Abstract

Generalizing results of Temperley [11], Brooks, Smith, Stone and Tutte [1] and others [10, 7] we describe a natural equivalence between three planar objects: weighted bipartite planar graphs; planar Markov chains; and tilings with convex polygons. This equivalence provides a measure-preserving bijection between dimer coverings of a weighted bipartite planar graph and spanning trees of the corresponding Markov chain. The tilings correspond to harmonic functions on the Markov chain and to “discrete analytic functions” on the bipartite graph.

The equivalence is extended to infinite periodic graphs, and we classify the resulting “almost periodic” tilings and harmonic functions.

1 Introduction

In [11], Temperley gave a bijection between the set of spanning trees of an $n \times n$ grid and the set of perfect matchings (dimer coverings) of a $(2n - 1) \times (2n - 1)$ grid with a corner removed. This bijection was generalized in [10] to a weight-preserving bijection (the KPW construction) from the set of in-directed spanning trees (also known as arborescences) on an arbitrary weighted, directed planar graph $G_T$ to the set of perfect matchings on a related graph $G_D$. The construction is useful in statistical mechanics because certain types of events in the spanning tree model can be easily computed using dimer technology, for example winding numbers of branches and local statistics. For dimer models arising from a spanning tree model, moreover, the spanning tree formulation provides other useful information. In particular Wilson’s algorithm [12] for generating spanning trees can be used to rapidly simulate dimer configurations. Moreover, the spanning tree formulation identifies natural boundary conditions (“Temperleyan” boundary conditions) for the dimer model which allows asymptotic computation of many properties, in particular conformal invariance properties of dimers [6]. However in the paper [10] it was not known if every dimer model on a bipartite planar graph corresponded to a spanning tree model on a related graph.

A seemingly unrelated construction is the construction of a “Smith diagram” from a planar resistor network [11]. This is a tiling of a plane region with squares of arbitrary
sizes, which is associated in a bijective way to a critical-point-free harmonic function on the network with unit resistances (there is a square in the tiling for each edge in the graph, whose size is proportional to the current flow through the edge). This construction was generalized in [7] to planar Markov chains (graphs with transition probabilities), where a harmonic function gives a tiling with trapezoids. It was not known at the time what if any graphical correspondence was natural for general polygonal tilings.

In the current paper we extend the above equivalences and describe a correspondence between these three types of objects: weighted bipartite planar graphs, planar Markov chains, and tilings with general convex polygons.

In particular from a weighted bipartite planar graph $\mathcal{G}_D$ we can construct a tiling $T$ of a plane region with convex polygonal tiles, and a planar Markov chain $\mathcal{G}_T$, in an essentially bijective way (that is, up to natural equivalences). There is a tile in $T$ for each “white” vertex of $\mathcal{G}_D$, whose shape is determined by a discrete analytic function (see definitions below). The graph $\mathcal{G}_T$ is a graph on the 1-skeleton of the tiles, with transitions determined by their Euclidean geometry. The tilings are therefore representations of discrete analytic functions on the bipartite planar graph $\mathcal{G}_D$, which correspond to harmonic functions on the Markov chain $\mathcal{G}_T$.

An important application of this construction is that it provides a converse to the Temperley-KPW construction. That is, starting with the finite weighted bipartite planar graph $\mathcal{G}_D$, one constructs a Markov chain $\mathcal{G}_T$ and a measure-preserving bijection from the dimer model on $\mathcal{G}_D$ to the spanning tree process on $\mathcal{G}_T$. This dimer/spanning tree correspondence has a number of important consequences. Firstly, it was used in [9] in a fundamental way to classify Gibbs measures on dimer models on infinite periodic planar graphs. Secondly, since spanning trees can be sampled efficiently [12], the construction provides a way to sample efficiently from bipartite planar dimer models. Previously the only (provably efficient) way to sample general planar bipartite dimer models was to do exact computations of joint edge probabilities. A third application [8] is that it allows one to compute the asymptotics of dimer correlations and height fluctuations in terms of the Green’s function on $\mathcal{G}_T$.

In Section 5 we discuss how the construction extends in the case of infinite periodic graphs. This is motivated by the study of the dimer model on periodic graphs, see [2, 9]. Given any periodic planar bipartite weighted graph $\mathcal{G}_D$, we produce an essentially unique “almost periodic” planar Markov chain $\mathcal{G}_T$, which extends the dimer/spanning tree correspondence. This unicity is an important element in the classification theorem of ergodic Gibbs measures on dimer coverings of $\mathcal{G}_D$ described in [9].

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2 Definitions

2.1 Dimers and measures

Let $\mathcal{G}_D = (V, E)$ be a finite bipartite planar graph. Bipartite means that the vertices $V$ can be 2-colored, that is, colored black and white so that black vertices are only adjacent to white vertices and vice versa. Let $\nu: E \to (0, \infty)$ be a weight function on the edges. A perfect matching, or dimer configuration $M \subset E$ is a set of edges with the property
that each vertex is contained in exactly one edge in $M$. The weight of a matching $M$ is $
u(M) = \prod_{e \in M} \nu(e)$. Let \( \mathcal{M}(G_D) \) denote the set of perfect matchings of $G_D$. Let \( \mu \) be the probability measure on \( \mathcal{M}(G_D) \) giving a matching a probability proportional to its weight:

\[
\mu(M) = \frac{1}{Z} \nu(M) \quad \text{where} \quad Z = \sum_{M \in \mathcal{M}(G_D)} \nu(M).
\]

### 2.2 Kasteleyn matrices

If $G_D$ has $n$ black and $n$ white vertices, a **Kasteleyn matrix** (see [4]) for $G_D$ is a real $n \times n$ matrix $K = (K_{i,j})$ whose rows index the black vertices and columns index the white vertices of $G_D$, defined as follows. The entry $K_{i,j}$ is zero if there is no edge from $b_i$ to $w_j$, and if there is an edge of weight $\nu(b_iw_j)$ then $K_{i,j} = \pm \nu(b_iw_j)$, where the signs are chosen so that the product of signs of edges around every interior face of $K$ is $(-1)^{d/2+1}$, where $d$ is the degree of the face. This property of signs is not changed if we multiply all elements in a particular column or row of $K$ by $-1$ (because each vertex of $G_D$ has an even number—zero or two—of edges on each face of $G_D$). Moreover, such a choice of signs always exists, and by Kasteleyn’s theorem, the determinant of $K$ is (up to sign) the sum of the weights of the matchings of $G_D$ ([4]). By a **discrete analytic function** we mean a function $f$ on black vertices (resp. white vertices) which satisfies $fK = 0$ (resp. $Kf = 0$). This generalizes the definition of discrete analytic function on $\mathbb{Z}^2$ defined in [3, 6]. These functions play a role implicitly in sections [4] and [7].

### 2.3 Gauge transformations

If we multiply the weights of all the edges in $G_D$ having a fixed vertex by a constant, the measure $\mu$ does not change, since exactly one of these weights is used in every configuration. More generally, two weight functions $\nu_1, \nu_2$ are said to be **gauge equivalent** if $\nu_1/\nu_2$ is a product of such operations, that is, if there are functions $f_1$ on white vertices and $f_2$ on black vertices so that for each edge $wb$, $\nu_1(wb)/\nu_2(wb) = f_1(w)f_2(b)$. Gauge equivalent weights define the same measure $\mu$.

Multiplying the $i$th row (resp., column) of a Kasteleyn matrix $K$ by a positive, non-zero constant $c$ is equivalent to multiplying by $c$ the weights of all of the edges of $G_D$ incident to $b_i$ (resp., $w_i$). In other words any matrix $\tilde{K}$ obtained from $K$ by multiplying the rows and columns of $K$ by non-zero constants will be a Kasteleyn matrix for a graph which is gauge equivalent to $G_D$.

### 3 T-graphs and corresponding dimer/spanning-forest models

In this section, we define a family of planar graphs called T-graphs and describe a weight-preserving correspondence between the spanning trees on a T-graph and dimer configurations on a bipartite graph derived from it. This is closely related to the result of [10], who also give a relation between spanning trees on a general planar graph and dimers on a derived graph.
In the present context however our derivation can be reversed: we will see in Section 4 that for every bipartite planar graph, which is non-degenerate in the sense that it contains no edges which fail to be used in any perfect matching of the graph (for the purposes of the dimer model, it makes sense to delete these edges), endowed with a generic choice of weights, there is a gauge-equivalent graph which can be derived from a T-graph in this way. By taking limits, the correspondence generalizes to the case when the weights are not assumed to be generic.

3.1 Complete edges that form T-graphs

The definition of T-graph on a torus—which we use in section 5—is quite simple. A disjoint collection \( L = \{L_1, L_2, \ldots, L_n\} \) of open line segments in the torus \( \mathbb{R}^2/\mathbb{Z}^2 \) forms a T-graph in the torus if \( \bigcup_{i=1}^{n} L_i \) is closed. The term “T-graph” refers to the fact each endpoint of a given \( L_i \) necessarily lies on the interior of some \( L_j \) with \( j \neq i \). In other words, each \( L_i \) “tees into” an \( L_j \) at each of its two endpoints.

We say a disjoint collection \( L_1, L_2, \ldots, L_n \) of open line segments in \( \mathbb{R}^2 \) forms a T-graph if \( \bigcup_{i=1}^{n} L_i \) is connected and contains all of its limit points except for some set \( R = \{r_1, \ldots, r_m\} \), where each \( r_i \) lies on the boundary of the infinite component of \( \mathbb{R}^2 \) minus the closure \( \overline{L} \) of \( \bigcup_{i=1}^{n} L_i \). Elements in \( R \) are called root vertices. For example, a single open line segment forms a T-graph with root vertices given by the two endpoints. A pair of open line segments—one of which tees into the other to make the letter “T”—forms a T-graph with three root vertices. The three open edges of a triangle also form a T-graph with three root vertices. A partitioning of a convex polygon \( P \) into convex polygonal tiles using a finite number of line segments will form a T-graph with root vertices given by the vertices of \( P \). (See Figure 1.)

Note that each endpoint of a given \( L_i \) is either a root vertex or an interior point of some \( L_j \). To distinguish the \( L_i \) from subsegments of the \( L_i \) (which we discuss later) we refer to the \( L_i \) as complete edges.

In subsequent subsections, we will use \( L \) to define a weighted, directed graph \( G_T(L) \) and a weighted, bipartite graph \( G_D(L) \). The ultimate goal of this section will be to derive a
weight-preserving bijection between directed spanning forests on \( G_T(L) \) (with specified roots) and perfect matchings of \( G_D(L) \). When the choice of \( L \) is clear from the context, we write \( G_T = G_T(L) \) and \( G_D = G_D(L) \).

### 3.2 T-graphs and their duals

The set \( V_T(L) \) of vertices of \( G_T \) (the graph we will call the tree-graph of \( L \)) is the set of points in \( \mathbb{R}^2 \) which are endpoints of at least one of the \( L_i \). A vertex \( v \) which is in the interior of a complete edge \( L_i \) (called an interior vertex) has exactly two edges in \( G_T \) directed outwards from it: these edges point towards the two immediate neighbors, \( v_1 \) and \( v_2 \), along \( L_i \) (one on each side of \( v \)). The weights on the edge from \( v \) to these two \( v_i \) are chosen in such a way that the two weights add up to one and are inversely proportional to the Euclidean distances \( |v - v_i| \). These weights correspond to the transition probabilities of a Markov chain on \( V_T(L) \). The root vertices are sinks of \( G_T \) (they have no outgoing edges in \( G_T \)) and are fixed points of the Markov chain. Note that (by our choice of transition probabilities) the expected change in Euclidean position during a step of the Markov chain is always zero; thus, a random walk on \( G_T \) — viewed as a Markov chain on positions in \( \mathbb{R}^2 \) — is a martingale. In other words, the coordinate functions on the vertices of \( G_T \) are harmonic functions on \( G_T \) away from the root vertices.

See Figure 2 for an example of a T-graph with three roots. Note that, by convention, when we have transitions both from \( i \) to \( j \) and from \( j \) to \( i \), rather than drawing two directed edges in the graph \( G_T \) we draw a single edge with two transition probabilities, one from each end.

![Figure 2: The directed Tree-graph \( G_T \). The root vertices are the corners of the triangle.](image)

We define \( G_T' = G_T'(L) \), an undirected dual graph of \( G_T \), as follows. Let \( C \) be an arbitrary simple closed curve that encircles the \( \bigcup_{i=1}^n L_i \) and contains each of the root vertices \( r_1, \ldots, r_m \) in clockwise order. The vertices of \( G_T' \) are the bounded faces of \( G_T \cup C \) (bounded connected components of \( \mathbb{R}^2 \backslash (\bigcup_{i=1}^n L_i \cup C) \)). Faces of \( G_T \cup C \) adjacent to \( C \) are called outer faces of \( G_T \): they correspond to outer vertices of \( G_T' \). Two vertices of \( G_T' \) are connected by an edge
Lemma 3.1 If $L_1, L_2, \ldots, L_n$ form a $T$-graph, then $G_T$ has exactly $n + 1$ faces (including outer faces). Hence $G'_T$ has $n + 1$ vertices.

Proof: This follows from Euler’s formula. The line segments $L_i$ decompose the interior of $C$ into some number $n_2$ of open faces (open 2-cells), $n_1 = n$ open complete edges, and $n_0 = 0$ vertices. Since $n_2 - n_1 + n_0 = 1$, the Euler characteristic of the disc, the result follows.

3.3 Spanning trees

A spanning tree of a graph $G$ is a subset of edges which is connected, contains no cycle, and passes through every vertex. If the edges of $G$ are directed, a directed spanning tree, or arborescence, is a spanning tree in which every vertex but one (called the root vertex) has a unique outgoing edge. Given a subset of vertices of $G$ called root vertices, a directed spanning forest is a set of edges with no cycles, passing through all vertices, each non-root vertex having a unique outgoing edge, and each component of which is connected to a unique root vertex.

We will employ the following correspondence between (non-directed) spanning trees in $G'_T$ and their (non-directed) dual spanning forests in $G_T$. Using the correspondence between edges of $G'_T$ and edges of $G_T$, we can think of edge subsets of both $G_T$ and $G'_T$ as subsets of the set of all edges of $G_T$. Using this interpretation, we state the following lemma (which is illustrated in Figure 3):

Lemma 3.2 The complement of a spanning tree $T$ of $G'_T$ is a spanning forest $F$ of $G_T$, with roots at the root vertices. Similarly, the complement of a spanning forest $F$ of $G_T$, with roots at the root vertices, is a spanning tree $T$ of $G'_T$.

Proof: We sketch the standard tree dualization argument. If $F$ is a spanning forest of $G_T$, with roots at root vertices, its complement $T$ cannot contain any cycles in $G'_T$ (since such a cycle would separate at least one interior vertex of $G_T$ from the root vertices), and it must be connected (since otherwise, the set of edges separating two components would either form a cycle in $G_T$ or a path connecting two root vertices in $G_T$); hence it is a spanning tree. Similarly, if $T$ is a spanning tree of $G'_T$, its complement $F$ cannot contain cycles of $G_T$ (since such a cycle would separate at least one inner face of $G_T$ from the outer faces) and each connected component of $F$ contains at least one root vertex (since otherwise the set of edges separating that component of $F$ from its complement would form a cycle in $G_T$).

3.4 Dimer graphs from $T$-graphs

Now we will define the weighted, bipartite (non-directed) graph $G_D = G_D(L)$. First, we define a slightly larger $G'_D = G'_D(L)$, whose black vertices are the $n$ complete edges $L_i$ and whose white vertices are the $n + 1$ faces of $G_T$ (including outer faces).
A white vertex $w$ of $\mathcal{G}_D$ is adjacent to a black vertex $b$ of $\mathcal{G}_D$ if the face $F$ corresponding to $w$ contains a portion of the $L_i$ corresponding to $b$ as its boundary. The weight $\nu((w, b))$ is then given by the Euclidean length of the portion of the line segment. The graph thus defined is planar. To see this, note that it can be drawn on top of the tiling $\mathcal{T}$ as follows: put a white vertex in the interior of each face, and a black vertex in the center of each complete edge. When $w$ and $b$ are connected, draw a line from $w$ inside the corresponding face towards the complete edge corresponding to $b$, and then along this complete edge, staying just to one side, until the center is reached. It is not hard to see that this can be done in such a way that the paths do not intersect.

The graph $\mathcal{G}_D$ is formed from $\mathcal{G}_D'$ by (arbitrarily) picking one of the outer white vertices of $\mathcal{G}_D'$ and removing it; we will refer to the removed vertex as the dual root of $\mathcal{G}_D$. Now, $\mathcal{G}_D$ is a weighted bipartite graph with $n$ white and $n$ black vertices. To every edge $e = (w, b)$ in a perfect matching of $\mathcal{G}_D$ (where $w$ corresponds to a face $F$ and $b$ to a complete edge $L_i$), we denote by $S_e$ the segment of the $L_i$ which borders $F$. Because of our choice of weights, $\mu(M)$ (where $\mu$ is the probability measure on perfect matchings defined in the introduction) is proportional to $\prod_{e \in M} |S_e|$ where $|S_e|$ is the Euclidean length of $S_e$. Now, the edge segment $S_e$ may have vertices of $\mathcal{G}_T$ in its interior; these vertices divide $S_e$ into subsegments, each of which has vertices of $\mathcal{G}_T$ as its endpoints and hence corresponds to an edge of $\mathcal{G}_T$. A marked matching of $\mathcal{G}_D$ is a matching $M$ of $\mathcal{G}_D$ together with a specified subsegment $S_e'$ of $S_e$ (which, again, we may interpret as an edge in $\mathcal{G}_T$) for each $e \in M$. We extend $\mu$ to give a measure on random marked matchings as follows: to sample a random marked matching, first choose a random matching. Then for each edge $e$, choose an $S_e'$ from among the subsegments of $S_e$, where probability of each subsegment is proportional to its length. If $M'$ is a marked matching, then $\mu(M')$ is proportional to $\prod_{e \in M} |S_e'|$.

| Graph | Vertex Set |
|-------|------------|
| $\mathcal{G}_T = \mathcal{G}_T(L)$ | Points that are endpoints of some $L_i$. |
| $\mathcal{G}_T'$ | Faces and outer faces of $\mathcal{G}_T$ (i.e., bounded components of $\mathbb{R}^2 \setminus (L \cup C)$). |
| $\mathcal{G}_D$ | Faces and outer faces of $\mathcal{G}_T$ (one partite class). |
| $\mathcal{G}_D'$ | Complete edges in $L$ (other partite class). |
| $\mathcal{G}_D''$ | Same as $\mathcal{G}_D'$ but with one outer face (the dual root) omitted. |
| $\mathcal{G}_D'''$ | Faces and some outer faces of $\mathcal{G}_D$ (which correspond to vertices of $\mathcal{G}_T$). |

Table 1: Summary of graphs constructed from a collection of edges $L$ that forms a $T$-graph. The graph $\mathcal{G}_D'''$ (which is not exactly the same as $\mathcal{G}_T$) is defined precisely in Section 4.2.

### 3.5 From dimers to trees

Let $\mathcal{T}_M' = \{ S_e' : e \in M \}$ be the set of of edges corresponding to a marked matching $M'$ of $\mathcal{G}_D$. Each $S_e'$ corresponds to an edge of $\mathcal{G}_T'$, so we can think of $\mathcal{T}_M'$ as a subgraph of $\mathcal{G}_T'$. We direct each such edge of $\mathcal{T}_M'$ (corresponding to some $S_e'$) of this graph from the face which corresponds to a vertex in $e$ towards the face which does not. We use this interpretation of $\mathcal{T}_M'$ (as a directed subgraph of $\mathcal{G}_T'$) in the following lemma:
Figure 3: (a) Edges of a T-graph \( L \) and the surrounding curve \( C \). (b) The graph \( \mathcal{G}_T \). This graph is the incidence graph for the set of complete edges of \( L \) (drawn as black vertices) and faces of \( L \) (drawn as white vertices). (c) A spanning forest of \( \mathcal{F} \) of \( \mathcal{G}_T \), drawn with thick arrows, and the dual spanning tree \( \mathcal{T} \) on \( \mathcal{G}'_T \), drawn with dotted lines connecting vertices of \( \mathcal{G}'_T \) (which are faces of \( \mathcal{G}_T \), represented as white vertices). Each such dotted line crosses a segment of a complete edge that is not used in \( \mathcal{F} \) (there is exactly one such segment for each complete edge). (d) The marked matching corresponding to \( \mathcal{F} \) when the dual root is taken to be the (unmatched) uppermost white vertex.
Lemma 3.3 If $M'$ is a marked matching, then $T_{M'}$ is an in-directed spanning tree of $G'_T$, rooted at the dual root. The dual $F_{M'}$ of $T_{M'}$ is thus a spanning forest of $G_T$ (when $G_T$ is viewed as an undirected graph).

Proof: It is sufficient to prove that $T_{M'}$ has no directed cycles, since it contains exactly one edge pointing away from each face of $G_T$ (excluding the dual root). This is accomplished using Euler’s formula. Suppose that $T_{M'}$ had a directed cycle $F_0, F_1, \ldots, F_j = F_0$ of faces of $G_T$. Let $S_i$ be the segment $S'_i$ separating $F_i$ and $F_{i+1}$. Let $C'$ be a simple closed curve which starts in the interior of $F_0$, passes through $S_0$ at one point, moves through the interior of $F_1$, passes through $S_1$ at a single point, etc. until it returns to $F_0$. Except for its intersections with the $S_i$’s, each at a single point, $C'$ is entirely contained in the union of the interiors of the $F_i$.

The intersection of the $L_i$ with the interior of $C'$ gives a decomposition of this interior into $n_2$ two-cells (where $n_2$ is the number of faces partial or completely contained inside the loop $C'$), $n_1$ open one-cells and $n_0 = 0$ vertices. Thus, by Euler’s formula $n_2 - n_1 + n_0 = n_2 - n_1 = 1$. In particular $n_2 + n_1$ is odd.

However, the sequence $w_0, b_0, w_1, b_1, \ldots, b_{j-1}, w_j = w_0$ (where $w_i$ is the white vertex of $G_D$ corresponding to $F_i$ and $b_i$ is the black vertex corresponding to the complete edge containing $S_i$) is a cycle in $G_D$, alternating edges of which are contained in $M$. The set of vertices in $G_D$ enclosed by this cycle must be matched only with each other in a perfect matching (since the cycle disconnects these vertices from the rest of the graph). This is a contradiction to the fact that $n_2 + n_1$ is odd. □

Let $\mu_F$ be the measure on directed spanning forests of $G_T$, rooted at the root vertices, for which $\mu_F(F)$ is proportional to the product of the weights of the edges in $F$. Since each of the two outgoing edges of a given interior vertex has weight (by construction) inversely proportional to its Euclidean length, $\mu_F(F)$ is inversely proportional to the product of the lengths of the edges of $F$; hence, $\mu_F(F)$ is also proportional to the product of the Euclidean lengths of all edges of $G_T$ which do not appear (directed one way or another) in $F$.

The following is the main result of this section.

Theorem 3.4 The map $M' \rightarrow F_{M'}$ gives a one-to-one correspondence between marked matchings of $G_D$ and in-directed spanning forests of $G_T$, rooted at $R$. The correspondence is measure preserving, i.e., $\mu(M') = \mu_F(F_{M'})$.

Proof: First, we would like to interpret $F_{M'}$ as a directed spanning forest of $G_T$ by orienting each edge of $F_{M'}$ towards its root vertex. In order to do this, we must check that if $M'$ is a perfect matching of $G_D$, then the directed path along $F_{M'}$, from a vertex $v$ to a root vertex, is a directed path of $G_T$. To see this, note first $F_{M'}$ contains all but one segment of each of the $L_i$; thus, for every interior vertex $v$ of $G_T$ (interior to some $L_i$), $F_{M'}$ includes a path from $v$ to exactly one of the endpoints of $L_i$. Call this vertex $v_1$; each of the directed edges in the directed path from $v$ to $v_1$ is a directed edge of $G_T$. If $v_1$ is also an interior vertex of some $L_j$, then there is a path of edges in $F_{M'}$ from $v_1$ to some endpoint $v_2$ of $L_j$. Iterating this process, we must eventually produce a directed path from $v$ to a root (since $F_{M'}$ has no cycles).

It now follows immediately from Lemma 3.3 and our choice of weights, that $\mu_F(F_{M'})$ is proportional to $\mu(M')$, since each is proportional to the same product of edge lengths. The proof will be complete once we show that the map $M' \rightarrow F_{M'}$ is invertible.
Let $F$ be an arbitrary directed spanning forest $F$, rooted in $R$. Since only the endpoints of a given $L_i$ have outgoing edges pointing to vertices not on $L_i$, each vertex of $L_i$ belongs to a path pointing to one of the two endpoints. It follows that $F$ must include all but one of the subsegments of $L_i$. By Lemma 3.2, the dual of $F$ is a spanning tree of $G_T$, which we may view as being directed towards the dual root. Each face $F$ of $G_T$ is (besides the dual root) is directed towards another face across an edge segment of one of the $L_i$. Pairing of $F$ with the edge segment produced in this way gives a marked matching $M'$ for which $F_{M'} = F$. □

3.6 T-graphs and dimers on the torus

If $L = \{L_1, \ldots, L_n\}$ forms a T-graph on the torus, then we can construct $G_T = G_T(L)$ exactly as above; in this case, $G_T(L)$ has no root vertices and no outer faces. Since the faces of $G_T$ and open edges $L_i$ give a decomposition of the torus into one-cells and two-cells, Euler’s formula implies that $G_T$ has exactly $n$ faces. We construct $G_D$ as above (with white vertices given by faces $F$ of $G_T(L)$, black vertices by the complete edges $L_i$, and edges occurring between $F$ and $L_i$ that share a line segment, weighted according to the length of that segment). We also construct $G_T'$ in a similar fashion.

A cycle-rooted spanning forest $F$ of $G_T$ is a (directed) subgraph of $G_T$—with one outgoing edge from each vertex of $G_T$—which has no null-homotopic (directed) cycles (i.e., no cycles which—when lifted to the universal cover of the torus—start and end at the same place). The “roots” of such an $F$ are the directed cycles of $F$. Clearly, every such $F$ has at least one (non-null-homotopic) directed cycle.

The dual of $F$ is a cycle-rooted spanning forest $F'$ on $G_T$. Now, if $F$ has exactly $j$ cycles, then it is not hard to see that $F'$ has $j$ cycles as well. We can view $F'$ as a directed cycle-rooted spanning forest by directing each edge not on a cycle towards its cycle root; and then orienting all of the edges in a given cycle one of the two possible directions (there are $2^j$ ways of doing this). The proof of the following is now similar to the proof of Theorem 3.4.

Theorem 3.5 There is a one-to-one weight preserving correspondence between perfect matchings on $G_D$ and in-directed cycle-rooted spanning forests $F'$ on $G_T'$ whose dual cycle-rooted spanning forests $F$ are in-directed, cycle-rooted spanning forests of $G_T$.

T-graphs in a torus can be extended to give periodic T-graphs on the plane, finite subsets of which correspond to finite subgraphs of infinite lattice graphs, such as the grid graph in Example 4.

4 T-graphs from dimer graphs

In this section, we describe a procedure for generating $G_T$ from $G_D$ that applies whenever the so-called Kastelyn matrix fails to have certain degeneracies. Before we begin the construction, we will define Kastelyn matrices and say a word about the kinds of graphs for which these degeneracies occur.
4.1 Cuts, and breakers

Say a square matrix $K$ is $k$-degenerate if it has an $(n-k) \times (n-k)$ minor whose determinant is zero; otherwise it is $k$-non-degenerate. The following lemma follows from the standard correspondence between determinants of $k$ minors of $K^{-1}$ and $(n-k)$ minors of $K$:

Lemma 4.1 $K$ is 0-non-degenerate if and only if it is invertible. Assuming $K$ is invertible, $K$ is $k$-non-degenerate if and only if $K^{-1}$ is $(n-k)$-non-degenerate.

Suppose now $K$ is a Kasteleyn matrix for a bipartite planar graph $G_D$. The following is immediate:

Lemma 4.2 If $K$ and $\tilde{K}$ are gauge equivalent, then $K$ is $k$-degenerate if and only if $\tilde{K}$ is $k$-degenerate

A bipartite graph is balanced if it contains an equal number of black and white vertices. A $k$-cut $A$ of a balanced bipartite graph $G_D$ is subset of the vertices for which:

1. $A$ contains at least one white vertex
2. $A$ contains $k$ more black vertices than white vertices
3. Each edge of $G_D$ that connects $A$ to its complement has a black vertex in $A$. 

Figure 4: A T-graph $G_T$ in the plane and the corresponding graph $G'_D$. 
Note that if $A$ is a $k$-cut, then its complement would be a $k$-cut if the colors black and white were reversed. In particular, the existence of $k$ cuts does not depend on which of the two ways we choose to color the vertices. Also, if $A$ is a $k$-cut of $G_D$, then by adding black vertices to $A$ and/or removing white vertices from $A$, we can construct $m$-cuts for any $k \leq m \leq n - 1$. An obvious parity argument implies the following:

**Lemma 4.3** If $A$ is a $k$-cut of $G_D$, then any perfect matching of $G_D$ contains exactly $k$ edges which connect $A$ to its complement; each of these edges matches a black vertex of $A$ and a white vertex of its complement.

A $k$-breaker is a subset $S$ of the vertices of $G_D$ with exactly $k$ white and $k$ black vertices for which the induced subgraph $G_D\setminus S$ of $G_D$ has no perfect matchings.

**Lemma 4.4** If $G_D$ is a connected, balanced, bipartite graph, then $G_D$

1. has a $(-1)$-cut if and only if it has no perfect matching.
2. has a 0-cut if and only if $G_D$ contains unused edges (i.e., edges which occur in no perfect matching of $G_D$).
3. generally has a $k$-cut if and only if has a $(k+1)$-breaker.

**Proof:** The first item is an immediate consequence of the Hall marriage theorem. That theorem states that $G_D$ has a perfect matching if and only if there is no set $B$ such that $B$ has $m$ more white vertices than black vertices and there are fewer than $m$ edges connecting white vertices of $B$ to its complement. A $(-1)$-cut is clearly such a set, with $m = 1$. Conversely, given $B$ as described above, construct $B'$ by removing from $B$ all of the (at most $m - 1$) white vertices of $B$ connected to the complement of $B$, and if necessary, some arbitrary additional white vertices (so that $m - 1$ vertices are removed in all). Then $B'$ is a $(-1)$-cut.

For the second item, first, it is clear that if $A$ is a 0-cut of $B$, then all of the edges connecting $A$ to its complement will be unused. Conversely, if $G_D$ has an unused edge $e$, then the graph $G$ formed by removing edge $e$ and its two vertices from $G_D$ will not have any perfect matching. Therefore it will have a $(-1)$-cut $A$ by part 1. The union of $A$ and the black vertex of $e$ is a thus a 0-cut. (Aside: if $G_D$ has a forced edge—i.e., an edge $e$ which occurs in every perfect matching of $G_D$—then all the edges that share vertices with $e$ will be unused.)

The same argument implies the third statement in the case $k = 0$. For larger $k$, if $G_D$ has a $k$-cut $A$, then any subset of $(k + 1)$ black vertices of $A$ and $(k + 1)$ white vertices of its complement is a $(k + 1)$-breaker (since the remaining set of vertices in $A$ contains more white than black vertices, but there are no edges connecting white vertices of this remaining set with its complement). Conversely, if $S$ is a $(k + 1)$-breaker, then $G_D\setminus S$ has a $(-1)$-cut $A$, and the union $A$ and the black vertices of $S$ is a $k$-cut of $G_D$. ❑

**Lemma 4.5** $G_D$ has no $k$-breaker (or, equivalently, no $(k - 1)$-cut) if and only if, for a generic choice of positive weights of the edges of $G_D$, the Kastelyn matrix $K = K(G_D)$ is $k$-non-degenerate.
**Proof:** The determinant of an \((n-k) \times (n-k)\) minor of the Kastelyn matrix is a polynomial of the edge weights. Clearly, this polynomial will be zero for a given minor precisely when the set of \(k\) white and \(k\) black vertices corresponding to rows and columns not in the minor is a \(k\)-breaker. The result follows from the fact that any non-zero polynomial in finitely many real variables is non-zero for a generic choice of inputs. \(\square\)

In this paper, we will mainly be interested in whether \(K\) is \(k\)-degenerate for \(k \in \{0, 1, 2\}\). But we know that whenever \(K\) is a Kastelyn matrix (of a graph having a perfect matching) it is \(k\)-non-degenerate for \(k = 0\). And assuming \(G_D\) has no unused edges (which we may always assume throughout, since the perfect matching model will be unchanged if we remove unused edges from \(G_D\)) \(K\) is generically 1-non-degenerate. We will address the potential failure of \(K\) to be 2-non-degenerate in a later section.

### 4.2 T-graphs: construction via integration of Kastelyn flow

Let \(G_D\) be a finite, weighted bipartite planar graph (with positive generic weight function \(\nu\)) with \(n\) black vertices \(b_1, b_2, \ldots, b_n\) and \(n\) white vertices \(w_1, w_2, \ldots, w_n\). Suppose \(G_D\) has a perfect matching and no unused edges. Suppose that \(G_D\) has no 1-cuts—and hence each of the entries and two-by-two minors of \(K^{-1}\) is non-zero (i.e., \(K\) is 1-non-degenerate and 2-non-degenerate).

We will now construct a T-graph corresponding to \(G_D\) in the case that \(K\) is 2-non-degenerate.

First, we may think of \(K\) as describing a linear map from the space \(\mathbb{R}^W\) of functions on white vertices to the space \(\mathbb{R}^B\) of functions on black vertices. Let \(b_0\) be a fixed vertex on the outer boundary of \(G_D\). Suppose that \(G_D\) has \(m\) black and \(m\) white vertices on its outer boundary face. Fix a generic convex \(m + 1\)-gon \(Q\) with edge vectors \(q_0, \ldots, q_m \in \mathbb{C}\) in cyclic order (and \(q_0 = -\sum_{i=1}^m q_i\)). Vertices of \(Q\) will be the root vertices of \(G_T\). Suppose that \(A_w \in \mathbb{R}^W\) assumes the values \(q_1, \ldots, q_m\) in cyclic order on the white vertices on the boundary face, and that \(A_w\) vanishes on all other white vertices of \(G_D\). Let \(A_b\) be the function on black vertices which is equal to 1 at \(b_0\) and 0 everywhere else. Denote by \(\mathbf{T}\) the all-ones column vector and by \(\mathbf{T}^t\) its transpose. View \(A_b\) as a column vector and \(A_w\) as a row vector.

We claim that there is a unique matrix \(\tilde{K}\), gauge equivalent to \(K\), for which \(\tilde{K}\mathbf{T}\) is a non-zero multiple of \(A_b\) and \(\mathbf{T}\tilde{K} = A_w\). The matrix \(\tilde{K}\) can be derived explicitly from \(K\) as follows. Since \(K\) is invertible, there exists a vector \(f\) for which \(KF = A_b\). Multiplying the \(i\)th column of \(K\) by the \(i\)th component of \(f\) (non-zero, because \(K\) is 1-non-degenerate) produces a \(K'\) for which \(K'\mathbf{T} = A_b\). Next, there exists a row vector \(g\) for which \(gK' = A_w\). Multiplying the \(j\)th row of \(K'\) by the \(j\)th component of \(g\) (also non-zero, since \((K')^{-1}\) is 1-non-degenerate and nonzero entries of \(A_w\) are generic) gives the desired \(\tilde{K}\).

We may think of \(\tilde{K}\) as describing a vector flow (2-component flow) on \(G_D\): sending \(\tilde{K}_{i,j}\) units of flow from \(b_i\) to \(w_j\). The net flow into each non-boundary white vertex and each black vertex (except \(b_0\)) is zero. Now, draw a dotted line from each white vertex on the outer face of \(G_D\) to infinity, and from \(b_0\) to infinity, so as to divide the outer face of \(G_D\) into \(m + 1\) outer faces; take these faces and the interior faces of \(G_D\) as the vertices of the dual graph \(G'_D\) of \(G_D\). Then \(\tilde{K}\) also describes a dual flow on \(G'_D\) (obtained by rotating each edge ninety degrees counter-clockwise) whose net flow around each non-boundary face of \(G'_D\) is zero; viewed in this light, \(\tilde{K}\) is the gradient of a function \(\psi : G'_D \to \mathbb{C}\).
Moreover, if \( v \) is a vertex of \( G \), then \( \psi(\{v\}) \) is a vertex of the T-graph; if \( v \) corresponds to an outer face of \( G \), then \( \psi(v) \) is a root vertex of the T-graph.

**Theorem 4.6** If \( K \) is 2-non-degenerate then the \( L = \{L_1, \ldots, L_n\} \) defined above forms a T-graph with root vertices at vertices of \( Q \), and \( G_D = G_D(L) \) (up to gauge equivalence). Moreover, if \( v \) is a vertex of \( G_D' \), then \( \psi(v) \) is a vertex of the T-graph; if \( v \) corresponds to an outer face of \( G_D \), then \( \psi(v) \) is a root vertex of the T-graph.

**Proof:** First, the change in \( \psi \), as one moves from outer face \( F \) of \( G_D \) around a vertex \( v \) to another outer face, is given by the flow of \( \tilde{K} \) into \( v \), which is given by \( q_i \), the \( i \)-th component of \( A_w \), whenever \( v \) is a white vertex \( w_i \), and zero when \( v \) is any black vertex besides \( b_0 \). By moving around the polygon in steps, it is clear that (up to an additive constant) \( \psi(F) \) assumes the values of the vertices of the convex polygon in cyclic order.

Let \( f \) be an interior vertex of \( G_D' \). We claim that for some black face incident to \( f \), with vertices \( f_1 \) and \( f_2 \) incident to \( f \), \( \psi(f_1) - \psi(f) \) and \( \psi(f_2) - \psi(f) \) point in opposite directions. Suppose otherwise. Then \( \tilde{K} \) would have to assume opposite signs on the entry corresponding to each such pair of edges \((f, f_1)\) and \((f, f_2)\). By the definition of a Kastelyn matrix, \( \tilde{K} \) has positive sign for an odd (resp., even) number of the edges incident to \( f \) if the total number of edges is 0 mod 4 (resp., 2 mod 4), so this is a contradiction. It follows that \( \psi(f) \) is an interior vertex of at least one \( L_i \). In particular, this implies that the endpoint of each \( L_i \) is either an interior vertex of some \( L_j \) or a root vertex.

It also implies a maximal principle, i.e., that for any vector \( u \) in \( \mathbb{R}^2 \), the function \( \psi_u(x) = (\psi(x), u) \) (an inner product computed with \( \psi(x) \) treated as a vector in \( \mathbb{R}^2 \)) has no local maxima or minima at interior faces of \( G_D \). That is, every interior face \( f \) (viewed as an interior vertex in \( G_D' \)) has neighbors \( f_1 \) and \( f_2 \) satisfying \( \psi_u(f_1) \leq \psi_u(f) \leq \psi_u(f_2) \). For generic \( u \) (i.e., any \( u \) whose slope is not parallel to one of the \( L_i \)'s), the inequality can be made strict.

Now, to show that the \( \{L_i\} \) form a T-graph, it remains only to show that they do not intersect one another; while proving this, we will also show that \( \psi(G_D') \) partitions the convex polygon \( Q \) into convex polygons (the white faces). First, the maximal principle immediately implies that \( \psi(G_D') \) lies in \( Q \). Furthermore, we claim that as one moves \( x \) clockwise around each a white interior face \( w \) of \( G_D' \), \( \psi(x) \) traces out a convex polygon in some fixed orientation (clockwise or counterclockwise; we refer to this direction as the orientation of \( w \) and denote
the polygon by $\psi(w)$). If this were not the case, then there would have to be vertices $f_1, f_2, f_3, f_4$, in clockwise order around $w$ and some generic $u$ for which $\psi_u(f_1)$ and $\psi_u(f_3)$ are less than both of $\psi_u(f_2)$ and $\psi_u(f_4)$. By the maximal principle, we can find paths in $p_2$ and $p_4$ in $G'_D$ from $f_2$ and $f_4$ to root vertices along which $\psi_u$ is strictly increasing and paths $p_1$ and $p_3$ from $f_1$ and $f_3$ to root vertices along which $\psi_u$ is strictly decreasing. Now, let $p$ be a path in $G'_D$ formed by concatenating $p_1$ (reversed), a dotted line from $f_1$ to $f_3$, and $p_3$. This path cannot intersect $p_2$ or $p_4$ (since $\psi_u$ at any point on these two paths is greater than $\psi_u$ at any point on $p_1$ or $p_3$). However, the Jordan curve theorem implies that $p$ separates its complement in $G'_D$ into at least two connected components and that $f_2$ and $f_4$ (which lie on either side of $p$ across the face $w$) are in separate components (this remains true even for the graph $(G'_D)'$ formed by adding to $G'_D$ the edges connecting each cyclically consecutive pair of outer vertices of $G'_D$). Now, the paths $p_2$ and $p_4$ both lead to root vertices at which $\psi_u$ assumes a larger value than it does at any point along $p$; and these points are in the same component of $(G'_D)'$, a contradiction. A similar argument shows that the outer faces $w$, joined with this, have this orientation. Another similar argument applies to black faces and shows that as one moves $x$ around a black interior face $b$ of $G'_D$, $\psi(x)$ traverses the corresponding $L_i$ exactly once in each direction.

Next, we argue that all white faces have the same orientation. It is enough to prove that any white faces of $G'_D$ (vertices of $G_D$) $w_1$ and $w_2$ incident to a common black $b$ have the same orientation. Now, as $x$ traverses the boundary of the face $b$ in $G'_D$, $\psi(x)$ traces out the corresponding $L_i$ once in each direction; divide the faces incident to $b$ into two categories according to the orientation of the edge shared with $b$. Clearly, if these faces do not all have the same orientation, we can find two of them, $w_1$ and $w_2$ in opposite categories that have opposite orientations. In this case, $\psi(w_1)$ and $\psi(w_2)$ will lie on the same side of $b$; let $u$ be vector orthogonal to $L_i$; assume without loss of generality that $\psi_u$ assumes a larger value on points on $L_i$ than on other points of $w_1$, $w_2$. Let $f_1$ and $f_3$ be the points in $G'_D$ incident to $b$ whose images are the endpoints of $b$, and let $f_2$ and $f_4$ be arbitrary points of $w_1$ and $w_2$ which do not lie on $b$. Let $p$ be formed by concatenating a path $p_1$ from $f_1$ to a root on which $\psi_u$ is strictly increasing (reversed), a dotted line from $f_1$ to $f_3$, and a path $p_3$ from $f_3$ to a root vertex along which $\psi_u$ is strictly increasing; observing that $f_2$ and $f_4$ are on opposite sides of $p$, we derive a contradiction through the Jordan curve argument described above.

Finally, suppose that two of the $L_i$ intersect. Then there must be two faces $w_1$ and $w_2$ for which $\psi(w_1)$ and $\psi(w_2)$ intersect. The outer boundary of $(G_D')'$ is mapped with some consistent orientation to $Q$. Now, let $h : Q \rightarrow \mathbb{Z}$ at $x$ be the number of white faces $\psi(w)$ which contain $x$ in their interiors. It is clear that $h$ assumes the value $1$ near the boundary. We claim that $h$ is equal to one throughout $Q \setminus \psi(G_D')'$; otherwise, there would be an $x$ in the interior of $Q$ (and not at the finitely many endpoints of any $L_i$ or intersections of pairs of $L_i$) on the boundary of regions at which $h$ assumes different values. Such an $x$ must lie on some $L_i$, and it is not hard to see that the two white faces incident to $x$ and $L_i$ must have opposite orientations. 

\[ \square \]

### 4.3 Flat-face degeneracy

Now, suppose that $K$ is merely 1-non-degenerate and not necessarily 2-non-degenerate; then we can formally construct $\psi$ exactly as above; in this case, however, we cannot rule out that
some of the $L_i$ may be parallel to one another—and in fact, some of the $L_i$ may overlap. However, the same arguments given above still imply that for each white $w$, $\psi(w)$ is either a convex face with some orientation (as described above) or a line segment traversed once in each direction (like the black faces). In the latter case, we say $\psi(w)$ is a degenerate face. In the presence of degenerate faces, we will consider $\psi(w)$ and $\psi(b)$ to be incident to one another along an edge if and only if $w$ and $b$ are adjacent vertices in $G'_D$.

It is clear that if a white vertex $w$ is degenerate, then $\psi(b)$ is parallel to $\psi(w)$ for each black $b$ adjacent to $w$. A maximal component of the subgraph of $G'_D$ consisting of vertices on which $\psi$ is parallel to a given line is called a parallel component of $G'_D$. Clearly, the neighbor set of any white vertex in a parallel component is also in the parallel component.

An extreme point of a degenerate face $w$ is a vertex $f$ incident to $w$ for which $\psi(f)$ is an endpoint of $\psi(w)$. The union of $\psi$-images of a parallel component is a segment which we call an extended complete edge. Now observe the following.

**Lemma 4.7** Each parallel component $P$ is a 1-cut.

**Proof:** Observe that every $f$ which is an interior vertex of a black edge of $G'_D$ in a parallel cluster is the extreme vertex for the same number of black and white faces of $G'_D$. The endpoints of the extended complete edge are extreme points of one more black vertex than white vertices. Since every face has exactly two extreme vertices, the result follows. □

Similar arguments to those given in the proof of Theorem 4.6 imply that as $x$ traverses the outside of a parallel component, $\psi(x)$ traverses the outside of the extended complete edge exactly once in each direction. Similar arguments to those of Theorem 4.6 imply that the extended complete edges form a T-graph. We say that $L = \{L_i\}$ forms a T-graph with overlaps if $L_i$ satisfies all of the T-graph conditions except that parallel pairs of $L_i$ are allowed to intersect (overlap) one another. The above analysis implies the following:

**Theorem 4.8** Theorem 4.6 still holds if $K$ is merely 1-non-degenerate and not necessarily 2-non-degenerate—except that in this case, some of the white faces may be degenerate (and so the T-graph may have overlaps). Theorem 3.4 still applies to T-graphs with overlaps.

Even though some of the white faces are flat in the overlapping T-graph $G_T$, we can define a dual to the overlapping T-graph, containing these faces, using the graph structure of $G_D$. After doing this, all of the arguments in the proof of Theorem 3.4 apply as before, so we still have a martingale on the T-graph and have a measure preserving correspondence between spanning forests and perfect matchings.

Recall that in any perfect matching, there is always exactly one edge connecting a given 1-cut to its complement, and that edge contains a black vertex of the 1-cut. It is perhaps not surprising that when we form the T-graph, 1-cuts, in some sense, play the same role as single black vertices. If we had simply replaced all 1-cuts in our original graph with single black vertices, then, for a generic choice of weights, the T-graph would not have any degenerate white faces.
4.4 Extending the correspondence to degenerate weighted graphs

Recall from Lemma 4.5 that if we remove the unused edges from $G_D$, then the Kastelyn matrix for $G_D$ is 1-non-degenerate (and hence Theorems 3.4, 4.8 apply) for a generic choice of weight functions $\nu$. Suppose, however, that the Kastelyn matrix for $G_D$ is not 1-non-degenerate for a particular choice of weight function $\nu$. Then we would like to take a generic sequence of weights $\nu_i$ converging to $\nu$, look at the limit (or some subsequential limit) of the corresponding T-graphs, and show that the measure-preserving correspondence described in Theorem 3.4 still holds for the limiting object. The problem is that, as Figure 1 makes clear, the limit of a sequence of T-graphs need not be a T-graph at all; in fact, some of the edge segments and faces may shrink to single points.

For practical computational applications, it may be sufficient to have the correspondence between dimers and spanning forests for a generic choice of weights. But a word of caution is in order. Consider the dimer model whose T-graph is given by the right diagram in Figure 1; if weights $\nu_i$ tend to a limit $\nu$ in such a way that the T-graphs have the graph on the left as a limit, then the shrinking small triangle in the center of the diagram will become a “trap” for the random walk on the T-graph, in that the expected amount of time that a walk spends on these three vertices before exiting towards a root vertex tends to infinity; sampling algorithms that rely on random walks will perform poorly for weights approximating $\nu$. In this case, however, one can simplify the limiting problem by reducing the three vertices in the small triangle at the center to a single vertex. The probability tends to one that only one of the “long” directed edges (i.e., edges whose lengths are not tending to zero) extending outward from these three vertices will appear in a random tree; given a spanning tree of the “reduced” graph, it is possible to work out which “short” edges appear in the graph. The details of this and more general versions of this reduction are left to the reader.

5 Periodic and almost periodic T-graphs

5.1 Definitions for almost periodic T-graphs

In this section, we prove some results about T-graphs which are motivated by the study of ergodic Gibbs measures on tilings of infinite periodic planar graphs. More on this subject can be found in [9], who cite the results of this section. Our first aim here is to construct from periodic bipartite planar graphs (and under certain conditions on the weights) infinite T-graphs with a property called “almost periodicity.”

Let $G_D$ be embedded in the torus $\mathbb{R}^2/\mathbb{Z}^2$ and let $G_D^\infty$ be the doubly periodic lift to $\mathbb{R}^2$ (we assume that $G_D^\infty$ is connected). As before, assume that $G_D$ has $n$ white and $n$ black vertices. Denote by $v_{j,k}$ the vertex of $G_D^\infty$ which lies in the square $[j,j+1) \times [k,k+1)$ and whose projection to the torus is the vertex $v \in G_D$. For the sake of simplicity we will assume throughout this section that $G_D$ has no unused edges and that it has generic weights. The non-generic weight case requires a slightly finer analysis which we choose not to go into here. See however [9].

A function $f$ on the vertices of $G_D$ is $\text{(\(\alpha,\beta\))-periodic}$ if $f(v_{j+x,k+y}) = \alpha^x \beta^y f(v_{j,k})$ for all $(v_{j,k}) \in G_D^\infty$. Say $f$ is $\text{almost periodic}$ if it is $(\alpha,\beta)$-periodic and $\alpha$ and $\beta$ have modulus one (but are not necessarily roots of unity). In this case, we write $\alpha = e^{2\pi i a}$ and $\beta = e^{2\pi i b}$. 
If $a$ and $b$ are rational, then $f$ is doubly periodic with some period.

For a fixed $(\alpha, \beta)$ the linear space of $(\alpha, \beta)$-periodic functions is $2n$-dimensional and is parametrized by the space of functions on one period of $G_D^\infty$—which we can represent as a single copy of $G_D$. It has a natural basis consisting of functions $\delta_v$ whose value is 1 at $v \in [0,1]^2$ and zero at other vertices in the fundamental domain. Let $K$ be a Kastelyn matrix for $G_D$ and $K^\infty$ an infinite-dimensional Kasteleyn matrix for $G_D^\infty$ which is a lift of $K$. We can think of $K^\infty$ as a linear function from the set of functions on the black vertices of $G_D^\infty$ to functions on the white vertices of $G_D^\infty$. Since this function maps $(\alpha, \beta)$-periodic functions to $(\alpha, \beta)$-periodic functions, it induces a linear map from the $n$-dimensional space of functions on the black vertices of $G_D$ to the $n$-dimensional space of functions on white vertices of $G_D$. Denote by $K_{\alpha, \beta}$ the matrix of this linear map in the basis $\{\delta_v\}$.

The determinant $\det K_{\alpha, \beta}$ is a polynomial function of $\alpha$ and $\beta$; in particular for certain $(\alpha, \beta)$ (corresponding to zeros of this polynomial function) the matrix $K_{\alpha, \beta}$ has a non-trivial null space, and hence we can find $(\alpha, \beta)$-periodic functions $f$ and $g$ satisfying $K^\infty f = 0$ and $gK^\infty = 0$. If the polynomial $\det K_{\alpha, \beta}$ happens to have a zero $(\alpha, \beta)$ that lies on the unit torus of complex variable pairs that both have modulus one, then $f$ and $g$ are almost periodic. If, furthermore, $f$ and $g$ happen to be nowhere zero, then we can define an infinite T-graph as follows. First, observe that the function $\tilde{K}_1^\infty(vw) = f(v)g(w)K^\infty(v, w)$ on edges $vw$ of $G_D^\infty$ is a nowhere zero flow. The dual of this flow is the gradient of a function $\psi_1$ on $G_D'$. Similarly the dual of $\tilde{K}_2^\infty(vw) = f(v)g(\overline{w})K^\infty(v, w)$ is the gradient of a function $\psi_2$ on $G_D'$ (where $\overline{g}$ denotes the complex conjugate of $g$). We may assume (multiplying $g(w)$ by a generic modulus one complex number if necessary) that $g(w) + g(\overline{w}) = 2\text{Re}g(w)$ is also nowhere zero. Then we can think of $\tilde{K} = \tilde{K}_1 + \tilde{K}_2$ as an infinite Kastelyn matrix and $\psi = \psi_1 + \psi_2$ as the corresponding T-graph. We will call a mapping $\psi$ from $(G_D^\infty)'$ to $\mathbb{R}^2$, constructed in this way, an almost periodic T-graph mapping. See Figure 5.
We remark that, given a fixed ν, the number of α, β on the unit torus for which det \( K_{\alpha,\beta} = 0 \) also plays a fundamental role in [9], where it is shown that the minimal specific free energy ergodic Gibbs measure on perfect matchings of the infinite weighted graph \( G_D^\infty \) is smooth if the corresponding polynomial \( K_{\alpha,\beta} \) polynomial has 0 roots on the unit torus and rough if it has 2 roots (necessarily complex conjugates) on the unit torus (in the non-generic case of a single root, it is rough only when \( \frac{d}{dx} \det K_{\alpha,\beta} = \frac{d}{dx} \det K_{\alpha,\beta} = 0 \)). The terms “smooth” and “rough” come from the statistical physics literature and are defined in [9]. The main goal of this section is to prove that when the choice of weights is generic, the number of modulus-one values of \((\alpha,\beta)\) that are roots of \( \det K_{\alpha,\beta} \) always belongs to the set \{0, 2\}.

### 5.2 Generic points on the variety of almost periodic T-graphs

Write \( \mathbb{R}_+ \) for the set of strictly positive real numbers, \( \mathbb{C}_+ \) for the set of non-zero complex numbers, and write \( \mathbb{P}^k \) for \( k \)-dimensional complex projective space. Suppose that \(|V| = 2n\), and define a variety \( X \subset \mathbb{R}_+^{|E|} \times \mathbb{C}_+^2 \times \mathbb{P}^{n-1} \) by:

\[ X = \{(\nu, \alpha, \beta, f) : K_{\alpha,\beta}f = 0\} \]

Here \( f \) is an element in \( \mathbb{P}^{n-1} \), which is a one-dimensional subspace of \( \mathbb{C}^n \), and by \( K_{\alpha,\beta}f = 0 \) we mean that this subspace lies in the null space of \( K_{\alpha,\beta} \). By abuse of notation, if \( f \) is a non-zero function on the black vertices of \( G_D \), we will also use \( f \) to denote the element of \( \mathbb{P}^{n-1} \) given by the linear span of \( f \). Denote by \( \tilde{X} \) the subset of \( X \) consisting of points for which \(|\alpha| = |\beta| = 1\). Denote by \( \text{Adj}(K_{\alpha,\beta}) \) the adjugate matrix of \( K_{\alpha,\beta} \), whose entries are the \((n-1) \times (n-1)\) minors of \( K_{\alpha,\beta} \) (so that \( K_{\alpha,\beta}\text{Adj}(K_{\alpha,\beta}) = \det K_{\alpha,\beta} \)). It is easily seen that \( \text{Adj}(K_{\alpha,\beta}) \) is identically equal to zero if and only if the rank of \( K_{\alpha,\beta} \) is less than \( n-1 \); and if the rank of \( \text{Adj}(K_{\alpha,\beta}) \) is exactly \( n-1 \), then at least one column of \( \text{Adj}(K_{\alpha,\beta}) \) is a non-zero vector whose span is the null space of \( \text{Adj}(K_{\alpha,\beta}) \). The following is the main result of this section:

**Theorem 5.1** The variety \( \tilde{X} \) is irreducible. For a generic choice of \( \nu \), there are either zero or two quadruples \((\nu, \alpha, \beta, f)\) in \( X \). When the latter is the case and \( \nu \) is generic, then the corresponding \( \alpha, \beta, f \) are such that \( \text{Adj}(K_{\alpha,\beta}) \) has rank \( n-1 \), all of its coordinates are non-zero, and \( f \) is given by any column of the (rank one) matrix \( \text{Adj}(K_{\alpha,\beta}) \).

Let us say a word about the significance of this theorem to T-graph classification before we prove it. By obvious symmetry, Theorem 5.1 implies that for generic \( \nu \), there are either zero or exactly two quintuples \((\nu, \alpha, \beta, f, g)\) with \( gK_{\alpha,\beta} = 0 \) and \( fK_{\alpha,\beta} \). Recall that our almost periodic T-graphs were defined to have gradient given by \( \tilde{K}^\infty(vw) = 2f(v)\text{Re}(g(w))K^\infty(v, w) \). Since \( f \) and \( g \) are uniquely determined up to complex conjugacy and multiplication by a constant factor, this implies that the almost periodic T-graph is completely determined up to rotations (which arise from multiplying \( f \) by a modulus one constant), constant rescalings (which arise from multiplying either \( g \) or \( f \) by a real constant), reflection (which comes from complex conjugacy), translations of the image space (which arise from the fact that \( \tilde{K}^\infty \) only determines the T-graph mapping up to an additive constant) and “translation of the domain, or a limit of such translations.” To explain the last symmetry, note that multiplying both \( f \) and \( g \) by \( \alpha^m\beta^n \) is equivalent to composing
the T-graph mapping with translation of the domain by \((m, n)\). If one of \(\alpha, \beta\) is irrational, then we can achieve any modulus one number as a limit of numbers of the form \(\alpha^m \beta^n\). We summarize these observations informally by saying that “the almost periodic T-graph mapping corresponding to \(\nu\) is unique up to affine orthogonal transformations of the image and translations of the domain.” We say two T-graphs are equivalent if one can be obtained from the other via a symmetry of this sort. Note, of course, that if \(\alpha\) and \(\beta\) are both rational, then multiplying \(f\) and \(g\) by a modulus one number is not necessarily the same as a domain translation, or even a limit of such translations. In this case, there is a one parameter family of T-graph equivalency classes.

We will now prove Theorem 5.1 in stages, beginning with the following lemma. First, denote by \(X'\) the projection of \(X\) onto its first three coordinates \((\nu, \alpha, \beta)\); i.e., \(X'\) is the zero set of the polynomial \(P(\nu, \alpha, \beta) = \det K_{\alpha, \beta}\).

**Lemma 5.2** The variety \(X'\) is irreducible. Moreover, for a generic point \((\nu, \alpha, \beta)\) on \(X'\), the matrix \(\text{Adj}(K_{\alpha, \beta})\) has no zero entries, and the \(f\) for which \((\nu, \alpha, \beta, f) \in X\) is unique.

**Proof:** Clearly, \(P\) is affine linear as a function of \(\nu(e)\), that is \(P = \nu(e)P_e + P_e'\), where \(P_e\) and \(P_e'\) do not involve \(\nu(e)\). If we could write \(P = P_1P_2\), then each \(\nu(e)\) must occur in either \(P_1\) or \(P_2\), but not both. Since the multiplicity of the \(\nu(e)\) terms determine the multiplicity of \(\alpha\) and \(\beta\) in each monomial, this implies that there is no cancellation when multiplying out \(P_1\) times \(P_2\) (i.e., there are no monomials that can represented as a product of a monomial in \(P_1\) and a monomial in \(P_2\) in two different ways). Thus, each monomial in \(P_1\) times a monomial of \(P_2\) corresponds to a matching. Let \(E_1, E_2\) be the set of edges represented in \(P_1, P_2\), respectively, and \(V_1, V_2\) their vertices. If an edge \(e\) connected a vertex \(v_1\) of \(V_1\) to a vertex \(v_2\) of \(V_2\), then its weight could not occur in either \(P_1\) or \(P_2\), since if it occurred in a monomial of, say, \(P_1\), then the product of that monomial with a monomial of \(P_2\) that included a factor of \(\nu(e')\) with \(e'\) incident to \(v_2\) (such a monomial exists by definition) would not correspond to a matching, since it would involve two edges incident to \(v_2\). Thus \(e\) must be unused, a contradiction. Thus, if \(P = P_1P_2\), then one of the \(P_i\)—say, \(P_2\)—must be a function of \(\alpha\) and \(\beta\) alone. Since each combination of edge weights corresponding to a matching occurs in exactly one monomial of \(P\), we conclude that \(P_2\) is a monomial in \(\alpha\) and \(\beta\).

Furthermore \(P\) is irreducible when considered as a polynomial in both the edge weights and \(\alpha, \beta\), except for a monomial factor in \(\alpha\) and \(\beta\). That is, if \(P = P_1(\nu, \alpha, \beta)P_2(\nu, \alpha, \beta)\) then one of the \(P_i\) consists of a single monomial in \(\alpha\) and \(\beta\). To see this, note that by the previous result, we may assume without loss of generality that \(P_2\) is a polynomial in \(\alpha, \beta\) alone; and since we are assuming \(\alpha \neq 0, \beta \neq 0\), the variety is not changed if we divide out by this term so that \(P\) is an irreducible polynomial.

Fix an edge \(e\) and consider the polynomial \(P_e\) as defined above. Since \(P\) is irreducible and \(e\) occurs in a proper subset of the set of all matchings, the zero set of \(\nu(e)P(e)\), intersected with \(X'\), forms a proper subvariety of \(X'\). In other words, on a generic subset of \(X'\), none of the entries of \(\text{Adj}(K_{\alpha, \beta})\) corresponding to an edge in \(G_D\) are zero. Since \(\text{Adj}(K_{\alpha, \beta})\) has rank at most one, and every row and column has a non-zero entry, we conclude that every entry of \(\text{Adj}(K_{\alpha, \beta})\) is non-zero and \(f\) is the span of any column of \(\text{Adj}(K_{\alpha, \beta})\).
Lemma 5.3 For a generic choice of weights $\nu$, every pair $\alpha, \beta$ for which $(\nu, \alpha, \beta) \in X$ is such that $\text{Adj}(K_{\alpha, \beta})$ has no zero entries, and the $f$ for which $(\nu, \alpha, \beta, f) \in X$ is unique.

Proof: Lemma 5.2 implies that for generic edge weights $\nu$, $P$ and $P_e$ have no common factor as functions of $\alpha$ and $\beta$ except for monomial factors. To see this, by irreducibility note that there exist polynomials $Q_1 = Q_1(\alpha, \beta, w)$ and $Q_2 = Q_2(\alpha, \beta, w)$ such that $PQ_1 + P_eQ_2 = Q(\alpha, w)$ where $Q$ is a nonzero polynomial depending only on $\alpha$ and the weights $w$, not on $\beta$. Similarly there exist $Q_3, Q_4$ such that $PQ_3 + P_eQ_4 = Q'(\beta, w)$ where $Q'$ is a nonzero polynomial independent of $\alpha$. Plugging in generic values for $w$, $Q$ and $Q'$ will still be nonzero, but any common factor of $P$ and $P_e$ is a common factor of $Q$ and $Q'$ which is impossible. So $P$ and $P_e$ have no common factor for generic $w$.

Therefore, when $\nu$ is fixed generically, by Bezout's theorem $P$ and $P_e$—viewed as polynomials in $\alpha$ and $\beta$—have a finite number of common zeros. By genericity none of these zeros lies on the unit torus (since for any positive real $x$, we can choose $\nu_x$ so that $P(\nu, \alpha, \beta) = P_{\nu_x, \alpha, \beta}$; and replacing $\nu$ with such a $\nu_x$, for a generic choice of $x$, preserves the genericity of the weights). \hfill \square

Lemma 5.4 Any almost periodic $T$-graph mapping $\psi$ is unbounded as a function of $(G_D')$. Moreover if $u$ is any vector in $\mathbb{R}^2 \setminus \{0\}$, then $(\psi, u)$ is unbounded if it is not identically equal to a constant (in which case $\psi$ is degenerate—i.e., its image is contained in a line).

Proof: Suppose that $f$ is $(\alpha, \beta)$-periodic and $g$ is $(\gamma, \delta)$-periodic with $\gamma = e^{2\pi i c}$ and $\delta = e^{2\pi i d}$. Then $K_1(\nu_{j,k}, w_{j,k})$ is a function of $\ell, m$ whose real and imaginary parts can both be written in the form $\cos(\ell a + bm + x)\cos(\ell c + dm + y)$ times a constant, for some $x$ and $y$.

If $\psi$ were bounded on $G_D'$, then the corresponding martingale on the $T$-graph would almost surely converge (by the martingale convergence theorem), and there would thus have to be a path of vertices $v_1, v_2, \ldots$ for which $\psi(v_i)$ converges to a constant. We claim that this is impossible. It is enough to show that for some $\epsilon$, the set of edges $(vw)^*$ for which $0 < K_1(v, w) < \epsilon$ has no infinite cluster. For some $N > |G_D|$, we can always find $\epsilon$ small enough so that the distance between any two clusters of $(\ell, m) \in \mathbb{Z}^2$ (viewed as points in $\mathbb{Z}^2$) on which $0 < \cos(\ell a + bm + x) < \epsilon^{1/2}$ is at least $2N$ times the diameter of the largest such cluster, and similarly for clusters on which $0 < \cos(\ell c + dm + y) < \epsilon^{1/2}$. (This is trivial if $a$ and $b$ are rational, since the function is periodic in that case; if they are irrational, then we can find $\epsilon_0$ for which there is no integer pair $(n_1, n_2)$ for which $n_1 a + n_2 b$ is less than $\epsilon_0$ (modulo $2\pi$) and $|n_1 + n_2| \leq 2N$. Choose $\epsilon_0$ small enough that there can't be two values differing by $\epsilon_0$ (modulo $2\pi$) with cosines $\epsilon$ apart.) Now, it is clear that the largest cluster of $\ell, m$ on which even one of these statements holds is at most $2N$; since the gradient of $\psi$ has norm at least $\epsilon$ when neither statement holds, we conclude that $\psi$ cannot be bounded.

The same argument shows that there cannot exist a non-zero vector $u \in \mathbb{R}^2$ for which the inner product $(\psi(v), u)$ is bounded as a function of $v$, unless $(\psi(v), u)$ is constant. \hfill \square

Lemma 5.5 If $\nu$ is generic, then the maximum number of linearly independent, almost periodic solutions to $K^\infty f = 0$ (or similarly, solutions to $gK^\infty = 0$) is two. If there are two solutions, which are $(\alpha, \beta)$- and $(\gamma, \delta)$-periodic, then $\alpha = \gamma$ and $\beta = \delta$. 

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Proof: For each $\alpha$ and $\beta$, the left null space of $K_{\alpha,\beta}$ has the same dimension as the right null space. Now, suppose that $f$ is $\alpha,\beta$-periodic and $g$ is $\gamma,\delta$-periodic with $\gamma = e^{2\pi ic}$ and $\delta = e^{2\pi id}$. Then as in the proof of Lemma 5.4, $\tilde{K}^\infty(v_{j,k}, w_{j,\ell,k+m})$ is a function of $\ell, m$ whose real and imaginary parts can both be written in the form $\cos(\alpha \ell + \beta m + x) \cos(\gamma \ell + \delta m + y)$ times a constant, for some $x$ and $y$. Let $S$ be a cycle in $G'_D$; if we lift it to $(G_D^\infty)'$, then its endpoints are its starting points plus an integer pair, $(n_1, n_2)$. Now, we would like to determine the asymptotics of $\psi_1$ and $\psi_2$ (whose derivative is the dual of $\tilde{K}^\infty$) along $S^\infty$ (a periodic lifting of $S$ to $G_D^\infty$). Expanding the cosines in exponentials, this involves adding up $|S|$ separate sequences (functions of $\ell$) of the form:

$$\sum_{\ell=1}^{n_1} e^{2\pi i [(x+\ell a)\pm(y+\ell c)]}$$

and $|S|$ sequences of the corresponding form for $m$.

Clearly, $\psi$ will remain bounded independently of $x$ and $y$, provided $a \neq \pm c \mod 2\pi$ and $b \neq \pm d \mod 2\pi$. In fact we must take the same sign for both equalities: unless $(a, b) = \pm (c, d) \mod 2\pi$ it is possible to find an independent pair of integer vectors $(m_1, n_1)$ and $(m_2, n_2)$ for which $am_1 + bn_1 \neq \pm (cm_1 + dn_1) \mod 2\pi$ and similarly $am_2 + bn_2 \neq \pm (cm_2 + dn_2) \mod 2\pi$. Taking $S_1$ and $S_2$ to be corresponding paths, we may deduce that $\psi$ is bounded unless $(\alpha, \beta)$ and $(\gamma, \delta)$ are either equal to one another or conjugates; by Lemma 5.4 $(\alpha, \beta)$ and $(\gamma, \delta)$ are either equal to one another or conjugates.

Now suppose we have $(a, b) = \pm (c, d)$. Then for the sums corresponding to steps in $S$,

$$\sum_{\ell=1}^{n_1} \cos(x + \ell a) \cos(y \pm \ell a)$$

is approximately linear as a function of $n_1$, that is, equal to a linear function plus a bounded function. If there were three linearly independent solutions $f_1, f_2, f_3$ to $Kf = 0$, and $\psi_1, \psi_2, \psi_3$ are formed using $g$ and $f_1, f_2, f_3$, then a linear combination of the $\psi_1, \psi_2, \psi_3$ would be approximately the linear function zero (i.e., bounded), a contradiction, by Lemma 5.4.

Finally, since it is clear that $(\alpha, \beta)$ is not real (i.e., not equal to $\pm 1$) for a generic choice of $\nu$, so any almost periodic $f$ or $g$ will be a strictly non-real function, that is, linearly independent from its complex conjugate, which is also a zero of $K^\infty$. $\square$

Now, Theorem 5.1 now follows immediately from Lemma 5.3 and Lemma 5.5.

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