UNIQUENESS OF MINIMAL MORPHISMS OF LOGARITHMIC SCHEMES

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Abstract. We give a sufficient condition under which the moduli space of morphisms between logarithmic schemes is quasifinite under the moduli space of morphisms between the underlying schemes. This implies that the moduli space of stable maps from logarithmic curves to a target logarithmic scheme is finite over the moduli space of stable maps, and therefore that it has a projective coarse moduli space when the target is projective.

1. Introduction

Chen [Che10], Abramovich and Chen [AC11], and Gross and Siebert [GS13] have recently constructed a moduli space of stable maps from logarithmic curves into logarithmic target schemes. In [Wis], we extended the existence results of those papers beyond logarithmic curves and eliminated some technical restrictions that applied even to curves. However, [Wis] only asserts that the moduli space of maps between logarithmic schemes exists as an algebraic space that is locally of finite presentation; Chen showed that when the target has a rank 1 logarithmic structure, the moduli space of logarithmic maps is a disjoint union of pieces, each of which is finite over the moduli space of maps between underlying schemes [Che10, Proposition 3.7.5]. This implies in particular that the moduli space of stable maps from logarithmic curves has a projective coarse moduli space when the target variety is projective. That result was extended to so-called generalized Deligne–Faltings logarithmic structures by Abramovich and Chen [AC11].

We will explain these results with a general criterion on the domain that applies to arbitrary logarithmic targets:

Theorem 1.1. Let $S$ be a fine logarithmic scheme. Let $X$ and $Y$ be fine logarithmic $S$-schemes, with $X$ also geometrically reduced, proper, and integral (in the logarithmic sense [Kat89, Definition (4.3)]) over $S$. If the relative characteristic monoid of $X/S$ is trivial then the projection from the space of logarithmic maps to the space of maps of underlying schemes factors as an injection followed by an étale map

$$\text{Hom}_{\text{LogSch}/S}(X, Y) \to \mathcal{T} \to \text{Hom}_{\text{Sch}/S}(X, Y)$$

where $\mathcal{T}$ is the space of types (see Definition 3.1).

The following corollary recovers and generalizes the earlier results of Chen, Abramovich–Chen, and Gross–Siebert, mentioned earlier:

Corollary 1.2. Let $Y$ be a fine, saturated logarithmic scheme over $S$. Let $M_u(Y)$ be the moduli space of stable maps from logarithmic curves to $Y$ of type $u$ and let $\overline{M}_u(Y)$ be its
underlying algebraic stack and let $M(Y)$ be the moduli space of stable maps to $Y$. Then $M_u(Y)$ is finite over $M(Y)$.

Proof. We have already seen that the moduli space of stable logarithmic maps is locally of finite type \cite[Theorem 1.1]{Wis}, bounded \cite[Proposition 1.5.6]{ACMW}, and satisfies the valuative criterion for properness \cite[Proposition 1.4.3]{ACMW} over stable maps, so it remains only to show that the geometric fibers are finite. Since logarithmic curves have generically trivial relative characteristic monoids, the finiteness is almost immediate from the theorem, which in fact has the stronger conclusion that the map

$$\text{Hom}_{\text{LogSch}/S}(X, Y) \to \text{Hom}_{\text{Sch}/S}(X, Y)$$

is injective. However, we are working in the category of fine logarithmic schemes, whereas prevailing convention dictates that the moduli space of stable maps from logarithmic curves to $Y$ be formed in the category of fine, saturated logarithmic schemes. It is easy to fix this by saturating $\text{Hom}_{\text{LogSch}/S}(X, Y)$. We conclude by remarking that the underlying scheme of the saturation of a fine logarithmic scheme $H$ is always quasifinite over $H$. \hfill $\square$

Conventions and notation. We have retained the notation of \cite{Wis} as much as seemed reasonable. In particular, $\underline{X}$ is the underlying space or stack of a logarithmic algebraic space or stack $X$. We have consistently used underlined roman characters to represent schemes, even when no logarithmic scheme is present for the schemes to underlie, but we have not applied the same convention to morphisms of schemes, lest the underlines become overwhelming.

When $A$ and $B$ are objects that vary with objects of some category $\mathcal{C}$, we write $\text{Hom}_\mathcal{C}(A, B)$ for the functor or fibered category on $\mathcal{C}$ of morphisms from $A$ to $B$. This convention conflicts with the more standard convention of using the subscript to indicate in which category the homomorphisms should be taken. Our perspective is that $\text{Hom}$ should only be applied to pairs of objects of the same type, and that it should be possible to infer the type from the arguments.

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I would also like to thank M. Chan for a suggestion that was very useful in our original approach to this problem, even though the method presented here circumvents it.

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2. A preliminary reduction

In order to simplify notation, we make a preliminary reduction. To prove Theorem 1.1, it is sufficient to work relative to $\text{Hom}_{\text{Sch}/S}(X, Y)$ and assume that an $S$-morphism $f : \underline{X} \to \underline{Y}$ has already been fixed. Then we have an equivalence of categories:

\begin{equation}
\text{Hom}_{\text{LogSch}/S}(X, Y) = \text{Hom}_{\text{LogSch}/S}(f^* M_Y, M_X)
\end{equation}
On the left side Hom should be interpreted as morphisms of logarithmic schemes over $S$; on the right side, Hom refers to morphisms of logarithmic structures commuting with the structural maps from $\pi^* M_S$.

Equation (1) informs us that we can dispense with $Y$ and work entirely on $X$, setting $M = f^* M_Y$. The following assumptions will remain in force for the rest of the paper:

(i) $S = (S, M_S)$ and $X = (X, M_X)$ are fine logarithmic schemes and $\pi : X \to S$ is a morphism of logarithmic schemes;
(ii) $M$ is a fine logarithmic structure on $X$, equipped with a homomorphism of logarithmic structures $\pi^* M_S \to M$.

Theorem 1.1 breaks up into the following two statements:

**Theorem 2.1.** Assume that $X$ is proper over $S$. Then the space of types (Definition 3.1) is étale over $S$.

**Theorem 2.2.** Assume that $S$ is the spectrum of an algebraically closed field, that $\pi^* M_S \to M_X$ is an integral homomorphism of monoids, that $X$ is reduced, and that $X$ is proper over $S$. If the sheaf of relative characteristic monoids $M_X/\pi^* M_S$ vanishes on a dense open subset of $X$ then for any type $u$ (see Definition 3.1), then the underlying algebraic space of $\text{Hom}_{\text{LogSch}/S}(f^* M_Y, M)$ has at most one point.

The first of these is treated in Section 3, and the other takes up the balance of the paper.

### 3. Types

**Definition 3.1.** Let notation be as in Section 2. A type consists of a homomorphism of sheaves of abelian groups:

$$u : M/\pi^* M_S \to M_X/\pi^* M_S$$

We define:

$$\mathcal{T} = \text{Hom}_{\text{Sch}/S}(M/\pi^* M_S, M_X/\pi^* M_S)$$

More explicitly, for any $S$-scheme $S'$, write $X' = X \times_S S'$, and then:

$$\mathcal{T}(S') = \text{Hom}(g^*(M/\pi^* M_S), g^*(M_X/\pi^* M_S))$$

**Proof of Theorem 2.1.** For the duration of the proof, we abbreviate $F = M/\pi^* M_S$ and $G = M_X/\pi^* M_S$. The object of the proof is to show that $\text{Hom}_{\text{Sch}/S}(F, G)$ is representable by an étale algebraic space over $S$. Since $F$ and $G$ are constructible, we have

$$\text{Hom}_{\text{et}(X')}(g^* F, g^* G) \simeq g^* \text{Hom}_{\text{et}(X)}(F, G)$$

for any morphism $g : X' \to X$. Let us establish (*): since these are sheaves it is sufficient to verify it at the stalks, so we can assume that $X'$ is the spectrum of an algebraically closed field. It is sufficient to prove this assertion in an affine neighborhood of each point of $X'$, so we can assume $X'$ is quasimacent and quasiseparated; therefore by [SGA73, Proposition IX.2.7], $F$ has a finite presentation, with $F_0$ and $F_1$ both free:

$$F_1 \Rightarrow F_0 \Rightarrow F$$

Then $\text{Hom}_{\text{et}(X)}(F, G)$ can be represented as an equalizer:

$$\text{Hom}_{\text{et}(X)}(F, G) \to \text{Hom}_{\text{et}(X)}(F_0, G) \Rightarrow \text{Hom}_{\text{et}(X)}(F_1, G)$$
But pullback preserves finite limits, and (*) holds by definition when $F$ is free, so we have what we need:

$$g^* \text{Hom}_{\text{ét}}(X)(F, G) = g^* \ker(\text{Hom}_{\text{ét}}(X)(F_0, G) \Rightarrow \text{Hom}_{\text{ét}}(X)(F_1, G))$$

$$= \ker(\text{Hom}_{\text{ét}}(X)(g^*F_0, G) \Rightarrow g^* \text{Hom}_{\text{ét}}(X)(g^*F_1, G))$$

$$= \text{Hom}_{\text{ét}}(X)(g^*F, G)$$

Thus the espace étalé of $\text{Hom}_{\text{ét}}(X)(F, G)$ represents $\text{Hom}_{\text{Sch}}/X(F, G)$.

To complete the proof of the theorem, we now observe that the espace étalé of $\pi_* \text{Hom}_{\text{ét}}(X)(F, G)$ represents $\text{Hom}_{\text{Sch}}/S(F, G)$. Indeed, for any $g : S' \rightarrow S$, we have:

$$\text{Hom}_{\text{Sch}}/S(F, G)(S') = \text{Hom}(g^*F, g^*G)$$

$$= \Gamma(S', \pi_* \text{Hom}_{\text{ét}}(X)(g^*F, g^*G))$$

$$= \Gamma(S', \pi_* g^* \text{Hom}_{\text{ét}}(X)(F, G)) \quad \text{by (*)}$$

$$= \Gamma(S', \pi_* \text{Hom}_{\text{ét}}(X)(F, G)) \quad \text{by proper base change}$$

[SGA73, Théorème 5.1 (i)]

This completes the proof. □

4. THE LEFT ADJOINT TO PULLBACK FOR ÉTALE SHEAVES

Throughout this section, $\pi : X \rightarrow S$ will be flat and locally of finite presentation, with reduced geometric fibers. It was shown in [Wis, Theorem 4.5] that, under these conditions, the pullback functor $\pi^*$ for étale sheaves has a left adjoint, $\pi_!$. In this section, we make some further observations about this functor.

We will use the notation $\pi_!$ for the left adjoint to $\pi^*$ in the category of integral monoids [Wis, Proposition 4.7], because this functor does not agree with $\pi_!$ upon passage to the underlying sheaf of sets. In later sections we will only be interested in $\pi_!$ and not in $\pi_!$, so we will discard the superscript from the former.

The functor $\pi^*$ does commute with passage from commutative monoids to their underlying sets, so it follows formally that its left adjoint respects passage from sets to their freely generated monoids: for any sheaf of sets $F$ on $X$, we have

$$\pi_!(F^\text{ét}/S)$$

and $\pi_!(F)^\text{ét}$ represent the same functor, hence are isomorphic. □

For the next few statements, we will use notation $\pi_!^N$ for the left adjoint to $\pi^*$ in the category of integral monoids [Wis, Proposition 4.7], because this functor does not agree with $\pi_!$ upon passage to the underlying sheaf of sets. In later sections we will only be interested in $\pi_!^N$ and not in $\pi_!$, so we will discard the superscript from the former.

The functor $\pi^*$ does commute with passage from commutative monoids to their underlying sets, so it follows formally that its left adjoint respects passage from sets to their freely generated monoids: for any sheaf of sets $F$ on $X$, we have

$$\pi_!^N(NF) = N\pi_!(F),$$

where we have written $NF$ for the monoid freely generated by $F$.

Lemma 4.2. (i) If $F$ is an étale sheaf of integral monoids on $X$ then $\pi_!^N F$ is generated by $\pi_! F$. 
(ii) The functor $\pi_1^N$ preserves surjections.

Proof. Let $G \subset \pi_1^N F$ be the submonoid generated by $\pi_1F$. Then $F \to \pi_1^* \pi_1^N F$ factors:

$$F \to \pi_1^* \pi_1^N F \to \pi_1^* G \subset \pi_1^* \pi_1^N F$$

Of course, the first map is just a morphism of sheaves of sets, but the composition is a monoid homomorphism, so upon applying $\pi_1^N$ again, we get a commutative diagram:

$$\begin{array}{ccc}
\pi_1^N F & \longrightarrow & \pi_1^N \pi_1^* G \\
\downarrow & & \downarrow \\
G & \longrightarrow & \pi_1^N F
\end{array}$$

Adjunction implies that the composition $\pi_1^N F \to \pi_1^N F$ is the identity, from which it follows that $G \to \pi_1^N F$ is surjective. This proves the first claim.

For the second, consider a surjection $H \to F$ of sheaves of integral monoids. Then $\pi_1 H \to \pi_1 F$ is surjective, and $\pi_1 F$ generates $\pi_1^N F$, so the image of $\pi_1 H$ in $\pi_1^N F$ generates $\pi_1^N F$. Therefore $\pi_1^N H \to \pi_1^N F$ is surjective. □

Lemma 4.3. Let $i: U \to X$ be the inclusion of an open subset such that $U_s \subset X_s$ is dense for every geometric point $s$ of $S$.

(i) For any étale sheaf of sets $F$ on $X$, the map $\pi_1^N i^* F \to \pi_1^N F$ is surjective.

(ii) For any étale sheaf of integral monoids $F$ on $X$, the homomorphism $\pi_1^N i^* F \to \pi_1^N F$ is surjective.

Proof. We begin with the statement about sheaves of sets. Since $\pi_1$ commutes with base change and surjectivity can be checked on the stalks, it is sufficient to assume that $S$ is the spectrum of an algebraically closed field. By Lemma 4.1, we have:

$$\pi_1 F = \pi_0(\text{F}_{\text{ét}})$$

$$\pi_1 i^* F = \pi_0 (i^{-1} \text{F}_{\text{ét}})$$

But $X$ is locally connected (since it is locally of finite type over a field) and $i^{-1} \text{F}_{\text{ét}} \subset \text{F}_{\text{ét}}$ is a dense open subset, so $\pi_0 (i^{-1} \text{F}_{\text{ét}})$ surjects onto $\pi_0 (\text{F}_{\text{ét}})$.

Now we prove the statement about monoids. As before, let $NF$ be the sheaf of monoids freely generated by the underlying sheaf of sets of $F$; note that $NF \to F$ is surjective. Consider the commutative diagram:

$$\begin{array}{ccc}
\pi_1^N i_1^* NF & \sim & N \pi_1^N i^* F \\
\downarrow & & \downarrow \\
\pi_1^N i_1^* F & \longrightarrow & \pi_1^N F
\end{array}$$

The upper arrow is surjective by the first part of the lemma and Lemma 4.2 implies that the right arrow is surjective, so the bottom arrow is surjective as well. □

5. Construction of minimal monoids

We recall the notation and main construction of [Wis]. Recall our assumptions from Section 2, to which we add that $\pi$ is flat with reduced geometric fibers and $\pi^* M_{\text{Spec}} X \to M_X$ is integral. We also specify a type, $u$. 


We think of fibered categories over \( \text{LogSch} \) as fibered categories over \( \text{Sch} \) via the projection \( \text{LogSch} \to \text{Sch} \). Thus, if \( H \) is a fibered category and \( \mathcal{S} \) is a scheme, the notation \( H(\mathcal{S}) \) refers to the category of all pairs \((M_\mathcal{S}, \xi)\) where \( M_\mathcal{S} \in \text{LogSch}(\mathcal{S}) \) is a logarithmic structure on \( \mathcal{S} \) and \( \xi \in H(\mathcal{S}, M_\mathcal{S}) \).

Define \( H = \text{Hom}_{\text{LogSch}/\mathcal{S}}(M, M_X) \). This is the fibered category over \( \text{LogSch}/\mathcal{S} \) whose fiber over \( f : (T, M_T) \to (S, M_S) \) is the set of morphisms of logarithmic structures

\[
f^*M \to f^*M_X
\]

on \( X \times_\mathcal{S} T \) that commute with the structural maps from \( f^*\pi^*M_S \) and such that the induced map

\[
f^*M^{\text{gp}} / f^*\pi^*M_S^{\text{gp}} \to f^*M_X^{\text{gp}} / f^*\pi^*M_S^{\text{gp}}
\]

coincides with the pullback \( f^*u \) of the type \( u \). Note that \( X \times_\mathcal{S} T \) is the underlying scheme of \( X \times_\mathcal{S} T \) because \( X \) is integral over \( S \).

We will often wish to refer to objects of the moduli space \( H \) with some additional fixed structure. For example, to fix a logarithmic base scheme \( T \), we write \( H(T) \); to fix the underlying scheme \( T \) of \( T \) and its characteristic monoid \( \overline{N}_T \) (but not the logarithmic structure \( N_T \)), we write \( H(T, \overline{N}_T) \).

It was shown in [Wis, Theorem 1.1] that if \( X \) is proper over \( \mathcal{S} \) then \( H \) is representable by an algebraic space \( H \) with a logarithmic structure \( M_H \), that is locally of finite presentation over \( \mathcal{S} \). By a theorem of Gillam, \( H(\mathcal{S}) \subset H(\mathcal{S}) \) may be characterized as the subcategory of minimal objects (see [Gil12] or [Wis, Appendix B]). This category is equivalent to a set by [Wis, Corollary 5.1.1], which shows minimal objects have no nontrivial automorphisms.

We will be particularly interested in the \( \mathcal{S} \)-points of \( H \), so we introduce some additional notation for handling them. For any morphism of logarithmic structures \( M_S \to N_S \), we write \( M_X \to N_X \) for morphism of logarithmic structures on \( X \) obtained by pushout along the morphism \( \pi^*M_S \to M_X \):

\[
\begin{array}{ccc}
\pi^*M_S & \to & M_X \\
\downarrow & & \downarrow \\
\pi^*N_S & \to & N_X
\end{array}
\]

In other words, \((X, N_X) = X \times_\mathcal{S}(\mathcal{S}, N_S)\).

Then \( H(\mathcal{S}) \) is the opposite of the category of pairs \((M_S \to N_S, M \xrightarrow{\varphi} N_X)\) where \( N_X \) is as above and and \( M \to N_X \) is a morphism of logarithmic structures commuting with the maps from \( \pi^*M_S \). Such an object is generally abbreviated to \((N_S, \varphi)\).

We now recall the construction of the logarithmic structure of \( H \) from [Wis]. This can be done without explicit reference to the underlying space of \( H \): given an \( \mathcal{S} \)-point \((N_S, \varphi)\) of \( H \), there is a corresponding map \( \alpha : \mathcal{S} \to H \); there must therefore be a logarithmic structure \( \alpha^*M_H \) on \( \mathcal{S} \) and a homomorphism of logarithmic structures \( \alpha^*M_H \to N_S \); we build the logarithmic structure \( \alpha^*M_H \). This is known as the minimal (or basic) logarithmic structure associated to \((N_S, \varphi)\).

In fact, what we do is find a sheaf of monoids (a quasilogarithmic structure, in the language of [Wis, Definition 1.2]) \( \overline{Q}_S \) on \( S \) such that \( \alpha^*M_H \) is a quotient of \( \overline{Q}_S \). We recall the construction of \( \overline{Q}_S \).
First we form a fiber product:

\[
R_0^{\text{gp}} = \overline{M}^{\text{gp}} \times_{\overline{M}^{\text{gp}}_{X/S}} \overline{M}^{\text{gp}}_X
\]

The first map in the product is the type \( u \) and the second is the tautological projection. The fiber product comes with two inclusions of \( \pi^*\overline{M}^{\text{gp}}_S \), one from each factor, and we take \( R^{\text{gp}}_0 \) to be their coequalizer. Then we define the submonoid \( R \subset R^{\text{gp}}_0 \) to be the smallest submonoid containing \( \pi^*\overline{M}^{\text{gp}}_S \) such that the pushout \( R_X \) in diagram (4)

\[
\begin{array}{ccc}
\pi^*\overline{M}^{\text{gp}}_S & \longrightarrow & M_X \\
\downarrow & & \downarrow \\
R & \longrightarrow & R_X \\
\end{array}
\]

contains the image of \( \overline{M} \) under the tautological map (see [Wis, Section 3.1] for the construction of this map). Next, we consider the pushout:

\[
\begin{array}{ccc}
\pi^*\overline{M}^{\text{gp}}_S & \longrightarrow & \pi^!R \\
\downarrow & & \downarrow \\
M_S & \longrightarrow & Q_S \\
\end{array}
\]

It was shown in [Wis, Section 4.2] that if \((N_S, \varphi) \in H(S)\) then there is a unique homomorphism of sheaves of monoids, commuting with maps from \( \overline{M}_S \) and inducing \( \varphi : \overline{M} \to \overline{N}_X \):

\[
\overline{Q}_S \to \overline{N}_S
\]

Pulling back the projection \( N_S \to \overline{N}_S \) and the map \( N_S \to \mathcal{O}_S \) induces a sheaf of monoids \( Q_S \) on \( S \) and a homomorphism \( Q_S \to \mathcal{O}_S \). This is not always a logarithmic structure on \( S \) (see, for example, the case of logarithmic points in Section 6), so we pass to the associated logarithmic structure to get \( \alpha^*\overline{M}_H \).

We record several basic consequences of the construction:

**Lemma 5.1.**

(i) \( Q_S^{\text{gp}} / \overline{M}^{\text{gp}}_S \simeq \pi^!\overline{R}_0^{\text{gp}} / \pi^!\pi^*\overline{M}^{\text{gp}}_S \simeq \pi^!((R^{\text{gp}} / \pi^*\overline{M}^{\text{gp}}_S)) \)

(ii) \( R^{\text{gp}} / \pi^*\overline{M}^{\text{gp}}_S \simeq R_0^{\text{gp}} / (\pi^*\overline{M}^{\text{gp}}_S \times \pi^*\overline{M}^{\text{gp}}_S) \simeq \overline{M}^{\text{gp}} / \pi^*\overline{M}^{\text{gp}}_S \)

(iii) The sharpening of \( \overline{Q}_S \to \overline{N}_S \) is \( \alpha^*\overline{H}_S \).

(iv) If \( M_X/S = 0 \) then \( \overline{R} = \overline{M} \), canonically.

**Proof.** Passage to the associated group is a left adjoint and therefore preserves cocartesian diagrams. Therefore diagram (5) remains cocartesian upon passage to the associated groups. It is then an easy exercise with universal properties to verify that the quotients along the horizontal direction are isomorphic. This gives the isomorphism on the left side of (i); for the right side, we observe that \( \pi^! \) is a left adjoint, hence respects quotients.

For (ii), observe that to go from the middle term to the left one, we quotient both \( R_0^{\text{gp}} \) and \( \pi^*\overline{M}^{\text{gp}}_S \times \pi^*\overline{M}^{\text{gp}}_S \) by the antidiagonal copy of \( \pi^*\overline{M}^{\text{gp}}_S \). To go from middle to right, we use the fiber product construction of \( R_0^{\text{gp}} \) in (3) and quotient by the right copy of \( \pi^*\overline{M}^{\text{gp}}_S \).

For (iii), first recall that the sharpening of \( \overline{Q}_S \to \overline{N}_S \) is the minimal quotient \( \overline{Q}_S \) of \( \overline{Q}_S \) through which the map to \( \overline{N}_S \) factors as a sharp homomorphism. It is constructed by dividing \( \overline{Q}_S \) by the set of elements that map to 0 in \( \overline{N}_S \).
Now, $Q_S$ is, by construction, an extension of $\overline{Q}_S$ by $\mathcal{O}_S^*$. By definition, the associated logarithmic structure $Q_S^*$ of $Q_S$ fits into a cocartesian diagram:

\[
\begin{array}{c}
\exp^{-1} \mathcal{O}_S^* \ar[r] & \mathcal{O}_S^* \\
\ar[d] & \ar[d] \\
Q_S & Q_S^*
\end{array}
\]

But note that $\exp^{-1} \mathcal{O}_S^* = \gamma^{-1} N_S^*$, where $\gamma : Q_S \to N_S$ is the tautological map. Dividing everything by $\mathcal{O}_S^*$ yields another cocartesian diagram:

\[
\begin{array}{c}
\gamma^{-1}(0) \ar[r] & 0 \\
\ar[d] & \ar[d] \\
\overline{Q}_S & \overline{Q}_S^a
\end{array}
\]

On the other hand, this is precisely the cocartesian diagram used to sharpen $\overline{Q}_S \to \overline{N}_S$.

For (iv), observe that when $M_{X/S} = 0$, the fiber product (3) becomes a product: $\overline{R}^{gp} = \overline{M}^{gp} \times \pi^* \overline{M}^{gp}$, so equalizing the two copies of $\pi^* \overline{M}^{gp}$ recovers $\overline{M}^{gp}$. Chasing the definitions\footnote{The reader who is so inclined may prefer to observe that $\overline{R}$ and $\overline{M} \to \overline{R}_X$ (where $\overline{R}_X$ is the pushout of $\pi^* \overline{M}_S \to \overline{M}_X$ along $\pi^* \overline{M}_S \to \overline{R}$) satisfy a universal property, and that $\overline{M}$ and $\text{id}: \overline{M} \to \overline{M}$ visibly satisfy this universal property when $M_{X/S} = 0$.} in [Wis, Section 3.1], one discovers that the map $\overline{M}^{gp} \to \overline{R}_X = \overline{M}^{gp}$ is the identity under this identification, and therefore that $\overline{R} = \overline{M}$.

6. A COUNTEREXAMPLE: LOGARITHMIC POINTS

It is perhaps easiest to appreciate how the criterion of Theorem 1.1 works by studying why quasifiniteness fails in an example where the criterion does not apply.

We consider the moduli space of logarithmic points valued in the standard logarithmic point, considered by Abramovich, Chen, Gillam, and Marcus [Gil12, ACGM10]: Let $k$ be the spectrum of an algebraically closed field, let $S = \text{Spec} k$ and $M_S = 0$, and let $X$ and $Y$ both be the standard logarithmic point over $k$, both regarded as logarithmic schemes over $S$. The space we are interested in is $\text{Hom}_{\text{LogSch}/S}(X, Y)$. Since the underlying map of schemes $X \to Y$ must be the identity, this may be identified with $\text{Hom}_{\text{LogSch}/S}(M, M)$. We have $M = M_X = N \times k^*$. An $S$-point of this moduli space is simply a map of logarithmic structures $M \to M_X$, and our choices are determined by where the generator $(1, 1)$ goes. That gives $N \times k^*$ for the $S$-points.

We could also look at the construction of the minimal monoid associated to one of these maps. Following the algorithm from Section 5, we should form $\overline{R}^{gp} = \overline{M}^{gp} \times_{M_{X/S}} \overline{M}^{gp}_{X}$. Since $M_S$ is trivial, this is just $\overline{M}^{gp}$. (There is an additional quotient by $\overline{M}^{gp}$ in the algorithm which doesn’t change anything since $\overline{M}_S = 0$.) Then we identify the smallest submonoid $\overline{R} \subset \overline{R}^{gp}$ such that $\overline{R}_X = \overline{M} \times N$ contains the image of $\overline{M}$ under the tautological map $(\text{id}, u) : \overline{M} \to \overline{M}^{gp} \times \mathbb{Z}$ (here $u$ is the type, also known as the contact order). This submonoid is obviously $\overline{M}$ itself, so $\overline{M}$ is the minimal characteristic monoid.

In order to obtain an actual logarithmic structure, we need to assume that $u$ came from an actual logarithmic map over some $(S, N_S)$. This induces a map $\overline{M} \to \overline{N}_S$ since $\overline{M}$ is minimal on the level of characteristic monoids (see Section 5). This gives a quasilogarithmic
structure (see [Wis, Definition 1.2]) by pulling back $N_S$ to $\overline{M}$, but in order to get a genuine logarithmic structure, we must also sharpen the map $\overline{M} \to \overline{N}_S$; that is, we must divide $\overline{M}$ by the preimage of 0.

In our example, we are working over $(S, \mathcal{O}_S)$ so that $\overline{N}_S = 0$. Therefore the sharpening of the minimal quasilogarithmic structure is trivial.

The main observation of this paper was that the sharpening process is responsible for the failure of quasifiniteness. More precisely, when the sharpening process does not change anything, there is at most one choice (up to unique isomorphism) of a minimal object (at least if the domain is proper). Indeed, it is a consequence of Corollary 7.2 and Lemma 7.3, below, that (under an assumption of properness) the automorphism group of the minimal quilogarithmic structure $Q_S$, as an extension of its characteristic monoid by $\mathcal{O}_S$—but not respecting the map to $\mathcal{O}_S$—precisely cancels the choices of maps $M \to Q_X$. This automorphism group agrees with the automorphism group of the minimal logarithmic structure exactly when $Q_S$ is the minimal logarithmic structure, and this occurs with the map $\pi_1 R \to \overline{N}_S$ is sharp (Lemma 5.1 (iii)).

Thus the analysis of the fiber of the moduli space of logarithmic maps comes down to the question of whether the minimal quasilogarithmic structure $Q_S$ is already a logarithmic structure. Lemma 7.1 shows that a sufficient condition is that the relative characteristic monoid $M_{X/S}$ be generically trivial.

7. Minimal characteristic monoids

Lemma 7.1. Let $\overline{R}$ be constructed as in Section 5. If the relative characteristic of $X/S$ is generically trivial on every geometric fiber of $X$ over $S$ then for any $(N_S, \varphi) \in H(\mathcal{S})$, the canonical map $\pi_1 \overline{R} \to \overline{N}_S$ is sharp.

Proof. Since the construction of $\pi_1 \overline{R}$ commutes with change of base, we can reduce to the case where $\mathcal{S}$ is the spectrum of an algebraically closed field. By Lemma 4.3, there is a surjection $\pi_1 \pi^* R \to \pi_1 R$, where $i$ is the inclusion of the dense open subset where $M_{X/S} = 0$. It is sufficient to show that $\pi_1 \pi^* R \to \overline{N}_S$ is sharp. We can therefore reduce to the case where $M_{X/S} = 0$, globally.

In that case, $\overline{R} = \overline{M}$ (Lemma 5.1 (iv)) and the map $\pi_1 \overline{R} \to \overline{N}_S$ is induced by adjunction from the map:

$$\varphi : \overline{M} \to \overline{N}_X = \pi^* \overline{N}_S$$

The equality on the right holds because $M_{X/S} = 0$. But now by Lemma 4.1 and Lemma 4.2 (i), any $a \in \pi_1 \overline{M}$ is the image of a sum of local sections $a_i$ defined over connected $U_i$ that are étale over $X$. If $a = \sum a_i$ maps to 0 in $\overline{N}_S$ then all $a_i$ map to 0 in $\overline{N}_S$ since $\overline{N}_S$ is sharp (as it is the characteristic monoid of a logarithmic structure). This means $a_i$ maps to 0 under $\varphi : \overline{M} \to \pi^* \overline{N}_S$. But $\varphi$ underlies the morphism of logarithmic structures $\varphi : M \to \pi^* N_S$, hence is sharp. Therefore all of the $a_i$ must be zero. \hfill \square

Corollary 7.2. Under the hypotheses of the lemma, if $(N_S, \varphi) \in H(\mathcal{S})$ then $\overline{N}_S = \overline{Q}_S$.

Proof. By Lemma 5.1 (iii), $\overline{N}_S$ is the sharpening of $\overline{Q}_S \to \overline{N}_S$, so the point is to show $\overline{Q}_S \to \overline{N}_S$ is sharp. We already know that $\pi_1 R \to \overline{N}_S$ is sharp by the lemma. When $X$ has connected geometric fibers over $\mathcal{S}$, this implies the corollary, since in that case $\pi_1 \overline{R} = \overline{Q}_S$.

\textsuperscript{2}Recall that by our notational conventions, $H(\mathcal{S})$ consists of all choices of logarithmic structure $N_S$ on $\mathcal{S}$ and all $\varphi \in H(\mathcal{S}, N_S)$. 
In general, we can proceed geometric fiber by geometric fiber and assume $S$ is the spectrum of an algebraically closed field. If $X = \emptyset$, the conclusion is obvious. Otherwise, the map $\pi_1 \pi^* \overline{M}_S \to \overline{M}_S$ is surjective, which implies that $\pi_1 \overline{R} \to \overline{Q}_S$ is surjective as well. Therefore the sharpness of $\pi_1 \overline{R} \to \overline{N}_S$ implies the sharpness of $\overline{Q}_S \to \overline{N}_S$, as required. \hfill $\square$

The corollary implies that all objects of $H(S)$ have the same characteristic monoid. This greatly simplifies the analysis of $H(S)$, since the following lemma gives a complete characterization of $H(S, \overline{N}_S)$ for any fixed characteristic monoid $\overline{N}_S$:

**Lemma 7.3.** Assume $S$ is the spectrum of an algebraically closed field. If $H(S, \overline{N}_S)$ is nonempty then it is isomorphic to the quotient groupoid$^3$

$$[\mathrm{Hom}(\overline{M}^{gp}/\pi^* \overline{M}^{gp}_S, \mathcal{O}_X^*)/\mathrm{Hom}(\overline{N}_S^{gp}/\overline{M}^{gp}_S, \mathcal{O}_S^*)].$$

The map $\mathrm{Hom}(\overline{N}_S/\overline{M}_S, \mathcal{O}_S^*) \to \mathrm{Hom}(\overline{M}/\pi^* \overline{M}_S, \mathcal{O}_X^*)$ used to construct the quotient is obtained from the canonical maps

$$\pi^* \mathcal{O}_S^* \to \mathcal{O}_X^*$$

$$\overline{M}^{gp}/\pi^* \overline{M}^{gp}_S \to \overline{N}_S^{gp}/\overline{M}_X^{gp} \simeq \pi^* \overline{N}_S^{gp}/\pi^* \overline{M}^{gp}_S,$$

the latter of which is induced from $\varphi$ and the cocartesian diagram (2).

**Proof.** We need to see how many ways there are to choose $(N_S, \varphi) \in H_u(S)$ with the same fixed characteristic monoid $\overline{N}_S$. Holding $N_S$ fixed, any two choices of $\varphi$ will differ by a uniquely determined homomorphism $M \to \mathcal{O}_X^*$ that vanishes on $\pi^* \overline{M}_S$. Therefore, for $N_S$ fixed, the choices of $\varphi$ form a torsor under $\mathrm{Hom}(\overline{M}/\pi^* \overline{M}_S, \mathcal{O}_X^*)$.

Since $S$ is the spectrum of an algebraically closed field, there is a unique choice of $N_S$ (up to nonunique isomorphism) once $\overline{N}_S$ is fixed. Making such a choice, we can now identify $H_u(S, \overline{N}_S)$ with the quotient of $\mathrm{Hom}(\overline{M}/\pi^* \overline{M}_S, \mathcal{O}_X^*)$ by the automorphism group of $N_S$ fixing $\overline{N}_S$ and $M_S$. That automorphism group is precisely $\mathrm{Hom}(\overline{N}_S/\overline{M}_S, \mathcal{O}_S^*)$, by the same calculation we made above. \hfill $\square$

Putting these two lemmas together, we discover first from Lemma 7.1 that if $H(\emptyset) \neq \emptyset$ then the characteristic monoid of any object of $H(\emptyset)$ is $Q_S$. Then Lemma 7.3 implies that $H(\emptyset)$ may be identified with the quotient of $\mathrm{Hom}(\overline{M}^{gp}/\pi^* \overline{M}^{gp}_S, \mathcal{O}_X^*)$ by

$$\mathrm{Hom}(\overline{Q}_S^{gp}/\overline{M}^{gp}_S, \mathcal{O}_S^*) = \mathrm{Hom}(\pi_1 \overline{R}^{gp}/\pi_1 \pi^* \overline{M}^{gp}_S, \mathcal{O}_S^*) \quad \text{(Lemma 5.1 (i))}$$

$$= \mathrm{Hom}(\overline{R}^{gp}/\pi^* \overline{M}^{gp}_S, \pi^* \mathcal{O}_S^*)$$

$$= \mathrm{Hom}(\overline{M}^{gp}/\pi^* \overline{M}^{gp}_S, \pi^* \mathcal{O}_S^*) \quad \text{(Lemma 5.1 (ii))}.$$  

We will therefore be able to conclude that the quotient $H(\emptyset)$ is trivial once we prove

$$\mathrm{Hom}(\overline{M}^{gp}/\pi^* \overline{M}^{gp}_S, \mathcal{O}_X^*) = \mathrm{Hom}(\overline{M}^{gp}/\pi^* \overline{M}^{gp}_S, \pi^* \mathcal{O}_S^*).$$

That will be done in the next section.

$^3$In fact, the groupoid is a 2-group and $H(S, \overline{N}_S)$ is a pseudotorsor under this 2-group.
8. Units

**Theorem 8.1.** Suppose that \( X \) is reduced, that \( X \) is proper over \( S \), and that \( M \) is a coherent logarithmic structure on \( X \). Then any homomorphism \( \overline{M} \to O_X^\ast \) factors through \( \pi^*O_S^\ast \).

**Remark.** Theorem 8.1 is obvious in the special case when \( \overline{M}^{sp} \) is generated by global sections, since maps \( \overline{M} \to O_X^\ast \) correspond to global sections of \( O_X^\ast \).

For any scheme \( X \), let \( \mathcal{U} \subset O_X^\ast \) be the \( \acute{e}tale \) subsheaf\(^4\) whose sections over an \( \acute{e}tale \) \( V \to X \) consist of all \( f \in O_X^\ast (V) \) such that, for every valuation ring \( R \) with field of fractions \( K \), and every commutative diagram

\[
\begin{array}{ccc}
\text{Spec } K & \xrightarrow{\varphi} & V \\
\downarrow & & \downarrow \\
\text{Spec } R & \xrightarrow{q} & X
\end{array}
\]

the restriction \( \varphi^*f \) of \( f \) to \( \text{Spec } K \) lies in \( R^\ast \). The idea is that \( \mathcal{U}(V) \) is the sheaf of units in \( O_X^\ast (V) \) that have no zeros or poles on the closure of \( V \).

**Lemma 8.2.** Suppose that \( M \) is a coherent logarithmic structure on \( X \). Then any homomorphism of sheaves of monoids \( \overline{M} \to O_X^\ast \) factors through \( \mathcal{U} \).

**Proof.** Let \( \alpha : \overline{M} \to O_X^\ast \) be a homomorphism. Consider a section \( f \in \overline{M}(V) \) for some \( \acute{e}tale \) \( V \to X \). We must show that for any commutative diagram \( (7) \), the image of \( \varphi^*\alpha \) in \( K^\ast \) lies in \( R^\ast \). We can therefore replace \( X \) with \( \text{Spec } R \) and \( V \) with \( \text{Spec } K \). Since \( K^\ast \) and \( R^\ast \) are sheaves in the \( \acute{e}tale \) topology on \( \text{Spec } R \) (namely, \( O_X^\ast \) and \( j_*O_{\text{Spec } K} \) where \( j \) is the inclusion of the generic point), we can work \( \acute{e}tale \)-locally in \( X \) and assume that \( M \) has a global chart. Then \( \alpha \) is the restriction of a global section \( \beta \) of \( \overline{M} \) and the image of \( \beta \) in \( O_X^\ast \) lies in \( \Gamma(X, O_X^\ast) = R^\ast \). Therefore the same holds for \( \alpha \). \( \square \)

**Lemma 8.3.** When \( \pi : X \to S \) is proper and \( X \) is reduced, the natural inclusion \( \pi^*O_S^\ast \subset \mathcal{U} \) is a bijection.

**Proof.** Suppose that \( f \) is a section of \( \mathcal{U} \) over an \( \acute{e}tale \) \( V \to X \). We can assume \( V \) is quasicompact. Let \( \overline{V} \) be the closure of the graph of \( f \) in \( X \times \mathbf{P}^1 \) (with its reduced structure) and let \( \overline{f} : \overline{V} \to \mathbf{P}^1 \) be the projection. For any point \( q \) of \( \overline{V} \), we can choose a valuation of \( O_{\overline{V}} \) whose center is \( q \) and whose generic point lies in \( V \); let \( R \) be the valuation ring. Then by definition of \( \mathcal{U} \), the restriction of \( \overline{f} \) to \( \text{Spec } R \) lies in \( R^\ast \). In particular, \( \overline{f}(q) \neq 0, \infty \). Therefore \( \overline{f} \) factors through \( \mathbf{G}_m \subset \mathbf{P}^1 \). This holds for any \( q \) of \( \overline{V} \), so \( \overline{f} \in \Gamma(\overline{V}, O_{\overline{V}}^\ast) \). But \( \overline{V} \) is reduced and proper over \( S \), so \( \Gamma(\overline{V}, O_{\overline{V}}^\ast) = \Gamma(\overline{V}, \pi^*O_S^\ast) \). But \( V \) is reduced (since it is \( \acute{e}tale \) over \( X \) and \( X \) is reduced) so \( V \subset \overline{V} \) as a scheme and \( f \) is the restriction of \( \overline{f} \) to \( V \). Thus \( f \in \Gamma(V, \pi^*O_S^\ast) \), as required. \( \square \)

**Proof of Theorem 8.1.** By Lemma 8.2, any homomorphism \( \overline{M} \to O_X^\ast \) factors through \( \mathcal{U} \). But because \( X \) is reduced and proper over \( S \), Lemma 8.3 implies \( \mathcal{U} = \pi^*O_S^\ast \). \( \square \)

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\(^4\)I do not know whether it is necessary to work in the \( \acute{e}tale \) topology, or if the sheaf is induced from the Zariski topology.
Proof of Theorem 2.2. By Corollary 7.2, any \((N_S, \varphi) \in H(S)\) must have \(N_S = Q_S\). Then by Lemma 7.3, if there are any objects of \(H(S)\) with characteristic monoid \(Q_S\) then they are parameterized by the quotient:

\[
\left[ \frac{\text{Hom}(\overline{M}^\text{gp}/\pi^*\overline{M}^\text{gp}_S, O_X^*)}{\text{Hom}(Q^\text{gp}_S/\pi^*\overline{M}^\text{gp}_S, O^*_S)} \right]
\]

The chain of equalities at the end of Section 7, along with Theorem 8.1, implies that

\[
\text{Hom}(\overline{Q}^\text{gp}_S/\pi^*\overline{M}^\text{gp}_S, O^*_S) = \text{Hom}(\overline{M}^\text{gp}/\pi^*\overline{M}^\text{gp}_S, O^*_X)
\]

so the quotient (8) is trivial. \(\square\)

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