Конечные группы с заданными системами обобщенных $\sigma$-перестановочных подгрупп

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Пусть $\sigma = \{\sigma_i | i \in I\}$ – разбиение множества всех простых чисел $\mathbb{P}$, а $G$ – конечная группа. Множество $\mathcal{H}$ подгрупп группы $G$ называется полным $\sigma$-множеством группы $G$, если каждый член $\neq 1$ из $\mathcal{H}$ является холловой подгруппой группы $G$ для некоторого $i \in I$ и $\mathcal{H}$ содержит ровно одну холлову $\sigma_i$-подгруппу группы $G$ для всех $i$ таких, что $\sigma_i \cap \pi(G) \neq \emptyset$. Группа считается $\sigma$-примарной, если она есть конечная $\sigma_i$-группа для некоторого $i$. Подгруппа $A$ группы $G$ называется $\sigma$-перестановочной в $G$, если $G$ содержит полное $\sigma$-множество $\mathcal{H}$ такое, что $AH = H'A$ для любого $H \in \mathcal{H}$ и любого $x \in G$; $\sigma$-субнормальная в $G$, если существует подгруппа цепи $A = A_0 \leq A_1 \leq \ldots \leq A_t = G$ такая, что либо $A_i = \sigma(A_i - 1)$, либо $A_i/\sigma(A_i - 1)$ является $\sigma$-примарной для всех $i = 1, \ldots, t$; $\sigma$-нормальной в $G$, если каждый главный фактор группы $G$ между $A_i$ и $A_i^G$ циклический. Мы говорим, что подгруппа $H$ группы $G$ является: (i) частично $\sigma$-перестановочной в $G$, если существуют $\sigma$-нормальная подгруппа $A$ и $\sigma$-перестановочная подгруппа $B$ из $G$ такие, что $H = <A, B>$; (ii) $\sigma$-вложенной в $G$, если существуют частично $\sigma$-перестановочная подгруппа $S$ и $\sigma$-субнормальная подгруппа $T$ из $G$ такие, что $G = HT$. 

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Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and $G$ be a finite group. A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. A group is said to be $\sigma$-primary if it is a finite $\sigma_i$-group for some $i$. A subgroup $A$ of $G$ is said to be $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; $\sigma$-subnormal in $G$ if there is a subgroup chain $A = A_0 \leq A_1 \leq \ldots \leq A_t = G$ such that either $A_{i-1} \leq A_i$ or $A_i/A_{i-1}$ is $\sigma$-primary for all $i = 1, \ldots, t$; $\sigma$-nilpotent in $G$ if every chief factor of $G$ between $A_G$ and $G^G$ is cyclic. We say that a subgroup $H$ of $G$ is: (i) partially $\sigma$-permutable in $G$ if there are a $\sigma_i$-normal subgroup $A$ and a $\sigma$-permutable subgroup $B$ of $G$ such that $H = A \times B$; (ii) $(\mathcal{U}, \sigma)$-embedded in $G$ if there are a partially $\sigma$-permutable subgroup $S$ and a $\sigma$-subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq S \leq H$. We study $G$ assuming that some subgroups of $G$ are partially $\sigma$-permutable or $(\mathcal{U}, \sigma)$-embedded in $G$. Some known results are generalised.

**Keywords:** finite group; $\sigma$-soluble groups; $\sigma$-nilpotent group; partially $\sigma$-permutable subgroup; $(\mathcal{U}, \sigma)$-embedded subgroup; $\mathcal{U}$-normal subgroup.

**Introduction**

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$.

A subgroup $A$ of $G$ is said to be $\mathcal{U}$-normal in $G$ [1] if either $A < G$ or $A_G \neq A^G$ and every chief factor of $G$ between $A_G$ and $G^G$ is cyclic.

Following L. Shemetkov [2], we use $\sigma$ to denote some partition of $\mathbb{P}$. Thus $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. The symbol $\pi(\sigma(n))$ denotes the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}; \sigma(G) = \sigma(|G|)$.

The group $G$ is said to be $[3–5]$: $\sigma$-primary if $G$ is a $\sigma_i$-group for some $i \in I$; $\sigma$-nilpotent if $G = G_1 \times \ldots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$; $\sigma$-soluble if every chief factor of $G$ is $\sigma$-primary.

A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ [6; 7] if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i$ such that $\sigma_i \cap \pi(G) \neq \emptyset$.

A subgroup $A$ of $G$ is said to be $[3]$: $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; $\sigma$-subnormal in $G$ if there is a subgroup chain $A = A_0 \leq A_1 \leq \ldots \leq A_t = G$ such that either $A_{i-1} \leq A_i$ or $A_i/A_{i-1}$ is $\sigma$-primary for all $i = 1, \ldots, t$.

Note that in the classical case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, $\sigma$-permutable subgroups are also called $S$-permutable [8; 9], and in this case $A$ is $\sigma$-subnormal in $G$ if and only if it is subnormal in $G$.

The $\sigma$-permutable and $\sigma$-subnormal subgroups were studied by a lot of authors (see, in particular, the papers [3–6; 10–29]).

In this paper we consider some applications of the following generalisation of $\sigma$-subnormal and $\sigma$-permutable subgroups.
Definition 1. We say that a subgroup $H$ of $G$ is
(i) partially $\sigma$-permutable in $G$ if there are a $\Omega$-normal subgroup $A$ and a $\sigma$-permutable subgroup $B$ of $G$
 such that $H = <A, B>$;
(ii) $(\Omega, \sigma)$-embedded in $G$ if there are a partially $\sigma$-permutable subgroup $S$ and a $\sigma$-subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq S \leq H$.

Note that every $\Omega$-normal subgroup $A = <A, 1>$ and every $\sigma$-permutable subgroup $B = <1, B>$ are partially $\sigma$-permutable in $G$. Moreover, every partially $\sigma$-permutable subgroup $S$ is $(\Omega, \sigma)$-embedded in $G$ since in this case we have $G = SG$ and $S \cap G = S \leq S$, where $G$ is a $\sigma$-subnormal subgroup of $G$ by definition.

Now we consider the following examples, which allow you to get various applications of the introduced concepts.

Example 1. (i) A subgroup $H$ of $G$ is said to be weakly $\sigma$-permutable [30] or weakly $\sigma$-quasinormal [31] in $G$ if there is a $\sigma$-subnormal subgroup $T$ and a $\sigma$-permutable subgroup $S$ of $G$ such that $G = HT$ and $H \cap T \leq S \leq H$.

Every weakly $\sigma$-quasinormal subgroup is $(\Omega, \sigma)$-embedded in the group.

(ii) A subgroup $H$ of $G$ is said to be weakly $\sigma$-permutable in $G$ [32] if there are an $S$-permutable subgroup $S$ and a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq S \leq H$. It is clear that every weakly $S$-permutable subgroup is $(\Omega, \sigma)$-embedded for every partition $\sigma$ of $G$.

(iii) Recall that a subgroup $M$ of $G$ is called modular in $G$ if $M$ is a modular element (in the sense of Kurosh [33, p. 43]) of the lattice $L(G)$ of all subgroups of $G$, that is $<X, M \cap Z> = <X, M> \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$, and $<M, M \cap Z> = <Y, M \cap Z>$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

A subgroup $H$ of $G$ is called $m$-$\sigma$-permutable in $G$ [34] if there are a modular subgroup $A$ and a $\sigma$-permutable subgroup $B$ of $G$ such that $H = <A, B>$. In view of [33, theorem 5.1.9], every modular subgroup is $\Omega$-normal in the group. Therefore, every $m$-$\sigma$-permutable subgroup is partially $\sigma$-permutable.

(iv) A subgroup $H$ of $G$ is called weakly $m$-$\sigma$-permutable in $G$ [34] if there are an $m$-$\sigma$-permutable subgroup $S$ and a $\sigma$-subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq S \leq H$. It is clear that every weakly $m$-$\sigma$-permutable subgroup is $(\Omega, \sigma)$-embedded.

(v) A subgroup $A$ of $G$ is said to be $c$-normal in $G$ [35] if for some normal subgroup $T$ of $G$ we have $AT = G$ and $A \cap T \leq A_G$. Every $c$-normal subgroup is $(\Omega, \sigma)$-embedded.

Our first observation generalises corresponding results in [34; 35].

Theorem A. (i) If every non-nilpotent maximal subgroup of $G$ is $(\Omega, \sigma)$-embedded in $G$, then $G$ is $\sigma$-soluble.

(ii) $G$ is soluble if and only if every maximal subgroup of $G$ is $(\Omega, \sigma)$-embedded in $G$ and $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ whose members are soluble groups.

In view of example 1 (iii), we get also from Theorem A the following corollary.

Corollary 1 [34, theorem B]. If every non-nilpotent maximal subgroup of $G$ is weakly $m$-$\sigma$-permutable in $G$, then $G$ is $\sigma$-soluble.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from theorem A (ii) the following know result.

Corollary 2 [35, theorem 3.1]. If every maximal subgroup of $G$ is $c$-normal in $G$, then $G$ is soluble.

Now, recall that if $M_2 < M_1 < G$ where $M_2$ is a maximal subgroup of $M_1$ and $M_1$ is a maximal subgroup of $G$, then $M_2$ is said to be a 2-maximal subgroup of $G$.

Our next theorem generalises a well-known Agrawal’s result on supersolubility of groups with $S$-permutable 2-maximal subgroups.

Theorem B. If every 2-maximal subgroup of $G$ is partially $\sigma$-permutable in $G$ and $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ whose members are supersoluble, then $G$ is supersoluble.

Corollary 3. If every 2-maximal subgroup of $G$ is $\sigma$-permutable in $G$ and $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ whose members are supersoluble, then $G$ is supersoluble.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from theorem B the following known results.

Corollary 4 [36; 37, chapter 1, theorem 6.5]. If every 2-maximal subgroup of $G$ is $S$-permutable in $G$, then $G$ is supersoluble.

Corollary 5 [38]. If every 2-maximal subgroup of $G$ is modular in $G$, then $G$ is supersoluble.

Recall that $G$ is meta-$\sigma$-nilpotent [7] if $G$ is an extension of a $\sigma$-nilpotent group by a $\sigma$-nilpotent group. An analysis of many open questions leads to the necessity of studying various classes of meta-$\sigma$-nilpotent groups (see, for example, the recent papers [3; 11–18; 30] and the survey [7]).

Our next result gives the following characterisation of meta-$\sigma$-nilpotent groups.

Theorem C. (i) The following conditions are equivalent:
(a) $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ whose members are $(\Omega, \sigma)$-embedded in $G$;
(b) G is meta-σ-nilpotent;
(c) G is σ-soluble and every σ-Hall subgroup H of G (that is σ(H) ∩ σ([G : H]) = ∅) is c-normal in G.

(ii) If G possesses a complete Hall σ-set ℍ whose members are partially σ-permutable in G, then the derived subgroup G′ of G is σ-nilpotent.

A group G is said to be: a Dσ-group if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E; a σ-full group of Sylow type [3] if every subgroup E of G is a Dσ-group for each σ ∈ σ(E).

In view of example 1 (ii) we get from theorem C the following known result.

**Corollary 6** [30, theorem 1.4]. Let G be a σ-full group of Sylow type. If every Hall σ-subgroup of G is weakly σ-permutable in G for all σ ∈ σ(G), then G is σ-soluble.

In the case when σ = {2}, {3}, ... we get from theorem C the following known result.

**Corollary 7** [39, chapter I, theorem 3.49]. G is metanilpotent if and only if every Sylow subgroup of G is c-normal.

**Proof of theorem A**

First we prove the following two lemmas.

**Lemma 1.** Let A, B and N be subgroups of G, where A is partially σ-permutable in G and N is normal in G. Then:

(1) AN/N is partially σ-permutable in G/N.
(2) If G is σ-full group of Sylow type and A ≤ B, then A is partially σ-permutable in B.
(3) If G is σ-full group of Sylow type, N ≤ B and B/N is partially σ-permutable in G/N, then B is partially σ-permutable in G.
(4) If G is σ-full group of Sylow type and B is partially σ-permutable in G, then <A, B> is partially σ-permutable in G.

**Proof.** Let A = <L, T>, where L is Λ-normal and T is σ-permutable subgroups of G.

(1) AN/N = <LN/N, TN/N>, where LN/N is Λ-normal in G/N by [40, lemma 2.8 (2)] and TN/N is σ-permutable in G/N by [3, lemma 2.8 (2)]. Hence AN/N is partially σ-permutable in G/N.

(2) This follows from [3, lemma 2.8 (1); 40 lemma 2.8].

(3) Let B/N = <V/N, W/N>, where V/N is Λ-normal in G/N and W/N is σ-permutable in G/N. Then B = <V, W>, where V is Λ-normal in G by [40 lemma 2.8 (3)] and W is σ-permutable in G. Hence B is partially σ-permutable in G.

(4) Let B = <V, W>, where V is Λ-normal and W is a σ-permutable subgroups of G. Then

<A, B> = <L, T>, <V, W> = <L, V>, <T, W>.

where <L, V> is Λ-normal in G by [40, lemma 2.8 (1)] and <T, W> is σ-permutable in G by [3, lemma 2.8 (4)]. Hence <A, B> is partially σ-permutable in G.

The lemma is proved.

**Lemma 2.** Let A, B and N be subgroups of G, where A is (Λ, σ)-embedded in G and N is normal in G.

(1) If either N ≤ A or |Λ, N| = 1, then AN/N is (Λ, σ)-embedded in G/N.
(2) If G is σ-full group of Sylow type and A ≤ B, then A is (Λ, σ)-embedded in B.
(3) If G is σ-full group of Sylow type, N ≤ B and B/N is (Λ, σ)-embedded in G/N, then B is (Λ, σ)-embedded in G.

**Proof.** Let T be a σ-subnormal subgroup of G such that AT = G and A ∩ T ≤ S ≤ A for some partially σ-permutable subgroup S of G.

(1) First note that NT ∩ NA = (T ∩ A)N. Therefore G/N = (AN/N)(TN/N) and

(AN/N) ∩ (TN/N) = (AN ∩ TN/N) = (A ∩ T)N/N ≤ SN/N,

where SN/N is a partially σ-permutable subgroup of G/N by lemma 1 (1). Hence AN/N is (Λ, σ)-embedded in G/N.

(2) B = A(B ∩ T) and (B ∩ T) ∩ A = T ∩ A ≤ S ≤ A, where S is partially σ-permutable in B by lemma 1 (2). Hence A is (Λ, σ)-embedded in B.

(3) See the proof of (1) and use lemma 1 (3).

The lemma is proved.

**Proof of theorem A.** (i) Assume that this assertion is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G.
(1) $G/R$ is $\sigma$-soluble. Hence $R$ is not $\sigma$-primary and it is a unique minimal normal subgroup of $G$.

Note that if $M/R$ is a non-nilpotent maximal subgroup of $G/R$, then $M$ is a non-nilpotent maximal subgroup of $G$ and so it is $(\mathfrak{U}, \sigma)$-embedded in $G$ by hypothesis. Hence $M/R$ is $(\mathfrak{U}, \sigma)$-embedded in $G/R$ by lemma 2 (1).

Therefore the hypothesis holds for $G/R$. Hence $G/R$ is $\sigma$-soluble and so $R$ is not $\sigma$-primary by the choice of $G$. Now assume that $G$ has a minimal normal subgroup $N \neq R$. Then $G/N$ is $\sigma$-soluble and $N$ is not $\sigma$-primary. But, in view of the $G$-isomorphism $RN/R \cong N$, the $\sigma$-solubility of $G/R$ implies that $N$ is $\sigma$-primary. This contradiction completes the proof of (1).

In view of claim (1), $R$ is not abelian. Hence $|\pi(R)| > 1$. Let $p$ be any odd prime dividing $|R|$ and $R_p$ a Sylow $p$-subgroup of $R$.

(2) If $G_p$ is a Sylow $p$-subgroup of $G$ with $R_p = G_p \cap R$, then there is a maximal subgroup $M$ of $G$ such that $RM = G$ and $G_p \leq N_G(R_p) \leq M$.

It is clear that $G_p \leq N_G(R_p)$. The Frattini argument implies that $G = RN_G(R_p)$. Conversely, claim (1) implies that $N_G(R_p) \neq G$, so for some maximal subgroup $M$ of $G$ we have $RM = G$ and $G_p \leq N_G(R_p) \leq M$.

(3) $M$ is not nilpotent and $M_{\sigma} = 1$. Hence $M$ is $(\mathfrak{U}, \sigma)$-embedded in $G$.

Assume that $M$ is nilpotent, and let $D = M \cap R$. Then $D$ is a normal subgroup of $M$ and $R_p$ is a Sylow $p$-subgroup of $D$ since $R_p \leq G_p \leq M$. Hence $R_p$ is characteristic in $D$ and so it is normal in $M$. Therefore $Z(J(R_p))$ is normal in $M$. Claims (1) and (2) imply that $M_{\sigma} = 1$. Hence $N_G(Z(J(R_p))) = M$ and so $N_p(Z(J(R_p))) = D$ is nilpotent. This implies that $R$ is $p$-nilpotent by the Glauberman – Thompson theorem on the normal $p$-complements. But then $R$ is a $p$-group, contrary to claim (1). Hence we have (3).

(4) There is a $\sigma$-subnormal subgroup $T$ of $G$ such that $MT = G$, $M \cap T = 1$ and $p$ does not divide $|T|$.

By claim (3), there are a partially $\sigma$-permutable subgroup $S$ and a $\sigma$-subnormal subgroup $T$ of $G$ such that $G = MT$ and $M \cap T \leq S \leq M$. Then $S = <A, B>$ for some $\mathfrak{U}$-normal subgroup $A$ and $\sigma$-permutable subgroup $B$ of $G$. Moreover, from the definition $\mathfrak{U}$-normality and claim (1) it follows that, in fact, $S = B$ and $S_G = 1$. Suppose that $S \neq 1$. Then for every $\sigma \in \sigma(S)$ we have $S = S_{\sigma}(S) \times S_{\sigma}(S)$ by [3, theorem B]. Therefore for every Hall $\sigma$-subgroup $H$ of $G$ from $SH = HS = S_{\sigma}(S)H$ we get that $1 < S_{\sigma}(S) \leq H_G$, contrary to claim (1). Therefore $S = 1$, so $T \cap M = 1$. Hence $|T| = |G : M|$, so $p$ does not divide $|T|$ since $G_p \leq M$ by claim (2).

The final contradiction for (i). Let $L$ be a minimal $\sigma$-subnormal subgroup of $G$ contained in $T$. Then $L$ is a simple group. If $L$ is a $\sigma_i$-group for some $i$, then $L \leq O_{\sigma_i}(G)$ by [12, lemma 2.2 (10)], which is impossible by claim (1).

Hence $L$ is non-abelian, so it is subnormal in $G$ by [12, lemma 2.2 (7)]. Suppose that $L \nsubseteq R$. Then $L \cap R = 1$. Conversely, $R \leq N_G(L)$ by [41, chapter A, theorem 14.3]. Hence $LR = L \times R$, so $L \leq C_G(R)$. But claim (1) implies that $R \nsubseteq C_G(R)$ and so $C_G(R) = 1$, a contradiction. Hence $L$ is a minimal normal subgroup of $R$. It follows that $p$ divides $|L|$ and hence $p$ divides $|T|$, contrary to claim (4). Therefore assertion (i) is true.

(ii) In view of theorem A, it is enough to show that if $G$ is soluble, then every maximal subgroup $M$ of $G$ is $(\mathfrak{U}, \sigma)$-embedded in $G$. If $M_G \neq 1$, then $M/M_G$ is $(\mathfrak{U}, \sigma)$-embedded in $G/M_G$ by induction, so $M$ is $(\mathfrak{U}, \sigma)$-embedded in $G$ by lemma 2 (3). Conversely, if $M_G = 1$ and $R$ is a minimal normal subgroup of $G$, then $R$ is abelian and so $G = R \rtimes M$. Hence $M$ is $(\mathfrak{U}, \sigma)$-embedded in $G$.

The theorem is proved.

Proof of theorem B

Lemma 3 [6, theorem A]. If $G$ is $\sigma$-soluble, then $G$ is a $\sigma$-full group of Sylow type.

Lemma 4. If $G$ is $\sigma$-soluble and $G$ possesses a complete Hall $\sigma$-set whose members are $p$-soluble, then $G$ is $p$-soluble.

Proof. Suppose that this lemma is false and let $G$ be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, ..., H_i\}$ be a complete Hall $\sigma$-set of $G$. Then $H_i$ is $p$-soluble by lemma 3 for all $i$.

First show that if $R$ is minimal normal subgroup of $G$, then $G/R$ is $p$-soluble. It is enough to show that the hypothesis holds for $G/R$.

Note that for every chief factor $(H/R)/(K/R)$ of $G/R$ we have that $(H/R)/(K/R) = G/HK$, where $H/K$ is a chief factor of $G$ and $H/K$ is $\sigma$-primary since $G$ is $\sigma$-soluble. So $(H/R)/(K/R)$ is $\sigma$-primary, hence $G/R$ is $\sigma$-soluble.

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Note also that \( \{H_iR/R, \ldots, H_iR/R\} \) is a complete Hall \( \sigma \)-set of \( G/R \), where \( H_iR/R = H_i/(H_i \cap R) \) is \( p \)-soluble since \( H_i \) is \( p \)-soluble. Therefore the hypothesis holds for \( G/R \), so \( G/R \) is \( p \)-soluble by the choice of \( G \).

Now show that \( R \) is \( p \)-soluble. Since \( G \) is \( \sigma \)-soluble, \( R \) is \( \sigma \)-primary, that is, \( \sigma \)-group for some \( i \). Also, for every Hall \( \sigma_i \)-subgroup \( H \) of \( G \) we have \( R \leq H \). So, \( R \) is \( p \)-soluble by the hypothesis, hence \( G \) is \( p \)-soluble.

The lemma is proved.

**Proof of theorem B.** Suppose that this theorem is false and let \( G \) be a counterexample of minimal order. Let \( \mathcal{H} = \{H_1, \ldots, H_t\} \). Then \( t > 1 \) since \( H_1 \) is \( p \)-soluble by hypothesis.

1. If \( R \) is minimal normal subgroup of \( G \), then \( G/R \) is supersoluble. Hence \( R \) is the unique minimal normal subgroup of \( G \), \( R \) is not cyclic and \( R \not\leq \Phi(G) \).

   It is enough to show that the hypothesis holds for \( G/R \). First note that \( \{H_iR/R, \ldots, H_iR/R\} \) is a complete Hall \( \sigma \)-set of \( G/R \), where \( H_iR/R = H_i/(H_i \cap R) \) is \( p \)-soluble since \( H_i \) is \( p \)-soluble by hypothesis.

   Now assume that statement (1) is false. Then \( G/R \) is not nilpotent, so every Sylow \( p \)-subgroup in \( G/R \) is proper. Then for every Sylow \( p \)-subgroup \( P \) of \( G/R \) it follows that \( P \) is contained in some maximal subgroup of \( G/R \). Hence \( R \) is contained in some 2-maximal subgroup \( T \) of \( G \) and so \( T \) is partially \( \sigma \)-permutable in \( G \) by the hypothesis. But then \( T/R \) is 2-maximal of \( G/R \) and partially \( \sigma \)-permutable in \( G/R \) by lemma 1 (1). Therefore the hypothesis holds for \( G/R \), so \( G/R \) is supersoluble by the choice of \( G \). Then we have a contradiction.

   Moreover, it is well-known that the class of all supersoluble groups is a saturated formation [42 chapter VI, definition 8.6]. Hence the choice of \( G \) implies that \( R \) is the unique minimal normal subgroup of \( G \), \( R \) is not cyclic and \( R \not\leq \Phi(G) \). Hence we have (1).

2. \( G \) is solvable.

   Every 2-maximal subgroup in \( G \) is partially \( \sigma \)-permutable and so, partially \( \sigma \)-subnormal in \( G \) by [3, theorem B]. Then, in view of theorem in [40], \( G \) is \( \sigma \)-soluble. Hence, from lemma 4 it follows that \( G \) is soluble. So we have (2).

3. \( R = O_p(G) \not\leq \Phi(G) \) for some prime \( p \in \sigma \). Hence for some maximal subgroup \( M \) of \( G \) we have \( G = R \rtimes M \) and \( M \neq M_1 = 1 \).

   By claim (2), \( G \) is solvable and so \( R \) is a \( p \)-group for some \( p \in \sigma \). Hence the choice of \( G \) and claim (1) imply that \( R \) is a unique minimal normal subgroup of \( G \). Moreover, \( R \not\leq \Phi(G) \) by claim (1), so \( R = C_G(R) = O_p(G) \) by [41, chapter A, lemma 15.2]. Hence for some maximal subgroup \( M \) of \( G \) we have \( G = R \rtimes M \) and \( M \neq M_1 = 1 \) by claim (1).

4. If \( 1 < H \leq M \), then \( H \) is not \( \Delta \)-normal in \( G \).

   Indeed, if \( H \) is \( \Delta \)-normal in \( G \), then \( H^G/H_G \leq Z_\Delta(G/H_G) \), where \( H_G = 1 \) by claim (3). Hence \( R \leq H^G \leq Z_\Delta(G) \) by claim (1). But then \( R \) is cyclic, contrary to claim (1). This contradiction completes the proof of the claim.

5. \( M \) is not a group of prime order:

   Suppose that \( |M| = q \) for some prime \( q \). Hence \( |M| = |G : R| \) is a prime and so \( R \) is a maximal subgroup of \( G \). Then every maximal subgroup \( V \) of \( R \) is 2-maximal in \( G \), so \( V \) is partially \( \sigma \)-permutable in \( G \) by hypothesis. So \( V = A \times B \), where \( A \) is \( \Delta \)-normal and \( B \) is \( \sigma \)-permutable in \( G \). Assume \( A \neq 1 \). Note \( A_G = 1 \) by the minimality of \( R \). Then \( R \leq A^G \leq Z_\Delta(G) \) and so \( R \) is cyclic, contrary to claim (1). Hence \( V = B \) is \( \sigma \)-permutable in \( G \). Therefore every maximal subgroups of \( R \) is \( \sigma \)-permutable in \( G \).

   Note that \( R \leq H_i \) since \( R \) is \( \sigma \)-group by claim (3) and \( H_i = R \rtimes (H_i \cap M) \), again by claim (3). Since \( H_i \) is super-soluble by hypothesis, some maximal subgroup \( W \) of \( R \) is normal in \( H_i \). In addition, \( W \) is \( \sigma \)-permutable in \( G \) since it is a maximal subgroup of \( R \). Hence for each \( j \neq i \) we have \( WH_j = H_jW_j \), which implies that \( H_j \leq N_G(W) \) since \( R \cap WH_j = W \cap H_j = W \). Therefore \( W \) is normal in \( G \), so the minimality of \( R \) implies that \( W = 1 \) and hence \( |R| = p \), which is impossible by claim (3). Hence we have (5).

6. If \( T \) is a maximal subgroup of \( M \), then \( T^G \) is a \( \sigma \)-subgroup of \( G \).

   Indeed, \( T \) is partially \( \sigma \)-permutable in \( G \) by hypothesis, so \( T = A \times B \) for some \( \Delta \)-normal subgroup \( A \) and some \( \sigma \)-permutable subgroup \( B \) of \( G \). Note that \( T \neq 1 \) by claim (5). Conversely, \( A = 1 \) by claim (4) and so \( T = B \) is \( \sigma \)-permutable in \( G \). Therefore \( T^G/T_G \) is \( \sigma \)-nilpotent [3, theorem B (ii)]. We have \( T_G \leq M_G = 1 \), so \( T^G/T_G \simeq T^G/T_G \simeq T^G/1 \simeq T^G \) is \( \sigma \)-nilpotent group. Hence the subgroup \( O_{\sigma_i}(T^G) \) is characteristic in \( T^G \), so it is normal in \( G \). By claim (3) we have that \( O_{\sigma_i}(T^G) = 1 \) for all \( k \neq i \). Hence \( T^G = O_{\sigma_i}(T^G) \) is a \( \sigma_i \)-subgroup of \( G \).

7. \( M \) is not \( \sigma_i \)-group.

   Suppose that this is false and let \( T \) be a maximal subgroup of \( M \). Then \( T \neq 1 \) by claim (5). Conversely, \( T \) is a \( \sigma_i \)-group by the hypothesis and \( T^G \) is a \( \sigma_i \)-subgroup of \( G \) by claim (6). Then we have a contradiction. Hence, \( M \) is not a \( \sigma_i \)-group.
(8) $M$ is not $\sigma_i$-group (this follows from the facts that $t > 1$ and $R$ is a $\sigma_i$-group).

**Final contradiction.**

Let $T$ be a maximal subgroup of $M$, containing a Hall $\sigma_j$-subgroup of $M$. Then $T^G$ is $\sigma_j$-group by claim (6). Therefore, a Hall $\sigma_j$-subgroup of $M$ is the identity group. Hence $M$ is $\sigma_j$-group, contrary to claim (8). This contradiction completes the proof of the result.

**Proof of theorem C**

We use $\mathfrak{H}_a$ to denote the class of all $\alpha$-nilpotent groups.

**Lemma 5** [3, corollary 2.4 and lemma 2.5]. (1) The class $\mathfrak{H}_a$ is closed under taking products of normal subgroups, homomorphic images and subgroups.

(2) If $G/N$ and $G/R$ are $\sigma_\alpha$-nilpotent, then $G/(N \cap R)$ is $\sigma_\alpha$-nilpotent.

(3) If $E$ is a normal subgroup of $G$ and $E/(E \cap \Phi(G))$ is $\sigma_\alpha$-nilpotent, then $E$ is $\sigma_\alpha$-nilpotent.

Recall that $G^{\sigma_\alpha}$ denotes the $\sigma_\alpha$-nilpotent residual of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma_\alpha$-nilpotent quotient $G/N$. In view of [43, proposition 2.2.8], we get from lemma 5 (1) the following result.

**Lemma 6.** If $N$ is a normal subgroup of $G$, then $(G/N)^{\sigma_\alpha} = G^{\sigma_\alpha} N/N$.

The next lemma is proved by the direct verifications on the basis of lemmas 5 and 6.

**Lemma 7.** (1) $G$ is meta-$\sigma_\alpha$-nilpotent if and only if $G^{\sigma_\alpha}$ is $\sigma_\alpha$-nilpotent.

(2) If $G$ is meta-$\sigma_\alpha$-nilpotent, then every quotient $G/N$ of $G$ is meta-$\sigma_\alpha$-nilpotent.

(3) If $G/N$ and $G/R$ are meta-$\sigma_\alpha$-nilpotent, then $G/(N \cap R)$ is meta-$\sigma_\alpha$-nilpotent.

(4) If $E$ is a normal subgroup of $G$ and $E/(E \cap \Phi(G))$ is meta-$\sigma_\alpha$-nilpotent, then $E$ is meta-$\sigma_\alpha$-nilpotent.

**Lemma 8.** Let $A$, $B$ and $N$ be subgroups of $G$, where $A$ is c-normal in $G$ and $N$ is normal in $G$.

(1) If $N \vartriangleleft A$ or $([A], [N]) = 1$, then $AN/N$ is $c$-normal in $G/N$.

(2) If $N \vartriangleleft B$ and $B/N$ is $c$-normal in $G/N$, then $B$ is $c$-normal in $G$.

**Proof.** See the proof of lemma 2.

A natural number $n$ is said to be a $\Pi$-number if $\sigma(n) \subseteq \Pi$. A subgroup $A$ of $G$ is said to be: a Hall $\Pi$-subgroup of $G$ [6; 7] if $[A]$ is a $\Pi$-number and $|G/A|$ is a $\Pi$-number; a $\sigma$-Hall subgroup of $G$ if $A$ is a Hall $\Pi$-subgroup of $G$ for some $\Pi \subseteq \sigma$.

Recall also that a normal subgroup $E$ of $G$ is called hypercyclically embedded in $G$ [33, p. 217] if every chief factor of $G$ below $E$ is cyclic.

**Proof of theorem C.** Let $D = G^{\sigma_\alpha}$ be the $\sigma_\alpha$-nilpotent residual of $G$.

(i) (a) $\Rightarrow$ (b). Assume that this is false and let $G$ be a counterexample of minimal order. Then $D$ is not $\sigma_\alpha$-nilpotent since $G/D$ is $\sigma_\alpha$-nilpotent by lemma 5 (2). Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that $H_i$ is a $\sigma_i$-group for all $i = 1, \ldots, t$. Let $S_i$ be a partially $\sigma_i$-permutable subgroup and $T_j$ be a subnormal subgroup of $G$ such that $S_i \leq H_j$, $H_j T_j = G$ and $H_j \cap T_j \leq S_i$ for all $i = 1, \ldots, t$. Then, for every $i$, $S_i = \langle A_i, B_i \rangle$ for some $\Phi$-normal subgroup $A_i$ and $\sigma_i$-permutable subgroup $B_i$ of $G$.

(1) If $R$ is a $\sigma$-primary minimal normal subgroup of $G$, then $G/R$ is meta-$\sigma_\alpha$-nilpotent and so $G$ is $\sigma$-soluble.

Moreover, $R$ is a unique minimal normal subgroup of $G$. $C_G(R) \vartriangleleft R$ and $R$ is not cyclic.

First we show that $G/R$ is meta-$\sigma_\alpha$-nilpotent. In view of the choice of $G$, it is enough to show that the hypothesis holds for $G/R$. Since $R$ is $\sigma$-primary, for some $i$ we have $R \vartriangleleft H_i$ and $|R|, |H_j| = 1$ for all $j \neq i$. Therefore $\{H_i/R, H_j/R\}$ is a complete Hall $\sigma$-set of $G/R$ whose members are $(\Phi, \sigma)$-embedded in $G/R$ by lemma 2 (1). Hence the hypothesis holds for $G/R$, so $G/R$ is meta-$\sigma_\alpha$-nilpotent and $G$ is $\sigma$-soluble. Hence every minimal normal subgroup of $G$ is $\sigma$-primary, so $R$ is a unique minimal normal subgroup of $G$ and $R \not\subseteq \Phi(G)$ by lemma 7 (4). Hence $C_G(R) \vartriangleleft R$ by [41, chapter A, lemma 15.6]. Finally, note that in the case when $R$ is cyclic we have $|R| = p$ for some prime $p$ and so $G/C_G(R) = G/R$ is cyclic, which implies that $G$ is metanilpotent and so it is meta-$\sigma_\alpha$-nilpotent. Therefore we have (1).

(2) For some $i$, $i = 1$ say, we have $S_i = S_i \neq 1$. Moreover, if for some $k$ we have $S_k = 1$, then $T_k$ is a normal complement to $H_k$ in $G$.

Assume that $S_i = 1$. Then $H_i \cap T_i = 1$, so $T_i$ is a $\sigma$-subnormal Hall $\sigma_i$-subgroup of $G$. Hence $T_i$ is a normal complement to $H_i$ in $G$ by [3, lemma 2.6 (10)]. Moreover, $G/T_i \simeq H_i$ is $\sigma$-nilpotent. Suppose that $S_i = 1$ for all $i = 1, \ldots, t$. Then $T_1 \cap \ldots \cap T_t = 1$ by [41, chapter A, theorem 1.6 (b)], so

$$G = G/1 = G/(T_1 \cap \ldots \cap T_t)$$

is $\sigma$-nilpotent by lemma 5 (2). Then we have a contradiction. Hence for some $i$ we have $S_i \neq 1$. This completes the proof.

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(3) If $S_i \neq 1$, then $(H_i)_{G_i} \neq 1$.

Assume that this is false. Then every non-identity subgroup $L$ of $H_i$ is not $\sigma$-permutable in $G$ since otherwise for every $x \in G$ we have $L^x = H_i^x L = H_i^y$ which implies that $1 < L \leq (H_i)_{G_i} = 1$.

Therefore $B_1 = 1$ and so $S_1 = A_i$ is a $\Delta$-normal subgroup of $G$ with $(S_1)_{G_1} = 1$. But then we have $1 < (S_i)_{G_i}$ is hypercyclically embedded in $G$ by the definition $\Delta$-normality and so $R$ is cyclic, contrary to claim (1). Hence we have (3).

(4) $G$ possesses a $\sigma$-primary minimal normal subgroup, $R$ say.

Claims (2) and (3) imply that $(H_i)_{G_i} \neq 1$. Therefore, if $R$ is a minimal normal subgroup of $G$ contained in $(H_i)_{G_i}$, then $R$ is $\sigma$-primary.

The final contradiction for the implication $(a) \Rightarrow (b)$. Claims (1) and (4) imply that $G$ is $\sigma$-soluble, so $R$ is a unique minimal normal subgroup of $G$ by claim (1). Hence claims (2) and (3) imply that $T_1, \ldots, T_t$ are normal subgroups of $G$ and $G/T_k \cong H_k$ for all $k = 2, \ldots, t$. Hence $G/(T_2 \cap \ldots \cap T_t)$ is $\sigma$-nilpotent by lemma 5 (2). Conversely, $T_2 \cap \ldots \cap T_t = H_i$ by [41, chapter A, theorem 1.6 (b)] and so $G$ is meta-$\sigma$-nilpotent, contrary to the choice of $G$. This contradiction completes the proof of the $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$. The subgroup $D$ is $\sigma$-nilpotent by lemma 7 (1). Let $\Pi = \sigma(H)$. Then $H$ is a Hall $\Pi$-subgroup of $G$.

Suppose that $H_G \neq 1$. Then $H/H_G$ is $c$-normal in $G$ by induction since the hypothesis holds for $G/H_G$ by lemma 7 (2). Hence $H$ is $c$-normal in $G$ by lemma 8 (2).

Now assume that $H_G = 1$. Then, since $D$ is $\sigma$-nilpotent, it follows that $D \cap H = 1$. Conversely, $G/D$ is $\sigma$-nilpotent by lemma 5 (2) and $H = HD/D$ is a Hall $\Pi$-subgroup of $G/D$, so $HD/D$ has a normal complement $T/D$ in $G/D$. Then $T$ is a normal subgroup of $G$ such that $HT = G$ and $T \cap H \leq T \cap HD \leq H \leq D \cap H = 1$. Hence $H$ is $c$-normal in $G$. Therefore the implication $(b) \Rightarrow (c)$ holds.

$(c) \Rightarrow (b)$. In view of example 1 (v), this application is a corollary of the implication $(a) \Rightarrow (b)$.

(ii) Suppose that this assertion is false and let $G$ be a counterexample of minimal order. Then $G$ is not $\sigma$-nilpotent, so $\sigma(G) > 1$. Moreover, from part (i) we know that $D$ is $\sigma$-nilpotent and so $G$ is $\sigma$-soluble. Let $H = \{H_i, \ldots, H_n\}$.

Let $R$ be a minimal normal subgroup of $G$. Then $R$ is a $\sigma_i$-group for some $i$, so the hypothesis holds for $G/R$ by lemma 1 (1). Hence $(G/R)_{c}(G/R) = G'/(G' \cap R)$ is $\sigma$-nilpotent by the choice of $G$. Therefore $R \leq G'$ and $R \not\leq \Phi(G)$ by lemma 5 (3). Moreover, if $G$ has a minimal normal subgroup $N \neq R$ of $G$, then $N \leq G'$ and $G'/1 = G'/1(R \cap N)$ is $\sigma$-nilpotent, contrary to the choice of $G$. Therefore $R$ is a unique minimal normal subgroup of $G$ and $C_G(R) \leq R$ by [41, chapter A, lemma 16.5]. We can assume without loss of generality that $R \leq H_i$.

Let $M$ be a maximal subgroup of $G$ such that $R \not\leq M$. Then $M_G = 1$ and $|G : M|$ is a $\sigma_i$-number. Therefore for some $x \in G$ we have $H = H_i^x \leq M$. Then $H = A_i = B_i$ for some $\Delta$-normal subgroup $A$ and $\sigma$-permutable subgroup $B$ of $G$. Moreover, $A_i \leq M_G = 1$, so $A_i$ is hypercyclically embedded in $G$ by the definition $\Delta$-normality. If $A \neq 1$, then $R \leq A^x$ and so $[R] = p$ for some prime $p$. But then $C_G(R) = R$ and $G/R = G/C_G(R)$ is cyclic. Hence $G'$ is nilpotent. This contradiction shows that $A = 1$, so $H = B$ is $\sigma$-permutable in $G$. But then $H/H_i = H'/H = H'$ for all $x \in G$ since $H$ is a Hall $\sigma_i$-subgroup of $G$ for some $i$. Hence $H$ is normal in $G$, so $1 < H \leq M_G$, a contradiction. Therefore assertion (ii) is true.

The theorem is proved.

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