Pseudospin, supersymmetry and the shell structure of atomic nuclei

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Abstract

The evolution of single-particle energies with varying isospin asymmetry in the shell model is an important issue when predicting changes in the shell structure for exotic nuclei. In many cases pseudospin partner levels, that are almost degenerate in energy for stable nuclei, are relevant in extracting the size of the shell gaps. A breaking of the pseudospin symmetry can affect the size of these gaps and change the magic numbers accordingly. The strength of the pseudospin splitting is expected to depend in particular on isovector-dependent and tensor contributions to the effective nuclear interaction. A description employing supersymmetric quantum mechanics allows to derive a pseudospin symmetry breaking potential that is regular in contrast to the pseudospin-orbit potential in the conventional relativistic treatment. The derived perturbation potential provides a measure to quantify the symmetry breaking and it can be employed to improve mean-field calculations in order to better reproduce the experimentally observed shell evolution. General potentials with exact pseudospin symmetry are obtained that can be used in relativistic mean-field Hamiltonians.

Key words: pseudospin symmetry, supersymmetry, shell model, relativistic mean field model, single-particle states

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1 Introduction

The single-particle shell model with strong central and spin-orbit potentials is a cornerstone of nuclear structure physics since about sixty years. It was introduced by Haxel, Jensen and Sueß [1,2,3,4] and Goeppert-Mayer [5] in order to explain the experimentally observed occurrence of magic numbers for nuclei close to the valley of β-stability. In this picture, nucleons move as independent particles in a spherical mean-field. The central potential usually has a shape similar to the nuclear density distribution and the spin-orbit potential
is peaked at the nuclear surface. The large spin-orbit potential explains the observed energy splittings between two spin-orbit partners with total angular momentum $j > = l + 1/2$ and parity $\pi = (-1)^l$ for a given orbital angular momentum $l$, see the left part of figure 1. This essential feature of the model generates the particular shell structure with the appearance of major shell closures at nucleon numbers 2, 8, 20, 28, 50, 82, 126, . . . and subshell closures at, e.g., 14, 40, 114. The single-particle wave functions of the mean-field calculation can serve as a basis in order to construct fully antisymmetrized many-body wave functions. Using a suitably chosen subset in the many-body space with an appropriate residual interaction, the excitation spectrum of a nucleus can be described with high accuracy by diagonalizing the corresponding many-body Hamiltonian, see, e.g., ref. [6].

Experiments with exotic nuclei suggest a change of the well-known shell structure outside the valley of stability and the appearance of new (sub-)shell closures often attributed to changes in the spin-orbit splitting or halo effects [7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26]. It is very important to understand the evolution of the single-particle energies with a change in isospin asymmetry and to identify the origin of this effect. A change in the size of shell gaps can have important consequences in r-process nucleosynthesis [27,28,29,30,31]. In recent years, contributions to the nucleon-nucleon potential that change the usual shell structure when the isospin asymmetry of the system increases have been discussed extensively in the literature [32,33,34,35,36,37,38,39,40,41,42,43,44]. There are mainly two contributions to the nucleon-nucleon potential that have been considered: 1. a central interaction of Majorana-type $V_M(r)\tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \tilde{\tau}_1 \cdot \tilde{\tau}_2$ (spin-isospin interaction) and 2. a non-central isospin-tensor interaction $V_{IT}(r)S_{12} \tilde{\tau}_1 \cdot \tilde{\tau}_2$ with the tensor operator $S_{12} = 3(\tilde{\sigma}_1 \cdot \tilde{r}_1)(\tilde{\sigma}_2 \cdot \tilde{r}_2)/(r_1 r_2) - \tilde{\sigma}_1 \cdot \tilde{\sigma}_2$, where the absolute strength of the potentials depends on the particular radial functions $V_M(r)$ and $V_{IT}(r)$ with $r = |\tilde{r}_1 - \tilde{r}_2|$. The monopole part of these interactions is found to be strongly attractive between protons and neutrons when the spins of the two nucleons are anti-aligned, i.e., for $\pi l_\downarrow$ and $\nu l'_\downarrow$ (or $\pi l_\uparrow$ and $\nu l'_\uparrow$) with $j_\downarrow = l - 1/2$, $j'_\downarrow = l' + 1/2$ (in case of the central spin-isospin interaction one has $l = l'$). Hence, the filling of one level with protons (neutrons) will lead to a strong shift of the corresponding neutron (proton) states changing the spin-orbit splittings and, depending on the involved levels, the size of the shell gaps.

Of particular importance is the behaviour of the levels that define the conventional shell gaps. It is interesting to note that for the (sub-)shell closures at 14, 20, 28, 50, 82, 114 and 126 at least one level defining the gap belongs a pair of so-called pseudospin partners. These states are marked by solid circles in the central level scheme in figure 1. The total and orbital angular quantum numbers of the partner states are related to the so-called pseudo orbital angular momentum $\tilde{l}$ by $j_\downarrow = \tilde{l} - 1/2$, $l_\downarrow = \tilde{l} - 1$ and $j_\downarrow = \tilde{l} + 1/2$, $l_\downarrow = \tilde{l} + 1$, respectively. Both levels have the same parity $\pi = (-1)^{\tilde{l}+1}$. In addition, the
principal quantum number $n$ of the state with $j_\prec = \tilde{l} - 1/2$ is larger than the principal quantum number of the state with $j_\succ = \tilde{l} + 1/2$ by one unit. For example, the doublets \{2s_{1/2}, 1d_{3/2}\}_{so} = \{1\tilde{p}_{1/2}, 1\tilde{p}_{3/2}\}_{ps}$ with $\tilde{n} = 1, \tilde{l} = 1$ and \{2p_{3/2}, 1f_{5/2}\}_{so} = \{1d_{3/2}, 1d_{5/2}\}_{ps}$ with $\tilde{n} = 1, \tilde{l} = 2$ are pseudospin partners. The indices $so$ and $ps$ denote the spin-orbit and pseudospin representation of the states, respectively. Many partner levels of pseudospin pairs are experimentally found to be almost degenerate in energy suggesting the existence of a particular symmetry in the mean-field Hamiltonian for the single-particle states. This feature has been noticed already more than forty years ago and the concept of pseudospin and pseudo-orbital angular momentum was introduced \[45,46\]. A breaking of the pseudospin symmetry in stable and exotic nuclei is potentially important for a change of the level structure and the appearance of new magic numbers. It is important to identify the contributions to the nuclear interaction that are responsible for a breaking or the restoration of the pseudospin symmetry. To this end, a quantitative measure of the pseudospin splitting is needed that can be used to compare and improve theoretical models.

Bohr, Hamamoto and Mottelson \[47\] discussed a transformation of the form $\vec{\sigma} \cdot \vec{r}/r$ that connects the angular momentum part of the pseudospin partner levels (see eq. (7)). However, it cannot account for the change of the principal quantum number $n$. Hence, a dynamical symmetry involving the momentum operator $\vec{p}$ is required. Also it was observed that the SU(3) and pseudo-SU(3) states of the shell model are related by transformations that are elements of the orthosymplectic superalgebra Osp(1/2) \[48,49\]. The many-particle pseudo-SU(3) theory is successfully used in the description of heavy deformed nuclei in the context of the interacting boson model (IBM) applying supersymmetry, see, e.g., ref. \[50\]. Eventually, the relevant momentum helicity operator $\vec{\sigma} \cdot \vec{p}/p$ was identified that accomplishes the pseudospin transformation and it was noticed that the Dirac equation of free or in-medium nucleons (in the Dirac-Brueckner approach) is invariant under this transformation \[51,52\]. The essential step in explaining the pseudospin symmetry was the application of a relativistic description. In this approach, e.g., in the form of relativistic mean-field (RMF) models, the single-particle states are described by Dirac spinor wave functions with large upper and small lower components. Ginocchio observed that the second-order differential equation for the lower component is approximately invariant with respect to the pseudospin symmetry transformation in RMF models with large scalar and vector potentials of nearly equal magnitude. In addition, the radial wave functions of this lower component are almost identical for two pseudospin partner states \[53,54,55,56,57,58,59,60\]. The pseudospin symmetry was studied in various models for nuclear structure without and with deformation \[61,62,63,64,65,66,67,68,69,70,71,72,73,74\] and in nucleon-scattering \[75,76,77,78,79\]. A symmetry breaking pseudospin-orbit potential can be derived from the wave equation of the lower component. In
principle, this potential should be a measure for the strength of the symmetry breaking. It should appear as a small perturbation in comparison with the dominating central potential in a mean-field description of nuclei. However, this potential exhibits a pole at finite radii and its position depends strongly on the level energy itself. Therefore, one cannot really estimate the strength of the splitting and test how the choice of the various contributions to the interaction influences the symmetry restoration or breaking as observed experimentally. The problems related to this peculiarity have been discussed in the literature, see, e.g., ref. [64].

Besides the single-particle levels that constitute pseudospin pairs there are several states that are (pseudo-)unpaired in a nucleus. There are all the \{np_{1/2}\}_{so} = \{(n - 1)s_{1/2}\}_{ps} states corresponding to \(\tilde{l} = 0\) irrespective of the principal quantum number \(n\) and the states for given pseudospin \(\tilde{l} > 0\) with (pseudo) principal quantum number \(n = 1\) (\(\tilde{n} = 0\)). E.g., for the pseudoangular momentum \(\tilde{l} = 1\) the 1\(s_{1/2}\) level has no pseudospin partner whereas for \(\tilde{n} = n > 1\) the pairs \{(n + 1)s_{1/2}, nd_{3/2}\}_{so} = \{\tilde{n}\tilde{p}_{1/2}, \tilde{n}\tilde{p}_{3/2}\}_{ps}\) appear. See the right part of figure 1. The latter feature is a typical observation for a system of levels respecting exact supersymmetry. It is therefore tempting to apply supersymmetric quantum mechanics to the nuclear single-particle shell model and to go beyond the harmonic oscillator case [48,49]. As will be shown below, the approach employing supersymmetry allows to extract a pseudospin breaking potential that is regular in contrast to the standard relativistic description. From the condition of vanishing perturbation potential, general potentials can be derived that exhibit exact pseudospin symmetry. In order to compare the standard description of pseudospin symmetry with the supersymmetric approach, the relativistic Hamiltonian will be used as a starting point in the theoretical considerations below. In principle, however, the supersymmetric approach can also be applied to non-relativistic Hamiltonians.

The paper is organized as follows. In section 2 the basic equations for describing single-particle states in relativistic models are given and the notation for quantum numbers, wave functions, potentials and Hamiltonians is established. The conventional description of pseudospin symmetry in the relativistic approach is discussed in section 3 and a simple example is introduced that will accompany the following theoretical presentation. In section 4 first the concept of supersymmetric quantum mechanics is presented as far as necessary to solve the problem considered here. More detailed accounts can be found, e.g., in refs. [80,81,82,83]. Then the supermomenta and supersymmetric partner Hamiltonians are defined. The form of the reduced supermomenta at small radii leads to a particular choice of energy shifts that relate the pseudospin partner Hamiltonians of the relativistic upper component wave functions to the partner Hamiltonians in the supersymmetric approach. Then, the connection between the wave functions in the standard relativistic and the supersymmetric description is given. A regular symmetry breaking potential is
derived that quantifies the pseudospin splitting. It can be used in the future to compare and improve nuclear structure calculations. In the final subsection, general potentials in relativistic Hamiltonians with exact pseudospin symmetry are discussed. The last section contains conclusions and an outlook.

2 Single-particle states in relativistic mean-field models

The motion of the nucleons in RMF models is governed by strong scalar and vector potentials (or self-energies) originating from the minimal coupling of scalar and vector mesons to the nucleons. In addition, a tensor potential can appear if vector mesons are coupled non-minimally to the meson field tensors, i.e. by a derivative coupling. In general, there are both isoscalar and isovector contributions to the nucleon self-energies. The details of the particular RMF model are not important in the present discussion. See, e.g., references [84,85,86,87,88,89,90,91] for a more extensive presentation. Here, it suffices to consider only the calculation of the single-particle states for given scalar, vector and tensor potentials in the Dirac equation.

In the following spherical symmetry will be assumed. In this case, single-particle levels are characterized by the principal quantum number \( n = 1, 2, \ldots \), the orbital angular momentum \( l = 0, 1, \ldots \), the parity \( \pi = (-1)^l \), the total angular momentum \( j = l \pm 1/2 \), and its projection \( \Omega = -j, -j+1, \ldots, j-1, j \).

In the relativistic description it is convenient to combine the quantum numbers \( l \) and \( j \) by introducing the quantity \( \kappa = \pm 1, \pm 2, \pm 3, \ldots \) where \( j(\kappa) = |\kappa| - 1/2 \) and

\[
l(\kappa) = \begin{cases} 
\kappa - 1 & \text{if } \kappa > 0 \\
-\kappa & \text{if } \kappa < 0 
\end{cases}.
\] 

(1)

See table 1 for the relation between the various quantum numbers. Levels with different \( \Omega \) for given \( n \) and \( \kappa \) are degenerate in energy.

A single-particle state \( \psi_{n\kappa \Omega} \) with energy \( E_{n\kappa} \) (including the rest mass \( m \)) of a nucleon is obtained by solving the time-independent Dirac equation

\[
\mathcal{H}\psi_{n\kappa \Omega} = E_{n\kappa}\psi_{n\kappa \Omega}
\] 

(2)

where

\[
\mathcal{H} = \vec{\alpha} \cdot \vec{p} + V(r) + \beta [m - S(r)] - i\beta \vec{\alpha} \cdot \vec{T}(r)
\] 

(3)
is the relativistic Hamiltonian with momentum operator $\vec{p}$ and Dirac matrices $\vec{\alpha}$ and $\beta$ in standard notation. The system of units is chosen such that $\hbar = c = 1$. The scalar, vector and tensor potentials are denoted by $S(r)$, $V(r)$ and $\vec{T}(r) = \vec{T}(r)/r$. Here, the sign convention with a positive scalar field $S$ is used. For spherical symmetry it is customary to write the Dirac spinor as

$$\psi_{n\kappa\Omega} = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) \mathcal{Y}_{\kappa\Omega}(\hat{r}) \\ iG_{n\kappa}(r) \mathcal{Y}_{-\kappa\Omega}(\hat{r}) \end{pmatrix}$$

(4)

with radial wave functions $F_{n\kappa}(r) = F_{nl,j}(r)$ and $G_{n\kappa}(r) = G_{nl,j}(r)$ in the upper and lower components, respectively, that are normalized according to

$$\int_0^\infty dr \left[ |F_{n\kappa}(r)|^2 + |G_{n\kappa}(r)|^2 \right] = 1.$$  

(5)

The spin and angular dependence is contained in the spinor spherical harmonics

$$\mathcal{Y}_{\kappa\Omega}(\hat{r}) = \sum_{m_l m_s} (l m_l s m_s | j \Omega) Y_{lm_l}(\hat{r}) \chi_{sm_s}$$

(6)

where the orbital angular momentum $l(\kappa)$ of the spherical harmonic $Y_{lm_l}$ is coupled with the nucleon spin $s = 1/2$ of the spinor $\chi_{sm_s}$ to the total angular momentum $j(\kappa)$. The upper and lower component in (4) have an opposite sign for the quantum number $\kappa$ in the spinor spherical harmonics and consequently opposite parity with a difference of one in the orbital angular momentum. By applying the Hamiltonian (3) to the Dirac spinor (4) and using the relations

$$\vec{\sigma} \cdot \vec{p} \mathcal{Y}_{\kappa\Omega}(\hat{r}) = -\mathcal{Y}_{-\kappa\Omega}(\hat{r})$$

(7)

and

$$\vec{\sigma} \cdot \vec{p} \frac{f(r)}{r} \mathcal{Y}_{\kappa\Omega}(\hat{r}) = \frac{i}{r} \left( \frac{d}{dr} - \frac{\kappa}{r} \right) f(r) \mathcal{Y}_{-\kappa\Omega}(\hat{r}),$$

(8)

the set of coupled first-order differential equations

$$(E_{n\kappa} - V + m - S) G_{n\kappa}(r) = \left( \frac{d}{dr} - \frac{\kappa}{r} - T \right) F_{n\kappa}(r)$$

(9)

$$(E_{n\kappa} - V - m + S) F_{n\kappa}(r) = -\left( \frac{d}{dr} + \frac{\kappa}{r} + T \right) G_{n\kappa}(r)$$

(10)
for the radial wave functions is obtained. A simple decoupling leads to the
independent wave equations

\[ H_F(\kappa) F_{n\kappa} = E_{n\kappa} F_{n\kappa} \]  
\[ H_G(\kappa) G_{n\kappa} = E_{n\kappa} G_{n\kappa} \]

with Schrödinger-like Hamiltonians

\[ H_F(\kappa) = -\left( \frac{d}{dr} + \frac{\kappa}{r} + T \right) \frac{1}{A} \left( \frac{d}{dr} - \frac{\kappa}{r} - T \right) + V - S + m \]
\[ = \frac{1}{A} \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} + 2T\frac{\kappa}{r} + T^2 + \frac{A'}{A} \left( \frac{d}{dr} - \frac{\kappa}{r} - T \right) \right] + V - S + m \]  
\[ H_G(\kappa) = \frac{1}{A} \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + 2T\frac{\kappa}{r} + T^2 + \frac{B'}{B} \left( \frac{d}{dr} + \frac{\kappa}{r} + T \right) \right] \]

respectively, that explicitly depend on the quantum number \( \kappa \). A prime denotes the derivative with respect to the radius \( r \). The quantities

\[ A(r) = E_{n\kappa} - V(r) + m - S(r) \]

and

\[ B(r) = E_{n\kappa} - V(r) - m + S(r) \]

depend on the single-particles energy \( E_{n\kappa} = E_{nlj} \) appearing as the eigenvalue in equations (11) and (12). The Hamiltonians \( H_F(\kappa) \) and \( H_G(\kappa) \) have the same spectrum, even though the potentials have a different form. \( A(r) \) can be considered as twice the position depending effective nucleon mass \( m_{\text{eff}} \). It is always positive in standard RMF models. The dependence on the energy \( E_{n\kappa} \) is a relativistic effect. Since \( E_{n\kappa} \) contains the rest mass \( m \) of the nucleon, the variation of \( A(r) \) with a change in the nucleon binding energy is very small. For vanishing scalar and vector fields \( S(r) \) and \( V(r) \) at large radii \( r \), \( B(r) \) reduces to the non-relativistic energy \( E_{n\kappa} = E_{n\kappa} - m < 0 \) of the bound nucleon. The Hamiltonian (11) for the upper component wave function \( F_{n\kappa}(r) \) can be written in the familiar non-relativistic form

\[ H_F(\kappa) = -\frac{d}{dr} \frac{1}{2m_{\text{eff}}} \frac{d}{dr} + \frac{l(l+1)}{2m_{\text{eff}}r^2} + V_c(r) + V_{so}(r)(\vec{l} \cdot \vec{s}) + m \]
with central and spin-orbit potentials

\[ V_c(r) = V - S + \frac{T}{A} \left( T - \frac{A'}{A} \right) + \frac{1}{Ar} \left( 2T - \frac{A'}{A} \right) \]

\[ = V - S - \frac{T^2}{2m_{\text{eff}}} + \frac{1 + T}{2} V_{so}(r) \]

\[ V_{so}(r) = \frac{2}{Ar} \left( 2T - \frac{A'}{A} \right) = \frac{1}{m_{\text{eff}}r} \left[ 2T + \frac{1}{2m_{\text{eff}}} \frac{d}{dr} (V + S) \right] \]

(18)

(19)

since \( \kappa(\kappa - 1) = l(l + 1) \) and \( \kappa - 1 = 2\langle \vec{l} \cdot \vec{s} \rangle \). The spin-orbit potential contains contributions from the tensor potential \( T \) and the derivative of the sum \( V + S \) whereas the main contribution to the central potential arises from the difference \( V - S \). With the typical strong vector and scalar potentials of almost similar magnitude inside nuclei, the usual depth of the central potential and the strong surface-peak spin-orbit potential are nicely explained in the relativistic approach. A tensor potential is usually omitted in traditional RMF calculations but it has a distinct effect on the spin-orbit splitting that can be relevant for the size of the shell gaps.

3 Relativistic description of pseudospin symmetry

The basic idea in the relativistic description of pseudospin symmetry is a comparison of the Hamiltonians \( H_G(\kappa), H_G(\kappa') \) and of the radial wave functions \( G_{(n+1)\kappa}(r), G_{n\kappa'}(r) \) for the two pseudospin partner levels of the doublet \( \{ (n+1)\kappa, n\kappa' \} \). It is easy to see that for every state with \( \kappa > 0 \) the pseudospin partner state has the quantum number \( \kappa' = -\kappa - 1 \). The corresponding orbital angular momenta are \( l = \kappa - 1 \) and \( l' = -(\kappa - 1) = \kappa + 1 \), respectively. Similarly, the total angular momenta are given by \( j = \kappa - 1/2 \) and \( j' = |\kappa - 1| + 1/2 = \kappa + 1/2 \), respectively. Hence, the relation between \( \tilde{l} \) and \( \kappa \) is given by

\[ \tilde{l}(\kappa) = \begin{cases} \kappa & \text{if } \kappa > 0 \\ -\kappa - 1 & \text{if } \kappa < 0 \end{cases} \]

(20)

The single-particle states in the relativistic model with \( \kappa = -1 \), i.e. the \( p_{1/2} \) states, have no pseudospin partner. In this case \( \tilde{l} = 0 \). See table [I] for a summary of the relations for the quantum numbers.

A simple example shows the main features in the relativistic description of pseudospin partner states. Here, the lowest pair of levels with \( \tilde{l} = 1 \), i.e.,
{2s_{1/2}, 1d_{3/2}}_{so}, will be considered. The shape of scalar and vector potentials in RMF models is often well approximated by a Woods-Saxon form, i.e., \( S(r) = S_0 f(r, R, a) \) and \( V(r) = V_0 f(r, R, a) \) with \( f(r, R, a) = 1/[1 + \exp[(r - R)/a]] \).

Assuming a radius of \( R = 3.8 \text{ fm} \), a diffuseness of \( a = 0.65 \text{ fm} \) and absolute magnitudes of \( S_0 = 450 \text{ MeV} \) and \( V_0 = 370 \text{ MeV} \), respectively, (corresponding to the the mean-field potentials of neutrons in a nucleus like \(^{40}\text{Ca}\) one finds single-particles energies of \( \tilde{E}_{2s_{1/2}} - m = -15.604 \text{ MeV} \) and \( \tilde{E}_{1d_{3/2}} - m = -15.424 \text{ MeV} \) with the \( 1d_{3/2} \) level lying only 0.180 MeV above the \( 2s_{1/2} \) level. The corresponding wave functions of the upper and of the lower component in the Dirac spinor are depicted in the top and bottom panel of figure [2].

The wave functions \( F_{2s_{1/2}}(r) \) and \( F_{1d_{3/2}}(r) \) are very different where the former has an additional node because there exists a stronger bound \( 1s_{1/2} \) state at \( \tilde{E}_{1s_{1/2}} - m = -56.848 \text{ MeV} \) without node (except at \( r = 0 \)). In contrast, the wave functions \( G_{2s_{1/2}}(r) \) and \( G_{1d_{3/2}}(r) \) resemble each other closely, a sign for the almost perfect pseudospin degeneracy of the two states. Due to the different orbital angular momenta and corresponding centrifugal potentials in \( H_F(\kappa) \) and \( H_F(\kappa') \) there is a clear difference in the radial dependence of the upper component wave functions at small \( r \). But for the lower component radial wave function the \( r \) dependence is very similar.

In fact, the orbital angular momenta in the Hamiltonians \( H_G(\kappa) \) and \( H_G(-\kappa-1) \) of a pseudospin pair \([((n+1)\kappa, n(-\kappa-1)]_{so}\) for \( \kappa = \tilde{l} > 0 \) are identical since \( \tilde{l}(-\kappa) = \kappa = \tilde{l}(\kappa + 1) \). The centrifugal potential \( \kappa(\kappa + 1)/(Ar^2) \) in (14) is invariant with respect to the replacement \( \kappa \rightarrow -\kappa-1 \). The difference between the potentials

\[
\Delta(\tilde{l}, r) = H_G(\kappa) - H_G(-\kappa-1) = \frac{2\kappa+1}{Ar} \left( \frac{B'}{B} + 2T \right)
\]

(21)

could be considered as a measure for the pseudospin splitting because for \( \Delta(\tilde{l}) = 0 \) the Hamiltonians \( H_G(\kappa) \) and \( H_G(-\kappa-1) \) are identical and yield the same energy eigenvalues.

However, there is a problem in this approach since the quantity \( B(r) \) has a zero at finite \( r \) inside the nucleus with usual scalar and vector potentials in RMF models. This is most easily seen in figure [3] where the potential difference \( V(r) - S(r) \) and the non-relativistic energy \( \tilde{E}_{2s_{1/2}} - m \) are plotted. They are identical at a certain radius \( r \) and the quantity (16) is zero. Consequently, the difference potential (21), shown as a red solid line in figure [3] has a pole. Its position is strongly energy dependent. Thus, \( \Delta(\tilde{l}) \) cannot be considered small as compared to the potential \( V - S \) in the differential equation and it is therefore not a good measure for the symmetry breaking.

For vanishing tensor potential \( T = 0 \), the pseudospin-orbit potential (21) is zero for \( B' = 0 \) or \( S(r) = V(r) + C \) with a constant \( C \). As a consequence of
this condition, the main contribution \( V - S \) to the central potential \( (18) \) is just an overall shift of the energy scale for exact pseudospin symmetry. But then there will be no sufficiently strong central potential to bind the states as required in a nucleus.

A case where exact pseudospin symmetry is found is the relativistic harmonic oscillator with \( V(r) = -S(r) = m\omega^2 r^2/4 \) and \( T = 0 \). The effective mass \( m_{\text{eff}} = A/2 = (E + m)/2 \) is a constant, \( B(r) = E - m - m\omega^2 r^2/2 \) but \( B'(r) \neq 0 \) and \( \Delta(l, r) \neq 0 \). Hence, the identification of \( \Delta(l, r) \) as the relevant quantity to measure the pseudospin splitting is doubtful. It would be convenient to have an independent measure of the symmetry breaking that has a regular behaviour for all radii and that vanishes for exact pseudospin symmetry.

4 Supersymmetric description of pseudospin symmetry

The problems related to the conventional relativistic explanation of pseudospin symmetry call for an alternative approach that allows to derive a regular symmetry breaking potential which is a small perturbation without a pole and useful for a quantitative comparison of different models. The main idea of the supersymmetric approach to describe the pseudospin symmetry in nuclei is similar to the standard relativistic description. In both cases one does not compare the original model Hamiltonians of the pseudospin partner systems but closely related Hamilton operators with the same spectra. In fact, it will be shown below that a comparison of the supersymmetric partner Hamiltonians provides a way to define a proper pseudospin splitting potential. These partner potentials are derived from the Hamiltonians \( H_F(\kappa) \) and \( H_F(\kappa') \) for the upper component wave functions of the pseudospin partners \( \kappa \) and \( \kappa' = -\kappa - 1 \), respectively. To this end, the Hamiltonian \( (13) \) is written as

\[
H_F(\kappa) = H_1(\kappa) + E(\kappa)
\]

(22)

with a Hamiltonian \( H_1(\kappa) \) of a supersymmetric pair and an energy shift \( E(\kappa) \) to be determined below. A corresponding relation holds for the pseudospin partner Hamiltonian \( H_F(\kappa') \).

4.1 General features of supersymmetric quantum mechanics

It is well known, that every second-order Hamiltonian can be factorized in a product of two Hermitian conjugate first-order operators. In the
present case, operators $B_\kappa^+$ and $B_\kappa^- = [B_\kappa^+]^\dagger$ are introduced such that
\[ H_1(\kappa) = B_\kappa^+ B_\kappa^- . \] (23)

Then, the Hermitian operators
\[ Q_1(\kappa) = \begin{pmatrix} 0 & B_\kappa^+ \\ -B_\kappa^- & 0 \end{pmatrix} \quad Q_2(\kappa) = iQ_1(\kappa)\tau = \begin{pmatrix} 0 & -iB_\kappa^+ \\ iB_\kappa^- & 0 \end{pmatrix} \] (24)

are formed that are so-called supercharges with respect to the involution
\[ \tau = \tau^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \tau^2 = 1 \] (25)

because \{Q_1, \tau\} = \{Q_2, \tau\} = 0. In the next step, the supersymmetric Hamiltonian
\[ H_S(\kappa) = [Q_1(\kappa)]^2 = [Q_2(\kappa)]^2 = \begin{pmatrix} H_1(\kappa) & 0 \\ 0 & H_2(\kappa) \end{pmatrix} \] (26)

is obtained with the supersymmetric partner Hamiltonians (23) and
\[ H_2(\kappa) = B_\kappa^- B_\kappa^+ \] (27)
on the diagonal. Obviously, $H_S(\kappa)$ commutes with $Q_1(\kappa)$ and $Q_2(\kappa)$, i.e.,
\[ [H_S(\kappa), Q_1(\kappa)] = [H_S(\kappa), Q_2(\kappa)] = 0 \] (28)

and the supercharges $Q_i(\kappa)$ ($i = 1, 2$) are generators of the symmetry transformation. The supercharges (24) and the Hamiltonian (26) with the commutators (28) and the anticommutator
\[ \{Q_1(\kappa), Q_2(\kappa)\} = 0 \] (29)

are the most simple example of a supersymmetric algebra. Since $H_S(\kappa)$ is the square of the Hermitian operators $Q_i(\kappa)$, all eigenvalues $E_S(\bar{n}\kappa)$ of the eigenvalue equation
\[ H_S(\kappa)\Psi_S(\bar{n}\kappa) = E_S(\bar{n}\kappa)\Psi_S(\bar{n}\kappa) \] (30)
with the two-component wave function

\[ \Psi_S(\bar{n}\kappa) = \begin{pmatrix} \psi_1(\bar{n}\kappa) \\ \psi_2(\bar{n}\kappa) \end{pmatrix} \]  

are non-negative. It is easily seen by applying \( Q_i(\kappa) \) to equation (30), that \( H_1(\kappa) \) and \( H_2(\kappa) \) have the same spectrum of positive energies \( E_S(\bar{n}\kappa) > 0 \) where the operators \( B^+\kappa \) and \( B^-\kappa \) connect the components of the wave function (31) by the transformations

\[ \psi_2(\bar{n}\kappa) = \frac{B^-\kappa}{\sqrt{E_S(\bar{n}\kappa)}} \psi_1(\bar{n}\kappa) \quad \psi_1(\bar{n}\kappa) = \frac{B^+\kappa}{\sqrt{E_S(\bar{n}\kappa)}} \psi_2(\bar{n}\kappa). \]  

If there is an eigenstate \( \Psi_S(0\kappa) \) (\( \bar{n} = 0 \)) with energy \( E_S(0\kappa) = 0 \), the supersymmetry is called exact because \( Q_i(\kappa)\Psi_S(0\kappa) = 0 \) and this ground state obeys the symmetry of the Hamiltonian \( H_S(\kappa) \). In this case,

\[ B^-\kappa \psi_1(0\kappa) = 0 \quad \psi_2(0\kappa) = 0 \]  

or

\[ B^+\kappa \psi_2(0\kappa) = 0 \quad \psi_1(0\kappa) = 0, \]  

i.e., the Hamiltonian \( H_1(\kappa) \) or \( H_2(\kappa) \) has an additional state at zero energy that is not appearing for its supersymmetric partner Hamiltonian. In this paper, the usual convention is chosen such that \( H_1(\kappa) \) has a ground state at zero energy if the supersymmetry is exact. If there is no state \( \Psi_S(0\kappa) \) at energy \( E_S(0\kappa) = 0 \), the supersymmetry is called broken since \( Q_i(\kappa)\Psi_S(\bar{n}\kappa) \neq 0 \) for all \( \bar{n} = 1, 2, \ldots \) and the partner Hamiltonians \( H_1(\kappa) \) and \( H_2(\kappa) \) have identical spectra.

4.2 The supermomentum and supersymmetric partner Hamiltonians

The main task in the supersymmetric description of pseudospin symmetry is the construction of the operators \( B^+\kappa \) and \( B^-\kappa \). The particular form of the Hamiltonian (13) suggests to use the ansatz

\[ B^-\kappa = \frac{1}{\sqrt{A_\kappa(r)}} \left[ Q_\kappa(r) + \frac{d}{dr} \right] \quad B^+\kappa = \left[ Q_\kappa(r) - \frac{d}{dr} \right] \frac{1}{\sqrt{A_\kappa(r)}} \]  

\[ (35) \]
with (the principal quantum number \( n \) is suppressed in the following if not explicitly needed)

\[
A_\kappa(r) = E_\kappa - V(r) + m - S(r) \tag{36}
\]

and a quantity \( Q_\kappa(r) \) that will be called supermomentum by dimensional reasons. The traditional superpotential is given by

\[
W_\kappa(r) = \sqrt{\frac{2}{A_\kappa(r)}} Q_\kappa(r) , \tag{37}
\]

however, it has the dimension energy\(^{1/2}\). The supersymmetric partner Hamiltonians are found as

\[
H_1(\kappa) = B_\kappa^+ B_\kappa^- = \frac{1}{A_\kappa} \left[ Q_\kappa^2 - Q_\kappa' - \frac{d^2}{dr^2} + \frac{A_\kappa'}{A_\kappa} \left( Q_\kappa + \frac{d}{dr} \right) \right] \tag{38}
\]

and

\[
H_2(\kappa) = B_\kappa^- B_\kappa^+ = \frac{1}{A_\kappa} \left[ Q_\kappa^2 + Q_\kappa' - \frac{d^2}{dr^2} + \frac{A_\kappa'}{A_\kappa} \frac{d}{dr} + \frac{A_\kappa''}{2A_\kappa} - \frac{3(A_\kappa')^2}{4A_\kappa^2} \right] \tag{39}
\]

with the symmetry

\[
H_1(\kappa, Q_\kappa) = H_2(\kappa, -Q_\kappa - \frac{A_\kappa'}{2A_\kappa}) . \tag{40}
\]

If the supersymmetry is exact, the supermomentum \( Q_\kappa \) can be found from the ground state wave function \( \psi_1(0\kappa) \) of the Hamiltonian \( H_1(\kappa) \) as

\[
Q_\kappa(r) = -\frac{d}{dr} \ln \psi_1(0\kappa) \tag{41}
\]

because of equation (33), i.e., it is given by the negative logarithmic derivative of the groundstate wave function. Comparing equation (38) via (22) with equation (11) leads to the defining equation for the supermomentum \( Q_\kappa(r) \) in general. However, it is useful to split off the dependence on \( \kappa \) and the tensor potential \( T \) first and to introduce the reduced supermomentum

\[
q_\kappa(r) = Q_\kappa(r) + \frac{\kappa}{r} + T(r) \tag{42}
\]

with the results
\[ H_1(\kappa) = \frac{1}{A_\kappa} \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa - 1)}{r^2} + q_\kappa^2 - 2q_\kappa \frac{\kappa}{r} - q'_\kappa - 2q_\kappa T + T^2 \right] \]

\[ + 2T \frac{\kappa}{r} + T' + \frac{A'_\kappa}{A_\kappa} \left( q_\kappa + \frac{d}{dr} - \frac{\kappa}{r} - T \right) \]}

and

\[ H_2(\kappa) = \frac{1}{A_\kappa} \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa + 1)}{r^2} + q_\kappa^2 - 2q_\kappa \frac{\kappa}{r} + q'_\kappa - 2q_\kappa T + T^2 \right] \]

\[ + 2T \frac{\kappa}{r} - T' + \frac{A'_\kappa}{A_\kappa} \frac{d}{dr} + \frac{A''_\kappa}{2A_\kappa} - \frac{3(A'_\kappa)^2}{4A^2_\kappa} \]}

These two Hamiltonians show the symmetry

\[ H_1(\kappa, q_\kappa, T) = H_2(-\kappa, -q_\kappa - \frac{A'_\kappa}{2A_\kappa}, -T) , \]

however, it is irrelevant for the present considerations. Subsequently, the defining equation for the reduced supermomentum

\[ q_\kappa^2 - \left( 2\frac{\kappa}{r} + 2T - \frac{A'_\kappa}{A_\kappa} \right) q_\kappa - q'_\kappa = -A_\kappa B_\kappa - T' \]

with

\[ B_\kappa(r) = E(\kappa) - V(r) + S(r) - m \]

is obtained. Note that \( B_\kappa(r) \) depends on the energy shift \( E(\kappa) \) and \( A_\kappa(r) \) on the eigenvalue \( E_\kappa \). Equation (46) is the central relation that connects the reduced supermomentum \( q_\kappa(r) \) with the relativistic potentials \( S(r), V(r) \) and \( T(r) \). In general, the reduced supermomentum \( q_\kappa(r) \) is not uniquely determined by equation (46) and suitable boundary conditions have to be specified. In the present case, it will be required that \( q_\kappa(r) = 0 \) for \( r = 0 \).

It is convenient, e.g., in numerical applications, to transform the Riccati equation (46), an inhomogeneous nonlinear first-order differential equation, into a linear second-order differential equation to determine the reduced supermomentum \( q_\kappa(r) \) for given scalar, vector and tensor potentials \( S(r), V(r) \) and \( T(r) \). To this end, a new function \( y_\kappa(r) \neq 0 \) is introduced that is related via

\[ q_\kappa = -\frac{y'_\kappa}{y_\kappa} \]
to the reduced supermomentum $q_\kappa(r)$. With

$$q'_\kappa = -\frac{y''_\kappa}{y_\kappa} + \left(\frac{y'_\kappa}{y_\kappa}\right)^2$$

(49)

one immediately obtains the homogeneous differential equation

$$y''_\kappa + \left(\frac{2\kappa}{r} + 2T - \frac{A'_\kappa}{A_\kappa}\right)y'_\kappa + \left(A_\kappa B_\kappa + T'\right)y_\kappa = 0$$

(50)

that can be solved with standard techniques.

Using equation (46) the supersymmetric partner Hamiltonian (44) assumes the form

$$H_2(\kappa) = \frac{1}{A_\kappa} \left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + 2q'_\kappa - \frac{A'_\kappa}{A_\kappa} \left( q_\kappa - \frac{d}{dr} \right) \right] + T^2 + 2T\frac{\kappa}{r} - 2T' + \frac{A''_\kappa}{2A_\kappa} - 3\left(\frac{A'_\kappa}{2A_\kappa}\right)^2 - B_\kappa.$$  

(51)

It will be useful for the comparison with the Hamiltonian $H_2(\kappa')$ of the pseudospin partner system.

### 4.3 Asymptotics of the supermomenta

The asymptotic behaviour of the reduced supermomentum for $r \to 0$ and $r \to \infty$ is easily derived from equation (46) assuming regular $q_\kappa(r)$ and regular potentials $S(r)$, $V(r)$ and $T(r)$. The reduced superpotential has to vanish for $r \to 0$ otherwise a diverging potential would appear in the Hamiltonian $H_F(\kappa)$.

For small radii the reduced supermomentum is given by

$$q_\kappa(r) = \frac{A_\kappa(0)B_\kappa(0) + T'(0)}{2\kappa + 1}r + O(r^2),$$

(52)

i.e., it increases almost linearly with the radius. At large radii it becomes a constant

$$q_\kappa(r) = \sqrt{(E_\kappa + m)[m - E(\kappa)]} + O(r^{-1})$$

(53)

for potentials $S(r)$, $V(r)$ and $T(r)$ that approach zero at large radii $r$. The energy shift $E(\kappa)$ with the condition $E(\kappa) < m$ has still to be determined.
The asymptotic form (53) translates to a dependence

\[ y_\kappa(r) \to \exp\left\{ -\sqrt{(E_\kappa + m)(m - E(\kappa))} r \right\} \] (54)

of the auxiliary function \( y_\kappa \) in equation (50) for \( r \to \infty \). Starting at large radii and integrating inwards a stable solution for \( y_\kappa(r) \) is found. Of course, if the relativistic potentials do not approach zero at large radii, a different asymptotic behaviour of the reduced supermomentum will be obtained.

Considering the relation (42), the full supermomentum \( Q_\kappa(r) \) approaches the same constant as \( q_\kappa(r) \) for large \( r \). In contrast, the asymptotic behaviour of \( Q_\kappa(r) \) for \( r \to 0 \) is determined by the angular momentum term \( \kappa/r \). For \( \kappa > 0 \), one finds \( \lim_{r \to 0} Q_\kappa(r) = -\infty \) and \( \lim_{r \to 0} Q_\kappa'(r) = \infty \) for \( \kappa' = -\kappa - 1 < 0 \). The standard superpotential (37) shows the same asymptotic limits as the full supermomentum. From general considerations of supersymmetric quantum mechanics it is known, that the type of supersymmetry is determined by the asymptotics of the superpotential. If there is a change of sign in \( W_\kappa(r) \) when comparing the limits \( r \to 0 \) with \( r \to \infty \), exact supersymmetry follows and a single non-degenerate state at zero energy exists. Conversely, if the superpotential \( W_\kappa(r) \) does not change the sign in the above limits the supersymmetry is broken and all eigenstates are doubly degenerate with positive energy.

### 4.4 Application of supersymmetry to pseudospin levels and energy shifts

With the foregoing considerations, the supersymmetric description can be applied to the system of pseudospin levels appearing in nuclei. For given pseudo orbital angular momentum \( \bar{l} = \kappa \geq 1 \) there is an unpaired state with total angular momentum \( j = \bar{l} - 1/2 \), orbital angular momentum \( l = \bar{l} - 1 \) and energy \( E_{1\kappa} \), cf. the left part of figure 4. Choosing the energy shift as

\[ E(\kappa) = E_{1\kappa} \] (55)

the ground state energy of the Hamiltonian \( H_1(\kappa) = H_F(\kappa) - E(\kappa) \) is placed at zero energy and the case of exact supersymmetry is obtained. The supersymmetric partner Hamiltonian \( H_2(\kappa) \) has the same energy eigenvalues as \( H_1(\kappa) \) except for the zero-energy level. The orbital angular momentum in \( H_2(\kappa) \) is \( l = \kappa = \bar{l} \) as seen in the centrifugal potential in equation (44).

The pseudospin partner levels with \( j' = \bar{l} + 1/2 \) and \( l' = \bar{l} + 1 \) are eigenstates of the Hamiltonian \( H_F(\kappa') \) with \( \kappa' = -\bar{l} - 1 < 0 \), see the right part of figure 4. The lowest state has a larger energy \( E_{1\kappa'} \) than the ground state of \( H_F(\kappa) \).
For this system the case of broken supersymmetry applies where \( H_F(\kappa') = H_1(\kappa') + E(\kappa') \) and the supersymmetric partner Hamiltonian \( H_2(\kappa') + E(\kappa') \) have identical eigenvalues. The orbital angular momentum in \( H_2(\kappa') \) is again given by \( l = \kappa = \tilde{l} \) due to \( \kappa(\kappa' + 1) = (\kappa - 1)(\kappa - 1 + 1) = \kappa(\kappa + 1) \). The energy shift \( E(\kappa') \) is now chosen such that the asymptotic form of the reduced supermomentum \( q_{\kappa'}(r) \) is the same as for the reduced supermomentum \( q_\kappa(r) \) in the limit \( r \to 0 \). This condition leads with equation (52) to the relation

\[
A_\kappa(0)B_\kappa(0) + T'(0) = -[A_{\kappa'}(0)B_{\kappa'}(0) + T'(0)]
\] (56)
since \( 2\kappa' + 1 = -(2\kappa + 1) \). For exact pseudospin symmetry, \( E_{(n+1)\kappa} = E_{nn'} \) and consequently \( A_\kappa(r) = A_{\kappa'}(r) \). Even if the pseudospin symmetry is broken, \( A_\kappa \) and \( A_{\kappa'} \) are almost identical and the difference can be neglected as a small relativistic correction. Without the contribution of the tensor potential \( T'(0) \), the energy shift \( E(\kappa') \) is determined by

\[
E(\kappa') - m + S(0) - V(0) = -[E(\kappa) - m + S(0) - V(0)]
\] . (57)

In the top panel of figure 5 the reduced supermomenta \( q_\kappa(r) \) and \( q_{\kappa'}(r) \) of the pseudospin pair levels with \( \kappa = 1 \) and \( \kappa' = -2 \) are shown for the example in section 3. With the present choice of the energy shifts \( E(1) - m = -56.849 \text{ MeV} \) and \( E(-2) - m = -102.680 \text{ MeV} \), the reduced supermomenta display the same approximately linear behaviour for \( r \to 0 \) as required. In fact, they are almost identical for radii below 5 fm. At large \( r \) they become constant and approach different values for \( r \to \infty \) as predicted by equation (53) with \( q_1(r) < q_{-2}(r) \) because \( E(1) > E(-2) \). The full supermomenta \( Q_1(r) \) and \( Q_{-2}(r) \) are shown in the bottom panel of figure 5. They contain the angular momentum contribution and, hence, they diverge for \( r \to 0 \). The change of sign for \( Q_1(r) \) indicates the case of exact supersymmetry with a single state at zero energy for \( H_1(1) \). In contrast, there is no change of sign for \( Q_{-2}(r) \) corresponding to the case of broken supersymmetry.

4.5 Relation of wave functions

The supersymmetric description of pseudospin degenerate states allows to establish relations between the wave functions in the Dirac spinor (4) and in the supersymmetric state vector (31). According to equation (9) the lower component wave function of the spinor is given by

\[
G_{nn}(r) = \frac{1}{A_{nn}} \left( \frac{d}{dr} - \frac{\kappa}{r} - T \right) F_{nn}(r)
\] (58)
for the pseudospin partner state \( n\kappa \) with \( \kappa > 0 \) and similarly for the state \( n'\kappa' \) with \( n' = n - 1 \) and \( \kappa' = -\kappa - 1 \). Using equation (32) the supersymmetric partner

\[
\psi_2(n\kappa, r) = \frac{1}{\sqrt{|E_{n\kappa} - E(\kappa)|} A_{n\kappa}} \left( q_{n\kappa} - \frac{\kappa}{r} - T + \frac{d}{dr} \right) \psi_1(n\kappa, r) \tag{59}
\]

of the upper component wave function \( \psi_1(n\kappa, r) = F_{n\kappa}(r) \) is obtained where the eigenenergy \( E_S(\bar{n}\kappa) \) in the supersymmetric formulation is identified with the energy \( E_{n\kappa} - E(\kappa) = E_{n\kappa} - E_{0\kappa} \). With equation (58) one finds

\[
\psi_2(n\kappa, r) = \frac{A_{n\kappa}}{E_{n\kappa} - E(\kappa)} \left[ \frac{q_{n\kappa}}{A_{n\kappa}} F_{n\kappa}(r) + G_{n\kappa}(r) \right] \tag{60}
\]

and

\[
\psi_2(n'\kappa', r) = \frac{A_{n'\kappa'}}{E_{n'\kappa'} - E(\kappa')} \left[ \frac{q_{n'\kappa'}}{A_{n'\kappa'}} F_{n'\kappa'}(r) + G_{n'\kappa'}(r) \right] \tag{61}
\]

for the two pseudospin partner states. These functions are normalized as

\[
\int_0^\infty dr \left| \psi_2(n\kappa, r) \right|^2 = \int_0^\infty dr \left| F_{n\kappa}(r) \right|^2 \tag{62}
\]

and correspondingly for the second pair. In the case of exact pseudospin symmetry, the wave functions \( \psi_2(n\kappa, r) \) and \( \psi_2(n'\kappa', r) \) are identical provided that the upper component wave functions \( F_{n\kappa}(r) \) and \( F_{n'\kappa'}(r) \) in the Dirac spinors are normalized to the same value. This is beautifully seen in figure 6 for the example of section 3 with a much better agreement of the two pseudospin partner wave functions than in the standard approach, cf. figure 2. In contrast to the wave functions \( G_{n\kappa}(r) \) and \( G_{n'\kappa'}(r) \), there is no zero in the functions \( \psi_2(n\kappa, r) \) and \( \psi_2(n'\kappa', r) \) because they are the lowest states of the supersymmetric partner Hamiltonians \( H_2(\kappa) \) and \( H_2(\kappa') \) with the same orbital angular momentum \( l = l' = \kappa = -\kappa' - 1 = l' \). The factors \( q_{n\kappa}(r)/A_{n\kappa}(r) \) and \( q_{n'\kappa'}(r)/A_{n'\kappa'}(r) \) in equations (60) and (61) are very small at small radii \( r \) explaining the observation of the close similarity of the wave functions \( G_{n\kappa}(r) \) and \( G_{n'\kappa'}(r) \) in the usual comparison. At larger radii, however, the contribution from the upper component wave functions \( F_{n\kappa}(r) \) and \( F_{n'\kappa'}(r) \), respectively, becomes more important.
4.6 Pseudospin symmetry breaking potential

The Hamiltonians $H_2(\kappa) + E(\kappa)$ and $H_2(\kappa') + E(\kappa')$ of the supersymmetric partners for the pseudospin partner levels are almost identical. The difference is given by the potential

$$\Delta_S(\tilde{l}, r) = H_2(\kappa) + E(\kappa) - [H_2(\kappa') + E(\kappa')]$$

$$= \frac{2}{\sqrt{A}} \frac{d}{dr} q_\kappa - q_{\kappa'} + 2T_2^{\kappa + 1} \frac{1}{Ar}$$

with $\tilde{l} = \kappa = -\kappa' - 1$ and $A = A_\kappa$ where equation (61) has been used and the difference between $A_\kappa$ and $A_{\kappa'}$ has been neglected. The quantity $\Delta_S(\tilde{l}, r)$ is a regular function for all radii $r$ with the same contribution from the tensor potential $T(r)$ as the previously defined pseudospin-orbit potential $\Delta(\tilde{l}, r)$ in (21). However, the pseudospin breaking potential (63) is a regular function for all $r$ without a pole as observed for $\Delta(\tilde{l}, r)$. Without tensor potential, the pseudospin symmetry breaking potential depends on the derivative containing the reduced supermomenta of the pseudospin partner systems instead on the derivative of the difference $V - S$ as in the traditional approach (21). From figure 3 it is evident that $\Delta_S(\tilde{l}, r)$ with $\tilde{l} = 1$ for the example in section 3 is really small as compared to $V(r) - S(r)$ and can be considered as a perturbation. In a relativistic model with tensor potential $T(r)$, both the non-relativistic spin-orbit potential (19) and the pseudospin symmetry breaking potential (63) are directly affected by $T(r)$ with the same functional form, only the prefactors $\kappa - 1$ and $2\kappa + 1$ are different. The dependence of $\Delta_S(\tilde{l}, r)$ on the scalar and vector potentials $S(r)$ and $V(r)$ is, however, more subtle through the Riccati equation for the reduced supermomenta.

4.7 Potentials in relativistic Hamiltonians with exact pseudospin symmetry

In general, there are three potentials, $S(r), V(r)$ and $T(r)$, in the relativistic Hamiltonian (3). With one constraint $\Delta_S = 0$ to achieve exact pseudospin symmetry, only two independent functions of these potentials can be specified freely (subject to some conditions). Since the difference potential (63) is a function of $A(r)$ (containing the sum $V(r) + S(r)$) and $T(r)$, it is reasonable to use these quantities as independent variables and to determine the reduced supermomenta, $q_\kappa(r)$ and $q_{\kappa'}(r)$ and finally the scalar and vector potentials, $S(r)$ and $V(r)$. This approach will be followed in the considerations below. As an alternative, the potentials in the relativistic Hamiltonian could be derived from two given supermomenta $q_\kappa(r)$ and $q_{\kappa'}(r)$. In order to simplify the notation, the quantities
\[ \omega = \frac{E(\kappa) - E(\kappa')}{2\kappa + 1} \]  
\[ E_0 = \frac{E(\kappa) + E(\kappa')}{2} \]  

are introduced replacing the energy shifts \( E(\kappa) \) and \( E(\kappa') \).

From the condition \( \Delta S(l, r) = 0 \) one obtains the general solution for the difference

\[ q_\kappa(r) - q_{\kappa'}(r) = -(2\kappa + 1)C(r) \]  

with the function

\[ C(r) = \sqrt{A(r)} \left( D + \int_0^r \frac{T(r')}{\sqrt{A(r')}} \frac{dr'}{r'} \right) \]  

where \( D \) is an integration constant. Writing

\[ q_\kappa(r) = q(r) + T(r) - \frac{2\kappa + 1}{2}C(r) \]  
\[ q_{\kappa'}(r) = q(r) + T(r) + \frac{2\kappa + 1}{2}C(r) \]

the difference of the defining equations for \( q_\kappa(r) \) and \( q_{\kappa'}(r) \) leads to the solution

\[ q(r) = \frac{1}{2} \left[ \frac{(A\omega - C')r - C}{1 + Cr} \right] = \frac{1}{2} \left[ \frac{A\omega r}{1 + Cr} - \frac{d}{dr} \ln(1 + Cr) \right] \]

for the auxiliary momentum \( q(r) \). The sum of the two defining equations gives in combination with the definition of \( A(r) \) the scalar and vector potentials

\[ S(r) = \frac{1}{2} \left[ E - E_0 - A(r) - U(r) \right] + m \]  
\[ V(r) = \frac{1}{2} \left[ E + E_0 - A(r) + U(r) \right] \]

with the auxiliary potential

\[ U(r) = V(r) - S(r) + m - E_0 \]  
\[ = \frac{1}{A} \left[ (q + T) \left( q - T + \frac{1}{r} + \frac{A'}{A} \right) - q' + (2\kappa + 1)^2 \left( \frac{C^2}{4} + \frac{C}{2r} \right) \right] \]
that is invariant under the replacement $\kappa \leftrightarrow \kappa' = -\kappa - 1$. For the determination of the potentials $S(r)$ and $V(r)$ in the relativistic Hamiltonian that exhibits exact pseudospin symmetry, the given functions $A(r)$ and $T(r)$ have to be regular, but not every choice is admissible. At least, one has to require that $A(r) > 0$ for $r > 0$ and that $T(r)$ is selected such that the integral in equation (67) is finite for all $r$. In addition, there is the constraint that $C(r)$ has to be zero at $r = 0$ because $q_\kappa(0) = q_{\kappa'}(0) = 0$ by construction of the supersymmetric partner Hamiltonians and that $C(r) \geq 0$, otherwise the momentum $q(r)$ is not regular for all radii. It is worthwhile to note that the potential (73) depends on the quantum number $\kappa$ if $C(r) \neq 0$ and that exact pseudospin symmetry is found only in the particular doublet of states with given $\kappa = \tilde{l} = \kappa$ and $\kappa' = -\tilde{l} - 1$ if the relativistic potentials are assumed to be identical for all states.

It is instructing to discuss the resulting potentials for a few particular choices of $A(r)$ and $T(r)$. There are two major cases depending on the value of $A(r)$ at $r = 0$.

1. $A(0) = 0$ but $A(r) > 0$ if $r > 0$. In this case, which might not be realized in a physical system, the integration constant $D$ in equation (67) can have any non-negative value as long as $C(r) \geq 0$ for all $r$. If $C(r)$ is not identically zero, the reduced supermomenta of the pseudospin partner system with different $\tilde{l}$ are different and the potential $U(r)$ explicitly depends on $\kappa$. This can occur even for vanishing tensor potential if $D > 0$.

2. $A(0) \neq 0$, the case usually encountered in physical systems. It follows that $D = 0$ and that $C(r) \neq 0$ only if there is a tensor interaction. See the discussion in the paragraph above.

In most relativistic Hamiltonians without a tensor interaction and everywhere finite effective mass of the nucleon there is still a large variation of potentials that exhibit exact pseudospin symmetry. For $T(r) = 0$ and then $C(r) = 0$ for all radii $r$, the reduced supermomenta of the pseudospin partners are identical with

$$q_\kappa(r) = q_{\kappa'}(r) = q(r) = \frac{A(r)}{2} \omega r$$

and the auxiliary potential becomes

$$U(r) = \frac{1}{A} \left[ q^2 + \frac{q}{r} + \frac{A'(r)}{A(r)} q - q' \right] = \frac{A(r)}{4} \omega^2 r^2 ,$$

i.e., a modified harmonic oscillator potential with position depending effective mass. With the scalar and vector potentials (71) and (72), respectively, the central and spin-orbit potentials (18) and (19) are found as
\[ V_c(r) = E_0 - m + U(r) + \frac{1}{2}V_{so}(r) \]  
\[ V_{so}(r) = -\frac{2A'}{A'^2r} \]

in the non-relativistic Hamiltonian \((17)\). The standard harmonic oscillator is obtained for constant \(A(r) = 2m, T(r) = 0\) and \(E(\kappa) = m + (2\kappa + 1)\omega/2\) such that \(E_0 = m\). Then the scalar and vector potentials are

\[ S(r) = -V(r) = -\frac{m}{4}\omega^2r^2 \]

leading to the non-relativistic central and spin-orbit potentials

\[ V_c(r) = \frac{m}{2}\omega^2r^2 \quad V_{so}(r) = 0. \]

In general, the choice of the position depending effective mass \(m_{\text{eff}}(r) = A(\kappa)/2\) has an important effect on the potentials that appear in the non-relativistic Hamiltonian. However, a reasonable effective mass should approach the free nucleon mass \(m\) for large radii \(r\). Then the potentials will approach the harmonic oscillator form in the limit \(r \to \infty\).

5 Conclusions and outlook

The occurrence of almost degenerate pseudospin partner levels in atomic nuclei is usually attributed to a symmetry of the underlying relativistic Hamiltonian as it appears, e.g., in RMF models. The lower-component radial wave functions in the Dirac spinors are found to be very similar. From the differential equation of these wave functions a pseudospin-orbit potential is extracted that breaks the pseudospin symmetry. However, it is not a regular function and it cannot be considered as a small perturbation. In addition, this potential does not vanish for the usual harmonic oscillator potential which exhibits exact pseudospin symmetry.

In the supersymmetric approach to pseudospin symmetry, the supersymmetric partner Hamiltonians of the upper-component wave functions are compared. It leads to a regular symmetry breaking potential that is small in comparison with the dominating central potential in the mean-field Hamiltonian. The corresponding partner wave functions are identical in the case of exact pseudospin symmetry. Employing both exact and broken supersymmetry, all levels in a sequence of states for given pseudo orbital angular momentum \(\tilde{l} > 0\), including the lowest unpaired state, fit into the scheme. This lowest level defines
the reduced supermomentum and consequently the potentials in the Hamiltonians for both pseudospin partner systems in the case of exact pseudospin symmetry. All higher lying pairs are degenerate in energy. From the condition of vanishing symmetry breaking potential, general relativistic potentials that exhibit exact pseudospin symmetry can be derived. Modified harmonic oscillator potentials with position depending effective mass are the most simple class.

The present description relies on the differential equations for the radial wave functions and the corresponding first-order differential operators in the factorization of the Hamiltonian for given quantum number $\kappa$. It is also possible to generalize the approach to a form independent of a particular $\kappa$. In this case the operators

$$B^+ = \left[ q - \vec{\sigma} \cdot \left( \vec{T} - i\vec{p} \right) \right] \frac{1}{\sqrt{A}} \quad B^- = \frac{1}{\sqrt{A}} \left[ q - \vec{\sigma} \cdot \left( \vec{T} + i\vec{p} \right) \right]$$

in the generators $\{21\}$ of the symmetry transformation act on the full wave functions in coordinate space. These and more formal aspects of the supersymmetric approach can be studied in the future.

The symmetry breaking potential depends on derivatives of the supermomenta for the two pseudospin partner states and the tensor potential. The supermomenta itself are determined via a Riccati equation by the scalar, vector and tensor potentials in the relativistic Hamiltonian. A simple example shows that the reduced supermomenta of pseudospin partner states are very similar rising approximately linear at small radii as in the case of the harmonic oscillator. It seems that all saturating potentials with a flat bottom will show an approximate pseudospin symmetry. This question can be explored in more detail by expanding the reduced supermomenta in a power series in terms of the radius $r$.

Calculating the supermomenta for actual selfconsistent RMF models will help to quantify the various sizes of the observed pseudospin symmetry breaking. The study of the symmetry breaking potential will eventually lead to improved RMF parametrizations since it can help to identify the relevant modifications of the model that are required to obtain a better description of the experimentally observed breaking and restoration of the pseudospin symmetry in nuclei. It will be interesting to see how the isospin dependence of the effective interaction changes the degree of pseudospin symmetry breaking. Of special importance is the tensor interaction (neglected in most RMF models) that affects both the spin-orbit and the pseudospin-orbit splitting.

The method that is developed in this paper is not restricted to a relativistic description of the single-particle states. In principle, it can also be applied
to the usual Schrödinger equation for the single-particle wave functions as it appears, e.g., in the Skyrme Hartree-Fock method, because it relies only on the Schrödinger-equivalent differential equation of the upper component in the Dirac spinor. Instead of the scalar, vector and tensor potentials in the relativistic Hamiltonian, there are also at least three independent quantities that characterize the mean-field in the non-relativistic Hamiltonian: the position depending effective mass, the central and the spin-orbit potential. The relation between these functions and the reduced supermomentum will be different as compared to the relativistic description, however, the same principles of the supersymmetric method can be applied.

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Pseudospin symmetry was found as a particular limit in the application of supersymmetry to the Dirac equation in Refs. [Lev04,Lev05] by directly factorizing the Dirac Hamiltonian.

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Tables

Table 1
Relation between the various quantum numbers in the relativistic description of single-particle states. Pseudospin partners with the same pseudoangular momentum \( \tilde{l} \) are placed in the same column.

| \( \tilde{l} \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|---|
| \( \kappa \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
| \( j^\pi \) | 1/2\(^+\) | 3/2\(^-\) | 5/2\(^+\) | 7/2\(^-\) | 9/2\(^+\) | 11/2\(^-\) | 13/2\(^+\) | ... |
| \( l \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
| \( \kappa \) | -1 | -2 | -3 | -4 | -5 | -6 | -7 | -8 | ... |
| \( j^\pi \) | 1/2\(^-\) | 3/2\(^+\) | 5/2\(^-\) | 7/2\(^+\) | 9/2\(^-\) | 11/2\(^+\) | 13/2\(^-\) | 15/2\(^+\) | ... |
| \( l \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
Fig. 1. Single-particles states in the conventional shell model description with spin-orbit partners $j = l \pm 1/2$ for given orbital angular momentum $l$ (left) and in the supersymmetric approach with pseudospin partners $j = \tilde{l} \pm 1/2$ for given pseudo-angular momentum $\tilde{l}$ (right). The pseudospin partner states are marked with a solid circle in the central level scheme. Additionally the magic numbers for shell closures are given. Subshell closures are enclosed in parentheses.
Fig. 2. Wave functions of the upper (top panel) and lower (bottom panel) component in the Dirac spinor for the $2s_{1/2}$ (red solid line) and $1d_{3/2}$ (blue dashed line) pseudospin partner levels.
Fig. 3. Pseudospin symmetry breaking potentials in the standard relativistic description (red solid line) and the supersymmetric approach (blue dashed line). The potential difference $V(r) - S(r)$ is denoted by a black dotted line and the energy of the $2s_{1/2}$ state by a horizontal green dotted line.

\[ \kappa = \tilde{l} > 0 \quad \kappa' = -\kappa - 1 \]

\[ j = \tilde{l} - 1/2 \quad j' = \tilde{l} + 1/2 \]

\[ l = \tilde{l} - 1 \quad l = \tilde{l} \]

\[ l' = 1 \quad l' = \tilde{l} + 1 \]

\[ E_{5\kappa} \quad E_{2\kappa} \quad E_{1\kappa} \quad A_{\kappa'}(0)[E(\kappa')-m+S(0)-V(0)] + T'(0) = - A_{\kappa}(0)[E(\kappa)-m+S(0)-V(0)] - T(0) \]

\[ E_{5\kappa} \quad E_{2\kappa} \quad E_{1\kappa} \quad E_{3\kappa} \quad E_{1\kappa'} \quad E_{2\kappa'} \quad B_{\kappa}^- \quad B_{\kappa}^+ \quad B_{\kappa'}^- \quad B_{\kappa'}^+ \]

\[ H_F(\kappa) = H_1(\kappa) + E(\kappa) \quad H_2(\kappa) + E(\kappa) \]

exact SUSY

broken SUSY

Fig. 4. Level scheme of pseudospin partner systems for given pseudoangular momentum $\tilde{l}$ in the supersymmetric description. See text for details.
Fig. 5. Reduced supermomenta $q_\kappa(r)$ (top panel) and full supermomenta $Q_\kappa(r) = q_\kappa(r) - \kappa/r$ (bottom panel) for the $2s_{1/2}$, $\kappa = 1$ (red solid lines) and $1d_{3/2}$, $\kappa' = -2$ (blue dashed lines) pseudospin partner levels.

Fig. 6. Wave functions of the supersymmetric partner states for the $2s_{1/2}$ (red solid line) and $1d_{3/2}$ (blue dashed line) pseudospin partner levels.