Black-hole entropy and minimal diffusion

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(Dated: July 23, 2013)

The density of states reproducing the Bekenstein-Hawking entropy-area scaling can be modeled via a nonlocal field theory. We define a diffusion process based on the kinematics of this theory and find a spectral dimension whose flow exhibits surprising properties. While it asymptotes four from above in the infrared, in the ultraviolet the spectral dimension diverges at a finite (Planckian) value of the diffusion length, signaling a breakdown of the notion of diffusion on a continuum spacetime below that scale. We comment on the implications of this minimal diffusion scale for the entropy bound in a holographic and field-theoretic context.

PACS numbers: 04.70.Dy, 04.60.Bc, 11.10.Lm

I. INTRODUCTION

Evidence has been accumulating in recent years suggesting that at small scales the dimensionality of spacetime might flow to values different than four due to quantum effects of the geometry [1]. The notion of spectral dimension can be used to explore spacetime geometries beyond the usual picture of smooth manifolds which can emerge in quantum gravity scenarios. Its running to lower values in the ultraviolet is a common feature of several approaches [2–8]. In general, whenever a suitable generalization of the Laplacian operator is available one can consider a diffusion process via a heat equation and the return probability allows one the definition of the spectral dimension. This probe of nonconventional degrees of freedom originates from quantum-gravitational degrees of freedom. In [10, 11], it was argued that black holes provide a nonlocal field theory which effectively models the de-
be probed in such an effective theory. In principle, the
smearing via the form factor $\Omega^2$ could also lead to a sort of
Planck-scale “fuzziness” of spacetime, where geometry
can still be described by conventional indicators such as
the spectral dimension. In other words, it is not obvious
whether the correlation length $\sim \sqrt{\sigma} = O(\ell_{Pl})$ acts as a
watershed between two spacetime regimes or as a lower
bound for length measurements. Here, we will show that
the model is constructed in such a way that there is no
manifold structure below the Planck scale. The Planck
length is a minimal physical scale to all purposes, and it
becomes meaningless to ask how spacetime is modified
at smaller distances.

Let us stress that there are only two key requirements
beyond our main result, both stated in [10]. The first
is the presence of a hypersurface with infinite redshift.
Roughly speaking, the event horizon “stretches” vir-
tual high-energy field excitations (representing the var-
iations of matter with quantum geometry) to
sub-Planckian energies and allows them to become real
modes, which then populate thermodynamical energy
levels. The second is Eq. (4). Any microscopic theory
of quantum gravity with the correct effective density of
states [11] for a black hole, i.e., predicting the entropy-
area law (string theory [13] and loop quantum gravity
[14] are examples), will be described (after some coarse-
graining approximation) by a nonlocal effective model of
the above or similar form near the horizon, with correla-
tion functions displaying a universal short-scale behavior.
As argued in [10], it is not necessary to know the details
of the ultimate theory, assuming it exists, to reproduce
some basic thermodynamical properties. In this sense,
our conclusions on the spectral dimension will not hold
in scenarios violating our working hypotheses, but other-
wise they will be quite general.

II. NONLOCALITY, DIFFUSION, AND
SPECTRAL DIMENSION

To probe the local structure of spacetime, we Eu-
clideanize coordinate time $t$, $t \rightarrow -i t = x_D$, and let a
pointwise test particle diffuse starting from some space-
time point $x' = (x_D', x')$. We then ask what is the proba-
bility $P$ to find the particle at another point $x = (x_D, x)$
after some abstract diffusion “time” $\sigma$ has elapsed.
This process is encoded in a diffusion equation. For instance,
the ordinary diffusion equation for Minkowski spacetime
reads $(\partial_\sigma - \Box_E) P = 0$, where $\Box_E = \partial_D^2 + \nabla^2$ and $\nabla^2$ is the spatial Laplacian; this yields an ordinary Brown-
ian motion with Gaussian probability density function $P$
and, eventually, a spectral dimension $d_S = D$.

In our case, we have to replace the standard Laplacian
with a nonlocal derivative operator, reproducing, in mo-
mentum space, the dispersion relation [3] with Eq. (4).
It is easy to convince oneself that the diffusion equation
should be of the form

$$[\partial_\sigma - F(i \partial_D, \nabla)] P(x, x', \sigma) = 0,$$

where

$$F(i \partial_D, \nabla) = \frac{\partial^2}{\partial x_D^2} - \frac{D - 1}{2 \sigma_+} \ln \left[ 1 - \frac{2 \sigma_+}{D - 1} \nabla^2 \right].$$

To see this, we notice that a nonlocal interaction can al-
ways be expressed as a nonlocal kinetic term [12]. In fact
(time dependence in $\phi$ omitted), $\int dy \Omega^2(y - x) \phi(y) =
\int dz \Omega^2(z) \phi(z + x) = \int [d z \Omega^2(z) e^{ik \cdot z}] \phi(x)$. Taking the
Fourier transform of $e^{ik \cdot z}$ and using the dispersion rela-
tion [3], we get

$$\int dz \Omega^2(z) e^{ik \cdot z} = \int dz \Omega^2(z) \int dk e^{i k x} \delta(k - i \nabla_x) = \int dk \omega^2(k) \delta(k - i \nabla_x)$$

$$= \omega^2(i \nabla_x),$$

which yields (7) when adopting Eq. (4). The operator $\omega$
mathematically well defined [14] and admits a series
representation with finite coefficients [in general, even
well-defined nonlocal operators do not, e.g., $(\nabla^2)^\alpha$]
with $\alpha$ complex]:

$$F(i \partial_D, \nabla) = \frac{\partial^2}{\partial x_D^2} + \frac{D - 1}{2 \sigma_+} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{2 \sigma_+}{D - 1} \nabla^2 \right)^m.$$ (9)

In the limit $\sigma_+ \rightarrow 0$ ($\ell_{Pl} \rightarrow 0$, large diffusion scales),
$F \rightarrow \Box_E$, as one can see from Eq. (9).

The solution of Eq. (6) is

$$P(x, x', \sigma) = \int \frac{d^D k}{(2 \pi)^D} e^{-\sigma F(i k, k)} e^{i k \cdot (x - x')}.$$ (10)

This expression can be computed exactly. The integral in $k^0$
 yields the usual Gaussian normalization $1/\sqrt{4 \pi \sigma}$,
while the integral in spatial momenta can be done in polar coordinates.
The result is

$$P(x, x', \sigma) = e^{-\frac{r^2}{4 \pi^2 \sigma}} \frac{2^{\frac{D+1}{2}}}{\Gamma \left( \frac{D+1}{4} \right)} \left( \frac{D-1}{4 \pi \sigma} \right)^{\frac{D-1}{2}}$$

$$\times \left( \frac{r}{r_\sigma} \right)^{\frac{D-1}{2} \left( \frac{D-1}{2} \right)} K_{\frac{D-1}{4}} \left( \frac{r}{r_\sigma} \right),$$

where $\Gamma$ is Euler’s function and $\sigma_+$ is given by Eq. (5).
Instead of imposing an initial condition for (10), we
fixed the normalization of $\int d^D x P$ to 1. At $\sigma = 0$
we do not get the usual delta, due to the smearing ef-
fect: $P(x, x', \sigma \sim 0) \sim -\delta(x_D - x_D') \sigma \Omega^2(r) \neq \delta(x - x')$.
From Eq. (11), we can extract much information. First,
$P$ defines probabilistic expectation values of the form
$$\langle f(x) \rangle = \int_{-\infty}^{\infty} d^D x P(x, 0, \sigma) f(x),$$

with (1) = 1. In particular, the mean squared displacement is

$$\langle x^2 \rangle = \langle x_D^2 + r^2 \rangle = 2 D \sigma, \quad x \neq 0,$$ (12)
and diffusion is nonanomalous (the walk dimension determined by \( \langle x^2 \rangle \propto t^{2D} \)) is equal to 2). However, it is not ordinary, either. The trace of \( P \) in position space is the return probability \( \mathcal{P}(\sigma) := \int d^3x \, P(x, x, \sigma) \). At small \( z, z^2 K_{\perp \nu}(z) \sim z^{2-\sigma/(\nu(n+1))} \) if \( \nu \neq 0 \), so that

\[
\mathcal{P}(\sigma) = \mathcal{A} \sqrt{\frac{2}{\sigma}} \Gamma\left(\frac{D-1}{2} \frac{\sigma - 1}{\sigma} \right), \quad \sigma > \sigma_*,
\]

where \( \mathcal{A} \) is a divergent constant proportional to the total volume \( V \). This expression diverges at \( \sigma = (1 - 2n)\sigma_* \), where \( n \in \mathbb{N} \). Since \( \sigma > 0 \), the only singular point of interest is \( \sigma = \sigma_* \) (\( n = 0 \)). At scales \( \sigma < \sigma_* \), the return probability is no longer positive semidefinite, implying that \( P \) is not a probability density function at coincident points \( x = x' \). Taking \( \sigma \to 0 \) means probing infinitely close points within an infinitely small diffusion distance \( \delta \sqrt{\sigma} \); but at scales \( \sigma < \sigma_* \), the diffusion process is ill defined. Consequently, in this range of scales there is no diffusive process by which the spectral dimension could be defined. As \( \mathcal{P} \) should also be continuous, it is natural to interpret the point \( P(x, x', \sigma) \) for the stochastic process is set a posteriori. Positivity of the solution of the diffusion equation at all initial points is a strong criterion characterizing an effective quantum geometry, which not only consolidates the determination of the number \( d_\text{WS} \) on physical grounds, but also constitutes a finer tool to classify geometries with the same spectral dimension [7,8].

The presence of a minimal diffusion length \( \ell_* = \sqrt{\sigma_*} \) suggests that physical happenings cannot be separated by time-space scales smaller than \( \ell_* \), and that we are actually facing a discreteness effect. Spacetime near a black hole shows an effective discrete structure at microscopic scales. It is exciting to notice that this picture is compatible with the holographic principle: a discrete structure facing a discreteness effect. Spacetime near a black hole is the Hausdorff dimension (in this case, \( d_\text{H} = D \)). This is expected, since the density of states \( [10,11] \) is not the one met in fractals [7].

From Eq. (13), we get the analytic expression for the spectral dimension:

\[
d_\Sigma(\sigma) := -2 \frac{\partial \ln \mathcal{P}(\sigma)}{\partial \ln \sigma} = 1 + (D - 1) \frac{\sigma}{\sigma_*} \psi\left(\frac{D - 1}{2} \frac{\sigma}{\sigma_*} \right) - \psi\left[\frac{D - 1}{2} \left( \frac{\sigma}{\sigma_*} - 1 \right) \right], \quad \sigma > \sigma_*,
\]

where \( \psi(a) = \partial_a \Gamma(a)/\Gamma(a) \) is the digamma function.

Figure 1 shows the whole profile (14) in \( D = 4 \) for \( \sigma > 0 \), with the understanding that the dashed part for \( \sigma < \sigma_* \) is reported only for illustrative purposes. Asymptotically,

\[
d_\Sigma \sim \begin{cases} +\infty \quad (\sigma = \sigma^*_*) \\ \frac{D}{\sigma} \quad (\sigma \gg \sigma_*) \end{cases}.
\]

While in the infrared \( d_\Sigma \) tends to the topological dimension, at the critical minimal length scale \( \ell_* \), it diverges. This confirms quantitatively the limitation in measuring times and lengths incorporated in the framework of [10,11]. Notice that in Eq. (6) we chose the diffusion coefficient \( \ell \) (a length) in front of \( \ell \) to be equal to 1, so that \( \sigma \) has dimension (length)\( ^2 \). If we had defined units so that the critical scale \( \sigma_* \to 4\pi\ell_\text{Pl}^2/\ell \) were a length, taking \( \ell = O(1)\ell_\text{Pl} \) would have led to the same minimal length \( \ell_* \) as above, modulo an immaterial \( O(1) \) prefactor which can always be reabsorbed in the diffusion parameter \( \sigma \).

An anomalous spectral dimension is compatible with the normal walk dimension \( d_\text{WS} = 2 \) obtained from Eq. (12) because, contrary to fractals, \( d_\text{WS} \neq 2d_\text{H}/d_\Sigma \), where \( d_\text{H} \) is the Hausdorff dimension (in this case, \( d_\text{H} = D \)). This is expected, since the density of states \( [10,11] \) is not the one met in fractals [7].

III. DISCUSSION

To the best of our knowledge, in the context of effective models of quantum gravity, this is the first example of a profile for the spectral dimension which stops at a minimal diffusion length. There are cases where geometry possesses some characteristic scale \( \ell_{\text{crit}} \), which, however, is not minimal. Models of “fuzzy manifolds” have a transition at a critical length \( \ell_{\text{crit}} \) where the Hausdorff dimension \( d_\text{H} = 2 - (D - d_\Sigma) \) becomes negative [13], a feature which may have some connection with results in multifractal geometry [18]. However, the spectral dimension \( d_\Sigma = \sigma D/(\sigma + \ell_{\text{crit}}) \) falls to \( d_\Sigma \sim 0^n \) all the way down to vanishing diffusion scale \( \sigma \to 0 \) [19]. Also in asymptotic safety, the intrinsic fuzziness of the quantum geometry [24] does not imply a minimal diffusion scale and the limit \( d_\Sigma(\sigma \to 0) \) is well defined [22,22]. Finally, in noncommutative spacetimes (where \( \ell_{\text{crit}} = \ell_\text{Pl} \)), the spectral dimension changes with the scale but geometry can be probed to arbitrarily small lengths [2], and \( \ell_\text{Pl} \) acts as...
a smearing length rather than a cutoff. The functional form of dispersion relations in noncommutative momentum spaces is similar to Eq. (1), but it does not have the same ultraviolet limit \[ \frac{\omega}{k} \rightarrow \sigma \].

Here, on the other hand, the Planck scale plays the role of a minimal rather than characteristic scale. Below it, we do not have a gradual loss of resolution of the diffusing probe, as in \[ \ell_{Pl} \] : simply, there is no diffusion at all. Whether one interprets it either as the absence of a continuum spacetime below \( \ell_{Pl} \) or as an operational limitation in measuring scales with accuracy greater than \( \ell_{Pl} \), the net result for physical observations is the same.

The loss of Lorentz invariance at microscopic scales [Eqs. (2) and (3)] is therefore expected in frameworks where \( \ell_{Pl} \) acts as a minimal diffusion scale. Forfeiting special relativity at these scales and recovering it in the infrared is not a unique feature of this model, and can be found also in other continuum scenarios with dimensional flow (such as Hořava-Lifshitz gravity \[ \frac{\ell}{\eta} \] or multiscale spacetimes with modified derivatives \[ \xi \]). On the other hand, nonlocal Laplacians are known to lead, in general, to unconventional geometric structures below a certain scale \( \xi \), but by itself nonlocality is not the cause of the unique behavior we found. Even small deviations from Eq. (1) give rise to a change in the density of states and, hence, to a spoiling of the entropy-area law. Thus, Eq. (4) is perhaps less a toy model than deemed in \[ \text{[10]} \].

The topic of black holes and dimensional flow has been previously considered in \[ \text{[22, 23]} \]. There is little intersection between those results and ours. In \[ \text{[23]} \], dimensional flow is simply used as a motivation to study lower-dimensional black holes. In \[ \text{[22]} \], dimensional flow was assumed to be monotonic from \( d_S \sim 4 \) in the infrared to some value smaller than 3 in the ultraviolet. In that case, it was argued that a Schwarzschild black hole, described in a local theory, stops evaporating when its radius shrinks to a minimal scale at which \( d_S \sim 3 \), below which the properties of the black hole can no longer be probed. Also, no observer, inside or outside the black hole, can see a value \( d_S < 2 \). Here, we did not postulate a profile for \( d_S \), but we started with a nonlocal theory realizing the density of states necessary to obtain the black-hole entropy-area law. The resulting spectral dimension is always greater than 4. We did find a minimal scale \( \ell_\ast \sim \ell_{Pl} \) below which it is not possible to check the spectral dimension of spacetime, but scales \( \sigma \sim \sigma_\ast \), correspond to a geometry with \( d_S \rightarrow \infty \).

The divergence of the spectral dimension is somewhat difficult to assess, as the heat kernel \[ \mathcal{K} \] neither resembles the case of ordinary manifolds nor reproduces the results for fractals. Still, we can advance an explanation by recalling that the definition of \( d_S \) expresses the heat kernel as an effective power law proportional to the volume \( V \) of the system, \[ \mathcal{P}(\sigma) \sim V \sigma^{-d_s(\sigma)/2} \]. In the limit \( \sigma \rightarrow \sigma_\ast \), \[ \mathcal{P} \sim V \sigma^{-d_s(\sigma)/2} \rightarrow 0 \] instead of diverging as usual. This is encouragingly compatible with the holographic principle: the volume of the system is no longer the leading contribution in the heat kernel.

We conclude with the following observation. In asymptotically flat spacetimes stable against gravitational collapse, the entropy \( S \) of the truncated Fock space of bosonic and fermionic local field theories is bounded from above by \( A^{3/4} \), where \( A \) is the boundary area of the region where the quantum fields live \[ \text{[24]} \]. This reproduces the bound by \'t Hooft \[ \text{[1]} \]. A key assumption to obtain this result is that the energy of the Fock states does not exceed an upper limit, conventionally fixed to be the Planck scale: \( E < E_{Pl} \). On the other hand, in the presence of a black hole the entropy follows the area law \( S \propto A \) and, according to the model discussed here, the correct description is in terms of a nonlocal field theory. Nonlocality implies that these fields do not represent particles: the propagator of the theory \[ \mathcal{K} \] is \( 1/F \), the off-shell inverse of the dispersion relation \[ \text{[13]} \], which has two branch cuts with branch points at \( k^0 = \pm \omega(k) \). (The presence of branch cuts in the propagator and the consequent loss of the particle interpretation often occur in nonlocal theories \[ \text{[13, 25]} \], but not always \[ \text{[26]} \].)

Here we point out a suggestive way to show that the two scenarios are, in fact, compatible. If we start from the nonlocal theory and truncate the dispersion relation for small momenta \( |k| \ll E_{Pl} \) to leading approximation we get a local theory with \( \omega^2(k) = |k|^2 + O(|k|^4) \). Then, we can consider the Fock space of this effective field theory but with field modes with momenta no larger than the Planck energy. This is precisely the situation where one meets the requirements for the nonholomorphic bound \( S \leq (A/E_{Pl}^2)^{3/4} \). Thus, we conjecture that the discrepancy between the \( A \) and \( A^{3/4} \) laws lies in the infinite number of degrees of freedom thrown away by the truncation of the nonlocal dispersion relation. How the quasiparticle states of the fully nonlocal effective theory contribute to the entropy of the system has been outlined already in \[ \text{[10]} \]. However, a nontrivial check of this conjecture would go beyond classical thermodynamical considerations and enter the realm of quantum field theory, linking with the results of \[ \text{[24]} \]. This study will entail the management of an infinite number of particle fields with techniques outside the scope of this paper.

ACKNOWLEDGMENTS

The work of M.A. is supported by the E.U. Marie Curie Actions through a Career Integration Grant and in part by the John Templeton Foundation. The work of G.C. is under a Ramón y Cajal contract.

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