Universality in scattering by large-scale potential fluctuations in two-dimensional conductors

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We study electron propagation through a random array of rare, opaque and large (compared the de Broglie wavelength of electrons) scatterers. It is shown that for any convex scatterer the ratio of the transport to quantum lifetimes \( \eta = \tau_{tr}/\tau_{q} \) does not depend on the shape of the scatterer but only on whether scattering is specular or diffuse and on the spatial dimensionality \( D \). In particular, for specular scattering, \( \eta \) is a universal constant determined only by the dimensionality of the system: \( \eta = 2 \) for \( D = 3 \) and \( \eta = 3/2 \) for \( D = 2 \). The crossover between classical and quantum regimes of scattering is discussed.

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At low temperatures, electron transport is controlled by disorder. The conventional model for disorder is an ensemble of relatively sparse impurities or surface roughness, whereas larger imperfections, greatly exceeding the electron wavelength, are assumed to be prevented by a reasonably advanced growth technique. This model is facing certain difficulties in view of mounting evidence for macroscopic inhomogeneities in a number of key materials, e.g., high-\( T_c \) superconductors, manganites, semiconductor heterostructures, etc. For example, nanoscale inhomogeneities in high-\( T_c \) materials and manganites have been observed via scanning tunnelling microscopy \cite{1, 2}. Also, the ionized impurities in very high-mobility GaAs heterostructures are engineered to be so far away from the 2D gas layer plane that it is not these impurities but rather large-scale potential fluctuations that provide the dominant scattering mechanism for electrons. Sometimes the large-scale potential fluctuations are introduced intentionally, as antidot arrays \cite{4}. Randomness in positions and/or shapes of antidots leads to additional scattering of electrons. On the theoretical side, the semiclassical motion of electrons in the presence of large-scale inhomogeneities has attracted a significant interest due to non-Boltzmann effects in magneto- and ac transport \cite{3}. In this communication, we study some very basic yet, to the best of our knowledge, unexplored universalities in the effective scattering cross-sections by large-scale fluctuations.

We assume that disorder is represented by an ensemble of large (of size \( a \) much larger than the electron de Broglie wavelength \( \lambda \)) objects of irregular but smooth shape, placed and oriented randomly along the conducting plane, see Fig. 1. An important parameter characterizing the spatial structure of disorder is the ratio of transport and “quantum” mean free times,

\[ \eta = \frac{\tau_{tr}}{\tau_q} = \frac{\sigma_{tot}}{\sigma_{tr}}, \]  

associated with the transport

\[ \sigma_{tr} = \int d\Omega \frac{d\sigma}{d\Omega} (1 - \cos \theta) \]  

and total

\[ \sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} \]  

scattering cross-sections, correspondingly.

Experimentally, \( \tau_{tr} \) is extracted from the conductivity whereas \( \tau_q \) is obtained as a damping rate of de Haas-van Alphen or Shubnikov-de Haas oscillations. For long-range disorder, \( \eta \gg 1 \). A classic example of such a system is a GaAs heterostructure with modulation doping \cite{3}. For isotropic impurities, \( \eta = 1 \). This is the case, e.g. for Si-based field-effect transistors in a certain range of densities \cite{4}. \( \eta < 1 \) corresponds to enhanced backscattering. A minimum value of \( \eta = 1/2 \) is achieved for the limiting case of strict backscattering, when \( d\sigma/d\Omega \propto \delta(\theta - \pi) \).

In the present paper, we show that for a wide class of randomly placed and oriented plane scatterers of mesoscopic size \( a \gg \lambda \), there is a surprising universality in the value of the parameter \( \eta \). Namely,

\[ \eta = \frac{3}{2} \]  

for the case of specular scattering and

\[ \eta = \frac{4}{3} \]  

for the case of diffuse scattering. The universality of \( \eta \) results from a combination of universal features of scattering at the true quantum or even classical levels of consideration. Given the mesoscopic size of the objects and smoothness of their shapes, we adopt first the classical mechanics to describe electron scattering. Later on, we discuss and implement the important modifications caused by the quantum nature of scattering particles.
the averaged differential cross-section as

\[
\frac{d\sigma^{cl}(\theta)}{d\theta} = dl \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{2\pi} \cos \alpha \\
\times \int_{-\pi/2}^{\pi/2} d\beta P(\beta; \alpha) \delta(\theta - \pi + \alpha + \beta). \tag{8}
\]

Assuming the distribution function \( P(\beta; \alpha) \) to be uniform along the scatterer boundary and integrating over \( dl \), we replace \( dl \) in Eq. (8) by the perimeter of the object, \( P \). Therefore, the total and transport cross-sections are given by

\[
\sigma^{cl}_\text{tot} = P \int_{-\pi/2}^{\pi/2} \frac{d\alpha d\beta}{2\pi} P(\beta; \alpha) \cos \alpha \tag{9}
\]

and

\[
\sigma^{cl}_\text{tr} = P \int_{-\pi/2}^{\pi/2} \frac{d\alpha d\beta}{2\pi} P(\beta; \alpha) [1 + \cos (\alpha + \beta)] \cos \alpha. \tag{10}
\]

The ratio of the two cross-sections, \( \eta^{cl} \), is determined entirely by function \( P(\beta; \alpha) \) and is independent of a particular geometry of the scattering object.

Consider two particular situations of special interest. For specular scattering, Eqs. (5,11) reduce to

\[
\frac{d\sigma^{cl}(\theta)}{d\theta} = \frac{P}{4\pi} \sin \frac{\theta}{2}, \tag{11}
\]

and

\[
\sigma^{cl}_\text{tot} = P/\pi, \quad \sigma^{cl}_\text{tr} = 4P/3\pi, \quad \text{and} \quad \eta^{cl} = 3/4. \tag{12}
\]

For a disk, these expressions coincide with Eqs. (6) and (7), respectively. For absolutely diffuse scattering, we arrive at

\[
\frac{d\sigma^{cl}(\theta)}{d\theta} = \frac{P}{\pi^2 \sin^2 \frac{\theta}{2}}, \tag{13}
\]

so that

\[
\sigma^{cl}_\text{tot} = P/\pi, \quad \sigma^{cl}_\text{tr} = 3P/2\pi, \quad \text{and} \quad \eta^{cl} = 2/3. \tag{14}
\]

The universality emerges as a result of the averaging over random orientation of scatterers and is the property of any scatterer of a convex geometry. The convexity condition prevents repeated scattering. To illustrate the importance of this requirement, we present a simple example of scatterers where the repeated scattering is not excluded. This is specular scattering by randomly oriented “right angles” of size \( a \). We have for averaged total and transport cross-sections:

\[
\sigma^{cl}_\text{tot} = a(2 + \sqrt{2})/\pi, \quad \sigma^{cl}_\text{tr} = a(4 + \sqrt{2})/\pi, \tag{15}
\]

\[
\eta^{cl} = (3 + \sqrt{2})/7 \approx 0.63. \tag{16}
\]

These values differ considerably from those for scatterers of a convex form.
The consideration that has lead to the above universal results is not restricted to a specific dimensionality. In 3D, averaging over random orientations of a convex scatterer is equivalent to averaging of a contribution of a particular surface patch which is rotated over the whole solid angle. For instance, for specular scattering in 3D we arrive at the universal expressions: $\sigma_{\text{tot}}^{\text{cl}} = \sigma_{\text{tr}} = S/4$ ($S$ is a surface area), and $\eta^{\text{cl}} = 1$. In fact, the universal connection $\sigma_{\text{geom}} = S/4$ between the geometrical cross-section and the surface area of randomly oriented convex scatterers is well known in the field of light scattering by dust particles (see e.g., Ref. [6]), and it is difficult to refer to the very first proof of this theorem.

Now we should account for modifications caused by the wave nature of scattering particles (in optical language, we are going beyond the geometrical optics). As long as $\lambda/a$ is much smaller than one, these modifications are negligible for the transport cross-section, but are very substantial for the total cross-section. For instance, the total cross-section of a sphere (of radius $a \gg \lambda$) is twice larger than the classical value $\sigma_{\text{tot}}^{\text{cl}} = \pi a^2$ determined simply by the geometrical cross-section (see, e.g., Ref. [6]). Such a dramatic discrepancy with the classical result (the so-called Extinction Paradox) stems from a sharp peak in the differential cross-section for forward scattering ($\theta \to 0$) which cannot be described semiclassically [6]. On the other hand, this peak does not contribute to the transport cross-section due to the factor $1 - \cos \theta \ (\to 0, \ at \ \theta \to 0)$ and $\sigma_{\text{tr}} \approx \sigma_{\text{tot}}^{\text{cl}}$.

It is important that the relations between the classical and quantum cross-sections, described in the previous paragraph, are not specific either to spherically symmetric scatterers or to a particular spatial dimensionality. To make evident the universality of these relations, we recall the underlying physical principle which governs scattering from an opaque object of size $a \gg \lambda$. Directly behind this object there is a shadow region with a vanishing amplitude of the wave field, $A = 0$. According to the superposition principle, $A = A_1 + A_s$, where $A_1$ and $A_s$ are the amplitudes of the incident and scattered waves, correspondingly. Therefore, $A_s = -A_1$ within the shadow region (Babinet principle, 1837). This means, that in addition to the flux, scattered in the directions outside the shadow region, an opaque object also scatters the incoming radiation in the forward direction, within a very narrow diffraction cone of angle $\theta \sim \lambda/a$. Obviously, the flux of the forward-scattered wave equals to the flux of the incident wave through the geometric cross-section $\sigma_{\text{geom}}$ of the scatterer. This leads to an additional contribution $\delta \sigma_{\text{tot}} = \sigma_{\text{geom}}$ to the total scattering cross-section. As a result, the true total scattering cross-section $\sigma_{\text{tot}}$ is a sum of its semi-classical value $\sigma_{\text{tot}}^{\text{cl}} = \sigma_{\text{geom}}$ and the forward scattering part $\delta \sigma_{\text{tot}} = \sigma_{\text{geom}}$. Thus, we arrive at the universal relationships

$$\sigma_{\text{tot}} = 2\sigma_{\text{tot}}^{\text{cl}},$$
$$\sigma_{\text{tr}} = \sigma_{\text{tot}}^{\text{cl}}.$$  

valid to the leading order in the small parameter $\lambda/a$ for an arbitrary opaque scatterer with a well-defined boundary (we will discuss the latter condition below). These relations refer to an arbitrary orientation of the scatterer, hence they sustain averaging over orientations. It needs to be emphasized that Eq. (17) refers to the total cross-section measured at distances larger then the Fraunhofer length, $L_F = a^2/\lambda$, from the scatterer, where quantum small-angle scattering smears the classical shadow.

Finally, combining Eqs. (17) and (18) with Eqs. (12) and (13), we obtain the announced results, $\eta = 3/2$ [Eq. (14)] and $\eta = 4/3$ [Eq. (15)], respectively.

Although we used a two-step (classical-to-quantum) derivation of Eqs. (4) and (5), it should be emphasized that their validity range is wider than that of the intermediate expressions, which involve the classical total cross-section $\sigma_{\text{tot}}^{\text{cl}}$. The point is that the latter quantity is well-defined only for scatterers with a sharp boundary. If, on the contrary, the scattering potential falls off continuously with the distance, the classical total cross-section $\sigma_{\text{tot}}^{\text{cl}}$ is infinite, no matter how small the potential is away from the center [7]. This makes the very notion of the classical total cross-section very restricted. On the contrary, the true quantum total cross-section accounts for the weakness of scattering by the potential tail and remains finite if the potential decays sufficiently fast [7]. For instance, if the scattering potential consists of a hard core of size $a \gg \lambda$ and a tail falling off as $\exp(-r-a)/h$ for $r \geq a$ with $h \ll a$, the quantum total and transport cross-sections differ from their values for the bare core only by small corrections of the relative order $h/a \ll 1$. This validates the robustness of the obtained universal relations with respect to smearing of the scatterer’s edge.
Experimentally, the transport scattering rate $1/\tau_{tr}$ is extracted from the conductivity. The quantum decay rate, $1/\tau_q$, may be obtained, in principle, by measuring attenuation of an electron beam propagating through a disordered stripe. However, it is more practical to extract $1/\tau_q$ from the amplitude of the de Haas-van Alfen or Shubnikov-de Haas oscillations in relatively weak magnetic fields (to avoid significant changes of the decay rate caused by the field itself). In both types of experiments, attenuation of the measured quantities (the beam amplitude or the amplitude of magneto-oscillations) is due to deflecting particles from their original trajectories by scattering, no matter in what direction. Consequently, $1/\tau_q$ is related to the total cross-section. If magneto-oscillations are measured in a very weak magnetic field, so that the cyclotron radius is larger than any other lengthscale of the problem (but still smaller than the system size), this cross-section is given by the quantum expression. The value of $\eta$, appropriate for this situation ($\eta = 3/2$) reflects a two-fold enhancement of the quantum cross-section compared to the classical one. The classical value of $\eta^{cl} = 3/4$, which is sometimes cited in the literature on chaos [3] and classical memory effects [9] in a system of disk scatterers, does not correspond to the ratio of the observable mean free times, as $\tau_{tot}$ is not observable under such experimental conditions. However, there is an interesting but still open question about the quantum-to-classical crossover in the effective total cross-section with an increase of the magnetic field (i.e., shortening of cyclotron orbits), in analogy with the crossover for the scattering by a single obstacle, observed when detectors are moved from the Fraunhofer ($r \gg L_F$) to Fresnel ($r \ll L_F$) regions.

To conclude, we have studied electron propagation through a random array of scatterers characterized by parameter $\eta = \tau_{tr}/\tau_q$ – the ratio of the transport, $1/\tau_{tr}$, and “quantum”, $1/\tau_q$, elastic scattering rates (associated with the transport, $\tau_{tr}$, and total, $\sigma_{tot}$, scattering cross-sections, respectively). For a given type of a disorder, the parameter $\eta$ describes the relative strength of backward and forward scattering. We have considered the case of strong mesoscopic scatterers (e.g., quantum antidots) of a typical size $a$ greater than the electron de Broglie wavelength $\lambda$. For a wide class of scatterer’s shapes, namely for convex ones, we have shown that $\eta$ does not depend on the scatterer’s shape and size. In particular, for specular scattering, $\eta$ is a universal constant determined only by the dimensionality $(D)$ of the system: $\eta = 2$ for $D = 3$ and $\eta = 3/2$ for $D = 2$.

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