The cozero-divisor graph relative to finitely generated modules

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THE COZERO-DIVISOR GRAPH RELATIVE TO FINITELY GENERATED MODULES

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Abstract. Let $R$ be a commutative ring and let $M$ be a finitely generated $R$-module. Let’s denote the cozero-divisor graph of $R$ by $\Gamma(R)$. In this paper, we introduce a certain subgraph $\Gamma_R(M)$ of $\Gamma(R)$, called cozero-divisor graph relative to $M$, and obtain some related results.

2010 Mathematics Subject Classification: 05C75; 13A99; 05C99

Keywords: cozero-divisor, complete graph, finitely generated

1. INTRODUCTION

Throughout this paper, $R$ will denote a commutative ring with identity. We denote the set of maximal ideals of $R$ by $\text{Max}(R)$.

A graph $G$ is defined as the pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of $G$ and $E(G)$ is the set of edges of $G$. For two distinct vertices $a$ and $b$ of $V(G)$, the notation $a \sim b$ means that $a$ and $b$ are adjacent. A graph $G$ is said to be complete if $a \sim b$ for all distinct $a, b \in V(G)$, and $G$ is said to be empty if $E(G) = \emptyset$. Note that by this definition a graph may be empty even if $V(G) \neq \emptyset$. If $|V(G)| \geq 2$, a path from $a$ to $b$ is a series of adjacent vertices $a = v_1 \sim v_2 \sim \cdots \sim v_n = b$. The length of a path is the number of edges it contains. A cycle is a path that begins and ends at the same vertex in which no edge is repeated, and all vertices other than the starting and ending vertex are distinct. If a graph $G$ has a cycle, the girth of $G$ (notated $g(G)$) is defined as the length of the shortest cycle of $G$; otherwise, $g(G) = \infty$. A graph $G$ is connected if for every pair of distinct vertices $a, b \in V(G)$, there exists a path from $a$ to $b$. If there is a path from $a$ to $b$ with $a, b \in V(G)$, then the distance from $a$ to $b$ is the length of the shortest path from $a$ to $b$ and is denoted $d(a, b)$. If there is not a path between $a$ and $b$, $d(a, b) = \infty$. The diameter of $G$ is $\text{diam}(G) = \sup \{d(a, b) | a, b \in V(G)\}$.

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [8]. He assumes that all elements of the ring are vertices of the graph and was mainly interested in colorings and then this investigation of coloring of a commutative ring was continued by Anderson and Naseer in [4]. Anderson and Livingston [7], studied the zero-divisor graph whose vertices are the nonzero zero-divisors.
Let $Z(R)$ be the set of zero-divisors of $R$. The zero-divisor graph of $R$ denoted by $\Gamma(R)$, is a graph with vertices $Z^*(R) = Z(R) \setminus \{0\}$ and for distinct $x, y \in Z^*(R)$ the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. This graph turns out to exhibit properties of the set of the zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us explore some interesting results in algebraic structures of rings. The zero-divisor graph of a commutative ring has also been studied by several other authors (e.g., [5, 6, 10]).

In [2], Afkhami and Khashyarmanesh introduced the cozero-divisor graph $\hat{\Gamma}(R)$ of $R$, in which the vertices are precisely the nonzero, non-unit elements of $R$, denoted $W^*(R)$, and two vertices $x$ and $y$ are adjacent if and only if $x \not\in yR$ and $y \not\in xR$.

Now let $M$ be a finitely generated $R$-module. The purpose of this paper is to introduce a certain subgraph $\hat{\Gamma}_R(M)$ of $\hat{\Gamma}(R)$, called the cozero-divisor graph relative to $M$ and obtain some results similar to those of [2] and [3]. This graph, with a different point of view, can be regarded as a reduction of $\hat{\Gamma}(R)$, namely, we have $\hat{\Gamma}_R(R) = \hat{\Gamma}(R)$.

2. Auxiliary results

Let $M$ be an $R$-module. The support of $M$ is denoted by $\text{Supp}(M)$ and it is defined by

$$\text{Supp}(M) = \{ P \in \text{Spec}(R) | \text{Ann}_R(N) \subseteq P \text{ for some cyclic submodule } N \text{ of } M \}.$$

In the rest of this paper Max($\text{Supp}(M)$) (i.e., the set of all maximal elements in $\text{Supp}(M)$) is denoted by Max($M$).

The Jacobson radical of $M$ is denoted by $J(M)$ and it is the intersection of all elements in Max($M$). Also, the union of all elements in Max($M$) is denoted by $N_R(M)$ [12].

$M$ is said to be a local module if $|\text{Max}(M)| = 1$ [12].

The subset $W_R(M)$ of $R$ is defined by $\{ r \in R | rM \neq M \}$ [12] and set $W_R^*(M) = W_R(M) \setminus \{0\}$.

$Z_R(M) = \{ r \in R \}$ the $R$-module endomorphism on $M$ defined by multiplication by $r$ is not injective $\}.$

Remark 1 (See [12]). Let $M$ be an $R$-module. Then $W_R(M) \subseteq N_R(M)$ and we have equality if $M$ is a finitely generated $R$-module.

Remark 2. Max($M$) $\subseteq$ Max($R$).

Proof. This follows immediately from the proof of [12, 1.4]. $\square$
3. MAIN RESULTS

In the rest of this paper $M$ is a finitely generated $R$-module.

**Definition 1.** We define the *cozero-divisor graph relative to $M$*, denoted by $\Gamma_R(M)$, as a graph with vertices $W^*_R(M) = W_R(M) \setminus \{0\}$ and two distinct vertices $r$ and $s$ are adjacent if and only if $r \not\in (sM :_RM)$ and $s \not\in (rM :_RM)$.

**Definition 2.** We define the *strongly cozero-divisor graph relative to $M$*, denoted by $\Gamma_R^*(M)$, as a graph with vertices $W^*_R(M) = W_R(M) \setminus \{0\}$ and two distinct vertices $r$ and $s$ are adjacent if and only if $r \not\in \sqrt{(sM :_RM)}$ and $s \not\in \sqrt{(rM :_RM)}$.

The following example shows that $\Gamma(R)$, $\Gamma_R(M)$, and $\Gamma_R^*(M)$ are different.

**Example 1.** Set $R = \mathbb{Z}$ (here $\mathbb{Z}$ denotes the ring of integers) and $M = \mathbb{Z}_{12}$. Then $W^*_R(R) = \mathbb{Z} \setminus \{-1, 1, 0\}$ and $W^*_R(M) = \mathbb{Z} \setminus \{m : (m, 12) = 1\} \cup \{0\}$, where $(m, 12)$ denotes the greatest common divisor of $m$ and 12. The elements 8 and 12 are adjacent in $\Gamma(R)$ but they are not adjacent in $\Gamma_R^*(M)$. Also, 6 and 8 are adjacent in $\Gamma_R^*(M)$ but they are not adjacent in $\Gamma_R(M)$. Moreover, 6 and 10 are adjacent in $\Gamma_R^*(M)$ but they are not adjacent in $\Gamma_R^*(M)$.

An $R$-module $L$ is said to be a *multiplication module* if for every submodule $N$ of $L$ there exists an ideal $I$ of $R$ such that $N = IL$.

**Theorem 1.**

(a) $\Gamma_R(M)$ is a subgraph of $\Gamma(R)$.

(b) $\Gamma_R^*(M)$ is a subgraph of $\Gamma^*(R)$.

(c) If $M$ is a faithful $R$-module, then $W^*_R(M) = W^*(R)$.

(d) If $M$ is a faithful $R$-module, then $\Gamma_R^*(M) = \Gamma_R(M)$.

(e) If $M$ is a faithful multiplication $R$-module, then $\Gamma_R(M) = \Gamma(R)$.

**Proof.** Parts (a) and (b) are clear.

(c) By part (a), $W^*_R(M) \subseteq W^*(R)$. Now let $r \in W^*(R)$ and $r \not\in W^*_R(M)$. Then $rM = M$. Thus by Nakayama’s Lemma, $1 + rt \in \text{Ann}_RM = 0$. Hence $Rr = R$, which is a contradiction.

(d) By part (c), $W^*_R(M) = W^*(R)$. Now let $r$ and $s$ be two distinct adjacent vertices of $\Gamma_R^*(M)$ and let $r \in \sqrt{(sM :_RM)}$. Then $r^nM \subseteq sM$ for some $n \in \mathbb{N}$. Thus by [11, Theorem 75], there exist $t \in R$ and $k \in \mathbb{N}$ such that $(r^{kn} + st)M = 0$. Since $M$ is faithful, $r^{kn} + st = 0$ and so $r \in \sqrt{sM}$. This contradiction shows that $E(\Gamma_R(M)) \subseteq E(\Gamma_R^*(M))$. The reverse inclusion is clear.

(e) By part (c), $W^*_R(M) = W^*(R)$. Now let $r$ and $s$ be two distinct adjacent vertices of $\Gamma(R)$ and let $r \in (sM :_RM)$. Then $rM \subseteq sM$. Thus by [1], $Rr \subseteq sR$, which is a contradiction. Hence $E(\Gamma(R)) \subseteq E(\Gamma_R(M))$. The reverse inclusion is clear.

**Remark 3.** By using part (e) of Theorem 1, if $M = R$, then $\Gamma_R(R) = \Gamma(R)$. 


We use the following lemma frequently.

**Lemma 1.** Let $M$ be an $R$-module and $P \in \text{Max}(M)$. Then $P = (PM :_R M)$.

**Proof.** Assume $(PM :_R M) = R$ so that $PM = M$. Since $M$ is finitely generated, there exists $x \in P$ such that $(1 + x)M = 0$. Thus $1 + x \in \text{Ann}_R(M)$ but by [12], $P \supseteq \text{Ann}_R(M)$. It follows that $1 \in P$, a contradiction. Now the results follows from $P \subseteq (PM :_R M)$ and Remark 2. □

**Proposition 1.**
(a) The graph $\hat{\Gamma}_R(M)$ is not complete if and only if there exists an element $s \in W^*_R(M)$ such that $|sM :_R M| > 2$.
(b) $\hat{\Gamma}_R(M)$ is complete if and only if $(sM :_R M) = \{0, s\}$ for all elements $s$ in $W^*_R(M)$.
(c) If $R$ is an integral domain, then $\hat{\Gamma}_R(M)$ is not complete.

**Proof.** Straightforward □

**Theorem 2.** $\hat{\Gamma}_R(M)$ is complete if and only if $\hat{\Gamma}_R(M)$ is complete.

**Proof.** The sufficiency is clear. Conversely, we assume that $\hat{\Gamma}_R(M)$ is complete and $r, s$ be arbitrary distinct elements in $W^*_R(M)$ and $r \in \sqrt{(sM :_R M)}$. Then $r^n M \subseteq sM$ for some $n \in \mathbb{N}$. Since $\hat{\Gamma}_R(M)$ is complete, $r^n$ and $s$ are adjacent. But this is a contradiction by the above arguments. □

We use the notation $\hat{\Gamma}_R(M) \setminus J(M)$ to denote a subgraph of $\hat{\Gamma}_R(M)$ with vertices $W^*_R(M) \setminus J(M)$.

**Theorem 3.**
(a) The graph $\hat{\Gamma}_R(M) \setminus J(M)$ is connected.
(b) If $M$ is a non-local module, then $\text{diam}(\hat{\Gamma}_R(M) \setminus J(M)) \leq 2$.

**Proof.** (a) If $M$ is a local module, then $W^*_R(M) \setminus J(M)$ is a empty set, which is connected. So we assume that $|\text{Max}(M)| > 1$. Let $r$ and $s$ be arbitrary distinct elements in $W^*_R(M) \setminus J(M)$. Suppose that $r$ is not adjacent to $s$. We may assume that $r \in (sM :_R M)$. Since $r \not\in J(M)$, there exists $P \in \text{Max}(M)$ such that $r \not\in P$. Thus $P \not\subseteq J(M)$, otherwise, $P \subseteq J(M)$ or $P \subseteq (sM :_R M)$. In first case, $J(M) = P$ so that $|\text{Max}(M)| = 1$. In second case, $P = (sM :_R M)$ by Lemma 1. In either case we have a contradiction. Choose $t$ in $P \setminus J(M) \cup (sM :_R M)$. Now by using Lemma 1, we see that $r - t - s$ is the required path.
(b) This follows from the proof of part (a). □

**Corollary 1.** Let $M$ be a non-local $R$-module with $J(M) = 0$. Then $\hat{\Gamma}_R(M)$ is connected and $\text{diam}(\hat{\Gamma}_R(M)) \leq 2$.

**Theorem 4.** Let $M$ be a non-local module such that for every element $r \in J(M)$, there exist $P \in \text{Max}(M)$ and $s \in P \setminus J(M)$ with $r \not\in (sM :_R M)$. Then $\hat{\Gamma}_R(M)$ is connected and $\text{diam}(\hat{\Gamma}_R(M)) \leq 3$. 
Suppose that \( r, s \in W_1^s(M) \) and \( r \) is not adjacent to \( s \). We may assume that \( r \in (sM :_R R) \). Then, we have the following cases:

Case 1. Suppose that \( s \in J(M) \). We claim that \( r \in J(M) \). Otherwise there exists \( P \in Max(M) \) such that \( r \notin P \). Then \( rM \subseteq sM \subseteq PM \). Thus by Lemma 1, \( r \in (PM :_R R) = P \), a contradiction. Thus by hypothesis, there exists \( t \in P \setminus J(M) \) for some \( P \in Max(M) \) with \( r \notin (tM :_R R) \). Also \( t \notin (rM :_R R) \); otherwise, we have \( tM \subseteq rM \subseteq sM \). Thus \( t \in (sM :_R R) \subseteq (PM :_R R) = P \) for each \( P \in J(M) \) so that \( t \in J(M) \), a contradiction. Thus \( r \) is adjacent to \( t \). By similar arguments, we see that \( t \) is adjacent to \( s \). Hence \( r - t - s \) is the required path.

Case 2. Suppose that \( r, s \notin J(M) \). Then \( r \notin P \), for some \( P \in Max(M) \). If \( P = (sM :_R R) \), then since \( r \in (sM :_R R) \), we have a contradiction. Choose \( p \) in \( P \setminus (sM :_R R) \). By similar arguments as in part (a), we see that \( r - p - s \) is the desired path.

Case 3. Assume that \( s \notin J(M) \) and \( r \in J(M) \). By our assumption, there exists \( q \in P \setminus J(M) \), for some \( P \in Max(M) \) such that \( r \notin (qM :_R R) \). We claim that \( q \notin (rM :_R R) \). Otherwise, \( qM \subseteq rM \subseteq PM \) for every \( P \in Max(M) \). Thus by Lemma 1, \( q \in (PM :_R R) = P \) for every \( P \in Max(M) \), a contradiction. Hence \( r \) is adjacent to \( q \). Further, \( s \notin (qM :_R R) \). If \( q \notin (sM :_R R) \), then we get the the path \( r - q - s \). Otherwise, we can apply case 2 for the elements \( q \) and \( s \) to get a path \( q - u - s \) for some \( u \in W_1^s(M) \). Hence we have \( r - q - u - s \). \( \square \)

**Theorem 5.** Let \( M \) be a non-local module. Then \( g(\hat{\Gamma}_R(M) \setminus J(M)) \leq 5 \) or \( g(\hat{\Gamma}_R(M) \setminus J(M)) = \infty \).

**Proof.** Use the technique of [2, 2.8] and apply Theorem 3. \( \square \)

**Theorem 6.** Let \( |Max(M)| \geq 3 \). Then \( g(\hat{\Gamma}_R(M)) = 3 \).

**Proof.** Clearly, \( g(\hat{\Gamma}_R(M)) \geq 3 \). Let \( P_1, P_2, \) and \( P_3 \) be distinct elements of \( Max(M) \). By Remark 2, \( Max(M) \subseteq Max(R) \). Choose \( a_i \in P_i \setminus \bigcup_{j=1}^{3} P_j \), \( 1 \leq i \leq 3 \) and \( j \neq i \). Then by using 1, we see that \( a_1 - a_2 - a_3 - a_1 \) is a cycle. Therefore \( g(\hat{\Gamma}_R(M)) = 3 \). \( \square \)

For a graph \( G \), let \( \chi(G) \) denote the **chromatic number of the graph** \( G \), i.e., the minimal number of colors which can be assigned to the vertices of \( G \) in such a way that every two adjacent vertices have different colors. A **clique** of a graph is its complete subgraph and the number of vertices in the largest clique of \( G \), denoted by \( \text{clique}(G) \), is called the clique number of \( G \).

**Theorem 7.** (a) Let \( R \) not be a field. Then if \( Max(M) \) has an infinite number of maximal ideals, then \( \text{clique}(\hat{\Gamma}_R(M)) \) is also infinite; otherwise \( \text{clique}(\hat{\Gamma}_R(M)) \geq |Max(M)| \).

(b) If \( \chi(\hat{\Gamma}_R(M)) < \infty \), then \( |Max(M)| < \infty \).

**Proof.** Use the technique of [2, 2.14]. \( \square \)
A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends.

**Theorem 8.** Assume that $|\text{Max}(M)| \geq 5$. Then $\hat{\Gamma}_R(M)$ is not planar.

**Proof.** Assume that $|\text{Max}(M)| \geq 5$. Choose $a_i \in m_i \setminus \cup_{j=1}^{5} m_j$, where $m_i \in \text{Max}(M)$, $1 \leq i \leq 5$, and $j \neq i$. Then $a_i \notin (a_j :_R M)$. Otherwise, $a_i \in (a_j :_R M) \subseteq (m_j :_R M) = m_j$ by Lemma 1. Similarly, $a_j \notin (a_i :_R M)$. Hence $m_1, m_2, m_3, m_4, m_5$ forms a complete subgraph of $\hat{\Gamma}_R(M)$ which is isomorphic to $K_5$. Thus by [9, p.153], $\hat{\Gamma}_R(M)$ is not planar.

For any vertex $x$ of a connected graph $G$, the eccentricity of $x$, denoted by $e(x)$, is the maximum of the distances from $x$ to the other vertices of $G$, and the minimum value of the eccentricity is the radius of $G$, which is denoted by $r(G)$.

**Theorem 9.** Let $M$ be a non-local module with $J(M) = 0$. Then $r(\hat{\Gamma}_R(M)) = 2$ if and only if for each $t \in W^*_R(M)$, there exists $s \in W^*_R(M)$ such that $t$ is not adjacent to $s$.

**Proof.** The proof is similar to that of [2, 3.14].

**Theorem 10.** Let $R$ be a Noetherian ring. If $\hat{\Gamma}_R(M)$ is totally disconnected, then $M$ is a local module with maximal ideal of the from $(xM :_R M)$ for some $x \in W^*_R(M)$.

**Proof.** It is easy to see that $M$ is a local module. Set $\text{Max}(M) = m$. Assume to contrary that $m$ is not the form of $(rM :_R M)$ for every $r \in W^*_R(M)$. Set $A = \{(rM :_R M), r \in W^*_R(M)\}$. Then $A$ has a maximal member, say $(\hat{r}M :_R M)$ for some $\hat{r} \in W^*_R(M)$. Choose $s \in m \setminus (\hat{r}M :_R M)$. We claim that $\hat{r} \notin (sM :_R M)$. Otherwise, we have $(\hat{r}M :_R M) \subseteq (sM :_R M)$, so $(\hat{r}M :_R M) = (sM :_R M)$ by maximality. Hence $s \in (\hat{r}M :_R M)$ so that $\hat{r}$ is adjacent to $s$, a contradiction.

**Theorem 11.** Assume that $M$ is a non-local module. Then the following conditions are equivalent.

(a) $\hat{\Gamma}_R(M) \setminus J(M)$ is complete bipartite.

(b) $\hat{\Gamma}_R(M) \setminus J(M)$ is bipartite.

(c) $\hat{\Gamma}_R(M) \setminus J(M)$ contains no triangles.

**Proof.** Use the technique of [3, 2.13].

**Proposition 2.** If the graph $\hat{\Gamma}_R(M) \setminus J(M)$ is $n$-partite for some positive integer $n$, then $|\text{Max}(M)| \leq n$.

**Proof.** Assume to the contrary that $|\text{Max}(M)| > n$. Since $\hat{\Gamma}_R(M) \setminus J(M)$ is an $n$-partite graph, there are maximal ideals $P_1$ and $P_2$ of $\text{Max}R(M)$ with $(rM :_R M) \subseteq P_1 \setminus P_2$ and $(sM :_R M) \subseteq P_2 \setminus P_1$, where $r, s$ belong to the same part. But this implies that $r$ is adjacent to $s$ which is a contradiction.
Theorem 12. Let $M$ be an $R$-module with $\text{Max}(M) = \{m_1, m_2\}$. Then $\hat{\Gamma}(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1, 2$, if and only if every pair of ideals $(rM :_R M)$, $(sM :_R M)$ contained in $(m_1 \setminus J(M))$ or $(m_2 \setminus J(M))$, where $r, s \in R$, are totally ordered.

Proof. Suppose that $\hat{\Gamma}(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1, 2$. Further assume to the contrary that there exist ideals $(rM :_R M)$, $(sM :_R M) \subseteq m_1 \setminus J(M)$ such that $(rM :_R M) \not\subseteq (sM :_R M)$ and $(sM :_R M) \not\subseteq (rM :_R M)$. We claim that $r$ is adjacent to $s$ in $m_1 \setminus J(M)$. Otherwise, without loss of generality, we assume that $r \in (sM :_R M)$ and $s \in m_1 \setminus J(M)$ and we have $rM \subseteq (sM :_R M)M$. Thus $(rM :_R M) \subseteq ((sM :_R M)M :_R M) = (sM :_R M)$, a contradiction. Hence $r$ is adjacent to $s$ in $m_1 \setminus J(M)$, which is again a contradiction by hypothesis. Conversely, assume that $i \in \{1, 2\}$ and $(rM :_R M), (sM :_R M) \subseteq m_i \setminus J(M)$. We may assume that $(rM :_R M) \subseteq (sM :_R M)$. Then clearly, $r, s \in m_i \setminus J(M)$ and $r$ is not adjacent. Now if $r \in m_1 \setminus m_2$ and $s \in m_2 \setminus m_1$, then by using 1, we see that $r$ is adjacent to $s$. Therefore $\hat{\Gamma}(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1, 2$.

Theorem 13. Let $M$ be a faithful $R$-module and $Z_R(M) \neq W_R(M)$. Then $\hat{\Gamma}_R(M)$ is finite if and only if $R$ is finite.

Proof. Clearly if $R$ is finite, then $\hat{\Gamma}_R(M)$ is finite. So we assume that $\hat{\Gamma}_R(M)$ is finite and show that $R$ is finite. Suppose that $R$ is infinite and look for a contradiction. By Remark 1, we have $Z_R(M) \subset W_R(M) = N_R(M)$. Choose $x \in W_R(M) \setminus Z_R(M)$. Since $Rx$ is a finite $R$-module and $R \setminus W_R(M)$ is an infinite set, there exist distinct elements $r_1, r_2 \in R \setminus W_R(M)$ such that $r_1x = r_2x$. Therefore $(r_1 - r_2)x = 0$. Then we have $x((r_1 - r_2)M) = 0$. Since $x$ is a nonzero-divisor on $M$, we have $(r_1 - r_2)M = 0$ so that $r_1 - r_2 \in \text{Ann}_R(M)$. Thus $r_1 = r_2$, a contradiction.

Corollary 2. Let $R$ be a domain and let $Z_R(M) = \{0\}$. If $\hat{\Gamma}_R(M)$ is a finite graph, then $R$ is a field.

Proof. If $W_R(M) \neq \{0\}$, then by Theorem 13, $R$ is finite so that $R$ is a field. Otherwise, if $W_R(M) = \{0\}$, then we have $W_R(M) = \bigcup_{p \in \text{Max}(M)} P = \{0\}$ by Remark 1. This implies that the zero ideal of $R$ is a maximal ideal and hence $R$ is a field.

Remark 4. One can see, by using the same technique, that the results about $\hat{\Gamma}_R(M)$ in this section is also true for $\Gamma_R(M)$.

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