ON THE TRANSITIVITY OF THE GROUP OF ORBIFOLD
DIFFEOMORPHISMS.

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Abstract. Consider a connected manifold of dimension at least two and the group of compactly supported diffeomorphisms that are compactly supported isotopic to the identity. This group acts $n$-transitive: Any tuple of $n$ points can be moved to any other tuple of $n$ points by a compactly supported diffeomorphism that is compactly supported isotopic to the identity. An orbifold diffeomorphism group is typically not $n$-transitive: simple obstructions are given by isomorphism classes of isotropy groups of points. In this paper we investigate the transitivity properties of the group of compactly supported diffeomorphisms of orbifolds that are compactly supported isotopic to the identity. We also study an example in the category of area preserving mappings.

1. Introduction

Given a connected manifold $M$ of dimension at least two and a natural number $n$, the group of compactly supported diffeomorphisms of $M$ that are compactly supported isotopic to the identity acts $n$-transitively, meaning that given two $n$-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of distinct points in $M$ there exists a compactly supported diffeomorphism $f$ of $M$, that is isotopic to the identity, such that $f(x_i) = y_i$ for all $i = 1, \ldots, n$. This result is fundamental for many applications in differential topology and holds also when the diffeomorphisms are required to preserve structure, e.g. symplectic forms and analytic structures. We refer to [6] for a discussion on this topic.

An orbifold diffeomorphism necessarily preserves (up to isomorphism) the isotropy groups of points, hence the compactly supported orbifold diffeomorphism group does not act transitively, let alone $n$-transitively. Nevertheless we show that the group of orbifold diffeomorphisms which are isotopic to the identity does act transitively on connected components of a given singular stratum. All the notions which appear in the theorem below will be properly introduced in the next section.

Theorem 1.1 ($n$-Transitivity). Let $\mathcal{O}$ be an orbifold. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two $n$-tuples of pairwise distinct points. Assume that for every $i$ the points $x_i$ and $y_i$ lie in the same connected component of the singular stratification and that their singular dimension is

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not equal to 1. Then there exists an orbifold diffeomorphism \( f : O \rightarrow O \), compactly supported isotopic to the identity, such that \( f(x_i) = y_i \) for all \( 1 \leq i \leq n \).

This has also ramifications in other categories of automorphisms: in Section [4] we consider an example in the category of orbifolds equipped with an area form. We also discuss a setting where \( n \)-transitivity might give rise to new invariants of orbifold mappings.

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2. Preliminaries

2.1. Orbifolds and orbifold maps. For us an (effective) orbifold \( O \) is a Hausdorff and paracompact topological space \( O \) together with a maximal atlas of orbifold charts around each point of \( O \). An orbifold chart centered around a point \( x \in O \) consists of an open set \( \tilde{U}_x \subseteq \mathbb{R}^n \) together with a smooth and effective action of the isotropy group, a finite group \( \Gamma_x \), fixing \( \tilde{x} \in \mathbb{R}^n \), and a homeomorphism \( \phi_x \) from \( \tilde{U}_x/\Gamma_x \) onto an open neighborhood of \( U_x \) of \( x \) such that \( \phi_x(\tilde{x}) = x \). Two charts \( \tilde{U}_x \) and \( \tilde{U}_y \) in the same atlas, with \( U_x \cap U_y \neq \emptyset \), need to satisfy the following compatibility condition: for every \( z \in U_x \cap U_y \) there exist an orbifold chart \( (\tilde{U}_z, \Gamma_z, \phi_z) \) with \( z \in \phi_z(\tilde{U}_z/\Gamma_z) \subseteq U_x \cap U_y \), injective group homomorphisms \( \rho_{zx} : \Gamma_z \rightarrow \Gamma_x \) and \( \rho_{zy} : \Gamma_z \rightarrow \Gamma_y \), and embeddings \( i_{zx} : \tilde{U}_z \rightarrow \tilde{U}_x \) and \( i_{zy} : \tilde{U}_z \rightarrow \tilde{U}_y \), such that \( i_{zx}(\gamma \tilde{w}) = \rho_{zx}(\gamma)i_{zx}(\tilde{w}) \) and \( i_{zy}(\gamma \tilde{w}) = \rho_{zy}(\gamma)i_{zy}(\tilde{w}) \) for all \( \gamma \in \Gamma_z \) and \( \tilde{w} \in \tilde{U}_z \).

Just as in the manifold case, every orbifold atlas lies in a unique maximal atlas and we will always assume our orbifolds to be equipped with a maximal atlas. The Bochner-Cartan linearization theorem shows that we may always choose charts in which the groups \( \Gamma_x \) act linearly on \( \tilde{U}_x = \mathbb{R}^n \), i.e., the chart is a representation of \( \Gamma_x \). We call such a chart a linear orbifold chart.

2.2. Singular set and singular dimension. The singular set \( \Sigma \) of \( O \) consists of the points with non-trivial isotropy group. Given \( x \in \Sigma \), let \( \tilde{x} \) be the lift of \( x \) to a chart \( \tilde{U}_x \) centered at \( x \). Then the action of \( \Gamma_x \) fixes \( \tilde{x} \) and the differential defines an action on \( T_{\tilde{x}}\tilde{U}_x \). This action fixes a subspace \( T_{\tilde{x}}\tilde{U}_x^{\Gamma_x} \), cf. [4]. The singular dimension of \( x \) is defined to be \( \text{sdim}(x) = \dim T_{\tilde{x}}\tilde{U}_x^{\Gamma_x} \) and does not depend on the choice of orbifold chart. The singular set is the union \( \Sigma = \bigcup_{k=0}^{n-1} \Sigma_k \) of singular strata \( \Sigma_k \), where \( \Sigma_k = \{ x \in \Sigma | \text{sdim}(x) = k \} \).

The following proposition, which is for example proven in [3] Proposition 3.4, uses the fact that that the fixed point set of a smooth finite group action on a manifold is a manifold.

**Proposition 2.1.** Let \( O \) be an orbifold. Then each \( \Sigma_k \) is naturally a manifold of dimension \( k \), whose tangent space at \( x \in \Sigma_k \) is modelled on \( T_{\tilde{x}}\tilde{U}_x^{\Gamma_x} \), where \( \tilde{U}_x \) is a chart centered at \( x \) and \( \tilde{x} \) is the lift of \( x \) to this chart.

We denote by \( \Sigma(x) \) the connected component of \( \Sigma_{\text{sdim}(x)} \) containing \( x \). The isomorphism class of the isotropy group \( \Gamma_y \) for \( y \in \Sigma(x) \) is constant and if \( O \) is compact there are finitely many connected components in each singular stratum \( \Sigma_k \).
Sometimes we need to consider multiple orbifolds at the same time. Whenever confusion might arise, we decorate the symbols for the singular set etc. with a sub/superscript specifying the orbifold in question. Given two orbifolds $O$ and $P$, a smooth complete orbifold map between $O$ and $P$ is a triple $f = (f, \{\tilde{f}_x\}, \{\Theta_x\})$ consisting of the following data:

- a continuous map $f: O \to P$ between the underlying topological spaces;
- for each $x \in O$, a group homomorphism $\Theta_x : \Gamma_x \to \Gamma_{f(x)}$;
- for each $x \in O$, given orbifold charts $(\tilde{U}_x, \Gamma_x, \phi_x)$ and $(\tilde{U}_{f(x)}, \Gamma_{f(x)}, \phi_{f(x)})$ around $x$ and $f(x)$, respectively, a smooth lift $\tilde{f}_x : \tilde{U}_x \to \tilde{U}_{f(x)}$ which is $\Theta_x$-equivariant, i.e. $
\tilde{f}_x(\gamma y) = \Theta_x(\gamma) \tilde{f}_x(y) \n$ for all $\gamma \in \Gamma_x$.

In what follows we will drop the adjectives smooth and complete, and just speak of an orbifold map.

We identify two orbifold maps $f = (f, \{\tilde{f}_x\}, \{\Theta^f_x\})$ and $g = (g, \{\tilde{g}_x\}, \{\Theta^g_x\})$ if for every $x \in O$ there exists an orbifold chart $(\tilde{U}_x, \Gamma_x, \phi_x)$ such that $\tilde{f}_x|_{\tilde{U}_x} = \tilde{g}_x|_{\tilde{U}_x}$ and $\Theta^f_x = \Theta^g_x$.

In particular, the maps $f$ and $g$ of the underlying topological spaces coincide for equivalent maps.

A smooth orbifold map $f: O \to P$ is an orbifold diffeomorphism if there exists a smooth orbifold map $f^{-1}: P \to O$ such that $f^{-1}f = id_O$ and $ff^{-1} = id_P$. The identity $id_O$ is the smooth orbifold map $O \to O$ defined by a triple where all maps involved are identities.

The support of an orbifold diffeomorphism $f: O \to O$ is the set

$$\text{supp}(f) = \{x \in O \mid f(x) \neq x\}.$$

In order to define homotopies of orbifold maps, we need to consider the product orbifold structure on $O \times [0,1]$ : its singular set consists of points of the form $(x,t)$, with $x \in \Sigma$ and $t \in [0,1]$, and for all such points the isotropy group $\Gamma_{(x,t)}$ is isomorphic to $\Gamma_x$. The time $t$ inclusion $i_t : O \to O \times [0,1]$ is a smooth orbifold map, where all the defining data are the obvious inclusions.

Two smooth orbifold maps $f, g : O \to P$ are called smoothly homotopic if there exists a smooth orbifold map $F : O \times [0,1] \to P$ such that

$$Fi_0 = f \quad \text{and} \quad Fi_1 = g.$$

Two orbifold diffeomorphisms are isotopic if they are homotopic through diffeomorphisms. They are compactly supported isotopic if the homotopy is compactly supported. The group of compactly supported diffeomorphisms of an orbifold $O$, which are compactly supported isotopic to the identity is denoted by $\text{Diff}_c(O)$.

3. Transitivity

In this section we will prove our main result. We start, though, by proving that an arbitrary orbifold diffeomorphism preserves the singular dimension, which shows that two points on an orbifold can only be related by a diffeomorphism if they have the same singular dimension.
Lemma 3.1. Suppose \( f : \mathcal{O} \to \mathcal{P} \) is an orbifold diffeomorphism: then for every \( x \in \mathcal{O} \) we have that \( \text{sdim}(x) = \text{sdim}(f(x)) \).

Proof. The isotropy group \( \Gamma_x \) acts on \( T\bar{U}_x \) via the differential. We denote this action by \( \gamma \cdot \bar{X} \), where \( \gamma \in \Gamma_x \) and \( \bar{X} \in T\bar{U}_x \). If \( \bar{X} \in T\bar{z}\bar{U}_{x}^{\Gamma_x} \), then \( g \cdot \bar{X} = \bar{X} \) and by equivariance \( \Theta_x(g)T\bar{z}\bar{f}_x(\bar{X}) = T\bar{z}\bar{f}_x(g \cdot \bar{X}) = T\bar{z}\bar{f}_x(\bar{X})X \). The group homomorphism \( \Theta_x : \Gamma_x \to \Gamma_{f(x)} \) is an isomorphism for orbifold diffeomorphisms, hence for every \( \mu \in \Gamma_{f(x)} \) is of the form \( \Theta_x(g) \) for some \( \gamma \in \Gamma_x \). Thus \( \mu \cdot T\bar{z}\bar{f}_x \bar{X} = T\bar{z}\bar{f}_x(\bar{X}) \) for all \( \mu \in \Gamma_{f(x)} \). We conclude that \( T\bar{z}\bar{f}_x(T\bar{z}\bar{U}_{x}^{\Gamma_x}) \subset T\bar{z}(\bar{f}_x(\bar{x})) \bar{U}_{f(x)}^{\Gamma_{f(x)}} \), which implies that \( \text{sdim}(x) \leq \text{sdim}(f(x)) \). The same argument applied to the inverse map \( f^{-1} \) then shows that \( \text{sdim}(f(x)) \leq \text{sdim}(x) \), hence \( \text{sdim}(x) = \text{sdim}(f(x)) \). \( \square \)

The next proposition is the crucial step in the proof of our result: it shows that locally we can carry a given point \( x \) to any sufficiently close point lying on the same connected component by a diffeomorphism which is isotopic to the identity via an isotopy having compact support in a prescribed open neighbourhood of \( x \).

Proposition 3.2. Let \( \mathcal{O} \) be an orbifold, \( x \) a point of \( \mathcal{O} \), and let \( U \) be an open neighbourhood of \( x \). Then there exists an open subset \( V \) of \( \Sigma(x) \), such that \( V \subset U \) and \( x \in V \) and for each \( y \in V \) there exists \( f \in \text{Diff}_x(\mathcal{O}) \) such that \( f(x) = y \) and \( f \) is compactly isotopic to the identity via an isotopy supported in \( U \).

Proof. Choose an orbifold chart \( (\bar{U}_x = \mathbb{R}^n, \Gamma_x, \phi_x) \) centered around \( x \), and such that \( U_x \subset U \). Let \( \bar{V} = \bar{U}_x^{\Gamma_x} \subset \bar{U}_x \) be the connected component of the fixed set of \( \Gamma_x \) containing \( \bar{z} = \phi_x^{-1}(x) \), and define \( V = \phi_x(\bar{V}/\Gamma_x) \). The map \( \phi_x|_{\bar{V}/\Gamma_x} \) is injective and \( \phi_x(\bar{V}/\Gamma_x) \subset \Sigma(x) \cap U_x \). Let \( \bar{y} \in \bar{V} \). We want to show that there exists a diffeomorphism \( \bar{f} \) of \( \bar{U}_x \) which satisfies the following three conditions:

(i) \( \bar{f} \) is equivariant with respect to the \( \Gamma_x \) action;
(ii) \( \bar{f} \) is isotopic to the identity, via equivariant diffeomorphisms compactly supported in \( U \);
(iii) \( \bar{f}(\bar{x}) = \bar{y} \).

Since \( \bar{V} \) is a connected manifold, there exists a vector field \( \bar{Y} \) on \( \bar{V} \), which is compactly supported\(^1\) and whose time-one flow maps \( \bar{x} \) to \( \bar{y} \). We can extend \( \bar{Y} \) to a compactly supported vector field defined everywhere on \( \bar{U}_x \), and we will denote this vector field by the same symbol. Recall that the action of \( \Gamma_x \) on \( \bar{U}_x \) induces a linearised action on \( T\bar{U}_x \), which we denote by \( \gamma \cdot \bar{X} \), with \( \gamma \in \Gamma_x \) and \( \bar{X} \in T\bar{U}_x \). Note that \( \gamma \cdot : T\bar{U}_x \to T\gamma \bar{U}_x \) and that, as \( \bar{Y} \) is tangent to \( \bar{V} \) and the action fixes \( \bar{V} \), \( \bar{Y}(\bar{z}) = \bar{Y}(\bar{z}) \) for all \( \bar{z} \in \bar{V} \). We need to average over the action of the group \( \Gamma_x \) in order to make our vector field equivariant, that is, for all \( \bar{z} \in \bar{U}_x \) we define \( \bar{X}(\bar{z}) \) via the formula:

\[
\bar{X}(\bar{z}) = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \gamma \cdot \bar{Y}(\gamma^{-1}\bar{z}).
\]

\(^1\)A vector field is compactly supported if it vanishes outside a compact set.
Since $\tilde{Y}$ is compactly supported, and $\Gamma_x$ is finite, the vector field $\tilde{X}$ is compactly supported. Then we check that, for $\mu \in \Gamma_x$, that

$$\mu \cdot \tilde{X}(\tilde{z}) = \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} \mu \cdot (\gamma \cdot \tilde{Y}(\gamma^{-1}\tilde{z}))$$

$$= \frac{1}{|\Gamma_x|} \sum_{\gamma \in \Gamma_x} (\mu\gamma) \cdot \tilde{Y}((\mu\gamma)^{-1}\mu\tilde{z}) = \frac{1}{|\Gamma_x|} \sum_{\gamma' \in \Gamma_x} \gamma' \cdot \tilde{Y}(\gamma'^{-1}\mu\tilde{z}) = \tilde{X}(\mu\tilde{z}),$$

i.e. $\tilde{X}$ is indeed equivariant with respect to the $\Gamma_x$ action. Note that for $\tilde{z} \in \tilde{V}$ we have that $\gamma \cdot X(\tilde{z}) = \tilde{Y}(\tilde{z})$, hence the flow of $\tilde{X}$ agrees with that of $\tilde{Y}$ when restricted to $V$. In particular the time-one flow $\tilde{f}$ of $\tilde{X}$ maps $\tilde{x}$ to $\tilde{y}$. Clearly $\tilde{f}$ extends to an complete orbifold diffeomorphism $\tilde{f}$ which is compactly supported isotopic to the identity, as it is compactly supported in the chart. The underlying map $f$ maps $x$ to $y = \phi_x([\tilde{y}])$. □

We can now move on to the proof of our main result.

**Theorem 3.3** (Transitivity). Let $\mathcal{O}$ be an orbifold. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two $n$-tuples of pairwise distinct points. Assume that for every $i$ the points $x_i$ and $y_i$ lie in the same connected component of the singular stratification and that their singular dimension is not equal to 1. Then there exists an orbifold diffeomorphism $f : \mathcal{O} \to \mathcal{O}$, compactly supported isotopic to the identity, such that $f(x_i) = y_i$ for all $1 \leq i \leq n$.

**Proof.** Consider the $n$-fold cartesian product $\mathcal{O}^n = \mathcal{O} \times \ldots \times \mathcal{O}$ and the (open) suborbifold $\mathcal{O}^{(n)} \subset \mathcal{O}^n$, whose underlying space consists of $n$-tuples of points which are pairwise distinct, i.e.

$$O^{(n)} = \{(x_1, \ldots, x_n) \in O^n | x_i \neq x_j \text{ if } i \neq j\}.$$  

The group of compactly supported diffeomorphisms which are isotopic to the identity, $\text{Diff}_c(\mathcal{O})$, acts on $\mathcal{O}^{(n)}$ diagonally. One verifies that $\Sigma^0(\mathcal{O}^{(n)}) = \Sigma^0(x_1) \times \ldots \times \Sigma^0(x_n)$. Notice that $\Sigma^0(x_i)$ is either zero dimensional or of dimension $\geq 2$ by assumption: since removing a codimension $\geq 2$ set does not change the connectivity, we find that

$$\Sigma^0(x_1, \ldots, x_n) = \Sigma^0(x_1, \ldots, x_n) \cap \mathcal{O}^{(n)} = (\Sigma^0(x_1) \times \ldots \times \Sigma^0(x_n)) \cap \mathcal{O}^{(n)}.$$  

We show next that each $\text{Diff}_c(\mathcal{O})$-orbit is open in $\Sigma^0(x_1, \ldots, x_n)$.

Let $(z_1, \ldots, z_n) \in \Sigma^0(x_1, \ldots, x_n)$ and choose for each $z_i$ an open subset $U_i \ni z_i$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Then $U_1 \times \ldots \times U_n \subset O^{(n)}$. Let $V_i$ be the open subset of $\Sigma(z_i)$ given by Proposition 3.2 where we choose $x = z_i$. Then for each $w_i \in V_i$ there exists $f_i \in \text{Diff}_c(\mathcal{O})$ such that $f_i(z_i) = w_i$. Let $g = f_1 \circ \ldots \circ f_n$. Since all $f_i$’s have distinct support, we have that $g(z_i) = w_i$. It follows that the $\text{Diff}_c(\mathcal{O})$ orbit through $(z_1, \ldots, z_n)$ is open in $\Sigma^0(x_1, \ldots, x_n)$. Since $\Sigma^0(x_1, \ldots, x_n)$ is connected, there can be only one orbit and hence there exists a compactly supported diffeomorphism $f$, compactly supported isotopic to the identity, such that $f(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. □
Remark 3.4. The statement can be modified to include the one-dimensional stratum, but it is slightly more technical to state. The reason is that even in the manifold case the diffeomorphism group does not act $k$-transitively on points if the manifold is one dimensional and $k > 1$. Consider for example the one-dimensional manifold $\mathbb{R}$. Then two tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are said to be ordered if $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$. There exists a compactly supported diffeomorphism $f$, isotopic to the identity with $f(x_i) = y_i$, if the tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are both ordered, but not if only one of the sets is ordered and the other is not. If one of the sets has the reverse order, e.g. $y_1 > y_2 > \ldots > y_n$, a diffeomorphism still exists, but it is not isotopic to the identity, as it reverses the orientation of $\mathbb{R}$.

The case for the circle $S^1$ is similar: one can lift a diffeomorphism of $S^1$ to a diffeomorphism of $\mathbb{R}$ by choosing a base-point in $S^1$ and an initial lift to $\mathbb{R}$. The points $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are then lifted to points $\tilde{x}_1, \ldots, \tilde{x}_n$ and $\tilde{y}_1, \ldots, \tilde{y}_n$. There exists a diffeomorphism that maps $x_i$ to $y_i$ if and only if there exists cyclic permutations $\sigma$ and $\mu$ such that $\tilde{x}_{\sigma(1)} < \ldots < \tilde{x}_{\sigma(n)}$ and $\tilde{y}_{\mu(1)} < \ldots < \tilde{y}_{\mu(n)}$.

In the orbifold case, if each component of the singular stratum $\Sigma$ of $\mathcal{O}$ is a one dimensional manifold, we can still make sense of the orderability condition above (this involves choosing and orientation of each component) by orienting these manifolds. But we do not think it worthwhile to make a full statement.

Corollary 3.5. Let $\mathcal{O}$ be an orbifold with $\dim \mathcal{O} \geq 2$ and let $x \in \mathcal{O}$. Then the orbit $\operatorname{Diff}_c(\mathcal{O})x$ coincides with $\Sigma(x)$. Hence the orbit $\operatorname{Diff}_c(\mathcal{O})x$ is dense in a connected orbifold $\mathcal{O}$ if and only if $\operatorname{sdim}(x) = \dim \mathcal{O}$.

4. Applications

4.1. Displacing curves on a two sphere, and its orbifold analogue. The non-transitivity of the orbifold diffeomorphism group has consequences outside of the smooth category, even if we are not concerned with the displacement of points: for instance, in the setting of area preserving maps. We give an example here, and we are confident that the reader can think of more implications.

Let us recall the following property of area preserving maps of the two-sphere. Equip the sphere $S^2$ with the standard area form $\omega$. Consider an embedded simple closed curve $i : S^1 \to S^2$, and let $u_1 : D^2 \to S^2$ and $u_2 : D^2 \to S^2$ be the two discs bounded by this curve.\footnote{It is clear that the curve divides $S^2$ into two compact orientable surfaces with boundary. A simple arument involving the Euler characteristic and the classification of orientable surfaces then shows that these must be discs.} If we denote by $D_1$ and $D_2$ their image on the sphere, that is, $D_i = u_i(D^2)$, $i = 1, 2$, then their area is given by $A_1 = \int_{D_2} u_1^* \omega$ and $A_2 = \int_{D_2} u_2^* \omega$, respectively. We are going to prove that any such simple closed curve corresponds, under an area preserving diffeomorphism of
Figure 1. On the right a simple closed curve on $S^2$ is drawn. Via a diffeomorphism $f$ this is pulled back to a parallel curve. Pulling the standard area form on the right back to the middle picture creates a non-standard area form. Via an additional diffeomorphism the area form can be brought back into standard form. The original curve is pulled back to a circle of constant latitude. The height is completely determined by the ratio of areas $A_1/A_2$.

If $A_1$ and $A_2$ are not equal, the original circle is mapped to a circle latitude strictly above or below the equator. Via a rotation, which is an area preserving diffeomorphism, the curve can be displaced from itself. Note that the disc with smaller area is mapped into the disc with larger area.

If $A_1 = A_2$, the embedded curve is called monotone (or balanced), and it cannot be displaced by an area preserving map. In fact, by the argument above, we may always assume the image of the curve to be the equator. If the equator could be displaced by an area preserving map, then one hemisphere would have to be mapped onto a proper subset of its interior or of the interior of the opposite hemisphere, contradicting in both cases the assumption that the map is area preserving. \(^3\)

This discussion can be summarized in the statement below:

**Proposition 4.1.** A simple closed curve on the two sphere can be displaced from itself by an area preserving diffeomorphism if and only if the curve is monotone.

\(^3\)A nice example of an area preserving map is the antipodal map $-\text{id}$. Thus any simple closed curve, diving $S^2$ into two regions of the same area, must have antipodal points. A nice visual explanation is given in the numberphile video “Antipodal Points”.

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There are various ways to generalize this example. For instance, this result fits into the framework of non-displaceable (monotone) Lagrangian submanifolds, which is due to Floer [5] (in the exact case) and Oh [8] in the monotone case. In this generality the proofs are much more difficult.

Let us now move on to the orbifold world. Let \( p \) and \( q \) be two coprime natural numbers. The \((p,q)\) spindle is the orbifold obtained as \( \mathbb{CP}^1(p,q) = S^3//S^1 \), where \( S^1 \subset \mathbb{C} \) acts on \( S^3 \subset \mathbb{C}^2 \) via \( t \cdot (z_0, z_1) = (t^p z_0, t^q z_1) \). As this action is isometric with respect to the standard Riemannian metric, the \((p,q)\) spindle is naturally a Riemannian orbifold, and therefore also carries an area form. The argument that follows does not depend on the specific area form. We denote the points on \( \mathbb{CP}^1(p,q) \) by the equivalence class \([z_0, z_1]\). The spindle has two isolated singular points at \( N = [0, z_1] \) and \( S = [z_0, 0] \), with isotropy \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \), respectively. Now let \( \gamma \) be any simple closed curve that does not pass through the singular points \( N \) and \( S \).

**Proposition 4.2.** Suppose \( p \) and \( q \) are coprime. If a simple closed curve on the spindle \( \mathbb{CP}^1(p,q) \) divides it into two components, each containing one of the singular points, then it cannot be displaced from itself by an area preserving orbifold diffeomorphism.

The interesting thing to note is that we are trying to displace a curve that lies entirely in the manifold part of the orbifold, but it still “feels” the singularities far away.

If both singular points lie in one region, say \( N, S \in D_1 \), then we can displace the curve by an area preserving map, provided the area of \( D_2 \) is smaller than that of \( D_1 \).

### 4.2. Homotopy classes of maps

Distinguishing homotopy classes of maps between connected closed manifolds \( M \) and \( N \) is an interesting and challenging question. One of the easiest invariants which can be defined to tackle this question is the (embedded) cobordism class of a regular value. A regular value \( y \) of a map \( f : M \to N \) defines a submanifold \( f^{-1}(y) \) of \( M \). If \( F : M \times [0,1] \to N \) is a homotopy between \( f \) and \( g \), and \( y \) is a regular value of \( F \), which is also a regular value of \( f \) and \( g \), then \( F \) defines an embedded cobordism \( F^{-1}(y) \) in \( M \times [0,1] \) between \( f^{-1}(y) \) and \( g^{-1}(y) \).

It follows from the fact that regular values are open and dense, and that the diffeomorphism group acts transitively on \( N \), that the cobordism class of a regular value of a map \( f : M \to N \) is independent of the regular value and the homotopy class of \( f \).

But this invariant is not strong enough to distinguish some important maps. Let us consider for instance the Hopf fibration, which is homotopically non-trivial. If we view \( S^3 \) as the unit sphere in \( \mathbb{C}^2 \), and \( S^2 \) as the unit sphere in \( \mathbb{C} \times \mathbb{R} \), then the Hopf fibration is the map \( f : S^3 \to S^2 \).
defined by \( f(z_0, z_1) = (z_0z_1, |z_0|^2 - |z_1|^2) \). This map is in fact the generator of \( \pi_3(S^2) \cong \mathbb{Z} \).

Every value is regular and the preimage of each regular value is an unknotted, embedded circle in \( S^3 \). The unknot is the boundary of an embedded disc in \( S^3 \times [0, 1] \), hence the cobordism class of the preimage of a regular value does not distinguish the homotopy class of the Hopf fibration from the homotopy class of the constant map.

Still we can try to obtain further information by considering a pair of distinct points and their preimages. In this setting, a cobordism will consist of a pair of disjoint, embedded surfaces, whose boundary consists of the preimages of different pairs of regular points. Given that the diffeomorphism group acts 2-transitively on \( S^2 \), the embedded cobordism class of two regular values is independent of the chosen regular values, and it is independent of the homotopy class. The preimages of two points \( p, q \in S^2 \) under the Hopf map are two embedded unknots in \( S^3 \), but they are linked. In particular, this implies that we cannot find a pair of disjoint, embedded surfaces in \( S^3 \times [0, 1] \) whose boundary consists of these two circles alone. It follows that the Hopf fibration is homotopically non-trivial. In this case this invariant captures the same information as the framed cobordism sets of \( S^3 \), but in general these invariants are different.

Let us formalize the argument above directly in the language of orbifolds.

**Definition 4.3.** Let \( \mathcal{O} \) be an orbifold. A \( k \)-dimensional \( n \)-colored suborbifold, is a tuple of \( n \) disjoint compact full suborbifolds (without boundary) \((\mathcal{N}_1, \ldots, \mathcal{N}_n)\) of \( \mathcal{O} \). A cobordism between the \( k \)-dimensional \( n \)-colored suborbifolds \((\mathcal{N}_1, \ldots, \mathcal{N}_n)\) and \((\mathcal{M}_1, \ldots, \mathcal{M}_n)\) is a tuple \((\mathcal{W}_1, \ldots, \mathcal{W}_n)\) of \((k + 1)\)-dimensional compact full suborbifolds of \( \mathcal{O} \times [0, 1] \) with boundary \((\mathcal{N}_1 \times \{0\}, \ldots, \mathcal{N}_n \times \{0\}) \cup (\mathcal{M}_1 \times \{1\}, \ldots, \mathcal{M}_n \times \{1\})\). We denote the set of \( k \)-dimensional \( n \)-colored suborbifolds modulo cobordism by \( \text{CEmb}_k^n(\mathcal{O}) \).

Note that the suborbifolds \( \mathcal{N}_i, \mathcal{M}_i \) and \( \mathcal{W}_i \) are not required to be connected. Now given a proper orbifold map \( f : \mathcal{O} \to \mathcal{P} \), and a choice of connected components of some singular strata \( \Sigma(x_1), \ldots, \Sigma(x_n) \), we would like to associate a proper homotopy invariant to \( f \) as follows. Perturb \( f \) to \( \tilde{f} \) such that there exist regular values \( y_i \in \Sigma(x_i) \), and define the \( n \)-colored suborbifold \((\mathcal{N}_1 = \tilde{f}^{-1}(y_1), \ldots, \mathcal{N}_n = \tilde{f}^{-1}(y_n))\). The invariance, up to \( n \)-colored cobordism, under different choices of regular values now depends on the \( n \)-transitivity of the compactly supported orbifold diffeomorphism group. But in this orbifold setting, we have to deal with the problem of equivariant transversality: even though the set of regular values of an orbifold mapping is still dense [2], if a point \( y \) lies in the singular stratum of \( P \) it is not clear that the map \( f : \mathcal{O} \to \mathcal{P} \) can be made transverse to \( y \) by a suitable perturbation. This will be the subject of future research.

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