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STONE-WEIERSTRASS THEOREM

G. LA VILLE and I.P. RAMADANOFF

Abstract.- It will be shown that the Stone-Weierstrass theorem for Clifford-valued functions is true for the case of even dimension. It remains valid for the odd dimension if we add a stability condition by principal automorphism.

Introduction.- Recall the classical Stone-Weierstrass theorem : let $Y$ be a metric space, $C(Y;\mathbb{R})$ the set of all continuous functions from $Y$ in $\mathbb{R}$, $B \subset C(Y;\mathbb{R})$ a subset such that $B$ contains the constant function 1 and separates the points of $Y$. Then the algebra $A_B(Y;\mathbb{R})$, generated by $B$ is dense in $C(Y;\mathbb{R})$ for the topology of the uniform convergence on every compact.

It is well-known that if one substitutes the field $\mathbb{R}$ by $\mathbb{C}$, then an additional hypothesis is needed, namely : $B$ should be stable with respect to complex conjugation. In case we are omitting this hypothesis and if we take, for example, $Y$ to be an open subset of $\mathbb{C}$ and $Y = \{1, z\}$, then we will get the algebra of holomorphic functions.

Let us mention that the case of functions taking values in the quaternionian field is known [2] and it is analogous to the real case.

Here, we will investigate the situation when $\mathbb{R}$ is replaced by $\mathbb{R}_{p,q}$ - an universal Clifford algebra of $\mathbb{R}^n$, $n = p+q$, with a quadratic form of signature $(p,q)$. This study is motivated by the theory of monogenic functions [1]. The present paper is organized as follows : in the $\S$1 we will recall some notations usually employed in Clifford algebras. The $\S$2 will deal with some elements of combinatorics. The essential part of the paper in the $\S$3 in which we give a formula allowing to compute the scalar part of a given Clifford number. As an application of this formula, we are able to prove in $\S$4 the following Stone-Weierstrass theorem for $C(Y;\mathbb{R}_{p,q})$:

Theorem.- Let $Y$ be a metric space and $C(Y;\mathbb{R}_{p,q})$ the set of all continuous functions from $Y$ to $\mathbb{R}_{p,q}$. Let $B \subset C(Y, \mathbb{R}_{p,q})$ be such that $B$ contains the constant function 1 and separates the points of $Y$. If $p+q$ is odd, suppose in addition that $B$ is stable with respect to the principal automorphism $\ast$. Then, the algebra $A_B(Y;\mathbb{R}_{p,q})$, generated by $B$, is dense in $C(Y;\mathbb{R}_{p,q})$ for the topology of uniform convergence on compact sets.
1. Notations

In a Clifford algebra $\mathbb{R}_{p,q} = C_0 \oplus C_1 \oplus \ldots \oplus C_n$, with $n = p + q$, the spaces $C_0, C_1, \ldots, C_n$ are supposed to be of respective basis $\{1\}, \{e_1, e_2, \ldots, e_n\}, \{e_{ij}\}_{i<j}, \ldots, \{e_{i_1\ldots i_k}\}_{i_1 < i_2 < \ldots < i_k}$, where $(i_1, \ldots, i_k)$ is a multi-index with $i_1, \ldots, i_k \in \{1, \ldots, n\}$, $1 \leq i_1 < \ldots < i_k \leq n$. The algebra obeys to the laws:

\[
\begin{align*}
e_i^2 &= 1, & i &= 1, \ldots, p \\
e_i^2 &= -1, & i &= p + 1, \ldots, n \\
e_i e_j &= -e_j e_i, & i &\neq j \\
e_{i_1\ldots i_k} &= e_{i_1} e_{i_2} \ldots e_{i_k}, & \text{for } i_1 < i_2 < \ldots < i_k
\end{align*}
\]

We will make use of the decomposition of a Clifford number $a$ in its scalar (real) part $\langle a \rangle_0$, its 1-vector $\langle a \rangle_1 \in C_1$, its bivector part $\langle a \rangle_2 \in C_2$, etc ... up to its pseudo-scalar part $\langle a \rangle_n \in C_n$, i.e:

$$a = \langle a \rangle_0 + \langle a \rangle_1 + \cdots + \langle a \rangle_n,$$

where:

$$\langle a \rangle_k = \sum_{|J| = k} a_J e_J.$$

Where $J = (j_1, \ldots, j_k)$ is a multi-index and $|J| = k$, $e_J = e_{j_1} \cdots e_{j_k}$.

Recall that the principal involution $*$, the anti-involution $^{*}$ and the reversion $\sim$ act on $a \in \mathbb{R}_{0,n}$ as follows:

$$a_* = \sum_{k=0}^{n} (-1)^k \langle a \rangle_k$$

$$a^{*} = \sum_{k=0}^{n} (-1)^{k(k+1)/2} \langle a \rangle_k$$

$$a^\sim = \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle a \rangle_k$$

Now, define

$$e^i = \begin{cases} 
e_i, & \text{if } 1 \leq i \leq p \\ -e_i, & \text{if } p + 1 \leq i \leq p + q \end{cases}$$

and $e^j = e_{j_k} \cdots e_{j_1}$.
2. Some combinatorics

Let us study the partition of the set \( \{1, \ldots, n\} \) in two strictly ordered subsets: \( I = \{i_1, \ldots, i_k\} \) and \( J = \{j_1, \ldots, j_p\} \). As for as the relative position of \( J \) with respect to \( I \) is concerned, we have different possible cases: \( J \cap I = \emptyset \); just one \( j_\alpha \) belongs to \( I; \ldots; \ell \) among the \( j_\alpha' \)'s belong to \( I; \ldots; \); the largest possible number of \( j_\alpha' \)'s belongs to \( I \). It is easy to compute the cardinals of the corresponding sets:

For the first case, the cardinal is \( \binom{p}{n-k} \binom{n}{k} \binom{0, p \,(n-k)}{\sup} \). If just one \( j_\alpha \) belongs to \( I \), then we will have \( \binom{p-1}{n-k} \binom{0, p \,(n-k)}{+1} \) and so on . . . In the last case, we will get \( \binom{0}{n-k} \binom{\inf}{p,k} \).

Now, recall the following result which is well-known in classical probability theory [3]:

**Lemma 1.-** For every \( k \), \( 0 \leq k \leq n \):

\[
\sum_{\ell=\sup\{0, p \,(n-k)\}}^{\inf\{p,k\}} \binom{p-\ell}{n-k} \binom{\ell}{k} = \binom{p}{n}.
\]

*In fact, this lemma will not be used here, but its elementary proof, which will be given below, is a source of inspiration for the next result (Lemma 2).*

**Proof of Lemma 1** – For every \( k \), \( 0 \leq k \leq n \), one has \( (1 + x)^{n-k}(1 + x)^k = (1 + x)^n \), which involves

\[
\sum_{\ell=0}^{k} (1 + x)^{n-k} \binom{\ell}{k} x^\ell = \sum_{p=0}^{n} \binom{p}{n} x^p,
\]

and again:

\[
\sum_{\ell=0}^{k} \sum_{n=0}^{n-k} \binom{n}{n-k} x^n \binom{\ell}{k} x^\ell = \sum_{p=0}^{n} \binom{p}{n} x^p.\]

Let us set \( n + \ell = p \), i.e. \( n = p - \ell \).

Then the double sum is equal to

\[
\sum_{\ell=0}^{k} \sum_{p=\ell}^{n-k+\ell} \binom{p-\ell}{n-k} \binom{\ell}{k} x^p = \sum_{p=0}^{n} \sum_{\ell=\sup\{0, p \,(n-k)\}}^{\inf\{p,k\}} \binom{p-\ell}{n-k} \binom{\ell}{k} x^p.
\]

It just remains to indentify the coefficients of \( x^p \). Now, we are in a position to formulate and prove the following:

**Lemma 2.-**

\[
\sum_{p=0}^{n} \sum_{\ell=\sup\{0, p \,(n-k)\}}^{\inf\{p,k\}} (-1)^{p+k+\ell} \binom{p-\ell}{n-k} \binom{\ell}{k} = \begin{cases} 
0, & \text{if } 1 \leq k \leq n - 1 \\
0, & \text{if } k = n, n \text{ even} \\
2^n, & \text{if } k = n, n \text{ odd} \\
2^n, & \text{if } k = 0.
\end{cases}
\]
Proof of Lemma 2 — Start from \((1 + (-1)^k x)^{n-k} (1 + (-1)^{k+1} x)^k = \)

\[
\sum_{\ell=0}^{k} (1 + (-1)^k x)^{n-k} (-1)^{k+1} \ell C_{k \ell} x^\ell =
\]

\[
= \sum_{\ell=0}^{n-k} \sum_{n=0}^{k} (-1)^{kn} C_{n-k}^n x^n (-1)^{k+1} \ell C_{k \ell} x^\ell =
\]

\[
= \sum_{p=0}^{n} \inf\{p, k\} \sum_{\ell=sup\{0, p-(n-k)\}}^{p\ell} (-1)^{pk+\ell} C_{n-k}^p C_k^\ell x^p,
\]

because \(kn + (k + 1)\ell = pk + \ell\). Thus it is enough to set \(x = 1\) and remark that:

\[
(1 + (-1)^k)^{n-k} (1 + (-1)^{k+1})^k = \begin{cases} 2^n, & \text{if } k = 0 \\ 0, & \text{if } 1 \leq k \leq n-1 \\ 2^n, & \text{if } k = n, n \text{ odd} \\ 0, & \text{if } k = n, n \text{ even} \end{cases}
\]

\[\Box\]

3. A formula for the real part of \(a \in \mathbb{R}_{p,q}\)

**Lemma 3.** — For every multiindice \(J\), we have \(e_J e^J = 1\).

**Lemma 4.** — Let \(I = (i_1, \ldots, i_k), \ |I| = k.\) \(J = (j_1, \ldots, j_p),\) \(|J| = p\) there is the following equality

\[
\sum_{p=0}^{n} \sum_{|J|=p} e_J e_I e^J = \begin{cases} 2^n, & \text{if } k = 0 \text{ or if } k = n \text{ with } n \text{ odd} \\ 0, & \text{in other cases} \end{cases}
\]

**Proof** — Decompose the sum

\[
\sum_{|J|=p} e_J e_I e^J
\]

following the relative position of \(J\) with respect to \(I\). If \(J \cap I = \phi\) we have \(C_{n-k}^p C_k^0\) such possibilities and the anticommutation gives \((-1)^{pk}\).

If only one \(j_\alpha \in I\) we have \(C_{n-k}^{p-1} C_k^1\) such possibilities and the anticommutation gives \((-1)^{(p-1)k} (-1)^{k-1}\) and so on. \ldots, if \(\ell j_\alpha \in I\) we have \(C_{n-k}^{(p-\ell)k} C_k^\ell\) such possibilities and the commutation gives \((-1)^{(p-\ell)k} (-1)^{\ell(k-1)}\).
The sum is equal to
\[
\inf_{\{p,k\}} \sum_{\ell = \sup\{0, p-(n-k)\}} (-1)^{(p-\ell)k} (-1)^{\ell(k-1)} C_{n-k}^{p-\ell} C_{k}^{\ell} e_I
\]

Thus we could apply lemma 2 and the result follows.

The next result is a formula for the scalar part of a Clifford number.

**Theorem 1.-** Let \( a \in \mathbb{R}_{p,q} \). Then :

a) if \( n \) is even,
\[
\langle a \rangle_0 = \frac{1}{2^n} \sum_{p=0}^{n} \sum_{|J|=p} e_J a e^J.
\]

b) if \( n \) is odd,
\[
\langle a \rangle_0 = \frac{1}{2^{n+1}} \sum_{p=0}^{n} \sum_{|J|=p} e_J a e^J + \frac{1}{2^{n+1}} \sum_{p=0}^{n} \sum_{|J|=p} e_J a^* e^J.
\]

**Proof** – When \( a \in \mathbb{R}_{0,n} \), then
\[
a = \sum_{k=0}^{n} \sum_{|I|=k} a_I e_I,
\]
where \( I = (i_1, \ldots, i_k), \ 1 \leq i_1 < i_2 < \ldots < i_k \leq n \). Take the sum
\[
\sum_{p=0}^{n} \sum_{|J|=p} e_J a e^J = \sum_{J \subseteq I} \sum_{I} a_I e_J e_I e^J.
\]

Now, apply lemma 4 :

a) if \( n \) is even, one gets :
\[
\sum_{p=0}^{n} \sum_{|J|=p} e_J a e^J = 2^n \langle a \rangle_0,
\]

b) if \( n \) is odd, one has :
\[
\sum_{p=0}^{n} \sum_{|J|=p} e_J a e^J = 2^n \langle a \rangle_0 + 2^n \langle a \rangle_n.
\]
But, in the case when \( n \) is odd, \( \langle a_+ \rangle_n = (-1)^n \langle a \rangle_n = -\langle a \rangle_n \). Thus, we get the part b) of the theorem.

\[\Box\]

**Remark.**- For \( n = 1 \), the preceding formula becomes to
\[4Re a = (a - iai) + (\alpha - i\alpha i)\] in \( \mathbb{R}_{0,1} = \mathbb{C} \) with the classical notations of \( \mathbb{C} \).

For \( n = 2 \), this means that \( 4Re a = a - iai - jaj - kank \) in \( \mathbb{R}_{0,2} = \mathbb{H} \) with the classical notations of \( \mathbb{H} \) \[2\].

**4. The Stone-Weierstrass theorem for \( C(Y; \mathbb{R}_{p,q}) \).**

**Theorem 3.**- Let \( Y \) be a metric space and \( C(Y; \mathbb{R}_{p,q}) \) the set of continuous functions from \( Y \) into \( \mathbb{R}_{p,q} \). Let \( B \subset C(Y; \mathbb{R}_{p,q}) \) be such that \( B \) contains the constant function 1 and separates the points of \( Y \). When \( p + q \) is even, nothing more is supposed. If \( p + q \) is odd, suppose \( B \) be stable with respect to the principal involution \( * \).

Then, the algebra \( A_B(Y; \mathbb{R}_{p,q}) \), generated by \( B \), is dense in \( C(Y; \mathbb{R}_{p,q}) \) for the topology of uniform convergence on compact.

**Proof** - Set \( A_B(Y; \mathbb{R}) \) for the subspace of \( A_B(Y; \mathbb{R}_{p,q}) \) consisting of those functions which take real values. This is a real algebra. Let \( A_B(Y; \mathbb{R})_I \) be the subspace of \( A_B(Y; \mathbb{R}_{p,q}) \) consisting of the \( I \)-components of functions from \( A_B(Y; \mathbb{R}_{p,q}) \). Thus, we have \( f_I = \langle f e^I \rangle_0 \) and \( A_B(Y; \mathbb{R})_I \subset A_B(Y; \mathbb{R}) \) by theorem 2.

In this way, \( A_B(Y; \mathbb{R}) \) satisfies to the hypothesis of the classical Stone-Weierstrass theorem for real functions. The algebra \( A_B(Y; \mathbb{R}) \) is consequently dense in \( C(Y; \mathbb{R}) \). Finally, one can conclude that:

\[A_B(Y; \mathbb{R}_{p,q}) = \bigoplus_I A_B(Y; \mathbb{R})e_I\]

is dense in \( C(Y; \mathbb{R}_{p,q}) \).

\[\Box\]

**5. A remark**

It should be noted that the computations of the scalar part it strongly related to formulas related to the Hestenes multivector derivative : see [4], chapter 2.
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