Computation of control for nonlinear approximately controllable system using Tikhonov regularization

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Abstract: For an approximately controllable nonlinear system, the task of finding a control for a given target state is ill-posed. By finding a regularized control of a linear approximately controllable system using Tikhonov regularization, we compute a regularized control for the nonlinear system under the assumption of Lipschitz continuity of the nonlinear function. The theory is substantiated with numerical experiments.

Keywords: approximate controllability; nonlinear control system; ill posed problem; Tikhonov regularization; regularized control

1. Introduction
Control theory is an interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems. Controllability is one of the qualitative property of a control system which plays an important role in mathematical control theory. The concept of controllability was introduced by Kalman (1963) and studied extensively by many researchers (Kumar & Sukavanam, 2015; Zhou, 1983). Controllable systems have many applications in different branches of science and engineering such as in Aerospace engineering, Chemical plants, Nuclear reactors, Robotics, Missiles, and Diffusion Systems. For more details on controllability (see Balachandran & Dauer, 2002; Barnett, 1975).

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PUBLIC INTEREST STATEMENT
Controllability is an important qualitative property of a control system which plays a crucial role in the development of science and engineering. Generally speaking, controllability implies that it is possible to steer the system from an arbitrary initial state to a given target state in a desired manner using a set of admissible inputs called controls. By the definition of approximate controllability we mean, there exists a control which steers the control system from an arbitrary initial state to an arbitrary neighbourhood of the given target state. For approximately controllable nonlinear systems, the problem of computing steering control for a given target state is ill-posed. By finding regularized control of a linear system, we compute a regularized control for the nonlinear system under simple assumptions.
Consider the semi linear control system

\[
\frac{dx}{dt} = Ax(t) + u(t) + f(t, x(t)),
\]

\[x(t_0) = x_0, \tag{1.1}\]

where \(A: D(A) \subseteq V \rightarrow V\) is a densely defined closed linear operator which generates a \(C_0\) semigroup \(T(t), \ t \geq 0\), and \(V\) is a Hilbert space. \(f: J \times V \rightarrow V\) is a nonlinear function and \(J = [t_0, r] \subseteq [0, \infty)\). If \(f \equiv 0\), then the resultant system is called the corresponding linear system which is denoted by Equation (1.1)*.

For a Hilbert space \(H\) and a closed interval \(J, L_2(J, H)\) denotes the Hilbert space of all measurable functions \(g: J \rightarrow H\) such that \(\int J \|g(t)\|^2 dt < \infty\). The inner product and the corresponding norm on a Hilbert space are denoted by \(|\cdot, \cdot|\), \(|\cdot|\), respectively.

For \(u \in L_2(J, V)\), the mild solution (see Pazy, 1983) of Equation (1.1) is given by

\[
x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - r)u(r)dr + \int_{t_0}^{r} T(t - r)f(r, x(r))dr. \tag{1.2}\]

By exact controllability of the system (Equation (1.1)), we mean that for every \(x_0\) and \(x_\epsilon \in V\), there exists \(u \in L_2(J, V)\) such that the mild solution \(x \in L_2(J, V)\) satisfies the condition \(x(r) = x_\epsilon\).

The semi linear control system (Equation (1.1)) is said to be approximately controllable if, for every \(\epsilon > 0\) and for every \(x_0\) and \(x_\epsilon \in V\), there exists \(u \in L_2(J, V)\) such that the corresponding mild solution \(x \in L_2(J, V)\) satisfies

\[
\|x(r) - x_\epsilon\| \leq \epsilon. \tag{1.3}\]

Assume that the semi linear control system (Equation (1.1)) is approximately controllable. The aim of this paper is to find a regularized control for the semi linear system (Equation (1.1)) for a given target state. To achieve this, we consider the linear control system as follows:

\[
\frac{dy}{dt} = Ay(t) + \hat{u}(t), \quad t \in J, \tag{1.4}\]

\[y(t_0) = x_0.\]

The mild solution of (Equation (1.4)) is given by

\[
y(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - r)\hat{u}(r)dr. \tag{2.5}\]

The system (Equation (1.4)) is approximately controllable under the following assumptions (Naito, 1987).

(i) The semigroup \(T(t)\) is compact for each \(t > 0\).

(ii) For every \(p \in L_2(J, V)\), there exists \(q \in V\) such that \(Lp = Lq\), where the operator \(L: L_2(J, V) \rightarrow V\) is defined as

\[
Lx = \int_{t_0}^{t} T(t - r)x(r)dr.
\]

Condition (i) implies that the system (Equation (1.4)) is never exactly controllable in finite time (Triggiani, 1977). Then the problem of finding the control for a given target state \(v \in V\) at the final time \(r\) is equivalent to the problem of solving the operator equation:
\[ L\hat{u} = v, \quad (1.6) \]
where \( L : L_2(J, V) \to V \) is the operator defined by
\[ L\hat{u} = \int_{t_0}^t T(\tau - r) \hat{u}(r) \, dr, \quad \hat{u} \in L_2(J, V), \quad (1.7) \]
and
\[ v = y(r) - T(r - t_0)x_0. \]

Since the linear system (Equation (1.4)) is approximately controllable, \( R(L) \), the range of \( L \), is dense in \( V \). Hence, Equation (1.6) is ill-posed in the sense of Hadamard (1923), i.e. small perturbations in \( v \) can produce large deviations in the solution, which is not desirable for practical problems. Therefore it is not advisable to solve (Equation (1.6)) directly to obtain \( \hat{u} \). One has to look for stable approximations \( \hat{u}_\lambda, \lambda > 0 \) such that
\[ \| L\hat{u}_\lambda - v \| \to 0 \text{ as } \lambda \to 0. \]

Thamban Nair, Sukavanam, and Ravinder (in press) constructed such a family for a linear approximately controllable system using Tikhonov regularization and the theory of ill-posed problems and also developed a method to select the regularization parameter \( \lambda \) which yielded some error estimates involved in the regularization procedure.

In this paper, we find a regularized control for the semi linear system (Equation 1.1). For this we shall use the theory of ill-posed problems and Tikhonov regularization.

The rest of the paper is organized as follows: In Section 2, some preliminaries are given which will be used in the sequel. The main result is presented in Section 3 and numerical results are presented in Section 4. Conclusions are made in Section 5.

2. Preliminaries

**Definition 2.1 (Well-posed problem)** Let \( \mathcal{U} \) and \( \mathcal{V} \) be normed linear spaces and \( L : \mathcal{U} \to \mathcal{V} \) be a linear operator. The equation
\[ Lu = v \quad (2.1) \]
is said to be well-posed if the following holds:

(i) For every \( v \in \mathcal{V} \), there exists a unique \( u \in \mathcal{U} \) such that \( Lu = v \).

(ii) \( L^{-1} \) is a bounded operator. Equivalently, for every \( v \in \mathcal{V} \) and for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) with the following properties: If \( \hat{v} \in \mathcal{V} \) with \( \| \hat{v} - v \| \leq \delta \) and if \( u, \tilde{u} \in \mathcal{U} \) are such that \( Lu = v \) and \( L\tilde{u} = \hat{v} \), then \( \| u - \tilde{u} \| \leq \varepsilon \).

**Definition 2.2 (Ill-posed problem)** Equation (2.1) is said to be ill-posed if \( L \) violates one of the conditions for well-posedness.

**Theorem 2.1 (Thamban Nair, 2009) (Tikhonov regularization)** Let \( \mathcal{U} \) and \( \mathcal{V} \) be Hilbert spaces and \( L : \mathcal{U} \to \mathcal{V} \) be a bounded linear operator. Then for each \( v \in \mathcal{V} \) and \( \lambda > 0 \), there exists a unique \( u_\lambda(v) \in \mathcal{U} \) which minimizes the map
\[ u \mapsto \| Lu - v \|^2 + \lambda \| u \|^2, \quad u \in \mathcal{U}. \]
Moreover, for each \( \lambda > 0 \), the map \( R_\lambda : \mathcal{V} \mapsto u_\lambda(v), \quad v \in \mathcal{V} \),
is a bounded linear operator from \( \mathcal{V} \) to \( \mathcal{U}^\prime \) and \( R_j v := (L^* L + \lambda I)^{-1} L^* v \) where \( L^* \) is the unique adjoint of the bounded linear operator \( L \).

**Theorem 2.2** *(Thamban Nair, 2009)* For \( \lambda > 0 \), the solution \( u_j \) of the operator equation

\[
(L^* L + \lambda I) u_j = L^* v \tag{2.2}
\]

minimizes the function \( u \mapsto \| Lu - v \|^2 + \lambda \| u \|^2 \), \( u \in \mathcal{U}' \) and

\( \| Lu_j - v \| \to 0 \) as \( \lambda \to 0 \).

**Definition 2.3** For \( v \in \mathcal{V} \) and \( \lambda > 0 \), the element \( u_j \in \mathcal{U}' \) as defined in Theorems 2.1 and 2.2 is called the Tikhonov regularized solution of \( Lu = v \).

For more details on ill-posed problems and regularization methods one can refer to Thamban Nair (2009), Engl, Hanke, and Neubauer (2003), Bakushinsky and Goncharsky (1994).

### 3. Main result

**Assumption H**

(i) The linear system (Equation (1.4)) is approximately controllable.

(ii) The nonlinear function \( f \) is Lipschitz continuous i.e. \( \exists c > 0 \) such that

\[ \| f(s, x) - f(s, y) \| \leq c \| x - y \|. \]

**Remark 3.1** The regularized control of nonlinear system (Equation (1.1)) is given by

\[ u_{\lambda}(t) = \hat{u}_{\lambda}(t) - f(t, y_{\lambda}(t)), \]

where \( \hat{u}_{\lambda}(t) \) is the regularized control of the linear system (Equation (1.4)) \( \forall t \in J \) and \( \lambda \) is the regularization parameter which plays a vital role in getting better approximations to the regularized control and the corresponding mild solution as well. We select \( \lambda \) as given in Thamban Nair et al. *(in press).*

Thamban Nair et al. *(in press)* proved the following theorem for a bounded linear operator \( L \) defined on Hilbert spaces \( \mathcal{U} \) and \( \mathcal{V} \) with slightly different notations.

**Theorem 3.1** *(Thamban Nair et al., in press)* For \( \lambda > 0 \), let \( u_{\lambda} \in \mathcal{U}' \) be the Tikhonov regularized solution of \( Lu = v \). Then we have the following.

(i) For \( \epsilon > 0 \), if \( v' \in R(L) \) is such that \( \| v - v' \| \leq \epsilon \) and if \( u' \in \mathcal{U}' \) is such that \( Lu' = v' \), then

\[
\| Lu_j - v \| \leq \frac{\sqrt{\lambda}}{2} \| u' \| + \epsilon \quad \forall \lambda > 0. \tag{3.1}
\]

(ii) If \( \lambda_\epsilon \in \left( 0, \frac{\epsilon^2}{\| u' \|^2} \right) \) then

\[
\| Lu_j - v \| \leq 2 \epsilon \quad \forall \lambda \in (0, \lambda_\epsilon).
\]

From part (ii) of Theorem 3.1, we select the regularization parameter \( \lambda \in (0, \lambda_\epsilon) \) where \( \lambda_\epsilon \in \left( 0, \frac{4 \epsilon^2}{\| u' \| \| u'' \|} \right) \). In this case we get

\[
\| Lu_j - v \| \leq 2 \epsilon \quad \forall \lambda \in (0, \lambda_\epsilon).
\]

Here \( u'' \) is the Tikhonov regularized solution of the ill-posed operator equation \( Lu'' = v' \) and \( u'' \) can be obtained by solving the well-posed operator equation...
\[ (L \cdot L + \mu I)u^{(\nu)} = L \cdot V. \]  

(3.2)

**Theorem 3.2** Under Assumption H, if \( u_j(t) = \hat{u}_j(t) - f(t, y_j(t)) \), then the solution of the nonlinear system (Equation (1.1)) is the same as that of the linear system (Equation (1.4)).

**Proof** The (regularized) mild solution of the nonlinear system (Equation (1.1)) is given by

\[ x_j(t) = T(t - t_0)X_0 + \int_{t_0}^{t} T(t - r)\left( u_j(r) + f(r, x_j(r)) \right) dr. \]  

(3.3)

Now, by substituting \( u_j(t) = \hat{u}_j(t) - f(t, y_j(t)) \) in (Equation (3.3)), we get

\[ x_j(t) = T(t - t_0)X_0 + \int_{t_0}^{t} T(t - r)\left( \hat{u}_j(r) - f(r, y_j(r)) + f(r, x_j(r)) \right) dr. \]  

(3.4)

Thus we have

\[ \| x_j(t) - y_j(t) \| = \left\| \int_{t_0}^{t} T(t - r)\left( f(r, x_j(r)) - f(r, y_j(r)) \right) dr \right\| \]  

(3.5)

\[ \leq M \int_{t_0}^{t} \| f(r, x_j(r)) - f(r, y_j(r)) \| dr \]  

(3.6)

\[ \leq M \int_{t_0}^{t} \| x_j(r) - y_j(r) \| dr. \]  

(3.7)

By using Grownwall’s inequality (Ruth & Curtain, 1977), we get \( x_j(t) = y_j(t) \forall t \in J \).

\[ \therefore x_j \equiv y_j. \]

This completes the proof.  

\[ \blacksquare \]

**4. Numerical results**

In this section, we give an application of the proposed theory to a heat control system. For a given target state, we compute the regularized control for the nonlinear approximately controllable system.

Consider a metal rod of length \( \ell \) that can be heated along its length according to the following mathematical model:

\[ \frac{\partial^2 z}{\partial t^2}(s, t) = \frac{\partial^2 z}{\partial s^2}(s, t) + u(s, t) + \sin(z(s, t)), s \in [0, \ell], \]  

(4.1)

\( z(0, t) = 0 = z(\ell, t), \)

\( z(s, 0) = g_0(s), \quad g_0 \in L_2[0, \ell], \)

where \( z(s, t) \) represents the temperature at position \( s \) at time \( t \), \( g_0(s) \) is the initial temperature profile, and \( u(s, t) \) is the heat input along the rod. Here, \( f(t, z(s, t)) = \sin(z(s, t)) \) is the nonlinear function and it is a Lipschitz continuous function with Lipschitz constant \( c = 1 \).

Let \( J = [0, r], \quad V = L_2[0, \ell], \quad U = L_2(J, V), \quad \nu = V. \)

Let \( \hat{g} \in V \) be the required distribution of the temperature at the final time \( r > 0 \). The question of whether one can heat the rod in such a way as to obtain \( z(s, r) = \hat{g}(s), \quad s \in [0, \ell] \) is identical to asking whether, for system (Equation (4.1)), the initial state \( g_0(.) \) can be transferred to \( \hat{g}(.) \) in time \( r \). As the
system (Equation (4.1)) is approximately controllable (and not exactly controllable), in general it cannot be transferred exactly to $\hat{g}(.)$ in time $\tau$. However it can be transferred arbitrarily closely to $\hat{g}(.)$.

As we explained in the main result, in order to find the regularized control of the nonlinear system (Equation (4.1)), consider the following linear control system:

$$\frac{\partial \hat{y}}{\partial t}(s, t) = \frac{\partial^2 \hat{y}}{\partial s^2}(s, t) + \hat{u}(s, t), s \in [0, \epsilon],$$
$$\hat{y}(0, t) = 0 = \hat{y}(\epsilon, t),$$
$$\hat{y}(s, 0) = g_0(s), g_0 \in L_2[0, \epsilon].$$

(4.2)

Define the operator $A$ by

$$Ah = \frac{d^2 h}{ds^2}, \quad h \in D(A),$$

where

$$D(A) = \{ h \in V; \hat{h} \in L_2[0, \epsilon], h(0) = h(\epsilon) \}.$$

By using the notation $y(t) = \hat{y}(., t)$ and $\hat{u}(t) = \hat{u}(., t)$, Equation (4.2) takes the form of the control system (Equation (1.4)), i.e.,

$$\frac{dy}{dt} = Ay(t) + \hat{u}(t), \quad t \in J,$$
$$y(0) = g_0.$$

(4.3)

For $U \subseteq U^\ast$, the mild solution of Equation (4.3) is given by

$$y(t) = T(t)g_0 + \int_0^t T(t - r)\hat{u}(r)dr,$$

(4.4)

where $\{T(t); t \geq 0\}$ is the $C_0$-semigroup generated by the operator $A$, which is given by

$$T(t)g = \sum_{j=1}^{\infty} e^{-\frac{1}{\epsilon} \frac{j \pi s}{\epsilon}} \langle g, \phi_j(s) \rangle \phi_j(s), \quad g \in L_2[0, \epsilon],$$

with $\phi_j(s) = \sqrt{2 \epsilon} \sin \left( \frac{j \pi s}{\epsilon} \right)$ for $s \in [0, \epsilon]$. It is known that the operator $T(t)$ is compact for each $t \in (0, \epsilon]$ (cf. Curtain & Zwart, 2012). In this case, for $\hat{u} \in U^\ast$, we have

$$L \hat{u} = \int_0^t T(t - r)\hat{u}(r)dr = \int_0^t \left[ \sum_{j=1}^{\infty} e^{-\frac{1}{\epsilon} \frac{j \pi s}{\epsilon}} \langle \hat{u}(r), \phi_j \rangle \phi_j(s) \right] dr.$$

Recall that the adjoint of $L$ is the unique bounded linear operator $L^\ast: V \rightarrow L_2(J, V)$ which satisfies

$$\langle Lu, v \rangle = \langle u, L^\ast v \rangle, \quad u \in L_2(J, V), v \in V.$$

Hence,

$$\langle L^\ast v \rangle(r) = \sum_{j=1}^{\infty} e^{-\frac{1}{\epsilon} \frac{j \pi s}{\epsilon}} \langle v, \phi_j(s) \rangle$$

(4.5)

and
\[
LL^*v = \sum_{j=1}^{\infty} \sigma_j^2 (v, \phi_j) \phi_j(s) \quad \text{where} \quad \sigma_j^2 = \left[ \frac{1 - e^{-\lambda \frac{2j^2}{\pi^2}}}{2j^2 \pi^2} \right] e^2. \tag{4.6}
\]

Since \(T(t)\) is compact for each \(t > 0\), the operators \(L^*\) and \(LL^*\) are also compact operators. Hence \(R(L)\) is a proper dense subspace of \(V\) (cf. Thamban Nair, 2002). Consequently, the system (Equation (4.3)) (equivalently, Equation (4.2)) is not exactly controllable, but is approximately controllable (Russell, 1978).

Here, \(\sigma_1^2, \sigma_2^2, \ldots\) are nonzero eigenvalues of \(LL^*\) with corresponding eigen vectors \(\phi_1, \phi_2, \ldots\)

Writing \(\psi_n := \frac{L^* \phi_n}{\sigma_n}, n = 1, 2, \ldots\), we see that \(\{\psi_n; n = 1, 2, \ldots\}\) is an orthonormal basis of \(R(L^*)\).

Hence, for every \(v \in V\),
\[
L^* v = \sum_{n=1}^{\infty} \psi_n < v, \psi_n > \psi_n = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} \psi_n < v, LL^* \psi_n > \psi_n = \sum_{n=1}^{\infty} \sigma_n \psi_n < v, \psi_n > \psi_n. \tag{4.7}
\]

To find a control corresponding to the final state \(y(r) = g\), we need to solve the operator equation

\[L \hat{u} = v \quad \text{with} \quad v = g - T(r)g_0.\]

Thus we have a representation for the regularized control \(\hat{u}_j\) for the linear approximately controllable system (Equation (4.3)) defined by

\[\hat{u}_j = L^*(LL^* + \lambda I)^{-1}v.\]

From Equations (4.6) and (4.7), we have

\[\hat{u}_j(r) = \sum_{j=1}^{\infty} \frac{e^{-\frac{\lambda \pi^2}{2j^2}}}{\sigma_j^2 + \lambda} (v, \phi_j) \phi_j(s).\]

Then, we have

\[v - L \hat{u}_j = \sum_{j=1}^{\infty} \frac{\lambda}{\sigma_j^2 + \lambda} < v, \phi_j > \phi_j(s) \]

so that

\[\|v - L \hat{u}_j\|^2 = \sum_{j=1}^{\infty} \frac{\lambda^2}{(\sigma_j^2 + \lambda)^2} | < v, \phi_j > |^2.\]

From this, it follows that \(\|v - L \hat{u}_j\| \to 0\) as \(\lambda \to 0\).

Hence, from Theorem 3.2, the regularized control of the nonlinear system (Equation (4.1)) for a given target state \(v\) is given by

\[u_j(s, t) = \hat{u}_j(s, t) - f(t, y_j(s, t)) = \sum_{j=1}^{\infty} \frac{e^{-\frac{\lambda \pi^2}{2j^2}}}{\sigma_j^2 + \lambda} (v, \phi_j) \sqrt{2} \sin(j\pi s) - f(y_j(s, t)), \tag{4.8}\]

where \(\hat{u}_j\), \(y_j\) are the regularized control and the corresponding mild solution of the system (Equation (4.2)), respectively.

For numerical experiments, we consider the system (Equation (4.1)) with \(\varepsilon = 1\), \(\tau = 10\), and \(\lambda_k = \frac{1}{k}, k \in \mathbb{N}\).
For $N = 1, 2, \ldots$, let

$$E(k, N) = \sqrt{\sum_{j=1}^{N} \frac{\lambda^2 < v, \phi_j >^2}{(\sigma^2_j + \lambda j^2)}}, \quad k = 1, 2, \ldots.$$  

Then we have

$$E(k) = \lim _{N \to \infty} E(k, N) = ||L \tilde{u}_k - v||.$$  

Here, $E(k)$ is the error involved in the regularization procedure.

**Example 1** Let

$$g_0(s) = \sin(\pi s), \quad g_j(s) = s^j(1 - s).$$

Then the target function $v_1$ is given by

$$v_1(s) = g_j(s) - [T(\tau)g_0](s) = s^j(1 - s) - \frac{e^{-10s}}{\sqrt{2}} \phi_j(s).$$

Thus we have,

$$\tilde{u}_k(t) = \sum_{j=1}^{\infty} \left[ \frac{k e^{-j^2(10 - t)}}{k\sigma_j^2 + 1} \right] < g, \phi_j > \sqrt{2} \sin(\pi s) - \frac{k e^{-j^2(10 - t)}}{k\sigma_j^2 + 1} \sin(\pi s),$$  

(4.9)

where

$$< g, \phi_j > = \sqrt{2} \left[ \frac{6(-1)^j}{(j\pi)^3} + \frac{24(-1)^j}{(j\pi)^5} - \frac{24}{(j\pi)^5} \right].$$

The regularized mild solution of the linear control system (Equation 4.3) corresponding to the regularized control $\tilde{u}_j$ is given by

Figure 1. Truncated regularized control graphs for (a) $\lambda = 0.01$, (b) $\lambda = 0.001$, (c) $\lambda = 0.0001$, (d) $\lambda = 0.00001$. 

\[ y_j(s, t) = e^{\frac{t}{2-j}} \sin(xs) + \sum_{j=1}^{\infty} \frac{12(-1)^j}{(j\pi)^3} + \frac{48(-1)^j}{(j\pi)^5} - \frac{48}{(j\pi)^7} \]
\[ \times \frac{e^{\frac{t}{2-j}}(10+4t) - e^{\frac{t}{2-j}}(10+4t+2x^2)}{\pi^2 + \lambda} \sin(j\pi s). \]

Hence, from Equation (4.8), the regularized control for the nonlinear system (Equation (4.1)) is given by
\[ u_j(s, t) = \tilde{u}_j(s, t) - \sin(y_j(s, t)), \]
where \( \tilde{u}_j(s, t) \) and \( y_j(s, t) \) are given by Equations (4.9) and (4.10), respectively.

We truncated the series \( u_j(s, t) \) at \( j = 100 \) and, for different \( \lambda \) values, the truncated regularized control graphs are shown in Figure 1.

5. Conclusion
In the control theory literature, Tikhonov regularization is not given much attention in the context of the problems related to approximately controllable systems. In this paper, we computed a regularized control for an approximately controllable nonlinear system by finding a regularized control of linear approximately controllable system using Tikhonov regularization under the assumption of Lipschitz continuity of the nonlinear function. The theory is substantiated with numerical results. The results can be further improved with other regularization parameter choice rules and regularization techniques.

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