A simplex-type algorithm for continuous linear programs with constant coefficients.

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Abstract

We consider continuous linear programs over a continuous finite time horizon \( T \), with a constant coefficient matrix, linear right hand side functions and linear cost coefficient functions, where we search for optimal solutions in the space of measures or of functions of bounded variation. These models generalize the separated continuous linear programming models and their various duals, as formulated in the past by Anderson, by Pullan, and by Weiss. In previous papers we have shown that these problems possess optimal strongly dual solutions. We also have presented a detailed description of optimal solutions and have defined a combinatorial analogue to basic solutions of standard LP. In this paper we present an algorithm which solves this class of problems in a finite bounded number of steps, using an analogue of the simplex method, in the space of measures.

1 Introduction

This paper presents a finite, exact, simplex-type algorithm for the solution of the pair of dual continuous linear programs of the form

\[
\begin{align*}
\text{max} & \quad \mu^T U(0) + \int_0^T (\gamma + (T - t)c)^T dU(t) \\
\text{M-CLP} & \quad \text{s.t.} \quad A U(t) + x(t) = \beta + bt, \quad 0 \leq t < T, \\
& \quad \quad \quad \quad A U(T) + x(T) = \beta + bT + \lambda, \\
& \quad \quad \quad \quad U(0-) = 0, \quad U(t) \text{ non-decreasing and right continuous on } [0,T], \quad x(t) \geq 0.
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad \lambda^T P(0) + \int_0^T (\beta + (T - t)b)^T dP(t) \\
\text{M-CLP*} & \quad \text{s.t.} \quad A^T P(t) - q(t) = \gamma + ct, \quad 0 \leq t < T, \\
& \quad \quad \quad \quad A^T P(T) - q(T) = \gamma + cT + \mu, \\
& \quad \quad \quad \quad P(0-) = 0, \quad P(t) \text{ non-decreasing and right continuous on } [0,T], \quad q(t) \geq 0.
\end{align*}
\]

Here \( A \) is a \( K \times J \) constant matrix, \( \beta, b, \gamma, c, \lambda \leq 0, \mu \geq 0 \) are constant vectors of corresponding dimensions, and the integrals are Lebesgue-Stieltjes. The unknowns are vectors of cumulative

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control functions $U, P$ and vectors of non-negative slack or state functions $x, q$, over the time horizon $[0, T]$. It is convenient to think of dual time as running backwards, so that $P(T - t)$ is the vector of dual variables that correspond to the constraints of (1) at time $t$, and $U(t)$ correspond to the constraints of (2) at time $T - t$.

These problems are special cases of continuous linear programs (CLP) formulated by Bellman in 1953 [4], and further discussed and investigated in [1, 2, 3, 6, 8, 10, 11, 12, 13, 14, 15, 19, 20, 23], and they generalize some of the separated continuous programs (SCLP) discussed by Anderson [1], Pullan [10] and Weiss [23].

This paper is based on [17, 18]. In [17] we showed that under feasibility and a Slater type condition, optimal strongly dual solutions exist. We also showed that there exist optimal solutions $U(t), P(t)$ that have jumps only at 0 and $T$, and are absolutely continuous in $(0, T)$, with densities $u(t), p(t)$ which are piecewise constant. In particular, $u(t), p(t), 0 < t < T$ are unique under the following non-degeneracy assumption.

**Assumption 1.1.** The vector $b$ is in general position to the matrix $[AI]$ (it is not a linear combination of any less than $K$ columns), and the vector $c$ is in general position to the matrix $[A^T I]$.

Next, in [18] we studied the form of optimal solutions. We found that when M-CLP/M-CLP* are feasible, they always have extreme point solutions which are characterized by a ‘base sequence’ $(\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}$ where $N$ is the number of pieces in the piecewise constant solution $u(t), p(t)$, and $\mathcal{K}_n, \mathcal{J}_n$ are subsets of $\{1, \ldots, K\}, \{1, \ldots, J\}$ respectively. These so called base sequences play a role analogue to basic solutions in standard linear programs (LP). We also showed that the Slater type condition is not necessary, and strong duality holds always.

In the current paper we consider a parametric formulation, where we fix $A, b, c$ and assume Assumption 1.1, and we consider the quantities $\rho = (\beta, \gamma, T, \lambda, \mu)$ as parameters. It is shown in [18] that with a proper base sequence there is associated a set of boundary parameters for which this base sequence is optimal; this is defined as the validity region of the base sequence.

Our algorithm solves a given M-CLP/M-CLP* as follows: Starting from an optimal solution for some initial boundary parameters $\rho_0$, it moves along a straight line in the parameter space, towards the optimal solution for the desired boundary parameters $\rho$, similar to the parametric self dual simplex algorithm (or Lemke’s algorithm) for standard LP (cf. [7, 21]). The parametric line passes through a finite number of validity regions. On the boundary of each validity region an M-CLP pivot is performed to obtain the optimal base sequence for the next validity region.

The paper is structured as follows. In the first part of the paper we extend some of the results of [18]: we give a succinct description of the solution (Section 2), then describe a decomposition of M-CLP and its dual to an Internal-SCLP problem and a pair of Boundary-LP problem (Section 3), and follow with a discussion of validity regions (Section 4).

To construct the algorithm, in Section 5 we discuss what happens on the boundaries of validity regions. When $\rho$ approaches the boundary of a validity region, at the boundary some of the values of the impulse controls, state variables or interval lengths which are positive inside the region reach 0, and the base sequence is no longer optimal outside the boundary, we call this a collision. We classify all possible types of collisions. In Section 6 we discuss M-CLP pivots. We consider two adjacent validity regions $\mathcal{V}, \mathcal{W}$, with base sequences $(\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}$ and $(\mathcal{K}'_n, \mathcal{J}'_n)_{n=0}^{N'+1}$. Given $(\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}$ and the type of collision as $\rho$ approaches the boundary of $\mathcal{V}$ we show how
to identify the type of collision as $\rho$ approaches the boundary of $\mathcal{W}$, from the other side of the boundary, and then construct the base sequence $(K^*_n, J^*_n)_{n=0}^{N+1}$ based on the identified collision.

Based on the pivot operation we then describe the algorithm in Section 7. In Section 8 we give an example of a small problem which we solve with the algorithm. The algorithm of Section 7 requires us to make some strong simplifying assumptions in order to work. However, in all cases these assumptions can be enforced by perturbation of the problem data. We discuss the necessary perturbation procedures in Section 9. Some of the proofs are delayed to the Appendix.

Remark In [17,18] M-CLP/M-CLP* are formulated with $\lambda = 0$, $\mu = 0$. The introduction of $\lambda \leq 0$, $\mu \geq 0$ here is done to avoid some inherent degeneracies that can inhibit our algorithm. We discuss the motivation and explain the sign restrictions, and prove that all the results of [17,18] extend directly to the current formulation in Appendix A.1.

Notes on presentation: Throughout the paper we quote results from [17,18]. We will refer to results from the duality paper [17] by appending D and from the structure paper [18] by appending S (e.g. D5.3, or (S4.2)). There is complete symmetry between M-CLP and M-CLP*.

Therefore when we formulate results for one or both of them, we will give a proof only for one of them. For easier reading we moved some of the proofs of Sections 4, 6 and 7 to Appendix A.2, A.3, A.4 respectively.

2 Structure of the Solution

We summarize some results from [18], as extended to $\lambda \leq 0$, $\mu \geq 0$. Optimal solutions of M-CLP,M-CLP* problems (1), (2) are strongly dual and satisfy complementary slackness conditions:

$$
\int_{0-}^{T} x(T-t)^{T}dP(t) = \int_{0-}^{T} q(T-t)^{T}dU(t) = 0.
$$

A necessary and sufficient condition for feasibility of M-CLP is feasibility of the following Test-LP (This is similar to Theorem S2.5, and based on ideas of Wang, Zhang and Yao [22]), with an analogous condition for M-CLP*.

$$
\max \quad z = (\gamma + cT + \mu)^{T}u + (\gamma + cT)^{T}U
$$

s.t. \quad Au \leq \beta,

Test-LP \quad Au + AU \leq \beta + bT + \lambda,

\quad u, U \geq 0.

Under non-degeneracy assumption [18] optimal solutions $U(t), P(t)$ have impulse controls $u^0 = U(0), u^N = U(T) - U(T-), p^0 = P(T) - P(T-), p^N = P(0)$ at 0 and $T$, piecewise constant control rates $u(t) = \frac{dU(t)}{dt}, p(t) = \frac{dP(t)}{dt}$, and continuous piecewise linear states $x(t) = \beta + bt - U(t), q(t) = \gamma + ct - P(t)$ with possible discontinuities at $T$. The time horizon $[0,T]$ is partitioned by $0 = t_0 < t_1 < \ldots < t_N = T$ which are the breakpoints in the rates $u,p$ and in the slopes of $x,q$. We denote the vectors of values of the states at the breakpoints by $x^n = x(t_n), n = 0, \ldots, N-1$, and $q^n = q(T-t_n), n = 1, \ldots, N$. Because there may be a discontinuity at $T$ we denote the values at $T$ itself by $x^N = x(T), q^0 = q(T)$, and let $x^N = x(T-), q^0 = q(T-)$ be the values of the limit as $t \nearrow T$. The constant slopes of the states and the constant values of the control rates for each interval are denoted $\dot{x}^n = \frac{dx(t)}{dt}, u^n = u(t), t_{n-1} < t < t_n$ and $\ddot{q}^n = \frac{dq(t)}{dt}, p^n = p(t), T - t_n < t < T - t_{n-1}$.
The entire solution, \( U(t), P(t), x(t), q(t) \) can be retrieved from vectors of the boundary values \( u^0, u^+, p^0, p^+, x^0, x^+, q^0, q^+ \), the rates: \( u^n, \dot{x}^n, p^n, q^n \), \( n = 1, \ldots, N \), and the values of interval lengths \( \tau_n = t_n - t_{n-1}, n = 1, \ldots, N \).

The rates \( u^n, \dot{x}^n, p^n, q^n \) are complementary slack basic solutions of the following dual pairs of Rates-LP problems

\[
\begin{align*}
\text{max} & \quad c^T u \\
\text{s.t.} & \quad A u + \dot{x} = b, \\
\text{Rates-LP}(\mathcal{K}_n, \mathcal{J}_n) & \quad u_j \in \mathbb{Z} \text{ for } j \in \mathcal{J}_n, \ u_j \in \mathbb{P} \text{ for } j \notin \mathcal{J}_n, \\
& \quad \dot{x}_k \in \mathbb{U} \text{ for } k \in \mathcal{K}_n, \ \dot{x}_k \in \mathbb{P} \text{ for } k \notin \mathcal{K}_n. \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad b^T p \\
\text{s.t.} & \quad A^T p - \dot{q} = c, \\
\text{Rates-LP}^*(\mathcal{K}_n, \mathcal{J}_n) & \quad p_k \in \mathbb{Z} \text{ for } k \in \mathcal{K}_n, \ p_k \in \mathbb{P} \text{ for } k \notin \mathcal{K}_n, \\
& \quad \dot{q}_j \in \mathbb{U} \text{ for } j \in \mathcal{J}_n, \ \dot{q}_j \in \mathbb{P} \text{ for } j \notin \mathcal{J}_n. \\
\end{align*}
\]

where for \( n = 1, \ldots, N \), \( B_1, \ldots, B_N \) denote the optimal bases, and \( \mathcal{K}_n, \mathcal{J}_n \) are the indexes of the basic \( \dot{x}_n^+, \dot{q}_n^+ \). By Assumption 1.1 all the bases are non-degenerate.

The bases \( B_1, \ldots, B_N \) are admissible in the sense that \( u^n, p^n \geq 0 \), and they are adjacent, so that \( B_n \to B_{n+1} \) involves a single pivot, with \( v_n \) leaving the basis, and \( w_n \) entering. We let \( (\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0, N+1\}} \) be the indexes of the non-zero boundary values of the primal and dual state variables at times 0 and \( T \). They satisfy a compatibility condition \( \mathcal{K}_0 \subseteq \mathcal{K}_1, \mathcal{J}_{N+1} \subseteq \mathcal{J}_N \). A base sequence \( (\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1} \) that is admissible, adjacent, and compatible is called a proper base sequence.

Given any proper base sequence, a solution can be constructed by solving the Rates-LP/LP* [4], [5] for bases \( B_n, n = 1, \ldots, N \) and using these values to formulate and solve the following coupled linear equations, which determine the interval lengths and the boundary values:

\textit{The time interval equations:}

\[
x_k(t_n) = x_k^0 + \sum_{m=1}^n \dot{x}_m^+ \tau_m = 0, \quad \text{if } v_n = \dot{x}_k, \\
q_j(T - t_n) = q_j^N + \sum_{m=n+1}^N \dot{q}_m^+ \tau_m = 0, \quad \text{if } v_n = u_j.
\]

\textit{The complementary slackness conditions:}

\[
\begin{align*}
\mathbf{u}^0_j = 0, & \quad j \in \mathcal{J}_0, & \mathbf{p}^N_k = 0, & \quad k \in \mathcal{K}_{N+1}, \\
x^0_k = 0, & \quad k \notin \mathcal{K}_0, & q^N_j = 0, & \quad j \notin \mathcal{J}_{N+1}, \\
\mathbf{p}^0_k = 0, & \quad k \in \mathcal{K}_0, & \mathbf{u}^N_j = 0, & \quad j \in \mathcal{J}_{N+1}, \\
q^0_j = 0, & \quad j \notin \mathcal{J}_0, & x^N_k = 0, & \quad k \notin \mathcal{K}_{N+1}.
\end{align*}
\]

\textit{The first boundary equations:}

\[
A \mathbf{u}^0 + x^0 = \beta, \quad A^T \mathbf{p}^N - q^N = \gamma.
\]

\textit{The second boundary equations:}

\[
A \mathbf{u}^N + x^N - x^N = \lambda, \quad A^T \mathbf{p}^0 - q^0 + q^0 = \mu.
\]
Where all the values of the state variables are obtained from:

\[ x^n_k = x^0_k + \sum_{m=1}^{n} x^m_k \tau_m, \quad k = 1, \ldots, K, \quad n = 1, \ldots, N, \]

\[ q^n_j = q^N_j + \sum_{m=n+1}^{N} q^m_j \tau_m \quad j = 1, \ldots, J, \quad n = 1, \ldots, N. \]  

(11)

Theorem S3.1 in [18] states that for a proper base sequence, if the solution to (6)–(11) is non-negative then it is an optimal solution, and every feasible M-CLP/M-CLP* problem has an optimal solution of this form.

Given a proper base sequence, after solving (4), (5), we can formulate all the equations (6)–(11) as

\[ MH = R. \]  

(12)

The matrix \( M \) is a square matrix of dimension \((J + K)(N + 4) + N\), defined in Section S3.5.²

The vector of unknowns (writing all vectors as row vectors) is:

\[ H = [ u^0 \ x^0 \ p^N \ q^N \ \tau \ \ x^1 \ \ldots \ x^N \ \ q^0 \ \ldots \ q^{N-1} \ u^N \ x^N \ p^0 \ q^0 ]^T \]

and the right hand side consists of the boundary parameters \( \rho \) interspersed by properly dimensioned 0 vectors:

\[ \rho \rightarrow R : R = \begin{bmatrix} \beta^T & 0 & 0 & \gamma^T & 0 & 0 & 0 & T & 0 & 0 & \lambda^T & 0 & 0 & \mu^T & 0 & 0 \end{bmatrix}^T \]  

(13)

Note, that equation (12) forces \( 2K + 2J \) boundary variables, \( N - 1 \) variables that are left-hand sides of (6) and \( N(K + J) - \sum_{n=1}^{N} (|K_n| + |J_n|) \) of the \( x_k^0 \in \{n, k : x_k^0 = 0\} \), \( q_j^0 \in \{n, j : q_j^0 = 0\} \) to be zero. We denote this set of zero-valued variables by \( H_z \). The set of remaining, non-negative variables, is denoted by \( H_p \). We say that the solution \( H \) is fully non-degenerate if all components of \( H_p \) are > 0.

3 Decomposition

We introduce the following distinction between the state variables that are indexed by \( K_0, J_{N+1} \):

Definition 3.1. Consider a fully non-degenerate solution, and the \( x_k(t) \) with \( k \in K_0 \). If for some \( t_n < T \), \( x_k(t) = 0 \) then we say that \( x_k \) is tied. On the other hand, if \( x_k(t) > 0 \) for \( 0 \leq t < T \) then we say that \( x_k \) is free. We denote by \( K^\tau \) the indexes of the tied variables, and by \( K^\uparrow \) the indexes of the free variables, so that \( K_0 = K^\tau \cup K^\uparrow \). We define \( J^\tau \), \( J^\uparrow \) analogously for \( q_j(t), j \in J_{N+1} \).

Note: There are exactly \( 2K + 2J \) boundary values which are 0, and up to \( 2K + 2J \) positive ones. However, the number of positive primal boundary values is \(|K_0| + |J_0| + |K_{N+1}| + |J_{N+1}| = 2K + L\), and the the number of positive dual boundary values \(|J_{N+1}| + |K_{N+1}| + |J_0| + |K_0| = 2J - L\) where \( L \) may be either 0 or positive or negative.

Given an optimal solution of an M-CLP/M-CLP* problem, with data \( A, b, c, \beta, \gamma, T, \lambda, \mu \), we introduce some more notations for various quantities:

\[ x_k^* = \min_{0 \leq t < T} x_k(t), \quad \bar{x}_k = x_k^0 - x_k^*, \quad k = 1, \ldots, K, \]  

(14)

\[ q_j^* = \min_{0 \leq t \leq T} q_j(t), \quad \bar{q}_j = q_j^N - q_j^*, \quad j = 1, \ldots, J. \]  

(15)

²There are some typos in the definition of \( M \) in [18]. They were corrected in [16].
Here \( x_k^\bullet \) is the minimum of \( x_k(t) \) over the range \( 0 \leq t < T \) (excluding the value \( x_k(T) = x^N \)), and one can also write \( \tilde{x}_k = -\min_{0 \leq n \leq N} \sum_{m=1}^{n} \tilde{x}_k^m \tau_m \) (where empty sum equals 0). For \( k \in \mathcal{K}^- \), \( x_k^\bullet = 0 \) and \( \tilde{x}_k = x_k^0 \). For \( k \in \mathcal{K}^+ \), \( x_k^\bullet > 0 \), and \( \tilde{x}_k = x_k^0 - x_k^\bullet \). For \( k \not\in \mathcal{K}_0 \), we have \( x_k^0 = x_k^\bullet = \tilde{x}_k = 0 \). Analogous statements hold for \( q_j(t) \).

We also introduce the following notation for the vectors of cumulative controls excluding the impulse controls:
\[
\tilde{U} = \int_0^T u(t) dt, \quad \tilde{P} = \int_0^T p(t) dt.
\]

**Proposition 3.2.** Under non-degeneracy assumption [1.1] M-CLP/M-CLP\(^*\) possess a unique set of decomposition parameters \( \tilde{U}, \tilde{P}, \tilde{x}, \tilde{q} \). They are all of them affine functions of elements of \( \rho \).

**Proof.** Uniqueness follows directly from Theorem D5.5(iii), which states that under assumption [1.1] \( u(t), p(t) \) are uniquely determined, and as a result so are \( \dot{x}(t), \dot{q}(t) \). Proof that they are affine functions of \( \rho \) follows similar to the proof of Corollary S3.8. \( \square \)

We are now ready to define a decomposition of M-CLP/M-CLP\(^*\) to two parametric families of problems. The first consists of SCLP/SCLP\(^*\) problems and concerns the solution in the interior of the time horizon. The second consists of pairs of Boundary-LP problems, and concerns the solution on the boundaries of the time horizon.

**Internal SCLP Problems:**

Consider M-CLP/M-CLP\(^*\) problems with data \( A, b, c \). We define a parametric family SCLP/SCLP\(^*\) with non-negative parameters \( x^0, q^N \)
\[
\max \int_0^T (-q^N + (T - t)c)^T u(t) dt
\]
SCLP\((x^0, q^N)\)
\[
s.t. \int_0^t A u(s) ds \leq x^0 + bt,
\]
\[
\begin{align*}
&u(t) \geq 0, \quad 0 \leq t \leq T. \\
&\min \int_0^T (x^0 + (T - t)b)^T p(t) dt
\end{align*}
\]
SCLP\(^*\)(\(x^0, q^N)\)
\[
s.t. \int_0^t A^T p(s) ds \geq -q^N + ct,
\]
\[
\begin{align*}
&p(t) \geq 0, \quad 0 \leq t \leq T.
\end{align*}
\]

These problem are a special case of separated continuous linear program (SCLP) of the form discussed in [23]. We call these problems *internal SCLP/SCLP\(^*\).*

**Boundary Problems**

Consider M-CLP/M-CLP\(^*\) problem with data \( A, b, c, \beta, \gamma, T, \lambda, \mu \). We define, similar to S15, S16 in [18], a family of pairs of LP problems, for parameter vectors \( \tilde{u}, \tilde{x}, \tilde{U}, \tilde{P}: \)
\[
\max (\gamma + cT + \mu)^T u^0 + \gamma^T u^N
\]
Boundary-LP\((\tilde{U}, \tilde{x})\)
\[
s.t. \begin{align*}
&A u^0 + x^\bullet = \beta - \tilde{x}, \\
&\begin{align*}
&u^0, u^N, x^\bullet, x^N \geq 0,
&\end{align*}
\end{align*}
\]
(19)
\[
\min (\beta + bT + \lambda)^T p^N + \beta^T p^0
\]
\[
\begin{align*}
\text{Boundary-LP}^* (\tilde{P}, \tilde{q}) & \quad \text{s.t.} \\
A^T p^N - q^* &= \gamma + \tilde{q}, \\
A^T p^N + A^T p^0 - q^0 &= \gamma + cT - A^T \tilde{P} + \mu,
\end{align*}
\] (20)

Note, that Boundary-LP(\(\tilde{U}, \tilde{x}\)) and Boundary-LP* (\(\tilde{P}, \tilde{q}\)) are not dual to each other. Note also that they use disjoint sets of decomposition parameters (\(\tilde{U}, \tilde{x}\) for Boundary-LP and \(\tilde{P}, \tilde{q}\) for Boundary-LP*).

**Theorem 3.3.** (i) An optimal solution of M-CLP/M-CLP* can be decomposed to optimal solutions of internal SCLP/SCLP*\((x^0, q^N)\) and feasible solutions of Boundary-LP(\(\tilde{U}, \tilde{x}\)) and Boundary-LP*(\(\tilde{P}, \tilde{q}\)) satisfying:

\[
x^0 = x^* + \tilde{x}, \quad q^N = q^* + \tilde{q}, \quad \tilde{U} = \int_0^T u(t)dt, \quad \tilde{P} = \int_0^T p(t)dt,
\]
\[
\tilde{x}_k = - \min_{1 \leq n \leq N} \left( \sum_{m=1}^n \dot{x}_k^m r_m \right), \quad \tilde{q}_j = - \min_{1 \leq n \leq N} \left( \sum_{m=1}^n \dot{q}_j^m r_m \right),
\]
\[
x^N t = x^N \quad p^0 = x^N t = q^N u = q^N t u = 0.
\] (21)

(ii) Conversely, a combination of optimal solutions of internal SCLP/SCLP*\((x^0, q^N)\), and of feasible solutions of Boundary-LP(\(\tilde{U}, \tilde{x}\)) and Boundary-LP*(\(\tilde{P}, \tilde{q}\)), that satisfy (21) can be composed into an optimal solution of M-CLP/M-CLP*.

(iii) The solutions of Boundary-LP(\(\tilde{U}, \tilde{x}\)) and Boundary-LP*(\(\tilde{P}, \tilde{q}\)) considered in (i) and (ii) are optimal.

**Proof.** (i) Let \(u(t), x(t), p(t), q(t), u^0, u^N, x^0, x^N, p^N, p^0, q^N, q^0\) be an optimal solution of M-CLP/M-CLP* as described in Theorem S3.1. Then it is immediate to see that the same \(u(t), x(t), p(t), q(t)\) solve the internal SCLP/SCLP*\((x^0, q^N)\).

Furthermore, one can see that \(u^0, u^N, x^* = x^0 - \tilde{x}, x^N\) is a feasible solution of Boundary-LP(\(\tilde{U}, \tilde{x}\)) and \(p^N, p^0, q^N - \tilde{q}, q^0\) is a feasible solution of Boundary-LP*(\(\tilde{P}, \tilde{q}\)).

(ii) On the other hand, optimal solution of SCLP/SCLP* provides \(u(t), p(t), x(t), q(t)\) and solutions of Boundary-LP/LP* provides \(u^0, u^N, x^*, x^N, p^0, p^N, q^*, q^0\). It is immediate to see that if these satisfy (21) then they provide a feasible complementary slack solution of M-CLP/M-CLP*.

(iii) Recall the proof of Corollary S3.3. □

**Corollary 3.4.** Consider a solution of internal SCLP/SCLP* and of Boundary-LP/LP* that satisfies (21). Then in the solution of Boundary-LP/LP* the following holds:

(i) The solutions of Boundary-LP(\(\tilde{U}, \tilde{x}\)) and of Boundary-LP*(\(\tilde{P}, \tilde{q}\)) are complementary slack.

(ii) For \(k \in K^\infty\) we have \(x_k^* = 0\), and for \(j \in J^\infty\) we have \(q_j^* = 0\).

(iii) For \(j \in J^\infty\) we have \(u_j^N = 0\), and for \(k \in K^\infty\) we have \(p_k^0 = 0\).

**Proof.** One can see that from (21) follows:

\[
x^* t p^0 = \tilde{x}^T p^0 = x^N t p^N = q^* t u^N = \tilde{q}^T u^N = q^0 t u^0 = 0.
\]

In particular this implies:

(i) Variables \(u_j^0, q_j^0\), variables \(p_k^N, x_k^N\), variables \(x_k^*, p_k^0\) and variables \(q_j^*, u_j^N\) are complementary slack.
Recall that the problems Boundary-LP(3.1 Boundary Simplex Dictionary further discussion of the two formulations see \[16\]. (19), (20) preferable to the more natural dual pair of boundary problems (S13), (S14). For and optimal complementary slack solutions to the Boundary-LP/LP corresponding Boundary-LP (19) has
\[ |K| \]
\[ \text{Proof.} \]
If M-CLP/M-CLP Theorem 3.5. Under the non-degeneracy assumption 1.1 there is a one to one correspondence between optimal solutions of M-CLP and Boundary-LP(\( \bar{U}, \bar{x} \)). The same holds for optimal solutions of M-CLP* and Boundary-LP*(\( \bar{P}, \bar{q} \)).
\[ \text{Proof.} \]
From Theorem 3.2 it follows that under non-degeneracy assumption 1.1 the formulation of Boundary-LP(\( \bar{U}, \bar{x} \)) problem is unique, whether M-CLP has a unique solution or not.
Furthermore, each optimal solution of M-CLP determines the boundary values \( u^0, u^N, x^N \) and the values of \( x^* = x^0 - \bar{x} \) that by Theorem 3.3 (iii) is an optimal solution of Boundary-LP(\( \bar{U}, \bar{x} \)). Hence, each optimal solution of M-CLP uniquely defines an optimal solution of the Boundary-LP(\( \bar{U}, \bar{x} \)).
Conversely, each optimal solution of Boundary-LP(\( \bar{U}, \bar{x} \)) together with \( \bar{x}(t), u(t) \) that are unique by Theorem D5.5(iii) produce a feasible solution of M-CLP. Moreover, objective values of these solutions satisfy:
\[ V(\text{M-CLP}) = V(\text{Boundary-LP}(\bar{U}, \bar{x})) + \int_{0+}^{T-} (\gamma + c(T-t))^T \text{u(t)dt} \]
That is, each optimal solution of Boundary-LP(\( \bar{U}, \bar{x} \)) produce a feasible solution of M-CLP with same objectıve value. However, by Theorem 3.3 one of optimal solutions of Boundary-LP(\( \bar{U}, \bar{x} \)) obtained by decomposition of optimal solution of M-CLP and hence each optimal solution of Boundary-LP(\( \bar{U}, \bar{x} \)) uniquely defines an optimal solution of M-CLP.
\[ \text{Proof.} \]
Note: It is this one to one correspondence between optimal solutions of M-CLP/M-CLP*, and optimal complementary slack solutions to the Boundary-LP/LP* that makes the formulation \[19, 20\] preferable to the more natural dual pair of boundary problems (S13), (S14). For further discussion of the two formulations see \[16\].

3.1 Boundary Simplex Dictionary
Recall that the problems Boundary-LP(\( \bar{U}, \bar{x} \)) and Boundary-LP*(\( \bar{P}, \bar{q} \)) are not dual to each other. However, optimal solutions of Boundary-LP(\( \bar{U}, \bar{x} \)) and Boundary-LP*(\( \bar{P}, \bar{q} \)) that correspond to optimal solutions of M-CLP/M-CLP* are unique and are complementary slack. If possible we wish to find basic solutions that are summarized by a single boundary simplex dictionary. This dictionary will enables us to define a boundary pivot operation in Section 6. The following theorem states this is possible.

Theorem 3.6. If M-CLP/M-CLP* has a unique solution with \( H_F > 0 \), then its optimal base sequence satisfies \[ |K^*| + |J_0| + |K_{N+1}| + |J_{N+1}| \leq 2K \] and \[ |J_0| + |K_0| + |\bar{J}^*| + |\bar{K}_{N+1}| \leq 2J. \]
\[ \text{Proof.} \]
If M-CLP/M-CLP* has a unique solution with \( H_F > 0 \), then the optimal solution of the corresponding Boundary-LP \[19\] has \[ |K^*| + |J_0| + |K_{N+1}| + |J_{N+1}| \] positive variables. But if
$|\mathcal{K}^\uparrow| + |\mathcal{T}_0| + |\mathcal{K}_{N+1}| + |\mathcal{J}_{N+1}| > 2K$ this implies that it has a non-basic optimal solution, and therefore its solution is non-unique. But then by Theorem 3.5 M-CLP has non-unique optimal solutions, which is a contradiction.

We now decompose the matrix of coefficients of (19) into a basic and non-basic part, with similar decomposition for the matrix of coefficients of (20). Let $\boldsymbol{B}$ be a basic submatrix of the former, $\mathbf{B}^*$ a basic matrix of the latter, such that they are complementary slack, and let $\mathbf{N}$, $\mathbf{N}^*$ be the matrices composed from the corresponding non-basic columns, so that (as seen for example in [21]) $\mathbf{B}^{-1}\mathbf{N} = (\mathbf{B}^{*-1}\mathbf{N}^*)^T$. We then say that $\mathbf{B}$ and $\mathbf{B}^*$ are compatible.

One can see that for a base sequence satisfying $|\mathcal{K}^\uparrow| + |\mathcal{T}_0| + |\mathcal{K}_{N+1}| + |\mathcal{J}_{N+1}| \leq 2K$ and $|\mathcal{J}^\uparrow| + |\mathcal{K}_{N+1}| + |\mathcal{J}_0| + |\mathcal{K}_0| \leq 2J$ we can build a pair of compatible bases using the following rules:

- define primal variables with indexes that are members of $\mathcal{K}^\uparrow$, $\mathcal{T}_0$, $\mathcal{K}_{N+1}$, $\mathcal{J}_{N+1}$ as basic variables for $\mathbf{B}$ and dual variables with indexes that are members of $\mathcal{J}^\uparrow$, $\mathcal{K}_{N+1}$, $\mathcal{J}_0$, $\mathcal{K}_0$ as basic variables for $\mathbf{B}^*$,
- if the solution of M-CLP possesses $\geq 2K$ strictly positive primal boundary values then choose an arbitrary subset of $|\mathcal{K}_0| + |\mathcal{T}_0| + |\mathcal{K}_{N+1}| + |\mathcal{J}_{N+1}| - 2K$ indexes out of $\mathcal{K}^=$ and define those $\mathbf{p}_k^*$ as dual basic variable in $\mathbf{B}^*$, and for the remaining indexes of $\mathcal{K}^=$ define $\mathbf{x}_k^*$ as primal basic variable in $\mathbf{B}$. Define all the variables $\mathbf{q}_j^*$, $j \in \mathcal{J}^=$ as dual basic variables in $\mathbf{B}^*$.
- similarly, if the solution of M-CLP* possesses $\geq 2J$ strictly positive boundary choose an arbitrary subset of $|\mathcal{J}_{N+1}| + |\mathcal{K}_{N+1}| + |\mathcal{J}_0| + |\mathcal{K}_0| - 2J$ indexes out of $\mathcal{J}^=$ and define those $\mathbf{u}_j^N$ as basic variables in $\mathbf{B}$, and for the remaining indexes of $\mathcal{J}^=$ define $\mathbf{q}_j^*$ as dual basic variable in $\mathbf{B}^*$. Define all the variables $\mathbf{x}_k^*$, $j \in \mathcal{K}^=$ as primal basic variables in $\mathbf{B}$. Using compatible boundary bases we define a boundary simplex dictionary $\mathcal{D}$ by:

$$\mathcal{D} = \begin{bmatrix} \mathbf{v}_B^* \\ \mathbf{v}_B \end{bmatrix} = \begin{bmatrix} \mathbf{v}^* \\ \mathbf{A} = \mathbf{B}^{-1}\mathbf{N} = (\mathbf{B}^{*-1}\mathbf{N}^*)^T \end{bmatrix}$$

where $\mathbf{v} = \{x^* = x^0 - \bar{x}, u^0, x^N, u^N\}$, $\mathbf{v}^* = \{q^* = q^N - \bar{q}, p^N, q^0, p^0\}$ are values of variables of Boundary-LP/LP* and $\mathbf{B}, \mathbf{B}^*$ are sets of indexes of the corresponding basic variables. Changes in $(\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0,N+1\}}$ will be achieved by pivots of this dictionary.

Examples of boundary simplex dictionaries will be shown later in Section 8.

### 4 Validity Regions

Throughout this section we assume that non-degeneracy assumption [11] holds. Similar to the definition of validity region 3.6 we define:

**Definition 4.1.** Let $(\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}$ be a proper base sequence. Let $\mathcal{V}$ be the set of all $\mathbf{\rho} = [\beta^T \gamma^T T \lambda^T \mu^T]$ for which this base sequence is optimal. Then $\mathcal{V}$ is called the validity region of $(\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}$. Let $\mathcal{F}$ be the union of all validity regions, then $\mathcal{F}$ is called the parametric feasible region.
As in Theorem S3.7, each validity region is a closed convex polyhedral cone (it may consist only of the origin). The parametric feasibility region is clearly convex, and it is a union of validity regions, hence $\mathcal{F}$ is a closed convex polyhedral cone.

Similar to Corollary S3.8 we can state:

**Corollary 4.2.** For a given optimal base sequence, within its validity region, the elements of $H_\varphi$ are affine functions of the non-zero elements of $R$. Conversely, consider all optimal solutions $H$ that belong to a given optimal base sequence. Then the non-zero elements of $R$ change linearly as a function of non-zero elements of $H_\varphi$.

**Proposition 4.3.** The number of validity regions is bounded by $\left(\frac{4(K+J)}{2(K+J)}\right)^{2(K+J)}$.

**Proof.** The total number of bases of Rates-LP/LP* is $\left(\frac{K+J}{2(K+J)}\right)^{2(K+J)}$. Moreover, for a proper base sequence, under the non-degeneracy assumption, the objective values of the Rates-LP/LP* are strictly decreasing in $n$ (see Corollary 4 in [23]) and hence the number of sequences $(K_n, J_n)_{n=1}^{N}$ is bounded by $2^{(K+J)}$. The number of $(K_n, J_n)_{n \in \{0, N+1\}}$ is bounded by $\left(\frac{4(K+J)}{2(K+J)}\right)^2$ and hence the total number of proper base sequences is bounded by $\left(\frac{4(K+J)}{2(K+J)}\right)^2$.

Note, a simpler somewhat larger bound is $2^{2K+J+1}$.

**Theorem 4.4.** For a proper base sequence $(K_n, J_n)_{n=0}^{N+1}$ with validity region $\mathcal{V}$, and for boundary parameters $\rho$ in the interior of $\mathcal{F}$, the following statements are equivalent:

(i) $\rho$ is an interior point of $\mathcal{V}$,
(ii) $\rho$ does not belong to the validity region of any other proper base sequence,
(iii) the solution of (12) for the point $\rho$ is unique and satisfies $H_\varphi > 0$.

**Proof.** See Appendix A.2

Note that Theorem 4.4 implies that interiors of validity regions are disjoint.

**Corollary 4.5.** If a proper base sequence has a validity region with non-empty interior, then the corresponding matrix $M$ is non-singular.

**Proof.** By Theorem 4.4 at an interior point of the validity region the solution of (12) is unique, and hence $M$ is non-singular.

**Corollary 4.6.** If a point $\rho$ is an interior point of some validity region then the solutions of M-CLP and M-CLP* for this point are unique.

**Proof.** By Theorem 4.4 the sequence of bases is unique, $u(t), p(t), \dot{x}(t), \dot{q}(t)$ are unique by Theorem D5.5(iii), and so $M$ is unique, and the solution is unique.

## 5 Collisions

In this section we describe how the solution changes as the boundary parameters move from an internal point of the validity region to a boundary point of the validity region. We consider a validity region $\mathcal{V}$ with non-empty interior and base sequence $(K_n, J_n)_{n=0}^{N+1}$. Let $\mathcal{L}(\theta)$ be a parametric line of boundary parameters, with $\mathcal{L}(\theta)$ in the interior of $\mathcal{V}$ for $\theta < \overline{\theta}$, $\mathcal{L}(\overline{\theta})$ on the boundary, with $H_\varphi(\theta)$ the solution at $\mathcal{L}(\theta)$. Then $H_\varphi(\theta) > 0$ for $\theta < \overline{\theta}$, but some elements of $H_\varphi$ shrink to 0 as $\theta \nearrow \overline{\theta}$. We call this a collision.

We now list all types of collisions. For convenience we number them (a), (b), . . . .
Internal collisions

(a) State collision. A single value \(x^n_k, n = 1, \ldots, N\), or \(q^j_n, n = 0, \ldots, N - 1\), shrinks to 0 (see fig. 1). This is classified in [23] as SCLP-III type collision.

Figure 1: Type (a) collision, \(x^2_k\) shrinks to 0.

(b) Interval collision between non-adjacent bases. A sequence of one or more internal interval lengths, \(\tau_{n_1}, \ldots, \tau_{n_s}\), shrink to 0, where \(1 < n_s \leq n_{ss} < N\) and \(|B_{n_s-1} \setminus B_{n_s+1}| = 2\), i.e. the basis preceding \(n_s\) and the basis succeeding \(n_{ss}\) are not adjacent. This means that at the collision point on the boundary, at time \(t = t_{n_s} = \cdots = t_{n_{ss}}\) two different state variables hit the value 0 (\(x_k(t)\) and \(x'_k(t)\), or \(q_j(T - t)\) and \(q'_j(T - t)\) or \(x_k(t)\) and \(q_j(T - t)\) (see fig. 2)). This is classified in [23] as SCLP-II type collision.

Figure 2: Type (b) collision, \(\tau_3\) shrinks to 0.

(c) Interval collision between adjacent bases or at 0 or \(T\), variable does not become free. Includes the following three cases: A sequence of one or more interval lengths, \(\tau_{n_1}, \ldots, \tau_{n_s}\), shrink to 0, where \(1 < n_s \leq n_{ss} < N\) and \(|B_{n_s-1} \setminus B_{n_s+1}| = 1\), or intervals \(\tau_{n_1}, \ldots, \tau_{n_{ss}}, 1 \leq n_{ss} < N\) shrink to zero, or intervals \(\tau_{n_1}, \ldots, \tau_{N}, 1 < n_s \leq N\) shrink to zero. In the first case, a state variable \(x_k\) (or a dual state variable \(q_j\)) is positive in the intervals \(t_{n_s-1} < t < t_{n_s}\) and \(t_{n_{ss}} < t < t_{n_{ss}+1}\), but it is 0 at \(t_{n_s}\) and at \(t_{n_{ss}}\). In the second case the collision is at \(t = 0\) and a dual state variable \(q_j\) in positive in \(t_{n_{ss}} < t < t_{n_{ss}+1}\) but it is 0 at \(t_{n_{ss}}\). In the third case the collision is at \(t = T\), and a state variable \(x_k\) in positive in \(t_{n_s-1} < t < t_{n_s}\) but it is 0 at \(t_{n_s}\). These collisions are classified as SCLP-I type collisions in [23]. The variable \(x_k\) (or \(q_j\)) remains tied in the sense that in the interior of the validity region \(V\) there is at least one time interval in \([0, t_{n_s}]\) or in \([t_{n_{ss}}, T]\) such that \(x_k(t) = 0\) (or \(q_j(T - t) = 0\)) in that interval.

Collisions that involve boundary values

(d) Interval collision between adjacent bases or at 0 or \(T\), variable become free. This is the same as (c) except that the variable \(x_k\) (or \(q_j\)) become free in sense that in the interior
Figure 3: Type (c) collision, $\tau_4$ shrinks to 0, $x_k$ remains tied.

![Figure 3](image)

of the validity region $V$, in the time intervals $(0, t_{n^*})$ and $(t_{n^*}, T)$, $x_k(t) > 0$ (or $q_j(T-t) > 0$) (see fig. 4).

Figure 4: Type (d) collision, $\tau_3$ shrinks to 0, $x_k$ becomes free.

![Figure 4](image)

(e) Boundary collision. One of the $>0$ boundary values including any of $u_j^b, u_j^N, p_k^b, p_k^N, x_k^N$, $q_j^0$ shrinks to zero, or one of the values $x_k^0, k \in K^{\uparrow \downarrow}$ or $q_j^N, j \in J^{\uparrow \downarrow}$, shrinks to 0, or one of the values of $x_k^0, k \in K=\{\}$ for which $\dot{x}_k^1 > 0$ (see fig. 5) or one of the values of $q_j^N, j \in J=\{\}$ for which $\dot{q}_j^N > 0$ shrinks to 0.

Figure 5: Type (e) collision, $x_k^0$ shrinks to 0.

![Figure 5](image)

(f) Joint collision. One of the values of $x_k^0, k \in K=\{\}$ for which $\dot{x}_k^1 < 0$ shrinks to 0, and as a result the intervals $\tau_1, \ldots, \tau_n$, shrink to zero, where in the interior of the validity region $x_k(t) > 0$ for $0 < t < t_{n^*}$, and $x_k(t_{n^*}) = 0$ (see fig. 6).

Figure 6: Type (f) collision, $x_k^0$ and $\tau_1$ shrink to 0.

![Figure 6](image)
Alternatively, one of the values of $q_j^N$, $j \in J^\tau$ for which $q_j^N < 0$ shrinks to 0, and as a result the intervals $\tau_{n_*+1}, \ldots, \tau_N$ shrink to zero, where in the interior of the validity region $q_j(T - t) > 0$ for $t_{n_*-1} < t < T$, and $q_j(T - t_{n_*-1}) = 0$.

**Multiple collisions**

Any collision that is not one of the types (a)–(f) is a *multiple collision*. Such collisions consists of a combination of several collisions of types (a)–(f). We will exclude the possibility of multiple collisions in Sections 6–8, but will return to discuss them in Section 9.

## 6 M-CLP pivots

An M-CLP pivot uses the solution in validity region $\mathcal{V}$ with non-empty interior to construct the solution in a neighboring validity region with non-empty interior $\mathcal{W}$. Let $\mathcal{L}(\theta)$, $0 < \theta < 1$ be a parametric line of boundary parameters, such that $\mathcal{L}(\theta) \in \text{interior of } \mathcal{V}$, $\theta < \theta_0$, $\mathcal{L}(\theta) \in \text{interior of } \mathcal{W}$, $\theta > \theta_0$, and $\mathcal{L}(\theta_0)$ is on the boundary of $\mathcal{V}$ and $\mathcal{W}$. We assume that at the collision point $\mathcal{L}(\theta_0)$, the collision from $\mathcal{V}$ as well as the collision from $\mathcal{W}$ are both of types (a)–(f). In an M-CLP pivot we will start from the region $\mathcal{V}$ with optimal base sequence $(\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}$, and with a specified type of collision, and we will show how to retrieve the collision from the other side, and from it construct the base sequence $(\mathcal{K}'_n, \mathcal{J}'_n)_{n=0}^{N'+1}$ optimal in $\mathcal{W}$.

There are three types of pivots:
- Pivots that involve only changes from $(\mathcal{K}_n, \mathcal{J}_n)_{n=1}^N$ to $(\mathcal{K}'_n, \mathcal{J}'_n)_{n=1}^{N'}$, which are pivots on the internal SCLP, similar to SCLP pivots described in [23]. We refer to these as internal pivots.
- Pivots that involve changes from $(\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0, N+1\}}$ to $(\mathcal{K}'_n, \mathcal{J}'_n)_{n \in \{0, N'+1\}}$ in which the solution at the collision point is unique. We refer to these as boundary pivots of type I.
- Pivots that involve changes from $(\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0, N+1\}}$ to $(\mathcal{K}'_n, \mathcal{J}'_n)_{n \in \{0, N'+1\}}$ in which the solution at the collision point is not unique. We refer to these as boundary pivots of type II.

We now describe all the possible types of pivots:

### Pure internal SCLP pivots

When the collision from validity region $\mathcal{V}$ is of types (a), (b), or (c) the pivot is an internal pivot.
- Collision of type (a): A single state variable $x_{k*}^*$ or $q_{j*}^*$ shrinks to 0. This is classified as SCLP-III type collision in [23]. In this pivot new internal bases $D_1, \ldots, D_M$ are inserted before $B_1$ (if $n = 0$), or after $B_N$ (if $n = N$), or between $B_n$ and $B_{n+1}$ (otherwise). At the collision point these new bases have interval length 0, and they become $> 0$ in the interior of $\mathcal{W}$. The collision from $\mathcal{W}$ is of type (c) if the shrinking state variable is tied, or of type (d) if it is free.
- Collision of type (b): Intervals $\tau_{n_*}, \ldots, \tau_{n**}$, $1 < n^* < n^{**} < N$ shrink to zero, with $|B_{n_*-1} \setminus B_{n_*+1}| = 2$. This is classified as SCLP-II type collision in [23]. The internal pivot involves removing the bases $B_{n_*}, \ldots, B_{n**}$, inserting new bases $D_1, \ldots, D_M$, $M \geq 1$ in their place. At the collision point these new bases have interval length 0, and they become $> 0$ in the interior of $\mathcal{W}$. The collision from $\mathcal{W}$ is also of type (b).
- Collision of type (c): Intervals $\tau_{n_*}, \ldots, \tau_{n**}$ shrink to zero, where $1 < n^* < n^{**} < N$ with $|B_{n_*-1} \setminus B_{n_*+1}| = 1$, or $n^* = 1$, or $n^{**}$ is $N$ but no tied state variable becomes free. This is classified as SCLP-I type collision in [23]. The internal pivot involves removing the bases $B_{n_*}, \ldots, B_{n**}$. The collision from $\mathcal{W}$ is of type (a).
M-CLP pivots that involve boundary pivots

This includes collisions of type (d), (e), (f) from the \( V \) side, and consists of up to three steps: it may require an initial internal SCLP pivot (pre-boundary step), it may then require a pivot on the boundary dictionary (boundary step), and it may finally require another internal SCLP pivot (post-boundary step).

The pre-boundary step is determined by the type of collision on the \( V \) side. Once pre-boundary SCLP pivot is complete the boundary pivot is carried out as follows: First the Boundary-LP simplex dictionary for \( V \) is constructed. Then the leaving variable is determined according to the type (d), (e) or (f). It is then determined if the pivot is type I or type II. If it is type I no pivot is performed, but the bases \( (K_n, J_n)_{n=1}^{N} \) may change. If it is type II, a pivot of the boundary dictionary is performed, and the new bases \( (K'_n, J'_n)_{n=1}^{N'} \) are obtained. This also determines the type of collision from the \( W \) side. Finally, according to the type of collision from the \( W \) side a post-boundary SCLP pivot may be performed. We now describe these steps.

**The pre-boundary step**

- Collision of type (d) or (f): The bases of intervals that shrink to 0 are removed.
- Collision of type (e): No pre-boundary pivot is required.

We denote by \( (\tilde{K}_n, \tilde{J}_n)_{n=1}^{N''} \) the base sequence obtained from \( (K_n, J_n)_{n=1}^{N} \) after the pre-boundary step, and refer to it as the intermediate base sequence.

**Constructing the Boundary-LP simplex dictionary for \( V \)** In all three cases we follow the rules detailed in Section 3.1. In case (d) we need some additional rules: If \( x_k \) became free then if \( |J^{\downarrow}| + |K_{N+1}| + |J_0| + |K_0| < 2J \) we choose \( p^0_k \) as a dual basic in \( B^* \), and if \( |J^{\downarrow}| + |K_{N+1}| + |J_0| + |K_0| = 2J \), we choose \( x_k^* \) as basic in \( B \). Similarly, if \( q_j \) becomes free then if \( |K^{\downarrow}| + |J_0| + |K_{N+1}| + |J_{N+1}| < 2K \) we choose \( u^N_j \) as basic in \( B \), and if \( |K^{\downarrow}| + |J_0| + |K_{N+1}| + |J_{N+1}| = 2K \) we choose \( q_j^* \) as dual basic in \( B^* \). We denote the Boundary-LP simplex dictionary for \( V \) by \( D \).

**Determining the leaving variable** In cases (e), (f) the variable leaving the basis is the boundary variable that has shrunk to 0 in the collision from \( V \). We denote it by \( v \), and it can be either a primal variable or a dual variable. We denote its dual variable by \( v^* \).

In case (d), if \( x_k \) has become free and \( x_k^* \) is basic, or if \( q_j \) has become free and \( q_j^* \) is basic, then no variable leaves the basis and there is no pivot. We classify this case as a pivot of type I. Otherwise \( v = p^0_k \) or \( v = u^N_j \) respectively.

**Determine the type of boundary pivot and the entering variable** We have following possibilities:

- If \( v \) is a primal (or a dual) variable, examine the dual (or the primal) variables which are \( w^* = 0 \). If any of them has a non-zero pivot element (i.e. if \( v \) is in row \( i \) and \( w^* \) is in column \( j \), the element \( \tilde{A}_{ij} \)) then this is a type I pivot.
- Otherwise the entering variable \( w \) is determined, as in standard LP, by the ratio test:
  - If there is a single candidate variable \( w \) to enter with ratio > 0 then we have a pivot of type II and \( w \) is the entering variable.
If there are multiple candidates to enter with ratio > 0, then there are several variables shrinking to 0 on the \( \mathcal{W} \) side. This indicates that there is a multiple collision from the \( \mathcal{W} \) side. We discuss multiple collisions in Section 3.2.

**Boundary-pivot of type I** The solution at the collision point is unique, and there is no need for a pivot on the dictionary. In the case of collision type (d), when \( \mathbf{v} = \mathbf{p}_k^0 \) (or \( \mathbf{v} = \mathbf{u}_j^N \)) we have \((\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0,N+1\}} = (\mathcal{K}^\prime_n, \mathcal{J}^\prime_n)_{n \in \{0,N^\prime+1\}}\), and the only change is that \( x_k \) (or \( q_j \)) is no longer tied but becomes free in \( \mathcal{W} \). Otherwise, the solution at the collision point is unique, but \((\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0,N+1\}} \neq (\mathcal{K}^\prime_n, \mathcal{J}^\prime_n)_{n \in \{0,N^\prime+1\}}\).

**Boundary pivot type II:** Assume \( \mathbf{v} \) leaving and \( \mathbf{w} \) entering. Let \( \mathbf{v}^*, \mathbf{w}^* \) be their dual variables. Then the effect of the pivot is as follows: Because the leaving variable \( \mathbf{v} = 0 \), after the pivot the new basic variable \( \mathbf{w} = 0 \). However, since the test ratio is > 0, the variable \( \mathbf{w}^* \) that leaves the dual basis is > 0, and when it leaves the basis its value will jump down to 0. Also, the variable entering the dual basis \( \mathbf{v}^* \) will jump from the non-basic value of 0 to a value > 0. In addition, all the basic boundary variables in the same category as \( \mathbf{w}^*, \mathbf{v}^* \) (that is in \( \mathbf{B}^* \) or in \( \mathbf{B} \)) according to whether \( \mathbf{v} \) is primal or dual) will change their values.

**Determine the variable shrinking to 0 from the \( \mathcal{W} \) side:** From the \( \mathcal{W} \) side we may have a single boundary variable shrinking to 0, which we denote by \( \mathbf{v}' \), or there may be no boundary value that shrinks to 0 and we then define \( \mathbf{v}' = 0 \).
- If the pivot is type I and the collision from \( \mathcal{V} \) is (d) then \( \mathbf{v}' = \emptyset \).
- If the pivot is type I and the collision from \( \mathcal{V} \) is (e) or (f) then \( \mathbf{v}' = \mathbf{v}^* \).
- In case of boundary pivot of type II, if \( \mathbf{w}^* = x_k^0, \bar{x}_k > 0 \) (or if \( \mathbf{w}^* = q_j^0, \bar{q}_j > 0 \)) then \( \mathbf{v}' = \emptyset \).
- Otherwise, for pivot of type II, \( \mathbf{v}' = \mathbf{w} \).

**Determine the new boundary basis:** To determine the new boundary bases \((\mathcal{K}^\prime_n, \mathcal{J}^\prime_n)_{n \in \{0,N^\prime+1\}}\) we first find the new values of \( x^0, q^N \), from the new values of \( x^*, q^* \) after the pivot, using \( x^0 = x^* + \bar{x}, q^N = q^* + \bar{q} \). Recall that \( \bar{x}, \bar{q} \) are uniquely determined at \( \mathcal{L}(\bar{\theta}) \) by Proposition 3.2.

Following this step we define \((\mathcal{K}^\prime_n, \mathcal{J}^\prime_n)_{n \in \{0,N^\prime+1\}}\) as follows:

\[
\begin{align*}
&j \in \mathcal{J}^\prime_0 \quad \text{if } a_j^0 > 0, \quad \text{or } \mathbf{v}' = a_j^0, \\
&k \in \mathcal{K}^\prime_0 \quad \text{if } x_k^0 > 0, \quad \text{or } \mathbf{v}' = x_k^*, \\
&j \in \mathcal{J}^\prime_{N^\prime+1} \quad \text{if } q_j^0 > 0, \quad \text{or } \mathbf{v}' = q_j^*. \\
&k \in \mathcal{K}^\prime_{N^\prime+1} \quad \text{if } x_k^N > 0, \quad \text{or } \mathbf{v}' = x_k^N.
\end{align*}
\]

This simply includes all > 0 boundary values, and the variable \( \mathbf{v}' \) (if \( \mathbf{v}' \neq \emptyset \)), and includes exactly \( 2(K + J) \) elements.

**Test for a multiple collision from the \( \mathcal{W} \) side:** If \( \mathbf{w}^* = x_k^* \) or \( \mathbf{w}^* = q_j^* \) and at \( \mathcal{L}(\bar{\theta}) \) the minimum of \( x_k(t) \) or \( q_j(t) \) is obtained in more than one point, as seen from \( \arg \min_{0 \leq n \leq N} x_k^n \) or \( \arg \min_{0 \leq n \leq N} q_j^n \) > 1, then there is a multiple collision from the \( \mathcal{W} \) side. This is illustrated in fig. 7 where \( x_k^*(\bar{\theta}) \) denotes the value of the optimal \( x_k^* \) for parameters \( \mathcal{L}(\bar{\theta}) \).

**Determine the type of collision from the \( \mathcal{W} \) side:** We assume that no multiple collision is discovered in the previous step.
- If \( \mathbf{v}' = \emptyset \) and the boundary pivot is of type I, the collision from the \( \mathcal{W} \) side is type (a).
If \( \mathbf{v}' = \emptyset \) and the boundary pivot is of type II, the collision form the \( W \) side is type (d).

- If \( \mathbf{v}' = x_k^* \) and \( \mathcal{K}_0' \) is not compatible with \( \bar{\mathcal{K}}_1 \) (of the intermediate base sequence), or if \( \mathbf{v}' = q_j^* \) and \( \mathcal{J}_{N+1}' \) is not compatible with \( \bar{\mathcal{J}}_N \), then the collision from the \( W \) side is of type (f).

- For any other \( \mathbf{v}' \neq \emptyset \), the collision from the \( W \) side is of type (e).

### The post-boundary step

In the post-boundary step we perform an SCLP pivot and build a base sequence \((\mathcal{K}_n', \mathcal{J}_n')_{n=1}^{N''}\) that is optimal on the \( W \) side.

- If the collision from the \( W \) side is of type (d), and \( \mathbf{w}^* = x_k^* \) and \( t_n \) is the minimum point of \( x_k(t) \) at \( L(\theta) \), or \( \mathbf{w}^* = q_j^* \) and \( t_n \) is the minimum point of \( q_j(T-t) \) at \( L(\theta) \), the bases \( D_1, \ldots, D_M \) need to be inserted between \( \tilde{B}_n, \tilde{B}_{n+1} \), in the sequence \((\tilde{\mathcal{K}}_n, \tilde{\mathcal{J}}_n)_{n=1}^{N''} \) as in SCLP-III type collision in [23].

- If the collision from the \( W \) side is of type (f), then if \( \mathbf{v}' = x_k^* \), then a number of bases \( D_1, \ldots, D_M \) are inserted before \( \tilde{B}_1 \), and if \( \mathbf{v}' = q_j^* \), then a number of bases \( D_1, \ldots, D_M \) are inserted after \( \tilde{B}_N' \), as in SCLP-IV type collision in [23].

We now formulate the following two theorems that state the conditions under which the pivot operation is well defined and constructs an optimal solution for the neighboring region \( W \).

**Theorem 6.1.** If the collisions on both the \( V \) and the \( W \) side are of types (a)–(f) then the M-CLP pivot constructs a unique base sequence \((\mathcal{K}_n', \mathcal{J}_n')_{n=0}^{N''+1}\).

**Proof.** See Appendix A.3

**Theorem 6.2.** If the collisions on both the \( V \) side as \( \theta \searrow \bar{\theta} \) and on the \( W \) side as \( \theta \nearrow \bar{\theta} \) are both of types (a)–(f) then the base sequence \((\mathcal{K}_n', \mathcal{J}_n')_{n=0}^{N''+1} \) constructed by the M-CLP pivot is the optimal base sequence in \( W \).

**Proof.** See Appendix A.3

### 7 Algorithm under simplifying assumption

Consider an M-CLP problem and let \( \beta_g, \gamma_g, \lambda_g, \mu_g, T_g \) be a goal set of boundary parameters in the feasible parametric region, i.e. the set of boundary parameters for which an optimal solution should be found. Let \( \beta_0, \gamma_0, \lambda_0, \mu_0, T_0 \) be another set of boundary parameters with a known optimal solution. Define the parametric line of boundary parameters as \( L(\theta) = (1 - \theta)(\beta_0, \gamma_0, \lambda_0, \mu_0, T_0) + \theta(\beta_g, \gamma_g, \lambda_g, \mu_g, T_g) \), \( 0 \leq \theta \leq 1 \). The algorithm described in this section requires the following assumption:
Assumption 7.1 (Simplifying assumption). Assume that all collisions along the parametric line $L(\theta)$ are of type (a)-(f).

We discuss the algorithm without this assumption in Section 9.

7.1 Properties of the parametric line

Theorem 7.2. Let $L(\theta), 0 \leq \theta \leq 1$ be a parametric line of boundary parameters. If M-CLP/M-CLP* are feasible for both goal set of parameters $L(1)$ and initial set of parameters $L(0)$, then:

(i) all points of the parametric line belong to the parametric-feasible region,

(ii) if $L(0)$ is an interior point of some validity region, then all points of the parametric line except possibly $L(1)$ belong to the interior of the parametric-feasible region,

(iii) the parametric line can be partitioned to a minimal finite set of closed intervals of positive length $[\theta_{\ell-1}, \theta_{\ell}]$, $\ell = 1, \ldots, L \leq \left[\frac{(K+J)}{2}K+J\right] 2^{(K+J)}$ such that for each interval there is a (possibly non-unique) proper base sequence which is optimal for every $L(\theta), \theta \in [\theta_{\ell-1}, \theta_{\ell}]$.

Proof. See Appendix A.3

Proposition 7.3. Consider a continuous parametric line of boundary parameters $L(\theta)$. Then the decomposition parameters $\tilde{U}(\theta), \tilde{x}(\theta), \tilde{P}(\theta), \tilde{q}(\theta)$ are continuous piecewise affine functions of $\theta$.

Proof. Within each closed validity region $\tilde{U}(\theta), \tilde{x}(\theta), \tilde{P}(\theta), \tilde{q}(\theta)$ are affine by Corollary 4.2. Moreover, by Proposition 3.2, for every set of boundary parameters $\tilde{U}, \tilde{x}, \tilde{P}, \tilde{q}$ are unique, and therefore $\tilde{U}(\theta), \tilde{x}(\theta), \tilde{P}(\theta), \tilde{q}(\theta)$ are continuous at the boundary points of the validity regions $L(\theta_{\ell})$.

7.1.1 Initial solution

The following theorem shows how to obtain a set of parameters for a simple initial optimal solution.

Theorem 7.4. Consider a set of boundary parameters $\beta_0, \gamma_0, \lambda_0, \mu_0, T_0$ that satisfy:

$$
\beta_0 > 0, \quad \lambda_0 < 0, \quad \beta_0 + bT_0 + \lambda_0 > 0, \quad \gamma_0 < 0, \quad \mu_0 > 0, \quad \gamma_0 + cT_0 + \mu_0 < 0, \quad (24)
$$

then M-CLP/M-CLP* have a single-interval optimal solution, with $U(t) = P(t) = 0$, and $x(t), -q(t)$ equal to the right hand sides of (7), (3), and this set of boundary values is an interior point of the validity region.

Proof. It is easy to see that under (24) $U(t) = 0, x(t) = \beta_0 + bt, 0 \leq t < T_0, x(T) = \beta_0 + bT_0 + \lambda_0, P(t) = 0, q(t) = -\gamma_0 - ct, 0 \leq t < T_0, q(T) = -\gamma_0 - cT_0 - \mu_0$ are feasible complementary slack solutions of M-CLP/M-CLP*, and hence are optimal. They define an optimal base sequence $(K_n, J_n)_{n=0}^2$, $K_0 = K_1 = K_2 = \{1, \ldots, K\}, J_0 = J_1 = J_2 = \{1, \ldots, J\}$, with a single interval solution. Moreover, one can see that the solution to the corresponding Boundary-LP/LP* is unique, and hence by Theorem 3.5 this solution of M-CLP/M-CLP* is unique. It is easy to see that $H_x > 0$ and so by Theorem 4.4 the parameter values are an interior point of its validity region.

□
7.2 The algorithm

We now describe a simplex-type algorithm to solve M-CLP/M-CLP* in a finite number of steps. For the algorithm to work we need the non-degeneracy assumption 1.1 and the simplifying assumption 7.1 to hold.

7.2.1 Input

Problem data consists of $A, b, c$, and goal parameters $L(1) = (\beta_g, \gamma_g, \lambda_g \leq 0, \mu_g \geq 0, T_g)$.

7.2.2 Output

Optimal solution of M-CLP/M-CLP* at $L(1)$: optimal base sequence, breakpoint times, impulse controls at 0 and $T_g$, rates $u, \dot{x}, p, \dot{q}$ in each interval, states $x(t), q(T_g - t)$ at each breakpoint.

7.2.3 Feasibility test

Solve Test-LP/LP* for $L(1)$ to determine whether both M-CLP/M-CLP* are feasible, or both are infeasible, or one is infeasible and the other unbounded. Proceed to solve the problems if both are feasible, otherwise stop.

7.2.4 Initialization

Choose $L(0) = (\beta_0, \gamma_0, \lambda_0, \mu_0, T_0)$ that satisfy:

$$
\beta_0 > 0, \quad \gamma_0 < 0, \quad \lambda_0 < 0, \quad \mu_0 > 0, \quad T_0 > 0,
$$

$$
\beta_0 + bT_0 + \lambda_0 > 0, \quad \gamma_0 + cT_0 + \mu_0 < 0.
$$

Define $\delta L = L(1) - L(0)$.

The initial solution has $U(t) = P(t) = 0, x(t), -q(t)$ are r.h.s. of $(1), (2)$ at $L(0)$, and the initial optimal base sequence is $(K_n, J_n)_{n=0}^2 = \{(K_n = \{1, \ldots, K\}, J_n = \{1, \ldots, J\})$, $n = 0, 1, 2\}$. Set: $\ell := 1, \quad \theta_\ell := 0, \quad (K, J)^{(\ell)} := (K_n, J_n)_{n=0}^2$.

7.2.5 Iteration

- Set up equations

Calculate rates for all bases in $(K, J)^{(\ell)}$, construct the coefficients of the matrix $M_\ell$ for $(K, J)^{(\ell)}$. Construct $R_\ell, \delta R_\ell$ from $L(\theta_\ell)$ and $\delta L$.

- Calculate current solution and its $\theta$ gradient

Solve the equations

$$
M_\ell H^\ell = R_\ell, \quad M_\ell \delta H^\ell = \delta R_\ell.
$$

- Find right endpoint of the validity region

Find

$$
1/\Delta = \max_{\{r: H^\ell_r \in H^\ell\}} \left(0, \frac{-\delta H^\ell_r}{H^\ell_r}\right).
$$

The right endpoint of the validity region is $\bar{\theta} := \theta_\ell + \Delta$. 

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• Termination test and conclusion

If \( \theta_\ell < 1 \leq \bar{\theta} \) terminate: \((K, J)^{(\ell)}\) is the optimal base-sequence, current rates are optimal, remaining output is obtained from \( H = H^\ell + (1 - \theta_\ell)\delta H^\ell \).

• Update solution

Calculate values of the solution at \( \bar{\theta} \), as \( H = H^\ell + \Delta \delta H^\ell \).

• Classify collision as \( \theta \nearrow \bar{\theta} \)

Find the set of elements of \( H^\ell \) that are 0 at \( L(\bar{\theta}) \). Split \( K_0, J_{N+1} \) to sets \( K^=, K^{\uparrow\downarrow}, J^=, J^{\uparrow\downarrow} \). Classify the collision to types (a)-(f) as discussed in Section 5.

• Pivot

Perform M-CLP pivot to determine a new base sequence, \( (K'_n, J'_n)_{n=0}^{N'+1} \) as discussed in Section 6. Set \( \ell := \ell + 1, \theta_\ell := \bar{\theta}, (K, J)^{(\ell)} := (K'_n, J'_n)_{n=0}^{N'+1} \) and move to the next iteration.

7.3 Verification of the algorithm

In this section we verify that under simplifying assumption (7.1) and non-degeneracy assumption (1.1) all calculations described in Section 7 are possible, and for any pair of feasible M-CLP/M-CLP* problems the algorithm produces an optimal solution in a finite bounded number of iterations. The main point here is to show that the new base sequence is optimal for \( \theta > \bar{\theta} \) small enough, and we show that this property follows from Theorem 6.2.

**Theorem 7.5.** Assume that M-CLP/M-CLP* are feasible for the goal set of parameters \( \beta_g, \gamma_g, T_g, \lambda_g, \mu_g \). Assume also non-degeneracy assumption (1.1) and simplifying assumption (7.1). Then the algorithm will compute the solution of M-CLP/M-CLP* \((\beta_g, \gamma_g, T_g, \lambda_g, \mu_g)\) in a finite number of iterations, bounded by \( L \leq \left( \frac{4(K+J)}{2(K+J)} \right) 2^{\left( \frac{K+J}{K} \right)} \).

**Proof.** The proof follows from the following sequence of Propositions.

**Proposition 7.6.** The feasibility test can be performed, to determine whether M-CLP is feasible and bounded, or is infeasible, or is unbounded.

**Proof.** Clearly, one can determine if Test-LP (4) is feasible and bounded, or is infeasible or is unbounded, which determines the same for M-CLP.

**Proposition 7.7.** Initialization phase of the algorithm yields a unique optimal solution with a unique optimal base sequence, and \( L(0) \) is an interior point of its validity region.

**Proof.** Follows from Theorem 7.4.

**Proposition 7.8.** Assume that \( L(\theta) \) for some \( \theta > \theta_\ell \) is in the interior of the validity region of \((K, J)^{(\ell)}\). Then setting up equations, calculation of the current solution and its gradient, and calculation of the right endpoint \( \bar{\theta} \), as well as the termination test can be carried out.
Proof. The base sequence \((\mathcal{K}, \mathcal{J})^{(l)}\) determines the structure of \(M_l\) and of \(R_l\). Solution of the Rates-LP/LP* for \((K_n, J_n)_{n=1}^N\) is unique by non-degeneracy assumption, and provides the non-zero coefficients of \(M_l\). The non-zero coefficients of \(R_l\) are given by \(\mathcal{L}(\theta_l)\). \(\delta R_l\) has the same structure as \(R_l\), and its non-zero elements are given by \(\delta \mathcal{L}\).

We assume in the statement of the proposition that the base sequence \((\mathcal{K}, \mathcal{J})^{(l)}\) has non-empty interior, and hence by Proposition 4.5 the matrix \(M_l\) is non-singular. Hence \(H^l\) is the unique solution of \(M_l H^l = R_l\) and \(\delta H^l\) is the unique solution of \(M_l(\delta H^l) = \delta R_l\). We denote by \(H(\theta)\) the solution at point \(\mathcal{L}(\theta)\). Then at the point \(\theta > \theta_l\) in the interior of the validity region the solution is determined by

\[
M_l H(\theta) = M_l(H^l + (\theta - \theta_l)\delta H^l) = R_l + (\theta - \theta_l)\delta R_l.
\]

That \(\Delta > 0\) follows because the interior of the validity region of \((\mathcal{K}, \mathcal{J})^{(l)}\) is non-empty and contains an interval of the line \(\mathcal{L}(\theta)\). It is possible to have \(1/\Delta = 0\) and \(\Delta = \infty\).

Finally, if any of \(\delta H_r < 0\) then by \(H_r(\theta) \geq 0\) it follows that \(H_r > 0\), and the ratio \(-\delta H_r / H_r > 0\) and it is finite, and so \(1/\Delta > 0\) and finite so that \(0 < \Delta < \infty\). Otherwise \(1/\Delta = 0\) and \(\Delta = \infty\).

\[\Box\]

Proposition 7.9. M-CLP pivot can be performed and is unique. There exists \(\bar{\theta}\) small enough, such that base sequence \((\mathcal{K}, \mathcal{J})^{(l+1)}\) is optimal for this \(\theta\) and \(\mathcal{L}(\theta)\) is an interior point of its validity region.

Proof. Clearly \(\bar{\theta} < 1\) and hence by Theorem 7.2 \(\mathcal{L}(\bar{\theta})\) is an interior point of the parametric-feasible region. We prove by induction that for iteration \(l\) the base sequence \((\mathcal{K}, \mathcal{J})^{(l)}\) has a validity region with non-empty interior, i.e. there is a \(\theta < \bar{\theta}\) big enough, such that the base sequence \((\mathcal{K}, \mathcal{J})^{(l)}\) is optimal for this \(\theta\) and \(\mathcal{L}(\theta)\) is an interior point of its validity region. Clearly this assumption holds for \(l = 0\). Assume the induction hypothesis for \(l\) which is optimal for \(V\). Then since validity regions are closed (Theorem S3.7), there exists another validity region with non-empty interior \(W\), such that \(\mathcal{L}(\bar{\theta}) \in W\). Furthermore, by the simplifying assumption 7.1 the collision from \(W\) is of type (a)–(f) and hence by Theorem 6.2 base sequence \((\mathcal{K}, \mathcal{J})^{(l+1)}\) that is optimal for \(W\) can be obtained from \((\mathcal{K}, \mathcal{J})^{(l)}\) by a single unique M-CLP pivot. Hence, there exists \(\theta > \bar{\theta}\) small enough, such that base sequence \((\mathcal{K}, \mathcal{J})^{(l+1)}\) is optimal for this \(\theta\) and \(\mathcal{L}(\theta)\) is an interior point of its validity region \(W\). This completes the proof.

\[\Box\]

Proposition 7.10. The total number of iterations is bounded by \(L \leq \binom{4(K+J)}{2} \binom{K+J}{K}\).

Proof. Follows from Theorem 7.2

\[\Box\]

8 Example

We use data from example S6.2 to solve a problem using our algorithm. We note that this is the smallest problem for which we can demonstrate most of the features of the algorithm, but even so, it involves 8 function variables, and necessitates quite a lot of data to describe it.

The solution requires four iterations with \(\theta\) moving through 0, \(\theta_1 = \frac{2}{27}\), \(\theta_2 = \frac{9}{27}\), \(\theta_3 = \frac{2}{3}\), \(\theta_4 = 1\). At these points the collision types from left and right are: ↘ \(\theta_1: (e); \downarrow \theta_1: (e); \uparrow \theta_2: (e); \downarrow \theta_2: (d); \uparrow \theta_3: (a); \downarrow \theta_3: (c); \uparrow \theta_4: \) multiple. Pivots at iterations 1 and 2 are type II boundary pivots, we present their boundary simplex dictionaries. The pivot of iteration 2 also requires a post-boundary SCLP pivot. Pivot at iteration 3 is an internal SCLP pivot.
A detailed description of the steps of the algorithm follows. In the figures we show the solutions at the collision points and also at points \( \theta \) between collisions, where \( H_\theta > 0 \). A guide for reading the figures appears after Figure 8.

**DATA** The problem data are:

\[
A = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \beta_g = \begin{pmatrix} 8 \\ 10 \end{pmatrix}, \quad \gamma_g = \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \quad T_g = 2.
\]

with \( \lambda = \mu = 0 \).

We consider following initial boundary parameters, which satisfy conditions (24) of Theorem 7.4.

\[
\beta_0 = \beta_g = \begin{pmatrix} 8 \\ 10 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \quad \lambda_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \mu_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T_0 = 0.1.
\]

The \( \theta \)-gradient \( \delta \mathcal{L} \) for the parametric line between the initial and the goal boundary parameters is:

\[
\delta \beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta \gamma = \begin{pmatrix} 7 \\ 8 \end{pmatrix}, \quad \delta \lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \delta \mu = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \delta T = 1.9.
\]

**INITIALIZATION STEP** By Theorem 7.4 the initial solution has a single interval base sequence \((\mathcal{K}, \mathcal{J})^{(1)}\) which is \((\mathcal{K}_n, \mathcal{J}_n)_{n=0}^2 = (\{1, 2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\).

**FIRST ITERATION** The rates values for the base sequence \((\mathcal{K}, \mathcal{J})^{(1)}\) are:

\[
\dot{x}^1 = b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \dot{u}^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dot{q}^1 = -c = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad \dot{p}^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Solution for the point \( \mathcal{L}(0) \) defined in the Section 7.1.1 and given by \( H^{(1)} \):

\[
x^0 = \beta_0 = \begin{pmatrix} 8 \\ 10 \end{pmatrix}, \quad x^1 = \beta_0 + bT_0 = \begin{pmatrix} 8.3 \\ 10.1 \end{pmatrix}, \quad x^N = \beta_0 + bT_0 + \lambda_0 = \begin{pmatrix} 7.3 \\ 9.1 \end{pmatrix}, \\
q^N = -\gamma_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad q^0 = -\gamma_0 - cT_0 = \begin{pmatrix} 1.9 \\ 1.8 \end{pmatrix}, \quad q^0 = -\gamma_0 - cT_0 - \mu_0 = \begin{pmatrix} 0.9 \\ 0.8 \end{pmatrix}, \\
\tau_1 = T_0 = 0.1, \quad u^0 = u^N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p^N = p^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The solution for \( \theta = 0 \) is displayed on Figure 8.

**GUIDE TO READ THE FIGURES** In all the figures (similar to graphic illustrations in [18]), primal states and dual controls are shown on the positive \( y \)-axis, dual states and primal controls on the negative \( y \)-axis, primal time is running left to right, dual time is running right to left, impulse controls are presented by arrows, and boundary values of \( x \) and \( q \) are presented as dots.

\( \delta H^{(1)} \) for the base sequence \((\mathcal{K}, \mathcal{J})^{(1)}\) is: \( \delta \tau_1 = 1.9, \)

\[
\delta x^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta x^1 = \begin{pmatrix} 5.7 \\ 1.9 \end{pmatrix}, \quad \delta x^N = \begin{pmatrix} 6.7 \\ 2.9 \end{pmatrix}, \quad \delta u^0 = \delta u^N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\delta q^N = \begin{pmatrix} -7 \\ -8 \end{pmatrix}, \quad \delta q^0 = \begin{pmatrix} -8.9 \\ -11.8 \end{pmatrix}, \quad \delta q^0 = \begin{pmatrix} -7.9 \\ -10.8 \end{pmatrix}, \quad \delta p^N = \delta p^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The right end-point of the validity region \( \bar{\theta} = \theta_1 + \bar{\Delta} = 0 + \frac{2}{\pi} = \frac{2}{\pi} \).
At $\theta = \frac{2}{27}$, $q_2^0$ shrinks to 0 and hence there is a collision of type (e) as $\theta \nearrow \frac{3}{4}$. The solution for $\theta \nearrow \frac{3}{4}$ is displayed on Figure 9.

At $\theta = \frac{2}{27}$, $q_2^0$ shrinks to 0 and hence there is a collision of type (e) as $\theta \nearrow \frac{3}{4}$. The solution for $\theta \nearrow \frac{3}{4}$ is displayed on Figure 9. At $\theta = \frac{2}{27}$, $q_2^0$ shrinks to 0 and hence there is a collision of type (e) as $\theta \nearrow \frac{3}{4}$. The solution for $\theta \nearrow \frac{3}{4}$ is displayed on Figure 9.

Figure 9: Solution for $\theta \nearrow \theta_1 = \frac{2}{27}$.

Figure 10: Solution for $\theta \searrow \theta_1 = \frac{2}{27}$. 
**Comment on non-unique solution at $\theta = \frac{2}{27}$:** We compare Figures 9 and 10 of the non-unique solution at $\bar{\theta}$. In the new validity region $p_N^N$ is basic with value 0, and will increase for $\theta > \theta_2$, and $x_N^N$ jumped down to 0 and is non-basic. A unit impulse $u_0^0 > 0$ replaces $q_0^0$ that is no longer basic. Both $x_0^1$ and $x_0^2$ and with them all of $x_1(t), x_2(t)$ jump down to lower but still positive values.

**Pivot for the first iteration** For the base sequence $(K, J)^{(1)}$ we have $J^* = K^* = \emptyset$, the values of the decomposition parameters at the collision point are: $\tilde{x} = [0, 0], \tilde{q} = [\frac{13}{54}, \frac{13}{27}]$, and the boundary simplex dictionary (which in this case is unique) is:

| $q_0^0$ | $q_0^2$ | $q_1^*$ | $q_2^*$ |
|---------|---------|---------|---------|
| $\frac{17}{54}$ | 0 | $\frac{67}{54}$ | $\frac{25}{27}$ |

$x_1^* = 8$ | $5$ | $2$ | $0$ | $0$

$x_2^* = 10$ | $3$ | $4$ | $0$ | $0$

$x_1^N = \frac{421}{54}$ | $5$ | $2$ | $5$ | $2$

$x_2^N = \frac{503}{54}$ | $3$ | $4$ | $3$ | $4$

For this dictionary we have a single candidate to enter $w = p_N^N$ with ratio $\frac{503}{216} > 0$ and hence we perform a boundary pivot of type II. Therefore, the solution of M-CLP at the collision point is not unique. At the boundary pivot, $q_0^0$ is exchanged with $p_N^N$, and $x_N^N$ is exchanged with $u_0^0$. $v' = w = p_N^N$ shrinking to 0 from the $W$ side and hence, the collision as $\theta \searrow \bar{\theta}$ is of type (e).

The new base sequence is $(K, J)^{(2)} = (\{1, 2\}, \{1\}), (\{1, 2\}, \{1, 2\}), (\{1\}, \{1, 2\})$.

**Second iteration.** The rates for the base sequence $(K, J)^{(2)}$ are the same as for the base sequence $(K, J)^{(1)}$, and the solution $H^{(2)}$ for $\theta = \frac{2}{27}$ is: $\tau_1 = \frac{13}{54}$,

$u^0 = \left( \begin{array}{c} 0 \\ \frac{503}{216} \end{array} \right), \hspace{1cm} x^0 = \left( \begin{array}{c} \frac{361}{108} \\ \frac{37}{54} \end{array} \right), \hspace{1cm} x^N = \left( \begin{array}{c} \frac{113}{36} \\ 0 \end{array} \right), \hspace{1cm} u^N = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \hspace{1cm} x^1 = \left( \begin{array}{c} \frac{439}{108} \\ \frac{25}{27} \end{array} \right), \hspace{1cm} q^0 = \left( \begin{array}{c} \frac{67}{54} \\ 0 \end{array} \right)$

$x^1 = \left( \begin{array}{c} \frac{40}{27} \\ \frac{38}{27} \end{array} \right), \hspace{1cm} q^0 = \left( \begin{array}{c} \frac{67}{54} \\ 0 \end{array} \right)$

The solution for $\theta \searrow \frac{2}{27}$ is displayed on Figure 10.

![Figure 11: Solution for $\theta = \frac{301}{1566}$, $\theta_1 < \theta < \theta_2$.](image-url)
δH(2) for the base sequence (ائها, ᾣ)2 is: δτ1 = \frac{19}{10}.

\delta u^0 = \begin{pmatrix} 0 \\ \frac{29}{10} \end{pmatrix}, \quad \delta x^0 = \begin{pmatrix} -\frac{29}{10} \\ -\frac{29}{10} \end{pmatrix}, \quad \delta x^N = \begin{pmatrix} \frac{21}{4} \\ 0 \end{pmatrix}, \quad \delta u^N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta x^1 = \begin{pmatrix} \frac{17}{4} \\ -1 \end{pmatrix},

\delta q^0 = \begin{pmatrix} \frac{2}{10} \\ 0 \end{pmatrix}, \quad \delta p^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta p^N = \begin{pmatrix} \frac{11}{10} \\ \frac{28}{10} \end{pmatrix}, \quad \delta q^N = \begin{pmatrix} \frac{8}{10} \\ -1 \end{pmatrix}.

The solution for an interior point of the validity region is displayed on Figure 11.

The right end-point of the validity region \( \bar{\theta} = \theta_2 + \Delta = \frac{2}{27} + \frac{185}{783} = \frac{9}{29} \).

At \( \theta = \frac{9}{29} \) \( x_2^0 \) shrinks to 0 and hence there is a collision of type (e) as \( \theta \nearrow \bar{\theta} \). The solution for \( \theta \nearrow \frac{9}{29} \) is displayed on Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{solution_theta_nearbartheta.png}
\caption{Solution for \( \theta \nearrow \theta_2 = \frac{9}{29} \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{solution_theta_decr.png}
\caption{Solution for \( \theta \searrow \theta_2 = \frac{9}{29} \).}
\end{figure}

**Pivot for the second iteration** For the base sequence (ائها, ᾣ)2 we have \( \mathcal{J}^1 = \mathcal{K}^1 = \emptyset \), the values of decomposition parameters at the collision point are: \( \tilde{x} = [0, 0], \tilde{q} = \left[ \frac{20}{29}, \frac{40}{29} \right] \) and the
boundary simplex dictionary (which in this case is again unique) is:

|   | \( q_0^1 \) | \( p_2^N \) | \( q_1^* \) | \( q_2^* \) |
|---|---|---|---|---|
| \( x_1^* \) | 3 | \( \frac{7}{2} \) | \(-\frac{1}{2}\) | \(-\frac{3}{2}\) | \(-2\) |
| \( x_2^* \) | 0 | 0 | \(-1\) | \(-3\) | \(-4\) |
| \( x_1^N \) | \( \frac{127}{29} \) | \( \frac{7}{2} \) | \(-\frac{1}{2}\) | \(\frac{7}{2}\) | 0 |
| \( u_0^0 \) | \( \frac{5}{2} \) | 0 | 0 | \(\frac{3}{4}\) | 1 |

For this dictionary we have a single candidate to enter \( w = u_2^N \) with ratio \( \frac{5}{29} > 0 \) and hence we perform a boundary pivot of type II. Therefore, the solution of M-CLP* at the collision point is not unique. In the boundary pivot \( x_1^* \) is exchanged with \( u_2^N \) and \( q_2^* \) is exchanged with \( p_2^0 \) and hence \( (\mathcal{K}_n, \mathcal{J}_n)_{n \in \{0, N+1\}} = (\{1\}, \{1\}), (\{1\}, \{1, 2\}) \). Moreover, \( w^* = q_2^*, \hat{q}_2 = \frac{40}{29} > 0 \) and hence \( \nu' = \emptyset \), i.e.
there is no boundary variable shrinking to 0 from the \( W \) side. Furthermore, we have a single \( t_n = 0 \) such that \( q_2(t_n) = q_2^* \), and the collision as \( \hat{\theta} \setminus \nu \) is of type (d). At the post-boundary step an SCLP pivot produces a new basis for a new interval, where \( \hat{q}_2 \) leaves and \( p_2 \) enters. This basis is adjacent to \( (\mathcal{K}_1, \mathcal{J}_1) \) and compatible to \( K_0 \) and hence there is no SCLP sub-problem. The new base sequence is: \( (\mathcal{K}, \mathcal{J})^{(3)} = (\{1\}, \{1\}), (\{1\}, \{1\}), (\{1, 2\}, \{1, 2\}), (\{1\}, \{1, 2\}) \).

**Third iteration.** The rates for the base sequence \( (\mathcal{K}, \mathcal{J})^{(3)} \) are:

\[
\begin{align*}
    u^1 &= \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}, &  \dot{x}^1 &= \begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}, & u^2 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \dot{x}^2 &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\
    \dot{q}^1 &= \begin{pmatrix} \frac{7}{2} \\ 0 \end{pmatrix}, & p^1 &= \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, & \dot{q}^2 &= \begin{pmatrix} -1 \\ -2 \end{pmatrix}, & p^2 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{align*}
\]

and the solution \( H^{(3)} \) for \( \theta = \frac{9}{29} \) is: \( \tau = \begin{pmatrix} 0 \\ \frac{20}{29} \end{pmatrix}, x^1 = x^0, q^1 = q^N, \)

\[
\begin{align*}
    u^0 &= \begin{pmatrix} 0 \\ \frac{5}{2} \end{pmatrix}, & x^0 &= \begin{pmatrix} 3 \\ 0 \end{pmatrix}, & x^N &= \begin{pmatrix} \frac{127}{29} \\ 0 \end{pmatrix}, & u^N &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x^2 &= \begin{pmatrix} \frac{147}{29} \\ \frac{20}{29} \end{pmatrix}, \\
    q^0 &= \begin{pmatrix} \frac{21}{58} \\ 0 \end{pmatrix}, & p^0 &= \begin{pmatrix} 0 \\ \frac{29}{29} \end{pmatrix}, & p^N &= \begin{pmatrix} \frac{71}{58} \\ \frac{40}{29} \end{pmatrix}, & q^N &= \begin{pmatrix} \frac{31}{58} \\ 0 \end{pmatrix}.
\end{align*}
\]

The solution for \( \hat{\theta} \setminus \nu = \frac{9}{29} \) is displayed on Figure 13

\( \delta H^{(3)} \) for the base sequence \( (\mathcal{K}, \mathcal{J})^{(3)} \) is: \( \delta \tau = \begin{pmatrix} \frac{29}{29} \\ -1 \end{pmatrix}, \)

\[
\begin{align*}
    \delta u^0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \delta x^0 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \delta x^N &= \begin{pmatrix} \frac{21}{4} \\ 0 \end{pmatrix}, & \delta u^N &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
    \delta q^0 &= \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}, & \delta p^0 &= \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix}, & \delta p^N &= \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}, & \delta q^N &= \begin{pmatrix} -\frac{5}{2} \\ -2 \end{pmatrix}, \\
    \delta x^1 &= \begin{pmatrix} \frac{29}{4} \\ 0 \end{pmatrix}, & \delta x^2 &= \begin{pmatrix} \frac{17}{4} \\ -1 \end{pmatrix}, & \delta q^1 &= \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}, & \delta q^0 &= \begin{pmatrix} -\frac{1}{20} \\ 0 \end{pmatrix}.
\end{align*}
\]

The solution for an interior point of the validity region is displayed on Figure 14. The right end-point of the validity region is \( \theta = \theta_3 + \Delta = \frac{9}{29} + \frac{31}{29} = \frac{2}{3}. \)
At \( \theta = \frac{2}{3} \), \( q_1^1 \) shrinks to 0 and hence there is a collision of type (a) as \( \theta \uparrow \bar{\theta} \). This is an internal collision and hence the solution for \( \theta = \frac{2}{3} \) is unique. The solution for \( \theta = \frac{2}{3} \) is displayed on Figure 15.

**Pivot for the third iteration**  The collision of type (a) is paired with a collision of type (c) as \( \theta \searrow \bar{\theta} \). The new interval(s) should be inserted at \( t_1 = \frac{31}{35} \). An SCLP pivot produce a new basis, where \( \dot{q}_1 \) leave and \( \dot{q}_2 \) enter. This basis is adjacent to the \((K_1, J_1)\) and to the \((K_2, J_2)\) and hence there is no SCLP sub-problem. The new base sequence is \((K, J)^{(4)} = (\{1\}, \{1\}), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{1\}, \{2\}), (\{1\}, \{1\}, \{2\})\).

**Fourth iteration.** The rates for the base sequence \((K, J)^{(4)}\) are:

\[
\begin{align*}
u^1 &= \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}, & \dot{x}^1 &= \begin{pmatrix} 0 \\ \frac{5}{2} \end{pmatrix}, & u^2 &= \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, & \dot{x}^2 &= \begin{pmatrix} \frac{1}{3} \end{pmatrix}, & u^3 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \dot{x}^3 &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\
\dot{q}^1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & p^1 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & \dot{q}^2 &= \begin{pmatrix} 0 \\ -2 \end{pmatrix}, & p^2 &= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}, & \dot{q}^3 &= \begin{pmatrix} -1 \\ -2 \end{pmatrix}, & p^3 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align*}
\]
and the solution \( H^{(4)} \) for \( \theta = \frac{2}{3} \) is: \( x^1 = x^2, q^2 = q^1 \),

\[
\begin{align*}
&\mathbf{u}^0 = \begin{pmatrix} 0 \\ \frac{5}{7} \end{pmatrix}, \quad x^0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \mathbf{x}^N = \begin{pmatrix} \frac{25}{7} \\ 0 \end{pmatrix}, \quad \mathbf{u}^N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&\mathbf{p}^N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad q^N = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \quad q^0 = \begin{pmatrix} \frac{13}{30} \\ 0 \end{pmatrix}, \quad p^0 = \begin{pmatrix} 0 \\ \frac{1}{12} \end{pmatrix}, \quad \tau = \begin{pmatrix} \frac{31}{30} \\ \frac{1}{3} \end{pmatrix}.
\end{align*}
\]

\( \delta H^{(4)} \) for the base sequence \((\mathcal{K}, \mathcal{J})^{(4)}\) is:

\[
\begin{align*}
&\delta \mathbf{u}^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta x^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta \mathbf{x}^N = \begin{pmatrix} \frac{7}{4} \\ 0 \end{pmatrix}, \quad \delta \mathbf{u}^N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&\delta \mathbf{q}^0 = \begin{pmatrix} \frac{1}{7} \\ 0 \end{pmatrix}, \quad \delta \mathbf{p}^0 = \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix}, \quad \delta \mathbf{p}^N = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \delta q^N = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\
&\delta x^1 = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix}, \quad \delta x^2 = \begin{pmatrix} \frac{15}{7} \\ 0 \end{pmatrix}, \quad \delta x^3 = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix}, \quad \delta \tau = \begin{pmatrix} -\frac{1}{10} \\ 3 \end{pmatrix}.
\end{align*}
\]

The solution for an interior point of validity region is displayed on Figure 16.

The right end-point of the validity region \( \overline{\theta} = \theta_4 + \Delta = \frac{2}{3} + \frac{1}{3} = 1 \).

At \( \theta = 1 \) \( \tau_3, p_3^0, x_3^0, q_1^N \) shrink to 0. The final solution contain two strictly positive intervals \( \tau_1 = \tau_2 = 1 \) with rates:

\[
\begin{align*}
&u^1 = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}, \quad \dot{x}^1 = \begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \quad \dot{x}^2 = \begin{pmatrix} \frac{4}{3} \\ 0 \end{pmatrix}, \\
&q^1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad p^1 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \quad q^2 = \begin{pmatrix} 0 \\ -\frac{2}{3} \end{pmatrix}, \quad p^2 = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}.
\end{align*}
\]

Figure 16: Solution for \( \theta = \frac{5}{6}, \theta_3 < \theta < \theta_4 \).
The boundary values for this solution are:

\[ u^0 = \begin{pmatrix} 0 \\ \frac{5}{2} \end{pmatrix}, \quad x^0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad x^N = \begin{pmatrix} \frac{41}{6} \\ 0 \end{pmatrix}, \quad u^N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

\[ q^0 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad p^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p^N = \begin{pmatrix} 0 \\ \frac{5}{3} \end{pmatrix}, \quad q^N = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}. \]

The values of state variables at breakpoints are:

\[ x^1 = \begin{pmatrix} \frac{11}{2} \\ 0 \end{pmatrix}, \quad x^2 = \begin{pmatrix} \frac{41}{6} \\ 0 \end{pmatrix}, \quad q^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad q^0 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \]

The solution for \( \theta = 1 \) is displayed on Figure 17.

![Figure 17: Solution for \( \theta = 1 \).](image)

**Comment:** The collision at this final point is a multiple collision, since we have reached the goal point at which \( \lambda = 0, \mu = 0 \).

9 **Algorithm under general settings**

Under the simplifying assumption, all collisions were single. We now discuss the algorithm without the simplifying assumption. The main idea is that whenever a multiple collision is discovered in step \( \ell \) of the algorithm, going from \( L(\theta^\ell) \) to \( L(\theta^{\ell+1}) \) through an interior of validity region \( V \), one moves to an interior point of \( V \) that is close to the current parametric line, and one restarts from this revised point on a new parametric line leading to \( L(1) \).

The general algorithm works exactly as the simplified algorithm described in Section 7 when collisions at point \( L(\theta^{\ell+1}) \) is of type (a)-(f) from both \( V \) and \( W \) sides. However, if collision from the \( V \) or the \( W \) side is a multiple collision there are some additional steps that build a new parametric line \( L'(\theta) \).

We extend the collision classification given in Section 5 by the following types:

- **Pre-\( \bar{\theta} \) multiple collision.** This is a multiple collision, that is discovered when the solution for the endpoint \( \bar{\theta} \) is calculated. It is then seen if the set of values of \( H^\ell_p \) that shrink to zero

\[ M = \arg \max_{\ell, H^\ell_p \in H^\ell_p} \left( 0, \frac{-\delta H^\ell_p}{H^\ell_r} \right) \]
indicates a multiple collision.

- **At-$\bar{\theta}$ multiple collision.** If there is no pre-$\bar{\theta}$ collision, and there is a collision of types (d)-(f) from the $V$ side, but in the boundary pivot in which $v$ leaves the boundary basis there are several boundary variables with equal ratio $> 0$ that are candidates to enter the basis, this is an at-$\bar{\theta}$ multiple collision. In that case, these candidates to enter are a set of boundary variables that shrink to zero from the $W$ side as $\theta \searrow \bar{\theta}$. The ratio test, that is a part of boundary pivot discussed in Section 6, performs discovery of this collision.

- **Post-$\bar{\theta}$ multiple collision.** If there is no pre-$\bar{\theta}$ or at-$\bar{\theta}$ multiple collision, then there could be a multiple collision from the $W$ side in which several local minima of the state variables shrink to zero. This can happen when there is a pivot of type II, and $w^* = x_k^*$ or $w^* = q_j^*$, and it is discovered by the test for multiple collision from $W$ side discussed in Section 6.

- **Single collision.** This is a collision that occurs under the simplifying assumption, i.e. it is a collision of one of the types (a)-(f) from both the $V$ and the $W$ side.

It is easy to see that these collisions cover all possible collisions that may occur along the parametric line.

For all these types of collisions one can choose an interior point $L(\theta) \in V$, a direction that is orthogonal to the current parametric line, denoted by $\vec{L}^\perp$, and a step size $\epsilon$ and build a new parametric line $L'(\theta)$ that goes through the two points $L(\theta) + \epsilon \vec{L}^\perp$ and $L(1)$ and this line can be chosen to satisfy the following properties:

- $L(\theta) + \epsilon \vec{L}^\perp$ is an interior point of $V$, and the line reaches the boundary of $V$ at $\bar{\theta}'$.
- Parametric line $L'(\theta)$ goes through all the validity regions that were crossed by the parametric line $L(\theta)$, from the point $L(0)$ up to the point $L(\bar{\theta})$.
- If there is a pre-$\bar{\theta}$ multiple collision at $L(\bar{\theta})$ from the $V$ side, then there is either at-$\bar{\theta}$, post-$\bar{\theta}$ or a single collision at $L'(\bar{\theta})$.
- If there is an at-$\bar{\theta}$ multiple collision at $L(\bar{\theta})$ from the $V$ side, then there is either post-$\bar{\theta}$ or a single collision at $L'(\bar{\theta})$.
- If there is a post-$\bar{\theta}$ multiple collision at $L(\bar{\theta})$ from the $V$ side, then there is a single collision at $L'(\bar{\theta})$.

One can see that under such a policy the algorithm cannot cycle and it will still find an optimal solution for the goal set of boundary parameters at the point $L(1)$ that is shared by all the parametric lines in a finite number of iterations bounded by $\left(\frac{4(K+J)}{2(K+J)}\right)^2 \left(\frac{K+J}{K}\right)$.

We leave the concrete calculations of an interior point $L'(\theta) \in V$, an orthogonal direction $\vec{L}^\perp$ and a step size $\epsilon$ that satisfy the requirements of this policy out of this paper. We state that this can be done, but it will be more appropriate to deal with the details when one creates an implementation of the algorithm.
A Appendix

A.1 Discussion of $\lambda$ and $\mu$

The motivation for including $\lambda$, $\mu$ in the M-CLP/M-CLP* formulation is that otherwise (i.e. when $\lambda = \mu = 0$), solutions in the interior of validity regions may not have $H_\pi > 0$, and validity regions may not have disjoint interiors.

Strong duality and structure properties of M-CLP/M-CLP* with $\lambda = \mu = 0$ remain valid for solutions of $\lambda, \mu \neq 0$, except for right continuity of solutions at $t = 0$. It is possible that optimal solutions now have $U(0^-) = 0 < U(0) < U(0^+)$ (or $P(0^-) = 0 < P(0) < P(0^+)$).

However, if we restrict the formulation to $\lambda \leq 0$, $\mu \geq 0$, this cannot happen as we state in the next theorem.

**Theorem A.1.** For feasible M-CLP/M-CLP* with $\lambda \leq 0, \mu \geq 0$, there exists a pair of optimal solutions $U_0(t), P_0(t)$ that are right-continuous at 0.

**Proof.** Assume first that the Slater type condition of [17] holds. Imitating the steps leading to Theorem D4.7 and D5.5(iii), M-CLP/M-CLP* of [1], [2] possess strongly dual optimal solutions $U_*(t), P_*(t)$, which are continuous piecewise linear on $(0, T)$. What remains to be checked is whether $U_*(t), P_*(t)$ are right continuous at 0. Let $u^0_0 = U_*(0)$ and $u^{0+}_0 = U_*(0^+) - U_*(0)$. Consider the following solution of M-CLP:

$$U_0(t) = \begin{cases} u^0_0 + u^{0+}_0 = U_*(0^+), & t = 0, \\ U_*(t), & t > 0. \end{cases}$$

We check that $U_0(t)$ is a feasible solution of M-CLP at $t = 0$. By feasibility of $U_*(t)$ we have $AU_0(0) = AU_0(0^+) = AU_*(0^+) \leq \beta$, and therefore $U_0(t)$ is a feasible solution of M-CLP. Next we compare the objective values produced by $U_*(t)$ and $U_0(t)$.

$$\mu^T U_0(0) + \int_{0}^{T} (\gamma + (T - t)c)^T dU_0(t) - \mu^T U_*(0) - \int_{0}^{T} (\gamma + (T - t)c)^T dU_*(t) =$$

$$= \mu^T (u^0_0 + u^{0+}_0) + (\gamma + cT)^T (u^0_0 + u^{0+}_0) + \int_{0}^{T} (\gamma + (T - t)c)^T dU_*(t) -$$

$$- \mu^T u^0_0 - (\gamma + cT)^T u^0_0 - (\gamma + cT)^T u^{0+}_0 - \int_{0}^{T} (\gamma + (T - t)c)^T dU_*(t) = \mu^T u^{0+}_0 \geq 0.$$

Hence, the right-continuous $U_0(t)$ is an optimal solution of M-CLP. The construction of $P_0(t)$ is similar. Note that $\mu \geq 0$ (and similarly $\lambda \leq 0$) is indeed necessary for the proof. To complete the proof we use the detailed structure of the solution described in [18], where the second boundary equations (S11) have been changed to the equations (10), and use Theorem S4.1 to show that the Slater-type condition is not necessary.

It is easy to check that all the results of [17], [18] hold for the extended formulation with $\lambda \leq 0$ and $\mu \geq 0$. Further discussion of the motivation and examples are included in [16].

A.2 Proof of Theorem 4.4

**Proof.** (i) $\implies$ (iii): Let $\rho$ be an interior point of $\mathcal{V}$ and let $M, R$ be the corresponding matrix and r.h.s. of (12) for the base sequence $(K_n, J_n)_{n=0}^{N+1}$ and the point $\rho$. Assume contrary to the theorem that the solution of (12) is not unique.

We now consider the solution of (12), to see which components may be not unique. By Theorem D5.5(iii), all $\tau_0$ and $x^0_k, k \in K^=, q^N_j, j \in J^= \text{ are unique. All the boundary values listed}
in Equations [8] are uniquely determined to be 0. Also, all of \( x_k(t), k \in \mathcal{K} = \) and \( q_j(t), j \in \mathcal{J} = \) are uniquely determined by the unique values of the \( x_k^0, k \in \mathcal{K} = \), \( q_j^N, j \in \mathcal{J} = \), \( \tau_n \). Hence, only components of \( \mathbf{u}^0, \mathbf{u}^N, \mathbf{p}^0, \mathbf{p}^N, \mathbf{x}^N, \mathbf{q}^0 \) and \( x_k^0, k \in \mathcal{K} = \), \( q_j^N, j \in \mathcal{J} = \) and \( x_k(t), q_j(t), k \in \mathcal{K} = \), \( j \in \mathcal{J} = \) may be non-unique. By [11], if \( x_k(t), q_j(t), k \in \mathcal{K} = \), \( j \in \mathcal{J} = \) are non-unique then \( x_k^0, k \in \mathcal{K} = \), \( q_j^N, j \in \mathcal{J} = \) must be non-unique.

We now note that \( \mathbf{u}^0, \mathbf{u}^N, \mathbf{p}^0, \mathbf{p}^N, \mathbf{x}^N, \mathbf{q}^0 \) and \( x_k^0, k \in \mathcal{K} = \), \( q_j^N, j \in \mathcal{J} = \) are determined by equations [9] and by equations [10] after substitution of [11]. But these equations have \( \beta, \gamma, \lambda, \mu \) on the r.h.s.

If the solution is non-unique then \( \mathcal{M} \) must be singular, and so we must have: \( \mathcal{M}_r = \alpha_r \mathcal{M}_1 + \cdots + \alpha_m \mathcal{M}_m \), for which also \( \mathcal{R}_r = \alpha_r \mathcal{R}_1 + \cdots + \alpha_m \mathcal{R}_m \) for some of the rows of \( \mathcal{M} \). But by the above argument, all these rows must have some component of \( \beta, \gamma, \lambda, \mu \) on the r.h.s.

Consider \( \delta \rho \) in which we change some of the values of \( \beta, \gamma, \lambda, \mu \) in such a way that \( \delta \mathcal{R}_r \neq \alpha_r \delta \mathcal{R}_1 + \cdots + \alpha_m \delta \mathcal{R}_m \). One can see that for any \( \Delta > 0 \) the equations [12] with r.h.s. \( \mathcal{R} + \Delta \delta \mathcal{R} \) have no solution, and therefore \( \rho \) cannot be an interior point. This shows that (i) implies that the solution must be unique.

We now consider the components of \( \mathcal{H}_p \). Look first at \( \tau_n \) and \( x_k(t), k \in \mathcal{K} = \), \( q_j(t), j \in \mathcal{J} = \). Any of the components of these which belong to \( \mathcal{H}_p \) can be changed by choosing an appropriate change in \( \beta, \gamma \) or \( T \). By the above discussion, all the remaining components of \( \mathcal{H}_p \) can be changed by appropriate choice of \( \beta, \gamma, \lambda, \mu \). Assume now that for some element \( \mathcal{H}_r \) of \( \mathcal{H}_p \), the value is \( \mathcal{H}_r = 0 \). Then we can choose \( \delta \rho \) such that for the corresponding \( \delta \mathcal{R} \) we have in the solution of \( \delta \mathcal{R} = \mathcal{M} \delta \mathcal{H} \) that \( \delta \mathcal{H}_r \neq 0 \). But in that case the solution of [12] for \( \rho + \Delta \delta \rho \) will have \( \mathcal{H}_r < 0 \) for all \( \Delta > 0 \) or for all \( \Delta < 0 \), in contradiction to the assumption that \( \rho \) is an interior point. This proves that (i) implies that all the components of \( \mathcal{H}_p \) are positive.

(iii) \( \Rightarrow \) (ii): Consider \( \rho \in \mathcal{V} \) and let \( \mathcal{M}, \mathcal{R} \) be the corresponding matrix and r.h.s. of [12] for the base sequence \( (\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1} \) and the point \( \rho \). Assume that the solution of [12] for \( \rho \) is unique and satisfies \( \mathcal{H}_p > 0 \). Hence, for this base sequence the boundary values are unique and strictly complementary slack, which implies the uniqueness of \((\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}\). Moreover, all interval lengths are \( > 0 \) and all values of \( x^n, q^n \) that are members of \( \mathcal{H}_p > 0 \), and hence at each \( t_n, n = 1, \ldots, N - 1 \) there is exactly one state variable that can leave or enter the basis, and so there cannot be another sequence of bases with zero length intervals between any of \( B_1, \ldots, B_N \). Furthermore, at time \( T \), none of the variables \( x_k^N, k \in \mathcal{K} = \), \( x_k \) is possible after \( B_N \), and similarly before \( B_1 \). Hence base sequence \((\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}\) is unique.

(ii) \( \Rightarrow \) (i): Assume \( \rho \in \mathcal{V} \) for the unique base sequence \((\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}\). Assume contrary to the theorem that \( \rho \) is a boundary point of the \( \mathcal{V} \). Then, as \( \rho \) is an interior point of the parametric-feasible region, from the convexity of the parametric-feasible region and closedness of validity regions (Theorem S3.7) it follow that \( \rho \) belongs to a boundary of another validity region \( \mathcal{V} \). This contradicts the uniqueness of \((\mathcal{K}_n, \mathcal{J}_n)_{n=0}^{N+1}\), and hence \( \rho \) is an interior point of the \( \mathcal{V} \).

\[ \square \]

A.3 Proofs for Section 6

**Proof of Theorem 6.4.** The rules for the M-CLP pivot define a unique \((\mathcal{K}'_n, \mathcal{J}'_n)_{n=0}^{N'+1}\), if we can show the following:

(i) There is no multiple collision discovered from the \( \mathcal{W} \) side.
(ii) If the boundary dictionary on the \( V \) side is not unique, then the results of the M-CLP pivot do not depend on the choice of boundary dictionary.

That (i) holds is true by the assumption that the collision from \( W \) is of type (a)–(f) i.e. it is not a multiple collision.

We now show (ii). Consider the case that the leaving variable is a primal variable \( v = v_i = 0 \) (in row \( i \) of the dictionary \( D \)). Assume first that for all the dual variables \( w^*_j \) such that \( w^*_j = 0 \), also \( \hat{A}_{i,l} = 0 \). If \( D \) is not a unique dictionary then we can pass from \( D \) to any other dictionary \( D'' \) by a series of pivots in which a primal variable with value 0 is exchanged for a dual variable of value 0. In that case there must be a pair \( v_k = 0 \) and \( w^*_j = 0 \) such that \( \hat{A}_{k,l} \neq 0 \). Otherwise the dictionary is unique. Consider then the pivot from \( D \) to \( D'' \) obtained by pivoting on \( A_{k,l} \neq 0 \).

We assumed that \( \hat{A}_{i,l} = 0 \), and as a result, the pivot will not change row \( i \). It will also not change the values of the dual variables. Hence the ratio test will yield the same \( w^*_j > 0 \) that will leave the dual basis, with \( w_r \) entering the primal basis, for all possible dictionaries.

Assume now that for some \( w^*_j = 0 \) the element \( \hat{A}_{i,l} \neq 0 \). Then by the above argument, for any other possible dictionary \( D'' \), there will be some \( w^*_j = 0 \) such that \( \hat{A}_{i,l} \neq 0 \). But in that case it follows that the pivot under all possible dictionaries will be a pivot of type I. But in that case all that will happen is that \( v \) will leave the primal basis, and \( v^* \) will enter the dual basis, independent of the dictionary chosen.

\( \square \)

**Proof of Theorem 6.2.** We distinguish two possibilities.

- Case I: If there are only two optimal base sequences at \( L(\bar{J}) \), then \((K_n, J_n)_{n=0}^{N+1} \) is optimal in \( \mathcal{V} \) and \((K'_n, J'_n)_{n=0}^{N'+1} \) is optimal in \( \mathcal{W} \).

- Case II: There is another (or several) optimal base sequence, \((K''_n, J''_n)_{n=0}^{N''+1} \) which is optimal at \( L(\bar{J}) \), but we show that it has a validity region with an empty interior, so again we have that \((K_n, J_n)_{n=0}^{N+1} \) is optimal in \( \mathcal{V} \) and \((K'_n, J'_n)_{n=0}^{N'+1} \) is optimal in \( \mathcal{W} \).

We consider all types of M-CLP pivots.

If the pivot is an internal SCLP pivot then all the boundary values are positive, so \((K'_n, J'_n)_{n=0}^{N'+1} = (K_n, J_n)_{n=0}^{N+1} \) is unique. Furthermore, by [23], if the collision is not a multiple collision then \((K_n, J_n)_{n=0}^{N+1} \) and \((K'_n, J'_n)_{n=0}^{N'+1} \) are the only two optimal base sequences at \( L(\bar{J}) \). Hence we are in case I.

If the pivot is a type I boundary pivot and the collision from \( V \) is type (d), then all the boundary values at \( L(\bar{J}) \) are positive and \((K'_n, J'_n)_{n=0}^{N'+1} = (K_n, J_n)_{n=0}^{N+1} \) is unique, and there are exactly two internal sequences \((K_n, J_n)_{n=1}^{N+1} \) and \((K'_n, J'_n)_{n=1}^{N'+1} \) optimal at \( L(\bar{J}) \). Hence we are in case I.

If the pivot is a type I boundary pivot and the collision from \( V \) is type (e) or (f), then the solution at \( L(\bar{J}) \) is unique, and away from \( L(\bar{J}) \) either \( v > 0 \) or \( v^* > 0 \) determine two unique boundary bases \((K_n, J_n)_{n=0}^{N+1} \) and \((K'_n, J'_n)_{n=0}^{N'+1} \). Further more for each or those there is a unique \((K_n, J_n)_{n=0}^{N+1} \) and \((K'_n, J'_n)_{n=0}^{N'+1} \), so again we are in case I.

Consider now the case of a type II pivot, in which \( v = 0 \) at the collision and is \( > 0 \) on the \( V \) side, and it leaves the primal basis and its dual \( v^* > 0 \) enters the dual basis, and \( w^* > 0 \) leaves the dual basis and its dual \( w = 0 \) enters the primal basis. Then in \((K_n, J_n)_{n=0}^{N+1} \), we have \( v = 0, w^* > 0 \), and in \((K'_n, J'_n)_{n=0}^{N'+1} \) we have \( v^* > 0, w = 0 \). Apart from these two solutions of the Boundary-LP/LP*, the only other solutions will be convex combinations of these two solutions, for which there is another base sequence \((K''_n, J''_n)_{n=0}^{N''+1} \) in which the variables \( v^*, w^* \) are positive, and at the point \( L(\bar{J}) \) they can have values on a whole interval.
Hence for the base sequence \((K''_n, J''_n)_{n=0}^{N''+1}\) at the point \(\bar{\theta}\) is not unique, and hence the matrix \(M\) is singular. This implies by Corollary \ref{corollary-4.5} that the interior of the validity region of the sequence \((K''_n, J''_n)_{n=0}^{N''+1}\) is empty.

\section*{A.4 Proof of Theorem \ref{theorem-7.2}}

\textit{Proof.}\ (i) By definition \(\mathcal{L}(0), \mathcal{L}(1) \in \mathcal{F}\) and the parametric-feasible region is a closed convex polyhedral cone.

(ii) If \(\mathcal{L}(0)\) is an interior point of a validity region then it is also an interior point of the parametric-feasible region. By convexity of \(\mathcal{F}\) its interior is convex, hence, if point \(\mathcal{L}(1)\) is an interior point of the parametric-feasible region then the whole line \(\mathcal{L}(\theta)\) belongs to the interior of the parametric-feasible region, otherwise all points of \(\mathcal{L}(\theta)\) except \(\mathcal{L}(1)\) are interior points of the parametric-feasible region.

(iii) The parametric-feasible region is the union of validity regions of proper base sequences. Hence \(\mathcal{L}(\theta)\) is a union of intervals belonging to these validity regions. Because validity regions are convex, no two intervals can belong to the same proper base sequence. By Proposition \ref{proposition-4.3} the number of proper base sequences is bounded by \(\left(\frac{4(K+J)}{2(K+J)}\right)^2\). Hence the number of intervals is finite. It is then possible to choose a minimal number of proper base sequences which define a minimal number of intervals.

\section*{References}

[1] Anderson, E. J. \textit{A new continuous model for job-shop scheduling.} International J. Systems Science, \textbf{12}, pp. 1469–1475, 1981.

[2] Anderson, E. J. and Nash, P. \textit{Linear Programming in Infinite Dimensional Spaces.} Wiley-Interscience, Chichester, 1987.

[3] Anderson, E. J., Philpott, A. B. \textit{A continuous time network simplex algorithm.} Networks, \textbf{19}, pp. 395–425, 1989.

[4] Bellman, R. \textit{Bottleneck problems and dynamic programming.} Proc. National Academy of Science \textbf{39}, pp. 947–951, 1953.

[5] Dantzig, G. B. \textit{Application of the simplex method to a transportation problem.} in T. Koopmans, ed., Activity Analysis of Production and Allocation John Wiley and Sons, New York, pp. 359–373, pp. 330–335, 1951.

[6] Grinold, R. C. \textit{Symmetric duality for continuous linear programs.} SIAM J. Applied Mathematics, \textbf{18}, pp. 32–51, 1970.

[7] Lemke, C. E. \textit{Bimatrix equilibrium points and mathematical programming.} Management science, \textbf{11(7)}, pp. 681–689, 1965.

[8] Levinson, N. \textit{A class of continuous linear programming problems.} J. of Mathematical Analysis and Applications, \textbf{16}, pp. 73–83, 1966.

[9] Papageorgiou, N. S. \textit{A class of infinite dimensional linear programming problems.} J. of Mathematical Analysis and Applications, \textbf{87}, pp. 228–245, 1982.
[10] Pullan, M. C. An algorithm for a class of continuous linear programs. SIAM J. Control and Optimization, 31, pp. 1558–1577, 1993.

[11] Pullan, M. C. Forms of optimal solutions for separated continuous linear programs. SIAM J. Control and Optimization, 33, pp. 1952–1977, 1995.

[12] Pullan, M. C. A duality theory for separated continuous linear programs. SIAM J. Control and Optimization, 34, pp. 931–965, 1996.

[13] Pullan, M. C. Existence and duality theory for separated continuous linear programs. Math. Model. Syst., 3, pp. 219–245, 1997

[14] Pullan, M. C. Convergence of a general class of algorithms for separated continuous linear programs. SIAM J. Control and Optimization, 10, pp. 722–731, 2000.

[15] Shapiro, A. On duality theory of conic linear problems. In: Goberna, M. A., Lopez, M. A. (eds.) Semi-Infinite Programming, Chap. 7, pp. 135–165, Kluwer, Netherlands, 2001

[16] Shindin, E. Symmetric Strong Duality and Simplex Type Algorithm for Continuous Linear Programs with Constant Coefficients. Ph.D. Thesis, University of Haifa, Israel, 2016.

[17] Shindin, E., Weiss G. Symmetric Strong Duality for a Class of Continuous Linear Programs with Constant Coefficients. SIAM J. Optimization, 24, pp. 1102–1121, 2014.

[18] Shindin, E., Weiss G. Structure of Solutions for Continuous Linear Programs with Constant Coefficients. SIAM J. Optimization, 25, pp. 1276–1297, 2015.

[19] Tyndall, W. F. A duality theorem for a class of continuous linear programming problems. SIAM J. Applied Mathematics, 13, pp. 644–666, 1965.

[20] Tyndall, W. F. An extended duality theorem for continuous linear programming problems. SIAM J. Applied Mathematics, 15, pp. 1294–1298, 1967.

[21] Vanderbei, R. J. Linear Programming, Foundations and Extensions. Springer, New York, 4th edition, 2014.

[22] Wang, X., Zhang, S., Yao, D. Separated Continuous Conic Programming: Strong Duality and an Approximation Algorithm. SIAM J. Control and Optimization, 48, pp. 2118–2138, 2009.

[23] Weiss, G. A simplex based algorithm to solve separated continuous linear programs. Mathematical Programming Series A, pp. 151–198, 2008.