Polynomial invariants of degree 4 for even-$n$ qubits and their applications in entanglement classification

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We develop a simple method for constructing polynomial invariants of degree 4 for even-$n$ qubits and give explicit expressions for these polynomial invariants. We demonstrate the invariance of the polynomials under stochastic local operations and classical communication and exemplify the use of the invariance in classifying entangled states. The absolute values of these polynomial invariants are entanglement monotones, thereby allowing entanglement measures to be built. Finally, we discuss the properties of these entanglement measures.

I. INTRODUCTION

In spite of the recent rapid progress in experimental realization of entangled states with large numbers of trapped ions [1] and photons [2], progress in theoretical studies of the entanglement quantification and classification of quantum states of a large number of qubits has been made through a gradual and continuous accumulation of bits of knowledge. The understanding of these fundamental features of the quantum world is utterly important, even if practical applications do not follow on a short time scale.

Quantum entanglement can be viewed as a crucial resource for quantum information tasks such as teleportation and cryptography. As different tasks require different resources, it is necessary to introduce an equivalence relation such that two quantum states belonging to the same equivalence class can perform the same tasks. Of particular importance is the equivalence under stochastic local operations and classical communication (SLOCC). For two qubits, there are only two SLOCC classes. For three qubits, six SLOCC classes have been distinguished [3]. For four or more qubits, there are infinite SLOCC classes and it is highly desirable to partition the infinite classes into a finite number of families. The key lies in finding criteria to determine which family an arbitrary quantum state belongs to. Extensive efforts have been devoted to the SLOCC entanglement classification of four qubits [4,11] and recently, a few attempts have been made toward the generalization to a higher number of qubits [12–18].

Polynomial functions in the coefficients of states which are invariant under SLOCC play a critical role in the investigation of entanglement measures and entanglement classification. For two and three qubits, the concurrence and the three-tangle are polynomial invariants of degrees 2 and 4, respectively, and they are the only polynomials in these cases. Considerable efforts have been undertaken over the last decade on the study of polynomial invariants for four or more qubits [3,11,12,13]. Although several approaches can potentially be used to construct polynomial invariants for more than four qubits [21–23], calculations become increasingly difficult as the number of qubits increases. Accordingly, the expressions of polynomial invariants have thus far been given only up to five qubits. Furthermore, while entanglement measures might be built from the absolute values of these polynomial invariants, the properties of those measures are very difficult to analyze.

In this paper, we first develop a method for constructing polynomial invariants of degree 2 for even-$n$ qubits. We then extend this method to construct polynomial invariants of degree 4 for even-$n$ qubits. The polynomial invariants are in the simple form of products of coefficient vectors, thereby allowing us to derive the explicit expressions. We also demonstrate that these polynomial invariants satisfy certain SLOCC equations. This leads to a classification under SLOCC depending on the vanishing or not of the polynomial invariants, as exemplified here for several even-$n$-qubit entangled states. The absolute values of the polynomial invariants are entanglement monotones, giving rise to a natural way to quantify entanglement of even-$n$-qubit states. In addition, having found the explicit expressions helps make it possible to explore the properties of these entanglement measures.
We first revisit the three well-known basis polynomial invariants of degree 4 for four qubits: $L$, $M$, and $N$ [20]. These three polynomial invariants are in an elegant form of determinants of coefficient matrices. In particular, $L$ is the determinant of coefficient matrix whose entries are the coefficients $a_0, a_1, \ldots, a_{15}$ arranged in ascending lexicographical order. The polynomial invariants $M$ and $N$ can be obtained by taking transpositions $(1,4)$ and $(1,3)$ to $L$, respectively (ignoring the sign). For even-$n$ ($n > 4$) qubits, determinants of coefficient matrices are polynomial invariants of degree $2^n/2$ [20, 28, 32]. Similar to the case of four qubits, a base polynomial invariant can be constructed as the determinant of coefficient matrix whose entries are the coefficients $a_0, a_1, \ldots, a_{2^n-1}$ arranged in ascending lexicographical order. Then a number of $\binom{n-1}{n/2-1}$ polynomial invariants of degree $2^n/2$ can be obtained by taking appropriate permutations of qubits to the base polynomial invariant. As has been previously noted, the above polynomial invariants of degree $2^n/2$ are closely related to reduced density matrices and have a direct physical meaning [17].

We begin by developing a method for constructing a polynomial invariant of degree 2 for even-$n$ qubits. The polynomial invariant is in the simple form of the scalar product of two coefficient vectors. More specifically, given an even-$n$-qubit state $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$, we split the coefficients $a_0, a_1, \ldots, a_{2^n-1}$ (in ascending order of their subscripts) into two halves: $H_0 = (a_0, a_1, \ldots, a_{2^{n-1}-1})$ and $H_1 = (a_{2^{n-1}}, a_{2^{n-1}+1}, \ldots, a_{2^n-1})$. Based on $H_0$ and $H_1$, we define the following two useful coefficient vectors: $H_0^R = ((-1)^{p(0)} a_0, (-1)^{p(1)} a_1, \ldots, (-1)^{p(2^{n-1}-1)} a_{2^{n-1}-1})$, where $p(\ell)$ is the parity of $\ell$; i.e., $p(\ell)$ is the sum of the bits in binary representation of $\ell$; and $H_1^R = (a_{2^{n-1}}, a_{2^n-2}, \ldots, a_{2^n-1})$, i.e., $H_1^R$ contains the elements of $H_1$ in reverse order. Let $G_n(|\psi\rangle)$ be the quadratic function defined by $G_n(|\psi\rangle) = H_0^R(H_1^R)^T$. Here superscript $T$ denotes the transpose. A simple calculation yields

$$G_n(|\psi\rangle) = \sum_{i=0}^{2^n-1} (-1)^{p(i)} a_i a_{2^n-1-i}.$$  

(1)

It turns out that $G_n(|\psi\rangle)$ is a polynomial invariant of degree 2 for even-$n$ qubits. Indeed, suppose that $|\psi\rangle$ and $|\psi'\rangle$ are two SLOCC equivalent states, i.e., there exist local invertible operators $A_1, A_2, \ldots, A_n$ such that

$$|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\psi'\rangle,$$

then $G_n(|\psi\rangle)$ and $G_n(|\psi'\rangle)$ satisfy the following SLOCC equation [25, 27, 33]:

$$G_n(|\psi\rangle) = G_n(|\psi'\rangle) \prod_{i=1}^{n} \det A_i.$$  

(3)

For example, letting $n = 2$, we have $G_4(|\psi\rangle) = a_0 a_3 - a_1 a_2$. Letting $n = 4$ gives

$$G_4(|\psi\rangle) = a_0 a_{15} - a_1 a_{14} - a_2 a_{13} + a_3 a_{12} - a_4 a_{11} + a_5 a_{10} + a_6 a_9 - a_7 a_8,$$

(4)

and this turns out to be equal to the polynomial invariant $H$ of degree 2 for four qubits in [20]. It is worth noting that $2G_n(|\psi\rangle)$ is an extension of the concurrence to even-$n$ qubits [33].

The above construction, then, may be extended to obtain polynomial invariants of degree 4 for even-$n$ qubits. Here we split the coefficients $a_0, a_1, \ldots, a_{2^n-1}$ (in ascending order of their subscripts) into four equal groups: $Q_0, Q_1, Q_2$, and $Q_3$, each comprising a quarter of the coefficients [for example, for four qubits, we have $Q_0 = (a_0, a_1, a_2, a_3)$, $Q_1 = (a_4, a_5, a_6, a_7)$, $Q_2 = (a_8, a_9, a_{10}, a_{11})$, and $Q_3 = (a_{12}, a_{13}, a_{14}, a_{15})$. In analogy with the definitions of $H_0^R$ and $H_1^R$, we may define $Q_i^R$ and $Q_i^T$ for $i = 1, \ldots, 4$. Let $P_n(|\psi\rangle)$ be the quartic function defined as

$$P_n(|\psi\rangle)$$

$$= \frac{1}{2} Q_0^R(Q_0^R)^T Q_3^R(Q_3^R)^T + \frac{1}{2} Q_1^R(Q_1^R)^T Q_2^R(Q_2^R)^T + Q_0^T(Q_1^R)^T Q_2^R(Q_3^R)^T + Q_0^T(Q_3^R)^T Q_2^R(Q_1^R)^T - Q_0^T(Q_2^R)^T Q_1^R(Q_3^R)^T.$$  

(5)

For $n = 2$, we have $P_2 = G_2^2/2$. For $n \geq 4$, $P_n(|\psi\rangle)$ is irreducible (as opposed to reducible polynomial invariants of degree 4 such as the $n$-tangle [19], which is just the square of the concurrence of even-$n$ qubits [28, 31, 33]) and the explicit expression of $P_n(|\psi\rangle)$ is given in the Appendix. In the following we will concentrate our attention on $n \geq 4$.

It is worth pointing out that $P_n(|\psi\rangle)$ is not in the form of determinants. Indeed, a close examination reveals that $P_4(|\psi\rangle) = 2N + L$, and it is clear that $P_4(|\psi\rangle)$ is not a determinant.

Next, we shall show that $P_n(|\psi\rangle)$ is invariant under SLOCC. We need the following lemma.
Lemma. For even \( n \geq 4 \) qubits, if \(|\psi\rangle\) and \(|\psi'\rangle\) are related by

\[
|\psi\rangle = I \otimes \cdots \otimes I \otimes A_k \otimes I \otimes \cdots \otimes I |\psi'\rangle,
\]

where \( I \) is the identity and \( A_k \) (\( 1 \leq k \leq n \)) is a local invertible operator, then

\[
P_n(|\psi\rangle) = P_n(|\psi'\rangle) \left[ \det(A_k) \right]^2.
\]

Proof. We only give the proof for \( \ell = 1 \). The proofs for other cases can be given analogously. Assume that \(|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle\) and \(|\psi'\rangle = \sum_{i=0}^{2^n-1} b_i |i\rangle\). Let

\[
A_1 = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}.
\]

In view of Lemma 1 in Ref. [32], we have

\[
a_s = \beta_1 b_s + \beta_2 b_{2^n-s+1},
\]

where \( 0 \leq s \leq 2^{n-1} - 1 \).

Substituting Eqs. (7) and (8) into \( P_n(|\psi\rangle) \) and combining like terms, all but the three terms disappear and we are left with (after some tedious calculation)

\[
P_n(|\psi'\rangle) = P_n(|\psi'\rangle) \beta_1^2 \beta_2^2 + P_n(|\psi'\rangle) \beta_3^2 \beta_4^2 - 2P_n(|\psi'\rangle) \beta_1 \beta_2 \beta_3 \beta_4 = P_n(|\psi'\rangle) \left[ \det(A_1) \right]^2.
\]

Q.E.D.

We now assert that the following SLOCC equation holds:

Theorem. Let \(|\psi\rangle\) and \(|\psi'\rangle\) be two SLOCC equivalent pure states of even \( n \geq 4 \) qubits, i.e., \(|\psi\rangle\) and \(|\psi'\rangle\) satisfy Eq. (2), then \( P_n(|\psi\rangle) \) and \( P_n(|\psi'\rangle) \) satisfy the following SLOCC equation:

\[
P_n(|\psi\rangle) = P_n(|\psi'\rangle) \left[ \prod_{i=1}^{n} \det A_i \right]^2.
\]

Proof. We use the induction principle.

Base case. Clearly, the theorem holds true for \( A_k = I \) (\( I \) is the identity), \( k = 1, \ldots, n \).

Inductive step. Suppose that the theorem holds for \( A_k = I, k = 1, \ldots, \ell \). Next let us consider the case where \( A_k = I, k = 1, \ldots, \ell - 1 \). Let \(|\psi\rangle\) and \(|\psi'\rangle\) be two states such that

\[
|\psi\rangle = (I \otimes \cdots \otimes I \otimes A_{\ell+1} \otimes \cdots \otimes I) \times (I \otimes \cdots \otimes I \otimes A_{\ell+1} \otimes \cdots \otimes A_n) |\psi'\rangle.
\]

Applying the above lemma to the first part on the right-hand side of Eq. (11) contributes a factor of \( \det A_{\ell+1}^2 \). This, together with the product of \( \det A_{\ell+1}^2 \) for \( i = \ell + 1, \ldots, n \) [invoking the inductive hypothesis to the second part on the right-hand side of Eq. (11)], completes the inductive step, and the proof of the theorem. Q.E.D.

As an immediate consequence of the theorem, we obtain the following result.

Corollary. For any two SLOCC equivalent pure states \(|\psi\rangle\) and \(|\psi'\rangle\) of even-\( n \) qubits, either \( P_n(|\psi\rangle) \) and \( P_n(|\psi'\rangle) \) both vanish or neither vanishes. In other words, clearly, if either \( P_n(|\psi\rangle) \) or \( P_n(|\psi'\rangle) \) vanishes while the other does not, then the two states \(|\psi\rangle\) and \(|\psi'\rangle\) are SLOCC inequivalent. Therefore, Eq. (11) can be used for SLOCC classification of even-\( n \) qubits.

More SLOCC polynomial invariants of degree 4 for even-\( n \) qubits can be constructed by taking permutations of qubits. This can be done as follows. Simply taking permutations \( \sigma \) to both sides of Eq. (11) yields

\[
\sigma P_n(|\psi\rangle) = \sigma P_n(|\psi'\rangle) \left[ \prod_{i=1}^{n} \det A_i \right]^2.
\]

It follows immediately from Eq. (12) that \( \sigma P_n(|\psi\rangle) \) is also a polynomial invariant of degree 4 and therefore it can be used for SLOCC classification for even-\( n \) qubits. Moreover, for four qubits, there are three polynomial invariants of degree 4, and for even \( n \geq 6 \) qubits, there are \( n(n-1)/2 \) polynomial invariants of degree 4.

III. POLYNOMIAL INVARIANTS OF DEGREE 4 FOR FOUR QUBITS

In addition to \( P_4(|\psi\rangle) \), we may obtain two more polynomial invariants of degree 4 for four qubits by taking permutations \((1,3)\) and \((1,4)\), respectively. We let \( P_4'(|\psi\rangle) = (1,3)P_4(|\psi\rangle) \) and \( P_4''(|\psi\rangle) = (1,4)P_4(|\psi\rangle) \). It is readily verified that, ignoring the sign, the set formed by \( P_4(|\psi\rangle), P_4'(|\psi\rangle), \) and \( P_4''(|\psi\rangle) \) is invariant with respect to permutations of qubits, i.e., applying any transposition to any one of the three polynomials always yields a polynomial in the same set (for details, see Table I). These three polynomials \( P_4(|\psi\rangle), P_4'(|\psi\rangle), \) and \( P_4''(|\psi\rangle) \) may be used as basis polynomials for degree-4 polynomial invariants of four qubits. Further inspection reveals that \( P_4, P_4', \) and \( P_4'' \) are related to \( L, M, \) and \( N \) via

\[
P_4(|\psi\rangle) = 2N + L,
\]

\[
P_4'(|\psi\rangle) = 2M + N,
\]

\[
P_4''(|\psi\rangle) = 2L + M.
\]
Also note that these three polynomials are pairwise linearly independent, but the three polynomials together are linearly dependent since \( P_4(|\psi\rangle) + P'_4(|\psi\rangle) + P''_4(|\psi\rangle) = 0 \).

**TABLE I. Polynomial invariants of degree 4 for four qubits under permutations of qubits.**

| Transpositions | \( P_4(|\psi\rangle) \) | \( P'_4(|\psi\rangle) \) | \( P''_4(|\psi\rangle) \) |
|----------------|----------------|----------------|----------------|
| (1,2)          | \( P_4(|\psi\rangle) \) | \( P'_4(|\psi\rangle) \) | \( P''_4(|\psi\rangle) \) |
| (1,3)          | \( P'_4(|\psi\rangle) \) | \( P_4(|\psi\rangle) \) | \( P''_4(|\psi\rangle) \) |
| (1,4)          | \( P''_4(|\psi\rangle) \) | \( P'_4(|\psi\rangle) \) | \( P_4(|\psi\rangle) \) |

**IV. SLOCC CLASSIFICATION OF EVEN-\( n \) QUBITS**

Consider, for example, the even-\( n \)-qubit cluster states

\[
|\text{CL}_1\rangle_n = \frac{1}{2}(|00...00\rangle + |0010...0\rangle - |1111...0\rangle), \tag{16}
\]

and

\[
|\text{CL}_2\rangle_n = \frac{1}{2}(|00...00\rangle + |0101...0\rangle + |1010...10\rangle - |1111...11\rangle). \tag{17}
\]

**TABLE II. Values of even-\( n \)-qubit polynomial invariants of degrees 2, 4, and 6 for some even-\( n \)-qubit states.**

| Degree | Poly. | \( |\text{GHZ}\rangle_n \) | \( |W\rangle_n \) | \( n/2, n \) | \( |\text{CL}_1\rangle_n \) | \( |\text{CL}_2\rangle_n \) |
|--------|-------|----------------|----------------|--------------|----------------|----------------|
| 2      | \( G_n \) | \( \neq 0 \) | \( 0 \) | \( \neq 0 \) | \( a \) | \( a \) |
| 4      | \( P_n \) | \( 0 \) | \( 0 \) | \( \neq 0 \) | \( 0^b \) | \( 0^c \) | \( \neq 0 \) |
| 6      | \( D^{(n)} \) | \( 0 \) | \( 0 \) | \( \neq 0 \) | \( 0 \) | \( 0 \) |

\( a \) Zero for \( n/2 \) even and nonzero for \( n/2 \) odd.
\( b \) With the exception of \( P_4(|2, 4\rangle) = 0 \).
\( c \) With the exception of \( P_4(|\text{CL}_1\rangle_n) = -1/8 \).

We list in Table II the values of even-\( n \)-qubit polynomial invariants of degree 2 (see Eq. [15]), degree 4 (see Eq. [14]), and degree 6 (see [22]) for some even-\( n \)-qubit states. We may determine whether two states in the table are inequivalent to each other under SLOCC via the vanishing or not of the polynomial invariants. For example, since \( P_n \) vanishes for the states \( |\text{GHZ}\rangle_n, |W\rangle_n, \) and \( |\text{CL}_1\rangle_n \) while \( P_n \) is nonzero for the state \( |\text{CL}_2\rangle_n \), we may conclude that \( |\text{CL}_2\rangle_n \) for \( n > 4 \) is SLOCC inequivalent to the states \( |\text{GHZ}\rangle_n, |W\rangle_n, \) and \( |\text{CL}_1\rangle_n \). Likewise, we may conclude that the Dicke state \( |n/2, n\rangle, (n > 4) \) is SLOCC inequivalent to the states \( |\text{GHZ}\rangle_n, |W\rangle_n, \) and \( |\text{CL}_1\rangle_n \).

As discussed above, the space of even-\( n \)-qubit states can be divided into two SLOCC inequivalent subspaces according to the vanishing or not of a polynomial invariant. Suppose that \( \sigma_i P_n (|\psi\rangle) \), \( i = 1, ..., m \) are different polynomial invariants. We define families \( F_{\sigma_i}^n = \{|\psi\rangle | \sigma_i P_n (|\psi\rangle) = 0 \} \) and \( F_{\sigma_i}^n = \{|\psi\rangle | \sigma_i P_n (|\psi\rangle) \neq 0 \} \). In view of Eq. [12], families \( F_{\sigma_i}^n \) and \( F_{\sigma_i}^n \) are SLOCC inequivalent. A more refined partition can be obtained by taking the intersection of the families: \( F_{\sigma_0, \sigma_1, ..., \sigma_m} = F_{\sigma_0}^n \cap F_{\sigma_1}^n \cap ... \cap F_{\sigma_m}^n \), where \( i_0, i_1, ..., i_m \in \{0, 1\} \). Clearly, families \( F_{\sigma_0, \sigma_1, ..., \sigma_m} \) and \( F_{\sigma_0', \sigma_1', ..., \sigma'_m} \) are SLOCC inequivalent when \( i_0, i_1, ..., i_m \neq i'_0, i'_1, ..., i'_m \).

**V. PROPERTIES OF THE ENTANGLEMENT MEASURE**

We call \( |P_n (|\psi\rangle) | \) a degree-4 entanglement measure. With the help of the explicit expression, we point out the following properties:

(i) \( 0 \leq |P_n (|\psi\rangle) | \leq 1 \).

(ii) \( |P_n (|\psi\rangle) | \) is a monotone function [27, 31].

(iii) \( |P_n (|\psi\rangle) | \) is invariant under determinant-1 SLOCC operations, especially under local unitary operations, and remains zero or non-zero under SLOCC.

(iv) Let \( |\phi\rangle \otimes |\omega\rangle_{n-\ell} \) be an even-\( n \)-qubit product state, where \( |\phi\rangle_{\ell} \) is a state of the first \( \ell \) qubits and \( |\omega\rangle_{n-\ell} \) is a state of the remaining \( n-\ell \) qubits. Then \( |P_n (|\phi\rangle_{\ell} \otimes |\omega\rangle_{n-\ell}) | \) vanishes for odd \( \ell \), especially for the full separate state.

(v) Assume the same product state as given in property iv. Then \( |P_n (|\phi\rangle_{\ell} \otimes |\omega\rangle_{n-\ell}) | \) is multiplicative for even \( \ell \).

We distinguish two cases.

**Case 1.** If \( \ell = 2 \), then \( |P_n (|\phi\rangle_2 \otimes |\omega\rangle_{n-2}) | = \frac{1}{2} |C_2 (|\phi\rangle_2) |^2 \times |C_{n-2} (|\omega\rangle_{n-2}) |^2 \),

where \( C_i (|\varphi\rangle) = 2 |G_i (|\varphi\rangle) | \) is the concurrence of \( i \) qubits. In other words, the degree-4 measure of the product state of even-\( n \) qubits is...
one-eighth of the product of the square of the concurrence of a two-qubit state and the square of the concurrence of an \((n - 2)\)-qubit state.

Case 2. If \(\ell \geq 4\), then \(|P_{\ell}((\phi)_{\ell} \otimes |\omega\rangle_{n-\ell})| = |P_{\ell}(\langle\phi\rangle_{\ell})| \cdot |C_{n-\ell}(|\omega\rangle_{n-\ell})|^2\). In other words, the degree-4 measure of the product state of even-\(n\) qubits is the product of the degree-4 measure of an \(\ell\)-qubit state and the square of the concurrence of an \((n - \ell)\)-qubit state.

In Tables III and IV we summarize the entanglement measure built upon the concurrence for even-\(n\) qubits (i.e., \(C_n\)), the degree-4 measure for even-\(n\) qubits (i.e., \(|P_n|\)), the degree-6 measure for even-\(n\) qubits (here denoted as \(|D^{(n)}|\), see [32] for details), and the degree-4* measure for odd-\(n\) qubits (here denoted as \(\tau\) and based on the polynomial invariant of degree 4 for odd-\(n\) qubits in [27]) and their properties on product states \(|\phi\rangle_{\ell} \otimes |\omega\rangle_{n-\ell}\). Consulting Tables III and IV we see that the entanglement measures built upon even-\(n\)-qubit polynomial invariants vanish for odd \(\ell\) and are multiplicative for even \(\ell\), whereas the entanglement measure built upon odd-\(n\)-qubit polynomial invariants is multiplicative for odd \(\ell \geq 3\) and vanishes for even \(\ell\) and \(\ell = 1\). Whether this holds for any entanglement measure built upon even (respectively, odd) \(n\)-qubit polynomial invariants remains open. In Table V we list the values of the measure \(|P_n|\) for some entangled states.

Finally, we extend the measure based on the polynomial invariant of degree 4 to mixed states via the convex roof construction [34]:

\[
\tau(\rho) = \min \sum_i p_i |P_n(\psi_i)|,
\]

where \(p_i \geq 0\) and \(\sum_i p_i = 1\), and the minimum is taken over all possible decompositions of \(\rho\) into pure states, i.e., \(\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|\).

### VI. CONCLUSION

We have presented a simple method for constructing polynomial invariants of degree 2 and 4 for even-\(n\) qubits. The polynomial invariants are in the form of products of coefficient vectors. As a result, the explicit expressions of the polynomial invariants can be easily calculated. We have shown that in the four-qubit case, these polynomial invariants are closely related to the known ones in the literature. We have also discussed the use of the polynomial invariants in entanglement classification and in the construction of entanglement measure: a SLOCC classification of even-\(n\)-qubit states can be achieved via the vanishing or not of the polynomial invariants; the absolute values of the polynomial invariants give rise to

| Measure | Degree | Qubits | \(|\phi\rangle_{\ell} \otimes |\omega\rangle_{n-\ell}\) |
|---------|--------|--------|------------------|
| Concurrence | 2 | Even | 0 | Mult.* |
| Degree-4 | 4 | Even | 0 | Mult.* |
| Degree-6 | 6 | Even | 0 | Mult.* |
| Degree-4* | 4 | Odd | Mult.* | 0 |

* Here "Mult." is an abbreviation for "multiplicative".

Table III. Properties of entanglement measures built upon polynomial invariants on product states.

| Measure | \(\ell\) | \(|\phi\rangle_{\ell} \otimes |\omega\rangle_{n-\ell}\) |
|---------|--------|------------------|
| Concurrence | Even \(\ell\) | \(C_{\ell}(\langle\phi\rangle_{\ell})C_{n-\ell}(|\omega\rangle_{n-\ell})\) |
| Degree-4 | \(\ell = 2\) | \(\frac{1}{4}[C_n(|\phi\rangle_2)^2[C_n-2(|\omega\rangle_{n-2})]^2\) |
| | Even \(\ell \geq 4\) | \(|P_{\ell}(\langle\phi\rangle_{\ell})|C_{n-\ell}(|\omega\rangle_{n-\ell})|^2\) |
| Degree-6 | \(\ell = 2\) | \(\frac{1}{4}\langle C_n(|\phi\rangle_2)^3[C_n-2(|\omega\rangle_{n-2})]^3\) |
| | Even \(\ell \geq 4\) | \(|D^{(\ell)}(\langle\phi\rangle_{\ell})|C_{n-\ell}(|\omega\rangle_{n-\ell})|^3\) |
| Degree-4* | Odd \(\ell \geq 3\) | \(\tau(\langle\phi\rangle_{\ell})C_{n-\ell}(|\omega\rangle_{n-\ell})^2\) |

Table IV. Multiplicative properties of the measures in Table III on product states.

| Measure | \(\ell\) | \(|\phi\rangle_{\ell} \otimes |\omega\rangle_{n-\ell}\) |
|---------|--------|------------------|
| Concurrence | Even \(\ell\) | \(C_{\ell}(\langle\phi\rangle_{\ell})C_{n-\ell}(|\omega\rangle_{n-\ell})\) |
| Degree-4 | \(\ell = 2\) | \(\frac{1}{4}[C_n(|\phi\rangle_2)^2[C_n-2(|\omega\rangle_{n-2})]^2\) |
| | Even \(\ell \geq 4\) | \(|P_{\ell}(\langle\phi\rangle_{\ell})|C_{n-\ell}(|\omega\rangle_{n-\ell})|^2\) |
| Degree-6 | \(\ell = 2\) | \(\frac{1}{4}\langle C_n(|\phi\rangle_2)^3[C_n-2(|\omega\rangle_{n-2})]^3\) |
| | Even \(\ell \geq 4\) | \(|D^{(\ell)}(\langle\phi\rangle_{\ell})|C_{n-\ell}(|\omega\rangle_{n-\ell})|^3\) |
| Degree-4* | Odd \(\ell \geq 3\) | \(\tau(\langle\phi\rangle_{\ell})C_{n-\ell}(|\omega\rangle_{n-\ell})^2\) |

Table V. Values of \(|P_n|\) for some entangled states.

| Qubits | States | \(|P_n|\) |
|--------|--------|--------|
| Even \(n \geq 4\) | \(|\text{GHZ}\rangle\) | 0 |
| Even \(n \geq 4\) | \(|\text{W}\rangle\) | 0 |
| Even \(n > 4\) | \(|\lambda/n, 2\rangle\) | \(\frac{2(n_{\lambda/2})^2 - (n_{\lambda/2})^2(n_{\lambda/2})^2}{(n_{\lambda/2})^2}\) |
| \(n = 4\) | \(|2, 4\rangle\) | 0 |
| Even \(n > 4\) | \(|\text{CL}_n\rangle\) | 0 |
| \(n = 4\) | \(|\text{CL}_4\rangle\) | \(\frac{1}{16}\) |
| Even \(n \geq 4\) | \(|\text{CL}_2\rangle\) | \(\frac{1}{16}\) |
a natural way to quantify the entanglement of even-n-qubit states. The explicit expressions of the polynomial invariants make it possible for us to investigate the properties of the built entanglement measures. We have conjectured that the entanglement measures built upon even-n-qubit and odd-n-qubit polynomial invariants have opposite vanishing and multiplicative properties on product states. Finally, we expect that the proposed method for constructing polynomial invariants may find further applications.

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APPENDIX: EXPLICIT EXPRESSION OF

\[ P_n(|\psi\rangle) \]

Let \( I_1, I_2, I_3, I_4, \) and \( I_5 \) denote the first, second, third, fourth, and fifth term of \( P_n(|\psi\rangle) \), respectively. A simple calculation yields

\[
I_1 = 2 \sum_{i=0}^{2^n-3-1} (-1)^{p(i)} a_i a_{2^n-1-i} \times \sum_{j=0}^{2^{n-2}-1} (-1)^{p(j)} a_{j+2^n-2} a_{2^n-1-j}, \quad (A1)
\]

\[
I_2 = 2 \sum_{i=0}^{2^n-3-1} (-1)^{p(i)} a_{i+2^n-2} a_{2^n-1-i} \times \sum_{j=0}^{2^{n-2}-1} (-1)^{p(j)} a_{j+2^n-2} a_{2^n-1-j}, \quad (A2)
\]

\[
I_3 = - \sum_{i=0}^{2^n-2-1} (-1)^{p(i)} a_i a_{2^n-1-i} \times \sum_{j=0}^{2^{n-2}-1} (-1)^{p(j)} a_{j+2^n-1} a_{2^n-1-j}, \quad (A3)
\]

\[
I_4 = - \sum_{i=0}^{2^n-2-1} (-1)^{p(i)} a_i a_{2^n-1+i} \times \sum_{j=0}^{2^{n-2}-1} (-1)^{p(j)} a_{j+2^n-2} a_{2^n-1-j}, \quad (A4)
\]

and

\[
I_5 = \sum_{i=0}^{2^n-2-1} (-1)^{p(i)} a_i a_{2^n-1-i} \times \sum_{j=0}^{2^{n-2}-1} (-1)^{p(j)} a_{j+2^n-2} a_{2^n-1-j}, \quad (A5)
\]

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