Topological transitions in electronic spectra: Crossover between Abrikosov and Josephson vortices

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The electronic structure of a vortex line trapped by a planar defect in a type-II superconductor is analyzed within the Bogoliubov-de Gennes theory. The normal reflection of electrons and holes at the defect plane results in the topological transition in the spectrum and formation of a new type of quasiparticle states skipping or gliding along the defect. This topological transition appears to be a hallmark of the initial stage of the crossover from the Abrikosov to the Josephson vortex type revealing in the specific behavior of the quantized quasiparticle levels and density of states. The increase in the resulting hard and soft gaps affects the vortex mobility along the defect plane and splitting of the zero bias anomaly in the tunneling spectral characteristics.

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I. INTRODUCTION

The most general definition of different vortex type solutions for the order parameter in superconducting and superfluid systems is based on the calculation of the so-called circulation of the gradient of the order parameter phase around the line of singularity. Provided this circulation equals to $2\pi$ we get a singly quantized vortex. The particular structure of the order parameter and magnetic field distributions strongly depends then on the specific system. In a homogeneous isotropic superconductor the vortex solution possessing a cylindrical symmetry is well known as an Abrikosov vortex while the presence of any anisotropy or inhomogeneity can strongly deform this vortex line in the plane perpendicular to its axis (see Fig.1). An extreme example of such anisotropic solution which does not even possess the normal core can be realized for a vortex pinned at the Josephson junction. Such quasi-one dimensional vortices are also called Josephson vortices (see Fig. 1a) and are known to play an important role in magnetic transport properties of layered and nano-structured systems. Provided the junction critical current density $j_c$ is much smaller than depairing current density

$$j_d = c\Phi_0/12\sqrt{3}\pi^2\lambda^2\xi,$$

the Josephson penetration depth

$$\lambda_J = \sqrt{c\Phi_0/16\pi^2j_c\lambda},$$

appears to be much larger than the London penetration depth $\lambda$. Here $\Phi_0 = \pi\hbar c/e$ is the magnetic flux quantum, and $\xi$ is the superconducting coherence length. Clearly, changing the electron transparency of the junction one can get a variety of intermediate vortex states corresponding to a crossover from the Josephson to the Abrikosov vortex. This situation with the intermediate transparencies naturally appears in many superconducting systems studied in experiments, e.g. in superconductors with twinning planes, low-angle grain boundaries or other types of defects. An appropriate theoretical treatment needed, for instance, for the interpretation of the experimental data on the magnetic field distribution can be well developed on the basis of the Ginzburg–Landau theory. Indeed, using a general expression for the critical current $I_c$ across the junction with a cross-section area $S$

$$I_c = j_cS = \pi\Delta_0/2cR_N,$$

and relation between the contact resistance and the angle-averaged transmission probability of the barrier $D$

$$R_N^{-1} = k_F^2S(2c^2/h)D,$$

we derive the following simple relation

$$\lambda_J^2 = \lambda\xi/12\pi^2D.$$

It is natural that the Josephson length $\lambda_J$ grows if the transmission probability of the barrier $D$ decreases. To satisfy the relation $\lambda_J \gg \lambda$, the barrier transparency should be small enough: $D \ll D_\lambda = 1/12\pi^2\kappa \ll 1$, where $\kappa = \lambda/\xi$ is the Ginzburg–Landau parameter. As the probability of electron transmission through the barrier grows above $D_\lambda$ the changes in the structure of the order parameter are controlled by the relation between the Josephson length $\lambda_J$, the London penetration depth $\lambda$ and the coherence length $\xi$. Keeping in mind type-II superconductors we should take $\xi \lesssim \lambda$. When the current density $j(r)$ in the vortex core ($r \lesssim \xi$) becomes of order of the depairing one $j_d$, the length $l$ of the core along the defect can be estimated from the continuity of currents flowing parallel and perpendicular to the defect within the core: $j_c \sim j_d\xi/j_c \sim \lambda_J^2/\lambda$. The case $D \gtrsim D_\lambda (\xi < l \lesssim \lambda \sim \lambda_J)$ corresponds to the limit of strong Josephson coupling with $j_c \gtrsim j_d/\kappa$, and we can no more consider the solution in the form of a
core free Josephson vortex having the size of the order $\lambda_J$. Instead, we get the crossover to the Abrikosov-like vortex having strongly deformed anisotropic core ($l \times \xi$), where the superconducting order parameter is suppressed (see Fig. 1b). The distributions of the magnetic field and circular screening currents outside the core ($r \gg l, \xi$) approach now with the ones for the Abrikosov vortex in a uniform superconductor. In the case of the extremely strong Josephson coupling $D \gtrsim D_\xi = 1/12\pi^2 (l \lesssim \xi)$ the anisotropy of the vortex core becomes negligible, and at this initial stage of the crossover (see Fig. 1c) the order parameter profile in the Abrikosov vortex core is almost insensitive to the defect.

Despite general correctness of the above qualitative picture there exist several important physical issues which definitely can not be described within the phenomenological model and demand a more careful microscopic consideration. This statement surely relates to the scanning tunneling microscopy (STM) and spectroscopy (STS) data which provide detailed spatially resolved excitation spectra$^{13-17}$ and also to the problem of the vortex dynamics and dissipation$^{18-23}$. In the latter case the crossover from the Abrikosov to the Josephson vortex is particularly important since it is accompanied by the disappearance of the normal vortex core which provides the dominating contribution to the dissipation and resulting vortex viscosity$^7$.

Considering the microscopic theory one should take into account the behavior of the subgap fermionic states bound to the Abrikosov vortex core which are known to determine both the structure and dynamics of vortex lines in the low temperature limit$^{18}$. These subgap states are known to form the so-called anomalous spectral branch crossing the Fermi level. For well separated vortices the behavior of the anomalous branches can be described by the Caroli-de Gennes–Matricon (CdGM) theory$^{24}$: for each individual vortex the energy $\varepsilon_{CdGM}(\mu)$ of subgap states varies from $-\Delta_0$ to $+\Delta_0$ as one changes the angular momentum $\mu$ defined with respect to the vortex axis. Here $\Delta_0$ is the superconducting gap value far from the vortex axis. At small energies $|\varepsilon| \ll \Delta_0$ the spectrum is a linear function of $\mu$: $\varepsilon_{CdGM}(\mu) \approx -\mu \omega_0$, where $\omega_0 \approx \Delta_0/(k_F \xi) = \Delta_0^2/2E_F \ll \Delta_0$ is the interlevel spacing, $\xi = hV_F/\Delta_0$, $k_F$, $V_F$ and $E_F$ are the Fermi momentum, velocity and energy, respectively. Neglecting the quantization of the angular momentum $\mu$ one can get the anomalous spectral branch crossing the Fermi level at $\mu = 0$ for all orientations of the momentum $k_F = k_F (\cos \theta_p, \sin \theta_p)$. Thus, in the space $(\mu - k_F)$ we obtain a Fermi surface (FS) for excitations localized within the vortex core (see Ref. 25 for review). For a fixed energy $\varepsilon$ we can define a quasiclassical orbit in the plane $(\mu - \theta_p)$: $\mu(\theta_p) = -\varepsilon/\hbar \omega_0$. Each point at this orbit corresponds to a straight trajectory passing through the vortex core (Fig. 2). The precession of quasiparticle trajectory along the orbit is described by the Hamilton equation: $\hbar \partial \theta_p/\partial t = \partial \varepsilon/\partial \theta_p$.

The wave functions of the subgap states are localized inside the vortex core because of the Andreev reflection of quasiparticles at the core boundary. Any additional normal scattering process should modify the behavior of the anomalous spectral branch. Such modification can be noticeable even for impurity atoms introduced in a vortex core$^{26}$ and becomes much more pronounced provided we consider a vortex pinned by a normal-metal$^{27,28}$ or an insulating$^{29-32}$ columnar defect of the size $R \ll \xi$ well exceeding the Fermi wavelength. In the last case the scattering at the defect is responsible for the opening of the minigap $\varepsilon_0 \sim \Delta_0 R/\xi$ in the spectrum of localized states and resulting suppression of the dissipation at low temperatures $T \ll \varepsilon_0^{18,33}$. For a vortex approaching a flat or curved sample boundary an appropriate spectrum transformation was studied in Refs. 34–37. Change in the anomalous spectral branch is accompanied by the changes in the topology of quasiclassical orbits in the $(\mu - \theta_p)$ plane. Such topological transitions in quasiparticle spectra of vortex systems are similar to the well-
known Lifshits transitions which occur in the band spectra of metals. The generic examples of such transitions in vortex matter including the opening of the closed segments of the orbits in the $(\mu - \theta_p)$ plane or merging and reconnection of the different segments via the Landau-Zener tunneling have been previously studied in Refs.\textsuperscript{36,40,41}. The basic properties of vortex matter such as pinning and transport characteristics, heat transport in the vortex state and peculiarities of the local density of states should be strongly affected by these changes in the topology of the subgap spectral branches.

It is the goal of the present work to develop a theoretical description of the changes in the electronic structure of the pinned vortex core which occur during the crossover between the Abrikosov and Josephson vortices and unveil a nontrivial topological nature of this vortex core transformation. We restrict ourselves to situations when the barrier is rather weak assuming $D \gtrsim D_\xi$, and focus on the modification of the anomalous energy branches which occurs in a vortex pinned by a planar defect due to the quasiparticle normal reflection at the defect boundary.

To elucidate our main findings we start from the simplified qualitative picture illustrating the effect of the barrier on the quasiparticle subgap states. First, considering the specular reflection of the quasiclassical trajectories at the plane defect in Fig. (2) one can clearly see that the scattering couples the wavefunctions with the opposite angular momenta $\pm \mu$. Phenomenologically one can describe this coupling by a standard two-level problem:

$$(\varepsilon - \varepsilon_{-\mu})(\varepsilon - \varepsilon_{+\mu}) \approx (V_{\text{gap}}(\theta_p))^2,$$

where $\varepsilon_{+\mu}$ denotes the anomalous spectral branch for a linear trajectory passing through the core of a free vortex. The scattering obviously can not couple the trajectories with $\theta_p = 0, \pm \pi$, which are parallel to the defect plane. Considering now the limit of small angles $\theta_p$ one can expect that even for the barriers with rather good transparency the tunneling probability should vanish in this angular interval. The splitting of the energy levels around $\varepsilon = 0$ should originate from the superconducting state energy in a one-dimensional Josephson junction we find: $\varepsilon = \pm \Delta_0 \cos(\pi/2 - \theta_p) \approx \pm \Delta_0 \theta_p$. This energy splitting gives us the estimate for the coupling coefficient in the above two-level problem: $V_{\text{gap}} \sim \Delta_0 \theta_p$.

As a result, one obtains a set of quasiclassical orbits in $(\mu - \theta_p)$ space

$$\mu(\theta_p) = \pm \frac{1}{\hbar \omega_0} \sqrt{\varepsilon^2 - \Delta_0^2 \theta_p^2}. \quad (6)$$

These orbits (6) corresponding to the precession of the quasiparticle trajectory are schematically shown in Fig. 3. For low energy levels one can clearly observe the formation of closed orbits near the points $\theta_p = 0, \pm \pi$, which are separated by the prohibited angular domains centered at $\theta_p = \pm \pi/2$. The closed orbits are nothing more but skipping (or gliding) quasiparticle states formed due to the scattering at the defect plane. The discrete subgap energy levels of quasiparticles can be obtained from the semiclassical Bohr-Sommerfeld quantization rule for canonically conjugate variables $\mu$ and $\theta_p$:\textsuperscript{43,44}

$$\Sigma(\varepsilon) = \int_{0}^{2\pi} \mu(\varepsilon, \theta_p) d\theta_p = 2\pi(n + \beta), \quad (7)$$

where $n$ is integer, $2\pi$ is the period of $\mu(\theta_p)$, and $\beta$ is of the order unity. Applying the Bohr-Sommerfeld rule (7) to the closed paths in $(\mu - \theta_p)$ space, we obtain the
spectrum in the form
\[ \varepsilon_n^2 = \frac{\Delta_0^4}{E_F} (n + \beta), \]
which is dramatically different from the CdGM spectrum \( \varepsilon_n = \hbar \omega_0 (n + 1/2) \) and reminds the square-root quantization of the quasiparticle spectra in different types of nodal problems \( \text{like graphene}^{45,46} \) or d-wave superconductors in magnetic fields \( \text{in}^{47} \). The novel minigap \( \varepsilon_0 \approx \Delta_0/\sqrt{E_F} \) determined by Eq. 8 well exceeds the CdGM interlevel spacing \( \hbar \omega_0 \). This minigap obviously manifests the partial suppression of the spectral flow which should give the origin to all the dissipation phenomena inside the vortex core during its motion. In this sense this spectrum change can be viewed as a precursor to the crossover to the Josephson vortex where all the subgap quasiparticle levels are repelled from the Fermi energy to the gap value \( \Delta_0 \). On the other hand, the limit of the moderate barrier strength studied here provides a possibility to observe a novel type of the vortex core with the peculiar quantization rule arising from the splitting of the orbit segments in the \( \mu - \theta_p \) plane. This splitting destroys the trajectory precession in the whole angular interval \( 0 < \theta_p < 2\pi \) changing, thus, the topology of the quasiclassical orbits. The precession region \( |\theta_p| < \delta \theta_p \) expands with an increase of the energy level \( n \). As a result, for rather high levels the prohibited angular domains shrink, the precession over the full region \( 0 < \theta_p < 2\pi \) restores, and we get the crossover to a CdGM type of spectrum \( \varepsilon_n \sim n \).

The paper is organized as follows. In Sec. II we introduce the basic equations used for the spectrum calculation. In Sec. III we study the quasiparticle spectrum transformation for a singly quantized vortex pinned at the planar defect and discuss the consequences for the vortex dynamics. The Sec. IV is devoted to the analysis of the peculiarities of the local density of states for a vortex pinned at the defect. We summarize our results in Sec. V.

II. BASIC EQUATIONS

Hereafter we consider a planar defect in the plane \( y = 0 \) as a \( \delta \)-function repulsive potential for quasiparticles, i.e. \( V(y) = \hbar \delta(y) \). The magnetic field \( B = B_0 \) is assumed to create a single quantum vortex line parallel to the \( z \)-axis trapped inside the attractive potential well within the defect\(^{48} \). The vortex center defined as a point of the order parameter phase singularity is positioned at the point \( x = y = 0 \).

We assume the system to be homogeneous along the \( z \)-axis, thus, the \( k_z \)-projection of the momentum is conserved. The quantum mechanics of quasiparticle excitations in a superconductor is governed by the two dimensional BdG equations for particlelike \( (u) \) and holelike \( (v) \) parts of the two-component quasiparticle wave functions

\[ \hat{\Psi}(r, z) = (u, v) \exp(i k_z z), \]

\[ \frac{\hbar^2}{2m} \left( \nabla^2 + k_z^2 \right) u + \Delta(r) v = \epsilon u \]

\[ \frac{\hbar^2}{2m} \left( \nabla^2 + k_z^2 \right) v + \Delta^*(r) u = \epsilon v. \]

Here \( \nabla = \partial_x x_0 + \partial_y y_0 \). \( r = (x, y) \) is a radius vector in the plane perpendicular to the cylinder axis, \( \Delta(r) \) is the gap function, \( k_z^2 = k^2 - k_x^2 \), \( k_z \) is the momentum projection on the vortex axis.

Following the procedure described in \( 35,36,41 \) we introduce the momentum representation:

\[ \hat{\psi}(r) = \left( \begin{array}{c} u \\ v \end{array} \right) = \frac{1}{(2\pi \hbar)^2} \int d^2p \, e^{i\mathbf{p}/\hbar} \hat{\psi}(\mathbf{p}) \]

where \( \mathbf{p} = p \mathbf{r} \). The unit vector \( \mathbf{p}_0 \) parametrized by the angle \( \theta_p \) defines the trajectory direction in the \( (x, y) \) plane. We assume that our solutions correspond to the momentum absolute values \( p \) close to the value \( \hbar k_z \): \( p = \hbar k_z + q \) \( \left( q \ll \hbar k_z \right) \). Within the quasiclassical approach the wave function in the momentum representation assumes the form

\[ \hat{\psi}(\mathbf{p}) = \frac{1}{\hbar k_z} \int_{-\infty}^{+\infty} ds \, e^{i (k_z - |\mathbf{p}| / \hbar) s} \hat{\psi}(s, \theta_p). \]

Finally, the slowly varying part of the wave function \( \hat{\psi} \) in the real space \( r = r(\cos \theta, \sin \theta) \) is expressed from Eqs. (10, 11) in the following way (see Ref. 35):

\[ \hat{\psi}(r, \theta) = \int_{0}^{2\pi} e^{i k_z r \cos(\theta_p - \theta)} \hat{\psi}(r \cos(\theta_p - \theta), \theta_p) \frac{d\theta_p}{2\pi}, \]

where \( r, \theta, z \) is a cylindrical coordinate system. The appropriate boundary conditions for wave function \( \hat{\psi} \) at \( y = 0 \) are follows\(^{49} \):

\[ \hat{\psi}(x, 0+) = \hat{\psi}(x, 0-) = \hat{\psi}_0, \]

\[ \partial_y \hat{\psi}(x, 0+) - \partial_y \hat{\psi}(x, 0-) = 2k_z \hat{\psi}_0, \]

where the dimensionless barrier strength \( Z = H/hV_z \) \( (mV_z = \hbar k_z) \) defines the transmission \( D = 1/(1 + Z^2) \) and reflection \( Z^2/(1 + Z^2) \) coefficients in the normal state.

For extremely weak barrier \( (D \gtrsim D_v) \) we can neglect the anisotropy of the order parameter \( \Delta(r) \) around the vortex and assume that

\[ \Delta(r) = \Delta_0 \delta_y(r) e^{i\theta}, \quad r = \sqrt{x^2 + y^2}. \]

Here \( \delta_y(r) \) is a normalized order parameter for a vortex centered at \( r = 0 \), such that \( \delta_y(r) = 1 \) for \( r \to \infty \). Nevertheless the solution (12) can not be characterized by a definite angular momentum \( \mu \) because of the normal reflection of quasiparticles at the defect results in interaction of angular harmonics with opposite momentum \( (\mu \text{ and } -\mu) \) (see Fig. 2). Thus, following Ref.\(^{50} \)
we introduce the angular momentum expansion for the solution (12):

\[ \hat{\psi}(s, \theta_p) = \sum_{\mu} e^{i\mu \theta_p + i \sigma_z \theta_p} / 2 \hat{f}_\mu(s), \]

where \( \mu = n + 1/2 \), and \( n \) is an integer. The function \( \hat{f}_\mu(s) \) satisfies the Andreev equation along the quasiclassical trajectory with the impact parameter \( b = -\mu/k_{\perp} \)

\[- i\hbar V_\perp \hat{\sigma}_z \hat{\sigma}_x \hat{f}_\mu + \hat{\Delta}_b(s) \hat{f}_\mu = \varepsilon \hat{f}_\mu, \]

where

\[ \hat{\Delta}_b(s) = \hat{\sigma}_x \Re D_b(s) - \hat{\sigma}_y \Im D_b(s) \]

is the gap operator, and \( \hat{\sigma}_i \) are the Pauli matrices. Taking into account the evident relations

\[ x = s \cos \theta_p - b \sin \theta_p, \quad y = s \sin \theta_p + b \cos \theta_p, \]

\[ x \pm iy = (s \pm ib) e^{\pm i \theta_p}. \]

one obtains from (14) the following expression for the order parameter \( \Delta(r) \) around the vortex in \( s, \theta_p \) variables:

\[ \Delta = D_b(s) e^{i \theta_p}, \quad D_b(s) = \Delta_0 \frac{\delta_b(\sqrt{s^2 + b^2})}{\sqrt{s^2 + b^2}} (s + ib). \]

Changing the sign of the coordinate \( s \) one can observe a useful symmetry property of the solution of Eq.(16):

\[ \hat{f}_\mu(-s) = \pm \hat{\sigma}_y \hat{f}_\mu(s). \]

A. General solution

To find the solution of Eqs. (16,17) we can use the results of Ref. 35. For low energies \( \mu \ll k_{\perp} \) we take the function \( \hat{f}_\mu \) as a sum

\[ \hat{f}_\mu = c_{\mu 1} \hat{G}_{\mu 1} + c_{\mu 2} \hat{G}_{\mu 2} \]

of the two linearly independent solutions

\[ \hat{G}_{\mu 1} = e^{i \sigma_z \pi / 4} \left( e^{-|D(s)|} - i \text{sgn}(s) \frac{\gamma_\mu}{2} \sigma_z e^{i|D(s)|} \right) \hat{\lambda}, \]

\[ \hat{G}_{\mu 2} = e^{i \sigma_z \pi / 4} e^{-|D(s)|} \sigma_z \hat{\lambda}, \]

where \( \hat{\lambda} = (1, 1)^T, \)

\[ D(s) = \frac{k_F}{k_{\perp} \xi} \int_0^s dt \delta \nu (\sqrt{t^2 + b^2}) \]

\[ \Lambda_\mu = \frac{2 k_F}{k_{\perp} \xi} \int_0^\infty ds e^{-2D(s)}, \]

\[ \gamma_\mu = \frac{\Lambda_\mu}{\Delta_0} (\varepsilon_\mu - \varepsilon) \]

and

\[ \varepsilon_\mu = -\frac{2 \Delta_0 k_F \mu}{k_{\perp}^2 \xi} \int_0^\infty ds \frac{\delta \nu (\sqrt{s^2 + b^2})}{\sqrt{s^2 + b^2}} e^{-2D(s)} \]

is the CdGM excitation spectrum. Here \( \xi = hV_F / \Delta_0 \) is the coherence length \( (V_F \) is the Fermi velocity).

B. Boundary condition.

As a next step we rewrite the boundary condition (13) for wave functions \( \hat{f}_{\pm \mu}(s) \) defined at the trajectories \( s_1 \) and \( s_2 \) (see Fig. 1). Due to normal reflection of quasiparticles at the defect the trajectories \( s_1 \) and \( s_2 \) with opposite momentum \( (\mu \) and \( -\mu) \) directions are coupled. Substituting the expressions (12,15) into the boundary condition (13), we obtain the following relation between the amplitudes of incident \( \hat{f}_{\pm \mu}(s) \) and transmitted \( \hat{f}_{\pm \mu}(s) \) two-component quasiparticle wave functions at the point \( s_0 = -b / \tan \theta_p \) where the trajectories cross the barrier:

\[ (\eta + i) \hat{f}_{\pm \mu} = \eta \hat{f}_{\pm \mu} - i e^{\mp i \sigma_z \theta_p} \hat{f}_{\mp \mu}, \]

where \( \eta = \sin \theta_p / Z \). Our further analysis of quasiparticle excitations is based on the solutions (20,21) which must be supplemented by the boundary conditions (26).

III. SPECTRUM OF THE VORTEX PINNED BY PLANAR DEFECT

We now proceed with the analysis of the subgap spectrum for a singly quantized vortex trapped by the planar defect. Hereafter in this section we assume the angular momentum to be positive, i.e. \( \mu > 0 \). The form of the two-component quasiparticle wave functions \( \hat{f}_{\pm \mu}(s) \) depends on a position of the point \( s_0 \) at the trajectory. If the coordinate \( s_0 \geq 0 \) than the general solution (20,21) takes the following form

\[ \hat{f}_{\pm \mu}(s) = \begin{cases} 
 c_{\pm \mu} e^{i(\sigma_z \pi / 4)} e^{-|D(s)|} \hat{\lambda}, & s \leq 0, \\
 c_{\pm \mu} e^{i(\sigma_z \pi / 4)} (e^{-|D(s)|} - i \gamma_\pm \sigma_z e^{i|D(s)|}) \hat{\lambda}, & 0 \leq s \leq s_0, \\
 c_{\pm \mu} e^{i(\sigma_z \pi / 4)} e^{-|D(s)|} \hat{\lambda}, & s \geq s_0, 
 \end{cases} \]

where

\[ \gamma_+ = -\frac{\Lambda_\mu}{\Delta_0} (|\varepsilon_\mu| + \varepsilon), \quad \gamma_- = \frac{\Lambda_\mu}{\Delta_0} (|\varepsilon_\mu| - \varepsilon). \]
Otherwise, if \( s_0 \leq 0 \)

\[
\hat{f}_{\pm \mu}(s) = \begin{cases} 
  c_{\pm \mu} \, e^{i (\sigma_\pi \pi / 4) e^{-|D(s)| \sigma_\mu}} 
  & \text{if } s \leq s_0, \\
  c_{\pm \mu} \, e^{i (\sigma_\pi \pi / 4) (e^{-|D(s)|} + i \gamma_{\pm \mu}^2 e^{i D(s)})} \, e^{-|D(s)| \sigma_\mu} 
  & \text{if } s_0 \leq s \leq 0, \\
  c_{\pm \mu} \, e^{i (\sigma_\pi \pi / 4) e^{-|D(s)| \sigma_\mu}} 
  & \text{if } s \geq 0,
\end{cases}
\]

The eigenfunction \( \hat{f}_{\pm \mu}(s) \) has to be normalized

\[
\int_{-\infty}^{\infty} ds \left( |\hat{f}_{+ \mu}(s)|^2 + |\hat{f}_{- \mu}(s)|^2 \right) = k_\perp.
\]  

Substituting the above expressions (27) or (28) into the boundary conditions (26), we obtain the following system of algebraic equations with respect to the amplitude \( c_{\pm \mu} \) of the incident waves

\[
\eta \gamma_{\pm \mu} \, c_{\pm \mu} + \left( \gamma_{\pm \mu} \cos \theta_p + e^{-2D_0} \sin \theta_p \right) \, c_{\pm \mu} = 0, \quad (29a)
\]

\[
\eta \gamma_{- \mu} \, c_{- \mu} - \left( \gamma_{\pm \mu} \cos \theta_p - e^{-2D_0} \sin \theta_p \right) \, c_{+ \mu} = 0. \quad (29b)
\]

The case \( s_0 > 0 \) (\( s_0 < 0 \)) corresponds to the choice of upper (lower) sign in Eqs. (29), \( D_0 = D(s_0) \) and the angle \( \theta_p \) defines the direction of the ray with the angular momentum \( + \mu \). To find the subgap quasiparticle excitation spectrum we should find the determinant of the algebraic system, and its zero give us the equation for the energy spectrum \( \varepsilon \):

\[
\varepsilon^2(\mu, \theta_p) = \varepsilon^2_{\mu} + \left( \frac{\Delta_0}{\Lambda_\mu} \right)^2 \frac{e^{-2D_0}}{\eta^2 + \cos^2 \theta_p} \times \left[ \frac{\Delta_0}{\Delta_\mu} \frac{\varepsilon_{\mu}}{\Lambda_\mu} \sin(2\theta_p) + e^{-2D_0} \sin^2 \theta_p \right].
\]  

Figure 4 shows the anomalous spectral branches as functions of the impact parameter \( b = -\mu/k_F \) for different values of the dimensionless barrier strength \( Z \) and the trajectory directions in the \((x, y)\) plane determined by the angle \( \theta_p \). The qualitative behavior of the spectrum is weakly sensitive to the concrete profile of the gap amplitude inside the core and we choose a simple model dependence

\[
\delta_{\nu}(r) = r/\sqrt{r^2 + \xi^2}
\]  

neglecting, thus, the influence of the defect on the behavior of the gap profile. Contrary to the CdGM case the spectrum branch (30) does not cross the Fermi level in the presence of the defect. For rather small \( Z \) the minigap in the quasiparticle spectrum

\[
\Delta_m(\theta_p) = \varepsilon(0, \theta_p) = \frac{\Delta_0}{\Lambda_\mu} \frac{Z}{\sqrt{1 + Z^2/\tan^2 \theta_p}}
\]

appears to be almost independent of \( \theta_p \) in a wide range of angles except the small angular intervals close to \( \theta_p = 0 \) and \( \theta_p = \pi \). It is natural to expect that in the patterns of the local density of states (LDOS) this angular independent quantity should reveal itself as a soft gap \( \Delta_{\text{soft}} \sim Z \Delta_0 \) growing with the increasing barrier strength \( Z \) (see the Section IV). We emphasize here the fact that this gap is soft since the spectrum (30) for small \( |\tan \theta_p| \ll Z \) is gapless and, thus, these angular intervals can contribute to the LDOS at the Fermi level. This nonzero contribution exists, of course, only in the quasiclassical limit when we completely neglect the quantum mechanical nature of the trajectory precession which

FIG. 4: (Color online) Quasiparticle spectra \( \varepsilon(b, \theta_p) \) calculated using Eq. (30) for different values of the dimensionless barrier strength \( Z \) and the trajectory direction \( \theta_p \) in the \((x, y)\) plane \((k_\pi = 0)\): (a) \( Z = 0.1 \); (b) \( Z = 0.2 \); (c) \( Z = 0.3 \). Dotted lines for \( \theta_p = 0 \) correspond to the CdGM branch of the spectrum. The dash blue lines show the dependence for \( \theta_p = \pi/4 \); solid red lines show the dependence for \( \theta_p = \pi/2 \).
should be responsible for the opening of the hard minigap for the energies below $\Delta_{soft}$. 

To derive the corresponding quantization rules in the limit $Z \ll 1$ we consider isoenergetic lines $\mu(\theta_p) = -k_z b(\theta_p)$ in $(\mu - \theta_b)$ plane. The resulting classical orbits are shown in Fig. 5. Generally, one can distinguish two types of the isoenergetic lines behavior. If the quasiparticle energy is of the order of the minigap ($\varepsilon \lesssim \Delta_{soft}$) there appear prohibited angular domains centered at the points $\theta_p = \pm \pi/2$ due to the normal reflection of quasiparticles at the defect. In this case classical orbits form close paths in $(\mu - \theta_b)$ space corresponding to the precession of the trajectory in the region with the width $2\delta\theta_p^{(1)}(\varepsilon)$ near the points $\theta_p = 0, \pm \pi$. The width $2\delta\theta_p$ of the precession region grows with an increase in energy level. For small $|\mu| \ll k_z \xi$ the value $\delta\theta_p$ can be estimated as follows:

$$\delta\theta_p \simeq \frac{\varepsilon \Lambda_0 / \Delta_0}{\sqrt{1 - (\varepsilon \Lambda_0 / Z \Delta_0)^2}}. \quad (32)$$

Shrinking of the prohibited angular domains and the crossover from the closed orbits to the open ones occur at the energy $\varepsilon^{*}$ satisfying the condition $\delta\theta_p^{(2)}(\varepsilon^{*}) = \pi/2$.

The low lying energy levels of quasiparticles can be obtained by applying the Bohr-Sommerfeld quantization rule (7) for closed paths in the plane of canonically conjugate variables $\mu$ and $\theta_p$. Figure 6 shows the typical dependence $\Sigma(\varepsilon)$ calculated using the spectrum (30). Taking $\varepsilon_p \simeq -\hbar \omega_0 \mu$ for small $\mu$ values and replacing the real classical orbits in $(\mu - \theta_b)$ plane by the model one (see the insert Fig. 6), one can obtain a reasonable fit (dashed curve) to the numerical results (solid curve):

$$\Sigma(\varepsilon) \approx 2 \frac{\varepsilon}{\hbar \omega_0} \delta\theta_p = \frac{2\varepsilon^2 \Lambda_0 / \Delta_0}{\hbar \omega_0 \sqrt{1 - (\varepsilon \Lambda_0 / Z \Delta_0)^2}}. \quad (33)$$

The above relation together with the Bohr-Sommerfeld rule (7) results in the following explicit expression for discrete subgap energy levels

$$\varepsilon_n \simeq \frac{\Delta_0 Z}{\Lambda_0} \left[ p_n \sqrt{1 + p_n^2} / 4 - p_n^2 / 2 \right]^{1/2} \quad (34)$$

$$p_n = \frac{\pi \Lambda_0 \Delta_0}{2 \hbar \omega_0 Z^2} (n + \beta),$$

which appears to be justified for $\varepsilon_n / \Delta_0 \lesssim Z^2 \ll 1$. The expression (34) can be strongly simplified provided $p_n \ll 1$ for low lying energy levels:

$$\varepsilon_n^2 \simeq \frac{\pi}{2 \Lambda_0 \hbar \omega_0 \omega_0} \frac{\Delta_0^3}{\Delta_{soft}} (n + \beta) \left[ 1 - \frac{\pi \Lambda_0 \Delta_0}{4 \hbar \omega_0 \omega_0^2} (n + \beta) \right].$$

The main term of the last relation appears to be in good agreement with the estimate (8) and describes qualitatively the new behavior of spectrum of subgap quasiparticle states ($\varepsilon_n \sim n^{1/2}$) due to the normal scattering at the planar defect. Both the hard minigap $\varepsilon_0 \lesssim \Delta_0 \sqrt{\Delta_0 / \hbar \omega_0} \ll \Delta_{soft}$ in the discrete spectrum (34) and the interlevel spacing $\hbar \omega = \varepsilon_n - \varepsilon_{n-1}$ grow with the increase in the barrier strength $Z$.

Besides its fundamental interest, the problem of pinned vortex spectrum important for understanding the nature of dissipation in the presence of planar defects. In particular, according to the spectral flow theory, it is the behavior of the anomalous branch which determines the high-frequency conductivity and Kerr effect. One can expect that the opening of the hard minigap $\varepsilon_0$ in discrete quasiparticle spectrum (34) and change in the slope $\varepsilon(\mu)$ dependence (30) can cause the suppression of the dissipation accompanying the vortex motion and the resulting changes in the relation between the Ohmic and...
Hall conductivities. As a result, the quasiparticle subgap spectrum can be tested by the measurements of the conductivity tensor at finite frequencies.

IV. LOCAL DENSITY OF STATES FOR A PINNED VORTEX

We now proceed with the calculations of the local density of states for a singly quantized vortex pinned at the planar defect. This quantity is known to be directly probed in the scanning tunneling microscopy/spectroscopy experiments\(^\text{17}\). For the sake of simplicity we assume here the Fermi surface to be a cylinder and neglect the dependence of the quasiparticle energy on the momentum component \(k_z\) along the cylinder axis \(z\) considering a motion of quasiparticles only in \((x, y)\) plane. The peculiarities of the LDOS are usually determined from the analysis of the local differential conductance (LDC):

\[
\frac{dI}{dV} \bigg|_N = \frac{1}{N_0} \int_{-\infty}^{\infty} d\varepsilon \frac{N(r, \varepsilon) \partial f(\varepsilon - eV)}{\partial V},
\]

where \(V\) is the applied voltage, \(\langle dI/dV \rangle_N\) is a conductance of the normal metal junction, and \(f(\varepsilon) = 1/(1 + \exp(\varepsilon/T))\) is a Fermi function. Within the quasiclassical approach the LDOS

\[
N(r, \varepsilon) = k_F \int db |u_b(r)|^2 \delta(\varepsilon - \varepsilon(b))
\]

can be expressed through the electron component \(u(r, \theta)\) of quasiparticle eigenfunctions (10) corresponding to the energy \(\varepsilon(b, \theta_p)\) determined by Eqs. (30),(22),(23),(24),(25). The wave function \(\hat{\psi}(r, \theta)\) parametrized by the impact parameter \(b = -\mu/k_F\)

\[
\hat{\psi}(r, \theta) = \frac{u(r, \theta)}{v(r, \theta)} = e^{i(2\mu + \pi)/2} \int_0^{2\pi} \frac{dr}{2\pi} e^{i\kappa r \cos \alpha + i(2\mu + \pi)/2} f_\mu(r) \cos \alpha
\]
in the limit \(k_F r \gg 1\) can be evaluated using the stationary phase method. For an impact parameter \(|b| \leq r\) the stationary phase points are given by the condition: \(\sin \alpha_{1,2} = -b/r\). Summing over two contributions in the vicinity of the stationary angles \(\alpha_1 = \theta_{p1} - \theta = \alpha_r\) and \(\alpha_2 = \theta_{p2} - \theta = \pi - \alpha_r\), we can write the electron component \(u(r, \theta)\) of quasiparticle eigenfunctions as follows:

\[
u(r, \theta) = \left(\frac{1}{2\pi\kappa_F s_r}\right)^{1/2} e^{i(2\mu + \pi)/2} f_\mu(s_r) e^{i\varphi_r} + f_\mu(-s_r) e^{-i\varphi_r + i(2\mu + \pi)/2},
\]

where \(s_r = r \cos \alpha_r = \sqrt{r^2 - b^2}\). The phase

\[
\varphi_r = k_F r \cos \alpha_r + |\mu| \alpha_r + \text{sgn}(\mu) \alpha_r = -\pi/4
\]
is determined by the trajectory orientation \(\alpha_r = -\arcsin(b/r)\). Neglecting the oscillations at the atomic length scale we obtain the following slowly varying enve-
lope function:
\[ |\mu(r, \theta)|^2 \approx \frac{1}{2\pi k_F s_p} \left[ |f^u(\mu r)|^2 + |f^u(-\mu r)|^2 \right], \tag{39} \]
where the function \( f^u(\pm \mu r) \) is determined by the relations (27) or (28).

We have calculated the differential conductance using Eqs. (35),(36),(39) for low temperature \( T/\Delta_0 = 0.02 \) for different values of the dimensionless barrier strength \( Z \).

The typical examples of dependence of the local differential conductance \( dI/dV \) vs the bias voltage \( eV \) at various distances \( r \) from the vortex axis are shown in Fig. 7. In order to compare our results with the standard CdGM ones, we present the dependence of the local \( dI/dV \) vs voltage at different distances \( r \) from the Abrikosov vortex axis in the absence of the barrier \( (Z = 0) \). One can clearly observe the disappearance of the zero bias peak in the core \( (r = 0) \) and opening of the soft spectral minigap \( \Delta_{\text{soft}} \) caused by the normal scattering at the defect (Fig. 7(a)). The barrier results in the anisotropy of the LDC structure in the plane \( (x, y) \) (Fig. 7(b,c)). The anisotropy of the LDC grows when barrier strength \( Z \) increases. Figure 8 illustrates the evolution of the local differential conductance \( dI/dV(eV, x, y) \) distribution in the plane \( (x, y) \) for several values of the bias voltage \( V \) and dimensionless barrier strength \( Z \). In Fig. 8(a,b) we can see the spread of the zero bias peak along the defect which appears to be another hallmark of the crossover from the Abrikosov to the Josephson vortex type. Due to the normal reflection of electrons and holes at the defect plane we get the azimuthal modulation of the LDC developing with the growth of the barrier strength \( Z \).

V. SUMMARY

To summarize, we have investigated the transformations of the subgap spectrum of quasiparticle excitations in the Abrikosov vortex pinned by the planar defect with a high transparency. We find that the normal scattering at the defect surface results in the opening of a soft minigap \( \Delta_{\text{soft}} \) in the elementary excitation spectrum near the Fermi level. The minigap size grows with the decrease in the transparency of the barrier. The increase in the resulting soft gap affects the splitting of the zero bias anomaly in the tunneling spectral characteristics and perturb the circular symmetry of the LDOS peaks. The normal reflection of electrons and holes at the defect plane changes the topology of the isoenergetic orbits in \( (\mu - \theta_p) \) space. This topological transition revealing in the specific behavior of the quantized quasiparticle levels and density of states, can be considered as a hallmark of the crossover from the Abrikosov to the Josephson vortex. As a result, there appears a new type of subgap quasiparticle states gliding along the defect, which reveal the qualitatively new behavior of discrete spectrum \( \varepsilon_n \sim n^{1/2} \). The hard minigap \( \varepsilon_0 \ll \Delta_{\text{soft}} \) in the spectrum of energy levels exceeds noticeably the value of the CdGM minigap \( \hbar \omega_0 \ll \varepsilon_0 \). The decrease in the barrier transparency is accompanied by the increase in the hard minigap \( \varepsilon_0 \) in the spectrum which can be observed in the measurements of the Ohmic and Hall conductivities at finite frequencies. The basic properties of the vortex such as pinning and mobility along the defect plane are strongly affected by these changes in the orbit topology. We have also analyzed the distinctive features of the quasiparticle density of states, which accompany the transformation of the subgap quasiparticle spectrum and the topology of the isoenergetic orbits for an Abrikosov vortex pinned by a planar defect with a perfect boundary. One can expect, however, that barrier imperfections and roughness should result in a partial smearing of both the hard and soft gap features similarly to the effect of the point impurity scattering.

Finally, we note that recently the vortices pinned by the defects are studied as the hosts for the Majorana states in the systems consisting of a primary superconductor with conventional pairing and a low dimensional layer with a nontrivial topology. The isolating inclusions in the vortex core in the primary superconductor allow to shift the low energy core spectrum from the Fermi level improving the topological protection of the Majorana states in the 2D topological superconductor. The vortex at the planar defect considered in our work can provide a perspective platform for such states since the hard minigap in the core can exhibit a strong increase even in the limit of the defect with high transparency when the shape of the gap inside the vortex core is only weakly perturbed by the scattering. Another advantage of this geometry is related to the possibility to move the vortices along the defects changing, thus, the positions of the Majorana states in the attached 2D layer without changing the minigap responsible for the desired topological protection.

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1 A. A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957) [Zh. Eksp. Teor. Fiz. 32, 1442 (1957)].
2 A. Barone and G. Paterno, Physics and Applications of...
FIG. 8: (Color online) Evolution of the local differential conductance $dI/dV(eV, x, y)$ corresponding to different bias voltages $(a),(b) eV = 0$; $(c),(d) eV/\Delta_0 = 0.1$; $(e),(f) eV/\Delta_0 = 0.2$ for different values of the dimensionless barrier strength $Z$: left column $Z = 0.1$; right column $Z = 0.3$. We put here $T/\Delta_0 = 0.02$.
11

(Clarendon Press, Oxford, 2001).

19 F. Guinea, Yu. F. Pogorelov, Phys. Rev. Lett. 74, 462 (1995).
20 M.V. Feigelman, M.A. Skvortsov, Phys. Rev. Lett. 78, 2640 (1997).
21 A. Larkin, Yu.N. Ovchinnikov, Phys. Rev. B 57, 5457 (1998).
22 M.A. Skvortsov, D.A. Ivanov, G. Blatter, Phys. Rev. B 67, 014521 (2003).
23 A. S. Mel’nikov and A. V. Samokhvalov, JETP Lett. 94, 759 (2011) [Pis’ma v ZhETF 94, 823 (2011)].
24 C. Caroli, P. G. de Gennes, J. Matricon, Phys. Lett. 9, 307 (1964).
25 G. E. Volovik, The Universe in a Helium Droplet, Clarendon Press, Oxford, 2003.
26 A. I. Larkin and Yu. N. Ovchinnikov, Phys. Rev. B 57, 5457 (1998).
27 Y. Tanaka, S. Kashiwaya, and H. Takayanagi, Jpn. J. Appl. Phys. Part 1 34, 4566 (1995).
28 M. Eschrig, D. Rainer, and J. A. Sauls: in Vortices in unconventional superconductors and superfluids, ed. R.P. Huebener, N. Schopohl and G.E. Volovik (Springer Verlag, Berlin, 2001), preprint cond-mat/0106546.
29 A.S. Mel’nikov, A.V. Samokhvalov, M.N. Zubarev, Phys. Rev. B 79, 134529 (2009).
30 B. Rosenstein, I. Shapiro, E. Deutch, B.Ya. Shapiro, Phys. Rev. B 84, 134521 (2011).
31 A.S. Mel’nikov, A.V. Samokhvalov, V.L. Vadimov, 102, 886 (2015).
32 V.L. Vadimov, A.S. Mel’nikov, J. Low Temp. Phys. 183, 342 (2016).
33 L.D. Landau, L.P. Pitaevskii, Statistical Physics, Part 2, (Oxford: Pergamon, Ch.5, 1980).
34 N. B. Kopnin, A. S. Melnikov, V. I. Pozdnyakova, D. A. Ryzhov, I. A. Shereshevskii, and V. M. Vinokur, Phys. Rev. Lett. 95, 197002 (2005).
35 N. B. Kopnin, A. S. Melnikov, V. I. Pozdnyakova, D. A. Ryzhov, I. A. Shereshevskii, and V. M. Vinokur, Phys. Rev. B 75, 024514 (2007).
36 A. S. Mel’nikov, D. A. Ryzhov, and M. A. Silaev, Phys. Rev. B 78, 064513 (2008).
37 A. S. Mel’nikov, D. A. Ryzhov, and M. A. Silaev, Phys. Rev. B 79, 134521 (2009).
38 I. M. Lifshits, Zh. Eksp. Teor. Fiz. 38, 1569 (1960) [Sov. Phys. JETP 11, 1130 (1960)].
39 Y. M. Blanter, M. I. Kaganov, A. V. Pantsulaya, A. A. Varlamov, Phys. Reports 245, 159 (1994).
40 G. E. Volovik, Pis’ma Zh. Eksp. Teor. Fiz. 49, 343 (1989) [JETP Lett. 49, (1989)].
41 S. Mel’nikov and M. A. Silaev, Pis’ma Zh. Eksp. Teor. Fiz. 83, 675 (2006) [JETP Lett. 83, 578 (2006)].
42 C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056, (1991); C. W. J. Beenakker, Phys. Rev. Lett. 67, 3836, (1991).
43 N. B. Kopnin and G. E. Volovik, Pisma Zh. Eksp. Teor. Fiz. 64, 641 (1996) [JETP Lett. 64, 690 (1996)]; N. B. Kopnin, Phys. Rev. B 57, 11775 (1998).
44 N. B. Kopnin and G. E. Volovik, Phys. Rev. Lett. 79, 1377 (1997).
45 D. N. Basov, M. M. Fogler, A. Lanzara, Feng Wang, Yuanbo Zhang, Rev. Mod. Phys. 86, 959 (2014).
46 A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov and A. K. Geim,Rev. Mod. Phys. 81, 109 (2009).
47 B. Janko, Phys. Rev. Lett. 82, 4703 (1999).
48 G. Blatter, M. V. Feigel’man, V. B. Geshkenbein, A. I. Larkin, V. M. Vinokur, Rev. Mod. Phys. 66, 1125 (1994).
49 G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B 25, 4515 (1982).
50 N. B. Kopnin, A. S. Melnikov, and V. M. Vinokur, Phys. Rev. B 68, 054528 (2003).
51 A. L. Rakhmanov, A. V. Rozhkov, and Franco Nori, Phys. Rev. B 84, 075141 (2011).
52 P. A. Ioselevich and M. V. Feigelman, Phys. Rev. Lett. 106, 077003 (2011).
53 P. A. Ioselevich, P. M. Ostrovsky, and M. V. Feigelman, Phys. Rev. B 86, 035441 (2012).
54 R. S. Akzyanov, A. V. Rozhkov, A. L. Rakhmanov, and Franco Nori, Phys. Rev. B 89, 085409 (2014).