THE \((S,\{2\})\)-IWASAWA THEORY

SU HU AND MIN-SOO KIM

Abstract. Iwasawa made the fundamental discovery that there is a close connection between the ideal class groups of \(\mathbb{Z}_p\)-extensions of cyclotomic fields and the \(p\)-adic analogue of Riemann’s zeta functions

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]

In this paper, we show that there may also exist a parallel Iwasawa’s theory corresponding to the \(p\)-adic analogue of Euler’s deformation of zeta functions

\[ \phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \]

1. Introduction

Throughout this paper we shall use the following notations.

- \(\mathbb{C}\) – the field of complex numbers.
- \(p\) – an odd rational prime number.
- \(\mathbb{Z}_p\) – the ring of \(p\)-adic integers.
- \(\mathbb{Q}_p\) – the field of fractions of \(\mathbb{Z}_p\).
- \(\mathbb{C}_p\) – the completion of a fixed algebraic closure \(\overline{\mathbb{Q}}_p\) of \(\mathbb{Q}_p\).

Before Kubota, Lepodlt and Iwasawa, all the zeta functions are considered in the complex field \(\mathbb{C}\).

For \(\text{Re}(s) > 1\), the Riemann zeta function is defined by

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]

This function can be analytic continuous to a meromorphic function in the complex plane with a simple pole at \(s = 1\).

For \(\text{Re}(s) > 0\), the alternative series (also called the Dirichlet eta function or Euler zeta function) is defined by

\[ \phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \]

This function can be analytic continuous to the complex plane without any pole.

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For $\text{Re}(s) > 1$, (1.1) and (1.2) are connected by the following equation
\begin{equation}
\phi(s) = (1 - 2^{1-s})\zeta(s).
\end{equation}

According to Weil’s history [38, p. 273–276] (also see a survey by Goss [7, Section 2]), Euler used (1.2) to investigate (1.1). In particular, he conjectured (“proved”)
\begin{equation}
\frac{\phi(1-s)}{\phi(s)} = \frac{-\Gamma(s)(2^s - 1)\cos(\pi s/2)}{(2^s - 1)\pi s},
\end{equation}
this leads to the functional equation of $\zeta(s)$.

For $0 < x \leq 1$, $\text{Re}(s) > 1$, in 1882, Hurwitz [9] defined the partial zeta functions
\begin{equation}
\zeta(s, x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^s}
\end{equation}
which generalized (1.1). As (1.1), this function can also be analytic continuous to a meromorphic function in the complex plane with a simple pole at $s = 1$.

For $0 < x \leq 1$, $\text{Re}(s) > 0$, Lerch [24] generalized (1.2) to define the so-called Lerch zeta functions. The following (we call it “Hurwitz-type Euler zeta function”) is a special case of Lerch’s definition
\begin{equation}
\zeta_E(s, x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+x)^s}.
\end{equation}
As (1.2), this function can be analytic continues to the complex plane without any pole.

Now we go on our story in the $p$-adic complex plane $\mathbb{C}_p$.

In 1964, Kubota and Leopoldt [12] first defined the $p$-adic analogue of (1.1). In fact, they defined the $p$-adic zeta functions by interpolating the special values of (1.1) at nonpositive integers.

In 1975, Katz [13, Section 1] defined the $p$-adic analogue of (1.2) by interpolating the special values of (1.2) at nonpositive integers.

In 1976, Washington [36] defined the $p$-adic analogue of (1.5) for $x \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, so called Hurwitz-Washinton functions (see Lang [23, p. 391]). This definition has been generalized to $\mathbb{C}_p$ by Cohen in his book [11 Chapter 11], and Tangedal-Young in [31]. Both Cohen, Tangedal-Young’s definitions are based on the following $p$-adic representation of Bernoulli polynomials by the Volkenborn integral
\begin{equation}
\int_{\mathbb{Z}_p} (x + a)^n dx = B_n(x),
\end{equation}
where the Bernoulli polynomials are defined by the following generating function
\begin{equation}
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\end{equation}
and the Volkenborn integral of any uniformly differentiable function \( f \) on \( \mathbb{Z}_p \) is defined by

\[
(1.9) \quad \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)
\]

(see [26, p. 264]). This integral was introduced by Volkenborn [34] and he also investigated many important properties of \( p \)-adic valued functions defined on the \( p \)-adic domain (see [34, 35]).

The Euler polynomials are defined by the following generating function

\[
(1.10) \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}
\]

(see [29, 20]). They are the special values of (1.6) at nonpositive integers (see Choi-Srivastava [2, p. 520, Corollary 3] and T. Kim [16, p. 4, (1.22)]) and can be representative by the fermionic \( p \)-adic integral as follows

\[
(1.11) \quad \int_{\mathbb{Z}_p} (x + a)^n d\mu_{-1}(a) = E_n(x),
\]

where the fermionic \( p \)-adic integral \( I_{-1}(f) \) on \( \mathbb{Z}_p \) is defined by

\[
(1.12) \quad I_{-1}(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-1}(a) = \lim_{N \to \infty} \sum_{a=0}^{p^N-1} f(a)(-1)^a.
\]

The above representation (1.11) and the fermionic \( p \)-adic integral (1.12) (in our notation, the \( \mu_{-1} \) measure) were independently founded by Katz [13, p. 486] (in Katz’s notation, the \( \mu^{(2)} \)-measure), Shiratani and Yamamoto [28], Osipov [25], Koblitz [14], Lang [23] (in Lang’s notation, the \( E_{1,2} \)-measure), T. Kim [19] from very different viewpoints. It seems that there is no simple connection as (1.6) between the fermionic and Volkenborn \( p \)-adic integrals [4].

Following Cohen [1, Chapter 11] and Tangedal-Young [31], using the fermionic \( p \)-adic integral instead of the Volkenborn integral, we [21] defined \( \zeta_{p,E}(s, x) \), the \( p \)-adic analogue of (1.6), which interpolates (1.6) at nonpositive integers (21 Theorem 3.8(2)), so called the \( p \)-adic Hurwitz-type Euler zeta functions. We also proved many fundamental results for the \( p \)-adic Hurwitz type Euler zeta functions, including the convergent Laurent series expansion, the distribution formula, the functional equation, the reflection formula, the derivative formula and the \( p \)-adic Raabe formula. Using these zeta function as building blocks, we have given a definition for the corresponding \( L \)-functions \( L_{p,E}(\chi, s) \), so called \( p \)-adic Euler \( L \)-functions (in fact, this \( L \)-function has already founded by Katz in [13, p. 483] using Kubta-Lepoldt’s methods on the interpolation of \( L \)-functions at special values). The Hurwitz-type Euler zeta functions interpolate Euler polynomials \( p \)-adically (21 Theorem 3.8(2)), while the \( p \)-adic Euler \( L \)-functions interpolate the generalized Euler numbers \( p \)-adically (21 Proposition 5.9(2))).

In a subsequent work [22], using the fermionic \( p \)-adic integral, we defined the corresponding \( p \)-adic Diamond Log Gamma functions. We call them the
$p$-adic Diamond-Euler Log Gamma functions. They share most proper ties of the original $p$-adic Diamond Log Gamma functions as stated in Lang’s book (see [23, p. 395–396, G$_p$,1-5 and Theorem 4.5]). Furthermore, using the $p$-adic Hurwitz-type Euler zeta functions, we found that the derivative of the $p$-adic Hurwitz-type Euler zeta functions $\zeta_{p,E}(\chi, s)$ at $s = 0$ may be represented by the $p$-adic Diamond-Euler Log Gamma functions. This led us to connect the $p$-adic Hurwitz-type Euler zeta functions to the $(S, \{2\})$-version of the abelian rank one Stark conjecture (see [22, Chapter 6]).

It has been pointed out that some properties for the $q$-analogue of $p$-adic Euler zeta and $L$-functions have also been obtained by T. Kim (see [15, 17, 18]).

The $p$-adic zeta ($L$-) functions become central themes in algebraic number theory after Iwasawa’s work. In [10], Iwasawa made the fundamental discovery that there is a close connection between his work on the ideal class groups of $\mathbb{Z}_p$-extensions of cyclotomic fields and the $p$-adic analogue of $L$-functions by Kubota-Leopoldt corresponding to (1.1).

Let $\mathbb{Q}(\mu_{p^{n+1}})$ denote the $p^{n+1}$-th cyclotomic field. In fact, Iwasawa [11] and Ferrero-Washington [5] proved the following results.

**Theorem 1.1** (See Lang [23, p. 260]). Let $h_n$ be the class number of $\mathbb{Q}(\mu_{p^{n+1}})$. There exist constants $\lambda$ and $c$ such that

\begin{equation}
\text{ord}_p h_n^- = \lambda n + c.
\end{equation}

for all sufficient large $n$.

Let $K$ be a number field, and choose a finite set $S$ of places $K$ containing all the archimedean places. Let $T$ be a finite set of places of $K$ disjoint from $S$. The $(S,T)$-class groups of global fields have been studied in detail by Rubin [27], Tate [30], Gross [8], Darmon [3], Vallieres [32, 33] (we shall recall some notations on the $(S,T)$-refined class groups of global fields in the next section). Let $K = \mathbb{Q}(\mu_{p^{n+1}})$ and $K^+ = \mathbb{Q}(\mu_{p^{n+1}})^+$ be the $p^{n+1}$-th cyclotomic field and its maximal real subfield, respectively. Let $S$ be the set of infinite places of $K$, $T$ be set of the places above 2, $h_{n,2}$ and $h_{n,2}^+$ be the $(S,T)$-refined class numbers of $K$ and $K^+$ respectively (the definition will be given in the next section), and $h_{n,2}^- = h_{n,2}/h_{n,2}^+$.

Using the $p$-adic analogue of $L$-functions corresponding to Euler’s deformation of zeta functions [17,2], We shall prove the following result (comparing with Theorem 1.1).

**Theorem 1.2** ($(S, \{2\})$-Iwasawa theory). There exist constants $m$, $\lambda$ and $c$ such that

\begin{equation}
\text{ord}_p h_{n,2}^- = mp^n + \lambda n + c
\end{equation}

for all sufficient large $n$.

Our paper is organized as follows.

In Section 2, we shall recall some notations and results on the $(S,T)$-refined class groups of global fields. In Section 3, from the Euler product decompositions of the $(S,T)$-Dedekind zeta functions, we shall express $h_{n,2}^-$
as the product of generalized Euler numbers. In section 4, we shall prove Theorem 1.2.

2. \((S,T)\)-refined class number formula ([8, Section 1])

In this section, we shall some notations and results on the \((S,T)\)-refined class groups of global fields following very closely the exposition of Gross in [8, Section 1].

Let \(k\) be a global field, and let \(S\) be a finite set of places of \(k\) which is non-empty and contains all archimedean places. Let \(A\) denote the \(S\)-integers of \(k\) and let \(U_S = A^*\) be the groups of \(S\)-units. The class group \(\text{Pic}(A)_S\) is finite of order \(h\), and the unit group \(U\) is finitely generated of rank \(n = \#S - 1\). The torsion subgroup of \(U_S\) is equal to the group of roots of unity \(\mu\) in \(k\); it is cyclic of order \(w\).

Let \(Y\) be the free abelian group generated by the places \(v \in S\) and \(X = \{\sum a_v \cdot v : \sum a_v = 0\}\) the subgroup of elements of degree zero in \(Y\). The \(S\)-regulator \(R\) is defined as the absolute value of the determinant of the map

\[
\lambda_R : U \to R \bigotimes X
\]

\[
\epsilon \mapsto \sum_S \log \|\epsilon\|_v \cdot v,
\]

taken with respect to \(\mathbb{Z}\)-bases of the free abelian groups \(U_S/\mu_S\) and \(X\).

The zeta-function of \(A\) is given by

\[
\zeta_S(s) = \prod_{p \notin S} \frac{1}{1 - Np^{-s}}
\]

in the half plane \(\text{Re}(s) > 1\). It has a meromorphic continuation to the \(s\)-plane, with a simple pole at \(s = 1\) and no other singularities. At \(s = 0\) the Taylor expansion begins:

\[
\zeta_S(s) \equiv \frac{-hR}{w} \cdot s^n \text{ (mod } s^{n+1}).
\]

Let \(T\) be a finite set of places of \(k\) which is disjoint from \(S\), and define

\[
\zeta_{S,T}(s) = \prod_{p \in T} (1 - Np^{1-s}) \cdot \zeta_S(s),
\]

we shall call it the \((S,T)\)-refined zeta function of \(k\) throughout this paper. Let \(U_{S,T}\) denote the subgroup of units which are \(\equiv 1 \text{ (mod } T\) and let \(\text{Pic}(A)_{S,T}\) be the group of invertible \(A\)-modules together with a trivialization at \(T\). We have an exact sequence

\[
1 \to U_T \to U \to \prod_{p \in T} F_p^* \to \text{Pic}(A)_{S,T} \to \text{Pic}(A) \to 1.
\]

Let \(h_{S,T}\) be the order of \(\text{Pic}(A)_{S,T}\) (we call it the \((S,T)\)-refined class number throughout this paper), \(R_{S,T}\) be the determinant of \(\lambda\) with respect to basis of \(U_{S,T}/\mu_{S,T}\) and \(X\), and \(w_{S,T}\) be the order of roots of unity \(\mu_{S,T}\) which are
\equiv 1 \pmod{T}, we have the following \((S,T)\)-refined class number formula due to Gross \[8\]

\(\zeta_{S,T} \equiv \frac{-h_{S,T}R_{S,T}}{w_{S,T}} \cdot s^n \pmod{s^{n+1}}.\)

3. Refined class number and the generalized Euler numbers

Let \(K = \mathbb{Q}(\mu_{p^{n+1}})\) and \(K^+ = \mathbb{Q}(\mu_{p^{n+1}})^+\) be the \(p^{n+1}\)-th cyclotomic field and its maximal real subfield, respectively. Let \(S\) be the set of infinite places of \(K\), \(T\) be set of the places above 2, \(h_{n,2}, h^+_{n,2}, U_{n,2}, U^+_{n,2}, \mu_{n,2}, \mu^+_{n,2}, w_{n,2}, w^+_{n,2}, R_{n,2}, R^+_{n,2}\) denote all the quantities or objects of \(K\) and \(K^+\) which are refined by \(T\) as in the above Section. Let \(\zeta_{K,2}(s)\) be the \((S,T)\)-zeta function of \(K\) (see (2.4)), and

\[
L_E(s, \chi) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^s}
\]

be the Dirichlet \(L\)-function corresponding to (1.2) (we call them the Euler \(L\)-functions throughout this paper). This function has close connection with the generalized Euler numbers. In \[21\] Section 5.3, using formal power series expansions, we recalled the definition and some results on generalized Euler numbers. The Propositions 5.2 and 5.3 of \[21\] correspond to properties (4) and (5) of the generalized Bernoulli numbers in Iwasawa’s book \[11\] p. 10–11 (for details we also refer to \[19\] Sections 1 and 2)). \[19\] Theorem 3.5 represents the special values of Euler \(L\)-functions at non-positive integers as the generalized Euler numbers which corresponds to Iwasawa’s book \[11\] p. 11, Theorem 1] for the relationship between the Dirichlet \(L\)-functions and the generalized Bernoulli numbers.

We have the following decomposition of \((S,\{2\})\)-refined Dedekind zeta functions as the Euler \(L\)-functions (comparing with the last formula on \[23\] p. 75]).

**Proposition 3.1.**

\[
\zeta_{K,2}(s) = \prod_{\chi}^{\frac{1}{2}} L_E(s, \chi),
\]

where the product is taken over all the primitive characters induced by the characters of \(\text{Gal}(K/\mathbb{Q})\).

**Proof.** From the last formula on \[23\] p. 75], we have

\[
\zeta_K(s) = \prod_{\chi} L(s, \chi).
\]

By \[24\], we have

\[
\zeta_{K,2}(s) = \prod_{p \in T} (1 - Np^{1-s}) \zeta_K(s).
\]
For any Dirichlet character $\chi$ of $\text{Gal}(K/\mathbb{Q})$, 
\[(3.5)\quad L(s, \chi) = \prod \left(1 - \frac{\chi(q)}{q^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},\]
where the product is taken over all primes $q$ such that $(q, p) = 1$ ([23, p. 76]).

By the following identity in [23, p. 76]:
\[(1 - t^f)^r = \prod \chi(1 - \chi(p)t),\]
we have
\[(3.6)\quad \prod_{p \in T} (1 - Np^{1-s}) = (1 - 2(1-s)f)^r = \prod_{\chi} (1 - \chi(2)2^{1-s}).\]

Combine (3.4), (3.5) and (3.6), we have
\[(3.7)\quad \zeta_{K,2}(s) = \prod_{p \in T} (1 - Np^{1-s})\zeta_K(s)
= \prod_{\chi} (1 - \chi(2)2^{1-s})L(s, \chi)
= \prod_{\chi} \left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} - \sum_{n=1}^{\infty} \frac{\chi(2n)}{2n^s}\right)
= \prod_{\chi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\chi(n)}{n^s}
= (-1)^{\varphi(p^{n+1})} \prod_{\chi} \frac{1}{2} \cdot 2 \sum_{n=1}^{\infty} (-1)^n \frac{\chi(n)}{n^s}
= \prod_{\chi} \frac{1}{2} L_E(s, \chi).\]

For the $(S, \{2\})$-refined zeta function of $K^+$, we have the following decomposition.

**Proposition 3.2.**
\[(3.8)\quad \zeta_{K^+,2}(s) = (-1)^{\varphi(p^{n+1})} \prod_{\chi} \frac{1}{2} L_E(s, \chi).\]

Now we express $h^{-n}_{n,2}$ as the product of generalized Euler numbers (comparing with [23, Theorem 3.2]).

**Proposition 3.3.**
\[(3.9)\quad h^{-n}_{n,2} = (-1)^{\varphi(p^{n+1})} 2^{1-\varphi(p^{n+1})} \prod_{\chi \text{ odd}} E_{0,\chi},\]
where $E_{0,\chi}$ are the generalized Euler numbers ([21, Section 5.1]).
Remark 3.4. From this, we also see that $E_{0,\chi} \neq 0$, when $\chi$ is an odd character. In fact, $E_{0,\chi} \neq 0$ if and only if $\chi$ is an odd character by [21, Proposition 5.1], this phenomenon is different from the generalized Bernoulli number $B_{0,\chi}$, since $B_{0,\chi} = 0$, for $\chi \neq \chi_0$, but corresponds to $B_{1,\chi}$, for details, we refer to [11, p.13, ii)].

Proof. By the exact sequence (2.5) above and $h_n^+ \mid h_n$, we know that $h_{n,2}^+ \mid h_{n,2}$.

By Propositions 3.1 and 3.2, we have

$$\zeta_{K,2}(s) = (-1)^{(p^{n+1}/2)} \prod_{\chi \text{ odd}} \frac{1}{2} L_E(s, \chi).$$

From the $(S,T)$-refined class number formula (2.6) and (3.10), we have

$$\frac{h_{n,2}R_{n,2}}{w_{n,2}} = \lim_{s \to 0} \frac{\zeta_{K,2}(s)}{\zeta_{K+2}(s)} = (-1)^{(p^{n+1}/2)} \prod_{\chi \text{ odd}} \frac{1}{2} L_E(0, \chi).$$

By Corollary 4.13 and Lemma 3.15 of [37], we have $R_{n,2}/R_{n,2}^+ = 2 \frac{\varphi(p^{n+1})}{2} - 1$.

It also easy to see $\mu_{n,2} = \mu_{n}^+ = (-1)$. By [19] Theorem 3.5, we have $L_E(0, \chi) = E_{0,\chi}$. Thus by (3.11), we have

$$\frac{h_{n,2}^+}{h_{n,2}} = (-1)^{(p^{n+1}/2)} 2^{1-\varphi(p^{n+1})} \prod_{\chi \text{ odd}} E_{0,\chi}.$$

This implies our result. \hfill \Box

4. Proof of the main result

Let $\chi$ be a Dirichlet character modulo $p^v$ for some $v$. We can extend the definition of $\chi$ to $\mathbb{Z}_p$ as in [11, p. 281], that is, if $a_n \in \mathbb{Z}$ and $a_n$ is a sequence tending to a $p$-adically, we have $v_p(a_n - a_m) \geq v$ for $n$ and $m$ sufficiently large, so $\chi(a_n)$ is an ultimately constant sequence, and we set $\chi(a) = \chi(a_n)$ for $v_p(a - a_n) \geq v$. $\chi$ is called a Dirichlet character on $\mathbb{Z}_p$.

As in Lang [23, p. 248]. For any $p$-adic measure $\mu$ on $\mathbb{Z}_p$, let

$$B(\chi, \mu) = \int_{\mathbb{Z}_p} \chi(x) d\mu(x),$$

by using Iwasawa power series, Lang gave the following two results.

Lemma 4.1 (See Lang [23, p. 248, Corollary 2]). There exists a positive integer $n_0$ such that if $n \geq n_0$ and $\text{Cond } \chi = p^n$, then

$$B(\chi, \mu) \sim p^m (\zeta - 1)^{\lambda},$$

where $\zeta$ is a primitive $p^n$th root of unity.
**Lemma 4.2** (See Lang [23, p. 249, Corollary 3]). *For some constant c, we have*

\[ \text{ord}_p \prod_{\text{Cond } \chi = p^n} B(\chi, \mu) = mp^n + \lambda n + c. \]

For \( n \geq 0 \), \( E_{n, \chi} \) be the generalized Euler numbers which was defined in [21, Section 5.1]. By [21, Proposition 5.4(2)], we have

\[ E_{0, \chi} = B(\chi, \mu-1) = \int_{\mathbb{Z}_p} \chi(x) d\mu_{-1}(x). \]

From the above lemmas, we have the following results.

**Proposition 4.3.** *There exists a positive integer \( n_0 \) such that if \( n \geq n_0 \) and \( \text{Cond } \chi = p^n \), then*

\[ E_{0, \chi} \sim p^m (\zeta - 1)\lambda, \]

*where \( \zeta \) is a primitive \( p^n \)th root of unity.*

**Proposition 4.4.** *For some constant c, we have*

\[ \text{ord}_p \prod_{\text{Cond } \chi = p^n} E_{0, \chi} = mp^n + \lambda n + c. \]

Finally, by Propositions 3.3 and 4.4, we obtain Theorem 1.2.

**References**

[1] H. Cohen, *Number Theory Vol. II: Analytic and Modern Tools*, Graduate Texts in Mathematics, 240, Springer, New York, 2007.

[2] J. Choi, H. M. Srivastava, *The multiple Hurwitz zeta function and the multiple Hurwitz-Euler eta function*, Taiwanese J. Math. 15 (2011), 501–522.

[3] H. Darmon, *Thaine’s method for circular units and a conjecture of Gross*, Canad. J. Math. 47 (1995), 302–317.

[4] D. Delbourgo, *The convergence of Euler products over \( p \)-adic number fields*, Proc. Edinb. Math. Soc. 52 (2009), 583–606.

[5] B. Ferrero, L. C. Washington, *The Iwasawa invariant \( \mu_p \) vanishes for abelian number fields*, Ann. of Math. 109 (1979), 377–395.

[6] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. 98 (1976), 263–284.

[7] D. Goss, *Zeros of \( L \)-series in characteristic \( p \)*, [http://arxiv.org/abs/math/0601717](http://arxiv.org/abs/math/0601717).

[8] B. Gross, *On the values of abelian \( L \)-functions at \( s = 0 \)*, J. Fac. Sci. Univ. Tokyo 35 (1988), 177–197.

[9] A. Hurwitz, *Einige Eigenschaften der Dirichletsehen Funktionen \( F(s) = \sum (\frac{1}{n}) \cdot \frac{1}{n^s} \), die bei der Bestimmung der Klassenzahlen Binärer quadratischer Formen auftreten*, Z. für Math. und Physik 27 (1882), 86–101 (in German).

[10] K. Iwasawa, *On \( p \)-adic \( L \)-functions*, Annals Math. 89 (1969), 198–205.

[11] K. Iwasawa, *Lectures on \( p \)-Adic \( L \)-Functions*, Ann. of Math. Stud. 74, Princeton Univ. Press, Princeton, 1972.

[12] T. Kubota and H.-W. Leopoldt, *Eine \( p \)-adische Theorie der Zetafunktion*, J. Reine Angew. Math. 214/215 (1964), 328–339 (in German).

[13] N. M. Katz, *\( p \)-adic \( L \)-functions via moduli of elliptic curves*, Algebraic geometry, Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974, pp. 479–506, Amer. Math. Soc., Providence, R. I., 1975.
[14] N. Koblitz, *A new proof of certain formulas for p-adic L-functions*, Duke Math. J. 46 (1979), 455–468.
[15] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on Z_p at q = −1*, J. Math. Anal. Appl. 331 (2007), 779–792.
[16] T. Kim, *Euler numbers and polynomials associated with Zeta functions*, Abstract and Applied Analysis, Article ID 581582, 2008.
[17] T. Kim, *A note on the q-analogue of p-adic log-gamma function*, http://arxiv.org/abs/0710.4981v1
[18] T. Kim, *On p-adic interpolating function for q-Euler numbers and its derivatives*, J. Math. Anal. Appl. 339 (2008), 598–608.
[19] M.-S. Kim, *On the behavior of p-adic Euler L-functions*, [arXiv:1010.1981](http://arxiv.org/abs/1010.1981)
[20] M.-S. Kim, *On Euler numbers, polynomials and related p-adic integrals*, J. Number Theory 129 (2008), 2166–2179.
[21] M.-S. Kim and S. Hu, *On p-adic Hurwitz-type Euler zeta functions*, J. Number Theory 132 (2012), 2977–3015.
[22] M.-S. Kim and S. Hu, *On p-adic Diamond-Euler Log Gamma functions*, J. Number Theory 133 (2013), 4233–4250.
[23] S. Lang, *Cyclotomic Fields I and II*, Combined 2nd ed., Springer-Verlag, New York, 1990.
[24] M. Lerch, *Note sur la fonction ζ(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(w+k)^s}* Acta Mathematica 11 (1887), 19–24 (in French).
[25] Ju. V. Osipov, *p-adic zeta functions*, Uspekhi Mat. Nauk 34 (1979), 209–210 (in Russian).
[26] A. M. Robert, *A course in p-adic analysis*, Graduate Texts in Mathematics, 198, Springer-Verlag, New York, 2000.
[27] K. Rubin, *A Stark conjecture over Z for abelian L-functions with multiple zeros*, Ann. Inst. Fourier (Grenoble) 46 (1996), 33–62.
[28] K. Shiratani and S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci. Kyushu Univ. Ser. A 39 (1985), 113–125.
[29] Z.-W. Sun, *Introduction to Bernoulli and Euler polynomials*, A Lecture Given in Taiwan on June 6, 2002. [http://math.nju.edu.cn/~zwsun/BerE.pdf](http://math.nju.edu.cn/~zwsun/BerE.pdf)
[30] J. Tate, *Les Conjectures de Stark sur les Fonctions L d’Artin en s = 0 (notes par D. Bernardi et N. Schappacher)*, Progr. Math., vol. 47, Birkhäuser, Boston, 1984 (in French).
[31] B. A. Tangedal and P. T. Young, *On p-adic multiple zeta and log gamma functions*, J. Number Theory 131 (2011), 1240–1257.
[32] D. Vallières, *On a generalization of the rank one Rubin-Stark conjecture*, Ph.D. Thesis, University of California, San Diego, 2011. [http://www.math.binghamton.edu/vallieres/](http://www.math.binghamton.edu/vallieres/)
[33] D. Vallières, *On a generalization of the rank one Rubin-Stark conjecture*, J. Number Theory 132 (2012), 2535–2567.
[34] A. Volkenborn, *Ein p-adisches Integral und seine Anwendungen I*, Manuscripta Math. 7 (1972), 341–373.
[35] A. Volkenborn, *Ein p-adisches Integral und seine Anwendungen II*, Manuscripta Math. 12 (1974), 17–46.
[36] L. C. Washington, *A note on p-adic L-functions*, J. Number Theory 8 (1976), 245–250.
[37] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Springer-Verlag, New York, 1997.
[38] A. Weil, *Number theory, An approach through history*, From Hammurapi to Legendre, Birkhäuser Boston, Inc., Boston, MA, 1984.
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Department of Mathematics and Statistics, McGill University, 805 Sherbrooke St. West, Montréal, Québec, H3A 2K6, Canada

E-mail address: hus04@mails.tsinghua.edu.cn, hu@math.mcgill.ca

Division of Cultural Education, Kyungnam University, 7(Woryeong-dong) Kyungnamdaehak-ro, Masanhappo-gu, Changwon-si, Gyeongsangnam-do 631-701, South Korea

E-mail address: mskim@kyungnam.ac.kr