FORMAL COMPLETION OF A CATEGORY ALONG A SUBCATEGORY

ALEXANDER I. EFIMOV

Abstract. Following an idea of Kontsevich, we introduce and study the notion of formal completion of a compactly generated (by a set of objects) enhanced triangulated category along a full thick essentially small triangulated subcategory.

In particular, we prove (answering a question of Kontsevich) that using categorical formal completion, one can obtain ordinary formal completions of Noetherian schemes along closed subschemes. Moreover, we show that Beilinson-Parshin adeles can be also obtained using categorical formal completion.

Contents

1. Introduction 1
2. Preliminaries 4
3. Definition of categorical formal completion 6
4. Properties of categorical formal completion 10
5. Relation to formal completions of Noetherian schemes 13
   5.1. Double centralizers and homotopy limits 13
   5.2. Algebraizable derived categories of formal completions of schemes 16
6. Beilinson-Parshin adeles and categorical formal completions. 20
References 27

1. Introduction

In this paper we introduce and study the notion of formal completion of a (compactly generated) triangulated category along a (full thick essentially small) triangulated subcategory. The original idea belongs to M. Kontsevich [Ko1, Ko2].

Our construction requires DG enhancement [BK] and is built on the notion of derived double centralizer. We illustrate it as follows.

The author was partially supported by the Moebius Contest Foundation for Young Scientists, and RFBR (grant 4713.2010.1).
Let $\mathcal{A}$ be a DG algebra, and $M \in D(\mathcal{A})$ be some object in the derived category of right $\mathcal{A}$-modules. Put $\mathcal{B}_M := \mathcal{R}\text{End}_\mathcal{A}(M)$, and consider the DG algebra

\begin{equation}
\hat{\mathcal{A}}_M := \mathcal{R}\text{End}_{\mathcal{B}_M^{op}}(M)^{op},
\end{equation}

the derived double centralizer of $M$. We have natural morphism $\mathcal{A} \to \hat{\mathcal{A}}_M$.

It turns out that (quasi-isomorphism class of) $\hat{\mathcal{A}}_M$ depends only on the subcategory $\mathcal{T} \subset D(\mathcal{A})$, classically generated by $M$ (this is special case of Proposition 3.2). We define

\begin{equation}
\hat{\mathcal{A}}_\mathcal{T} := \hat{\mathcal{A}}_M
\end{equation}

to be derived double centralizer of $\mathcal{T}$. Further, derived category $D(\hat{\mathcal{A}}_\mathcal{T})$ depends (up to equivalence) only on the (enhanced) triangulated category $D(\mathcal{A})$ and the full thick triangulated subcategory $\mathcal{T} \subset D(\mathcal{A})$ (this is special case of Proposition 3.4). We define

\begin{equation}
\hat{D}(\mathcal{A})_\mathcal{T} := D(\hat{\mathcal{A}}_\mathcal{T})
\end{equation}

to be the formal completion of $D(\mathcal{A})$ along $\mathcal{T}$.

In Section 3 we define, more generally, the notion of formal completion $\hat{D}_\mathcal{T}$ of a compactly generated enhanced triangulated category $D$ along full thick essentially small triangulated subcategory $\mathcal{T} \subset D$. This formal completion comes equipped with ”restriction functor” $\kappa^* : D \to \hat{D}_\mathcal{T}$.

One of the main results of this paper is the following theorem (see Theorem 5.4), which relates our construction with ordinary formal completions of Noetherian schemes. For a separated Noetherian scheme $X$, we denote by $D(X) := D(\text{QCoh} X)$ the derived category of quasi-coherent sheaves on $X$.

**Theorem 1.1.** Let $X$ be a separated Noetherian scheme, and $Y \subset X$ a closed subscheme. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
D(X) & \xrightarrow{\text{id}} & D(X) \\
\downarrow & & \downarrow \text{L}\kappa^* \\
\hat{D}(X)_{D_{\text{coh},Y}(X)} & \xrightarrow{\cong} & D_{\text{alg}}(\hat{X}_Y).
\end{array}
\]

Here $D_{\text{alg}}(\hat{X}_Y)$ is algebraizable derived category of $\hat{X}_Y$ (it is defined in Subsection 5.2). We have the following Corollaries (see Corollaries 5.6 and 5.7).

**Corollary 1.2.** Let $R$ be a regular commutative Noetherian $k$-algebra, and $M \in D_{f.g.}(R) \cong D_{\text{coh}}^b(\text{Spec} R)$ be a complex of $R$-modules with finitely generated cohomology.
Denote by $I \subset R$ the annihilator of $H(M)$, so that $V(\sqrt{I}) \subset \text{Spec } R$ is precisely the support of $H(M)$. Then we have an isomorphism
\[ \hat{R}_M \cong \hat{R}_I, \]
where the RHS is the ordinary $I$-adic completion.

**Corollary 1.3.** Let $R$ be commutative Noetherian $k$-algebra, and $I \subset R$ an ideal. Assume that either $R$ or $R/I$ is regular. Then we have an isomorphism
\[ \hat{R}_{R/I} \cong \hat{R}_I, \]
where the RHS is ordinary $I$-adic completion.

Moreover, Proposition 5.8 below shows that Corollary 1.3 fails to hold if we drop the regularity assumption.

We also relate our construction to Beilinson-Parshin adeles [Be, P] (see Section 6 for definitions and notation).

**Theorem 1.4.** Let $X$ be a separated Noetherian $k$-scheme of dimension $d$. Fix some sequence $d \geq k_0 > \cdots > k_p \geq 0$. If $p = 0$, then we have a commutative diagram,
\[ \begin{diagram}
D(X) \arrow{e}{\text{id}} \arrow{s}{\mathbb{A}(X,-)(k_0)} \arrow{s}{\sim} & D(X) \\
D(\mathbb{A}(X)(k_0)) \arrow{e}{\kappa} & (D(X)/\mathcal{D}_{\leq(k_0-1)}(X))_{\mathcal{D}_{\text{coh},\leq k_0}^b}.
\end{diagram} \]
For $p > 0$, there is a natural commutative diagram
\[ \begin{diagram}
D(\mathbb{A}(X)(k_1,\ldots,k_p)) \arrow{e}{\text{id}} \arrow{s}{\sim} & D(\mathbb{A}(X)(k_1,\ldots,k_p)) \\
D(\mathbb{A}(X)(k_0,\ldots,k_p)) \arrow{e}{\kappa} & (D(\mathbb{A}(X)(k_1,\ldots,k_p))/\mathcal{D}_{\leq(k_0-1)}(X))_{\mathcal{D}_{\text{coh},\leq k_0}^b}.
\end{diagram} \]

The paper is organized as follows.

In Section 2 we recall some preliminaries on DG categories.

Section 3 is devoted to the definition of categorical formal completion (and in particular derived double centralizers) and all necessary checkings which show that it is well-defined.

In Section 4 we investigate various properties of formal completions. In particular, we show (Theorem 4.1) that under some natural assumptions on $\mathcal{T} \subset \mathcal{D}$, the restriction of the functor $\kappa^* : \mathcal{D} \to \widehat{\mathcal{D}}_{\mathcal{T}}$ to the subcategory $\mathcal{T}$ is full and faithful. Moreover, under the same assumptions the functor
\[ \widehat{\mathcal{D}}_{\mathcal{T}} \to \widehat{\mathcal{D}}_{\mathcal{T}} \]
Section 5 is devoted to Theorem 1.1 (Theorem 5.4). In Subsection 5.1 we prove useful technical result, Lemma 5.2, which relates double centralizers with homotopy limits of DG algebras. In Subsection 5.2 we define algebraizable derived categories of formal completions of Noetherian schemes. Then we apply Lemma 5.2 to prove Theorem 5.4.

Section 6 is devoted to interpretation of Beilinson-Parshin adeles in terms of categorical formal completions and Drinfeld quotients (Theorem 6.1). Here our main technical tool is also Lemma 5.2.

2. Preliminaries

Fix some base commutative ring $k$. All DG categories under consideration will be over $k$. All DG modules which we consider will be right DG modules. In particular, for $A \in \text{dgcat}_k$, we denote by $D(A)$ the derived category of right DG $A$-modules. Also, denote by $A$-mod the DG category of right $A$-modules.

**Definition 2.1.** Let $A$ be a DG category. A DG module $M \in A$-mod is called $h$-projective (resp. $h$-injective) if for any acyclic DG module $N \in A$-mod the complex $\text{Hom}_A(M, N)$ (resp. $\text{Hom}_A(N, M)$) is acyclic. We denote by $h$-proj$(A) \subset A$-mod (resp. $h$-inj$(A) \subset A$-mod) the full DG subcategory which consists of $h$-projective (resp. $h$-injective) DG modules.

We also call $M \in A$-mod $h$-flat if for any acyclic $N \in A^{op}$-mod the complex $M \otimes_A N$ of $k$-modules is also acyclic.

It is easy to see that all $h$-projective DG modules are also $h$-flat.

Denote by $\text{dgcat}_k$ the category of small DG $k$-linear categories. It has natural model category structure $[\mathbb{T}]$, with weak equivalences being quasi-equivalences. All DG categories are fibrant in this model structure.

We call DG category $A \in \text{dgcat}_k$ $h$-flat (over $k$) if all complexes $\text{Hom}_A(X, Y)$, $X, Y \in \text{Ob}(A)$, are $h$-flat $k$-modules. We define $h$-projective (over $k$) DG categories in the same way. All cofibrant DG categories are $h$-projective, hence $h$-flat. In particular, each DG category is quasi-equivalent to an $h$-flat one.

**Definition 2.2.** Let $A \in \text{dgcat}_k$ be an $h$-flat DG category. We say that $A$ is smooth (over $k$) if $I_A \in \text{Perf}(A^{op} \otimes A)$, where

$$I_A \in D(A^{op} \otimes A), \quad I_A(X, Y) = \text{Hom}_A(Y, X).$$

An arbitrary DG category $A \in \text{dgcat}_k$ is said to be smooth if it is quasi-equivalent to smooth $h$-flat DG category.
There is an alternative nice well-known definition of smooth DG categories.

**Proposition 2.3.** Let $\mathcal{A} \in \text{dgcat}_k$ be a DG category. Then the following are equivalent:

(i) $\mathcal{A}$ is smooth;

(ii) For any h-flat $\mathcal{B} \in \text{dgcat}_k$, and any object $M \in D(\mathcal{A} \otimes \mathcal{B})$ such that $M(X, -) \in \text{Perf}(\mathcal{B})$ for all $X \in \text{Ob}(\mathcal{A})$, we have that $M \in \text{Perf}(\mathcal{A} \otimes \mathcal{B})$.

*Proof.* This is straightforward. $\Box$

**Corollary 2.4.** If $\mathcal{A}_1, \mathcal{A}_2 \in \text{dgcat}_k$ are Morita equivalent and $\mathcal{A}_1$ is smooth, then so is $\mathcal{A}_2$.

*Proof.* This follows directly from 2.3. $\Box$

**Lemma 2.5.** Let $\mathcal{A} \in \text{dgcat}_k$ be a smooth DG category. Then it is Morita equivalent to some (smooth) DG algebra.

*Proof.* It suffices to show that the category $D(\mathcal{A})$ is compactly generated by one object. We may and will assume that $\mathcal{A}$ is h-flat. By definition, there exists a finite collection of objects $X_1 \otimes Y_1, \ldots, X_n \otimes Y_n \in \mathcal{A}^{op} \otimes \mathcal{A}$, which generate the diagonal bimodule $\mathcal{I}_A \in D(\mathcal{A}^{op} \otimes \mathcal{A})$. It follows that each object $M \in D(\mathcal{A})$ is generated by $M(X_i) \otimes Y_i$, $1 \leq i \leq n$. Therefore, $\bigoplus_{i=1}^n Y_i \in \text{Perf}(\mathcal{A})$ is a compact generator of $D(\mathcal{A})$. $\Box$

**Definition 2.6.** Let $\mathcal{A} \in \text{dgcat}_k$ be a DG category. We say that $\mathcal{A}$ is proper (over $k$) if for any two objects $X, Y \in \text{Ob}(\mathcal{A})$, the complex $\text{Hom}_\mathcal{A}(X, Y)$ is a perfect $k$-module.

We have an analogue of Proposition 2.3.

**Proposition 2.7.** Let $\mathcal{A} \in \text{dgcat}_k$ be a DG category. Then the following are equivalent:

(i) $\mathcal{A}$ is proper;

(ii) For any h-flat $\mathcal{B} \in \text{dgcat}_k$, and any object $M \in \text{Perf}(\mathcal{A} \otimes \mathcal{B})$ we have that $M(X, -) \in \text{Perf}(\mathcal{B})$ for all $X \in \text{Ob}(\mathcal{A})$.

*Proof.* Evident. $\Box$

Finally, we recall the DG enhancement for the quotient of enhanced triangulated categories. Namely, let $\mathcal{D}$ be a compactly generated enhanced triangulated category, and $\mathcal{D}' \subset \mathcal{D}$ its localizing subcategory, and assume that $\mathcal{D}'$ is compactly generated by $\mathcal{D}' \cap \mathcal{D}^c$. According to [Ke2, Dr], the quotient category $\mathcal{D}/\mathcal{D}'$ is also enhanced (and compactly generated by the images of compact objects in $\mathcal{D}$).

Similarly, if $\mathcal{D}$ is essentially small enhanced triangulated category, and $\mathcal{D}' \subset \mathcal{D}$ a triangulated subcategory, then the quotient $\mathcal{D}/\mathcal{D}'$ is naturally enhanced.
3. Definition of categorical formal completion

Fix a base graded commutative ring \( k \).

Let \( \mathcal{A} \) be a small DG category. We may and will replace it by h-projective quasi-equivalent one. All tensor products below are assumed to be over \( k \) unless otherwise stated. It is well known that the category \( D(\mathcal{A}) \) is compactly generated by the set of objects \( \text{Ob}(\mathcal{A}) \), and we have that

\[
D(\mathcal{A})^c = \text{Perf}(\mathcal{A}),
\]

see [Ke1].

Now let \( S \subset D(\mathcal{A}) \) be a full small subcategory (not necessarily triangulated). Choosing an h-projective (resp. h-injective) resolution \( \widetilde{X} \) of each object \( X \in S \), we obtain a DG category \( \mathcal{B}_S \) with

\[
\text{Ob}(\mathcal{B}_S) = \text{Ob}(S), \quad \text{Hom}_{\mathcal{B}_S}(X, Y) := \text{Hom}_{\mathcal{A}}(\widetilde{X}, \widetilde{Y}).
\]

**Lemma 3.1.** The DG category \( \mathcal{B}_S \) is well-defined up to a quasi-equivalence.

**Proof.** Let \( S_1, S_2 \subset \mathcal{A}\text{-mod} \) be two full DG subcategories, such that for \( i = 1, 2 \) we have either \( S_i \subset \text{h-proj}(\mathcal{A}) \) or \( S_i \subset \text{h-inj}(\mathcal{A}) \). Moreover, let \( \Psi : \text{Ob}(S_1) \stackrel{\sim}{\rightarrow} \text{Ob}(S_2) \) be a bijection such that for each \( X \in \text{Ob}(S_1) \) the object \( \Psi(X) \) is quasi-isomorphic to \( X \).

We may and will assume that either \( S_1 \subset \text{h-proj}(\mathcal{A}) \), or \( S_2 \subset \text{h-inj}(\mathcal{A}) \). Then we may and will choose quasi-isomorphisms

\[
\alpha_X : X \rightarrow \Psi(X), \quad X \in S_1.
\]

Let \( \widetilde{S} \) be a DG category, defined as follows. First, \( \text{Ob}(\widetilde{S}) = \text{Ob}(S_1) \). Further, define

\[
\text{Hom}_{\widetilde{S}}(X, Y) \subset \text{Hom}_{\mathcal{A}}(\text{Cone}(\alpha_X), \text{Cone}(\alpha_Y))
\]

to be the subcomplex which consists of morphisms mapping \( \Psi(X) \) to \( \Psi(Y) \). Clearly, \( \widetilde{S} \) is a well-defined DG category. Further, we have obvious projection DG functors

\[
\pi_1 : \widetilde{S} \rightarrow S_1, \quad \pi_2 : \widetilde{S} \rightarrow S_2.
\]

We claim that both \( \pi_1 \) and \( \pi_2 \) are quasi-equivalences. Indeed, by our assumption, for any objects \( X \in S_1, \ Y \in S_2 \) we have that the complexes

\[
\text{Hom}_{\mathcal{A}}(X, \text{Cone}(\alpha_Y)), \quad \text{Hom}_{\mathcal{A}}(\text{Cone}(\alpha_X), Y)
\]
are acyclic. Therefore, the maps

\[
\pi_i : \text{Hom}_{\widetilde{S}}(X, Y) \rightarrow \text{Hom}_{S_i}(\pi_i(X), \pi_i(Y))
\]
are surjective with acyclic kernels, hence quasi-isomorphisms.

Lemma is proved. \( \square \)
We may consider $S$ as an object of $D(\mathcal{A} \otimes \mathcal{B}_S^{\text{op}})$. Namely, we put
\begin{equation}
S(U \otimes X) = \tilde{X}(U), \quad U \in \text{Ob}(\mathcal{A}), \quad X \in \text{Ob}(\mathcal{B}_S) = \text{Ob}(\mathcal{S}).
\end{equation}

Take some object $Q \in (\mathcal{A} \otimes \mathcal{B}_S^{\text{op}})\text{-mod}$, with an isomorphism $Q \cong S$ in $D(\mathcal{A} \otimes \mathcal{B}_S^{\text{op}})$, such that all DG $\mathcal{B}_S^{\text{op}}\text{-modules}$
\begin{equation}
Q(U, -) \in \mathcal{B}_S^{\text{op}}\text{-mod}, \quad U \in \mathcal{A},
\end{equation}
are h-projective (resp. h-injective). For instance, we can take $Q$ to be h-projective (resp. h-injective) itself. Further, define DG category $\hat{\mathcal{A}}_S$ as follows:
\begin{equation}
\text{Ob}(\hat{\mathcal{A}}_S) := \text{Ob}(\mathcal{A}), \quad \text{Hom}_{\hat{\mathcal{A}}_S}(X, Y) := \text{Hom}_{\mathcal{B}_S^{\text{op}}}(Q(Y, -), Q(X, -)).
\end{equation}

**Proposition 3.2.** 1) The DG category $\hat{\mathcal{A}}_S$ is well defined up to a natural isomorphism in $\text{Ho}(\text{dgcat}_k)$.

2) Moreover, if two subcategories $S_1, S_2 \subset D(\mathcal{A})$ split-generate each other, then we have a natural isomorphism
\begin{equation}
\hat{\mathcal{A}}_{S_1} \cong \hat{\mathcal{A}}_{S_2} \text{ in } \text{Ho}(\text{dgcat}_k).
\end{equation}

**Proof.** Statement 1) almost follows from Lemma 3.1. Indeed, let $Q_1, Q_2 \in (\mathcal{A} \otimes \mathcal{B}_S^{\text{op}})\text{-mod}$ be objects which are both quasi-isomorphic to $S$, $Q_1$ is h-projective, and $Q_2$ satisfies the assumptions for $Q$ above. Then we have a natural (up to homotopy) quasi-isomorphism $\alpha : Q_1 \to Q_2$, and we can repeat the proof of Lemma 3.1.

Now we prove 2). Let $Q_i \in (\mathcal{A} \otimes \mathcal{B}_S^{\text{op}})\text{-mod}$ be h-projective resolutions of $S_i$ for $i = 1, 2$. Further, define the bimodule $M \in D(\mathcal{B}_{S_1} \otimes \mathcal{B}_{S_2}^{\text{op}})$ by the formula
\begin{equation}
M(U \otimes V) := \text{Hom}_{\mathcal{A}^{\text{op}}}(Q_1(-, U), Q_2(-, V)), \quad U \in \mathcal{B}_{S_1}, V \in \mathcal{B}_{S_2}.
\end{equation}
Since $S_1$ and $S_2$ split-generate each other, we have that the bimodule $M$ induces an equivalence
\begin{equation}
- \otimes_{\mathcal{B}_{S_1}} M : D(\mathcal{B}_{S_1}^{\text{op}}) \to D(\mathcal{B}_{S_2}^{\text{op}}).
\end{equation}
Further, we have natural evaluation morphism
\begin{equation}
L_{\mathcal{B}_{S_1}} Q_1 \otimes_{\mathcal{B}_{S_1}} M = Q_1 \otimes_{\mathcal{B}_{S_1}} M \to Q_2 \text{ in } D(\mathcal{B}_{S_2}^{\text{op}}).
\end{equation}
We claim that this is an isomorphism. Before we prove this, we note that this would finish the proof of part 2) of Proposition.

Now, note that for each $N \in D(\mathcal{A})$, we have evaluation morphism
\begin{equation}
ev_N : L_{\mathcal{B}_{S_1}} Q_1 \otimes_{\mathcal{B}_{S_1}} \text{R Hom}_{\mathcal{A}^{\text{op}}}(Q_1, N) = Q_1 \otimes_{\mathcal{B}_{S_1}} \text{Hom}_{\mathcal{A}^{\text{op}}}(Q_1, N) \to N \text{ in } D(\mathcal{A}).
\end{equation}
Note that $ev_N$ is an isomorphism for $N \in S_1$. But $S_1$ split-generates $S_2$. Therefore, $ev_N$ is an isomorphism for each object of $S_2$. Hence, the map (3.14) is an isomorphism. Proposition is proved. □

Note that we have a natural DG functor $\iota_S : A \to \hat{A}_S$, which is identity on objects. It is easily seen from the proof of Proposition 3.2 2) that in the situation of Proposition 3.2 2), we have a commutative diagram in Ho(dgcat):

\[
\begin{array}{ccc}
A & \xrightarrow{\iota_S} & \hat{A}_{S_1} \\
\text{id} & \downarrow & \cong \\
A & \xrightarrow{\iota_S} & \hat{A}_{S_2}.
\end{array}
\]

Therefore, for any full thick essentially small triangulated subcategory $T \subset D(A)$ we have a naturally defined (up to quasi-equivalence) DG category $\hat{A}_T$, together with a morphism $\iota_T : A \to \hat{A}_T$. More precisely, one can choose any small subcategory $S \subset T$, which generates $T$, and put

\[
\hat{A}_T := \hat{A}_S, \quad \iota_T := \iota_S.
\]

**Definition 3.3.** For any small DG category $A \in \text{dgcat}_k$, and any full thick essentially small subcategory $T \subset D(A)$, we call the DG category $\hat{A}_T$ "derived double centralizer of $T".$

Next Proposition shows that the introduced notion of formal completion is Morita invariant.

**Proposition 3.4.** Suppose that DG categories $A_1$ and $A_2$ are Morita equivalent. Let $T_1 \subset D(A_1)$, $T_2 \subset D(A_2)$ be full thick essentially small triangulated subcategories, which correspond to each other under the equivalence $D(A_1) \cong D(A_2)$. Then the DG categories $\hat{A}_{1T_1}$ and $\hat{A}_{2T_2}$ are also Morita equivalent, and we have commutative diagram

\[
\begin{array}{ccc}
D(A_1) & \xrightarrow{L_{\iota_{T_1}}} & D(\hat{A}_{1T_1}) \\
\cong & \downarrow & \cong \\
D(A_2) & \xrightarrow{L\iota_{T_2}} & D(\hat{A}_{2T_2}).
\end{array}
\]

**Proof.** We may and will assume that $A_i \in \text{dgcat}_k$ are h-projective DG categories. Let $M \in D(A_1^{op} \otimes A_2)$ be a bimodule which defines an equivalence

\[
D(A_1) \xrightarrow{L} \otimes_{A_1} M : D(A_1) \to D(A_2).
\]

Then the category $D(A_2)$ is compactly generated by the set of objects $\{M(U, -) \in D(A_2), U \in A_1\}$. Thus, we may assume that $A_1 \subset \text{h-proj}(A_2)$, and the quasi-inverse
to (3.19) is given by the formula

\[(3.20) \quad F : D(A_2) \to D(A_1), \quad F(X)(U) = \text{Hom}_{A_2^{\text{op}}}(U, X).\]

Now choose any small subcategory \(S_2 \subset \mathcal{T}_2\), which split-generates \(\mathcal{T}_2\), and choose h-projective resolution \(\tilde{X} \to X\) of each object \(X \in S_2\). This choice defines the DG category \(B_{S_2}\) and the bimodule \(S_2 \in D(A_2 \otimes B_{S_2}^{\text{op}})\). Now, a choice of an h-projective resolution \(Q_2 \to S_2\) in \((A_2 \otimes B_{S_2}^{\text{op}})\)-mod defines the DG category \(\hat{A}_{2S_2} \cong \hat{A}_{2\mathcal{T}_2}\). Now, define the bimodule \(S_1 \in D(A_1 \otimes B_{S_2}^{\text{op}})\) by the formula

\[(3.21) \quad S_1(U, X) := \text{Hom}_{A_2}(U, \tilde{X}), \quad U \in A_1, X \in S_2.\]

Choose an h-projective resolution \(Q_1 \to S_1\). It defines the DG category \(\hat{A}_{1S_1} \cong \hat{A}_{1\mathcal{T}_1}\). Define the DG bimodule \(\hat{M} \in D(\hat{A}_{1S_1} \otimes \hat{A}_{2S_2})\) by the formula

\[(3.22) \quad \hat{M}(U, V) = \text{Hom}_{B_{S_2}^{\text{op}}}(Q_1(U, -), Q_2(V, -)), \quad U \in \hat{A}_{1S_1}, V \in \hat{A}_{2S_2}.\]

Since \(A_1\) and \(A_2\) split-generate each other in \(D(A_2)\), we have that the functor \((3.22)\) is an equivalence. It is straightforward to show that the following diagram commutes up to a natural isomorphism

\[(3.23) \quad \begin{array}{ccc}
D(A_1) & \xrightarrow{L_{S_1}} & D(\hat{A}_{1S_1}) \\
\downarrow \text{L}_{\otimes A_1 M} & & \downarrow \text{L}_{\otimes A_1 S_1} \hat{M} \\
D(A_2) & \xrightarrow{L_{S_2}} & D(\hat{A}_{2S_2}).
\end{array}
\]

Now we introduce the main notion of the paper.

**Definition 3.5.** Let \(\mathcal{D}\) be an enhanced triangulated category with infinite direct sums, which is compactly generated by a set of objects. Let \(\mathcal{T} \subset \mathcal{D}\) be an essentially small full thick triangulated subcategory. We define the formal completion \(\hat{\mathcal{D}}_{\mathcal{T}}\) of \(\mathcal{D}\) along \(\mathcal{T}\), together with a restriction functor \(\kappa^* : \mathcal{D} \to \hat{\mathcal{D}}_{\mathcal{T}}\), as follows. Choosing a set of compact generators in \(\mathcal{D}\), we may replace \(\mathcal{D}\) by \(D(A)\) for some small DG category \(A\). Then put

\[(3.24) \quad \hat{\mathcal{D}}_{\mathcal{T}} := D(\hat{A}_{\mathcal{T}}), \quad \kappa^* := L_{A_{\mathcal{T}}} : \mathcal{D} = D(A) \to D(\hat{A}_{\mathcal{T}}) = \hat{\mathcal{D}}_{\mathcal{T}}.\]

**Theorem 3.6.** In the notation of Definition 3.5, the category \(\hat{\mathcal{D}}_{\mathcal{T}}\) is well-defined up to an equivalence, compatible with the functor \(\kappa^* : \mathcal{D} \to \hat{\mathcal{D}}_{\mathcal{T}}\).

The category \(\hat{\mathcal{D}}_{\mathcal{T}}\) is enhanced, admits infinite direct sums, and is compactly generated by a set of objects. The functor \(\kappa^*\) commutes with infinite direct sums and preserves compact objects. If \(S \subset \text{Ob}(\mathcal{D})\) is a set of compact generators, then \(\kappa^*(S)\) is a set of compact generators in \(\hat{\mathcal{D}}_{\mathcal{T}}\).
Proof. The first statement follows from Proposition 3.3 since different sets of compact generators yield Morita equivalent DG categories.

The other statements follow directly from definition. □

It is convenient to introduce one more definition.

**Definition 3.7.** Let \( \mathcal{D} \) be an essentially small Karoubian complete enhanced triangulated category, and \( \mathcal{T} \subset \mathcal{D} \) be a full thick triangulated subcategory. Define the formal completion \( \hat{\mathcal{D}}_\mathcal{T} \) of \( \mathcal{D} \) along \( \mathcal{T} \), together with a restriction functor \( \kappa^* : \mathcal{D} \to \hat{\mathcal{D}}_\mathcal{T} \), as follows. Choosing a set of generators in \( \mathcal{D} \), we may replace \( \mathcal{D} \) by \( \text{Perf}(\mathcal{A}) \) for some small DG category \( \mathcal{A} \). Then put

\[
\hat{\mathcal{D}}_\mathcal{T} := \text{Perf}(\hat{\mathcal{A}}_\mathcal{T}), \quad \kappa^* := \text{L}^*_{\hat{\mathcal{T}}} : \mathcal{D} = \text{Perf}(\mathcal{A}) \to \text{Perf}(\hat{\mathcal{A}}_\mathcal{T}) = \hat{\mathcal{D}}_\mathcal{T}.
\]

**Remark 3.8.** If \( \mathcal{D} \) is a compactly generated triangulated category and \( \mathcal{T} \subset \mathcal{D}^c \) is an essentially small full thick subcategory, then we have

\[
(\hat{\mathcal{D}}_\mathcal{T})^c \cong \hat{\mathcal{D}}^c_\mathcal{T}.
\]

### 4. Properties of Categorical Formal Completion

In this section we study various properties of formal completions of categories along subcategories.

All categories are supposed to be enhanced. Further, by a "compactly generated triangulated category" we mean a "triangulated category with infinite direct sums, which is compactly generated by a set of objects".

**Theorem 4.1.** Let \( \mathcal{D} \) be a compactly generated triangulated category, \( \mathcal{T} \subset \mathcal{D} \) a full thick essentially small triangulated subcategory. Assume that \( \mathcal{T} \) is contained in the smallest localizing subcategory of \( \mathcal{D} \) containing \( \mathcal{T} \cap \mathcal{D}^c \). Then

(i) The restriction of the functor \( \kappa^* : \mathcal{D} \to \hat{\mathcal{D}}_\mathcal{T} \) on the subcategory \( \mathcal{T} \) is full and faithful (below we identify \( \mathcal{T} \) with its image under the functor \( \kappa^* \)).

(ii) The functor \( \kappa^* : \hat{\mathcal{D}}_\mathcal{T} \to \hat{\mathcal{D}}_{\mathcal{T} \mathcal{T}} \) is an equivalence.

(iii) Let \( \mathcal{T} \subset \mathcal{T} \) be a full thick triangulated subcategory. Then there is a natural equivalence \( \hat{\mathcal{D}}_{\mathcal{T}} \cong \hat{\mathcal{D}}_{\mathcal{T} \mathcal{T}} \).

**Proof.** We may and will assume that \( \mathcal{D} = D(\mathcal{A}) \) for some small h-flat DG category \( \mathcal{A} \), and the subcategory \( \mathcal{T} \cap \mathcal{D}^c \subset \mathcal{D} \) is split-generated by the full DG subcategory \( \mathcal{A}' \subset \mathcal{A} \). Let \( \mathcal{B} \subset \text{h-proj}(\mathcal{A}) \) be a small DG subcategory, which split-generates \( \mathcal{T} \). We may and will assume that \( \mathcal{A}' \subset \mathcal{B} \). We have the DG bimodule \( M \in D(\mathcal{A} \otimes \mathcal{B}^{opp}) \),

\[
M(U, V) = \text{Hom}_\mathcal{A}(U, V), \quad U \in \mathcal{A}, V \in \mathcal{B}.
\]
Choose an h-projective resolution \( Q \to M \). It gives the DG model for \( \tilde{A}_T \) :

\[
\text{(4.2) } \quad \text{Ob}(\tilde{A}_T) = \text{Ob}(A), \quad \text{Hom}_{\tilde{A}_T}(X, Y) = \text{Hom}_{B^{op}}(Q(Y, -), Q(X, -)).
\]

For \( X \in A, \ X' \in A' \), we have the following isomorphisms in \( D(k) \) :

\[
\text{(4.3) } \quad \text{Hom}_{\tilde{A}_T}(X, X') \cong \text{Hom}_{B^{op}}(Q(X', -), Q(X, -)) \cong \text{Hom}_{B^{op}}(\text{Hom}_B(X', -), \text{Hom}_A(X, -)) \cong \text{Hom}_A(X, X').
\]

Isomorphisms (4.3) imply in particular that the functor \( \kappa^* \) is full and faithful on \( T \cap D^c \). Moreover, since \( \kappa^* \) preserves compact objects, it is also full and faithful on the smallest localizing subcategory containing \( T \cap D^c \). In particular, by our assumption, it is full and faithful on \( T \). This proves (i).

Further, (4.3) also implies that the maps

\[
\text{(4.4) } \quad R\text{Hom}_D(X, Y) \to R\text{Hom}_{\tilde{A}_T}(\kappa^*(X), \kappa^*(Y))
\]

are isomorphisms (in \( D(k) \)) for \( X \in D^c, \ Y \in T \cap D^c \). Since \( X \) and \( \kappa^*(X) \) are compact, the maps (4.4) are also isomorphisms for \( Y \) in the smallest localizing subcategory containing \( T \cap D^c \), and in particular for \( Y \in T \). This easily implies both (ii) and (iii). Theorem is proved.

**Proposition 4.2.** Let \( A \) be a small DG category, and let \( \{ T_\beta \subset D(A) \}_{\beta \in B} \) be a (small) collection of mutually orthogonal full thick essentially small triangulated subcategories. Denote by \( T \subset D(A) \) the full thick triangulated subcategory classically generated by all \( T_\beta \). Then there is a natural isomorphism in \( \text{Ho}(\text{dgcat}_k) : \)

\[
\text{(4.5) } \quad \tilde{A}_T \cong \prod_{\beta \in B} \tilde{A}_{T_\beta}.
\]

**Proof.** This can be easily seen if we choose generating subset \( S \subset \text{Ob}(T) \) to be the disjoint union of generating subsets \( S_\beta \subset \text{Ob}(T_\beta) \).

**Proposition 4.3.** Let \( T \) be an essentially small Karoubian complete triangulated category, and suppose that we have a semi-orthogonal decomposition \( T = \langle S_1, S_2 \rangle \), so that \( \text{Hom}_T(S_2, S_1) = 0 \). Then we have natural equivalence \( \tilde{T}_{S_1} \cong S_1 \), and the corresponding functor \( \kappa^* : T \to S_1 = \tilde{T}_{S_1} \) is the semi-orthogonal projection.

**Proof.** By Lemma 2.5, we may (and will) assume that \( T = \text{Perf}(A) \) for some small DG category \( A \), and \( S_1 \subset T \) is generated by DG subcategory \( A_1 \subset A \).

We may assume that \( \text{Ob}(A) = \text{Ob}(A_1) \sqcup \text{Ob}(A_2) \). Further, we may assume that \( \text{Hom}_A(X, Y) = 0 \) for \( X \in A_2, \ Y \in A_1 \). With these assumptions, Proposition follows directly from definitions.
Proposition 4.4. 1) Let $\mathcal{T}$ be some smooth and proper pre-triangulated DG category, and $S \subset \text{Ho}(\mathcal{T})$ a full thick triangulated subcategory. Then we have a natural equivalence $(\widehat{T_S})^{\text{op}} \cong \widehat{T^{\text{op}} S_{\text{op}}}$. 

2) If we drop the assumption of either properness or smoothness, then Proposition fails to hold.

Proof. 1) We may assume that $\mathcal{T} = \text{Perf}(A)$ for smooth and proper DG algebra $A$. Then we have that $\text{Perf}(A) = D_{\text{fin}}(A)$, where $D_{\text{fin}}(A) \subset D(A)$ is the subcategory of DG modules which are perfect as $k$-modules. Therefore, we have an equivalence

$$(4.6) \quad (-)^* : \text{Perf}(A)^{\text{op}} \xrightarrow{\sim} \text{Perf}(A^{\text{op}}), \quad M \rightarrow M^* = R\text{Hom}_k(M, k).$$

Denote by $S^*$ the image of $S$ under this equivalence. Then, it is easy to see that

$$(4.7) \quad (\widehat{A_S})^{\text{op}} \cong \widehat{A^{\text{op}} S^*}.$$ 

This proves part 1) of Proposition.

2) To prove part 2), we first give an example when $\mathcal{T}$ is proper but not smooth, and Proposition does not hold. Define the DG category $A$ as follows. Put $\text{Ob}(A) := \{X_1, X_2\}$, and

$$(4.8) \quad \text{Hom}(X_1, X_1) = k[\epsilon]/(\epsilon^2), \quad \text{deg}(\epsilon) = 1, \quad \text{Hom}(X_1, X_2) = k[0],$$

$$\quad \text{Hom}(X_2, X_1) = 0, \quad \text{Hom}(X_2, X_2) = k.$$ 

The differential is identically zero and the composition is the only possible one. Put $\mathcal{T} = \text{Perf}(A)$, and take $S \subset \mathcal{T}$ to be subcategory generated by $X_1$. Then it is straightforward to check that

$$(4.9) \quad \widehat{T_S} \cong \text{Perf}(k[\epsilon]/(\epsilon^2)), \quad \widehat{T^{\text{op}} S_{\text{op}}} \cong \text{Perf}(k[[t]]).$$

Hence, there is no equivalence between $(\widehat{T_S})^{\text{op}}$ and $\widehat{T^{\text{op}} S_{\text{op}}}$. 

Now we give an example when $\mathcal{T}$ is smooth (and even homotopically finitely presented) but not proper, and Proposition does not hold.

Take the DG category $B$ with two objects $Y_1, Y_2$, which is a free $k$-linear category concentrated in degree zero with generators $s_{22} : Y_2 \rightarrow Y_2$, $s_{12} : Y_1 \rightarrow Y_2$. Put $\mathcal{T} = \text{Perf}(B)$. Take $S \subset \mathcal{T}$ to be subcategory generated by $Y_1$. Then we have that

$$(4.10) \quad \widehat{T_S} \cong \text{Perf}(k), \quad \widehat{T^{\text{op}} S_{\text{op}}} \cong \text{Perf}(M_{\infty}(k)),$$

where $M_{\infty}(k)$ is the endomorphism algebra of free countably generated $k$-module. Hence, there is no equivalence between $(\widehat{T_S})^{\text{op}}$ and $\widehat{T^{\text{op}} S_{\text{op}}}$. 

Proposition is proved. □
5. Relation to formal completions of Noetherian schemes

Before we formulate and prove main result of this section, we would like to proof a
general result which relates double centralizers and homotopy limits of DG algebras.

5.1. Double centralizers and homotopy limits. Let $I$ be a small category. Denote by
dgalg$_k^I$ the category of functors $I \to \text{dgalg}_k$. Take some \{$A_i$\}$_{i \in I} \in \text{dgalg}_k^I$.

Then there exists a homotopy limit

$$A = \lim_{\text{holim}_I} A_i.$$

We would like to write it in explicit form.

**Definition 5.1.** For a morphism $s : x \to y$ in the category $I$, we put $r(s) := y$, $l(s) = x$.
We denote by $\text{Mor}(I)$ the set of non-identical morphisms in $I$.

We put

$$A^i = \prod_{s_1, \ldots, s_p \in \text{Mor}(I), p > 0, \ l(s_{i+1}) = r(s_i)} A_{r(s_p)}^{i-p} \times \prod_{x \in I} A_x^i.$$

For $a \in A$, we denote by $a_{s_p, \ldots, s_1} \in A_{r(s_p)}$, $a_x \in A_x$ the corresponding components. It is convenient to consider components $a_x$ to be corresponding to empty paths in $I$, with final object $x$. With this in mind, the differential and the composition are defined as follows.

For homogeneous $a, b \in A$,

$$d(a)_{s_p, \ldots, s_1} = d(a_{s_p, \ldots, s_1}) + (-1)^{\bar{a} + 1}s_p(a_{s_p-1, \ldots, s_1}) +$$

$$\sum_{i=1}^{p-1} (-1)^{\bar{a} + 1 + p-j} a_{s_p, \ldots, s_{i+1}, \ldots, s_1} + (-1)^{\bar{a} + 1 + p} a_{s_p, \ldots, s_2},$$

$$a \cdot b = \sum_{s_0 \in I} (-1)^{\bar{a} - \bar{b}} a_{s_p, \ldots, s_{i+1}} \cdot s_p \ldots s_{i+1}(b_{s_i, \ldots, s_1}),$$

where $\bar{a}$ (resp. $\bar{b}$) denote the degree of $a$ (resp. $b$).

Now suppose that we have a compatible system of morphisms $f_x : B \to A_x$, $x \in I$, in \text{dgalg}_k (i.e. $sf_x = f_y$ for $s : x \to y$). Then we have natural morphism $f : B \to A = \lim_{\text{holim}_I} A_i$, given by the formula

$$f(b)_{s_p, \ldots, s_1} = 0 \quad \text{for } p > 0.$$
Now suppose that we have also a functor $I^{op} \to Z^0(\mathcal{C}\text{-Mod})$, $x \to M_x$, where $\mathcal{C}$ is some DG category, and $Z^0(\mathcal{C}\text{-Mod})$ is the abelian category of right DG $\mathcal{C}$-modules. Then there exists a homotopy colimit

\[(5.6) \quad M = \text{hocolim}_{I^{op}} M_x.\]

Again, we can write $M$ explicitly as follows:

\[(5.7) \quad M(X)^i = \bigoplus_{s_1, \ldots, s_p \in \text{Mor}(I), p > 0, \ i = \text{deg}(s_i)} M_{r(s_p)}(X)^{i+p} \oplus \bigoplus_{x \in I} M_x(X), \quad X \in \mathcal{C}.\]

For $m \in M_{r(s_p)}(X)$ (resp. $m \in M_x(X)$) we denote by $m_{s_p, \ldots, s_1} \in M(X)$ (resp. $m_x \in M(X)$) the corresponding elements with only one component. Again, it is convenient to consider $m_x$ to be corresponding to an empty path in $I$, with final object $x$. For a homogeneous $m$, we have that $\text{deg}(m_{s_p, \ldots, s_1}) = \text{deg}(m) - p$. For a homogeneous $m_{s_p, \ldots, s_1}$, we have

\[(5.8) \quad d(m_{s_p, \ldots, s_1}) = d(m)_{s_p, \ldots, s_1} + (-1)^m s_p(m)_{s_p-1, \ldots, s_1} + \sum_{i=1}^{p-1} (-1)^{m+p-i} m_{s_p, \ldots, s_{i+1}, \ldots, s_1} + (-1)^m p m_{s_p, \ldots, s_2}.\]

Further, for a homogeneous $f \in \text{Hom}_\mathcal{C}(Y, X)$, we have

\[(5.9) \quad m_{s_p, \ldots, s_1} \cdot f = (-1)^{p} f(m)_{s_p, \ldots, s_1}.\]

Suppose that we have a compatible system of morphisms $g : M_x \to N$ for some DG module $N$ (i.e. $g_x s = g_y$ for $s \in \text{Hom}_I(x, y)$). Then we have natural morphism $g : M = \text{hocolim}_{I^{op}} M_x \to N$, given by the formula

\[(5.10) \quad \begin{cases} g(m_x) = g_x(m) \quad \text{for} \quad x \in I; \\ f(m_{s_p, \ldots, s_1}) = 0 \quad \text{for} \quad p > 0. \end{cases}\]

Now, suppose that, with the above notation, we have a system of morphisms of DG algebras $\varphi_x : \mathcal{A}_x^{op} \to \text{End}_\mathcal{C}(M_x)$, $x \in I$, which are compatible in the following sense:

\[(5.11) \quad \varphi_x(a)(s(m)) = s(\varphi_y(a)(m)), \quad a \in \mathcal{A}_x, \quad m \in M_y, \quad s \in \text{Hom}_I(x, y).\]

Then we have a natural morphism

\[(5.12) \quad \mathcal{A}^{op} = (\text{hocolim}_I \mathcal{A}_x)^{op} \to \text{End}_\mathcal{C}(M) = \text{End}_\mathcal{C}(\text{hocolim}_{I^{op}} M_x).\]

Explicitly, for homogeneous $a \in \mathcal{A}$, $m_{s_p, \ldots, s_1} \in M(X)$, we have

\[(5.13) \quad a(m_{s_p, \ldots, s_1}) = \sum_{i=0}^{p} (-1)^{(p-i+\bar{a})} (s_p \ldots s_{i+1})(s)(a_{s_p, \ldots, s_{i+1}}(m))_{s_i, \ldots, s_1}.\]
Now we are ready to formulate and prove our main technical result.

**Lemma 5.2.** Let $\mathcal{A}$ be a DG algebra, $\mathcal{T} \subset D(\mathcal{A})$ a full thick essentially small triangulated subcategory. Suppose that $I$ is a small category, $\{\mathcal{A}_x\}_{x \in I} \in \text{dgalg}_k^I$, and we have a compatible system of morphisms $f : \mathcal{A} \to \mathcal{A}_x$, $x \in I$. Assume that all $\mathcal{A}_x$ lie in $\mathcal{T}$ as right DG $\mathcal{A}$-modules, and for any $E \in \mathcal{T}$ the natural map

\[(5.14) \quad \text{hocolim}_{I^{op}} R\text{Hom}_\mathcal{A}(\mathcal{A}_x, E) \to E\]

is an isomorphism in $D(k)$. Then we have natural commutative diagram in $\text{Ho}(\text{dgalg}_k)$:

\[(5.15) \quad \begin{array}{ccc}
\mathcal{A} & \xrightarrow{id} & \mathcal{A} \\
\downarrow \iota_{\mathcal{T}} & & \downarrow \\
\widehat{\mathcal{A}}_{\mathcal{T}} & \cong & \text{holim}_I \mathcal{A}_x.
\end{array}\]

**Proof.** Choose some set of $h$-injective $\mathcal{A}_{x}^{\text{op}}$-modules which generate $\mathcal{T}$, and denote by $\mathcal{D}$ the corresponding DG category. Then by our assumptions, we have natural quasi-isomorphism of DG $\mathcal{D}$-modules:

\[(5.16) \quad \text{hocolim}_{I^{op}} \text{Hom}_\mathcal{A}(\mathcal{A}_x, -) \to \text{Hom}_\mathcal{A}(\mathcal{A}, -).\]

Therefore, we have natural isomorphism in $\text{Ho}(\text{dgalg})$:

\[(5.17) \quad \widehat{\mathcal{A}}_{\mathcal{T}}^{\text{op}} \cong R\text{End}_{\mathcal{D}^{\text{op}}}(\text{hocolim}_{I^{op}} \text{Hom}_\mathcal{A}(\mathcal{A}_x, -)).\]

We have natural compatible system of morphisms of DG algebras:

\[(5.18) \quad \varphi_x : \mathcal{A}_x^{\text{op}} \to \text{End}_{\mathcal{D}^{\text{op}}}(\text{Hom}_\mathcal{A}(\mathcal{A}_x, -)).\]

Therefore, as in (5.12), we have natural morphism

\[(5.19) \quad \varphi : (\text{holim}_I \mathcal{A}_x)^{\text{op}} \to \text{End}_{\mathcal{D}^{\text{op}}}(\text{hocolim}_{I^{op}} \text{Hom}_\mathcal{A}(\mathcal{A}_x, -)).\]

Composing it with natural map from $\text{End}$ to $R\text{End}$ (in $\text{Ho}(\text{dgalg}_k)$) and (5.17), we obtain a natural morphism

\[(5.20) \quad \text{holim}_I \mathcal{A}_x \to \widehat{\mathcal{A}}_{\mathcal{T}}\]

in $\text{Ho}(\text{dgalg})$. Further, since $\mathcal{A}_x \in \mathcal{T}$, we have natural isomorphisms in $D(k)$:

\[(5.21) \quad \mathcal{A}_x \cong R\text{Hom}_{\mathcal{D}^{\text{op}}}(\text{Hom}_\mathcal{A}(\mathcal{A}_x, -), \text{Hom}_\mathcal{A}(\mathcal{A}, -)).\]
To conclude that \((5.20)\) is an isomorphism, it suffices to note the following chain of isomorphisms in \(D(\mathbb{k})\):

\[
\hat{A}_T \cong R \text{End}_{D^{\text{op}}}(\text{Hom}_A(A,-)) \cong \end{array}
\]

\[
\text{holim}_f R \text{Hom}_D(\text{Hom}_A(A_x,-), \text{Hom}_A(A,-)) \cong \text{holim}_f A_x.
\]

It is easy to check that the composition \((5.22)\) is inverse (in \(D(\mathbb{k})\)) to the morphism of DG algebras \((5.20)\), so we obtain the desired isomorphism in \(\text{Ho(dgalg)}\). Commutativity of \((5.15)\) is straightforward to check. □

5.2. Algebraizable derived categories of formal completions of schemes. Let \(X\) be a separated Noetherian \(\mathbb{k}\)-scheme. Recall that \([BvdB] \) \(D(X) = D(\text{Qcoh}(X))\), the derived category of quasi-coherent sheaves on \(X\), is compactly generated by one object, and \(D(X)^c = \text{Perf}(X)\). More precisely, they prove this for the category \(D_{\text{qch}}(X)\) of complexes of \(\mathcal{O}_X\)-modules with quasi-coherent cohomology, but for \(X\) separated the latter category is known to be equivalent to \(D(\text{Qcoh}(X))\) (see \([BvdB]\)).

Now let \(Y \subset X\) a closed subscheme. We would like to define the algebraizable derived category \(D_{\text{alg}}(\hat{X}_Y)\).

Let \(\mathcal{I}_Y \subset \mathcal{O}_X\) be ideal sheaf defining \(Y\). Denote by \(Y_n \subset X\) the \(n\)-th infinitesimal neighborhood of \(Y\), with ideal sheaf \(\mathcal{I}_n\). Denote by \(\iota_{n,n+1} : Y_n \to Y_{n+1}, \ i_n : Y_n \to X\) the natural inclusions. Choose some DG enhancements for \(\text{Perf}(X)\) and \(\text{Perf}(Y_n)\), with DG enhancements of functors \(L_{\iota_n}, L_{\iota_{n,n+1}}^*\) (we write the corresponding DG functors in the same way), so that we have equalities of DG functors \(L_{\iota_n}^* = L_{\iota_{n,n+1}}^* L_{\iota_{n+1}}^*\). Denote by \(R \text{Hom}(-,-)\) the complexes of morphisms in the corresponding DG enhancements.

Define the DG category \(\text{Perf}_{\text{alg}}(\hat{X}_Y)\) as follows. Its objects are the same as in \(\text{Perf}(X)\). Further, for \(\mathcal{E}, \mathcal{F} \in \text{Perf}(X)\), we put

\[
\text{Hom}_{\text{Perf}_{\text{alg}}(\hat{X}_Y)}(\mathcal{E}, \mathcal{F}) := \text{holim}_n R \text{Hom}(L_{\iota_n}^* \mathcal{E}, L_{\iota_n}^* \mathcal{F}).
\]

Composition are defined in the obvious way (as in the case of homotopy limits of DG algebras). Define algebraizable derived category by the formula

\[
D_{\text{alg}}(\hat{X}_Y) := D(\text{Perf}_{\text{alg}}(\hat{X}_Y)).
\]

We have an obvious DG functor

\[
\kappa^* : \text{Perf}(X) \to \text{Perf}_{\text{alg}}(\hat{X}_Y),
\]

and the corresponding functor

\[
L \kappa^* : D(X) \to D_{\text{alg}}(\hat{X}_Y).
\]
Theorem 5.4. Let \( \hat{A} \) the restriction of scalars for the natural morphism \( \hat{\kappa} \)
where \( \kappa^* \) in this case is just the restriction of scalars for the natural morphism \( A \rightarrow \hat{A} \).

Remark 5.3. In the case when \( X = \text{Spec}(A) \) is affine, and \( Y = \text{Spec}(A/I) \), we easily see that
\[
D_{alg}(\hat{X}_Y) \cong D(\hat{A}_I),
\]
where \( \hat{A}_I = \lim_n A/I^n \) is the \( I \)-adic completion of \( A \).

Theorem 5.4. Let \( X \) be a separated Noetherian scheme, and \( Y \subset X \) a closed subscheme. Then we have the following commutative diagram:
\[
\begin{array}{ccc}
D(X) & \xrightarrow{id} & D(X) \\
\downarrow & & \downarrow L\kappa^* \\
\hat{D}(X)_{D^{b, Y}(X)} & \xrightarrow{\cong} & D_{alg}(\hat{X}_Y).
\end{array}
\]

Proof. We follow notation above the theorem. Choose a generator \( F \in \text{Perf}(X) \). Then \( \kappa^*(F) \in \text{Perf}_{alg}(\hat{X}_Y) \) is a compact generator of \( D_{alg}(\hat{X}_Y) \). Put
\[
\mathcal{A} := R \text{End}(F), \quad \mathcal{A}_n := R \text{End}(L^n F).
\]
We have obvious morphisms \( L^{n+1} : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n \). Hence \( \{\mathcal{A}_n\}_{n \in \mathbb{N}} \in \text{dgAlg}_k^{\text{op}} \), where we treat \( \mathbb{N} \) as a category: \( \text{Ob}(\mathbb{N}) = \mathbb{Z}_{>0} \). \( \text{Hom}(i, j) = \emptyset \) for \( i > j \), and there is exactly one morphism \( i \rightarrow j \) for \( i \leq j \). Further, we have morphisms of DG algebras
\[
L_n^{\ast} : \mathcal{A} \rightarrow \mathcal{A}_n,
\]
which are compatible with \( L^{n+1} \) by our assumptions. Further, denote by \( T \subset D(\mathcal{A}) \) the essential image of \( D^{b, Y}(X) \) under the equivalence
\[
R \text{Hom}(F, -) : D(X) \rightarrow D(\mathcal{A}).
\]
By adjunction, \( \mathcal{A}_n \cong R \text{Hom}(F, t_n L^{\ast}_n F) \in T \). We claim that the data of \( \mathcal{A}_n \), \( \{\mathcal{A}_n\}_{n \in \mathbb{N}} \in \text{dgAlg}_k^{\text{op}}, \mathcal{T} \subset D(\mathcal{A}) \) and morphisms [5.30] satisfies the assumptions of Lemma 5.2. Indeed, by Grothendieck Theorem [Gr], for any \( G \in \text{Perf}(X), E \in D^Y(X) \) we have an isomorphism in \( D(X) \):
\[
hocolim_n R \text{Hom}(t_n L^{\ast}_n G, E) \cong R \text{Hom}(G, E).
\]
Moreover, since the functor \( R \Gamma \) commutes with infinite direct sums, we have a chain of isomorphisms in \( D(k) \):
\[
hocolim_n R \text{Hom}(t_n L^{\ast}_n G, E) \cong hocolim_n R \Gamma(R \text{Hom}(t_n L^{\ast}_n G, E)) \cong R \Gamma(hocolim_n R \text{Hom}(t_n L^{\ast}_n G, E)) \cong R \Gamma(R \text{Hom}(G, E)) \cong R \text{Hom}(G, E).
\]
Therefore, we have the following isomorphisms:

\[(5.34) \quad \text{hocolim}_n R\text{Hom}_A(A_n, R\text{Hom}(F, E)) \cong \text{hocolim}_n R\text{Hom}(\iota_n, L\iota_n^* F, E) \cong R\text{Hom}(F, E).\]

Hence, the assumptions of Lemma 5.2 are satisfied. Applying it, we obtain the chain of equivalences

\[(5.35) \quad \hat{D}(X)_{D_{\text{coh}, Y}(X)} \cong D(\text{A}_n) \cong D(\text{holim}_n \text{A}_n) \cong D_{\text{alg}}(X).\]

The last equivalence follows from the observation that $\kappa^*(F) \in \text{Perf}_{\text{alg}}(\hat{X}_Y)$ is a compact generator of $D_{\text{alg}}(X)$ and

\[(5.36) \quad \text{End}_{\text{Perf}_{\text{alg}}(\hat{X}_Y)}(\kappa^*(F)) \cong \text{holim}_n \text{A}_n.\]

Commutativity of (5.28) is straightforward. □

**Corollary 5.5.** Let $X$ be a separated Noetherian scheme, and $Y \subset X$ a closed subscheme. Then

1) The restriction of the functor $L\kappa^*: D(X) \to D_{\text{alg}}(\hat{X}_Y)$ to $D_{\text{coh}, Y}(X)$ is full and faithful;

2) The functor

\[(5.37) \quad D_{\text{alg}}(\hat{X}_Y) \to D_{\text{alg}}(\hat{X}_Y)_{D_{\text{coh}, Y}(X)}\]

is an equivalence.

**Proof.** Recall that the category $D_Y(X)$ is compactly generated by $\text{Perf}_Y(X) \subset D_{\text{coh}, Y}(X)$ [AJPV]. Hence, both 1) and 2) are direct consequences of Theorems 5.4 and 4.1. □

We have a nice corollary for completions of regular Noetherian $k$-algebras.

**Corollary 5.6.** Let $R$ be a regular commutative Noetherian $k$-algebra, and $M \in D_{f.g.}(R) \cong D_{\text{coh}}(\text{Spec } R)$ be a complex of $R$-modules with finitely generated cohomology. Denote by $I \subset R$ the annihilator of $H(M)$, so that $V(\sqrt{I}) \subset \text{Spec } R$ is precisely the support of $M$. Then we have an isomorphism

\[(5.38) \quad \hat{R}_M \cong \hat{R}_I,\]

where the RHS is the ordinary $I$-adic completion.

**Proof.** By a Theorem of Hopkins [Ho] and Neeman [Nee], all full thick triangulated subcategories of $\text{Perf}(R) \cong \text{Perf}(\text{Spec } R)$ generated by one object are of the form $\text{Perf}_Z(\text{Spec } R)$ (perfect complexes with cohomology supported on $Z$) for a closed subset $Z \subset \text{Spec } R$. Further, since $R$ is regular, we have that $D_{\text{coh}}(\text{Spec } R) = \text{Perf}(\text{Spec } R)$. It follows that
$M$ generates the subcategory $D^b_{\text{coh},V(\sqrt{T})}(\text{Spec } R) \subset D^b_{\text{coh}}(\text{Spec } R)$. It remains to apply Theorem \[5.4\]

**Corollary 5.7.** Let $R$ be commutative Noetherian $k$-algebra, and $I \subset R$ an ideal. Assume that either $R$ or $R/I$ is regular. Then we have an isomorphism

\[(5.39) \quad \hat{R}_{(R/I)} \cong \hat{R}_I,\]

where the RHS is ordinary $I$-adic completion.

**Proof.** If $R$ is regular, the isomorphism follows from Corollary \[5.6\]. Assume that $R/I$ is regular.

Put $X := \text{Spec } (R)$, and $Y := \text{Spec } (R/I) \subset X$. We claim that $\iota_*\mathcal{O}_Y$ is a generator of $D^b_{\text{coh},Y}(X)$. Indeed, $\iota_*\mathcal{O}_Y$ generates all objects $\iota_*\mathcal{F}, \mathcal{F} \in D^b_{\text{coh}}(Y)$, which generate the whole subcategory $D^b_{\text{coh},Y}(X)$.

Therefore, the assertion follows from Theorem \[5.4\].

The following Proposition shows that in the Corollary \[5.7\] one cannot drop the assumption of regularity.

**Proposition 5.8.** 1) Let $R$ be some commutative algebra over a field $k$, and $M$ an $R$-module. Denote by $\tilde{R}$ the split square-zero extension of $R$ by $M$. The following are equivalent:

(i) The map $\tilde{R} \to \hat{\tilde{R}}_R$ is an isomorphism in $\text{Ho}(\text{dgalg}_k)$;

(ii) The following are isomorphisms in $D(R)$:

\[(5.40) \quad M \sim M^\vee,\]

\[(5.41) \quad (M^\otimes n)^\vee \sim (M^\vee)^L, \quad n \geq 2.\]

Here tensor products are over $R$ and $(-)^\vee = R \text{Hom}_R(-, R)$.

2) In particular, if $R = k[x]/(x^2)$ and $M = k$, then the map $\tilde{R} \to \hat{\tilde{R}}_R$ is not an isomorphism.

**Proof.** 1) Let $A$ be any DG algebra, and $N$ a DG $A$-module. Then we can treat $A$ as an $A_\infty$-algebra, and $N$ as a right $A_\infty$-module over $A$. Denote by $A\text{-mod}_{\infty}$ the DG category of right $A_\infty$-modules over $A$. We put

\[(5.42) \quad B_N := \text{End}_{A\text{-mod}_{\infty}}(N) = \prod_{n \geq 0} \text{Hom}_k(N \otimes A^\otimes n, N)[-n].\]

Further, we have obvious projection morphism of DG algebras $B_N \to \text{End}_k(N)$, hence $N$ is naturally a DG module over $B_N^{\text{op}}$. We put

\[(5.43) \quad \hat{A}_M := (\text{End}_{B_M^{\text{op}}\text{-mod}_{\infty}}(M))^{\text{op}}.\]
Then $\hat{A}_M$ is a DG model for derived double centralizer of $M$. We have a natural $A_\infty$-morphism $A \to \hat{A}_M$.

Now put $A := \tilde{R}$, and $N := R$. Then DG algebra $B_N$ as a complex can be decomposed into the product of complexes:

\begin{equation}
B_M = \prod_{n \geq 0} C_n,
\end{equation}

\begin{equation}
C_n := \prod_{l_1 > 0, l_2, \ldots, l_{n+1} \geq 0} \text{Hom}_k(R^\otimes l_1 \otimes M \otimes \cdots \otimes M \otimes R^\otimes l_{n+1}, R)[-l_1 - \cdots - l_{n+1} - n + 1].
\end{equation}

Further, the DG algebra $\hat{\tilde{R}}_R$ as a complex can be decomposed into the product of complexes:

\begin{equation}
\hat{\tilde{R}}_R = \prod_{n \geq 0} D_n, \quad D_n := \prod_{m_1 + \cdots + m_l = n, m_i \geq 0} \text{Hom}_k(C_{m_1} \otimes \cdots \otimes C_{m_l} \otimes R, R)[-l].
\end{equation}

It is straightforward to observe the following isomorphisms in $D(k)$:

\begin{equation}
D_0 \cong R, \quad D_1 \cong M^\vee \vee.
\end{equation}

Thus, (i) holds iff the map \ref{5.40} is an isomorphism, and all the complexes $D_n, \ n \geq 2$, are acyclic. Further, it is straightforward to show by induction on $m \geq 2$ that the following are equivalent:

(iii) the map \ref{5.40} and \ref{5.41} for $2 \leq n \leq m$ are isomorphisms;

(iv) the map \ref{5.40} is an isomorphism and the complexes $D_n, \ 2 \leq n \leq m$, are acyclic.

This proves part 1) of Proposition.

2) We claim that in the case $R = k[x]/(x^2)$ and $M = k$ the map

\begin{equation}
(M \otimes_R M)^\vee \vee \to (M^\vee \otimes_R M^\vee)^\vee
\end{equation}

is not an isomorphism. Indeed, we have isomorphisms in $D(R)$:

\begin{equation}
(M \otimes_R M)^\vee \vee \cong \bigoplus_{n \geq 0} k[n], \quad (M^\vee \otimes_R M^\vee)^\vee \cong \bigoplus_{n \geq 0} k[-n].
\end{equation}

Therefore, according to 1), the map $\tilde{R} \to \hat{\tilde{R}}_R$ is not an isomorphism. Proposition is proved. 

\[ \square \]

6. Beilinson-Parshin adeles and categorical formal completions.

Let $X$ be a separated Noetherian $k$-scheme of finite Crull dimension $d$.

We first recall reduced Beilinson-Parshin adeles of $X$ \cite{Be, Pa}. Denote by $P(X)$ the set of all schematic points of $X$. Put

\begin{equation}
S(X)_{p}^{\text{red}} := \{(\eta_0, \ldots, \eta_p) : \eta_i \in P(X), \eta_i \neq \eta_{i-1}, \eta_i \in m_i^{-1} \text{ for } 1 \leq i \leq p\}
\end{equation}
For $T \subset S(X)_p$, $p > 0$, and $\eta \in P(X)$, put
\[
T_\eta := \left\{(\eta_1, \ldots, \eta_p) \in S(X)_{p-1} : (\eta, \eta_1, \ldots, \eta_p) \in T \right\} \subset S(X)_{p-1}.
\]
Also, for $\eta \in P(X)$, denote by $j_\eta : \text{Spec} (O_\eta) \to X$ the natural map. Denote by $m_\eta \subset O_\eta$ the unique maximal ideal.

For each subset $T \subset S(X)_p$, $0 \leq p \leq d = \dim X$, we will define a functor
\[
A_T(X, -) : \text{QCoh}(X) \to k\text{-Mod},
\]
extact and commuting with infinite direct sums (hence commuting with small colimits).

Since each quasi-coherent sheaf is a union of its coherent subsheaves, it suffices to define the functor $A_T(X, -)$ for coherent sheaves.

We define these functors by induction on $p$. For $p = 0$, $T \subset S(X)_0 = P(X)$, and $\mathcal{F} \in \text{Coh}(X)$, we put
\[
A_T(X, \mathcal{F}) := \prod_{\eta \in T} \hat{\mathcal{F}}_\eta.
\]

With above said, this defines uniquely the functor (6.3) for $p = 0$. It is easy to check that it is exact and commutes with small colimits.

Now let $T \subset S(X)_p$, $p > 0$. Suppose that all the functors $A_{T_\eta}(X, -)$ are already defined. For $\mathcal{F} \in \text{Coh}(X)$, put
\[
A_T(X, \mathcal{F}) := \prod_{\eta \in P(X)} \lim_{\eta_\eta} A_{T_\eta}(j_{\eta\eta}^*(\mathcal{F}/m_\eta^n)).
\]

This defines uniquely the functor (6.3) for all $T \subset P(X)$, $p > 0$, and by induction we see that $A_T(X, -)$ is exact and commutes with small colimits.

For $d \geq k_0 > \cdots > k_p \geq 0$, we put
\[
S(X)_{(k_0, \ldots, k_p)} := \{ (\eta_0, \ldots, \eta_p) \in S(X)_p : \dim \eta_i = k_i \text{ for } 0 \leq i \leq p \}.
\]

Further, put
\[
A(X, -)_p := A_{S(X)_p}(X, -), \quad A(X, -)_{(k_0, \ldots, k_p)} := A_{S(X)_{(k_0, \ldots, k_p)}}(X, -).
\]

Clearly, we have
\[
A(X, -)_p \cong \prod_{d \geq k_0 > \cdots > k_p \geq 0} A(X, -)_{(k_0, \ldots, k_p)}.
\]

It is easy to see that for all $T \subset S(X)_p$, the $k$-module $A_T(X, \mathcal{O}_X)$ is naturally a commutative $k$-algebra. Further, for all quasi-coherent $\mathcal{F}$ the $k$-module $A_T(X, \mathcal{F})$ is naturally an $A_T(X, \mathcal{O}_X)$-module. For convenience, we put
\[
A(X)_p := A(X, \mathcal{O}_X)_p, \quad A(X)_{(k_0, \ldots, k_p)} := A(X, \mathcal{O}_X)_{(k_0, \ldots, k_p)}.
\]
To formulate main result of this section, we would like to use the following notation. If $F : T_1 \to T_2$ is a functor between compactly generated triangulated categories, and $S \subset T_1$ is an essentially small Karoubian complete triangulated subcategory (resp. localizing subcategory), then we put

\[(6.10) \quad \hat{T}_2S := \hat{T}_2(F(S)), \quad \text{ (rep. } T_2/S := T_2/\langle F(S) \rangle), \]

where $\langle F(S) \rangle$ is subcategory classically generated by $F(S)$ (resp. smallest localizing subcategory containing $F(S)$).

Denote by $D_{coh, \leq p}(X) \subset D_{coh, k}(X)$ the full subcategory consisting of complexes, for which the dimension of support of cohomology is not greater than $p$. Further, Denote by $D_{\leq p}(X) \subset D(X)$ the smallest localizing subcategory, which contains $D_{coh, \leq k}(X)$. It is clear that $D_{\leq k}(X)$ is compactly generated by $\text{Perf}_{\leq k}(X) = \text{Perf}(X) \cap D_{coh, \leq k}(X)$.

**Theorem 6.1.** Let $X$ be a separated Noetherian $k$-scheme of dimension $d$. Fix some sequence $d \geq k_0 > \cdots > k_p \geq 0$. If $p = 0$, then we have a commutative diagram,

\[(6.11) \quad \begin{array}{ccc} D(X) & \xrightarrow{id} & D(X) \\ \downarrow \quad & & \downarrow \\ A(X, -)_{(k_0)} & \cong & (D(X)/D_{\leq (k_0 - 1)}(X))_{D_{coh, \leq k_0}}. \end{array} \]

For $p > 0$, there is a natural commutative diagram

\[(6.12) \quad \begin{array}{ccc} D(A(X)_{(k_1, \ldots, k_p)}) & \xrightarrow{id} & D(A(X)_{(k_1, \ldots, k_p)}) \\ \downarrow & & \downarrow \\ D(A(X)_{(k_0, \ldots, k_p)}) & \cong & (D(A(X)_{(k_1, \ldots, k_p)})/D_{\leq (k_0 - 1)}(X))_{D_{coh, \leq k_0}}. \end{array} \]

**Proof.** First we prove (6.11).

**Lemma 6.2.** 1) The functor

\[(6.13) \quad D(X)_{\leq k_0} \to D(X)_{\leq k_0}, \quad \mathcal{F} \mapsto \bigoplus_{\eta \in S(X)_{(k_0)}} j_{\eta*}(\mathcal{F}_{\eta}), \]

is the projection onto the right orthogonal to $D_{\leq (k_0 - 1)}(X)$. In particular, it induces a fully faithful embedding of $D(X)_{\leq k_0}/D_{\leq (k_0 - 1)}(X)$ into $D(X)$.

2) The images of $\mathcal{F} \in D_{coh, \leq k_0}(X)$ in $D(X)/D_{\leq (k_0 - 1)}(X)$, where $\eta_0 \in S(X)_{(k_0)}$, generate the essential image of $D_{coh, \leq k_0}(X)$.

3) If $\eta_1, \eta_2 \in S(X)_{(k_0)}$, $\eta_1 \neq \eta_2$, and $\mathcal{F}_i \in D_{coh, \leq k_0}(X) \text{ for } i = 1, 2$, then we have

\[(6.14) \quad \text{Hom}_{D(X)/D_{\leq (k_0 - 1)}(X)}(\mathcal{F}_1, \mathcal{F}_2) = 0. \]
According to (6.16), we have that \( j_{\eta*}(\mathcal{F}_{\eta}) \) is right orthogonal to \( D(X)_{(k_0-1)} \). It remains to note that the cone of the natural morphism (6.15) lies in \( D_{\leq(k_0-1)}(X) \).

2) We have that \( D_{coh,\leq k_0}(X) \) is itself generated by \( D_{coh}^{b\leq k_0}(X), \eta_0 \in S(X)_{(k_0)} \). This implies 2).

3) follow from 1) easily.

Let \( E \in \text{Perf}(X) \) be a generator. Put \( \mathcal{A} := \text{RHom}_{D(X)/D_{\leq(k_0-1)}(X)}(E, E) \). Let \( \mathcal{T} \subset D(\mathcal{A}) \) (resp. \( \mathcal{T}_{\eta} \subset D(\mathcal{A}) ; \eta \in S(X)_{(k_0)} \) ) be the subcategory classically generated by the image of \( D_{coh,\leq k_0}^{b}(X) \) (resp. \( D_{coh,\eta}^{b}(X) \)). Then, by Lemma 6.2 and Proposition 4.2 we have that (6.16)

\[
\hat{\mathcal{A}}_{\mathcal{T}} \cong \prod_{\eta \in S(X)_{(k_0)}} \hat{\mathcal{A}}_{\mathcal{T}_{\eta}}.
\]

Fix some \( \eta \in S(X)_{(k_0)} \) and put \( Y := \overline{\eta} \). We claim that \( \hat{\mathcal{A}}_{\mathcal{T}_{\eta}} \cong \text{REnd}_{\hat{\mathcal{O}}_{\eta}}(\hat{E}_{\eta}) \). This can be shown as follows. Denote by \( Y_{l} \) the infinitesimal neighborhoods of \( Y \), \( l_{t} : Y_{l} \to X \) the inclusions, and \( j_{\eta,l} : \text{Spec} (\mathcal{O}_{\eta}/m_{\eta}^{l}) \to X \) natural morphisms. Put \( \mathcal{A}_{l} := \text{REnd}(\text{L}j_{\eta,l}^{*}E) \). Then \( \{\mathcal{A}_{l}\}_{l \in \mathbb{N}} \in \text{dgalg}_{k}^{\text{op}} \), and we have a compatible system of morphisms \( \mathcal{A} \to \mathcal{A}_{l} \). Further, there are isomorphisms (6.17) \( \mathcal{A}_{l} \cong \text{RHom}_{D(X)}(E, j_{\eta,l*}\text{L}j_{\eta,l}^{*}E) \cong \text{RHom}_{D(X)/D_{\leq(k_0-1)}(X)}(E, t_{*}j_{*}^{\text{L}}E) \) in \( D(\mathcal{A}) \).

For each \( \mathcal{F} \in D_{coh,Y}(X) \), we have the following chain of isomorphisms:

(6.18) \( \text{holim}_{l} \text{RHom}_{\mathcal{A}_{l}}(\mathcal{A}_{l}, \text{RHom}_{D(X)/D_{\leq(k_0-1)}(X)}(E, \mathcal{F}) \cong \text{holim}_{l} \text{RHom}_{D(X)/D_{\leq(k_0-1)}(X)}(t_{*}Lj_{*}^{*}E, \mathcal{F}) \cong \text{holim}_{l} \text{RHom}_{D(X)}(t_{*}Lj_{*}^{*}E, j_{*}(\mathcal{F}_{\eta}) \cong \text{RHom}_{D(X)}(E, j_{*}(\mathcal{F}_{\eta}) \cong \text{RHom}_{D(X)/D_{\leq(k_0-1)}(X)}(E, \mathcal{F}).
\)

Hence, by Lemma 5.2 we have (6.19) \( \hat{\mathcal{A}}_{\mathcal{T}_{\eta}} \cong \text{holim}_{l} \text{REnd}(\text{L}j_{*}^{*}E) \cong \text{REnd}_{\hat{\mathcal{O}}_{\eta}}(\hat{E}_{\eta}). \)

According to (6.16), we have that (6.20) \( \hat{\mathcal{A}}_{\mathcal{T}} \cong \prod_{\eta \in S(X)_{(k_0)}} \text{REnd}_{\hat{\mathcal{O}}_{\eta}}(\hat{E}_{\eta}). \)
Further, since $\hat{E}_\eta, \hat{\mathcal{O}}_\eta \in \text{Perf}(\hat{\mathcal{O}}_\eta)$ generate each other in uniformly bounded number of steps, we have Morita equivalence

$$D\left( \prod_{\eta \in S(X)(k_0)} \mathbb{R} \text{End}_{\hat{\mathcal{O}}_\eta}(\hat{E}_\eta) \right) \cong D\left( \prod_{\eta \in S(X)(k_0)} \hat{\mathcal{O}}_\eta \right).$$

Further, by definition, $\prod_{\eta \in S(X)(k_0)} \hat{O}_\eta = \mathbb{A}(X)(k_0)$. Hence, we have equivalences

$$\left( D(X)/D_{\leq (k_0-1)}(X) \right)_{D_{\text{coh}, \leq k_0}} \cong D(\hat{\mathcal{A}}_T) \cong D(\mathbb{A}(X)(k_0)),$$

and commutativity of (6.11) is straightforward.

Now we prove (6.12). We have the morphisms of algebras

$$g_\eta : \mathbb{A}(X)(k_1, \ldots, k_p) \to \mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p), \quad \eta \in S(X)(k_0).$$

Note that we have the following isomorphisms of functors

$$\mathbb{A}(X, j_{\eta*}(-))(k_1, \ldots, k_p) \cong g_{\eta*}(\mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p) \otimes_{\mathcal{O}_\eta} -),$$

$$Lg^*_{\eta} \mathbb{A}(X, -)(k_1, \ldots, k_p) \cong \mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p) \otimes_{\mathcal{O}_\eta} Lj^*_\eta(-).$$

**Lemma 6.3.** 1) If $\mathcal{F} \in D_{\leq (k_0-1)}(X)$, $\mathcal{G} \in D_{\leq k_0}(X)$, and $\eta \in S(X)(k_0)$, then

$$\text{Hom}_{\mathbb{A}(X)(k_1, \ldots, k_p)}(\mathbb{A}(X, \mathcal{F})(k_1, \ldots, k_p), \mathbb{A}(X, j_{\eta*} \mathcal{G})(k_1, \ldots, k_p)) = 0.$$

2) The objects $\mathbb{A}(X, \mathcal{F})(k_1, \ldots, k_p)$, where $\mathcal{F} \in D_{\text{coh}, \leq k_0}(X)$, $\eta_0 \in S(X)(k_0)$, generate the essential image of $D_{\text{coh}, \leq k_0}(X)$ in $D(\mathbb{A}(X)(k_1, \ldots, k_p))/D_{\leq (k_0-1)}(X)$.

3) If $\eta_1, \eta_2 \in S(X)(k_0)$, $\eta_1 \neq \eta_2$, and $\mathcal{F}_i \in D_{\text{coh}, \leq k_0}(X)$ for $i = 1, 2$, then we have

$$\text{Hom}_{D(\mathbb{A}(X)(k_1, \ldots, k_p))/D_{\leq (k_0-1)}(X)}(\mathbb{A}(X, \mathcal{F}_1)(k_1, \ldots, k_p), \mathbb{A}(X, \mathcal{F}_2)(k_1, \ldots, k_p)) = 0.$$

**Proof.** 1) We have the following chain of isomorphisms

$$\text{Hom}_{\mathbb{A}(X)(k_1, \ldots, k_p)}(\mathbb{A}(X, \mathcal{F})(k_1, \ldots, k_p), \mathbb{A}(X, j_{\eta*} \mathcal{G})(k_1, \ldots, k_p)) \cong$$

$$\text{Hom}_{\mathbb{A}(X)(k_1, \ldots, k_p)}(\mathbb{A}(X, \mathcal{F})(k_1, \ldots, k_p), g_{\eta*}(\mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p) \otimes_{\mathcal{O}_\eta} \mathcal{G}_\eta)) \cong$$

$$\text{Hom}_{\mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p)}(Lg_{\eta*}^* \mathbb{A}(X, \mathcal{F})(k_1, \ldots, k_p), \mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p) \otimes_{\mathcal{O}_\eta} \mathcal{G}_\eta) \cong$$

$$\text{Hom}_{\mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p)}(\mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p) \otimes_{\mathcal{O}_\eta} Lj^*_\eta(\mathcal{F}), \mathbb{A}(X, j_{\eta*} \mathcal{O}_\eta)(k_1, \ldots, k_p) \otimes_{\mathcal{O}_\eta} \mathcal{G}_\eta) = 0,$$

since $Lj^*_\eta(\mathcal{F}) = 0$.

2) This is evident, as in Lemma 6.2 2).
3) Using 1) and the chain (6.28), we see that

\[(6.29) \quad \text{Hom}_{D(A)}(A(X_{k_1, \ldots, k_p}), A(X, F_1)_{(k_1, \ldots, k_p)}) \cong \text{Hom}_{A}(A(X_{k_1, \ldots, k_p}), A(X, F_1)_{(k_1, \ldots, k_p)}) \cong \text{Hom}_{A}(A(X, j_{n2}\ast O_{\eta 2})_{(k_1, \ldots, k_p)}, L_{j^*_{\eta 2}}(F_1), A(X, j_{n2}\ast O_{\eta 2})_{(k_1, \ldots, k_p)} \otimes_{O_{\eta 2}} L_{j^*_{\eta 2}}(F_2)) = 0,\]

since \( L_{j^*_{\eta 2}}(F_1) = 0 \).

\[\square\]

For convenience put \( B := \mathbf{R} \text{End}_{D(A)}(A(X_{k_1, \ldots, k_p}), A(X, F_1)_{(k_1, \ldots, k_p)}) \). Let \( T \subset D(B) \) (resp. \( T_\eta \subset D(B_\eta) \) be the subcategory classically generated by the image of \( D^b_{\text{coh}, \leq k_0}(X) \) (resp. \( D^b_{\text{coh}, \eta} \)). Then, by Lemma 6.3 and Proposition 4.2 we have that

\[(6.30) \quad \mathcal{B}_T \cong \prod_{\eta \in S(X)(k_0)} \mathcal{B}_{T_\eta}.\]

Fix some \( \eta \in S(X)(k_0) \). We claim that \( \mathcal{B}_{T_\eta} \cong \text{lim}_n A(X, j_{n}\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)} \). This can be shown as follows. Put \( B_n := A(X, j_{n}\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)} \). Then \( \{B_n\}_{n \in \mathbb{N}} \in \text{dgalg}^{\text{op}}_{k} \), and we have a compatible system of morphisms \( B \to B_n \). Denote by \( g_{n, \eta} : A(X, j_{n}\ast (O_\eta))_{(k_1, \ldots, k_p)} \to B_n \) the natural map. Put \( Y := T_\eta \). Denote by \( Y^l \) the infinitesimal neighborhoods of \( Y \), and \( t_l : Y^l \to Y \) the inclusions. We have natural isomorphisms

\[(6.31) \quad B_n \cong \mathbf{R} \text{Hom}_{A}(A(X), A(X, j_{n}\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)}), g_{n, \eta, n}\ast (A(X, j_n\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)}) \cong \mathbf{R} \text{Hom}_{D(A)}(A(X), A(X, j_{n}\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)}) \cong \mathbf{R} \text{Hom}_{D(A)}(A(X, j_{n}\ast (O_\eta))_{(k_1, \ldots, k_p)}, A(X, j_{n}\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)}) \cong \mathbf{R} \text{Hom}_{D(A)}(A(X), A(X, j_{n}\ast (O_\eta/m^*_n))_{(k_1, \ldots, k_p)}).\]

Further, denote by

\[(6.32) \quad \Phi : D(A(X))_{(k_1, \ldots, k_p)} \to D(X) \]

the functor which is right adjoint to \( A(X, -)_{(k_1, \ldots, k_p)} \).

Lemma 6.4. Let \( F \in D^b_{\text{coh}, Y}(X) \). Then

\[(6.33) \quad \Phi(A(X, j_\eta F))_{(k_1, \ldots, k_p)} \in D_Y(X).\]

Proof. We may assume that \( F = t_1 F' \) for some object \( F' \in D^b_{\text{coh}}(Y) \). Denote by \( \pi : A(X)_{(k_1, \ldots, k_p)} \to A(Y)_{(k_1, \ldots, k_p)} \) the natural projection, and let

\[(6.34) \quad \Psi : D(A(Y))_{(k_1, \ldots, k_p)} \to D(Y) \]
be right adjoint to $A(Y, -)(k_1, \ldots, k_p)$. Note the isomorphism of functors
\begin{equation}
L\pi^* A(X, -)(k_1, \ldots, k_p) \cong A(Y, L\pi^*(-))(k_1, \ldots, k_p).
\end{equation}
We have the following chain of isomorphisms:
\begin{equation}
\Phi(A(X, j_{\eta*}(F_{\eta}))(k_1, \ldots, k_p)) \cong \Phi(\pi_*(A(Y, \mathcal{F'} \otimes k(\eta))(k_1, \ldots, k_p))) \cong \\
\ell_1*\Phi(A(Y, \mathcal{F'} \otimes k(\eta))(k_1, \ldots, k_p)) \in D_Y(X).
\end{equation}
This proves Lemma. \hfill \square

Now, for each object $\mathcal{F} \in D^{b}_{coh,Y}(X)$ we have
\begin{equation}
\hocolim_n R \text{Hom}_{B_{n}}(\mathcal{B},)
\end{equation}
\begin{equation}
R \text{Hom}_{D(A(X)(k_1, \ldots, k_p)/D_{\leq(k_0-1)}(X)}(A(X)(k_1, \ldots, k_p), A(X, \mathcal{F})(k_1, \ldots, k_p)) \cong \\
\hocolim_n R \text{Hom}_{D(A(X)(k_1, \ldots, k_p)/D_{\leq(k_0-1)}(X)}(A(X, \pi_n*O_{Y_n})(k_1, \ldots, k_p), A(X, \mathcal{F})(k_1, \ldots, k_p)) \cong \\
\hocolim_n R \text{Hom}_{D(A(X)(k_1, \ldots, k_p))}(A(X, \pi_n*O_{Y_n})(k_1, \ldots, k_p), A(X, j_{\eta*}(F_{\eta}))(k_1, \ldots, k_p)) \cong \\
\hocolim_n R \text{Hom}_{D(X)(\pi_n*O_{Y_n}, \Phi(A(X, j_{\eta*}(F_{\eta}))(k_1, \ldots, k_p)))}
\end{equation}
By Lemma 5.2 the last object of $D(k)$ is isomorphic to
\begin{equation}
R \text{Hom}_{D(X)}(O_X, \Phi(A(X, j_{\eta*}(F_{\eta}))(k_1, \ldots, k_p))) \cong \\
R \text{Hom}_{A(X)(k_1, \ldots, k_p)}(A(X)(k_1, \ldots, k_p), A(X, j_{\eta*}(F_{\eta}))(k_1, \ldots, k_p)) \cong \\
R \text{Hom}_{D(A(X)(k_1, \ldots, k_p)/D_{\leq(k_0-1)}(X)}(A(X)(k_1, \ldots, k_p), A(X, \mathcal{F})(k_1, \ldots, k_p)).
\end{equation}
Therefore, by Lemma 5.2 we have that
\begin{equation}
\mathcal{B}_{\mathcal{T}} \cong \hocolim_n \mathcal{B}_{n} \cong \lim_n A(X, j_{\eta*}(O_{\eta}/m_{\eta}^n))(k_1, \ldots, k_p).
\end{equation}
According to (6.30), we have
\begin{equation}
\mathcal{B}_{\mathcal{T}} \cong \prod_{\eta \in S(X)(k_{0})} \lim_n A(X, j_{\eta*}(O_{\eta}/m_{\eta}^n))(k_1, \ldots, k_p) \cong A(X)(k_0, \ldots, k_p).
\end{equation}
Hence, we have equivalences
\begin{equation}
(D(A(X)(k_1, \ldots, k_p))/D_{\leq(k_0-1)}(X))_{D^{b}_{coh, \leq k_0}} \cong D(\mathcal{B}_{\mathcal{T}}) \cong D(A(X)(k_0, \ldots, k_p)),
\end{equation}
and commutativity of (6.12) is straightforward to check. \hfill \square
REFERENCES

[AJPV] Alonso Tarrío L., Jeremías López A., Pérez Rodríguez M., Vale Gonsalves M.J., On the existence of a compact generator in the derived category of a Noetherian formal scheme, Applied Categorical Structures (4 June 2009).

[Be] A.A. Beilinson, Residues and adeles, Funct. Anal. And Appl., 14 (1980), 3435.

[BK] A.I. Bondal, M.M. Kapranov, Enhanced triangulated categories, Mat. Sb., 181:5 (1990), 669683.

[BvdB] A. Bondal, M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J., 3:1 (2003), 136.

[Dr] V. Drinfeld, DG quotients of DG categories, J. Algebra 272 (2004), 643691.

[Gr] A. Grothendieck, Local cohomology, Springer-Verlag, Berlin, 1967, Course notes taken by R. Hartshorne, Harvard University, Fall, 1961. Lecture Notes in Mathematics, No. 41.

[Ho] M. Hopkins, Global methods in homotopy theory, in "Homotopy theory (Durham, 1985)" , 73-96, London Math. Soc. Lecture Note Ser., 117, Cambridge Univ. Press, 1987.

[Ke1] B. Keller, Deriving DG categories, Ann. Sci. E'cole Norm. Sup. (4) 27 (1994), no. 1, 63102.

[Ke2] B. Keller, On the cyclic homology category of exact categories, J. Pure Appl. Algebra, vol. 136 (1999), no. 1, 156.

[Ko1] M. Kontsevich, Symplectic geometry of homological algebra, preprint MPIM2009-40a.

[Ko2] M. Kontsevich, Discussion session at the Workshop on Homological Mirror Symmetry and Related Topics, Miami, January 18-23, 2010.

[Nee] A. Neeman, The chromatic tower for $D(R)$, in Topology 31 (1992), 519-532.

[P] A.N. Parshin, On the arithmetic of two-dimensional schemes I. Distributions and residues, Izv. Akad. Nauk SSSR, 40:4 (1976), 736773.

[T] G. Tabuada, Une structure de cate'gorie de mode'les de Quillen sur la cate'gorie des dg-cate'gories, C. R. Math. Acad. Sci. Paris 340 (2005), 15–19.

STEKLOV MATHEMATICAL INSTITUTE OF RAS, GUBKIN STR. 8, GSP-1, MOSCOW 119991, RUSSIA

INDEPENDENT UNIVERSITY OF MOSCOW, MOSCOW, RUSSIA

E-mail address: efimov@mccme.ru