AN INVERSE PROBLEM FOR THE STURM-LIOUVILLE PENCIL WITH ARBITRARY ENTIRE FUNCTIONS IN THE BOUNDARY CONDITION

CHUAN-FU YANG
Department of Applied Mathematics, School of Science
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, China

NATALIA PAVLOVNA BONDARENKO
Department of Applied Mathematics and Physics, Samara National Research University
Moskovskoye Shosse 34, Samara 443086, Russia
and
Department of Mechanics and Mathematics, Saratov State University
Astrakhanskaya 83, Saratov 410012, Russia

XIAO-CHUAN XU
School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China

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Abstract. The Sturm-Liouville pencil is studied with arbitrary entire functions of the spectral parameter, contained in one of the boundary conditions. We solve the inverse problem, that consists in recovering the pencil coefficients from a part of the spectrum satisfying some conditions. Our main results are 1) uniqueness, 2) constructive solution, 3) local solvability and stability of the inverse problem. Our method is based on the reduction to the Sturm-Liouville problem without the spectral parameter in the boundary conditions. We use a special vector-functional Riesz-basis for that reduction.

1. Introduction. In this paper, we consider the boundary value problem $L = L(q(x), h)$ for the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in (0, \pi),$$

with the boundary conditions

$$y'(0) - hy(0) = 0,$$

$$f_1(\lambda)y'(\pi) + f_2(\lambda)y(\pi) = 0,$$

where $\lambda$ is the spectral parameter; $q(x)$ is a complex-valued function, called the potential, $q \in L^2(0, \pi)$; $h \in \mathbb{C}$; $f_j(\lambda), j = 1, 2,$ are entire analytic functions.

The paper concerns an inverse spectral problem for $L$. Inverse problems of spectral analysis consist in recovering differential operators from their spectral characteristics. In applications, those spectral characteristics are usually related with
observed data, and operator coefficients being recovered are related with unknown properties of a medium.

The most complete results of inverse problem theory have been obtained for the Sturm-Liouville equations in the form (1.1)-(1.3) with the constant coefficients \( f_1 \) and \( f_2 \) (see, e.g., the monographs [17, 27, 28, 35]). Pencils of differential operators with eigenparameter-dependent boundary conditions are more difficult for investigation, since such problems are rarely self-adjoint. Inverse problems for pencils, having linear dependence on the spectral parameter in the boundary conditions, were studied in [9,10,21,44,46] and many other papers. The case of boundary conditions, containing rational Herglotz-Nevanlinna functions or arbitrary polynomials of \( \lambda \), also has been considered (see [3,4,13,15,18,19,22,33,45]).

In the present paper, we study the differential pencil (1.1)-(1.3) with arbitrary entire functions in the boundary condition. As far as we know, inverse problems for this class of pencils have not been investigated before. The behavior of the spectrum of the problem \( L \) substantially depends on the functions \( f_j(\lambda), j = 1,2 \). However, we assume that a subsequence of the eigenvalues is given, that satisfy some specific properties. This subsequence is used for recovering the potential \( q(x) \) and the coefficient \( h \). Such inverse problem statement generalizes various inverse spectral problems being actively studied in recent years, in particular:

- the Hochstadt-Lieberman problem [10,20,24–26,29,34,36];
- the inverse transmission eigenvalue problem [6,30–32];
- partial inverse problems for Sturm-Liouville operators with discontinuities [23, 38, 41, 42];
- partial inverse problems for quantum graphs [5,7,8,40,43].

The listed problems are applied in mechanics, geophysics, nanotechnology, acoustics and other branches of science and engineering. We discuss applications of our results to these inverse problems in Section 5 in more details.

Let us proceed to the problem formulation. Denote by \( \varphi(x,\lambda) \) the solution of equation (1.1), satisfying the initial conditions \( \varphi(0,\lambda) = 1, \varphi'(0,\lambda) = h \). Clearly, \( \varphi(x,\lambda) \) satisfies the boundary condition (1.2). For each fixed \( x \in [0,\pi] \), the functions \( \varphi^{(\nu)}(x,\lambda), \nu = 0,1, \) are entire in the \( \lambda \)-plane.

The eigenvalues of the problem \( L \) coincide with the zeros of the characteristic function

\[(1.4) \quad \Delta(\lambda) = f_1(\lambda)\varphi'(\pi,\lambda) + f_2(\lambda)\varphi(\pi,\lambda).\]

Let \( \{\lambda_n\}_{n \in \mathbb{N}} \) be a sequence of the eigenvalues of \( L \), satisfying the following conditions.

- (A1) The values \( \{\lambda_n\}_{n \geq 1} \) are distinct and nonzero.
- (A2) For every \( n \in \mathbb{N}, f_1(\lambda_n) \neq 0 \) or \( f_2(\lambda_n) \neq 0 \).

Note that the sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) may include only a part of all the eigenvalues. Suppose that \( \lambda_n \) is a zero of \( \Delta(\lambda) \) of multiplicity at least \( m_n \in \mathbb{N}, n \in \mathbb{N} \). The assumption \( \lambda_n \neq 0, n \in \mathbb{N} \), is imposed for simplicity. The case \( \lambda_n = 0 \) requires minor technical modifications.

Define \( \omega := h + \frac{1}{2} \int_0^{\pi} q(x) \, dx \). We study the following inverse problem.

**Inverse Problem 1.1.** Given eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}}, \) satisfying (A1) and (A2), the corresponding numbers \( \{m_n\}_{n \in \mathbb{N}} \) and \( \omega \), find the potential \( q(x) \) and the coefficient \( h \).
The functions $f_1(\lambda)$ and $f_2(\lambda)$ are supposed to be known a priori. In applications, the constant $\omega$ often can be found from the asymptotics of the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$.

In this paper, we prove the uniqueness theorem, local solvability and stability for Inverse Problem 1.1, and also provide a constructive algorithm for solution of this problem. Our technique develops the ideas of [5, 6, 42]. In order to solve the inverse problem, we reduce it to the classical Sturm-Liouville problem with boundary conditions independent of $\lambda$. For the reduction, a special vector-functional Riesz basis is used. The proof of the Riesz-basicty relies on the theory of entire functions and nonharmonic analysis of exponential sequences in the form $\{\exp(it\theta_n t)\}$. Note that our methods require no self-adjointness. We work with the complex-valued potential $q$ and possibly multiple eigenvalues. To deal with the inverse problem, having eigenparameter independent boundary conditions, we apply the Borg-type theorem, recently proved in [11] for the complex-valued potential.

The paper is organized as follows. In Section 2, the uniqueness theorem for Inverse Problem 1.1 is proved. We introduce a special exponential sequence, related to the given eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ and their multiplicities $\{m_n\}_{n \in \mathbb{N}}$. The proof of the uniqueness theorem is based on the completeness of that sequence and on the theory of entire functions. Section 3 is devoted to a constructive solution of Inverse Problem 1.1. We require the exponential sequence, introduced in Section 2, to be a Riesz basis. A crucial step of our algorithm for solving the inverse problem is recovering a vector-function from its coordinates by that Riesz basis. In Section 4, local solvability and stability are proved for Inverse Problem 1.1 under some additional requirements on the subspectrum $\{\lambda_n\}_{n \in \mathbb{N}}$ and the functions $f_1(\lambda)$ and $f_2(\lambda)$. In Section 5, we discuss applications of our results to various problems of the modern inverse spectral theory.

2. Uniqueness theorem. The goal of this section is to prove the uniqueness theorem for Inverse Problem 1.1. For uniqueness, we need an additional condition on the subspectrum $\{\lambda_n\}_{n \in \mathbb{N}}$. Define $\rho_n = -\rho_{-n} = \sqrt{-\lambda_n}$, $-\pi/2 \leq \arg \rho_n < \pi/2$, $m_{-n} := m_n$, $n \in \mathbb{N}$; $\mathbb{Z}_0 := \mathbb{Z}\setminus\{0\}$. Consider the sequence

$$\mathcal{E} := \{\exp(it\theta_n)\}_{n \in \mathbb{Z}_0, \nu = 0, m_0, \pi/2}.$$

Suppose that the following assumption holds.

(A4) The sequence $\mathcal{E}$ is complete in $L_2(-2\pi, 2\pi)$.

Along with the problem $L$, we consider the boundary value problem $\tilde{L} = L(\tilde{q}(x), h)$ of the same form as $L$, but with different coefficients $\tilde{q}(x)$ and $\tilde{h}$. The entire functions $f_1(\lambda)$ and $f_2(\lambda)$ for $\tilde{L}$ are the same as for $L$. Let us agree that, if a symbol $\gamma$ denotes an object related to $L$, the symbol $\tilde{\gamma}$ with tilde denotes the analogous object related to $\tilde{L}$. Now we are ready to formulate the uniqueness theorem for Inverse Problem 1.1.

**Theorem 2.1.** Suppose that the data $\{\lambda_n, m_n\}_{n \in \mathbb{N}}$ satisfy the assumptions (A1)-(A3), and that $\lambda_n = \tilde{\lambda}_n$, $m_n = \tilde{m}_n$ for $n \in \mathbb{N}$, and $\omega = \tilde{\omega}$. Then $q(x) = \tilde{q}(x)$ for a.a. $x \in (0, \pi)$ and $h = \tilde{h}$. Thus, the data $\{\lambda_n\}_{n \in \mathbb{N}}$, $\{m_n\}_{n \in \mathbb{N}}$ and $\omega$ uniquely specify the potential $q(x)$ and the boundary condition coefficient $h$.

In order to prove Theorem 2.1, we need some standard results. For any real $a > 0$, we denote by $B_{\infty}^2$ the Paley-Wiener class of entire functions of exponential type not greater than $a$, belonging to $L_2(\mathbb{R})$. The following proposition easily follows
from [37, Theorem 1.1]. For convenience, we formulate it directly for the sequence \( \mathcal{E} \), defined above.

**Proposition 2.2.** The sequence \( \mathcal{E} \) is incomplete in \( L_2(-a,a) \) if and only if there exists a function \( F(\rho) \in B^2_{a} \), \( F(\rho) \neq 0 \), such that

\[
\frac{d^n}{d\rho^n} F(\rho) \bigg|_{\rho=\rho_n} = 0, \quad n \in \mathbb{Z}_0, \quad \nu = 0, m_n - 1.
\]

**Lemma 2.3.** The following relations hold:

\[
\varphi(\pi, \lambda) = \cos(\rho \pi) + \frac{\omega \sin(\rho \pi)}{\rho} + \kappa_1(\rho),
\]

\[
\varphi'(\pi, \lambda) = -\rho \sin(\rho \pi) + \omega \cos(\rho \pi) + \kappa_2(\rho),
\]

where \( \rho = \sqrt{\lambda} \), \( \kappa_j \in B^2_{a} \), \( j = 1, 2 \).

One can easily prove Lemma 2.3, by using the transformation operator (see [28]) for the solution \( \varphi(x, \lambda) \).

The function \( M(\lambda) := \frac{\varphi'(\pi, \lambda)}{\varphi(\pi, \lambda)} \) is called the Weyl function of the Sturm-Liouville equation (1.1) with the boundary conditions \( y'(0) - hy(0) = y(\pi) = 0 \). Weyl functions and their generalizations play an important role in the spectral analysis of various differential operators (see [17, 28]). In particular, the following uniqueness result is valid (see [17, Section 1.4]).

**Proposition 2.4.** If \( M(\lambda) \equiv M(\lambda) \), then \( q(x) = \tilde{q}(x) \) for a.a. \( x \in (0, \pi) \) and \( h = \tilde{h} \).

The function \( q(x) \) and the constant \( h \) can be constructively recovered from \( M(\lambda) \) by the method of spectral mappings (see [17]).

**Proof of Theorem 2.1.** Suppose that \( f_1(\lambda_n) \neq 0 \) for some \( n \in \mathbb{N} \). The functions \( \varphi(\pi, \lambda) \) and \( \varphi'(\pi, \lambda) \) cannot both vanish at \( \lambda = \lambda_n \). Consequently, in view of the condition \((A_2)\) and the equality \( \Delta(\lambda_n) = 0 \), we have \( \varphi(\pi, \lambda_n) \neq 0 \).

Using (1.4), we conclude that the function

\[
d(\lambda) := \frac{\Delta(\lambda)}{f_1(\lambda)\varphi(\pi, \lambda)} = \frac{\varphi'(\pi, \lambda)}{\varphi(\pi, \lambda)} + \frac{f_2(\lambda)}{f_1(\lambda)}
\]

has the zero \( \lambda_n \) of multiplicity at least \( m_n \). The same is valid for the function

\[
\tilde{d}(\lambda) := \frac{\tilde{\varphi}'(\pi, \lambda)}{\tilde{\varphi}(\pi, \lambda)} + \frac{f_2(\lambda)}{f_1(\lambda)}.
\]

Considering the difference \( (d(\lambda) - \tilde{d}(\lambda)) \), we prove that the entire function

\[
H(\lambda) := \frac{\varphi'(\pi, \lambda)}{\varphi(\pi, \lambda)} - \varphi(\pi, \lambda)\varphi'(\pi, \lambda)
\]

has the zero \( \lambda_n \) of multiplicity at least \( m_n \). One can show that, if \( f_1(\lambda_n) = 0 \), then \( \varphi(\pi, \lambda_n) = \tilde{\varphi}(\pi, \lambda_n) = 0 \), and \( \lambda = \lambda_n \) is also the zero of \( H(\lambda) \) of multiplicity at least \( m_n \). Consequently, the function \( F(\rho) := H(\rho^2) \) has zeros \( \{\rho_n\}_{n \in \mathbb{Z}_0} \) of multiplicities at least \( \{m_n\}_{n \in \mathbb{Z}_0} \).

Substituting the relations of Lemma 2.3 into (2.1) and using the assertion \( \omega = \tilde{\omega} \), it is easy to check that \( F(\rho) \in B^2_{a} \). Thus, the function \( F(\rho) \) satisfies the conditions of Proposition 2.2 for \( a = 2\pi \). Since the sequence \( \mathcal{E} \) is complete according to \((A_3)\), Proposition 2.2 yields \( H(\lambda) \equiv 0 \). Hence

\[
\varphi'(\pi, \lambda) = \frac{\tilde{\varphi}'(\pi, \lambda)}{\tilde{\varphi}(\pi, \lambda)}
\]
Proposition 2.4 yields that \( q(x) = \tilde{q}(x) \) a.e. on \((0, \pi)\) and \( h = \tilde{h} \).

3. **Constructive solution.** In this section, we develop an algorithm for constructive solution of Inverse Problem 1.1. An important role in our solution is played by the vector-functional sequence \( \{v_n\} \), defined in (3.5). We show that this sequence is complete in an appropriate Hilbert space and, moreover, that it is an unconditional basis. The main equations (3.6) are derived, which allow us to reduce Inverse Problem 1.1 to the classical inverse problem without the spectral parameter in the boundary conditions. Finally, we obtain the constructive Algorithm 3.4 for solving Inverse Problem 1.1.

For simplicity, assume that \( m_n = 1, n \in \mathbb{N} \). The multiplicities of the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}} \) may be greater than 1, but we will not use this fact in our reconstruction procedure. In order to work with multiple eigenvalues, one can use the ideas of the paper [6] and of [5, Remark 2].

Note that the functions \( \varphi_j(\rho), j = 1, 2 \), from Lemma 2.3 have the form

\[
(3.1) \quad \varphi_1(\rho) = -i \int_{-\pi}^{\pi} K(t) \exp(i\rho t) \, dt, \quad \varphi_2(\rho) = \int_{-\pi}^{\pi} N(t) \exp(i\rho t) \, dt,
\]

where \( K, N \in L_2(-\pi, \pi), N(t) = N(-t), K(t) = -K(-t) \).

Substituting (3.1) together with the relations of Lemma 2.3 into (1.4) and taking \( \lambda = \rho_n^2 \), we get

\[
(3.2) \quad f_1(\rho_n^2) \left( -\rho_n \sin(\rho_n \pi) + \omega \cos(\rho_n \pi) + \int_{-\pi}^{\pi} N(t) \exp(i\rho_n t) \, dt \right) \\
+ f_2(\rho_n^2) \left( \cos(\rho_n \pi) + \frac{\omega}{\rho_n} \sin(\rho_n \pi) - \frac{i}{\rho_n} \int_{-\pi}^{\pi} K(t) \exp(i\rho_n t) \, dt \right) = 0, \quad n \in \mathbb{Z}_0.
\]

Let the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}} \) and the number \( \omega \) be given. Note that all the values in (3.2) can be easily constructed by the given data, except the functions \( K(t) \) and \( N(t) \). Consider (3.2) as a system of equations with respect to these functions. It is convenient to rewrite (3.2) in the form

\[
(3.3) \quad f_1(\rho_n^2) \int_{-\pi}^{\pi} N(t) \exp(i\rho_n t) \, dt - \frac{if_2(\rho_n^2)}{\rho_n} \int_{-\pi}^{\pi} K(t) \exp(i\rho_n t) \, dt = w_n, \quad n \in \mathbb{Z}_0,
\]

where

\[
(3.4) \quad w_n = f_1(\rho_n^2)(\rho_n \sin(\rho_n \pi) - \omega \cos(\rho_n \pi)) - f_2(\rho_n^2)(\cos(\rho_n \pi) + \omega \rho_n^{-1} \sin(\rho_n \pi)).
\]

Introduce the complex Hilbert space \( \mathcal{H} := L_2(-\pi, \pi) \oplus L_2(-\pi, \pi) \) of two-element vector-functions with elements from \( L_2(-\pi, \pi) \). The scalar product and the norm in \( \mathcal{H} \) are defined as follows:

\[
(u, v)_{\mathcal{H}} = \int_{-\pi}^{\pi} (u_1(t)v_1(t) + u_2(t)v_2(t)) \, dt, \quad \|u\|_{\mathcal{H}} = \sqrt{(u, u)_{\mathcal{H}}},
\]

\[
u, v \in \mathcal{H}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.
\]

Define the functions

\[
(3.5) \quad u(t) = \begin{bmatrix} N(t) \\ K(t) \end{bmatrix}, \quad v_n(t) = \begin{bmatrix} f_1(\rho_n^2) \\ -i\rho_n^{-1} f_2(\rho_n^2) \end{bmatrix} \exp(i\rho_n t) \quad n \in \mathbb{Z}_0.
\]
Clearly, \( u \in \mathcal{H} \) and \( v_n \in \mathcal{H} \) for \( n \in \mathbb{Z}_0 \). We rewrite the relation (3.3) in the following form:
\[
(u, v_n) = w_n, \quad n \in \mathbb{Z}_0.
\]
We call the relations (3.6) the main equations of Inverse Problem 1.1, since they play a crucial role in our solution.

Let us investigate the properties of the vector-functional sequence \( \{v_n\}_{n \in \mathbb{Z}_0} \).

**Lemma 3.1.** The relation \( v_n = c_n u_n \) holds for \( n \in \mathbb{Z}_0 \), where \( c_n \) are nonzero constants and
\[
(3.7) \quad u_n(t) = \begin{bmatrix} \varphi(\pi, \rho_n^2) \\ \rho_n^{-1} \varphi'(\pi, \rho_n^2) \end{bmatrix} \exp(i\rho_n t), \quad n \in \mathbb{Z}_0.
\]

**Proof.** Using (1.4), (3.5) and (3.7), we derive
\[
\begin{bmatrix} f_1(\rho_n^2) \\ -i \rho_n^{-1} f_2(\rho_n^2) \end{bmatrix} = \rho_n^{-1} \Delta(\rho_n^2) = 0.
\]

The assumption (A2) implies \( v_n \neq 0, \ n \in \mathbb{Z}_0 \). Since the values \( \varphi(\pi, \rho_n^2) \) and \( \varphi'(\pi, \rho_n^2) \) cannot both equal zero, we also have \( u_n \neq 0, \ n \in \mathbb{Z}_0 \). These arguments lead to the assertion of the lemma.

**Lemma 3.2.** Under the assumptions (A1)-(A3), the sequence \( \{v_n\}_{n \in \mathbb{Z}_0} \) is complete in \( \mathcal{H} \).

**Proof.** In view of Lemma 3.1, it is sufficient to prove the completeness of the sequence \( \{u_n\}_{n \in \mathbb{Z}_0} \). Consider an element \( z = \begin{bmatrix} \pi \\ \frac{\pi}{2} \end{bmatrix} \in \mathcal{H} \), such that \( (z, u_n)_\mathcal{H} = 0 \) for all \( n \in \mathbb{Z}_0 \). In the elementwise form, we have
\[
(3.8) \quad \int_{-\pi}^{\pi} (z_1(t)\varphi(\pi, \rho_n^2) + iz_2(t)\rho_n^{-1}\varphi'(\pi, \rho_n^2)) \exp(i\rho_n t) \, dt = 0, \quad n \in \mathbb{Z}_0.
\]

Consequently, the function
\[
Z(\rho) := \int_{-\pi}^{\pi} (z_1(t)\varphi(\pi, \rho^2) + iz_2(t)\rho^{-1}\varphi'(\pi, \rho^2)) \exp(i \rho t) \, dt
\]
has zeros \( \{\rho_n\}_{n \in \mathbb{Z}_0} \). Note that \( Z(\rho) \) can have a singularity at \( \rho = 0 \). However, the function
\[
\tilde{Z}(\rho) := \rho(\rho - \rho_1)^{-1} Z(\rho)
\]
is entire and belongs to \( B_{2\pi}^2 \).

Applying Proposition 2.2 to \( \tilde{Z}(\rho) \) and using the completeness of the sequence \( \mathcal{E} = \{\exp(i\rho_n t)\}_{n \in \mathbb{Z}_0} \), we show that \( Z(\rho) \equiv 0 \). Hence \( z_1(t) = z_2(t) = 0 \) for a.a. \( t \in (-\pi, \pi) \). Thus, the relation (3.8) implies \( z = 0 \) in \( \mathcal{H} \), so the sequences \( \{u_n\}_{n \in \mathbb{Z}_0} \) is complete in \( \mathcal{H} \). Hence \( \{v_n\}_{n \in \mathbb{Z}_0} \) is also complete.

Further we need some additional assumptions on the given part of the spectrum.

(A4) The sequence \( \mathcal{E} = \{\exp(i\rho_n t)\}_{n \in \mathbb{Z}_0} \) is a Riesz basis in \( L_2(-2\pi, 2\pi) \).

(A5) \( \text{Im} \, \rho_n = O(1) \) as \( |n| \to \infty \) and \( \{\rho_n^{-1}\}_{n \in \mathbb{Z}_0} \in l_2 \).

A reader can find more information about Riesz bases in [17, Section 1.8.5], [1] and [14]. Note that (A4) follows from (A4).

**Theorem 3.3.** Under the assumptions (A1), (A2), (A4) and (A5), the sequence \( \{v_n\}_{n \in \mathbb{Z}_0} \) is an unconditional basis in \( \mathcal{H} \), i.e. the normalized sequence \( \{v_n/\|v_n\|_\mathcal{H}\}_{n \in \mathbb{Z}_0} \) is a Riesz basis.
Proof. In order to prove the theorem, it is sufficient to show that \( \{u_n\}_{n \in \mathbb{Z}_0} \) is a Riesz basis in \( \mathcal{H} \). Using Lemma 2.3 and the relation (3.7), we obtain the asymptotics

\[
u_n(t) = u^0_n(t) + O\left(\frac{\exp(2|\text{Im}\rho_n|\pi)}{\rho_n}\right), \quad n \in \mathbb{Z}_0,
\]

where

\[(3.9) \quad u^0_n(t) := \left[\begin{array}{c}
\cos(\rho_n \pi) \\
- i \sin(\rho_n \pi)
\end{array}\right] \exp(i\rho_n t), \quad n \in \mathbb{Z}_0,
\]

and the \( O \)-estimate is uniform with respect to \( t \in [-\pi, \pi] \). Taking \( (A_3) \) into account, we conclude that the sequences \( \{u_n\}_{n \in \mathbb{Z}_0} \) and \( \{u^0_n\}_{n \in \mathbb{Z}_0} \) are quadratically close:

\[
\sum_{n \in \mathbb{Z}_0} \|u_n - u^0_n\|^2_{\mathcal{H}} < \infty.
\]

Let us prove that \( \{u^0_n\}_{n \in \mathbb{Z}_0} \) is a Riesz basis in \( \mathcal{H} \). It is sufficient to show, that this sequence is complete and there exist positive constants \( M_1 \) and \( M_2 \), such that for any sequence \( \{b_n\}_{n \in \mathbb{Z}_0} \) of complex numbers the two-side inequality holds (see [14, Theorem 3.6.6]):

\[(3.10) \quad M_1 \sum_{n \in \mathbb{Z}_0} |b_n|^2 \leq \left\| \sum_{n \in \mathbb{Z}_0} b_n u^n_0 \right\|_{\mathcal{H}}^2 \leq M_2 \sum_{n \in \mathbb{Z}_0} |b_n|^2.
\]

We emphasize that the constants \( M_1 \) and \( M_2 \) do not depend on the sequence \( \{b_n\}_{n \in \mathbb{Z}_0} \).

One can prove completeness of the sequence \( \{u^0_n\}_{n \in \mathbb{Z}_0} \) in \( \mathcal{H} \), by using arguments similar to the proof of Lemma 3.2.

Further we prove the inequality (3.10). Note that

\[
\left\| \sum_{n \in \mathbb{Z}_0} b_n u^n_0 \right\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{Z}_0} \sum_{k \in \mathbb{Z}_0} \overline{b_n} b_k (u^n_0, u^0_k)_{\mathcal{H}}.
\]

Using (3.9), we calculate

\[
(u^0_n, u^0_k)_\mathcal{H} = \left(\cos(\rho_n \pi) \cos(\rho_k \pi) + \sin(\rho_n \pi) \sin(\rho_k \pi)\right) \int_{-\pi}^{\pi} \exp(i(\rho_k - \overline{\rho_n}) t) dt
\]

\[
= \cos((\rho_n - \rho_k) \pi) \frac{\exp(i(\rho_k - \overline{\rho_n}) \pi) - \exp(-i(\rho_k - \overline{\rho_n}) \pi)}{i(\rho_k - \rho_n)}
\]

\[
= \frac{\sin(2(\rho_k - \overline{\rho_n}) \pi)}{\rho_k - \overline{\rho_n}}, \quad n, k \in \mathbb{Z}_0.
\]

Note that, for the functions \( e_n := \exp(i\rho_n t) \), we have

\[
(e_n, e_k)_{L^2(-2\pi, 2\pi)} = \int_{-2\pi}^{2\pi} \exp(i\rho_n t) \exp(i\rho_k t) dt = \frac{2 \sin(2(\rho_k - \overline{\rho_n}) \pi)}{\rho_k - \overline{\rho_n}}, \quad n, k \in \mathbb{Z}_0.
\]

Hence

\[
(u^0_n, u^0_k)_\mathcal{H} = 2(e_n, e_k)_{L^2(-2\pi, 2\pi)}, \quad n, k \in \mathbb{Z}_0.
\]

Consequently, the two-side inequality (3.10) holds for \( \{u^0_n\}_{n \in \mathbb{Z}_0} \) if and only if the similar inequality holds for the sequence \( \mathcal{E} = \{e_n\}_{n \in \mathbb{Z}_0} \) with the norm in \( L^2(-2\pi, 2\pi) \). By virtue of \( (A_4) \), the sequence \( \mathcal{E} \) is a Riesz basis in \( L^2(-2\pi, 2\pi) \), so the two-side inequality for \( \mathcal{E} \) is valid. Therefore \( \{u^0_n\}_{n \in \mathbb{Z}_0} \) is also a Riesz basis.

Thus, the sequence \( \{u_n\}_{n \in \mathbb{Z}_0} \) is complete by Lemma 3.2 and quadratically close to the Riesz basis \( \{u^0_n\}_{n \in \mathbb{Z}_0} \). By virtue of [17, Proposition 1.8.5], the sequence
\( \{u_n\}_{n \in \mathbb{Z}_0} \) is also a Riesz basis. Taking Lemma 3.1 into account, we conclude that \( \{v_n\}_{n \in \mathbb{Z}_0} \) is an unconditional basis in \( \mathcal{H} \).

Using the basisness of the sequence \( \{v_n\}_{n \in \mathbb{Z}_0} \), now we can constructively solve Inverse Problem 1.1 as follows.

**Algorithm 3.4.** Let eigenvalues \( \{\lambda_n\}_{n \in \mathbb{Z}_0} \) of the boundary value problem \( L \) and the constant \( \omega \) be given. Assume that \( (A_1), (A_2), (A_4) \) and \( (A_5) \) are fulfilled. We have to reconstruct the potential \( q(x) \) and the coefficient \( h \) of the boundary condition.

1. Find the numbers \( \rho_n = -\rho_{-n} = \sqrt{\lambda_n} \), \( -\pi/2 \leq \arg \rho_n < \pi/2 \), \( n \in \mathbb{N} \), and then construct the vector-functions \( \{v_n\}_{n \in \mathbb{Z}_0} \) and the numbers \( \{w_n\}_{n \in \mathbb{Z}_0} \) by using (3.5) and (3.4), respectively.
2. Find the vector-function \( u \in \mathcal{H} \) from the main equations (3.6) by the formula

\[
u = \sum_{n \in \mathbb{Z}_0} w_n v_n^*,\]

where \( \{v_n^*\}_{n \in \mathbb{Z}_0} \) is the sequence biorthonormal to \( \{v_n\}_{n \in \mathbb{Z}_0} \).
3. Recall that \( u(t) = \left[ \frac{N(t)}{K(t)} \right] \). Using the functions \( N \) and \( K \), construct \( \varphi(\pi, \lambda) \) and \( \varphi'(\pi, \lambda) \) by the formulas

\[
\varphi(\pi, \lambda) = \cos(\rho \pi) + \frac{\omega \sin(\rho \pi)}{\rho} - \frac{i}{\rho} \int_{-\pi}^{\pi} K(t) \exp(i \rho t) \, dt,
\]

\[
\varphi'(\pi, \lambda) = -\rho \sin(\rho \pi) + \omega \cos(\rho \pi) + \int_{-\pi}^{\pi} N(t) \exp(i \rho t) \, dt.
\]

4. Recover the coefficients \( q(x) \) and \( h \) from the Weyl function \( M(\lambda) = \frac{\varphi'(\pi, \lambda)}{\varphi(\pi, \lambda)} \), using the method of spectral mappings (see [17]).

4. **Local solvability and stability.** In this section, local solvability and stability of Inverse Problem 1.1 are investigated. We consider a small perturbation of the subspectrum \( \{\lambda_n\}_{n \in \mathbb{N}} \) of the problem \( L \). It is shown that, by applying Algorithm 3.4 to the perturbed sequence \( \{\tilde{\lambda}_n\}_{n \in \mathbb{N}} \), one can construct another problem \( \tilde{L} = L(\tilde{a}(x), \tilde{h}) \). The numbers \( \{\tilde{\lambda}_n\}_{n \in \mathbb{N}} \) will be among the eigenvalues of \( \tilde{L} \), and the coefficients \( \tilde{q} \) and \( \tilde{h} \) will be sufficiently close to \( q \) and \( h \), respectively, in an appropriate sense. Our results are rigorously formulated in Theorem 4.1 below. The results of such type for the standard Sturm-Liouville inverse problems are provided in the book [17] (see Theorems 1.6.4, 1.6.5, 1.8.1).

In order to study local solvability and stability, we impose the additional assumption on the functions \( f_j(\lambda) \), \( j = 1, 2 \).

(A6) For \( |\text{Im} \rho| \leq r_0 \), the following estimates are valid:

\[
|f_1(\rho^2)| \leq C|\rho|^\alpha, \quad |f_2(\rho^2)| \leq C|\rho|^{\alpha+1}.
\]

Furthermore,

\[
|f_1(\rho_n^2)|^2 + |\rho_n^{-1}f_2(\rho_n^2)|^2 \geq r_1|\rho_n|^{2\alpha}, \quad n \in \mathbb{Z}_0.
\]

Here \( C, r_0, r_1 \) and \( \alpha \) are some real constants, \( C, r_0, r_1 > 0 \).
The assumption \((A_6)\) is natural for applications. The estimate \((4.2)\) together with \((3.5)\) imply
\[
\|v_n\|_{H} \geq c_0|\rho|^n, \quad n \in \mathbb{Z}_0, \quad c_0 > 0.
\]

For \(j = 0, 1\), denote by \(L_j = L_j(q(x), h)\) the boundary value problem for equation \((1.1)\) with the boundary conditions \((1.2)\) and \(y^{(j)}(\pi) = 0\). Clearly, the problem \(L_j\) has the characteristic function \(\eta_j(\lambda) := \varphi^{(j)}(\pi, \lambda)\). It is well-known (see, e.g., [17]), that the functions \(\eta_j(\lambda), \ j = 0, 1\), have countable sets of zeros. Denote by \(\{\theta_{nj}\}_{n \geq 0}\) the zeros of \(\eta_j(\lambda)\) counted with their multiplicities. The numbers \(\{\theta_{nj}\}_{n \geq 0}\) are the eigenvalues of \(L_j\). Choose among \(\{\theta_{nj}\}_{n \geq 0, j = 0, 1}\) the eigenvalue with the largest multiplicity and denote this multiplicity by \(p\). Note that \(p < \infty\), since the eigenvalues of \(L_j\) are simple for large \(n\). In the special case, when \(q\) and \(h\) are real, we have \(p = 1\).

The next theorem is the main result of this section.

**Theorem 4.1.** Let the problem \(L = L(q(x), h)\) and its eigenvalues \(\{\lambda_n\}_{n \in \mathbb{N}}\) be fixed and satisfy the assumptions \((A_1)\), \((A_2)\), \((A_4)-(A_6)\). Then there exists \(\varepsilon > 0\), such that for any sequence \(\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}\), satisfying the estimate
\[
\Omega := \left(\sum_{n \in \mathbb{N}} |\rho_n|^2|\rho_n - \tilde{\rho}_n|^2\right)^{1/2} \leq \varepsilon, \quad \tilde{\rho}_n := \sqrt{\lambda_n},
\]

there exist a function \(\tilde{q} \in L_2(0, \pi)\) and a number \(\tilde{h} \in \mathbb{C}\), such that \(\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}\) are eigenvalues of the problem \(\tilde{L} = L(\tilde{q}(x), \tilde{h})\) and
\[
\|q - \tilde{q}\|_{L_2} \leq C\Omega^{1/p}, \quad |h - \tilde{h}| \leq C\Omega^{1/p},
\]

where the constant \(C\) depends only on the problem \(L\) and does not depend on the choice of \(\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}\).

Note that the problem \(\tilde{L}\) in the statement of Theorem 4.1 can have other eigenvalues together with \(\{\tilde{\lambda}_n\}_{n \in \mathbb{Z}_0}\).

Fix the problem \(L\) and its subspectrum \(\{\lambda_n\}_{n \in \mathbb{N}}\), satisfying the conditions of Theorem 4.1. In order to prove the theorem, we will apply Algorithm 3.4 to the data \(\{\{\lambda_n\}_{n \in \mathbb{N}}, \omega\}\). If a symbol \(\gamma\) denotes an object related to the problem \(L\), further we will denote by \(\tilde{\gamma}\) with tilde the similar object, constructed by \(\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}\) and \(\omega\). The same symbol \(C\) will be used for various positive constants, depending only on \(L\).

**Lemma 4.2.** There exists \(\varepsilon > 0\), such that for any sequence \(\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}\), satisfying \((4.4)\), the following estimates are valid:
\[
\left(\sum_{n \in \mathbb{Z}_0} \|v_n\|^2_H \|v_n - \tilde{v}_n\|^2_H\right)^{1/2} \leq C\Omega, \quad \left(\sum_{n \in \mathbb{Z}_0} \|v_n\|^2_H \|w_n - \tilde{w}_n\|^2_H\right)^{1/2} \leq C\Omega.
\]

**Proof.** We apply the standard approach, based on Schwarz’s Lemma (see [17, Section 1.6.1]). Taking the assumptions \((A_5)\) and \((A_6)\) into account, we obtain the estimates for all \(n \in \mathbb{Z}_0:\)
\[
|f_j(\rho_n^2) - f_j(\tilde{\rho}_n^2)| \leq C|\rho_n^2 - \tilde{\rho}_n^2|, \quad j = 1, 2,
\]
For every Riesz basis \(\{\xi_n\}\) of \([17, \text{Proposition 1.8.3}]\) and \([5, \text{Lemma 5}]\). Let \(162\) Chuan-Fu Yang, Natalia Pavlovna Bondarenko and Xiao-Chuan Xu

For the constructed functions \(\tilde{\lambda}_n\), there exists a unique element \(\tilde{\lambda}_n\) in \(B\) for all \(n\). Then for any sequence \(\{\tilde{T}_n\}\), such that

\[
\tilde{T} := \left( \sum_n |\tilde{T}_n - \tilde{T}|^2 \right)^{1/2} \leq \varepsilon,
\]

there exists a unique \(\tilde{T} \in B\), such that \(\tilde{T}_n = (\tilde{T}, \tilde{\xi}_n)_B\) for all \(n\). Moreover,

\[
|\tilde{T} - \tilde{T}|_B \leq C(X + T),
\]

where the constant \(C\) depends only on \(\{\xi_n\}\) and \(\tau\).

Using Proposition 4.3 together with Lemma 4.2 and the main equations (3.6), we arrive at the following assertion.

**Lemma 4.4.** There exists \(\varepsilon > 0\), such that for any sequence \(\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}\), satisfying (4.4), there exists a unique element \(\tilde{u} \in \mathcal{H}\), satisfying the relations \((\tilde{u}, \tilde{v}_n)_\mathcal{H} = \tilde{w}_n\), \(n \in \mathbb{Z}_0\). Moreover, \(\|u - \tilde{u}\|_{\mathcal{H}} \leq C\Omega\).

Represent \(\tilde{u}\) from Lemma 4.4 in the form

\[
\tilde{u}(t) = \begin{bmatrix} \tilde{N}(t) \\ \tilde{K}(t) \end{bmatrix}, \tilde{N}, \tilde{K} \in L^2(-\pi, \pi).
\]

Then

\[
\|K - \tilde{K}\|_{L^2} \leq C\Omega, \quad \|N - \tilde{N}\|_{L^2} \leq C\Omega,
\]

and the following relation holds, similar to (3.2):

\[
(9) \quad f_1(\tilde{\rho}_n^2) \left(-\tilde{\rho}_n \sin(\tilde{\rho}_n \pi) + \omega \cos(\tilde{\rho}_n \pi) + \int_{-\pi}^{\pi} \tilde{N}(t) \exp(i\tilde{\rho}_n t) \, dt \right)
+ f_2(\tilde{\rho}_n^2) \left(\cos(\tilde{\rho}_n \pi) \frac{\omega}{\tilde{\rho}_n} \sin(\tilde{\rho}_n \pi) - i \frac{i}{\tilde{\rho}_n} \int_{-\pi}^{\pi} \tilde{K}(t) \exp(i\tilde{\rho}_n t) \, dt \right) = 0, \quad n \in \mathbb{Z}_0.
\]

**Lemma 4.5.** For the constructed functions \(\tilde{K}(t)\) and \(\tilde{N}(t)\), we have \(\tilde{K}(t) = -\tilde{K}(-t)\), \(\tilde{N}(t) = \tilde{N}(-t)\), for a.a. \(t \in (-\pi, \pi)\).
Proof. Since $\rho_n = -\rho_{-n}$, $n \in \mathbb{Z}_0$, the functions $\tilde{N}(-t)$ and $-\tilde{K}(-t)$ satisfy (4.9). Consequently, $\tilde{u}^-(t) := \begin{bmatrix} \tilde{N}(-t) \\ -\tilde{K}(-t) \end{bmatrix}$ is a solution of the system $(\tilde{u}^-, \tilde{v}_n) = \tilde{w}_n$, $n \in \mathbb{Z}_0$. But, by virtue of Lemma 4.4, the solution of this system is unique in $\mathcal{H}$, Hence $\tilde{u} = \tilde{u}^-$, and the lemma is proved. \hfill \Box

Define the functions

\begin{equation}
\tilde{\eta}_0(\lambda) := \cos(\rho \pi) + \frac{\omega \sin(\rho \pi)}{\rho} - \frac{i}{\rho} \int_{-\pi}^{\pi} \tilde{K}(t) \exp(i\rho t) \, dt,
\end{equation}

\begin{equation}
\tilde{\eta}_1(\lambda) := -\rho \sin(\rho \pi) + \omega \cos(\rho \pi) + \int_{-\pi}^{\pi} \tilde{N}(t) \exp(i\rho t) \, dt.
\end{equation}

It follows from Lemma 4.5, that $\tilde{\eta}_j(\lambda)$ are entire functions of $\lambda$. One can easily show that, for $j = 0, 1$, $\tilde{\eta}_j(\lambda)$ have countable sets of zeros $\{\tilde{\theta}_{nj}\}_{n \geq 0}$.

**Lemma 4.6.** There exists $\varepsilon > 0$, such that for any complex-valued functions $\tilde{K}$ and $\tilde{N}$ from $L_2(-\pi, \pi)$, satisfying the estimate

$$\Xi := \max\{\|K - \tilde{K}\|_{L_2}, \|N - \tilde{N}\|_{L_2}\} \leq \varepsilon,$$

the zeros of the functions $\eta_j(\lambda)$ and $\tilde{\eta}_j(\lambda)$, $j = 0, 1$, can be numbered according to their multiplicities, so that

$$\left(\sum_{n=0}^{\infty} |\theta_{nj} - \tilde{\theta}_{nj}|^2\right)^{1/2} \leq C\Xi^{1/p}.$$  

**Proof.** One can prove this lemma similarly to [6, Lemma 3] and [42, Lemma 5], relying on the relations (3.11), (3.12), (4.10), (4.11). The only difference is, that the functions $K(t)$ and $N(t)$ are complex-valued, so the functions $\eta_j(\lambda)$, $j = 0, 1$, may have multiple zeros. However, there is only a finite number of multiple zeros, so it is sufficient to prove the following assertion.

For definiteness, consider $j = 0$. The case $j = 1$ is similar. Suppose that the function $\eta_0(\lambda)$ has a zero $\theta_0$ of multiplicity $r$. For definiteness, assume that $\theta_0 \neq 0$. Consider the contour $|\lambda - \theta_0| = R$ ($R > 0$), containing no other zeros of the function $\lambda \eta_0(\lambda)$ inside or on the boundary. Define the disk $D_R := \{\lambda \in \mathbb{C}: |\lambda - \theta_0| = R\}$. The relations (3.11) and (4.10) imply

$$|\eta^{(j)}_0(\lambda) - \tilde{\eta}^{(j)}_0(\lambda)| \leq C\|K - \tilde{K}\|_{L_2}, \quad \lambda \in D_R, \quad j = 1, r,$$

where the constant $C$ depends only on $K$, $\theta_0$ and $R$. Clearly, $|\eta_0(\lambda)| \geq C$ for $|\lambda - \theta_0| = R$. Applying Rouche’s Theorem, we conclude that the function $\tilde{\eta}_0(\lambda)$ has the same number of zeros (counting with multiplicities) as $\eta_0(\lambda)$ inside $D_R$, if $\|K - \tilde{K}\|_{L_2} \leq \varepsilon$ for sufficiently small $\varepsilon > 0$. Denote these zeros of $\eta_0(\lambda)$ by $\{\theta_j\}_{j=1}^{m}$ and the corresponding multiplicities by $\{r_j\}_{j=1}^{m}$, $\sum_{j=1}^{m} r_j = r$. In order to prove the estimate (4.12), it is sufficient to show that

$$|\theta_j - \theta_0| \leq C\|K - \tilde{K}\|^{1/r}_{L_2}, \quad j = 1, m.$$

For definiteness, we choose $\theta_1$. Consider the Taylor expansion

$$0 = \tilde{\eta}_0(\theta_1) = \tilde{\eta}_0(\theta_0) + \tilde{\eta}'_0(\theta_0)(\theta_1 - \theta_0) + \ldots + \frac{\tilde{\eta}^{(r-1)}_0(\theta_0)}{(r-1)!}(\theta_1 - \theta_0)^{r-1} + \frac{\tilde{\eta}^{(r)}_0(\theta_0)}{r!}(\theta_1 - \theta_0)^r, \quad \theta_* \in D_R.$$
Since $\theta_0$ is the zero of $\eta_0(\lambda)$ of multiplicity $r$, we have $v_0^{(j)}(\theta_0) = 0$ for $j = 0, r - 1$ and $|\eta_0^{(r)}(\lambda)| \geq C$ for $\lambda \in D_N$, where $C$ does not depend on $\lambda$. Using (4.13), we conclude that $|v_0^{(j)}(\theta_0)| \leq C\|K - K\|_{L^2}$, $j = 0, r - 1$, and $|\eta_0^{(r)}(\theta_0)| \geq C$. Substituting the latter estimates into (4.15), we derive (4.14) for $j = 1$.

Since $r \leq p$, we use (4.14) to obtain (4.12).

Proposition 4.7. For the boundary value problems $L_j$, $j = 0, 1$, there exists $\varepsilon > 0$ (which depends on $L_j$) such that if complex numbers $\{\tilde{\theta}_{n_j}\}_{n \geq 0}$, $j = 0, 1$, satisfy the condition

$$\Theta := \left( \sum_{n=0}^{\infty} |\theta_{n_0} - \tilde{\theta}_{n_0}|^2 \right)^{1/2} \leq \varepsilon,$$

then there exists a unique pair $q(x) \in L_2(0, \pi)$ and $h \in \mathbb{C}$, for which the numbers $\{\tilde{\theta}_{n_j}\}_{n \geq 0}$ are the eigenvalues of $L_j = L_j(\tilde{q}(x), \tilde{h})$, $j = 0, 1$. Moreover,

$$\|q - \tilde{q}\|_{L^2} \leq C\Theta, \quad |h - \tilde{h}| \leq C\Theta,$$

where the constant $C$ depends only on $L_j$, $j = 0, 1$.

Proposition 4.7 establishes local solvability and stability of the inverse Sturm-Liouville problem by the two spectra $\{\theta_{n_j}\}_{n \geq 0}$, $j = 0, 1$. For a real-valued potential $q(x)$ and $h \in \mathbb{R}$, Proposition 4.7 is the classical result by Borg (see [17, Theorem 1.8.1]). In the complex case, the similar proposition has been recently proved by Buterin and Kuznetsova (see [11]) for the Dirichlet boundary condition $y(0) = 0$ at the left. The case of the Robin boundary condition (1.2) has no principal differences. Note that, in the complex case, some of the eigenvalues $\{\theta_{n_j}\}_{n \geq 0}$ can be multiple, and they can split under a small perturbation. This case is included in Proposition 4.7.

Proof of Theorem 4.1. Lemma 4.6, Proposition 4.7 and the other arguments above yield the following conclusion. There exists $\varepsilon > 0$ (depending on $L$ and $\{\lambda_n\}_{n \in \mathbb{N}}$), such that for any sequence $\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}$, satisfying the condition $\Omega \leq \varepsilon$, there exist unique functions $\tilde{K}(x)$ and $\tilde{N}(x)$ from $L_2(-\pi, \pi)$, satisfying (4.9). Moreover, there exist a unique pair $(\tilde{q}(x), \tilde{h})$, such that the functions $\tilde{\eta}_j(\lambda)$, defined by (4.10) and (4.11), are the characteristic functions of the problems $\tilde{L}_j = L_j(\tilde{q}(x), \tilde{h})$, $j = 0, 1$. In addition, the estimate (4.5) holds.

Consider the boundary value problem $\tilde{L} := L(\tilde{q}(x), \tilde{h})$. It remains to prove that the numbers $\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}$ are eigenvalues of $\tilde{L}$. Define the solution $\tilde{\varphi}(x, \lambda)$ of the initial value problem

$$-\tilde{\varphi}''(x, \lambda) + \tilde{q}(x)\tilde{\varphi}(x, \lambda) = \lambda\tilde{\varphi}(x, \lambda), \quad \tilde{\varphi}(0, \lambda) = 0, \quad \tilde{\varphi}'(0, \lambda) = \tilde{h}.$$ 

Obviously, $\tilde{\eta}_j(\lambda) = \tilde{\varphi}^{(j)}(\pi, \lambda)$, $j = 0, 1$. In view of (4.9), (4.10) and (4.11), the following relation holds:

$$f_1(\tilde{\lambda}_n)\tilde{\varphi}'(\pi, \tilde{\lambda}_n) + f_2(\tilde{\lambda}_n)\tilde{\varphi}(\pi, \tilde{\lambda}_n) = 0, \quad n \in \mathbb{N}.$$ 

Thus, the characteristic function

$$\tilde{\Delta}(\lambda) = f_1(\lambda)\tilde{\varphi}'(\pi, \lambda) + f_2(\lambda)\tilde{\varphi}(\pi, \lambda)$$

of the constructed problem $\tilde{L}$ has zeros $\{\tilde{\lambda}_n\}_{n \in \mathbb{N}}$ ($\tilde{\Delta}(\lambda)$ can also have other zeros). Theorem 4.1 is proved.
5. **Applications.** In this section, we transform several important problems of spectral theory to the form similar to (1.1)-(1.3). In some applications, the Dirichlet boundary condition \( y(0) = 0 \) is imposed instead of the Robin condition (1.2). However, there is no significant difference between these two cases.

5.1. **Hochstadt-Lieberman problem.** Consider the boundary value problem

\[
\Delta(\lambda) = \psi\left(\frac{\pi}{2}, \lambda\right) \varphi'\left(\frac{\pi}{2}, \lambda\right) - \psi'\left(\frac{\pi}{2}, \lambda\right) \varphi\left(\frac{\pi}{2}, \lambda\right).
\]

The functions \( \psi\left(\frac{\pi}{2}, \lambda\right) \) and \( \psi'\left(\frac{\pi}{2}, \lambda\right) \) can be easily determined by the known potential \( q(x) \) on \( (\pi/2, \pi) \).

Thus, instead of the half-inverse problem for (5.1), one can study the equivalent problem for equation (1.1) on \( (0, \pi/2) \) with the boundary conditions (1.2) and

\[
f_1(\lambda)\varphi'\left(\frac{\pi}{2}, \lambda\right) + f_2(\lambda)\varphi\left(\frac{\pi}{2}, \lambda\right) = 0,
\]

where the entire functions \( f_1(\lambda) = \psi\left(\frac{\pi}{2}, \lambda\right) \) and \( f_2(\lambda) = -\psi'\left(\frac{\pi}{2}, \lambda\right) \) are known a priori. Consequently, the results of this paper can be adapted to the Hochstadt-Lieberman problem, even for the case of complex coefficients \( q(x) \), \( h \) and \( H \).

The similar problems with the potential known on the interval \( (a, \pi/2) \) for arbitrary \( a \in (0, \pi/2) \) have been studied in [20, 25] and other papers. Our results also can be applied to this case.

5.2. **Inverse transmission eigenvalue problem.** The transmission eigenvalue problem has the form

\[
\Delta(\lambda) = \psi\left(\frac{\pi}{2}, \lambda\right) \varphi'\left(\frac{\pi}{2}, \lambda\right) - \psi'\left(\frac{\pi}{2}, \lambda\right) \varphi\left(\frac{\pi}{2}, \lambda\right).
\]

The problem (5.2) attracted much attention of both mathematicians and physicists in connection with the inverse acoustic scattering problem (see [6,12,30–32,39] and references therein). In particular, McLaughlin and Polyakov [30] stated the problem of recovering the potential on the interval \( \left(0, \frac{a-1}{2}\right) \), \( a \neq 1 \), by a part of the spectrum, called an almost real subspectrum. In [30], the uniqueness of solution for
this problem has been established. Further the theory of the McLaughlin-Polyakov inverse problem has been developed in the studies [6,31,32]. The majority of those results are particular corollaries of the general approach of the present paper.

5.3. Partial inverse problems for Sturm-Liouville operators with discontinuities. Consider the discontinuous problem for the Sturm-Liouville equation

\[-y'' + q(x)y = \lambda y, \quad x \in (0, 1),\]

with the boundary conditions

\[y'(0) - h_1 y(0) = y'(1) + h_2 y(1) = 0\]

and with the jump conditions

\[y(d^+) = a_1 y(d^-), \quad y'(d^+) = a_1^{-1} y'(d^-) + a_2 y(d^-),\]

where \(\lambda\) is the spectral parameter; \(q \in L^2(0, 1)\) is a real-valued function; \(0 < d \leq 1/2\) is the discontinuity position; \(a_j\) and \(h_j\) are real numbers for \(j = 1, 2, a_1 > 0\).

Discontinuous Sturm-Liouville problems arise in geophysical models for oscillations of the Earth and in other applications (see, e.g., [17, Section 4.4]). Suppose that the potential \(q(x)\) is known on \((a, 1), a \leq d\), and our goal is to recover the potential on the remaining part of the interval by a part of the spectrum. Such partial inverse problems have been studied in [23,38,41,42].

Similarly to the Hochstadt-Lieberman problem, discontinuous partial inverse problems can be reduced to the form with entire functions in the right-hand side boundary condition. Uniqueness theorems, constructive algorithms, local solvability and stability for these problems can be easily obtained from the results of the present paper.

5.4. Partial inverse problems for quantum graphs. Consider the system of Sturm-Liouville equations

\[-y''_j(x_j) + q_j(x_j)y_j(x_j) = \lambda y_j(x_j), \quad x_j \in (0, \pi), \quad j = 1, m,\]

with the matching conditions

\[y_1(\pi) = y_j(\pi), \quad j = 2, m, \quad \sum_{j=1}^{m} y'_j(\pi) = 0,\]

\[y_j(0) = 0, \quad j = 1, m.\]

The potentials \(q_j\) belong to \(L^2(0, \pi), j = 1, m\). The system (5.3)-(5.5) corresponds to the Sturm-Liouville operator on a star-shaped graph (see [5,40]).

Differential operators on geometrical graphs are also called quantum graphs. There is an extensive literature, devoted to such operators (see, e.g., [2,16,47] and references therein). Quantum graphs have applications in organic chemistry, mesoscopic physics, nanotechnology, theory of waveguides, etc.

Suppose that the potentials \(\{q_j\}_{j=2}^{m}\) are known a priori, and it is required to find \(q_1\) by using a part of the spectrum of the problem (5.3)-(5.5). Problems of such kind are called partial inverse problems for differential operators on graphs.

For \(j = 1, m\), denote by \(S_j(x_j, \lambda)\) the solution of (5.3) under the initial conditions \(S_j(0, \lambda) = 0, S'_j(0, \lambda) = 1\). The solutions \(S_j(x_j, \lambda)\) for \(j = 2, m\) can be constructed
by the known potentials \{q_j\}_{j=2}^{m}. One can easily check, that the eigenvalues of (5.3)-(5.5) coincide with the eigenvalues of the boundary value problem for equation (5.3) for \( j = 1 \) with the boundary conditions
\[
y_1(0) = 0, \quad f_1(\lambda) y_1'(\pi) + f_2(\lambda) y_1(\pi) = 0,
\]
where \( f_j(\lambda), j = 1, 2 \), are entire functions of \( \lambda \), defined as follows:
\[
f_1(\lambda) := \prod_{k=2}^{m} S_j(\pi, \lambda), \quad f_2(\lambda) := \sum_{j=2}^{m} S'_j(\pi, \lambda) \prod_{k=2 \atop k \neq j}^{m} S_k(\pi, \lambda).
\]

Therefore the results of [5, 40] for the partial inverse problem of recovering \( q_1 \) follow from the results of the present paper. Note that the other partial inverse problems for quantum graphs, studied in [7, 8, 43], also can be reduced to the form similar to (1.1)-(1.3).

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E-mail address: chuanfuyang@njust.edu.cn
E-mail address: bondarenkonp@info.sgu.ru
E-mail address: xcxu@nuist.edu.cn