Optimal error estimates of Galerkin finite element methods for stochastic Allen-Cahn equation with additive noise

Ruisheng Qi\textsuperscript{a}, Xiaojie Wang\textsuperscript{b}

\textsuperscript{a} School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao, China
qirsh@neuq.edu.cn
\textsuperscript{b} School of Mathematics and Statistics, Central South University, Changsha, China
x.j.wang7@csu.edu.cn and x.j.wang7@gmail.com

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Abstract

Strong approximation errors of both finite element semi-discretization and spatio-temporal full discretization are analyzed for the stochastic Allen-Cahn equation driven by additive trace-class noise in space dimension \(d \leq 3\). The full discretization is realized by combining the standard finite element method with the backward Euler time-stepping scheme. Distinct from the globally Lipschitz setting, the error analysis becomes rather challenging and demanding, due to the presence of the cubic nonlinearity in the underlying model. By introducing two auxiliary approximation processes, we do appropriate decomposition of the considered error terms and propose a novel approach of error analysis, to successfully recover the expected convergence rates of the numerical schemes. The approach is ingenious and does not rely on high-order spatial regularity properties of the approximation processes. It is shown that the full discrete scheme possesses convergence rates of order \(h^\gamma\) in space and order \(\tau^{\frac{\gamma}{2}}\) in time, subject to the spatial correlation of the noise process, characterized by \(\|A^{\frac{\gamma}{2}}Q_1\|_{\mathcal{L}_2} < \infty, \gamma \in [1, 2]\). In particular, a classical convergence rate of order \(O(h^2 + \tau)\) is reachable, even in multiple spatial dimensions, when the aforementioned condition is fulfilled with \(\gamma = 2\). Numerical examples confirm the previous findings.

Keywords. Stochastic Allen-Cahn equation, additive noise, strong approximation, finite element method, backward Euler scheme.

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1 Introduction

Over the past decades, there have been plenty of research articles analyzing numerical discretizations of parabolic stochastic partial differential equations (SPDEs), see, e.g., monographs [21, 25] and references therein. In contrast to an overwhelming majority of literature focusing on numerical analysis of SPDEs with globally Lipschitz nonlinearity, only a limited number of papers investigated numerical SPDEs in the non-globally Lipschitz regime [1–5, 10, 11, 13, 15, 16, 19, 24] and it is still far from well-understood. As a typical example of parabolic SPDEs with non-globally Lipschitz nonlinearity, stochastic Allen-Cahn equations, perturbed by additive or multiplicative noises, have received increasing attention in the last few years. Recently, several research works were reported on numerical approximations of such equations [3, 5, 10, 17, 18, 23, 24, 26, 27, 30].

The present work makes further contributions in this direction, by successfully recovering optimal strong convergence rates for finite element semi-discretization and spatio-temporal full discretization of stochastic Allen-Cahn equations with additive trace-class noise.

Let $\mathcal{D} \subset \mathbb{R}^d, d \in \{1, 2, 3\}$ be a bounded open spatial domain with smooth boundary and let $H := L^2(\mathcal{D}, \mathbb{R})$ be the real separable Hilbert space endowed with usual inner product and norm. Throughout this article we are interested in the following semi-linear parabolic SPDE in $H$,

$$
\begin{align*}
\begin{cases}
    dX(t) + AX(t)dt = F(X(t))dt + dW(t), & (t, x) \in [0, T] \times \mathcal{D}, \\
    X(0) = X_0, & x \in \mathcal{D},
\end{cases}
\end{align*}
$$

where $A: D(A) \subset H \rightarrow H$ is a linear, densely defined, positive self-adjoint unbounded operator with compact inverse (e.g., $A = -\Delta$ with homogeneous Dirichlet boundary condition) in $H$, generating an analytic semigroup $S(t) = e^{-tA}$ in $H$. Moreover, $\{W(t)\}_{t \geq 0}$ is an $H$-valued $Q$-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The nonlinear mapping $F$ is assumed to be a Nemytskij operator, given by $F(u)(x) = f(u(x))$, $x \in \mathcal{D}$, with $f(v) = v - v^3$, $v \in \mathbb{R}$. Such problem is often referred to as stochastic Allen-Cahn equation. Under further assumptions specified later, particularly including

$$
\|A^{\frac{\gamma - 1}{2}} Q^{\frac{1}{2}}\|_{L_2} < \infty, \quad \text{for some } \gamma \in [1, 2], \quad (1.2)
$$

it is proved in Theorems 2.2, 2.6 that the problem (1.1) possesses a unique mild solution,

$$
X(t) = S(t)X_0 + \int_0^t S(t - s)F(X(s)) \, ds + \int_0^t S(t - s) \, dW(s), \quad t \in [0, T], \quad (1.3)
$$

which enjoys the Sobolev and Hölder regularity properties

$$
X \in L_\infty([0, T]; L^{2p}(\Omega; \dot{H}^{\gamma})), \quad \forall p \geq 1, \quad (1.4)
$$

and for $\forall p \geq 1$ and $0 \leq s < t \leq T$,

$$
\|X(t) - X(s)\|_{L^{2p}(\Omega; \dot{H}^{\beta})} \leq C(t - s)^\frac{\min\{1, \gamma - \beta\}}{2}, \quad \beta \in [0, \gamma]. \quad (1.5)
$$

Here $\| \cdot \|_{L_2}$ stands for the Hilbert-Schmidt norm, $\dot{H}^{\alpha} := D(A^\alpha)$, $\alpha \in \mathbb{R}$ and the parameter $\gamma \in [1, 2]$ coming from (1.2) quantifies the spatial regularity of the covariance operator $Q$ of the
driving noise process (Assumption 2.3). The obtained space-time regularity coincides with that in [22] for SPDEs with globally Lipschitz nonlinearity and is thus optimal in the spirit of [22].

Let \( D \subset \mathbb{R}^d, d \in \{1, 2, 3\} \), be an open convex polynomial domain and \( A = -\Delta \) with \( D(A) = H^2(D) \cap H_0^1(D) \). Let \( V_h \) be a finite element space of piecewise continuous linear functions and \( X_h \) the finite element spatial approximation of the mild solution \( X \), which can be represented by

\[
X_h(t) = S_h(t)P_hX_0 + \int_0^t S_h(t-s)P_hF(X_h(s)) \, ds + \int_0^t S_h(t-s)P_hW(s), \quad t \in [0, T].
\]  

(1.6)

Here \( S_h(t) := e^{-tA_h} \) is the strongly continuous semigroup generated by the discrete Laplace operator \( A_h \). The resulting spatial approximation error is measured as follows (Theorem 3.1)

\[
\|X(t) - X_h(t)\|_{L^2(\Omega; H)} = O(h^\gamma), \quad \gamma \in [1, 2],
\]

(1.7)

where \( \gamma \in [1, 2] \) is from assumption (1.2). The obtained convergence rate in space is called to be optimal since it exactly coincides with the order of optimal spatial regularity of the solution [28, Chapter 1]. Discretizing the semi-discrete problem by a backward Euler time-stepping scheme, we also investigate a full discrete discretization, given by

\[
X_{h,m} = S_{\tau,h}^m X_{h,0} + \tau \sum_{i=0}^{m-1} S_{\tau,h}^{m-i}P_hF(X_{h,i+1}) + \sum_{i=0}^{m-1} S_{\tau,h}^{m-i}P_h\Delta W_{i+1},
\]

(1.8)

where \( X_{h,m} \) is the full discrete approximations of \( X(t_m) \) and \( S_{\tau,h} := (I + \tau A_h)^{-1} \). As stated in Theorem 1.1, the corresponding strong approximation error reads,

\[
\|X(t_m) - X_{h,m}\|_{L^2(\Omega; H)} = O(h^\gamma + \tau^{\frac{\gamma}{2}}), \quad \gamma \in [1, 2].
\]

(1.9)

This indicates how the strong convergence rate of the full discretization relies on the regularity of the driven noise process. Particularly when the condition (1.2) is fulfilled with \( \gamma = 2 \), a classical convergence rate of order \( O(h^2 + \tau) \) for the backward Euler-finite element full discretization is reachable, even in multiple spatial dimensions (see Remark 2.9). These findings are identical to those in [29], where the strong convergence rate of the linear implicit Euler finite element scheme was analyzed for SPDEs with globally Lipschitz nonlinearity. Once the nonlinearity grows super-linearly, one can in general not expect the usual nonlinearity-explicit time-stepping schemes that work well in the globally Lipschitz setting converge in the strong sense (see comments following Theorem 2 in [14] and the relevant divergence result [12]). To address this issue, we therefore take the backward Euler, a nonlinearity-implicit scheme, for the temporal discretization. Although some error estimates are taken from [20, 22, 29], the presence of the non-globally Lipschitz (cubic) nonlinearity in the underlying model brings about essential difficulties in the error analysis (see the proof of Theorems 3.1, 4.1 and the error analysis becomes much more involved than that in the globally Lipschitz SPDE setting.

In the following, we take error estimates of the spatial semi-discretization to illuminate our approach of error analysis. By introducing an auxiliary approximation processes \( \tilde{X}_h \), defined by

\[
\tilde{X}_h(t) = S_h(t)P_hX_0 + \int_0^t S_h(t-s)P_hF(X(s)) \, ds + \int_0^t S_h(t-s)P_hW(s), \quad t \in [0, T],
\]

(1.10)
we separate the spatial error \( \| X(t) - X_h(t) \|_{L^p(\Omega; H)} \) into two parts,

\[
\| X(t) - X_h(t) \|_{L^p(\Omega; H)} \leq \| X(t) - \tilde{X}_h(t) \|_{L^p(\Omega; H)} + \| \tilde{X}_h(t) - X_h(t) \|_{L^p(\Omega; H)}. \tag{1.11}
\]

Subtracting (1.10) from (1.3), one can treat the first error term directly and show \( \| X(t) - \tilde{X}_h(t) \|_{L^p(\Omega; H)} = O(h^\gamma) \), with the aid of existing estimates for the error operators \( \Psi(t) := S(t) - S_h(t)P_h \) and regularity properties of the mild solution \( X(t) \) (see estimates of \( I_1, I_2, I_3, I_4 \) in the proof of Theorem 3.1 for details). To bound the remaining error term \( \tilde{e}(t) := \tilde{X}_h(t) - X_h(t) \), we subtract (1.6) from (1.10) to eliminate the stochastic convolution and thus \( \tilde{e}(t) \) is time differentiable and satisfies

\[
\frac{d}{dt} \tilde{e}(t) + A_h \tilde{e}(t) = P_h(f(X(t)) - F(X_h(t))), \quad t \in (0, T), \quad \tilde{e}_h(0) = 0. \tag{1.12}
\]

Deterministic calculus together with the monotonicity of the nonlinearity, regularity properties of \( \tilde{X}_h(t) \), the previous estimate of \( \| X(t) - \tilde{X}_h(t) \|_{L^p(\Omega; H)} \) and Gronwall’s inequality facilitates the derivation of \( \| \tilde{e}(t) \|_{L^p(\Omega; H)} = O(h^\gamma) \) (see (3.31)-(3.33)).

In the same manner as the semi-discrete case, we introduce an auxiliary processes

\[
\tilde{X}_{h,m} = \tilde{S}^m_{\tau,h}P_hX_0 + \tau \sum_{i=0}^{m-1} \tilde{S}^{m-i}_{\tau,h}P_hF(X(t_{i+1})) + \sum_{i=0}^{m-1} S^{m-i}_{\tau,h}P_h \Delta W_{t+i}. \tag{1.13}
\]

and decompose the full discretization error \( \| X(t) - X_h(t) \|_{L^p(\Omega; H)} \) as

\[
\| X(t_m) - X_{h,m} \|_{L^p(\Omega; H)} \leq \| X(t_m) - \tilde{X}_{h,m} \|_{L^p(\Omega; H)} + \| \tilde{X}_{h,m} - X_{h,m} \|_{L^p(\Omega; H)}. \tag{1.14}
\]

Then following the basic line as above, but with much more efforts made to call discrete versions of arguments as used in the semi-discrete scenario, enables us to attain the desired error bounds for the full discretization (cf. section 4).

As usual, the strong convergence rate analysis of numerical SPDEs with super-linearly growing nonlinearities are carried out based on appropriate uniform a priori moment \( L_\infty \)-bounds of approximations \( 1,10,10,24 \). In a different but ingenious way, we develop a new approach of error analysis here, which does not rely on high-order spatial regularity properties (e.g., a priori moment \( L_\infty \)-bounds) of approximation processes \( X_h(t), X_{h,m} \).

Before closing the introduction part, we recall a few closely relevant works. The backward Euler time semi-discretization was also examined in \( 18,19 \) for the problem \( 1.1 \), with no spatial discretization. Under assumption \( 1.2 \) taking \( \gamma = 2 \), i.e., \( \| A^{1/2}Q^1 \|_{L_2} < \infty \), only a strong convergence rate of order \( \frac{1}{2} \) was attained in \( 19 \). In \( 3,5 \), pure time semi-discretizations of splitting type were studied for \( 1.1 \). Particularly, the authors of \( 3 \) used exponential integrability properties of exact and numerical solutions to identify a strong convergence rate of order 1, but only valid in one space dimension, when the additive noise is moderately smooth, i.e., \( \gamma = 2 \) in \( 1.2 \). For the space-time white noise case, various discretizations were investigated in other contributions \( 1,2,24,30 \).

The outline of this paper is as follows. In the next section, some preliminaries are collected and the well-posedness and regularity properties of the considered problem are elaborated. Section 3 is devoted to error estimates of the finite element spatial semi-discretization and section 4 provides error estimates of backward Euler-finite element full discretization. At the end of the article, some numerical examples are presented, illustrating the above theoretical findings.
2 The stochastic Allen-Cahn equation

Given a separable $\mathbb{R}$-Hilbert space $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$, by $\mathcal{L}(H)$ we denote the Banach space of all linear bounded operators from $H$ into $H$. Also, we denote by $\mathcal{L}_2(H)$ the Hilbert space consisting of Hilbert-Schmidt operators from $H$ into $H$, equipped with the scalar product and the norm

$$
\langle \Gamma_1, \Gamma_2 \rangle_{\mathcal{L}_2(H)} = \sum_{i \in \mathbb{N}} \langle \Gamma_1 \phi_i, \Gamma_2 \phi_i \rangle, \quad \| \Gamma \|_{\mathcal{L}_2(H)} = \left( \sum_{i \in \mathbb{N}} \| \phi_i \|^2 \right)^{\frac{1}{2}},
$$

independent of the choice of orthonormal basis $\{ \phi_i \}$ of $H$. If $\Gamma \in \mathcal{L}_2(H)$ and $L \in \mathcal{L}(H)$, then $\Gamma L$, $L \Gamma \in \mathcal{L}_2(H)$ and

$$
\| \Gamma L \|_{\mathcal{L}_2(H)} \leq \| \Gamma \|_{\mathcal{L}_2(H)} \| L \|_{\mathcal{L}(H)}, \quad \| L \Gamma \|_{\mathcal{L}_2(H)} \leq \| \Gamma \|_{\mathcal{L}_2(H)} \| L \|_{\mathcal{L}(H)}.
$$

2.1 Abstract framework and main assumptions

In this subsection, we formulate main assumptions concerning the operators $A$ and $Q$, the nonlinear term $F(\cdot)$, the noise term $W(t)$ and the initial value $X_0$, which will be used throughout this paper.

**Assumption 2.1 (Linear operator $A$)** Let $\mathcal{D} \subset \mathbb{R}^d, d \in \{1, 2, 3\}$ be a bounded open spatial domain with smooth boundary and let $H := L^2(\mathcal{D}, \mathbb{R})$ be the real separable Hilbert space endowed with usual inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. Let $A : D(A) \subset H \to H$ be a densely defined, positive self-adjoint unbounded operator on $H$ with compact inverse.

Such assumptions imply the existence of a sequence of nondecreasing positive real numbers $\{\lambda_k\}_{k \geq 1}$ and an orthonormal basis $\{e_k\}_{k \geq 1}$ of $H$ such that

$$
A e_k = \lambda_k e_k, \quad \lim_{k \to \infty} \lambda_k = +\infty.
$$

Furthermore, it is known that $-A$ generates an analytic semigroup $S(t) = e^{-tA}$ satisfying

$$
\| A^\mu S(t) \| \leq Ct^{-\mu}, \quad t > 0, \quad \mu \geq 0,
$$

$$
\| A^{-\nu}(I - S(t)) \| \leq C t^\nu, \quad t \geq 0, \quad \nu \in [0, 1].
$$

Throughout this article, we use generic constants which may vary at each appearance but are always independent of discretization parameters. By means of the spectral decomposition of $A$, we can also define the fractional powers $\gamma \in \mathbb{R}$ of $A$ in a simple way, e.g., $A^\gamma v = \sum_{k=1}^\infty \lambda_k^\gamma \langle v, e_k \rangle e_k$. Then denote the Hilbert space $H^\gamma := D(A^\gamma)$ with the inner product $\langle A^{\frac{\gamma}{2}}, A^{\frac{\gamma}{2}} \rangle$ and the associated norm $\| \cdot \|_\gamma := \| A^{\frac{\gamma}{2}} \|$.\]

**Assumption 2.2 (Nonlinearity)** Let $F : L_6(D; \mathbb{R}) \to H$ be a deterministic mapping given by

$$
F(v)(x) = f(v(x)) = v(x) - v^3(x), \quad x \in \mathcal{D}, \quad v \in L_6(D; \mathbb{R}).
$$
Here and below, by $L^r(D;\mathbb{R}), r \geq 1$ ($L^r(D)$ or $L^r$ for short) we denote a Banach space consisting of $r$-times integrable functions. It is easy to check that, for any $v, \psi, \psi_1, \psi_2 \in L_6(D;\mathbb{R})$,

$$
(F'(v)(\psi))(x) = f'(v(x))\psi(x) = (1 - 3v^2)(x)\psi(x), \quad x \in D
$$
$$
(F''(v)(\psi_1, \psi_2))(x) = f''(v(x))\psi_1(x)\psi_2(x) = -6v(x)\psi_1(x)\psi_2(x), \quad x \in D.
$$

Moreover, there exists a constant $C$ such that

$$
\langle u - v, F(u) - F(v) \rangle \leq C\|u - v\|^2, \quad u, v \in L_6(D;\mathbb{R}).
$$

Assumption 2.3 (Noise process) Let $\{W(t)\}_{t \in [0,T]}$ be a standard $H$-valued $Q$-Wiener process on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})$, where the covariance operator $Q \in L(H)$ is bounded, self-adjoint and positive semi-definite. Assume

$$
\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{L_2(H)} < \infty, \quad \text{for some } \gamma \in [1, 2].
$$

Additionally, we assume that, for sufficiently large number $p_0 \in \mathbb{N}$

$$
\sup_{s \in [0,T]} \|W_A(s)\|_{L_{2p_0}(\Omega;L_{18})} < \infty, \quad \text{with } \ W_A(t) := \int_0^t S(t - s) \, dW(s).
$$

One can consult [8] for more details on properties of the $Q$-Wiener process and the stochastic convolution $W_A(t)$. Further comments regarding conditions (2.8)-(2.9) are added in Remark 2.9.

Assumption 2.4 (Initial value) Let $X_0 : \Omega \to H$ be $\mathcal{F}_0/\mathcal{B}(H)$-measurable and satisfies

$$
\mathbb{E}[\|X_0\|_{\gamma}^{p_0}] < \infty, \quad \text{for } \gamma \in [1, 2].
$$

We remark that the assumption on the initial value can be relaxed, but at the expense of having the constant $C$ later depending on $T^{-1}$, by exploring the smoothing effect of the semigroup $E(t), t \in [0,T]$ and standard non-smooth data error estimates [28].

2.2 Regularity results of the model

In this part, we focus on the well-posedness of the underlying problem and the space-time regularity properties of the mild solution. A preliminary theorem is stated as follows.

Theorem 2.5 Under Assumptions 2.1-2.4, the problem (1.1) admits a unique mild solution, given by (1.3), satisfying, for any $p \geq 1$,

$$
\sup_{s \in [0,T]} \|X(s)\|_{L_{2p}(\Omega;L_6)} \leq C \left(1 + \|X_0\|_{L_{2p}(\Omega;L_6)} + \sup_{s \in [0,T]} \|W_A(s)\|_{L_{2p}(\Omega;L_{18})}^2\right).
$$
To arrive at Theorem 2.5 one can simply adapt the proof of [6, Theorem 4.8], where the existence and uniqueness of the mild solution of the stochastic Allen-Cahn equation (1.1) was established in the special case $Q = A^{-s}$. There a basic tool for the proof is provided by the Yosida approximate arguments and the assumption $Q = A^{-s}$, $s > \frac{d}{2} - 1$ can be replaced by assumptions (2.8)-(2.9) instead. Indeed, the assumption $Q = A^{-s}$, $s > \frac{d}{2} - 1$ was simply used there to ensure $\|W_A(s)\|_{L^p(\Omega; C(D; \mathbb{R}))} < \infty$ and thus $\|W_A(s)\|_{L^{2p}(\Omega; L^6)} < \infty$. Also, one can consult the proof of [7, Theorem 5.5.8]. Evidently, the above estimate (2.11) suffices to ensure

$$\sup_{s\in[0,T]}\|F(X(s))\|_{L^{2p}(\Omega; H)} \leq C(1 + \sup_{s\in[0,T]}\|X(s)\|_{L^{2p}(\Omega; L_6)}) < \infty,$$

(2.12)

and similarly

$$\sup_{s\in[0,T]}\|f'(X(s))\|_{L^{2p}(\Omega; L_3)} + \sup_{s\in[0,T]}\|f''(X(s))\|_{L^{2p}(\Omega; L_6)} < \infty.$$

(2.13)

Equipped with Theorem 2.5 we can get the following further regularity results.

**Theorem 2.6** Under Assumptions 2.1-2.4, the mild solution (1.3) enjoys the following regularity,

$$\sup_{s\in[0,T]}\|X(s)\|_{L^{2p}(\Omega; H^\gamma)} < \infty, \quad \forall \gamma \geq 1,$$

(2.14)

and for any $\beta \in [0, \gamma]$

$$\|X(t) - X(s)\|_{L^{2p}(\Omega; H^\beta)} \leq C(t - s)^{\min(1, \gamma - \beta)}.$$

(2.15)

Before proving Theorem 2.6 we introduce some basic inequalities. Recall first two well-known Sobolev embedding inequalities,

$$\dot{H}^\delta \subset C(D; \mathbb{R}), \quad \text{for} \quad \delta > \frac{d}{2}, \quad d \in \{1, 2, 3\},$$

(2.16)

and

$$\dot{H}^1(D) \subset L_6(D), \quad \text{for} \quad d \in \{1, 2, 3\}.$$

(2.17)

With (2.16) at hand, one can show

$$\|A^{-\frac{\delta}{2}}x\| = \sup_{\|v\| = 1, v \in H} |\langle x, A^{-\frac{\delta}{2}}v \rangle| \leq \sup_{\|v\| = 1, v \in H} \|x\|_{L_1} \|A^{-\frac{\delta}{2}}v\|_{C(D; \mathbb{R})} \leq C \sup_{\|v\| = 1, v \in H} \|x\|_{L_1} \|v\| \leq C\|x\|_{L_1}, \quad \forall \delta \in (\frac{3}{2}, 2), x \in L_1(D).$$

(2.18)

Similarly, but with the help of (2.17), one can find

$$\|A^{-\frac{1}{2}}x\| = \sup_{\|x\| = 1, x \in H} |\langle x, A^{-\frac{1}{2}}\chi \rangle| \leq \sup_{\|x\| = 1, x \in H} \|x\|_{L_6} \|A^{-\frac{1}{2}}\chi\|_{L_6} \leq C \sup_{\|x\| = 1, x \in H} \|x\|_{L_6} \|\chi\| \leq C\|x\|_{L_6}, \quad \forall x \in L_6(D).$$

(2.19)

A slight modification of the proof of [22, Theorem 3.1,Corollary 5.2] gives the following lemma.
Lemma 2.7 If condition (2.8) from Assumption 2.3 is valid, then \( \forall \rho \geq 1 \),
\[
\sup_{s \in [0,T]} \| W_A(s) \|_{L^{2p}(\Omega; H^\gamma)} \leq C \| A^\frac{\gamma-1}{2} Q_{2}^{\frac{1}{2}} \|_{L^2(H)},
\]  
(2.20)
and, for any \( \alpha \in [0, \gamma] \) and for \( 0 \leq s < t \leq T \)
\[
\| W_A(t) - W_A(s) \|_{L^{2p}(\Omega; H^\alpha)} \leq C (t - s)^{\min(1, \gamma - \alpha)} \| A^\frac{\gamma-1}{2} Q_{2}^{\frac{1}{2}} \|_{L^2(H)}.
\]  
(2.21)
In addition to the above preparations, we also need a lemma quoted from [22, Lemma 3.2].

Lemma 2.8 For any \( \rho \in [0, 1] \) and all \( x \in H \)
\[
\int_{\tau_1}^{\tau_2} \| A^\frac{\rho}{2} S(\tau_2 - \sigma) x \|^2 d\sigma \leq C (\tau_2 - \tau_1)^{1-\rho} \| x \|^2, \quad 0 \leq \tau_1 < \tau_2,
\]  
(2.22)
\[
\left\| A^\rho \int_{\tau_1}^{\tau_2} S(\tau_2 - \sigma) x d\sigma \right\| \leq C (\tau_2 - \tau_1)^{1-\rho} \| x \|, \quad 0 \leq \tau_1 < \tau_2.
\]

At the moment, we are able to start the proof of Theorem 2.6.

Proof of Theorem 2.6. We take any fixed number \( \delta_0 \in (\frac{3}{2}; 2) \) and consider two possibilities: either \( \gamma \in [1, \delta_0] \) or \( \gamma \in (\delta_0, 2) \). When (2.8) is fulfilled with \( \gamma \in [1, \delta_0] \), we utilize (2.12), (2.20) and (2.4) with \( \mu = \gamma \) to show, for \( \gamma \in [1, \delta_0] \),
\[
\| X(t) \|_{L^{2p}(\Omega; H^\gamma)} \leq \| S(t) X_0 \|_{L^{2p}(\Omega; H^\gamma)} + \left\| \int_0^t S(t-s) F(X(s)) \, ds \right\|_{L^{2p}(\Omega; H^\gamma)} + \| W_A(t) \|_{L^{2p}(\Omega; H^\gamma)}
\]
\[
\leq C \| X_0 \|_{L^{2p}(\Omega; H^\gamma)} + C \int_0^t (t-s)^{-\frac{\gamma}{2}} \| F(X(s)) \|_{L^{2p}(\Omega; H)} \, ds + C \| A^\frac{\gamma-1}{2} Q_{2}^{\frac{1}{2}} \|_{L^2(H)}
\]
\[
\leq C \| X_0 \|_{L^{2p}(\Omega; H^\gamma)} + C \sup_{s \in [0,T]} \| F(X(s)) \|_{L^{2p}(\Omega; H)} + C \| A^\frac{\gamma-1}{2} Q_{2}^{\frac{1}{2}} \|_{L^2(H)} < \infty.
\]  
(2.23)
Concerning the temporal regularity of the mild solution, we apply (2.20), (2.21), (2.12), (2.4) and the Burkholder-Davis-Gundy-type inequality to obtain, for any \( \beta \in [0, \gamma] \)
\[
\| X(t) - X(s) \|_{L^{2p}(\Omega; H^\beta)} \leq \| (S(t-s) - I) X(s) \|_{L^{2p}(\Omega; H^\beta)} + \left\| \int_s^t S(t-r) F(X(r)) \, dr \right\|_{L^{2p}(\Omega; H^\beta)}
\]
\[
+ \| W_A(t) - W_A(s) \|_{L^{2p}(\Omega; H^\beta)} + \| (I - S(t-s)) W_A(s) \|_{L^{2p}(\Omega; H^\beta)}
\]
\[
\leq C (t-s)^{\frac{\gamma-\beta}{2}} \sup_{s \in [0,T]} \| X(s) \|_{L^{2p}(\Omega; H^\gamma)} + C \int_s^t (t-r)^{-\frac{\beta}{2}} \| F(X(r)) \|_{L^{2p}(\Omega; H)} \, dr
\]
\[
+ C (t-s)^{\min(1, \gamma-\beta)} \| A^\frac{\gamma-1}{2} Q_{2}^{\frac{1}{2}} \|_{L^2(H)} + C (t-s)^{\frac{\gamma-\beta}{2}} \| W_A(s) \|_{L^{2p}(\Omega; H^\gamma)}
\]
\[
\leq C (t-s)^{\min(1, \gamma-\beta)} \left( \sup_{s \in [0,T]} \| X(s) \|_{L^{2p}(\Omega; H^\gamma)}
\]
\[
+ \sup_{s \in [0,T]} \| F(X(s)) \|_{L^{2p}(\Omega; H)} + \| A^\frac{\gamma-1}{2} Q_{2}^{\frac{1}{2}} \|_{L^2(H)} \right)
\]
\[
\leq C (t-s)^{\min(1, \gamma-\beta)}.
\]  
(2.24)
Next, let us look at the other case $\gamma \in (\delta_0, 2]$. In this case, one can see $\sup_{s \in [0,T]} \|X(s)\|_{L^{8p}(\Omega; H^{\delta_0})} < \infty$, as already verified in the former case. Accordingly, applying (2.16) and (2.24) implies
\[
\|F(X(t)) - F(X(r))\|_{L^{8p}(\Omega; H^{\beta})} \leq \int_s^t \|S(t-r)F(X(t))\|_{L^{8p}(\Omega; H^{\beta})} + \|S(t-r)F(X(t)) - F(X(r))\|_{L^{8p}(\Omega; H^{\beta})} \, dr
\]
\[
\leq C(t-s)^{\frac{\alpha}{2}} \|F(X(t))\|_{L^{8p}(\Omega; H^{\beta})} + C \int_s^t (t-r)^{-\frac{\alpha}{2}} \|F(X(t)) - F(X(r))\|_{L^{8p}(\Omega; H^{\beta})} \, dr
\]
\[
\leq C(t-s)^{\frac{\alpha}{2}} \sup_{s \in [0,T]} \|F(X(s))\|_{L^{8p}(\Omega; H^{\beta})} + C \int_s^t (t-r)^{1+\frac{\alpha}{2}} \, dr
\]
\[
\leq C(t-s)^{\frac{\alpha}{2}}.
\]  

This together with (2.4), (2.22) and (2.12) leads to, for $\beta \in [0, \gamma]$
\[
\bigg\| \int_s^t S(t-r)F(X(r)) \, dr \bigg\|_{L^{8p}(\Omega; H^{\beta})}
\]
\[
\leq C(t-s)^{\frac{\alpha}{2}} \|F(X(t))\|_{L^{8p}(\Omega; H^{\beta})} + C \int_s^t (t-r)^{-\frac{\alpha}{2}} \|F(X(t)) - F(X(r))\|_{L^{8p}(\Omega; H^{\beta})} \, dr
\]
\[
\leq C(t-s)^{\frac{\alpha}{2}} \sup_{s \in [0,T]} \|F(X(s))\|_{L^{8p}(\Omega; H^{\beta})} + C \int_s^t (t-r)^{1+\frac{\alpha}{2}} \, dr
\]
\[
\leq C(t-s)^{\frac{\alpha}{2}}.
\]

Bearing this in mind and following the proof of (2.23) and (2.24), we can show (2.14) and (2.15) in the case $\gamma \in (\delta_0, 2]$. The proof of Theorem 2.6 is thus complete. \qed

To conclude this section, we make some useful comments on Assumption 2.3.

Remark 2.9 Note that the condition (2.9) is commonly used in the literature \[15, 20, 29, 31\] but the condition (2.9) not. For the special case $Q = A^{-s}$, it is not difficult to see (2.8) is fulfilled with $\gamma \in [1, 2]$ if $s > \frac{\gamma}{4} - \frac{1}{2}$, and (2.8) implies (2.9) since $\|W_A(s)\|_{L^{8p}(\Omega; \mathbb{L}_1)} \leq \|W_A(s)\|_{L^{8p}(\Omega; C(\mathbb{D}, \mathbb{R}))} < \infty$ by \[6, Proposition 4.3\]. For the general case when $A$ and $Q$ do not own the same eigenbasis, Sobolev embedding inequalities $H^1(\mathbb{D}) \subset L^{18}(\mathbb{D})$, $\mathbb{D} \subset \mathbb{R}^d, d = 1, 2$ and $H^3(\mathbb{D}) \subset L^{18}(\mathbb{D})$, $\mathbb{D} \subset \mathbb{R}^3, \delta > \frac{3}{2}$, together with (2.20) tell us that (2.9) is a direct consequence of (2.8) in space dimension $d \in \{1, 2\}$ and (2.9) holds true if (2.8) is satisfied with $\gamma > \frac{3}{2}$ in space dimension $d = 3$. Summarily, the condition (2.9) is, in general, necessary to promise the well-posedness of the nonlinear stochastic problem and estimate (2.11). However, the following error estimates for discretizations just utilize (2.8) and do not get involved with (2.9).

### 3 Error estimates of the spatial semi-discretization

This section is devoted to error estimates of the finite element approximation of the stochastic problem (1.1). For the sake of simplicity, from here to section 4, we always assume that $\mathbb{D} \subset \mathbb{R}^d, d = 1, 2, 3$, is an open convex polygonal domain and $A = -\Delta$ with $D(A) = H^2(\mathbb{D}) \cap H^1_0(\mathbb{D})$.

In order to introduce the semi-discrete finite element approximation, we present some notation and operators on the finite element space. Let $V_h \subset H^1_0(\mathbb{D}), h \in (0, 1]$ be the space of continuous
functions that are piecewise linear over the triangulation $T_h$ of $\mathcal{D}$. Then we introduce a discrete Laplace operator $A_h : V_h \rightarrow V_h$ defined by
\begin{equation}
(A_h v_h, \chi_h) = a(v_h, \chi_h) := \langle \nabla v_h, \nabla \chi_h \rangle, \quad \forall v_h, \chi_h \in V_h,
\end{equation}
and a generalized projection operator $P_h : \dot{H}^{-1} \rightarrow V_h$ given by
\begin{equation}
(P_h v, \chi_h) = \langle v, \chi_h \rangle, \quad \forall v \in \dot{H}^{-1}, \chi_h \in V_h.
\end{equation}

It is well-known that, the operators $A$ and $A_h$ obey
\begin{equation}
C_1 \| A_h^\frac{r}{2} P_h v \| \leq \| A^\frac{r}{2} v \| \leq C_2 \| A_h^\frac{r}{2} P_h v \|, \quad v \in \dot{H}^r, \quad r \in [-1, 1].
\end{equation}

The semi-discrete finite element method for the problem (1.1) is to find $S \in V_h$ such that
\begin{equation}
dX_h(t) + A_h X_h(t) \, dt = P_h F(X_h(t)) \, dt + P_h \, dw(t), \quad t \in (0, T], \quad X_h(0) = P_h X_0.
\end{equation}

Let $S_h(t)$ be the strongly continuous semigroup generated by the discrete Laplace operator $A_h$. Then it is easy to check that the semi-discrete problem (3.4) admits a unique solution in $V_h$, given by $X_h(0) = P_h X_0$ and
\begin{equation}
X_h(t) = S_h(t) P_h X_0 + \int_0^t S_h(t - s) P_h F(X_h(s)) \, ds + W_{A_h}(t), \quad t \in (0, T].
\end{equation}

The following results listed in [20, Lemmas 4.1, 4.2] on the error operator $\Psi_h(t)$ are crucial in the error estimates of the semi-discrete finite element approximation.

**Theorem 3.1** Let $X(t)$ and $X_h(t)$ be the mild solutions of (1.1) and (3.4), respectively. If Assumptions 2.1-2.4 are valid, then for $\forall p \in [1, \infty)$,
\begin{equation}
\| X(t) - X_h(t) \|_{L^p(\Omega;H)} \leq C h^\gamma, \quad \gamma \in [1, 2].
\end{equation}

**Lemma 3.2** Under Assumption 2.7, the following estimates for the error operator $\Psi_h(t)$ hold.
(i) For $0 \leq \nu \leq \mu \leq 2$, it holds that
\begin{equation}
\| \Psi_h(t)x \| \leq C h^\mu t^{-\frac{\mu - \nu}{2}} \| x \|_{\mu}, \quad \forall x \in \dot{H}^\nu, \quad t > 0.
\end{equation}
(ii) Let $0 \leq \rho \leq 1$. Then
\begin{equation}
\| \int_0^t \Psi_h(s)x \, ds \| \leq C h^{2-\rho} \| x \|_{-\rho}, \quad \forall x \in \dot{H}^{-\rho}, \quad t > 0.
\end{equation}
(iii) Let $0 \leq \varrho \leq 1$. Then
\begin{equation}
\left( \int_0^t \| \Psi_h(s)x \|^2 \, ds \right)^{\frac{1}{2}} \leq C h^{1+\varrho} \| x \|_{\varrho}, \quad \forall x \in \dot{H}^{\varrho}, \quad t > 0.
\end{equation}
Additionally, we need smoothing properties of the semigroup $S_h(t)$, as described below.

**Lemma 3.3** Under Assumption 2.7, the following estimates for the discrete semigroup $S_h(t)$ hold,

$$\|A_h^\frac{1}{2} S_h(t) P_h x\| \leq C t^{-\frac{1}{2}} \|x\|, \forall \mu \in [0,1], x \in H, \quad (3.11)$$

$$\int_0^t \|A_h^\frac{1}{2} S_h(s) P_h x\|^2 \, ds \leq C \|x\|^2, \forall x \in H. \quad (3.12)$$

The assertion (3.11) is obvious and (3.12) is derived by using (4.19) in 31.

**Lemma 3.4** Suppose Assumptions 2.7, 2.8 hold. Then

$$\sup_{s \in [0,T]} E[\|X_h(s)\|^{2p}] + \int_0^T E[\|\nabla X_h(s)\|^2] \, ds < \infty. \quad (3.13)$$

**Proof of Lemma 3.4** Define $Y_h(t) := X_h(t) - W_{A_h}(t) = S_h(t) P_h X_0 + \int_0^t S_h(t-s) P_h F(X_h(s)) \, ds$. Then $Y_h(t)$ is time differentiable and obeys

$$\frac{d}{dt} Y_h(t) + A_h Y_h(t) = P_h F(Y_h(t) + W_{A_h}(t)), \quad Y_h(0) = P_h X_0. \quad (3.14)$$

By multiplying both sides of (3.14) by $Y_h(t)$, taking the inner product and using (2.7), we obtain

$$\frac{d}{dt} \|Y_h(s)\|^2 + \|\nabla Y_h(s)\|^2 = \langle F(Y_h(s) + W_{A_h}(s)) - F(W_{A_h}(s)), Y_h(s) \rangle + \langle F(W_{A_h}(s)), Y_h(s) \rangle$$

$$\leq C \|Y_h(s)\|^2 + \frac{1}{2} \|F(W_{A_h}(s))\|^2 + \frac{1}{2} \|Y_h(s)\|^2$$

$$\leq C \|F(W_{A_h}(s))\|^2 + C \|Y_h(s)\|^2, \quad (3.15)$$

which, after integration over $[0, t]$ and using the Gronwall inequality, suggests that

$$\|Y_h(t)\|^2 + \int_0^t \|\nabla Y_h(s)\|^2 \, ds \leq C \left( \int_0^t \|F(W_{A_h}(s))\|^2 \, ds + \|P_h X_0\|^2 \right). \quad (3.16)$$

Noticing the fact that $X_h(t) = Y_h(t) + W_{A_h}(t)$ leads to

$$\|X_h(t)\|^2 + \int_0^t \|\nabla X_h(s)\|^2 \, ds \leq C \left( \int_0^t \|W_{A_h}(s)\|^2 \, ds + \int_0^t \|F(W_{A_h}(s))\|^2 \, ds \right.$$ \n
$$+ \|W_{A_h}(t)\|^2 + \|P_h X_0\|^2 \bigg). \quad (3.17)$$

Then, using (2.5), (2.17), (3.3), (3.12) and the Burkholder-Davis-Gundy-type inequality shows

$$\sup_{s \in [0,T]} \|F(W_{A_h}(s))\|_{L^{2p}(\Omega; H)} \leq C \left( 1 + \sup_{s \in [0,T]} \|W_{A_h}(s)\|_{L^p(\Omega; L^2)}^3 \right)$$

$$\leq C \left( 1 + \sup_{s \in [0,T]} \|W_{A_h}(s)\|_{L^p(\Omega; H^1)}^3 \right)$$

$$\leq C \left( 1 + \sup_{s \in [0,T]} \left( \int_0^s \|A_h^{\frac{1}{2}} S_h(s-r) P_h dW(r)\|_{L^p(\Omega; H)}^3 \right) \right)$$

$$\leq C \left( 1 + \sup_{s \in [0,T]} \left( \int_0^s \|A_h^{\frac{1}{2}} S_h(s-r) P_h Q_{\frac{1}{2}}^2 \|_{L^2}^2 \, dr \right)^{3/2} \right)$$

$$\leq C \left( 1 + \|Q_{\frac{1}{2}}^1\|_{L^2}^3 \right) < \infty,$$
where one can also realize that

\[
\sup_{s \in [0,T]} \|W_{A_h}(s)\|_{L^{2p}(\Omega;H^1)} \leq C\|Q^\frac{\gamma}{2}\|_{L_2} < \infty. \tag{3.19}
\]

This combined with Assumption 2.4 shows the desired assertion. □

We are now ready to rigorously prove Theorem 3.1.

\textbf{Proof of Theorem 3.1.} By introducing the following auxiliary process, we separate the considered error term \( \|X(t) - X_h(t)\|_{L^{2p}(\Omega;H)} \) as

\[
\|X(t) - X_h(t)\|_{L^{2p}(\Omega;H)} \leq \|X(t) - \tilde{X}_h(t)\|_{L^{2p}(\Omega;H)} + \|\tilde{X}_h(t) - X_h(t)\|_{L^{2p}(\Omega;H)}. \tag{3.21}
\]

In view of \((2.12), (2.17), (3.3), (3.11)\) and \((3.19)\) we acquire that, for any \(t \in [0,T]\),

\[
\begin{align*}
\|\tilde{X}_h(t)\|_{L^{2p}(\Omega;L_0)} & \leq \|\tilde{X}_h(t)\|_{L^{2p}(\Omega;H^1)} \\
& \leq \|S_h(t)P_hX_0\|_{L^{2p}(\Omega;H^1)} + \int_0^t \|S_h(t-s)P_hF(X(s))\|_{L^{2p}(\Omega;H^1)} ds + \|W_{A_h}(t)\|_{L^{2p}(\Omega;H^1)} \\
& \leq C\|X_0\|_{L^{2p}(\Omega;H^1)} + C \sup_{s \in [0,T]} \|F(X(s))\|_{L^{2p}(\Omega;H)} \int_0^t (t-s)^{-\frac{\gamma}{2}} ds + C\|Q^\frac{\gamma}{2}\|_{L_2} < \infty. \tag{3.22}
\end{align*}
\]

We are now prepared to bound the first error item in \((3.21)\). Subtracting \((3.20)\) from \((1.3)\) yields

\[
\begin{align*}
\|X(t) - \tilde{X}_h(t)\|_{L^{2p}(\Omega;H)} & \leq \|(S(t) - S_h(t)P_h)X_0\|_{L^{2p}(\Omega;H)} \\
& \quad + \left\| \int_0^t (S(t-s) - S_h(t-s)P_h)F(X(t)) ds \right\|_{L^{2p}(\Omega;H)} \\
& \quad + \left\| \int_0^t (S(t-s) - S_h(t-s)P_h)(F(X(t)) - F(X(s))) ds \right\|_{L^{2p}(\Omega;H)} \\
& \quad + \left\| \int_0^t (S(t-s) - S_h(t-s)P_h) dW(s) \right\|_{L^{2p}(\Omega;H)} \\
& := I_1 + I_2 + I_3 + I_4. \tag{3.23}
\end{align*}
\]

Subsequently \(I_1, I_2, I_3\) and \(I_4\) will be treated separately. For the first term \(I_1\), we utilize \((3.8)\) with \(\mu = \nu = \gamma\) to derive

\[
I_1 \leq Ch^\gamma \|X_0\|_{L^{2p}(\Omega;H^\gamma)}. \tag{3.24}
\]

Employing \((2.12)\) and \((3.9)\) with \(\rho = 0\) enables us to obtain

\[
I_2 \leq Ch^2 \|F(X(t))\|_{L^{2p}(\Omega;H)} \leq Ch^2. \tag{3.25}
\]
To handle $I_3$, we recall (2.25) and (2.12), which together imply, for any fixed number $\delta_0 \in \left(\frac{3}{2}, 2\right)$,

$$\|F(X(t)) - F(X(s))\|_{L^{2p}(\Omega; H)} \leq \left\{ \begin{array}{ll} C, & \gamma \in [1, \delta_0], \\ C|t - s|^\frac{1}{2}, & \gamma \in (\delta_0, 2]. \end{array} \right.$$ (3.26)

Therefore, using (3.8) with $\mu = \gamma$, $\nu = 0$ and also (3.26) give

$$I_3 \leq C h^\gamma \int_0^t (t - s)^{-\frac{1}{2}} \|F(X(t)) - F(X(s))\|_{L^{2p}(\Omega; H)} \, ds \leq C h^\gamma.$$ (3.27)

Now it remains to bound $I_4$. Combining the Burkholder-Davis-Gundy type inequality and (3.10) with $q = \gamma - 1$ results in

$$I_4 \leq C_p \left( \int_0^t \|(S(t) - S_h(t-s))P_hQ_t\|_{L^2}^2 \, ds \right)^{\frac{1}{2}} \leq C h^\gamma \|A^{-\frac{1}{2}}Q_t\|_{L^2}.$$ (3.28)

Finally, putting the above estimates together gives

$$\|X(t) - \tilde{X}_h(t)\|_{L^{2p}(\Omega; H)} \leq C h^\gamma.$$ (3.29)

Next we turn our attention to the error $\tilde{e}(t) := \tilde{X}_h(t) - X_h(t)$, which is time differentiable and

$$\frac{d}{dt}\tilde{e}(t) + A_h\tilde{e}(t) = P_h(F(X(t)) - F(X_h(t))), \quad t \in (0, T], \quad \tilde{e}(0) = 0.$$ (3.30)

Note that Lemma 3.4 and (3.22) guarantee that $\sup_{s \in [0, T]} E[\|\tilde{e}(s)\|^2 p] + \int_0^T E[\|\nabla \tilde{e}(s)\|^2] \, ds < \infty$. Multiplying both sides of (3.30) by $\tilde{e}(t)$, and applying (2.7), (2.19), (3.2) and (3.1) tell us

$$\frac{1}{2} \frac{d}{dt}\|\tilde{e}(s)\|^2 + \langle \nabla \tilde{e}(s), \nabla \tilde{e}(s) \rangle = \langle F(\tilde{X}_h(s)) - F(X_h(s)), \tilde{e}(s) \rangle + \langle F(X(s)) - F(\tilde{X}_h(s)), \tilde{e}(s) \rangle$$

$$\leq C\|\tilde{e}(s)\|^2 + \|A^{-\frac{1}{2}}(F(X(s)) - F(\tilde{X}_h(s)))\| \|\nabla \tilde{e}(s)\|$$

$$\leq C\|\tilde{e}(s)\|^2 + \frac{1}{2}\|F(X(s)) - F(\tilde{X}_h(s))\|_{L^2}^2 + \frac{1}{2}\|\nabla \tilde{e}(s)\|^2.$$ (3.31)

Then integrating over $[0, t]$ and using Hölder’s inequality suggest that

$$\|\tilde{e}(t)\|^2 \leq C \int_0^t \|\tilde{e}(s)\|^2 \, ds + C \int_0^t \|F(X(s)) - F(\tilde{X}_h(s))\|_{L^p}^2 \, ds$$

$$\leq C \int_0^t \|\tilde{e}(s)\|^2 \, ds + C \int_0^t \|X(s) - \tilde{X}_h(s)\|^2 (1 + \|X(s)\|_{L^6}^4 + \|\tilde{X}_h(s)\|_{L^6}^4) \, ds.$$ (3.32)

Using Gronwall’s inequality before employing (3.22), (3.29) and Theorem 2.5, one can arrive at

$$\|\tilde{e}(t)\|_{L^{2p}(\Omega; H)}^2 \leq C \int_0^t \|X(s) - \tilde{X}_h(s)\|_{L^{2p}(\Omega; H)}^2 (1 + \|X(s)\|_{L^{8p}(\Omega; L^6)}^4 + \|\tilde{X}_h(s)\|_{L^{8p}(\Omega; L^6)}^4) \, ds$$

$$\leq C h^{2\gamma},$$ (3.33)

which in a combination with (3.29) shows (3.6), as required. □
4 Error estimates of the spatio-temporal full discretization

In the present section, we proceed to study a full discretization based on the finite element semidiscretization. Let \( \tau := T/M, M \in \mathbb{N}^+ \) be a uniform time-step size and write \( t_m = m\tau \), for \( m \in \{1, 2 \cdots, M\} \). We discrete \([3,4]\) in time with a backward Euler scheme and the resulting full-discrete problem is to find \( F_{t_m} \)-adapted random variables \( X_{h,m}, m \in \{1, 2 \cdots, M\} \) such that,

\[
X_{h,m} = S_{\tau,h}X_{h,m-1} + \tau S_{\tau,h}P_hF(X_{h,m}) + S_{\tau,h}P_h\Delta W_m, \quad X_{h,0} = P_hX_0, \quad m \in \{1, 2 \cdots, M\},
\]

where we write \( \Delta W_m := W(t_m) - W(t_{m-1}) \), \( S_{\tau,h} := (I + \tau A_h)^{-1} \) for brevity. Equivalently,

\[
X_{h,m} = S_{\tau,h}X_{h,0} + \tau \sum_{i=0}^{m-1} S_{\tau,h}^{m-i}P_hF(X_{h,i+1}) + W_{A_h}^m, \quad \text{with} \quad W_{A_h}^m := \sum_{i=0}^{m-1} S_{\tau,h}^{m-i}P_h\Delta W_{i+1}.
\]

**Theorem 4.1** Let \( X(t) \) be the mild solutions of (1.1) and let \( X_{h,m} \) be produced by (4.1). If Assumptions 2.1, 2.4 are valid, then it holds that

\[
\|X(t_m) - X_{h,m}\|_{L^2(p;\Omega;H)} \leq C(h^\gamma + \tau^{\frac{\rho}{2}}), \quad \gamma \in [1, 2]. \tag{4.3}
\]

Its proof is also postponed. Define the full-discrete approximation operators \( \Psi_{\tau,h}(t), t \in [0, T] \) as

\[
\Psi_{\tau,h}(t) = S(t) - S_{\tau,h}P_h, \quad \forall t \in [t_{m-1}, t_m), \; m \in \{1, 2, \ldots, M\}. \tag{4.4}
\]

The forthcoming two lemmas, coming from [20, Lemmas 4.3, 4.4], are a temporal version of Lemmas 3.2, 3.3 and play a significant role in the error estimates of the full-discrete approximation.

**Lemma 4.2** Under Assumption 2.1, the following estimates hold.

(i) For \( 0 \leq \nu \leq \mu \leq 2 \), it holds that

\[
\|\Psi_{\tau,h}(t)x\| \leq C(h^\mu + \tau^{\frac{\mu}{2}})t^{-\frac{\mu-\nu}{2}}\|x\|_\nu, \quad \text{for all} \quad x \in \dot{H}^\nu. \tag{4.5}
\]

(ii) For \( 0 \leq \rho \leq 1 \), it holds that

\[
\left\|\int_0^t \Psi_{\tau,h}(s)x \, ds\right\| \leq C(h^{2-\rho} + \tau^{\frac{2-\rho}{2}})\|x\|_{-\rho}, \quad \text{for all} \quad x \in \dot{H}^{-\rho}. \tag{4.6}
\]

(iii) For \( 0 \leq \varrho \leq 1 \), it holds that

\[
\left(\int_0^t \|\Psi_{\tau,h}(s)x\|^2 \, ds\right)^{\frac{1}{2}} \leq C(h^{1+\varrho} + \tau^{\frac{1+\varrho}{2}})\|x\|_0, \quad \text{for all} \quad x \in \dot{H}^{\varrho}. \tag{4.7}
\]

**Lemma 4.3** Under Assumption 2.1, the following estimates for \( S_{\tau,h}^m \) hold, for any \( x \in H \)

\[
\|A_{\tau,h}^\frac{\mu}{2}S_{\tau,h}^mP_hx\| \leq Ct_m^{-\frac{\mu}{2}}\|x\|, \quad \mu \in [0, 1], \tag{4.8}
\]

\[
\tau \sum_{i=1}^m \|A_{\tau,h}^\frac{1}{2}S_{\tau,h}^iP_hx\|^2 \leq C\|x\|^2. \tag{4.9}
\]
Lemma 4.4  Suppose Assumptions 2.1-2.4 hold. Then there exists a constant $C$ such that

$$
\sup_{M \in \mathbb{N}} \sup_{m \in \{1, 2, \ldots, M\}} \left( E \left[ \|X_{h,m}\|^{2p} \right] + \tau \sum_{i=1}^{M} E[\|\nabla X_{h,i}\|^2] \right) < \infty. \tag{4.10}
$$

Proof of Lemma 4.4. Analogously to the proof of Lemma 3.4, we define

$$
Y_{h,m} := X_{h,m} - W_{A_h}^m = S_{\tau,h}^m Y_{h,0} + \tau \sum_{i=0}^{m-1} S_{\tau,h}^{m-i} P_h F(Y_{h,i+1} + W_{A_h}^{i+1}), \ m \in \{1, 2, \ldots, M\}. \tag{4.11}
$$

It is evident to verify that $Y_{h,m}$ satisfies

$$
\frac{Y_{h,m} - Y_{h,m-1}}{\tau} + A_h Y_{h,m} = P_h F(Y_{h,m} + W_{A_h}^m), \ Y_{h,0} = P_h X_0, \ m \in \{1, 2, \ldots, M\}. \tag{4.12}
$$

Multiplying this equation by $Y_{h,m}$ and using (3.1) imply

$$
\langle Y_{h,m} - Y_{h,m-1}, Y_{h,m} \rangle + \tau \langle \nabla Y_{h,m}, \nabla Y_{h,m} \rangle = \tau \langle F(Y_{h,m} + W_{A_h}^m) - F(W_{A_h}^m), Y_{h,m} \rangle + \tau \langle F(W_{A_h}^m), Y_{h,m} \rangle \tag{4.13}
$$

$$\leq C \tau \|Y_{h,m}\|^2 + \frac{\tau}{2} \|F(W_{A_h}^m)\|^2. $$

Further, using the fact $\frac{1}{2}(\|Y_{h,m}\|^2 - \|Y_{h,m-1}\|^2) \leq \langle Y_{h,m} - Y_{h,m-1}, Y_{h,m} \rangle$ and summation on $m$ shows

$$\frac{1}{2} \|Y_{h,m}\|^2 + \tau \sum_{i=1}^{m} \|\nabla Y_{h,i}\|^2 \leq \frac{1}{2} \|Y_{h,0}\|^2 + C \tau \sum_{i=1}^{m} \|Y_{h,i}\|^2 + \frac{\tau}{2} \sum_{i=1}^{m} \|F(W_{A_h}^i)\|^2. \tag{4.14}
$$

By virtue of the Gronwall inequality, we infer that

$$\|Y_{h,m}\|^2 + \tau \sum_{i=1}^{m} \|\nabla Y_{h,i}\|^2 \leq C \|Y_{h,0}\|^2 + C \tau \sum_{i=1}^{m} \|F(W_{A_h}^i)\|^2. \tag{4.15}
$$

Let $S_{\tau,h}(t) = S_{\tau,h}^i$, for $t \in \{t_i - 1, t_i\}$ and by $\chi_B$ we denote the characteristic function of a set $B \subset \mathbb{R}$. Then $W_{A_h}^m$ can be reformulated as $W_{A_h}^m = \int_{0}^{T} \chi_{[0,t_m]}(s) S_{\tau,h}(t_m - s) P_h dW(s)$. As in (3.18), employing (2.17), (3.3), (4.9) and Burkholder-Davis-Gundy-type inequality helps us to deduce

$$\|F(W_{A_h}^m)\|_{L^{2p}(\Omega; H)} \leq C \left(1 + \|W_{A_h}^m\|_{L^{6p}(\Omega; L_6)}^3 \right) \tag{4.16}
$$

$$\leq C \left(1 + \left( \int_{0}^{T} \|\chi_{[0,t_m]}(s) A_h^{1/2} S_{\tau,h}(t_m - s) P_h Q_{\tau}^{1/2}\|_{L_2}^2 ds \right)^{3/2} \right) \tag{4.16}
$$

$$\leq C \left(1 + \left( \tau \sum_{i=0}^{m-1} \|A_h^{1/2} S_{\tau,h}^{m-i} P_h Q_{\tau}^{1/2}\|_{L_2}^2 \right)^{3/2} \right) \leq C \left(1 + \|Q_{\tau}^{1/2}\|_{L_2}^3 \right) < \infty,$$

and

$$\|W_{A_h}^m\|_{L^{2p}(\Omega; H)} \leq C \|Q_{\tau}^{1/2}\|_{L_2} < \infty. \tag{4.17}$$
for any \( m \in \{1, 2, \cdots, M\} \). This together with the fact \( X_{h,m} = Y_{h,m} + W_{Ah}^m \) shows (4.10). □

We start to prove Theorem 4.1 rigorously.

**Proof of Theorem 4.1.** Similarly to the semi-discrete case, by introducing the auxiliary problem,

\[
\tilde{X}_{h,m} - \tilde{X}_{h,m-1} + \tau A_h \tilde{X}_{h,m} = \tau P_h F(X(t_m)) + P_h \Delta W_m, \quad \tilde{X}_{h,0} = P_h X_0,
\]

(4.18)

whose solution can be recasted as

\[
\tilde{X}_{h,m} = S_{\tau,h}^m P_h X_0 + \tau \sum_{i=0}^{m-1} S_{\tau,h}^{m-i} P_h F(X(t_{i+1})) + W_{Ah}^m,
\]

(4.19)

we decompose the considered error term \( \|X(t_m) - X_{h,m}\|_{L^2(\Omega; H)} \) into two parts:

\[
\|X(t_m) - X_{h,m}\|_{L^2(\Omega; H)} \leq \|X(t_m) - \tilde{X}_{h,m}\|_{L^2(\Omega; H)} + \|\tilde{X}_{h,m} - X_{h,m}\|_{L^2(\Omega; H)}.
\]

(4.20)

Resorting to (2.12), (2.17), (3.3), (4.8), (4.9) and (4.17), one can infer that, for any \( m \in \{1, 2, \cdots, M\} \),

\[
\|\tilde{X}_{h,m}\|_{L^2(\Omega; L_0)} \leq \|\tilde{X}_{h,m}\|_{L^2(\Omega; H^1)}
\]

\[
\leq \|S_{\tau,h}^m P_h X_0\|_{L^2(\Omega; H^1)} + \tau \sum_{i=0}^{m-1} \|S_{\tau,h}^{m-i} P_h F(X(t_{i+1}))\|_{L^2(\Omega; H)} + \|W_{Ah}^m\|_{L^2(\Omega; H^1)}
\]

\[
\leq C\|X_0\|_{L^2(\Omega; H^1)} + C \sup_{s \in [0,T]} \|F(X(s))\|_{L^2(\Omega; H)} \tau \sum_{i=1}^{m} \frac{1}{m} + C\|Q_{1/2}\|_{L_2} < \infty.
\]

(4.21)

As the first step, we aim to bound the error \( \|X(t_m) - \tilde{X}_{h,m}\|_{L^2(\Omega; H)} \). Subtracting (4.19) from (1.3), the error \( X(t_m) - \tilde{X}_{h,m} \) can be splitted into the following three terms:

\[
\|X(t_m) - \tilde{X}_{h,m}\|_{L^2(\Omega; H)} = \|(S(t_m) - S_{\tau,h}^m P_h)X_0\|_{L^2(\Omega; H)}
\]

\[
+ \left\| \int_0^{t_m} S(t_m - s) F(X(s)) \, ds - \tau \sum_{i=0}^{m-1} S_{\tau,h}^{m-i} P_h F(X(t_{i+1})) \right\|_{L^2(\Omega; H)}
\]

\[
+ \left\| \int_0^{t_m} S(t_m - s) \, dW(s) - \sum_{i=0}^{m-1} S_{\tau,h}^{m-i} P_h \Delta W_{i+1} \right\|_{L^2(\Omega; H)}
\]

\[
:= J_1 + J_2 + J_3.
\]

(4.22)

In the same manner as (3.21), the first term \( J_1 \) can be estimated with the aid of (1.5),

\[
J_1 \leq C(h^\gamma + \tau^{2/3})\|X_0\|_{L^2(\Omega; H^\gamma)}.
\]

(4.23)

To treat the term \( J_2 \), we decompose it into two terms as follows:

\[
J_2 \leq \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} S(t_m - s)(F(X(s)) - F(X(t_{i+1})) \, ds \right\|_{L^2(\Omega; H)}
\]

\[
+ \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (S(t_m - s) - S_{\tau,h}^{m-i} P_h) F(X(t_{i+1})) \, ds \right\|_{L^2(\Omega; H)}
\]

\[
:= J_{21} + J_{22}.
\]

(4.24)
Since the term $J_{22}$ is easy, we treat it first. Performing standard variable transformations $t_m - s = \sigma$, $m - i = j$ and using $2.12$, $3.26$, $4.5$ and $4.6$ yield

$$J_{22} = \left\| \sum_{j=1}^{m} \int_{t_{j-1}}^{t_{j}} (S(\sigma) - S_{\tau,h}^{\sigma}P_h)F(X(t_{m-j}+1)) \, d\sigma \right\|_{L^{2p}(\Omega;H)}$$

$$\leq \left\| \int_{0}^{t_m} \Psi_{\tau,h}(\sigma)F(X(t_m)) \, d\sigma \right\|_{L^{2p}(\Omega;H)} + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_{j}} \left\| \Psi_{\tau,h}(\sigma) (F(X(t_{m-j}+1)) - F(X(t_m))) \right\|_{L^{2p}(\Omega;H)} \, d\sigma$$

$$\leq C(h^2 + \tau)\|F(X(t_m))\|_{L^{2p}(\Omega;H)} + C \sum_{j=1}^{m} \int_{t_{j-1}}^{t_{j}} (h^{\gamma} + \tau^{\frac{\gamma}{2}})\, d\sigma$$

$$\leq C(h^2 + \tau) \sup_{\sigma \in [0,T]} \|F(X(\sigma))\|_{L^{2p}(\Omega;H)} + C \sum_{j=1}^{m} \int_{t_{j-1}}^{t_{j}} (h^{\gamma} + \tau^{\frac{\gamma}{2}})\, d\sigma$$

$$\leq C(h^2 + \tau^{\frac{\gamma}{2}}), \quad (4.25)$$

where for any fixed number $\delta_0 \in (\frac{3}{2}, 2)$, $\alpha_\gamma = 0$ for $\gamma \in [1, \delta_0]$ and $\alpha_\gamma = \frac{1}{2}$ for $\gamma \in (\delta_0, 2]$ by $3.26$. In the next step, we start the estimate of $J_{21}$. Noting that, for $s \in [t_i, t_{i+1})$

$$X(t_{i+1}) = S(t_{i+1} - s)X(s) + \int_{s}^{t_{i+1}} S(t_{i+1} - \sigma)F(X(\sigma)) \, d\sigma + \int_{s}^{t_{i+1}} S(t_{i+1} - \sigma) \, dW(\sigma), \quad (4.26)$$

and thus using the Taylor formula help us to split $J_{21}$ into four terms:

$$J_{21} \leq \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} S(t_m - s)F'(X(s))(S(t_{i+1} - s) - I)X(s) \, ds \right\|_{L^{2p}(\Omega;H)}$$

$$+ \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} S(t_m - s)F'(X(s)) \int_{s}^{t_{i+1}} S(t_{i+1} - \sigma)F(X(\sigma)) \, d\sigma \, ds \right\|_{L^{2p}(\Omega;H)}$$

$$+ \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} S(t_m - s)F'(X(s)) \int_{s}^{t_{i+1}} S(t_{i+1} - \sigma) \, dW(\sigma) \, ds \right\|_{L^{2p}(\Omega;H)}$$

$$+ \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} S(t_m - s)R_F(X(s), X(t_{i+1})) \, ds \right\|_{L^{2p}(\Omega;H)}$$

$$:= J_{21}^1 + J_{21}^2 + J_{21}^3 + J_{21}^4.$$ Here the remainder term $R_F$ reads,

$$R_F(X(s), X(t_{i+1}))$$

$$:= \int_{0}^{t_i} F''(X(s) + \lambda(X(t_{i+1}) - X(s)))(X(t_{i+1}) - X(s), X(t_{i+1}) - X(s))(1 - \lambda) \, d\lambda. \quad (4.28)$$
In the sequel we treat the above four terms one by one. Thanks to (2.4), (2.18), (2.14) and (2.13), we derive, for $\gamma \in [1, 2]$ and any fixed $\delta_0 \in (\frac{1}{2}, 2)$,

$$J_{21}^1 \leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} \left\| A^{-\frac{\delta_0}{2}} F'(X(s)) (S(t_{i+1} - s) - I) X(s) \right\|_{L^{2p}(\Omega; H)} ds$$

$$\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} \left\| F'(X(s)) (S(t_{i+1} - s) - I) X(s) \right\|_{L^{2p}(\Omega; L^1)} ds$$

$$\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} \left\| f'(X(s)) \right\|_{L^{4p}(\Omega; H)} \left\| (S(t_{i+1} - s) - I) X(s) \right\|_{L^{4p}(\Omega; H)} ds$$

$$\leq C T^2 \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} ds \sup_{s \in [0, T]} \left\| f'(X(s)) \right\|_{L^{4p}(\Omega; H)} \sup_{s \in [0, T]} \left\| X(s) \right\|_{L^{4p}(\Omega; H^\gamma)}$$

$$\leq C T^2.$$

For the second term $J_{21}^2$, using (2.4), (2.18) and (2.13) implies that, for any fixed $\delta_0 \in (\frac{1}{2}, 2)$

$$J_{21}^2 \leq \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_t^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} \left\| A^{-\frac{\delta_0}{2}} F'(X(s)) S(t_{i+1} - \sigma) F(X(\sigma)) \right\|_{L^{2p}(\Omega; H)} d\sigma ds$$

$$\leq \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_t^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} \left\| F'(X(s)) S(t_{i+1} - \sigma) F(X(\sigma)) \right\|_{L^{2p}(\Omega; L^1)} d\sigma ds$$

$$\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_t^{t_{i+1}} (t_m - s)^{-\frac{\delta_0}{2}} \left\| f'(X(s)) \right\|_{L^{4p}(\Omega; H)} \left\| F(X(\sigma)) \right\|_{L^{4p}(\Omega; H)} d\sigma ds$$

$$\leq C T \int_0^{t_m} (t_m - s)^{-\frac{\delta_0}{2}} ds \sup_{s \in [0, T]} \left\| f'(X(s)) \right\|_{L^{4p}(\Omega; H)} \sup_{s \in [0, T]} \left\| F(X(s)) \right\|_{L^{4p}(\Omega; H)}$$

$$\leq C T.$$

To estimate $J_{21}^3$, we first apply the stochastic Fubini theorem (e.g. see [8, Theorem 4.18]) and the Burkholder-Davis-Gundy-type inequality to obtain

$$J_{21}^3 = \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \chi_{[s, t_{i+1})}(\sigma) S(t_m - s) F'(X(s)) S(t_{i+1} - \sigma) dW(\sigma) ds \right\|_{L^{2p}(\Omega; H)}$$

$$= \left\| \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \chi_{[s, t_{i+1})}(\sigma) S(t_m - s) F'(X(s)) S(t_{i+1} - \sigma) ds dW(\sigma) \right\|_{L^{2p}(\Omega; H)}$$

$$\leq C \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} S(t_m - s) F'(X(s)) \chi_{[s, t_{i+1})}(\sigma) S(t_{i+1} - \sigma) Q^{\frac{1}{2}} ds \right)^{\frac{1}{2}}.$$
Further, we employ the Hölder inequality, \( (2.13) \) and \( (2.22) \) with \( \rho = 1 \) to get

\[
J_{21}^3 \leq C \tau^\frac{3}{2} \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( \sum_{j=1}^{m-1} \| S(t_m - s) F'(X(s)) S(t_{i+1} - \sigma) Q_j^2 \eta_j \|_{L^2 (\Omega; H)}^2 \right)^{\frac{1}{2}} \right)
\]

\[
\leq C \tau^\frac{3}{2} \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \| f'(X(s)) \|_{L^2 (\Omega; L_3)}^2 \right) \left( \sum_{j=1}^{m-1} \int_{t_i}^{t_{i+1}} \| S(t_{i+1} - \sigma) Q_j^2 \eta_j \|_{L_6}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \tau^\frac{3}{2} \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( \sup_{s \in [0,T]} \| f'(X(s)) \|_{L^2 (\Omega; L_3)} \right)^2 \right) \int_{t_i}^{t_{i+1}} \| A^\frac{1}{2} S(t_{i+1} - \sigma) Q_j^2 \|_{L_2}^2 \ ds \right)^{\frac{1}{2}}
\]

\[
\leq C \tau \left( \sup_{s \in [0,T]} \| f'(X(s)) \|_{L^2 (\Omega; L_3)} \cdot \| Q_j^2 \|_{L_2} \right)^{\frac{1}{2}}
\]

\[
\leq C \tau,
\]

where \( \eta_j, j \in \mathbb{N} \) is any ON-basis of \( H \). Now we are in a position to bound the term \( J_{21}^4 \). Owing to \( (2.14) \) with \( \nu = \delta_0 \in (\frac{3}{2}, 2) \), and applying \( (2.15) \) and \( (2.13) \), we learn that

\[
J_{21}^4 \leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( t_m - s \right)^{-\delta_0} \| A^{-\delta_0} R_F(X(s), X(t_{i+1})) \|_{L^2 (\Omega; H)} \ ds
\]

\[
\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( t_m - s \right)^{-\delta_0} \| R_F(X(s), X(t_{i+1})) \|_{L^2 (\Omega; L_1)} \ ds
\]

\[
\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( t_m - s \right)^{-\delta_0} \| X(t_{i+1}) - X(s) \|
\times \| f''(1 - \lambda) X(s) + \lambda X(t_{i+1}) \|_{L_4} \| X(t_{i+1}) - X(s) \|_{L^{2p} (\Omega; H)} \ ds
\]

\[
\leq C \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( t_m - s \right)^{-\delta_0} \| X(t_{i+1}) - X(s) \|_{L^p (\Omega; H)} \| X(t_{i+1}) - X(s) \|_{L^{2p} (\Omega; H)} \ ds
\]

\[
\times \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( t_m - s \right)^{-\delta_0} \ ds \right)^{\frac{1}{2}}
\]

\[
\leq C \tau^\frac{7}{2} \left( \sup_{s \in [0,T]} \| f''(X(s)) \|_{L^{4p} (\Omega; L_4)} \right) \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \left( t_m - s \right)^{-\delta_0} \ ds
\]

\[
\leq C \tau^\frac{7}{2}.
\]

Putting the above four estimates together results in

\[
J_{21} \leq C \tau^\frac{7}{2},
\]

which together with \( (4.24) \) and \( (4.25) \) shows

\[
J_2 \leq C (h^\gamma + \tau^\frac{7}{2}).
\]
Concerning the term $J_3$, \[ (4.7) \] (2.8) and the Burkholder-Davis-Gundy type inequality show
\[
J_3 = \left\| \int_0^{t_m} \Psi_{\tau,h}(t_m - s) \, dW(s) \right\|_{L^{2p}(\Omega;H)} 
\leq C_p \left( \int_0^{t_m} \left\| \Psi_{\tau,h}(t_m - s) Q^{\frac{1}{2}} \right\|_{L^2}^2 \, ds \right)^{\frac{1}{2}} 
= C_p \left( \int_0^{t_m} \left\| \Psi_{\tau,h}(s) Q^{\frac{1}{2}} \right\|_{L^2}^2 \, ds \right)^{\frac{1}{2}} 
\leq C(h^\gamma + \tau^\frac{3}{4}) Q^{\frac{1}{2}} \|A^{-\frac{1}{2}}\|_{L^2}. \quad (4.36)
\]
Gathering the above three estimates together implies
\[
\|X(t_m) - \tilde{X}_{h,m}\|_{L^{2p}(\Omega;H)} \leq C(h^\gamma + \tau^\frac{3}{4}). \quad (4.37)
\]
Next we turn our attention to the estimate of $\tilde{e}_m := \tilde{X}_{h,m} - X_{h,m}$, which obeys
\[
\frac{\tilde{e}_m - \tilde{e}_{m-1}}{\tau} + A_h \tilde{e}_m = P_h(F(X(t_m)) - F(X_{h,m})), \quad \tilde{e}_0 = 0. \quad (4.38)
\]
By multiplying this equation by $\tilde{e}_m$, one can observe
\[
\frac{1}{2}(\|\tilde{e}_m\|^2 - \|\tilde{e}_{m-1}\|^2) + \tau \langle \nabla \tilde{e}_m, \nabla \tilde{e}_m \rangle 
\leq \tau \langle F(\tilde{X}_{h,m}) - F(X_{h,m}), \tilde{e}_m \rangle + \tau \langle F(X(t_m)) - F(\tilde{X}_{h,m}), \tilde{e}_m \rangle. \quad (4.39)
\]
Here we also used the definition of $A_h$ in \[ (3.1) \] and the fact $\frac{1}{2}(\|\tilde{e}_m\|^2 - \|\tilde{e}_{m-1}\|^2) \leq \langle \tilde{e}_m - \tilde{e}_{m-1}, \tilde{e}_m \rangle$. Thanks to \[ (2.7) \] and \[ (2.19) \],
\[
\frac{1}{2}(\|\tilde{e}_m\|^2 - \|\tilde{e}_{m-1}\|^2) + \tau \langle \nabla \tilde{e}_m, \nabla \tilde{e}_m \rangle 
\leq \tau \|\tilde{e}_m\|^2 + \tau \|A^{-\frac{1}{2}}(F(X(t_m)) - F(\tilde{X}_{h,m}))\| \|\nabla \tilde{e}_m\| 
\leq \tau \|\tilde{e}_m\|^2 + \tau \|F(X(t_m)) - F(\tilde{X}_{h,m})\|_L^2 + \frac{\tau}{2} \|\nabla \tilde{e}_m\|^2 
\leq \tau \|\tilde{e}_m\|^2 + C\tau \|X(t_m) - \tilde{X}_{h,m}\|^2 (1 + \|X(t_m)\|_{L^6}^4 + \|\tilde{X}_{h,m}\|_{L^6}^4) + \frac{\tau}{2} \|\nabla \tilde{e}_m\|^2. \quad (4.40)
\]
Since Lemma \[ 4.3 \] and \[ 4.21 \] ensure $\mathbb{E}[\|\tilde{e}_m\|_{L^p}^p] + \tau \sum_{i=1}^m \mathbb{E}[\|\nabla \tilde{e}_i\|^2] < \infty$, by summation on $m$ and calling the Gronwall inequality and the fact $\tilde{e}_0 = 0$, it holds
\[
\|\tilde{e}_m\|^2 \leq C\tau \sum_{i=1}^m \|X(t_i) - \tilde{X}_{h,i}\|^2 (1 + \|X(t_i)\|_{L^6}^4 + \|\tilde{X}_{h,i}\|_{L^6}^4). \quad (4.41)
\]
Therefore,
\[
\|\tilde{e}_m\|_{L^{2p}(\Omega;H)} \leq C\tau \sum_{i=1}^m \|X(t_i) - \tilde{X}_{h,i}\|_{L^{4p}(\Omega;H)} (1 + \|X(t_i)\|_{L^{8p}(\Omega;L^6)}^2 + \|\tilde{X}_{h,i}\|_{L^{8p}(\Omega;L^6)}^2) \quad (4.42)
\]
which together with \[ 4.37 \] shows \[ 4.3 \] and thus finishes the proof. \[ \square \]
5 Numerical experiments

In this section, some numerical examples are included to visually illustrate the previous findings. To this end, we consider the following stochastic Allen-Cahn equation in one space dimension

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + u - u^3 + \dot{W} & \quad & t \in (0,1), \ x \in (0,1), \\
\frac{\partial u}{\partial t} &= \sin(\pi x), & x \in (0,1), \\
\frac{\partial u}{\partial t} &= u(t, 0), & t \in (0,1).
\end{aligned}
\]

(5.1)

Here \(\{W(t)\}_{t \in [0,1]}\) stands for a standard \(Q\)-Wiener process, with two simple choices of covariance operators \(Q = A^{-s}, s \in \{0.5005, 1.5005\}\). One can easily see that Assumption 2.3 is fulfilled with \(\gamma = 1\) for \(Q = A^{-0.5005}\) and \(\gamma = 2\) for \(Q = A^{-1.5005}\). According to Theorem 3.1 and Theorem 4.1, the mean-square (MS, \(p = 1\)) convergence rate in space reads \(O(h^\gamma)\) and the rate in time \(O(\tau^{\frac{3}{2}})\) for \(\gamma \in \{1, 2\}\). Since the exact solution is not available, we turn to fine numerical approximations for reference, using very small step-sizes \(h_{\text{exact}}\) and \(\tau_{\text{exact}}\). Also, error bounds are always measured in terms of mean-square discretization errors at the endpoint \(T = 1\) and the expectations are approximated by computing averages over 500 samples.

In Figure 1, one-path simulations with \(h = \tau = 2^{-8}\) are plotted. There one can observe that the numerical solution behaves more smoothly as the noise becomes more smoother. To test the convergence rate in space, we perform numerical simulations with four different space step-sizes \(h = 2^{-i}, i \in \{2, 3, 4, 5\}\). The ”true solutions” are computed using \(h_{\text{exact}} = 2^{-7}, \tau_{\text{exact}} = 2^{-15}\). In Figure 2, we depict the spatial errors against space step-sizes and one can detect the expected convergence rates in space, i.e., order 1 for \(Q = A^{-0.5005}\) and order 2 for \(Q = A^{-1.5005}\). Lastly, we test the convergence rate in time and take \(h_{\text{exact}} = 2^{-7}\) and \(\tau_{\text{exact}} = 2^{-12}\). Similarly, we do numerical approximations with four different time step-sizes \(\tau = 2^{-j}, j \in \{3, 4, 5, 6\}\) and present the resulting errors in Figure 3. Clearly, temporal approximation errors decrease at a slope close to \(\frac{1}{2}\) and 1 for the above two kinds of noises. This is consistent with previous theoretical results.

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Figure 2: MS convergence rates for spatial discretizations (Left: $Q = A^{-0.5005}$; right: $Q = A^{-1.5005}$)

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