On The Kantor Product

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Abstract. We study the algebra of bilinear multiplications of an $n$-dimensional vector space. In particular, we study the Kantor product of some well-known (associative, Lie, alternative, Novikov and some other) multiplications.

1. Introduction

Kantor introduced the class of conservative algebras in [11]. This class includes some well-known algebras: associative, Jordan, Lie, Leibniz and Zinbiel [13]. In the theory of conservative algebras of great importance is the conservative algebra $U(n)$ [12]. In the theory of Lie algebras $U(n)$ plays a role analogous to the role of $gl_n$.

The space of the algebra $L_n$ on $V_n$ isomorphic to its Kantor square; and discuss the coincidence of derivations and automorphisms of the cases; in particular, the associative algebras with identities, nilpotent and right-nilpotent algebras, associative Poisson, generalized Poisson and Novikov-Poisson algebras. Finally, we study the Kantor square in some special cases; in particular, the associative algebras with identities, nilpotent and right-nilpotent algebras, associative algebras isomorphic to its Kantor square; and discuss the coincidence of derivations and automorphisms of the algebra and its Kantor square. Here we can to formulate

Open problem. Is $K(V)$ a variety of algebras for some variety $V$?

2. The Kantor square

In this section we leave technical and trivial proofs of lemmas. We are using the standard notation:

$$
(a, b, c)_* = (a * b) * c - a * (b * c),
(a, b, c) = (ab)c - a(bc);
$$

$$
[a, b]_* = a * b - b * a,
[a, b] = ab - ba;
$$
\( \circ_{a,b,c} [f(a,b,c)] = f(a,b,c) + f(b,c,a) + f(c,a,b) \).

2.1. **Associative algebras.** The variety of associative algebras is defined by the identity

\[(xy)z = x(yz)\].

**Lemma 1.** Let \((A;\cdot)\) be an associative algebra. Then \((A;[\cdot,\cdot])\) is an associative algebra.

2.2. **(Anti)commutative algebras.** The variety of (anti)commutative algebras is defined by the identity

\[xy = \epsilon yx,\]

where \(\epsilon = 1\) in the commutative case and \(\epsilon = -1\) in the anticommutative case.

**Lemma 2.** Let \((A;\cdot)\) be an (anti)commutative algebra. Then \((A;[\cdot,\cdot])\) is an (anti)commutative algebra.

2.3. **Perm algebras.** The variety of Perm algebras (see, for example, \([3]\)) is defined by the identity

\[(xy)z = x(yz) = x(zy)\].

**Lemma 3.** Let \((A;\cdot)\) be a Perm algebra. Then \((A;[\cdot,\cdot])\) is a Perm algebra.

2.4. **Lie algebras.** The variety of Lie algebras is defined by the identities

\[xy = -yx, (xy)z + (yz)x + (zx)y = 0.\]

**Lemma 4.** Let \((A;\cdot)\) be a Lie algebra. Then \([\cdot,\cdot] = 0\).

2.5. **Leibniz algebras.** The variety of (left) Leibniz algebras (see, for example, \([6]\)) is defined by the identity

\[x(yz) = (xy)z + y(xz).\]

**Lemma 5.** Let \((A;\cdot)\) be a (left) Leibniz algebra. Then \([\cdot,\cdot] = 0\).

2.6. **Left-commutative algebras.** The variety of left-commutative algebras (see, for example, \([16]\)) includes commutative-associative, bicommutative, Novikov, Zinbiel algebras and some others. This variety is defined by the identity

\[x(yz) = y(xz).\]

**Lemma 6.** Let \((A;\cdot)\) be a left-commutative algebra. Then \((A;[\cdot,\cdot])\) is a left-commutative algebra.

2.7. **Bicommutative algebras.** The variety of bicommutative algebras (see, for example, \([8]\)) is defined by the identities

\[x(yz) = y(xz), (xy)z = (xz)y.\]

**Lemma 7.** Let \((A;\cdot)\) be a bicommutative algebra. Then \((A;[\cdot,\cdot])\) is an associative-commutative algebra.

2.8. **Zinbiel algebras.** The variety of (left) Zinbiel algebras (see, for example, \([7]\)) is defined by the identity

\[x(yz) = (xy)z + (yx)z.\]

**Lemma 8.** Let \((A;\cdot)\) be a (left) Zinbiel algebra. Then \((A;[\cdot,\cdot])\) is a (left) Zinbiel algebra.
2.9. Novikov algebras. The variety of (left) Novikov algebras (see, for example, [22]) is defined by the identities
\[ x(yz) = y(xz), \quad (x, y, z) = (x, z, y). \]

**Lemma 9.** Let \((A; \cdot)\) be a (left) Novikov algebra. Then \((A; [\cdot, \cdot])\) is a (left) Novikov algebra.

2.10. Alternative algebras. The variety of alternative algebras (see, for example, [15]) is defined by the identities
\[ x^2y = x(xy), \quad xy^2 = (xy)y. \] (2)
It is also well known (see, for example, [22]) that an alternative algebra is flexible:
\[ (xy)x = x(xy); \]
and satisfies the Moufang identities:
\[ x(yzy) = ((xy)z)y, \quad (yz)y = y(z(yx)); \]
and the following identities hold:
\[ (x, y, z) = -(y, x, z), \quad (x, y, z) = -(x, z, y), \]
\[ (x, xy, z) = (x, x, y)z. \]
The main example of a non-associative alternative algebra is a Cayley — Dickson algebra \(A\) [22]. Let \(F\) be a field of characteristic \(\neq 2\). It is an algebra \(C\) with the basis \(e_0 = 1, e_1, \ldots, e_7\) and the following multiplication table:

|   | \(e_0\) | \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(e_7\) |
|---|---|---|---|---|---|---|---|---|
| \(e_0\) | \(\alpha \cdot 1\) | \(e_3\) | \(\alpha e_2\) | \(e_5\) | \(\alpha e_4\) | \(-e_7\) | \(-e_7\) | \(-\alpha e_6\) |
| \(e_1\) | \(-e_3\) | \(\beta \cdot 1\) | \(-\beta e_1\) | \(e_6\) | \(e_7\) | \(\beta e_4\) | \(\beta e_5\) |
| \(e_2\) | \(-\alpha e_2\) | \(\beta e_1\) | \(-\alpha \beta \cdot 1\) | \(e_7\) | \(\alpha e_6\) | \(-\beta e_5\) | \(-\alpha \beta e_4\) |
| \(e_3\) | \(-e_5\) | \(-e_6\) | \(\gamma \cdot 1\) | \(-\gamma e_1\) | \(-\gamma e_2\) | \(-\gamma e_3\) | \(-\alpha \gamma e_2\) |
| \(e_4\) | \(-\alpha e_4\) | \(-e_7\) | \(-\alpha e_6\) | \(\gamma e_1\) | \(\gamma e_3\) | \(-\alpha \gamma e_3\) | \(-\alpha \gamma e_2\) |
| \(e_5\) | \(-e_6\) | \(\gamma \cdot 1\) | \(-\gamma e_1\) | \(\gamma e_3\) | \(-\alpha \gamma e_3\) | \(-\beta \gamma \cdot 1\) | \(-\beta \gamma e_1\) |
| \(e_6\) | \(\alpha e_6\) | \(-\beta e_5\) | \(\beta e_4\) | \(\gamma e_2\) | \(-\gamma e_3\) | \(-\beta \gamma \cdot 1\) | \(-\beta \gamma e_1\) |
| \(e_7\) | \(\alpha e_6\) | \(-\beta e_5\) | \(\beta e_4\) | \(\gamma e_2\) | \(-\gamma e_3\) | \(-\beta \gamma \cdot 1\) | \(-\beta \gamma e_1\) |

**Theorem 10.** Let \((A; \cdot)\) be an alternative algebra. Then \((A; [\cdot, \cdot])\) is a flexible algebra. Furthermore,
1) \((A; [\cdot, \cdot])\) is an alternative algebra if and only if \(A\) satisfies the identity
\[ (x, u, x, y) = 0; \] (4)
2) \((A; [\cdot, \cdot])\) is a noncommutative Jordan algebra if and only if \(A\) satisfies the identity
\[ [L_u L_x L_y, R_u R_x] = [L_y x u, R_u x]; \]
3) \((A; [\cdot, \cdot])\) is a Jordan algebra if \((A, \cdot)\) is a commutative alternative algebra;
4) \((C; [\cdot, \cdot])\) is alternative for a Cayley — Dickson algebra \(C\), if and only if \(u = u_0 \cdot 1\).

**Proof.** It is easy to see that
\[ a \ast b = u(ab) - (ua)b - a(ub) = (au)b - 2a(ub). \]
Now, we can see
\[ (x \ast y) \ast x - x \ast (y \ast x) = (((xy)y)x - 2((xy)y)(ux)) + x - x \ast ((yu)x - 2y(ux)) = \]
\[ (((xy)y)x - 2((xy)y)(ux)) + 2((xy)y)(ux) + 4(x(uy))(ux) -
\[ (xy)(uy)x + 2(xy)(uy)x) + 4x((uy)(ux) - 4x(u(uy)x)) = \]
\[ x(uy)x - 2((xy)x)u(x - 2x(uy)x - 2x((uy)x + 2((xy)x)u(x - 4x(uy)x = \]
\[ 2((xy)x)u(x - (xy)(y, u, x) - x(u(y, u, x)) = \]
\[ 2((xy)x)u(x - (xy)(y, u, x) - x(u(y, u, x)) = 0. \]
It follows that \((A, \ast)\) is a flexible algebra.

1) It is easy to see that a flexible algebra is alternative if and only if it satisfies the first identity from \((2)\). We have

\[
(x \ast x) \ast y - x \ast (x \ast y) =
\]

\[-(xuxu)y + 2(xux)(uy) - (xu)(xu)y + 2x(u((xu)y)) + 2(xu)(x(uy)) - 4x(u(uy)x)) =
\]

\[-2(xu)^2y - 2x(u(x(uy))) + 2x((ux)(uy)) + 2(xu)(x(uy)) = (x, u, (x, u, y)).
\]

Now, the multiplication \([\cdot, \cdot]\) is alternative if and only if \((x, u, (x, u, y)) = 0\).

2) Note that an algebra \(B\) is a non-commutative Jordan algebra if and only if it is flexible and it satisfies the Jordan identity: \((x^2, y, x) = 0\).

Obviously,

\[
((x \ast x) \ast y) \ast x - (x \ast x) \ast (y \ast x) =
\]

\[-((xuxu)y + 2(xux)(uy)) \ast x + (xux) \ast ((yu)x - 2y(ux)) =
\]

\[-(((xuxu)y)u)x + 2((xuxu)y)(ux) + 2(((xuxu)(uy))u)x - 4((xuxu)(uy))(ux) +
\]

\[(xuxu)((uy)x) - 2(uxu)(uy)(ux) - 2(xuxu)(uy)(ux) + 4(xux)(u(y)(ux)) =
\]

\[2(((xuxu)(uy)(ux) - (xuxu)(y)(ux)) + ((x(ux(uy)))u)x - x(u(x(uy)(ux)))].
\]

Now, \([\cdot, \cdot]\) is a noncommutative Jordan multiplication if and only if

\[\left[ L_2L_3L, R_3R_4 \right] = [L_{xuxu}, R_{uxx}].
\]

3) It is easy to see that if \(A\) is a commutative alternative algebra then we have

\[
((xuxu)y)(ux) - (xuxu)(y)(ux) + ((xuxu)(uy))u)x - (xuxu)(u((yu)x)) =
\]

\[(x)\((xuxu)(xuxu)) - (x)(xuxu)((xuxu)) + x(u(x(ux(uy)))) - x(u(x(ux(uy)))) = 0.
\]

It follows that \([\cdot, \cdot]\) is non-commutative Jordan and from Theorem \((2)\) we infer that \([\cdot, \cdot]\) is Jordan.

4) If \((C, \ast)\) is an alternative algebra for every \(u\) then \(A\) satisfies \((4)\). Note that for the elements \(e_{i1}, e_{i2}, e_{i3}\), where \(e_{i1}e_{i2} \neq e_{e_{i3}}\) (where \(e\) is some element from the ground field), we have

\[
(e_{i1}, e_{i2}, (e_{i1}, e_{i2}, e_{i3})) = 2(e_{i1})^2(e_{i2})^2e_{i1}.
\]

Such triple \((i_1, i_2, i_3)\) we call a g-triple. It is easy to see that if \((i, j, k)\) is not a g-triple then the subalgebra generated by \(e_i, e_j, e_k\) is a two-generated subalgebra, and by the Artin theorem this subalgebra is associative, i. e., \((e_i, e_j, (e_i, e_j, e_k)) = 0\). Now, for the element \(u = u_0 \cdot 1 + u_1e_1 + \ldots + u_7e_7\) we have \((e_i, u, (e_i, u, e_j)) = 0\) if and only if

\[
\sum (u_ke_k)^2 = 0.\]

It is equivalent to the following system

\begin{align*}
au_1^2 + \beta u_3^2 & = u_4^2 - \gamma u_6^2 - \beta\gamma u_6^2 = 0, \\
u_1^2 + \alpha u_3^2 & = u_4^2 - \alpha\gamma u_6^2 - \beta\gamma u_6^2 = 0, \\
u_1^2 + \beta u_3^2 & = u_4^2 - \beta\gamma u_6^2 - \alpha\gamma u_6^2 = 0, \\
\end{align*}

\[
\begin{align*}
\alpha u_2^2 - \alpha u_3^2 & = u_4^2 - \gamma u_6^2 - \beta\gamma u_6^2 = 0, \\
\beta u_2^2 - \alpha u_3^2 & = u_4^2 - \beta\gamma u_6^2 - \alpha\gamma u_6^2 = 0, \\
\gamma u_2^2 - \alpha u_3^2 & = u_4^2 - \alpha\gamma u_6^2 - \beta\gamma u_6^2 = 0, \\
\alpha u_2^2 - \beta u_3^2 & = u_4^2 - \alpha\gamma u_6^2 - \beta\gamma u_6^2 = 0, \\
\beta u_2^2 - \beta u_3^2 & = u_4^2 - \beta\gamma u_6^2 - \alpha\gamma u_6^2 = 0.
\end{align*}
\]

Calculating, we obtain \(u_1 = \sqrt{\beta\gamma}u_7, u_6 = \sqrt{\alpha\gamma}u_7, u_2 = u_3 = u_4 = u_5 = 0\).

Now, from the relation \((4)\) by simple calculations (for example, for \(x = e_1 + e_2, y = e_1\) and \(x = e_2 + e_6, y = e_6\)) we can find that \(u_7 = 0\) and \(u = u_0 \cdot 1\).

The theorem is proved.
2.11. Quasi-associative algebras. Quasi-associative algebras (see, for example, [3]) is defined by the identities
\[(x, y, z) + (y, z, x) + (z, x, y) = 0,\]
\[(x, y, z) = \alpha[y, x, z],\]
where \(\alpha\) is a fixed element in the ground field \(F\). It is known [3] that an algebra \((A, \cdot)\) is quasi-associative if and only if there exist an associative algebra \(A\) with the new multiplication, such that for some \(\lambda \in F\):
\[x \cdot y = \lambda xy + (1 - \lambda)yx.\]

Lemma 11. Let \((A; \cdot)\) be a quasi-associative algebra. Then \((A; [\cdot, \cdot])\) is a quasi-associative algebra.

2.12. Quasi-alternative algebras. Quasi-alternative algebras (see, for example, [3]) is defined by the identities
\[(x, y, x) = 0,\]
\[(x, x, y) = \alpha[x, y],\]
where \(\alpha\) is a fixed element from the ground field \(F\). It is known [3] that an algebra \((A, \cdot)\) is quasi-alternative if and only there exist an alternative algebra \(A\) with new multiplication, such that for some \(\lambda \in F\):
\[x \cdot y = \lambda xy + (1 - \lambda)yx.\]

Lemma 12. Let \((A; \cdot)\) be a quasi-alternative algebra. Then \((A; [\cdot, \cdot])\) is a flexible algebra.

2.13. Associative dialgebras. The variety of associative dialgebras (see, for example, [20]) is defined by the identities
\[(x \preceq y) \preceq z = (x \preceq y) \preceq z, x \preceq (y \preceq z) = x \preceq (y \preceq z),\]
\[(x \preceq y) \preceq z = x \preceq (y \preceq z), (x \preceq y) \preceq z = x \preceq (y \preceq z), (x \preceq y) \preceq z = x \preceq (y \preceq z).\]

Lemma 13. Let \((A; \preceq, \preceq)\) be an associative dialgebra. Then \((A; [\cdot, \cdot])\) is an associative algebra.

2.14. Duplicial algebras. The variety of duplicial algebras (see, for example, [18]) is defined by the identities
\[(x \prec y) \prec z = x \prec (y \prec z),\]
\[(x \succ y) \prec z = x \succ (y \prec z),\]
\[(x \succ y) \succ z = x \succ (y \succ z).\]

Lemma 14. Let \((A; \prec, \succ)\) be a duplicial algebra. Then \((A; [\succ, \prec])\) is an associative algebra.

2.15. Dual duplicial algebras. The variety of dual duplicial algebras (see, for example, [23]) is defined by the identities
\[(x \prec y) \prec z = x \prec (y \prec z), (x \succ y) \prec z = x \succ (y \prec z), (x \succ y) \succ z = x \succ (y \succ z),\]
\[x \prec (y \succ z) = (x \prec y) \succ z = 0.\]

Lemma 15. Let \((A; \prec, \succ)\) be a dual duplicial algebra. Then \([\succ, \prec] = 0\) and \((A; [\prec, \succ])\) is a 2-nilpotent algebra.

2.16. \(As^{(2)}\)-algebras. The variety of \(As^{(2)}\)-algebras (see, for example, [23]) is defined by the identities
\[(x \circ y) \circ z = x \circ (y \circ z), (x \circ y) \circ z = x \circ (y \circ z),\]
\[(x \circ y) \circ z = x \circ (y \circ z), (x \circ y) \circ z = x \circ (y \circ z).\]

Lemma 16. Let \((A; \circ)\) be a \(As^{(2)}\)-algebra. Then \((A; [\cdot, \circ])\) and \((A; [\circ, \cdot])\) are associative algebras.
2.17. **Commutative tridendriform algebra.** The variety of commutative tridendriform algebras (see, for example, [17]) is defined by the identities

\[
x \cdot y = y \cdot x, (x \cdot y) \cdot z = x \cdot (y \cdot z),
\]

\[
(x \prec y) \prec z = x \prec (y \prec z) + x \prec (z \prec y),
\]

\[
(x' \cdot y') \prec z = x' \cdot (y' \prec z).
\]

**Lemma 17.** Let \((A; \cdot, \prec)\) be a commutative tridendriform algebra. Then \((A; \cdot, \prec, \cdot)\) is a commutative algebra and \((A; \cdot, \prec, \cdot)\) is a right Zinbiel algebra.

2.18. **Poisson algebras.** The variety of Poisson algebras (see, for example, [19]) is defined by the identities

\[
xy = yx, (xy)z = x(yz), \{xy, z\} = \{x, y\}z + x\{y, z\},
\]

\[
\{x, y\} = -\{y, x\}, \{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0.
\]

**Theorem 18.** Let \((A; \cdot, \{\}, \cdot)\) be a Poisson algebra. Then \([\cdot, \cdot]\) is a bracket of \([\cdot, \cdot]\) and \((A; [\cdot, \cdot])\) is a Lie algebra.

**Proof.** In the first case, we have

\[
a \cdot b = \{a, ab\} - \{a, b\}b - a\{u, b\} = 0.
\]

In the second case,

\[
a \cdot b = u\{a, b\} - \{ua, b\} - \{a, ub\} = -a\{u, b\} - b\{a, u\} - a\{u, b\} = -b \cdot a.
\]
(\{a, u\}b\{u, c\} + b\{u, c\}\{u, a\}) + ((\{a, u\}, u)bc + cb\{u, a\}, u)\) + 
(\{a, u\}\{b, c\} + \{b, c\}u\{a, u\}) + ((\{a, b\}u\{u, c\} + \{c, u\}\{a, b\}u) + 
(\{a, b\}\{u, c\} + \{b, u\}\{u, c\}a) + (ba\{\{u, c\}, u\} + \{\{c, u\}, u\}ab) + 
(\{b, u\}\{u, a\} + \{c, u\}\{u, a\}b) + ((\{b, u\}\{c, a\})u + \{c, a\}u\{b, u\}) + 
(\{b, c\}\{u, a\}u + \{c, b\}\{u, a\}u) + ((\{c, u\}\{u, a\}b + \{c, u\}\{a, u\}b) + 
\{au\}\{u, b\}, c + au\{\{b, c\}, u\} + au\{\{c, u\}, b\}) + 
\{bu\}\{\{a, u\}, c\} + bu\{\{u, c\}, a\} + bu\{\{c, a\}, u\}) + 
\{cu\}\{\{a, b\}, u\} + cu\{\{u, a\}, b\} + cu\{\{b, u\}, a\}) + 
\{wu\}\{\{a, b\}, c\} + wu\{\{b, c\}, a\} + wu\{c, a\}, b\} = 0.

The theorem is proved.

2.19. Generalized Poisson algebras. The variety of unital generalized Poisson algebras (see, for example, [1]) is defined by the identities

\[ xy = yx, (xy)z = x(yz), \{xy, z\} = \{x, z\}y + x\{y, z\} + D(z)xy, D(x) = \{1, x\}, \]

\[ \{x, y\} = -\{y, x\}, \{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0. \]

Theorem 19. Let \((A; \cdot, \{,\})\) be a generalized Poisson algebra. Then \((A; [\{,\}, \cdot])\) is an associative-commutative algebra, and \((A; [\{,\}, \cdot])\) is a Lie algebra.

Proof. In the first case, we have

\[ a \times b = \{u, ab\} - \{u, a\}b - a\{u, b\} = -D(u)ab + b \times a \]

and

\[ (a \times b) \times c = D(u)^2abc = a \times (b \times c). \]

In the second case,

\[ a \times b = u\{a, b\} - \{ua, b\} - \{a, ub\} = 
- a\{u, b\} - b\{a, u\} - u\{a, b\} - D(b)ua + D(a)ub = -b \times a. \]

Here we use the proof of Theorem [18] It is easy to see that

\[ \circ_{a, b, c} \]

\[ (a \times b) \times c = -D(u)^2abc = a \times (b \times c). \]
By the proof of Theorem \[20\] we can conclude that the sum of all elements without \(D\) is zero. Now, we have

\[
\bigcirc_{a,b,c} [(a \ast b) \ast c] = \bigcirc_{a,b,c} [\{b, u\} D(c) au - \{b, u\} D(u) ac - \{b, u\} D(c) au + \{b, u\} D(a) cu + D(\{b, u\} acu - \{b, a\} D(c) u^2 - \{b, a\} D(u) ca + \{b, a\} D(c) u^2 + \{b, a\} D(u) cu + D(\{b, a\} cu^2 - \{u, a\} D(c) bu - \{u, a\} D(u) bc + \{u, a\} D(c) bu + \{u, a\} D(b) cu + \{c, u\} D(b) au - \{c, D(b)\} au^2 - \{c, u\} D(b) ua - \{c, a\} D(b) u^2 + 2D(b) D(c) au^2 - \{u, D(b)\} acu + 2D(b) D(u) acu + D(b) D(c) au^2 - D(b) D(u) acu - \{u, a\} D(b) cu - D(a) D(b) acu^2 + \{c, u\} D(a) bu + \{c, b\} D(a) u^2 - 2D(a) D(c) bu^2 + \{u, D(a)\} bcu + \{u, b\} D(a) cu - 2D(a) D(u) bu + D(a) D(c) bu^2 + D(a) D(b) bcu + D(a) D(u) bu^2 + D(a) D(b) bcu^2].
\]

It is easy to see that

\[
\bigcirc_{a,b,c} [D(D(a)) bcu^2 - D(D(b)) acu^2] = 0, \\
\bigcirc_{a,b,c} [D(\{b, u\}) acu + D(\{u, a\}) bcu] = 0, \\
\bigcirc_{a,b,c} [\{u, D(a)\} bcu - \{u, D(b)\} acu] = 0, \\
\bigcirc_{a,b,c} [\{D(\{b, u\}\}) cu^2 - \{c, D(b) au^2 + \{c, D(a)\} bu^2\} = 0, \\
\bigcirc_{a,b,c} [-2\{b, a\} D(c) u^2 - \{c, a\} D(b) u^2 + \{c, b\} D(a) u^2 = 0, \\
\bigcirc_{a,b,c} [\{b, u\} D(c) au + \{u, a\} D(c) bu + \{c, u\} D(b) au + \{u, c\} D(a) bu] = 0.
\]

Obviously, \(\bigcirc_{a,b,c} [(a \ast b) \ast c] = 0\) and \([\cdot, \cdot, \cdot]\) is a Lie algebra. The theorem is proved.

2.20. Novikov-Poisson algebras. The variety of left Novikov-Poisson algebras is defined by the identities

\[
xy = yx, (xy)z = x(yz), \\
x \circ (y \circ z) = y \circ (x \circ z), (x, y, z) = (x, y, z), \\
x \circ (yz) = (x \circ y)z, (xy) \circ z = x(y \circ z) = (xz) \circ y - x(z \circ y).
\]

Theorem 20. Let \((A; \ast, \circ)\) be a left Novikov-Poisson algebra. Then \((A; [\cdot, \cdot])\) is a left Novikov algebra and \((A; [\cdot, \cdot, \cdot])\) is an associative-commutative algebra.

Proof. Firstly, we have

\[
a \ast b = u(a \circ b) - (ua) \circ b - a \circ (ub) = -(ua) \circ b.
\]

Hence,

\[
a \ast (b \ast c) = (ua) \circ ((ub) \circ c) = (ub) \circ ((ua) \circ c) = b \ast (a \ast c),
\]

and

\[
(a, b, c) = (a \ast b) \ast c - a \ast (b \ast c) = (u((ua) \circ b)) \circ c - (ua) \circ ((ub) \circ c) =
\]

\[
((ua) \circ (ub)) \circ c - (ua) \circ ((ub) \circ c) = (ua, ub, c) = (ua, c, ub) =
\]

\[
((ua) \circ c) \circ (ub) = (ua) \circ (c \circ (ub)) =
\]

\[
u(ua, c, b) = u(ua, b, c) = (a \ast c) \ast b - a \ast (c \ast b) = (a, c, b).
\]

Secondly,

\[
a \ast b = u \circ (ab) - (u \circ a)b - a(u \circ b) = -u \circ (ab) = b \ast a.
\]

Therefore,

\[
(a \ast b) \ast c = u \circ ((u \circ (ab))c) = u \circ (u \circ (abc)) = u \circ (a(u \circ (bc))) = a \ast (b \ast c).
\]

The theorem is proved.

Similarly, the variety of right Novikov-Poisson algebras may be defined (see, for example, \[21\]). It is easy to prove the following theorem:
Theorem 21. Let \((A;\cdot,\circ)\) be a right Novikov-Poisson algebra. Then \((A;[\cdot,\circ])\) is a right Novikov algebra and \((A;[\circ,\cdot])\) is a commutative algebra.

3. The Kantor square of algebras of special type.

Here we study some special cases of the Kantor square. For an algebra \(A := (A;\cdot)\) its the Kantor square \((A;\wedge,\cdot)\) we denote by \((A,\ast)\). We denote the Kantor square for a fixed element \(u\) by \((A,\ast_u)\). We consider the relations between the ideals in \(A\) and \((A,\ast)\), the relations between an associative algebra \(A\) with polynomial identity and its the Kantor square. Moreover, the relations between the nilpotency and right nilpotency in \(A\) and \((A,\ast)\) are investigated.

3.1. Ideals in the Kantor product.

Theorem 22. Let \(I\) be an ideal of \(A\). Then \((I,\ast)\) is an ideal of \((A,\ast)\), but the converse statement is not true in general.

Proof. It is easy to see that if \(i \in I\) and \(a \in A\) then
\[
i \ast a = u(ia) - (ui)a - i(ua) \in I\text{ and }a \ast i = u(ai) - (ua)i - a(ui) \in I.
\]
It follows that \(I\) is an ideal of \((A,\ast)\).

Conversely, we can consider the trivial case, where for an algebra \(A\) has zero Kantor square (for example, Lie or Leibniz algebra) and every subspace of \(A\) is an ideal of \((A,\ast)\). For the non-trivial case (nonzero Kantor product), we can consider the following associative algebra: \(A_1 \oplus A_2\) is the direct sum of the matrix algebras of order 2. Here, if \(e_j\) is the unit of \(A_j\) then the subspace generated by \(A_1\) and \(e_2\) is an ideal of \((A,\ast_{e_1})\), but is not an ideal of \(A_1 \oplus A_2\). The theorem is proved.

3.2. Associative algebras with polynomial identity. Given a polynomial \(f\) in \(n\) variables, we define \(f_s(x_1,\ldots,x_n)\) as the value of \(f\) in \((A,\ast)\), where \(x_1,\ldots,x_n\) are some elements in \(A\).

Theorem 23. Let \((A;\cdot)\) be an associative algebra that satisfies the polynomial identity \(f(x_1,\ldots,x_n)\). Then there exists an identity \(g\) such that \(A\) and \((A,\ast)\) satisfy \(g\).

Proof. It is easy to see that if \(A\) satisfies the identity \(f(x_1,\ldots,x_n)\) then \(A\) satisfies the identity \(g(x_1,\ldots,x_n,z) = f(x_1,\ldots,x_n)z\). By Theorem 14, the multiplication in algebra \((A,\ast)\) is defined by \(x \ast y = -xuy\). Now, we can calculate the element \(g_s(x_1,\ldots,x_n,z)\) in \((A,\ast)\). Obviously, it is \((-1)^n f(x_1u,\ldots,x_nu)z\) which amounts to zero in \(A\). It follows that \((A,\ast)\) satisfies the identity \(g\). The theorem is proved.

One of the most popular identity in the associative algebras is the standard polynomial identity of degree \(n\):
\[
s_n(x_1,\ldots,x_n) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.
\]

Theorem 24. Let \((A;\cdot)\) be an associative algebra that satisfies \(s_n\). Then \((A,\ast)\) satisfies \(s_{n+1}\).

Proof. It is easy to see that the standard polynomial of degree \(n+1\) may be written as
\[
s_{n+1}(x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n} \left( \sum_{\sigma \in S_n, \sigma(n+1) = i} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \right) =
\]
\[
= \sum_{i=1}^{n} (\epsilon_i s_n(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1})x_i), \quad \epsilon_i = \pm 1.
\]
Now, by the proof of Theorem 23 \((A,\ast)\) satisfies the standard polynomial identity of degree \(n+1\). The theorem is proved.
3.3. Nilpotent algebras. For the nilpotent algebras, we can prove the following theorem.

Lemma 25. Let \( (A; \cdot) \) be a nilpotent algebra of nilpotency index \( n \). Then \( (A, \ast) \) is a nilpotent algebra of nilpotency index \( \leq \lceil n/2 \rceil + 1 \).

Proof. Obviously, every product of the form \( x_1 \cdot x_2 \cdot \ldots \cdot x_n \) (with some order of brackets) is a sum of multiplications of the form \( y_1 y_2 \cdot \ldots \cdot y_{n-1} \) (with some order of brackets). Now, it is easy to see that \( (A, \ast) \) is nilpotent and its index of nilpotency \( \leq \lceil n/2 \rceil + 1 \). The Lemma is proved.

3.4. Right-nilpotent algebras. An algebra \( A \) is called right-nilpotent (or left-nilpotent) of nilpotency index \( n \) if it satisfies the identity
\[
(x_1 \cdot x_2 \cdot \ldots \cdot x_n) = 0 \quad \text{or} \quad x_1 (x_2 \cdot x_3 \cdot \ldots \cdot x_n) = 0.
\]
Curiously, an analogue of the Theorem 25 is not true for the right-nilpotent algebras.

Theorem 26. There exists a right nilpotent algebra \( (A; \cdot) \) such that \( (A, \ast_u) \) is not right nilpotent, but \( (A, \ast_u) \) is solvable.

Proof. An algebra \( A \) is right alternative if the following identity holds in \( A \):
\[
(xy)_y = 0.
\]
It is interesting fact that in contrast to the algebras of many well-studied classes (Jordan, alternative, Lie and so on) a right nilpotent right alternative algebra need not be non-nilpotent. The corresponding example of a five-dimensional right nilpotent but not nilpotent algebra belongs to Dorofeev [5]. Its basis is \( \{a, b, c, d, e\} \), and the multiplication is given by (zero products of basis vectors are omitted)
\[
ab = -ba = ae = -ea = db = -bd = -c, \quad ac = d, \quad bc = e.
\]
It is easy to see that
\[
c \ast_u b = a(cb) - (ac)b - c(ab) = c.
\]
Obviously, \( c = (\ldots (c \ast_a b) \ast_a \ldots) \ast_a b \neq 0 \), and \( (A, \ast_u) \) is not right-nilpotent. It is easy to see that \( A^2 \subseteq \langle c, d, e \rangle \) and \( A \ast_u A \subseteq \langle c, d, e \rangle \), but \( (A \ast_u A) \ast_u (A \ast_u A) = 0 \), and \( (A, \ast_u) \) is solvable.

The theorem is proved.

3.5. Derivations. Remember that a linear mapping \( D \) of an algebra \( A \) is called a derivation if it satisfies the relation \( D(xy) = D(x)y + xD(y) \). By [12], an element \( a \) of an algebra \( A \) is called a Jacobi element if it satisfies the relation \( a(xy) = (ax)y + x(ay) \). All Jacobi elements of \( A \) form a vector space, which is called the Jacobi space of \( A \).

Lemma 27. Let \( D \) be a derivation of both \( A \) and \( (A, \ast) \). Then
1) If \( A \) has zero Jacobi space, then \( D = 0 \);
2) If \( D \) is invertible, then \( A \) is a left Leibniz algebra and \( (A, \ast) \) is a zero algebra. In particularly, if \( A \) is a finite-dimensional algebra over a field of zero characteristic, then \( A \) is nilpotent.

Proof. 1). By simple calculations, from \( D(x \ast y) = D(x) \ast y + x \ast D(y) \), we have
\[
D(u)(xy) = (D(u)x)y + x(D(u)y).
\]
By the definition of the Jacobi space, we have \( D = 0 \).

2). By invertibility of mapping \( D \) and arbitrarity of element \( u \), we infer that \( A \) is a left Leibniz algebra. By the Lemma [5] we imply that \( (A, \ast) \) is zero algebra.

In [10] it was proved that a finite-dimensional Leibniz algebra over a field of characteristic zero which admitting an invertible derivation is nilpotent. The Lemma is proved.
3.6. Automorphisms. Remember that an invertible linear mapping $\phi$ of an algebra is called an automorphism if it satisfies the relation $\phi(xy) = \phi(x)\phi(y)$.

**Lemma 28.** Let $\phi$ be an automorphism of both $A$ and $(A, \ast)$. If $A$ is an algebra with zero Jacobi space, then $\phi$ is the identity mapping.

**Proof.** By simple calculations from $\phi(x \ast y) = \phi(x) \ast \phi(y)$, we have
\[
(u - \phi(u))(xy) = ((u - \phi(u))x)y + x((u - \phi(u))y).
\]
By the definition of the Jacobi space, we have that $\phi = id$. The Lemma is proved.

3.7. Isomorphic Kantor squares. Here we talk about the situation where algebra $A$ and its Kantor square are isomorphic.

**Theorem 29.** Let $A$ be a finite-dimensional associative algebra. Then $A$ is isomorphic to $(A, \ast)$, if and only if $A$ is a skew field.

**Proof.** Let $f_u$ is an isomorphism between algebras $A$ and $(A, \ast_u)$ and $f_u(xy) = f_u(x) \ast u(y) = -f_u(x)u f_u(y)$. If in $A$ there are two elements $u$ and $v$ with zero product, for $x = f_u^{-1}(v)$ we have
\[
f_u(f_u(ab)) = -f_u(f_u(a)f_u(b)) = f_u(f_u(a)uvf_u(f_u(b))) = 0.
\]
Now, if there is a zero divisor, then the algebra $A$ has zero multiplication. It is Well known that every finite-dimensional algebra without zero divisors is a skew field.

On the other side, for some fixed nonzero element $u$ from a skew field $A$ we define $f_u(a) = -au^{-1}$. It is an isomorphism between algebras $A$ and $(A, \ast_u)$ for every nonzero element $u$. The theorem is proved.

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