Asymptotic iteration method for analytical solution of Klein-Gordon equation for trigonometric Pöschl-Teller potential in D-dimensions

Dewanta Arya Nugraha, A Suparmi, C Cari and Beta Nur Pratiwi
Physics Department, Graduate Program, Sebelas Maret University
E-mail: dewanta.an@gmail.com

Abstract. Klein-Gordon equation for Trigonometric Pöschl-Teller Potential in D-dimensions was obtained within framework of a centrifugal term approximation. Asymptotic iteration method was used to obtain the relativistic energy spectrum and wave functions. The value of relativistic energy was calculated numerically and the results have shown that in higher dimension the energy level is increased with positive energy states. The wave functions were expressed in hypergeometric term.

1. Introduction
Relativistic wave equation solution is very important in many aspects of modern physics. Wave equation of Klein-Gordon and Dirac are frequently used for describing the particle dynamics in relativistic quantum mechanics [1]. Klein-Gordon equation allows us to study spin-zero particles [2]. Klein-Gordon Solutions have been done by various techniques such as Standard Method [3], Supersymmetric Quantum Mechanics (SUSY)[4], Nikiforov Uvarov (NU) Method [5], Factorization Method [6], Asymptotic Iteration Method (AIM) [7][8] and others. In higher dimensional, Klein-Gordon solution have done for various potential, they are Poschl-Teller potential including centrifugal term [3], Hylleraas Potential[9], Hulthen potential [10], Manning- Rosen potential [8], Morse potential [11] and other.

In this paper, the radial part solution of D dimension Klein-Gordon equation is studied for Trigonometric Pöschl-Teller Potential in D-dimensional. In some physics area, the higher dimensional spaces extension for some physical problems is very important. The D-dimensional system has been constructed to explain the unification of gravitation and electromagnetic fields [12]. It is suspected that the dimensional system is applicable for gravitation field since it is involved in such huge universe. The relativistic and non-relativistic physical systems in D-dimensional have been investigated by many authors for various methods and potentials [13]–[16]. The Poschl-Teller potential can be applied to study the effect of the complex vibration-rotation energy structure of multi-electron atoms in the relativistic system,[17], [18]. We used the AIM to obtain the energy eigenvalues and radial wave functions for Klein-Gordon equation in D-dimensions.

This paper is organized as follows. In section 2 presented a brief review of D-dimensional Kein-Gordon equation. The methods we used are presented in Section 3 and the results and discussion are presented in Section 4. The conclusion is presented in Section 5.
2. Klein-Gordon Equation in D Dimensions

The D-Dimensional time independent KG equation with potentials of vector and scalar $V(r)$ and $S(r)$, respectively, with $r = |\mathbf{r}|$ describing a spinless particle which in the general write as [19], [20]

$$
\left\{ \nabla_D^2 + \frac{1}{\hbar^2 c^2} \left[ (E - V(r))^2 - \left[ Mc^2 + S(r) \right]^2 \right] \right\} \psi_{l_{1-\ell-2}}^{(\ell)}(x) = 0,
$$

(1)

With

$$
\nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \sum_{j=1}^{D-2} \frac{1}{\sin^2 \theta_{j+1} \sin^2 \theta_{j+2} \cdots \sin^2 \theta_{D-1}} \times \left\{ \frac{1}{\sin^{D-1} \theta_j} \left( \frac{\partial}{\partial \theta_j} \sin^{D-1} \theta_j \frac{\partial}{\partial \theta_j} \right) \right\}
$$

(2)

and

$$
\psi_{l_{1-\ell-2}}^{(\ell)}(x) = R_{\ell}(r) \mathbf{Y}_{l_{1-\ell-2}}^{(\ell)}(\theta_1, \theta_2, \ldots, \theta_{D-1})
$$

(3)

Where $E$, $M$, and $\nabla_D^2$ denoted KG energy, the mass and the Laplacian for D-dimension, respectively. And $x$ in equation (1) is a position vector for D-dimension. The wave function for radial part $R_{\ell}(r)$ is decomposed as

$$
R_{\ell}(r) = r^{-(D-1)/2} F(r),
$$

(4)

According to Alhaidari et al [21], only the choice $S = V$ can produce a non-trivial nonrelativistic limit with a potential function $2V$ and not $V$. Accordingly, it would be natural to scale the potential terms in Eq. (1) then the potential becomes $V$ in nonrelativistic limit the interaction and not $2V$. Thus equation (1) is reduced to D-Dimensional radial KG equation. (in the relativistic atomic units $\hbar = c = 1$)

$$
\nabla_D^2 \psi_{l_{1-\ell-2}}^{(\ell)}(x) + \left[ E^2 - M^2 - (E + M) V(r) \right] \psi_{l_{1-\ell-2}}^{(\ell)}(x) = 0,
$$

(5)

3. Asymptotic Iteration Method (AIM)

Asymptotic Iteration Methods (AIM) is an alternative method which has accuracy and high efficiency to determine eigen energies and eigen functions for analytically solvable hyperbolic like potential. Asymptotic Iteration Methods is also giving solution for exactly solvable problem. [22].

AIM is used to solve the second order homogeneous equation of form [7], [8], [23], [24].

$$
y_n''(x) = \dot{\lambda}_0(x)y_n'(x) + s_0(x)y_n(x)
$$

(6)

with $\dot{\lambda}_0(x) \neq 0$ and prime symbol denotes the derivative with respect to $x$. The others parameter $n$ is interpreted as the radial quantum number. The other variables, $s_0(x)$ and $\dot{\lambda}_0(x)$ are differentiable.

To get the solution, we have to differentiate Eq. (6) along $x$, and find

$$
y_n'''(x) = \ddot{\lambda}_0(x)y_n''(x) + s_0(x)y_n'(x)
$$

(7)

Where

$$
\ddot{\lambda}_0(x) = \dot{\lambda}_0'(x) + s_0(x) + \dot{\lambda}_0^2(x)
$$

(8)

$$
s_0(x) = s_0'(x) + s_0(x)\dot{\lambda}_0(x)
$$

(9)
where $\lambda_0(x) \neq 0$ and $s_0(x)$ is a function of $C_\infty$ (coefficient of the differential equation) Asymptotic Iteration Method and can be applied exactly in the different problem if the wave function has been known and fulfill boundary condition zero (0) and infinity ($\infty$).

Equation (6) can be simple iterated until $(k+1)$ and $(k+2), k = 1, 2, 3, ...$ and then we get

$$y^{k+1}_n(x) = \lambda_{k+1}(x)y'_n(x) + s_{k+1}(x)y_n(x)$$ (10)

$$y^{k+2}_n(x) = \lambda_k(x)y'_n(x) + s_k(x)y_n(x)$$ (11)

where

$$\lambda_k(x) = \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x)$$ (12)

$$s_k(x) = s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x)$$ (13)

Which is called recurrence relation. Eigen value can be found using this equation

$$\Delta_n(z) = \lambda_k(z)s_{k-1}(z) - \lambda_{k-1}(z)s_k(z) = 0$$ (14)

where $k = 1, 2, 3, ...$ is the iteration number and $n$ is the representation of radial quantum number.

Equation (6) is the second order of homogenous linear equation which is can be solved with comparison it with the second order linear equation as follow [8]:

$$y''(z) = 2\left(\frac{a^2}{1-bz^{N+2}} - \frac{t+1}{z}\right)y'(z) - \frac{wz^N}{1-bz^{N+2}}y(z)$$ (15)

where

$$\sigma = 2t + N + 3 \quad \rho = \frac{(2t+1)b + 2a}{(N+2)b}$$ (16)

The wave function of equation (15) can be determined by using the equation below:

$$y_n(z) = (-1)^nC_2(N+2)^n(\sigma; \rho; \sigma; b^2)F_1\left(-n, \rho+n; \sigma; b^2\right)$$ (17)

4. Results and Discussion

Pöschl Teller Potential is defined as[18]

$$V(r) = \frac{\kappa(k-1)}{\sin^2 \alpha r} + \frac{\eta(\eta-1)}{\cos^2 \alpha r}$$ (18)

Substituting equations (2-4, 18) into equation (5) we obtain

$$\frac{1}{r^{\alpha-1}} \frac{\partial}{\partial r} \left( r^{\alpha-1} \frac{\partial}{\partial r} \frac{F(r)}{r^{\alpha-1/2}} \right) - \left( E + M \right) \frac{F(r)}{r^{\alpha-1/2}} \left( \frac{k(k-1)}{\sin^2 \alpha r} + \frac{\eta(\eta-1)}{\cos^2 \alpha r} \right) = \lambda_{\alpha-1} \frac{F(r)}{r^{\alpha-1/2}}$$ (19)

Where $\lambda_{\alpha-1}$ separation constant. By using centrifugal approximations for $\alpha r \ll 1$, then

$$\frac{1}{r^2} \cong \frac{\alpha^2}{\sin^2 \alpha r}$$

for equation (19) become

$$\left[ \frac{d^2}{dr^2} - \frac{\alpha^2}{\sin^2 \alpha r} \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) \right] - \lambda_{\alpha-1} \left( \frac{k(k-1)(E+M)}{\alpha^2} \right) = \eta(\eta-1)(E+M) \left( E+M^2 \right) F(r) = 0$$ (20)

By setting,

$$A = \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) - \lambda_{\alpha-1} \frac{k(k-1)(E+M)}{\alpha^2}$$ (21)

$$B = \eta(\eta-1)(E+M)$$ (22)
$$E_s = \frac{(E^2 - M^2)}{\alpha^2}$$  \hspace{1cm} (23)

Equation (20) become

$$\frac{d^2 F(r)}{dr^2} - \frac{A\alpha^2}{\sin^2 \alpha r} F(r) - \frac{B\alpha^2}{\cos^2 \alpha r} F(r) + E_s \alpha^2 F(r) = 0$$  \hspace{1cm} (24)

We can solve equation (24) by letting \( \cos^2 \alpha r = z \), we get

$$z(1-z) \frac{\partial^2 F(r)}{\partial z^2} + \left( \frac{1}{2} - z \right) \frac{\partial F(r)}{\partial z} + A \frac{F(r)}{4(1-z)} + B \frac{F(r)}{4z} - E_s \frac{F(r)}{4} = 0$$  \hspace{1cm} (25)

Equation (25) is the hypergeometric differential equation which must be transformed to AIM type second differential equation. Let,

$$F(z) = z^\delta (1-z)^\gamma f_{n_z}(z)$$  \hspace{1cm} (26)

By substituting (26) to (25), we get

$$z(1-z) f''_{n_z}(z) + \left\{ 2\delta - \left( 2\delta + 2\gamma + 1 \right) z + \left( \frac{1}{2} \right) \right\} f'_{n_z}(z) + \left\{ \frac{4\delta(\delta - 1) + 2\delta + B}{4z} + \frac{4\gamma(\gamma - 1) + 2\gamma + A}{4(1-z)} - \delta^2 - 2\delta\gamma - \gamma^2 - \frac{E_s}{4} \right\} f_{n_z}(z) = 0$$  \hspace{1cm} (27)

By setting,

$$B = -4\delta^2 + 2\delta$$  \hspace{1cm} (28)

$$A = -4\gamma^2 + 2\gamma$$

Equation (28) are inserted into equation (27), we obtain

$$f''_{n_z}(z) = \frac{(2\delta + 2\gamma + 1)z - \left( 2\delta + \frac{1}{2} \right)}{z(1-z)} f'_{n_z}(z) + \frac{(\delta + \gamma)^2 + \frac{E_s}{4}}{z(1-z)} f_{n_z}(z) = 0$$  \hspace{1cm} (29)

Equation (29) is AIM type differential equation. Equation (29) compared with equation (6), we get

$$\lambda_0 = \frac{(2\delta + 2\gamma + 1)z - \left( 2\delta + \frac{1}{2} \right)}{z(1-z)}$$  \hspace{1cm} (30)

$$s_0(z) = \frac{(\delta + \gamma)^2 + \frac{E_s}{4}}{z(1-z)}$$  \hspace{1cm} (31)

From equations (12 – 14), we found the eigenvalue of equation (29) in general term as follows

$$E_{sn} = -4(\delta + \gamma + n)^2$$  \hspace{1cm} (32)

With

$$\delta = \frac{1}{4} + \frac{1}{2} \sqrt{\frac{1}{4} - B}$$  \hspace{1cm} (33)

$$\gamma = \frac{1}{4} + \frac{1}{2} \sqrt{\frac{1}{4} - A}$$  \hspace{1cm} (34)
Where $E_n$ is eigenvalue which $n$ is interpreted radial quantum number, with $n = 0, 1, 2, \ldots$. By inserting equation (21–23, 33–34) to equation (32) we get relativistic energy equation as follows

$$(M^2 - E_n^2) = \alpha^2 \left[ 2n + 1 - \frac{1}{4} \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) - \frac{\kappa (\kappa - 1)(E_n + M)}{\alpha^2} \right]$$

(35)

From equation (35) we can calculate the relativistic energy value numerically by using Matlab. The result is shown in Table 1.

| $E_n$ (eV) |
|-----------------|
| $M = \alpha = \lambda = 1$ |

| $D$ | $n$ | $\kappa = \eta = 1.5$ | $\kappa = \eta = 2$ | $\kappa = \eta = 2.5$ | $\kappa = \eta = 3$ |
|-----|-----|---------------------|---------------------|---------------------|---------------------|
| 0   | 0   | 4.006385            | 9.003597            | 16.00213            | 25.001479           |
| 1   | 1   | 4.009589            | 9.008375            | 16.005805           | 25.004080           |
| 2   | 2   | 4.016115            | 9.017870            | 16.012966           | 25.009275           |
| 3   | 3   | 4.026149            | 9.031961            | 16.023653           | 25.017047           |
| 4   | 4   | 4.039869            | 9.050475            | 16.037801           | 25.027376           |
| 5   | 5   | 4.057367            | 9.073195            | 16.055327           | 25.040229           |

From Table 1., it shows that the potential constant of trigonometric Pöschl-Teller potential gives the influent to the energy value which is increasing due to the larger potential constant. The value of relativistic energy in higher dimension is increase with positive energy states.

By comparing equation (15) and (29) and using equation (16) and insert it to (17)

$$F_n(z) = (-1)^n C_2 (1) \left( 2 \delta + \frac{1}{2} \right) \_2 F_1 \left( -n, 2 \delta + 2 \gamma + 1 + n, 2 \delta + \frac{1}{2}, z \right)$$

(36)

By inserting equation (36) to $F_n(z) = (z)^\gamma (1-z)^\delta f_n(z)$, we get

$$F_n(z) = (z)^\gamma (1-z)^\delta f_n(z)(-1)^n C' \left( 2 \delta + \frac{1}{2} \right) \_2 F_1 \left( -n, 2 \delta + 2 \gamma + 1 + n, 2 \delta + \frac{1}{2}, z \right)$$

(37)

where $C'$ is normalization constant and $\_2 F_1(a,b,c,z)$ is hypergeometry function. In Table 2., are expressed unnormalized radial wave functions in any states solved from equation (37) and in Figure 1 are the graphs of radial wave functions in a various value of quantum number $n$ following the Table 2.
Table 2. The wave functions \((n = 0, 1, 2)\)

| \(n\) | \(F_n(r)\) |
|-------|-------------|
| 0     | \(F_0(r) = (\cos^2 \alpha r)^\delta (\sin^2 \alpha r)^\gamma C^\nu\) |
| 1     | \(F_1(r) = -(\cos^2 \alpha r)^\delta (\sin^2 \alpha r)^\gamma C^\nu \left(2\delta + 1/2\right) \left(2\delta + 1/2\right) \left(2\delta + 3/2\right)\times\) |
|       | \(1 - \frac{2\left(2\delta + 2\gamma + 3\right)}{\left(2\delta + 1/2\right)} (\cos^2 \alpha r) + \frac{(2\delta + 2\gamma + 3)(2\delta + 2\gamma + 4)}{\left(2\delta + 1/2\right)\left(2\delta + 3/2\right)} (\cos^2 \alpha r)^2\) |
| 2     | \(F_2(r) = (\cos^2 \alpha r)^\delta (\sin^2 \alpha r)^\gamma C^\nu \left(2\delta + 1/2\right) \left(2\delta + 3/2\right)\times\) |

Figure 1. Four Dimensional Radial wave functions for \(M = 1; \lambda = 1; \alpha = 0.1; \kappa = 1.5; \eta = 1.5\)

(a) \(n=0\), (b) \(n=1\) and (c) \(n=2\)
Figure 1 are the graphs for various value of quantum number $n$. It is shown that for the higher value of $n$ give the increasing value for amplitude wave functions. The amplitude explains about the deviation of radial wave functions depend on radial dimension.

5. Conclusion
The radial relativistic energy of Klein-Gordon equation for Trigonometric Pöschl-Teller Potential in D-dimensional is obtained by using asymptotic iteration method. The results show that the relativistic energy is increasing with the increase of potential constant value. In higher dimension, the energy is increasing with positive energy.

Acknowledgement
This research was partly supported by SebelasMaret University Higher Education Project Grant with contract No. 632/UN27.21/LT/2016

References
[1] Ikhdair S M and Sever R, “Solution of the D-dimensional Klein-Gordon equation with equal scalar and vector ring-shaped pseudoharmonic potential,” arXiv:0808.1002v1 [quant-ph], pp. 1–14, 2008.
[2] Momtazi E, Rajabi A A, and Yazarloo B H, “Analytical solution of the Klein – Gordon equation under the Coulomb-like scalar,” Turkish J. Phys., vol. 38, no. 1, pp. 81–85, 2014.
[3] Xu Y, He S, and Jia C-S, “Approximate analytical solutions of the Klein–Gordon equation with the Pöschl–Teller potential including the centrifugal term,” Phys. Scr., vol. 81, no. 4, p. 45001, 2010.
[4] Setare M R and Nazari Z, “FIVE-PARAMETER EXPONENT-TYPE POTENTIAL,” Acta Phys. Pol. B, vol. 40, no. 10, pp. 2809–2824, 2009.
[5] Ikhdair S M, “Bound states of the Klein-Gordon equation in D-dimensions with some physical scalar and vector exponential-type potentials including orbital centrifugal term,” arXiv:1110.0943v1 [quant-ph], pp. 1–25, 2011.
[6] Wei G F, Liu X Y, and Chen W L, “The relativistic scattering states of the hulthén potential with an improved new approximate scheme to the centrifugal term,” Int. J. Theor. Phys., vol. 48, no. 6, pp. 1649–1658, 2009.
[7] Barakat T, “The asymptotic iteration method for dirac and klein gordon equations with a linear scalar potential,” Int. J. Mod. Phys. A, vol. 21, no. 19, pp. 4127–4135, 2006.
[8] Das T, “Exact Solutions of the Klein-Gordon Equation for q-Deformed Manning-Rosen Potential via Asymptotic Iteration Method,” arXiv:1409.1457v1 [quant-ph], pp. 1–11, 2014.
[9] Ikot A N, Awoga O A, and Ita B I, “Exact Solutions of the Klein-Gordon Equation with Hylleraas Potential,” Few-Body Syst., vol. 53, no. 3–4, pp. 539–548, 2012.
[10] Ikot A N, Akpablo L E, and Uwah E J, “Bound State Solutions of the Klein Gordon Equation with the Hulth ěn Potential,” Electron. J. Theor. Phys., vol. 25, no. 25, pp. 225–232, 2011.
[11] Soylu A, Bayrak O, and Boztosun I, “Exact Solutions of Klein – Gordon Equation with Scalar and Vector,” Chinese Phys. Lett., vol. 25, no. 8, pp. 2754–2757, 2008.
[12] Dong S H, “The Ansatz Method for Analyzing Schrödinger’s Equation with Three Anharmonic Potentials in D Dimensions,” Found. Phys. Lett., vol. 15, pp. 385–395, 2002.
[13] Ikot A N, Antia A D, Akpan I O, Awoga O A, and Group T P, “Bound state solutions of schrodinger equation with modified hylleraas plus exponential rosen morse potential,” Rev. Mex. Fis., vol. 59, pp. 46–53, 2013.
[14] Panahi H, Zarrinkamar S, and Baradaran M, “Solutions of the $D$ -dimensional Schrödinger equation with Killingbeck potential: Lie algebraic approach,” Chinese Phys. B, vol. 24, no. 6, p. 60301, 2015.
[15] Suparmi A, Cari C, and Pratiwi B N, “Thermodynamics properties study of diatomic molecules with q-deformed modified Poschl-Teller plus Manning Rosen non-central potential in D
dimensions using SUSYQM approach,” *J. Phys. Conf. Ser.*, vol. 710, no. 1, 2016.

[16] Suparmi A, Cari C, Pratiwi B N, and Deta U A, “Solution of D dimensional Dirac equation for hyperbolic tangent potential using NU method and its application in material properties,” *AIP Conf. Proc.*, vol. 1710, p. 30010, 2016.

[17] Suparmi S and Cari C, “Bound State Solution of Dirac Equation for Generalized Pöschl-Teller plus Trigonometric Pöschl-Teller Non- Central Potential Using SUSY Quantum Mechanics,” *J. Math. Fundam. Sci.*, vol. 46, no. 3, pp. 205–223, 2014.

[18] Liu X Y, Wei G F, Cao X W, and Bai H G, “Spin Symmetry for Dirac Equation with the Trigonometric Pöschl-Teller Potential,” *Int. J. Theor. Phys.*, vol. 49, no. 2, pp. 343–348, 2010.

[19] Bayrak O and Boztosun I, “Arbitrary l-state solutions of the rotating Morse potential by the asymptotic iteration method,” *J. Phys. A. Math. Gen.*, vol. 39, no. 22, p. 6955, 2006.

[20] Qiang W C and Dong S H, “Arbitrary l-state solutions of the rotating Morse potential through the exact quantization rule method,” *Phys. Lett. Sect. A Gen. At. Solid State Phys.*, vol. 363, no. 3, pp. 169–176, 2007.

[21] Alhaidari A D, Bahlouli H, and Al-Hasan A, “Dirac and Klein-Gordon equations with equal scalar and vector potentials,” *Phys. Lett. Sect. A Gen. At. Solid State Phys.*, vol. 349, no. 1–4, pp. 87–97, 2006.

[22] Suparmi A, Cari C, Deta U A, Husein A S, and Yuliani H, “Exact Solution of Dirac Equation for q-Deformed Trigonometric Scarf potential with q-Deformed Trigonometric Tensor Coupling Potential for Spin and Pseudospin Symmetries Using Romanovski Polynomial,” *J. Phys. Conf. Ser.*, vol. 539, p. 12004, 2014.

[23] Sari R A, Suparmi A, and Cari C, “Solution of Dirac equation for Eckart potential and trigonometric Manning Rosen potential using asymptotic iteration method,” *Chinese Phys. B*, vol. 25, no. 1, 2015.

[24] Pramono S, Suparmi A, and Cari C, “Relativistic Energy Analysis of Five-Dimensional q -Deformed Radial Rosen-Morse Potential Combined with q -Deformed Trigonometric Scarf Noncentral Potential Using Asymptotic Iteration Method,” *Adv. High Energy Phys.*, vol. 2016, 2016.