On The Edge Irregularity Strength of Firecracker Graphs $F_{2,m}$

Rismawati Ramdani$^{1,a}$ and Desi Laswati Suwandi$^{1,b}$

$^1$Department of Mathematics, UIN Sunan Gunung Djati Bandung, Indonesia

$a$email: rismawatiramdani@uinsgd.ac.id

$b$email: Desilaswati28@gmail.com

Abstract
Let $G = (V, E)$ be a graph and $k$ be a positive integer. A vertex $k$-labeling $f : V(G) \rightarrow \{1, 2, \cdots, k\}$ is called an edge irregular labeling if there are no two edges with the same weight, where the weight of an edge $uv$ is $f(u) + f(v)$. The edge irregularity strength of $G$, denoted by $es(G)$, is the minimum $k$ such that $G$ has an edge irregular $k$-labeling. This labeling was introduced by Ahmad, Al-Mushayt, and Bac’-a in 2014. An $(n,k)$-firecracker is a graph obtained by the concatenation of $n$ k-stars by linking one leaf from each. In this paper, we determine the edge irregularity strength of fireworks graphs $F_{2,m}$.

Keywords: edge irregular labeling, firecracker, the edge irregularity strength

Introduction
Graph labeling was first introduced by Sadjlačk (1964), then Stewart (1966), Kotzig and Rosa (1970). The process of labeling a graph includes assigning values (labels), represented by a set of positive integers to vertices, edges, or both. These numbers are called labels [1]. There are several types of labeling on graphs, including gracefull labeling, harmony labeling, total irregular labeling, magic labeling, and anti-magic labeling. The concept of irregular labeling on a graph was first introduced by Chartrand et al. in 1986 [2].

In 2014, Ahmad et al. [3] introduced edge irregular labeling of graphs, namely edge irregular labeling. For an integer $k$, a total labeling $f : V(G) \rightarrow \{1, 2, \cdots, k\}$ is called an edge irregular $k$-labeling of $G$ if every two distinct edges $e_2$ and $e_2$ in $E$ satisfy $w_f(e_1) \neq w_f(e_2)$, where $w_f(e_1 = uv) = f(u) + f(v)$. As an example, we have a graph $P_2 \cup C_3$ in the Figure 1.

![Figure 1](image)

Figure 1. Given a graph $P_2 \cup C_3$

Then, we give a vertex labeling of the graph. The labeling is can be seen in the Figure 2.
By the labeling in the Figure 2, the weight of edges of the graph are 3, 7, 8, and 9 and the maximum label used in this labeling is 5. Besides that, there are no two edges with the same weight, so the labeling is an edge irregular k-labeling of $P_2 \cup C_3$ with k=5.

The minimum k for which a graph $G$ has an edge irregular k-labeling, denoted by $es(G)$, is called the edge irregularity strength of $G$. For example, given an edge irregular 3-labeling of $P_2 \cup C_3$ in the Figure 3.

The labeling of the Figure 3 is in edge irregular 3-labeling of $P_2 \cup C_3$ because there are no two edges with the same weight and the maximum label used is 3. It is impossible to have an edge irregular k-labeling of $P_2 \cup C_3$ with maximum label 2. So, 3 is the minimum k for which $P_2 \cup C_3$ has an edge irregular k-labeling. We conclude that the edge irregularity strength of $P_2 \cup C_3$ is 3, denoted by $es(P_2 \cup C_3)$=3.

To get the exact value of $es$ of a graph $G$, we would previously determine a lower bound and an upper bound of $es(G)$ before. A lower bound on $es(G)$ is obtained by using the theorem from Martin Baca and Ali Ahmad in 2014. [3] as follows:

Theorem 1 [3] : Let $G = (V, E)$ be a graph with the maximum degree $\Delta$, then

$$es(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 1}{2} \right\rceil, \Delta(G) \right\}$$
Other results about computing the edge irregularity strength of graphs are given by Imran et al. in [4]. In the paper, Imran et al. determined the edge irregularity strength of caterpillars, n-star graphs, kite graphs, cycle chains and friendship graphs. Tarawneh et al. [5], determined the edge irregularity strength of corona product of cycle with isolated vertices. In [6], Tarawneh et al. determined the exact value of edge irregularity strength for triangular grid graph, zigzag graph and Cartesian product $P_n \times P_m \times P_2$.

In 2017, Ahmad et al determined the edge irregularity strengths of some chain graphs and the join of two graphs. They also introduced a conjecture and open problems for researchers for further research [7]. Ahmad et al. [6], gave computing of the edge irregularity strength of bipartite graphs and wheel related graphs. In [8], Asim et al gain an edge irregular k-labeling for several classes of trees. Asim et al also gained the edge irregularity strength of disjoint union of star graph and subdivision of star graph [9].

In 2020, Ahmad et al performed a computer based experiment dealing with the edge irregularity strength of complete bipartite graphs. They also presented some bounds on this parameter for wheel related graphs [6]. In [10], Tarawneh et al. gave the edge irregularity strength of some classes of plane graphs.

In this paper, we determined the exact value of $es$ of firecracker graphs $F_{2,m}$ with arbitrary $m$. A firecracker is a graph obtained by the concatenation of stars by linking one of leaf from each. If the number of stars is $n$ and the number of leaves in each star is $m$, then the firecracker is denoted by $F_{n,m}$.

![Figure 4. Firecracker graph $F_{4,8}$](image)

**Methods**

The method we use in this research is analytical method. To get the exact value of $es$ of firecracker graph, we consider a lower bound and an upper bound of $es(F_{n,m})$. Theorem 1 is used to have a lower bound of $es(F_{n,m})$. Besides that, an upper bound of $es(F_{n,m})$, we construct an edge irregular-k labeling with minimum $k$. 
Result and Discussion

The main result of our research is the edge irregularity strength of firecracker graphs $F_{2,m}$ is $m+1$. The result written in Theorem 2.

Theorem 2 : Let firecracker graphs $F_{2,m}$, for $m \geq 2$, we have edge irregularity strength by

$$es(F_{2,m}) = m + 1$$

Proof. 

We consider

$$es(F_{2,m}) \geq m + 1 \quad (1)$$

and

$$es(F_{2,m}) \leq m + 1 \quad (2)$$

To prove inequality (1), we use Theorem 1.

In the Figure 5, we can see an illustration of firecracker graph $F_{2,m}$

From the illustration of Figure 5, we have the graph $F_{2,m}$ has $2m+1$ edges and the maximum degree $\Delta = m$. By using Theorem 1, we have

$$es(F_{2,m}) \geq \lceil \frac{|E(F_{2,m})|+1}{2} \rceil, \Delta (F_{2,m}) = \text{max}\{\lceil \frac{2m+2}{2} \rceil, m\} = \text{max}\{m+1, m\} = m + 1.$$ 

So, we have

$$es(F_{2,m}) \geq m + 1$$

Next, we give an edge irregular $k$--labeling with $k = m + 1$ to get $es(F_{2,m}) \leq m + 1$ as follows.

For $m \geq 2$.

$$f(c_i) = \begin{cases} 1 & \text{for } i = 1 \\ m + 1 & \text{for } i = 2 \end{cases}$$

$$f(a_{i,j}) = \begin{cases} f(c_i) & \text{for } j = 1 \\ f(c_i) & \text{for } i = 1 \text{ and } 1 < j \leq m \text{ } f(a_{1,1}) + (j - 1), \text{ for } i = 2 \text{ and } 1 < j \leq m \end{cases}$$
From the labeling formula (3), we have the weight of edges of firecracker graphs $F_{2,m}$ as follows:

$$w_f(c_ia_{i,j}) = \begin{cases} 
  j + 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq m; \\
  mi + i, & \text{for } i = 2 \text{ and } j = 1; \\
  mi + i - (m + 1) + j & \text{for } i = 2 \text{ and } 1 < j \leq m;
\end{cases}$$

(4)

From the edge weight formula (4), there are no two edges with the same weight. The maximum label used in the labeling $f$ is $m + 1$. So, $f$ is an edge irregular-$(m + 1)$ of $F_{n,m}$. So, we can conclude that

$$es(F_{2,m}) \leq m + 1$$

From inequalities (1) and (2), we have an equality

$$es(F_{2,m}) = m + 1.$$  

For an illustration, in the Figure 6, we can see the edge irregular labeling $f$ of firecracker graph $F_{2,m}$ $m = 8$.

![Figure 6](image_url)

**Figure 6.** (a) An illustration of $F_{2,8}$; (b) An illustration of edge irregular-9 labeling $f$ of $F_{2,8}$

In the Figure 7, we can see the weight of edges of $F_{2,8}$ under the labeling in Figure 6.
Figure 7. The weight of edges of $F_{2,8}$ under the labeling $f$

From the illustration of the Figure 7, we can see that there are no two edges in $F_{2,8}$ with the same weight under the labeling $f$.

Conclusion

By the research, we have a lower bound of $es(F_{2,m})$ is $m + 1$, which is also an upper bound of $es(F_{2,m})$. So that, we can conclude the exact value of the edge irregularity strength of firecracker graphs $F_{2,m}$ is $m + 1$ for $m \geq 2$.

Acknowledgement

Author acknowledges the support from UIN Sunan Gunung Djati Bandung. The author wish to thank the referees for their thoughtful suggestions.

Reference

[1] M. Bača, M. Miller, and J. Ryan, “On irregular total labellings,” Discrete Math., vol. 307, no. 11–12, pp. 1378–1388, 2007.
[2] G. Chartrand, P. Erdős, and O. R. Oellermann, “How to define an irregular graph,” Coll. Math. J., vol. 19, no. 1, pp. 36–42, 1988.
[3] A. Ahmad, O. B. S. Al-Mushayt, and M. Bača, “On edge irregularity strength of graphs,” Appl. Math. Comput., vol. 243, pp. 607–610, 2014.
[4] M. Imran, A. Aslam, S. Zafar, and W. Nazeer, “Further results on edge irregularity strength of graphs,” Indones. J. Comb., vol. 1, no. 2, pp. 82–91, 2017.
[5] I. Tarawneh, R. Hasni, and A. Ahmad, “On the edge irregularity strength of corona product of cycle with isolated vertices,” AKCE Int. J. Graphs Comb., vol. 13, no. 3, pp. 213–217, 2016.
[6] A. Ahmad, M. A. Asim, B. Assiri, and A. Semaničová-Feňovčíková, “Computing the Edge Irregularity Strength of Bipartite Graphs and Wheel Related Graphs,” Fundam. Informaticae, vol. 174, no. 1, pp. 1–13, 2020.
[7] A. Ahmada, A. Guptaa, and R. Simanjuntakb, “Computing the edge irregularity strengths of chain graphs and the join of two graphs,” Electron. J. Graph Theory Appl., vol. 6, no. 1, pp. 201–207, 2018.
[8] M. A. Asim, A. Ahmad, and R. Hasni, “Edge irregular k-labeling for several classes of trees,” Util. Math., vol. 111, pp. 75–83, 2019.
[9] I. Tarawneh, R. Hasni, and M. A. Asim, “On the edge irregularity strength of disjoint union of star graph and subdivision of star graph,” *Ars Comb.*, vol. 141, pp. 93–100, 2018.

[10] I. Tarawneh, R. Hasni, A. Ahmad, and M. A. Asim, “On the edge irregularity strength for some classes of plane graphs,” *AIMS Math*, vol. 6, pp. 2724–2731, 2021.