The Refined Humbert Invariant for
Imprimitive Ternary Forms

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1 Introduction

Kani\cite{1} introduced the refined Humbert invariant \( q(A, \theta) \), which is a positive quadratic form that is intrinsically attached to a principally polarized abelian surface \((A, \theta)\); cf. \S 3 below. It is very useful because it encodes many special properties of \((A, \theta)\), and thus, several geometric properties of \((A, \theta)\) are translated into arithmetic properties of \( q(A, \theta) \). An elegant illustration of making use of the refined Humbert invariant can be found in Kani\cite{2, 3}.

From this point of view, it is interesting to classify the quadratic forms \( q \) which are equivalent to some refined Humbert invariant \( q(A, \theta) \). The refined Humbert invariants were classified when \( q \) is a binary form which represents a square by Kani\cite{5}. Recently, Kani\cite{9} gave a similar classification in the case that \( q \) is a primitive ternary quadratic form.

The main aim in this article is to give a similar classification in the case that \( f \) is an imprimitive ternary form, so the classification will be complete for the ternary forms. More precisely, we will prove two main results which are related to each other to fulfill our main aim.

We have the following first main result.

**Theorem 1.** Assume that an imprimitive ternary form \( f_1 \) is equivalent to a form \( q(A, \theta) \), for some \((A, \theta)\). If a form \( f_2 \) is genus-equivalent to \( f_1 \), then \( f_2 \) is equivalent to a form \( q(A, \tilde{\theta}) \), for some principal polarization \( \tilde{\theta} \) on \( A \).

This theorem has an interesting application that it can be used to give a formula for the number of isomorphism classes of smooth curves of genus 2 lying on an abelian surface \( A \) in many cases; cf. Kani\cite{6}.

Moreover, this result is a main tool to prove the second main theorem (cf. Theorem 2 below) which gives a classification of the imprimitive ternary forms \( f \) which are equivalent to a refined Humbert invariant \( q(A, \theta) \), for some
principally polarized abelian surface \((A, \theta)\). To this end, let us consider positive ternary quadratic forms \(f(x, y, z)\) satisfying the following two conditions:

\[ \frac{1}{2} f \text{ is an improperly primitive form; (cf. §2 below)} \]
\[ f(x_0, y_0, z_0) = (2n)^2 \text{ for some } x_0, y_0, z_0, n \in \mathbb{Z} \text{ with } \gcd(n, \text{disc}(f)) = 1, \]

where \(\text{disc}(f)\) is defined as in [20, p. 2] (or by equation (3) below).

The following is the main result of this article.

**Theorem 2.** Let \(f\) be an imprimitive positive ternary form. Then \(f\) is equivalent to a form \(q_{(A, \theta)}\), for some \((A, \theta)\) if and only if \(f\) satisfies conditions (1) and (2).

Although the proof of the necessary condition of Theorem 2 is easy, the proof of the sufficient condition is quite long and will occupy most of the article. Note that our strategy is similar to the one in [13]. Throughout the article, the same notations will be used and the same conventions will be assumed as in [13].

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## 2 Ternary quadratic forms

In this article, we mostly work on positive ternary integral quadratic forms as was mentioned in the introduction. For this, we want to use some results of Smith[13], Dickson[5], Jones[6], Brandt[1, 2] and Watson[20]. In particular, because it will be necessary for our aim to identify the assigned characters of a ternary form and its reciprocal and to calculate their values, we first mention some notations and concepts in sense of those authors.

Let \(R\) be a commutative ring with 1. A **quadratic form** over \(R\) in \(n\) variables is a polynomial \(f(x_1, \ldots, x_n)\) of the form

\[ f = f(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij}x_i x_j, \text{ where } a_{ij} \in R. \]
In particular, if \( f \) is a form over \( \mathbb{Z} \), then we say that \( f \) is an integral quadratic form as defined in [20] (and in [2]).

As in Watson [20, p. 2], we will use the notation \( A(f) \) denote the unique integral symmetric \( n \times n \) matrix such that \( f(x) = \frac{1}{2} x A(f) x^t \), for all \( x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \), where \( x^t \) denotes the transpose of \( x \). This matrix is called the coefficient matrix of \( f \).

By \( \text{disc}(f) \) and \( \det(f) \), we mean the discriminant and the determinant of a quadratic form \( f \) respectively as defined in [20, p. 2]. Recall that we have by [20], loc. cit., that \( \det(f) = \det(A(f)) \) and that

\[
\text{disc}(f) = \begin{cases} 
-\det(f) & \text{if } f \text{ is a binary form,} \\
-\frac{1}{2} \det(f) & \text{if } f \text{ is a ternary form.} 
\end{cases}
\]

Throughout the article, we study the quadratic forms whose discriminants are not zero.

Let \( f_1, f_2 \) be \( n \)-ary quadratic forms. We say that two forms \( f_1 \) and \( f_2 \) are \( \text{GL}_n(R) \)-equivalent if we have

\[
A(f_1) = M^t A(f_2) M, \quad \text{for some } M \in \text{GL}_n(R).
\]

If \( f_1 \) and \( f_2 \) are \( \text{GL}_n(\mathbb{Z}) \)-equivalent integral quadratic forms, then we shortly say that they are equivalent, and we use the symbol \( f_1 \sim f_2 \).

As in Jones [6, p. 82], if \( f_1 \) and \( f_2 \) are forms over the \( p \)-adic integers \( \mathbb{Z}_p \) and they are \( \text{GL}_n(\mathbb{Z}_p) \)-equivalent, then we say that they are \( p \)-adically equivalent. This is denoted by \( f_1 \sim_p f_2 \).

Finally, if \( f_1 \) and \( f_2 \) are forms such that they are \( \text{GL}_n(\mathbb{R}) \)-equivalent, we use the symbol \( f_1 \sim_\infty f_2 \).

Let \( f_1 \) and \( f_2 \) be two integral quadratic forms. If \( f_1 \sim_p f_2 \) for all primes \( p \) including \( p = \infty \), then we say that \( f_1 \) and \( f_2 \) are genus-equivalent (or semi-equivalent); cf. [6, p. 106] together with Theorem 40 of [6]. In this case, we say that they are in the same genus. Let \( \text{gen}(f) \) denote the set of integral quadratic forms \( f' \) which are genus-equivalent to a given integral quadratic form \( f \).

If a binary integral quadratic form \( \phi \) is properly represented by a ternary integral quadratic form \( f \) (as defined in [3, p. 25]), we will write \( f \to \phi \).

If a quadratic form \( f \) primitively represents a number \( n \) (as defined in [20, p. 8]), we will write \( f \to n \).

Throughout the article, we will use many useful results in [10]; cf. Proposition 11. In that paper, a slightly different definition of the symbol \( f \to \phi \)
was used, but the invariant factor theorem shows that both definitions are equivalent.

If \( q \) is a positive integral quadratic form, then let \( R(q) = \{ q(x) > 0 : x \in \mathbb{Z}^n \} \) denote the set of positive numbers represented by \( q \), and let \( R_{a,m}(q) = \{ r \in R(q) : r \equiv a \pmod{m} \} \), for given integers \( a \) and \( m \).

Since we will use some results from Dickson and Smith as we mentioned above, we have to connect our concepts related to the integral quadratic forms with the ones in those authors. At first, note that an integral quadratic form \( f \) (in the above sense) may not be an integral form in sense of Dickson [5] and Smith [18] because they also assume that the \textit{non-diagonal coefficients} of \( f \) are even, i.e., \( 2 | a_{ij} \), for all \( i < j \).

We next introduce a notation for the determinant in Dickson’s sense; this is denoted by \( \det^D \). If \( f(x, y) = ax^2 + 2txy + by^2 \) is a positive integral binary form, then we have that

\[
\det^D(f) = -\frac{\det(f)}{4} = t^2 - ab.
\]

From now on, we will shortly write a form instead of an integral form unless the context makes it necessary. In order to prove the main results we will study certain techniques for deciding when two positive ternary forms \( f_1 \) and \( f_2 \) are genus-equivalent. We can determine genus equivalence by using the method of Smith. For this aim, we introduce more some terminology for quadratic forms.

The \textit{content} of the form \( f \) is the greatest common divisor of its coefficients \( a_{ij} \), and is denoted by \( \text{cont}(f) \). If \( \text{cont}(f) = 1 \), we say that it is a \textit{primitive} form; otherwise, it is called \textit{imprimitive}. For the older terminology [2], [18], \( f \) is called \textit{properly primitive} if it is primitive (in the above sense) and if \( 2 | a_{ij} \), for all \( i < j \). Also, \( f \) is called \textit{improperly primitive} if \( \frac{1}{2}f \) is primitive (in the above sense) and if there exists some pair \((i, j)\) with \( i < j \) such that \( \frac{1}{2}a_{ij} \) is odd.

If \( f \) is properly or improperly primitive, then it has a \textit{reciprocal} quadratic form \( F_f \) in Smith’s sense as defined on [18, p. 455]. If \( f \) is a primitive form, then it has \textit{reciprocal} as defined by Brandt [2, p. 336]; this is denoted by \( F_{f}^P \). We will discuss the relations of these reciprocals when \( f \) is an improperly ternary form; cf. Proposition 8 below.

To decide when two positive ternary forms are genus-equivalent by using Smith’s method, we have to determine the \textit{assigned characters} of \( f_1 \) and \( F_{f_1} \), and those of \( f_2 \) and \( F_{f_2} \), and then we compare the values of the assigned
characters. In particular, we use Smith’s table \[18\] to decide the values of the supplementary assigned characters. For this, we have to show that the above definition of genus equivalence is equivalent to Smith’s definition. So, we will recall the definition of the genus equivalence in sense of Smith after we introduce some concepts and results.

Since we will mostly use the basic (genus) invariants $\Omega$ and $\Delta$ as in defined \[5, p. 7,\] \[18, p. 455,\] and the basic (genus) invariants $I_1$ and $I_2$ introduced by Brandt\[1, p. 316,\] \[2, p. 336,\] we now discuss how they are related to each other in the imprimitive case. Later, we will calculate them for $f = q(A, \theta)$; cf. Proposition 17 below.

**Proposition 3.** Let $f$ be a positive improperly primitive ternary form. Then $F_f$ is properly primitive, and $F_f = F_{f/2}^B$. Moreover, we have that

$$I_1(f/2) < 0 \text{ is odd and } 16 \mid I_2(f/2),$$

and

$$\text{disc}(f/2) = \frac{I_1(f/2)^2I_2(f/2)}{16},$$

and so,

$$I_1(f/2) = -\Omega_f \text{ and } I_2(f/2) = -8\Delta_f.$$

**Proof.** Since $f$ is an improperly primitive ternary form, we are in Case I of \[1, p. 316,\] and so $F_{f/2}^B = F_f$ is properly primitive and $I_1(f/2) < 0$ is odd and $16 \mid I_2(f/2)$ from the relations in \[1,\] loc. cit. By the identity on p. 316 of \[1,\] equation (7) holds. Moreover, we get from the relations of Type I of \[1,\] loc. cit., that $I_1(f/2) = -\Omega_f$ and $I_2(f/2) = -8\Delta_f$. Hence, all the assertions have been verified.

Thus, we can see in particular that for two positive improperly primitive ternary forms $f_1$ and $f_2$, we have that

$$\Omega_{f_1} = \Omega_{f_2} \text{ and } \Delta_{f_1} = \Delta_{f_2} \iff I_1(f_1/2) = I_1(f_2/2) \text{ and } I_2(f_1/2) = I_2(f_2/2).$$

Let $f_1$ and $f_2$ be both improperly or both properly primitive ternary forms. If their basic invariants $\Omega$ and $\Delta$ are equal and if their reciprocals $F_{f_1}, F_{f_2}$ are both properly or both improperly primitive, then we say that they belong to the same *order*; cf. \[18, p. 456,\].
Let us put

\[ \chi_\ell(x) = \left( \frac{x}{\ell} \right), \]

for \( x \) prime to \( \ell \), where \( \ell \) is an odd prime. If \( x \) is odd, let us define

\[ \chi_{-4}(x) = \left( \frac{-4}{x} \right) = (-1)^{(x-1)/2} \]

and

\[ \chi_8(x) = \left( \frac{8}{x} \right) = (-1)^{(x^2-1)/8}. \]

We know from Theorem 6 of [5] that any properly primitive ternary form represents an integer prime to any given integer, and from Theorem 7 of [5] that any improperly primitive ternary form represents the double of an odd integer prime to any given integer. Therefore, we can introduce the following important concept.

Given an integer \( \delta \), let \( \mathcal{X}(\delta) = \{ \chi_\ell : \ell \text{ odd prime with } \ell | \delta \} \cup \{ \chi_{-4}, \chi_8 \} \) be the indicated set of characters. If \( f \) is a primitive ternary form (as in Brandt[2]) or \( f \) is an improperly primitive ternary form (as in Smith[18]) and if \( \chi_n \in \mathcal{X}(\delta) \), for some \( \delta \), then \( \chi_n \) is called an **assigned character** of \( f \) if the following relation holds:

\[ \chi_n(r_1) = \chi_n(r_2), \]

for any \( r_1, r_2 \) represented by \( f \) with \( \gcd(r_i, n) = 1 \).

This common value is denoted by \( \chi_n(f) \). In particular, when this holds for some \( \chi_n \in \{ \chi_{-4}, \chi_8 \} \), we say that \( \chi_n \) is a **supplementary assigned character** of the form \( f \). When both \( \chi_{-4} \) and \( \chi_8 \) are supplementary assigned characters of the form \( f \), it is easy to see that equation (10) holds for \( \chi_{-4}\chi_8 \). Since it is customary to list only two of these characters in this case, we didn’t put \( \chi_{-4}\chi_8 \) in our set \( \mathcal{X}(\delta) \).

Let \( \mathcal{X}(f) \) denote the set of the assigned characters of the form \( f \). Let \( f \) be an improperly or properly primitive ternary form with the invariants \( \Omega \) and \( \Delta \). Then \( \chi_\ell \in \mathcal{X}(\Omega) \) is an assigned character of \( f \) for any odd (prime) \( \ell \) and \( \chi_\ell \in \mathcal{X}(\Delta) \) is an assigned character of \( F_f \) for any odd (prime) \( \ell \) by [18, p. 457].

If \( \Omega \) or \( \Delta \) is even, then we may have in certain cases supplementary assigned characters. Because we are interested in the case that \( f \) is improperly primitive, we will only discuss this case in detail. (For the other cases, see Table I of [18]). For this aim, let us take an improperly primitive ternary form \( f \). Thus, \( F_f \) is properly primitive ternary form by Proposition 3.

By Table I, Case B of [18, p. 459], we have the following supplementary assigned characters of \( F_f \).
Table 1

| Condition | Supplementary assigned characters |
|-----------|----------------------------------|
| $\Delta_f \equiv 2 \pmod{4}$ | $\chi_{-4}$ |
| $\Delta_f \equiv 0 \pmod{4}$ | $\chi_{-4}, \chi_8$ |

Moreover, we have from [18], loc. cit., that

\[(11) \quad \chi_{-4}(F_f) = -\chi_{-4}(\Omega_f).\]

Therefore, we get that $\chi_{-4}(r) = -\chi_{-4}(\Omega_f)$, for any odd number $r$ represented by $F_f$. When $\Delta_f \equiv 0 \pmod{4}$, we have also that $\chi_8(r_1) = \chi_8(r_2)$, for any odd numbers $r_1, r_2$ represented by $F_f$.

In conclusion, since $\Delta_f$ is odd when $f$ is a positive improperly primitive ternary form by Proposition 3, $f$ does not have any supplementary assigned characters, so we have that $X(f) = \{\chi_\ell : \ell \mid \Omega_f, \ell \text{ prime}\}$.

We have also that $X(F_f) = X(\Delta_f)$ if $\Delta_f \equiv 0 \pmod{4}$, and that $X(F_f) = X(\Delta_f) \setminus \{\chi_8\}$ if $\Delta_f \equiv 2 \pmod{4}$ by Table 1.

If two improperly primitive ternary forms $f_1$ and $f_2$ belong to the same order, and if the following equations hold

\[
\begin{align*}
\chi(f_1) &= \chi(f_2), \forall \chi \in X(f_1) = X(f_2) \quad \text{and} \\
\chi(F_f_1) &= \chi(F_f_2), \forall \chi \in X(F_f_1) = X(F_f_2),
\end{align*}
\]

then we say that $f_1$ and $f_2$ are genus-equivalent in Smith’s sense.

**Proposition 4.** Let $f_1$ and $f_2$ be two positive improperly primitive ternary forms. Then they are genus-equivalent if and only if they are genus-equivalent in Smith’s sense.

**Proof.** ($\Leftarrow$) Since $f_1$ and $f_2$ are genus-equivalent in Smith’s sense, we have that $\Omega := \Omega_{f_1} = \Omega_{f_2}$ and $\Delta := \Delta_{f_1} = \Delta_{f_2}$. Moreover, we have that $A(f_1) = M^T A(f_2) M$, for some $M \in \text{GL}_3(\mathbb{Q})$, where the denominator of $M$ is relatively prime to $2\Omega\Delta$ by §12 of [18]. Since the determinant $|f_i|$ of the form $f_i$ in Jones’s sense [6] is $\Omega^2 \Delta$ (cf. equation (28) of [5]), the denominator of $M$ is also relatively prime to $2|f_i|$ for $i = 1, 2$. By the equivalence of the conditions 1b and 2a of Theorem 40 of [6], it follows that $f_1$ and $f_2$ are genus-equivalent, so this part follows.

($\Rightarrow$) Assume that $f_1$ and $f_2$ are genus-equivalent. By the equivalence of the conditions 1a and 2a of Theorem 40 of [6], for any positive integer $q$, for any positive integer $q$,
there exists a matrix $M \in \text{GL}_3(\mathbb{Q})$ such that $A(f_1) = M^t A(f_2) M$, where the denominator of $M$ is relatively prime to $q$. In particular, there is such a matrix for $q = 2\Omega_{f_1} \Delta_{f_1}$. We therefore get that $f_1$ and $f_2$ are genus-equivalent in Smith’s sense by §12 of [18], so this part also follows.

Note that in the situation of Proposition [4] we can see that $f_1$ and $f_2$ are also genus-equivalent in sense of Watson by Theorem 50 of [20].

We will also discuss the theory of genus-equivalence of ternary quadratic forms in sense of Brandt [2], [1] because it has some advantages. The important one is that since he uses an integral ternary form without the assumption that the non-diagonal coefficients are even, it reduces to many case distinctions as was done in [1].

Let us take a positive primitive ternary form $f$ such that $2f$ is improperly primitive. It is useful to observe that $X(f) = X(2f)$ and $X(F_{2f}) = X(F_B^{f})$; cf. proof of Corollary 5 below.

**Corollary 5.** Assume that $f_1$ and $f_2$ are positive primitive ternary forms such that the $2f_i$ are improperly primitive. Then $f_1$ and $f_2$ are genus-equivalent if and only if the following equations hold

\[
I_1(f_1) = I_1(f_2) \text{ and } I_2(f_1) = I_2(f_2),
\]

\[
\chi(f_1) = \chi(f_2), \forall \chi \in X(f_1) \text{ and } \chi(F_B^{f_1}) = \chi(F_B^{f_2}), \forall \chi \in X(F_B^{f_1}) = X(F_B^{f_2}).
\]

**Proof.** Since the $2f_i$ are improperly primitive, we have by Proposition [3] that $I_1(f_i) = -\Omega_{2f_i}$ is odd. By Brandt [2], §9 and §19, we can see that the set of the assigned characters of $f_i$ is $X(f_i) = \{\chi_{\ell} : \ell \text{ odd prime with } \ell \mid I_1(f_i)\}$. Thus, we have that $X(f_i) = X(2f_i)$ since $I_1(f_i) = -\Omega_{2f_i}$ and by the above discussion. In addition, since $F_{2f_i} = F_B^{f_i}$ is properly primitive by Proposition [3] we have that $X(F_{2f_i}) = X(F_B^{f_i})$ as Brandt [2], §19 discussed. We now start to prove the assertion.

$(\Leftarrow)$ By equations (9) and (12), we have that $2f_1$ and $2f_2$ have the same basic invariants $\Delta$ and $\Omega$. From the definitions of the assigned characters (cf. equation (10)), we have the relation that

\[
\chi_{\ell}(2f_i) = \chi_{\ell}(2)\chi_{\ell}(f_i), \forall \chi_{\ell} \in X(f_i)
\]

since if $r \in R(f)$, then $2r \in R(2f)$. Hence, it follows that $\chi_{\ell}(2f_1) = \chi_{\ell}(2f_2), \forall \chi_{\ell} \in X(2f_i)$ by the first equation of (13). Since $F_{2f_i} = F_B^{f_i}$ by
Proposition 3, we have that the values of the assigned characters of $F_{2f_i}$ and $F_{2f_i}$ are the same, so $2f_1$ and $2f_2$ are genus-equivalent by Proposition 4. Hence, $f_1$ and $f_2$ are genus-equivalent, which proves this part.

(⇒) By the hypothesis, we get that $2f_1$ and $2f_2$ are genus-equivalent, so they are genus-equivalent in Smith’s sense by Proposition 4. By the definition and equations (9) and (14), this part follows. Indeed, equation (12) holds by the definition and equation (9). Since $\chi(2f_i) = \chi(2f'_i), \forall \chi \in X(2f_i) = X(2f'_i)$ and $X(2f_i) = X(f_i)$, for $i = 1, 2$, equation (14) gives the first equation of (13). Also, since we have $F_{2f_i} = F_{2f'_i}$ and $X(F_{2f_i}) = X(F_{2f'_i})$ by what was discussed above, the second equation of (13) follows as well. 

For later use, we prove some results, which will be used in the next sections.

**Proposition 6.** Assume that $f$ is a ternary form with $4 \mid \text{cont}(f)$. Then for an odd prime $p$, we have that $f \sim_p \frac{1}{4}f$.

*Proof.* Since 2 is unit in $\mathbb{Z}_p$, for any odd prime $p$, the assertion follows with the transformation matrix $2^{-1}I_{3\times3}$, where $I_{3\times3}$ identity matrix. 

The following result due to Kani (15) will be especially useful below.

**Proposition 7.** Let $f_1$ and $f_2$ be two primitive ternary quadratic forms. Assume that $f_1 \sim_p f_2$, for an odd prime $p$. Then we have that $v_p(I_1(f_1)) = v_p(I_1(f_2))$, where $v_p(n)$ is the exponent of the largest power of $p$ that divides $n$.

*Proof.* First of all, recall from [2, p. 336] that $|I_1(f_i)| = \text{cont(adj}(f_i))$, for $i = 1, 2$, where as in [20, p. 25], the adjoint $\text{adj}(f)$ of a ternary form $f$ is defined by the formula

\begin{equation}
A(\text{adj}(f)) = -2 \text{adj}(A(f)),
\end{equation}

where $\text{adj}(A(f))$ is the usual adjoint of the matrix $A(f)$, i.e., the transpose of the cofactor matrix of $A(f)$.

Secondly, we observe that

\begin{equation}
f_1 \sim_p f_2 \Rightarrow \text{adj}(f_1) \sim_p \text{adj}(f_2).
\end{equation}
To see this, note that since $f_1 \sim_p f_2$, there exists $M \in \text{GL}_3(\mathbb{Z}_p)$ such that

$$A(f_2) = M^t A(f_1) M.$$ 

Since $\text{adj}(A) = \det(A) A^{-1}$, we get that $\text{adj}(AB) = \text{adj}(B) \text{adj}(A)$, if $\det(AB) \neq 0$. Thus, it follows that

$$(17) \quad \text{adj}(A(f_2)) = \text{adj}(M^t A(f_1) M) = \text{adj}(M) \text{adj}(A(f_1)) \text{adj}(M)^t,$$

and since $\text{adj}(M)^t \in \text{GL}_3(\mathbb{Z}_p)$, the statement (16) follows from equation (15).

Since $p$ is odd, we have that $v_p(\text{cont}(f)) = \min(v_p(m_{ij}))$, where $A(f) = (m_{ij})$. Thus, if $I_p(A)$ denotes the $\mathbb{Z}_p$-ideal generated by the entries of a given matrix $A$, then we have that $p^{v_p(\text{cont}(f))} \mathbb{Z}_p = I_p(A(f))$.

We claim that $I_p(AM) = I_p(MA) = I_p(A)$, for any $(3 \times 3)$ integral $p$-adic matrix $A \in M_3(\mathbb{Z}_p)$ and any $M \in \text{GL}_3(\mathbb{Z}_p)$. Indeed, $I_p(AM) \subset I_p(A)$ because the entries of the matrix $AM$ are $\mathbb{Z}_p$-linear combinations of those of $A$, and since $M^{-1} \in \text{GL}_3(\mathbb{Z}_p)$, it follows that also $I_p(A) = I_p(AM M^{-1}) \subset I_p(AM)$. This proves the first equality of the claim and the second follows similarly. Note that the claim implies that $I_p(M^t AM) = I_p(A)$, for any $A \in M_3(\mathbb{Z}_p)$ and any $M \in \text{GL}_3(\mathbb{Z}_p)$.

Let us put $A_i := \text{adj}(A(f_i))$, for $i = 1, 2$. Thus, we have by equation (15) that $v_p(I_1(f_i)) = \min(v_p(a_{ij})), i, j$ and $v_p(I_1(f_2)) = \min(v_p(b_{ij})), i, j$, where $(a_{ij}) = A_1$ and $(b_{ij}) = A_2$. By what was proven above, we have that $A_1 = M^t A_2 M$, for some $M \in \text{GL}_3(\mathbb{Z}_p)$. Therefore, since we obtain by the claim that

$$p^{v_p(I_1(f_2))} \mathbb{Z}_p = I_p(A_1) = I_p(M^t A_2 M) = I_p(A_2) = p^{v_p(I_1(f_2))} \mathbb{Z}_p,$$

we get that $v_p(I_1(f_1)) = v_p(I_1(f_2))$. \hfill $\square$

**Proposition 8.** Assume that $f_1$ and $f_2$ are primitive ternary forms such that $f_1 \sim_p f_2$, for an odd prime $p$. If $p \mid I_1(f_1)$, then we have that $\chi_p(f_1) = \chi_p(f_2)$. Moreover, if $I_1(f_1) = 4^n I_1(f_2)$, for some $n \in \mathbb{Z}$, then we have that $F^{B}_{f_1} \sim_p F^{B}_{f_2}$. Therefore, if $p \mid I_2(f_1)$, then we have that $\chi_p(F^{B}_{f_1}) = \chi_p(F^{B}_{f_2})$.

**Proof.** Since $f_1 \sim_p f_2$, we have a matrix with integer elements $M$ with $p \nmid \det(M)$ such that $M^t A(f_1) M \equiv A(f_2) \pmod{p}$ by Theorem 10 of [6]. Thus, $f_1(Mx) \equiv f_2(x) \pmod{p}$, for any $x \in \mathbb{Z}^3$. Since $f_2$ is primitive, there exists $x$ such that $\gcd(f_2(x), p) = 1$ by Theorems 6 and 7 of [5]. So, we have two numbers $n, m$ represented by $f_1, f_2$, respectively, such that $n \equiv m \pmod{p}$
and $\gcd(nm, p) = 1$. Thus, it follows that $\chi_p(n) = \chi_p(m)$. If $p \mid I_1(f_1)$, then $p \mid I_1(f_2)$ by Proposition 7 since $f_1 \sim_p f_2$. Thus, $\chi_p$ is an assigned character of both $f_1$ and $f_2$, and so we have that $\chi_p(f_1) = \chi_p(f_2)$ by equation (10). Hence, the first assertion follows.

For the second assertion, first of all, recall from [2] p. 336 that $F^B_{f_i} = \text{adj}(f_i)/I_1(f_i)$, so we have that $A(F^B_{f_i}) = \frac{2}{I_1(f_i)} \text{adj}(A(f_i))$ for $i = 1, 2$, by equation (15).

Secondly, since $f_1 \sim_p f_2$, we have that $\text{adj}(f_1) \sim_p \text{adj}(f_2)$ by equation (16). Thus, we have that $\frac{\text{adj}(f_1)}{I_1(f_2)} \sim_p \frac{\text{adj}(f_2)}{I_1(f_2)} = F^B_{f_2}$, and as in the proof of Proposition 6 we have that $\frac{\text{adj}(f_1)}{I_1(f_2)} \sim_p \frac{\text{adj}(f_2)}{I_1(f_2)} = F^B_{f_1}$, and hence we obtain that $F^B_{f_1} \sim_p F^B_{f_2}$, which proves the second assertion.

The last assertion follows in a similar way. Indeed, observe first that $|I_2(f_i)| = I_1(F^B_{f_i})$, for $i = 1, 2$, by [2] p. 336. Since $F^B_{f_1} \sim_p F^B_{f_2}$, we similarly obtain that $\chi_p(F^B_{f_1}) = \chi_p(F^B_{f_2})$ when $\chi_p$ is an assigned character of $F^B_{f_1}$ and $F^B_{f_2}$; equivalently, when $p \mid \gcd(I_2(f_1), I_2(f_2))$. If $p \mid I_2(f_1)$, then $p \mid I_2(f_2)$ by Proposition 7 since $F^B_{f_1} \sim_p F^B_{f_2}$. Thus, the last assertion follows.

When we work on the assigned characters of the binary forms, we use the table of Cox [4, p. 55], who denotes $\chi_{-4}$ by $\delta$ and $\chi_8$ by $\epsilon$.

### 3 The refined Humbert invariant

In this section, we will recall the concept of the refined Humbert invariant which was introduced by Kani [7, 10]. This is a quadratic form that is intrinsically attached to a principally polarized abelian surface. It is a useful ingredient since it can be used to translate a geometric problem into an arithmetic one.

Let $A$ be an abelian surface over an algebraically closed field $K$ with $\text{char} K = 0$, and assume that $A$ has a principal polarization $\lambda : A \cong \hat{A}$. Thus, $\lambda = \phi_\theta$ for a (unique) $\theta \in \text{NS}(A) = \text{Div}(A)/\equiv$, where $\equiv$ denotes numerical equivalence; (see [16] p. 60 for the discussion of $\phi$). Here $\theta$ is the (numerical) equivalence class $\text{cl}(\Theta)$ of an ample divisor $\Theta$ in $\text{Div}(A)$. Since we have by the Riemann-Roch Theorem (cf. [16] p. 150) that $\deg(\phi_\theta) = \frac{(\Theta, \Theta)}{2}$ for any ample $\theta$, it follows that $(\Theta, \Theta) = 2$. For a principally polarized abelian surface $(A, \theta)$, let us put

$$q_{(A, \theta)}(D) = (D, \theta)^2 - 2(D, D), \text{ for } D \in \text{NS}(A),$$

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where (.) denotes the intersection number of divisors.

By \([7, \text{p. 200}]\), \(q_{(A, \theta)}\) defines a positive definite quadratic form on the quotient group

\[ NS(A, \theta) := \text{NS}(A)/\mathbb{Z}\theta. \]

Hence, we have a quadratic \(\mathbb{Z}\)-module \((\text{NS}(A, \theta), q_{(A, \theta)})\), which is called the \textit{refined Humbert invariant} of the principally polarized abelian surface \((A, \theta)\). Let \(\rho = \text{rank}(\text{NS}(A))\) be the Picard number of \(A\). Thus \(q_{(A, \theta)}\) gives rise to an equivalence class of integral quadratic forms with \(\rho - 1\) variables since \(\text{NS}(A, \theta) \cong \mathbb{Z}^{\rho - 1}\). Hence, when \(\rho = 4\), the (corresponding) refined Humbert invariant is actually a ternary form.

Let \(\mathcal{P}(A) \subset \text{NS}(A)\) denote the set of \textit{principal polarizations} of \(A\). Thus,

\[ \mathcal{P}(A) = \{ \text{cl}(D) \in \text{NS}(A) : D \in \text{Div}(A) \text{ is ample and } (D.D) = 2 \}. \]

Since we aim to classify the imprimitive ternary forms which are equivalent to some refined Humbert invariant, we actually work with a subset of \(\mathcal{P}(A)\). Before discussing it, we first observe that in the ternary case that \(A\) is a CM (complex multiplication) product surface, i.e., \(A \cong E_1 \times E_2\) for some CM elliptic curves \(E_1, E_2\) with \(E_1 \sim E_2\). For two isogenous elliptic curves \(E, E'\), let \(q_{E,E'}\) denote the \textit{degree} form on \(\text{Hom}(E, E')\) which is defined by \(q_{E,E'}(h) = \text{deg}(h)\), for \(h \in \text{Hom}(E, E')\).

**Proposition 9.** If \(q_{(A, \theta)}\) is a ternary form, then \(A \cong E_1 \times E_2\) is a CM product surface. Moreover, we have that

\[ \det(q_{(A, \theta)}) = 32 \det(q_{E_1, E_2}). \]

**Proof.** The first statement follows as in the proof of Theorem 1 of \([13]\). Indeed, since \(q_{(A, \theta)}\) is a ternary form, it follows that \(\text{rank}(\text{NS}(A)) = 4\) (as was discussed above). Thus, by the structure theorems for \(\text{End}(A)\) (cf. Proposition IX.1.2 of \([19]\)), it follows that \(A \sim E \times E\), for some CM elliptic curve \(E\). By Shioda and Mitani’s Theorem\([17]\) (or by Theorem 2 of \([8]\)), the first statement follows. The second statement follows from Lemma 30 of \([12]\) together with equation (37) of \([12]\). \(\square\)

We will state the useful numerical classification of the set \(\mathcal{P}(A)\) when \(A\) is a product surface. For this, we recall from \([10]\) some useful results and notations at first. In particular, we recall the binary forms of the type studied
in [10], which have important connections with the ternary forms which we want to classify in this article.

To state the (partial) result related to these binary forms, for a fixed $\delta \geq 1$, let us put

$$ P(\delta) := \{(n_1, n_2, k) \in \mathbb{Z}^3 : n_1, n_2 > 0 \text{ and } n_1n_2 - k^2\delta = 1\}, $$

and for $s = (n_1, n_2, k) \in P(\delta)$, let

$$ q_s(x, y) := n_2^2x^2 + 2k\delta(n_1n_2 + 2)xy + n_2^2\delta(n_1n_2 + 3)y^2. $$

By [10, p. 27], its discriminant is

$$ \text{disc}(q_s) = -16\delta. $$

If $A = E_1 \times E_2$ is a product surface, where $E_1$ and $E_2$ are two elliptic curves, then we have an isomorphism

$$ D = D_{E_1, E_2} : \mathbb{Z} \oplus \mathbb{Z} \oplus \text{Hom}(E_1, E_2) \to NS(A) $$

by Proposition 23 of [10].

Here is a characterization of the elements of the set $P(A)$ as given by Corollary 25 of [10].

**Proposition 10.** Let $D = D(a, b, h) \in NS(A)$, where $A = E_1 \times E_2$. Then $D \in P(A)$ if and only if $a > 0$ and $ab - \text{deg}(h) = 1$. Thus, every principal polarization of $A$ has the form $D_{s, h} = D(n_1, n_2, kh)$ with $h \in \text{Hom}(E_1, E_2)$ and $s = (n_1, n_2, k) \in P(\text{deg}(h))$.

There is a nice connection between the refined Humbert invariants $q_{(A, D_{s, h})}$ and the binary forms $q_s$. This is an important tool in the proofs below.

**Proposition 11.** Let $A = E_1 \times E_2$ and $\theta = D_{s, h} \in P(A)$, where $h$ is a cyclic isogeny of degree $\delta$ and $s \in P(\delta)$. Then $q_{(A, \theta)}$ properly represents $q_s$.

**Proof.** This follows from Proposition 29 of [10].

We will state one more result related to $q_s$ when it is an imprimitive binary form. For this, put

$$ P(\delta)^{ev} := \{(n_1, n_2, k) \in P(\delta) : 2 \mid \gcd(n_1, n_2)\}. $$

By Theorem 13 of [10] (and its proof), we obtain the following characterization of the imprimitive binary forms $q_s$.
**Proposition 12.** Let \( \delta \geq 1 \), and let \( q \) be a binary form. Then \( q = 4q' \), for some primitive form \( q' \) of discriminant \(-\delta \equiv 1 \pmod{4}\) which is in the principal genus \( \text{gen}(1-\delta) \) if and only if \( q \sim q_s \), for some \( s \in P(\delta)^{ev} \).

**Proof.** (\( \Rightarrow \)) Assume that the hypothesis holds in this part. By the equivalence of the conditions (ii) and (iii) of Theorem 13 of [10], we have that \( q \sim q_s \), for some \( s \in P(\delta)^{ev} \). Since \( \text{cont}(q) = \text{cont}(q_s) = 4 \), it is clear to see that \( n_1 \) is even by equation (19). Also, since \( 2 \mid n_2 \delta(n_1n_2 + 3) \) (cf. equation (19) again) and \( \delta \) is odd, it follows that \( n_2 \) is even, so \( s \in P(\delta)^{ev} \), which proves this part.

(\( \Leftarrow \)) Assume that \( q \sim q_s \), for some \( s \in P(\delta)^{ev} \). Let \( s = (n_1, n_2, k) \), so by equation (19), it is clear that \( 2 \mid \text{cont}(q_s) \) since \( n_1 \) and \( n_2 \) are even. Thus, since \( q_s \) is not primitive, it cannot be in the principal genus by the equivalence of the conditions (ii) and (iii) of Theorem 13 of [10], so this part follows as well.

When \( A \) is a CM product surface, Kani [9] proved the following useful result for the refined Humbert invariants.

**Proposition 13.** Let \( A = E_1 \times E_2 \) be a CM product surface, and let \( f_q := x^2 + 4q_{E_1,E_2} \). Then

(i) if \( \theta \in P(A) \), and \( p \) is an odd prime or \( p = \infty \), then \( q(\theta) \sim_q f_q \),

(ii) if \( \theta \in P(A)^{odd} \), where \( P(A)^{odd} = \{ \theta \in P(A) : 2 \nmid (D,\theta), \forall D \in \text{NS}(A) \} \), then \( q(\theta) \in \text{gen}(f_q) \).

**Proof.** The first assertion (i) follows from Corollary 19 of [9] because \( f_q = q_{\theta} \), where \( \theta = \mathbf{D}(1,1,0) \); cf. equation (29) of [13]. Thus, if we apply Theorem 20 of [9] to the quadratic module \( (\text{NS}(A), q_A) \), where \( q_A \) is the intersection pairing as defined in [9, p. 142], then we get that \( q(\theta) \in \text{gen}(f_q) \), and so the second assertion (ii) follows.

This result gives the following relation.

**Corollary 14.** If \( \theta \in P(A) \), where \( A \) is a CM product surface, then

\( q(\theta) \) is an imprimitive ternary form \( \iff \theta \in P(A)^{ev} := P(A) \setminus P(A)^{odd} \).

**Proof.** If we take \( \theta \in P(A)^{odd} \), then we see that \( q(\theta) \in \text{gen}(f_q) \), for some binary quadratic form \( q \) by Proposition 13(ii), and thus \( q(\theta) \) is primitive since clearly \( f_q \) is primitive.
Conversely, if we take $\theta \in P(A)^{ev}$, then we easily see that $4 \mid q_{(A,\theta)}(D), \forall D \in NS(A)$ by the definition (cf. the proof of Proposition 17 below), and so $q_{(A,\theta)}$ cannot be primitive. 

Note that if $\theta \in P(A)$, where $A$ is a CM product surface, then we obtain in a similar way that $q_{(A,\theta)}$ is a primitive ternary form if and only if $\theta \in P(A)^{odd}$. In this respect, Kani classified the ternary quadratic forms which are equivalent to some refined Humbert invariant $q_{(A,\theta)}$ for a CM product surface $A$ and $\theta \in P(A)^{odd}$, so this is the primitive ternary case. We will give the similar classification for the ternary quadratic forms which are equivalent to some refined Humbert invariant $q_{(A,\theta)}$ for a CM product surface $A$ and $\theta \in P(A)^{ev}$, so this is the imprimitive (ternary) case.

Let $A = E_1 \times E_2$ be a CM product surface. While $P(A)^{odd}$ is always nonempty, $P(A)^{ev}$ may be an empty set for certain cases. Fortunately, Kani gave a result which can serve a nice classification of many cases when $P(A)^{ev} \neq \emptyset$.

**Proposition 15.** Let $A = E_1 \times E_2$ be a CM product surface, and let $\Delta$ denote the discriminant of the form $q_{E_1,E_2}$, let $t$ denote the content of $q_{E_1,E_2}$. Then we have that

(i) $P(A)^{ev} \neq \emptyset$ if and only if $4 \mid \Delta$ and $R_3,4(q_{E_1,E_2}) \neq \emptyset$.

(ii) If $\Delta$ is odd or if $t$ is even, then $P(A)^{ev} = \emptyset$.

*Proof.* We apply Proposition 32 of [9] to the quadratic module $(NS(A), q_A)$. Then equation (6) of [9] shows that the hypothesis of that proposition holds and that $q' = q_{E_1,E_2}$, where $q'$ is as defined in that proposition. Then the first assertion (i) follows from Proposition 32 and Remark 33 of [9]. Also, (ii) easily follows from (i). 

For later use, we will prove the following useful result here.

**Proposition 16.** Let $A = E_1 \times E_2$ be a CM product surface. If $\theta = D(a,b,h) \in P(A)$, then we have that

(21) $\theta = D(a,b,h) \in P(A)^{ev} \Leftrightarrow 2 \mid a, \ 2 \mid b, \ 4 \mid \text{disc}(q_{E_1,E_2})$.

In particular, if $4 \mid \text{disc}(q_{E_1,E_2})$ and if there is a primitive $h \in \text{Hom}(E_1,E_2)$ such that $q_{E_1,E_2}(h) \equiv 3 \pmod{4}$, then $\theta' := D(2,(q_{E_1,E_2}(h)+1)/2,h) \in P(A)^{ev}$. 

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Proof. As in the proof of Proposition 15, if we apply Proposition 32 of [9] to the quadratic module \((\text{NS}(A), q_A)\), then we can take \(q' = q_{E_1, E_2}\), and so we see from equation (30) of [9] that \(\theta = D(a, b, h) \in \mathcal{P}(A)^{ev} \iff 2 \mid a, 2 \mid b, \) and \(\beta_{q'}(h, h') \equiv 0 \pmod{2}, \) for all \(h' \in \text{Hom}(E_1, E_2)\), where \(\beta_{q'}\) is the bilinear map associated to \(q'\); cf. equation (13) of [9]. By Remark 33 of [9], we see that \(\theta = D(a, b, h) \in \mathcal{P}(A)^{ev} \iff 2 \mid a, 2 \mid b, \beta_{q'}(h, h') \equiv 0 \pmod{2}, \) for all \(h' \in \text{Hom}(E_1, E_2)\), and thus equation (21) holds.

To prove the second assertion, note first that \(\theta' \in \mathcal{P}(A)\). Indeed, since 
\[
2 \left(\frac{q_{E_1, E_2}(h) + 1}{2}\right) - q_{E_1, E_2}(h) = 1,
\]
we have that \(\theta' \in \mathcal{P}(A)\) by Proposition 10. So, the second assertion easily follows from the first one. \(\square\)

4 Quadratic forms in \(\Theta(A)^{ev}\)

To prove our main results, we first need a substitute for Proposition 13(ii) in the imprimitive case. To be more precise, note that this result gives that \(\Theta(A)^{odd} := \{q_{(A, \theta')} : \theta' \in \mathcal{P}(A)^{odd}\} / \sim\) lies in a single genus. We will study the set
\[
\Theta(A)^{ev} := \{q_{(A, \theta')} : \theta' \in \mathcal{P}(A)^{ev}\} / \sim,
\]
and we will see in Theorem [21] below that this is also true in most cases when \(\theta \in \mathcal{P}(A)^{ev}\), but not in all cases. Theorem [21] is the main result in this section, and it will be a key tool for the main results of this article.

We have shown the relations between the basic invariants \(I_1, I_2\), and \(\Omega, \Delta\) and between the reciprocals \(F\) and \(F^B\) in Proposition 3. Now, we explicitly determine them in the imprimitive case, i.e., when \(q_{(A, \theta)} \in \Theta(A)^{ev}\).

Proposition 17. Let \(A = E_1 \times E_2\) be a CM product surface, and \(\theta \in \mathcal{P}(A)^{ev}\). Let us put \(f := \frac{1}{2}q_{(A, \theta)}, d := -\text{disc}(q_{E_1, E_2}), t := \text{cont}(q_{E_1, E_2})\) and \(d' := d/t^2\). Then \(f\) is an improperly primitive form and basic invariants of \(f/2\) are

\[
I_1(f/2) = -t \quad \text{and} \quad I_2(f/2) = -4d', \quad \text{and thus,} \quad \text{disc}(f/2) = \frac{-t^2 d'}{4}.
\]

Therefore, the following equations hold

\[
\Omega_f = t \quad \text{and} \quad \Delta_f = d'/2,
\]
and

\[
t \quad \text{is odd and} \quad 4 \mid d'.
\]
Proof. This proof comes from Prof. Kani’s handwritten notes.

Firstly, observe that $\text{cont}(q_{(A,θ)}) = 4$. Indeed, it is easy to see that $4 \mid q_{(A,θ)}(D)$, $\forall D \in \text{NS}(A)$ by the definition. So, it follows that $4 \mid \text{cont}(q_{(A,θ)})$. Let $θ =: \mathbf{D}(n_1, n_2, kh)$, where $h \in \text{Hom}(E_1, E_2)$ is primitive. Since $θ \in \mathcal{P}(A)^{\text{ev}}$, $n_1$ and $n_2$ are even by condition (21), and $n_1, n_2 > 0$ and $n_1n_2 - \\text{deg}(kh) = 1$ by Proposition 10. Hence, it gives that $s = (n_1, n_2, k) \in \mathcal{P}(\text{deg}(h))^{\text{ev}}$. By Proposition 11, we have that $q_{(A,θ)} \to q_s$, where $q_s$ is as in equation (19). Since $s \in \mathcal{P}(\text{deg}(h))^{\text{ev}}$, we have that $\text{cont}(q_s) = 4$ by Proposition 12. Thus, $\text{cont}(q_{(A,θ)}) \mid 4$, and so we have that $\text{cont}(q_{(A,θ)}) = 4$, which proves the observation.

Since $θ \in \mathcal{P}(A)^{\text{ev}}$, $\text{deg}(h)$ is odd. By equation (20), $\text{disc}(q_s) = -16 \text{deg}(h)$, so $\text{disc}(\frac{1}{2}q_s) = -\text{deg}(h)$ is odd. Hence, it follows that $\frac{1}{2}q_s$ is improperly primitive. Since $f = \frac{1}{2}q_{(A,θ)} \to \frac{1}{2}q_s$, it follows that $f$ is also improperly primitive. Indeed, we know that $f/2$ is a primitive form. If $f$ were not improperly primitive, then we would have that all the non-diagonal coefficients of $f$ would be divisible by 4. If we consider the coefficient matrices of the forms $f$ and $\frac{1}{2}q_s$, then we have an integral $3 \times 2$ matrix $T$ such that $T^tA(f)T = A(\frac{1}{2}q_s)$ since $f \to \frac{1}{2}q_s$. Since all the non-diagonal coefficients of $f$ are divisible by 4 and $\text{cont}(f) = 2$, it follows that $A(f)$ is divisible by 4, so $A(\frac{1}{2}q_s)$ is divisible by 4. But this would imply that the non-diagonal coefficient of the $\frac{1}{2}q_s$ would be divisible by 4, which is not possible since it is improperly primitive. This proves the first assertion.

Since $f$ is improperly primitive, $I_1(f/2)$ is a negative odd number by equation (5). We know that $f/2 = \frac{1}{4}q_{(A,θ)} \sim_p f_q$, for all odd primes $p$, where $f_q = x^2 + 4q_{E_1,E_2}$, by Propositions 6 and 13. Since $f/2 = \frac{1}{4}q_{(A,θ)}$ and $f_q$ are primitive forms, we get from Proposition 7 that $v_p(I_1(f/2)) = v_p(I_1(f_q))$ for odd primes $p$. Since $I_1(f_q) = -16t$ by equation (21) of 13, and $t$ is odd (cf. Proposition 15(ii)), we thus see that $I_1(f/2) = -t$.

Since $\text{disc}(f/2) = -\frac{1}{2} \text{det}(f/2)$ by equation (3) and since $\text{det}(q_{(A,θ)}) = 32d$, where $d = \text{det}(q_{E_1,E_2})$ by equation (18), it follows that

$$\text{disc}(f/2) = -\frac{1}{2} \text{det}\left(\frac{1}{4}q_{(A,θ)}\right) = -\frac{1}{2} \frac{1}{4^3} \text{det}(q_{(A,θ)}) = -\frac{1}{2} \frac{1}{64} 32d = \frac{-d}{4}.$$  

Thus, we see that $I_2(f/2) = 16(-d/4)/t^2 = -4d^l$ by equation (7), so equation (22) holds. Then equation (23) just follows from equation (22) together with equation (8). Lastly, equation (24) follows from equation (9).\[ \square \]

Note that this result also shows that condition (1) of a positive ternary
form $f$ (above) is necessary for $f$ to be equivalent to some refined Humbert invariant $q_{(A, \theta)}$.

The second nice application of the classification of the binary forms $q_s$ is that it can be used to prove the necessary condition \( (2) \) of the second main result.

**Proposition 18.** If an imprimitive ternary form $f$ is equivalent to some refined Humbert invariant $q_{(A, \theta)}$, then there is a number $n$ such that $f$ represents $\left(2n^2\right)$ with $\gcd(n, \text{disc}(f)) = 1$.

**Proof.** By Proposition 9, we have that $A = E_1 \times E_2$ is a CM product surface since $q_{(A, \theta)}$ is a ternary form. Let us put $\theta =: D(n_1, n_2, kh)$, where $h \in \text{Hom}(E_1, E_2)$ is primitive. Since $\theta \in \mathcal{P}(A)^{ev}$ by Corollary 14, we have that $q_{(A, \theta)} \rightarrow q_s$, where $s \in \mathcal{P}(\text{deg}(h))^{ev}$ as in the proof of Proposition 17.

Since $s \in \mathcal{P}(\text{deg}(h))^{ev}$, we have by Proposition 12 that $q_s = 4q$, for some primitive form $q$ lying in the principal genus $\text{gen}(1 - \text{deg}(h))$. By Proposition 14 of [13], we claim that $q$ represents $n^2$ with $\gcd(n, \text{disc}(f)) = 1$. Indeed, by Proposition 14 of [13] we see that there exists an integer $n \geq 1$ with $\gcd(n, \text{disc}(f)) = 1$ such that $n^2 R(1 - \text{deg}(h)) \subset R(q)$ (since $q \in \text{gen}(1 - \text{deg}(h))$).

Since a principal form represents 1, the claim follows. Therefore, since $q_{(A, \theta)} \rightarrow q_s$, we obtain that $4n^2 \in R(q_{(A, \theta)}) = R(f)$, which proves the assertion. \( \square \)

The following result from Kani[15] will be very useful to calculate the value of the assigned characters of the (corresponding) reciprocal form $F_f$.

**Proposition 19.** Assume that we are in the situation of Proposition 17. Let us put $\theta =: D(a, b, ch)$, where $h \in \text{Hom}(E_1, E_2)$ is primitive. Then $q_{E_1, E_2}(h)/t$ is represented by the reciprocal $F_f = F_{f/2}^B$ of $f/2$.

**Proof.** (Cf. [15].) As in the proof of Proposition 17 with the notations in there, we have that $f \rightarrow \frac{1}{2}q_s$. Also, by equation (23), we have that $\Omega_f = t$

Since $\text{disc}(\frac{1}{2}q_s) = -4 \text{deg}(h)$, we have that $\det^D(\frac{1}{2}q_s) = -\text{deg}(h)$ by equation [3]. So, it follows from Theorem 27 of [3] that $\text{deg}(h)/t \in R(F_f)$. By Proposition 3, the assertion follows because $F_f = F_{f/2}^B$. \( \square \)

We will see that the set $\Theta(A)^{ev}$ may not lie in a single genus as in the case $\Theta(A)^{odd}$. The reason why this is not the case is the following technical lemma.
Lemma 20. Assume that \( q \) is a positive binary form with \( R_{3,4}(q) \neq \emptyset \). In addition, assume that \( \text{disc}(q) \equiv 16 \pmod{32} \). Then the form \( q \) primitively represents two numbers \( r_1 \) and \( r_2 \) such that \( r_1 \equiv 3 \pmod{8} \) and \( r_2 \equiv 7 \pmod{8} \).

Proof. Since \( q \) primitively represents a number \( r_1 \equiv 3 \pmod{4} \) by the hypothesis, we have that \( q \sim r_1 x^2 + bxy + cy^2 \) for some \( b, c \in \mathbb{Z} \), by Lemma 2.3 of [4]. But this implies that \( 4r_1c - b^2 = -\text{disc}(q) \equiv 16 \pmod{32} \), so \( 4 | b^2 \), so \( b \) is even. Put \( b = 2b' \). Then this gives that \( b^2 - r_1 c \equiv 4 \pmod{8} \). To prove the statement, we will distinguish some cases.

Case 1: Assume that \( r_1 \equiv 3 \pmod{8} \).

By the hypothesis, we have \( b^2 \equiv 3c + 4 \pmod{8} \). We know that the residue classes of a square mod 8 are \( \{0, 1, 4\} \). So there are three cases for \( b^2 \) in mod 8.

If \( b^2 \equiv 0 \pmod{8} \), then \( b = 2b' \equiv 0 \pmod{8} \) and \( c \equiv 4 \pmod{8} \). Hence \( q(1, 1) = r_1 + b + c \equiv 7 \pmod{8} \).

If \( b^2 \equiv 4 \pmod{8} \), then \( b = 2b' \equiv 4 \pmod{8} \) and \( c \equiv 0 \pmod{8} \). Hence \( q(1, 1) = r_1 + b + c \equiv 7 \pmod{8} \).

If \( b^2 \equiv 1 \pmod{8} \), then \( b = 2, 6 \pmod{8} \) and \( c \equiv 7 \pmod{8} \). Hence \( q(0, 1) = c \equiv 7 \pmod{8} \). Hence, \( q \) primitively represents a number \( r_2 \equiv 7 \pmod{8} \) in these three cases.

Case 2: Assume that \( r_1 \equiv 7 \pmod{8} \). In this case, we have that \( b^2 \equiv 7c + 4 \pmod{8} \).

If \( b^2 \equiv 0 \pmod{8} \), then \( c \equiv 4 \pmod{8} \), and so \( q(1, 1) \equiv 3 \pmod{8} \).

If \( b^2 \equiv 4 \pmod{8} \), then \( c \equiv 0 \pmod{8} \), and so \( q(1, 1) \equiv 3 \pmod{8} \).

If \( b^2 \equiv 1 \pmod{8} \), then \( c \equiv 3 \pmod{8} \), and so \( q(0, 1) \equiv 3 \pmod{8} \). Hence, \( q \) primitively represents a number \( r_2 \equiv 3 \pmod{8} \) in these three cases, so the assertion follows.

The following result is the main result in this section, and it will be a key tool for the main results in this article.

Theorem 21. Let \( A = E_1 \times E_2 \) be a CM product surface, and let \( \mathcal{P}(A)^{ev} \neq \emptyset \). Let us put \( d := -\text{disc}(q_{E_1, E_2}) \). Then we have that

(i) if \( d \not\equiv 16 \pmod{32} \), then \( \Theta(A)^{ev} \) lies in a single genus.

(ii) if \( d \equiv 16 \pmod{32} \), then \( \Theta(A)^{ev} \) lies in exactly two genera.

Proof. Let us take two arbitrary principal polarizations \( \theta_1, \theta_2 \in \mathcal{P}(A)^{ev} \), and let \( f_i := \frac{1}{2q_{(A, \theta_i)}} \). We have by Proposition [17] that the \( f_i \) are improperly primitive, so let \( F_i := F_{f_i} \) be the reciprocals of \( f_i \)'s for \( i = 1, 2 \). Also, we
it suffices by Proposition 4 to show that the following equations hold:

\[ \chi(\ell) = \chi(t), \forall \chi \in \chi(\ell), \text{ and } \chi(F_{\ell}) = \chi(F_t), \forall \chi \in X(F_{\ell}). \]

We have that \( X(\ell) = \{ \chi_\ell : \ell | t, \ell \text{ prime} \} \), and that \( f_i \) has no supplementary assigned characters since \( t \) is odd; cf. equation (24). We also have that \( X(F_t) = \{ \chi_\ell : \ell | d', \ell \neq 2 \text{ prime} \} \cup X_s(F_t) \), where \( X_s(F_t) \) denotes the set of supplementary assigned characters of \( F_t \). Since \( 2 | d'/2 = \Delta_f \) (cf. equation (24)), \( X_s(F_t) \neq \emptyset \); cf. Table I.

Claim: \( \chi(f_i) = \chi(f_2), \forall \chi \in X(f_i), \text{ and } \chi(F_i) = \chi(F_2), \forall \chi \in X(F_i) \setminus \{ \chi_s \} \) (possibly \( \chi_s \not\in X(F_i) \)); cf. Table I.

Proof: Since we have by Propositions 8 and 13 that \( f_{i/2} \sim f_2/2 \) for all odd primes \( \ell \), it follows from Proposition 8 that \( \chi_\ell(f_{i/2}) = \chi_\ell(f_2/2) \), for all odd primes \( \ell \mid I_i(f_{i/2}) = -t \). By equation (14), it follows that \( \chi(f_i) = \chi(f_2), \forall \chi \in X(f_i) \). Thus, this verifies the first equation of the claim. In a similar way, we also have from Proposition 8 that \( \chi_\ell(F_{i/2}^B) = \chi_\ell(F_2^B) \), for all odd primes \( \ell \mid d' \). Since \( F_{i/2}^B = F_i \) by Proposition 8, we get that \( \chi_\ell(F_i) = \chi_\ell(F_2) \), for all odd primes \( \ell \mid d' \).

By Table I, \( \chi_{-4} \in X_s(F_1) \). By equation (11), we have that

\[ \chi_{-4}(F_i) = -\chi_{-4}(\Omega_{f_i}) = -\chi_{-4}(t), \]

for \( i = 1, 2 \). In particular, \( \chi_{-4}(F_1) = \chi_{-4}(F_2) \), so the Claim follows.

If \( \Delta_{f_i} \equiv 2 \pmod 4 \), then we can conclude from the Claim and Proposition 4 that \( f_1 \) and \( f_2 \) are genus-equivalent since \( \chi_{-4} \) is the only supplementary assigned character of \( F_i \) in this case. This proves the assertion (i) in this case. (Note that if \( \Delta_{f_i} \equiv 2 \pmod 4 \), then \( d \neq 16 \pmod {32} \).

Thus, assume that \( \Delta_{f_i} \equiv 0 \pmod 4 \), so \( d' \equiv 0 \pmod 8 \) and \( \chi_{-4} \in X_s(F_1) \). By [4, p. 55] with \( n = d'/4 \), we obtain the following supplementary assigned characters of the binary quadratic form \( q' := \frac{1}{4}q_{E_1,E_2} \).

Table 2
Let us put $\theta_i := D(n_i, m_i, k_i h_i)$, where the $h_i$ are primitive elements in $\text{Hom}(E_1, E_2)$, and let $\delta_i := q'(h_i)$ for $i = 1, 2$. Since $\theta_i \in \mathcal{P}(A)^{ev}$, we have that $n_i m_i - q_{E_1, E_2}(k_i h_i) = 1$ by Proposition 19, and $n_i m_i$ is even by equation (21), and so $q_{E_1, E_2}(k_i h_i) = k_i^2 q_{E_1, E_2}(h_i)$ is odd. Thus, it follows that the $\delta_i$ are odd.

Now, we can finish the proof of the assertion (i) when $d' \equiv 0 \pmod{8}$. Since we are in case (i), we have that $d \not\equiv 16 \pmod{32}$. Since $t$ is odd, it follows that $d' \not\equiv 16 \pmod{32}$.

**Case 1:** Assume that $d' \equiv 0$ or $24 \pmod{32}$.

By Table 2, $\chi_8 \in X(q')$ in these cases, where $X(q')$ denotes the set of the assigned characters of $q'$. Since $\delta_i$ is odd and $q' \rightarrow \delta_i$, we have that $\chi_8(q') = \chi_8(\delta_i)$. Moreover, since $\frac{1}{2} q_{E_1, E_2}(h_i) = q'(h_i) = \delta_i$, we obtain by Proposition 19 that $\delta_i$ is represented by the reciprocal $F_i$ for $i = 1, 2$, so $\chi_8(F_i) = \chi_8(\delta_i) = \chi_8(q')$. In particular, $\chi_8(F_1) = \chi_8(F_2)$. Hence, it follows by the Claim that equation (25) holds, so $f_1 = \frac{1}{2} q_{(A, \theta_1)}$ and $f_2 = \frac{1}{2} q_{(A, \theta_2)}$ are genus-equivalent, and so $q_{(A, \theta_1)}$ and $q_{(A, \theta_2)}$ are genus-equivalent. Since $\theta_1$ and $\theta_2$ are arbitrary elements in $\mathcal{P}(A)^{ev}$, the assertion (i) follows in this case.

**Case 2:** Assume that $d' \equiv 8 \pmod{32}$.

By Table 2, $\chi_{-4} \chi_8 \in X(q')$ in this case. As in Case 1, we obtain that $\chi_{-4} \chi_8(q') = \chi_{-4} \chi_8(\delta_i)$, for $i = 1, 2$, and $\chi_{-4} \chi_8(F_1) = \chi_{-4} \chi_8(F_2)$. Since $\chi_{-4}(F_1) = \chi_{-4}(F_2)$ by the Claim (or by equation (14)), it follows that $\chi_8(F_1) = \chi_8(F_2)$ again, so the assertion (i) follows in a similar way as in Case 1. Thus the assertion (i) follows all cases.

To prove the assertion (ii), assume that $d \equiv 16 \pmod{32}$. As above, since $t$ is odd, we have that $d' \equiv 16 \pmod{32}$. We clearly have that $q_{E_1, E_2}(k_i h_i) \equiv 3 \pmod{4}$. Thus, by Lemma 20, it follows that there are integers $r_1$ and $r_2$ such that $r_1 \equiv 3 \pmod{8}$ and $r_2 \equiv 7 \pmod{8}$, which are primitively represented by $q_{E_1, E_2}$. Let us put $q_{E_1, E_2}(h_i') := r_i$, for primitive $h_i' \in \text{Hom}(E_1, E_2)$, for $i = 1, 2$. Now, consider

$$ (26) \quad \theta_i' := D(2, (r_i + 1)/2, h_i'), \quad \text{for } i = 1, 2. $$

By Proposition 16, we have that $\theta_i' \in \mathcal{P}(A)^{ev}$, for $i = 1, 2$ (since $4 \mid t^2 d' = 48$).
\[ \gcd(q_{E_1,E_2}) \text{ and } r_i \equiv 3 \pmod{4} \]. We first observe that \( f'_1 := \frac{1}{2} q_{(A,\theta)} \) and \( f'_2 := \frac{1}{2} q_{(A,\theta)} \) are not genus-equivalent. Indeed, since \( \Delta' = d'/2 \) by equation (23), and since \( d' \equiv 16 \pmod{32} \), we have that \( \chi_8 \in X(F'_i) \), where \( F'_i \) is the reciprocal form of the \( f'_i \)'s, cf. Table 1. By Proposition 19, we have that \( F'_i \to r_i/t \). Since \( \chi_8(r_i/t) = \chi_8(t)^{-1} \chi_8(r_i) \), we get that \( \chi_8(r_1/t) \neq \chi_8(r_2/t) \), so \( \chi_8(F'_1) \neq \chi_8(F'_2) \), which proves the observation that \( q_{(A,\theta)} \) and \( q_{(A,\theta)} \) are not genus-equivalent.

By what was proven above, we have that \( \chi_8(F'_1) \neq \chi_8(F'_2) \), so by the Claim, we can see that for any \( \theta \in \mathcal{P}(A)^\omega \), equation (23) holds either for the two forms \( \frac{1}{2} q_{(A,\theta)} \) and \( f'_1 \) or for the two forms \( \frac{1}{2} q_{(A,\theta)} \) and \( f'_2 \). Hence, we have that either \( q_{(A,\theta)} \) and \( 2f'_1 \) or \( q_{(A,\theta)} \) and \( 2f'_2 \) are genus-equivalent (according to the value of the assigned character \( \chi_8 \)). Hence, it follows that any \( q_{(A,\theta)} \in \Theta(A)^\omega \) lies in either \( \text{gen}(2f'_1) \) or \( \text{gen}(2f'_2) \), so the assertion (ii) follows.

The following technical lemma is required in the proof of Lemma 23.

**Lemma 22.** Let \( q \) be a primitive binary form for which \( \text{disc}(q) = d \) is even. If \( q \) primitively represents an odd number \( a \), then \( q \) primitively represents a number \( a' \equiv a \pmod{4} \) with \( (a',d) = 1 \).

**Proof.** To prove the assertion we refine the argument of the proof of Proposition 4.2 of [3]. We may assume that \( q = ax^2 + bxy + cy^2 \) with \( b^2 - 4ac = -d \) since \( q \to a \) by Lemma 2.3 of [4]. Consider the following sets of prime divisors of \( d \):

\[
\mathcal{P}_1 := \{ p \mid \gcd(a,c,d) \}, \quad \mathcal{P}_2 := \{ p \mid \gcd(a,d), p \nmid c \},
\]

\[
\mathcal{P}_3 := \{ p \mid \gcd(c,d), p \nmid a \} \quad \text{and} \quad \mathcal{P}_4 := \{ p \mid d, p \nmid a, p \nmid c \}.
\]

It is clear that the union \( \bigcup_i \mathcal{P}_i \) is the set of the prime divisors of \( d \), and that the \( \mathcal{P}_i \)'s are disjoint.

Put \( x_i := \prod_{p \in \mathcal{P}_i} p \), and put \( a' := q(x_2,x_3x_4) \), so \( q \to a' \) since we have that \( \gcd(x_2,x_3x_4) = 1 \) by the construction. We first show that \( \gcd(a',d) = 1 \). Assume not, so there is a prime number \( p \mid \gcd(a',d) \). Hence, \( p \in \mathcal{P}_1 \), for some \( 1 \leq i \leq 4 \).

It is also clear that \( p \mid b \) for \( p \in \mathcal{P}_1 \), with \( i \in \{1,2,3\} \) since \( b^2 - 4ac = -d \). To get a contradiction we need to distinguish four cases.

**Case 1:** Assume that \( p \in \mathcal{P}_1 \). Since \( p \mid b \), this is not possible since \( q \) is primitive.

**Case 2:** Assume that \( p \in \mathcal{P}_2 \). Write

\[
a' = q(x_2,x_3x_4) = ax_2^2 + bx_2x_3x_4 + cx_3^2x_4^2.
\]
Since $p \mid x_2$ and $p \mid a'$, it follows that $p \mid cx_2^2x_4^2$. But, $p \nmid c$, so $p \mid x_3x_4$. By the construction, for any $p \mid x_3$ or $p \mid x_4$ we have that $p \nmid a$, which is a contradiction.

**Case 3:** Assume that $p \in \mathcal{P}_3$. As in case (ii), $p \mid ax_2^2$, so $p \mid x_2$. But, for any $p \mid x_2$ we have that $p \nmid c$, which is a contradiction.

**Case 4:** Assume that $p \in \mathcal{P}_4$. Since $p \mid a'$ and $p \mid x_4$, we have that $p \mid ax_2^2$, so $p \mid x_2$, which is not possible since for any $p \mid x_2$ we have that $p \nmid a$. Hence, we obtain that $\gcd(a', d) = 1$.

To complete the proof, observe that since $d$ is even, $b$ is also even. Consider two cases as follows.

**Case I:** Assume that $c$ is even. Then $2 \mid x_3$. So it follows that $a' = q(x_2, x_3x_4) = ax_2^2 + bx_2x_3x_4 + cx_2^2x_4^2 \equiv ax_2^2 \pmod 4 \equiv a \pmod 4$ since $x_2$ is an odd number. In this case, the assertion follows.

**Case II:** Assume that $c$ is odd. Then $2 \mid x_4$. So it follows that $a' = q(x_2, x_3x_4) = ax_2^2 + bx_2x_3x_4 + cx_2^2x_4^2 \equiv ax_2^2 \pmod 4 \equiv a \pmod 4$ by the same reason. Hence, this proves the assertion in all cases.

The following lemma will be useful in the proof of Proposition 23.

**Lemma 23.** Let $A = E_1 \times E_2$ be a CM product surface with $\mathcal{P}(A)^{ev} \neq \emptyset$. Let $t := \text{cont}(q_{E_1,E_2})$, and $q' := \frac{1}{2}q_{E_1,E_2}$. Then there exists $\theta \in \mathcal{P}(A)^{ev}$ such that

$R(q', \theta) := \{r \in R(q') \cap R(F_f) : (r, \text{disc}(q')) = 1\}$

is nonempty, where $f := \frac{1}{2}q_{(A,\theta)}$.

**Proof.** Since $\mathcal{P}(A)^{ev} \neq \emptyset$, there is $h^* \in \text{Hom}(E_1, E_2)$ with $q_{E_1,E_2}(h^*) = a^* \equiv 3 \pmod 4$ by Proposition 15(1). Write $h^* = ch$, with $h$ primitive. Since $q_{E_1,E_2}(h^*) = c^2q_{E_1,E_2}(h)$ is odd, $c^2 \equiv 1 \pmod 4$, so it follows that $a := q_{E_1,E_2}(h) \equiv 3 \pmod 4$. Let us put $a' := a/t$, so $q' \rightarrow a'$ (since $q_{E_1,E_2}$ primitively represents $a$). Since $4 \mid \text{disc}(q') = -d'$ by statement (24), and since $q'$ is primitive, we can apply Lemma 22 to $q'$. Thus, there is $a''$ such that $q' \rightarrow a'' \equiv a' \pmod 4$ with $(a'', d') = 1$. Thus, since $ta' = a \equiv 3 \pmod 4$, we get that $ta'' \equiv 3 \pmod 4$ and that $q_{E_1,E_2} \rightarrow ta''$.

Therefore, there is a primitive isogeny $h' \in \text{Hom}(E_1, E_2)$ such that $q_{E_1,E_2}(h) = ta''$. Now, consider $\theta = D(2, (ta'' + 1)/2, h') \in \mathcal{P}(A)^{ev}$ (cf. Proposition 16). Let $f := \frac{1}{2}q_{(A,\theta)}$, and let $F_f$ be its reciprocal. Then we know that $F_f$ represents $a''$ by Proposition 16. Hence $a'' \in R(q') \cap R(F_f)$ with $(a'', d') = 1$, which proves the assertion.

□

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The following result finds a binary form \( q \in \text{gen}(\frac{1}{2}q_{E_1,E_2}) \) such that \( q \) and \( F_f \) represent the same prime number when we are in the situation of Proposition 17.

**Proposition 24.** Let \( A = E_1 \times E_2 \) be a CM product surface. Assume that there exists a \( \theta \in \mathcal{P}(A)^{ev} \). Let \( d, d' \) and \( t \) be as in Proposition 17. Let \( f \in \text{gen}(\frac{1}{2}q_{\lambda,\theta}) \). Then there exists a prime number \( p \) represented by the reciprocal \( F_f \) of \( f \) with \( p \nmid d' \). Moreover, for such a number \( p \) there exists a binary form \( q' \in \text{gen}(q') \) which also represents \( p \), where \( q' := \frac{1}{4}q_{E_1,E_2} \).

**Proof.** First, we know that the content and the basic invariants \( I_1 \) and \( I_2 \) are genus invariants. Thus, since we have from Proposition 17 that \( f \) is improperly primitive, we have that \( F_f \) is properly primitive by Proposition 3 and so it represents infinitely many prime numbers by Corollary 12 of [13]. In particular, there is a prime number \( p \nmid d' \) represented by \( F_f \). This proves the first assertion.

Let \( X(q') \) be the set of assigned characters of \( q' \) as in defined in [4, p. 55]. Since \( q' \) is a primitive form, the second assertion follows from Theorem 2.26 (and Lemma 3.20) of [4], once we have shown that

\[
\chi(p) = \chi(q'), \quad \text{for all } \chi \in X(q').
\]

Let \( X_o(q') := \{ \chi_\ell : \ell \neq d', \ell \neq 2 \text{ prime} \} \), and \( X_s(q') \) denotes the set of the supplementary assigned characters of \( q' \) (cf. [4, p. 55]), so \( X(q') = X_o(q') \cup X_s(q') \). By Proposition 3 we have that \( F_f^{\beta/2} = F_f \). Let \( X_o(F_f) := \{ \chi_\ell : \Delta_f, \ell \neq 2 \text{ prime} \} \), and \( X_s(F_f) \) denotes the supplementary assigned characters of \( F_f \). Since \( \Delta_f = d'/2 \) by equation (23), we obtain that \( X_o(q') = X_o(F_f) \). Let \( q := q_{E_1,E_2} \), and let \( f_q := x^2 + 4q(y, z) \), and we have from Propositions 6 and 13 that \( f/2 \sim f_q \) for all odd primes \( \ell \). By Proposition 18 of [13], we have that \( I_1(f_q) = -16t \), and so \( I_1(f_q) = 16I_1(f/2) \) by Proposition 17. Since \( F_f^{\beta/2} = F_f \rightarrow p \), it thus follows from Proposition 8 that for any \( \chi_\ell \in X_o(q') = X_o(F_f) \)

\[
\chi_\ell(p) = \chi_\ell(F_f) = \chi_\ell(F_f^{\beta/2}).
\]

On the other hand, if \( r \in R(q') \) with \( (r, d't) = 1 \), then \( r \in R(F_f^{\beta/2}) \) by equation (44) of [9]. Hence, this yields the relation that

\[
\chi_\ell(q') = \chi_\ell(r) = \chi_\ell(F_f^{\beta/2}), \quad \text{for all } \chi_\ell \in X_o(q').
\]
By equations (28) and (29), we obtain that

\[ \chi(q') = \chi(p), \text{ for all } \chi \in X_o(q'). \]

In order to prove that \( \chi(q') = \chi(p) \), for all \( \chi \in X_s(q') \), we will use the supplementary assigned characters \( X_s(F_f) \) of \( F_f \) by making use of the Smith’s table; cf. Table 1. Since \( \Omega_f \) is an odd number (cf. assertion (24)) and \( f \) is improperly primitive, we have that equation (11) holds.

Claim: Let \( X_s^*(F_f) \) be the set of the supplementary assigned characters of \( F_f \), and also include \( \chi_4 \), \( \chi_8 \) if \( X_s(F_f) \) includes \( \chi_4 \) and \( \chi_8 \). If \( X_s(q') \subset X_s^*(F_f) \), then \( \chi(q') = \chi(p) \), for all \( \chi \in X_s(q') \).

Proof. It is easy to see that \( \chi_4 \chi_8 \) if \( X_s(F_f) \) includes \( \chi_4 \) and \( \chi_8 \). If \( X_s(q') \subset X_s^*(F_f) \), then \( \chi(q') = \chi(p) \), for all \( \chi \in X_s(q') \).

Case 1: Assume that \( d' \equiv 16 \mod 32 \).

In this case, we have that \( f' \) and \( f \) are genus-equivalent by Theorem 21, so we see from equation (13) that \( \chi(F_f') = \chi(F_f) \), for all \( \chi \in X(F_f) \). (Note that \( F_f = F_f^B \) and \( F'_f = F'_f^B \) by Proposition 3.) Since there is an odd number \( x \in R(q') \cap R(F_f') \) and \( X_s(q') \subset X_s^*(F_f) = X_s^*(F_f') \), the following equation holds

\[ \chi(x) = \chi(q') = \chi(F_f) = \chi(p), \text{ for all } \chi \in X_s(q'). \]

Hence, the Claim follows in this case.

Case 2: Assume that \( d' \equiv 16 \mod 32 \).

In this case, \( X_s(q') = \{ \chi_4 \}; \) cf. Table 2. Since \( \Omega_f = \Omega_{f'} \), we have that \( \chi_4(F_f) = \chi_4(F_f') \); cf. equation (11). As in Case 1, there is an odd number \( x \in R(q') \cap R(F_f') \) with \( (x, d') = 1 \), so it follows that

\[ \chi_4(x) = \chi_4(q') = \chi_4(F_f') = \chi_4(F_f) = \chi_4(p). \]

Thus the Claim follows in all cases.

By Table 4, there are two cases to classify the supplementary assigned characters of \( F_f \). These are as follows:

Case 1: Assume that \( \Delta_f \equiv 2 \mod 4 \), so \( d' \equiv 4 \mod 8 \). In this case, \( X_s^*(F_f) = \{ \chi_4 \}; \) cf. Table 4. By [4, p. 55] with \( n = d'/4 \), we have that
\(X_s(q') = \emptyset\) if \(d'/4 \equiv 3 \pmod{4}\), and \(X_s(q') = \{\chi_{-4}\}\) if \(d'/4 \equiv 1 \pmod{4}\). Therefore, equation (27) holds by the Claim and by equation (30) since \(X_s(q') \subset X_s^*(F_f)\) in this case. This proves the second statement in this case.

**Case II:** Assume that \(\Delta_f \equiv 0 \pmod{4}\), so \(8 \mid d'\). Then the supplementary assigned characters \(X_s(q')\) of \(q'\) and \(X_s^*(F_f)\) of \(F_f\) are as in the following table:

| Condition         | \(X_s(q')\)                      | \(X_s^*(F_f)\)       |
|-------------------|----------------------------------|-----------------------|
| \(d' \equiv 0\pmod{32}\) | \(\chi_{-4}, \chi_8\)          | \(\chi_{-4}, \chi_8, \chi_{-4}\chi_8\) |
| \(d' \equiv 8\pmod{32}\) | \(\chi_{-4}\chi_8\)          | \(\chi_{-4}, \chi_8, \chi_{-4}\chi_8\) |
| \(d' \equiv 16\pmod{32}\) | \(\chi_{-4}\)                  | \(\chi_{-4}, \chi_8, \chi_{-4}\chi_8\) |
| \(d' \equiv 24\pmod{32}\) | \(\chi_8\)                      | \(\chi_{-4}, \chi_8, \chi_{-4}\chi_8\) |

The second column follows from [4; p. 55] with \(n = d'/4\), and the third column follows from Table 1.

Since \(X_s(q') \subset X_s^*(F_f)\) in these cases, we obtain from the Claim together with equation (30) that equation (27) holds again, which proves the second assertion in these cases. Hence, the second assertion follows in all cases. \(\square\)

Given a positive ternary form \(f\) satisfying conditions (11) and (2), we will construct a principally polarized abelian surface \((A, \theta)\) such that \(f \sim q_{(A,\theta)}\). This will prove (the hard part of) the second main result. To this end, we will apply Theorem 34 of [5] which shows that two ternary forms with the same basic invariants \(\Omega\) and \(\Delta\) are equivalent if they represent a same binary form of discriminant \(-4\Omega C\), where \(C\) is a prime or the double of an odd prime. For this, we have to construct such a binary form. Hence, we will have to construct a suitable \((A, \theta)\) such that \(q_{(A,\theta)}\) represents such a binary form (which is also represented by \(f\)). We will see that this binary form can be of a form \(q_s\), for some \(s \in P(\Omega C)^{ev}\). Hence the following results aim to address this discussion.

**Lemma 25.** In the situation of Proposition [24] we have that \(tp \equiv 3 \pmod{4}\).

**Proof.** The assertion easily follows from equation (11). Indeed, since \(\Omega_f = t\) by equation (23), we get by equation (11) that \(\chi_{-4}(p) = \chi_{-4}(F_f) = -\chi_{-4}(t)\), so \(\chi_{-4}(tp) = -1\). Thus, the assertion follows from the relation that \(\left(\frac{-4}{tp}\right) = -1 \iff tp \equiv 3 \pmod{4}\). \(\square\)
Proposition 26. Let $A = E_1 \times E_2$ be a CM product surface. Let $d, t, d'$ be as in Proposition 17 and let $\theta \in \mathcal{P}(A)^{ev}$. For any $f \in \text{gen}(\frac{1}{2}q_{(A,\theta)})$, there is a prime number $p \nmid td'$ represented by $F_f$, and there exists a binary form $\phi_p$ of discriminant $-4\Omega_{fp} = -4tp$ such that $f \to \phi_p$. Moreover, $\text{cont}(\phi_p) = 2$.

Proof. The first assertion follows as in the proof of Proposition 24. More precisely, for any $f \in \text{gen}(\frac{1}{2}q_{(A,\theta)})$, we have by equation (23) that $\Omega_f = t$ and $\Delta_f = d'/2$, and that the reciprocal $F_f$ of $f$ is properly primitive by Proposition 3. By Corollary 12 of [13], $F_f$ represents infinitely many primes, so there is a prime number $p$ with $p \nmid td'$, which is represented by $F_f$. This proves the first assertion.

For any such $p$ we have by Theorem 38 of [5] that there is a binary form $\phi_p = ax^2 + 2bxy + cy^2$ with $\det^D(\phi_p) = b^2 - ac = \Omega_{fp} = -tp$ such that $f \to \phi_p$. Hence $\text{disc}(\phi_p) = -4tp$ (cf. equation (5)) with $p \nmid td'$, which proves the second assertion. Moreover, we see from Theorem 37 of [5] that $\phi_p$ is a properly or improperly primitive form since $\gcd(p, \Omega_f \Delta_f) = \gcd(p, td'/2) = 1$. Also, note that $\text{cont}(f) = 2$ by Proposition 17, so $2 \mid \text{cont}(\phi_p)$. Therefore, $\phi_p$ is improperly primitive, and hence, $\phi_p/2$ is primitive (in Watson’s sense), i.e., $\text{cont}(\phi_p) = 2$, which proves the last assertion.

The following result leads to a useful fact related to the forms $q_s$; cf. Corollary 28 below.

Theorem 27. Let $A, d, t, d'$ and $\theta$ as in Proposition 26. Let $f \in \text{gen}(\frac{1}{2}q_{(A,\theta)})$. If $f$ properly represents a binary form $\phi_p$ with $\text{cont}(\phi_p) = 2$ and $\text{disc}(\phi_p) = -4tp$, for some prime $p \nmid td'$, then $\phi' := \frac{1}{2}\phi_p \in \text{gen}(1_{-tp})$, i.e., $\phi'$ lies in the principal genus of discriminant $-tp$.

Proof. By Lemma 3.20 of [4], it suffices to show that $\chi(\phi') = 1$ for all assigned characters $\chi \in X(\phi')$. Since $tp$ is odd (cf. equation (24)), using the fact that $\text{disc}(\phi') = \text{disc}(\phi_p)/4 = -tp$ we obtain that the assigned characters of $\phi'$ are the $\chi_\ell$’s for odd primes $\ell \mid tp$; cf. [4, p. 55]. Moreover, we claim that the following equation holds.

\[
\chi_\ell(f/2) = 1, \text{ for any prime } \ell \mid t.
\]

To verify this, we first see that the assigned characters of $f/2$ are the $\chi_\ell$’s with primes $\ell \mid t$ since $I_1(f/2) = -t$ by equation (22). Let $q := q_{E_1, E_2}$. Since $f/2 \sim_\ell f_q$ for any odd prime $\ell$ by Propositions 6 and 13 and since $f_q \to 1$,
it follows from Proposition 8 that \( \chi_\ell(f/2) = \chi_\ell(f_q) = \chi_\ell(1) = 1 \), so equation (31) holds.

Let \( x \) be any integer represented by \( \phi' \) with \( (x, td') = 1 \). (There is such a number by Lemma 2.25 of [4].) Since \( f/2 \to \phi' \), \( x \) is also represented by \( f/2 \). By equation (31), we therefore obtain that

\[
(32) \quad 1 = \chi_\ell(f/2) = \chi_\ell(x) = \chi_\ell(\phi'),
\]

for all primes \( \ell \mid t \), where \( x \in R(\phi') \subset R(f/2) \) with \( (x, td') = 1 \).

So it remains to show that \( \chi_p(\phi') = 1 \). Since \( \text{disc}(\phi') = -tp \) is an odd number, there are infinitely many primes \( p' \equiv 1 \pmod{4} \) represented by \( \phi' \) by Corollary 4 of [13], so there is a prime \( p' \) with \( p' \not| tp \) and \( p' \equiv 1 \pmod{4} \) such that \( \phi' \to p' \). Then by Lemma 2.5 of [4], \( \text{disc}(\phi') \) is a quadratic residue modulo \( p' \), which means that \( \left( \frac{-tp}{p'} \right) = 1 \). Since \( f/2 \to \phi' \) and \( \phi' \to p \), it follows that \( f/2 \to p' \), so we get that

\[
(33) \quad \chi_\ell(p') = \chi_\ell(f/2) = 1 \quad \text{for all primes } \ell \mid t
\]

by equation (32). By the Quadratic Reciprocity Law, we have that

\[
(34) \quad \left( \frac{p'}{\ell} \right) = \left( \frac{\ell}{p'} \right), \text{ for all } \ell, \text{ and also } \left( \frac{p'}{p} \right) = \left( \frac{p}{p'} \right)
\]

because \( p' \equiv 1 \pmod{4} \). Also, since \( p' \equiv 1 \pmod{4} \), we have that \( \left( \frac{-1}{p'} \right) = 1 \).

Let us define the set \( S := \{ \ell \text{ prime} : v_\ell(t) \text{ is odd} \} \), i.e., \( S \) is the set of prime numbers which appear with an odd power in the factorization of \( t \). We therefore get that

\[
1 = \left( \frac{-tp}{p'} \right) = \left( \frac{tp}{p'} \right) = \left( \frac{t}{p'} \right) \left( \frac{p}{p'} \right) = \left( \frac{p}{p'} \right) \prod_{\ell \in S} \left( \frac{\ell}{p'} \right)
\]

because \( p' \equiv 1 \pmod{4} \). Also, since \( p' \equiv 1 \pmod{4} \), we have that \( \left( \frac{-1}{p'} \right) = 1 \).

Hence we have proved that \( \chi_p(\phi') = 1 \). This, together with equation (32), proves that \( \phi' \) lies in the principal genus of discriminant \( -tp \).

\[\square\]

**Corollary 28.** Assume that we are in the situation of Theorem 27. Then there exists \( s \in P(tp)^{ev} \) such that \( q_s \sim 2\phi_p \).
Proof. By Theorem 27, we have a binary quadratic form $\phi_p$ with $\text{cont}(2\phi_p) = 4$ and $\text{disc}(2\phi_p) = -16tp$. Moreover, $\frac{1}{2}\phi_p$ lies in the principal genus. By Lemma 25, the discriminant of $\frac{1}{2}\phi_p$ is $-tp \equiv 1 \pmod{4}$. We therefore get that $2\phi_p \sim q_s$, for some $s \in P(tp)^{ev}$ by Proposition 12. Hence, the assertion follows.

5 Classification of the refined Humbert invariant

In this section, the aim is to prove two main results in this article. Before we start to prove our main results, we recall from [13] the following very useful fact.

Lemma 29. Let $q$ be a positive binary quadratic form. Then there exist two CM elliptic curves $E_1$ and $E_2$ such that $q_{E_1,E_2} \sim q$.

Proof. See Lemma 26(a) of [13].

We are now ready to prove the first main result.

Proof of Theorem 1. Since $f_1$ is an imprimitive ternary form, which is equivalent to a form $q_{(A,\theta)}$, there are isogenous CM elliptic curves $E_1, E_2$ such that $A = E_1 \times E_2$ is a CM product surface and $\theta \in \mathcal{P}(A)^{ev}$ by Proposition 9 and Corollary 14. Let us put (as we did in the previous results)

$$f := \frac{1}{2} q_{(A,\theta)}, \quad q := q_{E_1,E_2}, \quad d := -\text{disc}(q), \quad t := \text{cont}(q), \quad d' := d/t^2.$$

Suppose that $f_2 \in \text{gen}(f_1)$. We first prove the hard part that

$$(35) \quad f_2 \sim q_{(A',\theta')}, \text{ for some } (A',\theta'),$$

where $A'$ is a CM product surface and $\theta' \in \mathcal{P}(A')^{ev}$. Since $f_1 \sim q_{(A,\theta)}$, we can apply Proposition 26 to $f_2/2 \in \text{gen}(f_1/2) = \text{gen}(\frac{1}{2} q_{(A,\theta)})$. Thus, there is a prime number $p \nmid td'$ represented by $F_{f_2/2}$, and there is a binary form $\phi_p$ of discriminant $-4\Omega_{f_2/2}p = -4tp$ such that $f_2/2 \rightarrow \phi_p$, and $\text{cont}(\phi_p) = 2$. By Corollary 28 we see that $2\phi_p$ has type $tp$, and that $2\phi_p \sim q_s$, for some $s \in P(tp)^{ev}$. Since $f_2/2 \rightarrow \phi_p$ and $\phi_p \sim q_s/2$, we have that $f_2/2 \rightarrow q_s/2$ by Theorem 28 of [5].
By Proposition \[24\] there exists a binary form \( \tilde{q} \in \text{gen}(q') \) with \( \tilde{q} \to p \), where \( q' := \frac{1}{2}q \). We also have that there exist two elliptic curves \( E_1', E_2' \) with \( q_{E_1', E_2'} \sim \tilde{q} := tq \) by Lemma \[29\]. Since \( \tilde{q} \) primitively represents \( p, \tilde{q} \) primitively represents \( tp \), and thus, there exists a primitive \( h \in \text{Hom}(E_1', E_2') \) such that \( q_{E_1', E_2'}(h) = tp \).

Let us put \( A' := E_1' \times E_2' \), and let \( \theta' := D_{s,h} := D(n, m, kh) \in \mathcal{P}(A') \), where \( s = (n, m, k) \in P(tp)^{ev} \); cf. Proposition \[10\]. Thus, we have that \( q_{(A', \theta')} \to q_s \) by Proposition \[11\].

We first observe that since \( s \in P(tp)^{ev} \), we see that both \( n \) and \( m \) are even, so \( \theta' \in \mathcal{P}(A')^{ev} \) by equation \[21\]. Since \( q_{E_1', E_2'} \sim \tilde{q} \), we have that \( \text{disc}(q_{E_1', E_2'}) = \text{disc}(\tilde{q}) = -t^2d' = -d \). If we apply Proposition \[17\] to \( (A', \theta') \), then we obtain that \( 4 \mid \text{disc}(q_{E_1', E_2'}) \) by assertion \[24\]. Moreover, if we let \( f' := \frac{1}{2}q_{(A', \theta')} \), then it follows from equation \[23\] that \( \Omega_{f'} = t \) and \( \Delta_{f'} = d'/2 \) since \( \text{cont}(q_{E_1', E_2'}) = \text{cont}(\tilde{q}) = t \) and \( \text{disc}(q_{E_1', E_2'}) = \text{disc}(\tilde{q}) = -t^2d' \). Moreover, we know that the basic invariants are genus invariants by Proposition \[4\] and thus \( f_1 \) and \( f_2 \) have the same basic invariants. Thus, since \( f_1 \sim q_{(A, \bar{\theta})} \), we see that \( \Omega_{f_2/2} = t \) and \( \Delta_{f_2/2} = d'/2 \) by equation \[23\] again, and so \( f' \) and \( f_2 \) have the same basic invariants \( \Omega \) and \( \Delta \).

Since \( \frac{1}{2}f_2 \) and \( f' \) are both improperly primitive (cf. Proposition \[17\]) and have the same invariants \( \Omega = t \) and \( \Delta = d'/2 \), and since both properly represent the binary form \( \frac{1}{2}q_s \) whose discriminant is \( -4tp \), we can apply Theorem 34 of \[5\] with \( C = p \). Note that the determinant of \( \frac{1}{2}q_s \) is \( \text{det}D(\frac{1}{2}q_s) = -tp \); cf. equation \[5\]. Therefore, we get from Theorem 34 of \[5\] that \( \frac{1}{2}f_2 \sim \frac{1}{2}q_{(A', \bar{\theta})} = f' \), so \( f_2 \sim q_{(A', \bar{\theta})} \), which verifies equation \[35\].

To complete the proof, we have to show that \( q_{(A', \theta')} \sim q_{(A, \bar{\theta})} \), for some \( \bar{\theta} \in \mathcal{P}(A)^{ev} \). To this end, note that we have that \( \bar{q} \in \text{gen}(q) \). Thus, it follows from Corollary 30 of \[13\] that there exists a principal polarization \( \tilde{\theta} \in \mathcal{P}(A) \) such that \( q_{(A, \tilde{\theta})} \sim q_{(A', \theta')} \). Since \( q_{(A', \theta')} \) is improper, \( q_{(A, \tilde{\theta})} \) is also improper, so \( \tilde{\theta} \in \mathcal{P}(A)^{ev} \) by Corollary \[14\]. Hence, \( f_2 \sim q_{(A', \theta')} \sim q_{(A, \tilde{\theta})} \), for some \( \bar{\theta} \in \mathcal{P}(A)^{ev} \), which proves the statement.

Next, we will prove (the hard part of) the second main result which gives a complete classification for improper ternary forms \( f \) for to be \( f \) equivalent to some refined Humbert invariant \( q_{(A, \theta)} \). To this end, we first prove the following proposition, which is actually an important step of the proof of this second main result.
Proposition 30. Assume that $f$ is a positive ternary form satisfying conditions (1) and (2). Let $\Omega$ and $\Delta$ be the basic invariants of $f/2$. In addition, assume that there is a prime number $p \nmid \Delta$ such that $(\frac{-2\Delta}{p}) = 1$ and $\Omega p \equiv 3 \pmod{4}$ and such that the reciprocal form $F_{f/2}$ represents either $p$ or $4p$. Then there exists a CM product surface $A = E_1 \times E_2$ and $\theta \in \mathcal{P}(A)^\text{ev}$ such that the following equations hold for $f' := q_{(A,\theta)}$:

\begin{align}
\Omega & = \Omega_{f/2} \text{ and } \Delta = \Delta_{f/2}, \\
\chi(f/4) & = \chi(f'/4), \forall \chi \in X(f/4), \\
\chi(F_{f/4}^B) & = \chi(F_{f'/4}^B), \forall \chi \in X(F_{f/4}^B) \setminus \{\chi_8\}.
\end{align}

In addition, we have that $F_{f'/2} \to p$.

Proof. We first construct a useful primitive binary form. Since $(\frac{-2\Delta}{p}) = 1$, there is an integer $b$ such that $-2\Delta \equiv b^2 \pmod{p}$. By replacing $b$ by $p - b$, if necessary, we may assume that $b$ is even. Since $f/2$ is improperly primitive by the condition (1), we have that $2 \mid \Delta$ by equations (6) and (8), so we have that $-2\Delta \equiv b^2 \pmod{4p}$.

Put $q'(x, y) = px^2 + bxy + \frac{b^2 + 2\Delta}{4p}y^2$. Clearly, $q'$ is an integral form with $\text{disc}(q') = -2\Delta$. Since $p \nmid \Delta$, $p \nmid b$, and thus $q'$ is primitive.

Next, put $q = \Omega q'$. We see that there exist two CM elliptic curves $E_1, E_2$ such that $q_{E_1,E_2} \sim q$ by Lemma 29. Let $A = E_1 \times E_2$, which is a CM product surface. By the construction, $\text{cont}(q) = \Omega$ and $\text{disc}(q) = -2\Omega^2\Delta$, and also by the hypothesis, $\Omega p \equiv 3 \pmod{4}$. Also, since $2 \mid \Delta$, we obtain that $4 \mid \text{disc}(q)$, and hence we have from Proposition 16 that $\theta := D(2, (\Omega p + 1)/2, h) \in \mathcal{P}(A)^\text{ev}$, where $q_{E_1,E_2}(h) = \Omega p$ for some primitive $h \in \text{Hom}(E_1, E_2)$. Thus, if we put $f' = q_{(A,\theta)}$, then it follows that $F_{f'/2} \to q_{E_1,E_2}(h)/\Omega = p$ by Proposition 19. This proves the last assertion. By equation (23), we have that $\Omega_{f'/2} = \text{cont}(q_{E_1,E_2}) = \Omega$ and $\Delta_{f'/2} = -\text{disc}(q_{E_1,E_2})/2\Omega^2 = \Delta$, so equation (30) holds.

Since $f/2$ is improperly primitive, $\Omega$ is odd; cf. equations (6) and (8). Thus, we have that $X(f/4) = \{\chi_\ell : \ell \mid \Omega, \ell \text{ prime}\}$. As we mentioned in the section we also have that $X(F_{f/2}) = \{\chi_{-4}, \chi_8, \chi_\ell : \ell \mid \Delta, \ell \neq 2 \text{ prime}\}$ if $\Delta \equiv 0 \pmod{4}$, and $X(F_{f/2}) = \{\chi_{-4}, \chi_{\ell} : \ell \mid \Delta, \ell \neq 2 \text{ prime}\}$ if $\Delta \equiv 2 \pmod{4}$.

Since $q_{(A,\theta)} \sim \ell q_{(A,\theta)}/4$ for all odd primes $\ell$ by Proposition 6 and since $f' = q_{(A,\theta)} \sim \ell f_q$ by Proposition 13 we obtain that $\chi_\ell(f'/4) = \chi_\ell(f_q)$, for any $\ell \mid \Omega$ by Proposition 8. On the one hand, we have that

$$\chi_\ell(f'/4) = \chi_\ell(f_q) = 1$$
since \( f_q \rightarrow 1 \). On the other hand, there is an integer \( m \) such that \( m^2 = \frac{1}{4}f(x, y, z) \) with \( \gcd(m, \text{disc}(f)) = 1 \) by the assumption (2). Since \( \Omega \mid \text{disc}(f) \) by equation (7) (and (8)), we have that \( \gcd(m, \Omega) = 1 \), so \( \ell \nmid m \). We now obtain that
\[
\chi_\ell(f/4) = \chi_\ell(m^2) = 1,
\]
and thus, this verifies equation (37).

Lastly, since \( F_{f/2} \rightarrow p \), we have that \( \chi_\ell(F_{f/2}) = \chi_\ell(p) \). By the hypothesis, we have that either \( \chi_\ell(F_{f/2}) = \chi_\ell(p) \) or \( \chi_\ell(F_{f/2}) = \chi_\ell(4p) \), for all odd primes \( \ell \mid \Delta \). But, this means that \( \chi_\ell(F_{f/2}) = \chi_\ell(p) \), and so it follows that \( \chi_\ell(F_{f/2}) = \chi_\ell(F_{f/2}') \), for all odd primes \( \ell \mid \Delta \).

Since \( f/2 \) and \( f'/2 \) are both improperly primitive forms and \( \Omega_{f/2} = \Omega'_{f'/2} \), we have that \( \chi_{-4}(F_{f/2}) = \chi_{-4}(F_{f'/2}) \) by equation (11). Hence, equation (38) holds since \( F_{f/2} = F_{f/4}^B \) by Proposition 3 and all the assertions have been proved.

Now, we are in position to prove the hard part of Theorem 2. For a positive ternary form \( f \) satisfying conditions (1) and (2), it suffices to find a principally polarized abelian surface \((A, \theta)\) such that \( f \) lies in \( \text{gen}(q(A, \theta)) \), because then we can conclude from Theorem 1 that \( f \) is equivalent to some refined Humbert invariant.

**Theorem 31.** Assume that \( f \) is a positive ternary form satisfying conditions (1) and (2). Then \( f \) is equivalent to some refined Humbert invariant \( q(A, \theta) \).

**Proof.** We first observe that it suffices to show that the hypotheses of Proposition 30 hold, and that either \( \chi_8 \not\in X(F_{f/4}^B) \) or that \( \chi_8(F_{f/4}^B) = \chi_8(F_{f'/4}^B) \), where \( f' = q_{(A, \theta)} \) is as constructed in Proposition 30. Indeed, if these conditions hold, then it follows from Corollary 5 and equation (9) that \( f \in \text{gen}(q(A, \theta)) \), and so we can apply Theorem 1 to conclude that \( f \sim q_{(A, \theta')} \), for some \( \theta' \in \mathcal{P}(A)^\text{ev} \).

By the assumption (2), there is an integer \( m \) with \( m^2 = \frac{1}{4}f(x, y, z) \) with \( \gcd(m, \text{disc}(f)) = 1 \). Put \( g = \gcd(x, y, z) \) and \( n = m/g \), so \( n^2 \) is primitively represented by \( f/4 \), and \( \gcd(n, \text{disc}(f)) = 1 \). By the condition (1), \( f/2 \) is improperly primitive. Since \( f/2 = F_{f/2} \) primitively represents \( 2n^2 \), we can apply Theorem 38 of [5] to \( F_{f/2} \), and thus conclude that there is a binary form \( \phi \) with \( F_{f/2} \rightarrow \phi \) and \( \det^D(\phi) = -\Omega_{f/2}2n^2 \). Hence, we see that \( \text{disc}(\phi) = -4\Omega_{f/2}2n^2 = -8\Delta_{f/2}n^2 \); cf. equation (3).
We claim that \( \text{cont}(\phi) \) is either 1 or 4. For this, we apply Proposition 10 of \([13]\) to \( F_{f/2} \) with \( C = 2n^2 \). Note first that \( F_{f/2} = F_{f/4}^B \) is properly primitive by Proposition \([8]\) since \( f/2 \) is improperly primitive. From the relation 
\[
\text{disc}(f/2) = \frac{-\Omega_{f/2}^2\Delta_{f/2}}{2} \quad \text{(cf. equations (11) and (12))}
\]
and \( \gcd(n^2, \text{disc}(f)) = 1 \), we see that \( \gcd(n^2, \Delta_{f/2}\Omega_{f/2}) = 1 \). (Note that \( n \) is odd since \( \gcd(n^2, \text{disc}(f)) = 1 \) and \( \text{disc}(f) \) is even). Thus, we get that \( \gcd(C, 2\Omega_{f/2}\Delta_{f/2}) = 2 \), and we therefore see that \( F_{f/2} \) represents a form \( \phi \) whose discriminant is \(-4\Omega_{f/2}C\), where \( \gcd(C, 2\Omega_{f/2}\Delta_{f/2}) = \gcd(C, 2\Delta_{f/2}\Omega_{f/2}) = 2 \). Hence, it follows from Proposition 10 of \([13]\) that \( \text{cont}(\phi) \) is either 1 or 4 as claimed.

**Case 1:** Assume that \( \text{cont}(\phi) = 1 \).

Since \( \phi \) is primitive, we have by Dirichlet’s theorem (cf. Theorem 9.12 of \([4]\)) that there exists a prime \( p \in R(\phi) \) with \( p \nmid -8\Delta_{f/2}n^2 \). Thus, \( \left( \frac{-8\Delta_{f/2}n^2}{p} \right) = 1 \) by Lemma 2.5 of \([4]\), because \( \text{disc}(\phi) = -8\Delta_{f/2}n^2 \), so we have

\[
\left( \frac{-2\Delta_{f/2}}{p} \right) = 1.
\]

Since \( f/2 \) is improperly primitive, we have that \( 2 \mid \Delta_{f/2} \) by equations (3) and (4). Thus, \( \chi_{-4} \) is an assigned character of \( F_{f/2} \) and \( \chi_{-4}(F_{f/2}) = -\chi_{-4}(\Omega_{f/2}) \) (cf. equation (11)). Moreover, the fact that \( \Delta_{f/2} \) is even implies that \( \frac{\text{disc}(\phi)}{4} = 2\Delta_{f/2}n^2 \equiv 0 \pmod{4} \), so \( \chi_{-4} \) is an assigned character of \( \phi \) by \([4]\, p. 55\). Also, since \( p \in R(\phi) \subset R(F_{f/2}) \), we have that \( \chi_{-4}(p) = \chi_{-4}(\phi) = \chi_{-4}(F_{f/2}) \). Thus, it follows that \( \chi_{-4}(p) = -\chi_{-4}(\Omega_{f/2}) \), and so

\[
\Omega_{f/2}p \equiv 3 \pmod{4}.
\]

Now, note that the condition \( F_{f/2} \rightarrow p \), together with equations (39) and (40), show that the hypotheses of Proposition 30 hold for \( f \). Hence, it follows from this proposition that there exists \((A, \theta)\), where \( A \) is a CM product surface and \( \theta \in \mathcal{P}(A)^\text{ev} \) such that equations (36)-(38) hold for the form \( f' = q_{(A,\theta)} \). In particular, it follows from equations (9) and (36) that \( f/4 \) and \( f'/4 \) have the same basic invariants \( I_1 \) and \( I_2 \). Therefore, if \( \Delta_{f/2} \equiv 2 \pmod{4} \), then we obtain from equations (37) and (38) together with Corollary \([5]\) that \( f/4 \) and \( f'/4 \) are genus-equivalent since \( \chi_8 \notin X(F_{f/4}^B) \) (cf. Table 1). Thus, also \( f \) and \( f' \) are genus-equivalent.

If \( \Delta_{f/2} \equiv 0 \pmod{4} \), then \( \frac{\text{disc}(\phi)}{-4} = 2\Delta_{f/2}n^2 \equiv 0 \pmod{8} \). Thus, \( \chi_8 \) is an assigned character of \( \phi \) by \([4]\, p. 55\). By Table 1 it is also an assigned
character of $F_{f/2}$. On the one hand, because $F_{f/2} \rightarrow \phi \rightarrow p$, we have that
\[
\chi_8(F_{f/2}) = \chi_8(\phi) = \chi_8(p).
\]
On the other hand, we see that
\[
\chi_8(F_{f'/2}) = \chi_8(F_{f/2}) = \chi_8(p)
\]
since $F_{f'/4} = F_{f'/2} \rightarrow p$ by Proposition 30. Thus, it follows that $\chi_8(F_{f/4}) = \chi_8(F_{f'/4})$. Hence, we similarly obtain that $f$ and $f'$ are genus-equivalent in this case as well. Thus, $f$ is equivalent to some refined Humbert invariant by Theorem 1, so the assertion follows in Case 1.

**Case 2:** Assume that $\operatorname{cont}(\phi) = 4$.

First of all, we claim that $\Omega_{F_{f/2}} = \Delta_{f/2} \equiv 2 \pmod{4}$. To see this, observe first that $C = 2n^2 \equiv \Omega_{F_{f/2}} + 4 \pmod{8}$. Indeed, if we had that $C \not\equiv \Omega_{F_{f/2}} + 4 \pmod{8}$, then we would have from equation (5) of [13] that $\phi$ is primitive, which is not possible. Hence, it follows that $C = 2n^2 \equiv \Delta_{f/2} + 4 \pmod{8}$. Also, since $n$ is odd, we have that $C \equiv 2 \pmod{8}$. Thus, the claim follows since $\Delta_{f/2}$ is even by what was proved above.

Secondly, let us put $\phi' := \phi/4$, so $\phi'$ is a primitive form of discriminant $-\Delta_{f/2}n^2$, and so $\operatorname{disc}(\phi')$ is odd. By Corollary 4 of [13], there exist prime numbers $p_1 \equiv 1 \pmod{4}$ and $p_2 \equiv 3 \pmod{4}$ with $p_i \nmid \Delta_{f/2}n^2$ such that $\phi' \rightarrow p_i$ for $i = 1, 2$. Hence, there is a prime number $p$ represented by $\phi'$ such that $p\Omega_{f/2} \equiv 3 \pmod{4}$ (since $\Omega_{f/2}$ is odd by Proposition 3). Since $\phi' \rightarrow p$, it follows that $\left(-\frac{\Delta_{f/2}n^2/2\ p}{p}\right) = 1$ by Lemma 2.5 of [11], and so we get that $\left(-\frac{2\Delta_{f/2}n^2/2\ p}{p}\right) = 1$. Since $4p \in R(\phi) \subset R(F_{f/2})$, we therefore conclude that we are in the situation of Proposition 30 and thus, there is a pair $(A, \theta)$ such that equations (36)-(38) hold for $q(A, \theta)$ and $f$. Since $\chi_8 \not\in X(F_{f/2})$ when $\Delta_{f/2} \equiv 2 \pmod{4}$ (cf. Table 1), it follows from equations (36)-(38) and Corollary 5 that $q(A, \theta)$ and $f$ are genus-equivalent in this case. Thus, the assertion follows by Theorem 1.

**Proof of Theorem 2.** $(\Leftarrow)$ This assertion follows from Theorem 31.

$(\Rightarrow)$ Assume that an imprimitive ternary form $f$ is equivalent to some refined Humbert invariant $q(A, \theta)$. We first see that the condition holds by Proposition 18. Secondly, it follows from Proposition 11 and Corollary 14 that $A$ is a CM product surface and $\theta \in \mathcal{P}(A)^{ev}$. Thus, the condition also holds by Proposition 17, and so the assertion follows.
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