HOLOMORPHIC RELATIVE HOPF MODULES OVER THE IRREDUCIBLE QUANTUM FLAG MANIFOLDS

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Abstract. We construct covariant $q$-deformed holomorphic structures for all finitely-generated relative Hopf modules over the irreducible quantum flag manifolds endowed with their Heckenberger–Kolb calculi. In the classical limit these reduce to modules of sections of holomorphic homogeneous vector bundles over irreducible flag manifolds. For the case of simple relative Hopf modules, we show that this covariant holomorphic structure is unique. This generalises earlier work of Majid, Khalkhali, Landi, and van Suijlekom for line modules of the Podleš sphere, and subsequent work of Khalkhali and Moatadelro for general quantum projective space.

1. Introduction

In this paper we consider noncommutative generalisations of homogeneous holomorphic vector bundles for the irreducible quantum flag manifolds. Ideas from classical complex geometry have, to a greater or lesser extent, always played a role in noncommutative geometry. This is not surprising, given that no examples can claim to be more influential than the noncommutative torus $\mathbb{T}^2_{\theta}$ and the Podleš sphere $\mathcal{O}_q(S^2)$. Both are noncommutative deformations of manifolds carrying a complex geometry in the classical limit. In fact, both $\mathbb{T}^2$ and $S^2$ are Kähler manifolds, the former being a Fano manifold and the latter a Calabi–Yau manifold. Much of the classical complex geometry of these examples survives deformation intact. Of particular relevance to this paper is the work of Polishchuk and Schwarz on $\theta$-deformed holomorphic vector bundles over the noncommutative torus [33, 32], and Majid’s description of the noncommutative complex geometry of the Podleš sphere in [25].

FDG is partially funded by Conacyt (Consejo Nacional de Ciencia y Tecnología, México). AK was supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund — the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004). RÓB acknowledges FNRS support through a postdoctoral fellowship within the framework of the MIS Grant “Antipode” grant number F.4502.18. RÓB and PS are partially supported from the Eduard Čech Institute within the framework of the grant GAČR 19–28628X, and by the grant GAČR 306–33/1906357. Research of KRS is supported by the GAČR project 20-17488Y and RVO: 67985840.
The notion of a noncommutative complex structure was introduced in [20] and [4] as an abstract framework in which to study the noncommutative complex geometry of both the noncommutative torus and the Podleś sphere. In particular, holomorphic modules, often called noncommutative holomorphic vector bundles, were introduced. The definition of a holomorphic module was motivated by two the classical Koszul–Malgrange equivalence between holomorphic vector bundles and smooth vector bundles endowed with a flat $(0, 1)$-connection [24]. Its prototypical examples were those in [33, 32] and [25] as mentioned above. In particular, homogeneous line modules over the Podleś sphere have canonical holomorphic structures in this sense. Moreover, they are known to satisfy a direct $q$-deformation of the classical Borel–Weil theorem [25, 20]. An extension of these results to the case of general projective space was established by Khalkhali and Moatadde [21] using a $q$-deformed Dolbeault anti-holomorphic complex originally introduced in [10] to construct spectral triples.

In [2], Beggs and Majid later introduced the notion of an Hermitian holomorphic module over an algebra $A$ (where they were called Hermitian holomorphic vector bundles) and showed that any such module admits a Chern connection [2, Theorem 8.53], see also [3, §4]. Around the same time, noncommutative Kähler structures were introduced by the third author as a framework for studying noncommutative Kähler geometry on quantum homogeneous spaces. In joint work of the third author with Šťovíček and van Roosmalen [31] it was shown that Hermitian holomorphic modules, defined over an algebra endowed with a noncommutative Kähler structure, have a remarkably rich theory extending the classical situation of Hermitian holomorphic vector bundles over Kähler manifolds. For example, the definition of a positive line bundle carries over directly, implying a natural definition of noncommutative Fano structure [31]. In this setting noncommutative generalisations of twisted Hodge theory, the Kodaira vanishing theorem, and Serre duality all hold[31], giving powerful results and techniques with which to study holomorphic modules and their associated Dolbeault cohomology.

The next step is to enlarge the family of examples beyond the rather specialised situation of line modules over quantum projective space. A natural choice is the quantum flag manifolds $O_q(G/L_S)$, the quantum counterpart of classical flag manifolds $G/L_S$, where $G$ is a compact Lie group and $L_S$ is a subgroup of $G$ indexed by some subset $S$ of the simple roots of $G$. The quantum flag manifolds form a far more general class of quantum homogeneous spaces whose theory is deeply rooted in the representation theory of Drinfeld–Jimbo quantum groups. For the irreducible case, which is to say, those quantum flag manifolds which are irreducible (or equivalently Hermitian symmetric spaces) in the classical limit, Heckenberger and Kolb established that each quantum space has an essentially unique covariant $q$-deformed de Rham complex $\Omega_q^\bullet(G/L_S)$ [15, 16]. These remarkable differential calculi are some of the most important objects in the study of the noncommutative geometry of quantum groups. As shown in [30, 26] each $\Omega_q^\bullet(G/L_S)$ comes endowed
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with a unique covariant noncommutative Kähler structure. For the special case of quantum projective space, the $q$-deformed anti-holomorphic Dolbeault complex of [10] can be realised as a subcomplex of the Heckenberger–Kolb calculus.

This raises the question of whether we can construct holomorphic structures for every finitely-generated relative Hopf module over each irreducible quantum flag manifold. Our main result, **Theorem 4.5**, shows that this is indeed possible. To establish the result, we need to establish the existence of covariant $(0, 1)$-connections and verify that they are flat. In [21] the $(0, 1)$-connections for higher order projective spaces $\mathcal{O}_q(\mathbb{C}P^n)$ were explicitly constructed and flatness was verified by direct calculation. Extending this approach to all irreducible quantum flag manifolds would be prohibitively lengthy and tedious. Instead, we show existence and flatness by turning to the general theory of principal comodule algebras, following an approach closer to the original constructions of Majid [25].

While the existence of holomorphic connections is highly interesting in its own right, we are especially interested in the implications of our main result. In particular, the existence of holomorphic connections for line modules is an essential ingredient in a number of associated works. In [13] the holomorphic line modules over the irreducible quantum flag manifolds are shown to satisfy a direct $q$-deformation of the classical Borel–Weil theorem. This extends the case of quantum projective space discussed above, as well as the more general quantum Grassmannian picture established in [27]. Building on the $q$-deformed Borel–Weil theorem, all non-trivial line modules over the irreducible quantum flag manifolds were identified in [12] as either positive or negative. This provides valuable information about the behaviour of their $q$-deformed Chern curvatures. These results in turn allowed Das and the third and fourth authors to establish the Fredholm property for any Dolbeault–Dirac operator twisted by a negative line module [11]. These operators then become natural candidates for spectral triples in the sense of Connes and Moscovici [9]. The existence of holomorphic structures also suggests a number of important new directions of research, such as the extension of Kostant’s theorem to the quantum setting as conjectured in §4.6.

The paper is organised as follows. In §2 we recall necessary preliminaries about differential calculi, complex structures, connections, holomorphic structures, principal comodule algebras, and strong principal connections.

In §3 we prove some general results about covariant connections for homogeneous vector bundles over quantum homogeneous spaces. In particular, we show that Takeuchi’s equivalence allows one to transfer flatness and uniqueness conditions for covariant connections into representation-theoretic conditions. We then observe that for a quantum homogeneous space $B = A^{co(H)}$, cosemisimplicity of $H$ implies the existence of a left $A$-covariant strong principal connection. Using this result, we are able to produce covariant connections for finitely-generated relative Hopf modules with respect to any covariant calculus.
In §4 the basic definitions and results of Drinfeld–Jimbo quantum groups and their quantised coordinate algebras are recalled. We then present the definition of the quantum flag manifolds, focusing on the special case of irreducible flag manifolds and their Heckenburger–Kolb calculi. We apply the general results of §3 to the finitely-generated relative Hopf modules over $O_q(G/L_S)$, and using a quantum Lie theoretic argument, prove flatness of the $(0,1)$-connections, as well as uniqueness in the irreducible case.

We would like to thank Edwin Beggs and Adam-Christaan van Roosmalen for useful discussions.

2. Preliminaries

In this section we recall the necessary preliminaries on differential calculi, complex structures, connections, holomorphic structures, and strong principal connections.

2.1. Calculi, Connections, and Holomorphic Modules.

2.1.1. Differential Calculi. A differential calculus $(\Omega^\bullet \simeq \bigoplus_{k \in \mathbb{N}_0} \Omega^k, d)$ is a differential graded algebra (dg-algebra) which is generated in degree 0 as a dg-algebra, that is to say, it is generated as an algebra by the elements $a, db$, for $a, b \in \Omega^0$. A differential $*$-calculus is a differential calculus equipped with a conjugate linear involutive map $*: \Omega^\bullet \to \Omega^\bullet$ satisfying $d(\omega^*) = (d\omega)^*$, and $(\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*$, for all $\omega \in \Omega^k, \nu \in \Omega^l$.

For a given algebra $B$, a differential calculus over $B$ is a differential calculus $(\Omega^\bullet, d)$ such that $\Omega^0 = B$. Note that if $(\Omega^\bullet, d)$ is a differential $*$-calculus over $B$, then $B$ is a $*$-algebra. We say that $\omega \in \Omega^\bullet$ is closed if $d\omega = 0$. See [3, §1] for a more detailed discussion of differential calculi.

2.1.2. First-Order Differential Calculi. A first-order differential calculus over an algebra $B$ is a pair $(\Omega^1, d)$, where $\Omega^1$ is a $B$-bimodule and $d : B \to \Omega^1$ is a linear map for which the Leibniz rule holds

$$d(ab) = a(db) + (da)b,$$

and for which $\Omega^1$ is generated as a left $B$-module by those elements of the form $db$, for $b \in B$. The universal first-order differential calculus over $B$ is the pair $(\Omega^1_u(B), d_u)$, where $\Omega^1_u(B)$ is the kernel of the multiplication map $m_B : B \otimes B \to B$ endowed with the obvious bimodule structure, and $d_u$ is the map defined by

$$d_u : B \to \Omega^1_u(B), \quad b \mapsto 1 \otimes b - b \otimes 1.$$

Every first-order differential calculus over $B$ is of the form $(\Omega^1_u(B)/N, \text{proj} \circ d_u)$, where $N$ is a $B$-sub-bimodule of $\Omega^1_u(B)$ and \text{proj} : $\Omega^1_u(B) \to \Omega^1_u(B)/N$
is the canonical quotient map. This gives a bijective correspondence between first-order differential calculi and sub-bimodules of \( \Omega^1_u(B) \).

We say that a differential calculus \((\Gamma^\bullet, d\Gamma)\) extends a first-order differential calculus \((\Omega^1, d\Omega)\) if there exists a bimodule isomorphism \( \phi : \Omega^1 \rightarrow \Gamma^1 \) such that \( d\Gamma = \phi \circ d\Omega \). It can be shown that any first-order differential calculus admits an extension \( \Gamma^\bullet \) which is maximal in the sense that there exists a unique differential map from \( \Omega^\bullet \) onto any other extension of \( \Omega^1 \), see [3, §1.5] for details. We call this extension the maximal prolongation of the first-order differential calculus.

2.1.3. Connections. Motivated by the Serre–Swan theorem, we think of a finitely-generated projective left \( B \)-module \( F \) as a noncommutative generalisation of a vector bundle. For \( \Omega^\bullet \) a differential calculus over an algebra \( B \) and \( F \) a finitely-generated projective left \( B \)-module, a connection on \( F \) is a \( C \)-linear map \( \nabla : F \rightarrow \Omega^1 \otimes_B F \) satisfying

\[
\nabla(bf) = db \otimes f + b \nabla f,
\]

for all \( b \in B, f \in F \).

An immediate but important consequence of the definition is that the difference of two connections \( \nabla - \nabla' \) is a left \( B \)-module map.

Any connection can be extended to a map \( \nabla : \Omega^\bullet \otimes_B F \rightarrow \Omega^\bullet \otimes_B F \) uniquely defined by

\[
\nabla(\omega \otimes f) = d\omega \otimes f + (-1)^{|\omega|} \omega \wedge \nabla f,
\]

where \( f \in F \), and \( \omega \) is a homogeneous element of \( \Omega^\bullet \) of degree \( |\omega| \). The curvature of a connection is the left \( B \)-module map \( \nabla^2 : F \rightarrow \Omega^2 \otimes_B F \). A connection is said to be flat if \( \nabla^2 = 0 \). Since \( \nabla^2(\omega \otimes f) = \omega \wedge \nabla^2(f) \), a connection is flat if and only if the pair \( (\Omega^\bullet \otimes_B F, \nabla) \) is a cochain complex.

2.1.4. Complex Structures. In this subsection we recall the definition of a complex structure for a differential calculus, as introduced in [20, 4], see also [3]. This gives an abstract characterisation of the properties of the de Rham complex of a classical complex manifold [18].

**Definition 2.1.** A complex structure \( \Omega^{\bullet, \bullet} \), for a differential \( * \)-calculus \((\Omega^\bullet, d)\), is an \( \mathbb{N}_0^2 \)-algebra grading \( \bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)} \) for \( \Omega^\bullet \) such that, for all \( (a, b) \in \mathbb{N}_0^2 \):

1. \( \Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)} \),
2. \( \mathsf{d}\mathsf{d} \Omega^{(a,b)} = \Omega^{(b,a)} \),
3. \( \mathsf{d}\mathsf{d}\mathsf{d} \Omega^{(a,b)} \subseteq \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)} \).

An element of \( \Omega^{(a,b)} \) is called an \((a, b)\)-form. For \( \mathsf{proj}_{\Omega^{(a+1, b)}} \) and \( \mathsf{proj}_{\Omega^{(a, b+1)}} \), the projections from \( \Omega^{a+b+1} \) to \( \Omega^{(a+1, b)} \), and \( \Omega^{(a, b+1)} \) respectively, we write

\[
\partial|_{\Omega^{(a,b)}} := \mathsf{proj}_{\Omega^{(a+1,b)}} \circ \mathsf{d}, \quad \overline{\partial}|_{\Omega^{(a,b)}} := \mathsf{proj}_{\Omega^{(a,b+1)}} \circ \mathsf{d}.
\]
It follows from Definition 2.1.3 that for any complex structure,
\[ d = \partial + \overline{\partial}, \quad \overline{\partial} \circ \partial = - \partial \circ \overline{\partial}, \quad \partial^2 = \overline{\partial}^2 = 0. \]
Thus \( (\bigoplus_{(a,b) \in \mathbb{N}_0} \Omega^{(a,b)}, \partial, \overline{\partial}) \) is a double complex. Both \( \partial \) and \( \overline{\partial} \) satisfy the graded Leibniz rule. Moreover,
\[ (2) \quad \partial(\omega^*) = (\overline{\partial} \omega)^*, \quad \overline{\partial}(\omega^*) = (\partial \omega)^*, \quad \text{for all } \omega \in \Omega^*. \]
See [3, §1] or [29] for a more detailed discussion of complex structures.

2.1.5. Holomorphic Modules. In this subsection we present the notion of a holomorphic left \( B \)-module for an algebra \( B \). Such a module should be thought of as a noncommutative holomorphic vector bundle, as has been considered in a number of previous papers, see for example [4], [33], and [20]. Indeed, the definition for holomorphic modules is motivated by the classical Koszul–Malgrange characterisation of holomorphic bundles [24]. See [31] for a more detailed discussion.

With respect to a choice \( \Omega^{(\bullet, \bullet)} \) of complex structure on \( \Omega^\bullet \), a \((0, 1)\)-connection on \( F \) is a connection with respect to the differential calculus \( (\Omega^{(0, \bullet)}, \partial) \).

Definition 2.2. Let \( (\Omega^\bullet, d) \) be a differential *-calculus over an algebra \( B \), equipped with a complex structure \( \Omega^{(\bullet, \bullet)} \). A holomorphic left \( B \)-module is a pair \( (F, \overline{\partial}_F) \), where \( F \) is a finitely-generated projective left \( B \)-module, and \( \overline{\partial}_F : F \rightarrow \Omega^{(0, 1)} \otimes_B F \) is a flat \((0, 1)\)-connection. We call \( \overline{\partial}_F \) the holomorphic structure of the holomorphic left \( B \)-module.

In the classical setting the kernel of the holomorphic structure map coincides with the space of holomorphic sections of a holomorphic vector bundle. This motivates us to call
\[ H^0_{\Omega}(F) = \ker \left( \overline{\partial}_F : F \rightarrow \Omega^{(0, 1)} \otimes_B F \right), \]
the space of holomorphic sections of \( (F, \overline{\partial}_F) \).

2.2. Quantum Homogeneous Spaces and Holomorphic Relative Hopf Modules. From this point in the paper \( A \) and \( H \) will always denote Hopf algebras defined over \( \mathbb{C} \), with coproduct, counit, and antipode denoted by \( \Delta, \epsilon \), and \( S \) respectively, without explicit reference to the Hopf algebra in question. Moreover, all antipodes are assumed to be invertible, and so, we always have an equivalence between the categories of right and left comodules of any Hopf algebra.

2.2.1. Comodule Algebras and Quantum Homogeneous Spaces. For \( H \) a Hopf algebra, and \( V \) a right \( H \)-comodule with coaction \( \Delta_R \), we say that an element \( v \in V \) is coinvariant if \( \Delta_R(v) = v \otimes 1 \). We denote the subspace of all coinvariant elements by \( V^{\text{co}(H)} \) and call it the coinvariant subspace of the coaction.

A right \( H \)-comodule algebra \( P \) is a right \( H \)-comodule which is also an algebra, such that the comodule structure map \( \Delta_R : P \rightarrow P \otimes H \) is an algebra map. Equivalently, it is a monoid object in \( \text{Mod}^{H} \), the category of right \( H \)-comodules.
Note that for a right $H$-comodule algebra $P$, its coinvariant subspace $B := P^{\text{co}(H)}$ is a subalgebra of $P$. In what follows we will always use $B$ in this sense.

If the functor $P \otimes_B - : B\text{Mod} \to C\text{Mod}$, from the category of left $B$-modules to the category of complex vector spaces, preserves and reflects exact sequences, then we say that $P$ is \textit{faithfully flat} as a right module over $B$. Faithful flatness for $P$ as a left $B$-module is defined analogously.

In this paper we are interested in a particular type of comodule algebra. Let $\pi : A \to H$ be a surjective Hopf algebra map between Hopf algebras $A$ and $H$. Then a \textit{homogeneous right $H$-coaction} is given by the map
\[
\Delta_R := (\text{id} \otimes \pi) \circ \Delta : A \to A \otimes H.
\]
Note that $\Delta_R$ gives $A$ the structure of a right $H$-comodule algebra. The associated \textit{quantum homogeneous space} is defined to be the space of coinvariant elements $A^{\text{co}(H)}$.

2.2.2. \textit{Takeuchi’s Equivalence}. Let $B := A^{\text{co}(H)}$ be a quantum homogeneous space. We denote by $A_B^\text{mod}$ the category of \textit{finitely-generated relative Hopf modules}, that is, the category whose objects are left $A$-comodules $\Delta_L : F \to A \otimes F$, endowed with a finitely-generated left $B$-module structure, such that
\[
\Delta_L(bf) = \Delta_L(b)\Delta_L(f), \quad \text{for all } f \in F, b \in B,
\]
and whose morphism sets $A_B^\text{Hom}(-,-)$ consist of left $A$-comodule, left $B$-module, maps. It is important to note that $B$ is naturally an object in $A_B^\text{mod}$.

We denote by $H^\text{mod}$ the category whose objects are finite-dimensional left $H$-comodules, and whose morphisms are left $H$-comodule maps. For a quantum homogeneous space $B := A^{\text{co}(H)}$, we denote $B^+ := B \cap \ker(\epsilon)$. Consider the functor
\[
\Phi : A_B^\text{mod} \to H^\text{mod}, \quad F \mapsto F/B^+F,
\]
where the left $H$-comodule structure of $\Phi(F)$ is given by $\Delta_L[f] := \pi(f_{(-1)}) \otimes [f_{(0)}]$ (with square brackets denoting the coset of an element in $\Phi(F)$). If $V \in H^\text{mod}$ with coaction $\Delta_L : V \to H \otimes V$, then the \textit{cotensor product} of $A$ and $V$ is given by
\[
A \square_H V := \ker(\Delta_R \otimes \text{id} - \text{id} \otimes \Delta_L : A \otimes V \to A \otimes H \otimes V).
\]
Using the cotensor product we can define the functor
\[
\Psi : H^\text{mod} \to A_B^\text{mod}, \quad V \mapsto A \square_H V,
\]
where the left $B$-module and left $A$-comodule structures of $\Psi(V)$ are defined on the first tensor factor, and if $\gamma$ is a morphism in $H^\text{mod}$, then $\Psi(\gamma) := \text{id} \otimes \gamma$. The following equivalence was established in [36, Theorem 1].

\textbf{Theorem 2.3} (Takeuchi’s Equivalence). Let $B = A^{\text{co}(H)}$ be a quantum homogeneous space such that $A$ is faithfully flat as a right $B$-module. An adjoint equivalence of categories between $A_B^\text{mod}$ and $H^\text{mod}$ is given by the functors $\Phi$ and $\Psi$ and
unit, and counit, natural isomorphisms

\[
U : \mathcal{F} \to \Psi \circ \Phi(\mathcal{F}), \quad f \mapsto f_{(-1)} \otimes [f(0)],
\]
\[
C : \Phi \circ \Psi(V) \to V, \quad \left[ \sum_i a^i \otimes v^i \right] \mapsto \sum_i \varepsilon(a^i)v^i.
\]

The usual tensor product of comodules gives \( \mathbb{H} \) \text{-} \text{mod} the structure of monoidal category. Every object \( F \in A^{B}_{\text{mod}} \) admits a right \( B \) \text{-} module structure uniquely defined by

\[
\mathcal{F} \times B \to \mathcal{F}, \quad (f, b) \mapsto f_{(-1)}bf(0),
\]
giving \( \mathcal{F} \) the structure of a bimodule. The usual tensor product of bimodules then endows \( A^{B}_{\text{mod}} \) with the structure of a monoidal category. It forms a monoidal subcategory of the category of \( B \)-bimodules, which for sake of clarity we denote by \( A^{B}_{\text{mod}} \). Takeuchi's equivalence can now be given the structure of a monoidal equivalence in the obvious way. In particular, this means that for any monoid object \( M \in A^{B}_{\text{mod}} \) the corresponding \( \Phi(M) \in \mathbb{H} \) \text{-} \text{mod} also has the structure of a monoid object. We will use this fact tacitly throughout the paper.

### 2.2.3. Relative Hopf Modules and Covariant Connections

Let \( \pi : A \to H \) be a surjective Hopf map and \( B = A^{\text{co}(H)} \) a quantum homogeneous space. A differential calculus \( \Omega^{\bullet} \) over \( B \) is said to be \textit{covariant} if the coaction \( \Delta_L : B \to A \otimes B \) extends to a (necessarily unique) map \( \Delta_L : \Omega^{\bullet} \to A \otimes \Omega^{\bullet} \) giving \( \Omega^{\bullet} \) the structure of a monoid object in \( A^{B}_{\text{mod}} \), and such that \( d \) is a left \( A \)-comodule map. For any \( \mathcal{F} \in A^{B}_{\text{mod}} \), a connection \( \nabla : \mathcal{F} \to \Omega^{1} \otimes_B \mathcal{F} \) is said to be \textit{covariant} if it is a left \( A \)-comodule map.

We say that a first-order differential calculus \( \Omega^{1}(B) \) over \( B \) is left \textit{covariant} if there exists a (necessarily unique) left \( A \)-coaction \( \Delta_L : \Omega^{1}(B) \to A \otimes \Omega^{1}(B) \) giving \( \Omega^{1}(B) \) the structure of an object in \( A^{B}_{\text{mod}} \) and such that \( d \) is a left \( A \)-comodule map. Note the universal calculus over \( B \) is left \( A \)-covariant. Moreover, any other first-order differential calculus over \( B \), with corresponding \( B \)-sub-bimodule \( N \subseteq \Omega^{1}(B) \), is covariant if and only if \( N \) is a left \( A \)-sub-comodule of \( \Omega^{1}(B) \). In particular, we note that the maximal prolongation of a covariant first-order differential calculus is covariant.

A complex structure \( \Omega^{\bullet}(\bullet, \bullet) \) for \( \Omega^{\bullet} \) is said to be \textit{covariant} if the \( \mathbb{N}^{2}_{0} \)-decomposition of \( \Omega^{\bullet} \) is a decomposition in the category \( A^{B}_{\text{mod}} \), or explicitly if the homogeneous subspace \( \Omega^{(a,b)} \) is a left \( A \)-sub-comodule of \( \Omega^{\bullet} \), for each \((a, b) \in \mathbb{N}^{2}_{0}\). This implies that \( \partial \) and \( \overline{\partial} \) are left \( A \)-comodule maps.

**Definition 2.4.** A \textit{holomorphic relative Hopf module} is a pair \((\mathcal{F}, \overline{\partial}_\mathcal{F})\) where \( \mathcal{F} \in A^{B}_{\text{mod}} \), \( \overline{\partial}_\mathcal{F} : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F} \) is a covariant \((0, 1)\)-connection, and \((\mathcal{F}, \overline{\partial}_\mathcal{F})\) is a holomorphic left \( B \)-module.
2.3. Principal Comodule Algebras and Strong Principal Connections. In this subsection we recall the basic theory of principal comodule algebras, structures of central importance in the paper.

2.3.1. General Case. We say that \( P \) is a \( H \)-Hopf–Galois extension of \( B \) if, for \( m_P : P \otimes_B P \to P \) the multiplication of \( P \), an isomorphism is given by
\[
\text{can} := (m_P \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) : P \otimes B \to P \otimes H.
\]

Definition 2.5. A principal right \( H \)-comodule algebra is a right \( H \)-comodule algebra \((P, \Delta_R)\) such that \( P \) is a Hopf–Galois extension of \( B := P^\text{col}(H) \) and \( P \) is faithfully flat as a right and left \( B \)-module.

We next recall the notion of a strong connection for a right \( H \)-comodule algebra, and its relationship with the definition of principal comodule algebras.

Definition 2.6. Let \( H \) be a Hopf algebra, \( P \) a right \( H \)-comodule algebra, and \( B := P^\text{col}(H) \). A principal connection for \( P \) is a left \( P \)-module right \( H \)-comodule projection \( \Pi : \Omega^1_u(P) \to \Omega^1_u(B)P \) satisfying
\[
\ker(\Pi) = P\Omega^1_u(B)P.
\]
A principal connection is said to be strong if
\[
(id - \Pi)dP \subseteq \Omega^1(B)P.
\]

As we now recall, the existence of a strong connection for a comodule algebra is equivalent to the comodule algebra being principal, see \( [6, \S 3.4] \) for details.

Theorem 2.7. A comodule algebra is principal if and only if it admits a strong connection.

2.3.2. Quantum Homogeneous Spaces. In this subsection we restrict to the special case of a quantum homogeneous space \( B = A^\text{col}(H) \) associated to a Hopf algebra surjection \( \pi : A \to H \). First we present a natural construction for strong principal connections. Consider \( H \) as a \( H \)-bicomodule in the obvious way, and consider \( A \) as a \( H \)-bicomodule with respect to the left and right \( H \)-coactions \( \Delta_L = (\pi \otimes \text{id}) \circ \Delta \) and \( \Delta_R = (\text{id} \otimes \pi) \circ \Delta \). Suppose that there exists a \( H \)-bicomodule map \( i : H \to A \) splitting \( \pi \), and such that \( i(1_H) = 1_A \). Then as a direct calculation confirms a left \( A \)-covariant strong connection \( \Pi_i : \Omega^1_u(A) \to \Omega^1_u(A) \) is given by
\[
\Pi_i(a'da) := a'(\pi(a_2))S(i(\pi(a_2)))_{(1)} \otimes i(\pi(a_2))_{(2)} - a' \otimes 1.
\]
Moreover, as shown in \( [7, \text{Proposition 4.4}] \), this gives an equivalence between left \( A \)-covariant strong principal connections and \( H \)-bicomodule splittings of \( \pi \) which send the unit of \( H \) to the unit of \( A \).

We are interested in strong principal connections because they allow us to construct covariant connections for any \( F \in \hat{A}^\text{mod} \). Consider first the isomorphism
\[
j : \Omega^1_u(B) \otimes_B F \simeq \Omega^1_u(B)A \Box_H \Phi(F), \quad \omega \otimes f \mapsto \omega f_{(-1)} \otimes [f(0)].
\]
We claim that a strong principal connection $\Pi$ defines a connection $\nabla$ on $\mathcal{F}$ by $\nabla : \mathcal{F} \to \Omega^1_u(B) \otimes_B \mathcal{F}$, $f \mapsto j^{-1}\left(\left((\text{id} - \Pi)d_u f(-1)\right) \otimes [f(0)]\right)$. Indeed, since $d_u$ and $\Pi$ are both right $H$-comodule maps, a right $H$-comodule map is also given by the composition $(\text{id} - \Pi) \circ d_u$. Hence $((\text{id} - \Pi)d_u f(-1)) \otimes [f(0)]$ is contained in $j(\Omega^1_u(B) \otimes_B \mathcal{F})$, meaning that $\nabla$ defines a connection.

3. Covariant Connections and Holomorphic Structures

In this section we use Takeuchi’s equivalence to convert questions about existence and uniqueness of connections into representation-theoretic statements. We also discuss principal comodule algebras and show how cosemisimplicity of a Hopf algebra $H$ can be used to construct left $A$-covariant strong principal connections. This sets up a general framework in terms of which we prove the main results of the paper in Section 4. Recall that $A$ and $H$ denote Hopf algebras and $B = A^{\text{co}(H)}$ a quantum homogeneous space.

3.1. Quotients of Connections. In this subsection we present some elementary technical results about producing connections for non-universal calculi from connections for universal calculi.

**Proposition 3.1.** For an algebra $B$ let $\mathcal{F}$ be a finitely-generated projective left $B$-module and let $\Omega^\bullet(B)$ be a differential calculus over $B$. Then the zero map $\mathcal{F} \to \Omega^1(B) \otimes_B \mathcal{F}$ is a connection if and only if $\Omega^\bullet$ is the zero calculus.

**Proof.** If the zero map were a connection, then we would necessarily have $db \otimes_B f = 0$, for all $b \in B$, $f \in \mathcal{F}$.

Since $\mathcal{F}$ is by assumption projective as a left $B$-module, this would imply that $db = 0$, for all $b \in \Omega^1(B)$, and hence that the calculus was trivial. The converse is clear, giving us the claimed equivalence. \(\square\)

**Corollary 3.2.** For any proper $B$-sub-bimodule $N \subseteq \Omega^1_u(B)$, let us denote $\text{proj}_N : \Omega^1_u(B) \to \Omega^1 := \Omega^1_u(B)/N$, $\omega \mapsto [\omega]$, where $[\omega]$ denotes the coset of $\omega$ in $\Omega^1_u(B)/N$. If $\nabla : \mathcal{F} \to \Omega^1(B) \otimes_B \mathcal{F}$ is a connection with respect to the universal calculus, then a non-zero connection is given by $\nabla' : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$, $f \mapsto (\text{proj}_N \otimes \text{id}) \circ \nabla(f)$.

**Proof.** It is clear from the definition of $\nabla'$ that it is a linear map satisfying the Leibniz rule (1), which is to say, it is clear that $\nabla'$ is a connection. The fact that it is non-zero follows from Proposition 3.1 and the assumption that $N$ is a proper $B$-sub-bimodule. \(\square\)
3.2. Covariant Connections and Takeuchi's Equivalence. In this subsection we make some novel observations about the flatness and uniqueness for covariant connections on a relative Hopf module $\mathcal{F} \in A \mod B$. The idea is to produce sufficient criteria in terms of the morphism sets of the category $A \mod B$. In practical cases, this allows these questions to be transferred to representation-theoretic form, allowing for a solution by direct calculation. We first give a criteria for flatness.

**Proposition 3.3.** If $A \Hom_B(\mathcal{F}, \Omega^2 \otimes_B \mathcal{F}) = 0$, then any left $A$-covariant connection $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ is necessarily flat.

**Proof.** Since the curvature of any connection is a module map, the curvature of a covariant connection is a morphism. Thus if $A \Hom_B(\mathcal{F}, \Omega^2 \otimes_B \mathcal{F})$ is trivial, $\nabla$ must be flat. □

The second proposition gives an analogous criteria for uniqueness of a covariant connection on a finitely-generated relative Hopf module.

**Proposition 3.4.** For $\mathcal{F} \in A \mod B$ such that $A \Hom_B(\mathcal{F}, \Omega^1 \otimes_B \mathcal{F}) = 0$, there exists at most one covariant connection for $\mathcal{F}$.

**Proof.** Since the difference of any two connections is a module map and the difference of two comodule maps is again a comodule map, the difference of two covariant connections is a morphism in $A \mod B$. Thus if $A \Hom_B(\mathcal{F}, \Omega^1 \otimes_B \mathcal{F})$ is trivial, then there exists at most one covariant connection $\mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$. □

We direct the interested reader to [12, §3.1] for a specialisation of these results to the case of factorisable irreducible CQH-Hermitian spaces, a general framework axiomatising properties of the irreducible quantum flag manifolds and their Heckenberger–Kolb calculi as presented in [7].

3.3. Principal Comodule Algebras and Cosemisimple Hopf Algebras. In this subsection we discuss comodule algebras $B = A^{\text{col}(H)}$ for which $H$ is a cosemisimple Hopf algebra. We begin by recalling the definition of cosemisimplicity.

**Definition 3.5.** A Hopf algebra $A$ is *cosemisimple* if it satisfies the following three equivalent conditions:

1. $A \cong \bigoplus_{V \in \hat{A}} \mathcal{C}(V)$, where summation is over $\hat{A}$, the set of all equivalence classes of irreducible left $A$-comodules,
2. the abelian category $A^{\text{Mod}}$ of left $A$-comodules is semisimple,
3. there exists a unique linear map $\mathbf{h} : A \to \mathbb{C}$, which we call the *Haar functional*, satisfying $\mathbf{h}(1) = 1$, and

\[
(id \otimes \mathbf{h}) \circ \Delta(a) = \mathbf{h}(a)1, \quad (\mathbf{h} \otimes \text{id}) \circ \Delta(a) = \mathbf{h}(a)1.
\]

For details about the equivalence of these three properties see [22]. Here we need only recall the implication from (i) to (iii): If $A$ decomposes as $A \cong \bigoplus_{V \in \hat{A}} \mathcal{C}(V)$,
then the associated Haar functional is given by projection onto the trivial sub-
coalgebra $C$.

Consider $^A\text{Mod}^A$ the category whose objects are $A$-bicomodules and whose mor-
phisms are $A$-bicomodule maps. In this paper, all Hopf algebras are assumed to
have invertible antipodes, so we have an equivalence between $^A\text{Mod}$ the category
of right $A$-comodules, and $\text{Mod}^A$ the category of left $A$-comodules. Hence we have
an equivalence of categories

$$^A\text{Mod}^A \simeq ^{A\otimes A}\text{Mod}$$

Denoting the Haar of $A$ by $h$, the linear map defined on simple tensors by

$$h_{A\otimes A} : A \otimes A \to C, \quad a \otimes a' \mapsto h(a)h(a'),$$

is readily seen to be a Haar functional for $A\otimes A$ in the sense of Definition 3.5. It
follows that $A\otimes A$ is a cosemisimple Hopf algebra. Hence $^{A\otimes A}\text{Mod}$ is a cosemisimple
abelian category, meaning that $^A\text{Mod}^A$ is a semisimple abelian category.

It is well known that cosemisimplicity of $H$ implies that $(A, \Delta_R)$ is a principal
comodule algebra. For example, it was shown in [28, Corollary 1.5] that cosemisim-
plcity of $H$ implies that $A$ is faithfully flat as a left and right $B$-module, and it
follows from [34, Corollary 2.6] that $A$ is a $H$-Hopf–Galois extension of $B$. In
fact cosemisimplicity implies a stronger result, namely the existence of a left $A$
-covariant strong principal connection. This easy observation is undoubtedly well
known to the experts, but we include a proof for the reader’s convenience.

**Lemma 3.6.** Let $\pi : A \to H$ be a Hopf algebra surjection and let $\Delta_R$ denote the
associated homogeneous right $H$-coaction on $A$. If $H$ is a cosemisimple Hopf algebra,
then $\Omega^1_u(A)$ admits a left $A$-comodule strong principal connection. In particular,
$(A, \Delta_R)$ is a principal comodule algebra.

**Proof.** Since $\pi : A \to H$ is a Hopf algebra map, it is necessarily a $H$-bicomodule
map. Since $H$ is cosemisimple, $^{H\otimes H}\text{Mod}$ is semisimple. Hence we can choose a
$H$-bicomodule map $i : H \to A$ splitting $\pi$ and satisfying $i(1_H) = 1_A$. It now
follows from the discussions of §2.3.2 that $(A, \Delta_R)$ is a principal comodule algebra
admitting a left $A$-comodule strong principal connection. □

**Proposition 3.7.** Assume that $H$ is cosemisimple. Let $F \in A^\text{mod}$ and let $\Omega^\bullet$
be a left $A$-covariant differential calculus over $B = A^{\text{co}(H)}$.

1. There exists a left $A$-covariant connection $\nabla : F \to \Omega^1 \otimes_B F$.
2. If $\Omega^\bullet$ is a differential $*$-calculus endowed with a covariant complex structure
   $\Omega^{(\bullet, \bullet)}$, there exists a left $A$-covariant $(0, 1)$-connection $\overline{\nabla}_F : F \to \Omega^{(0,1)} \otimes_B F$.

**Proof.** Since $A$ is a principal comodule algebra, we know from the discussions in
§2.3 that $F$ admits a universal connection $\nabla : F \to \Omega^1_u(B) \otimes_B F$. As discussed in
§2.1.3 composing $\nabla$ with the quotient map $\Omega^1_u(B) \otimes_B F \to \Omega^1 \otimes_B F$ will give a
covariant connection $\nabla : \mathcal{F} \to \Omega^* \otimes_B \mathcal{F}$ for the non-universal calculus. Finally we note that if $\Omega^*$ is endowed with a covariant complex structure $\Omega^{*,*}$, then

$$\overline{\partial}_F := (\text{proj}_{\Omega^{(0,1)}} \otimes \text{id}) \circ \nabla^* : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F}$$

is a left $A$-covariant $(0,1)$-connection.

\[\square\]

4. Holomorphic Relative Hopf Modules over Quantum Flag Manifolds

In this section we present the primary results of the paper, namely the existence of covariant holomorphic structures for any relative Hopf module over the irreducible quantum flag manifolds, and uniqueness of such structures in the irreducible case. We first recall the necessary definitions and results about Drinfeld–Jimbo quantum groups, quantum flag manifolds, and the Heckenberger–Kolb differential calculi over the irreducible quantum flag manifolds, and then follow with the existence and uniqueness results for holomorphic relative Hopf modules.

4.1. Drinfeld–Jimbo Quantum Groups. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank $r$. We fix a Cartan subalgebra $\mathfrak{h}$ with corresponding root system $\Delta \subseteq \mathfrak{h}^*$, where $\mathfrak{h}^*$ denotes the linear dual of $\mathfrak{h}$. With respect to a choice of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, denote by $(\cdot, \cdot)$ the symmetric bilinear form induced on $\mathfrak{h}^*$ by the Killing form of $\mathfrak{g}$, normalised so that any shortest simple root $\alpha_i$ satisfies $(\alpha_i, \alpha_i) = 2$. The coroot $\alpha_i^\vee$ of a simple root $\alpha_i$ is defined by

$$\alpha_i^\vee := \frac{\alpha_i}{d_i} = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \quad \text{where } d_i := \frac{(\alpha_i, \alpha_i)}{2}.$$

The Cartan matrix $A = (a_{ij})_{ij}$ of $\mathfrak{g}$ is the $(r \times r)$-matrix defined by $a_{ij} := (\alpha_i^\vee, \alpha_j)$. Let $\{\varpi_1, \ldots, \varpi_r\}$ denote the corresponding set of fundamental weights of $\mathfrak{g}$, which is to say, the dual basis of the coroots.

Let $q \in \mathbb{R}$ such that $q \notin \{-1, 0, 1\}$, and denote $q_i := q^{d_i}$. The quantised enveloping algebra $U_q(\mathfrak{g})$ is the noncommutative associative algebra generated by the elements $E_i, F_i, K_i,$ and $K_i^{-1},$ for $i = 1, \ldots, r,$ subject to the relations

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

along with the quantum Serre relations

\[
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - \frac{a_{ij}}{s} \right]_{q_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0, \quad \text{for } i \neq j,
\]

\[
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - \frac{a_{ij}}{s} \right]_{q_i} F_i^{1-a_{ij}-s} F_j E_i^s = 0, \quad \text{for } i \neq j;
\]
where we have used the $q$-binomial coefficients defined according to

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

where $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$.

A Hopf algebra structure is defined on $U_q(\mathfrak{g})$ by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

A Hopf $*$-algebra structure, called the compact real form of $U_q(\mathfrak{g})$, is defined by

$$K_i^* := K_i, \quad E_i^* := K_i F_i, \quad F_i^* := E_i K_i^{-1}.$$

Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$, and $\mathcal{P}^+$ its set of dominant integral weights. For each $\mu \in \mathcal{P}^+$ there exists an irreducible finite-dimensional $U_q(\mathfrak{g})$-module $V_\mu$, uniquely defined by the existence of a vector $v_\mu \in V_\mu$, which we call a highest weight vector, satisfying

$$E_i \triangleright v_\mu = 0, \quad K_i \triangleright v_\mu = q^{\langle \alpha, \mu \rangle} v_\mu, \quad \text{for all } i = 1, \ldots, r.$$

Moreover, $v_\mu$ is the uniquely such element up to scalar multiple. We call any finite direct sum of such $U_q(\mathfrak{g})$-representations a type-1 representation. In general, a vector $v \in V_\mu$ is called a weight vector of weight $\text{wt}(v) \in \mathcal{P}$ if

$$K_i \triangleright v = q^{\langle \text{wt}(v), \alpha \rangle} v, \quad \text{for all } i = 1, \ldots, r.$$

Finally, we note that since $U_q(\mathfrak{g})$ has an invertible antipode, we have an equivalence between $U_q(\mathfrak{g})\text{-Mod}$, the category of left $U_q(\mathfrak{g})$-modules, and $\text{Mod}_{U_q(\mathfrak{g})}$, the category of right $U_q(\mathfrak{g})$-modules, as induced by the antipode.

For further details on Drinfeld–Jimbo quantised enveloping algebras, we refer the reader to the standard texts [22, 8], or to the seminal papers [14, 19].

4.2. Quantum Coordinate Algebras. In this subsection we recall some necessary material about quantised coordinate algebras. Let $V$ be a finite-dimensional left $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, the $\mathbb{C}$-linear dual of $V$, endowed with its right $U_q(\mathfrak{g})$-module structure. Let us note that, with respect the equivalence between type-1 $U_q(\mathfrak{g})$-modules and finite-dimensional representations of $\mathfrak{g}$, the left module corresponding to $V_\mu^*$ is isomorphic to $V_{-\omega_0(\mu_S)}$, where $\omega_0$ denotes the longest element in the Weyl group of $\mathfrak{g}$.

Consider the function $c_{f,v}^V : U_q(\mathfrak{g}) \to \mathbb{C}$ defined by $c_{f,v}^V(X) := f(X \triangleright v)$. The coordinate ring of $V$ is the subspace

$$C(V) := \text{span}_{\mathbb{C}} \{c_{f,v}^V \mid v \in V, \ f \in V^*\} \subseteq U_q(\mathfrak{g})^*.$$

A $U_q(\mathfrak{g})$-bimodule structure on $C(V)$ is given by

$$(Y \triangleright c_{f,v}^V \triangleleft Z)(X) := f((ZXY) \triangleright v) = c_{f \circ Z,Y \triangleright v}^V(X).$$
Let $U_q(\mathfrak{g})^\circ$ denote the Hopf dual of $U_q(\mathfrak{g})$. It is easily checked that a Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ is given by

\begin{equation}
\mathcal{O}_q(G) := \bigoplus_{\mu \in \mathcal{P}^+} C(V_{\mu}).
\end{equation}

We call $\mathcal{O}_q(G)$ the quantum coordinate algebra of $G$, where $G$ is the compact, connected, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra. Note that $\mathcal{O}_q(G)$ is a cosemisimple Hopf algebra by construction.

4.3. Quantum Flag Manifolds. For $\{\alpha_i\}_{i \in S} \subseteq \Pi$ a subset of simple roots, consider the Hopf $*$-subalgebra $U_q(l_S) := \langle K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \rangle$. Just as for $U_q(\mathfrak{g})$, see for example [22, §7], the category of $U_q(l_S)$-modules is semisimple. The Hopf $*$-algebra embedding $\iota_S : U_q(l_S) \rightarrow U_q(\mathfrak{g})$ induces the dual Hopf $*$-algebra map $\iota_S^* : U_q(\mathfrak{g})^\circ \rightarrow U_q(l_S)^\circ$. By construction $\mathcal{O}_q(G) \subseteq U_q(\mathfrak{g})^\circ$, so we can consider the restriction map $\pi_S := \iota_S^*|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \rightarrow U_q(l_S)^\circ$, and the Hopf $*$-subalgebra $\mathcal{O}_q(L_S) := \pi_S(\mathcal{O}_q(G)) \subseteq U_q(l_S)^\circ$. The quantum flag manifold associated to $S$ is the quantum homogeneous space associated to the surjective Hopf $*$-algebra map $\pi_S : \mathcal{O}_q(G) \rightarrow \mathcal{O}_q(L_S)$. We denote it by

$$\mathcal{O}_q(G/L_S) := \mathcal{O}_q(G)^{\text{col}(\mathcal{O}_q(L_S))}.$$ 

Since the category of $U_q(l_S)$-modules is semisimple, $\mathcal{O}_q(L_S)$ must be a cosemisimple Hopf algebra. Thus by Proposition 3.3, the pair $(\mathcal{O}_q(G), \Delta_R)$ is a principal comodule algebra.

Denoting $\mu_S := \sum_{s \in S} \varpi_s$, choose for $V_{\mu_S}$ a weight basis $\{v_i\}_i$, with corresponding dual basis $\{f_i\}_i$. As shown in [15] Proposition 3.2], writing $N := \dim(V_{\mu_S})$, a set of generators for $\mathcal{O}_q(G/L_S)$ is given by

$$z_{ij} := c_{V_{\mu_S},V_{-\varpi_0(\mu_S)}}^{V_{ij},v_N} c_{V_{ij},f_N}$$

for $i, j = 1, \ldots, N$, where $v_N$ and $f_N$ are the highest weight basis elements of $V_{\mu_S}$ and $V_{-\varpi_0(\mu_S)}$, respectively.

4.4. The Heckenberger–Kolb Calculi and their Complex Structures. The construction and classification of covariant differential calculi over the quantum flag manifolds poses itself as a very important and challenging question. At present this question has only been addressed for the irreducible quantum flag manifolds, a distinguished sub-family whose definition we now recall.

Definition 4.1. A quantum flag manifold is irreducible if the defining subset of simple roots is of the form

$$S = \{1, \ldots, r\} \setminus \{s\}$$
where $\alpha_s$ has coefficient 1 in the expansion of the highest root of $\mathfrak{g}$.

In the classical limit of $q = 1$, these homogeneous spaces reduce to the compact Hermitian symmetric spaces, see for example [1, Table 10.1] or [17, §X.3]. For a convenient diagrammatic presentation of the explicit simple roots identified by this condition, as well as the dimensions of the classical differential manifolds, see [12, Appendix B].

The irreducible quantum flag manifolds are distinguished by the existence of an essentially unique $q$-deformation of their classical de Rham complex. The existence of such a canonical deformation is one of the most important results in the noncommutative geometry of quantum groups, establishing it as a solid base from which to investigate more general classes of quantum spaces. The following theorem is a direct consequence of results established in [15], [16], and [26]. See [11, §10] for a more detailed presentation.

**Theorem 4.2.** Over any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique finite-dimensional left $\mathcal{O}_q(G)$-covariant differential $\ast$-calculus

$$\Omega^\ast_q(G/L_S) \in \mathcal{O}_q(G/L_S)^{\text{mod}}_0,$$

of classical dimension, that is to say, satisfying

$$\dim \Phi(\Omega^k_q(G/L_S)) = \binom{2M}{k}, \quad \text{for all } k = 0, \ldots, 2M,$$

where $M$ is the complex dimension of the corresponding classical manifold.

The calculus $\Omega^\ast_q(G/L_S)$, which we refer to as the Heckenberger–Kolb calculus of $\mathcal{O}_q(G/L_S)$, has many remarkable properties. We recall here only the existence of a unique covariant complex structure, following from the results of [15], [16], and [26].

**Proposition 4.3.** Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold, and $\Omega^\ast_q(G/L_S)$ its Heckenberger–Kolb differential $\ast$-calculus. Then the following hold:

1. $\Omega^\ast_q(G/L_S)$ admits a unique left $\mathcal{O}_q(G)$-covariant complex structure,

$$\Omega^\ast_q(G/L_S) \simeq \bigoplus_{(a, b) \in \mathbb{N}_0^2} \Omega^{(a, b)} = \Omega^{(\ast, \ast)},$$

2. $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are simple objects in $\mathcal{O}_q(G/L_S)^{\text{mod}}_0$.

Complementing this abstract characterisation of the calculus is the original presentation of Heckenberger and Kolb in terms of the generators $z_{ij} \in \mathcal{O}_q(G/L_S)$ given in [16]. Here we need only recall the following: Consider the subset of the index set $J := \{1, \ldots, \dim(V_{\varpi_s})\}$ given by

$$J_{(1)} := \{i \in J \mid (\varpi_s, \varpi_s - \alpha_s - \text{wt}(v_i)) = 0\}.$$
In [16] Proposition 3.6] it was shown that a basis of $\Phi(\Omega^{(0,1)})$ is given by
\begin{equation}
\{ z^i_{N_i} \mid i \in J_{(1)} \}.
\end{equation}
4.5. Holomorphic Modules. Here we establish existence and uniqueness results of holomorphic structures for relative Hopf modules over the irreducible quantum flag manifolds. We begin by observing that the general theory of principal comodule algebras, together with cocompensimplicity of $\mathcal{O}_q(L_S)$, implies the existence of covariant connections.

**Lemma 4.4.** Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold endowed with its Heckenberger–Kolb calculus $\Omega^\bullet_q(G/L_S)$. Every $\mathcal{F} \in \mathcal{O}_q(G/L_S)^\text{mod}_0$ admits a left $\mathcal{O}_q(G)$-covariant connection $\nabla : \mathcal{F} \to \Omega^1_q(G/L_S) \otimes \mathcal{O}_q(G/L_S) \mathcal{F}$.

**Proof.** As observed in [16] each irreducible quantum flag manifold is a principal comodule algebra. Following the discussion of [23] this implies the existence of a covariant universal connection for each $\mathcal{F} \in \mathcal{O}_q(G/L_S)^\text{mod}_0$. Corollary 3.2 now implies that we can quotient this connection to produce a covariant connection with respect to the Heckenberger–Kolb calculus $\Omega^\bullet_q(G/L_S)$. □

We now use Proposition 3.4 to show uniqueness for covariant connections whenever $\mathcal{F}$ is simple. This is most easily done by considering $\Phi(\mathcal{F})$ as a module over the centre of $U_q(I_S)$. Recalling that the transpose of the Cartan matrix $A$ is the change of basis matrix taking fundamental weights to simple roots, we see that $\det(A)\varpi_x$ is contained in the root lattice of $\mathfrak{g}$. Denoting
\[ \det(A)\varpi_x =: a_1\alpha_1 + \cdots + a_r\alpha_r, \]
it follows directly from the commutation relations of $\mathfrak{g}$ that
\[ Z := K_1^{a_1} \cdots K_r^{a_r} \]
is a central element of $U_q(I_S)$. Recall that the elements of the centre $\mathfrak{z}(U_q(I_S))$ of $U_q(I_S)$ act on any irreducible $U_q(I_S)$-module $V$ by a corresponding central character $\chi_V \in \text{Hom}(\mathfrak{z}(U_q(I_S)), \mathbb{C})$ the algebra maps from $\mathfrak{z}(U_q(I_S))$ to $\mathbb{C}$. For the explicit case of $\Phi(\Omega^{(0,1)})$, it follows from the proof of [12] Theorem 4.11] that
\begin{equation}
\chi_{\Phi(\Omega^{(0,1)})}(Z) \neq 1.
\end{equation}
Note also that for $V$ and $W$ two irreducible $U_q(I_S)$-modules, since elements of the centre $\mathfrak{z}(U_q(I_S))$ are grouplike, $\mathfrak{z}(U_q(I_S))$ will act on $V \otimes W$ by a central character $\chi_{V \otimes W}$ according to
\[ \chi_{V \otimes W}(x) = \chi_V(x)\chi_W(x), \quad \text{for any } x \in \mathfrak{z}(U_q(I_S)). \]

**Theorem 4.5.** Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold endowed with its Heckenberger–Kolb calculus, and $\mathcal{F} \in \mathcal{O}_q(G/L_S)^\text{mod}_0$. It holds that
\begin{enumerate}
\item $\mathcal{F}$ admits a left $\mathcal{O}_q(G)$-covariant connection $\nabla : \mathcal{F} \to \Omega^1_q(G/L_S) \otimes \mathcal{O}_q(G/L_S) \mathcal{F}$, and this is the unique such connection if $\mathcal{F}$ is simple,
\end{enumerate}
2. \( \overline{\mathcal{D}}_\mathcal{F} := \text{proj}^{(0,1)} \circ \nabla \) is a left \( \mathcal{O}_q(G) \)-covariant holomorphic structure for \( \mathcal{F} \), and this is the unique such holomorphic structure if \( \mathcal{F} \) is simple.

**Proof.** 1. By Lemma 4.4 a covariant connection exists. Assuming that \( \mathcal{F} \) is simple, it follows from (9) that

\[
\chi_{\Phi(\Omega^{(0,1)} \otimes \Phi(\mathcal{F}))}(Z) = \chi_{\Phi(\Omega^{(0,1)})}(Z) \chi_{\Phi(\mathcal{F})}(Z) \neq \chi_{\Phi(\mathcal{F})}(Z).
\]

From this we can conclude that there are no non-zero \( U_q(I_S) \)-module maps from \( \Phi(\mathcal{F}) \) to \( \Phi(\Omega^{(0,1)}) \otimes \Phi(\mathcal{F}) \). Moreover, since we have a non-degenerate dual pairing between \( U_q(I_S) \) and \( \mathcal{O}_q(L_S) \), there are no non-zero \( \mathcal{O}_q(L_S) \)-comodule maps from \( \Phi(\mathcal{F}) \) to \( \Phi(\Omega^{(0,1)}) \otimes \Phi(\mathcal{F}) \). This in turn implies that there can exist no non-zero morphisms from \( \mathcal{F} \) to \( \Omega^{(0,1)} \otimes \mathcal{O}_q(G/L_S) \mathcal{F} \). Proposition 3.4 now implies that there exists at most one left \( \mathcal{O}_q(G) \)-covariant connection on \( \mathcal{F} \) for the calculus \( \Omega^{(0,*,0)} \). An analogous argument shows that there exists at most one left \( \mathcal{O}_q(G) \)-covariant connection on \( \mathcal{F} \) for the calculus \( \Omega^{(0,*,0)} \). Hence if \( \mathcal{F} \) is simple, then \( \nabla : \mathcal{F} \to \Omega_q^1(G/L_S) \otimes \mathcal{O}_q(G/L_S) \mathcal{F} \) is the unique such covariant connection.

2. Since \( \Omega^* \) is a monoid object in \( \mathcal{O}_q(G/L_S)^{\text{mod}_0} \), we see that \( \Phi(\Omega^*) \) has the structure of a monoid object in \( \mathcal{O}_q(L_S)^{\text{mod}} \), or in other words, it has the structure of a left \( \mathcal{O}_q(L_S) \)-comodule algebra. In particular, for any two forms \( \omega, \nu \in \Omega^* \), it holds that

\[
([\omega] \wedge [\nu]) \triangleleft Z = ([\omega] \triangleleft Z) \wedge ([\nu] \triangleleft Z).
\]

Thus we see that

\[
\chi_{\Phi(\Omega^{(0,2)})}(Z) = \left( \chi_{\Phi(\Omega^{(0,1)})}(Z) \right)^2.
\]

From this we see that, for any irreducible \( \mathcal{F} \),

\[
\chi_{\Phi(\Omega^{(0,2)} \otimes \Phi(\mathcal{F}))}(Z) = \left( \chi_{\Phi(\Omega^{(0,1)})}(Z) \right)^2 \chi_{\Phi(\mathcal{F})}(Z) \neq \chi_{\Phi(\mathcal{F})}(Z),
\]

where we have used (9). Following the same argument as for \((0,1)\)-forms in part 1 of the proof, this means that there are no non-zero \( \mathcal{O}_q(G) \)-comodule maps from \( \mathcal{F} \) to \( \Omega^{(0,2)} \otimes \mathcal{O}_q(G/L_S) \mathcal{F} \). Flatness of the \((0,1)\)-connection \( \overline{\mathcal{D}}_\mathcal{F} \) now follows from Proposition 3.3. Uniqueness was already established in 1.

For the case of a non-simple \( \mathcal{F} \), cosemisimplicity of \( \mathcal{O}_q(L_S) \) implies that \( \mathcal{F} \) is a direct sum of a finite number of simple objects \( \mathcal{F} \simeq \bigoplus_i \mathcal{F}_i \). The direct sum of the covariant holomorphic structures of the summands \( \mathcal{F}_i \) gives a covariant holomorphic structure for \( \mathcal{F} \).

4.6. **A Quantum Kostant Conjecture.** The existence of a holomorphic structure \( \overline{\mathcal{D}}_\mathcal{F} \), for each \( \mathcal{F} \in \mathcal{O}_q(G/L_S)^{\text{mod}_0} \), gives a complex

\[
\overline{\mathcal{D}}_\mathcal{F} : \Omega^k \otimes \mathcal{O}_q(G/L_S) \mathcal{F} \to \Omega^{k+1} \otimes \mathcal{O}_q(G/L_S) \mathcal{F},
\]
with associated cohomology groups

\[ H^k_{\partial}(\mathcal{F}) \cong \bigoplus_{(a,b) \in \mathbb{N}_0^2} H^{(a,b)}_{\partial}(\mathcal{F}). \]

In the classical setting, efforts to calculate these groups—a major undertaking—resulted in the celebrated Borel–Weil theorem for zero-th cohomology for line bundles \[35\], the Bott–Borel–Weil theorem for anti-holomorphic cohomologies for line bundles \[5\], and finally Kostant’s beautiful description of the general case \[23\]. In the quantum setting, for the irreducible quantum flag manifolds, the Borel–Weil theorem was shown in \[13\] to hold for all line bundles, that is, those homogeneous vector bundles \( \mathcal{E} \) satisfying \( \dim(\Phi(\mathcal{E})) = 1 \). The Bott–Borel–Weil theorem has been shown to hold for all positive line bundles over the irreducible quantum flag manifolds (see \[31, 12\] for the definition of a positive line bundle). This motivates us to make the following general conjecture.

**Conjecture 4.6.** For every \( \mathcal{F} \in \mathcal{O}_q(G)_{\text{mod}} \),

\[ \dim \left( H^{(a,b)}_{\partial}(\mathcal{F}) \right), \quad \text{for all } (a, b) \in \mathbb{N}_0^2, \]

has the same value as in the classical situation.

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