EXPONENTIAL STABILIZATION OF THE STOCHASTIC BURGERS EQUATION BY BOUNDARY PROPORTIONAL FEEDBACK

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Abstract. In the present paper it is designed a simple, finite-dimensional, linear deterministic stabilizing boundary feedback law for the stochastic Burgers equation with unbounded time-dependent coefficients. The stability of the system is guaranteed no matter how large the level of the noise is.

1. Presentation of the model. The subject of the present work is the 1−D stochastic Burgers equation with unbounded time-dependent coefficients, on the interval \([0, L]\)

\[
\begin{align*}
  dY(t, x) &= \nu \Delta Y(t, x) dt + b(t, x)Y(t, x)Y_x(t, x) dt + \theta Y(t, x) dW(t), \\
  Y(t, 0) &= \nu(t), \quad Y_x(t, L) = 0, \quad t > 0, \\
  Y(0, x) &= y_o(x), \quad x \in (0, L).
\end{align*}
\]

(1)

The unknown is the function \(Y); dW\) denotes a Gaussian time noise, that is usually understood as the distribution derivative of the Brownian sheet \(W(t)\) on a probability space \((\Omega, \mathcal{F}, P)\) with normal filtration \((\mathcal{F}_t)_{t \geq 0}; \ Y_x\) stands for the partial derivative with respect to the space variable, i.e. \(\partial Y / \partial x); \nu\) and \(\theta\) are positive constants known as the viscosity coefficient and the level of the noise, respectively.

The function \(b\) is such that there exist \(C_b > 0\) and

\[
0 \leq m_1 \leq m_2 \leq \ldots \leq m_S,
\]

for some \(S \in \mathbb{N}\), for which

\[
\sup_{x \in (0, L)} |b(t, x)| \leq C_b \left( \sum_{k=1}^{S} t^{m_k} + 1 \right), \quad \forall t > 0.
\]

(2)

Moreover, we assume that \(m_S\) and \(\theta\) are such that: \(\theta\) can be split as

\[
\frac{1}{2} \theta^2 = m_S + \frac{1}{4} + \theta_1.
\]

(3)

2010 Mathematics Subject Classification. 60H15, 76F70, 93B52, 60H20.

Key words and phrases. Stochastic Burgers equation, boundary control, stabilization, spectrum of the Laplace operator.

This work was supported by a grant of the “Alexandru Ioan Cuza” University of Iasi, within the Research Grants program, Grant UAIC, code GI-UAIC-2018-03.
where \( \theta_1 > 0 \).

Burgers equation is often refereed to as a one-dimensional “cartoon” of the Navier-Stokes equation because it does not exhibit turbulence. By contrary, it turns out that its stochastic version, (1), models turbulence, for details one can see [7, 9]. A principal tool in order to attenuate or even eliminate the turbulence is to control the equations. In this paper we propose a Neumann boundary-type control \( v \), that will be described with details below. The aim of the present paper is to find such a feedback \( v \) such that, once inserted into the equation (1), the corresponding solution of the closed-loop equation (1) satisfies

\[
e^{\gamma t} \int_0^L Y^2(t, x) dx < \text{const.}, \quad \forall t \geq 0, \mathbb{P} - \text{a.s.},
\]

for some positive constant \( \gamma \), provided that the initial data \( y_0 \) is small enough in the \( L^2 \)-norm (that is the main result stated in Theorem 2.2 below). (Note that this is an almost sure pathwise local boundary stabilization type result.)

In the literature there are plenty of results concerning the stabilization of the deterministic Burgers equation, for example we refer to [15] and [13]. The last one provides a global stabilization result, with some consequences on the stabilizability of the stochastic version. The control problem, associated to the stochastic version, has been addressed as-well in many works such as [7, 8, 9]. However, to our knowledge, the present work represents a first result on boundary stabilization for the stochastic Burgers equation. For more details about stochastic Burgers equation, we refer to [10, 12].

In (1), let us consider the transformation

\[
Y(t) = \Gamma(t)y(t), \quad t \in [0, \infty),
\]

where \( \Gamma(t) : L^2(0, L) \to L^2(0, L) \) is the linear continuous operator defined by the equations

\[
d\Gamma(t) = \theta \Gamma(t) dW(t), \quad t \geq 0, \quad \Gamma(0) = 1,
\]

that can be equivalently expressed as

\[
\Gamma(t) = e^{\theta W(t) - \frac{1}{2} \theta^2}, \quad t \geq 0.
\]

Frequently below we shall use the obvious inequality

\[
e^{-at} \leq t^{-a}, \quad \forall t > 0, a > 0.
\]

By the law of the iterated logarithm, arguing as in Lemma 3.4 in [3], it follows that there exists a constant \( C_\Gamma > 0 \) such that

\[
\Gamma(t) = e^{\theta W(t) - \theta_1 t} e^{-(m_S + \frac{1}{4})t} \leq C_\Gamma e^{-(m_S + \frac{1}{4})t}, \quad \forall t > 0, \mathbb{P} - \text{a.s.},
\]

where we have used that \( \frac{1}{2} \theta^2 = m_S + \frac{1}{4} + \theta_1 \). Then, by (2), we have that

\[
\Gamma(t) \sup_{x \in (0, L)} |b(t, x)| \leq C_\Gamma C_0 \left( \sum_{k=1}^s t^{m_k} e^{-(m_S + \frac{1}{4})t} + e^{-(m_S + \frac{1}{4})t} \right)
\]

(since \( 0 \leq m_1 \leq m_2 \leq \ldots \leq m_S \))
≤ C \left( \sum_{k=1}^{S} t^{m_k} e^{-(m_k + \frac{1}{4}) t} + e^{-\frac{1}{4} t} \right) 
\leq C \left( \sum_{k=1}^{S} t^{m_k} t^{-(m_k + \frac{1}{4}) + t^{-\frac{1}{4}}} \right) 
\leq (S + 1) C t^{-\frac{1}{4}}, \forall t > 0. \tag{7}

Applying Itô’s formula in (1) (the justification for this is as in [1]), we obtain for $y$ the random differential equation

\[
\begin{cases}
\frac{\partial y(t)}{\partial t} = \nu \Gamma^{-1}(t) \Delta (\Gamma(t)y(t)) + \Gamma^{-1}(t)b(t)(\Gamma(t)y(t))(\Gamma(t)y(t))_x, & t \in [0, \infty), \\
y_x(t, 0) = \Gamma^{-1}(t)v(t), & y_x(t, 1) = 0, \quad t \in [0, \infty), \\
y(0) = y_0.
\end{cases} \tag{8}
\]

Taking into account that $\Gamma(t)$ commutes with the space derivatives, it follows by (8) that $y$ obeys the equation

\[
\begin{cases}
\frac{\partial y(t)}{\partial t} = \nu \Delta y(t) + \Gamma(t)b(t)y(t)y_x(t), & t \in [0, \infty), \\
y_x(t, 0) = u(t) := \Gamma^{-1}(t)v(t), & y_x(t, L) = 0, \quad t \in [0, \infty), \\
y(0) = y_0.
\end{cases} \tag{9}
\]

The idea is to design a boundary stabilizer feedback $u(t)$ for the equation (9), then $v(t) = \Gamma(t)u(t)$ will achieve the path-wise stability desired. The method to design such an $u$ relies on the ideas in [17], where a proportional type stabilizer was proposed to stabilize, in mean, the stochastic heat equation. We emphasize that, unlike to the equation in [17], now we deal with a nonlinear one of order two. In order to overcome this complexity, we further develop the ideas in [17]. Roughly speaking, we design a similar feedback as in [17], whose simple form allows us to write the corresponding solution of the closed-loop system in a mild formulation, via a kernel. Then, in order to show the stability achieved, we use the estimates of the magnitude of the controller (similarly as in [17]) and, in addition, a fixed point argument. The idea to use fixed point arguments in order to show the stability of deterministic or stochastic equations has been previously used in papers like [6, 16]. Proportional type feedback, similar to the one we shall design here, has its origins in the works [2, 19], while in the papers [20, 21, 22, 23, 24, 18], it has been used to stabilize other important parabolic-type equations, such as the Navier-Stokes equations (also with delays), the Magnetohydrodynaminc equations, the phase filed equations, and even for stabilization to trajectories.

2. The boundary feedback stabilizer and the main result of the work. In what follows $L^2(0, L)$ stands for the space of all functions that are square Lebesgue integrable on $(0, L)$, and $\| \cdot \|_2, \langle \cdot , \cdot \rangle$ the classical norm and scalar product, respectively. Also, we denote by $\langle \cdot , \cdot \rangle_N$ the standard scalar product in $\mathbb{R}^N$, $N \in \mathbb{N}$. We shall denote by $C$ different constants that may change from line to line, though we keep denote them by the same letter $C$, for the sake of the simplicity of the writing.

Let us denote by

$$A y = -\nu y_{xx}, \forall y \in \mathcal{D}(A),$$
\(D(\mathcal{A}) = \{ y \in H^2(0, L) : y_x(0) = y_x(L) = 0 \}\),
the Neumann-Laplace operator on \((0, L)\). It is well known that it has a countable
set of eigenvalues, namely
\[
\mu_j = \frac{\nu(j - 1)^2 \pi^2}{L^2}, \ j = 1, 2, ...
\]
with the corresponding eigenfunctions
\[
\varphi_j(x) = \begin{cases} \frac{1}{\sqrt{L}} & , \ j = 1 \\ \frac{2}{\sqrt{L}} \cos \left( \frac{(j-1)\pi x}{L} \right) & , \ j = 2, 3, ... \end{cases}
\]
that form an orthonormal basis in \(L^2(0, L)\). Setting
\[
p_G(t, x, \xi) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-\xi|^2}{4t}}, \ t > 0, \ x, \xi \in \mathbb{R},
\]
the Gaussian heat kernel, it is well-known the relation
\[
p_H(t, x, \xi) \leq p_G(t, x, \xi), \ \forall t > 0, x, \xi \in (0, L),
\]
where \(p_H\) is the heat kernel
\[
p_H(t, x, \xi) = \sum_{j=2}^{\infty} e^{-\mu_j t} \varphi_j(x) \varphi_j(\xi).
\]
In particular, it follows that we have that
\[
\sum_{j=2}^{\infty} e^{-\mu_j t} \varphi_j^2(\xi) \leq C \frac{1}{\sqrt{t}}, \ t > 0, \xi \in (0, L).
\] (10)

Let \(N \in \mathbb{N}\) a large constant. Next, likewise in [17, Eq. (3)], we define the
Neumann operators as: \(D_{\gamma_k}, \ k = 1, 2, ..., N\), solution to the equation
\[
-\nu y_{xx}(x) - 2 \sum_{i=1}^{N} \mu_i \langle y, \varphi_i \rangle \varphi_i(x) + \gamma_k y(x) = 0
\] (11)
for \(x \in (0, L)\); \(y_x(0) = 1\) and \(y_x(L) = 0\).
Here,
\[
\gamma_k := \frac{\mu_N + \frac{k}{N} + N^{\alpha}}{N}, \ k = 1, 2, ..., N,
\] (12)
with \(\frac{7}{4} < \alpha < 2\).
We go on following the ideas in [17]. We denote by \(B\) the next square matrix
\[
B := \frac{1}{L} \begin{pmatrix} 1 & \sqrt{2} & \ldots & \sqrt{2} \\ \sqrt{2} & 2 & \ldots & 2 \\ \ldots & \ldots & \ldots & \ldots \\ \sqrt{2} & 2 & \ldots & 2 \end{pmatrix},
\] (13)
and multiply it on both sides by
\[
\Lambda_{\gamma_k} := \text{diag} \left( \frac{1}{\gamma_k - \mu_1}, \ldots, \frac{1}{\gamma_k - \mu_N} \right),
\]
to define
\[
B_k := \Lambda_{\gamma_k} B \Lambda_{\gamma_k}, \ k = 1, ..., N,
\] (14)
and finally, introduce the matrix
\[
A := (B_1 + B_2 + ... + B_N)^{-1}
\] (15)
which, as claimed in [17], is well-defined.

At this stage we are able to introduce the feedback law we propose here for stabilization. To this end, for each \( k = 1, \ldots, N \), we set the following feedback forms

\[
u_k(y) := \left( A \begin{pmatrix} \langle y, \varphi_1 \rangle \\ \langle y, \varphi_2 \rangle \\ \vdots \\ \langle y, \varphi_N \rangle \end{pmatrix} \right) \left( \begin{pmatrix} \frac{\varphi_1(0)}{\gamma_k - \mu_1} \\ \frac{\varphi_2(0)}{\gamma_k - \mu_2} \\ \vdots \\ \frac{\varphi_N(0)}{\gamma_k - \mu_N} \end{pmatrix} \right)_{N}, \tag{16}
\]

then, introduce \( u \) as

\[
u(y) := u_1(y) + \cdots + u_N(y). \tag{17}
\]

Finally, arguing similarly as in [17, Eqs. (17)-(19)], we equivalently rewrite (9) as an internal-type control problem:

\[
\partial_t y(t) = -Ay(t) + \sum_{i=1}^{N} u_i(y(t))((A + \gamma_i)D_{\gamma_i} - 2 \sum_{i,j=1}^{N} \mu_j \langle u_i(y(t))D_{\gamma_i}, \varphi_j \rangle \varphi_j \\
+ b(t)\Gamma(t)y(t); \quad y(0) = y_0. \tag{18}
\]

By [17, Lemma 3.1] (together with the details from its proof), we have the following result related to the linear operator that governs equation (18):

**Lemma 2.1.** The solution \( z \) of

\[
\partial_t z(t) = -Az(t) + \sum_{i=1}^{N} u_i(z(t))((A + \gamma_i)D_{\gamma_i} - 2 \sum_{i,j=1}^{N} \mu_j \langle u_i(z(s))D_{\gamma_i}, \varphi_j \rangle \varphi_j; \\
z(0) = z_0,
\]

can be written in a mild formulation as

\[
z(t, x) = \int_{0}^{L} p(t, x, \xi)z_{\omega}(\xi)d\xi,
\]

where

\[
p(t, x, \xi) := p_1(t, x, \xi) + p_2(t, x, \xi) + p_3(t, x, \xi), \tag{20}
\]

for \( t \geq 0, x, \xi \in (0, L) \). Here

\[
p_1(t, x, \xi) := \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ji}(t)\varphi_j(x) \varphi_i(\xi),
\]

\[
p_2(t, x, \xi) := \sum_{i=N+1}^{\infty} e^{-\mu_i t}\varphi_i(x)\varphi_i(\xi),
\]

and

\[
p_3(t, x, \xi) := \sum_{i=1}^{N} \left( \sum_{j=N+1}^{\infty} w_{ij}(t)\varphi_j(x) \right) \varphi_i(\xi).
\]
The quantities \( q_j(t) \) and \( w^j_i(t) \) involved in the definition of \( p \), obey the estimates: for some \( C_q > 0 \), depending on \( N \),

\[
|q_j(t)| \leq C_q e^{-c N^2 t}, \quad \forall t \geq 0,
\]

for all \( i, j = 1, 2, ..., N \), for some positive constant \( c < \frac{\sqrt{2}}{17} \) (because of later purpose); and for some \( C_w > 0 \), depending on \( N \),

\[
|w^j_i(t)| \leq C_w \frac{1}{\mu_j - c N^2} e^{-c N^2 t}, \quad \forall t \geq 0,
\]

for all \( i = 1, 2, ..., N \) and \( j = N + 1, N + 2, ... \). Moreover, for all \( z_o \in L^2(0, L) \), we have that

\[
\int_0^t e^{-\mu_j(t-s)} (B(u)(s))_j ds.
\]

It yields that

\[
\left\{ \sum_{j=N+1}^{\infty} \mu_j \left[ \sum_{i=1}^{N} w^j_i(t) \langle z_o, \varphi_i \rangle \right]^2 \right\}^{\frac{1}{2}} \leq C e^{-c N^2 t} \sup_{l=1,2,..,N} |\langle z_o, \varphi_l \rangle|, \quad \forall t \geq 0. \tag{23}
\]

Proof. We shall show only (23), since all the others items are exactly the results obtained already in [17, Lemma 3.1].

Setting

\[
B(u)(t) := \sum_{i=1}^{N} u_i(z(t))(\mathbb{A} + \gamma_i)\mathbb{D}_i - 2 \sum_{i,j=1}^{N} \mu_j \langle u_i(z(s))\mathbb{D}_i, \varphi_j \rangle \varphi_j,
\]

and denoting by

\[
(B(u)(t))_j := \langle B(u)(t), \varphi_j \rangle, \quad j = 1, 2, ..., \tag{24}
\]

we have, via [17, Eqs. (28)-(31)], that

\[
\sum_{i=1}^{N} w^j_i(t) \langle z_o, \varphi_i \rangle = \int_0^t e^{-\mu_j(t-s)} (B(u)(s))_j ds.
\]

Taking into account the form of \( B(u) \)

\[
\text{and the fact that } |u_i(t)| \leq C e^{-c N^2 t} \sup_{l=1,2,..,N} |\langle z_o, \varphi_l \rangle|, \quad \forall t \geq 0, \quad i = 1, ..., N
\]

\[
\leq C \int_0^t e^{-\mu N + c(t-s)} e^{-c N^2 t} ds \sup_{l=1,2,..,N} |\langle z_o, \varphi_l \rangle| \leq C e^{-c N^2 t} \sup_{l=1,2,..,N} |\langle z_o, \varphi_l \rangle|,
\]

\( \forall t \geq 0. \) That is exactly what we have claimed. \( \square \)

Relying on the above lemma, we may now proceed to state and prove the main existence & stabilization results of the present work.
Theorem 2.2. Let $\eta > 0$, depending on $\omega$ and sufficiently small. Then, for each $
yo \in L^2(0, L)$ with $|\nyo|^2 < \eta$, there exists a unique solution $\ny$ to the random deterministic equation (18) belonging to the space $\mathcal{Y}$,

$$\mathcal{Y} := \left\{ \ny \in C_b([0, \infty), H^1(0, L)) : \sup_{t \geq 0} \left[ e^{|\n|t}([\ny(t)] + t^{\frac{1}{2}}|\ny_x(t)|) \right] < \infty \right\}.$$  

In particular, the stochastic Burgers equation

$$d\ny(t,x) = \nu \Delta \ny(t,x)dt + b(t,x)\ny(t,x)\ny_x(t,x)dt + \theta \ny(t,x)dW(t), \ t > 0, \ x \in (0, L),$$

where $\nu$, $\theta$, and $b$ are as in Theorem 2.2. Then, for all $\ny \in \mathcal{Y}$, we have

$$\sup_{t \geq 0} \left[ e^{|\n|t}([\ny(t)] + t^{\frac{1}{2}}|\ny_x(t)|) \right] < \infty.$$  

It is clear that, for all $\ny \in \mathcal{Y}$, we have

$$e^{|\n|t}||\ny(t)|| \leq |\ny|_{\mathcal{Y}} \text{ and } e^{|\n|t}||\ny_x(t)|| \leq t^{-\frac{1}{2}}|\ny|_{\mathcal{Y}}, \forall t > 0. \hspace{1cm} (26)$$

We set $B_r(0) := \{ \ny \in \mathcal{Y} : |\ny|_{\mathcal{Y}} \leq r \}.$

Based on what we have discussed above, we may rewrite (18) in a mild formulation as

$$\ny(t,x) = \int_0^L p(t, x, \xi)\nyo(\xi)d\xi + \int_0^t \int_0^L p(t - s, x, \xi)b(s, \xi)\Gamma(s)y(s, \xi)\ny(s, \xi)d\xi ds,$$

where $p$ is defined in (20). Thus, existence of a solution $\ny$ is equivalent with the fact that the map $\mathcal{G} : \mathcal{Y} \to \mathcal{Y}$, defined as

$$\mathcal{G}y := \int_0^L p(t, x, \xi)\nyo(\xi)d\xi + \mathcal{F}y,$$

where

$$(\mathcal{F}y)(t) := \int_0^t \int_0^L p(t - s, x, \xi)b(s, \xi)\Gamma(s)y(s, \xi)\ny(s, \xi)d\xi ds,$$

has a fixed point.

In what follows we aim to show that $\mathcal{G}$ is a contraction on $B_r(0)$, that maps the ball $B_r(0)$ into itself, for $r > 0$ properly chosen. Then, via the contraction mappings theorem, we shall deduce that $\mathcal{G}$ has a unique fixed point $\ny \in B_r(0)$. Then, easily one arrives to the wanted conclusion claimed by the theorem.

We need to estimate the $|\cdot|_{\mathcal{Y}}$-norm of $G \ny$. So, in particular, we need to estimate the $|\cdot|_{\mathcal{Y}}$-norm of $\mathcal{F}y$, for $\ny \in \mathcal{Y}$. We begin with the $L^2$-norm of $\mathcal{F}y$. We aim to use
the Parseval identity, so, in order to do this, based on the kernel’s form (20), we conveniently rewrite the term $\mathcal{F}y$ as

$$\mathcal{F}y(t) = \int_0^t (\mathcal{F}_1(y(s)) + \mathcal{F}_2(y(s)) + \mathcal{F}_3(y(s))) \, ds,$$  \quad (27)

where

$$\mathcal{F}_1(y)(t, s, x) := \sum_{j=1}^{N} \left[ \sum_{i=1}^{N} q_{ji}(t-s) \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right] \varphi_j(x),$$

$$\mathcal{F}_2(y)(t, s, x) := \sum_{j=N+1}^{\infty} \left[ e^{-\mu_j(t-s)} \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right] \varphi_j(x),$$

$$\mathcal{F}_3(y)(t, s, x) := \sum_{j=N+1}^{\infty} \left[ \sum_{i=1}^{N} w_i^j(t-s) \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_i(\xi) \, d\xi \right] \varphi_j(x).$$

(28)

It follows via the Parseval identity, that

$$\|\mathcal{F}_1(y)\| = \left\{ \sum_{j=1}^{N} \left[ \sum_{i=1}^{N} q_{ji}(t-s) \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_i(\xi) \, d\xi \right]^2 \right\}^{\frac{1}{2}}$$

(using that the eigenfunctions are uniformly bounded and relation (7))

$$\leq C \sum_{i,j=1}^{N} |q_{ji}(t-s)| s^{-\frac{3}{4}} \int_0^L |y(s, \xi)||y_\xi(s, \xi)| \, d\xi$$

(involving relation (21) and the Schwarz’s inequality )

$$\leq Ce^{-cN^2(t-s)} s^{-\frac{1}{4}} \|y(s)||y_\xi(s)||$$

(by (26) and the fact that $-cN^2 + 2N + \frac{1}{4} < 0$ for $N$ large enough)

$$\leq Ce^{-N^2(t-s)} s^{-\frac{1}{4}} s^{-\frac{1}{4}} |y|^2_{L^2}$$

$$= Ce^{-N^2(t-s)} s^{-\frac{3}{4}} |y|^2_{L^2}, \quad \forall 0 < s < t.$$

We go on with

$$\|\mathcal{F}_2(y)\| = \left\{ \sum_{j=N+1}^{\infty} \left[ e^{-\mu_j(t-s)} \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right]^2 \right\}^{\frac{1}{2}}$$

$$= e^{-2Nt} \left\{ \sum_{j=N+1}^{\infty} \left[ e^{-(\mu_j-2N)(t-s)} \Gamma(s) e^{2Ns} \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right]^2 \right\}^{\frac{1}{2}}$$

$$\leq Ce^{-Nt}$$

$$\times \left\{ \sum_{j=N+1}^{\infty} \int_0^L \left( e^{-(\mu_j-2N)(t-s)} e^{2Ns} y(s, \xi) \varphi_j(\xi) \right) (\Gamma(s) b(s, \xi) e^{Nst} y_\xi(s, \xi)) \, d\xi \right\}^{\frac{1}{2}}$$
where (by Schwarz’s inequality)
\[
\|F\|_F \leq Ce^{-Nt} \left\{ \sum_{j=N+1}^{\infty} \int_0^L e^{-2(\mu_j-2N)(t-s)} \varphi_j^2(\xi)e^{2Ns}y^2(s,\xi)d\xi \right\}^{\frac{1}{2}}
\]
\[
\times \int_0^L \Gamma^2(s)b^2(s,\xi)e^{2Ns}y^2_s(s,\xi)d\xi \right\}^{\frac{1}{2}}
\]
\[
= Ce^{-Nt} \left\{ \int_0^L \left[ \sum_{j=N+1}^{\infty} e^{-2(\mu_j-2N)(t-s)} \varphi_j^2(\xi) \right] e^{2Ns}y^2(s,\xi)d\xi \right\}^{\frac{1}{2}}
\]
\[
\times \int_0^L \Gamma^2(s)b^2(s,\xi)e^{2Ns}y^2_s(s,\xi)d\xi \right\}^{\frac{1}{2}}
\]
\[
(30)
\]
(by the inequality between the heat kernel and the Gaussian kernel in (10))
\[
\leq Ce^{-Nt} \left\{ \int_0^L (t-s)^{-\frac{1}{4}} e^{2Ns}y^2(s,\xi)d\xi \times \int_0^L \Gamma^2(s)b^2(s,\xi)e^{2Ns}y^2_s(s,\xi)d\xi \right\}^{\frac{1}{2}}
\]
\[
(\text{by (7)})
\]
\[
\leq Ce^{-Nt}(t-s)^{-\frac{1}{4}} s^{-\frac{1}{2}} e^{Ns}\|y(s)\|e^{Nt}\|y_t(s)\|
\]
\[
(\text{using (26)})
\]
\[
\leq Ce^{-Nt}(t-s)^{-\frac{1}{4}} s^{-\frac{1}{2}} \|y\|_{L^2_y}, \quad \forall 0 < s < t.
\]
Finally, we deal with
\[
\|F_3(y)\| = \left\{ \sum_{j=N+1}^{\infty} \left[ \sum_{i=1}^{N} w_i^2(t-s)\Gamma(s) \int_0^L b(s,\xi)y(s,\xi)y_{\xi}(s,\xi)\varphi_i(\xi)d\xi \right] \right\}^{\frac{1}{2}}
\]
(by (7), (22) and the uniform boundedness of the eigenfunctions)
\[
\leq C \left( \sum_{j=N+1}^{\infty} \frac{1}{\mu_j - cN^2} \right) e^{-cN^2(t-s)}s^{-\frac{1}{2}} \|y(s)\|\|y_{\xi}(s)\|
\]
\[
(31)
\]
(clearly the series converge)
\[
\leq Ce^{-Nt} e^{(-cN^2+2N+\frac{1}{4})(t-s)} e^{-\frac{1}{2}(t-s)} s^{-\frac{1}{2}} e^{Ns}\|y(s)\|e^{Nt}\|y_{\xi}(s)\|
\]
\[
(\text{by (26) and the fact that } -cN^2 + 2N + \frac{1}{4} < 0 \text{ for } N \text{ large enough})
\]
\[
\leq Ce^{-Nt}(t-s)^{-\frac{1}{4}} s^{-\frac{1}{2}} \|y\|_{L^2_y}, \quad \forall 0 < s < t.
\]
We conclude that, (29)-(31) imply that
\[
\|F_3(y)(t)\| \leq Ce^{-Nt} \int_0^t s^{-\frac{3}{4}}(t-s)^{-\frac{1}{2}} ds = Ce^{-Nt}CB \left( \frac{1}{4}, \frac{3}{4} \right) \|y\|_{L^2_y}, \quad \forall t \geq 0, \quad (32)
\]
where \( B(x, y) \) is the classical beta function.
By the exponential semi-group property, we have as-well that
\[
\left\| \int_0^L p(t, x, \xi)y_o(\xi)d\xi \right\| \leq Ce^{-Nt}\|y_o\|. \quad (33)
\]
We go on with the estimates in the $H^1$-norm. Using the above notations, we have

$$
\|(F_1(y))_x\| = \left\{ \sum_{j=N+1}^{\infty} \mu_j \left[ e^{-\mu_j(t-s)} \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right]^2 \right\}^{\frac{1}{2}}
$$

(arguing as in (29))

$$
\leq Ce^{-cN^2(t-s)} s^{-\frac{1}{2}} \|y(s)\| \|y_\xi(s)\|
$$

$$
= Ce^{-2Nt} e^{-cN^2+2N+\frac{3}{4}} (t-s) e^{-\frac{3}{4}(t-s)} s^{-\frac{1}{2}} e^{N^2 s} \|y(s)\| \|c^N\| \|y_\xi(s)\|
$$

(by (26) and the fact that $-cN^2 + 2N + \frac{3}{4} < 0$ for $N$ large enough)

$$
\leq Ce^{-Nt} (t-s)^{-\frac{3}{2}} s^{-\frac{1}{2}} |y|^2, \ \forall 0 < s < t.
$$

(34)

Next,

$$
\|(F_2(y))_x\| = \left\{ \sum_{j=N+1}^{\infty} \mu_j \left[ e^{-\mu_j(t-s)} \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right]^2 \right\}^{\frac{1}{2}}
$$

$$
= (t-s)^{-\frac{1}{2}} \left\{ \sum_{j=N+1}^{\infty} \frac{1}{2} \mu_j \frac{1}{2} e^{-\mu_j(t-s)} \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right\}^{\frac{1}{2}}
$$

(using the obvious inequality $[(t-s)\mu_j]^{-\frac{1}{2}} \leq e^{\frac{1}{2} \mu_j(t-s)}$)

$$
\leq (t-s)^{-\frac{1}{2}} \left\{ \sum_{j=N+1}^{\infty} \frac{1}{2} \mu_j \frac{1}{2} e^{-\mu_j(t-s)} \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_j(\xi) \, d\xi \right\}^{\frac{1}{2}}
$$

(arguing as in (30))

$$
\leq C(t-s)^{-\frac{1}{2}} e^{-Nt} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} |y|^2
$$

$$
= Ce^{-Nt} (t-s)^{-\frac{3}{2}} s^{-\frac{1}{2}} |y|^2, \ \forall 0 < s < t.
$$

(35)

Finally,

$$
\|(F_3(y))_x\| = \left\{ \sum_{j=N+1}^{\infty} \mu_j \left[ \sum_{i=1}^{N} w_i^j (t-s) \Gamma(s) \int_0^L b(s, \xi) y(s, \xi) y_\xi(s, \xi) \varphi_i(\xi) \, d\xi \right]^2 \right\}^{\frac{1}{2}}
$$

(by (23))

$$
\leq Ce^{-cN^2(t-s)} \sup_{t=1,2,\ldots,N} \|\Gamma(s) b(s, \cdot) y(s, \cdot) y_\xi(s, \cdot, \varphi_i(\cdot))\|.
$$

(with similar arguments as before)

$$
\leq Ce^{-Nt} (t-s)^{-\frac{3}{2}} s^{-\frac{1}{2}} |y|^2, \ \forall 0 < s < t.
$$

(36)

Therefore, (34)-(36) imply that

$$
\|(F(y))(t)\| \leq Ce^{-Nt} \int_0^t (t-s)^{-\frac{3}{2}} s^{-\frac{1}{2}} ds |y|^2 = e^{-Nt} t^{-\frac{1}{4}} CB \left( \frac{1}{2}, \frac{1}{4} \right) |y|^2, \ \forall t > 0.
$$

(37)
Heading towards the end of the proof, we note that
\[ \int_0^\infty e^{Nt} t^\frac{1}{2} \left( \int_0^L \left( \frac{\partial p}{\partial x}(t, x, \xi) \right)^2 d\xi \right)^\frac{1}{2} dt < \infty, \]
since the presence of the \( \mu_j \), in the infinite summation, is controlled as in (35) by
the presence of \( t^\frac{1}{2} \).

Consequently, via the semi-group property, we deduce that
\[ \left\| \int_0^L \frac{\partial p}{\partial x}(t, x, \xi) d\xi \right\| \leq C e^{-Nt} t^{-\frac{1}{2}} \| y_0 \|. \tag{38} \]

Now, gathering together the relations (31), (33), (37) and (38), we arrive to the
fact that, there exists a constant \( C_1 > 0 \) such that
\[ |Gy|_Y \leq C_1 (\| y_0 \| + |y|^2_Y), \tag{39} \]
for all \( y \in Y \).

Easily seen, similar arguments as above lead as-well to
\[ |Gy - G\tilde{y}|_Y \leq C_2 (|y|_Y + |\tilde{y}|_Y) |y - \tilde{y}|_Y, \forall y, \tilde{y} \in Y, \tag{40} \]
for some constant \( C_2 > 0 \).

Recall that \( \| y_0 \| < \eta \). It then follows that, if \( \eta \) is small enough such that
\[ \eta < \min \left\{ \frac{1}{4C_1^2}, \frac{1}{4C_1C_2} \right\}, \]
taking \( r = 2C_1 \eta \), we get from (40) that \( G \) is a contraction and by (39) that \( G \)
maps the ball \( B_r(0) \) into itself, as claimed.

Note that, \( C_1, C_2 \) depend on \( \omega \), since in the above, \( \omega \)-estimations for \( \Gamma \)
were used. Thus, \( \eta \) should depend on \( \omega \) too. This means that, in fact, \( y_0 \) must depend
on \( \omega \).

Since the solution of the system, \( y \), is a fixed point of the map \( G \), we deduce from
(39) that
\[ |y|_Y \leq C_1 (\| y_0 \| + |y|^2_Y). \]

Seeing this as a second order inequality in \( |y|_Y \), we immediately obtain that
\[
|y|_Y \leq \frac{1 - \sqrt{1 - 4C_1^2 \| y_0 \|}}{2C_1} \left( \frac{4C_1^2 \| y_0 \|}{2C_1^2 (1 + \sqrt{1 - 4C_1^2 \| y_0 \|})} \right) \]
\[
\leq 2C_1 \| y_0 \|. \tag{41}
\]

Now, recalling the norm in \( Y \), we readily see that
\[ \| y(t) \| \leq 2C_1 e^{-Nt} \| y_0 \|, \forall t \geq 0. \]

To conclude with the proof, in virtue of the relation between the solution \( Y \)
of the stochastic Burgers equation (25) and the solution \( y \) of the random equation
(18), namely
\[ Y = \Gamma^{-1} y, \]
the above relation yields
\[ \| Y(t) \| \leq 2C_1 e^{-\frac{N}{T} t} e^{-\frac{N}{T} \Gamma^{-1}(t)} \| y_0 \|, \forall t \geq 0, \mathbb{P} \text{- a.s..} \]
Using again the law of the iterated logarithm, we can be sure with probability 1 that 
\( e^{-\frac{2}{N} \Gamma^{-1}(t)} \) (for large enough \( N \)) is bounded. Thereby, completing the proof. \( \square \)

3. Conclusions. In this work, based on the ideas of constructing proportional type 
stabilizing feedbacks in [17] associated to the stochastic heat equation, together with 
a fixed point argument, we managed to obtain a first result of boundary stabilization 
for the stochastic Burgers equation. We emphasize that, the proposed feedback 
is linear, of finite-dimensional structure, involving only the eigenfunctions of the 
Laplace operator, being easily to manipulate from the computational point of view. 
Regarding the stabilization of deterministic Navier-Stokes equations, with finitely 
many controllers, we mention the works [4, 5], and emphasize that the ideas used 
in the stochastic context bear a lot of similarities.

Due to the presence of the space-time function \( b \), the system (9) is not stabilizable 
by collocated boundary feedback like in [13]. Indeed, let us consider, as in [13], 
equation (9) with the boundary conditions

\[
y_x(t, 0) = u_0 \text{ and } y_x(t, L) = u_1.
\]
(Note that, in this case equation is controlled at both ends.) When computing the 
time derivative of the Lyapunov function

\[
V(t) = \frac{1}{2} \int_0^L y(t, x)^2 dx,
\]
we get in virtue of (9) that

\[
\frac{d}{dt} V = -\nu \int_0^L (y_x)^2 dx 
- \nu y(t, 0) u_0 + \frac{1}{3} \Gamma(t) b(t, 0) y(t, 0)^3 
+ \nu y(t, L) u_1 - \frac{1}{3} \Gamma(t) b(t, L) y(t, L)^3 - \frac{1}{3} \Gamma(t) \int_0^L b_x(t, x) y(t, x)^3 dx.
\]

It is clear that the presence of the integral term \( \frac{1}{3} \Gamma(t) \int_0^L b_x(t, x) y(t, x)^3 dx \) destroys 
the argument in [13]. Consequently the collocated feedback laws \( u_0, u_1 \), proposed 
in [13, Eqs. (3.2)-(3.3)], do not imply the stability of the system.

Indeed, the feedback law, we propose here, requires full state knowledge. However, taking into account the numerical results in [15], where a similar feedback is 
proposed for the stabilization of the Fischer equation, we may expect that only the 
knowledge on a part of the domain is enough. In [15] it is shown that measurements 
are needed only on \((a, 1)\) instead of the whole interval \((0, 1)\), where \( a \leq 0.25\).

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Received May 2018; 1st revision August 2018; 2nd revision September 2018.

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