The Vernon transform and its use in quantum thermodynamics

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The thermodynamics of a quantum system interacting with an environment that can be assimilated to a harmonic oscillator bath has been extensively investigated theoretically. In recent experiments, the system under study however does not interact directly with the bath, but through a cavity or a transmission line. The influence on the system from the bath is therefore seen through an intermediate system, which modifies the characteristics of this influence. Here we first show that this problem is elegantly solved by a transform, which we call the Vernon transform, mapping influence action kernels on influence action kernels. We also show that the Vernon transform takes a particularly simple form in the Fourier domain, though it then must be interpreted with some care. Second, leveraging results in quantum thermodynamics we show how the Vernon transform can also be used to compute the generating function of energy changes in the environment. We work out the example of a system interacting with two baths of the Caldeira-Leggett type, each of them seen through a cavity.

I. INTRODUCTION

The study of of heat released to or absorbed from a bath (or baths) by a driven quantum system took off with the path-breaking contribution by Alicki now 40 years ago [1]. For recent reviews of the context and later developments, see [2, 3]. In Alicki’s approach the dynamics of a system is modeled as a quantum Markov process [4–6], and energy exchange between the system and the bath (or baths) are expressed in the Lindblad operators. The quantum heat, defined as expected energy change in the bath, is then the reverse of the expected dissipative energy change of the system, which is determined by the Lindblad operators acting on the system quantum state.

The setting has been extended in several directions, both recently and less recently. First, higher moments or the entire distribution of bath energy changes may be of interest. They are not necessarily the same as higher moments and distribution of system energy changes, but at least in a formal sense these quantities remain quantum functionals of the system history [7–9]. When modelling the dynamics of a qubit in a supercomputing circuit as very low temperature [10, 11], the quantum Markov process assumption is questionable. Indeed, a wealth of phenomena have been worked out for the problem of a qubit interacting with a bath when the drive of the qubit changes on a time scale comparable to or faster than the bath [12]. Other types of explicit results have been obtained in the spin-boson problem [13], in the “non-interacting blip approximation” (NIBA). The relaxation time scale of the bath is then assumed shorter than the times between system jumps, but not zero, and far longer than the time over which a jump a takes place [8, 14–17]. In a related direction, when the interaction between the system and the bath is strong, meaning that in a process the typical energy stored in system-bath interactions is comparable to the variations in system energy, the concept of heat is delicate even classically, see e.g. [18–22]. In the quantum domain these questions have been actively investigated by many groups with different techniques [23–35].

Our goal here is a different one, and motivated by experimental set-ups when investigating heat flow through superconducting qubits [36]. In such devices the bath (or baths) with which the qubit eventually exchanges energy are (small) normal-metal components held at fixed temperature. The influence of such baths, which are physically comprised of a relatively small number of conduction-band electrons (fermions) excited above the ground state may be assimilated to a bath of harmonic oscillators that would classically act as friction/resistance [37]. However, in the experimental set up, this (these) bath(s) do not interact directly with the qubit, but through a single-mode transmission line. The relevant physical model for most experiments on quantum thermodynamics is hence that of (system)-(cavity)-(bath). Even if it is valuable to know how a given bath influences a given system through a direct linear coupling this is hence not always directly applicable. The system only experiences the bath as seen through (filtered by) the oscillator, and this changes the effects on the system dramatically. While several

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that (B) is described by the kernels of the real and imaginary parts of the Feynman-Vernon action, \(k^{I-}_{b \rightarrow c}\) and \(k^{R+}_{c \rightarrow b}\). This means that whatever its actual physical constituents, as to its influence on C, the environment B behaves as a harmonic oscillator bath. C is described as one oscillator degree of freedom (a mode in an actual cavity, or a transmission line) with mode frequency \(\omega_c\), which interacts linearly with both (S) and (B). The influence on (S) from its total environment, (C) and (B) considered together, is then described Feynman-Vernon action kernels \(k^{I-}_{b \rightarrow c}\) and \(k^{R+}_{c \rightarrow b}\). The Vernon transform maps \((k^{I-}_{b \rightarrow c}, k^{R+}_{c \rightarrow b})\) on \((k^{c-}_{c \rightarrow s}, k^{s-}_{s \rightarrow c})\). It is described in text how this is used to express quantum heat (energy changes in the bath) as functionals of system history.

Theoretical investigations have been performed, e.g. recently in [38] and [39], we believe the perspective taken here is sufficiently different to motivate another investigation on the topic.

In particular, outside the specialized literature it does not seem to be generally appreciated that the bath seen through the oscillator constitutes a systematic transformation of the bath. It is further not generally known that this transformation was written down by F. Vernon in what may be the very first publication on open quantum systems, his 1958 Caltech PhD thesis [40]. Vernon’s supervisor was Feynman, and much of the material in Vernon’s thesis can also be found in their famous later joint paper [41]. However, the transformation which will be the main tool in the following is not found in [41]. In Vernon’s honor we thus refer to it as the Vernon transformation.

Combining recent development in quantum thermodynamics and the Vernon transform, we will further show that we can compute the generating function of energy change in the total environment, i.e. the bath and the cavity considered together. We will show that these formula are particularly transparent for the first moment that gives quantum heat, and further simplifies for quantum power, the quantum heat per unit time. They can be given in almost closed form when the bath is of the Caldeira-Leggett model [42]. Although technically more involved, we can also compute moments of the energy changes in the bath only; these details are presented in an appendix.

The paper is organized as follows. In Section II, we introduce the model of a system, an oscillator and bath, the oscillator interacting linearly with both the system and the bath, and we derive the Vernon transform for the real part of the Feynman-Vernon action kernel. Details are given in Appendices A and B. Thermodynamics and generating function of quantum heat in a system interacting to more than one bath are introduced in Section III. In Section IV we apply this method to a system interacting with two cavities, each of them interacting with its own bath as in the Caldeira-Leggett model. In Section V we sum up and discuss our results. In Appendix C we discuss how to estimate the energy deposited in the bath only (not in the cavity). In that case we can only compute moments of the energy change and not the complete generating function, and in practice only the first moment. A bath initially in equilibrium with constant, time-independent interactions between the bath and the system has issues previously addressed by Caldeira and co-workers [43], and Hänggi and co-workers [44]. For completeness we summarize these issues in Appendix D.

II. THE VERNON TRANSFORM

We are interested in Hamiltonian operators of the form

\[
\hat{H}(t) = \hat{H}_S + \hat{H}_C + \hat{H}_B + \hat{H}_{SB}(t) + \hat{H}_{BC}(t),
\]

where \(\hat{H}_S\) is a system Hamiltonian, possibly time-dependent, the bath is a collection of harmonic oscillators \(\hat{H}_B = \sum_k \omega_k b_k \dagger b_k\) and the cavity consists of one harmonic oscillator \(\hat{H}_C = \omega_c a_c \dagger a_c\). Initially the bath is in a state of thermal equilibrium with respect to \(\hat{H}_B\). The cavity interacts linearly with both the bath and the system. In the following discussion it will at some points be convenient to allow these interactions to be time-dependent, so that we have

\[
\hat{H}_{SB}(t) = C_{SC}(t) \hat{Q}_S \otimes \hat{X}_C
\]

\[
\hat{H}_{BC}(t) = C_{CB}(t) \hat{X}_C \otimes \sum_k \hat{Y}_B
\]

with possibly time-dependent coefficients \(C_{SC}(t)\) and \(C_{CB}(t)\), and where \(\hat{Y}_B = \sum_k \frac{1}{2} \left(\hat{b}_k + \hat{b}_k \dagger\right)\). It is well known that in this situation the influence of the bath on the cavity is completely described by the Feynman-Vernon action [4] [5] [41]. In path integral language this is a quadratic functional of forward and backward cavity paths, representing unitary evolution operators \(U\) and \(U^\dagger\) acting on the cavity after tracing out the bath. It is also well known that the influence on the system of the cavity and the bath together is described by another Feynman-Vernon action, a quadratic functional of forward and backward system paths. The Vernon transform [40].
(Appendix 5) expresses the kernels of the second Feynman-Vernon action in terms of the kernels of the first. In this way one sees how the system experiences the bath when its influence is transmitted through the cavity.

We start by stating an intermediate result in Vernon’s derivation of the Feynman-Vernon action itself. For convenience we repeat that derivation in Appendix A. The central observation is that a double path integral for a harmonic oscillator can be re-written in terms of the of the sums and differences of forward and backward paths, and thus gives rise to an auxiliary function satisfying

$$\Delta y_0 = \Delta y_0^* = 0 \quad \Delta Y^* = -\omega_k^2 \Delta Y^* + C_{CB}(t) \Delta X = 0$$

This is the equation of motion of a harmonic oscillator with an external drive starting from rest at the final time $t_f$, and evolving backwards in time to the initial time $t_i$. The value of $\Delta Y^*$ at time $t$ hence depends on the values of $\Delta X$ at times $s$ larger than $t$, but not on the values of $\Delta X$ at times $s$ less than or equal to $t$. The Feynman-Vernon influence functional from one bath oscillator on the cavity is then

$$\mathcal{F}^{k \rightarrow C} = e^{i \int_{t_i}^{t_f} C_{CB}(t) \Delta X^* dt} . e^{-\frac{i}{\hbar} \coth \left( \frac{\omega_k t}{\hbar} \right) \left( \frac{1}{2} \Delta y_0^* \right)^2 + \omega_k \Delta y_0^* \Delta y_0^*}$$

The dependence of $\Delta Y(t)$ on $\Delta X(s)$ can be written

$$\Delta Y^*(t) = \int_{t_i}^{t_f} R_k(t, s) C_{CB}(s) \Delta X(s) ds$$

where $R_k = \frac{1}{\omega_k} \sin \omega_k (s-t) \theta(s-t)$ is the response function of a harmonic oscillator. Inserting this in the imaginary term in the exponent it becomes a double integral

$$S^{k \rightarrow C} = \int_{t_i}^{t_f} \int_{t_i}^{t_f} \frac{C_{CB}(t) C_{CB}(s)}{2} \bar{X}(t) R_k(t, s) \Delta X(s) dt ds$$

which is the standard form of the real part of the Feynman-Vernon action from one bath oscillator (see below).

Now we consider the system-cavity-bath situation and assume that the cavity starts in equilibrium with respect to $\hat{H}_C$. The influence of all the bath oscillators on the cavity is expressed as the real and imaginary parts of the bath-cavity Feynman-Vernon action

$$S^{B \rightarrow C}_r = \frac{i}{\hbar} \int_{t_i}^{t_f} \int_{t_i}^{t_f} k^{B \rightarrow C}(t, s) \Delta X(t) \bar{X}(s) ds dt \quad (7)$$

$$S^{B \rightarrow C}_i = -\frac{1}{2\hbar} \int_{t_i}^{t_f} \int_{t_i}^{t_f} k^{B \rightarrow C}(t, s) \Delta X(t) \Delta X(s) ds dt \quad (8)$$

$$k^{B \rightarrow C}_r = \sum_k \frac{C_{CB}(t) C_{CB}(s)}{2 \omega_k} \cos \omega_k (t-s) \coth \frac{\omega_k \hbar \beta}{2}$$

The real action kernel $(S^{B \rightarrow C}_r)$ has been derived above and the imaginary action kernel $(S^{B \rightarrow C}_i)$ is derived in Appendix A. We are here interested in the real action, as only this one depends on $\bar{X}$.

The problem of integrating the bath and the cavity is solved by a new auxiliary function which satisfies

$$\Delta \bar{X}^* + \omega_k^2 \Delta X^* = C_{SC}(t) \Delta Q + 2 \int_{t_i}^{t_f} k^{B \rightarrow C}(s, t) \Delta X^*(s) \quad (9)$$

$$\Delta X^*_r = \Delta X^*_i = 0 \quad (10)$$

In above $\Delta Q$ is the difference of the forward and backward paths of the system, and $C_{SC}(t)$ is the system-cavity coupling. The equation is again that of a harmonic oscillator with an external drive starting from rest at the final time $t_f$, and evolving backwards in time to the initial time $t_i$. However, there is now also a damping term which contains the effects of the bath. It is well known that for an Ohmic bath (Caldeira-Leggett model), the kernel $k^{B \rightarrow C}(s, t)$ is $-\eta \bar{s}(t-s)$ where $\eta$ is a classical friction coefficient; the damping integral is then an ordinary friction term $-\eta \Delta X^*$. The example is treated in Section IV.

Whatever the influence from the bath, the value of $\Delta X^*$ at time $t$ depends on the values of $\Delta Q$ at times $s$ larger than $t$. We can write that as

$$\Delta X^*(t) = \int_{t_i}^{t_f} R_C(t, s) C_{SC}(s) \Delta Q(s) ds \quad (11)$$

where $R_C(t, s)$ is the response function of the damped harmonic oscillator describing the cavity. The real Feynman-Vernon action of the cavity and the bath on the system is hence expressed as as double integral analogous to (6):

$$S^{C \rightarrow S}_C = \int_{t_i}^{t_f} \int_{t_i}^{t_f} \frac{C_{SC}(t) C_{SC}(s)}{2} \bar{Q}(t) R_C(t, s) \Delta Q(s) dt ds$$

We are now in a position to state the Vernon transform. For clarity we do it first for the general case and then (below) under simplifying assumptions and in the Fourier domain.

**Definition 1** The Vernon transform $\mathcal{V}$ of a Feynman-Vernon kernel $k^{B \rightarrow C}$ on another Feynman-Vernon kernel $k^{C \rightarrow S}$ is given by

$$k^{C \rightarrow S}_r(t, s) = \mathcal{V} \left[ k^{B \rightarrow C} \right] = \frac{C_{SC}(t) C_{SC}(s)}{2} R_C(t, s) \quad (12)$$

where the cavity response function is defined by (10) and the auxiliary function $\Delta X^*$ satisfies (9).

The Vernon transform is especially convenient on the Fourier side, and when assuming that all interactions are time-independent and the process goes on for all time. There are physical issues with such a model which we discuss in Appendix D but in this paper we will mostly leave
these aside. The Feynman-Vernon kernel $k^{B \to C}_{i}(t, s)$ then only depends on the time difference $\tau = s - t$ and is represented by its Fourier transform

$$\hat{k}^{B \to C}_{i}(\nu) = \int_{-\infty}^{\infty} e^{i\nu \tau} k^{B \to C}_{i}(\tau) d\tau$$  \hspace{1cm} (13)$$

The auxiliary function $\Delta X^*$ is hence Fourier domain given by

$$\Delta X^*(\nu) = \frac{C_{SC} \Delta Q(\nu)}{-\nu^2 + \omega_C^2 - 2k^{B \to C}_{i}(-\nu)}$$  \hspace{1cm} (14)$$

where the denominator is the response function $\hat{R}_C$ in the Fourier domain. From this we have

**Definition 2** The Vernon transform $V^{\infty}$ on an infinite time interval of a Feynman-Vernon kernel $k^{B \to C}$ which only depends on the time difference and where all interactions are time-independent is in the Fourier domain given by

$$\hat{k}^{C \to S}_{C}(\nu) = \frac{1}{2} C_{SC} k^{B \to C}_{i}(-\nu) \hat{R}_C(\nu)$$  \hspace{1cm} (15)$$

The Vernon transform is non-linear. For this reason we separate it from the analogous mapping of the real domain. From assumptions of Definition 2 and excepting boundary terms $a$ only depends on the time difference and where all interactions are time-independent and in the Fourier domain given by

$$\hat{k}^{C \to S}_{C}(\nu) = \frac{1}{2} C_{SC} k^{B \to C}_{i}(-\nu) \hat{R}_C(\nu)$$  \hspace{1cm} (16)$$

where $\hat{R}(\nu)$ is the Fourier transform of the response function. The real Vernon transform $W$ hence depends quadratically on $R_C$.

**III. THE VERNON TRANSFORM IN QUANTUM THERMODYNAMICS**

In the previous Section we showed how the system state is influenced by a bath when the interaction between them is mediated via the cavity. In this Section we are interested in thermodynamics of such setups. The central quantity of interest is heat flow through a system between reservoirs. The set-up in this section will hence be that of a system coupled to two baths with one cavity between on each side.

Conceptually we are faced with the question what is heat. A natural definition would be to say that heat on side 1(2) is the energy change in bath B1(B2), while energy change in cavity C1(C2) counts as a kind of interaction energy which is not necessarily lost to the system, but which can be returned and do useful work. The generating function of such a bath energy change is analogous to the generating function of a system interacting directly a bath which has been discussed multiple times e.g. \[7–9, 31, 45–48\], and other papers cited therein. For the simpler one-sided case it reads

$$G_{i}(\nu) = \text{Tr}_{CB} (f | e^{i\nu H_B} U(t) e^{-i\nu H_B} | f)$$  \hspace{1cm} (17)$$

where $| i \rangle (| f \rangle)$ denotes the initial (final) state of the system, $\rho_C$ and $\rho_{B}$ are initial states of the cavity and the bath (each one), respectively, and $U(t)$ is the evolution of the system, cavity, and the bath generated by the Hamiltonian $H$. It turns out that while moments of this quantity can be computed, see Appendix C, the resulting expressions in the system variables, after integrating out the cavity, contain several terms, and are not very transparent. The full generating function (C) can also not be simply expressed in the Vernon transform, which is the theme of this paper.

In the following we have instead taken the route of defining heat given to side 1(2) to be the total energy change of the cavity and the bath on side 1(2). In the long time limit, and barring the case where energy is built up in the cavity this should to leading order give the same behaviour as counting heat in the bath only. The generating functions we will consider are thus

$$G_{i}(\nu_1, \nu_2) = \text{Tr}_{CB} (f | e^{i\nu_1 (H_{B1}+H_{C1})+i\nu_2 (H_{B2}+H_{C2})} U(t) e^{-i\nu_1 (H_{B1}+H_{C1})-i\nu_2 (H_{B2}+H_{C2})}$$

$$\times (\rho_C \otimes \rho_{B} \otimes | i \rangle \langle i |) U^\dagger(t) | f \rangle)$$  \hspace{1cm} (18)$$

where $\text{Tr}_{CB}$ stands for the trace of both baths and cavities, $\rho_C \otimes \rho_{B}$ stand for the initial product state of both
baths and cavities, and where we have assumed that the cavity-bath interaction vanishes at the beginning and the end of the process. From general results obtained in \[9\] it follows that the generating function (18) is given by a modified Feynman-Vernon action

\[ G_{\text{f}}(\kappa_1, \kappa_2) = \int_{t_f}^{t_i} DQDQ' e^{i \int_{t_i}^{t_f} dt (S_Q - S_Q')} \rho_S(Q, Q') \]

where the only change in is the substitution of the Feynman-Vernon functional \( \mathcal{F}_{\kappa_1}^{C \rightarrow S}(Q, Q') \) (either side) with

\[ \mathcal{F}_{\kappa_1}^{C \rightarrow S}(Q, Q') = e^{i S_{\kappa_1}^{C \rightarrow S}(Q, Q') - i S_{\kappa_1}^{C \rightarrow S}(Q, Q')} \]

For simplicity we will in the following drop \( \kappa_2 \) and only consider the generating function of energy changes on side 1 only. The imaginary and real part of the modified Feynman-Vernon cavity-to-system action depend on \( \kappa \). For the case when the bath-cavity and cavity-system coupling constants coefficient they read

\[ S_{\text{C}, \kappa}^{C \rightarrow S}(Q, Q') = \int_{t_1}^{t_f} dt \int_{t_1}^{t'_f} ds \left[ k_{\text{C}}^{C \rightarrow S}(t - s)(Q(t)Q(s) - Q'(t)Q'(s)) + k_{\text{C}}^{C \rightarrow S}(t - s + \kappa)(Q(t)Q'(s) - Q'(t)Q'(s)) \right] \]

with the same kernels as in \[19\]. In the setting of the bath interacting with the cavity the form of the new kernels is still that of a modification of \[19\] when the coupling constants are time dependent, but the time shifts then only pertain to the arguments of the sines and the cosines see e.g. \[9\].

The first moment of the energy change follows from the generating function by differentiating with respect to parameter \( \nu \) and then setting it to zero, which gives

\[ \frac{d}{d(i\kappa)} \mathcal{F}_{\kappa}^{C \rightarrow S}(Q, Q') \bigg|_{\kappa=0} = \mathcal{F}_{\kappa}^{C \rightarrow S}(Q, Q') \int_{t_f}^{t_i} dt ds \left[ (Q(t)Q'(s) + Q(s)Q'(t)) \mathcal{I}_{C \rightarrow S}(t - s) \right. \]

\[ \left. (Q(t)Q'(s) - Q(s)Q'(t)) \mathcal{J}_{C \rightarrow S}(t - s) \right] \]

It is convenient to extend these kernels, analogous to the ones introduced in \[20\] for the situation where the system interacts directly with the bath(s), to negative time arguments. \( \mathcal{I}_{\kappa}^{C \rightarrow S} \) is then an even function and \( \mathcal{J}_{\kappa}^{C \rightarrow S} \) is odd.

## IV. A CALDEIRA-LEGGETT BATH SEEN THROUGH A CAVITY

Many approaches have been developed to deal with open quantum system dynamics, numerically and analytically, and several of them can be adapted to to quite complex interactions. Here we focus instead on what is special when inserting a single oscillator mode between a system and a real bath. We therefore consider what is arguably the simplest but still realistic and interesting setting, which is a system interacting via cavities with two Caldeira-Leggett baths. In this section we will thus assume that in both baths the oscillators form a continuous spectrum with frequency-dependent bath-cavity interaction coefficients \( C_{\omega} \) and spectral density \( f(\omega) \) such that

\[ f(\omega)C_{\omega}^2 = \begin{cases} \frac{2\eta^2}{\pi} & \text{if } \omega < \Omega \\ 0 & \text{if } \omega > \Omega \end{cases} \]

The equation for the auxiliary function \( \Delta X^* \), in one of the cavities, is now that of a driven damped oscillator

\[ \Delta \ddot{x}^* + \omega_0^2 \Delta X^* = C_{SC}(t)\Delta Q - \eta \Delta \dot{x}^* \]

\[ \Delta x^*_f = \Delta x^*_0 = 0 \]

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\[ \Delta x^*_f = \Delta x^*_0 = 0 \]

For time-independent interactions that go on for all time, \( \Delta X^* \) is hence a filtered version of \( \Delta Q \) i.e.

\[ \Delta \ddot{x}^* = \frac{C_{SC}(\nu)}{-\nu^2 + \omega_0^2 + i\eta} \]

The denominator is the response function in the Fourier domain. The simple formulae \[23\] or \[24\] can be inserted.
directly in (15) and (16) to give
\[ \tilde{k}_{i}^{C \rightarrow S}(\nu) = \frac{1}{2} \frac{C_{SC}^{2}}{-\nu^{2} + \omega_{C}^{2} + i\eta \nu} \]
\[ \tilde{k}_{r}^{C \rightarrow S}(\nu) \approx \frac{\eta \nu}{2} \frac{C_{SC}^{2} \coth \left( \frac{\nu \eta h}{2} \right)}{(-\nu^{2} + \omega_{C}^{2})^{2} + \eta^{2} \nu^{2}} \]
(29)
(30)

The approximation symbol in the second equation is to emphasize that boundary terms which are in principle also present, have been neglected. Obviously both \( \tilde{k}_{i}^{C \rightarrow S} \) and \( \tilde{k}_{r}^{B \rightarrow C(S)} \) are largest at frequencies \( \nu \approx \omega_{C} \) and go down away from this resonance. This means out of all the possible frequencies of the motion of the system, it is only those around the cavity frequency which feel the bath strongly, all the others only experience the bath indirectly.

We now turn to thermodynamics. As shown in Section III the expected energy change in the cavity and the bath on side 1, in a process that goes on for a long time, is

\[ \langle \Delta E_{B1} + \Delta E_{C1} \rangle_{if} = \int_{i} dQ dQ' e^{i \tau} \left( \mathcal{I}^{S}(Q)[\mathcal{I}^{S}(Q')] + \mathcal{J}^{C \rightarrow S}(Q') \mathcal{J}^{C \rightarrow S}(Q) \right) \]
\[ \left. \right|_{t_{f}}^{t_{i}} \right]_{t_{i}}^{t_{f}} dtds \left[ Q(t)Q'(s) \left( \mathcal{I}^{C \rightarrow S}(t-s) + \mathcal{J}^{C \rightarrow S}(t-s) \right) \right], \]
(31)

where the two kernels in the second line are the derivatives of the of the Feynman-Vernon kernels on side 1 with respect to the time argument. When the bath is as Caldeira-Leggett model, on the Fourier side this means
\[ \hat{I}^{C \rightarrow S}(\nu) = i \frac{\nu}{\Delta} \frac{C_{SC}^{2}}{(-\nu^{2} + \omega_{C}^{2})^{2} + \eta^{2} \nu^{2}} \]
\[ \hat{J}^{C \rightarrow S}(\nu) = \frac{\eta \nu}{2} \frac{C_{SC}^{2} \coth \left( \frac{\nu \eta h}{2} \right)}{(-\nu^{2} + \omega_{C}^{2})^{2} + \eta^{2} \nu^{2}} \]
(32)
(33)
The long first line in (31) expresses a quantum expectation value of the system process. This will depend on the nature of the system and its own dynamics as much as how it interacts with the cavities and through them with the baths. However, if the system reaches a stationary state \( \dot{\hat{\rho}}_{S}^{Stat} \) then we can take that as initial state at \( t'_{i} \) sufficiently before \( t \) and \( s \), and integrate out the process from \( t_{i} \) to \( t_{f} \). There is then no memory of the initial system state \( i \). Likewise we can integrate out the process from the largest of \( t \) and \( s \) to \( t'_{f} \) such that at the end we only find the probability to observe the system in state \( f \) at the final time, \( \langle f | \dot{\hat{\rho}}_{S}^{Stat} | f \rangle \). The path integrals in (31) are hence averages in stationary state of the super-operators which the path integral variables \( Q(t) \) and \( Q'(s) \) represent: \( Q(t) \) means acting with the operator \( \hat{Q}(t) \) from the left, and \( Q'(s) \) means acting with the operator \( \hat{Q}(s) \) from the right. If the final state \( f \) is summed over we can write the averages as an open system correlation function
\[ C_{Q}(\tau) = \text{Tr} \left[ \langle \hat{Q}(\tau) \dot{\hat{\rho}}_{S}^{Stat} \hat{Q}(0) \rangle \right] \]
(34)
where \( \tau = t - s \), \( \dot{\hat{\rho}}_{S}^{Stat} \) is inserted long before \( t \) and \( s \), the process is evolved inserting whichever comes first of \( \hat{Q}(0) \) and \( \hat{Q}(\tau) \), and then the other operator is inserted and the trace is taken at the later time. By the cyclic property of the trace the later of the two operators can be moved over to the other side, that is \( \text{Tr} \left[ \langle \hat{Q}(0) \hat{Q}(\tau) \rangle \dot{\hat{\rho}}_{S}^{Stat} \right] \) if \( \tau \leq 0 \), and \( \text{Tr} \left[ \langle \dot{\hat{\rho}}_{S}^{Stat} \hat{Q}(0) \hat{Q}(\tau) \rangle \right] \) if \( \tau \geq 0 \).

The analytical/numerical determination of quantities as in (34) is not a trivial task even in simple open quantum systems, for recent investigations using different methods, see e.g. [49–51]. Theoretically such quantities seem to have been first considered by Lindblad in [52], and more recently by several groups [53–54]. The open systems aspect is hence not here the crucial one: the central problem is given operational meaning to operators inserted at different times on the left and on the right of a density matrix. One approach is to use pre- and post-selection, following [52], and more recently [55]. The proposal in [54] is that auxiliary systems can be coupled to the system at different times after which (34) can be realized as joint measurements of the auxiliary systems at the end of the process, and after tracing out the system.

When the process goes on for a long time the expected energy change in the cavity-and-bath per unit time is quantum thermal power, which can hence be written
\[ \Pi = \int C_{Q}(\tau) \left( \mathcal{I}^{C \rightarrow S}(\tau) + \mathcal{J}^{C \rightarrow S}(\tau) \right) d\tau \]
\[ = \int \frac{1}{2} \left( C_{Q}(\tau) + C_{Q}(-\tau) \right) \mathcal{I}^{C \rightarrow S}(\tau) d\tau \]
\[ + \int \Theta(\tau) (C_{Q}(\tau) - C_{Q}(-\tau)) \mathcal{J}^{C \rightarrow S}(\tau) d\tau \]
(35)

where we have used that \( \mathcal{I}^{C \rightarrow S} \) is even and \( \mathcal{J}^{C \rightarrow S} \) is odd and rewritten the last line so that it looks as a response function. If both environments are at the same temperature the above simplifies considerably. For the
s and differences we can set

\[ S(\tau) = \text{Tr} \left[ \rho_{\hat{S}}^{\beta} \{ \hat{Q}(\tau), \hat{Q}(0) \} \right] \] (36)

\[ \chi(\tau) = \text{Tr} \left[ \rho_{\hat{S}}^{\beta} \Theta(\tau) \left[ \hat{Q}(\tau), \hat{Q}(0) \right] \right] \] (37)

the Fourier transforms of which are related by the fluctuation-dissipation theorem. Adding and subtracting term \( \text{thermal power} \) is then zero.

V. DISCUSSION

In this work we have considered the problem of estimating energy taken from or given to a system when it is connected to a bath through a cavity. The cavity is a harmonic oscillator degree of freedom interacting linearly with both the bath and the system. This setting describes a superconducting qubit connected to two normal-metal baths through transmission lines as investigated experimentally and theoretically in [36] and other publications from the same group. We have considered the system connected to one bath or to two baths, possibly at different temperatures.

The paper has two main points. The first is that the influence from a bath on a cavity is transformed in a systematic manner to an influence from the cavity to a bath. Although not a new result this is very useful. We have called this transformation the Vernon transform (Vernon 1959 PhD thesis from Caltech. On the Fourier side the transform is similar to a band-pass filter, such that the system mostly experiences the influences of the bath(s) at the resonance frequencies of the cavities. The form of this filter can be described in a precise manner both for dissipation and for quantum noise. We hope to have helped bring light again to this nice classical result. The second point is that one can combine the Vernon transform with recent results in quantum thermodynamics to get compact expressions for the generating function of quantum heat (energy change in the environment). To do so in a simple way one has to consider the bath and the cavity as one environment; the moments of energy changes in the bath only can also be determined by the same techniques, but involve considerably more complicated expressions.

As an example we have considered thermal power (expected energy change in the environment per unit time) when the bath is of the Caldeira-Leggett type. Thermal power is a convolution of a kernel describing the bath given by the Vernon transform, and an open system unequal-time correlation function. The latter is not straightforward to compute analytically – there is no free lunch – but could be determined experimentally using methods of modern quantum science, or estimated from numerical solutions of the open system dynamics. In any case, we hope to have shown that the problem of heat transfer through a system connected to two baths via two cavities admits a systematic theory for all strengths of the interactions.

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Appendix A: Vernon’s derivation of Feynman-Vernon action

In this section we give for completeness details omitted in Section [1]. This derivation Feynman-Vernon action kernels can be found in Appendix I of Vernon’s PhD thesis [40], as well as elsewhere in the later literature e.g. [42], though not in [1]. We start by only considering the cavity and the bath. The evolution operator corresponding to $\hat{H}_C + \hat{H}_B + \hat{H}_{BC}(t)$ acting on cavity-bath wave function can be written as a (multi-variable) path integral

$$\hat{U} = \int DXDY e^{\frac{\partial}{\partial X} (SC[X] + SB[Y] + SCB[X,Y])}$$

(A1)

The evolution operator acting on cavity-bath density matrices is similarly a double path integral over “forward paths” and a “backward paths”

$$\hat{U} \cdot \hat{U}^\dagger = \int DXDX’DYDY’ e^{\frac{\partial}{\partial X} (SC[X] + SB[Y] + SCB[X,Y])}$$

\[ \cdot e^{-\frac{\partial}{\partial X} (SC[X'] + SB[X'] + SCB[X',Y'])} \]

(A2)

where the slot marks where the initial density matrix is to be inserted. When the cavity and the bath are initially independent and the bath is initially in a thermal state, the path integrals for each mode of the bath can be done independently. When the final state of a bath oscillator is traced over, the outcome is the Feynman-Vernon influence function $\mathcal{F}^{k \to C}[X,X']$, where by $k \to C$ we here mean the influence of bath oscillator $k$ on the cavity.

Vernon’s approach to compute $\mathcal{F}^{k \to C}[X,X']$ starts by re-writing the double path integral in terms of of sums and differences of forward and backward paths i.e. in terms of $Y = Y + Y', \Delta Y = Y - Y'$.
\[ X = X + X' \text{ and } \Delta X = X - X'. \] The initial thermal state of bath oscillator \( k \) is then
\[ \rho_B^{(k)}(y, \Delta y) = \frac{1}{N} \exp \left( -\frac{w}{2\hbar} \left( \frac{\omega_k \beta}{2} + \Delta y^2 \coth \frac{\omega_k \beta}{2} \right) \right), \]
where \( N \) is a normalization. The actions over the paths are for the terms that involve the bath variable
\[ \frac{1}{2} \int_{t_i}^{t_f} \dot{Y} \Delta Y - \omega_k y \Delta Y + C_{CB}(t) (\dot{X} \Delta Y + \Delta X \dot{Y}) \, dt \]
where the first term can be integrated by parts to
\[ -\frac{1}{2} \int_{t_i}^{t_f} \dot{Y} \Delta Y \, dt + \frac{1}{2} (\bar{y} \Delta y_f - \bar{y} \Delta y_i). \]

Tracing over the final state of the bath oscillator means that \( \Delta y_f = \Delta Y(t_f) \) has to be zero, while \( \bar{y} = \bar{Y}(t_f) \) is integrated over. The terms linear in \( \bar{y} \) and \( \Delta y \) in the action over paths is then \( \frac{1}{2} \bar{y} \Delta y \) and
\[ \frac{1}{2} \int_{t_i}^{t_f} \dot{Y} \left( -\Delta Y - \omega_k^2 \Delta Y + C_{CB}(t) \Delta X \right) \, dt \]
(A3)
The initial condition and the integration over \( \bar{y} \) and \( \dot{y} \) means that path integral is non-zero only when \( \Delta Y \) agrees with a deterministic auxiliary function defined in the main text, and for convenience repeated here
\[ \Delta y^*_f = \Delta y^*_i = 0 \text{ and } \Delta Y^* = -\omega_k^2 \Delta Y^* + C_{CB}(t) \Delta X = 0 \]
As discussed in main text this is the equation of a harmonic oscillator with an external drive starting from rest at \( t_f \) and the solution can be expressed as
\[ \Delta Y^*(t) = \int_{t_i}^{t_f} R_k(t, s) C_{CB}(s) \Delta X(s) \, ds \]
(A4)
where \( R_k \) is a response function.

On the other hand, \( \rho_B^{(k)} \) can be seen as \( \int_{t_i}^{t_f} \dot{Y} (L \Delta Y) \, dt \) where \( L \) is a linear operator acting on \( \Delta Y \), hence the functional integral also formally gives a factor \( \frac{1}{\sqrt{|\det L|}} \) in the ordinary path integral of the harmonic oscillator, and should therefore be interpreted as undetermined constant times \( \sin^{-1} \omega_k(t_f - t_i) \). The last integral over \( \bar{y} \) yields
\[ \exp \left( -\frac{1}{\omega_k \hbar} (\Delta y^*)^2 \coth \frac{\omega_k \beta}{2} \right) \] (cancelling the normalization factor \( N \) in above). What remains of the actions involving the sum of forward and backward paths \( \Delta y^*_f, \Delta y^*_i \) and \( \bar{y} \) is hence this term in \( \Delta y^* \), the other remaining term from \( \rho_B^{(k)} \) which is
\[ \exp \left( -\frac{w}{2\hbar} \left( (\Delta y^*)^2 \coth \frac{\omega_k \beta}{2} \right) \right), \]
and
\[ \text{Const.} \cdot \frac{1}{|\det L|} \cdot \frac{1}{\det L} \cdot e^{\frac{1}{2\hbar} \int_{t_i}^{t_f} C_{CB}(t) \Delta Y \, dt}. \]
(\delta(\Delta y_f - \Delta y^*_f) \delta(\Delta y_f - \Delta y^*_f) \delta(\Delta Y - \Delta Y^*) \) (A5)
We can trade the integral over \( \Delta y \) against an integral over \( \Delta y_f \) at the price of Jacobian which cancels \( \frac{1}{\det L} \). Hence the Feynman-Vernon action from the bath on the cavity can be expressed as
\[ \mathcal{F}^{k \rightarrow C} = e^{\frac{1}{2h} \int_{t_i}^{t_f} C_{CB}(t) \Delta Y^* \, dt} e^{-\frac{1}{2h} \coth \frac{\omega_k \beta}{2} \left( (\Delta y^*)^2 + \omega_k (\Delta y^*)^2 \right)} \]
(A6)
The imaginary term in the exponent in above (real part of Feynman-Vernon action) is discussed in the main text. Here we will continue on the real terms in the exponent (imaginary part of Feynman-Vernon action).

Using (A4) and \( R_k = \frac{1}{\omega_k} \sin \omega_k(s - t) \) we have
\[ \Delta y^* = \int_{t_i}^{t_f} \frac{1}{\omega_k} \sin \omega_k(t - t_i) C_{CB}(t) \Delta X(t) \, dt \]
(A7)
and hence
\[ \left( \frac{1}{\omega_k^2} \Delta y^* \right)^2 + (\Delta y^*)^2 = \int_{t_i}^{t_f} \Delta X(t) \Delta X(t) \left( \cos \omega_k(t - t_i) \cos \omega_k(s - t_i) + \sin \omega_k(t - t_i) \sin \omega_k(s - t_i) \right) C_{CB}(t) C_{CB}(s) \, dt \, ds \]
(A8)
where the inner parenthesis is \( \cos \omega_k(t - s) \). This hence gives the imaginary part of the Feynman-Vernon action as in the main text (8).

**Appendix B: The real part of the Vernon transform**

In this appendix we show the real parts of the Vernon transform, found in Appendix 5 of Vernon’s PhD thesis[10].

From section 11 we know how the Vernon transform transforms the Feynman-Vernon kernel \( k^{B \rightarrow C} \) on the Feynman-Vernon kernel \( k^{C \rightarrow S} \). Collecting the various terms of that derivation we will now describe the real Vernon transform \( \mathcal{W} \) which maps \( k^{B \rightarrow C} \) on \( k^{C \rightarrow S} \). This will depend on \( k^{C \rightarrow S} \), hence \( \mathcal{W} \) depends on \( \mathcal{V} \).

The first term to consider is the imaginary part of the Feynman-Vernon action of the bath on the cavity, given as (8) in the main text, but expressing the cavity variable in the system variable through (11). This will be
\[ S_r^{C \rightarrow C} = \left. -\frac{1}{2\hbar} \int_{t_i}^{t_f} k^{B \rightarrow C}(t, s) \left( \int_{t_i}^{t_f} R_C(t, t') C_{SC}(t') \Delta Q(t') \cdot \int_{s}^{t_f} R_C(s, s') C_{SC}(s') \Delta Q(s') \right) \, ds \, dt \right| \]
(B1)
which can be re-written as
\[ S_r^{C \rightarrow S} = \left. -\frac{1}{2\hbar} \int_{t_i}^{t_f} k^{C \rightarrow S}(t', s') \Delta Q(t') \Delta Q(s') \, ds' \, dt' \right| \]
(B2)
with the new combined kernel

\[ k_{r}^{C \rightarrow S}(t', s') = \int_{t_{i}}^{s'} \int_{t_{i}}^{t'} k_{r}^{B \rightarrow C}(t, s) C_{SC}(t') C_{SC}(s') R_{C}(t', t) R_{C}(t, s') \, ds \, dt \]  

(B3)

The response functions can then further be expressed in terms the kernel \( k_{i}^{C \rightarrow S} \) through (12) so that we have also

\[ k_{i}^{C \rightarrow S}(t', s') = \int_{t_{i}}^{s'} \int_{t_{i}}^{t'} k_{i}^{C \rightarrow S}(t, s') C_{SC}(t') C_{SC}(s) \frac{R_{C}(t', t) R_{C}(t, t')}{k_{r}^{B \rightarrow C}(t, s) ds \, dt} \]  

(B4)

Note that \( k_{i}^{C \rightarrow S} \) depends on two interaction coefficients, hence the fraction in above is in total quadratic in in the cavity-system interaction. The above is in fact the only term proportional to the total duration of the process, and the only one considered in the main body of the paper.

The second term to consider is a boundary term, the analogy of the real exponent in (A6), but for the cavity. This will give

\[ S_{r}^{C \rightarrow S(S)} = -\frac{1}{4\hbar} \coth \frac{\omega_{C}\beta\hbar}{2} \left( \frac{1}{\omega_{C}}(\Delta \dot{x})^2 + \omega_{C}(\Delta x)^2 \right) \]  

where again the cavity variable should be expressed in terms of the system variable through (10). We write this as

\[ \Delta x^* = \int_{t_{i}}^{t_{f}} R_{C}(t_{i}, t) C_{SC}(t) \Delta Q(t) \, dt \]  

(B5)

\[ \Delta \dot{x}^* = \int_{t_{i}}^{t_{f}} \dot{R}_{C}(t_{i}, t) C_{SC}(t) \Delta Q(t) \, dt \]  

(B6)

where \( \dot{R}_{C}(t_{i}, t) = \frac{d}{ds} R_{C}(s, t)|_{s=t_{i}} \). This then gives

\[ S_{r}^{C \rightarrow S(S)} = -\frac{1}{2\hbar} \int_{t_{i}}^{t_{f}} \left( k_{r}^{C \rightarrow S(S, 1)} + k_{r}^{C \rightarrow S(S, 2)} \right) \Delta Q(t') \Delta Q(s') ds' \, dt' \]  

(B7)

with two kernels

\[ k_{r}^{C \rightarrow S(S, 1)}(t', s') = \frac{\omega_{C}}{2} \coth \frac{\omega_{C}\beta\hbar}{2} C_{SC}(t') C_{SC}(s') \frac{R_{C}(t_{i}, t') R_{C}(t', s')}{k_{r}^{B \rightarrow C}(t, s)} \]  

(B8)

\[ k_{r}^{C \rightarrow S(S, 2)}(t', s') = \frac{1}{2\omega_{C}} \coth \frac{\omega_{C}\beta\hbar}{2} C_{SC}(t') C_{SC}(s') \dot{R}_{C}(t_{i}, t') \dot{R}_{C}(t', s') \]  

(B9)

We summarize this as

**Definition 4** The real Vernon transform \( \mathcal{W} \) is a linear mapping of the Feynman-Vernon kernel \( k_{r}^{B \rightarrow C} \) on the Feynman-Vernon kernel \( k_{r}^{C \rightarrow S} \) given by the the integral transform (B3) which is proportional to \( k_{r}^{B \rightarrow C} \), and two boundary terms (B8) and (B9) which do not depend on \( k_{r}^{B \rightarrow C} \).

We end this appendix by discussing simplifications of (B3). If the interaction is time-independent for all time and \( k_{r}^{B \rightarrow C} \) and the response function \( R_{C} \) only depend on the time differences, \( k_{r}^{C \rightarrow S} \) will also only depend on the time difference, and we have

\[ k_{r}^{C \rightarrow S}(t) = \int_{t}^{0} R_{C}(t') R_{C}(t - s) ds \, dt \]  

(B10)

On the Fourier side that gives

\[ S_{r}^{B \rightarrow C(S)} = -\frac{1}{2\hbar} \int k_{r}^{B \rightarrow C(S)}(\nu) \Delta \hat{Q}(\nu) \Delta \hat{Q}(\nu) \, d\nu \]  

(B11)

with

\[ k_{r}^{B \rightarrow C(S)}(\nu) = \hat{R}_{C}(\nu) \hat{R}_{C}(\nu) k_{r}^{B \rightarrow C}(\nu) \]  

(B12)

In other words, on the Fourier side this transformed imaginary-action Feynman-Vernon kernel from the cavity to the system is proportional to the one from the bath to the cavity at minus the frequency, the proportionality being the spectral energy of the response function.

The two terms (B8), and (B9) stem from the parenthesis in (B5), which is like a final energy for the auxiliary process. If there is no damping this energy could grow indefinitely, but with damping it will remain finite. These two terms hence gives a contribution which does not depend on the duration of the process, and can hence be ignored when considering quantities per unit time such as quantum power.

**Appendix C: Vernon transforms for the energy changes in a bath alone**

We will show how the Vernon transform of Section (11) can be used to compute moments of generating functions of energy changes in the bath, in this context also called quantum heat. For simplicity we focus on the case in which the system is coupled via the cavity to only one bath. Generating functions for energy changes in multiple baths have been considered several times, e.g. recently in [8]. For the simpler one-sided case it reads

\[ G_{i f}(\nu) = \text{Tr}_{CB} \langle \langle |e^{i \nu H_{B} U(t)} | e^{-i \nu H_{B} U(t)} | \otimes (\rho_{C} \otimes | i \rangle \langle i | \otimes U_{i}(t) | f \rangle \rangle \rangle \]  

(C1)

where \( | i \rangle \langle f \rangle \) denotes the initial (final) state of the system, \( \rho_{C} \) and \( \rho_{B} \) are initial states of the cavity and the bath (one each), respectively, and \( U(t) \) is the evolution of the system, cavity, and the bath generated by the Hamiltonian (1).

Moments of the energy change are generated by taking derivatives of the generating function with respect to \( \nu \), e.g. the change of average energy of the bath is given by the first derivative

\[ \langle \Delta E_{B} \rangle = \frac{d}{d(\nu)} G_{i f}(\nu) \bigg|_{\nu=0} . \]  

(C2)
The path integral formulation of generating functions and moments of heat and work has been studied in [9].

The most important change with respect to the setting of Section II is that the path integral formulation of the generating function involves modified Feynman-Vernon kernels $k^{B\to C}_i, k^{B\to C}_r$. Following the notation of the previous Section one has

$$G_{ij}(\nu) = \int_{if} DQDQ' e^{i\int_{if} dt (S_{ij}[Q] - S_{ij}[Q'])} \rho_S(Q, Q')$$

$$\int DXDX' e^{i\int_{if} dt (S_{ij}[Q] - S_{ij}[Q'])} e^{i\int_{if} dt (S_{ij}[Q] - S_{ij}[Q'])} \rho_{BC}(X, X'). \ (C3)$$

where the only change in is the substitution of the Feynman-Vernon functional $F^{B\to C}[X, X']$ with

$$F^{B\to C}_\nu[X, X'] = e^{iS^B_{\nu}[X, X'] - iS^B_{\nu}[X, X']} \ (C4)$$

The imaginary and real part of this modified Feynman-Vernon action depend on $\nu$. For the case when the bath-cavity coupling coefficients are constant they read

$$S^B_{i,\nu}[X, X'] = \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} ds \left[ k^B_i(t - s)(X(t)X(s) - X'(t)X'(s) + k^B_i(t - s + \nu)X(t)X'(s) - k^B_i(t - s - \nu)X(s)X'(t) \right] \ (C5)$$

$$S^B_{r,\nu}[X, X'] = \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} ds \left[ k^B_r(t - s)(X(t)X(s) + X'(t)X'(s) - k^B_r(t - s + \nu)X(t)X'(s) - k^B_r(t - s - \nu)X(s)X'(t) \right]. \ (C6)$$

with the same kernels as in (73). When the coupling constants are time dependent, the form of the new kernels is still that of a modification of (73) but with time shifts only to the arguments of the sine and the cosine respectively; see e.g. [9].

The task is now to integrate out the cavity degrees of freedom with the modified Feynman-Vernon actions from the bath on the cavity. This cannot be done with the Vernon transform directly, because in terms of the sums and differences of the forward and backward cavity paths ($\bar{X}$ and $\Delta X$), the modified Feynman-Vernon action is quadratic in $\bar{X}$. This is in contrast to the situation in Section II where it was only linear in $\bar{X}$, and where the path integral therefore gave a functional delta. Although the full generation function could also be addressed, here we will focus on the moments of the energy changes where this issue does not appear.

For the first moment, computed by the formula (C2) and expressed separately in $\bar{X}$ and $\Delta X$ we have three terms

$$\frac{d}{d(t)} F^{\nu \to S}[\bar{Q}, \Delta Q] \bigg|_{\nu=0} = \int DXDX' \int_{t_i}^{t_f} dt ds \left[ \Delta X(t)\Delta X(s) I^{B\to C}(t, s) + \bar{X}(t)\Delta X(s) J^{B\to C}(t, s) + \bar{X}(t)\bar{X}(s) \right] F^{\nu \to C}[\bar{X}, \Delta X] \rho_C(\bar{x}, \Delta x_i). \ (C7)$$

with two kernels

$$I^{B\to C}(t, s) \equiv \frac{1}{4} \sum_k C_{CB}(t) C_{CB}(s) \cos[\omega_k(t - s)] \ (C8)$$

$$J^{B\to C}(t, s) \equiv \frac{i}{2} \sum_k C_{CB}(t) C_{CB}(s) \sin[\omega_k(t - s)] \coth \frac{\omega_k B}{2} \ (C9)$$

In above we have for completeness reintroduced the time dependence of the interaction coefficients. The modified Feynman-Vernon kernels $\mathcal{I}$ and $\mathcal{J}$ were derived in [20].

The $\Delta X$ terms in (C7) will by the path integral over $\bar{X}$ be transformed into the auxiliary function $\Delta X^*$ which by [9] and [10] is a linear functional of the system difference $\Delta Q$. The $\bar{X}$ terms in (C7) can similarly be expressed as functional derivatives at zero with respect to a new auxiliary field $L(t)$, by adding a new coupling term $\frac{1}{2} \int_{t_i}^{t_f} dt' L(t') \bar{X}(t')$ to the action. The path integral with respect to $\bar{X}$ then gives a new auxiliary function $\Delta X^*_L$ which is a linear functional of both $\Delta Q$ and $L$.

The structure of the expected heat functional in terms of the system variables will hence be

$$\frac{d}{d(t)} G_{ij}(\nu) \bigg|_{\nu=0} = \int_{if} DQDQ' e^{i\int_{if} dt (S_{ij}[Q] - S_{ij}[Q'])}$$

$$\cdot F^{C\to S}[\bar{Q}, \Delta Q] \cdot \int_{t_i}^{t_f} dt' \left[ \Delta Q(t)\Delta Q(s) \mathcal{L}^{C\to S}_{\Delta Q, \Delta Q}(t, s) + \mathcal{Q}(t)\Delta Q(s) \mathcal{L}^{C\to S}_{\Delta Q, \Delta Q}(t, s) + \mathcal{Q}(t)\mathcal{Q}(s) \mathcal{L}^{C\to S}_{\Delta Q, \Delta Q}(t, s) \right] ds \ dt \ (C10)$$

The only qualitative difference to what the situation would be if the system would only interact with the cavity (which then would take the role of the bath) are that the kernels multiplying $\Delta Q(t)\Delta Q(s)$, $\bar{Q}(t)\Delta Q(s)$ and $\bar{Q}(t)\mathcal{Q}(s)$, which are

$$\mathcal{L}^{C\to S}_{\Delta Q, \Delta Q}(t, s) = \mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}(t, s) + \mathcal{J}^{C\to S}_{\Delta Q, \Delta Q}(t, s) + \mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}(t, s) \ (C11)$$

where the kernels $\mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}$, $\mathcal{J}^{C\to S}_{\Delta Q, \Delta Q}$, and $\mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}$ are given by Eqs (C16), (C25), and (C32). The second kernel

$$\mathcal{L}^{C\to S}_{\Delta Q, \Delta Q}(t, s) = \mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}(t, s) + \mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}(t, s) \ (C12)$$

where the kernels $\mathcal{J}^{C\to S}_{\Delta Q, \Delta Q}$ and $\mathcal{I}^{C\to S}_{\Delta Q, \Delta Q}$ are given by Eqs (C23), and (C30).
The last kernel $\mathcal{I}_{Q, \overline{Q}}^{C \to S}(t, s)$ is given by Eq. ([28]).

It is clear that in principle the above procedure can also be used to compute the higher moments of the generating function. It is also clear that the resulting expressions will be increasingly more complicated convolutions involving bath-to-cavity quantities and response functions.

1. Calculation details

Here we sketch the most important steps in performing the path integral over the cavity degree of freedom in Eq. ([17]). As that expression consist of three terms we will describe the computation term by term. The first term reads

$$
\int \mathcal{D}X \mathcal{D}X \int \int dtds \mathcal{I}^{B \to C}(t, s) \Delta X(t) \Delta X(s) \tag{13}
$$

$$
e^{\frac{i}{\hbar} \mathcal{S}_C[\bar{X}, \Delta X] + \frac{i}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) + \mathcal{C}(t) \bar{X}(t) \overline{X}(t) \times \frac{i}{\hbar} \int \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) \Delta X(s) + i \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X(s) \rho_C(\bar{X}, \Delta X)
$$

The path integrals over $\Delta X$, $\bar{X}$ can be easily done

$$
\int \int \int dtds \mathcal{Q}(t) \mathcal{Q}(s) \mathcal{I}_{\Delta Q, \overline{\Delta Q}}^{C \to S}(t, s) \tag{14}
$$

$$e^{\frac{\pi}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) - \frac{i}{\hbar} \int dtds \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)},
$$

what translates into

$$
\int \int \int dtds \mathcal{Q}(t) \mathcal{Q}(s) \mathcal{I}_{\Delta Q, \overline{\Delta Q}}^{C \to S}(t, s) \mathcal{I}^{B \to C}(t, s)
$$

$$e^{\frac{\pi}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) - \frac{i}{\hbar} \int dtds \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)},
$$

where

$$
\mathcal{I}_{\Delta Q, \overline{\Delta Q}}^{C \to S}(t, s) = \int \int \int \int dtds \mathcal{C}(s) \mathcal{C}(t) \mathcal{I}^{B \to C}(t, s) \mathcal{I}^{C \to B}(s, t).
$$

Now we move to the second term, which is

$$
\int \int \int \int dtds \mathcal{J}^{B \to C}(t, s) \bar{X}(t) \Delta X(s) \tag{17}
$$

$$e^{\frac{\pi}{\hbar} \mathcal{S}_C[\bar{X}, \Delta X] + \frac{i}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) + \mathcal{C}(t) \bar{X}(t) \overline{X}(t) \times \frac{i}{\hbar} \int \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) \Delta X(s) + i \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X(s) \rho_C(\bar{X}, \Delta X)
$$

We rewrite it using a functional derivative with respect to an auxiliary driving force $L(t)$ as

$$
\frac{\hbar}{1} \int \int \int \int dtds \mathcal{J}^{B \to C}(t, s) \Delta X(s) \frac{\delta}{\delta L(t)} \Delta X(s) \tag{18}
$$

$$e^{\frac{\pi}{\hbar} \mathcal{S}_C[\bar{X}, \Delta X] + \frac{i}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) + \mathcal{C}(t) \bar{X}(t) \overline{X}(t) \times \frac{i}{\hbar} \int \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) \Delta X(s) + i \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X(s) \rho_C(\bar{X}, \Delta X)
$$

Now we can perform the integral over cavity degrees of freedom

$$
\frac{\hbar}{1} \int \int \int \int dtds \mathcal{J}^{B \to C}(t, s) \frac{\delta}{\delta L(t)} \Delta X(s) \tag{19}
$$

$$e^{\frac{\pi}{\hbar} \mathcal{S}_s[\bar{Q}, \Delta Q] + \frac{i}{\hbar} \int dtds \mathcal{C}(t) \bar{Q}(t) \Delta X(t) \times \frac{i}{\hbar} \int \int dtds \Delta X^*(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)}
$$

Now we need to take the functional derivative with respect to the auxiliary field. It will result in three terms. The first contribution comes from the term multiplying the exponent

$$
\frac{\delta \Delta X^*(s)}{\delta L(t)} = R_C(s, t) \tag{20}
$$

We are going to neglect this term. The second term comes from the derivative of coupling term between the cavity and the system

$$
\mathcal{J}_{Q, \overline{Q}}^{C \to S}(t, s) = \frac{1}{2} C_{SC}(t) C_{SC}(s) \int \int dtds \mathcal{C}(t) \mathcal{C}(s) \mathcal{I}^{C \to B}(t, s) \mathcal{I}^{B \to C}(s, t) \frac{\delta}{\delta L(t)} \Delta X^*(s) \tag{22}
$$

$$e^{\frac{\pi}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) - \frac{i}{\hbar} \int dtds \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)}
$$

Moreover, the derivative of the term involving the real kernel will result in

$$
\int \int dtds \mathcal{Q}(t) \mathcal{Q}(s) \mathcal{J}_{Q, \overline{Q}}^{C \to S}(t, s) \mathcal{I}^{B \to C}(t, s) \tag{24}
$$

$$e^{\frac{\pi}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) - \frac{i}{\hbar} \int dtds \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)}
$$

where

$$
\mathcal{J}_{Q, \overline{Q}}^{C \to S}(t, s) = 2i C_{SC}(t) C_{SC}(s) \int \int \int dtds \mathcal{C}(t) \mathcal{C}(s) \mathcal{I}^{C \to B}(t, s) \mathcal{I}^{B \to C}(s, t)
$$

$$R_C(p, t) R_C(r, w) k_{\overline{B} \to C}(p-r) \mathcal{J}^{B \to C}(u, w) \tag{25}
$$

The third term involves the double functional derivative.

$$
\int \int dtds \mathcal{I}^{B \to C}(t, s) \frac{\delta^2}{\delta L(t) \delta L(s)} \tag{26}
$$

$$e^{\frac{\pi}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(s) \times \frac{i}{\hbar} \int dtds \Delta X^*(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)}
$$

It will result in three contributions. The first one is

$$
\int \int dtds \mathcal{Q}(t) \mathcal{Q}(s) \mathcal{I}_{\Delta Q, \overline{\Delta Q}}^{C \to S}(t, s) \tag{27}
$$

$$e^{\frac{\pi}{\hbar} \int dtds \mathcal{C}(t) \bar{X}(t) \Delta X(t) - \frac{i}{\hbar} \int dtds \Delta X(t) k_{\overline{B} \to C}(t-s) \Delta X^*(s)}
$$
where
\[ I_{Q,Q}^{t_1} = \frac{1}{4} C_{SC}(t) C_{SC}(s) \int_{t_1} I_{Q,Q}^{t_1} dt ds Q(t) \Delta Q(s) \]

The second one stems from the first derivative of the coupling term and the real kernel term
\[ \int_{t_1} I_{Q,Q}^{t_1} dt ds Q(t) \Delta Q(s) \]
and the second derivative with respect to the real kernel
\[ \int_{t_1} I_{Q,Q}^{t_1} dt ds Q(t) \Delta Q(s) \]
where
\[ I_{Q,Q}^{t_1} \Delta Q(t, s) = \int_{t_1} I_{Q,Q}^{t_1} dt ds Q(t) \Delta Q(s) \]

Hence, the Vernon transform has to be interpreted with care when the simplifying assumption is made that the Feynman-Vernon kernels \((k_{I}^{b+c}, k_{R}^{b+c})\) only depends on the second time argument. One approach is to reintroduce a time-dependent system-cavity coupling which vanishes in the beginning of the process, and which leads to Feynman-Vernon kernels that depend on both times. The output of the Vernon transform, i.e. the Feynman-Vernon kernels \((k_{I}^{b+c}, k_{R}^{b+c})\), will then also depend on both times. If now the process goes on for a long time and the bath-cavity coupling stays constant most of that time, the kernels \((k_{I}^{b+c}, k_{R}^{b+c})\) will approximately only depend on the second time argument when the bath-cavity is constant. This property is inherited by the output kernels. Focusing for simplicity on just the imaginary kernel we can thus compare two different Fourier transforms:
\[ \tilde{k}_{I}(t, \nu) = \int_{t}^{\infty} k_{I}(s, t) e^{i\nu s} ds \quad "true Fourier" \]
\[ = \int_{0}^{\infty} e^{i\nu t} k_{I}(t, \tau) e^{i\nu \tau} d\tau \quad (D1) \]
and
\[ \tilde{k}_{I}^{t}(\nu) = \int_{t}^{\infty} k_{I}^{t}(s, t) e^{i\nu s} ds \quad "truncated Fourier" \]
\[ = \int_{0}^{\infty} e^{i\nu t} k_{I}^{t}(\tau) e^{i\nu \tau} d\tau \quad (D2) \]
where we assume that \(k_{I}(t, s - t)\) only depends on the first argument for small and large \(t\), and \(k_{I}^{t}(s - t)\) is that \(t\)-independent function (independent of its first argument) extended to all \(t\). In both equations we have in the last equality used that \(k_{I}(t, s - t) = 0\) if \(s < t\). If the dependence of \(k_{I}(t, \tau)\) on its second argument is essentially finite range we have \(\tilde{k}_{I}(t, \nu) \approx \tilde{k}_{I}^{t}(\nu)\) except for small and large \(t\).