TOPOLOGICAL RADICAL OF BANACH MODULE

O. YU. ARISTOV

Abstract. We introduce the concept of topological radical of a Banach module. This closed submodule have two description: as the intersection of ranges of maximal contractive monomorphism (from outside) and as the union of ranges of small morphisms (from inside). This concept is a functional analytic analogue for radical of module over a unital ring and has the similar categorical properties.

Consideration of projective covers in [1] induce us to seek some analogue for a notion of a small submodule in the Banach module context. Let us remind that a submodule \( Y \) in a module \( X \) over a ring is called small (other nicks are ‘superfluous’ and ‘coessential’) if for every submodule \( Z \), \( Y + Z = X \) implies \( Z = X \). A generalization of Dixon’s theorem on topologically nilpotent Banach algebras (see Theorem [12]) leads us to the definition of a small morphism. The ranges of small morphisms are submodules in Banach modules and can be considered as functional analytic analogues of small submodules.

Our main aim is to extend the concept of Jacobson radical from Banach algebras to Banach modules. As a pattern we take the notion of radical of a module from Rings Theory. But our approach offers some functional analytic modifications. The Jacobson radical of a unital ring can be described as the intersection of all maximal left ideals (from outside) or as the set of all \( r \) such that \( 1 + ar \) is invertible for every \( a \) (from inside). This concept applies well to unital Banach algebra \( A \) because every maximal left ideal is closed and \( 1 + ar \) is invertible for every \( a \in A \) iff \( ar \) is topologically nilpotent (i.e. \( \|\!(ar)^n\!\|^1/n \to 0 \) for every \( a \in A \)).

On the other hand, the notion of radical can be extended to modules. The radical of a unital module \( X \) over unital ring is the intersection of all maximal submodules and the union of all small submodules (the notation is \( \text{rad} X \)). Note that for an element \( r \) of a ring \( A \), \( Ar \) is small iff \( 1 + ar \) is invertible for every \( a \). The pure algebraic notion of the radical of a module is useful in Banach Module Theory only in particular cases, for example for finitely-generated modules [1]. In general, neither maximal submodules nor small submodules in a Banach module need to be closed. But then again we can not restrict ourselves to some classes of closed submodules because submodules of the form \( A \cdot x \) (that are potentially not closed) play an important role in the basic theory of module radicals. As we see below the right way is to consider ranges of bounded modules morphisms as an intermediate class between closed submodules and all submodules. But it seems more appropriate from both ideological and technical points of view to work with morphisms themselves instead of their ranges.

In this article we introduce the concept of topological radical of a Banach module. This closed submodule have two description: as the intersection of ranges of

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maximal contractive monomorphism (from outside) or as the union of ranges of small morphisms (from inside).

1. **Small morphism of Banach modules**

Let $R$ be a Banach algebra. We suppose that the norm of multiplication in $R$ is not greater than 1. For $n \in \mathbb{N}$ set

$$S(n) := \sup \| r_1 r_2 \cdots r_n \|^{1/n},$$

where $r_1, r_2, \ldots, r_n$ run over the unit ball of $R$. If $\lim_{n \to \infty} S(n) = 0$ then $R$ is called topologically nilpotent. Note that $R$ is topologically nilpotent if and only if for every bounded sequence $(r_n) \subset R$,

$$\lim_{n \to \infty} \| r_1 r_2 \cdots r_n \|^{1/n} = 0.$$

Obviously, a topologically nilpotent Banach algebra is radical.

Remind that $C[0, 1]$ and $L^1[0, 1]$ are Banach algebras with respect to the cut-off convolution. The first algebra is topologically nilpotent but the second algebra is not topologically nilpotent [6, Section 4.8.8].

In [3] P. G. Dixon shows that $R \cdot X \neq X$ for every non-trivial left Banach module $X$ over a topologically nilpotent Banach algebra $R$ (see the proof in [6, Theorem 4.8.9] also). But in fact his argument gives a stronger assertion. Let us set

$$\pi_X^R : R \hat{\otimes} X \to X : r \otimes x \mapsto r \cdot x$$

for a left Banach $R$-module $X$, where $\hat{\otimes}$ denotes the projective tensor product of Banach spaces. (We suppose that the norm of multiplication in $X$ is not greater than 1 also.)

**Theorem 1.1** (Dixon). If $X$ is a non-trivial left Banach module over a topologically nilpotent Banach algebra $R$, then $\text{Im} \pi_X^R \neq X$.

We need a more general result.

**Theorem 1.2.** Let $R$ be a topologically nilpotent Banach algebra, and let $\varphi : Y \to X$ be a morphism of left Banach modules such that $X = \text{Im} \varphi + \text{Im} \pi_X^R$. Then $\varphi$ is surjective.

**Proof.** The assumption of the theorem means that the morphism

$$Y \oplus (R \hat{\otimes} X) \to X : (y, u) \mapsto \varphi(y) + \pi_X^R(u)$$

is surjective. (Here the sum is endowed with the $\ell^1$-norm.) By the open mapping theorem there is $C > 0$ with the following property. For every $x$ in $X$ there exist $y \in Y$, $r_i \in R$, and $x_i \in X$ such that

$$x = \varphi(y) + \sum_{i=1}^{\infty} r_i \cdot x_i \quad \text{and} \quad \|y\| + \sum_{i} \|r_i\| \|x_i\| \leq C \|x\|.$$

Now we fix $x$ in $X$ and choose by induction sequences $(y_n) \subset Y$ and $(v_n) \subset X$ such that

$$x = \varphi(y_n) + v_n,$$

where $v_n$ can be represented as

$$v_n = \sum_{i=1}^{\infty} r_{1,i} \cdots r_{n,i} \cdot x_i \quad \text{for some} \ r_{1,i}, \ldots, r_{n,i} \in R \ \text{and} \ x_i \in X; \ i \in \mathbb{N},$$

$$\lim_{n \to \infty} \| r_1 r_2 \cdots r_n \|^{1/n} = 0.$$
and the following two conditions are satisfied.

(4) \[ \|y_{n+1} - y_n\| \leq C \sum_i \|r_{1,i} \cdots r_{n,i}\| \|x_i\|; \]

(5) \[ \sum_i \|r_{1,i}\| \cdots \|r_{n,i}\| \|x_i\| \leq C^n \|x\|. \]

Suppose that for \( n \in \mathbb{N} \) we have elements \( y_1, \ldots, y_n \) and \( r_1, \ldots, r_n \) that satisfy the above conditions, in particular, the condition (4) satisfies up to \( n - 1 \). Fix decompositions in (2) and (3). Set \( t_i := r_{1,i} \cdots r_{n,i} \). Applying the open mapping theorem as described before we can write every \( x_i \) as

(6) \[ x_i = \varphi(y'_i) + \sum_j s_{ji, i} x'_{ji}, \quad \text{where} \quad \|y'_i\| + \sum_j \|s_{ji, i}\| \|x'_{ji}\| \leq C \|x_i\|. \]

Then

\[ x = \varphi(y_n) + v_n = \varphi(y_n) + \sum_i t_i \cdot \varphi(y'_i) + \sum_{i,j} t_is_{ji} \cdot x'_{ji}. \]

Now set \( y_{n+1} := y_n + \sum_i t_i \cdot y'_i \) and \( v_{n+1} := \sum_{i,j} t_is_{ji} \cdot x'_{ji} \). It follows from (6) that \( \|y'_i\| \leq C \|x_i\| \). Hence,

\[ \|y_{n+1} - y_n\| \leq \sum_i \|t_i\| \|y'_i\| \leq C \sum_i \|t_i\| \|x_i\|, \]

i.e. we obtain (4). By (3) and (5) we get

\[ \sum_{i,j} \|r_{1,i}\| \cdots \|r_{n,i}\| \|s_{ji}\| \|x'_{ji}\| \leq C^{n+1} \|x\|, \]

i.e. after an obvious change of notation we have (3) and (5) for \( n + 1 \). By induction, there exist sequences with the desired properties.

Note that (6) implies

\[ \sum_i \|r_{1,i} \cdots r_{n,i}\| \|x_i\| \leq \sum_i S(n)^n \|r_{1,i}\| \cdots \|r_{n,i}\| \|x_i\| \leq S(n)^n C^n \|x\| \]

for every \( n \). Therefore \( \|v_n\| \leq S(n)^n C^n \|x\| \) and \( \|y_{n+1} - y_n\| \leq S(n)^n C^{n+1} \|x\| \) by (4). Hence we have for \( m > n \)

\[ \|y_m - y_n\| \leq \sum_{k=n}^{m-1} S(k)^k C^{k+1} \|x\|. \]

Since \( S(n) \to 0 \), it follows that \( y_n \) is a fundamental sequence and \( v_n \to 0 \). Finally, from \( x = \varphi(y_n) + v_n \) we get \( x = \varphi(\lim_n y_n) \), i.e. \( x \in \text{Im} \varphi \).

**Notation 1.3.** Let \( \varphi : Y \to X \) and \( \psi : Z \to X \) be morphisms of Banach modules. Denote by \( \varphi + \psi \) the morphism

\[ Y \oplus Z \to X : (y,z) \mapsto \varphi(y) + \psi(z). \]

**Definition 1.4.** We say that a morphism \( \psi : X_0 \to X \) of Banach modules is *small* if for every morphism \( \varphi : Y \to X \) such that \( \varphi + \psi \) is surjective \( \varphi \) is surjective also.

Thus, Theorem 1.2 asserts that for every left Banach module \( X \) over a topologically nilpotent Banach algebra \( R \) the morphism \( \pi_X^R \) is small.

**Proposition 1.5.** If \( \psi : X_0 \to X \) is a small morphism then \( \tau \psi \) is small for every \( \tau \).
Proof. Let \( \varphi : X \to V \) and \( \varphi : Y \to V \) be morphisms of Banach modules such that \( \varphi \) is surjective. Consider the pullback diagram

\[
\begin{array}{c}
Y \times_V X \\
\downarrow \varphi' \\
X \\
\downarrow \tau \\
V
\end{array}
\]

associated with \( \varphi \) and \( \tau \).

For every \( x \in X \) there are \( y \in Y \) and \( z \in X_0 \) such that \( \tau(x) = \varphi(y) + \tau(z) \). Then \( \varphi(y) = \tau(x - \psi(z)) \). By explicit construction of \( Y \times_V X \) this means that \( w = (y, x - \psi(z)) \in Y \times_X V \) and \( \varphi(w) = x - \psi(z) \). Hence, \( \varphi' + \psi \) is surjective. Since \( \psi \) is small, \( \varphi' \) is surjective also. Therefore for every \( x \in X \) there exists \( y \in Y \) such that \( \varphi(y) = \tau(x) \). The assumption implies that \( \varphi' + \tau \) is surjective. Thus, \( \varphi \) is surjective also.

Proposition 1.6. Let \( \psi : X_1 \to X \) be a morphism of Banach modules, and let \( \varepsilon : X_0 \to X_1 \) be a surjective morphism of Banach modules such that \( \psi \varepsilon \) is small. Then \( \psi \) is small.

Proof. Suppose that \( \varphi : Y \to X \) is a morphism such that \( \varphi + \psi \) is surjective. Then \( \varphi + \psi \varepsilon \) is surjective also. Since \( \varepsilon \psi \) is small, \( \varphi \) is surjective. \( \square \)

Remind that left Banach \( A \)-module \( P \) is called strictly projective if for every surjective morphism of Banach \( A \)-modules \( \varepsilon : Y \to P \) there exists a morphism \( \rho \) such that \( \varepsilon \rho = 1 \). Denote by \( \ell^1 \) the infinite Banach \( \ell^1 \)-space with a countable basis.

Theorem 1.7. (sf. [2] Th.11.5.5). Let \( I \) be a closed left ideal in a unital Banach algebra \( A \), and let \( \iota : I \to A \) be the natural inclusion. The following conditions are equivalent.

(A) \( I \) is topologically nilpotent;
(B) for every unital left Banach \( A \)-module \( X \) the morphism of Banach \( A \)-modules \( I \hat{\otimes} A X \to X : a \otimes_A x \mapsto a \cdot x \) is small;
(C) for every strictly projective unital left Banach \( A \)-module \( P \) the morphism of Banach \( A \)-modules \( I \hat{\otimes} A P \to P : a \otimes_A x \mapsto a \cdot x \) is small;
(D) the morphism of left Banach \( A \)-modules \( (\iota \otimes 1) : I \hat{\otimes} \ell^1 \to A \hat{\otimes} \ell^1 \) is small.

Proof. (A) \( \Rightarrow \) (B) If \( I \) is topologically nilpotent and \( X \) is a left Banach \( A \)-module then by Theorem 1.2, \( \pi^A_X \) is small as a morphism of left Banach \( I \)-modules. Hence, it is small as a morphism of left Banach \( A \)-modules. Since \( \pi^A_X \) is a composition of a surjective morphism \( I \hat{\otimes} X \to I \hat{\otimes} A X \) and a morphism \( I \hat{\otimes} A X \to X \), Proposition 1.6 implies (B).

(B) \( \Rightarrow \) (C) It is obvious.

(C) \( \Rightarrow \) (D) It is easy to see that \( A \hat{\otimes} \ell^1 \) is strictly projective. By assumption

\[ I \hat{\otimes} A \hat{\otimes} \ell^1 \to \hat{\otimes} \ell^1 : a \otimes_A b \otimes x \mapsto ab \otimes x \]

is a small morphism of left Banach \( A \)-modules. Since \( A \) is unital, \( I \hat{\otimes} A \to I \) and we have (D).

(D) \( \Rightarrow \) (A). Let \( (a_n) \) be a bounded sequence in \( I \), and let \( \{ e_i \}_{i \in \mathbb{N}} \) be the canonical basis in \( \ell^1 \). Consider

\[ \varphi : A \hat{\otimes} \ell^1 \to A \hat{\otimes} \ell^1 : \sum_{i=1}^{\infty} b_i \otimes e_i \mapsto \sum_{i=1}^{\infty} b_i a_i \otimes e_{i+1} . \]
It is obvious that \( \varphi \) is a morphism of left Banach modules. Fix \( \lambda \in \mathbb{C} \). Since
\[
\sum_{i=1}^{\infty} b_i a_i \otimes e_{i+1} \in I \otimes \ell^1,
\]
we have \( I \otimes \ell^1 + \text{Im}(1 + \lambda \varphi) = A \otimes \ell^1 \). Since \((t \otimes 1)\) is small, \(1 + \lambda \varphi\) is surjective. If \((1 + \lambda \varphi)(u) = 0\) for some \(u = \sum_i b_i \otimes e_i\), then \(b_1 = 0\) and \(b_{i+1} - \lambda b_i a_i = 0\) for all \(i\). It follows that \(b_i = 0\) for all \(i\), so that \(1 + \lambda \varphi\) is injective. Thus, \(1 + \lambda \varphi\) is an isomorphism for every \(\lambda \in \mathbb{C}\). This implies that \(\varphi\) is a topologically nilpotent operator, i.e. \(\lim_{n \to \infty} \|\varphi^n\|^{1/n} = 0\).

It is clear that
\[
\|\varphi^n(1 \otimes e_1)\| = \|a_1 a_2 \cdots a_n \otimes e_{n+1}\| = \|a_1 a_2 \cdots a_n\|.
\]
Therefore \(\|a_1 a_2 \cdots a_n\| \leq \|\varphi^n\|\). The rest is obvious. \(\square\)

Considering every Banach algebra as an ideal in the unitization we have

**Corollary 1.8.** A Banach algebra \(R\) is topologically nilpotent if and only if for every Banach \(R\)-module \(X\) the morphism \(R \otimes_R X \to X\) is small if and only if for every strictly projective left Banach \(A\)-module \(P\) the morphism \(R \otimes_R P \to P\) is small.

Remark that the definition of \(S(n)\) is invariant under replacement of the left multiplication to the right multiplication. So all results above can be applied to right Banach modules.

If \(X\) and \(Y\) are left Banach \(A\)-modules we denote by \(A\text{h}(X,Y)\) the set of all bounded morphisms of \(A\)-modules from \(X\) to \(Y\).

**Proposition 1.9.** Let \(A\) be a Banach algebra, \(X\) and \(Y\) unital left Banach \(A\)-modules, and \(\alpha\) in \(A\text{h}(X,Y)\). The following conditions are equivalent.

1. \(1 - \alpha \varphi\) is right invertible in \(A\text{h}(Y)\) for every \(\varphi \in A\text{h}(Y,X)\).
2. \(\alpha \circ A\text{h}(Y,X)\) is a small right \(A\text{h}(Y)\)-submodule in \(A\text{h}(Y)\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(L\) be a right submodule in \(A\text{h}(Y)\) such that \(\alpha \circ A\text{h}(Y,X) + L = A\text{h}(Y)\). Then there are \(\varphi \in A\text{h}(Y,X)\) and \(\psi \in L\) satisfying \(\alpha \varphi + \psi = 1\). By assumption \(\psi\) is right invertible, hence, \(L = A\text{h}(Y)\).

(2) \(\Rightarrow\) (1) Let \(\varphi \in A\text{h}(Y,X)\). Set \(L := (1 - \alpha \varphi) \circ A\text{h}(Y)\). Then \(1 \in \alpha \circ A\text{h}(Y,X) + L\). Therefore \(\alpha \circ A\text{h}(Y,X) + L = A\text{h}(Y)\). Since \(\alpha \circ A\text{h}(Y,X)\) is small, \(L = A\text{h}(Y)\). This implies that \(1 - \alpha \varphi\) is right invertible. \(\square\)

**Theorem 1.10.** Let \(X\) and \(P\) be unital left Banach \(A\)-modules. Suppose that \(P\) is strictly projective and \(\alpha \in A\text{h}(X,P)\). The following conditions are equivalent.

1. \(\alpha\) is small.
2. \(1 - \alpha \varphi\) is right invertible in \(A\text{h}(P,X)\) for all \(\varphi \in A\text{h}(P,X)\).
3. \(\alpha \circ A\text{h}(P,X)\) is a small right \(A\text{h}(P)\)-submodule in \(A\text{h}(P)\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(\varphi \in A\text{h}(P,X)\). Then \((1 - \alpha \varphi) + \alpha : P \oplus X \to P\) is obviously surjective. Since \(\alpha\) is small, \(1 - \alpha \varphi\) is surjective. Since \(P\) is strictly projective, \(1 - \alpha \varphi\) admits a right inverse morphism.

(2) \(\Rightarrow\) (1) Suppose \(\eta \in A\text{h}(Y,P)\) for some \(Y\) and \(\eta + \alpha\) is surjective. Since \(P\) is strictly projective, \(\eta + \alpha\) is right invertible, i.e. there exist \(\psi_1 \in A\text{h}(P,Y)\) and \(\psi_2 \in A\text{h}(P,X)\) such that \(\eta \psi_1 + \alpha \psi_2 = 1\). By assumption \(\eta \psi_1\) is right invertible. Hence, \(\eta\) is surjective.

\(\eta\) is surjective.

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A surjective morphism \( \varepsilon : X \to V \) of Banach \( A \)-modules is said to be a cover if a morphism \( \varphi : Y \to X \) of Banach \( A \)-modules is a surjective morphism, whenever \( \varepsilon \varphi \) is so [1].

**Proposition 1.11.** A surjective morphism \( \varepsilon : X \to V \) of Banach modules is a cover if and only if \( \ker \varepsilon \) is a small morphism.

**Proof.** Suppose that \( \varepsilon \) is a cover. Let \( \varphi : Y \to X \) be a morphism of Banach modules such that \( \varphi + \ker \varphi \) is surjective. Note that \( \varepsilon(\varphi + \ker \varepsilon) = \varepsilon \varphi \) is surjective also. Since \( \varepsilon \) is a cover, \( \varphi \) is surjective. Thus, \( \ker \varepsilon \) is a small morphism.

Suppose that \( \ker \varepsilon \) is small. Let \( \varphi : Y \to X \) be a morphism of Banach modules such that \( \varepsilon \varphi \) is surjective. Then for every \( x \in X \) there is \( y \) such that \( \varepsilon(x) = \varepsilon \varphi(y) \).

Thus, \( \ker \varepsilon \) is surjective. Note that \( \varepsilon \) is surjective. Thus, \( \ker \varepsilon \) is a small morphism.

Since \( \ker \varepsilon \) is small, \( \varphi \) is surjective. Thus, \( \varepsilon \) is a cover. \( \square \)

2. **Maximal contractive monomorphisms**

Fix a unital Banach algebra \( A \) and a left unital Banach \( A \)-module \( X \). Consider a pre-order on the set of contractive monomorphisms with range in \( X \). We set \( \beta \geq \gamma \) for \( \beta \) and \( \gamma \) if there exists a contractive morphism \( \kappa \) such that \( \gamma = \beta \kappa \). We say that \( \beta \) and \( \gamma \) is equivalent if \( \kappa \) is an isometric isomorphism. The pre-order induces an order on the set of equivalence classes of contractive monomorphisms.

**Remark 2.1.** If \( X \) is unital and \( \beta : Y \to X \) is a monomorphism then \( Y \) is also unital. To see this consider the decomposition \( Y = Y_0 \oplus Y_1 \), where \( Y_0 = \{ y \in Y : 1 \cdot y = 0 \} \) and \( Y_0 = \{ y \in Y : 1 \cdot y = y \} \). Since \( X \) is unital, \( \lambda h(Y_0, X) = 0 \). Therefore \( Y_0 = 0 \). Thus, we does not need a restriction on initial module of a monomorphism.

**Definition 2.2.** Let \( \beta : Y \to X \) and \( \gamma : Z \to X \) be contractive monomorphisms.

(1) Denote by \( \beta \vee \gamma \) the natural morphism \( (Y \oplus Z)/\ker(\beta + \gamma) \to X \) associated with \( \beta + \gamma \).

(2) Denote by \( \beta \wedge \gamma \) the natural morphism \( Y \times_X Z \to X \), where \( Y \times_X Z \) is the pullback of \( \beta \) and \( \gamma \).

It is not hard to check that \( \beta \vee \gamma \) and \( \beta \wedge \gamma \) are contractive monomorphisms. For equivalence classes \([\beta]\) and \([\gamma]\) of \( \beta \) and \( \gamma \) we set \([\beta] \vee [\gamma] := [\beta \vee \gamma]\) and \([\beta] \wedge [\gamma] := [\beta \wedge \gamma]\). It is easy to see that these operations are well-defined.

**Proposition 2.3.** Let \( \beta \) and \( \gamma \) be contractive monomorphisms. Then, with respect to the order define above, \([\beta] \vee [\gamma]\) and \([\beta] \wedge [\gamma]\) are the supremum and the infimum of \([\beta]\) and \([\gamma]\), respectively.

The proof is standard.

**Definition 2.4.** We say that a contractive monomorphism of left unital Banach \( A \)-modules \( \alpha : Y \to X \) is maximal if \( \alpha \) is not surjective and for every non-surjective contractive monomorphism \( \beta \) and every contractive morphism \( \kappa \) the equality \( \alpha = \beta \kappa \) implies that \( \kappa \) is an isometric isomorphism.

Thus, \( \alpha \) is maximal iff \([\alpha]\) is maximal in the set of equivalence classes of all non-surjective monomorphisms with range in \( X \).

Remind that a morphism \( \varepsilon : Y \to X \) is called a \( C \)-epimorphism for some \( C \geq 1 \) if for every \( x \in X \) there exist \( y \in Y \) such that \( x = \varphi(y) \) and \( \|y\| \leq C\|x\| \).
Proposition 2.5. Set
\[ \tau : A \to X : a \mapsto a \cdot x_0 \]
where \( x_0 \in X \). Suppose that \( \varphi : Y \to X \) is a morphism such that \( x_0 \notin \text{Im} \varphi \) and \( \varphi + \tau \) is a \( C \)-epimorphism for \( C \geq 1 \). Then \( \text{dist}(x_0, \text{Im} \varphi) \geq 1/C \).

Proof. Assume that \( \|x_0 - \varphi(y)\| < 1/C \). Since \( \varphi + \tau \) is \( C \)-epimorphism, there exist \( y' \in Y \) and \( a \in A \) such that \( x_0 - \varphi(y) = \varphi(y') + a \cdot x_0 \) and
\[
\|y'\| + \|a\| \leq C \|x_0 - \varphi(y)\| < 1.
\]
Thus, \( \|a\| < 1 \), hence \( 1 - a \) is invertible in \( A \). Therefore \( x_0 = \varphi((1 - a)^{-1} \cdot (y' + y)) \).
Hence, \( x_0 \in \text{Im} \varphi \). We get a contradiction. \( \square \)

Theorem 2.6. Every maximal contractive monomorphism is an isometry.

Proof. Let \( \alpha : Y \to X \) be a maximal contractive monomorphism. Suppose that \( x_0 \in X \setminus \text{Im} \alpha \). Define \( \tau \) as in Proposition 2.5. Since \( \alpha \) is maximal, \( [\alpha] \lor [\text{im} \tau] = 1 \).
Hence, \( \alpha + \tau \) is surjective. Therefore \( \alpha + \tau \) is \( C \)-epimorphism for some \( C \geq 1 \). It follows Proposition 2.5 that \( \text{dist}(x_0, \text{Im} \varphi) \geq 1/C \). Since \( x_0 \) is arbitrary, \( \text{Im} \alpha \) is closed. Let \( \kappa : \text{Im} \alpha \to X \) be the natural embedding. Since \( \alpha = \kappa \alpha \) and \( \alpha \) is maximal, \( \alpha \) is an isometry. \( \square \)

Theorem 2.7. Let \( X' \) be a closed submodule of \( X \). Then the natural embedding \( \iota : X' \to X \) is a maximal contractive monomorphism if and only if \( X/X' \) is an irreducible module.

Proof. \( \Rightarrow \) Assume that \( \iota \) is maximal. Let \( x_0 \in X \setminus X' \) and \( x_1 \in X \). Since \( \alpha \) is maximal, \( X' + A \cdot x_0 = X \). In particular, there is \( a \in A \) such that \( x_1 - a \cdot x_0 \in X' \). Therefore \( x_0 + X' \) is a cyclic element of \( X/X' \). Hence, \( X/X' \) is irreducible.
\( \Leftarrow \) Assume that \( X/X' \) is irreducible. Suppose that there are a non-surjective contractive monomorphism \( \beta \) and a contractive morphism \( \kappa \) such that \( \iota = \beta \kappa \).

Since \( X' \subset \text{Im} \beta \), \( \text{Im} \beta \neq X \) and \( X/X' \) is irreducible, \( \text{Im} \beta = X' \). Therefore, \( \beta \kappa = 1 \). Since \( \beta \) is a monomorphism, it is an isomorphism. Since \( \beta \) and \( \kappa \) are contractive, \( \kappa \) is isometric. \( \square \)

Note that \( X/X' \) is irreducible iff \( X' \) is a maximal submodule in the algebraic sense. Thus, maximal monomorphisms can be described as embeddings of closed maximal submodules.

Below we need the lifting Lemma for maximal monomorphisms.

Lemma 2.8. Let \( Z \) be a closed submodule in \( X \) and \( \alpha : Y \to X/Z \) a maximal contractive monomorphism. Denote the projection \( X \to X/Z \) by \( \sigma \). Then there exists a commutative diagram
\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & Y \\
\downarrow{\beta} & & \downarrow{\alpha} \\
X & \xrightarrow{\sigma} & X/Z \\
\end{array}
\]
where \( \beta \) is a maximal contractive monomorphism.
Proof. Set \(W := Y \times_{X/Z} X\) or, more precisely, \(W = \{(y, x) \in Y \times X : \alpha(y) = \sigma(x)\}\). Denote by \(\beta\) and \(\mu\) the morphisms \((y, x) \mapsto x\) and \((y, x) \mapsto y\), respectively. Note that \(\mu\) is surjective and \(Z \subset \text{Im} \beta\). It is obvious that \(\beta\) is a contractive monomorphism.

Suppose that \(\gamma : V \to X\) is a non-surjective contractive monomorphism and \(\kappa : W \to V\) is a contractive morphism such that \(\beta = \gamma \kappa\).

Assume that \(\text{Im} \alpha + \text{Im}(\sigma \gamma) = X/Z\). Since \(\text{Im} \alpha = \text{Im}(\alpha \mu) = \text{Im}(\sigma \beta) = \text{Im}(\sigma \gamma \kappa)\), we have \(\text{Im}(\sigma \gamma) = X/Z\). It follows from \(Z \subset \text{Im} \beta \subset \text{Im} \gamma\) that \(\gamma\) is surjective. We get a contradiction. Hence, \(\text{Im} \alpha + \text{Im}(\sigma \gamma) \neq X/Z\).

Since \(\alpha\) is maximal, \(\text{Im}(\sigma \gamma) \subset \text{Im} \alpha\). By Theorem 2.6 \(\alpha\) is an isometry, therefore there is a well-defined contractive morphism \(\delta : V \to Y\) such that \(\alpha \delta = \sigma \gamma\). The pull-back property implies that there is a contractive morphism \(\rho : V \to W\) such that \(\beta \rho = \gamma\). Then \([\beta] = [\gamma]\). Thus, \(\beta\) is maximal. \(\square\)

**Lemma 2.9.** Let \(\alpha\) and \(\beta\) be non-surjective contractive monomorphisms with ranges in \(X\) such that \(\beta \geq \alpha\), and let \(\varphi\) be a morphism such that \(\alpha + \varphi\) is a \(C\)-epimorphism for some \(C \geq 1\). Then \(\beta + \varphi\) is a \(C\)-epimorphism.

**Proof.** Suppose that \(\kappa : Y \to Z\) is a contractive morphism such that \(\alpha = \beta \kappa\). Since \(\alpha + \varphi\) is a \(C\)-epimorphism, for every \(x \in X\) there exist \(x_0 \in X_0\) and \(y \in Y\) such that \(x = \varphi(x_0) + \alpha(y)\) and \(\|x_0\| + \|y\| \leq C\|x\|\). Denote \(\kappa(y)\) by \(z\). Then \(x = \varphi(x_0) + \beta(z)\) and \(\|x_0\| + \|z\| \leq \|x_0\| + \|y\| \leq C\|x\|\). Therefore, \(\varphi + \beta\) is a \(C\)-epimorphism. \(\square\)

**Lemma 2.10.** Let \(C \geq 1\), and let \(\varphi\) be a contractive morphism with range in \(X\). Denote by \(\Gamma\) a family of all contractive monomorphisms \(\alpha\) with range in \(X\) such that

1. \(\alpha\) is not surjective;
2. \(\alpha + \varphi\) is a \(C\)-epimorphism.

Suppose that there are \(\delta > 0\) and \(x_0 \in X\) such that \(\text{dist}(x_0, \text{Im} \alpha) \geq \delta\) for every \(\alpha \in \Gamma\). Then for every \(\alpha_0 \in \Gamma\) there exists a maximal contractive monomorphism \(\gamma\) such that \(\gamma \in \Gamma\) and \(\gamma \geq \alpha_0\).

**Proof.** Set \(\Gamma' := \{\alpha \in \Gamma : \alpha \geq \alpha_0\}\). Suppose that \(\Gamma_0\) is a linear ordered subset of \(\Gamma'\). We claim that that \(\Gamma_0\) admits an upper bound.

Denote by \(Y_\alpha\) the initial module of \(\alpha \in \Gamma_0\) and by \(\kappa_{\alpha \alpha'}\) the connecting contractive morphism for \(\alpha\) and \(\alpha'\) in \(\Gamma_0\) such that \(\alpha' \geq \alpha\). Then there exists an inductive limit \(Y\) of a spectral family \((\kappa_{\alpha \alpha'})\) in the category of contractive morphisms. In particular, there is a family \((\kappa_\alpha : Y_\alpha \to Y)\) of contractive morphisms and \(\beta : Y \to X\) such that \(\alpha = \beta \kappa_\alpha\) for every \(\alpha\). Note that \(\bigcup_{\Gamma_0} \text{Im} \kappa_\alpha\) is dense in \(Y\), hence \(\bigcup_{\Gamma_0} \text{Im} \alpha\) is dense in \(\text{Im} \beta\). Since \(\text{dist}(x_0, \text{Im} \alpha) \geq \delta\) for all \(\alpha\), we have \(\text{dist}(x_0, \text{Im} \beta) \geq \delta\). Hence, \(\beta\) is not surjective. Applying Lemma 2.9 we get that \(\beta + \varphi\) is a \(C\)-epimorphism. Therefore, we have that \(\beta \in \Gamma'\) and \(\beta \geq \alpha\) for every \(\alpha \in \Gamma_0\).

Since \(\Gamma'\) is not empty and every linear ordered subset in \(\Gamma'\) admits an upper bound, there is a maximal element \(\gamma\) in \(\Gamma'\). Now we claim that \(\gamma\) is a maximal contractive monomorphism. Suppose that \(\gamma' : Z \to X\) is a non-surjective contractive monomorphism and \(\kappa : Y \to Z\) is a contractive morphism such that \(\gamma = \gamma' \kappa\). It follows from Lemma 2.9 that \(\gamma' + \varphi\) is a \(C\)-epimorphism. Hence, \(\gamma' \in \Gamma'\). Since \(\gamma\) is maximal in \(\Gamma'\), \(\kappa\) is an isometric isomorphism. Thus \(\gamma\) is a maximal contractive monomorphism. By construction \(\gamma \geq \alpha_0\). \(\square\)
Proof. Let \( \exists \) such that \( \gamma \) exists. Since \( \alpha \) satisfies. Hence, there exists a maximal contractive monomorphism such that \( \alpha \geq \alpha_0 \).

Proposition 2.12. Suppose that \( X \) is finitely-generated. Then for every non-surjective contractive monomorphisms \( \alpha_0 \) with range in \( X \) there exists a maximal contractive monomorphisms \( \gamma \) such that \( \gamma \geq \alpha_0 \).

Proof. Let \( x_1, \ldots, x_n \) be generators of \( X \). Consider morphisms

\[
\tau_i : A \to X : a \to a \cdot x_i \quad (i = 1, \ldots, n).
\]

Since \( \alpha_0 \) is not surjective and \( \alpha_0 + \tau_1 + \cdots + \tau_n \) is surjective, there exist minimal \( k \) in \( \{0, \ldots, n-1\} \) such that \( \alpha_0 + \tau_1 + \cdots + \tau_k \) is surjective. This implies that there exists \( C \geq 1 \) such that \( \alpha_0 + \tau_1 + \cdots + \tau_k \) is a \( C \)-epimorphism.

Denote by \( \Gamma \) a family of all contractive monomorphisms \( \beta \) with range in \( X \) such that \( \beta \) is not surjective and \( \beta + \tau_k \) is a \( C \)-epimorphism. Let us set \( \beta_0 = \alpha_0 \vee \alpha_1 \vee \cdots \vee \alpha_{k-1} \), where \( \alpha_i \) is a contractive monomorphism such that \( \text{Im} \alpha_i = \text{Im} \tau_i \). Note that \( x_k \notin \text{Im} \beta \) for every \( \beta \in \Gamma \). It follows from Proposition 2.5 that \( \text{dist}(x_k, \text{Im} \beta) \geq 1/C \) for every \( \beta \in \Gamma \). Thus, the conditions of Lemma 2.10 are satisfied. Hence, there exists a maximal contractive monomorphism \( \gamma \) such that \( \gamma \geq \beta_0 \). Therefore \( \gamma \geq \alpha_0 \).

\[ \square \]

3. Topological radical of a Banach module

Remark that equivalence classes of contractive morphism form a lattice with respect to the operations \( \vee \) and \( \wedge \). Under some conditions there is a standard way to define a radical in a lattice using small and maximal elements (see, for example, [2, Ch.9, Exersises]). But on this way we meet two difficulties. First, we define small and maximal morphism in different categories of Banach modules (topological and metric). Second, there are no sufficiently many compact elements in our lattice. Using basic Proposition 2.5 and its corollaries from Section 2 we can find a topological interplay between small and maximal morphism.

Proposition 3.1. Let \( X \) be a left unital Banach \( A \)-module and let \( x_0 \in X \). If

\[
\tau : A \to X : a \to a \cdot x_0
\]

is not small then there exists a maximal contractive monomorphism \( \gamma \) such that \( x_0 \notin \text{Im} \gamma \).

Proof. Since \( \tau \) is not small there exists a non-surjective morphism \( \alpha_0 : Y \to X \) such that \( \alpha_0 + \tau \) is surjective. Therefore, there is \( C \geq 1 \) such that \( \alpha_0 + \tau \) is \( C \)-epimorphism. We can assume \( \alpha_0 \) is a contractive monomorphism.

Denote by \( \Gamma \) a family of all contractive monomorphisms \( \alpha \) with range in \( X \) such that \( \alpha \) is not surjective and \( \alpha + \tau \) is a \( C \)-epimorphism. Note that \( x_0 \notin \text{Im} \alpha \) for every \( \alpha \in \Gamma \). Proposition 2.5 implies that \( \text{dist}(x_0, \text{Im} \alpha) \geq 1/C \) for every \( \alpha \in \Gamma \). It follows from Lemma 2.10 that there exists a maximal contractive monomorphism \( \gamma \) such that \( \gamma \in \Gamma \). Hence, \( x_0 \notin \text{Im} \gamma \).

\[ \square \]

Theorem 3.2. Let \( X \) be a left unital Banach \( A \)-module. Denote \( \bigcup \text{Im} \psi \), where \( \psi \) runs all small morphism with the range in \( X \), by \( X_1 \) and denote \( \bigcap \text{Im} \iota \), where \( \iota \) runs all maximal contractive monomorphisms with the range in \( X \), by \( X_2 \). Then \( X_1 = X_2 \) and this submodule of \( X \) is closed.
Theorem 3.6. Let \( X \) be a unital Banach \( A \)-module.

(1) Suppose that \( x_0 \in X \). Assume that \( \tau: A \to X : a \mapsto a \cdot x_0 \) is not small. By Proposition 3.1 there exists a maximal contractive monomorphism \( \gamma \) such that \( x_0 \notin \text{Im} \gamma \subset X \). This contradiction implies that \( \tau \) is small. Thus, \( X_2 \subset X_1 \).

(2) Suppose that \( \psi \) is a small morphism with the range in \( X \). We can assume \( \psi \) is a contractive monomorphism. Suppose that there is a maximal contractive monomorphism \( \gamma \) such that \( \text{Im} \gamma \subset X \). Then \( \gamma \) is not a subset of \( \text{Im} \gamma \). Therefore, \( \text{Im} \gamma + \text{Im} \psi = \text{Im}(\gamma \vee \psi) = X \). Since \( \psi \) is small, \( \gamma \) is surjective. This contradiction implies that \( \text{Im} \psi \subset \text{Im} \gamma \). Thus, \( X_1 \subset X_2 \).

It follows from Theorem 2.6 that \( X_2 \) is closed. \( \blacksquare \)

Definition 3.3. Let \( X \) be a left unital Banach \( A \)-module. We say that the closed submodule of \( X \) from Theorem 3.2 is a topological radical of \( X \) and denote it by \( t\text{-rad} X \).

Proposition 3.4. The topological radical of every irreducible Banach module is trivial.

Proof. Let \( X \) be an irreducible Banach module, and let \( \varphi \) be a small morphism with range in \( X \). Then \( \text{Im} \varphi = X \) or \( \text{Im} \varphi = 0 \). Since \( \varphi \) is small, \( \varphi \) is not surjective. Hence, \( \varphi = 0 \). This implies that \( t\text{-rad} X = 0 \). \( \blacksquare \)

Proposition 3.5. Let \( X \) be a finitely-generated Banach module. Then the natural embedding \( \iota: t\text{-rad} X \to X \) is a small morphism.

Proof. Let \( \varphi \) be a contractive monomorphism such that \( \varphi + \iota \) is surjective. If \( \varphi \) is not surjective it follows from Proposition 2.12 that there is a maximal contractive monomorphism \( \gamma \) such that \( \gamma \) is surjective and \( \gamma \) is small. This contradiction implies that \( \varphi \) is surjective. \( \blacksquare \)

Now we can establish main properties of topological radical, which are similar to the algebraic case (sf. [2 Sec.9.1, 9.2]).

Theorem 3.6. Let \( X \) be a unital Banach \( A \)-module.

(1) If \( \varphi \in \mathcal{A}(X,Y) \) then \( \varphi(t\text{-rad}(X)) \subset t\text{-rad} Y \).

(2) \( \tau: A \to X : a \mapsto a \cdot x_0 \) is small iff \( x_0 \in t\text{-rad} X \).

(3) \( \overline{R \cdot X} \subset t\text{-rad} X \).

(4) \( t\text{-rad}(X/t\text{-rad} X) = 0 \).

(5) If \( Z \) is a closed submodule in \( X \) such that \( t\text{-rad}(X/Z) = 0 \) then \( t\text{-rad} X \subset Z \).

Proof. (1) It follows from the definition and Proposition 1.5.

(2) See the proof of Theorem 2.2.

(3) Let \( x_0 \in X \). It is sufficient to show that \( \tau': R \to X : r \mapsto r \cdot x_0 \) is small. Since \( \tau' \) is a composition of \( R \to A \), which is small by Proposition 3.5 and \( \tau: A \to X \), Proposition 1.5 implies that \( \tau' \) is small.

(4) Suppose that \( x \in X \) such that \( x + t\text{-rad} X \in t\text{-rad}(X/t\text{-rad} X) \). By the definition \( x + t\text{-rad} X \in \text{Im} \alpha \) for every maximal contractive monomorphism \( \alpha: Y \to X/t\text{-rad} X \). By Lemma 2.8 there exists a maximal contractive monomorphism \( \beta: W \to X \) such that \( x \in \text{Im} \beta \). This implies that \( x \in t\text{-rad} X \).

(5) Denote by \( \sigma \) the projection \( X \to X/Z \). It follows from (1) that \( \sigma(t\text{-rad} X) = 0 \). Therefore \( t\text{-rad} X \subset Z \). \( \blacksquare \)

Proposition 3.7. If \( X \) is a unital finitely-generated Banach module then \( t\text{-rad} X = \text{rad} X \). In particular, \( t\text{-rad} A = \text{Rad} A \) for a unital Banach module.
The proposition follows immediately Theorem 2.6 and the lemma below.

**Lemma 3.8.** Every algebraically maximal submodule in a finitely-generated Banach module is closed.

**Proof.** Let $X_0$ be an algebraically maximal submodule in a finitely-generated Banach module $X$. Then there exists a minimal $k$ such that there are $x_1, \ldots, x_k \in X_0$ and $x_{k+1}, \ldots, x_n \in X \setminus X_0$ such that \{ $x_1, \ldots, x_n$ \} generates $X$. Fix generators of $X$ with this property. Denote by $U$ the set of all elements of the form $x = \sum_{i>1} a_i \cdot x_i + (1 - a_1) \cdot x_1$, where $\sum_i \|a_i\| < 1$.

If there is $x_0 \in X_0 \cap U$, then $x_0 = \sum_{i>1} a_i \cdot x_i + (1 - a_1) \cdot x_1$, where $\|a_1\| < 1$. Therefore, $1 - a_1$ is invertible and

$$x_1 = (1 - a_1)^{-1}(x_0 - \sum_{i>1} a_i \cdot x_i)$$

Hence, \{ $x_0, x_2, \ldots, x_n$ \} generates $X$ but only $k - 1$ generators are not in $X_0$. This contradiction with minimality of $k$ implies that $X_0 \cap U = \emptyset$.

It follows from the open mapping theorem that the surjective map $A \otimes \ell_1 \to X$: $e_i \mapsto x_i$ is open. Therefore $U$ is open. Since $x_1 \in U$, we have $x_1 \notin \overline{X_0}$.

Let $y \in \overline{X_0} \setminus X_0$. Since $X_0$ is maximal, $X_0 + A \cdot y = X$. Therefore there are $a \in A$ and $x_0 \in X_0$ such that $x_0 + a \cdot y = x_1$. Note that $x_1 + a \cdot (y - y') = x_0 - a \cdot y' \in X_0$ for every $y' \in X_0$. But we can take $y' \in X_0$ sufficiently close to $y$ to satisfy $x_1 + a \cdot (y - y') \notin X_0$. This contradiction implies that $X_0$ is closed. \qed

Remind that a left Banach $A$-module $P$ is called projective if a morphism of Banach $A$-modules with range in $P$ admits a right inverse morphism provided it admits a right inverse bounded operator.

**Proposition 3.9.** If $P$ is a unital projective module with the approximation property, then $\mathrm{t-rad} \, P = \overline{R \cdot P}$.

**Proof.** By Theorem 3.6(3) $\overline{R \cdot P} \subset \mathrm{t-rad} \, P$.

On the other side, suppose that $x_0 \in \mathrm{t-rad} \, P$. Since $P$ is projective and has the approximation property, \[ \textmd{Theorem 1(3)]} \implies that $x_0$ can be approximated in the norm topology by elements of the form $\sum_{i=1}^n \chi_i(x_0) \cdot y_i$ where $\chi_1, \ldots, \chi_n \in \mathrm{Ah}(P, A)$ and $y_1, \ldots, y_n \in P$. It follows from Theorem 3.6(2) and Proposition 1.5 that $a \to a \cdot \chi_i(x_0)$ is small for every $i$. Therefore $\chi_i(x_0) \in R$. Hence, $x_0 \in \overline{R \cdot P}$. \qed

**Remark 3.10.** It is not hard to check that in the case when $P$ is free, i.e. has the form $A \otimes E$ for some Banach space $E$, the argument of Proposition 3.9 can be applied to the case when $A$ or $E$ has the approximation property.

**References**

[1] O. Yu. Aristov, *Projective covers of finitely generated Banach modules and the structure of some Banach algebras*// Extracta mathematicae V 21, N. 1, 2006, P. 1–26.

[2] F. Kasch, Modules and rings. Academic Press, London, 1982.

[3] P. G. Dixon, *Topologically nilpotent Banach algebras and factorization*// Proc. Roy. Soc. Edinburgh Sect A, V. 119. 1991. P. 329–341.

[4] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*. Moscow University Press, 1986 (in Russian); English translation: Dordrecht: Kluwer Academic Publishers 1989.

[5] Yu. V. Selivanov *Projective Frechet modules with the approximation property*// Russian Mathematical Surveys, V 50:1, 1995, P. 211–213.

[6] T. W. Palmer, Banach Algebras and the General Theory of *-algebras. Vol.I, Cambridge University Press, 1994.