PARABOLIC CONSTRUCTIONS OF ASYMPTOTICALLY FLAT 3-METRICS OF PRESCRIBED SCALAR CURVATURE

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Abstract. In 1993, Bartnik [3] introduced a quasi-spherical construction of metrics of prescribed scalar curvature on 3-manifolds. Under quasi-spherical ansatz, the problem is converted into the initial value problem for a semi-linear parabolic equation of the lapse function. The original ansatz of Bartnik started with a background foliation with round metrics on the 2-sphere leaves. This has been generalized by several authors [10, 14, 12] under various assumptions on the background foliation. In this article, we consider background foliations given by conformal round metrics, and by the Ricci flow on 2-spheres. We discuss conditions on the scalar curvature function and on the foliation that guarantee the solvability of the parabolic equation, and thus the existence of asymptotically flat 3-metrics with a prescribed inner boundary. In particular, many examples of asymptotically flat-scalar flat 3-metrics with outermost minimal surfaces are obtained.

1. Introduction

Einstein’s field equation of a space-time \((V, \gamma)\) is

\[
R^V_{ab} - \frac{1}{2} R^V \gamma_{ab} = 8 \pi T_{ab}, \quad a, b = 0, 1, 2, 3,
\]

where \(T_{ab}\) is the space-time energy momentum tensor. The equation admits Cauchy data formulation and initial data cannot be chosen arbitrarily. Let \((V, \gamma)\) be a solution and consider a space-like hypersurface \((N, g)\). From the Gauss and Codazzi equations the scalar curvature \(\bar{R}\) and the second fundamental form \(k\) of \((N, g)\) will satisfy the following constraint equations [15]:

\[
\bar{R} + (\text{tr} k)^2 - |k|^2_g = 16 \pi T_{00},
\]

\[
\nabla_j (k_{ij} - \text{tr} g_{ij}) = 8 \pi T_{0i} \quad i, j = 1, 2, 3,
\]

where \(e_0\) is the future time-like unit normal vector of the hypersurface \(N\). When \(T = 0\), these equations are called the vacuum constraint equations. There are various ways to construct solutions of the constraint equations. In 1993 Bartnik [3] introduced a new construction of 3-metrics of prescribed scalar curvature and prescribed the inner boundary using a quasi-spherical ansatz. A manifold \(N\) is called quasi-spherical if it can be foliated by round spheres. Under quasi-spherical ansatz, the equation for the scalar curvature \(\bar{R}\) can be written as a semilinear parabolic equation. Let \(\Sigma\) be a smooth compact surface without boundary. Let \(N = [a, \infty) \times \Sigma\) be equipped with a quasi-spherical metric

\[
\bar{g} = u^2 dt^2 + \sum_{i=1}^{2} (\beta_i dt + t \sigma_i)^2
\]
for some functions $u$ and $\beta_i$, $i = 1, 2$, where $\sigma_i^2$ is the standard metric on the unit 2-sphere. By viewing $u$ as an unknown function and the scalar curvature $\bar{R}$ of $(N, \bar{g})$ and $\beta_i$ as prescribed fields, Bartnik [3] observed that the function $u$ is given by

$$2\frac{\partial u}{\partial t} - 2\beta_i u_i = \gamma u^2 \Delta u + (1 + \gamma B) u - \gamma \left(1 - \frac{1}{2} \bar{R} t^2\right) u^3,$$

where $\gamma_i = \nabla_i$, denotes the covariant derivative of $du^2$, $\gamma = \left(1 - \frac{1}{2} \text{div} \beta\right)^{-1}$, and $B = \frac{3}{2} |\text{div} \beta|^2 + \frac{3}{2} [\beta_{ij} + \beta_{ji}]^2 - t \partial_t (\text{div} \beta) + \beta_i (\text{div} \beta)_i - \frac{3}{2} \text{div} \beta$.

Bartnik’s parabolic method under quasi-spherical ansatz has been generalized by several authors. In 2002, Shi and Tam [10] used the foliation by level sets of the distance function to a convex hypersurface in $\mathbb{R}^n$. Let $\Sigma_0$ be a smooth compact strictly convex hypersurface in $\mathbb{R}^n$, and $t$ the distance function from $\Sigma_0$. The metric $\bar{g}$ on $N = [a, \infty) \times \Sigma$ is of the form

$$\bar{g} = u^2 dt^2 + g_t,$$

where $g_t$ is the induced metric on $\Sigma_t$, which is the hypersurface with distance $t$ from $\Sigma_0$. The function $u$ with prescribed flat-scalar curvature $\bar{R} = 0$ satisfies the equation

$$2H_0 \frac{\partial u}{\partial t} = 2u^2 \Delta_t u + (u - u^3) R_t,$$

where $H_0$ is the mean curvature of $\Sigma_t$ in $\mathbb{R}^n$, $R_t$ is the scalar curvature of $\Sigma_t$, and $\Delta_t$ is the Laplacian on $\Sigma_t$. Shi and Tam showed that for a smooth compact strictly convex hypersurface $\Sigma_0$ in $\mathbb{R}^n$, a positive function $u$ can be arbitrarily prescribed initially.

Weinstein and Smith studied this parabolic method under quasi-convex foliations in [13, 14]. A topological 2-sphere is said to be quasi-convex if its Gauss and mean curvature are positive. A foliation is quasi-convex if its leaves are quasi-convex spheres. Using the Poincaré Uniformization, $\bar{g}$ on $N = [a, \infty) \times \Sigma$ can be written as

$$\bar{g} = u^2 dt^2 + e^\theta g_{ij} \left(\beta^i dt + t d\theta^i\right) \left(\beta^j dt + t d\theta^j\right),$$

where $(\theta^1, \theta^2)$ are local coordinates on a topological 2-sphere $\Sigma$ and $g_{ij}$ is a fixed round metric of area $4\pi$. The parabolic equation for $u$ on $\Sigma$ is given by

$$t \frac{\partial u}{\partial t} - \beta \cdot \nabla u = \Gamma u^2 \Delta u + A u - B u^3,$$

where $\beta = e^{2 \nu} \beta^i \partial_i$, and $\Gamma$, $A$, and $B$ are functions that can be calculated in terms of only $\beta, \nu$, and $t$. Weinstein and Smith derived conditions on the source functions $\beta, \Gamma, A$, and $B$ from above that guarantee the existence of a global positive solution on the interval $[a, \infty)$. However, sometimes the decay conditions may not be verified.

For initial data with an apparent horizon, the parabolicity of the parabolic equation breaks down on the horizon. To overcome this, Smith [12] considered the metrics of the form

$$\bar{g} = \frac{u^2}{1 - \frac{r}{t}} dt^2 + \bar{g}(t),$$

on $N = [a, \infty) \times \Sigma$, where $(\Sigma, h)$ is a given Riemannian surface, $g(t)$ satisfies

$$a^2 g(a) = h, \quad \frac{\partial g_{ij}}{\partial t} g^{ij} > -4,$$
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$$g(t) = g(a) \text{ for } t \in [a, a + \epsilon),$$

$$g(t) = \sigma \text{ for } t \text{ large},$$

where $\sigma$ is the standard metric on $S^2$, and the scalar curvature $R(t)$ of $g(t)$ is positive. The second condition, $g(t) = g(a)$ for $t \in [a, a + \epsilon)$, allows the separation of variables so that solving the parabolic scalar curvature equation reduces to solving the elliptic equation

$$\Delta_g u - \frac{R}{2} u + \frac{1}{u} = 0$$

on the region $[a, a + \epsilon] \times \Sigma$, which is referred as the collar region. He then obtained asymptotically flat time symmetric initial data on $N$ by constructing the metric on the collar region, and the metric exterior to the collar region using the parabolic method.

This parabolic construction also provides an insight into the extension problem, which is suggested by the definition of quasi-local mass [2]. Here one hopes to extend a bounded Riemannian 3-manifold $(\Omega, g_0)$ to an asymptotically flat 3-manifold $(M, g)$ with nonnegative scalar curvature containing $(\Omega, g_0)$ isometrically. The condition that the scalar curvature can be defined distributionally and bounded across $\partial(M \setminus \Omega)$ leads to the geometric boundary conditions

$$g|_{\partial(M \setminus \Omega)} = g_0|_{\partial \Omega}, \ H|_{\partial(M \setminus \Omega), g} = H|_{\partial \Omega, g_0},$$

where the metrics and the mean curvatures in $(\Omega, g_0)$ and $(M, g)$ match along the boundary $\partial(M \setminus \Omega)$. Note that this extension is not possible with the traditional conformal method [4, 9]. Specifying both the boundary metric and the mean curvature leads to simultaneous Dirichlet and Neumann boundary conditions, which are ill-posed for the elliptic equation of the conformal factor.

In this paper, we let $e^{2f}\sigma$ be a family of metrics on a topological sphere $\Sigma$ where $\sigma$ is the standard metric on the sphere. Let $N = [1, \infty) \times \Sigma$, $R$ be a given function on $N$, and

$$\bar{g} = u^2 dt^2 + t^2 e^{2f} \sigma$$

be a metric on $N$ with scalar curvature $\bar{R}$. The metric $\bar{g}$ has the scalar curvature $\bar{R}$ if and only if $u$ satisfies the parabolic equation

$$\left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \frac{\partial u}{\partial t} = \frac{1}{2t^2} u^2 \Delta_f u + \left( \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + \frac{3}{2} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2 \right) u$$

$$- \frac{1}{4t^2} \left( R_f - t^2 \bar{R} \right) u^3,$$

where $R_f$ and $\Delta_f$ denote the scalar curvature and the Laplacian with respect to $e^{2f}\sigma$, respectively.

First, we study decay conditions of the foliation and prescribed scalar curvature which ensure the solution $u$ gives an asymptotically flat metric with prescribed scalar curvature (Theorem 1). Second, with suitable decay conditions we show that there exists a solution $u^{-1} \in C^{2+\alpha}(N)$ so that the metric $\bar{g}$ has outermost totally geodesic boundary. Instead of assuming a collar region, we use the dilation invariance of weighted Hölder norms together with suitable curvature conditions to obtain uniform bounds of solutions to the initial value problem (1) with initial condition $u^{-1}(1 + \epsilon_\epsilon, \cdot) \cdot 1 + \epsilon, \infty)$. By Arzela-Ascoli Theorem, there exists a weak solution to (1) with $u^{-1}(1, \cdot) = 0$ (Theorem 2). Since the mean curvature...
\( H = \frac{2}{u} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \) of \( \Sigma_t = \{ t \} \times \Sigma \) stays positive, by the maximum principle the boundary surface is the outermost minimal surface. Theorem 1 and Theorem 2 together give an initial data set of prescribed geometry with a horizon. Last, we study existence results under Ricci flow foliation. It is known by the work of R. Hamilton \[6\] and B. Chow \[5\] that the evolution under Ricci flow of an arbitrary initial metric on a topological 2-sphere, suitably normalized, exists for all time and converges exponentially to a round metric. Given a compact Riemannian surface \((\Sigma, g_1)\), let

\[ \bar{g} = u^2 dt^2 + t^2 g(t) \]

be a metric on \( N = [1, \infty) \times \Sigma \) where \( g(t) \) evolves by the Ricci flow defined in \[5\]. Using the fast convergence property, we have corresponding existence results (Theorem 3 and Theorem 4) under Ricci flow foliation. Since \( \Sigma_t \) are nearly round, the ADM mass of the asymptotically flat manifold \( N \) is approached by the Hawking mass \( m_H(\Sigma_t) \) as \( t \to \infty \) (see \[11\]). If in addition the prescribed scalar curvature \( \bar{R} \) is nonnegative, by the equation (11) of \( w = u - 2 \), a direct computation shows that

\[ \frac{d}{dt} m_H(\Sigma_t) = \frac{1}{8\pi} \int w|\nabla u|^2 + \frac{t^2}{2} |M_{ij}|^2 w + \frac{t^2}{2} \bar{R} d\sigma \geq 0, \]

where \( M_{ij} \) is the trace-free part of the Ricci potential. We obtain an interesting byproduct, the Hawking mass \( m_H(\Sigma_t) \) is nondecreasing in \( t \). If we impose the initial condition \( u(1, \cdot)^{-1} = 0 \), i.e., minimal boundary surface, and assume that \( \bar{R} \geq 0 \), then ADM mass is bounded below by

\[ m_{ADM}(N) \geq \frac{1}{2} \sqrt{\frac{A(\Sigma)}{4\pi}} = \frac{1}{2}. \]

In particular, if we start with the standard metric \((\Sigma, \sigma)\) and prescribe flat scalar curvature \( \bar{R} \equiv 0 \), then the metric \( \bar{g} \) obtained from above would be exactly a Schwarzschild metric with ADM mass \( m_{ADM} = \frac{1}{2} \) (Corollary 14).

Let \( A_I = I \times \Sigma \) for any interval \( I \subset [1, \infty) \). For compact intervals \( I \subset [1, \infty) \), the parabolic Hölder space \( C^{k+\alpha}(A_I) \) is the Banach space of continuous functions on \( A_I \) with finite weighted \( \| \cdot \|_{k+\alpha, I} \) norm, and for \( I \) noncompact, \( C^{k+\alpha}(A_I) \) is defined as the space of continuous functions which are norm-bounded on compact subsets of \( I \). \( C^{k,\alpha}(\Sigma) \) is the Hölder space on \( \Sigma \) with norm \( \| \cdot \|_{k,\alpha} \). For any \( \xi \in C^0(A_I) \), we define functions \( \xi_*(t) \) and \( \xi^*(t) \) by

\[ \xi_*(t) = \inf \{ \xi(t, x) : x \in \Sigma \}, \quad \xi^*(t) = \sup \{ \xi(t, x) : x \in \Sigma \}. \]

Our main theorems in this paper are the following:

**Theorem 1.** Assume \( \bar{R} \in C^\alpha(N) \) and \( f \in C^{4+\alpha}(N) \) such that

\[ 0 < 1 + t \frac{\partial f}{\partial t} < \infty \text{ for all } 1 \leq t \leq \infty, \]

\[ \left( t \frac{\partial f}{\partial t} \right)^* \in L^1([1, \infty)), \]

\[ t \left( \frac{\partial}{\partial t} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \right)^* - t \frac{d}{dt} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^* \in L^1([1, \infty)), \]

and

\[ \int_1^\infty |R_f - 2|^* + |\bar{R}t^2|^* dt < \infty. \]
Suppose there exists a constant $C > 0$ such that for all $t \geq 2$ and $I_t = [t/2, 2t],
\|\tilde{R}^2\|_{\alpha, I_t} + \|\frac{\partial f}{\partial t}\|_{2+\alpha, I_t} + \|1 - e^{-2f}\|_{2+\alpha, I_t} \leq \frac{C}{t}.

Further assume the nonnegative constant $K$ defined by
\[
K = \sup_{1 \leq t < \infty} \left\{ -\int_1^t \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \tilde{R}}{1 + \frac{\partial f}{\partial \tau}} \right) \right\}_e e^{\int_1^t \left( 2\frac{\tau}{\sigma} \ln \left( \frac{1 + \frac{\partial f}{\partial \sigma}}{1 + \frac{\partial f}{\partial \tau}} \right) + 3 \left( \frac{1 + \frac{\partial f}{\partial \sigma}}{1 + \frac{\partial f}{\partial \tau}} \right) \right) ds \, d\tau \}
\]
satisfies
\[
K < \infty.
\]
Then for every $\varphi(x) \in C^{2+\alpha}(\Sigma)$ satisfying
\[
0 < \varphi(x) < \frac{1}{\sqrt{K}} \quad \text{for all } x \in \Sigma,
\]
there is a unique positive solution $u \in C^{2+\alpha}(N)$ of (1) with the initial condition
\[
(2) \quad u(1, \cdot) = \varphi(\cdot)
\]
such that the metric $\bar{g} = u^2dt^2 + t^2e^{2f}\sigma$ on $N$ satisfies the asymptotically flat condition
\[
(3) \quad |\bar{g}_{ab} - \delta_{ab}| + t |\partial_a \bar{g}_{bc}| < \frac{C}{t}, \quad a, b, c = 1, 2, 3
\]
with ADM mass of $(N, \bar{g})$ that can be expressed as
\[
m_{ADM} = \frac{1}{4\pi} \lim_{t \to \infty} \int_{S_t} \frac{t}{2} (1 - u^{-2}) \, d\sigma.
\]
Moreover the Riemannian curvature $\tilde{R}m$ of the 3-metric $\bar{g}$ on $N$ is Hölder continuous and decays as $|\tilde{R}m| < \frac{C}{t^3}.$

**Theorem 2.** Let $t \frac{\partial f}{\partial t} \in C^{2+\alpha}(N)$ and $\tilde{R} \in C^{\alpha}(N)$ be given such that
\[
\delta_*(t) = \int_1^t \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \tilde{R}}{1 + \frac{\partial f}{\partial \tau}} \right) e^{\int_1^t \left( 2\frac{\tau}{\sigma} \ln \left( \frac{1 + \frac{\partial f}{\partial \sigma}}{1 + \frac{\partial f}{\partial \tau}} \right) + 3 \left( \frac{1 + \frac{\partial f}{\partial \sigma}}{1 + \frac{\partial f}{\partial \tau}} \right) \right) ds \, d\tau,
\]
and
\[
\delta^*(t) = \int_1^t \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \tilde{R}}{1 + \frac{\partial f}{\partial \tau}} \right) e^{\int_1^t \left( 2\frac{\tau}{\sigma} \ln \left( \frac{1 + \frac{\partial f}{\partial \sigma}}{1 + \frac{\partial f}{\partial \tau}} \right) + 3 \left( \frac{1 + \frac{\partial f}{\partial \sigma}}{1 + \frac{\partial f}{\partial \tau}} \right) \right) ds \, d\tau,
\]
are finite on $[1, \infty)$. Further suppose that for all $1 \leq t < \infty,$
\[
0 < 1 + t \frac{\partial f}{\partial \tau} < \infty,
\]
and
\[
t^2 \tilde{R} < R_f.
\]
Then there is $u^{-1} \in C^{2+\alpha}(A_{1, \infty})$ such that the metric $\bar{g}$ on $N$ has curvature uniformly bounded on $A_{[1, 2]}$ with totally geodesic boundary.

Let $0 < \eta < 1$ be such that
\[
1 - \eta < R_f - t^2 \tilde{R}|_{t=1} < (1 - \eta)^{-1}.
\]
Then there is $t_0 > 1$ such that $1 < t < t_0,$
\[
\frac{t - 1}{t} (1 - \eta) < u^{-2}(t) < \frac{t - 1}{t} (1 - \eta)^{-1},
\]
which gives \( 1 - \eta \frac{t}{1 - \eta} (t - 1) \leq 2m \leq 1 + \eta (t - 1) \).

The modified Ricci flow is defined by the geometric evolution equation
\[
\frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij} + 2 D_i D_j F = 2 M_{ij},
\]
where \( R \) is the scalar curvature, \( r = \int R d\mu / \int 1 d\mu \) is the mean scalar curvature, and the Ricci potential \( F \) is a solution of the equation \( \Delta F = R - r \) with mean value zero. The equation of \( F \) satisfies
\[
\frac{\partial F}{\partial t} = \Delta F + r F - \int |D F|^2 d\mu / \int 1 d\mu.
\]

\( M_{ij} = (r - R) g_{ij} / 2 + D_i D_j F \) is the trace-free part of \( \text{Hess} (F) \). The solution under the modified Ricci flow of an arbitrary initial metric on a topological 2-sphere \( \Sigma \) exists for all time and converges exponentially to the round metric, and \( |M|_{ij} \to 0 \) exponentially \([5, 6]\). Moreover if \( R \geq 0 \) at the start, it remains so for all time. The modified Ricci flow also preserves area. For convenience we normalize the area so that \( A(\Sigma) = 4\pi \). By the Gauss-Bonnet formula the mean scalar curvature \( r = \int R d\mu / \int 1 d\mu = 2 \). The solution to the modified Ricci flow provides a canonical foliation on \( N = [1, \infty) \times \Sigma \).

\section*{Theorem 3.}
Assume that \( \bar{R} \in C^\alpha (N) \) and the constant \( K \) is defined by
\[
K = \sup_{t \leq t < \infty} \left\{ - \int_1^t \left( \frac{R}{2} - \frac{\tau^2}{2} \bar{R} \right) + \exp \left( \int_1^\tau \frac{s |M|^2}{2} ds \right) d\tau \right\} < \infty.
\]
Suppose there is a constant \( C > 0 \) such that for all \( t \geq 1 \) and \( I_t = [t, 4t] \),
\[
\| \bar{R} t^2 \|_{\alpha, I_t} \leq C / t, \quad \text{and} \quad \int_1^\infty |\bar{R}|^2 t^2 dt < \infty.
\]
Then for any function \( \varphi \in C^{2, \alpha} (\Sigma) \) satisfying
\[
0 < \varphi < \frac{1}{\sqrt{R}}.
\]
there is a unique positive solution \( u \in C^{2+\alpha} (N) \) of the parabolic equation
\[
\frac{\partial u}{\partial t} = \frac{1}{2} u^2 \Delta u + \frac{t^2}{4} |M|^2 u + \frac{1}{2} u - \frac{1}{4} (R - t^2 \bar{R}) u^3
\]
with initial condition \( u(1, \cdot) = \varphi (\cdot) \) such that the metric \( \bar{g} = u^2 dt^2 + t^2 g(t) \) on \( N \) satisfies the asymptotically flat condition with finite ADM mass and
\[
m_{\text{ADM}} = \lim_{t \to \infty} \frac{1}{4\pi} \int_{\Sigma_t} \frac{t}{2} (1 - u^{-2}) d\sigma.
\]
Moreover, the Riemannian curvature \( \bar{R} m \) of the 3-metric \( \bar{g} \) on \( N \) is Hölder continuous and decays as \( |\bar{R} m| < \frac{C}{t^3} \).
Theorem 4. Let $\bar{R} \in C^\alpha(N)$. Suppose that $\bar{R}t^2 < R$ for $1 \leq t < \infty$. Let $0 < \eta < 1$ be such that

$$1 - \eta < R - \bar{R}|_{t=1} < (1 - \eta)^{-1}.$$ 

Then there is a solution $u^{-1} \in C^{2+\alpha}(N)$ such that the constructed metric on $N$ has curvature uniformly bounded on $A_{[1,2]}$ with totally geodesic boundary $\Sigma$.

The outline of the paper is as follows: In Section 2, we derive the parabolic equation for $u$ and its equivalent forms. In Section 3, we prove Theorem 1 in two steps. First, we prove the existence of $u$ (Theorem 7). Second, we discuss decay conditions for $\bar{R}$ and metrics $e^{2f}g$ which ensure asymptotic flatness of $\bar{g}$ (Theorem 9). After we prove Theorem 1, we show there exists a solution $u$ such that the boundary surface is the outermost minimal surface (Theorem 2). In Section 4, we prove similar existence results under Ricci flow foliations.

2. Curvature calculations

For now we use $g(t)$ to denote a family of metrics on $\Sigma$. Let $\bar{g} = u^2 dt^2 + g(t)$ be a metric on $N = [1, \infty) \times \Sigma$, and $\bar{R}$ and $R$ denote the scalar curvatures of $\bar{g}(t)$ and $g$ respectively. Let $H$ and $|A|$ denote the mean curvature and the norm squared of the second fundamental form of $\Sigma_t = \{t\} \times \Sigma$. A direct computation shows that the Ricci curvature of $g$ is given by

$$g^{ij} R^3_{3ij} = -\frac{1}{u} \frac{\partial}{\partial t} H - \frac{1}{u} \Delta u - |A|^2,$$

where $\Delta$ is the Laplacian with respect to $g(t)$. The Gauss equation gives that

$$g^{ik} g^{jl} \bar{R}_{ijkl} = R - H^2 + |A|^2$$

where $i, j, k, l = 1, 2$.

Combining the above two equations, the scalar curvature $\bar{R}$ with a metric of the form

$$\bar{g} = u^2 dt^2 + g(t)$$

is given by

$$\bar{R} = -\frac{2}{u} \frac{\partial}{\partial t} H - \frac{2}{u} \Delta u - |A|^2 + R - H^2.$$ 

The second fundamental forms $h_{ij}, i, j = 1, 2$ measure the change of the metric along the normal direction $\nu$.

When $g(t) = t^2 e^{2f}g$, the second fundamental forms are

$$h_{ij} = \frac{1}{u} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \sigma_{ij},$$

where $i, j = 1, 2$.

In particular,

$$H = \frac{2}{u} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \quad \text{and} \quad |A|^2 = \frac{2}{u^2} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2.$$ 

From (6) and above, we have

$$\bar{R} = \frac{4}{u^2} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \frac{\partial u}{\partial t} - \frac{4}{u^2} \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) - \frac{2}{t^2 u^2} \Delta f u + \frac{1}{t^2} R_f - \frac{6}{u^2} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2.$$
where $\Delta_f$ is the Laplacian with respect to $e^{2f}$. We can rewrite it as
\[
\left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \frac{\partial u}{\partial t} = \frac{1}{2t^2} u^2 \Delta_f u + \left( \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + \frac{3}{2} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2 \right) u
- \frac{1}{4t^2} (R_f - t^2 \bar{R}) u^3.
\]

Introducing $w = u^{-2}$ and $m = \frac{t}{2} (1 - u^{-2})$, we have the equivalent forms
\[
\left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \frac{\partial w}{\partial t} = \frac{1}{2t^2} \Delta_f w + \frac{3}{2t^2} u \nabla u \cdot \nabla w \quad \text{and} \quad \frac{1}{2t^2} \Delta_f w + \frac{3}{2t^2} u \nabla u \cdot \nabla w = \frac{1}{2t^2} \left( R_f - t^2 \bar{R} \right) w^3.
\]

and
\[
\left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \frac{\partial m}{\partial t} = \frac{u^2}{2t^2} \Delta_f m + \frac{3u}{2t^2} \nabla u \cdot \nabla m \quad \text{and} \quad \frac{u^2}{2t^2} \Delta_f m + \frac{3u}{2t^2} \nabla u \cdot \nabla m = \frac{1}{2t^2} \left( R_f - 2 - t^2 \bar{R} - 4t^2 \frac{\partial^2 f}{\partial t^2} - 12t \frac{\partial f}{\partial t} - 6 \left( \frac{\partial f}{\partial t} \right)^2 \right) m.
\]

where $\Delta_f$ and $\nabla$ are the Laplacian and the covariant derivatives on $\Sigma_t = \{ t \} \times \Sigma$ with respect to $e^{2f}$. 

For Ricci flow ansatz, we consider $N = [1, \infty) \times \Sigma$ equipped with the metric
\[
\bar{g} = u^2 dt^2 + t^2 g_{ij}(t,x) dx^i dx^j,
\]

where $g(t,x)$ is the solution to the modified Ricci flow (5). Direct computation shows that the second fundamental forms $h_{ij}$ on $\Sigma_t$ with respect to the normal $\nu = \frac{1}{u} \frac{\partial}{\partial t}$ are given by
\[
h_{ij} = \frac{1}{u} \left( \frac{1}{t} \bar{g}_{ij} + t^2 M_{ij} \right), \quad i, j = 1, 2;
\]

the mean curvature and the norm squared of the second fundamental form $|A|$ are
\[
H = \frac{2}{tu} \quad \text{and} \quad |A|^2 = \frac{2}{t^2 u^2} + \frac{|M_{ij}|^2}{u^2},
\]

respectively. By equation (6), the scalar curvature $\bar{R}$ of $\bar{g}$ is given by
\[
\bar{R} = \frac{4}{tu^3} \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} + \frac{2}{t^2 u} \Delta u + \frac{R}{t^2} - \frac{2}{t^2 u^2} - \frac{|M|^2}{u^2},
\]

where $\Delta$ is the Laplacian with respect to $g(t)$. The metric $\bar{g} = u^2 dt^2 + t^2 g(t,x)$ has the scalar curvature $\bar{R}$ if and only if $u$ satisfies the parabolic equation
\[
\frac{t \partial u}{\partial t} = \frac{1}{2} u^2 \Delta u + \left( \frac{t^2}{4} |M|^2 + \frac{1}{2} \right) u - \frac{1}{4} (R - t^2 \bar{R}) u^3.
\]

The equations for the corresponding terms $w = u^{-2}$ and $m = \frac{t}{2} (1 - u^{-2})$ are as follows
\[
t \partial_t w = \frac{1}{2w} \Delta w + \frac{3}{2} \frac{u}{u^2} \nabla u \cdot \nabla w - \left( \frac{t^2}{2} |M|^2 + 1 \right) w + \frac{R}{2} - \frac{t^2}{2} \bar{R},
\]
Proposition 5. We basically follow the argument in [3], see also [10].

u_\ast (Proposition 5), and prove Schauder estimates for u that under suitable decay conditions of the foliation and the scalar curvature \( \bar{R} \) with respect to \( g \).

If we further assume that

\begin{equation}
R(t, x) = -2 \tau \frac{\partial}{\partial \tau} \ln (\frac{1}{t} + \frac{\partial f}{\partial t}) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) - t \frac{\partial R}{\partial t} + t^3 \frac{\partial \bar{R}}{\partial \tau},
\end{equation}

where \( \Delta \) and \( \nabla \) are the Laplacian and the covariant derivatives on \( \Sigma_t = \{ t \} \times \Sigma \) with respect to \( g(t) \).

3. Existence

In this section we use the maximum principle to obtain \( C^0 \) estimates for \( u^{-2} \) (Proposition \( 4 \)), and prove Schauder estimates for \( u \) and \( m \) (Proposition \( 5 \)). Using these a priori estimates, we prove long-time existence of solution \( u \). Then we show that under suitable decay conditions of the foliation and the scalar curvature \( \bar{R} \), the metric \( \bar{g} = u^2 dt^2 + t^2 g(t, x) \) is asymptotically flat and has finite ADM mass. We basically follow the argument in [9], see also [10].

**Proposition 5.** Suppose \( u \in C^{2+\alpha} (A_{[t_0, t_1]} \bar{g}) \), \( 1 \leq t_0 < t_1 \) is a positive solution to \( \bar{g} \). Then for \( t_0 \leq t \leq t_1 \) we have

\[
w^{-2}(t, x) \geq t_0 \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \bar{R}}{1 + \frac{\partial f}{\partial t}} \right)_* e^{-t_0 \int_0^t (2 \frac{\partial}{\partial \tau} \ln \left( \frac{1}{t} \frac{\partial}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right))} ds dt,
\]

and

\[
w^{-2}(t, x) \leq t_0 \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \bar{R}}{1 + \frac{\partial f}{\partial t}} \right)_* e^{-t_0 \int_0^t (2 \frac{\partial}{\partial \tau} \ln \left( \frac{1}{t} \frac{\partial}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right))} ds dt.
\]

If we further assume that \( \bar{R} \) is defined on \( N \) such that the functions

\[
\delta_\ast (t) = \int_1^t \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \bar{R}}{1 + \frac{\partial f}{\partial t}} \right)_* e^{-t_0 \int_0^t (2 \frac{\partial}{\partial \tau} \ln \left( \frac{1}{t} \frac{\partial}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right))} ds dt,
\]

and

\[
\delta^\ast (t) = \int_1^t \frac{1}{2\tau^2} \left( \frac{R_f - \tau^2 \bar{R}}{1 + \frac{\partial f}{\partial t}} \right)_* e^{-t_0 \int_0^t (2 \frac{\partial}{\partial \tau} \ln \left( \frac{1}{t} \frac{\partial}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right))} ds dt
\]

are defined and finite for all \( t \in [1, \infty) \), then the estimates may be rewritten as

\[
w^{-2}(t, x) \geq \delta_\ast (t) + (u^{-2}(t_0) - \delta_\ast (t_0)) e^{-t_0 \int_0^t (2 \frac{\partial}{\partial \tau} \ln \left( \frac{1}{t} \frac{\partial}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right))} ds,
\]

and

\[
w^{-2}(t, x) \leq \delta^\ast (t) + (u^{-2}(t_0) - \delta^\ast (t_0)) e^{-t_0 \int_0^t (2 \frac{\partial}{\partial \tau} \ln \left( \frac{1}{t} \frac{\partial}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right))} ds.
\]

**Proof.** Applying the parabolic maximum principle to \( \bar{g} \) for \( w = u^{-2} \) gives

\[
w_\ast (t) \geq - \left( 2 \frac{\partial}{\partial t} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \right)_* w_\ast
\]

\[
+ \frac{1}{2t^2} \left( (R_f - t^2 \bar{R}) \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^{-1} \right)_*.
\]
Solving the associated O.D.E., we have

\[
w_\ast(t) \geq \int_{t_0}^{t} \left( \frac{1}{2 \tau^2} \left( R_\tau - \frac{\tau^2}{2} \right) \right) e^{\int_{0}^{\tau} \left( 2 \frac{\partial f}{\partial t} \ln \left( \frac{1}{2 \tau^2} \right) + 3 \left( \frac{1}{2 \tau^2} \right) \right) ds} d\tau \\
\times e^{\int_{0}^{t_0} \left( 2 \frac{\partial f}{\partial t} \ln \left( \frac{1}{2 \tau^2} \right) + 3 \left( \frac{1}{2 \tau^2} \right) \right) ds} + w_\ast(t_0) e^{- \int_{0}^{t_0} \left( 2 \frac{\partial f}{\partial t} \ln \left( \frac{1}{2 \tau^2} \right) + 3 \left( \frac{1}{2 \tau^2} \right) \right) ds} \\
= \int_{t_0}^{t} \left( \frac{1}{2 \tau^2} \left( R_\tau - \frac{\tau^2}{2} \right) \right) e^{- \int_{t_0}^{\tau} \left( 2 \frac{\partial f}{\partial t} \ln \left( \frac{1}{2 \tau^2} \right) + 3 \left( \frac{1}{2 \tau^2} \right) \right) ds} d\tau \\
+ w_\ast(t_0) e^{- \int_{0}^{t_0} \left( 2 \frac{\partial f}{\partial t} \ln \left( \frac{1}{2 \tau^2} \right) + 3 \left( \frac{1}{2 \tau^2} \right) \right) ds}.
\]

Therefore

\[w(t) \geq \delta_\ast(t) + (w_\ast(t_0) - \delta_\ast(t_0)) e^{- \int_{0}^{t_0} \left( 2 \frac{\partial f}{\partial t} \ln \left( \frac{1}{2 \tau^2} \right) + 3 \left( \frac{1}{2 \tau^2} \right) \right) ds}.
\]

Similarly, applying the maximum principle to \(w_\ast\), we get the upper bound for \(u^{-2}\).

\[\square\]

**Proposition 6.** Let \(I = [1, t_1]\) and \(I' = [t_0, t_1]\) with \(1 < t_0 < t_1\), and suppose \(u \in C^{2+\alpha}(A_I)\) is a solution to (1) on \(A_I\) with source functions \(R\) and \(f\) such that

\[0 < f_0 \leq \frac{1}{t} + \frac{\partial f}{\partial t} \leq f_0^{-1}\]

for all \((t, x) \in A_I\),

for some constant \(f_0 > 0\). Further suppose there is a constant \(\delta_0 > 0\) such that

\[0 < \delta_0 \leq u^{-2}(x, t) \leq \delta_0^{-1}\]

for all \((t, x) \in A_I\).

Then

\[\|u\|_{2+\alpha, I'} \leq C \left( \|\frac{\partial f}{\partial t}\|_{\alpha, I}, \|\frac{\partial^2 f}{\partial t^2}\|_{\alpha, I}, \|R_f\|_{\alpha, I}, \|\bar{R}\|_{\alpha, I} \right),\]

where \(C\) is a constant dependent on \(t_0, t_1, f_0, \delta_0, \|\nabla f\|_{0, I}\).

With \(m = \frac{t}{2} (1 - u^{-2})\), there is a constant \(C\) such that

\[\|m\|_{2+\alpha, I'} \leq C \left( \|\frac{\partial f}{\partial t}\|_{\alpha, I}, \|\frac{\partial^2 f}{\partial t^2}\|_{\alpha, I}, \left\|1 - \frac{1}{2} R_f\right\|_{\alpha, I}, \|\bar{R}\|_{\alpha, I} \right),\]

where \(C\) depends on \(t_0, t_1, f_0, \delta_0, \|\nabla f\|_{0, I}\).
Proof. Let $s = \ln t$. Then $\frac{\partial}{\partial s} = t \frac{\partial}{\partial t}$. Let $\gamma = \frac{1}{2} \left( 1 + t \frac{\partial f}{\partial t} \right)^{-1}$.

\[
\frac{\partial u}{\partial s} = \gamma u^2 \Delta f + t^2 \gamma \left( 2 \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2 \right) u - \frac{1}{2} \gamma \left( R_f - t^2 \bar{R} \right) u^3
\]

\[
= \frac{\partial}{\partial x^i} \left( \gamma u^2 e^{-2f} \sigma^{ij} \frac{\partial u}{\partial x^j} \right) - \frac{\partial}{\partial x^i} \left( \frac{\gamma}{e^f \sqrt{\det \sigma}} \right) e^{-f} \sqrt{\det \sigma} \sigma^{ij} \frac{\partial u}{\partial x^j} u^2
\]

\[
-2 \gamma e^{-2f} \sigma^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} u + t^2 \gamma \left( 2 \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2 \right) u
\]

\[
- \frac{1}{2} \gamma \left( R_f - t^2 \bar{R} \right) u^3
\]

\[
= \frac{\partial}{\partial x^i} a^i(x, t, u, \partial u) - a(x, t, u, \partial u),
\]

where $a(x, t, u, p)$ and $a^i(x, t, u, p)$ are functions defined by

\[
a^i(x, t, u, p) = \gamma u^2 e^{-2f} \sigma^{ij} p_j
\]

and

\[
a(x, t, u, p) = \frac{\partial}{\partial x^i} \left( \frac{\gamma}{e^f \sqrt{\det \sigma}} \right) e^{-f} \sqrt{\det \sigma} \sigma^{ij} p_j u^2 + 2 \gamma e^{-2f} \sigma^{ij} p_i p_j u
\]

\[
- t^2 \gamma \left( 2 \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^2 \right) u + \frac{1}{2} \gamma \left( R_f - t^2 \bar{R} \right) u^3
\]

respectively. By the assumption, the functions $a$ and $a^i$ satisfy that

\[
a^i p_i \geq C|p|^2,
\]

\[
|a^i| \leq C'||p|,
\]

and

\[
|a| \leq C''(1 + |p|^2),
\]

for some positive constants $C, C'$, and $C''$. Applying [8, Theorem V.1.1], we have

\[
||u||_{\alpha', I''} \leq C_1,
\]

for some $0 < \alpha' < 1, \alpha' = \alpha(\delta_0)$ and $C_1$ dependent on $t_0, t_1, f_0, \delta_0, \left\| \frac{\partial f}{\partial t} \right\|_{0, I}$, \left\| \frac{\partial^2 f}{\partial t^2} \right\|_{0, I}$, $||\nabla f||_{0, I}$, $||R_f||_{0, I}$, and $||\bar{R}||_{0, I}$ where $I' \subset I'' \subset I$.

Without loss of generality, we may assume $\alpha' \leq \alpha$. The usual Schauder interior estimates [8, Theorem IV. 10.1] give

\[
||u||_{2 + \alpha', I'} \leq C_2 \left( C_1, \left\| \frac{\partial f}{\partial t} \right\|_{\alpha, I}, \left\| \frac{\partial^2 f}{\partial t^2} \right\|_{\alpha, I}, ||R_f||_{\alpha, I}, ||\bar{R}||_{\alpha, I} \right).
\]
Theorem 7. Assume \( \bar{R} \in C^\alpha(A_{[t_0,\infty)}) \) and \( f \in C^{2+\alpha}(A_{[t_0,\infty)}) \), \( t_0 > 1 \) such that the function satisfies

\[
0 < 1 + t \frac{\partial f}{\partial t} < \infty \quad \text{for all} \quad t_0 \leq t < \infty.
\]

Further assume the nonnegative constant \( K \) defined by

\[
K = \sup_{t_0 \leq t < \infty} \left\{ -t \int_{t_0}^t \frac{1}{2t^2} \left( \frac{R_f - \tau^2 \bar{R}}{\frac{1}{t} + \frac{\partial f}{\partial t}} \right) e^{\int_t^\tau \left( 2 \frac{\partial f}{\partial s} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial s} \right) + 3 \left( \frac{\partial f}{\partial s} \right)^2 \right) ds} d\tau \right\}
\]

satisfies \( K < \infty \). Then for every \( \varphi(x) \in C^{2+\alpha}(\Sigma) \) such that

\[
0 < \varphi(x) < \frac{1}{\sqrt{K}} \quad \text{for all} \quad x \in \Sigma,
\]

there is a unique positive solution \( u \in C^{2+\alpha}(A_{[t_0,\infty)}) \) to (1) with initial condition

\[
u(t_0, \cdot) = \varphi(\cdot).
\]

Proof. Define \( \tilde{u}(t, x) = u(t_0 t, x) \). Observe that \( u \in C^{2+\alpha}(A_{[t_0,\infty)}) \) satisfies (1) if and only if \( \tilde{u} \) on the interval \([1, \infty)\) satisfies

\[
\left( \frac{1}{t} + \frac{\partial \tilde{f}}{\partial t} \right) \frac{\partial \tilde{u}}{\partial t} = \frac{1}{2t^2} \tilde{u}^2 \Delta f \tilde{u} + \left[ \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial \tilde{f}}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial \tilde{f}}{\partial t} \right)^2 \right] \tilde{u}
\]

\[
-\frac{1}{4t^2} \left( \bar{R}_f - t^2 \bar{R} \right) \tilde{u}^3,
\]

where

\[
\tilde{f}(t, x) = f(t_0 t, x), \quad \bar{R}_f(t, x) = R_f(t_0 t, x), \quad \bar{R}(t, x) = t_0^2 \bar{R}(t_0 t, x).
\]

Denoting the estimating functions of Proposition 5 by \( \hat{\delta}^* (t) \) and \( \hat{\delta}_* (t) \), we have that

\[
\hat{\delta}^*(t) = \delta^*(t_0 t), \quad \hat{\delta}_*(t) = \delta_*(t_0 t) \quad \text{for all} \quad 1 \leq t < \infty.
\]

It can be verified that

\[
K = \sup_{1 \leq t < \infty} \left\{ -t \int_{1}^t \frac{1}{2t^2} \left( \frac{R_f - \tau^2 \bar{R}}{\frac{1}{t} + \frac{\partial f}{\partial t}} \right) e^{\int_t^\tau \left( 2 \frac{\partial f}{\partial s} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial s} \right) + 3 \left( \frac{\partial f}{\partial s} \right)^2 \right) ds} d\tau \right\}.
\]

The upper bound

\[
\varphi(x) < \frac{1}{\sqrt{K}} \quad \text{for all} \quad x \in \Sigma
\]
implies

\begin{equation}
\tilde{\delta}_i(t) + \tilde{\alpha}^{-2}(1) \exp \left( \int_1^t \left( 2 \frac{\partial}{\partial t} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) + 3 \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \right) ds \right) > 0
\end{equation}

for all $t \geq 1$. Since the foliation satisfies \([15]\), the equation \([16]\) for $\tilde{u}$ is parabolic. The short-time existence of solutions can be obtained from Schauder theory and the implicit function theorem. By Propositions \([5]\) and \([6]\) there is $T > 0$ and $\tilde{u} \in C^{2+\alpha}(A_{[1,1+T]})$ satisfying \([10]\) and the initial condition $\tilde{u}(1, x) = \varphi(x)$ on $[1, 1+T]$. Moreover by Proposition \([5]\) and \([17]\), there are functions $0 < \delta_1(t) \leq \delta_2(t) < \infty$, $t \geq 1$, independent of $T$, such that

$$0 < \delta_1(t) \leq \tilde{u}^{-2}(t, x) \leq \delta_2(t) < \infty \quad \text{for } 1 \leq t \leq 1 + T.$$

Let $U = \{ t \in \mathbb{R}^+ : \exists \tilde{u} \in C^{2+\alpha}(A_{[1,1+t]}) \text{ satisfying } (10) \text{ and } \tilde{u}(1, x) = \varphi(x) \}$. The local existence guarantees that $U$ is open in $\mathbb{R}^+$. Since $[1, 1+t]$ is compact, there is a constant $\delta_0$ such that $0 < \delta_0 \leq \tilde{u}^{-2}(t, x) \leq \delta_0^{-1}$ for all $(t, x) \in A_{[1,1+t]}$. From the interior estimates \([13]\) of Proposition \([4]\) we have an a priori estimate for $||\tilde{u}(1+t, \cdot)||_{2,\alpha}$. By the local existence, the solution can be extended to $A_{[1,1+t+T]}$ for some $T$ independent of $\tilde{u}$, which shows that $U$ is closed. Hence $\tilde{u}$ extends to a global solution $\tilde{u} \in C^{2+\alpha}(A_{[1,\infty)})$ and the function $u(t, x) = \tilde{u}(t/t_0, x)$ is the required solution.

Theorem \([7]\) gives the existence of the initial value problem \([1]\) and \([2]\); however, the asymptotic behavior of the solution is still not controlled yet. In Theorem \([8]\) we will describe decay conditions on the source functions $R$ and $f(t, x)$ which ensure existence of solutions satisfying the boundary behavior.

From now on we use $C$ to denote a constant, but it may vary from line to line.

**Lemma 8.** Assume that $f$ satisfies

$$0 < 1 + t \frac{\partial f}{\partial t} < \infty \text{ for all } 1 \leq t < \infty.$$

Suppose

\begin{equation}
\frac{1}{t} \left( \frac{\partial f}{\partial t} \right) \in L^1([1, \infty)) \tag{18}
\end{equation}

and

\begin{equation}
\frac{1}{t} \left( \frac{\partial \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)}{\partial t} \right) - t \frac{d}{dt} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \in L^1([1, \infty)) \tag{19}
\end{equation}

Then there is a constant $C$ such that

\begin{equation}
1 - \frac{C}{t} \leq \int_1^t \frac{1}{\tau^2} \left( \frac{1}{\tau} + \frac{\partial f}{\partial \tau} \right)^{-1} e^{-\int_{\tau}^{1} \left( \frac{1}{\tau} + \frac{\partial f}{\partial \tau} \right) ds d\tau} \leq 1 + \frac{C}{t}, \tag{20}
\end{equation}

and

\begin{equation}
1 - \frac{C}{t} \leq \int_1^t \frac{1}{\tau^2} \left( \frac{1}{\tau} + \frac{\partial f}{\partial \tau} \right)^{-1} e^{-\int_{\tau}^{1} \left( \frac{1}{\tau} + \frac{\partial f}{\partial \tau} \right) ds d\tau} \leq 1 + \frac{C}{t}, \tag{21}
\end{equation}

for all $t \geq 1$. 

Proof. To show (20), it suffices to show that

\[
\left| \int_1^t \frac{t}{\tau^2} \left( 1 + \frac{\partial f}{\partial \tau} \right)^{t-1} e^{-\int_1^\tau (2 \frac{\partial f}{\partial \tau} \ln(\frac{1}{s} + \frac{\partial f}{\partial s})) + 3(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds \, d\tau - t \right| \\
\leq \left| \int_1^t \frac{t}{\tau^2} \left( 1 + \frac{\partial f}{\partial \tau} \right)^{t-1} e^{-\int_1^\tau 3(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds \right| \\
+ \left| \int_1^t \frac{t}{\tau^2} \left( 1 + \frac{\partial f}{\partial \tau} \right)^{t-1} e^{-\int_1^\tau 2 \frac{\partial f}{\partial s} \ln(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds \right| \\
\times \left( e^{-\int_1^\tau (2 \frac{\partial f}{\partial \tau} \ln(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds - e^{-\int_1^\tau 2 \frac{\partial f}{\partial s} \ln(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, d\tau \right) \\
= I + II \\
< C.
\]

From (18), there is \( t_0 > 1 \) such that \( \int_{t_0}^\infty 3t \left( \frac{\partial f}{\partial t} \right)^{t} \, dt < 1 \). Using

\[ |e^\eta - 1| \leq 2 |\eta| \] for \( |\eta| \leq 1 \),

we see

\[
I = \left| \int_1^t \frac{t}{\tau^2} \left( 1 + \frac{\partial f}{\partial \tau} \right)^{t-1} e^{-\int_1^\tau 3(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds \, d\tau - t \right| \\
= \left| \int_1^t \left( 1 + \tau \frac{\partial f}{\partial \tau} \right)^* e^{-\int_1^\tau 3(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds \, d\tau - t \right| \\
\leq C + \int_{t_0}^t \frac{1}{(1 + t \frac{\partial f}{\partial \tau})^{t-1}} e^{-\int_1^\tau 3(\frac{1}{s} + \frac{\partial f}{\partial s}))} \, ds \, d\tau - 1 \, d\tau + \int_{t_0}^t \frac{1}{(1 + t \frac{\partial f}{\partial \tau})^{t-1}} \, d\tau \\\n\leq C + C \int_{t_0}^t \int_1^t \frac{\partial f}{\partial s} \, ds \, d\tau \\
+ \frac{1}{(1 + t \frac{\partial f}{\partial \tau})^{t-1}} \int_{t_0}^t \left| \partial \frac{\partial f}{\partial \tau} \right| - 2t \left( \frac{\partial f}{\partial t} \right)^* + t^2 \left( \frac{\partial f}{\partial t} \right)^{t} \, d\tau \\
= C + C \int_{t_0}^t \frac{\partial f}{\partial s} \, ds \, d\tau \leq (s - t_0) \, ds < C.
\]

Similarly, from (19), there is \( t_0 > 1 \) such that

\[
\int_{t_0}^\infty \left| 2 \frac{\partial f}{\partial s} \ln \left( \frac{1}{s} + \frac{\partial f}{\partial s} \right) \right| - 2 \frac{d}{ds} \ln \left( \frac{1}{s} + \frac{\partial f}{\partial s} \right)^* \, ds < 1.
\]

We see
\[ II = \left| \int_1^t \frac{t}{\tau^2} \left( \frac{1}{\tau} + \frac{\partial f}{\partial \tau} \right)^{-1} e^{-\int_1^\tau (\frac{s}{\tau} + \frac{\partial f}{\partial s})^* ds} \times \left( e^{-\int_1^\tau (\frac{s}{\tau} + \frac{\partial f}{\partial s})^* ds} \right) \right| d\tau \]
\[ \leq C + C \int_1^t \left| e^{-\int_1^\tau (\frac{s}{\tau} + \frac{\partial f}{\partial s})^* - 2 \frac{d}{ds} \ln (\frac{1}{s} + \frac{\partial f}{\partial s})^* ds - 1} \right| d\tau \]
\[ \leq C + C \int_1^t \int_1^\tau \left( \frac{\partial}{\partial s} \ln \left( \frac{1}{s} + \frac{\partial f}{\partial s} \right) \right)^* - \frac{d}{ds} \ln \left( \frac{1}{s} + \frac{\partial f}{\partial s} \right)^* ds d\tau \]
\[ = C + C \int_1^t (s - t_0) \left( \frac{\partial}{\partial s} \ln \left( \frac{1}{s} + \frac{\partial f}{\partial s} \right) \right)^* - \frac{d}{ds} \ln \left( \frac{1}{s} + \frac{\partial f}{\partial s} \right)^* ds < C. \]

**Theorem 9.** Let \( u \in C^{2+\alpha}(N) \) be a solution of (1). Suppose that \( f \in C^{4+\alpha}(N) \) satisfies

(22) \[ \left( \frac{\partial f}{\partial t} \right)^* \in L^1([1, \infty)) \]

and

(23) \[ t \left( \frac{\partial}{\partial t} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right) \right)^* - \frac{d}{dt} \ln \left( \frac{1}{t} + \frac{\partial f}{\partial t} \right)^* \in L^1([1, \infty)). \]

Further assume there is a constant \( C > 0 \) such that for all \( t \geq 2 \) and \( I_t = [t/2, 2t] \),

(24) \[ \| R_t \|_{\alpha, I_t} + \left\| \frac{\partial f}{\partial t} \right\|_{2+\alpha, I_t} + \left\| 1 - e^{-2f} \right\|_{2+\alpha, I_t} \leq \frac{C}{t} \]

and

(25) \[ \int_1^\infty |R_f - 2|^* + |\tilde{R}t|^* dt < \infty. \]

Then \( \tilde{g} \) satisfies the asymptotically flat condition (3). Moreover the Riemannian curvature \( \tilde{R}m \) of the 3-metric \( \tilde{g} \) on \( N \) is Hölder continuous and decays as \( |\tilde{R}m| < \frac{C}{t^\delta} \), and the ADM mass of \((N, \tilde{g})\) can be expressed as

\[ m_{ADM} = \frac{1}{4\pi} \lim_{t \to \infty} \int_{S_t} \frac{t}{2} (1 - u^{-2}) \, d\sigma. \]

**Proof.** By Lemma 8 and the decay conditions (22), (23), and (25), we have

\[ 1 - \frac{C}{t} \leq \delta_*(t) \leq \delta^*(t) \leq 1 + \frac{C}{t} \]

for all \( t \geq t_0 \), for some \( t_0 \) large.
Define $\tilde{u}(t, x) = u(\tau t, x)$. Observe that $u \in C^{2+\alpha}(A_{1/2, 2\tau})$ satisfies (1) if and only if $\tilde{u}$ on the interval $[1/2, 2]$ satisfies

\[
\left(1 + \frac{\partial \tilde{f}}{\partial t}\right) \frac{\partial \tilde{u}}{\partial t} = \frac{1}{2t^2} \tilde{u}^2 \Delta \tilde{f} \tilde{u} + \left[ \frac{\partial}{\partial t} \left(1 + \frac{\partial \tilde{f}}{\partial t}\right) + \frac{1}{2} \left(1 + \frac{\partial \tilde{f}}{\partial t}\right)^2 \right] \tilde{u}
\]

\[
- \frac{1}{4t^2} \left(\tilde{R}_f - t^2 \tilde{R}\right) \tilde{u}^3,
\]

where

\[
\tilde{f}(t, x) = f(\tau t, x), \quad \tilde{R}_f(t, x) = R_f(\tau t, x), \quad \text{and} \quad \tilde{R}(t, x) = \tau^2 \tilde{R}(\tau t, x).
\]

Applying Proposition [3] to $\tilde{u}$ on the interval $[1/2, 2]$ and then rescaling back, we obtain the estimates $\|u\|_{2+\alpha, I_t} \leq C$. Also, $m$ scales as $m(t, x) = m(\tau t, x)/\tau$. The scaling argument and (14) yield

\[
\|m\|_{2+\alpha, I_{t'}} \leq C t \left( \left\|\frac{\partial f}{\partial t}\right\|_{\alpha, I_t} + \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{\alpha, I_t} + \left\|1 - \frac{1}{2} R_f\right\|_{\alpha, I_t} + \left\|\tau^2 \tilde{R}\right\|_{\alpha, I_t} \right)
\]

\[
+ C \|m\|_{0, I_t},
\]

where $I_{t'} = [t, 2t]$, $I_t = [t/2, 2t]$, and $C$ is some constant independent of $u$ and $t$. The bounds (3) control $\|m\|_{0, I_t}$. Under conformal change, the scalar curvature $R_f$ satisfies

\[
\frac{R_f}{2} = e^{-2f} \left(1 - \Delta_{\sigma} f\right).
\]

Thus the decay condition (24) controls the remaining terms, and

\[
\|m\|_{2+\alpha, I_{t'}} \leq C
\]

is uniformly bounded for all $t \geq 1$. Expressing this in terms of $u$ and derivatives of $u$ gives

\[
\|1 - u^{-2}\|_{\alpha, I_{t'}} + \|t \partial_t u\|_{\alpha, I_{t'}} + \|\nabla u\|_{\alpha, I_{t'}} + \|\nabla^2 u\|_{\alpha, I_{t'}} \leq C /
t,
\]

which shows that $\hat{g}$ is asymptotically Euclidean. The estimate for $\nabla^2 u$ shows that the Ricci curvature on $N$, $\hat{Rc} \in C^{0, \alpha}(N)$ and $\|\hat{Rc}\| \leq C/t^3$. Since $N$ is of dimension 3, this controls the full curvature tensor.

From (1) the ADM mass is uniquely defined. We compute the flux integral in terms of spherical coordinates.

\[
m_{ADM} = \frac{1}{16\pi} \int_{S^2} \left( g^{ab} \hat{\nabla}_a \hat{g}_{bc} - \hat{\nabla}_c (g^{ab} \hat{g}_{ab}) \right) d\sigma^c
\]

\[
= \frac{1}{16\pi} \lim_{t \to \infty} \int_{S^2} \left( g^{ab} \hat{\nabla}_a \hat{g}_{b3} - \hat{\nabla}_3 (g^{ab} \hat{g}_{ab}) \right) u^{-2} t^2 d\sigma,
\]

where $\hat{g} = dt^2 + t^2 \sigma$ is the flat metric, $\hat{\nabla}$ denotes covariant derivative of $\hat{g}$, and $d\sigma$ is the area element on $S^2$. For $i, j = 1, 2$, the connections of $\hat{g}$ are $\hat{\Gamma}^{3}_{33} = 0$, and...
we need to solve for $u$ and (2). Theorem 9 shows the asymptotical flatness of $\bar{g}$ together give us Theorem 1. Next we prove Theorem 2. Since the mean curvature $\delta = 2H$ approaches $\infty$ from decay conditions (22), (24), and (25), it can be shown that the right hand side of the equation is integrable on $(t, \infty)$.

\begin{align*}
\frac{\partial^2 f}{\partial t^2} - \frac{2}{t} \frac{\partial f}{\partial t} + \frac{1}{t^2} f &= 0, \\
\frac{\partial^2 m}{\partial t^2} - \frac{1}{t} \frac{\partial m}{\partial t} + \frac{1}{t^2} m &= 0.
\end{align*}

From the decay condition (24), we see that the mass integral reduces to

$$m_{ADM} = \frac{1}{4\pi} \lim_{t \to \infty} \int_{S_t} m d\sigma.$$ 

To show that the limit exists, it suffices to show the following is integrable on $(t_0, \infty)$,

$$\int_{S_t} \left(1 + t \frac{\partial f}{\partial t}\right) \frac{\partial m}{\partial t} d\sigma = \int_{S_t} \frac{u^2}{2t^2} \Delta f + \frac{3u}{2t^2} \nabla u \cdot \nabla m - \left(\frac{2}{t} \frac{\partial^2 f}{\partial t^2} + \frac{5}{t} \frac{\partial f}{\partial t} + 3 \left(\frac{\partial f}{\partial t}\right)^2\right) m - \frac{1}{4t} \left(R - 2 - t^2 \bar{R} - 4t^2 \frac{\partial^2 f}{\partial t^2} - 12t \frac{\partial f}{\partial t} - 6 \left(\frac{\partial f}{\partial t}\right)^2\right) d\sigma.$$ 

From decay conditions (22), (24), and (25), it can be shown that the right hand side of the the equation is integrable on $(t_0, \infty)$ for some large $t_0$ since $m$ and $\nabla m$ are bounded.

Theorem 7 shows the existence of the solution of the initial value problem (11) and (2). Theorem 9 shows the the asymptotical flatness of $\bar{g} = u^2 dt^2 + t^2 e^{2f}$. They together give us Theorem 1. Next we prove Theorem 2. Since the mean curvature $H = \frac{2}{u} \left(\frac{1}{t} \frac{\partial f}{\partial t}\right)$ and $\frac{1}{t} \frac{\partial f}{\partial t} > 0$, in order to have minimal boundary surface $\Sigma$, we need to solve for $u$ so that the initial condition $u^{-1}(1, \cdot) = 0$.

**Theorem 2** Let $\frac{\partial f}{\partial t} \in C^{2+\alpha}(N)$ and $\bar{R} \in C^\alpha(N)$ be given such that $\delta(t)$ and $\delta^*(t)$ defined in Proposition 5 are finite on $[1, \infty)$. Further suppose that for all $1 \leq t < \infty$,

$$0 < 1 + t \frac{\partial f}{\partial t} < \infty,$$
and
\[ t^2 \bar{R} < R_f. \]
Then there is \( u^{-1} \in C^{2+\alpha}(A_{[1,\infty)}) \) such that the metric \( \bar{g} \) on \( N \) has curvature uniformly bounded on \( A_{[1,2]} \) with totally geodesic boundary. Let \( 0 < \eta < 1 \) be such that
\[
(27) \quad 1 - \eta < R_f - t^2 \bar{R}|_{t=1} < (1 - \eta)^{-1}.
\]
Then there is \( t_0 > 1 \) such that \( 1 < t < t_0, \)
\[
\frac{t-1}{t} (1 - \eta) < u^{-2}(t) < \frac{t-1}{t} (1 - \eta)^{-1},
\]
which gives
\[
1 - \frac{\eta}{1 - \eta} (t - 1) \leq 2m \leq 1 + \eta (t - 1).
\]
Proof. We introduce the scaling transformation
\[
\tilde{u}(t) = \sqrt{\frac{t}{t+1}} u(t + 1) \quad \text{where} \quad t \in (0, \infty).
\]
The evolution equation for \( \tilde{u} \) is
\[
\left( \frac{1}{t} + \frac{\partial \tilde{f}}{\partial t} \right) \partial \tilde{u} = \frac{1}{2t^2} \tilde{u}^2 \Delta \tilde{f} \tilde{u} + \left( \frac{\partial}{\partial t} \left( \frac{1}{t} + \frac{\partial \tilde{f}}{\partial t} \right) + \frac{3}{2} \left( \frac{1}{t} + \frac{\partial \tilde{f}}{\partial t} \right) \left( \frac{1}{t} + \frac{t}{t+1} \frac{\partial \tilde{f}}{\partial t} \right) \right) \tilde{u}
\]
\[
(28) \quad - \frac{1}{4t^2} \left( \tilde{R}_f - t^2 \bar{R} \right) \tilde{u}^3
\]
where
\[
\frac{\partial \tilde{f}}{\partial t}(t) = \frac{t+1}{t} \frac{\partial f}{\partial t}(t + 1),
\]
\[
\tilde{R}_f(t) = R_f(t + 1),
\]
\[
t^2 \bar{R}(t) = (t+1)^2 \bar{R}(t + 1).
\]
Observe that the \( \tilde{u} \)-equation has a similar form as the \( u \)-equation (1). The assumptions \( 0 < 1 + t^2 \frac{\partial f}{\partial t} < \infty \) and \( t^2 \bar{R} < R_f \) imply that
\[
K = \sup_{0 < t < \infty} \left\{ - \int_0^1 \frac{1}{2t^2} \left( \frac{\tilde{R}_f - \tau^2 \bar{R}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) e^{f(t) \left( 2 \frac{\partial f}{\partial \tau} \ln \left( \frac{1 + \frac{\partial f}{\partial \tau}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) + 3 \left( \frac{1 + \frac{\partial f}{\partial \tau}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) \right) \ast ds d\tau \right\}
\]
\[
= 0.
\]
By the existence theorem, it can be shown that for any positive initial condition \( \tilde{u}(t_0, \cdot) = \varphi(\cdot), t_0 > 0, \) the solution \( u^{-2}(t, x) > 0 \) satisfying \( \tilde{u}(t_0, \cdot) = \varphi(\cdot) \) exists on \( [t_0, \infty) \). Let \( \varphi_\epsilon \in C^{2,\alpha}(\Sigma), 0 < \epsilon < 1, \) be a family of functions satisfying
\[
\delta_\ast_\epsilon (\epsilon) \leq \varphi_\epsilon^2 \leq \delta^\ast_\epsilon (\epsilon),
\]
where
\[
\delta_\ast_\epsilon (t) = \int_0^t \frac{1}{2t^2} \left( \frac{\tilde{R}_f - \tau^2 \bar{R}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) e^{-f(t) \left( 2 \frac{\partial f}{\partial \tau} \ln \left( \frac{1 + \frac{\partial f}{\partial \tau}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) + 3 \left( \frac{1 + \frac{\partial f}{\partial \tau}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) \right) \ast ds d\tau
\]
and
\[
\delta^\ast_\epsilon (t) = \int_0^t \frac{1}{2t^2} \left( \frac{\tilde{R}_f - \tau^2 \bar{R}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) \ast e^{-f(t) \left( 2 \frac{\partial f}{\partial \tau} \ln \left( \frac{1 + \frac{\partial f}{\partial \tau}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) + 3 \left( \frac{1 + \frac{\partial f}{\partial \tau}}{1 + \frac{\partial \tilde{f}}{\partial \tau}} \right) \right) \ast ds d\tau.
\]
Applying the rescaling argument and the Schauder estimates to $\tilde{u}$, we can obtain an uniform bound
$$\|\tilde{u}\|_{2+\alpha,I} < C,$$
where $C$ is independent of $\tilde{u}$ and $I$ is a compact interval in $\mathbb{R}^+$. Applying Arzelá-Ascoli Theorem, we thus obtain a solution $\tilde{u}^{-1} \in C^{2+\alpha}(A(0,\infty))$ bounded by
$$0 < \tilde{\delta}_*(t) \leq \tilde{u}^{-2} \leq \tilde{\delta}^*(t).$$
It follows from the definition of the $C^{k+\alpha}$ norm that
$$\left\|\tilde{R}_f\right\|_{2+\alpha,[a,b]} \leq \|R_f\|_{2+\alpha,[a+1,b+1]}$$
for any $0 \leq a < b$, and we find that
$$\frac{\partial \tilde{f}}{\partial t} \in C^{2+\alpha}(A[0,\infty)), \quad \tilde{R}_f \in C^{\alpha}(A[0,\infty)), \quad t^2 \tilde{R} \in C^{\alpha}(A[0,\infty]).$$
Since $0 < 1 + (t + 1) \frac{\partial f}{\partial t}(t + 1) < \infty$ for all $t > 0$, $\frac{1 + \partial \tilde{f}}{t + 1} = \frac{\partial \tilde{f}}{\partial t}(t + 1) < C$. So $\tilde{\delta}_*(t)$ and $\tilde{\delta}^*(t)$ can be estimated on $(0,\epsilon)$, for some small $\epsilon > 0$, using (27):
$$(1 - \eta) \leq \tilde{\delta}_*(t) < \tilde{\delta}^*(t) < (1 - \eta)^{-1},$$
which gives the bounds on $u^{-2}$ and $m$, and also shows that for $0 < t < \epsilon$,
$$-\frac{\eta t}{1 - \eta} < 2\tilde{m}(t) < \eta t,$$
where $\tilde{m}(t) = \frac{t}{2} (1 - \tilde{u}^{-2}(t)) = m(t + 1) - 1/2$. The rescaling estimate (26) applied to $\tilde{m}$ shows that the covariant derivatives of $m$ decay,
$$|\nabla m(t)| + |\nabla^2 m(t)| \leq C(t - 1),$$
from which it follows that the curvature of $\tilde{g}$ is bounded on $A[1,2]$.

4. Under Ricci Flow Foliation

Let $(\Sigma, g_1)$ be a given 2-sphere with area $A(\Sigma) = 4\pi$, and $N = [1,\infty) \times \Sigma$ equipped with the metric
$$\tilde{g} = u^2 dt^2 + t^2 g_{ij}(t,x) dx^i dx^j,$$
where $g_{ij}(t,x)$ is the solution of the modified Ricci flow. Direct computation shows that the metric $\tilde{g} = u^2 dt^2 + t^2 g(t,x)$ has the scalar curvature $\tilde{R}$ if and only if $u$ satisfies the parabolic equation (10)
$$t \frac{\partial u}{\partial t} = \frac{1}{2} u^2 \Delta u + \frac{t^2}{2} |M|^2 u + \frac{1}{2} u - \frac{1}{4} \left(R - t^2 \tilde{R}\right) u^3,$$
where $\Delta$ is the Laplacian with respect to $g$, $R$ and $\tilde{R}$ are the scalar curvatures with respect to $g$ and $\tilde{g}$ respectively, and $|M|^2 = M_{ij} M^{il} g^{kj} g^{jl}$. In this section we will prove existence results, Theorem 3 and Theorem 4.
Lemma 10. Suppose \( u \in C^{2+\alpha}(A_{[t_0, t_1]}), \ 1 \leq t_0 < t_1, \) is a positive solution. If we further assume that \( R \) is defined on \( A_{[1, \infty)} \) such that the functions

\[
\delta_*(t) = \frac{1}{t} \int_1^t \left( \frac{R}{2} - \frac{\tau^2}{2} R \right)^* (\tau) \exp \left( - \int_\tau^t \frac{s |M|^2}{2} ds \right) d\tau
\]

and

\[
\delta^*(t) = \frac{1}{t} \int_1^t \left( \frac{R}{2} - \frac{\tau^2}{2} R \right)^* (\tau) \exp \left( - \int_\tau^t \frac{s |M|^2}{2} ds \right) d\tau
\]

are defined and finite for all \( t \in [t_0, \infty) \). Then for \( t_0 \leq t \leq t_1 \), we have

\[
u^2(t, x) \geq \delta_*(t) + \frac{t_0}{t} (u^*(t_0)^{-2} - \delta_*(t_0)) \exp \left( - \int_{t_0}^t \frac{s |M|^2}{2} ds \right)
\]

and

\[
u^2(t, x) \leq \delta^*(t) + \frac{t_0}{t} (u_*(t_0)^{-2} - \delta^*(t_0)) \exp \left( - \int_{t_0}^t \frac{s |M|^2}{2} ds \right).
\]

Proof. Apply the maximum principle to (11)

\[
t \partial_t w = \frac{1}{2w} \Delta w + \frac{3}{2} u \nabla u \cdot \nabla w - \left( \frac{t^2}{2} |M|^2 + 1 \right) w + \frac{R}{2} - \frac{t^2}{2} R.
\]

We have, at the maximum of \( u(t, x) \),

\[
t \frac{d\bar{w}}{dt} \geq - \left( \frac{t^2}{2} |M|^2 + 1 \right)^* \bar{w} + \left( \frac{R}{2} - \frac{t^2}{2} R \right)^*.
\]

Solving the associated ordinary differential equation,

\[
u^2 \geq \frac{1}{t} \int_{t_0}^t \left( \frac{R}{2} - \frac{\tau^2}{2} R \right)^* \exp(- \int_\tau^t \frac{s |M|^2}{2} ds) d\tau
\]

\[
+ \frac{1}{t} \int_{t_0}^t \frac{t_0}{t} (u^*(t_0) - \delta_*(t_0)) \exp \left( - \int_{t_0}^t \frac{s |M|^2}{2} ds \right)
\]

\[
= \delta_*(t) + \frac{t_0}{t} (u^*(t_0) - \delta_*(t_0)) \exp \left( - \int_{t_0}^t \frac{s |M|^2}{2} ds \right).
\]

Similarly, applying the maximum principle to \( w^* \), we get the upper bound of \( u^{-2} \). \( \square \)

Since \( M \) converges to 0 exponentially, we can obtain the interior Schauder estimates. Let \( I = [1, t_1] \) and \( I' = [t_0, t_1] \) with \( 1 < t_0 < t_1 \). For a solution \( u \in C^{2+\alpha}(A_I) \) with a source function \( \bar{R} \in C^\alpha(A_I) \) and satisfying

\[
0 < \delta_0 \leq u^{-2}(t, x) \leq \delta_0^{-1} \quad \text{for all } (t, x) \in A_I,
\]

for some constant \( \delta_0 \), we have

\[
\|u\|_{2+\alpha', I'} \leq C \left( t_0, t_1, \delta_0, \|R\|_{\alpha, I}, \|M\|_{\alpha, I}, \|\bar{R}\|_{\alpha, I} \right),
\]

for some \( 0 < \alpha' < 1, \alpha' = \alpha'(\delta_0) \).
Theorem 11. Assume that $\bar{R} \in C^\alpha(N)$ and the constant $K$ is defined by

$$K = \sup_{1 \leq t < \infty} \left\{ - \int_1^t \left( \frac{R}{2} - \frac{\tau^2}{2} \bar{R} \right) \exp \left( \int_1^\tau |M|^2 \frac{ds}{2} \right) d\tau \right\} < \infty.$$  

Then for every $\varphi \in C^{2,\alpha}(\Sigma)$ such that

$$0 < \varphi(x) < \frac{1}{\sqrt{K}}, \quad \text{for all } x \in \Sigma,$$

there is a unique positive solution $u \in C^{2,\alpha}(N)$ of (10) with the initial condition

$$u(1, \cdot) = \varphi(\cdot).$$

Proof. The upper bound of (29) implies that

$$\delta_1(t) + \frac{1}{t} (u^*(1))^{-2} \exp \left( - \int_{t_0}^t s |M|^2 \frac{ds}{2} \right) > 0 \quad \text{for all } t \geq 1.$$

By the short time existence of parabolic equations and the Schauder estimates, there is $\epsilon > 0$ and $u \in C^{2+\alpha}(A_{1,1+\epsilon})$ satisfying the initial condition $u(1) = \varphi$ on $[1,1+\epsilon]$ for some $\epsilon > 0$. By Lemma 10 there are functions $0 < \delta_1(t) \leq \delta_2(t) < \infty$, $1 \leq t$, independent of $\epsilon$, such that

$$0 < \delta_1(t) \leq u^{-2}(t,x) \leq \delta_2(t) \quad \text{for all } t \in [1,1+\epsilon].$$

Let $U = \{ t \in \mathbb{R}^+ : \exists u \in C^{2+\alpha}(A_{1,1+\epsilon}) \text{ satisfying (10) and (33)} \}$. The short time existence guarantees $U$ is open in $\mathbb{R}^+$. From the interior estimate (31), we have an a priori estimate for $||u(1+t,\cdot)||_2$. By the short time existence again, the solution can be extended to $A_{1,1+t+T}$ for some $T$ independent of $u$, which shows that $U$ is closed. Hence $u$ extends to a global solution $u \in C^{2+\alpha}(N)$. \qed

Corollary 12. Let $\bar{R}$ and $K$ be as given in Theorem 11. Suppose $H \in C^{2,\alpha}(\Sigma)$ satisfies

$$H(x) > 2\sqrt{K} \quad \text{for all } x \in \Sigma.$$ 

Then there is a metric $\bar{g}$ with scalar curvature $\bar{R}$ having boundary $\Sigma_1$ with mean curvature $H$.

Proof. Let $\varphi(x) = \frac{2}{H(x)}$. Then the assumption

$$H(x) = \frac{2}{\varphi(x)} > 2\sqrt{K} \quad \text{for all } x \in \Sigma$$

is equivalent to $\varphi(x) < \frac{1}{\sqrt{K}}$. Theorem 11 shows that there exists a unique solution $u \in C^{2+\alpha}(N)$ to the initial value problem, and the resulting metric has boundary $\Sigma$ with mean curvature $\Sigma$ by (9). \qed

Theorem 13. Let $u \in C^{2+\alpha}(N)$ be a solution of (11). Suppose that $\bar{R}$ is given such that

$$\int_1^\infty |\bar{R}|^2 t \, dt < \infty,$$

and suppose there is a constant $C > 0$ such that for all $t \geq 1$

$$||\bar{R}^2||_{\alpha, t} \leq \frac{C}{t},$$
where $I_t = [t, 4t]$. Then $\bar{g}$ satisfies the asymptotically flat condition for $t > t_0$, where $t_0$ is some fixed constant. Moreover, the Riemannian curvature $\tilde{R}$ of the 3-metric $\bar{g}$ on $N$ is H"{o}lder continuous and decays as $|\tilde{R}m| < \frac{C}{t^3}$, and ADM mass can be expressed as
\begin{equation}
\label{eqn:adm_mass}
m_{ADM} = \lim_{t \to \infty} \frac{1}{4\pi} \int_{\Sigma} \frac{t}{2} (1 - u^{-2})d\sigma.
\end{equation}

**Proof.** We compare $\bar{g}$ with the flat metric $\hat{g} = dt^2 + t^2\sigma$, and get
\[
\bar{g} - \hat{g} = (u^2 - 1)dt^2 + t^2(g(t) - \sigma).
\]
Since $g(t)$ converges to the round metric $\sigma$ exponentially, to prove the asymptotic flatness of the metric $\bar{g}$, it suffices to show that
\[
|u^2 - 1| + |t\partial_t u| + |\nabla_i u| \leq \frac{C}{t} \quad i = 1, 2.
\]
Define $\tilde{u}(t, x) = u(\tau t, x)$. Observe that $u \in C^{2+\alpha}(A_{[\tau, 4\tau]} \sigma)$ satisfies (11) if and only if $\tilde{u}$ on the interval $[1, 4]$ satisfies
\[
\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} \tilde{u}^2 \Delta \tilde{u} + \frac{t^2}{4} |\tilde{M}|^2 \tilde{u} + \frac{1}{2} \tilde{u} - \tilde{R} \tilde{u} + \frac{t^2}{4} \tilde{R} \tilde{u}^3,
\]
where $\tilde{M}(t, x) = \tau M(\tau t, x)$, $\tilde{R}(t, x) = R(\tau t, x)$, and $\tilde{R}(t, x) = \tau^2 \tilde{R}(\tau t, x)$.

Applying the Schauder estimates to $\tilde{u}$ on the interval $[1, 4]$, and then rescaling back, we have $||u||_{2+\alpha, I_\tau} \leq C$, and
\begin{equation}
\label{eqn:u_bound}
||m||_{2+\alpha, I_\tau} \leq C||m||_{0, I_\tau} + C\tau \left(||\tau M||_{0, I_\tau} + ||R - 2||_{0, I_\tau} + ||\tilde{R}||_{0, I_\tau}\right),
\end{equation}
where $I_\tau = [2\tau, 4\tau]$, $I_\tau = [\tau, 4\tau]$, and $C$ is independent of $\tau$ and $\sigma$. $M$ converges to 0, and $R$ converges to 2 exponentially fast under Ricci flow. By the assumption $\int_1^\infty |\tilde{R}|^2 t^2 dt < \infty$, we have
\[
1 - \frac{C}{\tau} \leq \delta_\tau(t) \leq \delta^\tau(t) \leq 1 + \frac{C}{\tau} \quad \text{for all } t \geq 1,
\]
which controls $||m||_{0, I_\tau}$. Exponential convergence of $M$ and $R - 2$, together with the decay assumption $||\tilde{R}^2||_{0, I_\tau} \leq \frac{C}{\tau}$, controls the second term of (35). Hence there is a uniform bound
\[
||m||_{2+\alpha, I_\tau} \leq C \quad \text{for all } \tau \geq 1.
\]
Expressing this in terms of $u$ and derivatives of $u$ gives
\[
||1 - u^{-2}||_{0, I_\tau} + ||\partial_\tau u||_{0, I_\tau} + ||\nabla u||_{0, I_\tau} + ||\nabla^2 u||_{0, I_\tau} \leq \frac{C}{\tau}.
\]
The estimates for $1 - u^{-2}$ and $\nabla u$ show that $\bar{g}$ is asymptotically Euclidean. The estimate for $\nabla^2 u$, together with the expression of the Ricci curvature of $\bar{g}$ in terms of $u$, shows that $\tilde{R} \in C^{0,\alpha}(N)$ and $|\tilde{R}(t, x)| \leq \frac{C}{\tau^3}$. Thus, the Riemannian curvature on $N$ is H"{o}lder continuous and decays as $|\tilde{R}m| < \frac{C}{\tau^3}$.

Let $(\Omega, g) \rightarrow M$ be a compact three manifold with smooth boundary $\Sigma$. The Hawking quasi local mass $m_H(\Sigma)$ is defined by [7]
\[
m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int H^2 d\sigma\right).
\]
Although \( \Sigma_t = \{t\} \times \Sigma \) are not round spheres, the sequence \( \Sigma_t \) approaches the round sphere exponentially. Thus the Hawking mass \( m_H(\Sigma_t) \) approaches ADM mass of the asymptotically flat 3-manifold \( N \) as \( t \to \infty \) (see [11]). Since the area of \( \Sigma \) is normalized to \( 4\pi \) and the Ricci flow preserves the area, we have the area 
\[
A(\Sigma_t) = 4\pi t^2,
\]
the mean curvature \( H = 2tu \) for each leaf \( \Sigma_t \). The Hawking mass \( m_H(\Sigma_t) \) is given by
\[
m_H(\Sigma_t) = \sqrt{\frac{A(\Sigma_t)}{16\pi}} \left( 1 - \frac{1}{16\pi} \int H^2 d\sigma \right)
\]
and then the ADM mass of \( N \) is
\[
m_{\text{ADM}} = \lim_{t \to \infty} \frac{1}{4\pi} \int_{\Sigma_t} \frac{t}{2} (1 - u^{-2}) d\sigma.
\]

Theorems [11] and [13] together thus give Theorem [3]. Last, we prove Theorem [4].

**Theorem 4.** Let \( \bar{R} \in C^\alpha(N) \). Suppose that \( \bar{R} t^2 < R \) for \( 1 \leq t < \infty \). Let \( 0 < \eta < 1 \) be such that
\[
1 - \eta < R - \bar{R}|_{t=1} < (1 - \eta)^{-1}.
\]
Then there is a solution \( u^{-1} \in C^{2+\alpha}(N) \) such that the constructed metric on \( N \) has curvature uniformly bounded on \( A_{[1,2]} \) with totally geodesic boundary \( \Sigma \).

**Proof.** To prove Theorem [4] we define
\[
\tilde{u}(t) = \sqrt{\frac{t}{t+1}} u(t+1) \quad \text{where } t \in (0, \infty),
\]
and obtain a parabolic equation for \( \tilde{u} \). Secondly, the equation of \( \tilde{u} \) has the same form as the \( u \)-equation
\[
\frac{\partial \tilde{u}}{\partial t}(t) = \frac{1}{2} \tilde{u}^2 \Delta \tilde{u} + \frac{t^2}{2} |\tilde{M}|^2 \tilde{u} + \frac{1}{2} \tilde{u} - \frac{\bar{R}}{4} \tilde{u}^3 + \frac{t^2 \bar{R}_N}{4} \tilde{u}^3,
\]
where the fields \( \tilde{M} \), \( \bar{R} \), and \( \bar{R}_N \) are defined by
\[
|\tilde{M}(t)|^2 = \frac{t+1}{t} |M(t+1)|^2,
\]
\[
\bar{R}(t) = R(t+1),
\]
\[
t^2 \bar{R}_N(t) = (t+1)^2 \bar{R}(t+1).
\]
The equation for the corresponding term \( \tilde{w} = \tilde{u}^{-2} \) is
\[
t \partial_t \tilde{w} = \frac{1}{2\tilde{w}} \Delta \tilde{w} + \frac{3}{2} \tilde{u} \nabla \tilde{u} \cdot \nabla \tilde{w} - \left( \frac{t^2}{2} |\tilde{M}|^2 + 1 \right) \tilde{w} + \frac{\bar{R}}{2} - \frac{t^2}{2} \bar{R}_N.
\]
The curvature assumption \( R t^2 < R \) for \( 1 \leq t < \infty \) implies that there exist functions
\[
\delta_\epsilon(t) = \frac{1}{t} \int_0^t \left( \frac{\bar{R}}{2} - \frac{\tau^2}{2} \bar{R}_N \right) (\tau) \exp \left( - \int_\tau^t s |\bar{M}|^2 ds \right) d\tau
\]
and
\[
\delta^\ast(t) = \frac{1}{t} \int_0^t \left( \frac{\bar{R}}{2} - \frac{\tau^2}{2} \bar{R}_N \right) (\tau) \exp \left( - \int_\tau^t s |\bar{M}|^2 ds \right) d\tau
\]
so that
\[
0 < \delta_\epsilon(t) \leq \delta^\ast(t) < \infty \quad \text{for all} \quad t > 0.
\]
Let \( \varphi_\epsilon \in C^{2,\alpha}(\Sigma), 0 < \epsilon < 1 \) be any family of functions satisfying
\[
\delta_\epsilon(\epsilon) \leq \varphi^{-2}_\epsilon(x) \leq \delta^\ast(\epsilon).
\]
Applying the parabolic maximum principle to (38) for \( \tilde{w} = \tilde{u}^{-2} \), we can show that the solution \( \tilde{u}_\epsilon \) exists for all time \([\epsilon, \infty)\) with initial condition \( \varphi_\epsilon \) and
\[
\delta_\epsilon(t) \leq \tilde{u}_\epsilon^{-2}(t,x) \leq \delta^\ast(t), \quad \epsilon \leq t < \infty,
\]
for all \( 0 < \epsilon < 1 \).

Now suppose \( t_0 > 0 \) and \( v \in C^{2+\alpha}(A_I), I = [t_0, 4t_0], \) is a solution of (37) satisfying
\[
\dot{\delta}_\epsilon(t) \leq v^{-2}(t,x) \leq \dot{\delta}^\ast(t), \quad \text{for all} \quad t \in I,
\]
and define \( \bar{v}(t,x) = v(t/t_0, x) \). Applying the Schauder estimates to \( \bar{v} \) on the interval \([1, 4]\) and rescaling back, we obtain an estimate of the form
\[
\|v\|_{2+\alpha, I'} \leq C, \quad I' = [2t_0, 4t_0],
\]
where \( C \) is a constant independent of \( v \). Applying (39) to \( \tilde{u}_\epsilon \), shows, by Arzela-Ascoli theorem, there is a sequence \( \epsilon_j \to 0 \) such that \( \{\tilde{u}_\epsilon\} \) converges uniformly in \( C^{2+\alpha}(A_I) \) for any compact interval \( I \subset \mathbb{R}^+ \) to the desired solution \( \tilde{u} \in C^{2+\alpha}(A_{[0,\infty)}) \).

Moreover, since \( |M| \leq Ce^{-ct} \) for some constants \( c \) and \( C \), and (36), there is a small \( \epsilon \) such that on \((0, \epsilon), 1 - \eta < R - t^2 \bar{R}_N < (1 - \eta)^{-1}, \)
\[
\frac{1}{t} \int_0^t \exp \left( - \int_\tau^t s |\bar{M}|^2 ds \right) d\tau \quad < \quad 1,
\]
and
\[
\frac{1}{t} \int_0^t \exp \left( - \int_\tau^t s |\bar{M}|^2 ds \right) d\tau \quad < \quad 1.
\]
It shows that on \((0, \epsilon)\)
\[
1 - \eta \leq \tilde{\delta}_\epsilon(t) \quad \leq \quad \tilde{\delta}^\ast(t) \quad < \quad (1 - \eta)^{-1},
\]
and
\[
\frac{-\eta t}{1 - \eta} \quad < \quad 2\tilde{m}(t) \quad < \quad \eta t,
\]
where \( \tilde{m}(t) = \frac{t}{2} (1 - \tilde{u}^{-2}(t)) = m(1 + t) - \frac{1}{2} \). The rescaling Schauder estimate shows that the covariant derivatives of \( m \) decay,
\[
|\nabla m(t)| + |\nabla^2 m(t)| \leq C(t - 1).
\]
It follows that the curvature of \( \tilde{g} \) is bounded on \( A_{[1,2]} \). \( \square \)
Corollary 14. If we start with the standard metric $(\Sigma, \sigma)$ and prescribe the scalar curvature $\bar{R} \equiv 0$, then the metric $\bar{g}$ obtained from above is exactly a Schwarzschild metric with ADM mass $m_{ADM} = \frac{1}{2}$.

Proof. Since the initial metric is the standard round metric, the modified Ricci flow doesn’t change the metric and $g(t) = \sigma$, $R \equiv 2$, and $M_{ij} \equiv 0$ for all $t \geq 1$. Since $\bar{R} \equiv 0$, the rescaling fields are

$$\bar{R} = 2, \quad \bar{M}_{ij} \equiv 0, \quad \text{and} \quad \bar{R}_N \equiv 0$$

for all $t \geq 0$. Moreover the a priori bounds are $\bar{\delta}_s(t) = 1$ and $\bar{\delta}^*(t) = 1$ for all $t \geq 0$. Hence we have the solution $\bar{u} \equiv 1$ and the metric

$$\bar{g} = \frac{1}{1 - \frac{1}{t}} dt^2 + t^2 g_{S^2}.$$

\[\square\]

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