Abstract In this paper we analyze the photoproduction of heavy quarkonia pairs which include $b$-quarks, such as $B^+_c B^-_c$-mesons or charmonium-bottomonium pairs. Compared to charmonia pair production, these channels get contributions only from some subsets of diagrams, and thus allow for a better theoretical understanding of different production mechanisms. In contrast to the production of hidden-flavor quarkonia, for the production of $B^+_c B^-_c$-meson pairs there are no restrictions on internal quantum numbers in the suggested mechanisms. Using the Color Glass Condensate approach, we estimated numerically the production cross-sections in the kinematics of the future electron–hadron colliders and the ultraperipheral collisions at LHC. We found that the production of $J/\psi \eta_c$ and $B^+_c B^-_c$ meson pairs are the most promising channels for studies of quarkonia pair production.

1 Introduction

Since the early days of QCD, heavy quarkonia have been used for the study of the gluonic field in high energy interactions. Due to their heavy masses, the heavy quarks, which constitute the quarkonia, might be described in a perturbative approach [1,2]. Nowadays the heavy quarks processes are described in the so-called NRQCD framework, which allows to incorporate systematically various perturbative corrections [3–14]. The studies of quarkonia usually focus on charmonia, since they have significantly larger cross-sections. However, it is known that in the charm sector there are certain technical challenges, such as for example the “non-universality” of the long-distance matrix elements, which potentially might be due to sizable corrections to the heavy quark mass limit. In contrast, for quarkonia including bottom quarks it is expected that the heavy quark mass approximation is much more reliable, and thus the expected corrections should be much smaller. A special place in these studies occupy the $B^\pm_c$-mesons, which are made of a $b$- and $c$-quarks. As of now these states are poorly understood, and only two states are included in the Particle Data Group’s listings. Due to difference of constituent heavy flavors, the hadronic decays of these mesons are forbidden, which implies quite a large mean lifetime. For the same reason, they have completely different production mechanisms compared to the hidden-flavor states: the $B^\pm_c$ might be produced hadronically only in hard subprocesses, which include both $\bar{b}b$ and $\bar{c}c$ quark pairs at the partonic level. Due to this fact, their production cross-sections are very small. Nevertheless, studies of $B^\pm_c$ production are important for the confirmation of our current understanding of heavy quarkonia in general, as well as providing a potential gateway for the study of various exotic multiquark states in their decay products.

The exclusive production of heavy quarkonia is one of the cleanest channels for their study. Most of the previous work on this topic focused on single quarkonia states [15–19], since they have the largest production cross-sections. However, the production of multiple heavy quarkonia (e.g. heavy quarkonia pairs) has been a subject of theoretical attention almost since inception of QCD [20–24]. The interest in this channel has drastically increased recently due to the forthcoming launch of high-luminosity accelerator facilities and a recent discovery of all-heavy tetraquarks, which might be molecular states of quarkonia pairs [25–35]. Nowadays such processes might be studied both in ultraperipheral collisions at the LHC, as well as in electron–proton collisions at the forthcoming Electron Ion Collider (EIC) [36–39], the future Large Hadron electron Collider (LHeC) [40], and the Future Circular Collider (FCC-he) [41–43].

Most of the previous studies of exclusive double quarkonia production focused on the so-called two-photon mechanism, $\gamma \gamma \rightarrow M_1 M_2$, which gives the dominant contribution to the production of quarkonia pairs with the same $C$-parity [44–49]. Numerically, the cross-section of this process is very
small due to smallness of the fine structure constant. In this paper we analyze an alternative mechanism, which has significantly larger cross-section, although leads to the production of quarkonia pairs with opposite $C$-parities. We consider in detail the production of quarkonia including both $c$- and $b$-quarks. For all-charmonia and all-bottononia pairs the theoretical analyses are very similar, though numerically the bottomonia cross-sections are significantly suppressed due to heavier mass quark mass and smaller size of bottomonium. A special interest present mixed charmonium-bottonium pairs, such as for example $J/\psi \eta_b$ or $\Upsilon \eta_c$, as well as the production of $B_c^+ B_c^-$ meson pairs. The production of these states obtains contributions only from some of the diagrams which are relevant for the production of all-charm or all-bottom quarkonia. In the case of $B_c^+ B_c^-$ production, there are no restrictions on internal quantum numbers of the produced mesons, which presents an important advantage over the charmonia pairs. According to theoretical expectations, the production cross-sections in these channels are maximal in the near-threshold region, which is relevant for searches of different exotic states, like e.g. $b\bar{b}$-containing tetraquarks [50].

Both in ultraperipheral collisions at LHC and in $ep$ collisions at future colliders the dominant contribution stems from quasi-real photons with small virtuality $Q^2 \approx 0$. At central rapidities it is expected that the produced quarkonia will carry just a small fraction of the colliding hadron momenta, $x_i \ll 1$. In this kinematics the saturation effects should play an important role in the dynamics of partons, and thus should be properly accounted for in the theoretical models of interaction. In what follows we’ll use the color dipole framework, also known as Color Glass Condensate or CGC framework [51–62], which naturally incorporates the saturation effects and provides a phenomenologically successful description of both hadron–hadron and lepton–hadron collisions [63–70].

The paper is structured as follows. Below in Sect. 2 we present the theoretical results for the exclusive production of heavy quarkonia pairs in the CGC approach. In Sect. 3 we present our numerical estimates for meson pairs which include at least one $b$ or $b$-quark, and analyze the dependence on quantum numbers of produced quarkonia. Finally, in Sect. 4 we draw conclusions.

2 Theoretical framework

Nowadays, photoproduction processes might be studied in electron–proton, proton–proton and proton–nuclear collisions in ultraperipheral kinematics. The corresponding cross-sections of these processes are related to photoproduction cross-section as

$$\frac{d\sigma_{ep \rightarrow cM_1M_2p}}{d Q^2 dy_1 d^2 k_1^+ dy_2 d^2 k_2^+} \approx \frac{\alpha_{em}}{\pi Q^2} \left(1 - \gamma + \frac{\gamma^2}{2}\right) \left|\frac{d\sigma_T (\gamma + p \rightarrow \gamma + p + M_1 + M_2)}{dy_1 d^2 p_1^+ dy_2 d^2 p_2^+}\right|_{p_I^+ = k_2^+}, \quad (1)$$

$$\frac{d\sigma_{pA \rightarrow pAM_1M_2}}{dy_1 d^2 k_1^+ dy_2 d^2 k_2^+} = \int dn_\gamma (\omega \equiv E_\gamma, \mathbf{q}_\perp) d\sigma_T (\gamma + p \rightarrow \gamma + p + M_1 + M_2) \left|\frac{d\sigma_T}{dy_1 d^2 p_1^+ dy_2 d^2 p_2^+}\right|_{p_I^+ = k_2^+ - q^+} \quad (2)$$

where in (1) we use standard DIS notation in which $\gamma$ is the inelasticity (fraction of electron energy which passes to the photon), and $(y_a, k_a^+)$, with $a = 1, 2$, are the rapidities and transverse momenta of the produced quarkonia with respect to electron–proton or hadron–hadron collision axis. The expression $dn_\gamma (\omega \equiv E_\gamma, \mathbf{q}_\perp)$ in (2) is the spectral density of the flux of equivalent photons with energy $E_\gamma$ and transverse momentum $\mathbf{q}_\perp$ with respect to the nucleus, which was found explicitly in [71]. The momenta $p_a^+ = k_a^+ - q^+$ are the transverse parts of the quarkonia momenta with respect to the produced photon. The nuclear form factors strongly suppress the transverse momenta $\mathbf{q}_\perp$ larger than the inverse nuclear radius $R_A^{-1}$. For this reason the average values of $\mathbf{q}_\perp$ are quite small, $|\mathbf{q}_\perp| \sim \langle Q^2 \rangle \sim \langle R_A^2 \rangle^{-1} \lesssim (0.2 \text{ GeV}/A^{1/3})^2$, and the $p_\perp$-dependence of the cross-sections in the left-hand side of (2) almost coincides with the $p_\perp$-dependence of the cross-section in the integrand in the right-hand side. The subscript letter $T$ in the right-hand side of (1,2) reminds us that the dominant contribution comes from quasiareal transversely polarized photons. The corresponding cross-section $d\sigma_T$ is related to the amplitude as

$$\frac{d\sigma_T}{dy_1 d \mathbf{p}_1^+ d^2 dy_2 d \mathbf{p}_2^+ d\phi} \approx \frac{1}{256\pi^3} \left|\mathcal{A}_{Y\gamma p \rightarrow M_1M_2p}\right|^2 \delta \left(\mathbf{p}_1^+ + \mathbf{p}_2^+ - q^+ - 1\right), \quad (3)$$

where $\mathcal{A}_{Y\gamma p \rightarrow M_1M_2p}$ is the amplitude of the exclusive process, induced by a transversely polarized photon, and $\phi$ is the angle between the vectors $\mathbf{p}_1^+$ and $\mathbf{p}_2^+$ in transverse plane. The variable $q^+$ is the light-cone momentum of the photon, and $\mathbf{p}_1^+$, $\mathbf{p}_2^+$ are the light-cone momenta of the produced quarkonia. As we will show below, it is possible to express them via quarkonia kinematic variables $(y_a, \mathbf{p}_a^+)$. For further evaluations of the amplitude $\mathcal{A}_{Y\gamma p \rightarrow M_1M_2p}$ it is necessary to fix the reference frame and write out explicit light-cone decomposition of momenta of the participating hadrons. In what follows
we will use the notations: $q$ for the photon momentum, $P$ and $P'$ for the momentum of the proton before and after the collision, and $p_1$, $p_2$ for the 4-momenta of produced heavy quarkonia. We will also use the notation $\Delta$ for the momentum transfer to the proton, $\Delta = P' - P$, and $t$ for its square, $t \equiv \Delta^2$. The light-cone expansion of the above-mentioned momenta in the lab frame is given by

$$q = (q^+, 0, \mathbf{0}_\perp), \quad q^+ \approx 2E_\gamma$$ (4)

$$P = \left(\frac{m_N^2}{2P^2}, P^-, \mathbf{0}_\perp\right), \quad P^- = E_p + \sqrt{E_p^2 - m_N^2} \approx 2E_p$$ (5)

$$P' \approx \left(\frac{m_N^2 + (p_1^+ + p_2^+)^2}{2P^2 - M_1^+ e^{-\gamma_1} + M_2^+ e^{-\gamma_2}}, p_1^-, -p_1^+ - p_2^+\right),$$ (6)

$$p_a = \left(\frac{M_{a^+} e^{\gamma_a}}{2}, p_a^\perp\right), \quad a = 1, 2, \quad M_a^+ \equiv \sqrt{M_a^2 + (p_a^+)^2},$$ (7)

where $m_N$ is the mass of the nucleon, and $M_1, M_2$ are the masses of produced quarkonia. In the high-energy collider kinematics, when $q^+, P^- \gg \{Q, M_a, m_N, \sqrt{t}\}$, there is an approximative relation between the energy (component $q^+$) of the photon and the light-cone momenta of the produced quarkonia,

$$q^+ \approx 2E_\gamma \approx M_1^+ e^{\gamma_1} + M_2^+ e^{\gamma_2},$$ (9)

which in essence reflects the fact that the change of the light-cone plus-component proton momentum, $(P')^+ - P^+$, is negligibly small, in agreement with the eikonal picture expectations. The relations (4–9) allow to link the quarkonia kinematic variables $(y_a, p_a^\perp)$ to conventional DIS variables, such as the Bjorken variable $x_B$ or invariant energy $W = \sqrt{s_{\gamma p}}$. In what follows we will use these variables $(y_a, p_a^\perp)$, since they allow to keep an explicit symmetry w.r.t. permutation of quarkonia, and are directly measurable in experiments. In terms of these variables, the invariant energy $W$ of the $\gamma p$ collision and the invariant mass $M_{12}$ of the produced heavy quarkonia pair are given by

$$W^2 \equiv s_{\gamma p} = (q + P)^2 = -Q^2 + m_N^2 + 2q \cdot P$$

$$\approx -m_N^2 + P^- \left(M_1^+ e^{\gamma_1} + M_2^+ e^{\gamma_2}\right),$$ (10)

and

$$M_{12}^2 = (p_1 + p_2)^2$$

$$= M_1^2 + M_2^2 + 2\left(M_1^+ M_2^+ \cosh \Delta y - p_1^+ \cdot p_2^+\right)$$ (11)

respectively. The photoproduction amplitude $A_{\gamma p \rightarrow M_1 M_2 p}$, which appears in (3), is the central quantity of interest for our study. Since the formation time of quarkonia is larger than the typical size of the proton, the amplitude of the process might be factorized and written as a convolution of the quarkonia wave functions with hard amplitudes which describe photoproduction of two quark–antiquark pairs in the gluonic field of the target. In what follows we will refer to the heavy quarks produced in such hard subprocess as “final state” quarks, and will use the color dipole (Color Glass Condensate) approach [51–62] for evaluation of the $\bar{Q}Q\bar{Q}Q$ amplitude. In the following Sect. 2.1 we briefly discuss the algorithm and main assumptions of the theoretical framework, and in Sect. 2.2 we present final theoretical results of this evaluation.

2.1 High energy scattering in CGC/color dipole picture

In this subsection, for the sake of completeness we will briefly remind the general procedure which allows to express different hard amplitudes in terms of the color singlet forward dipole scattering amplitude. For the sake of definiteness, we will use the Iancu–Mueller approach [72] (equivalent formulations might be found in [51,53–61]).

For heavy quarkonia production, the heavy quark mass $m_Q$ plays the role of the natural hard scale, which determines the interaction strength of heavy quarks with the gluonic field. In the heavy quark mass limit it is formally possible to develop a systematic expansion over the strong coupling $\alpha_s(m_Q) \ll 1$. However, such expansion might be not very reliable in the small-$x$ limit, when the gluon fields are enhanced due to saturation effects and reach values $A_\mu^a \sim 1/\alpha_s$. Fortunately, for some processes which get the dominant contribution from short-distance interactions, a destructive interference of various amplitudes might lead to an additional suppression of the strength of gluonic interaction with the ensemble of heavy partons. For example, for single quarkonia photoproduction the typical separation between the partons $r \sim 1/m_Q$, so the interaction of singlet dipoles with gluons remains suppressed as $\sim Q_s(x)/m_Q$, where $Q_s$ is the saturation scale. However, the interaction of gluons with each other, as well as with light quarks, remains strongly nonperturbative, so the dynamics of the dipole amplitudes still should satisfy the nonlinear Balitsky–Kovchegov equation.

The dynamics of high energy partons in the gluonic field of the target might be described in the eikonal approximation, disregarding parton motion in the transverse direction and the change of its helicity due to eikonal interaction [73–76]. In this picture the interaction of heavy partons with the gluonic field of the target is described by Wilson lines $V(x_\perp)$ [62, 66, 72]

$$V(x_\perp) = P \exp \left( ig \int dx^- A_\mu^+(x^-, x_\perp) \tau^\mu \right),$$ (12)
where $x_{\perp}$ is the impact parameter of the parton, $t_\perp$ are the ordinary color group generators of pQCD in corresponding representation (fundamental or adjoint for quarks or gluons respectively), and $A_{\mu}^a(x) = -\frac{1}{\nu_{\perp}} \rho_\perp(x^-)$ is the gluonic field in a hadron created by the color charge density $\rho_\perp$.

According to classical CGC picture [52–54], for multiparton scattering processes the amplitude should be averaged with a weight function $W[\rho]$ which describes the probability of a given charge distribution $\rho$ inside the target. Fortunately, for many processes it is possible to simplify significantly the evaluations and express physical observables via color singlet dipole scattering amplitudes known from Deep Inelastic Scattering. Indeed, a dipole interaction with the target is described by the $S$-matrix element [62,66,72]

$$S_2(y, x_Q, x_\perp) = \frac{1}{N_c} \left( \text{tr} \left( V^\dagger(y, x_Q) V(y, x_\perp) \right) \right), \quad (13)$$

where the notation $y = \ln(1/x)$ is used for the dipole rapidity, $x_Q, x_\perp$ are the transverse coordinates of the partons (quark or antiquark), and the factors $V^\dagger(x_Q)$ and $V(x_\perp)$ in (13) are the Wilson lines which describe the interaction of the partons with the color field of a hadron. The impact parameter dependent color dipole amplitude $N(x, r, b)$ can be related to $S \left( y, x_Q, x_\perp \right)$ as

$$N(x, r, b) = 1 - S_2 \left( y, x_Q, x_\perp \right), \quad (14)$$

where the variable $r \equiv x_Q - x_\perp$ is the transverse size of the dipole, $b \equiv z x_Q + (1 - z) x_\perp$ is the transverse position of the dipole center of mass, and $z$ is the fraction of the light-cone momentum of a dipole which is carried by the quark $Q$. For some processes it could be possible to express physical observables in terms of pairwise combinations $\sim \left( \text{tr} \left( V^\dagger(y, x_j) V(y, x_j) \right) \right)$, and later use (14) to express them via color singlet dipole amplitudes. In the following two subsections we will discuss how this might be done in $m_Q \rightarrow \infty$ limit for exclusive heavy quarkonia pair production, which requires evaluation of the $\bar{Q} Q \bar{Q} Q$ scattering in the gluonic field of the target. While in general the interaction of such system with the target cannot be expressed in terms of the color singlet amplitude $N(x, r, b)$, we expect that for small-size color singlet ensemble the dominant contribution comes from dipole term in multipolar expansion, and evaluate explicitly this contribution. First, in Sect. 2.1.1 we consider the dilute (non-saturated) limit, relying on smallness of strong coupling $\alpha_s(m_Q)$ and making expansion over it. Next, in Sect. 2.1.2 we discuss how to redraw these results making expansion over short distances between partons rather than smallness of $\alpha_s(m_Q)$. The formal expansion parameter in this case is $Q_s(x)/m_Q$, where $Q_s$ is the saturation scale, and $m_Q \approx \infty$ is the heavy quark mass. Finally, in Sect. 2.1.3 we comment briefly on possible contribution of final state soft interactions.

### 2.1.1 Dilute scattering limit

In the dilute (non-saturated) regime, the weakness of interaction between heavy quarks and gluons allows to make a Taylor expansion of Wilson line (12), yielding

$$V(x_{\perp}) \approx 1 + ig \int dx^- A^+_a(x^-, x_{\perp}) t^a + \frac{(ig)^2}{2!} \times P \int dx^- A^+_a(x^-_1, x_{\perp}) \int dx^- A^+_a(x^-_2, x_{\perp}) \times (x^\perp_1, x^\perp_2) t^a t^{a_2} + \cdots \quad (15)$$

While formally we make the expansion over $\alpha_s(m_Q) \sim g^2$, we will show a bit later that for observables studied in the manuscript, the actual suppression parameter is $\sim \alpha_s(m_Q)/m_Q$, which suggests that the final result also remains valid in the strong gluonic field $A_{\mu} \sim 1/\alpha_s$ [77–79].

The expression (15) demonstrates that the effective interaction of the quark or antiquark with the gluonic field of the target is described by a factor $\pm i t^a \gamma_0 (x_{\perp})$, where $x_{\perp}$ is the transverse coordinate of the quark, $\gamma_0(x) = g \int dx^- A^+_a(x^-, x) = g^2 \int dx^- \frac{1}{\nu_{\perp}} \rho_\perp(x^-)$.

$$\gamma_0(x) = g \int dx^- A^+_a(x^-, x) = g^2 \int dx^- \frac{1}{\nu_{\perp}} \rho_\perp(x^-), \quad (16)$$

For the dipole scattering amplitude (14), using (13, 16), we may obtain

$$N(x, r, b) \approx \frac{1}{2N_c} \left[ \left( \gamma_0(x_Q) - \gamma_0(x_\perp) \right)^2 \right] + O(\alpha_s). \quad (17)$$

For further evaluations it is convenient to rewrite this result in the form

$$\frac{1}{N_c} \langle \gamma_0(x_1) \gamma_0(x_2) \rangle = -N(x, r_{12}, b_{12}) + \frac{\Gamma(x_1) + \Gamma(x_2)}{2}, \quad (18)$$

where we introduced a shorthand notation $\Gamma(x_i) \equiv |\gamma_0(x_i)|^2$, and $r_{ij}, b_{ij}$ are the transverse size and impact parameter of the center of mass of the quark–antiquark pair, which consist of partons located at points $x_i, x_j$. For many processes the contributions $\sim \Gamma(x_i)$ cancel, so the amplitudes eventually might be represented as a linear superposition of the dipole amplitudes $N(x, r, b)$. In what follows, we will see that the amplitude of the process considered in this manuscript might be represented as a bilinear combination of terms with structure $\sim \left[ \gamma_0(x_i) - \gamma_0(x_j) \right]$. For this special case substitution of (18) allows to get a few important identities between the bilinear expressions.
\[
\frac{1}{N_c} \left( [\gamma_0 (x_1) - \gamma (x_2)] [\gamma_0 (x_3) - \gamma_0 (x_4)] \right) \\
= N(x, r_{23}, b_{23}) + N(x, r_{14}, b_{14}) \\
- N(x, r_{13}, b_{13}) - N(x, r_{24}, b_{24}) ,
\]
(19)

\[
\frac{1}{N_c} \left( [\gamma_0 (x_1) - \gamma_0 (x_2)] [\gamma_0 (x_3) + \gamma_0 (x_4) - 2\gamma_0 (x_5)] \right) \\
= N(x, r_{23}, b_{23}) + N(x, r_{24}, b_{24}) - N(x, r_{13}, b_{13}) \\
+ 2 [N(x, r_{15}, b_{15}) - N(x, r_{25}, b_{25})] ,
\]
(20)

\[
\frac{1}{N_c} \left( [\gamma_0 (x_1) + \gamma_0 (x_2) - 2\gamma_0 (x_3)]^2 \right) \\
= 2N(x, r_{13}, b_{13}) + 2N(x, r_{23}, b_{23}) - N(x, r_{12}, b_{12}) .
\]
(21)

For the impact parameter independent \( (b\)-integrated) cross-section the results (17–19) might be rewritten in a simpler form,

\[
N(x, r) = \frac{1}{2N_c} \int d^2 b \left( |\gamma_0 (x, b - zr) - \gamma_0 (x, b + zr)|^2 \right) .
\]
(22)

\[
\frac{1}{N_c} \int d^2 b \langle \gamma_0 (x, b) \gamma_0 (x, b + r) \rangle \\
= -N(x, r) + \int d^2 b \left| \gamma_0 (x, b) \right|^2 .
\]
(23)

The value of the constant term in the second line of (23) is related to the infrared behavior of the theory, and for the observables which we consider in this paper, it cancels exactly.

The formation of quarkonia pairs from the photon in photoproduction process might be considered as a sequence of several elementary steps: formation of color singlet \( Q \bar{Q} \) dipole via \( \gamma \rightarrow Q \bar{Q} \) subprocess, subsequent emission of gluon \( Q \bar{Q} \rightarrow Q \bar{Q} g \) and formation of the secondary dipole via \( g \rightarrow Q \bar{Q} \) subprocess. The high-energy interaction with the target at each stage is described by the corresponding matrix elements \( S_2, S_3, S_4 \), which include products of Wilson lines of the corresponding partons. Since the gluon Wilson line \( V_{ab} \) in the adjoint representation is related to the Wilson lines in the fundamental representation via well-known group theoretical relation

\[
V_{ab} (x) = 2 tr_c \left( V^\dagger (x) t_a V (x) t_b \right) ,
\]
(24)

the \( S \)-matrix element \( S_3 \) which describes interaction of color singlet \( Q \bar{Q} \) with the target might be expressed through the \( S \)-matrix element \( S_4 \) of the \( Q \bar{Q} Q \bar{Q} \) ensemble. The matrix element \( S_4 \) in the CGC approach is given by

\[
S_4 \left( y, x_Q, x_{\bar{Q}}, z_Q, z_{\bar{Q}} \right) \\
= \frac{1}{N_c} \left( \left( V^\dagger (x_Q) \otimes V (x_{\bar{Q}}) \otimes V (z_Q) \otimes V (z_{\bar{Q}}) \right) \right) ,
\]
(25)

where we use the notations \( x_Q, x_{\bar{Q}}, z_Q, z_{\bar{Q}} \) for the transverse coordinates of quarks and antiquarks. In the dilute limit, using the expansion (15), we may see that the amplitude is dominated by \( O (\langle y (x_1) y (x_2) \rangle) \sim O (a_s) \) contribution, which might be related to dipole amplitudes using the identities (19–21).

Now we would like to explain why physically such reduction is possible, and motivate that the actual expansion parameter is \( \sim a_s (m_Q) / m_Q \) rather than \( \sim a_s (m_Q) \).

Any quark–antiquark pair might be either in color singlet (1) or in adjoint irreducible representation of dimension \( N_c^2 - 1 \). In what follows we will refer to the latter as color octet (8) states, tacitly assuming that the limit \( N_c = 3 \) should be taken everywhere in final expressions. The net color of the 4-quark ensemble is zero, for this reason we may represent it as a superposition of states \( (\bar{Q}_1 Q_2) \) \( (\bar{Q}_3 Q_4) \) and \( (\bar{Q}_1^t Q_2^t Q_3 Q_4^t) \). Using the Fierz identity

\[
(i_\mu^a \beta_\rho) (i_\nu^b \delta_\rho) = \frac{1}{2} \delta_\mu^a \delta_\nu^b - \frac{1}{2N_c} \delta_\mu^a \delta_\nu^b ,
\]
(26)

we may rewrite the latter octet-octet term as

\[
(\bar{Q}_1 t^a Q_2) (\bar{Q}_3 t^d Q_4) \\
= \frac{1}{2} (\bar{Q}_1 Q_4) (\bar{Q}_3 Q_2) - \frac{1}{2N_c} (\bar{Q}_1 Q_2) (\bar{Q}_3 Q_4) ,
\]
(27)

i.e. the Fock state of any 4-quark ensemble might be represented as a linear superposition of the color singlet pairs \( (\bar{Q}_1 Q_2) (\bar{Q}_3 Q_4) \) and \( (\bar{Q}_1 Q_4) (\bar{Q}_3 Q_2) \). Since the interaction of small-size color singlet dipoles with gluons is suppressed in the heavy quark mass limit by \( r Q_s \sim Q_s / m_Q \), the scattering amplitude (25) in the heavy quark mass limit will be dominated by color dipole scattering amplitudes,

\[
S_4 \left( y, x_Q, x_{\bar{Q}}, z_Q, z_{\bar{Q}} \right) \\
= \alpha_s (m_Q) \sum_{i j k l} c_{ijkl} \left( \gamma (x_i) - \gamma (x_j) \right) \left( \gamma (x_k) - \gamma (x_l) \right) ,
\]
(28)

where the transverse coordinates \( x_i, x_j, x_k, x_l \) correspond to different permutations of quark–antiquark coordinates \( x_Q, x_{\bar{Q}}, z_Q, z_{\bar{Q}} \), and the coefficients \( c_{ijkl} \) depend on color state of partons \( i j k l \). Using the identities (19–21), it is possible to express (28) as a linear superposition of the forward color dipole amplitudes.

At higher orders in \( \alpha_s (m_Q) / m_Q \) there are contributions which include multipoint correlators \( \langle \gamma_{a_1} (x_1) \ldots \gamma_{a_n} (x_n) \rangle \).
which in general cannot be reduced to a mere superposition of dipole amplitudes (see Fig. 1 for example). Indeed, such contributions, if taken into account in expansion in the right-hand side of 17, break the relation 18 which is used to express the amplitude of physical process in terms of color dipole amplitude.

Fortunately, in exclusive photoproduction such contributions are suppressed. This happens because the quarks formed from the photon have net zero color charge, and, as discussed earlier, the corresponding Fock state might be represented as a superposition of color singlet dipole pairs. At high energies, the interaction is dominated by colorless pomeron exchanges in t-channel (see Fig. 1). The pomeron coupling to colorless dipole is suppressed by dipole size \( r \sim m_Q^{-1} \). For this reason, the amplitude of the process gets dominant contribution from configurations with minimal number of pomerons in t-channel. We need to emphasize that the suppression of higher order contributions is a peculiar feature of the exclusive photoproduction, which is due to confluence of net zero color charge and small size of partonic ensemble. It is not universal: for example, in inclusive hadroproduction of quarkonia pairs, the color dipoles at intermediate stages might have nonzero net color charge; in diffractive hadroproduction of quarkonia pairs \( pp \to ggX p \to M_1 M_2 X p \), which has similar color structure, the distance between the produced gluons/quarkonia pairs is not suppressed in heavy quark mass limit, and, as was argued in [80], the quadrupole contributions should be taken into account in these channels.

2.1.2 Short-distance expansion

Now we would like to discuss how the results of previous section could be derived assuming smallness of the distances between quarks inside the multiquark system. This expansion does not assume smallness of strong coupling \( \alpha_s (m_Q) \) and thus might have wider applicability than the method developed in the previous section, at least for the moderately strong fields \( A_{\alpha s}^n \sim 1/\alpha_s (m_Q) \). A formal expansion parameter in this case is \( r Q_s (x) \sim Q_s (x) / m_Q \), where \( r \) is the typical distance, and \( Q_s (x) \) is the saturation scale. The main focus of our study is the EIC kinematics, where according to most phenomenological estimates typical \( Q_s (x) \lesssim 0.5-0.7 \text{ GeV} \). For the quark mass \( m_Q \) bound by \( m_c \lesssim m_Q \lesssim m_b \), the ratio \( Q_s (x) / m_Q \) is indeed a small parameter, so the expansion over \( Q_s / m_Q \) is well justified. However, in the abstract limit of large \( Q_s (x) \) (strongly saturated regime) potentially the ratio \( Q_s (x) / m_Q \) might not be small, so the derivation suggested in this section should not be considered as attempt to justify our approach in deeply saturated regime.

The quarks and antiquarks belong to fundamental irreducible representations of dimension \( N_c \) of the color group \((3, \bar{3}) \) for \( N_c = 3 \) respectively, both before and after interaction. For this reason, the quark scattering matrices \( V^\dagger (x \bar{Q}) \), \( V (x Q) \) in color space may be represented as a sum of matrices belonging to color singlet and adjoint irreducible representation of dimension \( N_c^2 - 1 \) (color octet in case of \( N_c = 3 \)).

\[
V (x) = V^{(1)} (x) + i V_a^{(adj)} (x) r^a. \tag{29}
\]

Due to the unitarity constraint \( V^\dagger V = 1 \), the functions \( V^{(1)} (x) \), \( V_a^{(adj)} (x) \) might be related to each other as

\[
\left( V^{(1)} \right)^2 + \frac{1}{2N_c} \left( V_a^{(adj)} \right)^2 = 1. \tag{30}
\]

For the evaluation of the color singlet dipole scattering amplitude (14) we need to evaluate \( V^\dagger (x \bar{Q}) V (x Q) \) with close arguments. Using the dipole size \( r \) and impact parameter \( b \) defined in the text below (14), we may rewrite the matrices \( V, V^\dagger \) as

\[
V (x \bar{Q}) = V (b) - V (b) \left( 1 - V^\dagger (b) V (x \bar{Q}) \right)
\]

\[
= V (b) - V (b) \delta V (b, x \bar{Q}), \tag{31}
\]

\[
V^\dagger (x Q) = V^\dagger (b) - \left( 1 - V^{\dagger (b)} V (x Q) \right) V^\dagger (b)
\]

\[
= V^\dagger (b) - \delta V (x Q, b) V^\dagger (b), \tag{32}
\]

where

\[
\delta V (b, x) \equiv \left( 1 - V^\dagger (b) V (x) \right), \tag{33}
\]

\[
\delta V (x_1, x_2) = \delta V (x_2, x_1^\dagger). \tag{34}
\]
Making a Taylor expansion and using unitarity of $V$, we may rewrite a product of $V$, $V^\dagger$ with close arguments as

$$V^\dagger (b) \, V (b + \delta x) \approx V^\dagger (b) \times \left[ V (b) + \delta x^\mu \partial_\mu V (b) + \frac{\delta x_\mu \delta x_\nu}{2} \partial_\mu \partial_\nu V (b) \right]$$

$$\approx 1 + \frac{\delta x^\mu}{2} \left[ V^\dagger (b) \partial_\mu V (b) \right] + \frac{\delta x_\mu \delta x_\nu}{2} \left[ V^\dagger (b) \partial_\mu \partial_\nu V (b) \right].$$

The coefficient in front of $O (\delta x)$-term in (35) can be expressed via $V^{(1)}$, $V^{(adj)}$ as

$$V^\dagger (\partial_\mu V) = - \left( \partial_\mu V^\dagger \right) V$$

$$= i \gamma^\mu \left( V^{(1)} \partial_\mu V^{(adj)} - V^{(adj)} \partial_\mu V^{(1)} \right).$$

and corresponds to color octet exchange in the $t$-channel. From (33–36) we may see that the last terms of (31, 32) are suppressed at least as $O (\delta x_a)$, where $\delta x_a \equiv x_a - b$ is a distance of a parton (heavy quark) from the center of mass of the dipole. The dipole amplitude (14) reduces to

$$N (x, r, b) = \left\{ \delta V (x_{\bar{Q}}, b) + \delta V (b, x_{\bar{Q}}) \right\} - \delta V (x_{\bar{Q}}, b) \delta V (b, x_{\bar{Q}}).$$

We can see from (35, 36) that the first two terms in the right-hand side of (37) might contribute to color-white $N (x, r, b)$ only via the second-order $O (\delta x_a^2)$-terms, and for this reason all terms in (37) have the same $O (\delta x_a^2)$-suppression. We may repeat the same procedure for the evaluation of the quadrupole matrix element $S$ defined in (25). Defining the impact parameter

$$b = \sum_{a=1}^4 \alpha_a x_a,$$

and using (31, 32), we can see that the dominant contribution for small-size quadrupole comes from the second-order $O (\delta V (b, x_j) \delta V (b, x_j))$ terms, which eventually might be expressed in terms of the linear combinations of color singlet dipole amplitudes (37). The final result of this evaluation coincides with findings in the dilute approximation, since for the dominant (leading order) terms, the substitutions

$$V^{(adj)}_a (x) \to \gamma_a (x), \quad V^{(1)}_a (x) \to 1 - \frac{1}{2N_c} \gamma_a^2 (x),$$

$$\delta V (b, x) \approx i \left[ \gamma_a (b) - \gamma_a (x) \right] t^a$$

$$+ \frac{1}{2} \left[ \gamma_a (b) - \gamma_a (x) \right]^2 + O (|\delta x|^3),$$

eventually convert short-distance expansion into expansion over small-$\gamma_a$.

2.1.3 Soft corrections

Finally, we would like to comment briefly on final state soft interactions between quarkonia, which might be schematically represented by the diagram shown in the Fig. 2. In view of the discussion of previous subsection, formally such corrections should be suppressed by $\sim 1/m_Q$. However, when relative momentum of hadrons is small, this estimate might be not reliable due to possible formation of shallow bound state (tetraquark) and the associated behaviour of the amplitude. In order to exclude contributions of this domain, we will restrict our predictions to the region of large relative momenta $|p_{rel}| \gg \alpha_s (m_Q) m_Q \sim 1$ GeV, where contributions of such soft interactions are suppressed.

2.2 Amplitude of meson pair photoproduction

The photoproduction amplitude $A_{\gamma \gamma p \to M_1 M_2 p}$, which appears in (3), is the central quantity of interest for our study. For analysis of hadronization of these quark pairs into quarkonia, we need to pay special attention to contributions of the color octet mechanism [8, 9]. This mechanism has been introduced to improve the description of inclusive production of single quarkonia, and is forbidden for exclusive photoproduction of single quarkonia states due to quantum numbers [81–84]. However, this is no longer so in case of double quarkonia production, and for this reason we will estimate it explicitly.

The color singlet contribution to the amplitude of the double quarkonia photoproduction has been evaluated in [85] using the framework discussed in previous section. That evaluation was performed for the charm sector, focusing on the production of $J/\psi \eta_c$ pairs. In this paper we are going to extend those results, taking into account color octet contributions, and extending our analysis for the case in which the final state quarkonia include both $b$ and $c$-quarks. For the production of mixed states, such as $J/\psi \eta_c$, $\Upsilon \eta_c$ and $B^+ \bar{B}^-$ meson pairs, only some subsets of diagrams contribute to the
total cross-section, thus providing the possibility to understand the relative contribution of different mechanisms. In the heavy quark mass limit, all the leading order diagrams might be grouped into two main classes, shown schematically in Fig. 3. In what follows we will call them “type-A” and “type-B” respectively, and take into account that the amplitude of the whole process might be written as an additive sum,

$$A(y_1, p_{T1}, y_2, p_{T2}) = A^{(A)}(y_1, p_{T1}, y_2, p_{T2}) + A^{(B)}(y_1, p_{T1}, y_2, p_{T2}),$$

(41)

where $A^{(A)}$ and $A^{(B)}$ are the contributions of the respective classes. The relative motion of the heavy quarks inside quarkonia might be neglected in the heavy quark mass limit, so we may assume that each final quark carries approximately half of the quarkonium momentum. In this approximation, the gluon connecting different quark lines in Fig. 3, has a large virtuality $\sim M^2_{12}/4$ for the type-A diagrams, and $\sim M^2_1, M^2_2$ for type-B diagrams. This finding justifies perturbative treatment for its description. In the target rest frame the process might be considered as a subsequent evolution of a colorless photon into colorless $\bar{Q}Q$, $g\bar{Q}Q$ and $\bar{Q}\bar{Q}QQ$ ensembles, all separated by small distance $\sim 1/m_Q$. As discussed in previous Sect. 2, the interaction of such ensembles with the gluonic field of the target is suppressed by $\sim \alpha_s(m_Q)/m_Q$, and the amplitude should be dominated by diagrams with minimal number of possible $t$-channel gluon exchanges. For this reason in what follows we will refer to the $t$-channel interactions as $t$-channel gluon exchanges, tacitly assuming that such gluon exchanges can be described in terms of the dipole scattering amplitudes.

For the production of all-charm or all-bottom quarkonia pairs, both $A^{(A)}$ and $A^{(B)}$ give nonzero contribution. In this case, $C$-parity conservation indicates that the produced quarkonia must have opposite $C$-parities. For the production of mixed $B_c^+B_c^-$ meson pairs, the amplitude $A^{(B)} \equiv 0$, so only the type-A diagrams contribute. The $C$-parity conservation in this case does not impose any constraints on the produced $B_c$-quarkonia internal quantum numbers, although imposes constraints on the angular momentum $L$ of the relative motion, which should take odd values for pairs with the same $C$-parity. This constraint is relaxed if the produced $B_c$ mesons have different spins, like e.g. $B_c^{1+}B_c^{-}$ or $B_c^{1+}B_c^{8-}$, where $B_c^{0\pm}$ is the (so far undiscovered) vector state. Finally, for the production of charmonium-bottomonium pairs, such as $J/\psi \eta_b$ or $\Upsilon \eta_c$, we can see that $A^{(A)} \equiv 0$, so the amplitude only gets contributions of the type-B diagrams. Further analysis of the type-B diagrams allows to reach some conclusions about the relative size of mixed charmonium-bottomonium production cross-sections. Analysis of quantum numbers suggests that a vector particle ($J^P = 1^-$) might be produced only in the upper loop, whereas scalar particles might originate from the quark loop in lower part of the diagram. This observation allows to understand the behavior of the cross-section under permutation of charm and bottom flavors. Since in the heavy quark mass limit each gluon attachment is suppressed and the natural scale for heavy quark is its mass, we may immediately conclude that in channels with charmonium-bottomonium production, the states with vector bottomonia are suppressed significantly stronger than the states with vector charmonia. In the next section we will corroborate this expectation by explicit comparison of numerical predictions for $\Upsilon(1S)\eta_c$ and $J/\psi \eta_b$ production cross-sections.
In the eikonal picture the impact parameter of the parton is conserved during interaction with the target. The interaction of the colored dipole with the target might be described as a linear combination of the color singlet dipole scattering amplitudes, which are known from Deep Inelastic Scattering. For this reason, it becomes possible to rewrite both types of diagrams as a mere convolution of the four quark component of the photon wave function $\psi^{(y)}_{QQQQ}$ with final state quarkonia wave functions and a linear combination of color singlet dipole scattering amplitudes,

$$
A^{(A)}(y_1, p_{T1}, y_2, p_{T2}) = \prod_{i=1}^{4} \left( \frac{d \alpha_i d^2 x_i}{\hat{A}} \right) \delta \left( \sum_k \alpha_k - 1 \right) \mathcal{A}^{(A)} \times (\alpha, x_1; \alpha, x_2; \alpha, x_3; \alpha, x_4)
$$

$$
\times \left[ \psi_{M_1}^{\dagger} (\alpha_{14}, r_{14}) \psi_{M_2}^{\dagger} (\alpha_{23}, r_{23}) e^{(p_{14}^+ b_{14} + p_{23}^+ b_{23})} \right]
$$

$$
\times (y_1 - y_{14}) \delta (y_2 - y_{12}) + \psi_{M_1}^{\dagger} (\alpha_{23}, r_{23}) \psi_{M_2}^{\dagger} (\alpha_{14}, r_{14}) e^{(p_{23}^+ b_{23} + p_{14}^+ b_{14})} \delta (y_1 - y_{12}) \delta (y_2 - y_{14})
$$

$$
\times \psi^{(y)}_{QQQQ} (\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4; q), \quad (42)
$$

$$
A^{(B)}(y_1, p_{T1}, y_2, p_{T2}) = \prod_{i=1}^{4} \left( \frac{d \alpha_i d^2 x_i}{\hat{A}} \right) \delta \left( \sum_k \alpha_k - 1 \right) \mathcal{A}^{(B)} \times (\alpha, x_1; \alpha, x_2; \alpha, x_3; \alpha, x_4)
$$

$$
\times (\alpha_{14}, r_{14} e^{(p_{14}^+ b_{14} + p_{23}^+ b_{23})} \delta (y_1 - y_{12}) \delta (y_2 - y_{14})
$$

$$
+ \psi_{M_1}^{\dagger} (\alpha_{34}, r_{34}) \psi_{M_2}^{\dagger} (\alpha_{12}, r_{12}) e^{(p_{23}^+ b_{23} + p_{14}^+ b_{14})} \delta (y_1 - y_{14}) \delta (y_2 - y_{13}) \psi^{(y)}_{QQQQ}
$$

$$
\times (\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4; q), \quad (43)
$$

where $r_{ij} = x_i - x_j$ is the relative distance between partons $i$ and $j$; $\alpha_{ij} = \alpha_i / (\alpha_i + \alpha_j)$ is the light-cone fraction carried by the quark in the pair $(ij)$, and $b_{ij} = (\alpha_i x_i + \alpha_j x_j) / (\alpha_i + \alpha_j)$ is the (transverse) position of the center of mass of $(i, j)$ pair. The notations $\psi_{M_1}, \psi_{M_2}$ are used for the wave functions of the final state quarkonia $M_1$ and $M_2$ (for a moment we disregard completely their spin indices), and $\psi^{(y)}_{QQQQ} (\alpha_i, x_i; q)$ is the 4-quark light-cone wave function of the virtual photon $\gamma^a$ which is given explicitly in Appendix A. The latter wave function includes instantaneous and non-instantaneous parts. In this paper we mostly interested in production of $S$-wave vector and pseudoscalar quarkonia. For these states, the convolution with instantaneous part of $\psi^{(y)}_{QQQQ}$ cancels, so only non-instantaneous contributions matter. We do not include the color-related indices into definitions of the wave functions $\psi_{M_1}, \psi_{M_2}$, and $\psi^{(y)}_{QQQQ}$; instead of this we take them into account in evaluation of the amplitudes $\mathcal{A}^{(A)}$ and $\mathcal{A}^{(B)}$, which include resummation over all possible connections of $t$-channel gluons to quark and gluon lines. As we discussed in Sect. 2.1, at high energies the interaction reduces to a Wilson lines, which are predominantly sensitive to the component $A_{\mu}^+$ of the gluonic field. It is possible to demonstrate that the $t$-channel gluons with this polarization cannot connect to instantaneous partons nor can flip the polarization of non-instantaneous partons, and thus cannot affect cancellation of instantaneous contributions, which will be disregarded in what follows. For non-instantaneous contributions, as was shown in [85], all the $t$-channel exchanges can be rewritten as a linear superposition of the color singlet dipole amplitudes $N (x, r_{ij}, b_{ij})$.

$$
\mathcal{A}^{(A)} (\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4)
$$

$$
= \left\{ \begin{array}{l}
2 - N_2^2 \frac{2}{4 N_c} N (x, r_{14}, b_{14}) - \frac{1}{2} N_c N (x, r_{34}, b_{34}) \\
+ 3 + 5 N_2^2 \frac{2}{4 N_c} N (x, r_{12}, b_{12}) + \frac{1}{2} N_c N (x, r_{23}, b_{23}) \\
- \frac{1}{2} N_c N \left( x, \frac{\alpha_1 r_{14} + \alpha_2 r_{34}}{1 - \alpha_2}, b_{1344} \right) \\
+ N_2^2 - 2 \frac{2}{4 N_c} N \left( x, \frac{\alpha_3 r_{13} + \alpha_4 r_{14}}{1 - \alpha_2}, b_{134} \right) + \frac{3}{2} N_2^2 - 2 \frac{2}{4 N_c} N \left( x, \frac{\alpha_3 r_{13} + \alpha_4 r_{14}}{1 - \alpha_2}, b_{134} \right) + 2 N_c N \left( x, \frac{\alpha_3 r_{23} + \alpha_4 r_{24}}{1 - \alpha_2}, b_{234} \right) \\
+ N_2^2 + 1 \frac{2}{4 N_c} N \left( x, \frac{\alpha_3 r_{13} + \alpha_4 r_{14}}{1 - \alpha_2}, b_{134} \right) - \frac{N_c}{2} N \left( x, \frac{\alpha_3 r_{13} + \alpha_4 r_{14}}{1 - \alpha_2}, b_{134} \right) \\
- \frac{N_c}{2} N \left( x, \frac{\alpha_3 r_{13} + \alpha_4 r_{14}}{1 - \alpha_2}, b_{134} \right) \\
- \frac{N_c}{2} N \left( x, \frac{\alpha_3 r_{13} + \alpha_4 r_{14}}{1 - \alpha_2}, b_{134} \right) \right\}, \quad (44)
$$

$$
\mathcal{A}^{(B)} (\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4)
$$

$$
= \left\{ \begin{array}{l}
N (x, r_{23}, b_{23}) - N (x, r_{24}, b_{24}) \\
+ N (x, r_{34}, b_{34}) - N (x, r_{24}, b_{23}) + 2 N (x, r_{14}, b_{12}) - 2 N (x, r_{13}, b_{12}) \right\}. \quad (45)
$$
The variables \( Y_{ij} \) in (42, 43) stand for the lab-frame rapidity of quark–antiquark pair made of partons \( i, j \). Explicitly it is given by

\[
Y_{ij} = \ln \left( \frac{(a_i + a_j)}{M_\perp} \right), \tag{46}
\]

where \( a_i \) and \( a_j \) are light-cone fractions of the heavy quarks which form a given quarkonium.

The color octet contributions take into account the possibility of quarkonia formation from a \( \bar{Q}Q \) pair in a color octet state. Such contributions might be understood as emissions of soft (nonperturbative) gluons which eventually lead to formation of color singlet charmonium state. The contribution of the color octet mechanism to the cross-section is given by

\[
\frac{d\sigma_{\text{octet}}}{dy_1 d|p_1^+|^2 d|p_2^+|^2 d\phi} \approx \frac{1}{256\pi^\delta} \left( \frac{p_1^+ + p_2^+}{q^+} - 1 \right) \times \sum_{ij} \left| \langle \gamma_i(M_1) | \gamma_j(M_2) \rangle A_{\gamma \gamma p-} [\bar{Q}Q], [\bar{Q}Q], \rho \right|^2 \tag{47}
\]

where \( \langle \gamma_i(M_1) \rangle \) are the color octet Long Distance Matrix Elements (LDMEs) corresponding to a given state \( i \) of the \( \bar{Q}Q \). According to NRQCD [3–14], the series is expected to converge rapidly in the heavy quark mass limit, so for numerical evaluations usually only the first few terms are relevant. The corresponding amplitudes \( A_{\gamma \gamma p-} [\bar{Q}Q], [\bar{Q}Q], \rho \) are given by a sum of contributions which correspond to type-\( A \) and type-\( B \) diagrams,

\[
A_{\gamma \gamma p-} [\bar{Q}Q], [\bar{Q}Q], \rho = A_{ij}^{(A)}(y_1, p_{T1}, y_2, p_{T2}) + A_{ij}^{(B)}(y_1, p_{T1}, y_2, p_{T2}), \tag{48}
\]

where

\[
A_{ij}^{(A)}(y_1, p_{T1}, y_2, p_{T2}) = \frac{4}{i=1} \left( \int d\alpha_i d^2x_i \right) \delta \left( \sum_k \alpha_k - 1 \right) \tilde{N}^{(A)}(\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4) \times \\
\times [\Gamma_{M1}^{(ij)^+} (\alpha_{14}, r_{14}) \Gamma_{M2}^{(ij)^+} (\alpha_{23}, r_{23}) e^{i(p_1^+ b_{14} + p_2^+ b_{23})} \times \delta(y_1 - \gamma_{14}) \delta(y_2 - \gamma_{23}) + \Gamma_{M1}^{(ij)^+} (\alpha_{23}, r_{23}) \Gamma_{M2}^{(ij)^+} (\alpha_{14}, r_{14}) e^{i(p_1^+ b_{23} + p_2^+ b_{14})} \times \delta(y_1 - \gamma_{23}) \delta(y_2 - \gamma_{14}) \right] \\
\times \psi_{\bar{Q}Q\bar{Q}Q}^{(y)}(\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4; q), \tag{49}
\]

\[
A_{ij}^{(B)}(y_1, p_{T1}, y_2, p_{T2}) = \frac{4}{i=1} \left( \int d\alpha_i d^2x_i \right) \delta \left( \sum_k \alpha_k - 1 \right) \tilde{N}^{(B)}(\alpha_1, x_1; \alpha_2, x_2; \alpha_3, x_3; \alpha_4, x_4; q), \tag{50}
\]

for the mechanisms \( A \) and \( B \) respectively. In the heavy quark mass limit the largest LDMEs are associated with the \( ^3S_1 \), \( ^1S_0 \) and \( ^3P_0 \) states, for which the corresponding vertices are given by [13]

\[
\Gamma \left[ ^3S_0 \right] \approx g_S (\hat{p}_Q + m_Q) / \sqrt{2m_Q}, \tag{53}
\]

\[
\Gamma \left[ ^3S_1 \right] \approx \hat{e} (S_c) (\hat{p}_Q + m_Q) / \sqrt{2m_Q}, \tag{54}
\]

\[
\Gamma \left[ ^3S_1 \right] \approx (\hat{p}_Q - m_Q) \hat{e} (S_c) (\hat{p}_Q + m_Q) \sqrt{2m_Q}, \tag{55}
\]

where \( p_Q, p_{\bar{Q}} \) are the 4-momenta of the corresponding quarks.
Fig. 4 Left plot: sensitivity of the $J/\psi \eta_c$ production cross-section to the choice of the wave function. We compare results with the LC-Gauss parametrization of the wave function [86–89] and the wave functions evaluated in potential models [90–93]. In the lower panel of the left figure we show the ratio of the cross-sections from the upper panel to the "LC-Gauss" curve. Right plot: sensitivity of the $J/\psi \eta_c$ production cross-section to the choice of parametrization of the dipole cross-section. In the lower panel of the figure we show the ratio of the cross-sections in $b$-Sat and $b$-CGC models (for $b$-Sat we also consider a linearized version "$b$-Sat-Lin" given by (56)). In both plots, for the sake of definiteness, we considered the case when both quarkonia are produced at central rapidities ($y_1 = y_2 = 0$) in the lab frame; for other rapidities and quarkonia pairs the $p_T$-dependence has similar shape.

3 Numerical results

The framework presented in the previous section allows to make unambiguous predictions for the cross-sections. We would like to start the presentation of numerical results with a brief discussion of different uncertainties which are present in our evaluations. For the sake of definiteness, we’ll consider the all-charm sector and focus on $J/\psi + \eta_c$ production, for which the production cross-section is the largest, and which might be the best channel for experimental studies of exclusive quarkonia pair production.

The largest uncertainty in our estimates is due to the wave function of the quarkonia, which might be reformulated as uncertainty of the Long Distance Matrix Elements (LDMEs) (see [94–97] for more details). A popular choice used in phenomenological estimates is the so-called light-cone Gaussian (LC-Gauss) parametrization [86–89]. This parametrization depends on unknown parameters, which must be fixed from phenomenology. While for $J/\psi$ and $\Upsilon$ mesons these parameters are known or might be fixed from existing experimental data, for heavier mesons, especially for $B_c$ quarkonia, this procedure cannot be applied due to lack of experimental data, thus making it almost impossible to make predictions for heavier mesons. A more systematic approach requires to use the wave functions of the quarkonia evaluated in potential models, and using the well-known Brodsky-Huang-Lepage-Terentevy (BHLT) prescription [98–100] to convert the rest frame wave function $\psi_{RF}$ into a light-cone wave function $\psi_{LC}$. In the small-$r$ region, which is relevant for estimates, the wave functions of the $S$-wave heavy quarkonia in different schemes are quite close to each other [101–104], so the uncertainty due to the choice of the potential model should be minimal for physical observables. In order to illustrate this for heavy quarkonia production, in the left panel of Fig. 4 we compare predictions for the cross-sections obtained with the LC-Gauss parametrization and various potential models [90–93]. The uncertainty due to the wave function does not exceed 30%, on par with expectations based on $\alpha_s(m_c)$-counting.

Another source of uncertainty in our evaluations is the choice of parametrization of the dipole amplitude. In the right panel of the Fig. 4 we compare predictions obtained with impact parameter ($b$) dependent "$b$-CGC" [82,83] and "$b$-Sat" [84] parametrizations of the dipole cross-section. In the region of small $p_T$ both parametrizations give very close results. In the region of very large $p_T$, the difference between the two models increases due to different small-$r$ behavior implemented in "$b$-CGC" and "$b$-Sat" parametrizations: in the former, the dipole amplitude behaves like $\sim r^2$, whereas in the latter the dependence is much more complicated due to built-in DGLAP evolution of gluon densities in dipole cross-section. In order to estimate the role of the nonlinear terms in the chosen kinematics, we also added in this plot evaluations with the linearized "$b$-Sat", which might be understood as the first $O(r^2)$-term in the expansion of the "$b$-Sat" color dipole amplitude,
Fig. 5  Sensitivity of the $J/\psi \eta_c$ production cross-section to the choice of to the parametrization of the dipole cross-section. In the lower panel of the figure we show the ratio of the cross-sections in $b$-Sat and $b$-CGC models (for $b$-Sat we also consider a linearized version "$b$-Sat-Lin" given by (56)). For the sake of definiteness we assume that rapidity of one of the quarkonia is fixed ($y_2 = 0$), and study dependence on rapidity of the other.

The phenomenological parametrization of the gluon density $g(x, \mu)$, scale dependence on dipole size $\mu(r)$, and impact parameter profile $T_G(b)$ are taken the same as in $b$-Sat and might be found in [84]. Closeness of predictions obtained with "$b$-Sat" and its linearized version corroborates smallness of the higher order nonlinear $O(r^4)$ corrections and consistency of the framework developed in previous Sect. 2. In the Fig. 5 we show the model dependence as a function of rapidity. The typical values of Bjorken variable $x_B$ in this plot roughly overlap with values which were studied at HERA and used for fits of $b$-CGC and $b$-Sat parametrizations in [83,84], for this reason predictions done with $b$-CGC and $b$-Sat agree with each other within a factor of two and have similar dependence on rapidity. From the lower panel of the Fig. 5 we can see that the difference between $b$-Sat and $b$-Sat-Lin tends to grow (in relative terms) as a function of rapidity, which signals increasing role of the saturation effects. While being negligible at central rapidities, their contribution might be pronounced at very forward rapidities. In what follows we will use the impact parameter ($b$) dependent "$b$-CGC" parameterization of the dipole cross-section [82,83].

Now we would like to discuss the role of the color octet contributions. We’ll consider for illustration the production of $J/\psi \eta_c$ pairs, where the contribution of the color octet LDMEs should be the largest. In the heavy quark mass limit it is expected that LDMEs of the $\eta_c$ should be related to those of $J/\psi$ as [3]

$$
N_{b-{Sat-Lin}}(x,r,b) \approx \frac{\pi^2 r^2}{2N_c} \alpha_s(\mu(r)) x g(x, \mu(r)) T_G(b).
$$

(56)
For the sake of definiteness, we will use for our estimates the potentially challenge the expected universality of LDMEs. Unfortunately, for this reason knowledge of $J/\psi$ LDMEs is sufficient for estimates of the color octet contributions. Unfortunately, phenomenological extractions of these LDMEs done by different authors differ quite significantly from each other [6, 13, 14, 105], which might signal presence of some unaccounted systematic uncertainties in existing fits, or even potentially challenge the expected universality of LDMEs. For the sake of definiteness, we will use for our estimates the set of LDMEs from [13],

$$\langle O^{\eta_c} \left( 1_{0}^{[1]} \right) \rangle = \frac{1}{3} \langle O^{J/\psi} \left( 3_{1}^{[1]} \right) \rangle, \quad a = 1, 8, \quad (57)$$

$$\langle O^{\eta_c} \left( 3_{1}^{[8]} \right) \rangle = \langle O^{J/\psi} \left( 1_{0}^{[8]} \right) \rangle, \quad (58)$$

$$\langle O^{\eta_c} \left( 1_{1}^{[8]} \right) \rangle = 3 \langle O^{J/\psi} \left( 3_{0}^{[8]} \right) \rangle, \quad (59)$$

for this reason knowledge of $J/\psi$ LDMEs is sufficient for estimates of the color octet contributions. In the Fig. 6 we show the relative contribution of the color octet mechanism. These findings demonstrate that the color octet corrections are indeed very small even for $J/\psi$ $\eta_c$, where they should give the largest contribution. This smallness agrees with experimentally observed smallness of color octet contributions in inclusive $\eta_c$ production [105, 106], and the fact that in double quarkonia production the color octet LDMEs of $J/\psi$ contribute multiplied by color octet LDMEs of $\eta_c$. Due to lack of phenomenological LDMEs we cannot repeat this analysis for all quarkonia pairs, yet we expect that color octet contributions should be even smaller in those channels.

In Figs. 7 and 8 we illustrate the $p_T$-dependence of the cross-section for different quarkonia states (for the sake of definiteness we considered that both quarkonia are produced with the same absolute value of transverse momenta $p_T$). The strong mass dependence can be understood in the dipole picture: all gluon interactions with dipoles of small size $\sim 1/m_Q$ are strongly suppressed in the heavy quark mass limit, leading to a strong suppression of the cross-sections. As explained in the previous section, the production of $B_c^+B_c^-$ and charmonium-bottomonium pairs get contributions from different classes of diagrams, which explains the significant differences in the cross-sections. The production of bottomonium-bottomonium pairs has significantly smaller cross-sections and does not present any practical interest. For the $B_s^+B_s^-$ meson pairs, the $C$-parity does not impose constraints on internal quantum numbers, and for this reason the suggested mechanism might lead to production of both scalar and vector mesons. In the lower panel of Fig. 7 we can see that the scalar and vector $B_s$ quarkonia should have similar cross-sections at very large $p_T$, although might differ substantially in the region of small momenta $p_T$. Potentially this channel might present special interest for searches of the (so far undiscovered) vector mesons $B_s^{\pm}$. From the Fig. 8 we can see that in the kinematics of the LHeC and FCC, the cross-section maintains its $p_T$- and quarkonia mass dependence, though the magnitude of the cross-section scales up by a factor 2–5, in agreement with implemented energy dependence of the color dipole cross-section.

In Fig. 9 we show the dependence of the cross-sections on the azimuthal angle $\phi$ between the transverse momenta of $J/\psi$ and $\eta_c$ mesons. For the sake of definiteness, we assumed that transverse momenta $p_{j/\psi}^{\perp}$, $p_{\eta}^{\perp}$ of both quarkonia have equal absolute values. In order to make meaningful comparison of the cross-sections, which differ by orders of magnitude, in the upper row of Fig. 9 we plotted the normalized ratio

$$R(\phi) = \frac{d\sigma (..., \phi) / dy_1 dp_1^2 dy_2 dp_2^2 d\phi}{d\sigma (..., \phi = \pi) / dy_1 dp_1^2 dy_2 dp_2^2 d\phi}, \quad (63)$$

$$R(\phi = \pi) \equiv 1 \quad (64)$$
Fig. 9  Upper row: dependence of the normalized ratio \( R(\phi) \), defined in (63), on the angle \( \phi \) (difference between azimuthal angles of both quarkonia). The left plot corresponds to \( J/\psi \eta_c \) pair production, but with different transverse momenta. The right plot corresponds to different quarkonia states, and fixed absolute values of the transverse momentum \( p_T \). Lower row: similar dependence of \( p_T \)-integrated ratio \( \mathcal{R}(\phi) \), defined in (65), on the angle \( \phi \). In the left plot we compare the predictions for \( J/\psi \eta_c \) with different rapidities; in the right plot we compare the predictions for different quarkonia pairs, at fixed central rapidity in the lab frame. The appearance of the sharp peak in back-to-back kinematics is explained in the text. For other rapidities the \( p_T \)-dependence has similar shape.

We can see that the ratio has a sharp peak in the back-to-back kinematics (\( \phi = \pi \)), which minimizes the momentum transfer to the target \( |t| = |\Delta^2| \). In contrast, for the angle \( \phi \approx 0 \), which maximizes the variable \( |t| = |\Delta^2| \), the ratio has a pronounced dip. The increase of the peak-to-trough ratio with \( p_T \) is due to the higher values of \( |t| \) achievable in \( \phi \approx 0 \) kinematics. For \( p_1 \neq p_2 \) the dependence on \( \phi \) is qualitatively similar, although the maximum and minimum are less pronounced. The dependence on \( \phi \) has very similar shape for all quarkonia states. Due to smallness of the cross-sections at large \( p_T \), it could be challenging to measure the ratio (63). For this reason, we also estimated the ratio of the \( p_T \)-integrated cross-sections

\[
\mathcal{R}(\phi) = \frac{d\sigma(..., \phi)}{d\sigma(..., \phi = \pi)} \frac{dy_1 dy_2 d\phi}{dy_1 dy_2 d\phi} \quad \mathcal{R}(\phi = \pi) \equiv 1
\]

which might be easier to study experimentally. The \( \phi \)-dependence of \( \mathcal{R}(\phi) \) is qualitatively similar to that of unintegrated ratio (63), although it is much milder, since in the small-\( p_T \) kinematics the variation of momentum transfer \( t \) as a function of angle \( \phi \) is not very large. Since the angle \( \phi \) also might affect the cross-section via other mechanisms (e.g., via distances \( r_{ij} \) between different quark–antiquark pairs in (44, 45), for some mesons the dependence on \( \phi \) might be even inhomogeneous function.

We expect that experimental study of the ratios (63, 65) could help to understand possible correlations between orientations of the dipole separation vector \( r \) and dipole impact parameter \( b \) in the color singlet dipole amplitude \( N(x, r, b) \). Such dependence is frequently neglected in phenomenological parametrizations, like \( b \)-CGC and \( b \)-Sat, and for many channels (e.g., DIS, DVCS, DVMP) this simplification is justified, since the corresponding cross-sections are not sensitive to the \( \phi \)-dependence. However, in different theoretical models it has been demonstrated that such dependence might exist, and its extraction from data becomes possible if the final state includes two hadrons in addition to recoil proton (see [107, 108] for more details). While all previous studies of this dependence focused on exclusive dijet production, the exclusive production of heavy quarkonia pairs might be also used for this purpose and presents a lot of interest in view.
For this reason, extraction of the constant dipole amplitude temporarily we’ll assume that such dependence is given by the \( \phi \) of such measurement, we analyze the modification of the of its very clean final state. In order to illustrate feasibility of such measurement, we analyze the modification of the \( \phi \)-dependence of the ratios \( 63, 65 \) due to possible angular dependence of the dipole amplitude. Following \( 107 \), temporarily we’ll assume that such dependence is given by the dipole amplitude

\[
N(x, r, b) \approx N_{b-CGC}(x, r, b) (1 + 2v_2 \cos(2\theta_{r,b})) \tag{66}
\]

where \( \theta_{r,b} \) is the angle between vectors \( r \) and \( b \), and \( v_2 \) is a numerical constant which characterizes the size of angular dependent term. In the upper panel of the Fig. 10 we illustrate the \( \phi \)-dependence of the ratio \( R(\phi) \) for different values of \( v_2 \). Since expected values of \( v_2 \) are very small (of order a few percent), the shape of \( R(\phi) \) experiences only small changes. For this reason, extraction of the constant \( v_2 \) from quarkonia pair production requires to use special observables which would enhance sensitivity to \( v_2 \). We suggest to use for this purpose the geometric mean

\[
G(\phi) = \sqrt{R(\phi)R(\pi - \phi)}. \tag{67}
\]

The strong \( \phi \)-dependence of the cross-sections, which is due to increase of momentum transfer \( t \) to the recoil proton, largely cancels in \( G(\phi) \), and thus extraction of \( v_2 \) from this observable might be done with better precision. Indeed, for small \( t \), the dependence of the cross-sections on \( t \) might be approximated with exponent, so

\[
R(\phi) \sim e^{Bt} \sim e^{-B(p_1^+ + p_2^+)^2} \approx e^{-B(p_1^+)^2} e^{-2Bp_1^+ p_2^+ \cos \phi} \tag{68}
\]

In the product \( R(\phi) R(\pi - \phi) \) the exponents with pronounced \( \phi \)-dependence cancel, thus giving possibility to study the “residual” \( \phi \)-dependence due to \( O(v_2) \)-terms in prefactors. The extension of this proof for the \( p_T \)-integrated ratios, which appear in \( \text{(67)} \), is straightforward. As we can see from the lower panel of the Fig. 10, the observable \( G(\phi) \) indeed has significantly milder dependence on \( \phi \), and thus is much better suited for extraction of \( v_2 \).

Finally, in the Fig. 11 we show the dependence of the photoproduction cross-section on the quarkonia rapidities, integrated over the transverse momenta of both heavy mesons. The dependence on \( y_1 \) in this setup merely reflects the dependence on the invariant photon–proton energy, as could be seen from \( 10 \). In the upper row we consider the special case when both quarkonia are produced with the same rapidities \( y_1 = y_2 \) in the lab frame. For electroproduction of charmonia, the corresponding values of the Bjorken variable \( x_B \) decrease from \( \sim 0.2 \) down to \( \sim 5 \times 10^{-4} \) for the rapidity range shown in the figure. For all other states (\( B^{+}B^{-} \), \( J/\psi \eta_b \) and \( \Upsilon(1S) \eta_c \)) the corresponding values of \( x_B \) similarly decrease, being approximately a factor of two larger than for \( J/\psi \eta_c \) pairs. In the lower row of the same Fig. 11 we show the dependence of the cross-section on the rapidity difference \( \Delta y \) between the two heavy mesons. For the sake of definiteness we consider that both quarkonia have opposite rapidities in the lab frame, \( y_1 = -y_2 = \Delta y / 2 \). In this setup the variable \( \Delta y \) might be related to the invariant mass of the heavy quarkonia pair using \( 11 \). The suppression of the cross-section as a function of \( \Delta y \) implies that the quarkonia are predominantly produced with the same rapidities. The corresponding values of the variable \( x_B \) for this case increase from \( \sim 10^{-2} \) up to \( \sim 7 \times 10^{-2} \) for electroproduction of charmonia in the rapidity range shown in the Figure, so the decrease of the cross-section might be partially attributed to decrease of the gluon densities. Similar to the previous plot, for all other states (\( B^{+}B^{-} \), \( J/\psi \eta_b \) and \( \Upsilon(1S) \eta_c \)) the corresponding values of \( x_B \) also increase as a function of \( \Delta y \), being approximately a factor of two larger than for \( J/\psi \eta_c \) pairs.
The rapidity dependence of the photoproduction cross-section in EIC kinematics. In all plots we consider $Q^2 = 0$. The plots in the upper row correspond to a configuration with equal rapidities of the produced quarkonia, $y_1 = y_2$, whereas the lower row corresponds to rapidities which differ by a sign in the lab frame, $y_1 = -y_2$. In both rows the left column corresponds to cross-section of photoproduction subprocess, $\gamma p \rightarrow M_1 M_2 p$, whereas the right column corresponds to a cross-section of the electroproduction process $e^+ e^- \rightarrow M_1 M_2 e^+ e^-$ and takes into account an additional leptonic factor, as defined in (1). In all plots we consider $Q^2 = 0$. For electroproduction of charmonia, the corresponding values of the Bjorken variable $x_B$ decrease from $\sim 0.2$ down to $\sim 5 \times 10^{-4}$ for $y_1 = y_2$ kinematics (upper row), and increase from $\sim 10^{-2}$ up to $\sim 7 \times 10^{-2}$ in the $y_1 = -y_2$ kinematics (lower row). For all other states ($B^{+}c B^{-}c$, $J/\psi \eta_c$, and $\Upsilon(1S) \eta_c$) the corresponding values of $x_B$ should be multiplied by factor of 2.

In Fig. 12 we show predictions for the pair production in the kinematics of ultraperipheral collisions at LHC. For the sake of definiteness, we consider proton-lead collisions. Qualitatively the behavior of the cross-section is similar to that of $e p$ production: the cross-section grows as a function of average rapidity $Y = (y_1 + y_2)/2$, yet decreases rapidly as a function of rapidly difference $\Delta y = y_1 - y_2$. In the last two rows of the Fig. 12 we also show the dependence on rapidity of one of the quarkonia ($y_1$) when the rapidity of the other ($y_2$) is fixed: the cross-section reaches its maximum near the point $y_1 \approx y_2$.

Finally, in the Table 1 we provide tentative estimates of the so-called fiducial cross-sections, which might be obtained integrating the differential cross-section over the central ($|y_{1,2}| < 2.5$) and forward ($2.5 \leq y_{1,2} \leq 4$) rapidity regions.

4 Conclusions

In this manuscript we have studied in detail the exclusive photoproduction of heavy quarkonia pairs, which include bottom mesons. We focused on the leading order contribution, which leads to production of charmonia and bottomonia pairs with opposite $C$-parities. For $B^+_c B^-_c$ pairs, the $C$-parity does not impose any constraints on the internal quantum numbers of quarkonia, so the suggested mechanism might be used as a clean channel for studies of (so far undiscovered) $B_c$ mesons with different internal quantum numbers. The analysis of mixed charm-bottom pairs (e.g. $B^+_c B^-_c$, $J/\psi \eta_c$, and $\Upsilon(1S) \eta_c$ pairs) allows to single out contributions of two main classes of diagrams in the suggested production mechanism. In all cases the quarkonia are produced with relatively small opposite transverse momenta $p_T$, and small separation in rapidity: the kinematic which minimizes the momentum transfer to the recoil proton and the invariant mass of the produced pair. The dependence of the cross-section on azimuthal angle between transverse momenta of produced quarkonia might present special interest, since it allows to test the dependence of the dipole amplitude $N(x, r, b)$ on the relative angle between the dipole separation $r$ and impact parameter $b$. We estimated numerically the cross-sections in the kinematics of ultraperipheral collisions at LHC and the kinematics of the
Fig. 12 The rapidity dependence of the photoproduction cross-section in the kinematics of the ultraperipheral collisions at LHC. The plots in the upper row correspond to a configuration with equal rapidities of the produced quarkonia, $y_1 = y_2$, the second row corresponds to rapidities which differ by a sign in lab frame, $y_1 = -y_2$, and the last two rows correspond to fixed rapidity of one of the quarkonia ($y_2 = 0$ and $y_2 = 3$ respectively). In all rows we show the cross-section of the photoproduction subprocess in the left, and the cross-section of the full process $pA \rightarrow pA M_1 M_2$, as defined in (2), in the right.
forthcoming Electron-Ion Collider. We found that $J/\psi \eta_c$ and $B_c^+ B_c^−$ might be studied with reasonable precision in forthcoming experiments. The production cross-sections of other quarkonia pairs, especially from the all-bottom states (like e.g. $\Upsilon \eta_c$) are numerically significantly smaller due to extra suppression by the heavy mass and a different production mechanism.

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Data availability statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical study and does not include new experimental data. All generated data (theoretical predictions) are contained in this published article and shown in the Figs. 4-12.].

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Appendix A: Evaluation of the photon wave function

The evaluation of the photon wave function follows the standard light–cone rules formulated in [21,109]. The result for the $\bar{Q}Q$ component is well-known in the literature [86,110]. The wave function of the $\bar{Q}Q\bar{Q}Q$-component might be expressed in terms of the wave function of $\bar{Q}Q$-component. The dominant contribution$^2$ to the electroproduction and photoproduction in ultraperipheral kinematics comes from quasireal transversely polarized photons, for this reason in what follows we’ll focus on the contribution of on-shell photons. The expression for the momentum of the photon (4) simplifies in this case and has only light-cone components in the direction of plus-axis,

$$q \approx (q^+, 0, 0_\perp) .$$

The polarization vector of the transversely polarized photon is given by

$$\varepsilon^\mu(q) \equiv \left(0, \frac{q_+}{q^+}, \varepsilon_\gamma \right) \approx \left(0, 0, \varepsilon_\gamma \right),$$

$$\varepsilon_\gamma = \frac{1}{\sqrt{2}} \left(1 \pm i \right), \quad \gamma = \pm 1 .$$

where in (70) we took into account that $q_\perp = 0$. Before interaction with the target, the photon might fluctuate into a virtual quark–antiquark pairs, as well as into gluons. In configuration space the wave function of the $\bar{Q}Q$ state is given by [86,110]

$$\Psi_{\bar{q}q}(z, r_{12}, m_q, a) = \frac{2}{(2\pi)^2} \left( z_\lambda z_\bar{\lambda} - (1 - z) \delta_{\lambda, -\bar{\lambda}} \right) \delta_{h, -\bar{h}} i \varepsilon_\lambda \cdot \nabla \left( \frac{m_q}{\sqrt{2}} \text{sign}(h) \delta_{h, \bar{h}} K_0 (a r_{12}) \right) .$$

where $z$ is the fraction of the photon momentum carried by the quark, and $r_{12}$ is the transverse distance between quark and antiquark.

The $\bar{Q}Q\bar{Q}Q$ component of the photon wave function has been evaluated in detail in our earlier paper [85], and the final result will be given here for the sake of completeness. In leading order over $\alpha_s$ the wave functions obtain contributions from the two diagrams shown in the Fig. 3. For the sake of generality we will assume that the produced quark–antiquark pairs have different flavors, and will use the notations $m_1$ for the current mass of the quark line connected to a photon, and $m_2$ for the current masses of the quark–antiquark pair produced from the virtual gluon. The evaluation of the diagrams follows the standard rules of the light–cone perturbation theory [21,109]. The wave function might be represented as a sum

$$\psi^{(y)}_{\bar{Q}Q\bar{Q}Q} = \psi^{(y,\text{noninst})}_{\bar{Q}Q\bar{Q}Q} + \psi^{(y,\text{inst})}_{\bar{Q}Q\bar{Q}Q}$$

where the first and the second terms correspond to contributions of non-instantaneous and instantaneous parts of propagators of all virtual particles, and for the sake of brevity we

$^2$ We assume that there is no special cuts which select or enhance contributions of events with large virtuality $Q^2$. 

Table 1 The fiducial cross-section of the quarkonia pair production in ultraperipheral kinematics at LHC. The imposed rapidity cuts roughly correspond to the central ($|y_{1,2}| < 2.5$) and forward ($2.5 \leq y_{1,2} \leq 4$) production regions

|                  | Central $|y_{1,2}| < 2.5$ | Forward $2.5 \leq y_{1,2} \leq 4$ |
|------------------|--------------------------|-----------------------------------|
| $J/\psi \eta_c$  | 21.4 nb                  | 0.15 nb                           |
| $B_c^+ B_c^-$    | 8.7 pb                   | 0.09 pb                           |
| $\Upsilon(1S)\eta_c$ | 46 fb              | 2.7 fb                            |
omitted color and helicity indices of heavy quarks ($c_j$ and $a_i$ respectively). The non-instantaneous contribution is given by the sum

$$ \psi^{(y, \text{noninst})}_{\bar{Q}Q\bar{Q}} (\{a_i, x_i\}) = A (\{a_i, x_i\}) + B (\{a_i, x_i\}), \quad (74) $$

where the function $A (\{a_i, r_i\})$ is defined as

$$ A (\{a_i, r_i\}) = -2e_q \alpha_s (\mu) (t_a)_{c_1 c_2} \otimes (t_a)_{c_1 c_2} \int \frac{q_1 dq_1 k_2 dk_2}{\alpha \nu (\alpha_1 - \alpha_2)^2 \sqrt{\alpha_1 \alpha_2}} \times $$

$$ \times \left[ \frac{1}{\alpha_1 (1 - \alpha_1 - \alpha_2)} + \frac{m_f^2 (\alpha_1 + \alpha_2)}{(\alpha_1 \alpha_2)} + \frac{k_2^2}{\alpha_2 \alpha_2} \right] \frac{1}{2} \sqrt{\alpha_1} \times $$

$$ \times \left[ \frac{(\alpha_2 \delta_{\gamma, a_2} - \alpha_2 \delta_{\gamma, a_1}) (\alpha_2 \delta_{\gamma, a_1} + \alpha_1 \delta_{\gamma, a_1}) \delta_{a_1, -a_2}}{n_{1,34} \cdot \epsilon_\gamma} \eta_k (q_1 | x_1 - b_{34}) \right] J_0 (k_2 | x_2 - b_{134}) $$

$$ \times (q_1 | x_1 - b_{34}) \frac{(1 - \alpha_1 - \alpha_2)^2}{1 - \alpha_2} - \frac{i m_q}{\sqrt{2}} \text{sign} (a_2) \times $$

$$ \times \eta_k (q_1 | x_1 - b_{134}) J_0 (q_1 | x_1 - b_{34}) \right] $$

$$ \times \psi^{(y, \text{inst})}_{a_3 a_4} \left( \frac{a_3}{a_4}, r_{34}, m_2, \right) $$

$$ \times \left[ \bar{m}_2^2 + \alpha_3 a_3 + a_4 \right] \left[ \frac{\bar{m}_2^2 (\alpha_1 - \alpha_2)}{\alpha_1 (1 - \alpha_1 - \alpha_2)} + \frac{m_f^2 (\alpha_1 + \alpha_2)}{\alpha_1 \alpha_2} + \frac{k_2^2}{\alpha_2 \alpha_2} \right] \right) \frac{1}{2} \frac{1}{5} \frac{1}{3} \frac{1}{2} \right) $$

(75)

and the function $B (\alpha_1, x_1, \alpha_2, x_2, \alpha_3, x_3, \alpha_4, x_4)$ is related to $A (\alpha_1, x_1, \alpha_2, x_2, \alpha_3, x_3, \alpha_4, x_4)$ as

$$ B (\alpha_1, x_1, \alpha_2, x_2, \alpha_3, x_3, \alpha_4, x_4) $$

$$ = -A (\alpha_2, x_2, \alpha_1, x_1, \alpha_4, x_4, \alpha_3, x_3), \quad (76) $$

The variable $\mathbf{b}_{j_1 \ldots j_n}$ corresponds to the center of mass position of the $n$ partons $j_1, \ldots, j_n$. The variable $\mathbf{n}_{j_1 \ldots j_n} = (x_i - \mathbf{b}_{j_1 \ldots j_n}) / |x_i - \mathbf{b}_{j_1 \ldots j_n}|$ is a unit vector pointing from quark $i$ towards the center-of-mass of the system of quarks $j_1 \ldots j_n$. The tree-like structure of the leading order diagrams 1, 2 in Fig. 13 and iterative evaluation of the coordinate of the center of mass of two partons $\mathbf{b}_{ij} = (\alpha_i r_i + \alpha_j r_j) / (\alpha_i + \alpha_j)$, allows to reconstruct the transverse coordinates of all intermediate partons. The variables $r_1 - \mathbf{b}_{34}$ and $r_2 - \mathbf{b}_{34}$ physically have the meaning of the relative distance between the recoil quark or antiquark and the emitted gluon. Similarly, the variables $r_1 - \mathbf{b}_{234}$ and $r_2 - \mathbf{b}_{134}$ might be interpreted as the size of $\bar{Q}Q$ pair produced right after splitting of the incident photon.

Similarly, for the instantaneous contributions it is possible to get

$$ \psi^{(y, \text{inst})}_{\bar{Q}Q\bar{Q}} (\{a_i, r_i\}) = A_q (\{a_i, r_i\}) + B_q (\{a_i, r_i\}) $$

$$ + A_g (\{a_i, r_i\}) + B_g (\{a_i, r_i\}), \quad (77) $$

where the subscript indices $q, g$ in the right-hand side denote the parton propagator which should be taken instantaneous ($q$ for quark, $g$ for gluon), and

$$ A_q (\{a_i, r_i\}) $$

$$ = -e_q \alpha_s (m_q) (t_a)_{c_1 c_2} \otimes (t_a)_{c_1 c_2} \int \frac{q_1 dq_1 k_2 dk_2}{\alpha \nu (\alpha_1 - \alpha_2)^3} \times $$

$$ \times J_0 (q_1 | r_1 - \mathbf{b}_{34}) \frac{1}{k_2^2 + m_f^2} \left[ \alpha_2 \delta_{\gamma, a_1} - \alpha_2 \delta_{\gamma, a_1} \right] $$

(78)
(78)

In the text of the paper we work with a wave function

\[ \psi^{(y)}_{QQQQ} \]

which by definition might be obtained from \( \psi^{(y)}_{QQQQ} \) stripping common color factors \( (t_3)_{c1c2} \otimes (t_3)_{c1c4} \), i.e.

\[ \psi^{(y)}_{QQQQ} = \psi^{(y)}_{QQQQ} (t_3)_{c1c2} \otimes (t_3)_{c1c4}. \]  

(79)

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\[ \times \delta_{a_3, a_4} n_{2,134} \cdot \varepsilon_y k_2 J_1 (k_2 | r_2 - b_{134}|) \]

\[ + \frac{m_y}{\sqrt{2}} \text{sign}(a_1) \delta_{a_1, a_2} J_0 (k_2 | r_2 - b_{134}|) \]

\[ \times \alpha \alpha a_3 a_4 a_1, a_2, K_0 (a_34 r_{34}). \]

\[ A_q \left( (a_1, r_1) \right) = - \frac{e_q a_q \left( m_q \right) (t_3)_{c1c2} \otimes (t_3)_{c1c4} \delta_{a_1, a_2} \delta_{\gamma_1, a_1} \delta_{\gamma_2, a_2}}{2 \pi^2 \left( 1 - a_1 - a_2 \right)^2 a_2}
\]

\[ \times \int q_1 d q_1 k_2 d k_2 J_0 (q_1 | r_1 - b_{34}|) J_0 (k_2 | k_2 - b_{134}|) \]

\[ \frac{D_2 (a_1, a_2, a_3, a_4)}{a_1 a_2 a_3 a_4} K_1 (a_34 r_{34}) \]

\[ a_34 (q_1, k_2) \]

\[ = \sqrt{m_2^2 + \frac{a_2 a_4}{a_3 + a_4} \left( \frac{a_2 a_4}{a_1 (1 - a_1 - a_2)} + \frac{m_q^2 (a_1 + a_2)}{a_1 a_2} + \frac{k_2^2}{a_1 a_2} \right)} \]

and the functions \( B_q, B_g \) might be obtained from \( A_q, A_g \) using

\[ B_q \left( (a_1, x_1, a_2, x_2, a_3, x_3, a_4, x_4) \right) = - A_q \left( (a_2, x_2, a_1, x_1, a_4, x_4, a_3, x_3) \right), \quad i = q, g. \]  

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