The ratio $\mathcal{R}(D_s)$ for $B_s \to D_s \ell \nu_\ell$ by using the QCD light-cone sum rules within the framework of heavy quark effective field theory

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In the paper, we study the $B_s \to D_s$ transition form factors by using the light-cone sum rules within the framework of heavy quark effective field theory. We adopt a chiral current correlation function to do the calculation, the resultant transition form factors $f_{B_s \to D_s}^D(q^2)$ and $f_{B_s \to D_s}^D(q^2)$ are dominated by the contribution of $D_s$-meson leading-twist distribution amplitude, while the contributions from less certain $D_s$-meson twist-3 distribution amplitudes are greatly suppressed. At the largest recoil point, we obtain $f_{B_s \to D_s}^D(0) = 0.53^{+0.112}_{-0.054}$. By further extrapolating the transition form factors into all the physically allowable $q^2$ region with the help of the $z$-series parametrization approach, we calculate the branching fractions $\mathcal{B}(B_s \to D_s \ell \nu_\ell)$ with $(\ell = e, \mu)$ and $\mathcal{B}(B_s \to D_s \tau \nu_\tau)$, which gives $\mathcal{R}(D_s) = 0.334 \pm 0.017$.

I. INTRODUCTION

It is one of the most attractive research topics in the field of high energy physics to accurately test the standard model (SM) and to search new physics effects beyond the SM. The $B \to D^{(*)}$ semileptonic decay provides such an example. The ratio $\mathcal{R}(D^{(*)}) = B(B \to D^{(*)}\tau \nu_\tau)/B(B \to D^{(*)}\ell \nu_\ell)$ with $(\ell = e, \mu)$ has been measured by various groups, e.g. the BaBar Collaboration firstly reported $\mathcal{R}_{\text{exp}}^{B s \to D s} = 0.440 \pm 0.058 \pm 0.042$ and $\mathcal{R}_{\text{exp}}^{D s} = 0.332 \pm 0.024 \pm 0.018$ [1, 2], the BELLE Collaboration subsequently given $\mathcal{R}_{\text{exp}}^{D s} = 0.375 \pm 0.064 \pm 0.026$ and $\mathcal{R}_{\text{exp}}^{D s} = 0.293 \pm 0.038 \pm 0.015$ in year 2015 [3], $\mathcal{R}_{\text{exp}}^{D s} = 0.270 \pm 0.035 \pm 0.028$ in year 2016 [4, 5], and $\mathcal{R}_{\text{exp}}^{D s} = 0.307 \pm 0.037 \pm 0.016$ and $\mathcal{R}_{\text{exp}}^{D s} = 0.283 \pm 0.018 \pm 0.014$ in year 2019 [6], and the LHCb Collaboration reported $\mathcal{R}_{\text{exp}}^{D s} = 0.336 \pm 0.027 \pm 0.030$ in year 2015 [7] and $\mathcal{R}_{\text{exp}}^{D s} = 0.283 \pm 0.018 \pm 0.014$ in year 2017 [8, 9]. The Heavy Flavor Averaging Group (HFLAG) gave the weighted average of those measurements, i.e. $\mathcal{R}_{\text{exp}}^{D s} = 0.339 \pm 0.026 \pm 0.014$ and $\mathcal{R}_{\text{exp}}^{D s} = 0.295 \pm 0.010 \pm 0.010$ [10], where they also gave the averages of theoretical predictions $\mathcal{R}_{\text{th}}^{D s} = 0.298 \pm 0.003$ and $\mathcal{R}_{\text{th}}^{D s} = 0.252 \pm 0.005$ from Refs. [11–13]. Those theoretical values are consistent with other predictions calculated using various approaches, such as the heavy quark effective theory (HQET) [14, 15], the lattice QCD (LQCD) [16–18], the light-cone sum rules (LCSR) [19, 20]. Since the theoretical predictions are generally smaller than the measured ones, this difference may indicate new physics beyond the SM [21–24].

The LHCb collaboration has measured the branching fraction $\mathcal{B}(B^0 \to D^+_s \mu^+ \nu_\mu) = (2.49 \pm 0.12 \pm 0.14 \pm 0.16) \times 10^{-2}$ [25] and gave the ratio of the branching fractions $\mathcal{B}(B^0 \to D^-_s \mu^- \nu_\mu)$ and $\mathcal{B}(B^0 \to D^-_s \mu^- \nu_\mu)$, i.e., $\mathcal{R} = 1.09 \pm 0.05 \pm 0.06 \pm 0.05$. This indicates $B_s \to D_s \ell \nu_\ell$ could behave closely to $B \to D \ell \nu$. Therefore, it is meaningful to make a detailed study on the similar ratio $\mathcal{R}(D_s)$.

At present, there is still no published data on the ratio $\mathcal{R}(D_s)$, while many theoretical studies on it have been done in Refs.[26–34]. As the key components of calculating the ratio $\mathcal{R}(D_s)$, the $B_s \to D_s$ transition form factors (TFFs) $f_{B_s \to D_s}^D(q^2)$ and $f_{0_B \to D_s}^D(q^2)$ have been studied under various approaches, e.g. the QCD sum rules (QCDSR) [35], the constituent quark model (CQM) [36], the light-cone sum rules (LCSR) [37], the Bethe-Salpeter equation (BSE) [38], and the lattice QCD (LQCD) [30–33]. Different approaches are applicable in various energy scale regions, for example, the LCSR is applicable in the largest low and intermediate $q^2$-region; and in the present paper, as the same as the previous treatment of $B \to \pi$ TFFs [39], we will adopt the LCSR approach within the framework of heavy quark effective field theory (HQEFT) [40–45] to calculate $B_s \to D_s$ TFFs. The HQEFT separates the non-perturbative long-distance terms from the short-distance dynamics via a systematic way, and the long-distance terms can be decreased to a series over the non-perturbative wave functions or transition form factors. It has been pointed out that by choosing a proper chiral correlator, as will be adopted in this paper, one can suppress the uncertainties from the high-twist LCDAs and achieve a more accurate LCSR prediction of the $B_s \to D_s$ TFFs.

The remaining parts of the paper are organized as follows. In Sec. II, we present the calculation technologies...
for the two TFFs of the $B_s \to D_s \ell \bar{v}_\ell$ semileptonic decays by using the light-cone sum rules within the framework of HQEFT. In Sec. III, we present our numerical results and discussions. Sec. IV is reserved for a summary.

II. CALCULATION TECHNOLOGY

A. $B_s \to D_s$ Transition Matrix Element

For the $B_s \to D_s \ell \bar{v}_\ell$ decays, the transition matrix element can be parameterized as follows:

$$\langle D_s(p)|\bar{c}\gamma_\mu b|B_s(p+q)\rangle = 2 f^B_{+\to D_s}(q^2) q_\mu + \left[f^B_{+\to D_s}(q^2) + f^B_{-\to D_s}(q^2)\right] q_j$$

and

$$f^B_{0\to D_s}(q^2) = f^B_{+\to D_s}(q^2) + \frac{i}{m^2_{B_s} - m^2_{D_s}} f^B_{-\to D_s}(q^2),$$

where $p$ is the momentum of the $D_s$-meson and $(p+q)$ is the momentum of $B_s$-meson. At the maximum recoil point, we have $f^B_{0\to D_s}(0) = f^B_{+\to D_s}(0)$. The transition matrix element can be expanded as 1/m$_0$-power series within the framework of HQEFT. Based on the heavy quark symmetry, the transition matrix element of heavy quark in the effective theory is parameterized as [43–45]:

$$\langle D_s(p)|\bar{c}\gamma_\mu b|B_s(p+q)\rangle = \frac{\sqrt{m_{B_s}}}{\sqrt{\Lambda_{B_s}}} \langle D_s(p)|\bar{u}\gamma_\mu b^+|B_{sv}\rangle$$

where

$$\Lambda_{B_s} = m_{D_s} - m_b,$$

$$D_s(v, p) = \gamma_5 [A(v \cdot p) + \bar{p}B(v \cdot p)],$$

$$M_{sv} = -\sqrt{\Lambda}(1 + \gamma_5 \gamma_5)/2,$$

where $b^+_s$ is the effective $b$-quark field and $v$ is the $B_s$-meson velocity, $\bar{p} = p^\mu / (v \cdot p)$. $A(v \cdot p)$ and $B(v \cdot p)$ are leading-order heavy flavor-spin independent coefficient functions. $\Lambda = \lim_{m_b \to \infty} \Lambda_{B_s}$, which is the heavy flavor independent binding energy that reflects the effects of light degrees of freedom in the heavy hadron. Using those formulas, we obtain the $B_s \to D_s \ell \bar{v}_\ell$ TFFs $f^B_{\pm}(q^2)$, which are

$$f^B_{\pm\to D_s}(q^2) = \frac{\sqrt{\Lambda}}{\sqrt{m_{B_s} \Lambda_{B_s}}} \left[A(y) \pm \frac{m_{B_s}}{y} B(y)\right] + \cdots,$$

with

$$y = v \cdot p = (m^2_{B_s} + m^2_{D_s} - q^2)/(2m_{B_s}),$$

where “…” denotes the higher-order $O(1/m_0)$ contributions that will not be taken into consideration here.

B. Light-Cone Sum Rule For $f^{B_s\to D_s}(q^2)$

To derive the sum rules of the two leading order heavy-flavor-spin independent coefficient functions $A(y)$ and $B(y)$, we construct the following correlator:

$$F_{\mu}(p, q) = i \int d^4 x e^{iq\cdot x} \langle D_s(p)\{j_\mu(x), j_\nu(0)\}\rangle,$$

where the currents

$$j_\mu(x) = \bar{c}(x)\gamma_\mu(1 + \gamma_5)b(x),$$

$$j_\nu(0) = \bar{b}(0)i(1 + \gamma_5)s(0).$$

Following the standard procedure of LCSR approach, we first deal with the hadronic representation for the correlation function. One can insert a complete series of the intermediate hadronic states in the correlator (13) in the physical $q^2$-region and isolate the pole term of the lowest pseudoscalar state from the hadronic representation. Then the correlator $F_{\mu}(p, q)$ becomes:

$$F_{\mu}^{\text{Had.}}(p, q) = \frac{\langle D_s(p)|\bar{c}\gamma_\mu b|B_s\rangle\langle B_s|\bar{b}\gamma_5 s\rangle}{m^2_{B_s} - (p + q)^2}$$

$$+ \sum_{B_H} \frac{\langle D_s(p)|\bar{c}\gamma_\mu(1 + \gamma_5)b_B|B_H\rangle\langle B_H|\bar{b}\gamma_5 s\rangle}{m^2_{B_H} - (p + q)^2},$$

In the effective theory of heavy quark, the hadronic representation (10) can be expanded in powers of 1/m$_0$. Taking the transition matrix element (3) into consideration and neglecting the contributions from higher 1/m$_b$ order, we can further write the hadronic representation as:

$$F_{\mu}^{\text{Had.}}(p, q) = 2F\frac{A(y)v^\mu + B(y)\bar{p}^\mu}{2\Lambda_{B_s} - 2v \cdot k} + \int_{s_0}^{\infty} ds \frac{\rho(y, s)}{s - 2v \cdot k} + \text{Subtractions},$$

with the matrix element [46]

$$\langle B_s|\bar{b}^+_s\gamma_5 d|0\rangle = \frac{i}{2} F\sqrt{\Lambda}\gamma_5 M_{sv},$$

where $F$ is the leading-order decay constant of the $B_s$-meson [47, 48]. $k$ is the residual momentum of the heavy hadronic. Using the ansatz of the quark-hadron duality the spectral density $\rho(y, s)$ can be obtained [49, 50].

On the other hand, we apply the operator product expansion (OPE) to the correlator in the deep Euclidean region. The correlator (13) can be explicitly written as

$$F_{\mu}(p, q) = i \int d^4 x e^{i(q - m_0)\cdot x} \langle D_s(p)\{\bar{c}(x)\gamma_\mu(1 + \gamma_5)b^+_s(x), \bar{b}^+_s(0)i(1 + \gamma_5)s(0)\}\rangle,$$

Using the $B$-meson heavy-quark propagator $S(x, v) = (1 + \gamma_5) \times \int_0^{\infty} dt \delta(x - vt)/2$ [47], the correlator can be
expanded as a complex power series over the $D_s$-meson LCDAs. Due to the chiral suppressions, it is noted that the main contribution to the correlator comes from the leading-twist LCDA, and the contributions from all the twist-3 LCDAs are exactly zero.

Through the dispersion relation, the OPE in deep Euclidean region and the hadron expression in physical region can be matched. And by further applying the Borel transformation to suppress the contributions from power-suppressed terms, the LCSRs for the coefficient functions $A(y)$ and $B(y)$ are

$$A(y) = -\frac{f_{D_s}}{2F} \int_0^{s_B} ds e^{(2\lambda_B - s)/T} \left. \frac{\partial}{\partial y} g_2(u) \right|_{u=1-\frac{m_B}{y}}$$ (14)

$$B(y) = -\frac{f_{D_s}}{2F} \int_0^{s_B} ds e^{(2\lambda_B - s)/T} \left[ -\phi_{2;D_s}(u) + \left( \frac{1}{y} \frac{\partial}{\partial u} \right)^2 g_1(u) - \left( \frac{1}{y} \frac{\partial}{\partial u} \right)^2 g_2(u) \right] \right|_{u=1-\frac{m_B}{y}}$$ (15)

where $T$ is the Borel parameter and $s_B$ is the continuum threshold, $g_1$ and $g_2$ are twist-four LCDAs. Since the contributions from the twist-4 LCDAs are only several percent, so we shall directly adopt the light pseudo-scale ones to do the calculations, whose explicit forms can be found in Refs.[51]. Substituting them into Eqs.(2,5), we obtain

$$f_{B_s \rightarrow D_s}(q^2) = -\frac{f_{D_s}}{2F} \int_0^{s_B} ds e^{(2\lambda_B - s)/T}$$

$$\times \left\{ \left( \frac{q^2}{m_{D_s}^2 - m_{D_s}^2} \right) \left( \frac{1}{y^2} \frac{\partial}{\partial u} \right) g_2(u) + \left( \frac{1}{y} \frac{\partial}{\partial u} \right)^2 g_1(u) \right\} \right|_{u=1-\frac{m_B}{y}}$$ (16)

$$f_{0 \rightarrow D_s}(q^2) = -\frac{f_{D_s}}{2F} \int_0^{s_B} ds e^{(2\lambda_B - s)/T}$$

$$\times \left\{ \left( \frac{q^2}{m_{D_s}^2 - m_{D_s}^2} \right) \left( \frac{1}{y^2} \frac{\partial}{\partial u} \right) g_2(u) + \left( \frac{1}{y} \frac{\partial}{\partial u} \right)^2 g_1(u) \right\} \right|_{u=1-\frac{m_B}{y}}$$ (17)

The Borel parameter $T$ and the continuum threshold $s_B$ shall be fixed such that the resulting TFFs do not depend too much on the precise values of those parameters. In addition, the continuum contribution, which is the part of the dispersive integral from $s_B$ to $\infty$ that is subtracted from both sides of the equation, should not be too large.

### III. NUMERICAL ANALYSIS

#### A. Input parameters

To determine the TFFs $f_{B_s \rightarrow D_s}(q^2)$ of the exclusive process $B_s \rightarrow D_s \bar{\nu}_\tau$, we take $[43, 52]$:

$$m_{B_s} = 5.367 \pm 0.00014\text{GeV},$$

$$m_{D_s} = 1.968 \pm 0.00007\text{GeV},$$

$$f_{D_s} = 0.256 \pm 0.0042\text{GeV},$$

$$F = 0.30 \pm 0.04\text{GeV}^{3/2}.$$

| TABLE I: The $D_s$-meson leading-twist LCDA parameters at the scale $\mu = 3\text{GeV}$. The values in the second row are for the central values, and the values in third and fourth rows are uncertainties. |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| $A_{D_s}$(GeV$^{-1}$) | $B_{1s}$ | $B_{2s}$ | $B_{3s}$ | $B_{4s}$ | $\beta_{D_s}$(GeV)$^-1$ |
|---------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1.246 | -0.214 | -0.167 | 0.055 | 0.005 | 5.521 |
| 11.001 | -0.165 | 0.014 | -0.004 | 0.003 | 1.046 |
| 1.184 | -0.189 | -0.163 | 0.047 | 0.008 | 6.970 |

#### FIG. 1: The behavior of LCDA $\phi_{2;D_s}(u, \mu = 3\text{GeV})$, the shaded area indicates LCDA’s uncertainties.

For the $D_s$-meson leading-twist LCDA $\phi_{2;D_s}(u, \mu)$ in the LCSRs of TFFs (16) and (17), we adopt the light cone harmonic oscillator (LCHO) model suggested in Ref.[53], which has a better end-point behavior and takes the fol-
\begin{equation}
\phi_{2,D_s}(u,\mu) = \frac{\sqrt{6}A_{D_s} \beta_{D_s}^2}{\pi^2 f_{D_s}} u \bar{u} \varphi_{2,D_s}(u) 
\times \exp \left[ -\frac{\mu^2}{8\beta_{D_s}^2} \frac{(\bar{u} u + m_c^2 - \bar{m}_c^2)}{\bar{u} u} \right] 
\times \left\{ 1 - \exp \left[ -\frac{\mu^2}{8\beta_{D_s}^2} \frac{((\bar{u} u + m_c^2 - \bar{m}_c^2))}{\bar{u} u} \right] \right\},
\end{equation}

where \( \tilde{u} = 1 - u \) and \( \varphi_{2,D_s}(u) = 1 + \sum_{n=1}^{4} B_{D_s}^{D_s} C_n^3 (\xi) \) with \( \xi = u - \tilde{u} \). \( \bar{m}_c \) and \( \bar{m}_s \) are \( c \) - and \( s \)-constituent quark masses, whose values are taken as \( \bar{m}_c \simeq 1.5 \) GeV and \( \bar{m}_s \simeq 0.5 \) GeV. \( A_{D_s}, \beta_{D_s}, B_{D_s}^{D_s}, B_{D_s}^{B_s}, B_{D_s}^{B_s} \) and \( B_{D_s}^{D_s} \) are model parameters, whose initial values at the scale \( \mu = 2 \) GeV have been given in Ref. \[53\]. For the present process, the typical factorization scale \( \mu = (m_{B_s} - m_b)^{1/2} = 3 \) GeV. The input parameters at the scale \( \mu = 3 \) GeV can be achieved by using the conventional one-loop evolution equation \[54\], and these values are given in Table I. Figure 1 shows the behavior of the \( D_s \)-meson leading-twist LCDA \( \phi_{2,D_s}(u,\mu = 3 \) GeV) with the typical values exhibited in Table I, where the solid line is the central value and the shaded band shows its uncertainty given in Table I.

**B. The \( B_s \rightarrow D_s \) TFFs**

Next, we calculate the \( B_s \rightarrow D_s \) TFFs \( f_{+0}^{B_s \rightarrow D_s}(q^2) \) and \( f_{0}^{B_s \rightarrow D_s}(q^2) \) by using the LCSRs \( (16) \) and \( (17) \) in the \( q^2 \)-region when the LCSR approach is applicable, i.e., \( 0 < q^2 < 7 \) GeV. For the purpose, we first determine the continuum threshold \( s_0^{B_s} \) and the Borel parameter \( T \). Within the framework of HQEFT, the continuum threshold \( s_0^{B_s} \equiv 2 \Lambda_{B_s} = 2(m_{B_s} - m_b) \) with \( B_s \) being the \( B_s \)-meson first excited state \[47\], and we take \( s_0^{B_s} = 3.85 \pm 0.15 \) GeV. To determine the Borel window, as suggested in Refs. \[47, 48\], we require the TFFs \( f_{+0}^{B_s \rightarrow D_s}(q^2) \) to be as stable as possible within corresponding Borel windows. Figure 2 shows the TFFs \( f_{+0}^{B_s \rightarrow D_s}(q^2) \) versus the Borel parameter \( T \) at several typical squared momenta transfer, in which the solid, the dashed, the dot-dashed and the dotted lines are for \( q^2 = 0, 3, 5, 7 \) GeV \(^2\), respectively. One can find that, the TFFs \( f_{+0}^{B_s \rightarrow D_s}(q^2) \) are stable for a large \( T \), e.g. the uncertainty caused by \( T \) is less than 5\% when \( T \geq 10 \) GeV for all those \( q^2 \) values. Therefore, we take the Borel window as \( 10 \) GeV \( < T < 20 \) GeV.

At the maximum recoil point \( q^2 = 0 \), we have

\begin{equation}
f_{+0}^{B_s \rightarrow D_s}(0) = 0.533^{+0.082}_{-0.063} \varphi_{2,D_s}^{+0.007}_{-0.014} \beta_{D_s}^{+0.065 \beta_{D_s}^{0.014}} \beta_{D_s}^{0.007}_{+0.004} |T|^{-0.037} \beta_{D_s}^{0.004} |m_s|.
\end{equation}

There are also errors caused by the uncertainties of \( m_{B_s} \) and \( m_{D_s} \), which are negligibly small. It is found that the uncertainties of the \( D_s \)-meson leading-twist LCDA \( \phi_{2,D_s} \) and the continuum threshold \( s_0^{B_s} \) are main errors of \( f_{+0}^{B_s \rightarrow D_s}(0) \). By adding all those errors in quadrature, we obtain \( f_{+0}^{B_s \rightarrow D_s}(0) = 0.533^{+0.160}_{-0.128} \). We present the theoretical predictions of \( f_{+0}^{B_s \rightarrow D_s}(q^2) \) at the maximum recoil point \( q^2 = 0 \) in Table II, where the predictions under the pQCD approach \[26\], the pQCD+LQCD approach \[27\], the LQCD approach \[30, 31\], the QCD SR approach \[35, 55\], the LCSR approach \[37\], the BSE approach \[38\] and the RQM approach \[28\] are also presented. Our present prediction of \( f_{+0}^{B_s \rightarrow D_s}(0) \) is in good agreement with the values calculated with the pQCD prediction \[26\], the pQCD+LQCD prediction \[27\] and the BSE prediction \[38\].

As mentioned above, the LCSRs \( (16) \) and \( (17) \) for TFFs \( f_{+0}^{B_s \rightarrow D_s}(q^2) \) are only reliable in low and intermediate regions, i.e., \( 0 < q^2 < 7 \) GeV. To estimate the total
TABLE II: Theoretical predictions of the TFF $f_{+0}^{B_{s} 	o D_s}(q^2)$ at the maximum recoil point under various approaches.

| Methods                  | $f_{+0}^{B_{s} 	o D_s}(0)$ |
|--------------------------|----------------------------|
| This work (HQEFT)        | 0.533 $^{+0.112}_{-0.094}$ |
| pQCD [26]                | 0.55 $^{+0.15}_{-0.12}$    |
| pQCD+LQCD [27]           | 0.52 $^{+0.10}_{-0.08}$    |
| RQM [28]                 | 0.74 $^{+0.02}_{-0.02}$    |
| LQCD [30]                | 0.656 $^{+0.031}_{-0.024}$ |
| LQCD [31]                | 0.661 $^{+0.042}_{-0.039}$ |
| QCDSR [35]               | 0.7 $^{+0.1}_{-0.1}$       |
| QCDSR [55]               | 0.24                       |
| LCSR [37]                | 0.43 $^{+0.09}_{-0.08}$    |
| BSE [38]                 | 0.57 $^{+0.02}_{-0.02}$    |

decay width of the semi-leptonic decay $B_s \to D_s \ell \bar{\nu}_\ell$, we extrapolate the TFFs to the whole physically allowable $q^2$-region, $0 < q^2 < (m_{B_s} - m_{D_s})^2 = 11.50 \text{GeV}^2$, via the $z$-series parametrization [56, 57]:

\[
\left. \begin{array}{l}
 f_{+0}^{B_{s} \to D_s}(q^2) = \frac{f_{+0}^{B_{s} \to D_s}(0)}{1 - q^2/m_{B_s}^2} \left(1 + \sum_{k=1}^{N-1} b_k \left[ z(q^2)^k - z(0)^k \right] \right) \\
 f_{0}^{B_{s} \to D_s}(q^2) = f_{0}^{B_{s} \to D_s}(0) \left(1 + \sum_{k=1}^{N-1} b_k (z(q^2)^k - z(0)^k) \right),
\end{array} \right\}
\]

(21)

where

\[
z(q^2) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}};
\]

\[
t_0 = t_+ (1 - \sqrt{1 - t_+/t_-});
\]

\[
t_\pm = (m_{B_s} \pm m_{D_s})^2.
\]

Then, by fitting the values of the TFFs in low and intermediate regions calculated via the LCSRIs (16) and (17), the coefficients $b_1$, $b_2$ and $b_3$ in extrapolation formula (20) and (21) can be determined, and which have been exhibited in Table III. The quality-of-fit is defined as:

\[
\Delta = \frac{\sum_t |f_t - f_{\text{fit}}|}{\sum_t |f_t|} \times 100\%, t \in \left\{0, \frac{1}{2}, \ldots, \frac{23}{2}, 12 \right\} \text{ GeV}^2.
\]

(22)

The coefficients $b_i$ are determined such that the quality-of-fit ($\Delta$) is no more than 1%. The $\Delta$ values for the central, the upper and lower TFFs are shown in Table III. These quality-of-fits are much smaller than 1%, indicating that our present extrapolations are of high accuracy. We present the extrapolated TFFs $f_{+0}^{B_{s} \to D_s}(q^2)$ in Figure 3, where the shaded hands are theoretical uncertainties from all the mentioned error sources. For comparison, we present the results of the pQCD+LQCD approach [27], the pQCD approach [27], the RQM approach [29] and the LQCD approach [30].

TABLE III: The fitted parameters and the quality-of-fit for the extrapolated TFFs $f_{+0}^{B_{s} \to D_s}(q^2)$.

| $f_{+0}^{B_{s} \to D_s}(q^2)$ | $b_1$ | $b_2$ | $b_3$ | $\Delta$ |
|------------------------------|-------|-------|-------|---------|
| $0.533$                      | $-2.378$ | $-19.414$ | $196.397$ | $0.006\%$   |
| $+0.112$                     | $-3.586$ | $-15.843$ | $236.929$ | $0.006\%$   |
| $-0.094$                     | $-4.646$ | $-12.450$ | $281.244$ | $0.006\%$   |

C. The $B_s \to D_s \ell \bar{\nu}_\ell$ branching fractions and the ratio $\mathcal{R}(D_s)$

The branching fraction of the semi-leptonic decay $B_s \to D_s \ell \bar{\nu}_\ell$ is defined as

\[
\mathcal{B}(B_s \to D_s \ell \bar{\nu}_\ell) = \tau_{B_s} \times \int_0^{(m_{B_s} - m_{D_s})^2} dq^2 \frac{d\Gamma(B_s \to D_s \ell \bar{\nu}_\ell)}{dq^2},
\]

(23)

where $q^2_{\text{max}} = (m_{B_s} - m_{D_s})^2$ and $\tau_{B_s}$ is the $B_s$-meson lifetime. Here the differential decay widths is

\[
\frac{d\Gamma(B_s \to D_s \ell \bar{\nu}_\ell)}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_{B_s}^3} \left(1 - \frac{m_Q^2}{q^2} \right)^2 \left(1 + \frac{m_Q^2}{2q^2} \right) \times \lambda^2(q^2)f_{+0}^{B_s \to D_s}(q^2)^2,
\]

(24)

where $\lambda(q^2) = (m_{B_s}^2 + m_{D_s}^2 - q^2)^2 - 4m_{B_s}^2 m_{D_s}^2$, which is the phase-space factor. $|V_{cb}|$, $G_F$ and $m_e$ are CKM matrix element, the Fermi-coupling constant and the lepton mass, respectively, and we take [52]: $\tau_{B_s} = (1.510 \pm 0.004) \times 10^{-12}$s, $|V_{cb}| = (0.05 \pm 1.5) \times 10^{-3}$, $G_F = 1.166378(6) \times 10^{-5}$GeV$^{-2}$ and $\tau$-lepton mass $m_\tau = 1.776 \pm 0.0012$GeV. For lepton $\ell' = e$ or $\mu$, its mass is negligible, and then the above differential decay width can be simplified as

\[
\frac{d\Gamma(B_s \to D_s \ell' \bar{\nu}_{\ell'})}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_{B_s}^3} \lambda^{3/2}(q^2)f_{+0}^{B_s \to D_s}(q^2)^2,
\]

(25)

where $f_{+0}^{B_s \to D_s}(q^2)$ has zero contribution due to chiral suppression.

We present the differential decay widths of $B_s \to D_s \tau \bar{\nu}_\tau$ and $B_s \to D_s \ell' \bar{\nu}_{\ell'}$ in Figure 4, in which the solid lines are for the central choices of input parameters, and the shaded bands are uncertainties by adding all the errors caused by the error sources such as $f_{+0}^{B_s \to D_s}(q^2)$, $m_{B_s}$, $m_{D_s}$, $|V_{cb}|$, $G_F$ and $m_\ell$, etc., in quadrature. In addition, the predictions under the RQM approach [29] and the LQCD approach [33] are also given. One may observe
FIG. 3: The extrapolated TFFs $f_{B_s^{*0} \rightarrow D_s^{*0}}(q^2)$ versus $q^2$. The solid line are central values and the shaded bands are corresponding uncertainties. As a comparison, the predictions under the pQCD+LQCD approach [27], the pQCD approach [27], and the RQM approach [29] and the LQCD approach [30] are also presented.

FIG. 4: The differential decay widths of $B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell$, where the uncertainties are squared averages of those from all the mentioned error sources. The predictions under the LQCD approach [33] and the RQM approach [29] are also presented.

that our prediction of $d\Gamma(B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell)/dq^2$ is consistent with the LQCD and RQM predictions in Refs. [29, 33]; And for $d\Gamma(B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell)/dq^2$, our prediction agrees with the LQCD and RQM predictions [29, 33] in larger $q^2$ region, but is smaller than those predictions in lower $q^2$ region.

We present the branching fractions $\mathcal{B}(B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell)$ and $\mathcal{B}(B_s \rightarrow D_s^{*} \tau \bar{\nu}_\ell)$ in Table IV, where the predictions under various approaches are also presented as a comparison. It is noted that our present predictions are consistent with most of the previous predictions within errors. Especially, our prediction of $\mathcal{B}(B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell)$ is in good agreement with the pQCD prediction of Refs. [26, 27] and the pQCD+LQCD approach [27], and our prediction of $\mathcal{B}(B_s \rightarrow D_s^{*} \tau \bar{\nu}_\ell)$ is in good agreement with the pQCD+LQCD prediction [27] and the RQM predictions of Refs. [28, 29].

Combining Eqs. (23), (24) and (25), we can obtain the ratio $\mathcal{R}(D_s)$

$$\mathcal{R}(D_s) = \frac{\int_{m_{D_s}^2}^{q_{max}^2} d\Gamma(B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell)/dq^2}{\int_{0}^{q_{max}^2} d\Gamma(B_s \rightarrow D_s^{*} \ell \bar{\nu}_\ell)/dq^2}, \quad (26)$$

which leads to

$$\mathcal{R}(D_s) = 0.334 \pm 0.017. \quad (27)$$

We present the ratios under various approaches in Table V. And to be consistent with the above branching fractions, our ratio $\mathcal{R}(D_s)$ is in good agreement with prediction under the pQCD+LQCD approach [27].
TABLE IV: Theoretical predictions of the branching fractions $B(B_s \to D_s^{(*)} \ell \bar{\nu}_\ell)$ and $B(B_s \to D_s \tau \bar{\nu}_\tau)$ (in unit: $10^{-6}$).

| Methods | $B(B_s \to D_s^{(*)} \ell \bar{\nu}_\ell)$ | $B(B_s \to D_s \tau \bar{\nu}_\tau)$ |
|---------|---------------------------------|---------------------------------|
| This work (HQEFT) | $1.817^{+0.392}_{-0.571}$ | $0.606^{+0.266}_{-0.211}$ |
| pQCD [27] | $1.97^{+0.89}_{-0.51}$ | $0.72^{+0.32}_{-0.23}$ |
| pQCD+LQCD [27] | $1.84^{+0.77}_{-0.51}$ | $0.63^{+0.13}_{-0.17}$ |
| pQCD [26] | $2.13^{+1.12}_{-0.77}$ | $0.84^{+0.38}_{-0.28}$ |
| RQM [29] | $2.1 \pm 0.2$ | $0.62 \pm 0.05$ |
| RQM [28] | $2.54^{+0.28}_{-0.27}$ | $0.695^{+0.085}_{-0.075}$ |
| CQM [36] | $2.73 \pm 3.00$ | $-$ |
| QCDSR [35] | $2.46 \pm 0.38$ | $-$ |
| QCDSR [55] | $2.8 \pm 3.5$ | $-$ |
| LCSR [53] | $2.03^{+0.35}_{-0.49}$ | $-$ |
| LCSR [37] | $1.0^{+0.4}_{-0.3}$ | $0.33^{+0.14}_{-0.11}$ |
| LQCD [32] | $2.013 \pm 2.469$ | $0.619 \pm 0.724$ |
| BSE [38] | $1.4 \pm 1.7$ | $0.47 \pm 0.55$ |

TABLE V: The ratios $\mathcal{R}(D_s)$ under various approaches.

| Methods | $\mathcal{R}(D_s)$ |
|---------|-------------------|
| This work (HQEFT) | $0.334 \pm 0.017$ |
| pQCD [27] | $0.365^{+0.009}_{-0.012}$ |
| pQCD+LQCD [27] | $0.341^{+0.024}_{-0.025}$ |
| pQCD [26] | $0.392 \pm 0.022$ |
| RQM [29] | $0.295$ |
| RQM [28] | $0.274^{+0.020}_{-0.019}$ |
| LQCD [32] | $0.299^{+0.027}_{-0.022}$ |
| LQCD [30] | $0.314 \pm 0.006$ |
| CCQM [34] | $0.271 \pm 0.069$ |
| LCSR [37] | $0.33$ |

IV. SUMMARY

In the present paper, we make a detailed study on the TFFs of the semileptonic decay $B_s \to D_s \ell \bar{\nu}_\ell$ under the LCSR approach within the framework of HQEFT. By using the chiral correlator, the TFFs $f_{B_s \to D_s}^{B_s \to D_s} (q^2)$ are dominated by the leading-twist contributions and the accuracy of the LCSR prediction is improved. At the maximum recoil point, we have $f_{B_s \to D_s}^{B_s \to D_s} (0) = 0.533^{+0.112}_{-0.094}$. After applying the $s$-series extrapolation, we obtain the TFFs in the whole physical $q^2$-region. Figure 3 and Figure 4 show the extrapolated TFFs $f_{B_s \to D_s}^{B_s \to D_s} (q^2)$ and the differential decay widths of $B_s \to D_s \ell \bar{\nu}_\ell$, respectively. Furthermore, we derive the branching fractions $B(B_s \to D_s \ell \bar{\nu}_\ell) = (1.817^{+0.890}_{-0.571}) \times 10^{-6}$ and $B(B_s \to D_s \tau \bar{\nu}_\tau) = (6.061^{+2.608}_{-2.114}) \times 10^{-3}$. The resultant ratio $\mathcal{R}(D_s) = 0.334 \pm 0.017$ agrees well with the previous prediction under a combined approach of pQCD+LQCD [27]. This could be treated as a good example of showing the consistency of the TFFs under various approaches [58]. Analyzing the data in Tab. V, we can find that the predictions of $\mathcal{R}(D_s)$ through various methods are not in good agreement with each other, which needs more reasonable and accurate research in the future. At the same time, we also look forward to the experimental measurements of $\mathcal{R}(D_s)$, so as to test the theoretical prediction for $\mathcal{R}(D_s)$ in the framework of SM.

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