Exact analytical solution of Entanglement of Formation and Quantum Discord for Werner state and Generalized Werner-Like states

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We obtained analytical expressions for Entanglement of Formation (EoF) and Quantum Discord (QD) of Werner states and Generalized Werner-Like states. The optimization problem involved under the exact analytical form is obtained for both quantities. In order to illustrate the importance of our results we studied the EoF and the QD of these states. Using discrete formulation of continue states with the $f-$deformed coherents states obtained as deformed annihilation operator coherent states and as deformed displacement operator coherent states. The EoF and QD of bipartite Werner-Like states $f-$deformed coherents states are studied for the Pöschl-Teller, Morse and quantum dot deformed potentials. The result obtained are compared with the case of bipartite Werner-Like coherents states.

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I. INTRODUCTION

Quantum correlations lie in the foundation of quantum mechanics and are the heart of quantum information science. They are important to study the differences between the classical and quantum worlds because, in general, the quantum systems can be correlated in ways inaccessible to classical objects. The research on quantum correlation measures were initially developed on the entanglement-separability paradigm (and the references therein). However, it is well known that entanglement does not account for all quantum correlations and that even correlations of separable states are not completely classical. Entanglement is an inevitable feature of not only quantum theory but also any non-classical theory, and this is necessary for emergent classicality in all physical theories. The study of quantum correlation quantifiers other than entanglement, such as QD, has a crucial importance for the full development of new quantum technologies because it is more robusted than entanglement against the effects of decoherence and can be among others a resource in quantum computation, quantum non-locality, quantum key distribution, remote state preparation, quantum cryptography and quantum coherence.

The QD, as a quantum correlation of a bipartite system, initially introduced by Olliver and Zurek and by Henderson and Vedral, is a more general concept to measure quantum correlations than quantum entanglement, since separable mixed states can have nonzero QD. This measures the fraction of the pairwise mutual information that is locally inaccessible in a multipartite system. The QD is defined as the difference between the total and classical correlations coded in the same state, given by $\mathcal{I}_{AB} = S[\rho_A] + S[\rho_B] - S[\rho_{AB}]$ and $\mathcal{J}_{AB} = S[\rho_B] - S_{X[|\Pi_A^m\rangle]}(\rho_{AB})$, respectively, where $X = \{A, B\}$ with $Y = \{B, A\}$, and $\Pi_m^Y$ is a measurement carried out in the partition $A$ or $B$. These correlations are also known as quantum mutual information and conditional mutual information, respectively. In this correlations, $S[\rho] = -\text{tr}[\rho \log_2 \rho] = -\sum \lambda_i \log_2 \lambda_i$ is the von-Neumann entropy, where the $\lambda_i$‘s are the eigenvalues of the density operator $\rho$; the density operator $\rho_A = \text{tr}_B[\rho_{AB}]$ and $\rho_B = \text{tr}_A[\rho_{AB}]$ is the reduced state in the partition $A$ ($B$) and $S_{X[|\Pi_A^m\rangle]}(\rho_{AB})$ is the conditional entropy of $X$ due to a measure in $Y$. The classical correlations measured in the partition $A$ and $B$ are written as $\mathcal{J}_{AB}^c$ and $\mathcal{J}_{BA}^c$, respectively. The QD is also called the locally inaccessible information (LII)

$$\delta_{AB} = \mathcal{I}_{AB} - \mathcal{J}_{AB}$$

and when performing measured on the partition $B$ it is given by

$$\delta_{AB} = \mathcal{I}_{AB} - \mathcal{J}_{AB}$$

The difficult step is to find the conditional entropy because it requires a process of minimization. The information in the unmeasured partition can be evaluated by
the quantum conditional entropy
\[ S_{B|\{\Pi^A_m\}}(\rho_{AB}) = \min_{\{\Pi^A_m\}} \sum_m S_{B|\{\Pi^A_m\}}(\rho_{AB}), \quad (2a) \]
\[ S_{A|\{\Pi^B_m\}}(\rho_{AB}) = \min_{\{\Pi^B_m\}} \sum_m S_{A|\{\Pi^B_m\}}(\rho_{AB}), \quad (2b) \]
where \( S_{B|\{\Pi^A_m\}}(\rho_{AB}) \) and \( S_{A|\{\Pi^B_m\}}(\rho_{AB}) \) are the von-Neumann entropy of the partition \( B \) and \( A \) of \( \rho_{AB} \) obtained after the projective measurements \( \{\Pi^A_m\} \) or \( \{\Pi^B_m\} \), respectively. The QD is not always larger than the entanglement [21, 22], and there is not clear evidence of the relationship between entanglement and quantum discord, in general, since they seem to capture different properties of the states.

Experimentally, it is difficult to prepare pure states. In general, the states are mixed since they characterize the interaction of the system with its surrounding environment. The study of the quantum information properties of mixed states is more complicated and lesser understood than that of pure states. The set of Werner states [22] is an important type of mixed states, derived in 1989, which plays a fundamental role in the foundations of quantum mechanics and quantum information theory. Since these states admit a hidden variable model without violating Bell’s inequalities, then the correlation measured that are generated with these states can also be described by a local model, despite of being entangled. Moreover, these states are used as quantum channels with noise that do not maintain the additivity, they are also in the study of deterministic purifications (see references in [23]).

It is important to clarify the main differences between the Werner states (Ws) and the Generalized Werner-Like states (GWLS). The bipartite Werner states of qubits are self-adjoint operators, bounded and class trace that act onto the composite space \( \mathcal{H}_2 \otimes \mathcal{H}_2 \), where \( \mathcal{H}_2 \) is the Hilbert space of dimension two, formed by an admixture convex of the exchange operator previously normalized
\[ \frac{1}{2} \hat{1}_4 = \frac{1}{2} \sum_{i,j=0}^3 |ij\rangle\langle ij|, \quad (3) \]
with the maximally mixed state, also is designated as white noise, given for
\[ \hat{1}_4 = \frac{1}{2} \sum_{i,j=0}^3 |ij\rangle\langle ij|. \quad (4) \]
So that the Ws are written as
\[ \rho_W(p) = \frac{1+p}{2} \hat{1}_4 + \frac{1-p}{2} \hat{P}_4, \quad \text{where} \quad p \in [-1, \frac{1}{3}]. \quad (5) \]
The range of variation of the mixing parameter \( p \) guarantees the positivity of \( \rho_W \); furthermore, Volberech and Werner show in [24] that the EoF for the states defined by the equation (5) is
\[ \text{EoF}_W(p) = \begin{cases} 0 & \text{if } -\frac{1}{3} \leq p \leq \frac{1}{3} \\text{and } \frac{1}{3} \leq p \leq 1, \\text{if } -1 \leq p < -\frac{1}{3} \end{cases} \]
where \( H_2(x) = -x \log_2(x) - (1-x) \log_2(1-x) \) is the Shannon binary entropy function. The Ws, given in the equation (5), are invariant under any unitary operator of the form \( \hat{U} \otimes \hat{U} \) and admit a model of hidden variables if \( -1 \leq p \leq -\frac{1}{3} \) (see [22]), being still entangled. The Ws is pure only when \( p = -\frac{1}{3} \), being the same as the Bell state \( |\Phi_-\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \), this is
\[ \rho_W(-\frac{1}{3}) = \frac{1}{2} (\hat{1}_4 - \hat{F}_4) = |\Phi_-\rangle \langle \Phi_-| \quad (7) \]
For all \( p \neq -\frac{1}{3} \) the Ws are mixed.

On the other hand, the GWLS, Quasi-Werner states [23] or Werner-Popescu states [24], for qubits, are a one-parametric family of mixed states, being the sum convex between the maximally mixed state (4) and an pure state \( |\psi\rangle \), where the density matrix of order 4 for the GWLS has the form
\[ \rho_{GWL}(\psi, p) = \frac{1-p}{2} \hat{1}_4 + p \hat{P}_\psi, \quad \text{where} \quad \hat{P}_\psi = |\psi\rangle \langle \psi|. \quad (8) \]
The range of variation of the mixing parameter \( p \) is in this case \( -\frac{1}{3} \leq p \leq 1 \), which guarantees the positivity of the GWLS, shown in equation (8). Here the parameter \( p \) is considered as a probability when the range of variation is \( 0 \leq p \leq 1 \), in this case equation (8) represents a convex sum of a pure state \( |\psi\rangle \) and white noise (3), with probabilities \( p \) and \( 1 - p \), respectively. The fundamental difference between the states (5) and (8) is that \( \hat{P} \) is an involutive operator \( (\hat{P}^2 = \hat{1}) \) while \( \hat{P}_\psi \) is an idempotent operator \( (\hat{P}_\psi^2 = \hat{P}_\psi) \), generating different correlations since the replacing of \( \frac{1}{2} \hat{P} \) by \( \hat{P}_\psi \) in equation (5) makes \( \rho_W(p) \) not unitarily equivalent to \( \rho_{GWL}(\psi, p) \).

Otherwise, the Bell Werner-Like states (BWLS), also called noisy singlets [24], are obtained by using the Bell states \( |\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \) and \( |\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \) as projectors in equation (8). This states are maximally entangled and have been studied widely as a fundamental resource for the quantum information processing, and also in the study of non-local properties in quantum mechanics. The noisy singlets and the Ws are connected by the transformation
\[ \rho_{GWL}(\Phi_-, p) = \frac{1+p}{2} \hat{1}_4 - p \hat{P}_\Phi \equiv \rho_W \quad (9) \]
This equality is exact only in four dimensions. In other dimensions it is impossible obtain this equality. The BWLS obtained from \( |\Psi_+\rangle \) and \( |\Phi_+\rangle \) have been employed in the study of the QD [16, 17], and they can be considered as a particular case of two-qubit X-states [25, 26]. Nevertheless, any unitary transformation applied on GWLS leaves them invariant in shape, without changing the mixing parameter, is
\[ \rho_{GWL}(\psi, p) \xrightarrow{\hat{U}} \hat{U} \rho_{GWL}(\psi, p) \hat{U}^\dagger = \frac{1-p}{2} \hat{1}_4 + p \hat{P}_\psi \]
\[ = \rho_{GWL}(\psi, p) \quad (10) \]
where $|\psi\rangle = \hat{U}|\psi\rangle$. The GWLs changed by unitary transformations are called Werner derivative states [30], however, the study described in reference [30] is incomplete since it only considers a particular class of unitary transformations. In this article, we show that Werner derivative states they have the same EoF and QD.

Until now, due to the quantitative evaluation of QD involves an optimization procedure over all possible measurement on one of the subsystems under study the explicit expression of QD only has been obtained for a few special classes of two-qubit $X$-states [29, 31–33], and generally this is determined numerically [34]. The principal aim of this paper is to derive analytical solutions of EoF and QD for the GWLs built with generalized pure states in bipartite systems of qubits.

In order to illustrate the relevance of our results, we study the QD of the GWLs associated to one bipartite entangled $f-$deformed (or nonlinear) coherent state. In this context, Man’ko and collaborators [35] introduced the $f-$deformed oscillators as a generalization of $q-$oscillators [36, 37]. These nonlinear coherent states exhibit some nonclassical features such as quadrature squeezing [38, 39], second order squeezing [40], sub-Poissonian [38, 39] and super-Poissonian statistics [41, 42], antibunching effect [43], and negativity of Wigner function in parts of the phase space [40]. The $f-$deformed coherent states have been used: to evaluate the statistical behavior of nonlinear coherent states associated to the Morse and Pöschl-Teller Hamiltonians [41], to describe the center-of-mass motion of a trapped ion [44–47], to study quantum dot exciton states [32], the nonclassical properties of deformed photon-added nonlinear coherent states [48] and $f-$deformed intelligent states [49], to produce the superposition of nonlinear coherent states and entangled coherent states [50, 51], to describe non-linear coherent states by photonic lattices [42], among other applications. In this work, we studied the analytical results obtained for the QD and EoF associated to bipartite Werner-Like $f-$deformed coherent states in the following cases: the center-of-mass motion of trapped ions with Pöschl-Teller potential, the entangled exciton states in a quantum dot, and the entangled diatomic molecules using the deformed Morse potential function.

The paper is organized as follows. In Sec. II y IV we present an analytical approach to obtain the exact solutions of EoF and the QD for the GWLs, while in Sec. III we present a technique that allows obtaining exact solutions of the QD for the WSs. In Sec V we present some applications, first, we consider the case of discrete states, where we illustrate the monotonous behavior of the QD with the concurrence of a pure state. In this section, also we present an algebraic review to the bipartite entangled $f-$deformed coherent states when they are obtained as eigenstates of the deformed annihilation opera-

tor, as well as when they are obtained by the application of the deformed displacement operator on the vacuum state. Several deformation functions that we use later in the paper are presented in this section. In this section, also is devoted to illustrate the behavior of QD and EoF of bipartite entangled $f-$deformed GWLs. Finally in Sec. VI y VII we present the analysis and conclusions drawn from our results.

II. ENTANGLEMENT OF FORMATION OF WS AND GWLs

A good measure to quantify the entanglement of a pure state $|\psi\rangle$ is the von-Neumann entropy, since a pure state can be constructed from a set of maximally entangled singlet states and the number of these states is proportional to the entropy of the reduced states of any partitions [52, 53]. So the EoF for a pure state $|\psi\rangle$ is given by

$$EoF_\psi = S[\rho_A] = S[\rho_B], \quad (11)$$

where $\rho_X = tr_X [|\psi\rangle\langle\psi|]$ with $X = A$ or $B$. It is clear that the EoF does not change under local unitary operations, so it is not possible to create or destroy entanglement using these transformations. However, the von-Neumann entropy is not a good measure of the degree of entanglement for mixed states because there are product states whose partitions may have entropies different from zero, for example, $\rho = \rho_1 \otimes \rho_2$ with $S[\rho_1] \neq 0$. To quantify the degree of entanglement for any states belonging to $\mathcal{H}_2 \otimes \mathcal{H}_2$, Wooters [52, 53] proposed that the EoF for any states $\rho$ (pure or mixing) is

$$EoF_\rho = H_2 \left( \frac{1 + \sqrt{1 - C^2[\rho]}}{2} \right), \quad (12)$$

where $C[\rho]$ is the concurrence function of the state $\rho$, defined as $C[\rho] = \max(0, \sqrt{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4})$. The $\lambda_i$’s are the eigenvalues of the positive operator $\rho$, arranged in decreasing order. The operator $\tilde{\rho}$ is the spin-flip operation on the conjugate of the state $\rho$, i.e. $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho (\sigma_y \otimes \sigma_y)$, being $\bar{\rho}$ the conjugate complex of $\rho$. Here the difficult step is to evaluate the concurrence of the state. In this section we find the eigenvalues of $\rho \tilde{\rho}$ for the GWLs.

In the case of a pure state $|\psi\rangle$, the spin-flip operation onto the conjugate complex of the state is given by $\tilde{\rho} = (\sigma_y \otimes \sigma_y)|\psi\rangle\langle\psi| (\sigma_y \otimes \sigma_y) = |\tilde{\psi}\rangle\langle\tilde{\psi}|$, so $\rho \tilde{\rho} = |\psi\rangle\langle\tilde{\psi}| [\tilde{\psi}|\psi\rangle\langle\psi|]$, and the characteristic equation $\rho \tilde{\rho}|\lambda\rangle = \lambda|\lambda\rangle$ leads to $\lambda = |\langle\tilde{\psi}|\psi\rangle|^2$, after projecting this equation on $|\tilde{\psi}\rangle$. Also, the determinant of $|\psi\rangle\langle\tilde{\psi}|$ is zero and therefore $\rho \tilde{\rho}$ has a null eigenvalue with multiplicity three which corresponds to the orthogonal projection to the state $|\tilde{\psi}\rangle$. In this sense, $\sqrt{\lambda_1} = |\langle\tilde{\psi}|\psi\rangle|$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$, being the concurrence for a pure state $|\psi\rangle$

$$C[|\psi\rangle] = |\langle\tilde{\psi}|\psi\rangle| = |\langle\psi|\sigma_y \otimes \sigma_y|\tilde{\psi}\rangle|. \quad (13)$$
This result is known as the Wooters [52] formula for pure states.

In the case of GWLs, the spin-flip operation applied on the conjugate complex of the states defined by equation (5) is given by

$$\tilde{\rho}_{\text{GWL}}(\psi, p) = \frac{1-p}{2} \mathbb{1}_4 + p \lvert \tilde{\psi} \rangle \langle \tilde{\psi} \rvert \equiv \rho_{\text{GWL}}(\tilde{\psi}, p),$$  \hspace{0.5cm} (14)

while

$$\rho_{\text{GWL}}(\psi, p) \tilde{\rho}_{\text{GWL}}(\psi, p) = \left( \frac{1-p}{4} \right)^2 \mathbb{1}_4 + \hat{A},$$  \hspace{0.5cm} (15)

where

$$\hat{A} = p^2 C[\psi] e^{i\phi} \lvert \tilde{\psi} \rangle + \frac{p(1-p)}{4} \left( \lvert \psi \rangle \langle \psi \rvert + \lvert \tilde{\psi} \rangle \langle \tilde{\psi} \rvert \right).$$  \hspace{0.5cm} (16)

Here we have replaced $\langle \psi \vert \tilde{\psi} \rangle$ by $C[\psi] e^{i\phi}$, where $\phi$ is the argument of $\langle \psi \vert \tilde{\psi} \rangle$. On the other hand, the eigenvectors of the matrix $\hat{A}$ are equal to the eigenvectors of $\rho_{\text{GWL}}(\psi, p) \tilde{\rho}_{\text{GWL}}(\psi, p)$, so we will focus on finding the eigenvalues of this matrix. It is clear from equation (16) that the domain of $\hat{A}$ is the linear capsule generated or expanded by $\{ \lvert \psi \rangle, \lvert \tilde{\psi} \rangle \}$, which means that the eigenvectors of $\hat{A}$ belong to this capsule and they can be written as $\lvert \lambda \rangle = \lambda_{1} \lvert \psi \rangle + \lambda_{2} \lvert \tilde{\psi} \rangle$. Projecting the equation $\hat{A} \lvert \lambda \rangle = \lambda \lvert \lambda \rangle$ into the linear capsule, we obtain an equation system for $\langle \psi \vert \lambda \rangle$ and $\langle \tilde{\psi} \vert \lambda \rangle$ from which a straightforward calculation yields

$$\begin{align*}
\begin{bmatrix}
\frac{p(1-p)}{4} - \lambda & \frac{p(1+3p)}{4} C[\psi] e^{i\phi} \\
\frac{p(1-p)}{4} C[\psi] e^{-i\phi} & \frac{p(1-p)}{4} + p^2 C[\psi] - \lambda
\end{bmatrix} 
\begin{bmatrix}
\langle \psi \vert \lambda \rangle \\
\langle \tilde{\psi} \vert \lambda \rangle
\end{bmatrix}
= 0.
\end{align*}
$$  \hspace{0.5cm} (17)

To determine a solution other than the trivial one, we impose that the determinant of the equation system is zero and obtain the following eigenvalue equation

$$\lambda^2 - \frac{p^2(1-2\Delta_0^2)}{2} + \lambda + \frac{p^2(1-p)^2}{16} \Delta_0^2 = 0,$$  \hspace{0.5cm} (18)

where $\Delta_0 \overset{\text{def}}{=} \sqrt{1-C^2[\lvert \psi \rangle \rangle]}$. From this equation, two eigenvalues are determined. The other two eigenvalues of the operator $\hat{A}$ that correspond to the eigenvector expanded into the linear capsule orthogonal to $\{ \lvert \psi \rangle, \lvert \tilde{\psi} \rangle \}$ are zero because $\det(\hat{A}) = 0$. Finally, the eigenvalues of equation (18) are in decreasing order

$$\begin{align*}
\lambda_1 &= \left( \frac{1-p}{4} \right)^2 + \frac{p(1-p+2pC^2[\psi]) + p^2 C[\psi] \sqrt{(1+p)^2 - 4p^2 \Delta_0^2}}{4}, \\
\lambda_2 &= \left( \frac{1-p}{4} \right)^2 + \frac{p(1-p-2pC^2[\psi]) - p^2 C[\psi] \sqrt{(1+p)^2 - 4p^2 \Delta_0^2}}{4},
\end{align*}$$  \hspace{0.5cm} (19a-b)

and

$$\lambda_3 = \lambda_4 = \left( \frac{1-p}{4} \right)^2,$$  \hspace{0.5cm} (19c)

so that the concurrence for GWLs is given by

$$C[\rho_{\text{GWL}}] = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \frac{1-p}{2} \right\}.$$  \hspace{0.5cm} (20)

This shows that GWLs are separable when $\frac{1}{1+2C[\psi]} \leq p \leq \frac{1}{1+2C[\psi]}$ and entangled when $\frac{1}{1+2C[\psi]} < p \leq 1$. In particular, for BWLs we have the usual result [25], i.e., they are entangled if $1/3 < p \leq 1$ and classically correlated if $-1/3 \leq p \leq 1/3$, since $C[\psi]$ is maximally entangled, i.e. $C[\psi] = 1$. When the pure state $\lvert \psi \rangle$ is a product state ($C[\psi] = 0$) then all the GWLs are a convex sum of product states. In Fig. 1 is shown the EoF as a function of the mixing parameter $p$, for Ws and GWLs associated to the pure states $\lvert \psi_1 \rangle$, $\lvert \psi_2 \rangle$, $\lvert \psi_3 \rangle$ and $\lvert \psi_{\text{max}} \rangle$, with the values of concurrence $C_1 = 1/4$, $C_2 = 1/2$, $C_3 = 3/4$ and $C_{\text{max}} = 1$, respectively. The EoF of Ws is given by equation (6). We observe that the EoF of the GWLs increase with the concurrence of the pure state associated to the GWLs. The BWLs are the ones that have the maximum entanglement. The Ws and GWLs are entangled in different regions, and the maximum value of $p$, where the EoF is zero, for the pure states $\lvert \psi_1 \rangle$, $\lvert \psi_2 \rangle$, $\lvert \psi_3 \rangle$ and $\lvert \psi_{\text{max}} \rangle$ are given by $2/3, 1/2, 2/5$ and $1/3$, respectively.

### III. QUANTUM DISCORD OF WERNER STATES

The fundamental amount for the study of quantum information is the von-Neumann entropy, namely, the information in terms of its uncertainty. This quantity measures the expected value of quantum information content [54]. For pure states, the von-Neumann entropy is zero, because the density operator is a projector of rank one, and represent full knowledge about the state of quantum system. For the 4-dimensional Hilbert space $\mathcal{H}_3 \otimes \mathcal{H}_2$ maximal uncertainty is represented by the completely mixed density operator [4], with a value for the von-Neumann entropy of 2, in bits. In these systems one has $0 \leq S[\rho] \leq 2$. Thus, the entropy for the Ws given in (5) will be bounded between these two values, being zero when $p = -1$ and maximum when $p = 0$.

The starting point is to obtain the entropy of the Ws
to find the eigenvalues of the states given under the equation (3). Is easily demonstrated that the eigenvalues del exchange operator are \( \pm 1 \), since

\[
\hat{F}_4 |f \rangle = f |f \rangle \quad \Rightarrow \quad \hat{F}_4^2 |f \rangle = f^2 |f \rangle = \hat{I}_4 |f \rangle = f^2 |f \rangle \\
\Rightarrow \quad |f \rangle = f^2 |f \rangle \quad : \quad f = \pm 1.
\] (21)

However, the exchange operator \( \hat{F}_4 \) acts onto \( \mathcal{H}_2 \otimes \mathcal{H}_2 \) so it must have 4 eigenvalues, which are not simple. Let \( |\lambda^+_W \rangle \) be the eigenvectors of the exchange operator (that are same to eigenstates of the werner states) with eigenvalues 1 and \(-1\), respectively, then,

\[
\hat{F}_4 |\lambda^+_W \rangle = +|\lambda^+_W \rangle \quad \Rightarrow \quad \langle ji |\lambda^+_W \rangle = \langle ij |\lambda^+_W \rangle. \quad (22a)
\]

\[
\hat{F}_4 |\lambda^-_W \rangle = -|\lambda^-_W \rangle \quad \Rightarrow \quad \langle ji |\lambda^-_W \rangle = -\langle ij |\lambda^-_W \rangle. \quad (22b)
\]

This proves that the eigenstates \( |\lambda^+_W \rangle \) and \( |\lambda^-_W \rangle \) belong to spaces of dimensions \( \frac{2(2+1)}{2} = 3 \) and \( \frac{2(2-1)}{2} = 1 \), respectively. The multiplicity of this eigenvalues is the dimensions of those spaces. The Ws has one simple eigenvalue, given for \( \frac{1+\sqrt{5}}{4} \), and 3 degenerates eigenvalues with value \( \frac{1-\sqrt{5}}{4} \). This allows to obtain the von-Neumann entropy for the Ws, given for,

\[
S_{AB}(p) \overset{\text{def}}{=} S[\rho_W] = -\text{tr} [\rho_W \log_2 \rho_W] = 2 - \frac{1-3p}{4} \log_2 (1-3p) - \frac{3(1+p)}{4} \log_2 (1+p). \quad (23)
\]

This expression is a concave function of the mixing parameter, being zero for \( p = -1 \) and two for \( p = 0 \).

The partial trace of the exchange operator in any partition is equal to identity operator in two dimensions, this is, \( \text{tr}_X \hat{F}_4 = \mathbb{I}_2 \) with \( X = A \) or \( X = B \). This allows determine the quantum information of each partition of the system \( \mathcal{H}_2 \otimes \mathcal{H}_2 \) contained in the Ws. Therefore, the reduced state of the Ws is a maximally mixed state and the entropy of those states is the logarithm of the dimension of the reduced space, this is, for the state

\[
\rho^X_W = \text{tr}_Y [\rho_W] = \frac{1}{2} \mathbb{I}_2 \tag{24a}
\]

we have,

\[
S_X(p) = -\text{tr} [\rho^X_W \log_2 \rho^X_W] = 1. \quad (24b)
\]

Where \( X = \{ A, B \} \) when \( Y = \{ B, A \} \) in the equation (24a).

In order to quantify the QD, the entropy condicional is required, for this we perform a projective measurement \( \{\hat{\Pi}_m\} \) on one partition of the subsystem. In the partition \( A \) we have

\[
\hat{\Pi}^A_m = \hat{\Pi}_m \otimes \mathbb{I}_2 = \frac{1}{2} [\mathbb{I}_2 + (-1)^m \hat{n} \cdot \hat{\sigma}] \otimes \mathbb{I}_2, \tag{25}
\]

with \( m = 0 \) or \( m = 1 \). Here \( \hat{n} = \sin(2\theta) \cos(\phi)\hat{i} + \sin(2\theta) \sin(\phi)\hat{j} + \cos(2\theta)\hat{k} \), is a unitary vector on the Bloch sphere, and \( \hat{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k} \) is the Pauli vector. After a local measurement \( \hat{\Pi}^A_m \), on the density matrix \( \rho_W \), the state of the system becomes a hybrid quasi-classical state [4], this is, applying the Lüder rule [5] to the Ws is obtained the post-measurement states \( \rho_{W|\Pi^A_m} \) (see appendix A)

\[
\rho_W(p) \rightarrow \hat{\Pi}^A_m \rho_{W|\Pi^A_m} = \frac{(\hat{\Pi}^A_m) \rho_W(\hat{\Pi}^A_m)^+}{\rho^A_m}, \tag{26a}
\]

\[
\rho_{W|\Pi^A_m} = \hat{\Pi}_m \otimes \left\{ \frac{1-p}{2} \mathbb{I}_2 + p\hat{\Pi}_m \right\}, \tag{26b}
\]

Where \( \rho^A_m \) corresponds to the probability of reaching the state measured, which can be evaluate as

\[
\rho^A_m = \langle \hat{\Pi}^A_m | \rho_W \rangle = \text{tr} \left[ \hat{\Pi}^A_m \rho_W \right] = \frac{1}{2}. \tag{27}
\]

The states in the partition \( B \) of the equation (26b) have the form of GWLs in \( \mathcal{H}_2 \) with mixing parameter \( p \). The eigenvalues of these states are \( \frac{1+\sqrt{5}}{2} \), and they are independent of the measure. For this reason, the conditional entropy is same that the entropy of the reduced states in the partition \( B \) of equation (26b). Using (25) we obtain

\[
S_{A|\Pi^B_m}(p) = \min_{\{\hat{\Pi}^B_m\}} \sum_{m} \rho^A_m S_{B|\Pi^B_m}(\rho_W) = H_2 \left( \frac{1+p}{2} \right). \tag{28}
\]

A straightforward calculator shows that the conditional QD in both partitions are the same. Therefore, the QD is symmetric and it’s given by

\[
\delta_{AB}(p) = \delta_{AB}(p) = \delta_{BA}(p) = H_2 \left( \frac{1+p}{2} \right) - 1 + \frac{1-3p}{4} \log_2 (1-3p) + \frac{3(1+p)}{4} \log_2 (1+p). \tag{29}
\]

In Fig. 2 it shown the graph of QD and EoF for Ws. It is noted that there is correlations, although the states not are entanglement. For \(-1 \leq p < -0.88 \) the EoF is more big that the QD, in the rest of interval the relation it is reversed. The quantum discord is only zero only for \( p = 0 \) and for values of mixing parameter near of the origin is very small.
IV. QUANTUM DISCORD OF GWLS

Firstly we evaluate the von-Neumann entropy. The GWLs given in the equation \( \text{[3]} \), has a simple eigenvalue given by \( \frac{1+3p}{4} \), and three degenerate eigenvalues with value \( \frac{1-p}{2} \), this allows to obtain the von-Neumann entropy, given by

\[
S_{AB}[\psi, p] = S[\rho_{GWL}] = -\text{tr} \left[ \rho_{GWL} \log_2 \rho_{GWL} \right] = 2 - \frac{3(1-p)}{4} \log_2(1-p) - \frac{1+3p}{4} \log_2(1+3p).
\]

This expression is independent of the pure state \( |\psi\rangle \), in addition to being a monotonic function of the mixing parameter. Is clear from \( \text{[30]} \) that the information provided by the GWLs is minimal (maximum entropy) when \( p = 0 \), corresponding to a maximally mixed quantum state (white noise), while that the information is maximal (minimum entropy) when \( p = 1 \), value for which \( \text{[3]} \) is pure.

To determine the quantum information (in terms of its uncertainty) of each partition of the system \( \mathcal{H}_2 \otimes \mathcal{H}_2 \) contained in GWLs, given the equation \( \text{[3]} \), its is sufficient to take its partial traces, so that we have

\[
\rho^A_{GWL} = \text{tr}_B [\rho_{GWL}] = \frac{1+p}{2} \mathbb{1}_2 + p \hat{\mathbb{W}} \hat{\mathbb{W}}^\dagger,
\]

\[
\rho^B_{GWL} = \text{tr}_A [\rho_{GWL}] = \frac{1-p}{2} \mathbb{1}_2 + p \hat{\mathbb{W}}^T \hat{\mathbb{W}}^\dagger.
\]

Where \( \hat{\mathbb{W}} \) is the matrix constructed with the components of the pure state \( |\psi\rangle \) in the computational base \( |ij\rangle \), namely \( \psi_{ij} = (i|\psi) \), while \( \hat{\mathbb{W}}^T \) is the transposed matrix of \( \hat{\mathbb{W}} \). The normalization condition of the state \( |\psi\rangle \) in term of matrix \( \hat{\mathbb{W}} \) is \( \text{tr} \left[ \hat{\mathbb{W}} \hat{\mathbb{W}}^\dagger \right] = 1 \) (see appendix \( \text{A} \) for details). In this context the transpose connects the density operator of both partitions and this operation does not modify the eigenvalues of the reduce states. For this reason, the expressions \( \text{[31]} \) show that the entropies of the reduced states are equal, so that

\[
-\text{tr} \left[ \rho^A_{GWL} \log_2 \rho^A_{GWL} \right] = -\text{tr} \left[ \rho^B_{GWL} \log_2 \rho^B_{GWL} \right],
\]

\[
S_A(\psi, p) = S_B(\psi, p).
\]

The two eigenvalues of \( \hat{\mathbb{W}}^\dagger \hat{\mathbb{W}} \) are \( \frac{1}{2} \left( 1 \pm \Delta_0 \right) \) with \( \Delta_0 = \sqrt{1 - C^2[\psi]} \), where \( C[\psi] \) is the concurrence of the pure state \( |\psi\rangle \), belonging to \( \mathcal{H}_2 \otimes \mathcal{H}_2 \). This quantity is given by the formula of Wooters \( \text{[13]} \)

\[
C[\psi] = 2|\psi_{00}\psi_{11} - \psi_{01}\psi_{10}| = 2|\det \hat{\mathbb{W}}|.
\]

These results show that the two eigenvalues of the reduced states \( \text{[31]} \) are \( \frac{1}{2} \left( 1 \pm \Delta_0 \right) \), so the entropy \( \text{[32]} \) takes the following form

\[
S_A(\psi, p) = S_B(\psi, p) = H_2 \left( \frac{1+p\Delta_0}{2} \right).
\]

When \( p = 1 \) in the equation \( \text{[31]} \) one has the EoF given in the equation \( \text{[12]} \) of pure state \( |\psi\rangle \); on the other hand, when \( p = 0 \) the entropy of the reduced state is maximal, take the value of one, which corresponds to a maximally mixed state in \( \mathcal{H}_2 \).

In order to quantify the QD, the entropy conditional is required. We performed a projective measurement \( \text{[25]} \) on one partition \( A \) of the subsystem. After of this local measurement on the density matrix \( \rho_{GWL} \), the state of the system becomes a hybrid quasi-classical state \( \text{[4]} \), this is, using the Lüder rule \( \text{[55]} \) we obtain (see appendix \( \text{A} \))

\[
\rho_{GWL} \xrightarrow{\hat{\Pi}_m} \rho_{GWL|A} = \frac{\langle \hat{\Pi}_m \rangle \rho_{GWL} \hat{\Pi}_m^\dagger}{\langle \hat{\Pi}_m \rangle^2},
\]

where \( \rho_{GWL|A} \) is the post-measurement state and \( \rho^p_m \) corresponds to the probability of reaching that state, which can be evaluated as

\[
p_m^A = \langle \hat{\Pi}_m \rangle \rho_{GWL} = \text{tr} \left[ \hat{\Pi}_m \rho_{GWL} \right] = \frac{1-p}{2} + p \langle \hat{\Pi}_m \rangle \psi.
\]

The amount \( x_m(p) \) shown in \( \text{[26]} \) is equivalent to a new mixing parameter of the GWLs in the partition \( B \), given by (see appendix \( \text{A} \))

\[
x_m(p) = \frac{p \langle \hat{\Pi}_m \rangle \psi}{\frac{1-p}{2} + p \langle \hat{\Pi}_m \rangle \psi}.
\]

Noteworthy is that \( x_m(p) \) is an injective function of the mixing parameters \( p \), so both parameters \( p \) and \( x_m(p) \) present the same variation range. It is important to see that all \( x_m(p) \) are not independent, since the sum over all probabilities \( \sum_m \langle \hat{\Pi}_m \rangle \psi = 1 \) impose a restriction on the \( x_m(p) \), given by

\[
\sum_m x_m(p) = \frac{2p}{1-p}.
\]

In order to simplify the result \( \text{[26]} \) we define the projector \( |\tilde{\psi}\rangle \langle \tilde{\psi}| \) as

\[
|\tilde{\psi}\rangle \langle \tilde{\psi}| = \sum_{i,j} \langle i | \hat{\mathbb{W}}^\dagger \hat{\mathbb{W}}^\dagger | j \rangle \langle i | \hat{\mathbb{W}}^\dagger \hat{\mathbb{W}}^\dagger | j \rangle.
\]

Thus, a projective measurement on the subsystem \( A \) projects the system into a statistical ensemble \( \{p^A_m, \rho_{AB|A} \} \) quantifies the information in the unmeasured partition as the quantum conditional entropy, given by

\[
S_{B|A}(\rho^A_m)\psi, p = \min_{\langle \hat{\Pi}_m \rangle} \sum_m p_m^A S_{B\rho_{AB|A}^m} \psi, p = \frac{1}{2} \left( \frac{1-p}{1-x_m(p)} \right) H_2 \left( \frac{1+x_m(p)}{2} \right).
\]
Here the probability $p_\alpha^A$ is replaced by the expression $\langle \hat{\sigma}_\alpha^A \rangle$, while the probability $\langle \hat{\Pi}^A_\psi \rangle$ is written in terms of the mixing parameter $x_m(p)$ using \[38\]. $S_B|\Pi^A_\psi(\psi;p)$ is the von-Neumann entropy of the partition $B$ of $\rho_{GW\ell}(\psi;p)$ obtained after the projective measurements $(\hat{\Pi}^A_\psi)$. Since the measurement might give different results depending on the basis choice, a minimization is taken over all possible rank-1 measurement $(\hat{\Pi}^A_\psi)$, applied on the subsystem $B$. Minimizing chooses the measurement of $A$ that extracts as much information as possible of $B$. The hard step in the evaluation quantum conditional entropy is usually the optimization of the conditional entropy $S_B|\Pi^A_\psi$ over all projective measurements. However, it is clear that the process of minimizing the conditional entropy is ineffect to find the values of $x_m(p)$ that minimize the probability $\langle \hat{\Pi}^A_\psi \rangle$. In the Appendix \[13\] we show that the conditional entropy of the partition $B$, have the form

$$S_B|\Pi^A_\psi(\psi;p) = F_p(\underline{y}_0) + F_p(\underline{y}_1), \quad (42)$$

where

$$F_p(x) = \frac{1-p}{2} + p(1 - x)H_2\left(\frac{1+x}{2}\right), \quad (43)$$

and the values $\{\underline{y}_0, \underline{y}_1\}$ are such that minimize the conditional entropy, for which $\underline{y}_0$ minimizes $F_p(x)$ but $\underline{y}_1$ maximizes it (see appendix \[13\]). In fact, $\underline{y}_0$ is obtained when the probability $\langle \hat{\Pi}^A_\psi \rangle$ is minimized, while $\underline{y}_1$ is obtained by $\langle \hat{\Pi}^A_\psi \rangle$.

$$\underline{y}_0 = \frac{p(\langle \hat{\Pi}^A_\psi \rangle)_{\min}^{\min}}{1 - 2pA} = \frac{p(1 - 2A)}{1 - 2pA}, \quad (44a)$$

and

$$\underline{y}_1 = \frac{2p}{1 + 2p}x_0(p) = \frac{p(1 + 2A)}{1 + 2pA}. \quad (44b)$$

Can also be written as

$$\underline{y}_1 = \frac{p(\langle \hat{\Pi}^A_\psi \rangle)_{\max}^{\max}}{1 - 2pA} = \frac{p(1 - (\langle \hat{\Pi}^A_\psi \rangle)_{\min}^{\min})}{1 - 2pA}, \quad (45)$$

which also is consistent with $\langle \hat{\Pi}^A_\psi \rangle$. The value of $A$ showed in $\langle 44 \rangle$ (see appendix \[13\]) is given by

$$A = \frac{1}{2} \left[ \sum_{i=1}^{3} \left( \text{tr} \left( \hat{W}^\dagger_\psi \sigma_i \hat{W}_\psi \right) \right)^2 \right]. \quad (46)$$

The equation $\langle 42 \rangle$ is an analytical expression for the conditional entropy after a measurement in partition $A$. The aforementioned procedure can be applied to obtain the conditional entropy $S_A|\Pi^B_\psi(\psi;p)$, after a measurement in partition $B$. The same result is obtained, except that instead of the matrix $\hat{W}_\psi$, its transpose is used, namely,

$$S_A|\Pi^B_\psi(\psi;p) = F_p(\underline{y}_0) + F_p(\underline{y}_1), \quad (47)$$

with

$$\underline{y}_0 = \frac{p(1 - 2B)}{1 - 2pB} \quad \text{and} \quad \underline{y}_1 = \frac{p(1 + 2B)}{1 + 2pB}, \quad (48)$$

and

$$B = \frac{1}{2} \left[ \sum_{i=1}^{3} \left( \text{tr} \left( \hat{W}^\dagger_\psi \sigma_i \hat{W}_\psi \right) \right)^2 \right]. \quad (49)$$

Generally the QD is asymmetric and $\delta_{AB} \neq \delta_{BA}$. We can study the average of LII, defined as $\overline{\omega}^{+}_{AB} = (\delta_{AB} + \delta_{BA})/2$, and the balance of LII, defined as $\overline{\omega}^{-}_{AB} = (\delta_{AB} - \delta_{BA})/2$ (see reference \[56\]). Nevertheless, a straightforward calculation showed that $\langle 40 \rangle$ and $\langle 41 \rangle$ coincide, with which the QD in both partitions are equal, therefore the balance is zero. If we take the explicit forms of the entropy given in the Eqs. \[30\], \[31\] and \[12\], we can obtain the analytical expressions of the QD $\delta_{AB}(\psi;p)$ for the GWLs, after a measurement in partition $A$ or $B$. The exact analytical solutions are

$$\delta_{AB}(\psi;p) = \delta_{AB}(\psi;p) = \delta_{AB}(\psi;p)$$

where

$$= H_2\left(\frac{1-p\Delta_0}{2}\right) + F_p(\underline{y}_0) + F_p(\underline{y}_1) +$$

$$+ \log_2\left(\frac{1-p\Delta_0}{2}\right) + \log_2\left(\frac{1+3p}{4}\right). \quad (50)$$

with $\Delta_0 = \sqrt{1 - C^2[\psi]}$, being $C[\psi]$ the concurrence of the pure state $|\psi\rangle$ associated with the GWLs. The QD given in the equation $\langle 50 \rangle$, in addition to being symmetrical is a monotonous function of the concurrence $C[\psi]$. So, all the pure states with the same concurrence $C[\psi]$ have equal QD, in the same way as the EoF of the GWLs; this forms classes of equivalence among pure states $|\psi\rangle$ with equal concurrence.
V. Applications

In this section we show the EoF and QD for GWLs built with discrete and continue pure states. For the discrete case we present four pure states with different entanglement, and both measures of correlations are compared. In the other case, two mode $f$-deformed coherent states are considered, and again the EoF and QD are compared.

A. Discrete States

To illustrate the monotone behavior of the QD for the GWLs with respect to the concurrence of the pure state with which are built, it is enough consider the following states (in the representation of the matrix $W_\psi$, given in the Eqs. 31)

$$W_{\psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_{\psi_3} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 3 & \sqrt{6} \\ 2\sqrt{3} & -1 \end{pmatrix},$$

$$W_{\psi_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -3 & -3\sqrt{2} \\ 2\sqrt{2} & 1 \end{pmatrix}, \quad W_{\psi_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{7} & \sqrt{5} \\ 3\sqrt{5} & \sqrt{7} \end{pmatrix}.$$  (51)

These matrices are representative of the equivalence classes corresponding to the concurrence 1, 2, 3, and 4, respectively. The first matrix corresponds to the Bell state $|\Psi^+\rangle$ (maximal entanglement), while the states $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ have less entanglement. In the Fig. 3 the EoF [12] and QD [60] have been sketched for the four states given in (51). It can be observed that the QD is a monotonous function that grows with the increase of the concurrence of the pure state associated to the GWLs, but the QD not is a monotonous function of your own EoF, as it happens for pure states. On the other hand, exists a region in which the QD is bigger than the EoF.

B. $f$-Deformed Coherent States

The coherent states for the electromagnetic field, introduced by Glauber in 1963 [57, 58], have played an important role, not only in quantum optics, but in many fields of the physics. In terms of their evolution, they remain localized about the corresponding classical trajectory when acted on by harmonic interactions and do not change their functional form with the time [60]. Glauber showed that these states can be obtained from any one of these three mathematical definitions:

1. As the eigenstate of the annihilation bosonic operator $\hat{a}$, i.e., $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, being $\alpha$ a complex number.

2. As those that can be obtained by the application of the displacement operator $\hat{D}(\alpha) = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}$ on the vacuum state of the harmonic oscillator, i.e., $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$.

3. As the quantum states with a minimum uncertainty relationship $(\Delta \hat{Q})_\alpha^2(\Delta \hat{P})_\alpha^2 = 1/4$, with $\hat{Q}$ and $\hat{P}$ the position and momentum operators, respectively.

Each of these definitions lead to equivalent coherent states for the harmonic oscillators. On the other hand, two different coherent states are not orthogonal but they form an complete basis, with which one can decompose any state [61].

The $f$-deformed oscillators are nonlinear oscillators with a specific kind of nonlinearity for which the frequency depends on the oscillator energy, these single mode nonlinear states are essentially based on the deformation of bosonic annihilation and creation operators, according to the relations

$$\hat{A} = \hat{a}f(\tilde{n}) = f(\tilde{n} + \hat{1})\hat{a},$$  (52a)

$$\hat{A}^\dagger = f(\tilde{n})\hat{a}^\dagger = \hat{a}^\dagger f(\tilde{n} + \hat{1}),$$  (52b)

where $\tilde{n} = \hat{a}^\dagger\hat{a}$ is the bosonic number operator, with actions on the Fox space as

$$\hat{A}(n) = \sqrt{n}f(n)|n - 1\rangle,$$  (53a)

$$\hat{A}^\dagger(n) = \sqrt{n + 1}f(n + 1)|n + 1\rangle.$$  (53b)

These deformed boson creator $\hat{A}$ and annihilator $\hat{A}^\dagger$ differ from the usual harmonic operators $\hat{a}$ and annihilation $\hat{a}^\dagger$ by a deformation function $f(\tilde{n})$. This deformation function is real and non-negative, and it is convenient to assume that is a continuous function with $f(0) = 1$. The commutation relations between the deformed operators are given by

$$[\tilde{n}, \hat{A}] = - \hat{A}, \quad [\tilde{n}, \hat{A}^\dagger] = - \hat{A}^\dagger, \quad [\hat{A}, \hat{A}^\dagger] = \hat{1} + \phi(\tilde{n}).$$  (54)

Where $\phi(\tilde{n}) = (\tilde{n} + 1)f^2(\tilde{n} + \hat{1}) - \tilde{n}f^2(\tilde{n}) - \hat{1}$. The deformation becomes fixed when one choose the explicit form of the function $f(\tilde{n})$, and the harmonic case is recovered when $f(\tilde{n}) = \hat{1}$. The Hamiltonian of these deformed oscillators can be written in terms of annihilation and creation deformed operators $\hat{A}$ and $\hat{A}^\dagger$ as

$$\hat{H} = \frac{\hbar \Omega}{2}[\hat{A}\hat{A}^\dagger + \hat{A}^\dagger\hat{A}],$$  (55)

and the spectrum of $\hat{H}$ is given by

$$E_n = \frac{\hbar \Omega}{2} [(n + 1)f^2(n + 1) + nf^2(n)].$$  (56)

Man’ko and collaborators [35] introduced the nonlinear coherent states $|\alpha\rangle_A$ of an $f$-oscillator algebra, as the eigenstates of the annihilation operator $\hat{A}$, such as $\hat{A}|\alpha\rangle_A = \alpha|\alpha\rangle_A$. The explicit form of normalized states $|\alpha\rangle_A$, in the number state representation, is given by

$$|\alpha\rangle_A = N_A \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$  (57a)
with
\[ N_A = \left( \sum_{n=0}^{\infty} \frac{|\alpha|^2 n! f(n)!^2}{n!} \right)^{-1/2}, \]  
where \( f(n)! = f(0)f(1)f(2) \cdots f(n) \). When using the
displacement operator method to generate the coherent
states for deformed algebra, one faces the problem that
the commutation between the deformed operators \( A \) and
\( A^\dagger \) is not a number, as a consequence, the displacement
operator obtained by the replacement of the usual
displacement operator by their deformed counter parts can not be
written in a product form \[44\].

Récamier and collaborators \[62\] proposed an approach
to generate the \( f_-\) deformed coherent state \( |\alpha\rangle_D \) by the
application of a deformed operator \( \hat{D}(\alpha, f) \) acting upon
the vacuum state, such that \( |\alpha\rangle_D = \hat{D}(\alpha, f)|0\rangle \). The
Récamier deformed displacement operator can be written as
\[ \hat{D}(\alpha, f) = \mathsf{e}^{\alpha \hat{A}^\dagger} \mathsf{e}^{\alpha^* \hat{A}} \mathsf{e}^{\alpha^2 (1+\phi(n))}. \]
This displacement operator is nearly unitary and dis-
places \( A \) and \( A^\dagger \) whenever
\[ \frac{1}{2} |\alpha|^2 \phi(n) \ll 1, \]
\[ \frac{1}{2} |\alpha|^2 \left[ (n+1)f^2(n+1) - n f^2(n) - 1 \right] \ll 1. \]  
This restriction requires that the values of \( n \) do not
belong to the range \([0, \infty)\), but rather \( n \) has a maximum
value \( n_{\text{max}} \) determined by this restriction on \( |\alpha|^2 \).
The normalized \( f_-\) deformed coherent states obtained by
application of the approximately displacement operator
upon the vacuum state are
\[ |\alpha\rangle_D = N_D \sum_{n=0}^{n_{\text{max}}} \alpha^n \frac{f(n)!}{\sqrt{n!}} |n\rangle, \]  
with
\[ N_D = \left( \sum_{n=0}^{n_{\text{max}}} \frac{|\alpha|^2 n! f(n)!^2}{n!} \right)^{-1/2}. \]

The deformed states \( |\alpha\rangle_A \) and \( |\alpha\rangle_D \) are not equivalents, in
general, they present a similar quantum evolution and however they show a different statistical behavior \[41\].

The QD and the EoF of the BWLS in the orthogonal
basis \(|0\rangle \) and \(|1\rangle \) are well known \[24, 31\]. In this work,
we are interested in studying the behaviour of QD of
the GWLS if we have bipartite entangled \( f_-\) deformed
coherent states. We use the \( |\alpha\rangle \), \( |\alpha\rangle_A \) and \( |\alpha\rangle_D \) coherent
states and \( f_-\) deformed coherent states to encode qubits
when they are superposed with \(|-\alpha\rangle \), \(|-\alpha\rangle_A \) and \(|-\alpha\rangle_D \),
respectively. The states \( |\alpha\rangle_X \) and \(|-\alpha\rangle_X \) correspond
to non-orthogonal coherent states (when \( X = C \)) and
\( f_-\) deformed coherent states (when \( X = A \) or \( X = D \))
with opposite phases. We choose an orthogonal basis by
considering even and odd superpositions of \( |\alpha\rangle_X \) and
\(|-\alpha\rangle_X \), such that \[63, 64\]
\[ |\pm\rangle_X = N_{\pm}^X \left( |\alpha\rangle X \pm |-\alpha\rangle X \right), \]  
where \( n_{\text{max}} \) is taken according to the convergence of the
expression \[59\] for \( X = D \). The + and - signs in the
expression \[61b\] are taken when \( X = D \) or \( X = A \), respec-
tively. When \( X = C \) then the deformation function is
\( f(n) = 1 \), with which \( |\alpha\rangle_C \) is a coherent states \(|\alpha\rangle \). These quantum superpositions \(|\pm\rangle_X \) can be consid-
ered as realization of a \( f_-\) deformed Schrödinger cat \[65\]. In this paper, we are interested in to study
the Quasi-Bell entangled \( f_-\) deformed coherent states even and odd and they are, respectively
\[ |\Psi_+\rangle_X = n_X \left( |\alpha, \alpha\rangle_X + |-\alpha, -\alpha\rangle_X \right), \]  
\[ |\Psi_-\rangle_X = n_X \left( |\alpha, -\alpha\rangle_X + |-\alpha, \alpha\rangle_X \right), \]  
\[ n_X = \left[ 2(1 + |x| \langle \alpha | -\alpha \rangle_X^2 ) \right]^{-1/2}. \]  
If we express \( |\Psi_{\pm}\rangle_X \) in the orthogonal basis of states
\(|\pm\rangle_X \) given in \[61\], we can write
\[ |\Psi_{\pm}\rangle_X = \frac{n_X}{2} \left[ |+, +\rangle_X \pm |-, -\rangle_X \right]. \]  
These states are non-maximally entangled and are mutually non-orthogonal. When \( f(n) = 1 \) and in the limit of large mean photon number \( |\alpha|^2 \) these states form a complete orthogonal basis just like standard Bell states \(|\Psi_+\rangle \) and \(|\Psi_-\rangle \). For the complete study of entanglement and quantum discord of GWLSs the matrix \( \hat{W}_{\Psi_X} \) of the
pure state \[63\] is requiered, which is diagonal and contains in the constants \[61b\] and \[62c\] all the effects of
the deformations. So that,
\[ \hat{W}_{\Psi_X} = \frac{n_X}{2} \left[ \frac{(N_{X}^X \pm \alpha)}{N_{X}^X \pm \alpha} \right] \]  
\[ \text{det} \hat{W}_{\Psi_X} = \frac{1}{2} \left( \frac{n_X}{N_{X}^X \pm \alpha} \right)^2. \]
The QD of the GWLs corresponding to the pure states $|\Psi_X \rangle$ and $|\Psi_+ \rangle$ are the same. As consequence, in this work only we take $|\Psi_+ \rangle$ for study EoF and QD of GWLs corresponding to several $f$-deformed functions.

In order to illustrate the relevance of our results, we considered some deformation functions that are important in quantum computing and in quantum information processing. Ours first example, we considered the center-of-mass motion of trapped ion. In such model, both the “center-of-mass motional states” and the electronic-states can be simultaneously coupled and manipulated by light fields [45, 46]. The trapped ion systems are useful to study the quantum optical and quantum dynamical properties of quantum systems that are approximately isolated from the environment, and the strong Coulomb forces between the ions can be used to realize logical gate operations by coupling different qubits. For this reason, trapped atomic ions are one of the leading candidate systems to construct a robust quantum computer [47]. Recently, experimental entanglement between remote ions in different ion traps modules has been reported [66]. The trapped ion in a modifie Pöschl-Teller potential can be considered as a $f$-deformed oscillator with a specific kind of the $f$-deformed Heisenberg-Weyl algebra [46], in which the corresponding deformation function has the form

$$f(n) \to f_N(n) = \left( \frac{\sqrt{N^2 + 1} - n}{N} \right)^{1/2}$$

(66)

where $N$ is a dimensionless positive parameter that is associated with the depth of the trap. The values of $N$ and $n$ are not independent and they satisfy the relation $n < \sqrt{N^2 + 1}$. In the limiting case where $N \rightarrow \infty$, we obtain $f_N(n) = 1$, therefore the energy levels are related by the deep of the trap, and finite rang trap has a finite dimensional Hilbert space, which is important because it seems possible realize experiments in order to study the Hilbert space size effects on these systems. In Fig. 4 the EoF and QD for this trapped ion in a modified Pöschl-Teller potential are shown.

Due to the modern semiconductor microfabrication technology, the quantum dots are other of the promising candidates for a solid-state quantum computer. These solid-state quantum systems are especially attractive because of their good scalabilibity and stability. Exciton in coupled quantum dots are being used in the preparation of entangled states in solid-states systems [67], and entangled of exciton states in a single quantum dot, or in a quantum dot molecule, have been experimentally demonstrated [67–70]. Harouni and collaborators [39] proposed an $f$-deformed oscillator approach, to study the confinement conditions of an exciton with definite angular momentum in a wide quantum dot interacting with two lasers beams. Under this approach the deformation function takes the form

$$f(n) \to f_\kappa(n) = e^{-\kappa^2} \frac{L^n_1(\kappa^2)}{(n+1)L^n_0(\kappa^2)},$$

(67)

where $\kappa$ is similar to the Lamb-Dicke parameter in trapped ion systems, and is defined as the ratio of the quantum dot is radius to the wavelength of the driving laser, and the $L^n_m(x)$ are the associated Laguerre Polynomials. This deformation function is similar to the one that appears in the center-of-mass motion of a trapped ion confined in a harmonic trap [44], but in this case $\kappa$ is the Lamb-Dicke parameter which depend on the laser wavelength and the quantum fluctuation of the ion position in the ground vibrational state. For this deformation function the parameters $\kappa$ and $n$ not are independent, they satisfy inequality $\kappa^2(2n + 1) << 1$ in the Lamb-Dicke regime; this inequality bound to maximum value of $n$ for $\kappa$ given. In Fig. 5 the EoF and QD for the exciton potential in coupled quantum dots are shown.

Finally, we consider the entanglement of diatomic molecules. The entanglement may play a crucial role in explaining the relations of electronic and vibrational degrees of freedom in molecules [71], and some diatomic molecules are the best candidates for multiple molecular
quantum bits in diatomic molecular quantum computers [72]. Molecular Quantum computers may be more advantageous than atomic ion traps because the internal degrees of freedom utilized as quantum bit for the former are much larger than those for the latter.

In order to model the entanglement between diatomic molecules, we consider the Morse potential deformation function introduced by Récamier and collaborators [41, 62, 63]. The Morse potential is an interatomic model for the potential energy of a diatomic model, it is a better approximation for the vibrational structure of these molecules than the quantum harmonic oscillator. The deformation function of Morse potential has the form

\[ f(n) \rightarrow f_N(n) = \sqrt{1 + \frac{1 - n}{2N}}, \]  

where \( N \) is an integer number determined by the number of bound states. The values of \( N \) and \( n \) are not independent of each other, satisfying the relationship \( n < \sqrt{2N+1} \). This deformation function reproduce the spectra of the Morse and Pöschl-Teller Hamiltonian [41]. In Fig. 5(a) the EoF and QD for Morse potential are shown.

VI. RESULTS

In the previous section, we studied the behavior of the QD and EoF for continuous variable states in their discrete formulations, taking as examples the coherent states and type \( A \) and \( D \) \( f \)-deformed coherent states, associated to deformations functions: the Pöschl-Teller (Fig. 3); excitons coupled to a quantum dot (Fig. 5); Morse potential (Fig. 5). The QD and EoF are plotted for average numbers of photons \(|\alpha|\) smaller and greater than unity, observing that the EoF is greater than the QD for certain values of \(|\alpha|\). The graphs of the QD and EoF for states the type \( D \) with \(|\alpha|\) greater than unity are not shown, because they do not meet the inequality (69) for the values of \( N, \kappa \) and \( n_{\text{max}} \) indicated in each figure. On the other hand, the difference between our results and the numerical results of QD obtained using method in the reference [33], for all the deformations functions considered, presented a percentage relative error of 12 significant digit. Moreover, for guarantee the convergence of the QD and EoF are taken the value of \( n_{\text{max}} \) for which the QD (or the EoF) in \( n_{\text{max}} \) and \( n_{\text{max}}+1 \) be the same.

In this case of deformation function associated to Pöschl-Teller the value of \( N \) is fixed to 10, implies that the value of \( n_{\text{max}} \) must be less than 20. However, the value taken for \( n_{\text{max}} \) is 9, because it guarantees the convergence of the QD and EoF; it means that, for this value of 9 the QD (or the EoF) is same to that obtained with the value of 10. Using the values of \(|\alpha| = 0.65, N = 10 \) and \( n_{\text{max}} = 9 \) the restriction condition (69) takes the value of 0.4. In other hand, the QD is greater that the EoF for the states \(|\Psi^A_+\rangle \) and \(|\Psi^C_+\rangle \) when \(|\alpha| > 0.8785 \) and \(|\alpha| > 0.8807 \), respectively. Furthermore, for values with \( 0 < |\alpha| < 1.3 \) the QD of the state \(|\Psi^A_+\rangle \) is greater than the QD of the coherent state \(|\Psi^C_+\rangle \), and the QD of this state are greater than the QD of the state \(|\Psi^D_+\rangle \), so that for this values of \(|\alpha|\) the QD satisfies the inequality (see Fig. 4(b))

\[ \delta_{AB}(\Psi^A_+,\rho) > \delta_{AB}(\Psi^C_+,\rho) > \delta_{AB}(\Psi^D_+,\rho). \]  

(69)

And for \(|\alpha| > 1.3 \) we have (see Fig. 4(b))

\[ \delta_{AB}(\Psi^C_+,\rho) > \delta_{AB}(\Psi^A_+,\rho). \]  

(70)

For the case of the deformation function associated to excitons coupled to a quantum dot the value of \( \alpha \) is fixed to 0.3 to consider Lamb-Dicke regime, with which the value of \( n_{\text{max}} \) must be less than 5.06. Using the values of \(|\alpha| = 0.65, \kappa = .3 \) and \( n_{\text{max}} = 5 \) the restriction condition (69) takes the value of 0.4. On the other hand, the QD is greater that the EoF for the states \(|\Psi^A_+\rangle \) and \(|\Psi^C_+\rangle \) when \(|\alpha| > 0.8786 \) and \(|\alpha| > 0.8804 \), respectively. Also,
the inequality \( \delta_{AB} \geq 0 \) is hold for all the range of values of average number of photons \( |\alpha| \) (see Fig. 6).

Finally, for case of the deformation function associated Morse potential the value of \( N \) is fixed to a number maximum of the confined levels in one molecule. In this work we considered \( N = 18 \) and \( n_{\text{max}} \), and must be lesser than 6.08. With the values \( |\alpha| = 0.65, N = 18 \) and \( n_{\text{max}} = 6 \) the restriction condition \( 59 \) takes the value of 0.07. Using this values we find that the QD is robust (no deformations to the coherent states) for values \( |\alpha| \) less than one (see Fig. 6(a)), nevertheless, there are deformations for values of \( |\alpha| \) greater than one (see Fig. 6(b)).

VII. CONCLUSIONS

This work, permitted to obtained the exact analytical solutions of QD and EoF for the WSs and GWLs. The WSs and GWLs are different quantum states, since we can not obtain \( \rho_N \) from \( \rho_{GW|L} \), and vice versa, using a unitary transformation between them. The values of the mixture parameter \( p \), for which we can make comparisons of the correlations present in the WSs and GWLs, are bounded by \(-1/3 < p < 1/3\), finding that in this region both states are a convex sum of product states. The QD and the EoF, for the GWLs are monotonic functions of the concurrence of the pure state \( \tilde{\rho}_\psi = |\psi\rangle\langle\psi| \), and for pure states with the same concurrence the QD \( = \text{EoF} \) for GWLs. On the other hand, the QD of the WSs and GWLs are symmetric because their balance is zero.

To illustrated the relevance of our analytical results we calculated the QD and EoF for \( f \)-deformed coherent states, in a discrete formulation of GWLs, applied to three different deformed function: Post-teller, quantum dot excitons and Morse potentials. We find that the difference between our results and the numerical results of QD, in all the considered cases, presented a percentage relative error of 12 significant figures. Regardless of the deformation function used, Quasi-Bell states entangled \( f \)-deformed coherent states even and odd have the same QD.

For the case of Pöschl-Teller and the Morse potentials, there is a critical value for the quantum number of photons \( \alpha_c \), for which if \( \alpha < \alpha_c \) the QD satisfies the inequality \( \delta_{AB}(\Psi^A_\alpha, \rho) > \delta_{AB}(\Psi_\alpha, \rho) \), and for \( \alpha > \alpha_c \) the QD satisfies \( \delta_{AB}(\Psi_\alpha, \rho) > \delta_{AB}(\Psi^A_\alpha, \rho) \). However, in the case of the quantum dot potential, the QD satisfies the inequality \( \delta_{AB}(\Psi^A_\alpha, \rho) > \delta_{AB}(\Psi_\alpha, \rho) \), for all \( \alpha \) values. The QD of the \( f \)-deformed coherent states \( \alpha D \), obtained from the displacement operator \( 58 \) do not present QD for large values of \( \alpha \) because the inequality \( 59 \) is not met.

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Appendix A: Projective measurement onto pure state, WSs and GWLs

Let \( U = [U_{ij}] \) be unitary transformations, the base \( \{|\pi_m\}_\{m\} \) is unitarily equivalent to the computational base \( \{|i\}_\{i\} \) if \( |\pi_m\rangle = \sum U_{mi}|i\rangle \). The projector, associated to these measurement are

\[
\hat{\Pi}_m = |\pi_m\rangle\langle\pi_m| = \sum_{ij} U_{im}^* U_{jm}|i\rangle\langle j|, \tag{A1}
\]

where \( U_{jm}^* \) is the complex conjugate of \( U_{jm} \). The projectors associated to local projective measurement in the partition \( A \) of bipartite systems are

\[
\hat{\Pi}_{mA} = \hat{\Pi}_m \otimes \hat{1} = \sum_{ijk} U_{im} U_{jm}|ik\rangle\langle jk|, \tag{A2}
\]
where the identity operator \( \mathbb{I} \) has been replaced by the sum of projectors \( \sum_k |k\langle k| \). In the other hand, any pure state \( |\psi\rangle \) belonging to \( \mathcal{H} \otimes \mathcal{H} \) can be written in term of

\[
|\psi\rangle = \sum_{ij} \psi_{ij}|ij\rangle \quad \text{with} \quad \sum_{ij} \psi_{ij} = 1. \tag{A3}
\]

In order to simplify our results we define the matrix \( W_{ij} \), whose elements are \( \psi_{ij} \), so the normalization condition can be written as

\[
\sum_{ij} \psi_{ij} \bar{\psi}_{ij} = 1 \implies \sum_{ij} W_{ij} \bar{W}_{ij} = 1 \tag{A4}
\]

The representation of pure states in terms of density matrix is given by the following projector of rank one,

\[
|\psi\rangle\langle\psi| = \sum_{ijk\ell} \psi_{ij} \bar{\psi}_{k\ell} |ij\rangle \langle k\ell|, \tag{A5}
\]

the reduced state are obtained taking partial trace over both partitions, so for partition \( A \) we have that

\[
\rho_A(\psi) = \text{tr}_B [ |\psi\rangle\langle\psi| ] = \sum_{ijk\ell} \psi_{ij} \bar{\psi}_{k\ell} \text{tr}_B [ |ij\rangle \langle k\ell| ] = \sum_{ijk\ell} \psi_{ij} \bar{\psi}_{k\ell} \delta_{ij} |k\ell\rangle \langle k\ell|,
\]

\[
\rho_A(\psi) = \hat{W}_{ij} \hat{W}_{ij}^\dagger \tag{A6}
\]

and for partition \( B \) we have,

\[
\rho_B(\psi) = \text{tr}_A [ |\psi\rangle\langle\psi| ] = \sum_{ijk\ell} \psi_{ij} \bar{\psi}_{k\ell} \text{tr}_A [ |ij\rangle \langle k\ell| ] = \sum_{ijk\ell} \psi_{ij} \bar{\psi}_{k\ell} \delta_{jk} |i\ell\rangle \langle i\ell|,
\]

\[
\rho_B(\psi) = (\hat{W}_{ij}^\dagger)(\hat{W}_{ij}^\dagger) \tag{A7}
\]

This shows that partition \( B \) can be accesses through the transpose operation. On the other hand, the probability of obtaining a result after the local projective measurement \( A \) when the system is initially in the pure state \( |\psi\rangle \) is give by

\[
\langle \bar{\Pi}^A \psi \rangle = \langle \psi | \bar{\Pi}^A | \psi \rangle = \sum_{ijk} \psi_{ik} \bar{\psi}_{jk} U_{im} U_{jm}, \tag{A8}
\]

in this expression have been replace \( \{2\} \). The last expression can be written in matrix form as,

\[
\langle \bar{\Pi}^A \psi \rangle = \sum_{ijk} \bar{\psi}_{ik} U_{im} U_{jm} \psi_{jk} \tag{A9}
\]

where we have used the expressions \( \{1\} \) and \( \{6\} \). If the measurement is performed on partition \( B \) then

\[
\langle \bar{\Pi}^B \psi \rangle = \langle \bar{\Pi}^B \psi \rangle_{\rho_B} = \text{tr} \left[ (\hat{W}_{ij}^\dagger)(\hat{W}_{ij}^\dagger) \bar{\Pi} \right] = \langle \bar{\Pi} \rho_B \psi \rangle, \tag{A10}
\]

We perform a projective measurement on the pure state on partition \( A \). The state after of the measure is obtains by Lüders rule \( \{5\} \), so

\[
|\psi\rangle \langle\psi| \bigg|_{\Pi_m^A} = \frac{\hat{\Pi}_m^A |\psi\rangle \langle\psi| (\hat{\Pi}_m^A)^\dagger}{(\hat{\Pi}_m^A \psi) \langle\psi| \hat{\Pi}_m^A \psi) \rangle} \tag{A11}
\]

and for partition \( B \) we have

\[
\rho_B|\Pi_m^A \rangle = \sum_{ij} \psi_{ij} \bar{\psi}_{ij} U_{im} U_{jm} U_{rm} |ij\rangle \langle rt| \tag{A12}
\]

where we have defined

\[
\rho_B|\Pi_m^A \rangle = \frac{1}{(\hat{\Pi}_m^A \psi) \langle\psi| \hat{\Pi}_m^A \psi) \rangle} \sum_{ij} \psi_{ij} \bar{\psi}_{ij} U_{im} U_{jm} U_{rm} |ij\rangle \langle rt| \tag{A13}
\]

In the equation \( \{12\} \) it has been replace in the equation \( \{9\} \). We can show that \( \{12\} \) is pure state, since it is projector operator of rank one. A straightforward calculator leads to

\[
\text{tr} [ \rho_B|\Pi_m^A \rangle ] = 1 \quad \text{and} \quad \rho_B^2|\Pi_m^A \rangle = \rho_B|\Pi_m^A \rangle. \tag{A14}
\]

In the case of projective measurement in the partition \( B \) the results are similar, except for the transpose operation in the matrix \( \hat{W}_{ij} \).

Now we perform a local projective measure in the partition \( A \) to the \( W \)s given in \( \{9\} \). According to \( \{20\} \) we
have that

\[ \rho_{\text{GW}}(\Pi^4_m) = \frac{(\Pi^4_m \rho_{\text{GW}})(\Pi^4_m)'}{p_m^4} \]

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{1}{p_m^4} \left[ \frac{1}{4} \hat{\Pi}^4_m + \frac{p}{4} \hat{\Pi}^\psi_m \right] (\Pi^4_m)'
\]

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{1}{p_m^4} \left[ \frac{1}{4} \hat{\Pi}^4_m + \frac{p}{4} \sum_i \langle i | \hat{\Pi}^4_m | i \rangle \langle i | \hat{\Pi}^4_m \rangle \right] \]

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{1}{p_m^4} \left[ \frac{1}{4} \hat{\Pi}^4_m + \frac{p}{4} \sum_i \langle i | \hat{\Pi}^4_m | i \rangle \langle i \rangle \langle i | \hat{\Pi}^4_m \rangle \right] \]

Using (A15) in the equation (A15) we obtain the equation (A15).

Finally, we perform a local projective measurement on the GWLs (8) in the partition A. According to the equation (A15) we have

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{(\Pi^A_m \rho_{\text{GW}})(\Pi^A_m)'}{p_m^4} \]

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{1}{p_m^4} \left[ \frac{1}{4} \hat{\Pi}^A_m + \frac{p}{4} \hat{\Pi}^\psi_m \right] (\Pi^A_m)'
\]

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{1}{p_m^4} \left[ \frac{1}{4} \hat{\Pi}^A_m + \frac{p}{4} \sum_i \langle i | \hat{\Pi}^A_m | i \rangle \langle i \rangle \langle i | \hat{\Pi}^A_m \rangle \right] \]

\[ \rho_{\text{GW}}(\Pi^A_m) = \frac{1}{p_m^4} \left[ \frac{1}{4} \hat{\Pi}^A_m + \frac{p}{4} \sum_i \langle i | \hat{\Pi}^A_m | i \rangle \langle i \rangle \langle i | \hat{\Pi}^A_m \rangle \right] \]

Where we have used Eqs. (A11), (A12) and (A13). Defining the mixing parameter in the partition B as

\[ x_m(p) = \frac{p(\Pi^4_m \rho_{\text{GW}})(\Pi^4_m)'}{p_m^4} = \frac{p(\Pi^A_m \rho_{\text{GW}})(\Pi^A_m)'}{p_m^4} \]

While the term that accompanies the identity matrix in (A18) can be written as

\[ \frac{1}{4p_m^4} = \frac{1 - x_m(p)}{4p_m^4} \]

These showed the equation (A18). This result is very important since the projective measurement does not alter the structure of the GWLs, but modifies the mixing parameter p by x_m(p).

Appendix B: Calculation of condicional entropy for Werner-like states

For the optimization process, it is convenient to define (A3), which is a positive an monotonically increasing function of the mixing parameter x_m(p), of partition B. So that the conditional entropy (A11) is given by

\[ S_B(\Pi^A_m) \langle \psi, p \rangle = \min \left( \sum_m F(x_m(p)) \right). \]  

(B1)

The minimum is obtained when there set of existence of a value for the mixing parameter x_m(p) such that the function F is minimal, subject to restriction (39). For the case n = 2, it is sufficient find the value x_m for which F is minimal, while \( z_m \) is obtained from (39). Deriving F(z_m) with respect to z_m and after a simple calculation, we can obtain

\[ dF(x_m) = -\frac{(1 - p) \log_2 \left( \frac{1 - x_m}{2} \right)}{2(1 - z_m)^2} dx_m. \]  

(B2)

Using the values of x_m given in (25), it is easy to show that

\[ dF(x_m) = p \log_2 \left( \frac{1 - p}{4} + \frac{p}{4} \frac{\hat{\Pi}^A_m}{\hat{\Pi}^A_m} \right) d(\hat{\Pi}^A_m). \]  

(B3)

It is clear from (B3) that the process of minimizing the conditional entropy is relegated to finding the values of x_m that minimize the probability \( \langle \hat{\Pi}^A_m | \psi \rangle \), which in turn minimize the function F(x_m). This probability presents oscillations around the uniform distribution, which allows us to evaluate its minimum quickly. Considering the local projective measurement (25) and after straightforward calculation, we obtain the simplified result

\[ \langle \hat{\Pi}^A_m | \psi \rangle = \frac{1}{2} \left[ 1 + (\sigma_z)_{\rho_A(\psi)} \cos(2\theta + m\pi) \right] \]

\[ + (|e^{i\sigma_z}|)_{\rho_A(\psi)} \sin(2\theta + m\pi) \]  

(B4)

where \( \rho_A(\psi) \) is given by (A6) and the explicit expressions for the coefficients of the trigonometric functions are

\[ \langle \sigma_z \rangle_{\rho_A(\psi)} = \sqrt{\psi_00^2 + |\psi_1|^2} - |\psi_{11}|^2, \]  

\[ \langle e^{i\sigma_z} \rangle_{\rho_A(\psi)} = 2 \text{Re} \left[ (\psi_{00}^{*}\psi_{11} - \psi_{01}^{*}\psi_{10}) e^{-i\phi} \right]. \]  

(B5)

Taking into account that \( \text{Re} \left[ z e^{i\phi} \right] \leq |z| \) and

\[ 2|\psi_{00}^{*}\psi_{11} - \psi_{01}^{*}\psi_{10}| = \sqrt{\langle \sigma_z \rangle_{\rho_A(\psi)}^2 + \langle \sigma_y \rangle_{\rho_A(\psi)}^2}, \]

we have that the amplitude of the oscillations presented in (B5) is given by

\[ A = \frac{1}{2} \sqrt{\langle \sigma_z \rangle_{\rho_A(\psi)}^2 + \langle \sigma_y \rangle_{\rho_A(\psi)}^2}. \]  

(B6)
This result coincides with (16). So the minimum probability value is

$$
\langle \hat{I}^A_{m_0} \rangle_{\text{min}} = \frac{1}{2} - A.
$$

(B7)

Sintetizing, the value that minimize the function $F(x_m(p))$, and therefore minimize the conditional entropy (B1), given by (44).

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