\( \mathcal{O}(\alpha_s^2) \) Contributions to the asymmetric fragmentation function in \( e^+e^- \) annihilation

P.J. Rijken and W.L. van Neerven

Instituut–Lorentz
University of Leiden
P.O. Box 9506
2300 RA Leiden
The Netherlands

August 1996

Abstract

The order \( \alpha_s^2 \) contributions to the coefficient functions corresponding to the asymmetric fragmentation function \( F_A(x, Q^2) \) in \( e^+e^- \) annihilation are calculated. From this calculation we infer that the order \( (\alpha_s/4\pi)^2 \) correction to the flavour asymmetry sum rule is non vanishing and amounts to \(-12\beta_0 C_F \zeta(3)\). We also study the effect of the higher order QCD corrections on \( F_A(x, Q^2) \) and compare them with the OPAL data. The latter put a strong constraint on the valence part of the fragmentation densities \( D_q^H(x, \mu^2) \).
The measurement of the fragmentation functions in the process

\[ e^+e^- \rightarrow \gamma, Z \rightarrow H + "X", \]

provides us in addition to other experiments like deep inelastic lepton-hadron scattering, with a new test of scaling violation as predicted by perturbative quantum chromodynamics (QCD). Here “X” denotes any inclusive hadronic final state and H represents either a specific charged outgoing hadron or a sum over all charged hadron species. This process has been studied over a wide range of energies of many different \(e^+e^-\)-colliders. Data have been collected from TASSO [1] (\(\sqrt{s} = 22, 35, 45 \text{ GeV}\)), MARK II [2] and TPC/2 [3] (\(\sqrt{s} = 29 \text{ GeV}\)), CELLO [4] (\(\sqrt{s} = 35 \text{ GeV}\)), AMY [5] (\(\sqrt{s} = 55 \text{ GeV}\)) and DELPHI [3], ALEPH [7], OPAL [8] (\(\sqrt{s} = 91.2 \text{ GeV}\)).

Following the notation in [9] the unpolarized differential cross section of process (1) is given by

\[
\frac{d^2\sigma_H}{dx d\cos \theta} = \frac{3}{8} (1 + \cos^2 \theta) \frac{d\sigma_T^H}{dx} + \frac{3}{4} \sin^2 \theta \frac{d\sigma_L^H}{dx} + \frac{3}{4} \cos \theta \frac{d\sigma_A^H}{dx},
\]

where \(x\) stands for the Bjørken scaling variable

\[
x = \frac{2pq}{Q^2}, \quad 0 < x \leq 1,
\]

and \(p\) and \(q\) \((q^2 = Q^2 > 0)\) are the four-momenta of the produced particle \(H\) and the virtual vector boson \((\gamma, Z)\) respectively. The variable \(\theta\) denotes the angle of emission of particle \(H\) with respect to the electron beam direction in the center of mass (CM) frame. The transverse, longitudinal and asymmetric cross section in (2) are defined by \(\sigma_T, \sigma_L\) and \(\sigma_A\) respectively. The latter only shows up if the intermediate vector boson is given by the \(Z\)-boson and is absent in purely electromagnetic annihilation. Before the advent of LEP1 the CM energies were so low \((\sqrt{s} < M_Z)\) that \(\sigma_A\) could not be measured and no effort was made to separate \(\sigma_L\) from \(\sigma_T\) so that only data for \(d^2\sigma_H/dxd\cos \theta\) were available. Recently, after LEP1 came into operation. ALEPH [7] and OPAL [8] obtained data for \(\sigma_L\) and \(\sigma_T\) separately and the latter collaboration even made a measurement of \(\sigma_A\) for the first time. The separation of \(\sigma_L\) and \(\sigma_T\) is important because the former cross section enables us to extract the strong coupling constant \(\alpha_s\) and allows us to determine the gluon fragmentation density \(D_g(x)\) with a much higher degree of accuracy as could be done before. Furthermore the measurement of \(\sigma_A\) provides us with information on hadronization effects [9] since the QCD corrections are very small.

As far as the theoretical achievements are concerned the order \(\alpha_s\) QCD corrections to the coefficient functions \(C_{k,\ell}\) \((k = T, L, A, \ell = q, g)\), corresponding to the cross sections \(\sigma_k\) [2], have been calculated in the past in [10] [11] (see also [9]). Also computed are the NLO corrections to the DGLAP timelike splitting functions \(P_{ij}(x)\) \((i, j = q, g)\) in [12]. Recently the order \(\alpha_s^2\) contributions to the coefficient functions \(C_{L,i}\) [13] and \(C_{T,i}\) [16] became available. From the latter one obtains the order \(\alpha_s^2\) corrections to the total longitudinal and transverse cross sections defined by

\[
\sigma_k(Q^2) = \frac{1}{2} \sum_H \int_0^1 dx \int_0^1 dx' \frac{d\sigma_k^H(x, Q^2)}{dx},
\]
\[ \sigma_k(Q^2) = \sigma^{(0)}(Q^2) \int_0^1 dz \ z \left[ C_{k,q}^S(z, Q^2/M^2) + \frac{1}{2} C_{k,g}(z, Q^2/M^2) \right]. \] 

Here \( \sigma^{(0)}(Q^2) \) is the zeroth order annihilation cross section of process (1) which is identical to \( \sum_f \sigma_{0,f} \) in eq. (2.14) of [9] where \( f \) denotes a specific flavour \( (f = u, d, s, c, b) \). Furthermore \( \sigma_{\text{tot}}(e^+e^- \rightarrow X) = \sigma_T + \sigma_L \) which has been calculated up to order \( \alpha_s^2 \) in [17]. In [15, 16] it was shown that the order \( \alpha_s^2 \) contributions to the coefficient functions are necessary to get agreement between the OPAL-data and the theoretical predictions. Since the OPAL-collaboration also measured the asymmetric cross section \( d\sigma^H_A/dx \) in (2) it will be of interest to compute the order \( \alpha_s^2 \) corrections to this quantity too. In the QCD-improved parton model it can be written as

\[ \frac{d\sigma^H_A}{dx}(x, Q^2) = \sum_f \int_x^1 \frac{dz}{z} A_f(Q^2) \left( D^H_f \left( \frac{x}{z}, \mu^2 \right) - D^H_g \left( \frac{x}{z}, \mu^2 \right) \right) C^\text{NS}_{A,q}(z, Q^2/\mu^2), \]

where \( f \) denotes the flavour of the quark. The asymmetry factor \( A_f \), which is defined in eq. (2.12) of [15], contains the products of the vector and axial vector electroweak couplings appearing in the \( \gamma - Z \) and \( Z - Z \) interference term. It also includes the contribution of the photon and Z-boson propagators. The parton fragmentation densities denoted by \( D^H_f(z, \mu^2) \) depend in addition to the partonic scaling variable \( z \) also on the factorization scale \( \mu \). Because of charge conjugation symmetry of the strong interactions we have used in (1) the identities

\[ C^\text{NS}_{A,f} = -C^\text{NS}_{A,f} = C^\text{NS}_{A,q}, \quad C^S_{A,q} = C^\text{NS}_{A,g} = 0. \]

Up to order \( \alpha_s^2 \) the non-singlet coefficient function \( C^\text{NS}_{A,q} \) receives contributions from the following parton subprocesses where all quarks are taken to be massless

\[ \mathcal{O}(\alpha_s^0) : \ V \rightarrow \ "q" + \bar{q}, \]
\[ \mathcal{O}(\alpha_s) : \ V \rightarrow \ "q" + \bar{q} + g, \]
\[ \mathcal{O}(\alpha_s^2) : \ V \rightarrow \ "q" + \bar{q} + g + g, \]
\[ V \rightarrow \ "q" + \bar{q} + q + \bar{q}. \]

Here the detected quark, which fragments into the hadron, is indicated by \"q\" and \( V = \gamma, Z \). The above reactions also include the one- and two-loop corrections to process (8) and the one-loop corrections to process (6). Further in reaction (11) the two anti-quarks can be identical as well as non-identical. In the case the anti-quark is detected one has to interchange \( q \) and \( \bar{q} \) in eqs. (8)-(11). The computation of the parton cross sections proceeds in the same way as has been done for the longitudinal \( C_{L,i} \) and transverse coefficient functions \( C_{T,i} \) presented in [13] and [16] respectively. In the calculation one has to deal with the presence of ultra-violet (UV), infrared (IR) and collinear (C) divergences which have to be regularized.
using the method of \( n \)-dimensional regularization. However there is one difference between the calculation of \( C_{k,i} \) \( (k = T, L) \) on one hand and the computation of \( C_{A,i} \) on the other hand. This difference is due to the appearance of the \( \gamma_5 \)-matrix in the interference term \( M_V M_A^* + M_A M_V^* \) where \( M_V \) and \( M_A \) stand for the vector and axial-vector amplitude of the above processes. Here one has to find an \( n \)-dimensional extension for the \( \gamma_5 \)-matrix occurring in \( M_A \). For our calculation we have adopted the prescription for \( \gamma_5 \) given by ‘t Hooft and Veltman [19] (see also Breitenlohner and Maison [20]). Since the axial vector vertex is represented by \( \gamma_\mu \gamma_5 \) one can simplify the traces using the identification

\[
\gamma_\mu \gamma_5 = -\frac{i}{6} \epsilon_{\mu\alpha\beta\sigma} \gamma^\alpha \gamma^\beta \gamma^\sigma,
\]

which yields the same result as the prescription of ‘t Hooft and Veltman as is shown in [21, 24]. Although this prescription is consistent it has one drawback namely that the non-singlet axial vector current is renormalized in spite of the fact that it is conserved. Hence for each virtual correction where the \( \gamma_5 \)-matrix appears in the loop one needs an additional renormalization constant to undo this unwanted effect. This constant has been calculated in [24] and reads up to order \( \alpha_s^2 \)

\[
Z_A = 1 - \frac{\alpha_s}{4\pi} C_F \left[ 4 - 5\varepsilon \right] + \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left\{ 22 \right\} + C_A C_F \left\{ - \frac{44}{3\varepsilon} - \frac{107}{9} \right\} + n_f C_F T_f \left\{ \frac{16}{3\varepsilon} + \frac{4}{9} \right\},
\]

where the colour factors in QCD are given by \( C_F = (N^2 - 1)/2N \), \( C_A = N \), and \( T_f = 1/2 \) with \( N = 3 \) and the number of light flavours is denoted by \( n_f \). The rest of the calculation proceeds in the same way as performed for the deep inelastic coefficient functions \( C^{NS}_{3,q} \) which is the analogue of \( C^{NS}_{A,q} \). Apart from the check on the procedure outlined in [20] we will add a new one which has the advantage that we can get rid of the renormalization constant \( Z_A \) in (13) which is needed when the \( \gamma_5 \)-matrix appears in the loop of the virtual Feynman graph. This check is based on the observation that the difference between the asymmetric and transverse parton cross sections does not contain the distributions denoted by \( \delta(1 - z) \) and \( (\ln^k(1 - z)/(1 - z))^+ \). These singular functions originate from the one- and two-loop corrections to the Born-process (8) and the contributions due to soft gluon and collinear fermion pair production in reactions (9)-(11). Hence these distributions cancel in \( C^{NS}_{A,q}(z, Q^2/\mu^2) - C^{NS}_{T,q}(z, Q^2/\mu^2) \). Since the transverse coefficient function \( C^{NS}_{T,q} \) is known [16] we can obtain \( C^{NS}_{A,q} \) from the difference \( C^{NS}_{A,q} - C^{NS}_{T,q} \). The latter is only determined by the one-loop corrections to the regular part of process (9) (hard gluon radiative part) and the regular part of (11), (12) (hard gluon radiation plus quark anti-quark production). Hence we only have to deal with the \( \gamma_5 \)-matrix in the one-loop corrections to (9). However we have now checked that the following identity holds

\[
Z_A \left\{ \left\{ M_V^{(1)*} M_A^{(1)} + M_A^{(1)*} M_V^{(1)} \right\} + \left\{ M_V^{(1)*} M_A^{(3)} + M_A^{(3)*} M_V^{(1)} + M_A^{(1)*} M_V^{(3)} \right\} \right\}
\]
where $M_k^{(\ell)}$ is the order $g^\ell (\alpha_s = g^2/4\pi)$ contribution to the amplitude $M_k \ (k = V, A)$. Here $M_V^{(\ell)}$ and $M_A^{(\ell)}$ denote the amplitudes where the quark is attached to the vector- and axial-vector current respectively so that $M_A^{(\ell)}$ contains the $\gamma_5$-matrix. The term between the first pair of curly brackets on the left-hand side of (14) originates from the purely radiative process (9) whereas the term in the second pair of curly brackets refers to the interference between process (9) and the virtual corrections to (9). The latter is represented by the amplitude $M_k^{(3)}(k = V, A)$. Equation (14) reveals that one can get rid of the renormalization constant $Z_A$ by shifting the $\gamma_5$-matrix from $M_A^{(3)}$ to the amplitude $M_A^{(1)}$ of the radiative process (9) so that this matrix becomes harmless. Actually one can now also choose the naive $\gamma_5$-prescription without altering the final result. The order $\alpha_s$ corrections to $C_{T,q}^{NS}$ are calculated in [14] (see also eqs. (2.15), (2.16) in [9]). The order $\alpha_s^2$ corrections calculated in this paper are presented as follows. First we split the coefficient functions $C_{k,q}^{NS}(k = T, A)$ in two parts i.e.

$$C_{T,q}^{NS} = C_{T,q}^{NS,\text{nid}} + C_{T,q}^{NS,\text{id}},$$

$$C_{A,q}^{NS} = C_{A,q}^{NS,\text{nid}} - C_{A,q}^{NS,\text{id}},$$

where the last term in the above equations is only due to the contribution from identical anti-quarks in reaction (11). The second order contributions to both parts can be now obtained from $C_{T,q}^{NS,\text{nid}}$ and $C_{T,q}^{NS,\text{id}}$ (see appendix in A in [14]) as follows

$$C_{A,q}^{NS,\text{nid}}, (2) - C_{T,q}^{NS,\text{nid},(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[ C_F^2 \left\{ 2(1 - z)(2\ln z - 4\ln(1 - z) + 1) \right\} \right.\
\left. \cdot \ln\frac{Q^2}{\mu^2} + 4(1 - z)(4S_{1,2}(1 - z) - 8Li_3(-z) - 4\zeta(2)\ln(1 - z) + 4\ln z\zeta_2(-z) + 3\zeta_2(1 - z) - \ln^2(1 - z) - \ln z\ln(1 - z) + \frac{25}{2}\\ln(1 - z)) + \left(\frac{24}{5z^2} + \frac{8}{z}\right)\right.\
\left. - 16 - 16z + 8z^2 + \frac{24}{5}z^3)(\zeta_2(-z) + \ln z\ln(1 + z)) + (-24 + 8z + 8z^2 + \frac{24}{5}z^3)\zeta(2) + (10 - 2z - 4z^2 - \frac{12}{5}z^3)\ln^2 z + \left(-\frac{24}{5z^2} + \frac{2}{5} + \frac{202}{5}z - \frac{24}{5}z^2\right)\right.\
\left. \cdot \ln z + \frac{24}{5}z + \frac{9}{5}z - \frac{24}{5}z^2 \right\} \right)$$

$$+ C_A C_F \left\{ \frac{22}{3}(1 - z)\ln\frac{Q^2}{\mu^2} + 4(1 - z)(4Li_3(-z) - 2S_{1,2}(1 - z)$$

4
where the Riemann zeta-function $\zeta$ of a specific flavour $f$ via the axial-vector coupling constant representing the weak isospin component $Z_{\alpha}$ over all flavours in one family. This is because the $Z_{\alpha}$ defined in eq. (2.23) of [9]. It is given by

$$\Sigma_{A,q}^{NS,\text{id},(2)} - \Sigma_{T,q}^{NS,\text{id},(2)} = \left( C_{A}^{2} - \frac{1}{2} C_{A} C_{F} \right) \left( \frac{Q_s}{4\pi} \right)^2 \left[ 8(1 + z)(4S_{1,2}(-z) - 2\text{Li}_3(-z)
+ 4\ln(1 + z)\text{Li}_2(-z) + 2\zeta(2)\ln(1 + z) + 2\ln z\ln^2(1 + z) - \ln^2 z\ln(1 + z)
- 2\zeta(3)) + \left( \frac{24}{5z^2} - \frac{8}{z} - 8z^2 + \frac{24}{5} z^3 \right)(\text{Li}_2(-z) + \ln z\ln(1 + z)) + (8 - 8z
- 8z^2 + \frac{24}{5} z^3)\zeta(2) + (-4 + 4z + 4z^2 - \frac{12}{5} z^3)\ln^2 z + \left( \frac{24}{5z} + \frac{72}{5} 
+ \frac{72}{5} - \frac{24}{5} z^2 \right) \ln z + \frac{24}{5z} + \frac{104}{5} - \frac{104}{5} z - \frac{24}{5} z^2 \right],$$

(18)

where the Riemann zeta-function $\zeta(n)$ and the polylogarithms $\text{Li}_n(z)$, $S_{n,p}(z)$ can be found in [27].

Notice that in $\Sigma_{k,q}^{NS,\text{id}}$ ($k = T, A$) we have omitted contributions as represented by the cut graphs in fig. [1]. The photon cannot couple to the cut fermion triangle because of charge conjugation invariance. However the $Z$-boson decouples too if one sums over all flavours in one family. This is because the $Z$ is connected to the quarks via the axial-vector coupling constant representing the weak isospin component $I_s^{(f)}$ of a specific flavour $f$ with the property $\sum_{f=u,d} I_s^{(f)} = 0$. From now on we will assume that in the inclusive state one sums over all members in one family so that the contribution due to fig. [1] can be dropped.

The first quantity we would like to study is the flavour asymmetry sum rule which is defined in eq. (2.23) of [3]. It is given by

$$\Sigma_A^Q = \sum_{H,f} A_f(Q^2) \int_0^1 dz_1 Q_H^{(f)} \left( D^H_f(z_1, \mu^2) - D^H_f(z_1, \mu^2) \right) \cdot \int_0^1 dz_2 \Sigma_{A,q}^{NS}(z_2, Q^2/\mu^2),$$

(19)

where $Q_H^{(f)}$ is a conserved additive quantity. The first moment of the non-singlet
coefficient function calculated up to order $\alpha_s^2$ is equal to

$$\int_0^1 dz_2 \mathbb{C}^{\text{NS}}_{A,q}(z_2, Q^2/\mu^2) = 1 - \left( \frac{\alpha_s(Q^2)}{4\pi} \right)^2 \left[ 12\beta_0 C_F \zeta(3) \right] + \left( \frac{\alpha_s(Q^2)}{4\pi} \right)^3 \left[ c_{A,q}^{(3)} \right],$$

where $\beta_0$ is the lowest order coefficient of the beta-function given by

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_f n_f.$$

Notice that the first moments of $C_{A,q}^{\text{NS}}$ and $D_f^H - D_{\bar{f}}^H$ are separately scheme independent. Further there are no order $\alpha_s$ corrections to the first moment of $C_{A,q}^{\text{NS}}$ and the order $\alpha_s^2$ correction is proportional to $\beta_0$. A comparison between (20) and the order $\alpha_s^2$ corrected $R_{ee} = \sigma_{\text{tot}}(e^+e^- \rightarrow X)/\sigma_{(0)}$ reveals that the coefficients of the Riemann zeta-functions (here $\zeta(3)$ only) are exactly the same. Furthermore if one drops all rational numbers in $R_{ee}$ one obtains exactly (20). Following the arguments in [30] one can make an interesting conjecture about the third order term $c_{A,q}^{(3)}$ which has not been calculated yet. Suppose that all rational numbers in $c_{A,q}^{(3)}$ are zero and that the coefficients of the Riemann zeta-functions $\zeta(n)$ (here $\zeta(3)$ and $\zeta(5)$) are the same as in $R_{ee}$ then we can make the following conjecture

$$c_{A,q}^{(3)} = C_A C_F^2 \left[ -572\zeta(3) + 880\zeta(5) \right] + C_F C_A^2 \left[ -\frac{10948}{9}\zeta(3) - \frac{440}{3}\zeta(5) \right] + C_F^2 T_f n_f \left[ 304\zeta(3) - 320\zeta(5) \right] + C_A C_F T_f n_f \left[ -\frac{7168}{9}\zeta(3) + \frac{160}{3}\zeta(5) \right] + C_F T_f^2 n_f^2 \left[ -\frac{1216}{9}\zeta(3) \right] + \frac{n_f}{N} d_{abc} d_{abc} \left[ -8\zeta(3) \right],$$

where $d_{abc}$ denote the structure constants which emerge from the anti commutation relations of the generators of the group $SU(N)$. We now want to study the effect of the order $\alpha_s^2$ correction on the asymmetric fragmentation function and compare the result with the OPAL data [8]. The fragmentation functions $F_k^H$ will be defined by (see [8])

$$F_k^H(x, Q^2) = \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma_k^H(x, Q^2)}{dx}, \quad (k = T, L, A).$$

If we sum over all hadrons of species $H$ we obtain the quantities

$$F_k(x, Q^2) = \sum_H F_k^H(x, Q^2), \quad (k = L, T),$$

$$F_A(x, Q^2) = \sum_H Q_H F_A^H(x, Q^2),$$

(25)
where the sum in (23) is taken over all charged hadrons. From (3) and (24) we infer that $F_A$ gets only contributions from the valence fragmentation densities $D_{f,j}^H = D_f^H - D_f^H$. Hence the measurement of $F_A$ provides us with information about the $x$-behaviour of the valence fragmentation functions. The latter are available for $H = \pi^\pm, K^\pm, p, \bar{p}$ in [32] where they are parametrized in leading log (LL) and in next-to-leading log (NLL, MS-scheme). For $H = \pi^+, K^+, p$ we obtain from (3) and (23)

$$F^H_A(x, Q^2) = \frac{1}{\sigma_{\text{tot}}} \int_x^1 \frac{dz}{z} \left[ A_U(Q^2) D_{V,U}^H \left( \frac{x}{z}, \mu^2 \right) - A_D(Q^2) D_{V,D}^H \left( \frac{x}{z}, \mu^2 \right) \right] \cdot C_{A,4}(z, Q^2/\mu^2),$$

with $U = u (\pi^+, K^+)$ and $D = d (\pi^+)$ or $D = s (K^+)$. The proton contribution is given by $F_p^A = 0.16 F_{A,U}^{\pi^+}$ where the factor 0.16 originates from [32] where one has estimated $F_{p,\bar{p}}^{\pi^+} = F_{L,\bar{L}}^{\pi^+} + F_{T,\bar{T}}^{\pi^+}$ by putting $F_{p,\bar{p}}^{\pi^+} = 0.16 F_{\pi^+,\pi^-}$. Since in [32] one has taken $D_{V,U}^H = D_{V,D}^H$ we observe that $F(A, Q^2)$ (26) is negative over the whole $x$-region. This property can be traced back to the value of the electroweak angle leading to $A_D(M_Z^2)/A_U(M_Z^2) \approx 2$. Therefore all hadrons with positive charge ($Q_H > 0$) give a negative contribution to $F_A(x, M_Z^2)$. If $H$ is the anti-particle of $H$ we have the relation $F_A^H = -F_A^H$. Because of $Q_H = -Q_D$ in (23) the anti-particles ($\pi^-, K^-, \bar{p}$) also give a negative contribution to $F_A(x, Q^2)$. Therefore the parametrization in [32] predicts a negative $F_A(x, Q^2)$ (25) over the whole $x$-region at $Q^2 = M_Z^2$.

In our plots discussed below a comparison will be made with the OPAL data [8] so that we have to choose $Q^2 = M_Z^2$. Further we take $\mu^2 = Q^2$ in (6) and $n_f = 5$. The running coupling constant is chosen to be $\alpha_s(M_Z^2) = 0.126$. Finally we want to emphasize that a full next-to-next-to-leading (NNLO) analysis of $F_T$ and $F_A$ is not possible yet because of the missing three-loop contributions to the DGLAP splitting functions. Therefore the order $\alpha_s^2$ correction, which can be only attributed to the coefficient functions in (17), (18) and (19), have to be considered as an estimate. The order $\alpha_s^2$ corrected $F_L$ is complete because here the NLL fragmentation densities and the order $\alpha_s^2$ corrected coefficient functions are available (see [15]).

In fig. 2 we have plotted $F_A^{LO}, F_A^{NLO}$ and $F_A^{N^{NLO}}$ together with the OPAL data (see also fig. 4 in [8]). There is a difference between $F_A^{LO}$ and $F_A^{NLO}$ but the order $\alpha_s^2$ corrections shown by $F_A^{N^{NLO}}$ are unobservable. Furthermore the theoretical curves are above the data. In fig. 8 of [8] the OPAL-collaboration also presented the data for the ratio

$$R_A(x, Q^2) = \frac{F_A(x, Q^2)}{F(x, Q^2)}, \quad F(x, Q^2) = F_T(x, Q^2) = F_L(x, Q^2).$$

In fig. 3 these data are compared with the theoretical predictions $R_A^{LO}, R_A^{NLO}$, and $R_A^{N^{NLO}}$. Here we see the same features as has been observed for $F_A$ in fig. 2. There is no difference between $R_A^{NLO}$ and $R_A^{N^{NLO}}$ and only the order $\alpha_s$ corrections, represented by $R_A^{N^{LO}}$, are visible. Also in this case the data are below the theoretical predictions.

From the data one can infer the integrated fragmentation function for which the
theoretical predictions corrected up to order $\alpha_s^2$ are given below

$$\int_{0.1}^{1} dx \, F_A^{N\text{NLO}}(x, M_Z^2) = -0.016 \ ( -0.023), \quad (28)$$

$$\int_{0.1}^{1} dx \, \frac{1}{2} x \, F_A^{N\text{LO}}(x, M_Z^2) = -0.0020 \ ( -0.0027). \quad (29)$$

The experimental values for (28) and (29) are -0.0229 $\pm$ 0.0044 and -0.00369 $\pm$ 0.00046 respectively. Since the fragmentation densities in [32] have a limited range of validity we have imposed a lower bound on the integration which is given by $x = 0.1$. Between the brackets in (28), (29) we have quoted the LO results. It turns out that the latter are in better agreement with experiment than the NLO and NNLO numbers. Further the values of the integrals also hold in NLO since the order $\alpha_s^2$ corrections are extremely small.

The OPAL-data indicate that at low $x$ $F_A(x, M_Z^2)$ might become positive. If this is the case one has to assume that in this region $D_H^{V,U}(x, \mu^2) > D_H^{V,D}(x, \mu^2)$ provided the zeroth order contribution to $C_{NS, A,q}$ which is given by $\delta(1 - z)$ dominates the integral. Summarizing the above we conclude that the order $\alpha_s^2$ corrections to $F_A$ are negligible and we do not expect that this will change when the effect of the three-loop DGLAP splitting functions are taken into account. Furthermore the above results reveal that the measurement of $F_A$ puts some constraints on the valence fragmentation densities.

In particular it means that the NLL parametrizations in [32] have to be modified in order to get agreement with the OPAL data.
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1 Figure captions

Fig. 1 Diagrams with four quarks in the final state containing a cut triangular quark loop.

Fig. 2 Contributions to the asymmetry fragmentation function $F_A(x, Q^2)$ (25) at $Q = M_Z$ using the fragmentation density set of [32]. Solid line: LO. Dashed line: NLO. Dotted line: NNLO. The experimental data are taken from OPAL [8].

Fig. 3 The ratio $R_A(x, Q^2)$ (27) at $Q = M_Z$ using the fragmentation density set of [32]. Short dashed line: LO. Solid line: NLO. Dotted line: NNLO. The experimental data are taken from OPAL [8].
Fig. 1
Fig. 2
Fig. 3