Holographic symmetries and generalized order parameters for topological matter

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We introduce a universally applicable method, based on the bond-algebraic theory of dualities, to search for generalized order parameters in disparate systems including non-Landau systems with topological order. A key notion that we advance is that of holographic symmetry. It reflects situations wherein global (or bulk) symmetries become, under a duality mapping, symmetries that act solely on the system’s boundary. Holographic symmetries are naturally related to edge modes and localization. The utility of our approach is illustrated by systematically deriving generalized order parameters for pure and matter-coupled Abelian gauge theories, and for some models of topological matter.

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Introduction. Landau’s concept of an order parameter (OP) and spontaneous symmetry breaking are central in physics.1 In systems with long-range Landau orders, two-point correlation functions of an OP field $\mathcal{O}(r)$, in their large-distance limit, tend to a finite (i.e., nonzero) value, $\lim_{r \to -\infty} \lim_{N \to \infty} \langle \mathcal{O}(r) \mathcal{O}(r') \rangle \neq 0$, where $N$ is the linear size of the $d$-dimensional system, and $\mathcal{O}(r)$ is local in the (spatial) variable $r$. It is in Landau’s spirit to use the OP as a macroscopic variable characterizing the ordered phase and as an indicator of a possible phase transition (classical or quantum) to a disordered state where the OP becomes zero.

There is much experience, including systematic methods,2,3 for deriving Landau OPs and their effective-field theories.1 Landau’s ideas of a (local) OP cannot be extended to topological states of matter because, by definition,4,5 these lie beyond Landau’s paradigm. However, the notion of long-range order or the design of a witness correlator (i.e., a correlator discerning the existence of various phases and related transitions) can be extended to topological phases—phases that can only be meaningfully examined by nonlocal probes.5 Topological orders appear in gauge theories, quantum Hall and spin liquid states (when defined as deconfined phases of emergent gauge theories6), including well-studied exactly solvable models.7,8

In this paper we demonstrate that generalized nonlocal OPs may diagnose topological phases of matter. Most importantly, we outline a method based on bond-algebraic duality mappings to search systematically for generalized OPs. Dualities have the striking capability of mapping Landau to topological orders and vice versa for essentially two reasons: First, dualities in general represent nonlocal transformations of elementary degrees of freedom9 and may even perform transmutation of statistics.10 Second, bond-algebra techniques10–12 allow for the generation of dualities in finite- and infinite-size systems. As we will show, in systems with a boundary, dualities realize a form of holography13 capable of transforming a global symmetry, that may be spontaneously broken, into a boundary symmetry. We term these distinguished boundary symmetries holographic. They are, under suitable further conditions, connected to edge (boundary) states. To illustrate the method, we derive explicitly a (nonlocal) witness correlator and a generalized OP, suited to diagnose the transition between deconfined and confined phases of matter-coupled gauge theories, undetectable by standard OPs or Wilson loops. Other examples are reported in Ref. 14.

The search for generalized order parameters. A natural mathematical language to describe a physical system is that for which the system’s degrees of freedom couple locally. This simple observation is key to understanding that topological order is a property of a state(s) relative to the algebra of observables (defining the language) used to probe the system experimentally.5 In the language in which the system is topologically ordered, it is also robust (at zero temperature15) against perturbations local in that language. Spectral properties are invariant under unitary transformations of the local Hamiltonian $H$ governing the system: $H \mapsto U H U \dagger$. If $U H U \dagger$ corresponds to a sensible local theory then the unitary transformation $U$ establishes a duality.10 A duality may map a system that displays topological order to one that does not.5 Dualities for several of Kitaev’s models7,8,16 epitomize this idea.5,12,15

Since dualities are unitary transformations (or, more generally, partial isometries)10 they cannot in general change a phase diagram, only its interpretation. This leads to a central point of our work: A duality mapping a Landau to a topologically ordered system must map the Landau OP to a generalized OP characterizing the topological order. Our method for searching for generalized OPs combines this observation with the advantages of the bond-algebraic theory of dualities.10 In this framework, dualities in arbitrary size (finite or infinite) systems can be systematically searched for as alternative local representations of bond algebras of interactions associated to a Hamiltonian $H$. Hence it is possible for any system possessing topological order to systematically search for a duality mapping it to a Landau order. When a dual Landau theory is found, the dual system’s OP can be mapped back to obtain a generalized OP for the topologically ordered system. In what follows and in Ref. 14, we study various quantum gauge and topologically ordered theories, and their duals, to illustrate our ideas.

Holographic symmetries and edge states: the gauged Kitaev wire. We next illustrate the concept of holographic symmetry and its relation to generalized OPs and edge modes. Consider the Kitaev wire Hamiltonian16 with open boundary conditions, here generalized to include a Z2 gauge field (termed the gauged
Kitaev wire),
\[ H_{GK} = -ih \sum_{m=1}^{N} b_m a_m - \sum_{m=1}^{N-1} [i J b_m a_m^z + \kappa a_{m+1}^z], \]
where \( a_m = \phi_d^*, b_m = b_m^* \) denote two Majorana fermions \((a_m, a_{m+1}) = 2 \delta_{m0} = (b_m, b_{m+1}), (a_m, b_n) = 0\) placed on each site of an open chain with \( N \) sites. The Pauli matrices \( \sigma_{\alpha}^{m,n}, \alpha = x, z, \) placed on the links \((m,1)\) connecting sites \( m \) and \( m+1 \) represent a \( Z_2 \) gauge field. For the gauged Kitaev wire, fermionic parity is obtained as the product of the local (gauge) \( \sigma_{\alpha}^{N,m} \) and \( \sigma_{\alpha}^{N-1,m+1} \), where \( m \leq \frac{N}{2} \). Just like the standard Kitaev wire, \( H_{GK}[h = 0] \) has two free edge modes \( a_1 \) and \( b_N \).

The gauged Kitaev wire holds two important dualities. It is dual to the one-dimensional \( Z_2 \) Higgs model \( H_0 \)
\[ H_0 = -\sum_{i=1}^{N} \sigma_i^+ - \sum_{i=1}^{N-1} \left[ J \sigma_i^z \sigma_{i+1}^z + \kappa \sigma_i^z \right], \]
with Pauli matrices \( \sigma_{\alpha}^i \) placed on sites \( i \). Moreover, the gauge-reducing \( \Phi_d \) duality mapping \( \Phi_d \)
\[ \begin{align*}
ib_m a_m & \xrightarrow{\Phi_d} \sigma_m^+ \sigma_{m+1}, \\
ib_m \sigma_{m+1}^z & \xrightarrow{\Phi_d} \sigma_{m+1}^+. 
\end{align*} \]
transforms \( H_{GK} \) into a spin-\( \frac{1}{2} \) system
\[ H_0^{\text{sp}}_{GK} = -\sum_{m=1}^{N} \sigma_m^+ \sigma_{m+1}^z - \sum_{m=1}^{N} J \sigma_m^z + \kappa \sigma_{m+1}^z. \]

The language providing the most local operator description of the ground-state manifold is the one realizing the edge modes, which are expected to be exponentially localized to the boundary. Thus, as long as the thermodynamic-limit degeneracy remains, a suitable local probe will detect localization on the boundary for those states. Conversely, noncommuting edge-mode operators in a gapped phase reflect the existence of low-energy many-body states with energy splittings vanishing exponentially with system size. Many-body (zero-energy) edge states are thus simply a natural consequence of a degenerate ground-state manifold in a gapped system. They are witnesses of an ordered (degenerate) phase described in a most local language. Note that boundary operators that commute with the Hamiltonian at special values of the coupling(s) are a necessary but not sufficient condition to realize exact (zero-energy) edge modes.

The duality \( H_{0} \leftrightarrow H_{GK} \) maps a global symmetry of \( H_{0} \) to a boundary symmetry of \( H_{GK} \).

The zero-energy modes disappear together with the boundary degeneracy. For \( \kappa > 0 \), they disappear despite the fact that fermionic parity remains an exact symmetry and cannot be spontaneously broken. Consider now \( H_{0}^{\text{sp}}_{GK} \) of Eq. (4). At \( h = 0 \), it has zero-energy edge-mode operators \( \sigma_i^+ \), \( \sigma_i^\dagger \), \( \sigma_i^\sigma_i \), and \( \sigma_i^\sigma_i^\dagger \). For \( h > 0 \) and \( \kappa = 0 \), these remain unchanged, and the other two evolve into \( \Sigma_1 = \sigma_i^z + \sum_{m=1}^{N-1} (h/J) \sigma_i^z \sigma_m^z \) and \( \Sigma_2 = \sum_{m=1}^{N} (h/J) \sigma_i^z \sigma_m^z \). These behave just as their Majorana relatives, yet they are recognized as nonlocal. The Majorana language distinguishes itself as the most local one for zero modes.

To obtain a generalized OP for the gauged Kitaev wire, notice that \( H_{0}^{\text{sp}}_{GK} \) reduces to the transverse-field Ising (TFI) chain. Hence it exhibits a second-order phase transition at \( h = h_c \). Consider now \( H_{0}^{\text{sp}}_{GK} \) of Eq. (4). At \( h = 0 \), it has zero-energy edge-mode operators \( \sigma_i^+ \), \( \sigma_i^\dagger \), \( \sigma_i^\sigma_i \), and \( \sigma_i^\sigma_i^\dagger \). For \( h > 0 \) and \( \kappa = 0 \), these remain unchanged, and the other two evolve into \( \Sigma_1 = \sigma_i^z + \sum_{m=1}^{N-1} (h/J) \sigma_i^z \sigma_m^z \) and \( \Sigma_2 = \sum_{m=1}^{N} (h/J) \sigma_i^z \sigma_m^z \). These behave just as their Majorana relatives, yet they are recognized as nonlocal. The Majorana language distinguishes itself as the most local one for zero modes.

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We will call system indices the link indices \((i,j); \mu = 1.2\) labeling \(R\) operators that explicitly appear in \(H_{ZG}\), and extra indices the remaining link indices. In the bulk, the plaquette operator \(b_{(i,j)}\) reads

\[
b_{(i,j)} \equiv X_{(i,j;1)} + X_{(i+1,j;2)} - X_{(i,j+1;3)} - X_{(i,j;2)}.
\]

On the lattice boundary, the plaquette operators are set by two rules: (i) \(b_{(1,N)} = X_{(1,N;1)} - X_{(2,N;2)} - X_{(1,N+1;1)}\). Thus, \(b_{(1,N)}\) involves one degree of freedom \(X_{(1,N+1;1)}\) labelled by an extra link index. (ii) The remaining boundary plaquettes are determined by Eq. (7) provided operators labelled by extra link indices are omitted. With these definitions in tow, the mapping of bonds

\[
b_{(i,j)} \xrightarrow{\phi_d} L_{(i,j)}, \quad 1 \leq i, j \leq N,
\]

\[
R_{(i,j;1)} \xrightarrow{\phi_d} S_{(i,j;1)}, \quad 1 \leq i \leq N, \quad 2 \leq j \leq N,
\]

\[
R_{(i,j;2)} \xrightarrow{\phi_d} S_{(i,j;2)}, \quad 2 \leq i \leq N, \quad 1 \leq j \leq N,
\]

implements the duality transformation \(H_{ZG} \xrightarrow{\phi_d} H_{XY}\). Because the operators \(R_{(1,N+1;1)}, R_{(1,N;2)}\) do not appear in \(H_{ZG}\), the operator \(X_{(1,N+1;1)}\) constitutes a boundary symmetry of \(H_{ZG}\). Similar to the duality between the one-dimensional theories of Eqs. (1) and (4), this is a gauge-reducing duality. The gauge symmetries of \(H_{ZG}\), given by \(A_{(i,j)} = R_{(i,j;1)} R_{(i,j;2)} R_{(i,j;1)} R_{(i,j;2)}\), \(2 \leq i, j \leq N\), are removed by the mapping since \(A_{(i,j)} \xrightarrow{\phi_d} 1\).

In the thermodynamic \((N \to \infty)\) limit, the strongly-coupled \((J \gg h)\) phase of the \(XY\) model displays spontaneous symmetry breakdown of its global \(U(1)\) symmetry with generator \(L_{XY} = \sum_{i,j=1}^{N} L_{(i,j)}\) as evinced by a nonvanishing \(\langle XY|S_{(i,j)}|XY\rangle\) in the limit \(|r - r'| \to \infty\). By virtue of being dual to the \(XY\) system, the gauge theory displays a nonanalyticity in its ground-state energy as \(h\) is varied and its symmetry is broken. However, the phase transition in the gauge theory cannot be characterized by a local OP. So, how can the duality connecting the two models bridge the drastic gap separating the physical interpretation of their common phase diagram? The answer lies in our notion of holography, since

\[
-X_{(1,N+1;2)} = \sum_{i,j=1}^{N} b_{(i,j)} \xrightarrow{\phi_d} L_{XY}.
\]

Thus, the global symmetry of the \(XY\) model is holographically dual to the (local) boundary symmetry \(X_{(1,N+1;2)}\) of its dual gauge theory and cannot be spontaneously broken in this dual theory. This is how holographic symmetries explain the non-Landau nature of critical transitions in the \(Z\) gauge theory. There are no edge modes nor localization associated with this holographic symmetry as the ordered phase of the \(XY\) model is gapless.

We now derive a generalized OP for the \(Z\) gauge theory. Let \(\Gamma\) be an oriented path from \(r\) to \(r'\) made of directed links \(I \in \Gamma\), and we adopt the convention that \(S_I \equiv S_{(r,r')}\) if \(I\) points...
from \( \mathbf{r} \) to \( \mathbf{r} + \mathbf{e}_\mu \), or \( S_l \equiv \hat{S}_l \), if \( \mathbf{l} \) points oppositely from \( \mathbf{r} + \mathbf{e}_\mu \) to \( \mathbf{r} \). Then \( S_{\mathbf{l}} \equiv \prod_{\mathbf{l} \in \Gamma^*} S_l \). Also let \( \Gamma^* \) denote the set of links \( \mathbf{l}^* \) such that \( \Phi_{\mathbf{d}}(R_{\mathbf{l}}) = S_l \) (\( \Gamma^* \) need not be continuous, as for instance in Fig. 2). Then

\[
\langle ZG | \prod_{\mathbf{l} \in \Gamma^*} R_{\mathbf{l}} | ZG \rangle \xrightarrow{\Phi_{\mathbf{d}}} \langle XY | s^1_l s^\dagger_l | XY \rangle, \tag{10}
\]

and so the string correlator on the left-hand side is a generalized OP for the \( Z \) gauge theory, displaying long-range order in the ordered phase. On a closed path, \( \prod_{\mathbf{l} \in \Gamma^*} R_{\mathbf{l}} \) reduces to a product of gauge symmetries.

Finally, we couple the \( Z \) gauge theory to a \( Z \) matter field (defined on sites \( \mathbf{r} \)), \( H_{Z\mathbf{M}} = H_{ZG} + H_{\mathbf{M}} \), with

\[
H_{\mathbf{M}} = \sum_{\mathbf{l}} \left[ \lambda (R_{\mathbf{l}} + R_{\mathbf{l}}^*) + \kappa \sum_{\mu=1,2} l^2 (r_{\mu;l}) \right], \tag{11}
\]

and \( l(r_{\mu;l}) \equiv X_{r_{\mu}+r_{\mu}^*} - q X(r_{\mu}) - X_r \). The resulting matter-coupled theory \( H_{Z\mathbf{M}} \) is dual to the Abelian Higgs model Hamiltonian

\[
H_{\text{AH}} = \sum_{\mathbf{l}} \left[ \lambda (B_{\mathbf{l}} + B_{\mathbf{l}}^*) + h L_{\mathbf{l}} \right]
+ \sum_{\mu=1,2} \kappa L^2_{\mathbf{l}} (r_{\mu;l}) + \frac{j}{2} (s_{\mathbf{l}}(r_{\mu;l}) + s_{\mathbf{l}}(r_{\mu;l})) \right] \tag{12}
\]

Here \( s_{\mathbf{l}}(r_{\mu;l}) \equiv \hat{s}^1_{\mathbf{l}}(r_{\mu;l}) + \hat{s}^1_{\mathbf{l}}(r_{\mu;l}) \) includes a coupling with integer charge \( q \) to the \( U(1) \) gauge field \( s(r_{\mu;l}) \equiv e^{-iq \theta_{\mathbf{r}}}, \hat{s}^q_{\mathbf{l}} \equiv e^{-iq \Theta_{\mathbf{r}}}, \) and \( B_{\mathbf{l}} \equiv \hat{s}^q_{\mathbf{l}}(r_{\mu;l}) \). The correspondence between the two models, established by the mapping of bonds

\[
R_{\mathbf{r}} \xrightarrow{\Phi_{\mathbf{d}}} R_{(r;1)} \xrightarrow{\Phi_{\mathbf{d}}} R_{(r;2)}, \quad R_{(r;1)} \xrightarrow{\Phi_{\mathbf{d}}} \frac{\langle \text{AH} | s^1_{\mathbf{l}} \prod_{\mathbf{l} \in \Gamma} s^q_{\mathbf{l}} | \text{AH} \rangle}{\langle \text{AH} | \prod_{\mathbf{l} \in \Gamma^*} s^{q\dagger}_{\mathbf{l}} | \text{AH} \rangle}, \tag{13}
\]

which holds only on physical gauge-invariant states. The reason is that \( \Phi_{\mathbf{d}} \) preserves all commutation relations while “trivializing” all gauge symmetries. More precisely, \( H_{Z\mathbf{M}} \) ’s gauge symmetries \( G_{\mathbf{r}} = R_{\mathbf{r}} A_{\mathbf{r}} \) map to \( \Phi_{\mathbf{d}}(G_{\mathbf{r}}) = 1 \), while \( H_{\text{AH}} \)’s gauge generators \( g_{\mathbf{r}} = L_{(r;1)} + L_{(r;2)} - L_{(r;1)} - L_{(r;2)} + q L_r \) map to \( \Phi_{\mathbf{d}}(g_{\mathbf{r}}) = 0 \) as follows from Eqs. (13) \( [\Phi_{\mathbf{d}}] \) is the mapping obtained from Eqs. (13) by reversing all the arrows.

If the \( Z \) matter field is weakly coupled to the \( Z \) gauge field, the string correlator of Eq. (10) will still change analytic behavior across transitions. Then, from Eqs. (13),

\[
\langle ZG | \prod_{\mathbf{l} \in \Gamma^*} R_{\mathbf{l}} | ZG \rangle \xrightarrow{\Phi_{\mathbf{d}}} \langle \text{AH} | s^1_{\mathbf{l}} \prod_{\mathbf{l} \in \Gamma} s^q_{\mathbf{l}} | \text{AH} \rangle, \tag{14}
\]

we obtain a witness correlator for the Abelian Higgs model that reduces to a Wilson loop on closed contours (\( \mathbf{r} = r' \)) (here \( s^q_{\mathbf{l}} = s^q_{\mathbf{r}(r;\mu)} \) if a link \( \mathbf{l} \) points from \( \mathbf{r} + \mathbf{e}_\mu \) and \( s^q_{\mathbf{l}} = s^q_{\mathbf{r}(\mu;\mathbf{r})} \) otherwise). This nonlocal correlator is directly related to intuitively motivated generalized OPs like \( \langle \text{AH} | s^1_{\mathbf{l}} \prod_{\mathbf{l} \in \Gamma} s^q_{\mathbf{l}} | \text{AH} \rangle \prod_{\mathbf{l} \in \Gamma^*} s^{q\dagger}_{\mathbf{l}} | \text{AH} \rangle \) conjectured in earlier work \(^6,20,21 \) (\( \Gamma_C \) denotes a closed loop roughly twice as long as \( \Gamma \) and containing \( \Gamma \) as a proper segment).

Outlook. As demonstrated, holographic symmetries and generalized OPs appear in numerous systems once boundary conditions are properly accounted for in the framework of bond-algebraic dualities. By providing a systematic methodology and many examples, our results might bring the theory of generalized OPs and topological orders to a new level of development closer to that of Landau’s theory. More key problems need to be tackled. First, the sufficient conditions under which a given topological order may be mapped to a Landau order and vice versa should be understood. Second, the problem of associating effective field theories to generalized OPs should be studied systematically.

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holographic symmetries and a derivation of generalized order OPs for the extended toric code and other topologically-ordered models.

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