RIGOROUS “RICH ARGUMENT” IN MICROLENSING PARALLAX

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Abstract: I show that when the observables \((\pi_E, \theta_E, \pi_s, \mu_s)\) are well measured up to a discrete degeneracy in the microlensing parallax vector \(\pi_E\), the relative likelihood of the different solutions can be written in closed form \(P_i = K H_i B_i\), where \(H_i\) is the number of stars (potential lenses) having the mass and kinematics of the inferred parameters of solution \(i\) and \(B_i\) is an additional factor that is formally derived from the Jacobian of the transformation from Galactic to microlensing parameters. The Jacobian term \(B_i\) constitutes an explicit evaluation of the “Rich Argument”, i.e., that there is an extra geometric factor disfavoring large-parallax solutions in addition to the reduced frequency of lenses given by \(H\). Here \(\theta_E\) is the Einstein timescale, \(\theta_E\) is the angular Einstein radius, and \(\pi_s, \mu_s\) are (the parallax, proper motion) of the microlensed source. I also discuss how this analytic expression degrades in the presence of finite errors in the measured observables.

Key words: gravitational microlensing

1. INTRODUCTION

The “Rich argument” played an important role in motivating space-based microlensing studies. When Refsdal (1966) introduced space-based microlensing parallax, he already realized that it would yield four degenerate solutions in what we now call the microlensing parallax vector,

\[
\pi_E = \frac{\pi_E}{\mu}, \quad \pi_{\text{rel}} = \frac{\pi_{\text{rel}}}{\theta_E},
\]

(1)

where \(\theta_E\) is the Einstein radius and \((\pi_E, \mu)\) are the lens-source relative (parallax, proper motion). See Figure 1 of Gould (1994) for an illustration of how this degeneracy arises and Figure 1 of Yee et al. (2015a) for the first practical example.

This problem initially appeared as quite severe: in the great majority of cases for which the actual value of \(\pi_E\) was small (e.g., \(\pi_E \lesssim 0.1\)), there would be an alternate solution in which it was large (e.g., \(\pi_E \sim 1\)). That is, the microlens parallax is given by

\[
\pi_E = \frac{\text{AU}}{D_L} \left( \frac{t_{0,\text{sat}} - t_{0,\oplus}}{t_E}, u_{0,\text{sat}} - u_{0,\oplus} \right),
\]

(2)

where \((t_0, u_0)\) are the time of peak and impact parameter as seen from either Earth or the satellite, \(t_E = \theta_E/\mu\) is the Einstein timescale, and \(D_L\) is the projected satellite-Earth separation vector. Because \(u_0\) is a signed quantity for which only the absolute value is normally measured, events whose true \(u_0\) values are \((u_{0,\text{sat}}, u_{0,\oplus}) = (0.5, 0.4)\) can also be interpreted as \((u_{0,\text{sat}}, u_{0,\oplus}) = (0.5, -0.4)\) and hence (according to Equation (2)) with a second parallax component that is nine times larger. Then, because \(\pi_E\) enters directly into the mass and distance estimates,

\[
M = \frac{\theta_E}{\kappa \pi_E}; \quad \pi_{\text{rel}} = \theta_E \pi_E; \quad \kappa = \frac{4G}{c^2} \text{AU} \simeq 8.14 \text{ mas} \frac{\text{AU}}{M_\odot}
\]

(3)

this degeneracy appeared to pose a major obstacle to the interpretation of any space-based microlensing experiment.

Refsdal (1966) had already proposed a “simple” solution: launch a second satellite into solar orbit to take simultaneous observations. See also Figure 4 of Gould (1995). However, given the challenges (in the first place, the expense) of launching even one such satellite, this did not appear as a practical approach.

James Rich (circa 1997, private communication) argued that these alternate solutions were geometrically improbable. Hence, while they could not be ruled out in any particular case, their presence would not interfere with the statistical interpretation of a parallax-satellite experiment. This insight had an important motivating impact on early workers who were investigating the mathematical and physical basis of microlensing parallax.

However, this argument only became widely known when, following the first large-scale satellite-parallax campaign (Udalski et al. 2013; Yee et al. 2015a) using the Spitzer satellite in solar orbit, Calchi Novati et al. (2015) explicitly gave this argument and made the first attempt to quantify it in the course of analyzing 21 Spitzer events from 2014. Based on purely geometric reasoning, they argued that larger-parallax solutions were disfavored by \(\pi_E^2\).

It was always known that, in the absence of any other information, large parallax (i.e., nearby-lens) solutions were disfavored simply because of the smaller volume available. And also that this effect was often augmented...
by the lower space density of stars for the more nearby solution. However, the “Rich argument” was regarded (correctly, as we will see) as additionally disfavoring the large-parallax solutions.

In fact, Batista et al. (2011) already explicitly noted such an effect in their analysis of MOA-2009-BLG-387, for which purely ground-based data yielded a measurement of $\pi_E$ with large error bars. When they estimated $\pi_E$ by weighting the microlensing likelihood of each $\pi_E$ value according to a prior based on a Galactic model, they found additional purely geometric terms in the Jacobian arising from the transformation from physical coordinates to microlensing parameters. See their Equations (17)–(18). They then showed (lower-left panel of their Figure 6) that the combination of Galactic and Jacobian factors drove the solution 2–3 $\sigma$ from the best fit based only on the $\chi^2$ of the microlensing fit.

Mathematically, the “discrete degeneracies” (multiple isolated maxima in the likelihood function) that appear routinely in space-based microlensing parallax are simply a special case of a more general likelihood function, such as the one analyzed by Batista et al. (2011). Therefore, exactly the same Galactic factors and Jacobian factors should appear in both. However, while this statement would probably have appeared obvious if it had been so formulated, it was not made initially. Hence, for example, Zhu et al. (2017) considered two prescriptions for weighting discrete $\pi_E$ solutions in their analysis of 50 Spitzer events from 2015. In one, they calculated the likelihood of solutions using a product of the light-curve likelihood and Galactic-model likelihood. In the second, they further multiplied by $\pi_E^{-2}$ for the “Rich argument”. Their Galactic model was implemented by numerical integration over physical parameters, and hence it implicitly contained the Jacobian factors discussed above. However, as a practical matter they found that their statistical results depended only weakly on this choice. In their study of the Zhu et al. (2017) results, Koshimoto & Bennett (2018) argued that the “Rich argument” was simply an ad hoc way of evaluating the Galactic prior. Nevertheless, for completeness they likewise considered both cases (with and without the extra factor derived from the “Rich argument”), and they likewise found that the choice had only a weak effect on their statistical conclusions.

Here I evaluate analytically (in closed form) the relative likelihood of discretely degenerate microlens parallax solutions for the case that the observables are well measured. I show that the relative probability $P_i$ takes the form

$$P_i = KH_iB_i,$$  \(\text{(4)}\)

where $H_i$ may be thought of as the number of Galactic stars with the physical properties (mass, distance, transverse velocity) of the $i$-th inferred solution and $B_i = D_{\nu,i}/\pi_{E,i}$ is an additional factor coming from the Jacobian. The latter should be associated with the “Rich argument”, although it differs somewhat from the $\pi_{E}^{-2}$ factor that had been originally proposed. I also discuss how this exact formula evolves as the assumption of perfect measurement of the observables is relaxed.

### 2. Derivation

I assume that the Einstein timescale $t_E$ and the angular Einstein radius $\theta_E$ are precisely measured, but that the microlens parallax $\pi_E$ suffers from a discrete degeneracy. While we will be most interested in the case that each of these local solutions is also precisely determined, it will also be important to consider that these measurements have finite, and possibly different, error ellipses (or more generalized error distributions). In addition, consideration of these finite error distributions will allow us to better understand how the “Rich factor” behaves in the face of deteriorating errors.

For simplicity, I will initially assume that the source proper motion $\mu_s$ and source parallax $\pi_s$ are also known precisely. While, this is sometimes true of $\mu_s$, it is essentially never true of $\pi_s$, so I will later discuss how the results are affected when these assumptions are relaxed.

In the usual formulation of the problem, there are then four observables,

$$(\theta_E, t_E, \pi_{E,N}, \pi_{E,E}) \quad (\text{“standard” observables})$$  \(\text{(5)}\)

Hence, for example, when the errors in one or more of these quantities is poorly constrained (or unconstrained), one carries out a Monte Carlo simulation of many events drawn from a Galactic model, and one then derives values and error bars for various physical properties, such as the lens mass $M = \theta_E/\kappa\pi_E$, by summing over the simulated events that are consistent with these measured observables.

A key point of principle is that $(\pi_{E,N}, \pi_{E,E})$ are not in fact “observable”. Rather what is observed is $(\Delta \tau, \ln |u_{0,\text{sat}}|, |u_{0,0}|)$. Then there are four different combinations of $(\pi_{E,N}, \pi_{E,E})$ that are consistent with these observables. Hence in the Monte Carlo integration imagined in the previous paragraph, the vast majority of parameter space would contribute essentially nothing to the integral, while the integrand would be finite in four small regions where the values of $\pi_E$ reproduced the “true observables”: $(\Delta \tau, \ln |u_{0,\text{sat}}|, |u_{0,0}|)$.

Nevertheless, for simplicity of exposition, I will treat $(\pi_{E,N}, \pi_{E,E})$ as observables, keeping in mind that they are a short hand that is applicable only locally for the quantities that are actually observed. I will also substitute $\mu = \theta_E/t_E$ for $t_E$ as an observable. Then the four observables become

$$(\mu, \theta_E, \pi_{E,N}, \pi_{E,E}) \quad (4 \text{ adopted observables})$$  \(\text{(6)}\)

In general, if we want to estimate the Bayesian expectation of some quantity $Z$, that is a function of observables, we would evaluate the ratio of integrals

$$\langle Z \rangle = \frac{\int d^2\mu \ln D_L H(\mu, D_L, M) \theta_E \mu E(\theta) Z(\theta)}{\int d^2\mu \ln D_L H(\mu, D_L, M) \theta_E \mu E(\theta)},$$  \(\text{(7)}\)

where

$$H(\mu, D_L, M) \equiv f(\mu)\rho(D_L) D_L^2 \Phi(M)$$  \(\text{(8)}\)

is the “effective number” of potential lenses with Galaxy parameters $(\mu, D_L, M)$, and where $\theta \equiv$
The first term in brackets can be regarded as the naive “Galactic model term”, which simply records the local frequency of lenses with the physical properties inferred from the solutions. That is, this is the total number of such lenses ($\rho D_L^2$) times the fraction of such lenses with the inferred mass $M$ and relative proper motion $\mu$. The second term in brackets is the suppression of nearby (small $D_L$, large $\pi_E$) solutions, i.e., the “Rich argument”. Note that this second factor is stronger (i.e., larger for larger $D_L$) by $D_L \pi_E \propto (1 - D_L/D_S)$ (for fixed $\theta_E$) than the factor derived using more qualitative arguments by Calehi Novati et al. (2013).

3. RELAXING ASSUMPTIONS

I had assumed that $(\pi_s, \mu_s)$ are known precisely. The assumption about $\mu_s$ plays almost no role. This vector only enters via $f(\mu)$, where $\mu = \mu_1 - \mu_2$. The assumption regarding $f(\mu)$, was only that it was effectively linear in $\mu$ over the space of allowed solutions. For small error bars in $D_L$, $\pi_E = \mu/\pi_E$, then this is likely to be true even when $\mu$ is known. For the case that $\mu_s$ is not known, the distribution is more “smeared out” and therefore even more consistent with being linear over small regions. By contrast, $\pi_s$ is rarely if ever known precisely. More typically, it is estimated with a 10% error. Still, if one makes the evaluation at the average value of $\pi_s$, then the error in the ratio of $D_L$ terms is quadratic in the fractional error in $\pi_s$, so no more than a few percent. This is not likely to enter in a material way into quantitative probability arguments.

A more serious issue is the assumption that the joint likelihood distribution of the two components of $\pi_E$ is Gaussian. Actually, the argument given does not really require that it be Gaussian, but only that it be symmetric in reflections through the best fit $\pi_E$. The problem is that even this weaker condition is not met for many events in the Spitzer microlensing survey. In particular, events for which the Spitzer data do not cover the peak (or, more accurately, do not probe an approach to the peak), the parallax solutions tend to form an arc (Gould, 2013). In severe cases, the two arcs can even merge to form a large part of a circle (Gould, 2013; Zang et al., 2020a). In such cases, it does not even make sense to talk about separate solutions and so relative probability of separate solutions. However, there can be intermediate cases for which the arcs deviate considerably from an ellipse but still form two separate solutions. Depending on the precision required, one might have to abandon the analytic result given in Equation (15) and carry out a numerical evaluation. However, one should keep in mind that the default procedure for such numerical Bayesian analyses is often to represent the parallax error distribution as a Gaussian. If such an approximation is made, the final result will essentially reproduce the analytic results given here (unless the error ellipse is so large as to violate the linearity assumption for $\eta$, defined in Equation (14)).

Another, much more common, deviation from the assumptions of Section 2 is that $\theta_E$ is not measured. This is relatively rare for planetary and binary events, for
which caustic crossings are usually observed, leading to measurements of $\rho$ and so $\theta_E = \theta_\ast / \rho$. These events are of exceptional interest and so generally lead to the most detailed investigations.

However, non-planetary (point-lens) events are also very important, if only because they form the comparison sample for planetary events. Moreover, some planetary events do not have caustic crossings, and in these cases $\theta_E$ is generally not measured.

In this case, the formalism of Section 2 can still be used, but significant caution is required. The key problem is that in these cases $D_L$ (for each degenerate parallax solution) can only be estimated with the aid of a Bayesian analysis. The physical basis for such $D_L$ determinations was already recognized by Han & Gould (1995): the projected velocities $\tilde{v} = AU/\pi \theta_E t_E$ of disk lenses is roughly proportional to $D_L$, and those of bulge lenses are very high. Thus, in most cases, such Bayesian analyses will yield a relatively well localized distance. And with this distance (provided that it is relatively well localized), one can directly apply Equation (15).

A “problem” with this approach is that the Bayesian analysis, in addition to yielding a distance estimate (via Equation (7) with $Z \rightarrow D_L$) will also yield the probability of this solution (via the denominator of this same equation). Moreover, it does so without the simplifying assumptions that went into the analytic results of Equation (15).

Nevertheless, even in these cases, Equation (15) provides an important sanity check on the results of the numerical integration. That is, while the logic of the Bayesian integration over a Galactic model is transparent, its output can be somewhat opaque. Hence, it is quite useful to have a simple consistency check on this output.

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1Note that this concern does not apply to another class of non-planetary events: isolated-star mass measurements. For these, by definition, $\theta_E$ is measured.