A study on the scattering of matter waves through slits

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Abstract Scattering of matter waves through slits has been explored using the Feynman Path Integral formalism. We explicitly plot the near-zero probability densities to analyse the behaviour near the slit. Upon doing so, intriguing patterns emerge, most notably the braid-like structure in the case of double slits, whose complexity increases as one increases the number of slits. Furthermore, the plot shows the existence of a transition region, where the distribution of near-zero probability points changes from the braided to the fringe-like structure, which has been analysed by explicitly expressing the wavefunction as a hypergeometric function. These patterns are analysed while considering the continuity equation and its consequences for the regions with zero probability density.

1 Introduction

The famed double-slit experiment, introduced originally by Thomas Young in his lectures at the Royal Society in 1802 [1], is unarguably one of the most beautiful experiments ever performed in the history of science. Originally designed to test the corpuscular behaviour of the light advocated by Newton, the double-slit experiment has been performed in the recent times using atoms and electrons to test the Quantum Mechanical principles [2]. Originally thought to be impossible, a version of the experiment has been performed even with a single electron demonstrating the wave properties associated with particles in the microscopic domain [3]. The experiment has taken an important role in the pursuit of understanding the Quantum Mechanics itself, one such example being the well-known delayed-choice experiment [4], and as recently by Aharonov in which he used the idea of double-slit experiment to discuss deterministic perspective of quantum mechanics through Heisenberg picture [5]. Scattering of matter waves composed of large molecules such as \( C_{60} \) from multiple slits has been demonstrated experimentally by Anton Zeilinger and collaborators [6, 7], which demonstrated that the molecule as a whole act as one quantum object. The double-slit experiment thus has a very rich history, yet it is young as ever.

In this paper, we discuss the similar scenario, but with matter waves, which allows us to employ Feynman Path Integral formalism [8]. This method has been used number of times by various authors to describe scattering through slits [9–12]. Here, we present an alternate perspective of the analysis of scattering through slits, where we examine the distribution of points having probability densities so small that they can be considered to be zero. As we shall demonstrate in the subsequent sections, this approach yields surprising insight into the behaviour of such systems in the regions near as well as far away from the slit plane.

Quantum Mechanics in conjugation with the equation of continuity, allows one to think of the evolution of a quantum particle following a well-defined trajectory, although the precise determination of these trajectories require another set of guiding equations. This idea has been used by Bohm in his formalism of Quantum Mechanics, which is often known as Bohmian Mechanics [13]. There exist other formalisms which use the same core idea but follow a different approach to Quantum Mechanics, one prominent example being Quantum-hydrodynamics [14]. In this paper, the trajectory picture that has been adopted (Sect. 3) is assumed to strictly follow the continuity equation.

The paper is organized as follows. A detailed formalism is developed in Sect. 2, along with a few preliminary numerical results. A possible existence of scaling symmetry in the above scenario has been demonstrated there. The notions of Nulls (points with probability density equal to zero) along with Null maps are developed and analysed in Sect. 3. We then present the detailed analytical approach to the present problem including representation of wavefunction as hypergeometric functions as well as vectors on the complex plane defined by Fresnel integral is presented in Sect. 5. The summary of this investigation and conclusions are contained in the last section.

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2 Formalism

Consider a source that generates matter waves. The wave propagates forward and encounters a barrier with slits through which it will scatter and eventually will be incident on the screen. A schematic representation of this setup is shown in Fig. 1.

To analyse the details of the scattering phenomenon, one can attempt to solve the time-dependent Schrödinger’s equation, i.e.

\[ i\hbar \frac{\partial}{\partial t} \psi(x, y, t) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + V(x, y, t) \right] \psi(x, y, t) \]  

(1)

here the potential is infinite at the position(s) of the barrier and is zero elsewhere. Alternatively, one can use the well-known Feynman path integral (FPI) approach where the total transition amplitude is computed by summing over all the paths (see, for example, [15]). These paths have the same amplitudes but they differ in phase, which can be computed from action corresponding to the individual paths. This idea of developing quantum mechanics from the Lagrangian perspective was initially introduced by Dirac [16], but it was Feynman who developed it in its full glory [15] (also, see [8]). Both the FPI approach and Schrödinger’s wave mechanics are known to be equivalent [8, 15, 17]. However, due to the inherent simplicity and intuitive character of the former in the context of the present problem, we have adopted the FPI approach here.

2.1 One dimensional analysis

For the sake of simplicity, one of the spatial directions (namely \(y\)) has been taken to be temporal (see, for example, ref [8, 9]), as illustrated in the Figs. 2 and 3.

If approached through Schrödinger’s equation, it is equivalent to solving the 1-D time-dependent equation:

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \]  

(2)

with the initial wavefunction being the Dirac-delta at \(x = x_0\), which is evolved in time. Then we introduce a sudden perturbation at the time \(t = t_1\), which will kill all the matter present outside the slit. The remnant wavefunction is further evolved in time to obtain the interference pattern due to multiple slits.

Alternatively in the case of FPI, we evolve the wavefunction in a similar manner but at time \(t = t_1\), we only consider the paths that are present inside the slits, rest of the paths are ignored. We shall now present and discuss the relevant details of the mathematical analysis of the scattering process in one-dimensional (1-D) scenario within the FPI framework. Consider a localized particle with mass \(m\) at a position \(x_0\) at time \(t = t_0\). The spatial wavefunction of such a particle is given by (see, for example, ref [18]):

\[ \psi(x, t = t_0) = \delta(x - x_0) \]  

(3)

The evolution of the wave function of the free particle as per Feynman Path Integral approach with a given initial wavefunction \(\psi(x, t_0)\) is given as:

\[ \psi(x_1, t_1) = \int_{-\infty}^{\infty} K(x_1, t_1; x, t_0)\psi(x, t_0) \, dx \]  

(4)
The equation Eq. (10) can easily be employed for numerical computation. It can be shown that the same governing equation applies to the two-dimensional case to a good degree of accuracy, which has been demonstrated in the appendix A. Henceforth, we will proceed with a 1-dimensional study in this paper.

2.2 Preliminary numerical results

As all the parameters in Eq. (10) have either the dimensions of $L$ or $L^2$, we express all the parameters in the units of $\lambda$ and $\lambda^2$, respectively. For the sake of convenience, $\lambda$ is chosen to be 1. The factor $1/\pi$ is omitted from the expression as well, as a constant phase does not impact the probability density at any given point. The simplified expression thus obtained reads:

$$
\psi(x_2, t_2) = \frac{1}{\sqrt{\epsilon' \epsilon''}} \int_{S_0} \exp \left\{ i \frac{\pi}{\lambda} \left( \frac{D'}{\epsilon'} + \frac{D''}{\epsilon''} \right) \right\} dx_1
$$
Fig. 4  Evolution of the probability density through a single slit present at \( x = 0 \) with slit width \( 10 \lambda \). In the bottom map, it is noticeable that the minima is at positions \( x_2 = m \times 100 \), where \( m = \pm 1, \pm 2 \). This conforms to the known relation of minima for single slit diffraction. The colour box corresponding to probability density presented here is in logarithmic scale.

We first demonstrate the diffraction only through a single slit. One expects to have a characteristic central global maximum followed by local maxima on either side. The minima obtained from the diffraction from a slit obey the following relation:

\[
a \sin(\theta) = m\lambda
\]

(12)

\( a \) is the slit-width, \( \theta \) is the angle subtended by the line joining the centre of the slit to the point on the screen with respect to the perpendicular on the slit, \( \lambda \) is the wavelength of the wave, and \( m = \pm 1, \pm 2 \ldots \)

If we observe the region with \( \theta \ll 1 \), then we can approximate Eq. (12) and write

\[
\theta = \frac{m\lambda}{a}
\]

(13)

The position on the screen will be described by \( x_2 = \varepsilon \theta \). Further, upon inputting the values: \( a = 10, \lambda = 1 \) and \( \varepsilon = 1000 \), we obtain

\[
x_2 = m \times 100
\]

(14)

This agrees with the numerical results obtained in Fig. 4.

Similarly, in the case of double slits, the plot of double-slit interference is demonstrated in Fig. 5, which depicts the characteristic evolution of the double-slit pattern. For double slit, the minima obeys the following relation

\[
d \sin(\theta) = \left(m + \frac{1}{2}\right)\lambda
\]

(15)

\( d \) is the distance between the two slits, \( m = 0, \pm 1, \pm 2 \ldots \), and the \( \theta \) is measured from the middle of the two slits. Upon using the same approximation as in the case of the single slit and making substitutions: \( d = 40, \lambda = 1 \) and \( \varepsilon = 1000 \), we obtain

\[
\theta = m \times 25 + 12.5
\]

(16)

Inspection of the two Figs. 4 and 5 reveals conformity with the predicted values of the minima, which gives fair confidence in the validity of the numerical procedures adopted here.
Fig. 5 Evolution of the probability density through the double slit present at \( x = -20 \) and \( x = 20 \). In the bottom map, it is noticeable that the minima is at positions \( x_2 = m \times 25 + 12.5 \), where \( m = 0, \pm 1, \pm 2 \cdots \). This conforms to the known relation of minima for double-slit interference. The colour box corresponding to probability density presented here is in logarithmic scale.

2.3 Scaling symmetry

Here we make an approximation that \( z' \gg x' \). Since \( z' \) is a function of \( t' \) (see Eq. 9), the above condition is equivalent to \( t' \to \infty \). In the case of 2-D, this essentially means that the source has been kept at infinity, thereby obtaining a plane waveform at the slits. Therefore, the first term inside the exponent in Eq. 11 can be dropped. Further, through the evolution beyond the slit plane, \( z' \) is a constant. Thus, it can be safely omitted from the factor outside the integral, without affecting the shape of the wavefunction. Therefore, the simplified form of the integral becomes:

\[
\psi(x_2, t_2) = \frac{1}{\sqrt{z''}} \int_{S'} \exp\left\{i\pi \frac{x'^2}{z''} \right\} dx_1
\]

This situation is similar to the well-known Kirchhoff approximation (for a detailed discussion on the validity of the Kirchhoff’s approximation, see Ref. [19]).

Scaling symmetry can directly be observed from this form. Let the slit-width be \( a \) and the inter-slit distance be \( d \). On preserving the ratio between \( a \) and \( d \), and noting that the set \( S' \) can be defined with \( a \) and \( d \) alone, it follows that scaling them is equivalent to scaling \( x_1 \). Therefore, the form of the integrand is preserved if one scales \( z'' \) as \( x'^2 \). However, recall that \( x'^2 = x_2 - x_1 \), which indicates that if \( x_2 \) is scaled as \( x_1 \), then \( x'^2 \) is scaled as square of the chosen scaling factor. Thus, if one wishes to preserve the shape of the probability density, then \( z'' \) must scale as \( x_2^2 \).

To summarize the argument, if ratio \( a/d \) is preserved, then \( z'': x_2^2 : a^2 \) should also be preserved. This symmetry is demonstrated numerically in Fig. 6.

It is important to note that this scaling symmetry is independent of the \( \lambda \). Thus, it can help extensively to study the close slit structures in macro scales. We can essentially scale the apparatus thus magnifying the structures and making it experimentally perceivable.
3 Behaviour of Null

Zero probability density points, referred to as ‘Null’ in this work, and also known as nodes in literature, are of great importance. They are well-studied in the context of standing waves on strings [20], waveguides [21], cavities [21], Chladni plates [22, 23], Quantum Chemistry [24], Quantum Billiards [23] etc. From a pedagogical perspective, drawing nodal lines is a powerful tool because it allows to quickly sketch the behaviour of a wave qualitatively [25]. In acoustical engineering, study of nodes in specified geometries is important to design setups/rooms that produce nodes at certain places, or prevent formation of nodes altogether at other places [26, 27]. Nodes are also found to be important in computational many-body physics. For example, in the fixed-node Diffusion Monte Carlo technique, an understanding of the structure of nodes can lead to a better estimation of the ground state energies [28]. Given the importance of the nodes/nulls across various disciplines of science and engineering, it is natural to wonder how the null points or boundaries might be forming in scattering through slits. It is known that the middle of the screen is the maximum, which has minima on either side. These minima seemingly appear to be Null; if they extend through the slit plane, then it is only natural to ponder can a trajectory be constructed which can connect from the slit to the central maximum. With this inspiration, it felt only appropriate that we should plot and analyse the points in space that are Null. Before proceeding allow us to analyse some properties akin to Null.

3.1 Properties of Null

Any trajectory picture where the particle number is conserved must satisfy the continuity equation of the following form:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}
\]  

(18)

As the model used for the study is in 1-D. Thus, the appropriate continuity equation will be:

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho \mathbf{v})}{\partial x}
\]  

(19)

Integrating over the domain with the boundary of zero probability density:

\[
\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} \, dx = -\int_{x_1}^{x_2} \frac{d(\rho \mathbf{v})}{dx} \, dx = -\rho \mathbf{v} \bigg|_{x_1}^{x_2} = 0
\]  

(20)
Here $x_2$ and $x_1$ are the boundaries, and in the case of dynamical boundaries, they are the functions of $t$. Using Leibniz integral rule:

$$\frac{d}{dt} \left( \int_{x_1}^{x_2} \rho(x, t) \, dx \right) = \rho(x_2, t) \cdot \frac{dx_2}{dt} - \rho(x_1, t) \cdot \frac{dx_1}{dt} + \int_{x_1}^{x_2} \frac{\partial \rho(x, t)}{\partial t} \, dx$$  \hspace{1cm} (21)

As $x_1$ and $x_2$ are defined, $\rho$ is zero on them at all times, i.e. $\rho(x_1, t)$ and $\rho(x_2, t)$ are zero. Therefore, even if $x_1$ and $x_2$ are functions of time; the first two terms on the right-hand side will be zero.

Hence,

$$\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} \, dx = \frac{d}{dt} \left( \int_{x_1}^{x_2} \rho \, dx \right) = 0$$  \hspace{1cm} (22)

This relation implies that the total probability density enclosed by the boundary having zero probability density remains unchanged. In other words, the trajectories are trapped within.

In the domain of quantum mechanics, one has the notion of the continuity equation derivable from the Schrödinger’s equation, which is given as:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left( \frac{\hbar}{2m} \left[ \psi^* \nabla (\psi) - \psi \nabla (\psi^*) \right] \right)$$  \hspace{1cm} (23)

where $\frac{\hbar}{2m} \left[ \psi^* \nabla (\psi) - \psi \nabla (\psi^*) \right] = j$ is the probability density current.

To get further insight into the probability density current, we write the wavefunction in polar form:

$$\psi = R e^{iS/\hbar}$$  \hspace{1cm} (24)

Both $R$ and $S$ are real functions. The quantity $S$ can be written as:

$$S = -\frac{i\hbar}{2} \ln \left( \frac{\psi}{\psi^*} \right)$$  \hspace{1cm} (25)

Upon taking the gradient of $S$, one obtains:

$$\nabla S = -\frac{i\hbar}{2} \left[ \nabla \frac{\psi}{\psi^*} - \frac{\psi \nabla \psi^*}{\psi^*} \right]$$  \hspace{1cm} (26)

Multiplying with $\rho$ on both sides, we get:

$$\rho \nabla S = -\frac{i\hbar}{2} \left[ \psi^* \nabla \psi - \psi \nabla \psi^* \right] = mj$$  \hspace{1cm} (27)

Therefore, in the trajectory picture, $\nabla S/m$ can be identified with velocity of particles along the possible trajectories and $\nabla S$ as the momentum of the particle, as has been interpreted by Bohm in the Bohmian Mechanics [13]. It is to be noted that the momentum ($\nabla S$) identified here is not the same as the momentum eigenvalue ($\hbar k$) of the system (for detailed discussion look [13, 29]). Thus, $\nabla S = 0$ does not correspond to the state with $\hbar k = 0$.

In the context of probability density, scattering through slits present an interesting case to study. We are aware that far away from the slit, the probability density distribution has consecutive maxima and minima. If one assumes the minima correspond to the Null, then one essentially knows that the probability density is trapped between the consecutive minima. Thus, one can pose the question, how a probability density current must have led the matter inside the trap, and how Null points evolve beyond the slits.

4 Null maps

In this section we present the plots where the loci of Nulls are marked. It has been shown in the previous section that they have an interesting consequence for the trajectory picture. We shall call these loci as Null maps. In the maps presented in this study, the points where the probability density falls below $10^{-14}$ have been marked. To speed up the process, Monte-Carlo sampling has been employed on a CUDA platform [30], in which points are randomly generated using the standard C library function.

First, we investigate the Null maps obtained for the scattering through a single slit with $a = 0.1\lambda$, which is shown in Fig. 7. The sub-$\lambda$ slit width is chosen to reduce the Gaussian points required to perform the numerical integration. However, the result will be identical to scaled slit-widths, if all the other parameters are scaled appropriately as was demonstrated explicitly in the Sect. 2.3 (scaling symmetry).
Fig. 7 The Null map is obtained from the scattering through a single slit with \( a = 0.1\lambda \). Null seems to be originating from a parabola like curve.

Fig. 8 Null map of scattering through two slits of \( a = 0.02\lambda \) and \( d = 40\lambda \) has been shown. The Null appears to form braid like structure closer to the slit plane.

Fig. 9 Null map plotted for same scenario as Fig. 8 but on log scale along \( z'' \). One notices a smooth transition occurring in the range \( \approx 1 \) to \( \approx 8 \) along \( z'' \).

As observed in Fig. 7, the Null maps have the diverging behaviour. However, they do not form closed boundaries, which shows how the probability density can seep into the local maxima, which are surrounded by apparent Nulls.

The Null map of scattering through double slit is demonstrated in Figs. 8 and 9, which reveals intriguing features, most notably are the braids like formation during the convolution of the nulls from the respective slits. At the current plotting scale the braids appear to be closed structures, but, it has been demonstrated in the previous subsection that having closed Null is essentially trapping a trajectory within it. However, in the scattering process, the trapped particle in what essentially looks like a bubble in the configuration space is counter-intuitive. Hence, it will require further investigation. We shall call the region enclosed by the Null boundary a Bubble.¹

¹ One might notice Moiré patterns [31] in Fig. 8 because of the overlaps of (a) limited density of the random point generator, (b) the underlying structure of Nulls, and (c) the chosen resolution of the plot (Fig 9).
Fig. 10 The zoom in of a braid of Fig. 8. The braids which appeared to be closed are found to be perforated.

Upon zooming in Fig. 8 we find that the braids form perforated boundaries (see Fig. 10). However, a closer inspection of the region lying in the middle of the two slits reveals a bubble-like structure. It forms where the braids from the individual slit convolute with the ray of the Null originating from the centre (Fig. 11).

The Null map far away from the slit plane undergoes a transition where the probability density from the individual slits merge together, which has been demonstrated in a log plot (Fig. 9), where transition is seen to be occurring somewhere in the interval $\approx 1$ to $\approx 8$ along $z''$. Analysis to obtain the transition has been done in the next section. Before proceeding into the analysis, the case of increasing the number of slits is demonstrated in set of Fig. 12, where one observes that adding slits makes the structure increasingly smeared and quite peculiar.
Fig. 11 The zoom in the centre of Fig. 8. An interesting structure is revealed in a region whose boundaries appear to be continuous closed nulls

5 Analysis

5.1 Hypergeometric functions

To understand the structures that have been obtained by numerical integration, we evaluate the integral by expanding it into an infinite series. Let us begin with Eq. (17). That is:

\[
\psi = \frac{1}{\sqrt{\pi}} \int_{S'} \exp\left\{ \frac{\pi x''^2}{\sqrt{\pi}} \right\} dx_1
\]  (28)

Although the integral is pretty similar to the Gaussian integral, it has complex argument, which makes it an error function. However, complex error functions are not easy to analyse. It is simpler to expand it into the Taylor series and conduct the analysis. Foremost, we make some simple substitutions. Using \( q = \sqrt{\frac{\pi}{\pi}} x'' \), where \( x'' = x_1 - x_2 \), \( dx_1 = \sqrt{\frac{\pi}{\pi}} dq \). Note that as \( x'' \in \mathbb{R} \), which implies \( q \in \mathbb{R} \). Thus, the above integral becomes:

\[
\psi = \frac{1}{\sqrt{\pi}} \int_{S'} \exp\{iq^2\} dq
\]  (29)
The above figures demonstrate increasing peculiarity for the increasing number of slits.

Fig. 12 The above figures demonstrate increasing peculiarity for the increasing number of slits

The set $S'$ is appropriately scaled with the suggested substitution. Expanding the integrand in Taylor series:

$$\psi = \frac{1}{\sqrt{\pi}} \int_{S'} \sum_{k=0}^{\infty} \frac{(iq^2)^k}{k!} \, dq$$  \hspace{1cm} (30)
And, after evaluating the integral, one gets:

\[ \psi = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} q^{2k+1} \bigg|_{\partial S'} \]  

(31)

Upon separating the real and imaginary parts, we get:

\[ \psi = \frac{1}{\sqrt{\pi}} \left[ \ell \sum_{n=0}^{\infty} (-\pi)^n \frac{q^{4n+3}}{(2n+1)! (4n+3)} + \sum_{n=0}^{\infty} (-\pi)^n \frac{q^{4n+1}}{(2n)! (4n+1)} \right] \bigg|_{\partial S'} \]

(32)

Consider

\[ T_1 = \sum_{n=0}^{\infty} (-\pi)^n \frac{q^{4n+3}}{(2n+1)! (4n+3)} \]

(33)

\[ T_2 = \sum_{n=0}^{\infty} (-\pi)^n \frac{q^{4n+1}}{(2n)! (4n+1)} \]

(34)

The above expression can be written in terms of hypergeometric series. Foremost, we define shifted factorials as: \( \forall \alpha \in \mathbb{C} \) and \( k \geq 0, \ k \in \mathbb{N} \)

\[ (\alpha)_k = \prod_{j=0}^{k-1} (\alpha + j) \]

(35)

\[ (\alpha)_0 = 1 \]

(36)

By this definition, it is easy to see that:

\[ \frac{4n+3}{3} = \left(\frac{7}{4}\right)_n \]

(37)

Identifying:

\[ (2n+1)! = 4^n (3/2)_n n! \]

(38)

\[ (4n+3) = \frac{(7/4)_n}{(1/3)(3/4)_n} \]

(39)

Substituting the above in Eqn: 33 gives us:

\[ T_1 = \frac{q^3}{3} \sum_{n=0}^{\infty} \frac{(3/4)_n (-q^4/4)^n}{n! (3/2)_n (7/4)_n} \]

(40)

The above sum is a hypergeometric series:

\[ _1 F_2(3/4; 3/2, 7/4; -q^4/4) = \sum_{n=0}^{\infty} \frac{(3/4)_n (-q^4/4)^n}{n! (3/2)_n (7/4)_n} \]

(41)

This series is convergent for all \( q \in \mathbb{C} \) [32]. Thus, \( T_1 \) in terms of hypergeometric series is written as:

\[ T_1 = \frac{q^3}{3} _1 F_2(3/4; 3/2, 7/4; -q^4/4) \]

(42)

By similar procedures, \( T_2 \) can also be expressed as:

\[ T_2 = q_1 F_1(1/4; 1/2, 5/4, -q^4/4) \]

(43)

Therefore:

\[ \psi = \frac{1}{\sqrt{\pi}} \left[ q_1 F_1(1/4; 1/2, 5/4, -q^4/4) + \frac{q^3}{3} _1 F_2(3/4; 3/2, 7/4; -q^4/4) \right] \bigg|_{\partial S'} \]

(44)

The hypergeometric functions obtained above are oscillatory for argument less than zero. Consider the case of two slits for simplicity. We assume that the slits are symmetric with respect to the point \( x_1 = 0 \). Then the interval over which integration is carried out can be expressed as \( x_1 \in [-\beta, -\alpha] \cup [\alpha, \beta] \), here, \( -\beta, -\alpha, \alpha \) and \( \beta \) are the edges of the slits. Correspondingly the integration limits are then expressed as:

\[ q_1 = \sqrt{\frac{2\pi}{z^2}} (-\beta - x_2) \]

(45)
Fig. 13 $T_1(q)$ and $T_2(q)$ exhibit oscillatory behaviour, with oscillation period getting shorter for large value of $q$

\begin{align}
q_2 &= \sqrt{\frac{\pi}{z''}}(-\alpha - x^2) \\
q_3 &= \sqrt{\frac{\pi}{z''}}(\alpha - x^2) \\
q_4 &= \sqrt{\frac{\pi}{z''}}(\beta - x^2)
\end{align}

With the hypergeometric analysis, we can also convince ourselves how the Null map is exhibiting peculiarity with increasing number of slits. Let $\alpha_i$ be an edge of a given slit. Hence, $q_i = \sqrt{\frac{\pi}{z''}}(\alpha_i - x^2)$. Therefore, we can express the wavefunction (From Eq. (32)):

$$\psi = \frac{1}{\sqrt{\pi}} \sum_i (-)^i [T_1(q_i) + iT_2(q_i)]$$

Let us consider a path, such that one of the $q_i$ is a constant on it. Let us call the constant $q_k$. Thus one can write:

$$x^2 = \alpha_k - \frac{q_k}{\sqrt{\pi}} \sqrt{z''}$$

Upon substituting $x^2$ thus obtained in the other $q_i$’s we get:

$$q_i = \sqrt{\frac{\pi}{z''}}(\alpha_i - \alpha_k) + q_k$$

For the case of single slit, the $i$ index has only two values. Upon fixing $q_k$, the other $q_i$ varies along the path (Eq. 50) to contribute to the fluctuations. Hence, we observe fairly simple oscillations. It is also interesting to note that the Eq. (50) represent a parabolic path, which can be traced near the origin in Fig. 7. However, upon adding a second slit ($i$ index runs from 1 to 4), and fixing one $q_k$ we are left with three variable $q_i$’s. However, it is not straightforward how having more variable $q_i$’s leads to the peculiar behaviour. This can be understood if we consider that the slit-width is much smaller than inter-slit distance. This consideration implies that the contribution from any individual slit comes from a particular region of $T_1$ and $T_2$ functions. A closer inspection of Fig. 13 reveals that the period of oscillation of $T_1$ and $T_2$ functions varies with $q$. It is known that the superposition of different periods leads to beats; this suggests the reason for observing such a quasi-periodic fluctuation in the probability density as found in Fig. 8. Furthermore, if we include a third slit, we will have a contribution from another set of points, thus, adding another period onto the chosen path. This is what appears to be happening in Fig. 12, which seems to account for the peculiar behaviour that is reported here.

5.2 Fresnel functions

The parametric plot obtained by plotting $T_1$ and $T_2$ yields a Cornu like spiral. The Cornu spiral has been applied extensively in the theory of Fresnel diffraction (see, for example, [33]). With this motivation, we next analyse the patterns obtained here by directly using the Cornu spiral and its properties.
Conventionally the Cornu spiral is a parametric plot generated by using the Fresnel functions, which are defined as:

\[
S(z) = \int_0^z \sin\left(\frac{\pi u^2}{2}\right) \, du \tag{52}
\]

\[
C(z) = \int_0^z \cos\left(\frac{\pi u^2}{2}\right) \, du \tag{53}
\]

\[
S(z) + iC(z) = \int_0^z \exp\left(\frac{i\pi u^2}{2}\right) \, du \tag{54}
\]

These functions are identical to \(T_1\) and \(T_2\) up to an overall multiplicative factor. Therefore, the conclusions drawn using the Cornu spiral are applicable to the case under consideration.

In terms of the Fresnel functions, the wavefunction, up to an overall factor of \(1/\sqrt{2}\), can be expressed as:

\[
\psi = [S(u) + iC(u)] \bigg|_{\alpha' y'} \tag{55}
\]

Expressing the wavefunction in the form of Fresnel functions presents an interesting way of analysis, as it enables us to comment on its behaviour while looking at the Cornu spiral directly. The Fresnel functions along with Cornu spiral for ready reference are shown in Fig. 14.

Given \(u_i = \sqrt{2}(a_i - x_2)/\sqrt{z'}\), the wavefunction is written as:

\[
\psi = \sum_i (-1)^{i+1} [S(u_i) + iC(u_i)] \tag{56}
\]

A bit of reflection reveals that \([S(u_i) + iC(u_i)]\) can be represented as a vector from origin to a point on the Cornu spiral. This representation will imply that \(\psi\) is the sum of vectors on the parametric plot with alternating signs. In order for \(\psi\) to be a Null, these vectors should sum up to zero.

In the case of a single slit, we have only two vectors with opposite signs. For them to form a Null, they are required to be identical. However, every \(u_i\) corresponds to a unique point on the Cornu spiral. Given that the Cornu spiral has no self intersections, it follows that these vectors never add to form a Null. Hence, we conclude that there is no true Null in the case of the single slit. Thus, in Fig. 7, the Null maps are not made of true Nulls but they represent points at which the probability density is smaller than \(10^{-14}\) but non-zero.

In the case of the double slit, we are dealing with four vectors, whose sum has to lead to a Null. We can divide them into two sets, each set containing vectors from each slit. Thus, to have a net Null, vector sum of both sets must yield equal vectors, but with opposite signs, which is not forbidden on the parametric plot. Hence, \(\psi\) can be zero at some points in the case of the double slit (see Fig. 15c for illustration). Thus, Null points mapped in Fig. 8 can be actual Nulls.

Now, we begin the analysis of the bubble found in Fig. 11. First, we explore the exact nature of the bubble at the central slice described by \(x_2 = 0\). This simplifies \(u_i\), i.e. \(u_i = \sqrt{2/z'} a_i\). Since the \(a_i\) are symmetric with respect to the origin, one obtains, \(a_1 = -a_4\) and \(a_2 = -a_3\). Noting further that Fresnel integrals are odd functions, one gets:

\[
S(u_4) + iC(u_4) = -(S(u_1) + iC(u_1)) \tag{57}
\]
(a) Sum of two vectors onto the Cornu spiral will represent the Probability amplitude at a given point after getting scattered through a single slit.

(b) Vectors from origin are represented by only dots for brevity. The Probability amplitude at a given point will be determined by the vector drawn between the two dots as shown above.

(c) In the case of double-slit, there will be four vector dots on the Cornu spiral. The difference in the vectors shown above will determine the probability amplitude at a given point.

**Fig. 15** Vector representation onto the Cornu spiral has been shown in the above figures, and a method of estimating the probability amplitude has been demonstrated.
Fig. 16 The Null map of the single slit whose edges are at $\alpha_1$ and $\alpha_2$ is illustrated above, if another slit is introduced at $-\alpha_1$ and $-\alpha_2$, then the marked slice in the centre will get scaled. However, the relative behaviour along the slice will remain unchanged as it was in the presence of single slit.

Fig. 17 An illustration of the arrangement of vectors near the centre of the viewing plane has been shown above. As they lie in the opposite quadrants, a slight shift from the centre can move the difference vector facing opposite to each other, thus creating a null. A path conserving this relative orientation will form a restrictive boundary for the trajectory.

\[ S(u_3) + iC(u_3) = -(S(u_2) + iC(u_2)) \]  
\[ \psi = 2[(S(u_1) + iC(u_1)) - (S(u_2) + iC(u_2))] \]

Therefore, the wavefunction along the slice $x_2 = 0$ is:

which is identical to the wavefunction of single-slit taken along the same slice (illustrated in Fig. 16). However, it has been demonstrated that no Null exists in the case of a single slit; thus, the horizontal boundary that we encounter on this path is not true Null. Hence, establishing that the bubble observed in Fig. 11 is not completely enclosed, and the probability density flow is permitted through the upper and lower edges of the bubble. We shall call these structures quasi-bubbles for the sake of brevity. Since $\rho$ is a continuous function, having a point at which $\rho$ is nonzero guarantees that it is non-zero over a finite region.

However for sides (Fig. 11d), it can be argued that there can exist a true Null, as demonstrated below.

We proceed by dividing the wavefunction into two parts:

\[ \psi = \psi_l + \psi_r \]

Here, $\psi_l$ and $\psi_r$ represents the wavefunction contributions from the left and right sides, respectively, with respect to the centre of the view plane. For one side $x_2$ will be greater than $\alpha_i$ and for the other side, it will be smaller than $\alpha_i$, thus suggesting that respective vectors of $\psi_l$ and $\psi_r$ are in the opposite quadrants of the parametric plot. In such a case, Null can exist, as it is possible to arrange the four vectors to give a zero (as illustrated in Fig. 17).

6 Bubble and continuity

We were able to show that two sides are permeable, i.e. sides a and d in Fig. 18b. However, the permeability of the other two sides namely b and c is still indecisive. We will carry out the analysis by assuming that the probability densities are zero along the sides b and c.
An example case is schematically illustrated in Fig. 19, in which the probability density at two different times \( t_1 \) and \( t_2 \) has been shown. We assume that \( |\psi(y_1(t))|^2 = |\psi(y_2(t))|^2 = 0 \) for all the values of \( t \). At time \( t = t_1 \),

\[
\int_{y_1}^{y_2} |\psi(y, t)|^2 \, dy = C_1, \quad t = t_1 \tag{61}
\]

As per the quantum equilibrium hypothesis or Born’s rule, \( \rho = |\psi|^2 \) at any given position and time. If the quantity \( \rho \) is identified with the particle density (for example considering ensemble interpretation of the current \([18]\)), it must follow the continuity equation.

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{62}
\]

for which it has been shown in Sect. 3.1 that:

\[
\frac{d}{dt} \left( \int_{V_t} \rho \, dV \right) = 0 \tag{63}
\]

Here, \( V_t \) is the boundary which can be a function of time, but \( \rho \) is zero on the time variable boundary. However, at another time \( t = t_2 \),

\[
\int_{y_1}^{y_2} |\psi(y)|^2 \, dy = C_2, \quad t = t_2 \tag{64}
\]

If \( C_1 \neq C_2 \), which is apparent from Fig. 18a and illustrated in Fig. 19, implies that:

\[
\frac{d}{dt} \left( \int_{y_1}^{y_2} |\psi(y)|^2 \, dy \right) \neq 0 \tag{65}
\]

which contradicts Eq. 63. One conclusion can be that Quantum Equilibrium Hypothesis or Born’s rule cannot hold in the given scenarios, i.e.

\[
|\psi(y, t)|^2 \neq \rho(y, t) \tag{66}
\]

**Fig. 18** Figure a shows the bubble as has been obtained from the numerical integration. Figure b shows the Null map of the same region. The bubble appears to be closed. The integration is exact for 1-D, where the \( z' \) is the temporal direction. Thus, one observes the change in probability density with time which is enclosed between the Null

**Fig. 19** An illustration where the \( |\psi|^2 \) is changing between the nulls, as appears to be happening in Fig. 18a
However, the continuity equation is directly derivable from the Schrödinger equation, as discussed in Sect. 3.1, which means that no situation satisfying the Schrödinger equation can violate the equation of continuity. Further, it is well known that Feynman Path Integral formalism and Schrödinger’s wave mechanics are equivalent [8, 18], thus, only logical conclusion is that the side boundaries are not closed as well.

In fact, a careful numerical analysis of the above scenario reveals that the probability density at the Null boundary is quite small but finite and about six orders of magnitude less than the probability density at the peak of the bubble.

This prompts one to explore the flow of probability density at the Null boundary. The glimpse of the change with time is shown in Fig. 20, where the light region represent the increment in probability density with time, whereas the dark region represents the decrement in probability density. It can be observed that the null lies at the transition of the two regions and can be stated to have stationary probability density. However, this region clearly requires more detailed analysis, for example probing the behaviour of current in the region, which is presently underway and will be reported later.

7 Summary and conclusions

The scattering of matter waves through slits is not a new phenomenon and has been discussed and studied extensively throughout the history. However, closer inspection of probability density near the slit plane reveals intriguing structures, which seems to not to have been studied in detail in the literature.

To study such a rich structure, we seek to look at the evolution of probability densities through a different perspective. We plot regions with near-zero probability density in the quest to get any insights into the structures found near the slits. Upon plotting Null maps, peculiar structures were revealed in the multi-slit scenario. Most notably, braids were observed in the presence of double slit, whose complexity was found to increase with increasing number of slits.

The Null map appears to have a transition zone where braided structure disappears and fringe-like structure makes its appearance as seen in Fig. 9. It has been demonstrated explicitly that the origin of transition zone can be understood on the basis of the hypergeometric structure of the wavefunction. By making use of the oscillatory nature of the hypergeometric function, the transition region has been estimated, which is in agreement with the one obtained by explicit numerical integration.

Upon a closer examination probability density in the near slit region, we found existence of regular structures separated by boundaries of zero or near zero probability densities, which we called bubbles. As per the trajectory picture, the probability density should be trapped in a bubble, as it must follow the continuity equation. A detailed analysis using Cornu spiral revealed that the two edges (namely a and d in Fig. 18) have to be permeable. A simple argument based on the continuity equation suggested that the side boundaries (namely b and c in Fig. 18) have to be permeable as well.

The study of scattering through Null maps provides an intricate perspective to analyse certain details of the system, which are otherwise difficult to visualize. Specifically, this analysis has revealed the existence of intricate behaviour of the otherwise well-studied multi-slit systems.

In the future, it will be interesting to study evolution of probability current density around the “quasi-Bubbles” reported in this study, whose size can easily be controlled by changing parameters of the system suitably and can be made much larger than the wavelength used. The increasing “fuzziness” of the Null maps in the case of multi-slit scenario opens the possibility of studying the existence of complexity in such systems. Investigations along these lines are under progress.

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Appendix A: Two-dimensional analysis

One can attempt to write the wavefunction in two-dimensional case (2-D) by solving the time-dependent Schrödinger’s equation with an appropriate boundary condition at \( y = y_1 \) describing the slit and the barrier. Alternatively, one may proceed with a similar set of arguments as discussed in Sect. 2.1 using the FPI and demonstrate how similar it is to the 1-D wavefunction.

For the initial state, consider a collapsed state:

\[
\psi(x, y, t = t_0) = \delta(x - x_0)\delta(y - y_0)
\]  

(67)

The kernel in 2-D is slightly different from that in 1D. By taking Lagrangian of free particle in 2-D, i.e.

\[
\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)
\]  

(68)

and following the procedure as given in [8, 18], one obtains the following Kernel for a particle going from position \((x_a, y_a)\) at time \(t_a\) to \((x_b, y_b)\) at time \(t_b\):

\[
K(b, a) = \frac{m}{2\pi\hbar(t_b - t_a)} \exp\left\{ \frac{im[(x_b - x_a)^2 + (y_b - y_a)^2]}{2\hbar(t_b - t_a)} \right\}
\]  

(69)

Introducing a slit at a specific value of \(y\), let us choose it to be \(y = y_1\), then at the position of the slits, the wavefunction will be

\[
\psi(x_1, y_1, t_1) = \frac{m}{2\pi\hbar(t_1 - t_0)} \exp\left\{ \frac{im[(x_1 - x_0)^2 + (y_1 - y_0)^2]}{2\hbar(t_1 - t_0)} \right\}
\]  

(70)

It is not straightforward to compute the evolution beyond the slit in higher dimensional cases. One reason is that one has to consider the paths contributed from the wavefunction present before the slit, and there is a possibility of backflow of paths as well as looped paths around the slits [10]. In order to understand the salient feature of the scenario, the analysis can be carried out by introducing certain approximations, leading to a considerable simplification of the analysis.

Note that the initial condition (collapsed state) is represented by Dirac delta function, by the uncertainty principle the wavefunction is in the superposition of all the possible momentum states. However, our desired state of analysis in 2-D is a plane-wave having a specific momentum \(p\) corresponding to wavelength \(\lambda\), given by the de Broglie relation \(p = h/\lambda\). One proposed solution is to slice the wavefunction originating from the collapsed state only to select the part which has the effective wavelength \(\lambda\). Consider the kernel of the free particle:

\[
K = F(t) \exp\left\{ \frac{imy^2}{2\hbar t} \right\}
\]  

(71)

\(F(t)\) is the normalization function, which depends on the dimensions of the space in the system. To find the effective wavelength, we propose to increment \(y\) by \(\lambda\). This increment should introduce a phase difference of \(2\pi\) in the argument of the complex exponential. Mathematically it is expressed as:

\[
2\pi = \frac{m(y + \lambda)^2}{2\hbar t} - \frac{my^2}{2\hbar t} = \frac{my\lambda}{\hbar t} + \frac{m\lambda^2}{2\hbar t}
\]  

(72)

If one goes sufficiently far away from the point of origin such that \(y \gg \lambda\), then one can essentially ignore \(\lambda^2\) contribution. Therefore:

\[
\lambda = \frac{2\pi\hbar}{my/t}
\]  

(73)
We can interpret this form as following: If one takes the ratio \( y/t \) as constant, one is essentially tracing the sliced wavefunction whose effective wavelength is \( \lambda \). Therefore, we look only at those paths that conserve this ratio. This idealization allows us to deal with the issue of backflow, as the selected paths encounter the slit only once at specific \( y/t \). Using this ratio, we can make the substitution.

\[
t = \frac{\hbar m_y}{2\pi \hbar}
\]  

The above relation strikingly resembles the dimensional substitution we made in 1-D (Eq. 9). Thus, we can proceed to write the evolution wavefunction beyond the slit at the point \((x_2, y_2, t_2)\).

\[
\psi(x_2, y_2, t_2) = \int_{S'} K(b, a) \psi(x_1, y_1, t_1) \, dx_1
\]  

The kernel \( K(b, a) \) and wave function \( \psi(x_1, y_1, t_1) \) appearing here in this expression has been explicitly define in Eqs. (69) and (70) respectively, and \( S' \) is the set of points lying inside the slit. Upon changing the variables as \( x' = x_1 - x_0, y' = y_1 - y_0, t' = t_1 - t_0, x'' = x_2 - x_1, y'' = y_2 - y_1 \) and \( t'' = t_2 - t_1 \), Eq. (75) becomes;

\[
\psi(x_2, y_2, t_2) = \int_{S'} F(t') \exp\left\{ \frac{i m(x'^2 + y'^2)}{2\hbar t'} \right\} 
\]

\[
F(t'') \exp\left\{ \frac{i m(x''^2 + y''^2)}{2\hbar t''} \right\} \, dx_1
\]  

Now, making use of the fact that \( y/t \) is a constant (Eq. 73), we get:

\[
\psi(x_2, y_2, t_2) = \tilde{F}(y') \tilde{F}(y'') \exp\left\{ \frac{t \pi}{\lambda} \left( y' + y'' \right) \right\}
\]

\[
\int_{S'} \exp\left\{ \frac{t \pi}{\lambda} \left( \frac{x'^2}{y'} + \frac{x''^2}{y''} \right) \right\} \, dx_1
\]  

The factors \( \tilde{F}(y') \) and \( \tilde{F}(y'') \) are obtained by transforming \( F(t') \) and \( F(t'') \), respectively, under the transformation \( t \mapsto y \). Further substituting

\[
T(y', y'') = \tilde{F}(y') \tilde{F}(y'') \exp\left\{ \frac{t \pi}{\lambda} \left( y' + y'' \right) \right\}
\]

the wavefunction gets the similar form as was in the case of 1-D.

\[
\psi(x_2, y_2, t_2) = T(y', y'') \int_{S'} \exp\left\{ \frac{t \pi}{\lambda} \left( \frac{x'^2}{y'} + \frac{x''^2}{y''} \right) \right\} \, dx_1
\]  

This suggests that the shape of probability density obtained in the idealized 2-D case is identical to that in 1-D.

A1: Steady-state analysis of the 2-D case

For a steady-state solution, the probability density must have ‘settled’ in the region and will not change with time. This will be equivalent to solving the time-independent Schrödinger’s equation with appropriate boundary conditions at \( y = y_1 \), which can further be written in the form of the Helmholtz equation

\[
\frac{d^2 \psi}{dx^2} = -\frac{d^2 \psi}{dy^2} - k^2 \psi
\]  

and the relevant boundary condition would be a Sommerfeld condition (see, for example, [34] for details).

For the case of FPI, in order to introduce a steady-state, one needs to have a continuously emitting source which can be implemented through a boundary condition as follows

\[
\psi_0(x, y_0, t) = \delta(x - x_0) \exp\left\{ i \frac{Et}{\hbar} \right\}
\]  

Here, \( E \) is the energy of the non-interacting particles that are introduced into the system.

With this modification, the source is continuously injecting particles in the system. Therefore, to compute the probability density at a point in space, it is required to consider the \( \psi \) introduced at previous times as well. To simplify the analysis we employ the time slicing method, where the entire time-interval \([0, \infty)\) has been sliced into countable number of bins, each with a fixed size \( \epsilon \).
Let \( l \) be the time elapsed since a wavefunction has been introduced to the system, where \( l \) is a positive integer. For the sake of brevity, we use \( D = x^2 + y^2 \), here the symbols have the usual meaning as defined before. Hence we can write the steady-state wavefunction as:

\[
\psi(D) = \sum_{l=1}^{\infty} \psi(D, l\epsilon)
\]  

(81)

where for a free particle,

\[
\psi(D, l\epsilon) = F(l\epsilon) \exp \left( \frac{imD}{2h\epsilon} \right) \exp \left( \frac{iE_l\epsilon}{\hbar} \right)
\]  

(82)

Noting that \( F(l\epsilon) \propto 1/l\epsilon \), one can effectively ignore wavefunctions for which time elapsed is large such that \( F(l\epsilon) \approx 0 \). Also, the wavefunctions corresponding to the very small time elapses will have very rapid variations of phase. Thus, their overall contribution will be effectively negated.

Consider a wavefunction for which time elapsed is \( t' \) and look at the wavefunction from neighbouring times, i.e. \( t' \pm \Delta t \), where \( \Delta t \ll t' \). These are given by:

\[
\psi(D, t' \pm \Delta t) = F(t' \pm \Delta t) \exp \left( \frac{imD}{2h(t' \pm \Delta t)} \right) \exp \left( \frac{iE(t' \pm \Delta t)}{\hbar} \right)
\]  

(84)

Since \( \Delta t \ll t' \), we can write

\[
\frac{1}{t' \pm \Delta t} \simeq \frac{1 \pm (\Delta t/t' )}{t'}
\]

(85)

Given this and the fact that \( F \) is inversely proportional to \( t \) we can write

\[
F(t' \pm \Delta t) \simeq F(t') \left[ 1 \mp (\Delta t/t') \right].
\]

Using these approximations, one obtains:

\[
\psi(D, t' \pm \Delta t) = F(t') \left[ 1 \mp (\Delta t/t') \right] \exp \left( 2mD t'/2h \right) \exp \left( \frac{iE(t' \pm \Delta t)}{\hbar} \right)
\]  

(86)

After a simple rearrangement of terms, we get:

\[
\psi(D, t' \pm \Delta t) = F(t') \exp \left( \frac{imD}{2ht'} \right) \exp \left( \frac{iE(t' \pm \Delta t)}{\hbar} \right)\left[ 1 \mp (\Delta t/t') \right] \exp \left( \mp \frac{t(\Delta t/t')}{\hbar} \left[ \frac{mD}{2t'} - E(t' \pm \Delta t/t') \right] \right)
\]

(87)

Evidently it can be re-expressed using Eq. (83) as:

\[
\psi(D, t' \pm \Delta t) = \psi(D, t') \left[ 1 \mp (\Delta t/t') \right] \exp \left( \mp \frac{i(\Delta t/t')}{\hbar} \left[ \frac{mD}{2t'} - E(t') \right] \right)
\]  

(88)

Recall that \( E \) is the energy of the particles emitted by the source and is a constant. In the classical scenario, it is given as \( mv^2/2 \). Furthermore, \( v \) can be taken as the constant ratio of position and time variable. If we take \( v \) to be \( y'/t' \), then it is evident to write down \( Et' = mv^2/2t' \). Under these conditions, Eq. (87) can be written as:

\[
\psi(D, t' \pm \Delta t) = \psi(D, t') \left[ 1 \mp (\Delta t/t') \right] \exp \left( \mp \frac{i(\Delta t/t')}{\hbar} \left( \frac{mv^2}{2t'} \right) \right)
\]

(89)

If the argument inside the exponent is taken to be small as well, given \( x' \) is comparable with \( y' \), then it is possible to write:

\[
\psi(D, t' \pm \Delta t) = \psi(D, t') \left[ 1 \mp (\Delta t/t') \right] \left[ 1 \mp \frac{i(\Delta t/t')}{\hbar} \left( \frac{mx'^2}{2t'} \right) \right]
\]

(90)

Expanding the square brackets and retaining the terms only up to first-order in \( (\Delta t/t') \), one gets:
Therefore,
\[ \psi(D, t' + \Delta t) + \psi(D, t' - \Delta t) \approx 2\psi(D, t') \]  
(91)

Hence, we can write using Eq. (81):
\[ \psi(D) \approx k\psi(D, t') \]  
(92)

With the condition that \( y'/t' \) is constant, such that they are related to energy as, \( E = mv^2/2t'^2 \). Therefore, one concludes that the idealization of the 2-D case that has been developed in the previous sub-section is a good approximation to the steady state scenario.

A2: Validity of the 1-D approximation in the near field region

In the steady state analysis (Appendix A.1), we used the approximation that the \( x' \) and \( y' \) are comparable. However, at the location of bubbles \( x' \gg y' \) (for the description of the bubble refer to Sect. 4). Therefore, one cannot approximate Eqs. (88, 89).

To check the validity of the approximation in the near field region, we have to proceed with the Eq. (88) without the linear approximation.

Consider:
\[ T = \exp\left\{ \frac{i(\Delta t/t') mx'^2}{\hbar} \right\} \]  
(93)
\[ G = \psi(D, t' + \Delta t) + \psi(D, t' - \Delta t) \]  
(94)

Therefore, using the Eq. (88) we can write:
\[ G = \psi(D, t') \left[ 1 - \frac{(\Delta t/t')}{T^*} \right] T + \left[ 1 + \frac{(\Delta t/t')}{T^*} \right] T \]  
(95)
\[ \therefore G = \psi(D, t') \left[ (T + T^*) + \frac{\Delta t}{t'} (T - T^*) \right] \]  
(96)

Assume the argument inside the exponential (Eq. 93) to be \( \tilde{T} \), that is:
\[ \tilde{T} = \frac{(\Delta t/t')}{t'^*} \]  
(97)
\[ \Rightarrow G = \psi(D, t') \left[ \cos(\tilde{T}) + \frac{\Delta t}{t'} \sin(\tilde{T}) \right] \]  
(98)

It is clear that the neighbouring wavefunctions are not simply adding up to the 1-D wavefunction, but have an additional factor to them. To see the cumulative effect of all the neighbouring wavefunctions, we must integrate them over \( \frac{\Delta t}{t'} \). Foremost, substituting
\[ z = \frac{\Delta t}{t'}; k = \frac{mx'^2}{2\hbar t'} \]  
(99)

Implying that the sum described in Eq. (81) can be written as:
\[ \Rightarrow \psi(D) = \int_{0}^{a} Gdz \]  
(100)

Here, \( a \) is upper limit to the integral, which is less than 1 and greater than zero.
\[ \therefore \psi(D) = \psi(D, t') \int_{0}^{a} \cos(kz) + iz \sin(kz)dz \]  
(101)

Upon evaluating the integral we will get:
\[ \therefore \psi(D) = \psi(D, t') \left[ \frac{1}{k} \left[ \sin(kz) + \frac{t}{k} \left[ \sin(kz) - kz \cos(kz) \right] \right] \right]_{0}^{a} \]  
(102)

Observing the above equation we can comment that all the zeroes of the \( \psi(D) \) will be identical to that of the zeroes of \( \psi(D, t') \), as the other part of the expression does not have any zeroes. Thus, the Null map for the case of a single source will be considered identical to that obtained in the 1-D formalism. When considering the double slit or dual source, the additional factor will change the Null map in the region. However, near the middle of the slits, the map will nearly be the same as that of 1-D, for similar reasons as discussed in Sect. 5.2.
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