A Local Strong Solution of the Navier-Stokes Problem in $L^2(\Omega)$

Maoting Tong

Department of Mathematical Science, Xi’anJiaotong-Liverpool University, Suzhou, 215123, P.R.China, E-mail address: Maoting.Tong18@student.xjtlu.edu.cn

Daorong Ton

Department of Mathematics, Hohai University, Nanjing, 210098, P.R.China, E-mail address: 1760724097@qq.com, Current address: 1-3306 Moonlight Square, Nanjing, 210036, P.R.China.

Abstract

In this paper we prove that the Navier-Stokes initial value problem (1) has a unique smooth local strong solution $u(t,x)$ and $p(t,x)$ if the following condition are satisfied

(1) $f \in DL_2(\Omega)$ and $f$ is Hölder continuous about $t$ on $(0,\infty)$,

(2) The initial value $u_0 \in D((-\Delta)^{\frac{1}{2}})$.

Mathematics Subject Classification (2010). Primary 35Q30, 76N10; Secondary 47D06

Keywords. Navier-Stokes equation, Existence and uniqueness, local solution, Semigroup of operators, Invariance, Banach lattice, Fractional powers.

The Navier-Stokes initial value problem can be written in its classical form as

\begin{align}
\frac{\partial u}{\partial t} &= \Delta u - \nabla p - (u \cdot \nabla)u + f, x \in \Omega, t \in (0,\infty) \\
\nabla \cdot u &= div u = 0 \\
\frac{\partial u}{\partial \nu} &= 0, t \in (0,\infty) \\
u|_{t=0} &= u_0, x \in \Omega
\end{align}

where $\Omega$ is a bounded domain in $R^3$ with smooth boundary $\partial \Omega$ of class $C^3$, $u = u(t,x) = (u_1(t,x),u_2(t,x),u_3(t,x))$ is the velocity field, $u_0 = u_0(x)$ is the initial velocity, $p = p(t,x)$ is the pressure, $f = f(t,x)$ is the external force,

$$
\Delta u = \left( \sum_{i=1}^{3} \frac{\partial^2 u_1}{\partial x_i^2} \right) + \left( \sum_{i=1}^{3} \frac{\partial^2 u_2}{\partial x_i^2} \right) + \left( \sum_{i=1}^{3} \frac{\partial^2 u_3}{\partial x_i^2} \right).
$$
In these four equations $u, p$ are unknown and $f, u_0$ are given. The boundary condition imposed on the velocity at $\Omega$ is homogeneous.

The existence, uniqueness and regularity properties of solutions for the Navier-Stokes problems are extensively studied. There is an extensive literature on the solvability of the initial value problem for Navier-Stokes equations.

Let $L_2(\Omega)$ be the Banach space of real vector functions in $L^2(\Omega)$. That is

$$L_2(\Omega) = \{ h : \Omega \to R^3, h = (h_1, h_2, h_3), h_i \in L^2(\Omega) (i = 1,2,3) \}.$$

For $u = (u_1, u_2, u_3) \in L_2(\Omega)$, we define the norm

$$\|u\|_{L_2(\Omega)} = \left( \sum_{i=1}^{3} \|u_i\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

then $L_2(\Omega)$ is a Banach space. The set of all real vector functions $u$ such that $\text{div} \ u = 0$ and $u \in C^\infty_0(\Omega)$ is denoted by $C^\infty_{0,\sigma}(\Omega)$. Let $DL_2(\Omega)$ be the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L_2(\Omega)$. If $u \in C^\infty(\Omega)$ then

(2) $u \in DL_2(\Omega)$ if and only if $\text{div} \ u = 0$ in $\Omega$ and $u_n = 0$ on $\partial \Omega$. (p.270 in [3]).

In this paper we always consider the spaces of vector value functions on $\Omega$.

We have

$$DL_2(\Omega) \subseteq L_2(\Omega) = W^{0,2}(\Omega), \quad \|u\|_{DL_2(\Omega)} = \|u\|_{L_2(\Omega)},$$

$$L_2(\Omega) = DL_2(\Omega) \oplus DL_2(\Omega)^\perp.$$

From [2] and [3] we see that $DL_2(\Omega)^\perp = \{ \nabla h; h \in W^{1,2}(\Omega) \}$. Let $P$ be the orthogonal projection from $L_2(\Omega)$ onto $DL_2(\Omega)$. $A = -P\Delta$ is called the Stokes operator. By applying $P$ to the first equation of (1.1) and taking account of the other equations, we are let the following abstract initial value problem, Pr.II

(3) \[
\begin{cases}
\frac{du}{dt} = P\Delta u + Fu + Pf, & t \in (0,\infty) \\
u \big|_{t=0} = u_0, & x \in \Omega
\end{cases}
\]

where $Fu = -P(u \cdot \nabla)u$. 

2
We consider equation (3) in integral form Pr.III

(4) \[ u(t) = e^{tp}u_0 + \int_0^t e^{(t-s)p}Fu(s)ds. \]

In Theorem 1.6 of [3] H.Fujita and T.Kato proved that if \( u \) is a solution of (3) then \( u \) is of the class \( C^\infty \) as \( L_2(\Omega) \) – valued functions. They gave some sufficient conditions for (3) having a solution. In Theorem 3.4 of [2] Y.Giga and T.Miyakawa proved that the solution of (4) belongs to \( (C^\infty(\Omega \times (0,T)))^n \). They gave some sufficient conditions for (3) and (4) having a solution. In [8] Mukhtarbay.Otelbaev proved that if all \( \frac{\partial u}{\partial t}, \Delta u, (u \cdot \nabla)u, \text{div } p \in L^p((0,T) \times \Omega) \), then (1) have the unique local solution. But he did not prove that this solution is smooth. In [10] Veli B.Shakhmurov discussed nonlocal Navier-Stokes problems in abstract function space \( DL_2(\Omega) \).

He gave some sufficient conditions for (4) having a solution. In this paper we will prove that the Navier-Stokes initial value problem (1) have the unique strong solution \( u(t,x) \in (C^\infty([0,t] \times \Omega))^n \), \( p(t,x) \in C^\infty([0,t] \times \Omega) \) in the state space \( DL_2(\Omega) \) by using the theory of semigroup of operators.

For \( u = (u_1, u_2, u_3) \in L_2(\Omega) \) we define \( \Delta u = (\Delta u_1, \Delta u_2, \Delta u_3) \) and \( \nabla u = (\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}) \).

Since the operator \( -\nabla = -\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \) is strongly elliptic of order 2. Theorem 7.3.6 in [11] implies that \( \Delta \) is the infinitesimal generator of an analytic semigroup of contractions on \( L^2(\Omega) \) with \( D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \). Hence \( \Delta \) is also the infinitesimal generator of an analytic semigroup of contraction on \( L_2(\Omega) \) with \( D(\Delta) = H_2(\Omega) \cap H_{1,0}(\Omega) \), where \( H_2(\Omega) \) and \( H_{1,0}(\Omega) \) are the Sobolev spaces of vector value in \( H^2(\Omega) \) and \( H_0^1(\Omega) \) respectively. We will prove that \( \Delta \) is also the infinitesimal generator of an analytic semigroup of contraction on \( DL_2(\Omega) \).

A operator \( A \) is called preserving divergence-free on a vector value functions space \( X \) if \( A \) maps every \( u \in X \) with \( \text{div } u = 0 \) to an \( Au \) with \( \text{div } Au = 0 \).

**Lemma 1.** For every \( u \in L_2(\Omega) \), \( \text{div } u = 0 \) if and only if \( \text{div } (\lambda I - \Delta)u = 0 \) for \( \lambda \in \Sigma_\beta = \{ \lambda : \beta - \pi < \arg \lambda < \pi - \beta, |\lambda| \geq r \} \).
where $0<\theta<\frac{\pi}{2}$.

Proof. Let $u=(u_1,u_2,u_3) \in L_2(\Omega)$. Then

$$
\Delta u = \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_3^2}, \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_3^2}, \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right),
$$

$$
\text{div}(\Delta u) = \frac{\partial^2 u_1}{\partial x_1^2} \frac{\partial u_1}{\partial x_1} + \frac{\partial^2 u_1}{\partial x_2^2} \frac{\partial u_1}{\partial x_2} + \frac{\partial^2 u_1}{\partial x_3^2} \frac{\partial u_1}{\partial x_3} + \frac{\partial^2 u_2}{\partial x_2^2} \frac{\partial u_2}{\partial x_2} + \frac{\partial^2 u_2}{\partial x_3^2} \frac{\partial u_2}{\partial x_3} + \frac{\partial^2 u_3}{\partial x_3^2} \frac{\partial u_3}{\partial x_3} + \frac{\partial^2 u_1}{\partial x_1^2} \frac{\partial u_2}{\partial x_1} + \frac{\partial^2 u_2}{\partial x_2^2} \frac{\partial u_2}{\partial x_2} + \frac{\partial^2 u_3}{\partial x_3^2} \frac{\partial u_3}{\partial x_3} + \frac{\partial^2 u_1}{\partial x_1^2} \frac{\partial u_3}{\partial x_1} + \frac{\partial^2 u_2}{\partial x_2^2} \frac{\partial u_3}{\partial x_2} + \frac{\partial^2 u_3}{\partial x_3^2} \frac{\partial u_3}{\partial x_3} = \Delta(\text{div} u).
$$

So we have

$$
(5) \quad \text{div}[(\lambda I - \Delta)u] = (\lambda I - \Delta)(\text{div} u).
$$

From (5) it is clear that $\text{div} u = 0$ implies that $\text{div} (\lambda I - \Delta)u = 0$.

Since $-\Delta$ is a strongly elliptic operator of order 2 on $\Omega$. From Theorem 7.3.2 in [11] it follows that there exist constant $C>0$, $R \geq 0$ and $0<\theta<\frac{\pi}{2}$ such that

$$
||u||_{L_2(\Omega)} \leq C ||(\lambda I - \Delta)u||_{L_2(\Omega)}
$$

for $u \in D(\Delta) = H_2(\Omega) \cap H_{1,0}(\Omega) \subset L_2(\Omega)$ and

$$
\lambda \in \Sigma_\theta = \{ \lambda: \theta - \pi < \arg \lambda < \pi - \theta, |\lambda| \geq r \}.
$$

From (6) it follows that for every $\lambda \in \Sigma_\theta$ the operator $\lambda I - \Delta$ is injective from $D(\Delta)$ into $L_2(\Omega)$. From (5) it follows that $\text{div}(\lambda I - \Delta)u = 0$ implies that $\text{div} u = 0$. \Box

**Lemma 2.** (1.5.12 in [5]) Let $\{ T(t) : t \geq 0 \}$ be a $C_0$-semigroup on a Banach space $X$. If $Y$ is a closed subspace of $X$ such that $T(t)Y \subset Y$ for all $t \geq 0$, i.e., if $Y$ is $T(t)_{t \geq 0}$-invariant, then the restrictions

$$
T(t)_Y := T(t)_Y
$$

form a $C_0$-semigroup $\{ T(t)_Y : t \geq 0 \}$, called the subspace semigroup, on the Banach space $Y$. 

4
Lemma 3. (Proposition 2.2.3 in [5]) Let \((A, D(A))\) be the generator of a \(C_0\) -semigroup \(\{T(t) : t \geq 0\}\) on a Banach space \(X\) and assume that the restricted semigroup (subspace semigroup) \(\{T(t)_1 : t \geq 0\}\) is a \(C_0\) -semigroup on some \((T(t))_{t \geq 0}\) - invariant Banach space \(Y \rightarrow X\). Then the generator of \(\{T(t)_1 : t \geq 0\}\) is the part \((A_1, D(A_1))\) of \(A\) in \(Y\).

Lemma 4. The operator \(\Delta\) with \(D(\Delta) \subset DL_2(\Omega)\) is the infinitesimal generator of an analytic semigroup of contractions on \(DL_2(\Omega)\).

Proof. From Theorem 7.3.6 in [11] \(\Delta\) is the infinitesimal generator of an analytic semigroup of contractions on \(L^2(\Omega)\). Then \(\Delta\) is also the infinitesimal generator of an analytic semigroup of contractions on \(L_2(\Omega)\). Let \(\{T(t) : t \geq 0\}\) be the restriction of the analytic semigroup generated by \(\Delta\) on \(L_2(\Omega)\) to the real axis. \(\{T(t) : t \geq 0\}\) is a \(C_0\) semigroup of contractions by Theorem 7.2.5 and Theorem 3.1.1 in [11]. We have already noted that \(DL_2(\Omega)\) is a closed subspace of \(L_2(\Omega)\) and is also a Hilbert space. We want to show that \(DL_2(\Omega)\) is \(T(t)_{t \geq 0}\) - invariant. For every \(u \in L_2(\Omega)\) with \(\text{div} u = 0\) and \(\lambda \in \rho(\Delta) \cap \Sigma_\phi = \{\lambda : \theta - \pi < \arg \lambda - \pi - \theta, |\lambda| \geq r\}\) we have \((\lambda I - \Delta)[R(\lambda : \Delta)u] = u\) where \(\Sigma_\phi\) is the same as in the proof of lemma 1. From Lemma 1 it follows that \(\text{div} R(\lambda : \Delta)u = 0\). That is to say that \((R(\lambda : \Delta)\) is preserving divergence-free for \(\lambda \in \rho(\Delta) \cap \Sigma_\phi\). From Theorem 2.5.2 (c) in [11] it follows that \(\rho(\Delta) \supset R^+\), and so \(\rho(\Delta) \cap \Sigma_\phi \supset \{\lambda : |\lambda| \geq r\}\). Hence \((R(\lambda : \Delta)\) is preserving divergence free for every \(\lambda \geq r\). Let \(u \in DL_2(\Omega)\) then there exists a sequence \(u_n\) such that \(\lim_{n \to \infty} u_n = u\) and \(\text{div} u_n = 0\) for \(n = 1, 2, \ldots\). Since \(R(\lambda : \Delta)\) is bounded and so is continuous. Hence \(\lim_{n \to \infty} R(\lambda : \Delta)u_n = R(\lambda : \Delta)u\) and \(\text{div} R(\lambda : \Delta)u_n = 0\) for every \(\lambda \geq r\). Therefore \(R(\lambda : \Delta)u \in DL_2(\Omega)\) for every \(\lambda \geq r\). It follows that \(DL_2(\Omega)\) is \(R(\lambda : \Delta)\) - invariant for every \(\lambda \geq r\). Now the Theorem 4.5.1 in [11] implies that \(DL_2(\Omega)\) is \(T(t)_{t \geq 0}\) - invariant.
From Lemma 2 and Lemma 3 it follows that \( \Delta_{\rho|DL_2(\Omega)} \) with \( D(\Delta_{\rho|DL_2(\Omega)}) = D(\Delta) \cap DL_2(\Omega) \) is the infinitesimal generator of the \( C_0 \) semigroup \( \{T(t)_{|DL_2(\Omega)} : t \geq 0 \} \) of contractions on \( DL_2(\Omega) \).

We will prove that \( \{T(t)_{|DL_2(\Omega)} : t \geq 0 \} \) can also be extended to an analytic semigroup on \( DL_2(\Omega) \). Suppose that \( \lambda \in \rho(\Delta) \cap \Sigma_\theta \), i.e. there exists \( R(\lambda : \Delta) \) from \( L_2(\Omega) \) into \( D(\Delta) \). Then for any \( u \in DL_2(\Omega) \subset L_2(\Omega) \), \( R(\lambda : \Delta)u \in DL_2(\Omega) \). We have

\[
(\lambda I - \Delta)R(\lambda : \Delta)u = u \quad \text{and} \quad R(\lambda : \Delta)(\lambda I - \Delta)u = u.
\]

Since \( R(\lambda : \Delta) \) is preserving divergence-free we have

\[
R(\lambda : \Delta)u \in DL_2(\Omega) \quad \text{and} \quad (\lambda I - \Delta)u \in DL_2(\Omega).
\]

Thus the formula (7) becomes

\[
(\lambda I - \Delta_{\rho|DL_2(\Omega)})R(\lambda : \Delta)u = u \quad \text{and} \quad R(\lambda : \Delta)(\lambda I - \Delta_{\rho|DL_2(\Omega)})u = u.
\]

Hence \( (\lambda I - \Delta_{\rho|DL_2(\Omega)})(\lambda I - \Delta_{\rho|DL_2(\Omega)})^{-1} = R(\lambda : \Delta)_{|DL_2(\Omega)} \).

From the formula (8) and Theorem 2.5.2(c) in [11] we have

\[
\rho(\Delta_{\rho|DL_2(\Omega)}) \supset \rho(\Delta) \cap \Sigma_\theta \supset \Sigma = \{ \lambda : |\arg \lambda| < \frac{\pi}{2} \cup \{0\} \} \cap \Sigma_\theta
\]

\[
= \Sigma_\theta = \{ \lambda : |\arg \lambda| < \delta, 0 \leq |\lambda| \leq r \}
\]

where \( 0 < \delta < \frac{\pi}{2} \) and \( \theta = \min\{\frac{\pi}{2} + \delta, \pi - \theta\} \). Thus, for \( \lambda \in \Sigma_\theta \), \( \lambda I - \Delta_{\rho|DL_2(\Omega)} \) is invertible. From Theorem 2.5.2(c) in [11] we have

\[
\|R(\lambda : \Delta_{|DL_2(\Omega)})\| = \sup_{u \in DL_2(\Omega)} \|R(\lambda : \Delta_{|DL_2(\Omega)})u\|_{L_2(\Omega)}^{-1} \leq \frac{M}{|\lambda|}
\]

Now Theorem 2.5.2(c) in [11] implies that \( \{T(t)_{|DL_2(\Omega)} : t \geq 0 \} \) can also be extended to an
analitic semigroup on $DL_2 (\Omega)$. Therefor $\Delta^{\frac{3}{2}}_{DL_2 (\Omega)}$ is a infinitesimal generator of an analytic semigroup of ccontraction on $DL_2 (\Omega)$. □

Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space $X$. From the results of section 2.6 in [11] we can define the fraction powers $A^\alpha$ for $0 \leq \alpha \leq 1$ and $A^\alpha$ is a closed linear invertible operator with domain $D(A^\alpha)$ dense in $X$. $D(A^\alpha)$ equipped with the norm $\|x\|_\alpha = \|A^\alpha x\|$ is a Banach space denoted by $X_\alpha$. It is clear that $0 < \alpha < \beta$ implies $X_\alpha \supset X_\beta$ and that the embedding of $X_\beta$ into $X_\alpha$ is continuous. If $-A = \Delta$ and $\gamma > \frac{1}{2}$ then $X_\gamma \subset X_\frac{1}{2}$ and $D((\Delta)^{\gamma}) \subset D((\Delta)^{\frac{1}{2}})$, so the norms $\|\cdot\|_{DL_2 (\Omega)}$ and $\|\cdot\|_{DL_2 (\Omega)}$ are equivalent (see p291 in [6]), i.e. there exists $L_0 > 0$ such that for any $u \in D((\Delta)^{\gamma})\|\cdot\|_{DL_2 (\Omega)}$

$$\|(-\Delta)^{\gamma} u\|_{DL_2 (\Omega)} = \|u\|_{DL_2 (\Omega)} \leq L_0 \|x\|_{DL_2 (\Omega)}^{\frac{1}{2}}.$$

For $u \in D(\Delta)$ we have

$$\|\nabla u\|_{DL_2 (\Omega)} = \|\nabla u\|_{DL_2 (\Omega)} = \|(-\Delta)^{\frac{1}{2}} u\|_{DL_2 (\Omega)} = \|\cdot\|_{DL_2 (\Omega)}^{\frac{1}{2}}.$$

In [2] Giga proved the following result:

**Lemma 5.** (Lemma 2.2 in [2]) Let $0 \leq \delta < \frac{1}{2}$, $n(1-r^{-1}) / 2$. Then

$$\|A^{-\delta} P(u, \nabla) v\|_{L_r} \leq M \|A^{\delta} u\|_{L_r} \|A^n v\|_{L_r}$$

with some constant $M = M (\delta, \theta, \omega, r)$, provided $\delta + \theta + \omega \geq n / 2r + 1 / 2$, $\theta > 0, \omega > 0, \omega + \delta > 1 / 2$.

From the Lemma 5 and the formula (10) we see that if take $n = 3, r = 2$

$$\delta = 0, \theta = 3 / 4 \text{ and } \omega = 3 / 4,$$ then
with some constant $M$ for any $u, v \in DL_2(\Omega)$. Hence we have

**Lemma 6.** Suppose that $u, v \in DL_2(\Omega)$ are velocity fields and $(u \cdot \nabla)v \in DL_2(\Omega)$, then

$$\| (u \cdot \nabla)v \|_{DL_2(\Omega)} \leq M_0 \| u \|_{DL_2(\Omega)^{1/2}} \| v \|_{DL_2(\Omega)^{1/2}}.$$ 

**Assumption (F).** Let $X = DL_2(\Omega)$ and $U$ be an open subset in $R^+ \times X_\alpha$ ($0 < \alpha < 1$). The function $f: U \to X$ satisfies the assumption (F) if for every $(t, u) \in U$ there is a neighborhood $V \subset U$ and constants $L \geq 0$, $0 \leq \theta \leq 1$ such that for all $t_i, u_i \in V (i = 1, 2)$

$$\| f(t_1, u_1) - f(t_2, u_2) \| \leq L \| t_1 - t_2 \| + \| u_1 - u_2 \|.$$ 

**Lemma 7.** (Theorem 6.3.1 in [11]) Let $-A$ be the generator of an analytic semigroup $T(t)$ satisfying $\| T(t) \| \leq M$ and assume that $0 \in \rho(-A)$. If, $0 < \alpha < 1$ and $f$ satisfies the assumption (F) then for every initial date $(t_0, u_0) \in U$ the initial value problem

\[
\begin{aligned}
\frac{du(t)}{dt} + Au(t) &= f(t, u(t)), t \in (t_0, \infty) \\
u(t_0) &= u_0
\end{aligned}
\] 

has a unique local solution $u \in C([t_0, t_1); X) \cap C^1([t_0, t_1); X)$ where $t_1 = t_1(t_0, u_0) > t_0$.

**Lemma 8.** Suppose that $u, v \in D((-\Delta)^{1/2}) \subset DL_2(\Omega)$, then $(u \cdot \nabla)v \in DL_2(\Omega)$. Proof. From (3) we have

$$\int_{\Omega} u \cdot \nabla h dx = 0 \quad \text{and} \quad \int_{\Omega} v \cdot \nabla h dx = 0$$
for all $h \in W^{1,2}(\Omega)$. So $\int_{\Omega} u \frac{\partial h}{\partial x_i} \, dx = 0$ and $\int_{\Omega} v_i \frac{\partial h}{\partial x_j} \, dx = 0$ for $i = 1, 2, 3$. $u, v \in D$

$((-\Delta)^{\frac{3}{2}}) \in H^1(\Omega)$ imply $u, \frac{\partial u}{\partial x_i}$ and $v_i, \frac{\partial v_i}{\partial x_j}$ are all bounded on $\Omega$, $\frac{\partial u}{\partial x_i} \leq L_1$, $\frac{\partial v_i}{\partial x_j} \leq L_2$

for $i = 1, 2, 3$. Hence

$$\int_{\Omega} \left( \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i} \right) v_j \frac{\partial h}{\partial x_j} \, dx \leq \int_{\Omega} \left| u_j \right| \frac{\partial v_j}{\partial x_j} \frac{\partial h}{\partial x_j} \, dx \leq L_2 \int_{\Omega} \left| u_j \right| \frac{\partial h}{\partial x_j} \, dx = 0, j = 1, 2, 3,$$

$$\int_{\Omega} (u \cdot \nabla) v \cdot \nabla h \, dx = 0, h \in W^{1,2}(\Omega),$$

$$(u \cdot \nabla) v \in DL_2(\Omega).$$

Now we study the Navier-Stokes initial value problem (1).

A function $u$ which is differentiable almost everywhere on $[0, T]$ such that $u \in L^1[0,T:DL_2(\Omega)]$ is called a strong solution of the initial value problem (1) if $u(0) = u_0$ and $u$ satisfies (1) a.e. on $[0, T]$.

Let $C^\mu([0, \infty); DL_2(\Omega)_{\frac{1}{2}})$ denote the space of all Hölder continuous functions on $[0, T]$ with exponent $\mu$ and with values in a Banach space $DL_2(\Omega)_{\frac{1}{2}}$.

**Theorem.** The initial value problem (1) has a unique smooth local strong solution if the initial value $u_0 \in D((-\Delta)^{\frac{3}{2}})$, $f$ is Hölder continuous about $t$ on $(0, \infty)$ with exponent $\beta_1$, i.e. there exist constant $C_1$ and $0 < \beta_1 < 1$ such that

$$\|f(s, x) - f(t, x)\| \leq C_1 |s - t|^{\beta_1}, \text{ for } s, t \in (0, \infty) \text{ and } x \in \Omega. \quad (13)$$

Proof. We will find that by incorporating the divergence-free condition, we can remove the pressure term from our equation. (see p. 271 in [3], p. 234, and p. 239 in [9]) So first we can rewrite (1) into a abstract initial value problem on $DL_2(\Omega)$
\[
\begin{aligned}
\frac{du}{dt} &= \Delta u + F(t, u(t)), t \in (0, \infty) \\
\left. u \right|_{t=0} &= u_0, x \in \Omega
\end{aligned}
\]

(14)

where \( F(t, u(t)) = -(u \bullet \nabla)u + f \) is a abstract function. From Lemma 4 \( \Delta \) is the generator of an analytic semigroup \( T(t) \) of contraction on \( DL_2(\Omega) \) and \( \|T(t)\| \leq 1 \). From Theorem 2.5.2(c) in [11] \( 0 \in \rho(\Delta) \).

Let \( U \) be the subset of \( R^+ \times DL_2(\Omega) \) such that \( (t_1, u_1) \in U \) iff \( u_1 = u(t_1) \) for some \( u(t) \in C^H([0, \infty); DL_2(\Omega)) \) Let \( u_k(t, x) = (k, k, k) \ (t \in [0, \infty), x \in \Omega, k \in R) \). Then \( (t, u_k) \in U \) for \( t \in [0, \infty) \) and all \( k \in R \). For any \( u \in D((-\Delta)^2) \) let \( u(t) \equiv u \ (t \in [0, \infty)) \). Then \( u(t) \in C^H([0, \infty); DL_2(\Omega)) \) Hence \( U \) is not empty.

Let \( U_1 \) be the open kernel of \( U \). From Lemma 8 \( F(t, u(t)) = -(u \bullet \nabla)u + f \) is a function:

\( U_1 \rightarrow DL_2(\Omega) \). For any \( (t_1, u_1), (t_2, u_2) \in U_1 \) we have \((u_1 \bullet \nabla)u_1, (u_2 \bullet \nabla)u_2 \in DL_2(\Omega)\) and

\[
\begin{aligned}
\left\| (u_1 \bullet \nabla)u_1 - (u_2 \bullet \nabla)u_2 \right\|_{DL_2(\Omega)} \\
&= \left\| (u_1 \bullet \nabla)u_1 - (u_1 \bullet \nabla)u_2 \right\|_{DL_2(\Omega)} + \left\| (u_1 \bullet \nabla)u_2 - (u_2 \bullet \nabla)u_2 \right\|_{DL_2(\Omega)} \\
&= \left\| (u_1 \bullet \nabla)(u_1 - u_2) \right\|_{DL_2(\Omega)} + \left\| (u_1 - u_2 \bullet \nabla)u_2 \right\|_{DL_2(\Omega)} \\
&\leq ML_0 \left( \left\| u_1 \right\|_{DL_2(\Omega)} \right) \left\| u_1 - u_2 \right\|_{DL_2(\Omega)} + \left\| u_1 - u_2 \right\|_{DL_2(\Omega)} \left\| u_2 \right\|_{DL_2(\Omega)} \\
&= ML_0 \left( \left\| u_1 \right\|_{DL_2(\Omega)} + \left\| u_2 \right\|_{DL_2(\Omega)} \right) \left\| u_1 - u_2 \right\|_{DL_2(\Omega)}.
\end{aligned}
\]

(15)

We used lemma 6 in the above third step. Therefore \( (u \bullet \nabla)u \) is local Lipschitz continuous.

Since \( u(t) \) is Hölder continuous about \( t \) on \( [0, \infty) \) in \( DL_2(\Omega) \), so there is a constant \( C_2 \) and \( 0 < \beta_2 < 1 \) such that

\[
\left\| u(s, x) - u(t, x) \right\|_{DL_2(\Omega)} \leq C_2 |s - t|^\beta_2 \quad \text{for} \quad s, t \in [0, \infty).
\]

(16)

Hence
\[(\|u(t_1) \bullet \nabla u(t_1) - (u(t_2) \bullet \nabla)u(t_2)\|_{L^2(\Omega)} \leq \|u(t_1) \bullet \nabla u(t_1) - (u(t_2) \bullet \nabla)u(t_2)\|_{L^2(\Omega)} + \|u(t_1) \bullet \nabla (u(t_1) - u(t_2)) \bullet \nabla (u(t_1) - u(t_2))\|_{L^2(\Omega)} \]
\[= \|u(t_1) \bullet \nabla u(t_1) - u(t_2)\|_{L^2(\Omega)} + \|u(t_1) - u(t_2)\|_{L^2(\Omega)} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} \]
\[(17) \leq ML_0 \|u(t_1)\|_{L^2(\Omega)} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} + \|u(t_1) - u(t_2)\|_{L^2(\Omega)} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} \]
\[= ML_0 \|u(t_1)\|_{L^2(\Omega)} + \|u(t_2)\|_{L^2(\Omega)} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} \]
\[\leq ML_0 C_2 \left(\|u(t_1)\|_{L^2(\Omega)} + \|u(t_2)\|_{L^2(\Omega)} \right) \|t_1 - t_2\|^\beta_1 . \]

We used the Lemma 6 in the above third step, and the formula (16) in fifth step.

If \( u_0 \in D((-\Delta)^{1/2}) \), then \( 0, u_0 \in U_1 \). Set
\[ V = B_\varepsilon (0, u_0) = \left\{ (t, u) \in U : \|t - 0\| \leq \varepsilon, \|u - u_0\|_{L^2(\Omega)} \leq \varepsilon \right\} . \]

Then for \( (t, u) \in V \),
\[ \|u\|_{L^2(\Omega)} \|t\| \leq \|u - u_0\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)} \leq \varepsilon + \|u_0\|_{L^2(\Omega)} . \]

Let \( L = \varepsilon + \|u_0\|_{L^2(\Omega)} \), \( L_1 = 2ML_0 L \), \( L_2 = 2ML_0 LC_2 \) and \( L_3 = Max(C_1, L_2, L_1) \),
\( \beta = Max(\beta_1, \beta_2) \) then from (13), (15) and (17) it follows that for all \( (t, u_1) \in V \)
\[ \|F(t, u_1)=F(t, u_2)\|_{L^2(\Omega)} \leq \|F(t, u_1) = F(t, u_2)\|_{L^2(\Omega)} + \|F(t_1, u_2(t_1)) - F(t_2, u_2(t_2))\|_{L^2(\Omega)} \]
\[= \|(u_1(t_1) \bullet \nabla u_1(t_1) - (u_2(t_1) \bullet \nabla)u_2(t_1)\|_{L^2(\Omega)} + \|(u_2(t_1) \bullet \nabla)u_2(t_1) - (u_2(t_2) \bullet \nabla)u_2(t_2)\|_{L^2(\Omega)} \]
\[+ \|f(t_1, x) \bullet \nabla u_1(t_1) - f(t_2, x) \bullet \nabla u_2(t_2)\|_{L^2(\Omega)} \]
\[\leq 2ML_0 L \|u_1 - u_2\|_{L^2(\Omega)} + 2ML_0 LC_2 \|t_1 - t_2\|^{\beta_1} + C_1 \|t_1 - t_2\|^{\beta_1} \]
\[\leq L_3 \|u_1 - u_2\|_{L^2(\Omega)} + L_2 \|t_1 - t_2\|^{\beta_1} + C_1 \|t_1 - t_2\|^{\beta_1} \]
\[\leq L_3 (\|t_1 - t_2\|^{\beta_1} + \|u_1 - u_2\|_{L^2(\Omega)}) . \]

Hence \( F(t, u(t)) \) satisfies the assumption \( (F) \), then by Lemma 7 for every initial data
(0, u₀) ∈ U₁, the initial value problem (14) has a unique local solution

\[ u(t) ∈ C(0, t₁): DL₁(Ω)) \cap C²((0, t₁): DL₁(Ω)) \]

(18)

where \( t₁ = t₁(u₀) \).

We consider equation (3) in integral form

\[ u(t) = e^{tα}u₀ + \int_0^t e^{(t-s)α}Fu(s)ds. \]

(19)

In a similar induction way as Theorem 3.9 in [2] or as Theorem 5.1 in [10] we can prove that the solution \( u(t, x) ∈ C^∞((0, t₁] × Ω) \). (see Appendix) We can also prove directly that \( u(t, x) \) is smooth. In fact, the solution (18) of (14) is also the solution of (19). The Theorem 3.4 in [2] mean that as long as the solution of (19) exists, this solution is smooth. From Theorem 3.4 in [2] we have the solution \( u(t, x) ∈ C^∞((0, t₁] × Ω) \). Substituting \( u(t, x) \) into (1) we get the solution \( p(t, x) \). We also have \( p(t, x) ∈ C^∞((0, t₁] × Ω) \). It follows from the formula (2) and \( u ∈ DL₁(Ω) \) that the solution \( u(t, x) \) is divergence-free. Changing the value of \( u \) on \( ∂Ω \) to zero we get a unique smooth local strong solution \( u = u(t, x), \ p(t, x) \) of the Navier-Stokes initial value problem (1).

The above Theorem extents Theorem 1.2 in [3] and Theorem 2.5 in [2]. In another paper we will prove that if we take \( f(t, x) \) to be identically zero and assume that any initial value \( u₀ ∈ D((-Δ)₁) \cap C^∞([0, ∞) × R³) \) satisfies \( |∂ₓ u₀(x)| \leq C_{₁₁} (1 + |x|)^{-k} \) on \( R³ \) for any \( α \) and \( k \), then the Navier-Stokes initial value problem have a global smooth solution with bounded energy.

Appendix

**Lemma 1.** (Lemma 3.1 in [2]) Let \( u ∈ D(-Δ) \) and \( -Δu ∈ W^{m,r}(Ω) \) for some integer \( m \geq 0 \), then \( u ∈ W^{m+2,r}(Ω) \) and satisfies

\[ \|u\|_{m+2,r} \leq C_{₁₁} \|−Δu\|_{m,r} \]
with a constant $C_m > 0$ independent of $u$ and $-\Delta u$.

Let $C^\mu([0, T]; X)$ denote the space of Hölder continuous functions on $[0, T]$ with exponent $\mu$ and with values in a Banach space $X$. Similarly let $C^\mu((0, T], X)$ denote the space of functions which are Hölder continuous on every subinterval $[\varepsilon, T]$ of $(0, T]$, with exponent $\mu$.

**Lemma 2.** (Lemma 3.2 in [2]) Let $f(t) \in C^\mu([0, T]; DL_2(\Omega))$, for some $0 < \mu < 1$. Then the function

$$v(t) = \int_0^t e^{-(t-s)\Delta} f(s)ds \in C^\mu((0, T]; D(-\Delta)) \cap C^{1+\nu}((0, T]; DL_2(\Omega))$$

for every $\nu$ such that $0 < \nu < \mu$.

Let $P_r$ be the continuous projection from $L^2(\Omega)$ to $DL_2(\Omega)$.

**Lemma 3.** (Lemma 3.3 in [2]) (i) For $m > 3/r$, there exists a constant $C_{m,r} > 0$ such that

$$\|P_r(u \cdot V)\|_{m,r} \leq C_{m,r} \|u\|_{m,r} \|V\|_{m+1,r}$$

for every $u \in (W^{m,r}(\Omega))^3$, $v \in (W^{m+1,r}(\Omega))^3 (1 < r < \infty)$.

(ii) When $r > 3$, we have

$$\|P_r(u \cdot V)\|_{r,r} \leq C_r \|u\|_{r,r} \|V\|_{r,r}$$

for $u, v \in (W^{1,r}(\Omega))^3$.

We will say that $u(t)$ has property $(P_m)(m \geq 1)$ if

$$u^{(m)} \in C^\mu([0, T]; D((-\Delta)^{\frac{m}{2}}))$$

$$u^{(j)} \in C^\mu([0, T]; (W^{m+1-j,2}(\Omega))^3) \quad 1 \leq j \leq m-1,$$

$$u \in C^\mu([0, T]; (W^{m+2,2}(\Omega))^3),$$

for all $\mu, 0 < \mu < \frac{3}{2}$. Here $u^{(j)} = \frac{d^ju}{dt^j}$.

Let the solution in the above Theorem be

$$u(t, x) = T(t)u_0 + \int_0^t T(t-s)F(s, (-\Delta)^{\frac{1}{2}})y_0(s)ds$$

Lemma 2 and Lemma 3(i) now imply

**Lemma 4.** (Lemma 3.6 in [2]) $u \in C^\mu((0, T]; D(\Delta))$ and $u' = du/ dt \in C^\mu((0, T];DL_2(\Omega))$. 

13
\( DL^2(\Omega) \) for all \( \mu, 0 < \mu < \frac{\gamma}{2} \). Moreover \( F(s, (-\Delta)^{\frac{\gamma}{2}} y_0(s)) \in C^\mu \left( (0, T) \left( W^{1,2}(\Omega) \right)^3 \right) \)

**Lemma 5.** (Lemma 3.7 in [2]) We have \( u' = \partial u / \partial x \in C^\mu \left( (0, T) D((-\Delta)^{\frac{\gamma}{2}}) \right) \) for all \( \mu, 0 < \mu < \frac{\gamma}{2} \).

The proof is similar to Lemma 3.7 in [2].

Since \( D((-\Delta)^{\frac{\gamma}{2}}) \subset (W^{1,\gamma}(\Omega))^3 \), Lemma 1, Lemma 4, Lemma 4 and the identity \( u = (-\Delta)^{-\frac{\gamma}{2}}(Fu-u') \) show that

\[
\begin{align*}
(20) & \\
& u \in C^\mu \left( (0, T) \left( W^{3,2}(\Omega) \right)^3 \right)
\end{align*}
\]

Lemma 5 and the above formula (20) show that \( u(t) \) has property \( (P)_1 \).

**Lemma 6.** (Lemma 3.8 in [2]) \( (P)_m \) implies \( (P)_{m+1} \).

The proof is the same to Lemma 3.8 in [2].

Therefore we can prove the following theorem in a similar way as Theorem 3.9 in [2] or as Theorem 5.1 in [10].

**Theorem.** The solution in the above Theorem is smooth.

---

**References**

1. Andras Batkai, Marjeta Kramar Fijavz, Abdelaziz Rhandi, Positive Operator Semigroups, Birkhauser (2017).

2. Y.Giga.T.Miyakava, Solutions in \( L_r \) of the Vavier-Stokes Initial Value Problem, Arch.Ration.Mech.Anal. 89(1985)267-281.

3. H.Fujita, T.Kato, On the Navier-Stokes initial value problem I, Arch. Ration. Mech. Anal. 16(1964),269-315

4. T.Kato, Strong \( L_p \)-solution of the Navier-Stokes equation in \( \mathbb{R}^m \), with application to weak solutions, Math. Z. 187(1984)471-480.

5. Engel Klaus-Jochen , Rainer Nagel. One-Parameter Semigroups for Linear Evolution Equations, Springer, (2000).

6. E.Kreyszig, Introduction functional analysis with applications, John Wiley &Sons (1978).

7. Peter Meyer-Nieberg, Banach Lattices, Springer-Verlag, (1991).
8. M.Otelbaev, Existence of a strong solution of the Navier-Stokes equation, Mathematical Journal, 13(4)(2013),5-104.

9. James C. Robinson, Infinite-Dimensional Dynamical Systems, Cambridge University Press, (2001).

10. Veli B.Shakhmurov, Nonlocal Navier-Stokes problems in abstract function, Nonlinear Analysis:Real World Applications, 26(2015)19-43.

11. A.Pazy, Semigroups of linear operators and applications to partial differential equations, Springer Verlag (1983, reprint in China in 2006).

12. K.Yosida, Functional analysis (6th edition), Springer Verlag (1980).