1. Introduction

In this article, we introduce an affine analogue \( f^{\hat{g}_{\mathbb{L}}}(x, p|s, \kappa|q, t) \) of the asymptotically free eigenfunction \( f^{g_{\mathbb{L}}}(x|q, t) \) [1–3] for the Macdonald operator [4]. We call \( f^{\hat{g}_{\mathbb{L}}}(x, p|s, \kappa|q, t) \) the non-stationary Ruijsenaars function. We derive it from a construction based on the algebra of the affine (or toroidal) screening operators [5–7]. (See Theorem 2.13 in Section 2.) Then, comparing the characters, we identify \( f^{\hat{g}_{\mathbb{L}}}(x, p|s, \kappa|q, 1/t) \) with the Euler characteristics of the affine Laumon spaces [8]. (See Theorem 3.3 and Proposition 3.4 in Section 3.) Most probably, when the parameters are chosen as in Definition 1.6 below (corresponding to the dominant integrable representations of \( \mathfrak{g}_{\mathbb{L}}(\chi) \)), \( f^{\hat{g}_{\mathbb{L}}}(x, p|s, \kappa|q, q/t) \) coincides with Etingof and Kirillov Jr.’s affine Macdonald polynomial [9], up to some normalization factor. Based on the same philosophy as in the work of Atai and Langmann for the non-stationary Heun and Lamé equations [10], we present several conjectures which support the idea that the \( f^{\hat{g}_{\mathbb{L}}}(x, p|s, \kappa|q, t) \) could be applied for the eigenvalue problems associated with the elliptic Ruijsenaars operator [11], and whose particular degeneration limits including the quasi-difference affine Toda system, elliptic Calogero–Sutherland system.

Let \( N \in \mathbb{Z}_{\geq 2} \). Introduce the collections of independent indeterminates

\[
(x, p) = (x_1, x_2, \ldots, x_N, p), \quad (s, \kappa) = (s_1, s_2, \ldots, s_N, \kappa).
\]
Extend the indices of $x$ and $s$ to $\mathbb{Z}$, assuming the cyclic identifications $x_{i+N} = x_i$ and $s_{i+N} = s_i$. Let $\omega$ be the permutation acting on $(x, p)$ and $(s, \kappa)$ by $\omega x_i = x_{i+1}$, $\omega p = p$, $\omega s_i = s_{i+1}$, $\omega \kappa = \kappa$. A sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-increasing non-negative integers with finitely many positive parts is called a partition, i.e. $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_1 \geq \lambda_2 \geq \cdots$, and $|\lambda| = \sum \lambda_i < \infty$. Let $\mathcal{P}$ be the set of all partitions. Transposition of $\lambda$ is denoted by $\lambda'$. We use the standard notation for the shifted products as in (7) below, see [12] as for the detail.

**Definition 1.1** For $k \in \mathbb{Z}/N\mathbb{Z}$, and $\lambda, \mu \in \mathcal{P}$, set

$$N_{\lambda, \mu}^{(k)}(u|q, \kappa) = N_{\lambda, \mu}^{(k)}(u|q, \kappa) = \prod_{j \geq i \geq 1} (uq^{-\mu_j+i-j-1}k^{-i+j}; q)_{\lambda_j - \lambda_{j-1}} \cdot \prod_{\beta \geq \alpha \geq 1} (uq^{\lambda_\beta - \mu_\beta + 1}; q)_{\mu_\beta - \mu_{\beta+1}}.$$

Note that the ordinary $K$-theoretic Nekrasov factor [13] reads

$$N_{\lambda, \mu}(u|q, \kappa) = \prod_{(i, j) \in \lambda} (1 - uq^{-\mu_i+i-j-1}k^{-i+j-i}); \prod_{(k, j) \in \mu} (1 - uq^{\lambda_k-k-1}k^{-\lambda_k+k+1}),$$

or equivalently

$$N_{\lambda, \mu}(u|q, \kappa) = \prod_{j \geq i \geq 1} (uq^{-\mu_j+i-j-1}k^{-i+j}; q)_{\lambda_j - \lambda_{j-1}} \cdot \prod_{\beta \geq \alpha \geq 1} (uq^{\lambda_\beta - \mu_\beta + 1}; q)_{\mu_\beta - \mu_{\beta+1}}.$$

We have the factorization $N_{\lambda, \mu}(u|q, \kappa) = \prod_{k=1}^N N_{\lambda, \mu}^{(k)}(u|q, \kappa)$.

**Definition 1.2** Let $f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t)$ be the formal power series

$$f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t) \in \mathbb{Q}(s, \kappa, q, t)[[px_1/x_1, \ldots, px_N/x_{N-1}, px_1/x_N]],$$

$$f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t) = \sum_{\lambda \in \mathcal{P}} \prod_{i=1}^N N_{\lambda, \mu}^{(j-i)(N)}(i; s_i|q, \kappa) \cdot N_{\lambda, \mu}^{(j-i)(N)}(s_j|s_i|q, \kappa) \cdot \prod_{\beta=1}^N (px_{\alpha+\beta}/lx_{\alpha+\beta-1})^{(j)}\cdot$$

We call $f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t)$ the non-stationary Ruijsenaars function.

Note that we have $e_0 f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t) = f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t)$. In Section 2, a derivation is presented for the series $f^{\tilde{\mathfrak{g}}_N}(x, p|s, \kappa|q, t)$ as a matrix element of a composition of certain screened vertex operators associated with the affine (or toroidal) screening operators found in [5–7]. See Theorem 2.13. For the moment, the role of the deformed $W$-algebras associated with the affine screening operators remains unclear.

A simple calculation using the $q$-binomial formula [12] gives us the following factorization formula. See Remark 4.5 below.
Proposition 1.3 Setting $\kappa = 0$, we have

$$f^{\widehat{g}_N}(x, p|s, 0|q, t) = \prod_{1 \leq i < j \leq N} \frac{(p^{j-i}q x_j/x_i; q, p_N)_{\infty}}{(p^{j-i}t x_j/x_i; q, p_N)_{\infty}} \cdot \prod_{1 \leq i \leq j \leq N} \frac{(p^{N-j+i}q x_i/x_j; q, p_N)_{\infty}}{(p^{N-j+i}t x_i/x_j; q, p_N)_{\infty}}.$$  

Dividing $f^{\widehat{g}_N}(x, p|s, \kappa|q, t)$ by $f^{\widehat{g}_N}(x, p|s, 0|q, t)$, we introduce the normalized version $\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t)$ as follows.

Definition 1.4 Let $\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t)$ be the formal power series

$$\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t) \in \mathbb{Q}(q, t)[[px_1/x_1, \ldots, px_N/x_N, \kappa s_1/s_1, \ldots, \kappa s_N/s_N]] \setminus \{0\},$$

$$\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t) = \prod_{1 \leq i < j \leq N} \frac{(p^{j-i}q x_j/x_i; q, p_N)_{\infty}}{(p^{j-i}q x_j/x_i; q, p_N)_{\infty}} \cdot \prod_{1 \leq i \leq j \leq N} \frac{(p^{N-j+i}q x_i/x_j; q, p_N)_{\infty}}{(p^{N-j+i}q x_i/x_j; q, p_N)_{\infty}} f^{\widehat{g}_N}(x, p|s, \kappa|q, t),$$

(1)

where the coefficients $\prod_{i=1}^{N} N_{\lambda^{(i)}(\emptyset)}^{i(\lambda^{(i-1)(\emptyset)})} (s_i/s_i|q, \kappa)/N_{\lambda^{(i+1)(\emptyset)}}^{i(\lambda^{(i)(\emptyset)})} (s_i/s_i|q, \kappa)$ in $f^{\widehat{g}_N}(x, p|s, \kappa|q, t)$ are Taylor expanded in $\kappa$ at $\kappa = 0$.

We have $\omega \varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t) = \varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t)$.

Conjecture 1.5 We have the duality properties

$$\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t) = \varphi^{\widehat{g}_N}(x, \kappa|p|q, t) \quad \text{(bispectral duality)},$$

(2)

$$\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t) = \varphi^{\widehat{g}_N}(x, p|s, \kappa|q/t) \quad \text{(Poincaré duality)}.$$  

(3)

These bispectral and the Poincaré duality conjectures for $\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, t)$ are regarded as affine analogues of the ones for the Macdonald function $\varphi^g_N(x|s|q, t)$ established in Noumi and the present author’s paper [2]. See Proposition 4.7 in Section 4 below.

We show that the series $f^{\widehat{g}_N}(x, p|s, \kappa|q, 1/t)$ is the generating function for the Euler characteristic $\sum_{\lambda} (-1)^{\text{length} \lambda} f^\iota([P^1_1(\Omega)]_{\lambda})$ of the de Rham complex of the affine Laumon space $P_d$ studied in [8]. See Theorem 3.3 and Proposition 3.4 in Section 3. The key for this identification between $f^{\widehat{g}_N}(x, p|s, \kappa|q, 1/t)$ and the geometric object (the Euler characteristics of the affine Laumon spaces), is a comparison of the combinatorial identities given in Propositions 3.1 and 3.2.

We remark that in [14, 15] Braverman’s conjecture [16] was proved, showing that the generating function of the Chern polynomials of the (affine) Laumon spaces satisfies the (elliptic) Calogero–Sutherland equation (Theorem 7.1 in [14] and Theorem 1.5 in [15]). In view of Conjecture 7.2 below, it seems plausible that the normalized series $\varphi^{\widehat{g}_N}(x, p|s, \kappa|q, 1/t)$ admits a similar interpretation in terms of the Chern polynomials of the affine Laumon spaces, or Hirzebruch–Riemann–Roch theorem.

The Schur polynomials are obtained from the Macdonald polynomials by taking the limit $t \to q$. In the same manner, we have the $\widehat{g}_N$ dominant integrable characters (up to the character of $\widehat{g}_N$) from $f^{\widehat{g}_N}(x, p|s, \kappa|q, t)$ by considering the limit $t \to q$. Set $\delta = (N-1, N-2, \ldots, 1, 0)$. Here and hereafter, we use the standard notation as $t^d = (t^{N-1}s_1, t^{N-2}s_2, \ldots, t s_N, s_N).$
Let $K$ be a non-negative integer. We call $K$ the level. Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition satisfying the condition $K + \mu_N - \mu_1 \geq 0$. Then set
\[ s = (\kappa t)^{\delta} q^{\mu} = q^{-Ks/N+\mu}, \quad \kappa = q^{-K/\ell}. \tag{4} \]
i.e. for $s$, we set $s_i = q^{-K(N-i)/N+\mu_i} \ (1 \leq i \leq N)$.

For such $K$ and $\mu$, we have the level $K$ dominant integrable weight $\Lambda(K, \mu) = (K + \mu_N - \mu_1)\Lambda_0 + \sum_{i=1}^{N-1}(\mu_i - \mu_{i+1})\Lambda_i$, and the dominant integrable representation $L(\Lambda(K, \mu))$ of $\widehat{\mathfrak{sl}}_N$, where $\Lambda_0, \ldots, \Lambda_{N-1}$ denote the fundamental weights. Denote by $\text{ch}_{L(\Lambda(K, \mu))}$ the character of $L(\Lambda(K, \mu))$ associated with the principal gradation.

**Theorem 1.7** Let $K, \mu, s, \kappa$ be fixed as in (4). We have
\[ \lim_{t \to q} x^\mu f_{\widehat{\mathfrak{sl}}_N}(x, p|q^{-Ks/N+\mu}, q^{-K/\ell}|q, q/t) = \frac{1}{(p^N; p^N)_{\infty}} \cdot \text{ch}_{L(\Lambda(K, \mu))}. \]

Note that the factor $1/(p^N; p^N)_{\infty}$ is interpreted as the $\widehat{\mathfrak{sl}}_1$ character. A proof of this is given in Section 3.4 based on the affine Gelfand–Tsetlin pattern obtained in [8], which we can regard as Tingley’s $\mathfrak{sl}_N$-crystal [17].

**Proposition 1.8** Let $K = 0, \mu = \emptyset$. Then (4) means $s_i = 1 (1 \leq i \leq N)$ and $\kappa = t^{-1}$. Let $q$ and $t$ be arbitrary. In this case, we have
\[ f_{\widehat{\mathfrak{sl}}_N}(x, p|1, \ldots, 1, t^{-1}|q, q/t) = \frac{1}{(p^N; p^N)_{\infty}}. \]

Proofs of Theorem 1.7 and Proposition 1.8 are given in Section 3.4.

Proposition 1.8 and the Poincaré duality in Conjecture 1.5 imply the following evaluation formula.

**Conjecture 1.9** Set $x = (1, \ldots, 1)$ and $p = 1/t$. We have the identity (evaluation formula) in $\mathbb{Q}(q)[s_2/s_1, s_3/s_2, \ldots, s_1/s_N][[1/t, \kappa]]$ as
\[ f_{\widehat{\mathfrak{sl}}_N}(1, \ldots, 1, 1/t|s, \kappa|q, q/t) = \frac{1}{(\kappa^N; \kappa^N)_{\infty}} \frac{1}{(q/t; q)_{\infty}^N} \prod_{1 \leq i < j \leq N} (\kappa^{N-j}q^{s_j/t}s_i; q, q^N)_{\infty} (\kappa^{N-j}q^{s_i/t}s_j; q, q^N)_{\infty}. \tag{5} \]

Setting our parameters as in (4) and letting $t = q^k \ (k \in \mathbb{Z}_{\geq 0})$, one finds that (5) looks very close to the specialization formula for the affine Macdonald polynomials based on the representation theories of quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_N)$ in [9]. See Conjecture 11.3 in [9], and Conjecture 4.3 in [18] as for the improved version. In view of this correspondence and Conjecture 7.2 (for elliptic Calogero–Sutherland limit) in Section 7 below, we strongly expect that Etingov and Kirillov Jr.’s affine Macdonald polynomial $\widehat{\mathcal{P}}_\lambda$ coincide with $f_{\widehat{\mathfrak{sl}}_N}(x, p|s, \kappa|q, q/t)$ up to a normalization factor.
Since our description of $\hat{f}^{\varphi_N}(x, p|s, \kappa|q, t)$ (via the affine screening operators or the affine Laumon spaces) is quite different from the ‘trace of intertwiner’ construction in [9], we have not been able to compare two objects, unfortunately. It is an intriguing problem to establish the connection between them.

Now, we turn to the eigenvalue problem associated with the elliptic Ruijsenaars operator [11], from the point of view of the series $f^{\varphi_N}(x, p|s, \kappa|q, t)$. We use the multiplicative notation for the elliptic theta function as $\Theta_p(z) = (z; p)_\infty(p/z; p)_\infty(p; p)_\infty$.

**Definition 1.10** Let $D_\lambda(p) = D_\lambda(p|q, t)$ denotes the Ruijsenaars operator [11]

\[
D_\lambda(p) = \sum_{i=1}^{N} \prod_{j \neq i} \frac{\Theta_p(t_{x_i}/x_j)}{\Theta_p(x_i/x_j)} T_{q, x_i},
\]

where $T_{q, x_i}$ is the $q$-shift operator $q^{\varphi/q x_i}$.

Naively speaking, we take the ‘stationary limit’ $\kappa \to 1$ of $f^{\varphi_N}(x, p|s, \kappa|q, t)$. Such a limit, however, does not exist. It seems that we need to normalize $f^{\varphi_N}$, before taking the limit $\kappa \to 1$. The simplest way might be to divide $f^{\varphi_N}$ by its constant term in $x$.

We closely follow the method developed in Atai and Langmann’s paper [10] for the non-stationary Heun and Lamé equations. Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ be an $N$-tuple of partitions. Set

\[
|\lambda| = \sum_{i=1}^{N} |\lambda^{(i)}|, \quad m_i = m_i(\lambda) = \sum_{\beta=1}^{N} \sum_{q \equiv 1 \pmod{N}} \lambda^{(\beta)}_q - \lambda^{(\beta+1)}_q.
\]

Then we have $\prod_{\beta=1}^{N} \prod_{q \equiv 1 \pmod{N}} (p x_{\alpha} + \beta / t x_{\alpha} + \beta - 1)^{\lambda^{(\beta)}_q} = (p / t)^{|\lambda|} \prod_{i=1}^{N} x_i^{-m_i}$. Note that when $m_1 = \ldots = m_N = 0$, we have $|\lambda| \equiv 0 \pmod{N}$.

**Definition 1.11** Let $\alpha(p|s, \kappa|q, t) = \sum_{d \geq 0} p^{d N} \alpha_d(s, \kappa|q, t)$ be the constant term of the series $f^{\varphi_N}(x, p|s, \kappa|q, t)$ with respect to $x_i$’s. Namely,

\[
\alpha(p|s, \kappa|q, t) = \sum_{\lambda^{(1)}, \ldots, \lambda^{(N)} \in \mathcal{P}} (p / t)^{|\lambda|} \prod_{i=1}^{N} \frac{N_{\lambda^{(1)}, \lambda^{(N)}}(s_j / s_i|q, \kappa)}{N_{\lambda^{(1)}, \lambda^{(N)}}(s_j / s_i|q, \kappa)}.
\]

**Conjecture 1.12** We have the properties:

1. The series $f^{\varphi_N}(x, p|s, \kappa|q, t)$ is convergent on a certain domain. With respect to $\kappa$, it is regular on a certain punctured disk $\kappa \in \mathbb{C} | \kappa - 1 | < r, \kappa \neq 1$.

2. The $f^{\varphi_N}(x, p|s, \kappa|q, t)$ and $\alpha(p|s, \kappa|q, t)$ are essential singular at $\kappa = 1$. (The coefficient $\alpha_d(s, \kappa|q, t)$ has a pole of degree $d$ in $\kappa$ at $\kappa = 1$.)

3. The ratio $f^{\varphi_N}(x, p|s, \kappa|q, t)/\alpha(p|s, \kappa|q, t)$ is regular at $\kappa = 1$. 

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Definition 1.13 Assuming Conjecture 1.12, set

\[
f^{s,t,N}(x,p|s,q,t) = \left. \frac{f^{s,t,N}(x,p|s,\kappa|q,t)}{\alpha(p|s,\kappa|q,t)} \right|_{\kappa=1}.\]

We call \(f^{s,t,N}(x,p|s,q,t)\) the stationary Ruijsenaars function.

Now, we are ready to state our main conjecture.

Conjecture 1.14 (Main Conjecture) Let \(s = t^i q^{\lambda}(s_i = t^{N-i} q^{\lambda_i})\). Denote by \(p^{\lambda/N}\) the collection of the shifted coordinates \(p^{(N-i)/N} x_i\). The stationary Ruijsenaars function \(x^{s} f^{s,t,N}(p^{\lambda/N} x, p^{\lambda/N} | s, q, q/t)\) is an eigenfunction of the Ruijsenaars operator:

\[
D_\lambda(p) x^s f^{s,t,N}(p^{\lambda/N} x, p^{\lambda/N} | s, q, q/t) = \epsilon(p|s,q,t) x^s f^{s,t,N}(p^{\lambda/N} x, p^{\lambda/N} | s, q, q/t),
\]

\[
\epsilon(p|s,q,t) = \sum_{i=1}^{N} s_i + \sum_{d>0} \epsilon_d(s|q,t)p^d.
\]

To check the conjecture, we first need to formulate the eigenvalue problem associated with the Ruijsenaars operator on the space of formal series \(\mathbb{Q}(q,t,s)[[px_2/x_1, \ldots, px_N/x_1]]\), clarifying the meaning of the perturbation in \(p\). Then we can make a systematic check. The detail will be reported elsewhere [19] (a joint work with E. Langmann and M. Noumi).

We investigate the implications of Conjecture 1.14, in the Macdonald \((p \rightarrow 0, \text{in Section 4})\), the affine \(q\)-Toda \((t \rightarrow 0, \text{in Section 6})\) and the elliptic Calogero–Sutherland \((q,t \rightarrow 1, \text{in Section 7})\) limits. For the moment, unfortunately, we have not been able to find a non-stationary analogue (a \(\kappa\)-deformation) of the Ruijsenaars operator \(D_\lambda(p)\), or \(t\)-deformation of the non-stationary \(q\)-affine Toda operator \(T^{\lambda,N}(\kappa)\) in Definition 6.1 below, for which \(f^{s,t,N}(x,p|s,\kappa|q,q/t)\) should give us the eigenfunction. We strongly expect that Felder and Varchenko’s \(q\)-deformed KZB heat equation provides us with the answer [20–22].

This article is organized as follows. In Section 2, we introduce the screened vertex operator \(\Phi(z|x,p)\) associated with the algebra of the affine screening operators. In Section 3, we recall some basic facts concerning the affine Laumon spaces, and identify \(f^{s,t,N}(x,p|s,\kappa|q,1/t)\) with the generating function for the Euler characteristics of the affine Laumon spaces. It is shown that we have the dominant integrable characters from \(f^{s,t,N}(x,p|s,\kappa|q,q/t)\) by taking the limit \(t \rightarrow q\) with \(s\) and \(\kappa\) being chosen as in Definition 1.6. In Section 4, we study the Macdonald limit \(p \rightarrow 0\). Section 5 is devoted to the derivations of the Macdonald operator, the elliptic Calogero–Sutherland operator and the affine \(q\)-Toda operator. This technical section is necessary to have a unified picture about the family of elliptic integrable systems based on the elliptic Ruijsenaars operator, on which we can argue the validity of Conjecture 1.5. In Section 6, we introduce the operator \(T^{\lambda,N}(q,\tilde{p},\kappa)\) representing the non-stationary affine Toda in the \(q\)-difference setting. Then we explain what does Conjecture 1.5 imply in the Toda limit \(t \rightarrow 0\). In Section 7, we test Conjecture 1.5 in the elliptic Calogero–Sutherland limit \(q,t \rightarrow 1\).

In this article, we state several Propositions, Lemmas, etc. without proof; all such results are standard and can be proved by straightforward computations using definitions.
We use the standard notation for the $q$-shifted factorials and the double infinite products such as

\[
(u; q)_\infty = \prod_{i=0}^{\infty} (1 - q^i u), \quad (u; q)_n = (u; q)_{\infty} / (q^n u; q)_{\infty}, \quad (u; q, p)_{\infty} = \prod_{i,j=0}^{\infty} (1 - q^i p^j u).
\]

As for the detail, see [12].

2. Affine screening operators and $f_N^\omega(x, p^{1/N}|s, k^{1/N}|q, t)$

2.1 Heisenberg algebra $h_N$

As for the Heisenberg algebra and the affine (or toroidal) screening operators, we basically follow the construction in [5–7].

**Definition 2.1** Let $N \in \mathbb{Z}_{\geq 2}$. Let $h_N$ be the Heisenberg algebra generated by $\beta^i_n, \beta^i_n (n \in \mathbb{Z}_{\neq 0})$ satisfying the commutation relations

\[
[\beta^i_n, \beta^j_m] = n \frac{1 - t^n q^{n-i} q^{-nm} \delta_{n+m,0}}{1 - q^{n} \delta_{n,m,0}} \quad (i = 0, \ldots, N - 1),
\]

\[
[\beta^i_n, \beta^j_m] = n \frac{1 - t^n q^{n-i} q^{-nm} \delta_{n+m,0}}{1 - q^{n} \delta_{n,m,0}} \quad (0 \leq i < j \leq N - 1),
\]

\[
[\beta^i_n, \beta^j_m] = n \frac{1 - t^n q^{n-i} q^{-nm} \delta_{n+m,0}}{1 - q^{n} \delta_{n,m,0}} \quad (0 \leq j < i \leq N - 1).
\]

Let $F_N$ be the Fock space associated with $h_N$, and let $|0\rangle$ be the Fock vacuum satisfying $\beta^i_n |0\rangle = 0$ $(1 \leq i \leq N, n > 0)$. Let $\langle 0|$ be the dual vacuum satisfying $\langle 0| \beta^i_n = 0 (1 \leq i \leq N, n < 0)$ and $\langle 0|0\rangle = 1$.

By an abuse of notation, let $\omega$ be the automorphism of $h_N$ defined by the cyclic permutation $\omega \beta^i_n = \beta^{i+1}_n$ $(0 \leq i \leq N - 2)$, and $\omega \beta^{N-1}_n = \beta^0_n$. Note that we have $\omega^N = \text{id}$.

**Definition 2.2** Set $\alpha^i_n = \kappa^{-n/N} \beta^{i-1}_n - \beta^i_n (1 \leq i \leq N - 1)$, and $\alpha^0_n = \kappa^{-n/N} \beta^{N-1}_n - \beta^0_n$.

We have $\omega \alpha^i_n = \alpha^{i+1}_n$, where $\alpha^N_n = \alpha^0_n$.

**Proposition 2.3** For $0 \leq i \leq N - 1$, we have

\[
[\alpha^i_n, \alpha^j_m] = n(1 + q^n t^{-n}) \frac{1 - t^n}{1 - q^n} \delta_{n+m,0},
\]

\[
[\alpha^i_n, \alpha^{i+1}_m] = -n \kappa^{n/N} q^n \frac{1 - t^n}{1 - q^n} \delta_{n+m,0}, \quad [\alpha^i_n, \alpha^{i-1}_m] = -n \kappa^{-n/N} \frac{1 - t^n}{1 - q^n} \delta_{n+m,0},
\]

and $[\alpha^i_n, \alpha^j_m] = 0$ otherwise.
Proposition 2.4  For \(0 \leq i, j \leq N - 1\), we have
\[
[\beta_m^i, \alpha_m^{i+1}] = n \kappa^{n/N} q^n t^{-n} \frac{1 - t^n}{1 - q^n} \delta_{n+m, 0}, \quad [\beta_m^i, \alpha_m^i] = -n \frac{1 - t^n}{1 - q^n} \delta_{n+m, 0},
\]
and \([\beta_m^i, \alpha_m^j] = 0\) otherwise.

2.2 Affine screening operators \(S_i(z)\)

In this article, we only work out the case \(N \geq 3\). The case \(N = 2\) has to be treated separately, which we omit. We remark that the final result given in Theorem 2.13 below also applies for the case \(N = 2\).

Definition 2.5  For \(0 \leq i \leq N - 1\), set
\[
S_i(z) = :\exp \left( -\sum_{n \neq 0} \frac{1}{n} \alpha_n^i z^{n} \right) : = \exp \left( \sum_{n > 0} \frac{1}{n} \alpha_n^i z^{n} \right) \exp \left( -\sum_{n > 0} \frac{1}{n} \alpha_n^i z^{n} \right).
\]

Here and hereafter, we use the standard notation for the normal ordered product \(\cdot : \cdot\), i.e., we put all the annihilation operators \(\beta_n^i (0 \leq i \leq N - 1, n > 0)\) to the right of the creation ones \(\beta_n^{1-n} (0 \leq i \leq N - 1, n > 0)\). Note that we have \(\omega S_i(z) = S_{i+1}(z)\). We call \(S_i(z)\)'s the affine screening operators.

Proposition 2.6  For \(0 \leq i, j \leq N - 1\), we have
\[
S_i(z)S_j(w) = \frac{(w/z; q)_{\infty}}{(tw/z; q)_{\infty}} \frac{(qw/z; q)_{\infty}}{(qw/tz; q)_{\infty}} : S_i(z)S_j(w) :,
\]
\[
S_i(z)S_{i+1}(w) = \frac{(k^{1/N} qw/z; q)_{\infty}}{(k^{1/N} qtw/z; q)_{\infty}} : S_i(z)S_{i+1}(w) :,
\]
\[
S_{i+1}(z)S_i(w) = \frac{(k^{-1/N} tw/z; q)_{\infty}}{(k^{-1/N} w/z; q)_{\infty}} : S_{i+1}(z)S_i(w) :,
\]
and \(S_i(z)S_j(w) = S_j(w)S_i(z) = : S_i(z)S_j(w) :\) otherwise.

2.3 Vertex operators \(\phi_i(z)\)

Definition 2.7  For \(0 \leq i \leq N - 1\), set
\[
\phi_i(z) = : \exp \left( \sum_{n \neq 0} \frac{1}{n} \beta_n^i z^{n} \right) : = \exp \left( -\sum_{n > 0} \frac{1}{n} \beta_n^{1-n} z^{n} \right) \exp \left( \sum_{n > 0} \frac{1}{n} \beta_n^{1-n} z^{n} \right).
\]

Note that we have \(\omega \phi_i(z) = \phi_{i+1}(z)\).

Proposition 2.8  We have
\[
\phi_i(z)\phi_i(w) = \frac{(w/z; q, \kappa)_{\infty}}{(tw/z; q, \kappa)_{\infty}} \frac{(kqw/z; q, \kappa)_{\infty}}{(kqw/tz; q, \kappa)_{\infty}} : \phi_i(z)\phi_i(w) : \quad (0 \leq i \leq N - 1),
\]
\[
\phi_i(z) \phi_j(w) = \frac{(k^{(-i+j)/N}w/z; q, \kappa)_\infty}{(k^{(-i+j)/N}tw/z; q, \kappa)_\infty} \frac{(k^{(-i+j)/N}qw/z; q, \kappa)_\infty}{(k^{(-i+j)/N}qw/tz; q, \kappa)_\infty} : \phi_i(z) \phi_j(w) : \quad (0 \leq i < j \leq N - 1),
\]
\[
\phi_j(z) \phi_i(w) = \frac{(k^{(-i+j+N)/N}w/z; q, \kappa)_\infty}{(k^{(-i+j+N)/N}tw/z; q, \kappa)_\infty} \frac{(k^{(-i+j+N)/N}qw/z; q, \kappa)_\infty}{(k^{(-i+j+N)/N}qw/tz; q, \kappa)_\infty} : \phi_j(z) \phi_i(w) : \quad (0 \leq j < i \leq N - 1).
\]

**Proposition 2.9** For \(0 \leq i, j \leq N - 1\) we have
\[
\phi_i(z) S_{i+1}(w) = \frac{(k^{1/N}qw/z; q)_\infty}{(k^{1/N}qw/tz; q)_\infty} : \phi_i(z) S_{i+1}(w) : ,
\]
\[
S_{i+1}(w) \phi_i(z) = \frac{(k^{-1/N}tz/w; q)_\infty}{(k^{-1/N}z/w; q)_\infty} : \phi_i(z) S_{i+1}(w) : ,
\]
\[
\phi_i(z) S_i(w) = \frac{(w/z; q)_\infty}{(tw/z; q)_\infty} : \phi_i(z) S_i(w) : , \quad S_i(w) \phi_i(z) = \frac{(qz/tw; q)_\infty}{(qz/w; q)_\infty} : \phi_i(z) S_i(w) : ,
\]
\[
\phi_i(z) S_j(w) = : \phi_i(z) S_j(w) : , \quad S_j(w) \phi_i(z) = : \phi_i(z) S_j(w) : \quad (j \neq i, i+1).
\]

**Proposition 2.10** For \(0 \leq i \leq N - 1\), we have the fusion properties
\[
\phi_i(z) = : \phi_{i-1}(k^{1/N}z) S_i(z) : .
\]

### 2.4 Screened vertex operators

We assume that the indices of \(S_i(z)\) and \(\phi_i(z)\) are extended to \(\mathbb{Z}\) assuming the cyclic identifications \(S_i(z) = S_{i+N}(z)\), \(\phi_i(z) = \phi_{i+N}(z)\). We use the following notation for the ordered products: \(\prod_{1 \leq j \leq \ell} A_j := A_1 \cdots A_{\ell} A_1\).

Let \(0 \leq i \leq N - 1\), and \(\lambda = (\lambda_1, \ldots, \lambda_{\ell}) \in P\), where \(\ell = \ell(\lambda)\) denotes the length of \(\lambda\). Set
\[
\phi_i^\lambda(z) = \phi_{i-\ell}(k^{(\ell+1)/N}z) \prod_{1 \leq j \leq \ell} S_{i-j+1}(k^{1/N}q^{\lambda_j}z).
\]

Then we introduce the stabilized version \(\Phi_i^\lambda(z)\) as follows.

**Definition 2.11** For \(1 \leq i \leq N - 1\) and \(\lambda \in P\), set
\[
\Phi_i^\lambda(z) = \left(\frac{(q/1; q)_\infty}{(q; q)_\infty}\right)^{\ell(\lambda)} \phi_i^\lambda(z).
\]

Thanks to the properties in Propositions 2.9 and 2.10, the operators \(\Phi_i^\lambda(z)\) are consistently defined for all \(\lambda \in P\).

Let \((x, p) = (x_1, \ldots, x_N, p)\) be a collection of parameters. Extend it to \(x = (x_i)_{i \in \mathbb{Z}}\) assuming the cyclic identification \(x_i = x_{i+N}\).
**Definition 2.12** Define the screened vertex operator \( \Phi^i(z|x,p) \) by the infinite series

\[
\Phi^i(z|x,p) = \sum_{\lambda \in \mathcal{P}} \Phi_\lambda^i(z) \prod_{k \geq 1} (p^{1/N} x_{N-i+k} / x_{N-i+k-1})^{\lambda_k}.
\]

Set \( \omega^i = (\omega^i_1, \ldots, \omega^i_N) = (0, \ldots, 0, 1, \ldots, 1) \in \mathbb{Z}^N \) for \( 1 \leq i \leq N \). Write \( t^a x = (t^a_1 x_1, \ldots, t^a_N x_N) \) for simplicity. Let \( (s, \kappa) = (s_1, \ldots, s_N, \kappa) \) be another collection of parameters.

**Theorem 2.13** Let \( N \in \mathbb{Z}_{\geq 2} \). We have

\[
\langle 0| \Phi^0(s_1|t^{a_1} x_1, p) \Phi^1(s_2|t^{a_1} x_1, p) \cdots \Phi^{N-1}(s_N|t^{a_1} x_1, p)|0 \rangle = \prod_{1 \leq j < i \leq N} \frac{(k^{N-1} s_j/s_i; q, \kappa)^{\infty}}{(k^{N-1} s_i/s_j; q, \kappa)^{\infty}} \cdot f^{2N}(x, p^{1/N}|s, \kappa^{1/N}|q, t).
\]

**Proof.** For \( i = 0, 1, \ldots, N-1 \), set \( |\lambda|^{(i)} = \sum_{j=i+1 \pmod{N}} \lambda_j \). It follows from Lemmas 2.14, 2.15 and 2.16 below, we have

\[
\text{LHS} = \sum_{\lambda^{(1)}, \ldots, \lambda^{(N)} \in \mathcal{P}} \langle 0| \Phi^0(\lambda^{(N)})(s_1) \Phi^1(\lambda^{(N-1)})(s_2) \cdots \Phi^{N-1}(\lambda^{(1)})(s_N)|0 \rangle \times \prod_{j=1}^N \prod_{i=1}^{N-1} (p^{1/N} x_{j+1}/x_{j+1-1})^{\lambda_j^{(i)}} \cdot \prod_{0 \leq j \leq N-1} \prod_{i=1}^{N-1} t^{p^{1/N}(N+i-j+1)|\lambda^{(N-1)}(N+i-j)|(-i+j-1)} = \text{RHS}.
\]

**Lemma 2.14** For \( \alpha = 0, 1, \ldots, N-1 \), we have

\[
\sum_{\substack{j \geq 1 \atop j \equiv \alpha \pmod{N}}} \lambda_j - \lambda_{j+1} = \sum_{j \geq 1} \left[ j + N - 1 - \alpha \right] / N \lambda_j - \left[ j + N - 1 - \alpha \right] / N \lambda_{j+1} = |\lambda|^{(\alpha)}.
\]

**Lemma 2.15** For \( k = 0, \ldots, N-1 \), we have

\[
\Phi^k(z) = t^{-|\lambda|^{(0)}} N^{(0)}_{\lambda\lambda}(t|q, \kappa^{1/N}) / N^{(0)}_{\lambda\lambda}(1|q, \kappa^{1/N}) : \phi^k(z) :.
\]

**Lemma 2.16** Let \( 0 \leq \alpha < \beta \leq N-1 \), we have

\[
: \Phi^\alpha(z) :: \Phi^\beta(w) : = \sum_{\mu \neq \lambda} \frac{N^{(\beta-\alpha)}_{\mu\lambda}(tw|q, \kappa^{1/N}) N^{(\beta-\alpha)}_{\mu\lambda}(z/w|q, \kappa^{1/N})}{(k^{\beta-\alpha}w/z; q, \kappa)^{\infty}} \cdot \frac{(k^{\beta-\alpha}tw/z; q, \kappa)^{\infty}}{(k^{\beta-\alpha}tw/z; q, \kappa)^{\infty}} \cdot \frac{(k^{\beta-\alpha}w/z; q, \kappa)^{\infty}}{(k^{\beta-\alpha}w/z; q, \kappa)^{\infty}} : \Phi^\alpha(z) \Phi^\beta(w) :.
\]

Proofs of Lemmas 2.15 and 2.16 will be given in the next subsection.
2.5 Proofs of Lemmas 2.15 and 2.16

Proof of Lemma 2.15. Let $\ell = \ell(\lambda)$. Write $z_i = k_i^Iq^\lambda_i z$ for short. We have

\[
\Phi^k(z) = \frac{\left(\frac{q}{t_I}; q\right)_\infty}{\left(\frac{q}{q}; q\right)_\infty} \sum_{1 \leq i \leq \ell} S_{k-i+1}(z_i)
\]

\[
= \phi_{k-\ell}(z_{\ell+1}) \prod_{1 \leq i \leq \ell} S_{k-i+1}(z_i) \frac{(q^I/t_I; q\infty)}{(q^I; q\infty)}
\]

\[
	imes \prod_{1 \leq j \leq \ell} \frac{(k^I^j q^I\lambda_j z_j; q\infty)}{(k^I^j q^I z_j; q\infty)} \cdot \prod_{1 \leq j \leq \ell} \frac{(k^{-I^j} t^I_j z_j; q\infty)}{(k^{-I^j} z^I_j; q\infty)} \cdot \prod_{1 \leq j \leq \ell} \frac{(z_j/z_j q\infty)}{(t^I_j/z_j q\infty)} \frac{(q^I z_j/t^I_j z_j; q\infty)}{(q^I z_j/z_j; q\infty)}
\]

We separate the factors in two groups, and simplify each of them as follows. First, we have

\[
\prod_{1 \leq j \leq \ell} \frac{(z_j/z_j+1; q\infty)}{(t^I_j/z_j+1; q\infty)} \cdot \prod_{1 \leq j \leq \ell} \frac{(k^{-I^j} t^I_j z_j; q\infty)}{(k^{-I^j} z^I_j; q\infty)} = \prod_{1 \leq j \leq \ell} \frac{(k^{-I^j} z^I_j; q\infty)}{(z^I_j; q\infty)} = \prod_{1 \leq j \leq \ell} \frac{(k^{-I^j} t^I_j z_j; q\infty)}{(t^I_j z^I_j; q\infty)} = \prod_{1 \leq j \leq \ell} \frac{(k^{-I^j} z^I_j; q\infty)}{(t^I_j z^I_j; q\infty)}
\]

Next, we have

\[
\left(\frac{q}{t_I}; q\infty\right)^\ell \prod_{1 \leq j \leq \ell} \frac{(k^I^j q^I\lambda_j z_j; q\infty)}{(k^I^j q^I z_j; q\infty)} \cdot \prod_{1 \leq j \leq \ell} \frac{(q^I z_j/t^I_j z_j q\infty)}{(q^I z_j/z_j q\infty)}
\]

\[
= \prod_{1 \leq j \leq \ell} \frac{(k^I^j q^I\lambda_j z_j+1; q\infty)}{(k^I^j q^I z_j+1; q\infty)} \cdot \prod_{1 \leq j \leq \ell} \frac{(q^I z_j/t^I_j z_j q\infty)}{(q^I z_j/z_j q\infty)} = \prod_{1 \leq j \leq \ell} \frac{(k^I^j q^I\lambda_j z_j+1; q\infty)}{(k^I^j q^I z_j+1; q\infty)} \cdot \prod_{1 \leq j \leq \ell} \frac{(q^I z_j/t^I_j z_j q\infty)}{(q^I z_j/z_j q\infty)}
\]

Using Lemma 2.14, we have (8).

Proof of Lemma 2.16. It is sufficient to consider the case $\ell = \ell(\lambda) = \ell(\mu)$. For simplicity, set $z_i = k_i^I q^\lambda_i z$, and $w_i = k_i^I q^\mu_i z$, meaning : $\phi^k(z) = : \phi_{a-I}(z_{\ell+1}) \prod_{1 \leq i \leq \ell} S_{a-i+1}(z_i) :$, and : $\phi^\delta(w) = :
Next, we have

\[
\phi_{\beta-\ell}(w_{\ell+1}) \prod_{1 \leq j \leq \ell} S_{\beta-j+1}(w_j) \; : \; \text{We have}
\]

\[
(k^{(\beta-\alpha)/N} w_{\ell+1}/z_i, q, \kappa)_\infty \; (k^{(\beta-\alpha)/N} w_{\ell+1}/z_i, q, \kappa)_\infty
\]

\[
\phi_\alpha^\beta(z) \phi_\mu^\beta(w) : \]

\[
= \prod_{1 \leq j \leq \ell} (k^{1/N} w_{j+1}/z_{\ell+1} ; q)_\infty \prod_{1 \leq j \leq \ell} (w_{j+1}/z_{\ell+1} ; q)_\infty
\]

\[
\times \prod_{1 \leq j \leq \ell} \frac{(k^{-1/N} w_{j+1}/z_i ; q)_\infty}{(k^{-1/N} w_{j+1}/z_i ; q)_\infty} \prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty}
\]

\[
\times \prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty}
\]

Separate the factors in two groups, and simplify each of them as follows. First, we have

\[
\prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_{\ell+1} ; q)_\infty}{(w_{j+1}/z_{\ell+1} ; q)_\infty} \prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty}
\]

\[
\times \prod_{1 \leq j \leq \ell} \frac{(k^{-1/N} w_{j+1}/z_i ; q)_\infty}{(k^{-1/N} w_{j+1}/z_i ; q)_\infty} \prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty}
\]

\[
= \prod_{1 \leq j \leq \ell} \frac{(k^{(j-i-1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}{(k^{(j-i-1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}
\]

\[
\times \prod_{1 \leq j \leq \ell} \frac{(k^{(j-i-1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}{(k^{(j-i-1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}
\]

Next, we have

\[
\prod_{1 \leq j \leq \ell} \frac{(k^{1/N} w_{j+1}/z_{\ell+1} ; q)_\infty}{(k^{1/N} w_{j+1}/z_{\ell+1} ; q)_\infty} \prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty}
\]

\[
\times \prod_{1 \leq j \leq \ell} \frac{(k^{1/N} w_{j+1}/z_i ; q)_\infty}{(k^{1/N} w_{j+1}/z_i ; q)_\infty} \prod_{1 \leq j \leq \ell} \frac{(w_{j+1}/z_i ; q)_\infty}{(w_{j+1}/z_i ; q)_\infty}
\]

\[
= \prod_{1 \leq j \leq \ell} \frac{(k^{(j-i+1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}{(k^{(j-i+1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}
\]

\[
\times \prod_{1 \leq j \leq \ell} \frac{(k^{(j-i+1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}{(k^{(j-i+1)/N} q^{\mu_j-i-\lambda_j} w_{j+1}/z_i ; q)_\infty}
\]
where we have used Lemma 2.14. Hence we have (9).

\[ \square \]

### 3. Affine Laumon spaces

#### 3.1 Parabolic sheaves and affine Laumon spaces

We briefly recall the basic facts concerning the affine Laumon spaces studied in [8]. Let $C$ and $X$ be smooth projective curves of genus zero. Fix a coordinate $z$ (resp. $y$) on $C$ (resp. $X$) and consider the action of $C^*$ on $C$ (resp. $X$) such that $v(z) = vz^{-2}$ (resp. $c(y) = c^2y$). We have $C^{*c} = \{0, \infty \}$ and $X^{*c} = \{0, \infty \}$.

Let $S = C \times X$, $D_\infty = C \times \infty \times X \cup \infty \times C \times X$, and $D_0 = C \times 0_X$. Let $W$ be an $N$-dimensional vector space with a basis $w_1, \ldots, w_N$. Let $T$ be the Cartan torus acting on $W$ as follows: for $t = (t_1, \ldots, t_n) \in T$ we have $t(w_i) = t_i w_i$.

Let $d = (d_0, \ldots, d_{N-1}) \in \mathbb{Z}^N_\geq 0$. A parabolic sheaf $\mathcal{F}$ of degree $d$ is an infinite flag of torsion free coherent sheaves of rank $N$ on $S$ satisfying:

- $(a)$ $\mathcal{F}_{k+N} = \mathcal{F}_k(D_0)$ for any $k$,
- $(b)$ $ch_1(\mathcal{F}_k) = k[D_0]$ for any $k$,
- $(c)$ $ch_2(\mathcal{F}_k) = d_i$ for $i \equiv k \pmod{N}$,
- $(d)$ $\mathcal{F}_0$ is locally free at $D_\infty$ and trivialized at $D_\infty : \mathcal{F}_0|_{D_\infty} = W \otimes \mathcal{O}_{D_\infty}$.
- $(e)$ For $-N \leq k \leq 0$ the sheaf $\mathcal{F}_k$ is locally free at $D_\infty$, and the quotient sheaves $\mathcal{F}_k/\mathcal{F}_{-N}, \mathcal{F}_0/\mathcal{F}_k$ (both supported at $D_0$) are both locally free at the point $\infty_X \times 0_X$; moreover the local sections of $\mathcal{F}_k|_{\infty_X \times X}$ are those sections of $\mathcal{F}_0|_{\infty_X \times X} = W \otimes \mathcal{O}_X$ which take value in $\langle w_1, \ldots, w_{N+k} \rangle \subset W$ at $0_X \in X$.

The fine moduli space $\mathcal{P}_d$ of degree $d$ parabolic sheaves exists and is a smooth connected quasiprojective variety of dimension $2d_0 + \cdots + 2d_{N-1}$. The $\mathcal{P}_d$ is called the affine Laumon space.

#### 3.2 Fixed points in $\mathcal{P}_d$

The group $\tilde{T} \times C^* \times C^*$ acts on $\mathcal{P}_d$, with the fixed point set being finite. The fixed points associated with the action of $C^* \times C^*$ on the Hilbert scheme of $(C-\infty_C) \times (X-\infty_X)$ are parametrized by the partitions. Namely for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$, we have the corresponding ideal $J_\lambda = C[z] \cdot (Cy_0^{-1} \oplus Cy_1^{-1} \oplus \cdots)$. Write $\lambda \supseteq \mu$ for indicating $\lambda_i \geq \mu_i (i \geq 1)$, and write $\lambda \supseteq \mu$ for $\lambda_i \geq \mu_{i+1} (i \geq 1)$.

Let $\lambda = (\lambda^k)_{1 \leq k \leq N}$ be a collection of partitions satisfying

\[ \lambda^{11} \supseteq \lambda^{21} \supseteq \cdots \supseteq \lambda^{N1} \supseteq \lambda^{12} \supseteq \cdots \supseteq \lambda^{22} \supseteq \cdots \supseteq \lambda^{NN} \supseteq \lambda^{1N} \supseteq \cdots \supseteq \lambda^{N-1N} \supseteq \lambda^{NN}. \tag{10} \]

Set $d_\lambda(\lambda) = \sum_{i=1}^N |\lambda^k|$, and $d = (d_0(\lambda), \ldots, d_{N-1}(\lambda))$, where $d_0(\lambda) := d_N(\lambda)$. 
For a collection $\lambda$ satisfying (10), let $\mathcal{F}_* = \mathcal{F}_*(\lambda)$ be the parabolic sheaf

$$\mathcal{F}_{k-N} = \bigoplus_{1 \leq i \leq k} J_{k,i}w_i \oplus \bigoplus_{k \leq i \leq N} J_{k,i}(-D_0)w_i.$$ 

The correspondence $\lambda \mapsto \mathcal{F}_*(\lambda)$ is a bijection between the set of collections $\lambda$ satisfying (10) and $\mathcal{F}(\lambda) = d$, and the set of $\tilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$-fixed points in $P_d$.

We have the bijection between the set of collection of partitions satisfying (10) and the set of collection of partitions $(\lambda^{(1)}, \ldots, \lambda^{(N)})$, given by $\lambda^{(i)}_{N(j-1)-N(l(k-i+1))} = \lambda^{(i)}_l$. By an abuse of notation we also write $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$.

### 3.3 Character associated with the series $f^\partial(N, x, p|s, \kappa|q, t)$

For any product of the form $P = \prod_{i \neq j \geq 0}(1 - q^{a+b})u^a$, set $L(P) = \sum_{a,b \geq 0} a^{a+b}u^{-1}$.

**Proposition 3.1** Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ be an $N$-tuple of partitions. Assume the cyclic identification as $\lambda^{(i)} = \lambda^{(i+N)}$ $(i \in \mathbb{Z})$. We have the character $\text{ch}(\lambda)$ of the denominator in the series $f^\partial(N, x, p|s, \kappa|q, t)$, i.e.

$$\text{ch}(\lambda) = \sum_{i} \sum \frac{q^{1 - \lambda^{(i)}_l} - q^{1 - \lambda^{(i)}_l+1}}{1 - q} \frac{1 - q^{1 - \lambda^{(i)}_l+2i-1}}{1 - q}.$$ 

Introduce a collection $(d_{ij}(\lambda)) = \tilde{d}(\lambda)$ by setting $d_{k,i}(\lambda) = \lambda^{(i)}_{k-i+1}$.

**Proposition 3.2** The $\text{ch}(\lambda)$ can be recast as

$$\text{ch}(\lambda) = \sum_{i} \sum_{l' \leq i} \sum_{l \leq i} \frac{k^{l'} - q^{a+b} - d_{i,l}}{1 - q} + \sum_{l' \leq i} \sum_{l \leq i} \frac{k^{l'} - q^{a+b} - d_{i,l}}{1 - q}$$

$$- \sum_{l \leq i} \sum_{l' \leq i} \frac{k^{l'} - q^{a+b} - d_{i,l}}{1 - q} - \sum_{l \leq i} \sum_{l' \leq i} \frac{k^{l'} - q^{a+b} - d_{i,l}}{1 - q}.$$ 

Note that we have $k^{l'} - q^{a+b} - d_{i,l}$.

**Theorem 3.3** With the identification $l^2 = k^{N-i}s_i, q' = k^{N}(q$ being the same for both), the $\text{ch}(\lambda)$ coincides with the torus character in a fixed tangent space to $P_d$. (See Proposition 4.15 and Remark 4.17 in [8].) Hence the Euler characteristic $\sum(s, \kappa|q, t) := [H^*(P_d, \Omega^*_u)]$ of the de Rham complex on $P_d$ is given via
the Atiyah–Bott–Lefschetz localization technique as

$$3_d(s, \kappa | q, t) = \sum_{i,j} (-1)^{i+j} \cdot i^j \cdot H^i(P_d, \Omega^d_{P_d}) \bigg| \sum_{d=2}^{d=N} \prod_{i,j=1}^{\lambda} \frac{\mathbb{N}_{i,j}^{(N)}}{\mathbb{N}_{i,j}^{(N)}(s_j/t_s | q, \kappa)}.$$

where $[H^i(P_d, \Omega^d_{P_d})]$ denotes the character of $H^i(P_d, \Omega^d_{P_d})$ as a representation of $\widetilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$.

**Proposition 3.4** The non-stationary Ruijsenaars function is the generating function for the Euler characteristics of the affine Laumon spaces

$$f_{\lambda}^{\beta}(x, p | s, \kappa | q, 1/t) = \sum_{d} 3_d(s, \kappa | q, t) \prod_{i=1}^{N} (p \lambda_{i+1} / \lambda_i)^{d_i}.$$

A geometric construction is given in [1] for the Macdonald functions based on the Laumon spaces. Theorem 3.3 is an affine analogue of it.

### 3.4 Proofs of Theorem 1.7 and Proposition 1.8

Let $K$ and $\mu = (\mu_1, \ldots, \mu_N)$ be as in Definition 1.6. Set $s = (tk)^{\delta} q^\mu = q^{-K/N + \mu}$ and $\kappa = q^{-K/N - 1}$. Then we want to show that

$$\lim_{t \to q} x^\mu f_{\lambda}^{\beta}(x, p | q^{-K/N + \mu}, q^{-K/N - 1} | q, q/t) = \frac{1}{(p^N / p^K)_\infty} \cdot \chi_{L(\lambda, (K, \mu))}.$$  

**Proof of Theorem 1.7.** First we impose the conditions (4) while $q$ and $t$ still being independent. We need to investigate the vanishing conditions for the numerators of the series $f_{\lambda}^{\beta}(x, p | q^{-K/N + \mu}, q^{-K/N - 1} | q, q/t)$. We have

$$\prod_{i,j=1}^{N} \frac{\mathbb{N}_{i,j}^{(N)}}{\mathbb{N}_{i,j}^{(N)}(s_j/t_s | q, \kappa)}$$

$$= \prod_{i,j=1}^{N} \prod_{\beta \geq \alpha \equiv 1 \mod N} (q^{-N/j} \beta + \mu_j - \mu_i - \gamma_0^{(j)} \beta + \gamma_1^{(j)} \beta + \gamma_2^{(j)} \beta - \beta - 1 ; q)_\beta^{\gamma_0^{(j)} \beta + \gamma_1^{(j)} \beta + \gamma_2^{(j)} \beta} 
\times \prod_{\beta \geq \alpha \equiv N/j + i - 1 \mod N} (q^{-N/j} \beta + \mu_j - \mu_i - \gamma_0^{(j)} \beta + \gamma_1^{(j)} \beta + \gamma_2^{(j)} \beta + \gamma_3^{(j)} \beta - \beta - 1 ; q)_\beta^{\gamma_0^{(j)} \beta + \gamma_1^{(j)} \beta + \gamma_2^{(j)} \beta + \gamma_3^{(j)} \beta}.$$

Note that the terms containing $t^\alpha$ with $\alpha \neq 0$ do not contribute for vanishing. Then we find that the vanishing is caused only from

$$\prod_{\alpha=1}^{\infty} (q^{K/N} | - \mu_1 - \gamma_0^{(1)} \alpha + \gamma_1^{(1)} \alpha + \gamma_2^{(1)} \alpha + \gamma_3^{(1)} \alpha ; q)_\alpha^{\gamma_0^{(1)} \alpha + \gamma_1^{(1)} \alpha + \gamma_2^{(1)} \alpha + \gamma_3^{(1)} \alpha} \times \prod_{j=1}^{N-1} (q^{K/N} | - \mu_{j+1} - \gamma_0^{(j)} \alpha + \gamma_1^{(j)} \alpha + \gamma_2^{(j)} \alpha + \gamma_3^{(j)} \alpha ; q)_\alpha^{\gamma_0^{(j)} \alpha + \gamma_1^{(j)} \alpha + \gamma_2^{(j)} \alpha + \gamma_3^{(j)} \alpha}.$$
Hence the condition for the vanishing is equivalent to the set of inequalities

\[ \lambda_a^{(N)} - \lambda_a^{(1)} \leq K + \mu_N - \mu_1 \quad (\alpha \geq 1), \]
\[ \lambda_a^{(j)} - \lambda_a^{(j+1)} \leq \mu_j - \mu_{j+1} \quad (1 \leq j < N, \alpha \geq 1). \]

This is equivalent to the conditions for the affine Gelfand–Tsetlin patterns \( D(\mu) \)

\[ \tilde{d} \in D(\mu) \iff d_{ij} - \tilde{\mu}_j \leq d_{i+j+1} - \tilde{\mu}_{j+1} \quad (j \leq i, l \geq 0), \]

where \( \tilde{\mu} = (\tilde{\mu}_i)_{i \in \mathbb{Z}} \) is the non-increasing sequence \( \tilde{\mu}_i = \mu_i (\text{mod } N) + \lfloor \frac{i}{N} \rfloor K \). It is proved in [8] that there is a weight preserving bijection between the set of affine Gelfand–Tsetlin patterns and the basis vectors of \( L(\Lambda(K, \mu)) \otimes \mathcal{F} \) (\( \mathcal{F} \) stands for the space spanned by partitions). Note that their proof is based on the \( \hat{\mathfrak{gl}}_N \)-crystal of Tingley constructed on the set of cylindrical plane partitions [17].

Next we take the limit \( t \to q \). Note that all the ratios of the Nekrasov factors in the limit \( t \to q \) become one, whenever \( \tilde{d} \in D(\mu) \). Hence we have

\[
\lim_{t \to q} x^\mu f^{\hat{\Omega}_N}(x, p|q^{-K/\mu}, q^{-K/N} t^{-1}|q, q/t) = x^\mu \sum_{\tilde{d} \in D(\mu)} \prod_{i=1}^N (px_{i+1}/x_i)^{d_i} = \frac{1}{(p^N; p^N)_\infty} \cdot \text{ch}_{L(\Lambda(K, \mu))}^{\hat{\mathfrak{gl}}_N}. \quad \square
\]

Next, let \( K = 0, \mu = \emptyset \), indicating that have \( s_i = 1 \) \( (1 \leq i \leq N) \) and \( \kappa = t^{-1} \). In this case, we show that

\[
f^{\hat{\Omega}_N}(x, p|1, \ldots, 1, t^{-1}|q, q/t) = \frac{1}{(p^N; p^N)_\infty}.
\]

**Proof of Proposition 1.8.** When \( K = 0, \mu = \emptyset \), we have the restriction \( \lambda_a^{(0)} = \lambda_a^{(1)} \) \( (1 \leq i, j \leq N) \). Write \( \lambda = \lambda^{(0)} \) for short. Note that we have \( \prod_{i=1}^N N_{\lambda, \lambda}^{(i)}(u|q, 1/t) = N_{\lambda, \lambda}(u|q, 1/t) \) and \( N_{\lambda, \lambda}(1|q, 1/t) = (t/q)^{|\lambda|} N_{\lambda, \lambda}(q/t|q, 1/t) \). Hence \( f^{\hat{\Omega}_N}(x, p|1, \ldots, 1, t^{-1}|q, q/t) \) is written as

\[
\sum_{\lambda \in \mathcal{P}} \left( \frac{N_{\lambda, \lambda}(q/t|q, 1/t)}{N_{\lambda, \lambda}(1|q, 1/t)} \right)^N = \sum_{\lambda \in \mathcal{P}} p^{N_{\lambda}} = \frac{1}{(p^N; p^N)_\infty}. \quad \square
\]

4. Macdonald functions: the limit \( p \to 0 \)

4.1 Macdonald functions

We recall some facts about the Macdonald functions [1–3].
**Definition 4.1** Let $D^B_N(x) = D^B_N(q,t)$ be the Macdonald operator [4] of type $gl_N$

$$D^B_N = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q,x_i},$$

where $T_{q,x_i}$ denotes the $q$-shift operator $T_{q,x_i}f(x_1, \ldots, x_i, \ldots, x_N) = f(x_1, \ldots, q x_i, \ldots, x_N)$.

Let $M^{(N)}$ be the set of strictly upper triangular matrices with non-negative integer entries: $M^{(N)} = \{ \theta = (\theta_{ij})_{1 \leq i,j \leq N} | \theta_{ij} \in \mathbb{Z}_{\geq 0}, \theta_{ij} = 0 \text{ if } i \geq j \}$. Define recursively $c_N(\theta; s, q, t) \in \mathbb{Q}(q,t,s_1, \cdots, s_N)$ by $c_1(-; s_1; q, t) = 1$, and

$$c_N(\theta \in M^{(N)}; s_1, \cdots, s_N; q, t) = c_{N-1}(\theta' \in M^{(N-1)}; q^{-\theta_1} s_1, \cdots, q^{-\theta_N} s_{N-1}; q, t) \times \prod_{1 \leq i < j \leq N} \frac{(ts_{i+1}/s_i; q)_{\theta_{i,j}} (q^{-\theta_i} s_i/s_j; q)_{\theta_{i,j}}}{(qs_{j+1}/s_i; q)_{\theta_{i,j}} (q^{-\theta_j} s_j/s_i; q)_{\theta_{i,j}}}.$$ We have

$$c_N(\theta; s_1, \cdots, s_N; q, t) = \prod_{k=2}^{N} \prod_{1 \leq i < j \leq k-1} \frac{(q^{-\sum_{a=k+1}^{N} (\theta_{i,a}-\theta_{j,a})} ts_{j+1}/s_i; q)_{\theta_{i,k}} (q^{-\sum_{a=k+1}^{N} (\theta_{i,a}-\theta_{j,a})} qs_{j+1}/s_i; q)_{\theta_{i,k}}}{(q^{-\sum_{a=k+1}^{N} (\theta_{i,a}-\theta_{j,a})} s_j/s_i; q)_{\theta_{i,k}}}.$$**

**Definition 4.2** Define $f^{B_N}(x|s|q,t) \in \mathbb{Q}(s,q,t)[[x_2/x_1, \ldots, x_N/x_{N-1}]]$ by

$$f^{B_N}(x|s|q,t) = \sum_{\theta \in M^{(N)}} c_N(\theta; s_1, q, t) \prod_{1 \leq i < j \leq N} (x_j/x_i)^{\theta_{ij}}.$$**

**Proposition 4.3** ([1, 2]) Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$, and set $s = t^{\lambda} = (s_i = t^{\lambda_i})$. Then we have

$$D^B_N(x^{\lambda} f^{B_N}(x|s|q,t) = \sum_{i=1}^{N} \lambda_i x^{\lambda} f^{B_N}(x|s|q,t).$$**

**Lemma 4.4** We have

$$\lim_{\epsilon \to 0} f^{B_N}(x|\epsilon^{-s}|q,t) = \prod_{1 \leq i < j \leq N} (qs_j/x_i; q)_{\infty} (qs_j/tx_i; q)_{\infty}.$$**

**Proof.** In the limit $\epsilon \to 0$, we have $\epsilon^{-s_i}/s_i \to 0$ for $1 \leq i < j \leq N$. Hence we have

$$\text{LHS} = \sum_{\theta \in M^{(N)}} \prod_{1 \leq i < j \leq N} \frac{(t; q)_{\theta_{ij}} (qs_j/tx_i; q)_{\theta_{ij}}}{(q; q)_{\theta_{ij}}} = \text{RHS}. \quad \square$$
Remark 4.5. We remark that Proposition 1.3 is obtained in the same way as above.

Definition 4.6. Let \( \phi^{N}(x|s|q,t) \in \mathbb{Q}(q,t)[[x_1/x_2, \ldots, x_N/x_{N-1}, s_2/s_1, \ldots, s_N/s_{N-1}]] \) be

\[
\phi^{N}(x|s|q,t) = \prod_{1 \leq i < j \leq N} \frac{(x_j/x_i; q)_{\infty}}{(q_j/x_i; q)_{\infty}} f^{N}(x|s|q,t),
\]

where \( c_N(\theta; s|q,t) \)'s are expanded in \( \mathbb{Q}(q,t)[[s_2/s_1, \ldots, s_N/s_{N-1}]] \).

Proposition 4.7 ([2]). We have

\[
\phi^{N}(x|s|q,t) = \phi^{N}(x|x|q,t) \quad \text{(bipspectral duality)},
\]

\[
\phi^{N}(x|s|q,t) = \phi^{N}(x|s|q,q/t) \quad \text{(Poincaré duality)}.
\]

4.2 Macdonald limit of \( f^{N}(x, p|s, \kappa|q,t); p \to 0 \)

Proposition 4.8. We have\( \lim_{p \to 0} f^{N}(p^{\delta}x, p|\kappa^{\delta}s, \kappa|q,t) = f^{N}(x|s|q,t) \).

Remark 4.9. Note that we have defined \( f^{N}(x, p|s, \kappa|q,t) \) in such a way that we obtain \( f^{N}(x|s|q,t) \) in the limit \( p \to 0 \), instead of \( f^{N}(x|s|q,t) \). Some explanations about this inconvenient definition is in order. (1) We started our construction based on the affine screening operators with the Heisenberg algebra given in Definition 2.1. This notation seems rather standard in the context of the deformed \( W \)-algebras, and it seems safe and reasonable to stick to this convention. One may also expect that there are some representation theoretical interpretations for having the exchange of the parameters \( t \leftrightarrow q/t \). (2) We regard \( f^{N}(x, p|s, \kappa|q,t) \) as the generating function for the Euler characteristics of the affine Laumon space (see Proposition 3.4), where the parameter \( 1/t \) counts the degrees of the differential forms. Hence in this geometric context, the use of the parameter \( t \) (not \( q/t \)) seems more natural.

Proof. While taking the limit \( p \to 0 \) of \( f^{N}(p^{\delta}x, p|\kappa^{\delta}s, \kappa|q,t) \), the partitions producing non-vanishing contribution to the summation satisfy \( \ell(\lambda^{(i)}) \leq N-i \) \((1 \leq i \leq N)\). Hence we can parametrize them by using the set \( M^{(N)} \) as \( \lambda^{(i)} = \sum_{k=j+i}^{N} \theta_{i,k} \). Namely \( \theta_{i,j} = \lambda^{(i)} - \lambda^{(j)} \). Assuming the restriction condition \( \ell(\lambda^{(i)}) \leq N-i \), we have for \( 1 \leq i \leq j \leq N \)

\[
N^{(j-i)(N)}_{\lambda^{(i)}, \lambda^{(j)}}(u|q, \kappa) = \prod_{1 \leq f \leq N-i} \frac{(uq^{-\lambda^{(i)}+\lambda^{(j)}+i+1-\lambda^{(j)}-\lambda^{(i)}}; q)_{\delta}}{(uq^{-\lambda^{(i)}+\lambda^{(j)}-i+1-\lambda^{(j)}-\lambda^{(i)}}; q)_{\delta}},
\]

and for \( 1 \leq j < i \leq N \)

\[
N^{(j-i)(N)}_{\lambda^{(i)}, \lambda^{(j)}}(u|q, \kappa) = \prod_{1 \leq f \leq N-j} \frac{(uq^{-\lambda^{(i)}+\lambda^{(j)}-i+1-\lambda^{(j)}-\lambda^{(i)}}; q)_{\delta}}{(uq^{-\lambda^{(i)}+\lambda^{(j)}+i+1-\lambda^{(j)}-\lambda^{(i)}}; q)_{\delta}}.
\]
\[ = \prod_{k=1}^{N} (uq^{\sum_{a=k+1}^{N} \theta_a - \sum_{a=k}^{N} \theta_a} x^{l-i}; q)_{\theta,k} \cdot \]

Note that we have the rules for changing the order of products \( \prod_{1 \leq i \leq N} \prod_{k=i+1}^{N} (x_{a+\beta}/tx_{a+\beta-1})^{(\theta)} = c_N(\theta; s, q, q/t) \cdot \prod_{1 \leq i < j \leq N} (x_i/x_j)^{\theta_{ij}}. \)

### 5. Ruijsenaars operator and its particular limits

In this section, starting from the Ruijsenaars operator \( D_s(p) \), we take several particular limits, deriving the Macdonald operator, the elliptic Calogero–Sutherland operator and the \( q \)-difference affine Toda operator.

#### 5.1 Ruijsenaars operator

Let \( D_s(p) = D_s(p|q, t) \) be the Ruijsenaars operator in (6).

**Definition 5.1** Changing the base as \( p \to p^{N} \), and making the shift in \( x \) as \( x \to p^{-\delta} x \), set \( D_s(p) = D_s(p|q, t) \) by

\[
D_s(p) = D_{p^{-\delta}}(p^N) = \sum_{i=1}^{N} t^{N-i} \prod_{j=1}^{i-1} \frac{\Theta_{p^N}(p^{i-j}tx_i/x_j)}{\Theta_{p^N}(p^{i-j}x_i/x_j)} \cdot \prod_{k=i+1}^{N} \frac{\Theta_{p^N}(p^{k-i}x_k/x_i)}{\Theta_{p^N}(p^{k-i}x_i/x_k)} \cdot T_{q,x_i}.
\]

Note that in terms of the modified Ruijsenaars operator \( \tilde{D}_s(p) \), the main conjecture (Conjecture 1.14) is recast as

\[
\tilde{D}_s(p) x^{\delta} f^{st,\tilde{g}N}(x, p|s, q, q/t) = \epsilon(p^N|s, q, q/t) x^{\delta} f^{st,\tilde{g}N}(x, p|s, q, q/t),
\]

where \( s = t^\delta q^\lambda (s_i = t^{N-i}q^\lambda) \).

**Definition 5.2** For simplifying our calculation in the limit \( q \to 1 \), set \( \tilde{D}_s(p) = \tilde{D}_s(p|q, t) \) by

\[
\tilde{D}_s(p) = x^{\delta} D_s(p|q, t)x^{-\delta} = \sum_{i=1}^{N} \prod_{j=1}^{i-1} \frac{\Theta_{p^N}(p^{i-j}tx_i/x_j)}{\Theta_{p^N}(p^{i-j}x_i/x_j)} \cdot \prod_{k=i+1}^{N} \frac{\Theta_{p^N}(p^{k-i}x_k/x_i)}{\Theta_{p^N}(p^{k-i}x_i/x_k)} \cdot T_{q,x_i}.
\]

#### 5.2 Macdonald operator: \( p \to 0 \)

We have the Macdonald operator \( D_s^{glN}(q, t) \) in the limit \( p \to 0 \)

\[
D_s^{glN}(q, t) = \lim_{p \to 0} D_s(p) = \sum_{i=1}^{N} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}.
\]
5.3 Elliptic Calogero–Sutherland operator: limit $q \to 1$

We set $q = e^{ih}, t = e^{\beta h}$ and consider the limit $h \to 0$ of the Ruijsenaars operator $\tilde{D}(p)$, while fixing $\beta$.

As for the details, we refer the readers to the work of Langmann [23] where the kernel function identity for the non-stationary Calogero–Sutherland model was introduced.

5.3.1 Derivatives of theta function and elliptic potential $V(z|p)$

For studying the $h$ expansion of $\tilde{D}(p)$, we collect some simple facts concerning the derivatives of the theta function $\Theta_p(z)$.

**Definition 5.3** Set $\Theta_p^{(1)}(z) = z \frac{\partial}{\partial z} \Theta_p(z)$, and $\Theta_p^{(2)}(z) = \left( z \frac{\partial}{\partial z} \right)^2 \Theta_p(z)$.

**Lemma 5.4** We have

\[ \Theta_p(1/z) = \Theta_p(pz), \quad \Theta_p^{(1)}(1/z) = -\Theta_p^{(1)}(pz), \quad \Theta_p^{(2)}(1/z) = \Theta_p^{(2)}(pz), \]

\[ \Theta_p(pz) = -z^{-1} \Theta_p(z), \quad \Theta_p^{(1)}(pz) = z^{-1} \Theta_p(z) - z^{-1} \Theta_p^{(1)}(z), \]

\[ \Theta_p^{(2)}(pz) = -z^{-1} \Theta_p(z) + 2z^{-1} \Theta_p^{(1)}(z) - z^{-1} \Theta_p^{(2)}(z), \quad \Theta_p^{(1)}(1) = \Theta_p^{(2)}(1) = -(p; p)_\infty^3, \]

\[ -p \frac{\partial}{\partial p} \Theta_p(z) + \left( z \frac{\partial}{\partial z} \right)^2 \Theta_p^2(z) = 0. \]

**Definition 5.5** Set $V(z|p) = \frac{\Theta_p^{(2)}(z)}{\Theta_p(z)} - \left( \frac{\Theta_p^{(1)}(z)}{\Theta_p(z)} \right)^2$.

**Proposition 5.6** The potential $V(z|p)$ is an elliptic function satisfying the conditions $V(pz|p) = V(z|p)$,

\[ V(z|p) = -\frac{1}{(1 - z)^2} + \frac{1}{1 - z} + V_0(p) + O(1 - z), \]

\[ V_0(p) = 2 \frac{1}{(p; p)_\infty} p \frac{\partial (p; p)_\infty}{\partial p} = -2p - 6p^2 - 8p^3 - 14p^4 - 12p^5 - 24p^6 - 16p^7 - \cdots. \]

**Remark 5.7** Note that $V(z|p)$ is essentially the Weierstrass' $\wp(u)$

\[ V(e^{iu}|p) = \frac{1}{u^2} + \frac{1}{12} + V_0(p) + O(u) = \wp(u) + \frac{1}{12} + V_0(p). \]

**Lemma 5.8** We have

\[ V_0(p^N) = \frac{2}{3N} \frac{1}{\Theta_{p^N}^{(1)}(1)} p \frac{\partial}{\partial p} \Theta_{p^N}^{(1)}(1). \]

5.3.2 Elliptic Sutherland Hamiltonian $H_\beta(p)$

**Definition 5.9** Let $\vartheta_i = x_i \partial / \partial x_i$. Set $H_\beta(p)$ as

\[ H_\beta(p) = \frac{1}{2} \sum_{i=1}^N \vartheta_i^2 - \beta \sum_{1 \leq i < j \leq N} \frac{\Theta_{p^N}^{(1)}(p^{j-i}x_j/x_i)}{\Theta_{p^N}(p^{j-i}x_j/x_i)} (\vartheta_i - \vartheta_j) + \beta^2 \sum_{1 \leq i < j \leq N} \frac{\Theta_{p^N}^{(2)}(p^{j-i}x_j/x_i)}{\Theta_{p^N}(p^{j-i}x_j/x_i)}. \]
\[ + \beta^2 \sum_{1 \leq i < j \leq N} \left( \frac{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)} \frac{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)} - \frac{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)} \frac{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)} \right) \]

**Proposition 5.10** Set \( q = e^b, t = e^{\beta h} \). We have

\[
\bar{D}_i(p) = N + h \sum_{i=1}^{N} \partial_i + h^2 H_{\bar{\beta}}(p) + O(h^3).
\]

**Lemma 5.11** We have the identity

\[
\frac{\Theta^{(1)}_{pN}(x_j/x_i)}{\Theta^{(1)}_{pN}(x_j/x_i)} \frac{\Theta^{(1)}_{pN}(x_k/x_i)}{\Theta^{(1)}_{pN}(x_k/x_i)} - \frac{\Theta^{(1)}_{pN}(x_j/x_i)}{\Theta^{(1)}_{pN}(x_j/x_i)} \frac{\Theta^{(1)}_{pN}(x_k/x_i)}{\Theta^{(1)}_{pN}(x_k/x_i)} + \frac{\Theta^{(1)}_{pN}(x_k/x_i)}{\Theta^{(1)}_{pN}(x_k/x_i)} \frac{\Theta^{(1)}_{pN}(x_k/x_i)}{\Theta^{(1)}_{pN}(x_k/x_i)} = \frac{1}{2} \left( \frac{\Theta^{(2)}_{pN}(x_j/x_i)}{\Theta^{(2)}_{pN}(x_j/x_i)} + \frac{\Theta^{(2)}_{pN}(x_k/x_i)}{\Theta^{(2)}_{pN}(x_k/x_i)} \right) - \frac{1}{N} \frac{1}{\Theta^{(1)}_{pN}(1)} \frac{\partial}{\partial p} \Theta^{(1)}_{pN}(1).
\]

**Lemma 5.12** Using the identity in Lemma 5.11 above, we can recast the three body interaction terms in \( H_{\bar{\beta}} \) as

\[
\sum_{1 \leq i < j < k \leq N} \left( \frac{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)} \frac{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)} - \frac{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)} \frac{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)} \right)
+ \frac{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)} \frac{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}{\Theta^{(1)}_{pN}(p^{j-k}x_k/x_i)}
= \sum_{1 \leq i < j \leq N} \left( \frac{N - 2}{2} \frac{\Theta^{(2)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(2)}_{pN}(p^{i-j}x_j/x_i)} - \frac{N - 2}{2} \frac{\Theta^{(2)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(2)}_{pN}(p^{i-j}x_j/x_i)} \right) - \frac{(N - 1)(N - 2)}{6} \frac{1}{\Theta^{(1)}_{pN}(1)} \frac{\partial}{\partial p} \Theta^{(1)}_{pN}(1).
\]

**Proposition 5.13** We have \( H_{\bar{\beta}}(p) \) written only with two body interaction terms as

\[
H_{\bar{\beta}}(p) = \frac{1}{2} \sum_{i=1}^{N} \partial_i^2 - \beta \sum_{1 \leq i < j \leq N} \frac{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)}{\Theta^{(1)}_{pN}(p^{i-j}x_j/x_i)} \frac{\Theta^{(1)}_{pN}(p^{j-i}x_i/x_j)}{\Theta^{(1)}_{pN}(p^{j-i}x_i/x_j)} (\partial_i - \partial_j).
\]
5.3.3 Quasi-ground state \( \psi_0(x, p) \) and elliptic Calogero–Sutherland Hamiltonian \( H_{CS}(p) \)

**Definition 5.14** Set \( q = e^{\hbar}, t = e^{\beta \hbar} \). In view of Definition 1.4, we introduce the quasi-ground state \( \psi_0(x, p|\beta) \) as follows.

\[
\psi_0(x, p|\beta) = \lim_{h \to 0} \prod_{1 \leq i < j \leq N} \frac{(p^{-i}q x_j/x_i; q, p_N)^\infty}{(p^{-i}q x_j/x_i; q, p_N^\infty)} \prod_{1 \leq i < j \leq N} \frac{(p^{N-i+j}q x_j/x_i; q, p_N^\infty)}{(p^{-i}q x_j/x_i; q, p_N^\infty)} \equiv \left( (p_N^-; p_N^\infty)^{N-N(N-1)/2} \prod_{1 \leq i < j \leq N} \Theta_{p_N}(p^{-i}x_j/x_i) \right)^\beta.
\]

**Definition 5.15** Let \( H_{CS}(p) = H_{CS}(p|\beta) \) be the elliptic Calogero–Sutherland operator

\[
H_{CS}(p) = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial p_i^2} + \beta(\beta - 1) \sum_{1 \leq i < j \leq N} V(p^{-i}x_j/x_i|p_N^\infty) + \frac{\beta(\beta - 1)N}{2} V_0(p_N^\infty).
\]

**Proposition 5.16** We have

\[
\psi_0(x, p) \left( H_{\beta}(p) + \frac{\beta(\beta - 1)N}{2} V_0(p_N^\infty) \right) \psi_0^{-1}(x, p) = H_{CS}(p).
\]

5.4 Affine \( q \)-Toda operator: \( t \to 0 \)

**Definition 5.17** Define the \( \hat{g}_N \) affine \( q \)-Toda operator \( \hat{D}_{x}^{q_N \text{Toda}} = \hat{D}_{x}^{q_N \text{Toda}}(q, \tilde{p}) \) by

\[
\hat{D}_{x}^{q_N \text{Toda}} = (1 - \tilde{p}x_1/x_1)T_{q,x_1} + (1 - \tilde{p}x_3/x_2)T_{q,x_2} + \cdots
\]

\[
\cdots + (1 - \tilde{p}x_{N-1}/x_{N-1})T_{q,x_{N-1}} + (1 - \tilde{p}x_1/x_N)T_{q,x_N}.
\]

**Proposition 5.18** Putting \( p = \tilde{p}t \) in \( \hat{D}_{x}(p) \), we have the affine \( q \)-difference Toda operator \( D_{x}^{q_N \text{Toda}}(\tilde{p}) \) in the limit \( t \to 0 \) with \( \tilde{p} \) being fixed

\[
\lim_{t \to 0} \hat{D}_{x}(\tilde{p}t) = D_{x}^{q_N \text{Toda}}(q, \tilde{p}).
\]
6. Non-stationary and stationary \( \widehat{gl}_N \) \( q \)-difference affine Toda equation: the limit \( p, t \to 0 \)

6.1 \( \widehat{gl}_N \) non-stationary affine \( q \)-Toda operator \( T_{\widehat{\phi}^N}(\kappa) \)

We study the limit \( p, t \to 0 \) while the ratio \( \tilde{p} = p/t \) being fixed. Let \( D_{\kappa}^{\hat{\phi}^N_{\text{Toda}}} \) be as in Definition 5.17.

**Definition 6.1** Denote the Euler operators by \( \partial_i = x_i / \partial x_i \) (\( 1 \leq i \leq N \)). Let \( \Delta \) be the Laplacian \( \Delta = \frac{1}{2} \sum_{i=1}^{N} \partial_i^2 \). Introduce the operator \( T_{\widehat{\phi}^N}(\kappa) = T_{\widehat{\phi}^N}(q, \tilde{p}, \kappa) \) defined by

\[
T_{\widehat{\phi}^N}(\kappa) = \prod_{i=1}^{N} \frac{1}{(\tilde{p}q_{x_i+1}/x_i; q)_{\infty}} \cdot q^\Delta T_{\kappa, \tilde{p}},
\]

where \( T_{\kappa, \tilde{p}}(\tilde{p}) = f(\kappa \tilde{p}) \) or equivalently \( T_{\kappa, \tilde{p}} = \kappa^{\Delta}/\tilde{p} \). We call \( T_{\widehat{\phi}^N}(\kappa) \) the non-stationary affine \( q \)-Toda operator.

**Proposition 6.2** If \( \kappa = 1 \), then we have the commutativity \([T_{\widehat{\phi}^N}(1), D_{\kappa}^{\hat{\phi}^N_{\text{Toda}}}] = 0 \). On the other hand, we have \([T_{\widehat{\phi}^N}(\kappa), D_{\kappa}^{\hat{\phi}^N_{\text{Toda}}} \neq 0 \) when \( \kappa \neq 1 \).

**Remark 6.3** It seems a difficult problem to find such a \( \kappa \)-deformation of \( D_{\kappa}^{\hat{\phi}^N_{\text{Toda}}} \) that which commutes with \( T_{\widehat{\phi}^N}(\kappa) \).

We call \( D_{\kappa}^{\hat{\phi}^N_{\text{Toda}}} \psi(x, \tilde{p})|s; q) = \varepsilon(\tilde{p})|s; q) \psi(x, \tilde{p})|s; q) \) the stationary problem for the affine \( q \)-Toda system. Note that the eigenvalue \( \varepsilon(\tilde{p})|s; q) \) depends on the variable \( \tilde{p} \). We call the eigenvalue equation \( T_{\widehat{\phi}^N}(\kappa) \psi(x, \tilde{p})|s; \kappa; q) = \varepsilon(s)|q) \psi(x, \tilde{p})|s; \kappa; q) \), the non-stationary problem for the affine \( q \)-Toda system. One may regard this problem as a \( q \)-difference analogue of the Heat equation (or a time dependent Schrödinger equation) in an affine Toda potential. Note that the eigenvalue \( \varepsilon(s)|q) \) in the non-stationary case does not depend in \( \tilde{p} \).

6.2 Conjecture concerning non-stationary \( \widehat{gl}_N \) \( q \)-difference affine Toda equation

**Definition 6.4** Set

\[
f_{\hat{\phi}^N_{\text{Toda}}}(x, \tilde{p}, \kappa|q) = \lim_{t \to 0} f_{\hat{\phi}^N}(x, t\tilde{p}, \kappa|q, q/t).
\]

**Conjecture 6.5** Let \( s_i = q^\lambda_i \). We have

\[
T_{\widehat{\phi}^N}(\kappa) x^\lambda f_{\hat{\phi}^N_{\text{Toda}}}(x, \tilde{p}, \kappa|q) = q^\frac{1}{2} \sum_{j=1}^{N} x_j^\lambda x^\lambda f_{\hat{\phi}^N_{\text{Toda}}}(x, \tilde{p}, \kappa|q).
\] (11)

**Proposition 6.6** Let \( s_i = q^{\lambda_i} \). The Poincaré duality conjecture (3) in Conjecture 1.5 implies the non-stationary eigenvalue equation (11).

Recall that we have set \( m_i = m_i(\lambda) = \sum_{\beta=1}^{N} \sum_{\alpha \geq 1, \alpha + \beta = \lambda (\text{mod} N)} \lambda \beta) - \lambda (\beta+1) \). Hence we have

\[
\prod_{\beta=1}^{N} \prod_{\alpha \geq 1} (x_{\alpha + \beta}/x_{\alpha + \beta - 1})^{\lambda \beta) = \prod_{i=1}^{N} x_i^{m_i}.
\]
Proposition 6.7. We have
\[ f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q) = \sum_{(i_1, \ldots, i_N) \in \mathcal{P}} \prod_{i=1}^{N} s_{i}^{-m_{i} q^{-m_{i}^2 / 2} \kappa^{-\lambda(0)}} \cdot \prod_{i,j=1}^{N} \frac{1}{N_{\lambda(0), \lambda(j)}(s_{j}/s_{i}|q, \kappa)} \cdot \prod_{\beta=1}^{N-\beta} \prod_{a=1}^{\beta} (\tilde{\phi}_{\lambda_{a}+\beta}/\lambda_{a}+\beta-1)^{\lambda(0)}. \] (12)

6.3 Proof of Propositions 6.6 and 6.7

The Poincaré duality conjecture (3) in Conjecture 1.5 is recast as
\[ \prod_{1 \leq i < j \leq N} \frac{(p^{i-j} x_i/x_j; q, p^N)_{\infty}}{(p^{i-j} x_i/x_j; q, p^N)_{\infty}} \cdot \prod_{1 \leq i < j \leq N} \frac{(p^{N-i+j} x_i/x_j; q, p^N)_{\infty}}{(p^{N-i+j} x_i/x_j; q, p^N)_{\infty}} f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q, t) = f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q). \]

Setting \( p = \tilde{p} t \) and taking the limit \( t \to 0 \), we have
\[ \lim_{t \to 0} \prod_{i=1}^{N} \frac{1}{(\tilde{p} q x_{i+1}/x_{i}; q)_{\infty}} f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q, t) = f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q). \] (13)

Lemma 6.8. We have
\[ \lim_{t \to 0} \prod_{i,j=1}^{N} \frac{N_{\lambda(0), \lambda(j)}(t s_{j}/s_{i}|q, \kappa)}{N_{\lambda(0), \lambda(j)}(s_{j}/s_{i}|q, \kappa)} = \prod_{i,j=1}^{N} \frac{1}{N_{\lambda(0), \lambda(j)}(s_{j}/s_{i}|q, \kappa)}, \]
\[ \lim_{t \to 0} \frac{N_{\lambda(0), \lambda(j)}(q s_{j}/t s_{i}|q, \kappa)}{N_{\lambda(0), \lambda(j)}(s_{j}/s_{i}|q, \kappa)} = \prod_{i=1}^{N} s_{i}^{-m_{i} q^{-m_{i}^2 / 2} \kappa^{-\lambda(0)}} \cdot \prod_{i,j=1}^{N} \frac{1}{N_{\lambda(0), \lambda(j)}(s_{j}/s_{i}|q, \kappa)}. \]

Lemma 6.9. We have \( \Delta \chi^{\lambda} \prod_{i=1}^{N} x_{i}^{m_{i}} = \sum_{i=1}^{N} \left( \frac{1}{2} \lambda x_{i} + \lambda m_{i} + \frac{m_{i}^2}{2} \right) \chi^{\lambda} \prod_{i=1}^{N} x_{i}^{m_{i}} \), namely \( q^{\Delta} \chi^{\lambda} \prod_{i=1}^{N} x_{i}^{m_{i}} = q^{\frac{1}{2} \sum_{i} \lambda_{i}^2} \prod_{i=1}^{N} s_{i}^{-m_{i} q^{-m_{i}^2 / 2} \kappa^{-\lambda(0)}} \cdot \chi^{\lambda} \prod_{i=1}^{N} x_{i}^{m_{i}}. \)

Lemma 6.10. From Lemmas 6.8 and 6.9, we have
\[ q^{\frac{1}{2} \sum_{i} \lambda_{i}^2} x^{\lambda} \lim_{t \to 0} f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q, t) = q^{\Delta} T_{x_{i}^{\lambda}} x^{\lambda} \lim_{t \to 0} f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q, t/q) = q^{\Delta} T_{x_{i}^{\lambda}} x^{\lambda} f_{\tilde{\phi} N}^{\text{Toda}}(x, \tilde{p}|s, \kappa|q). \]
6.4 Limit from non-stationary affine $q$-Toda to stationary affine $q$-Toda: the limit $\kappa \to 1$.

Even though the operator $T^{\hat{g}_N}(\kappa)$ itself has no singularity at $\kappa = 1$, and the commutativity $[T^{\hat{g}_N}(1), D^{g_N}_{\text{affine} q-\text{Toda}}] = 0$ takes place, the behaviour of $f^{\hat{g}_N}_{\text{affine} q-\text{Toda}}$ in the vicinity of $\kappa = 1$ is quite subtle. This is because the operator $T^{\hat{g}_N}(1)$ commutes with multiplication by any function in $\hat{\rho}$, producing drastic degenerations in the spectrum, resulting Jordan blocks of infinite size lacking any eigenspaces. However, we still find some nice structure.

**Definition 6.11** Let $\alpha(\hat{\rho}) = \alpha(\hat{\rho}|s, \kappa|q) = \sum_{d \geq 0} \hat{\rho}^d \alpha_d(s, \kappa|q)$ be the constant term of $f^{\hat{g}_N}_{\text{affine} q-\text{Toda}}(x, \hat{\rho}|s, \kappa|q)$ with respect to $x_i$'s. Namely,

$$\alpha(\hat{\rho}) = \sum_{\lambda(1) \cdots \lambda(N)} (\hat{\rho}/\kappa)^{|\lambda|} \prod_{i,j=1}^N N_{\lambda(i) \lambda(j)}^{\hat{g}_N}(s_j/s_i|q, \kappa).$$

**Conjecture 6.12** We have the properties:

1. The series $f^{\hat{g}_N}_{\text{affine} q-\text{Toda}}$ is convergent on a certain domain. With respect to $\kappa$, it is regular on a certain punctured disk $\{ \kappa \in \mathbb{C} | |\kappa - 1| < r, \kappa \neq 1 \}$.
2. The $f^{\hat{g}_N}_{\text{affine} q-\text{Toda}}$ and $\alpha(\hat{\rho})$ are essential singular at $\kappa = 1$. The $\alpha_d(s, \kappa|q)$ has a pole of degree $d$ in $\kappa$ at $\kappa = 1$.
3. The ratio $f^{\hat{g}_N}_{\text{affine} q-\text{Toda}}/\alpha(\hat{\rho})$ is regular at $\kappa = 1$.
4. The ratio $\alpha(\hat{\rho})/\alpha(\kappa \hat{\rho})$ is regular at $\kappa = 1$. The limit $\lim_{\kappa \to 1} \alpha(\hat{\rho})/\alpha(\kappa \hat{\rho})$ is a nontrivial function, which is Taylor expanded in $\hat{\rho}^N$.

Conjecture 6.12 suggests the following scheme for analysing the eigenvalue problem of stationary $q$-affine Toda equation.

**Definition 6.13** Assuming Conjecture 6.12, set

$$f^{\text{affine} q-\text{Toda}}(x, \hat{\rho}|s|q) = \frac{f^{\hat{g}_N}_{\text{affine} q-\text{Toda}}(x, \hat{\rho}|s, \kappa|q)}{\alpha(\hat{\rho}|s, \kappa|q)} \bigg|_{\kappa = 1}, \quad \epsilon(\hat{\rho}|s|q) = q^{\frac{1}{2} \sum_{i=1}^n x_i^2} \frac{\alpha(\hat{\rho}|s, \kappa|q)}{\alpha(\kappa \hat{\rho}|s, \kappa|q)} \bigg|_{\kappa = 1}.$$
Because of the commutativity \([T \tilde{g}^N, D_{\tilde{g}^N}^{\text{affToda}}] = 0\), the \(x^i f_{\tilde{g}^N}^{\text{affToda}}(x, \tilde{p}|s|q)\) should also be the eigenfunction of the stationary affine Toda operator \(D_{\tilde{g}^N}^{\text{affToda}}\).

**Conjecture 6.15** We have
\[
D_{\tilde{g}^N}^{\text{affToda}} x^i f_{\tilde{g}^N}^{\text{affToda}}(x, \tilde{p}|s|q) = \epsilon D(\tilde{g}|s|q) x^i f_{\tilde{g}^N}^{\text{affToda}}(x, \tilde{p}|s|q),
\]
\[
\epsilon D(\tilde{g}|s|q) = \sum_{i=1}^n s_i + \sum_{k=1}^\infty \epsilon_k^D(s|q)\tilde{p}^N_k.
\]

7. Non-stationary and stationary elliptic Calogero–Sutherland equation: limit \(q, t \to 1\)

7.1 Non-stationary elliptic Calogero–Sutherland equation

Let \(H^{\text{eCS}}(p)\) be the elliptic Calogero–Sutherland operator in Definition 5.15. The non-stationary elliptic Calogero equation is defined to be the eigenvalue equation
\[
\left( k p \frac{\partial}{\partial p} + H^{\text{eCS}}(p|\beta) \right) \psi(x, p|\lambda, k|\beta) = \epsilon(\lambda) \psi(x, p|\lambda, k|\beta).
\]

**Definition 7.1** Set \(q = e^h, t = e^{\beta h}, s_i = q^i (1 \leq i \leq N)\) and \(\kappa = e^{\beta h}\). Define \(f^{\text{eCS}}(x, p|\lambda, k|\beta)\) and \(\varphi^{\text{eCS}}(x, p|\lambda, k|\beta)\) by the limits
\[
f^{\text{eCS}}(x, p|\lambda, k|\beta) = \lim_{h \to 0} f^{\tilde{g}^N}(x, p|s|\kappa|q/t),
\]
\[
\varphi^{\text{eCS}}(x, p|\lambda, k|\beta) = \lim_{h \to 0} \varphi^{\tilde{g}^N}(x, p|s|\kappa|q/t).
\]

Note that \(\varphi^{\text{eCS}}(x, p|\lambda, k|\beta) = \psi_0(x, p|\beta)f^{\text{eCS}}(x, p|\lambda, k|\beta)\).

**Conjecture 7.2** We have
\[
\left( k p \frac{\partial}{\partial p} + H^{\text{eCS}}(p|\beta) \right) x^i \varphi^{\text{eCS}}(x, p|\lambda, k|\beta) = \frac{1}{2} \sum_{i=1}^N \lambda_i^2 x^i \varphi^{\text{eCS}}(x, p|\lambda, k|\beta).
\]

Or equivalently that
\[
\left( k \left( \frac{1}{\psi_0} \frac{\partial \psi_0}{\partial p} \right) + k p \frac{\partial}{\partial p} + H_\beta(p) + \frac{\beta(\beta - 1)N}{2} V_0(p^N) \right) x^i f^{\text{eCS}}(x, p|\lambda, k|\beta)
\]
\[
= \frac{1}{2} \sum_{i=1}^N \lambda_i^2 x^i f^{\text{eCS}}(x, p|\lambda, k|\beta).
\]

7.2 Explicit formula for \(f^{\text{eCS}}(x, p|\lambda, k|\beta)\)

Denote by \((a)_n = a(a + 1) \cdots (a + n - 1)\) the ordinary shifted product.
DEFINITION 7.3 For \( l \in \mathbb{Z}/N\mathbb{Z} \), and \( \lambda, \mu \in \mathbb{P} \), set
\[
N^{(l)}_{\lambda, \mu}(v|k) = \prod_{j \neq i \in \mathbb{Z}/l \mathbb{Z} \atop j - i = l (\text{mod} \, N)} (v - \mu_i + \lambda_{j+1} - (i - j)k)_{\lambda_j - \lambda_{j+1}} 
\times \prod_{\beta \geq \gamma \geq 1 \atop \beta - \alpha = - l - 1 (\text{mod} \, N)} (v + \lambda_\alpha - \mu_\beta + (\alpha - \beta - 1)k)_{\mu_\beta - \mu_\beta}.
\]

LEMMA 7.4 Let \( q = e^h, t = e^{\beta h}, s_i = q^h (1 \leq i \leq N) \) and \( \kappa = e^{ih} \). We have
\[
f^{\text{CS}}(x, p|\lambda, k|\beta) = \lim_{h \to 0} f^{\text{CS}}(x, p|e^{ih}, e^{ih}k|e^{ih(1-\beta)}) = \sum_{\sigma(1), \ldots, \sigma(N) \in \mathbb{P}} \sum_{i=1}^N N^{(i-j)}_{\mu(\sigma(1)), \mu(\sigma(2))}(1 - \beta + \lambda_j - \lambda_i | k)_{\lambda_j - \lambda_i} \cdot \prod_{\beta = 1}^{N} (p_{x_{\alpha+\beta}/x_{\alpha+\beta-1}})^{\mu(\beta)}. \]

7.3 \( \hat{\mathfrak{sl}}_N \) dominant integrable character case

Let \( K, \mu \) be as in Definition 1.6. This in the differential case means that we set \( k = -\frac{K}{N} - 1, \beta = 1 \). Note that we have the \( \hat{\mathfrak{sl}}_N \) character as \( x^\mu f^{\text{CS}}(x, p|\mu, -(K + N)/N|1) = ch^{\hat{\mathfrak{sl}}_N}_{\mathcal{L}(\Lambda(K, \mu))}/(p^N; p^N)_{\infty} \). Note also that when \( \beta = 1 \), the quasi-ground state \( \psi_0(x, p|\beta = 1) \) (given in Definition 5.14) divided by \( (p^N; p^N)_{\infty} \), i.e. \( \psi_0(x, p|1)/(p^N; p^N)_{\infty} \), is nothing but Weyl–Kac’s denominator for the dominant integrable characters of \( \mathfrak{sl}_N \). Then the case \( \beta = 1 \) of Conjecture 7.2 reduces to the heat equation for the numerator of the Weyl–Kac formula, i.e. for \( \psi_0(x, p|1) \times ch^{\hat{\mathfrak{sl}}_N}_{\mathcal{L}(\Lambda(K, \mu))}/(p^N; p^N)_{\infty} = x^\mu \varphi^{\text{CS}}(x, p|\mu, -(K + N)/N|1). \)

PROPOSITION 7.5 Let \( \beta = 1 \), and \( K, \mu \) be as in Definition 1.6. We have the heat equation
\[
\left( -\frac{K + N}{N} p \frac{\partial}{\partial p} + \frac{1}{2} \Delta \right) x^\mu \varphi^{\text{CS}}(x, p|\mu, -(K + N)/N|1) = \frac{1}{2} \sum_{i=1}^N \mu_i^2 x^\mu \varphi^{\text{CS}}(x, p|\mu, -(K + N)/N|1).
\]

Next, consider the case \( K = 0 \) (i.e. \( k = -\beta \)), and \( \mu = \emptyset \). Note that we have \( f^{\text{CS}}(x, p|0, \ldots, 0, -N\beta|\beta) = 1/(p^N; p^N)_{\infty} \). This case gives another non-trivial check (for arbitrary \( \beta \)) of Conjecture 7.2.

PROPOSITION 7.6 We have
\[
\left( -\beta p \frac{\partial}{\partial p} + H^{\text{CS}}(p|\beta) \right) \psi_0(x, p|\beta) \frac{1}{(p^N; p^N)_{\infty}} = 0. \tag{14}
\]

We remark that this is a special case of Langmann’s kernel function identity [23].
Proof. Equation (14) is equivalent to the following $\beta$ independent equation

$$
\sum_{1 \leq i < j \leq N} \frac{1}{\Theta_{pN}(p^{j-i}x_j/x_i)} \frac{\partial}{\partial p} \Theta_{pN}(p^{j-i}x_j/x_i)
= \sum_{1 \leq i < j \leq N} \left( \frac{N}{2} \Theta_{pN}^{(2)}(p^{j-i}x_j/x_i) - \frac{N - 2j + 2i}{2} \Theta_{pN}^{(1)}(p^{j-i}x_j/x_i) \right),
$$

which follows from the heat equations for the theta function for $1 \leq i < j \leq N$

$$
-\frac{\partial}{\partial p} \Theta_{pN}(p^{j-i}x_j/x_i) + \frac{N}{2} \Theta_{pN}^{(2)}(p^{j-i}x_j/x_i) - \frac{N - 2j + 2i}{2} \Theta_{pN}^{(1)}(p^{j-i}x_j/x_i) = 0. \quad \Box
$$

7.4 Stationary elliptic Calogero–Sutherland equation: case $k = 0$

Definition 7.7 Let $\alpha(p|\lambda, k|\beta) = \sum_{d \geq 0} p^{Nd} \alpha_d(p|\lambda, k|\beta)$ be the constant term of the series $f^{eCS}(x, p|\lambda, k|\beta)$ with respect to $x_i$'s. Namely,

$$
\alpha(p|\lambda, k|\beta) = \sum_{\lambda(1), \ldots, \lambda(N) \in \mathbb{P}} \sum_{m_1 = \ldots = m_N = 0}^{N} \prod_{i=1}^{N} N_{k^{(i)}(\lambda)}^{(i)(N)} (1 - \beta + \mu_j - \mu_i|k).
$$

Conjecture 7.8 We have the properties:

1. The series $f^{eCS}$ is convergent on a certain domain. With respect to $k$, it is regular on a certain punctured disk $\{k \in \mathbb{C}| |k| < r, k \neq 0 \}$.
2. The $f^{eCS}$ and $\alpha(p|\lambda, k|\beta)$ are essential singular at $k = 0$. The $\alpha_d(p|\lambda, k|\beta)$ has a pole of degree $d$ in $k$ at $k = 0$.
3. The ratio $f^{eCS}/\alpha(p|\lambda, k|\beta)$ is regular at $k = 0$.
4. The derivative $\frac{1}{k} \frac{\partial}{\partial p} \log \alpha(p|\lambda, k|\beta)$ is regular at $k = 0$, the limit $k \to 0$ of which being a non-trivial Taylor series in $p^{N}$.

Definition 7.9 Assuming Conjecture 7.8, set

$$
\varphi^{\text{st.eCS}}(x, p|\lambda, k|\beta) = \left. \frac{\varphi^{eCS}(x, p|\lambda, k|\beta)}{\alpha(p|\lambda, k|\beta)} \right|_{k=0}, \quad \varepsilon(p|\lambda, k|\beta) = \frac{1}{2} \sum_{i=1}^{N} \lambda_i^2 + \left. \frac{1}{k} \frac{\partial}{\partial p} \log \alpha(p|\lambda, k|\beta) \right|_{k=0}.
$$

Conjectures 7.2 and 7.8 assert the following conjecture for the stationary elliptic Calogero–Sutherland equation.

Conjecture 7.10 We have $H^{eCS}(p) x^{\frac{1}{2}} \varphi^{\text{st.eCS}}(x, p|\lambda|\beta) = \varepsilon(p|\lambda|\beta) x^{\frac{1}{2}} \varphi^{\text{st.eCS}}(x, p|\lambda|\beta)$. 
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