$p$-Adic and Adelic Quantum Mechanics*

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Abstract

$p$-Adic mathematical physics emerged as a result of efforts to find a non-Archimedean approach to the spacetime and string dynamics at the Planck scale. One of its main achievements is a successful formulation and development of $p$-adic and adelic quantum mechanics, which have complex-valued wave functions of $p$-adic and adelic arguments, respectively. Various aspects of these quantum mechanics are reviewed here. In particular, the corresponding Feynman's path integrals, some minisuperspace cosmological models, and relevant approach to string theory, are presented. As a result of adelic approach, $p$-adic effects exhibit a spacetime and some other discreteness, which depend on the adelic quantum state of the physical system under consideration. Besides review, this article contains also some new results.

1 Introduction

At the transition from 19th to 20th century two great parts of fundamental science were born: Quantum Physics and $p$-Adic Mathematics. Developing quite independently till the last two decades, they started to interact rather successfully so that not only some quantum but also classical $p$-adic models have been constructed and investigated. As a result, in 1987 emerged $p$-Adic Mathematical Physics, which is a basis to explore various $p$-adic aspects of modern theoretical physics. Among the main achievements of this new branch of contemporary mathematical physics are $p$-Adic and Adelic Quantum Mechanics.

There are many physical and mathematical motivations to employ $p$-adic numbers and adeles in investigation of mathematical and theoretical aspects of modern quantum physics. Some primary of them are: (i) the field of rational numbers $\mathbb{Q}$, which contains all observational and experimental numerical data, is a dense subfield not only in the field of real numbers $\mathbb{R}$ but also in the

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fields of $p$-adic numbers $\mathbb{Q}_p$, (ii) there is a sufficiently well developed analysis $[1]$ within and over $\mathbb{Q}_p$ analogous to that one related to $\mathbb{R}$, (iii) local-global (Hasse-Minkowski) principle which states that usually when something is valid on all local fields ($\mathbb{R}$, $\mathbb{Q}_p$) is also valid on the global field ($\mathbb{Q}$), (iv) fundamental physical laws and relevant general mathematical methods should be invariant $[2]$ under an interchange of the number fields $\mathbb{R}$ and $\mathbb{Q}_p$, (v) question "Which aspects of the Universe cannot be described without use of $p$-adic numbers ?", (vi) there is a generic quantum gravity uncertainty $\Delta x$ (see (1)) for possible measurements of distances approaching to the Planck length $\ell_0$, which restricts priority of Archimedean geometry based on real numbers and gives rise to employment of the non-Archimedean one related to $p$-adic numbers, (vii) it seems to be quite reasonable to extend standard Feynman’s path integral over real space to $p$-adic spaces, and (viii) adelic quantum mechanics $[3]$, which is quantum mechanics on an adelic space and contains standard as well as all $p$-adic quantum mechanics, is consistent with all the above assertions.

It is worth to explain in some details the above motivation (vi). Namely, according to various considerations, which take together standard quantum and gravitational principles, it follows a strong generic restriction on experimental investigation of the space-time structure at very short distances due to the relation

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} \text{cm},$$

(1)

where $\Delta x$ is an uncertainty measuring a distance, $\ell_0$ is the Planck length, $\hbar = \frac{\hbar}{2\pi}$ is the reduced Planck constant, $G$ is Newton’s gravitational constant and $c$ is the speed of light in vacuum. The uncertainty (1) means that one cannot measure distances smaller than $\ell_0$ and $\ell_0$ can be regarded as a minimal (fundamental) length. However, this result is derived assuming that whole space-time can be described only by real numbers and Archimedean geometry. But we cannot a priori exclude $p$-adic numbers and their non-Archimedean geometric properties. To get a more complete insight into structure of space-time at the Planck scale it is quite natural to use adelic approach, which treats simultaneously and at an equal footing real (Archimedean) and $p$-adic (non-Archimedean) aspects.

According to (1), approach based only on Archimedean geometry and real numbers predicts its own breakdown at the Planck scale and gives rise to include $p$-adic non-Archimedean sector of possible geometries. Namely, recall that having two segments on straight line of different lengths $a$ and $b$, where $a < b$, one can overpass the longer $b$ by applying the smaller $a$ some $n$-times along $b$. In other words, if $a$ and $b$ are two positive real numbers and $a < b$ then there exists an enough large natural number $n$ such that $na > b$. This is an evident property of the Euclidean spaces (and the field of real numbers), which is known as Archimedean postulate, and can be extended to the standard Riemannian spaces. One of the axioms of the metric spaces is the triangle
inequality which reads:

\[ d(x, y) \leq d(x, z) + d(z, y), \]  

(2)

where \( d(x, y) \) is a distance between points \( x \) and \( y \). However, there is a subclass of metric spaces for which triangle inequality is stronger in such way that:

\[ d(x, y) \leq \max\{d(x, z), d(z, y)\} \leq d(x, z) + d(z, y). \]  

(3)

Metric spaces with strong triangle inequality (3) are called non-Archimedean or ultrametric spaces. Since a measurement means quantitative comparison of a given observable with respect to a fixed value taken as its unit, it follows that a realization of the Archimedean postulate is practically equivalent to the measurements of distances. According to the uncertainty (1), it is not possible to handle distances shorter than \( 10^{-33} \text{cm} \) and consequently one cannot apply only Archimedean geometry beyond the Planck length. Hence, for mathematical modelling of space-time as well as matter (strings and branes) when approaching to the Planck scale it is necessary to employ adeles.

Before a systematic investigation of a possible adelic quantum theory at the Planck scale it is useful to explore various aspects of \( p \)-adic and adelic quantum mechanics. These quantum mechanics are well formulated and so far elaborated at the level which promises their successful extension towards Adelic Superstring/M-theory. This article contains a brief systematic presentation of quantum mechanics on real, \( p \)-adic and adelic spaces.

For a necessary information on usual properties of \( p \)-adic numbers and related analysis one can see [1, 4, 5, 6, 7, 8].

2 Quantum mechanics on a real space

This is ordinary (or standard) quantum mechanics (OQM). It has four main sectors: Hilbert space, Quantization, Evolution and Interpretation. Some of them can be formulated in a few different ways, which are equivalent. Hilbert space of OQM consists of square integrable complex-valued functions of real arguments, which are mainly coordinates of \( D \)-dimensional space and time, and is usually denoted by \( L^2(\mathbb{R}^D) \).

To physical observables correspond linear self-adjoint operators in \( L^2(\mathbb{R}^D) \). Classical dynamical expressions, which depend on canonical variables \( x_i, k_j \) of phase space, become operators by quantization procedure usually initiated by the Heisenberg algebra with commutation relations

\[ [\hat{x}_i, \hat{k}_j] = i \hbar \delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{k}_i, \hat{k}_j] = 0, \]  

(4)

where \( i, j = 1, 2, \ldots, D \). Note that instead of (4) one can use an equivalent quantization based on group relations (\( h = 1 \))

\[ \chi_\infty(-\alpha_i \hat{x}_i) \chi_\infty(-\beta_j \hat{k}_j) = \chi_\infty(\alpha_i \beta_j \delta_{ij}) \chi_\infty(-\beta_j \hat{k}_j) \chi_\infty(-\alpha_i \hat{x}_i), \]  

(5)
\[
\begin{align}
\chi_\infty(-\alpha_i \hat{x}_i) \chi_\infty(-\alpha_j \hat{x}_j) &= \chi_\infty(-\alpha_j \hat{x}_j) \chi_\infty(-\alpha_i \hat{x}_i), \\
\chi_\infty(-\beta_i \hat{k}_i) \chi_\infty(-\beta_j \hat{k}_j) &= \chi_\infty(-\beta_j \hat{k}_j) \chi_\infty(-\beta_i \hat{k}_i),
\end{align}
\]
where \(\chi_\infty(u) = \exp(-2\pi i u)\) is real additive character and \((\alpha_i, \beta_j)\) is a point of classical phase space. Quantization of expressions which contain products of \(x_i\) and \(k_j\) is not unique. According to the Weyl quantization \cite{11} any function \(f(k, x)\), of classical canonical variables \(k\) and \(x\), which has the Fourier transform \(\hat{f}(\alpha, \beta)\) becomes a self-adjoint operator in \(L_2(\mathbb{R}^D)\) in the following way:

\[
\hat{f}(\hat{k}, \hat{x}) = \int \chi_\infty(-\alpha \hat{x} - \beta \hat{k}) \hat{f}(\alpha, \beta) d^D \alpha d^D \beta.
\]

Evolution of the elements \(\Psi(x, t)\) of \(L_2(\mathbb{R}^D)\) is usually given by the Schrödinger equation

\[
i \hbar \frac{\partial}{\partial t} \Psi(x, t) = H(\hat{k}, x) \Psi(x, t),
\]
where \(H\) is a Hamiltonian and \(\hat{k} = -i \hbar \hat{\partial}/\hat{\partial} \hat{x}\). Besides this differential equation there is also the following integral form:

\[
\psi(x'', t'') = \int K_\infty(x'', t''; x', t') \psi(x', t') d^D x',
\]
where \(K_\infty(x'', t''; x', t')\) is the kernel of the unitary representation of the evolution operator \(U_\infty(t'', t')\) and is postulated by Feynman to be a path integral \cite{12}

\[
K_\infty(x'', t''; x', t') = \int_{(x', t')}^{(x'', t'')} \chi_\infty(-S[q]) Dq,
\]
where functional \(S[q] = \int_{t'}^{t''} L(\dot{q}, q, t) dt\) is the action for a path \(q(t)\) in the classical Lagrangian \(L(\dot{q}, q, t)\), and \(x'' = q(t'')\), \(x' = q(t')\) are end points with the notation \(x = (x_1, x_2, \ldots, x_D)\) and \(q = (q_1, q_2, \ldots, q_D)\). The kernel \(K_\infty(x'', t''; x', t')\) is also known as the probability amplitude for a quantum particle to pass from position \(x'\) at time \(t'\) to another point \(x''\) at \(t''\), and is closely related to the quantum-mechanical propagator and Green’s function. The integral in \cite{11} has an intuitive meaning that a quantum-mechanical particle may propagate from \(x'\) to \(x''\) using infinitely many paths which connect these two points and that one has to sum probability amplitudes over all of them. Thus the Feynman path integral means a continual (functional) summation of single transition amplitudes \(\exp \left( \frac{i}{\hbar} S[q] \right)\) over all possible continual paths \(q(t)\) connecting \(x' = q(t')\) and \(x'' = q(t'')\). In Feynman’s formulation, the path integral \cite{11} is the limit of an ordinary multiple integral over \(N\) variables \(q_i = q(t_i)\) when \(N \rightarrow \infty\). Namely, the time interval \(t'' - t'\) is divided into \(N+1\) equal subintervals and integration is performed for every \(q_i \in (-\infty, +\infty)\) at fixed time \(t_i\). We will consider Feynman’s path integral also in next sections and especially in Sec. 4.

Interpretation of OQM is related to the scalar products of complex-valued functions in \(L_2(\mathbb{R}^D)\) and will not be discussed here, but can be found in standard books on quantum mechanics including Ref. \cite{11}. 

[11]
3 Quantum mechanics on $p$-adic and adelic spaces

It is remarkable that quantum mechanics on a real space can be generalized on $p$-adic spaces for any prime number $p$. However, there is not a unique way to perform generalization. As a result, there are two main approaches - with complex-valued and $p$-adic valued elements of the Hilbert space on $\mathbb{Q}_p$. Approach with $p$-adic valued wave functions has been introduced by Vladimirov and Volovich [13] and developed by Khrennikov [14]. P-adic quantum mechanics with complex-valued wave functions is more suitable for connection with ordinary quantum mechanics, and in the sequel, we will refer only to this kind of quantum mechanics on $p$-adic spaces.

Since wave functions are complex-valued, one cannot construct a direct analogue of the Schrödinger equation (9) with a $p$-adic version of Heisenberg algebra [4]. According to the Weyl approach to quantization, canonical non-commutativity in $p$-adic case should be introduced by operators ($\hbar = 1$)

$$\hat{Q}_p(\alpha) \psi_p(x) = \chi_p(-\alpha x) \psi_p(x),$$
$$\hat{K}_p(\beta) \psi_p(x) = \psi_p(x + \beta)$$

(12)

which satisfy

$$\hat{Q}_p(\alpha) \hat{K}_p(\beta) = \chi_p(\alpha \beta) \hat{K}_p(\beta) \hat{Q}_p(\alpha),$$

(13)

where $\chi_p(u) = \exp(2\pi i u)_p$ is additive character on the field $\mathbb{Q}_p$ and $\{u\}_p$ is the fractional part of $u \in \mathbb{Q}_p$.

Let $\hat{x}$ and $\hat{k}$ be some operators of position $x$ and momentum $k$, respectively. Let us define operators $\chi_v(\alpha \hat{x})$ and $\chi_v(\beta \hat{k})$ by formulas

$$\chi_v(\alpha \hat{x}) \chi_v(\alpha x) = \chi_v(\alpha x) \chi_v(\alpha x), \quad \chi_v(\beta \hat{k}) \chi_v(bk) = \chi_v(\beta k) \chi_v(bk),$$

(14)

where index $v$ denotes real ($v = \infty$) and any $p$-adic case, i.e. $v = \infty, 2, \cdots, p, \cdots$ taking into account all non-trivial and inequivalent valuations on $\mathbb{Q}$. It is evident that these operators also act on a function $\chi_v(x)$, which has the Fourier transform $\tilde{\psi}(k)$, in the following way:

$$\chi_v(-\alpha \hat{x}) \chi_v(\alpha x) = \chi_v(-\alpha \hat{x}) \int \chi_v(-k x) \chi_v(\alpha x) \psi_v(k) d^k k = \chi_v(-\alpha x) \psi_v(x),$$

(15)

$$\chi_v(-\beta \hat{k}) \psi_v(x) = \int \chi_v(-\beta k) \chi_v(-k x) \psi_v(k) d^k k = \psi_v(x + \beta),$$

(16)

where integration in $p$-adic case is with respect to the Haar measure $dk$ with the properties: $d(k + a) = dk$, $d(ak) = |a|_p dk$ and $\int_{|k|_p \leq 1} dk = 1$. Comparing (15) with (14) and (16) we conclude that $\hat{Q}_p(\alpha) = \chi_p(-\alpha \hat{x})$, $\hat{K}_p(\beta) = \chi_p(-\beta \hat{k})$. Now group relations (3), (6), (7) can be straightforwardly generalized, including $p$-adic cases, by replacing formally index $\infty$ by $v$. Thus, we have

$$\chi_v(-\alpha_i \hat{x}_i) \chi_v(-\beta_j \hat{k}_j) = \chi_v(\alpha_i \hat{x}_i) \chi_v(-\beta_j \hat{k}_j),$$

(17)

$$\chi_v(-\alpha_i \hat{x}_i) \chi_v(-\alpha_j \hat{x}_j) = \chi_v(-\alpha_i \hat{x}_i) \chi_v(-\alpha_j \hat{x}_j),$$

(18)
\[ \chi_v(-\beta_i \hat{k}_i) \chi_v(-\beta_j \hat{k}_j) = \chi_v(-\beta_j \hat{k}_j) \chi_v(-\beta_i \hat{k}_i). \]  \quad (19) 

One can introduce the unitary operator

\[ W_v(\alpha \hat{x}, \beta \hat{k}) = \chi_v(\frac{1}{2} \alpha \beta) \chi_v(-\beta \hat{k}) \chi_v(-\alpha \hat{x}), \]  \quad (20) 

which satisfies the Weyl relation

\[ W_v(\alpha \hat{x}, \beta \hat{k}) W_v(\alpha' \hat{x}, \beta' \hat{k}) = \chi_v(\frac{1}{2}(\alpha \beta' - \alpha' \beta)) W_v((\alpha + \alpha') \hat{x}, (\beta + \beta') \hat{k}) \]  \quad (21) 

and is a unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements \((z, \eta)\) with the group product

\[ (z, \eta) \cdot (z', \eta') = (z + z', \eta + \eta' + \frac{1}{2} B(z, z')), \]  \quad (22) 

where \( z = (\alpha, \beta) \in \mathbb{Q}_v \times \mathbb{Q}_v \) and \( B(z, z') = \alpha \beta' - \beta \alpha' \) is a skew-symmetric bilinear form on the phase space. Using operator \( W_v(\alpha \hat{x}, \beta \hat{k}) \) one can generalize Weyl formula for quantization (8) and it reads

\[ \hat{f}_v(\hat{k}, \hat{x}) = \int W_v(\alpha \hat{x}, \beta \hat{k}) \tilde{f}_v(\alpha, \beta) \, d^D \alpha d^D \beta. \]  \quad (23) 

It is worth noting that equation (16) suggests to introduce

\[ \{ \beta \hat{k} \}_p^p \psi_p(x) = \int \{ \beta \hat{k} \}_p^p \chi_p(\alpha \hat{x}) \tilde{\psi}_p(k) \, d^D k \]  \quad (24) 

which may be regarded as a new kind of the \( p \)-adic pseudodifferential operator (for a successful Vladimirov pseudodifferential operator, see [1]). Also equation (17) suggests a \( p \)-adic analogue of the Heisenberg algebra in the form \((h = 1)\)

\[ \{ \alpha_i \hat{x}_i \}_p \{ \beta_j \hat{k}_j \}_p - \{ \beta_j \hat{k}_j \}_p \{ \alpha_i \hat{x}_i \}_p = -\frac{i}{2\pi} \delta_{ij} \{ \alpha \beta \}_p. \]  \quad (25) 

As a basic instrument to treat dynamics of a \( p \)-adic quantum model is natural to take the kernel \( \mathcal{K}_p(x'', t''; x', t') \) of the evolution operator \( U_p(t'', t') \). This kernel obtains by generalization of its real analogue starting from (10) and (11), i.e.

\[ \psi_v(x'', t'') = \int \mathcal{K}_v(x'', t''; x', t') \psi_v(x', t') \, d^D x', \]  \quad (26) 

and

\[ \mathcal{K}_v(x'', t''; x', t') = \int_{(x', t')}^{(x'', t'')} \chi_v(-\int_t^{t''} L(\dot{q}, q, t) \, dt) \, Dq. \]  \quad (27) 

According to Vladimirov and Volovich [13, 16, 1], \( p \)-adic quantum mechanics is given by a triple

\[ (L_2(\mathbb{Q}_p), W_p(z), U_p(t)), \]  \quad (28) 

where $W_p(z)$ corresponds to our $W_\alpha (\alpha \hat{x}, \beta \hat{k})$. A similar formulation is done in [17], where evolution operator for one-dimensional systems is presented by a unitary representation of an Abelian subgroup of $SL(2, \mathbb{Q}_p)$ instead of the path integral for the kernel $K_p(x'', t''; x', t')$ (see also [18]).

Adelic quantum mechanics [3] is a natural generalization of the above formulation of ordinary and $p$-adic quantum mechanics. Recall that an adele $x$ [8, 9, 10] is an infinite sequence

$$x = (x_\infty, x_2, \cdots, x_p, \cdots),$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $S$ of primes $p$ one has $x_p \in \mathbb{Z}_p = \{x_p \in \mathbb{Q}_p : |x_p|_p \leq 1\}$. Componentwise addition and multiplication are standard arithmetical operations on the ring of adeles $\mathcal{A}$, which can be defined as

$$\mathcal{A} = \bigcup_S \mathcal{A}(S), \quad \mathcal{A}(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \not\in S} \mathbb{Z}_p.$$  

Rational numbers are naturally embedded in the space of adeles. $\mathcal{A}$ is a locally compact topological space.

There are two kinds of analysis over topological ring of adeles $\mathcal{A}$, which are generalizations of the corresponding analyses over $\mathbb{R}$ and $\mathbb{Q}_p$. The first one is related to the mapping $\mathcal{A} \to \mathcal{A}$ and the other one to $\mathcal{A} \to \mathbb{C}$. In complex-valued adelic analysis it is worth mentioning an additive character

$$\chi(x) = \chi_\infty(x_\infty) \prod_p \chi_p(x_p),$$

a multiplicative character

$$|x|^s = |x_\infty|^s_\infty \prod_p |x_p|^s_p, \quad s \in \mathbb{C},$$

and elementary functions of the form

$$\varphi_S(x) = \varphi_\infty(x_\infty) \prod_{p \in S} \varphi_p(x_p) \prod_{p \not\in S} \Omega(|x_p|_p),$$

where $\varphi_\infty(x_\infty)$ is an infinitely differentiable function on $\mathbb{R}$ and $|x_\infty|^n_\infty \varphi_\infty(x_\infty) \to 0$ as $|x_\infty|_\infty \to \infty$ for any $n \in \{0, 1, 2, \cdots\}$, $\varphi_p(x_p)$ are some locally constant functions with compact support, and

$$\Omega(|x_p|_p) = \begin{cases} 1, & |x_p|_p \leq 1, \\ 0, & |x_p|_p > 1. \end{cases}$$

All finite linear combinations of elementary functions (33) make the set $S(\mathcal{A})$ of the Schwartz-Bruhat adelic functions. The Fourier transform of $\varphi(x) \in S(\mathcal{A})$, which maps $S(\mathcal{A})$ onto $S(\mathcal{A})$, is

$$\hat{\varphi}(y) = \int_{\mathcal{A}} \varphi(x) \chi(xy) dx,$$
where $\chi(xy)$ is defined by (31) and $dx = dx_\infty dx_2 dx_3 \cdots$ is the Haar measure on $\mathcal{A}$.

The Hilbert space $L_2(\mathcal{A})$ contains complex-valued functions of adelic argument with the following scalar product and norm:

$$
(\psi_1, \psi_2) = \int_\mathcal{A} \psi_1^*(x) \psi_2(x) \, dx, \quad ||\psi|| = (\psi, \psi)^{\frac{1}{2}} < \infty.
$$

A basis of $L_2(\mathcal{A})$ may be given by the set of orthonormal eigenfunctions in a spectral problem of the evolution operator $U(t)$, where $t \in \mathcal{A}$. Such eigenfunctions have the form

$$
\psi_{S, \alpha}(x, t) = \psi_n^{(\infty)}(x_\infty, t_\infty) \prod_{p \in S} \psi^{(p)}_{\alpha_p}(x_p, t_p) \prod_{p \notin S} \Omega(|x_p|_p), \quad (36)
$$

where $\psi_n^{(\infty)} \in L_2(\mathbb{R})$ and $\psi^{(p)}_{\alpha_p} \in L_2(\mathbb{Q}_p)$ are eigenfunctions in ordinary and $p$-adic cases, respectively. Indices $n, \alpha_2, \ldots, \alpha_p, \ldots$ are related to the corresponding real and $p$-adic eigenvalues of the same observable in a physical system. $\Omega(|x_p|_p)$ is an element of $L_2(\mathbb{Q}_p)$, defined by (31), which is invariant under transformation of an evolution operator $U_p(t_p)$ and provides convergence of the infinite product (36). For a fixed $S$, states $\psi_{S, \alpha}(x, t)$ in (36) are eigenfunctions of $L_2(\mathcal{A}(S))$, where $\mathcal{A}(S)$ is a subset of adeles $\mathcal{A}$ defined by (31). Elements of $L_2(\mathcal{A})$ may be regarded as superpositions $\psi(x) = \sum_{S, \alpha} C(S, \alpha) \psi_{S, \alpha}(x)$, where $\psi_{S, \alpha}(x) \in L_2(\mathcal{A}(S))$ (36) and $\sum_{S, \alpha} |C(S, \alpha)|_\infty^2 = 1$.

Theory of $p$-adic generalized functions is presented in Ref. [1]. There is not yet a theory of generalized functions on adelic spaces, but there is some progress within adelic quantum mechanics [19] (see also [20]).

Adelic evolution operator $U(t)$ is defined by

$$
U(t'') \psi(x'') = \int_\mathcal{A} \mathcal{K}(x'', t''; x', t') \psi(x', t') \, dx'
$$

$$
= \prod_v \int_{\mathbb{Q}_v} \mathcal{K}_v(x''_v, t''_v; x'_v, t'_v) \psi_v(x'_v, t'_v) \, dx'_v,
$$

where $v = \infty, 2, 3, \ldots, p, \ldots$. The eigenvalue problem for $U(t)$ reads

$$
U(t) \psi_{S, \alpha}(x) = \chi(E, t) \psi_{S, \alpha}(x),
$$

where $\psi_{S, \alpha}(x)$ are adelic eigenfunctions [36], and $E = (E_\infty, E_2, \ldots, E_p, \ldots)$ is the corresponding adelic energy.

Adelic quantum mechanics takes into account ordinary as well as $p$-adic quantum effects and may be regarded as a starting point for construction of a more complete quantum cosmology, quantum field theory and string/M-theory. In the limit of large distances adelic quantum mechanics effectively becomes the ordinary one [21].
4  

$p$-Adic and adelic path integrals, and some simple quantum models

$p$-Adic path integral was initiated in [16] and by subdivision of the time interval was computed for the harmonic oscillator [22] and for a particle in a constant field [23]. Analytic evaluation of path integral for quantum-mechanical systems with general form of quadratic Lagrangians in the same way for real and $p$-adic cases is performed in [24, 25].

Starting from (26) one can easily derive the following three general properties:

\[ \int K_v(x'', t''; x, t)K_v(x, t; x', t')dx = K_v(x'', t''; x', t'), \]  

(40)

\[ \int \bar{K}_v(x'', t''; x', t')K_v(y, t''; x', t')dx' = \delta_v(x'' - y), \]  

(41)

\[ K_v(x'', t''; x', t') = \lim_{v \to v''} K_v(x'', t''; x', t') = \delta_v(x'' - x'). \]  

(42)

Quantum fluctuations lead to deformations of classical trajectory and any quantum history may be presented as $q(t) = x(t) + y(t)$, where $y' = y(t') = 0$ and $y'' = y(t'') = 0$. For Lagrangians $L(\dot{q}, q, t)$ which are quadratic polynomials in $\dot{q}$ and $q$, the corresponding Taylor expansion of $S[q]$ around classical path $x(t)$ is

\[ S[q] = S[x] + \frac{1}{2!} \delta^2 S[x] = S[x] + \frac{1}{2} \int_{t'}^{t''} \left( y \frac{\partial}{\partial \dot{q}} + y \frac{\partial}{\partial q} \right)^2 L(\dot{q}, q, t)dt, \]  

(43)

where we used $\delta S[x] = 0$. Hence we get

\[ K_v(x'', t''; x', t') = \chi_v \left( -\frac{1}{\hbar} \bar{S}(x'', t''; x', t') \right) \]

\times \int_{(y' \to 0, t')}^{(y'' \to 0, t'')} \chi_v \left( -\frac{1}{2\hbar} \int_{t'}^{t''} \left( y \frac{\partial}{\partial \dot{q}} + y \frac{\partial}{\partial q} \right)^2 L(\dot{q}, q, t)dt \right) Dy, \]  

(44)

where $\bar{S}(x'', t''; x', t') = S[x]$.

From (43) follows that $K_v(x'', t''; x', t')$ has the form

\[ K_v(x'', t''; x', t') = N_v(t'', t') \chi_v \left( -\frac{1}{\hbar} \bar{S}(x'', t''; x', t') \right), \]  

(45)

where $N_v(t'', t')$ does not depend on end points $x''$ and $x'$.

To calculate $N_v(t'', t')$ one can use [10] and [11] (see, e.g. [25]). Then one obtains that $v$-adic kernel $K_v(x'', t''; x', t')$ of the unitary evolution operator for one-dimensional systems with quadratic Lagrangians has the form

\[ K_v(x'', t''; x', t') = \lambda_v \left( -\frac{1}{2\hbar} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}(x'', t''; x', t') \right) \]
\[ \prod_v \int_{(x_v', t_v')} \chi_v \left( -\frac{1}{\hbar} \bar{S}(x''', t'''; x', t') \right) Dq_v. \]  

As an adelic Lagrangian one understands an infinite sequence

\[ L_A(\dot{q}, q, t) = (L(\dot{q}_\infty, q_\infty, t_\infty), L(\dot{q}_2, q_2, t_2), L(\dot{q}_3, q_3, t_3), \ldots, L(\dot{q}_p, q_p, t_p), \ldots), \]  

where \(|L(\dot{q}_p, q_p, t_p)|_p \leq 1\) for all primes \(p\) but a finite set \(S\) of them.

When one has a system with more than one dimension and coordinates are uncoupled, then the total \(v\)-adic path integral is product of the corresponding one-dimensional path integrals. Investigation of the coupled case is in progress.

As an illustration of \(p\)-adic and adelic quantum-mechanical models, the following one-dimensional systems with the quadratic Lagrangians were considered: 1) \(L(\dot{q}, q) = \frac{m}{2} \dot{q}^2\), a free particle \([1, 3]\), 2) \(L(\dot{q}, q) = \frac{m}{2} \dot{q}^2 + aq\), a particle in a constant field \([23]\), 3) \(L(\dot{q}, q) = \frac{m}{2} \dot{q}^2 - \frac{m \omega^2}{2} q^2\), a harmonic oscillator \([1, 3]\), 4) \(L(\dot{q}, q) = -mc^2 \sqrt{\bar{q}_\mu \bar{q}^\mu}\), a free relativistic particle \([21]\) and 5) \(L(\dot{q}, q) = \frac{m}{2} \dot{q}^2 - \frac{m \omega^2}{2} q^2\), a harmonic oscillator with time-dependent frequency \([26]\).

Let us mention that when time is real and trajectories are \(p\)-adic, and vice versa, possible functional integrals are considered by Parisi \([27]\). There is another proposal for a path integral formulation of some evolution operators with \(p\)-adic time \([28]\). For an approach to adelic path space integrals with real time, see \([29]\).

## 5 \(p\)-Adic and adelic quantum cosmology

The main task of quantum cosmology is to describe the very early stage in the evolution of the Universe. At this stage, the Universe was in a quantum state, which should be described by a wave function. Usually one takes that this wave function is complex-valued and depends on some real parameters. Since quantum cosmology is related to the Planck scale phenomena it is natural to reconsider its foundations. We maintain here the standard point of view that the wave function takes complex values, but we treat its arguments (space-time coordinates, gravitational and matter fields) to be not only real but also \(p\)-adic and adelic.

There is not \(p\)-adic generalization of the Wheeler - De Witt equation for cosmological models. Instead of differential approach, Feynman’s path inte-
The method was exploited and minisuperspace cosmological models are investigated as models of adelic quantum mechanics.

\( p \)-Adic and adelic minisuperspace quantum cosmology is an application of \( p \)-adic and adelic quantum mechanics to the cosmological models, respectively. In the path integral approach to standard quantum cosmology, the starting point is Feynman’s path integral method, i.e. the amplitude to go from one state with intrinsic metric \( h'_{ij} \) and matter configuration \( \phi' \) on an initial hypersurface \( \Sigma' \) to another state with metric \( h''_{ij} \) and matter configuration \( \phi'' \) on a final hypersurface \( \Sigma'' \) is given by the path integral

\[
\mathcal{K}_\infty(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]) \mathcal{D}(g_{\mu\nu}) \mathcal{D}(\Phi)
\]

over all four-geometries \( g_{\mu\nu} \) and matter configurations \( \Phi \), which interpolate between the initial and final configurations. In (49) \( S_\infty[g_{\mu\nu}, \Phi] \) is an Einstein-Hilbert action for the gravitational and matter fields. This action can be calculated using metric in the standard 3+1 decomposition

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_iN^i)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j,
\]

where \( N \) and \( N_i \) are the lapse and shift functions, respectively. To perform \( p \)-adic and adelic generalization we make first \( p \)-adic counterpart of the action using form-invariance under change of real to the \( p \)-adic number fields. Then we generalize (49) and introduce \( p \)-adic complex-valued cosmological amplitude

\[
\mathcal{K}_p(h''_{ij}, \phi'', \Sigma''; h'_{ij}, \phi', \Sigma') = \int \chi_p(-S_p[g_{\mu\nu}, \Phi]) \mathcal{D}(g_{\mu\nu}) \mathcal{D}(\Phi).
\]

Since the space of all three-metrics and matter field configurations on a three-surface, called superspace, has infinitely many dimensions, in computation one takes an approximation. A useful approximation is to truncate the infinite degrees of freedom to a finite number \( q_\alpha(t) \), \( (\alpha = 1, 2, ..., n) \). In this way, one obtains a minisuperspace model. Usually, one restricts the four-metric to be of the form (50), with \( N^i = 0 \) and \( h_{ij} \) approximated by \( q_\alpha(t) \). For the homogeneous and isotropic cosmologies the metric is a Robertson-Walker one, which spatial sector reads

\[
h_{ij} dx^i dx^j = a^2(t) d\Omega_3^2 = a^2(t) \left[ d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \right],
\]

where \( a(t) \) is a scale factor. If we use also a single scalar field \( \phi \), as a matter content of the model, minisuperspace coordinates are \( a \) and \( \phi \). Models can be also homogeneous and anisotropic.

For the boundary condition \( q_\alpha(t'') = q''_\alpha, \ q_\alpha(t') = q'_\alpha \) in the gauge \( N = 1 \), we have \( v \)-adic minisuperspace propagator

\[
\mathcal{K}_v(q''_\alpha|q'_\alpha) = \int dt \mathcal{K}_v(q''_\alpha, t''; q'_\alpha, t'), \quad t = t'' - t',
\]
where
\[
\mathcal{K}_v(q'_\alpha, t''; q''_\alpha, t') = \int \chi_v(-S_v[q_\alpha]) \mathcal{D}q_\alpha
\]
is an ordinary quantum-mechanical propagator between fixed minisuperspace coordinates \((q'_\alpha, q''_\alpha)\) in fixed times. \(S_v\) is the \(v\)-adic valued action of the minisuperspace model which has the form
\[
S_v[q_\alpha] = \int_{t'}^{t''} dt \left[ \frac{1}{2} f_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta - U(q) \right],
\]
where \(f_{\alpha\beta}\) is a minisuperspace metric \((ds_m^2 = f_{\alpha\beta} dq^\alpha dq^\beta)\) with a signature \((-,+,+,...)\). This metric includes gravitational and matter degrees of freedom.

The standard minisuperspace ground-state wave function in the Hartle-Hawking (no-boundary) proposal [30] is defined by functional integration in the Euclidean version of
\[
\psi_\infty[h_{ij}] = \int \chi_\infty(-S_\infty[g_{\mu\nu}, \Phi]) \mathcal{D}(g_{\mu\nu})_\infty \mathcal{D}(\Phi)_\infty,
\]
over all compact four-geometries \(g_{\mu\nu}\) which induce \(h_{ij}\) at the compact three-manifold. This three-manifold is the only boundary of the all four-manifolds. Extending Hartle-Hawking proposal to the \(p\)-adic minisuperspace [31], an adelic Hartle-Hawking wave function is the infinite product
\[
\psi[h_{ij}] = \prod_v \int \chi_v(-S_v[g_{\mu\nu}, \Phi]) \mathcal{D}(g_{\mu\nu})_v \mathcal{D}(\Phi)_v,
\]
where path integration must be performed over both, Archimedean and non-Archimedean geometries. When evaluation of the corresponding functional integrals for a minisuperspace model yields \(\psi(q_\alpha)\) in the form (36), then we say that such cosmological model is a Hartle-Hawking adelic one. It is shown [32] that the de Sitter minisuperspace model in \(D = 4\) space-time dimensions is the Hartle-Hawking adelic one.

It is shown in [33, 34] that \(p\)-adic and adelic generalization of the minisuperspace cosmological models can be successfully performed in the framework of \(p\)-adic and adelic quantum mechanics [3] without use of the Hartle-Hawking approach. The following cosmological models are investigated: the de Sitter model, model with a homogeneous scalar field, anisotropic Bianchi model with three scale factors and some two-dimensional minisuperspace models. As a result of \(p\)-adic effects and adelic approach, in these models there is some discreteness of minisuperspace and cosmological constant. This kind of discreteness was obtained for the first time in the context of the Hartle-Hawking adelic de Sitter quantum model [32].
6 Towards adelic string theory

A notion of $p$-adic string was introduced in [35], where the hypothesis on the existence of non-Archimedean geometry at the Planck scale was made, and string theory with $p$-adic numbers was initiated. In particular, generalization of the usual Veneziano and Virasoro-Shapiro amplitudes with complex-valued multiplicative characters over various number fields was proposed and $p$-adic valued Veneziano amplitude was constructed by means of $p$-adic interpolation. Very successful $p$-adic analogues of the Veneziano and Virasoro-Shapiro amplitudes were proposed in [36] as the corresponding Gel’fand-Graev [8] beta functions. Using this approach, Freund and Witten obtained [37] an attractive adelic formula

$$A_\infty(a, b) \prod_p A_p(a, b) = 1, \quad (58)$$

which states that the product of the crossing symmetric Veneziano (or Virasoro-Shapiro) amplitude and its all $p$-adic counterparts equals unit (or a definite constant). This gives possibility to consider an ordinary four-point function, which is rather complicate, as an infinite product of its inverse $p$-adic analogues, which have simpler forms. These first papers induced an interest in various aspects of $p$-adic string theory (for a review, see [38, 1]). A recent interest in $p$-adic string theory has been mainly related to the tachyon condensation [39], nonlinear dynamics [40] and an extension of $p$-adic and adelic path integrals to string amplitudes [41].

Like in the ordinary string theory, the starting point in $p$-adic string theory is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following four forms:

$$A_\infty(k_1, \cdots, k_4) \equiv A_\infty(a, b)$$

$$= g^2 \int R |x|^{a-1}_\infty |1 - x|^{b-1}_\infty dx$$

$$= g^2 \left[ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right]$$

$$= g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}$$

$$= g^2 \int D X \exp \left( -\frac{i}{2\pi} \int d^2 \sigma \partial^\mu X_\mu \partial_\alpha X^\alpha \right) \prod_j \int d^2 \sigma_j \exp \left( i k^{(j)}_p X^\mu \right), \quad (62)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - 2$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s + t + u = -8$, i.e. $a + b + c = 1$.

To introduce a $p$-adic Veneziano amplitude one can consider a $p$-adic analogue of any of the above four expressions. $p$-Adic generalization of the first expression was proposed in [36] and it reads

$$A_p(a, b) = g_p^2 \int_{Q_p} |x|^{a-1}_p |1 - x|^{b-1}_p dx, \quad (63)$$
where $| \cdot |_p$ denotes $p$-adic absolute value. In this case only string world-sheet parameter $x$ is treated as $p$-adic variable, and all other quantities maintain their usual (real) valuation. An attractive adelic formula [37] was found. A similar product formula holds also for the Virasoro-Shapiro amplitude. These infinite products are divergent, but they can be successfully regularized. Unfortunately, there is a problem to extend this formula to the higher-point functions.

$p$-Adic analogues of (59) and (60) were also proposed in [35] and [42], respectively. In these cases, world-sheet, string momenta and amplitudes are manifestly $p$-adic. Since string amplitudes are $p$-adic valued functions, it is not so far enough clear their physical interpretation.

Expression (61) is based on Feynman’s path integral method, which is generic for all quantum systems and has successful $p$-adic generalization. $p$-Adic counterpart of (61) is proposed in [41] and has been partially elaborated in [43] and [44]. Note that in this approach, $p$-adic string amplitude is complex-valued, while not only the world-sheet parameters but also target space coordinates and string momenta are $p$-adic variables. Such $p$-adic generalization is a natural extension of the formalism of $p$-adic [16] and adelic [3] quantum mechanics to string theory. This is a promising subject and should be investigated in detail, and applied to the branes and M-theory, which is presently the best candidate for the fundamental physical theory at the Planck scale.

7 Concluding remarks

Among the very interesting and fruitful recent developments have been noncommutative geometry and noncommutative field theory, which may be regarded as a deformation of the ordinary one in which standard field multiplication is replaced by the Moyal (star) product

$$f \ast g(x) = \exp \left[ \frac{i \hbar}{2} \theta^{i j} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right] f(y)g(z) \bigg|_{y=z=x},$$

(64)

where $x = (x^1, x^2, \cdots, x^d)$ is a spatial point, and $\theta^{ij} = -\theta^{ji}$ are noncommutativity parameters. Replacing the ordinary product between noncommutative coordinates by the Moyal product (64) one obtains

$$x^i \ast x^j - x^j \ast x^i = i \hbar \theta^{ij}.$$  

(65)

It is worth noting that one can introduce [44] the Moyal product in $p$-adic quantum mechanics and it reads

$$f \ast g(x) = \int_{\mathbb{Q}_p^D} \int_{\mathbb{Q}_p^D} d^D k d^D k' \chi_p(- (x^i k_i + x^j k'_j) + \frac{1}{2} k_i k'_j \theta^{ij}) \tilde{f}(k) \tilde{g}(k'),$$

(66)

where $D$ denotes spatial dimensionality, and $\tilde{f}(k)$, $\tilde{g}(k')$ denote the Fourier transforms of $f(x)$ and $g(x)$. Some real, $p$-adic and adelic aspects of the noncommutative scalar solitons are investigated in Ref. [45].
An extension of the above formalism with Feynman’s path integral to \( p \)-adic and adelic quantum field theory is considered in [46]. For a review on non-Archimedean Geometry and Physics on Adelic Spaces, see [47]. Some remarks on arithmetic quantum physics are presented in [48].

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References

[1] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, \textit{p-Adic Analysis and Mathematical Physics}, World Scientific, Singapore, 1994.

[2] I.V. Volovich, \textit{Number theory as the ultimate physical theory}, CERN preprint, CERN-TH.4781/87 (July 1987).

[3] B. Dragovich, Adelic Model of Harmonic Oscillator, \textit{Theor. Math. Phys.} \textbf{101} (1994) 1404-1412; Adelic Harmonic Oscillator, \textit{Int. J. Mod. Phys.} \textbf{A10} (1995) 2349-2365.

[4] F.Q. Gouvea, \textit{p-adic Numbers : An introduction}, Universitext, Springer-Verlag, 1993.

[5] W.H. Schikhof, \textit{Ultrametric Calculus : an introduction to p-adic analysis}, Cambridge U.P., Cambridge, 1984.

[6] K. Mahler, \textit{p-adic numbers and their functions}, Cambridge tracts in mathematics \textbf{76}, Cambridge U.P., Cambridge, 1980.

[7] N. Koblitz, \textit{p-adic numbers, p-adic analysis and zeta functions}, London Mathematical Society Lecture Notes Series \textbf{46}, Cambridge U.P., Cambridge, 1980.

[8] I.M. Gel’fand, M.I. Graev and I.I. Piatetskii-Shapiro, \textit{Representation Theory and Automorphic Functions} (in Russian), Nauka, Moscow, 1966.

[9] A. Weil, \textit{Adeles and Algebraic Geometry}, Progress in Mathematics \textbf{23}, Birkhäuser, 1982.

[10] V.P. Platonov and A.S. Rapinchuk, \textit{Algebraic Groups and Number Theory} (in Russian), Nauka, Moscow, 1991.

[11] H. Weyl, \textit{The theory of groups and quantum mechanics}, Dover Publ., 1931.

[12] R.P. Feynman and A.R. Hibbs, \textit{Quantum Mechanics and Path Integrals}, McGraw-Hill Book Company, New York, 1965.
[13] V.S. Vladimirov and I.V. Volovich, *p*-Adic Quantum Mechanics, *Dokl. Acad. Nauk USSR* **302** (1988) 320-323.

[14] A. Khrennikov, *p*-Adic Distributions in Mathematical Physics, Kluwer AP, Dordrecht, 1994.

[15] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer AP, Dordrecht, 1997.

[16] V.S. Vladimirov and I.V. Volovich, *p*-Adic Quantum Mechanics, *Commun. Math. Phys.* **123** (1989) 659-676.

[17] Ph. Ruelle, E. Thiran, D. Verstegen and J. Weyers, Quantum mechanics on *p*-adic fields, *J. Math. Phys.* **30** (1989) 2854-2874.

[18] Y. Meurice, Quantum Mechanics with *p*-Adic Numbers, *Int. J. Mod. Phys.* **A4** (1989) 5133-5147.

[19] B. Dragovich, On Generalized Functions in Adelic Quantum Mechanics, *Integral Transforms and Special Functions* **6** (1998) 197-203.

[20] Ya.M. Radyna and Ya.V. Radyno, Fourier Transform of Distribution on Adeles, in these Proceedings.

[21] G.S. Djordjevi´c, B. Dragovich, Lj. Nešić, *p*-Adic and Adelic Free Relativistic Particle, *Mod. Phys. Lett.* **A14** (1999) 317-325.

[22] E. I. Zelenov, *p*-adic path integrals, *J. Math. Phys.* **32** (1991) 147-152.

[23] G.S. Djordjevi´c and B. Dragovich, On *p*-adic functional integration, *Proc. of the II Mathematical Conference*, Priština, Yugoslavia (1997) 221-228.

[24] G.S. Djordjevi´c and B. Dragovich, *p*-Adic Path Integrals for Quadratic Actions, *Mod. Phys. Lett.* **A12** (1997) 1455-1463.

[25] G.S. Djordjevi´c, B. Dragovich and Lj. Nešić, Adelic Path Integrals for Quadratic Lagrangians, *Infin. Dim. Anal. Quantum Probab. Related Topics*, **6** (2003) 179-195.

[26] G.S. Djordjevi´c and B. Dragovich, *p*-Adic and Adelic Harmonic Oscillator with a Time-dependent Frequency, *Theor. Math. Phys.* **124** (2000) 1059-1067.

[27] G. Parisi, On *p*-Adic Functional Integrals, *Mod. Phys. Lett.* **3** (1988) 639-643.

[28] Y. Meurice, A path integral formulation of *p*-adic quantum mechanics, *Phys. Lett.* **B245** (1990) 99-104.

[29] A.D. Blair, Adelic Path Space Integrals, *Rev. Math. Phys.* **7** (1995) 21-49.
[30] J.B. Hartle and S.W. Hawking, Wave function of the Universe, *Phys. Rev.* 28 (1983) 2960-2075.

[31] I.Ya. Aref’eva, B. Dragovich, P.H. Frampton and I.V. Volovich, The Wave Function of the Universe and p-Adic Gravity, *Int. J. Mod. Phys.* A6 (1991) 4341-4358.

[32] B. Dragovich, Adelic Wave Function of the Universe, *Proc. of the Third A. Friedmann Int. Seminar on Gravitation and Cosmology*, Friedmann Lab. Publishing, St. Petersburg, 1995, pp. 311-321.

[33] B. Dragovich and Lj. Nešić, p-Adic and Adelic Generalization of Quantum Cosmology, *Gravitation and Cosmology* 5 (1999) 222-228.

[34] G.S. Djordjević, B. Dragovich, Lj. Nešić and I.V. Volovich, p-Adic and Adelic Minisuperspace Quantum Cosmology, *Int. J. Mod. Phys.* A17 (2002) 1413-1433.

[35] I.V. Volovich, p-Adic string, *Class. Quantum Grav.* 4 (1987) L83-L87.

[36] P.G.O. Freund and M. Olson, Non-Archimedean strings, *Phys. Lett.* B 199 (1987) 186-190.

[37] P.G.O. Freund and E. Witten, Adelic string amplitudes, *Phys. Lett.* B 199 (1987) 191-194.

[38] L. Brekke and P.G.O. Freund, p-Adic Numbers in Physics, *Phys. Rep.* 233 (1993) 1-63.

[39] D. Ghoshal and A. Sen, Tachyon Condensation and Brane Descent Relations in p-adic String Theory, *Nucl. Phys.* B584 (2000) 300.

[40] V.S. Vladimirov and Ya.I. Volovich, On the Nonlinear Dynamical Equation in the p-adic String Theory, *math-ph/0306018*.

[41] B. Dragovich, On Adelic Strings, *hep-th/0005200*.

[42] I.Ya. Aref’eva, B. Dragovich and I.V. Volovich, On the adelic string amplitudes, *Phys. Lett.* B 209 (1988) 445-450.

[43] B. Dragovich, p-Adic and Adelic Strings, *Proc. Int. Conference dedicated to the memory of Prof. E. Fradkin: Quantization, Gauge Theory and Strings*, Scientific World, Moscow, 2001, pp. 108-114.

[44] B. Dragovich and I.V. Volovich, p-Adic Strings and Noncommutativity, *Proc. Workshop on Noncommutative Structures in Mathematics and Physics*, NATO Science Series: II. Mathematics, Physics and Chemistry - Vol. 22., Kluwer AP, 2001, pp. 391-399.
[45] B. Dragovich and B. Sazdović, Real, $p$-Adic and Adelic Scalar Solitons, *Summer School in Modern Mathematical Physics*, Institute of Physics, Belgrade, SFIN **A3** (2002) 283-296.

[46] B. Dragovich, On $p$-Adic and Adelic Generalization of Quantum Field Theory, *Nucl. Phys. B (Proc. Suppl.*) 102, 103 (2001) 150-155.

[47] B. Dragovich, *Non-Archimedean Geometry and Physics on Adelic Spaces*, math-ph/0306023

[48] V.S. Varadarajan, *Some remarks on arithmetic quantum physics*, in these Proceedings.