Irreducible SU(3) Schwinger Bosons

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We develop simple computational techniques for constructing all possible SU(3) representations in terms of irreducible SU(3) Schwinger bosons. We show that these irreducible Schwinger oscillators make SU(3) representation theory as simple as SU(2). The new Schwinger oscillators satisfy certain Sp(2,R) constraints and solve the multiplicity problem as well. These SU(3) techniques can be generalized to SU(N).

I. INTRODUCTION

It is well known that the simple features of SU(2) Lie algebra or angular momentum algebra and it’s representations are lost when we study SU(3) or higher SU(N) groups. In the past, considerable work has been done in this direction [2, 3, 4, 5]. A detailed discussion with an extensive list of references in this context can be found in [3]. The purpose and motivation of this work is to address these issues and make the construction of all SU(3) irreducible representations (irreps.) as simple and accessible as SU(2). Moreover, the techniques can be generalized to higher SU(N) groups in a straightforward manner. We use Schwinger boson representation of SU(2) and SU(3) Lie algebra to illustrate our results. Infact, Schwinger boson representation of SU(2) Lie algebra has played many important and diverse roles in different areas of physics due to its intrinsic simplicity. More explicitly, the simplicity is because the Schwinger analysis of the SU(2) Lie algebra is in terms of it’s smallest (spin half) constituents instead of (spin one) angular momentum operators themselves. These spin half operators are two quantum mechanical harmonic oscillators \((j = \frac{1}{2}, m = \pm \frac{1}{2})\) which are called Schwinger bosons. The angular momentum operators are composites of Schwinger bosons and belong to (higher) spin one representation. This leads

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to enormous computational simplifications in the representation theory of SU(2) as well as it's various coupling coefficients \[1\]. Besides this group theory advantages, the Schwinger SU(2) construction has also been exploited in nuclear physics \[6\], strongly correlated systems \[7\], supersymmetry and supergravity algebras \[10\], lattice gauge theories \[11\], loop quantum gravity \[12\] etc.

In the context of SU(2) group theory, the Schwinger bosons, each carrying basic (half) unit of spin angular momentum flux, provide an explicit and simple realization of the angular momentum algebra as well as all it’s representations \[1\]. In particular, the Hilbert space created by the two Schwinger oscillators is isomorphic to the representation space of SU(2) group (see section II). Thus the Schwinger boson representation of SU(2) group is simple, economical as well as complete. However, all these features are lost when we generalize the Schwinger boson construction to SU(N) with \(N \geq 3\) (see section III). The origin of these problems is the existence of certain SU(N) invariants which can be constructed for \(N \geq 3\). Any two states which differ by the overall presence of such an invariants will transform in the same way under SU(N). This leads to the problem of multiplicity which in turn makes the representation theory of SU(N) \((N \geq 3)\) much more involved compared to SU(2) (compare representations \[13\] for SU(2) with \[12\] for SU(3)). These issues have also been extensively addressed in the past \[2, 3, 4, 5\]. In the context of SU(3) Schwinger boson analysis, a systematic group theoretic procedure based on noncompact group Sp(2,R) is given in \[5\] to label the multiplicity of SU(3). In this work, exploiting this Sp(2,R) labeling in \[5\], we define irreducible SU(3) Schwinger bosons in terms of which construction of SU(3) representations is as simple as SU(2) (compare \[5\] for SU(2) with \[26\], instead of \[12\], for SU(3)). Further like in SU(2) case, the representations in terms of irreducible Schwinger bosons are multiplicity free.

The plan of the paper is as follows. In section II, we start with a brief introduction to SU(2) Schwinger bosons. This section makes the presentation self contained and also allows us to compare our SU(3) results (section IV) with the corresponding SU(2) results explicitly. In section III we discuss SU(3) Lie algebra and it’s representations in terms of Schwinger bosons. We also summarize the construction of Sp(2,R) group in \[5\] which commutes with SU(3). In section IV, we solve certain Sp(2,R) constraints in terms of irreducible Schwinger bosons leading to all SU(3) representations with the simplicity of SU(2). The appendix A describes the construction of SU(3) projection operators to construct the SU(3) irreps. discussed in section III. In appendix B we show that the irreducible Schwinger bosons in \((1,0)\) and \((0,1)\) representations acting on \((n,m)\) SU(3) irrep. directly produce \((n+1,m)\) and \((n,m+1)\) irreps. respectively.
The SU(2) Lie algebra is given by a set of three angular momentum operators \( \{ \vec{J} \} \equiv \{ J_1, J_2, J_3 \} \) or equivalently by \( \{ J_+, J_-, J_3 \} \), \( J_\pm \equiv J_1 \pm i J_2 \) satisfying
\[
[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3.
\]
(1)

The SU(2) group has a Casimir operator given by \( \vec{J} \cdot \vec{J} \), and the different irreducible representations are characterized by its eigenvalues \( j(j+1) \), where \( j \) is an integer or half-odd-integer.

A given basis vector in representation \( j \) is labeled by the eigenvalue \( m \) of \( J_3 \) as \( |j, m\rangle \).

We now define a doublet of quantum mechanical oscillators or equivalently Schwinger bosons, \( \vec{a} \equiv (a_1, a_2) \) and \( \vec{a}^\dagger \equiv (a_1^\dagger, a_2^\dagger) \) respectively [1]. They satisfy the simpler bosonic commutation relation \( [a_\alpha, a^\dagger_\beta] = \delta^\beta_\alpha \) with \( \alpha, \beta = 1, 2 \). The angular momentum operators in (1) are constructed out of Schwinger bosons as:
\[
J^a \equiv a_\alpha^\dagger \sigma^a \sigma^\alpha \quad \text{with} \quad \alpha, \beta = 1, 2.
\]
(2)

where \( \sigma^a \) denote the Pauli matrices. It is easy to check that the operators in (2) satisfy the SU(2) Lie algebra with the SU(2) Casimir:
\[
\vec{J} \cdot \vec{J} \equiv \frac{\vec{a}_1 \cdot \vec{a}_2}{2} \left( \frac{\vec{a}_1 \cdot \vec{a}_2}{2} + 1 \right).
\]
(3)

Thus the representations of SU(2) can be characterized by the eigenvalues of the total occupation number operator with the angular momentum satisfying,
\[
j = \frac{(n_1 + n_2)}{2} \equiv \frac{n}{2}
\]
(4)

where \( n_1 \) and \( n_2 \) are the eigenvalues of \( a_1^\dagger a_1 \) and \( a_2^\dagger a_2 \) respectively.

A general irreducible representation of SU(2) with \( n = n_1 + n_2 = 2j \) can be written as,
\[
|\psi\rangle_{\alpha_1 \alpha_2 \ldots \alpha_n} = a_1^{\dagger \alpha_1} a_2^{\dagger \alpha_2} \ldots a_n^{\dagger \alpha_n} |0\rangle \equiv O^{\alpha_1 \alpha_2 \ldots \alpha_n} |0\rangle.
\]
(5)

Above we have defined \( O^{\alpha_1 \alpha_2 \ldots \alpha_n} = a_1^{\dagger \alpha_1} a_2^{\dagger \alpha_2} \ldots a_n^{\dagger \alpha_n} \) for later convenience. The state \( |0\rangle \) denotes the vacuum state of both the Schwinger bosons, i.e., \( a_\alpha |0\rangle = 0 \), \( \alpha = 1, 2 \). The states in (5) are completely symmetric in all the \( n = 2j \) number of \( \alpha \) indices. Graphically, the representation (5) is given by Young tableau in Fig. 1.

As mentioned in the introduction, the aim of the present work is to define irreducible SU(3) Schwinger bosons in terms of which all SU(3) representations retain the simplicity of (5). However, before going into technical details, we briefly summarize the essential preliminary ideas of SU(3) Lie algebra and it’s representations in terms of SU(3) Schwinger bosons.
\[ \begin{array}{c|c|c|c}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\end{array} \]
\[ n = 2j \text{ boxes} \]

FIG. 1: The SU(2) representation in terms of Young Tableau. Each Schwinger boson \( a^{\dagger \alpha} \) in \([5]\) creates a Young tableau box.

### III. SU(3) SCHWINGER BOSONS

The rank of the \( SU(3) \) group is two. Therefore, to cover all \( SU(3) \) irreducible representations we need two independent harmonic oscillator triplets belonging to the two fundamental representations 3 and 3*. Let’s denote them by \( \{ a^{\dagger \lambda} \} \) and \( \{ b^{\dagger \alpha} \} \) respectively with \( \alpha = 1, 2, 3 \). Now the generator of \( SU(3) \) group are written as \([8]\):

\[
Q^a = a^{\dagger \lambda} \frac{\lambda^a}{2} a - b^{\dagger \tilde{\lambda}} \frac{\tilde{\lambda}^a}{2} b
\]

In \([8]\), \( a = 1, 2, \ldots, 8 \), \( \lambda^a \) are the Gell Mann matrices for the triplet (3) representation, \(-\tilde{\lambda}^a\) are the corresponding matrices for the anti-triplet \( 3^* \) representation where \( \tilde{\lambda} \) denotes the transpose of \( \lambda \). Throughout the paper, the upper and lower indices are in the conjugate 3 (i.e, transforming as \( a^{\dagger \alpha} \)) and \( 3^* \) (i.e, transforming as \( b^{\dagger \alpha} \)) representations respectively. The operators \( Q^a \) satisfy the \( SU(3) \) algebra amongst themselves, i.e, \([Q^a, Q^b] = if^{abc}Q^c\) where \( f^{abc} \) are the \( SU(3) \) structure constants \([8]\). The defining relations \([6]\) also imply:

\[
\begin{align*}
[Q^a, (a^{\dagger})^\alpha] &= \frac{1}{2}(a^{\dagger})^\beta (\lambda^a)^\alpha_{\beta}, \\
[Q^a, b^{\dagger \alpha}] &= -\frac{1}{2}(\lambda^a)^\alpha_{\beta} b^{\dagger \beta}.
\end{align*}
\]

Thus the operators \( (a^{\dagger})^\alpha \) and \( (b^{\dagger})^\alpha \) transform according to 3 and \( 3^* \) representations. The corresponding Young tableau are given by one box and two vertical boxes respectively. As \( Q^a, (a = 1, 2, \ldots, 8) \) in \([6]\) involve both creation and annihilation operators of \( a \) and \( b \) types, it is clear that:

\[
[Q^a, N_a] = 0, \quad [Q^a, N_b] = 0.
\]

The \( SU(3) \) Casimirs are given by the total occupation numbers of \( a \) and \( b \) type oscillators:

\[
N_a = a^{\dagger} \cdot a, \quad N_b = b^{\dagger} \cdot b.
\]

We represent their eigenvalues by \( n \) and \( m \) respectively and the \( SU(3) \) vacuum \((n = 0, m = 0)\) by the state \( |0\rangle \).
A. Symmetric vs. Mixed representations

A general SU(3) irreducible representation is characterized by \((n, m)\). Note that \(n\) and \(m\) are the number of single and double boxes in a Young tableau diagram. At this stage, it is convenient to define the most basic SU(3) tensor operator:

\[
O_{\alpha_1\alpha_2...\alpha_n}^{\alpha_1\alpha_2...\alpha_n} \equiv (a^\dagger)^{\alpha_1}(a^\dagger)^{\alpha_2}...(a^\dagger)^{\alpha_n}(b^\dagger)^{\beta_1}(b^\dagger)^{\beta_2}...(b^\dagger)^{\beta_m},
\]

with \(O_{\alpha_1\alpha_2...\alpha_n}^{\alpha_1\alpha_2...\alpha_n} \equiv (a^\dagger)^{\alpha_1}(a^\dagger)^{\alpha_2}...(a^\dagger)^{\alpha_n}\) and \(O_{\beta_1\beta_2...\beta_m}^{\beta_1\beta_2...\beta_m} \equiv (b^\dagger)^{\beta_1}(b^\dagger)^{\beta_2}...(b^\dagger)^{\beta_m}\). A general irreducible \((n, m)\) representation of SU(3), denoted by \(|\psi\rangle_{\beta_1\beta_2...\beta_m}^{\alpha_1\alpha_2...\alpha_n}\), satisfies the following three conditions [13]:

C1: symmetry in all upper \((\alpha)\) indices.

C2: symmetry in all lower \((\beta)\) indices.

C3: tracelessness in any of it’s upper \((\alpha)\) and lower \((\beta)\) indices.

Let us first consider the simpler pure irreps. of type \((n, 0)\) and \((0, m)\) respectively:

\[
|\psi\rangle_{\alpha_1\alpha_2...\alpha_n}^{\alpha_1\alpha_2...\alpha_n} = O_{\alpha_1\alpha_2...\alpha_n}^{\alpha_1\alpha_2...\alpha_n}|0\rangle; \quad |\psi\rangle_{\beta_1\beta_2...\beta_m}^{\beta_1\beta_2...\beta_m} = O_{\beta_1\beta_2...\beta_m}^{\beta_1\beta_2...\beta_m}|0\rangle
\]

These pure representations satisfy C1 and C2 as the Schwinger boson creation operators commute amongst themselves and the condition C3 is redundant. The \((n, 0)\) and \((0, m)\) Young tableau are given by:

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\end{array} \quad \begin{array}{c}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_m \\
\end{array}
\]

\(n\) boxes \(\in 3\) \(m\) boxes \(\in 3^*\)

FIG. 2: The two symmetric representations \((n, 0)\) and \((0, m)\) respectively. Like in SU(2) case, each single (double) box corresponds to \(a^\dagger_\alpha\) \(\left(b_\alpha^\dagger\right)\).

Therefore, as far as symmetric representations of SU(3) are concerned, each single (double) Young tableau box represents a Schwinger boson creation operator \(a^\dagger \in 3\) \(\left(b^\dagger \in 3^*\right)\). This construction is simple and retain the simplicity of SU(2). However, this simplicity is lost when we consider mixed representations \((n \neq 0, m \neq 0)\). A \((n, m)\) representation of SU(3) has to satisfy C3 in addition to C1 and C2. The states in \((n, m)\) irrep. are given by [4]:

\[
|\psi\rangle_{\beta_1\beta_2...\beta_m}^{\alpha_1\alpha_2...\alpha_n} \equiv O_{\beta_1\beta_2...\beta_m}^{\alpha_1\alpha_2...\alpha_n} + L_1 \sum_{l_1=1}^{n} \sum_{k_1=1}^{m} \delta_{\beta_{k_1}}^{\alpha_{l_1}} O_{\beta_1\beta_2...\beta_{k_1-1}\beta_{k_1+1}...\beta_m}^{\alpha_1\alpha_2...\alpha_{l_1-1}\alpha_{l_1+1}...\alpha_n} + L_2 \sum_{l_1,l_2=1}^{n} \sum_{k_1,k_2=1}^{m} \delta_{\beta_{k_1}}^{\alpha_{l_1}} \delta_{\beta_{k_2}}^{\alpha_{l_2}}
\]
\[
O_{\beta_1 \cdots \beta_k \cdots \beta_{m+1} \cdots \alpha_1 \cdots \alpha_n} + L_3 \sum_{(l_3=1)}^{n} \sum_{(k_1,k_2=1)}^{m} \delta_{\beta_{l_3}}^{\alpha_1} \delta_{\beta_{k_1} \beta_{k_2} \beta_3} O_{\beta_1 \cdots \beta_{k_1+1} \cdots \beta_{k_2+1} \cdots \beta_{m+1} \cdots \alpha_1 \cdots \alpha_n} \]

\[
\quad + \cdots + L_q \sum_{l_1, \ldots, l_q=1}^{m} \sum_{k_1, \ldots, k_q=1}^{m} \delta_{\beta_{l_1} \beta_{k_1} \beta_{l_2} \beta_{k_2}} O_{\beta_1 \cdots \beta_{l_1+1} \cdots \beta_{l_2+1} \cdots \beta_{m+1} \cdots \alpha_1 \cdots \alpha_n} \]

where \( q = \min(n, m), \delta_{\beta_1 \beta_2 \cdots \beta_r} \equiv \delta_{\beta_1}^{\alpha_1} \delta_{\beta_2}^{\alpha_2} \cdots \delta_{\beta_r}^{\alpha_r} \) and all the sums in (12) are over different indices, i.e. \( l_1 \neq l_2 \neq \cdots \neq l_q \) and \( k_1 \neq k_2 \neq \cdots \neq k_q \). The coefficients \( L_r \) are given by (9):

\[
L_r \equiv \frac{(-1)^r (a^\dagger \cdot b^\dagger)^r}{(n + m + 1)(n + m)(n + m - 1) \cdots (n + m + 2 - r)},
\]

The coefficients in (13) are chosen to satisfy the condition C3:

\[
\sum_{i, j=1}^{3} \delta_{\beta_i} \langle \psi \rangle_{\beta_1 \beta_2 \cdots \beta_m} = 0, \quad \text{for all } l = 1, 2 \ldots n, \quad \text{and } k = 1, 2 \ldots m.
\]

The projection operators implementing (14) in the Hilbert space of Schwinger bosons are constructed in Appendix A. The Young tableau for the \((n, m)\) irrep. (12) is shown in Fig. 3.

```
| \beta_1 | \beta_2 | \cdots | \beta_m | \alpha_1 | \alpha_2 | \cdots | \alpha_n |
|---|---|---|---|---|---|---|---|
|   |   |   |   |   |   |   |   |
|\text{n boxes} \in 3^r |
```

**FIG. 3:** A mixed \((n, m)\) representation Young tableau. Like in SU(2) case, each single (double) box corresponds to the irreducible Schwinger boson \(A^{\alpha} (B^{\dagger}_{\alpha}) \) in (26).

It is clear that the tracelessness condition C3 makes the mixed irreps. in (12) much more involved and complicated compared to the symmetric ones (11). As a result, like in SU(2) case, a chain of \( n \) number of \( a^\dagger \) and \( m \) number of \( b^\dagger \) operators acting on the vacuum does not serve the purpose for SU(3). In the next section, we make the construction of all SU(3) representation as simple as SU(2). In other words, we restore 1 - 1 correspondence between Young tableau boxes and the (irreducible) Schwinger boson operators. Infact, the origin of these problems is \(a \cdot b\) and \(a^\dagger \cdot b^\dagger\) which are SU(3) invariant operators. Another related issue is the problem of multiplicity \( \delta \) arising due to the above invariants. Given a state \( |\psi\rangle_{\beta_1 \beta_2 \cdots \beta_m} \) in (12), we consider the following tower of states:

\[
(a^\dagger \cdot b^\dagger)^\rho |\psi\rangle_{\beta_1 \beta_2 \cdots \beta_m}, \quad \rho = 0, 1, 2 \ldots \infty.
\]
All the infinite states in (15) transform like \((n, m)\) irrep. as they differ by different powers of the SU(3) invariant operators. In [5], it is shown that these infinite number of SU(3) degenerate states can be uniquely labeled by the quantum numbers of the group Sp(2,R) which commutes with SU(3). More explicitly, if we define the following SU(3) invariant operators,

\[
k_+ \equiv a^\dagger \cdot b^\dagger, \quad k_- \equiv a \cdot b, \quad k_0 \equiv \frac{1}{2} \left(N_a + N_b + 3\right),
\]

then it is easy to check that they satisfy Sp(2,R) or SU(1,1) algebra:

\[
[k_-, k_+] = 2k_0, \quad [k_0, k_+] = k_+, \quad [k_0, k_-] = -k_-.
\] (16)

It is shown in [5] that the states in (15) are in 1−1 correspondence with the the positive discrete family \(D^+_k\), labeled by \(|k, m\rangle\) with \(k = \frac{1}{2}(n + m + 3)(= \frac{3}{2}, 2, \frac{5}{2}, ...\) and \(m' = k + \rho\). As \(k_-|k, m' = k\rangle = 0\), the states in (12) are all annihilated by \(k_-\), i.e:

\[
k_-|\psi\rangle_{\alpha_1, \beta_1, ..., \beta_m} = a \cdot b |\psi\rangle_{\alpha_2, ..., \beta_m} \equiv 0.
\] (17)

Note that the symmetric states in (11) are trivially annihilated by \(a \cdot b\) as they contain either \(a^\dagger\)s or \(b^\dagger\)s only. The mixed states are also annihilated by \(k_-\). As an example, let us consider the simplest mixed state \(|\psi\rangle_8^\alpha \in 8\) representation:

\[
k_-|\psi\rangle_8^\alpha = (a_\gamma b_\gamma) \left(a^{i_\alpha\beta} b^{i_\alpha\gamma} - \frac{1}{3} \delta^\alpha_\beta a^\dagger \cdot b^\dagger\right) |0\rangle = 0.
\]

Infact, the tracelessness condition (14) and the \(k_-\) annihilation condition (17) are exactly equivalent (see Appendix A).

IV. THE IRREDUCIBLE SU(3) SCHWINGER BOSONS

We have already seen that all symmetric SU(3) representations (11) retain the simplicity of SU(2). In this next section we define irreducible SU(3) Schwinger bosons in terms of which:

- all mixed representations also remain as simple as SU(2) (compare (12) with (26) of this section).

- the representations are multiplicity free (see eqn. (29) and (30)).

With the above motivation in mind, we need to define new Schwinger bosons \(A^\dagger_\alpha\) and \(B^\dagger_\alpha\) which satisfy the following properties:
(i) $A^\dagger$ and $B^\dagger$ increase $N_a$ and $N_b$ by 1 with $A^\dagger \in 3$ and $B^\dagger \in 3^*$,
(ii) they commute amongst themselves to maintain the symmetry properties C1 and C2,
(iii) the tracelessness property C3 is obtained by demanding the equivalent (appendix A) Sp(2,R) constraint. More explicitly:

$$k_-(A^\dagger |\psi)^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = [k_-, A^\dagger] |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = 0.$$  (18)

The most general form of $A^\dagger_\alpha$ consistent with (i) and (iii) is:

$$A^\dagger_\alpha = a^\dagger_\alpha + L(N_a, N_b) k_+ b^\alpha$$  (19)

Now, $L(N_a, N_b)$ is fixed by:

$$k_- A^\dagger_\alpha |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = (1 + (n + m + 3)L(n + 1, m + 1)) b^\alpha |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = 0,$$  (20)

In (22) we have made use of (16) and (17). This fixes

$$L(n, m) = -\frac{1}{(n + m + 1)}$$  (21)

and we get:

$$A^\dagger_\alpha = a^\dagger_\alpha - \frac{1}{N_a + N_b + 1} k_+ b^\alpha, \quad A_\alpha = a_\alpha - b^\dagger_\alpha k_- \frac{1}{N_a + N_b + 1}$$  (22)

Similarly,

$$B^\dagger_\alpha = b^\dagger_\alpha - \frac{1}{N_a + N_b + 1} k_+ a_\alpha, \quad B^\alpha = b^\alpha - a^\dagger_\alpha k_- \frac{1}{N_a + N_b + 1}$$  (23)

It is easy to check that the irreducible Schwinger boson creation operators commute amongst themselves:

$$[A^\dagger_\alpha, A^{\dagger \beta}] = 0, \quad [B^\dagger_\alpha, B^{\dagger \beta}] = 0, \quad [A^\dagger_\alpha, B^{\dagger \beta}] = 0.$$  (24)

The other commutation relations acting on the SU(3) irreps. are:

$$[A_\alpha, A^{\dagger \beta}] |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = \left(\delta^\beta_\beta - \frac{1}{N_a + N_b + 2} B^{\dagger \beta}_\alpha B^\beta \right) |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m},$$

$$[A_\alpha, B^{\dagger \beta}] |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = -\frac{1}{N_a + N_b + 2} B^{\dagger \alpha}_\beta A^\beta |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m},$$

$$[B^\alpha, B^{\dagger \beta}] |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m} = \left(\delta^\alpha_\alpha - \frac{1}{N_a + N_b + 2} A^\alpha_\beta A^\beta \right) |\psi^{\alpha_1 \alpha_2 \ldots \alpha_n}_{\beta_1 \beta_2 \ldots \beta_m}. $$  (25)
A. SU(3) Representations

Hence a general \((n,m)\) irreducible representation of \(SU(3)\) can be written in terms of these irreducible Schwinger bosons as:

\[
|\Psi\rangle_{\alpha_1\alpha_2...\alpha_n} \equiv A^\dagger_{\alpha_1} A^\dagger_{\alpha_2} ... A^\dagger_{\alpha_n} B^\dagger_{\beta_1} B^\dagger_{\beta_2} ... B^\dagger_{\beta_m} |0\rangle
\]  

(26)

The simple construction (26) is equivalent to (12). To see the equivalence, it is instructive to first consider some simple examples. The two simplest fundamental representations ((1,0) and (0,1)) are:

\[
|\Psi\rangle_{\alpha} = A^\dagger_{\alpha} |0\rangle = (a^\dagger_{\alpha} - \frac{1}{N_a + N_b + 1} k_+ b^\alpha) |0\rangle = a^\dagger_{\alpha} |0\rangle \equiv |\psi\rangle_{\alpha}
\]

\[
|\Psi\rangle_{\beta} = B^\dagger_{\beta} |0\rangle = (b^\dagger_{\beta} - \frac{1}{N_a + N_b + 1} k_+ a^\beta) |0\rangle = b^\dagger_{\beta} |0\rangle \equiv |\psi\rangle_{\beta}.
\]  

(27)

The equations (27) also demonstrate that all the symmetric (i.e \((n,0), (0,m)\)) representations in terms of irreducible Schwinger boson are exactly same as before. This is of course trivial. The simplest mixed (1,1) states are:

\[
|\Psi\rangle_{\alpha\beta} = A^\dagger_{\alpha} B^\dagger_{\beta} |0\rangle = \left(a^\dagger_{\alpha} - \frac{1}{N_a + N_b + 1} k_+ b^\alpha\right) \left(b^\dagger_{\beta} - \frac{1}{N_a + N_b + 1} k_+ a^\beta\right) |0\rangle = a^\dagger_{\alpha} b^\dagger_{\beta} |0\rangle \equiv |\psi\rangle_{\alpha\beta}.
\]  

(28)

The equivalence between (12) and (26) is explicitly established using the method of induction in Appendix B. Infact, it is easy to see that the states (26) satisfy all the three conditions: C1, C2 and C3 mentioned in section III. The symmetry properties C1 and C2 of (26) follow from the commutation relations (24). The tracelessness property C3 of all the mixed states \((n,m)\) with \(n,m = 1, 2, ..., \infty\) in (26) also follows (in fact obvious) from the tracelessness of the octet state (28). To see this, we consider:

\[
|\Psi\rangle_{\gamma\beta_2...\beta_m} = A^\dagger \cdot B^\dagger |\Psi\rangle_{\alpha_2...\alpha_n} = A^\dagger_{\alpha_2} ... A^\dagger_{\alpha_n} B^\dagger_{\beta_2} ... B^\dagger_{\beta_m} |\Psi\rangle_{\gamma} = 0
\]  

(29)

In (29), we have used the fact that all the \(A^\dagger\)s and \(B^\dagger\)s commute amongst themselves (24) and the octet state (28) is traceless. Therefore, the \(SU(3)\) \((n,m)\) representation states in (26) are exactly same as the states in (12): \(|\Psi\rangle_{\alpha_1\alpha_2...\alpha_n} \equiv |\psi\rangle_{\alpha_1\alpha_2...\alpha_n}\). We further note that:

\[
A \cdot B \ |\Psi\rangle_{\beta_1\beta_2...\beta_m} = a \cdot b \ |\Psi\rangle_{\beta_1\beta_2...\beta_m} = 0.
\]  

(30)

Hence, the present construction in terms of irreducible Schwinger bosons also solves the multiplicity problem. The eqns. (29) and (30) show that it is no longer possible to construct
the infinite tower (15) in terms of irreducible Schwinger bosons. We again emphasize that each Young tableau single (double) box $\in 3 \ (3^*)$ representation in Figure 3 corresponds to the irreducible Schwinger boson creation operator $A^{\dagger \alpha} \ (B^{\dagger \alpha})$. This SU(3) representation feature is again like SU(2) representations.

V. SUMMARY & DISCUSSION

We conclude that the irreducible Schwinger bosons make the SU(3) representation theory as simple as SU(2). By constructing irreducible Schwinger bosons commuting with $k_-$, we are able to produce all the SU(3) irreps. with the ease of SU(2). In fact, the irreducible Schwinger bosons play important physical role in lattice gauge theories. In SU(2) lattice gauge theories the SU(2) Schwinger bosons create the electric or angular momentum fluxes along the links [11]. In SU(3) lattice gauge theory, the corresponding role is played by irreducible SU(3) Schwinger bosons [15].

We now briefly discuss the extension of the ideas in this paper to the SU(N) group. The SU(N) generalization of the SU(2) Schwinger mapping was done in [14] in the context of SU(N) coherent states. The SU(N) group has rank $(N-1)$. Therefore, we now define the SU(N) generators as:

$$Q^a = \sum_{r=1}^{N-1} Q^a[r] = \sum_{r=1}^{N-1} a^\dagger[r] \chi^a_r / 2 a[r]. \quad (31)$$

In [31], the index $r(=1, 2, ..., (N-1))$ covers all the $(N-1)$ fundamental representations of SU(N) group. The invariance group is now bigger than Sp(2,R) and can again be used to define irreducible SU(N) Schwinger bosons. This will again lead to a simplified representation theory of SU(N). The work in this direction is in progress and will be reported elsewhere.

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APPENDIX A: SU(3) PROJECTION OPERATORS

We now construct projection operators $P(n, m)$ such that,

$$P_{(n, m)} \ O^{\alpha_1 \alpha_2...\alpha_n}_{\beta_1 \beta_2...\beta_m} |0\rangle = |\psi^{\alpha_1 \alpha_2...\alpha_n}_{\beta_1 \beta_2...\beta_m} \rangle \quad (A1)$$

where, $O^{\alpha_1 \alpha_2...\alpha_n}_{\beta_1 \beta_2...\beta_m}$ is defined in [10]. It is clear that the state $|\psi^{\alpha_1 \alpha_2...\alpha_n}_{\beta_1 \beta_2...\beta_m} \rangle$ will transform in the same way as $O^{\alpha_1 \alpha_2...\alpha_n}_{\beta_1 \beta_2...\beta_m} |0\rangle$. Hence the projection operator can contain only SU(3) invariant
operators. Thus the most general form of \( P_{(n,m)} \) is given by:

\[
P_{(n,m)} \equiv \sum_{r=0}^{\infty} l_r(n,m)(k_+)^r(k_-)^r = \sum_{r=0}^{q} l_r(n,m)(k_+)^r(k_-)^r
\]  

(A2)

where, \( q = \min(n,m) \).

Applying \( k_- \) on (A1) and equating it to zero, we get the recurrence relation:

\[
\frac{(n + m + 2 - r)}{(r - 1)} l_r(n,m) = -l_{r-1}(n,m)
\]  

(A3)

Choosing the overall normalization \( l_0 = 1 \), the solution of (A3) is

\[
l_r(n,m) = \frac{(-1)^r}{r!(n+m+1) \ldots (n+m+2-r)} = \frac{(-1)^r (n + m + 1 - r)!}{r! (n + m + 1)!},
\]  

(A4)

leading to:

\[
P_{(n,m)} = \frac{1}{(n + m + 1)!} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} (n + m + 1 - r)! (k_+)^r(k_-)^r
\]  

(A5)

The action of the projection operator on the state \( O_{\alpha_1 \alpha_2 \ldots \alpha_n}^{\beta_1 \beta_2 \ldots \beta_m} |0 \rangle \) leads to (A2). In fact the projection operator in (A5) is idempotent, i.e, it satisfies:

\[
P_{(n,m)} \psi_{\beta_1 \beta_2 \ldots \beta_m}^{\alpha_1 \alpha_2 \ldots \alpha_n} = \psi_{\beta_1 \beta_2 \ldots \beta_m}^{\alpha_1 \alpha_2 \ldots \alpha_n}
\]  

(A6)

The above property is obvious as \( k_- \) annihilates the states \( \psi_{\beta_1 \beta_2 \ldots \beta_m}^{\alpha_1 \alpha_2 \ldots \alpha_n} \) and therefore only the identity (i.e, \( r = 0 \) term) in (A5) contributes.

**APPENDIX B: ACTION OF IRREDUCIBLE SCHWINGER BOSONS**

In this appendix we show that the states in (26) are same as the SU(3) irreps. in (12). These results are obvious for all the symmetric representations as shown in section IV A. We have also seen this equivalence for the octet \((1, 1)\) representation. We now use the method of induction for the general case. Let us assume the equivalence for \((n, m)\) representation:

\[
|\Psi\rangle_{\beta_1 \beta_2 \ldots \beta_m}^{\alpha_1 \alpha_2 \ldots \alpha_n} \equiv |\psi\rangle_{\beta_1 \beta_2 \ldots \beta_m}^{\alpha_1 \alpha_2 \ldots \alpha_n}.
\]  

(B1)

We now need to prove:

\[
|\Psi\rangle_{\beta_1 \beta_2 \ldots \beta_{m+1}}^{\alpha_1 \alpha_2 \ldots \alpha_{n+1}} = |\psi\rangle_{\beta_1 \beta_2 \ldots \beta_{m+1}}^{\alpha_1 \alpha_2 \ldots \alpha_{n+1}}
\]  

(B2)
Let us first consider the case \( n \to n + 1 \) and \( m \to m \). The l.h.s. of (B2) is:

\[
|\Psi\rangle^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} = A^{\dagger \alpha} |\Psi\rangle^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} = A^{\dagger \alpha} |\psi\rangle^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m}
\]

\[
= \left( a^{\dagger \alpha} - \frac{1}{N_a + N_b + 1} k_+ b^\alpha \right) \left( 1 + \sum_{r=1}^{\infty} l_r(n, m) k_+^r k_-^r \right) O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} |0\rangle
\]

\[
= \left( \frac{k_+ b^\alpha}{(n + m + 2)} O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} + \sum_{r=0}^{\infty} l_r(n, m) k_+^r a^{\dagger \alpha} k^-_r O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} \right)
\]

\[
- \frac{1}{(n + m + 2)} \sum_{r=0}^{\infty} l_r(n, m) k_+ b^\alpha k_+^r k_-^r O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} |0\rangle
\]

We use:

\[
[a^{\dagger \alpha}, k^-] = -r k_-^{r-1} b_\alpha, \quad [b^\alpha, k_+] = r k_+^{r-1} a^{\dagger \alpha},
\]

to write the third term \( T_3 \) and the fourth term \( -T_4 \) in (B3) as:

\[
T_3 = \sum_{r=1}^{\infty} l_r(n, m) k_-^r k_+^r O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} - \sum_{r=1}^{\infty} r l_r(n, m) k_+^{r-1} b^\alpha O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m}
\]

\[
T_4 = \sum_{r=1}^{\infty} r l_r(n, m) k_+^{r+1} k_-^r \frac{1}{n + m + 2} k_+^r a^{\dagger \alpha} O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m} - \sum_{r=1}^{\infty} r^2 l_r(n, m) k_+^r k_-^{r-1} b^\alpha O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m}
\]

\[
+ \sum_{r=1}^{\infty} l_r(n, m) k_+^{r+1} k_-^r b^\alpha O^{\alpha_1 \alpha_2 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m}
\]

The defining eqn. (A4) implies:

\[
l_r(n + 1, m) \equiv \frac{n + m + 2 - r}{n + m + 2} l_r(n, m).
\]

Using (B5), (A5) and (A1), we get:

\[
(T_1 + T_{31} - T_{41}) |0\rangle = |\psi\rangle^{\alpha_1 \cdots \alpha_n}_{\beta_1 \beta_2 \cdots \beta_m}.
\]

We now need to show:

\[-T_2 + T_{32} - T_{42} - T_{43} = 0.
\]

Using (B3) and

\[
- \frac{l_r(n, m)}{n + m + 2} = (r + 1) l_{r+1}(n + 1, m), \quad l_1(n + 1, m) = -1 \frac{1}{n + m + 2}.
\]
we get:

\[ T_{32} - T_{42} = - \sum_{r=1}^{\infty} rl_r(n+1,m)k_r^+ k_r^{-1} b^\alpha O_{\beta_1 \beta_2 ... \beta_m}^{\alpha_1 \alpha_2 ... \alpha_n} \]  \hspace{1cm} (B7)

\[-T_2 - T_{43} = \left( \sum_{r=1}^{\infty} (r+1)l_{r+1}(n+1,m)k_{r+}^+ k_r^{-1} + l_1(n+1,m)k_{+}^+ \right) b^\alpha O_{\beta_1 \beta_2 ... \beta_m}^{\alpha_1 \alpha_2 ... \alpha_n} \]

\[ = \sum_{r=1}^{\infty} rl_r(n+1,m)k_r^+ k_r^{-1} b^\alpha O_{\beta_1 \beta_2 ... \beta_m}^{\alpha_1 \alpha_2 ... \alpha_n} \equiv T_{42} - T_{32} \]  \hspace{1cm} (B8)

Similarly, we can prove the case: \( m \rightarrow m + 1 \) and \( n \rightarrow n \). Thus we have also explicitly proved that the simple SU(3) Schwinger boson states \([26]\) (which are exact SU(3) analogues of the SU(2) construction \([5]\)) are indeed the SU(3) irreps states \([12]\).

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