Abstract

This paper deals with the exact controllability to the trajectories of the one-phase Stefan problem in one spatial dimension. This is a free-boundary problem that models solidification and melting processes. It is assumed that the physical domain is filled by a medium whose state is liquid on the left and solid, with constant temperature, on the right. In between we find a free-boundary (the interface that separates the liquid from the solid). In the liquid domain, a parabolic equation completed with initial and boundary conditions must be satisfied by the temperature. On the interface, an additional free-boundary requirement, called the Stefan condition, is imposed. We prove the local exact controllability to the (smooth) trajectories. To this purpose, we first reformulate the problem as the local null controllability of a coupled PDE-ODE system with distributed controls. Then, a new Carleman inequality for the adjoint of the linearized PDE-ODE system, coupled on the boundary through nonlocal in space and memory terms, is presented. This leads to the null controllability of an appropriate linear system. Finally, a local result is obtained via local inversion, by using Liusternik-Graves’ Theorem. As a byproduct of our approach, we find that some parabolic equations which contains memory terms localized on the boundary are null-controllable.

Keywords: Free-boundary problems, one-phase Stefan problem, exact controllability to the trajectories, global Carleman inequalities, Inverse Function Theorem.

Mathematics Subject Classification: 35R35, 80A22, 93B05, 93C20

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1 Introduction

The phenomena of melting and solidification occurs in a plenty of situations in nature and industry, from melting and freezing of polar ice sheets to the continuous casting of steel, see for instance [LSTY83]. The mathematical formulation describing this thermodynamical model of liquid-solid phase transition is known as the Stefan problem, named after the work of the Slovene physicist and mathematician Josef Stefan. In such a problem, the model involves a moving free boundary, i.e. the spatial physical domain is time-dependent. Physically, in the Stefan problem the dynamics of the liquid-solid interface is influenced by the heat flux induced by melting or solidification. Mathematically, the time-evolution of the liquid-solid interface is modeled through a nonlinear ordinary differential equation. Among other situations, Stefan problem has also been employed to model population dynamics that describe tumor growth process [FR99] and information diffusion in online social networks [LLW13].

For the sake of completeness, we will give a short description of the mathematical formulation of the Stefan problem. A detailed presentation is given for instance in [Gup03].

Let \( \ell_0 \in \mathbb{R}_{>0} \) be given. At each time \( t \), the material domain is separated in two parts: the set \((0, \ell(t))\) (the liquid phase domain) and the set \((\ell(t), +\infty)\) (the solid phase domain). Here, \( \ell = \ell(t) \) indicates the position of the interface; it must satisfy \( \ell(0) = \ell_0 \) and \( \ell(t) \in (\ell_*, +\infty) \) at least for all small times, where \( \ell_0 \) and \( \ell_* \) are given and \( \ell_0 > \ell_* \). Hereafter, for any \( T > 0 \) and any \( \ell \in C^0([0, T]; \mathbb{R}_{>0}) \), we set

\[
Q_\ell := \{(x, t) : t \in (0, T), \quad x \in (0, \ell(t))\} \quad \text{and} \quad H^{1,2}(Q_\ell) := \{u \in L^2(Q_\ell) : u_x, u_{xx}, u_t \in L^2(Q_\ell)\}.
\]

This paper deals with the controllability properties of the following one-phase Stefan problem:

\[
\begin{aligned}
&u_t - u_{xx} = 0 \quad \text{in} \quad Q_\ell, \\
&u(0, t) = v(t) \quad \text{in} \quad (0, T), \\
&u(\ell(t), t) = 0 \quad \text{in} \quad (0, T), \\
&\beta \ell_t(t) = -u_x(\ell(t), t) \quad \text{in} \quad (0, T), \\
&\ell(0) = \ell_0, \\
&u(\cdot, 0) = u_0 \quad \text{in} \quad (0, \ell_0).
\end{aligned}
\]
Here, \( \beta \) is the so called Stefan number (a positive constant) and the initial state \( u_0 \in H^1(0, \ell_0) \) satisfies \( u_0(x) \geq 0 \) for all \( x \in [0, \ell_0] \) and \( u_0(\ell_0) = 0 \). The functions \( u = u(x, t) \) and \( v = v(t) \) may be respectively viewed as the temperature of the liquid phase and the imposed temperature on the left. In (1.1), \( v \) is the control (devised for heating or freezing the liquid) and \( (u, \ell) \) is the state.

In this paper, the objective is to prove the local exact controllability of (1.1) to the (smooth) trajectories at time \( T > 0 \). By definition, a trajectory of (1.1) is a triplet \((\bar{u}, \bar{\ell}, \bar{v})\) belonging to \( H^{1,2}(Q_T) \times H^1(0, T) \times H^{3/4}(0, T) \) satisfying

\[
\begin{align*}
\bar{u}_t - \bar{u}_{xx} &= 0 \quad \text{in} \quad Q_{\bar{\ell}}, \\
\bar{u}(0, t) &= \bar{v}(t) \quad \text{in} \quad (0, T), \\
\bar{u}(\bar{\ell}(t), t) &= 0 \quad \text{in} \quad (0, T), \\
\beta \bar{\ell}_t(t) &= -\bar{u}_x(\bar{\ell}(t), t) \quad \text{in} \quad (0, T), \\
\bar{\ell}(0) &= \bar{\ell}_0, \\
\bar{u}(\cdot, 0) &= \bar{u}_0 \quad \text{in} \quad (0, \bar{\ell}_0),
\end{align*}
\]

(1.2)

where \( \bar{\ell}_0 > \ell_*, \bar{\ell}(t) \in (\ell_*, +\infty) \) for all \( t \in [0, T] \), \( \bar{u}_0 \in H^1(0, \bar{\ell}_0) \), \( \bar{u}_0(x) \geq 0 \) for all \( x \in [0, \ell_0] \), \( \bar{u}_0(\ell_0) = 0 \), \( \bar{v}(t) \geq 0 \) for all \( t \in [0, T] \) and the compatibility condition \( \bar{u}_0(0) = \bar{v}(0) \) holds.

We will denote by \( T \) the space of triplets \((\bar{u}, \bar{\ell}, \bar{v})\) \( \in H^{1,2}(Q_T) \times H^1(0, T) \times H^{3/4}(0, T) \) such that the function \((y, t) \mapsto \bar{u}(y, \bar{\ell}(t), t)\) belongs to \( W^{1,\infty}(0, T; H^1(0, 1)) \) and \( \bar{\ell} \in W^{1,\infty}(0, T) \). Our main result is the following:

**Theorem 1.1.** Let \((\bar{u}, \bar{\ell}, \bar{v})\) be a trajectory of (1.1) with \((\bar{u}, \bar{\ell}) \in T \) and \( \bar{v}(t) > 0 \) for all \( t \in [0, T] \). Then, there exists \( \delta > 0 \) with the following property: for any \( \ell_0 \in (\ell_*, +\infty) \) and any \( u_0 \in H^1(0, \ell_0) \) with \( u_0(x) \geq 0 \) for all \( x \in [0, \ell_0] \) satisfying

\[
|\ell_0 - \bar{\ell}_0| + \|\ell_0 u_0(\cdot, \ell_0) - \bar{\ell}_0 \bar{u}_0(\cdot, \bar{\ell}_0)\|_{H^1(0, 1)} \leq \delta,
\]

(1.3)

there exist nonnegative controls \( v \in H^{3/4}(0, T) \) and associated states \((u, \ell)\) with

\[
u \in H^{1,2}(Q_T), \quad \ell \in H^1(0, T) \quad \text{and} \quad \ell(t) \in (\ell_*, +\infty) \quad \forall t \in [0, T]
\]

such that

\[
\ell(T) = \bar{\ell}(T) \quad \text{and} \quad u(\cdot, T) = \bar{u}(\cdot, T) \quad \text{in} \quad (0, \bar{\ell}(T)).
\]

(1.4)

Note that the norm in (1.3) is given by

\[
\|\ell_0 u_0(\cdot, \ell_0) - \bar{\ell}_0 \bar{u}_0(\cdot, \bar{\ell}_0)\|_{H^1(0, 1)} = \left( \int_0^1 |\ell_0 u_0(y, \ell_0) - \bar{\ell}_0 \bar{u}_0(y, \bar{\ell}_0)|^2 + |\ell_0 u_{0,x}(y, \ell_0) - \bar{\ell}_0 \bar{u}_{0,x}(y, \bar{\ell}_0)|^2 \, dy \right)^{1/2}.
\]

**Remark 1.2.** The assumption \( \bar{\ell}(t) > 0 \) on the left edge is a natural assumption as the fluid is liquid, being solid only on the right edge.

**Remark 1.3.** Theorem 1.1 also holds if we just assume that \( \bar{v} \) is nonnegative and \( \bar{v} \not\equiv 0 \).

**Remark 1.4.** Thanks to the regularizing effect, Theorem 1.1 still holds if we only assume that the trajectories belong to \( H^{1,2}(Q_T) \times H^1(0, T) \times H^{3/4}(0, T) \).

Let us mention some previous works on the control of (1.1) and other similar models.

The analysis of the controllability properties for linear and nonlinear parabolic PDEs in cylindrical parabolic domains is nowadays a classical problem in control theory and some relevant contributions are in [FPZ95, FR71, FCZ00, FR98, LR95] and the references therein. On the other hand, the study of the control and stabilization properties of free-boundary problems for PDEs has not been explored too much, although some important results have been obtained recently; see [DFC18, FCLdM16, FChSI74, FCHL13, WLL22] and [KK20, AFCS21], respectively for one-phase and two-phase Stefan problems. In [G721], the authors study the controllability of
free-boundary viscous Burgers equation with one moving endpoint; a similar problem was considered for a 1D fluid-structure system, with modified equations for the interface, see [DFC05, FCDS17b, LTT13].

In this paper, we are going to consider a different situation, which leads to several new difficulties and novelties, not found in the previous works on free-boundary problems and fluid-structure models. Let us give more details:

- From our knowledge, our result is the first one concerning the exact control to the trajectories in the context of a parabolic system where the spatial domain changes with time and starts from a different location. Up to now, the available results have dealt with null controllability. Indeed, in the context of Stefan problems, the physical meaning of solutions in the previous works is limited due to the fact that the controlled solutions do not preserve positivity. Our findings provide progress in that direction, because our solutions preserve positivity, and thus have a proper physical meaning.

- In fact, we control both components of the state (the final temperature and the final position of the liquid-solid interface). This will bring an additional difficulty.

- After a suitable change of variable and some additional arguments, it will be seen that the free-boundary control problem is equivalent to the null controllability of a nonlinear parabolic PDE-ODE system, which can also be viewed as a nonlinear parabolic equation with nonlocal and memory terms on the boundary. To establish this property, we will use two main tools: a new global Carleman inequality (with weights chosen to handle satisfactorily the boundary terms) and Lyusternik–Graves’ Inverse Function Theorem.

The rest of this paper is organized as follows. In Section 2 we will reformulate the free-boundary problem as a nonlinear parabolic system in a cylindrical domain and we will establish some well-posedness results. In Section 3 we will present a new Carleman inequality for an adjoint system which leads to the null controllability of a related linearized PDE-ODE system and we will give the proof of Theorem 1.1. Finally, the proofs of several results will be presented in Appendices A, B and C.

2 Preliminaries

2.1 Reformulation of the free-boundary problem in a cylindrical domain

In order to study the controllability of (1.1), it is convenient to get a reformulation as a nonlinear parabolic equation in a cylindrical domain. More precisely, let us set

\[ p(y, t) := u(y \ell(t), t) \quad \text{and} \quad q(t) := \ell(t)^2 \]

for \((y, t) \in Q_1 := (0, 1) \times (0, T)\).

After this transformation, (1.1) reads

\[
\begin{cases}
qp_t - p_{yy} + \frac{y}{\beta}p_y(1, \cdot)p_y = 0 & \text{in } Q_1, \\
p(0, \cdot) = v & \text{in } (0, T), \\
p(1, \cdot) = 0 & \text{in } (0, T), \\
p(\cdot, 0) = p_0 & \text{in } (0, 1), \\
\beta q_t + 2p_y(1, \cdot) = 0 & \text{in } (0, T), \\
q(0) = q_0,
\end{cases}
\]

where \(q_0 := \ell_0^2\) and \(p_0(y) := u_0(y \ell_0)\) in \((0, 1)\).
Remark 2.1. By introducing the square of \( \ell(t) \), the Stefan condition on the interface becomes a linear constraint on \( q_t \) and \( p_y(1, \cdot) \). Otherwise, we would have
\[
\beta \ell(t) = \frac{1}{\ell(t)} p_y(1, t).
\]

Since \( \ell \) has a strictly positive lower bound \( \ell_* \), squaring is a diffeomorphism. □

With a similar change of variables, (1.2) is transformed into
\[
\begin{aligned}
\bar{q} t - \bar{q}_{yy} + y \beta \bar{p}_y(1, \cdot) \bar{p}_y &= 0 \quad \text{in } Q_1, \\
\bar{p}(0, \cdot) &= \bar{v} \quad \text{in } (0, T), \\
\bar{p}(1, \cdot) &= 0 \quad \text{in } (0, T), \\
\bar{p}(\cdot, 0) &= \bar{p}_0 \quad \text{in } (0, 1), \\
\beta \bar{q}_t + 2 \bar{p}_y(\cdot, \cdot) &= 0 \quad \text{in } (0, T), \\
\bar{q}(0) &= \bar{q}_0,
\end{aligned}
\]

where \( \bar{p}_0(y) := \bar{u}_0(y\ell_0), \bar{q}_0 := \ell_0^2 \) and \( \bar{p}(y, t) = \bar{u}(\ell(t)y, t) \) and \( \bar{q}(t) := \ell(t)^2 \) for \( (y, t) \in Q_1 \). Note that, by assumption, \( \bar{q}(t) \in (q_*, +\infty) \) for all \( t \in [0, T] \) with \( q_* = \ell_*^2 \).

Thus, to prove that (1.1) is locally exactly controllable to the trajectory \((\bar{u}, \bar{\ell})\) is equivalent to prove that (2.1) is locally exactly controllable to \((\bar{p}, \bar{q})\). Consequently, Theorem 1.1 will be a direct consequence of the following result:

**Proposition 2.2.** Let \((\bar{p}, \bar{q}, \bar{v}) \in [W^{1,\infty}(0, T; H^1(0, 1)) \cap H^{1,2}(Q_1)] \times W^{1,\infty}(0, T) \times H^{3/4}(0, T)\) satisfy (2.2), with \( \bar{v}(t) > 0 \) for all \( t \in [0, T] \). Then, there exists \( \delta > 0 \) with the following property: for any \( p_0 \in H^1_0(0, 1) \) and any \( q_0 \in (q_*, +\infty) \) satisfying
\[
|q_0 - \bar{q}_0| + \|p_0 - \bar{p}_0\|_{H^1_0(0, 1)} \leq \delta,
\]
there exist nonnegative controls \( v \in H^{3/4}(0, T) \) and associated solutions \((p, q)\) to (2.1), with
\[
p \in H^{1,2}(Q_1), \quad q \in H^1(0, T) \quad \text{and} \quad q(t) \in (q_*, +\infty) \quad \forall t \in [0, T]
\]
such that
\[
q(T) = \bar{q}(T) \quad \text{and} \quad p(\cdot, T) = \bar{p}(\cdot, T) \quad \text{in } (0, 1).
\]

### 2.2 Reformulation as a null controllability problem

Now, we will reformulate the desired control property as a null controllability problem.

To do this, let us introduce the change of variable \( z = p - \bar{p} \) and \( h = \beta(q - \bar{q})/2 \). Then, the local exact controllability of \((\bar{p}, \bar{q})\) for (2.1) is reduced to the local null controllability of the following system, where we have denoted again by \( x \) the spatial variable:
\[
\begin{aligned}
\bar{q} z_t - z_{xx} + \frac{x}{\beta} \bar{p}_x(1, \cdot) z_{xx} + \frac{x}{\beta} \bar{p}_x z_x(1, \cdot) + \frac{2}{\beta} \bar{p}_t h + \frac{2}{\beta} h z_t + \frac{x}{\beta} z_x(1, \cdot) z_x &= 0 \quad \text{in } Q_1, \\
z(0, \cdot) &= \bar{v} \quad \text{in } (0, T), \\
z(1, \cdot) &= 0 \quad \text{in } (0, T), \\
z(\cdot, 0) &= z_0 \quad \text{in } (0, 1), \\
h(z) + \bar{z}_x(1, \cdot) &= 0 \quad \text{in } (0, 1), \\
h(0) &= h_0,
\end{aligned}
\]

where \( z_0 := p_0 - \bar{p}_0, h_0 := \beta(q_0 - \bar{q}_0)/2, \bar{v} = v - \bar{v} \) and \((2/\beta)h(t) + \bar{q}(t) \in (q_*, +\infty) \) for all \( t \in [0, T] \). Here, we have used (2.2) to simplify some terms.

Consequently, Proposition 2.2 is obviously equivalent to the following result:
Proposition 2.3. Let \((\bar{p}, \bar{q}, \bar{v}) \in [W^{1,\infty}(0, T; H^1(0, 1)) \cap H^{1,2}(Q_1)] \times W^{1,\infty}(0, T) \times H^{3/4}(0, T)\) satisfy \((2.2)\), with \(\bar{v}(t) > 0\) for all \(t \in [0, T]\). There exists \(\delta > 0\) with the following property: for any \(p_0 \in H^1_0(0, 1)\) and any \(q_0 \in (q_*, +\infty)\) satisfying
\[
|q_0 - \bar{q}_0| + \|p_0 - \bar{p}_0\|_{H^1_0(0, 1)} \leq \delta,
\]
there exist nonnegative controls \(v \in H^{3/4}(0, T)\) and associated solutions \((z, h)\) to \((2.4)\) where we have taken \(z_0 := p_0 - \bar{p}_0, h_0 := \beta(q_0 - \bar{q}_0)/2\) and \(\bar{v} = v - \bar{v}\) such that
\[
z \in H^{1,2}(Q_1), \quad h \in H^1(0, T) \quad \text{and} \quad (2/\beta)h(t) + \bar{q}(t) \in (q_*, +\infty) \; \forall t \in [0, T],
\]
\[
h(T) = 0 \quad \text{and} \quad z(\cdot, T) = 0 \; \text{in} \; (0, 1).
\]

(2.5)

2.3 Reformulation as a distributed control problem

Let us establish a result similar to Proposition 2.3 for a distributed control system.

Thus, let us set
\[
Q := (-1, 1) \times (0, T) \quad \text{and} \quad H^1_0(0, T) := \{z \in H^{1,2}(Q) : z(-1, \cdot) = z(1, \cdot) = 0 \; \text{in} \; (0, T)\}
\]
and let us consider a non-empty open set \(\omega \subset \subset (-1, 0)\). The following holds:

Proposition 2.4. Assume that \((\bar{p}, \bar{q}) \in [W^{1,\infty}(0, T; H^1(-1, 1)) \cap H^{1,2}(Q_1)] \times W^{1,\infty}(0, T)\), with \(\bar{q}(t) \in (q_*, +\infty)\) for all \(t \in [0, T]\). There exists \(\delta > 0\) with the following property: for any \(z_0 \in H^1_0(-1, 1)\) and any \(h_0 \in \mathbb{R}\) satisfying
\[
|h_0| + \|z_0\|_{H^1_0(-1, 1)} \leq \delta,
\]
there exist controls \(w \in L^2(\omega \times (0, T))\) and associated solutions to the system
\[
\begin{aligned}
\begin{cases}
\bar{q}z_t - z_{xx} + \frac{x}{\beta} \bar{p}_x(1, \cdot)z_x + \frac{x}{\beta} \bar{p}_x z_x(1, \cdot) + \frac{2}{\beta} \bar{p}_t h + \frac{x}{\beta} h z_t + x z_x(1, \cdot) z_x = w1_\omega & \text{in} \; Q, \\
z(-1, \cdot) = 0 & \text{in} \; (0, T), \\
z(1, \cdot) = 0 & \text{in} \; (0, T), \\
z(\cdot, 0) = z_0 & \text{in} \; (-1, 1), \\
h_t + z_x(1, \cdot) = 0 & \text{in} \; (0, T), \\
h(0) = h_0
\end{cases}
\end{aligned}
\]
with \((z, h) \in H^{1,2}_0(Q) \times H^1(0, T)\), satisfying \(\|(z, h)\|_{H^{1,2}_0(Q) \times H^1(0, T)} \leq C\|z_0, h_0\|_{H^1_0(-1, 1) \times \mathbb{R}}\), such that
\[
h(T) = 0 \quad \text{and} \quad z(\cdot, T) = 0 \; \text{in} \; (-1, 1),
\]
for some constant \(C > 0\).

The proof of Proposition 2.4 will be given in Section 3.2. The main reason to consider this extended problem is that the boundary controls obtained with the help of Carleman estimates are not sufficiently regular for our purposes, just \(L^2(0, T)\), while we need at least \(H^{3/4}(0, T)\). On the other hand, with distributed controls, local parabolic results can be used to improve the regularity of the control.

Obviously, Proposition 2.3 follows from Proposition 2.4 by restricting to \(Q_1\) and accepting that the boundary control \(\bar{v} = \bar{v}(t)\) is just the lateral trace of \(z\) at \(x = 0\).

Also, note that we can take \(\delta\) small enough to have \((2/\beta)h(t) + \bar{q}(t) \in (q_*, +\infty)\) for all \(t \in [0, T]\). Since \(\bar{v}(t) > 0\) for all \(t \in [0, T]\), by taking \(\delta\) sufficiently small, we can ensure that the control \(v := \bar{v} + \bar{v}\) is nonnegative.
2.4 Linearization

Now, our aim is to linearize (2.6) in a neighborhood of (0, 0) and analyze the null controllability properties of the resulting system. Thus, let us consider the non-homogeneous linear equation

\[
\begin{cases}
    qz_t - z_{xx} + \frac{x}{\beta} \bar{p}_x(1, \cdot)z_x + \frac{x}{\beta} \bar{p}_x z_x(1, \cdot) + \frac{2}{\beta} \bar{p}_t h = f_1 + w1_{\omega}, & \text{in } Q, \\
    z(-1, \cdot) = 0 & \text{in } (0, T), \\
    z(1, \cdot) = 0 & \text{in } (0, T), \\
    z(\cdot, 0) = z_0 & \text{in } (0, T), \\
    h_t + z_x(1, \cdot) = f_2 & \text{in } (0, T), \\
    h(0) = h_0, & \text{in } (0, T),
\end{cases}
\]

where \(f_1\) and \(f_2\) belong to appropriate spaces of functions that decay exponentially as \(t \to T^-\) and will be made precise below.

In order to prove the null controllability of (2.8), we are going to use the Hilbert Uniqueness Method (see [Lio88]). Accordingly, we will first deduce an observability inequality for the adjoint, which is the following:

\[
\begin{cases}
    -q\varphi_t - \varphi_{xx} - \frac{x}{\beta} \bar{p}_x(1, \cdot)\varphi_x + \frac{1}{\beta} \bar{p}_x(1, \cdot)\varphi = g_1 & \text{in } Q, \\
    \varphi(-1, \cdot) = 0 & \text{in } (0, T), \\
    \varphi(1, \cdot) = \gamma + \int_{-1}^1 \frac{x}{\beta} \bar{p}_x(x, \cdot)\varphi(x, \cdot) \, dx & \text{in } (0, T), \\
    \varphi(\cdot, T) = \varphi_T & \text{in } (-1, 1), \\
    \gamma_t = \int_{-1}^1 \frac{2}{\beta} \bar{p}_t(x, \cdot)\varphi(x, \cdot) \, dx + g_2 & \text{in } (0, T), \\
    \gamma(T) = \gamma_T.
\end{cases}
\]

(2.9)

It is worth to mention that, in [GZ21], the authors point out that the exact controllability to the trajectories for the free-boundary viscous Burgers equation is an open problem. They also linearize that problem and compute its adjoint system (similar to (2.9)).

2.5 Well-posedness of the adjoint system

Henceforth, we will denote by \((\cdot, \cdot)_2\) the usual scalar product in \(L^2(-1, 1)\) and \(\| \cdot \|_2\) will stand for the associated norm.

For clarity, we will provisionally change (2.9) by a similar in time system with general coefficients:

\[
\begin{cases}
    -\bar{q}(t)\varphi_t - \varphi_{xx} - a\varphi_x - b\varphi = f & \text{in } Q, \\
    \varphi(-1, \cdot) = 0 & \text{in } (0, T), \\
    \varphi(1, t) = \gamma(t) + (N(\cdot, t), \varphi(\cdot, t))_2 & \text{in } (0, T), \\
    \varphi(\cdot, T) = \varphi_T & \text{in } (-1, 1), \\
    \gamma'(t) = (R(\cdot, t), \varphi(\cdot, t))_2 + g(t) & \text{in } (0, T), \\
    \gamma(T) = \gamma_T.
\end{cases}
\]

(2.10)

Note that the boundary condition on \(\varphi\) at \(x = 1\) involves \(\gamma\), that is essentially a primitive in time of a spatial integral of \(\varphi\) and an additional spatial integral of \(\varphi\). Thus, in this system, we find nonlocal in space and memory boundary terms.
Lemma 2.6. Let us assume that $R \in L^2(Q)$, $N \in H^1(0,T; L^2(-1,1))$, $a, b \in L^2(0,T; L^\infty(-1,1))$ and $\bar{q} \in C^0([0,T])$ with $\bar{q}(t) \in (q_*, +\infty)$ for all $t \in [0,T]$. Let $f \in L^2(Q)$, $g \in L^2(0,T)$, $\varphi_T \in H^1(-1,1)$ and $\gamma_T \in \mathbb{R}$ be given and assume that

$$\varphi_T(-1) = 0 \text{ and } \varphi_T(1) = \gamma_T + (N(\cdot, T), \varphi_T)_2.$$  \hspace{1cm} (2.11)

Then, there exists a unique strong solution in $H^{1,2}(Q) \times H^1(0,T)$ to (2.11) such that the following estimate holds:

$$\|\varphi\|^2_{H^{1,2}(Q)} + \|\gamma\|^2_{H^1(0,T)} \leq e^{C(1+T)} \left( \|f\|^2_{L^2(Q)} + \|g\|^2_{L^2(0,T)} + \|\varphi_0\|^2_{H^1(0,1)} + |\gamma_0|^2 \right), \hspace{1cm} (2.12)$$

where $C$ is a positive constant depending on $a$, $b$, $R$, $N$ and $\bar{q}$ but independent of $T$.

The proof is given in Appendix A.

### 2.6 Carleman estimates for parabolic equations with nonlocal boundary conditions

Let us recall the definition of several classical weights, frequently used in connection with global Carleman inequalities for parabolic equations, see [FI96].

Let $\omega_0$ be a non-empty open set, with $\omega_0 \subset \subset \omega$ and let be a function $\eta$ in $C^2([-1,1])$ satisfying

$$\eta > 0 \text{ in } [-1,1], \min_{x \in [-1,1] \setminus \omega_0} \eta(x) > 0, \eta(-1) = \eta(1) = \min_{x \in [-1,1]} \eta(x). \hspace{1cm} (2.13)$$

Let us introduce the following associated weights:

$$\alpha(x, t) := \frac{e^{2\lambda m\|\eta\|_{\infty} - e^{\lambda (m\|\eta\|_{\infty} + \eta(x))}}}{t(T-t)} \quad \forall (x, t) \in Q,$$

$$\xi(x, t) := \frac{e^{\lambda (m\|\eta\|_{\infty} + \eta(x))}}{t(T-t)} \quad \forall (x, t) \in Q,$$

$$\tilde{\alpha}(t) := \max_{x \in [-1,1]} \alpha(x, t) = \alpha(1, t) = \alpha(-1, t) \quad \forall t \in (0, T),$$

$$\tilde{\xi}(t) := \min_{x \in [-1,1]} \xi(x, t) = \xi(1, t) = \xi(-1, t) \quad \forall t \in (0, T),$$

where $m > 1$ and $\lambda > 0$ is a sufficiently large constant (to be chosen later).

We will establish a Carleman inequality that holds for the solutions to a simplified version of (2.10). This will be later extended to the solutions to (2.10) and, consequently, to the adjoint states in (2.4).

**Lemma 2.6.** Let us assume that $R \in L^\infty(0,T; L^2(-1,1))$, $N \in W^{1,\infty}(0,T; L^2(-1,1))$ and $d \in C^1([0,T])$ with $d(t) > d_* > 0$ for all $t \in [0,T]$. There exist constants $\lambda_0 \geq 1$, $s_0 \geq 1$ and $C_0 > 0$ such that, for any $\lambda \geq \lambda_0$, any $s \geq s_0(T + T^2)$, any $(\psi_T, \gamma_T) \in H^1(-1,1) \times \mathbb{R}$ satisfying (2.11) and any source terms $f \in L^2(Q)$ and $g \in L^2(0,T)$, the strong solution to

$$\begin{cases}
\psi_t + d(t)\psi_{xx} = f & \text{in } Q, \\
\psi(-1,\cdot) = 0 & \text{in } (0,T), \\
\psi(1,t) = \gamma(t) + (N(\cdot, t), \psi(\cdot, t))_2 & \text{in } (0,T), \\
\psi(\cdot, T) = \psi_T & \text{in } (-1,1), \\
\gamma(t) = (R(\cdot, t), \psi(\cdot, t))_2 = g & \text{in } (0,T), \\
\gamma(T) = \gamma_T & \text{in } (0,T),
\end{cases} \hspace{1cm} (2.14)$$

satisfies
\[
\int_Q \left[ (s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_x|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2 \right] e^{-2s\alpha} \, dx \, dt \\
+ \int_0^T \left[ \lambda^3(s\xi)^3|\psi(1,t)|^2 + \lambda^5(s\xi)(|\psi_x(-1,t)|^2 + |\psi_x(1,t)|^2) \right] e^{-2s\delta} \, dt \\
\leq C_0 \left( s^3\lambda^4 \int_0^T \xi^3|\psi|^2 e^{-2s\alpha} \, dx \, dt + \int_Q |f|^2 e^{-2s\alpha} \, dx \, dt + \int_0^T |g|^2 e^{-2s\delta} \, dt \right).
\] (2.15)

Note that, in view of (2.14) and (2.14)_5, we can also include weighted integrals of \( \gamma \) and \( \gamma_t \) in the left hand side of (2.15).

The proof of Lemma 2.6 is given in Appendix C. As already mentioned, this Carleman inequality is new. It is one of the main contributions in the paper. The main difficulty to overcome is that we have to deal with a non-local term on the boundary, both in the space and time variables.

### 2.7 Well-posedness of the linearized system

The aim of this section is to prove the existence and uniqueness of a global solution to (2.8).

For convenience, we will establish the existence and uniqueness of a strong solution to a similar, where (again) we have introduced general coefficients.

More precisely, we have the following result:

**Proposition 2.7.** Assume that \((a, R, N)\) belongs to the space \(L^2(0,T;L^\infty(-1,1)) \times L^2(Q) \times L^\infty(0,T;L^2(-1,1))\) and \(\overline{z} \in W^{1,\infty}(0,T)\), with \(\overline{z}(t) \in (q_*, +\infty)\) for all \(t \in [0,T]\). Let \(F \in L^2(Q), G \in L^2(0,T), z_0 \in H^1_0(-1,1)\) and \(h_0 \in \mathbb{R}\) be given. There exists a unique strong solution in \(H^1_0(Q) \times H^1(0,T)\) to the system

\[
\begin{cases}
\bar{q}(t)z_t - z_{xx} + a z_x + Rh + N z_x(1, \cdot) = F & \text{in } Q, \\
z(-1, \cdot) = 0 & \text{in } (0, T), \\
z(1, \cdot) = 0 & \text{in } (0, T), \\
z(\cdot, 0) = z_0 & \text{in } (-1, 1), \\
h_t + z_x(1, \cdot) = G & \text{in } (0, T), \\
h(0) = h_0,
\end{cases}
\] (2.16)

such that the following inequality holds:

\[
\|z\|_{H^1_0(Q)}^2 + \|h\|_{H^1(0,T)}^2 \leq e^{C(1+T)} \left( \|F\|_{L^2(Q)}^2 + \|G\|_{L^2(0,T)}^2 + \|z_0\|_{H^1_0(-1,1)}^2 + |h_0|^2 \right),
\] (2.17)

where \(C\) is a positive constant depending on \(a, R, N\) and \(\overline{z}\) but independent of \(T\).

The proof is given in Appendix C.

At this point, we will introduce the definition of solution by transposition to (2.16):

**Definition 2.8.** It will be said that \((z, h) \in L^2(Q) \times L^2(0,T)\) is a solution by transposition to (2.16) if

\[
\int_Q z(x,t)f(x,t) \, dx \, dt + \int_0^T h(t)g(t) \, dt = M(f, g) \quad \forall (f, g) \in L^2(Q) \times L^2(0,T),
\] (2.18)

where the linear form \(M : L^2(Q) \times L^2(0,T) \to \mathbb{R}\) is given by

\[
M(f, g) := \int_Q F(x,t) \varphi(x,t) \, dx \, dt + \bar{q}(0)(z_0, \varphi(\cdot, 0))_2 + h_0 \gamma(0) + \int_0^T G(t)\gamma(t) \, dt
\]
and \((\varphi, \gamma)\) is the unique strong solution to
\[
\begin{aligned}
-(\bar{q}\varphi)_t - \varphi_{xx} - (a\varphi)_x &= f & \text{in} & \ Q, \\
\varphi(-1, \cdot) &= 0 & \text{in} & \ (0, T), \\
\varphi(1, t) &= \gamma(t) + (N(\cdot, t), \varphi(\cdot, t))_2 & \text{in} & \ (0, T), \\
\varphi(\cdot, T) &= 0 & \text{in} & \ (-1, 1), \\
\gamma'(t) &= (R(\cdot, t), \varphi(\cdot, t))_2 + g & \text{in} & \ (0, T), \\
\gamma(T) &= 0.
\end{aligned}
\]
\hspace{1cm} (2.19)

Since the boundary and final conditions in (2.19) satisfy the appropriate compatibility conditions (2.11), Proposition 2.5 guarantees the existence and uniqueness of a strong solution to (2.19). Consequently, Definition 2.8 makes sense.

Proposition 2.9. Let the assumptions in Proposition 2.7 be satisfied and, additionally, consider that
\[ a \in L^2(0, T; W^{1,\infty}(-1, 1)) \text{ and } N \in H^1(0, T; L^2(-1, 1)) \]
Then, there exists a unique solution by transposition to (2.16).

Proof. Note that \( M : L^2(Q) \times L^2(0, T) \mapsto \mathbb{R} \) is a continuous linear form. Indeed, since \((\varphi, \gamma)\) is the unique strong solution, from Proposition 2.5 we have
\[ ||\varphi||^2_{H^1(Q)} + ||\gamma||^2_{H^1(0, T)} \leq C ||(f, g)||^2_{L^2(Q) \times L^2(0, T)}.\]
Therefore, we deduce from Riesz Representation Theorem that there exists exactly one solution by transposition to (2.16).

Notice that strong solutions to (2.16) are solutions by transposition.

3 Exact controllability to the trajectories

This section is devoted to prove the null controllability of the linear system (2.8) and the local null controllability of the nonlinear PDE-ODE system (2.6).

3.1 Controllability of the linearized problem

We will present a suitable Carleman inequality for the solutions to a properly chosen adjoint system. This will imply the null controllability of the linearized system (2.8) (see Proposition 3.5 below). This result will be essential for the proof of Proposition 2.4 (the local null controllability of (2.6).)

3.1.1 A Carleman inequality

The following holds:

Theorem 3.1. Assume that \((\bar{p}, \bar{q})\) belong to the space \([W^{1,\infty}(0, T; H^1(-1, 1)) \cap H^1_0(Q)] \times W^{1,\infty}(0, T)\) with \(\bar{p}(t) \in (q_*, +\infty)\) for all \(t \in [0, T]\). There exist constants \(\lambda_0 \geq 1, s_0 \geq 1\) and \(C_0 > 0\) such that, for any \(\lambda \geq \lambda_0,\) any \(s \geq s_0(T + T^2),\) any \(\varphi_T \in H^1(-1, 1)\) any \(\gamma_T \in \mathbb{R}\) with
\[ \varphi_T(-1) = 0 \quad \text{and} \quad \varphi_T(1) = 2\gamma_T + \frac{1}{\beta} \int_{-1}^1 \bar{p}_x(x, T)x\varphi_T(x) \, dx \]
\hspace{1cm} (3.1)
and any right hand sides \( g_1 \in L^2(Q) \) and \( g_2 \in L^2(0, T) \), the strong solution to (2.9) satisfies:

\[
\int_Q \left[ (s\xi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\xi)|\varphi_x|^2 + \lambda^4(s\xi)^3|\varphi|^2 \right] e^{-2s\alpha} \, dx \, dt \\
+ \int_0^T \left[ |\gamma|^2 + \lambda(s\xi)^2(|\varphi_x(-1, t)|^2 + |\varphi_x(1, t)|^2) + \lambda^3(s\xi)^3(|\varphi(1, t)|^2 + |\varphi|^2) \right] e^{-2s\tilde{\alpha}} \, dt \\
\leq C_0 \left( \int_Q |g_1|^2 e^{-2s\alpha} \, dx \, dt + \int_0^T |g_2|^2 e^{-2s\tilde{\alpha}} \, dt + s^3 \lambda^4 \int_0^T \int_\omega \xi^3|\varphi|^2 e^{-2s\alpha} \, dx \, dt \right). 
\]

(3.2)

**Proof.** Let us apply Lemma 2.6 with the following data:

\[
d = \frac{1}{q}, \quad f = -\frac{1}{q} \left[ g_1 + \frac{\bar{p}_e}{\beta}(x\varphi_x - \varphi) \right], \quad N(x, t) = \frac{x}{\beta}\bar{p}_e(x, t), \quad R = \frac{2}{\beta}\bar{p}_e \quad \text{and} \quad g = g_2.
\]

We obtain:

\[
\int_Q \left[ (s\xi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\xi)|\varphi_x|^2 + \lambda^4(s\xi)^3|\varphi|^2 \right] e^{-2s\alpha} \, dx \, dt \\
+ \int_0^T \left[ |\gamma|^2 + \lambda(s\xi)^2(|\varphi_x(-1, t)|^2 + |\varphi_x(1, t)|^2) + \lambda^3(s\xi)^3(|\varphi(1, t)|^2 + |\varphi|^2) \right] e^{-2s\tilde{\alpha}} \, dt \\
\leq C_0 \left( s^3 \lambda^4 \int_0^T \int_\omega \xi^3|\varphi|^2 e^{-2s\alpha} \, dx \, dt + \int_Q |f|^2 e^{-2s\alpha} \, dx \, dt + \int_0^T |g|^2 e^{-2s\tilde{\alpha}} \, dt \right).
\]

Clearly, one can absorb the lower order terms in \( f \) and obtain (3.2). \hfill \Box

### 3.1.2 Null controllability with nonhomogeneities

In this section we prove the null controllability property of (2.8) with source terms that decay exponentially as \( t \to T^- \). As we will see below, this result will be useful to prove the local null controllability of (2.4).

Before, it will be convenient to deduce a second Carleman inequality with weights that do not vanish at \( t = 0 \).

More precisely, let the function \( r = r(t) \) be given by

\[
r(t) = \begin{cases} 
T^2/4 & \text{in } [0, T/2], \\
T - t & \text{in } [T/2, T]. 
\end{cases}
\]

(3.3)

and let us set \( D_1 = (-1, 1) \times (0, T/2), D_2 = (-1, 1) \times (T/2, T), \)

\[
\zeta(x, t) := \frac{e^{2\lambda m \|\eta\|_{\infty}} - e^{\lambda m (|\eta|_{\infty} + \eta(x))}}{r(t)} \quad \text{and} \quad \mu(x, t) := \frac{e^{\lambda m (|\eta|_{\infty} + \eta(x))}}{r(t)} \quad \forall (x, t) \in Q.
\]

(3.4)

Let us also introduce the notation:

\[
\tilde{\zeta}(t) := \max_{x \in [-1,1]} \zeta(x, t), \quad \tilde{\mu}(t) := \min_{x \in [-1,1]} \mu(x, t), \quad \zeta^*(t) := \min_{x \in [-1,1]} \zeta(x, t), \quad \mu^*(t) := \max_{x \in [-1,1]} \mu(x, t) \quad \forall t \in (0, T)
\]

(3.5)

and

\[
\rho_0(t) := e^{\zeta^*(t)}, \quad \rho_1(t) := e^{\tilde{\zeta}(t)}, \quad \rho_2(t) := \mu^* e^{3/2(t)} e^{\zeta^*(t)}, \quad \rho_3(t) := e^{\tilde{\zeta}(t)} \tilde{\mu}^{-3/2(t)}, \quad \rho_4(t) := \rho_3^{1/2}(t) \quad \forall t \in (0, T).
\]

**Remark 3.2.** Notice that \( e^{\tilde{\zeta}} \) and \( e^{\zeta^*} \) blow up exponentially as \( t \to T^- \) and \( \tilde{\mu} \) and \( \mu^* \) blow up polynomially as \( t \to T^- \).
Remark 3.3. It is not difficult to deduce the following:

- Since $\rho_4^{-1} \in L^\infty(0, T)$, we have that $\rho_4 \rho_3^{-1} = \rho_4^{-1} \in L^\infty(0, T)$.
- If we take $\lambda_0$ large enough, for instance $\lambda_0 \geq \frac{\ln 2}{||\eta||_{\infty}(m-1)}$, we have that $e^{\lambda_0 ||\eta||_{\infty}} - 2e^{\lambda_0 ||\eta||_{\infty}} + e^{\lambda_0(1)} > 0$.
- Therefore, $\rho_4^{-1} \rho_3^{-1} \in L^\infty(0, T)$.
- From $\rho_{4,t} := e^\kappa \rho^{-3/4} \eta - \frac{3}{2} \rho^{-7/4} \eta$ and by taking $\lambda_0$ large enough, we have that $\rho_4 \rho_0^{-1} \in L^\infty(0, T)$.

An estimate with such weights is given in the following result. In the proof, we will use Theorem 3.1 and classical energy estimates.

Proposition 3.4. Under the conditions in Theorem 3.7, the unique strong solution to (2.9) satisfies:

$$
\begin{align*}
&\int_0^T \left[|\gamma'|^2 + \tilde{\mu} (|\varphi_x(-1, t)|^2 + |\varphi_x(1, t)|^2) + \tilde{\mu}^3 (|\gamma|^2 + |\varphi(1, t)|^2) \right] e^{-2\kappa \gamma} dt \\
&+ \int_Q \left[\mu^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \mu|\varphi|^2 + \mu^3|\varphi|^2 \right] e^{-2\kappa \gamma} dx dt + ||\varphi(\cdot, 0)||_{H^1(-1, 1)}^2 + ||\gamma(0)||^2
\end{align*}
$$

for a positive constant $C_2$ depending on $T$, $s$ and $\lambda$, with $s$ and $\lambda$ as in Theorem 3.7.

Proof. It suffices to start from (3.2) and split the left hand side in two parts, respectively corresponding to the restrictions of $\varphi$ to $D_1$ and $D_2$ and the corresponding restrictions of $\gamma$ to $(0, T/2)$ and $(T/2, T)$.

Let us start by proving the following estimate for system (2.9):

$$
\begin{align*}
&\|\gamma\|_{H^1(0,T/2)}^2 + \|\varphi\|_{L^2(0,T/2;H^2(-1, 1))}^2 + \|\varphi_t\|_{L^2(D_1)}^2 \\
&\leq e^{C(1+T)} \left( \| (g_1, g_2) \|_{L^2(0,3T/4;L^2(-1, 1))}^2 \times L^2(0,3T/4) \\
&+ \frac{1}{T^2} \| (\varphi, \gamma) \|_{L^2(3T/4, 2T;L^2(-1, 1))}^2 \times L^2(3T/4, 2T, 3T/4) \right).
\end{align*}
$$

To do that, let us introduce a function $\kappa \in C^1([0, T])$ with $\kappa \equiv 1$ in $[0, T/2]$, $\kappa \equiv 0$ in $[3T/4, T]$ and $|\kappa'| \leq C/T$, for some $C > 0$. Using classical energy estimates for the system satisfied by $(\kappa \varphi, \kappa \gamma)$ (see Proposition 2.3), we obtain:

$$
\|\kappa \gamma\|_{H^1(0,T)}^2 + \|\kappa \varphi\|_{L^2(Q)}^2 \leq e^{C(1+T)} \left( \| (\kappa g_1, \kappa g_2) \|_{L^2(Q)}^2 \times L^2(0, T) + \| (\kappa \varphi, \kappa \gamma') \|_{L^2(Q)}^2 \times L^2(0, T) \right),
$$

which leads to (3.3).

Since the weights are bounded from above and from below, using (3.6) we obtain a first estimate in $D_1$:

$$
\begin{align*}
&\int_0^{T/2} \left[|\gamma'|^2 + \tilde{\mu} (|\varphi_x(-1, t)|^2 + |\varphi_x(1, t)|^2) + \tilde{\mu}^3 (|\gamma|^2 + |\varphi(1, t)|^2) \right] e^{-2\kappa \gamma} dt \\
&+ \int_Q \left[\mu^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \mu|\varphi|^2 + \mu^3|\varphi|^2 \right] e^{-2\kappa \gamma} dx dt + \gamma(0)^2
\end{align*}
$$

(3.7)
where \( C \) is a positive constant depending on \( s, \lambda \) and \( T \).

On the other hand, since \( \alpha = \zeta \) and \( \xi = \mu \) in \( D_2 \), thanks to Theorem 3.4 we have:

\[
\begin{align*}
\int_{D_2} (s\mu)^{-1} \left( |\varphi_t|^2 + |\varphi_{xx}|^2 + \lambda^2 (s\mu) |\varphi_x|^2 + \lambda^4 (s\mu)^3 |\varphi|^2 \right) e^{-2s\zeta} \, dx \, dt \\
+ \int_{T/2}^T \left[ |\gamma_t|^2 + \bar{\mu} \left( |\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2 \right) + \bar{\mu}^3 \left( |\gamma|^2 + |\varphi(1,t)|^2 \right) \right] e^{-2s\hat{\zeta}} \, dt \\
\leq \int_{Q} (s\xi)^{-1} \left( |\varphi_t|^2 + |\varphi_{xx}|^2 + \lambda^2 (s\xi) |\varphi_x|^2 + \lambda^4 (s\xi)^3 |\varphi|^2 \right) e^{-2s\alpha} \, dx \, dt \\
+ \int_0^T \left[ |\gamma_t|^2 + \lambda(s\xi) \left( |\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2 \right) + \lambda^3 (s\xi)^3 \left( |\gamma|^2 + |\varphi(1,t)|^2 \right) \right] e^{-2s\hat{\zeta}} \, dt \\
\leq C_0 \left( \int_{Q} |g_1|^2 e^{-2s\alpha} \, dx \, dt + \int_0^T |g_2|^2 e^{-2s\hat{\zeta}} \, dt + \int_0^T \mu^3 |\varphi|^2 e^{-2s\alpha} \, dx \, dt \right),
\end{align*}
\]

which, combined with (3.7), provides (3.5).

\( \square \)

In the sequel, we will use the notation

\[
C^k_{\rho}([0,T];B) := \{ v : \rho v \in C^k([0,T];B) \} \quad \text{and} \quad W^r,p(0,T;B) := \{ v : \rho v \in W^{r,p}(0,T;B) \}.
\]

Here, it is assumed that \( B \) is a Banach space, \( \rho : [0,T] \mapsto \mathbb{R} \) is a positive measurable function, \( k \in \mathbb{N}, r \in \mathbb{R}_{\geq 0} \) and \( p \in [1, +\infty) \). Accordingly, we set

\[
\|v\|_{C^k_{\rho}([0,T];B)} := \|\rho v\|_{C^k([0,T];B)} \quad \text{and} \quad \|v\|_{W^r,p(0,T;B)} := \|\rho v\|_{W^{r,p}(0,T;B)}.
\]

In particular, when \( B = \mathbb{R} \), we simply write \( C^k_{\rho}([0,T]) \) and \( W^{r,p}(0,T) \); when \( p = 2 \), we use the notation \( H^r(0,T) := W^{r,2}(0,T) \) and \( H^r(0,T) := W^{r,2}(0,T) \).

We will also need the spaces \( H^{1,2}_{\rho}(Q) := \{ v : \rho v \in H^{1,2}(Q) \} \) and \( H^{1,2}_{\rho}(Q) := \{ v : \rho v \in H^{1,2}(Q) \} \), endowed with the norm \( \|v\|_{H^{1,2}_{\rho}(Q)} := \|\rho v\|_{H^{1,2}(Q)} \).

Let us establish the null controllability of (2.2) with a right hand side which decays exponentially as \( t \to T^- \). As we will see in the next section, this will be crucial to deduce the local null controllability of (2.6).

Let us introduce the linear operators

\[
L_1(z,h) := \bar{q}z_t - z_{xx} + \frac{x}{\beta} p_x(1,\cdot)z_x + \frac{x}{\beta} p_x z_x(1,\cdot) + \frac{2}{\beta} h \quad \text{and} \quad L_2(z,h) := h_t + z_x(1,\cdot)
\]

and the space \( E \), given by

\[
E := \{ (z,h,w) \in L^2_{\rho_1}(Q) \times L^2_{\rho_1}(0,T) \times L^2_{\rho_2}(\omega \times (0,T)) : L_1(z,h) - w 1_\omega \in L^2_{\rho_3}(Q), L_2(z,h) \in L^2_{\rho_4}(0,T) \}
\]

\( h \in H^{1,2}_{\rho_1}(0,T) \) and \( z \in H^{1,2}_{\rho_4}(Q) \).
It is clear that $E$ is a Hilbert space for the norm $\| \cdot \|_E$, where

$$
\|(z, h, w)\|_E := \left( \|(z, h, w\omega)\|_{L^2_{\rho_0}(Q) \times L^2_{\rho_1}(0,T) \times L^2_{\rho_2}(Q)} + \|\mathcal{L}_1(z, h) - w\omega\|_{L^2_{\rho_3}(Q)}^2 \\
+ \|\mathcal{L}_2(z, h)\|_{L^2_{\rho_3}(0,T)}^2 + \|h\|_{H^1_{\rho_4}(0,T)}^2 + \|\rho_4\|_{H^1_{\rho_4}(Q)}^2 \right)^{1/2}.
$$

The null controllability of the linearized system is guaranteed by the following result:

**Proposition 3.5.** Assume that $(f_1, f_2) \in L^2_{\rho_0}(Q) \times L^2_{\rho_1}(0,T)$ and that $(z_0, h_0) \in H^1_0(-1, 1) \times \mathbb{R}$. Then, there exists a solution to (2.8) satisfying $(z, h) \in \hat{E}$.

**Proof.** Let us consider the following subspace of $H^{1,2}(Q) \times H^1(0,T)$:

$$
P_0 = \{ (\varphi, \gamma) \in H^{1,2}(Q) \times H^1(0,T) : \varphi(\cdot, -1) = 0, \ \varphi(1, \cdot) - \gamma = -\frac{1}{\beta} \int_{-1}^1 \tilde{p}_x(x, \cdot) x\varphi(x, \cdot) \, dx = 0 \text{ in } (0,T) \}.
$$

Let $A : P_0 \times P_0 \mapsto \mathbb{R}$ be the bilinear form

$$
A((\tilde{\varphi}, \tilde{\gamma}), (\varphi, \gamma)) := \int_0^T \int_\Omega \rho_2^{-2} \tilde{\varphi} \varphi \, dx \, dt + \int_0^T \int_\Omega \rho_0^{-2} \mathcal{L}_1(\tilde{\varphi}, \tilde{\gamma}) \mathcal{L}_1^*(\varphi, \gamma) \, dx \, dt + \int_0^T \int_\Omega \rho_1^{-2} \mathcal{L}_2(\tilde{\varphi}, \tilde{\gamma}) \mathcal{L}_2^*(\varphi, \gamma) \, dt
$$

and let $F : P_0 \mapsto \mathbb{R}$ be the linear form

$$
F(\varphi, \gamma) := \bar{q}(0) \int_0^1 z_0(x) \cdot \varphi(x, 0) \, dx + \beta h_0 \gamma(0) + \int_0^T f_1 \varphi \, dx \, dt + \int_0^T f_2 \gamma \, dt,
$$

where

$$
\mathcal{L}_1^*(\phi, \gamma) := -\tilde{q}_t \varphi - \varphi_{xx} - \frac{x}{\beta} \tilde{p}_x(x, \cdot) \varphi + \frac{1}{\beta} \tilde{p}_x(1, \cdot) \varphi \quad \text{and} \quad \mathcal{L}_2^*(\phi, \gamma) := \gamma_t - \int_{-1}^1 \frac{2}{\beta} \tilde{p}_1(x, \cdot) \varphi(x, \cdot) \, dx.
$$

Note that the observability inequality (3.5) holds for every $(\phi, \kappa) \in P_0$. Consequently, $A(\cdot, \cdot)$ is a scalar product in $P_0$ and there exists $C > 0$ such that, for all $(\varphi, \kappa) \in P_0$, the following estimate holds:

$$
|F(\varphi, \gamma)| \leq C \left( \|z_0\|_{L^2(-1, 1)} + \|h_0\| + \|f_1\|_{L^2_{\rho_2}(Q)} + \|f_2\|_{L^2_{\rho_2}(0,T)} \right) \sqrt{A((\varphi, \gamma), (\varphi, \gamma))}.
$$

In the sequel, we will denote by $P$ the completion of $P_0$ for the scalar product $A$. We will still denote by $A$ and $F$ the corresponding continuous extensions. Note that $P$ can be identified with the Hilbert space

$$
\{(\varphi, \gamma) \in L^2_{loc}(Q_T) \times L^2_{loc}(0,T) : A((\varphi, \gamma), (\varphi, \gamma)) < +\infty, \ \varphi|_{(-1) \times (0,T)} = 0, \ \varphi(1, \cdot) - \gamma = -\frac{1}{\beta} \int_{-1}^1 \tilde{p}_x(x, \cdot) x\varphi(x, \cdot) \, dx = 0 \text{ in } (0,T) \text{ and } (\varphi, \gamma) \text{ satisfies (3.5)} \}.
$$

From Lax-Milgram Theorem, there exists a unique $(\tilde{\varphi}, \tilde{\gamma}) \in P$ such that

$$
A((\tilde{\varphi}, \tilde{\gamma}), (\varphi, \gamma)) = F(\varphi, \gamma) \quad \forall (\varphi, \gamma) \in P.
$$

Let us introduce $(\tilde{\varphi}, \tilde{\gamma}, \tilde{\vartheta})$, with

$$
(\tilde{\varphi}, \tilde{\gamma}) := (\rho_0^{-2} \mathcal{L}_1^*(\tilde{\varphi}, \tilde{\gamma}), \rho_1^{-2} \mathcal{L}_2^*(\tilde{\varphi}, \tilde{\gamma})), \quad \tilde{\vartheta} = -\rho_2^{-2} \tilde{\vartheta}_1 \omega.
$$
From (3.8), we get:

\[ \left\| \int_Q \rho_0^2 |\tilde{z}|^2 \, dx \, dt + \int_0^T \rho_1^2 |\tilde{h}|^2 \, dt + \int_0^T \int_\omega \rho_2^2 |\tilde{w}|^2 \, dx \, dt \right\| = A((\tilde{\varphi}, \tilde{\gamma}), (\tilde{\varphi}, \tilde{\gamma})) = \mathcal{F}(\tilde{\varphi}, \tilde{\gamma}). \]

Therefore, taking into account the continuity of \( \mathcal{F} \), we have:

\[ \left\| \int_Q \rho_0^2 |\tilde{z}|^2 \, dx \, dt + \int_0^T \rho_1^2 |\tilde{h}|^2 \, dt + \int_0^T \int_\omega \rho_2^2 |\tilde{w}|^2 \, dx \, dt \right\| \leq C \left( \|z_0\|^2_{L^2(-1,1)} + \|b_0\|^2 + \|f_1\|^2_{L^2_\rho_3(q)} + \|f_2\|^2_{L^2_\rho_3(0,T)} \right). \quad (3.9) \]

Note that, in particular, \((\tilde{z}, \tilde{h}, \tilde{w})\) is the unique solution by transposition of (2.8) with \( w = \tilde{w} \), see Proposition 2.9. Thanks to the fact that the \( z_0, \tilde{w}, f_1 \) and \( f_2 \) are sufficient regular, Proposition 2.7 guarantees that \((\tilde{z}, \tilde{h})\) is indeed the strong solution of (2.8) in \( H^1_\rho(Q) \times H^1(0,T) \).

Let us finally prove that \((\tilde{z}, \tilde{h}, \tilde{w}) \in E\).

Using (2.8) and (3.9), we can easily check that \( \tilde{z} \in L^2_{\rho_0}(Q), \tilde{h} \in L^2_{\rho_2}(0,T), \tilde{w} \in L^2_{\rho_2}(\omega \times (0,T)), \mathcal{L}_1(\tilde{z}, \tilde{h}) - \tilde{w}1_\omega \in L^2_{\rho_3}(Q) \) and \( \mathcal{L}_2(\tilde{z}, \tilde{h}) \in L^2_{\rho_3}(0,T) \).

It remains to check that \( \tilde{h} \in H^1(0,T) \) and \( \tilde{z} \in H^2_{0,\rho_4}(Q) \). With that purpose, we define \( \hat{z} = \rho_4 \tilde{z} \) and \( \hat{h} = \rho_4 \tilde{h} \).

Then, \((\hat{z}, \hat{h})\) is the solution to the system:

\[
\begin{align*}
\mathcal{L}_1(\hat{z}, \hat{h}) &= (\rho_4 \rho_3^{-1}) \rho_3 f_1 + (\rho_4 \rho_2^{-1}) \rho_2 w_1 \omega + (\rho_4 \rho_0^{-1}) \rho_0 \hat{z} \quad \text{in} \quad Q, \\
\hat{z}(-1, \cdot) &= 0 \quad \text{in} \quad (0,T), \\
\hat{z}(1, \cdot) &= 0 \quad \text{in} \quad (0,T), \\
\hat{z}(\cdot, 0) &= \rho_4(0) z_0 \quad \text{in} \quad (-1, 1), \\
\hat{L}_2(\hat{z}, \hat{h}) &= \rho_4 f_2 + \rho_4 h \quad \text{in} \quad (0,T), \\
\hat{h}(0) &= \rho_4(0) h_0.
\end{align*}
\]

(3.10)

Consequently, thanks to Remark 3.8 and Proposition 2.7, we obtain the desired estimates and \( (z, h, w) \in E \), as desired.

### 3.2 Controllability of the nonlinear system

We now prove the controllability of (2.10) by applying a local inversion theorem. More precisely, we are going to use the following result, whose proof can be found for instance in [ATF87, Chapter 2, p. 107]:

**Theorem 3.6** (Liusternik-Graves’ Theorem). Let \( B_1 \) and \( B_2 \) be two Banach spaces and let \( \Lambda : B_1 \to B_2 \) be of class \( C^1 \) in a neighborhood of \( b_{1,0} \in B_1 \). Assume that \( \Lambda(b_{1,0}) = b_{2,0} \) and \( \Lambda'(b_{1,0}) : B_1 \to B_2 \) is surjective. Then, there exists \( \delta > 0 \) such that, for every \( b_2 \in B_2 \) satisfying \( \|b_2 - b_{2,0}\|_{B_2} \leq \delta \), there exists at least one solution \( b_1 \in B_1 \) to the equation \( \Lambda(b_1) = b_2 \).

We shall apply this result with \( B_1 = E, B_2 = F_1 \times F_2 \) and

\[
\Lambda(z, h, w) = \left( \mathcal{L}_1(z, h) - w_1 \omega + \frac{2}{B} h z_t + \frac{x}{B} z_x(1, \cdot) z_x, \mathcal{L}_2(z, h), z(\cdot, 0), h(0) \right) \tag{3.11}
\]

for every \((z, h, w) \in E \). Here, we have introduced the Hilbert spaces \( F_1 := L^2_{\rho_3}(Q) \times L^2_{\rho_2}(0,T) \) for the right hand sides and \( F_2 := H^1_0(-1,1) \times \mathbb{R} \) for the initial conditions.
Since $\Lambda$ contains linear and bilinear terms and thanks to the definition of $E$, it is not difficult to check that $\Lambda$ is continuous. Indeed, we only have to prove that the bilinear form
\[
((z_1, h_1, w_2), (z_2, h_2, w_2)) \rightarrow \frac{2}{\beta}h_1 z_{2,t} + \frac{x}{\beta}z_{1,x}(1,\cdot)z_{2,x}
\]
is bounded from $E \times E$ to $L^2_\rho(Q)$. This is true because $h_1 \in H^1_\rho(0, T)$ and $z_1, z_2 \in H^{1,2}_\rho(Q)$ and, in particular, we have $\rho_4 h_1 \in H^1(0, T)$, $\rho_4 z_{2,t} \in L^2(Q)$, $z_{1,x}(1,\cdot) \in L^2(0, T)$ and $\rho_4 z_2 \in C^0([0, T]; H^1_\rho(-1, 1))$.

Therefore, $\Lambda \in C^1(B_1; B_2)$.

On the other hand, note that $\Lambda'(0, 0, 0): B_1 \mapsto B_2$ is given by
\[
\Lambda'(0, 0, 0) (z, h, v) = (L_1(z, h, w), L_2(z, h, w), z(\cdot, 0), h(0)) \quad \forall (z, h, v) \in B_1.
\]
In view of the null controllability result for (2.8) given in Proposition 3.5 $\Lambda'(0, 0, 0)$ is surjective.

Consequently, we can apply Theorem 3.6 with these data and the proof of Proposition 2.4 is achieved.

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Appendix A  Proof of Proposition 2.5

Recall that $(\cdot, \cdot)_2$ and $\| \cdot \|_2$ stand for the usual scalar product and norm in $L^2(-1, 1)$. On the other hand, we will denote by $\| \cdot \|_\infty$ the usual norm in $L^\infty(Q)$.

The proof of existence relies on Leray-Schauder’s Fixed-Point Principle (see for instance [Zei86]). For convenience, let us recall this important result:

**Theorem A.1.** Let $B$ be a Banach space and let $\Lambda: B \times [0, 1] \mapsto B$ be a continuous and compact mapping such that

- $\Lambda(x, 0) = 0$ for all $x \in B$.
- There exists $M > 0$ such that, for any pair $(x, \sigma) \in B \times [0, 1]$ satisfying $x = \Lambda(x, \sigma)$, one has $\|x\|_B \leq M$.

Then, there exists at least one fixed-point of the mapping $\Lambda_1: B \mapsto B$, given by
\[
\Lambda_1(x) = \Lambda(x, 1) \quad \forall x \in B.
\]

Let us consider the mapping $\Lambda: B \times [0, 1] \mapsto B$, given by $\Lambda((\hat{\varphi}, \hat{\gamma}), \sigma) = (\varphi, \gamma)$, where
\[
B = \{(\hat{\varphi}, \hat{\gamma}) \in H^{3/4}(0, T; L^2(-1, 1)) \times H^{3/4}(0, T) : \hat{\varphi}(\cdot, T) = \varphi_T, \quad \hat{\gamma}(T) = \gamma_T\}
\]
and $(\varphi, \gamma)$ is the unique solution to
\[
\begin{aligned}
-\bar{q}(t)\varphi_t - \varphi_{xx} - a\varphi_x - b\varphi &= \sigma f & \text{in } Q, \\
\varphi(-1, \cdot) &= 0 & \text{in } (0, T), \\
\varphi(1, t) &= \sigma (\hat{\gamma}(t) + (N(\cdot, t), \hat{\varphi}(\cdot, t))) & \text{in } (0, T), \\
\varphi(T) &= \sigma \varphi_T & \text{in } (-1, 1), \\
\gamma'(t) &= (R(\cdot, t), \varphi(\cdot, t)) + \sigma g(t) & \text{in } (0, T), \\
\gamma(T) &= \sigma \gamma_T.
\end{aligned}
\]

(A.1)
Remark A.2. Since \( H^s(0, T) \to C^{0,s-1/2}([0, T]) \) for any \( s \in (1/2, 1) \), we see that the initial conditions in the definition of \( \mathcal{B} \) make sense.

Remark A.3. The unique solution \((\varphi, \gamma)\) to (A.1) belongs to \( H^{1,2}(Q) \times H^1(0, T) \) and depends continuously in this space with respect to \( \varphi \) and \( \gamma \). Indeed, note that \( \hat{\gamma} \in H^{3/4}(0, T) \) by definition and \( t \mapsto (N(\cdot, t), \hat{\varphi}(\cdot, t))_2 \) belongs to \( H^{3/4}(0, T) \), since \( H^{3/4}(0, T) \) is a Banach algebra (see for instance [DNPV12], though the results date back to [Str67]). Thanks to the compatibility condition (2.11) and the equation satisfied by \( \varphi \), we have

\[
\sigma \varphi_T(1) = \varphi(1, T) = \sigma [\gamma_T + (N(\cdot, T), \varphi_T)_2].
\]

This implies that \((\varphi, \gamma) \in H^{1,2}(Q) \times H^1(0, T)\) and, moreover, there exists a constant \( C > 0 \) such that

\[
\|(\varphi, \gamma)\|_{H^{1,2}(Q) \times H^1(0, T)}^2 \leq \sigma^2 C^{(1+T)} \left( \|(f, g)\|_{L^2(Q) \times L^2(0, T)}^2 + \|(\varphi_T, \gamma_T)\|_{H^{1/2}(-1, 1) \times \mathbb{R}}^2 \right),
\]

whence the continuous dependence is ensured.

As in [BFCMIRM08], we are going to prove that \( \Lambda \) fulfills the assumptions in Theorem [A.1]

- \( \Lambda : \mathcal{B} \times [0, 1] \to \mathcal{B} \) is well-defined and continuous. Indeed, this follows from Remark A.3 and the fact that \( H^{1,2}(Q) \times H^1(0, T) \) is compactly embedded in \( \mathcal{B} \): note that \( H^{s_2}(0, T) \) is compactly embedded in \( H^{s_1}(0, T) \) for any \( s_1 < s_2 \), \( H^{1,2}(Q) \) is continuously embedded in \( H^r(0, T; H^{2(1-r)}(-1, 1)) \) for any \( r \in [0, 1] \) and, also, \( H^r(0, T; H^{2(1-r)}(-1, 1)) \) is compactly embedded in \( H^{3/4}(0, T; L^2(-1, 1)) \) for any \( r \in (3/4, 1) \). We deduce that \( \Lambda \) is compact.

- Obviously, \( \Lambda((\hat{\varphi}, \hat{\gamma}), 0) = (0, 0) \) for all \((\hat{\varphi}, \hat{\gamma}) \in \mathcal{B} \).

- There exists \( C > 0 \) (depending on \( R, N \) and \( \tilde{q} \), but independent of \( \sigma \)) such that, for any \((\varphi, \gamma) \in \mathcal{B} \times [0, 1] \) satisfying \((\varphi, \gamma) = \Lambda((\varphi, \gamma), \sigma)\), one has

\[
\|(\varphi, \gamma)\|_{H^{1,2}(Q) \times H^1(0, T)}^2 \leq e^{C(1+T)} \left( \|(f, g)\|_{L^2(Q) \times L^2(0, T)}^2 + \|(\varphi_T, \gamma_T)\|_{H^{1/2}(-1, 1) \times \mathbb{R}}^2 \right).
\]

Let us prove this. Let \((\varphi, \gamma) \in H^{1,2}(Q) \times H^1(0, T)\) be a solution to

\[
\begin{aligned}
-\tilde{q}(t) \varphi_t - \varphi_{xx} - a \varphi_x - b \varphi &= \sigma f & \text{in } Q, \\
\varphi(-1, \cdot) &= 0 & \text{in } (0, T), \\
\varphi(1, t) &= \sigma [\gamma(t) + (N(\cdot, t), \varphi(\cdot, t))_2] & \text{in } (0, T), \\
\varphi(\cdot, T) &= \sigma \varphi_T & \text{in } (-1, 1), \\
\gamma_t(\cdot) &= (R(\cdot, t), \varphi(\cdot, t))_2 + \sigma g(t) & \text{in } (0, T), \\
\gamma(T) &= \sigma \gamma_T & \gamma(\cdot) \in \mathcal{B}.
\end{aligned}
\]

Let us multiply (A.4) successively by \( \varphi \) and \( \varphi_t \) and let us integrate in \([\tau, T] \times [-1, 1] \), with \( \tau \in [0, T] \). Then, from Young and Cauchy-Schwarz inequalities, we obtain:

\[
\begin{aligned}
\tilde{q}(\tau) \|(\cdot, \tau)\|_2^2 + \int_\tau^T \|\varphi_x(\cdot, t)\|_2^2 \, dt &\leq \sigma^2 \tilde{q}(T) \|(\varphi_T)\|_2^2 + \|f\|_2^2 + 2 \int_\tau^T \varphi(1, t) \varphi_x(1, t) \, dt \\
&\quad + \int_\tau^T (2 + \|a(\cdot, t)\|_\infty^2 + \|b(\cdot, t)\|_\infty^2) \|\varphi(\cdot, t)\|_2^2 \, dt
\end{aligned}
\]

(4.5)
Consequently, combining (A.4), (A.5) and (A.6) holds:
\[ \|H \| H_{2\tau(−1,1)} \leq 2 \int_{\tau}^{T} \varphi_{x}(1,t) \varphi_{t}(1,t) dt + \sigma^{2} \|\varphi_{T,x}\|_{2}^{2} + \frac{2\sigma^{2}}{q_{x}} \|\varphi\|_{2}^{2} + \frac{4}{q_{x}} \int_{\tau}^{T} (\|a(\cdot,t)\|_{2\infty}^{2} + \|b(\cdot,t)\|_{2\infty}^{2}) \|\varphi(\cdot,t)\|_{H_{2\tau(−1,1)}}^{2} dt. \] (A.6)

Consequently, combining (2.10), (A.7), (A.8) and (A.9) and using that the $H^{1}$ norm can be interpolated by the $L^{2}$ and $H^{2}$ norms and the fact that $\|\varphi\|_{\bar{H}^{2}(−1,1)}^{2}$ for any $\|\varphi\|_{\bar{H}^{1}(−1,1)}^{2}$ and $\|\varphi\|_{\bar{H}^{2}(−1,1)}^{2}$ for any $\|\varphi\|_{\bar{H}^{1}(−1,1)}^{2}$ with $s > \frac{1}{2}$ and, also, that the following interpolation inequality holds: $\|\varphi\|_{\bar{H}^{r}(−1,1)}^{2} \leq C_{r} \|\varphi\|_{L^{2}(−1,1)}^{2} \|\varphi\|_{H^{2}}^{2}$ for all $r \in [0, 2]$. This gives
\[ \int_{\tau}^{T} \|\varphi(1,t)\|_{H_{2\tau(−1,1)}}^{2} dt \leq \delta \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt + C_{\delta} \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \] (A.8)

In order to conclude, we have to estimate the boundary terms. The first one can be easily bounded with the help of trace interpolation. Indeed, recall that $|\phi(1)| \leq C_{0,s} \|\phi\|_{\bar{H}^{r}(−1,1)}$ for any $\phi \in H^{s}(−1, 1)$, $|\phi(1)| \leq C_{1,s} \|\phi\|_{\bar{H}^{s+1}(−1,1)}$ for any $\phi \in H^{s+1}(−1, 1)$ with $s > \frac{1}{2}$ and, also, that the following interpolation inequality holds: $\|\phi\|_{\bar{H}^{r}(−1,1)}^{2} \leq C_{r} \|\phi\|_{L^{2}(−1,1)}^{2} \|\phi\|_{H^{2}}^{2}$ for all $r \in [0, 2]$. This gives
\[ \int_{\tau}^{T} |\varphi(1,t)||\varphi_{x}(1,t)| dt \leq \delta \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt + C_{\delta} \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \]

for any $\delta > 0$. For the second boundary term we can use (A.4), (A.5) and obtain:
\[ \int_{\tau}^{T} |\varphi_{x}(1,t)||\varphi_{t}(1,t)| dt = \int_{\tau}^{T} |\varphi_{x}(1,t)| \left| \frac{\partial}{\partial t} (R(\cdot,t), \varphi(\cdot,t))_{2} + \sigma \left| \frac{\partial}{\partial t} (N(\cdot,t), \varphi(\cdot,t))_{2} \right| dt 

\leq \int_{\tau}^{T} |\varphi_{x}(1,t)| \left| \frac{\partial}{\partial t} (R(\cdot,t), \varphi(\cdot,t))_{2} + \sigma \left| \frac{\partial}{\partial t} (N(\cdot,t), \varphi(\cdot,t))_{2} \right| dt 
\]

\[ + \int_{\tau}^{T} |\varphi_{x}(1,t)| \left| \frac{\partial}{\partial t} (N(\cdot,t), \varphi(\cdot,t))_{2} \right| dt \]

\[ \leq \delta \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt + \|g\|_{L^{2}(0,T)}^{2} G_{\delta} \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \]

\[ + \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \]

\[ \leq \delta \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt + \|g\|_{L^{2}(0,T)}^{2} G_{\delta} \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \]

\[ + \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \]

\[ \leq \delta \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt + \|g\|_{L^{2}(0,T)}^{2} G_{\delta} \int_{\tau}^{T} \|\varphi(\cdot,t)\|_{2}^{2} dt \]

Finally, combining (2.10), (A.7), (A.8) and (A.9) and using Gronwall’s inequality, (A.3) is found.

Therefore, in view of the Leray-Schauder’s Fixed Point Theorem, we have that (2.10) possesses at least one solution.

Now, let us see that the solution we have found is unique. Let $(\varphi_{1}, \gamma_{1})$ and $(\varphi_{2}, \gamma_{2})$ be two solutions
Finally, there is no difficult to estimate the boundary terms and apply Gronwall’s inequality to deduce that

$$\Phi(\tau) = 0 \quad \text{at any } \tau \in (0, T)$$

Proceeding as in the previous step, we obtain

$$\int_0^T \left( \| \Phi(\cdot, t) \|^2_{L^2((-1, 1))} + \| \Phi_\tau(\cdot, t) \|^2_{L^2((-1, 1))} \right) dt + \| \Phi(\cdot, \tau) \|^2_{L^2((-1, 1))}$$

$$\leq C \left( \int_0^T (1 + \| a(\cdot, t) \|^2_{L^2} + \| b(\cdot, t) \|^2_{L^2}) \| \Phi(\cdot, t) \|^2_{L^2((-1, 1))} dt \right)$$

$$+ \int_0^T (|\Phi(1, t)| + |\Phi_\tau(1, t)|)|\Phi_x(1, t)| dt.$$  \hspace{1cm} \text{(A.11)}$$

Finally, there is no difficult to estimate the boundary terms and apply Gronwall’s inequality to deduce that

$$\Phi \equiv 0$$

and, consequently, $$\Gamma \equiv 0.$$  

\section*{Appendix B \hspace{1cm} Proof of Lemma 2.6}

For brevity, the Lebesgue integration elements $dx$ and $dt$ will be omitted in this section. On the other hand, $(\cdot, \cdot)$ and $\| \cdot \|$ will stand for the usual scalar product and norm in $L^2(Q).$

The main difficulties in the proof are that we have to work with non-local terms both in the time and the space variables. In order to deal with the nonlocal in time terms, we have started the computations using that the time derivatives do not exhibit nonlocal behavior in time. In addition, we will take advantage of the fact that the nonlocal in space terms are written on the boundary, at $x = 1,$ just where $-\alpha$ and $\xi$ attain their respective minima.

We start by noting that

$$\alpha_x = -\lambda \xi \eta_x,$$

$$\alpha_{xx} = -\lambda^2 \xi \eta_x^2 - \lambda \xi \eta_{xx},$$

$$\alpha_t = -\xi^2 \left[ e^{-2\lambda \eta} - e^{-\lambda(m \| \eta \|_\infty + \eta)} \right] (T - 2t),$$

$$\alpha_{tt} = 2\xi^2 \left[ e^{-2\lambda \eta} - e^{-\lambda(m \| \eta \|_\infty + \eta)} \right]^2 + 2(T - 2t)^2 \xi^3 \left[ e^{-\lambda(m \| \eta \|_\infty + 3\eta)} - e^{-2\lambda(m \| \eta \|_\infty + \eta)} \right].$$  \hspace{1cm} \text{(B.1)}$$

It follows that there exists $C > 0$ such that the following pointwise estimates are satisfied for sufficiently large $\lambda$ at any $(x, t) \in \mathcal{Q}.$

$$|\alpha_t| \leq CT \xi^2,$$

$$|\alpha_{xx}| \leq T \lambda \xi^2,$$

$$|\alpha_{tt}| \leq C(\xi^2 + T^2 \xi^3) \leq CT^2 \xi^3.  \hspace{1cm} \text{(B.2)}$$

We set $w = e^{-\alpha \tau} \psi$ and, using the boundary conditions satisfied by $\psi,$ we observe that $w(-1, \cdot) = 0.$ Also, note that, from the definitions of $\alpha$ and $w,$ we get:

$$\lim_{t \to 0^+} t^{-2}(T - t)^{-2} w(\cdot, t) = \lim_{t \to T^-} t^{-2}(T - t)^{-2} w(\cdot, t) = 0$$

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and
\[ w_x(\cdot, T) = w_x(\cdot, 0) = 0. \]

Let us introduce the partial differential operator \( P := \partial_t + d\partial_{xx} \). We have the following decomposition
\[ e^{-sa} f = e^{-sa} P(e^{sa} w) = P_e w + P_k w, \]
where \( P_e w := dw_{xx} + (sa_t + s^2d^2a_x^2)w \) is the self-adjoint part of \( P \) and \( P_k w := w_t + 2sda_xw_x + sda_{xx}w \) is the skew-adjoint part. It follows that
\[ P_e w + (P_k w - sda_{xx}w) = e^{-sa} f - sda_{xx}w \]
and, consequently,
\[ \|e^{-sa} f - sda_{xx}w\|^2 = \|P_e w\|^2 + \|P_k w - sda_{xx}w\|^2 + 2(P_e w, P_k w - sda_{xx}w). \]

The rest of the proof is devoted to analyzing the term \( (P_e w, P_k w - sda_{xx}w) \). Thus, from the above definition of the operators \( P_e \) and \( P_k \), it follows that
\[ 2(P_e w, P_k w - sda_{xx}w) = 2(dw_{xx}, w_t) + 2(dw_{xx}, 2sda_xw_x) + 2(sa_t w + s^2da_x^2, w_t) + 2(sa_t w + s^2da_x^2, 2sda_xw_x) \]
\[ =: I_1 + I_2 + I_3 + I_4. \]

For the first integral term \( I_1 \), we integrate by parts in space and obtain that
\[ I_1 = -2 \int_Q dw_x w_{xt} + 2 \int_0^T \frac{dw_x w_x}{x=0} \]
\[ = \int_Q d_t w_x^2 - \int_{-1}^1 \frac{dw_x^2}{x=0} + 2 \int_0^T [dw_x w_x]_{x=0}. \]

For the second one, we integrate again by parts in space and deduce that
\[ I_2 = -2s \int_Q d^2a_{xx}|w_x|^2 + 2s \int_0^T [d^2a_x|w_x|^2]_{x=0}. \]

For the third term, we integrate by parts in time. The following is found:
\[ I_3 = -s \int_Q a_t|w|^2 - s^2 \int_Q (da_x^2)|w|^2 + \int_{-1}^1 [(sa_t + s^2da_x^2)|w|^2]_{t=0}. \]

Then, for the fourth term, we see that
\[ I_4 = -\int_Q d(2s^2(a_t a_x) + 6s^3da_x^2a_{xx}) |w|^2 + 2 \int_0^T [d(s^2a_t a_x + 6s^3da_x^3)|w|^2]_{x=0}. \]

From (B.3)-(B.9), we get:
\[ 2(P_e w, P_k w - sda_{xx}w) = \int_Q (-2sd^2a_{xx} + d_t)|w_x|^2 \]
\[ + \int_Q (-sa_t - s^2(da_x^2)_t - 2s^2d(a_xa_t) - 6s^3d^2a_x^2a_{xx}) |w|^2 \]
\[ + \int_0^T [2dw_x w_x + 2sd^2a_x|w_x|^2 + 2s^2dax(a_t + sda_x^2)|w|^2]_{x=0} \]
\[ - \int_{-1}^1 [dw_x^2 - (sa_t + s^2da_x^2)|w|^2]_{t=0} \]
\[ = I_{D1} + I_{D2} + I_{BS} + I_{BT}, \]
where $I_{D1}$ and $I_{D2}$ correspond to distributed terms, $I_{BS}$ is related to the boundary terms and $I_{BT}$ contains initial and final terms. Obviously, $I_{BT} = 0$.

Let us estimate the distributed terms. Thanks to (B.11) and from (2.13), we have

$$ I_{D1} = 2s\lambda^2 \int_Q d^2 \eta^2 \xi |w_x|^2 + 2s\lambda \int_Q d^2 \eta_{xx} \xi |w_x|^2 + \int_Q d_4 |w_x|^2 $$

$$ \geq C \lambda^2 \int_Q \xi |w_x|^2 - C \lambda^2 \int_Q^T \xi |w_x|^2 - C \left( s \lambda \int_Q \xi |w_x|^2 + \int_Q |w_x|^2 \right). $$

Hence, using the fact that $(s\xi)^{-1} \leq 1/(4s_0)$ and $\lambda \geq \lambda_0$ and taking $s_0$ and $\lambda_0$ large enough, we obtain:

$$ C \lambda^2 \int_Q \xi |w_x|^2 + I_{D1} \geq C \lambda^2 \int_Q \xi |w_x|^2. \quad (B.10) $$

Also, in order to get an estimate for $I_{D2}$, we use (2.13), (B.11), (3.2) and the fact that $s \geq s_0(T + T^2)$ and $\lambda \geq \lambda_0$. This gives:

$$ C \lambda^4 \int_Q^T \xi^3 |w|^2 + I_{D2} \geq C \lambda^4 \int_Q \xi^3 |w|^2. \quad (B.11) $$

Finally, let us estimate the integral containing boundary terms. Recalling that $w(-1, \cdot) = 0$ in $(0, T)$, we deduce that:

$$ I_{BS} = 2s^2 \int_0^T \partial_x (\alpha t + s\alpha^2) |w|^2 \bigg|_{x=1} + 2s \int_0^T \left[ d^2 \alpha_x |w_x|^2 \right]_{x=-1} + 2 \int_0^T d w_t w_x \bigg|_{x=1} $$

$$ =: I_{BS1} + I_{BS2} + I_{BS3}. $$

Thanks to (B.11), (B.2) and the fact that $s \geq s_0(T + T^2)$ and $w_t = -s\alpha w + e^{-s\alpha} \psi_t$, we see that

$$ I_{BS1} \geq -2s^3 \lambda^3 \int_0^T d^2 \eta^2 \xi |w|^2 \bigg|_{x=1} - C \lambda \int_0^T \xi^3 |w|^2 \bigg|_{x=1}, $$

$$ I_{BS2} = -2s \lambda \int_0^T d^2 \eta_{xx} \xi |w_x|^2 \bigg|_{x=-1}, $$

$$ I_{BS3} \geq 2 \int_0^T d \psi_t w_x e^{-s\alpha} \bigg|_{x=1} - C \lambda \int_0^T \xi^3 |w|^2 \bigg|_{x=1} - C \lambda \int_0^T \xi |w_x|^2 \bigg|_{x=1}. $$

Using again (2.13), the fact that $(s\xi)^{-1} \leq 1/(4s_0)$ and $\lambda \geq \lambda_0$, taking $s_0$ and $\lambda_0$ large enough and recalling the Cauchy-Schwarz inequality, we find from the previous estimate that

$$ I_{BS} \geq C \int_0^T (s^3 \lambda^3 \xi^3 |w|^2 + s\lambda \xi |w_x|^2) \bigg|_{x=1} + C \lambda \int_0^T \xi |w_x|^2 \bigg|_{x=1} - C \int_0^T (s\lambda \xi)^{-1} e^{-2s\alpha} |\psi_t|^2 \bigg|_{x=1}. \quad (B.13) $$

From (B.13), (B.10), (B.11) and (B.13) and the fact that $(s\xi)^{-1} \leq 1/(4s_0)$ and $\lambda \geq \lambda_0$, taking $s_0$ and $\lambda_0$ large enough, we conclude that

$$ \| P e w \|^2 + \| P k w - s d_4 x w \|^2 $$

$$ + s^3 \lambda^4 \int_Q \xi^3 |w|^2 + s^2 \lambda^2 \int_Q \xi |w_x|^2 + \int_0^T \left( s^3 \lambda^3 \xi^3 |w|^2 + s\lambda \xi |w_x|^2 \right) \bigg|_{x=1} + s\lambda \int_0^T \xi |w_x|^2 \bigg|_{x=-1} $$

$$ \leq C \left( \| e^{-s\alpha} f \|^2_2 + s^3 \lambda^4 \int_0^T \xi^3 |w|^2 + s^2 \lambda^2 \int_0^T \xi |w_x|^2 + \int_0^T (s\lambda \xi)^{-1} e^{-2s\alpha} |\psi_t|^2 \bigg|_{x=1} \right). \quad (B.14) $$
Now, using that $P_c w = w_{xx} + (s\alpha_t + s^2\alpha_x^2)w$, we get:

$$s^{-1} \iint_Q \xi^{-1}|w_{xx}|^2 = s^{-1} \iint_Q \xi^{-1}|P_c w - (s\alpha_t + s^2\alpha_x^2)w|^2 \leq Cs^{-1} \iint_Q \xi^{-1}(|P_c w|^2 + s^2\lambda^2\xi^4|w|^2 + s^4\lambda^4\xi^4|w|^2$$

$$\leq C \left(s^{-1} \iint_Q \xi^{-1}|P_c w|^2 + \iint_Q s^3\lambda^4\xi^3|w|^2\right).$$

(B.15)

We can do the same for $P_k w - s\alpha_{xx} w = w_t + 2s\alpha_x w_x$. Then,

$$s^{-1} \iint_Q \xi^{-1}|w_t|^2 = s^{-1} \iint_Q \xi^{-1}(|P_k w - s\alpha_{xx} w| - 2s\alpha_x w_x|^2$$

$$\leq C s^{-1} \iint_Q \xi^{-1}( |P_k w - s\alpha_{xx} w|^2 + s^2\lambda^2\xi^2|w_x|^2$$

$$\leq C \left(s^{-1} \iint_Q \xi^{-1}|P_k w - s\alpha_{xx} w|^2 + \iint_Q s\lambda^2\xi|w_x|^2 \right).$$

(B.16)

From (B.14), (B.15) and (B.16), by introducing a cut-off function to estimate the local gradient integral and performing the usual integration by parts, the following holds

$$\iint_Q s^{-1} \xi^{-1}(|w_t|^2 + |w_{xx}|^2) + \iint_Q s\lambda^2\xi|w_x|^2 + s^3\lambda^4 \iint_Q \xi^3|w|^2$$

$$+ s\lambda T \iint_0^T \xi|w_x|^2 \big|_{x=1} + \iint_Q \left(s^3\lambda^3\xi^3|w|^2 + s\lambda^2\xi|w_x|^2 \right) ig|_{x=1}$$

$$\leq C \left(\|e^{-s\alpha t} f\|_{L^2(Q)}^2 + s^3\lambda^4 \iint_Q \xi^3|w|^2 + s\lambda^2 \int_0^T \xi^3|w|^2 + s^{-1}\lambda^{-1} \iint_Q \xi^{-1}e^{-2s\alpha t}\psi_t|^2 \big|_{x=1} \right).$$

(B.17)

Notice that $w_x \big|_{x=1} = e^{-s\tilde{\alpha}}\psi_x |_{x=1}$ since $\psi(-1, \cdot) = 0$ and $w_x |_{x=1} = e^{-s\tilde{\alpha}}\psi_x |_{x=1} + s\lambda^2\xi \eta |_{x=1}$. Thus, we can come back to $\psi$ and deduce that

$$I(s, \lambda, \psi) \leq C \left(\iint_Q e^{-2s\alpha t}|f|^2 + s^3\lambda^4 \iint_Q e^{-2s\alpha t}\xi^3|\psi|^2 + s^{-1}\lambda^{-1} \iint_Q \xi^{-1}e^{-2s\alpha t}|\psi_t|^2 \big|_{x=1} \right),$$

(B.18)

where we have set

$$I(s, \lambda, \psi) := \iint_Q e^{-2s\alpha t} \left([s\xi]^{-1}(|\psi_t|^2 + |\psi_{xx}|^2) + s\lambda^2\xi|\psi_x|^2 + s^3\lambda^4\xi^3|\psi|^2 \right) + s^3\lambda^3 \iint_Q e^{-2s\alpha t}\xi^3|\psi|^2 \big|_{x=1}$$

$$+ s\lambda T \iint_0^T e^{-2s\alpha t}\xi|\psi_x|^2 \big|_{x=1} + s\lambda \iint_0^T e^{-2s\alpha t}\xi^3|\psi_x|^2 \big|_{x=1}.$$

To conclude the proof, we have to eliminate the last term in (B.18). Using (2.13), we find that

$$\psi_t \big|_{x=1} = (R(\cdot, t) + N_t(\cdot, t), \psi(\cdot, t) )_2 + (N(\cdot, t), \psi_t(\cdot, t))_2 + g.$$

Then, using the fact that $R \in L^\infty(0, T; L^2(-1, 1))$ and $N \in W^{1, \infty}(0, T; L^2(-1, 1))$ and performing some immediate estimates, we obtain:

$$\iint_0^T (s\lambda\xi)^{-1}e^{-2s\tilde{\alpha}t}\psi_t|^2 \big|_{x=1} \leq C \iint_Q (s\lambda\xi)^{-1}e^{-2s\tilde{\alpha}t}(|\psi_t|^2 + |\psi_x|^2) + \iint_0^T (s\lambda\xi)^{-1}e^{-2s\tilde{\alpha}t}|g|^2.$$
Note that \(\hat{\xi}(t)^{-1} e^{-2s\hat{a}(t)} \leq \xi(x, t)^{-1} e^{-2s\alpha(x, t)}\) for all \((x, t) \in Q\). Accordingly, we have from (B.19) that
\[
\begin{align*}
s^{-1}\lambda^{-1} & \int_{0}^{T} \hat{\xi}^{-1} e^{-2s\hat{a}} |\psi_{t}|^2 \bigg|_{x=1} \leq C s^{-1}\lambda^{-1} \int_{Q} \xi^{-1} e^{-2\alpha} (|\psi|^2 + |\psi_{t}|^2) + s^{-1}\lambda^{-1} \int_{0}^{T} \hat{\xi}^{-1} e^{-2\hat{a}} |g|^2 \\
& \leq C\lambda^{-1} s^{-1} \int_{Q} \xi^{-1} e^{-2\alpha} |\psi_{t}|^2 + \frac{C}{256 s_{0}^{3}\lambda_{0}} s^{3} \lambda^{4} \int_{Q} \xi^{3} e^{-2\alpha} |\psi|^2 \\
& + s^{-1}\lambda^{-1} \int_{0}^{T} \hat{\xi}^{-1} e^{-2\hat{a}} |g|^2.
\end{align*}
\]
This estimate, used together with (B.18) and taking \(s_{0}\) and \(\lambda_{0}\) large enough, leads to (2.15). This ends the proof.

**Appendix C  Proof of Proposition 2.7**

The proof of existence can be achieved via the *Faedo-Galerkin method*. It will be divided into several steps.

1. **Galerkin approximations**

   Let \(\{w_{1}, w_{2}, \ldots\}\) be the “special” basis of \(H_{0}^{1}(-1, 1)\), formed by the eigenfunctions of the Dirichlet Laplacian, orthogonal in this space and orthonormal in \(L^{2}(-1, 1)\). For each \(n \geq 1\), we will look for a pair \((z_{n}, h_{n}) : [0, T] \mapsto H_{0}^{1}(-1, 1) \times \mathbb{R}\) with
   \[
   z_{n}(t) = \sum_{k=1}^{n} a_{n}^{k}(t) w_{k} \quad \text{(C.1)}
   \]
satisfying
   \[
   \begin{cases}
   \tilde{q}(z_{n}', z_{n}) + (z_{n}, x, z_{n})_{2} + (a z_{n}, x, x, z_{n})_{2} + (h_{n}(R, z_{n}) + z_{n, x}(1, \cdot)(N, z_{n})_{2} = (F, z_{n})_{2} & \forall k = 1, \ldots, n \\
   h_{n}'(t) + z_{n, x}(1, t) = G, & (0 \leq t \leq T) \\
   a_{n}^{k}(0) = (z_{0}, w_{k}) & (k = 1, \ldots, n) \quad \text{and} \quad h_{n}(0) = h_{0}. \quad \text{(C.2)}
   \end{cases}
   \]

   Obviously, (C.2)–(C.3) is a Cauchy problem for a first order linear system of ODEs. Consequently, the existence and uniqueness of absolutely continuous functions \(a_{n}^{k}\) and \(h_{n}\) on \([0, T]\) is ensured.

2. **A priori estimates**

   Now, the goal is to get some uniform estimates of the couples \((z_{n}, h_{n})\). To do this, let us multiply the first equation of (C.2) by \(a_{n}^{k}(t)\) and sum from 1 to \(n\). Let us also multiply the second equation by \(h_{n}\). The following is found in \((0, T)\):
   \[
   \begin{cases}
   \tilde{q}(z_{n}', z_{n}) + (z_{n, x}, z_{n, x})_{2} + (a z_{n, x}, x, z_{n})_{2} + h_{n}(R, z_{n})_{2} + z_{n, x}(1, \cdot)(N, z_{n})_{2} = (F, z_{n})_{2}, \\
   h_{n}'(t) h_{n}(t) + z_{n, x}(1, t) h_{n}(t) = G h_{n}(t).
   \end{cases}
   \]

   Summing these identities, we easily obtain:
   \[
   \frac{d}{dt} \left( \tilde{q} \right| z_{n}(t) \left|_{t}^{2} + |h_{n}|_{2}^{2} \right) \leq (\tilde{q}') + |a(\cdot, t)|_{2}^{2} + |R(\cdot, t)|_{2}^{2} + |N(\cdot, t)|_{2}^{2} |z_{n}(t)|_{2}^{2} \\
   + 3|h_{n}(t)|_{2}^{2} + |F(\cdot, t)|_{2}^{2} + |G(t)|^{2}.
   \]
   Let us also multiply the first \(n\) identities of (C.2) by the corresponding \(a_{n, t}^{k}(t)\) and \(\lambda_{k} a_{n}^{k}(t)\) and let us sum from 1 to \(n\). Then,
   \[
   \begin{cases}
   \tilde{q} \left( z_{n}' \right|_{2}^{2} + (z_{n, x}, z_{n, x})_{2} + (a z_{n, x}, x, z_{n})_{2} + h_{n}(R, z_{n})_{2} + z_{n, x}(1, \cdot)(N, z_{n})_{2} = (F, z_{n})_{2}, \\
   \tilde{q}(z_{n, x}', z_{n, x})_{2} + \left| z_{n, x} \right|_{2}^{2} + (a z_{n, x}, x, x, z_{n})_{2} + h_{n}(R, -z_{n, x})_{2} + z_{n, x}(1, \cdot)(N, -z_{n, x})_{2} = (F, -z_{n, x})_{2}.
   \end{cases}
   \]

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Similarly, since

\[
\frac{d}{dt} \left[ \|z_n(t)\|_2^2 + \|z_{n,x}(t)\|_2^2 \right] + \|z_{n,x}(t)\|_2^2 + |z_n(t)|^2 + (\|z_{n,x}(t)\|_2^2 + q\|z_n'(t)\|_2^2 + \|z_{n,xx}(t)\|_2^2) \\
\leq C(1 + \|q\| + \|a(·, t)\|_2^2 + \|R(·, t)\|_2^2 + \|N(·, t)\|_2^2)z_n(t)\|_{H^1_{0,1,1}}^2 + |h_n(t)|^2) \\
+ C(1 + \|N(·, t)\|_2^2)z_n(1, t)^2 + C(\|F(·, t)\|_2^2 + |G(t)|^2) \\
\leq 6\|z_{n,x}(t)\|_2^2 + C(\|F(·, t)\|_2^2 + |G(t)|^2) \\
+ C(1 + \|q\| + \|a(·, t)\|_2^2 + \|R(·, t)\|_2^2 + \|N(·, t)\|_2^2)\|z_n(t)\|_{H^1_{0,1,1}}^2 + |h_n(t)|^2),
\]
whence

\[
\frac{d}{dt} \left[ \|z_n(t)\|_2^2 + \|z_{n,x}(t)\|_2^2 \right] + \|z_{n,x}(t)\|_2^2 + |z_n(t)|^2 + (\|z_{n,x}(t)\|_2^2 + q\|z_n'(t)\|_2^2 + \|z_{n,xx}(t)\|_2^2) \\
\leq C(\|q\| + \|a(·, t)\|_2^2 + \|R(·, t)\|_2^2 + \|N(·, t)\|_2^2)z_n(1, t)^2 + \|z_n(t)\|_{H^1_{0,1,1}}^2 + |h_n(t)|^2.
\]

Then, from Gronwall’s Lemma, we deduce that

\[
\|z_n\|_{L^\infty(0,T;H^1_{0,1,1})}^2 + |h_n|_{L^\infty(0,T)}^2 \leq e^{C(1+T)} \left( \|z_0\|_{H^1_{0,1,1}}^2 + \|f_1\|_{L^2(Q)}^2 + \|f_2\|_{L^2(Q)}^2 \right).
\]
(C.4)

Therefore, one has:

\[
\|z_n\|_{H^1_{0,2}(Q)}^2 + |h_n|_{H^1(0,T)}^2 \leq e^{C(1+T)} \left( \|z_0\|_{H^1_{0,1,1}}^2 + \|F\|_{L^2(Q)}^2 + \|G\|_{L^2(Q)}^2 \right).
\]
(C.5)

3. The existence of a strong solution

Let us take limits in (a subsequence of) the sequence \((z_n, h_n)\).

In view of the a priori estimates \((C.4)\) and \((C.5)\), there exists a subsequence (again indexed by \(n\)) and functions \(h \in L^2(0,T)\) and \(z \in H^1_{0,2}(Q)\) such that

\[
\begin{align*}
z_n & \to z \text{ weakly in } L^2(0,T;H^1_{0,1,1}) \cap H^2(-1,1), \\
z_{n,x}(1,·) & \to z_x(1,·) \text{ weakly in } L^2(0,T), \\
z_n'(·) & \to z(·) \text{ weakly in } L^2(Q), \\
h_n & \to h \text{ weakly in } H^1(0,T).
\end{align*}
\]

Then, following standard and well known arguments, we can deduce that \((z, h)\) satisfies \((2.16)_1, (2.16)_2, (2.16)_3, (2.16)_5\) and \((2.16)_6\).

4. Checking the initial conditions

Thanks to the well known Aubin-Lions’ Lemma, we have that \(H^1_{0,2}(Q)\) is compactly embedded in \(C^0([0,T];L^2(-1,1))\). Then, since \(z_n \to z\) weakly in \(H^1_{0,2}(Q)\), we also have that

\[
z(·,0) = \lim_{n \to \infty} z_n(·,0) = z_0.
\]

Similarly, since \(h_n\) converges weakly in \(H^1(0,T)\), we deduce that

\[
h(0) = \lim_{n \to \infty} h_n(0) = h_0.
\]

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The uniqueness of the solution is an almost direct consequence of energy estimates. Indeed, let \((z_1, h_1)\) and \((z_2, h_2)\) be two solutions (in \(H_0^{1,2}(Q) \times H^1(0,T)\)) to (2.16) and let us set \(z = z_1 - z_2\) and \(h = h_1 - h_2\). Then, \((z, h)\) is a solution to
\[
\begin{align*}
\bar{q}z_t - z_{xx} + az_x + Rh + N z_x(1, \cdot) &= 0 & \text{in } & Q, \\
z(-1, \cdot) &= 0 & \text{in } & (0, T), \\
z(1, \cdot) &= 0 & \text{in } & (0, T), \\
z(\cdot, 0) &= 0 & \text{in } & (-1, 1), \\
h_t + z_x(1, \cdot) &= 0 & \text{in } & (0, T), \\
h(0) &= 0
\end{align*}
\]
and, from well known arguments, this shows that \(z \equiv 0\) and \(h \equiv 0\).

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