On the mean curvature type flow for convex capillary hypersurfaces in the ball

Yingxiang Hu¹ · Yong Wei² · Bo Yang³ · Tailong Zhou²,⁴

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Abstract
In this paper, we study the mean curvature type flow for hypersurfaces in the unit Euclidean ball with capillary boundary, which was introduced by Wang and Xia (Commun Anal Geom, 2019) and Wang and Weng (Calc Var Partial Differ Equ 59(5):149–174, 2020). We show that if the initial hypersurface is strictly convex, then the solution of this flow is strictly convex for \( t > 0 \), exists for all positive time and converges smoothly to a spherical cap. As an application, we prove a family of new Alexandrov–Fenchel inequalities for convex hypersurfaces in the unit Euclidean ball with capillary boundary.

Mathematics Subject Classification 53C44 · 53C21 · 35K93 · 52A40

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Yong Wei
yongwei@ustc.edu.cn

Yingxiang Hu
huyingxiang@buaa.edu.cn

Bo Yang
boyang16@amss.ac.cn

Tailong Zhou
ztl20@ustc.edu.cn; zhoutailong@scu.edu.cn

1 School of Mathematics, Beihang University, Beijing 100191, People’s Republic of China
2 School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, People’s Republic of China
3 Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
4 School of Mathematics, Sichuan University, Chengdu 610065, Sichuan, People’s Republic of China
1 Introduction

In this paper, we are interested in the mean curvature type flow for hypersurfaces in the unit Euclidean ball with capillary boundary, which was introduced recently by Wang-Xia [25] and Wang-Weng [23]. We first describe some definitions and notations for such hypersurfaces. The readers are referred to [28] for more details. Let $\Sigma \subset \bar{B}^{n+1}(n \geq 2)$ be a properly embedded smooth hypersurface in the unit Euclidean ball $\bar{B}^{n+1}$, given by an embedding $x: \bar{B}^n \to \bar{B}^{n+1}$ satisfying

$$\Sigma = x(\bar{B}^n) \subset \bar{B}^{n+1}, \quad \partial \Sigma = x(\partial \bar{B}^n) \subset \mathbb{S}^n.$$  

Let $\bar{N}$ be the unit outward normal of $\mathbb{S}^n = \partial \bar{B}^{n+1}$, and $\nu$ be a smooth choice of the unit normal of $\Sigma$. For $\theta \in (0, \pi)$, we call that $\Sigma$ has $\theta$-capillary boundary $\partial \Sigma$ on $\mathbb{S}^n$ if $\Sigma$ intersects $\mathbb{S}^n$ at the constant contact angle $\theta$, that is,

$$\langle \bar{N} \circ x, \nu \rangle = -\cos \theta, \quad \text{along } \partial \bar{B}^{n+1}.$$  

In particular, if $\theta = \frac{\pi}{2}$, i.e., $\Sigma$ intersects $\mathbb{S}^n$ orthogonally, we call that $\Sigma$ has free boundary. Two model examples are the spherical cap of radius $r$ around a constant unit vector $e \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ with $\theta$-capillary boundary, given by

$$C_{\theta, r}(e) := \left\{ x \in \bar{B}^{n+1} : \left| x - \sqrt{r^2 + 2r \cos \theta + 1} e \right| = r \right\}, \quad (1.1)$$

and the flat ball around $e$ with $\theta$-capillary boundary, given by

$$C_{\theta, \infty}(e) := \{ x \in \bar{B}^{n+1} : \langle x, e \rangle = \cos \theta \}. \quad (1.2)$$

Denote the principal curvatures of $\Sigma$ by $\kappa = (\kappa_1, \cdots, \kappa_n)$. For $k = 1, \cdots, n$, we denote by $H_k$ the normalized $k$th mean curvature of $\Sigma$, which is defined as the normalized $k$th elementary symmetric polynomial of $\kappa$:

$$H_k = \binom{n}{k}^{-1} \sigma_k(\kappa), \quad k = 1, \cdots, n.$$  

We also denote the mean curvature as $H = n H_1$. We say that $\Sigma$ is convex (resp. strictly convex) if its principal curvatures $\kappa_i \geq 0$ (resp. $\kappa_i > 0$). In this paper, we always assume that $\Sigma$ is convex. Denote by $\hat{\Sigma}$ the convex body in $\mathbb{S}^n$ enclosed by $\partial \Sigma$, and $\tilde{\Sigma}$ the convex domain in $\bar{B}^{n+1}$ enclosed by $\partial \Sigma$ and $\Sigma$. See Fig. 1. We choose the unit normal $\nu$ of $\Sigma$ as the one pointing outward of $\hat{\Sigma}$. For each $e \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, the smooth vector field $X_e$ in $\mathbb{R}^{n+1}$ defined by

$$X_e = \langle x, e \rangle x - \frac{1}{2} (|x|^2 + 1) e \quad (1.3)$$

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is a conformal Killing vector field in $\mathbb{R}^{n+1}$ and its restriction on $\partial B^{n+1}$ is a tangential vector field on $\partial B^{n+1}$ (see [24, Prop.3.1]). Using $X_e$, the following Minkowski type formula (see [28, Prop. 2.8]) holds:

$$\int_{\Sigma} H_k - 1 (x + \cos \theta v, e) dA = \int_{\Sigma} H_k (X_e, v) dA, \quad k = 1, \ldots, n.$$  (1.4)

Based on the formula (1.4), the following mean curvature type flow with $\theta$-capillary boundary was introduced by Wang and Xia [25] for the case $\theta = \pi/2$, and later by Wang and Weng [23] for general case $\theta \in (0, \pi)$:

$$\begin{cases} 
(\partial_t x) = \left( n (x + \cos \theta v, e) - H (X_e, v) \right) v & \text{in } \bar{B}^n \times [0, T), \\
(\bar{N} \circ x, v) = - \cos \theta & \text{on } \partial B^n \times [0, T), \\
x(\cdot, 0) = x_0(\cdot) & \text{on } \bar{B}^n,
\end{cases}$$  (1.5)

where $(\cdot)^\perp$ denotes the projection to the normal bundle of $\Sigma$. Denote $\Sigma_t = x(\bar{B}^n, t)$ as the solution of the flow (1.5). The flow (1.5) has a property that the volume $|\hat{\Sigma}_t|$ is preserved which follows from (1.4) for $k = 1$. The convergence results [23, 25] of the flow (1.5) for star-shaped initial hypersurface $\Sigma$, i.e., $\langle X_e, v \rangle > 0$ holds everywhere on $\Sigma$, can be stated as follows.

**Theorem A** Let $\Sigma \subset \bar{B}^{n+1} (n \geq 2)$ be a properly embedded smooth hypersurface with $\theta$-capillary boundary, given by an embedding: $x : \bar{B}^n \to \Sigma \subset \bar{B}^{n+1}$. Assume that $\Sigma$ is star-shaped with respect to $e \in S^n$, i.e., $\langle X_e, v \rangle > 0$ holds everywhere in $\Sigma$.

(i) [25] If $\theta = \pi/2$, then the solution $\Sigma_t$ of the flow (1.5) exists for all positive time and converges smoothly to a spherical cap with free boundary as $t \to \infty$, whose enclosed domain has the same volume as $|\hat{\Sigma}|$.

(ii) [23] If $\theta$ satisfies $|\cos \theta| < \frac{3n+1}{5n-1}$, then the solution $\Sigma_t$ of the flow (1.5) exists for all positive time and subsequently converges smoothly to a spherical cap with $\theta$-capillary boundary as $t \to \infty$, whose enclosed domain has the same volume as $|\hat{\Sigma}|$.

The first aim of this paper is to prove the preservation of the strict convexity along the flow (1.5). The main tool is the tensor maximum principle in Theorem 4.1, which was developed by the authors in [13]. With help of the convexity preserving, we establish the convergence result of the flow (1.5) for convex hypersurfaces with $\theta$-capillary boundary in the unit Euclidean ball.

**Theorem 1.1** Let $\Sigma \subset \bar{B}^{n+1} (n \geq 2)$ be a properly embedded, strictly convex smooth hypersurface with $\theta$-capillary boundary ($\theta \in (0, \pi/2]$), given by an embedding: $x : \bar{B}^n \to \Sigma \subset \bar{B}^{n+1}$, where $(\cdot)^\perp$ denotes the projection to the normal bundle of $\Sigma$. Denote $\Sigma_t = x(\bar{B}^n, t)$ as the solution of the flow (1.5). The flow (1.5) has a property that the volume $|\hat{\Sigma}_t|$ is preserved which follows from (1.4) for $k = 1$. The convergence results [23, 25] of the flow (1.5) for star-shaped initial hypersurface $\Sigma$, i.e., $\langle X_e, v \rangle > 0$ holds everywhere on $\Sigma$, can be stated as follows.

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\( \hat{B}^{n+1} \). Then there exists a constant unit vector \( e \in S^n \) such that \( \langle X_e, v \rangle > 0 \) holds everywhere on \( \Sigma \), and the solution \( \Sigma_t \) of the flow (1.5) starting from \( \Sigma \) is strictly convex and exists for all time \( t > 0 \). Moreover, \( \Sigma_t \) converges smoothly to a spherical cap \( C_{\theta, r_{\infty}}(e) \) as \( t \to \infty \), where \( r_{\infty} \) is determined by \( |C_{\theta, r_{\infty}}(e)| = |\hat{\Sigma}| \).

The mean curvature type flow (1.5) is a locally constrained curvature flow, which was previously considered by Guan and Li [8] for closed hypersurfaces in space forms. Other kinds of locally constrained curvature flows have been investigated by many authors, see [3, 9, 10, 12, 16, 18, 20, 26, 27], etc. A main motivation of studying these locally constrained curvature flows is their powerful applications in proving geometric inequalities including the Alexandrov–Fenchel type inequalities for quermassintegrals.

The quermassintegrals for convex hypersurfaces with \( \theta \)-capillary boundary in \( \hat{B}^{n+1} \) were first introduced by Scheuer, Wang and Xia [19] for \( \theta = \frac{\pi}{2} \), and later by Weng and Xia [28] for \( \theta \in (0, \frac{\pi}{2}] \). Let \( \Sigma \subset \hat{B}^{n+1} (n \geq 2) \) be a properly embedded, smooth hypersurface with \( \theta \)-capillary boundary \( \partial \Sigma \subset \mathbb{S}^n \), where \( \theta \in (0, \frac{\pi}{2}] \). Then the quermassintegrals \( W_{k, \theta} (\hat{\Sigma}) \) for \( \hat{\Sigma} \) are defined by (2.2)–(2.4). In particular,

\[
W_{0, \theta} (\hat{\Sigma}) = |\hat{\Sigma}|, \quad W_{1, \theta} (\hat{\Sigma}) = \frac{1}{n+1} (|\Sigma| - \cos \theta |\partial \Sigma|).
\]

For strictly convex hypersurfaces in \( \hat{B}^{n+1} \) with \( \theta \)-capillary boundary, the following flow

\[
\begin{cases}
(\partial_t x)^\perp = \left( \frac{\langle x, e \rangle + \cos \theta \langle v, e \rangle}{H_n/H_{n-1}} - \langle X_e, v \rangle \right) v & \text{in } \hat{B}^n \times [0, T), \\
\langle \tilde{N} \circ x, v \rangle = -\cos \theta & \text{on } \partial \hat{B}^n \times [0, T), \\
x(\cdot, 0) = x_0(\cdot) & \text{on } \hat{B}^n,
\end{cases}
\]

was studied in [19, 28], and as an application they proved that for any \( \ell = 0, 1, \cdots, n-1 \), there holds

\[
W_{n, \theta} (\hat{\Sigma}) \geq (f_n \circ f_\ell^{-1})(W_{\ell, \theta} (\hat{\Sigma})),
\]

where \( f_\ell (r) \) is the strictly increasing function given by \( f_\ell (r) = W_{k, \theta} (C_{\theta, r}) \). Equality holds in (1.7) if and only if \( \Sigma \) is a spherical cap or a flat ball with \( \theta \)-capillary boundary.

As an application of our Theorem 1.1, we prove new sharp Alexandrov–Fenchel type inequalities relating \( W_{k, \theta} (\hat{\Sigma}) \) and \( W_{0, \theta} (\hat{\Sigma}) \) for \( k \geq 1 \).

**Theorem 1.2** Let \( \Sigma \subset \hat{B}^{n+1} (n \geq 2) \) be a properly embedded, convex smooth hypersurface in the unit ball with \( \theta \)-capillary boundary, where \( \theta \in (0, \frac{\pi}{2}] \). Then for any \( k = 1, \cdots, n-1 \), there holds

\[
W_{k, \theta} (\hat{\Sigma}) \geq (f_k \circ f_0^{-1})(W_{0, \theta} (\hat{\Sigma})).
\]

Equality holds in (1.8) if and only if \( \Sigma \) is a spherical cap or a flat ball with \( \theta \)-capillary boundary.

**Remark 1.3** The inequality (1.8) for \( k = n \) is the inequality (1.7) for \( \ell = 0 \) and has been proved in [19, 28].

The paper is organized as follows: In Sect. 2, we give some preliminaries for convex capillary hypersurfaces in the unit ball, including some estimates on the geometric quantities on the hypersurfaces. We also recall the definition of quermassintegrals for such hypersurfaces.
In Sect. 3, the evolution equations along the flow (1.5) will be deduced. In Sect. 4, we apply the tensor maximum principle to prove the preservation of convexity along the flow (1.5). In Sect. 5, we give the proofs of Theorems 1.1 and 1.2.

2 Convex hypersurfaces with capillary boundary

In this section, we collect some preliminaries on smooth hypersurfaces in the unit ball with capillary boundary, the quermassintegrals, and some estimates on convex hypersurfaces with capillary boundary.

2.1 Hypersurfaces in the ball with capillary boundary

Let \( \theta \in (0, \frac{\pi}{2}] \). Suppose that \( \Sigma \subset \overline{B}^{n+1} \) is a smooth, properly embedded, convex hypersurface with \( \theta \)-capillary boundary, which is given by an embedding \( x : \overline{B}^n \rightarrow \Sigma \) such that

\[
\Sigma = x(\mathbb{B}^n) \subset \mathbb{H}^{n+1}, \quad \partial \Sigma = x(\partial \mathbb{B}^n) \subset S^n.
\]

Then \( \partial \Sigma \subset S^n \) is a convex hypersurface of the unit sphere and bounds a convex body in \( S^n \), which we denote by \( \hat{\partial} \Sigma \), cf. [5, Theorem 1.1]. Denote by \( \hat{\Sigma} \) the bounded domain in \( \overline{B}^{n+1} \) enclosed by \( \Sigma \) and \( \hat{\Sigma} \). We denote by \( \nu \) the unit normal field of \( \Sigma \) and \( \bar{N} \) the position vector of \( \partial \Sigma \subset S^n \), see Fig. 1.

It follows from \( \langle \bar{N} \circ x, \nu \rangle = -\cos \theta \) that

\[
\begin{align*}
\hat{\bar{N}} \circ x &= \sin \theta \mu - \cos \theta \nu, \\
\bar{\nu} &= \cos \theta \mu + \sin \theta \nu.
\end{align*}
\] (2.1)

We denote by \( D \) the Levi-Civita connection of \( \mathbb{R}^{n+1} \) with respect to the Euclidean metric \( \delta \), and \( \nabla \) the Levi-Civita connection on \( \Sigma \) with respect to the induced metric from the embedding \( x : \mathbb{B}^n \rightarrow \Sigma \subset \mathbb{R}^{n+1} \), respectively. The second fundamental of \( \Sigma \) in \( \mathbb{R}^{n+1} \) is given by

\[
h(X, Y) := -\langle DX Y, \nu \rangle, \quad X, Y \in T(\Sigma).
\]

Note that \( \partial \Sigma \) can be viewed as a smooth closed hypersurface in \( S^n \) and \( \Sigma \), respectively. The second fundamental form of \( \partial \Sigma \) in \( S^n \) is given by

\[
\hat{h}(X, Y) := -\langle \nabla^{S^n} X Y, \bar{v} \rangle = -\langle DX Y, \bar{v} \rangle, \quad X, Y \in T(\partial \Sigma).
\]

The second fundamental form of \( \partial \Sigma \) in \( \Sigma \) is given by

\[
\tilde{h}(X, Y) := -\langle \nabla X Y, \mu \rangle = -\langle DX Y, \mu \rangle, \quad X, Y \in T(\partial \Sigma).
\]

Proposition 2.1 [28] Let \( \Sigma \subset \overline{B}^{n+1} \) be a \( \theta \)-capillary hypersurface. Let \( \{e_\alpha\}_{\alpha=1}^{n-1} \) be an orthonormal frame of \( \partial \Sigma \). Then the following relations hold on \( \partial \Sigma \):

1. \( \mu \) is a principal direction of \( \Sigma \), i.e., \( h(\mu, e_\alpha) = 0 \).
2. \( h_{\alpha\beta} = \sin \theta \hat{h}_{\alpha\beta} - \cos \theta \delta_{\alpha\beta} \).
3. \( \hat{h}_{\alpha\beta} = \cos \theta \hat{h}_{\alpha\beta} + \sin \theta \delta_{\alpha\beta} = \cot \theta h_{\alpha\beta} + \frac{1}{\sin \theta} \delta_{\alpha\beta} \).
4. \( \nabla_\mu h_{\alpha\beta} = \tilde{h}_{\mu\gamma} (h_{\mu\mu} \delta_{\alpha\gamma} - h_{\alpha\gamma}) \).
2.2 Quermassintegrals

The quermassintegrals for $\theta$-capillary hypersurfaces in the unit ball were first introduced for $\theta = \frac{\pi}{2}$ by Scheuer, Wang and Xia [19], and later generalized by Weng and Xia [28] to $\theta \in (0, \frac{\pi}{2})$. Let $\Sigma \subset \mathbb{B}^{n+1}$ be a smooth convex, embedded hypersurface with $\theta$-capillary boundary $\partial \Sigma \subset S^n$. The quermassintegrals for $\hat{\Sigma}$ are defined by

\[
W_{0,\theta}(\hat{\Sigma}) = |\hat{\Sigma}|,
\]

\[
W_{1,\theta}(\hat{\Sigma}) = \frac{1}{n+1} \left( |\hat{\Sigma}| - \cos \theta W_{0}^{S^n}(\hat{\partial \Sigma}) \right),
\]

and for $1 \leq k \leq n - 1$,

\[
W_{k+1,\theta}(\hat{\Sigma}) = \frac{1}{n+1} \left\{ \int_{\Sigma} H_k dA - \cos \theta \sin^k \theta W_{k}^{S^n}(\hat{\partial \Sigma}) \right. \\
- \cos^{k-1} \theta \sum_{\ell=0}^{k-1} \frac{(-1)^{k+\ell}}{n-k-\ell} \binom{k}{\ell} \left[ (n-k) \cos^2 \theta + k - \ell \right] \tan^\ell \theta W_{\ell}^{S^n}(\hat{\partial \Sigma}) \left. \right\}.
\]

For the free boundary case $\theta = \pi/2$, the definition (2.4) simplifies as

\[
W_{k+1,\pi/2}(\hat{\Sigma}) = \frac{1}{n+1} \left\{ \int_{\Sigma} H_k dA - \frac{k}{n-k+1} W_{k-1}^{S^n}(\hat{\partial \Sigma}) \right\}.
\]

Here $dA$ is the area element and $H_k$ is the $k$th normalized mean curvature of the hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ respectively, and for a $k$-dimensional submanifold $M \subset \mathbb{R}^{n+1}$ (with or without boundary), $|M|$ denotes the $k$-dimensional Hausdorff measure of $M$. The quermassintegrals $W_k^{S^n}(\hat{\partial \Sigma})$ in $S^n$ are defined by (see [4, 21])

\[
W_{0}^{S^n}(\hat{\partial \Sigma}) = |\hat{\partial \Sigma}|,
\]

\[
W_{1}^{S^n}(\hat{\partial \Sigma}) = \frac{1}{n} |\hat{\partial \Sigma}|,
\]

\[
W_{k+1}(\hat{\partial \Sigma}) = \frac{1}{n} \int_{\hat{\partial \Sigma}} H_k^{S^n} ds + \frac{k}{n-k+1} W_{k-1}^{S^n}(\hat{\partial \Sigma}), \quad 1 \leq k \leq n - 1,
\]

where $ds$ is the area element and $H_k^{S^n}$ is the $k$th normalized mean curvature of the hypersurface $\hat{\partial \Sigma}$ in $S^n$ respectively.

The quermassintegrals $W_{k,\theta}(\hat{\Sigma})$ defined in (2.2)–(2.4) are viewed as a natural counterparts to the quermassintegrals for smooth closed hypersurfaces in $\mathbb{R}^{n+1}$, as evidenced by the following nice variational formula:

**Proposition 2.2** [28] Let $\Sigma_t \subset \mathbb{B}^{n+1}$ be a family of smooth, properly embedded hypersurfaces with $\theta$-capillary boundary on $S^n$, given by $x(\cdot, t) : \mathbb{B}^n \rightarrow \Sigma_t \subset \mathbb{B}^{n+1}$, such that

\[
(\partial_t x)^\perp = f v
\]

for some normal speed $f$. Then there holds

\[
\frac{d}{dt} W_{k,\theta}(\hat{\Sigma}_t) = \frac{n+1-k}{n+1} \int_{\Sigma_t} H_k f dA, \quad 0 \leq k \leq n.
\]

Using the Minkowski formula (1.4) and the variational formula (2.6), we have the following monotonicity of quermassintegrals along the flow (1.5).
Lemma 2.3 Let $\Sigma_t$ be a smooth strictly convex solution of the flow (1.5). Then along the flow, $W_{0,\theta}(\Sigma_t)$ is preserved and $W_{k,\theta}(\Sigma_t)$ is non-increasing in time for all $1 \leq k \leq n - 1$. Moreover, $\frac{d}{dt} W_{k,\theta}(\Sigma_t) = 0$ if and only if $\Sigma_t$ is a spherical cap.

Proof Firstly, by the variational formula (2.6) and the Minkowski formula (1.4) for $k = 1$, we have

$$\frac{d}{dt} W_{0,\theta}(\Sigma_t) = \frac{n}{n+1} \int_{\Sigma_t} \left( \langle x + \cos \theta v, e \rangle - H_1 \langle X_e, v \rangle \right) dA = 0.$$ As the solution $\Sigma_t$ is strictly convex for $t > 0$, using the Newton’s inequality $H_{k+1} \leq H_k H_1$ and the Minkowski formula (1.4) again, we obtain

$$\frac{d}{dt} W_{k,\theta}(\Sigma_t) = \frac{n(n+1-k)}{n+1} \int_{\Sigma_t} \left( H_k \langle x + \cos \theta v, e \rangle - \langle X_e, v \rangle H_1 H_k \right) dA$$

$$= \frac{n(n+1-k)}{n+1} \int_{\Sigma_t} (H_{k+1} - H_1 H_k) \langle X_e, v \rangle dA$$

$$\leq 0,$$ (2.7)

for $k = 1, \ldots, n-1$. If equality holds in (2.7), then $\Sigma_t$ are umbilic and hence it is a spherical cap with $\theta$-capillary boundary. \hfill \Box

2.3 Estimates on convex capillary hypersurfaces

In this subsection, we collect some estimates on strictly convex hypersurfaces with $\theta$-capillary boundary in the unit ball, which were obtained by Weng and Xia [28].

Proposition 2.4 [28, Prop. 2.15] Let $\Sigma \subset \mathbb{B}^{n+1}$ be a strictly convex hypersurface with $\theta$-capillary boundary for $\theta \in (0, \frac{\pi}{2})$. Then for any $p \in \Sigma$, $\Sigma$ lies on one side of $T_p \Sigma$.

Let $\Sigma \subset \mathbb{B}^{n+1}$ be a strictly convex hypersurface with $\theta$-capillary boundary. Then it is proved in [28, Prop. 2.16] that there exists a point $e \in \mathrm{int}(\partial \Sigma)$ and a constant $0 < \delta_0 < 1 - \cos \theta$ depending only on $\Sigma$ such that

$$\langle x, e \rangle \geq \cos \theta + \delta_0,$$ (2.8)

$$\langle x, v \rangle \leq -\cos \theta.$$ (2.9)

Recall that the foliations $C_{\theta,s}(e), s \geq 0$ are spherical caps of radius $s$ around $e$ with $\theta$-capillary boundary such that $C_{\theta,\infty}(e)$ is the flat ball around $e$ with $\theta$-capillary boundary. Then there exists some $0 < R_1 < R_2 < \infty$ such that

$$\Sigma \subset C_{\theta,R_2}(e) \setminus C_{\theta,R_1}(e).$$

Proposition 2.5 (cf. Prop.2.16 of [28]) Let $\Sigma \subset \mathbb{B}^{n+1}$ be a strictly convex hypersurface with $\theta$-capillary boundary such that $\Sigma \subset C_{\theta,R_2}(e) \setminus C_{\theta,R_1}(e)$. Then the following estimates hold:

1. $\langle x - e, v \rangle \geq \delta_1 > 0$;
2. $\langle X_e, v \rangle \geq \delta_2 > 0$;
3. $\cos \theta + \delta_0 \leq \langle x, e \rangle \leq 1 - \delta_3$;
4. $\langle e, v \rangle \leq -\delta_4$.

Here $\delta_i, i = 1, 2, 3, 4$ are positive constants depending only on $R_1, \theta,$ and $\delta_0$ is a positive constant depending only on $R_2, \theta$ such that $\cos \theta + \delta_0 \leq 1 - \delta_3.$
Proof The estimates are proved in Prop. 2.16 of [28]. Here we provide a proof by clarifying that the constants $\delta_i, i = 0, 1, \ldots, 4$ can be chosen explicitly depending only on $\theta, R_1$ and $R_2$. These estimates will be used in the proof of preserving of the strict convexity of $\Sigma_1$ in Theorem 4.3. Let $S$ be a flat ball in $\mathbb{R}^{n+1}$ such that $\partial S = \partial C_{\theta, R_1}(e)$. Since $S \subset C_{\theta, R_2}(e) \subset \Sigma$, we know that $S$ lies in the upper half-space determined by $T_x \Sigma$ in the direction of $-v$, see Fig. 2.

Let $d^R$ be the distance in $\mathbb{R}^{n+1}$. Since $\langle x - e, v \rangle$ measures the distance from $e$ to $T_x \Sigma$, which can be bounded from below by the distance from $e$ to $S$, i.e., $\langle x - e, v \rangle \geq d^R(e, S)$. We calculate $d^R(e, S)$ explicitly as follows:

$$1 - (1 - d^R(e, S))^2 = R_1^2 - (\sqrt{R_1^2 + 2R_1 \cos \theta} + 1 - 1 + d^R(e, S))^2,$$

which gives

$$d^R(e, S) = 1 - \frac{1 + R_1 \cos \theta}{\sqrt{R_1^2 + 2R_1 \cos \theta} + 1} =: \delta_1(R_1, \theta) > 0.$$

Then we get $\langle x - e, v \rangle \geq \delta_1$. It follows that

$$\langle X_e, v \rangle = \frac{1 + |x|^2}{2} \langle x - e, v \rangle - \langle x, v \rangle \frac{|e - x|^2}{2} \geq \frac{1}{2} \delta_1 =: \delta_2(R_1, \theta),$$

where we used $\langle x, v \rangle \leq -\cos \theta \leq 0$ by (2.9).

Note that $\Sigma \subset C_{\theta, R_2}(e)$, i.e., $\Sigma$ lies above $C_{\theta, R_2}(e)$ in the direction of $e$. Then we get

$$\min_{\Sigma} \langle x, e \rangle \geq \min_{C_{\theta, R_2}(e)} \langle x, e \rangle = \sqrt{R_2^2 + 2R_2 \cos \theta} + 1 - R_2 = \cos \theta + \frac{\sin^2 \theta}{\sqrt{R_2^2 + 2R_2 \cos \theta} + 1 + R_2 + \cos \theta} =: \cos \theta + \delta_0(R_2, \theta).$$

Similarly,

$$\max_{\Sigma} \langle x, e \rangle \leq \max_{C_{\theta, R_1}(e)} \langle x, e \rangle = \frac{1 + R_1 \cos \theta}{\sqrt{R_1^2 + 2R_1 \cos \theta} + 1} =: 1 - \delta_3(R_1, \theta).$$
Since $\partial C_{\theta, R_1}(e)$ is enclosed by $\partial \Sigma$ in $\mathbb{S}^n$, we know that $\nu|_{\partial \Sigma}$ is strictly contained in $B_{\frac{\pi}{2} - \beta}(-e)$ in $\mathbb{S}^n$, where $\beta$ is given by

$$
\cos \beta = \frac{1 + R_1 \cos \theta}{\sqrt{R_1^2 + 2R_1 \cos \theta + 1}}.
$$

The convexity of $\Sigma$ implies that $\nu|_{\Sigma}$ also lies in $B_{\frac{\pi}{2} - \beta}(-e)$ and hence

$$
\langle \nu, e \rangle \leq -\sin \beta = -\frac{R_1 \sin \theta}{\sqrt{R_1^2 + 2R_1 \cos \theta + 1}} = -\delta_4(R_1, \theta).
$$

\[\Box\]

### 3 Evolution equations

In this section, we derive the evolution equations along the flow (1.5). Denote the normal speed of the flow by

$$
f = n\langle x + \cos \theta v, e \rangle - H\langle Xe, v \rangle,
$$

where we choose $e \in \text{int}(\partial \Sigma_0)$ as in Lemma A.1. As in [28, §2.4], the $\theta$-capillary boundary condition along the flow requires the tangential component $V = (\partial_t x)^\top \in T \Sigma_t$ must satisfy $V|_{\partial \Sigma_t} = f \cot \theta \mu + \tilde{V}$, where $\tilde{V} \in T (\partial \Sigma_t)$. Hence up to a time-dependent diffeomorphism of $\partial \mathbb{B}^n$, we can assume that $\tilde{V} = 0$ and hence

$$
V|_{\partial \Sigma_t} = f \cot \theta \mu.
$$

The $\theta$-capillary boundary condition also ensures that (see [28, Prop. 2.12])

$$
\nabla_\mu f = \left(\frac{1}{\sin \theta} + \cot \theta h_{\mu \mu}\right) f \text{ along } \partial \Sigma_t.
$$

So we rewrite the flow (1.5) equivalently as the following form:

$$
\begin{cases}
\partial_t x = f v + V & \text{in } \mathbb{R}^n \times [0, T), \\
\langle \tilde{N} \circ \partial_t x, v \rangle = -\cos \theta & \text{on } \mathbb{R}^n \times [0, T), \\
x(\cdot, 0) = x_0(\cdot) & \text{on } \mathbb{R}^n,
\end{cases}
$$

where $f$ is given by (3.1) and the tangential component $V = (\partial_t x)^\top \in T \Sigma_t$ satisfies (3.2) on the boundary $\partial \Sigma_t$. By Proposition 2.5, we have $\langle Xe, v \rangle \geq \delta_2 > 0$ on $\Sigma_0$. Then the short time existence of the flow (3.4) follows from rewriting it as a strictly parabolic scalar equation with oblique boundary condition, using Möbius coordinate transformation (see [23, §3] for details).

It is easy to check that the spherical cap $C_{\theta, r}(e)$ of radius $r$ satisfies

$$
\langle x, e \rangle + \cos \theta \langle v, e \rangle = \frac{1}{r} \langle Xe, v \rangle,
$$

and hence it is stationary along the flow (3.4). Using the spherical caps as barriers gives the following $C^0$ estimate.
Lemma 3.1 Assume that the initial hypersurface $\Sigma$ satisfies

$$\Sigma \subset C_0R_2(e) \setminus C_0R_1(e),$$

for some $0 < R_1 < R_2 < \infty$. Then this property also holds for the solution $\Sigma_t$ along the flow (3.4) for $t > 0$.

Let $g_{ij}, h_{ij}$ and $H = g^{ij}h_{ij}$ be the induced metric, the second fundamental form and the mean curvature of the flow hypersurface $\Sigma_t$. We recall the following general evolution equations:

**Proposition 3.2** [28, Proposition 2.11] Along a general flow satisfying

$$\partial_t x = f v + V,$$

where $V \in T\Sigma_t$, we have

\begin{align*}
\partial_t g_{ij} &= 2fh_{ij} + \nabla_i V_j + \nabla_j V_i, \quad (3.5) \\
\partial_t h_{ij} &= -\nabla_i \nabla_j f + f(h^2)_{ij} + \nabla V h_{ij} + h_k^i \nabla_i V_k + h_k^j \nabla_j V_k, \quad (3.6) \\
\partial_t H &= -\Delta f - |A|^2 f + \nabla V H, \quad (3.7)
\end{align*}

where $(h^2)_{ij} = h_i^k h_{kj}$ and $|A|^2 = g^{ij} h_i^k h_{kj}$.

We also need the following lemma.

**Lemma 3.3** [19] There hold

\begin{equation}
\nabla_i \langle X_e, v \rangle = \langle e_i, e \rangle \langle x, v \rangle - \langle e_i, x \rangle \langle e, v \rangle + h_i^k \langle X_e, e_k \rangle \tag{3.8}
\end{equation}

and

\begin{align*}
\nabla_i \nabla_j \langle X_e, v \rangle = & \langle X_e, \nabla h_{ij} \rangle + \langle x, e \rangle h_{ij} - (h^2)_{ij} \langle X_e, v \rangle - g_{ij} \langle e, v \rangle \\
& + h_i^k ((e_i, e) \langle x, e_k \rangle - (e_i, x) \langle e, e_k \rangle) \\
& + h_j^k ((e_j, e) \langle x, e_k \rangle - (e_j, x) \langle e, e_k \rangle). \tag{3.9}
\end{align*}

Now, we calculate the evolution equations along the flow (3.4).

**Lemma 3.4** Along the flow (3.4) with $f$ given by (3.1), we have

(i) The induced metric $g_{ij}$ evolves by

\begin{equation}
\frac{\partial}{\partial t} g_{ij} = 2(n\langle x + \cos \theta v, e \rangle - H \langle X_e, v \rangle) h_{ij} + \nabla_i V_j + \nabla_j V_i. \tag{3.10}
\end{equation}

(ii) The mean curvature $H$ evolves by

\begin{equation}
\frac{\partial}{\partial t} H = \langle X_e, v \rangle \Delta H + \left( H X_e - n \cos \theta e + V, \nabla H \right) \\
+ 2(\nabla \langle X_e, v \rangle, \nabla H) + (H^2 - n|A|^2) \langle x, e \rangle, \tag{3.11}
\end{equation}

and

\begin{equation}
\nabla_\mu H = 0, \quad \text{on } \partial \Sigma_t. \tag{3.12}
\end{equation}
(iii) The second fundamental form \( h = (h_{ij}) \) evolves by

\[
\frac{\partial}{\partial t} h_{ij} = (X_e, v) \Delta h_{ij} + \left( H X_e - n \cos \theta e + V, \nabla h_{ij} \right) \\
+ \nabla_i H \left( (e_j, e)(x, v) - (e_j, x)(e, v) + h_{ij}^k X_e, e_k \right) \\
+ \nabla_j H \left( (e_i, e)(x, v) - (e_i, x)(e, v) + h_{ij}^k X_e, e_k \right) \\
+ (h^2)_{ij} \left( n(x, e) + 2n \cos \theta (e, v) - 3(X_e, v) H \right) \\
+ h_{ij} \left( n(e, v) + (X_e, v)|A|^2 + H (x, e) \right) \\
+ h_{ij}^k \nabla_i V_k + h_{ij}^k \nabla_j V_k - g_{ij} H (e, v). \\
\tag{3.13}
\]

**Proof**

(1) Equation (3.10) follows from (3.1) and (3.5).

(2) Substituting (3.1) into (3.7), we get

\[
\begin{align*}
\partial_t H &= -n \Delta (x, e) - n \cos \theta \Delta \langle v, e \rangle \\
&\quad + \Delta H \langle X_e, v \rangle + 2 \langle \nabla H, \nabla \langle X_e, v \rangle \rangle + H \Delta \langle X_e, v \rangle \\
&\quad - |A|^2 (n \langle x + \cos \theta v, e \rangle - H \langle X_e, v \rangle) + \nabla_v H. \\
&\tag{3.14}
\end{align*}
\]

Recall that

\[
\begin{align*}
-\Delta \langle x, e \rangle &= H \langle v, e \rangle, \\
\Delta \langle v, e \rangle &= \langle \nabla^i (h_{ij}^k e_k), e \rangle = \langle \nabla H, e \rangle - |A|^2 \langle v, e \rangle. \\
&\tag{3.15, 3.16}
\end{align*}
\]

Substituting (3.15), (3.16) and (3.9) into (3.14), we obtain the equation (3.11).

For the boundary condition (3.12), by [24, Proposition 3.3], along \( \partial \Sigma_t \) we have

\[
\nabla_\mu \langle X_e, v \rangle = \left( \frac{1}{\sin \theta} + \cot \theta h_{\mu\mu} \right) \langle X_e, v \rangle,
\]

and

\[
\nabla_\mu \langle x + \cos \theta v, e \rangle = \left( \frac{1}{\sin \theta} + \cot \theta h_{\mu\mu} \right) \langle x + \cos \theta v, e \rangle.
\]

Substituting the above two equalities into (3.3), we obtain \( \nabla_\mu H = 0. \)

(3) Substituting (3.1) into (3.6), we have

\[
\begin{align*}
\frac{\partial}{\partial t} h_{ij} &= -n \nabla_i \nabla_j \langle x, e \rangle - n \cos \theta \nabla_i \nabla_j \langle v, e \rangle \\
&\quad + \nabla_i \nabla_j H \langle X_e, v \rangle + H \nabla_i \nabla_j \langle X_e, v \rangle + \nabla_i H \nabla_j \langle X_e, v \rangle + \nabla_j H \nabla_i \langle X_e, v \rangle \\
&\quad + \left( n \langle x, e \rangle + n \cos \theta \langle v, e \rangle - H \langle X_e, v \rangle \right) (h^2)_{ij} \\
&\quad + \nabla_v h_{ij} + h_{ij}^k \nabla_i V_k + h_{ij}^k \nabla_j V_k.
\end{align*}
\]

For the first two terms, we have

\[
\begin{align*}
-\nabla_i \nabla_j \langle x, e \rangle &= h_{ij} \langle v, e \rangle, \\
\nabla_i \nabla_j \langle v, e \rangle &= \nabla_i \langle h_{ij}^k e_k, e \rangle = \langle \nabla h_{ij}, e \rangle - (h^2)_{ij} \langle v, e \rangle.
\end{align*}
\]

By the Simons’ identity

\[
\nabla_i \nabla_j H = \Delta h_{ij} + |A|^2 h_{ij} - H (h^2)_{ij},
\]
and the Eqs. (3.8)–(3.9), we get
\[
\frac{\partial}{\partial t} h_{ij} = nh_{ij} \langle e, v \rangle + n \cos \theta (h^2)_{ij} \langle e, v \rangle - n \cos \theta \langle \nabla h_{ij}, e \rangle
\]
\[
+ \nabla_i H (\langle e_j, e \rangle \langle x, v \rangle - \langle e_j, x \rangle \langle e, v \rangle + h_j^k \langle e, e_k \rangle)
\]
\[
+ (X_e, v) (\Delta h_{ij} + |A|^2 h_{ij} - H (h^2)_{ij})
\]
\[
+ H \left( (X_e, \nabla h_{ij}) + h_{ij} \langle x, e \rangle - (h^2)_{ij} \langle X_e, v \rangle - g_{ij} \langle e, v \rangle \right)
\]
\[
+ h^k_j (\langle e_j, e \rangle \langle x, e_k \rangle - \langle e_j, x \rangle \langle e, e_k \rangle)
\]
\[
+ (n (x, e) + n \cos \theta \langle v, e \rangle - H (X_e, v)) (h^2)_{ij}
\]
\[
+ \nabla \nabla h_{ij} + h^k_j \nabla_i V_k + h^k_i \nabla_j V_k.
\]

Then (3.13) follows by rearranging the terms. □

4 Preserving of convexity

We first recall the following tensor maximum principle which was recently developed by
the authors [13] and can be viewed as a generalization of the tensor maximum principles of
Hamilton [11], Stahl [22] and Andrews [2].

**Theorem 4.1** [13] Let \( \Sigma \) be a smooth compact manifold with boundary \( \partial \Sigma \) and \( \mu \) be the
outward pointing unit normal vector field of \( \partial \Sigma \) in \( \Sigma \). Assume that \( S_{ij} \) is a smooth time-varying
symmetric tensor field on \( \Sigma \) satisfying
\[
\frac{\partial}{\partial t} S_{ij} = a^{k\ell} \nabla_k \nabla_\ell S_{ij} + b^k \nabla_k S_{ij} + N_{ij}
\]
on \( \Sigma \times [0, T] \), where the coefficients \( a^{k\ell} \) and \( b^k \) are smooth, \( \nabla \) is a (possibly time-dependent)
smooth symmetric connection, and \( (a^{k\ell}) \) is positively definite everywhere. Suppose that
\[
N_{ij} \xi^i \xi^j + \sup_{\Gamma} 2a^{k\ell} \left( 2\Gamma^p_k \nabla_\ell S_{ip} \xi^i - \Gamma^p_k \Gamma^q_\ell S_{pq} \right) \geq 0 \quad \text{on } \Sigma \times (0, T],
\]
\[
(\nabla_\mu S_{ij}) \xi^i \xi^j \geq 0 \quad \text{on } \partial \Sigma \times (0, T]
\]
whenever \( S_{ij} \geq 0 \) and \( S_{ij} \xi^i = 0 \), where \( \Gamma = (\Gamma^p_k) \) is a \( n \times n \) matrix. If \( S_{ij} \geq 0 \) everywhere on
\( \Sigma \times \{0\} \), then it remains so on \( \Sigma \times [0, T] \).

We also need the following useful lemma.

**Lemma 4.2** [13] Let \( \Sigma \) be a smooth compact manifold with boundary \( \partial \Sigma \) and \( \mu \) be the
outward pointing unit normal vector field of \( \partial \Sigma \) in \( \Sigma \). Suppose that \( S_{ij} \) is a smooth symmetric
tensor satisfying \( S_{ij} \geq 0 \) on \( \Sigma \) and
\[
(\nabla_\mu S_{ij}) \xi^i \xi^j \geq 0, \quad \text{on } \partial \Sigma
\]
whenever \( S_{ij} \xi^j = 0 \) for some tangent vector \( \xi \). Define a function
\[
Z(p, \xi) = S(p)(\xi, \xi)
\]
where \( p \in \Sigma \) and \( \xi \in T_p \Sigma \). If \( Z \) attains its minimum at some \((p, \xi) \in T \Sigma , \) then \( DZ(p, \xi) = 0 \) and \( D^2 Z(p, \xi) \) is non-negative definite.

Theorem 4.1 has been used in [13] to prove the preservation of convexity along the locally constrained inverse curvature flows for capillary hypersurfaces in the half-space. In the following, we shall apply Theorem 4.1 to show that the convexity is preserved along the flow (3.4).

**Theorem 4.3** Assume that the initial hypersurface \( \Sigma \subset \mathbb{R}^{n+1} \) is strictly convex with \( \theta \)-capillary boundary \( \partial \Sigma \subset \mathbb{S}^n \) such that \( \Sigma \subset C_{\alpha, R_0}(e) \setminus C_{\theta, R_1}(e) \). Let \( \Sigma_t, t \in [0, T), \) be the smooth solution of the flow (3.4) starting from \( \Sigma \). Then \( \Sigma_t \) is strictly convex and there exists a constant \( \varepsilon > 0 \) depending only on \( \Sigma \) such that the principal curvatures \( \kappa_i \) of \( \Sigma_t \) satisfy

\[
\kappa_i \geq \varepsilon, \quad i = 1, \cdots, n
\]

for all \( t \in [0, T) \).

**Proof** We consider the tensor

\[
S_{ij} = h_{ij} - \varepsilon g_{ij},
\]

where \( \varepsilon > 0 \) is a small constant to be determined later. Since the initial hypersurface \( \Sigma \) is strictly convex, we can first choose \( \varepsilon > 0 \) small enough such that \( S_{ij} \geq 0 \) on \( \Sigma \). Combining the equations (3.10), (3.13), we deduce that

\[
\begin{aligned}
\frac{\partial}{\partial t} S_{ij} &= \frac{\partial}{\partial t} h_{ij} - \varepsilon \frac{\partial}{\partial t} g_{ij} \\
&= \langle X_e, \nu \rangle \Delta S_{ij} + \langle H X_e + n \cos \theta e + V, \nabla S_{ij} \rangle + S^k_j \nabla_i V_k + S^k_i \nabla_j V_k \\
&+ \nabla_i H \left( \langle e_j, e \rangle \langle x, \nu \rangle - \langle e_j, \nu \rangle \langle x, e \rangle + h^k_j \langle X_e, e_k \rangle \right) \\
&+ \nabla_j H \left( \langle e_i, e \rangle \langle x, \nu \rangle - \langle e_i, \nu \rangle \langle x, e \rangle + h^k_i \langle X_e, e_k \rangle \right) \\
&+ \left( (S^2)_{ij} + 2 \varepsilon S_{ij} + \varepsilon^2 g_{ij} \right) \langle (x, e) \rangle + 2 n \cos \theta \langle \nu, e \rangle - 3 \langle X_e, \nu \rangle H \\
&+ (S_{ij} + \varepsilon g_{ij}) \langle n, \nu \rangle + \langle X_e, \nu \rangle |A|^2 + H \langle x, e \rangle \rangle - g_{ij} H \langle \nu, e \rangle \\
&- 2 \varepsilon \langle S_{ij} + \varepsilon g_{ij} \rangle \langle x + \cos \theta \nu, e \rangle - \langle X_e, \nu \rangle H.
\end{aligned}
\]

(4.4)

We first check that the boundary condition (4.2) is satisfied whenever \( S_{ij} \geq 0 \) and \( S_{ij} \xi^j = 0 \) at some point \( p_0 \in \partial \Sigma_{\theta_0} \) for some vector \( \xi \in T_{p_0} \Sigma_{\theta_0} \). Let \( \{e_\alpha\}_{\alpha = 1}^{n-1} \) be a local orthonormal frame of \( \partial \Sigma_\alpha \subset \Sigma_\alpha \) so that \( \{\mu\} \cup \{e_\alpha\}_{\alpha = 1}^{n-1} \) forms an orthonormal frame of \( \Sigma_\alpha \). We have either \( \xi = \mu \) or \( \xi = e_\alpha \) for some \( \alpha \) on \( \partial \Sigma_{\theta_0} \). By Proposition 2.1, we have:

(i) If \( \xi = e_\alpha \), then \( h_{\mu\mu} \geq h_{\alpha\alpha} = \varepsilon > 0 \) and

\[
\begin{aligned}
\nabla_{\mu} S_{ij} \cdot \xi^j &= \nabla_{\mu} h_{\alpha\alpha} = \tilde{h}_{\alpha\alpha} (h_{\mu\mu} - h_{\alpha\alpha}) \\
&= \left( \frac{1}{\sin \theta} + \cot \theta h_{\alpha\alpha} \right) (h_{\mu\mu} - \varepsilon) \geq 0.
\end{aligned}
\]

(ii) If \( \xi = \mu \), then \( h_{\alpha\alpha} \geq h_{\mu\mu} = \varepsilon > 0 \) for all \( \alpha = 1, \cdots, n - 1 \). By \( \nabla_{\mu} H = 0 \), we get

\[
0 = \nabla_{\mu} h_{\mu\mu} + \sum_\alpha \nabla_{\mu} h_{\alpha\alpha} = \nabla_{\mu} h_{\mu\mu} + \sum_\alpha (h_{\mu\mu} - h_{\alpha\alpha}) \tilde{h}_{\alpha\alpha}
\]

\[
= \nabla_{\mu} h_{\mu\mu} + \sum_\alpha (\varepsilon - h_{\alpha\alpha}) \left( \frac{1}{\sin \theta} + \cot \theta h_{\alpha\alpha} \right),
\]

\( \varepsilon \) Springer
and hence
\[
(\nabla_{\mu} S_{ij}) \xi^{i} \xi^{j} = \nabla_{\mu} h_{\mu \mu} = \sum_{\alpha} (h_{\alpha \alpha} - \varepsilon) \left( \frac{1}{\sin \theta} + \cot \theta h_{\alpha \alpha} \right) \geq 0.
\]

Therefore, in both cases, the boundary condition (4.2) is satisfied.

We next check the condition (4.1) whenever \( S_{ij} \geq 0 \) and \( S_{ij} \xi^{l} = 0 \) at a point \((p_{0}, t_{0})\). We choose an orthonormal frame \( \{ e_{1}, \ldots, e_{n} \} \) around \( p_{0} \) such that \( h_{ij} \) is diagonal at \((p_{0}, t_{0})\) with principal curvatures \( \kappa_{1} \leq \kappa_{2} \leq \cdots \leq \kappa_{n} \) in the increasing order and \( e_{i} \) corresponds to the principal direction of \( \kappa_{i} \), and the null vector \( \xi = e_{1} \). The lower order terms in (4.4) involving \( S_{ij} \) and \((S^{2})_{ij}\) can be ignored due to the null vector condition. To apply Theorem 4.1, we need to check that

\[
Q_{1} := 2 \nabla_{1} H \left( (e_{1}, e) \langle x, v \rangle - (e_{1}, x) \langle e, v \rangle + \varepsilon \langle X_{e}, e_{1} \rangle \right)
- (H - n \varepsilon) \langle e, v \rangle + \varepsilon \left( (|A|^{2} - \varepsilon H) \langle X_{e}, v \rangle + (H - n \varepsilon) \langle x, e \rangle \right)
+ 2 \langle X_{e}, v \rangle \sup_{k} g^{k\ell} (2 \Gamma_{k}^{p} \nabla_{\ell} S_{1p} - \Gamma_{k}^{p} \Gamma_{\ell}^{q} S_{pq}) \geq 0.
\]

(4.5)

There are two cases:

(1) \( \varepsilon = \kappa_{1} = \cdots = \kappa_{n} \) at \((p_{0}, t_{0})\);
(2) there exists an \( 1 \leq m < n \) such that \( \varepsilon = \kappa_{1} = \cdots = \kappa_{m} < \kappa_{m+1} \leq \cdots \leq \kappa_{n} \) at \((p_{0}, t_{0})\).

We consider the two cases separately.

For case (1), we have \( H = n \varepsilon \) and \(|A|^{2} - \varepsilon H = 0\). So the second line of (4.5) vanishes. Moreover, by Lemma 4.2, \( \nabla_{k} S_{\ell \ell} = \nabla_{k} h_{\ell \ell} = 0 \) for all \( 1 \leq k, \ell \leq n \). This implies that \( \nabla_{1} H = 0 \). Then \( Q_{1} \geq 0 \) by simply choosing \( \Gamma_{k}^{p} = 0 \) for all \( k, p = 1, \ldots, n \).

For case (2), we have \( S_{11} = \cdots S_{mm} = \kappa_{1} - \varepsilon = 0 \) at \((x_{0}, t_{0})\) and \( S_{\ell \ell} > 0 \) for \( \ell = m + 1, \ldots, n \). Then

\[
|A|^{2} - \varepsilon H > 0, \quad H - n \varepsilon > 0.
\]

Again by Lemma 4.2, the minimality of \( S_{11}, \ldots, S_{mm} \) implies that \( \nabla_{k} S_{\ell \ell} = 0 \) for all \( k = 1, \ldots, n \) and \( \ell = 1, \ldots, m \). Then the last term in (4.5) can be explicitly computed as follows

\[
\sup_{k} g^{k\ell} \left( 2 \Gamma_{k}^{p} \nabla_{\ell} S_{1p} - \Gamma_{k}^{p} \Gamma_{\ell}^{q} S_{pq} \right)
\geq \sum_{k=1}^{n} \sum_{\ell=m+1}^{n} \frac{\nabla_{k} S_{1\ell}}{S_{\ell \ell}} \geq \sum_{\ell=m+1}^{n} \frac{\nabla_{1} h_{\ell \ell}}{\kappa_{\ell} - \varepsilon},
\]

(4.6)

where in the first inequality we have chosen \( \Gamma_{k}^{\ell} = \frac{\nabla_{k} S_{1\ell}}{S_{1\ell}} \) for \( \ell = m + 1, \cdots, n \) and \( \Gamma_{k}^{\ell} = 0 \) for \( \ell = 1, \cdots, m \), and in the last inequality we used the Codazzi equation \( \nabla_{k} h_{1\ell} = \nabla_{1} h_{k\ell} \) and discarded the terms for \( k \neq \ell \).
For the first term of (4.5), we calculate that
\[
\left\langle e_1, e \langle x, v \rangle - x \langle e, v \rangle + \varepsilon X_e \right\rangle^2
\]
\[
\leq |e \langle x, v \rangle - x (e, v) + \varepsilon X_e|^2 - \langle v, e \langle x, v \rangle - x \langle e, v \rangle + \varepsilon X_e \rangle^2
\]
\[
= (x, v)^2 + |x|^2 (e, v)^2 - 2 \langle x, v \rangle \langle e, v \rangle (x, e)
\]
\[
+ 2 \varepsilon \langle X_e, e (x, v) - x \langle e, v \rangle \rangle + \varepsilon^2 (|X_e|^2 - \langle X_e, v \rangle^2)
\]
\[
= -2 \langle X_e, v \rangle \langle e, v \rangle + \langle x, v \rangle^2 - \langle e, v \rangle^2
\]
\[
+ 2 \varepsilon \langle X_e, e \langle x, v \rangle - x \langle e, v \rangle \rangle + \varepsilon^2 (|X_e|^2 - \langle X_e, v \rangle^2).
\] (4.7)

By (2.9) and Proposition 2.5, we have
\[
\langle x, v \rangle^2 - \langle e, v \rangle^2 = \langle x + e, v \rangle \langle x - e, v \rangle \leq -\delta_1 \delta_4,
\] (4.8)

where \(\delta_1, \delta_4\) are constants in Proposition 2.5. Since
\[
\langle X_e, v \rangle \leq |X_e| \leq 2,
\]
\[
|e \langle x, v \rangle - x \langle e, v \rangle| \leq 2,
\] (4.9)

it follows from (4.7), (4.8) and (4.9) that
\[
\langle e_1, e \langle x, v \rangle - x \langle e, v \rangle + \varepsilon X_e \rangle^2 \leq -2 \langle X_e, v \rangle \langle e, v \rangle - \delta_1 \delta_4 + 4 \varepsilon^2 + 8 \varepsilon.
\] Note that \(\delta_1, \delta_4\) depend only on \(R_1, R_2\) and \(\theta\). By choosing \(\varepsilon > 0\) small enough such that \(4 \varepsilon^2 + 8 \varepsilon \leq \delta_1 \delta_4\), we have
\[
\langle e_1, e \langle x, v \rangle - x \langle e, v \rangle + \varepsilon X_e \rangle^2 \leq -2 \langle X_e, v \rangle \langle e, v \rangle.
\] (4.10)

Then using (4.6) and (4.10), we obtain the following estimate on \(Q_1\):
\[
Q_1 \geq -2 |\sum_{\ell=m+1}^n \nabla_1 h_{\ell \ell} \sqrt{-2 \langle X_e, v \rangle \langle e, v \rangle} - \sum_{\ell=m+1}^n (\kappa_\ell - \varepsilon) \langle e, v \rangle |
\]
\[
+ 2 \langle X_e, v \rangle \sum_{\ell=m+1}^n \frac{|\nabla_1 h_{\ell \ell}|^2}{\kappa_\ell - \varepsilon}
\]
\[
\geq \sum_{\ell=m+1}^n \left( \sqrt{\kappa_\ell - \varepsilon} \sqrt{-\langle e, v \rangle} - \sqrt{2 \langle X_e, v \rangle} \frac{|\nabla_1 h_{\ell \ell}|}{\sqrt{\kappa_\ell - \varepsilon}} \right)^2
\]
\[
\geq 0.
\] (4.11)

In summary, in both two cases, we have \(Q_1 \geq 0\) and thus the condition (4.1) in Theorem 4.1 is satisfied. Therefore, we conclude that \(S_{ij} \geq 0\) is preserved for \(t > 0\). This implies that \(h_{ij} \geq \varepsilon g_{ij} > 0\) for \(t \in [0, T)\).

5 Proofs of Theorem 1.1 and Theorem 1.2

In this section, we prove the long time existence and smooth convergence of the flow (1.5) for strictly convex hypersurfaces in the unit ball with \(\theta\)-capillary boundary. As an application, we prove a family of new inequalities for quermassintegrals \(W_{k, \theta}(\Sigma)\).
5.1 Long-time existence

Let \( \Sigma \subset \overline{\mathbb{B}^{n+1}} (n \geq 2) \) be a smooth properly embedded, strictly convex hypersurface with \( \theta \)-capillary boundary \( (\theta \in (0, \frac{\pi}{2})) \), given by an embedding: \( x : \mathbb{B}^n \rightarrow \Sigma \subset \overline{\mathbb{B}^{n+1}} \). By Theorem 4.3, the solution \( \Sigma_t \) of the flow (3.4) starting from \( \Sigma \) remains to be strictly convex for \( t > 0 \). Proposition 2.5 implies that \( (X_e, \nu) \geq \delta_2 > 0 \) and hence \( \Sigma_t \) is star-shaped for \( t \geq 0 \). Then we can reduce the flow (3.4) to a scalar parabolic equation with oblique boundary condition as in [23, 25].

We briefly review the transformation of the flow (1.5) to a scalar parabolic equation on the hemisphere \( \overline{S^n}_+ \), and refer the readers to [23, 25] for more details. Assume that \( e \) is the \((n + 1)\)-th coordinate vector \( e_{n+1} \). Consider the following Möbius transformation:

\[
\varphi : \mathbb{B}^{n+1} \rightarrow \overline{\mathbb{R}^{n+1}}, \\
(x', x_{n+1}) \mapsto \frac{2(x', 0) + (1 - |x'|^2 - x_{n+1}^2)e}{|x'|^2 + (x_{n+1} - 1)^2} := (y', y_{n+1}) = \tilde{y},
\]

where \( x = (x', x_{n+1}) \) with \( x' = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( x_{n+1} = \langle x, e \rangle \). Then \( \varphi \) maps \( S^n = \partial \mathbb{B}^{n+1} \) to \( \partial \mathbb{R}^{n+1} = \{(y', y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} = 0\} \) and

\[
\varphi^* (\delta_{\mathbb{R}^{n+1}}) = \frac{4}{(|x'|^2 + (x_{n+1} - 1)^2)^2} \delta_{\mathbb{B}^{n+1}} := e^{-2w} \delta_{\mathbb{B}^{n+1}},
\]

which implies that \( \varphi \) is a conformal transformation from \( (\mathbb{B}^{n+1}, \delta_{\mathbb{B}^{n+1}}) \) to \( (\mathbb{R}^{n+1}, \delta_{\mathbb{R}^{n+1}}) \). Then a properly embedded hypersurface \( \Sigma_t = x(\mathbb{B}^n, t) \) in \( (\mathbb{B}^{n+1}, \delta_{\mathbb{B}^{n+1}}) \) can be identified with \( \tilde{\Sigma}_t := \tilde{y}(\mathbb{B}^n, t) \) in \( (\mathbb{R}^{n+1}, (\varphi^{-1})^* \delta_{\mathbb{R}^{n+1}}) \), where \( \tilde{y} = \varphi \circ x \).

Note that \( X_e \) is a conformal vector field such that \( \varphi_* (X_e) = -\tilde{v} \). For a hypersurface \( \Sigma \subset \overline{\mathbb{B}^{n+1}} \) with capillary boundary \( \partial \Sigma \subset S^n \), one has

\[
e^{-2w} (X_e, \nu) = \langle \varphi_* (X_e), \varphi_* (\nu) \rangle = |\varphi_* (\nu)| \langle \tilde{y}, \tilde{v} \rangle,
\]

where \( |\varphi_* (\nu)| = \frac{1}{2}(|y'|^2 + (y_{n+1} + 1)^2) \) and \( \tilde{v} := -\frac{\varphi_* (\nu)}{|\varphi_* (\nu)|} \). Hence, the hypersurface \( \varphi (\Sigma) \) is star-shaped in \( \mathbb{R}^{n+1} \) with respect to the origin, i.e., \( \langle \tilde{y}, \tilde{v} \rangle > 0 \) on \( \varphi (\Sigma) \), if and only if \( \langle X_e, \nu \rangle > 0 \) on \( \Sigma \). In particular, since \( \langle X_e, \nu \rangle > 0 \) on \( \Sigma_t \) by Proposition 2.5, the hypersurface \( \tilde{\Sigma}_t \) in \( (\mathbb{R}^{n+1}, (\varphi^{-1})^* \delta_{\mathbb{R}^{n+1}}) \) can be written as a radial graph over \( S^n_+ \).

In \( \mathbb{R}^{n+1}_+ \), we use the polar coordinate \( \tilde{y} = (\rho, z) \in [0, \infty) \times S^n_+ \), where \( \rho \) is the distance from \( \tilde{y} \) to the origin and we write \( z = (\beta, \xi) \in [0, \frac{\pi}{2}) \times S^{n-1} \) for the spherical polar coordinate of \( z \in S^n \). Then

\[
\begin{align*}
\rho^2 &= |y'|^2 + y_{n+1}^2, \\
y_{n+1} &= \rho \cos \beta, \quad |y'| = \rho \sin \beta.
\end{align*}
\]

Let \( u := \log \rho \) and \( v := \sqrt{1 + |\nabla_S u|^2} \), where \( \nabla_S \) is the Levi-Civita connection on \( S^n_+ \) with respect to the round metric \( \sigma \). We have

\[
(X_e, v) = e^w \frac{\rho}{v} = \frac{2\rho}{\sqrt{1 + |\nabla_S u|^2} (\rho^2 + 2\rho \cos \beta + 1)}.
\]
Furthermore, up to a time-dependent tangential diffeomorphism, one can rewrite the flow (1.5) equivalently as the following scalar parabolic equation on $\overline{S}_+^n$:

$$
\begin{cases}
\frac{\partial}{\partial t} u = F(\nabla^S_+ u, \nabla^S u, \rho, \beta) & \text{in } S_+^n \times [0, T), \\
\nabla^S_+ u = -\cos \theta \sqrt{1 + |\nabla^S u|^2} & \text{on } \partial S_+^n \times [0, T), \\
u(\cdot, 0) = u_0(\cdot) & \text{on } S_+^n,
\end{cases}
$$

where $u_0 = \log \rho_0$, $\rho_0$ is radial function of the initial hypersurface $\varphi(\Sigma_0)$, $\partial_\nu$ is the unit outward normal of $\partial S_+^n$ on $\overline{S}_+^n$ and

$$
F(\nabla^S_+ u, \nabla^S u, \rho, \beta) := \text{div}_{S_+^n} \left( \frac{\nabla^S_+ u}{\rho e^u v} \right) - \frac{n + 1}{v} \sigma \left( \nabla^S u, \nabla^S \left( \frac{1}{\rho e^u} \right) \right) \frac{n \cos \theta}{2} \frac{\rho^2 - 1}{\rho} \sin \beta \sigma(\nabla^S u, \partial_\nu) + \frac{n \cos \theta}{2} \frac{\rho^2 \cos \beta + 2 \rho + \cos \beta}{\rho}.
$$

As $\theta \in (0, \frac{\pi}{2})$, we have $\cos \theta \in [0, 1)$. Then we have a uniformly oblique boundary condition in (5.4).

The barrier estimate in Lemma 3.1 and the convexity of $\Sigma_t$ imply the uniform $C^1$ estimate of $\Sigma_t$. In fact, under the transformation (5.1), using (5.2) we have

$$
\rho^2 = \frac{|x'|^4 + (1 - x_{n+1}^2)^2 + 2(1 + x_{n+1}^2)|x'|^2}{|x'|^4 + (1 - x_{n+1}^2)^4 + 2(1 - x_{n+1}^2)|x'|^2} = 1 + \frac{4x_{n+1}}{|x'|^2 + (1 - x_{n+1}^2)^2}.
$$

By Proposition 2.5, we have $x_{n+1} = \langle x, e \rangle \geq \cos \theta + \delta_0$ and $\langle x - e, v \rangle \geq \delta_1 > 0$. Then

$$
\rho^2 \geq 1 + 2(\cos \theta + \delta_0) \geq 1 + 2\delta_0, \\
\rho^2 \leq 1 + \frac{4}{|x - e|^2} \leq 1 + \frac{4}{|\langle x - e, v \rangle|^2} \leq 1 + \frac{4}{\delta_1^2}.
$$

Since $u = \log \rho$, we obtain the uniform $C^0$ estimate of $u$, i.e.,

$$
0 < c_1 \leq u \leq c_2 < \infty
$$

for some constants $c_1, c_2$ depending on $\Sigma_0$. For the $C^1$ estimate, using $\langle X_e, v \rangle \geq \delta_2 > 0$ on $\Sigma_t$ and (5.3) we obtain that

$$
|\nabla^S_+ u| \leq \sqrt{1 + |\nabla^S u|^2} \leq \frac{2\rho}{\delta_2\sqrt{\rho^2 + 2\rho \cos \beta + 1}},
$$

where $c_3$ is a positive constant depending on $\Sigma_0$. It follows that the scalar flow equation (5.4) is a quasilinear parabolic equation with uniform oblique boundary condition. The higher
order estimates of $u$ follows from the classical parabolic theory for quasilinear parabolic equations [14, Theorem 7.4] (see also [17, §13]). As a consequence, the solution $u$ of (5.4) exists for all time $t \in [0, \infty)$. Equivalently, we have

**Proposition 5.1** Let $\Sigma \subset \mathbb{R}^{n+1}(n \geq 2)$ be a properly embedded, strictly convex hypersurface with $\theta$-capillary boundary $\partial \Sigma \subset S^n (\theta \in (0, \pi/2))$, given by an embedding: $x : \overline{B}^n \to \Sigma \subset \mathbb{R}^{n+1}$. Then the solution $\Sigma_t$ of the flow (1.5) starting from $\Sigma$ has uniform $C^k$ regularity estimates for all $k \geq 0$ and exists for all time $t \in [0, \infty)$.

### 5.2 Convergence to a spherical cap

To prove the smooth convergence to the spherical cap, we explore the monotonicity of the flow (3.4). By the case $k = 1$ of (2.7), we have

$$
\frac{d}{dt} W_{1, \theta}(\Sigma_t) = \frac{n^2}{n+1} \int_{\Sigma_t} (H_2 - H_1^2)(X_e, v) dA_t \leq 0.
$$

(5.5)

The $C^0$ estimate of $\Sigma_t$ implies that $W_{1, \theta}(\Sigma_t)$ is uniformly bounded from below. Then the long-time existence and uniform $C^\infty$-estimates imply that

$$
\int_{\Sigma_t} (H_2 - H_1^2)(X_e, v) dA_t \to 0, \quad \text{as } t \to \infty.
$$

Since $\langle X_e, v \rangle \geq \delta_2 > 0$ along the flow, any limit of $\Sigma_t$ is totally umbilical and hence must be a spherical cap.

The limit spherical cap is uniquely determined and is independent of the subsequence of times. In fact, for any sequence of times $t_j \to \infty$ such that $\Sigma_{t_j}$ converges smoothly to a $\theta$-capillary spherical cap $C_{\theta, r_\infty}(e_\infty)$, the radius $r_\infty$ is uniquely determined by the fact $W_{0, \theta}(C_{\theta, r_\infty}(e_\infty)) = W_{0, \theta}(\Sigma)$. Furthermore, we must have $e_\infty = e$, by using a similar argument as in [19, 28] for the locally constrained inverse curvature flow (1.6). The idea is simply that any spherical cap $C_{\theta, r_\infty}(e_\infty)$ with the center $e_\infty \neq e$ is not stationary and must evolve towards $e$ along the flow (3.4). We refer the readers to [19, 28] for the similar justification of this fact along the flow (1.6).

Therefore, we conclude that along the flow (3.4), the solution $\Sigma_t$ converges smoothly to the unique spherical cap $C_{\theta, r_\infty}(e)$. This completes the proof of Theorem 1.1.

### 5.3 Proof of Theorem 1.2

Firstly, we assume that $\Sigma$ is strictly convex, then by Proposition 2.5, there exists a constant vector $e \in S^n$ such that $\langle X_e, v \rangle > 0$ holds everywhere on $\Sigma$. Start the flow (3.4) from $\Sigma$. By Theorem 1.1, the solution $\Sigma_t$ is strictly convex and converges smoothly to a spherical cap $C_{\theta, r_\infty}(e)$ as $t \to \infty$. By the monotonicity in Proposition 2.3, we have

$$
W_{0, \theta}(\Sigma) = W_{0, \theta}(C_{\theta, r_\infty}(e)) = f_0(r_\infty)
$$

$$
W_{k, \theta}(\Sigma) \geq W_{k, \theta}(C_{\theta, r_\infty}(e)) = f_k(r_\infty).
$$

Since $f_k$ is strictly increasing, we can rewrite the above two equations as

$$
W_{k, \theta}(\Sigma) \geq f_k \circ f_0^{-1}(W_{0, \theta}(\Sigma)).
$$

(5.6)
If equality holds in (5.6), then all $\Sigma_t$ are spherical caps with $\theta$-capillary boundary for $t > 0$ and in particular the initial hypersurface $\Sigma$ is also a spherical cap with $\theta$-capillary boundary.

When $\Sigma$ is convex, we may assume that it is not a flat ball, otherwise the statement is trivially true. By Theorem A.3, we can approximate $\Sigma$ by a family of strictly convex hypersurfaces $\Sigma_\varepsilon$ as $\varepsilon \to 0$. Then the inequality (1.8) follows by approximation. The equality characterization can be proved by using an argument of [7] and noting the fact that $\Sigma$ has an interior strictly convex point.

5.4 Further question

To establish the Alexandrov–Fenchel type inequalities comparing quermassintegrals $W_{k,\theta}$ and $W_{\ell,\theta}$ for $k > \ell$, it’s also natural to study the following fully nonlinear locally constrained curvature flow

$$
\begin{cases}
(\partial_t x)_\perp = \left( (x + \cos \theta \nu, e) - \frac{H_k}{H_{k-1}} (X_e, v) \right) v & \text{in } \mathbb{B}^n \times [0, T),

\langle \tilde{N} \circ x, v \rangle = -\cos \theta & \text{on } \partial \mathbb{B}^n \times [0, T),

x(\cdot, 0) = x_0(\cdot) & \text{on } \mathbb{B}^n,
\end{cases}
$$

(5.7)

for $k = 2, \cdots, n$. Along the flow (5.7), $W_{k-1,\theta}$ is preserved while $W_{k,\theta}$ is decreasing. Using the maximum principle, the two-sided positive bounds on the curvature function $H_k / H_{k-1}$ can be proved. Furthermore, we can prove that the strict convexity is preserved along the flow (5.7) by using the similar argument as in Sect. 4. The curvature upper bound is not yet available. However, we still expect that this is true and then this would give the smooth convergence of the flow to a spherical cap.

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A. Approximation result

In this appendix, we prove that a convex hypersurface $\Sigma$ with $\theta$-capillary boundary in the unit ball can be approximated by a sequence of strictly convex hypersurfaces with $\theta$-capillary boundary in the unit ball in the $C^{2,\alpha}$ sense. The free boundary case (i.e. $\theta = \frac{\pi}{2}$) has been treated by Lambert and Scheuer in [15]. We adapt their idea and include a proof for the capillary case.

Firstly, we show the following lemma which could be considered as a natural generalization of [15, Lemma 3.1].

Lemma A.1 Let $\Sigma \subset \mathbb{B}^{n+1}$ be a convex hypersurface with $\theta$-capillary boundary for $\theta \in (0, \frac{\pi}{2}]$. Then there exists a point $e \in \text{int}(\partial \Sigma)$, such that $\partial \Sigma \subset \partial C_{\theta, \infty}(e)$. Moreover, either $\partial \Sigma = \partial C_{\theta, \infty}(e)$, or $\partial \Sigma \subset \text{int}(\partial C_{\theta, \infty}(e))$ and in this case there exists an interior point in $\Sigma$ such that $h_{ij} > 0$ at this point.

Proof The case $\theta = \frac{\pi}{2}$ is treated in [15, Lemma 3.1]. In the following, we assume $\theta \in (0, \frac{\pi}{2})$.

Step 1. We first show that there exists $e \in \text{int}(\partial \Sigma)$ such that $\partial \Sigma \subset \partial C_{\theta, \infty}(e)$. 

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We next show that either $q \angle \partial \Sigma_{1}$ or $\hat{\partial} \Sigma_{1}$. Then we get $\gamma \subset \partial \Sigma_{1}$ such that it touches $\partial \Sigma_{1}$ tangentially at $q$. On the other hand, by the convexity of $\partial \Sigma$, we have $\hat{\partial} \Sigma = \bigcap_{q \in \partial \Sigma} \hat{S}_{q, \theta}$. Therefore, we have $\hat{\partial} \Sigma = \bigcap_{q \in \partial \Sigma} \hat{S}_{q, \theta}$. Let $B_{r}(e) \subset S^{n}$ be the closed geodesic ball with the smallest radius $r$ and center $e$ such that $\hat{\partial} \Sigma \subset B_{r}(e)$. It is obvious that $r \leq \theta$. We show that $e \in \text{int}(\hat{\partial} \Sigma)$. Let $C = S_{r}(e) \cap \partial \Sigma$, where $S_{r}(e) = \partial B_{r}(e)$. The minimality of $r$ implies that $C$ must support $B_{r}(e)$. From each hyperplane of $S^{n}$ which passes through $e$, $S_{r}(e)$ is cut into two hemispheres $S_{r}(e) = S_{1} \cup S_{2}$ and must satisfies $S_{1} \cap C \neq \emptyset$, $i = 1, 2$. This leads to the following two cases.

Case 1: If two points $q_{1}, q_{2}$ of $C$ are antipodal on $S_{r}(e)$, then $e$ is the midpoint of the minimal geodesic $\gamma$ joining $q_{1}$ and $q_{2}$. Then $e \in \gamma \subset \hat{\partial} \Sigma$ by convexity. If $e \in \partial \Sigma$, then $\gamma \subset \hat{S}_{e, \theta}$ implies that $\theta = \frac{\pi}{2}$ and $\gamma \subset \partial \Sigma$ by the convexity of $\partial \Sigma$, contradicting with $\theta < \frac{\pi}{2}$. Therefore, we have $e \in \text{int}(\hat{\partial} \Sigma)$.

Case 2: If $C$ contains no pair of antipodal points of $S_{r}(e)$, since $C$ support $B_{r}(e)$, then $C$ contains at least three points $q_{1}, q_{2}, q_{3}$ which would not lie simultaneously in any hemisphere of $S_{r}(e)$. Then the convex hull of $q_{1}, q_{2}, q_{3}$, which is a spherical triangle, would not be contained in any half ball of $B_{r}(e)$. Then we must have $e \in \text{int}((\text{conv}(q_{1}, q_{2}, q_{3})) \subset \text{int}(\hat{\partial} \Sigma)$.

**Step 2.** We next show that either $\partial \Sigma = \partial C_{\theta, \infty}(e)$ or $\partial \Sigma \subset \text{int}(\partial C_{\theta, \infty}(e))$.

For any $q_{1}, q_{2} \in \partial \Sigma$, we take $p_{1}, p_{2} \in S^{n}$ such that $\hat{S}_{q_{i}, \theta} = \hat{B}_{\theta}(p_{i})$ for $i = 1, 2$. If $p_{1} \neq p_{2}$, then

$$\hat{\partial} \Sigma \subset \hat{B}_{\theta}(p_{1}) \cap \hat{B}_{\theta}(p_{2}).$$

Let $o$ be the midpoint of the geodesic between $p_{1}$ and $p_{2}$. For any $z \in \partial(\hat{B}_{\theta}(p_{1}) \cap \hat{B}_{\theta}(p_{2}))$, we draw the geodesics from the point $z$ to the points $o, p_{1}$ and $p_{2}$, respectively. Denote by $\angle z o p_{i}$ the angle of the spherical triangle $\Delta z o p_{i}$ at the vertex $o$ for $i = 1, 2, 3$. Without loss of generality, we assume that $z \in \partial B_{\theta}(p_{1})$, then $\angle z o p_{1} \geq \pi/2$, see Fig. 3.

Let $d^{\Sigma}$ be the distance function on $S^{n}$. By the Law of Cosines on the sphere [29], we have

$$\cos d^{\Sigma}(z, o) \cos d^{\Sigma}(o, p_{1}) + \cos \angle z o p_{1} \sin d^{\Sigma}(z, o) \sin d^{\Sigma}(o, p_{1}) = \cos d^{\Sigma}(z, p_{1}).$$

Then we get

$$\cos d^{\Sigma}(z, o) > \cos d^{\Sigma}(z, o) \cos d^{\Sigma}(o, p_{1}) \geq \cos d^{\Sigma}(z, p_{1}) = \cos \theta,$$

and hence $d^{\Sigma}(z, o) < \theta$. Thus, the minimality of $\hat{B}_{r}(e)$ implies that either $r = \theta$ or $r < \theta$. 

![Fig. 3](image-url)
In the latter case, we have \( \delta \Sigma \subset B_r(e) \) with \( r < \theta \). Then we have \( \langle x, e \rangle \geq \cos r \) and \( 0 \geq \langle \tilde{v}, e \rangle \geq -\sin r \) on \( \partial \mathbb{B}^n \). Using the relation (2.1), we obtain

\[
D_\mu \langle x, e \rangle = \langle \mu, e \rangle = \sin \theta \langle \tilde{N} \circ x, e \rangle + \cos \theta \langle \tilde{v}, e \rangle \\
\geq \sin \theta \cos r - \cos \theta \sin r \\
= \sin(\theta - r) > 0, \quad \text{on } \partial \mathbb{B}^n.
\]

Then the strict minimum of \( \langle x, e \rangle \) is attained at some interior point in \( \mathbb{B}^n \). By attaching a large supporting sphere to \( \Sigma \) from below, we obtain an interior strictly convex point in \( \mathbb{B}^n \).

Let \( \Sigma \subset \mathbb{B}^{n+1} \) be a convex hypersurface with capillary boundary supported on \( S^n \) at a contact angle \( \theta \in (0, \frac{\pi}{2}] \), which is given by the embedding \( x_0 : \mathbb{B}^n \rightarrow \Sigma \subset \mathbb{B}^{n+1} \). We consider the mean curvature flow:

\[
\begin{aligned}
\partial_t x &= -H \nu + V \quad \text{in } \mathbb{B}^n \times [0, T), \\
\langle \tilde{N} \circ x, \nu \rangle &= -\cos \theta \quad \text{on } \partial \mathbb{B}^n \times [0, T), \\
x(\cdot, 0) &= x_0(\cdot) \quad \text{on } \mathbb{B}^n,
\end{aligned}
\]

(A.1)

and denote \( \Sigma_t = x(\mathbb{B}^n, t) \), where \( V \) is the tangential component of \( \partial_t x \) and satisfies \( V|_{\partial \Sigma_t} = -H \cot \theta \mu \).

The short time existence of the flow (A.1) can be obtained by a similar argument as in [22] by Stahl.

**Theorem A.2** For any \( \alpha \in (0, 1) \), the flow (A.1) admits a unique solution:

\[
x(\cdot, t) \in C^\infty(\mathbb{B}^n \times (0, \delta)) \cap C^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{B}^n \times [0, \delta]),
\]

where \( \delta > 0 \) is a small constant.

We prove the following approximation result:

**Theorem A.3** Let \( \Sigma \subset \mathbb{B}^{n+1} \) be a convex hypersurface with capillary boundary supported on \( S^n \) at a contact angle \( \theta \in (0, \frac{\pi}{2}] \). Suppose that \( \Sigma_t, t \in [0, \delta] \) is a solution to the flow (A.1) starting from \( \Sigma \). Then either \( \Sigma \) is a flat ball \( C_{\theta, \infty} \), or \( \Sigma_t \) is strictly convex for all time \( t > 0 \) as long as the flow exists.

**Proof** By Lemma A.1, there exists a point \( e \in \text{int}(\partial \Sigma) \) such that \( \partial \Sigma \subset \partial C_{\theta, \infty}(e) \), and either \( \partial \Sigma = \partial C_{\theta, \infty}(e) \) or \( \partial \Sigma \subset \text{int}(\partial C_{\theta, \infty}(e)) \).

For the first case \( \partial \Sigma = \partial C_{\theta, \infty}(e) \), by item (2) of Proposition 2.1, we have \( h_{\alpha \beta} = 0 \) for \( \alpha, \beta = 1, \ldots, n-1 \) on the boundary \( \partial \Sigma \). If \( h_{\mu \mu} > 0 \) on \( \partial \Sigma \), then item (4) of Proposition 2.1 implies that \( \nabla_{\hat{\nu}} h_{\alpha \beta} > 0 \) on \( \partial \Sigma \) and then \( h_{\alpha \beta} < 0 \) at some interior points sufficiently close to the boundary \( \partial \Sigma \). So we must have \( h_{\mu \mu} = 0 \) on \( \partial \Sigma \). This implies that if \( \Sigma \) is not \( C_{\theta, \infty}(e) \), then there exists an interior point where the second fundamental form is positive definite. For the second case \( \partial \Sigma \subset \text{int}(\partial C_{\theta, \infty}(e)) \), by Lemma A.1, there is also a strictly convex point in the interior of \( \Sigma \).

In the following, we assume that \( \Sigma \) is not the flat ball \( C_{\theta, \infty}(e) \). Let

\[
\chi(z, t) = \min_{|\xi|=1} h_{ij} \xi^i \xi^j, \quad z \in \mathbb{B}^n
\]

be the smallest principal curvature at the point \( x(z, t) \in \Sigma_t \). Since \( h_{ij} \) is smooth, the function \( \chi(z, t) \) is Lipschitz continuous in space and by a cut-off function argument, we find a smooth
function \( \phi_0 : \mathbb{R}^n \to \mathbb{R} \) such that \( 0 \leq \phi_0 \leq \chi(\cdot, 0) \) and there exists an interior point \( y \) such that \( \phi_0(y) > 0 \). We extend the function \( \phi_0 \) to \( \phi : \mathbb{R}^n \times [0, \delta') \to \mathbb{R} \) by solving a linear parabolic PDE:

\[
\begin{align*}
\frac{\partial}{\partial t} \phi &= \Delta \phi + \nabla_V \phi, & \text{in } \mathbb{R}^n \times [0, \delta'), \\
\nabla_\mu \phi &= 0 & \text{on } \partial \mathbb{R}^n \times [0, \delta'), \\
\phi(\cdot, 0) &= \phi_0(\cdot) & \text{on } \mathbb{R}^n,
\end{align*}
\]

(A.2)

where \( \Delta \) and \( V \) are Laplacian operator and Levi-Civita connection with respect to the induced metric on \( \Sigma_t = x(\mathbb{R}^n, t) \) of the flow (A.1). The solution \( \phi \) of (A.2) exists at least for a short time interval \([0, \delta')\). By the strong maximum principle for scalar functions (see [22, Corollary 3.2]), we have \( \phi(\cdot, t') > 0 \) in \( \mathbb{R}^n \) and \( t \in (0, \delta') \).

We take \( \tau = \frac{1}{2} \min{\delta, \delta'} \) and consider the tensor:

\[
M_{ij} = h_{ij} - \phi g_{ij} \quad (A.3)
\]

for time \( t \in [0, \tau) \). By the construction of \( \phi_0 \), we see that \( M_{ij} \geq 0 \) at time \( t = 0 \). We now apply the tensor maximum principle (i.e. Theorem 4.1) to deduce that \( M_{ij} \geq 0 \) is preserved along the flow (A.1).

By a direct computation using Proposition 3.2 for \( f = -H \), we have:

\[
\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + \nabla_V M_{ij} + N_{ij}, \quad (A.4)
\]

where

\[
N_{ij} = |A|^2 M_{ij} + |A|^2 \phi g_{ij} - 2HM_{i}^{\ell} M_{j\ell} - 2H\phi M_{ij} + M_{j}^{\ell} \nabla_i V_k + M_{i}^{\ell} \nabla_j V_k. \quad (A.5)
\]

We easily see that whenever \( M_{ij} \geq 0 \) and \( M_{ij} \xi^i \xi^j = 0 \) at a point, we have

\[
N_{ij} \xi^i \xi^j = |A|^2 \phi \geq 0 \quad (A.6)
\]

and thus the condition (4.1) is satisfied. The boundary condition (4.2) can be checked similarly as in Theorem 4.3. Hence Theorem 4.1 implies that \( M_{ij} \geq 0 \) is preserved along the flow (A.1) for time \( t \in [0, \tau) \), and it follows that \( \Sigma_t \) is strictly convex for time interval \((0, \tau)\).

To show the strict convexity for the whole time interval \((0, \delta)\), we fix a time \( t_0 \in (0, \tau) \). Since \( \Sigma_t \) is strictly convex at time \( t_0 \), then there exists a constant \( \epsilon > 0 \), such that \( h_{ij} \geq \epsilon g_{ij} \) holds everywhere on \( \Sigma_{t_0} \). A similar procedure as above can be used to show that \( \tilde{M}_{ij} = h_{ij} - \epsilon g_{ij} \geq 0 \) is preserved along the flow (A.1) for all time \( t > t_0 \) as long as the flow (A.1) exists, which finishes the proof. \( \square \)

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