DelayDiffEq: Generating Delay Differential Equation Solvers via Recursive Embedding of Ordinary Differential Equation Solvers

David Widmann and Chris Rackauckas

1Uppsala University, Sweden
2Massachusetts Institute of Technology, USA
3Julia Computing
4Pumas-AI

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Abstract

Traditional solvers for delay differential equations (DDEs) are designed around only a single method and do not effectively use the infrastructure of their more-developed ordinary differential equation (ODE) counterparts. In this work we present DelayDiffEq, a Julia package for numerically solving delay differential equations (DDEs) which leverages the multitude of numerical algorithms in OrdinaryDiffEq for solving both stiff and non-stiff ODEs, and manages to solve challenging stiff DDEs. We describe how compiling the ODE integrator within itself, and accounting for discontinuity propagation, leads to a design that is effective for DDEs while using all of the ODE internals. We highlight some difficulties that a numerical DDE solver has to address, and explain how DelayDiffEq deals with these problems. We show how DelayDiffEq is able to solve difficult equations, how its stiff DDE solvers give efficiency on problems with time-scale separation, and how the design allows for generality and flexibility in usage such as being repurposed for generating solvers for stochastic delay differential equations.

Keywords Julia, Delay Differential Equation, Scientific Computing

1 Introduction

In nature many changes do not occur instantaneously, as prominent examples such as gestation times and incubation periods indicate. This lead to the rise of delay differential equations (DDEs) in many areas of mathematical modeling, such as population dynamics, pharmacokinetics, and control problems [2]. In contrast to ordinary differential equations (ODEs), DDEs allow the rate of change to depend on past conditions of the considered system.

However, while DDEs are a small conceptual change to ODEs, there is a larger mathematical difference that arises. Specifically, many DDEs have a discontinuity in the initial condition which, unlike ODEs, will continue to be in the system and propagate as we describe further in Section 2.3. Handling these propagations is crucial to performance and accuracy, and thus the software of this field, such as the MATLAB dde23 [14] and the classic RADAR5 [9], are completely separate software from those of ODE integrators. This gives rise to many development and maintenance issues: while in theory an advance in linear solvers (Newton-Krylov methods, preconditioners, etc.) can be used in DDE solvers, in practice none of these solvers support these features while their more-developed ODE counterparts do. This raises the question as to whether DDE software could be designed to more effectively make use of the ODE infrastructure so that all developments of the ODE solvers are inherited by the DDE solvers.

To answer this challenge we developed DelayDiffEq, a DDE solver package in Julia [4] which uses the ODE solvers of OrdinaryDiffEq [13] to develop the DDE solvers. We describe a form of “compilation trick”, compiling an ODE integrator inside of an ODE integrator, which is used to generate the DDE solvers from the ODE solvers. This alone however does not guarantee the accurate handling of discontinuities, and thus DelayDiffEq then modifies the stepping to account for these issues. This gives a DDE solver where all methods of the ODE solver, ones for non-stiff and stiff equations, are directly inherited by the DDE solver with all of the available options. We show that this generated solver is capable of handling known difficult equations to verify its accuracy.

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1 The source code of the paper and its examples are available on Github.
2 Background

2.1 Definition

Loosely speaking, a delay differential equation (DDE) is an ordinary differential equation (ODE) in which the derivative depends on past values of the state. More concretely, let $t \in \mathbb{R}$ be the independent variable, which usually denotes time, and $x(t) \in \mathbb{R}^n$ be the dependent state at time $t$. Then the differential equation

$$x'(t) = f(t, x(t), x(t - \tau)), \quad (1)$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\tau > 0$, is called a DDE with constant delay $\tau$. In general, a delay $\tau$ may also depend on $t$ or even on both $t$ and $x(t)$. In this case we call the DDE time-dependent or state-dependent, respectively. Additionally, DDEs may contain multiple delays. A further generalization is possible by allowing the state to depend also on its derivative at previous time points, yielding so-called neutral DDEs.

2.2 History function

For ODEs, so-called initial value problems (IVPs) can be formulated by specifying an initial condition $x(t) = x_0$ at an initial time point $t_0$. For DDEs, usually declaring an initial value only at time point $t_0$ is not sufficient. Instead typically a so-called history function, or initial function, has to be provided for some times $t \leq t_0$. Thus commonly an IVP formulation for the DDE in eq. (1) is given by

$$x'(t) = f(t, x(t), x(t - \tau)), \quad t \geq t_0,$n

$$x(t) = x_0(t), \quad t \leq t_0. \quad (2)$$

2.3 Propagated discontinuities

Even if $x_0$, $f$, and $\tau(t, x(t))$ are smooth, typically the solution $x$ is non-smooth since it has a derivative jump discontinuity at the initial time point $t_0$, i.e.,

$$\lim_{t \to t_0^-} x'(t) \neq \lim_{t \to t_0^+} x'(t).$$

If there exist time points $t > t_0$ such that the deviated argument $t - \tau(t, x(t)) < t_0$, then the discontinuity at time point $t_0$ will propagate in time and induce discontinuities at subsequent time points. For non-neutral DDEs these propagated discontinuities are usually discontinuities of higher-order derivatives, leading to a so-called smoothing of the solution, whereas in general for neutral DDEs smoothing does not occur [2, 3].

2.4 Example: Dynamical structure

Mackey and Glass’ model of circulating blood cells [8] is a scalar DDE system but still for certain parameter values we can observe a chaotic behavior. The model is given by

$$x'(t) = \lambda \theta^n x(t - \tau) - \gamma x(t), \quad t \geq 0, \quad (3)$$

$$x(t) = x_0, \quad t \leq 0.$$n

For the parameter values $\lambda = 2$, $\theta = 1$, $n = 9.65$, $\tau = 2$, $\gamma = 1$, and $x_0 = 0.5$ the system exhibits a chaotic solution [8, Figure 6(e)]. In Figure 1 the chaotic solution computed with DelayDiffeq and its time delay embedding are shown for time $300 \leq t \leq 600$.

2.5 Method of steps

Consider an IVP for a DDE with one constant delay, given by eq. (2). Assuming that a solution to this problem exists, a solution can be found by computing solutions of an iterative sequence of IVPs for different ODEs. More concretely, for $k = 1, 2, \ldots$ one computes a solution $x_k : [t_{k-1}, t_k - \tau] \to \mathbb{R}^n$ of the IVP

$$x'_k(t) = f(t, x_k(t), x_{k-1}(t - \tau)), \quad t \in [t_{k-1}, t_k],$$

$$x_k(t_{k-1}) = x_{k-1}(t_{k-1}), \quad (4)$$

where $t_k := t_0 + k \tau$. Then the function $x$ defined by $x(t) := x_k(t) \iff t \in [t_{k-1}, t_k]$ is a solution to the original IVP in eq. (2).

This algorithm for solving DDEs is called method of steps. It can be applied more generally also to IVPs of
DDEs with multiple state-dependent delays, as long as the delays are uniformly strictly greater than 0. The same iterative approach can be used to transfer results from the theory of ODEs such as existence, uniqueness, and boundedness of solutions to DDEs without vanishing delays.

3 Numerical challenges

For DDEs, the computation of numerical solutions of IVPs can be challenging. In theory the method of steps allows to reduce the problem to the computation of a sequence of numerical solutions to IVPs of ODEs. A crucial requirement for this approach to work are so-called continuous ODE methods that provide a numerical solution not only for a grid of discrete time points but for all time points in the integration interval.

A disadvantage of the method of steps that it can be inefficient. The algorithm requires that the step size is upper bounded by the minimally occurring delay, and hence the number of time steps can become large if the maximally allowed step size is small relative to the integration time interval. Additionally, problems with vanishing delays cannot be solved by the method of steps.

Another difficulty is posed by the fact that usually numerical methods for solving ODEs expect the solution to be sufficiently smooth in each step. To ensure this, propagated discontinuities have to be considered and included in the grid of time points that the numerical method steps to \[3\]. For constant delays the relevant discontinuities can be computed a priori but for state-dependent delays this is not possible. Even for constant delays numerical difficulties can arise, as Shampine and Thompson \[14\] illustrate: two constant delays of 1/3 and 1, albeit both inducing propagated discontinuities at time point \(t_0 + 1\), might seem to cause discontinuities at different time points numerically since 1/3 cannot be represented exactly by the floating point arithmetic of computers.

4 How DelayDiffEq works

So far one of the core design principles of DelayDiffEq has been to exploit the large amount of numerical ODE algorithms readily available in the Julia package OrdinaryDiffEq. The ODE solvers in OrdinaryDiffEq support both stiff and non-stiff ODEs\[2\]. Moreover, the numerical methods are continuous and come with specialized interpolations.

The design of DelayDiffEq is as follows. In order for a user to describe a DDE, we need to have a history function which the user can access. This history function is a continuous function of the states of the DDE solution, which, if we only had an ODE, would be given by the dense output (continuous interpolation) of the ODE solver. Thus what we do is the following:

1. Create an ODE integrator from OrdinaryDiffEq with state size matching the DDE we wish to solve. The integrator comes with the property that integ(t) computes the interpolation at time point \(t\) given the current solution of the ODE.

2. Generate an ODE by enclosing this interpolation object into the user-defined DDE: \( f_{ODE}(t, x(t)) = f(t, x(t), \text{integ}) \).

3. Create a new ODE integrator to solve the \( f_{ODE} \) system. If the interpolation integ is correct, then the solution of this new ODE will give the solution of the DDE.

The second ODE integrator will have the correct state values \(x(t)\) and stage calculation if the function integ is correct. On the other hand, integ is only a correct interpolation if it has the correct state values \(x(t)\) and stage calculations. Thus we effectively alias the caches between the two versions of the ODE integrators, so that a step of the second ODE integrator updates its history function, which then allows for an accurate step of the integrator.

Notably, this scheme does not require implementing any new numerical integrators and simply relies on the ODE integration itself. However, as described this scheme is correct only if a new step uses only the history of previous steps, i.e., only when the step size is smaller than the minimum delay time. Thus while this generates an ODE solver, more work must be done to make this routine efficient.

4.1 Unconstrained time stepping

As discussed above, it is desirable to not upper bound the step size that the numerical solver is allowed to take. However, unconstrained time stepping comes at a cost. Allowing unconstrained steps implies that the numerical solution of the DDE is only available as the solution to an implicit problem, even for explicit ODE methods. Essentially, one must know that the next step is correct, but if the time step is larger than the minimum delay then the integrator will use an inaccurate extrapolation from the previous time step. In order to correct for this behavior, we transform this problem into an implicit problem which can be solved via fixed-point iteration.

Formally, in the \(k\)th integration step we want to compute an approximation \(\hat{x}_k: [t_{k-1}, t_k] \rightarrow \mathbb{R}^n\) of the solution to the DDE in eq. \([1]\) with approximate solution \(\hat{x}(t)\) for \(t \leq t_{k-1}\) as history function. In DelayDiffEq, we

\[\text{Although it “is extremely difficult to classify stiffness in a completely rigorous manner” }\[3\], it is commonly characterized by the existence of processes with significantly different time scales \[4,5\],}
perform a fixed-point iteration to compute \( \hat{x}_k \) if it is only available implicitly. Hereby we proceed in the following way:

1. Extend the dense solution \( \hat{x} \) up to time \( t_{k-1} \) to the time interval \([t_{k-1}, t_k]\) by extrapolating the specialized collocation polynomials available in OrdinaryDiffEq.

2. Based on the approximation up to time \( t_k \), compute a discrete approximation of \( x(t_k) \) and additional collocation stages in the time interval \([t_{k-1}, t_k]\) by using an ODE method from OrdinaryDiffEq.

3. Obtain a new dense approximation of the solution on time interval \([t_{k-1}, t_k]\) by completing this discrete approximation using the specialized collocation polynomials in OrdinaryDiffEq.

4. If in steps 2 and 3 the "old" dense approximation was only evaluated at time points \( \leq t_{k-1} \), we stop. Otherwise we repeat steps 2 and 3 iteratively based on the newly calculated approximations, until some stopping criterion is met.

The fixed-point iteration can be performed efficiently since OrdinaryDiffEq allows us to specify that the Jacobian and its factorization should only be evaluated once. Optionally, Anderson acceleration \cite{Anderson1965} can be used to accelerate the convergence of the fixed-point iteration. We have observed that, e.g., for the one-dimensional IVP

\[
\begin{align*}
x'(t) &= -x(t-1/3) - x(t-1/5), & t \in [0, 100], \\
x(t) &= \delta_0(t), & t \leq 0,
\end{align*}
\]

Anderson acceleration reduces the number of evaluations of the model, fixed-point iterations, and convergence failures from 5517, 720, and 58 to 2829, 308, and 4, respectively.

### 4.2 Discontinuities

User-provided discontinuities and discontinuities arising from constant time delays up to the order of the ODE method are added as grid points. Existing functionality in OrdinaryDiffEq for handling discontinuities ensures that the numerical solver steps to these discontinuities exactly and integration is restarted appropriately at these time points.

Discontinuities due to state-dependent delays are handled by locating time points \( \zeta \) that satisfy

\[
\zeta = \xi - \tau(\hat{x}(\xi), \xi), \quad (5)
\]

where \( \zeta \) is a known discontinuity and \( \hat{x} \) is the current approximation of the solution \( x \), and adding them as grid points. We search for such discontinuities only if a step is rejected and rely on the error estimator otherwise. If a step is rejected, we check for sign changes of \( \xi - \tau(\hat{x}(\xi), \xi) - \zeta \) at pre-defined number of equally spaced time points \( \xi \) in the current time interval \([t_{k-1}, t_k]\), where \( \zeta \leq t_{k-1} \) are known discontinuities. If a sign change is detected, a solution \( \xi \) of eq. (5) is calculated by a root finding algorithm and added as grid point, and the step size is reduced accordingly. We do not try to find the earliest time point that satisfies eq. (5) but stop as soon as one discontinuity is found in the time interval \([t_{k-1}, t_k]\).

A very similar algorithm is used by the DDE solver RADAR5 by Guglielmi and Hairer, however, they additionally locate discontinuities if the error estimates increases above a certain threshold in subsequent time steps.

Users may also specify no time delays at all. In that case, the choice of a method with residual error estimator such as RK4 or OwrenZen5 is strongly recommended because the residual error estimator attempts to ensure accuracy over the entire solution interval instead of simply at step points. This may be required for the history function (i.e., the ODE integrator’s interpolation) to have the relevant accuracy.

### 4.3 Interface

We do not present all features and options of DelayDiffEq but only provide one example for how DDEs can be declared and solved with DelayDiffEq. For a more thorough explanation we refer the reader to the extensive documentation of the DifferentialEquations framework.

The interface of DelayDiffEq is similar to the one for ODEs in OrdinaryDiffEq or stochastic differential equations (SDEs) in StochasticDiffEq \cite{StochasticDiffEq}. The code snippet below shows how a version of the so-called Hutchinson’s equation \cite{Hutchinson1958}, a population growth model also known as delay logistic equation, can be formulated with DelayDiffEq. The model is given by

\[
\begin{align*}
x'(t) &= -x(t-1), & t \geq 0, \\
x(t) &= 1, & t \leq 0.
\end{align*}
\]

In DelayDiffEq, a DDE problem consists of the differential equation, the history function, the integration time span, and the time delays. Optionally a value at the initial time point and model parameters can be provided.

```plaintext
using DelayDiffEq

# Hutchinson's equation
f(x, h, p, t) = - h(p, t - 1)
```

\(^{4}\)The documentation is available online at [http://docs.sciml.ai/stable/](http://docs.sciml.ai/stable/)
Although not useful in this case, for illustrative purposes we demonstrate how a problem with state-dependent delay can be formulated.

```julia
# Complete DDE problem formulation
prob = DDEProblem(f, h, tspan;
    constant_lags = (1.0,))
```

In both cases presented above, the `solve` function is used to compute a solution to the problem with the ODE method `Tsit5`, using the default solver options. `DelayDiffEq` supports DDEs with mass matrices and neutral DDEs as well. Similar to the ODE case, singular mass matrix DDEs generate differential-algebraic equations (DAEs) which are then solved via the `OrdinaryDiffEq` DAE solvers.

Next we consider a DDE model of antigen stimulated antibody production by Waltman \[17\]. The model is given by

\[
\begin{align*}
x_1'(t) &= -r x_1(t) x_2(t) - s x_1(t) x_4(t), \\
x_2'(t) &= -r x_1(t) x_2(t) + \alpha r x_1(x_5(t)) x_2(x_5(t)) H(t - t_1), \\
x_3'(t) &= r x_1(t) x_2(t), \\
x_4'(t) &= -s x_1(t) x_4(t) - \gamma x_4(t) + \beta r x_1(x_6(t)) x_2(x_6(t)) H(t - t_2), \\
x_5'(t) &= H(t - t_1) \frac{x_1(t) x_2(t) + x_3(t)}{x_1(x_5(t)) x_2(x_5(t)) + x_3(x_5(t))}, \\
x_6'(t) &= H(t - t_2) \frac{10^{-12} + x_2(t) + x_3(t)}{10^{-12} + x_2(x_6(t)) + x_3(x_6(t))},
\end{align*}
\]

where \( H \) is the heavyside step function, i.e., \( H(t) = 0 \) if \( t < 0 \) and \( H(t) = 1 \) if \( t \geq 0 \). As Guglielmi and Hairer \[9\], we consider the model on the time interval \( t \in [0, 300] \) with parameter values \( \alpha = 1.8, \beta = 20, \gamma = 0.002, r = 5 \times 10^4, s = 10^5, t_1 = 35, t_2 = 197, \) and history functions \( x_1(t) = 5 \times 10^{-6}, x_2(t) = 10^{-15}, \) and \( x_3 = x_4 = x_5 = x_6 = 0 \) for all \( t \leq 0 \).

This problem is very stiff, contains vanishing state-dependent delays, and displays steep increases in components \( x_2, x_4, \) and \( x_6 \) at time points \( t_1 \) and \( t_2 \) \[9\] \[17\]. Guglielmi and Hairer \[9\] compared their code RADAR5 with five existing numerical solvers, and only RADAR5 was able to solve problem \[9\] \[17\]. As shown in Figure 2, `DelayDiffEq` manages to reproduce Figure 3 in \[9\] using the stiff ODE solver `KenCarp5`. As Guglielmi and Hairer \[9\], we used relative tolerances of \( 10^{-9} \) and absolute tolerances \( 10^{-21} \) and \( 10^{-9} \) for the first four and the last two components, respectively. We are pleased to see

4 A later produced blog post showed that Maple was able to successfully solve the problem, while Mathematica and MATLAB solvers were not.

5 Examples

5.1 Stiff problem: Antibody production

We note in passing that the quasi-Newton schemes used in the ODE solvers only require the Jacobian and its factorization at the beginning of the step for the non-FIRK implicit methods. This means that the entire handling of the unconstrained time stepping of Section 4.3 can be done without recalculating the Jacobian or refactorizing the Jacobian. As these steps are commonly the most costly aspect for solving large stiff equations, this greatly improves the efficiency of the implementation over using a naive Newton solve.
that DelayDiffEq can cope with the discontinuities at the time points \( t_1 = 35 \) and \( t_2 = 197 \), even though they are not declared as grid points as in [9].

### 5.2 Sensitivity analysis via automatic differentiation

The SciMLSensitivity tutorials highlight that DelayDiffEq is compatible with forward- and reverse-mode automatic differentiation. For this example, we define a Lotka-Volterra model with delay and constant history function:

```plaintext
using DelayDiffEq
using SciMLSensitivity, ForwardDiff, Zygote
#
#
function lv!(du, u, h, p, t)
    x, y = u
    α, β, δ, γ = p
    xτ = h(p, t - 0.1)[1]
    du[1] = dx = (α - β * y) * xτ
    du[2] = dy = (δ * x - γ) * y
    return nothing
end
#
#
# Constant history function
history_lv(p, t) = ones(2)
```

The following function computes the solution of the DDE model on an equally-spaced time grid:

```plaintext
function predict_lv(p)
    prob = DDEProblem(lv!, history_lv, (0.0, 5.0), p;
    constantlags = (0.1,))
    sol = solve(prob, MethodOfSteps(Tsit5());
    saveat = 0.1)
    return Array(sol)
end
```

This function includes the DDE solver and is compatible with forward- and reverse mode automatic differentiation:

```plaintext
# Parameters
p = [2.2, 1.0, 2.0, 0.4]
# Compute Jacobian
jacFD = ForwardDiff.jacobian(predict_lv, p)
jacZygote = Zygote.jacobian(predict_lv, p)
```

### 5.3 Algorithm development and benchmarking

To facilitate the research and development of efficient DDE solving, DelayDiffEq ties into the SciMLBenchmarks ecosystem to provide an easy way to benchmark its DDE solvers. A demonstration of a work-precision diagram generated by DiffEqDevTools on a delay differential equation model of quorum sensing [5] is shown in Figure 3.

```plaintext
using DelayDiffEq
using DiffEqDevTools
using DDEProblemLibrary
function benchmark()
    # Compute reference solution of third state
    # with low tolerances
    fpsolve = NLFunctional(; max_iter = 1000)
    alg = MethodOfSteps(Vern9());
    sol = TestSolution(solve(prob_dde_qs, alg;
        reltol = 1e-14,
        abstol = 1e-14,
        save_idxs = 3))
    # Benchmark different algorithms
    setups = [Dict(:alg => MethodOfSteps(RK4())),
        Dict(:alg => MethodOfSteps(DP5())),
        Dict(:alg => MethodOfSteps(OwrenZen3())),
        Dict(:alg => MethodOfSteps(OwrenZen5())),
        Dict(:alg => MethodOfSteps(Vern6())),
        Dict(:alg => MethodOfSteps(Vern7())),
        Dict(:alg => MethodOfSteps(Vern8())),
        Dict(:alg => MethodOfSteps(Vern9())),
        Dict(:alg => MethodOfSteps(Rosenbrock23())),
        Dict(:alg => MethodOfSteps(Rodas4())),
        Dict(:alg => MethodOfSteps(Rodas5P())),
    ]
    abstols = exp10.(-4:11)
    reltols = exp10.(-1:8)
    return WorkPrecisionSet(prob_dde_qs,
        abstols,
        reltols,
        setps;
        save_idxs = 3,
        appxsol = sol,
        maxiters = Int(1e5),
        error_estimate = :l2)
end
```

Notice how with very little code, DDE integrators for non-stiff and stiff DDEs can be compared to help optimize the implementations.
5.4 Stochastic delay differential equations

This same infrastructure of DelayDiffEq was used on StochasticDiffEq for SDEs to generate convergent solvers for stochastic delay differential equations, as was demonstrated in [15].

6 Conclusion

In this paper we presented DelayDiffEq, a Julia software package for numerically solving DDEs. We explained the employed numerical algorithm and showed that it can successfully solve very stiff systems with state-dependent delays. But most importantly, the design of DelayDiffEq allows it to inherit all of the development work from the OrdinaryDiffEq solvers of DifferentialEquations, a design which in turns leads to rapid development of DelayDiffEq even without developers having to focus on DDEs. This is thus an example of how composable software designs can change the efficiency of scientific computing.

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