The Pimsner-Voiculescu sequence for coactions of compact Lie groups

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Abstract

The Pimsner-Voiculescu sequence is generalized to a Pimsner-Voiculescu tower describing theKK-category equivariant with respect to coactions of a compact Lie group satisfying the Hodgkin condition. A dual Pimsner-Voiculescu tower is used to show that coactions of a compact Hodgkin-Lie group satisfy the Baum-Connes property.

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Introduction

When \( G \) is a second countable, locally compact group and \( A \) is a separable \( \mathcal{C}^* \)-algebra with a continuous \( G \)-action, the Baum-Connes conjecture states that the \( K \)-theory of the reduced crossed product \( A \rtimes_r G \) can be calculated by means of geometric and representation theoretical properties of \( G \) and \( A \), see more in [4]. To be more precise, the Baum-Connes conjecture states that the assembly mapping \( \mu_A : K_\ast^G (\mathcal{E}G; A) \to K_\ast (A \rtimes_r G) \) is an isomorphism. The space \( \mathcal{E}G \) is the universal proper \( G \)-space and \( K_\ast^G (\mathcal{E}G; A) \) is the proper equivariant \( K \)-homology with coefficients in \( A \). There are known counterexamples when \( \mu_A \) is not an isomorphism, so it is more natural to speak of groups having the Baum-Connes property. In [10], the equivariant \( K \)-homology with coefficients in \( A \) was proved to be the left derived functor of \( F(A) = K_\ast (A \rtimes_r G) \) and the assembly mapping being the natural transformation from \( LF \) to \( F \). The approach to the Baum-Connes property using triangulated categories can be generalized to discrete quantum groups, see [9], which indicates that geometric techniques such as universal proper \( G \)-spaces can be generalized to discrete quantum groups.

The generalization of the Baum-Connes property to quantum groups has been studied in for instance [11] and [17]. The case studied in [11] is that of quantum group actions of the dual of a compact Lie group which correspond to coactions of the Lie group. In [11] duals of compact Lie groups were shown to satisfy the strong Baum-Connes property, i.e. the embedding of the triangulated category generated by proper coactions, the \( \mathcal{C}^* \)-algebras that are Baaj-Skandalis dual to trivial \( G \)-actions, into the \( KK \)-category equivariant with respect to coactions is essentially surjective. In this paper we construct an analogue of the Pimsner-Voiculescu sequence for coactions of a compact Hodgkin-Lie group \( G \) that describes how the \( KK \)-category equivariant with respect to coactions of \( G \)
is built up from the $C^*$-algebras with coactions of $G$ which are proper in the sense of [11].

The starting point is to express the Pimsner-Voiculescu sequence for $\mathbb{Z}$-actions in terms of a property of the representation ring of a rank one torus. Using the Universal Coefficient Theorem, the Pimsner-Voiculescu sequence can be constructed from a Koszul complex

$$0 \to \mathcal{R}(T) \overset{\alpha}{\to} \mathcal{R}(T) \to 0,$$

where $\alpha$ is defined as multiplication by $1-t$ under the isomorphism $\mathcal{R}(T) \cong \mathbb{Z}[t, t^{-1}]$. When $A$ has a coaction of $T$, i.e., a $\mathbb{Z}$-action, the tensor product over $\mathcal{R}(T)$ between this Koszul complex and $K_T^*(A \rtimes_r \mathbb{Z})$ gives the Pimsner-Voiculescu sequence. In the generalization to higher rank, when $T$ is a torus of rank $n$ we consider the Koszul complex

$$0 \to \Lambda^n \mathcal{R}(T)^n \to \Lambda^{n-1} \mathcal{R}(T)^n \to \ldots \to \Lambda^2 \mathcal{R}(T)^n \to \mathcal{R}(T)^n \to \mathcal{R}(T) \to 0.$$

The boundary mappings in this complex are defined from interior multiplication with the element $\sum (1-t_i)e_i^* \in \text{Hom}_{(R(T))}(\mathcal{R}(T)^n, \mathcal{R}(T))$. If $G$ is a compact Hodgkin-Lie group with maximal torus $T$, the representation ring $\mathcal{R}(T)$ is a free $R(G)$-module by [15], so the generalization from a torus to compact Hodgkin-Lie groups goes in a straightforward fashion. Just as when the rank is 1, the Koszul complex above can be used to produce sequence of distinguished triangles which is the analogue of a Pimsner-Voiculescu sequence for the $K$-theory of crossed products by coactions of $G$.

We will give a geometric description of a sequence of distinguished triangles in the $KK$-category equivariant with respect to coactions of $G$ that corresponds to the above Koszul complex under the Universal Coefficient Theorem. As for the Pimsner-Voiculescu sequence for $\mathbb{Z}$ we will obtain a projective resolution of the crossed product by a coaction in the sense of triangulated categories rather than exact sequences. Using suitable tensor products we produce in Theorem 3.2 a sequence of distinguished triangles in the $KK$-category equivariant with respect to coactions of $G$ that we call the generalized Pimsner-Voiculescu tower for $A$: 

\[
\begin{array}{cccccccc}
\mathbb{C}^\omega \otimes A & \to & \Sigma^n D_{n-1}(A) & \to & \Sigma^n D_{n-2}(A) & \to & \cdots \\
\Sigma \mathbb{C}^\omega \otimes A & \to & \Sigma^2 \mathbb{C}^\omega \otimes A & \to & \cdots \\
\cdots & \to & \Sigma^n D_2(A) & \to & \Sigma^n D_1(A) & \to & t(A \rtimes_r G) \\
\cdots & \to & \Sigma^{n-1} \mathbb{C}^\omega \otimes A & \to & \Sigma^n \mathbb{C}^\omega \otimes A \\
\end{array}
\]

Here $t(A \rtimes_r G)$ denotes the $C^*$-algebra $A \rtimes_r G$ equipped with the trivial $G$-action and the terms $D_i(A)$ can be explicitly described as a braided tensor product. Taking $K$-theory of the lower row will give a complex similar to the Koszul complex that in a sense forms a projective resolution of the $K$-theory of $A \rtimes_r G$. The dual Pimsner-Voiculescu gives a more precise description of the results of
by a sequence of distinguished triangles in $KK^G$ that describes the crossed product $A \rtimes \hat{G}$ in terms of $G - C^*$-algebras with trivial $G$-action, thus giving a direct route to the strong Baum-Connes property of $\hat{G}$.

The paper is organized as follows; the first section consists of a review of $KK$-theory of actions and coactions. In particular we gather some known results about the braided tensor product and the Drinfeld double which plays a mayor role in constructing the dual Pimsner-Voiculescu tower. The main references of this section are [1], [2], [3], [7], [10], [12] and [15]. In the second section a geometric construction of the Pimsner-Voiculescu sequence for $\mathcal{CI}$-actions is presented and generalized to higher rank via a Koszul complex. In the third section the restriction functor for coactions is used to generalize the Pimsner-Voiculescu sequence to coactions of compact Hodgkin-Lie groups $G$. As an example of this we calculate the $K$-theory of some compact homogeneous spaces. By similar methods, a dual Pimsner-Voivulescu tower is constructed in $KK^G$, following the ideas of [10]. At the end of the paper we discuss some possible generalizations to duals of Woronowicz deformations.

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1 Actions and coactions of compact groups

The standard approach to equivariant $K$-theory is to introduce equivariant $KK$-theory. If $A$ and $B$ are two separable $C^*$-algebras with a continuous action of a locally compact group $G$, the equivariant $KK$-group $KK^G(A,B)$ is defined as the set of homotopy classes of $G$-equivariant $A-B$-Kasparov modules which forms an abelian group under direct sum. The $KK$-groups can be equipped with a product such that if $C$ is a third separable $C^*$-algebra with a continuous $G$-action there is an additive pairing called the Kasparov product

$$KK^G(A,B) \times KK^G(B,C) \to KK^G(A,C).$$

Following the standard construction, we let $KK^G$ denote the additive category of all separable $C^*$-algebras with a continuous $G$-action and a morphism in $KK^G$ from $A$ to $B$ is an element of $KK^G(A,B)$. The composition of two $KK^G$-morphisms is defined to be their Kasparov product. The group $KK^G(\mathbb{C},A)$ coincides with the equivariant $K$-theory of $A$. In particular, if $G$ is compact $KK^G(\mathbb{C},\mathbb{C}) = R(G)$, the representation ring of $G$. The action of $R(G)$ on equivariant $K$-theory generalizes to an $R(G)$-module structure on the bivariant groups $KK^G(A,B)$.

The category $KK^G$ can be equipped with a triangulated structure with a mapping cone coming from the mapping cone construction of a $*$-homomorphism. The triangulated structure on $KK^G$ is universal in the sense that any homotopy invariant, stable, split-exact functor on the category of $C^*$-algebras with a continuous $G$-action defines a homological functor on $KK^G$. The construction of the triangulated structure and its universality are thoroughly explained in [10]. Let us just recall the basics of the construction of the triangulated structure on $KK^G$. The suspension $\Sigma A$ of a $G - C^*$-algebra is defined by $C_0(\mathbb{R}) \otimes A$. By Bott periodicity $\Sigma^2 \cong \text{id}$. A distinguished triangle in $KK^G$ is a triangle isomorphic to
one of the form

\[
\begin{array}{c}
C(f) \\
\downarrow f \\
B,
\end{array}
\]

where \( C(f) \) is the mapping cone of the equivariant \(*\)-homomorphism \( f : A \to B \).

In particular, if \( f : A \to B \) is a surjection and admits an equivariant completely positive splitting the natural mapping \( \ker(f) \to C(f) \) defines an equivariant \( \text{KK} \)-isomorphism, so under suitable assumptions a distinguished triangle is isomorphic to a short exact sequence.

How to construct \( \text{KK} \)-theory of coactions of groups is easiest seen in the simpler case when \( G \) is an abelian group. If \( A \) is a \( C^* \)-algebra equipped with an action \( \alpha \) of the abelian group \( G \), the crossed product \( A \rtimes_r G \) carries a natural action of the Pontryagin dual \( \hat{\mathcal{C}} \). This action is called the dual action of \( \hat{\mathcal{C}} \).

Since abelian groups are exact, the crossed product by an abelian group defines a triangulated functor \( \text{KK}^G \to \text{KK}^G \). The crossed product by the dual action is described by Takesaki-Takai duality which states that there is an equivariant isomorphism

\[
A \rtimes_r G \rtimes_r \hat{\mathcal{C}} \cong A \otimes \mathcal{X}(L^2(G)),
\]

where \( A \rtimes_r G \rtimes_r \hat{\mathcal{C}} \) is equipped with the dual action of \( G \) and the \( G \)-action on \( A \otimes \mathcal{X}(L^2(G)) \) is defined as \( a \otimes \text{Ad} \). Takesaki-Takai duality implies that the crossed product defines a triangulated equivalence \( \text{KK}^G \to \text{KK}^G \).

An action \( \alpha \) of a group \( G \) on \( A \) defines a \(*\)-homomorphism \( \Delta_\alpha : A \to \mathcal{M}(A \otimes C_\alpha(G)) \) by letting \( \Delta_\alpha(a) \) be the function \( g \mapsto a \cdot (g) \). When \( G \) is abelian there is a natural isomorphism \( C_\alpha(G) \cong C^*_\alpha(G) \) and a \( G \)-action corresponds to a non-degenerate \(*\)-homomorphism \( \Delta_\alpha : A \to \mathcal{M}(A \otimes \text{min} C^*_\alpha(G)) \) satisfying certain conditions. The first instance of a coaction of a group \( G \) is on \( C^*_\alpha(G) \). Using the universal property of \( C^*_\alpha(G) \), one can construct a non-degenerate mapping \( \Delta : C^*_\alpha(G) \to \mathcal{M}(C^*_\alpha(G) \otimes \text{min} C^*_\alpha(G)) \) called the comultiplication and is induced from the diagonal homomorphism \( G \to G \times G \). Clearly, the mapping \( \Delta \) satisfies:

\[
(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta,
\]

so we say that \( \Delta \) is coassociative. Since \( \Delta_{21} = \Delta \) the comultiplication \( \Delta \) is cocommutative, so if we interpret \( C^*_\alpha(G) \) as the functions on a reduced locally compact quantum group \( \hat{\mathcal{C}} \) then \( \hat{\mathcal{C}} \) can be thought of as abelian, see more in [7]. With the abelian setting as motivation, we say that a separable \( C^* \)-algebra \( A \) has a coaction of the locally compact second countable group \( G \) if there is a non-degenerate \(*\)-homomorphism \( \Delta_\alpha : A \to \mathcal{M}(A \otimes \text{min} C^*_\alpha(G)) \) satisfying the two conditions that \( \Delta_\alpha(A) \cdot 1_A \otimes \text{min} C^*_r(G) \) is a dense subspace of \( A \otimes \text{min} C^*_r(G) \) and that \( \Delta_\alpha \) is coassociative in the sense that

\[
(\Delta_\alpha \otimes \text{id}_{C^*_r(G)}) \Delta_\alpha = (\text{id}_A \otimes \Delta) \Delta_\alpha.
\]

A separable \( C^* \)-algebra equipped with a coaction of \( G \) will be called a \( \hat{\mathcal{C}} - C^* \)-algebra. Sometimes we will abuse the notation and call a coaction of \( G \) a \( G \)-action. An example of a coaction is the dual coaction on \( C^* \)-algebras of the form \( A = B \rtimes_r G \), for some \( G - C^* \)-algebra \( B \). When \( G \) is discrete we can decompose \( B \rtimes_r G \) by means of the dense subspace \( \oplus_{g \in G} B \lambda_g \) and the dual coaction is defined
by \( \Delta_\lambda(b \lambda_x) := b \lambda_x \otimes \lambda_x \). In the general setting, the construction of the dual coaction goes analogously and we refer the reader to [1].

Much of the theory for group actions also holds for group coactions, the crossed product will as for abelian groups be a stepping stone back and forth between actions and coactions. In [1], the KK-theory equivariant with respect to a bi-C*-algebras and the corresponding Kasparov product was constructed. In [12] it was proved that the KK-theory equivariant with respect to a locally compact quantum group has a triangulated structure defined in the same fashion as for a group.

Let us explain the setting of [1] more explicitly in the case of coactions of a group. An \( A \rtimes B \)-Hilbert bimodule \( \mathcal{E} \) is called \( \hat{G} \)-equivariant if there is a coaction \( \hat{\delta}_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{L}_{\text{Ber-min}} C_r^*(G)(\mathcal{B} \otimes_{\text{min}} C_r^*(G), \mathcal{E} \otimes_{\mathcal{E}} C_r^*(G)) \) satisfying a coassociativity condition similar to (1) and \( \delta_\mathcal{E} \) should commute with the \( A \)-action and \( B \)-action in the obvious ways. By Proposition 2.4 of [1], the coaction \( \delta_\mathcal{E} \) is uniquely determined by a unitary \( V_\mathcal{E} \in \mathcal{L}(\mathcal{E} \otimes_{\Delta} \mathcal{E}(B \otimes_{\text{min}} C_r^*(G)), \mathcal{E} \otimes_{\mathcal{E}} C_r^*(G)) \) via the equation \( \delta_\mathcal{E}(x)y = V_\mathcal{E}(x \otimes_{\Delta} y) \) for \( x \in \mathcal{E} \) and \( y \in B \otimes_{\text{min}} C_r^*(G) \). A \( \hat{G} \)-equivariant \( A \rtimes B \)-Kasparov module is an \( A \rtimes B \)-Kasparov module \( (\mathcal{E}, \mathcal{F}) \) such that \( \mathcal{E} \) is a \( \hat{G} \)-equivariant \( A \rtimes B \)-Hilbert module and the operator \( F \) commutes with the unitary \( V_\mathcal{E} \) up to a compact operator. The group \( KK^\hat{G}(A,B) \) is defined as the homotopy classes of \( \hat{G} \)-equivariant \( A \rtimes B \)-Kasparov modules. The additive category \( KK^\hat{G} \) is defined by taking the objects to be separable \( \hat{G} \)-C*-algebras and the group of morphisms from \( A \) to \( B \) is \( KK^\hat{G}(A,B) \). The composition in \( KK^\hat{G} \) is Kasparov product of \( G \)-equivariant Kasparov modules.

To a closed subgroup \( H \) of \( G \), the restriction of a \( G \)-action to \( H \) defines a restriction functor \( \text{Res}_H^G : KK^G \rightarrow KK^H \) and its right adjoint is the induction functor \( \text{Ind}_H^G : KK^H \rightarrow KK^G \). However the restriction goes in the other direction for coactions. When \( H \) is a closed subgroup of \( G \), there is a non-degenerate embedding \( C^*(H) \subseteq \mathcal{M}(C^*(G)) \) so a coaction of \( H \) can be restricted to a coaction of \( G \). This construction defines a triangulated functor \( \text{Res}_H^G : KK^H \rightarrow KK^G \).

The crossed product \( B \rtimes \hat{G} \) sends a \( G \)-C*-algebra to a \( \hat{G} \)-C*-algebra and if \( G \) is exact the crossed product induces a triangulated functor \( KK^\hat{G} \rightarrow KK^G \).

In order to construct a duality similar to Takesaki-Takai duality one introduces the crossed product by a coaction. If \( A \) is a \( \hat{G} \)-C*-algebra we define

\[
A \rtimes \hat{G} := [\Delta_\lambda(A) \cdot 1_A \otimes C_\rho(G)] \subseteq \mathcal{M}(\hat{A} \otimes \mathcal{X}(L^2(G))).
\]

It follows from Lemma 7.2 of [2] that \( A \rtimes \hat{G} \) forms a C*-algebra. For a thorough introduction to crossed products by coactions see [13]. The C*-algebra \( A \rtimes \hat{G} \) carries a continuous \( G \)-action defined in the dense subspace \( \Delta_\lambda(A) \cdot 1_A \otimes C_\rho(G) \) by

\[
g.(\Delta_\lambda(a) \cdot 1_A \otimes f) := \Delta_\lambda(a) \cdot 1_A \otimes g.f.
\]

Similarly to the abelian setting, Takesaki-Takai duality holds so there are equivariant isomorphisms \( A \rtimes \hat{G} \cong B \otimes \mathcal{X}(L^2(G)) \) and \( A \rtimes \hat{G} \cong \hat{A} \otimes \mathcal{X}(L^2(G)) \) which ensures that the crossed product defines an equivalence of triangulated categories known as Baaj-Skandalis duality.

The tensor product on the category of \( G \)-C*-algebras is well defined. If \( A \) and \( B \) have actions \( \alpha \) respectively \( \beta \) of \( G \) the tensor product \( A \otimes_{\text{min}} B \) can be equipped with the action \( A \otimes \beta : G \rightarrow \text{Aut}(A \otimes_{\text{min}} B) \). However, for a non-abelian group \( G \) the construction of a tensor product of \( \hat{G} \)-C*-algebras can not be done
by just taking tensor products of the $C^*$-algebras. The tensor product relevant for $\hat{G} - C^*$-algebras is the braided tensor product over $\hat{G}$ which requires one further structure. Suppose that $A$ is a $\hat{G}$-algebra with a continuous $G$-action $\alpha$. If the action $\alpha$ satisfies that
\[
\Delta_A \circ \alpha_g = (\alpha_g \otimes \text{Ad}(g)) \Delta_A
\]
we say that $A$ is a Yetter-Drinfeld algebra. An example of a Yetter-Drinfeld algebra is $C^*_r(G)$ with $G$-action defined by the adjoint action $G \to \text{Aut}(G)$. It is much easier to construct a Yetter-Drinfeld algebra from a $G - C^*$-algebra, if $A$ is a $G - C^*$-algebra we can in a functorial way define a coaction of $G$ on $A$ by setting $\Delta_A(a) := a \otimes 1$. When $A$ is a Yetter-Drinfeld algebra, the $C^*$-algebra $A \rtimes G$ is also a Yetter-Drinfeld algebra since the morphism $\Delta_A$ is covariant with respect to the $G$-action and $\Delta_A$ extends to a coaction of $G$ on $A \rtimes G$, see more in \cite{12}. This construction is functorial and the crossed product can be seen as a functor on the category of Yetter-Drinfeld algebras.

When $A$ is a Yetter-Drinfeld algebra and $B$ is a $\hat{G} - C^*$-algebra we define the mappings
\[
t_A : A \to \mathcal{M}(A \otimes_{\min} B \otimes \mathcal{X}(L^2(G))), \quad \iota(a) := \Delta_A(a)_{13}
\]
\[
t_B : B \to \mathcal{M}(A \otimes_{\min} B \otimes \mathcal{X}(L^2(G))), \quad \iota(b) := \Delta_B(b)_{23}.
\]

Following \cite{12}, the braided tensor product $A \boxtimes_G B$ is defined as the closed linear span of $t_A(a) \cdot t_B(b)$. By Proposition 8.3 of \cite{16}, $A \boxtimes_G B$ forms a $*$-subalgebra of $\mathcal{M}(A \otimes_{\min} B \otimes \mathcal{X}(L^2(G)))$ so the braided tensor product is a $C^*$-algebra. The coaction of $G$ on $A \boxtimes_G B$ is defined by
\[
\Delta_A \boxtimes_G \Delta_B(t_A(a) \cdot t_B(b)) := (t_A \otimes \text{id})(\Delta_A(a)) \cdot (t_B \otimes \text{id})(\Delta_B(b)).
\]

Observe that since $C^*_r(G)$ is cocommutative, the adjoint $\hat{G}$-action is trivial and a similar construction of a braided tensor product over $G$ between $G - C^*$-algebras with trivial $\hat{G}$-actions coincides with the usual tensor product. In general, the braided tensor product over $\hat{G}$ does not need to coincide with the usual tensor product. By Lemma 3.5 of \cite{12} there is a $G$-equivariant isomorphism
\[
(A \boxtimes_G B) \rtimes \hat{G} \cong (A \rtimes G) \boxtimes_G B
\]
where the $G$-coaction on the right hand side is the trivial one on $B$. More generally, this identity holds for any quantum group and in particular also for braided tensor products over $G$. We will prove this statement in special case of braided tensor products over a compact group $G$ with $C(G)$ below in Lemma 6.3.

If we interpret the structure of a Yetter-Drinfeld algebra as two actions of the quantum groups $G$ and $\hat{G}$ satisfying a certain cocycle relation, the cocycle defines a quantum group by means of a double crossed product such that Yetter-Drinfeld algebras are precisely the $C^*$-algebras with an action of this double crossed product. The right quantum group to look at is the Drinfeld double $D(G)$. Using the notations of quantum groups, the algebra of functions on $D(G)$ is $C_0(G, C^*_r(G)) = C_0(G) \otimes C^*_r(G)$ with the obvious action and coaction of $G$. The action and coaction define a comultiplication
\[
\Delta_{D(G)} : C_0(D(G)) \to \mathcal{M}(C_0(D(G)) \otimes C_0(D(G)))
\]
by \( \Delta_{D(G)} := \sigma_{23} \text{Ad}(W_{23})(\Delta_{C(G)} \otimes \Delta_{C(G)}) \) where \( W \in \mathcal{B}(L^2(G) \otimes L^2(G)) \) is the multiplicative unitary of \( G \) defined by \( Wf(g_1, g_2) = f(g_1, g_1g_2) \). The comultiplication \( \Delta_{D(G)} \) makes \( D(G) \) into a quantum group by Theorem 5.3 of [3]. A Yetter-Drinfeld algebra \( A \) with the action \( \alpha \) and coaction \( \Delta_A \) correspond to a \( D(G) \)-coaction by defining the \( D(G) \)-coaction

\[
\Delta^D_A := (\Delta_\alpha \otimes \text{id})\Delta_A : A \to \mathcal{M}(A \otimes_{\text{min}} C_0(D(G))),
\]

see more in Proposition 3.2 of [12]. Therefore we can consider the braided tensor product as a tensor product between \( D(G) \)-algebras and \( \hat{G} \)-algebras. The braided tensor product induces a biadditive functor

\[ \otimes : KK\mathcal{D}(G) \times KK\hat{G} \to KK\hat{G}. \]

Much of the theory of coactions can be done without introducing any quantum groups, but in order to construct the Pimsner-Voiculescu sequence for coactions of compact Hodgkin-Lie groups we will need the braided tensor product as a biadditive functor between \( KK \)-categories.

2 The Pimsner-Voiculescu sequence from the viewpoint of representation rings

In this section we will study the Pimsner-Voiculescu sequence for \( \mathbb{Z} \) and generalize to a Pimsner-Voiculescu tower for \( \mathbb{Z}^n \). We will use representation theory to calculate all the mappings explicitly. These calculations will in a surprisingly straightforward way give a natural route to a Pimsner-Voiculescu tower for coactions of compact Lie groups.

Consider the evaluation mapping \( l : C_0(\mathbb{R}) \to C_0(\mathbb{Z}) \). This mapping fits into a \( \mathbb{Z} \)-equivariant short exact sequence

\[
0 \to \Sigma C_0(\mathbb{Z}) \to C_0(\mathbb{R}) \xrightarrow{l} C_0(\mathbb{Z}) \to 0.
\]

The \( \mathbb{Z} \)-equivariant Dirac operator \( \mathcal{D} \) on \( \mathbb{R} \) defines a \( \mathbb{Z} \)-equivariant odd unbounded \( K \)-homology class, thus an element \( [\mathcal{D}] \in KK^\mathbb{Z}(C_0(\mathbb{R}), \Sigma \mathbb{C}) \). While \( \mathbb{R} \) is the universal proper \( \mathbb{Z} \)-space the element \( [\mathcal{D}] \) is the Dirac element of \( \mathbb{Z} \) and the strong Baum-Connes property of \( \mathbb{Z} \) implies that \( [\mathcal{D}] \) is a \( KK^\mathbb{Z} \)-isomorphism. The exact sequence induces a distinguished triangle in \( KK^\mathbb{Z} \) which after using the isomorphism \( C_0(\mathbb{R}) \cong \Sigma \mathbb{C} \) and rotation 4 steps to the left becomes

\[
\begin{tikzcd}
C_0(\mathbb{Z}) \arrow[r] & C_0(\mathbb{Z}) \arrow[swap]{d} & C_0(\mathbb{Z}) \arrow[swap]{dl}
\end{tikzcd}
\]

In a certain sense, the distinguished triangle captures the entire behavior of the Pimsner-Voiculescu sequence. If \( A \) is a \( \mathbb{Z} \)-algebra we can apply Baaj-Skandalis duality to \( A \) and tensor with \( A \times_{\mathbb{Z}} \mathbb{Z} \). If we apply Baaj-Skandalis
duality again, we obtain a distinguished triangle in $KK^\mathbb{Z}$:

$$
\begin{array}{ccc}
A & \rightarrow & A \\
\downarrow & & \downarrow \\
A \rtimes_r \mathbb{Z} & \rightarrow & \mathbb{Z}
\end{array}
$$

where $A \rtimes_r \mathbb{Z}$ is given the trivial $\mathbb{Z}$-action. Taking $K$-theory of this distinguished triangle gives back the classical Pimsner-Voiculescu sequence due to the following lemma:

**Proposition 2.1.** When $T$ is a torus of rank 1 and the element $\kappa \in KK^T(\mathbb{C}, \mathbb{C})$ is defined using the isomorphisms $KK^T(\mathbb{C}, \mathbb{C}) \cong \text{Hom}_{\text{R}(T)}(\mathbb{R}(T), \mathbb{R}(T))$ and $\mathbb{R}(T) \cong \mathbb{Z}[t, t^{-1}]$ as

$$
\kappa f(t, t^{-1}) = (1 - t)f(t, t^{-1}),
$$

the $KK$-morphism $\kappa$ is Baaj-Skandalis dual to the $KK$-morphism $C_0(\mathbb{Z}) \rightarrow C_0(\mathbb{Z})$ defined by $[4]$.

Observe that the $K$-theory of the exact sequence (4) is described from the exact sequence:

$$
0 \rightarrow R(T) \xrightarrow{1-t} R(T) \rightarrow \mathbb{Z} \rightarrow 0,
$$

by Proposition 2.1. The first terms in this exact sequence is the Koszul complex defined by $1 - t \in \text{Hom}_{\text{R}(T)}(\mathbb{R}(T), \mathbb{R}(T))$ and $\mathbb{Z}$ is the cohomology of the Koszul complex.

**Proof.** Let $\kappa_0 \in \text{Hom}_{\text{R}(T)}(\mathbb{R}(T), \mathbb{R}(T))$ denote the Baaj-Skandalis dual of the $KK$-morphism induced from (4). It follows directly from the construction that the mapping $\mathbb{R}(T) \rightarrow \mathbb{Z}$ induced from $\Sigma C_0(\mathbb{Z}) \rightarrow C_0(\mathbb{R})$ is the augmentation mapping $\mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Z}$ onto the generator of $K_1(C_0(\mathbb{R}))$. Therefore the image of $\kappa_0$ is the ideal generated by either $1 + t$ of $1 - t$ so $\kappa_0$ is of the form $u \cdot (1 \pm t)$ for some unit $u \in \mathbb{Z}[t, t^{-1}]$. The sign and $u = 1$ is found by either a direct calculation or by considering the Pimsner-Voiculescu sequence for $C_0(\mathbb{Z})$.

We will return to the Koszul complexes later on. First we will construct a geometric interpretation of the higher rank situation. Assume that $T$ is a torus of rank $n$ and consider the semi-open unit cube $I = [0, 1]^n \subseteq \mathbb{R}^n$. For $i = 1, \ldots, n$ we define $\tilde{X}_i$ as the set of open $i - 1$-dimensional faces of $I$. The union satisfies

$$
\bigcup_{i=1}^n \tilde{X}_i = \partial I \cap I.
$$

We let $k_i$, for $i = 1, 2, \ldots n$, denote the integers

$$
k_i := \left(\begin{array}{c} n \\ i - 1 \end{array}\right).
$$

The set $\tilde{X}_i$ has $k_i$ connected components so if we choose a homeomorphism $\mathbb{R}^0 \cong \mathcal{R}$ there are homeomorphisms

$$
\tilde{X}_i \cong \bigcup_{j=1}^{k_i} \mathbb{R}^{i-1} \quad \text{for} \quad i = 1, 2, \ldots, n,
$$

where we interpret $\mathbb{R}^0$ as the one-point space. We take $X_i$ to be the $\mathbb{Z}^n$-translates of $\bigcup_{j \in \mathbb{Z}} \tilde{X}_j$ and define $Y_i := \mathbb{R}^n \setminus X_i$ for $i = 1, 2, \ldots, n$ and $Y_0 := \mathbb{R}^n$.
Proposition 2.2. For $i = 1, 2, \ldots, n$ there are $\mathbb{Z}^n$-equivariant isomorphisms

$$C_0(Y_{i-1})/C_0(Y_i) \cong \mathbb{C}^k \otimes \Sigma^{i-1} C_0(\mathbb{Z}^n).$$

Proof. By equation (6) there is a $\mathbb{Z}^n$-equivariant homeomorphism

$$Y_{i-1} \setminus Y_i \cong \bigsqcup_{m \in \mathbb{Z}^n} \left( \prod_{j=1}^k \mathbb{R}^{i-1} \right),$$

where $\mathbb{Z}^n$ acts by translation on the first disjoint union. Therefore

$$C_0(Y_{i-1})/C_0(Y_i) \cong C_0(Y_{i-1} \setminus Y_i) \cong C_0 \left( \bigsqcup_{m \in \mathbb{Z}^n} \left( \prod_{j=1}^k \mathbb{R}^{i-1} \right) \right) \cong$$

$$\cong \mathbb{C}^k \otimes C_0(\mathbb{Z}^n \times \mathbb{R}^{i-1}) \cong \mathbb{C}^k \otimes \Sigma^{i-1} C_0(\mathbb{Z}^n).$$

By equation (6).}

Consider the classifying space $\mathbb{R}^n$ for proper actions of $\mathbb{Z}^n$. Since $\mathbb{Z}^n$ has the strong Baum-Connes property, the Dirac element $[\mathcal{D}]$ induces a $KK^{\mathbb{Z}^n}$-isomorphism $C_0(\mathbb{R}^n) \cong \Sigma^n \mathbb{C}$. An alternative approach to constructing this isomorphism is the Julg theorem which implies that for any $T$-$C^*$-algebra $A$ there is an isomorphism $K^T_n(A) \cong K_n(A \rtimes T)$. Therefore $K^T_n(\Sigma^n \mathbb{C} \rtimes \mathbb{Z}^n) \cong K^T_n(C_0(\mathbb{R}^n) \rtimes \mathbb{Z}^n)$ and the statement follows from the Universal Coefficient Theorem for the compact Hodgkin-Lie group $T$, see more in [14].

For $i = 1, 2, \ldots, n$, Proposition 2.2 implies that there is a $\mathbb{Z}^n$-equivariant short exact sequence

$$0 \to C_0(Y_i) \to C_0(Y_{i-1}) \to \mathbb{C}^k \otimes \Sigma^{i-1} C_0(\mathbb{Z}^n) \to 0. \tag{7}$$

We will by $k_i \in KK^{\mathbb{Z}^n}(\mathbb{C} \otimes C_0(\mathbb{Z}^n), \mathbb{C} \otimes C_0(\mathbb{Z}^n))$ denote the $\mathbb{Z}^n$-equivariant $KK$-morphism defined in such a way that the extension class defined by (7) composed with the restriction mapping $C_0(Y_i) \to \mathbb{C} \otimes \Sigma^{i-1} C_0(\mathbb{Z}^n)$ coincides with $\Sigma^{i-1} k_i$. Notice that $Y_n = \mathbb{Z}^n \times [0, 1^n]$ and $Y_0 = \mathbb{R}^n$ so we have that $C_0(Y_n) = \Sigma^n C_0(\mathbb{Z}^n)$ and $C_0(Y_0) = C_0(\mathbb{R}^n)$, the latter being $KK^{\mathbb{Z}^n}$-isomorphic to $\Sigma^n \mathbb{C}$. Thus we get a sequence of distinguished triangles in $KK^{\mathbb{Z}^n}$:

$$\Sigma^n C_0(\mathbb{Z}^n) \to C_0(Y_{n-1}) \to \cdots \to C_0(Y_2) \to C_0(Y_1) \to \Sigma^n \mathbb{C} \tag{8}$$

$$\cdots \to C_0(Y_2) \to C_0(Y_1) \to \cdots \to \Sigma^n \mathbb{C} \to C_0(\mathbb{Z}^n) \to 0.$$

A sequence of distinguished triangles of this type will be called a tower. The tower (8) in $KK^{\mathbb{Z}^n}$ is the higher rank analogue of the distinguished triangle (5). The tower (8) can be generalized to contain any coefficient ring.
Therefore there exists sequences of numbers \( \kappa_i \). Let \( R \) denote a commutative ring and \( E \) an \( R \)-module. For simplicity we will assume that \( E \) is free and finitely generated, let us say of rank \( N \). For an element \( \nu \in \text{Hom}_R(E,R) \), the Koszul complex of \( E \) with respect to \( \nu \) is the complex

\[
0 \to \Lambda^{n}E \overset{\delta_1}{\to} \Lambda^{n-1}E \overset{\delta_2}{\to} \cdots \overset{\delta_{n-1}}{\to} \Lambda^{2}E \overset{\delta_{n-1}}{\to} \Lambda^0E \overset{\nu}{\to} R \to 0,
\]

where each \( \delta_k \) is defined as interior multiplication by \( \nu \). Since we have assumed \( E \) to be free, we may write \( \nu = \sum v_i e_i^* \) for some \( v_1, v_2, \ldots, v_N \in R \) and the dual basis \( e_i^* \) of a basis \( e_i \), \( i = 1, 2, \ldots, N \), of \( E \). If the sequence \( v_1, v_2, \ldots, v_N \) is a regular sequence the Koszul complex is exact except at \( R \). The cohomology of the Koszul complex is in this case \( R/\nu(E) \) at \( R \). See more in [5].

The Koszul complex of interest to us is constructed from the module \( E := R(T)^a \) over the representation ring of the torus \( T \) which has the following form:

\[
R(T) \cong \mathbb{Z}[t_1^\pm 1, \ldots, t_a^\pm 1].
\]

Observe that Baaj-Skandalis duality and the Universal Coefficient Theorem implies that

\[
KK^Z(\mathbb{C}^{b_1} \otimes C_0(\mathbb{Z}^n), \mathbb{C}^{b_{k+1}} \otimes C_0(\mathbb{Z}^n)) \cong KK^T(\mathbb{C}^{b_1}, \mathbb{C}^{b_{k+1}}) \\
\cong \text{Hom}_{R(T)}(R(T)^{b_1}, R(T)^{b_{k+1}}).
\]

We have that \( R(T)^{b_1} \cong \Lambda^{n-i+1}E \) so the lower row in (9) have the right ranks for coinciding with a Koszul complex. Let \( f_i \in \text{Hom}_{R(T)}(\Lambda^{n-i+1}E, \Lambda^{n-i}E) \) denote the image of \( \kappa_i \) under the isomorphisms above. To simplify notations, we will by \((e_i)_{i=1}^n\) denote the \( R(T) \)-basis of \( E \) coming from the isomorphism \( E \cong R(T) \otimes_{\mathbb{Z}} \mathbb{Z}^n \) and by \((e^*_i)_{i=1}^n\) denote the dual basis.

**Theorem 2.3.** Under the isomorphisms \( R(T)^{b_1} \cong \Lambda^{n-i+1}E \) the mappings \( f_i \) coincide with interior multiplication by the element \( \nu := \sum_{i=1}^n (1 - t_i) e^*_i \). Therefore the sequence

\[
0 \to \Lambda^{n}E \overset{f_1}{\to} \Lambda^{n-1}E \overset{f_2}{\to} \cdots \overset{f_{n-1}}{\to} \Lambda^{2}E \overset{f_{n-1}}{\to} \Lambda^0E \overset{f_n}{\to} R(T) \to 0
\]

defines a complex isomorphic to the Koszul complex of \( E \) whose cohomology at \( R(T) \) is \( \mathbb{Z} \).

**Proof.** While both \( f_i \) and the mapping defined by interior multiplication by \( \nu \) are \( R(T) \)-linear it is sufficient to prove that \( f_i(u) = \nu_u \) for elements of the form \( u = e_{m_1} \wedge \cdots \wedge e_{m_{n+1}} \in \Lambda^{n-i+1}E \), where \( m_1, \ldots, m_{n+1} \in \{1, 2, \ldots, n\} \). Let \((m_p)_{p=1}^{n-i+1}\) be an enumeration of all \( j = 1, 2, \ldots, n \) such that \( j \notin (m_p)_{p=1}^{n-i+1} \). If we view \( \mathbb{Z}^n \) as a subset of \( \mathbb{R}^n \) we can define \( X_u \subseteq \tilde{X}_i \) as the open face in \( \mathbb{R}^n \) spanned by the vectors \( e_{m_1}, e_{m_2}, \ldots, e_{m_{n+1}} \).

Under the isomorphism \( \Lambda^{n-i+1}E \cong \mathcal{K}_{-1}(\mathbb{C}^{b_1} \otimes \mathcal{S}^{i-1}C_0(\mathbb{Z}^n)) \) the element \( u \) corresponds to a \( K \)-theory class on \( \tilde{X}_i \) which is trivial except on the face \( X_u \). Therefore there exists sequences of numbers \( (a_j)_{j=1}^{n-i+1}, (b_j)_{j=1}^{n-i+1} \subseteq \mathbb{Z} \) such that

\[
f_i(u) = \sum_{j=1}^{n-i+1} (a_j + b_j t_j) e_{m_j} u.
\]
If \( j = 1, 2, \ldots, n-i+1 \), we will let \( X_{n,j} \) denote the open face spanned by the vectors \( e_{m_1}, e_{m_2}, \ldots, e_{m_i} \). It follows from restricting to \( X_{n,j} \) that \( a_j = 1 \) since Bott periodicity implies that the index mapping \( K_{n-i}(C_0(X_{n,j})) \to K_i(C_0(X_{n,j})) \) is an isomorphism. In a similar fashion it follows that \( b_1 = -1 \).

While \( \nu(E) \) is the ideal generated by the regular sequence \( 1 - t_1, 1 - t_2, \ldots, 1 - t_n \), the cohomology of the Koszul complex is \( R(T)/\nu(E) = \mathbb{Z} \) and the quotient mapping \( R(T) \to \mathbb{Z} \) coincides with the augmentation mapping.

Consider the tower Baaj-Skandalis dual to \( \mathfrak{s} \). Given \( A, B \in KK^T \) we can apply the homological functor \( KK^T(A, - \otimes_{min} B) \) to this tower. This functor is only homological on the bootstrap category if \( B \) is not exact, but all objects in the tower Baaj-Skandalis dual to \( \mathfrak{s} \) are in the bootstrap category. The lowest row of the corresponding tower in the category of \( R(T) \)-modules is a Koszul complex:

\[
0 \to \Lambda^n \mathbb{Z}^n \otimes KK^T_s(A, B) \xrightarrow{\nu_n} \Lambda^{n-1} \mathbb{Z}^n \otimes KK^T_s(A, B) \xrightarrow{\nu_{n-1}} \cdots \xrightarrow{\nu_1} \mathbb{Z}^n \otimes KK^T_s(A, B) \xrightarrow{\nu_0} KK^T_s(A, B) \to 0
\]

where \( \nu_i := \sum_{i=1}^n (1 - \beta_i)e_i \in Hom_{R(T)}(KK^T_s(A, B)^n, KK^T_s(A, B)) \)

and \( (\beta_i)_{i=1}^n \) are the commuting equivariant automorphisms of \( A \) that are Baaj-Skandalis to the \( \mathbb{Z}^n \)-action on \( B \times T \). The cohomology of this Koszul complex can be calculated from \( KK^T_s(A, B) \). We will return to this subject in the next section in the more general case of Hodgkin-Lie groups and explain this procedure further.

### 3 The generalized Pimsner-Voiculescu-towers

As mentioned in the introduction, the representation ring \( R(T) \) is free over \( R(G) \) when \( G \) is a Hodgkin-Lie group, so the step to coactions of a compact Hodgkin-Lie group will not be too large. We will throughout this section assume that \( G \) is a compact Hodgkin-Lie group of rank \( n \) with maximal torus \( T \). Recall that a group satisfies the Hodgkin condition if it is connected and the fundamental group is torsion-free.

The embedding \( T \subseteq G \) induces a restriction functor \( KK^T \to KK^G \). Using the isomorphism \( T \cong \mathbb{Z}^n \), the tower \( \mathfrak{s} \) can be restricted to a \( KK^G \)-tower:
In order to work with this $KK^G$-tower we need to describe the terms $C^*(T)$ in the second row.

**Lemma 3.1.** If $G$ is a compact Hodgkin-Lie group with Weyl group of order $w$ there is an isomorphism

$$C^*(T) \cong \mathbb{C}^w \otimes C^*(G) \quad \text{in} \quad KK^G.$$

Observe that the condition on $G$ to be a Hodgkin group is equivalent to $\hat{G}$ being a torsion-free quantum group in the sense of Meyer, see [9]. The torsion-free quantum groups are the only non-classical discrete quantum groups for which there is a general formulation of the Baum-Connes property in terms of triangulated categories. In [11], coactions of compact non-Hodgkin Lie groups were considered and the "torsion" turned out to be the torsion elements of $H^2(G,S^1)$. The less precise statement $C(G/T) \cong \mathbb{C}^k$ in $KK^G$ for some $k$ is stated and proved in [11]. An explicit calculation that $k = |W|$ can be found in [15]. We will review the conceptually important part of the proof of a Proposition in [11] which proves Lemma 3.1 aside from the calculation of $k$.

**Proof.** By [15], the representation ring $R(T)$ is free of rank $w$ over the representation ring $R(G)$ if $\pi_1(G)$ is torsion-free. If we let $\mathcal{S}$ denote the localizing subcategory of $KK^G$ generated by $\mathbb{C}$ and $C(G,T)$, Lemma 11 of [11] states that for $A \in \mathcal{S}$ the natural homomorphism

$$R(T) \otimes_{R(G)} KK^G(A, \mathbb{C}) \to KK^T(A, \mathbb{C})$$

is an isomorphism. Thus the representable functor on $\mathcal{S}$

$$A \to KK^G(A, \mathbb{C}^w) \cong R(T) \otimes_{R(G)} KK^G(A, \mathbb{C})$$

coincides with the representable functor

$$A \to KK^G(A, C(G/T)) \cong KK^T(A, \mathbb{C}).$$

The last isomorphism exists as a consequence of the fact that the induction functor $Ind^T_G$ is the right adjoint of the restriction functor from $G$ to $T$. So the Yoneda lemma implies that $C(G/T) \cong \mathbb{C}^w$ in $\mathcal{S}$ and therefore in $KK^G$. Applying Baaj-Skandalis duality it follows that there is an equivariant $KK$-isomorphism $C^*(T) \cong \mathbb{C}^w \otimes C^*(G)$.

Using Lemma 3.1 the tower (8) takes the form:

\[ \Sigma^n \mathbb{C}^w \otimes C^*(G) \rightarrow \cdots \rightarrow C_0(Y_{n-1}) \rightarrow \cdots \]  
\[ \Sigma^{n-1} \mathbb{C}^w \otimes C^*(G) \rightarrow \cdots \rightarrow C_0(Y_2) \rightarrow \cdots \rightarrow C_0(Y_1) \rightarrow \cdots \rightarrow \Sigma^n \mathbb{C} \]
\[ \cdots \rightarrow \Sigma \mathbb{C}^w \otimes C^*(G) \rightarrow \cdots \rightarrow \mathbb{C}^w \otimes C^*(G) \]
\[ \cdots \rightarrow \Sigma \mathbb{C} \otimes C^*(G) \rightarrow \cdots \rightarrow \mathbb{C} \otimes C^*(G) \]
We will call this $KK^G$, tower the fundamental $G$–PV-tower. The dual fundamental $G$–PV-tower is defined to be the $KK^G$-tower which is Baaj-Skandalis dual to the fundamental $G$–PV-tower:

\[
\begin{array}{ccccccccccc}
\Sigma^n C^w & \rightarrow & D_{n-1} & \rightarrow & \cdots \\
\downarrow & & & & & \downarrow \\
\Sigma^{n-1} C^w & \rightarrow & \cdots & \cdots \\
\downarrow & & & & & \downarrow \\
& \cdots & \cdots & \cdots \\
\end{array}
\]

where $D_1 := C_0(Y) \rtimes_r \hat{G}$.

As mentioned above, if $A$ is a $G$–$C^*$-algebra, the trivial coaction of $G$ on $A$ makes $A$ into a Yetter-Drinfeld algebra. This follows from that $C(G)$ is commutative so we can extend a $G$-action via the $D(G)$-equivariant $*$-monomorphism $C(G) \rightarrow \mathcal{M}(C_0(D(G)))$. Clearly, a $G$-equivariant mapping is equivariant in this new $D(G)$-action. Furthermore, since mapping cones does not depend on the action, the trivial extension of a $G$-action to a $D(G)$-action is functorial and respects mapping cones. The following proposition follows from universality.

**Proposition 3.2.** If $G$ is a locally compact group, the functor mapping a $G$–$C^*$-algebra to a $G$-Yetter-Drinfeld algebra with trivial $\hat{G}$-action defines a triangulated functor $KK^G \rightarrow KK^{D(G)}$.

Using the triangulated functor of Proposition 3.2, we may consider the tower (11) as a tower in $KK^{D(G)}$. Applying a crossed product by $G$ we obtain that also the tower (10) is a tower in $KK^{D(G)}$. For a $C^*$-algebra $B$ we will use the notation $t(B)$ for the $\hat{G}$–$C^*$-algebra with trivial coaction, or in the context of $G$–$C^*$-algebras $t(B)$ will denote the $G$–$C^*$-algebra with trivial action. Let us state and prove the corresponding version of (3) in a simple case of a braided tensor product over $G$ with $C(G)$, a more general proof can be found in [12].

**Lemma 3.3.** When $B$ has a continuous $G$-action, there is a $\hat{G}$-equivariant Morita equivalence

$$(C(G) \otimes B) \rtimes_r G \sim_M t(B).$$

**Proof.** By Baaj-Skandalis duality, it suffices to prove that there is a $\hat{G}$-equivariant isomorphism $(C(G) \otimes B) \rtimes_r G \cong (C(G) \rtimes_r G) \otimes t(B)$. Denote the $G$-action on $B$ by $\beta$ and define the equivariant mapping $\varphi_0 : L^1(G, C(G, B)) \rightarrow (C(G) \rtimes_r G) \otimes t(B)$ by setting

$$\varphi_0(f)(g_1, g_2) := \beta_{g_1}^{-1} f(g_1, g_2).$$

The linear mapping $\varphi_0$ is a $*$-homomorphism when $L^1(G, C(G, B))$ is equipped with the convolution twisted by the $G$-action on $C(G) \otimes B$. It is straightforward to verify that $\varphi_0$ is bounded in $C^*$-norm so we can define $\varphi : (C(G) \otimes B) \rtimes_r G \rightarrow (C(G) \rtimes_r G) \otimes t(B)$ by continuity. The $*$-homomorphism $\varphi$ is an equivariant isomorphism since an inverse can be constructed by extending

$$\varphi^{-1}(f \otimes b)(g_1, g_2) := f(g_1, g_2) \beta_{g_1}^r(b)$$

to a $*$-homomorphism $\varphi^{-1} : (C(G) \rtimes_r G) \otimes t(B) \rightarrow (C(G) \otimes B) \rtimes_r G$. \hfill \Box
Theorem 3.4 (The Pimsner-Voiculescu tower). Let $G$ be a compact Hodgkin-Lie group of rank $n$ and Weyl group of order $w$. For any separable $\hat{G} - C^*$-algebra $A$ there is a $KK^G$-tower

$$
\begin{array}{cccccc}
\mathbb{C}^n \otimes A & \longrightarrow & \Sigma^n D_{n-1}(A) & \longrightarrow & \Sigma^n D_{n-2}(A) & \cdots \\
\Sigma \mathbb{C}^w \otimes A & \longleftarrow & \Sigma^w \mathbb{C}_n \otimes A & \longleftarrow & \cdots \\
\end{array}
$$

where $D_i(A) := (C_0(\hat{Y}_i) \otimes \mathcal{K}(L^2(G))) \boxtimes_G (A \rtimes \hat{G})$ and is equipped with the $\hat{G}$-action induced from the diagonal $G$-action on $C_0(\hat{Y}_i) \otimes \mathcal{K}(L^2(G))$. 

Observe that the $D(G)$-actions on the $C^*$-algebras $C_0(\hat{Y}_i) \otimes \mathcal{K}(L^2(G))$ is defined to come from those on their Baaj-Skandalis duals $C_0(\hat{Y}_i) \rtimes \hat{G}$, which are $D(G) - C^*$-algebras in the dual $G$-actions on the crossed products and the trivial $\hat{G}$-actions. So in general, $D_i(A)$ is not the tensor product of $C_0(\hat{Y}_i) \otimes \mathcal{K}(L^2(G))$ and $A \rtimes \hat{G}$.

**Proof.** By Lemma 3.3 the $\hat{G} - C^*$-algebra $A$ admits the equivariant Morita equivalence:

$$
(C(\hat{G}) \otimes (A \rtimes \hat{G})) \rtimes_r G \cong t(A \rtimes \hat{G}).
$$

Furthermore, the isomorphism of equation (3) holds for braided tensor products over $\hat{G}$ so while the $\hat{G}$-actions on $D_i = C_0(\hat{Y}_i) \rtimes \hat{G}$ are trivial there are equivariant isomorphisms

$$
(D_i \otimes (A \rtimes \hat{G})) \rtimes_r G \cong ((C_0(\hat{Y}_i) \rtimes \hat{G}) \boxtimes_G (A \rtimes \hat{G})) \rtimes_r G \cong (C_0(\hat{Y}_i) \otimes \mathcal{K}(L^2(G))) \boxtimes_G (A \rtimes \hat{G}).
$$

Thus if we tensor the dual fundamental $G$-PV-tower (11) by the $G - C^*$-algebra $A \rtimes \hat{G}$ we obtain a new $KK^G$-tower which becomes the Pimsner-Voiculescu tower of $A$ after applying Baaj-Skandalis duality, using the Morita equivalence (13) and the isomorphisms (14).

The Pimsner-Voiculescu tower (12) is the generalization of the resolution in (9) to compact Hodgkin-Lie groups. Applying the cohomological functor $KK(\cdot, B)$ to the Pimsner-Voiculescu tower we obtain a similar resolution of $KK_n(A \rtimes \hat{G}, B)$ in terms of $KK_0(A, B)$ as in (9). Similarly, the homological functor $KK(B, \cdot)$ applied to the Pimsner-Voiculescu tower gives a resolution of $KK(B, A \rtimes \hat{G})$ in terms of $KK(B, A)$. Observe that since $A$ has a $\hat{G}$-action, the groups $KK(\mathbb{C}^w \otimes A, B)$ and $KK(B, \mathbb{C}^w \otimes A)$ will always have an $R(G)$-module structure and since $R(G)$ is free over $R(G)$ also an $R(T)$-module structure.

As an example of this, we will use the Pimsner-Voiculescu tower to calculate the $K$-theory of the homogeneous space $G/H$ when $H \subseteq G$ is a Lie subgroup. More generally, this technique can be used to calculate $K_n(A \rtimes \hat{G})$ for any $\hat{G} - C^*$-algebra $A$ when one knows $K_n(A)$ and its $R(G)$-module structure coming.
from the Julg isomorphism $K_*(A) \cong K_0^G(A \rtimes \hat{G})$. To calculate $K^*(G/H)$, consider
the $C^*$-algebra $A := C^*(H)$ equipped with the $\hat{G}$-action induced from the natural
mapping $C^*(H) \to \mathcal{M}(C^*(G))$. Green’s imprimitivity theorem implies that
$C^*(H) \rtimes \hat{G}$ is $KK$-equivalent with $C(G/H)$. Thus, if we take the $K$-theory of the
Pimsner-Voiculescu tower of $C^*(H)$ we obtain a tower of abelian groups of the form

\[ R(T) \otimes_{R(G)} R(H) \xrightarrow{\Sigma v \otimes 1} K_{-n}(D_{n-1}(C^*(H))) \to \cdots (15) \]

\[ \cdots \xrightarrow{\Sigma v \otimes 1} K_{-n}(D_1(C^*(H))) \xrightarrow{\Sigma v \otimes 1} K^*(G/H) \]

\[ \cdots \xrightarrow{\Sigma v \otimes 1} K_{-n}(D_0(C^*(H))) \xrightarrow{\Sigma v \otimes 1} K^*(G/H) \]

We use $\Sigma$ to denote degree shift in the category of $\mathbb{Z}/\mathbb{Z}$-graded abelian groups.
Here we have used that $R(T)$ is a free $R(G)$-module of rank $\mathfrak{k}$ so $K_*(C^w \otimes
C^*(H)) \cong R(T) \otimes_{R(G)} R(H)$. Thus the lowest row is the tensor product of $R(H)$
with the Koszul complex of $R(T)$ that is associated with the regular sequence
$1 - t_1, 1 - t_2, \ldots, 1 - t_n$ under the isomorphism $R(T) \cong \mathbb{Z}[[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]]$.

If we restrict our attention to simple compact Lie groups we can perform an
explicit calculation of all the groups in (15). Assume that $G = G_n$ is a simple
compact Hodgkin-Lie group in the classical $A, B, C,$- or $D$-series of rank $n$ and assume
that $H = G_k \subseteq G_n$ is a simple simply connected compact Lie group in the same
classical serie being of rank $k < n$. We may take a maximal torus $T_n \subseteq G_n$ such
that $T_k := T_n \cap G_k$ is a maximal torus in $G_k$. In this case we may consider $R(T_k)$
as an ideal in $R(T_n)$ and $R(T_n) \otimes_{R(G_k)} R(G_k) \cong R(T_k)$ as $R(T_n)$-modules. Under the
isomorphisms $R(T_k) \cong \mathbb{Z}[[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_k^{\pm 1}]]$ and $R(T_n) \cong \mathbb{Z}[[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]]$, the
Koszul vector $v$ is identified with $\sum_{i=1}^k (1 - t_i) e_i^* \in \text{Hom}(R(T_k)^n, R(T_k))$. Thus we
arrive at the tower

\[ R(T_k) \xrightarrow{\Sigma \partial} K_{-n}(D_{n-1}(C^*(G_k))) \to \cdots \]

\[ \cdots \xrightarrow{\Sigma \partial} K_{-n}(D_1(C^*(G_k))) \xrightarrow{\Sigma \partial} K^*(G_n/G_k) \]

Let us use the notation $E^*$ for the complex $\Lambda^{n-k} \mathbb{Z}^n \otimes R(T_k)$ equipped with the
Koszul differential from the vector $\sum_{i=1}^k (1 - t_i) e_i^*$ which we as above denote by
$\partial_i : E^{i-1} \to E^i$. After some simpler calculations in this Koszul complex we arrive
at the conclusion that
\[
K_{*}(D_{1}(C^{*}(G_{k}))) \cong \ker(\partial_{i+1}) \oplus \bigoplus_{j=i+2}^{n+1} \Sigma^{n-j}H^{i}(E). 
\]

Hence we obtain the isomorphism \(K^{*}(G_{n}/G_{k}) \cong \bigoplus_{j=0}^{n-k} \Sigma^{n-j}\mathbb{Z}^{k(j)} \cong \mathbb{Z}^{2^{k-1}} \oplus \Sigma \mathbb{Z}^{2^{n-k-1}}\).

**Theorem 3.5** (The dual Pimsner-Voiculescu tower). Under the assumptions of Theorem 3.4 there is a \(KK\)-tower

\[
\begin{array}{ccccccc}
\mathbb{C}^{w} \otimes t(A) & \rightarrow & \Sigma^{n} \tilde{D}_{n-1}(A) & \rightarrow & \cdots \\
\Sigma \mathbb{C}^{wn} \otimes t(A) & \rightarrow & \Sigma^{2} \mathbb{C}^{wk_{n-1}} \otimes t(A) & \rightarrow & \cdots \\
\vdots & & \Sigma^{n} \tilde{D}_{2}(A) & \rightarrow & \Sigma^{n} \tilde{D}_{1}(A) & \rightarrow & A \rtimes \tilde{G} \\
\vdots & & \Sigma^{n-1} \mathbb{C}^{wk_{2}} \otimes t(A) & \rightarrow & \Sigma^{n} \mathbb{C}^{w} \otimes t(A) & \rightarrow & \\
\end{array}
\]

where \(\tilde{D}_{i}(A) := D_{i} \otimes G A\).

For a homological functor \(F : \text{KK} \rightarrow \text{Ab}\), the dual Pimsner-Voiculescu tower of \(A\) allows us to calculate \(F(A)\) in terms of the objects \(F(C^{*}(G) \otimes t(A))\). As we shall see below, \(\tilde{G} - C^{*}\)-algebras of the form \(C^{*}(G) \otimes t(A)\) behaves similarly to proper actions. Compare this result to Theorem 4.4 of [5].

**Proof.** Consider the braided tensor product by \(\Sigma^{n}A\) and the tower (10):

\[
\begin{array}{ccccccc}
\mathbb{C}^{w} \otimes C^{*}(G) \otimes G A & \rightarrow & \Sigma^{n} C_{0}(Y_{n-1}) \otimes G A & \rightarrow & \cdots \\
\Sigma \mathbb{C}^{wn} \otimes C^{*}(G) \otimes G A & \rightarrow & \cdots \\
\vdots & & \Sigma^{n} C_{0}(Y_{1}) \otimes G A & \rightarrow & A \\
\vdots & & \Sigma^{n} \mathbb{C}^{w} \otimes C^{*}(G) \otimes G A & \rightarrow & \\
\end{array}
\]

Taking crossed product between this tower and \(\tilde{G}\) implies the Theorem since the following equivariant Morita equivalences follows from (3)

\[
\begin{align*}
(C^{*}(G) \otimes G A) \rtimes \tilde{G} & \sim \text{M} t(A) \\
(C_{0}(Y) \otimes G A) \rtimes \tilde{G} & \sim \text{M} (C_{0}(Y_{1}) \rtimes \tilde{G}) \otimes G A = D_{1} \otimes G A.
\end{align*}
\]
One of the main motivations behind this paper was to give a precise description of the Baum-Connes property of duals of Hodgkin-Lie groups. The Baum-Connes property for coactions of compact Lie groups was given meaning to and was proved to hold in [11]. More generally, this fits into the program of generalizing the Baum-Connes property to quantum groups. So far, it is not known what a suitable property the Baum-Connes property should be for a general locally compact quantum group. For discrete quantum groups which are torsion-free, in the sense of [9], there is a formulation and as mentioned above duals of compact Hodgkin-Lie groups are torsion-free.

The problem that arises when one tries to define the Baum-Connes assembly mapping for a quantum group is that there is no natural notion of a proper action and there are in general too many quantum homogeneous spaces. It is much easier to generalize certain notions of free actions than proper actions of a quantum group by just saying that an action of a discrete quantum group $\Gamma$ on a $C^*$-algebra $A$ is truly free if there is a $C^*$-algebra $A_0$ and an equivariant $*$-isomorphism $A \cong A_0 \otimes_{\min} C_0(\Gamma)$ with $\Gamma$ only acting on the second leg. In the case of a group, there are many free actions that are not truly free but this stronger notion of a free action will suffice for our purposes.

Restricting one’s attention to generalizing the Baum-Connes property of the simpler class of torsion-free discrete groups to the quantum setting, when proper actions are free, Meyer introduced a class of quantum groups known as torsion-free in [9]. Following [9], we say that a discrete quantum group $\Gamma$ is torsion-free if every coaction of the compact quantum group $\hat{\Gamma}$ on a finite-dimensional $C^*$-algebra is Morita equivalent to a trivial coaction on a direct sum of $C$'s. This fact implies that any finite-dimensional projective representation of the dual compact quantum group is equivalent to a representation. If $\Gamma$ is a discrete group, coactions of the dual compact quantum group on finite-dimensional $C^*$-algebras that are not Morita equivalent to a trivial coaction on a direct sum of $C$'s correspond to finite subgroups so a discrete group is torsion-free if and only if it is torsion-free in the sense of [9].

For a torsion-free quantum group a proper action should correspond to a free action. Under Baaj-Skandalis duality, a truly free $\Gamma-C^*$-algebra corresponds to a trivial $\Gamma$-action. Let $\mathcal{C}\mathcal{S}_\Gamma$ denote the image of $t: KK \to KK^\Gamma$. The triangulated category $\langle\mathcal{C}\mathcal{S}_\Gamma\rangle$ is defined as the localizing subcategory generated by $\mathcal{C}\mathcal{S}_\Gamma$. Following the formulation of [9], $\Gamma$ is said to satisfy the strong Baum-Connes property if the embedding of triangulated categories $\langle\mathcal{C}\mathcal{S}_\Gamma\rangle \to KK^\Gamma$ is essentially surjective. The strong Baum-Connes property of $\Gamma$ is equivalent to that any $\Gamma-C^*$-algebra is in the localizing category generated by all truly free actions. So regardless of what notion of a proper action we choose, the strong Baum-Connes conjecture will imply that the localizing category generated by all such proper actions will be $KK^\Gamma$. The quantum group is said to satisfy the Baum-Connes property if the same statement holds after localizing with respect to the kernel of equivariant $K$-theory.

In [11] the Baum-Connes property was formulated in the slightly more general setting of duals of compact Lie groups. The finite-dimensional projective representations of a compact Lie group $G$ correspond to the torsion classes of $H^2(G,S^1)$, which can be thought of as the torsion of $G$. When $G$ is Hodgkin, $H^2(G,S^1)$ is torsion-free so $G$ is torsion-free. In this case a “proper” action is an object of the additive category generated by $G$-algebras that are Baaj-Skandalis
dual to $A_0 \otimes C_{\omega}$, with $C_{\omega}$ denoting the endomorphisms of a projective representation $\omega$ and $A_0$ having trivial $G$-action. So the substitute in the setting of [11] for proper actions is the category of tensor products between Baaj-Skandalis duals of coactions on finite-dimensional $C^*$-algebras and trivial actions, just as the truly free actions form a substitute for proper actions of torsion-free quantum groups. The Baum-Connes property of coactions of a compact Hodgkin-Lie group is a direct consequence of Theorem 3.5. The method of proof of Proposition 2.1 of [11] can be used to generalize both Theorem 3.4 and Theorem 3.5 to arbitrary compact Lie group.

Finally, let us mention a promising generalization of Theorem 3.5 to Woronowicz deformations. It was proved in [12] that the compact quantum group $SU_q(2)$ satisfies that $C(SU_q(2)/T)$ is $KK^{D(SU_q(2))}$-isomorphic to $C^2$ for $q \in ]0,1[$. So if we apply the induction functor $Ind^{SU_q(2)}_T: KK_T \to KK^{SU_q(2)}$ to the distinguished triangle Baaj-Skandalis dual to (5) and use the isomorphism of Nest-Voigt we arrive at the distinguished triangle in $KK^{D(SU_q(2))}$:

\[
\begin{array}{c}
C^2 \\
\vee
\end{array}
\begin{array}{c}
C^2 \\
\vee
\end{array}
\begin{array}{c}
C(SU_q(2)).
\end{array}
\]

Using the technique from the proof of Theorem 3.5 any $A \in KK^{SU_q(2)}$ fits into a distinguished triangle

\[
\begin{array}{c}
C^2 \otimes t(A) \\
\vee
\end{array}
\begin{array}{c}
C^2 \otimes t(A) \\
\vee
\end{array}
\begin{array}{c}
A \rtimes SU_q(2).
\end{array}
\]

This distinguished triangle gives an alternative proof of the strong Baum-Connes property for $SU_q(2)$, a result first proved in [17]. The interesting part about this proof is that it only relies on the isomorphism $C(G_q/T) \cong C^w$ in $KK^{D(G_q)}$. So if such an isomorphism exists for a simply connected semi-simple compact Lie group $G$, the strong Baum-Connes conjecture holds for $\hat{G}_q$, the quantum dual of the Woronowicz deformation of $G$. To formulate the Baum-Connes property for $\hat{G}_q$ we must of course know that it is torsion-free, a statement proved in [17] for $G = SU(2)$ and the general case was proved in [6]. Another striking application of such an isomorphism is that the method above for calculating $K$-theory of homogeneous spaces can be generalized to classical quantum homogeneous spaces of the Woronowicz deformations.
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