LONG-TIME BEHAVIOR OF A CLASS OF VISCOELASTIC PLATE EQUATIONS

YANG LIU

ABSTRACT. This paper is concerned with the initial-boundary value problem for a class of viscoelastic plate equations on an arbitrary dimensional bounded domain. Under certain assumptions on the memory kernel and the source term, the global well-posedness of solutions and the existence of global attractors are obtained.

1. Introduction

In this paper, we study the following initial-boundary value problem for nonlinear viscoelastic plate equations

\begin{align}
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - \int_{-\infty}^{t} g(t-\tau) \Delta^2 u(\tau) \, d\tau - \Delta u_t &= f(u) + h(x), \quad x \in \Omega, \quad t > 0, \\
u(x, t) &= u_0(x, t), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad t \leq 0, \\
u(x, t) = \Delta u(x, t) &= 0, \quad x \in \partial \Omega, \quad t \in \mathbb{R},
\end{align}

where \(\Omega\) is a bounded domain of \(\mathbb{R}^N\) with a smooth boundary \(\partial \Omega\), the memory kernel \(g\) and the external forces \(f, h\) will be specified later.

Problem 1-3 can be used to describe the vibrations of viscoelastic materials possessing a capacity of storage and dissipation of mechanical energy, see [17] for the details. And \(u(x, t)\) represents the displacement at time \(t\) of a particle having position \(x\) in a given reference configuration with the prescribed past history \(u_0 : \Omega \times (-\infty, 0] \rightarrow \mathbb{R}\). In view of the main results of [5], we see that the viscoelastic term (namely the memory term) produces the effect of strong dissipation, which prevails the effect of weak damping term on the decay of solutions in time.

There have been many works on the long-time behavior of viscoelastic plate equations, we refer the readers to [1, 2, 4, 6, 18, 20, 21, 25] and the references therein. As for viscoelastic plate equations with past history, Pata [23] studied

\begin{align}
\frac{\partial^2 u}{\partial t^2} + \alpha \Delta u - \int_0^{\infty} g(\tau) \Delta u(t-\tau) \, d\tau + \mu u_t &= 0, \quad t > 0, \\
u(t) &= u_0(t), \quad u_t(0) = u_1, \quad t \leq 0,
\end{align}

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* Corresponding author: Yang Liu.
where $A$ is a self-adjoint and strictly positive linear operator, $\alpha$ and $\mu$ are positive constants. Based on certain assumptions on $g$, he analyzed the exponential stability of the related semigroup. Guesmia and Messaoudi [13] investigated

$$u_{tt} + Au - \int_0^\infty g(\tau)Au(t-\tau)\,d\tau = 0, \quad t > 0,$$

$$u(-t) = u_0(t), \quad u_t(0) = u_1, \quad t \geq 0.$$  

Under some assumptions on $A$ and $g$, they established a general decay result which depends on the behavior of $g$. Jorge Silva and Ma [15] considered

$$u_{tt} + \alpha \Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,d\tau$$

$$- \text{div}(|\nabla u|^{p-2}\nabla u) - \Delta u_t + f(u) = h(x), \quad x \in \Omega, \quad t > 0,$$

$$u(x,t) = u_0(x,t), \quad u_t(x,t) = \partial_t u_0(x,t), \quad x \in \Omega, \quad t \leq 0,$$

$$u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in \mathbb{R},$$

where $h \in L^2(\Omega)$, and $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$. Under some assumptions on $f$ and $g$, they obtained the global well-posedness and regularity of weak solutions. Moreover, they proved the exponential decay of energy. Recently, Conti and Geredeli [9] studied

$$u_{tt} + \alpha \Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,d\tau + f_1(u_t) + f_2(u) = h(x), \quad x \in \Omega, \quad t > 0,$$

$$\begin{cases}
  u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
  u(x,-t) = \phi_0(x,t), \quad x \in \Omega, \quad t > 0,
\end{cases}$$

$$u(x,t) = \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0,$$

where $h \in L^2(\Omega)$, $\Omega$ is a bounded domain of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Under some assumptions on $f_1$, $f_2$ and $g$, they obtained the existence and regularity of global attractors.

In the works mentioned above, authors introduced a variable which reflects the relative displacement history so that the corresponding problem could be turned into an autonomous system. This scheme is so-called the past history approach [12] which suggests to consider some past history variables as additional components of the phase space corresponding to the equation under study.

In the present paper, in order to study the long-time behaviour of solutions of problem 1-3, we employ the past history approach and the operator technique so that Eq. 1 can be transformed into an abstract system in the history phase space. And thus the operator technique combined with the energy estimates becomes a crucial tool for the proof of the existence of global attractors.

This paper is organized as follows. In Section 2 some notations and assumptions on $f$ and $g$ are displayed. Moreover, 1-3 is transformed into a generalized problem, and the main results of this paper are stated. In Section 3 the global well-posedness of regular solutions is obtained. And the global well-posedness of weak solutions is established by the density arguments [6]. In Section 4 the existence of global attractors is derived by means of the existence of an absorbing set and the semigroup decomposition [14, 16, 26].
2. Preliminaries and main results

2.1. Notations and assumptions. Throughout the paper, in order to simplify the notations, we denote
\[ \| \cdot \|_p := \| \cdot \|_{L^p(\Omega)}, \quad \| \cdot \| := \| \cdot \|_{L^2(\Omega)}. \]

\((\cdot, \cdot)\) denotes either the \(L^2\)-inner product or a duality pairing between a space and its dual space. Moreover, \(|\Omega|\) stands for the Lebesgue measure of \(\Omega\), \(C_i, i = 1, 2, 3, \cdots\) denote some different positive constants, and \(C_i, i = 1, 2, 3\) represent the positive constants for inequalities
\[ \|u\| \leq C_1 \|\nabla u\|, \quad \|u\| \leq C_2 \|\Delta u\|, \quad \|\nabla u\| \leq C_3 \|\Delta u\|. \]

As in [3, 10, 27], we give the following assumptions on \(f\) in order to state the main results of this paper.

(A1): \(f(0) = 0\), and there exists a constant \(b > 0\) such that
\[ |f(u) - f(v)| \leq b(|u|^{p-2} + |v|^{p-2})|u - v|, \quad \forall u, v \in \mathbb{R}, \]
where
\[ 2 \leq p < \infty \text{ if } N \leq 4, \quad 2 \leq p \leq \frac{2N - 4}{N - 4} \text{ if } N > 4. \]

Moreover,
\[ \limsup_{|u| \to \infty} \frac{F(u)}{|u|^2} \leq 0, \]  
and
\[ \limsup_{|u| \to \infty} \frac{uf(u) - \varrho F(u)}{|u|^2} \leq 0, \]
where \(0 < \varrho < 1\) and
\[ F(u) = \int_0^u f(s) \, ds. \]

In addition, as in [7, 19, 22], we assume that \(g\) satisfies the following conditions.

(A2): \(g \in C^1(\mathbb{R}^+ \cap L^1(\mathbb{R}^+)), \quad g(t) \geq 0, \quad g'(t) \leq 0, \quad t \in [0, \infty), \) and
\[ \kappa := 1 - \int_0^\infty g(t) \, dt > 0. \]

2.2. Reformulation of the problem. As in [2, 3, 8, 10, 27], we define the operator
\[ A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega) \]
\[ Au = \Delta^2 u, \quad \forall u \in D(A), \]
where the dense domain
\[ D(A) = \{u \in H^4(\Omega) \cap H^1_0(\Omega) | \Delta u \in H^2(\Omega) \cap H^1_0(\Omega)\}. \]

It is easy to verify that \(A\) is self-adjoint and strictly positive. Thus \(A^\dagger\) is also self-adjoint and strictly positive for any \(\gamma > 0\). Denote \(V_{\gamma} := D(A^\gamma)\) and \(V_{-\gamma} := V_{\gamma}^\prime\). Then, for any \(\gamma \in \mathbb{R}, \) \(V_{\gamma}\) and \(L_{\gamma, \gamma}\) are Hilbert spaces equipped with inner products and norms
\[ (u, w)_{V_{\gamma}} = (A^\dagger u, A^\dagger w), \quad \|u\|_{V_{\gamma}} = \|A^\dagger u\|, \]
\[ (v, w)_{g, \gamma} = \int_0^\infty g(\tau) (v(\tau), w(\tau))_{V_{\gamma}} \, d\tau, \quad \|v\|^2_{g, \gamma} = \int_0^\infty g(\tau) \|v(\tau)\|^2_{V_{\gamma}} \, d\tau, \]
where
\[ L_{g, \gamma} := L^2_{g}(\mathbb{R}^+; V_\gamma) = \left\{ v : \mathbb{R}^+ \to V_\gamma \bigg| \int_0^{\infty} g(\tau)\|v(\tau)\|_{V_\gamma}^2 \, d\tau < \infty \right\}. \]

Thus
\[ V_3 := H_3(\Omega) = \{ u \in H^3(\Omega) \cap H_0^1(\Omega) | \Delta u \in H_0^1(\Omega) \}, \]
\[ V_2 := H^2(\Omega) \cap H_0^1(\Omega), \quad V_1 := H_0^1(\Omega), \quad V_0 := L^2(\Omega). \]

In this way, problem 1-3 can be seen as
\[ u_{tt} + Au - \int_{-\infty}^{t} g(t - \tau)Au(\tau) \, d\tau + A^{\frac{1}{2}} u_t = f(u) + h, \quad t > 0, \]
\[ u(t) = u_0(t), \quad u_t(0) = u_1, \quad t \leq 0. \]

Now we are in a position to define the auxiliary function
\[ v^t(\tau) = u(t) - u(t - \tau), \quad \tau > 0, \quad t \geq 0. \]

Thus the viscoelastic dissipation in (7) can be rewritten as
\[ - \int_{-\infty}^{t} g(t - \tau)Au(\tau) \, d\tau = - \int_0^{\infty} g(\tau)Au(t - \tau) \, d\tau \]
\[ = - (1 - \kappa)Au + \int_0^{\infty} g(\tau)A^{\frac{1}{2}}v^t(\tau) \, d\tau. \]

Therefore, problem 1-3 is transformed into the following system
\[ \begin{cases} u_{tt} + \kappa Au + \int_0^{\infty} g(\tau)A^{\frac{1}{2}}v^t(\tau) \, d\tau + A^{\frac{1}{2}} u_t = f(u) + h, \quad t > 0, \\ v^t(\tau) = u(t) - u(t - \tau), \quad \tau > 0, \quad t > 0, \end{cases} \]
with
\[ \begin{cases} u(0) = u_0, \quad u_t(0) = u_1, \\ v^0(\tau) = v_0(\tau). \end{cases} \]

where
\[ u_0 = u_0(0), \]
\[ v_0(\tau) = u_0(0) - u_0(-\tau), \quad \tau > 0. \]

**Definition 2.1.** \((u(t), u_t(t), v^t)\) is called a weak solution of problem 8, 9 if \(u \in C([0,T]; V_2), \) \(u_t \in C([0,T]; V_0), \) \(v^t \in C([0,T]; L_{g,2}), \) \(u(0) = u_0 \) in \(V_2, \) \(u_t(0) = u_1 \) in \(V_0, \) \(v^0 = v_0 \) in \(L_{g,2}, \) and
\[ \begin{cases} \langle u_t, w_1 \rangle + \kappa \int_0^{t} \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} w_1 \rangle \, d\tau + \int_0^{t} \langle v^s, w_1 \rangle_{g,2} \, ds \\ - \int \langle f(u) + h, w_1 \rangle \, d\tau + \langle u_1, w_1 \rangle + \langle A^{\frac{1}{2}} u_0, A^{\frac{1}{2}} w_1 \rangle, \end{cases} \]
\[ \begin{cases} \langle v^t, w_2 \rangle_{g,2} = \langle u, w_2 \rangle_{g,2} - \langle u_0, w_2 \rangle_{g,2} - \int_0^{t} \langle v^s, w_2 \rangle_{g,2} \, ds + \langle v_0, w_2 \rangle_{g,2}. \end{cases} \]
for any \(w_1 \in V_2, w_2 \in L_{g,2}\) and a.e. \(t \in (0, T].\)
Thus, in order to deal with problem 1-3, we study the modified problem 8, 9. In fact, for a solution \((u, u_t, v')\) of problem 8, 9, we have
\[
(u_{tt}, w_1) + \kappa (A^\frac{1}{2} u, A^\frac{1}{2} w_1) + (v', w_1), g, 2 + (A^\frac{1}{2} u_t, A^\frac{1}{2} w_1) = (f(u) + h, w_1).
\]
In view of [17, Chapter 2, Section 4], we see that \(g(t) := -G'(t)\), where \(G(t)\) is the viscoelastic flexural rigidity. From
\[
\kappa A^\frac{1}{2} u = \left(1 + \int_0^\infty G'(\tau) d\tau\right) A^\frac{1}{2} u = A^\frac{1}{2} u + \left(\lim_{\tau \to \infty} G(\tau) - G(0)\right) A^\frac{1}{2} u,
\]
it follows that
\[
(u_{tt}, w_1) + (A^\frac{1}{2} u, A^\frac{1}{2} w_1) + \left(\lim_{\tau \to \infty} G(\tau) - G(0)\right) (A^\frac{1}{2} u, A^\frac{1}{2} w_1)
\]
\[
+ (v', w_1), g, 2 + (A^\frac{1}{2} u_t, A^\frac{1}{2} w_1) = (f(u) + h, w_1).
\]
Since
\[
\left(\lim_{\tau \to \infty} G(\tau) - G(0)\right) (A^\frac{1}{2} u, A^\frac{1}{2} w_1) + (v', w_1), g, 2
\]
\[
= \left(\lim_{\tau \to \infty} G(\tau) - G(0)\right) (A^\frac{1}{2} u, A^\frac{1}{2} w_1) + \int_0^t g(\tau)(A^\frac{1}{2} v'(\tau), A^\frac{1}{2} w_1) d\tau
\]
\[
+ \int_t^\infty g(\tau)(A^\frac{1}{2} v'(\tau), A^\frac{1}{2} w_1) d\tau,
\]
and
\[
v'(\tau) = \begin{cases} u(t) - u_0(t - \tau), & \tau \geq t, \\
u(t) - u(t - \tau), & \tau < t, \end{cases}
\]
we deduce that
\[
\left(\lim_{\tau \to \infty} G(\tau) - G(0)\right) (A^\frac{1}{2} u, A^\frac{1}{2} w_1) + (v', w_1), g, 2
\]
\[
= \left(\lim_{\tau \to \infty} G(\tau) - G(0)\right) (A^\frac{1}{2} u, A^\frac{1}{2} w_1) - \int_0^t G'(\tau)(A^\frac{1}{2} u(t), A^\frac{1}{2} w_1) d\tau
\]
\[
+ \int_0^t G'(\tau)(A^\frac{1}{2} u(t - \tau), A^\frac{1}{2} w_1) d\tau - \int_t^\infty G'(\tau)(A^\frac{1}{2} u(t), A^\frac{1}{2} w_1) d\tau
\]
\[
+ \int_0^t G'(\tau)(A^\frac{1}{2} u_0(t - \tau), A^\frac{1}{2} w_1) d\tau
\]
\[
= \int_0^t G'(\tau)(A^\frac{1}{2} u(t - \tau), A^\frac{1}{2} w_1) d\tau + \int_t^\infty G'(\tau)(A^\frac{1}{2} u_0(t - \tau), A^\frac{1}{2} w_1) d\tau.
\]
Substituting this into 10, we obtain
\[
(u_{tt}, w_1) + (A^\frac{1}{2} u, A^\frac{1}{2} w_1) + \int_0^t G'(\tau)(A^\frac{1}{2} u(t - \tau), A^\frac{1}{2} w_1) d\tau
\]
\[
+ \int_t^\infty G'(\tau)(A^\frac{1}{2} u_0(t - \tau), A^\frac{1}{2} w_1) d\tau + (A^\frac{1}{2} u_t, A^\frac{1}{2} w_1) = (f(u) + h, w_1).
\]
Due to
\[
\int_0^t G'(\tau)(A^{\frac{1}{2}}u(t - \tau), A^{\frac{1}{2}}w_1) \, d\tau + \int_t^\infty G'(\tau)(A^{\frac{1}{2}}u_0(t - \tau), A^{\frac{1}{2}}w_1) \, d\tau
= - \int_{-\infty}^t g(t - \tau)(A^{\frac{1}{2}}u(\tau), A^{\frac{1}{2}}w_1) \, d\tau,
\]
we conclude that
\[
(u_{tt}, w_1) + (A^{\frac{1}{2}}u_t, A^{\frac{1}{2}}w_1) - \int_t^\infty g(t - \tau)(A^{\frac{1}{2}}u(\tau), A^{\frac{1}{2}}w_1) \, d\tau
+ (A^{\frac{1}{2}}u_t, A^{\frac{1}{2}}w_1) = (f(u) + h, w_1),
\]
which shows that \((u, u_t)\) is a solution of problem 1-3.

2.3. **Statement of main results.** The main results of this paper are stated as follows.

**Theorem 2.2.** Let \((A_1)\) and \((A_2)\) be fulfilled. Assume that \(h \in V_0\) and \((u_0, u_1, v_0) \in Z := V_2 \times V_0 \times L_{g,2}\). Then problem 8, 9 admits a unique solution \((u, u_t, v^t) \in C([0, \infty); Z)\) depending continuously on initial data.

Define the mapping \(S(t) : Z \to Z\) by
\[
S(t)(u_0, u_1, v_0) = (u(t), u_t(t), v^t).
\]
Then it is easy to see from Theorem 2.2 that \(\{S(t)\}_{t \geq 0}\) is a \(C^0\)-semigroup generated by problem 8, 9.

**Theorem 2.3.** Let \((A_1)\) and \((A_2)\) be fulfilled. And there exists a constant \(\rho > 0\) such that \(g'(t) + \rho g(t) \leq 0\) for all \(t \in [0, \infty)\). Assume that \(h \in V_0\) and \((u_0, u_1, v_0) \in Z\). Then \(S(t)\) possesses a global attractor in \(Z\).

3. **Proof of Theorem 2.2**

**Theorem 3.1.** Let \((A_1)\) and \((A_2)\) be fulfilled. Assume that \(h \in V_0, u_0 \in V_3, u_1 \in V_1, v_0 \in L_{g,3}\). Then problem 8, 9 admits a unique solution \(u \in L^\infty(0, \infty; V_3), u_t \in L^\infty(0, \infty; V_1) \cap L^2(0, \infty; V_2), v^t \in L^\infty(0, \infty; L_{g,3})\), which depends continuously on initial data.

**Proof.** Let \(\{\omega_j\}_{j=1}^\infty\) be an orthogonal basis of \(V_2\) and an orthonormal basis of \(V_0\) given by eigenfunctions of \(A\). As in [11, 15], we select \(\{e_j\}_{j=1}^\infty\) of the form
\[
\{l_k \omega_j\}_{k=1}^\infty, \text{ where } \{l_k\}_{k=1}^\infty \text{ is an orthonormal basis of } L^2_g(\mathbb{R}^+) \cap C_0^\infty(\mathbb{R}^+) \text{ and } \tilde{\omega}_j = \frac{|\omega_j||\nu_j|}{|v_2|}.
\]
Then \(\{e_j\}_{j=1}^\infty\) is an orthonormal basis of \(L_{g,2}\).

We construct the approximate solutions of problem 8, 9
\[
\begin{align*}
    u_n(t) &= \sum_{j=1}^n \zeta_{jn}(t) \omega_j, \\
    v_n^t(\tau) &= \sum_{j=1}^n \zeta_{jn}(t) e_j(\tau), \quad n = 1, 2, \cdots,
\end{align*}
\]
which satisfy
\[
\begin{cases}
    (u_{nrt}, \omega_j) + \kappa(A^{\frac{1}{2}} u_n, A^{\frac{1}{2}} \omega_j) + (v_n^t, \omega_j)_{g,2} \\
    + (A^{\frac{1}{2}} u_{nt}, A^{\frac{1}{2}} \omega_j) = (f(u_n), \omega_j) + (h, \omega_j), \\
    (v_n^t, e_j)_{g,2} = (u_{nt}, e_j)_{g,2} - (v_n^t, e_j)_{g,2}, \quad j = 1, 2, \cdots, n,
\end{cases}
\]
with

\[
\begin{align*}
  u_n(0) &= \sum_{j=1}^{n} \xi_{jn}(0) \omega_j \to u_0 \text{ in } V_3, \\
  u_{nt}(0) &= \sum_{j=1}^{n} \xi'_{jn}(0) \omega_j \to u_1 \text{ in } V_1, \\
  v^0 &= \sum_{j=1}^{n} \zeta_{jn}(0) e_j \to v_0 \text{ in } L_{g,3}.
\end{align*}
\]  

(12)

The approximate problem 11, 12 can be reduced to an ordinary differential system in the variables $\xi_{jn}(t)$ and $\zeta_{jn}(t)$. In terms of standard theory for ODEs, there exists a solution $(u_n(t), u_{nt}(t), v^0_n)$ on some interval $[0, T_n)$ with $T_n \leq T$. The following estimates will allow us to extend the local solutions to $[0, T]$ with any $T > 0$.

Replacing $\omega_j$ in 11 with $u_{nj}$ and $e_j$ in 112 with $v^0_n$, summing for $j$ and adding the two results, we obtain

\[
E_n(t) + \|A^{\frac{1}{2}} u_{nt}\|^2 = -(v^t_n, v^t_n)_{g,2},
\]

where

\[
E_n(t) = \frac{1}{2} \|u_{nt}\|^2 + \kappa \frac{1}{2} \|A^{\frac{1}{2}} u_n\|^2 + \frac{1}{2} \|v^t_n\|^2_{g,2} - \int_{\Omega} F(u_n) \, dx - (h, u_n).
\]

(14)

Since $\lim_{\tau \to 0} v^t_n(\tau) = 0$, we deduce from (A2) that

\[
(v^t_n, v^t_n)_{g,2} = \frac{1}{2} \int_{0}^{\infty} \frac{\partial}{\partial \tau} \left( g(\tau) \|A^{\frac{1}{2}} v^t_n(\tau)\|^2 \right) \, d\tau - \frac{1}{2} \int_{0}^{\infty} g'(\tau) \|A^{\frac{1}{2}} v^t_n(\tau)\|^2 \, d\tau \geq 0.
\]

Hence, by integrating 13 with respect to $t$ from 0 to $t$, we get

\[
E_n(t) + \int_{0}^{t} \|A^{\frac{1}{2}} u_{n\tau}\|^2 \, d\tau \leq E_n(0).
\]

(15)

It follows from 4 in (A1) that, for any $\eta > 0$, there exists a constant $C_\eta > 0$ such that

\[
\int_{\Omega} F(u_n) \, dx \leq \eta \|u_n\|^2 + C_\eta |\Omega|.
\]

By virtue of Cauchy’s inequality with $\epsilon > 0$, we get

\[
(h, u_n) \leq \|h\| \|u_n\| \leq \epsilon \mathcal{C}_2^2 \|A^{\frac{1}{2}} u_n\|^2 + \frac{1}{4\epsilon} \|h\|^2.
\]

Consequently, taking sufficiently small $\eta$ and $\epsilon$ such that

\[
C_1 := \frac{\kappa}{2} - \eta \mathcal{C}_2^2 - \epsilon \mathcal{C}_2^2 > 0,
\]

we deduce from 14 that

\[
E_n(t) \geq \frac{1}{2} \|u_{nt}\|^2 + C_1 \|A^{\frac{1}{2}} u_n\|^2 + \frac{1}{2} \|v^t_n\|^2_{g,2} - C_2 (\|h\|^2 + |\Omega|).
\]

(16)

Hence, from 15, 16 and 12, it follows that

\[
\|u_{nt}\|^2 + \|A^{\frac{1}{2}} u_n\|^2 + \|v^t_n\|^2_{g,2} + \int_{0}^{t} \|A^{\frac{1}{2}} u_{n\tau}\|^2 \, d\tau \leq C_3.
\]

(17)
Replacing $\omega_j$ in (11) with $A^\frac{1}{2} u_{nt}$ and $v_j$ in (11) with $A^\frac{1}{2} v'_n$, summing for $j$ and adding the two results, we obtain

\[
\frac{1}{2} \frac{d}{d t} \left( \| A^\frac{1}{2} u_{nt} \|^2 + \kappa \| A^\frac{1}{2} u_n \|^2 + \| v'_n \|^2_{g,3} \right) + \| A^\frac{1}{2} u_{nt} \|^2 = (f(u_n), A^\frac{1}{2} u_{nt}) + (h, A^\frac{1}{2} u_{nt}) - (v'_n, v'_n)_{g,3}.
\]

Noting that

\[-(v'_n, v'_n)_{g,3} \leq 0,
\]

\[(f(u_n), A^\frac{1}{2} u_{nt}) \leq C_4 \| A^\frac{1}{2} u_n \|^{2p-2} + \frac{1}{4} \| A^\frac{1}{2} u_{nt} \|^2,
\]

and

\[(h, A^\frac{1}{2} u_{nt}) \leq \| h \|^2 + \frac{1}{4} \| A^\frac{1}{2} u_{nt} \|^2,
\]

we conclude from (17) that

\[
\| A^\frac{1}{2} u_{nt} \|^2 + \| A^\frac{1}{2} u_n \|^2 + \| v'_n \|^2_{g,3} + \int_0^t \| A^\frac{1}{2} u_{nt} \|^2 d \tau \leq C_5.
\]

Therefore, there exist $u$, $v'$ and subsequences of $\{u_n\}$, $\{v'_n\}$, still represented by the same notations and we shall not repeat, such that, as $n \to \infty$,

\[
u_n \rightarrow u \text{ weakly star in } L^\infty(0, T; V_3),
\]

\[
u_{nt} \rightarrow v' \text{ weakly star in } L^\infty(0, T; L_{g,3}),
\]

for any $T > 0$. According to the Aubin-Lions lemma, we have

\[u_n \rightarrow u \text{ in } L^2(0, T; V_2).
\]

Moreover, from 18-20, it follows that

\[
u_n \rightarrow u \text{ in } C([0, T]; V_2).
\]

We now claim that for any $t \in [0, T]$ and fixed $j$,

\[
\int_0^t (f(u_n), \omega_j) \, d \tau \to \int_0^t (f(u), \omega_j) \, d \tau,
\]

as $n \to \infty$.

Indeed, for any $w \in V_2$, we have

\[
(f(u_n) - f(u), w) \leq b((|u_n|^p - 2 + |u|^p - 2) |u_n - u|, w).
\]

If $p > 2$, then when $N \leq 4$,

\[
(f(u_n) - f(u), w) \leq b \left( \| u_n \|^{p-2}_{4(p-2)} + \| u \|^{p-2}_{4(p-2)} \right) \| u_n - u \| \| w \|,
\]

when $N > 4$,

\[
(f(u_n) - f(u), w) \leq b \left( \| u_n \|^{p-2}_{\frac{N}{2}(p-2)} + \| u \|^{p-2}_{\frac{N}{2}(p-2)} \right) \| u_n - u \| \| w \|^{\frac{N}{2}} \| w \|^{\frac{N}{2}}.
\]

Hence

\[
(f(u_n) - f(u), w) \leq C_6 \left( \| u_n \|^p_{V_2} + \| w \|^p_{V_2} \right) \| u_n - u \| \| w \| \| V_2
\]

\[
\leq C_7 \| u_n - u \| \| V_1 \| \| w \| \| V_2.
\]

(23)
If $p = 2$, then it is clear that (23) remains valid. Therefore,
\[ \left| \int_0^t (f(u_n) - f(u), w_j) \, d\tau \right| \leq C_8 \int_0^t \| u_n - u \|_{V_1} \, d\tau. \]
Thus assertion (22) follows from (21).
For fixed $j$,
\[ (v_{\text{tr}}, e_j)_{g, 2} = - \int_0^\infty g'(\tau)(A^{\frac{2}{n}} v_2'(\tau), A^{\frac{2}{n}} e_j(\tau)) \, d\tau - \int_0^\infty g(\tau)(A^{\frac{2}{n}} v_n'(\tau), A^{\frac{2}{n}} e_j(\tau)) \, d\tau. \]
Hence
\[ \lim_{n \to \infty} (v_{\text{tr}}, e_j)_{g, 2} = (v_{\text{tr}}, e_j)_{g, 2}. \]
Consequently, for fixed $j$, integrating (11) with respect to $t$ and taking $n \to \infty$, we get
\[
\begin{cases}
(u_t, \omega_j) + \kappa \int_0^t (A^{\frac{2}{n}} u, A^{\frac{2}{n}} \omega_j) \, d\tau + \int_0^t (v^s, \omega_j)_{g, 2} \, ds \\
+ (A^{\frac{2}{n}} u, A^{\frac{2}{n}} \omega_j) = \int_0^t (f(u) + h, \omega_j) \, d\tau + (u_1, \omega_j) + (A^{\frac{2}{n}} u_0, A^{\frac{2}{n}} \omega_j), \\
(v^s, e_j)_{g, 2} = (u, e_j)_{g, 2} - (u_0, e_j)_{g, 2} - \int_0^t (v^s, e_j)_{g, 2} \, ds + (v, e_j)_{g, 2}.
\end{cases}
\]
Moreover, it is easy to see from (12) that $u(0) = u_0$ in $V_3$, $u_t(0) = u_1$ in $V_1$, $v^0 = v_0$ in $L^2$. Therefore, $(u, u_t, v^t)$ is a solution of problem 8, 9.

Next we prove continuous dependence of $(u(t), u_t(t), v^t)$ on $(u_0, u_1, v_0)$. Suppose that $(u, u_t, v^t)$ and $(\tilde{u}, \tilde{u}_t, \tilde{v}^t)$ are two regular solutions of problem 8, 9 with initial data $(u_0, u_1, v_0)$ and $(\tilde{u}_0, \tilde{u}_1, \tilde{v}_0)$, respectively. Set $\hat{u} = \tilde{u} - u$ and $\hat{v}^t = \tilde{v}^t - v^t$. Then
\[
\begin{cases}
\hat{u}_{tt} + \kappa A \hat{u} + \int_0^\infty g(\tau) A^{\frac{2}{n}} \hat{u} \, d\tau + A^{\frac{2}{n}} \hat{u}_t = (f(\hat{u}) - f(u), \hat{u}_t) - (\tilde{v}^t - v^t), \\
\hat{v}^t = \tilde{v}^t - v^t,
\end{cases}
\]
By the arguments similar to [7, Lemma 4.9], we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|\hat{u}_t\|^2 + \kappa \|A^{\frac{2}{n}} \hat{u}\|^2 + \|\hat{v}^t\|^2_{g, 2} \right) + \|A^{\frac{2}{n}} \hat{u}_t\|^2
= (f(\hat{u}) - f(u), \hat{u}_t) - (\tilde{v}^t, \tilde{v}^t)_{g, 2}.
\]
By the arguments similar to the proof of (23), we have
\[
(f(\hat{u}) - f(u), \hat{u}_t) \leq C_9 \|A^{\frac{2}{n}} \hat{u}\| \|A^{\frac{2}{n}} \hat{u}_t\|
\leq \frac{C_9}{4\varepsilon} \|A^{\frac{2}{n}} \hat{u}\|^2 + C_9 \varepsilon \|A^{\frac{2}{n}} \hat{u}_t\|^2.
\]
Note that $-(\tilde{v}^t, \tilde{v}^t)_{g, 2} \leq 0$. Hence, by taking $\varepsilon = \frac{1}{C_9}$, we deduce from (25) that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\hat{u}_t\|^2 + \kappa \|A^{\frac{2}{n}} \hat{u}\|^2 + \|\hat{v}^t\|^2_{g, 2} \right) \leq \frac{C_9^2}{4} \|A^{\frac{2}{n}} \hat{u}\|^2.
\]
As a consequence, by Gronwall’s inequality, we obtain
\[
\|\hat{u}_t\|^2 + \|A^{\frac{2}{n}} \hat{u}\|^2 + \|\hat{v}^t\|^2_{g, 2} \leq C_{10} \left( \|\hat{u}_1\|^2 + \|A^{\frac{2}{n}} \hat{u}_0\|^2 + \|\tilde{v}_0\|^2_{g, 2} \right).
\]
In particular, by taking \((u_0, u_1, v_0) = (\bar{u}_0, \bar{u}_1, \bar{v}_0)\), it is obvious that \((u, u_t, v^t)\) is the unique solution of problem 8, 9. \(\square\)

**Proof of Theorem 2.2.** For \(u_0 \in V_2, u_1 \in V_0, v_0 \in \mathcal{L}_{g,2}\), there exist \(u_{0m} \subset V_3, u_{1m} \subset V_1, v_{0m} \subset \mathcal{L}_{g,3}\) such that
\[
u_{0m} \rightarrow u_0 \text{ in } V_2, \ u_{1m} \rightarrow u_1 \text{ in } V_0, \ v_{0m} \rightarrow v_0 \text{ in } \mathcal{L}_{g,2}.
\]

According to Theorem 3.1, for any \(m \in \mathbb{N}^+\), problem 8, 9 admits a unique regular solution \((u_m, u_{mt}, v^t_m)\) satisfying
\[
\begin{cases}
u_{mtt} + \kappa A u_m + \int_0^\infty g(\tau) A v^t_m(\tau) d\tau + A^\frac{1}{2} u_{mt} \\
= f(u_m) + h, \ t \in (0, \infty), \\
v^t_m(\tau) = u_{mt}(t) - v^t_m(\tau), \ \tau \in (0, \infty), \ t \in (0, \infty),
\end{cases}
\]

Hence \(u_m \in C([0, T]; V_2), u_{mt} \in C([0, T]; \mathcal{L}_{g,2})\). Moreover, according to [24, Theorem 3.2], we have \(v^t_m \in C([0, T]; \mathcal{L}_{g,2})\).

Set \(y_m = u_{m_2} - u_{m_1}\) and \(z_m = v^t_{m_2} - v^t_{m_1}\). Then, by the arguments similar to the proof of 26, we get
\[
\|y_{mt}\|^2 + \|A^\frac{1}{2} y_m\|^2 + \|z_{m}\|^2 \leq C_{11}(\|y_{mt}(0)\|^2 + \|A^\frac{1}{2} y_m(0)\|^2 + \|z_{m}(0)\|^2)_{g,2}.
\]

By 27 and the arguments similar to the proof of 17, we have
\[
u_m \rightarrow u \text{ in } C([0, T]; V_2),
\]
\[
u_{mt} \rightarrow u_t \text{ in } C([0, T]; V_0),
\]
\[
v^t_m \rightarrow v^t \text{ in } C([0, T]; \mathcal{L}_{g,2}).
\]

Thus \((u, u_t, v^t)\) is a global weak solution of problem 8, 9.

Suppose that \((u, u_t, v^t)\) and \((\tilde{u}, \tilde{u}_t, \tilde{v}^t)\) are two solutions of problem 8, 9 with initial data \((u_0, u_1, v_0)\), \((\tilde{u}_0, \tilde{u}_1, \tilde{v}_0)\), respectively. Then there exist
\[
(u_{0m}, u_{1m}, v_{0m}) \subset V_3 \times V_1 \times \mathcal{L}_{g,3}, \ (\tilde{u}_{0m}, \tilde{u}_{1m}, \tilde{v}_{0m}) \subset V_3 \times V_1 \times \mathcal{L}_{g,3},
\]
such that
\[
u_{0m} \rightarrow (u_0, u_1, v_0) \text{ in } V_2 \times V_0 \times \mathcal{L}_{g,2},
\]
\[
u_{0m} \rightarrow (\tilde{u}_0, \tilde{u}_1, \tilde{v}_0) \text{ in } V_2 \times V_0 \times \mathcal{L}_{g,2}.
\]

Set \(\tilde{u}_m = \tilde{u}_m - u_m\) and \(\tilde{v}^t_m = \tilde{v}^t_m - v^t_m\). Then, on account of 26, we obtain
\[
\|\tilde{u}_{mt}\|^2 + \|A^\frac{1}{2} \tilde{u}_m\|^2 + \|\tilde{v}^t_m\|^2 \leq C_{12} \left(\|\tilde{u}_{1m}\|^2 + \|A^\frac{1}{2} \tilde{u}_{0m}\|^2 + \|\tilde{v}_{0m}\|^2\right)_{g,2}.
\]
Therefore, in terms of 28-32, the conclusions of Theorem 2.2 are derived immediately. \(\square\)
4. Proof of Theorem 2.3

In this section, for the sake of convenience, we denote
\[ \|S(t)(u_0, u_1, v_0)\|_Z^2 := \|u\|_{Y_2}^2 + \|u\|_{V_0}^2 + \|v\|_{g,2}^2. \]

**Lemma 4.1.** Under the conditions of Theorem 2.3, \( S(t) \) possesses an absorbing set in \( Z \).

**Proof.** Let \( U = u_t + \varepsilon u, \ t \in [0, \infty) \), where \( \varepsilon \) is a positive constant to be determined later. Then \( 8_1 \) becomes

\[ U_t + A^{\frac{1}{2}} U - \varepsilon U - \varepsilon A^{\frac{1}{2}} u + \varepsilon^2 u + \kappa Au + \int_0^\infty g(\tau)A^t \, d\tau = f(u) + h. \]  

Note that
\[(v_t, v^t)_{g,2} \geq \frac{\rho}{2} \|v^t\|_{g,2}^2. \]

Multiplying 33 by \( U \) in \( V_0 \) and \( 8_2 \) by \( v^t \) in \( L_{g,2} \), integrating over \( \Omega \) and adding the two results, we obtain

\[ E_1(t) + E_2(t) \leq 0, \]

where
\[ E_1(t) = \frac{1}{2} \left( \|U\|^2 + \kappa \|A^{\frac{1}{2}} u\|^2 + \|v^t\|_{g,2}^2 + \varepsilon^2 \|u\|^2 + \varepsilon \|A^{\frac{1}{2}} u\|^2 \right. \]
\[ \left. - 2 \int_\Omega F(u) \, dx - 2(h, u) \right), \]

and
\[ E_2(t) = \|A^{\frac{1}{2}} U\|^2 - \varepsilon \|U\|^2 - \varepsilon^2 \|A^{\frac{1}{2}} u\|^2 + \varepsilon^3 \|u\|^2 + \varepsilon \kappa \|A^{\frac{1}{2}} u\|^2 \]
\[ + \varepsilon \int_0^\infty g(\tau)(A^{\frac{1}{2}} v^t(\tau), A^{\frac{1}{2}} u(t)) \, d\tau - \varepsilon (f(u), u) - \varepsilon (h, u) + \frac{\rho}{2} \|v^t\|_{g,2}^2. \]

Hence
\[ E_2(t) - \varepsilon \varrho E_1(t) = \varepsilon \kappa \left( \frac{2 - \varrho}{2} \right) \|A^{\frac{1}{2}} u\|^2 - \varepsilon^2 \left( 1 - \frac{\varrho}{2} \right) \|A^{\frac{1}{2}} u\|^2 \]
\[ + \varepsilon^3 \left( 1 - \frac{\varrho}{2} \right) \|u\|^2 + \sum_{i=1}^4 \lambda_i, \]

where
\[ \lambda_1 = \|A^{\frac{1}{2}} U\|^2 - \varepsilon \left( 1 + \frac{\varrho}{2} \right) \|U\|^2; \]
\[ \lambda_2 = \frac{\rho}{2} \|v^t\|_{g,2}^2 + \varepsilon \int_0^\infty g(\tau)(A^{\frac{1}{2}} v^t(\tau), A^{\frac{1}{2}} u(t)) \, d\tau - \frac{\varrho \varepsilon}{2} \|v^t\|_{g,2}^2; \]
\[ \lambda_3 = \varepsilon \left( \varrho \int_\Omega F(u) \, dx - (f(u), u) \right), \]
\[ \lambda_4 = -\varepsilon (1 - \varrho)(h, u). \]

Applying Cauchy’s inequality with \( \varepsilon_1 > 0 \), we get
\[ \lambda_2 \geq \frac{\rho}{2} \|v^t\|_{g,2}^2 - \varepsilon_1 \varepsilon (1 - \kappa) \|A^{\frac{1}{2}} u\|^2 - \frac{\varepsilon}{4 \varepsilon_1} \|v^t\|_{g,2}^2 - \frac{\varrho \varepsilon}{2} \|v^t\|_{g,2}^2. \]
It follows from 5 that, for any η > 0, there exists a constant $C_\eta > 0$ such that
\[
\lambda_3 \geq -\varepsilon (\eta \|u\|^2 + C_\eta |\Omega|) \\
\geq -\varepsilon \eta \mathcal{C}_2^2 \|A^{\frac{1}{2}} u\|^2 - \varepsilon C_\eta |\Omega|.
\]

Moreover,
\[
\lambda_1 \geq \frac{1}{\mathcal{C}_1^2} \|U\|^2 - \varepsilon \left(1 + \frac{\rho}{2}\right) \|U\|^2,
\]
and
\[
\lambda_4 \geq -\varepsilon (1 - \varrho) \left(\varepsilon \mathcal{C}_2^2 \|A^{\frac{1}{2}} u\|^2 + \frac{1}{4\varepsilon_2} \|h\|^2\right).
\]

Consequently, by taking sufficiently small $\epsilon_1$, $\eta$ and $\varepsilon_2$ such that

\[
C_{13} := \frac{\kappa(2 - \varrho)}{2} - \epsilon_1 (1 - \kappa) - \eta \mathcal{C}_2^2 - \epsilon_2 (1 - \varrho) \mathcal{C}_2^2 > 0,
\]
we deduce that
\[
E_2(t) - \varepsilon \varrho E_1(t) \geq \varepsilon C_{13}\|A^{\frac{1}{2}} u\|^2 - \varepsilon^2 \left(1 - \frac{\varrho}{2}\right) \|A^{\frac{1}{2}} u\|^2
\]
\[
+ \left[\frac{1}{\mathcal{C}_1^2} - \varepsilon \left(1 + \frac{\varrho}{2}\right)\right] \|U\|^2 + \left(\frac{\rho}{2} - \varepsilon - \frac{\rho \varepsilon}{2}\right) \|v'|^2_{g,2}
\]
\[- \varepsilon C_\eta |\Omega| - \frac{\varepsilon (1 - \varrho)}{4\varepsilon_2} \|h\|^2.
\]

Choosing
\[
\varepsilon \leq \min\left\{\frac{2C_{13}}{(2 - \varrho)\mathcal{C}_2^2}, \frac{2}{(2 + \varrho)\mathcal{C}_1^2}, \frac{2\epsilon_1 \rho}{1 + 2\epsilon_1 \varrho}\right\},
\]
we obtain
(35) \[
E_2(t) - \varepsilon \varrho E_1(t) \geq -C_{14} (\|h\|^2 + |\Omega|)
\]
and
(36) \[
\kappa \|A^{\frac{1}{2}} u\|^2 - \varepsilon \|A^{\frac{1}{2}} u\|^2 \geq C_{15} \|A^{\frac{1}{2}} u\|^2.
\]

Since
\[
\|U\|^2 \geq \|u_t\|^2 - \varepsilon^2 \|u\|^2,
\]
we conclude from 36 and the arguments similar to the proof of 16 that
(37) \[
E_1(t) \geq C_{16} \left(\|u_t\|^2 + \|A^{\frac{1}{2}} u\|^2 + \|v'|^2_{g,2}\right) - C_{17} (\|h\|^2 + |\Omega|).
\]

It follows from 34 and 35 that
\[
E_1'(t) + \varepsilon \varrho E_1(t) \leq C_{14} (\|h\|^2 + |\Omega|),
\]
which yields
\[
E_1(t) \leq E_1(0) e^{-\varepsilon \varrho t} + \frac{C_{14}}{\varepsilon \varrho} (\|h\|^2 + |\Omega|).
\]

This, together with 37, gives
\[
\|S(t)(u_0, u_1, v_0)\|_{2}^2 \leq \frac{E_1(0)}{C_{16}} e^{-\varepsilon \varrho t} + \frac{C_{14} + \varepsilon \varrho C_{17}}{\varepsilon \varrho C_{16}} (\|h\|^2 + |\Omega|).
\]

Hence $S(t)$ possesses an absorbing set with the radius $R > \sqrt{\frac{C_{14} + \varepsilon \varrho C_{17}}{\varepsilon \varrho C_{16}} (\|h\|^2 + |\Omega|)}$. □
Proof of Theorem 2.3. We decompose $u = \dot{u} + \bar{u}$ and $v^t = \dot{v} + \bar{v}^t$ satisfying

$$
\begin{align*}
\dot{u}_{tt} + \kappa A\dot{u} + A^{\frac{1}{2}} \dot{u} + \int_0^\infty g(\tau) A\dot{v}^t(\tau) d\tau &= 0, \\
\dot{v}^t &= \dot{u}_t - \dot{\bar{v}}^t, \\
\bar{u}(0) &= u_0, \quad \bar{u}_t(0) = u_1, \quad \bar{v}^0(\tau) = v_0(\tau),
\end{align*}
$$

(38) where

$$
\Phi = f(u) + h.
$$

Let $\psi = A^\delta \bar{u}$, $\varphi^t = A^\delta \bar{v}^t$, $0 < \delta \leq \frac{1}{4}$. Then it follows from 39 that

$$
\begin{align*}
\psi_{tt} + \kappa A\psi + A^{\frac{1}{2}} \psi_t + \int_0^\infty g(\tau) A\varphi^t(\tau) d\tau &= A^\delta \Phi, \\
\varphi^t_t &= \psi_t - \varphi^t, \\
\psi(0) &= 0, \quad \psi_t(0) = 0, \quad \varphi^0(\tau) = 0.
\end{align*}
$$

(40)

Let $\Psi = \psi_t + \varepsilon \psi$, where $\varepsilon$ is a positive constant to be determined later. Then 40 can be written in the form

$$
\begin{align*}
\Psi_t + A^{\frac{1}{2}} \Psi - \varepsilon A^{\frac{1}{2}} \psi + \varepsilon^2 \psi + \kappa A\psi + \int_0^\infty g(\tau) A\varphi^t(\tau) d\tau &= A^\delta \Phi.
\end{align*}
$$

(41)

Multiplying 41 by $\Psi$ in $V_0$ and 40 by $\varphi^t$ in $L_{g,2}$, integrating over $\Omega$ and adding the two results, we obtain

$$
\begin{align*}
E_3'(t) + 2\|A^{\frac{1}{2}} \Psi\|^2 - 2\varepsilon \|\Psi\|^2 - 2\varepsilon^2 \|A^{\frac{1}{2}} \psi\|^2 + 2\varepsilon^3 \|\psi\|^2 + 2\varepsilon \kappa \|A^{\frac{1}{2}} \psi\|^2
\end{align*}
$$

$$
+ 2\varepsilon \varepsilon \|\varphi^t(\tau), \psi(t)\|_{g,2} = 2(A^\delta \Phi, \Psi) - 2(\varphi^t, \varphi^t)_{g,2},
$$

where

$$
\begin{align*}
E_3(t) &= \|\Psi\|^2 + \kappa \|A^{\frac{1}{2}} \psi\|^2 - \varepsilon \|A^{\frac{1}{2}} \psi\|^2 + \varepsilon^2 \|\psi\|^2 + \|\varphi^t\|^2_{g,2}.
\end{align*}
$$

Hence

$$
\begin{align*}
E_3'(t) + \sum_{i=1}^2 \Lambda_i - 2\varepsilon^2 \|A^{\frac{1}{2}} \psi\|^2 + 2\varepsilon^3 \|\psi\|^2 + 2\varepsilon \kappa \|A^{\frac{1}{2}} \psi\|^2 &\leq 2(A^\delta \Phi, \Psi),
\end{align*}
$$

(42)

where

$$
\Lambda_1 = 2\|A^{\frac{1}{2}} \Psi\|^2 - 2\varepsilon \|\Psi\|^2,
$$

and

$$
\Lambda_2 = 2\varepsilon \varepsilon \|\varphi^t(\tau), \psi(t)\|_{g,2} + \rho \|\varphi^t\|^2_{g,2}.
$$

Note that

$$
\Lambda_1 \geq \|A^{\frac{1}{2}} \Psi\|^2 + \left( \frac{1}{\varepsilon^2} - 2\varepsilon \right) \|\Psi\|^2,
$$

and

$$
\Lambda_2 \geq \left( \rho - \frac{\varepsilon}{2\varepsilon} \right) \|\varphi^t\|^2_{g,2} - 2\varepsilon \varepsilon \kappa \|A^{\frac{1}{2}} \psi\|^2.
$$

Consequently, taking

$$
\varepsilon \leq \frac{\kappa(2 - \sigma_1)}{2(1 - \kappa)},
$$
with some constant $0 < \sigma_1 < 2$, we deduce from (42) that
\[
E_3'(t) + \|A^2 \Psi\|^2 + \left(\frac{1}{c_4^2} - 2\varepsilon\right) \|\Psi\|^2 - 2\varepsilon^2 \|A^2 \psi\|^2 + 2\varepsilon^3 \|\psi\|^2
+ \sigma_1 \varepsilon \kappa \|A^2 \psi\|^2 + \left(\rho - \frac{\varepsilon}{2\varepsilon}\right) \|\varphi\|^2_{L_2} \leq 2(A^4 \Phi, \Psi).
\]
We further choose
\[
\varepsilon < \min \left\{ \frac{1}{(\sigma_2 + 2)c_2^2}, \frac{2\varepsilon \rho}{1 + 2\varepsilon \sigma_2}, \frac{(\sigma_1 - \sigma_2)\kappa}{2c_3^2}, \frac{\kappa}{c_3^2} \right\},
\]
with some constant $0 < \sigma_2 < \sigma_1$. Thus
\[
E_3'(t) + \sigma_2 \varepsilon E_3(t) + \|A^2 \Psi\|^2 \leq 2(A^4 \Phi, \Psi)
\]
and
\[
E_3(t) \geq \|\psi_t\|^2 + (\kappa - \varepsilon c_3^2) \|A^2 \psi\|^2 + \|\varphi\|^2_{L_2}.
\]
Applying Hölder’s inequality and Cauchy’s inequality to the right side of (43), we get
\[
E_3'(t) + \sigma_2 \varepsilon E_3(t) \leq \|\Phi\|^2_{V_{4\delta - 1}}.
\]
For any $w \in V_{1 - 4\delta}$, we have
\[
(f(u), w) \leq b \|u\|^{p-1} \|\phi\|_{V_2} \|\psi\|_{V_{4\delta - 1}} \leq C_{18} \|u\|^{p-1} \|\psi\|_{V_{4\delta - 1}}.
\]
Since
\[
\|\Phi\|_{V_{4\delta - 1}} \leq \|f(u)\|_{V_{4\delta - 1}} + \|h\|_{V_{4\delta - 1}},
\]
we deduce from (45) that
\[
E_3(t) \leq E_3(0) e^{-\sigma_2 \varepsilon t} + C_{19}.
\]
Combining this with (40) and (44), we obtain
\[
\|\tilde{u}\|^2_{V_{2+4\delta}} + \|\tilde{u}_t\|^2_{V_{4\delta}} + \|\tilde{v}\|^2_{V_{2+4\delta}} \leq C_{20}, \forall t \in [0, \infty).
\]
Taking into account
\[
\tilde{v}^t(\tau) = \begin{cases} 
\tilde{u}(t), & \tau \geq t, \\
\tilde{u}(t) - \tilde{u}(t - \tau), & 0 < \tau < t,
\end{cases}
\]
we get
\[
\tilde{u}^t(\tau) = \begin{cases} 
0, & \tau \geq t, \\
\tilde{u}(t - \tau), & 0 < \tau < t.
\end{cases}
\]
Let $\Upsilon = \bigcup_{t \geq 0} \tilde{v}^t$. Then $\Upsilon$ is bounded in $L_{g,2+4\delta} \cap H^1_g(\mathbb{R}^+, V_{4\delta})$ due to (46), where
$H^1_g(\mathbb{R}^+, V_{4\delta})$ is a Hilbert space of $V_{4\delta}$-valued functions $v$ on $\mathbb{R}^+$ such that both $v$ and $v_t$ belong to $L_{g,4\delta}$. Note that $\sup_{v \in \Upsilon} \|v(\tau)\|^2_{V_2} \in L^1_g(\mathbb{R}^+)$. Hence, in view of [24, Lemma 5.5], we see that $\Upsilon$ is relatively compact in $L_{g,2}$. Since $V_{2+4\delta} \hookrightarrow V_2 \times V_0$, we conclude from (46) that there exists a $t_0 = t_0(B)$ such that $\bigcup_{t \geq t_0} S_2(t)B$ is relatively compact. Thus the operators $S_2(t)$ are uniformly compact for $t$ large. Furthermore, as for problem (38), it is easy to check that
\[
\|\tilde{u}\|^2_{V_2} + \|\tilde{u}_t\|^2_{V_0} + \|\tilde{v}\|^2_{g,2} \leq C_{21} e^{-\sigma_2 \varepsilon t}, \forall t \in [0, \infty).
\]
This means that $S_1(t)$ is a continuous mapping from $Z$ into itself such that

$$\sup_{(u_0, u_1, v_0) \in B} \|S_1(t)(u_0, u_1, v_0)\|_Z \to 0,$$

as $t \to \infty$. Therefore, by virtue of [26, Chapter I, Theorem 1.1] and Lemma 4.1, the proof of Theorem 2.3 is complete. □

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YANG LIU, COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, NORTHWEST MINZU UNIVERSITY, LANZhou 730124, CHINA; COLLEGE OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU 610065, CHINA

Email address: liuyang@nufn5@163.com