Provable Multi-Objective Reinforcement Learning with Generative Models

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Abstract

Multi-objective reinforcement learning (MORL) is an extension of ordinary, single-objective reinforcement learning (RL) that is applicable to many real world tasks where multiple objectives exist without known relative costs. We study the problem of single policy MORL, which learns an optimal policy given the preference of objectives. Existing methods require strong assumptions such as exact knowledge of the multi-objective Markov decision process, and are analyzed in the limit of infinite data and time. We propose a new algorithm called model-based envelop value iteration (EVI), which generalizes the enveloped multi-objective \(Q\)-learning algorithm in Yang et al. [19]. Our method can learn a near-optimal value function with polynomial sample complexity and linear convergence speed. To the best of our knowledge, this is the first finite-sample analysis of MORL algorithms.

1 Introduction

Real-world decision-making systems pose many practical challenges for using reinforcement learning (RL) \[2\]. In this paper, we focus on just two. First, real-world decision-making systems must handle multiple conflicting objectives simultaneously, yet without obvious preference for any one objective. For example, a bank may wish to use RL techniques in making credit decisions, to produce models that adapt changing market structure and account for the historical outcomes of past deals when making future decisions. To be profitable, a credit decisioning model will need to consider an applicant’s credit risk. However, the bank also needs to consider other risks, such as reputational risk and counterfactual lost revenue risk associated with falsely declined applications, and fair lending regulatory risk associated with apparent bias in credit decisions with regard to race, gender, age, or other protected classes \[5, 10\]. It is difficult, if not impossible, to assign precise monetary values to these risks, and therefore the optimal policy for loan approvals cannot be expressed as a straightforward optimization problem to maximize profit. Ordinary RL algorithms, which work on a single objective function that assigns fixed relative costs to each type of risk, are therefore unsuitable for these problems. Generalizations of RL, known as multi-objective reinforcement learning (MORL), have been proposed to address such challenges. In MORL, the agent has to make decisions not under a single objective, but under multiple objectives, and can choose different policies flexibly based on different preferences for the objectives. However, these methods are generally costly and/or require strong assumptions on what is known about the problem.

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Second, real-world RL does not have access to the underlying, unknown dynamics of the problem, thus necessitating learning strategies for the optimal policy that can succeed on finite limited data. In the credit decisioning example, it is unrealistic to assume that the bank knows perfectly well the outcome of each loan application, or the precise mechanics of how today’s credit needs will affect tomorrow’s demands for credit.

Combining these challenges leads to the following question:

Can we devise a MORL algorithm with provable finite sampling properties?

To the best of our knowledge, no such algorithm currently exists.

Our contributions We answer the question above: yes. We propose a new algorithm, which we call model-based envelope value iteration (model-based EVI), to learn the optimal multi-objective Q-function (MOQ). Our algorithm is based on the envelope Q-learning algorithm proposed in Yang et al. [19]. We show that with access to a generative model or simulator, model-based EVI exhibits \( \tilde{O}(mSA/(1-\gamma)^3k^2) \) sample complexity and \( \tilde{O}(1/(1-\gamma)) \) convergence rate to learn an \( \epsilon \)-suboptimal MOQ function, where \( m \) is the number of reward functions (objectives), \( S \) is the cardinality of the state space, \( A \) is the cardinality of the action space, and \( \gamma \in (0,1) \) is the discount factor. Therefore, the MOQ-learning problem has essentially the same cost as learning an optimal \( Q \)-function for each objective separately, and MORL is hence about as complex as \( m \) separate RL problems.

Notation We use lowercase letters for scalars, lowercase bold letters for vectors, and uppercase bold letters for matrices. For a vector \( x \in \mathbb{R}^d \) and matrix \( \Sigma \in \mathbb{R}^{d \times d} \), we denote by \( \|x\|_2 \) the Euclidean norm and denote by \( \|x\|_\Sigma = \sqrt{x^\top \Sigma x} \). For two sequences \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n = O(b_n) \) if there exists an absolute constant \( C \) such that \( a_n \leq Cb_n \), and we write \( a_n = \Omega(b_n) \) if there exists an absolute constant \( C \) such that \( a_n \geq Cb_n \). We use \( \tilde{O}() \) to further hide the logarithmic factors.

2 Related Work

The literature on MORL is relatively sparse compared to the extensive body of work on ordinary RL, and can be grouped into two main approaches [11].

Multi-policy algorithms These methods build and maintain a Pareto-optimal set of optimal policies, and scale poorly due to the intrinsic growth of the optimality frontier. White [18] uses dynamic programming to compute a Pareto-optimal set of nonstationary policies. However, the size of this set increases exponentially with the horizon, making this method impractical. Barrett and Narayanan [2] proposed the convex hull value-iteration method, which only computes the stationary policies on the convex hull of the Pareto front. Castelletti et al. [3, 4] proposed multi-objective fitted Q-learning (MOFQL), which constructs the \( Q \)-function approximator with embedded preferences to learn the optimal policy for any given preference during testing. Wang and Sebag [17] introduced multi-objective Monte-Carlo tree search using the hypervolume indicator [8] to define an action selection criterion that is similar to the upper confidence bound (UCB) in ordinary RL. The hypervolume indicator is maximized for any policy on the optimality frontier, but is still expensive to compute, with the best known practical algorithms requiring a typical complexity of approximately \( \Theta(\gamma^{1/2}m) \), where \( n \) is the number of Pareto-optimal policies.

Single-policy algorithms These methods scalarize the vector of multiple rewards, collapsing them into a single scalar-valued function using some specification of their preferences, then apply ordinary RL methods to solve the resulting problem, which is now single-objective. Single-policy algorithms use less memory and are easier to implement. However, at each time, a single-policy algorithm finds the optimal policy with respect to some specific preference parameter, which hinders generalization to other unseen preferences [12, 15, 9, 16]. The simplest of these methods use linear scalarization functions [2, 14], which compute the weighted sum of the values for each objective. Nonlinear scalarizations have also been proposed [16] to address the limits of the scalarized representation using linear functions. [4] studied the regret minimization problem with vectorial feedback and complex objectives, while they need the access to the adapted preference vector. In the next section, we will review the method of Yang et al. [19], as it forms the starting point for our work.
3 Preliminaries

Discounted multi-objective Markov decision processes (MOMDPs) We denote a discounted MOMDP by the tuple \( (S, A, \gamma, r, P, \Omega) \), where \( S \) is the state space (possibly infinite), \( A \) is the action space, \( \gamma \in [0, 1) \) is the discount factor, \( r: S \times A \rightarrow [0, 1]^m \) is the vector-valued reward function, and \( m \) is the number of reward functions. For simplicity, we assume the reward function \( r \) is deterministic and known. \( P(s'|s, a) \) is the transition probability function which denotes the probability for state \( s \) to transfer to state \( s' \) under the action \( a \), \( \Omega \subseteq \mathbb{R}^m \) is the set of the preference vectors \( w \in \Omega \) which represent how to utilize the reward functions. A policy \( \pi: S \rightarrow A \) is a function which maps a state \( s \) to an action \( a \). We define the action-value function \( Q^\pi(s, a) \) and its corresponding value function \( V^\pi(s) \) as follows:

\[
Q^\pi(s, a) = \mathbb{E}\left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \middle| s_0 = s, a_0 = a, \forall t \geq 1, a_t = \pi(s_t) \right], \quad V^\pi(s) = Q^\pi(s, \pi(s)).
\]

We define the optimal value function \( V^* \) and the optimal action-value function \( Q^* \) with respect to some weight parameter \( w \in \mathbb{R}^m \) as follows:

\[
V^*(s; w) = \arg\max_{\pi} \mathbb{E}^\pi V^\pi(s), \quad Q^*(s, a; w) = \arg\max_{\pi} \mathbb{E}^\pi Q^\pi(s, a),
\]

(3.1)

where \( \arg\max_{\pi} \mathbb{E}^\pi V^\pi(s) \) and \( \arg\max_{\pi} \mathbb{E}^\pi Q^\pi(s, a) \) take the vectors of \( V \) or \( Q \)-values that attain the maximum. For simplicity, we denote \( [P^v](s, a) = \mathbb{E}_{s' \sim P(s, a)} V(s') \) for any function \( V: S \times \Omega \rightarrow \mathbb{R}^m \). Therefore, we have the following Bellman equation:

\[
Q^\pi(s, a) = r(s, a) + \gamma \cdot [P^\pi V^\pi](s, a).
\]

Problem statement In this work, let \( \mathcal{R} \) denote the set of all possible returns for some policy, where

\[
\mathcal{R} := \left\{ \tilde{r} \in \mathbb{R}^m : \exists \pi, \tilde{r} = \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t), s_t \sim P(\cdot | s_{t-1}, a_{t-1}), a_t = \pi(s_t) \right\}.
\]

We aim to find the following possible accumulated return \( \tilde{r} \in \mathcal{R} \) from a MOMDP belongs to a Pareto frontier \( F^* \), that is \( F^* := \{ \tilde{r} \in \mathcal{R} : \exists \tilde{r}' \in \mathcal{R} \text{ such that } \tilde{r}' \succeq \tilde{r} \} \). For all preferences in \( \Omega \), we define the following convex converage set (CCS) of \( F^* \) as

\[
\{ \tilde{r} \in F^* | \exists w \in \Omega, \text{such that } \forall \tilde{r}' \in F^*, w^\top \tilde{r} \geq w^\top \tilde{r}' \},
\]

which includes the returns that maximizes the reward corresponding to some specific preference \( w \), and effectively convexifies the starting Pareto-dominance operator \( \succeq \) to the operator \( w^\top \cdot \geq w^\top \cdot \). Our goal is to recover all policies for CCS of any given MOMDP.

Enveloped Q-learning [19] We now review enveloped Q-learning, which was proposed in Yang et al. [19] to solve this problem. At each time step, enveloped Q-learning uses the convex envelope of the solution frontier to update the parameters. More specifically, the agent initializes the multi-objective Q-value function (MOQ) \( Q_0(s, a; w) \) at the beginning of the algorithm. At each round \( t \), the agent defines the optimality filter for any MOQ function \( Q: S \times A \times \Omega \rightarrow \mathbb{R}^m \) as follows:

\[
[H\mathcal{Q}](s; w) = \arg\max_{a \in A, w' \in \Omega} w^\top \mathcal{Q}(s, a; w'),
\]

(3.2)

When multiple solutions to (3.2) exist, it suffices to choose any one solution arbitrarily.

Yang et al. [19] introduced two key concepts, the first being the multi-objective optimality operator \( T \), which is defined following (3.2) as:

\[
T \mathcal{Q}(s, a; w) := r(s, a) + \gamma [P[H\mathcal{Q}]](s, a; w),
\]

(3.3)

where \( T \) does not depend on \( w \). The second is the following definition of distance between MOQs. For any two MOQs \( \mathcal{Q} \) and \( \mathcal{Q}' \), the distance between them is

\[
d(\mathcal{Q}, \mathcal{Q}') = \sup_{(s, a) \in S \times A, w \in \Omega} |w^\top \mathcal{Q}(s, a; w) - w^\top \mathcal{Q}'(s, a; w)|.
\]

(3.4)

Due to the nonuniqueness of solutions to (3.2), \( d(\mathcal{Q}, \mathcal{Q}') = 0 \) does not imply that \( \mathcal{Q} = \mathcal{Q}' \), and thus \( d \) is not a true metric since it violates the axiom of identity of indiscernibles. Nevertheless, the \( d \)
satisfies the weaker axioms of a pseudometric or semimetric, since it is nonnegative, vanishing for all \(d(Q, Q') = 0\), symmetric, and satisfies the triangle inequality, since

\[
d(Q, Q') = \sup_{(s, a) \in S \times A, w \in \Omega} |w^T Q(s, a; w) - w^T Q'(s, a; w)|
\]

\[
\leq \sup_{(s, a) \in S \times A, w \in \Omega} |w^T Q(s, a; w) - w^T Q''(s, a; w)|
\]

\[
+ \sup_{(s, a) \in S \times A, w \in \Omega} |w^T Q''(s, a; w) - w^T Q'(s, a; w)|
\]

\[
= d(Q, Q'') + d(Q', Q'').
\]

The multi-objective optimality operator \(\mathcal{T}\) has two important properties.

**Proposition 3.1** (Fixed point \([19]\)). The optimal \(Q\)-function \(Q^*\) is the fixed point of the multi-objective optimality operator \(\mathcal{T}\), i.e., \(Q^* = \mathcal{T}Q^*\).

**Proposition 3.2** (\(\gamma\)-contraction \([19]\)). The multi-objective optimality operator \(\mathcal{T}\) is a \(\gamma\)-contraction operator, where \(\gamma\) is the contraction factor. That suggests that the distance between any two MOQs after applying \(\mathcal{T}\) to both of them is less than \(\gamma\) times their original distance. In other words, let \(Q, Q'\) be any two MOQs, then \(d(\mathcal{T}Q, \mathcal{T}Q') \leq \gamma d(Q, Q')\). In the context of the discounted MOMDP, the contraction factor is simply the discount factor.

Enveloped Q-learning presumes that the transition probability function \(P\) is fully known. Then, the envelop value iteration (EVI) rule suggests that \(Q_{t+1} = \mathcal{T}Q_t\). By Propositions 3.1 and 3.2, a generalized form of Banach’s fixed-point theorem yields \(\mathcal{T}^\infty Q = Q^*\) for any MOQ \(Q\). In contrast, we will now present an alternative algorithm that does not require exact knowledge of the transition probabilities \(P\), and show that it has favorable finite sampling properties.

### 4 Model-based envelop value iteration (model-based EVI)

We now present our method, which we call model-based envelop value iteration (model-based EVI), in Algorithm\([1]\). Model-based EVI aims to learn a MOQ function which is close to the optimal MOQ function \(Q^*\) given a finite number of samples and finite time. Model-based EVI can be divided into two phases: the data collection phase and the evaluation phase.

**Data collection phase** Model-based EVI first aims to collect enough data to learn the unknown transition probability. We assume that model-based EVI has an access to a generative model or simulator, such that for any state–action pair \((s, a) \in S \times A\), model-based EVI can sample independent next states \(s'\) generated by the underlying transition dynamics. This is similar to experience replay\([13]\) used in many RL applications to collect the samples, which randomly collects training samples from a batch of previous visited states and actions generated from a stationary distribution. Model-based EVI samples \(N\) next states for each state–action pair \((s, a)\), then builds an empirical transition probability estimate \(\hat{P}(\cdot|\cdot) : S \times A \times S \rightarrow [0, 1]\) from these samples:

\[
\hat{P}(s'|s, a) = \frac{N(s'|s, a)}{N},
\]

where \(N(s'|s, a)\) denotes the number of next states \(s'\) sampled starting from \((s, a)\).

**Evaluation phase** Next, model-based EVI will learn the optimal MOQ function based on the empirical model \(\hat{P}\). During this phase, model-based EVI evaluates the optimal MOQ function based on the empirical model obtained in data collection phase. At the beginning, model-based EVI initializes \(Q_0 \leftarrow \sum_{n=0}^{\infty} \gamma^n = 1/(1 - \gamma)\), based on a presumed reward of 1. Similar to the estimated model \(\hat{P}\), model-based EVI also builds the empirical version of multi-objective optimality operator \(\hat{\mathcal{T}}\), which is an estimator for \(\mathcal{T}\) in Equation \((3.2)\), as follows:

\[
\hat{\mathcal{T}} Q(s, a; w) := r(s, a) + \gamma [\hat{P}(\mathcal{H}Q)](s, a; w).
\]

Then, model-based EVI updates the estimated MOQ function by iteratively applying \(\hat{\mathcal{T}}\), obtaining the next MOQ by \(Q_{t+1} \leftarrow \hat{\mathcal{T}} Q_t\). Equation \((4.2)\) is the empirical analogue of Equation \((3.2)\), and defines the optimality filter over the empirical model \(\hat{P}\) using a finite number of samples. Importantly,
we do not need to know the underlying true model $P$, which is unaccesible in the practice, and is in sharp contrast to the requirements of the original envelope $Q$-learning algorithm of Yang et al. [19] described in Section 5.

We now analyze the convergence behavior of model-based EVI, which is summarized by the following theorem.

**Theorem 1.** Suppose that the set of preference vectors $\Omega \subseteq \{w : \|w\|_1 \leq 1\}$, and that as $N \uparrow \infty$, the limit $\hat{P} \rightarrow P$ exists. Then, there exists some constants $\{c_i\}_{i=1}^4$ such that for all $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$, if we set the sampling number $N$ and the number of iterations $T$ to be

$$N = \left\lceil \frac{c_1 m}{e^2(1-\gamma)^3} \log \frac{c_2 m S A}{\delta (1-\gamma) \epsilon} \right\rceil, \quad T = \left\lceil \frac{c_3}{1-\gamma} \log \frac{c_4}{(1-\gamma) \epsilon} \right\rceil,$$

then with probability at least $1-\delta$, $Q_T$ satisfies $d(Q_T, Q^*) \leq \epsilon$, where $Q^*$ is the optimal MOQ function as defined in (3.1), and $d$ is the pseudometric over MOQs as defined in (3.4).

In other words, the solution computed by Algorithm 1 while being a solution to the empirical discounted MOMDP $(S, A, \gamma, r, \hat{P}, \Omega)$, converges to the solution to the true discounted MOMDP $(S, A, \gamma, r, P, \Omega)$, given sufficiently many samples and time steps. This is the main result of the paper.

**Remark 4.1.** Proposition 1 implies that the dependence of the sample complexity on the number of objectives is almost linear $(m \log m)$, which suggests that learning an optimal MOQ function is essentially as hard as to learn these objective functions separately. Meanwhile, the number of iterations $T$ does not depend on $m$, which suggests that MORL is essentially the same as ordinary RL in terms of convergence rate.

**Remark 4.2.** A trivial approach to solve $Q^*(s, a; w)$ for any $(s, a)$ and $w$ is to enumerate all possible $w \in \Omega$ and calculate the optimal MOQ function for each possible $w$. However, this naive approach would lead to a dependence on the cardinality of $\Omega$ in the sample complexity since we need to repeat the data collection phase $|\Omega|$ times. In contrast, the sample complexity in Theorem 1 is independent of $|\Omega|$, which suggests that the use of the optimality operator can make MORL more sample efficient.

**Remark 4.3.** When $m = 1$, the MORL problem degenerates to a single-objective ordinary RL problem, and Proposition 1 suggests a total $NSA = \tilde{O}(SA/(\epsilon^2(1-\gamma)^3))$ sample complexity and $\tilde{O}((1-\gamma)^{-1})$ number of iterations. These match the sample complexity and time complexity of Azar et al. [1] for the ordinary RL case.

Furthermore, when $m = 1$, the MORL problem degenerates to a single-objective RL problem. Therefore, existing lower bound for RL problem also yields a lower bound for our case. To illustrate the lower bound, we first define the $(\epsilon, \delta)$-correct RL algorithm as follows.

**Definition 4.4** (Azar et al. [1]). We call an algorithm $A$ is an $(\epsilon, \delta)$-correct RL algorithm if there exists a class of MOMDPs $M_1, \ldots, M_n$ such that with probability at least $1-\delta$, $d(Q^A, Q^*) \leq \epsilon$ holds for all $M_i$, where $Q^A$ is the Q function output by $A$. 

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**Algorithm 1 Model-based envelop value iteration (model-based EVI)**

**Require:** State space $S$, action space $A$, discount factor $\gamma \in (0, 1)$, reward function $r$, preference vector $w \in \Omega$, a generative model $S \times A \rightarrow S$ for the next state $s'$ from any state–action pair $(s, a)$, and the number of time steps, $T$.

1: Let $Q_0 \leftarrow 1/(1-\gamma)$.
2: For each $(s, a) \in S \times A$, sample $N$ next states from the generative model.
3: Construct the empirical model for the transition probabilities $\hat{P}$ defined in (4.1).
4: Construct the empirical multi-objective optimality operator $\hat{T}$ defined in (4.2).
5: for each time step $t = 1, \ldots, T$ do
6: $Q_t \leftarrow \hat{T}Q_{t-1}$
7: end for
8: return $Q_T(s, a; w)$
We have proposed a new MORL algorithm, model-based EVI, to address two real-world challenges in RL: multiple objectives with unknown weights, and learning from finite samples. We show that in order to find an $\epsilon$-suboptimal MOQ function, it suffices to use $O(mSA/((1-\gamma)^3\epsilon^2))$ samples and $O(1/(1-\gamma))$ time steps as described in Theorem 3 which implies that learning an optimal MOQ function is essentially as hard as learning $m$ separate objective functions. Comparing with the lower bound result for single-objective RL suggests that our method is nearly optimal. We will leave the lower bound of MORL to future work.

## 5 Conclusion

We have proposed a new MORL algorithm, model-based EVI, to address two real-world challenges in RL: multiple objectives with unknown weights, and learning from finite samples. We show that in order to find an $\epsilon$-suboptimal MOQ function, it suffices to use $O(mSA/((1-\gamma)^3\epsilon^2))$ samples and $O(1/(1-\gamma))$ time steps as described in Theorem 3, which implies that learning an optimal MOQ function is essentially as hard as learning $m$ separate objective functions. Comparing with the lower bound result for single-objective RL suggests that our method is nearly optimal. We will leave the lower bound of MORL to future work.

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### A Proof of Theorem 3

Let $\hat{Q}^*$ be the optimal action-value function over the empirical transition probability $\hat{P}$. We have the following lemmas.

#### Lemma A.1

We have $d(Q_t, \hat{Q}^*) \leq \gamma^t/(1-\gamma)$.

**Proof of Lemma A.1.** We prove that by induction. For all $t > 0$ we have

$$d(Q_t, \hat{Q}^*) = d(\bar{T}Q_{t-1}, \hat{T}\hat{Q}^*) \leq \gamma d(Q_{t-1}, \hat{Q}^*),$$

where the equality holds due to the update rule in Line 6 of Algorithm 1 and the fact $\hat{Q}^* = \hat{T}\hat{Q}^*$ by Proposition 3.1, the inequality holds due to Proposition 3.2. Therefore, recursively applying (A.1), we have

$$d(Q_t, \hat{Q}^*) \leq \gamma^t d(Q_0, \hat{Q}^*)$$

$$\leq \gamma^t \sup_{(s,a)\in S\times A, w\in \Omega} |w^TQ_0(s, a; w) - w^T\hat{Q}^*(s, a; w)|$$

$$\leq \gamma^t \sup_{(s,a)\in S\times A, w\in \Omega} \|Q_0(s, a; w) - \hat{Q}^*(s, a; w)\|_\infty$$

$$\leq \gamma^t/(1-\gamma),$$

where the second inequality holds due to the fact $\|w\|_1 \leq 1$ and Cauchy-Schwarz inequality $(a, b) \leq \|a\|_\infty \|b\|_1$, and the last inequality holds since $1/(1-\gamma)1 = Q_0(s, a; w) \geq \hat{Q}^*(s, a; w) \geq 0$. That ends our proof.

The next lemma provides the upper bound between $Q^*$ and $\hat{Q}^*$.

By Definition 4.4, we propose a lower bound of the sample complexity.

**Proposition 4.5** (Theorem 3, Azar et al. [1]). There exist some constants \(c_i\) such that for any \(\epsilon \in (0, c_1), \delta \in (0, c_2/(SA)),\) and for any \((\epsilon, \delta)\)-correct RL algorithm \(\mathcal{A}\), there exists a tabular MDP \(M(S, A, \gamma, r, P, \Omega)\) such that the total number of samples that \(\mathcal{A}\) needs is at least

$$\frac{c_3SA}{\epsilon^2(1-\gamma)^3} \cdot \log \frac{c_4SA}{\delta}.$$
Lemma A.2. For any $\xi, \delta \in (0, 1)$, with probability at least $1 - \delta$, we have
\[
d(Q^*, \hat{Q}^*) \leq \frac{4m \log(8SA/(\xi \delta))}{N(1 - \gamma)^3} + \left(\frac{5(\gamma/(1 - \gamma)^2)^{4/3}m \log(12SA/(\xi \delta))}{N}\right)^{3/4} + 3m \log(24SA/(\xi \delta)) + 2m/(1 - \gamma).
\]

Proof of Lemma A.2. We first show that $w^\top Q^*(s, a; w)$ is Lipschitz continuous w.r.t. $w$. Select $w_1, w_2 \in \Omega$. Let $\pi_i$ be the optimal policies corresponding to $w_i$ satisfying $w_i^\top Q^*(s, a; w_i) = w_i^\top Q^{\pi_i}(s, a), i = 1, 2$. Then for any $(s, a) \in S \times A$, we have
\[
w_1^\top Q^*(s, a; w_1) - w_2^\top Q^*(s, a; w_2)
= w_1^\top Q^{\pi_1}(s, a) - w_2^\top Q^{\pi_2}(s, a)
= w_1^\top Q^{\pi_1}(s, a) - w_2^\top Q^{\pi_1}(s, a) + w_2^\top Q^{\pi_1}(s, a) - w_2^\top Q^{\pi_2}(s, a)
\leq w_1^\top Q^{\pi_1}(s, a) - w_2^\top Q^{\pi_1}(s, a)
\leq ||w_1 - w_2||_\infty ||Q^{\pi_1}(s, a)||_1
\leq ||w_1 - w_2||_\infty \cdot m/(1 - \gamma),
\]
where the first inequality holds since $\pi_2$ is the optimal policy corresponding to $w_2$, the second inequality holds due to Cauchy-Schwarz inequality, the last one holds since $||Q^{\pi_1}(s, a)||_1 \leq m/||Q^*(s, a)||_1 \leq m/(1 - \gamma)$. Similarly, we have $w_1^\top Q^*(s, a; w_1) - w_1^\top Q^*(s, a; w_1) \leq ||w_1 - w_2||_\infty \cdot m/(1 - \gamma)$. Therefore, taking maximum over $(s, a) \in S \times A$, we have
\[
\max_{(s, a) \in S \times A} |w_1^\top Q^*(s, a; w_1) - w_2^\top Q^*(s, a; w_2)| \leq ||w_1 - w_2||_\infty \cdot m/(1 - \gamma). \tag{A.2}
\]
The same argument also holds for $\hat{Q}^*(s, a; w)$, thus taking maximum over $(s, a) \in S \times A$, we have
\[
\max_{(s, a) \in S \times A} |w_1^\top \hat{Q}^*(s, a; w_1) - w_2^\top \hat{Q}^*(s, a; w_2)| \leq ||w_1 - w_2||_\infty \cdot m/(1 - \gamma). \tag{A.3}
\]
Let $C_\xi$ be the $\xi$-covering set of the $\ell_1$ ball with respect to $\ell_\infty$ norm. It is easy to verify that $|C_\xi| \leq (2/\xi)^m$. For any $w \in C_\xi$, let $r = w^\top r$ be the scalar reward function corresponding to the preference $w$. Let $Q^*$ be the optimal action-value function with respect to reward function $r$ and transition probability $P$, and $\hat{Q}^*$ be the optimal action-value function with respect to reward function $\hat{r}$ and transition probability $\hat{P}$. Then we have $r \in [-1, 1]$ since $||w||_1 \leq 1$ and $||r||_\infty \leq 1$. By Lemma 8, Azar et al. [1], with probability at least $1 - \delta$, we have
\[
\max_{(s, a) \in S \times A} |Q^*(s, a) - \hat{Q}^*(s, a)|
\leq \frac{4 \log(4SA/\delta)}{N(1 - \gamma)^3} + \left(\frac{5(\gamma/(1 - \gamma)^2)^{4/3} \log(6SA/\delta)}{N}\right)^{3/4} + \frac{3 \log(12SA/\delta)}{(1 - \gamma)^3 N}. \tag{A.4}
\]
Meanwhile, note that by the definition of $Q^*$ and $\hat{Q}^*$, we have
\[
Q^*(s, a) = w^\top Q^*(s, a; w), \quad \hat{Q}^*(s, a) = w^\top \hat{Q}^*(s, a; w).
\]
Then substituting the definitions of $Q^*$ and $\hat{Q}^*$ into A.4 and taking an union bound over all $w \in C_\xi$, replacing $\delta$ with $\delta/|C_\xi|$, we have that with probability at least $1 - \bar{\delta}$, for all $w \in C_\xi$,
\[
\max_{(s, a) \in S \times A} |w^\top Q^*(s, a; w) - w^\top \hat{Q}^*(s, a; w)|
\leq \frac{4 \log(4SA|C_\xi|/\delta)}{N(1 - \gamma)^3} + \left(\frac{5(\gamma/(1 - \gamma)^2)^{4/3} \log(6SA|C_\xi|/\delta)}{N}\right)^{3/4} + \frac{3 \log(12SA|C_\xi|/\delta)}{(1 - \gamma)^3 N}. \tag{A.5}
\]
Finally, we use the fact that for any \( w \in \Omega \), there exists \( w_\xi \in C_\xi \) such that \( \| w - w_\xi \|_\infty \leq \xi \). Then with probability at least 1 - \( \delta \), for all \( w \in \Omega \), we have

\[
\max_{(s,a) \in S \times A} \left| w^T Q^*(s,a;w) - \hat{w}^T \hat{Q}^*(s,a;w) \right| = \max_{(s,a) \in S \times A} \left| w^T Q^*(s,a;w_\xi) - \hat{w}^T \hat{Q}^*(s,a;w_\xi) + \hat{w}^T \hat{Q}^*(s,a;w) - \hat{w}^T \hat{Q}^*(s,a;w_\xi) \right| \\
\leq \max_{(s,a) \in S \times A} \left| w^T Q^*(s,a;w_\xi) - \hat{w}^T \hat{Q}^*(s,a;w_\xi) \right| + \max_{(s,a) \in S \times A} \left| \hat{w}^T \hat{Q}^*(s,a;w) - \hat{w}^T \hat{Q}^*(s,a;w_\xi) \right| \\
\leq \sqrt{\frac{4 \log(4SA|C_\xi|/\delta)}{N(1-\gamma)^3}} + \left( \frac{5(\gamma/(1-\gamma))^2/3 \log(6SA|C_\xi|/\delta)}{N} \right)^{3/4} + 3 \log(12SA|C_\xi|/\delta) \\
+ \frac{3 \log(12SA|C_\xi|/\delta)}{(1-\gamma)^3N} + \frac{3 \log(24SA/(\xi \delta))}{(1-\gamma)^3N} + 2 \xi m/(1-\gamma),
\]

(A.6)

where the first inequality holds due to triangle inequality, the second one holds due to (A.2), (A.3), the fact that \( \| w - w_\xi \|_\infty \leq \xi \) and (A.5), (A.6) suggests that with probability at least 1 - \( \delta \),

\[
d(Q^*, \hat{Q}^*) = \max_{(s,a) \in S \times A, w \in \Omega} \left| w^T Q^*(s,a;w) - \hat{w}^T \hat{Q}^*(s,a;w) \right| \\
\leq \sqrt{\frac{4 \log(4SA|C_\xi|/\delta)}{N(1-\gamma)^3}} + \left( \frac{5(\gamma/(1-\gamma))^2/3 \log(6SA|C_\xi|/\delta)}{N} \right)^{3/4} + 3 \log(12SA|C_\xi|/\delta) \\
+ \frac{3 \log(12SA|C_\xi|/\delta)}{(1-\gamma)^3N} + \frac{3 \log(24SA/(\xi \delta))}{(1-\gamma)^3N} + 2 \xi m/(1-\gamma).
\]

Now we prove Theorem 1.

**Proof of Theorem 1** By triangle inequality we have

\[
d(Q_T, Q^*) \leq d(Q_T, \hat{Q}^*) + d(\hat{Q}^*, Q^*) \\
\leq \gamma^T/(1-\gamma) + \sqrt{\frac{4m \log(8SA/(\xi \delta))}{N(1-\gamma)^3}} + \left( \frac{5(\gamma/(1-\gamma))^2/3m \log(12SA/(\xi \delta))}{N} \right)^{3/4} \\
+ \frac{3m \log(24SA/(\xi \delta))}{(1-\gamma)^3N} + 2 \xi m/(1-\gamma),
\]

(A.7)

where the last inequality holds due to Lemma A.1 and A.2. Therefore, set \( T = \lceil \log(5/(1-\gamma)\epsilon)/(1-\gamma) \rceil, \xi = (1-\gamma)\epsilon/(10m), \) and select \( N \) to make sure that

\[
\sqrt{\frac{4m \log(8SA/(\xi \delta))}{N(1-\gamma)^3}} + \left( \frac{5(\gamma/(1-\gamma))^2/3m \log(12SA/(\xi \delta))}{N} \right)^{3/4} + \frac{3m \log(24SA/(\xi \delta))}{(1-\gamma)^3N} \leq \epsilon/5,
\]

(A.8)

we have \( d(Q_T, Q^*) \leq \epsilon \). Solving out \( N \) ends our proof. \( \square \)
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