Loop Expansion in Light-Cone $\phi^4$ Field Theory

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Abstract

Based on the path integral quantization of the light-cone $\phi^4$ field theory in 1+1 dimensions, a loop expansion is implemented. The effective potential as a function of the zero-mode field $\omega$ is calculated up to two loop order and its derivative with respect to $\omega$ is used to determine the vacuum expectation value of the field $\phi$. The critical coupling constant at the occurrence of the spontaneous symmetry breaking is consistent with that obtained in the ordinary instant-form field theory. The critical exponents which describe the behavior of the susceptibility and the vacuum expectation value of $\phi$ near the critical point are evaluated from the effective potential. The one loop diagrams for the connected Green’s function are calculated in momentum space. The relevant equal-time correlation function is shown to be closely related.

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1. Introduction

For many years the light-cone vacuum was considered to be simple: It carries no longitudinal momentum ($k^+ = 0$), while Fock operators for fermions have $k^+ > 0$, and the vacuum state of the free Hamiltonian is an eigenstate of the light-cone Hamiltonian with interactions in many theories. Based on this physical vacuum a Fock basis can be constructed to produce the hadron spectrum [1]. However, it was stressed early [2, 3] that zero modes may change this simple picture drastically, (the zero mode of a field $\phi$ is defined as $\omega = \lim_{L \to \infty} \frac{1}{L} \int_{-L}^{+L} dx^- \phi(x)$ to project $\phi$ onto $k^+ = 0$) and the resulting complexity of the vacuum has been exploited since then [4-17]. Due to the nontrivial nature of the vacuum the bosonic zero modes have a large effect on the spectra of the Hamiltonian and field operators such as $\phi$ [13], they enter into internal lines of Feynman diagrams of any order (for the $\phi^3$ field theory, see [12]), and they remove certain noncovariant and quadratically divergent terms in the fermion self-energy in the discretized light-cone quantization of Yukawa theory [10]; moreover, $\theta$-vacua exist and exhibit non-vanishing fermion condensates [8, 17].

The quantization of the $\phi^4$ theory has been achieved [3, 4, 8, 13] by applying the Dirac-Bergmann algorithm [18] for constrained systems. The vacuum expectation value $\langle 0 | \phi | 0 \rangle$ (VEV) of the scalar field $\phi$ has been calculated by solving a secondary constraint $\theta_3$ (see Eq. (3)) related to spontaneous (reflection) symmetry breaking [8, 9, 11, 13]. The constraint equation is most easily obtained from the integration of the field equation of motion [11]. The vacuum expectation value of the $\phi$ field was obtained in the tree approximation plus a lowest order correction [8, 11]. In [9] the critical coupling constant $\lambda_c \approx 40 m^2$ is calculated from the curvature of a potential which has quadratic and quartic terms of the VEV (two ansätze are used for the zero mode field operator). In a series of studies [13] a so-called $\delta$ expansion combined with numerical calculations was used to find the VEV as a function of the coupling constant; a critical coupling constant $\lambda_c \approx 59.5 m^2$ was determined by searching for a converging solution of the constraint equation.

For the $\phi^4$ theory the $\phi$ field is decomposed into the zero modes and nonzero modes and the two types of modes couple to each other by a cubic interaction term. This means they do not move individually but evolve with mutual interactions. Via the constraint $\theta_3$ the nonzero modes influence the zero modes not only in tree level interactions but also in higher order diagrams, i.e. loop graphs, and the zero modes enter into an internal propagator at any order of diagrams describing the propagation of the nonzero modes [12]. Motivated by this, we first take into account the vacuum fluctuation based on the classical background field, i.e. make a loop expansion beyond the tree approximation. The loop diagrams show a much richer nonlinear structure than the classical approximation and are responsible for the phase transition between
the regime of spontaneously broken symmetry and its restoration at finite temperature [19]. Secondly, we
calculate the Green’s function for the nonzero modes to one loop order. Following the Green’s function,
another important quantity relevant to the nonvanishing VEV of the field \( \phi \), the equal-time correlation
function, is briefly discussed. This currently interesting physical quantity measures the size of the domain
of disordered chiral condensates formed possibly in ultra-high energy hadronic or heavy nucleus collisions.
In 3+1 dimensions, the zero modes depend on the other two coordinates \( x^1 \) and \( x^2 \). This extra coordinate
dependence substantially increases the difficulty in evaluating vacuum graphs. Thus for simplicity, the \( \phi^4 \)
field theory is studied here in 1+1 dimensions with a single spin-0 field \( \phi \).

In [19] a WKB-type expansion is developed for a potential from an effective action in order to study
nonlinear phenomena including spontaneous symmetry breaking, bound states and resonances, etc. We use
this loop expansion to deal with the nonperturbative effects of \( \phi^4 \) field theory in 1+1 dimensions. Before
starting the loop expansion, the Lagrangian is shifted by a constant field (in a static case). The propagator
then contains the constant field in its denominator and the coupling constant in the cubic interaction depends
linearly on the constant field (see, e.g. the third term in Eq.12). When the constant field is taken as the
vacuum expectation value of a quantum field, the loop expansion includes the nonperturbative effects of the
vacuum represented by the VEV parameter [20].

This paper is organized as follows. In section 2, we first prove that the ”shift” field technique is applicable
in the vacuum persistence amplitude for the purpose of calculating the VEV. Based on this justification,
we use the loop expansion given by Jackiw [19] and present the calculation of vacuum diagrams up to two
loops. In section 3 the connected Green’s function for nonzero modes at the one-loop order is calculated in
momentum space and the equal-time correlation function is evaluated for the case of a non-vanishing VEV.
In the final section 4, we briefly discuss our results for the critical coupling constant and critical exponents
which describe the behavior of the susceptibility and VEV near the critical coupling constant.

2. Effective Potential

The Lagrangian of the light-cone \( \phi^4 \) field theory in 1+1 dimensions is

\[
\mathcal{L} (\phi) = \frac{1}{2} \partial^+ \phi \partial^- \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4
\]

(1)

where \( m \) is the mass and \( \lambda \) is the coupling constant. The field \( \phi \) is separated into the zero modes \( \omega \)
and nonzero modes \( \varphi: \varphi = \phi - \omega \), where \( \omega = \frac{1}{2 \pi} \int L dx^- \phi (x) \) and \( L \) is the boundary of a spatial box in which
the field is quantized. In terms of \( \omega \) and \( \varphi \) the Lagrangian is written as

\[
\mathcal{L} (\varphi, \omega) = \frac{1}{2} \partial^+ \varphi \partial^- \varphi - \frac{1}{2} m^2 (\varphi + \omega)^2 - \frac{\lambda}{4!} (\varphi + \omega)^4
\]

(2)

Since the light-cone Lagrangian is maximally singular, two primary constraints are obtained from the definition
of the canonical momenta with respect to \( \varphi \) and \( \omega \). Applying the Dirac-Bergmann algorithm, three
constraints of second class (denoted by \( \theta_1 \), \( \theta_2 \) and \( \theta_3 \)) and their Dirac brackets were obtained [8]. Alternatively,
a symplectic method [21] was applied to get the same Dirac brackets without the classification of primary constraints,
first and second constraints [22]. The constraint \( \theta_3 \) is the derivative of the potential with respect to \( \omega \) [9],

\[
\theta_3 = m^2 \omega + \frac{\lambda}{3!} \varphi^3 + \frac{1}{2L} \int L dx^- \frac{\lambda}{3!} \varphi^3 (x) + 3 \varphi^2 (x) \omega \approx 0
\]

(3)

From these results we write down the vacuum persistence amplitude in the absence of the external
source according to the formalism of path integral quantization of field theories with second class constraints
[23],

\[
Z (0) = |2 \text{det} (\partial^+ \delta (x - y)) |^{1/2} \int \mathcal{D} \varphi \exp \left[ \frac{i}{\hbar} \int d^2 x \mathcal{L} (\varphi, \omega) \right]
\]

(4)

The determinant before the functional integral is simply a constant, and not important in calculations of
conventional Feynman diagrams. In later calculations it is neglected. The connected generating functional
is

\[
W (0) = -i \hbar \ln Z (0)
\]

(5)

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and the effective action \( \Gamma(\omega) \) is just \( W(0) \). Taking the derivative of \( \Gamma(\omega) \) with respective to \( \omega \) gives

\[
\frac{d\Gamma(\omega)}{d\omega} = -\frac{1}{Z[0]} \int D\varphi \exp[-i\int dx^+ H_c] \int dx^+ 2L\theta_3,
\]

where \( H_c = \int_L^+ dx^- [\frac{1}{2}m^2(\varphi + \omega)^2 + \frac{3}{4} (\varphi + \omega)^4] \) is the canonical Hamiltonian. It is obvious that \( \frac{d\Gamma(\omega)}{d\omega} = 0 \) is equivalent to \( \theta_3 = 0 \). The estimate of the vacuum expectation value \( \langle \phi \rangle \) from the constraint \( \theta_3 = 0 \) is translated into a calculation of \( \frac{d\Gamma(\omega)}{d\omega} = 0 \). Moreover, it is well-known that the path integral formalism provides us with a rich knowledge of the field motion and interactions. We thus expect to obtain more information on the zero-mode component of the field \( \varphi \) by carrying out the loop expansion beyond the classical (tree) approximation. We employ the frame-independent loop expansion formalism given by Jackiw [19] to get the VEV and calculate vacuum graphs up to two loop levels. In Ref. [19], an expansion in powers of \( \hbar \) was made for the effective action \( \Gamma(\bar{\phi}) \) by shifting the field \( \phi \) by a constant \( \bar{\phi} \). In this spirit, Eq. (2) suggests a "shift" of the field \( \varphi \) by a constant field \( \omega \). So in any calculation with the formulas given by Jackiw, it must be kept in mind that the point \( k^+ = 0 \) in the integration over \( k^+ \) must be left out.

In [12] the secondary constraint equation for the zero modes and nonzero modes was first solved to express \( \omega \) in terms of \( \varphi \), then it was expanded in a power series of the coupling constant, and finally an expansion in powers of the coupling constant for the interaction Hamiltonian of the field \( \varphi \) was obtained. In such an expansion the zero modes propagate along the internal lines in any order of the Feynman diagrams for the field \( \varphi \). In contrast to this procedure, we first expand the effective action in an \( \hbar \) power series, then invoke the constraint \( \theta_3 = 0 \) in this expansion series, or equivalently, implement \( \frac{d\Gamma(\omega)}{d\omega} = 0 \). In other words, according to Eqs. (3.7)-(3.11) in the second paper of Ref. [8], we first make an expansion of \( Z(0) \) before integrating \( \omega \), then carry out the integration over \( \omega \) and thus implement the constraint \( \theta_3 = 0 \).

In the case of an external source \( J(x) \), a term \( J(x)\varphi(x) \) is added to the Lagrangian in Eq. (4) to get the vacuum persistence amplitude \( Z(J) \). In terms of \( Z(J) \) the connected generating functional \( W(J) \) is defined by

\[
W(J) = -i\hbar \ln Z(J)
\]

The effective action is obtained from \( W(J) \) by a Legendre transformation

\[
\Gamma(\bar{\varphi}) = W(J) - \int d^2x \bar{\varphi}(x)J(x)
\]

with \( \bar{\varphi}(x) = \frac{\delta W(J)}{\delta J(x)} \). The effective potential \( V(\omega) \) is defined from the effective action by setting \( \bar{\varphi}(x) \) to be a constant field \( \omega \), which is reached by "shifting" the \( \varphi(x) \) with the \( \omega \) in the Lagrangian, and extracting an over-all factor of 1+1 dimension volume,

\[
\Gamma(\omega) = -V(\omega) \int d^2x
\]

The \( \frac{d\Gamma(\omega)}{d\omega} = 0 \), i.e. \( \frac{dV(\omega)}{d\omega} = 0 \) produces exactly the VEV of the field \( \phi \). By expanding \( W(J) \) and \( \frac{\delta W(J)}{\delta J(x)} \) as a power series in \( \hbar \), the effective potential \( V(\omega) \) in a loop expansion is given by [19]

\[
V(\omega) = V_{tree}(\omega) - \frac{1}{2}i\hbar \int [d^2k]' \ln \det iD^{-1}(\omega; k) + i\hbar <exp\frac{i}{\hbar} \int d^2x L(\varphi, \omega) >
\]

where \( [d^2k]' \) is the measure of the light-cone momentum in 1+1 dimensions without the point \( k^+ = 0 \) since we make the loop expansion for the field \( \varphi \). The first term \( V_{tree}(\omega) \) is the classical potential (tree approximation). The second term is the one-loop effective potential which involves a logarithm and the third term expresses the effective potential generated from multi-loop diagrams. The \( L(\varphi, \omega) \) is composed of terms cubic and quartic in \( \varphi(x) \). The \( D(\omega; k) \) is the propagator in momentum space and its explicit dependence on \( \omega \) results from the field \( \varphi \) "shifted" by \( \omega \). The practical calculation as outlined above starts from the Lagrangian for the spin-0 \( \phi^4 \) field,

\[
\mathcal{L}(\phi) = \frac{1}{2} \phi^+ \phi^0 - \frac{1}{2} (m_0^2 + \delta m^2) \phi^2 - \frac{\lambda_0 + \delta \lambda}{4!} \phi^4
\]
where the $m_0$ and $\lambda_0$ are the finite but underdetermined mass and coupling constant. The counterterms $\delta m^2$ and $\delta \lambda$ are given in $\hbar$ power series form. We "shift" the field $\phi$ by a constant field $\omega$. The "shifted" Lagrangian is

$$L(\phi, \omega) = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda_0 + \delta \lambda}{6} \omega \phi^3 - \frac{\lambda_0 + \delta \lambda}{4!} \phi^4$$  \hspace{1cm} (12)

with

$$\mu^2 = m_0^2 + \delta m^2 + \frac{\lambda_0 + \delta \lambda}{2} \omega^2$$  \hspace{1cm} (13)

We see in the shifted Lagrangian the induced mass $\mu$ depends on $\omega$ and an $\omega$-dependent cubic interaction is obtained. The $\omega$-dependent propagator in the momentum space is defined by the new quadratic term,

$$D(\omega; p) = \frac{i}{p^+ p^- - \mu^2 + i\epsilon}$$  \hspace{1cm} (14)

and the classical potential is

$$V_{\text{tree}}(\omega) = \frac{m_0^2 + \delta m^2}{2} \omega^2 + \frac{\lambda_0 + \delta \lambda}{4!} \omega^4$$  \hspace{1cm} (15)

For a single field $\phi$, the propagator $D(\omega, p)$ is diagonal in momentum space and the determinant is thus removed. The one-loop effective potential corresponding to Fig. 1(a) is written as

$$V_1(\omega) = -\frac{\hbar}{2} \int [d^2 k] \ln iD^{-1}(\omega; k)$$

$$= -\frac{\hbar}{8\pi} \{(2\Lambda^2 + \mu^2) \ln(2\Lambda^2 + \mu^2) - \mu^2 \ln(\mu^2)\}$$  \hspace{1cm} (16)

where $\Lambda$ is a cut-off of the high momenta $k^+$ and $k^-$. The two-loop effective potential is

$$V_2(\omega) = \frac{\hbar^2 \lambda_0}{8} \int [d^2 k][d^2 l] D(\omega; k) D(\omega; l)$$

$$- \frac{i\hbar^2 \lambda_0^2}{12} \omega^2 \int [d^2 k][d^2 l] D(\omega; k) D(\omega; l)$$  \hspace{1cm} (17)

where the first term corresponds to the Fig. 1(b) and the second term the Fig. 1(c). We obtain

$$V_2(\omega) = \frac{\hbar^2 \lambda_0}{128\pi^2} \ln^2(1 + \frac{2\Lambda^2}{\mu^2}) + \frac{\hbar^2 \lambda_0^2 \omega^2}{96\pi^2} \{(\frac{1}{4\Lambda^2} - \frac{\pi}{\mu^2 \sqrt{3 + \frac{8\Lambda^2}{\mu^2}}}) \ln(1 + \frac{2\Lambda^2}{\mu^2}) + \ln(1 + \frac{\mu^2}{2\Lambda^2}) + \frac{2\pi}{3\sqrt{3}\mu^2} \ln(1 + \frac{\mu^2}{2\Lambda^2})\}$$  \hspace{1cm} (18)

Expanding the counterterms $\delta m^2$ and $\delta \lambda$ in powers of $\hbar$ gives

$$\delta m^2 = \hbar^2 \delta m_1^2 + \hbar^2 \delta m_2^2 + \cdots$$

$$\delta \lambda = \hbar^2 \delta \lambda_1 + \hbar^2 \delta \lambda_2 + \cdots$$  \hspace{1cm} (19)

The values of these counterterms are obtained by cancelling divergent terms appearing in $V_1(\omega)$ and $V_2(\omega)$ which are expanded to the order of $\hbar^2$. We find

$$\delta m_1^2 = \frac{\lambda_0}{8\pi} \ln(2\pi\Lambda^2) + \delta \tilde{m}_1^2$$  \hspace{1cm} (20)

where $\delta \tilde{m}_1^2$ is a finite but arbitrary quantity. This result resembles other low dimensional calculations, for example, in Ref. [24]. Since no divergent terms need to be cancelled by $\delta m_2^2$, $\delta \lambda_1$ and $\delta \lambda_2$, we leave them finite but arbitrary. They might be determined by some renormalization conditions [25] relating to the spin-0 meson mass and empirical coupling constant. In terms of these quantities, the finite effective potential up to two-loop orders is

$$V(\omega) = V_0(\omega) + V_1(\omega) + V_2(\omega)$$

$$= \frac{1}{2} m_0^2 \omega^2 + \frac{1}{4!} \lambda_0 \omega^4 + \frac{\hbar^2 \lambda_0}{128\pi^2} \ln a + \frac{\hbar^2 \lambda_0^2}{64\pi^2} \ln a + \frac{\hbar^2 \lambda_0^2}{64\pi^2} \ln a + \frac{\hbar^2 (\delta \tilde{m}_1^2 + \frac{\delta \lambda_1}{2})}{8\pi} \ln a$$  \hspace{1cm} (21)
with \( a = m_0^2 + \frac{\lambda_0}{2} \omega^2 \), renormalized mass square \( m_R^2 = m_0^2 + \hbar \delta m_0^2 + \hbar^2 \delta m_0^2 \) and renormalized coupling constant \( \lambda_R = \lambda_0 + \hbar \delta \lambda_1 + \hbar^2 \delta \lambda_2 \). The nonlinear logarithmic structure coming from the loop graphs is explicitly exhibited.

3. Correlation Function

The nonvanishing VEV of the field \( \phi \) is related to the equal-time correlation function. To see this we discuss the spin-0 particle field with the momentum \( k^+ \geq 0 \). The equal-time correlation function in this case is calculated from

\[
S(x^- - y^-) = \langle \varphi(x) \varphi(y) \rangle = \langle \omega \rangle \langle \varphi(x) \varphi(y) \rangle
\]

\[
= \langle \phi \rangle < \langle \phi \rangle + \lim_{x^+ \to y^+ + 0^+} \int_0^{+\infty} \int_{-\infty}^{+\infty} d^2 p' \left( -i G_2^c(\omega; p) \right) e^{-i \frac{\omega}{2} (x^- - y^-) - i \frac{\omega}{2} (x^+ - y^+)}
\]

where the \( G_2^c(\omega; p) \) is the connected two-point Green’s function in momentum space. Fig. 2 exhibits three one-loop diagrams which contribute

\[
T_{2a} = -\frac{(\lambda_0 + \delta \lambda) i \hbar}{8 \pi} \ln(1 + \frac{2 \Lambda^2}{\mu^2}), \quad T_{2b} = 0, \quad T_{2c} = \frac{(\lambda_0 + \delta \lambda)^2 \omega^2 i \hbar}{16 \pi \mu^2 (1 + \frac{\mu^2}{\Lambda^2})} \delta a p^- - \delta b p^-
\]

to the connected Green’s function

\[
G_2^c(\omega; p) = D(\omega; p) - (T_1 + T_2 + T_3) D^2(\omega; p)
\]

In the expression for \( T_{2c} \), the \( \delta a p^- = 1 \) when \( p^- = 0 \) and vanishes otherwise. This indicates that the \( T_3 \) at the momentum \( p^- = 0 \) gives a nonzero contribution to the connected Green’s function at the order \( \hbar \) when the \( \Lambda \) goes to infinity. To order \( \hbar \), by cancelling divergent terms with the counterterms, the connected Green’s function is

\[
G_2^c(\omega; p) = \frac{i}{p^+ p^- - a} + \frac{i \hbar}{(p^+ p^- - a)^2} \left( \frac{\lambda_0}{8 \pi} \ln a + \frac{\lambda_0^2 \omega^2}{16 \pi a} \delta a p^- \right)
\]

and the equal-time correlation function is

\[
S(x^- - y^-) = \langle \varphi \rangle < \langle \phi \rangle > - \frac{i}{4 \pi} \int_0^{+\infty} \frac{d p^+}{p^+} e^{-i \frac{\omega}{2} (x^- - y^-)}
\]

Here the \( S(x^- - y^-) \) is time-independent since the field is in equilibrium.

4. Discussions and Conclusions

Up to this point, we have obtained the effective potential, the connected two-point Green’s function and the equal-time correlation function. In our calculations, the cut-off of the \( k^+ = 0 \) in the integration over internal momenta does not induce infrared divergences since the field is massive. The ultraviolet divergence is cancelled by the counterterms. By solving the \( \frac{dV(\omega)}{d\omega} = 0 \), the VEV of the field \( \phi \) can be obtained. Numerical calculations are required by the nonlinear formalism of \( V(\omega) \). Moreover, \( m_0^2 \) may be adjusted in the renormalization condition [25] so that the constant \( a \) in Eq. (21) is always positive to ensure that \( V(\omega) \) is real. Translation invariance of the theory [3, 12] ensures that the VEV is a constant. Since the loop expansion involves integrating over the field \( \varphi \), it is somewhat different from the instant-form field theory. This is a result of the separation of the zero and nonzero modes and the vacuum structure and implies that the tadpole diagram is actually excluded, while Figs. 2(a) and 2(c) remain to contribute to the Green’s function, but not to the equal-time correlation function. The reason is that the equal-time condition \( x^+ - y^+ = 0 \) and \( \delta b p^- \) eliminate their contributions, respectively. The two-loop order diagrams are expected to contribute a term to Eq. (26), but are difficult to evaluate. The first term in Eq. (26) is a background provided by the VEV and is not cancelled by the second term which is produced by the nonzero modes.
Before the path integral formalism (4) is applied, the limiting procedure \( L \to \infty \) is taken [8]. In this limit the definition for \( \omega \) gives the \( L \)-independent zero-mode field. For a stable field \( \phi \), the zero-mode field \( \omega \) is a constant field which is independent of space-time. Notice, however, that \( \omega \) is functionally dependent on the \( \varphi \) field through the constraint \( \theta_3 \), and both \( \omega \) and \( \varphi \) are operators which include quantum fluctuations. Then \( < \phi > \) is determined from the loop expansion, Eq.10, based on the path integral formalism while the \( \omega \) and \( \varphi \) are taken as classical fields. Nonetheless our result differs from [9] since the effective potential up to two loop orders contains the richer nonlinear structure, but is close to a numerical calculation [13] which is equivalent to using an effective potential of polynomial form.

The effective potentials in the tree approximation, one loop and two loops are plotted in fig. 3 for a single scalar field \( \phi \) with mass \( m_0 \) for which \( m_R^2 = -m_0^2 \) and \( \lambda_R = 4m_0^2 \). We plot the \( \omega \) values corresponding to the minimum of the effective potential which is the VEV of the \( \phi \) field. In the tree approximation the minimum of the effective potential is about -0.007GeV\(^2\) at \( \omega \approx 1.2 \). The one-loop and two-loop effective potentials give almost the same minimum value -0.018 at \( \omega \approx 1.3 \). The one loop contribution \( V_1(\omega) \) is important while the two loop correction \( V_2(\omega) \) is small. At \( \omega = 0 \), the three curves reach their local maximum. If the renormalized mass \( m_R = 0 \), the maximum and the two minima coincide at \( \omega = 0 \). In this case the effective potential is a parabola with its minimum located at \( \omega = 0 \), and \( < \phi > = 0 \). The dash-dotted curve in the one loop approximation almost coincides with the solid curve in the two loop approximation. This indicates that the loop expansion series converges. There is also convergence at the critical point.

The effective potential becomes imaginary at \( m_0^2 < 0 \) and infinite at \( m_0^2 = 0 \) when \( \omega \) approaches zero. Therefore, we choose \( m_0^2 > 0 \). To start we also assume \( \delta m_0^2 = -(m_0^2 + m_2^2) \), then we extract the critical coupling constant where the curvature of the potential at \( \omega = 0 \) changes sign. The second derivative of the effective potential with respect to \( \omega \) is

\[
\frac{d^2V(\omega)}{d\omega^2} \bigg|_{\omega=0} = m_R^2 + \frac{h\lambda_0}{8\pi} + \left( \frac{\lambda_0}{64\pi^2} + \frac{\lambda_R}{8\pi} \right) \frac{h^2}{m_0^2} + \frac{h\lambda_0 + h\delta\lambda}{8\pi} \ln m_0^2 + \frac{h^2\lambda_0^2 \ln m_0^2}{64\pi^2 m_0^2} \tag{27}
\]

Setting \( \frac{d^2V(\omega)}{d\omega^2} \bigg|_{\omega=0} = 0 \), the critical coupling constant \( \lambda_c \) is

\[
\lambda_c = -\frac{1}{\ln m_0^2} \left[ 8\pi m_R^2 + h\lambda_0 + \frac{h^2\lambda_0}{m_0^2} (\frac{\lambda_0}{8\pi} + \delta m_1^2) + \frac{h^2\lambda_0^2 \ln m_0^2}{8\pi m_0^2} \right] \tag{28}
\]

We calculate the critical coupling constant after choosing the values of \( \lambda_0 \) and \( m_0^2 \). The results are listed in Table 1. Clearly, this effective potential gives critical coupling constants that are consistent with those in [9, 13] and the values from 22\( m_0^2 \) to 55\( m_0^2 \) reported for the instant-form field theory [26].

The susceptibility above the critical coupling constant is defined as

\[
\chi^{-1} = \frac{d^2V}{d\omega^2} \bigg|_{\omega=0} \tag{29}
\]

Near the critical point, the susceptibility and the VEV behave like

\[
\chi^{-1} \propto (\lambda - \lambda_c)^{\gamma} \tag{30}
\]

\[
< \phi > \propto (\lambda_c - \lambda)^{\beta} \tag{31}
\]

The values of the critical exponents \( \gamma \) and \( \beta \) are shown in Table 1. If the \( \phi^4 \) field theory is taken as a representation of the Ising model in \( 1+1 \) dimensions, then \( \gamma \) and \( \beta \) are 1.75 and 0.125, respectively [27]. The critical exponents shown in the table are almost independent of the mass and coupling constant. The critical exponent \( \gamma \) is somewhat lower than 1.75. The value of \( \beta \) is close to 0.5 obtained in [13], but much higher than 0.125. We can attribute this discrepancy to the use of \( \delta m_1^2 = -(m_0^2 + m_2^2) \) and \( \delta m_2^2 = 0 \). If we choose \( m_0^2 = 70m_0^2 \), \( \delta m_1^2 = -50m_0^2 \) and \( \delta m_2^2 = -m_0^2 - \delta m_1^2 - m_2^2 \) leading to \( m_R^2 = -m_0^2 \), then the critical values become \( \lambda_c \approx 30.618m_0^2 \), \( \gamma = 1.235 \) and \( \beta = 0.276 \), the latter being an improvement over the 0.5 obtained in mean field theory [27].

In conclusion, we first show the equivalence between the \( \frac{dV(\omega)}{d\omega} = 0 \) and the constraint \( \theta_3 = 0 \); then we apply the loop expansion given by Jackiw to calculate the effective potential up to two loop orders; third, we
calculate the equal-time correlation function. The effective potential is shown to have a nonlinear logarithmic structure. The one loop contribution $V_1(\omega)$ is much bigger than the two loop $V_2(\omega)$. We have applied the effective potential to calculate the critical coupling constant and two critical exponents which are consistent with other theories. For a stable field we have given a static description for spontaneous symmetry breaking in 1+1 dimensions.

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Figure Captions

Fig. 1. (a) one loop. (b) double bubble. (c) “radiatively” corrected single bubble.

Fig. 2. Feynman diagrams for (a) $T_{2a}$, (b) $T_{2b}$, (c) $T_{2c}$.

Fig. 3. The effective potential as a function of $\omega$ is calculated with $m_0^2 = 0.1 \text{GeV}^2$, $\delta m_1^2 = -(m_0^2 + m_\phi^2)$, $\delta m_2^2 = 0$, $\lambda_0 = 4m_\phi^2$, $\delta \lambda_1 = 0$ and $\delta \lambda_2 = 0$ where $m_\phi = m_\pi$ is taken to be the observed pion mass. The dashed (dash-dotted, solid) curve is the tree (one loop, two loop) approximation.
Caption for the table

Table 1. The $\lambda_0$ and $m_0^2$ are arbitrary but finite quantities. Results are the critical coupling constant $\lambda_c$ and critical exponents $\gamma$ and $\beta$. 
| $\lambda_0$ | $m_0^2$ | $\lambda_c$ | $\gamma$ | $\beta$ |
|----------|---------|-------------|---------|--------|
| $45m_0^2$ | $90m_0^2$ | $46.29m_0^2$ | 1.35 | 0.676 |
| $45m_0^2$ | $100m_0^2$ | $38.55m_0^2$ | 1.33 | 0.675 |
| $30m_0^2$ | $90m_0^2$ | $47.42m_0^2$ | 1.35 | 0.676 |
| $60m_0^2$ | $90m_0^2$ | $44.58m_0^2$ | 1.34 | 0.675 |
This figure "fig1-1.png" is available in "png" format from:

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This figure "fig1-2.png" is available in "png" format from:

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