Free Malcev algebra of rank three.

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Abstract

We find a basis of the free Malcev algebra on three free generators over a field of characteristic zero. The specialty and semiprimity of this algebra are proved. In addition, we prove the decomposability of this algebra into subdirect sum of the free Lie algebra rank three and the free algebra of rank three of variety of Malcev algebras generated by a simple seven-dimensional Malcev algebra.

The problem of finding of a basis of a free algebra is important for different varieties. For free Malcev algebras this problem is posed by Shirshov in [1, the problem 1.160]. For alternative algebras with three generators similar problem is solved in [2]. Recall that the Malcev algebra is called special if it is a subalgebra of a commutator algebra $A^-$ for some alternative algebra $A$. The question of the speciality of a Malcev algebras was posed by Malcev in [3]. In this paper we find the basis of the free Malcev algebra with three free generators, and prove the specialty of this algebra. In addition, we prove decomposition of this algebra into subdirect sum of free Lie algebra of rank three and a the free algebra of rank three of the variety of Malcev algebras generated by a simple seven-dimensional Malcev algebra.

Shestakov [4] in 1976 proved that a free Maltsev algebra of $n > 8$ generators over commutative ring $\Phi$ is not semiprime provided $7! \neq 0$ in $\Phi$. Filippov [5] in 1979 then proved that in fact a free Maltsev $\Phi$–algebra of $n > 4$ generators is not semiprime if $6\Phi \neq 0$. We prove that the free Malcev of rank three is semiprime.

For brevity, we omit the brackets in the terms of the following type $((x_1 x_2) x_3) \ldots x_n$. In addition, the products of the form $a x x \ldots x$ we denote as $a x^n$.

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Linear algebra $M$ over a field $F$, which satisfies following identities

$$x^2 = 0,$$

$$J(x, y, xz) = J(x, y, z)x,$$

where $J(x, y, z) = xyz + zxy + yzx$, is called a Malcev algebra. In what follows, the characteristic of $F$ is assumed to be zero. Let $R_a$ be the operator of right multiplication by element $a$ of the algebra $M$ and $R(M)$ is the algebra generated by all $R_a$ where $a$ is the element of the algebra $M$. We will adhere to following notations:

$$L_{a,b} = 1/2(R_aR_b + R_bR_a),$$

$$[R_a, R_b] = R_aR_b - R_bR_a.$$

In addition, by $\Delta^i_a(b)$ we denote the operator defined in [6, Chapter 1, §4], by $Z(M)$ we denote the center of the algebra $M$ and by $G(a, b, c, d)$ - the function defined in [7] with equality

$$G(a, b, c, d) = J(ab, c, d) - bJ(a, c, d) - J(b, c, d)a.$$

Let $X = \{x, y, z\}$ and $M$ be the free Malcev algebra with the set of the free generators $X$. For brevity, the expressions of the form $G(...G(t, x, y, z), x, y, z)...), x, y, z$ will be denoted as $tG^n$.

By $Alt[X]$ denote the free alternative algebra generated by the set of free generators $X$ and by $Ass[X]$ denote the free associative algebra, generated by the set of free generators $X$. Furthermore, for $a, b \in Alt[X]$ we denote $a \circ b = \frac{1}{2}(ba + ab)$ and for $a \in Alt[X]$ denote by $R_a^+$ the operator in the algebra $Alt[X]$, defined as follows: $xR_a^+ = x \circ a$ for any $x \in Alt[X]$. And $L_{a,b}^+$ is the operator in the algebra $Alt[X]$, defined as follows: $xL_{a,b}^+ = R_a^+R_b^+ - R_{a\circ b}^+$ for any $x \in Alt[X]$ and $a, b \in Alt[X]$. If $B$ is the alternative algebra, then by $B^-$ denote the commutator algebra of the algebra $B$.

The main result:

**Theorem.** Let $M$ be the free Malcev algebra with free generators $X = \{x, y, z\}$

Let $U = \{J(x, y, z)G^kL_{x,x}^kL_{y,y}^mL_{z,z}^nL_{x,y}^pL_{y,x}^qL_{y,z}^r | k, l, m, n, p, q, r \in \mathbb{N} \cup \{0\}\}$. Then the set of the vectors $U \cup Ux \cup Uy \cup Uz \cup Uxy \cup Uxz \cup Uyz$ forms the basis of the space $J(M, M, M)$. Besides $M$ is special.

The algebras $M$ and $R(M)$ satisfy the following identities:

$$(ab)(cd) = acbd + dabc + bdac + cbda,$$  \hspace{1cm} (1)
\[3J(wa, b, c) = J(a, b, c)w - J(b, c, w)a - 2J(c, w, a)b + 2J(w, a, b)c, \quad (2)\]
\[G(t, a, b, c) = \frac{2}{3}(J(t, b, c)a + J(a, t, c)b + J(a, b, t)c - J(a, b, c)t), \quad (3)\]
\[G(t, a, b, c) = 2(J(ta, b, c) + J(t, a, bc)), \quad (4)\]
\[J(J(a, b, c), a, b) = 3(ab)J(a, b, c), \quad (5)\]
\[J(c, ba^{2k-1}, b) = J(c, a, ba^{2k-2}b, (k \geq 1), \quad (6)\]
\[uL_{a,b}t = utL_{a,b} + uL_{at,b} - uL_{a,tb}. \quad (7)\]

The identities (1), (2), (3), (4) and (5) are proved in [7], (6) is the identity (5) of [8, §1] and identity (7) is the identity (12) of [8, §1] rewritten in our notation. Moreover, from the identity (3) it is clear that the function \(G\) is a skew-symmetric for any two arguments.

**Lemma 1.** The algebras \(M\) and \(R(M)\) satisfy the following identities

\[(ta)J(a, b, c) = -\frac{1}{2}J(a, t, c)ab - J(b, c, ta)a - \frac{1}{2}J(t, a, b)ac - \frac{3}{2}J(a, t, cb)a \quad (8)\]
\[J(a, b, tac) = -\frac{1}{2}J(a, t, c)[R_a, R_b] + J(a, b, t)L_{a,c}, \quad (9)\]
\[J(a, b, c)L_{b,b}^k a = J(a, b, c)aL_{b,b}^k, \quad (10)\]
\[J(a, b, c)L_{a,a}^k L_{b,b}^l = J(a, b, c)L_{a,a}^k L_{b,b}^l \quad (11)\]

**Proof.** Applying the operator \(\Delta^1(h)\) to the identity \(bJ(a, b, c) = J(a, b, cb)\) obtain

\[hJ(a, b, c) = J(a, h, c)b + J(a, h, cb) + J(a, b, ch). \]

From the identity (2):

\[hJ(a, b, c) = \frac{1}{3}J(a, h, c)b + J(a, h, cb) + \frac{1}{3}J(a, b, h)c - \frac{2}{3}J(b, c, h)a + \frac{1}{3}hJ(a, b, c). \]

That is

\[hJ(a, b, c) = \frac{1}{2}J(a, h, c)b - J(b, c, t)a + \frac{1}{2}J(a, b, h)c + \frac{3}{2}J(a, h, cb). \]

Replacing now \(h\) by \(ta\) obtain the identity (8).
Twice applying to the first assertion of Lemma to the identity \((5)\), obtain identity
\[
(ta)J(a, b, c) = -\frac{1}{2}J(a, t, c)ab - \frac{1}{6}J(a, b, t)ca - \frac{1}{6}J(a, t, c)ba - \frac{2}{3}J(t, b, c)aa + \frac{2}{3}J(a, b, c)ta - \frac{1}{2}J(a, b, t)ac.
\]
From the identity \((2)\) obtain
\[
J(a, b, tac) = \frac{2}{3}J(ta, c, a)b - \frac{2}{3}J(b, ta, c)a - \frac{1}{3}J(a, b, ta)c + \frac{1}{3}J(c, a, b)(ta) = \]
\[
= -\frac{2}{3}J(t, c, a)ab + \frac{4}{9}J(a, b, t)ca + \frac{4}{9}J(a, t, c)ba - \frac{2}{9}J(t, b, c)aa + \frac{2}{9}J(a, b, c)ta + \frac{1}{3}J(a, b, t)ac - \frac{1}{3}(ta)J(a, b, c)
\]
Applying now the previous identity, we obtain \((9)\). The identity \((10)\) shall prove by induction on \(k\). From the identity \((7)\), replacing \(a\) by \(b, t\) by \(a\) and \(u\) by \(J(a, b, c)\), we have \(J(a, b, c)L_{b,b}a = J(a, b, c)aL_{b,b} - 2J(a, b, c)L_{b,ab}\). From the identity \((3)\): \(J(J(a, b, c), b, ab) = -3J(a, b, c)(ab)b\), \(J(J(a, b, c), b, ab) = -J(J(a, b, bc), a, b) = 3J(a, b, bc)(ab) = 3J(a, b, c)b(ab)\). That is, \(-3J(a, b, c)(ab)b = 3J(a, b, c)b(ab)\). Thus, \(J(a, b, c)\) is obtained "by induction" on \(k\) and \(J(a, b, c)\) is obtained "by induction" on \(k\).

The identity \((11)\). We apply \((6)\) and identity \((10)\).
\[
J(a, b, c)L_{a,a}L_{b,b}^k = -J(ab^{2k+1}, b, cb^{2l-1}) = -J(a, b, cb^{2l-1})L_{a,a}^k b = \]
\[
= -J(a, b, cb^{2l-1})bL_{a,a}^k = J(a, b, c)L_{b,b}^k L_{a,a}^k.
\]

**Lemma 2.** For an arbitrary polynomial \(f\) of degree \(n\) from the subalgebra generated by the elements \(a\) and \(b\) of Malcev algebra \(M\), the following equalities are true:
1) \(J(a, b, J(a, f, c)) + (-1)^nJ(a, f, J(a, b, c)) = 0\),
2) \(J(a, b, J(f, b, c)) + (-1)^nJ(f, b, J(a, b, c)) = 0\),
3) \( J(a, b, fc) = (-1)^n fJ(a, b, c) \).

**Proof.** We shall prove the conjunction of all three statements by induction on \( n \).

For \( n = 1 \) all identities are obvious. Suppose, they are correct when \( n = k \). Prove them for \( n = k + 1 \). Since \( M \) is binary Lie algebra, then we can assume that \( f = f_1 a \) or \( f = f_1 b \).

1) If \( f = f_1 a \), the proof is obvious. Let \( f = f_1 b \). Applying operator \( \Delta_b^1(f_1) \) to the identity \( J(a, cb, c) = J(a, b, c)c \) obtain

\[
J(a, f_1 b, c) = J(a, f_1, cb) + J(a, b, c)f_1
\]

Next, using this equation and the induction hypothesis, obtain

\[
J(a, b, J(a, f_1 b, c)) + (-1)^{k+1} J(a, f_1 b, J(a, b, c)) = J(a, b, J(a, f_1, cb)) + J(a, b, J(a, b, c)f_1) +
\]

\[
+ (-1)^{k+1} J(a, f_1, J(a, b, c)b) + (-1)^{k+1} J(a, b, J(a, b, c))f_1 = 0
\]

2) The proof is similar to 1).

3) Let \( f = f_1 a \).

\[
J(a, b, f_1 ac) = J(a, b, J(f_1, a, c) - cf_1 a - acf_1) = -J(a, b, J(a, f_1, c)) +
\]

\[
+ (-1)^{k+1}(f_1 J(a, b, c)a + J(a, b, c)af_1) = -J(a, b, J(a, f_1, c)) + (-1)^{k+1}(J(J(a, b, c), a, f_1) -
\]

\[
- af_1 J(a, b, c)) = (-1)^{k+1} f_1 a J(a, b, c)
\]

If \( f = f_1 b \), then the arguments are similar and used the equality 2).

**Corollary 1.** Under the conditions of Lemma 2 the equalities are true:

\[
J(a, f, cb) - J(b, f, ca) = \frac{3(-1)^n + 1}{2} fJ(a, b, c)
\]

(12)

\[
J(a, fb) = -J(a, f, c)b + (-1)^n fJ(a, b, c)
\]

(13)

**Proof.** From Lemma 2: \( J(a, b, fc) = (-1)^n fJ(a, b, c) \). From the identity \( [2] \):

\[
J(a, b, fc) = \frac{2}{3} J(f, c, a)b - \frac{2}{3} J(b, f, c)a + \frac{1}{3} J(a, b, f)c + \frac{1}{3} J(c, a, b) f.
\]

Combining, we obtain the first equality. Second equality follows from the first after the application of identity \( [2] \) to \( J(a, fb, c) \).

**Proposition 1.** Let \( u \in J(M, M, M) \). There are \( \alpha_i \) from \( F \) for which

\[
u = \sum \alpha_i J(x, y, z)x_{i1}x_{i2}...x_{ik_i} x_{i, j} \in X, k_i \in \mathbb{N} \cup \{0\}.
\]
**Proof.** In algebra $M$ the following identity is fulfilled:

$$J(a, b, c)abv = J(a, b, v)cba + J(a, b, c)vba + J(a, b, v)bca +$$

$$+ J(a, b, c)bva - J(a, b, v)abc - J(a, b, c)avb - J(a, b, v)acb \quad (a)$$

Applying the operator $\Delta_1^k(c)$ to identity (10) for $k = 1$ obtain: $J(a, b, c)cba + J(a, b, c)abc + J(a, b, c)acv$. After the application of the operator $\Delta_1^k(v)$ we obtain the desired identity.

The proposition we prove by induction of the degree $d$ of the element $u$. The corollary 2 [6, chapter 1, §3] imply the homogeneity of any variety of Malcev algebras over a field $F$. Therefore, the proposition is valid for $d = 3$. Let the proposition be true for all $d \leq k$. From the induction hypothesis it follows that $u$ can be written in the form

$$u = \sum_{i} \alpha_i J(x, y, z)x_{i1}x_{i2} \ldots x_{ik}v_i,$$

where $x_{i,j} \in X, k_i \in \mathbb{N} \cup \{0\}, v_i \in M$. It is obvious that the elements $\overline{v_i} = v_i + J(M, M, M)$ of Lie algebra $M/J(M, M, M)$ are linear combinations of the monomials of the form $\overline{y_{i1} y_{i2} \ldots y_{il}}$, where $y_{i,j} \in X$ and $l \in \mathbb{N}$. Hence $v_i$ can be written as $v_i = y_{i1}y_{i2} \ldots y_{il} + u_i$, where $u_i \in J(M, M, M)$. From the homogeneity, it follows that the degrees of elements $u_i$ does not exceed $k$, therefore, the induction hypothesis implies that $v_i$ can be represented as a linear combination of the monomials of the form $t_{i1}t_{i2} \ldots t_{is}$, where $t_{i,j} \in X$ and $s \leq k$. Thus, to prove the proposition it is sufficient show that the expression of the form $J(x, y, z)x_{i1}x_{i2} \ldots x_{ir}(wt)$, where $x_j, t \in X$ and $w \in M$ belong to the subspace generated by the set $J(M, M, M)X$. Prove this by induction induction on $r$. From the identity (8) it follows that this is true for $r = 0$. Let $r = 1$. There are two cases.

1. $x_1 = t$. Without loss of generality we can assume that $x_1 = t = x$. We have

$$J(x, y, z)x(wx) = J(J(x, y, z), x, wx) - (wx)J(x, y, z)x + (wx)J(x, y, z) =$$

$$= -J(J(x, y, z), x, w)x - (wx)J(x, y, z)x + (wx)J(x, y, z).$$

From the identity (a) follows that the third term, and therefore all expression belongs $J(M, M, M)X$.

2. $x_1 \neq t$. Without loss of generality, we may assume that $x_1 = x, t = y$. Applying the identity (11), obtain

$$J(x, y, z)x(wy) = J(x, y, z)wxy + yJ(x, y, z)wx + xyJ(x, y, z)w + wxyJ(x, y, z) =$$

$$= J(x, y, z)wxy + yJ(x, y, z)wx + xyJ(x, y, z)w + wxyJ(x, y, z).$$
\[ J(x, y, z)wx + yJ(x, y, z)w + wxyJ(x, y, z) + \frac{1}{2}J(x, y, z)xyw - \frac{1}{2}J(x, y, z)yxw. \]

The identities (8) and (a) imply the required result.

Let \( r \geq 1 \) and \( J(x, y, z)w_{x_1...x_{r-1}} \in J(M, M, M)X \). The identity (1) gives

\[ J(x, y, z)x_{1...x_{r-1}}w = -wtJ(x, y, z)x_{r-1} - x_r(wt)(J(x, y, z)x_{1...x_{r-2}}) - x_r(wt)(J(x, y, z)x_{1...x_{r-2}})x_{r-1} - J(x, y, z)x_{1...x_{r-2}}x_r(wt)(J(x, y, z)x_{1...x_{r-2}}) - J(x, y, z)x_{1...x_{r-2}}x_r(wt). \]

\[ \square \]

**Corollary 1.** Let \( f, g, h \) be the polynomials from the subalgebra, generated by elements \( x \) and \( y \) of the Malcev algebra and \( \deg f = n \). Then

1) \( J(yx, f, z) = (-1)^{n-1}fJ(x, y, z) \),

2) \( J(x, f, yz) = J(x, f, z)y + ((-1)^n + 1)fJ(x, y, z) \),

3) if \( f \) has even degree, then \( J(z, g, h)L_{f,y} = 0 \).

**Proof.** 1) Applying the operator \( \Delta^1_y(f) \) to the identity \( J(yx, y, z) = J(x, y, z)y \) we obtain

\[ J(yx, f, z) = J(y, f, z) + J(x, f, z)y - fJ(x, y, z) \]

From the identity (13):

\[ J(yx, f, z) = -J(y, f, z)x - (-1)^n fJ(x, y, z) + J(x, f, z)y - fJ(x, y, z), \]

whence follows 1).

2) Applying the operator \( \Delta^1_z(f) \) to identity \( J(x, zy, z) = J(x, y, z)z \) we obtain

\[ J(x, fy, z) + J(x, zy, f) = J(x, y, z)f. \]

Using now \( (12) \) and \( (13) \), we obtain the required result.

3) We first show that \( J(x, y, z)L_{f,y} = 0 \). Combining \( (12) \) and \( (13) \) obtain

\[ J(x, f, z)y = J(f, x, y, z) + \frac{(-1)^n + 1}{2}fJ(x, y, z) \]

Applying this equality and equity 2) of this corollary, we obtain

\[ J(x, y, z)fy = -J(x, f, z)yy + J(f, x, y, z)y = -J(x, f, yz)y - 2fJ(x, y, z)y + J(f, x, y, z) \]
That is
\[ J(x, y, z)fy = J(x, f, yz)y - J(fx, y, yz). \]

From the identities (12) and (13):
\[ J(x, y, z)fy = J(y, f, yz)x + 2fJ(x, y, yz) + J(y, fx, yz) = -fJ(x, y, yz) + 2fJ(x, y, yz) = -J(x, y, z)yx. \]

Then, from Proposition 1 it follows that \( J(z, g, h) \) can be represented as
\[ \sum_i \alpha_i J(z, x, y)x_{i1}x_{i2}...x_{ik_i}, \]
where \( k_i \in \mathbb{N} \cup \{0\}, \alpha_i \in F \). From the homogeneity follows that \( x_{ij} \in \{x, y\} \). Thus, it is clear that
\[ J(z, g, h) = J(\pm \sum_i \alpha_i zx_{i1}x_{i2}...x_{ik_i}, x, y). \]

This implies the required assertion \( \blacksquare \).

Corollary 2. If \( f \) and \( h \) are the polynomials of even degree from the subalgebra, generated by elements \( x \) and \( y \) of the Malcev algebra \( M \), then \( fh \) belongs to \( Z(M) \).

Proof. The induction on the degree \( k \) polynomials \( h \). When \( k = 2 \) the assertion obviously follows from the equality 1) of Corollary 1. Now suppose that statement is true in the case when the degree polynomial is equal to \( k = 2l \). Prove it for a polynomial \( h_1 \) of degree \( k = 2l + 2 \).

1) Let \( h_1 = hxy \). We rewrite the identity (9) as
\[ J(z, x, txy) = \frac{1}{2}J(x, t, y)yx - \frac{1}{2}J(x, t, y)xy - J(x, z, t)L_{x,y} \]

Apply the operator \( \Delta_x^1(\mathbf{f}) \) to this equation and replace \( t \) by \( h \)
\[ J(z, f, hxy) + J(z, x, hfy) = \frac{1}{2}J(f, h, y)yx + \frac{1}{2}J(x, h, y)fz - \frac{1}{2}J(x, h, y)z - \frac{1}{2}J(x, y, h)Jx, y - J(x, z, h)L_{f,y} \]

That is \( J(z, f, hxy) = J(z, x, h)L_{f,x} = 0 \).

1) Let \( h_1 = hxh \). Apply the operator \( \Delta_x^1(f) \) to the equation \( J(z, x, txx) = J(z, x, t)xx \)
and replace \( t \) by \( h \)
\[ J(z, f, hxh) + J(z, x, hfh) + J(z, x, hxh) = J(z, f, hxx) + J(z, x, h)fx + J(z, x, h)xf. \]
Hence, $J(z, f, hxx) = J(z, x, h)L_{f,x} = 0$.

The proofs of the cases $h_1 = hyy$ and $h_1 = hyx$ are similar. ■

**Lemma 3.** In algebras $M$ and $R(M)$ the following identity is fulfilled:

$$J(ac^{2k+1}(RcRa)^{n-1}, b, c) = J(a, b, c)L^k_{a,a}L^n_{a,c}, n \geq 1, k \geq 0.$$  \hspace{1cm} (14)

**Proof.** We write the identity (6) of \[8, \S 1\]

$$J(ac^{2k+2}, b, c) = J(a, b, c)a^{2k}(ca) - J(a, b, c)a^{2k+1}c.$$  

Applying (8) we obtain

$$J(ac^{2k+1}, b, c) = -\frac{1}{2}J(a, ba^{2k}, c)ac + \frac{1}{2}J(a, ba^{2k}, c)ca + J(a, b, c)a^{2k+1}c.$$

That is $J(ac^{2k+1}, b, c) = J(a, b, c)L^k_{a,a}L^n_{a,c}$. Using the identities (3) and (9):

$$J(a, b, c)L^k_{a,a}L^n_{a,c} = J(ac^{2k+1}, b, c) = J(ac^{2k+1}(RcRa)^{n-1}, b, c).$$  ■

We denote $L_{zy,zy} + L_{y,y}L_{z,z} - L^2_{y,z}$ by $d(z, y)$.

**Proposition 2.** Let $T = \{L_{x,x}, L_{y,y}, L_{z,z}, L_{x,y}, L_{y,z}, L_{x,z}\}$. For all $S_i, T_i$ from $T$ and for any $n$ from $\mathbb{N} \cup \{0\}$ it is fulfilled:

1) $J(x, y, z)T_1T_2...T_n[S_1, S_2] = 0$,
2) $J(x, y, z)T_1T_2...T_nL_{y,z} = 0$,
3) $J(x, y, z)T_1T_2...T_nL_{d(z, y)} = 0$.

**Proof.** We first prove the same equalities in the assumption that $S_i$ and $T_i$ belong to the set $T_0 = T\{L_{x,y}\}$. For this we prove the identity

$$J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,x}, L_{y,y}] = 0.$$  \hspace{1cm} (a)

1) $k = l = m = 0$. Follows from the identity (11).
2) $k = l = 0, m \neq 0$. Apply (8), (10) and (11).

$$J(x, y, z)L^m_{z,z}L_{x,x}L_{y,y} = J(x, y, xz^{2m+1})xL_{y,y} = J(x, xy^3, xz^{2m+1}) = J(x, y^3z, yz^{2m+1}) = J(x, y, z)L^m_{z,z}L_{y,y}L_{x,x}.$$

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3) \( k = m = 0, \ l \neq 0 \). It is obvious from (11).
4) \( k = 0, \ l \neq 0, \ m \neq 0 \). From the identity (11):
\[
J(x, y, z) L_{x,y}^k L_{y,z}^m [L_{x,z}, L_{y,y}] = -J(x, y, y z^{2m+1} y^{2l-1}) [L_{x,x}, L_{y,y}] = 0
\]
5) \( k \neq 0, \ l = m = 0 \). Follows from the identity (11).
6) \( k \neq 0, \ l = 0, \ m \neq 0 \). It is similar to the case 4).
7) \( k \neq 0, \ l \neq 0, \ m = 0 \). Follows from (11).
8) \( k \neq 0, \ l \neq 0, \ m \neq 0 \).
\[
J(x, y, z) L_{x,x}^k L_{y,y}^l L_{z,z}^m [L_{x,x}, L_{y,y}] = J(x, y, z) L_{x,x}^k L_{y,y}^l L_{z,z}^m L_{x,x} L_{y,y} - \\
-J(x, y, z) L_{x,x}^l L_{y,y}^l L_{z,z}^m L_{y,y} L_{x,x} = -J(x, x y^{2l+1} y^{2k-1}, z) L_{z,z}^m L_{x,x} L_{y,y} + \\
+J(y x^{2k+1} y^{2l-1}, y, z) L_{z,z}^m L_{y,y} L_{x,x} = J(x, x y^{2l+1}, x z^{2m+1} y^{2k}) L_{y,y} - \\
-J(y x^{2k+1} y^{2l-1}, y, y z^{2m+1}) L_{x,x} = ( -J(x, x y^{2l+3}, z) L_{x,x} + J(y x^{2k+1} y^{2l-1}, y, z) L_{y,y} ) L_{z,z}^m = 0.
\]
Prove now that
\[
J(x, y, z) L_{x,x}^k L_{y,y}^l L_{z,z}^m [L_{x,y}, L_{x,x}] = 0 \quad (b)
\]
1) \( k = l = m = 0 \). Follows from the identity (9):
\[
J(x, y, z) L_{x,y} L_{x,x} = J(x, y x y, z) L_{x,x} = J(x, y x y, z x^2) = J(x, y, z) L_{x,x} L_{x,y}
\]
2) \( m = 0 \).
\[
J(x, y, z) L_{x,x}^k L_{y,y}^l [L_{x,y}, L_{x,x}] = J(x, y, z x^{2k} y^{2l}) [L_{x,y}, L_{x,x}] = 0
\]
3) \( m \neq 0, \ k = l = 0 \). From the identity (a):
\[
J(x, y, z) L_{z,z}^m [L_{x,x}, L_{y,y}] = 0.
\]
Using the operator \( \Delta_x^l(y) \) obtain
\[
J(x, y, z) L_{z,z}^m [L_{x,y}, L_{x,x}] = 0.
\]
4) \( m \neq 0 \) and, for example \( k \neq 0 \). From the identity (a):
\[
J(x, y, z) L_{x,x}^k L_{y,y}^l L_{z,z}^m [L_{x,y}, L_{x,x}] = -J(x, y, z x^{2m+1} y^{2k-1} y^{2l}) [L_{x,y}, L_{x,x}] = 0.
\]
Prove now that
\[
J(x, y, z) L_{x,x}^k L_{y,y}^l L_{z,z}^m [L_{x,y}, L_{z,z}] = 0 \quad (c)
\]
1) $k = l = m = 0$. From the identity (b) follows $J(x, y, z)[L_{x,z}, L_{z,z}] = 0$. Applying the operator $\Delta^1_z(y)$ obtain

\[ J(x, y, z)[L_{x,y}, L_{z,z}] + 2J(x, y, z)[L_{x,z}, L_{y,z}] = 0 \]

\[ J(x, y, z)[L_{x,z}, L_{y,z}] = J(x, y, z)L_{x,y}L_{z,z} - J(x, y, z)L_{y,z}L_{x,z} = J(xz, y, z)L_{y,z} - J(x, yz, z)L_{x,z} = J(xxz, y, z)L_{y,z} - J(x, yzy, z)L_{x,z} = 0. \]

Hence, $J(x, y, z)[L_{x,y}, L_{z,z}] = 0$.

2) $k = l = 0$, $m \neq 0$. From the identity (a) follows $J(x, y, z)L^m_{z,z}[L_{x,x}, L_{z,z}] = 0$. Applying the operator $\Delta^1_z(y)$ obtain $J(x, y, z)L^m_{z,z}[L_{x,y}, L_{z,z}] = 0$.

3) $k = 0$, $l \neq 0$, $m = 0$. From the identity (a) follows $J(x, y, z)L^l_{y,y}[L_{x,x}, L_{z,z}] = 0$. Applying the operator $\Delta^1_z(y)$ we have $J(x, y, z)L^l_{y,y}[L_{x,y}, L_{z,z}] = 0$.

4) $k = 0$, $l \neq 0$, $m \neq 0$. From the identity (a) follows $J(x, y, z)L^l_{y,y}L^m_{z,z}[L_{x,x}, L_{z,z}] = 0$. Applying the operator $\Delta^1_z(y)$ obtain $J(x, y, z)L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] = 0$.

5) $k \neq 0$, $l = m = 0$. From the identity (a) follows $J(x, y, z)L^k_{x,x}L^m_{z,z}[L_{x,y}, L_{z,z}] = 0$. Applying the operator $\Delta^1_z(x)$ we have $J(x, y, z)L^k_{x,x}L^m_{z,z}[L_{x,y}, L_{z,z}] = 0$.

6) $k \neq 0$, $l = 0$, $m \neq 0$. From the identity (a) follows $J(x, y, z)L^k_{x,x}L^m_{z,z}[L_{y,y}, L_{z,z}] = 0$. Applying the operator $\Delta^1_z(x)$ we have $J(x, y, z)L^k_{x,x}L^m_{z,z}[L_{y,y}, L_{z,z}] = 0$.

7) $k \neq 0$, $l \neq 0$, $m = 0$. Using the identities (b), (14) and (a) obtain

\[ J(x, y, z)L^k_{x,x}L^l_{y,y}[L_{x,y}, L_{z,z}] = J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} - J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} - J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} = J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} - J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} = J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} = J(x, y, z)L^k_{x,x}L^l_{y,y}L_{x,y}L_{z,z} = 0. \]

8) $k \neq 0$, $l \neq 0$, $m \neq 0$. Using (a) obtain

\[ J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] = J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] - J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] = J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] - J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] = J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] - J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}[L_{x,y}, L_{z,z}] = 0. \]

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We prove now the identity
\[ J(x, y, z) L^{k}_{x,x} L^{l}_{y,y} L^{m}_{z,z} [L_{x,y}, L_{x,z}] = 0 \]

1) \( m = 0 \). From (c) using the operator \( \Delta_{x}^{1} (x) \).

2) \( l = 0 \). From the identity (c):
\[ J(x, y, z) L^{k}_{x,x} L^{m}_{z,z} [L_{y,y}, L_{x,z}] = 0. \]

Applying the operator \( \Delta_{x}^{1} (y) \), obtain the required result.

3) \( k = 0 \). The proof is similar to the cases 1) and 2).

4) \( k \neq 0 \), \( l \neq 0 \), \( m \neq 0 \). Using the identities (b), (14) and (a):
\[ J(x, y, z) L^{k}_{x,x} L^{l}_{y,y} L^{m}_{z,z} [S_{1}, S_{2}] = 0 \]
where \( S_{i} \in T_{0} \). Hence, with operator \( \Delta_{x}^{1} (y) \) it is easy to prove the identity
\[ J(x, y, z) L^{n}_{x,y} L^{k}_{x,x} L^{l}_{y,y} L^{m}_{z,z} [S_{1}, S_{2}] = 0, \]
using the induction on \( n \). The induction on \( p \) and operator \( \Delta_{x}^{1} (y) \) give the identity
\[ J(x, y, z) L^{n}_{x,y} L^{p}_{y,z} L^{k}_{x,x} L^{l}_{y,y} L^{m}_{z,z} [S_{1}, S_{2}] = 0, \]
. Finally, the induction on \( q \) and the operator \( \Delta_{x}^{1} (z) \) give the identity
\[ J(x, y, z) L^{n}_{x,y} L^{p}_{y,z} L^{q}_{z,z} L^{k}_{x,x} L^{l}_{y,y} L^{m}_{z,z} [S_{1}, S_{2}] = 0. \]
This is the equality 1) assuming that the \( S_{i} \) and \( T_{i} \) belong to the set \( T_{0} = T \setminus \{ L_{x,y} \} \).
We now prove the equality of 2) assuming that \( S_{i} \) and \( T_{i} \) belong to the set \( T_{0} = T \setminus \{ L_{x,z} \} \). For this we prove
\[ J(x, y, z) L^{n}_{x,y} L^{p}_{y,z} L^{q}_{z,z} L^{k}_{x,x} L^{l}_{y,y} L^{m}_{z,z} L_{y,z,y} = 0 \]
1) \( n = p = k = l = m = 0 \), Obviously follows from the equality 3) of Corollary 1 and from Proposition 1.

2) \( n = p = l = m = 0, \ k \neq 0 \). Applying the identity (6) and proved in this Proposition identities we obtain

\[
J(x, y, z)L^{k}_{x,x}yL_{y,z} = J(yx^{2k+1}, y, z)L_{y,z} = J(yx^{2k+1}, y, zy) = J(x, y, zy)L^{k}_{x,x}y =
\]

\[= J(x, y, z)L_{y,z}L^{k}_{x,x}y = J(x, y, z)L^{k}_{x,x}L_{y,y}y.\]

That is, \( J(x, y, z)L^{k}_{x,x}L_{y,y}y - J(x, y, z)L^{k}_{x,x}yL_{y,z} = 0 \). From the identity (7):

\[
J(x, y, z)L^{k}_{x,x}L_{y,y}y = J(x, y, z)L^{k}_{x,x}L_{y,y,z} = 0.
\]

3) \( n = l = m = 0, \ k \neq 0, \ p \neq 0 \). Using the identity (7) obtain

\[
J(x, y, z)L^{p}_{y,z}L^{k}_{x,x}L_{y,z} = J(x, y, z)L^{p}_{y,z}L^{k}_{x,x}L_{y,z} = -\frac{1}{2}J(yx^{2k+1}L^{p-1}y, y, z)L_{y,z} +
\]

\[+ \frac{1}{2}J(zx^{2k+1}L^{p-1}y, y, z)L_{y,z} = 0.\]

In other cases, using the identities (6) and (14) it is easy to show that

\[
J(x, y, z)L^{p}_{y,z}L^{k}_{x,x}L^{m}_{y,z}L_{y,z} \in J(M, y, z)L_{y,z} = 0.
\]

Applying the operator \( \Delta_{z}^{1}(z) \) to the resulting identity and using the induction on \( q \) is easy to prove the identity

\[
J(x, y, z)L^{p}_{x,y}L^{q}_{y,z}L^{m}_{z,y}L_{z,z}L_{y,z}L_{y,z} = 0.
\]

We now prove equality 3) assuming that the \( S_{i} \) and \( T_{i} \) belong to the set \( T_{0} = T\{L_{x,y}\} \).

1) \( n = p = k = l = m = 0 \). From the identity (8) and the operator \( L_{y,z} \):

\[
J(x, y, z)d(z, y) = J(x, y, z)L_{y,z} + J(x, y, z)L_{y,z} - J(x, y, z)L_{y,z} =
\]

\[= J(x, y, z)(zy)(zy) + J(x, y, z)L_{y,z} - J(x, y, z)L_{y,z} = J(\frac{1}{2}xyz - \frac{1}{2}xzy, y, z)(zy) +
\]

\[+ J(x, y, z)L_{y,z} - J(x, y, z)L_{y,z} = J(\frac{1}{4}xyz - \frac{1}{4}xzy, y, z)(zy) +
\]

\[+ J(x, y, z)L_{y,z} - J(x, y, z)L_{y,z} = 0.\]
2) \( n = p = l = m = 0, k \neq 0 \). Apply to the identity \( J(x, y, z)L_{x,z}^kL_{y,y} = J(x, y, z)L_{x,z}^kL_{y,y} \) the operator \( \Delta^2_{z}(zy) \):

\[
J(x, y, z)L_{x,z}^kL_{z,y,zy} + 2J(x, y, z)L_{x,z}^kL_{y,zy} = J(x, y, z)L_{z,y,zy}L_{x,z}^k + 2J(x, y, z)L_{y,zy}L_{x,z}^k
\]

\[
J(x, y, z)L_{x,z}^kL_{z,y,zy} + 2J(x, y, z)L_{y,zy} = J(x, y, z)L_{z,y,zy}L_{x,z}^k + 2J(x, y, z)L_{y,zy}L_{x,z}^k
\]

\[
J(x, y, z)L_{x,z}^kL_{z,y,zy} = J(x, y, z)L_{z,y,zy}L_{x,z}^k
\]

From the proved identities \( J(x, y, z)L_{x,z}^kL_{z,y,zy} = J(x, y, z)L_{z,y,zy}L_{x,z}^k \) that required. The remaining cases are proved similarly to the equality 2).

We now prove that

\[
J(x, y, z)T_1T_2...T_n[S_1, S_2] = 0.
\]

for all \( T_i \) from \( T_0 \) and \( S_j \) from \( T \). Let none of the operators \( T_i \) be not equal \( L_{x,z} \).

From proved equalities of this proposition follows

\[
(x, y, z)T_1T_2...T_n[L_{x,x}, L_{y,y}] = 0.
\]

Apply the operator \( \Delta^1_{z}(zy) \). Obtain \( L_{x,y}\Delta^1_{z}(zy) = L_{z,y,y}\Delta^1_{z}(zy) = 0, L_{z,z}\Delta^1_{z}(zy) = 0 \).

\[
J(x, y, z)T_1T_2...T_n[L_{x,z}, L_{y,y}] = 0,
\]

where \( T_i \neq L_{x,z} \). The proof of this identity for all \( T_i \in T \) is obtained by induction on degree of the operator \( L_{x,z} \) using the operator \( \Delta^1_{z}(x) \). Besides \( L_{x,z}\Delta^1_{z}(x) = L_{x,z} \).

The remaining terms arising from the action of the operator are equal to zero from a previously proven identities. Applying to the resulting identity of the corresponding operator \( \Delta^1_{x}(x_j) \), \( x_i, x_j \in X \) obtain

\[
J(x, y, z)T_1T_2...T_n[S_1, S_2] = 0.
\]

For all \( T_i \) from \( T_0 \) and \( S_j \) from \( T \).

We now prove all statements of this Proposition without restrictions on \( T_i \) and \( S_j \).

We shall prove the conjunction of all three statements using induction on \( l \), where \( l \) is number of operators \( L_{x,z} \) in the sequence \( T_1, T_2, ..., T_n \). For \( l = 0 \) all three identities are proved. Assume now that all three identities proved for all \( l \), do not exceed \( k \). We show that these identities are fulfilled for \( l = k + 1 \).

First, we note that induction hypothesis for 1) implies that if we have any sequence of operators from \( T \) and it contain not more than \( k + 1 \) copies of \( L_{x,z} \) then we can permute this operators acting on \( J(x, y, z) \). Suppose now that among the
Applying to this identity the operator \(\Delta\) obtain

\[J(x, y, z)T_1T_2...L_{x,xy}...T_n[S_1, S_2] = 0.\]

Applying to this identity the operator \(\Delta^1_z(zy)\) and taking into account equalities:

\[L_{x,xy}\Delta^1_z(zy) = 2L_{x,zy}, \quad L_{x,zy}\Delta^1_z(zy) = L_{zy,zy}, \quad L_{zy,zy}\Delta^1_z(zy) = 2L_{z,zy}, \quad L_{z,zy}\Delta^1_z(zy) = L_{y,z}\Delta^1_z(zy) = 0,\]

and conditions 2) and 3) we obtain

\[J(x, y, z)T_1T_2...L_{x,zy}...T_n[S_1, S_2] = 0.\]

Similar arguments prove the assertion 2) and 3) for a sequence \(T_1, T_2, ..., T_n\) containing \(k+1\) copies of \(L_{x,xy}\).

\[\Box\]

**Corollary 1.** Let \(U = J(x, y, z)T_1T_2...T_n\), for some

\[T_i \in T = \{L_{x,xy}, L_{xy,y}, L_{z,zy}, L_{x,zy}, L_{y,z}, L_{z,xy}\},\]

\(k, l, m, n \in \mathbb{N} \cup \{0\}\). The following equalities are fulfilled:

\[UL_{x,xy} = UL_{y,xy} = UL_{z,xy} \quad (15)\]

\[U[L_{x,xy,y}, R_{x,y}] = 0, x_i \in X \quad (16)\]

\[J(x, y, z)L^{k}_{x,xy}L^{l}_{y,y}L^{m}_{z,xy}L_{x,xy} = J(x, y, z)L^{k}_{x,xy}L^{l}_{y,y}L^{m}_{z,xy}L_{x,xy} = 0. \quad (17)\]

\[J(x, y, z)L^{k}_{x,xy}L^{l}_{y,y}L^{m}_{z,xy}L_{x,xy} = 0. \quad (18)\]

**Proof.** The equality (15). From the Proposition 2: \(UL_{x,xy} = 0\). Apply the operator \(\Delta^1_z(z)\) to this equality. From Proposition 2 obtain \(UL_{z,xy} + UL_{x,xy} = 0\). That is, \(UL_{y,xy} = UL_{x,xy}\). Similarly, the operator \(\Delta^1_y(z)\), applied to the identity \(UL_{y,xy} = 0\) gives \(UL_{z,xy} = UL_{y,xy}\).

The equality (16). The equalities \(U[L_{x,xy,y}, R_x] = U[L_{x,xy,y}, R_y] = 0, U[L_{x,xy}, R_x] = U[L_{x,xy}, R_y] = U[L_{x,xy}] = 0\) follows from the identity (17) and Proposition 2. The equality \(U[L_{x,xy}, R_z] = U[L_{y,xy}, R_z] = U[L_{xy,xy}, R_y] = 0\) follows from the identity (17) and from the equality (15).

The identity (17). From the Proposition 2 and the equality (16):

\[J(x, y, z)L^{k}_{x,xy}L^{l}_{y,y}L^{m}_{z,xy}L_{x,xy} = J(x, y, z)L^{k}_{x,xy}L^{l}_{y,y}L^{m}_{z,xy}L_{x,xy}.\]
Apply to this identity the operator $\Delta_g^2(xy)$. From the Proposition 2 we have:

$$J(x, y, z) L_{y,y}^{l-i} L_{y,xy}^{i-1} L_{y,z}^{m} z L_{x,xy}^{k} L_{x,xx} = J(x, y, z) L_{x,xy}^{l-i} L_{y,xy}^{i-1} L_{y,z}^{m} z L_{x,xx}^{k} = 0$$

$$J(x, y, z) L_{y,y}^{l-1} L_{y,xy}^{i-1} L_{y,z}^{m} z z L_{x,xx}^{k} L_{x,xy} = J(x, y, z) L_{x,xx}^{l-1} L_{y,xy}^{i-1} L_{y,z}^{m} z z L_{x,xx}^{k} = 0$$

$$J(x, y, z) L_{y,y}^{l-i} L_{y,xy,xy}^{i-2} L_{y,y}^{m} z L_{x,xx}^{k} L_{x,xy} = J(x, y, z) L_{x,xy,xy}^{l-i} L_{y,xy,xy}^{i-2} L_{y,y}^{m} z L_{x,xx}^{k}$$

Therefore, the application of the operator gives

$$J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z L_{x,xx}^{k} L_{x,xy} = 0. \quad \text{(a)}$$

Furthermore,

$$0 = J(x, y, z) L_{l,i}^{l} L_{m,i}^{m} z L_{x,xx}^{k} L_{x,xy} = J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z L_{x,xx}^{k} L_{x,xy} =$$

$$= -J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z x L_{x,xx}^{k} L_{x,xy} + 2J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z L_{x,xx}^{k} L_{x,xy} =$$

$$= -J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z x L_{x,xx}^{k} L_{x,xy} - J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z x 2 L_{x,xx}^{k} L_{x,xy} =$$

That is,

$$J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z L_{x,xx}^{k} L_{x,xy} = J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} z L_{x,xx}^{k} L_{x,xy} = 0$$

If $m > 0$, then

$$J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} L_{x,xx}^{k} (xz) L_{x,xy} = J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} L_{x,xx}^{k} (xz) L_{x,xy} =$$

$$= -\frac{1}{2} J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} L_{x,xx}^{k} (xz) L_{x,xy} + \frac{1}{2} J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} L_{x,xx}^{k} (xz) L_{x,xy} =$$

$$= \frac{1}{2} J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} L_{x,xx}^{k} (xz) L_{x,xy} - \frac{1}{2} J(x, y, z) L_{y,y}^{l} L_{y,z}^{m} L_{x,xx}^{k} (xz) L_{x,xy} = 0$$

Similarly for $k > 0$.

Let $k = l = m = 0$.

$$J(x, y, z) (xz) L_{x,xy} = \frac{1}{2} J(x, y, z) (xz) L_{x,xy} - \frac{1}{2} J(x, y, z) (xz) L_{x,xy} = - J(x, y, z) (xz) L_{x,xy} = - J(x, y, z) (xz) L_{x,xy}$$

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Suppose \( k = m = 0, \ l \neq 0 \). From (6) and proposition 2:
\[
J(x, y, z)zxL_{y,y}^l z - 2J(x, y, z)zL_{y,y}^l x =
\]
\[
= J(x, y, z)xzL_{y,y}^l z
\]
Applying the operator \( \Delta_{1z}^l(zx) \) to identity
\[
J(x, y, z)zxL_{y,y}^l z = J(x, y, z)zL_{y,y}^l x
\]
obtain
\[
J(x, y, z)L_{y,y}^l z + J(x, y, z)L_{y,y}^l (zx) = J(x, y, z)(zx)L_{y,y}^l z + J(x, y, z)L_{y,y}^l (yx)z
\]
\[
- J(x, y, z)L_{y,y}^l xzL_{x,xy}^l + J(x, y, z)L_{y,y}^l (zx)L_{x,xy}^l = J(x, y, z)(zx)L_{y,y}^l (zx) = J(x, y, z)zL_{y,y}^l L_{x,xy}^l
\]
From the identities (a) and (b) obtain:
\[
J(x, y, z)L_{y,y}^l (zx)L_{x,xy}^l = 0
\]
The identity (18) follows from (17) and (7).

**Lemma 4.** In the algebras \( M \) and \( R(M) \) the following identities are fulfilled:

\[
G(tx, x, y, z) = G(t, x, y, z) - 2J(x, y, z)(tx) \tag{19}
\]
\[
G(tx^2, x, y, z) = G(t, x, y, z)x^2 + 2J(x, t, z)L_{x,xy} + 2J(x, y, t)L_{x,xz} \tag{20}
\]
\[
t(y)xz + t(zy)x + t(x)yz = tL_{x,zy} + tL_{y,xz} + tL_{z,yx} - \frac{1}{2}J(x, y, z)t \tag{21}
\]
\[
G(t, x, y, z) = \frac{2}{3}(tyzx - txy) + \frac{2}{3}(tzxy - txyz) + \frac{2}{3}(txyz - tzyx) +
\]
\[
+ \frac{2}{3}tL_{x,zy} + \frac{2}{3}tL_{y,xz} + \frac{2}{3}tL_{z,yx} - J(x, y, z)t \tag{22}
\]
**Proof.** The identity (19). From the identities (3), (4) and the definition of the function \( G \):
\[
\frac{3}{2}G(tx, x, y, z) = J(tx, y, z) + J(x, tx, z)y + J(x, y, tx)z - J(x, y, z)(tx) = J(tx, y, z)x +
\]
\[ + J(yz, x, tx) - G(y, z, x, tx) - J(x, y, z)(tx) = J(tx, y, z)x - J(yz, x, t)x + G(tx, x, y, z) - \]
\[ - J(x, y, z)(tx) = G(tx, x, y, z) - J(x, y, z)(tx) + \frac{1}{2}G(t, x, y, z)x, \]

whence follows the desired.

The identity (20). From the identities (8), (3), (4) and the definition of the function \( G \):

\[ (tx)J(x, y, z) = -\frac{1}{2}J(x, t, z)xy - \frac{1}{2}J(t, x, y)xz - J(y, z, tx)x - \frac{3}{2}J(x, t, zy)x = \]
\[ = \frac{1}{2}J(x, t, z)yx + \frac{1}{2}J(t, x, y)zx - J(y, z, tx)x - \frac{3}{2}J(x, t, zy)x - J(x, t, z)L_{x,y} - \]
\[ - J(t, x, y)L_{x,z} = \frac{1}{2}(J(x, z, t, x) + J(t, x, y)z + J(t, x, zy)x) - \frac{1}{2}G(t, x, y, z)x - J(x, t, z)L_{x,y} - \]
\[ - J(t, x, y)L_{x,z} = - J(t, x, z)L_{x,y} - J(t, x, y)L_{x,z} = J(t, x, yz)x. \]

Further, from the identity (19) we have:

\[ G(tx, x, y, z) = G(t, x, y, z)x - 2J(x, y, z)(tx) = G(t, x, y, z)x - 2J(x, t, z)L_{x,y} - \]
\[ - 2J(x, y, t)L_{x,z} - 2J(t, x, yz)x \]

Thus, from this identity and the identity (7) obtain:

\[ G(tx^2, x, y, z) = G(t, x, y, z)x - 2J(x, t, z)L_{x,y} - 2J(t, x, y)L_{x,z} - 2J(t, x, y)tx = \]
\[ = G(t, x, y, z)x^2 - 2J(x, t, z)L_{x,y}x - 2J(t, x, y)L_{x,z}x - 2J(t, x, yz)x^2 + 2J(x, t, z)L_{x,y} + \]
\[ + 2J(t, x, y)L_{x,z} + 2J(t, x, yz)x^2 = G(t, x, y, z)x^2 + 2J(x, t, z)L_{x,xy} + 2J(x, y, t)L_{x,xz}. \]

It implies (21). From the identity (11) we have:

\[ t(xy)z + t(yz)x + t(zx)y = yztx + txyz - txz - at(xy) + xyt + txyz - txyz - y(tz)x + \]
\[ + zxyt + txz - tzxy - xt(yz) = ty(zx) + tx(yz) + ty(zx) + J(x, y, z)t. \]

That is,

\[ t(xy)z + t(yz)x + t(zx)y = ty(zx) + tx(yz) + ty(zx) + J(x, y, z)t. \]
Hence follows the identity \((21)\).
Further, from this identity and the identities \((11)\) and \((3)\) we obtain:

\[
G(t, x, y, z) = \frac{2}{3}(J(t, y, z)x + J(x, t, z)y + J(x, y, t)z - J(x, y, z)t) = \\
= tyzx + ztyx + yzx + xtz + ztxy + zty + txz + ytxz - J(x, y, z)t = \\
= \frac{2}{3}(tyzx - txzy) + \frac{2}{3}(tzxy - tyxz) + \frac{2}{3}(txyz - tzyx) + \frac{2}{3}tL_{x,y} + \\
\quad + \frac{2}{3}tL_{y,x} - J(x, y, z)t.
\]

\[\blacksquare\]

**Lemma 5.** Let \(T = \{L_{x,x}, L_{y,y}, L_{z,z}, L_{x,y}, L_{x,z}, L_{y,z}, L_{x,y,z}\}\), and \(T_i \in T, n \in \mathbb{N} \cup \{0\}\). In the algebras \(M\) and \(R(M)\) the following relations are fulfilled:

\[
J(x, y, z)T_1T_2...T_nJ(x, y, z) = 0 \quad (23)
\]

\[
J(x, y, z)T_1T_2...T_n(J(x, y, z)L_{x,x}) = 0 \quad (24)
\]

\[
J(x, y, z)T_1T_2...T_n(x(J(x, y, z)x) = 0 \quad (25)
\]

**Proof.** We first prove that

\[
J(x, y, z)L_{x,x}^k L_{y,y}^l L_{z,z}^m J(x, y, z) = 0.
\]

From the identity \((20)\) for \(t = J(x, y, z)L_{x,x}^k L_{y,y}^l L_{z,z}^m:\)

\[
G(tL_{x,x}, x, y, z) - G(t, x, y, z)L_{x,x} = 2J(x, t, z)L_{x,xy} + 2J(x, y, t)L_{x,xz}.
\]

If \(k + m \neq 0\) and \(k + l \neq 0\), then from identity \((17)\) we obtain:

\[
G(tL_{x,x}, x, y, z) - G(t, x, y, z)L_{x,x} = 2J(x, J(x, y, z)L_{y,y}^l L_{z,z}^m, z)L_{x,xy} + \\
\quad + 2J(x, J(x, y, z)L_{x,x}^k L_{y,y}^l L_{z,z}^m)L_{x,xz} = -2J(x, J(x, y, z)L_{y,y}^l L_{z,z}^m, z)L_{x,xy} - \\
\quad - 2J(x, J(x, y, z)L_{x,x}^k L_{y,y}^l L_{z,z}^m, z)L_{x,xy} = 6J(x, y, z)L_{x,xz} + 6J(x, y, z)L_{x,x}^k L_{y,y}^l L_{z,z}^m(x z)L_{x,xy} + \\
\quad + 6J(x, y, z)L_{x,x}^k L_{y,y}^l L_{z,z}^m(xy)L_{x,xz} = 0.
\]
Suppose, for example $k + l = 0$, that is $t = J(x, y, z)L_{x,z,z}^m$.

$$J(x, y, J(x, y, z)L_{y,y}^m)L_{x,x,z} = J(x, y, J(x, y, zy^2m))L_{x,x,z} = 3J(x, y, z)L_{y,y}^m(xy)L_{x,x,z} = 0.$$  

That is, $J(x, y, J(x, y, z)L_{y,y}^m)L_{x,x,z} = 0$. Applying the operator $\Delta^m_{y}(z)$ to this identity we obtain $J(x, y, z)L_{x,z,z} = 0$. Further,

$$G(tL_{x,x}, x, y, z) - G(t, x, y, z)L_{x,x} = 2J(x, t, z)L_{x,x} + 2J(x, y, t)L_{x,x} = 0.$$  

Thus, we have $G(tL_{x,x}, x, y, z) = G(t, x, y, z)L_{x,x}$, for all $t$ of the form $t = J(x, y, z)L_{x,x}L_{y,y}L_{z,z}^m$. Using this equality and induction on $k + l + m$ obtain:

$$J(x, y, z)L_{x,z}^kL_{y,y}^lL_{z,z}^m G = J(x, y, z)GL_{x,x}^kL_{y,y}^lL_{z,z}^m$$  

(a)

Further, we denote $J(x, y, z)L_{x,x}^kL_{y,y}^lL_{z,z}^m$ by $U$. We have:

$$UL_{x,x}xyz = Uxyz = -Uzxyz + 2UzxL_{x,z,y} + 2UzL_{x,z,y} = -Uzxyz + 2UzxL_{x,z} - 2UxL_{x,z,y} + 2UL_{x,z}L_{x,y}y -$$

$$-2UzL_{x,z} + 2UL_{x,z}L_{x,y}y + 2UL_{x,z}L_{x,y}y + 2UL_{x,z}L_{x,y}y + 2UL_{x,z}L_{x,y}y + 2UL_{x,z}L_{x,y}y -$$

Therefore, from (18) it follows that sum of the last two terms is $0$. That is, $UL_{x,x}xyz = UxyzL_{z,z}$. Similarly can be verified the equality $UL_{y,y}xyz = UxyzL_{y,y}y$ and $UL_{x,x}xyz = UxyzL_{x,x}$. Therefore,

$$J(x, y, z)L_{x,x}^kL_{y,y}^lL_{z,z}^mxyz = J(x, y, z)L_{x,x}^kL_{y,y}^lL_{z,z}^m$$  

(b)

From the identities (22), (a), (b) and Proposition 2 for $U = J(x, y, z)L_{x,x}^kL_{y,y}^lL_{z,z}^m$ we have:

$$UJ(x, y, z) = +UG - \frac{2}{3}Uyxy - \frac{2}{3}Uxyx + \frac{2}{3}Uyxy - \frac{2}{3}Uxyx - \frac{2}{3}Uyxy - \frac{2}{3}UL_{x,y} -$$

$$- \frac{2}{3}UL_{x,y} - \frac{2}{3}UL_{y,x} = - \frac{2}{3}[UG + (J(x, y, z)yxz + xzy) - zxy + yxz - xzy + yzx] +$$

$$+ J(x, y, z)(L_{x,y} + L_{y,x} + L_{x,y} + L_{y,x}))(L_{x,x}^kL_{y,y}^lL_{z,z}^m) = J(x, y, z)J(x, y, z)L_{x,x}^kL_{y,y}^lL_{z,z}^m = 0$$

Induction on the degree of operator $L_{x,y}$ and application of $\Delta^m_{z}(y)$ give:

$$J(x, y, z)L_{x,y}^nL_{x,x}^kL_{y,y}^lL_{z,z}^m J(x, y, z) = 0.$$

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Induction on the degree of operator $L_{y,z}$ and application of $\Delta^1_y(y)$ give:

$$J(x, y, z)L^p_{y,z}L^n_{x,y}L^k_{x,x}L^l_{y,y}L^m_{z,z}J(x, y, z) = 0.$$ 

Induction on the degree of operator $L_{x,z}$ and application of $\Delta^1_x(x)$ give:

$$J(x, y, z)L^p_{x,z}L^q_{y,z}L^n_{x,y}L^k_{x,x}L^l_{y,y}L^m_{z,z}J(x, y, z) = 0.$$ 

Induction on the degree of operator $L_{x,zy}$ and application of $\Delta^1_z(zy)$ give:

$$J(x, y, z)L^p_{x,zy}L^q_{y,z}L^n_{x,y}L^k_{x,x}L^l_{y,y}L^m_{z,z}J(x, y, z) = 0,$$

for all $k, l, m, n, p, q, s \in \mathbb{N} \cup \{0\}$.

Prove (24). First, we prove that

$$J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}(J(x, y, z)L_{x,z}) = 0.$$ 

Applying to (23) the operator $\Delta^1_y(yx^2)$. From (7) and Proposition 2 obtain:

$$J(x, y, z)L^k_{x,x}L^l_{y,y}L^m_{z,z}(J(x, y, z)L_{x,z}) = 0.$$ 

Therefore, the application of the operator $\Delta^1_y(yx^2)$ with Proposition 2 and the equality (15) give the equality (24).
Prove (25). The following equality is obvious: \( L_{z,yx} \Delta^1_1(zz) = L_{zx,yx} \).

From Proposition 2:

\[
J(x, y, z)T_1 T_2 \ldots T_i d(z, x) = J(x, y, z)T_1 T_2 \ldots T_i (L_{zx, zx} + L_{x,x} L_{z,z} - L_{z,x}^2) = 0,
\]

That is,

\[
J(x, y, z)T_1 T_2 \ldots T_i L_{zx} = J(x, y, z)T_1 T_2 \ldots T_i L_{x,x} L_{z,y} - J(x, y, z)T_1 T_2 \ldots T_i L_{x,z} L_{x,y}
\]

From (23) and Proposition 2 the identity follows:

\[
J(x, y, z)T_1 T_2 \ldots T_m J(x, y, z) = 0,
\]

where \( T_i \in T \cup \{L_{x,x}, L_{z,z}\} \). The application of the operator \( \Delta^1_1(zx) \) to this identity gives:

\[
J(x, y, zx)T_1 T_2 \ldots T_m J(x, y, z) + J(x, y, z)T_1 T_2 \ldots T_m J(x, y, zx) = 0,
\]

for all \( T_i \in T \cup \{L_{x,x}, L_{z,z}\} \). Now write (23) as follows:

\[
J(x, y, z)L^s_{z,yx} L^q_{x,z} L^p_{y,z} L^m_{x,y} J(x, y, z) = 0
\]

for any \( k, l, m, n, p, q, s \in \mathbb{N} \cup \{0\} \).

Applying to it the operator \( \Delta^2_2(zx) \), grouping corresponding terms and considering the previous identity, we obtain (24).

**Lemma 6.** In the algebras \( M \) and \( R(M) \) the following relations are fulfilled:

\[
J(y, z, x(R_{zy} R_x)^n) = J(y, z, x)L^n_{x,zy} \tag{26}
\]

\[
J(x, y, z)G^n = 6^n J(x, y, z)L^n_{x,zy} \tag{27}
\]

**Proof.** Prove first the identity

\[
J(x, x(R_{zy} R_x)^n R_{zy}, y) = J(x, y, z)yx L^n_{x,zy}
\]

by induction on \( n \). For \( n = 0 \) this identity is obvious. Suppose that it fulfilled for \( n = k \). Prove it for \( n = k + 1 \). Applying the operator \( \Delta^1_1(zy) \) to the identity

\[
J(x, txy, y) = J(x, t, y)xy
\]
\[ J(x, tx(zy), y) + J(x, txy, zy) = J(x, t, yz)xy + J(x, t, y)x(yz) \]

If \( t = x(R_{zy}R_x)^kR_{zy} \), then

\[ J(x, tx(zy), y) = -J(x, txy, zy) + J(x, t, y)x(yz) \]

And from the identity (9) obtain:

\[ J(x, tx(zy), y) = -J(x, txy, zy) + J(x, t, y)x(yz) = \]

\[ = J(x, t, y)L_{x,zy} = J(x, y, z)yzL_{x,zy}. \]

Prove now the identity (26) by induction on \( n \). Let \( n = 1 \). Apply the operator \( \Delta_1 \) to the identity

\[ J(y, z, xyx) = J(y, z, x) \]

We have

\[ J(y, z, x(zy)) = J(y, z, x) \]

Suppose the identity holds for \( n = k \). Prove that for \( n = k + 1 \). Applying the operator \( \Delta_1 \) to the identity (9) we have

\[ J(z, y, txy) = J(z, y, t) \]

Let \( t = x(R_{zy}R_x)^k \). From (16) and the induction hypothesis:

\[ J(z, y, tx(zy)) - J(z, y, t)L_{x,zy} = \frac{1}{2}J(y, t, x)y^2 + \frac{1}{2}J(y, t, x)xyz + \frac{1}{2}J(y, t, x)[R_z, R_{zy}] - \]

\[ + \frac{1}{2}J(y, t, x)[R_z, R_{zy}] = \frac{1}{2}J(y, t, x)y^2 + \frac{1}{2}J(y, t, x)xyz + \frac{1}{2}J(y, t, x)L_{x,zy}z + \frac{1}{2}J(y, t, x)zL_{x,zy} - \]

\[ + \frac{1}{2}J(z, y, t)[R_z, R_{zy}] + \frac{1}{2}J(y, t, x)[R_z, R_{zy}]. \]

Let \( t = x(R_{zy}R_x)^k \). From (16) and the induction hypothesis:

\[ J(z, y, t(zy)) - J(z, y, t)L_{x,zy} = \frac{1}{2}J(y, t, x)y^2 - \frac{1}{2}J(y, t, x)xyz + \frac{1}{2}J(y, t, x)[R_z, R_{zy}]. \]

From the proved identity, the Proposition 2 and identity (9):

\[ J(z, y, t(zy)) - J(z, y, t)L_{x,zy} = \frac{1}{2}J(z, y, x)L_{x,zy}^kxyz - \frac{1}{2}J(z, y, x)L_{x,zy}^kxyz + \]

\[ + \frac{1}{2}J(z, y, x)L_{x,zy}^k[x, R_{zy}] = \frac{1}{2}J(z, y, x)L_{x,zy}^kxyz - \frac{1}{2}J(z, y, x)L_{x,zy}^kxyz - \]

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obtain:

\[ G(J(x, y, z), x, y, z) = \frac{2}{3} J(J(x, y, z), y, z)x + \frac{2}{3} J(x, J(x, y, z), z)y + \]

\[ + \frac{2}{3} J(x, y, J(x, y, z))z - \frac{2}{3} J(x, y, z)J(x, y, z)z = 2J(x, y, z)(zy)x + 2J(x, y, z)(xz)y + \]

\[ + J(x, y, z)(yx)z = 2J(x, y, z)L_{x,zy} + 2J(x, y, z)L_{y,xz} + 2J(x, y, z)L_{z,yx} = 6J(x, y, z)L_{x,zy}. \]

Suppose, the identity holds for \( n = k \). Prove it for \( n = k + 1 \). From the identities (8), (15), (23) and (26):

\[ 6^k G(J(x, y, z)L_{x,zy}^k, x, y, z) = \frac{2}{3} 6^k(J(J(x, y, z)L_{x,zy}^k, y, z)x + J(x, J(x, y, z)L_{x,zy}^k, z)y + \]

\[ + J(x, y, J(x, y, z)L_{x,zy}^k, y, z)x + J(x, J(x, y, z)L_{x,zy}^k, z)x + \]

\[ + J(x, y, L_{y,xz}^k, z)y + J(x, y, J(x, y, z)L_{x,zy}^k, z)x + \]

\[ + J(x, y, (R_{xy}R_x^k, z))y + J(x, y, z(R_{xy}R_z^k))y + J(x, y, z(R_{yz}R_z^k))y \]

\[ = 2 \cdot 6^k(J(x, y, z(L_{x,zy}^k, y, z)x + \]

\[ + J(x, y, z(R_{xy}R_z^k, z)x + J(x, y, z(R_{yz}R_z^k))y + J(x, y, z(R_{yz}R_z^k))y \]

\[ + J(x, y, L_{y,xz}^k, z)y + J(x, y, L_{y,xz}^k, z)y \]

\[ + J(x, y, L_{y,xz}^k, z)y + J(x, y, L_{y,xz}^k, z)y \]

\[ = 2 \cdot 6^k(J(x, y, z(L_{x,zy}^k, y, z)x + \]

\[ + J(x, y, z(R_{xy}R_z^k, z)x + J(x, y, z(R_{yz}R_z^k))y + J(x, y, z(R_{yz}R_z^k))y \]

\[ + J(x, y, L_{y,xz}^k, z)y + J(x, y, L_{y,xz}^k, z)y \]

\[ + J(x, y, L_{y,xz}^k, z)y + J(x, y, L_{y,xz}^k, z)y \]

\[ = 6^{k+1} J(x, y, z)L_{x,zy}^{k+1}. \]

Lemma 7. The set

\[ U \cup U_x \cup U_y \cup U_z \cup U_{xy} \cup U_{xz} \cup U_{yz} \]

generates a linear space \( J(M, M, M) \) over field \( F \).

Proof. We first prove that if

\[ U \in U = \{ J(x, y, z)G^k L_{x,zy}^l L_{y,xz}^m L_{x,yz}^n L_{y,xy}^p L_{z,zy}^q L_{x,yz}^r | k, l, m, n, p, q, r \in \mathbb{N} \cup \{0\} \}, \]

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then

\[ UG = 6UL_{x,zy} = 3U(xyz - zyx) = 3U(yzx - xzy) = 3U(zxy - yxz) \]  \hspace{1cm} (28)

\[ Uxyz = 1/6UG + UL_{y,zx} - UL_{x,zy} + UL_{x,yz} \]  \hspace{1cm} (29)

From the identity (19):

\[ G(U_{L_{x,x}}, x, y, z) - G(U, x, y, z)L_{x,x} = -2J(x, y, z)(Ux)x - 2J(x, y, z)(UL_{x,x}) \]

On the other hand, from the identity (1):

\[ (Ux)(J(x, y, z)x) = UJ(x, y, z)xx + xUJ(x, y, z)x + J(x, y, z)xxU. \]

From (22), (24), (25):

\[ UxJ(x, y, z)x = -(Ux)(J(x, y, z)x) = 0. \]

Thus,

\[ G(U_{L_{x,x}}, x, y, z) - G(U, x, y, z)L_{x,x} = 0 \]

and the operators \( \Delta_{x_1}^1(x_2), x_i \in X \) give the identities:

\[ G(U_{L_{x_1,x_2}}, x, y, z) - G(U, x, y, z)L_{x_1,x_2} = 0, \]

for all \( x_i \in X \). Let \( U = J(x, y, z)T_1T_2...T_n \), where

\[ T_i \in T = \{L_{x,x}, L_{y,y}, L_{z,z}, L_{x,y}, L_{x,z}, L_{y,z}, L_{x,zy}\}, \]

and \( n \in \mathbb{N} \cup \{0\} \). The Proposition 2 implies that \( U \) can be written

\[ U = J(x, y, z)L^k_{x,zy}S_1S_2...S_{n-k}, \]

where \( S_i \in T \setminus \{L_{x,zy}\} \). Therefore, using the proven identity and the identity (27) and (15) obtain:

\[ G(U, x, y, z) = G(J(x, y, z)L^k_{x,zy}S_1S_2...S_{n-k}, x, y, z) = G(J(x, y, z)L^k_{x,zy}, x, y, z)S_1S_2...S_{n-k} = \]

\[ = 6J(x, y, z)L^k_{x,zy}S_1S_2...S_{n-k} = 6UL_{x,zy} = 6UL_{y,xz} = 6UL_{z,yx}. \]

From the identities (22), (15), (23) and (16):

\[ G(U, x, y, z) = \frac{2}{3}(Uyxz - Uxzy) + \frac{2}{3}(Uzxy - Uyxz) + \frac{2}{3}(Uxyz - Uzyx) + \]

\[ 25 \]
That is,
$$ UG = U_{xyz} - U_{zyx} + U_{yxt} - U_{xty} = U_{xyz} - U_{zyx} - U_{ytx} + 2UL_{yzt}x +$$
$$ + U_{xyz} - 2UXL_{yzt} - U_{zyt} + 2UL_{yzt} + U_{xyz} - 2UL_{yzt} = 3U_{xyz} - 3U_{zyx}. $$

The remaining equalities in (28) can be proved similarly. Further, from (28) and (16):
$$ U_{xyz} = \frac{1}{3}UG + U_{zyx} = \frac{1}{3}UG - U_{zyx} + 2UL_{xt} + 2UL_{yt} - 2UL_{zt} =$$
$$ = \frac{1}{3}UG - U_{xyz} + 2UL_{yzt}x - 2UL_{yzt} + 2UL_{yzt}, $$

that implies (29).

Let $u \in J(M, M, M)$. Proposition 1 implies that there are $\alpha_i$ from $F$ for which
$$ u = \sum_i \alpha_i J(x, y, z)x_{i1}x_{i2}...x_{ik}, x_{i,j} \in X, k_i \in \mathbb{N} \cup \{0\} $$

Therefore, it suffices to show that the polynomials of the form $J(x, y, z)x_{i1}x_{i2}...x_{ik}, x_{i} \in X, k \in \mathbb{N} \cup \{0\}$ are the linear combinations of the polynomials from the set
$$ U \cup U_x \cup U_y \cup U_z \cup U_{xy} \cup U_{xz} \cup U_{yz}. $$

The proof by induction on $k$ obviously follows from the identities (16) and (29). ■

**Proof of Theorem.** Prove the specialty of the algebra $M$. That is, we show that free Malcev algebra on the free generators $X$ is isomorphic to the subalgebra of the algebra $(Alt[X])^{-}$, generated by the set $X$. Let $f$ be the canonical homomorphism $f : M \rightarrow (Alt[X])^{-}$. From Lemma 7 follows that it suffices to show that the set $f(U \cup U_x \cup U_y \cup U_z \cup U_{xy} \cup U_{xz} \cup U_{yz})$ is linearly independent in $Alt[X]$. Prove first the linear independence of the set $f(U)$. As in [2] define in $Alt[X]$ the following subsets:
$$ W_0 = \{(x, y, z)(L_{xy}^+)^{n_1}(L_{yz}^+)^{n_2}(L_{zx}^+)^{n_3}(L_{xy}^+)^{n_4}(L_{yx}^+)^{n_5}(L_{zx}^+)^{n_6}(L_{xy}^+)^{n_7} | n_i \in \mathbb{N} \cup \{0\}\}; $$

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\[ W_1 = \{ [w, a] \mid w \in W_0, a \in X \}; \]
\[ W_2 = \{ (w, x, y), (w, y, z), (w, z, x) \mid w \in W_0 \}; \]
\[ W = W_0 \cup W_1 \cup W_2. \]

We prove that for each \( u \) from \( U \) exists such \( \alpha \in F, \alpha \neq 0 \), that \( f(u) = \alpha w \), where \( w \in W \). Let \( u = J(x, y, z)T_1...T_n \), where \( T_i \in \{ L_{x,y}, L_{x,z}, L_{y,z}, L_{x,zy} \} \). The proof is by induction on \( n \). For \( n = 0 \) the assertion is obvious. Assume that the assertion is proved for \( u = J(x, y, z)T_1...T_{n-1} \) and assume that \( T_n = L_{x,y} \). It is easy to show that then
\[
 f : uL_{x,y} \mapsto \frac{1}{2}([[f(u), x], y] - [[f(u), y], x]) = \\
= \frac{1}{2}(f(u)yx - xf(u)y - yf(u)x + yxf(u) + f(u)yx - yf(u)x - xf(u)y + yxf(u)) = \\
= -2f(u)L_{x,y}^+ - 2f(u)L_{y,x}^+. 
\]
From the identities (24), (8) and (9) from \([2]\) and induction hypothesis follows that \( f(u)L_{x,y}^+ = f(u)L_{y,x}^+ \), therefore,
\[
 f : uL_{x,y} \mapsto -4f(u)L_{x,y}^+. 
\]
Similarly,
\[
 f : uL_{x,zy} \mapsto 4f(u)L_{x,y}^+. 
\]
From \([27]\) we obtain
\[
 f : uG \mapsto 24f(u)L_{x,y}^+. 
\]
Finally, direct calculation gives for any \( t \) from \( M \): \( f : tL_{x,x} \mapsto [[f(t), x], x] = x(xf(t)) + f(t)x - xf(t)x - x(f(t)x) = -4(f(t) \circ x \circ x - f(t) \circ (x \circ x)) = -4f(t)L_{x,x}^+. \)
That is,
\[
 f : tL_{x,x} \mapsto [[f(t), x], x] = -4f(t)L_{x,x}^+. 
\]
From the proved and \([2]\) follows the linear independence of \( f(U) \).
Note that for any \( u \in U \) holds
\[
 f : uxy \mapsto -4f(u)L_{x,y}^+ + 2(f(u), x, y). 
\]
Now suppose that there exist \( u_i \in F \) (\( U \)) that the following relations are fulfilled.
\[
 f(u_0 + u_1x + u_2y + u_3z + u_4xy + u_5xz + u_6yz) = 0 
\]
Denoting \( f(u_i) = w_i \) we obtain

\[
\begin{align*}
w_0 + [w_1, x] + [w_2, y] + [w_3, z] & - 4w_4 L_{x,y}^+ + 2(f(u_4), x, y) - 4w_5 L_{z,x}^+ + 2(x, z, f(u_5)) - 4w_6 L_{y,z}^+ + \\
+ 2(y, z, f(u_6)) & = (w_0 - 4w_4 L_{x,y}^+ - 4w_5 L_{z,x}^+ - 4w_6 L_{y,z}^+) + [w_1, x] + [w_2, y] + [w_3, z] + (f(u_4), x, y) + \\
+ 2(x, z, f(u_5)) + 2(y, z, f(u_6)).
\end{align*}
\]

Hence, from the linear independence of the set \( W \), proved in [2] we have: \( w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0 \) and \( w_0 - 4w_4 L_{x,y}^+ - 4w_5 L_{z,x}^+ - 4w_6 L_{y,z}^+ = 0 \). Therefore, \( w_0 = 0 \) from proven we have \( u_i = 0, \ i \in \{0, 1, 2, 3, 4, 5, 6\} \).

Obviously, from the proved and Lemma 7 follows that the set

\[
\mathbb{U} \cup \mathbb{U}x \cup \mathbb{U}y \cup \mathbb{U}z \cup \mathbb{U}xy \cup \mathbb{U}xz \cup \mathbb{U}yz
\]

is a basis of the space \( J(M, M, M) \).

\[\blacksquare\]

**Corollary 1.** Let \( S \) be the free algebra of rank three of varieties of Malcev algebras generated by simple seven-dimensional Malcev algebra. Free Malcev algebra \( M \) of rank three is a subdirect sum of the free Lie algebra \( L \) of rank three and the free algebra \( S \).

**Proof.** Let \( K \) be the free algebra with set of free generators \( X \) of the variety generated Cayley-Dickson algebra over a field \( F \). It is easy to understand that the subalgebra of the Malcev algebra \( K^{-} \), generated by the \( X \) is isomorphic to \( S \). From the Theorem 1 of [2] we obtain that \( M \) is the subalgebra in \( K^{-} \oplus (\text{Ass}[X])^{-} \).

The projection \( K^{-} \oplus (\text{Ass}[X])^{-} \rightarrow K^{-} \) induces a homomorphism \( g_1 : M \rightarrow S \), which obviously is a surjective. In addition, it is similar that the homomorphism \( g_2 : M \rightarrow L \), induced by the projection \( K^{-} \oplus (\text{Ass}[X])^{-} \rightarrow (\text{Ass}[X])^{-} \) is surjective.

\[\blacksquare\]

In [9] particularly was shown that the free algebra of the variety generated by the Cayley-Dickson algebra is prime. From this result and the last corollary can be easily obtained

**Corollary 2.** Free Malcev algebra of rank three is semiprime.

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