1. Introduction

The Néron model of the Jacobian [33, 37] is fundamental in the theory of semi-stable reduction of curves and plays a crucial role in the study of compactified Jacobians. Indeed the Jacobian of a smooth curve over the field of fractions of a discrete valuation ring $R$ is a proper variety $\text{Pic}^0 C_K$, naturally equipped with the structure of an Abelian variety. In broad terms, the problem of compactified Jacobians aims at finding a proper model for $\text{Pic}^0 C_K$. In general, one cannot expect a proper model which shares the same group properties as $\text{Pic}^0 C_K$; however, by relaxing the condition of properness and concentrating on the smoothness, A. Néron discovered during 1961–1963 that a canonical $R$-model exists for any abelian variety $A_K$; this is the Néron model $\text{Né}(A_K)$. The case of the Jacobian was completely elucidated in the late sixties by M. Raynaud [37] to whom we also owe the translation of Néron’s construction in the language of schemes.

The geometry of $\text{Né}(\text{Pic}^0 C_K)$ is rich and interesting in its own right. For instance, whenever $C_K$ admits a semi-stable reduction on $R$, the group of components of the special fibre is the critical group $\mathcal{K}(\Gamma)$ of the dual graph $\Gamma$ of the special fibre of the semi-stable reduction. The critical group is the group governing sandpile dynamics discovered numerous times...
in different areas [7, 9, 18, 28] and probably making their first appearance precisely in this context, Raynaud [37, Prop. 8.1.2]. We recall the main definitions in Section 4.

Furthermore, Nér(Pic⁰ C_K) sheds new light on the geometry of compactified Jacobians, e.g. [6, 13, 20, 31]. We will assume for simplicity that R is complete with an algebraically closed residue field k. There is a common feature to well-behaved compactifications of the Jacobian Pic⁰ C_K: a stratification of the special fibre indicating that Néron models should be regarded as building blocks of the compactified Jacobians. Indeed, following [13] or [20] and [31], if we admit that the curve C_K has stable reduction C_R over R, we can provide Pic⁰ C_K with a proper model J of Pic⁰ C_K whose special fibre J_C admits a stratification

\[ J_C = \bigsqcup_{S \subseteq \text{Sing}_C, C^S \text{ connected}} J_{C^S}, \]

where C is the stable curve, special fibre of C_R, and the union runs over all possible sets of nodes S ⊆ Sing_C whose corresponding normalization C^S → C is connected. The spaces J_{C^S} have co-dimension #S, satisfy \( \overline{J_{C^S}} \supseteq J_{C^S'} \) for \( S \subseteq S' \), and can be identified with special fibres of Néron models of Jacobians of curves; namely we have

\[ J_{C^S} \cong \text{Nér}(\text{Pic}^0 C_K^S)_k. \]

where \( C_K^S \) is the generic fibre of any smoothing of \( C^S \) along Spec R and \( k = R/(\pi) \) is the residue field. The points of the compactification can be regarded as parametrising isomorphism classes of line bundles on nodal curves; shortly in this introduction we will illustrate the approach of Caporaso to which we return in §5.3.

Notice that \( J_{C^S} \cong \text{Nér}(\text{Pic}^0 C_K^S)_k \) is only an identification of schemes (the algebraic structure of Néron models is not involved) and the right hand side of the isomorphism above is, as a scheme, merely the disconnected union of c copies of Pic⁰ C^S where \( c = \#K(\Gamma_S) \) is the complexity of the dual graph \( \Gamma_S \) of \( C^S \). However, this decomposition arises the following natural question: is there a universal family, possibly equipped with a group structure, over a compactification of \( \mathcal{M}_g \) whose fibre over each isomorphism class \( [C] \) is isomorphic to \( J_C \)?

Caporaso’s approach yields the positive answer to the analogue question where, instead of the Jacobian, we start from the stack \( \mathcal{P}ic^d_g \) representing the relative Picard functor Pic^d over \( \mathcal{M}_g \) for \( \text{gcd}(2g - 2, d - g + 1) = 1 \) and \( g \geq 3 \) (in particular for \( d \neq 0 \)). Indeed for these degrees Caporaso constructs a compactification over \( \overline{\mathcal{M}}_g \) which satisfies the decomposition (1) after restriction to a curve over R. For the functor Pic⁰ the approach can only be used in presence of a distinguished marking (e.g. either locally over R or globally over \( \overline{\mathcal{M}}_{g,n>1} \) where the marking labelled by 1 allows a trivialisation of Pic^d C_K as a Pic⁰ C_K-torsor and the use of the result in degree-d). See further discussion in Section 5.3.

When it comes to assembling the special fibres of Néron models into a group scheme, it should be noticed that Caporaso’s approach can never admit a group structure compatible with that of the Jacobian over \( \mathcal{M}_g \). Within the theory of nodal curves, one should rather refer to the group scheme introduced by Holmes [23] which is fibred over a stack \( \tilde{\mathcal{M}}_g \) lying over \( \overline{\mathcal{M}}_g \) and in general not quasi-comapact over it. This stack, which is only locally of finite presentation, has the striking advantage of satisfying the universal Néron mapping property on a higher dimensional basis. The stack \( \mathcal{M}_g \) is a sort of completion of \( \mathcal{M}_g \) because it satisfies the valuative criterion of properness for traits of \( \overline{\mathcal{M}}_g \) whose generic points maps to \( \mathcal{M}_g \). We refer to Section 5.3 for further discussion.
In this paper, we show that we can indeed assemble the special fibres of the Néron models of all Jacobians once we hit upon the right Picard functor over the right moduli space of curves. We describe a universal group scheme over a slight variation of Deligne and Mumford’s compactification of $M_g$: the proper moduli stack $\overline{M}_g$ of of $\ell$-stable twisted curves, i.e. stack-theoretic nodal curves with stabilisers of order $\ell$ at all nodes. This compactification is equipped with a stack $\mathcal{P}ic^0_g$ representing the relative Picard functor $\text{Pic}^0$. Within $\mathcal{P}ic^0_g$ the stack $\mathcal{P}ic^0_{g, \ell}$ is locally closed and represents line bundles, invariant with respect to the so-called ghost automorphisms fixing all geometric points and operating nontrivially as $\zeta \cdot (x, y) = (\zeta x, y)$ at all nodes $\{xy = 0\}$ for $\zeta \in \mu_\ell$ (see (14)). Below, $R_\ell$ is the discrete valuation ring extracting an $\ell$th root from the uniformizer $\pi$: we set $R_\ell = R[\pi]/(\pi^\ell = \pi)$.

**Theorem.** The stack $\mathcal{P}ic^0_{g, \ell}$ is a group scheme over $\overline{M}_g^\ell$ and is a separated model of the universal Jacobian representing the relative Picard functor $\text{Pic}^0$ over $M_g$.

Furthermore, assume that $\ell$ is a multiple of the exponent of the critical group $K(\Gamma)$ of any stable graph $\Gamma$ of genus $g$. Then, for any trait $\text{Spec} R \rightarrow \overline{M}_g$ transversal to the boundary $\overline{M}_g \setminus M_g$, there exists a lift $\text{Spec} R_\ell \rightarrow \overline{M}_g^\ell$ such that $\mathcal{P}ic^0_{R_\ell} = \mathcal{P}ic^0_{g, \ell} \otimes R_\ell$ descends to $\mathcal{P}ic^0_{R}$ on $R$ and the Néron model of $\mathcal{P}ic^0_{R, \ell} = \mathcal{P}ic^0_{g, \ell} \otimes K$ satisfies

$$\text{Néron}(\mathcal{P}ic^0_{R, \ell}) = \mathcal{P}ic^0_{R}.$$ 

**Remark 1.1.** We refer to Corollary 5.1 describing explicitly the lift $\text{Spec} R_\ell \rightarrow \overline{M}_g^\ell$.

**Remark 1.2.** The statement (2) holds for a fixed trait $\text{Spec} R \rightarrow \overline{M}_g$ if and only if $\ell$ is a multiple of the exponent of the critical group $K(\Gamma_k)$, where $\Gamma_k$ is the dual graph of the curve identified by the special closed point $\text{Spec} k \rightarrow \overline{M}_g$ in $\text{Spec} R$.

Clearly, the statement (2) holds for any trait $\text{Spec} R \rightarrow \overline{M}_g$ if $\ell$ is a multiple of the complexity $c(\Gamma)$ of all stable graphs $\Gamma$ of genus $g$ (this follows immediately from $\# K(\Gamma) = c(\Gamma)$).

In Corollary 5.6, we can relax the transversality condition and consider a trait whose generic point still lies in $M_g$ and whose strict Henselization at the special point $\text{Spec} k \rightarrow [C] \in \overline{M}_g$

maps to $\pi^n f$ the local parameter $f$ defining the Cartier divisor $D = (f) \subset \text{Def}(C)$ parametrising curves where the node $n \in C$ persists. Then $\ell$ is the exponent of the critical group of the $t$-weighted graph $K(\Gamma_c)$ where each edge is decorated with the corresponding index $t_n$, see Section 4 (the possibility of extending this formalism in the presence of weighted edges is alluded to in [7, §9, Rem. 2]).

These considerations led us to a closer study of critical groups. We refer to Section 4, where we revisit the theory of critical groups, Abel theorems, Kirchhoff matrix-tree theorem, and complexity in the context of graph with weights assigned to each edge. Some formulae simplify computations of the standard invariants, see (12) and Theorem 4.1.

We can now step back to $\text{Spec} R$ and compare the classical presentation of the Néron model of the Jacobian $\text{Pic}^0(C_K)$ by Raynaud to the new one arising from (2).

First, recall that, on $R$, the functor $\text{Pic}^0$ of line bundles of degree zero on all irreducible components, provides a separated model, but fails to satisfy the universal Néron property. The classical solution, provided by Raynaud in 1970 [37], is to enlarge $\text{Pic}^0$ to a non-separated functor $\text{Pic}^{\text{tot}0}$ of line bundles of total degree $0$ and to take the quotient by the closure $\overline{E}_K$ of
the identity section $E_K$ over $K$. Summarizing, if we write $k$ for the residue field and $C_k$ for the special fibre we have

\[ \text{Né{r}(Pic}_0^0\mathcal{K}) = \text{Pic}^{\text{tot}0}(C_R/R)/E_K. \]

The passage to the quotient by $E_K$ clearly prevents us from defining a modular functor over $M_g$ starting from $\text{Pic}^{\text{tot}0}_0$ and $E_K$; indeed, the group structure defining $E_K$ depends on the chosen smoothing. Interestingly, in 2014, Holmes [23] shows how $E_K$ fails to be flat when we work of the moduli space of universal deformations of curves. As Holmes further illustrates, the problem persists on essentially any proper modification of the base space.

In this paper, we get back to $\text{Pic}_0^0$, pointing out that it becomes sufficiently large to satisfy the Néron property when we consider a reduction over $R_\ell$ given by a twisted curve, i.e. when we allow line bundles on stack-theoretic nodes with stabilizer $\mu_\ell$ of a twisted curve $C_k$ over $C_k$. Then (2) may be rephrased as saying that the points of the special fibre of the Néron model are all represented by line bundles of degree zero on every irreducible components. According to (3), the group of components of the special fibre is usually phrased in terms of multi-degrees of line bundles on reducible curves and coincides with the determinant group of the lattice of integral cuts (functions on the vertex set). Instead, in (2), we have

\[ \text{Né{r}(Pic}_0^0\mathcal{C}_K)_k = (\text{Pic}_0^0\mathcal{C}_k)^{\mu_\ell} \]

and the connected components correspond to characters at each node; this is naturally related to the determinant group of the lattice of integral flows (functions on the edge set). As a byproduct we get a new version of the so-called Abel theorem for graphs, see §4.4.

We can now rewrite the above decomposition of the special fibre of the compactified Jacobian as a union of Picard groups

\[ \mathcal{J}_C = \bigsqcup_{S \subseteq C^{\text{sing}}} (\text{Pic}_0^0\mathcal{C}_S)^{\mu_\ell}, \]

where $C_S$ is the twisted curve coarsely represented by $C_S$ with stabilisers of order $\ell$ over the nodes $C_S$ and $\ell$ is a multiple of all complexities of the curves $C_S$ (we can take for instance $\ell = c(\Gamma)!$).

**Structure of the paper.** After setting up things in Section 2, we prove the above statement (2) and some variants in Section 3. Then in Section 4 we develop the combinatorial consequences and in Section 5 we prove the theorem stated above.

**Acknowledgements.** I am are grateful to Lucia Caporaso, Eduardo Esteves, David Holmes, Johannes Nicaise, Cédric Pepin, Michel Rayanud, Matthieu Romagny, Angelo Vistoli, Filippo Viviani for many useful conversations. A special thank goes to André Hirschowitz, who first encouraged me in this research direction.

## 2. Curves and Picard functors

### 2.1. Assumptions and notations.

Unless otherwise specified, we work with schemes locally of finite type over an algebraically closed field $k$. $R$ denotes a complete discrete valuation ring with algebraically closed residue field $k$ and field of fractions $K$. We denote by $\pi$ the uniformizer of $R$. For any $R$-scheme $T_R$, we write $T_K$ for its generic fibre and $T_k$ for its special fibre. In general for a scheme $U \to X$ and for any $X$-scheme $S \to X$ we write $U_S \to S$ for the corresponding base change.
We often use strict Henselizations in order to describe a scheme or a morphism between schemes locally at a closed point: by “local picture of $X$ at $x$” we mean the strict Henselization of $X$ at $x$. We systematically need to extract $\ell$th roots of unity from the uniformizer and from the local parameters at the branches of the nodes we consider. In these cases we will need to assume that the residual characteristic is prime to $\ell$ and we will write

$$R_\ell = R[\overline{\pi}]/(\overline{\pi}^\ell = \pi).$$

In Theorems 3.7, 3.4 and 5.1, this will force us to assume that char $k$ is prime to the value of $\ell$ specified in the statement.

Whenever we work with Deligne–Mumford stacks $X$ we rely on the existence of a coarse space $X$ and a morphism $p_X: X \rightarrow X$ universal with respect to morphisms to algebraic spaces. This allows us to associate to any morphism between Deligne–Mumford stacks $f: X \rightarrow Y$ a morphism between algebraic spaces $\overline{f}: \overline{X} \rightarrow \overline{Y}$.

2.2. Curves, twisted curves and families of curves. A curve $C_k$ over $k$ always denotes a reduced, connected, proper, one-dimensional scheme over $k$ whose only singularities are nodes. We refer to $h^1(C_k, \mathcal{O}_{C_k})$ as the genus $g = g(C_k)$ of $C_k$.

2.2.1. Dual graph. The dual graph $\Gamma$ of $C_k$ is a connected nonoriented graph, possibly containing multiple edges (edges linking the same two vertices) and loops (edges starting and ending at the same vertex). Consider the normalisation $\text{nor}: C_k^\text{nor} \rightarrow C_k$ and the connected components $C_v$ of the normalisation, it coincides with the set of irreducible components of $C_k$. The edge set is the set of the nodes of $C_k$. A node identifies the connected components of $C_k^\text{nor}$ where its preimages lie, in this way an edge links two (possibly equal) vertices.

2.2.2. Families of curves. A family of curves $C_S \rightarrow S$ is a proper and flat morphism whose fibres are curves. We consider a family of curves $C_R$ over a trait, the spectrum of a discrete valuation ring $R$. We assume that $C_K$ is smooth. Then, the local picture at a node $n$ of the special fibre is $\text{Spec } R[x, y]/(xy = \pi^t_n)$ where $\pi$ is a uniformizer of $R$ and $t_n$ is a positive integer. We refer to $t_n$ as the thickness of the node. Note that, if $t_n = 1$ at all nodes, then $C_R$ is regular. Geometrically, we can regard the family as a morphism $\text{Spec } R \rightarrow \mathcal{M}_g$; the condition $t_v = 1 \forall v$ is a condition of transversality to the boundary $\partial \mathcal{M}_g = \mathcal{M}_g \setminus \mathcal{M}_g$.

In this way, the special fibre of $C_R \rightarrow R$ yields a decorated graph equipped with a function $V \rightarrow \mathbb{N}$, $v \mapsto g_v := g(C_v)$ and a function $E \rightarrow \mathbb{Z}_{\geq 1}$, $e \mapsto t_e$, where $t_e$ is the thickness of the node corresponding to $e$. We read off the decorated graph the genus $g = \sum_v g_v + b(\Gamma)$, where $b(\Gamma)$ is the first Betti number $1 - |V| + |E|$ of the dual graph $\Gamma$.

2.2.3. Twisted curves. A twisted curve $C_k$ (in the sense of Abramovich and Vistoli) is a Deligne–Mumford stack whose coarse scheme $C_k$ is a curve. We only consider the case where the smooth locus is represented by a scheme (we do not allow non-trivial stabilisers on smooth points). At the nodes the local picture is $[(\text{Spec } k[x, y]/(xy))/\mu_\ell]$ with $\zeta \in \mu_\ell$ operating as $\zeta(x, y) = (\zeta x, \zeta^{-1} y)$. The notion of family and of dual graph generalises word for word for a twisted curve. We only consider twisted curves whose coarse space is a stable curve.

2.2.4. The regular twisted model. We consider a stable curve $C_R \rightarrow \text{Spec } R$ with $C_K$ smooth; it may be regarded as a stable reduction of $C_K$ over $R$. Take an étale neighbourhood of a node $p \in C_k$ of thickness $t$: the local picture is $\overline{U} = \text{Spec } R[x, y]/(xy = \pi^t)$. Consider the quotient
stack \([U/\mu]\) where \(U = \text{Spec } R[z, w]/(zw = \pi)\) and \(\zeta \in \mu_t\) acts as \((z, w) \mapsto (\zeta z, \zeta^{-1} w)\). The morphism \([U/\mu_t] \to U, x \mapsto z^t, y \mapsto w^t\) is invertible away from the origin

\[ p = [(z = w = 0)/\mu_t] \longrightarrow p = (x = y = 0) \in U. \]

We define a twisted curve by gluing \([U/\mu]\) to \(C_R \setminus \{p\}\) along the isomorphism \([U/\mu_t] \setminus \{p\} \to U \setminus \{p\}\). We repeat this procedure at all nodes of the special fibre \(C_k\) and we get \(C^\text{tw}\) over \(R\). Note that \(C^\text{tw}\) is regular (as a stack over \(k\)).

**Definition 2.1.** We refer to \(C^\text{tw} \to \text{Spec } R\) as the regular twisted model associated to a stable reduction \(C_R\) over \(R\) of a smooth curve \(C_K\) over \(K\).

**Remark 2.2.** In this context, the regular twisted model plays a similar role than the regular minimal semistable model. In particular it is not stable with respect to base change; we have chosen the notation \(C^\text{tw}\) (and we will avoid a notation of the form “\(C_Rt\)” precisely because the construction of this model over \(R'\) after base change of \(C_R\) via an extension \(R \subseteq R'\) is not a simple base change.

We illustrate it by observing how the regular twisted model behaves when we pullback all data to the discrete valuation ring \(R_\ell\) obtained by extracting an \(\ell\)th root \(\pi'\) from the uniformizer \(\pi \in R\). Then the stable curve \(C_{R_\ell}\), pullback of \(C_R\) to \(R_\ell\), is a stable reduction of the curve \(C_K\) pulled back to the field of fraction \(K_\ell = \text{Frac}(R_\ell)\). However, the regular twisted model associated to the stable reduction \(C_{R_\ell}\) of \(C_K\) is not the pullback of the regular twisted model of the stable reduction \(C_R\) of \(C_K\). Whereas the respective coarse spaces are indeed related by a simple pullback, the stacks differ: where the pullback of \(C^\text{tw}\) to \(R_\ell\) has a stabiliser of order \(\ell t\), the regular twisted model of \(C_{R_\ell}\) has a stabiliser of order \(\ell t\). For \(\ell = 1, 2, \ldots\) we write \(C^\text{tw} = C^\text{tw}(1), C^\text{tw}(2), \ldots\) for these regular twisted models and we stress that the twisted curve

\[ C^\text{tw}(\ell) \longrightarrow \text{Spec } R_\ell \]

depends on the parameter \(\ell \geq 1\) and is not the pullback of \(C^\text{tw}\) from \(R\) to \(R_\ell\).

2.3. **The relative Picard functors.** For any stack \(X\) over \(k\), we denote by \(\text{LB}(X)\) the category of line bundles on \(X\) and by \(\text{Pic}(X)\) the group of isomorphism classes of line bundles on \(X\)

\[ \text{Pic}(X) = H^1(X, \mathcal{O}^\times). \]

2.3.1. **Multi-degrees.** Notice that, for \(X = C_k\) a stable curve or — more generally — for \(X = C_k\), we can decompose \(\text{LB}(X)\) and \(\text{Pic}(X)\) into sub-loci with fixed degree on all irreducible components. In the case of \(C_k\) we get a multi-degree whose entries are rational numbers (and indeed elements of \(\frac{1}{n}\mathbb{Z}\) where \(n\) is the lowest common multiple of the orders of the stabilisers of the points of \(C_k\)).

2.3.2. **Relative functors.** For a family of curves \(C \to B\) there exist relative versions of these notions. We consider the fibred category \(\text{LB}(C/B)\) whose objects are \((S, L)\) formed by \(B\)-schemes \(S \to B\) paired with a line bundle \(L\) on \(C_S\). This is represented by an Artin stack, [30, Lem. 2.3.1]. Furthermore, for any object \(\tau = (S, L)\) of \(\text{LB}(X/B)\) there is an embedding \(\mathbb{G}_m(S) \hookrightarrow \text{Aut}_S(\tau)\) compatible with pullbacks. Then, by [3, Thm. 5.1.5] and [38, I. Prop. 3.0.2, (2)] there exists a stack \(\text{LB}(C/B) \sslash \mathbb{G}_m\) which coincides on each fibre with the Picard group. We take this as a definition for the relative Picard functor

\[ \text{Pic}(C/B) = \text{LB}(C/B) \sslash \mathbb{G}_m. \]

In the case of a family of twisted curves \(C \to B\), each point of the stack \(\text{LB}(C/B) \sslash \mathbb{G}_m\) has trivial automorphism group. Therefore \(\text{Pic}(C/B)\) is represented by a group scheme. When
we work with $C_R \to \text{Spec } R$, $C_K \to \text{Spec } K$ and $C_k \to \text{Spec } k$ (as we usually do in this paper), we get three group schemes which we denote by Pic$_R$, Pic$_K$ and Pic$_k$.

**Definition 2.3.** Within the relative Picard functor of any twisted curve $X$ over any base scheme $B$ we can identify the following remarkable sub-functors by imposing the following conditions to the restrictions of line bundles to the fibres of $X$

Pic$^0 = \text{degree zero on all irreducible components of each fibre } X_b$ with $b \in B$;

Pic$^{tot \cdot d} = \text{total degree } d \text{ on all irreducible components of each fibre } X_b$ with $b \in B$.

2.3.3. **The separated sub-group Pic$^0$ of the Picard group.** When $C_R \to \text{Spec } R$ is a twisted curve the group scheme Pic$_R := \text{Pic}(C_R/R)$ contains the above mentioned sub-group scheme Pic$^0_R := \text{Pic}^0(C_R/R)$ representing line bundles of degree zero on every irreducible component of the fibres.

**Proposition 2.4.** Let $C_R$ be the regular twisted model of a stable curve $C_R$ over $R$. Then, Pic$^0(C_R/R)$ is a separated group scheme.

**Proof.** The proof is the same, word for word, as in [13, Lem. 3.4.,(i)]. We assume that $L$ is a line bundle on $C_R$ extending $O$ on $C_K$ with all degrees equal to 0 on all components of $C_k$. Since $C_R$ is a regular Deligne–Mumford stack we present it in terms of a divisor supported on the special fibre. This can be written as $\sum_{n \in \mathbb{Z}} nD_n$ with $\cup D_n = C_k$ and two distinct $D_n$ and $D_n''$ overlapping only at a set of nodes. The claim is that for $m = \min \{n \mid D_n \neq \emptyset\}$ we have $D_m = C_k$. This claim is enough to prove that $L$ is a pullback from Spec $R$, i.e. it represents the identity section of Pic$(C_R/R)$. Indeed, this claim follows from computing $\text{deg } L |D_m| \geq \sum_{n > m} D_n \cdot D_m \geq 0$. Notice that $L$ has degree zero when restricted to $D_n$; we get $\sum_{n > m} D_n \cdot D_m = 0$, which means $D_n = \emptyset$ all $n > m$ as desired. \hfill $\square$

2.3.4. **The ghost action on Pic$^0(C^w(\ell)/R_{\ell})$.** The regular twisted model $C^w(\ell)$ attached to $C_{R_{\ell}}$ is equipped with a $\mu_\ell$-action operating on the base ring $R_{\ell}$ by multiplication on $\pi$ and, equivariantly, on $C^w(\ell)$. We consider the local picture at a node $[U/\mu_\ell]$ with $U = \text{Spec } R_{\ell}[z, w]/(zw = \pi)$ and $\zeta \in \mu_{\ell t}$ operating as $\zeta \cdot (z, w, \pi') = (\zeta z, \zeta^{-1}w, \pi')$. The action of $\eta \in \mu_\ell \subseteq \mu_{\ell t}$ on $[U/\mu_{\ell t}]$ is $\eta \cdot (z, w, \pi') = (\eta z, \eta w, \eta \pi')$. This action is 2-isomorphic to $(z, \eta w, \eta \pi')$; so, no coordinate has been privileged (the 2-isomorphism is realised for instance by the natural transformation $(z, w, \pi') \mapsto (\zeta^{-t}z, \zeta^{t}w, \pi')$ if $\zeta^{t} = \eta$). This action restricts on the special fibre to an automorphism which fixes the coarse space of $C_k$; therefore, it is usually referred to as a ghost automorphism.

We consider the $\mu_\ell$-action on the special fibre of $\text{Pic}^0(C^w(\ell)/R_{\ell}) \to \text{Spec } R_{\ell}$.

We describe the connected components of the special fibre and among them we identify those who are formed by fixed points. In fact, their union is the fixed set, with respect to the $\mu_\ell$-action of the special fibre. Before, we need to develop the combinatorics of the decorated graph $\Gamma$.

2.4. **The combinatorics of the graph $\Gamma$.** We consider the dual graph, its vertex set $V$ and its edge set $E$. Let $E$ be the double cover of $E$ formed by oriented edges. For any $e \in E$, we write $\overline{e}$ for the oriented edge obtained from $e$ by reversing its orientation. For $e \in E$ we write $e_+$ and $e_-$ for its tip and its tail in $V$. We recall that $\Gamma$ is decorated by the genus function $v \mapsto g_v$ and by the thickness function $e \mapsto t_e$, it has genus $g = \sum_v g_v + b(\Gamma)$ and is stable if for any vertex $2g_v - 2 + n_v$ is positive.
2.4.1. The cochain differential. For any group $G$, $C^0(\Gamma; G) = \{ f : V \to G \}$ denotes the group of $G$-valued 0-cochains, $C^1(\Gamma, G) = \{ g : E \to G \mid g(e) = -g(\delta) \}$ the group of $\mathbb{Q}$-valued 1-cochains. $\delta$ denotes the differential

$$\delta : C^0(\Gamma; G) \to C^1(\Gamma; G), \quad f \mapsto (e \mapsto f(e_+) - f(e_-)).$$

2.4.2. Pairings. When the group $G$ is equal to the field of rational numbers $\mathbb{Q}$, we have the following perfect pairings which depend on the thicknesses

$$\langle f_1, f_2 \rangle_0 = \sum_{v \in V} f_1(v)f_2(v) \quad \text{and} \quad \langle g_1, g_2 \rangle_0 = \frac{1}{2} \sum_{e \in E} \frac{1}{t_e} (g_1(e)g_2(e)).$$

The adjoint of $\delta$ with respect to the above bilinear forms is the differential

$$\partial^t : C^1(\Gamma; \mathbb{Q}) \to C^0(\Gamma; \mathbb{Q}), \quad g \mapsto \left( v \mapsto \frac{1}{2} \sum_{e \in E} \frac{g(e)}{t_e} \right).$$

We can make the above definition explicit by choosing an orientation of each edge, i.e. a lift $E \to \mathbb{E}$ (we systematically declare when this choice is made). In this way $C^1(\Gamma; \mathbb{Q})$ is simply the set of $\mathbb{Q}$-valued functions on $E$. Then, there is a distinguished basis of $C^0$ and of $C^1$ formed by characteristic functions $\chi_v : v' \to \delta_{v,v'}$ and $\chi_e : e' \to \delta_{e,e'}$. Abusing the notation we identify $v$ to $\chi_v$ and $e$ to $\chi_e$ and we get

$$\delta : \mathbb{Q}^{|V|} \to \mathbb{Q}^{|E|}, \quad \delta v = \sum_{e_+ = v} e - \sum_{e_- = v} e$$

and

$$\partial^t : \mathbb{Q}^{|E|} \to \mathbb{Q}^{|V|}, \quad \partial e = \frac{1}{t_e} (e_+ - e_-).$$

2.4.3. Laplacian and Jacobian of the graph $\Gamma$ with thicknesses. As mentioned in [7, §9, Rem. 2], there is no difficulty in extending the theory of the Laplacian in the present setup where the pairings depend on the thicknesses. We consider the Laplacian $\partial^t \delta$ and, following the most widely used notation, we refer to the group

$$K_t(\Gamma) = \frac{\partial^t C^1(\Gamma; \mathbb{Z})}{\partial^t \delta C^0(\Gamma; \mathbb{Z})}$$

as the critical group of the graph $\Gamma$ decorated with the thicknesses $t$. This group is related—but not isomorphic in general—to the group of components of the special fibre of the Néron model. We refer to Proposition 3.3.

2.4.4. Ordinary differentials. When $G$ is the ring $\mathbb{Z}$, the bilinear forms still make sense and are non-degenerate. They induce a perfect pairing only if $t = 1$. The corresponding differential $\partial^1$ matches the standard homology differential $\partial$ from 1-chains to 0-chains via the canonical identification $C_i = C^i$. By tensoring with any group $A$ we recover the differentials

$$\partial_A : C^1(\Gamma; A) \to C^0(\Gamma; A) \quad \text{and} \quad \delta_A : C^0(\Gamma; A) \to C^1(\Gamma; A)$$

between $A$-valued chains and cochains. Here we will use the above structure with $A = \mathbb{G}_m$ and $A = \mathbb{Q}/\mathbb{Z}$.

2.5. The special fibre of $\text{Pic}^0$. Below, in Proposition 2.6, we completely describe the connected components of the special fibre.
2.5.1. The differential $\partial_t$. The combinatorics introduced above specialises as follows. By restricting $\partial^t$ from (5) to $\mathbb{Z}$ we get
$$\partial^t|_\mathbb{Z} : C^1(\Gamma; \mathbb{Z}) \to C^0(\Gamma; \mathbb{Q}).$$
Notice that $\partial^t|_\mathbb{Z}$ maps the sub-group of $\mathbb{Z}$-valued 1-cochains satisfying $g(e) \in t_e \mathbb{Z}$ to $C^0(\Gamma; \mathbb{Z})$. By taking the quotient $C^1(\Gamma; \oplus_e \mathbb{Z}/t_e) = C^1(\Gamma; \mathbb{Z})/C^1(\Gamma; \oplus_e(t_e))$ we get the reduced differential
$$\partial_t : C^1(\Gamma; \oplus_e \mathbb{Z}/t_e) \to C^0(\Gamma; \mathbb{Q}/\mathbb{Z}).$$
After a choice of an orientation, $\partial_t$ is simply
$$\partial_t : \bigoplus_{e \in E} \mathbb{Z}/t_e \to (\mathbb{Q}/\mathbb{Z})^{|V|}, \quad e \mapsto \frac{1}{t_e}(e_+ - e_-).$$

2.5.2. Characters at the nodes. Let us return to the regular twisted model $C^\text{tw}(\ell)$. Every choice of an orientation of an edge of the dual graph of the special fibre, allows us to choose a distinguished branch of the corresponding node. The fibre of a line bundle $L$ over the node can be written for a unique choice of $a_e \in \mathbb{Z}/t_e$ as the $a_e$-th tensor power of the line tangent along this distinguished branch. Note that $a_e(L)$ changes sign if we change the orientation of $e$. In this way, to any line bundle $L$ on the special fibre, we can attach an element $a(L)$ of the above $C^1(\Gamma; \oplus_e \mathbb{Z}/t_e) = \bigoplus_{e \in E} \mathbb{Z}/t_e$. We have the following definition.

**Definition 2.5.** Consider a twisted curve $X$ over a discrete valuation ring $R$. We assume that the generic fibre over $K$ is smooth, that $X$ is regular, and that the stabilisers at the nodes of the special fibre have order $t_e$.

We have a morphism
$$\text{ev} : \text{Pic}(X/S) \to C^1(\Gamma; \oplus_e \mathbb{Z}/t_e)$$
$$L \mapsto a(L),$$
For any $a \in C^1(\Gamma; \oplus_e \mathbb{Z}/t_e)$ we write $\text{Pic}_a(X/S)$ and for the sub-loci of line bundles mapping to $a$.

2.5.3. The component group of the special fibre of $\text{Pic}^0$. For a twisted curve $X$ over a discrete valuation ring $R$ with smooth generic fibres and stabilisers of order $t_e$ on the nodes of the special fibres the connected components of the special fibre of $\text{Pic}^0(X/R)$ are classified by the kernel of characters $\ker \partial_t$. We state this in terms of the regular twisted model $C^\text{tw}(\ell)$, which is our main focus here. The thicknesses of $C^\text{tw}(\ell)$ are given by $\ell t$; we write $p$ for the morphism to the coarse space.

**Proposition 2.6.** We have the exact sequence
$$0 \to \text{Pic}^0(C_k) \xrightarrow{\partial^t} \text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k \xrightarrow{\text{ev}} \ker \partial_t \to 0,$$
where $\text{Pic}^0(C_k) = \text{Pic}^0(C_k)$ is connected.

**Remark 2.7.** The above statement may be regarded as a decomposition of the special fibre of the relative Picard functor $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)$ into non-empty connected components isomorphic to $\text{Pic}^0 = \text{Pic}^0(C_k)$ parametrized by $\ker \partial_t$
$$\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k = \bigsqcup_{a \in \ker \partial_t} \text{Pic}^0_a(C^\text{tw}(\ell)/R_\ell)_k.$$
Proof of Proposition 2.6. The group $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k$ decomposes into open and closed subloci $\text{Pic}^0_n(C^\text{tw}(\ell)/R_\ell)_k$ parametrised by characters $a \in C^1(\Gamma; \oplus_v \mathbb{Z}/t_v)$. The identification of $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k$ with $\text{Pic}^0(C_k)$ is obvious, the line bundles that can be obtained as pullback from the coarse space are those and only those whose characters are trivial.

We need to check that $a$ lies in $\ker(\partial_1)$; at any vertex $v$ the induced 0-chain $\partial_1(a)$ should vanish. Consider a line bundle $L$ of $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k$, we write $L_X$ for the induced bundle on the normalisation $X$ of $C_v$. The special points $p_e$ of $X$ are the pre-images of the nodes and are naturally attached to an oriented edge $e$ directed toward the vertex $v$. By definition each special point contributes $a_e/t_e \in \mathbb{Q}/\mathbb{Z}$ to $\partial_1(a)$. The value of $\partial_1(a)$ at $v$ is $\sum_{e \in X} a_e/t_e$, which coincides with $\deg D - [D]$ for $L_X = \mathcal{O}_X(D)$. Here, $[D]$ is the divisor on the coarse moduli space $p: X \to \mathcal{X}$ attached to $p_*L_X$; clearly $\deg [D] \in \mathbb{Z}$. The claim follows from the fact that $\deg D = 0$ because $L$ is in $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k$.

2.5.4. The kernel of characters $\ker(\partial_1)$. For any decoration $e \mapsto t_e$, we compute $\ker(\partial_1)$. For any power of prime number $p^l$, consider the edges $e$ whose thickness $t_e$ is a multiple of $p^l$. This set of edges alongside with the adjacent vertices determines a sub-graph $\Gamma_p^l$ of $\Gamma$. For any prime number, we have

$$\Gamma = \Gamma_p^0 \supset \Gamma_p^1 \supset \cdots \supset \Gamma_p^l \supset \cdots \supset \Gamma_p^{\text{max}_p} = \emptyset,$$

where $\text{max}_p$ is the maximum $p$-adic valuation of the thicknesses $t_e$. If we set $b_{p,l} = b(\Gamma_p^l)$ we have a sequence $b = b_{p,0} \geq b_{p,1} \geq \cdots b_{p,\text{max}_p} = 0$. We have

$$\ker(\partial_1) \cong \bigoplus_{p^l} \bigoplus_{1 \leq l \leq \text{max}_p} (\mathbb{Z}/p^l)^{b_{p,l} - b_{p,l+1}}.$$

2.6. The Galois action on the special fibre of $\text{Pic}^0$. As mentioned above, the combinatorial analysis above allows us to describe the $\mu_\ell$-action on the special fibre of $\text{Pic}^0$.

For any choice of $\zeta \in \mu_\ell$, each vector $a \in \bigoplus \mathbb{Z}/\ell t_e$ is naturally identified to a $\mathbb{G}_m$-valued 1-cochain $a(\zeta)$. This happens because $a_e/\ell t_e = \text{Hom}(\mu_\ell, \mathbb{G}_m)$ maps $\zeta$ to $\mathbb{G}_m$.

Proposition 2.8. The connected component $\text{Pic}^0_n(C^\text{tw}(\ell)/R_\ell)_k$ within the special fibre $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k$ is fixed by $\mu_\ell$ if and only if $a(\zeta) \in \text{im}\delta^{\mu_\ell}$ for all $\zeta$ in $\mu_\ell$. Furthermore, within the remaining connected components of $\text{Pic}^0(C^\text{tw}(\ell)/R_\ell)_k$, no point is fixed.

Proof. This follows immediately from the fact that $\text{im}(\delta: C^0(\Gamma; \mathbb{G}_m) \to C^1(\Gamma; \mathbb{G}_m))$ equals $\ker(\tau: C^1(\Gamma; \mathbb{G}_m) \to \text{Pic}C_k)$ and from the observation that $\zeta^*L = L \otimes \tau(a(\zeta))$. See [16, eq.(29)] and [15, Prop. 2.18].

3. The Néron model of the Jacobian

We give two presentations of the Néron model in §3.1 and §3.2. The first presentation is over $R$ and is in the same spirit of Raynaud’s theorem [37]; here, however, we use the regular twisted model instead of the regular semi-stable model. Since the special fibre of the former is more often irreducible than the latter, the presentation of Néron-$\text{Pic}^0(C_k)$ given here is already a worth-mentioning improvement. Then, in §3.2, we work over a base $R_\ell$ and the Néron model descends to $R$ from a group $R_\ell$-scheme representing line bundles of degree 0 on every component of the regular twisted model.
3.1. **The Néron model via the regular twisted model.** Consider $G = \text{Pic}^{\text{tot}0}(C^\text{tw})$. Because $C^\text{tw}$ is smooth, the morphism $G(R) \to G(K)$ is surjective; this is very useful in view of the criterion [10, 7.1/1], which guarantees that a finite-type $R$-group scheme is the Néron model of its generic fibre as soon as $G(R^\text{sh}) \to G(K^\text{sh})$ is bijective where $\text{sh}$ denotes the strict Henselization.

Here, we can interchange $R$ and its strict Henselization $R^\text{sh}$ because all objects involved in the statement are compatible with étale base changes (of course $C^{\text{tw}}$ remains regular under any such base change). Recall also that Néron models descend from the strict Henselization of $R$ to $R$ itself ([10, 6.5/3]).

We notice, however, that the group $\text{Pic}^{\text{tot}0}(C^{\text{tw}})$ is not of finite type. On the one hand, its generic fibre is of finite type and coincides with $\text{Pic}^0(C_K)$. On the other hand, its special fibre $\text{Pic}^{\text{tot}0}(C^{\text{tw}})_k$ coincides with $\text{Pic}^{\text{tot}0}(C_k)$, where $C_k$ is the special fibre of $C^{\text{tw}}$. The following decomposition of the Picard group $\text{Pic}^{\text{tot}0}(C^{\text{tw}})_k$ into disconnected components shows that the special fibre is not finite as soon as the curve is reducible. Since the regular twisted model is reducible more often than the usual regular semi-stable model, it is worthwhile to point out a simple proposition summarizing the above discussion.

**Proposition 3.1.** The following conditions are equivalent:

1. $C_K$ admits an irreducible stable model;
2. $C_K$ admits an irreducible regular twisted model;
3. $\text{Pic}^{\text{tot}0}(C^{\text{tw}})$ is of finite type.

Furthermore, under the above conditions, we have

$$N(\text{Pic}^0(C_K)) = \text{Pic}^0_r(C^{\text{tw}}),$$

where the functor $\text{Pic}^0$, in this case, coincides with $\text{Pic}^{\text{tot}0}$ on both sides. \hfill \Box

We generalise the above proposition by removing the condition (1) (or their equivalent versions (2) or (3)). We pass to the quotient

$$G := \text{Pic}^{\text{tot}0}(C^{\text{tw}})/E,$$

where $E$ is the scheme-theoretic closure within $\text{Pic}^{\text{tot}0}(C^{\text{tw}})$ of the zero-section of $\text{Pic}^0$. The group $E$ is formed by line bundles of the form $O(D)$ where is $D$ is supported on the special fibre of $C^{\text{tw}}$. This quotient insures the bijectivity of $G(R^\text{sh}) \to G(K^\text{sh})$. It remains to determine that $G$ is of finite type. Again, this concerns the special fibre $G_k$ and amounts to determine the finiteness of the group of connected components of $G_k$.

Proposition 3.2 is a preliminary step; it computes the group of connected components of $\text{Pic}^{\text{tot}0}(C_k^{\text{tw}})$ in terms of the image of the homomorphism

$$\tilde{\partial}^t = (\vartheta^t \times q_t): C^1(\Gamma; Z) \to C^0(\Gamma; \mathbb{Q}) \times C^1(\Gamma; \oplus e\mathbb{Z}/te),$$

where on the second factor, for each $Z$-valued 1-cochain $g: e \mapsto g(e)$, the morphism $q_t$ is simply the quotient by $(te)$. Below, recall that $\text{ev}$ is the evaluation homomorphism (7) and $\text{deg}$ is the (possibly rational) multi-degree.

**Proposition 3.2.** The image of $(\text{deg}, \text{ev}): \text{Pic}^{\text{tot}0}(C_k^{\text{tw}}) \to C^0(\Gamma; \mathbb{Q}) \times C^1(\Gamma; \oplus e\mathbb{Z}/te)$ lies in $\tilde{\partial}^t C^1(\Gamma; Z)$ and we have the exact sequence

$$0 \to \text{Pic}^0(C_k) \xrightarrow{p^*} \text{Pic}^{\text{tot}0}(C_k^{\text{tw}}) \xrightarrow{(\text{deg}, \text{ev})} \tilde{\partial}^t C^1(\Gamma; Z) \to 0.$$
Proof. Let us write $C := C^w_k$, $C := C_k$, and $p : C \to C$. We only need to show that $(\deg, \ev)$ maps to $\partial^t C^1(\Gamma, \mathbb{Z})$ and onto it. We do so, by providing a geometrical interpretation to $(\deg, \ev)$.

The exact sequence

$$0 \to \text{Pic}^{\text{tot}0}(C) \to \text{Pic}^{\text{tot}0}(C) \xrightarrow{\text{ev}} C^1(\Gamma, \otimes \mathbb{Z}/t_e) \to 0$$

(which may be regarded as the long exact sequence attached to $1 \to \mathbb{G}_m \to R\pi_*\mathbb{G}_m \to R\pi_*\mathbb{G}_m/\mathbb{G}_m$, see for instance [15]), allows us to express every line bundle $L \to C$ of total degree 0 as

$$p^* M_C \otimes \bigotimes_{0 \leq a_e < t_e} M_e^{\otimes a_e},$$

where $M_C$ is a line bundle of total degree 0 on $C$ and $M_e$ is the gluing of the structure sheaf $\mathcal{O}$ on the regular locus $C^\text{reg} = C^\text{reg}$ and the $\mu_{t_e}$-linearized line bundle $\mathcal{O}$ on $\{zw = 0\}$ given by the three characters $1, t_e - 1, 1 \in \mathbb{Z}/t_e$ acting on $z, w$, and the fibre parameter $\lambda$, respectively. Note that the restriction of $p_* M_e$ to the $z^{t_e}$-branch without the origin $\text{Spec} k[z^{t_e}, z^{-t_e}]$ is the structure sheaf $\mathcal{O}$ via the identification of the parameter along the fibre with the $\mu_{t_e}$-invariant parameter $\lambda z^{-1}$. Similarly the restriction of $\pi_* M_e$ to the $w^{t_e}$-branch without the origin $\text{Spec} k[w^{t_e}, w^{-t_e}]$ is the structure sheaf $\mathcal{O}$ via the identification of the local parameter along the fibre with the $\mu_{t_e}$-invariant parameter $\lambda w$. Then, the gluing is given by the canonical identification of the structure sheaves on $C^\text{reg} = C^\text{reg}$ on the $z$-branch $\text{Spec} k[z^{t_e}, z^{-t_e}]$ and on the $w$-branch $\text{Spec} k[w^{t_e}, w^{-t_e}]$.

The local analysis above shows that $M_e$ is a line bundle on $C$ of multi-degree $\partial^t(\chi_e)$ and that the bundles $M_e^{\otimes e}$ span $p^* \text{Pic}^{\text{tot}0}(C)$ (their $\mu_{t_e}$-linearization is trivial). This allows us to rewrite

$$L \cong p^* M_C \otimes \bigotimes_{0 \leq a_e < t_e} M_e^{\otimes a_e} = p^* M_C \otimes \bigotimes_{e \in E \atop 0 \leq a_e < t_e} M_e^{\otimes a_e}, \quad (\text{with } M'_C \in \text{Pic}^0(C)),
$$

and to define the homomorphism

$$\tau : \text{Pic}^{\text{tot}0}(C) \to \text{Pic}^{\text{tot}0}(C)/p^* \text{Pic}^0(C)
\quad L \mapsto \bigotimes_{e \in E \atop 0 \leq a_e < t_e} M_e^{\otimes a_e},$$

whenever $(m_e)_{e \in E}$ fit in (10). Indeed, since any other choice $m'_e$ of $m_e \in \mathbb{Z}$ should be compatible with (10), we have $m'_e - m_e \in t_e \mathbb{Z}$. Now, notice that any line bundle of the form $\bigotimes_{e \in E} M_e^{\otimes b_e}$ for $b \in \mathbb{Z}$ is a pullback from $C$.

We can now identify the map $(\deg, \ev)$ with $\partial^t \circ \tau$. Indeed $\deg(\bigotimes_{e \in E} M_e^{\otimes m_e}) = t^e(m_e)$ and $\ev(\bigotimes_{e \in E} M_e^{\otimes m_e}) = q_t(m_e)$. The surjectivity follows immediately, any element of the form $(\partial^t(m_e), q_t(m_e))$ is the image of $\bigotimes_{e \in E} M_e^{\otimes m_e}$ via $(\deg, \ev)$. \hfill \square

**Proposition 3.3.** The group of connected components of the special fibre of the quotient $\text{Pic}^{\text{tot}0}(C^w)/\mathbb{E}$, is finite and it is given by the group of components

$$\Phi_t(\Gamma) = \frac{\partial^t C^1(\Gamma; \mathbb{Z})}{\partial^t \delta C^0(\Gamma; \mathbb{Z})}$$

which is an extension of the critical group $K_t(\Gamma)$ by the kernel of characters $\ker(\partial_t)$

$$0 \to \ker(\partial_t) \to \Phi_t(\Gamma) \to K_t(\Gamma) \to 0.$$
Proof. We need to identify $(\deg, ev)(E) \subseteq \tilde{\partial}C^1(\Gamma, \mathbb{Z})$. Any line bundle of the form of the form $\mathcal{O}(D)$ where is $D = \sum_{v \in V} n_v C_v$, where $n_v \in \mathbb{Z}$ and $C_v$ is the irreducible component of the special fibre of $C^{tw}$ corresponding to the vertex $v \in V$. We set

$$L_v = \mathcal{O}(C_v)(C^{tw})$$

and show

$$(\deg, ev)(L_v) = (\deg, ev)(\bigotimes_{e \vdash v} M_e) = (\tilde{\partial}^\ell, q_e) \sum_{e \vdash v} \chi_e = \tilde{\partial}^\ell \delta(\chi_v).$$

We have

$$(\deg, ev)(L_v) = (\deg, ev)(L_v) = (\deg, ev)(\bigotimes_{e \vdash v} M_e) = (\tilde{\partial}^\ell, q_e) \sum_{e \vdash v} \chi_e = \tilde{\partial}^\ell \delta(\chi_v).$$

Finally, $\tilde{\partial}^\ell C^1(\Gamma; \mathbb{Z})$ contains $\partial C^1(\Gamma; \mathbb{Z})$ and $\tilde{\partial}^\ell \delta C^0(\Gamma; \mathbb{Z})$ contains $\partial \delta C^0(\Gamma; \mathbb{Z})$. Furthermore they are both contained in the kernel of $C^0(\Gamma; \mathbb{Q}) \to \mathbb{Q}; (\sum_v n_v[v]) \to \sum v$. Therefore their rank equals that of $\partial C^1(\Gamma; \mathbb{Z})$ and $\partial \delta C^0(\Gamma; \mathbb{Z})$, which is $|V| - 1$. This allows us to conclude that the quotient group is finite. \hfill \Box

Theorem 3.4. Let $C_R$ be a stable curve with smooth generic fibre over a complete discrete valuation ring $R$. Then, the Néron model of the Jacobian of $C_K$ is

$$\text{Néron}(\text{Pic}^0_K) = \text{Pic}^{\text{tot}}(C^{tw})/E,$$

where $E$ is the scheme-theoretic closure within $\text{Pic}^{\text{tot}}(C^{tw})$ of the zero-section of $\text{Pic}^0$.

Proof. Write $G$ for $\text{Pic}^{\text{tot}}(C^{tw})/E$. The proposition above implies that $G$ is of finite type. Then, we can apply [10, 7.1/1] and conclude that $G$ is a Néron model of its generic fibre by [10, 7.1/1]. Indeed $G(R^\text{sh}) \to G(K^\text{sh})$ is bijective (see [10, 7.1/1]). \hfill \Box

3.2. Néron model of $\text{Pic}^0$ via $\text{Pic}^0$. We introduce the subgroup $\text{Pic}^{0,\ell}$ of the group scheme $\text{Pic}^0(C^{tw}(\ell)/R\ell)$. Then we show that it singles out within $\text{Pic}^{0,\ell}(C^{tw}(\ell)/R\ell)$ the part of the Picard functor that descends on $R$ and, when $\ell$ is conveniently chosen, coincides over $R$ with the Néron model of the Jacobian of $C_K$.

Definition 3.5. We denote by $\text{Pic}^{0,\ell}_R$ the complement within $\text{Pic}^{0,\ell}(C^{tw}(\ell)/R\ell)$ of the connected components of the special fibre which are not fixed by $\mu_\ell = \text{Gal}(K', K)$.

Remark 3.6. The restriction of the map $m$: $\text{Pic}^0 \times_R \text{Pic}^0 \to \text{Pic}^0$ to $\text{Pic}^{0,\ell} \times_R \text{Pic}^{0,\ell}$ factors through $\text{Pic}^{0,\ell}$. Indeed, the pre-image of the connected components of the special fibre which are not fixed by $\mu_\ell$ lie within the special fibre of $\text{Pic}^0 \times_R \text{Pic}^0$ and does not intersect the special fibre $\text{Pic}^{0,\ell}_K \times_K \text{Pic}^{0,\ell}_K$ because the tensor product of $\mu_\ell$-fixed line bundles is again a $\mu_\ell$-fixed line bundle. Therefore $\text{Pic}^{0,\ell}_R$ is an open sub-group scheme of $\text{Pic}^{0}(C^{tw}(\ell)/R\ell)$.

Theorem 3.7. Let $C_R$ be a stable curve, with smooth generic fibre, over a complete discrete valuation ring $R$. Then, for any multiple $\ell$ of the exponent of $K_\ell(\Gamma)$, the group scheme $\text{Pic}^{0,\ell}_R$ descends to $\text{Pic}^{0,\ell}_R$ over $R$ and the Néron model of the Jacobian $\text{Pic}^{0}(C_K) = \text{Pic}^{0,\ell}_R$ satisfies

$$\text{Néron}(\text{Pic}^{0,\ell}_K) = \text{Pic}^{0,\ell}_R.$$

Proof. Following [13], over $R\ell$, we may write the pull-back of $\text{Néron}(\text{Pic}^{0,\ell}_K)$ as

$$\text{Néron}(\text{Pic}^{0,\ell}_K) \otimes R\ell = (\text{Pic}^{\text{tot}}(C^{tw})/E) \otimes R\ell = \bigsqcup_{(d,a)} \text{Pic}^0_d(C^{tw}(\ell)/R\ell).$$

where $\sqcup$ denotes a scheme-theoretic disjoint union over the indices $(d,a) \in \Phi_\ell(\Gamma)$ and “$\sim K$” denotes the identification along the isomorphic generic fibres.
We can rewrite the above group scheme by letting the the indices \((d, a)\) vary in \(\delta^t C^1(\Gamma; \mathbb{Z})\) and by modding out the action of \(s \in C^0(\Gamma; \mathbb{Z}).\) Indeed any \(\mathbb{Z}\)-valued cochain \(s\) can be identified to the divisor \(\sum_{v \in V} \ell s(v)C_v\) in \(C^w(\ell).\) Then we have the isomorphism
\[
\eta_{d, a} : \text{Pic}^{\ell t}_a \to \text{Pic}^{\ell t}_a(\mathbb{C}^w(\ell)/R_{\ell})
\]
where \(d' = d + \delta^t \delta s\) and \(a' = a + q_\ell \delta s.\) We write
\[
\text{Né}r(\text{Pic}^{\ell t}_{R_{\ell}}) \otimes R_{\ell} = \frac{\bigcup_{(d, a) \in \delta^t C^1} \text{Pic}^{\ell t}_a(\mathbb{C}^w(\ell)/R_{\ell})}{\sim K}.
\]
where the scheme-theoretic disjoint union runs over \(a \in \ker \partial_{\ell t} \cap \im \delta_{G_m}.\)

In order to show the isomorphism between the above group schemes over \(R_{\ell}\) we need to use the condition that \(\ell\) is a multiple of the exponent of \(K_1(\Gamma)\) and the following lemma.

**Lemma 3.8.** If \(\ell\) is the exponent of \(K_1(\Gamma),\) we have an exact sequence
\[
0 \to \delta^t C^0(\Gamma; \mathbb{Z}) \to \delta^t C^1(\Gamma; \mathbb{Z}) \xrightarrow{\iota} \ker \partial_{\ell t} \cap \im \delta_{G_m} \to 0.
\]
where \(\iota\) maps \(\delta^t c \in \delta^t C^1(\Gamma; \mathbb{Z})\) to \(\iota c - \delta b,\) where \(b\) is the unique element of \(\delta C^0(\Gamma; \mathbb{Z}) \subseteq C^1(\Gamma; \mathbb{Q})\) satisfying \(\delta^t c = \delta^t \delta b\) in \(C^0(\Gamma; \mathbb{Q}).\)

**Proof.** Indeed for any element \((\delta^t c, q_\ell c) \in \delta^t C^1\) there exists a unique \(\delta b \in C^1(\Gamma; \mathbb{Q})\) such that \(\delta^t c = \delta^t \delta b.\) This happens because \(\ell\) is a multiple of the exponent of \(K_1(\Gamma) = \delta^t C^1/\delta^t \delta C^0\) and, therefore, \(\ell K_1(\Gamma) = \delta^t C^1/\delta^t \delta C^0\) vanishes; hence, any element of the form \(\delta^t c\) lies in \(\delta^t \delta C^0.\) The uniqueness follows because \(\delta^t\) is the adjoint of \(\delta.\)

For \(\delta^t c = (d, a),\) we set
\[
\iota(d, a) = \ell c - \delta b \in \ker \partial_{\ell t} \cap \im \delta_{G_m},
\]
and we notice that any other choice \(c'\) in \(C^1(\Gamma; \mathbb{Z})\) lifting \((d, a)\) via \(\delta^t\) yields the same 1-cochain \(\ell c' = \ell c\) with values in \(\ell t_{c'}\mathbb{Z}\) because the reduction modulo \(t_{c'}\) and \(c'\) coincides with \(a.\) Furthermore we have \(\delta^t c' = \delta^t c = d,\) so \(c'\) yields the same element \(\delta b' = \delta b,\) uniquely determined by the relation \(\delta^t c' = \delta^t \delta b'.\) The uniqueness of \(\delta b\) also guarantees that \(\iota\) is an homomorphism of groups.

It is now obvious that \(\iota\) vanishes on \(\delta^t \delta C^0.\) Furthermore, any element in \(\ker \partial_{\ell t} \cap \im \delta_{G_m}\) can be written as the sum of a 1-cochain of the form \(\ell c\) and a 1-cochain lying in \(\im \delta.\) Here, each co-ordinate is identified up to a multiple \(\ell t_{c_e} \in \ell t_{c_e}\mathbb{Z}\); hence, we can represent each element of \(\ker \partial_{\ell t} \cap \im \delta_{G_m}\) as the image of \(c - (t_{c_e} d_e)_{e \in \mathbb{E}}\) via \(\iota.\)

Now \(\ker \iota\) equals \(\delta^t \delta C^0(\Gamma; \mathbb{Z}).\) Consider \(\iota(\delta^t c) = \ell c - \delta b \in \ker \partial^t.\) If this 1-cochain is congruent to 0 modulo \(\ell t_e\) on each vertex \(e,\) then we can assume \(\ell c - \delta b = \ell (t_{c_e} d_e)_{e \in \mathbb{E}}\) and this implies that \(b,\) whose image via \(\delta\) lies in \(C^0(\Gamma, \mathbb{Z}),\) can be chosen in \(C^0(\Gamma, \ell t_e)\). Then, the conditions \(\delta^t \ell c = \delta b = 0\) and \(\delta c = \delta b\) imply \(\delta^t c = \delta^t \delta(b/\ell)\) and \(q_\ell c = q_\ell \delta(b/\ell),\) respectively; in other words, we have \(\delta^t \delta(b/\ell) = \delta^t c.\) \(\square\)
For each \((d, a) \in \overline{\partial} C^1(\Gamma; \mathbb{Z})\) we consider \(m = \iota(d, a)\) and the isomorphism
\[
\pi_{d,a} : \text{Pic}^d_a \rightarrow \text{Pic}^0_m(C^\text{tw}(\ell)/R_\ell)
\]
\[
L \mapsto L \otimes \mathcal{O}(\sum_{v \in V} -b(v)\mathcal{C}_v).
\]
where \(b\) is the unique element of \(\delta C^0(\Gamma; \mathbb{Z}) \subseteq C^1(\Gamma; \mathbb{Q})\) satisfying \(\partial^* c = \partial^* \delta b\) in \(C^0(\Gamma; \mathbb{Q})\). Notice that for any \(e \in C^0(\Gamma; \mathbb{Z})\) we have \(\pi_{d', a'} \circ \eta_{d, a} = \pi_{d, a}\). This yields the isomorphism of group schemes
\[
\text{Nèr}(\text{Pic}_K^{0, d}) \otimes R_\ell \cong \text{Pic}_{R_\ell}^{0, d}.
\]
We conclude that \(\text{Pic}_{R_\ell}^{0, d}\) descends to \(\text{Nèr}(\text{Pic}_K^{0, d})\).

\[\square\]

4. The component group of the special fibre of \(\text{Pic}^0\)

The group of components of the special fibre of the Néron model of the Jacobian is interesting in its own right. It can be regarded as a purely combinatorial object. It is the quotient of the group of 0-cochains \(b\) with vanishing total value \(\varepsilon(b) = \sum_v b_v\) modulo the image of the Laplacian \(\Delta = \partial \delta\) of any integer valued 0-cochain (recall that \(\partial\) is the adjoint of \(\delta\)). It is convenient to express \(\partial, \delta\) and \(\Delta\) in terms of matrices. Once we fix an orientation \(\partial\) is the incidence matrix \(I = (i_{v, e})\) with \#\(V\) rows and \#\(E\) columns: the entry \(i_{v, e}\) equals \(\pm 1\) when \(v = e_\pm\) and vanishes otherwise. Then \(\delta\) is given by the transposed matrix \(I^T\) and the Laplacian is given by \(M = -I I^T\). The quotient \(\ker \varepsilon/\im(\Delta)\) is the critical group \(K(\Gamma)\).

Sandpile dynamics offers a different point of view. Let \(\chi_v^w\) be the characteristic function attached to a vertex: \(\chi_v^w = 0\) if \(w \neq v\) and \(\chi_v^v = 1\). Summing a 0-cochain \(\partial \delta \chi_v\) to an integer valued 0-cochain \(b\) produces the only 0-cochain whose total number of values equals \(\sum_v b_v\) and whose value on a vertex \(w \neq v\) is the same as \(b_w\) plus the number of oriented edges going from \(v\) to \(w\). This relates the group of components of the special fibre of the Néron model to the dynamics of abelian sandpiles which studies the collection of indistinguishable chips distributed among the vertices of a graph. In this model, a vertex \(v\) topples when it sends one chip to each neighbouring vertex; this is described by the above operation \(b \leadsto b + \partial \delta \chi_v\). This justifies the terminology sandpile or chip-firing. In this context the group \(\partial C^1(\Gamma; \mathbb{Z})/\partial \delta C^0(\Gamma; \mathbb{Z})\) is referred to as critical group.

The group is also analogous to the algebro-geometric notion Picard group \(\text{Pic}^{\text{tot}^0}\) of an algebraic curve. We refer to [7] and [8] and where \(\partial C^1(\Gamma; \mathbb{Z})\) is regarded as a group of divisors of total degree 0. Modding out \(\partial \delta C^0(\Gamma; \mathbb{Z})\) is often described in this context as the analogous as passing to linear equivalence classes. For this reasons the critical group \(K(\Gamma)\) is also referred to as the “Picard group” of the graph.

In [7] we can also find a combinatorial definition of the analogue of the Jacobian of the curve, regarded analytically as the quotient of the dual space of abelian differentials \(H^0(C, \omega_C)^\vee\) modulo \(H^1(C; \mathbb{Z}) \cong H^0(C, \omega_C)^\vee\). Indeed, we can consider 1-chains with values in \(\mathbb{Q}\). Then the lattice \(H^1(\Gamma; \mathbb{Z})\) is the analogue of \(H^1(C; \mathbb{Z})\), whereas \(H^1(\Gamma; \mathbb{Z})^\# = \{x \in H^1(\Gamma; \mathbb{Q}) | \langle x, \lambda \rangle \in \mathbb{Z}\text{ for all } \lambda \in H^1(\Gamma; \mathbb{Z})\}\) is the analogue of \(H^0(C, \omega_C)^\vee\). The Jacobian is the quotient
\[
\mathcal{J}(\Gamma) = \frac{H^1(\Gamma; \mathbb{Z})^\#}{H^1(\Gamma; \mathbb{Z})}.
\]

All the above groups are different presentations of the same finite Abelian group \(K(\Gamma)\) whose cardinality equals the complexity \(c(\Gamma)\): the number of spanning trees within \(\Gamma\). These are subgraphs of \(\Gamma\) with the same vertex set \(V\) as \(\Gamma\) and of type tree (the Betti number \(b\)
The complexity is efficiently computed by Kirchhoff’s matrix-tree theorem
\[ c(\Gamma) = (-1)^{s+t+#V-1} \det M_{s,t} \]
where \( M_{s,t} \) is obtained by suppressing the \( s \)th row and the \( t \)th line.

The previous results allow us to go through some natural generalisations of these notion and theorems when each edge \( e \) is equipped with a thicknesses \( t_e \). There are several interesting consequences even for the standard case \( t_e = 1 \) for all \( e \).

4.1. **The \( t \)-critical group** \( K_t(\Gamma) \). In the previous sections we encountered the following generalisations of the above mentioned objects. The \( t \)-critical group of \( \Gamma \) is
\[ K_t(\Gamma) = \frac{\partial^t C^1(\Gamma; \mathbb{Z})}{\partial t \delta C^0(\Gamma; \mathbb{Z})} \]
Notice how modding our \( \partial t \delta C^0(\Gamma; \mathbb{Z}) \) is the analogue for thick edges of the operation defining the equivalence between abelian sandpiles. In the present model one should think of fractional sandpiles, where \( v \) topples when it sends a fractional chip \( 1/t_e \) to the neighbouring vertex at the opposite extreme of the edge \( e \).

4.2. **The group of components** \( \Phi_t(\Gamma) \). The group of components is
\[ \Phi_t(\Gamma) = \frac{\partial^t C^1(\Gamma; \mathbb{Z})}{\partial t \delta C^0(\Gamma; \mathbb{Z})} \]
and, by Theorem 3.7 is indeed the group of components of the special fibre of the Néron model of the Jacobian of a smooth curve whose stable model has thicknesses \( t \). The group of characters is \( \ker \partial t \) is the kernel of \( \Phi_t(\Gamma) \to K_t(\Gamma) \).

4.3. **The Jacobian** \( J_t(\Gamma) \) of a graph. The group of characters attached to the thicknesses \( \ell t \) contain a copy of the group of components \( \Phi_t(\Gamma) \) for suitable choices of \( \ell \) (the multiples \( \ell \) of the exponent of the \( t \)-critical group). The subgroup isomorphic to \( \Phi_t(\Gamma) \) is the preimage of \( \text{im} \delta_\ell \) under the reduction modulo \( t_e \) of all values at each edge \( e \). Generalising the above notion of Jacobian of a graph \( \Gamma \), we can present the group \( \lim_{\ell} (\ker \partial_\ell \cap \text{im} \delta_\ell) \) as an extension
\[ 0 \to \Lambda_\ell \to \Lambda^\# \to \lim_{\ell} (\ker \partial_\ell \cap \text{im} \delta_\ell) \to 0, \]
where \( \Lambda \) is the lattice within \( \ker \partial_\ell \) whose values at the edge \( e \) lie in \( t_e \mathbb{Z} \) (as above we have \( \Lambda^\# = \{ x \in \ker \partial_\ell \mid \langle x, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \} \)). We set the notation
\[ J_t(\Gamma) = \lim_{\ell} (\ker \partial_\ell \cap \text{im} \delta_\ell) = \frac{\Lambda^\#}{\Lambda_\ell}. \]

4.4. **Abel theorem for a graph with thickness \( t \)**. Lemma 3.8 may be regarded as Abel’s theorem for a graph with thicknesses \( t \). Indeed we have an isomorphism between the \( t \)-critical group (playing the role of a sort of Picard group) and the Jacobian \( J_t(\Gamma) \) of the graph:
\[ \Phi_t(\Gamma) \cong J_t(\Gamma). \]
Notice how this provides a somewhat different proof even in the known case \( t = 1 \).
4.5. Complexity for a graph with thickness t. For $t = 1$, the complexity counts the number of elements of the critical group, which coincides with the group of components. In the general setup the complexity admits a natural generalisation yielding a positive integer counting the number of elements of the group of components.

Let $S$ be the set of spanning trees of $\Gamma$. Each element of $\Gamma$ is determined by a subset of $E$ of cardinality $#V - 1$ (this happens because the Betti number $b$ vanishes if and only if $1 - #V + #E$ vanishes). We can define the complexity $c_t(\Gamma)$ as the following weighted sum

$$c_t(\Gamma) = \sum_{S \in S} \prod_{e \not\in S} t_e,$$

where $e \not\in s$ means that $e$ is not one of the $#V - 1$ edges of $S$ (regarded as a set). Notice that any spanning tree of $\Gamma$ which does not contain the edges $e_1, \ldots, e_r$ yields $t_{e_1} \cdots t_{e_r}$ spanning trees of the “blown up” graph $\text{Bl}_t \Gamma$ obtained by subdividing the edge $e$ into $t_e$ edges. This is the graph of the special fibre of the regular semi-stable model. We have $c_t(\Gamma) = c(\text{Bl}_t \Gamma) = |\Phi(\text{Bl}_t \Gamma)|$. On the other hand, by Theorem 3.7, $\Phi_k(\Gamma)$ is isomorphic to $\Phi(\text{Bl}_t \Gamma)$; hence we have

$$c_t(\Gamma) = |\Phi_t(\Gamma)| = |\Phi(\text{Bl}_t \Gamma)| = c(\text{Bl}_t \Gamma),$$

which can be used to simplify the computation of ordinary graphs after contraction of all subchains. Now, the size of the $t$-critical group $\mathcal{K}_t(\Gamma)$ can be computed via (8)

$$|\mathcal{K}_t(\Gamma)| = c_t(\Gamma) \prod_{p} \prod_{l=0}^{\max_p} p^{-b_{p,l}}.$$

Notice that this is also a suitable index $\ell$ in Theorem 3.4.

4.6. Kirchhoff’s matrix-tree theorem for graph with thickness $t$. Kirchhoff theorem matrix-tree theorem extends straightforwardly as follows.

**Theorem 4.1.** We have

$$c_t(\Gamma) = (-1)^{a+b+\#V-1} \det M_{a,b} \prod_{e \in E} t_e,$$

where $M$ is the matrix representing the Laplacian $\partial^t \delta$ as a above and $M_{a,b}$ is the sub-matrix obtained by suppressing the $a$th row and the $b$th column.

**Proof.** Let $I_t$ be the incidence matrix of the graph with thicknesses. It is given by multiplying by $1/t_e$ the column corresponding to $e$. Notice that $I_t$ represents $\partial^t$, $\delta$ is still represented by $I^T$, and the Laplacian $\partial^t \delta$ is represented by $M = -I_t I^T$.

It is enough to prove the identity for $a = b$. Let $I^*$ and $I_t^*$ be the sub-matrices of the incidence matrix $I$ and $I_t$ obtained by deleting the row $a$. Notice that $M_{a,a} = -I^*(I^*)^T$. We have

$$c_t(\Gamma) = \sum_{S \in S} \prod_{e \in S} t_e = \sum_{S \in S} (\det I(S))^2 \prod_{e \in S} t_e,$$
where \( I(S) \) (resp. \( I_t(S) \)) is the square sub-matrix of \( I^* \) determined by suppressing all columns attached to the vertices \( e \notin S \). Notice that \( \det I(S) = \det I_t(S) \prod_{e \in S} t_e \). Then we have

\[
c_t(\Gamma) = \sum_{S \subseteq \mathcal{S}} \det I_t(S) \det I(S) \prod_{e \in E} t_e = \sum_{S \subseteq E} \det I_t(S) \det I(S) \prod_{e \in E} t_e = (-1)^{\#V-1} \det M_{a,a} \prod_{e \in E} t_e,
\]

where, in the last identity, we used the Binet–Cauchy formula.

\[\square\]

5. A Universal Group Scheme

5.1. The stack of \( \ell \)-stable curves. Let \( \overline{\mathcal{M}}^\ell_g \) be the stack of twisted curves with all stabilisers of order \( \ell \) and stable moduli space. We call its objects \( \ell \)-stable curves because the condition imposing the same order on the stabilisers is indeed a stability condition (in the sense that a unique stable reduction exists). Every stable dual graph \( \Gamma \) determines a locally closed substack \( \mathcal{M}^\ell_\Gamma \), the full subcategory of \( \ell \)-stable curves whose dual graph is \( \Gamma \).

We have a stratification

\[\overline{\mathcal{M}}^\ell_g = \bigsqcup_{\Gamma} \mathcal{M}^\ell_\Gamma,\]

where \( \Gamma \) runs over all stable genus-\( g \) graphs and \( \mathcal{M}^\ell_\Gamma \) is in the closure of \( \mathcal{M}^\ell_{\Gamma'} \) if \( \Gamma' \) can be obtained from \( \Gamma \) via edge-contraction. Consider the analogue stratification \( \mathcal{M}^\ell_{\Gamma} \subset \overline{\mathcal{M}}^\ell_g \) within the Deligne–Mumford stack of stable curves; then the forgetful morphism \( \mathcal{M}^\ell_{\Gamma} \to \mathcal{M}^\ell_{\Gamma'} \) is a constant \( (\mu_\ell)^E \)-gerbe; in other words the automorphism group \( \text{Aut}_{\mathcal{C}}(\mathcal{C}) \) classifying ghost automorphisms of \( \mathcal{C} \) that fix the coarse space is canonically isomorphic to \( (\mu_\ell)^E \) for any \( \ell \)-stable curve \( \mathcal{C} \) with coarse space \( \mathcal{C} \) and the dual graph \( \Gamma \), see [3, §7]. The ghost automorphism group scheme, which is trivial over \( \mathcal{M}^\ell_{\Gamma} \), contains the distinguished subgroup scheme \( \Delta^\ell_{\Gamma} \) of diagonal automorphisms. This consists of the ghost automorphisms \( \zeta \in \mu_\ell \) acting simultaneously at all nodes \( \{(xy = 0)/\mu_\ell\} \) as \( \zeta \cdot (x,y) = (\zeta x, \zeta y) \equiv (x,\zeta y) \). We have \( \Delta^\ell_{\Gamma} = \mu_\ell \) over \( \mathcal{M}^\ell_{\Gamma} \).

5.2. The Picard group scheme over \( \overline{\mathcal{M}}^\ell_g \). Consider the separated group scheme \( \mathcal{P}ic^0_g \) representing the functor \( \text{Pic}^0 \) over the moduli stack \( \overline{\mathcal{M}}^\ell_g \) (the proof of the separatedness follows almost immediately from Lemma 2.4. On each substack \( \mathcal{M}^\ell_{\Gamma} \) we consider \( \mathcal{P}ic^0_{\Gamma,a} \), which is a disjoint union of components \( \mathcal{P}ic^0_{\Gamma,a} \) parametrised by \( a \in \ker \partial_{\ell} \)

\[\mathcal{P}ic^0_{\Gamma,a} = \bigsqcup_{a \in \ker \partial_{\ell}} \mathcal{P}ic^0_{\ell,a}.\]

By §2.6 the the components labelled by \( a \in \ker \partial_{\ell} \cap \text{im} \delta_{\ell} \) are fixed by \( \Delta \) whereas in the remaining components no point is fixed. Consider

\[\mathcal{P}ic^0_{\ell} := \mathcal{P}ic^0_g \setminus \bigsqcup_{\Gamma,a \in \text{im} \delta_{\ell}} \mathcal{P}ic^0_{\Gamma,a}.\]

It is a separated subgroup scheme of \( \mathcal{P}ic^0_g \) (the invariance with respect to \( \Delta^\ell_{\Gamma} \) is closed under multiplication).

We can now state Theorem 3.4 in a global form. We recall that a smoothing of \( C_k \) is a family \( C_R \) over \( R \) whose generic fibre is smooth and whose special fibre is isomorphic to \( C_k \). The base change via \( R \subseteq R_{\ell} \) of \( C_R \) is not a smoothing of the generic fibre, but is associated
to to a unique regular stable twisted model $\mathcal{C}^{tw}(\ell) \to \text{Spec} R_\ell$. This is an $\ell$-stable curve and we may regard it as a morphism $\text{Spec} R_\ell \to \overline{\mathcal{M}}_g^\ell$.

**Corollary 5.1.** For any stable curve $C_k$ and for any of its smoothings $C_R$ over $R$, the Néron model of the Jacobian Pic$_K$ of the generic fibre coincides, after pullback to $R_\ell$, with the base change of Pic$_g^0,\ell \to \overline{\mathcal{M}}_g^\ell$ via $\text{Spec} R_\ell \to \overline{\mathcal{M}}_g^\ell$, the morphism induced by the regular twisted model $\text{Spec} R_\ell \to \overline{\mathcal{M}}_g^\ell$ associated to $C_R$. Summarising, we have the following fibre diagrams

\[
\begin{array}{cccc}
\text{Pic}_K & \xrightarrow{N(\text{Pic}_K)} & \text{Pic}_R^0,\ell & \xrightarrow{\text{Pic}_g^0,\ell} \\
\downarrow & & \downarrow & \\
\text{Spec} K & \to & \text{Spec} R & \to \overline{\mathcal{M}}_g^\ell.
\end{array}
\]

\[\square\]

**5.3. The different approaches of Caporaso and of Holmes.** We discuss here two different approaches to the problem of assembling into a single universal family the special fibres of the Néron model of the Jacobian Pic$^0 C_K$.

Caporaso’s approach [13] is a subfunctor of the Picard functor. It is given by singling out within the Picard functor the points representing the so-called “balanced” line bundles which we recall below. As a first step, we switch to the study of the variety Pic$^d C_K$, where $C_K$ is the smooth curve, generic fibre of a stable curve over $R$. We assume $g \geq 3$ and that $2g - 2$ is prime to $d - g + 1$, which rules out the case of the Jacobian Pic$^0 C_K$. However there will be consequences for the initial problem of Néron models of Jacobians in some special cases where we can rely on a trivialization of the torsor Pic$^d C_K \cong$ Pic$^0 C_K$.

Let us illustrate first the notion of balanced multidegree. In rough terms a line bundle $L$ on a stable curve $C$ is balanced of degree $d$ essentially if it has degree $d$ and if its multi-degree $d = (\deg L|_{Z_1}, \ldots, \deg L|_{Z_r})$ is “not too far” from

\[
\left(\frac{\deg \omega|_{Z_1}}{\deg \omega}, \ldots, \frac{\deg \omega|_{Z_r}}{\deg \omega}\right),
\]

where $Z_1, \ldots, Z_r$ are the irreducible components of $C$ and $\omega$ is the canonical bundle of $C$. In precise terms we impose

\[
\left|\deg L|_Z - d \frac{\deg \omega|_Z}{\deg \omega_C}\right| < \frac{\#(Z \cap C \setminus Z)}{2}
\]

for any proper subcurve $Z \subset C$. Caporaso restricts the Picard functor of line bundles of total degree $d$ by imposing the above balancing condition: we get in this way $\mathcal{N}_g^d \to \overline{\mathcal{M}}_g$. It is a remarkable fact that the above inequality, originating naturally in Geometric Invariant Theory, under the hypotheses $(d - g + 1, 2g - 2) = 1, g \geq 3$, identifies a representable morphism $\mathcal{N}_g^d \to \overline{\mathcal{M}}_g$ whose restriction on each trait Spec $R \to \overline{\mathcal{M}}_g$ transversal to the boundary $\partial \overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is the Néron model of its generic fibre, the torsor Pic$^d(C_K)$.

**Example 5.2.** Consider a smooth curve $C_K$ of genus 3 degenerating at the special point of Spec $R$ to a curve $C_k$ with two smooth components $C_1$ and $C_2$ of genus 1 and two nodes. We can consider the above functor $\mathcal{N}_3^1$ which, over the special fibre, allows only multidegrees $(0, 1)$ and $(1, 0)$. Indeed Pic$^{(0,1)} C_k$ and Pic$^{(1,0)} C_k$ are the two components of the special fibre of the Néron model of Pic$^1 C_K$. 

In this example we can describe the scheme \( \Pic^{(0,1)} C_k \sqcup \Pic^{(1,0)} C_k \) without ordering the two components with the indices “1” and “2”; for this, we can use the expression “balanced line bundles”, or we can unravel this notion in this case by saying that \( \mathcal{N}_g^1 \otimes R \) is the scheme parametrising line bundles of total degree-1 and nonnegative degree on all components. As we see in the next example, this can be done only because the hypotheses gcd\((d-g+1, 2g-2) = 1\) and \(g \geq 3\) are satisfied.

Over \( \overline{\mathcal{M}}_{g,n} \) with \( n \geq 1 \) we can exploit the functor of balanced line bundle of total degree \( d \) to form \( \mathcal{N}^d_{g,n} \). Then, via the isomorphism \( L \to L \otimes \mathcal{O}(-d[x_1]) \), where \( x_1 \) denotes the first marking, we provide a separable family whose restriction on every trait Spec \( R \to \overline{\mathcal{M}}_{g,n} \) transversal to the boundary locus \( \partial \mathcal{M}_{g,n} = \mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n} \) is the Néron model of its generic fibre, the Jacobian \( \Pic^0 C_K \).

**Example 5.3.** We examine again the previous example where the curve \( C_K \), which we now assume equipped with a distinguished marking \( x_1 \), degenerates to the 2-noded curve \( C_1 \cup C_2 \), where we assume that \( x_1 \) lies on \( C_1 \). The isomorphism \( \phi: L \to L \otimes \mathcal{O}(-[x_1]) \) yields \( \phi \mathcal{N}^3_{3,1} \) which, over the special fibre, consists of points parametrizing line bundles of multidegrees \((-1, 1)\) and \((0, 0)\). Indeed \( \Pic^{(-1,1)} C \) and \( \Pic^{(0,0)} C \) are the two components of the special fibre of the Néron model of \( \Pic^0 C_K \).

Notice that this approach uses in a crucial way the marking \( x_1 \) which distinguishes one, privileged, irreducible component. The scheme \( \phi \mathcal{N}^3_{3,1} \) can never admit a group structure compatible with that of the Jacobian over \( \mathcal{M}_{g,n} \).

We now apply Corollary 5.1 and consider the functor \( \Pic^{(0,1)} C_k \), whose fibre over a point of \( \partial \mathcal{M}_g = \mathcal{M}_g \setminus \mathcal{M}_g \) is a group of line bundles on an \( \ell \)-stable curve \( C \) of degree zero on every component. Furthermore this group is isomorphic to the special fibre of Néron model of a smoothing of the coarse space \( C \) of \( C \). Recall that \( \ell \) should be a multiple of the exponent of any critical group of any genus-3 stable graph. In particular since the dual graph of \( C_1 \cup C_2 \) is the 2-cycle graph whose critical group is \( \mathbb{Z}/2 \), we should assume \( \ell \in 2 \mathbb{Z} \).

**Example 5.4.** Consider again the smooth genus-3 curve \( C_K \) degenerating at the special point of Spec \( R \) to a curve \( C_k \) with two smooth genus-1 components \( C_1 \) and \( C_2 \). Over \( R_\ell \) we consider the corresponding regular twisted model whose special fibre \( C \) has stabilisers of order \( \ell \) at the two nodes \( n \) and \( n' \) locally described as \( \{x y = 0\} \) and \( \{x' y' = 0\} \). We can consider \( \Pic^{(0,1)} \) which, over the special fibre, represents line bundles whose degree vanishes on each component and such that \( \gamma^* L \cong L \), where \( \gamma \) is the ghost automorphism \( \gamma(x, y) = (\xi x, y) \) and \( \gamma(x', y') = (\xi x', y') \).

We get a new description of the two components forming the special fibre of the Néron model of the Jacobian of \( C_K \). One component parametrizes line bundles on \( C \) arising as pull-backs from the coarse space \( C \); namely, locally at the node, the \( \mu_\ell \)-action is

\[
\frac{1}{\ell}(1, \ell - 1, 0)
\]

where the entries 1 and \( \ell - 1 \) describe the action on the curve and the third entry describes the trivial action on the fibre of the line bundle \( L \). The second component parametrizes line bundles which still have degree 0 on both components ad whose \( \mu_\ell \)-action on each node is given by

\[
\frac{1}{\ell}(1, \ell - 1, \frac{\ell}{2})
\]
As observed in Remark 1.2, when we are only interested in a local picture over $R$, we can choose $\ell = 2$. The advantage of the approach via $\Pic^\ell_g$ is that we do not need to privilege one component among $C_1$ and $C_2$.

We can now place the attention on a universal deformation of $C_1 \cup C_2$. The deformation space $U$ contains a normal crossing divisor with two irreducible components $D' = (u = 0)$ and $D'' = (v = 0)$ parametrising the deformation along which $n_2$ is smoothed and $n_1$ persists and the deformation along which $n_1$ is smoothed and $n_2$ persists. Caporaso’s functor of balanced Picard groups allows us to patch together the Néron models along each trait $u = \lambda v$.

We can also apply Theorem 3.4. The scheme $\tilde{\mathcal{N}}^1$ cannot be equipped with a group structure. On the other hand, when we consider $\tilde{U}$ the spectrum of the ring obtained by extracting a $\ell$th roots $\tilde{u}$ and $\tilde{v}$ from the parameters $u$ and $v$, we can consider a map $\tilde{U}$ to $\tilde{\mathcal{M}}^\ell_g$ and a family $C$ of $\ell$th twisted curves over $\tilde{U}$ whose coarse space descends to a family $C$ of stable curves over $U$. There we can consider the functor $\Pic^00\ell$ and its restriction $\Pic^00\ell_U$ on the scheme $\tilde{U}$. We get a group scheme whose restriction on each trait $\Spec R \to \tilde{U}$ mapping $\pi \mapsto (\pi, \lambda\pi)$ with $\lambda \neq 0$ is the pullback of the Néron model of the Jacobian of the smooth curve $\Spec(K^H) \times_U C$.

One can now address a more ambitious question: namely, can we provide a group scheme over a universal deformation of $C_1 \cup C_2$ containing the Néron models of any Jacobian $\text{Jac}C_K$ for any injective trait $\Spec R \to U$ whose generic point maps into the open locus of $U \setminus (D \cup D')$? It is not difficult to see that such a group scheme cannot be of finite type; indeed we can approach the special point $u = v = 0$ by mapping $\pi$ to $(\pi^n, \lambda\pi^m)$ with $\lambda \neq 0 \gcd(n, m) = 1$ and the special fibre of the Néron model of the Jacobian of the generic curve is a group scheme with $n + m$ components. The work of Holmes [23] provides a positive answer by modifying $U$ as follows.

**Example 5.5.** Let $U_0$ be $U$, let $D_0$ be $D \cup D'$, and let $B_0$ be $\text{Sing}(D_0) = \{u = 0, v = 0\}$. Then, we set $U_1 = \text{Bl}_{B_0}(U_0)$ mapping via $f_1$ to $U_0$, $D_1 = f^*D_0$, and $B_1 = \text{Sing}D_1$. We iterate by blowing up $B_1$ within $U_1$, etc. We now consider the co-limit $V$ induced by $V_0 = U_0 \setminus B_0 \to V_1 = U_1 \setminus B_1 \leftarrow V_2 = U_2 \setminus B_2 \leftarrow \ldots$ with a divisor induced by the co-limit $D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \ldots$ Any trait $\Spec R \to U$ mapping the generic point to $U_0 \setminus D_0$ naturally lifts to $V$ and Holmes provides a group scheme $\tilde{N}_V$ over $V$ such that the Néron model of $\text{Jac}(C_U \otimes K) = N_V \otimes K$ is $N_V \otimes R$.

This construction is all the more satisfactory because it arises as the special fibre over the deformation space $U$ of a global group scheme $\tilde{\mathcal{N}}_g$, which is a model of the the universal Jacobian $\Pic^0$ over $\mathcal{M}_g$ and satisfy an analogue of the Néron property over a birational non-proper modification $\tilde{\mathcal{M}}_g \to \mathcal{M}_g$ restricting to an isomorphism on $\mathcal{M}_g$. Indeed in [23], Holmes produces $\tilde{\mathcal{N}}_g$ as the terminal object in a category of “Néron admitting morphisms”, see Defn. 1.1 and 1.2 in [23].

In the above construction both $\tilde{\mathcal{N}}_g$ and the base $\tilde{\mathcal{M}}_g$ are not of finite type. We point out that Olsson’s stack $\mathcal{M}_g^{\text{tw}}$ of all twisted curves (see [35]) can be also equipped with a group scheme $\Pic^{0\text{tw}}_g$ parametrising line bundles of degree zero on all components. It should be noticed that $\mathcal{M}_g^{\text{tw}}$ is not of finite type nor separated (the stack of $\ell$-stable curves was introduced precisely in order to identify a proper substack of $\mathcal{M}_g^{\text{tw}}$). Reformulating 3.4, we can obtain Néron models for any trait as follows.
Corollary 5.6. Consider a stable curve $C_R$ over a discrete valuation ring $R$ whose fibre over $K$ is the smooth curve $C_K$ and whose special fibre $C_k$ has thicknesses $t$, dual graph $\Gamma_k$ and $t$-critical group $K_t(\Gamma_k)$.

For any positive integer $\ell$, we consider the regular twisted model $C^{tw}$ over $R$ and the corresponding morphism $\text{Spec} R_\ell \to \mathcal{M}_g^{tw}$. The $\text{Pic}^0$ functor is a group scheme over $\mathcal{M}_g^{tw}$ and the pullback over $R_\ell$ is a $\mu_\ell$-equivariant morphism $\text{Pic}^0_{R_\ell} \to \text{Spec} R_\ell$ whose special fibre splits into the disjoint union of a $\mu_\ell$-fixed part $F_\ell$ and a $\mu_\ell$-moving part $M_\ell$.

Then, as soon as $\ell$ is a multiple of the exponent of the $t$-critical group $K_t(\Gamma_k)$, the Néron model of the Jacobian $\text{Pic}_K$ of the generic fibre coincides, after pullback to $R_\ell$, with the complement of the moving part $M_\ell$ of the base change of $\text{Pic}_0^{tw} \to \mathcal{M}_g^{tw}$ via $\text{Spec} R_\ell \to \mathcal{M}_g^{tw}$.

Summarising, we have the following fibre diagrams

\[
\text{Pic}_K \quad \xrightarrow{\quad N(\text{Pic}_K) \quad} \quad \text{Pic}^0_{R_\ell} = \text{Pic}^0_{R_\ell} \setminus M_\ell \quad \subseteq \quad \text{Pic}^0_{R_\ell} \quad \xrightarrow{\quad \mathcal{M}_g^{tw} \quad} \quad \mathcal{M}_g^{tw}.
\]

Remark 5.7. The statement above does not hold as soon as the index $\ell$ is not a multiple of the exponent of $K_t(\Gamma_k)$. Furthermore we point out that (13) provides an explicit formula for an index $\ell$ for which the statement holds — a multiple of the exponent of the $t$-critical group.

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