot)$ is the logarithm function in base 2. However, when $X$ is supported on a set of $d$-ary strings, we find it more convenient to use the logarithm function in base $d$ and measure the entropy in $d$-ary symbols instead of bits. The **Shannon** entropy of the distribution, on the other hand, is defined as

$$H(X) := \sum_{x \in S} (-\Pr_X(x) \log \Pr_X(x)).$$

When a distribution defined on the set of $n$-bit strings has min-entropy $k$, the quantity $k/n$ defines the **entropy rate** of the distribution. Note that the above definition immediately implies that the min-entropy of a distribution is upper bounded by its Shannon entropy (which is in fact the expectation of the logarithm of the probabilities). Hence, if the min-entropy of a distribution is at least $h$, its Shannon-entropy is also at least $h$. These two measures however coincide for uniform distributions.

**Definition 22.** The **statistical distance** (or total variation distance) of two distributions $X$ and $Y$ defined on the same finite space $S$ is given by

$$\frac{1}{2} \sum_{s \in S} |\Pr_X(s) - \Pr_Y(s)|,$$

and is denoted by $\text{dist}(X, Y)$. Note that this is half the $\ell_1$ distance of the two distributions when regarded as vectors of probabilities over $S$.

It can be shown that the statistical distance of the two distributions is at most $\epsilon$ if and only if for every $T \subseteq S$, we have $|\Pr_X[T] - \Pr_Y[T]| \leq \epsilon$. When the statistical distance of $X$ and $Y$ is at most $\epsilon$, we denote it by $X \sim_{\epsilon} Y$.

While we defined the above terms for probability distributions, with a slight abuse of notation we may use them interchangeably for random variables as well.

The following proposition quantifies the Shannon entropy of a distribution that is close to uniform:

**Proposition 23.** Let $X$ be a probability distribution on a finite set $S$, $|S| > 4$, that is $\epsilon$-close to the uniform distribution on $S$, for some $\epsilon \leq 1/4$. Then $H(X) \geq \log |S|(1 - \epsilon)$

**Proof.** Let $n := |S|$, and let $f(x) := -x \log x$. The function $f(x)$ is concave, passes through the origin and is strictly increasing in the range $[0, 1/e]$. From the definition, we have $H(X) =$
∑_{s ∈ S} f(Pr_{X}(s)). For each term s in this summation, the probability that X assigns to s is either at least 1/n, which makes the corresponding term at least log n/n (due to the particular range of |S| and ε), or is equal to 1/n − ε_s, for some ε_s > 0, in which case the term corresponding to s is less than log n/n by at most ε_s log n (this follows by observing that the slope of the line connecting the origin to the point (1/n, f(1/n)) is log n). The bound on the statistical distance implies that the differences ε_s add up to at most ε. Hence, the Shannon entropy of X can be less than log n by at most ε log n.

The following (easy to verify) proposition shows that any function maps close distributions to close distributions:

**Proposition 24.** Let Ω and Γ be finite sets and f be a function from Ω to Γ. Suppose that X and Y are probability distributions on Ω and Γ, respectively, and let X' be a probability distribution on Ω which is δ-close to X. Then if f(X) ∼_ε Y, then f(X') ∼_ε+δ Y.

We will use the following proposition in our construction of wiretap protocols from invertible extractors in Section 2:

The q-ary entropy function h_q (that is used in the introduction) is defined as

\[ h_q(x) := x \log_q(q - 1) - x \log_q(x) - (1 - x) \log_q(1 - x). \]

### A.1 Preliminaries on Extractors

Now we are ready to define the basic notions related to randomness extractors that we will use. We refer the reader to [40] for a more detailed account of these notions.

**Definition 25.** Let Σ be a finite set of size d > 1. An \((n,k)_d\) family of q-ary randomness sources of length n and min-entropy k is a set \(F\) of probability distributions on \(\Sigma^n\) such that every \(X ∈ F\) has d-ary min-entropy at least k.

There are numerous natural families of sources that have been introduced and studied in the theory of randomness extractors. In this work, besides the general family of distributions with high min-entropy, we will focus on the family of symbol-fixing and affine sources, defined below.

**Definition 26.** An \((n,k)_d\) symbol-fixing source is the distribution of a random variable \(X = (X_1, X_2, \ldots, X_n) ∈ \Sigma^n\), for some set Σ of size d, in which at least k of the coordinates (chosen arbitrarily) are uniformly and independently distributed on Σ and the rest take deterministic values.

When d = 2, we will have a binary symbol-fixing source, or simply a bit-fixing source. In this case Σ = \{0, 1\}, and the subscript d is dropped from the notation.

**Definition 27.** For a prime power \(q\), the family of q-ary k-dimensional affine sources of length n is the set of distributions on \(\mathbb{F}_q^n\), each uniformly distributed on an affine translation of some k-dimensional sub-space of \(\mathbb{F}_q^n\).

Affine sources are natural generalizations of symbol-fixing sources when the alphabet size is a prime power. It is easy to see that the q-ary min-entropy of a k-dimensional affine source is k.

**Definition 28.** A function \(E: \mathbb{F}_q^{n} × \mathbb{F}_2^{d} → \mathbb{F}_q^{n}\) is a strong seeded \((k,ε)\)-extractor if for every distribution \(X\) on \(\mathbb{F}_2^n\) with min-entropy at least k, random variable \(X ∼ X\) and a seed \(Y ∼ U_d\), the distribution of \((E(X,Y), Y)\) is ε-close to \(U_{m+d}\). An extractor is explicit if it is polynomial-time computable.
A strong extractor $E(x, y)$ for a source $X$ with error $\epsilon$ satisfies the property that for all but an $\epsilon$ fraction of the choices of the seed $y$, the distribution of $E(X, y)$ is $\epsilon$-close to uniform.

For more restricted sources (in particular, symbol-fixing and affine sources), seedless (or deterministic) extraction is possible.

**Definition 29.** Let $\Sigma$ be a finite alphabet of size $d > 1$. A function $E: \Sigma^n \rightarrow \Sigma^m$ is a (seedless) $(k, \epsilon)_d$-extractor for a family $F$ of $(n, k)_d$ sources (defined on $\Sigma^n$) if for every distribution $X \in F$ with min-entropy at least $k$, the distribution $E(X)$ is $\epsilon$-close to $U_{\Sigma^m}$. A seedless extractor is **explicit** if it is polynomial-time constructible.

### A.2 Preliminaries on Expander Graphs

For the wiretap protocol given in Section 3 we need essentially the same tools used for the symbol-fixing extractor construction of [23], that we briefly review here.

We will always work with directed regular expander graphs that are obtained from undirected graphs by replacing each undirected edge with two directed edges in opposite directions. Let $G = (V, E)$ be a $d$-regular graph. Then a labeling of the edges of $G$ is a function $L: V \times [d] \rightarrow V$ such that for every $u \in V$ and $t \in [d]$, $(u, L(u, t))$ is in $E$. The labeling is **consistent** if whenever $L(u, t) = L(v, t)$, then $u = v$. Note that the natural labeling of a Cayley graph is in fact consistent.

We call a family of expander graphs **constructible** if all the graphs in the family have a consistent labeling that is efficiently computable.

Let $A$ denote the normalized adjacency matrix of a $d$-regular graph $G$. We denote by $\lambda_G$ the second largest eigenvalue of $A$ in absolute value. The **spectral gap** of $G$ is given by $1 - \lambda_G$. Starting from a probability distribution $p$ on the set of vertices, represented as a real vector, performing a single-step random walk on $G$ leads to the distribution defined by $pA$. The following is a well known lemma on the convergence of the distributions resulting from random walks (cf. [29] for a proof):

**Lemma 30.** Let $G = (V, E)$ be a $d$-regular undirected graph, and $A$ be its normalized adjacency matrix. Then for any probability vector $p$, we have $\|pA - U_V\|_2 \leq \lambda_G \|p - U_V\|_2$, where $\|\cdot\|_2$ denotes the $\ell_2$ norm.

### B Proofs

#### B.1 Proof that Definition 2 Solves Wiretap II Problem

Suppose that $(E, D)$ is an encoder/decoder pair as we modeled in Definition 2. Let $W := Y|S$ be the intruder’s observation, and denote by $W'$ the set of good observations, namely,

$$W' := \{w \in \Sigma^t: \text{dist}(\mathcal{X}_{S, w}, U_{\Sigma^m}) \leq \epsilon\}.$$ 

Let $H(\cdot)$ denote the Shannon entropy in $d$-ary symbols. Then we will have

$$H(X|W) = \sum_{w \in \Sigma^t} \Pr(W = w)H(X|W = w) \geq \sum_{w \in W'} \Pr(W = w)H(X|W = w) \geq \sum_{w \in W'} \Pr(W = w)(1 - \epsilon)m \geq (1 - \gamma)(1 - \epsilon)m \geq (1 - \gamma - \epsilon)m.$$
The inequality (a) follows from the definition of $W'$ combined with Proposition 23 and (b) by the definition of leakage parameter.

**B.2 Proof of Lemma 3**

In this appendix we show that, up to a loss in error parameters, the notion of wiretap protocols in Definition 2 is equivalent to the notion of average-case AONTs. This easily follows from the following Proposition:

**Proposition 31.** Let $(X,Y)$ be a pair of random variables jointly distributed on a finite set $\Omega \times \Gamma$. Then

$$\mathbb{E}_Y[\operatorname{dist}(X|Y,X)] = \mathbb{E}_X[\operatorname{dist}(Y|X,Y)].$$

**Proof.** For $x \in \Omega$ and $y \in \Gamma$, we will use shorthands $p_x, p_y, p_{xy}$ to denote $\Pr[X = x], \Pr[Y = y], \Pr[X = x, Y = y]$, respectively. Then we have

$$\mathbb{E}_Y[\operatorname{dist}(X|Y,X)] = \sum_{y \in \Gamma} p_y \operatorname{dist}(X|(Y = y), X) = \frac{1}{2} \sum_{y \in \Gamma} p_y \sum_{x \in \Omega} |p_{xy}/p_y - p_x|$$

$$= \frac{1}{2} \sum_{y \in \Gamma} \sum_{x \in \Omega} |p_{xy} - p_x p_y| = \frac{1}{2} \sum_{x \in \Omega} p_x \sum_{y \in \Gamma} |p_{xy}/p_x - p_y|$$

$$= \sum_{x \in \Omega} p_x \operatorname{dist}(Y|(X = x), Y) = \mathbb{E}_X[\operatorname{dist}(Y|X,Y)].$$

Now, consider a $(t, \epsilon, \gamma)_q$-resilient wiretap protocol as in Definition 2 and accordingly, let the random variable $Y = E(X, R)$ denote the encoding of $X$ with a random seed $R$. For a set $S \subseteq [n]$ of size at most $t$, denote by $W := Y|_S$ the intruder’s observation. The resiliency condition implies that, the set of bad observations $B_S$ has a probability mass of at most $\gamma$ and hence, the expected distance $\operatorname{dist}(X|W,X)$ taken over the distribution of $W$ is at most $\epsilon + \gamma$. Now we can apply Proposition 31 above to the jointly distributed pair of random variables $(W,X)$, and conclude that the expected distance $\operatorname{dist}(W|X,W)$ over the distribution of $X$ (which is uniform) is at most $\epsilon + \gamma$. This implies that the encoder is an average-case $t$-AONT with error at most $2(\epsilon + \gamma)$. Conversely, the same argument combined with Markov’s bound shows that an average-case $t$-AONT with error $\eta^2$ can be seen as $(t, \eta, \eta)$-resilient wiretap protocol.

**B.3 Proof of Lemma 6**

Let $E$ and $D$ denote the wiretap encoder and decoder, respectively. Hence, $E$ is the $(\gamma^2/2)$-inverter for $f$, and $D$ is the extractor $f$ itself. From the definition of the inverter, for every $x \in \Sigma^n$ and every random seed $r$, we have $D(E(x, r)) = x$. Hence it is sufficient to show that the pair satisfies the resiliency condition. We will use the following Proposition in our proof:

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\[^{10} \text{Here we are abusing the notation and denote by } Y \text{ the marginal distribution of the random variable } Y, \text{ and by } Y|(X = a) \text{ the distribution of the random variable } Y \text{ conditioned on the event } X = a. \]
Proposition 32. Let $\Omega$ be a finite set that is partitioned into subsets $S_1, \ldots, S_k$ and suppose that $\mathcal{X}$ is a distribution on $\Omega$ that is $\gamma$-close to uniform. Denote by $p_i$, $i = 1, \ldots, k$, the probability assigned to the event $S_i$ by $\mathcal{X}$. Then

$$\sum_{i \in [k]} p_i \cdot \text{dist}(\mathcal{X}|S_i, U_{S_i}) \leq 2\gamma.$$

Proof. Let $N := |\Omega|$, and define for each $i$, $\gamma_i := \sum_{s \in S_i} |\Pr_\mathcal{X}(s) - \frac{1}{N}|$, so that $\gamma_1 + \cdots + \gamma_k \leq 2\gamma$. Observe that by triangle’s inequality, for every $i$ we must have $|p_i - |S_i|/N| \leq \gamma_i$. To conclude the claim, it is enough to show that for every $i$, we have $\text{dist}(\mathcal{X}|S_i, U_{S_i}) \leq \gamma_i/p_i$. This is shown in the following.

$$p_i \cdot \text{dist}(\mathcal{X}|S_i, U_{S_i}) = \frac{p_i}{2} \sum_{s \in S_i} \left| \frac{\Pr_\mathcal{X}(s)}{p_i} - \frac{1}{|S_i|} \right|$$

$$= \frac{1}{2} \sum_{s \in S_i} \left| \frac{\Pr_\mathcal{X}(s) - p_i}{|S_i|} \right|$$

$$= \frac{1}{2} \sum_{s \in S_i} \left( \frac{\Pr_\mathcal{X}(s) - 1}{N} \right) + \frac{1}{2|S_i|} \left( \frac{|S_i|}{N} - p_i \right)$$

$$\leq \frac{1}{2} \sum_{s \in S_i} \left| \Pr_\mathcal{X}(s) - \frac{1}{N} \right| + \frac{1}{2|S_i|} \sum_{s \in S_i} \left| \frac{|S_i|}{N} - p_i \right|$$

$$\leq \frac{\gamma_i}{2} + \frac{1}{2|S_i|} \cdot |S_i| \gamma_i = \gamma_i.$$

Now, let the random variable $X$ be uniformly distributed on $\Sigma^k$ and the seed $R \in \mathbb{F}_2^r$ be chosen uniformly at random. Denote the encoding of $X$ by $Y := E(X, R)$. Fix any $S \subseteq [n]$ of size at most $n - k$.

For every $w \in \Sigma^{|S|}$, let $Y_w$ denote the set $\{y \in \Sigma^n : y|_S = w\}$. Note that the sets $Y_w$ partition the space $\Sigma^n$ into $|\Sigma^{|S|}|$ disjoint sets.

Let $\mathcal{Y}$ and $\mathcal{Y}_S$ denote the distribution of $Y$ and $Y|_S$, respectively. The inverter guarantees that $\mathcal{Y}$ is $(\gamma^2/2)$-close to uniform. Applying Proposition 32, we get that

$$\sum_{w \in \Sigma^{|S|}} \Pr[Y|_S = w] \cdot \text{dist}(\mathcal{Y}|Y_w, U_{Y_w}) \leq \gamma^2.$$

The left hand side is the expectation of $\text{dist}(\mathcal{Y}|Y_w, U_{Y_w})$. Denote by $W$ the set of all bad outcomes of $Y|_S$, i.e., $W := \{w \in \Sigma^{|S|} \mid \text{dist}(\mathcal{Y}|Y_w, U_{Y_w}) > \gamma\}$. By Markov’s inequality, we conclude that

$$\Pr[Y|_S \in W] \leq \gamma.$$

For every $w \in W$, the distribution of $Y$ conditioned on the event $Y|_S = w$ is $\gamma$-close to a symbol-fixing source with $n - |S| \geq k$ random symbols. The fact that $D$ is a symbol-fixing extractor for this entropy and Proposition 24 imply that, for any such $w$, the conditional distribution of $D(Y)|Y_S = w$ is $(\gamma + \epsilon)$-close to uniform. Hence with probability at least $1 - \gamma$ the distribution of $X$ conditioned on the outcome of $Y|_S$ is $(\gamma + \epsilon)$-close to uniform. This ensures the resiliency of the protocol.
B.4 Omitted Details of the Proof of Theorem 7

In this section we show that SFExt, as defined in Theorem 7, is a symbol-fixing extractor. Let $(x, w) \in [d]^m \times [d]^{n-m}$ be a vector sampled from an $(n, k)_d$ symbol-fixing source, and let $u := \text{SFExt}(x, w)$. Recall that $u$ can be seen as the vertex of $G$ reached at the end of the walk described by $w$ starting from $x$. Let $p_i$ denote the probability vector corresponding to the walk right after the $i$th step, for $i = 0, \ldots, n - m$, and denote by $p$ the uniform probability vector on the vertices of $G$. Our goal is to bound the error $\epsilon$ of the extractor, which is half the $\ell_1$ norm of $p_{n-m} - p$.

Suppose that $x$ contains $k_1$ random symbols and the remaining $k_2 := k - k_1$ random symbols are in $w$. Then $p_0$ has the value $d^{-k_1}$ at $d^{k_1}$ of the coordinates and zeros elsewhere, hence

$$\|p_0 - p\|_2^2 = d^{k_1}(d^{-k_1} - d^{-m})^2 + (d^m - d^{k_1})d^{-2m} = d^{-k_1} - d^{-m} \leq d^{-k_1}.$$ 

Now for each $i \in [n - m]$, if the $i$th step of the walk corresponds to a random symbol in $w$ the $\ell_2$ distance is multiplied by $\lambda_G$ by Lemma 30. Otherwise the distance remains the same due to the fact that the labeling of $G$ is consistent. Hence we obtain $\|p_{n-m} - p\|_2^2 \leq d^{-k_1} \lambda_G^{2k_2}$. Translating this into the $\ell_1$ norm by using the Cauchy-Schwarz inequality, we obtain $\epsilon$, namely,

$$\epsilon \leq \frac{1}{2}d^{m-k_1}/2 \lambda_G^{k_2} \leq 2^{((m-k_1)\log d + k_2 \log \lambda_G^2)/2}.$$ 

By our assumption, $\lambda_G \geq 1/\sqrt{d}$. Hence, everything but $k_1$ and $k_2$ being fixed, the above bound is maximized when $k_1$ is minimized. When $k \leq n - m$, this corresponds to the case $k_1 = 0$, and otherwise to the case $k_1 = k - n + m$. This gives us the desired upper bound on $\epsilon$. \hfill \box

B.5 Omitted Details of the Proof of Corollary 8

We prove Corollary 8 for the case $c > 1$. The construction is similar to the case $c = 1$, and in particular the choice of $m$ and $k$ will remain the same. However, a subtle complication is that the expander family may not have a graph with $d^m$ vertices and we need to adapt the extractor of Theorem 7 to support our parameters, still with exponentially small error. To do so, we pick a graph $G$ in the family with $N$ vertices, such that $c^m d^m \leq N \leq c^{m+1} d^m$, for a small absolute constant $\eta > 0$ that we are free to choose. The assumption on the expander family guarantees that such a graph exists. Let $m'$ be the smallest integer such that $d^{m'} \geq c^{mN}$. Index the vertices of $G$ by integers in $[N]$. Note that $m'$ will be larger than $m$ by a constant multiplicative factor that approaches 1 as $\eta \to 0$.

For positive integers $q, p \leq q$, define the function $\text{Mod}_{q,p} : [q] \to [p]$ by $\text{Mod}_{q,p}(x) := 1 + (x \mod p)$. The extractor SFExt interprets the first $m'$ symbols of the input as an integer $u$, $0 \leq u < d^{m'}$ and performs a walk on $G$ starting from the vertex $\text{Mod}_{d^{m'},N}(u + 1)$, the walk being defined by the remaining input symbols. If the walk reaches a vertex $v$ at the end, the extractor outputs $\text{Mod}_{N,d^{m'}}(v) - 1$, encoded as a $d$-ary string of length $m$. A similar argument as in Theorem 7 can show that with our choice of the parameters, the extractor has an exponentially small error, where the error exponent is now inferior to that of Theorem 7 by $O(m)$, but the constant behind $O(\cdot)$ can be made arbitrarily small by choosing a sufficiently small $\eta$.

The real difficulty lies with the inverter because $\text{Mod}$ is not a balanced function (that is, all images do not have the same number of preimages), thus we will not be able to obtain a perfect inverter. Nevertheless, it is possible to construct an inverter with a close-to-uniform output in $\ell_\infty$.

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norm. This turns out to be as good as having a perfect inverter, and thanks to the following lemma, we will still be able to use it to construct a wiretap protocol with zero leakage:

**Lemma 33.** Suppose that \( f : [d]^n \to [d]^m \) is a \((k, 2^{-\Omega(m)})_d\) symbol-fixing extractor and that \( \mathcal{X} \) is a distribution on \([d]^n\) such that \( \| \mathcal{X} - \mathcal{U}_d|n\|_\infty \leq 2^{-\Omega(m)/d^n} \). Denote by \( \mathcal{X}' \) the distribution \( \mathcal{X} \) conditioned on any fixing of at most \( n - k \) coordinates. Then \( f(\mathcal{X}') \sim_{2^{-\Omega(m)}} \mathcal{U}_d|n\).

**Proof.** By Proposition 24, it suffices to show that \( \mathcal{X}' \) is \( 2^{-\Omega(m)} \)-close to an \((n, k)_d\) symbol-fixing source. Let \( S \subseteq [d]^m \) denote the support of \( \mathcal{X}' \), and let \( \epsilon/d^n \) be the \( \ell_\infty \) distance between \( \mathcal{X} \) and \( \mathcal{U}_d|n\), so that by our assumption, \( \epsilon = 2^{-\Omega(m)} \). By the bound on the \( \ell_\infty \) distance, we know that \( \Pr_X(S) \) is between \( \frac{|S|}{d^n}(1 - \epsilon) \) and \( \frac{|S|}{d^n}(1 + \epsilon) \). Hence for any \( x \in S \), \( \Pr_{\mathcal{X}'(x)} \), which is \( \Pr_{\mathcal{X}(x)} / \Pr_S(S) \), is between \( \frac{1}{|S|} \cdot \frac{1 + \epsilon}{1 + \epsilon} \) and \( \frac{1}{|S|} \cdot \frac{1 + \epsilon}{1 + \epsilon} \). This differs from \( 1/|S| \) by at most \( O(\epsilon)/|S| \). Hence, \( \mathcal{X}' \) is \( 2^{-\Omega(m)} \)-close to \( \mathcal{U}_S \).

In order to invert our new construction, we will need to construct an inverter \( \text{Inv}_{q,p} \) for the function \( \text{Mod}_{q,p} \). For that, given \( x \in [p] \) we will just sample uniformly in its preimages. This is where the non-balancedness of \( \text{Mod} \) causes problems, since if \( p \) does not divide \( q \) the distribution \( \text{Inv}_{q,p}(\mathcal{U}_p) \) is not uniform on \([q]\).

**Lemma 34.** Suppose that \( q > p \). Given a distribution \( \mathcal{X} \) on \([p]\) such that \( \| \mathcal{X} - \mathcal{U}_p\|_\infty \leq \frac{\epsilon}{p} \), we have \( \| \text{Inv}_{q,p}(\mathcal{X}) - \mathcal{U}_q\|_\infty \leq \frac{1}{q} \cdot \frac{p + \epsilon q}{q - p} \).

**Proof.** Let \( X \sim \mathcal{X} \) and \( Y \sim \text{Inv}_{q,p}(\mathcal{X}) \). Since we invert the modulo function by taking for a given output a random preimage uniformly, \( \Pr[Y = y] \) is equal to \( \Pr[X = \text{Mod}_{q,p}(y)] \) divided by the number of \( y \) with the same value for \( \text{Mod}_{q,p}(y) \). The latter number is either \( \lceil q/p \rceil \) or \( \lfloor q/p \rfloor \), so

\[
\frac{1 - \epsilon}{p|q/p|} \leq \Pr(Y = y) \leq \frac{1 + \epsilon}{p|q/p|}
\]

Bounding the floor and ceiling functions by \( q/p \pm 1 \), we obtain

\[
\frac{1 - \epsilon}{q + p} \leq \Pr(Y = y) \leq \frac{1 + \epsilon}{q - p}
\]

That is

\[
\frac{p - \epsilon q}{q(q + p)} \leq \Pr(Y = y) - \frac{1}{q} \leq \frac{p + \epsilon q}{q(q - p)}
\]

which concludes the proof since this is true for all \( y \).

Now we describe the inverter \( \text{Inv}(x) \) for the extractor, again abusing the notation. First the inverter calls \( \text{Inv}_{N,d^m}(x) \) to obtain \( x_1 \in [N] \). Then it performs a random walk on the graph, starting from \( x_1 \), to reach a vertex \( x_2 \) at the end which is inverted to obtain \( x_3 = \text{Inv}_{d^m \cdot N}(x_2) \) as a \( d \)-ary string of length \( m' \). Finally, the inverter outputs \( y = (x_3, w) \), where \( w \) corresponds the inverse of the random walk of length \( n - m' \). It is obvious that this procedure yields a valid preimage of \( x \).

Using the previous lemma, if \( x \) is chosen uniformly, \( x_1 \) will be at the \( \ell_\infty \)-distance \( \epsilon_1 := \frac{1}{N} \cdot \frac{d^m}{N - d^m} = \frac{1}{N} O(c^{-nm}) \). For a given walk, the distribution of \( x_2 \) will just be a permutation of the distribution of \( x_1 \) and applying the lemma again, we see that the \( \ell_\infty \)-distance of \( x_3 \) from the uniform distribution is \( \epsilon_2 := \frac{1}{d^{m'}} \cdot \frac{N + 1}{d^m \cdot N} = \frac{1}{d^m} O(c^{-nm}) \). This is true for all the \( d^m \cdot N \) possible walks so the \( \ell_\infty \)-distance of the distribution of \( y \) from uniform is bounded by \( \frac{1}{d^m} O(c^{-nm}) \). Applying Lemma 33 in an argument similar to Lemma \( \square \) concludes the proof.
B.6 Proof of Theorem 20

We will need the following Proposition in our proof, which is easy to verify:

**Proposition 35.** Let \( f : \mathbb{F}_2^n \to \mathbb{F}_2^\delta \) be a Boolean function. Then for every \( \alpha > 0 \), and \( X \sim \mathcal{U}_n \), the probability that \( f(X) \) has fewer than \( 2^n(1 - \delta - \alpha) \) preimages is at most \( 2^{-\alpha n} \).

Let \( \text{Ext} \) be the linear seeded extractor of Theorem 10, set up for input length \( n \), seed length \( d = O(\log^3(n/\epsilon)) \), min-entropy \( n(1 - \delta - \alpha) \), and output length \( m = n(1 - \delta - \alpha) - O(d) \). Then the encoder chooses a seed \( Z \) for the extractor uniformly at random and sends it through the side channel. For the chosen value of \( Z \), the extractor is a linear function, and as before, given a message \( x \in \mathbb{F}_2^m \), the encoder picks a random vector in the affine subspace that is mapped by this linear function to \( x \) and sends it through the public channel. Then the decoder applies the extractor to the seed received from the secure channel and the transmitted string. The resiliency of the protocol can be shown in a similar manner as in Theorem 14. Specifically, note that by Proposition 35, with probability at least \( 1 - 2^{-\alpha n} \), the string transmitted through the main channel, conditioned on the observation of the intruder from the main channel, has a distribution \( Y \) with min-entropy at least \( n(1 - \delta - \alpha) \). Now in addition suppose that the seed \( z \) is entirely revealed to the intruder. As the extractor is strong, with probability at least \( 1 - \epsilon \), \( z \) is a good seed for \( Y \), meaning that the output of the extractor applied to \( Y \) and seed \( z \) is \( \epsilon \)-close to uniform, and hence the view of the intruder on the original message remains \( \epsilon \)-close to uniform.