Exploring dark energy using the Statefinder

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Abstract. Observations of high redshift supernovae indicate that the universe is accelerating. The hypothesis of ‘Dark energy’ (cosmological constant, scalar field tracker potentials, braneworld models, etc.) has been advanced to explain this phenomenon. Sensitive tests of dark energy which can differentiate between rival models are clearly the need of the hour. The statefinder pair \( \{r, s\} \) is a geometrical diagnostic which can play this role. \( r \) \\& \( s \) depend upon the third time derivative of the scale factor \( \dddot{a} \) and provide the next logical step in the hierarchy of the cosmological parameter set after \( H \) and \( q \). The statefinder pair \( \{r, s\} \) can be determined to high accuracy from a SNAP type experiment and allows us to successfully differentiate between dark energy models having constant as well as time-varying equations of state.

Type Ia supernovae, treated as standard candles, lead to a compelling scenario in which the universe is accelerating, driven by a form of ‘dark energy’ which could have large negative pressure [5, 6]. Many candidates for dark energy have been advanced [8, 9] and several are in good agreement with current observational constraints. Rapid advances in the observational situation and prospects for the deployment of a space-telescope dedicated to supernova research (SNAP) add momentum to our quest for determining the nature of dark energy. Clearly the need of the hour is to combine high-z Sn searches with diagnostic tools which are sensitive to the properties of dark energy and could thereby help in distinguishing between rival models. The Statefinder, recently introduced in [7], promises to do just that. The Statefinder pair \( \{r, s\} \) is a diagnostic of dark energy which is constructed from second and third derivatives of the scale factor. As a result it contains information about both the equation of state \( w \) as well as its time derivative \( \dot{w} \).

Geometrical parameters have traditionally played a key role in cosmology. The first of these is the Hubble parameter \( H_0 = (\ddot{a}/a)_0 \) and the second is the deceleration parameter \( q_0 = -H_0^{-2}(\dddot{a}/a)_0 \). In order to be able to differentiate between different forms of dark energy we add to this hierarchy a third geometrical parameter

\[
    r = \frac{\dddot{a}}{aH^3}. \tag{1}
\]

For models of dark energy with equation of state \( w \) and density parameter \( \Omega_X \), the first Statefinder \( r \) has the form

\[
    r = 1 + \frac{9w}{2} \Omega_X (1 + w) - \frac{3}{2} \Omega_X \frac{\dot{w}}{H}. \tag{2}
\]
Combining $r$ and $q$ leads to the second Statefinder

$$s = \frac{r - 1}{3(q - 1/2)} \equiv 1 + w - \frac{1}{3} \frac{\dot{w}}{wH}. \quad (3)$$

An interesting property of $s$ is that it does not explicitly depend upon the dark energy density $\Omega_X$.

![Figure 1](image-url)

Figure 1: The Statefinder pair $(r, s)$ for different models of dark energy. Quiessence (Q) models have a constant equation of state ($w =$constant $\neq -1$). In them the value of $s$ remains fixed at $s = 1 + w$ while the value of $r$ asymptotically declines to $r(t \gg t_0) \simeq 1 + \frac{2w}{3}(1 + w)$. Two models of Quiessence corresponding to $w_Q = -0.25, -0.5$ are shown. Kinessence (K) is a scalar field rolling down the potential $V(\phi) \propto \phi^{-\alpha}$ with $\alpha = 2, 4$. K models commence their evolution on a tracker trajectory (at $t \ll t_0$) and asymptotically approach $\Lambda$CDM at late times. $\Lambda$CDM ($r = 1, s = 0$) and SCDM without radiation ($r = 1, s = 1$) are fixed points of the system. The hatched region is disallowed in Quiessence models and in the Kinessence model which we consider. The filled circles show the current values of the Statefinder pair $(r_0, s_0)$ for the Q and K models ($\Omega_{0m} = 0.3$). Reproduced from [7].

The Statefinder pair $\{r, s\}$ has several useful properties. For instance $\{r, s\} = \{1, 0\}$ in a universe containing a cosmological constant and non-relativistic matter ($\Lambda$CDM) while $\{r, s\} = \{1, 1\}$ for the standard cold dark matter model containing no radiation.

As illustrated in figure 1, the fixed point $\{r, s\} = \{1, 0\}$ is a late time attractor for a large class of dark energy models including those in which acceleration is caused by the slow roll of a scalar field down a ‘tracker potential’. However the current value $\{r_0, s_0\}$ in tracker models can differ significantly from $\{1, 0\}$ and it is this feature which allows us to successfully differentiate between different tracker models of dark energy. Figure 1 also shows ‘Quiessence’ models for...
which the equation of state remains constant as the universe expands. These models follow vertical trajectories in the plane defined by \( \{r(t), s(t)\} \) and do not approach the \( \Lambda \)CDM fixed point at late times. Figure 1 clearly shows that models with evolving as well as unevolving equations of state can be easily differentiated by the Statefinder statistic.

An important property of the statefinder diagnostic is that it is geometrical in nature since it depends upon the expansion factor and its derivatives. This distinguishes it from physical descriptors of dark energy including \( \Omega_X \), (or \( T_0^0 \) and \( T_\alpha^\alpha \)). Geometrical and physical parameters are related to one another through the field equations of cosmology; a summary is given in table 1.

Table 1: Relationship between geometrical and physical parameters characterizing the observable Universe

| Geometrical parameters | Related physical parameters |
|------------------------|-----------------------------|
| \( H = \dot{a}/a \)    | \( \Omega_{\text{total}}, \Omega_{\text{curvature}} \) |
| \( q = -\ddot{a}/aH^2 \) | \( \Omega_i, w_i \) |
| \( r = \ddot{a}/aH^3 \) | \( \Omega_i, w_i, \dot{w}_i \) |
| \( s = (r - 1)/3(q - 1/2) \) | \( w_i, \dot{w}_i \) |
| \( r_c = \int dt/a \) | \( \Omega_i, w_i \) |

Figure 2 shows confidence levels for the ‘mean statefinders’

\[
\bar{r} = \frac{1}{z_{\text{max}}} \int_0^{z_{\text{max}}} r(z) \, dz, \quad \bar{s} = \frac{1}{z_{\text{max}}} \int_0^{z_{\text{max}}} s(z) \, dz.
\]

computed from a SNAP-type experiment which probes a \( \Lambda \)CDM fiducial model with \( \Omega_{0m} = 0.3, \Omega_{0\Lambda} = 0.7 \) (see [7] for details). The discriminatory power of the Statefinder diagnostic is demonstrated by the fact that inverse power law models of dark energy lie well outside of the three sigma contour centered around \( \Lambda \)CDM.

Finally we would like to emphasize that the Statefinder diagnostic can be applied with advantage to an extremely general category of models including several for which the notion of equation of state is not directly applicable. These include models in which acceleration is caused by a change in the theory describing gravitation (hence space-time expansion) and not due to the presence of a fluid with negative pressure; examples include braneworld models [3, 10, 1] and scalar-tensor theories [2]. Interesting results using Statefinders have recently been obtained for the Chaplygin gas [4].

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Figure 2: 1σ, 2σ, 3σ confidence levels of $\bar{r}$ and $\bar{s}$ computed from 1000 SNAP-type experiments. The fiducial model is $\Lambda$CDM with $\Omega_m = 0.3, \Omega_\Lambda = 0.7$. Filled circles show values of $\bar{r}$ and $\bar{s}$ for the ‘tracker’ potential $V(\phi) \propto \phi^{-\alpha}$ with $\alpha = 1, 2, 3, 4, 5, 6$ (bottom to top). Quiessence models ($w \equiv \text{constant} \neq -1$) lie on the dashed line, filled triangles on this line show $w = -2/3, -1/2, -1/3, 0$ (bottom to top). We note that all inverse power-law models (and most Quiessence models) lie well outside of the three sigma contour centered around the $\Lambda$CDM model and so does the braneworld model [3] for which $\bar{r} = 0.7, \bar{s} = 0.27$. Reproduced from [7].

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