Supplementary Material

Linking turbulent waves and bubble diffusion in self-aerated open-channel flows: Two-state air concentration

Matthias Kramer\(^1\)\(^*\), Daniel Valero\(^2,3\)

\(^1\)UNSW Canberra, School of Engineering and Information Technology (SEIT), Canberra, ACT 2610, Australia
\(^2\)Karlsruhe Institute of Technology (KIT), Institute for Water and River Basin Management (IWG), Karlsruhe, Germany
\(^3\)IHE Delft, Water Resources and Ecosystems dept., Delft, the Netherlands

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1. Presentation

This supplementary material presents the application of the two-state model to more than 500 concentration profiles from different data sets, comprising smooth chute data from Straub and Anderson (1958, 74 profiles), Killen (1968, 17 profiles), Bung (2009, 28 profiles), Severi (2018, 261 profiles), and stepped chute data from Bung (2009, 151 profiles), Zhang (2017, 6 profiles), and Kramer and Chanson (2018, 34 profiles). A summary of key parameters is given in Table 1.

In the following, the chute angle ($\theta$), specific discharge ($q$), and streamwise location ($x$) of the respective profile are indicated in the top left corner of each subfigure, while the background of the first profile of each series of measurements is shaded in gray. The parameter $x$ is presented as distance (in m) from the upstream crest for smooth chutes or as step edge (SE) number for stepped chutes.

Laboratory flows with depth-averaged air concentrations ($\bar{c}$) $\lesssim 0.25$ are dominated by free-surface instabilities and turbulent waves (air transported between wave crests and troughs), and an analytical solution for the air concentration ($\bar{c}$) involves the Gaussian error function [Eq. (2.2); blue lines]. In general, Eq. (2.2) is applicable to flows where the bubbly flow layer does not protrude to the channel bottom, which holds true for small chute angles ($\theta < 15^\circ$) or for locations upstream or immediately downstream of the inception point of air entrainment. For $\bar{c} \gtrsim 0.25$, air bubbles are diffused deeper into the water column, and the time-averaged air concentration is described by the two-state principle, involving a convolution of the Rouse profile and the Gaussian error function with the interface probability [Eq. (2.8); red lines].

\(^*\) Email address for correspondence: m.kramer@adfa.edu.au
Table 1: Key parameters of the re-analysed data sets - smooth chutes (upper part) and stepped chutes (lower part)

| Reference | Straub and Anderson (1958) | Killen (1968) | Bung (2009) | Severi (2018) |
|-----------|---------------------------|--------------|-------------|---------------|
| chute type| smooth                    | smooth       | smooth      | smooth        |
| $q$ (m$^2$/s) | 0.13 to 0.92 | 0.39 and 0.78 | 0.07 to 0.11 | 0.03 to 0.38 |
| $\theta$ (°) | 7.5 to 75 | 30 and 52.5 | 18.4 | 10.8 |
| $k_s$ (mm) | 0.71 | 0.71 | 8 | 1.6 to 9.5 |
| $L_{chute}$ (m) | 15.24 | 15.24 | 7.6 | 8 |
| $B_{chute}$ (m) | 0.46 | 0.46 | 0.3 | 0.8 |
| comment | granular roughness | granular roughness | micro-roughness | micro-roughness |

| Reference | Bung (2009) | Zhang (2017) | Kramer and Chanson (2018) |
|-----------|-------------|--------------|---------------------------|
| chute type| stepped | stepped | stepped |
| $q$ (m$^2$/s) | 0.07 to 0.11 | 0.14 | 0.032 to 0.11 |
| $\theta$ (°) | 18.4 and 26.6 | 45.0 | 45.0 |
| $h$ (m) | 0.03 to 0.06 | 0.1 | 0.1 |
| $k_s$ (mm) | 26.8 to 56.9 | 70.7 | 70.7 |
| $L_{chute}$ (m) | 5.4 to 7.6 | 1.7 | 1.7 |
| $B_{chute}$ (m) | 0.3 | 0.99 | 0.99 |
| comment | skimming flow | skimming flow | transition + skimming flow |

$q$ = specific discharge; $\theta$ = chute angle; $k_s$ = roughness height; $h$ = step height (stepped spillways); $L_{chute}$ = chute length; $B_{chute}$ = chute width
2. Smooth chute data sets

2.1. Straub and Anderson (1958, 74 profiles)
\( \theta = 75.0^\circ \)
\( q = 0.26 \text{ m}^2/\text{s} \)
\( x = 13.88 \text{ m} \)

\( \theta = 75.0^\circ \)
\( q = 0.32 \text{ m}^2/\text{s} \)
\( x = 13.88 \text{ m} \)

\( \theta = 75.0^\circ \)
\( q = 0.39 \text{ m}^2/\text{s} \)
\( x = 13.88 \text{ m} \)

\( \theta = 75.0^\circ \)
\( q = 0.44 \text{ m}^2/\text{s} \)
\( x = 13.88 \text{ m} \)

\( \theta = 75.0^\circ \)
\( q = 0.50 \text{ m}^2/\text{s} \)
\( x = 13.88 \text{ m} \)

\( \theta = 75.0^\circ \)
\( q = 0.59 \text{ m}^2/\text{s} \)
\( x = 13.88 \text{ m} \)
2.2. Killen (1968, 17 profiles)
2.3. Bung (2009, 28 profiles)
\begin{align*}
\theta &= 18.4^\circ \\
q &= 0.11 \text{ m}^2/\text{s} \\
x &= \text{SE12} \\
\theta &= 18.4^\circ \\
q &= 0.11 \text{ m}^2/\text{s} \\
x &= \text{SE14} \\
\theta &= 18.4^\circ \\
q &= 0.09 \text{ m}^2/\text{s} \\
x &= \text{SE16} \\
\theta &= 18.4^\circ \\
q &= 0.09 \text{ m}^2/\text{s} \\
x &= \text{SE18} \\
\theta &= 18.4^\circ \\
q &= 0.09 \text{ m}^2/\text{s} \\
x &= \text{SE20} \\
\theta &= 18.4^\circ \\
q &= 0.09 \text{ m}^2/\text{s} \\
x &= \text{SE22} \\
\theta &= 18.4^\circ \\
q &= 0.09 \text{ m}^2/\text{s} \\
x &= \text{SE24} \\
\theta &= 18.4^\circ \\
q &= 0.09 \text{ m}^2/\text{s} \\
x &= \text{SE26}
\end{align*}
2.4. Severi (2018, 261 profiles)
\( \theta = 10.8^\circ \)
\( q = 0.375 \text{ m}^3/\text{s} \)
\( x = 7.76 \text{ m} \)
3. Stepped chute data sets

3.1. **Bung (2009, 151 profiles)**

![Graphs showing stepped chute data sets with various angles and flow rates](image-url)
3.2. Zhang (2017, 6 profiles)

\[(1) \theta = 45.0^\circ, \quad q = 0.143 \text{ m}^3/\text{s}, \quad x = \text{SE6} \]

\[(2) \theta = 45.0^\circ, \quad q = 0.143 \text{ m}^3/\text{s}, \quad x = \text{SE7} \]

\[(3) \theta = 45.0^\circ, \quad q = 0.143 \text{ m}^3/\text{s}, \quad x = \text{SE8} \]

\[(4) \theta = 45.0^\circ, \quad q = 0.143 \text{ m}^3/\text{s}, \quad x = \text{SE9} \]

\[(5) \theta = 45.0^\circ, \quad q = 0.032 \text{ m}^3/\text{s}, \quad x = \text{SE10} \]

\[(6) \theta = 45.0^\circ, \quad q = 0.143 \text{ m}^3/\text{s}, \quad x = \text{SE11} \]
3.3. Kramer and Chanson (2018, 34 profiles)
\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE5} \)

\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE6} \)

\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE7} \)

\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE8} \)

\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE9} \)

\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE10} \)

\( \theta = 45.0^\circ \)  
\( q = 0.067 \text{ m}^2/\text{s} \)  
\( x = \text{SE11} \)
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