A NECESSARY CONDITION OF PONTRYAGIN TYPE FOR FUZZY FRACTIONAL OPTIMAL CONTROL PROBLEMS

OMID S. FARD AND JAVAD SOOLAKI
Department of Applied Mathematics
School of Mathematics and Computer Science
Damghan University, Damghan, Iran

DELFIM F. M. TORRES*
Center for Research & Development in Mathematics and Applications (CIDMA)
Department of Mathematics, University of Aveiro
3810–193 Aveiro, Portugal

Abstract. We prove necessary optimality conditions of Pontryagin type for a class of fuzzy fractional optimal control problems with the fuzzy fractional derivative described in the Caputo sense. The new results are illustrated by computing the extremals of three fuzzy optimal control systems, which improve recent results of Najariyan and Farahi.

1. Introduction. Optimal control problems are usually solved with the help of the famous Pontryagin Maximum Principle (PMP), which provides a generalization of the classical Euler–Lagrange and Weierstrass necessary optimality conditions of the calculus of variations and is one of the central results of the mathematics of the XX century [30, 32]. On the other hand, fractional (noninteger order) derivatives play an increasing role in mathematics, physics and engineering [17, 19, 23, 31]. The two subjects have recently been put together and a theory of the calculus of variations and optimal control that deals with more general systems containing noninteger order derivatives is now available: see the books [2, 21, 29]. In particular, the fractional Hamiltonian perspective is a very active subject, being investigated in a series of publications: see, e.g., [4, 7, 13, 24, 25, 33, 39, 40].

Uncertainty is inherent to most real world systems and fuzziness is a kind of uncertainty very common in real world problems [16]. In recent years, the notion of fuzzy set has been widely spread to various research areas, such as linear programming, optimization, differential equations and even fractional differential equations [34]. Thus, the study of a fuzzy optimal control theory forms a suitable setting for the mathematical modelling of real world problems in which uncertainties or vagueness pervade [14]. In the past few decades, the interest in the field of fuzzy optimal control has increased and fuzzy optimal control problems have attracted a great deal of attention. A large number of existing schemes of fuzzy optimal control for

2010 Mathematics Subject Classification. Primary: 26A33, 93C42; Secondary: 49K05.
Key words and phrases. Fractional calculus, fuzzy control systems, fuzzy optimal control, fuzzy fractional Hamiltonian function, Pontryagin maximum principle.

This work is part of second author’s PhD project. It was partially supported by Damghan University, Iran; and CIDMA–FCT, Portugal, under project UID/MAT/04106/2013.

*Corresponding author: Delfim F. M. Torres (delfim@ua.pt).
nonlinear systems are proposed based on the framework of the Takagi–Sugeno (T-S) fuzzy model originated from fuzzy identification \[38\]. Moreover, for most of the T-S modelled nonlinear systems, fuzzy control design is carried out by the aid of the parallel distributed compensation (PDC) approach \[42\]. However, it is still possible to enumerate all works that establish necessary optimality conditions for the fuzzy calculus of variations or fuzzy optimal control: see \[8, 10, 11, 12, 26, 27, 36, 37\].

In \[26, 27\], Najariyan and Farahi obtain necessary optimality conditions of Pontryagin type for a very special case of fuzzy optimal control problems, using \(\alpha\)-cuts and presentation of numbers in a more compact form by moving to the field of complex numbers. The authors of \[26, 27\] study the following fuzzy optimal control problem subject to a time-invariant linear control system:

\[
\mathcal{J}(\tilde{u}) = \int_{a}^{b} \tilde{L}(\tilde{u}(t), t) \, dt \longrightarrow \min,
\]

\[
\dot{\tilde{x}}(t) = \tilde{A} \odot \tilde{x}(t) + \tilde{C} \odot \tilde{u}(t),
\]

\[
\tilde{x}(a) = \tilde{x}_a, \quad \tilde{x}(b) = \tilde{x}_b.
\]

In \[12\], Farhadinia applies the fuzzy variational approach of \[11\] to fuzzy optimal control problems and derives necessary optimality conditions for fuzzy optimal control problems that depend on the Buckley and Feuring derivative (a Hukuhara derivative) \[6\]. In \[8, 10\], the generalized Hukuhara derivative is used for a fuzzy-number-valued function, leading to solutions with decreasing length on their supports. Salahshour et al. \[34\] and Mazandarani and Kamyad \[22\] proposed, respectively, the concepts of Riemann–Liouville and Caputo fuzzy fractional differentiability, based on the Hukuhara difference, which strongly generalizes fuzzy differentiability. In \[1, 35\], the generalized Hukuhara fractional Riemann–Liouville and Caputo concepts for fuzzy-valued functions are further investigated. For a Hukuhara approach valid on arbitrary nonempty closed sets of the real numbers (time scales) see \[9\]. In \[8\], Fard and Salehi investigate fuzzy fractional Euler–Lagrange equations for fuzzy fractional variational problems defined via generalized fuzzy fractional Caputo type derivatives. In \[37\], Soolaki et al. present necessary optimality conditions of Euler–Lagrange type for variational problems with natural boundary conditions and problems with holonomic constraints, where the fuzzy fractional derivative is described in a combined sense. Here, using the PMP and a novel form of the Hamiltonian approach, we achieve fuzzy solutions (state and control) by solving an appropriate system of differential equations. The proposed method is not limited to just optimal fuzzy linear time-invariant controlled systems, which were previously studied in \[26, 27\] for integer-order problems. Since the Buckley and Feuring concept of differentiability \[6\] or even the Hukuhara notion of differentiability are not able to guarantee that the obtained solutions are fuzzy functions, in the present work we focus on the generalized Hukuhara differentiation. If the order of the derivatives appearing in the formulation of our problems approach integer values, then one obtains via our results the extremals of fuzzy optimal control problems investigated in \[12, 26, 27\].

The paper is organized as follows. Section 2 introduces necessary notations on fuzzy numbers and differentiability and integrability of fuzzy mappings. The notion of Caputo generalized Hukuhara fuzzy fractional derivative is recalled in Section 3. In Section 4 we establish our main result, Theorem 4.1, that provides Pontryagin conditions for fuzzy fractional optimal control problems. In Section 5 we consider three problems, illustrating the proposed method. In particular, it is shown that
the candidates to minimizers given in [26, Example 4.2] and [28, Example 3] are not solutions to the considered problems. We end with Section 6 of conclusions and future work.

2. Preliminaries. Let us denote by $\mathbb{R}_f$ the class of fuzzy numbers, i.e., normal, convex, upper semicontinuous and compactly supported fuzzy subsets of the real numbers. For $0 < r \leq 1$, let $[\tilde{u}]^r = \{x \in \mathbb{R}; \tilde{u}(x) \geq r\}$ and $[\tilde{u}]^0 = \{x \in \mathbb{R}; \tilde{u}(x) \geq 0\}$. Then, it is well known that $[\tilde{u}]^r$ is a bounded closed interval for any $r \in [0, 1]$.

Lemma 2.1 (See Theorem 1.1 of [15] and Lemma 2.1 of [41]). If $\tilde{u}^r : [0, 1] \to \mathbb{R}$ and $\overline{\pi}^r : [0, 1] \to \mathbb{R}$ satisfy the conditions

(i) $\tilde{u}^r : [0, 1] \to \mathbb{R}$ is a bounded nondecreasing function,
(ii) $\overline{\pi}^r : [0, 1] \to \mathbb{R}$ is a bounded nonincreasing function,
(iii) $\tilde{u}^0 \leq \overline{\pi}^0$,
(iv) for $0 < k \leq 1$, $\lim_{r \to k} \overline{\pi}^r = \overline{\pi}^k$ and $\lim_{r \to k} \overline{\pi}^r = \overline{\pi}^k$,
(v) $\lim_{r \to 0} \overline{\pi}^r = \overline{\pi}^0$ and $\lim_{r \to 0} \overline{\pi}^r = \overline{\pi}^0$,
then $\tilde{u} : \mathbb{R} \to [0, 1]$, characterized by $\tilde{u}(t) = \sup\{r | \tilde{u}^r \leq t \leq \overline{\pi}^r\}$, is a fuzzy number with $[\tilde{u}]^r = [\tilde{u}^r, \overline{\pi}^r]$. The converse is also true: if $\tilde{u}(t) = \sup\{r | \tilde{u}^r \leq t \leq \overline{\pi}^r\}$ is a fuzzy number with parametrization given by $[\tilde{u}]^r = [\tilde{u}^r, \overline{\pi}^r]$, then functions $\tilde{u}^r$ and $\overline{\pi}^r$ satisfy conditions (i)-(v).

For $\tilde{u}, \tilde{v} \in \mathbb{R}_f$ and $\lambda \in \mathbb{R}$, the sum $\tilde{u} + \tilde{v}$ and the product $\lambda \cdot \tilde{u}$ are defined by $[\tilde{u} + \tilde{v}]^r = [\tilde{u}^r] + [\tilde{v}^r]$ and $[\lambda \cdot \tilde{u}]^r = \lambda [\tilde{u}]^r$ for all $r \in [0, 1]$, where $[\tilde{u}^r] + [\tilde{v}^r]$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [\tilde{u}]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$. The product $\tilde{u} \circ \tilde{v}$ of fuzzy numbers $\tilde{u}$ and $\tilde{v}$, is defined by

$$[\tilde{u} \circ \tilde{v}]^r = [\min\{\tilde{u}^r, \tilde{v}^r, \tilde{v}^r \circ \tilde{v}^r, \tilde{v}^r \circ \tilde{v}^r\}, \max\{\tilde{u}^r \circ \tilde{v}^r, \tilde{u}^r \circ \tilde{v}^r, \tilde{u}^r \circ \tilde{v}^r, \tilde{u}^r \circ \tilde{v}^r\}]$$

The metric structure is given by the Hausdorff distance $D : \mathbb{R}_f \times \mathbb{R}_f \to \mathbb{R}_+ \cup \{0\}$,

$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0, 1]} \max\{|\tilde{u}^r - \tilde{v}^r|, |\tilde{u}^r - \overline{\pi}^r|\}.$$ 

We say that the fuzzy number $\tilde{u}$ is triangular if $\tilde{u}^1 = \overline{\pi}^1$, $\tilde{u}^r = \tilde{u}^1 - (1 - r)(\tilde{u}^1 - \tilde{u}^0)$ and $\overline{\pi}^r = \tilde{u}^1 - (1 - r)(\tilde{u}^0 - \tilde{u}^1)$. The triangular fuzzy number $\tilde{u}$ is generally denoted by $\tilde{u} = \langle \tilde{u}^0, \tilde{u}^1, \tilde{u}^0 \rangle$. We define the fuzzy zero $\tilde{0}_x$ as

$$\tilde{0}_x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Definition 2.2 (See [11]). We say that $\tilde{f} : [a, b] \to \mathbb{R}_f$ is continuous at $t \in [a, b]$, if both $\tilde{f}^r(t)$ and $\overline{\tilde{f}}(t)$ are continuous functions of $t \in [a, b]$ for all $r \in [0, 1]$.

Definition 2.3 (See [5]). The generalized Hukuhara difference of two fuzzy numbers $\tilde{x}, \tilde{y} \in \mathbb{R}_f$ ($gH$-difference for short) is defined as follows:

$$\tilde{x} \ominus_{gH} \tilde{y} = \tilde{z} \Leftrightarrow \tilde{x} = \tilde{y} + \tilde{z} \text{ or } \tilde{y} = \tilde{x} + (-1)\tilde{z}.$$ 

If $\tilde{z} = \tilde{x} \ominus_{gH} \tilde{y}$ exists as a fuzzy number, then its level cuts $[\tilde{z}^r, \overline{\pi}^r]$ are obtained by

$$[\tilde{z}^r, \overline{\pi}^r] = \min\{[\tilde{z}^r, \overline{\pi}^r], [\tilde{y}^r, \overline{\pi}^r] - [\tilde{x}^r, \overline{\pi}^r]\}, \overline{\pi}^r = \max\{[\tilde{z}^r, \overline{\pi}^r], [\tilde{y}^r, \overline{\pi}^r] - [\tilde{x}^r, \overline{\pi}^r]\}$$

for all $r \in [0, 1]$. 


Definition 2.4 (See [18]). Let \( t \in (a, b) \) and \( h \) be such that \( t + h \in (a, b) \). The generalized Hukuhara derivative of a fuzzy-valued function \( \tilde{x} : (a, b) \to \mathbb{R}_f \) at \( t \) is defined by

\[
D_{gH}\tilde{x}(t) = \lim_{h \to 0} \frac{\tilde{x}(t + h) \ominus_{gH} \tilde{x}(t)}{h}.
\]

(1)

If \( D_{gH}\tilde{x}(t) \in \mathbb{R}_f \) satisfying (1) exists, then we say that \( \tilde{x} \) is generalized Hukuhara differentiable \((gH\text{-}differentiable \text{ for short})\) at \( t \). Also, we say that \( \tilde{x} \) is \([(1) - gH]\)-differentiable at \( t \) (denoted by \( D_{1,gH}\tilde{x}(t) \)) if \( [D_{gH}\tilde{x}(t)]^r = [\dot{\tilde{x}}^r(t), \tilde{x}^r(t)] \), and that \( \tilde{x} \) is \([(2) - gH]\)-differentiable at \( t \) (denoted by \( D_{2,gH}\tilde{x}(t) \)) if

\[
[D_{gH}\tilde{x}(t)]^r = [\dot{\tilde{x}}^r(t), \tilde{x}^r(t)], \quad r \in [0, 1].
\]

If the fuzzy function \( \tilde{f}(t) \) is continuous in the metric \( D \), then its definite integral exists. Furthermore,

\[
\left( \int_a^b \tilde{f}(t)dt \right)^r = \int_a^b \tilde{f}^r(t)dt, \quad \left( \int_a^b \tilde{f}(t)dt \right)^r = \int_a^b \tilde{F}(t)dt.
\]

Definition 2.5 (See [11]). Let \( \tilde{a}, \tilde{b} \in \mathbb{R}_f \). We write \( \tilde{a} \preceq \tilde{b} \), if \( \tilde{a}^r \preceq \tilde{b}^r \) and \( \tilde{a}^r \preceq \tilde{b}^r \) for all \( r \in [0, 1] \). We also write \( \tilde{a} \prec \tilde{b} \), if \( \tilde{a} \preceq \tilde{b} \) and there exists an \( r' \in [0, 1] \) so that \( \tilde{a}^r < \tilde{b}^r \) and \( \tilde{a}^r < \tilde{b}^r \). Moreover, \( \tilde{a} \approx \tilde{b} \) if \( \tilde{a} \preceq \tilde{b} \) and \( \tilde{a} \succeq \tilde{b} \), that is, \([\tilde{a}]^r = [\tilde{b}]^r \) for all \( r \in [0, 1] \).

We say that \( \tilde{a}, \tilde{b} \in \mathbb{R}_f \) are comparable if either \( \tilde{a} \preceq \tilde{b} \) or \( \tilde{a} \succeq \tilde{b} \); and noncomparable otherwise.

3. The fuzzy fractional calculus. The Riemann–Liouville fractional derivative has one disadvantage when modelling real world phenomena: the fractional derivative of a constant is not zero. To eliminate this problem, one often considers fractional derivatives in the sense of Caputo. For this reason, in our work we restrict ourselves to problems defined by generalized Hukuhara fractional Caputo derivatives. Analogous results are, however, easily obtained for generalized Hukuhara fractional derivatives in the Riemann–Liouville sense.

The fuzzy \( gH\)-fractional Caputo derivative of a fuzzy valued function was introduced in [1]. Following [1], we denote the space of all continuous fuzzy valued functions on \( [a, b] \subset \mathbb{R} \) by \( C^F[a, b] \); the class of fuzzy functions with continuous first derivatives on \( [a, b] \subset \mathbb{R} \) by \( C^{F1}[a, b] \); and the space of all Lebesgue integrable fuzzy valued functions on the bounded interval \( [a, b] \) by \( L^F[a, b] \).

Definition 3.1 (See [3]). Let \( \tilde{f}(x) \in C^F[a, b] \cap L^F[a, b] \) be a fuzzy valued function and \( \alpha > 0 \). Then the Riemann–Liouville fractional integral of order \( \alpha \) is defined by

\[
\int_a^\alpha \tilde{f}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \tilde{f}(t)(x - t)^{\alpha - 1}dt,
\]

where \( \Gamma(\alpha) \) is the Gamma function and \( x > a \).

Definition 3.2 (See [3]). Let \( \tilde{f}(x) \in C^F[a, b] \cap L^F[a, b] \) be a fuzzy valued function. The fuzzy (left) Riemann–Liouville integral of \( \tilde{f}(x) \), based on its \( r \)-level representation, can be expressed as follows:

\[
[a \int_a^\alpha \tilde{f}(x)]^r = [a \int_a^\alpha \tilde{f}^r(x), a \int_a^\alpha \tilde{F}(x)], \quad 0 \leq r \leq 1,
\]
where
\[
_{a}I_{x}^{\alpha}F(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} F(t)(x-t)^{\alpha-1} dt,
\]
\[
_{a}I_{x}^{\alpha}F(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{d}{dt} F(t)(x-t)^{\alpha-1} dt.
\]

Following [1, 18], we now recall the definition of Caputo-type fuzzy fractional derivative under the gH-difference. The definition is similar to the concept of Caputo derivative in the crisp case [31] and gives a direct extension of gH-differentiability to the fractional context [5].

**Definition 3.3** (See [18]). Let \( \hat{x}(t) \in C^{r}[a, b] \cap L^{r}[a, b] \). The fuzzy \( gH \)-fractional Caputo derivative of the fuzzy-valued function \( \tilde{x} \) ([\( gH \)]_{\alpha}^{C}-differentiability for short) is defined by
\[
g^{H-C}_{a}D_{t}^{\alpha}\tilde{x}(t) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-s)^{-\alpha-(m-\alpha)} (D_{gH}^{m}(\tilde{x})(s)) ds,
\]
where \( m-1 < \alpha < m, t > a \). If \( \alpha \in (0, 1) \), then
\[
g^{H-C}_{a}D_{t}^{\alpha}\tilde{x}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} (D_{gH}\tilde{x})(s) ds.
\]

**Theorem 3.4** (See [1]). Let \( \hat{x}(t) \in C^{r}[a, b] \cap L^{r}[a, b] \) and \( [\hat{x}(t)]^{r} = [\tilde{x}^{r}(t), \tilde{x}''(t)] \) for \( r \in [0, 1], t \in (a, b) \) and \( \alpha \in (0, 1) \). The function \( \tilde{x}(t) \) is \( [gH]_{\alpha}^{C} \)-differentiable if and only if both \( \tilde{x}^{r}(t) \) and \( \tilde{x}''(t) \) are Caputo fractional differentiable functions. Furthermore,
\[
[g^{H-C}_{a}D_{t}^{\alpha}\tilde{x}(t)]^{r} = \left[ \min \left\{ a_{D_{t}^{\alpha}\tilde{x}^{r}(t)}, a_{D_{t}^{\alpha}\tilde{x}''(t)} \right\}, \max \left\{ a_{D_{t}^{\alpha}\tilde{x}^{r}(t)}, a_{D_{t}^{\alpha}\tilde{x}''(t)} \right\} \right].
\]

where
\[
a_{D_{t}^{\alpha}\tilde{x}^{r}(t)} = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} \frac{d}{ds} \tilde{x}^{r}(s) ds,
\]
\[
a_{D_{t}^{\alpha}\tilde{x}''(t)} = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} (t-s)^{-\alpha} \frac{d}{ds} \tilde{x}''(s) ds.
\]

**Definition 3.5** (See [1]). Let \( \alpha \in [0, 1] \) and \( \tilde{x} : [a, b] \rightarrow \mathbb{R} \) be \( [gH]_{\alpha}^{C} \)-differentiable at \( t \in [a, b] \). We say that \( \tilde{x} \) is \( [(1) - gH]_{\alpha}^{C} \)-differentiable at \( t \in [a, b] \) if
\[
[g^{H-C}_{a}D_{t}^{\alpha}\tilde{x}(t)]^{r} = \left[ a_{D_{t}^{\alpha}\tilde{x}^{r}(t)}, a_{D_{t}^{\alpha}\tilde{x}''(t)} \right], \quad 0 \leq r \leq 1,
\]
and that \( \tilde{x} \) is \( [(2) - gH]_{\alpha}^{C} \)-differentiable at \( t \) if
\[
[g^{H-C}_{a}D_{t}^{\alpha}\tilde{x}(t)]^{r} = \left[ a_{D_{t}^{\alpha}\tilde{x}''(t)}, a_{D_{t}^{\alpha}\tilde{x}^{r}(t)} \right], \quad 0 \leq r \leq 1.
\]

**Remark 1.** We use the notation \( g^{H-C}_{a}D_{t}^{\alpha}\tilde{x} \) when the fuzzy-valued function \( \tilde{x} \) is \( [(i) - gH]_{\alpha}^{C} \)-differentiable with respect to the independent variable \( t, i \in \{1, 2\} \).

The definitions for the right fuzzy fractional operators \( _{a}I_{x}^{\alpha}, _{i}D_{t}^{\alpha} \) and \( g^{H-C}_{i}D_{t}^{\alpha} \) of order \( \alpha \), are completely analogous.
4. Optimality of fuzzy fractional optimal control problems. The fuzzy fractional optimal control problem in the sense of Caputo is introduced, without loss of generality, in Lagrange form:

\[
\tilde{J}(\tilde{x}, \tilde{u}) = \int_a^b \tilde{L}(\tilde{x}(t), \tilde{u}(t), t) dt \longrightarrow \min,
\]

\[
\gamma H - C_a^\beta \tilde{D}_t^\beta \tilde{x}(t) = \tilde{\varphi}(\tilde{x}(t), \tilde{u}(t), t), \quad i = 1, 2,
\]

(2)

where \(\tilde{x} : [a, b] \rightarrow \mathbb{R}_F^n\) satisfies appropriate boundary conditions, \(\tilde{u}^r(t)\) and \(\tilde{w}(t)\) are piecewise continuous, and \(\beta \in (0, 1)\). The Lagrangian \(\tilde{L} : \mathbb{R}_F^n \times \mathbb{R}_F^m \times [a, b] \rightarrow \mathbb{R}_F\) and the velocity vector \(\tilde{\varphi} : \mathbb{R}_F^n \times \mathbb{R}_F^m \times [a, b] \rightarrow \mathbb{R}_F\) are assumed to be functions of class \(C^F\) with respect to all their arguments and

\[
\gamma H - C_a^\beta \tilde{D}_t^\beta \tilde{x}(t) = \frac{1}{\Gamma(1-\beta)} \int_a^b (t - \tau)^{-\beta}(\tilde{D}_{\tau}^\beta \tilde{x})(\tau) d\tau, \quad i = 1, 2.
\]

We say that an admissible fuzzy curve \((\tilde{x}^*, \tilde{u}^*)\) is solution of problem (2), if for all admissible curve \((\tilde{x}, \tilde{u})\) of problem (2),

\[
\tilde{J}(\tilde{x}^*, \tilde{u}^*) \leq \tilde{J}(\tilde{x}, \tilde{u}).
\]

It follows, from the definition of partial ordering given in Definition 2.5, that the inequality \(\tilde{J}(\tilde{x}^*, \tilde{u}^*) \leq \tilde{J}(\tilde{x}, \tilde{u})\) holds if and only if

\[
\tilde{J}^r(\tilde{x}^*, \tilde{x}^*, \tilde{u}^*, \tilde{w}^*) \leq \tilde{J}^r(\tilde{x}, \tilde{x}, \tilde{u}, \tilde{w})
\]

and

\[
\tilde{J}^r(\tilde{x}^*, \tilde{x}^*, \tilde{u}^*, \tilde{w}^*) \leq \tilde{J}^r(\tilde{x}, \tilde{x}, \tilde{u}, \tilde{w})
\]

for all \(r \in [0, 1]\), where the \(r\)-level set of fuzzy curves \(\tilde{x}^*, \tilde{u}^*, \tilde{x}\) and \(\tilde{u}\) are

\[
[\tilde{x}^*]_r = [\tilde{x}^*, \tilde{x}^*], \quad [\tilde{u}^*]_r = [\tilde{u}^*, \tilde{u}^*], \quad [\tilde{x}]_r = [\tilde{x}, \tilde{x}], \quad [\tilde{u}]_r = [\tilde{u}, \tilde{u}],
\]

respectively.

**Remark 2.** Choosing \(\beta = 1\), problem (2) is reduced to the fuzzy optimal control problem

\[
\tilde{J}(\tilde{x}(\cdot), \tilde{u}(\cdot)) = \int_a^b \tilde{L}(\tilde{x}(t), \tilde{u}(t), t) dt \longrightarrow \min,
\]

\[
\tilde{D}_{\tau}^\beta \tilde{x}(t) = \tilde{\varphi}(\tilde{x}(t), \tilde{u}(t), t)
\]

\[
\tilde{x}(a) = \tilde{x}_a, \quad \tilde{x}(b) = \tilde{x}_b,
\]

which is studied in [12].

**Remark 3.** The fuzzy fractional problem of the calculus of variations in the sense of Caputo,

\[
\tilde{J}(\tilde{x}(\cdot)) = \int_a^b \tilde{L}(\tilde{x}(t), \gamma H - C_a^\beta \tilde{D}_t^\beta \tilde{x}(t), t) dt \longrightarrow \min,
\]

\[
\tilde{x}(a) = \tilde{x}_a, \quad \tilde{x}(b) = \tilde{x}_b,
\]

was first introduced in [8] and is a particular case of our problem (2): one just need to choose \(\tilde{\varphi}(\tilde{x}, \tilde{u}, t) = \tilde{u}\).

**Theorem 4.1** (Pontryagin Maximum Principle for problem (2)). Let control \(\tilde{u}^*\) have the lower and upper bounds \(\tilde{u}^{*r}\) and \(\tilde{u}^{r}\), and \(\tilde{x}^*\) be the corresponding state with lower and upper bounds \(\tilde{x}^{*r}\) and \(\tilde{x}^r\). If \((\tilde{x}^*, \tilde{u}^*)\) is solution to (2), then there exist costate functions \(p_1\) and \(p_2\) such that the quadruple \((\tilde{x}^{*r}, \tilde{x}^{r}, \tilde{u}^{*r}, \tilde{u}^r)\) satisfies
- the Hamiltonian adjoint system
  \[ i D_b^2 p_1^r(t) = \frac{\partial H}{\partial x^r}, \quad i D_b^2 p_2^r(t) = \frac{\partial H}{\partial \pi^r}, \]

- and the stationary conditions
  \[ \frac{\partial H}{\partial x^r} = 0, \quad \frac{\partial H}{\partial \pi^r} = 0, \]

where the partial derivatives are evaluated at
\[ (x^r(t), \pi^r(t), u^r(t), \pi^r(t), p_1(t), p_2(t), t) \]

with the Hamiltonian \( H \) defined as follows: if \( \tilde{x}^* \) is \([(1)-gH] \)-differentiable, then
\[
H(x^r, \pi^r, u^r, \pi^r, p_1, p_2, t) = -(L^r(x^r, \pi^r, u^r, \pi^r, t) + L^r(x^r, \pi^r, u^r, \pi^r, t)) + p_1 \cdot \iota^r(x^r, \pi^r, u^r, \pi^r, t) + p_2 \cdot \pi^r(x^r, \pi^r, u^r, \pi^r, t); \quad (3)
\]

if \( \tilde{x}^* \) is \([(2)-gH] \)-differentiable, then
\[
H(x^r, \pi^r, u^r, \pi^r, p_1, p_2, t) = -(L^r(x^r, \pi^r, u^r, \pi^r, t) + L^r(x^r, \pi^r, u^r, \pi^r, t)) + p_1 \cdot \iota^r(x^r, \pi^r, u^r, \pi^r, t) + p_2 \cdot \pi^r(x^r, \pi^r, u^r, \pi^r, t). \quad (4)
\]

**Proof.** Consider a variation \( x^r = x^r + \delta x^r \) and a variation \( \pi^r = \pi^r + \delta \pi^r \) of \( x^r \) and \( \pi^r \), respectively, with corresponding state \( (x^r + \delta x^r, \pi^r + \delta \pi^r, t) \). The consequent change \( \Delta(\tilde{J}) \) in \( \tilde{J} \) is
\[
\Delta(\tilde{J}) = \int_a^b \tilde{L}(\tilde{x}^* + \delta \tilde{x}, \tilde{u}^* + \delta \tilde{u}, t) dt \leq gH \int_a^b \tilde{L}(\tilde{x}^*, \tilde{u}^*, t) dt.
\]

Denote \( [\Delta \tilde{J}]^r = [\Delta \tilde{J}^r, \Delta \tilde{J}^r] \). Using the gH-difference, one gets
\[
\Delta \tilde{J}^r = \min \left\{ \int_a^b L^r[\tilde{x} + \delta \tilde{x}, \tilde{u} + \delta \tilde{u}] dt - \int_a^b L^r[\tilde{x}, \tilde{u}] dt, \right. \\
\left. \int_a^b L^r[\tilde{x} + \delta \tilde{x}, \tilde{u} + \delta \tilde{u}] dt - \int_a^b L^r[\tilde{x}, \tilde{u}] dt \right\},
\]

\[
\Delta \tilde{J}^r = \max \left\{ \int_a^b L^r[\tilde{x} + \delta \tilde{x}, \tilde{u} + \delta \tilde{u}] dt - \int_a^b L^r[\tilde{x}, \tilde{u}] dt, \right. \\
\left. \int_a^b L^r[\tilde{x} + \delta \tilde{x}, \tilde{u} + \delta \tilde{u}] dt - \int_a^b L^r[\tilde{x}, \tilde{u}] dt \right\},
\]

where
\[
[\tilde{x} + \delta \tilde{x}, \tilde{u} + \delta \tilde{u}]^r = (x^r + \delta x^r, \pi^r + \delta \pi^r, u^r + \delta u^r, \pi^r + \delta \pi^r, t),
\]
\[
[\tilde{x}, \tilde{u}]^r = (x^r, \pi^r, u^r, \pi^r, t).
\]

Without loss of generality, we consider
\[
\Delta \tilde{J}^r = \int_a^b L^r(x^r + \delta x^r, \pi^r + \delta \pi^r, u^r + \delta u^r, \pi^r + \delta \pi^r, t) dt
\]
\[
- \int_a^b L^r(x^r, \pi^r, u^r, \pi^r, t) dt
\]
and
\[
\Delta J = \int_a^b \left[ \frac{\partial L}{\partial x^r} \delta x^r + \frac{\partial L}{\partial u^r} \delta u^r + \frac{\partial L}{\partial \pi^r} \delta \pi^r \right] dt \quad - \int_a^b L(\phi^r, \pi^r, t) dt.
\]
If we evaluate the derivatives in the integrand along the optimal trajectory, then we arrive at
\[
\Delta J = \int_a^b \left[ \frac{\partial L}{\partial x^r} \delta x^r + \frac{\partial L}{\partial u^r} \delta u^r + \frac{\partial L}{\partial \pi^r} \delta \pi^r \right] dt + O((\delta u^r)^2) + O((\delta \pi^r)^2)
\]
and
\[
\overline{\Delta J} = \int_a^b \left[ \frac{\partial L}{\partial x^r} \delta x^r + \frac{\partial L}{\partial u^r} \delta u^r + \frac{\partial L}{\partial \pi^r} \delta \pi^r \right] dt + O((\delta u^r)^2) + O((\delta \pi^r)^2).
\]
Since \(\mathcal{J}(\tilde{x}^*, \tilde{u}^*) \leq \mathcal{J}(\bar{x}, \bar{u})\) if and only if \(\mathcal{J}'[\tilde{x}^*, \tilde{u}^*] \leq \mathcal{J}'[\bar{x}, \bar{u}]\) and \(\mathcal{J}'[\tilde{x}^*, \tilde{u}^*] \leq \mathcal{J}'[\bar{x}, \bar{u}]\) for all \(r \in [0, 1]\), so [\(\tilde{x}^*, \tilde{u}^*]\] is an optimal solution for the crisp functions \(\mathcal{J}'\) and \(\mathcal{J}\). Let \(\delta \mathcal{J}'\) and \(\delta \mathcal{J}\) denote the first variation. If \(\delta u^r\) and \(\delta \pi^r\) are optimal, from the classical theory of optimal control, it is necessary that the first variation \(\delta \mathcal{J}'\) and \(\delta \mathcal{J}\) are zero. Thus, on optimal trajectories, one has
\[
\delta \mathcal{J} = \int_a^b \left[ \frac{\partial L}{\partial x^r} \delta x^r + \frac{\partial L}{\partial u^r} \delta u^r + \frac{\partial L}{\partial \pi^r} \delta \pi^r \right] dt = 0
\]
and
\[
\overline{\delta \mathcal{J}} = \int_a^b \left[ \frac{\partial L}{\partial x^r} \delta x^r + \frac{\partial L}{\partial u^r} \delta u^r + \frac{\partial L}{\partial \pi^r} \delta \pi^r \right] dt = 0
\]
for all variations. Now, we simply need to introduce two Lagrange multipliers \(p_1(t)\) and \(p_2(t)\). If \(\tilde{x}\) is \([1-(gH)]\)-differentiable, then we consider the integrals
\[
\bar{\phi} = \int_a^b p_1 \cdot (\tilde{C} D \tilde{x}^r - \varphi^r) dt
\]
(5)
and
\[
\overline{\phi} = \int_a^b p_2 \cdot (\tilde{C} D \tilde{x}^r - \overline{\varphi}^r) dt.
\]
(6)
If \(\tilde{x}\) is \([2-(gH)]\)-differentiable, then we consider the integrals
\[
\bar{\phi} = \int_a^b p_1 \cdot (\tilde{C} D \tilde{x}^r - \varphi^r) dt
\]
and
\[
\overline{\phi} = \int_a^b p_2 \cdot (\tilde{C} D \tilde{x}^r - \overline{\varphi}^r) dt.
\]
Let us assume \([1-(gH)]\)-differentiability of \(\tilde{x}^*\). The proof for the other case is completely similar, so it is here omitted. We begin by computing the variation \(\delta \phi^r\) of
functional (5):

$$\delta \phi^r = \int_a^b \delta p_1 (C \int_a^b D_t^\alpha x^r - \varphi^r)$$

$$+ p_1 \left[ \delta (C \int_a^b D_t^\alpha x^r) - \left( \frac{\partial \varphi^r}{\partial x^r} \delta x^r + \sum_{\ell} \frac{\partial \varphi^r}{\partial u_\ell} \delta u_\ell + \frac{\partial \varphi^r}{\partial \pi} \delta \pi \right) \right] dt \tag{7}$$

$$= \int_a^b p_1 \left[ \delta \left( C \int_a^b D_t^\alpha x^r \right) - \left( \frac{\partial \varphi^r}{\partial x^r} \delta x^r + \frac{\partial \varphi^r}{\partial u_\ell} \delta u_\ell + \frac{\partial \varphi^r}{\partial \pi} \delta \pi \right) \right] dt.$$

Because $\hat{x}(a)$ and $\hat{x}(b)$ are specified, we have $\delta x^r(a) = \delta x^r(b) = \delta x^r(a) = \delta x^r(b) = 0$. Using fractional integration by parts [21], equation (7) is equivalent to

$$\frac{\delta \phi^r}{\delta \hat{x}} = \int_a^b \left( \int_a^b \frac{\delta J^r}{\delta \hat{x}} + p_1 \frac{\delta \varphi^r}{\delta \hat{x}} \right) dx^r - \left( \frac{\partial \varphi^r}{\partial x^r} \delta x^r + \frac{\partial \varphi^r}{\partial u_\ell} \delta u_\ell + \frac{\partial \varphi^r}{\partial \pi} \delta \pi \right) p_1 dt$$

since $\delta \hat{x}$ is for all $x^r$ and $\pi$ and $\delta \phi^r = 0$. Therefore, the condition $\delta J^r = 0$ can now be replaced by $\delta \hat{x} = 0$. With substitutions of $\delta J^r$ and $\delta \phi^r$, we have

$$\int_a^b \left( \frac{\partial \hat{x}}{\partial x^r} + D_\beta p_1 - p_1 \frac{\partial \varphi^r}{\partial \hat{x}} \right) \delta x^r + \left( \frac{\partial \hat{x}}{\partial u_\ell} - p_1 \frac{\partial \varphi^r}{\partial \hat{x}} \right) \delta u_\ell + \left( \frac{\partial \hat{x}}{\partial \pi} - p_1 \frac{\partial \varphi^r}{\partial \hat{x}} \right) \delta \pi \right) \right] dt = 0.

(8)

Now, following the scheme of obtaining Eq. (8), and adapting it to the case under consideration involving Eq. (6), the condition $\delta J^r = 0$ can be replaced by $\delta \hat{x} = 0$. So we have

$$\int_a^b \left( \frac{\partial \hat{x}}{\partial x^r} - p_1 \frac{\partial \varphi^r}{\partial \hat{x}} \right) \delta x^r + \left( \frac{\partial \hat{x}}{\partial u_\ell} - p_1 \frac{\partial \varphi^r}{\partial \hat{x}} \right) \delta u_\ell + \left( \frac{\partial \hat{x}}{\partial \pi} - p_1 \frac{\partial \varphi^r}{\partial \hat{x}} \right) \delta \pi \right) \right] dt = 0.

\tag{9}

If we use the Hamiltonian function as in (3), then by summing Eqs. (8) and (9) we arrive at

$$\int_a^b \left( \int_a^b D_\beta p_1 - \frac{\partial H}{\partial \hat{x}} \right) \delta x^r + \left( \int_a^b D_\beta p_2 - \frac{\partial H}{\partial \pi} \right) \delta x^r - \frac{\partial H}{\partial u_\ell} \delta u_\ell - \frac{\partial H}{\partial \pi} \delta \pi \right) \right] dt = 0.$$ 

The intended necessary conditions follow.

A pair $(\hat{x}^r, \hat{u}^r)$ satisfying Theorem 4.1 is said to be an extremal for problem (2).

5. **Illustrative examples.** In this section, we apply the necessary conditions of Pontryagin type given by Theorem 4.1 to three fuzzy optimal control problems.

5.1. A non-autonomous fuzzy fractional optimal control problem. We begin with a non-autonomous fuzzy fractional optimal control problem.
Example 1. Consider the following problem:

\[ \int_1^2 \dddot{u}^2(t) dt \rightarrow \min, \]

\[ gH - C \dot{D}_t^\beta \dddot{x}(t) = (2t - 1)\dddot{x}(t) \odot_{gH} \sin(t)\dddot{u}(t), \]  
\[ \dddot{x}(1) = (0, 1, 2), \quad \dddot{x}(2) = (-2, -1, 1). \]  

We assume that \((2t - 1)\dddot{x}(t) \odot_{gH} \sin(t)\dddot{u}(t)\) exists and

\[ [(2t - 1)\dddot{x}(t) \odot_{gH} \sin(t)\dddot{u}(t)]^r = [(2t - 1)\dddot{x}^r - \sin(t)\dddot{u}^r, (2t - 1)\dddot{x}^r - \sin(t)\dddot{u}^r]. \]

Using Theorem 4.1, we consider two cases to obtain the extremals of (10).

(i) Suppose that \(\dddot{x}\) is a \([1 - gH]^{\beta}\)-differentiable function. Then, the Hamiltonian is given by

\[ \mathcal{H} = -\left((\dddot{u}^r)^2 + (\dddot{u}^r)^2\right) + p_1((2t - 1)\dddot{x}^r - \sin(t)\dddot{u}^r) + p_2((2t - 1)\dddot{x}^r - \sin(t)\dddot{u}^r). \]

The optimality conditions of Theorem 4.1, the initial conditions, and the control system of (10) assert that

\[ \begin{aligned}
    &\{\dot{D}_t^\beta p_1(t) = (2t - 1)p_1(t), \\
    &\dot{D}_t^\beta p_2(t) = (2t - 1)p_2(t), \\
    &\dddot{u}^r(t) = -\frac{p_1(t)\sin(t)}{2}, \\
    &\dddot{u}^r(t) = -\frac{p_2(t)\sin(t)}{2}, \\
    &\dddot{u}^r(1) = r, \\
    &\dddot{u}^r(1) = 2 - r, \\
    &\dddot{u}^r(2) = -2 + r, \\
    &\dddot{u}^r(2) = 1 - 2r.
\end{aligned} \]  

(11)

Note that it is difficult to solve the above fractional equations to get the extremals. For \(0 < \beta < 1\), a numerical method should be used [2, 20]. When \(\beta\) goes to 1, problem (10) reduces to

\[ \int_{1}^{2} \dddot{u}^2(t) dt \rightarrow \min, \]

\[ \mathcal{D}_{gH} \dddot{x}(t) = (2t - 1) \odot \dddot{x}(t) \odot_{gH} \sin(t)\dddot{u}(t), \]

\[ \dddot{x}(1) = (0, 1, 2), \quad \dddot{x}(2) = (-2, -1, 1). \]  

The extremals for (12) are obtained from (11) by considering \(\beta \to 1\):

\[ \begin{aligned}
    &\dot{p}_1(t) = (1 - 2t)p_1(t), \\
    &\dot{p}_2(t) = (1 - 2t)p_2(t), \\
    &\dddot{u}^r(t) = -\frac{p_1(t)\sin(t)}{2}, \\
    &\dddot{u}^r(t) = -\frac{p_2(t)\sin(t)}{2}, \\
    &\dddot{u}^r(1) = r, \\
    &\dddot{u}^r(1) = 2 - r, \\
    &\dddot{u}^r(2) = -2 + r, \\
    &\dddot{u}^r(2) = 1 - 2r.
\end{aligned} \]  

(13)

We solved (13) numerically, with MATLAB’s built-in solver \texttt{bvp4c}. Figure 1 shows the control and state extremals, where the solid lines in the center corresponds to
A NECESSARY CONDITION OF PONTRYAGIN TYPE

1.2
1.4
1.6
1.8

0
1
2
3
4
5
6
7

\( t \)

\( u(t) \)

(a) Fuzzy control extremal

(b) Fuzzy state extremal

Figure 1. The fuzzy extremals for the fuzzy optimal control problem (12) of Example 1 under \((1) - gH|_C^\beta\)-differentiability of \( \dot{x} \).

For \( r = 1 \), the dashed lines are the upper bounds and the doted lines are the lower bounds, for both fuzzy control and state functions, correspondent to \( r = 0 \).

(ii) Suppose now that \( \dot{x} \) is \((2) - gH|_C^\beta\)-differentiable. This leads to

\[ \mathcal{H} = -(\dot{x}^r)^2 + (\dot{\beta})^2 + p_1((2t-1)x^r - \sin(t)x^r) + p_2((2t-1)x^r - \sin(t)u^r). \]

The optimality conditions of Theorem 4.1, the initial conditions and the control system assert that

\[
\begin{align*}
\dot{p}_1(t) &= (2t-1)p_2(t), \\
\dot{p}_2(t) &= (2t-1)p_1(t), \\
\frac{\ddot{x}^r(t)}{\dot{\beta}} &= \frac{-p_2(t)\sin(t)}{2}, \\
\frac{\ddot{x}^r(t)}{\dot{x}^r} &= \frac{-p_1(t)\sin(t)}{2}, \\
\dot{x}^r(1) &= r, \\
x^r(1) &= 2 - r, \\
x^r(2) &= -2 + r, \\
x^r(2) &= 1 - 2r.
\end{align*}
\]

Note that it is difficult to solve the above fractional equations to get the extremals. For \( 0 < \beta < 1 \), a numerical method should be used. When \( \beta \) goes to 1, problem (10) reduces to problem (12). The extremals for (12) are obtained from (14) and considering \( \beta \to 1 \):

\[
\begin{align*}
\dot{p}_1(t) &= (1-2t)p_2(t), \\
\dot{p}_2(t) &= (1-2t)p_1(t), \\
\frac{\ddot{x}^r(t)}{\dot{\beta}} &= \frac{(2t-1)x^r - \sin(t)x^r}{2}, \\
\frac{\ddot{x}^r(t)}{\dot{x}^r} &= \frac{(2t-1)x^r - \sin(t)u^r}{2}, \\
x^r(1) &= r, \\
x^r(1) &= 2 - r, \\
x^r(2) &= -2 + r, \\
x^r(2) &= 1 - 2r.
\end{align*}
\]

Similarly as before, we solved (15) with MATLAB’s built-in solver bvp4c. Figure 2 shows the graphic of the control and state extremals, where the solid lines at the center correspond to \( r = 1 \), the dashed lines are the upper bounds, and the doted lines are the lower bounds for fuzzy control and state functions for \( r = 0 \). Comparing
Figures 1 and 2, we see that using \([(2) - gH]\)-differentiability of \(\tilde{x}\) the length of support of \(\tilde{u}(t)\) is decreasing.

5.2. On two examples of Najariyan and Farahi. In the recent paper [26], Najariyan and Farahi characterize extremals for fuzzy linear time-invariant (autonomous) optimal control systems. Precisely, they investigate a method for solving the following time-invariant fuzzy optimal control problem:

\[
\int_a^b \tilde{u}(t)^2 dt \to \min,
\]

\[
\mathcal{D}_{gH} \tilde{x}(t) = A \circ \tilde{x}(t) + C \circ \tilde{u}(t),
\]

\[
\tilde{x}(a) = \tilde{x}_a, \quad \tilde{x}(b) = \tilde{x}_b.
\]

Main result of [26] asserts that the fuzzy optimal control problem (16) is equivalent to the crisp complex optimal control system

\[
\int_a^b (|\tilde{u}(t)|^2 + |\tilde{\omega}(t)|^2) dt \to \min,
\]

\[
\tilde{x}^r(t) + i\tilde{x}^i(t) = B(\tilde{x}^r(t) + i\tilde{x}^i(t)) + D(\tilde{u}^r(t) + i\tilde{u}^i(t)),
\]

\[
\tilde{x}^r(a) + i\tilde{x}^i(a) = \tilde{x}^r_a + i\tilde{x}^i_a, \quad \tilde{x}^r(b) + i\tilde{x}^i(b) = \tilde{x}^r_b + i\tilde{x}^i_b,
\]

where the elements of the matrices \(B\) and \(D\) are determined from those of \(A\) and \(C\) as follows:

\[
b_{ij} = \begin{cases} e_a_{ij} & \text{if } a_{ij} \geq 0, \\ g_a_{ij} & \text{if } a_{ij} < 0, \end{cases} \quad d_{ij} = \begin{cases} e_c_{ij} & \text{if } c_{ij} \geq 0, \\ g_c_{ij} & \text{if } c_{ij} < 0, \end{cases}
\]

with \(e : a + bi \to a + bi\) and \(g : a + bi \to b + ai\). The extremals for the crisp optimal control problem (17) are given by the classical PMP [32].
Example 2 (Example 4.2 of Najariyan and Farahi [26]). Consider the following problem:

\[
\int_0^1 \tilde{u}^2(t)dt \longrightarrow \min,
\]

\[
\begin{aligned}
\dot{x}_1(t) &= -2x_2(t) + \tilde{u}(t), \\
\dot{x}_2(t) &= 2x_1(t), \\
x_1(0) &= x_2(0) = 1, \\
x_1(1) &= x_2(1) = 0.
\end{aligned}
\]  

(18)

In [26] the authors provide a figure (see [26, Figure 2]) with what they claim to be the fuzzy control and state extremals for problem (18). It turns out that the provided functions are not extremals for the optimal control problem (18). Indeed, in the crisp case, i.e., when the variables \(\tilde{x}_1(t), \tilde{x}_2(t)\) and \(\tilde{u}(t)\) and \(\tilde{2} = (1, 2, 3)\) and \(\tilde{0} = (-0.5, 0, 0.5)\) are crisp quantities, the fuzzy optimal control problem (18) is transformed into the following crisp optimal control problem:

\[
\int_0^1 u^2(t)dt \longrightarrow \min,
\]

\[
\begin{aligned}
\dot{x}_1(t) &= -2x_2(t) + u(t), \\
\dot{x}_2(t) &= 2x_1(t), \\
x_1(0) &= x_2(0) = 2, \\
x_1(1) &= x_2(1) = 0.
\end{aligned}
\]  

(19)

The extremals for (19) are easily obtained from the classical PMP [30, 32]. Figure 3 shows the graphics of the control and state extremals for problem (19). Comparing these functions with the ones given in [26, Example 4.2], one may conclude that there is an inconsistency in [26, Example 4.2]. Let us use Theorem 4.1 to obtain the extremals for (18). Suppose that \(\tilde{x}_1\) is a \([1]-gH\)-differentiable function and \(\tilde{x}_2\) is a \([2]-gH\)-differentiable function. The analysis of the other three cases are similar and are left to the reader. Our assumption leads to

\[
\mathcal{H} = -((w_r)^2 + (\mathcal{W})^2) + p_1(-2\mathcal{W}_2 + w_r^r) + p_2(-2w_2^r + \mathcal{W}) + p_3(2\mathcal{W}_1) + p_4(2w_1^r).
\]

From the optimality conditions of Theorem 4.1, the initial conditions and the control system of problem (18), and considering \(\beta = 1\), we obtain that

\[
\begin{aligned}
\dot{p}_1(t) &= -2p_4(t), \\
\dot{p}_2(t) &= -2p_3(t), \\
\dot{p}_3(t) &= 2p_2(t), \\
\dot{p}_4(t) &= 2p_1(t), \\
u^r(t) &= \frac{p_1(t)}{p_2(t)}, \\
\mathcal{W}_1(t) &= \frac{2}{p_2(t)}.
\end{aligned}
\]  

(20)

By solving (20), the control and state extremals can be found straightforwardly. Figure 4 shows the graphics of the fuzzy control and state extremals, where the continuous lines in the center correspond to \(r = 1\). We clearly see from Figures 3 and 4 that the fuzzy extremals of the time-invariant linear optimal control problem (18) are related with the extremals of the crisp optimal control problem (19), which is in agreement with the results of [26].
The control extremal $u(t)$

The state extremal $x_1(t)$

The state extremal $x_2(t)$

Figure 3. The extremals for the crisp optimal control problem (19) of Example 2.

In [28], Najariyan and Farahi also propose a method to find extremals for linear non-autonomous fuzzy optimal control problems with fuzzy boundary conditions. Here we show that fuzzy minimizers for [28, Example 3] do not exist.

Example 3 (Example 3 of Najariyan and Farahi [28]). Consider the following problem:

$$
\int_0^2 \tilde{u}(t) \odot \tilde{u}(t) dt \rightarrow \min,
$$

$$
\mathcal{D}_{gH} \tilde{x}(t) = (2t - 1) \odot \tilde{x}(t) + \sin(t) \tilde{u}(t),
$$

$$
\tilde{x}(0) = (1, 2, 3), \quad \tilde{x}(2) = (-1, 0, 1).
$$

In [28, Example 3], the authors claim to have found the control and state extremals for problem (21) (cf. [28, Figure 3]). Here we show that in fact the fuzzy state and control minimizers do not exist for problem (21). Since $\text{diam}(\tilde{x}(0)) = \text{diam}(\tilde{x}(2)) = 2 - 2r$, we have $\tilde{x}(t) = \tilde{x}(0) + \tilde{f}(t)$, where $\text{diam}(\tilde{f}(t)) = 0$. Hence, $\mathcal{D}_{gH} \tilde{x}(t) = \mathcal{D}_{gH} \tilde{f}(t)$, that is, $(2t - 1) \odot \tilde{x}(t) + \sin(t) \tilde{u}(t) = \mathcal{D}_{gH} \tilde{f}(t)$. Consequently, $\text{diam}((2t - 1) \odot \tilde{x}(t) + \sin(t) \tilde{u}(t)) = 0$. Then, $\sin(t(\overline{r} - \underline{u}^r)) + (2t - 1)(\overline{x} - \underline{x}^r) = 0$ for every $t \in [0, 2]$ and $r \in [0, 1]$. Since $\sin(t) > 0$ and $2t - 1 > 0$ for $t \in \left(\frac{1}{2}, 2\right]$, we arrive at $(\overline{x} - \underline{x}^r) = (\overline{x} - \underline{x}^r) = 0$. Therefore, $\text{diam}(\tilde{x}(t)) = \text{diam}(\tilde{u}(t)) = 0$, which is
impossible, and the fuzzy state and control minimizers do not exist. Moreover, note that, in the crisp case, that is, when the variables \( \tilde{x}(t) \) and \( \tilde{u}(t) \) and \( \tilde{2} = (1, 2, 3) \) and \( \tilde{0} = (-1, 0, 1) \) are crisp quantities, the fuzzy optimal control problem \((21)\) is transformed into the following crisp optimal control problem:

\[
\int_0^2 u^2(t)dt \longrightarrow \min,
\]

\[
\dot{x}(t) = (2t - 1)x(t) + \sin(t)u(t),
\]

\[
x(0) = 2, \quad x(2) = 0.
\]

The extremals for \((22)\) are easily obtained from the classical PMP \([32]\). Figure 5 plots the control and state extremals for problem \((22)\). We conclude that the extremals of the crisp optimal control problem are not necessarily solution to the original fuzzy optimal control problem when \(r = 1\). This gives new insights to the results of \([28]\).

6. **Conclusion.** In this paper, a novel technique has been presented to solve a class of fuzzy fractional optimal control problems, where the coefficients of the system can be time-dependent. More precisely, we established a weak Pontryagin Maximum
Principle (PMP) for fuzzy fractional optimal control problems depending on generalized Hukuhara fractional Caputo derivatives (Theorem 4.1). The results improve those of [26, 27], where fuzzy optimal controls subject to time-invariant control systems are considered. See also Remarks 2 and 3, showing that our result easily generalizes those previously obtained in [8, 12]. The main features of our optimality conditions were summarized and highlighted with three illustrative examples. Two of the examples give interesting insights to the results of [26, 28].

We have just discussed necessary optimality conditions. Much remains to be done and we end by mentioning some possible lines of research. The obtained fuzzy fractional optimality conditions are, in general, difficult to solve and it would be good to develop specific numerical methods to address the issue. To obtain second order necessary optimality conditions is presently a big challenge. Other open lines of research consist to prove sufficient optimality conditions and existence results. While here we have assumed that the optimal solution exists, and necessary optimality conditions have been obtained under such assumption, as Example 3 shows, this is not always the case. As future work, we intend to prove conditions assuring the existence of optimal solutions to fuzzy fractional optimal control problems.

Acknowledgements. The authors are grateful to Catherine Choquet and to an anonymous Referee for their helpful and constructive suggestions.

REFERENCES

[1] T. Allahviranloo, A. Armand and Z. Gouyandeh, Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, J. Intell. Fuzzy Systems 26 (2014), no. 3, 1481–1490.

[2] R. Almeida, S. Pooseh and D. F. M. Torres, Computational methods in the fractional calculus of variations, Imp. Coll. Press, London, 2015.

[3] S. Arshad and V. Lupulescu, On the fractional differential equations with uncertainty, Nonlinear Anal. 74 (2011), no. 11, 3685–3693.

[4] D. Baleanu and Om. P. Agrawal, Fractional Hamilton formalism within Caputo’s derivative, Czechoslovak J. Phys. 56 (2006), no. 10-11, 1087–1092.

[5] B. Bede and L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems 230 (2013), 119–141.

[6] J. J. Buckley and T. Feuring, Introduction to fuzzy partial differential equations, Fuzzy Sets and Systems 105 (1999), no. 2, 241–248.
A NECESSARY CONDITION OF PONTRYAGIN TYPE

[7] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order \((\alpha, \beta)\), *Math. Methods Appl. Sci.* 30 (2007), no. 15, 1931–1939. arXiv:math-ph/0702099

[8] O. S. Fard and M. Salehi, A survey on fuzzy fractional variational problems, *J. Comput. Appl. Math.* 271 (2014), 71–82.

[9] O. S. Fard, D. F. M. Torres and M. R. Zadeh, A Hukuhara approach to the study of hybrid fuzzy systems on time scales, *Appl. Anal. Discrete Math.* 10 (2016), no. 1, 152–167. arXiv:1603.03737

[10] O. S. Fard and M. S. Zadeh, Note on “Necessary optimality conditions for fuzzy variational problems”, *J. Adv. Res. Dyn. Control Syst.* 4 (2012), no. 3, 1–9.

[11] B. Farhadinia, Necessary optimality conditions for fuzzy variational problems, *Inform. Sci.* 181 (2011), no. 7, 1348–1357.

[12] B. Farhadinia, Pontryagin’s minimum principle for fuzzy optimal control problems, *Iran. J. Fuzzy Syst.* 11 (2014), no. 2, 27–43.

[13] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, *Nonlinear Dynam.* 53 (2008), no. 3, 215–222. arXiv:0711.0609

[14] Y. Gao and Y.-J. Liu, Adaptive fuzzy optimal control using direct heuristic dynamic programming for chaotic discrete-time system, *J. Vib. Control* 22 (2016), no. 2, 595–603.

[15] R. Goetschel, Jr. and W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems* 18 (1986), no. 1, 31–43.

[16] J. Y. Halpern, *Reasoning about uncertainty*, MIT Press, Cambridge, MA, 2003.

[17] R. Hilfer, *Applications of fractional calculus in physics*, World Sci. Publishing, River Edge, NJ, 2000.

[18] N. V. Hoa, Fuzzy fractional functional differential equations under Caputo gH-differentiability, *Commun. Nonlinear Sci. Numer. Simul.* 22 (2015), no. 1-3, 1134–1157.

[19] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[20] C. Li and F. Zeng, *Numerical methods for fractional calculus*, Chapman & Hall/CRC Numerical Analysis and Scientific Computing, CRC, Boca Raton, FL, 2015.

[21] A. B. Malinowska and D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imp. Coll. Press, London, 2012.

[22] M. Mazandarani and A. V. Kamyad, Modified fractional Euler method for solving fuzzy fractional initial value problem, *Commun. Nonlinear Sci. Numer. Simul.* 18 (2013), no. 1, 12–21.

[23] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, Wiley, New York, 1993.

[24] S. I. Muslih and D. Baleanu, Formulation of Hamiltonian equations for fractional variational problems, *Czechoslovak J. Phys.* 55 (2005), no. 6, 633–642.

[25] S. I. Muslih, D. Baleanu and E. Rabei, Hamiltonian formulation of classical fields within Riemann-Liouville fractional derivatives, *Phys. Scr.* 73 (2006), no. 5, 436–438.

[26] M. Najariyan and M. H. Farahi, Optimal control of fuzzy linear controlled system with fuzzy initial conditions, *Iran. J. Fuzzy Syst.* 10 (2013), no. 3, 21–35.

[27] M. Najariyan and M. H. Farahi, A new approach for the optimal fuzzy linear time invariant controlled system with fuzzy coefficients, *J. Comput. Appl. Math.* 259 (2014), part B, 682–694.

[28] M. Najariyan and M. H. Farahi, A new approach for solving a class of fuzzy optimal control systems under generalized Hukuhara differentiability, *J. Franklin Inst.* 352 (2015), no. 5, 1836–1849.

[29] A. B. Malinowska, T. Odzijewicz and D. F. M. Torres, *Advanced methods in the fractional calculus of variations*, Springer Briefs in Applied Sciences and Technology, Springer, Cham, 2015.

[30] E. R. Pinch, *Optimal control and the calculus of variations*, Oxford Science Publications, Oxford Univ. Press, New York, 1993.

[31] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.

[32] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The mathematical theory of optimal processes*, Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt, Interscience Publishers John Wiley & Sons, Inc. New York, 1962.
[33] S. Poosheh, R. Almeida and D. F. M. Torres, Fractional order optimal control problems with free terminal time, *J. Ind. Manag. Optim.* 10 (2014), no. 2, 363–381. [arXiv:1302.1717](/)

[34] S. Salahshour, T. Allahviranloo and S. Abbasbandy, Solving fuzzy fractional differential equations by fuzzy Laplace transforms, *Commun. Nonlinear Sci. Numer. Simul.* 17 (2012), no. 3, 1372–1381.

[35] S. Salahshour, T. Allahviranloo, S. Abbasbandy and D. Baleanu, Existence and uniqueness results for fractional differential equations with uncertainty, *Adv. Difference Equ.* 2012, 2012:112, 12 pp.

[36] J. Soolaki, O. S. Fard and A. H. Borzabadi, Generalized Euler-Lagrange equations for fuzzy variational problems, *SeMA Journal* 73 (2016), no. 2, 131–148.

[37] J. Soolaki, O. S. Fard and A. H. Borzabadi, Generalized Euler-Lagrange equations for fuzzy fractional variational calculus, *Math. Commun.* 21 (2016), 199–218.

[38] T. Takagi and M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, *IEEE Trans. on SMC*, 15 (1985), 116–132.

[39] V. E. Tarasov, Fractional variations for dynamical systems: Hamilton and Lagrange approaches, *J. Phys. A* 39 (2006), no. 26, 8409–8425.

[40] G. S. Taverna and D. F. M. Torres, Generalized fractional operators for nonstandard Lagrangians, *Math. Meth. Appl. Sci.* 38 (2015), no. 9, 1808–1812. [arXiv:1404.6483](/)

[41] J. Xu, Z. Liao and J. J. Nieto, A class of linear differential dynamical systems with fuzzy matrices, *J. Math. Anal. Appl.* 368 (2010), no. 1, 54–68.

[42] D. Yang and K.-Y. Cai, Finite-time quantized guaranteed cost fuzzy control for continuous-time nonlinear systems, *Expert Systems with Applications* 37 (2010), no. 10, 6963–6967.

Submitted Jun 6, 2016; revised Nov 27, 2016; accepted Feb 3, 2017.

E-mail address: osfard@du.ac.ir
E-mail address: javad.soolaki@gmail.com
E-mail address: delfim@ua.pt