Deriving Einstein’s Field Equation (EFE) and Modified Gravity by Statistical Mechanics and Quantization of Anti-Commuting Space

R. Hassannejad and S. Navid Mousavi (Dated: November 13, 2018)

Department of Physics, Shiraz University, Shiraz 71454, Iran.

In this paper, we derive the Einstein’s field equation (EFE) by considering an anti-commuting two dimensional quantized space, which can be excited by absorbing energy. Any change of the energy in space will result in a change in the quanta’s area state. This means the geometry of space depends on the energy in it. We also show that in temperatures far below the Planck temperature $T \ll T_p$, our model results in Newton’s law of gravity.

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I. INTRODUCTION

After introduction of general relativity by Einstein, and its experimental affirmations, together with development of quantum mechanics and it’s great achievements, a large number of efforts were made in order to connect these two theories to answer questions about the nature of gravity. In this process, physicists made great theories such as string theory [1–3], and loop quantum gravity which was first done by Beckenstein [4, 5].

Also, many tried to find a connection between Thermodynamics and gravity, which was first done by Beckenstein in 1973 [6], and Hawking in 1974 [7]. They compared the mass, electric charge, and angular momentum of black-holes with thermodynamical parameters, and found relations between temperature $T$, and entropy $S$, of black-holes. In fact just like the study of statistical mechanics in the black-body radiation, which ended up in emergent of quantum mechanics, the study of statistical mechanics of black-holes had a great impact in understanding quantum gravity, which finally resulted in formulating the Holographic principle [8].

In addition to these, theorists tried to drive the Einstein’s field equation using thermodynamics, and we can point out the work done by Jacobson in 1995 [9], which uses the first law of thermodynamics to derive the Einstein’s field equation. Also, Verlinde in a brilliant paper in 2011 [10], used Holographic principle and statistical mechanics to derive Newton’s law of gravity and Einstein’s equation.

Considering the things done in order to connect quantum mechanics and gravitation, quantization of space seems to be a good approach. There exist different methods to quantize the space, but in general some variable which are commutative in the continuous limit, assumed to be non commuting for the sake of forming a different lie algebra [11]. This idea was first introduced by Heisenberg to solve the infinities in the quantum field theory [11]. The first concrete example of a Lorentz discrete space time were made by Snyder in 1947 [12], and after that in 1980s Connes developed the non commutative differential geometry [13]. More detailed history of the subject is written in the reference [14].

Romero et al. in a paper in 2003 [15], calculated the area of a two dimensional anti commuting space with anti commutation relation

$$[\hat{x}_1, \hat{x}_2] = i\hbar \Theta, \quad \Theta > 0.$$ (1)

They found the below result for the area of the space

$$A_n = 2\pi \hbar \Theta (n + \frac{1}{2}), \quad n = 0, 1, 2, \cdots$$ (2)

They compared their result with the area calculated in the reference [16] for the black hole horizon, which can be written as

$$A_n = \gamma l_p^2 n, \quad n = 1, 2, 3, \cdots$$ (3)

and also the loop quantum gravity theory result for the space area [17], which is in the form

$$A(j_i) = \gamma l_p^2 (j_i + \frac{1}{2}).$$ (4)

With the similarity of the results given by Romero et al. [15], and Alekseev et al. [17], which are driven with different approaches, we can conclude that anti commuting geometry and quantum gravity must be connected [15].

The main idea behind this paper is the quantization of anti commuting space and the assumption of excitation in space quanta when some sort of energy is in the space. ie. we consider the space containing $N$ quanta, where each quanta can be excited by absorbing energy. This excitation changes the quanta’s area state, and the ultimate space geometry should be determined by the sum over area states of the space quanta.

When there exist no energy or mass in the space, all the quanta will have the same area (Eq. (2) with $n = 0$). This situation can be thought of similar to the ground state of a system of harmonic oscillators, which all of them have the same energy state ($n = 0$).

$$E = \frac{1}{2} \hbar \omega$$ (5)
Just by inserting some energy to the space, space quanta will become excited and the area of each quanta would change as the Eq. (2) dictates. Each quanta will have a specific area, which is dependent on its energy. Fig. (1), presents a schematic view of a flat space with all quanta in the ground area state \((n = 0)\) with area equal to \(\hbar \Theta\), but in Fig. (2) you can see the space quanta which are excited and each quanta has a specific area.

With these descriptions in mind, we derive the EFE, modified gravity equation, and Newton’s law of gravity. The procedure is presented in three parts. First in part (II A), we calculate the entropy and the internal energy of the space, then in part (II B), the derivation of the EFE and modified gravity will be presented, and finally in part (II C), we show that our model yields the Newton’s law of gravity in the limit \(T \ll T_p\), where \(T_p\) is the Planck’s temperature.

II. PROCEDURE

A. Entropy and internal energy of space

As mentioned before, we start by using the area law (Eq. (2)) which is derived in the reference [15]. The only difference is that we omit the constant \(\pi\) as a matter of convenience. We assume each quanta of space can have a specific value of area which depends on its energy. So the area of each quanta of space can be written as

\[
A_n = 2\hbar \Theta (n + \frac{1}{2}).
\]

\(n = 0\) corresponds to the ground state, which means there exist no energy in the space. In result the space would be flat. Also, the space quanta area will all be the same when \(n = 0\), and they have a value equal to \(\hbar \Theta\) (Fig. (1)). Just by inserting some energy or mass in space, it’s quanta would become excited, and instead of having quanta with the same area, we will see quanta with different areas (Fig. (2)) depending on their energy.

Since there exist no exact knowledge of the state of the space quanta, we should perform an ensemble average on all area states to find the expectation value of area \(\langle A \rangle\) of each quanta in the excited case. The expectation value for space quanta area reads

\[
\langle A \rangle = \sum_{n=0}^{\infty} A_n e^{-\beta A_n} \sum_{n=0}^{\infty} e^{-\beta A_n}.
\]

As the dimension of \(A_n\) in Eq. (6), is Area \([L^2]\), in order to have \((\beta A_n)\) dimensionless we define \(\beta = \frac{\alpha}{k_B T}\), where \(k_B\) is the Boltzmann constant, \(T\) is the temperature, and \(\alpha\) is a constant to set the dimension of \(\beta\) equivalent to \(\frac{1}{L^2}\). In Eq. (7), \(\sum_{n=0}^{\infty} e^{-\beta A_n}\), is the partition function and by simplifying Eq. (7), the area can be written as follows

\[
\langle A \rangle = \hbar \Theta \left( 1 + \frac{2}{e^{2\beta (\hbar \Theta)} - 1} \right).
\]

Also, if we assume no interaction between space quanta, and \(N\) as the total number of quanta in the space, the total area \(A_x\) can be denoted as below

\[
A_x = N \langle A \rangle.
\]

Now, using partition function, we can calculate the entropy of space. To do so, we calculate first the partition function of one space quanta in the form

\[
Z_1 = \sum_{n=0}^{\infty} e^{-\beta A_n} = \frac{e^{-\beta \hbar \Theta}}{1 - e^{-2\beta \hbar \Theta}}.
\]

As we assumed no interaction between quanta of space, the partition function of the whole space would be

\[
Z = (Z_1)^N = \left( \frac{e^{-\beta \hbar \Theta}}{1 - e^{-2\beta \hbar \Theta}} \right)^N.
\]

By using the system partition function, the Helmholtz free energy can be written as

\[
F = -k_B T \ln (Z) = k_B T N \ln (e^{\beta \Theta} - e^{-\beta \hbar \Theta}).
\]

In the other hand, there exist the following relation between thermodynamical variables of the system and Helmholtz free energy

\[
dF = -SdT + \sum_i X_i dY_i
\]
where for constant $Y_i$'s relation between free energy $F$ and entropy $S$ is
\[ S = -\frac{\partial F}{\partial T}. \] (14)

With latter relation in hand, and Helmholtz free energy, entropy of space can be calculated as follows.
\[ S = -k_B \ln(e^{\beta \theta} - e^{-\beta \theta}) + \frac{\alpha N \Theta}{T} \left( \frac{e^{2\theta\beta} - 1}{e^{2\theta\beta} - 1} \right) \] (15)

by substituting Eq.(8) in Eq.(9), and finally putting the result in Eq.(15) we get
\[ S = \frac{\alpha A_T}{T} - A_T \left[ \frac{k_B}{\hbar \Theta} \left( \frac{e^{2\theta\beta} - 1}{e^{2\theta\beta} + 1} \right) \right] \ln(e^{\beta \theta} - e^{-\beta \theta}). \] (16)

To continue, we fix the dimension by multiplying RHS of Eq.(16), by
\[ \left( \frac{k_B c^3}{4\hbar G} \right) \left( \frac{4\hbar G}{k_B c^3} \right) \] (17)
and we get
\[ S \left( \frac{k_B c^3 A_T}{4\hbar G} \right) \left( \frac{4\hbar G}{k_B c^3} \right) \left( \frac{\alpha}{T} - \frac{k_B M}{\hbar \Theta} \right) \] (18)

where $M$ is defined as a dimensionless form of temperature as
\[ M = \left( \frac{e^{2\theta\beta} - 1}{e^{2\theta\beta} + 1} \right) \ln(e^{\beta \theta} - e^{-\beta \theta}). \] (19)

Since \( \left( \frac{k_B c^3 A_T}{4\hbar G} \right) \) has the dimension of entropy \([JK^{-1}]\), then \( \left( \frac{4\hbar G}{k_B c^3} \right) \left( \frac{\alpha}{T} - \frac{k_B M}{\hbar \Theta} \right) \), must be dimensionless. Also, from Eq.(9), we see $\Theta$ must have the dimension of \([TM^{-1}]\). As a result $\alpha$ must have dimension of \([MT^{-2}]\). In Short, using quantum mechanics and statistical mechanics we calculated the entropy of two dimensional quantized anti-commuting space as below
\[ S = \left( \frac{k_B c^3 A_T}{4\hbar G} \right) \gamma(T) \] (20)

where $\gamma(T)$, is defined as
\[ \gamma(T) = \left( \frac{4\hbar G}{k_B c^3} \right) \left( \frac{\alpha}{T} - \frac{k_B M}{\hbar \Theta} \right). \] (21)

Considering third law of thermodynamics, when temperature goes to zero, the entropy must show the same behavior. So we are going to check this law for the Eq.(20).

With a small calculation we can easily show that, when $T \to 0$, in Eq.(21), $\gamma(T) \to 0$, which means the entropy tends to zero and Eq.(21) obeys the third law of thermodynamics.

Now we proceed to calculate the constants $\alpha$ and $\Theta$ in terms of Planck units. Since $\alpha$ has he dimension of \([MT^{-2}]\), We choose $\alpha = \frac{m_p}{t_p^2}$. Where $m_p$ is the Planck mass and $t_p$ is the Planck time. Also, considering the dimension of $\Theta$, which is \([TM^{-1}]\), we choose $\Theta = \frac{t_p}{m_p}$, where $t_p$ and $m_p$, are again the Planck time and mass respectively and have the value
\[ t_p = \sqrt{\frac{\hbar G}{c^3}}, \quad m_p = \sqrt{\frac{\hbar c}{G}} \] (22)

Using Eq.(19), we can write the constants $\alpha$ and $\Theta$ as
\[ \alpha = \sqrt{\frac{c^4}{\hbar G}}, \quad \Theta = \frac{G}{c^3}. \] (23)

In continue we rewrite Eq.(19) and Eq.(21) in a simpler form, using the constants we found as
\[ M = \left( \frac{\alpha}{T} - \frac{k_B M}{\hbar \Theta} \right) \ln(e^{\frac{T_p}{T}} - e^{-\frac{T_p}{T}}) \] (24)
\[ \gamma(T) = 4\left( \frac{T}{T_p} - M \right) \] (25)

where $T_p$ is the Planck temperature. Now by using Eq.(24), we can write the following inequality
\[ e^{\frac{T_p}{T}} - e^{-\frac{T_p}{T}} > 0. \] (26)

By solving the latter inequality we can finally obtain the below condition for temperature
\[ T > 0. \] (27)

Besides, even in temperatures close to Planck temperature $\frac{T}{T_p} \gg 1$, and we can expand Eq.(24), and by substituting in Eq.(25), find the below relation for entropy
\[ S = \left( \frac{k_B c^3 A_T}{4\hbar G} \right) \gamma(T) \] (28)

with
\[ \gamma(T) = \frac{4T_p}{T} \left[ 1 - \ln(\frac{2T}{T_p}) \right]. \] (29)

Since Eq.(29), must have a positive value, then \( (1 - \ln(\frac{2T}{T_p})) > 0 \) always holds, and by a little algebra we can find
\[ T \geq \frac{2T_p}{e}. \] (30)

Lastly, we have to mention that the calculated entropy is very similar to entropy of the photon gas black-body. The entropy of the photon gas black-body can be written as below [13].
\[ s = \left( \frac{4\pi^2 k^4}{45 c^3 \hbar^3} \right) VT^3. \] (31)

Now we would use the Legendre transform between Helmholtz free energy and system's internal energy to
calculate the internal energy $U$. The following relation exists between Helmholtz free energy $F$ and internal energy $U$

$$F = U - TS$$  \hspace{1cm} (32)$$

also, we have the first law of thermodynamics as

$$dU = \partial W + \partial Q.$$  \hspace{1cm} (33)$$

Now using Eq. (33), Eq. (12) and Eq. (20), The internal energy $U$ can be written in the form

$$U = \left( \frac{A_o k_B c^3}{h G} T_p \right).$$  \hspace{1cm} (34)$$

B. Deriving EFE and Modified gravity equation

In recent years, some scientists, tried to derive the EFE using the first law of thermodynamics, where we can name the references [9, 19] as most important efforts. In this section we proceed to derive the EFE by using Eq. (34), and equations in references [9, 19]. Since we did not assume any specific horizon, and the only condition we put on the space quantas, is that quantas can be excited in presence of energy, and turn the flat space to a curved one, then our result is general and holds for any two dimensional anti commuting space. So we are allowed to use the results derived in References [9, 19], which have derived the EFE by considering local Rindler casual horizons. To derive the EFE, we first differentiate the Eq. (34), and use the relations in references [9, 19]. In reference [9], $\partial Q$ and $\partial A_o$ are defined as

$$\partial A_o = - \int \lambda R_{ab} k^a k^b d\lambda dA$$  \hspace{1cm} (35)$$

$$\partial Q = - \int \lambda T_{ab} k^a k^b d\lambda dA$$  \hspace{1cm} (36)$$

where $k^a$ is the Killing vector, and $T_{ab}$ and $R_{ab}$, are the energy-momentum and Ricci tensors respectively. By substituting Eq. (34), Eq. (35) and Eq. (36) in Eq. (33), we reach to

$$\left( \frac{T_p k_B c^3}{h G} \right)(- \int \lambda R_{ab} k^a k^b d\lambda dA) = - \int \lambda T_{ab} k^a k^b d\lambda dA + \partial W.$$  \hspace{1cm} (37)$$

To continue we use the relation between pressure $P$ and work $W$ which is in the form

$$\partial W = -PDV.$$  \hspace{1cm} (38)$$

In the other hand, considering the relation between pressure $P$ and cosmological constant $\Lambda$ which can be written as [20]

$$P = - \frac{\Lambda}{8\pi}$$  \hspace{1cm} (39)$$

we can conclude that there must be a relation between work $W$, and cosmological constant $\Lambda$, which can be written in the following form

$$\partial W = \left( \frac{T_p k_B c^3}{h G} \right)(\int \lambda \left( \frac{R}{2} - \Lambda \right) (g_{ab} k^a k^b d\lambda dA) \right).$$  \hspace{1cm} (40)$$

Considering the reference [21], the term $g_{ab} k^a k^b = 0$, so the work term in EFE is equivalent to zero, which might not be zero in modified gravity models. At last by substituting Eq. (40) in Eq. (37), we find the next result

$$R_{ab} - \frac{1}{2}g_{ab} R + \Lambda g_{ab} = \frac{h G}{T_p k_B c^3} T_{ab}$$  \hspace{1cm} (41)$$

where the Planck temperature $T_p$ is equal to

$$T_p = \sqrt{\frac{\hbar c^5}{G k_B^3}}.$$  \hspace{1cm} (42)$$

This leads to below equation, by substituting Eq. (42) in Eq. (41)

$$R_{ab} - \frac{1}{2}g_{ab} R + \Lambda g_{ab} = \sqrt{\frac{\hbar G^3}{c^3}} T_{ab}.$$  \hspace{1cm} (43)$$

Now we are going to compare the Eq. (43) with EFE which is in the form

$$R_{ab} - \frac{1}{2}g_{ab} R + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$  \hspace{1cm} (44)$$

Eq. (44), is derived using Lagrangian and with a classical approach, however, we derived Eq. (43), just by using quantum and statistical mechanics, and it contains Planck constant $\hbar$, which is the signature of quantum mechanics. So we assert that gravitation is a consequence of statistical and quantum mechanics, then our approach is complete.

In recent years, different models of modified gravity have been suggested, which seem to be practical in solving new problems in cosmology, such as dark energy and dark matter. In continue, we try to investigate these modifications. Considering Eq. (33), (35) and (36), we can say that internal energy $U$, and thermal energy $Q$, has unique values, but this does not hold for work $W$. So we can use work $W$ in order to obtain different modification of gravity.

Let’s rewrite the Eq. (33) in the form

$$\partial W = \left( \frac{T_p k_B c^3}{h G} \right)(\int \lambda \left( \frac{R}{2} - \Lambda \right) (g_{ab} k^a k^b d\lambda dA) + \zeta \int \lambda \chi_{ab} k^a k^b d\lambda dA)$$  \hspace{1cm} (45)$$

where $\zeta$ is a constant and $\chi_{ab} k^a k^b \neq 0$, is a parameter to describe equation of motion of different modifications of gravity like Lovelock gravity [21], massive gravity model
or other models. So in result we can write our field equation as follows

$$R_{ab} = -\frac{1}{2} g_{ab} R + \Lambda g_{ab} + \Phi \chi_{ab} = \frac{\hbar G^5}{c^4 T^4} T_{ab}$$  \hspace{1cm} (46)$$

where $\Phi$ in Eq.\((46)\), is a constant.

We can conclude that each time physicists have tried to modify gravity to solve the new problems in Theoretical Physics, they have changed the work $W$ term in Eq.\((33)\), and maybe we could find a unique and complete work term for gravity some day.


C. Deriving Newton’s law of gravity

In this section we are going to investigate the proposed model in the low temperature limit $T \ll T_p$, which yields the Newton’s law of gravity. For this sake we start by using Eq.\((8)\), and expand it in the low temperature limit which gives the below result for the expectation value of each quanta are $\langle A \rangle$.

$$\langle A \rangle = \hbar \Theta$$  \hspace{1cm} (47)$$

Eq.\((17)\) tells us that the size of space quantas are equal and constant in this limit, which is just the assumption that Erik Verlinde used in his paper in 2011 to derive Newton’s gravity equation and gravity Poisson equation \[10\]. Now using the relations in reference \[10\], we derive the Newton’s equation for gravity. Considering space quantas having the same constant size, each quanta of space will have the same energy, equal to $\frac{k_B T}{2}$, then the total energy of $N$ quantas will be

$$E = \frac{1}{2} N k_B T$$  \hspace{1cm} (48)$$

Where $N$ is the number of quantas in space. Now using Eq.\((9)\) and Eq.\((47)\), we can write

$$N = \frac{A_p c^3}{G \hbar}$$  \hspace{1cm} (49)$$

By substitution of Eq.\((49)\) in Eq.\((48)\), and replacing $E = M c^2$ and $A_p = 4 \pi R^2$, in the obtained result we find the below relation

$$M c^2 = \frac{1}{2} \frac{4 \pi R^2 c^3}{G \hbar} k_B T.$$  \hspace{1cm} (50)$$

Now by substitution of $F = ma$ and by using Unruh temperature \[23\] we can replace the $k_B T = \frac{\hbar a}{2 \pi c}$ in Eq.\((50)\) we can find the Newton equation of gravity.

$$F = \frac{G m M}{R^2}$$  \hspace{1cm} (51)$$

III. CONCLUSION

In this paper, without any tough and complicated calculations of high energy Physics, and just by using elementary equations of thermodynamics and anti-commuting quantized space, we found a relation for entropy $S$ of space. Then by calculating the internal energy $U$, and using references \[9\] \[19\], we derived the Einstein’s field equation (EFE) and modified gravity equation. It is evident that all the efforts made in order to modify the Einstein’s gravity has changed the work term $W$ in the Eq.\((33)\). We also showed that in the low temperature limits $T \ll T_p$, our model leads to the Newton’s law of gravity. So we can assert that gravity is not a fundamental phenomenon, just as Einstein claimed. It depends on other fundamental variables such as energy of the space. Here we can ask the question, What is Gravity? To answer this question, we can say that gravity is nothing but the excitation of space quantas in result of inserting energy to the space. This excitation will cause the space quantas have different areas and in the result the space will become curved.

We also have to mention, that it is necessary to redefine the conservation of energy principle. As it does not hold only for objects in space, but also for the objects in space plus the space itself. In fact, when some amount of energy is transmitted to quantas of space, it does not annihilate, but it is conserved in the system consisting the space and the objects in space.

We have to say, that our calculations are made in two dimensional anti-commuting space, without considering time dimension, but the same procedure can be done in three dimensional space if we find a similar relation to Eq.\((1)\) for the three dimensional space.

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