LOCAL COHOMOLOGY BOUNDS AND TEST IDEALS

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Abstract. We find new classes of prime characteristic rings whose finitistic test ideal and test ideal are equal. In particular, we equate the notions of weakly F-regular and strongly F-regular for four dimensional rings whose anti-canonical algebra is Noetherian on the punctured spectrum.

1. Introduction

Introduced and developed by Hochster and Huneke in [HH89, HH90, HH91, HH93, HH94a, HH94b], the theory of tight closure is a central topic in the study of Noetherian rings of prime characteristic $p > 0$. Suppose $R$ is a Noetherian ring of prime characteristic $p > 0$ and let $R^e$ be the set of elements which avoid all minimal primes of $R$. Let $I \subseteq R$ be an ideal of $R$ and denote by $I^{[p^e]}$ the expansion of $I$ along the $e$th iterate of the Frobenius endomorphism. The tight closure of $I$ is the ideal $I^*$ consisting of elements $x \in R$ such that there exists an element $c \in R^e$ with the property that $cx^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. A defining problem of tight closure theory was the question of whether or not tight closure commutes with localization: If $W$ is a multiplicative set and $I \subseteq R$ an ideal, is $I^* R_W = (I R_W)^*$? There are scenarios when tight closure does commute with localization, e.g., [AHH93] and [Yao05]. However, there exist hypersurfaces for which tight closure does not commute with localization, [BM10]. Brenner’s and Monsky’s counterexample to the localization problem leaves open the intriguing problem if the property of tight closure being a trivial operation on ideals commutes with localization.

Continue to let $R$ be a Noetherian ring of prime characteristic $p > 0$. The ring $R$ is called weakly F-regular if every ideal is tight closed, that is $I = I^*$ for every ideal $I$.\footnote{A defining property of tight closure theory is that every regular ring is weakly F-regular.} A ring is called F-regular if every localization of $R$ is weakly F-regular. Let $F^*_e R$ denote the restriction of scalars of $R$ along the $e$th iterate Frobenius endomorphism $F^e : R \to R$. We say that $R$ is strongly F-regular if for each nonzero element $c \in R$ there exists $e \in \mathbb{N}$ such that the $R$-linear map $R \to F^*_e R$ defined by $1 \mapsto F^*_e c$ is pure. Every strongly F-regular ring is weakly F-regular and the property of being strongly F-regular passes to localization. It is conjectured that all three notions of F-regularity agree. Our main contribution towards this problem is the following:

**Theorem A.** Let $(R, \mathfrak{m}, k)$ be a weakly F-regular ring of Krull dimension no more than 4 which has a canonical module. If the anti-canonical algebra of $R$ is Noetherian on the punctured spectrum then $R$ is strongly F-regular.

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Theorem A provides a substantial step forward to equating the notions of $F$-regular and strongly $F$-regular for 4-dimensional rings. Suppose $(R, \mathfrak{m}, k)$ is a local strongly $F$-regular ring of Krull dimension, e.g., $R$ is the localization of a 4-dimensional $F$-regular ring. Then there exists an effective boundary divisor $\Delta$ such that $(\text{Spec}(R), \Delta)$ is globally $F$-regular (or just $F$-regular since Spec($R$) is affine), [SS10, Corollary 6.9], and therefore has KLT singularities by [HW02, Theorem 3.3]. Thus if one could show every 3-dimensional KLT singularity in prime characteristic has finitely generated anti-canonical algebra then the notions of $F$-regular and strongly $F$-regular would coincide.

There has been tremendous effort to equate the various notions of $F$-regularity since the theory of tight closure was introduced. Efforts to equate at least two of the notions of $F$-regularity have typically required making desirable geometric assumptions on the ring $R$. For example:

1. Hochster and Huneke showed weak implies strong for Gorenstein rings, [HH89];
2. Weakly $F$-regular is equivalent to strongly $F$-regular whenever $R$ is $\mathbb{N}$-graded over a field by work on Lyubeznik and Smith, [LS99];
3. MacCrimmon showed weakly $F$-regular is equivalent to strongly $F$-regular if $R$ is assumed to be $\mathbb{Q}$-Gorenstein at non-maximal points of Spec($R$), [Mac96];
4. Murthy proved that weakly $F$-regular and $F$-regular are equivalent conditions whenever the ring $R$ is finite type over an uncountable field, see [Hun96, Theorem 12.2]. Hochster and Huneke extended Murthy’s result in [HH94a, Theorem 8.1] to rings finite type over a field which has infinite transcendence degree over its prime field $\mathbb{F}_p$;
5. The conjectured equivalence of weak and strong follows by an unpublished result of Singh, provided the symbolic Rees ring associated with an anti-canonical ideal is Noetherian, see [CEMS18, Corollary 5.9]. See also [Abe02] for a related assumption on the anti-canonical ideal from which validity of the weak implies strong conjecture can be derived.

There has been limited progress on equating the various notions of $F$-regularity without making conjecturely unnecessary assumptions. Williams’ theorem, see [Wil95], equates the notions of weakly $F$-regular and strongly $F$-regular for rings of dimension no more than 3. Williams proof of the weak implies strong conjecture in dimension 3 relies on Lipman’s theorem that the divisor class groups of the local rings of an excellent surface with at worst rational singularities is finite, [Lip69]. Specifically, Williams uses that the canonical class of a local three dimensional weakly $F$-regular ring is torsion on the punctured spectrum, an assumption MacCrimmon imposed on larger dimensional rings in [Mac96] in order to extend William’s methodology to a large class of rings of arbitrarily large dimension. Lipman’s theorem on divisor class groups requires an understanding of minimal resolutions of rational surface singularities by quadratic transforms.

In the spirit of MacCrimmon’s theorem, we do not limit ourselves to low dimensions. We instead impose desirable geometric conditions on higher dimensional rings to find a new and interesting class of rings for which the finitistic test ideal and the test ideal agree. To motivate our most general result we rephrase MacCrimmon’s theorem from [Mac96] in terms of analytic spread of an anti-canonical ideal. Let $(R, \mathfrak{m}, k)$ be a normal Cohen-Macaulay domain of prime characteristic $p > 0$ and $\omega_R \cong J_1 \subseteq R$ a canonical ideal. The ideal $J_1$ is of
pure height 1 and so there exists an element \( a \in R \) such that \( a \) generates \( J_1 \) at its components. We can then write \( (a) = J_1 \cap K_1 \) where \( K_1 \) is pure height one and the components of \( K_1 \) are disjoint from the components of \( J_1 \). The ideal \( K_1 \) is an anti-canonical ideal of \( R \) and is the inverse of \( J_1 \) as an element of the divisor class group of \( R \). To assume \( R \) is \( \mathbb{Q} \)-Gorenstein on the punctured spectrum is equivalent to assuming that for some natural number \( N \geq 1 \) the \( N \)th symbolic power of \( K_1, K_1^{(N)} \), has analytic spread 1 on the punctured spectrum.

We recover MacCrimmon’s result by proving every weakly \( F \)-regular ring is strongly \( F \)-regular under the milder hypothesis that some symbolic power of an anti-canonical ideal has analytic spread at most 2 on the punctured spectrum.

**Theorem B.** Let \((R, m, k)\) be a local Cohen-Macaulay normal domain of prime characteristic \( p \), Krull dimension \( d \), and \( \mathbb{Q} \)-Gorenstein in codimension 2. Suppose further that \( R \) has a canonical ideal and that some symbolic power of the corresponding anti-canonical ideal has analytic spread at most 2 on the punctured spectrum. Let \( E_R(k) \) be the injective hull of the residue field. Then the finitistic tight closure and the tight closure of the zero submodule of \( E_R(k) \) agree. In particular, the finitistic test ideal and the test ideal of \( R \) agree. Therefore if \( R \) is weakly \( F \)-regular then \( R \) is strongly \( F \)-regular.

Techniques introduced in this article allow us to equate the finitistic tight closure and the tight closure of the zero submodule of the injective hull of the residue field through a careful analysis of the maps of Koszul cohomologies defining certain local cohomology modules. Our analysis of local cohomology is centered around the notion of a local cohomology bound defined in Section 3.

The relationship between \( F \)-signature and relative Hilbert-Kunz multiplicity is also explored. See [Hun13, PT18] for introductions to the theory of prime characteristic invariants, such as Hilbert-Kunz multiplicity and \( F \)-signature, in local rings. The \( F \)-signature of an \( F \)-finite ring\(^2\) is the limit

\[
\lim_{e \to \infty} \frac{\text{frk}(F_e^e R)}{\text{rank}(F_e^e R)}
\]

where \( \text{frk}(F_e^e R) \) is the largest rank of an \( R \)-free summand appearing in a direct sum decomposition of \( F_e^e R \). The invariant \( F \)-signature was shown to exist under the local hypothesis in [Tuc12]. If \((R, m, k)\) is local of Krull dimension \( d \) then the Hilbert-Kunz multiplicity of an \( m \)-primary ideal \( I \) is the limit

\[
e_{\text{HK}}(I) = \lim_{e \to \infty} \frac{\lambda(R/I[I^e])}{p^{ed}}
\]

where \( \lambda(R/I[I^e]) \) denotes the length of \( R/I[I^e] \). Existence of Hilbert-Kunz multiplicity was established by Monsky, [Mon83].

If \((R, m, k)\) is local, \( I \) an \( m \)-primary ideal, and \( u \notin I \) then

\[
s(R) \leq e_{\text{HK}}(I) - e_{\text{HK}}((I, u))
\]

[HL02, Proposition 15]. Watanabe and Yoshida explored the notion of minimal relative Hilbert-Kunz multiplicity and its relation with \( F \)-signature in [WY04]. They suspected that the \( F \)-signature of \( R \) is realized as the minimum of all relative Hilbert-Kunz multiplicities.

\(^2\)The ring \( R \) is \( F \)-finite if \( F_e^e R \) is a finitely generated module for some, equivalently each, \( e \in \mathbb{N} \).
For example, if \((R, m, k)\) is Gorenstein, \(I\) a parameter ideal, and \(u \in R\) generates the socle mod \(I\), then \(s(R) = e_{\text{HK}}(I) - e_{\text{HK}}(I, u)\) by \([\text{HL}02, \text{Theorem 11}]\). More generally, it is known that the \(F\)-signature of a local ring agrees with the infimum of all relative Hilbert-Kunz multiplicities by work of the second author and Tucker in \([\text{PT}18, \text{Theorem A}]\).

Relating \(F\)-signature with relative Hilbert-Kunz multiplicities is closely connected with the weak implies strong conjecture. Under mild hypotheses, a local ring \(R\) is weakly \(F\)-regular if and only if \(e_{\text{HK}}(I) - e_{\text{HK}}((I, u)) > 0\) for every \(m\)-primary ideal \(I\) and \(u \in R - I\) by \([\text{HH}90, \text{Proposition 4.16 and Theorem 8.17}]\), and \(R\) is strongly \(F\)-regular if and only if \(s(R) > 0\) by \([\text{AL}03]\). In particular, if it is known that \(F\)-signature of a weakly \(F\)-regular ring can be realized as a relative Hilbert-Kunz multiplicity then the conjecture of weak implies strong would follow. The techniques of this article are used to equate \(F\)-signature with a relative Hilbert-Kunz multiplicity for strongly \(F\)-regular rings which satisfy the hypotheses of Theorem \(B\).\(^3\)

**Theorem C.** Let \((R, m, k)\) be a local \(F\)-regular and \(F\)-finite ring of prime characteristic \(p > 0\) such that some symbolic power of the anti-canonical ideal has analytic spread at most 2 on the punctured spectrum. There exists an irreducible \(m\)-primary ideal \(I\) and socle generator \(u \mod I\) such that

\[
a_e(R) = \frac{\lambda(R/I^p^e)}{p^e \dim(R)} - \frac{\lambda(R/(I, u)^p^e)}{p^e \dim(R)}
\]

for every \(e \in \mathbb{N}\). In particular,

\[
s(R) = e_{\text{HK}}(I) - e_{\text{HK}}((I, u)).
\]

Section 2 is devoted to background and preliminary results. Central to this paper is the notion of a local cohomology bound. Local cohomology bounds are of independent interest and are defined and discussed in Section 3. Section 4 is the technical heart of the paper and is where proofs of Theorem A and Theorem B are given. The proof of Theorem C is found in Section 5. In Section 6 we list some open problems of interest.

2. **Background and preliminary results**

2.1. **Tight closure.** Let \(R\) be a ring of prime characteristic \(p > 0\) and let \(R^e\) be complement of the union of the minimal primes of \(R\). The \(\alpha\)th Frobenius functor, or the \(\alpha\)th Peskine-Szpiro functor, is the functor \(F^\alpha : \text{Mod}(R) \rightarrow \text{Mod}(R)\) obtained by extending scalars along the \(\alpha\)th iterate of the Frobenius endomorphism. If \(N \subseteq M\) are \(R\)-modules and \(m \in M\), then \(m\) is in the tight closure of \(N\) relative to \(M\) if there exists a \(c \in R^e\) such that for all \(e \gg 0\) the element \(m\) is in the kernel of the following composition of maps:

\[
M \rightarrow M/N \rightarrow F^\alpha(M/N) \rightarrow F^\alpha(M/N).
\]

In particular, if we consider an inclusion of \(R\)-modules of the form \(I \subseteq R\) then \(F^\alpha(R/I) \cong R/I^{p^\alpha}\) where \(I^{p^\alpha} = (r^{p^\alpha} \mid r \in I)\), and an element \(r \in R\) is in the tight closure of \(I\) relative to \(R\) if there exists a \(c \in R^e\) such that \(c^{p^\alpha} \in I^{p^\alpha}\) for all \(e \gg 0\). The tight closure of the

\(^3\)The only property in the hypotheses of Theorem \(B\) which is not enjoyed by every strongly \(F\)-regular ring is the property that some symbolic power of the anti-canonical ideal has analytic spread at most 2 on the punctured spectrum.
module $N$ relative to the module $M$ is denoted $N^*_M$. In the case that $M = R$ and $N = I$ is an ideal then we denote the tight closure of $I$ relative to $R$ as $I^*$. We say that $N$ is tightly closed in $M$ if $N = N^*_M$. If an ideal is tightly closed in $R$ then we simply say that the ideal is tightly closed. The finitistic tight closure of $N \subseteq M$ is denoted $N^*_{Mf}$ and is the union of $(N \cap M')^*_{M'}$ where $M'$ runs over all finitely generated submodules of $M$.

The notions of weak $F$-regularity and strong $F$-regularity can be compared by studying the finitistic tight closure and tight closure of the zero submodule of the injective hull of a local ring by [HH90, Proposition 8.23] and [Smi93, Proposition 7.1.2]. Suppose that $(R, \mathfrak{m}, k)$ is complete local and $E_R(k)$ is the injective hull of the residue field. The finitistic test ideal of $R$ is $\tau_{fg}(R) = \bigcap_{I \subseteq R} \text{Ann}_R(I^*/I)$ and agrees with $\text{Ann}_R(0^*_{E_R(k)})$. The test ideal of $R$ is $\tau(R) = \bigcap_{N \subseteq \mathcal{M}_{\text{Mod}(R)}} \text{Ann}_R(N^*_M/N)$ and agrees with $\text{Ann}_R(0^*_{E_R(k)})$. The ring $R$ is weakly $F$-regular if and only if $\tau_{fg}(R) = R$ and $R$ is strongly $F$-regular if and only if $\tau(R) = R$. Thus to prove the conjectured equivalence of weak and strong $F$-regularity it is enough to show $0^*_{E_R(k)} = 0^*_{E_R(k)}$ under hypotheses satisfied by rings which are weakly $F$-regular.

To explore the tight closure of the zero submodule of $E_R(k)$ we exploit the structure of $E_R(k)$ as direct limit of 0-dimensional Gorenstein quotients of $R$ described in [Hoc77]. Suppose $(R, \mathfrak{m}, k)$ is a complete local Cohen-Macaulay domain of Krull dimension $d$ and $J_1 \subseteq R$ a canonical ideal. Let $0 \neq x_1 \in J_1, x_2, \ldots, x_d \in R$ a parameter sequence on $R/J_1$, and for each $t \in \mathbb{N}$ let $I_t = (x_1^{t-1}J_1, x_2, \ldots, x_d)$. The sequences of ideals $\{I_t\}$ form a decreasing sequence of irreducible $\mathfrak{m}$-primary ideals cofinal with $\{\mathfrak{m}^t\}$. Moreover, the direct limit system $\lim_{t \to \infty} R/I_t \xrightarrow{\cdot x_1 \cdots x_d} R/I_{t+1}$ is isomorphic to $E_R(k)$. There is flexibility in choosing parameters when realizing the injective hull as a direct limit just described and it will be beneficial to choose the parameter sequence to satisfy some additional properties.

**Definition 2.1.** Let $(R, \mathfrak{m}, k)$ be a local catenary domain of dimension $d$, and let $J$ be an ideal of $R$ of pure height 1. We say that the sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ is suitable with respect to $J$ (or merely suitable, if $J$ is clear) if

1. $x_1, \ldots, x_d$ is a system of parameter for $R$,
2. $x_1 \in J$ and $x_2, \ldots, x_d$ are parameters for $R/J$,
3. if $J_p$ is principal for all minimal primes of $J$, then $J_{x_2}$ is principal,
4. if $J$ is principal in codimension 2, then $J_{x_3}$ is principal.

Observe that if $J \subseteq R$ is an ideal of pure height 1 which is principal in codimension 2 then there exists a parameter sequence which is suitable with respect to $J$.

**Lemma 2.2.** Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay local ring of prime characteristic $p > 0$ and of Krull dimension $d$. Let $J_1 \subseteq R$ be a choice of canonical ideal and $x_1, \ldots, x_d$ a suitable system of parameters. Make the following identifications of $E_R(k)$ and $H_{\mathfrak{m}}^{d-1}(R/J_1)$:

$$E_R(k) \cong \lim_{\to} \left( \frac{R}{(x_1^{t-1}J_1, x_2^1, \ldots, x_d^t)} \xrightarrow{\cdot x_1 \cdots x_d} \frac{R}{(x_1^{t}J_1, x_2^{t+1}, \ldots, x_d^{t+1})} \right)$$

$$H_{\mathfrak{m}}^{d-1}(R/J_1) \cong \lim_{\to} \left( \frac{R}{(J_1, x_2^1, \ldots, x_d^t)} \xrightarrow{\cdot x_2 \cdots x_d} \frac{R}{(J_1, x_2^{t+1}, \ldots, x_d^{t+1})} \right)$$
Then under the above identifications of \( E_R(k) \) and \( H_m^{d-1}(R/J_1) \) we have that
\[
0^*f_g^{E_R(k)} \simeq \lim_{\xrightarrow{t \to \infty}} \frac{R}{{(x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)^*}} \xrightarrow{x_1 \cdots x_d} \frac{R}{{(x_1^tJ_1, x_2^{t+1}, \ldots, x_d^{t+1})^*}}
\]
and
\[
0^*f_g^{H_m^{d-1}(R/J)} \simeq \lim_{\xrightarrow{t \to \infty}} \frac{R}{{(J_1, x_2^t, \ldots, x_d^t)^*}} \xrightarrow{x_2 \cdots x_d} \frac{R}{{(J_1, x_2^{t+1}, \ldots, x_d^{t+1})^*}}
\]

**Proof.** The containments \( \supseteq \) are clear by definition of finitistic tight closure. The containments \( \subseteq \) are also straightforward since the maps in the direct limit systems above are injective under the Cohen-Macaulay assumption. \( \square \)

**Lemma 2.3.** Let \((R, m, k)\) be a complete Cohen-Macaulay local normal domain of Krull dimension \(d\) and of prime characteristic \(p > 0\). Let \(J_1 \subset R\) be a choice of canonical ideal. Then \(0^*E_R(k) = 0^*E_R(k)\) if and only if \(0^*H_m^{d-1}(R/J_1) = 0^*H_m^{d-1}(R/J_1)\).

**Proof.** Let \(x_1, \ldots, x_d \in R\) be a suitable system of parameters with respect to \(J_1\), and identify the injective hull of the residue field as
\[
E_R(k) = \lim_{\xrightarrow{t \to \infty}} \frac{R}{{(x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)}} \xrightarrow{x_1 \cdots x_d} \frac{R}{{(x_1^tJ_1, x_2^{t+1}, \ldots, x_d^{t+1})}}.
\]

Suppose that \(0^*H_m^{d-1}(R/J_1) = 0^*f_g^{H_m^{d-1}(R/J_1)}\). Let \(e \in 0^*E_R(k)\). Without loss of generality suppose that \(e = r + (J_1, x_2, \ldots, x_d)\). Then there exists a \(c \in R^e\) such that \(c^p \cdot e^e = 0\) for all \(e \geq 1\). Equivalently, for every \(e \in \mathbb{N}\) there exists a \(t \in \mathbb{N}\) such that
\[
c^p (x_1 \cdots x_d)^{(t-1)p^e} \in (x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)^{[p^e]},
\]
in which case there exists an element \(s \in J_1^{[p^e]}\) such that
\[
(c^p (x_2 \cdots x_d)^{(t-1)p^e} - s)^p \cdot e^e \in (x_2^t, \ldots, x_d^t)^{[p^e]}.
\]
But \(x_1, x_2, \ldots, x_d\) is a regular sequence and therefore
\[
c^p (x_2 \cdots x_d)^{(t-1)p^e} - s \in (x_2^t, \ldots, x_d^t)^{[p^e]}
\]
and hence
\[
c^p (x_2 \cdots x_d)^{(t-1)p^e} \in (J_1, x_2^t, \ldots, x_d^t)^{[p^e]}.
\]
If we identify \(H_m^{d-1}(R/J_1)\) as
\[
\lim_{\xrightarrow{t \to \infty}} \frac{R}{{(J_1, x_2^t, \ldots, x_d^t)}} \xrightarrow{x_2 \cdots x_d} \frac{R}{{(J_1, x_2^{t+1}, \ldots, x_d^{t+1})}}
\]
then the above shows that the class of \(r + (J_1, x_2, \ldots, x_d)\) is an element of \(0^*H_m^{d-1}(R/J_1)\). Moreover, under the direct limit identification of \(H_m^{d-1}(R/J_1)\) we have by Lemma 2.2 that
\[
0^*H_m^{d-1}(R/J_1) = 0^*f_g^{H_m^{d-1}(R/J_1)} \simeq \lim_{\xrightarrow{t \to \infty}} \frac{R}{{(J_1, x_2^t, \ldots, x_d^t)^*}} \xrightarrow{x_2 \cdots x_d} \frac{R}{{(J_1, x_2^{t+1}, \ldots, x_d^{t+1})^*}}
\]
In particular, there exists a \(t \in \mathbb{N}\) such that \((x_2 \cdots x_d)^t r \in (J_1, x_2^{t+1}, \ldots, x_d^{t+1})^*\). It follows that \((x_1 x_2 \cdots x_d)^t r \in (x_1^t J_1, x_2^{t+1}, \ldots, x_d^{t+1})^*\) and therefore \(e \in 0^*E_R(k)\).
Conversely, suppose that $0^*_{E_R(k)} = 0^*_{E_R(k)}$ and let $\eta \in 0^*_{H^d_{m-1}(R/J_1)}$. Without loss of generality we may assume $\eta = r + (J_1, x_2, \ldots, x_d)$. Then there exists a $c \in R^\circ$ such that for every $e \in \mathbb{N}$ there exists a $t \in \mathbb{N}$ such that

$$c^{r^e}(x_2 \cdots x_d)^{(t-1)p^e} \in (J_1, x_2^t, \ldots, x_d^t)[p^e].$$

It then follows that

$$c^{r^e}(x_1x_2 \cdots x_d)^{(t-1)p^e} \in (x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)[p^e]$$

and therefore the element $r + (J_1, x_2, \ldots, x_d)$ of $E_R(k)$ is an element of $0^*_{E_R(k)}$. Under the direct limit identification of $E_R(k)$ we have by Lemma 2.2 that

$$0^*_{E_R(k)} = 0^*_{E_R(k)} \simeq \lim_{\rightarrow} \left( \left( \frac{x_1^{t-1}J_1, x_2^t, \ldots, x_d^t}{x_1^{t-1}J_1, x_2^t, \ldots, x_d^t} \right)_{\rightarrow} \left( \frac{x_1^tJ_1, x_2^{t+1}, \ldots, x_d^{t+1}}{x_1^tJ_1, x_2^{t+1}, \ldots, x_d^{t+1}} \right) \right).$$

Therefore there exists a $t \in \mathbb{N}$ such that $(x_1 \cdots x_d)^{tr} \in (x_1^tJ_1, x_2^{t+1}, \ldots, x_d^{t+1})^*$, i.e., there exists a $c \in R^\circ$ such that

$$c((x_1 \cdots x_d)^{tr})^{p^e} \in (x_1^tJ_1, x_2^{t+1}, \ldots, x_d^{t+1})[p^e]$$

for every $e \in \mathbb{N}$. Thus for every $e \in \mathbb{N}$ there exists a $s \in J[p^e]$ such that

$$c((x_2 \cdots x_d)^{tr})^{p^e} - s \in (x_1^{t+1}, \ldots, x_d^{t+1})[p^e].$$

But $x_1, \ldots, x_d$ is a regular sequence and it follows that

$$c((x_2 \cdots x_d)^{tr})^{p^e} \in (J, x_2^{t+1}, \ldots, x_d^{t+1})[p^e]$$

for every $e \in \mathbb{N}$. In particular, $(x_2 \cdots x_d)^{tr} \in (J_1, x_2^{t+1}, \ldots, x_d^{t+1})^*$ and therefore $\eta = (x_2 \cdots x_d)^{tr} + (J_1, x_2^{t+1}, \ldots, x_d^{t+1})$ is an element of $0^*_{H^d_{m-1}(R/J_1)}$. \hfill \Box

2.2. $S_2$-ification and higher Ext-modules. Though we do not directly use the results of [Dut13, Dut16], we would like to mention that important aspects of our techniques are inspired by these two articles. For example, suppose $(S, n, k)$ is a Cohen-Macaulay local domain of dimension $d$ and $M$ a finitely generated $S$-module such that $ht(\text{Ann}_S(M)) = h$. Let $(F_\bullet, \partial_\bullet)$ be the minimal free resolution of $M$, let $(-)^*$ denote $\text{Hom}_S(-, S)$, and consider the dual complex $(F^*_\bullet, \partial^*_\bullet)$. Because $ht(\text{Ann}_S(M)) = h$ we have that the following complex is exact:

$$0 \to F_0^* \xrightarrow{\partial_0^*} F_1^* \to \cdots \to F_{n-1}^* \xrightarrow{\partial_{n-1}^*} F_n^* \to \text{Coker}(\partial_n^*) \to 0.$$ 

In particular, $\text{depth}(\text{Coker}(\partial_n^*)) = d - h$. Moreover, there is a short exact sequence

$$0 \to \text{Ext}^h_S(M, S) \to \text{Coker}(\partial_n^*) \to \text{Im}(\partial_{n+1}^*) \to 0.$$ 

The module $\text{Im}(\partial_{n+1}^*)$ is torsion-free and therefore has depth at least 1. If $d - h \geq 2$ then $\text{Ext}^h_S(M, S)$ has depth at least 2. If $d - h = 1$ then the depth of $\text{Ext}^h_S(M, S)$ is 1. If $d - h = 0$ then $M$ is 0-dimensional. Therefore if $ht(\text{Ann}_S(M)) = h$ then $\text{Ext}^h_S(M, S)$ is an $(S_2)$-module over its support, an observation we record for future reference.

**Lemma 2.4.** Let $(S, n, k)$ be a Cohen-Macaulay local domain and $M$ a finitely generated $S$-module such that $ht(\text{Ann}_S(M)) = h$. Then $\text{Ext}^h_S(M, S)$ is an $(S_2)$-module over its support.
Continue to consider the ring $S$, the module $M$, and the resolution $(F_\bullet, \partial_\bullet)$ as above. Also consider the minimal free resolution $(G_\bullet, \delta_\bullet)$ of $\text{Ext}^h_S(M, S)$. If $\text{depth}(M) = d - h$ is maximal, then $\text{Ext}^h_S(M, S) = \text{Coker}(\delta_h)$ and therefore $(G_\bullet, \delta_\bullet)$ is the complex

$$0 \to F^*_0 \xrightarrow{\partial^*_0} F^*_1 \to \ldots \to F^*_{h-1} \xrightarrow{\partial^*_h} F^*_h \to 0.$$  

In particular, if $\text{depth}(M) = d - h$ then $\text{Ext}^h_S(\text{Ext}^h_S(M, S), S) \cong M$. Suppose $\text{depth}(M) < d - h$ and let $(F_\bullet^*, \partial_\bullet^*)_{tr}$ be the complex obtained by truncating $(F_\bullet^*, \partial_\bullet^*)$ at the $h$th spot. That is $(F_\bullet^*, \partial_1^*)_{tr}$ is the minimal free resolution of $\text{Coker}(\partial_h^*)$. Then the natural inclusion $\text{Ext}^h_S(M, S) \subseteq \text{Coker}(\partial_h^*)$ lifts to a map of complexes $(G_\bullet, \delta_\bullet) \to (F_\bullet^*, \partial_\bullet^*)_{tr}$ and therefore there is an induced map $M \to \text{Ext}^h_S(\text{Ext}^h_S(M, S), S)$.

**Lemma 2.5.** Let $(R, m, k)$ be a complete local domain of dimension at least 3 and $J \subseteq R$ a pure height 1 ideal. Suppose $(S, n, k)$ is a regular local ring mapping onto $R$, $R \cong S/P$, and $\text{ht}(P) = h$. Then for every integer $i$ the kernel of the natural map $R/J^i \to \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^i, S), S)$ is $J^{(i)}/J^i$. In particular, for every integer $i$ there is a natural inclusion $R/J^{(i)} \subseteq \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^i, S), S)$. Moreover, the natural inclusion $R/J^{(i)} \subseteq \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^i, S), S)$ is an isomorphism whenever localized at prime ideal $p \in V(J)$ such that $(R/J^{(i)})_p$ is Cohen-Macaulay.

**Proof.** It only remains to show that the kernel of $R/J^i \to \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^i, S), S)$ is $J^{(i)}/J^i$. But this follows from the observation that the map

$$R/J^i \to \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^i, S), S)$$

is an isomorphism when localized at any minimal component of $J$ by the discussion proceeding the statement of the lemma. □

We record a corollary of Lemma 2.5 for future reference.

**Corollary 2.6.** Let $(R, m, k)$ be a complete local Cohen-Macaulay domain, which is $\mathbb{Q}$-Gorenstein in codimension 2, and $J_1 \subseteq R$ a choice of canonical ideal. Let $m \in \mathbb{N}$ be an integer such that $J_1^{(m)}$ is principal in codimension 2. Suppose $(S, n, k)$ is a regular local ring mapping onto $R$, $R \cong S/P$, and $\text{ht}(P) = h$. Then for every integer $i$ the natural inclusion $R/J_1^{(m+i)} \to \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J_1^{(m+i)}, S), S)$ is an isomorphism whenever localized at a prime ideal of $R$ of height 2 or less.

**Proof.** Immediate by Lemma 2.5 since $J_1^{(m+i)} \cong J_1 R_p$ is a canonical ideal whenever $p$ is a prime of $R$ of height 2 or less. □

### 2.3. Rees algebras, symbolic Rees algebras, and analytic spread

Let $R$ be a Noetherian domain and $I \subseteq R$ an ideal. The Rees ring of $I$ is the blowup algebra

$$R[It] = R \oplus I \oplus I^2 \oplus \cdots.$$  

If all associated primes of $I$ are minimal and $W$ is the complement of the union of the prime components of $I$, then the $N$th symbolic power of $I$ is the ideal $I^{(N)} = I^N R_W \cap R$. The symbolic Rees ring of $I$ is the $R$-algebra

$$\mathcal{R}_I := R \oplus I \oplus I^{(2)} \oplus \cdots,$$
an $R$-algebra with the potential of being non-Noetherian, [Cut88, Ree58, Rob85]. We will typically be interested symbolic Rees rings associated to ideals of pure height 1.

Suppose further that $(R, m, k)$ is local. Then the analytic spread of $I$ is the Krull dimension of the fiber cone

$$k \otimes_R R[It] \cong k \oplus \frac{I}{mI} \oplus \frac{I^2}{m^2I} \oplus \cdots.$$ 

The analytic spread of a nonzero proper ideal $I$ is a natural number between 1 and dim($R$). If all associated primes of the ideal $I$ are minimal and the symbolic Rees ring $R_I$ is Noetherian then we can compare the analytic spread of $I^{(N)}$ with the analytic spread of $I$.

**Proposition 2.7.** Let $(R, m, k)$ be a excellent local Noetherian normal domain and $I \subseteq R$ an ideal without embedded components. Suppose that the analytic spread of $IR_P$ is no more than $\text{ht}(P) - 1$ for each prime $P \supseteq I$ which is not an associated prime of $I$. If the analytic spread of $I$ is $\ell$ then for each integer $N \in \mathbb{N}$ the analytic spread of $I^{(N)}$ is no more than $\ell$.

**Proof.** Under the assumptions of the proposition the symbolic Rees ring $R_I$ is a graded subalgebra of the normalization of $R[It]$, [CHS10, Theorem 1.1]. In particular, $R[It] \to R_I$ is finite. Hence the maps of the $N$th Veronese subalgebras

$$R[I^N t] \to R_I^{(N)}$$

are finite for each integer $N$. Observe that the $R$-algebra map above can be factored as

$$R[I^N t] \to R[I^{(N)} t] \to R_I^{(N)}.$$ 

Therefore the induced map of fiber cones

$$k \otimes R[I^N t] \to k \otimes R[I^{(N)} t]$$

are finite for each integer $N$. In particular, $k \otimes R[I^{(N)} t]$ has Krull dimension no more than the analytic spread of $I^N$ and the analytic spread of $I^N$ is equal to the analytic spread of $I$.\[4\]

Finite generation of symbolic Rees rings away from the maximal ideal of a local ring allows us to effectively compare ordinary and symbolic powers of an ideal.

**Proposition 2.8.** Let $(R, m, k)$ be a local domain and $I \subseteq R$ an ideal without embedded components. Suppose that for each $P \in \text{Spec}(R) - \{m\}$ that $R_P \otimes R_I$ is a Noetherian $R_P$-algebra. Then there exists an integer $N \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ the inclusion of ideals $I^{(N)i} \subseteq I^{(Ni)}$ agree when localized at any point of $\text{Spec}(R) - \{m\}$.

**Proof.** We are assuming $R \to R_I$ is finite type on each point of $\text{Spec}(R) - \{m\}$. But being finite type is an open condition on the scheme $\text{Spec}(R) - \{m\}$. Therefore for each point $P \in \text{Spec}(R) - \{m\}$ there is an $f \in m - P$ such that $R_f \to R_I \otimes R_f$ is finite type. Consider a cover $\text{Spec}(R_{f_1}) \cup \text{Spec}(R_{f_2}) \cup \cdots \cup \text{Spec}(R_{f_n})$ of $\text{Spec}(R) - \{m\}$. We can then choose integers $N_j$ such that $I^{(N_ji)} R_{f_j} = I^{(N_i)} R_{f_j}$ for each $1 \leq j \leq n$ and this is accomplished

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\[4\]Jonathan Montaño has shown to us Proposition 2.7 can be significantly generalized. It is possible to adapt the proof technique of Proposition 2.7 under the weaker assumptions that $R$ is assumed to be a domain which is analytically unramified and formally equidimensional. Under these assumptions the normalization of $R[It]$ is Noetherian and one can adapt the proof of [CHS10, Theorem 1.1] to this scenario.
by choosing integers $N_j$ so that the $N_j$th Veronese subalgebra of $R_{f_j} \otimes R_I$ is a standard graded $R_{f_j}$-algebra. We then take $N$ to be the least common multiple of the set of integers $\{N_j\}_{j=1}^n$. $\square$

Let $I \subseteq R$ be an ideal whose components have the same height. The collection of associated primes of the set of ideals $\{I^n\}_{n \in \mathbb{N}}$ is finite by [Bro79], see also [HS15]. The finite set of associated primes of the collection of ideals $\{I^n\}_{n \in \mathbb{N}}$ are known as the asymptotic associated primes of $I$. Suppose that $P_1, \ldots, P_n$ are the finitely many non-minimal asymptotic associated primes of $I$ and let $a = P_1 \cap \cdots \cap P_n$. Then for each integer $N \in \mathbb{N}$ we have that $I^{(N)} = (I^N : a^{\infty}) := \{r \in R \mid a^r \subseteq I^N \forall i \gg 0\}$. The analytic spreads of the collection of ideals $\{I^{(N)}R_P\}_{N \in \mathbb{N}, P \in V(a)}$ and finite generation of the symbolic Rees ring $R_I$ have an interesting connection.

**Theorem 2.9.** [CHS10, Theorem 1.1 and Theorem 1.5] Let $R$ be an excellent Noetherian normal domain of Krull dimension $d$ and $I \subseteq R$ an ideal without embedded components. Suppose $a \subseteq R$ is a reduced ideal of height at least 2. Then the following are equivalent:

1. The ring $\bigoplus (I^N : a^{\infty})$ is Noetherian;
2. There exists an integer $m$ so that for all $P \in V(a)$ the analytic spread of $(I^m : a^{\infty})R_P$ is no more than $\text{ht} P - 1$;
3. There is a containment of $R$-algebras $\bigoplus (I^N : a^{\infty}) \subseteq \overline{R[I]}$ where $\overline{R[I]}$ is the normalization of the Rees ring $R[I]$.

In particular, if $a$ is the intersection of the non-minimal asymptotic primes of $I$ then the symbolic Rees ring $R_I$ is Noetherian if and only if there exists an integer $m \in \mathbb{N}$ such that the analytic spread of $I^{(m)}R_P$ is no more than $\text{ht} P - 1$ at each $P \in V(a)$.

The criterion described in Theorem 2.9 to determine finite generation of symbolic Rees rings is originally due to Katz and Ratliff, [KR86, Theorem A and Corollary 1]. The reader interested in learning more about connections between finite generation of symbolic Rees rings and analytic spread will also be interested in [Sch86] and [DM19]. We also remark that finite generation of symbolic Rees rings is deeply rooted to progress in the minimal model program. This is because finite generation of certain symbolic Rees rings is equivalent to the existence of flips, see [KM98, Lemma 6.2 and Remark 6.3].

The following is a consequence of Theorem 2.9 and will be used in Section 4.

**Proposition 2.10.** Let $R$ be an excellent Noetherian normal domain. Suppose that $I \subseteq R$ an ideal of pure height 1 with analytic spread at most 2 and suppose that as an element of the divisor class group of $R$ the ideal $I$ is torsion in codimension 2. Then the symbolic Rees ring $R_I$ is Noetherian.

**Proof.** Let $a$ be the intersection of the asymptotic primes of $I$ of height at least 3 and let $b$ be the intersection of the asymptotic primes of $I$ of height 2. Then the $N$th symbolic power of the ideal $I$ is realized as $(I^N : a^{\infty}) : b^{\infty}$. The analytic spread of $I$ is at most 2 and the analytic spread of $I$ does not increase under localization. Therefore the $R$-algebra $\bigoplus (I^N : a^{\infty})$ is Noetherian by Theorem 2.9. Hence there exists an integer $m \in \mathbb{N}$ such that $\bigoplus (I^{mN} : a^{\infty})$ is a standard graded $R$-algebra. Equivalently, for each integer $N \in \mathbb{N}$ we have that $(I^m : a^{\infty})^N = (I^{mN} : a^{\infty})$. Let $\bar{I} = I^m : a^{\infty}$. Because we are assuming $I$ is torsion as
an element in the divisor class group in codimension 2 we can choose an integer \( n \) such that \( I^{(mn)} = \mathfrak{I}^n : \mathfrak{b}^\infty \) is principal in codimension 2. In particular, the analytic spread of \( \mathfrak{I}^n : \mathfrak{b}^\infty \) is 1 at each of the components of \( \mathfrak{b} \). Therefore the symbolic Rees ring \( \mathcal{R}_{I^{(mn)}} = \bigoplus_{N \in \mathbb{N}} \mathfrak{I}^N : \mathfrak{b}^\infty \) is Noetherian by a second application of Theorem 2.9. It then follows that the symbolic Rees ring \( \mathcal{R}_I \) is Noetherian since the \( mn \)-th Veronese embedding of \( \mathcal{R}_I \) is Noetherian, see the proof of [HHT07, Theorem 2.1].

Another important concept surrounding the theory of analytic spread and reductions is the notion of a reduction number. Let \( R \) be a Noetherian ring and \( J \subseteq I \) ideals such that \( J \) forms a reduction of \( I \). The reduction number of \( I \) with respect \( J \) is the least integer \( N \) such that \( JI^N = I^{N+1} \). A theorem of Hoa allows us to relate reduction numbers with the analytic spread of an ideal via understanding properties of the graded ring \( \text{gr}_I(R) = \oplus I^i/I^{i+1} \). But first, recall that if \( S = S_0 \oplus S_1 \oplus \cdots \) is a graded ring and \( S_i \) is the irrelevant ideal then the \( i \)th \( a \)-invariant of \( S \) is denoted by \( a_i(S) \) and is the largest degree of support of the local cohomology module \( H^i_{S_k}(S) \).

**Theorem 2.11.** [Hoa93, Theorem 2.1] Let \( (R, m, k) \) be a Noetherian local ring and \( I \subseteq R \) an ideal. Let \( \ell \) be the analytic spread of \( I \) and suppose that \( a_\ell(\text{gr}_I(R)) < 0 \). Then for all integers \( n \gg 0 \) and reductions \( J \) of \( I^n \) the ideal \( I^n \) has reduction number with respect to \( J \) no more than \( \ell - 1 \).

As a consequence to Theorem 2.11 we can effectively estimate the reduction numbers of large powers of pure height 1 ideals of a strongly \( F \)-regular ring.

**Theorem 2.12.** Let \( (R, m, k) \) be a strongly \( F \)-regular and \( F \)-finite local ring of prime characteristic \( p > 0 \) and dimension \( d \geq 2 \). Suppose further that \( I \subseteq R \) is an ideal of pure height 1 with the property that \( I^n = I^{(n)} \) for all \( n \in \mathbb{N} \). If \( I \) has analytic spread \( \ell \geq 2 \) then for all \( n \gg 0 \) the reduction number of \( I^n \) with respect to any reduction is no more than \( \ell - 1 \).

**Proof.** By Theorem 2.11 it is enough to show that \( a_\ell(\text{gr}_I(R)) < 0 \). In fact, we will show that \( a_i(\text{gr}_I(R)) < 0 \) for all \( 2 \leq i \leq d \). But first, we will show \( a_i(R[I]) < 0 \) for all \( 2 \leq i \leq d \). Because \( R[I] = \mathcal{R}_I \) we have that \( S := R[I] \) is a strongly \( F \)-regular graded \( R \)-algebra by [CEMS18, Lemma 3.1], see also [Wat94, Theorem 0.1] and [MPST19, Main Theorem]. The cohomology groups \( H^i_{S_k}(S) \) are only supported in finitely many positive degrees. Indeed, let \( X = \text{Proj}(S) \) so that \( H^i_{S_k}(S) \cong H^{i-1}(X, \mathcal{O}_X) \) for all \( i \geq 2 \), see [ILL+07, Theorem 12.41], and therefore \( [H^i_{S_k}(S)]_N = H^{i-1}(X, \mathcal{O}_X(N)) = 0 \) for all \( N \gg 0 \) by Serre vanishing, [Har77, Theorem 5.2]. It follows that there exists a homogeneous positive degree element \( c \in S \) such that \( c[H^i_{S_k}(S)]_\geq 0 = 0 \). Because \( S \) is strongly \( F \)-regular the \( S \)-linear maps \( S \rightarrow F_c^e \rightarrow F_c^e S \) are pure for all \( e \gg 0 \). Therefore the \( e \)th Frobenius action on \( H^i_{S_k}(S) \) followed by multiplying by \( c \), which is the map realized by tensoring the pure map \( S \rightarrow F_c^e \rightarrow F_c^e S \) with \( H^i_{S_k}(S) \), are injective. But the \( e \)th Frobenius action of \( H^i_{S_k}(S) \) maps elements of degree \( n \) to elements of degree \( np^e \). Furthermore, \( c \) was chosen to annihilate elements of non-negative degree and therefore \( H^i_{S_k}(S) \) can only be supported in negative degree.

The ring \( S = R[I] \) is Cohen-Macaulay and therefore \( a_d(\text{gr}_I(R)) < 0 \) by [Hoa93, Theorem 3.1]. By [Tru98, Theorem 3.1 (ii)] we have that \( a_i(\text{gr}_I(R)) = a_i(S) \) whenever \( a_i(\text{gr}_I(R)) \geq
An easy descending induction argument now tells us that $a_i(\text{gr}_f(R)) < 0$ for all $2 \leq i \leq d$ and this completes the proof of the theorem.

3. Koszul cohomology, local cohomology, and local cohomology bounds

In this section $R$ denotes a commutative Noetherian ring. Unless stated otherwise, we do not make any assumptions on the characteristic of $R$. Our study of local cohomology modules is centered around the realization of local cohomology as a direct limit system of Koszul cohomologies. We are interested in understanding at what point in a direct limit system that an element of a Koszul cohomology group representing the zero element of a local cohomology group becomes zero. Key to our study of local cohomology is the notion of a local cohomology group relative to a sequence of elements defined below in Definition 3.1.

3.1. Definition of local cohomology bound. Suppose $M$ is a module over a ring $R$ and $\underline{x} = x_1, \ldots, x_d$ a sequence of elements. Then for each integer $j \in \mathbb{N}$ we let $x_j = x_1^j, \ldots, x_d^j$ and for each pair of integers $j_1 \leq j_2$ let $\alpha_{M;\underline{x};j_1;j_2}$ denote the natural map of Koszul cocomplexes

$$K^\bullet(\underline{x}^{j_1}; M) \xrightarrow{\alpha_{M;\underline{x};j_1;j_2}} K^\bullet(\underline{x}^{j_2}; M).$$

The map of cocomplexes $\alpha_{M;\underline{x};j_1;j_2}$ is realized as the following tensor product of maps of Koszul cocomplexes on one element:

$$\alpha_{M;\underline{x};j_1;j_2} \cong \alpha_{R;\underline{x}^{j_1};j_1;j_2} \otimes \alpha_{R;\underline{x}^{j_2};j_1;j_2} \otimes \cdots \otimes \alpha_{R;\underline{x}^{j_2};j_1;j_2} \otimes M.$$

We let $\alpha_{M;\underline{x};j_1;j_2}$ denote the induced map of Koszul cohomologies

$$H^i(\underline{x}^{j_1}; M) \xrightarrow{\alpha_{M;\underline{x};j_1;j_2}} H^i(\underline{x}^{j_2}; M).$$

More specifically, suppose $j_1 = j$ and $j_2 = j + k$ and consider the Koszul cocomplexes $K^\bullet(\underline{x}^j; M)$ and $K^\bullet(\underline{x}^{j+k}; M)$. Then the cokernel of the $d$th map of these cocomplexes are $M/(\underline{x}^j)M$ and $M/(\underline{x}^{j+k})M$ respectively. Let $\alpha_{j,k}^i : K^\bullet(\underline{x}^j; M) \to K^\bullet(\underline{x}^{j+k}; M)$ be the natural choice of map of cocomplexes lifting the map $M/(\underline{x}^j)M \xrightarrow{\{x_1, \ldots, x_d\}^k} M/(\underline{x}^{j+k})M$. Then $\alpha_{M;\underline{x};j;j+k}$ is the induced map $\alpha_{j,k}$ on Koszul cohomology. In particular,

$$\lim_{j_1 \leq j_2} \left( H^i(\underline{x}^{j_1}; M) \xrightarrow{\alpha_{M;\underline{x};j_1;j_2}} H^i(\underline{x}^{j_2}; M) \right) \cong H^i_{\underline{x}}(A)(M)$$

by [BH93, Theorem 3.5.6].

Denote by $\alpha_{M;\underline{x};j;\infty}$ the natural map

$$H^i(\underline{x}^j; M) \xrightarrow{\alpha_{M;\underline{x};j;\infty}} H^i_{\underline{x}}(A)(M).$$

Observe that $\eta \in \text{Ker}(\alpha_{M;\underline{x};j;\infty})$ if and only if there exists some $k \geq 0$ such that $\eta \in \text{Ker}(\alpha_{M;\underline{x};j;j+k})$. If $\eta \in \text{Ker}(\alpha_{M;\underline{x};j;\infty})$ we let

$$\epsilon^i_{\underline{x}}(\eta) = \min \{k \mid \eta \in \text{Ker}(\alpha_{M;\underline{x};j;j+k})\}.$$
Definition 3.1. Let $R$ be a ring, $\underline{x} = x_1, \ldots, x_d$ a sequence of elements in $R$, and $M$ an $R$-module. The $i$th local cohomology bound of $M$ with respect to the sequence of elements $\underline{x}$ is

$$\text{lcb}_i(\underline{x}; M) = \sup\{\epsilon^i_{\underline{x}}(\eta) \mid \eta \in \text{Ker}(\alpha^i_{M}[x_1, \ldots, x_d]) \text{ for some } j\} \in \mathbb{N} \cup \{\infty\}.$$ 

Observe that if $M$ is an $R$-module and $\underline{x}$ is a sequence of elements, then $\text{lcb}_i(\underline{x}; M) = N < \infty$ simply means that if $\eta \in H^i(\underline{x}; M)$ represents the 0-element in the direct limit

$$\lim_{j_1 \leq j_2} \left( H^i(\underline{x}_1; M) \xrightarrow{\alpha^i_{M[x_1, x_2, \ldots, x_d]}} H^i(\underline{x}_2; M) \right) \cong H^i(\underline{x}; M)$$

then $\alpha^i_{M[x_1, \ldots, x_d]}(\eta)$ is the 0-element of the Koszul cohomology group $H^i(\underline{x}; M)$. Therefore finite local cohomology bounds correspond to a uniform bound of annihilation of zero elements in a choice of direct limit system defining a local cohomology module. It would be interesting to know when local cohomology bounds are finite.

3.2. Basic properties of local cohomology bounds. Our study of local cohomology bounds begins with two elementary, yet useful, observations.

Lemma 3.2. Let $R$ be a commutative Noetherian ring, $M$ an $R$-module, and $\underline{x} = x_1, \ldots, x_d$ a sequence of elements. Then $\text{lcb}_i(\underline{x}; M) \leq jm$ for some integers $j, m$ if and only if $\text{lcb}_i(\underline{x}; M) \leq m$ where $\underline{x}_j$ is the sequence of elements $x_1^j, \ldots, x_d^j$.

Proof. One only has to observe that $\alpha^i_{M[\underline{x}_1, x_2, \ldots, x_d]; \underline{x}_2, \ldots, x_d]}$.

If $x_1, \ldots, x_d$ is a sequence of elements in a ring $R$ and if $x_1 M = 0$ for some $R$-module $M$ then the short exact sequence of Koszul complexes

$$0 \to K^*(x_1^d, \ldots, x_d^d; M) \to K^*(x_1, x_2, \ldots, x_d; M) \to K^*(x_2, \ldots, x_d; M) \to 0$$

is split and therefore $H^i(x_1, x_2, \ldots, x_d; M) \cong H^i(x_2, \ldots, x_d; M) \oplus H^{i-1}(x_2, \ldots, x_d; M)$. The content of the following lemma is a description of the behavior of the maps $\alpha^i_{M[x_1, x_2, \ldots, x_d]; x_1, x_2, \ldots, x_d]}$ with respect to these isomorphisms of Koszul cohomologies.

Lemma 3.3. Let $R$ be a commutative Noetherian ring, $M$ an $R$-module, and $x_1, x_2, \ldots, x_d$ a sequence of elements such that $x_1 M = 0$. If $i, j, k \in \mathbb{N}$ then

$$H^i(x_1^j, x_2^j, \ldots, x_d^j; M) \cong H^i(x_2^j, \ldots, x_d^j, M) \oplus H^{i-1}(x_2, \ldots, x_d; M)$$

and the map $\alpha_{M[x_1, x_2, \ldots, x_d]; x_1, x_2, \ldots, x_d]}$ is the direct sum of $\alpha^i_{M[x_2, \ldots, x_d]; x_1, x_2, \ldots, x_d]}$ and the 0-map.

Proof. Let $(F^\bullet, \partial^\bullet)$ be the Koszul complex $K^*(x_2^d, \ldots, y_d R)$ and let $(G^\bullet, \delta^\bullet)$ be the Koszul complex $K^*(x_1^d; R)$. Let

$$(L^\bullet, \gamma^\bullet) = K^*(x_1^d, x_2^j, \ldots, x_d^j; R) \cong K^*(x_2^j, \ldots, x_d^j, M) \otimes K^*(x_1^d; R).$$

Then $L^i \cong (F^i \otimes G^0) \oplus (F^{i-1} \otimes G^1) \cong F^i \oplus F^{i-1}$. We abuse notation and let $\cdot x_1^i$ denote the multiplication map on $F^i$. Then up to sign on $\cdot x_1^i$ the map $\epsilon^i$ can be thought of as

$$\gamma^i = \begin{pmatrix} \partial^i & 0 \\ \cdot x_1^i & \partial^i \end{pmatrix} : F^i \oplus F^{i-1} \to F^{i+1} \oplus F^i.$$
In particular, if we apply $- \otimes_R M$ the map $\cdot x^j \otimes M$ is the 0-map and therefore $i$th map of the Koszul cocomplex $K^i(x_1^j, x_2^j, \ldots, x_d^j; M)$ is the direct sum of maps $(\partial^i \otimes M) \oplus (\partial^{i-1} \otimes M)$.

In particular
\[ H^i(x_1^j, x_2^j, \ldots, x_d^j; M) \cong H^i(x_2^j, \ldots, x_d^j; M) \oplus H^{i-1}(x_2^j, \ldots, x_d^j; M). \]

To see that $\alpha_{M; x_1, x_2, \ldots, x_d; j; j+k}$ is the direct sum of $\alpha_{M; x_2, \ldots, x_d; j; j+k}$ and the 0-map is similar to above argument but uses the fact that
\[ \tilde{\alpha}^j_{M; x_1, x_2, \ldots, x_d; j; j+k} = \alpha^j_{R; x_2, \ldots, x_d; j; j+k} \otimes \alpha^j_{R; x_1; j; j+k} \otimes M \]
and $\tilde{\alpha}^j_{R; x_1; j; j+k} \otimes M = 0.$

A particularly useful corollary of Lemma 3.3 is the following:

**Corollary 3.4.** Let $R$ be a commutative Noetherian ring and $M$ an $R$-module. Suppose $x_1, \ldots, x_d$ is a sequence of elements and $(x_1, \ldots, x_{d-i})M = 0$. If $j, k \in \mathbb{N}$ then
\[ \alpha^j_{M; x_1, \ldots, x_d; j; j+k} : H^j(x_1^j, \ldots, x_d^j; M) \to H^j(x_1^{j+k}, \ldots, x_d^{j+k}; M) \]
is the 0-map for all $\ell \geq i + 1$. In particular, $\text{lcb}_\ell(x_1, \ldots, x_d; M) = 1$ for all $\ell \geq i + 1$.

**Proof.** By multiple applications of Lemma 3.3 it is enough to observe that
\[ H^\ell(x_{d-i+1}^j, \ldots, x_d^j; M) = 0. \]

This is clearly the case since $x_{d-i+1}^j, \ldots, x_d^j$ is a list of $i$ elements and we are examining an $\ell \geq i + 1$ Koszul cohomology of $M$ with respect to this sequence. \hfill \Box

Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of $R$-modules. The next two properties of local cohomology bounds we record allow us to compare the local cohomology bounds of the modules appearing in the short exact sequence. Proposition 3.5 allows us to effectively compare the local cohomology bounds of two of the terms in the sequence provided a subset of the elements in the sequence of elements defining Koszul cohomology annihilates the third. Proposition 3.6 compares the the local cohomology bounds of two of the terms in the short exact whenever the sequence of elements defining Koszul cohomology is a regular sequence on the third module.

**Proposition 3.5.** Let $(R, m, k)$ be a local ring and
\[ 0 \to M_1 \to M_2 \to M_3 \to 0 \]
a short exact sequence of finitely generated $R$-modules. Let $x = x_1, \ldots, x_d$ be a sequence of elements of $R$.

(1) If $(x_1, \ldots, x_{d-j})M_1 = 0$ then for all $\ell \geq j + 1$
\[ \text{lcb}_\ell(x; M_2) \leq \text{lcb}_\ell(x; M_3) + 1. \]

(2) If $(x_1, \ldots, x_{d-j})M_2 = 0$ then for all $\ell \geq j + 1$
\[ \text{lcb}_\ell(x; M_3) \leq \text{lcb}_{\ell+1}(x; M_1) + 1. \]

(3) If $(x_1, \ldots, x_{d-j})M_3 = 0$ then for all $\ell \geq j + 1$
\[ \text{lcb}_\ell(x; M_1) \leq \text{lcb}_\ell(x; M_2) + 1. \]
Proof. For each integer \( j \in \mathbb{N} \) let \( \mathcal{x}^j \) denote the sequence of elements \( x_1^j, x_2^j, \ldots, x_d^j \). For (1) we consider the following commutative diagram, whose middle row is exact:

\[
\begin{array}{c}
H^\ell(\mathcal{x}; M_2) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}; M_3) \\
\downarrow \alpha_{M_2, M_3}^j \downarrow \uparrow \alpha_{M_3, M_3}^j \\
H^\ell(\mathcal{x}^{j+k}; M_1) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}^{j+k}; M_2) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}^{j+k}; M_3) \\
\downarrow \alpha_{M_1, M_3}^{j+k} \downarrow \uparrow \alpha_{M_2, M_3}^{j+k} \downarrow \\
H^\ell(\mathcal{x}^{j+k+1}; M_1) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}^{j+k+1}; M_2)
\end{array}
\]

By Corollary 3.4 the map \( \alpha_{M_1, M_3}^{j+k+1} \) is the 0-map for all \( \ell \geq j + 1 \). A straightforward diagram chase of the above diagram, which follows an element \( \eta \in \text{Ker}(\alpha_{M_2, M_3}^j) \) for some \( k \), shows that \( \eta \in \text{Ker}(\alpha_{M_2, M_3}^j) \) whenever \( k \geq \text{lcb}_d(\mathcal{x}; M_3) \). In particular, \( \text{lcb}_d(\mathcal{x}; M_2) \leq \text{lcb}_d(\mathcal{x}; M_3) + 1 \).

Statements (2) and (3) follow in a similar manner. For (2) one needs to consider the commutative diagrams

\[
\begin{array}{c}
H^\ell(\mathcal{x}; M_3) \rightarrow \rightarrow \rightarrow H^{\ell+1}(\mathcal{x}; M_1) \\
\downarrow \alpha_{M_3, M_1}^j \downarrow \uparrow \alpha_{M_3, M_3}^j \\
H^\ell(\mathcal{x}^{j+k}; M_2) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}^{j+k}; M_3) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}^{j+k}; M_1) \\
\downarrow \alpha_{M_2, M_3}^{j+k} \downarrow \uparrow \alpha_{M_2, M_2}^{j+k} \downarrow \\
H^\ell(\mathcal{x}^{j+k+1}; M_2) \rightarrow \rightarrow \rightarrow H^\ell(\mathcal{x}^{j+k+1}; M_3)
\end{array}
\]

and invoke Corollary 3.4 to know that \( \alpha_{M_2, M_3}^{j+k+1} \) is the 0-map for all \( \ell \geq j + 1 \).

For (3) a diagram chase of the commutative diagram

\[
\begin{array}{c}
H^\ell(\mathcal{x}; M_1) \rightarrow \rightarrow \rightarrow H^{\ell-1}(\mathcal{x}; M_2) \\
\downarrow \alpha_{M_1, M_2}^j \downarrow \uparrow \alpha_{M_2, M_2}^j \\
H^{\ell-1}(\mathcal{x}^{j+k}; M_3) \rightarrow \rightarrow \rightarrow H^{\ell-1}(\mathcal{x}^{j+k}; M_1) \rightarrow \rightarrow \rightarrow H^{\ell-1}(\mathcal{x}^{j+k}; M_2) \\
\downarrow \alpha_{M_3, M_2}^{j+k} \downarrow \uparrow \alpha_{M_3, M_3}^{j+k} \downarrow \\
H^{\ell-1}(\mathcal{x}^{j+k+1}; M_3) \rightarrow \rightarrow \rightarrow H^{\ell-1}(\mathcal{x}^{j+k+1}; M_1)
\end{array}
\]

and knowing \( \alpha_{M_3, M_3}^{j+k+1} \) is the 0-map whenever \( \ell - 1 \geq j \) is all that is needed. \( \square \)

**Proposition 3.6.** Let \( R \) be a commutative Noetherian ring, \( 0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \) a short exact sequence of \( R \)-modules, and \( \mathcal{x} = x_1, \ldots, x_d \) a sequence of elements in \( R \).

1. If \( \mathcal{x} \) is a regular sequence on \( M_1 \) then \( \text{lcb}_i(\mathcal{x}; M_2) = \text{lcb}_i(\mathcal{x}; M_3) \) for all \( i \leq d - 1 \).
2. If \( \mathcal{x} \) is a regular sequence on \( M_2 \) then \( \text{lcb}_i(\mathcal{x}; M_3) = \text{lcb}_{i+1}(\mathcal{x}; M_1) \) for all \( i \leq d - 1 \).
3. If \( \mathcal{x} \) is a regular sequence on \( M_3 \) then \( \text{lcb}_i(\mathcal{x}; M_1) = \text{lcb}_i(\mathcal{x}; M_2) \) for all \( i \leq d \).
Proof. Proof of (1): For $i < d$ we have $H^i(x^j; M_1) = 0$ and therefore if $i \leq d - 2$ there are commutative diagrams

$$
H^i(x^j; M_2) \xrightarrow{\cong} H^i(x^j; M_3)
$$

whose horizontal arrows are isomorphisms. It readily follows that $\text{lcb}_i(x^j; M_2) = \text{lcb}_i(x^j; M_3)$ whenever $i \leq d - 2$. Because $x$ is a regular sequence on $M_1$ we have that the maps $\alpha^d_{M_1, x^i,j,j+k}$ are injective. Consider the following commutative diagrams whose rows are exact:

$$
0 \xrightarrow{} H^{d-1}(x^j; M_2) \xrightarrow{\pi_j} H^{d-1}(x^j; M_3) \xrightarrow{\delta_j} H^d(x^j; M_1)
$$

If $\eta \in \text{Ker}(\alpha^{d-1}_{M_2, x^j,j,j+k})$ then $\pi_j(\eta) \in \text{Ker}(\alpha^{d-1}_{M_3, x^j,j,j+k})$. The maps $\pi_j$ are injective. Therefore $\alpha^{d-1}_{M_2, x^j,j,j+k}(\eta) = 0$ whenever $k \geq \text{lcb}_{d-1}(x; M_3)$ and hence $\text{lcb}_{d-1}(x; M_2) \leq \text{lcb}_{d-1}(x; M_3)$.

To show that $\text{lcb}_{d-1}(x; M_2) \geq \text{lcb}_{d-1}(x; M_3)$ consider an element $\eta \in \text{Ker}(\alpha^{d-1}_{M_3, x^j,j,j+k})$. Then $\delta_j(\eta) \in \text{Ker}(\alpha^{d}_{M_1, x^j,j,j+k})$. But the maps $\alpha^{d}_{M_1, x^j,j,j+k}$ are injective and therefore $\delta_j(\eta) = 0$. In particular, $\eta = \pi_j(\eta')$ for some $\eta' \in H^{d-1}(x^j; M_2)$. The maps $\pi_j$ are all injective. Therefore $\eta' \in \text{Ker}(\alpha^{d-1}_{M_1, x^j,j,j+k})$ and it follows that $\alpha^{d-1}_{M_1, x^j,j,j+k}(\eta) = 0$ whenever $k \geq \text{lcb}_{d-1}(x; M_2)$. Therefore $\text{lcb}_{d-1}(x; M_2) \geq \text{lcb}_{d-1}(x; M_3)$ and hence $\text{lcb}_{d-1}(x; M_2) = \text{lcb}_{d-1}(x; M_3)$. This completes the proof of (1).

Proof of (2): Because we are assuming that $x$ is a regular sequence on $M_2$ it follows that $H^i(x^j; M_2) = 0$ whenever $i \leq d - 1$ and therefore if $i \leq d - 2$ there are commutative diagrams

$$
H^i(x^j; M_3) \xrightarrow{\cong} H^{i+1}(x^j; M_1)
$$

whose horizontal arrows are isomorphisms. It easily follows that $\text{lcb}_i(x^j; M_3) = \text{lcb}_{i+1}(x^j; M_1)$ whenever $i \leq d - 2$. To verify that $\text{lcb}_{d-1}(x; M_3) = \text{lcb}_d(x; M_1)$ consider the following commutative diagrams:

$$
0 \xrightarrow{} H^{d-1}(x^j; M_3) \xrightarrow{\delta_j} H^d(x^j; M_1) \xrightarrow{i_j} H^d(x^j; M_2)
$$

$$
0 \xrightarrow{} H^{d-1}(x^{j+k}; M_3) \xrightarrow{\delta_{j+k}} H^d(x^{j+k}; M_1) \xrightarrow{i_{j+k}} H^d(x^{j+k}; M_2)
$$

Similar to the proof of (1), a simple diagram chase and utilizing the injectivity of the maps $\delta_j, \delta_{j+k}$, and $\alpha^d_{M_2, x^j,j,j+k}$ will imply $\text{lcb}_{d-1}(x; M_3) = \text{lcb}_d(x; M_1)$.
Proof of (3): Similar to the proofs of (1) and (2), if $i \leq d-1$ there are commutative squares

$$H^i(x^j; M_1) \xrightarrow{\alpha} H^i(x^j; M_2)$$

whose horizontal arrows are isomorphisms. There will also be commutative diagrams

$$0 \longrightarrow H^d(x^j; M_1) \xrightarrow{i_j} H^d(x^j; M_2) \xrightarrow{\pi_j} H^d(x^j; M_3)$$

Utilizing the commutative square above will show $lcb_i(x^j; M_1) = lcb_i(x^j; M_2)$ whenever $i \leq d-1$. A simple diagram chase of the second diagram and utilizing the injectivity of the maps $i_j, i_{j+k}$, and $\alpha_{M_3,x^j;j+k}$ imply $lcb_d(x^j; M_1) = lcb_d(x^j; M_2)$. \hfill \Box

4. Equality of test ideals

The proof of Theorem B goes as follows: Theorem 4.2 shows that the test ideals of a local ring agree provided there exists a system of parameters for $R$ which satisfy some technical conditions. Proposition 4.3, Proposition 4.4, and Proposition 4.5 can be put together to show that a system of parameters satisfies the hypotheses of Theorem 4.2 provided that the parameter sequence annihilates a family of Ext-modules in a controlled way. Theorem 4.6 provides to us a suitable system of parameters so that the desired annihilation properties of the previous propositions are met under the assumptions of Theorem B.

4.1. Sufficient conditions which imply equality of test ideals. The content of the following lemma can be pieced together by work of the first author in [Abe02]. We refer the reader to [PT18, Lemma 6.7] for a direct presentation of the lemma.\footnote{In [PT18, Lemma 6.7] there is an assumption that $R$ is complete. But observe that since $R \rightarrow \hat{R}$ is faithfully flat the claims of the lemma can be checked after completion.}

Lemma 4.1. Suppose that $(R, m, k)$ is a Cohen-Macaulay local normal domain of dimension $d$, and $J \subseteq R$ an ideal of pure height $1$. Let $x_1, \ldots, x_d \in R$ be a suitable system of parameters for $R$ with respect to $J$, and fix $e \in \mathbb{N}$.

(1) If $x_2J \subseteq a_2R$ for some $a_2 \in J$, then for any non-negative integers $N_2, \ldots, N_d$ with $N_2 \geq 2$, we have that

$$((J^p), x_2^{N_2p^e}, x_3^{N_3p^e}, \ldots, x_d^{Ndp^e}) : x_2^{(N_2-1)p^e} = ((J^p), x_2^{2p^e}, x_3^{np^e}, \ldots, x_d^{dp^e}) : x_2^{(N_2-1)p^e} = ((J^p), x_2^{2p^e}, x_3^{np^e}, \ldots, x_d^{dp^e}) : x_2^{2p^e}.$$
Suppose \( x_1, \ldots, x_d \) is a suitable system of parameters with respect to \( J_1 \) such that the following conditions are met:

- The localized ideals \( J_1R_{x_2} \) and \( J_1^{(m)}R_{x_3} \) are principal ideals in their respective localizations;
- For each \( i, s \in \mathbb{N} \) there exists an integer \( \ell \) such that
  \[
  (J_1^{(m+1)}, x_2^i, x_3^i, x_4^i, \ldots, x_d^i) : x_d^{i(s-1)} \subseteq (J_1^{(m+1)}, x_2^i, x_3^i, x_4^i, \ldots, x_d^i) : x_d^{i(s-1)}.
  \]

Then \( 0^s_{E_R(k)} = 0^s_{E_R(k)} \).

**Proof.** Identify \( H_{m-1}^d(R/J_1) \) as

\[
\lim_{s} \left( H_{m-1}^d \left( x_2^s, x_3^s, \ldots, x_d^s ; R/J_1 \right) \xrightarrow{\cdot \cdot \cdot \cdot \cdot} \right) H_{m-1}^d \left( x_2^{s+1}, x_3^{s+1}, \ldots, x_d^{s+1} ; R/J_1 \right) .
\]

In particular, \( F_R^d(H_{m-1}^d(R/J_1)) \) is isomorphic to the following direct limit:

\[
\lim_{s} \left( H_{m-1}^d \left( x_2^s, x_3^s, \ldots, x_d^s ; R/J_1 \right) \xrightarrow{\cdot \cdot \cdot \cdot \cdot} \right) H_{m-1}^d \left( x_2^{s+1}, x_3^{s+1}, \ldots, x_d^{s+1} ; R/J_1 \right) .
\]

Suppose that \( \eta \in 0^h_{H_{m-1}^d(R/J_1)} \). Any sequence of the form \( x_2^N, x_3^N, \ldots, x_d^N \) will still satisfy the above conditions. So without loss of generality we may assume that \( \eta \) is represented by the class of the element \( r + (x_2, x_3, \ldots, x_d) \). To show that \( \eta \in 0^h_{H_{m-1}^d(R/J_1)} \) it is enough to show there exists an integer \( N \) such that \( (x_2x_3 \ldots x_d)^N r \in (J_1, x_2^N, x_3^N, \ldots, x_d^N)^s \) by Lemma 2.2.

Suppose \( c \in R^e \) is a test element. Then for every \( e \in \mathbb{N} \) there exists a \( s \in \mathbb{N} \) such that

\[
( c_2^p (x_2x_3 \ldots x_d)^{(s-1)p} ) \in (J_1, x_2^s, x_3^s, \ldots, x_d^s)^{p^e}.
\]

By (2) of Lemma 4.1 we have that

\[
( c_2^{m} (r x)^p (x_2x_4 \ldots x_d)^{(s-1)p} ) \in (J_1^{(p^e)}, x_2^{2p^e}, x_3^{2p^e}, x_4^{2p^e}, \ldots, x_d^{2p^e}).
\]

It then follows by (1) of Lemma 4.1 that

\[
( c_2^{m} (x_2x_3r)^p (x_4 \ldots x_d)^{(s-1)p} ) \in (J_1^{[p^e]}, x_2^{p^e}, x_3^{p^e}, x_4^{p^e}, \ldots, x_d^{p^e}).
\]

Observe that \( p^e \geq m(\lfloor \frac{e}{2} \rfloor - 1) + 1 \) and hence \( J_1^{[p^e]} \subseteq J_1^{(m(\lfloor \frac{e}{2} \rfloor - 1) + 1)} \). Therefore

\[
( c_2^{m} (x_2x_3r)^p (x_4 \ldots x_d)^{(s-1)p} ) \in (J_1^{(m(\lfloor \frac{e}{2} \rfloor - 1) + 1)}, x_2^{p^e}, x_3^{p^e}, x_4^{p^e}, \ldots, x_d^{p^e}).
\]
By assumption there exists an integer \( \ell \), which depends on \( p^e \), such that
\[
\ell x_1^m (x_2 x_3 r)^p \cdot (x_2 x_3^2)^{(p^e+3)-p^e} \cdot (x_4 \cdots x_d)^{3(p^e+3)-p^e} \in \\
(J_1^{m(\ell p^e+1)} x_2^{2(p^e+3)}, x_3^{2(p^e+3)}, x_4^{2(p^e+3)}, \ldots, x_d^{2(p^e+3)}).
\]

Multiplying by \((x_4 \cdots x_d)^{2(p^e-3)}\) we find that
\[
\ell x_1^m (x_2 x_3 (x_4 \cdots x_3)^3 r)^{p^e} (x_2 x_3^2)^{(p^e+4)-p^e} \in \\
(J_1^{m(\ell p^e+1)} x_2^{2(p^e+4)}, x_3^{2(p^e+4)}, x_4^{4p^e}, \ldots, x_d^{4p^e}).
\]

Observe that \( m | \frac{p^e}{m} | = m(p^e - m) \) and hence \( m(\ell p^e+1) + 1 \geq p^e - (2m - 1) \).

In particular,
\[
\ell x_1^m (x_2 x_3 (x_4 \cdots x_3)^3 r)^{p^e} (x_2 x_3^2)^{(p^e+4)-p^e} \in \\
(J_1^{p^e-(2m-1)} x_2^{2(p^e+4)}, x_3^{2(p^e+4)}, x_4^{4p^e}, \ldots, x_d^{4p^e}).
\]

Multiplying by \( x_1^{3m-1} \) we arrive at
\[
\ell x_1^{3m-1} (x_2 x_3 (x_4 \cdots x_3)^3 r)^{p^e} (x_2 x_3^2)^{(p^e+4)-p^e} \in \\
(J_1^{p^e} x_2^{2(p^e+4)}, x_3^{2(p^e+4)}, x_4^{4p^e}, \ldots, x_d^{4p^e}).
\]

Multiplying by \( (x_2 x_3^2)^{(p^e-4)} \)
\[
\ell x_1^{3m-1} (x_2 x_3 (x_4 \cdots x_3)^3 r)^{p^e} (x_2 x_3^2)^{(2(1)-1)p^e} \in \\
(J_1^{p^e} x_2^{2(2p^e)}, x_3^{2(2p^e)}, x_4^{4p^e}, \ldots, x_d^{4p^e}).
\]

Applying (2) of Lemma 4.1 to the element \( x_2^2 \) we arrive at
\[
\ell x_1^{4m-1} (x_2 x_3 (x_4 \cdots x_3)^3 r)^{p^e} (x_2 x_3^2)^{(2(1)-1)p^e} \in \\
(J_1^{p^e} x_2^{2(2p^e)}, x_3^{4p^e}, x_4^{4p^e}, \ldots, x_d^{4p^e}).
\]

Applying (1) of Lemma 4.1 to the element \( x_2^2 \) we arrive at
\[
\ell x_1^{4m-1} (x_2 x_3 (x_4 \cdots x_3)^3 r)^{p^e} \in \\
(J_1^{p^e} x_2^{4p^e}, x_3^{4p^e}, x_4^{4p^e}, \ldots, x_d^{4p^e}).
\]

The integer \( m \) does not depend on \( e \). Therefore
\[
x_2^3 x_3^3 (x_4 \cdots x_d)^3 r \in (J_1, x_2^4, x_3^4, x_4^4, \ldots, x_d^4)^*.
\]

In particular, the element \( \eta = x_2^3 x_3^3 x_4^3 r + (x_2^4, x_3^4, x_4^4, \ldots, x_d^4) \) of \( H_m^{d-1}(R/J_1) \) is an element of \( 0^{i,f_{J_1}} H_m^{d-1}(R/J_1) \).

4.2. Local cohomology bounds and colon ideals. Theorem 4.2 establishes the equality of finitistic tight closure and tight closure of the zero submodule of the injective hull of the residue field of a local ring provided a family of colon ideals satisfy uniform containment properties. Let \( (R, m, k), m, \) and \( J_1 \) be as in Theorem 4.2. Our next proposition provides the desired uniform containment properties of the family of the described colon ideals whenever the top local cohomology bounds with respect to a suitable system of parameters of the family of cyclic modules \( \{ R/J_1^{4m+1} \} \) is bounded linearly in \( i \).
Proposition 4.3. Let \((R, m, k)\) be a local normal Cohen-Macaulay domain of Krull dimension \(d \geq 4\), \(\mathbb{Q}\)-Gorenstein in codimension 2, and of prime characteristic \(p > 0\). Assume that \(R\) has a test element. Let \(J_1 \subseteq R\) be a choice of canonical ideal and \(m \in \mathbb{N}\) such that \(J_1^{(m)}\) is principal in codimension 2. Let \(x_1, \ldots, x_d\) be a suitable system of parameters with respect to \(J_1\) such that the ideals \(J_1^{(m)}R_{x_2}\) and \(J_2^{(m)}R_{x_3}\) are principal ideals in their respective localizations. For each integer \(i \in \mathbb{N}\) suppose there exists integer \(\ell\) such that

\[
lcb_{d-1}(x_2^i, x_3^\ell, x_4, \ldots, x_d; R/J_1^{(mi+1)}) \leq i + 1.
\]

Then for all \(i, s \in \mathbb{N}\) there exists an integer \(\ell\) such that

\[
(J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}, \ldots, x_d^{is}) : x_d^{is} \subseteq (J_1^{(mi+1)}, x_2^\ell(i+3), x_3^\ell(i+3), x_4^{2(i+3)}, \ldots, x_d^{2(i+3)}).
\]

Proof. To ease notation we will write \(x\) to be denote the parameter sequence \(x_4, \ldots, x_d\) and denote by \(y\) the product \(x_1 \cdots x_d\). For each integer \(n \in \mathbb{N}\) we write \(x^n\) to denote the parameter sequence \(x_4^n, \ldots, x_d^n\).

Let \(r \in (J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}) : y^i(s-1)\) and consider the element

\[
\eta = r + (x_2^i, x_3^\ell, x_4^{is}) \in H^{d-1}(x_2^i, x_3^\ell, x_4^{is}, R/J_1^{(mi+1)}) \cong R/(J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}).
\]

Because \(y^i(s-1)r \in (J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is})\) we have that \((x_2x_3y)^i(s-1)r \in (J_1^{(mi+1)}, x_2^is, x_3^is, x_4^{is}).\) In particular,

\[
\alpha_{R/J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}}(\eta) = (x_2x_3y)^i(s-1)r + (J_1^{(mi+1)}, x_2^is, x_3^is, x_4^{is})
\]

is the 0-element of \(H^{d-1}(x_2^i, x_3^\ell, x_4^{is}, R/J_1^{(mi+1)}) \cong R/(J_1^{(mi+1)}, x_2^is, x_3^is, x_4^{is}).\) Let \(\alpha\) be the natural map

\[
H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}) \rightarrow H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}).
\]

Specifically,

\[
\alpha(z + (x_2^i, x_3^\ell, x_4^{is})) = (x_2^i, x_3^\ell, x_4^{is})^\ell z + (x_2^i, x_3^\ell, x_4^{is}).
\]

Similarly, let \(\bar{\alpha}\) be the natural map

\[
H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}) \rightarrow H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}).
\]

That is

\[
\bar{\alpha}(z + (x_2^i, x_3^\ell, x_4^{is})) = (x_2^i, x_3^\ell, x_4^{is})^\ell z + (x_2^i, x_3^\ell, x_4^{is}).
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}) & \xrightarrow{\alpha} & H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}) \\
\downarrow{\alpha_{R/J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}}} & & \downarrow{\alpha_{R/J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}}} \\
H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)}) & \xrightarrow{\bar{\alpha}} & H^{d-1}(x_2^i, x_3^\ell, x_4^{is}; R/J_1^{(mi+1)})
\end{array}
\]

Because \(\eta \in \text{Ker}(\alpha_{R/J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}})\) we must have \(\alpha(\eta) \in \text{Ker}(\alpha_{R/J_1^{(mi+1)}, x_2^i, x_3^\ell, x_4^{is}}).\) Because we are assuming

\[
lcb_{d-1}(x_2^i, x_3^\ell, x_4, \ldots, x_d; R/J_1^{(mi+1)}) \leq i + 1
\]
we have that
\[ \alpha(\eta) \in \ker(\alpha_{R/J_1^{(m+1)}; x_2^i x_3^j x_d^{2(i+1)}}). \]
Therefore
\[ 0 = \alpha_{R/J_1^{(m+1)}; x_2^i x_3^j x_d^{2(i+1)}}(\alpha(\eta)) = (x_2 x_3)^{\ell(i+3) - i} y^{i+1} r + (x_2^{\ell(i+3)}, x_3^{\ell(i+3)}, x_d^{2(i+1)}). \]
Equivalently,
\[ (x_2 x_3)^{\ell(i+3) - i} y^{i+1} r \in (J_1^{(m+1)}, x_2^{\ell(i+3)}, x_3^{\ell(i+3)}, x_d^{2(i+3)}). \]
Multiplying by \( y^5 \) we see that
\[ (x_2 x_3)^{\ell(i+3) - i} y^{2(i+3) - i} r \in (J_1^{(m+1)}, x_2^{\ell(i+3)}, x_3^{\ell(i+3)}, x_d^{2(i+3)}). \]
Therefore
\[ r \in (J_1^{(m+1)}, x_2^{\ell(i+3)}, x_3^{\ell(i+3)}, x_d^{2(i+3)}):(x_2 x_3)^{\ell(i+3) - i} y^{2(i+3) - i} \]
as claimed. \( \square \)

The next two propositions provide the linear bound of top local cohomology bounds of the family of \( R \)-modules \( \{R/J_1^{(m+1)}\} \) described in Proposition 4.3 whenever there exists a suitable system of parameters which annihilates a family of Ext-modules in a controlled manner.

**Proposition 4.4.** Let \((R, m, k)\) be a local normal Cohen-Macaulay domain of Krull dimension \( d \) and \( \mathbb{Q} \)-Gorenstein in codimension 2. Assume that \( R \) has a test element. Let \( J_1 \subseteq R \) be a choice of canonical ideal and \( m \in \mathbb{N} \) such that \( J_1^{(m)} \) is principal in codimension 2. Suppose \( S \) is a regular local ring mapping onto \( R \), \( R \cong S/P \), and \( \text{ht}(P) = h \). Let \( x_1, \ldots, x_d \) be a suitable system of parameters with respect to \( J_1 \) such that for each integer \( i \in \mathbb{N} \) and \( 2 \leq j \leq d - 2 \)
\[ (x_1^i, x_3^i, \ldots, x_{j+2}^i) \operatorname{Ext}^{h+j}_S(\operatorname{Ext}^{h+1}_S(R/J_1^{mi+1}, S), S) = 0. \]

Then for each integer \( i \in \mathbb{N} \)
\[ \operatorname{lcb}_{d-1}(x_2^{d-1}, x_3^{d-1}, \ldots, x_d^{d-1}, \operatorname{Ext}^{h+1}_S(\operatorname{Ext}^{h+1}_S(R/J_1^{mi+1}, S), S)) \leq i. \]

**Proof.** Let \( J_i = J_1^{mi+1} \) and let \((F_*, \partial_*)\) be the minimal free \( S \)-resolution of \( \operatorname{Ext}^{h+1}_S(R/J_i, S) \). Denote by \((-)^*\) the functor \( \operatorname{Hom}_S(-, S) \) and consider the dualized complex \((F_*, \partial_*)^*\). For every \( j \geq 1 \) there are short exact sequences
\[ 0 \to \operatorname{Ext}^{h+j}_S(\operatorname{Ext}^{h+1}_S(R/J_i, S), S) \to \operatorname{Coker}(\partial_{h+j}^*) \to \operatorname{Im}(\partial_{h+j+1}^*) \to 0 \]
and
\[ 0 \to \operatorname{Im}(\partial_{h+j+1}^*) \to F_{h+j+1}^* \to \operatorname{Coker}(\partial_{h+j+1}^*) \to 0. \]
The \( S \)-module \( \operatorname{Coker}(\partial_{h+1}^*) \) has projective dimension \( h + 1 \) and the height \( h + 1 \) ideal \( J_i \) annihilates the submodule \( \operatorname{Ext}^{h+1}_S(\operatorname{Ext}^{h+1}_S(R/J_i, S), S) \). By a simple prime avoidance argument we may lift \( x = x_2, \ldots, x_d \) to elements of \( S \) and assume that \( x \) is a regular sequence on \( \operatorname{Coker}(\partial_{h+1}^*) \) and the free \( S \)-modules \( F_i^* \).

The module \( \operatorname{Ext}^{h+1}_S(R/J_1, S) \) is an \((S_2)\)-module over its support, see Lemma 2.4. In particular,
\[ \operatorname{Ext}^{h+d}_S(\operatorname{Ext}^{h+1}_S(R/J_1, S), S) = \operatorname{Ext}^{h+d-1}_S(\operatorname{Ext}^{h+1}_S(R/J_1, S), S) = 0 \]
and
\[ \text{Coker}(\partial_{h+d-2}^*) \cong \text{Ext}_S^{h+d-2}(\text{Ext}_{S}^{h+1}(R/J_i, S), S). \]

Consider the short exact sequence
\[ 0 \to \text{Im}(\partial_{h+d-2}^*) \to F_{h+d-2}^* \to \text{Ext}_S^{h+d-2}(\text{Ext}_{S}^{h+1}(R/J_i, S), S) \to 0. \]
We are assuming \((x_2^i, x_3^i, \ldots, x_d^i) \text{ Ext}_S^{h+d-2}(\text{Ext}_{S}^{h+1}(R/J_i, S), S) = 0 \) for every \( i \in \mathbb{N} \). By (2) of Proposition 3.6 and (3) of Proposition 3.5 we have that \( \text{lcb}_2(x; \text{Im}(\partial_{h+d-2}^*)) \leq i \). Next, we consider the short exact sequence
\[ 0 \to \text{Ext}_S^{h+d-3}(\text{Ext}_{S}^{h+1}(R/J_i, S), S) \to \text{Coker}(\partial_{h+d-3}^*) \to \text{Im}(\partial_{h+d-2}^*) \to 0. \]
We established \( \text{lcb}_2(x; \text{Im}(\partial_{h+d-2}^*)) \leq i \) and we are assuming
\[ (x_2^i, \ldots, x_{d-1}^i) \text{ Ext}_S^{h+d-3}(\text{Ext}_{S}^{h+1}(R/J_i, S), S) = 0 \]
for every \( i \in \mathbb{N} \). By (1) of Proposition 3.5 we have
\[ \text{lcb}_2(x; \text{Coker}(\partial_{h+d-3}^*)) \leq i + i = 2i. \]
Next consider the short exact sequence
\[ 0 \to \text{Im}(\partial_{h+d-3}^*) \to F_{h+d-3}^* \to \text{Coker}(\partial_{h+d-3}^*) \to 0. \]
By (2) of Proposition 3.6 and knowing that \( \text{lcb}_2(x; \text{Coker}(\partial_{h+d-3}^*)) \leq 2i \) we see that
\[ \text{lcb}_3(x; \text{Im}(\partial_{h+d-3}^*)) \leq 2i. \]
Inductively, we find that
\[ \text{lcb}_j(x; \text{Im}(\partial_{h+d-j}^*)) \leq (j - 1)i \]
and
\[ \text{lcb}_j(x; \text{Coker}(\partial_{h+d-j-1}^*)) \leq ji \]
for each \( 2 \leq j \leq d - 1 \). In particular,
\[ \text{lcb}_{d-1}(x; \text{Ext}_S^{h+1}(\text{Ext}_{S}^{h+1}(R/J_i, S), S)) \leq (d - 1)i. \]
By Lemma 3.2 the parameter sequence \( x_{d-1}^d = x_{d-1}^2, \ldots, x_{d-1}^d \) on \( R/J_1 \) satisfies
\[ \text{lcb}_{d-1}(x_{d-1}^d; \text{Ext}_S^{h+1}(\text{Ext}_{S}^{h+1}(R/J_i, S), S)) \leq i \]
for each integer \( i \in \mathbb{N} \).

**Proposition 4.5.** Let \((R, m, k)\) be a local normal Cohen-Macaulay domain of Krull dimension \( d \geq 4 \) which is \( \mathbb{Q} \)-Gorenstein in codimension 2. Let \( J_1 \subseteq R \) be a choice of canonical ideal and \( m \in \mathbb{N} \) such that \( J_1^{(m)} \) is principal in codimension 2. Suppose \( S \) is a regular local ring mapping onto \( R, R \cong S/P \), and \( \text{ht}(P) = h \). Let \( x_1, \ldots, x_d \) be a suitable system of parameters with respect to \( J_1 \) such that:

- The ideals \( J_1^{(m)} R_{x_2} \) and \( J_1^{(m)} R_{x_3} \) are principal ideals in their respective localizations;
- For each integer \( i \in \mathbb{N} \) and \( 2 \leq j \leq d - 2 \)
  \[ (x_2^i, x_3^i, \ldots, x_{j+2}^i) \text{ Ext}_S^{h+j}(\text{Ext}_{S}^{h+1}(R/J^{mi+1}, S), S) = 0. \]

Then the following hold:
(1) For each integer \( i \in \mathbb{N} \) there exists an integer \( \ell \) such that
\[
\ellcb_{d-1}(x_2^\ell, x_3^\ell, x_4^{d-1}, \ldots, x_d^{d-1}; R/J^{mi+1}) \leq i + 1;
\]
(2) For each integer \( i \in \mathbb{N} \) there exists an integer \( \ell \) such that
\[
\ellcb_{d-1}(x_2^\ell, x_3^\ell, x_4^{d-1}, \ldots, x_d^{d-1}; R/J^{mi+1}) \leq i + 2.
\]

**Proof.** For each \( i \in \mathbb{N} \) let \( C_i \) be the cokernel of
\[
R/J^{mi+1} \to \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^{mi+1}, S), S)
\]
and consider the short exact sequences
\[
0 \to R/J^{mi+1} \to \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^{mi+1}, S), S) \to C_i \to 0,
\]
see Lemma 2.5 for details.

By Lemma 2.5 the module \( C_i \) is 0 when either \( x_2 \) or \( x_3 \) is inverted. Hence for each \( i \in \mathbb{N} \) there exists an integer \( \ell \) such that \((x_2^\ell, x_3^\ell)C_i = 0\). Because \( d \geq 4 \) we have that \( d - 1 \geq 3 \) and (3) of Proposition 3.5 is applicable and implies
\[
\ellcb_{d-1}(x_2^\ell, x_3^\ell, x_4^{d-1}, \ldots, x_d^{d-1}; R/J^{mi+1}) \leq \ellcb_{d-1}(x_2^\ell, x_3^\ell, x_4^{d-1}, \ldots, x_d^{d-1}; \text{Ext}^{h+1}_S(\text{Ext}^{h+1}_S(R/J^{mi+1}, S), S)) + 1.
\]

Statement (1) follows by Proposition 4.4.

To prove (2) let \( K_i = J^{mi+1}_1/J^{mi+1}_1 \) and consider the short exact sequences
\[
0 \to K_i \to R/J^{mi+1} \to R/J^{mi+1}_1 \to 0.
\]
The module \( K_i \) is 0 when either \( x_2 \) or \( x_3 \) are inverted. Hence for each \( i \in \mathbb{N} \) there exists an integer \( \ell \) such that \((x_2^\ell, x_3^\ell)K_i = 0\). By (1) of Proposition 3.5 we have that
\[
\ellcb_{d-1}(x_2^\ell, x_3^\ell, x_4^{d-1}, \ldots, x_d^{d-1}; R/J^{mi+1}) \leq \ellcb_{d-1}(x_2^\ell, x_3^\ell, x_4^{d-1}, \ldots, x_d^{d-1}; R/J^{mi+1}) + 1 \leq i + 2.
\]

\( \square \)

**4.3. Main results.** We have arrived at the main theorem of the article. Theorem A and Theorem B are consequences of the next theorem. Theorem 4.6 below gives the existence of suitable system of parameters satisfying the annihilation properties of Proposition 4.4 and Proposition 4.5 whenever an anti-canonical ideal has analytic spread at most 2 and reduction number 1 on the punctured spectrum.

**Theorem 4.6.** Let \((R, m, k)\) be an excellent local normal Cohen-Macaulay domain of Krull dimension \( d \geq 4 \) which is \( \mathbb{Q} \)-Gorenstein in codimension 2. Let \( J_1 \subsetneq R \) be a choice of a canonical ideal and \( x_1 \in J_1 \) a generic generator of \( J_1 \). Suppose \((x_1) = J_1 \cap K_1 \) so that \( K_1 \) is an anti-canonical ideal of \( R \). Suppose further that there exists integer \( m' \) such that \( K_1^{m'} \) has analytic spread at most 2 and reduction number 1 with respect to some reduction on the punctured spectrum. Then there exists an integer \( m \in \mathbb{N} \) and suitable parameters \( x_2, \ldots, x_d \) on \( R/J_1 \) such that
\[
(x_2^i, x_3^i, \ldots, x_{j+2}^i)\text{Ext}^{h+j}_S(\text{Ext}^{h+1}_S(R/J^{mi+1}_1, S), S) = 0
\]
for every integer \( i \in \mathbb{N} \) and \( 2 \leq j \leq d - 2 \).
Proof. We can choose $m'' \in \mathbb{N}$ so that $K = K_1^{(m'')}$ is principal in codimension 2. If $m$ is any multiple of $m'$ and $m''$ then $K_1^{(m)}$ is principal in codimension 2 and has analytic spread at most 2 on $\text{Spec}(R) - \{m\}$, see Proposition 2.10 to know that the symbolic Rees ring $\mathcal{R}_{K_1^{(m')}}$ is Noetherian on the punctured spectrum and Proposition 2.7 to insure that the analytic spread of $K^{(m)}$ is no more than the analytic spread of $K_1^{(m')}$ on the punctured spectrum. By Proposition 2.8 we can choose $m$ to be a multiple of $m'$ and $m''$ and such that the containment of ideals $K^i \subseteq K^{(i)}$ is an equality on $\text{Spec}(R) - \{m\}$ for each $i \in \mathbb{N}$. Let $K = K_1^{(m)}$ and let $x = x_1^{m}$.

Claim 4.7. For each integer $i \in \mathbb{N}$

$$\text{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \cong x_1 K_1^{(mi)}/x_1^{mi+1} J_1 \cong K_1^{(mi)}/x_1^{mi} J_1 = K^{(i)}/x^i J_1.$$  

Proof of claim. For each integer $i$ consider the following short exact sequence:

$$0 \to \frac{J_1^{mi+1}}{x_1^{mi+1} J_1} \to \frac{R}{x_1^{mi+1} J_1} \to \frac{R}{J_1^{mi+1}} \to 0$$

The ideal $x_1^{mi+1} J_1$ is isomorphic to the canonical module of $R$, therefore

$$\text{Ext}_S^{h+1}(R/x_1^{mi+1} J_1, S) \cong R/x_1^{mi+1} J_1,$$

and there are exact sequences

$$0 \to \text{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \to \frac{R}{x_1^{mi+1} J_1} \to \text{Ext}_S^{h+1}(J_1^{mi+1}/x_1^{mi+1} J_1, S).$$

Therefore $\text{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \cong L_i/x_1^{mi+1} J_1$ for some ideal $L_i \subseteq R$. Moreover, $R/L_i \cong \text{Ext}_S^{h+1}(J_1^{mi+1}/x_1^{mi+1} J_1, S)$. Because $\text{Ext}_S^{h+1}(J_1^{mi+1}/x_1^{mi+1} J_1, S)$ is an $(S_2)$-module over its support, see Lemma 2.4, it follows that $R/L_i$ is an $(S_1)$-module over its support. Hence $L_i$, as an ideal of $R$, is unmixed of height 1. Moreover, every component of $L_i$ is a component of $x_1 R$. Localizing at a component of $J_1$ we see that $L_i$ agrees with $x_1 R$ and localizing at a component of $K_1$ we see that $L_i$ agrees with $x_1^{mi+1}$. Therefore $L_i$ agrees with the unmixed ideal $x_1 K_1^{(mi)}$ and so

$$\text{Ext}_S^{h+1}(R/J_1^{mi+1}, S) \cong x_1 K_1^{(mi)}/x_1^{mi+1} J_1.$$  

The second isomorphism

$$x_1 K_1^{(mi)}/x_1^{mi+1} J_1 \cong K_1^{(mi)}/x_1^{mi} J_1 = K^{(i)}/x^i J_1$$

is division by $x_1$. 

Claim 4.8. For all integers $i, j \in \mathbb{N}$ and $j \geq 2$

$$\text{Ext}_S^{h+j}(K^{(i)}/x^i J_1, S) \cong \text{Ext}_S^{h+j+1}(R/K^{(i)}, S).$$

Proof of claim. For each integer $i \in \mathbb{N}$ consider the short exact sequence

$$0 \to K^{(i)}/x^i J_1 \to R/x^i J_1 \to R/K^{(i)} \to 0.$$  

The cyclic $R$-module $R/x^i J_1$ is Cohen-Macaulay of dimension $d - 1$ and therefore

$$\text{Ext}_S^{h+j}(R/x^i J_1, S) = 0.$$
for all \( j \geq 2 \) and hence \( \text{Ext}^{h+j} \left( \frac{K(i)}{x^iJ_1}, S \right) \cong \text{Ext}^{h+j+1} \left( \frac{R}{K(i)}, S \right) \).

To prove the theorem it is now enough to find parameters \( x_2, x_3, \ldots, x_d \) on \( R/J_1 \) such that

\[
(x^i_2, x^i_3, \ldots, x^i_{j+2}) \text{Ext}^{h+j+1} \left( \frac{R}{K(i)}, S \right)
\]

for every integer \( i \in \mathbb{N} \) and \( 2 \leq j \leq d-2 \).

**Claim 4.9.** For every integer \( i \in \mathbb{N} \) and \( 2 \leq j \leq d-2 \)

\[
\text{Ext}^{h+j} \left( \frac{R}{K(i)}, S \right) \cong \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right)
\]

**Proof of claim.** Consider the short exact sequences

\[
0 \to \frac{K(i)}{K^j} \to \frac{R}{K^i} \to \frac{R}{K(k)} \to 0.
\]

For each \( i \in \mathbb{N} \) the modules \( K(i)/K^j \) are supported only at the maximal ideal. In particular, \( \text{Ext}^{\ell} \left( \frac{K(i)}{K^j}, S \right) = 0 \) for all \( \ell \leq d + h - 1 \) and the claim follows.

To prove the theorem it is now enough to find parameters \( x_2, x_3, \ldots, x_d \) on \( R/J_1 \) such that

\[
(x^i_2, x^i_3, \ldots, x^i_{j+2}) \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right)
\]

for every integer \( i \in \mathbb{N} \) and \( 2 \leq j \leq d-2 \).

We can choose parameters \( x_2 \) and \( x_3 \) on \( R/J_1 \) such that \( KR_{x_2} \) and \( KR_{x_3} \) are principal ideals in their respective localizations. Suppose \( x_2 \) has been chosen such that \( KR_{x_2} = (a)R_{x_2}, a \in K \), and \( x_2K \subseteq (a^j) \). Then \( x_2K(i) \subseteq (a_i^i) \) and therefore the left term of the following short exact sequence is annihilated by \( x_2^i \):

\[
0 \to \frac{K(i)}{(a^i)} \to \frac{R}{(a_i)} \to \frac{R}{K^j} \to 0.
\]

It follows that \( \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right) \) is isomorphic to \( \text{Ext}^{h+j} \left( \frac{R}{K^i/(a^i)}, S \right) \) for every \( 2 \leq j \leq d-2 \) and therefore

\[
x^i_2 \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right) = 0
\]

for every \( i \) and \( 2 \leq j \leq d-2 \).

Similarly, we can find \( x_3 \) a parameter on \( R/(J_1, x_2) \) such that

\[
x^i_3 \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right) = 0
\]

for every \( i \) and \( 2 \leq j \leq d-2 \).

Assume we have found parameters \( x_2, x_3, \ldots, x_\ell \) on \( R/J_1 \) such that

\[
x^i_m \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right) = 0
\]

for every \( 2 \leq m \leq \ell, i \geq 1, \) and \( m-2 \leq j \leq d-2 \). We wish to find parameter element \( x_{\ell+1} \) of \( R/(J_1, x_2, \ldots, x_\ell) \) such that

\[
x^i_{\ell+1} \text{Ext}^{h+j+1} \left( \frac{R}{K^i}, S \right) = 0
\]

for every \( i \in \mathbb{N} \) and \( \ell - 1 \leq j \leq d-2 \).
Claim 4.10. Let $\Lambda = \{ P_1, \ldots, P_m \}$ be the collection of minimal prime ideals of the pure height $\ell$ ideal $(J_1, x_2, x_3, \ldots, x_\ell)$. If necessary, enlarge the set of height $\ell$ primes $\Lambda$ so that every component of $K$ is contained in a prime ideal of $\Lambda$. Let $W_\ell$ be the multiplicative set $R - \bigcup_{P \in \Lambda} P$. There exist elements $a, c \in K$ such that

1. $(a, c)R_{W_\ell}$ forms a reduction of $KR_{W_\ell}$;
2. the element $a$ generates $K$ at its minimal components;
3. as an ideal of $R$, the principal ideal $(a) = K \cap K'$ where $K'$ is of pure height 1 whose components are disjoint from $K$, and the element $c$ avoids all components of $K'$.

Proof of claim. We are assuming the ideal $K$ has analytic spread at most 2 at each of the localizations $R_{P_i}$ as $P_i$ varies among the prime ideals in $\Lambda = \{ P_1, \ldots, P_m \}$. So for each $1 \leq i \leq m$ there exist $a_i, c_i \in K$ such that $(a_i, c_i)R_{P_i}$ forms a reduction of $KR_{P_i}$. For each $1 \leq i \leq m$ choose $r_i \in \bigcap_{P \in \Lambda \setminus \{ P_i \}} P - P_i$ and set $a' = \sum r_ia_i$ and $c' = \sum r_ic_i$. We claim $(a', c')R_{W_\ell}$ is a reduction of $KR_{W_\ell}$. By [HS06, Proposition 8.1.1] it is enough to check $(a', c')$ forms a reduction of $K$ at each of the localizations $R_{P_i}$ for $1 \leq i \leq m$. By [HS06, Proposition 8.2.4] it is enough to check that the fiber cone $R_{P_i}/PR_{P_i} \otimes R[K] \cong \bigoplus K^nR_{P_i}/P_iK^n$ is finite over the subalgebra spanned by $(a', c')R_{P_i}$, $P_iK$, $c' \equiv r_ic_i \bmod P_iK$, $r_i$ is a unit of $R_{P_i}$, and therefore $(a', c')R_{W_\ell}$ does indeed form a reduction of $KR_{W_\ell}$ by a second application of [HS06, Proposition 8.2.4].

Now consider the set of primes $\Gamma = \{ Q_1, \ldots, Q_n \}$ which are the minimal components of $K$. The purpose of enlarging the set of height $\ell$ primes in the statement of the claim was to insure that each $Q_j \in \Gamma$ is a prime ideal of the localization $R_{W_\ell}$. In particular, $(a', c')R_{Q_j}$ forms a reduction of $KR_{Q_j}$ for each $1 \leq i \leq n$. But $R_{Q_j}$ is a discrete valuation ring and therefore for each $1 \leq i \leq \ell$ either $KR_{Q_j} = (a')R_{Q_j}$ or $KR_{Q_j} = (c')R_{Q_j}$. Without loss of generality we assume $KR_{Q_j} = (a')R_{Q_j}$ for at least one value of $i$ and relabel the primes $\Gamma$ so that $KR_{Q_j} = (a')R_{Q_j}$ for each $1 \leq i \leq j$ and $KR_{Q_i} \neq (a')R_{Q_i}$ for each $j+1 \leq i \leq n$. Choose $r \in Q_1 \cap \cdots \cap Q_j \setminus \bigcup_{j+1}^{\ell} Q_i$ and consider the element $a' + rc'$. We claim that $a' + rc'$ generates $KR_{Q_j}$ for each $1 \leq i \leq n$. First consider a localization at a prime $Q_i \in \Gamma$ with $1 \leq i \leq j$. Then $(a', c')R_{Q_i} = (a')R_{Q_i}$ by assumption and so $(c')R_{Q_i} \subseteq (a')R_{Q_i}$. Because $r \notin Q_i$ there is a strict containment of principal ideal $(rc')R_{Q_i} \subsetneq (a')R_{Q_i}$ and it follows that $(a')R_{Q_i} = (a' + rc')R_{Q_i}$. Now consider a localization $R_{Q_j}$ with $j+1 \leq i \leq n$. We are assuming that $a'$ does not generate $KR_{Q_j}$, and therefore $(a')R_{Q_j} \subseteq (c')R_{Q_j} = KR_{Q_j}$. Moreover, $r$ is a unit of $R_{Q_j}$ and therefore $(c')R_{Q_j} = (a' + rc')R_{Q_j}$.

Let $a = a' + rc'$. Then $(a, c')R_{W_\ell} = (a', c')R_{W_\ell}$ forms a reduction of $KR_{W_\ell}$ and the element $a$ generates $K$ at each of its minimal components as desired. Suppose as an ideal of $R$ the principal ideal $(a)$ has decomposition $(a) = K \cap K' \cap K''$ so that

1. $K, K', K''$ are pure height 1 ideals whose components are disjoint from one another;
2. the components of $K'$ are height 1 prime ideals which do not contain $c$;
3. the components of $K''$ are height 1 prime ideals which do contain $c$.

We take $K'$ or $K''$ to be $R$ if no such components of $(a)$ exist. If $K'' = R$ then we let $c = c'$ and the elements $a, c$ satisfy the conclusions of the claim. If $K'' \neq R$ then first observe that, because $(a, c')R_{W_\ell}$ forms a reduction of $KR_{W_\ell}$ and $a, c' \in K''$, we must have that $(a)R_{W_\ell} = (K \cap K')R_{W_\ell}$. Choose an element $r \in K \cap K'$ which avoids all components in $K''$.
and consider the element \( c = c' + r \). Then \((a, c)R_{W_\ell} = (a, c')R_{W_\ell}\) forms a reduction of \( KR_{W_\ell}\). Moreover, the element \( c \) avoids all minimal components of \( K' \) and \( K'' \) by construction. \( \square \)

By assumption there exists a natural number \( n_\ell \) so that \( K'^{n_\ell}R_{W_\ell}\) has reduction number at 1 with respect to any reduction. Recall that \( K'^{n_\ell} \) and \( K''^{n_\ell} \) agree on the punctured spectrum. So we may replace \( K \) by \( K'^{n_\ell}, x_2, x_3, \ldots, x_\ell \) by \( x_2^{n_\ell}, \ldots, x_\ell^{n_\ell} \), and \( a, c \) by \( a^{n_\ell}, c^{n_\ell} \) and assume further that \((a, c)KR_{W_\ell} = K^2R_{W_\ell}\).

**Claim 4.11.** There exists a parameter element \( x_{\ell+1} \) of \( R/(J_1, x_2, x_3, \ldots, x_\ell) \) such that the following hold:

1. \( x_{\ell+1}^{i-1} \) annihilates \( K^i/(a, c)^{i-1}K \) for every integer \( i \);
2. \( x_{\ell+1} \) annihilates \( \text{Ext}^{h+j+1}_S(R/(a, c)K, S) \) for every \( \ell - 1 \leq j \leq d - 2 \);
3. \( x_{\ell+1} \) annihilates \( \text{Ext}^{h+j+1}_S(R/K, S) \) for every \( \ell - 1 \leq j \leq d - 2 \).

**Proof.** Consider \( W_\ell \) as a multiplicative set of \( S \). Then \( S_{W_\ell} \) has dimension \( h + \ell \), \( K^iR_{W_\ell} = K^{(i)}S_{W_\ell} \), and \((a, c)KS_{W_\ell} = K^{(2)}S_{W_\ell} \). Because \( K^{(i)} \) is an unmixed ideal we have that \( R_{W_\ell}/K^{(i)}R_{W_\ell} \) has positive depth and therefore the Ext-modules

\[
\text{Ext}^{h+j+1}_S(R/K) \otimes R_{W_\ell} \text{ and } \text{Ext}^{h+j+1}_S(R/(a, c)K) \otimes R_{W_\ell}
\]

are 0 for each \( \ell - 1 \leq j \leq d - 2 \). It follows that we can choose \( x_{\ell+1} \) a parameter on \( R/(J_1, x_2, x_3, \ldots, x_\ell) \) such that \( x_{\ell+1}^{i-1}K^i \subseteq (a, c)K \) and \( x_{\ell+1} \) satisfies (2) and (3). Because \( x_{\ell+1}^{i-1}K^i \subseteq (a, c)K \) it follows that for every \( i \geq 1 \) that \( x_{\ell+1}^{i-1}K^i \subseteq (a, c)^{i-1}K \) and therefore (1) is satisfied as well. \( \square \)

The element \( x_{\ell+1}^{i-1} \) annihilates the left term of the following short exact sequence:

\[
0 \to \frac{K^i}{(a, c)^{i-1}K} \to \frac{R}{(a, c)^{i-1}K} \to \frac{R}{K^i} \to 0.
\]

In particular, there are exact sequences

\[
\text{Ext}^{h+j}_S(K^i/(a, c)^{i-1}K, S) \to \text{Ext}^{h+j+1}_S(R/K^i, S) \to \text{Ext}^{h+j+1}_S(R/(a, c)^{i-1}K, S)
\]

and the left term is annihilated by \( x_{\ell+1}^{i-1} \). We will show that \( x_{\ell+1} \text{Ext}^{h+j+1}_S(R/(a, c)^{i-1}K, S) = 0 \) for every \( i \geq 2 \) and \( \ell - 1 \leq j \leq d - 2 \). It will then follow that \( x_{\ell+1}^{i-1} \) annihilates \( \text{Ext}^{h+j+1}_S(R/K^i, S) \) for every \( i \) and \( \ell - 1 \leq j \leq d - 2 \) as desired.

**Claim 4.12.** For every integer \( i \) there is short exact sequence

\[
0 \to \frac{R}{(ac^j)} \to \frac{R}{a(ac^j)K} \oplus \frac{R}{cK} \to \frac{R}{(a, c)^iK} \to 0.
\]

**Proof of claim.** For any two ideals \( I, J \subseteq R \) there is a short exact sequence

\[
0 \to \frac{R}{I \cap J} \to \frac{R}{I} \oplus \frac{R}{J} \to \frac{R}{I + J} \to 0.
\]

Therefore it is enough to show that

\[
a(ac^j)^{i-1}K \cap c^jK = (ac^j).
\]

Clearly \( ac^j \in a(ac^j)^{i-1}K \cap c^jK \). Now consider an element of the form \( c^jy \) with \( y \in K \) and \( c^jy \in a(ac^j)^{i-1}K \). To show \( c^jy \in (ac^j) \) we only need to show \( y \in (a) \). Recall that by
Claim 4.10 we have that \((a) = K \cap K'\) and \(c\) avoids all components of \(K'\). We already know that \(y \in K\). Localizing at a component \(P\) of \(K'\) we have that
\[
\hat{c} y \in a(a, c)^{i-1} K R_P.
\]
However, \(c\) is a unit of \(R_P, c \in K\), and therefore \(y \in a R_P\). \(\square\)

**Claim 4.13.** For each \(2 \leq j \leq d - 2\) there are isomorphisms
\[
\text{Ext}^{h+j+1}(R/(a, c)^{i-1} K, S) \cong \text{Ext}^{h+j+1}(R/(a, c)^{i-1} K, S)
\]
and
\[
\text{Ext}^{h+j+1}(R/c^i K, S) \cong \text{Ext}^{h+j+1}(R/K, S).
\]

**Proof of claim.** For the first isomorphism consider the long exact sequence of Ext-modules induced from the short exact sequence
\[
0 \to \frac{R}{(a, c)^{i-1} K} \to \frac{R}{a(a, c)^{i-1} K} \to \frac{R}{(a)} \to 0
\]
and for the second isomorphism consider the long exact sequence of Ext-modules induced from the short exact sequence
\[
0 \to \frac{R}{K} \to \frac{R}{c^i K} \to \frac{R}{(c^i)} \to 0.
\]
\(\square\)

Observe that by Claim 4.12 there are isomorphisms
\[
\text{Ext}^{h+j+1}(R/(a, c)^{i} K, S) \cong \text{Ext}^{h+j+1}(R/(a, c)^{i-1} K, S) \oplus \text{Ext}^{h+j+1}(R/c^i K, S)
\]
for all \(2 \leq j \leq d - 2\). Therefore Claim 4.13 and induction we find that there isomorphisms
\[
\text{Ext}^{h+j+1}(R/(a, c)^{i} K, S) \cong \bigoplus \text{Ext}^{h+j+1}(R/K, S)
\]
The element \(x_{\ell+1}\) has the property that it annihilates the modules appearing the direct sum decompositions above. Therefore \(x_{\ell+1}\) annihilates each \(\text{Ext}^{h+j+1}(R/(a, c)^{i} K, S)\) for each \(\ell - 1 \leq j \leq d - 2\) as desired. \(\square\)

Theorem B is established by piecing together Theorem 4.2, Proposition 4.3, Proposition 4.5, and Theorem 4.6.

**Corollary 4.14.** Let \((R, m, k)\) be an excellent local normal Cohen-Macaulay domain of prime characteristic \(p > 0\), of Krull dimension at least 4, and \(\mathbb{Q}\)-Gorenstein in codimension 2. Suppose that some symbolic power of the anti-canonical ideal of \(R\) has analytic spread no more than 2 on the punctured spectrum. Then \(0^{*}_{E_R(k)} = 0^{*}_{E_R(k)}\).

**Proof.** We wish to invoke Theorem 4.6. Therefore if we denote by \(K \subseteq R\) an anti-canonical ideal we must prove the existence of an integer \(n\) such that for all \(P \in \text{Spec}(R) \setminus \{m\}\) that \(K^{(n)} R_P\) has a reduction by 2 elements with reduction number 1.

The anti-canonical algebra \(R_K\) is Noetherian on the punctured spectrum by Theorem 2.9. Hence \(R_P\) is strongly \(F\)-regular by [CEMS18, Corollary 5.9] for each \(P \in \text{Spec}(R) \setminus \{m\}\). Therefore at each prime ideal \(P \in \text{Spec}(R) \setminus \{m\}\) there exists an integer \(n_P\) so that \(K^{(n_P)} R_P\) has analytic spread 2 and reduction number 1 with respect to any reduction by Theorem 2.12.
For a choice of reduction of $K^{(n_p)} R_P$ it is easy to see there is an open neighborhood of $P \in \text{Spec}(R) \setminus \{m\}$ so that $K^{(n_p)}$ has a reduction by 2 elements with reduction number 1. By a simple quasi-compactness argument there exists an integer $n$ such that $K^{(n)} R_P$ has a reduction by 2 elements with reduction number 1 for each $P \in \text{Spec}(R) \setminus \{m\}$ and therefore Theorem 4.6 is applicable.

Let $J_1 \subseteq R$ be a choice of a canonical ideal and let $x_1 \in J_1$ be a generic generator. By Theorem 4.6 we may extend $x_1$ to a suitable system of parameters $x_1, x_2, \ldots, x_d$ such that

$$(x_2^i, x_3^i, \ldots, x_{d+2}^i) \text{Ext}_S^{h+j}(\text{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S) = 0$$

for every integer $i \in \mathbb{N}$ and $2 \leq j \leq d - 2$. By Proposition 4.5 for every integer $i \in \mathbb{N}$ there exists an integer $\ell$ such that

$$\text{lcb}_{d-1}(x_2^{\ell(d-1)}, x_3^{\ell(d-1)}, x_4^{d-1}, \ldots, x_d^{d-1}, R/J_1^{(mi+1)}) \leq i + 1.$$  

We replace $x_2, \ldots, x_d$ by the sequence of elements $x_2^{d-1}, \ldots, x_d^{d-1}$ and have now have that for all $i \in \mathbb{N}$ there exists an integer $\ell$ such that

$$\text{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \ldots, x_d; R/J_1^{(mi+1)}) \leq i + 1.$$  

The corollary now follows by Proposition 4.3 and Theorem 4.2.

\textbf{Corollary 4.15.} Let $R$ be a locally excellent weakly $F$-regular ring of prime characteristic $p$ which has a canonical ideal. Suppose further that at each non-closed point of $\text{Spec}(R)$ there is a symbolic power of the anti-canonical ideal which has analytic spread at most 2. Then $R$ is strongly $F$-regular.

\textbf{Proof.} It is well known that the properties of being weakly $F$-regular and strongly $F$-regular can be checked at localizations at the maximal ideals of $R$, see [HH90, Corollary 4.15]. The properties of weakly $F$-regular and strongly $F$-regular for a local ring can be checked after completion. In which case, the property of being weakly $F$-regular is equivalent to $0^{*f}_J$, $E_{R(k)}$ being 0 and the property of being strongly $F$-regular is equivalent to $0^{*f}_J$ being 0. Every complete local weakly $F$-regular ring is normal by [HH90, Lemma 5.9], Cohen-Macualay by [HH90, Theorem 4.9], and Gorenstein in codimension 2 by [Smi97, Theorem 3.1] and [Lip69, Proposition 17.1]. In particular, Corollary 4.14 is applicable and the result follows.

\section{5. $F$-signature and relative Hilbert-Kunz multiplicity}

\textbf{5.1. Background on $F$-signature and Hilbert-Kunz multiplicity.} We summarize some basic properties of Frobenius splitting numbers, $F$-signature, and Hilbert-Kunz multiplicity. For an introduction to these concepts we refer the reader to [Hum13, PT18]. Let $(R, m)$ be a local $F$-finite domain of prime characteristic $p > 0$ and Krull dimension $d$. For each $e \in \mathbb{N}$ let $a_e(R)$ be the largest rank of a free $R$-module $G$ for which there exists an onto $R$-linear map $F_*^e R \to G$. The $F$-signature of $R$ is the limit

$$s(R) = \lim_{e \to \infty} \frac{a_e(R)}{\text{rank}_R(F_*^e R)^{\frac{1}{e}}}.$$
a limit which always exists by [Tuc12, Main Result]. The ring $R$ is strongly $F$-regular if and only if $s(R) > 0$ by [AL03, Main Theorem]. For each integer $e \in \mathbb{N}$ we denote by $I_e$ the $e$th Frobenius degeneracy ideal. Specifically,

$$I_e = \{ r \in R \mid \varphi(F^e_r) \in m, \forall \varphi \in \text{Hom}_R(F^e_R, R) \}.$$ 

The ideals $I_e$ satisfy the following properties:

1. $m^{[p^e]} \subseteq I_e$;
2. For each integer $e_0 \in \mathbb{N}$, $I_e^{[p^{e_0}]} \subseteq I_{e+e_0}$;
3. $\frac{a_e(R)}{\text{rank}(F^e_R)} = \frac{\lambda(R/I_e)}{p^{ed}}$;
4. $s(R) = \lim_{e \to \infty} \frac{\lambda(R/I_e)}{p^{ed}}$.

Suppose $I \subseteq R$ is an $m$-primary ideal. The Hilbert-Kunz multiplicity of the ideal $I \subseteq R$ is the limit

$$e_{HK}(I) = \lim_{e \to \infty} \frac{\lambda(R/I^{[p^e]})}{p^{ed}}.$$ 

a limit which exists by [Mon83, Theorem 1.8]. By [Kun76, Proposition 2.1] we have that for each $m$-primary ideal $I \subseteq R$,

$$\frac{\lambda(R/I^{[p^e]})}{p^{ed}} = \frac{\lambda(F^e_R/IF^e_R)}{\text{rank}(F^e_R)}.$$ 

Therefore the Hilbert-Kunz multiplicity of an $m$-primary ideal agrees with the limit

$$e_{HK}(I) = \lim_{e \to \infty} \frac{\lambda(F^e_R/IF^e_R)}{\text{rank}(F^e_R)}.$$ 

Suppose that $F^e_R \cong R^{a_e(R)} \oplus M_e$. Then for each $m$-primary ideal $\lambda(F^e_R/IF^e_R) = a_e(R)\lambda(R/I) + \lambda(M_e/IM_e)$. If $I \not\subseteq J$ are $m$-primary it is then easy to see that

$$a_e(R)\lambda(J/I) \leq \lambda(F^e_R/IF^e_R) - \lambda(F^e_R/JF^e_R)$$

and therefore for each pair of $m$-primary ideals $I \not\subseteq J$ we have that

$$s(R) \leq e_{HK}(I) - e_{HK}(J) - \frac{\lambda(J/I)}{\lambda(J/I)}.$$ 

Work of the second author and Tucker show that $F$-signature of a local ring is realized as the infimum of relative Hilbert-Kunz multiplicities.

**Theorem 5.1.** [PT18, Theorem A] If $(R, m, k)$ is an $F$-finite local ring, then

$$s(R) = \inf_{I \subseteq J \subseteq R, \lambda(R/I) < \infty, I \not\subseteq J} \frac{e_{HK}(I) - e_{HK}(J)}{\lambda(J/I)} = \inf_{I \subseteq R, \lambda(R/I) < \infty} \frac{e_{HK}(I) - e_{HK}((I, x))}{\lambda(J/I)}.$$
5.2. F-signature and relative Hilbert-Kunz multiplicity. Our proof of Theorem C begins with the following well known lemma concerning the Frobenius splitting numbers of a local ring. We refer the reader to [PT18, Lemma 6.2] for a direct proof.

Lemma 5.2. Let \((R, m, k)\) be an F-finite local domain of prime characteristic \(p > 0\) and Krull dimension \(d\). Suppose that \(J_1 \subsetneq R\) is a choice of canonical ideal, \(0 \neq x_1 \in J_1, x_2, \ldots, x_d\) parameters on \(R/J_1\), and \(u \in R\) generates the socle \((J_1, x_2, \ldots, x_d)\). For each integer \(t \in \mathbb{N}\) let \(I_t = (x_1^{t-1}J_1, x_2, \ldots, x_d^t)\) and \(u_t = u(x_1 \cdots x_d)^{t-1}\). Then for each \(e \in \mathbb{N}\) the sequence of ideals \(\{(I_t^p^e : u_t^p)\}_{t \in \mathbb{N}}\) forms an ascending chain of ideals which stabilizes at the Frobenius degeneracy ideal \(I_e\). In particular, for each \(e \in \mathbb{N}\) there exists a \(t \in \mathbb{N}\) such that

\[
\frac{a_e(R)}{\text{rank}(F^e R)} = \frac{\lambda(R/(I_t, u_t)^{[p^e]}) - \lambda(R/(I_t, u_t)^{[p^e]})}{p^ed}.
\]

Theorem 5.3. Let \((R, m, k)\) be a local strongly F-regular F-finite domain of prime characteristic \(p > 0\) such that some symbolic power of the anti-canonical ideal has analytic spread at most 2 on the punctured spectrum. Then there exists an irreducible \(m\)-primary ideal \(I\) and \(u \in R\) which generates the socle mod \(I\) such that for each integer \(e \in \mathbb{N}\)

\[
I_e = (I^p^e : u^p).
\]

It follows that for all \(e \in \mathbb{N}\)

\[
\frac{a_e(R)}{\text{rank}(F^e R)} = \frac{\lambda(R/I^p^e) - \lambda(R/(I, u)^{[p^e]})}{p^{\text{dim} R}}
\]

and therefore

\[
s(R) = e_{\text{HK}}(I) - e_{\text{HK}}((I, u)).
\]

Proof. Following the proof of Theorem 4.2 and utilizing Theorem 4.6, Proposition 4.5, Proposition 4.4, and Proposition 4.3, if \(J_1 \subsetneq R\) is a choice of canonical ideal there exists 0 \(\neq x_1 \in J_1\), parameters \(x_2, \ldots, x_d\) on \(R/J_1\) and \(m \in \mathbb{N}\) such that if we let \(\{I_t\}, \{u_t\}\) be as in Lemma 5.2 then for each integer \(t \in \mathbb{N}\)

\[
(I_t^p^e : u_t^p) = (J_1^p^e, x_2^p^e, \ldots, x_d^p^e : u(x_2 \cdots x_d)^{(t-1)p^e} \subset\]

\[
(J_1^{[p^e]}, x_2^{[p^e]}, x_3^{[p^e]}, x_4^{[p^e]}, \ldots, x_d^{[p^e]} : x_1^{4m-1}u(x_2x_3^3(x_4 \cdots x_d)^3)^p^e) = (I_3^{[p^e]} : u_3^{[p^e]} : x_1^{4m-1}).
\]

Because the sequence of ideals \(\{(I_t^p^e : u_t^p)\}\) is an ascending chain of ideals which stabilizes at the Frobenius degeneracy ideal \(I_e\) we see that there are containments

\[
(I_3^{[p^e]} : u_3^{[p^e]}) \subset I_e \subset (I_3^{[p^e]} : u_3^{[p^e]}) : x_1^{4m-1}
\]

We claim that the inclusion \((I_3^{[p^e]} : u_3^{[p^e]}) \subset I_e\) is an equality for each \(e \in \mathbb{N}\). Suppose \(r \in I_e\), then \(r^{p^e} \in I_3^{[p^e]} \subset I_{e+e_0} \subset (I_3^{[p^e]+e_0]} : u_3^{[p^e]+e_0]} : x_1^{4m-1}\). Therefore \(x_1^{4m-1}(u_3^{[p^e]} r)^{p^e} \in (I_3^{[p^e]} : u_3^{[p^e]})\) for all \(e_0 \in \mathbb{N}\). Hence \(u_3^{[p^e]} r \in (I_3^{[p^e]} : u_3^{[p^e]})\), i.e. \(r \in (I_3^{[p^e]} : u_3^{[p^e]})\) as claimed. \(\square\)

Suppose \((R, m, k)\) is an F-finite normal domain of Krull dimension \(d\). Then for each \(m\)-primary ideal \(I \subseteq R\) there is a real number \(\beta_I\) such that \(\lambda(R/I^p^e) = e_{\text{HK}}(R)p^ed + \beta_I p^{ed-1} + O(p^{ed-2})\) by [HMM04, Theorem 1].
Corollary 5.4. Let \((R, \mathfrak{m}, k)\) be a local strongly \(F\)-regular \(F\)-finite domain of prime characteristic \(p > 0\), of Krull dimension \(d\), and such that some symbolic power of the anti-canonical ideal has analytic spread at most 2 on the punctured spectrum. Then there exists a real number \(\tau \in \mathbb{R}\) such that
\[
\lambda(R/I_e) = s(R)p^{ed} + \tau p^{e(d-1)} + O(p^{e(d-2)}).
\]

Proof. By Theorem 5.3 there exists \(m\)-primary ideal \(I \subseteq R\) and \(u \in R\) such that
\[
\frac{a_e(R)}{\text{rank}(F_e^e R)} = \frac{\lambda(R/I_e)}{p^{ed}} = \frac{\lambda(R/I[u^{p^e}]) - \lambda(R/(I, u)[u^{p^e}])}{p^{ed}}
\]
for all \(e \in \mathbb{N}\). Every strongly \(F\)-regular local ring is normal and therefore the results of [HMM04] are applicable. □

6. Questions

Lyubeznik and Smith proved that if \(R\) is an \(F\)-finite \(\mathbb{N}\)-graded ring then the finitistic test ideal and test ideal of \(R\) agree, [LS99, Corollary 3.4]. This article shows equality of test ideals for local rings whenever a certain family of Ext-modules are annihilated in a controlled way. It is therefore natural to ask when the Ext-annihilation properties established in Theorem 4.6 hold for graded rings. For example, we ask the following:

Question 1. Let \(S = k[T_1, \ldots, T_n]\) be a polynomial ring over a field \(k\) of prime characteristic \(p > 0\), \(P \subseteq R\) a homogeneous prime ideal of height \(h\), and \(S = R/P\). Suppose that the Krull dimension of \(R\) is \(d\) and \(J_1 \subseteq R\) a canonical ideal. Does there exist an integer \(m \in \mathbb{N}\) and parameters \(x_2, \ldots, x_d\) on \(R/J_1\) such that
\[
(x_2^i, x_3^i, \ldots, x_d^i) \Ext^h_{S}(\Ext^{h+1}_{S}(R/J_1^{mi+1}, S), S) = 0 \quad \text{for every integer } i \in \mathbb{N} \text{ and } 2 \leq j \leq d - 2?
\]

Under mild hypotheses, this article equates the finitistic test ideal and test ideal of a ring under the assumption that the anti-canonical ideal has analytic spread at most 2 on the punctured spectrum. For rings of Krull dimension at most 4 this is equivalent to the anti-canonical algebra being Noetherian on the punctured spectrum.

Question 2. Can the techniques of this article be extended to show equality of test ideals whenever the anti-canonical algebra of a ring is assumed to be Noetherian on the punctured spectrum?

The critical point of the argument where the analytic spread 2 assumption is being used is in Claim 4.12. In Claim 4.12 we find families of ideals which intersect principally, so that the higher Ext-modules of the cyclic modules defined by these ideals vanish.

There are interesting connections between the theory of multiplier ideals in the birational geometry of complex varieties and test ideals of varieties defined over a field of prime characteristic. Suppose \(R\) is an \(F\)-finite normal domain. Following the methods of [HY03, Tak04] one can develop a tight closure theory of triples \((R, \Delta, a^t)\) where \(\Delta \geq 0\) is an effective \(Q\)-divisor, \(a \subseteq R\) an ideal, and \(t \geq 0\) a real number. Suppose that \(K\) is the fraction field of \(R\). Then for each \(e \in \mathbb{N}\) consider the fractional ideal \(R((p^e - 1)\Delta) \subseteq K\) generated by nonzero elements \(f \in K\) such that \(\text{div}(f) + (p^e - 1)\Delta\) is effective. For each \(e \in \mathbb{N}\) we consider the extension
of scalars functors $F^e_\Delta : \text{Mod}(R) \to \text{Mod}(R)$ sending a module $M \mapsto {}^e R((p^e - 1)\Delta) \otimes_R M$. An element $m \in M$ is mapped to $F^e_\Delta(m) = m^{p^e} := 1 \otimes m \in {}^e R((p^e - 1)\Delta) \otimes_R M$. If $N \subseteq M$ are $R$-modules we say that an element $m$ is in the $(\Delta, a^t)$-tight closure of $N$, denoted by $N^t_M(\Delta, a^t)$, if there exists $c \in R^{>0}$ such that the submodule $a^{(c)p^e}m$ of $M$ is contained in the kernel of the following maps for all $e \gg 0$:

$$M \to M/N \to F^e_\Delta(M/N) \to F^e_\Delta(M/N).$$

The finitistic $(\Delta, a^t)$-tight closure of $N \subseteq M$ is denoted by $N^t_M(\Delta, a^t)^{fg}$ and is $\bigcup(N \cap M')^{t(\Delta, a^t)^{*}}$ where the union is taken over all finitely generated submodules $M'$ of $M$. If $\Delta = 0$ and $a = R$ then $(\Delta, a^t)$-tight closure agrees with the usual tight closure and finitistic $(\Delta, a^t)$-tight closure agrees with the usual finitistic tight closure.

**Question 3.** To what extent can the results of this article be extended to show equality of test ideals of pairs or triples? Specifically, if $(R, \Delta, a^t)$ is a triple and $(R, m, k)$ is local, then when may we conclude that

$$0^{(\Delta, a^t)^{*}}_{E_R(k)} = 0^{(\Delta, a^t)^{*}, fg}_{E_R(k)}?$$

For a partial answer to the above question see [Tak04, Theorem 2.8] for a proof that $0^{(\Delta, a^t)^{*}}_{E_R(k)} = 0^{(\Delta, a^t)^{*}, fg}_{E_R(k)}$ when $a = R$ and $K_X + \Delta$ is assumed to be a $\mathbb{Q}$-Cartier divisor, where $K_X$ is a canonical divisor on $X = \text{Spec}(R)$.

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References

[Abe02] Ian M. Aberbach. Some conditions for the equivalence of weak and strong $F$-regularity. *Comm. Algebra*, 30(4):1635–1651, 2002.

[AHH93] Ian M. Aberbach, Melvin Hochster, and Craig Huneke. Localization of tight closure and modules of finite phantom projective dimension. *J. Reine Angew. Math.*, 434:67–114, 1993.

[AL03] Ian M. Aberbach and Graham J. Leuschke. The $F$-signature and strong $F$-regularity. *Math. Res. Lett.*, 10(1):51–56, 2003.

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[BM10] Holger Brenner and Paul Monsky. Tight closure does not commute with localization. *Ann. of Math. (2)*, 171(1):571–588, 2010.

[Bro79] M. Brodmann. Asymptotic stability of $\text{Ass}(M/\mathfrak{m}^nM)$. *Proc. Amer. Math. Soc.*, 74(1):16–18, 1979.

[CEMS18] Alberto Chiecchio, Florian Enescu, Lance Edward Miller, and Karl Schwede. Test ideals in rings with finitely generated anti-canonical algebras—corrigendum [MR3742559]. *J. Inst. Math. Jussieu*, 17(4):979–980, 2018.

[CHS10] Steven Dale Cutkosky, Jürgen Herzog, and Hema Srinivasan. Asymptotic growth of algebras associated to powers of ideals. *Math. Proc. Cambridge Philos. Soc.*, 148(1):55–72, 2010.

[Cut88] S. Cutkosky. Weil divisors and symbolic algebras. *Duke Math. J.*, 57(1):175–183, 1988.

[DM19] Hailong Dao and Jonathan Montaño. Symbolic analytic spread: upper bounds and applications. *arXiv e-prints*, page arXiv:1907.07081, Jul 2019.
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[Dut13] S. P. Dutta. The monomial conjecture and order ideals. *J. Algebra*, 383:232–241, 2013.

[Dut16] S. P. Dutta. The monomial conjecture and order ideals II. *J. Algebra*, 454:123–138, 2016.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HH89] Melvin Hochster and Craig Huneke. Tight closure and strong $F$-regularity. *Mém. Soc. Math. France (N.S.)*, (38):119–133, 1989. Colloque en l’honneur de Pierre Samuel (Orsay, 1987).

[HH90] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990.

[HH91] Melvin Hochster and Craig Huneke. Tight closure and elements of small order in integral extensions. *J. Pure Appl. Algebra*, 71(2-3):233–247, 1991.

[HH93] Melvin Hochster and Craig Huneke. Phantom homology. *Mem. Amer. Math. Soc.*, 103(490):vi+91, 1993.

[HH94a] Melvin Hochster and Craig Huneke. $F$-regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.*, 346(1):1–62, 1994.

[HH94b] Melvin Hochster and Craig Huneke. Tight closure of parameter ideals and splitting in module-finite extensions. *J. Algebraic Geom.*, 3(4):599–670, 1994.

[HHT07] Jürgen Herzog, Takayuki Hibi, and Ngô Viêt Trung. Symbolic powers of monomial ideals and vertex cover algebras. *Adv. Math.*, 210(1):304–322, 2007.

[HL02] Craig Huneke and Graham J. Leuschke. Two theorems about maximal Cohen-Macaulay modules. *Math. Ann.*, 324(2):391–404, 2002.

[HMM04] Craig Huneke, Moira A. McDermott, and Paul Monsky. Hilbert-Kunz functions for normal rings. *Math. Res. Lett.*, 11(4):539–546, 2004.

[Hoa93] Le Tuan Hoa. Reduction numbers and Rees algebras of powers of an ideal. *Proc. Amer. Math. Soc.*, 119(2):415–422, 1993.

[Hoc77] Melvin Hochster. Cyclic purity versus purity in excellent Noetherian rings. *Trans. Amer. Math. Soc.*, 231(2):463–488, 1977.

[HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006.

[HS15] Craig Huneke and Ilya Smirnov. Prime filtrations of the powers of an ideal. *Bull. Lond. Math. Soc.*, 47(4):585–592, 2015.

[Hun96] Craig Huneke. *Tight closure and its applications*, volume 88 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With an appendix by Melvin Hochster.

[Hun13] Craig Huneke. Hilbert-Kunz multiplicity and the F-signature. In *Commutative algebra*, pages 485–525. Springer, New York, 2013.

[HW02] Nobuo Hara and Kei-Ichi Watanabe. F-regular and F-pure rings vs. log terminal and log canonical singularities. *J. Algebraic Geom.*, 11(2):363–392, 2002.

[HY03] Nobuo Hara and Ken-Ichi Yoshida. A generalization of tight closure and multiplier ideals. *Trans. Amer. Math. Soc.*, 358(8):3143–3174, 2003.

[ILL+07] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther. *Twenty-four hours of local cohomology*, volume 87 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2007.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original.

[KR86] Daniel Katz and Louis J. Ratliff, Jr. On the symbolic Rees ring of a primary ideal. *Comm. Algebra*, 14(5):959–970, 1986.

[Kun76] Ernst Kunz. On Noetherian rings of characteristic $p$. *Amer. J. Math.*, 98(4):999–1013, 1976.
[Lip69] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36):195–279, 1969. 2, 29

[LS99] Gennady Lyubeznik and Karen E. Smith. Strong and weak F-regularity are equivalent for graded rings. *Amer. J. Math.*, 121(6):1279–1290, 1999. 2, 32

[Mac96] Brian Cameron Macrimmon. Strong F-regularity and boundedness questions in tight closure. ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–University of Michigan. 2

[Mon83] P. Monsky. The Hilbert-Kunz function. *Math. Ann.*, 263(1):43–49, 1983. 3, 30

[MPST19] Linquan Ma, Thomas Polstra, Karl Schwede, and Kevin Tucker. F-signature under birational morphisms. *Forum Math. Sigma*, 7:e11, 20, 2019. 11

[PT18] Thomas Polstra and Kevin Tucker. F-signature and Hilbert-Kunz multiplicity: a combined approach and comparison. *Algebra Number Theory*, 12(1):61–97, 2018. 3, 4, 17, 29, 30, 31

[Ree58] D. Rees. On a problem of Zariski. *Illinois J. Math.*, 2:145–149, 1958. 9

[Rob85] Paul C. Roberts. A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian. *Proc. Amer. Math. Soc.*, 94(4):589–592, 1985. 9

[Sch86] Peter Schenzel. Finiteness of relative Rees rings and asymptotic prime divisors. *Math. Nachr.*, 129:123–148, 1986. 10

[Smi93] Karen Ellen Smith. *Tight closure of parameter ideals and F-regularity*. ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–University of Michigan. 5

[Smi97] Karen E. Smith. F-rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997. 29

[SS10] Karl Schwede and Karen E. Smith. Globally F-regular and log Fano varieties. *Adv. Math.*, 224(3):863–894, 2010. 2

[Tak04] Shunsuke Takagi. An interpretation of multiplier ideals via tight closure. *J. Algebraic Geom.*, 13(2):393–415, 2004. 32, 33

[Tru98] Ngô Viêt Trung. The Castelnuovo regularity of the Rees algebra and the associated graded ring. *Trans. Amer. Math. Soc.*, 350(7):2813–2832, 1998. 11

[Tuc12] Kevin Tucker. F-signature exists. *Invent. Math.*, 190(3):743–765, 2012. 3, 30

[Wat94] Keiichi Watanabe. Infinite cyclic covers of strongly F-regular rings. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, volume 159 of *Contemp. Math.*, pages 423–432. Amer. Math. Soc., Providence, RI, 1994. 11

[Wil95] Lori J. Williams. Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism. *J. Algebra*, 172(3):721–743, 1995. 2

[WY04] Kei-ichi Watanabe and Ken-ichi Yoshida. Minimal relative Hilbert-Kunz multiplicity. *Illinois J. Math.*, 48(1):273–294, 2004. 3

[Yao05] Yongwei Yao. Modules with finite F-representation type. *J. London Math. Soc. (2)*, 72(1):53–72, 2005. 1

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