1. Introduction

In [CG89], Coleman and Gross proposed a definition of a $p$-adic height pairing on curves over number fields with good reduction at primes above $p$. The pairing was defined as a sum of local terms and the most interesting terms are the ones corresponding to primes above $p$ where the definition depends on Coleman’s theory of $p$-adic integration. Later, Nekovář constructed in [Nek93] a general $p$-adic height pairing for Galois representations which are cristalline at primes above $p$, and satisfying certain additional technical assumptions.

Wintenberger raised the question of the equality of the two height pairings for curves. In other words: does the Nekovář height pairing applied to $H^1_{\et}(X)$ of a curve recover the Coleman-Gross pairing. The purpose of this small note is to answer this in the affirmative. More precisely, let $F$ be a number field and let $X$ be a smooth and proper curve over $F$, with good reduction at all places $v$ above a fixed prime $p$. To define either the height of Coleman and Gross or the height of Nekovář the following choices must be made.

- A “global log” - a continuous idele class character
  \[ \ell : \mathbb{A}_F^\times/F^\times \to \mathbb{Q}_p. \]

- for each $v | p$ a choice of a subspace $W_v \in H^1_{\dR}(X \otimes F_v/F_v)$ complementary to the space of holomorphic forms.

For the definition of the Coleman-Gross height we must insist that the local characters $\ell_v$, for $v | p$, are ramified in the sense that they do not vanish on the units in $F_v$ (this seems to be overlooked in [CG89]).

With these choices the Coleman-Gross height pairing is a pairing
\[ h_{\CG} : \text{Div}_0(X) \times \text{Div}_0(X) \to \mathbb{Q}_p, \]
where $\text{Div}_0(X)$ denotes the group of zero divisors on $X$ (defined over $F$), while the Nekovář height pairing is a pairing
\[ h_{\Nek} : H^1_{\ell}(F,V) \times H^1_{\ell}(F,V) \to \mathbb{Q}_p, \]
where $V = V_p(J_X) \cong H^1_{\et}(X, \mathbb{Q}_p(1))$ is the Tate-module of the jacobian of $X$ and $H^1_{\ell}$ is finite cohomology in the sense of Bloch and Kato. The relation between the two pairings is provided by the étale Abel-Jacobi map
\[ \alpha_X : \text{Div}_0(X) \to H^1_{\ell}(F,V). \]

Our main result is the following.

**Theorem 1.1.** *With the same choices of $(\ell, W_v)$ we have*
\[ h_{\Nek}(\alpha_X(y), \alpha_X(z)) = h_{\CG}(y, z). \]
The proof goes as follows. We first recall the construction of the Coleman-Gross height in Section 2. This is expressed in terms of local heights. Likewise, the Nekovář height can be expressed as a sum of local heights, although it also has a global description which we will not need here. The decomposition into local heights depends on a choice of a “mixed extension”. We recall this in Section 3 as well as the particular mixed extension we will use. With this mixed extension, it is known that the local heights at places not dividing \( p \) are equal to the local heights of Coleman-Gross. This leaves the comparison of the local heights above \( p \). This is done in the rest of the paper.

We would like to thank Wintenberger and Nekovář for suggesting this problem to us on various occasions.

2. The Coleman-Gross height pairing

We first recall the theory of \( p \)-adic height pairings due to Coleman and Gross \[CG89\].

Recall that we have the character \( \ell : \mathbb{A} \times F^\times \rightarrow \mathbb{Q}_p \). One deduces from \( \ell \) the following data:

- For any place \( v \not| p \) we have \( \ell_v(\mathbb{O}_{F_v}^\times) = 0 \) for continuity reasons, which implies that \( \ell_v \) is completely determined by the number \( \ell_v(\pi_v) \), where \( \pi_v \) is any uniformizer in \( F_v \).
- For any place \( v| p \) we can decompose \( \mathbb{O}_{F_v}^\times \rightarrow \mathbb{Q}_p \) where \( t_v \) is \( \mathbb{Q}_p \)-linear. Since we assume that \( \ell_v \) is ramified it is then possible to extend \( \log \) to \( F_v^\times \) in such a way that the diagram remain commutative.

The height pairing is a sum of local terms, \( h_{CG}(y, z) = \sum_v h_v^{CG}(y, z) \) over all finite places \( v \). When \( v \not| p \) the local term is given \[CG89\ (1.3)] by

\[
(2.1) \quad h_v^{CG}(y, z) = \ell_v(\pi_v) \cdot (y, z)
\]

where \( (y, z) \) denotes the intersection multiplicity of the extension of \( y \) and \( z \) to a regular model of \( X \) over \( \mathbb{O}_{F_v} \).

We now describe the local contribution at a place \( v| p \). We modify the definitions of Coleman and Gross to work with integration theory over \( \mathbb{Q}_p \) instead of over \( \mathbb{C}_p \). Let \( X \) be a curve over \( K = F_v \) with good reduction. Let \( \Omega_{log}^1(X) \) be the space of one forms on \( X \) (defined over \( K \)) with at most logarithmic singularities. For \( \omega \in \Omega_{log}^1(X) \) we let \( \text{Res}(\omega) \) be the residue divisor. This is a divisor of degree 0 defined over \( K \) such that over \( \overline{K} \) it becomes \( \sum_{P \in X} \text{Res}_P(\omega)P \). Let \( \Omega^1(X) \) be the space of global holomorphic forms on \( X \). We note that if \( \omega_1 \) and \( \omega_2 \in \Omega_{log}^1(X) \) and \( \text{Res}(\omega_1) = \text{Res}(\omega_2) \) then \( \omega_1 - \omega_2 \in \Omega^1(X) \). There is a canonical projection

\[
(2.2) \quad \Psi : \Omega_{log}^1(X) \rightarrow H^1_{dR}(X).
\]

In \[CG89\] it is given by certain requirements. Following \[Bes02\] we can describe it as follows:
Proposition 2.1. Let $\omega \in \Omega^1_{\log}(X)$ and let $U \subset X$ be a wide open subspace on which $\omega$ is holomorphic. The class $\Psi(\omega)$ is the image of $\omega|_U \in H^1_{\text{dR}}(U)$ under the unique Frobenius equivariant section to the map $H^1_{\text{dR}}(X) \to H^1_{\text{dR}}(U)$.

Now recall that we have at our disposal the complementary subspace $W = W_v$.

Definition 2.2. For any divisor $y$ of degree 0 on $X$ we let $\omega_W(y) \in \Omega^1_{\log}(X)$ be the unique form satisfying $\text{Res}(\omega_W(y)) = y$ and $\Psi(\omega_W(y)) \in W$.

Now we need to use the theory of $p$-adic integration. In [CG89] the theory of Coleman [CdS88] is used. Several other $p$-adic integration theories have been developed in recent years [Zar96, Col98, Bes02a, Vol01, Bes02b]. In most of these theories one can at least integrate algebraic differentials on all smooth algebraic varieties over a finite extension of $\mathbb{Q}_p$ and in this domain of integration they are conjectured to be identical. In [Bes02a] it is shown that the theory developed there using [Vol01], as well as the theory of [Col98], coincide for curves with Coleman’s integration theory. For our needs these theories are equivalent.

The integration theory depends on a branch of the $p$-adic log. For this we take the branch $\log_v$, defined by $\ell_v$, and we can extend it in a unique way to $\overline{K}$. The properties of $p$-adic integration, which are valid for all versions except [Bes02a, CdS88] (which work in the rigid setting but require good or close to good reduction), can be summarized as follows.

Proposition 2.3. For any smooth algebraic variety $X/K$ and a holomorphic differential $\omega$ on $X$ defined over $K$ the theory associates a function $\int \omega : X(\overline{K}) \to \overline{K}$ unique up to an additive constant, such that the following properties are satisfied.

- The function $\int \omega$ is locally analytic and satisfies $d \int \omega = \omega$.
- The integration is functorial with respect to arbitrary morphisms in the sense that if $f : Y \to X$ is defined over $K$, then we have $f^* \omega = f^* \int \omega$ up to a constant.
- if $\sigma \in \text{Gal}(\overline{K}/K)$ and $x, y \in X(\overline{K})$, then $\int_{\sigma(x)}^y \omega = \sigma(\int_x^y \omega)$.

The following easy corollary of the above properties is well known (and is in fact used by Zarhin and Colmez to define the integral)

Corollary 2.4. Let $J$ be a commutative algebraic group over $K$, let $\omega$ be an invariant differential on $J$ and let $\omega_0$ be its value at the identity element. Let $\log : J(K) \to T_0(J)$ be the logarithm for the $p$-adic Lie group $J$, depending on the choice of a branch of the logarithm, taking values in the tangent space at 0 to $J$. Then we have $\int_P \omega = \langle \omega_0, \log(P) \rangle$, where $\langle , \rangle$ is the duality between the cotangent and tangent space at the identity.

Suppose again that $X$ is a curve as before and let $\omega \in \Omega^1_{\log}(X)$. Then we obtain an integral $\int \omega$ defined on $X(\overline{K})$ minus the singular points of the form. The sum of the values of $\int \omega$ on $z \in \text{Div}_0(X)$, disjoint from the singular points, is well defined independent of the constant of integration and denoted $\int_z \omega$. Furthermore, since $z$ and $\omega$ are defined over $K$ we have $\int_z \omega \in K$.

Definition 2.5. The local height pairing at $v|p$ of Coleman and Gross is defined as follows: Let $y$ and $z$ be two divisors of degree 0 on $X$ with disjoint supports. Then their pairing is given by $h_{CG}^v(y, z) := t_v(\int_z \omega_{\mathcal{W}_v}(y))$. 


3. The Nekovář height in terms of mixed extensions

Nekovář gives two expressions for the height pairing. The one which is relevant for us is given in terms of mixed extensions and is expressed as a sum of local terms. As we saw in the previous section the Coleman-Gross height pairing is also given as a sum of local terms. That the terms at places not dividing \( p \) are the same is well-known and we recall this here.

The height pairing of Nekovář is a map

\[
h_{\text{Nek}} : H^1_1(F, V) \times H^1_1(F, V^* \otimes(1)) \rightarrow \mathbb{Q}_p,
\]

where \( V \) is a continuous \( G = \text{Gal}(\mathcal{F}/F) \)-representation satisfying certain conditions (see [Nek93, 2.1.2]) and \( H_1 \) is finite cohomology in the sense of Bloch and Kato [BK90]. To describe the height pairing we assume that we are given extensions of continuous \( G \)-representations,

\[
0 \rightarrow V \rightarrow E_1 \rightarrow A \rightarrow 0
\]

and

\[
0 \rightarrow B(1) \rightarrow E_2 \rightarrow V \rightarrow 0.
\]

Here, \( A \) and \( B \) are finite dimensional \( \mathbb{Q}_p \)-representations of \( G \) which are trivialized by a finite extension (this is more general then in [Nek93] and is inspired by [Sch94]). Suppose for now that \( G \) acts trivially on \( A \) and \( B \). Then, such sequences yield (the second by first dualizing) elements \([E_1] \in \text{Hom}(A, H^1(F, V))\) and \([E_2] \in B \otimes H^1(F, V^* \otimes(1))\). We assume that these in fact belong to \( \text{Hom}(A, H^1_1(F, V)) \) and \( B \otimes H^1_1(F, V^* \otimes(1)) \) respectively. The height pairing produces out of these two extensions an element \( h(E_1, E_2) \in \text{Hom}(A, B) \). The construction is functorial with respect to maps \( A' \rightarrow A \) and \( B \rightarrow B' \). To get the height pairing itself one restricts to the case \( A = B = \mathbb{Q}_p \). The more general setup is convenient for example in geometric situations, as we will see.

To describe the height pairing one chooses a mixed extension. This is an embedding of the sequences (3.1) and (3.2) inside a commutative diagram with exact rows and columns as follows [Nek93, p. 159]:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B(1) & E_2 & \pi & V & 0 \\
0 & B(1) & E & E_1 & 0 \\
A & A & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Having fixed such a mixed extension \( E \) we obtain local mixed extensions \( E_v \) by restricting to the decomposition groups at the various places \( v \). Each of these local mixed extensions gives a local contribution \( h_{E_v} \) and the sum \( h_E = \sum_v h_{E_v} \) is shown
to depend only on $E_1$ and $E_2$ so we can define $h(E_1, E_2) = h_E$. We will only need to describe the local contribution at places above $p$, which we will do in the next section.

Suppose now that $X$ is a curve as in the statement of the theorem and that $y$ and $z$ are two zero divisors on $X$ with disjoint supports. Let $Y$ and $Z$ be the supports of $y$ and $z$ respectively. We will use an overline to denote extension of scalars to the algebraic closure of the corresponding field. Then we obtain our mixed extension as follows (compare [Nek93, 5.6]. Let $\tilde{A} = (\mathbb{Q}_p^Y)_0$ be the subspace of the space of functions from $Y$ to $\mathbb{Q}_p$ where the sum of values is 0, and let $\tilde{B} = (\mathbb{Q}_p^Z)_0$ be the quotient of the corresponding space for $Z$ by the subspace of constant functions.

**Definition 3.1.** The semi geometric mixed extension associated with the cycles $y$ and $z$ is the mixed extension in (3.3) with $V = H^1_{\text{et}}(X, \mathbb{Q}_p(1))$, $E_1 = H^1_{\text{et}}(X - Y, \mathbb{Q}_p(1))$, $E_2 = H^1_{\text{et}}(X - Z, \mathbb{Q}_p(1))$ (étale cohomology of $X$ relative to $Z$), and finally

$$E = H^1_{\text{et}}(X - Y - Z, \mathbb{Q}_p(1)).$$

The associated geometric mixed extension is the extension obtained from the semi geometric mixed extension by the maps $A = A^G \hookrightarrow \tilde{A}$ and $B \rightarrow \tilde{B} = \tilde{B}^G$.

In this mixed extension, $\tilde{A}$ is identified with the kernel of $H^2_{\text{et}}(X, \mathbb{Q}_p(1)) \rightarrow H^2_{\text{et}}(X, \mathbb{Q}_p(1))$ and $\tilde{B}$ with the cokernel of $H^0_{\text{et}}(X, \mathbb{Q}_p(0)) \rightarrow H^0_{\text{et}}(X, \mathbb{Q}_p(0))$.

Consider now $y, z \in \text{Div}_0(X)$ as $y \in A$ and $z \in B^* (\mathbb{Q}_p\text{-dual})$ in the obvious manner. Then it is well known that $[E_1](y) = \alpha_X(y)$ and $((z \otimes 1) \circ [E_2])(1) = \alpha_X(z)$. We thus find

$$h_{\text{Nek}}(y, z) = z(h_E(y)) = \sum_v z(h_{E_v}(y)).$$

**Proposition 3.2.** For $v \nmid p$ we have $z(h_{E_v}(y)) = h_{CG}^v(y, z)$, where $E$ is the mixed extension described above and $h_{CG}$ is the local height pairing of Coleman and Gross described in (2.1).

**Proof.** This is well known. See for example [Nek93, Proposition 2.16] where a more general result is proved. 

To prove Theorem 1.1 it thus remains to prove the following result.

**Theorem 3.3.** For $v | p$ we have, with the same choice of the space $W_v$, $z(h_{E_v}(y)) = h_{CG}^v(y, z)$.

The proof of this theorem will occupy the rest of this paper. Before proving this we want to make the following comment: Since we will dualize several times with respect to cup products, one has to check the signs very carefully here. We did not do this, hence, apriori, Theorem 1.1 is only true up to a global sign. This sign can not be $-1$, however, as can be seen from the fact that both the Nekovář height pairing and the Coleman-Gross height pairing have the property that they vanish if either $y$ or $z$ are principal divisors.
4. The local height at a place above $p$

We now begin to compute the local contribution to the height pairing at a place $v | p$ coming from the mixed extension described in the previous section. In this section we do this using a certain splitting (the map $w$ below), which will be described in Section 6.

We begin by recalling the general construction of Nekovár in [Nek93, 4.7]. We make several modifications to bring it to a form suitable for comparison with the Coleman-Gross construction.

Let $K = F_v$. The Kummer map,

\begin{equation}
K^\times \otimes \mathbb{Q}_p \xrightarrow{\alpha_K} H^1(K, \mathbb{Q}_p(1)),
\end{equation}

is an isomorphism and identifies $H^1_f(K, \mathbb{Q}_p(1))$ with $\mathbb{Q}_p \times K^\times \otimes \mathbb{Q}_p$, so that there exists a short exact sequence

\begin{equation}
0 \rightarrow H^1_f(K, \mathbb{Q}_p(1)) \rightarrow H^1(K, \mathbb{Q}_p(1)) \xrightarrow{\text{ord}} \mathbb{Q}_p \rightarrow 0.
\end{equation}

The $p$-adic logarithm $\log : \mathbb{Q}_p \rightarrow K$ defines an isomorphism $H^1_f(K, \mathbb{Q}_p(1)) \cong K$. The branches of the $p$-adic logarithms, i.e., extensions of $\log$ to $K^\times$, are therefore in one to one correspondence with splittings of (4.2). We will denote the splitting induced by a branch $\log^\prime \chi$ by $\log^\prime \chi$.

We can view the local component of the idele class character $\ell_v$ as a map $\ell_v : H^1(K, \mathbb{Q}_p(1)) \rightarrow \mathbb{Q}_p$. Then, the character $\ell_v$ factors as $\ell_v = \ell_v \circ \log^\prime \chi$ if and only if we have $\ell_v = \ell_v \circ \log^\prime \chi$. Note that in this case we have

\begin{equation}
(K \rightarrow H^1_f(K, \mathbb{Q}_p(1)) \xrightarrow{\ell_v} \mathbb{Q}_p) = t_v.
\end{equation}

From now onward we assume that such a choice of $\log^\prime \chi$ has been made.

We now consider the diagram (3.3) as a diagram of representations of $\text{Gal}(\overline{K}/K)$ and assume that $A$ and $B$ are trivial representations. Recall that $\pi : E_2 \rightarrow V$. We then have the following diagram

\begin{equation}
\begin{array}{c}
\begin{array}{ccccccc}
0 & \rightarrow & B \otimes H^1_f(K, \mathbb{Q}_p(1)) & \xrightarrow{\text{ord}} & H^1_f(K, E_2) & \xrightarrow{\pi} & H^1_f(K, V) & \rightarrow & 0 \\
& \downarrow & 1 \otimes \log^\prime \chi & & \downarrow i & & \downarrow \pi & & \\
0 & \rightarrow & B \otimes H^1_f(K, \mathbb{Q}_p(1)) & \xrightarrow{\text{ord}} & \pi^{-1}(H^1_f(K, V)) & \xrightarrow{\pi} & H^1_f(K, V) & \rightarrow & 0 \\
& \downarrow 1 \otimes \ell_v & & & \downarrow w & & \downarrow \phi & & \\
0 & \rightarrow & B & \rightarrow & B & \rightarrow & 0 & & \\
& 0 & \rightarrow & 0 & & 0 & & \\
\end{array}
\end{array}
\end{equation}

This diagram is mostly copied from [Nek93, 4.8]. The dotted lines in the diagram denote various splittings. We have already discussed $\log^\prime \chi$. The map $w$ depends on the choice of the complementary subspace $W_v$ from the introduction (it does not depend on a choice of $\log^\prime \chi$) and will be discussed later. The two other splittings are induced, $w'$ by $w$ and $\Theta$ by $\log^\prime \chi$ in the following formal way.
Definition 4.1. Let \( x \in \pi^{-1}(H^1_f(K,V)) \) and choose \( y \in H^1_f(K,E_2) \) such that \( \pi(y) = \pi(x) \). Then \( \pi(x - i(y)) = 0 \) so \( x - i(y) = j(z) \) with \( z \in B \otimes H^1(K,\mathbb{Q}_p(1)) \). Then we define
\[
w'(x) = u(w(y)) + z
\]
and
\[
\Theta(x) = y + j \circ (1 \otimes \log'_\mathbb{A})(z)
\]

The following is an easy diagram chase away.

Lemma 4.2. The maps \( w' \) and \( \Theta \) are well defined and constitute splittings of the respective diagrams. Furthermore, in diagram (1.4) the two possible ways of connecting \( H^1_f(K,E_2) \) with \( B \otimes H^1(K,\mathbb{Q}_p(1)) \) (in both directions) give the same map.

In [Nek93, 4.8] Nekovář describes the local height pairing separately in the two cases \( \ell_v \) ramified or not. However it is easy to see that his two definitions coincide with the following.

Definition 4.3. The local height pairing corresponding to the mixed extension \( E \) is the element of \( h_E \in \text{Hom}(A,B) \) obtained as follows: We have \([E] \in \text{Hom}(A, H^1_f(K,E_2))\), but in fact lies in \( \text{Hom}(A, \pi^{-1}(H^1_f(K,V)) \). Then \( h_E \) is the composition
\[
A \xrightarrow{[E]} \pi^{-1}(H^1_f(K,V)) \xrightarrow{w'} B \otimes H^1(K,\mathbb{Q}_p(1)) \xrightarrow{1 \otimes \ell_v} B.
\]

Since we are assuming the existence of a \( \log_\mathbb{A} \) through which \( \ell_v \) factors, we can write this in a different way.

Lemma 4.4. We have \((1 \otimes \ell_v) \circ w' = (1 \otimes \ell_v) \circ u \circ w \circ \Theta\).

Proof. With \( x, y \) and \( z \) as in Definition 4.1 we have
\[
u \circ w \circ \Theta(x) = u(w(y + (1 \otimes \log'_\mathbb{A})(z))) = u(w(y) + (1 \otimes \log'_\mathbb{A})(z))
\]
so that \( u \circ w \circ \Theta(x) - w'(x) = u((1 \otimes \log'_\mathbb{A})(z)) - z \). But the compatibility of \( \log_\mathbb{A} \) with \( \ell_v \) precisely means that \( \ell_v \) vanishes on the image of \( u \circ \log_\mathbb{A} \circ \text{Id} \).

As a consequence we have obtained the description of the local height pairing as the composition
\[
h_E = (1 \otimes \ell_v) \circ u \circ w \circ \Theta \circ [E].
\]

5. The connection with \( p \)-adic integration

In this section we go back to the geometric mixed extension of Definition 4.1 and we show that the composition \( \Theta \circ [E] : A \to H^1_f(K,E_2) \) can be described in terms of Coleman integration.

Assume that we are given a curve \( X \) over \( K \), zero divisors \( y \) and \( z \) with disjoint supports \( \mathcal{Y} \) and \( \mathcal{Z} \) respectively. We assume that \( y \) and \( z \) are sums of \( K \)-rational points. This implies that \( A = \hat{A} \) in the notation of Section 3. This restriction will be eliminated in Section 3.

In the situation we are considering, we have a natural isomorphism, given by the exponential map of Bloch and Kato,
\[
D_{\text{dr}}(V)/F^0 \xrightarrow{\exp} H^1_f(K,V),
\]
Lemma 5.2. the Galois representation $E_2 = H^1_{dR}(X; \mathbb{Z}, Q_p(1))$ is isomorphic to $V_p(J)$, the $p$-adic Tate-module of $J$ tensored with $Q_p$. Furthermore, the exact sequence (3.2) is induced by the short exact sequence (5.3).

Proposition 5.5. Let $D$ be a zero-divisor on $X$. Then we have $[E](D) = \alpha_J([D]) \in H^1(K, V_p(J))$ where $\alpha_J$ is the Kummer map of $J$ and $[D]$ is the class of $D$ in $J(K)$.

Proof. This follows by the same argument as the one used in the proof of [Ras95, Appendix, Lemma].

The following is an easy extension of the well known identification of the tangent space to the Picard scheme (probably, also well known).
Lemma 5.4. There is a canonical isomorphism \( T_0(J) \rightarrow H^1_{\text{dR}}(X; \mathbb{Z}/K)/F^1 \).

By [BK90, Example 3.10.1] there exist a commutative diagram
\[
\begin{array}{ccc}
H^1_{\text{dR}}(X; \mathbb{Z}/K)/F^1 = T_0(J) & \xrightarrow{\text{exp}_J} & J(K) \otimes \mathbb{Q}_p \\
& \searrow \alpha_J & \\
& & H^1_f(K, E_2)
\end{array}
\]

Where \( \text{exp}_J \) is the exponential map of the \( p \)-adic Lie group \( J(K) \) while \( \text{exp} \) is the Bloch-Kato exponential.

Proposition 5.5. Assume that \( X \) has a \( K \)-rational point \( P_0 \) and let \( g : X - Z \rightarrow J \) be the map \( P \mapsto P - P_0 \). Then the map \( g^* : \Omega^1(J) \rightarrow F^1 H^1_{\text{dR}}(X - Z/K) \) is an isomorphism (recall that the space on the right is the space of forms with logarithmic singularities along \( Z \)). Furthermore, this isomorphism fits in a commutative diagram
\[
\begin{array}{ccc}
\Omega^1(J) & \xrightarrow{g^*} & F^1 H^1_{\text{dR}}(X - Z/K) \\
\downarrow \text{ev}_0 & & \downarrow \text{Poincaré duality} \\
T_0(J)^* & \xrightarrow{} & H^1_{\text{dR}}(X; \mathbb{Z}/K)^*
\end{array}
\]

where \( \text{ev}_0 \) is evaluation at 0 and the bottom map is the dual of the isomorphism of Lemma 5.4.

Proof. The first statement is [Ser59, Proposition V.5]. The second statement is again well known for the usual jacobian (see, e.g., the proof of Proposition 2.2 in [Mil80]) and the proof is the same. \( \square \)

Proof of Proposition 5.4. The short exact sequence (5.3) induces a short exact sequence of tangent spaces compatible with exponential maps. In addition, the exponential map for \( J_X \) is onto. Even more precisely, an integer multiple of every class in \( J_X(K) \) is in the image of the exponential map, without tensoring with \( \mathbb{Q}_p \). This implies that any divisor \( D \) on \( X \), can be, after multiplying by a sufficiently large integer, written as a sum of a divisor, whose class, \([D] \in J(K)\), is in the image of \( \text{exp}_J \), and a principal divisor, both disjoint from \( Z \). It is thus sufficient to check the result for these two types of divisors.

We now identify \( T_0(J) \) with \( H^1_{\text{dR}}(X, \mathbb{Z}/K) \) and \( \Omega^1(J) \) with \( F^1 H^1_{\text{dR}}(X - Z/K) \). The duality between these spaces is given by the cup product.

For the first type the proof is essentially the same as the proof of [Bes00a, Proposition 9.2]: By assumption we have \([D] = \alpha_J(\eta)\) for some \( \eta \in H^1_{\text{dR}}(X; \mathbb{Z}/K)/F^1 \). We have
\[
[E](D) = \alpha_J([D]) \quad \text{by Proposition 5.3}
= \alpha_J(\text{exp}_J(\eta))
= \text{exp}(\eta) \quad \text{by (5.4)}.
\]

In particular we have \([E](D) \in H^1_f(K, E_2)\) so that \( \Theta \circ [E](D) = [E](D) = \text{exp}(\eta) \) as well. It follows that \([E]_D(\omega) = \omega \cup \eta\). By Proposition 5.4 there exists a holomorphic invariant differential \( \omega' \) on \( J \) such that \( g^* \omega' = \omega \). By functoriality of
the $p$-adic integral (Proposition 2.3) we have $\int_D \omega = \int_0^{[D]} \omega'$, where $[D]$ is the image of $D$ in $J(K)$. Let $\log_J : J(K) \to T_0(J) = H^1_{\text{dR}}(X, \mathcal{O}_K)$ be the logarithm for the group $J(K)$ (depending on the choice of $\log_{\mathcal{O}_K}$). By Corollary 2.4

$$\int_0^{[D]} \omega' = \langle \omega', \log_J([D]) \rangle = \omega \cup \log_J([D]),$$

where $\langle , \rangle$ is the duality between the tangent space and holomorphic forms on the jacobian. But $\log_J([D]) = \eta$ so the proof in this case is complete.

Consider now the case where $D$ is the divisor of a rational function $f$. If $f$ take values in $O_K$ on $Z$, then $D$ belongs to the image of $\exp_D$. Thus we may assume that $\log(f) = 0$ on $Z$. From the description of the maps in (5.3) and Lemma 5.2 it follows that $[E](D) = j(\oplus_{x \in Z} \mathcal{O}_K(f(x)))$. The definition of $\log_{\mathcal{O}_K}$ and the fact that $\log(f) = 0$ on $Z$ implies that $1 \otimes \log_{\mathcal{O}_K}(\oplus_{x \in Z} \mathcal{O}_K(f(x))) = 0$. It follows from Lemma 4.2 that $\Theta \circ [E](D) = 0$. Now we apply the theory of the double index in the algebraic version of [Bes02] (in this case it can also be deduced with a bit of effort from the rigid theory of [Bes00]). We take the sum of the double indices $\langle f \omega, \log(f) \rangle_x$ of the two $p$-adic integrals $\log(f)$ and $\int (\omega)$ over all points $x$. A general result (in the rigid case see [Bes00], Corollary 4.11) says that the sum of these indices for two functions, one of which is a log, is always 0. By definition, the sum of the local indices is $\int (f) \omega = \log(f) \omega = 0$ so this last expression is 0. But by our assumptions $\log(f) \omega = 0$ so also $\int (f) \omega = 0$ and the proof is complete. \hfill $\square$

6. END OF THE PROOF

To finish the description of the height pairing and complete the proof we must describe the map $w : H^1_{\text{dR}}(K, E_2) \to H^1_{\text{dR}}(K, B(1))$, giving a splitting of the first line of (4.4). To describe this splitting is equivalent to describing a splitting $H^1_{\text{dR}}(K, V) \to H^1_{\text{dR}}(K, E_2)$. By the fact that the exponential map (3.1) is an isomorphism for $V$ and $E_2$ this is the same as describing a section $D_{\text{dR}}(V)/F^0 \to D_{\text{dR}}(E_2)/F^0$ to the natural map in the other direction.

Under the assumptions in [Nek93], which are satisfied in our case, there is a unique Frobenius equivariant splitting of $D_{\text{dR}}(E_2) \to D_{\text{dR}}(V)$. We now throw in the space $W = W_v \in D_{\text{dR}}(V)$, complementary to $F^0$. This gives the required splitting.

$$D_{\text{dR}}(V)/F^0 \xrightarrow{\text{along } W} D_{\text{dR}}(V) \xrightarrow{\text{Frob equivariant}} D_{\text{dR}}(E_2) \to D_{\text{dR}}(E_2)/F^0.$$

This completes the description of the local height pairing.

In our situation we must describe the map

$$H^1_{\text{dR}}(X/K)/F^1 \xrightarrow{\text{along } W} H^1_{\text{dR}}(X/K) \xrightarrow{\text{Frob equivariant}} H^1_{\text{dR}}(X; \mathcal{O}_K) \to H^1_{\text{dR}}(X; \mathcal{O}_K)/F^1.$$

Dualizing, we have the splitting

$$F^1 H^1_{\text{dR}}(X-K) \to H^1_{\text{dR}}(X-K) \xrightarrow{\text{Frob equivariant}} H^1_{\text{dR}}(X/K) \to F^1 H^1_{\text{dR}}(X/K).$$

Let $W'$ be the annihilator of $W$ with respect to the cup product. It is easy to check that the rightmost map is the projection with respect to the direct sum decomposition $H^1_{\text{dR}}(X/K) = W' \oplus F^1 H^1_{\text{dR}}(X/K)$.

Proposition 6.1. The composition of the two leftmost maps in (6.3) is induced by the map $\Psi$ of (2.2).
Lemma 6.2. Let $X_s$ be the closed fiber of $X$. Let $S = \{s_1, \ldots, s_n\}$ be a finite set of closed points of $X_s$ such that every point of $Z$ reduces to a point of $S$. Then, the map $H^1_{\text{dR}}(X - Z/K) \to H^1_{\text{rig}}(X_s - S/K)$ is Frobenius equivariant.

Proof. Note that the Frobenius structure on $\mathcal{H}^1_{\text{dR}}(X; \mathcal{O}_K)$ is induced via its identification with $D_{\text{dR}}$ of étale cohomology and is not directly related to any cohomology on which Frobenius acts naturally. However, as both de Rham and rigid cohomology have Poincaré duality (but not crystalline cohomology, which is badly behaved for open schemes), we can dualize and must show that the natural map $H^1_{\text{rig}}(X_s; S/K) \to H^1_{\text{dR}}(X; \mathcal{O}_K)$ is Frobenius equivariant. Lift each $z_i \in Z$ to a section $z_i : \text{Spec}(\mathcal{O}_K) \to X$, and let $z_i^r : \text{Spec}(\overline{\mathcal{O}}) \to X_s$ be the reduction. Then, the comparison isomorphisms in a relative situation (see for example [Kis04]) imply that the the Frobenius structure on $H^1_{\text{dR}}(X; \mathcal{O}_K)$ is induced via the isomorphism $H^1_{\text{dR}}(X; \mathcal{O}_K) \cong H^1_{\text{dR}}(X_s; \cup_i \text{Spec}(\overline{k})/K) \otimes K$, where $H^1_{\text{dR}}(X_s; \cup_i \text{Spec}(\overline{k})/K)$ is the cohomology relative to the maps $z_i^r$. As both $X_s$ and $\text{Spec}(\overline{k})$ are proper the relative crystalline cohomology is the same as the relative rigid cohomology and the lemma follows easily. □

Corollary 6.3. The induced splitting $B^* \otimes K \to F^1H^1_{\text{dR}}(X - Z/K)$ is given by $z \mapsto \omega_{W^*(z)}$, where $\omega_{W^*(z)}$ is the form defined by Definition 2.2.

Proof. The induced splitting is given by the following recipe: for $z \in B^*$ take $\omega \in F^1H^1_{\text{dR}}(X - Z/K)$ whose residue divisor is $z$, apply the map (6.3) and subtract the result from $\omega$. The corollary follows immediately. □

Proof of Theorem 3.3. We are still assuming that $Y$ is split over $K$. Let $y \in A$. By (4.3) we need to apply $z \in B^*$ to $h_E(y) = (1 \otimes \ell_v) \circ u \circ \Theta \circ (|E|)(y)$. By Proposition 5.1, $\Theta \circ (|E|)(y)$ is a functional on $H^1_{\text{dR}}(X - Z/K)$ defined by integrating a form on $y$. According to Corollary 6.3, this is mapped by $w$ to the functional on $B^* \otimes H^1_{\text{dR}}(K, \mathcal{O}_p(1)) \cong B^* \otimes K$ obtained by applying this functional to the form $\omega_{W^*(z)}$. Finally, we apply to the result $(1 \otimes \ell_v) \circ u$, which by (4.3) is just $1 \otimes t_v$. This gives $z(h_E(y)) = t_v(f_z \omega_{W^*(z)})$. By the proof of Proposition 5.2 in [CG89] we have $f_z \omega_{W^*(z)} = f_z \omega_{W^*(y)}$, so $z(h_E(y)) = \omega_{\text{CG}}(y, z)$ as required.

To finish the proof we must remove the assumption that $Y$ is split over $K$. For this it is sufficient to observe that the two local height pairings behave in the same with respect to finite extensions. Indeed, suppose that $K_1$ is a finite extension of $K$. Let $\ell_1$ be an extension of $\ell_v$ to $K_1^\times$. Such an extension can be given by $t_1 \circ \log_{K_1}$, where $\log_{K_1}$ extends uniquely to $K_1^\times$ and $t_1 : K_1 \to \mathbb{Q}_p$ extends $t_v$. Let $W_1 = W_v \cdot K_1$. Let $E$ be a mixed extension over $K$. Write $h_{E/K_1}$ for the height pairing corresponding to the mixed extension $E$ restricted to $K_1$, taken with respect to $\ell_1$ and $W_1$. Then it is straightforward to see that $h_{E/K_1} = h_E$. Thus, the local height pairing is unchanged when we extend the field to $K_1$, provided that we choose the data $(\ell_1, W_1)$ as above. If we now compare the Coleman-Gross height pairing we see that the form $\omega_z$ is unchanged. Since the log is also unchanged the integral $\int_y \omega_z$ is unchanged. Finally, since this integral is in fact in $K$ and $\ell_1$ coincides with $\ell_v$ on $K$ we see that the entire local height pairing is unchanged by our extension of scalars. This completes the proof. □
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