ON VARIATIONAL INEQUALITIES WITH MULTIVALED OPERATORS WITH SEMI-BOUNDED VARIATION

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In this paper we explore some problems for the steady-state variational inequalities with multivalued operators (VIMO). As far as we know, in this variant the term VIMO had been first introduced in [1]. The results of this paper with respect to VIMO extend and/or improve analogous ones from [1-7]. We refused the regularity conditions of the monotonic disturbance of the multivalued mapping ([1]) and some other properties of the objects ([2]). Besides we considered a wider class of operators with respect to [3,4]. We are studying the connections between the class of radially semi-continuous operators with semi-bounded variation, the class of pseudo-monotone mappings, which is used earlier (for example, in [2]) on selector's language, and the class of monotone mappings. Moreover, for new class of operators the property of local boundedness is substituted for a weaker one with respect to [1,2,5,7,8]. Also we refuse the condition that \( A(y) \) is a convex closet set owing to the forms of support functions.

Let \( X \) be a reflexive Banach space, \( X^* \) be its topological dual space, by \( \langle \cdot, \cdot \rangle \) we denote the dual pairing on \( X \times X^* \), \( 2^{X^*} \) be the totality of all nonempty subsets of the space \( X^* \), \( A : X \to 2^{X^*} \) be a multivalued mapping with \( \text{Dom} A = \{ y \in X : A(y) \neq \emptyset \} \). \( A : X \to 2^{X^*} \) is called strong iff \( \text{Dom} A = X \). Further for simplicity we will consider only strong mappings \( A \). Let us consider the upper and lower support functions which are associated to \( A \):

\[
[A(y), \xi]_+ = \sup_{d \in A(y)} \langle d, \xi \rangle, \quad [A(y), \xi]_- = \inf_{d \in A(y)} \langle d, \xi \rangle,
\]

and norms:

\[
[Ay]_+ = \sup_{d \in A(y)} \| d \|_{X^*}, \quad [Ay]_- = \inf_{d \in A(y)} \| d \|_{X^*}.
\]

We will consider the following VIMO

\[
[A(y), \xi - y]_+ + \varphi(\xi) - \varphi(y) \geq \langle f, \xi - y \rangle \quad \forall \xi \in \text{dom } \varphi \cap K, \quad (1)
\]

where \( f \) is a fixed element from \( X^* \), \( \varphi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \) is a proper convex lower semi-continuous function, \( \text{dom } \varphi = \{ y \in X : \varphi(y) < \infty \} \), \( K \) is a convex weakly closed set.

**Definition.** \( L : X \to \overline{\mathbb{R}} \) is called lower semi-continuous, if the following is satisfied: if \( X \ni y_n \to y \) in \( X \) then \( \lim_{n \to \infty} L(y_n) \geq L(y) \).

**Definition[1].** Operator \( A : X \to 2^{X^*} \) is called
a) radially semi-continuous, if for each \( y, \xi, h \in X \) the following inequality holds:

\[
\lim_{t \to +0} [A(y + t\xi), h]_+ \geq [A(y), h]_-
\]

b) operator with semi-bounded variation, if for each \( R > 0 \) and arbitrary \( y_1, y_2 \in X \) such that \( \|y_i\|_X \leq R \) \((i = 1, 2)\) the following inequality holds:

\[
[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|_X^r),
\]

where \( C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) is continuous, \( \tau^{-1}C(r_1, \tau r_2) \to 0 \) as \( \tau \downarrow 0 \) for each \( r_1, r_2 > 0, \| \cdot \|_X^r \) is a compact norm with respect to the initial norm \( \| \cdot \|_X \);

c) coercive operator, if \( \exists y_0 \in K \) such that

\[
\|y\|^{-1}_X [A(y), y - y_0]_- \to \infty \text{ as } \|y\|_X \to \infty;
\]

d) locally bounded on \( X \), if for each \( y \in X \) there exist \( \varepsilon > 0 \) and \( M > 0 \) such that \( [A(\xi)]_+ \leq M \) for each \( \xi \) such that \( \|\xi - y\|_X \leq \varepsilon \);

e) monotone, if for each \( y_1, y_2 \in X \) the following inequality holds:

\[
[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+.
\]

**Definition [2].** Operator \( A : X \to 2^{X^*} \) is called pseudo-monotone, if

i) the set \( A(y) \) is nonempty, bounded, closed and convex at each \( y \in X \);

ii) \( A : F \to 2^{X^*} \) is locally bounded on each finite-dimensional subspace \( F \subset X \);

iii) if \( y_j \to y \) weakly in \( X \), \( w_j \in A(y_j) \) and \( \lim_{j \to \infty} \langle w_j, y_j - y \rangle \leq 0 \), then for each element \( v \in X \) there exists \( w(v) \in A(y) \) with the property

\[
\lim_{j \to \infty} \langle w_j, y_j - v \rangle \geq \langle w(v), y - v \rangle.
\]

**Definition [2].** Operator \( A : X \to 2^{X^*} \) is called generalized pseudo-monotone, if from \( y_j \to y \) weakly in \( X \), \( A(y_j) \ni w_j \to w \) \( \ast \)-weakly in \( X^* \) and \( \lim_{j \to \infty} \langle w_j, y_j - y \rangle \leq 0 \) it follows that \( w \in A(y) \) and \( \langle w_j, y_j \rangle \to \langle w, y \rangle \).

Each pseudo-monotone operator is generalized pseudo-monotone one ([2]).

It is easy to see that each monotone operator is an operator with semi-bounded variation, the next result is connecting the classes of operators with semi-bounded variation and of pseudo-monotone operators. Simultaneously, we showed the interconnection between monotone and pseudo-monotone operators.

**Definition.** Operator \( A : X \to 2^{X^*} \) is called sequentially weakly locally bounded, if for each \( y \in X \) if \( y_n \to y \) weakly in \( X \) then there exist a finite number \( N \) and a constant \( M > 0 \) such that \( [A(y_n)]_+ \leq M \) for each \( n \geq N \).

**Theorem 1.** Let \( A \) be a radially semi-continuous operator with semi-bounded variation. Then \( \overline{\text{co}} A \) is pseudo-monotone, locally bounded and sequentially weakly locally bounded.

**Remark.** It is enough to consider the weaker condition of the radially semi-continuity:

\[
\lim_{t \to +0} [A(y + t\xi), -\xi]_+ \geq [A(y), -\xi]_-.
\]

Let us consider solvability theorems.

**Theorem 2.** Let \( K \) be a bounded convex weakly closed set and \( A : X \to 2^{X^*} \) be a radially semi-continuous operator with semi-bounded variation. Then for each \( f \in X^* \) the solution set of the inequality

\[
[A(y), \xi - y]_+ \geq \langle f, \xi - y \rangle \quad \forall \xi \in K
\]

is nonempty and weakly compact in \( X \). Moreover, there exists the element \( w \in \overline{\text{co}} A(y) \) such that

\[
\langle w, \xi - y \rangle \geq \langle f, \xi - y \rangle \quad \forall \xi \in K.
\]
Proof. Let us consider the filter $\mathcal{F}$ of the finite-dimensional subspaces $F$ of $X$. We construct the auxiliary operator $L_c(\lambda, y) = \{ (1-\lambda)P_c(y) + \lambda (I_{F_c} - I_{F_c}^*)A(I_{F_c}y) \}$, where $I_{F_c} : X \to F$ is the inclusion map, $K_{F} = K \cap F$, $P_c(y) = [K_{F} \cap (\lambda N_{K_{F}}(y))] \setminus B_{\varepsilon}(y) - y$, $N_{K_{F}}(y)$ is the normal cone, $B_{\varepsilon}(y) = \{ \xi \in K_{F} : ||\xi - y||_F < \varepsilon \}$ and $\varepsilon > 0$ such that $K_{F} \setminus B_{\varepsilon}(y) \neq \emptyset$ for each $y$. We can show that $L_c(\lambda, y)$ is upper semi-continuous on $F$. By construction for each $y \in \partial K_{F}$ we have that $L_c(0, y) \setminus T_{K_{F}}(y) \neq \emptyset$, where $T_{K_{F}}$ is the tangential cone. If $\exists y \in \partial K_{F}$ and $\lambda \in [0, 1]$ such that $0 \in L_c(\lambda, y)$ then this $y \in \partial K_{F}$ is a solution of VIMO on $F$. Else by Leray–Schauder theorem the inclusion $0 \in L_c(0, y)$ has a solution on $int K_{F}$. Thus, we have the bounded sequence $\{ y_F \} \subset K$. Using the generalized pseudo-monotonicity and the sequentially weakly locally boundedness of the operator $\overline{\partial}_\mathcal{F}A$ we can find some limit element $y$ which is a solution of (2). □

**Theorem 3.** Let the conditions of Theorem 2 be satisfied without $K$ be a bounded set. If in this case operator $A$ is coercive, then the statement of Theorem 2 holds.

Proof. On each bounded set $K_R = K \cap B_R$ the solution $y_R$ exists, by the coercivity of the operator $\overline{\partial}_\mathcal{F}A$ under some $R$ $y_R$ is a solution of (2). □

From Theorem 3 we can obtain following statement:

**Theorem 4.** Let $\Phi : X \to 2^{X^*}$ be a radially semi-continuous coercive operator with semi-bounded variation. Then $\forall f \in X^*$ the solution set of the inclusion

$$\overline{\partial}_\mathcal{F}A(y) \ni f$$

is nonempty and weakly compact in $X$.

Now we consider the based inequality (1) and the corresponding inclusion

$$\overline{\partial}_\mathcal{F}A(y) + \partial \varphi(y) \ni f, \quad (3)$$

where $\partial \varphi(y)$ is the subdifferential of the function $\varphi : X \to \mathbb{R}$ at the point $y \in X$.

**Proposition.** Each solution of (3) satisfies VIMO (1). If $y$ is a solution of (1) and belong to $int K \cap \text{dom} \partial \varphi$, then $y$ is a solution of (3).

This simple statement allows to study VIMO (1) using the inclusion (3).

**Theorem 5.** Let $\Phi : X \to 2^{X^*}$ be a radially semi-continuous operator with semi-bounded variation, $\varphi : X \to \mathbb{R}$ be a proper convex lower semi-continuous function and the following coercivity condition satisfies:

$$\exists y_0 \in \text{dom} \varphi \cap K \text{ such that } \|y\|_X^{-1}(\|A(u, y) - y_0\|_X - \varphi(y_0)) \to +\infty \text{ as } \|y\|_X \to \infty.$$  

Then $\forall f \in X^*$ the solution set of (1) is nonempty and weakly compact in $X$.

Proof. Let us construct the auxiliary objects:

$$\mathcal{X} = X \times \mathbb{R}, \quad \tilde{y} = (y, \mu) \in \mathcal{X}, \quad \tilde{A}(\tilde{y}) = (A(y), 0) \quad \forall \tilde{y} \in \mathcal{X},$$

$$\mathcal{K} = \{ (y, \mu) \in (K \cap \text{dom} \varphi) \times \mathbb{R} \mid \mu \geq \varphi(y) \}, \quad f = (f, -1).$$

We can prove that these objects satisfy all conditions of Theorem 2. Thus, the solution $\tilde{y}$ exists and its first coordinate is a solution of (1). □

**Example.** Let us consider the free boundary problem on Sobolev space $W^2_p(\Omega)$, $p \geq 2$:

$$- \sum_{i,j=1}^n a_{ij}(x, y, Dy) \frac{\partial^2 y}{\partial x_i \partial x_j} = f \text{ on } \Omega,$$

$$y \geq 0, \quad \frac{\partial y}{\partial n} \geq 0, \quad y \frac{\partial y}{\partial n} = 0 \quad \text{ on } \Gamma,$$
where $\Omega$ is a sufficiently smooth simple connected domain of $\mathbb{R}^n$, $\Gamma$ is the bound of $\Omega$, the normal vector $\nu$ is defined at each $x \in \Gamma$, $\frac{\partial y}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x, y, D_y) \frac{\partial y}{\partial x_j} \cos(x, \nu_i)$, $Dy = (\frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n})$. This problem can have not a classical solution, but we can find a weak solution on $W^{2}_{p}(\Omega)$. Let us assume that $a_{ij}(x, y, \xi)$ satisfy the following conditions:

1. for each $y, \xi$ the functions $a_{ij}$ are continuous with respect to $x$,
2. $\forall x \in \overline{\Omega}$ the functions $a_{ij}$ are bounded with respect to $\xi$ and $y$, and the following estimation holds: $|a_{ij}(x, y, \xi_1, \ldots, \xi_n)| \leq g(x) + k_0 |y|^{p-2} + \sum_{i=1}^{n} k_i |\xi_i|^{p-2}$, where $k_i > 0$ ($i = \overline{1,n}$), if $p = 2$ then $g \in C(\Omega)$, and if $p > 2$ then $g \in L^{q'}(\Omega)$, $q' = \frac{p}{p-2}$,
3. $\sum_{i,j=1}^{n} a_{ij}(x, y, \xi) \xi_i \xi_j \geq \gamma(R) R$, where $R = |y| + \sum_{i=1}^{n} |\xi_i|$ and $\gamma(R) \to +\infty$ as $R \to +\infty$.

Then the free boundary problem conforms to the following inequality:

$$ [A(y, \xi - y)]_+ = a_1(y, \xi - y) + [a_2(y), \xi - y]_+ \geq \langle f, \xi - y \rangle \quad \forall \xi \in K. \quad (4) $$

where $a_1(y, \xi) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x, y, D_y) \frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_i} \, dx$, $A_2(y) = \left( \frac{\partial}{\partial x_i} a_{ij}(x, y, D_y) \right) \frac{\partial y}{\partial x_j}$, is the subdifferential of $a_{ij}$, $K = \{y \in W^{2}_{p}(\Omega) : \frac{\partial y}{\partial \nu} \geq 0 \}$ is a convex weakly closed set. We can prove that $A$ is a radially semi-continuous coercive operator with semi-bounded variation. Thus, the inequality (4) has a solution.

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