EXISTENCE OF MINIMIZERS FOR THE SDRI MODEL

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Abstract. The SDRI model introduced in [35] in the framework of the theory on Stress-Driven Rearrangement Instabilities for the morphology of crystalline material under stress [2, 33] is considered. The existence of solutions is established in dimension two in the absence of graph-like assumptions and of the restriction to a finite number m of connected components for the free boundary of the region occupied by the crystalline material, thus extending previous available results for epitaxially-strained thin films and material cavities [26, 27, 35]. Due to the lack of compactness and lower semicontinuity even for sequences of m-minimizers, i.e., energy minimizers among configurations with a fixed number m of connected boundary-components, the minimizing candidate of the SDRI model is directly constructed. By means of uniform density estimates for the local decay of the energy at the m-minimizers’ boundaries, such candidate is then shown to be a minimizer also in view of the convergence of the energy at m-minimizers to the energy infimum as m → ∞. Finally, regularity properties for the morphology of any minimizer are deduced.

1. Introduction

In this paper we establish existence and regularity properties for the solutions of the model for Stress-Driven Rearrangement Instabilities (SDRI) [2, 18, 33] that was introduced in [35]. Under the name of Stress-Driven Rearrangement Instabilities are included all those material morphologies, such as boundary irregularities, cracks, filaments, and surface patterns, which a crystalline material may exhibit in the presence of external forces, such as in particular the chemical bonding forces with adjacent materials. In order to release the induced stresses, atoms rearrange from the material optimal crystalline order and instabilities may develop.

The main leap forward of the results contained in this manuscript with respect to [35] is the absence of the unphysical restriction on the number of connected components for the boundary of the region occupied by the crystalline material, by both avoiding graph-like assumptions for such boundaries assumed for the specific settings of epitaxially-strained thin films in [11, 26] and material voids in [27], and the extension to unphysical bulk displacements, namely displacements allowed to attain infinite value on sets of positive measure (and technically assigning a zero cost to the elastic-energy contribution related to those sets), considered in [11].

The SDRI model of [35] is a variational model introduced in the framework of the SDRI theory initiated in the seminal papers of [2] and [33], and on the basis of the subsequent analytical descriptions provided in context of epitaxially-strained thin films [19, 20, 26], crystal cavities [5, 27], capillarity droplets [21, 22], fractures...
[8, 12, 14], and boundary debonding and delamination [3, 37]. All such settings are included and can be treated simultaneously in the SDRI model [35]. In agreement with [2, 33] since SDRI morphologies relates to the boundary of crystalline materials and depend on the bulk rearrangements, the energy $F$ characterizing the SDRI model displays both an elastic bulk energy and a surface energy denoted by $W$ and $S$, respectively. More precisely, the energy $F$ is defined as

$$F(A, u) := S(A, u) + W(A, u)$$

(1.1)

for any admissible configurational pair $(A, u)$ consisting of a set $A$ that represents the region occupied by the crystalline material in a fixed container $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, i.e., $A \subset \Omega$, such that the topological boundary $\partial A$ is $H^{d-1}$-rectifiable with finite area, and of a displacement function of the bulk materials (with respect to the optimal crystal arrangement)

$$u \in GSBD^2(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^d) \cap H^{1}_{\text{loc}}(\text{Int}(A \cup S); \mathbb{R}^d),$$

where $S \subset \mathbb{R}^d \setminus \Omega$ represents the region occupied by possibly a different material and it is denoted as substrate in analogy with the setting of thin film, while

$$\Sigma := \partial S \cap \partial \Omega$$

is the contact surface between the container $\Omega$ and the substrate $S$. In the following we refer to $C$ as the configurational space and to any configuration $(A, u) \in C$ as a free crystal with $A$ and $u$ as the free-crystal region and the free-crystal displacement, respectively (see Figure 1).

![Figure 1](image-url)

**Figure 1.** An admissible free-crystal region $A$ is displayed in light blue in the container $\Omega$, while the substrate $S$ is represented in dark blue. The boundary of $A$ (with the cracks) is depicted in black, the container boundary in green, the contact surface $\Sigma$ in red (thicker line) while the free-crystal delamination region $J_u$ with a white dashed line.

We consider the case $d = 2$ as in [35], with the fixed sets $\Omega$ and $S$ bounded Lipschitz open, connected, and such that $\Sigma$ is a Lipschitz 1-manifold. For $d \geq 3$ there
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are presently no available existence results for the SDRI model. They are available for the isotropic Griffith model with $L^p$-fidelity term (of the type (2.8)) in [8] and with Dirichlet conditions for the displacements at the boundary in [9]. Moreover, the SDRI energy introduced in [35] was subsequently found for the wetting regime (i.e., the case where free crystals are assumed to always cover the substrate) in [11] as a relaxation formula for both thin films and material voids (without external filaments). The existence results in [11] are achieved by working with displacements in the functional space $\text{GSBD}^p_{\infty}$, $p > 1$, that includes displacements attaining the infinite value in a set of finite perimeter (on which their strain $e(u)$ is defined to be zero, see [11, Page 1055]), which appear to go beyond the classical framework of small displacements of linearized elasticity, and for the thin-film setting (where indeed minimizing displacements are proven to be in $\text{GSBD}^p$) under a bounded-graph assumption on film profiles and without considering delamination.

The bulk elastic energy $W$ in (1.1) is defined by

$$W(A, u) = \int_{A \cup S} W(z, e(u) - M_0) \, dz,$$

where the elastic density $W$ is given by

$$W(z, M) := C(z) M : M$$

for any $z \in \Omega \cup S$ and any $2 \times 2$-symmetric matrix $M \in \mathbb{M}^{2 \times 2}_{\text{sym}}$, and for a positive-definite elasticity tensor $C$, and attains its minimum value zero for any $z$ at a fixed strain $M_0 \in \mathbb{M}^{2 \times 2}_{\text{sym}}$ in the following referred to as mismatch strain. The inclusion in (1.2) of $M_0$ that is defined by

$$M_0 := \begin{cases} e(u_0) & \text{in } \Omega, \\ 0 & \text{in } S, \end{cases}$$

for a fixed $u_0 \in H^1(\Omega; \mathbb{R}^2)$, together with the fact that both $M_0$ and $C$ are let free of jumping across $\Sigma$, allows to model the presence of two different materials in the substrate and in the free crystals, and in particular to take into account the lattice mismatch that can be present between their optimal crystalline lattices that is crucial, e.g., in the setting of heteroepitaxy [19, 20].

The surface energy $S$ in (1.1) is defined as

$$S(A, u) = \int_{\partial A} \psi(z, u, \nu) \, d\mathcal{H}^{d-1},$$

where the surface tension $\psi$ is given by

$$\psi(z, u, \nu) := \begin{cases} \varphi(z, \nu_A(z)) & z \in \Omega \cap \partial^* A, \\ 2\varphi(z, \nu_A(z)) & z \in \Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A, \\ \varphi(z, \nu_S(z)) + \beta(z) & z \in \Sigma \cap A^{(0)} \cap \partial A, \\ \beta(z) & z \in \Sigma \cap \partial^* A \setminus J_u, \\ \varphi(z, \nu_S(z)) & J_u, \end{cases}$$

with $\varphi \in C(\overline{\Omega} \times \mathbb{R}^d; [0, +\infty))$ being a Finsler norm such that $c_1 |\xi| \leq \varphi(x, \xi) \leq c_2 |\xi|$ for some $c_1, c_2 > 0$ and representing the anisotropy of the free-crystal material, $\beta$ denoting the relative adhesion coefficient on $\Sigma$ such that, as for capillarity problems [21, 22],

$$|\beta(z)| \leq \varphi(z, \nu_S(z))$$

for $z \in \Sigma$, $\nu$ coinciding with the exterior normal on the reduced boundary $\partial^* A$, and $A^{(\delta)}$ denoting the set of points of $A$ with density $\delta \in [0, 1]$. 

The anisotropic form of $\psi$ in (1.3) distinguishes the various portions of the free-crystal topological boundary $\partial A$, that are the free boundary $\partial^* A \cap \Omega$, the set of internal cracks $A^{(1)} \cap \Omega \cap \partial A$, the set of external filaments $A^{(0)} \cap \Omega \cap \partial A$, the delaminated region $J_u$ that coincides with the debonding area, i.e., the portion on the contact surface $\Sigma$ where there is no bonding between the free crystal and the substrate (even if they are adjacent), the adhesion area where the free-crystal displacement is continuous through $\Sigma$, i.e., $\Sigma \cap \partial^* A \setminus J_u$, and the wetting layer represented by the filaments on $\Sigma$, i.e., $\Sigma \cap A^{(0)}$. In particular, $\psi$ weights the different portions of $\partial A$ in relation to the active chemical bondings present at each portion: $\varphi$ when there is no extra chemical bonding, such as at the free profile and at the delaminated region, and $\beta$ at the adhesion contact area with the substrate, while both the cracks and at external filaments are counted $2 \varphi$ and the wetting layer sees the contribution of both $\psi$ and $\beta$.

As shown in [35, Theorem 2.8] such specific weights are crucial to obtain the lower semicontinuity of the energy $F$ in the subfamily $C_m$ of configurations with free-crystal regions presenting a fixed number $m \in \mathbb{N}$ of boundary connected components, namely

$$C_m := \{(A, u) \in C : \partial A \text{ has at most } m \text{ connected components}\},$$

with respect to a properly selected topology $\tau_C$, i.e.,

$$\liminf_{k \to \infty} F(A_k, u_k) \geq F(A, u)$$

for every $\{(A_k, u_k)\} \subset C_m$ and $(A, u) \in C_m$ such that $H^1(\partial A_k)$ are equibounded, $\text{sdist}(\cdot, \partial A_k) \to \text{sdist}(\cdot, \partial A)$ locally uniformly in $\mathbb{R}^2$ with $\text{sdist}$ representing the signed distance function (recall definition at (2.1)), and $u_n \to u$ a.e. in $\text{Int}(A) \cup S$. In particular, the restriction to the subfamily $C_m$, which represented already an extension of the more restrictive graph condition assumed in [26] for the particular setting of epitaxially-strained thin films and the starshapedness condition in [27] for material cavities, was needed in [35] not only to prove lower semicontinuity, but also to prove compactness with respect to $\tau_C$, which fails in $C$ (see Remark 2.3). This enabled in [35, Theorem 2.6] to prove the existence of minimizers $(A_m, u_m) \in C_m$ of $F$ among all configurations in $C_m$, in the following referred to as $m$-minimizer of $F$, by means of the direct method of the calculus of variations.

The aim of the investigation in this paper is to recover the full generality avoiding any extra hypothesis on the admissible free-crystal regions apart from having an $H^1$-rectifiable and finite-length topological boundary (or displacements $u$ allowed to be infinite on sets with positive measure as in [11]). This is achieved by retrieving compactness with respect to the free-crystal regions at least for any sequence of $m$-minimizers $(A_m, u_m) \in C_m$, and by combining the strategies of [17] and [35]. More precisely, the use in [35] of the Golab-type Theorem [30] is avoided for the compactness of the free-crystal regions by adapting to our setting classical density-estimate arguments first introduced for surface energies and the Mumford-Shah functional (see, e.g., [1, 39]), and then extended to the Griffith functional [9, 14]. We notice that even though our approach stems from the strategy employed in [17], in our setting there is the further difficulty that compactness and lower semicontinuity along sequences of $m$-minimizers (with respect to the topology used to find such $m$-minimizers through the direct method) are missing. Therefore, we directly construct a minimizing candidate, prove that it belongs to $C$, and finally show a “lower-semicontinuity inequality” (see (1.6) below) with respect to the $m$-minimizers (see Subsection 1.1 for more details).
The result of this paper are twofold: An existence result in Theorem 2.4, and regularity results in Theorem 2.5. More precisely, in Theorem 2.4 we prove the existence of a minimum configuration of $\mathcal{F}$ among all configurations in $\mathcal{C}$ with free-crystal region satisfying a volume constraint, i.e., we solve the minimum problem

$$\inf_{(A,u) \in \mathcal{C}, |A| = v} \mathcal{F}(A, u)$$

for a fixed volume parameter $v \in (0, |\Omega|)$, and we show that the unconstrained problem consisting in minimizing in $\mathcal{C}$ the volume-penalized functional

$$\mathcal{F}^\lambda(A, u) := \mathcal{F}(A, u) + \lambda |A| - v$$

for a penalization constant $\lambda > 0$, is equivalent to the minimization problem (1.4) provided that $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$ (independent of $(A, u)$).

In Theorem 2.5 some regularity properties of the minimizers of (1.4) are determined, such as that minimizing free-crystal regions are open sets with cracks coinciding in $\Omega$ with the jump set of the corresponding minimizing free-crystal displacements, and that their boundary satisfies uniform upper and lower density estimates. Furthermore, we also observe that, given a minimizer $(A, u)$ any connected component $E$ of $A$ that do not intersect $\Sigma \setminus J_u$ (up to $\mathcal{H}^1$-negligible sets), must have a sufficiently large area, i.e.,

$$|E| \geq (c_1 \sqrt{4\pi/\lambda_0})^2,$$

and must satisfy $u = u_0$ in $E$.

1.1. **Paper organization and detail of the proofs.** The paper is organized in 5 sections. In Section 2 we introduce the mathematical setting, recall the SDRI model from [35], and carefully state the main results of the paper.

In Section 3 we prove the upper and lower density estimates for the local decay of the energy $\mathcal{F}$ on any sequence of $m$-minimizers $(A_m, u_m) \in \mathcal{C}_m$ (see Theorem 3.1) by considering a local version of $\mathcal{F}^\lambda$ (see (3.4)) and adapting arguments of [1, 9, 14] to our setting with not only displacements, but displacements paired with free-crystal regions, and paying extra care to the fact that $\mathcal{C}$ is possibly not constant (but in $L^\infty(\Omega \cup S) \cap C^0(\Omega)$).

In Section 4 we begin by establishing in Proposition 4.1 the compactness (up to extracting a subsequence) of the free-crystal regions $A_m$ of the $m$-minimizers $(A_m, u_m)$ with limit a set of finite perimeter $A \subset \Omega$ by means of both the Blaschke-type selection principle [35, Proposition 3.1] and the density estimates established in Section 3. Then, in Proposition 4.3, we further extend the (already generalized) Golab-type Theorem [30, Theorem 4.2] to a priori not connected $\mathcal{H}^1$-measurable sets satisfying uniform density estimates (see [17] for the isotropic case). Finally, we prove Theorem 2.4. This is achieved by first considering the free-crystal configuration $(\tilde{A}, \tilde{u})$ with free-crystal region $\tilde{A}$ defined as

$$\tilde{A} := \text{Int}(A \cup \{x \in A^{(1)} \cap \partial A : \text{one-sided traces of } \tilde{u} \text{ at } x \text{ exist and are equal}\}),$$

where the displacement $\tilde{u}$ is defined as the weak $H^1_{\text{loc}}$-limit of the displacements $u_m$ in the regular substrate $S$ and in those components of $A$ where such limit exists in $H^1_{\text{loc}}$ (up to extracting subsequences and adding infinitesimal rigid displacements), and coincides with the function $u_0$ on all other components $E$ of $A$, where we observe that

$$\liminf_{m \to \infty} |u_m(x)| = +\infty$$
for a.e. \( x \) in \( E \). The configuration \((\tilde{A}, \tilde{u}) \in \mathcal{C}\) (not necessarily to any \( C_m \)) and the property
\[
\liminf_{h \to \infty} \mathcal{F}(A_{m_h}, u_{m_h}) \geq \mathcal{F}(\tilde{A}, \tilde{u}). \tag{1.6}
\]
follows from the blow-up method differently performed for each portion of the \( \partial \tilde{A} \).
In particular extra care is needed for \( \tilde{A} \)-cracks (where by definition of \( \tilde{A} \) there is no bound on the number of connected components) where we need to extend some ideas from [35, Proposition 4.1]. This yields the assertion in view of property no bound on the number of connected components) where we need to extend some ideas from [35, Proposition 4.1]. This yields the assertion in view of property (b)
\[
\inf_{(A,u) \in \mathcal{C}, |A|=\nu} \mathcal{F}(A,u) = \inf_{(A,u) \in \mathcal{C}} \mathcal{F}(A,u) = \lim_{m \to \infty} \inf_{(A,u) \in \mathcal{C}_m, |A|=\nu} \mathcal{F}(A,u).
\]
following from [35, Theorem 2.6].

In Section 5 we prove Theorem 2.5, whose regularity properties are direct consequence of the density estimates of Section 3, comparison arguments, the isoperimetric inequality in \( \mathbb{R}^2 \), and the equivalence of the constrained minimum problem (1.4) and the unconstrained penalized minimum problem related to (1.5).

2. Mathematical setting

In this section we recall the SDRI model from [35], collect all the definitions and hypothesis and state the main results of the paper. Since our model is two-dimensional, unless otherwise stated, all sets we consider are subsets of \( \mathbb{R}^2 \). We choose the standard basis \( \{e_1 = (1, 0), e_2 = (0, 1)\} \) in \( \mathbb{R}^2 \) and denote the coordinates of \( x \in \mathbb{R}^2 \) with respect to this basis by \((x_1, x_2)\). We denote by \( \text{Int}(A) \) the interior of \( A \subset \mathbb{R}^2 \). Given a Lebesgue measurable set \( E \), we denote by \( \chi_E \) its characteristic function and by \( |E| \) its Lebesgue measure. The set
\[
E^{(\alpha)} := \left\{ x \in \mathbb{R}^2 : \lim_{r \to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \alpha \right\}, \quad \alpha \in [0, 1],
\]
where \( B_r(x) \) denotes the ball in \( \mathbb{R}^2 \) centered at \( x \) of radius \( r > 0 \), is called the set of points of density \( \alpha \) of \( E \). Clearly, \( E^{(\alpha)} \subset \partial E \) for any \( \alpha \in (0, 1) \), where
\[
\partial E := \{ x \in \mathbb{R}^2 : B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \setminus E \neq \emptyset \text{ for any } r > 0 \}
\]
is the topological boundary. The set \( E^{(0)} \) is the Lebesgue set of \( E \) and \( |E^{(1)}| \Delta E| = 0 \). We denote by \( \partial^* E \) the reduced boundary of a finite perimeter set \( E \) [1, 31], i.e.,
\[
\partial^* E := \left\{ x \in \mathbb{R}^2 : \exists \nu_E(x) := -\lim_{r \to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E(B_r(x))|}, \quad |\nu_E(x)| = 1 \right\}.
\]
The vector \( \nu_E(x) \) is called the measure-theoretic normal to \( \partial E \).

The symbol \( \mathcal{H}^s \), \( s \geq 0 \), stands for the \( s \)-dimensional Hausdorff measure. An \( \mathcal{H}^1 \)-measurable set \( K \) with \( 0 < \mathcal{H}^1(K) < \infty \) is called \( \mathcal{H}^1 \)-rectifiable if \( \theta^*(K, x) = 0 \) for \( \mathcal{H}^1 \)-a.e. \( x \in K \), where
\[
\theta^*(K, x) := \limsup_{r \to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r}, \quad \theta_*(K, x) := \liminf_{r \to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r}.
\]
By [25, Theorem 2.3] any \( \mathcal{H}^1 \)-measurable set \( K \) with \( 0 < \mathcal{H}^1(K) < \infty \) satisfies \( \theta^*(K, x) = 1 \) for \( \mathcal{H}^1 \)-a.e. \( x \in K \).

**Remark 2.1** ([31]). If \( E \) is a finite perimeter set, then
\begin{enumerate}[(a)]
\item \( \overline{\partial^* E} = \partial E^{(1)} \);
\item \( \partial^* E \subseteq E^{(1/2)} \) and \( \mathcal{H}^1(E^{(1/2)} \setminus \partial^* E) = 0 \);
\item \( P(E, B) = \mathcal{H}^1(B \cap \partial^* E) = \mathcal{H}^1(B \cap E^{(1/2)}) \) for any Borel set \( E \).
\end{enumerate}
The notation \(\text{dist}(\cdot, E)\) stands for the distance function from the set \(E \subset \mathbb{R}^2\) with the convention that \(\text{dist}(\cdot, \emptyset) \equiv +\infty\). Given a set \(A \subset \mathbb{R}^2\), we consider also signed distance function from \(\partial A\), negative inside, defined as

\[
\text{sdist}(x, \partial A) := \begin{cases} 
\text{dist}(x, A) & \text{if } x \in \mathbb{R}^2 \setminus A, \\
-\text{dist}(x, \mathbb{R}^2 \setminus A) & \text{if } x \in A.
\end{cases}
\] (2.1)

**Remark 2.2.** The following assertions are equivalent:

(a) \(\text{sdist}(x, \partial E_k) \to \text{sdist}(x, \partial E)\) locally uniformly in \(\mathbb{R}^2\);

(b) \(E_k \overset{K}{\to} E\) and \(\mathbb{R}^2 \setminus E_k \overset{K}{\to} \mathbb{R}^2 \setminus \text{Int}(E)\), where \(K\)-Kuratowski convergence of sets [15].

Moreover, either assumption implies \(\partial E_k \overset{K}{\to} \partial E\).

Given \(r > 0\), \(\nu \in S^1\) and \(x \in \mathbb{R}^2\) we denote by \(U_{r, \nu}(x)\) the square of sidelength \(2r\) centered at \(x\) whose sides are either parallel or perpendicular to \(\nu\). When \(\nu = e_2\) or \(\nu = e_1\), drop the dependence on \(\nu\) and write \(U_r(x)\) if in addition \(x = 0\), we write just \(U_r\). Given \(x \in \mathbb{R}^2\) and \(r > 0\), the blow-up map \(\sigma_{x,r}\) is defined as

\[
\sigma_{x,r}(y) = \frac{y - x}{r}.
\] (2.2)

The blow-up \(K \subset \mathbb{R}^2\) is defined as \(\sigma_{x,r}(K)\).

### 2.1. The SDRI model.

Given two nonempty open sets \(\Omega \subset \mathbb{R}^2\) and \(S \subset \mathbb{R}^2 \setminus \Omega\), we define the family of admissible regions for the free crystal and the space of admissible configurations by

\[
\mathcal{A} := \{ A \subset \overline{\Omega} : \partial A \text{ is } \mathcal{H}^1\text{-rectifiable and } \mathcal{H}^1(\partial A) < \infty\}
\]

and

\[
\mathcal{C} := \{ (A, u) : A \in \mathcal{A}, \quad u \in \text{GSBD}^2(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap \mathcal{H}^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^2) \},
\]

respectively, where \(\Sigma := \partial S \cap \partial \Omega\) and \(\text{GSBD}^2(E, \mathbb{R}^2)\) is the collection of all generalized special functions of bounded deformation [10, 16]. Given a displacement field \(u \in \text{GSBD}^2(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap \mathcal{H}^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^2)\) we denote by \(e(u(\cdot))\) the density of \(e(u) = (Du + (Du)^T)/2\) with respect to Lebesgue measure \(\mathcal{L}^2\) and by \(J_u\) the jump set of \(u\). Recall that \(e(u) \in L^2(A \cup S)\) and \(J_u \subset \mathcal{H}^1\text{-rectifiable. Notice also that assumption } u \in \mathcal{H}^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^2)\) implies \(J_u \subset \Sigma \cap \partial^* A\).

Unless otherwise stated, in what follows \(\Omega\) is a bounded Lipschitz open set, \(\Sigma \subseteq \partial \Omega\) is a Lipschitz 1-manifold and \(S \subset \mathbb{R}^2 \setminus \Omega\) is a nonempty connected bounded Lipschitz open set.

The energy of admissible configurations is given by \(\mathcal{F} : \mathcal{C} \to [-\infty, +\infty]\),

\[
\mathcal{F} := \mathcal{S} + \mathcal{W},
\]

where \(\mathcal{S}\) and \(\mathcal{W}\) are the surface and elastic energies of the configuration, respectively. The surface energy of \((A, u) \in \mathcal{C}\) is defined as

\[
\mathcal{S}(A, u) := \int_{\Omega \cap \partial^* A} \varphi(x, \nu_A(x)) d\mathcal{H}^1(x)
+ \int_{\Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A} (\varphi(x, \nu_A(x)) + \varphi(x, -\nu_A(x))) d\mathcal{H}^1(x)
+ \int_{\Sigma \cap A^{(0)} \cap \partial A} (\varphi(x, \nu_\Sigma(x)) + \beta(x)) d\mathcal{H}^1(x)
\]
where \( \varphi : \Omega \times S^1 \to [0, +\infty) \) and \( \beta : \Sigma \to \mathbb{R} \) are Borel functions denoting the anisotropy of crystal and the relative adhesion coefficient of the substrate, respectively, and \( \nu_S := \nu_S \). In the following we refer to the first term in (2.3) as the free-boundary energy, to the second as the energy of internal cracks and external filaments, to the third as the wetting-layer energy, to the fourth as the contact energy, and to the last as the delamination energy. In applications instead of \( \varphi(x, \cdot) \) it is more convenient to use its positively one-homogeneous extension \( |\xi| \varphi(x, \xi/|\xi|) \).

With a slight abuse of notation we denote this extension also by \( \varphi \).

The reason to introduce \( (A, u) \in \mathcal{C} \) is defined as

\[
W(A, u) := \int_{A \cup S} W(x, e(u(x)) - M_0(x)) dx,
\]

where the elastic density \( W \) is determined as the quadratic form

\[
W(x, M) := \mathbb{C}(x) M : M,
\]

by the so-called stress-tensor, a measurable function \( x \in \Omega \cup S \to \mathbb{C}(x) \), where \( \mathbb{C}(x) \) is a nonnegative fourth-order tensor in the Hilbert space \( M_{2 \times 2}^{\text{sym}} \) of all \( 2 \times 2 \)-symmetric matrices with the natural inner product

\[
M : N = \sum_{i,j=1}^2 M_{ij} N_{ij}
\]

for \( M = (M_{ij})_{1 \leq i,j \leq 2}, N = (N_{ij})_{1 \leq i,j \leq 2} \in M_{2 \times 2}^{\text{sym}} \).

The mismatch strain \( x \in \Omega \cup S \mapsto M_0(x) \in M_{2 \times 2}^{\text{sym}} \) is given by

\[
M_0 := \begin{cases} 
  e(u_0) & \text{in } \Omega, \\
  0 & \text{in } S,
\end{cases}
\]

for a fixed \( u_0 \in H^1(\Omega; \mathbb{R}^2) \).

Given \( m \geq 1 \), let \( \mathcal{A}_m \) be a collection of all subsets \( A \) of \( \overline{\Omega} \) such that \( \partial A \) has at most \( m \) connected components and let

\[
\mathcal{C}_m := \left\{ (A, u) \in \mathcal{C} : A \in \mathcal{A}_m \right\}
\]

to be the set of constrained admissible configurations.

**Remark 2.3.** The reason to introduce \( \mathcal{C}_m \) is that \( \mathcal{C}_m \) is both closed under \( \tau_C \)-convergence (see [35, Definition 2.5]) and \( \mathcal{F} \) is lower semicontinuous with respect to \( \tau_C \) in \( \mathcal{C}_m \) (see [35, Theorems 2.7 and 2.8]). Such two properties do not apply instead to \( \mathcal{C} \) as the following examples show.

We begin by recalling that a sequence \( \{(A_k, u_k)\} \subset \mathcal{C} \) is said to \( \tau_C \)-converge to \( (A, u) \subset \mathcal{C} \) and we denote by \( (A_k, u_k) \overset{\tau_C}{\rightharpoonup} (A, u) \), if

- \( \sup_{k \geq 1} \mathcal{H}^1(\partial A_k) < \infty \),
- \( \text{sdist}(\cdot, \partial A_k) \to \text{sdist}(\cdot, \partial A) \) locally uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \),
- \( u_k \to u \) a.e. in \( \text{Int}(A) \cup S \).

Let \( X := \{x_n\} \) be a countable dense set in \( \Omega \) and \( A \in \mathcal{A} \) such that \( |A| = v \in (0, |\Omega|] \). Then the sets \( A_k := A \setminus \{x_1, \dotsc, x_k\} \in \mathcal{A}, k \in \mathbb{N}, \) are such that \( |A_k| = v \in (0, |\Omega|) \), \( \mathcal{H}^1(\partial A_k) = \mathcal{H}^1(\partial A) \), and \( (A_k, 0) \overset{\tau_C}{\rightharpoonup} (A \setminus X, 0) \) as \( k \to \infty \), but \( A \setminus X \notin \mathcal{A} \) since \( \partial(A \setminus X) = \overline{A} \). Therefore, compactness with respect to \( \tau_C \) fails in \( \mathcal{C} \).
Furthermore, let $\Gamma \subset A$ be a segment such that $\mathcal{H}^1(\Gamma) > 0$, $B := A \setminus \Gamma$, $B_k := A \setminus (\Gamma \cap \{x_1, \ldots, x_k\})$ for every $k \in \mathbb{N}$, and assume that $X$ is dense in $\Gamma$. We notice that $\{(B_k, 0)\} \subset C$, $(B, 0) \in C$, $|B_k| = |B| = |A|$, $(B_k, 0) \xrightarrow{k} (B, 0)$ as $k \to \infty$. However, 
\[ \mathcal{F}(B_k, 0) = \mathcal{F}(A, 0) < \mathcal{F}(A \setminus \Gamma, 0) = \mathcal{F}(B, 0). \]
Therefore, lower semicontinuity of $\mathcal{F}$ with respect to $\tau_C$ fails in $C$.

2.2. **Main results.** We begin by stating the hypotheses which will be assumed throughout the paper:

(H1) $\varphi \in C(\overline{\Omega} \times \mathbb{R}^2)$ and is a Finsler norm, i.e., there exist $c_2 \geq c_1 > 0$ such that for every $x \in \overline{\Omega}$, $\varphi(x, \cdot)$ is a norm in $\mathbb{R}^2$ satisfying 
\[ c_1|\xi| \leq \varphi(x, \xi) \leq c_2|\xi| \quad (2.4) \]
for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^2$;

(H2) $\beta \in L^\infty(\Sigma)$ and satisfies 
\[ -\varphi(x, \nu_\Sigma(x)) \leq \beta(x) \leq \varphi(x, \nu_\Sigma(x)) \quad (2.5) \]
for $\mathcal{H}^1$-a.e. $x \in \Sigma$;

(H3) $C \in L^\infty(\Omega \cup S) \cap C^0(\Omega)$ and there exists $c_4 \geq c_3 > 0$ such that 
\[ 2c_3 M : M \leq C(x)M : M \leq 2c_4 M : M \quad (2.6) \]
for any $x \in \Omega \cup S$ and $M \in \mathbb{M}^{2x2}_{\text{sym}}$.

The first result of the paper is the **global existence** in $C$.

**Theorem 2.4 (Existence).** Assume (H1)-(H3). Let either $v \in (0, |\Omega|)$ or $S = \emptyset$. Then the minimum problem 
\[ \inf_{(A,u) \in C, \ |A|=v} \mathcal{F}(A,u) \quad (2.7) \]
admits a solution. Moreover, there exists $\lambda_0 > 0$ such that $(A,u) \in C$ is a solution to (2.7) if and only if it solves 
\[ \inf_{(A,u) \in C} \mathcal{F}^\lambda(A,u) \]
for any $\lambda \geq \lambda_0$, where 
\[ \mathcal{F}^\lambda(A,u) := \mathcal{F}(A,u) + \lambda|\lambda| - v|; \]

For simplicity we call the solutions of (2.7) the **global minimizers**.

The second result of the paper is a regularity result.

**Theorem 2.5 (Properties of global minimizers).** Assume (H1)-(H3). Let either $v \in (0, |\Omega|)$ or $S = \emptyset$, and let $(A,u) \in C$ be a solution of (2.7). Then:

(A) there exists an open set $G \subset A$ such that $\mathcal{H}^1(\partial A \Delta \partial G) = 0$ and $\mathcal{F}(G,u) = \mathcal{F}(A,u)$, hence $A$ can be supposed to be open;

(B) the closure of the set of all $x \in A^{(1)} \cap \partial A$ at which one-sided traces $u^+(x)$ and $u^-(x)$ exist and are equal, is $\mathcal{H}^1$-negligible, hence cracks are jump set for the deformation $u$;

(C) for any $x \in \Omega$ and $r \in (0, \min\{1, \text{dist}(x, \partial \Omega)\})$, 
\[ \frac{\mathcal{H}^1(U_r(x) \cap \partial A)}{r} \leq \frac{16c_2 + 4\lambda_0}{c_1}, \]
where $c_1, c_2$ are given by (2.4);
(D) there exist \( q_0 = q_0(c_3, c_4) \in (0, 1) \) and \( R_0 = R_0(c_1, c_2, c_3, c_4) > 0 \), where \( c_3, c_4 \) are given by (2.6), with the following property: if \( x \in \Omega \cap \partial A \) is such that \( \theta_*(\partial A, x) > 0 \), then

\[
\frac{\mathcal{H}^1(U_r(x) \cap \partial A)}{r} \geq q_0
\]

for any ball \( U_r(x) \subset \subset \Omega \) with \( r \in (0, R_0) \);

(E) if \( E \subset A \) is any connected component of \( A \) with \( \mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0 \), then \( |E| \geq (c_1 \sqrt{4\pi/\lambda_0})^2 \) and \( u = u_0 \) in \( E \), where \( \lambda_0 > 0 \) is given in Theorem 2.4.

In what follows we refer to the estimates in (C) and (D) as the (uniform) upper and lower density estimate, respectively.

### 2.3. Applications

We recall that the SDRI model introduced in [35] includes the settings of various free boundary problems, some of which are outlined below.

- **Epitaxially-strained thin films** [5, 19, 20, 26, 29]: \( \Omega := (a, b) \times (0, +\infty) \), \( S := (a, b) \times (-\infty, 0) \) for some \( a < b \), free crystals in the subfamily
  \[
  \mathcal{A}_{\text{subgraph}} := \{ A \subset \Omega : \exists h \in BV(\Sigma; [0, \infty)) \text{ and l.s.c. such that } A = A_h \subset A_1, \]
  where \( A_h := \{(x^1, x^2) : 0 < x^2 < h(x^1)\} \), and admissible configurations in the subspace
  \[
  \mathcal{C}_{\text{subgraph}} := \{(A, u) : A \in \mathcal{A}_{\text{subgraph}}, u \in H^1_{\text{loc}}(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset C_1
  \]
  (see also [4, 32]).

- **Crystal cavities** [27, 28, 41, 42]: \( \Omega \subset \mathbb{R}^2 \) smooth set containing the origin, \( S := \mathbb{R}^2 \setminus \Omega \), free crystals in the subfamily
  \[
  \mathcal{A}_{\text{starshaped}} := \{ A \subset \Omega : \text{open, starshaped w.r.t. } (0, 0), \text{ and } \partial \Omega \subset \partial A \} \subset A_1,
  \]
  and the space of admissible configurations
  \[
  \mathcal{C}_{\text{starshaped}} := \{(A, u) : A \in \mathcal{A}_{\text{starshaped}}, u \in H^1_{\text{loc}}(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset C_1.
  \]

- **Capillarity droplets**, e.g., [6, 21, 22]: \( \Omega \subset \mathbb{R}^2 \) is a bounded open set (or a cylinder), \( C = 0, S = \emptyset \), and admissible configurations in the collection
  \[
  \mathcal{C}_{\text{capillary}} := \{(A, 0) : A \in \mathcal{A}\} \subset C.
  \]

- **Griffith fracture model**, e.g., [7, 8, 12, 14]: \( S = \Sigma = \emptyset, E_0 \equiv 0 \), and the space of configurations
  \[
  \mathcal{C}_{\text{Griffith}} := \{(\Omega \setminus K, u) : K \text{ closed, } \mathcal{H}^1\text{-rectifiable, } u \in H^1_{\text{loc}}(\Omega \setminus K; \mathbb{R}^2)\} \subset C.
  \]

We notice that the same arguments employed to prove Theorem 2.4 seems to be adaptable to the case where a *fidelity term*, namely

\[
\kappa \int_{\Omega \setminus \partial A} |u - g|^p \, dx
\]

for \( p \in (1, \infty), \kappa > 0 \), and \( g \in L^\infty(\Omega; \mathbb{R}^2) \) if \( p \in (1, 2] \), and \( g \in W^{1,p}(\Omega; \mathbb{R}^2) \) if \( p \in (2, \infty) \), is added to the energy \( \mathcal{F} \), providing, in particular, a different proof (and an extension to the anisotropic case) of [14, Theorem 1.2].

- **Mumford-Shah model**, e.g., [1, 17, 38]: \( S = \Sigma = \emptyset, E_0 = 0, C \) is such that the elastic energy \( \mathcal{W} \) reduces to the Dirichlet energy, and the space of configurations
  \[
  \mathcal{C}_{\text{Mumford–Shah}} := \{(\Omega \setminus K, u) \in \mathcal{C}_{\text{Griffith}} : u = (u_1, 0)\} \subset C.
  \]
Boundary delaminations [3, 23, 34, 36, 37, 43]: the SDRI model includes also the setting of debonding and edge delamination in composites [43]. The focus is here on the 2-dimensional film and substrate vertical section, while in [3, 36, 37] a reduced model for the horizontal interface between the film and the substrate is derived.

3. DECAY ESTIMATES

By [35, Theorem 2.6] under the hypotheses (H1)-(H3) for any $m \in \mathbb{N}$ and $v \in (0,|\Omega|)$ both the volume-contained minimum problem

$$\inf_{(A,u) \in C_m, |A|=v} F(A,u),$$

and the unconstrained minimum problem

$$\inf_{(A,u) \in C_m} F^\lambda(A,u)$$

have a solution. Moreover, there exists a universal constant $\lambda_0 > 0$ such that for any $v \in (0,|\Omega|)$ and $\lambda \geq \lambda_0$,

$$\inf_{(A,u) \in C_m, |A|=v} F(A,u) = \inf_{(A,u) \in C} F^\lambda(A,u) = \lim_{m \to \infty} \inf_{(A,u) \in C_m, |A|=v} F(A,u). \quad (3.1)$$

The main result of this section is the following density estimates for minimizers of $F$ in $C_m$. Without loss of generality we assume throughout this section that $u_0 = 0$.

**Theorem 3.1 (Density estimates).** There exist $\varsigma = \varsigma(c_3, c_4) \in (0, 1)$ and $R_\ast = R_\ast(c_1, c_2, c_3, c_4) > 0$, where $c_i$ are given by (2.6), such that if $(A,u) \in C_m$ is a minimizer of $F$ in $C_m$, then for any $x \in \Omega$ and $r \in (0, \min\{1, \text{dist}(x, \partial\Omega)\})$,

$$\frac{\mathcal{H}^1(U_r(x) \cap \partial A)}{r} \leq \frac{16c_2 + 4\lambda_0}{c_1}. \quad (3.2)$$

Moreover, if $x \in \Omega \cap \partial A$ is such that $\theta_\ast(\partial A, x) > 0$, then

$$\frac{\mathcal{H}^1(U_r(x) \cap \partial A)}{r} \geq \varsigma \ast \quad (3.3)$$

for any ball $U_r(x) \subset \subset \Omega$ with $r \in (0, R_\ast)$.

To prove Theorem 3.1 we need the notion of quasi minimizers of $F$ in $\Omega$ and the local version $F(\cdot; O) : C_m \to \mathbb{R}$ of $F$ in open sets $O \subset \Omega$ defined as

$$F(A, u; O) := S(A; O) + W(A, u; O), \quad (3.4)$$

where $S(\cdot; O)$ and $W(\cdot; O)$ are the local versions of the surface and the elastic energy, respectively. Since $F(\cdot; O)$ does not “see” the substrate $S$, $S(\cdot; O)$ does not depend on $u$ and without loss of generality, we assume $u_0 = 0$ in the definition of $W(\cdot; O)$, i.e.,

$$S(A; O) := \int_{O \cap \partial A} \varphi(y, \nu_A) d\mathcal{H}^1 + 2 \int_{O \cap (A^{(1)} \cup A^{(0)}) \cap \partial A} \varphi(y, \nu_A) d\mathcal{H}^1$$

and

$$W(A, u; O) = \int_{O \cap A} C(y) e(u - u_0) : e(u - u_0) dy.$$

**Definition 3.2 ($\theta$-minimizers).** Given $\theta \geq 0$, the configuration $(A, u) \in C_m$ is a local $\theta$-minimizer of $F : C_m \to \mathbb{R}$ in $O$ if

$$F(A, u; O) \leq F(B, v; O) + \theta |A\Delta B| \quad (3.5)$$

whenever $(B, v) \in C_m$ with $A\Delta B \subset \subset O$ and supp $(u - v) \subset \subset O$. 

For any \((A,u) \in \mathcal{C}_m\) and any open set \(O \subset \Omega\) let
\[
\Phi(A,u;O) := \inf \left\{ \mathcal{F}(B,v;O) : (B,v) \in \mathcal{C}_m, \quad B\Delta A \subset O, \supp (u-v) \subset O \right\}
\] (3.6)
and let
\[
\Psi(A,u;O) := \mathcal{F}(A,u;O) - \Phi(A,u;O)
\] (3.7)
be the deviation of \((A,u)\) from minimality in \(O\).

The following proposition is a generalization to our setting of [8, Theorem 4] established for the Griffith model.

**Proposition 3.3.** Let \(U_R(x_0) \subset \Omega\) be a square of side length \(2R > 0\). Consider sequences of integers \(\{m_h\} \subset \mathbb{N}\), Finsler norms \(\varphi_h\) and ellipticity tensors \(\{C_h\}\) such that \(\{C_h\}\) is equi-continuous in \(\bar{U}_R(x_0)\) and there exist \(d_3, d_4 > 0\) with
\[
d_3 M : M \leq C_h(x)M : M \leq d_4 M : M \quad \text{for all} \ (x,M) \in \bar{U}_R(x_0) \times M_{\text{sym}}^{2x2},
\]
and define \(\mathcal{F}_h\) and \(\Psi_h\) in \(\mathcal{C}_{m_h}\) as in (3.4) and (3.7), respectively, with \(\varphi_h, C_h\) and \(m_h\) in places of \(\varphi\), \(C\) and \(m\). Let \(\{(A_h,u_h)\} \subset \mathcal{C}_{m_h}\) be such that
\[
\lim_{h \to \infty} \Psi_h(A_h,u_h;U_R(x_0)) = 0,
\]
\[
\lim_{h \to \infty} \mathcal{H}^1(U_R(x_0) \cap \partial A_h) = 0,
\]
\[
\sup_{h \geq 1} \mathcal{F}_h(A_h,u_h;U_R(x_0)) < \infty.
\]
Then there exist \(u \in H^1(U_R(x_0))\), an elasticity tensor \(C \in C^{0,1}(\bar{U}_R(x_0); M_{\text{sym}}^{2x2})\), sequences \(\{\xi_j\} \subset (0,1)^2\) of vectors and \(\{a_j\}\) of rigid displacements and subsequences \(\{A_{h_j}, u_{h_j}\}\), \(\{\varphi_{h_j}\}\) and \(\{C_{h_j}\}\) such that
(a) \(C_{h_j} \to C\) uniformly in \(U_R(x_0)\) and \(w_j := u_{h_j} \chi_{U_R(x_0) \cap \partial A_{h_j}} + \xi_j \chi_{U_R(x_0) \setminus A_{h_j}} - a_j \to u\) pointwise a.e. in \(U_R(x_0)\), and \(e(w_j) \to e(u)\) in \(L^2(U_R(x_0))\) as \(j \to \infty\);
(b) for all \(v \in u + H^1_0(U_R(x_0))\)
\[
\int_{U_R(x_0)} C(y)e(u) : e(v) \, dy \leq \int_{U_R(x_0)} C(y)e(v) : e(v) \, dy;
\]
(c) for any \(r \in (0,R]\)
\[
\lim_{j \to \infty} \mathcal{F}_h(A_{h_j}, u_{h_j}; U_r(x_0)) = \int_{U_r(x_0)} C(x)e(u) : e(u) \, dx.
\]

**Proof.** Without loss of generality, we suppose \(R = 1\). For shortness, given \(r > 0\), we write \(U_r\) in place of \(U_r(x_0)\). Since \(\sup_h \mathcal{H}^1(U_1 \cap \partial A_h) < \infty\), for every \(h \geq 1\) there exists \(\xi_h \in (0,1)^2\) such that
\[
\mathcal{H}^1\{y \in U_1 \cap \partial A_h : \text{trace of } u_h \text{ exists and equals to } \xi_h \text{ at } y\} = 0
\]
(see e.g. [39, Proposition 2.16]). Therefore
\[
\tilde{u}_h := \begin{cases} u_h & \text{in } U_1 \cap A_h, \\ \xi_h & \text{in } U_1 \setminus A_h \end{cases}
\]
(3.14)
belongs to \(GSD^2(U_1; \mathbb{R}^2)\) with \(J_{\tilde{u}_h} \subset U_1 \cap \partial A_h\) and
\[
\lim_{h \to \infty} \mathcal{H}^1(J_{\tilde{u}_h}) = 0
\]
(3.15)
in view of (3.10). Further we suppose \(\mathcal{H}^1(J_{\tilde{u}_h}) < 1/4\) for any \(h \geq 1\).
By [7, Proposition 2] and (3.8), there exist a constant $c$ (depending only on $d_3$) and sequences $\{\omega_h\}$ of a Lebesgue measurable subsets $U_1$ with $|\omega_h| \leq cH^1(B_1 \cap \partial A_h)$ and $\{a_h\}$ of rigid motions such that
\[
\int_{U_1 \setminus \omega_h} |\tilde{u}_h - a_h|^2 dx \leq c \int_{U_1} C_h(x)e(\tilde{u}_h) : e(\tilde{u}_h) dx.
\]
By (3.8) and (3.11), there exists $u \in L^2(U_1)$ such that up to a subsequence $(\tilde{u}_h - a_h)\chi_{U_1 \setminus \omega_h} \rightharpoonup u$ weakly in $L^2(U_1)$. By definition, $|\omega_h| \to 0$, hence we may also suppose that
\[
\tilde{u}_h - a_h \to u \quad \text{a.e. in } U_1.
\]
Furthermore from (3.8) and (3.11) we obtain
\[
\sup_{h \geq 1} \int_{U_1} |e(\tilde{u}_h - a_h)|^2 dx + H^1(J_{\tilde{u}_h}) < \infty,
\]
and hence, by [10, Theorem 1.1] there exist a subsequence still denoted by $\{\tilde{u}_h - a_h\}$ for which the set
\[
E := \{y \in U_1 : \limsup_{h \to \infty} |\tilde{u}_h(y)| \to \infty\}
\]
has finite perimeter and $\tilde{u} \in GSBD^2(U_1 \setminus E; \mathbb{R}^2)$ with $\tilde{u} = 0$ in $E$ such that
\[
\tilde{u}_h - a_h \to \tilde{u} \quad \text{a.e. in } U_1 \setminus E
\]
and
\[
\int_{U_1 \setminus \omega_h} |\tilde{u}_h - a_h|^2 dx + H^1(J_{\tilde{u}_h}) \leq \liminf_{h \to +\infty} H^1(J_{\tilde{u}_h}) = 0.
\]
In view of (3.16), $\tilde{u} = u$ a.e., and since $u \in L^2(U_1)$, the set $E$ has zero Lebesgue measure, i.e., all relations in (3.17) hold in $U_1$. In particular, since $H^1(J_u) = 0$, by the Poincaré-Korn inequality $u \in H^1(U_1; \mathbb{R}^2)$. In view of the fact that our elastic energy is invariant under rigid deformations, we suppose $a_h = 0$ for any $h \geq 1$.

Next we prove (3.12). Let $v \in H^1(U_1; \mathbb{R}^2)$ be such that $\text{supp}(u - v) \subset U_r$ for some $r \in (0, 1)$. Let $\psi \in C^1_c(U_r; [0, 1])$ be a cut-off function with $0 < \psi < 1$ \subset \{u = v\} \cap U_r, and \text{supp}(u - v) \subset \{\psi = 1\} \subset U_r$ for some $r'' < r' < r$. By (3.15) and [8, Theorem 3] there exist a positive constant $c > 0$ depending only on $d_3$ and $d_4$, $\tilde{v}_h \in GSBD^2(U_1), r_h \in (r - \delta_h, r)$ with $\delta_h := \sqrt[4]{H^1(J_{\tilde{u}_h})}$, and a Lebesgue measurable set $\tilde{\omega}_h \subset U_{r_h}$ such that
\[
(\text{a1}) \quad \tilde{v}_h \in C^\infty(U_{r - \delta_h}; \mathbb{R}^2), \quad \tilde{v}_h = \tilde{u}_h \quad \text{in } U_1 \setminus U_{r_h}, \quad \text{and}
\]
\[
\mathcal{H}^1(J_{\tilde{u}_h} \cap \partial U_{r_h}) = \mathcal{H}^1(\tilde{v}_h \cap \partial U_{r_h}) = 0;
\]
\[
(\text{a2}) \quad |\tilde{\omega}_h| \leq c\delta^2_h \mathcal{H}^1(U_{r_h} \cap J_{\tilde{u}_h}) \quad \text{and by (3.8),}
\]
\[
\int_{U_1 \setminus \omega_h} |\tilde{v}_h - \tilde{u}_h|^2 dx \leq c\delta^4_h \int_{U_r} C_h(x)e(\tilde{u}_h) : e(\tilde{u}_h) dx;
\]
\[
(\text{a3}) \quad \text{if } \eta \in \text{Lip}(U_1; [0, 1]), \text{ then}
\]
\[
\int_{U_r} \eta C_h(x)e(\tilde{v}_h) : e(\tilde{v}_h) dx \leq \int_{U_r} \eta C_h(x)e(\tilde{u}_h) : e(\tilde{u}_h) dx + c\delta^4_h(1 + \text{Lip}(\eta)) \int_{U_r} C_h(x)e(\tilde{u}_h) : e(\tilde{u}_h) dx
\]
for some universal constant $s \in (0, 1)$. 
By the choice of \( \tilde{v}_h \), we have \( \tilde{v}_h \in H^1(U_{\rho}; \mathbb{R}^2) \) and \( \text{supp}(\tilde{v}_h - \tilde{u}_h) \subset U_r \) for all sufficiently large \( h \). Define

\[
v_h := (1 - \psi)\tilde{v}_h + \psi v.\tag{3.20}\]

By (a1) and (3.14), \( \text{supp}(u_h - v_h) \cap A_h \subset U_1 \), and hence \( (A_h, v_h|_{A_h}) \) is an admissible configuration in (3.6). Therefore from (3.9) and the definition of deviation it follows that

\[
F_h(A_h, u_h; U_1) \leq F_h(A_h, v_h; U_1) + o(1), \tag{3.21}
\]

where \( o(1) \to 0 \) as \( h \to \infty \). Note that

\[
e(v_h)|_{A_h} = (1 - \psi)e(\tilde{v}_h)|_{A_h} + \psi e(v)|_{A_h} + \nabla \psi \odot (v - \tilde{v}_h)|_{A_h},
\]

where \( X \odot Y = (X \otimes Y + Y \otimes X)/2 \). By (3.15), (3.17) and (3.18), \( \tilde{v}_h \to u \) a.e. in \( U_1 \).

We claim that \( \tilde{v}_h \to u \) strongly in \( L^2_{\text{loc}}(U_r) \). Indeed, from (3.8), (3.11) and (3.19) as well as the Poincaré-Korn inequality we deduce that for any \( \rho \in (0, r) \) there exists \( h_\rho > 1 \) such that

\[
\sup_{h > h_\rho} \int_{U_\rho} |\nabla \tilde{v}_h|^2 dx < \infty.
\]

Since \( \tilde{v}_h \in H^1(U_{\rho}; \mathbb{R}^2) \), by the Poincaré inequality and Rellich-Kondrachov Theorem, there exist \( z \in H^1(U_{\rho}; \mathbb{R}^2) \) and not relabelled subsequence such that \( \tilde{v}_h - C_h \to z \) in \( L^2(U_{\rho}; \mathbb{R}^2) \), where

\[
C_h = |U_{\rho}|^{-1} \int_{U_{\rho}} \tilde{v}_h dx.
\]

Since \( \tilde{v}_h \to u \) a.e. in \( U_\rho \), and \( u \in H^1(U_1; \mathbb{R}^2) \), one has \( C_h \to C \) some constant \( C \in \mathbb{R} \). Therefore,

\[
\lim_{h \to \infty} \sup_{h > h_\rho} \|\tilde{v}_h - u\|_{L^2(U_\rho)} \leq \lim_{h \to \infty} \sup_{h > h_\rho} \|\tilde{v}_h - C_h - u + C\|_{L^2(U_{\rho})} + \lim_{h \to \infty} \sup_{h > h_\rho} \|C_h - C\|_{L^2(U_{\rho})} = 0,
\]

and the claim follows.

Since \( u = v \) out of \( \{ \psi = 1 \} \), the claim implies \( \tilde{v}_h \to v \) strongly in \( L^2(\{0 < \psi < 1\}) \), and hence,

\[
\lim_{h \to \infty} \int_{U_r} |\nabla \psi \odot (v - \tilde{v}_h)|_{A_h}^2 \leq \liminf_{h \to \infty} \int_{\{0 < \psi < 1\}} |\nabla \psi \odot (v - \tilde{v}_h)|^2 = 0. \tag{3.22}
\]

Thus, by the definition of \( v_h \) we get

\[
\int_{U_r \cap A_h} C_h e(v_h) : e(v_h) dx \\
= \int_{U_r \cap A_h} (1 - \psi)^2 C_h e(\tilde{v}_h) : e(\tilde{v}_h) dx + \int_{U_r \cap A_h} \psi^2 C_h e(v) : e(v) dx \\
+ \int_{U_r \cap A_h} C_h (\nabla \psi \odot (v - \tilde{v}_h)) : (\nabla \psi \odot (v - \tilde{v}_h)) dx \\
+ \int_{U_r \cap A_h} (1 - \psi) C_h e(\tilde{v}_h) : (\nabla \psi \odot (v - \tilde{v}_h)) dx \\
+ \int_{U_r} \psi C_h e(v) : (\nabla \psi \odot (v - \tilde{v}_h)) dx \\
= \int_{U_r \cap A_h} (1 - \psi)^2 C_h e(\tilde{v}_h) : e(\tilde{v}_h) dx + \int_{U_r \cap A_h} \psi^2 C_h e(v) : e(v) dx + o(1) \\
\leq \int_{U_r \cap A_h} (1 - \psi)^2 C_h e(u_h) : e(u_h) dx + \int_{U_r \cap A_h} \psi^2 C_h e(v) : e(v) dx + o(1), \tag{3.23}
\]
where in the second equality we use (3.11), (3.19) with $\eta \equiv 1$, (3.22), (3.8) and the Hölder inequality, while in the last inequality we use (3.19) with $\eta = (1 - \psi)^2$ and (3.14). Now (3.14), (3.21) and (3.23) imply
\[
\int_{U_r} (2\psi - \psi^2) C_h e(\tilde{u}_h) : e(\tilde{u}_h) dx \leq \int_{U_r} \psi^2 C_h e(v) : e(v) dx + o(1). \tag{3.24}
\]
Since $\{C_h\}$ is equibounded (see (3.8)) and equicontinuous, by the Arzela-Ascoli Theorem, there exist a (not relabelled) subsequence and an elasticity tensor $C \in C^0(U_1; M_{2\times 2}^{\text{sym}})$ such that $C_h \rightarrow C$ uniformly in $U_1$. Hence, letting $h \rightarrow \infty$ in (3.24) and using the convexity of the elastic energy and (3.17), we obtain
\[
\int_{U_r} (2\psi - \psi^2) C(y)e(u) : e(u) dy \leq \int_{U_r} \psi^2 C(y)e(v) : e(v) dy. \tag{3.25}
\]
By the choice of $\psi$, (3.25) implies
\[
\int_{U_{r'}} C(y)e(u) : e(u) dy \leq \int_{U_r} C(y)e(v) : e(v) dy. \tag{3.26}
\]
Since $r''$ is arbitrary, letting $r'' \nearrow r$ we deduce that (3.26) holds also with $r'' = r$. Since $\text{supp}(u - v) \subset \subset U_r$, this implies (3.12).

It remains to prove (3.13). If we take $v = u$ in (3.24) and use $0 \leq \psi \leq 1$ and $\psi = 1$ in $U_{r''}$ we get
\[
\int_{U_{r''}} C e(u) : e(u) dx \leq \liminf_{h \rightarrow \infty} \int_{U_{r''}} C_h e(\tilde{u}_h) : e(\tilde{u}_h) dx
\]
\[
\leq \limsup_{h \rightarrow \infty} \int_{U_{r''}} C_h e(\tilde{u}_h) : e(\tilde{u}_h) dx \leq \int_{U_r} C e(u) : e(u) dx.
\]
Since $r''$ is arbitrary, letting $r'' \nearrow r$ we deduce
\[
\lim_{h \rightarrow \infty} \int_{U_r} C_h e(\tilde{u}_h) : e(\tilde{u}_h) dx = \int_{U_r} C e(u) : e(u) dx.
\]

Now we prove that
\[
\lim_{h \rightarrow \infty} S_h(A_h; U_r) = 0 \tag{3.27}
\]
for any $r \in (0, 1)$.

Indeed, by (3.10) and the relative isoperimetric inequality, passing to further subsequence we may suppose that either
\[
\lim_{h \rightarrow \infty} |U_1 \cap A_h| = 0 \tag{3.28}
\]
or
\[
\lim_{h \rightarrow \infty} |U_1 \setminus A_h| = 0. \tag{3.29}
\]

If (3.28) holds, by (3.10), we can can find $h_0 > 0$ such that $\mathcal{H}^1(U_1 \cap \partial A_h) < \frac{1 - r}{5}$ for any $h > h_0$, and hence there is no connected component of $\partial A_h$ intersecting both $\partial U_r$ and $\partial U_1$. Let $E_h$ denote the union of connected components of $A_h$ not intersecting $\partial U_1$. Note that $(A_h \setminus E_h, u_h|_{A_h \setminus E_h})$ is an admissible configuration in (3.6), thus,
\[
\mathcal{F}_h(A_h, u_h; U_1) \leq \Phi_h(A_h, u_h; U_1) + o(1) \leq \mathcal{F}_h(A_h \setminus E_h, u_h; U_1) + o(1), \tag{3.30}
\]
where in the first inequality we use (3.9) and in the second we use the definition of $\Phi_h$. From the definition of $E_h$ and (3.30) it follows that
\[
S_h(E_h; U_1) \leq o(1). \tag{3.31}
\]
Since $S_h(A_h; U_r) \leq S_h(E_h; U_1)$, (3.31) implies (3.27).
Now suppose that (3.29) holds. Let \( \delta_h := \sqrt[4]{\mathcal{H}^1(F_{\tilde{u}_h})} \) and \( \psi \), and \( r'' < r' < r \) and \( v_h \) be as in (3.20) with \( v = u \). Fix any \( \rho \in (0, r) \). By (3.10), we can find \( h_{r, \rho} > 0 \) such that \( \mathcal{H}^1(U_1 \cap \partial A_h) < \min \{ \frac{1}{r^2}, \frac{1}{r'^2} \} \) for any \( h > h_{r, \rho} \). Since \( \partial A_h \in A_{mn} \), no connected component of \( \partial A_h \) intersect both \( \partial U_r \) and \( \partial U_\rho \). Let \( E_h \) be the union of all connected components \( F \) of \( U_1 \setminus A_h \) (if exists) such that \( \partial F \subset \partial A_h \) and lying strictly inside \( U_r \) (so \( E_h \) is the finite union of “holes”). Setting \( A_h' := A_h \cup \overline{E_h} \), we observe that \( (A_h', v_h|_{A_h'}) \) is an admissible configuration for \( \Phi_h(A_h, u_h; U_1) \) in (3.6).

Thus from (3.9) and (3.29),
\[
\mathcal{F}_h(A_h, u_h; U_1) \leq \mathcal{F}_h(A_h', v_h; U_1) + o(1).
\]

By the definition of \( \mathcal{F}_h \), as in the proof of (3.24) we establish
\[
\mathcal{S}_h(E_h; U_1) + \int_{U_r} (2\psi - \psi^2)\mathcal{C}_h e(\tilde{u}_h) : e(\tilde{u}_h) dx \\
\leq \int_{U_r} \psi^2 \mathcal{C}_h e(u) : e(u) dx + o(1).
\]

Thus, letting \( h \to \infty \) we get
\[
\limsup_{h \to \infty} \mathcal{S}_h(E_h; U_1) + \liminf_{h \to \infty} \int_{U_r} (2\psi - \psi^2)\mathcal{C}_h e(\tilde{u}_h) : e(\tilde{u}_h) dx \\
\leq \limsup_{h \to \infty} \int_{U_r} \psi^2 \mathcal{C}_h e(u) : e(u) dx.
\]  

(3.32)

Since \( \mathcal{S}_h(E_h; U_1) \geq \mathcal{S}_h(A_h; U_\rho) \), \( \psi = 1 \) in \( U_{r''} \), and \( |\psi| \leq 1 \), from (3.32) it follows that
\[
\limsup_{h \to \infty} \mathcal{S}_h(A_h; U_\rho) + \int_{U_{r''}} \mathcal{C}e(u) : e(u) dx \leq \int_{U_r} \mathcal{C}e(u) : e(u) dx.
\]

Now letting \( r'' \to r \) we get
\[
\limsup_{h \to \infty} \mathcal{S}_h(A_h; U_\rho) = 0.
\]  

(3.33)

Since the function \( B \to \mathcal{S}_h(A_h; B) \) defined for Borel sets \( B \subset U_1 \) extends to a bounded nonnegative Radon measure \( \mu_h \) in \( U_1 \). Since (3.33) holds for any \( \rho \in (0, r) \), \( \mu_h \) converges to 0 in the weak* sense, and thus (3.27) follows. \( \Box \)

Recall that by [13, Proposition 3.4] if the elasticity tensor \( \mathcal{C} \) is constant, then for any \( \gamma \in (0, 2) \) there exists \( c_\gamma := c_\gamma(c_3, c_4) > 0 \) such that for every local minimizer \( (\Omega, u) \in \mathcal{C} \) of \( \mathcal{F}(\cdot; O) \), \( u \) is analytic in \( O \) and for any square \( U_R(x) \subset O \) and \( r \in (0, R) \),
\[
\int_{U_R(x)} \mathcal{C}e(u) : e(u) dx \leq c_\gamma \left( \frac{r}{R} \right)^{2-\gamma} \int_{U_R(x)} \mathcal{C}e(u) : e(u) dx.
\]  

(3.34)

Given \( \gamma \in (0, 1) \) let
\[
\tau_0 = \tau_0(\gamma, c_3, c_4) := \min \{ 1, \frac{1}{2} c_\gamma \left( \frac{1}{\sqrt{2}} \right)^{\frac{1}{3-2\gamma}} \},
\]
where \( c_\gamma \) is the constant appearing in (3.34). Using Proposition 3.3 and repeating similar arguments of [9, 14] we get the following decay property of the functional \( \mathcal{F} \).

**Proposition 3.4.** Assume (H1)-(H3). For any \( \tau \in (0, \tau_0) \) there exist \( \varsigma = \varsigma(\tau) \in (0, 1) \), \( \vartheta := \vartheta(\tau) > 0 \) and \( \hat{\vartheta} := \hat{\vartheta}(\tau) > 0 \) such that if \( (A, u) \in \mathcal{C}_m \) satisfies
\[
\mathcal{H}^1(U_\rho(x) \cap \partial A) < 2\varsigma \rho \quad \text{and} \quad \mathcal{F}(A, u; U_\rho(x)) \leq (1 + \vartheta)\Phi(A, u; U_\rho(x))
\]
for some \( m \geq 1 \) and \( U_\rho(x) \subset \subset \Omega \) with \( 0 < \rho < R \), then
\[
\mathcal{F}(A, u; U_\tau(x)) \leq \tau^{2-\gamma} \mathcal{F}(A, u; U_\rho(x)).
\]
**Proof.** Assume by contradiction that there exist \( \tau \in (0, \tau_0) \), positive real numbers \( c_\tau, \theta, \rho \to 0 \), natural numbers \( m_h \in \mathbb{N} \), squares \( U_{\rho_h}(x_h) \subset \subset \Omega \), and admissible configurations \((A_h, u_h) \in \mathcal{C}_{m_h}\) such that

\[
\mathcal{H}^1(U_{\rho_h}(x_h) \cap \partial A_h) \leq 2c_\tau \rho_h \tag{3.35}
\]
\[
\mathcal{F}(A_h, u_h; U_{\rho_h}(x_h)) \leq (1 + \theta) \Phi(A_h, u_h; U_{\rho_h}(x_h)), \tag{3.36}
\]

but

\[
\mathcal{F}(A_h, u_h; U_{\rho_h}(x_h)) > \tau^{2-\gamma} \mathcal{F}(A_h, u_h; U_{\rho_h}(x_h)) \tag{3.37}
\]

for any \( h \). Note that \( \mathcal{F}(A_h, u_h; U_{\rho_h}(x_h)) > 0 \). Let us define the rescaled energy \( \mathcal{F}_h(\cdot; U_1) : \mathcal{C}_{m_h} \to \mathbb{R} \) as in (3.4) with

\[
\varphi_h(y, \nu) := \frac{\rho_h \varphi(x_h + \rho_h y, \nu)}{\mathcal{F}(A_h, u_h; U_{\rho_h}(x_h))}
\]

in place of \( \varphi(y, \nu) \) and

\[
\mathcal{C}_h(y) := \mathcal{C}(x_h + \rho_h y)
\]

in place of \( \mathcal{C}(y) \), for \( y \in U_1 \). We notice that

\[
\mathcal{F}_h(E_h, v_h; U_1) = 1 \tag{3.38}
\]

for

\[
E_h := \sigma_{x_h, \rho_h}(A_h)
\]

(see definition of blow-up map \( \sigma_{x, r} \) at (2.2)) and

\[
v_h(y) := \frac{\rho_h u_h(x_h + \rho_h y)}{\sqrt{\mathcal{F}(A_h, u_h; U_{\rho_h}(x_h))}}.
\]

By (3.35) we obtain

\[
\mathcal{H}^1(U_1 \cap \partial E_h) < 2c_\tau
\]

while (3.36) and (3.38) entails

\[
\Psi_h(E_h, v_h; U_1) \leq \theta h \Phi_h(E_h, v_h; U_1) \leq \theta \mathcal{F}_h(E_h, v_h; U_1) = \theta h,
\]

where \( \Phi_h \) and \( \Psi_h \) are defined as in (3.6) and (3.7) (again with \( \varphi_h \) and \( \mathcal{C}_h \) in places of \( \varphi \) and \( \mathcal{C} \), respectively). By (2.6) \( \{\mathcal{C}_h\} \) is equibounded. Since \( \Omega \) is bounded, there exists \( x_0 \in \Omega \) such that, up to extracting a subsequence, \( x_h \to x_0 \) as \( h \to +\infty \). As \( \rho_h \to 0 \), one has \( x_h + \rho_h y \to x_0 \) for every \( y \in U_1 \). Thus \( \{\mathcal{C}_h\} \) is also equicontinuous and \( \mathcal{C}_h \to \mathcal{C}_0 := \mathcal{C}(x_0) \) uniformly in \( U_1 \). In view of (3.35), (3.36) and (3.38), we can apply Proposition 3.3 to find \( u \in H^1(U_1; \mathbb{R}^2) \), vectors \( \xi_h \in (0, 1)^2 \), and infinitesimal rigid displacements \( a_h \) such that, up to a subsequence,

\[
 w_h := v_h \chi_{U_1 \cap E_h} + \xi_h \chi_{U_1 \setminus E_h} - a_h \to v
\]

pointwise a.e. in \( U_1 \), \( e(w_h) \to e(v) \) in \( L^2(U_1) \) as \( h \to +\infty \), and

\[
\lim_{h \to +\infty} \mathcal{F}_h(E_h, v_h; U_r) = \lim_{h \to +\infty} \mathcal{F}(E_h, w_h; U_r) = \int_{U_r} \mathcal{C}_0(x)e(v) : e(v)dx \tag{3.39}
\]

for any \( r \in (0, 1] \). In particular, from (3.39) and (3.37) it follows that

\[
\int_{U_r} \mathcal{C}_0(x)e(v) : e(v)dx = \lim_{h \to +\infty} \mathcal{F}(E_h, v_h; U_r)
\]

\[
\geq \lim_{h \to +\infty} \tau^{2-\gamma} \mathcal{F}(E_h, v_h; U_1) = \tau^{2-\gamma} \int_{U_1} \mathcal{C}_0(x)e(v) : e(v)dx.
\]

Since \( \mathcal{C}_0 \) is constant, applying (3.34) with \( r := \tau \) and \( R := 1 \) we get

\[
c_{\tau} \tau^{2-\gamma} \int_{U_1} \mathcal{C}_0(x)e(v) : e(v)dx \geq \int_{U_r} \mathcal{C}_0(x)e(v) : e(v)dx,
\]
for any ball $U_r$ where in the first inequality we used the isoperimetric inequality, in the second

$$\text{Note that for any } (A, u) \in C_m \text{ is a } \theta \text{-minimizer of } \mathcal{F} \text{ in } U_{r_0}(x_0) \text{ for some } r_0 > 0, \text{ then for any } x \in U_{r_0}(x_0) \cap \partial A \text{ with } \theta_*(\partial A, x) > 0 \text{ one has}
$$

$$\mathcal{F}(A, u; U_\rho(x)) \geq 2c_1 \varsigma \rho \quad (3.40)$$

for any ball $U_\rho(x) \subset U_{r_0}(x_0)$ with $\rho \in (0, R_0)$, where

$$R_0 := R_0(\theta, \tau, c_1) := \min \left\{ R(\tau), \frac{\sqrt{\pi c_1} \theta}{\theta (2 + \theta)} \right\}.$$ 

Proof. Note that for any $(C, w), (D, v) \in C_m$ and $O \subset \Omega$ with $C \Delta D \subset \subset O$

$$\sqrt{4\pi |C \Delta D|^{1/2}} \leq H^1(\partial^*(C \Delta D)) \leq H^1(O \cap \partial^* C) + H^1(O \cap \partial^* D)$$

$$\leq \frac{S(C, O) + S(D, O)}{c_1} \leq \frac{\mathcal{F}(C, w; O) + \mathcal{F}(D, v; O)}{c_1}, \quad (3.41)$$

where in the first inequality we used the isoperimetric inequality, in the second $\partial^*(C \Delta D) \subset O \cap (\partial^* C \cup \partial^* D)$, in the third (2.4) and the definition of $S(\cdot; O)$ and in the last the nonnegativity of $W(\cdot; O)$. Thus, from the $\theta$-minimality of $(A, u)$ in $U_{r_0}(x_0)$ we deduce that

$$\mathcal{F}(A, u; U_r(x)) \leq \mathcal{F}(B, v; U_r(x)) + \theta |A \Delta B|^{1/2} |A \Delta B|^{1/2}$$

$$\leq \mathcal{F}(B, v; U_r(x)) + \frac{\theta r}{\sqrt{\pi c_1}} \left( \mathcal{F}(A, u; U_r(x)) + \mathcal{F}(B, v; U_r(x)) \right) \quad (3.42)$$

for any $U_\rho(x) \subset U_{r_0}(x_0)$ and $(B, v) \in C_m$ with $A \Delta B \subset \subset U_r(x)$ and supp $(u - v) \subset \subset U_r(x)$, where in the last inequality we used (3.41) and the inequality $|A \Delta B| \leq |U_r| = 4r^2$. Let $r > 0$ be small enough so that $\frac{\theta r}{\sqrt{\pi c_1}} \leq \frac{\theta}{2 + \theta}$, where $\theta := \theta(\tau) \in (0, 1)$ is given by Proposition 3.4. From (3.42) we obtain

$$\mathcal{F}(A, u; U_r(x)) \leq (1 + \theta) \Phi(B, v; U_r(x)),$$

which by the arbitrariness of $(B, v)$ is equivalent to

$$\mathcal{F}(A, u; U_r(x)) \leq (1 + \theta) \Phi(A, u; U_r(x)). \quad (3.43)$$

We are ready to prove (3.40). Let

$$J_A^* := \{ y \in U_{r_0}(x_0) \cap \partial A : \theta^*(\partial A, y) = \theta_*(\partial A, y) = 1 \}.$$

and $x \in J_A^*$. For simplicity we suppose that $x = 0$. Assume by contradiction that

$$\mathcal{F}(A, u; U_\rho) < 2c_1 \varsigma \rho$$

for some $U_\rho \subset \subset U_{r_0}(x_0)$ with $\rho \in (0, R_0)$. Then by the nonnegativity of the elastic energy and (2.4),

$$2c_1 \varsigma \rho > \mathcal{F}(A, u; U_\rho) \geq \int_{U_\rho \cap \partial A} \varphi(x, \nu_A) dH^1 \geq c_1 H^1(U_\rho \cap \partial A)$$

so that

$$H^1(U_\rho \cap \partial A) < 2 \varsigma \rho. \quad (3.44)$$
By (3.44) and (3.43) we can apply Proposition 3.4 and obtain that
\[ \mathcal{F}(A, u; U_{\tau}) \leq \tau^{2-\gamma} \mathcal{F}(A, u; U_{\rho}) \leq 2c_1 \tau^{2-\gamma} \rho < 2c_1 \tau \rho. \]
since \( \gamma, \tau \in (0, 1) \), so that
\[ \mathcal{H}^1(U_{\tau_{\rho}} \cap \partial A) < 2\zeta \tau \rho. \]
Thus, by induction,
\[ \mathcal{H}^1(U_{\tau_{\rho}} \cap \partial A) < 2\zeta \tau^n \rho. \]
However, by the choice of \( x \)
\[ 1 = \lim_{n \to +\infty} \frac{\mathcal{H}^1(U_{\tau_{\rho}} \cap \partial A)}{2\tau^n \rho} \leq \frac{2c_1}{2c_1} = \zeta < 1, \]
a contradiction. This contradiction implies (3.40) for \( x \in J^*_A \). Note that the map
\( \mathcal{F}(A, u; \cdot) \), defined for open sets \( O \subset \subset U_{\tau_{\rho}}(x_0) \) extends to a positive Borel measure in \( U_{\tau_{\rho}}(x_0) \), hence, (3.40) is valid also for all \( x \in U_{\tau_{\rho}}(x_0) \cap \mathcal{J}_A^\tau \). Since \( \partial A \) has at most \( m \) connected components, \( \theta_*(\partial A, x) = 0 \) if and only if \( x \in \partial A \) is isolated point. Since \( (A, u) \) is a minimizer, points \( x \) with \( \theta_*(\partial A, x) = 0 \) are removable singularities for \( u \) so that \( u \) is a \( H^1 \)-function in a small neighborhood of \( x \), and hence,
\[ U_{\tau_{\rho}} \cap \mathcal{J}_A = \{ x \in U_{\tau_{\rho}}(x_0) \cap \partial A : \theta_*(\partial A, x) > 0 \}. \]

Now we are ready to prove (3.2) and (3.3).

**Proof of Theorem 3.1.** We begin by recalling that there exists \( \lambda_0 > 0 \) such that (3.1) holds. Since \( (A, u) \) is a minimizer of \( \mathcal{F}^{\lambda_0} \) in \( C_m \), for any open set \( O \subset \subset O \) and \( (B, v) \in C_m \) with \( A \Delta B \subset \subset O \) and \( \text{supp} (u - v) \subset \subset O \) we have
\[ \mathcal{F}(A, u; O) \leq \mathcal{F}(B, v; O) + \lambda_0 |A| - |B| \leq \mathcal{F}(B, v; O) + \lambda_0 |A \Delta B|. \]
Hence, \( (A, u) \) is \( \lambda_0 \)-minimizer of \( \mathcal{F}(\cdot; \Omega) \) in \( \Omega \).

We begin by establishing (3.2). Let \( x \in \Omega \), \( r \in (0, \min\{1, \text{dist}(x, \partial \Omega)\}) \), and \( U_r := U_r(x) \). Since \( E := (A \setminus \overline{U}_r) \cup \partial U_r \in \mathcal{A}_m \), by the \( \lambda_0 \)-minimality of \( (A, u) \) we obtain
\[ \mathcal{F}(A, u; U_r) \leq \mathcal{F}(E, u; U_r) + \lambda_0 |U_r|, \]
so that
\[ \int_{U_r \cap \partial A} \varphi(x, \nu_A) dH^1 \leq 2 \int_{\partial U_r} \varphi(x, \nu_{U_r}) dH^1 + 4 \lambda_0 r^2, \]
in view of the nonnegativity of \( W(A, u; U_r) \), which by (2.4) entails (3.2). In particular, since \( E \Delta A \subset \subset U_r \) for every \( \rho \in (r, \text{dist}(x, \partial \Omega)) \), we also have
\[ \mathcal{F}(A, u; U_{\rho}) \leq \mathcal{F}(E, u; U_{\rho}) + \lambda_0 |U_r| = \mathcal{F}(E, u; U_{\rho} \setminus \overline{U}_r) + S(E, u; U_r) + 4 \lambda_0 r^2 \]
\[ \leq \mathcal{F}(E, u; U_{\rho} \setminus \overline{U}_r) + 2 \int_{\partial U_r} \varphi(x, \nu_{U_r}) dH^1 + 4 \lambda_0 r^2 \]
and hence, letting \( \rho \searrow r \) and using \( r \leq 1 \) we get
\[ \mathcal{F}(A, u; U_{\overline{r}}) \leq (16c_2 + 4 \lambda_0) r. \]

Now we prove (3.3). Let \( \tau_r := \frac{\tau}{2} \), let \( \varsigma_0 = \varsigma(\tau_0) \in (0, 1) \) and \( R_0 = R_0(\tau_0, \phi, \lambda_0) > 0 \) be as in Proposition 3.5. Then by (3.40),
\[ \mathcal{F}(A, u; U_{\tau_r}) \geq 2c_1 \varsigma_0 \tau r \]
for \( \tau \in (0, 1] \) and for any square \( U_r \subset \Omega \) with \( r \in (0, R_0) \). We consider \( \varsigma_* := \varsigma(\tau_*), \theta_* := \theta(\tau^*), \) and \( R_* := \min\{R(\tau_*), R_0\} \) as given by Proposition 3.4 for \( \tau_* := \)}
\[
\min \left\{ \frac{\varepsilon}{2}, \left( \frac{-c_1 \gamma \omega}{16c_2 + 4\lambda_0} \right)^\frac{1}{1 - \gamma} \right\}.
\]

By contradiction, if \( \mathcal{H}^1(U_r \cap \partial A) < \varsigma_* r \), then by applying (3.43) with \( \tau = \tau_* \) we obtain
\[
\mathcal{F}(A, u; U_r) \leq (1 + \vartheta_*) \Phi(A, u; U_r).
\]

Hence, by Proposition 3.4,
\[
\mathcal{F}(A, u; U_{\tau_* r}) \leq \tau_*^{2-\gamma} \mathcal{F}(A, u; U_r)
\]
so that by (3.46) and (3.45)
\[
\tau_*^{1-\gamma} \geq \frac{2c_1 \gamma \omega}{16c_2 + 4\lambda_0},
\]
which is a contradiction. \( \square \)

4. PROOF OF THEOREM 2.4

We postpone the proof after several propositions.

Proposition 4.1 (Compactness of free crystals). Assume (H1)-(H3). For any \( m \in \mathbb{N} \) let \( (A_m, u_m) \in C_m \) be a minimizer of \( \mathcal{F} \) in \( C_m \) such that \( \partial A_m \) does not contain isolated points. Then there exist \( A \in A \) and a sequence \( \{A_{m_n}\} \) such that \( \text{sdist}(\cdot, \partial A_{m_n}) \to \text{sdist}(\cdot, \partial A) \) locally uniformly in \( \mathbb{R}^2 \) as \( h \to \infty \). Moreover, for any \( x \in \Omega \cap \partial A \) and \( r \in (0, \min\{R_*, \text{dist}(x, \partial \Omega)\}) \)
\[
\varsigma_* \leq \frac{\mathcal{H}^1(U_r(x)) \cap \partial A}{r} \leq 2\pi c_2,
\]
where \( \varsigma_* := \varsigma_*(c_3, c_4, \text{Lip}(\mathbb{C})) \in (0, 1) \) and \( R_* := R_*(c_1, c_2, c_3, c_4, \text{Lip}(\mathbb{C})) > 0 \) are given in Theorem 3.1.

Proof. By [35, Proposition 3.1], there exist \( A \subset \Omega \) and a subsequence \( \{(A_{m_n}, u_{m_n})\} \) such that \( \text{sdist}(\cdot, \partial A_{m_n}) \to \text{sdist}(\cdot, \partial A) \) as \( h \to \infty \). Consider the sequence \( \mu_h := H^1 \mathcal{L} \partial A_{m_n} \) of positive Radon measures. By Theorem 3.1

\[
\frac{\varsigma_*}{2} \leq \frac{\mu_h(U_r(x))}{2r} \leq \frac{2\pi c_2}{c_1}
\]
for any \( x \in \Omega \cap \partial A_{m_n} \) and \( U_r(x) \subset \subset \Omega \) with \( r \in (0, R_*) \), where \( c_1, c_2 \) are given by (2.4) and \( c_3, c_4 \) is given by (2.6). By (2.4), (2.5) and (3.1),
\[
\mu_h(\mathbb{R}^2) = \mathcal{H}^1(\partial A_{m_n}) \leq \mathcal{H}^1(\partial \Omega) + \frac{\mathcal{F}(A_{m_n}, u_{m_n}) + 2c_2 \mathcal{H}^1(\Sigma)}{c_1}
\]
\[
\leq \mathcal{H}^1(\partial \Omega) + \frac{\mathcal{F}(A_1, u_1) + 2c_2 \mathcal{H}^1(\Sigma)}{c_1},
\]
hence, by compactness, there exist a not relabelled subsequence and a positive Radon measure \( \mu \) in \( \mathbb{R}^2 \) such that \( \mu_h \rightharpoonup^* \mu \) as \( h \to \infty \). We claim that
\[
\Omega \cap \partial A \subseteq \text{supp} \mu \subseteq \partial A.
\]
Indeed, let \( x \in \Omega \cap \partial A \) and \( r \in (0, \min\{\text{dist}(x, \partial \Omega), R_*\}) \). By the sdist-convergence, there exists \( x_h \in U_r(x) \cap \partial A_{m_n} \) with \( x_h \to x \) such that \( U_{r/2}(x_h) \subset B_r(x) \) and hence, by the weak* convergence and (4.1),
\[
\mu(U_r(x)) \geq \limsup_{h \to \infty} \mu_h(U_r(x)) \geq \limsup_{h \to \infty} \mu_h(U_{r/2}(x_h)) \geq \varsigma_* r.
\]
This implies \( x \in \text{supp}\mu \). Conversely, if, by contradiction, there exists \( x \in \text{supp}\mu \setminus \partial A \), then we can find \( r > 0 \) such that \( U_r(x) \cap \partial A = \emptyset \). From the sdist-convergence it follows that \( U_{r/2}(x) \cap \partial A_m = \emptyset \) for \( h \) large enough, and hence,

\[
0 < \mu(U_{r/2}(x)) \leq \lim_{h \to \infty} \mu_h(U_{r/2}(x)) = 0,
\]

which is a contradiction.

From (4.1) it follows that

\[
\frac{\varsigma_s}{2} \leq \mu(U_r(x)) \leq \frac{2\pi c_2}{c_1} \tag{4.3}
\]

for any \( x \in \Omega \cap \text{supp}\mu \) any \( r \in (0, R_\star) \) with \( U_r(x) \subset \subset \Omega \). Indeed, let \( x \in \Omega \cap \text{supp}\mu \) and \( r \in (0, \min\{R_\star, \text{dist}(x, \partial\Omega)\}) \). Then by the sdist-convergence for any \( \delta \in (0, \frac{R_\star-r}{2}) \) there exists \( x_h \in U_r(x) \cap \partial A_{m_h} \) with \( x_h \to x \) such that \( U_r(x) \subset U_{r+\delta}(x_h) \) so that by weak* convergence and (4.1),

\[
\mu(U_r(x)) \leq \lim_{h \to \infty} \mu_h(U_r(x)) \leq \limsup_{h \to \infty} \mu_h(U_{r+\delta}(x_h)) \leq \frac{4\pi c_2}{c_1} (r + \delta).
\]

Now letting \( \delta \to 0 \) implies the upper density estimate in (4.3). On the other hand, for any \( \delta \in (0, \frac{r}{2}) \) there exists \( x_h \in U_r(x) \cap \partial A_{m_h} \) with \( x_h \to x \) such that \( U_{r-\delta}(x_h) \subset U_r(x) \) so that by weak* convergence and (4.1),

\[
\mu(U_r(x)) \geq \liminf_{h \to \infty} \mu_h(U_r(x)) \geq \liminf_{h \to \infty} \mu_h(U_{r-\delta}(x_h)) \geq \varsigma_s (r - \delta).
\]

Hence, letting \( \delta \to 0 \) yields \( \mu(U_r(x)) \geq \varsigma_s r \). Since \( \mu \) is a finite Radon measure, this implies the lower density estimate in (4.3).

From (4.3) and [1, Theorem 2.56] it follows that

\[
\varsigma_s \mathcal{H}^1(\Omega \cap \text{supp}\mu) \leq \mu \mathcal{L} \Omega \leq \frac{4\pi c_2}{c_1} \mathcal{H}^1(\Omega \cap \text{supp}\mu). \tag{4.4}
\]

Thus, \( \mu \) is absolutely continuous with respect to \( \mathcal{H}^1(\Omega \cap \text{supp}\mu) \), \( \text{supp}\mu \) is \( \mathcal{H}^1 \)-rectifiable, and \( \mathcal{H}^1(\text{supp}\mu) < \infty \). By (4.4),

\[
\mathcal{H}^1(\partial A) \leq \mathcal{H}^1(\Omega \cap \partial A) + \mathcal{H}^1(\partial \Omega \cap \partial A) \leq \frac{1}{\varsigma_s} \mu(\Omega) + \mathcal{H}^1(\partial \Omega) < \infty.
\]

Since \( \partial \Omega \) is Lipschitz, (4.2) implies that \( \partial A \) is \( \mathcal{H}^1 \)-rectifiable. Therefore, \( A \in \mathcal{A} \). \( \square \)

**Lemma 4.2.** Let \( \{A_{m_h}\} \) and \( A \) be as in Proposition 4.1. Then \( A_{m_h} \to A \) in \( L^1(\mathbb{R}^2) \) as \( h \to \infty \).

**Proof.** Since \( \mathcal{H}^1(\partial A) < \infty \) and \( A_{m_h} \xrightarrow{K} A \) as \( h \to \infty \), one has \( \chi_{A_{m_h}}(x) \to \chi_A(x) \) as \( h \to \infty \) for a.e. \( x \in \mathbb{R}^2 \). Now Lemma 4.2 follows from the Dominated Convergence Theorem. \( \square \)

The following result generalizes [30, Theorem 4.2] since it applies to set \( \Gamma \) a priori not connected but satisfying uniform density estimates.

**Proposition 4.3.** Let \( \Gamma \subset \mathbb{R}^2 \) be a \( \mathcal{H}^1 \)-rectifiable closed set such that for some \( r_0, c, C > 0 \) and for all \( x \in \Gamma \)

\[
c \leq \frac{\mathcal{H}^1(U_r(x))}{2r} \leq C, \quad r \in (0, r_0).
\]

Then for a.e. \( x \in \Gamma \),

\[
\overline{U_{1,\nu(x)}(x)} \cap \sigma_{x,\varrho}(\Gamma) \xrightarrow{K} \overline{U_{1,\nu(x)}(x)} \cap T_x \tag{4.6}
\]
and

\[ H^1(\sigma_{x,\rho}(\Gamma)) \xrightarrow{\ast} H^1(T_x) \quad (4.7) \]

as \( \rho \to 0 \), where \( \sigma_{x,\rho} \) denotes the blow-up map defined in (2.2) and \( T_x \) is the generalized tangent line to \( \Gamma \) at \( x \). Moreover, for any \( H^1 \)-measurable \( \Gamma' \subset \Gamma \) and \( H^1 \)-a.e. \( x \in \Gamma' \) the relations (4.6) and (4.7) hold with \( \Gamma' \) in place of \( \Gamma \).

**Proof.** By [25, Theorem 3.3], \( \Gamma \) (and hence \( \Gamma' \)) has an approximate tangent line at \( H^1 \)-a.e. \( x \), therefore, (4.7) follows from [1, Remark 2.80]. To prove (4.6) with \( \Gamma' \) choose \( x \in \Gamma \) such that \( \theta_x(\Gamma, x) = \theta^*(\Gamma, x) = 1 \) and \( T_x \) exists. Without loss of generality we assume that \( x = 0 \), \( H^1(\Gamma) < \infty \) and \( v_T(x) = e_2 \) is the unit normal to \( T_x \). First we prove

\[ \sigma_{0,r}(\Gamma) \xrightarrow{K} T_0 \quad (4.8) \]

as \( r \searrow 0 \). Indeed, let \( \mu_r := H^1(\sigma_{0,r}(\Gamma)) \) and \( \mu_0 := H^1(T_0) \). Given \( r > 0 \), since \( \mu_r(U_\rho(x)) = \frac{H^1(U_{\rho/(r)}(x))}{\rho} \), by (4.5) for all \( x \in \sigma_{0,r}(\Gamma) \) and \( \rho \in (0, r_0/r) \) one has

\[ c \leq \frac{\mu_r(U_\rho(x))}{2\rho} \leq C. \quad (4.9) \]

Let \( r_k \searrow 0 \) be any sequence. By compactness of sets in the Kuratowski convergence, passing to a further not relabelled subsequence if necessary we suppose that

\[ \sigma_{0,r_k}(\Gamma) \xrightarrow{K} L \quad (4.10) \]

for some closed set \( L \subset \mathbb{R}^2 \) as \( k \to \infty \). We claim that \( L = T_0 \). If there exists \( x \in T_0 \setminus L \), then for some \( \rho > 0 \), \( U_\rho(x) \cap L = \emptyset \). By (4.10), \( U_{\rho/2}(x) \cap \sigma_{0,r_k}(\Gamma) = \emptyset \) for all large \( k \) so that \( \mu_{r_k}(U_{\rho/2}(x)) = 0 \). Then by (4.7)

\[ 0 = \lim_{k \to \infty} \mu_{r_k}(U_{\rho/2}(x)) \geq \mu_0(U_{\rho/2}(x)) > \rho, \]

a contradiction. If there exists \( x \in L \setminus T_0 \), then for some \( U_\rho(x) \subset T_0 = \emptyset \) for some \( \rho > 0 \) and there exists a sequence \( x_k \in \sigma_{0,r_k}(\Gamma) \) such that \( x_k \to x \). Then \( U_{\rho/2}(x_k) \subset U_\rho(x) \) for all large \( k \) so that by (4.7) and (4.9),

\[ 0 = \mu_0(U_\rho(x)) \leq \limsup_{k \to \infty} \mu_{r_k}(U_\rho(x)) \geq \limsup_{k \to \infty} \mu_{r_k}(U_{\rho/2}(x_k)) \geq c_0, \]

a contradiction. Thus, \( L = T_0 \). Since the sequence \( r_k \searrow 0 \) is arbitrary, (4.8) follows. Now (4.6) is obvious. \( \square \)

Now we are ready to prove the existence of global minimizers of \( F \).

**Proof of Theorem 2.4.** The proof of the second assertion can be done using the first one and also following the arguments of [24, Theorem 1.1] and [35, Proposition A.6]. Hence, we prove only the first assertion. Let \( (A_{m_i}, u_{m_i}) \in C_{m_i} \) and \( A \in C \) be as in Proposition 4.1. By Lemma 4.2, \( |A| = v \). Since \( S \) is connected and Lipschitz, by the Korn-Poincaré inequality and the Rellich-Kondrachov Theorem there exists a further not relabelled subsequence \{\( \{u_i\} \), a sequence \{\( \{a_i\} \) of infinitesimal rigid displacements and \( v_0 \in H^1(S; \mathbb{R}^2) \) such that \( u_{m_i} + a_h \to v_0 \) weakly in \( H^1_{\text{loc}}(S; \mathbb{R}^2) \) and a.e. in \( S \). Let \( \{E_i\}_{i \in \Lambda} \) be all connected components of \( \text{Int}(A) \). By a diagonal argument we choose a further not relabelled subsequence \{\( \{u_{m_i} \) \} and the subset \( \Lambda' \) of indices \( i \in \Lambda \) such that there exists a ball \( B_i \subset E_i \) such that the limit of \( u_{m_i}(x) + a_h(x) \) is finite for a.e. \( x \in B_i \). Then by [35, Proposition 3.7] for all \( i \in \Lambda' \) there exists \( v_i \in H^1_{\text{loc}}(E_i; \mathbb{R}^2) \) such that \( u_{m_i} + a_h \to v_i \) weakly in \( H^1_{\text{loc}}(E_i; \mathbb{R}^2) \) and
a.e. in $E_i$ as $h \to \infty$. By the choice of $\Lambda'$, for any $j \in \Lambda \backslash \Lambda'$ one has $|u_{m_h} + a_h| \to +\infty$ a.e. in $E_j$ as $h \to +\infty$. Define
\[ u := v_0 \chi_S + \sum_{i \in \Lambda'} v_i \chi_{E_i} + \sum_{j \in \Lambda \backslash \Lambda'} u_0 \chi_{E_j}. \] (4.11)

By construction, $u_{m_h} + a_h \to u$ weakly in $H^1_{\text{loc}}(\cup_{i \in \Lambda'} E_i; \mathbb{R}^2)$ and a.e. in $\cup_{i \in \Lambda'} E_i$, and $|u_{m_h} + a_h| \to +\infty$ a.e. in $\cup_{j \in \Lambda \backslash \Lambda'} E_j$. In particular, $e(u_{m_h}) = e(u_{m_h} + a_h) \to e(u)$ in $L^2_{\text{loc}}(\cup_{i \in \Lambda'} E_i; \mathbb{R}^2)$. Therefore, by the convexity of the elastic energy density $W(x, M)$ in $M$, for any $D \subset \subset \text{Int}(A) \cup S$ we have
\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) \
\geq \liminf_{h \to \infty} \left( \int_{D \cap S} W(x, e(u_{m_h}) - M_0) \, dx + \sum_{j \in \Lambda'} \int_{D \cap E_j} W(x, e(u_{m_h}) - M_0) \, dx \right) \
\geq \int_{D \cap S} W(x, e(u) - M_0) \, dx + \sum_{j \in \Lambda'} \int_{D \cap E_j} W(x, e(u) - M_0) \, dx.
\]
Taking into account $e(u) - E_0 = 0$ a.e. in $\cup_{j \in \Lambda \backslash \Lambda'} E_j$, this inequality can also be rewritten as
\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) \geq \int_{D \cap (A \cup S)} W(x, e(u) - M_0) \, dx.
\]
In particular, letting $D \subset \subset \text{Int}(A) \cup S$ and using $|A \backslash \text{Int}(A)| = 0$ we get
\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) \geq W(A, u). \tag{4.12}
\]

We now introduce the open subset $G$ of $A^{(1)}$, which we refer as reduced set $G$ constructed from $A$. Let
\[ \Gamma := \{ x \in A^{(1)} \cap \partial A : \text{one-sided traces of } u \text{ at } x \text{ exist and are equal} \}. \]
By Proposition 4.1 all points of $\Gamma$ satisfy the uniform density estimates, and therefore, $\theta^*(\Gamma, x) = \theta_*(\Gamma, x) = 1$ at $\mathcal{H}^1$-a.e. $x \in \Gamma$. In particular, $\mathcal{H}^1(\overline{\Gamma} \backslash \Gamma) = 0$. Let
\[ G := \text{Int}(A \cup \overline{\Gamma}). \]

We claim that $(G, u)$ is a minimizer of $\mathcal{F}$. Indeed, note that $|G| = |A|$ and $(G, u) \in \mathcal{C}$. In view of (3.1), it suffices to prove that
\[
\liminf_{h \to \infty} \mathcal{F}(A_{m_h}, u_{m_h}) \geq \mathcal{F}(G, u). \tag{4.13}
\]
Since $|G \Delta A| = 0$, by (4.12) we have
\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) \geq W(G, u),
\]
thus to prove (4.13) it suffices to establish
\[
\liminf_{h \to \infty} \mathcal{S}(A_{m_h}, u_{m_h} + a_h) \geq \mathcal{S}(G, u). \tag{4.14}
\]
To prove (4.14) we follow the arguments of [35, Proposition 4.1]. Let $g \in L^\infty(\Sigma \times \{0, 1\})$ be such that
\[ g(x, s) := \varphi(x, \nu_\Sigma(x)) + s\beta(x) \]
and define
\[ J_{A_{m_h}} := J_{u_{m_h}} = J_{u_{m_h} + a_h} \quad \text{and} \quad J_{G} := J_u \cup \bigcup_{j \in \Lambda \backslash \Lambda'} (\Sigma \cap \partial E_j). \]
Then by (2.5), $g \geq 0$ and
\[ |g(x, 1) - g(x, 0)| \leq \varphi(x, \nu_\Sigma(x)) \quad \text{for a.e. } x \in \Sigma. \tag{4.15} \]
Let \( \mu_h \) be the sequence of positive Radon measures defined at Borel sets \( B \subset \mathbb{R}^2 \) as
\[
\mu_h(B) := \int_{B \cap A_{\Omega}^{0} - A_{\Omega}^{1}} \varphi(x, \nu_{A_{\Omega}^{1}}) d\mu^1 + 2 \int_{B \cap A_{\Omega}^{1} \cup A_{\Omega}^{2} \cap \partial A_{\Omega}^{2}} \varphi(x, \nu_{A_{\Omega}^{1}}) d\mu^1 + \int_{B \cap \Sigma \cap \partial A_{\Omega}^{2}} [\varphi(x, \nu_{\Sigma}) + g(x, 1)] d\mu^1(x) + \int_{B \cap \Sigma \cap \partial A_{\Omega}^{1}} g(x, 0) d\mu^1 + \int_{B \cap \Sigma \cap \partial A_{\Omega}^{2}} [g(x, 0) + \varphi(x, \nu_{\Sigma})] d\mu^1
\]
and let \( \mu \) be the positive measure defined at Borel sets \( B \subset \mathbb{R}^2 \) as
\[
\mu(B) := \int_{B \cap A_{\Omega}^{0} - A_{\Omega}^{1}} \varphi(x, \nu_{\Omega}) d\mu^1 + 2 \int_{B \cap A_{\Omega}^{1} \cup A_{\Omega}^{2} \cap \partial A_{\Omega}^{2}} \varphi(x, \nu_{\Omega}) d\mu^1 + \int_{B \cap \Sigma \cap \partial A_{\Omega}^{2}} g(x, 0) d\mu^1 + \int_{B \cap \Sigma \cap \partial A_{\Omega}^{1}} g(x, 1) d\mu^1 + \int_{B \cap \Sigma \cap \partial A_{\Omega}^{2}} [g(x, 0) + \varphi(x, \nu_{\Sigma})] d\mu^1.
\]
Recalling \( \mu^1(G^{(0)} \cap \partial G) = 0 \) we note that \( \mu \) is defined as \( \mu_h \) with \( A_{\Omega}^{2} \) replaced by \( E \). Note that
\[
\mu_h(\mathbb{R}^2) = S(A_{\Omega}^{2}, u_{A_{\Omega}^{2}}) + \int_{\Sigma} \varphi(x, \nu_{\Sigma}) d\mu^1
\]
and
\[
\mu(\mathbb{R}^2) = S(G, u) + \int_{\Sigma} \varphi(x, \nu_{\Sigma}) d\mu^1.
\]
Hence, it suffices to prove
\[
\liminf_{h \to \infty} \mu_h(\mathbb{R}^2) \geq \mu(\mathbb{R}^2).
\] (4.16)
Note that since \( \sup_{h} \mu_h(\mathbb{R}^2) < +\infty \), by compactness, there exists a positive Radon measure \( \mu_0 \) in \( \mathbb{R}^2 \) such that (up to a subsequence) \( \mu_h \rightharpoonup \mu_0 \) as \( h \to \infty \). We prove
\[
\mu_0 \geq \mu
\] (4.17)
and we observe that (4.16) immediately follows from (4.17). To establish (4.17) it suffices to prove
\[
\frac{d\mu_0}{d\mu^1}[\Omega \cap \partial^* G](x) \geq \varphi(x, \nu_G(x)) \quad \text{for a.e. } x \in \Omega \cap \partial^* G,
\] (4.18a)
\[
\frac{d\mu_0}{d\mu^1}[\Omega \cap \partial^* G](x) \geq 2\varphi(x, \nu_G(x)) \quad \text{for a.e. } x \in \Omega \cap G^{(1)} \cap \partial G,
\] (4.18b)
\[
\frac{d\mu_0}{d\mu^1}[\Sigma \cap \partial^* G](x) \geq g(x, 1) \quad \text{for a.e. } x \in \Sigma \cap \partial^* G,
\] (4.18c)
\[
\frac{d\mu_0}{d\mu^1}[\Sigma \cap \partial G](x) \geq \varphi(x, \nu_\Sigma(x)) + g(x, 0) \quad \text{for a.e. } x \in \Sigma \cap \partial G,
\] (4.18d)
\[
\frac{d\mu_0}{d\mu^1}[\Sigma \setminus \partial G](x) \geq g(x, 0) \quad \text{for a.e. } x \in \Sigma \setminus \partial G.
\] (4.18e)
Since \( \partial^* A = \partial^* G \) (up to a \( \mu^1 \)-negligible set) and \( A_k \to G \) in \( L^1(\mathbb{R}^2) \), the proofs of (4.18a) and (4.18e) can be done following the arguments of [35, Eq. 4.40a] and [35, Eq. 4.40d], respectively. Moreover, for any \( x_0 \in \Sigma \setminus \partial G \) and a square \( U_r := U_{r, \nu_{\Sigma}(x_0)}(x_0) \subset \mathbb{R}^2 \setminus \overline{G} \) one has \( U_r \cap \text{Int}(A_{\Omega}^{2}) = \emptyset \) for all large \( h \), and thus, by the nonnegativity of \( g \) and (4.15),
\[
\mu_h(U_r) \geq \int_{U_r} g(x, 0) d\mu^1,
\]
and hence (4.18c) follows. Furthermore, the proof of (4.18d) for \( x \in J_G \cap J_u \) can be done following [35, Eq. 4.40g] and for \( x \in J_G \setminus J_u \) can be done adapting the arguments of [35, Proposition 3.9].

It remains to show (4.18b). We notice that the argument of [35, Eq. 4.40c] cannot be used in this setting, as it hinges on the uniform bound on the number of connected components which we do not have here. We instead adapt the arguments employed in [35, Eq. 4.40g]. We prove (4.18b) for all \( x \in G^{(1)} \cap \partial G \) satisfying:

(b1) \( \theta_\pm(\partial G, x) = \theta^*(\partial G, x) = \theta_\pm(G^{(1)} \cap \partial G, x) = \theta^*(G^{(1)} \cap \partial G, x) = 1 \) and \( \nu_\partial G(x) \) exists;
(b2) \( \theta_\pm(A, x) = \theta^*(A, x) = \theta_\pm(A^{(1)} \cap \partial A, x) = \theta^*(A^{(1)} \cap \partial A, x) = 1 \) and \( \nu_{\partial A}(x) \) exists;
(b3) \( U_{1, \nu_\partial G}(x) \cap \sigma_{\rho,x}(\partial G) \to K U_{1, \nu_\partial G}(x) \cap T_x \) as \( \rho \to 0 \), where \( T_x \) is the generalized
tangent line to \( \partial G \) at \( x \);
(b4) one-sided traces \( u^+(x) \) and \( u^-(x) \) of \( u \) w.r.t. \( G^{(1)} \cap \partial G \) exist and are not equal;
(b5) \( \frac{d\mu_0}{d\mathcal{H}^1(\{G^{(1)} \cap \partial G\})}(x) \) exists and is finite.

We notice that the set of points of \( G^{(1)} \cap \partial G \) not satisfying (b1)-(b5) is \( \mathcal{H}^1 \)-negligible,
since \( \partial G \) and \( \partial A \) are \( \mathcal{H}^1 \)-rectifiable, the definition and \( \mathcal{H}^1 \)-rectifiability of \( J_u \) [16, Definition 2.4 and Theorem 6.2], both the statements \( G^{(1)} \cap \partial G = J_u \) and \( G^{(1)} \cap \partial G \subseteq A^{(1)} \cap \partial A \) hold up to a \( \mathcal{H}^1 \)-negligible set, and the fact that we can apply Proposition 4.3 with \( \Gamma = \partial A \) and \( \Gamma' = \partial G \), and the Besicovitch Differentiation Theorem. Notice that \( \nu_{\partial G}(x) = \nu_{\partial A}(x) \) for every \( x \in \partial G \), and assume, without loss of generality, that \( x = 0 \) and \( \nu_{\partial G}(x) = \mathbf{e}_2 \). By the construction of \( G \), the definition of \( G^{(1)} \cap \partial G \neq 0 \), and the definition (4.11) of \( u \) we have three cases:

(a) there exists \( i_0 \in \Lambda' \) such that
\[
\theta^*(E_{i_0}^{(1)} \cap \partial E_{i_0}, 0) = \theta_\pm(E_{i_0}^{(1)} \cap \partial E_{i_0}, 0) = 1
\]
and \( u_{m_h} + a_{m_h} \to u \) a.e. in \( E \);
(b) there exist \( i_0 \in \Lambda' \) and \( j_0 \in \Lambda \setminus \Lambda' \) such that \( 0 \in \partial^* E_{i_0} \cap \partial^* E_{j_0}^* \) and \( u_{m_h} + a_{m_h} \to u \) a.e. in \( E_{i_0} \) and \( |u_{m_h} + a_{m_h}| \to \infty \) a.e. in \( E_{j_0} \). Without loss of generality we assume that \( \mathbf{e}_2 \) is the inner normal of \( E_{i_0} \);
(c) there exist \( i_1, i_2 \in \Lambda' \) such that \( 0 \in \partial^* E_{i_1} \cap \partial^* E_{i_2}^* \) and \( u_{m_h} + a_{m_h} \to u \) a.e. in \( E_{i_1} \cup E_{i_2} \).

First we observe the following. Let \( 4r_0 := \text{dist}(0, \partial \Omega) \). By weak convergence,
\[
\lim_{h \to \infty} \mu_h(U_r) = \mu_0(U_r)
\] (4.19)
for a.e. \( r \in (0,r_0) \). Let \( \xi \in (0,1)^2 \) be such that the set of all \( x \in \partial A_{m_h} \) (resp. \( x \in \partial G \)) with one-sided traces of \( u_{m_h} + a_h \) (resp. of \( u \)) existing and equal to \( \xi \) is \( \mathcal{H}^1 \)-negligible. Define
\[
w_h := (u_{m_h} + a_h)\chi_{A_{m_h}} + \xi\chi_{\Omega \setminus A_{m_h}}
\]
and
\[
w := u\chi_G + \xi\chi_{\Omega \setminus G}.
\]
We recall that by the definition (4.11) of \( u \) the set of \( x \in \Omega \) such that \( w_h(x) \) does not converge to \( w(x) \) could have positive measure.

By the construction of \( w_h \), \( J_{w_h} \subset \partial A_{m_h} \) and
\[
\sup_{h \geq 1} \mathcal{H}^1(J_{w_h}) + \int_\Omega |e(w_h)|^2 dx < \infty.
\]
Moreover, by (b4), [16, Definition 2.4] and [16, Remark 2.2] separately applied to $U_1^+$ and $U_1 \setminus U_1^+$ we have
\[
\lim_{r \to 0} \int_{U_1} |\tau(w(rx)) - \tau(w_0(x))| dx = 0, \quad (4.20)
\]
where $\tau(x_1, x_2) = (\arctan x_1, \arctan x_2)$,
\[
w_0 := u^+(0)\chi_{U_1^+} + u^-(0)\chi_{U_1 \setminus U_1^+},
\]
and $u^+(0)$ and $u^-(0)$ are given by (b4). For every $r \in (0, r_0)$ let
\[
U_r^\infty := \{ x \in U_1 : \liminf_{h \to \infty} |w_h(rx)| = +\infty \}.
\]
Unlike the proof of [35, Eq. 4.40g], $U_r^\infty$ can be non-empty; more precisely, $rU_r^\infty \subset \cup_{j \in A \setminus A^2} E_j$ (see the definition (4.11) of $u$). Since $w_h(rx) \to w(rx)$ a.e. in $U_1 \setminus U_r^\infty$, one has
\[
\lim_{h \to \infty} \int_{U_1 \setminus U_r^\infty} |\tau(w_h(rx)) - \tau(w(rx))| dx = 0. \quad (4.21)
\]
By Proposition 4.3
\[
\begin{align*}
U_4 \cap \sigma_r(\partial A) &\xrightarrow{K} I_4 \quad \text{and} \quad \mathcal{H}^1(U_4 \cap \sigma_r(\partial A)) \xrightarrow{\ast} \mathcal{H}^1 I_4, \\
U_4 \cap \sigma_r(\partial E_{i_0}) &\xrightarrow{K} I_4 \quad \text{and} \quad \mathcal{H}^1(U_4 \cap \sigma_r(\partial E_{i_0})) \xrightarrow{\ast} \mathcal{H}^1 I_4, \\
U_4 \cap \sigma_r(\partial(E_{i_0} \cup E_{j_0})) &\xrightarrow{K} I_4 \quad \text{and} \quad \mathcal{H}^1(U_4 \cap \sigma_r(\partial(E_{i_0} \cup E_{j_0}))) \xrightarrow{\ast} \mathcal{H}^1 I_4 \quad (4.22c) \\
U_4 \cap \sigma_r(\partial(E_{i_1} \cup E_{i_2})) &\xrightarrow{K} I_4 \quad \text{and} \quad \mathcal{H}^1(U_4 \cap \sigma_r(\partial(E_{i_1} \cup E_{i_2}))) \xrightarrow{\ast} \mathcal{H}^1 I_4 \quad (4.22d)
\end{align*}
\]
as $r \to 0$, where we apply Proposition 4.3 with $\Gamma := \partial A$ and with $\Gamma' := \partial E_{i_0}$ in (4.22b), with $\Gamma' := \partial(E_{i_0} \cup E_{j_0}) = \partial E_{i_0} \cup \partial E_{j_0}$ in (4.22c) and with $\Gamma' := \partial(E_{i_1} \cup E_{i_2}) = \partial E_{i_1} \cup \partial E_{i_2}$ in (4.22d). Furthermore, by (4.22a)-(4.22d) and [35, Proposition A.5]
\[
dsdist(\cdot, \sigma_r(\partial A)) \to -\text{dist}(\cdot, T_0), \quad (4.23a)
dsdist(\cdot, \sigma_r(\partial E_{i_0})) \to -\text{dist}(\cdot, T_0), \quad (4.23b)
dsdist(\cdot, \sigma_r(\partial(E_{i_0} \cup E_{j_0}))) \to -\text{dist}(\cdot, T_0), \quad (4.23c)
dsdist(\cdot, \sigma_r(\partial(E_{i_1} \cup E_{i_2}))) \to -\text{dist}(\cdot, T_0) \quad (4.23d)
\]
locally uniformly in $\overline{U}_4$ as $r \to 0$. Hence,
\[
\lim_{r \to 0} |U_r^\infty \Delta U_0^\infty| = 0, \quad (4.24)
\]
where by (4.23b)-(4.23d)
\[
U_0^\infty = \begin{cases} 
\emptyset & \text{in cases (a) and (c)}, \\
U_1 \setminus U_1^+ & \text{in case (b)}.
\end{cases} \quad (4.25)
\]
Now we choose sequences $h_k \to \infty$ and $r_k \searrow 0$ as follows. By (4.19), (4.23a), (4.20) and (4.24) for any $k \in \mathbb{N}$ there exists $r_k \in (0, \frac{1}{k})$ such that (4.19) holds with $r = r_k$ and
\[
\|\text{sdist}(\cdot, \sigma_{r_k}(U_{4r_k} \cap \partial A)) + \text{dist}(\cdot, U_4 \cap T_x)\|_{L^\infty(U_{3/2})} < \frac{1}{k^2}, \quad (4.26a)
\]
\[
\int_{U_1} |\tau(w(r_kx)) - \tau(w_0(x))| dx < \frac{1}{k^2}, \quad (4.26b)
\]
\[
|U_{r_k}^\infty \Delta U_{0_k}^\infty| < \frac{1}{k^2}. \quad (4.26c)
\]
\footnote{Note that if if $P, Q \subset \mathbb{R}^n$ are open and disjoint, then $\partial(P \cup Q) = \partial P \cup \partial Q$.}
Given $k \geq 1$ and $r_k$, since $A_{mh_k}$ $r$-converges to $A$ and $\tau$ is bounded, by (4.21), (4.26c) and (4.19) we can choose $h_k$ such that
\[
\frac{1}{h_k r_k} < \frac{1}{k},
\] (4.27a)
and
\[
\| \text{sdist}(\cdot, U_4 \cap \sigma_{r_k}(\partial A_{mh_k})) - \text{sdist}(\cdot, U_4 \cap \sigma_{r_k}(\partial A)) \|_{L^\infty(U_{3/2})} < \frac{1}{k},
\] (4.27b)
\[
\int_{U_{1/2}} \tau(w_{h_k}(r_k x)) - \tau(w(r_k x))dx < \frac{1}{k},
\] (4.27c)
\[
\mu_{h_k}(U_{r_k}) \leq \mu_0(U_{r_k}) + r_k^2.
\] (4.27d)
Notice that by (4.27a), $h_k \to \infty$ as $k \to \infty$.
Let
\[
D_k := \sigma_{r_k}(U_{4 r_k} \cap A_{mh_k}) = U_4 \cap \sigma_{r_k}(A_{mh_k}).
\]
By (4.27b) and (4.26a),
\[
U_{3/2} \cap \partial D_k \overset{L^2}{=\to} I_{3/2} \cap T_0 \quad \text{as } k \to \infty.
\]
In particular, for any $\delta \in (0, 1)$ there exists $k_\delta > 1$ such that $U_1 \cap \partial D_k \subset [-1, 1] \times [-\delta, \delta]$. Furthermore, by (4.27d) and finiteness of
\[
\frac{d\mu_0}{\mathcal{H}^1(G \cap \partial G)(U_{1/2})} = \lim_{k \to \infty} \frac{\mu_0(U_{r_k})}{2 r_k}
\]
we may suppose for $C := \frac{d\mu_0}{\mathcal{H}^1(G \cap \partial G)(U_{1/2})} + 1$,
\[
\mathcal{H}^1(U_1 \cap \partial D_k) = \frac{\mathcal{H}^1(U_{r_k} \cap \partial A_{mh_k})}{r_k} \leq \frac{\mu_{h_k}(U_{r_k})}{r_k} \leq C
\] (4.28)
for all $k \geq 1$. Furthermore, since $\varphi$ is uniformly continuous, we suppose also that
\[
|\varphi(x, \nu) - \varphi(0, \nu)| < \delta, \quad x \in U_{r_k}, \ \nu \in \mathbb{S}^1,
\]
thus, by the definition of $\mu_{h_k}$ and $D_k$ and (4.28),
\[
\frac{\mu_{h_k}(U_{r_k})}{r_k} \geq \int_{U_1 \cap \partial A_{D_k}} \phi(\nu_{D_k})d\mathcal{H}^1 + 2 \int_{U_1 \cap D_k \cap \partial A D_k} \phi(\nu_{D_k})d\mathcal{H}^1 - 2 C \delta,
\] (4.29)
where $\phi(\nu) = \varphi(0, \nu)$. Also, since the number of connected components of $\partial A_{mh_k}$ is at most $m_{h_k}$, the number of connected components of $[U_1 \cap \partial D_k] \cup \{ \{0\} \times [-\delta, \delta] \}$ does not exceed $m_{h_k} + 2$. Let us index them with the index set $\Lambda''$ and denote their closures by $\{ L^k_i \}_{i \in \Lambda''}$. As in the proof of [35, Lemma 4.7] we will replace $L^k_i$ with a pairwise disjoint finite family $\{ V^k_j \}_{j \in \Lambda''}$ of closed connected sets containing them.

First we partition $\Lambda''$ into finitely many pairwise disjoint non-empty subsets $\{ \Lambda_j^k \}_{j=1}^{N_k}$ by induction as follows. In the following we denote by $\text{co}(S)$ the closed convex hull of a set $S$.

We begin by defining $\Lambda_1^k$ as the set of indices $\{i_1, \ldots, i_{n_1^k} \}$ in $\Lambda''$ for some $n_1^k \geq 1$ such that:

(c1) $i_1 := 1$;
(c2) $i_l$ for any $1 < l \leq n_1^k$ is the smallest index $q \in \Lambda'' \setminus \{i_1, \ldots, i_{l-1} \}$ such that $\text{co}(L^k_i)$ intersects $\text{co}(\bigcup_{j=1}^{l-1} L^k_i)$,
(c3) $n_1^k$ is maximal, i.e., given $q \in \Lambda'' \setminus \Lambda_1^k$ such that $\text{co}(L^k_i)$ does not intersects $\text{co}(\bigcup_{j=1}^{n_1^k} L^k_i)$.
Suppose that for \( j \geq 2 \) the non-empty sets \( \Lambda^j_k, \ldots, \Lambda^k_{k-1} \) are defined. If \( \Lambda_j := \Lambda^j \setminus \bigcup_{i=1}^{j-1} \Lambda^i \neq \emptyset \), then we define \( \Lambda^j_k := \{ i^j, \ldots, i^{j}_{m_h} \} \) as the set with \( i^j \) the smallest element of \( \Lambda_j \) and satisfying (c2) and (c3) with \( i^j, m^j_h \) and \( \Lambda^j \setminus \Lambda^i_j \) replaced by \( i^j, m^j_h \) and \( \Lambda_j \), respectively. Since \( \Lambda'' \subset \{ 1, \ldots, m^j_h + 2 \} \), the cardinality of \( \{ \Lambda^j_k \}_j \), which we denote by \( N_k \), satisfies \( N_k \leq m^j_h + 2 \).

Now define
\[
V^j := \operatorname{co} \left( \bigcup_{i \in \Lambda^j_k} L^j_i \right), \quad j = 1, \ldots, N_k.
\]
Notice that if \( C^j_k \) denotes the convex hull of \( L^j_i \), then \( V^j \) is also the convex hull of \( \bigcup_{i \in \Lambda^j_k} C^j_k \). By applying (recursively if needed) the anisotropic minimality of segments, for any \( i \geq 1 \) and
\[
\int_{L^j_i \cap \partial D_k} \phi(\nu_{L^j_i}) dH^1 + 2 \int_{L^j_i \cap D^{(1)}_k \cap \partial D_k} \phi(\nu_{L^j_i}) dH^1 + \int_{L^j_i \cap \{ \pm 1 \} \times [-\delta, \delta]} \phi(\nu_{L^j_i}) dH^1 \geq \int_{\partial C^j_k} \phi(\nu_{C^j_k}) dH^1
\]
for any \( i \geq 1 \) and
\[
\sum_{i \in \Lambda^j_k} \int_{\partial C^j_k} \phi(\nu_{C^j_k}) dH^1 \geq \int_{\partial V^j} \phi(\nu_{V^j}) dH^1
\]
for any \( j \in \{ 1, \ldots, N_k \} \). Note that if \( V^j = \text{segment} \), then \( \partial V^j \) and \( V^j \), and in this case we replace \( V^j \) with a very thin rectangle \( \hat{V}^j \) such that \( V^j \subset \hat{V}^j \), \( \text{dist}(\hat{V}^j, V^j) > 0 \) for \( i \neq j \) and
\[
2 \int_{V^j} \phi(\nu_{V^j}) dH^1 \geq \int_{\partial V^j} \phi(\nu_{V^j}) dH^1 - \frac{c_2 \rho_k}{2^j}
\]
for some sequence \( N_k \rho_k \searrow 0 \). Such \( \hat{V}^j \) always exists since \( N_k < \infty \). For shortness we use the notation \( V^j \) also for \( \hat{V}^j \). Then by (4.29),
\[
\frac{\mu_h(U^j_{r_h})}{r_h} \geq \sum_{i \in \Lambda''} \left( \int_{L^j_i \cap \partial D_k} \phi(\nu_{D_k}) dH^1 + 2 \int_{D^{(1)}_k \cap \partial D_k} \phi(\nu_{D_k}) dH^1 \right) - 2c \delta
\]
\[
\geq \sum_{j=1}^{N_j} \int_{\partial V^j} \phi(\nu_{V^j}) dH^1 - (5c_2 + 2C) \delta,
\]
where \( 5c_2 \delta \) is related to the surface energy of vertical segments \( \{ \pm 1 \} \times [-\delta, \delta] \) and the error in (4.30).

Notice that \( x \in U_1 \mapsto w_{h_k}(r_k x) \) belongs to \( H^1(U_1 \setminus \bigcup_{j=1}^{N_k} V^j_k; \mathbb{R}^2) \). Hence, for any fixed \( \xi_1 \in (0, 1)^2 \) if we define
\[
v_k(x) := w_{h_k}(r_k x) \chi_{U_1 \setminus \bigcup_{j=1}^{N_k} V^j_k} + \xi_1 x \chi_{\bigcup_{j=1}^{N_k} V^j_k}, \quad x \in U_1,
\]
then \( v_k \in GSB2D(U_1; \mathbb{R}^2) \). Further we choose \( \xi_1 \in (0, 1)^2 \) such that \( J_{v_k} = \bigcup_{i} \partial V^j_i \) (up to a \( H^1 \)-negligible set). Since
\[
\int_{U_1} |e(v_k)|^2 dx \leq \int_{U_{r_h} \cap \Lambda_{mb_h}} |e(u_{mb_h})|^2 dx
\]
and \(\{(A_m, u_m)\}\) is a minimizing sequence, it follows that
\[
\sup_{k \geq 1} \int_{U_1} |e(v_k)|^2 dx + \mathcal{H}^1(J_{v_k}) < \infty.
\]
By (4.27c) and (4.26b) and from the estimate \(|\tau(y)| \leq 2\pi\) it follows that
\[
\int_{U_1 \setminus U_0^\infty} |\tau(v_k(x)) - \tau(w_0(x))| dx \leq \frac{2}{k} + 2\pi \sum_{j \geq 1} |V_j^k|.
\] (4.32)
As \(\overline{U_1} \cap \partial D_k \to I_1\),
\[
\lim_{k \to \infty} \sum_{j \geq 1} |V_j^k| = 0.
\]
Thus, from (4.32) passing to further not relabelled subsequence, we get \(v_k \to w_0\) a.e. in \(U_1 \setminus U_0^\infty\) and \(|v_k| \to +\infty\) a.e. in \(U_0^\infty\). By (4.25), \(U_0^\infty\) is either empty or the lower part of the unit square. Thus repeating the same arguments of the proof of [35, Eq. 4.35] we get
\[
\liminf_{k \to \infty} \int_{J_{v_k}} \phi(v_{J_{v_k}}) d\mathcal{H}^1 = \liminf_{k \to \infty} \sum_{i \geq 1} \int_{\partial V_i^k} \phi(v_{V_i^k}) d\mathcal{H}^1
\]
\[
\geq 2 \int_{U_1 \cap (J_{w_0} \cup \partial^* U_0^\infty)} \phi(e_2) d\mathcal{H}^1 = 2 \int_{I_1} \phi(e_2) d\mathcal{H}^1.
\]
In particular, there exists \(k_\delta > 0\) such that
\[
\sum_{i \geq 1} \int_{\partial V_i^k} \phi(v_{V_i^k}) d\mathcal{H}^1 > 2 \int_{I_1} \phi(e_2) d\mathcal{H}^1 - \delta
\]
for all \(k > k_\delta\). Furthermore, from (4.31) and (4.27d) it follows that
\[
\frac{\mu_0(U_{r_k})}{2r_k} + \frac{r_k}{2} \geq \int_{I_1} \phi(e_2) d\mathcal{H}^1 - \frac{5c_2 + 2C + 1}{2} \delta.
\]
Now first letting \(k \to \infty\) and then \(\delta \to 0\) we get (4.18b). \(\Box\)

5. PROOF OF THEOREM 2.5

In this section we study the properties of the minimizers of \(\mathcal{F}\) in \(\mathcal{C}\) provided by Theorem 2.4.

Proof of Theorem 2.5. The proofs of (C) and (D) can be done along the lines of the proof of Theorem 3.1. Since \(\partial A\) satisfies the uniform density estimates, the reduced set \(G\) constructed in the proof of Theorem 2.4 satisfies (A) and (B). Hence, we can suppose that \(A = G\).

Notice that if \(E \subset A\) is a connected component of \(A\) with \(\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0\), then for \((A, u')\) with \(u' = u\chi_{(A \cup S) \setminus E} + u_0\chi_E\) we have
\[
\mathcal{S}(A, u) \geq \mathcal{S}(A, u')
\]
and
\[
\mathcal{W}(A, u) \geq \mathcal{W}(A, u'),
\] (5.1)
where in (5.1) equality holds if and only of \(u = u_0\) in \(E\). Therefore, by the minimality of \((A, u)\) it follows that \(u = u_0\) in \(E\). It remains to prove
\[
|E| \geq 4\pi \left(\frac{c_1}{\lambda_0}\right)^2.
\] (5.2)
Consider the competitor \((A \setminus E, u) \in C\). By minimality and Theorem 2.4, \(\mathcal{F}^\lambda(A, u) \leq \mathcal{F}^\lambda(A \setminus E, u)\), so that by (5.1) and the additivity of the surface energy, \(\mathcal{S}(E, u) \leq \lambda|E|\). Then by (2.4) and the isoperimetric inequality in \(\mathbb{R}^2\)

\[
\lambda|E| \geq c_1 \mathcal{H}^1(\partial E) \geq c_1 \sqrt{4\pi|E|^{1/2}}.
\]

Hence, \(|E| \geq (c_1 \sqrt{4\pi}/\lambda)^2\). Since \(\lambda > \lambda_0\) is arbitrary, (5.2) follows. \(\square\)

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