Rate of convergence in the law of large numbers for supercritical general multi-type branching processes

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Abstract

Motivated by an application to the theory of fixed points of smoothing transformations, we provide sufficient conditions for polynomial rate of convergence in the weak law of large numbers for supercritical general indecomposable multi-type branching processes. The main result is derived by investigating the embedded single-type process composed of all individuals having the same type as the ancestor. As an important intermediate step, we determine the (exact) polynomial rate of convergence of Nerman’s martingale in continuous time to its limit. The techniques used also allow us to give streamlined proofs of the weak law of large numbers and ratio convergence for the processes in focus.

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1 Introduction

In the present paper, we derive sufficient conditions for polynomial rate of convergence in the weak law of large numbers for supercritical general indecomposable multi-type branching processes. The main source of motivation for us are fixed-point equations of smoothing transforms (see Section 6 for more details), i.e., equations for the distribution of a random variable \(X\) of the type

\[
X \overset{d}{=} C + \sum_{j \geq 1} T_j X_j
\]

where “\(\overset{d}{=}\)” denotes equality in distribution, \((C, T_1, T_2, \ldots)\) is a given sequence of random variables and \(X_1, X_2, \ldots\) is a sequence of i.i.d. copies of \(X\) independent of \((C, T_1, T_2, \ldots)\). Recently, there has been progress in solving such equations via techniques from single-type general branching processes [1, 5, 9, 11]. The cited papers consider the situation when \(X\) is nonnegative or real-valued and the \(T_j, j \geq 1\) are nonnegative. To extend the methods employed in [1, 5] to higher-dimensional situations or the situation where the \(T_j, j \geq 1\) take both positive and negative values, limit theorems for multi-type processes are required along with the corresponding rate of convergence results.

1.1 Model description

Let \(\mathbb{N} := \{1, 2, \ldots\}\) denote the set of positive integers and \(\mathbb{N}^0 = \{\emptyset\}\) the set that contains the empty tuple only. Define \(\mathcal{I} := \bigcup_{n \geq 0} \mathbb{N}^n\) to be the set of finite tuples of positive integers.
Members of \( \mathcal{I} \) are called (potential) individuals and are typically denoted by the letters \( x, y, z \). If \( x = (x_1, \ldots, x_n) \), we write \( |x| = n \) and call \( n \) the generation of \( x \). If \( y = (y_1, \ldots, y_m) \), then we write \( xy \) for \( (x_1, \ldots, x_n, y_1, \ldots, y_m) \). \( xk \) is defined as the ‘ancestor of \( x \) in the \( k \)th generation’, that is, \( x \) if \( k \leq |x| \). \( x \vartriangleleft y \) means that \( |x| < |y| \) and \( y|x| = x \) while \( x \preceq y \) means that either \( x < y \) or \( x = y \). If \( J \subseteq \mathcal{I} \) is a set of individuals, we write \( x < J \) (\( x \preceq J \)) if \( y \vartriangleleft x \) (resp., \( y \preceq x \)) for all \( y \in J \). In words, \( x < J \) if \( x \) has no ancestor in \( J \).

Let \((\Omega, \mathcal{A}, P)\) be a probability space on which point processes \( i(\mathcal{Z}) = \sum_{k=1}^{iN} \delta(\tau_k, X_k) \) on \( \{1, \ldots, p\} \times \mathbb{R}_{\geq 0}, \) \( i = 1, \ldots, p \) are defined where here and in the remainder of the paper, \( \delta_x \) denotes the Dirac measure with a point at \( x \) and \( \mathbb{R}_{\geq 0} := [0, \infty) \). Notice that \( P(\mathcal{N} = \infty) > 0 \) is not excluded. For \( i = 1, \ldots, p \), the basic probability space is defined to be the product space

\[
(\Omega, \mathcal{A}, \mathbb{P}) := (\{1, \ldots, p\}, \mathfrak{P}(\{1, \ldots, p\}), \delta_i) \otimes \prod_{x \in \mathcal{I}} (\Omega_x, \mathcal{A}_x, P_x),
\]

where \((\Omega_x, \mathcal{A}_x, P_x), x \in \mathcal{I} \) are copies of \((\Omega, \mathcal{A}, P)\) and \( \mathfrak{P}(\{1, \ldots, p\}) \) is the set of all subsets of \( \{1, \ldots, p\} \). In particular, each space \((\Omega_x, \mathcal{A}_x, P_x)\) carries copies \( i(\mathcal{Z}) = \sum_{k=1}^{iN} \delta(\tau_k, X_k) \) of the point processes \( i(\mathcal{Z}), j = 1, \ldots, p \). We slightly abuse notation and interpret the \( i(\mathcal{Z})(x)_j \) as \( \delta_i(\tau(x), X(x)) \) offspring. The \( i(\mathcal{Z}) \) and \( \tau(x) \) of \( x \) are defined by \( \tau(x) := \tau_x(\mathcal{Z}) \) and \( S(x) := X(x) \) respectively. \( \mathcal{G}_1 = \{1, 2, \ldots, \mathcal{N}\} \) is the first generation of the process. Further, an individual \( x = x_1 \ldots x_n \in \mathbb{N}^n \) of the \( n \)th generation with type \( \tau(x) = j \) and birth time \( S(x) \) produces at time \( n + 1 \) a random number \( \mathcal{N}(x) \) offspring. The offspring are labeled \( x_1, \ldots, x_{\mathcal{N}(x)} \). For \( y \in \mathbb{N}, y \leq \mathcal{N}(x) \), type and birth time of particle \( xy \) are given by \( \tau_y(x) \) and \( S(x) + \mathcal{N}(x)y \), respectively. The \( (n + 1) \)st generation \( \mathcal{G}_{n+1} \) is defined by

\[
\{xy : x \in \mathcal{G}_n \text{ and } y \in \mathbb{N}, y \leq \tau(x)N(x)\}.
\]

We set \( \mathcal{G} := \bigcup_{n \in \mathbb{N}} \mathcal{G}_n \). The point process of types and positions of the \( n \)th generation individuals will be denoted by

\[
\mathcal{Z}_n := \sum_{|x| = n} \delta(\tau(x), S(x))
\]

where here and in what follows summation over \( |x| = n \) means summation over \( x \in \mathcal{G}_n \). The sequence \((\mathcal{Z}_n)_{n \geq 0}\) forms a multi-type branching random walk.

We further assume the existence of a product-measurable, separable random characteristic \( \phi : \Omega \times \mathbb{R} \to [0, \infty) \) with \( \phi(t) = 0 \) for all \( t < 0 \). For \( t \in \mathbb{R} \), we write \( \phi(t) \) for the random variable \( \omega \mapsto \phi(\omega, t) \). Notice that \( \phi \) may depend on the types of all individuals in \( \mathcal{I} \), in particular on the type of the ancestor.

To define the general branching process counted with characteristic \( \phi \), we need to introduce further notation. An element \( \omega \in \Omega \) is of the form \( \omega = (i, (\omega_x)_{x \in \mathcal{I}}) \). For each \( x \in \mathcal{I} \), let \( \sigma_x : \Omega \to \Omega, \omega = (\omega_y)_{y \in \mathcal{I}} \mapsto \sigma_x \omega := (\tau(x), (\omega_{xy})_{y \in \mathcal{I}}) \) be the shift operator. Whenever \( \Psi \) is a function from \((\Omega, \mathcal{A})\) into another measurable space, we denote by \( [\Psi]_{\varphi} \) the function \( \omega \mapsto \Psi(\sigma_x \omega) \). The general (multi-type) branching process counted with characteristic \( \phi \) is
there exists some $\alpha > 0$, $e^{-\alpha t}Z^\theta(t)$ converges in probability to a limit which is not degenerate at 0.\footnote{Notice that in \cite{16, 23} the more general situation of an abstract type space is considered.} The main result of the paper at hand is Theorem 2.11, in which sufficient conditions are provided for $e^{-\alpha t}Z^\theta(t)$ to converge to its limit in probability at a polynomial rate. Additionally, the methods employed here allow us to give simple proofs of the convergence in probability of $e^{-\alpha t}Z^\theta(t)$ and the a.s. convergence of $Z^\theta(t)/Z^\varphi(t)$ which are stated as Theorems 2.1 and 2.4, respectively. As far as we know, the latter result improves on an earlier result by Norman \cite[Theorem 6.7]{20}.

\section{Main results}

It is known from \cite[Theorem 6.5]{20}, \cite[Theorem 7.2]{16} and \cite[Theorem 2.1]{23} that under appropriate assumptions which include the existence of a Malthusian parameter $\alpha > 0$, the following assumptions will only be in force when explicitly stated:

- (A1) For all $h > 0$, $h \in \mathbb{Z}$, and $j \in \{1, \ldots, p\}$, let $\nu_{ij} = \pi_{ij}(\nu)$ denote the intensity measure of the point process $\nu$.

- (A2) Either $\nu_{ij} > 0$, for all $i, j \in \{1, \ldots, p\}$.

- (A3) There exists a Malthusian parameter $\rho > 0$.

- (A4) $\nu_{ij} > 0$, for all $i, j \in \{1, \ldots, p\}$.

- (A5) $\nu_{ij} > 0$, for all $i, j \in \{1, \ldots, p\}$.

The main result of the paper at hand is Theorem 2.11, in which sufficient conditions are provided for $e^{-\alpha t}Z^\theta(t)$ to converge to its limit in probability at a polynomial rate. Additionally, the methods employed here allow us to give simple proofs of the convergence in probability of $e^{-\alpha t}Z^\theta(t)$ and the a.s. convergence of $Z^\theta(t)/Z^\varphi(t)$ which are stated as Theorems 2.1 and 2.4, respectively. As far as we know, the latter result improves on an earlier result by Norman \cite[Theorem 6.7]{20}.

\subsection{Preliminaries and assumptions}

For $i, j \in \{1, \ldots, p\}$, let $\nu_{ij}(d\nu \times dt)$ denote the intensity measure of the point process $\nu$ on $\mathbb{Z} \times \mathbb{R}^+$. By $\nu_{ij}$ we denote the Laplace transform of $\nu_{ij}(d\nu \times dt)$:

\begin{equation}
\nu_{ij}(\theta) := \int_{[0, \infty]} e^{-\theta t} \nu_{ij}(d\nu \times dt) = \mathbb{E}^i \left[ \sum_{x \in A \cap \{1, \ldots, p\}} e^{-\theta x} \right], \quad \theta > 0.
\end{equation}

Let $M(\theta)$ denote the matrix with entries $M(\theta)_{ij} = \nu_{ij}(\theta)$, $i, j = 1, \ldots, p$. Each $M(\theta)$ is a nonnegative matrix that may have entries $+\infty$. Throughout the paper, we make the following assumptions:

(A1) For all $h > 0$ and all $h_1, \ldots, h_p \in [0, h)$, $\nu_{ij}(d\nu \times dt) > 0$ for some $i, j \in \{1, \ldots, p\}$.

(A2) Either $M(0)$ has an infinite entry or $M(0)$ has finite entries only and Perron-Frobenius eigenvalue $\rho > 1$.

(A3) There exists some $\alpha > 0$ such that $M(\alpha)$ has finite entries only and 1 is the Perron-Frobenius eigenvalue of $M(\alpha)$ with left and right eigenvectors $u = (u_1, \ldots, u_p)$ and $v = (v_1, \ldots, v_p)$.

(A4) $\nu_{ij} \in \nu_{ij}(d\nu \times dt) > 0$, for all $i, j \in \{1, \ldots, p\}$.

The following assumption will only be in force when explicitly stated:

(A5) $\mathbb{E} \left[ \int_{[0, \infty]} e^{-\alpha t} \nu_{ij}(d\nu \times dt) \right] < \infty$, for all $i, j \in \{1, \ldots, p\}$.

Although the nonlattice assumption which forms a part of (A1) may appear restrictive at first sight, it is not, for it holds whenever one of the $\nu_{ij}(d\nu \times dt)$ has a nontrivial continuous component. With little effort, the results of the paper can be extended to the lattice case. While (A2) entails supercriticality, (A3) demands the existence of a Malthusian parameter.
By convention, we assume that \( u \) and \( v \) are such that \( u \cdot v^T = 1 \cdot v^T = 1 \) (\( T \) for transpose) or, more explicitly,
\[
\sum_{i=1}^{p} u_i v_i = \sum_{i=1}^{p} v_i = 1. \tag{2.2}
\]
Finally we note that (A4) is a drift condition, whereas (A5) is the classical \((Z \log Z)\)-condition for the multi-type branching random walk.

For \( i = 1, \ldots, p \), define
\[
{^i}W_n := \sum_{|x|=n} \frac{v_r(x)}{v_i} e^{-\alpha S(x)}, \quad n \in \mathbb{N}_0. \tag{2.3}
\]

It can be checked that \((^iW_n)_{n \geq 0}\) is a nonnegative mean-one martingale under \( P^i \) with respect to the canonical filtration. Hence, it converges \( P^i\)-a.s. to some finite random variable \( ^iW \geq 0 \). If (A5) holds, then \( P^i(^iW > 0) > 0 \), see Section 4.1 for details.

### 2.2 Convergence in probability and \( L^1 \)

Our starting point is the following weak law of large numbers.

**Theorem 2.1.** Assume that \( t \mapsto e^{-\alpha t} E^i[\phi(t)] \) is directly Riemann integrable and that, for each \( i = 1, \ldots, p \),
\[
E^i \left[ \sup_{0 \leq s \leq t} \phi(s) \right] < \infty \quad \text{for all } t \geq 0 \tag{2.4}
\]

Then, for each \( i = 1, \ldots, p \),
\[
e^{-\alpha t} Z^\phi(t) \rightarrow ^iW \frac{v_i \sum_{j=1}^{p} u_j \int_{0}^{\infty} e^{-\alpha s} E^j[\phi(s)] \, ds}{\sum_{j,k=1}^{p} u_j v_k (-j m_k)'(\alpha)} \quad \text{in } P^i\text{-probability as } t \to \infty. \tag{2.5}
\]

If (A5) is valid, then the above convergence also holds in \( L^1(P^i) \).

This result has been derived before by Nerman [20] (using more restrictive assumptions than those here), Jagers [16] and Olofsson [23] (the last two in the more general situation of an abstract type space). We include a new proof since the methods employed in the proof of our main result, the rate of convergence in (2.5), lead to a short and simple derivation of (2.5).

### 2.3 Ratio convergence

Theorem 2.4 below provides sufficient conditions for the convergence of the ratio of two processes \( Z^\phi \) and \( Z^\psi \). The theorem is interesting mainly when the \((Z \log Z)\)-condition (A5) fails. It is an extension of Theorem 6.3 in [21] to the multi-type case and follows quite easily from the methods used here. A similar result can be found in [20]; however, there convergence is shown under assumptions that are too restrictive for the applications in connection with (1.1) that we have in mind.

**Condition 2.2.** There is some \( \theta < \alpha \) such that \( M(\theta) \) has finite entries only.

**Condition 2.3.** \( \phi \) is not identically 0 with positive probability and has paths in the Skorokhod space \( D := D(\mathbb{R}) \) of right-continuous functions with finite left limits and there exists a \( \theta < \alpha \) such that, for \( i = 1, \ldots, p \),
\[
E^i \left[ \sup_{t \geq 0} e^{-\theta t} \phi(t) \right] < \infty.
\]
Theorem 2.4. Assume that Condition 2.2 holds and that φ and ψ are characteristics satisfying Condition 2.3. Then, on \( S = \{ |G_n| > 0 \text{ for all } n \in \mathbb{N}_0 \} \), for \( i = 1, \ldots, p \),
\[
\frac{Z^\phi(t)}{Z^\psi(t)} \to \frac{\sum_{j=1}^p u_j \int_0^\infty e^{-\alpha s} E^j[\phi(s)] \, ds}{\sum_{j=1}^p u_j \int_0^\infty e^{-\alpha s} E^j[\psi(s)] \, ds} \quad \mathbb{P}^i\text{-almost surely as } t \to \infty. \tag{2.6}
\]

2.4 Rate of convergence

Let \( \delta > 0 \). The following conditions are needed to formulate our main result, Theorem 2.11.

Condition 2.5.
\[
\mathbb{E}^i \left[ \left( \frac{W_1}{\log^+ W_1} \right)^{1+\delta} \right] < \infty \quad \text{for } i = 1, \ldots, p.
\]

Condition 2.6. Assume that there is a finite sequence \( i_0, \ldots, i_n \in \{1, \ldots, p\} \) such that the convolution
\[
i_0 \mu \{ \{ i_1 \} \times \cdot \} * \ldots * i_n \mu \{ \{ i_n \} \times \cdot \}
\]
possesses a nontrivial component which is absolutely continuous with respect to the Lebesgue measure.

Remark 2.7. In the single-type case \((p = 1)\), Condition 2.6 says that the distribution \( e^{-\alpha t} \mu \{ \{ 1 \} \times dt \} \) is spread-out.\(^2\)

Condition 2.8. There exists an eventually increasing function\(^3\) \( h : \mathbb{R}_{\geq 0} \to (0, \infty) \) that is regularly varying of index 1 at \( \infty \) with the properties that (i) \( t \to t/h(t) \) is eventually decreasing, (ii) \( t \to t^2/h(t) \) is eventually increasing, and (iii) \( t (\log t)^{2\delta} = o(h(t)) \) as \( t \to \infty \) such that, for \( i = 1, \ldots, p \),
\[
\sup_{t \geq 0} \mathbb{E}^i \left[ h(e^{-\alpha t} Z^\phi(t)) \right] < \infty. \tag{2.7}
\]

For a particular \( \phi \) sufficient conditions for (2.7) to hold are given in the proof of Theorem 6.1. For general \( \phi \) finding such sufficient conditions is a problem on its own which does not seem simple, and we refrain from investigating it here.

Remark 2.9. \( h(t) = t (\log t)^{2\delta} \log(\log t) \) (for large \( t \)) is a typical example of the function \( h \) in Condition 2.8.

Condition 2.10. For \( i = 1, \ldots, p \), the mapping \( t \to e^{-\alpha t} \mathbb{E}^i[\phi(t)] \) is bounded and Lebesgue integrable with
\[
\lim_{t \to \infty} t^\delta \int_1^\infty e^{-\alpha s} \mathbb{E}^i[\phi(s)] \, ds = 0 \quad \text{and} \quad \lim_{t \to \infty} t^\delta \sup_{s \geq t} e^{-\alpha s} \mathbb{E}^i[\phi(s)] = 0. \tag{2.8}
\]

Theorem 2.11. Assume that, for some \( \delta > 0 \), Conditions 2.5, 2.6, 2.8 and 2.10 are valid and that, for each \( i = 1, \ldots, p \),
\[
\mathbb{E}^i \left[ \sum_{|x|=1} e^{-\alpha S(x)} S(x)^{1+\delta} \right] < \infty. \tag{2.9}
\]

Then, for each \( i = 1, \ldots, p \), in \( \mathbb{P}^i \)-probability,
\[
\lim_{t \to \infty} t^\delta \left| e^{-\alpha t} Z^\phi(t) - \left( \frac{\mathcal{W}_1 \sum_{j=1}^p u_j \int_0^\infty e^{-\alpha s} E^j[\phi(s)] \, ds}{\sum_{j,k=1}^p u_j v_k (-j m_k)'(\alpha)} \right) \right| = 0. \tag{2.10}
\]

\(^2\) A finite measure \( \mu \) on \( \mathbb{R} \) is called spread-out if some convolution power \( \mu^{*n} \) of \( \mu \) has a nontrivial component which is absolutely continuous with respect to Lebesgue measure.

\(^3\) A function \( h : \mathbb{R}_{\geq 0} \to \mathbb{R} \) is called increasing if \( s \leq t \) implies \( h(s) \leq h(t) \) for all \( s, t \geq 0 \). It is called eventually increasing if for some \( a \geq 0 \), \( h \) is increasing on \([a, \infty)\). \( h \) is called decreasing or eventually decreasing if \(-h \) is increasing or eventually increasing, respectively.
The rest of the paper is organized as follows. The proofs of the main results are given in Section 5. They are based on an embedding technique that is set out in Section 3. In Section 4, we derive auxiliary results concerning two martingales that are important in our analysis, namely, the additive martingale in the multi-type branching random walk and Nerman’s martingale in continuous time. For the former, we prove log-type moment results, for the latter, we derive the exact polynomial rate of convergence. Section 6 contains an application of our main results to a situation that is relevant in the context of (1.1).

3 The embedded single-type process

The basic idea in this paper is to derive the results in the multi-type case from the corresponding single-type ones by considering the embedded process of type-\(i\) individuals. In this section we prove some auxiliary results which are needed to construct the latter process.

3.1 Change of measure

For \(i = 1, \ldots, p\), we define the finite-dimensional distributions of a sequence \((M_n, S_n)_{n \geq 0}\) under \(\mathbb{P}^i\) on \(\{1, \ldots, p\} \times \mathbb{R}_{\geq 0}\) via the identity

\[
\mathbb{E}^i[h((M_0, S_0), \ldots, (M_n, S_n))] = \mathbb{E}^i\left[\sum_{|x|=n} e^{-\alpha S(x)} \frac{v_x}{v_i} h((i, 0), (\tau(x)_1, S(x)_1)), \ldots, (\tau(x), S(x)))\right],
\]

where \(h : (\{1, \ldots, p\} \times \mathbb{R}_{\geq 0})^{n+1} \to [0, \infty)\) is (Borel-\)measurable. The right-hand side of (3.1) equals 1 for \(h \equiv 1\) because the sequence \((^iW_n)_{n \geq 0}\) defined in (2.3) is a mean-one martingale. \(M(\alpha) v^T = v^T\) guarantees that (3.1) defines a consistent family of finite-dimensional distributions. One can further check using induction on \(n\) that \(((M_n, S_n))_{n \geq 0}\) is a Markov random walk\(^4\) with initial distribution \(\mathbb{P}^i((M_0, S_0) = (i, 0)) = 1\) and transition kernel

\[
\mathbb{P}^i((M_{n+1}, S_{n+1} - S_n) \in \{k\} \times B| M_n = j) = \frac{v_k}{v_j} \int_B e^{-\alpha t} j \{k\} \times dt = \frac{v_k}{v_j} \mathbb{E}^j\left[\sum_{|x|=1} e^{-\alpha S(x)} 1_{\{\tau(x) = k, S(x) \in B\}}\right]
\]

for \(j, k \in \{1, \ldots, p\}\) and \(B \subseteq \mathbb{R}_{\geq 0}\) Borel. For later use, we list a few properties of \((M_n)_{n \geq 0}\).

**Lemma 3.1.** Fix \(i \in \{1, \ldots, p\}\) and let \(\sigma^i := \inf\{n > 0 : M_n = i\}\).

(a) Under \(\mathbb{P}^i\), \((M_n)_{n \geq 0}\) is a Markov chain with probability of transition from \(j\) to \(k\) given by \(j m_k(\alpha) v_k/v_j\) and stationary distribution \(\pi = (\pi_1, \ldots, \pi_p)\) where \(\pi_j = u_j v_j, j = 1, \ldots, p\).

(b) \(\mathbb{E}^i[\#\{0 < k < \sigma^i : M_k = j\}] = \mathbb{E}^i[\sigma^i] u_j v_j\) and \(\mathbb{E}^i[\sigma^i] = (u_i v_i)^{-1}\).

(c) For some \(\gamma > 0\), \(\mathbb{E}^i[e^{\gamma \sigma^i}] < \infty\) for \(i = 1, \ldots, p\).

\(^4((M_n, S_n))_{n \geq 0}\) is a Markov random walk or Markov additive process on \(\{1, \ldots, p\} \times \mathbb{R}\) if \(((M_n, S_{n+1} - S_n))_{n \geq 0}\) is a time-homogeneous Markov chain on \(\{1, \ldots, p\} \times \mathbb{R}\) for which the transition probabilities depend only on the first coordinate, cf. [24].
Proof. The first statement in (a) follows from (3.2), the second from

$$\sum_{j=1}^{p} \frac{j m_k(\alpha) v_k}{v_j} = \sum_{j=1}^{p} u_j j m_k(\alpha) v_k = (uM(\alpha))v_k = u_k v_k = \pi_k$$

and the fact that \( \pi \) is normed by convention, see (2.2). For the proof of (b), define \( \tilde{\pi}_j := E^{j}[\{0 \leq k < \sigma^j : M_k = j\}] \), \( j = 1, \ldots, p \), and \( \tilde{\pi} := (\tilde{\pi}_j)_{j=1,\ldots,p} \). It is known that \( \tilde{\pi} \) is a left eigenvector to the eigenvalue \( 1 \) for the transition matrix \( (m_k(\alpha) v_k/v_j)_{j,k=1,\ldots,p} \). Hence, \( \tilde{\pi} = c \pi \) for some \( c > 0 \). Further, \( \sum_{j=1}^{p} \tilde{\pi}_j = E^{i}[\sigma^i] \), \( \sum_{j=1}^{p} \pi_j = 1 \) and \( \tilde{\pi}_i = 1 \) imply \( c = E^{i}[\sigma^i] = \pi_i^{-1} \) and hence assertion (b). Finally, by (A1), there exists an \( n \in \mathbb{N} \) such that \( M(\alpha)^n \) has positive entries only. Let \( d \) be the minimal entry of the matrix \( M(\alpha)^n \). Then \( \mathbb{P}^i(\sigma^i > kn) \leq (1 - d)^k \) for all \( k \in \mathbb{N}_0 \). From this, assertion (c) is easily deduced.

\[\square\]

3.2 Optional lines

We make use of particular optional lines (see [10, 16] for a general treatment).

Fix \( i \in \{1, \ldots, p\} \) and let \( \sigma^i \) be defined as in Lemma 3.1, i.e., \( \sigma^i := \inf\{k > 0 : M_k = i\} \). Associated with \( \sigma^i \) is an optional line \( J^i \subseteq \mathcal{G} \) defined by

\[ J^i := \{x \in \mathcal{G} \setminus \{\emptyset\} : \tau(x) = i, \tau(x_j) \neq i \text{ for } 0 < j < |x|\}. \]

Further, let \( \sigma^i_n \) be the \( n \)th consecutive application of \( \sigma^i \), i.e., \( \sigma^i_0 := 0 \) and \( \sigma^i_n := \inf\{k > \sigma^i_{n-1} : M_k = i\} \). The optional lines associated with the \( \sigma^i_n \) are denoted by \( J^i_n \), i.e., \( J^i_0 := \{\emptyset\} \) and

\[ J^i_n := \bigcup_{x \in J^i_{n-1}} \{xy : y \in |J^i|_x\}. \]

Notice that the \( J^i_n \) as defined here are optional lines in the sense of [16] and very simple lines in the sense of [10, Section 6]. Jagers [16, Theorem 4.14] established the strong Markov property for branching processes along optional lines, a result that is crucial for our arguments here.

One can check using (3.1) that

\[ \mathbb{P}^i[(M_0, S_0), \ldots, (M_n, S_n)] \in B, \sigma^i = n] = \mathbb{E}^i \left[ \sum_{|x|=n : x \in J^i} e^{-\alpha S(x) v_i} \delta_{T \otimes S(x)}(B) \right] \tag{3.3} \]

and

\[ \mathbb{P}^i[(M_0, S_0), \ldots, (M_n, S_n)] \in B, \sigma^i > n] = \mathbb{E}^i \left[ \sum_{|x|=n : x \in J^i} e^{-\alpha S(x) v_i} \delta_{T \otimes S(x)}(B) \right] \tag{3.4} \]

for \( B \subseteq \{1, \ldots, p\} \times \mathbb{R}_{\geq 0} \) Borel and \( T \otimes S(x) := ((\tau(x|_k), S(x|_k)))_{0 \leq k \leq |x|} \). Summation over \( n \geq 0 \) and a standard approximation argument give

\[ \mathbb{E}^i[f(M_{\sigma^i}, S_{\sigma^i})] = \mathbb{E}^i \left[ \sum_{x \in J^i} e^{-\alpha S(x) v_i} f(\tau(x), S(x)) \right] \tag{3.5} \]

and

\[ \mathbb{E}^i \left[ \sum_{k=0}^{\sigma^{i-1}} f(M_k, S_k) \right] = \mathbb{E}^i \left[ \sum_{x \in J^i} e^{-\alpha S(x) v_i} f(\tau(x), S(x)) \right] \tag{3.6} \]

for every measurable function \( f : \{1, \ldots, p\} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \).
For ease of notation, in the subsequent proofs, we shall fix \( i = 1 \) (the type of the ancestor). This constitutes no loss of generality. We shall write \( \mathbb{P} \) for \( \mathbb{P}^1 \), \( \mathbb{E} \) for \( \mathbb{E}^1 \), \( \sigma_n \) for \( \sigma^1_n \), \( J \) for \( J^1 \), \( J_n \) for \( J^1_n \), etc.

By \( Z_{J_n} \), we denote the point process \( \sum_{x \in J_n} \delta_{s(x)} \), by \( \mu_n \) its associated intensity measure, and by \( m_n \) the Laplace transform of \( \mu_n \). We write \( Z_{J_n} \), \( \mu \) and \( m \) when \( J = J_1 \). Further, for \( n \in \mathbb{N}_0 \), we define

\[
V_n := \int_{(0,\infty)} e^{-at} Z_{J_n}(dt) = \sum_{x \in J_n} e^{-\alpha S(x)}. \tag{3.7}
\]

(\( V_n \))_{n \geq 0} is a nonnegative martingale w.r.t. the canonical filtration and converges a.s. to a limit variable \( V \geq 0 \). In the following proposition, we establish that \( (Z_{J_n})_{n \in \mathbb{N}_0} \) fulfills the standing assumptions given on p. 366 of [21] which correspond to the assumptions (A1)–(A4) in the case \( p = 1 \) here.

**Proposition 3.2.** Assume that (A1)–(A4) hold. Then:

(a) \( \mu \) is not concentrated on any lattice \( h\mathbb{Z} \) for \( h > 0 \).

(b) \( m(0) > 1 \) and \( m(\alpha) = 1 \).

(c) \( -m'(\alpha) := \int_{[0,\infty)} u e^{-\alpha u} \mu(du) = (u_1 v_1)^{-1} \sum_{i,j=1}^p u_i v_j (\gamma m_j)'(\alpha) < \infty \).

**Proof.** By (3.5), (a) is equivalent to \( \mathbb{P}(S_\sigma \in h\mathbb{Z}) < 1 \) for all \( h > 0 \). On the other hand, \( \mathbb{P}(S_\sigma \in h\mathbb{Z}) = 1 \) for \( h > 0 \) is equivalent to the existence of \( h_1, \ldots, h_p \in [0, h) \) with \( \mathbb{P}(S_1 \in h_{M_1} - h_i + h\mathbb{Z}) = 1 \) for all \( i = 1, \ldots, p \), see [26]. The latter is excluded by (A1).

Regarding (b), observe that by (3.3) and the recurrence of the Markov chain \( (M_n)_{n \geq 0} \)

\[
m(\alpha) = \mathbb{E} \left[ \sum_{x \in J} e^{-\alpha S(x)} \right] = \mathbb{P}(\sigma < \infty) = 1.
\]

Further, the function \( \theta \mapsto m(\theta) \) is strictly decreasing in \( \theta \), hence \( m(0) > 1 \).

Regarding the proof of (c), first notice that

\[
-m'(\alpha) = \mathbb{E} \left[ \sum_{x \in J} e^{-\alpha S(x)} S(x) \right] = \mathbb{E}[S_\sigma]
\]

having utilized (3.5) for the second equality. The latter can be rewritten using standard Markov renewal theory:

\[
\mathbb{E}[S_\sigma] = \mathbb{E}[\sigma] \sum_{i=1}^p \pi_i \mathbb{E}'[S_1] = \pi_1^{-1} \sum_{i=1}^p \pi_i \mathbb{E}'[S_1]
\]

where \( \pi_i = u_i v_i, \quad i = 1, \ldots, p \) (see Lemma 3.1). Using (3.1), \( \mathbb{E}'[S_1] \) can be written as

\[
\mathbb{E}'[S_1] = \sum_{j=1}^p \mathbb{E}'[S_1 1_{\{M_1 = j\}}] = \frac{1}{v_i} \sum_{j=1}^p v_j (-m_j)'(\alpha)
\]

which yields

\[
-m'(\alpha) = (u_1 v_1)^{-1} \sum_{i=1}^p \sum_{j=1}^p u_i v_j (-m_j)'(\alpha).
\]

\[\square\]
4 Martingale convergence

For the proofs of our main results we need certain results on the martingales \( (V_n)_{n \geq 1} \), \( (iW_n)_{n \geq 1} \) and Nerman’s martingale, and the relations between them.

4.1 Basic martingale convergence results

In this section, for the reader’s convenience, we review the basic convergence theorems for the martingales \( (V_n)_{n \geq 1} \) and \( (iW_n)_{n \geq 1} \). Let \( S = \{|G_n| > 0 \text{ for all } n \in \mathbb{N}\} \) denote the survival set of the multi-type branching random walk.

**Proposition 4.1.** Fix \( i \in \{1, \ldots, p\} \). Then the following assertions are equivalent:

1. \( (iW_n)_{n \geq 1} \) is uniformly integrable w.r.t. \( \mathbb{P}^{i} \).
2. \( \mathbb{P}^{i}(iW > 0) > 0 \).
3. \( \{iW > 0\} = S \mathbb{P}^{i}\text{-a.s.} \).
4. \( (A5) \) holds.

**Source.** The equivalence between (b) and (f) follows from Theorem 1 in [17]. Note that in the cited reference Kyprianou and Sani assume Condition 2.2 to hold, that is, that \( M(\beta) \) has finite entries only for some \( \beta < \alpha \). However, their proof also works when this assumption is replaced by the present (weaker) assumption (A4).\(^5\) The remaining equivalences follow from standard arguments.

5Condition 2.2 is assumed in [17] in order to conclude that the spinal walk, which corresponds to \( (S_n)_{n \geq 0} \) here, has finite-mean increments. The latter property follows from the proof of our Proposition 3.2(c). Further, note that the drift condition in Theorem 1 of [17], \( \log \rho(\theta) - \theta \rho(\theta)/\rho(\theta) > 0 \) (retaining their notation), is used only to show that the drift of \( (S_n)_{n \geq 0} \), \( \mathbb{E}[S_1] \) is positive. The latter is clear here since \( S_1 > 0 \text{ a.s.} \).

When \( p = 1 \), Proposition 4.1 is known as Biggins’ martingale convergence theorem. Versions of this theorem have been derived by Biggins [7], Lyons [18] and Alsmeyer and Iksanov [2] (in increasing generality).

**Proposition 4.2.** Let \( b : (0, \infty) \to (0, \infty) \) be a measurable, locally bounded function that is regularly varying at \( +\infty \) of positive index. Then for \( \mathbb{E}^i [iW b(\log^+ iW)] < \infty \) for \( i = 1, \ldots, p \)

to hold it is sufficient that

\[
\mathbb{E}^i \left[ iW_1 b(\log^+ iW_1) \log^+ iW_1 \right] < \infty \text{ for } i = 1, \ldots, p.
\]

Proposition 4.2 is the multi-type analogue of one implication of Theorem 1.4 in [2] and the proof given below follows closely the proof given in [2]. It is likely that the converse implication of the proposition also holds true as it is the case for \( p = 1 \). However, we refrained from investigating it since we only need the converse implication in the single-type case.

**Proof.** We shall not make use of the fact that the point processes \( iZ(\{j\} \times \cdot) \) are concentrated on \( \mathbb{R}_{\geq 0} \) thereby proving the proposition in a greater generality than it is stated.

The following recursive construction of the modified multi-type branching random walk with a distinguished ray \( (\xi_n)_{n \in \mathbb{N}_0} \), called spine, is based on the presentations in [2, 17] and, therefore, kept short here. Start with \( \xi_0 := \emptyset \) and suppose that the first \( n \) generations have been constructed with \( \xi_k \) being the spinal individual in the \( k \)th generation, \( k \leq n \). Now,
while $\xi_n$ has child the displacements of which relative to $\xi_n$ are given by a point process whose law has Radon-Nikodym derivative $\sum_{x \in N(\xi_n)} v_{x}^{\xi_n} e^{-\alpha(S(x) - S(\xi_n))}$ with respect to the law of $\tau(\xi_n)Z$, where $N(x) := \{xy : y \in [G_1]\}$ denotes the set of children of $x$, all other individuals of the $n$th generation produce and spread offspring according to independent copies of $^iZ$, $i = 1, \ldots, p$ (i.e., in the same way as in the original multi-type BRW). All children of the individuals of the $n$th generation form the $(n + 1)$st generation, and among the children of $\xi_n$ the next spinal individual $\xi_{n+1}$ is picked with probability proportional to $v_k e^{-\alpha s}$ if $s$ is the displacement of $\xi_{n+1}$ relative to $\xi_n$ and $k$ is the type of $\xi_{n+1}$.

Let $\hat{Z}_n$ denote the point process describing the positions of all members of the $n$th generation as well as their types. We call $(\hat{Z}_n)_{n \geq 0}$ modified multi-type branching random walk associated with the original multi-type branching random walk $(Z_n)_{n \geq 0}$. Both, $(Z_n)_{n \geq 0}$ and $(\hat{Z}_n)_{n \geq 0}$, may be viewed as a random weighted tree with an additional distinguished ray (the spine) for $(\hat{Z}_n)_{n \geq 0}$. On an appropriate measurable space $(X, B)$ specified below, they can be realized as the same random element under two different probability measures $P^i$ and $\hat{P}^i$, respectively.

Let $R = \{(0, \xi_1, \xi_2, \ldots) : \xi_k \in N \text{ for all } k \in N\}$ denote the set of infinite rays (starting at 0) and, for a subtree $t \subset I$, let $\mathcal{G}(t)$ be the set of functions $s : I \to R \cup \{\infty\}$ assigning position $s(x) \in R$ to $x \in t$ (with $s(\emptyset) = 0$) and $s(x) = \infty$ to $x \not\in t$. Further, let $\Sigma(t)$ denote the set of functions $q : I \to \{0, 1, \ldots, p\}$ assigning type $q(x) \in \{1, \ldots, p\}$ to $x \in t$ and $q(x) = 0$ to $x \not\in t$. Then let

$$X := \{(t, s, q, \xi) : t \subset I, s \in \mathcal{G}(t), q \in \Sigma(t), \xi \in R\}$$

be the space of weighted rooted subtrees of $I$ with a distinguished ray (spine). Endow this space with the $\sigma$-field $B := \sigma(B_n : n = 0, 1, \ldots)$, where $B_n$ is the $\sigma$-field generated by the sets

$$\{(t', s', q', \xi') \in X : t' = t_n, s'|_{t_n} \in B, q'|_{t_n} = q_{t_n} \text{ and } \xi'|_{t_n} = \xi_{t_n}\}$$

where $t' = \{x \in t' : |x| \leq n\}$, $t_n$ ranges over the subtrees $\subseteq I$ with $\max\{|x| : x \in t_n\} \leq n$, $q$ ranges over $\Sigma(t)$, $B$ over the Borel sets $\subseteq \mathbb{R}^n$ and $\xi$ over $R$. The subscript $t_n$ means restriction to the coordinates in $t_n$ while the subscript $t_n$ means restriction to all coordinates up to the $n$th. Similarly, let $\mathcal{F}_n \subset B_n$ denote the $\sigma$-field generated by the sets

$$\{(t', s', q', \xi') \in X : t' = t_n, s'|_{t_n} \in B, q'|_{t_n} = q_{t_n}\}$$

where again $t_n$ ranges over the subtrees $\subseteq I$ with $\max\{|x| : x \in t_n\} \leq n$, $q$ ranges over $\Sigma(t)$ and $B$ over the Borel sets $\subseteq \mathbb{R}^n$. Then under $\hat{P}^i$ the identity map $(G, S, \tau, \xi) = (G, (S(x))_{x \in I}, (\tau(x))_{x \in I}, (\xi_n)_{n \geq 0})$ represents the modified multi-type branching random walk with its spine, while $(G, S, \tau)$ under $P^i$ represents the original branching random walk (the way how $P^i$ picks a spine does not matter and thus remains unspecified). Finally, the random variable $\hat{W}_n : X \to \mathbb{R}_{\geq 0}$, defined as

$$\hat{W}_n(t, s, q, \xi) := \sum_{x \in I : |x| = n} v_{x}^{q} e^{-\alpha s(x)}$$

is $\mathcal{F}_n$-measurable for each $n \geq 0$ and satisfies $\hat{W}_n = \sum_{|x| = n} \frac{v_{x}^{q}}{v_{1}} e^{-\alpha s(x)}$. $W_n$ is the Radon-Nikodym derivative of $\hat{P}^i$ w.r.t. $P^i$ on $\mathcal{F}_n$, see formula (4) in [17]. Standard theory (cf. Lemmas 5.1 and 5.2 in [2]) yields that the martingale $(\hat{W}_n)_{n \in \mathbb{N}_0}$ is uniformly $P^i$-integrable if and

---

Theorem 1.2. There is a slight abuse of notation in interpreting $P^i$ as a distribution on $(X, B)$ rather than on the product space $(\Omega, A)$. However, we think that introducing a new notation for this proof would be distracting rather than clarifying.
only if \( \hat{\mathbb{P}}^i \{ 'W < \infty \} = 1 \) and, in the case of uniform integrability,

\[
\mathbb{E}^i[ 'W h('W)] = \hat{\mathbb{E}}^i[h('W)]
\]

(4.1)

for each nonnegative Borel function \( h \) on \([0, \infty)\).

For \( x \in \mathcal{G} \), put \( L(x) = \frac{v_{\tau(x)} - a_S(x)}{v_{\tau(x)}} \) and notice that, if \( |x| = k \),

\[
[\tau(\emptyset) W_n]_x = \sum_{y: xy \in G_{k+n}} \frac{L(xy)}{L(x)}, \quad n = 0, 1, \ldots
\]

Since all individuals off the spine reproduce and spread as in the original multi-type BRW, we have that, under \( \mathbb{P}^i \) and \( \hat{\mathbb{P}}^i \), the \([\tau(\emptyset) W_n]_x \) for \( x \) off the spine and of type \( j \) have the same distributions as \( j W_n \) under \( \mathbb{P}^j \), in particular,

\[
\hat{\mathbb{E}}^i[ [\tau(\emptyset) W_n]_x ] = \mathbb{E}^i[ [\tau(\emptyset) W_n]_x ] = \mathbb{E}^j[ j W_n ] = 1.
\]

(4.2)

Let \( \mathcal{C} \) be the \( \sigma \)-field generated by the family of types of the children of the \( \xi_n \) and displacements of these relative to their mother, i.e., by the family \( \{ \tau(x), S(x) - S(\xi_n) : x \in \mathcal{N}(\xi_n), n = 0, 1, \ldots \} \). For \( n \geq 1 \) and \( k = 1, \ldots, n \), put

\[
R_{n,k} := \sum_{x \in \mathcal{N}(\xi_{k-1}) \setminus \{ \xi_k \}} \frac{L(x)}{L(\xi_{k-1})} ([\tau(\emptyset) W_{n-k}]_x - 1).
\]

With these definitions we can rewrite \( W_n \) as follows

\[
W_n = L(\xi_n) + \sum_{k=1}^{n} \sum_{x \in \mathcal{N}(\xi_{k-1}) \setminus \{ \xi_k \}} L(x) ([\tau(\emptyset) W_{n-k}]_x
\]

\[
= L(\xi_n) + \sum_{k=1}^{n} L(\xi_{k-1}) \left( \sum_{x \in \mathcal{N}(\xi_{k-1})} \frac{L(x)}{L(\xi_{k-1})} - \frac{L(\xi_k)}{L(\xi_{k-1})} + R_{n,k} \right)
\]

\[
= L(\xi_n) + \sum_{k=1}^{n} \left( L(\xi_{k-1}) \left( \sum_{x \in \mathcal{N}(\xi_{k-1})} \frac{L(x)}{L(\xi_{k-1})} + R_{n,k} \right) - L(\xi_k) \right) \hat{\mathbb{P}}^i \text{-a.s.}
\]

This implies

\[
\hat{\mathbb{E}}^i[ W_n | \mathcal{C} ] = \sum_{k=0}^{n-1} \left( L(\xi_k) \sum_{x \in \mathcal{N}(\xi_k)} \frac{L(x)}{L(\xi_k)} \right) - \sum_{k=1}^{n-1} L(\xi_k)
\]

\[
\leq \sum_{k=0}^{n-1} \left( L(\xi_k) \sum_{x \in \mathcal{N}(\xi_k)} \frac{L(x)}{L(\xi_k)} \right) \hat{\mathbb{P}}^i \text{-a.s.},
\]

(4.3)

for \( \hat{\mathbb{E}}^i[ L(\xi_{k-1}) R_{n,k} | \mathcal{C} ] = L(\xi_{k-1}) \hat{\mathbb{E}}^i[ R_{n,k} | \mathcal{C} ] = 0 \hat{\mathbb{P}}^i \text{-a.s.} \) as a consequence of (4.2) and since the \([\tau(\emptyset) W_{n-k}]_x \) for \( x \) off the spine are independent of \( \mathcal{C} \).

According to Proposition 4.1, the assumptions of the proposition ensure that the martingale \( \{ W_n \}_{n \in \mathbb{N}_0} \) is uniformly \( \mathbb{P}^i \)-integrable, hence \( \hat{\mathbb{P}}^i \{ 'W < \infty \} = 1 \). Passing to the limit as \( n \to \infty \) in (4.3) and using Fatou’s lemma, we get

\[
\hat{\mathbb{E}}^i[ W | \mathcal{C} ] \leq \bar{v} \sum_{k \geq 0} \left( e^{-\alpha S(\xi_k)} \sum_{x \in \mathcal{N}(\xi_k)} \frac{L(x)}{L(\xi_k)} \right) \hat{\mathbb{P}}^i \text{-a.s.}
\]

(4.4)
where \( \bar{v} := \max(v_1, \ldots, v_p)/\min(v_1, \ldots, v_p) \in [1, \infty) \).

Let \((A_n^{(k)}, B_n^{(k)}), k = 1, \ldots, p, n \geq 1\) be independent under \(\hat{P}^i\) with

\[
\hat{P}^i((A_n^{(k)}, B_n^{(k)}) \in B) = \hat{P}^i\left(\left. e^{-\alpha(S(\xi_n) - S(\xi_{n-1}))}, \sum_{x \in \mathcal{N}(\xi_{n-1})} \frac{v_r(x)}{v_r(\xi_{n-1})} e^{-\alpha(S(x) - S(\xi_{n-1}))} \right| \tau(\xi_{n-1}) = k \right) \in B \bigg| \tau(\xi_{n-1}) = k \bigg]
\]

\[
= \mathbb{E}^k \left[ \sum_{|x| = 1} \frac{v_r(x)}{v_k} e^{-\alpha S(x)} \mathbb{1}_B \left( e^{-\alpha S(x)}, \sum_{|y| = 1} \frac{v_r(y)}{v_k} e^{-\alpha S(y)} \right) \right],
\]

(4.5)

where \(B\) is a Borel subset of \(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\). Now define

\[
(C_n, D_n) := \left( \max_{k=1, \ldots, p} A_n^{(k)}, \max_{k=1, \ldots, p} B_n^{(k)} \right), \quad n \geq 1.
\]

Then the vectors \((C_n, D_n)_{n \geq 1}\) are i.i.d. Further, for \(x, y \geq 0\) and \(n \geq 1\),

\[
\hat{P}^i\left(\left. e^{-\alpha(S(\xi_n) - S(\xi_{n-1}))} \leq x, \sum_{x \in \mathcal{N}(\xi_{n-1})} \frac{v_r(x)}{v_r(\xi_{n-1})} e^{-\alpha(S(x) - S(\xi_{n-1}))} \leq y \right| \tau(\xi_{n-1}) = k \right) \hat{P}^i(A_n^{(k)} \leq x, B_n^{(k)} \leq y)
\]

\[
\geq \hat{P}^i\left(\left. \max_{k=1, \ldots, p} A_n^{(k)} \leq x, \max_{k=1, \ldots, p} B_n^{(k)} \leq y \right| \tau(\xi_{n-1}) = k \right) \hat{P}^i(C_n \leq x, D_n \leq y).
\]

Hence,

\[
\sum_{k \geq 0} \left( e^{-\alpha S(\xi_k)} \sum_{x \in \mathcal{N}(\xi_k)} \frac{L(x)}{L(\xi_k)} \right) \leq \sum_{k \geq 1} \left( \prod_{j < k} C_j \right) D_k
\]

(4.6)

where "\(\leq\)" denotes stochastic domination (w.r.t. \(\hat{P}^i\) here).

By Lemma 2.1 in [2], there exist nondecreasing and concave functions \(f\) and \(g\) on \([0, \infty)\) with \(f(0) = g(0) = 0\) such that \(b(\log x) \sim f(x)\) and \(b(\log x) \log x \sim g(x)\) as \(x \to \infty\). Therefore it suffices to prove that

\[
\mathbb{E}^i[I_W^i g(I_W^i)] < \infty \quad \text{for } i = 1, \ldots, p
\]

(4.7)

entails

\[
\mathbb{E}^i[I_W^i Wf(I_W^i)] < \infty \quad \text{for } i = 1, \ldots, p.
\]

Using (4.5) we have \(\hat{E}^i[g(B_1^{(k)})] = \mathbb{E}^k[I_W^i g(I_W^i)] < \infty\) for \(k = 1, \ldots, p\), where the finiteness is secured by (4.7). Hence

\[
\hat{E}^i[g(D_1)] = \hat{E}^i[g(\max_{k=1, \ldots, p} B_1^{(k)})]
\]

\[
\leq \hat{E}^i[g(B_1^{(1)} + \ldots + B_1^{(p)})] \leq \sum_{k=1}^p \hat{E}^i[g(B_1^{(k)})] < \infty,
\]

(4.8)

the penultimate inequality following by subadditivity. Since

\[
\frac{v_r e^{-\alpha(S(\xi_n) - S(\xi_{n-1}))}}{v_r(\xi_{n-1})} \leq \sum_{x \in \mathcal{N}(\xi_{n-1})} \frac{v_r(x)}{v_r(\xi_{n-1})} e^{-\alpha(S(x) - S(\xi_{n-1}))}
\]

\(\hat{P}^i\)-a.s.
for each $n \geq 1$, where $\underline{V} := \min(v_1, \ldots, v_p)/\max(v_1, \ldots, v_p) \in (0, 1]$, we infer with the help of (4.5) that, for $n \geq 1$, under $\hat{\mathbb{P}}^i$,

$$
\underline{V}C_n \leq D_n.
$$

The latter inequality in combination with (4.8) and the concavity of $g$ implies

$$
\underline{V}\hat{\mathbb{E}}^i[g(C_1)] \leq \hat{\mathbb{E}}^i[g(C_1)] \leq \hat{\mathbb{E}}^i[g(D_1)] < \infty. \tag{4.9}
$$

By Theorem 1.2 in [2] (4.8) and (4.9) are sufficient for

$$
\hat{\mathbb{E}}^i\left[f\left(\sum_{k \geq 1} \left(\prod_{j<k} C_j\right)D_k\right)\right] < \infty
$$

to hold. This together with (4.4) and (4.6) yields

$$
\mathbb{E}^i[Wf(tW)] = \hat{\mathbb{E}}^i[f(tW)] \leq \hat{\mathbb{E}}^i\left[f\left(\overline{v}\sum_{k \geq 1} \left(\prod_{j<k} C_j\right)D_k\right)\right]
$$

$$
\leq \bar{v}\hat{\mathbb{E}}^i\left[f\left(\sum_{k \geq 1} \left(\prod_{j<k} C_j\right)D_k\right)\right] < \infty, \tag{4.10}
$$

where the equality is a consequence of (4.1), the first inequality is justified by an application of Jensen’s inequality for conditional expectations, while the second follows from the inequality $f(bx) \leq bf(x)$ which holds for fixed $b \geq 1$ and any $x > 0$.

4.2 Rate of convergence of Nerman’s martingale

In this section, we assume that $p = 1$, i.e., we are in the single-type case. Then the martingales $(V_n)_{n \geq 0}$ and $(W_n)_{n \geq 0}$ are identical. There is a continuous-time analogue of the martingale $(V_n)_{n \geq 0}$ which is important in the study of the asymptotic behavior of the general branching process. Let

$$
\mathcal{J}(t) := \{x \in \mathcal{G} : S(x) > t, S(x|_k) \leq t \text{ for all } k < |x|\} \tag{4.11}
$$

and define

$$
V(t) := \sum_{x \in \mathcal{J}(t)} e^{-\alpha S(x)}, \quad t \geq 0. \tag{4.12}
$$

The family $(V(t))_{t \geq 0}$ can be viewed as Nerman’s martingale evaluated at certain random times. We now make this connection precise. Order the individuals according to their times of birth: $x_1$ is the ancestor, $x_2$ its first-born child etc. We let $t_n := S(x_n)$ be the time of birth of the $n$th individual in the process. For $k \in \mathbb{N}$, let $Y_k := [V_1]_{x_k} - 1$ and $\mathcal{H}_k := \sigma(\mathcal{Z}(x_1), \ldots, \mathcal{Z}(x_k))$. Further, define

$$
R_n := 1 + \sum_{k=1}^{n} e^{-\alpha x_k}Y_k, \quad n \in \mathbb{N}.
$$

Then $V(t) = R_{T_t}$, $t \geq 0$ where $T_t = \#\{x \in \mathcal{G} : S(x) \leq t\}$ is the total number of births up to and including time $t$. It is known (see Lemma 2.3 and Proposition 2.4 in [21] or Theorem 4.1 on p. 371 in [6]) that $(R_n, \mathcal{H}_n)_{n \in \mathbb{N}}$ and $(V(t), \mathcal{H}_{T_t})_{t \geq 0}$ are nonnegative martingales. Furthermore, $V(t)$ and $R_n$ converge a.s., as $t \to \infty$ and $n \to \infty$, respectively, to the random variable $V$, the a.s. limit of Biggins’ martingale $(V_n)_{n \geq 0}$, see e.g. [12, Theorem 3.3].

For the proof of Theorem 2.11, we need information about the rate of convergence of $V(t)$ to $V$. While various results for the rate of convergence of Biggins’ martingale to its limit have been established [3, 13, 14], we are not aware of a corresponding result for Nerman’s martingale. The following proposition provides such a result.

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Proposition 4.3. Suppose that $\mathbb{E}[V_i \log^+ V_i] < \infty$ and that, for some $\varepsilon > 0$, 

$$
\mathbb{E}\left[\sum_{|x|=1} e^{-\alpha S(x)} S(x)(\log^+ (S(x)))^{1+\varepsilon}\right] < \infty. \tag{4.13}
$$

Let $\delta > 0$. Then

$$
\lim_{t \to \infty} (\log t)^{\delta} \mathbb{E}[V_i (\log V_i - \log t) \mathbb{1}_{\{V_i > t\}}] = 0 \tag{4.14}
$$

is necessary and sufficient for

$$
\lim_{t \to \infty} t^{\delta}|V(t) - V| = 0 \quad \text{a.s.} \tag{4.15}
$$

to hold. In particular, the simpler condition $\mathbb{E}[V_i (\log^+ V_i)^{1+\delta}] < \infty$ is sufficient for (4.15) to hold.

Remark 4.4. Assumption (4.13) enables us to apply Theorem 5.4 in [21] which, with $\phi(t) = \mathbb{1}_{(0,\infty)}(t)$, implies that $e^{-\alpha t}T_i \to dV$ a.s. as $t \to \infty$ for some constant $d > 0$. In particular, (4.13) guarantees that

$$
T_i \asymp e^{\alpha t} \quad \text{a.s. on } S \text{ as } t \to \infty \tag{4.16}
$$

where $S = \{|G_n| > 0 \text{ for all } n \in \mathbb{N}_n\}$ is the survival set, and that

$$
e^{-\alpha t_n} \asymp n^{-1} \quad \text{a.s. on } S \text{ as } n \to \infty \tag{4.17}
$$

which follows on substituting $t = t_n$ in (4.16). Actually, (4.13) in Proposition 4.3 may be replaced by the weaker Condition 5.1 in [21] or any other assumption which ensures (4.16).

Before we prove the proposition, we recall a technical result stated as Lemma 4.2 on p. 37 in [6].

Lemma 4.5. Let $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}$ be sequences of real numbers with $0 < \beta_n \uparrow \infty$ as $n \to \infty$. If $\sum_{n \geq 1} \alpha_n \beta_n$ converges, then $\lim_{n \to \infty} \beta_n \sum_{k \geq n} \alpha_k = 0$.

The proof of the proposition is based on two lemmas.

Lemma 4.6. Suppose that $\mathbb{E}[V_i (\log^+ V_i)^\gamma] < \infty$ for some $\gamma \geq 1$ and let $\theta \in (0, \gamma]$. Then $\lim_{n \to \infty} (\log n)^\theta (V - R_n) = 0$ a.s. on $S$ is equivalent to

$$
\lim_{n \to \infty} (\log n)^\theta \sum_{k \geq n} e^{-\alpha k - 1} \mathbb{E}[V_i \mathbb{1}_{\{V_i > k (\log k)^{-\gamma})\}} = 0 \quad \text{a.s. on } S. \tag{4.18}
$$

Proof. Put $\tilde{Y}_n := Y_n \mathbb{1}_{\{Y_n \leq n (\log n)^{-\gamma}\}}$ and $\varepsilon_n := -\mathbb{E}[e^{-\alpha n + 1} \tilde{Y}_{n+1} | \mathcal{H}_n]$, $n \in \mathbb{N}$. We first show that the condition $\mathbb{E}[V_i (\log^+ V_i)^\theta] < \infty$ implies a.s. convergence of $\sum_{n \geq 2} (\log n)^\theta (R_{n+1} - R_n + \varepsilon_n)$ on $S$. To this end, it suffices to check that

(a) $\sum_{n \geq 2} \mathbb{P}(Y_n \neq Y_{n+1}) < \infty$;

(b) $\sum_{n \geq 2} (\log n)^{2\theta} \text{Var}[e^{-\alpha n + 1} \tilde{Y}_{n+1} + \varepsilon_n | \mathcal{H}_n] < \infty$ a.s. on $S$.

Indeed, (b) implies that $\sum_{n \geq 2} (\log n)^\theta (e^{-\alpha n + 1} \tilde{Y}_{n+1} + \varepsilon_n)$ converges a.s. on $S$ because the partial sums of this series constitute an $\mathcal{L}^2$-martingale. Invoking (a) and the Borel-Cantelli lemma, we infer a.s. convergence of

$$
\sum_{n \geq 2} (\log n)^\theta (e^{-\alpha n + 1} Y_{n+1} + \varepsilon_n) = \sum_{n \geq 2} (\log n)^\theta (R_{n+1} - R_n + \varepsilon_n)
$$

on $S$. 

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For $x \geq e^\theta$, $f_\theta(x) := x(\log x)^{-\theta}$ is strictly increasing and continuous and hence possesses an inverse function which we denote by $g_\theta(x)$, $x \geq (e/\theta)^\theta$. Since $g_\theta(x) \sim x(\log x)^\theta$ as $x \to \infty$, we have, for some appropriate $c > 0$,

$$
\sum_{n \geq 2} \mathbb{P}(Y_n \neq \tilde{Y}_n) = \sum_{n \geq 2} \mathbb{P}(|Y_1| > n(\log n)^{-\theta}) < \infty
$$

iff

$$
\sum_{n \geq 2} \mathbb{P}(g_\theta(|Y_1| \vee c) > n) < \infty
$$

iff

$$
\mathbb{E}[V_1(\log^+ V_1)^\theta] < \infty.
$$

This implies (a). According to Lemma 4.2(v) on p. 372 in [6], $\sup_{n \geq 1} ne^{-\alpha t_n} < \infty$ a.s. on $S$.\footnote{The cited lemma says that $e^{-\alpha t_n} \asymp n^{-\alpha}$ as $n \to \infty$ a.s. on $S$ under the assumption $m(\beta) < \infty$ for some $\beta < \alpha$. An inspection of the proof reveals that the latter assumption is only needed for $\inf_{n \geq 1} ne^{-\alpha t_n} > 0$ a.s. on $S$ to hold.} With this at hand, we obtain

$$
\text{Var}[e^{-\alpha t_{n+1}} \tilde{Y}_{n+1} + \varepsilon_n | \mathcal{H}_n] = e^{-2\alpha t_{n+1}} \mathbb{E}[Y_1^2 \mathbb{1}_{(|Y_1| \leq n(\log n)^{-\theta})}] = O(n^{-2} \mathbb{E}[Y_1^2 \mathbb{1}_{(|Y_1| \leq n(\log n)^{-\theta})}])
$$

as $n \to \infty$ a.s. on $S$. Hence, for an appropriate $c > 0$, a.s. on $S$,

$$
\sum_{n \geq 2} (\log n)^{2\theta} \text{Var}[e^{-\alpha t_{n+1}} \tilde{Y}_{n+1} + \varepsilon_n | \mathcal{H}_n] < \infty
$$

iff

$$
\sum_{n \geq 2} n^{-2}(\log n)^{2\theta} \mathbb{E}[Y_1^2 \mathbb{1}_{(|Y_1| \leq n(\log n)^{-\theta})}] < \infty
$$

iff

$$
\mathbb{E}\left[Y_1^2 \sum_{n \geq g_\theta(|Y_1| \vee c)} n^{-2}(\log n)^{2\theta}\right] < \infty
$$

iff

$$
\mathbb{E}\left[Y_1^2 (\log g_\theta(|Y_1| \vee c))^{2\theta}\right] < \infty
$$

iff

$$
\mathbb{E}[|Y_1|(\log^+ |Y_1|)^\theta] < \infty
$$

iff

$$
\mathbb{E}[V_1(\log^+ V_1)^\theta] < \infty.
$$

This proves (b).

Now a.s. convergence of the series $\sum_{n \geq 2} (\log n)^{\theta}(R_{n+1} - R_n + \varepsilon_n)$ on $S$ together with Lemma 4.5 for $\alpha_n = R_{n+1} - R_n + \varepsilon_n$ and $\beta_n = (\log n)^{\theta}$ imply

$$
\lim_{n \to \infty} (\log n)^{\theta}(V - R_n + \sum_{k \geq n} \varepsilon_k) = 0 \quad \text{a.s. on } S.
$$

Thus $\lim_{n \to \infty} (\log n)^{\theta}(V - R_n) = 0$ a.s. on $S$ is equivalent to

$$
\lim_{n \to \infty} (\log n)^{\theta}\sum_{k \geq n} \varepsilon_k = 0 \quad \text{a.s. on } S. \quad (4.18)
$$

Here, for sufficiently large $k$, we have $\{|Y_1| \leq k(\log k)^{-\theta}\} = \{Y_1 \leq k(\log k)^{-\theta}\}$. Hence, for these $k$, using that $\mathbb{E}[Y_1] = 0$ we obtain

$$
\varepsilon_{k-1} = -\mathbb{E}[e^{-\alpha t_k} \tilde{Y}_{k-1}| \mathcal{H}_{k-1}] = -e^{-\alpha t_k} \mathbb{E}[Y_1 \mathbb{1}_{(|Y_1| \leq k(\log k)^{-\theta})}]
$$

$$
= -e^{-\alpha t_k} \mathbb{E}[Y_1 \mathbb{1}_{(Y_1 \leq k(\log k)^{-\theta})}]
$$

$$
= -e^{-\alpha t_k} \mathbb{E}[Y_1 \mathbb{1}_{(Y_1 > k(\log k)^{-\theta})}].
$$
Further, $\mathbb{E}[Y_1 \mathbb{1}_{\{Y_1 > k(\log k)^{-\theta}\}}] \sim \mathbb{E}[V_1 \mathbb{1}_{\{V_1 > k(\log k)^{-\theta} + 1\}}]$ as $k \to \infty$. Hence, in order to deduce that (4.18) is equivalent to
\[
\lim_{n \to \infty} (\log n)^{\theta} \sum_{k \geq n} e^{-\alpha t_k} \mathbb{E}[V_1 \mathbb{1}_{\{V_1 > k(\log k)^{-\theta}\}}] = 0 \quad \text{a.s. on } S
\]

it remains to check that
\[
\lim_{n \to \infty} (\log n)^{\theta} \sum_{k \geq n} e^{-\alpha t_k} \mathbb{E}[V_1 \mathbb{1}_{\{V_1 \in (k(\log k)^{-\theta}, k(\log k)^{-\theta} + 1)\}}] = 0 \quad \text{a.s. on } S.
\]

Validity of the latter relation can be seen from $e^{-\alpha t_k} = O(k^{-1})$ a.s. on $S$ and the following rough estimate
\[
\limsup_{n \to \infty} (\log n)^{\theta} \sum_{k \geq n} \frac{1}{k} \mathbb{E}[V_1 \mathbb{1}_{\{V_1 \in (k(\log k)^{-\theta}, k(\log k)^{-\theta} + 1)\}}]
\leq 2 \limsup_{n \to \infty} \sum_{k \geq n} \mathbb{P}(V_1 \in (k(\log k)^{-\theta}, k(\log k)^{-\theta} + 1))
\leq 2 \limsup_{n \to \infty} \sum_{k \geq n} \mathbb{P}(V_1 > k(\log k)^{-\theta})
\leq 2 \limsup_{n \to \infty} \sum_{k \geq n} \mathbb{P}(g_\delta(V_1 \vee c) > k) = 0
\]

for appropriate $c > 0$. \qed

**Lemma 4.7.** Assume that (4.13) holds and let $\gamma > 0$. Then (4.14) with $\delta$ replaced by $\gamma$ is necessary and sufficient for
\[
\lim_{n \to \infty} (\log n)^{\gamma} \sum_{k \geq n} e^{-\alpha t_{k+1}} \mathbb{E}[V_1 \mathbb{1}_{\{V_1 > k(\log k)^{-\gamma}\}}] = 0 \quad \text{a.s. on } S. \tag{4.19}
\]

**Proof.** In view of (4.17),
\[
\lim_{n \to \infty} (\log n)^{\gamma} \sum_{k \geq n} \frac{1}{k} \mathbb{E}[V_1 \mathbb{1}_{\{V_1 > k(\log k)^{-\gamma}\}}] = 0 \quad \text{a.s. on } S \tag{4.20}
\]
is necessary and sufficient for (4.19) to hold. For large enough $n$ and appropriate $c > 0$
\[
\sum_{k \geq n} \frac{1}{k} \mathbb{E}[V_1 \mathbb{1}_{\{V_1 > k(\log k)^{-\gamma}\}}] = \mathbb{E} \left[ V_1 \sum_{k \geq n} \frac{1}{k} \mathbb{1}_{\{g_\delta(V_1 \vee c) > k\}} \right].
\]

Since $\sum_{k=1}^{n} k^{-1} = \log n + O(1)$ as $n \to \infty$, (4.20) is equivalent to
\[
\lim_{n \to \infty} (\log n)^{\gamma} \mathbb{E}[V_1 (\log g_\gamma(V_1 \vee c) - \log n) \mathbb{1}_{\{g_\delta(V_1 \vee c) > n\}}] = 0
\]
and hence to $\lim_{n \to \infty} (\log n)^{\gamma} \mathbb{E}[V_1 (\log g_\gamma(V_1) - \log g_\gamma(n)) \mathbb{1}_{\{V_1 > n\}}] = 0$ on substituting $g_\gamma(n)$ instead of $n$ and then simplifying. Finally, the latter relation is equivalent to
\[
\lim_{t \to \infty} (\log t)^{\gamma} \mathbb{E}[V_1 (\log g_\gamma(V_1) - \log g_\gamma(t)) \mathbb{1}_{\{V_1 > t\}}] = 0
\]
by a monotonicity argument. It remains to note that, as $t \to \infty$, $\log g_\gamma(t) = \log t + \gamma \log(\log t) + o(1)$ and that $\log(\log u) - \log(\log v) = o(\log u - \log v)$ as $u, v \to \infty$ to complete the proof. \qed
Proof of Proposition 4.3. We first prove that

$$\mathbb{E}[V_1 (\log^+ V_1)^{1+\delta}] < \infty \Rightarrow (4.14) \Rightarrow \mathbb{E}[V_1 (\log^+ V_1)^{1+\delta-\varepsilon}] < \infty \quad (4.21)$$

for any $\varepsilon \in (0, 1 + \delta)$. In particular, the first implication justifies the last statement of the proposition.

Suppose $\mathbb{E}[V_1 (\log^+ V_1)^{1+\delta}] < \infty$. Then

$$\lim_{t \to \infty} (\log t)^{\delta} \mathbb{E}[V_1 (\log V_1 - \log t) 1_{\{V_1 > t\}}] \leq \lim_{t \to \infty} \mathbb{E}[V_1 (\log V_1)^{1+\delta} 1_{\{V_1 > t\}}] = 0,$$

that is, (4.14) holds.

Suppose (4.14) and let $\varepsilon \in (0, \delta)$. Then

$$t^{-1}(\log t)^{\delta-\varepsilon-1}\mathbb{E}[V_1 (\log V_1 - \log t) 1_{\{V_1 > t\}}] \leq \text{const} t^{-1}(\log t)^{-\varepsilon-1}$$

for large enough $t$ whence

$$\infty > (\delta - \varepsilon) \int_1^\infty t^{-1}(\log t)^{\delta-\varepsilon-1}\mathbb{E}[V_1 (\log V_1 - \log t) 1_{\{V_1 > t\}}] dt$$

$$= \mathbb{E} \left[ V_1 \log V_1 \int_1^{V_1} (\delta - \varepsilon)t^{-1}(\log t)^{\delta-\varepsilon-1}dt 1_{\{V_1 > 1\}} \right]$$

$$- \mathbb{E} \left[ V_1 \int_1^{V_1} (\delta - \varepsilon)t^{-1}(\log t)^{\delta-\varepsilon}dt 1_{\{V_1 > 1\}} \right]$$

$$= (1 + \delta - \varepsilon)^{-1}\mathbb{E}[V_1 (\log^+ V_1)^{1+\delta-\varepsilon}]$$

which completes the proof of (4.21).

$$\lim_{t \to \infty} t^\delta |V(t) - V| = 0 \ a.s. \ on \ S$$

holds trivially. In view of (4.17), for $\lim_{t \to \infty} t^\delta |V(t) - V| = 0 \ a.s. \ on \ S$ to hold it is necessary and sufficient that

$$\lim_{n \to \infty} (\log n)^\delta (V - R_n) = 0 \ a.s. \ on \ S. \quad (4.22)$$

Therefore we work towards proving that condition (4.14) is equivalent to (4.22). While doing so we argue as in the proof of Theorem 4.1(ii) on p. 36 in [6].

Case 1: $\delta \in (0, 1]$. The result follows from Lemma 4.6 with $\gamma = 1$ and $\theta = \delta$ which is applicable because $\mathbb{E}[V_1 (\log^+ V_1) < \infty$ by the assumption and Lemma 4.7 with $\gamma = \delta$.

Case 2: $\delta > 1$. First assume that (4.14) holds. Then $\mathbb{E}[V_1 (\log^+ V_1)] < \infty$ by (4.21). The result now follows from Lemmas 4.6 and 4.7 with $\gamma = \theta = \delta$.

Now assume that (4.14) fails. If $\mathbb{E}[V_1 (\log^+ V_1)^\delta] < \infty$, then the result follows from Lemmas 4.6 and 4.7 with $\gamma = \theta = \delta$. Suppose $\mathbb{E}[V_1 (\log^+ V_1)] = \infty$. We have to prove that the relation $\lim_{n \to \infty} (\log n)^\delta (V - R_n) = 0 \ a.s. \ on \ S$ fails to hold. Pick $\gamma \in [1, \delta)$ such that $\mathbb{E}[V_1 (\log^+ V_1)^\gamma] < \infty$, yet $\mathbb{E}[V_1 (\log^+ V_1)^{\gamma+1/2}] = \infty$. Using (4.21) with $\delta$ replaced by $\gamma$ we infer that $\lim_{t \to \infty} (\log t)^\gamma\mathbb{E}[V_1 (\log V_1 - \log t) 1_{\{V_1 > t\}}] = 0$ does not hold. According to Lemma 4.6 and Lemma 4.7 the relation $\lim_{n \to \infty} (\log n)^\gamma (V - R_n) = 0 \ a.s. \ on \ S$ does not hold. This finishes this part of the proof, for $\gamma < \delta$.

4.3 Reduction to the single-type case

Recall that $\mathbb{P}$, $\mathbb{E}$, $\sigma$ and $\mathcal{J}$ are shorthand notation for $\mathbb{P}^1$, $\mathbb{E}^1$, $\sigma^1$ and $\mathcal{J}^1$, respectively.

**Proposition 4.8.** Assume that (A1)-(A4) hold. Then

(a) If (A5) holds, then $V = \text{W} \ \mathbb{P}$-a.s. and $\mathbb{E}[V_1 \log^+ V_1] < \infty$. 

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(b) If Condition 2.2 holds, then \( m(\beta) < \infty \) for some \( \beta < \alpha \).

(c) If, for some \( \delta > 0 \), (2.9) holds, i.e., if
\[
\mathbb{E}
\left[\sum_{x=1}^{\infty} e^{-\alpha S(x)} S(x)^{1+\delta}\right] < \infty \quad \text{for } i = 1, \ldots, p,
\]
then
\[
\mathbb{E}
\left[\sum_{x \in \mathcal{J}} e^{-\alpha S(x)} S(x)^{1+\delta}\right] < \infty.
\]

(d) If, for some \( \delta > 0 \), Condition 2.5 holds, then \( \mathbb{E}[V_1(\log^+ V_1)^{1+\delta}] < \infty \).

Proof. (a) If (A5) holds, then \( \mathbb{E}[1W] = 1 \) by Proposition 4.1. We only need to prove that \( V = 1W \) \( \mathbb{P}\text{-a.s.} \) because then \( \mathbb{E}[V] = 1 \) and hence, by Proposition 4.1 (for \( p = 1 \)), \( \mathbb{E}[V_1 \log^+ V_1] < \infty \). But \( V = 1W \) \( \mathbb{P}\text{-a.s.} \) follows from \([10, \text{Theorem 6.1}]\) if we can check that
\[
1W_n = \lim_{k \to \infty} \sum_{x \in \mathcal{J}_k \land n} \frac{v(x)}{v_1} e^{-\alpha S(x)} \quad \mathbb{P}\text{-a.s.}
\]
where \( \mathcal{J}_k = \mathcal{J}_k^1 \), and \( \sum_{x \in \mathcal{J}_k \land n} \) means summation over the set
\[
\mathcal{J}_k \land n := \{ x \in \mathcal{G} : \text{either } x \in \mathcal{J}_k \text{ and } |x| \leq n \text{ or } x < \mathcal{J}_k \text{ and } |x| = n \}.
\]
(In words, these are the \( x \) in \( \mathcal{J}_k \) in the first \( n \) generations and the \( x \) in the \( n \)th generation with no ancestor in \( \mathcal{J}_k \).) Relation (4.23) holds true since by definition of \( \mathcal{J}_k \), we have \( |x| \geq k \) for all \( x \in \mathcal{J}_k \) and, therefore, \( \mathcal{J}_k \land n = \mathcal{G}_n \) for \( k \geq n \). The proof of assertion (a) is complete.

For the proof of (b), assume that Condition 2.2 holds, that is, \( m_j(\theta) < \infty \) for all \( i, j = 1, \ldots, p \) and some \( \theta < \alpha \). This implies that
\[
c(\varepsilon) := \max_{i=1, \ldots, p} \mathbb{E}[\varepsilon^{S_1}] < \infty
\]
for \( 0 \leq \varepsilon \leq \alpha - \theta \). Further, \( c(\varepsilon) \to 1 \) as \( \varepsilon \downarrow 0 \). By Lemma 3.1(c), \( \mathbb{P}(\sigma = n) \leq C^2 e^{-\gamma n} \) for all \( n \geq 0 \) and some \( C, \gamma > 0 \). Now pick \( \beta \in (\theta, \alpha) \) such that \( c(2(\alpha - \beta)) < e^\gamma \). By (3.5), \( m(\beta) < \infty \) is equivalent to \( \mathbb{E}[e^{(\alpha - \beta)S_n}] < \infty \). For the latter expectation, we obtain using the Cauchy-Schwarz inequality,
\[
\mathbb{E}[e^{(\alpha - \beta)S_n}] = \sum_{n \geq 0} \mathbb{E}[\mathbb{1}_{\{\sigma = n\}} e^{(\alpha - \beta)S_n}] \leq \sum_{n \geq 0} \mathbb{P}(\sigma = n)^{1/2} \left( \mathbb{E}[e^{2(\alpha - \beta)S_n}] \right)^{1/2}
\]
\[
\leq C \sum_{n \geq 0} e^{-\gamma n/2} c(2(\alpha - \beta))^{n/2} < \infty.
\]

For the proof of (c), assume that (2.9) holds. Then (3.1) yields
\[
C := \max_{i,j=1, \ldots, p} \mathbb{E}[1^{1+\delta} |M_1 = j] < \infty
\]
where the maximum is over those \( i, j \) only with \( \mathbb{P}(M_1 = j) > 0 \). Fix \( i_1, \ldots, i_{n-1} \in \{1, \ldots, p\} \) with \( \mathbb{P}(M_1 = i_1, \ldots, M_{n-1} = i_{n-1}, M_n = 1) > 0 \) and observe that, by Minkowski’s inequality,
\[
\mathbb{E}[S_n^{1+\delta} \mathbb{1}_{\{M_1=i_1, \ldots, M_{n-1}=i_{n-1}, M_n=1\}}] = \left\| \sum_{k=1}^{n} (S_k - S_{k-1}) \mathbb{1}_{\{M_1=i_1, \ldots, M_{n-1}=i_{n-1}, M_n=1\}} \right\|_{1+\delta}^{1+\delta}
\]
\[
\leq \left( \sum_{k=1}^{n} \left\| (S_k - S_{k-1}) \mathbb{1}_{\{M_1=i_1, \ldots, M_{n-1}=i_{n-1}, M_n=1\}} \right\|_{1+\delta} \right)^{1+\delta}.
\]
Conditioning with respect to \((M_n)_{n \geq 0}\) yields

\[
\mathbb{E}[(S_k - S_{k-1})^{1+\delta} \mathbb{1}_{\{M_1 = \ldots = M_{n-1} = i, \text{ } M_n = 1\}}] \leq C \mathbb{P}(M_1 = \ldots = M_{n-1} = i, \text{ } M_n = 1)
\]

for \(k = 1, \ldots, n\). Hence, using that \(\{\sigma = n\} = \bigcup_{2 \leq i_1, \ldots, i_{n-1} \leq p} \{M_1 = \ldots = M_{i-1} = i, \text{ } M_n = 1\}\), (3.5) and Lemma 3.1(c), we infer

\[
\mathbb{E}\left[ \sum_{x \in J} e^{-\alpha S(x)} S(x)^{1+\delta} \right] = \mathbb{E}[S_\sigma^{1+\delta}]
\]

\[
= \sum_{n \geq 1} \sum_{2 \leq i_1, \ldots, i_{n-1} \leq p} \mathbb{E}[S_n^{1+\delta} \mathbb{1}_{\{M_1 = \ldots = M_{n-1} = i, \text{ } M_n = 1\}}]
\]

\[
\leq \sum_{n \geq 1} \sum_{2 \leq i_1, \ldots, i_{n-1} \leq p} n^{1+\delta} C \mathbb{P}(M_1 = \ldots = M_{n-1} = i, \text{ } M_n = 1)
\]

\[
= C \sum_{n \geq 1} n^{1+\delta} \mathbb{P}(\sigma = n) < \infty.
\]

Finally, for the proof of (d), assume that, for some \(\delta > 0\), Condition 2.5 is valid. Proposition 4.2 then implies that \(\mathbb{E}[V(\log^+ V)^\delta] < \infty\) and thereupon \(\mathbb{E}[V(\log^+ V)^\delta] < \infty\) because \(V = 1W \mathbb{P}\text{-a.s. by part (a). Hence } \mathbb{E}[V_i(\log^+ V_i)^{\delta+1}] < \infty\) by Theorem 1.2 in [2].

**Lemma 4.9.** For \(\phi : \Omega \times \mathbb{R} \to [0, \infty)\) a product-measurable, separable random characteristic, define, for \(t \in \mathbb{R}\),

\[
\phi_J(t) := \sum_{x \in J} \phi_x(t - S(x)).
\]

The following assertions hold.

(a) Suppose that \(\mathbb{E}[\sup_{0 \leq s \leq t} \phi(s)] < \infty\) for \(i = 1, \ldots, p\) and all \(t \geq 0\). Then \(\phi_J\) is product-measurable, \(\sup_{0 \leq s \leq t} \phi_J(s) \leq Y_t\) for random variables \(Y_t, \ t \geq 0\) with \(\mathbb{E}[Y_t] < \infty\). Further, if \(\phi\) has \(D\)-valued paths, so has \(\phi_J\).

(b) Suppose that \(\mathbb{E}[\sup_{0 \leq s \leq t} \phi(s)] < \infty\) for \(i = 1, \ldots, p\) and all \(t \geq 0\) and that \(t \mapsto e^{-\alpha t} \mathbb{E}[\phi(t)]\) is directly Riemann integrable\(^8\) on \(\mathbb{R}_{\geq 0}\) for \(i = 1, \ldots, p\). Then \(t \mapsto e^{-\alpha t} \mathbb{E}[\phi_J(t)]\) is directly Riemann integrable on \(\mathbb{R}_{\geq 0}\).

(c) Assume that Condition 2.2 is satisfied. If Condition 2.3 holds for \(\phi\), then there exists a \(\beta < \alpha\) such that \(\mathbb{E}[\sup_{t \geq 0} e^{-\beta t} \phi_J(t)] < \infty\).

(d) If Condition 2.10 holds for \(\phi\), then it also holds for \(\phi_J\).

**Proof.** (a) From the representation

\[
\phi_J(t) = \sum_{x \in I} \mathbb{1}_{\{x \in J\}} \mathbb{1}_{\{x < J\}} [\phi_x(t - S(x))]
\]

it can be concluded that \(\phi_J\) is product-measurable. In fact, it suffices to check that each summand is product-measurable. Fix any \(x \in I\). The factors \(\mathbb{1}_{\{x \in J\}}\) and \(\mathbb{1}_{\{x < J\}}\) are \(\mathcal{A}\)-measurable. Since they do not depend on \(t\), they are product-measurable. The shift \(\omega \mapsto \sigma_x \omega\) is measurable, thus the mapping \((\omega, t) \mapsto (\sigma_x \omega, t)\) is product-measurable and hence so is \([\phi_x]_x\). Finally, since \((\omega, t) \mapsto t - S(x, \omega)\) is product-measurable, so is \([\phi_x]_x(t - S(x))\). Further,

\[
\sup_{0 \leq s \leq t} \phi_J(s) \leq Y_t := e^{\alpha t} \sum_{x < J} e^{-\alpha S(x)} \sup_{0 \leq s \leq t} [\phi_x]_x(s)
\]

\(^8\)See p. 232 in [25] for the definition of direct Riemann integrability.
where it should be recalled that \( \sum_{x \prec J} \) means summation over the \( x \in \mathcal{G} \) with \( x \prec J \). \( Y_t \) is a random variable since \( \phi \) is separable. In view of (3.6),

\[
\mathbb{E}[Y_t] = e^{\alpha t} \mathbb{E} \left[ \sum_{x \prec J} e^{-\alpha S(x)} \sup_{0 \leq s \leq t} [\phi]_x(s) \right] = e^{\alpha t} \mathbb{E} \left[ \sum_{k=0}^{\sigma-1} \frac{v_1}{v_{M_k}} E_{M_k} \left[ \sup_{0 \leq s \leq t} \phi(s) \right] \right] \\
\leq e^{\alpha t} v_1 \mathbb{E}[\sigma] \max_{j=1, \ldots, p} \frac{E_j \left[ \sup_{0 \leq s \leq t} \phi(s) \right]}{v_j} < \infty. \tag{4.25}
\]

In particular, \( \phi_J(t) \) is finite for all \( t \geq 0 \) (simultaneously) a.s. If \( (x_k)_{k \geq 1} \) is an enumeration of \( \mathcal{I} \), then \( \phi_J \) is the almost sure limit of

\[
\phi_n(t) := \sum_{k=1}^{n} 1_{(x_k \in \mathcal{G})} 1_{\{x_k < J\}} [\phi]_{x_k}(t - S_k).
\]

What is more, the almost sure convergence of \( \phi_n \) to \( \phi_J \) is locally uniform since

\[
\sup_{0 \leq s \leq t} \left| \phi_n(s) - \phi_J(s) \right| \leq \sum_{k>n} 1_{(x_k \in \mathcal{G})} 1_{\{x_k < J\}} \sup_{0 \leq s \leq t} [\phi]_{x_k}(s) \leq Y_t
\]

and

\[
\mathbb{E} \left[ \sum_{k>n} 1_{(x_k \in \mathcal{G})} 1_{\{x_k < J\}} \sup_{0 \leq s \leq t} [\phi]_{x_k}(s) \right] \rightarrow 0 \text{ as } n \rightarrow \infty \tag{4.26}
\]

by the dominated convergence theorem. Hence, if \( \phi \) is \( D \)-valued, so is \( \phi_J \), as the locally uniform limit of \( D \)-valued functions.

(b) (4.26) implies that \( \tilde{\phi}_n = \mathbb{E} \left[ \phi_n \right] \) converges locally uniformly to \( \tilde{\phi}_J = \mathbb{E} \left[ \phi_J \right] \). In particular, \( \tilde{\phi}_J \) is continuous at each point in which all \( \tilde{\phi}_n \) are continuous. Consequently, \( \tilde{\phi}_J \) is continuous almost everywhere with respect to Lebesgue measure. Using (3.6) we obtain

\[
e^{-\alpha t} \tilde{\phi}_J(t) = e^{-\alpha t} \mathbb{E} \left[ \sum_{x \prec J} [\phi]_x(t - S(x)) \right] \\
= \mathbb{E} \left[ \sum_{x \prec J} e^{-\alpha S(x)} e^{-\alpha(t-S(x))} [\phi]_x(t - S(x)) \right] \\
= \mathbb{E} \left[ \sum_{k=0}^{\sigma-1} \frac{v_1}{v_{M_k}} e^{-\alpha(t-S_k)} M_k \tilde{\phi}(t - S_k) \right] \\
\leq \mathbb{E} \left[ \sum_{k=1}^{\sigma-1} \max_{i=1, \ldots, p} \frac{v_1}{v_i} e^{-\alpha(t-S_k)} \tilde{\phi}(t - S_k) \right] \tag{4.27}
\]

where \( \tilde{\phi}(t) = \mathbb{E}^t [\phi(t)] \), \( i = 1, \ldots, p \). The latter implies that since \( t \mapsto \max_{i=1, \ldots, p} \frac{v_i}{v_1} e^{-\alpha t} \tilde{\phi}(t) \) is bounded (as directly Riemann integrable), so is \( t \mapsto e^{-\alpha t} \tilde{\phi}_J(t) \). Set

\[
A := \sum_{k \geq 0} \sup_{k \leq t < k+1} \max_{i=1, \ldots, p} \frac{v_i}{v_1} e^{-\alpha t} \tilde{\phi}(t)
\]

and note that \( A < \infty \) because of direct Riemann integrability. Using (4.27) we infer

\[
\sum_{k \geq 0} \sup_{k \leq t < k+1} e^{-\alpha t} \tilde{\phi}_J(t) \leq A \mathbb{E}[\sigma] < \infty,
\]

and according to Remark 3.10.4 on p. 236 in [25] the direct Riemann integrability of \( t \mapsto e^{-\alpha t} \tilde{\phi}_J(t) \) follows.
(c) Condition 2.3 implies that there exists \( \theta < \alpha \) such that

\[
C_\theta := \max_{i=1, \ldots, p} \frac{E_i}{v_i} \left[ \sup_{t \geq 0} e^{-\theta t} \phi(t) \right] < \infty.
\]

By Lemma 3.1(c) there are \( C, \gamma > 0 \) such that \( \mathbb{P}(\sigma > n) \leq C^2 e^{-\gamma n} \) for all \( n \geq 0 \). Condition 2.2 ensures that \( c(2(\alpha - \beta)) = \max_{i=1, \ldots, p} E_i e^{2(\alpha - \beta) S_1} \leq e^\gamma \) for some \( \beta \in (\theta, \alpha) \) (see the proof of Proposition 4.8b). Now the claim follows from

\[
\mathbb{E} \left[ \sup_{t \geq 0} e^{-\beta S(t)} \right] = \mathbb{E} \left[ \sup_{t \geq 0} e^{-\beta(t - S(x))} \phi_x(t - S(x)) \right]
\]

\[
\leq C_\beta \mathbb{E} \left[ \sum_{k=0}^{\infty} e^{(\alpha - \beta) S_k} \right]
\]

\[
= C_\beta \sum_{k=0}^{\infty} \mathbb{P}(\sigma > k)^{1/2} \mathbb{E}[e^{2(\alpha - \beta) S_k}]^{1/2}
\]

\[
\leq C_\beta C \sum_{k=0}^{\infty} e^{-\gamma k/2} c(2(\alpha - \beta))^{k/2} < \infty,
\]

where the Cauchy-Schwarz inequality has been used for the 5th line.

(d) While the Lebesgue integrability of \( t \mapsto e^{-\alpha t} \mathbb{E}[\phi_J(t)] \) follows from

\[
\int_0^\infty e^{-\alpha t} \mathbb{E}[\phi_J(t)] dt \leq \mathbb{E}[\sigma] \max_{i=1, \ldots, p} \frac{v_i}{v_1} \int_0^\infty e^{-\alpha t} \phi(t) dt < \infty
\]

(4.28)

which is a consequence of (4.27) and the integrability of \( t \mapsto \max_{i=1, \ldots, p} \frac{v_i}{v_1} e^{-\alpha t} \phi(t) \), its boundedness follows from the boundedness of \( t \mapsto \max_{i=1, \ldots, p} \frac{v_i}{v_1} e^{-\alpha t} \phi(t) \), (4.27) and \( \mathbb{E}[\sigma] < \infty \) (see Lemma 3.1(b)).

It remains to show that (2.8) holds for \( \phi_J \), that is,

\[
t^\delta \int_t^\infty e^{-\alpha s} \mathbb{E}[\phi_J(s)] ds \to 0 \quad \text{and} \quad t^\delta \sup_{s \geq t} e^{-\alpha s} \mathbb{E}[\phi_J(s)] \to 0 \quad \text{as} \quad t \to \infty.
\]

As to the first relation, notice that by a variant of the argument leading to (4.28), (2.8) and the dominated convergence theorem, we have

\[
t^\delta \int_t^\infty e^{-\alpha s} \mathbb{E}[\phi_J(s)] ds \leq \mathbb{E} \left[ \sum_{k=0}^{\infty} v_i t^\delta \max_{i=1, \ldots, p} \frac{\alpha(t - S_k)}{v_i} \right] \quad \text{as} \quad t \to \infty.
\]

Similarly, the second relation holds since

\[
t^\delta \sup_{s \geq t} e^{-\alpha s} \mathbb{E}[\phi_J(s)] \leq v_i t^\delta \sup_{s \geq t} \mathbb{E} \left[ \sum_{k=0}^{\infty} e^{-\alpha(s - S_k)} \max_{i=1, \ldots, p} \frac{\phi(s - S_k)}{v_i} \right]
\]

\[
\leq v_i \mathbb{E} \left[ \sum_{k=0}^{\infty} \max_{i=1, \ldots, p} t^\delta \sup_{s \geq t - S_k} e^{-\alpha s} \phi(s) \right] \quad \text{as} \quad t \to \infty
\]

by the dominated convergence theorem (using that \( \sup_{t \geq 0} e^{-\alpha t} \phi(t) < \infty \) for \( i = 1, \ldots, p \)).
5 Proofs of the main results

The proofs of the main results rely on a decomposition of \( Z^\phi(t) \) along the optional lines \( J_n \), \( n \geq 0 \). For a random characteristic \( \psi \), define \( Z^{1,\psi} \) by

\[
Z^{1,\psi}(t) := \sum_{x \in G^1} \left[ \psi \right]_x (t - S(x))
\]  

(5.1)

where \( G^1 := \{ x \in G : \tau(x) = 1 \} \). We choose \( \psi := \phi_J \) as defined in (4.24). Then

\[
Z^{1,\psi}(t) = \sum_{x \in G^1} \left[ \phi_J \right]_x (t - S(x)) = \sum_{x \in G^1} \sum_{y \prec J} (\phi_{xy}(t - S(xy)))
\]  

(5.2)

Using this connection, limit theorems for the multi-type process will be derived from the corresponding single-type ones\(^9\).

5.1 Convergence in probability

Proof of Theorem 2.1. By (5.2), the first part of the theorem follows if we can check that \( Z_J \) and \( \phi_J \) satisfy the assumptions of Theorem 3.1 in [21]. That \( Z_J \) satisfies the standing assumptions given on p. 366 of [21] is established in Proposition 3.2. That \( \phi_J \) satisfies the conditions of Theorem 3.1 in [21] is secured by Lemma 4.9 with the exception that it cannot be guaranteed that \( \phi_J \) is separable. On the other hand, the perusal of the proof of Theorem 3.1 in [21] makes it clear that the assumption of separability can be omitted as long as \( \sup_{s \leq t} \phi_J(s) \) is dominated by an integrable random variable. This is indeed the case here, see Lemma 4.9(a). Consequently, Theorem 3.1 in [21] yields

\[
e^{-\alpha t} Z^\phi(t) = e^{-\alpha t} Z^{1,\phi_J}(t) \to \frac{W}{-m'(\alpha)} \int_0^\infty e^{-\alpha t} \mathbb{E}[\phi_J(t)] \, dt
\]  

(5.3)

in probability as \( t \to \infty \). From Proposition 3.2(c), we know that

\[
-m'(\alpha) = (u_1v_1)^{-1} \sum_{i,j=1}^p u_i v_j (-1)^{m_j}(\alpha).
\]

Left with the calculation of the integral, we write, recalling (4.24) and using (3.5) and Lemma 3.1(b),

\[
\int_0^\infty e^{-\alpha t} \mathbb{E}[\phi_J(t)] \, dt = \int_0^\infty \mathbb{E} \left[ \sum_{x \in J} e^{-\alpha S(x)} e^{-\alpha(t - S(x))} [\phi_J]_x (t - S(x)) \right] \, dt
\]

\[
= \int_0^\infty \mathbb{E} \left[ \sum_{k=0}^{\sigma - 1} e^{-\alpha(t - S_k)} \frac{v_1}{\nu_{M_k}} \nu_t \phi_J(t - S_k) \right] \, dt
\]

\[
= \sum_{i=1}^p \mathbb{E} \left[ \# \{ k < \sigma : M_k = i \} \right] \frac{v_1}{v_i} \int_0^\infty e^{-\alpha t} \mathbb{E}[\phi(t)] \, dt
\]

\[
= \sum_{i=1}^p \frac{u_i}{u_1} \int_0^\infty e^{-\alpha t} \mathbb{E}[\phi(t)] \, dt
\]

---

\(^9\)In [19], along similar lines limit theorems for branching random walks on the line are obtained from the corresponding results for branching random walks with positive steps only.
where \( \psi(t) = \mathbb{E}^i[\phi(t)] \).

As for convergence in mean, observe that (A5) implies \( \mathbb{E}[V_t \log^+ V_t] < \infty \) by Proposition 4.8 (a) and apply Corollary 3.3 in [21]. \( \square \)

### 5.2 Ratio convergence

**Proof of Theorem 2.4.** The scheme of proof is identical to that of Theorem 2.1 but it is based on an application of Theorem 6.3 in [21] rather than Theorem 3.1 in the same source. Hence, we have to check that the assumptions of Theorem 6.3 in [21] are fulfilled. Since Condition 2.2 holds, we conclude that \( m(\beta) < \infty \) for some \( \beta < \alpha \) by Proposition 4.8(b). Further \( \phi_J \) and \( \psi_J \) have \( D \)-valued paths by Lemma 4.9(a) which applies because Condition 2.3 for \( \phi \) and \( \psi \) entails \( \mathbb{E}[\sup_{0 \leq s \leq t} \phi(s)] < \infty \) and \( \mathbb{E}[\sup_{0 \leq s \leq t} \psi(s)] < \infty \) for \( i = 1, \ldots, p \) and all \( t \geq 0 \). Finally, invoking Lemma 4.9(c) gives \( \mathbb{E}[\sup_{t \geq 0} e^{-\beta t} \phi_J(t)] < \infty \) and \( \mathbb{E}[\sup_{t \geq 0} e^{-\beta t} \psi_J(t)] < \infty \), where we assume w.l.o.g. that the \( \beta \) from Lemma 4.9 is the same \( \beta \) for which \( m(\beta) < \infty \). Theorem 6.3 in [21] now yields that, on \( \mathbb{S} \),

\[
\frac{Z^\phi(t)}{Z^\psi(t)} \to \frac{\int_0^\infty e^{-at}\mathbb{E}[\phi_J(t)] \, dt}{\int_0^\infty e^{-at}\mathbb{E}[\psi_J(t)] \, dt} \quad \text{a.s. as } t \to \infty.
\]

Numerator and denominator of the fraction on the right-hand side have been calculated in the proof of Theorem 2.1. \( \square \)

### 5.3 Rate of convergence

We first prove Theorem 2.11 in the case \( p = 1 \), that is, in the single-type case in which \( u = v = 1 \). Recall the definition of \( J(t) \):

\[
J(t) := \{ x \in \mathcal{G} : S(x) > t, S(x|_k) \leq t \text{ for all } k < |x| \}.
\]

**Proof of Theorem 2.11: The single-type case.** We have to show that

\[
t^\delta |e^{-at} Z^\phi(t) - V m^\phi_\infty| \to 0 \quad \text{in } \mathbb{P}\text{-probability as } t \to \infty \tag{2.10}
\]

where \( m^\phi_i := e^{-a\alpha(t)} \mathbb{E}[Z^\phi(t)] \) and

\[
m^\phi_\infty = \lim_{t \to \infty} m^\phi_t = \frac{1}{-m'(\alpha)} \int_0^\infty e^{-a\alpha u} \mathbb{E}[\phi(u)] \, du.
\]

This limit exists since \( m^\phi_t \) solves a renewal equation, see formula (2.4) in [21]. To prove (2.10), we truncate \( \phi \) at some \( c > 0 \) and set \( \phi_c(t) := \phi(t) \mathbb{1}_{[0,c]}(t), t \geq 0 \). For \( 0 \leq c < s \leq t \), we consider (2.10) at \( t + s \) and use the triangular inequality to obtain

\[
(t + s)^\delta |e^{-a(t+s)} Z^\phi(t + s) - V m^\phi_\infty|
\]

\[
\leq (t + s)^\delta (e^{-a(t+s)}(Z^\phi(t + s) - Z^\phi(t)) + V(m^\phi_c - m^\phi_\infty))
\]

\[
+ (t + s)^\delta |e^{-a(t+s)} Z^\phi(t + s) - V m^\phi_c|.
\]

The expectation of the first summand on the right-hand side is equal to

\[
(t + s)^\delta (m^\phi_{t+s} - m^\phi_{t+s} + m^\phi_\infty - m^\phi_\infty)
\]

\[
= (t + s)^\delta (m^\phi_{t+s} + m^\phi_\infty - m^\phi_\infty)
\]

\[
\leq (t + s)^\delta (|m^\phi_{t+s} - m^\phi_\infty| + 2m^\phi_\infty).
\]

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Here, when choosing $2c = s = t$, we have
\[(t + s)^\delta m^{\phi - \phi_\infty}_\infty = \frac{(t + s)^\delta}{-m'(\alpha)} \int_c^\infty e^{-\alpha u} \mathbb{E}[\phi(u)] \, du \to 0 \quad \text{as} \quad t \to \infty\]
by (2.8). Assumption (2.9) is equivalent to $\mathbb{E}[S_1^{1+\delta}] < \infty$. This together with Condition 2.6 (see, in particular, Remark 2.7) and Condition 2.10 allows us to apply Theorem 4.2(ii) in [22] (with $\psi(t) = t^\delta$ and $g(s) = e^{-\alpha s} \mathbb{E}[\phi(s)]$, the notation of [22]) which gives
\[(t + s)^\delta |m^{\phi - \phi_\infty}_{t+s} - m^{\phi - \phi_\infty}_\infty| \leq (t + s)^\delta \sup_f \left| f \ast U(t + s) - \frac{1}{-m'(\alpha)} \int_0^\infty f(u) \, du \right| \to 0\]
as $t \to \infty$ where $U$ denotes the renewal measure of the random walk $(S_n)_{n \geq 0}$ and the supremum is over all Lebesgue integrable functions $f \geq 0$ satisfying $f(u) \leq e^{-\alpha u} \mathbb{E}[\phi(u)]$ for all $u \in \mathbb{R}$. It thus remains to show that
\[(t + s)^\delta |e^{-\alpha(t+s)} Z^{\phi_\infty}(t + s) - V m^{\phi_\infty}_\infty| \mathbb{P} \to 0 \quad \text{as} \quad t \to \infty \tag{5.4}\]
where $2c = s = t$. To this end, we choose $0 < a < 1$ and estimate as follows
\[
\begin{align*}
(t + s)^\delta |e^{-\alpha(t+s)} Z^{\phi_\infty}(t + s) - V m^{\phi_\infty}_\infty| &\leq (t + s)^\delta \left| \sum_{x \in \mathcal{J}(t)} e^{-\alpha S(x)} (e^{-\alpha(t+s-S(x))}[Z^{\phi_\infty}]_x(t+s-S(x)) - m^{\phi_\infty}_{t+s-S(x)}) \right| \\
&+ (t + s)^\delta \sum_{x \in \mathcal{J}(t): S(x) \leq t+at} e^{-\alpha S(x)} |m^{\phi_\infty}_{t+s-S(x)} - m^{\phi_\infty}_\infty| \\
&+ (t + s)^\delta \sum_{x \in \mathcal{J}(t): S(x) > t+at} e^{-\alpha S(x)} |m^{\phi_\infty}_{t+s-S(x)} - m^{\phi_\infty}_\infty| \\
&+ (t + s)^\delta |V(t) - V| m^{\phi_\infty}_\infty \\
&=: \sum_{j=1}^4 |I_j(t)|.
\end{align*}
\]
Since (4.13) is a consequence of (2.9) and Condition 2.5 holds, Proposition 4.3 applies and gives
\[|I_4(t)| = (2t)^\delta |V(t) - V| m^{\phi_\infty}_\infty \to 0 \quad \text{a.s. as} \quad t \to \infty.\]
Further,
\[
\mathbb{E}[|I_3(t)|] \leq 2 \left( \sup_{u \geq 0} m^{\phi_\infty}_u \right) (t + s)^\delta \mathbb{E} \left[ \sum_{x \in \mathcal{J}(t): S(x) > t+at} e^{-\alpha S(x)} \right] \\
\leq 2 \left( \sup_{u \geq 0} m^{\phi}_u \right) (2t)^\delta \mathbb{P}(R_t > at)
\]
where $R_t = S_{\nu(t)} - t$ and $\nu(t) = \inf\{n \in \mathbb{N}_0 : S_n > t\}$. An application of Lemma A.1 yields $\mathbb{E}[|I_3(t)|] \to 0$ when $t \to \infty$. Turning to $I_2$, we see that
\[
\mathbb{E}[|I_2(t)|] = \mathbb{E} \left[ \sum_{x \in \mathcal{J}(t): S(x) \leq t+at} e^{-\alpha S(x)} (t + s)^\delta |m^{\phi_\infty}_{t+s-S(x)} - m^{\phi_\infty}_\infty| \right] \\
\leq (2t)^\delta \sup_{u \geq (1-a)t} |m^{\phi_\infty}_u - m^{\phi_\infty}_\infty| \to 0
\]
\[24\]
as $t \to \infty$ by Theorem 4.2(ii) in [22] (applicability of the cited result has already been justified).

It remains to show that $I_1(t) \to 0$ in $\mathbb{P}$-probability when $t \to \infty$. For fixed $t$, define $Z_x := e^{-\alpha(t+s-S(x))}[Z_{\phi_c}]_x(t+s-S(x))$. Then, given $\mathcal{H}_T$, the $Z_x$, $x \in \mathcal{J}(t)$ are independent. In order to use Chebyshev’s inequality below, we truncate the $Z_x$. Define $Z_x' := Z_x I\{Z_x \leq e^{\alpha S(x)}\}$ and $m_x' := \mathbb{E}[Z_x']$. Notice that $Z_x' \leq Z_x$ and hence $m_x' \leq m_{t+s-S(x)}^\phi_c$. Let $I_1'(t)$ be defined as $I_1(t)$ but with $Z_x$ replaced by $Z_x'$ and $m_{t+s-S(x)}^\phi_c$ replaced by $m_x'$. Then

$$I_1(t) = I_1'(t) + (t + s)^\delta \sum_{x \in \mathcal{J}(t)} e^{-\alpha S(x)}(m_x' - m_{t+s-S(x)}^\phi_c)$$
onumber

on $\{Z_x \leq e^{\alpha S(x)}\}$ for all $x \in \mathcal{J}(t)$). Using this, we infer for arbitrary $\eta > 0$,

$$\mathbb{P}(I_1(t) \geq \eta) = \mathbb{E}[\mathbb{P}(I_1(t) \geq \eta | \mathcal{H}_T)] \leq \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} \mathbb{P}(Z_x > e^{\alpha S(x)} | \mathcal{H}_T) \right] + \mathbb{E}[\mathbb{P}(I_1'(t) \geq \eta/2 | \mathcal{H}_T)] + \frac{2(t + s)^\delta}{\eta} \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} e^{-\alpha S(x)}(m_x' - m_{t+s-S(x)}^\phi_c) \right].$$

We consider the last three terms separately. Using Markov’s inequality, we obtain for the first term that

$$\mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} \mathbb{P}(Z_x > e^{\alpha S(x)} | \mathcal{H}_T) \right] \leq \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} e^{-\alpha S(x)} \mathbb{E}[Z_x I\{Z_x \leq e^{\alpha S(x)}\} | \mathcal{H}_T] \right] \leq \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} e^{-\alpha S(x)} \sup_{u \geq 0} \mathbb{E}[e^{-\alpha u Z_{\phi}(u) I\{e^{-\alpha u Z_{\phi}(u)} > e^{\alpha t}\}}] \right] = \sup_{u \geq 0} \mathbb{E}[e^{-\alpha u Z_{\phi}(u) I\{e^{-\alpha u Z_{\phi}(u)} > e^{\alpha t}\}}] \to 0$$

as $t \to \infty$ by the uniform integrability of the family $e^{-\alpha u Z_{\phi}(u)}$, $u \geq 0$ (which is a consequence of Condition 2.8). Next, we estimate the second term

$$\mathbb{E}[\mathbb{P}(I_1'(t) \geq \eta/2 | \mathcal{H}_T)] \leq \frac{4(t + s)^{2\delta}}{\eta^2} \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} e^{-2\alpha S(x)} \operatorname{Var}[Z_x | \mathcal{H}_T] \right] \leq \frac{4(t + s)^{2\delta}}{\eta^2} \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} e^{-2\alpha S(x)} \mathbb{E}[Z_x^2 I\{Z_x \leq e^{\alpha S(x)}\} | \mathcal{H}_T] \right] \leq \frac{4(t + s)^{2\delta}}{\eta^2} \mathbb{E}\left[ \sum_{x \in \mathcal{J}(t)} e^{-2\alpha S(x)} \frac{e^{\alpha S(x)}}{h(e^{\alpha S(x)})} \sup_{u \geq 0} \mathbb{E}[h(e^{-\alpha u Z_{\phi}(u)})] \right] \leq \frac{4^{1+\delta} e^{\alpha t} t^{2\delta}}{\eta^2} \sup_{u \geq 0} \mathbb{E}[h(e^{-\alpha u Z_{\phi}(u)})] \to 0 \text{ as } t \to \infty$$

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where we have used the independence of $Z_x$ and $H_{T_k}$. Chebyshev’s inequality given $H_{T_k}$ and the facts that $t \mapsto t^2/h(t)$ and $t \mapsto t/h(t)$ are increasing and decreasing, respectively, for large $t$, $(t\log t)^{\delta}/h(t) \to 0$ as $t \to \infty$, and the finiteness of the supremum (see Condition 2.8). Finally, the third term can be estimated as follows:

$$
\frac{2(t+s)^{\delta}}{\eta} \mathbb{E} \left[ \sum_{x \in J(t)} e^{-\alpha S(x)} (m_x^{\delta} - m_x') \right] = \frac{2^{1+\delta} t^{\delta}}{\eta} \mathbb{E} \left[ \sum_{x \in J(t)} e^{-\alpha S(x)} \mathbb{E} \left[ Z_x 1_{\{Z_x > e^{\alpha S(x)}\}} \big| H_{T_k} \right] \right] \\
\leq \frac{2^{1+\delta} t^{\delta}}{\eta} \mathbb{E} \left[ \sum_{x \in J(t)} e^{-\alpha S(x)} \frac{e^{\alpha S(x)}}{h(e^{\alpha S(x)})} \right] \sup_{u \geq 0} \mathbb{E} \left[ h(e^{-\alpha Z}\phi(u)) \right] \\
\leq \frac{2^{1+\delta} t^{\delta}}{\eta} \frac{e^{\alpha t}}{h(e^{\alpha t})} \sup_{u \geq 0} \mathbb{E} \left[ h(e^{-\alpha Z}\phi(u)) \right] \to 0
$$
as $t \to \infty$ using the same argument as above.

**Proof of Theorem 2.11:** The general (multi-type) case. In view of the embedding technique and key identity (5.2), it is enough to show that the embedded single-type process $(Z_{J,x})_{n \geq 0}$ and the characteristic $\phi_J$ fulfill the assumptions of the single-type version of this theorem. According to Proposition 3.2, $(Z_{J,x})_{n \geq 0}$ satisfies the counterparts of the standing assumptions (A1)-(A4). Validity of (2.9) and of Condition 2.5 for the embedded process follow from Proposition 4.8(c) and (d), respectively. Condition 2.6 carries over immediately to the embedded process. Condition 2.8 is a condition on $Z^\phi$. It is thus identical for the original and the embedded process (via (5.2)). Condition 2.10 holds for the embedded process according to Lemma 4.9.

**6 Application of the main result**

Iteration of (1.1) leads to a signed weighted branching process, the asymptotic behavior of which plays a crucial role when solving (1.1), see [15] for more details. As an application of our results, in this section, we prove that this signed weighted branching process converges to zero in probability and investigate the rate of convergence.

Let $(T_1, T_2, \ldots)$ be a sequence of real-valued random variables and let $(\Omega, \mathcal{A}, \mathbb{P})$ be the canonical product space on which the $(T_1(x), T_2(x), \ldots)$, $x \in I$ are i.i.d. with the same distribution as $(T_1, T_2, \ldots)$. For simplicity, suppose that $|T_j| < 1$ a.s. for all $j \in N$. Define multiplicative weights by

$$
L(x) := \prod_{k=1}^{n} T_{x_k}(x_{k-1}) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{N}^n, \ n \in \mathbb{N}_0.
$$

The family $(L(x))_{x \in I}$ forms a weighted branching process. It can be interpreted as the multiplicative analogue of the multi-type general branching process with type space $\{ -1, 1 \}$ based on the point processes $'J(x)$, $x \in I$ where $'J(x)(\{ j \} \times \cdot) = \sum_{k \geq 1} 1_{\{ j \in T_k(x) > 0 \}} \delta_{-\log |T_k(x)|}$. Then, position and type of individual $x$ are given by

$$
S(x) := -\log |L(x)| \quad \text{and} \quad \tau(x) := \text{sign}(L(x)) \in \{1, -1\}, \quad (6.1)
$$
respectively. Notice that, by definition, the ancestor has type 1. We write $\mathbb{P}^1$ for $\mathbb{P}$ when we want to emphasize this and we write $\mathbb{P}^{-1}$ for the same probability measure but under which the ancestor has type $-1$ a.s.

In [15] where the set of all solutions to (1.1) is determined, the most intricate case in which $\mathbb{E} \left[ \sum_{j \geq 1} |T_j| \right] = 1$ while $\mathbb{E} \left[ \sum_{j \geq 1} T_j \right] \in (-1, 1)$ is dealt with by using the results obtained below on the asymptotic behavior of $\sum_{x \in J(t)} L(x)$, where $J(t)$ is defined in (4.11).

Define $q := \mathbb{E} \left[ \sum_{j \geq 1} T_j 1_{(T_j > 0)} \right]$. Then $q \in (0, 1)$ and

$$
\mathbf{M} := \mathbf{M}(1) = \begin{pmatrix} q & 1 - q \\ 1 - q & q \end{pmatrix}.
$$

It is readily checked that 1 is the Perron-Frobenius eigenvalue of $\mathbf{M}$ with left and right eigenvectors $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (\frac{1}{2}, \frac{1}{2})$, respectively. This verifies (A2) and (A3) for the general branching process defined by (6.1). Now additionally assume that (A1) and (A4) hold.

We rewrite $\sum_{x \in J(t)} L(x)$ as a difference of general branching processes counted with appropriate characteristics. Let

$$
\phi^1(t) := e^t \sum_{|x|=1} e^{-S(x)} 1_{[0,S(x))}(t) 1_{\{\tau(x) = \tau(x)\}}, \quad t \geq 0
$$

and

$$
\phi^{-1}(t) := e^t \sum_{|x|=1} e^{-S(x)} 1_{[0,S(x))}(t) 1_{\{\tau(x) = -\tau(x)\}}, \quad t \geq 0.
$$

Then $[\phi^1](t - S(x)) = e^t \sum_{|y|=1, \tau(y) = 1} e^{-S(xy)} 1_{\{S(x)t < S(xy)\}}$, analogously for $[\phi^{-1}](t - S(x))$. Consequently, for $t \geq 0$

$$
e^{-t} \mathcal{Z}^\phi(t) = \sum_{x \in J(t); \tau(x) = 1} e^{-S(x)} \quad \text{and} \quad e^{-t} \mathcal{Z}^{-\phi}(t) := \sum_{x \in J(t); \tau(x) = -1} e^{-S(x)}.
$$

For $i, j = 1, -1, t \mapsto e^{-t} \mathbb{E}^i[\phi^j(t)]$ is decreasing and

$$
0 \leq e^{-t} \mathbb{E}^i[\phi^j(t)] \leq \mathbb{E} \left[ \sum_{|x|=1} e^{-S(x)} 1_{\{S(x)t > 0\}} \right] = \mathbb{P}(S_1 > t)
$$

where the last equality is a consequence of (3.1). The direct Riemann integrability of $t \mapsto e^{-t} \mathbb{E}^i[\phi^j(t)]$ follows because (A4) implies $\int_0^\infty \mathbb{P}(S_1 > t) \, dt = \mathbb{E}[S_1] < \infty$.

Further, $\sup_{0 \leq s \leq t} \phi^j(s) \leq e^t \sum_{|x|=1} e^{-S(x)}$ is $\mathbb{P}^i$ integrable for all $t \geq 0, i, j = -1, 1$. Consequently, Theorem 2.1 applies and gives

$$
\sum_{x \in J(t); \tau(x) = 1} e^{-S(x)} \to V \frac{\sum_{i=1, -1} u_i \int_0^\infty e^{-s} \mathbb{E}^i[\phi^1(s)] \, ds}{\sum_{i,j=1, -1} u_i v_j (-m_{ij})'(1)} \quad \text{in } \mathbb{P}^1\text{-probability}
$$

as $t \to \infty$. Analogously, $\sum_{x \in J(t); \tau(x) = -1} e^{-S(x)} \to V/2$ in $\mathbb{P}^1\text{-probability as } t \to \infty$. In particular,

$$
\sum_{x \in J(t)} L(x) = \sum_{x \in J(t); \tau(x) = 1} e^{-S(x)} - \sum_{x \in J(t); \tau(x) = -1} e^{-S(x)} \to 0 \quad \text{in } \mathbb{P}^1\text{-probability as } t \to \infty.
$$

The rate of convergence to zero is discussed next.
Theorem 6.1. Assume that the general branching process defined in (6.1) satisfies (A1)--(A4) with $\alpha = 1$ and that Condition 2.6 holds. Let $\delta > 0$. If
\[
\mathbb{E}\left[\sum_{|x|=1} e^{-S(x)} S(x)^{1+\delta}\right] < \infty \tag{2.9}
\]
and $\mathbb{E}[h(t^{1}W_{1}) \log^{+} 1W_{1}] < \infty$ for $h(t) = t(\log t)^{\delta_{0}} \log(\log t) \mathbbm{1}_{[e,\infty)}(t)$, then
\[
t^{\delta}\left|\sum_{x \in \mathcal{J}(t)} L(x)\right| \to 0 \quad \text{in } \mathbb{P}\text{-probability}. \tag{6.2}
\]

Proof. If we can check that the assumptions of Theorem 2.11 hold, then we can conclude that
\[
t^{\delta}\left|\sum_{x \in \mathcal{J}(t)} L(x)\right| = t^{\delta}\left|e^{-tZ^{\phi^{1}}(t)} - e^{-tZ^{\phi^{-1}}(t)}\right|
\leq t^{\delta}\left|e^{-tZ^{\phi^{1}}(t)} - \frac{V}{2}\right| + t^{\delta}\left|e^{-tZ^{\phi^{-1}}(t)} - \frac{V}{2}\right| \to 0
\]
in $\mathbb{P}$-probability as $t \to \infty$. Condition 2.5 follows from $\mathbb{E}[h(t^{1}W_{1})] < \infty$. Condition 2.6 and (2.9) hold by assumption. In order to check that Condition 2.10 holds, first notice that $\mathbb{E}[\phi^{j}(s)] \leq e^{s}\mathbb{P}(S_{1} > t)$ for all $t \geq 0$ and $i, j \in \{1, -1\}$. Here, (2.9) implies that $\mathbb{E}[S_{1}^{1+\delta}] < \infty$ and, therefore,
\[
t^{\delta}\int_{t}^{\infty} e^{-s}\mathbb{E}[\phi^{j}(s)] \, ds \leq \int_{t}^{\infty} s^{\delta}\mathbb{P}(S_{1} > s) \, ds \to 0 \quad \text{as } t \to \infty.
\]
Similarly,
\[
t^{\delta}\sup_{s \geq t} e^{-s}\mathbb{E}[\phi^{j}(s)] \leq t^{\delta}\mathbb{P}(S_{1} > t) \to 0 \quad \text{as } t \to \infty.
\]
This implies validity of Condition 2.10.

It remains to check that (2.7) holds. To this end, using $e^{-tZ^{\phi^{i}}(t)} \leq e^{-t} \sum_{x \in \mathcal{J}(t)} e^{-S(x)} = V(t)$ for $i = 1, -1$, it suffices to verify that
\[
\mathbb{E}\left[h\left(\sup_{t \geq 0} V(t)\right)\right] < \infty. \tag{6.3}
\]
To this end, we invoke Lemma 8.1 of [4] (which is an extension of an observation in [8]). The cited lemma gives that for any $0 < a < 1$, there is a finite constant $C(a) > 0$ such that
\[
\mathbb{P}(V > at) \geq C(a)\mathbb{P}\left(\sup_{s \geq 0} V(s) > t\right) \quad \text{for all } t > 1.
\]
In view of this inequality and since $h$ is regularly varying at $+\infty$, we conclude that for $\mathbb{E}[h(\sup_{s \geq 0} V(s))] < \infty$ to hold it is necessary and sufficient that $\mathbb{E}[h(V)] < \infty$. According to Proposition 4.2, $\mathbb{E}[h(t^{1}W_{1}) \log^{+} 1W_{1}] < \infty$ entails the finiteness of $\mathbb{E}[h(t^{1}W)]$ which is equal to $\mathbb{E}[h(V)]$ by Proposition 4.8(a).

A Auxiliary results

Lemma A.1. Assume that $(S_{n})_{n \geq 0}$ is a zero-delayed renewal process and that $\mathbb{E}[S_{1}^{1+\delta}] < \infty$ for some $\delta > 0$. If $\nu(t) := \inf\{n \in \mathbb{N}_{0} : S_{n} > t\}$ is the first passage time into $(t, \infty)$ and $R_{t} = S_{\nu(t)} - t$ is the excess at time $t$, then
\[
t^{\delta}\mathbb{P}(R_{t} > at) \to 0 \quad \text{as } t \to \infty
\]
for every $a > 0$. 28
Proof. With $U$ denoting the renewal function, write
\[
t^δ P(R_t > at) = t^δ \int_{[0,t]} P(S_1 > (a+1)t - y) U(dy) \leq t^{1+δ} P(S_1 > at) \frac{U(t)}{t}
\]
and observe that $\lim_{t \to \infty} t^{1+δ} P(S_1 > at) = 0$ in view of $E[S_1^{1+δ}] < \infty$, while $\lim_{t \to \infty} t^{-1} U(t) = E[S_1] < \infty$ by the elementary renewal theorem.

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