Almost Parity Structure, 
Connections and 
Vielbeins in BV Geometry

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Abstract

We observe that an anti-symplectic manifold locally always admits a parity structure. The parity structure can be viewed as a complex-like structure on the manifold. This induces an odd metric and its Levi-Civita connection, and thereby a new notion of an odd Kähler geometry. Oversimplified, just to capture the idea, the bosonic variables are “holomorphic”, while the fermionic variables are “anti-holomorphic”. We find that an odd Kähler manifold in this new “complex” sense has a nilpotent odd Laplacian iff it is Ricci-form-flat. The local cohomology of the odd Laplacian is derived. An odd Calabi-Yau manifold has locally a canonical volume form. We suggest that an odd Calabi-Yau manifold is the natural geometric notion to appear in covariant BV-quantization. Finally, we give a vielbein formulation of anti-symplectic manifolds.

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1 Introduction

Odd symplectic geometry was first introduced in 1981 by Batalin and Vilkovisky [1] in antifield formulation of gauge theories. It was soon realized that many of the constructions in even symplectic geometry, could be transferred into odd symplectic geometry. However, there are important exceptions: For instance, there is no canonical volume element in the odd case. The main applications of odd symplectic geometry has up to now been quantization of gauge theories and covariant formulations of string field theory [2]. Many authors Khudaverdian and Nersessian [3], Batalin and Tyutin [4], Schwarz [5], and Hata and Zwiebach [6], have contributed to a covariant formulation of odd symplectic geometry.

The paper is roughly organized as follows: After a short review of anti-symplectic geometry, we introduce in Section 2 the notion of an almost parity structure. The main idea is that whereas the Grassmann parity in general is a property of the local coordinate charts only, we would like to ask when it is as a property of the manifold. The appropriate tool is an almost parity structure. With an almost parity structure at hand we may introduce various new geometric constructions. In Section 4 we discuss a connection in anti-symplectic geometry. In Section 5 we restrict ourselves to consider odd Kähler manifolds. Finally, we give a vielbein formulation in Section 7. Super conventions are written down in an Appendix.

1.1 Basic Settings

Let the number $2n$ of variables be finite. The anti-symplectic phase space $M$ is a $(n|n)$ real superspace with a non-degenerate antibracket,

$$\langle F, G \rangle = (F \partial^A G) E^{AB} (\partial_A G),$$

$$E^{AB} E_{BC} = \delta^A_C = E_{CB} E^{BA},$$

or equivalently an anti-symplectic two-form

$$E = \frac{1}{2} dz^A E_{AB} \wedge dz^B = - \frac{1}{2} E_{AB} dz^B \wedge dz^A. \quad (1.2)$$

The Jacobi identity can neatly be restated as that the two-form $E$ is closed

$$dE = 0. \quad (1.3)$$

The antibracket has the following symmetry

$$\langle G, F \rangle = -(-1)^{(\epsilon_F+1)(\epsilon_G+1)}(F, G),$$

$$E^{BA} = -(-1)^{(\epsilon_A+1)(\epsilon_B+1)} E^{AB}, \quad \epsilon(E^{AB}) = \epsilon_A + \epsilon_B + 1,$$

$$E_{BA} = -(-1)^{\epsilon_A \epsilon_B} E_{AB}, \quad \epsilon(E_{AB}) = \epsilon_A + \epsilon_B + 1. \quad (1.5)$$

Locally there exists an anti-symplectic potential $\vartheta = \vartheta_A dz^A$ such that $E = d\vartheta$. Written out

$$E_{AB} = (\partial_A \vartheta_B) + (-1)^{\epsilon_A \epsilon_B} (\partial_B \vartheta_A). \quad (1.6)$$

Locally one may resort to anti-symplectic Darboux coordinates $z^A = (\phi^\alpha, \phi^*_\alpha)$, where $\epsilon(\phi^*_\alpha) = \epsilon(\phi^\alpha) + 1$, so that

$$E^\alpha_{\beta} = \delta^\alpha_\beta = - E^*_\beta, \quad E^{\alpha\beta} = 0 = E^{*_\alpha*_\beta}. \quad (1.7)$$
or in terms of the fundamental antibrackets

\[ (\phi^\alpha, \phi^*_\beta) = \delta^\alpha_\beta, \quad (\phi^\alpha, \phi^\beta) = 0, \quad (\phi^*_\alpha, \phi^*_\beta) = 0. \]  

### 1.2 Odd Pfaffian

Perhaps one of the most important differences between even versus odd symplectic geometry is that there is no canonical Liouville measure\(^1\) in odd symplectic geometry. The naive guess would probably be an odd Pfaffian \(\text{Pf}(E_{AB})\) of \(E_{AB}\). But it is easy to see that the volume element cannot be a function of \(E_{AB}\). This is because an anti-symplectic coordinate change \(z^A \to z'^A(z)\), which by definition leaves \(E_{AB}\) invariant, may not be volume preserving, i.e. the volume density \(\rho\) may change although \(E_{AB}\) is not being changed.

On the other hand, there is a closed one-form

\[ C \equiv \frac{1}{2} \overrightarrow{dz^A} (\partial_A E_{BC}) E^{CB} (-1)^{\epsilon_B} = 0, \]  

which is zero by the Darboux Theorem. A reasonable definition of the odd Pfaffian \(\text{Pf}(E_{..})\) should evidently satisfy \(C = d \ln \text{Pf}(E_{..})\). So according to this identification \(\text{Pf}(E_{..})\) is a constant. It therefore transforms as a scalar. This is in striking contrast to the even symplectic case, where the Pfaffian of the symplectic metric transforms as a scalar density.

### 2 Almost Parity Structure

#### 2.1 Almost Darboux Coordinates

Let us introduce the notion of “almost Darboux coordinates”. These are coordinates where all non-vanishing entries of the anti-symplectic metric are *bosonic*, i.e.

\[ \epsilon(E^{AB}) \equiv \epsilon_A + \epsilon_B + 1 = 0. \]

A set of Darboux coordinates is an example of almost Darboux coordinates, as the name indicates. Let us for completeness mention that under the assumption of invertibility, the following conditions are equivalent:

1. \(E^{AB}\) is given in almost Darboux coordinates.
2. \(E^{AB}\) is bosonic.
3. \(E^{BA} = -E^{AB}\) is antisymmetric.
4. \(E_{BA} = -E_{AB}\) is antisymmetric.
5. \(E_{AB}\) is bosonic.
6. \(E^{AB}\) anticommutes with the Grassmann parity operator \((-1)^{\epsilon_A} E^{AB} (-1)^{\epsilon_B} = 0\) .

\(^1\)In this paper all measure densities \(\rho\) are *signed* single-valued densities. This means that the supermanifold (and in particular the body) has to be orientable.
Almost Darboux coordinates are clearly stable under Grassmann preserving coordinate changes $z^A \to z'^B(z)$, i.e. when the non-vanishing entries of the Jacobian-matrix

$$
z'^B \left\langle \frac{\partial}{\partial z^A} \right\rangle \quad \text{(2.3)}$$

has even Grassmann-grading

$$
\epsilon(z'^B \left\langle \frac{\partial}{\partial z^A} \right\rangle) \equiv \epsilon_A + \epsilon_B = 0.
\quad \text{(2.4)}
$$

### 2.2 Almost Parity Structure

The Grassmann parity operator $(-1)^{\epsilon_A}$ is invariant under Grassmann preserving changes of coordinates (2.4) but not under general coordinate transformations. Also let us note that the Grassmann parity is a property of the coordinates, not the manifold itself. Let us generalize the Grassmann parity operator $(-1)^{\epsilon_A}$ to a covariant object in the following way. Consider a general $(n_+|n_-)$-supermanifold $M$. An almost parity structure $P : TM \to TM$ is a Grassmann-even $(1,1)$-tensor, whose square is the identity,

$$
P = \partial^r_A P^B_A \otimes dz^B, \quad \epsilon(P^B_A) = \epsilon_A + \epsilon_B,
\quad \text{(2.5)}$$

(In the first equality $\partial^r_A$ does not differentiate the $P^B_A$-functions, see the Appendix.) We will furthermore assume that the supertrace of $P$ is equal to the dimension of the manifold:

$$
(-1)^{\epsilon_A} P^A_A = \text{str}(P) = n_+ + n_-.
\quad \text{(2.6)}$$

It is convenient to introduce the idempotent projection operators

$$
P_\pm = \frac{1}{2}(\text{Id} \pm P), \quad \text{Id} = P_+ + P_-, \quad P = P_+ - P_-, \quad P_\pm P_\mp = P_\pm, \quad P_\pm P_\mp = 0.
\quad \text{(2.7)}$$

Consider a point $m \in M$ on the manifold, and a coordinate system which cover this point $m$. The natural basis of tangent vectors in $T_mM$ wrt. the chosen coordinate system is $(\partial^r_A)_{A=1,\ldots,n_++n_-}$. However $P_m : T_mM \to T_mM$ may not be diagonal in this basis. To diagonalize the almost parity structure $P_m$, one should perform a change of the basis to a new basis $(e^r_A)_{A=1,\ldots,n_++n_-}$ for the tangent space $T_mM$:

$$
\partial^r_A = e^r_B \Lambda^B_A,
\quad \text{(2.8)}$$

where $\Lambda^B_A$ is an invertible matrix. We will restrict the allowed basis shift to shift that carries definite Grassmann parity $\epsilon(\Lambda^B_A) = \epsilon_A + \epsilon_B$. The almost parity structure $P : T_mM \to T_mM$ is diagonalizable in this restricted sense with eigenvalues $\sigma = \pm 1$. The eigenspaces for $P$ are $P_\pm(TM)$. Moreover, we see from the condition (2.6), that the eigenspace $P_+(TM)$ ($P_-(TM)$) has multiplicity $n_+$ ($n_-$) and their bases carry Grassmann parity 0 (1), respectively. As a consequence the superdeterminant of $P$ is

$$
\text{sdet}(P) = (-1)^{n_-}.
\quad \text{(2.9)}$$

**Proposition.** Two almost parity structures $P_{(1)}$ and $P_{(2)}$ are globally related via similarity transformations, i.e. there exists a global automorphism $\Lambda : TM \to TM$, such that $P_{(1)} \Lambda = \Lambda P_{(2)}$.

**Sketched proof:** To prove the statement locally, we choose a coordinate system. It is enough to show that $P$ is related via a similarity transformation to the Grassmann parity. This follows from the discussion above. Finally, the global statement follows by use of a partition of the unity.
Now consider an odd $(n|n)$ supermanifold. An almost parity structure $P$ and an anti-symplectic structure $E$ are compatible with the anti-symplectic structure $E$ iff

$$P^A_B E^{BC} (P^T)_C^D = - E^{AD},$$

(2.10)
or equivalently

$$(P^T)_A^B E_{BC} P^C_D = - E_{AD}, \quad [P \wedge E] = 0, \quad E(PX, Y) = - E(X, PY),$$

(2.11)

where $X$ and $Y$ are vector fields. Here the super transposed $P^T = dz^B (P^T)_B^A \overset{\rightarrow}{\iota}_A : T^*M \to T^*M$ is

$$(P^T)_B^A = (-1)^{(\kappa_A+1)\kappa_B} P^A_B.$$ (2.12)

Moreover, $E(P_{\pm}X, Y) = E(X, P_{\pm}Y)$. The eigenspaces $P_\sigma(TM)$ are Lagrangian subspaces of the tangent space $TM$:

$$E(P_{\pm}X, P_{\pm}Y) = 0, \quad (P^T)_\sigma^A B E_{BC} (P_{\sigma})^C_D = 0.$$

(2.13)

Note that the compatibility condition (2.10) is a non-trivial condition even for a cohomology-exact odd (possibly degenerate) two-form. Therefore, the condition makes some of the representants in a cohomology class $[E]$ more preferable.

The almost parity structure $P$ is clearly analogous to the almost complex structure $J$ in complex differential geometry. Let us search for the counterparts of the complex conjugation, the Nijenhuis tensor $N$, the Kähler potential $K$, the Dolbeault differentials, etc. A different approach binding together Kähler and BV geometry has been studied by Aoyama and Vandoren [7] and Khurdaverdian and Nersessian [8].

2.3 Canonical Examples

Darboux Coordinates. It is perhaps useful to be a bit more explicit in case of an anti-symplectic $(n|n)$ supermanifold with an almost parity structure $P$. Consider Darboux coordinates. In the usual $2 \times 2$ Grassmann block representation, we have

$$P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad P_{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

(2.14)

where $P_{(0)}$ denote the Grassmann parity. From the compatible condition (2.10) we get that

$$P_{11} = - P^T_{00}, \quad P^T_{01} = - P_{01}, \quad P^T_{10} = P_{10}.$$ (2.15)

The supertrace condition (2.6) and $P^2 = \text{Id}$ yields that $P_{00} = 1 + S$, where $S$ denotes the soul part of $P_{00}$. It is straightforward to show by use of $P^2 = \text{Id}$ that

$$\begin{bmatrix} 1 + S & P_{01} \\ P_{10} & -1 - S^T \end{bmatrix} \begin{bmatrix} 1 + \frac{1}{2} S & \frac{1}{2} P_{01} \\ \frac{1}{2} P_{10} & -1 - \frac{1}{2} S^T \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{2} S & - \frac{1}{2} P_{01} \\ \frac{1}{2} P_{10} & 1 + \frac{1}{2} S^T \end{bmatrix}$$
1 + \frac{1}{2}S & \frac{1}{2}P_{01} \\
\frac{1}{2}P_{10} & -1 - \frac{1}{2}S^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\
0 & -1 \end{bmatrix}.

(2.16)

So the diagonalizing transformation is \( P\Lambda = \Lambda P(0) \) with

\[
\Lambda = \begin{bmatrix}
1 + \frac{1}{2}S & \frac{1}{2}P_{01} \\
\frac{1}{2}P_{10} & -1 - \frac{1}{2}S^T 
\end{bmatrix}.
\]

(2.17)

**Almost Darboux Coordinates.** Let us next consider an example of almost Darboux coordinates \( z^A = (x^\alpha; \theta^{\bar{\alpha}}) \), where \( \alpha, \bar{\alpha} = 1, \ldots, n \), and where \( x^\alpha \) are bosons and \( \theta^{\bar{\alpha}} \) are fermions. The almost parity structure \( P \) is assumed to be equal to the Grassmann parity \( P^{A,B} = (-1)^{\epsilon_A \delta_B^A} \). As a consequence the Nijenhuis tensor \( N \) for the almost parity structure \( P \) vanishes (see Section 3.4). Because the coordinates are almost Darboux, the block diagonal pieces of the anti-symplectic metric vanish,

\[
E_{\alpha\beta} = 0 = E_{\bar{\alpha}\bar{\beta}},
\]

cf. (2.13). From the closeness of the anti-symplectic two-form \( E \), one may prove that there exists locally a corresponding odd parity potential \( K = K(x, \theta) \) such that

\[
-E_{\alpha\bar{\beta}} = E_{\bar{\beta}\alpha} = (\frac{\partial}{\partial \theta^{\bar{\beta}}} K \frac{\partial}{\partial x^\alpha}).
\]

(2.19)

Moreover, any other odd parity potential \( K' \) differs from \( K \) in the following way:

\[
K'(x, \theta) - K(x, \theta) = F(x) + \bar{F}(\theta).
\]

(2.20)

The parity potential \( K \) is quite similar to the Kähler potential \( K \) appearing in standard complex differential geometry. When there is no confusion, we shall simply call \( K \) a Kähler potential. The parity preserving coordinate transformations are transformations of the form

\[
x^\alpha \rightarrow x'^\alpha(x), \quad \theta^{\bar{\alpha}} \rightarrow \theta'^{\bar{\alpha}}(\theta).
\]

(2.21)

In this example the eigenspaces are:

\[
P_+(TM) = \text{span}(\frac{\partial}{\partial x^\alpha}), \quad P_-(TM) = \text{span}(\frac{\partial}{\partial \theta^{\bar{\alpha}}}).
\]

(2.22)

The submanifolds

\[
\{(x^\alpha; \theta^{\bar{\alpha}})| x^\alpha = x^\alpha_{(0)} \}, \quad \{(x^\alpha; \theta^{\bar{\alpha}})| \theta^{\bar{\alpha}} = \theta^{\bar{\alpha}}_{(0)} \}
\]

(2.23)

are two Lagrangian submanifolds of Grassmann parity \( n|0 \) and \( 0|n \), respectively, that intersect the point \( z^A_{(0)} = (x^\alpha_{(0)}; \theta^{\bar{\alpha}}_{(0)}) \).

Let us for simplicity restrict to the case of constant almost Darboux coordinates

\[
E = E_{\alpha\bar{\alpha}}(0) d\theta^{\bar{\alpha}} \wedge dx^\alpha.
\]

(2.24)
The group $G$ of anti-symplectic, parity preserving coordinate transformations is a subgroup of the rigid (“global”, “point independent”) transformations. In fact, it is the semidirect product of the rigid translations and the rigid $GL(n)$ group of linear transformations of the form
\[
\begin{align*}
\alpha' &= \lambda^{\alpha}_\beta \alpha, \\
\bar{\alpha}' &= E^{\bar{\alpha}}_{(0)} (\lambda^{-1}T)_{\bar{\beta}}^\beta E^{(0)}_{\beta} \bar{\beta}.
\end{align*}
\]
(2.25)

The volume density changes as $\rho' = s\det(\lambda) = \rho$. We shall later see that the group $G$ of anti-symplectic, parity preserving coordinate transformations may equally well be characterized as the group of ortho-symplectic transformations.

3 Parity Structure

3.1 Definition of Parity Structure

Consider a $(n_+|n_-)$ supermanifold $M$ equipped with an almost parity structure $P$. A coordinate system is said to adapt an almost parity structure $P$ if the natural basis of tangent vectors $\partial_A^r$ are eigenvectors for the almost parity structure $P$ with eigenvalue $(-1)^{\epsilon_A}$:
\[
\partial_B^r P_{BA} = (-1)^{\epsilon_A} \partial_A^r.
\]
(3.26)
or equivalently
\[
(P^T)_{BA} \partial_B^r = (-1)^{\epsilon_A} \partial_A^r.
\]
(3.27)

We shall see in the Section 3.4 that the Nijenhuis tensor $N$ corresponding to the almost parity structure $P$ vanishes in regions of the manifold that are covered by $P$-adapted coordinate patches. We will also see that the two eigenspaces $P_\epsilon(TM)$ are stable under the Lie bracket $[,]$ in these regions.

$P$-adapted coordinates are also almost Darboux coordinates, cf. Section 2.1, but the opposite need not be the case.

A parity preserving coordinate transformation is a coordinate transformation between two local coordinate system such that the non-vanishing entries of the Jacobian-matrix
\[
\begin{align*}
z'^B \frac{\partial}{\partial z^A}
\end{align*}
\]
has positive $P$-parity, i.e. is bosonic. So a parity preserving coordinate transformation is the same as a Grassmann parity preserving coordinate transformation, cf. 2.4. Note that a coordinate transformation $(x^\alpha; \theta^\alpha) \to (x'^\alpha; \theta'^\alpha)$ between two $P$-adapted coordinate patches is a parity preserving coordinate transformation. Therefore $x'^\alpha = x^\alpha(x)$ and $\theta'^\alpha = \theta^\alpha(\theta)$. So the only coordinate transformations among $P$-adapted coordinate charts are a “holomorphic” change of the “holomorphic” variables $x^\alpha$ and an independent “anti-holomorphic” change of the “anti-holomorphic” variables $\theta^\alpha$.

**Definition.** An almost parity structure $P$ is a parity structure iff there exists an atlas of $P$-adapted coordinate charts.

**Definition.** An odd almost Kähler manifold $M, P, E$ is a $(n|n)$-manifold $M$ with a non-degenerate anti-symplectic structure $E$ and an almost parity structure $P$ that is compatible with $E$, cf. (2.10).
(Actually, this is what one would normally call an odd almost pseudo-Kähler manifold, but we shall drop the prefix pseudo for convenience.) We will later show that an odd almost Kähler manifold can be equipped with an odd metric (i.e. the odd Kähler metric) and its Levi-Cevita connection.

**Definition.** An odd Kähler manifold \( M, P, E \) is an odd almost Kähler manifold, where the almost parity structure \( P \) is a parity structure.

We conclude that an anti-symplectic \((n|n)\)-manifold \( M \) possesses a compatible Grassmann parity structure iff there exists an atlas of almost Darboux charts, which are mutually connected via Grassmann parity preserving transformations, see also (2.4). This means that every point in the manifold is intersected by a \( P \)-adapted \((n|0)\) and a \((0|n)\) Lagrangian submanifold, cf. (2.13).

Below follows an analysis of the integrability of an almost parity structure \( P \).

### 3.2 Characteristic one-forms

It is convenient to locally introduce a (double) overcomplete generating set of vectors \( X_{(\sigma,A)} \) for the eigenspace \( P_\sigma(TM) \), \( \sigma = \pm 1 \).

\[
X_{(\sigma,A)} = \partial_B (P_\sigma)^B_A = (-1)^{\epsilon_A + \epsilon_B} (P_\sigma^T)^B_A \partial_B , \quad \epsilon(X_{(\sigma,A)}) = \epsilon_A .
\]

(3.29)

Analogously we can define the (double) overcomplete generating set of characteristic one-forms for the eigenspace \( P_\sigma(TM) \)

\[
\eta^{(\sigma,A)} = (P_\sigma)^A_B \overrightarrow{dz}^B , \quad \epsilon(\eta^{(\sigma,A)}) = \epsilon_A .
\]

(3.30)

Obviously, we have

\[
\eta^{(\sigma,A)}(X_{(\tau,B)}) = \delta^\sigma_\tau (P_\sigma)^A_B , \quad \eta^{(+,.A)} + \eta^{(-,.A)} = dz^A ,
\]

(3.31)

and

\[
\bigcap_A \text{Ker}(\eta^{(\sigma,A)}) = P_\sigma(TM) .
\]

(3.32)

We furthermore define two-forms

\[
a^{(A)} = \pm 2d\eta^{(\pm,.A)} = (P^A_B \overleftarrow{\partial_C}) dz^C \wedge dz^B = \frac{1}{2} a^{(A)}_{BC} dz^C \wedge dz^B ,
\]

(3.33)

where

\[
a^{(A)}_{BC} = (P^A_B \overleftarrow{\partial_C}) - (-1)^{\epsilon_B \epsilon_C} (P^A_B \overleftarrow{\partial_C}) .
\]

(3.34)

### 3.3 An Odd Parity Conjugation

Consider an \((n|n)\) anti-symplectic supermanifold \( M \). What should be the analogue of complex conjugation? Loosely speaking, it should be the operation that takes fields \( \phi^a \) into its corresponding antifield \( \phi^a_\ast \). Let us start by defining the concept of odd parity conjugation in the frame bundle over the manifold. Let \( (e_{(A)})_{A=1,...,2n} \) be a basis for the tangent space \( TM \). Then the parity conjugated basis \( (e_{(A)}^\ast)_{A=1,...,2n} \) (of opposite Grassmann parity) is the unique basis such that

\[
E(e_{(A)}, e_{(B)}^\ast) = \delta_A^B .
\]

(3.35)
Applying conjugation twice yields a minus
\[ e^{**}(A) = - e(A) \, . \] (3.36)

For a tangent vector \( X = X^A e(A) \) we define a parity conjugated tangent vector \( X^* = X^A e^*(A) \). This of course depends strongly on the choice of basis. Also the notion of real and imaginary part has no counterpart. The only nice things to say is that no matter which basis we choose, the conjugation is linear, \( X^{**} = -X \). If \( E \) and \( P \) are compatible, and \( (e(A))_{A=1,...,2n} \) are eigenvectors for \( P \), then \( (e^*(A))_{A=1,...,2n} \) are also eigenvectors for \( P \) (with opposite eigenvalue), so that in this case the \(*\)-conjugation gives a bijection between \( P_+(TM) \) and \( P_-(TM) \). We emphasize that \( [X,Y]^* = [X^*,Y^*] \) does not hold in general.

For an one-form \( \eta \) the parity conjugated one-form \( \eta^* \) is defined via \( \eta^*(X) = \eta(X^*) \). The above definitions leads to the convenient rules
\[
\left( \frac{\partial}{\partial \phi^\alpha} \right)^* = \frac{\partial}{\partial \phi^{\alpha^*}} \, , \quad (d\phi^\alpha)^* = \phi^{\alpha^*} \, ,
\]
(3.37)
in Darboux coordinates \( E = d\phi^\alpha \wedge d\phi^{\alpha^*} \).

### 3.4 Nijenhuis Tensor

We start by defining two tensors \( N_\sigma : TM \times TM \to TM \),
\[
N_\pm(X,Y) = P_\mp[P_\pm X, P_\pm Y] = -(-1)^{\epsilon(X)e(Y)} N_\pm(Y,X) \, .
\]
(3.38)

Note that \( N_\pm(X,PY) = \pm N_\pm(X,Y) = N_\pm(PX,Y) \). One can write the Lie-bracket symbol \([ , ] : TM \times TM \to TM\) in the following way
\[
[ , ] = \frac{\partial^A}{\partial z^A} \times \frac{\partial^B}{\partial z^B} - (-1)^{\epsilon(X)e(Y)} \frac{\partial^A}{\partial z^A} \times \frac{\partial^B}{\partial z^B} \, .
\]
(3.39)

Then we can write \( N_\pm \in \Gamma(TM \otimes \Lambda^2(T^*M)) \) as
\[
N_\pm = P_\mp [ , ](P_\pm \times P_\pm) = \frac{1}{2} \partial_A N_{BC}^{AB} dz^C \wedge dz^B \, .
\]
(3.40)

We now define the Nijenhuis tensor as
\[
N = 4(N_+ + N_-) \, ,
\]
(3.41)
or
\[
N(X,Y) = [X,Y] + [PX,PY] - P[X, PY] - P[PX,Y] = -(-1)^{\epsilon(X)e(Y)} N(Y,X) \, .
\]
(3.42)

It satisfies \( N(PX,Y) = N(X,PY) \). In components the Nijenhuis tensor
\[
N = \frac{1}{2} \partial_A N_{BC}^{AB} \wedge dz^C \wedge dz^B \in \Gamma(TM \otimes \Lambda^2(T^*M))
\]
(3.43)
reads
\[
\vec{dz}^A (N(\partial_B^r, \partial_C^r)) = -N^A_{BC} \\
= \left( P^A_D \left( P^D_B \partial_C^r \right) - \left( P^A_B \partial_D^r \right) P^D_C \right) - (-1)^{\epsilon B\epsilon C} (B \leftrightarrow C).
\]  
(3.44)

The relation can be inverted to give
\[
8 N_{\pm}(X, Y) = N(X, Y) \pm N(X, PY) = 2N(X, P\sigma Y), \\
8 N^A_{\sigma BC} = N^A_{\sigma DB} (P_{\sigma}^D C) - (-1)^{\epsilon B\epsilon C} (B \leftrightarrow C).
\]  
(3.45)

We observe that \((-1)^{\epsilon A} N^A_{AB} = 0\) and \((-1)^{\epsilon A} N^A_{BAC} = 0\).

### 3.5 Integrability Condition

By the Frobenius Theorem, an almost parity structure is locally a parity structure if one of the following equivalent integrability criteria is satisfied:

1. The Nijenhuis tensor \(N = 0\) vanishes.

2. Both of the tensors \(N_{\pm} = 0\) vanishes.

3. Both of the eigenspaces \(P_{\sigma}(TM)\) is stable under the Lie-bracket operation \([\cdot, \cdot]\).

4. \(\forall \sigma: \) The ideal \(I(\eta^{(\sigma, A)})\) in the exterior algebra of forms, generated by the characteristic one-forms for \(P_{\sigma}(TM)\), is stable under the action of the exterior derivative \(d\).

We conjecture that a vanishing Nijenhuis tensor actually ensures that the manifold admits the parity structure globally. The corresponding statement for bosonic complex manifolds was proven by Newlander and Nirenberg [9].

### 3.6 Dolbeault Bi-Complex

In this Subsection we review some basic facts about Dolbeault bi-complexes, that we need later. Consider some \(P\)-adapted coordinates \((x^\alpha; \theta^{\bar{\alpha}})\) on a supermanifold \(M\) with a parity structure \(P\). We can then split the exterior derivative \(d = \partial + \bar{\partial}\) into its Dolbeault components

\[
\partial = dx^\alpha \frac{\partial^j}{\partial x^\alpha}, \quad \bar{\partial} = d\theta^{\bar{\alpha}} \frac{\partial^j}{\partial \theta^{\bar{\alpha}}}.
\]  
(3.46)

Now let us mention some basic applications of a bigraded complex.

**Proposition.** Consider a zero-form \(F\), satisfying \(\bar{\partial} \wedge \partial F = 0\). Then \(F(x, \theta) = f(x) + \bar{f}(\theta)\) splits locally into a "holomorphic" and an "anti-holomorphic" piece. This split can be made global if the first Čech cohomology class \(\check{H}^1_{\text{const}}(M, \mathbb{R}) = 0\) vanishes.

(The function \(\bar{f}\) has no relation to \(f\).) The space \(\check{H}^1_{\text{const}}(M, \mathbb{R})\) refers to the Čech cohomology in terms of (locally) constant sections. Moreover, we will work in a fixed atlas of open neighbourhoods \((U_i)_{i\in I}\) covering \(M\). It is clear that a manifest geometric treatment has to be independent of the atlas. But it turns out that this is merely a technical complication that we will not discuss here.
Proof. Locally two intersecting neighbourhoods $U_i$ and $U_j$ defines a one-cochain $c^{(1)} = \frac{1}{2}\bar{c}_{ij} di \wedge dj$ by a “separation of variables” argument

$$f_i(x) + \bar{f}_i(\theta) = F(x, \theta) = f_j(x) + \bar{f}_j(\theta)$$

so that

$$f_i(x) - f_j(x) = c_{ij} = -\bar{f}_i(\theta) + \bar{f}_j(\theta).$$

The $c^{(1)}$ is a one-cochain both in terms of sheaf of general sections and in the sense of sheaf of constant sections. We will mainly work in the latter. However in the former, $c^{(1)}$ is also a one-coboundary, so that it is trivially a one-cocycle. Therefore, it is a (locally) constant one-cycle. By the assumptions it is then a one-coboundary in the constant sense, i.e. there exists a constant zero-cochain $c^{(0)} = c_i di$ such that $c_{ij} = c_i - c_j$. We may then define $f(x) = f_i(x) - c_i$.

\[\]

Proposition. Consider an exact $(1, 1)$ two-form $\omega^{(11)} = d\eta$, where $\eta = \eta^{(10)} + \eta^{(01)}$ is a global one-form. If the Čech cohomology classes $\check{H}^1(M, \mathbb{R}) = 0$ and $\check{H}^2_{\text{const}}(M, \mathbb{R}) = 0$ vanish, then there exists a global zero-form $F$ such that $\omega^{(11)} = (\partial \wedge \partial F)$.

Proof: Locally, there exist zero-forms $h_i$ and $\bar{h}_i$ such that $\eta^{(10)} = \partial h_i$ and $\eta^{(01)} = \partial \bar{h}_i$. So locally the zero-form $F_i = h_i - \bar{h}_i$ will be a solution to $d\eta = \partial \wedge \partial F_i$. Furthermore, two intersecting neighbourhoods $U_i$ and $U_j$ defines a one-coboundary $F^{(1)} = \frac{1}{2}F_{ij} di \wedge dj$ by $F_{ij} = F_i - F_j$. Note that the functions $F_{ij}$ satisfy $\partial \wedge \partial F_{ij} = 0$. So locally they split $F_{ij}(x, \theta) = f_{ij}(x) + \bar{f}_{ij}(\theta)$ into “holomorphic” and “anti-holomorphic” parts. By maybe going to a finer atlas we may assume that the splitting is defined on the whole $U_i \cap U_j$. Therefore, we have one-cochains $f^{(1)} = \frac{1}{2}f_{ij} di \wedge dj$ and $\bar{f}^{(1)} = \frac{1}{2}\bar{f}_{ij} di \wedge dj$. Unfortunately, they are not one-cocycles. But this turns out to be easy to fix. Let us define a two-cochain

$$c^{(2)} = \frac{1}{3!}c_{ijk} di \wedge dj \wedge dk = df^{(1)} = -d\bar{f}^{(1)},$$

that is $c_{ijk} = f_{ij}(x) + f_{jk}(x) + f_{ki}(x)$. Note that $c^{(2)}$ is constant by a “separation of variable” argument. $c^{(2)}$ is actually a constant two-cocycle, and therefore by assumption a two-coboundary in the constant sense, i.e. there exists a constant one-cocycle $c^{(1)} = \frac{1}{2}c_{ij} di \wedge dj$ such that $c^{(2)} = d\bar{c}^{(1)}$ or written out $c_{ijk} = c_{ij} + c_{jk} + c_{ki}$. So we may construct “holomorphic” and “anti-holomorphic” one-cocycles $f^{(1)} = f^{(1)} - c^{(1)}$ and $\bar{f}^{(1)} = \bar{f}^{(1)} + c^{(1)}$. By assumption there exist zero-cochains $f^{(0)} = f_i(x, \theta) di$ and $\bar{f}^{(0)} = \bar{f}_i(x, \theta) di$, such that $f^{(1)} = df^{(0)}$ and $\bar{f}^{(1)} = d\bar{f}^{(0)}$. Now let $F = F_i - f_i - \bar{f}_i$.

\[\]

4 Connection

4.1 Odd Metric

Unless otherwise stated, we consider in this Section 4 a $(n|n)$ supermanifold $M$ with a non-degenerate anti-symplectic structure $E$ and an almost parity structure $P$, that are mutually compatible, cf. \[2.10\] Let us introduce a non-degenerate odd metric $(0,2)$-tensor

$$g = \frac{1}{2}dz^A g_{AB} \vee dz^B = \frac{1}{2}g_{AB} dz^B \vee dz^A,$$
by
\[ g_{AB} = E_{AC} P^C_B = (-1)^{\epsilon_A \epsilon_B} g_{BA} , \quad \epsilon(g_{AB}) = \epsilon_A + \epsilon_B + 1 , \]

\[ g = \frac{1}{2}[E \, \gamma \, P] . \]  

(4.2)

The symbol $\gamma$ is the super-symmetrized tensor product, $dz^B \gamma dz^A = (-1)^{\epsilon_A \epsilon_B} dz^A \gamma dz^B$. The above symmetry property of $g$ follows from (2.10). (As we are not demanding any positivity properties of $g$, we should properly refer to $g$ as an odd \textit{pseudo}-Riemannian metric.) The inverse metric satisfies
\[ g^{BA} = (-1)^{(\epsilon_A + 1)(\epsilon_B + 1)} g^{AB} , \quad \epsilon(g^{AB}) = \epsilon_A + \epsilon_B + 1 . \]  

(4.3)

On the other hand we get
\[ (P^T)_A^B g_{BC} P^C_D = - g_{AD} , \quad [P \, \gamma \, g] = 0 . \]  

(4.4)

and
\[ (P_\sigma)_A^B g^{BC} (P^T_\sigma)_C^D = 0 . \]  

(4.5)

Let us note for later that $g(\partial^\sigma_A \partial_B) = g_{AB}$ and $(-1)^{\epsilon_B} N^A_{BC} g^{CB} = 0$. The metric $g$ leads to a $g$-bracket (which we also denote by $g$):
\[ g(F,G) = (F \frac{\partial}{\partial z^A})g^{AB}(\frac{\partial}{\partial z^A}G) , \]  

(4.6)

where $F,G \in C^\infty(M)$ are functions.

4.2 Connection

We define a Grassmann-even linear connection $\nabla : TM \times TM \to TM$ in the tangent bundle by
\[ \nabla_X = X^A \, \nabla_A , \quad \nabla^l_A = \frac{\partial^l}{\partial z^A} + \partial_B^l \, \Gamma^B_{AC} \, \frac{dz^C}{z^A} , \]  

(4.7)

or equivalently $\nabla = d + \Gamma = dz^A \otimes \nabla^l_A : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ where
\[ \Gamma = dz^A \otimes \partial_B^l \, \Gamma^B_{AC} \, \frac{dz^C}{z^A} . \]  

(4.8)

Indices $B$ and $C$ are bundle indices, while index $A$ is a base manifold index. This distinction is in general useful but becomes somewhat blurred for connections over the tangent bundle. Acting on forms it reads
\[ \nabla^l_A = \frac{\partial^l}{\partial z^A} + \frac{dz^B}{z^A} \, \Gamma^B_{AC} \, \frac{dz^C}{z^A} : = \frac{\partial^l}{\partial z^A} - (-1)^{\epsilon_A \epsilon_B} \Gamma^B_{AC} \, \frac{dz^C}{z^A} . \]  

(4.9)

The existence of two different connections $\nabla^{(1)}$ and $\nabla^{(2)}$ reflects the existence of a global non-trivial $(1,2)$ tensor. The torsion tensor $T : TM \times TM \to TM$ of a connection $\nabla$

\[ T(X,Y) = \nabla_X Y - (-1)^{\epsilon(X)\epsilon(Y)} \nabla_Y X - [X,Y] = - (-1)^{\epsilon(X)\epsilon(Y)} T(Y,X) \]  

(4.10)

can be viewed as an element
\[ T = [\nabla \wedge \text{Id}] = [\nabla \wedge \partial_A^l \otimes dz^A] \in \Gamma(TM \otimes \Lambda^2(T^*M)) . \]  

(4.11)
In terms of the Christoffel symbols this means
\[ \Gamma^C_{BA} = -(-1)^{(\epsilon_A+1)(\epsilon_B+1)} \Gamma^C_{AB}. \] (4.12)

Perhaps the various relations between a connection and an almost parity structure are best summarized as
\[ \nabla P = 0 \Rightarrow P_\pm \nabla_{P_\pm X} P_\pm Y = 0, \] (4.13)

\[ P \nabla_X Y = \nabla_X P Y \iff P_\pm \nabla_{X P_\pm Y} = 0 \]
\[ \nabla P = 0 \]
\[ \nabla[P \nabla_X Y] = 0 \equiv P_\pm \nabla_{P_\pm X} P_\pm Y = 0 \]
\[ \nabla P = 0 \iff [P X, Y] = 0 \iff [P_{\pm X}, P_{\pm Y}] = 0, \] (4.14)

and
\[ N = 0 \]
\[ \nabla P = 0 \land T = 0 \]
\[ \nabla P = 0 \land \nabla_{P_\pm X} Y = \nabla_{X P Y} \land T = 0 \]
\[ \nabla P = 0 \land \nabla_{P_\pm X} Y = \nabla_{X P Y} \]
\[ P_\rho \nabla_{P_\sigma X} P_\tau Y = 0 \text{ vanishes for mixed signs } \rho, \sigma, \tau \]
\[ \text{Holonomy Group } G \subseteq GL(n). \]
In a $P$-adapted coordinate system the next-to-last condition simply state that the only non-vanishing components of the Christoffel symbols $\Gamma^B_{AC}$ are the bose-bose-bose and the fermi-fermi-fermi components.

### 4.3 Symplectic Connection

A connection is called *symplectic* iff it preserves the anti-symplectic structure: 

$$\nabla E = 0,$$

or in components

$$(-1)^{\epsilon_A} (\partial_A E_{BC}) = (-1)^{\epsilon_A \epsilon_B} E_{BD} \Gamma^D_{AC} - (-1)^{\epsilon_C (\epsilon_A + \epsilon_B)} E_{CD} \Gamma^D_{AB}.$$

This condition does not determine uniquely the connection, not even if we impose that the connection should be torsion-free. It clearly implies the weaker super-symmetrized condition:

$$[\nabla \wedge, E] = 0.$$

Applying the Jacobi identity ($dE = 0$), this becomes

$$\sum_{\text{cycl.} \; A,B,C} (-1)^{\epsilon_A \epsilon_C} E_{AD} \left((-1)^{\epsilon_B} \Gamma^D_{BC} - (-1)^{\epsilon_B + \epsilon_C} \Gamma^D_{CB}\right) = 0.$$

Note that the last equation is satisfied for a torsion-free connection. We also have that

$$\nabla P = 0 \iff \nabla g = 0 \Rightarrow [\nabla \gamma, g] = 0.$$

### 4.4 Levi-Civita Connection

We are led to consider the Levi-Civita connection. It is the unique connection that satisfies

$$T = 0, \quad \nabla g = 0,$$

where $g$ in principle could be any odd, non-degenerate, symmetric metric. In terms of the Christoffel symbols the metric condition reads

$$(-1)^{\epsilon_A} (\partial_A g_{BC}) = (-1)^{\epsilon_C (\epsilon_A + \epsilon_B)} g_{CD} \Gamma^D_{AB} + (-1)^{\epsilon_A \epsilon_B} g_{BD} \Gamma^D_{AC}.$$

This can easily be inverted to yield the familiar formula

$$\Gamma^D_{AB} = (-1)^{\epsilon_A} g^{DC} \Gamma_{C,AB}, \quad \Gamma_{C,AB} = (-1)^{\epsilon_A \epsilon_B} \Gamma_{C,BA},$$

$$2 \Gamma_{C,AB} = (-1)^{\epsilon_C \epsilon_A} (\partial_A \rightarrow g_{CB}) + (-1)^{\epsilon_B (\epsilon_A + \epsilon_C)} (\partial_B \rightarrow g_{CA}) - (\partial_C \rightarrow g_{AB}).$$

In a $P$-adapted coordinate system the parity-mixed components of the Christoffel symbols $\Gamma^D_{AB}$ vanish. The following formulas apply to the non-vanishing components only:

$$\Gamma^D_{AB} = (-1)^{\epsilon_A} g^{DC} \frac{\partial_C}{\partial_A} g_{CB} = (-1)^{\epsilon_A (\epsilon_C + 1)} g^{DC} \frac{\partial_C}{\partial_A} g_{CB},$$

$$g_{AB} = (\partial_A K \frac{\partial_B}{\partial_A} g_{CB}),$$

$$\Gamma_{C,AB} = (K \frac{\partial_C}{\partial_A} \frac{\partial_B}{\partial_A} g_{CB}),$$

where $K$ is a local parity potential. A connection in the field-sector of the field-antifield space has previously been discussed by Alfaro and Damgaard [10] in the context of BV quantization of quantum field theories using Schwinger-Dyson equations.
4.5 Divergence

We define the divergence $\text{div}(X)$ of a bosonic vector field $X$ as

$$\text{div}(X) = \text{str}(\nabla(X)) = (-1)^{\epsilon_A} (\frac{\partial}{\partial z^A} + F_A) X^A ,$$  \hspace{1cm} (4.26)

where the contracted Christoffel symbols $F_A$ for a general connection on the tangent bundle read

$$(-1)^{\epsilon_A} F_A = \Gamma^B_{BA} .$$  \hspace{1cm} (4.27)

$F_A$ is not a tensor. From the gauge transformation like property of the Christoffel symbols it follows that under a coordinate transformation $z^A \rightarrow z'^A(z)$ the contracted Christoffel symbols transforms as

$$F_A = (\frac{\partial}{\partial z^A} z'^A) F'_B + (\frac{\partial}{\partial z^A} \ln \text{sdet}(\frac{\partial}{\partial z^B} z'^A)) .$$  \hspace{1cm} (4.28)

So $\nabla^F = d + F$ with $F = dz^A F_A$ is an (Abelian) connection in the superdeterminant bundle over the manifold. $d - F$ is a connection in the inverse superdeterminant bundle, or equivalently the bundle of volume densities. In general, the presence of two different superdeterminant connection fields $F^{(1)}$ and $F^{(2)}$ witness the existence of a non-trivial global Grassmann-even one-form $\eta = F^{(2)} - F^{(1)}$. Notice that global Grassmann-even one-forms $\eta$ by the symplectic metric is in one-to-one correspondence with global Grassmann-odd vector-fields $V$. We may form another superdeterminant connection $\nabla^F = d + F = d + dz^A F_A$ with connection field

$$F_A = (-1)^{\epsilon_A + 1} \epsilon_B \Gamma^B_{AB} .$$  \hspace{1cm} (4.29)

So $F - F$ is a globally defined one-form. In the torsion-free case, this one-form vanishes identically $F_A = F_A$.

In the case of a Levi-Civita connection, we can rewrite the contracted Christoffel symbols $F_A$ as

$$(-1)^{\epsilon_A} F_A = - (-1)^{\epsilon_B} g^{BC} (\frac{\partial}{\partial z^B} g_{BA}) = - (-1)^{\epsilon_C} (\frac{\partial}{\partial z^C} g^{CB}) g_{BA} .$$  \hspace{1cm} (4.30)

Therefore the divergence takes the form

$$\text{div}(X) = - (-1)^{\epsilon_A} g^{AB} \left( \frac{\partial}{\partial z^B} (g_{AC} X^C) \right) .$$  \hspace{1cm} (4.31)

Thus the Levi-Civita divergence

$$\text{div}(Y_F) = 0$$  \hspace{1cm} (4.32)

vanishes identically for a vector field of the form

$$Y_F = g(F, \cdot) = g(\cdot, F) ,$$  \hspace{1cm} (4.33)

where $F$ is an odd function. However, contrary to even Riemannian geometry where a canonical volume density is $\text{Pf}(g_{AB})$, a volume density in odd Riemannian geometry is not a function of the metric $g_{AB}$. There are Killing symmetries which are not volume preserving. See also the analogous discussion in Section 1.2.
4.6 Odd Laplacian

The odd Laplacian $\Delta(F)$ of an odd function $F$ is

$$\Delta(F) = -\frac{1}{2} \text{div}(X_F), \quad (4.34)$$

where $X_F = (F, \cdot) = -\langle \cdot, F \rangle$ denotes the Hamiltonian vector field for $F$. We have the following commuting diagram

$$C^\infty_{\text{odd}}(M) \xrightarrow{X} \Gamma_{\text{even}}(TM) \xrightarrow{\nabla} \Gamma_{\text{even}}(T^*M \otimes TM)$$

$$\downarrow \text{div} \quad \uparrow \text{str}$$

$$-2\Delta \quad \downarrow \text{div}$$

$$C^\infty_{\text{even}}(M) \quad (4.35)$$

In components $\Delta$ reads

$$\Delta = \frac{1}{4}(-1)^{\epsilon_A}(\overrightarrow{\partial_A} + F_A)E^{AB} \overrightarrow{\partial_B} = \frac{1}{4}(-1)^{\epsilon_A}E^{AB}(F_B^{(0)} + \overrightarrow{\partial_B})\overrightarrow{\partial_A}, \quad (4.36)$$

where $F_A^{(0)}$ is defined as

$$F_A^{(0)} = -E_{AB} \Gamma^{B}_{CD} E^{DC}. \quad (4.37)$$

The last equality in (4.36) does in general only hold for a symplectic connection. To derive it, we used the following contracted version of (4.18):

$$(-1)^{\epsilon_A}((\overrightarrow{\partial_A} + F_A)E^{AB}) = (-1)^{\epsilon_A}F_A^{(0)}E^{AB} = (-1)^{\epsilon_B}E^{RA}F_A^{(0)}. \quad (4.38)$$

Let us mention that $F_A^{(0)}$ has non-trivial transformation properties. However, it vanishes identically in $P$-adapted coordinates. This fact is essential for odd Kähler manifolds (see Section 5).

4.7 Analysis of Odd Laplacian

The odd Laplacian $\Delta$ is a second order operator, or equivalently

$$[[[\Delta, A_1], A_2], A_3] = 0, \quad (4.39)$$

where $A_1$, $A_2$ and $A_3$ are functions (i.e. operators of order 0). The supercommutator $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ of the second order operator $\Delta$ is at most of order $2 + 2 - 1 = 3$. In fact, the Jacobi identity guarantees that it is at most of order 2. We can give a proof which does not use the explicit form of $\Delta$, but merely that it is of second order and Grassmann-odd. First, note that

$$[[[\Delta, A_1], A_2], A_3] = \sum_{\pi \in S_3} (-1)^{\epsilon_\pi}[[\Delta, [\Delta, A_{\pi(1)}], A_{\pi(2)}], A_{\pi(3)}] = 0. \quad (4.40)$$

Here $\epsilon_\pi$ is a Grassmann factor arising when permuting

$$A_1 \ A_2 \ A_3 = (-1)^{\epsilon_\pi}A_{\pi(1)} \ A_{\pi(2)} \ A_{\pi(3)}. \quad (4.41)$$

Recalling that the antibracket can be expressed as a multiple commutator

$$(A, B) = (-1)^{\epsilon(A)}[[\Delta, A], B]1, \quad (4.42)$$

we recognize the right hand side of (4.40) as the Jacobi identity.
4.8 Curvature

The curvature tensor $R$ reads

\[
R = \frac{1}{4} [\nabla \wedge \nabla] = - \frac{1}{2} dz^D \wedge dz^A \otimes [\nabla_A, \nabla_D] = - \frac{1}{2} dz^D \wedge dz^A \otimes \partial_B R^B_{ADF} \, dz^F
\]

\[
R^B_{ADF} = (-1)^{\epsilon_A \epsilon_B} (\partial_A \Gamma^B_{DF}) + \Gamma^B_{AC} \Gamma^C_{DF} - (-1)^{\epsilon_B \epsilon_D} (A \leftrightarrow D) ,
\]

where it is implicitly understood that there is no contractions among the base manifold indices, in this case index $A$ and index $D$. An alternative way of saying this is that one should project to the appropriated space of base manifold two-forms. The Ricci tensor is usually defined as a contraction of a bundle and a base manifold index with the metric

\[
Ric = dz^A \otimes \partial_B \, Ric^B_A , \quad Ric^B_A = R^B_{ADF} \, g^F_D ,
\]

It is Grassmann odd. The Levi-Civita Ricci tensor $Ric$ vanishes on manifolds which possesses a compatible parity structure. There is many possibilities of contracting the curvature tensor with the the metric and the symplectic metric. One way, which actually works for arbitrary vector bundles, is to contract the two base manifold indices with the metric:

\[
Rig \equiv \frac{1}{4} (\epsilon_D g^{DA}) [\nabla_A, \nabla_D] = \frac{1}{4} \epsilon_D g^{DA} \partial_B R^B_{ADF} \otimes dz^F : TM \rightarrow TM .
\]

Much more central for our considerations is the Ricci two-form $R$, that only depends on the connection itself. It is by definition the curvature two-form for the superdeterminant bundle

\[
\frac{1}{2} dz^A R_{AB} \wedge dz^B = R = \frac{1}{2} [\nabla^F \wedge \nabla^F] = - dz^A \wedge d z^B (\partial_B \, \mathcal{F}_A) ,
\]

so that

\[
R_{AB} = (\partial^A_{\mathcal{F}_B} \, (-1)^{\epsilon_B} - (-1)^{(\epsilon_A+1)\epsilon_B} (\partial^A_{\mathcal{F}_B} \, \mathcal{F}_A) = (-1)^{(\epsilon_A+1)(\epsilon_B+1)} R_{BA} .
\]

It could equivally well be defined as the contraction of the bundle indices of the curvature tensor $R$:

\[
R = \text{str} \, R , \quad (-1)^{\epsilon_D} R_{AD} = (-1)^{\epsilon_B (\epsilon_A+\epsilon_D+1)} R^B_{ADB} .
\]

The Ricci two-form $R$ is closed because of the Bianchi identity, so it defines a cohomology class $[R]$, which up to a normalization constant is the first Chern class. On a Ricci-form-flat manifold we can locally find a bosonic function $ln \rho$ such that

\[
\mathcal{F}_A = (\partial^A_{\ln \rho}) .
\]

The function $\rho$ transforms as a volume density. The $\rho$ is determined up to a multiplicative constant.

4.9 Ricci Bracket

We can introduce an even Ricci bracket

\[
(F,G)_{(2)} = [[\Delta^2, F], G]_1 = \frac{1}{4} (F \frac{\partial \mathcal{F}_A}{\partial z^A}) R^{AB} (\partial^A \mathcal{F}_B) = (-1)^{\epsilon_F \epsilon_G} (G, F)_{(2)} ,
\]

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where \( F, G \in C^\infty(M) \) are functions. Here we have introduce a contravariant version of the Ricci-form tensor
\[
\mathcal{R}^{AD} = E^{AB} \mathcal{R}_{BC} E^{CD} = (-1)^{\epsilon_A \epsilon_B} \mathcal{R}^{DA} .
\] (4.51)
The Ricci-bracket satisfies a Jacobi identity
\[
\sum_{\text{cycl. } F, G, H} (-1)^{\epsilon_F \epsilon_H} ((F, G)_{(2)}, H)_{(2)} = 0 .
\] (4.52)
This of course reflects the Bianchi identity. Moreover, the Ricci-bracket has the Poisson property
\[
(F, GH)_{(2)} = (F, G)_{(2)} H + (F, H)_{(2)} G - (-1)^{\epsilon_F \epsilon_H} .
\] (4.53)
The relations with the anti-symplectic structure is as follows \[11\]
\[
[\Delta, \Delta^2] = 0 ,
\]
\[
\Delta(F, G)_{(2)} - (\Delta F, G)_{(2)} - (-1)^{\epsilon_F} (F, \Delta G)_{(2)} = (-1)^{\epsilon_F} \Delta^2 (F, G)
\]
\[
\sum_{\text{cycl. } F, G, H} (-1)^{\epsilon_F (\epsilon_H + 1)} ((F, G), H)_{(2)} = \sum_{\text{cycl. } F, G, H} (-1)^{\epsilon_F (\epsilon_H + 1) + \epsilon_G} ((F, G)_{(2)}, H) .
\] (4.54)

We collect the following equivalent conditions in case of a torsion-free connection:

1. The connection \( \nabla \) is Ricci-form-flat.
2. The superdeterminant connection \( \nabla^F \) is flat.
3. \( (\partial_B^I F_A) = (-1)^{\epsilon_A \epsilon_B} (\partial_A^I F_B) \).
4. There exists locally a volume density \( \rho \) such that \( F_A = (\partial_A^I \ln \rho) \).
5. \( \Delta^2 \) is a linear operator:
\[
\Delta^2 (FG) = \Delta^2 (F) \cdot G + F \cdot \Delta^2 (G) .
\] (4.55)
6. Leibnitz rule for \( \Delta \) and the antibracket:
\[
\Delta(F, G) = (\Delta F, G) - (-1)^{\epsilon_F} (F, \Delta G) .
\] (4.56)

### 4.10 Almost Tensors

Let us define differentiation operators
\[
\partial_A = (P^I)_A^B \partial_B , \quad (\partial^I)_A = (P^I)_A^B \partial_B .
\] (4.57)
A contravariant almost vector \( \tilde{X}^A \) transforms as
\[
\tilde{X}^{IB} = \tilde{X}^A (\tilde{\partial}_A z^B) ,
\] (4.58)
under change of coordinates \( z^A \rightarrow z^A(z) \). An \((r, s)\) almost tensor is the obvious generalization. Almost tensors behaves as tensors under parity preserving transformations. With this notation the Levi-Civita \( \Delta \) can neatly be written as
\[
\Delta = \frac{1}{2} (-1)^{\epsilon_A} g^{AB} \partial_B (P^I)_A^C \partial^I = -\frac{1}{2} (-1)^{\epsilon_A} E^{AB} \partial_B \partial_A = \frac{1}{2} (-1)^{\epsilon_A} E^{AB} \partial_B \partial_A .
\] (4.59)
The last equality in general only holds for \( P \)-adapted coordinates.
4.11 Determinant Density

In this Subsection we consider only the Levi-Civita connection. By use of the Jacobi identity, one can write the contracted Christoffel symbol $F_A$ as a sum of two contributions:

$$F_A = (P^T)_A^B \tilde{F}_B + p_A , \tag{4.60}$$

where

$$\tilde{F}_A \equiv P^B C \Gamma^C_{BA}(-1)^{\varepsilon} = -\frac{1}{2}(-1)^{\varepsilon} B (\overrightarrow{\partial_A} E_{BC}) g^{CB} = -\frac{1}{2}(-1)^{\varepsilon} (\overrightarrow{\partial_A} g^{BC}) E_{CB}$$

and

$$(-1)^{\varepsilon} p_A \equiv (-1)^{\varepsilon} B (P^T)_B^C (\overrightarrow{\partial_A} P^B_A) = -(-1)^{\varepsilon} B (\overrightarrow{\partial_B} P^B_C) P^C_A . \tag{4.62}$$

$\tilde{F}_A$ is zero in Darboux coordinates. It transforms under coordinate transformations $z^A \to z'^A(z)$ as a connection-like field:

$$\tilde{F}_A = (\overrightarrow{\partial_A} z^{A'}) \tilde{F}_{A'} + (\overrightarrow{\partial_A} z^{B'}) (\overrightarrow{\partial_B} z^{C'}) P^A_{B'} (-1)^{\varepsilon}. \tag{4.63}$$

Let us now impose that $P$ is a parity structure. Note that among $P$-adapted coordinates (4.63) reduces to

$$\tilde{F}_A = (\overrightarrow{\partial_A} z^{A'}) \tilde{F}_{A'} + (\overrightarrow{\partial_A} \ln det(\overrightarrow{\partial^l} z^r)) . \tag{4.64}$$

So in that case $\nabla \tilde{F} = d + \tilde{F}$, where $\tilde{F} = dz^A \tilde{F}_A$ is just a connection in the determinant bundle over the manifold. The corresponding curvature two-form is

$$R^\tilde{F} = \frac{1}{2}[\nabla \tilde{F} \backslash \nabla \tilde{F}] = -dz^A \wedge dz^{B'} (\overrightarrow{\partial_B} \tilde{F}_A) \equiv 0 . \tag{4.65}$$

The second equality is true also in non-$P$-adapted coordinates. $R^\tilde{F}$ is identically zero because $\tilde{F}_A$ vanishes in Darboux coordinates. So the connection $\nabla \tilde{F}$ is flat. Hence, there exists locally a density-like function $\tilde{\rho}$, such that

$$\tilde{F}_A = (\overrightarrow{\partial_A} \ln \tilde{\rho}) . \tag{4.66}$$

$\tilde{\rho}$ is uniquely determined up to a multiplicative constant. Moreover, $\tilde{\rho}$ is a constant in Darboux coordinates. Notice that in $P$-adapted coordinates $p_A = 0$ vanishes, so that in these coordinates

$$F_A = (P^T)_A^B \tilde{F}_B , \quad \tilde{F}_A = (\overrightarrow{\partial_A} \ln \tilde{\rho}) . \tag{4.67}$$

If there are two $P$-compatible anti-symplectic structures $E^{(1)}$ and $E^{(2)}$, then the difference of the connection fields $\tilde{F}^{(2)} - \tilde{F}^{(1)}$ is a global closed Grassmann-even one-form. Locally,

$$\tilde{F}^{(2)} - \tilde{F}^{(1)} = (\overrightarrow{\partial_A} \ln \frac{\tilde{\rho}^{(2)}}{\tilde{\rho}^{(1)}}) . \tag{4.68}$$

We shall see in the next Section that for a so-called odd Kähler manifold, many of the above local statements holds globally.

Let us note for later that we can rewrite the divergence as

$$\text{div}(X) = ((-1)^{\varepsilon} A (P^T)_A^B \overrightarrow{\partial_B} P^A_C + (-1)^{\varepsilon} C (P^T)_C^B \tilde{F}_B) X^C . \tag{4.69}$$

By previous definition we get

$$p_A = (\overrightarrow{\partial_A} \ln \rho) - (\overrightarrow{\partial_A} \ln \tilde{\rho}) . \tag{4.70}$$

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5 Odd Kähler Manifolds

5.1 Definition

Recall from Section 3.1 that an odd Kähler manifold \((M, P, E)\) is a \((n|n)\) supermanifold \(M\) with a non-degenerate anti-symplectic structure \(E\) and a parity structure \(P\) that are mutual compatible. The corresponding metric \(g\) is called an odd Kähler metric. An odd Kähler manifold \((M, P)\) is a \((n|n)\) supermanifold \(M\) with a parity structure \(P\), that admits an odd Kähler metric.

In the next two Sections we consider an odd Kähler manifold \((M, P, E)\). As a consequence of the definition the Levi-Civita connection \(\nabla\) preserves the parity structure \(P\) and it is a symplectic connection, cf. Subsection 4.3. The holonomy group is \(G \subseteq GL(n)\). This should be compared with the holonomy group \(G \subseteq U(n)\) of the usual Kähler manifolds. Also note that the Lie-bracket between a “holomorphic” and an “anti-holomorphic” vector field vanishes. In a \(P\)-adapted coordinate system the algebra of vector fields

\[
\left[ \frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B} \right] = c_{ABC}^C(z) \frac{\partial}{\partial z^C}
\]

reduces into two independent algebras, i.e. the only non-vanishing components of the structure functions \(c_{ABC}^C(z)\) are the boso-bose-bose and the fermi-fermi-fermi components.

5.2 Kähler Potential

Note that although a Kähler potential \(K\) may only be locally defined, the formula (2.19) (= (5.4) below) still holds for arbitrary \(P\)-adapted coordinate systems. In other words, it makes sense to view \(K\) as a local odd scalar function. Moreover, the odd Laplacian on \(K\) is equal to half the dimension of the manifold:

\[
(\Delta K) = n.
\]

5.3 Canonical Determinant density

There is a globally defined determinant density \(\tilde{\rho}\), cf. Subsection 4.11. In \(P\)-adapted coordinate systems it is defined as the determinant

\[
\tilde{\rho} = \det(E_{\dot{\alpha}\alpha})
\]

of the purely bosonic matrix

\[
E_{\dot{\alpha}\alpha} = \left( \frac{\partial}{\partial \theta^{\dot{\alpha}}} K \frac{\partial}{\partial x^\alpha} \right).
\]

Note that \(\tilde{\rho}\) within \(P\)-adapted coordinate systems behaves as a determinant density. It is straightforward to see that \(\tilde{\rho}\) up to a multiplicative constant is equal to the locally defined density-like object of Section 4.11. Thus the locally defined \(\tilde{\rho}\) of Section 4.11 can be patched together to yield a globally defined density-like object. Let us (as always in this paper) assume that the supermanifold \(M\) is orientable, so that there exists an atlas of \(P\)-adapted charts such that \(\text{body}(\tilde{\rho}) > 0\) everywhere. (This should be compared with the situation for (bosonic) complex manifolds. These are automatically orientable.) We also concluded in Section 4.11, cf. (4.67), that

\[
F_A = (P^T)_A^B \tilde{F}_B, \quad \tilde{F}_B = \left( \frac{\partial}{\partial B} \ln \tilde{\rho} \right).
\]
From (5.5) follows directly that the Ricci-form $\mathcal{R}$ is a $(1,1)$-form wrt. the Dolbeault bi-complex.

$$\mathcal{R} = 2(\bar{\partial} \wedge \partial \ln \tilde{\rho}) , \quad (5.6)$$

However, $\mathcal{R}$ is not necessary exact, because $\ln \tilde{\rho}$ is not a scalar object. Note that if $E'$ is another $P$-compatible anti-symplectic structure, the corresponding Ricci-form $\mathcal{R}'$ belongs to the same cohomology class

$$\mathcal{R}' - \mathcal{R} = 2(\bar{\partial} \wedge \partial \ln \tilde{\rho}') . \quad (5.7)$$

Thus the cohomology class $[\mathcal{R}]$ of the Ricci-form, i.e. the first Chern class up to a proportionality factor, is independent of the anti-symplectic structure. So if $M$ admits a Ricci-form-flat anti-symplectic structure, then the first Chern class vanishes.

## 6 Odd Calabi-Yau Manifolds

### 6.1 Definition

Let us define an odd Calabi-Yau manifold $(M, P)$ to be a Kähler manifold $(M, P)$ with a vanishing first Chern class.

If we naively should translate Calabi’s conjecture [12] into the odd case, we are asking whether there exists a $P$-compatible anti-symplectic two-form $E'$ in the same Kähler class $[E]$, such that the corresponding Levi-Cevita Ricci-form $R' \equiv 0$ vanishes identically? Calabi managed (in the case of usual Kähler manifolds) to prove the uniqueness of the Ricci-form-flat Kähler 2-form $[\omega']$ using the very powerful maximum-modulus principle of complex analysis. But we don’t have such a principle in the odd case. And in fact we shall soon see that uniqueness does not hold in the odd case. The existence question (the analogue of Calabi’s conjecture, i.e. Yau’s Theorem [12]) is an interesting question, that we however shall not address in this paper.

Let us mention that an important consequence of the Ricci-form flatness condition is that the holonomy group is $G \subseteq SL(n)$. This should be compared with the holonomy group $G \subseteq SU(n)$ for the usual Calabi-Yau manifold.

### 6.2 Holomorphic $n$-form

Let us assume that the Ricci-form-tensor is of the form $\mathcal{R} = 2(\bar{\partial} \wedge \partial \ln s)$, where $s = s(x, \theta)$ is a global scalar function with body$(s) > 0$. Locally, this is of course true. (In Subsection 3.6 we gave a sufficient condition for this to be true globally. However, the assumptions given there are pretty restrictive, so we prefer just to assume the global existence. After all, the most interesting case is $s \equiv 1$.) So

$$\frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial \theta^\alpha} \ln \frac{\tilde{\rho}}{s} = 0 . \quad (6.1)$$

The full solution is that $\tilde{\rho}s^{-1}$ factorizes

$$\tilde{\rho}(x, \theta) s(x, \theta)^{-1} = \rho_+(x) \rho_-(\theta)^{-1} , \quad (6.2)$$

2Often in the Literature compactness is included in the definition of a Calabi-Yau manifold, but we shall not do so.
where $\rho_+(x)$ and $\rho_-(\theta)$ are invertible Grassmann-even functions. Clearly, they are uniquely determined up to a multiplicative constant. If the first Čech cohomology class $\check{H}^{1}_{\text{const}}(M, \mathbb{R}) = 0$ vanishes, the $\rho_+(x)$ and $\rho_-(\theta)$ are globally defined, see Subsection 3.3. Then there exists a globally defined nowhere-vanishing “holomorphic” $n$-form

$$\Omega^{(+)}_{\text{vol}} = d^n x \rho_+ = \frac{1}{n!} \Omega^{(+)}_{\alpha_1 \alpha_2 \ldots \alpha_n} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_n}.$$  \hspace{1cm} (6.3)

It is straightforward to derive that

$$\nabla \Omega^{(+)}_{\text{vol}} = 0 \Leftrightarrow (\partial s) = 0.$$  \hspace{1cm} (6.4)

So the “holomorphic” $n$-form is covariantly preserved iff the anti-symplectic structure is Ricci-form-flat. If $\Omega^{(+)}_{\text{vol}} = d^n x \rho'_+$ is another nowhere-vanishing “holomorphic” $n$-form, then $s' = \frac{\rho'_+}{\rho_+}$ is a nowhere-vanishing “holomorphic” scalar. If both $n$-forms are covariantly preserved, then $s'$ is a constant on each connected components of $M$. In particular we have proven:

**Proposition.** Given a connected odd Kähler manifold $M$ that admits an $P$-compatible Ricci-form-flat anti-symplectic structure. A nowhere-vanishing, covariantly preserved, “holomorphic” $n$-form is unique up to a multiplicative constant. If the first Čech cohomology class $\check{H}^{1}_{\text{const}}(M, \mathbb{R}) = 0$ vanishes, then there exists a globally defined nowhere-vanishing, covariantly preserved, “holomorphic” $n$-form $\Omega^{(+)}_{\text{vol}}$.

### 6.3 Canonical Volume Form

Let $M$ be a Ricci-form-flat odd Kähler manifold. If the nowhere-vanishing, covariantly preserved, “holomorphic” $n$-form $\Omega^{(+)}_{\text{vol}}$ exists, we may define an “anti-holomorphic” pendant

$$\Omega^{(-)} = \frac{1}{n!} \Omega^{(-)}_{\alpha_1 \alpha_2 \ldots \alpha_n} d\theta^{\alpha_1} \vee d\theta^{\alpha_2} \vee \ldots \vee d\theta^{\alpha_n}, \hspace{1cm} \frac{1}{n!} \Omega^{(-)}_{\bar{\alpha}_1 \bar{\alpha}_2 \ldots \bar{\alpha}_n} = \rho_-(\theta)^{-1} = \frac{\tilde{\rho}(x, \theta)}{\rho_+(x)}.$$  \hspace{1cm} (6.5)

Recall that for Grassmann-odd variables differential forms are not directly related to integration theory and volume forms. However in the case at hand, we may easily construct a volume form $\Omega_{\text{vol}}^{(M)} = d^{2n} z \rho$. In a local $P$-adapted coordinate system the density $\rho$ is simply

$$\rho = \rho_+(x) \rho_-(\theta).$$  \hspace{1cm} (6.6)

### 6.4 Nilpotency

In the case of an odd Kähler manifold $M$, we can write the square of the odd Levi-Civita Laplacian in $P$-adapted coordinates as

$$\Delta^2 = \frac{i}{2} \bar{R}^{\bar{\alpha}_\alpha} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial \theta^{\bar{\alpha}}}. \hspace{1cm} (6.7)$$

This fact follows from (4.50) and the observation that $\Delta^2$ has no monomial with less than two differentiations when $\Delta^2$ is normal-ordered, i.e. differential operators ordered to the right, cf. expression (4.36) or expression (4.59). As a consequence we have:

**Theorem.** Let $M$ be an odd Kähler manifold. $M$ is Ricci-form-flat if and only if the odd Levi-Civita Laplacian $\Delta$ is nilpotent.
6.5 Standard Example

The standard example of an odd Calabi-Yau manifold is the cotangent bundle $M = \Omega^1(L)$ of a bosonic $n$-dimensional manifold $L$ that satisfies certain extra properties listed below. We identify $\theta^\alpha = dx^\alpha$. The Grassmann parity (i.e. the form parity) is here a global parity structure $P$. The manifold $L$ should be equipped with a covariant non-degenerate two-tensor

$$g = dx^\alpha g_{\alpha\bar{\alpha}}(x) \otimes \theta^{\bar{\alpha}} \in \Gamma(T^*(L) \otimes T^*(L))$$

such that

$$\overrightarrow{\partial_\alpha^I g_{\beta\bar{\alpha}}} = \overrightarrow{\partial_\beta^J g_{\alpha\bar{\alpha}}} .$$

(6.9)

So locally we can introduce a Kähler potential $K = \eta_\alpha(x) \theta^{\bar{\alpha}}$, where

$$\overrightarrow{\partial_\alpha^I \eta_\beta} = g_{\alpha\bar{\alpha}} .$$

(6.10)

We introduce for convenience the “transposed” matrix $g^{\bar{\alpha}\alpha} = g_{\alpha\bar{\alpha}}$. The Ricci-form-flat anti-symplectic structure is $E = d\theta^{\alpha} g_{\alpha\bar{\alpha}} \wedge dx^\alpha$. In the generic case, the Ricci-form-flat structure (already within a given cohomology class) is not unique. So Calabi’s Uniqueness Theorem does not hold for the odd case.

6.6 Lagrangian Density

Now fix a point $m \in M$ in the manifold and consider the two Lagrangian surfaces $L_\pm$ that intersect $m$. Contrary to even symplectic geometry the canonical volume form $\Omega^{(M)}_{\text{vol}}(L)$ actually induces canonical volume forms $\Omega^{(\pm)}_{\text{vol}}$ on the Lagrangian surfaces $L_\pm$, respectively. They are defined via

$$\Omega^{(\pm)}_{\text{vol}}(e(1), \ldots, e(n)) = \sqrt{\Omega^{(M)}_{\text{vol}}(e(1), \ldots, e(n), c^+_1, \ldots, c^+_n)} ,$$

$$\Omega^{(\pm)}_{\text{vol}}(e(1), \ldots, e(\bar{n})) = \sqrt{\Omega^{(M)}_{\text{vol}}(e(1), \ldots, e(\bar{n}), c^-_1, \ldots, c^-_n)} .$$

(6.11)

Here $(e(\alpha))_{\alpha=1,\ldots,n}$ and $(e(\bar{\alpha}))_{\bar{\alpha} = \bar{1}, \ldots, \bar{n}}$ denotes a basis for the tangent spaces $TL_\pm$, respectively, and $\ast$ denotes the odd parity conjugation, cf. Section 3.3. From the formulas

$$\partial^+_{A} = E^{a\alpha} \partial_\alpha^I , \quad \partial^-_{A} = E^{a\alpha} \partial_\bar{\alpha}^I ,$$

(6.12)

we conclude (up to an inessential over-all sign convention) that

$$\Omega^{(\pm)}_{\text{vol}} = d^n x \rho_+ , \quad \Omega^{(\pm)}_{\text{vol}} = d^n \theta \rho_- .$$

(6.13)

In other words, the bosonic Lagrangian volume density $\Omega^{(\pm)}_{\text{vol}}$ is precisely the previously mentioned “holomorphic” $n$-form.

6.7 Odd Symplectic Potential

There are three natural choices of a symplectic potential. Perhaps the most symmetric choice is $\vartheta = \vartheta'$, where

$$\vartheta' = \vartheta'_A dz^A \quad \vartheta'_A = \frac{1}{2} (P^T)_{AB} \overrightarrow{\partial_\beta^B} K ,$$

$$\vartheta'_{\bar{\alpha}} = \frac{1}{2} (\overrightarrow{\partial_{\bar{\alpha}}} K) \quad \vartheta'_{\bar{\alpha}} = -\frac{1}{2} (\overrightarrow{\partial_{\bar{\alpha}}} K) .$$

(6.14)
Note that the first two formulas are covariant. They hold in arbitrary coordinates. So if the Kähler potential $K$ is globally defined, so is the odd symplectic potential $\vartheta'$. A second choice is

$$\vartheta = \vartheta_A dz^A, \quad \vartheta'_A = (P^T_A) B(\frac{\partial f}{\partial \varpi^A}) K,$$  

(6.15)

$$\vartheta_\alpha = \left(\frac{\partial}{\partial x^\alpha} K\right), \quad \vartheta_{\bar{\alpha}} = 0.$$  

We can change coordinates $(x^\alpha; \theta^\beta) \rightarrow (x^\alpha; \vartheta_\alpha)$ to Darboux coordinates $(x^\alpha; \vartheta_\alpha)$,

$$(x^\alpha, x^\beta) = 0, \quad (x^\alpha, \vartheta_\beta) = \delta^\alpha_\beta, \quad (\vartheta_\alpha, \vartheta_\beta) = 0.$$  

(6.16)

The super Jacobian of the coordinate transformation is $J = \det(E_{\bar{\alpha} \alpha})^{-1} = \rho_+ \rho_-^{-1}$, so the canonical volume density in these Darboux variables reads

$$\rho = \rho_+^2.$$  

(6.17)

Hence the $\Delta$ operator becomes

$$\Delta = E^{\bar{\alpha} \bar{\beta}} \frac{\partial}{\partial \theta^\alpha}(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial \vartheta_\alpha} \rho_+(\frac{\partial}{\partial x^\alpha}) \vartheta_{\bar{\beta}}.$$  

(6.18)

The new coordinates $(x^\alpha; \vartheta_\alpha)$ does not adapt the $P$-structure. In a Grassmann $2 \times 2$ block representation the parity structure reads

$$P = \begin{bmatrix} 1 & 0 \\ 2K & -1 \end{bmatrix},$$  

(6.19)

where $K$ denote the $n \times n$ matrix with entries

$$K_{\alpha \beta} = \left(\frac{\partial}{\partial x^\alpha} K \frac{\partial}{\partial x^\beta}\right).$$  

(6.20)

Despite this fact, they are related to a Fourier transform, see next Section. Moreover, in a mixed notation, where we use both coordinate systems $(x^\alpha; \vartheta^\beta)$ and $(x^\alpha; \vartheta_\alpha)$, the odd Laplacian $\Delta$ operator acquires a Darboux-like form

$$\Delta = \frac{\partial}{\partial \vartheta_\alpha}(\frac{\partial}{\partial x^\alpha}) \vartheta_{\bar{\beta}}, \quad (\frac{\partial}{\partial x^\alpha}) = \left(\frac{\partial}{\partial x^\alpha} K \frac{\partial}{\partial x^\beta}\right) \frac{\partial}{\partial \vartheta_{\bar{\beta}}}.$$  

(6.21)

Let us for completeness list the third natural choice $\vartheta = \chi$

$$\chi = \chi_A dz^A, \quad \chi_A = (P^T_A) B(\frac{\partial f}{\partial \varpi^A}) K,$$  

(6.22)

$$\chi_\alpha = 0, \quad \chi_{\bar{\alpha}} = -\left(\frac{\partial}{\partial x^\alpha} K\right).$$

We now choose Darboux coordinates $(\chi_{\bar{\alpha}}; \theta^\beta)$

$$(\chi_{\bar{\alpha}}, \chi_{\bar{\beta}}) = 0, \quad (\chi_{\bar{\alpha}}, \theta^\beta) = \delta^\beta_{\bar{\alpha}}, \quad (\theta^\alpha, \theta^\beta) = 0.$$  

(6.23)

The super Jacobian of the coordinate transformation $(x^\alpha; \theta^\beta) \rightarrow (\chi_{\bar{\alpha}}; \theta^\beta)$ is $J = \det(E_{\bar{\alpha} \alpha}) = \rho_+ \rho_-^{-1}$, so the canonical volume density is $\rho = \rho_+^2$. 

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6.8 Fourier Transform

Let there be given a \(P\)-adapted bosonic \((n|0)\) Lagrangian surface \(L\) in \(M\) and a globally defined Kähler potential \(K\). We equip \(L\) with the canonical volume density \(\rho_L = \rho_+\) on \(L\). In a \(P\)-adapted coordinate system, we may write

\[
L = \{(x^\alpha; \theta^\alpha)|\theta^\alpha = \theta^\alpha_{(0)}\}.
\]

We may assume that the local \(P\)-adapted charts in the atlas are of the box-type \(U \times \mathbb{R}^n_+\), where \(U \subseteq \mathbb{R}^n_+\) and \(\mathbb{R}^n_+\) denotes the set of Grassmann-even (Grassmann-odd) real supernumbers, respectively, see [13]. In other words, we may assume that the \(P\)-adapted fermionic \((0|n)\) Lagrangian surfaces are covered by a single chart.

\[
L^{-}(x_{(0)}) = \{(x^\alpha; \theta^\alpha)|x^\alpha = x^\alpha_{(0)}\}.
\]

We now define a Fourier transform \(\wedge: \mathcal{C}^\infty(M) \to \Omega^*(L)\) from the space of functions \(f(x, \theta)\) on \(M\) to the exterior algebra of forms \(\hat{f}(x, \theta)\) on the Lagrangian surface \(L\).

\[
\hat{f}(x, c) = \int_{L^{-}_{(0)}} \exp \left[ c^\alpha (\frac{\partial}{\partial x^\alpha} f(x, \theta) - \theta^\alpha_{(0)}) \right] f(x, \theta) \Omega^{\wedge}_{\text{vol}}(\theta) = \int_{L^{-}_{(0)}} \exp \left[ c^\alpha (\frac{\partial}{\partial x^\alpha} f(x, \hat{\theta}) - \theta^\alpha_{(0)}) \right] f(x, \hat{\theta}) \, d^n \hat{\theta} \cdot \rho_-(\theta + \theta_{(0)})
\]

\[
= \rho_+(x) \int_{L^{-}_{(0)}} \exp \left[ c^\alpha (\frac{\partial}{\partial x^\alpha} f(x, \theta)) + \theta^\alpha_{(0)}) \right] f(x, \theta) \, d^n \theta.
\]

Note that the Fourier transform acts from the right. Here the Grassmann-odd variables \(c^\alpha\) play the role of the natural basis of one-forms \(dx^\alpha\), i.e. they transform in the same way with the form-degree replaced by the Grassmann-degree. In the second equality, we substituted

\[
\tilde{\theta} = (\frac{\partial}{\partial x^\alpha} K)_{\theta} (x, \theta)\cdot (\theta^\alpha_{(0)}).
\]

Clearly \(\tilde{\theta} \to \hat{\theta}\) is a bijection, whose inverse we denote by \(\hat{\theta} = \tilde{\theta}(\theta, \tilde{\theta})\). We have

\[
(\frac{\partial}{\partial x^\alpha} f) = c^\alpha \hat{f},
\]

\[
\wedge \rho_+(\frac{\partial}{\partial x^\alpha} \theta^\alpha_{(0)} f) = (\frac{\partial}{\partial x^\alpha} \hat{f}),
\]

\[
\wedge (\frac{\partial}{\partial x^\alpha} \theta^\alpha_{(0)} f) = (\frac{\partial}{\partial x^\alpha} \hat{f}).
\]

In particular, the Fourier transform of the odd Laplacian is the exterior derivative [14].

\[
\hat{\Delta} f = d \hat{f}, \quad d = c^\alpha \frac{\partial}{\partial x^\alpha}.
\]

The inverse Fourier transform reads

\[
f(x, \theta) = \rho_+(x)^{-1} \int \exp \left[ (\frac{\partial}{\partial x^\alpha} (x, \theta) - \theta^\alpha_{(0)}) c^\alpha \right] \hat{f}(x, c) \, d^n c.
\]

Note that the fermionic top-monomial

\[
\prod_{\alpha} (\theta^\alpha_{(0)} - \theta^\alpha_{(0)}) \propto \frac{\prod_{\alpha} (\theta^\alpha_{(0)} - \theta^\alpha_{(0)})}{\rho_-(\theta)} \quad \Rightarrow \quad 1.
\]
is mapped to a constant by the Fourier transform. But the Poincaré Lemma states that the only non-trivial local DeRahm cohomology is the constant zero-forms. We therefore have:

**Local Cohomology Theorem.** Given a Ricci-form-flat odd Kähler manifold $M$. Locally, the solutions $f$ to the equation

$$（\Delta f）= 0 .$$

are of the form

$$f = (\Delta \Psi) + c \Theta ,$$

where $\Psi$ is a function of opposite Grassmann parity, $c$ is a constant and $\Theta$ is the fermionic top-monomial:

$$\Theta \equiv \prod _{\alpha} \theta^{\alpha} .$$

Whereas $\Theta$ is not a covariant object the one-dimensional $\Delta$-cohomology class $\{[c\Theta]|c \in \mathbb{R}\}$ is. The fact that $\Delta$ has non-trivil cohomology given by the fermionic top-monomial has previously been reported in [15].

7 **Vielbein Formulation**

7.1 **Vielbeins**

In general, there is no canonical way of introducing an almost parity structure $P$. We shall see that a vielbein formulation overcome this difficulty.

We consider an $(n|n)$ supermanifold $M$ equipped with an anti-symplectic vielbein $e^a_A$, of Grassmann parity $ε(e^a_A) = ε_a + 1$, i.e. a diffeomorphism

$$e = ∂^\alpha_a e^a_A dz^A : TM → TW .$$

(7.1)

Here “$w$-space” $TW = W$ is an anti-symplectic vector space, with a constant almost Darboux metric $E_{(0)}^{ab}$. We denote the basis $∂^\alpha_a$ and dual basis for $dw^a$, both of Grassmann parity $ε_a$, although we will not necessarily give any sense to a $w^a$ coordinate. The inverse vielbein map is denoted

$$e^{-1} = ∂^\alpha_a e^A_a dw^a : TW → TM .$$

(7.2)

We have the orthonormality relations

$$e^a_A e^b_A = δ^a_b , \quad e^a_A e^a_B = δ^A_B , \quad [e,e^{-1}] = \text{Id}_{TW} - \text{Id}_{TM} , \quad [e,e] = 0 , \quad [e^{-1},e^{-1}] = 0 .$$

(7.3)

The anti-symplectic structure is given as

$$E^{AB} = e^A_a E_{(0)}^{ab} (e^T)^b_B , \quad E_{AB} = (e^T)^a_A E_{ab}^{(0)} e^b_B , \quad E = -\frac{1}{2}[e \wedge [e \wedge E^{(0)}]] .$$

(7.4)

where the supertransposed vielbein $e^t = dz^A (e^T)^a_A i^i_a : T^*W → T^*M$ is

$$(e^T)^a_A = (-1)^{(ε_a+1)ε_A} e^a_A .$$

(7.5)
Let us also introduce the vielbein one-forms

\[ e^a = e^a_A \, dz^A = dz^A (e^T)_A^a . \]  

(7.6)

We will restrict our attention to vielbein formulations where the vielbeins \( \partial_a^{(z)} = (e^T)_a^A \partial_A \) spans a unit volume-cell of \( TM \) (up to a sign):

\[ \text{vol}(\partial_a^{(z)}) = \pm 1 , \]  

(7.7)

where "vol" denotes the *signed* volume of a frame. This is a non-trivial but very reasonable requirement for a vielbein formulation, thereby linking in a natural way the notion of volume density and the notion of anti-symplectic metric. A canonical volume density is then given as

\[ \rho = s\det(e) \text{ vol}(\partial_a^{(z)}|a=1,\ldots,2n) = \pm s\det(e) . \]  

(7.8)

Let us for simplicity assume that the manifold is orientable, so that we can treat "vol" as being single-valued.

### 7.2 Canonical Almost Parity Structure

The manifold possesses a *canonical almost parity structure* \( P \),

\[ P^A_B = e^A_a (-1)^{\epsilon_a} e^a_B . \]  

(7.9)

\( P \) is compatible with the anti-symplectic structure \( E \) in the sense of (2.10). In \( P \)-adapted coordinates, the vielbein \( e^a_A \) acquires a block-diagonal form

\[ e^a_A \neq 0 \Rightarrow \epsilon_a = \epsilon_A . \]  

(7.10)

Moreover, the vielbein \( \partial_a^{(z)} = (e^T)_a^A \partial_A \) separates into "holomorphic" and "anti-holomorphic" directions for the canonical \( P \)-structure. This can be seen by noting that the vielbein is the diagonalizing transformation in the tangent space \( TM \) for the almost parity structure \( P \). A coordinate system is called a *Grassmann preserving coordinate system*, iff all the non-vanishing entries of the vielbein is bosonic. Grassmann preserving coordinates are therefore the same as \( P \)-adapted coordinates. The manifold \( M \) is an odd pre-Kähler manifold wrt. the canonical almost parity structure \( P \), if it admits an atlas of Grassmann preserving coordinates.

We may introduce the "flat" bosonic (fermionic) volume factor

\[ \rho^{(0)}_\sigma = \text{vol}(\partial_a^{(z)}) , \quad \sigma = \pm 1 , \]  

(7.11)

spanned by the \( n \) bosonic (fermionic) vielbeins \( \partial_a^{(z)} \), \( \epsilon_a = 0 \) (\( \epsilon_a = 1 \)), respectively. A vielbein is called *special*, iff

\[ \text{vol}(\partial_a^{(z)}) \text{ vol}(\partial_a^{(z)}) = \text{vol}(\partial_a^{(z)}) . \]  

(7.12)

For \( P \)-adapted coordinates the canonical volume form factorizes. In case of a special vielbein we may write:

\[ \rho = \rho_+ \rho_- , \quad \rho_\pm = s\det(e^a_{\pm A}) \rho^{(0)}_\pm = \det(e^a_{\pm A})^{\pm 1} \rho^{(0)}_\pm . \]  

(7.13)
7.3 Gauge Group

There is a structure-preserving gauge group acting on the flat indices of the vielbeins. In other words, the group is a subgroup of $GL(TW)$. (We shall be more explicit about the group below).

A group element
\[ \Lambda = \partial_a^* \Lambda^a_b \xrightarrow{\text{dw}^b} : TW \to TW \]  
acts on the vielbein as
\[ e^A_a \to (\Lambda e)^A_a = \Lambda^a_b e^b_A , \]
\[ e^A_a = (e^{-1})^A_a \to ((\Lambda e)^{-1})^A_a = e^A_b (\Lambda^{-1})^b_a . \]  

The group action reflects the gauge freedom in choice of vielbein. The above requirement that the vielbein should represent the volume element reduces the gauge group to the subgroup $G$ whose elements has superdeterminant $\pm 1$. In the light of the canonical almost parity structure $P$, we could just as well define the vielbein map $e : TM \to TW$ in terms of the canonical odd metrics
\[ g_{AB} = (e^T)_A^a g^{(0)}_{ab} e^b_B , \quad g = -\frac{1}{2} [e \gamma [e \gamma g^{(0)}] . \]

where as usual the odd metric is
\[ g_{AB} = E_{AC} P_{CB} , \quad g^{(0)}_{ab} = E^{(0)}_{ab} (-1)^{\epsilon_b} . \]

We see that the gauge group $G \subseteq GL(TW)$ for the flat indices should preserve 1) the anti-symplectic metric, 2) the metric and 3) the volume up to a sign. This means the group elements $\Lambda \in G$ satisfy
\[ E_{\text{ad}}^{(0)} = \Lambda^a_b E^{bc}_{(0)} (\Lambda^T)_c^d , \quad g_{\text{ad}}^{(0)} = \Lambda^a_b g^{bc}_{(0)} (\Lambda^T)_c^d , \quad \text{sdet}(\Lambda) = \pm 1. \]

$G$ is the subgroup of the orthosymplectic group whose elements has superdeterminant $\pm 1$. This is isomorphic to $GL(n) \cap \text{det}^{-1}(\{\pm 1\})$.
\[ G = Osp(n|n) \cap \text{sdet}^{-1}(\{\pm 1\}) \cong GL(n) \cap \text{det}^{-1}(\{\pm 1\}) . \]

To see this, consider the usual Grassmann $2 \times 2$ block representation of the matrices. Let us fix notation
\[ E_{(0)}^- = \begin{pmatrix} 0 & \eta^{-1,T}_{(0)} \\ -\eta_{(0)} & 0 \end{pmatrix} , \quad E_{(0)}^+ = \begin{pmatrix} 0 & -\eta_{(0)} \\ \eta^T_{(0)} & 0 \end{pmatrix} , \]
\[ g^-_{(0)} = \begin{pmatrix} 0 & \eta^{-1,T}_{(0)} \\ \eta_{(0)} & 0 \end{pmatrix} , \quad g^+_{(0)} = \begin{pmatrix} 0 & \eta_{(0)} \\ \eta^T_{(0)} & 0 \end{pmatrix} , \]
\[ P_{(0)}^- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]

The group elements $\Lambda \cdot$ are of the form
\[ \Lambda \cdot = \begin{pmatrix} \lambda & 0 \\ 0 & \eta^{-1,T}_{(0)} \eta_{(0)} \end{pmatrix} . \]

We shall only discuss real representations. Then the superdeterminant is positive
\[ 0 \leq \text{det}(\lambda)^2 = \text{sdet}(\Lambda) = \pm 1 , \]
so the $N \times N$ bose-bose block $\lambda$ has determinant

$$\det(\lambda) = \pm 1.$$  

(7.23)

Hence we have a non-rigid $GL(n) \cap \det^{-1}(\{\pm 1\})$ gauge symmetry in each point. We see that the bosonic and the fermionic volume factors $\rho_\sigma^{(0)}$ is preserved up to a sign under gauge transformation. And the product $\rho_+^{(0)} \rho_-^{(0)}$ is invariant.

### 7.4 Levi-Civita Connection

Having a canonical odd metric $g$ at our disposal, we can construct the Levi-Civita connection

$$\nabla^T = d + \Gamma : TM \times TM \to TM ,$$  

(7.24)

where

$$\Gamma = dz^A \partial_B \Gamma^B_{AC} \, dz^C = e^a \partial_b^{(z)} \Gamma^b_{ac} \, e^c ,$$  

(7.25)

and where

$$\Gamma^b_{ac} \equiv (-1)^{(\epsilon_a + \epsilon_c + \epsilon_b)} (e^T)_a^A \, \partial_B e^b \Gamma^B_{AC} \, e^C_c = -(1)^{(\epsilon_a + 1)(\epsilon_c + 1)} e^b \Gamma^B_{CA} e^A_a \, e^C_c .$$  

(7.26)

In the second equality of (7.24), we used the symmetry property (1.12) between the lower indices of the Levi-Civita Christoffel symbols. The upper flat index is lowered with the flat metric:

$$\Gamma^d_{ab} = (-1)^{\epsilon_a} g^{d(c)} \Gamma_{c,ab} ,$$  

$$\Gamma_{c,ab} = (e^T)_c^A \Gamma^B_{CA} \, e^A_a \, e^B_b (-1)^{\epsilon_a \epsilon_B}.$$  

(7.27)

We stress the fact that the flat Christoffel symbols depend on the choice of the curved coordinate system. To facilitate writing down the formulas we will introduce short hand notation for the most common combinations of vielbeins. We introduce structure functions $f_{a}^{b} c$, $f_{a b}^{c}$, $\gamma_{a b c}$ and $a_{a b c}$, mutually related as indicated below:

$$f_{a}^{b} c \equiv (e^T)_a^A \partial_A (\partial^D e^b) \, e^D c = (\partial^D e^b) ((\ln e)^b_c) ,$$  

$$f_{a b}^{c} \equiv (e^T)_a^A \partial_A (\partial^D (e^T)_b^D) \, e^D e^c = (\partial^D e^b) ((\ln e^{-1} T)_b^c) = -(1)^{\epsilon_b(\epsilon_c + 1)} f_{a b}^c ,$$  

$$\gamma_{a b c} \equiv (e^T)_a^A (\partial^D (e^T)_b^D) \, g_{d b}^{(0)} \, e^D c (-1)^{\epsilon_b \epsilon_d} = -(1)^{\epsilon_b(\epsilon_c + 1)} f_{a b}^c g_{d b}^{(0)} ,$$  

$$a_{a b c} \equiv -(1)^{\epsilon_a(\epsilon_d + 1)} g_{d b}^{(0)} f_{a b}^c = -1)^{\epsilon_a \epsilon_b \gamma_{a b c}}.$$  

(7.28)

The Grassmann parity is

$$\epsilon(f_{a}^{b} c) = \epsilon(f_{a b}^{c}) = \epsilon_a + \epsilon_b + \epsilon_c = \epsilon(\gamma_{a b c}) + 1 = \epsilon(a_{a b c}) + 1.$$  

(7.29)

With the help of the identity

$$(e^T)_a^A (\partial^D (g_{C B}) \, e^B \, e^C (-1)^{\epsilon_b \epsilon_C}) = \gamma_{a b c} + (-1)^{\epsilon_b \epsilon_C} \gamma_{a b c} \equiv \gamma_{a b c} ,$$  

(7.30)

the Levi-Civita formula (1.24) translates into

$$2 \Gamma_{c,ab} = -(1)^{\epsilon_a \epsilon_c} \gamma_{a b c} + (-1)^{\epsilon_b(\epsilon_a + \epsilon_c)} g_{b d}^{(0)} - \gamma_{a b c}.$$  

28
\[ (-1)^{\epsilon_a \epsilon_b} \gamma_{abc} + (-1)^{\epsilon_a (\epsilon_a + \epsilon_b)} \gamma_{abc} + (-1)^{\epsilon_a \epsilon_b + \epsilon_a \epsilon_c + \epsilon_b \epsilon_c} \gamma_{bac} + (-1)^{\epsilon_b (\epsilon_a + \epsilon_c)} \gamma_{bca} \]
\[ -\gamma_{cab} - (-1)^{\epsilon_a \epsilon_b} \gamma_{cua} . \]  

(7.31)

If \( M \) is an odd Kähler manifold wrt. the canonical \( P \)-structure, the only non-vanishing components of the raised Christoffel symbols \( \Gamma^b_{ac} \) are the pure bosonic and the pure fermionic components. We assume for the rest of this paper the symplectic condition (4.16).

### 7.5 Jacobi Identity

Certain (partially) antisymmetrized versions of the abovementioned structure functions are independent of the curved coordinate system that was used to define them in the first place. We mention, most notably

\[ f_{[ab]}^c = f_{ab}^c - (-1)^{\epsilon_a \epsilon_b} f_{bc}^c, \quad a_{b[ac]} = a_{bac} - (-1)^{\epsilon_a \epsilon_c} a_{bca} . \]

(7.32)

The Jacobi identity for the anti-bracket, or equivalently the closeness of \( E \) yields Jacobi identities for the structure functions

\[ \sum_{\text{cycl.} \ a,b,c} (-1)^{\epsilon_a \epsilon_c} f_{[ab]}^d E_{dc}^{(0)} = 0 , \]
\[ \sum_{\text{cycl.} \ a,b,c} (-1)^{\epsilon_b (\epsilon_c + 1)} a_{b[ac]} = 0 . \]

(7.33)

The differential operators \( \vec{\partial}_a^{(z)} \equiv (e^T)_a^A \vec{\partial}_A \) form an algebra

\[ [\vec{\partial}_a^{(z)}, \vec{\partial}_b^{(z)}] = f_{[ab]}^c \vec{\partial}_c^{(z)} . \]

(7.34)

The corresponding Jacobi identity is

\[ \sum_{\text{cycl.} \ a,b,c} (-1)^{\epsilon_a \epsilon_c} \left( (\vec{\partial}_a^{(z)} f_{[bc]}^e) - f_{[ab]}^d f_{[dc]}^e \right) = 0 . \]

(7.35)

In case of \( P \)-adapted curved coordinates in an odd Kähler manifold, the structure functions \( f_{ab}^c \) has only “holomorphic” and “anti-holomorphic” components. To see this, first note that this is true independent of the coordinate system for the anti-symmetrized structure functions \( f_{[ab]}^c \). This yields that the algebra of differential operators \( \vec{\partial}_a^{(z)} \equiv (e^T)_a^A \vec{\partial}_A \) reduces to a “holomorphic” and an “anti-holomorphic” algebra, that are mutually commutative, cf. (7.34). For \( P \)-adapted curved coordinates, it follows from the very definition of the structure functions, cf. (7.20), that

\[ \epsilon_b \neq \epsilon_c \Rightarrow f_{a \ c}^b = 0 = f_{ab}^c , \]
\[ \epsilon_b = \epsilon_c \Rightarrow \gamma_{abc} = 0 = a_{bac} . \]

(7.36)

We obtain the claim by combining these arguments.
7.6 Spin Connection

Let us introduce a spin connection \( \nabla^A = d + A : TM \times TW \to TW \), where

\[
A = \text{dz}^A \partial^r B^a_{Ac} \overset{\to}{dw}^c = \epsilon^a \partial^r B^b_{ac} \overset{\to}{dw}^c ,
\]

and where

\[
A^b_{Ac} = (-1)^{(\epsilon_A + \epsilon_a)\epsilon_b} (e^T)_A^a B^b_{ac} .
\]

The gauge transformations reads

\[
(-1)^{\epsilon_A \epsilon_b} A^b_{Ac} = (\partial^r (\Lambda^T)_c^d (\Lambda^{-1,T})_d^b (-1)^{\epsilon_c (\epsilon_b + 1)}) + (-1)^{\epsilon_A \epsilon_b} (\Lambda^{-1})^b_d A^d_{Af} \Lambda^f_c .
\]

It is assumed that \( A \) does not depend on the choice of the curved coordinate system. Of course, the full connection is \( \nabla = d + \Gamma + A \), where \( \Gamma \) is the Levi-Civita Christoffel symbols. The spin connection acts trivially on objects that carry no flat indices. The torsion two-form

\[
T \in \Gamma(TW \otimes \Lambda^2(T^*M))
\]

is by definition

\[
\frac{1}{2}\text{dz}^A \wedge \partial^r B^b_{AC} \text{dz}^C = \frac{1}{2}\epsilon^a \wedge \partial^r B^b_{ac} e^c = T = [\nabla \wedge e] .
\]

\( T \) only depends on \( A \), because \( \Gamma \) has no torsion, cf. (4.22). In fact, we have

\[
T(X,Y) = \nabla^A_X e Y - (-1)^{\epsilon(X)\epsilon(Y)} \nabla^A_Y e X - e[X,Y] = - (-1)^{\epsilon(X)\epsilon(Y)} T(Y,X) .
\]

In components

\[
(-1)^{\epsilon_A} \overset{\to}{dw}^b (T(\partial^r_A, \partial^r_C)) = T^b_{AC} = (-1)^{\epsilon_A \epsilon_b} (\partial^r_A e^c B^b_{ac}) + A^b_{Ac} e^c + (-1)^{\epsilon_A (\epsilon_b + 1)} (A \leftrightarrow C) ,
\]

or

\[
T^b_{ac} = (-1)^{\epsilon_A \epsilon_b} (A^A_{T})^a e^c B^b_{ac} e^C + A^b_{ac} + (-1)^{\epsilon_A (\epsilon_b + 1)} (a \leftrightarrow c) ,
\]

It is convenient to lower the first index with the flat metric \( g^{(0)}_{ab} \):

\[
A^b_{ac} = (-1)^{\epsilon_A} g^{(0)}_{bd} A^d_{ac} ,
\]

\[
T^b_{ac} = (-1)^{\epsilon_A} g^{(0)}_{bd} T^d_{ac} .
\]

We then have

\[
T_{bac} = a_{bac} + A_{bac} - (-1)^{\epsilon_A \epsilon_c} (a \leftrightarrow c) = a_{b[ac]} + A_{b[ac]} .
\]

7.7 Curvature

The curvature is

\[
R = \frac{1}{2} [\nabla \wedge \nabla] = \frac{1}{2} [\nabla^T \wedge \nabla^T] + \frac{1}{2} [\nabla^A \wedge \nabla^A] = R^T + R^A .
\]

The Bianchi identity is a trivial consequence of the supercommutator version of the Jacobi identity:

\[
[\nabla \wedge R] = 0 .
\]
The curvature of the spin connection is

$$ R^A = -e^d \wedge e^a \otimes \left( (-1)^{\epsilon_a \epsilon_b} \partial_b (\overrightarrow{\partial^l(z)} A^b_d f) + \partial_b A^b_a c d f - f_{ad} e A^b_c d f \right) \otimes \, dw^f, \quad (7.49) $$

The Ricci two-form $\mathcal{R}^A$ thus reads

$$ \mathcal{R}^A = -e^b \wedge e^a \otimes \left( (\overrightarrow{\partial^l(z)} A_b) - f_{ab}^c A_c \right) \otimes \, dw^f, \quad (7.50) $$

where the superdeterminant gauge field reads

$$ A_a = (-1)^{(\epsilon_a + 1) \epsilon_b} A^b_{ab}. \quad (7.51) $$

It transforms under gauge transformations $\Lambda^a_{\, b}$ as a tensor

$$ A_a = (\Lambda^T)^a_{\, b} A'_b + (\overrightarrow{\partial^l(z)} \ln \text{sdet}(\Lambda^\cdot)) = (\Lambda^T)^a_{\, b} A'_b, \quad (7.52) $$

because $\text{sdet}(\Lambda^\cdot) = 1$. The condition for Ricci-form-flatness reads

$$ (\overrightarrow{\partial^l(z)} A_b) - f_{ab}^c A_c = 0. \quad (7.53) $$

This is the closeness-condition in a non-Abelian basis that by the Poincaré Lemma ensures that $A_a$ is locally exact. We may also form a determinant gauge field

$$ \tilde{A}_a = (-1)^{\epsilon_a \epsilon_b} A^b_{ab}. \quad (7.54) $$

It also transforms under gauge transformations $\Lambda^a_{\, b}$ as a tensor

$$ \tilde{A}_a = (\Lambda^T)^a_{\, b} \tilde{A}'_b + (\overrightarrow{\partial^l(z)} \ln \text{det}(\Lambda^\cdot)) = (\Lambda^T)^a_{\, b} \tilde{A}'_b, \quad (7.55) $$

because $\text{tr}((\ln \Lambda^\cdot)) = \ln \text{det}(\Lambda^\cdot) = 0$. Here we made use of the fact that the matrices $\Lambda$ have vanishing bose-fermi blocks.

### 7.8 Levi-Civita Spin Connection

As in the usual bosonic case we define the Levi-Civita spin connection to be the unique spin connection that satisfies

$$ T = 0, \quad \nabla g^{(0)} = 0. \quad (7.56) $$

The last equation yields

$$ A_{bac} + (-1)^{\epsilon_b \epsilon_c + \epsilon_a (\epsilon_b + \epsilon_c)} A_{cab} = 0. \quad (7.57) $$

Together with the condition of no torsion, this implies that the $A_{bac}$ can be expressed purely in terms of the structure functions $a_{b[ac]}$:

$$ 2A_{bac} = -(-1)^{\epsilon_a \epsilon_b} a_{b[a|c]} - (-1)^{\epsilon_c (\epsilon_a + \epsilon_b)} a_{c[b|a]} - a_{b[a|c]} \quad (7.58) $$

$$ = (-1)^{\epsilon_b (\epsilon_a + \epsilon_c)} a_{acb} - (-1)^{\epsilon_a \epsilon_b} a_{abc} $$

$$ + (-1)^{\epsilon_a \epsilon_b + \epsilon_a \epsilon_c + \epsilon_b \epsilon_c} a_{cab} - (-1)^{\epsilon_c (\epsilon_a + \epsilon_b)} a_{cba} $$

$$ + (-1)^{\epsilon_a \epsilon_c} a_{bca} - a_{bac}. \quad (7.58) $$
Evidently, \( A_{bac} \) does not transform under change of the curved coordinate frame. On the other hand we can construct a spin connection by conjugating with the vielbein

\[
\nabla_X^A = e \nabla_X^\Gamma e^{-1}.
\]

(7.59)

It satisfies (7.56), so it is the Levi-Civita spin connection. It follows that

\[
\Gamma_{bac} = A_{bac} + a_{bac},
\]

\[
A_{bac} - \Gamma_{bac} = (-1)^\epsilon_b (\epsilon_a + \epsilon_c) + \epsilon_c f_{ac}^b = - (-1)^\epsilon_b \epsilon_a f_{a c}^b.
\]

(7.60)

Conjugation with vielbeins does not change the curvature:

\[
R^A \cong R^\Gamma.
\]

(7.61)

Here we have identified the \( w \)-basis with the vielbein-basis.

From the conjugation formula (7.59) it follows that the Levi-Civita spin connection respects the flat anti-symplectic structure \( \nabla E^{(0)} = 0 \) and the flat canonical almost parity structure \( \nabla P^{(0)} = 0 \), where \( P^{(0)} = (-1)^{a} \delta^b_a \). Explicitly, the condition \( \nabla P^{(0)} = 0 \) reads

\[
\epsilon_b = \epsilon_c \Rightarrow A_{bac} = 0.
\]

(7.62)

This condition is of course completely superseded, if we furthermore assume that the manifold \( M \) is an odd Kähler manifold wrt. the canonical \( P \)-structure. Then only the bose-bose-bose and the fermi-fermi-fermi components of \( A_{bac} \) survive. This follows from arguments similar to those presented in Section 4.2 for a tangent bundle connection \( \nabla^\Gamma \), which yields the analogous conclusion for the Christoffel symbols \( \Gamma^b_{ac} \). Or one could use the relation (7.60) and the fact that the structure functions \( f^b_{a c} \) and \( f^b_{ab} \) has also only “holomorphic” and “anti-holomorphic” components.

### 7.9 Connection Fields

We translate the superdeterminant connection field \( F_A \) into the flat indices as follows:

\[
(-1)^\epsilon_a F_a = \Gamma^b_{ba} = (-1)^\epsilon_A F_A^a e^A_a = (-1)^\epsilon_b f_{ab}^b + (-1)^\epsilon c f_{a c}^0 .
\]

(7.63)

Here \( f_{a}^{(0)} \) is

\[
(-1)^\epsilon_a f_{a}^{(0)} = A^b_{ba} = P_{(0)}^{cd} f_{cd}^b g_{ba}^{(0)} = \frac{1}{2} (-1)^\epsilon_a g_{(0)}^{cd} f_{[cd]}^b g_{ba}^{(0)} = - f_{[ab]}^b .
\]

(7.64)

The last equality follows from the Jacobi identity (7.33). From (7.60) follows a relation between the two superdeterminant connection fields

\[
F_a - A_a = (-1)^\epsilon_b f_{ab}^b = (-1)^\epsilon_s (\delta^l_{a} e^B) e^B_b = (\delta^l_{a} \ln sdet(e)) .
\]

(7.65)

So for the Levi-Civita connection we have that

\[
F_a = (\delta^l_{a} \ln sdet(e)) \iff A_a = 0 .
\]

(7.66)

We have that the quantity \( F_A^{(0)} \) defined in (4.37) transforms into

\[
F_a^{(0)} = (e^T)_a A^{(0)} F_A^{(0)} = - E_{ab} \Gamma^b_{cd} E^{dc} .
\]
\[
= -(-1)^{\epsilon a} f_{[ab]}^b + f^{(0)}_a = f^{(0)}_a + f^{(0)}_a .
\]
Here \( f^{(0)}_a \) is
\[
f^{(0)}_a = g^{dc} f_{cd}^b g_{ba}^{(0)} = \frac{1}{2} (-1)^{\epsilon d} F^{dc}_{(0)} f_{[cd]}^b g_{ba}^{(0)} .
\]
The connection field \( \tilde{F}_A \) is
\[
\tilde{F}_A = (\partial_A e^b_B) e^B_b = (e^T)_A a f_{a}^b .
\]
In the flat indices it reads
\[
\tilde{F}_a = (-1)^{\epsilon a} + \epsilon_b \Gamma_{ba} = (e^T)_a A \tilde{F}_A = f_{a}^b .
\]
From (7.60) follows a relation between the two determinant connection fields
\[
\tilde{F}_a - \tilde{A}_a = f_{a}^b = (\partial_a (z) e^b_B e^B_b = (\partial_a (\ln \rho) .
\]
This is in agreement with the fact that \( \tilde{A}_a = 0 \) for the Levi-Civita spin connection. The determinant density is
\[
\ln \rho = (\ln e)_a .
\]
The odd Laplacian in flat indices reads
\[
\Delta = \frac{1}{2} g^{ab}_{(0)} (f^0_b + \tilde{f}^{(0)}_b) \partial_a (z) \partial_a (z) = \frac{1}{2} g^{ab}_{(0)} \partial_b (z) \partial_a (z) : + \frac{1}{2} g^{ab}_{(0)} F^0_b \partial_a (z) .
\]
We note the identity
\[
(-1)^{\epsilon a} (\partial_A e^A_a) = -(-1)^{\epsilon a} \partial_a (z) .
\]
7.10 Odd Kähler Manifolds
Finally, let us just list the case of an odd Kähler manifold \( M \). Then
\[
\Delta = \frac{1}{2} g^{ab}_{(0)} \partial_b (z) \partial_a (z) : = \frac{1}{2} g^{ab}_{(0)} \partial_b (z) \partial_a (z) ,
\]
\[
F_a^{(0)} = 0 , \quad f_a^{(0)} = 0 , \quad \tilde{f}_a^{(0)} = 0 .
\]
The following formulas apply to the non-vanishing components only:
\[
g^{(0)}_{ab} = (\partial_a (z) \partial_b (z) : K) ,
\]
\[
\Gamma_{c,ab} = (K : \partial_c (z) \partial_a (z) \partial_b (z) : ) ,
\]
where \( K \) is the odd Kähler potential. In case of a Ricci-form-flat manifold \( M \) we have
\[
A_a = 0 , \quad F_a = (\partial_a (z) \ln \rho) , \quad \rho = sdet(e) .
\]
A Super conventions

Derivatives have two kinds of attributes. First, a superscript “r” or “l” indicates a left or right derivative

\[ \frac{\partial^l}{\partial z^A} = (-1)^{\epsilon_A} \frac{\partial^r}{\partial z^A} . \]  

(A.1)

Secondly, arrows indicate, on which objects the derivatives should act. The derivatives is uniquely specified through its action on the fundamental variables

\[ (\frac{\partial^r}{\partial z^A} z^B) = \delta^B_A = (z^B \frac{\partial^r}{\partial z^A}) . \]  

(A.2)

If a derivative carries no arrow it does not act on anything present in the formula. It is then merely understood as the natural basis for the tangent space in the sense that we are only interested in immitating its transformation properties under change of coordinates.

\[ \frac{\partial^r}{\partial z^A} = \frac{\partial^r}{\partial z^B} (z^B \frac{\partial^r}{\partial z^A}) = (-1)^{\epsilon_A+\epsilon_B} (\frac{\partial^l}{\partial z^A} z^B) \frac{\partial^r}{\partial z^B} . \]  

(A.3)

and

\[ \frac{\partial^l}{\partial z^A} = (\frac{\partial^r}{\partial z^A} z^B) \frac{\partial^l}{\partial z^B} . \]  

(A.4)

The exterior derivative \( d \) is

\[ d = dz^A \frac{\partial^l}{\partial z^A} . \]  

(A.5)

One-forms act on vectors according to

\[ (-1)^{\epsilon_B} dz^A (\frac{\partial^r}{\partial B}) = dz^A (\frac{\partial^l}{\partial B}) = (-1)^{\epsilon_A+\epsilon_B} (\frac{\partial^l}{\partial z^A} z^B) = (-1)^{\epsilon_A} \delta^A_B . \]  

(A.6)

Note that this definition is stable under change of coordinates as it should be. The contraction \( i_X \) with a bosonic vector field \( X, \epsilon(X) = 0, \)

\[ i_X = X^A i^l_A = i^r_A X^A \]  

(A.7)

is defined via its action on the natural basis of one-forms

\[ (-1)^{\epsilon_A} i^r_A (dz^B) = i^l_A (dz^B) = \delta^A_B . \]  

(A.8)

In fact, we can symbolically write

\[ i^r_A = \frac{\partial^l}{\partial (dz^A)} . \]  

(A.9)

Both the exterior derivative \( d \) and the contraction \( i_X \) carries odd “form-parity”

\[ p(d) = 1 = p(i_X) . \]  

(A.10)

The wedge \( \wedge \) will in this respect be a total redundant symbol, \( i.e. \) for two forms \( \omega \) and \( \eta \)

\[ \omega \wedge \eta = (-1)^{\epsilon(\omega)+\epsilon(\eta)+p(\omega)p(\eta)} \eta \wedge \omega . \]  

(A.11)
In the same spirit we define the supercommutator $[A,B]$ of two operators $A$ and $B$ to be

$$[A,B] = AB - (-1)^{c(A)c(B) + p(A)p(B)} BA$$ \hspace{1cm} (A.12)

For instance

$$d^2 = \frac{i}{2}[d,d] = 0, \quad [i_X, i_Y] = 0, \quad \mathcal{L}_X = [i_X, d].$$ \hspace{1cm} (A.13)

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