SELF-DUAL MODULES IN CHARACTERISTIC TWO AND NORMAL SUBGROUPS

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ABSTRACT. We prove Clifford theoretic results on the representations of finite groups which only hold in characteristic 2.

Let $G$ be a finite group, let $N$ be a normal subgroup of $G$ and let $\varphi$ be an irreducible 2-Brauer character of $N$ which is self-dual. We prove that there is a unique self-dual irreducible Brauer character $\theta$ of $G$ such that $\varphi$ occurs with odd multiplicity in the restriction of $\theta$ to $N$. Moreover this multiplicity is 1.

Conversely if $\theta$ is an irreducible 2-Brauer character of $G$ which is self-dual but not of quadratic type, the restriction of $\theta$ to $N$ is a sum of distinct self-dual irreducible Brauer character of $N$, none of which have quadratic type.

Let $b$ be a real 2-block of $N$. We show that there is a unique real 2-block of $G$ covering $b$ which is weakly regular.

1. Statement of results

Throughout the paper $G$ is a finite group and $N$ is a normal subgroup of $G$. We fix a 2-modular system $(K, R, F)$ for $G$. So $R$ is a complete discrete valuation ring which has field of fractions $K$ of characteristic 0 and residue field $F = R/J(R)$ of characteristic 2. We will assume that $K$ and $F$ are splitting fields for all subgroups of $G$. For example, this holds if $K$ contains a primitive $|G|$-th root of unity. We use $r^*$ to denote the image of $r \in R$ in $F$. Each integer $m$ can be factored as $m = m_2m_2'$, where $m_2$ is a power of 2 and $m_2'$ is odd.

We use $\text{Irr}(G)$ to denote the irreducible $K$-characters of $G$. These have values in a cyclotomic subfield of $K$ which can be identified with a subfield of $\mathbb{C}$. So $\text{Irr}(G)$ can be identified with the irreducible complex characters of $G$. Next recall that the Brauer character of an $FG$-module is an $R$-valued class function defined on the 2-regular (odd order) elements of $G$. The Brauer characters of the irreducible $FG$-modules are called the irreducible 2-Brauer characters of $G$. We use $\text{IBr}(G)$ to denote all such characters. The dual of a character $\theta$ is the character $\overline{\theta}$ defined by $\overline{\theta}(g) := \theta(g^{-1})$, for all $g \in G$. We say that $\theta$ is self-dual if $\theta = \overline{\theta}$. This holds if and only if $\theta$ is the character of some self-dual module.

Let $\theta$ be an irreducible Brauer character of $G$ and let $\varphi$ be an irreducible Brauer character of a subgroup $H$ of $G$. We say that $\theta$ lies over $\varphi$ if $\theta$ is constituent of the induced character $\varphi^G$. Likewise we say that $\varphi$ lies under $\theta$ if $\varphi$ is a constituent of the restricted
character $\theta^H$. There is no analogue of Frobenius reciprocity for Brauer characters. So the fact that $\theta$ lies over $\varphi$ does not imply that $\varphi$ lies under $\theta$, and conversely. However these implications do hold if $H$ is a normal subgroup of $G$. See [N, p.155 and (8.7)]. Here is our first result:

**Theorem 1.** Let $\varphi$ be an irreducible 2-Brauer character of $N$. Then $\varphi$ lies under some self-dual irreducible 2-Brauer character $\theta$ of $G$ if and only if $\varphi$ is $G$-conjugate to $\overline{\varphi}$. If such a self-dual $\theta$ exists, it can be chosen so that $\varphi$ occurs with odd multiplicity in $\theta^N$.

**Example:** Let $G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3 \rangle$ be the semi-dihedral group of order 16. The two faithful irreducible $K$-characters $\mu \neq \overline{\mu}$ of $\langle a^2 \rangle$ are $G$-conjugate. Now the irreducible $K$-characters of $G$ lying over $\mu$ consist of an irreducible $K$-character and its dual. So Theorem 1 does not generalize to irreducible $K$-characters nor to irreducible $p$-Brauer characters, for primes $p \neq 2$.

Our second result is:

**Theorem 2.** Let $\varphi$ be a self-dual irreducible 2-Brauer character of $N$. Then

(i) $\varphi$ extends to its stabilizer in $G$, and exactly one such extension is self-dual.

(ii) $G$ has a unique self-dual irreducible 2-Brauer character $\theta$ such that $\varphi$ occurs with odd multiplicity in $\theta^N$.

(iii) $\varphi$ occurs with multiplicity 1 in $\theta^N$.

We will refer to $\theta$ as the canonical irreducible Brauer character of $G$ lying over $\varphi$. The module form of this theorem is stated and proved in Theorem 9 below.

Theorem 2 is similar in flavour to a result of I. M. Richards. In [R] he proved that when $G/N$ has odd order, each self-dual irreducible $K$-character of $N$ extends to its stabilizer in $G$, and has a unique self-dual extension.

**Example:** Let $G = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$ be the dihedral group of order 8. Then the non-trivial irreducible $K$-character of $\langle a^2 \rangle$ is self-dual and $G$-invariant, but it does not extend to $D_8$. So Theorem 2 does not generalize to self-dual irreducible $K$-characters nor to irreducible $p$-Brauer characters, for primes $p \neq 2$.

Next recall that a non-trivial irreducible 2-Brauer character of $G$ is said to have quadratic type if the corresponding $FG$-module affords a non-zero $G$-invariant quadratic form. Our first application is:

**Theorem 3.** Let $\theta$ be a non-quadratic type self-dual irreducible 2-Brauer character of $G$ which does not lie over the trivial character of $N$. Then $\theta^N$ is a sum of non-quadratic type self-dual irreducible Brauer characters of $N$, each occurring with multiplicity 1.

The second application is to blocks. For undefined notation, see Section 6 below and for a full exposition of block theory, see Chapter 5 of [NT].

Let $B$ be a 2-block of irreducible $K$-characters of $G$. Then the duals of the characters in $B$ form another 2-block $B^\circ$, called the contragredient of $B$. We say that $B$ is real if $B = B^\circ$. Recall that $B$ is said to cover a 2-block $b$ of $N$ if the restriction of an irreducible...
character in $B$ contains an irreducible character in $b$. Also we say that $B$ is weakly regular (with respect to $N$) if it has maximal defect among the blocks of $G$ which cover $b$.

**Theorem 4.** Let $b$ be a 2-block of $N$. Then

(i) $G$ has a real weakly regular block covering $b$ if and only if $b$ and $b^o$ are $G$-conjugate.
(ii) if $b = b^o$, then $G$ has a unique real weakly regular block of $G$ covering $b$.

We prove (i) in Lemma 18 and (ii) in Lemma 20.

Recall that corresponding to each irreducible 2-Brauer character $\theta$, $G$ has a principal indecomposable character $\Phi_\theta$. Then $\Phi_\theta$ is a $K$-character of $G$ which vanishes off the 2-regular elements of $G$. We use the following result, which is implicit in [GW93, 1.4], to prove Theorem 4. As it may be of independent interest, we include a short proof here:

**Lemma 5.** Let $B$ be a 2-block of $G$. Then $B$ has an odd number of height 0 irreducible Brauer characters $\theta$ such that $\Phi_\theta(1)_2 = |G|_2$.

**Proof.** Let $D$ be a defect group of $B$. Then Brauer showed that $\frac{\dim(B)}{|G||G:D|}$ is a unit in $R$. See [NT, 5.10.1]. Now $\dim(B) = \sum_{\theta \in IBr(B)} \Phi_\theta(1)\theta(1)$. It is known that $\Phi_\theta(1)/|G|$ and $\theta(1)/|G : D|$ belong to $R$, for all $\theta \in IBr(B)$. So Brauer’s result gives us an identity in $F$:

$$\sum_{\theta \in IBr(B)} \left(\frac{\Phi_\theta(1)}{|G|}\right)^* \left(\frac{\theta(1)}{|G : D|}\right)^* = \left(\frac{\dim(B)}{|G||G : D|}\right)^* = 1_F.$$

The contribution of $\theta$ to the left hand side is $1_F$, if $\theta$ has height 0 and $\Phi_\theta(1)_2 = |G|_2$. Otherwise the contribution is $0_F$. So the lemma follows directly from the above equality. $\Box$

Many 2-blocks have an odd number of height 0 irreducible Brauer characters. For example, the main result of [KOW] is that each 2-block of a symmetric group has a unique height 0 irreducible Brauer character. Furthermore, it is known that each principal indecomposable character $\Phi$ of a finite solvable group satisfies $\Phi(1)_2 = |G|_2$. So Lemma 5 implies that each 2-block of a finite solvable group has an odd number of height 0 irreducible Brauer characters. However, as B. Sambale has pointed out, the faithful 2-block of $3.Suz.2$ has four height 0 irreducible Brauer characters. Three of these satisfy the condition of Lemma 5 on their principal indecomposable character degree.

2. **Real orbits of irreducible Brauer characters**

Recall that $g \in G$ is said to be real in $G$ if $xgx^{-1} = g^{-1}$, for some $x \in G$. Similarly a conjugacy class of $G$ is real if its elements are real, and 2-regular if its elements have odd order. Now $G$ acts on the conjugacy classes, the irreducible $K$-characters and the Brauer characters of its normal subgroup $N$. We say that a $G$-orbit of conjugacy classes of $N$ is real if its union is a real conjugacy class of $G$. Likewise we say that a $G$-orbit of irreducible Brauer characters of $N$ is real if it contains the duals of its characters.
Proof of Theorem 4.4. It is clear that each self-dual irreducible Brauer character of $G$ lies over a real $G$-orbit of irreducible Brauer characters of $N$.

Suppose that $G$ has $\ell$ conjugacy classes of 2-regular elements, with representatives $g_1, \ldots, g_{\ell}$. Let $\theta_1, \ldots, \theta_{\ell}$ be the irreducible Brauer characters of $G$ and let $\Phi_1, \ldots, \Phi_{\ell}$ be the corresponding principal indecomposable characters of $G$. The second orthogonality relations give

$$\sum_{\chi \in \text{Irr}(G)} \chi(g_i) \chi(g_j) = \delta_{i,j} |C_G(g_i)|, \quad \text{for all } i, j \in \{1, \ldots, \ell\}.$$ 

Now for all $\chi \in \text{Irr}(G)$, we have $\chi(g_j) = \sum_{u=1}^{\ell} d_{\chi,\theta_u} \theta_u(g_j)$, where the $d_{\chi,\theta_u}$ are non-negative integers, called decomposition numbers. Then for all $u = 1, \ldots, \ell$, we have $\Phi_u(g_i) = \sum_{\chi \in \text{Irr}(G)} d_{\chi,\theta_u} \chi(g_i)$. It is known that $\frac{\Phi_u(g_i)}{|C_G(g_i)|} \in R$, for all $u, i$. So the above displayed equation can be rewritten in $R$ as

$$(1) \quad \sum_{u=1}^{\ell} \frac{\Phi_u(g_i^{-1})}{|C_G(g_i)|} \theta_u(g_j) = \delta_{i,j}, \quad \text{for all } i, j \in \{1, \ldots, \ell\}.$$ 

In particular the Brauer character table $[\theta_i(g_j)]$ of $G$ is a non-singular $\ell \times \ell$ matrix.

(We note that the proof of [N], (2.18)] shows that $\det[\theta_i(g_j)]^2 = \pm \prod_{j=1}^{\ell} |C_G(g_j)|^{2^r}.$)

Suppose that $G$ has $r$ real conjugacy classes of 2-regular elements, which we may assume have representatives $g_1, \ldots, g_r$. We choose notation so that $\theta_1, \ldots, \theta_r$ are the self-dual irreducible Brauer characters of $G$. Then the self-dual Brauer character table of $G$ is the $r \times r$ submatrix $T := [\theta_i(g_j)]$ of the Brauer character table. Suppose that $u \in \{r+1, \ldots, \ell\}$. Then there is a unique $\overline{u} \in \{r+1, \ldots, \ell\}$, with $\overline{u} \neq u$, such that $\theta_{\overline{u}} = \overline{\theta_u}$. So $\frac{\Phi_u(g_i^{-1})}{|C_G(g_i)|} \theta_u(g_j) = \frac{\Phi_{\overline{u}}(g_i^{-1})}{|C_G(g_i)|} \theta_{\overline{u}}(g_j)$, for all $i, j$. So the contribution of the summands indexed by $u$ and $\overline{u}$ to (1) is 0 mod $J(R)$, and we deduce that

$$\sum_{u=1}^{r} \frac{\Phi_u(g_i^{-1})}{|C_G(g_i)|} \theta_u(g_j) \equiv \delta_{i,j} \mod J(R), \quad \text{for all } i, j \in \{1, \ldots, r\}.$$ 

As $R$ is a local ring, it follows that $T$ is invertible, with inverse congruent to the $r \times r$-matrix $[\frac{\Phi_u(g_i^{-1})}{|C_G(g_i)|}]$ mod $J(R)$. In particular $\det(T) \notin J(R)$.

Now suppose that $G$ has $t$ real conjugacy classes of 2-regular elements which are contained in $N$, with representatives $n_1, \ldots, n_t$. We relabel the $\theta_1, \ldots, \theta_r$ so that the $t \times t$ submatrix $S := [\theta_i(n_j)]$ of $T$ satisfies $\det(S) \notin J(R)$.

For $i = 1, \ldots, t$, let $\varphi_i$ be an irreducible Brauer character of $N$ which is a constituent of $\theta_{i \downarrow N}$, and set $\hat{\varphi}_i$ as the sum of the distinct $G$-conjugates of $\varphi_i$. Then $\theta_{i \downarrow N} = e_i \hat{\varphi}_i$, for some positive integer $e_i$. The non-singularity of $S$ implies that all the multiplicities $e_1, \ldots, e_t$ are odd and $\varphi_1, \ldots, \varphi_t$ lie in distinct $G$-orbits. Moreover each of these orbits is real, as each $\theta_i$ is self-dual.
By the non-singularity of the Brauer character table of $N$ and Brauer’s permutation lemma, $G$ has $t$ real orbits on the irreducible Brauer characters of $N$. So $\varphi_1, \ldots, \varphi_t$ represent all real $G$-orbits of irreducible Brauer characters of $N$.

Our work above shows that if $\varphi$ is an irreducible Brauer character of $N$ which is $G$-conjugate to $\overline{\varphi}$, then $G$ has a self-dual irreducible Brauer character $\theta$ such that $\varphi$ occurs with odd multiplicity in $\theta_{\downarrow N}$. This concludes the proof of Theorem 1.

□

3. Clifford Theory for self-dual irreducible modules

We prove Theorem 2 in this section. Recall that the dual of an $FG$-module $V$ is the $FG$-module $V^* := \text{Hom}_F(V, F)$. We say that $V$ is self-dual if $V \cong V^*$ as $FG$-modules.

For the reader’s convenience, we begin by stating a module version of Clifford’s Theorem:

**Lemma 6** (Clifford 1937). Let $F$ be an arbitrary field, let $V$ be an irreducible $FG$-module and let $W$ be an irreducible submodule of $V_{\downarrow N}$. Set $T$ as the stabilizer of $W$ in $G$. Then

(i) $V_{\downarrow N} \cong e(W_1 \oplus \cdots \oplus W_n)$ for some integer $e > 0$, where $W_1, \ldots, W_n$ are the distinct $G$-conjugates of $W$.

(ii) Let $U$ be the sum of all submodules of $V_{\downarrow N}$ which are isomorphic to $W$. Then $U$ is an irreducible $FT$-module, $U_{\downarrow N} = eW$ and $U^G = V$.

(iii) If $W$ is absolutely irreducible, it extends to a projective $FT$-module $\hat{W}$ and $U \cong P \otimes \hat{W}$, for some projective $F(T/N)$-module $P$.

(iv) If $W$ extends to an $FT$-module $\hat{W}$, then the distinct irreducible $FG$-modules lying over $W$ are $(S_1 \otimes \hat{W})^G, \ldots, (S_t \otimes \hat{W})^G$ where $S_1, \ldots, S_t$ are the distinct irreducible $F(T/N)$-modules.

*Proof.* See for example Huppert and Blackburn, Finite Groups II, VII, 9.12. □

We would next like to point out that Fong’s Lemma holds for all self-dual irreducible $FG$-modules $V$ over all fields $F$ of characteristic 2. In particular $F$ need not be perfect:

**Lemma 7** (Fong’s Lemma). Let $G$ be a finite group, let $F$ be an arbitrary field of characteristic 2 and let $V$ be a non-trivial self-dual irreducible $FG$-module. Then $V$ affords a non-degenerate $G$-invariant alternating bilinear form. In particular dim($S$) is even.

*Proof.* As $V$ is self-dual, it affords a non-degenerate $G$-invariant bilinear form $B$. It may happen that $B$ is symmetric. If not, set $\tilde{B}(v_1, v_2) = B(v_1, v_2) + B(v_2, v_1)$, for all $v_1, v_2 \in V$. Then $\tilde{B}$ is a non-zero $G$-invariant symmetric bilinear form. Now the radical of $\tilde{B}$ is a submodule of $V$. As $V$ is irreducible, it follows that the radical is zero. So $\tilde{B}$ is non-degenerate.

By the previous paragraph $V$ affords a $G$-invariant non-degenerate symmetric bilinear form, henceforth denoted $B$. We claim that $B$ is alternating, meaning $B(v, v) = 0$, for all $v \in V$. For suppose otherwise. Set $Q(v) := B(v, v)$, for all $v \in V$. Then $Q$ is a non-zero
G-invariant quadratic form on V. Now Q is additive, as for all \( v_1, v_2 \in V \), we have
\[
Q(v_1 + v_2) = B(v_1 + v_2, v_1 + v_2) \\
= B(v_1, v_1) + B(v_1, v_2) + B(v_2, v_1) + B(v_2, v_2) \\
= Q(v_1) + Q(v_2), \text{ using } B(v_1, v_2) + B(v_2, v_1) = 0.
\]
Moreover \( Q(\lambda v) = \lambda^2 Q(v) \), for all \( \lambda \in F \) and \( v \in V \). Define \( U := \{ v \in V \mid Q(v) = 0 \} \). Then our work shows that \( U \) is a submodule of \( V \). But \( U \neq V \), as \( Q \neq 0 \). So \( U = 0 \), by irreducibility of \( V \). Let \( v \in V \) and \( g \in G \). Then \( Q(gv + v) = Q(gv) + Q(v) = 0 \), as \( Q \) is additive and \( G \)-invariant. So \( gv + v \in U \), whence \( gv = v \). But then \( G \) acts trivially on \( V \), contrary to hypothesis. This proves our claim.

The final statement follows as every symplectic vector space has even dimension. \( \square \)

**Corollary 8.** Let \( G \) be a finite group and let \( F \) be an arbitrary field of characteristic 2. Then the radical of \( FG \) has odd codimension in \( FG \).

**Proof.** We use \( \text{rad}(FG) \) to denote the radical of \( FG \), which is the intersection of the annihilators of all irreducible \( FG \)-modules. Suppose first that \( F \) is a splitting field for \( G \). Let \( \theta_1, \ldots, \theta_r \) be the irreducible Brauer characters of \( G \), with \( \theta_1, \ldots, \theta_r \) precisely the self-dual characters, and \( \theta_1 \) the trivial Brauer character. We have
\[
\text{dim}(FG) - \text{dim}(\text{rad}(FG)) = \theta_1(1)^2 + \cdots + \theta_r(1)^2.
\]
Now \( \theta_1(1) = 1 \) and \( \theta_i(1) \) is even for \( 2 \leq i \leq r \), by Fong’s Lemma. If \( i > r \), we may pair \( \theta_i \) with its dual, and these two characters have the same degree. It is now clear that
\[
\theta_1(1)^2 + \cdots + \theta_r(1)^2
\]
is odd, and the result follows in this case.

Now suppose that \( F \) is any field of characteristic 2. Set \( E := F(\omega) \), where \( \omega \) is a primitive \( |G|_2 \)-th root of unity in an extension field of \( F \). Then \( E \) is a splitting field for \( G \) and a finite separable extension of \( F \). As \( FG \) contains an \( E \)-basis of \( EG \), it is a standard fact that \( \text{rad}(EG) \) is the \( E \)-span of \( \text{rad}(FG) \). In particular \( \text{dim}_E(\text{rad}(EG)) = \text{dim}_F(\text{rad}(FG)) \). The first part shows that \( \text{dim}_E(EG) - \text{dim}_E(\text{rad}(EG)) \) is odd. So \( \text{dim}_F(EG) - \text{dim}_F(\text{rad}(FG)) \) is odd in this case also. \( \square \)

For the rest of this section \( F \) is a perfect field of characteristic 2. Here is the module version of Theorem 2.

**Theorem 9.** Let \( W \) be a self-dual irreducible \( FN \)-module. Then \( W \) extends to its stabilizer in \( G \), and there is a unique extension \( \tilde{W} \) which is self-dual.

Set \( V := \tilde{W}^G \). Then \( V \) is a self-dual irreducible \( FG \)-module, and \( V\downarrow_N \cong W_1 \oplus \cdots \oplus W_n \), where \( W_1, \ldots, W_n \) are the distinct \( G \)-conjugates of \( W \). Moreover \( V \) is the unique self-dual irreducible \( FG \)-module such that \( W \) occurs with odd multiplicity in \( V\downarrow_N \).

**Proof.** We may assume that \( W \) is non-trivial and \( G \)-invariant. As \( W \) is a self-dual \( FN \)-module, it affords a non-degenerate \( N \)-invariant bilinear form \( B : W \times W \to F \). An application of Schur’s Lemma shows that \( B \) is unique up to scalars.
Let $X : N \to \text{GL}(W)$ be the $F$-representation given by $W$. For each $g \in G$, we define the conjugate representation $X^g$ of $N$ by

$$X^g(n) = X(gng^{-1}), \quad \text{for all } n \in N.$$ 

Then $X$ and $X^g$ are equivalent representations, as $W$ is $G$-invariant. So there is $Y(g) \in \text{GL}(W)$ such that

$$Y(g)X(n) = X^g(n)Y(g), \quad \text{for all } n \in N.$$ 

We choose $Y(g) = X(g)$, whenever $g \in N$. Now for all $g, h \in G$, we have

$$[Y(gh)^{-1}Y(g)Y(h)] X(n) = X(n) [Y(gh)^{-1}Y(g)Y(h)].$$

So by Schur’s Lemma there is a non-zero $\alpha(g, h) \in F$ such that

$$(2) \quad Y(gh) = \alpha(g, h)Y(g)Y(h).$$

Then $\alpha : G \times G \to F^\times$ is an $F$-valued cocycle and $Y$ is a projective representation of $G$ which extends $X$.

Next, for all $g \in G$, we define the bilinear form $B^g : W \times W \to F$ by

$$B^g(u, v) = B(Y(g)u, Y(g)v), \quad \text{for all } u, v \in W.$$

Then for all $n \in N$ we have

$$B^g(X(n)u, X(n)v) = B(Y(g)X(n)u, Y(g)X(n)v) = B(X(gng^{-1})Y(g)u, X(gng^{-1})Y(g)v) = B(Y(g)u, Y(g)v),$$

as $B$ is $X$-invariant

$$= B^g(u, v).$$

This shows that $B^g$ is $X$-invariant. As $B$ is unique up to scalars

$$(3) \quad B^g = \lambda(g)B, \quad \text{for some } \lambda(g) \in F^\times.$$ 

As $B$ is $N$-invariant, we have $\lambda(n) = 1$, for all $n \in N$.

Now for all $g, h \in G$ we have $B^{gh} = \lambda(gh)B$. On the other hand

$$B^{gh}(u, v) = B(Y(gh)u, Y(gh)v) = B(\alpha(g, h)Y(g)Y(h)u, \alpha(g, h)Y(g)Y(h)v), \quad \text{by (2)},$$

$$= \alpha(g, h)^2 \lambda(g)\lambda(h)B(u, v), \quad \text{by (3)}.$$ 

Comparing these expressions, we see that

$$(4) \quad \lambda(gh) = \alpha(g, h)^2 \lambda(g)\lambda(h).$$

Since $F$ is perfect, for each $g$ in $G$, there exists $\mu(g) \in F$ such that $\mu(g)^2 = \lambda(g)$. Set $\hat{Y}(g) = \mu(g)^{-1}Y(g)$ for all $g \in G$. Then $\hat{Y}$ is a projective representation of $G$ which extends $X$. Moreover $\hat{Y}$ corresponds to the cocycle $\beta(g, h) := \mu(g)\mu(h)\mu(gh)^{-1}\alpha(g, h)$.

Now $\beta(g, h)^2 = \lambda(g)\lambda(h)\lambda(gh)^{-1}\alpha(g, h)^2 = 1$ and char$(F) = 2$. So $\beta(g, h) = 1$, for all $g, h \in G$. This means that $\hat{Y}$ is an $F$-representation of $G$ which extends $X$. 

If we now consider the action of the elements $\hat{Y}(g)$ on the bilinear form $B$, a repetition of an earlier argument shows that for all $u$ and $v$ in $W$, and all $g \in G$,

$$B(\hat{Y}(g)u, \hat{Y}(g)v) = \epsilon(g)B(u, v)$$

for some nonzero scalar $\epsilon(g)$. The fact that $\hat{Y}$ is a representation of $G$, and $B$ is $N$-invariant now implies that $\epsilon$ is a homomorphism $G/N \to F^\times$.

Finally, as $F$ has characteristic 2, $\epsilon$ has odd order in the character group of $G/N$. So as $F$ is perfect, $\epsilon = \delta^2$ for a unique homomorphism $\delta : G/N \to F^\times$. Then if we set $\hat{X}(g) = \delta(g)^{-1}\hat{Y}(g)$, we find that $\hat{X}$ is also an $F$-representation of $G$ which extends $X$. Moreover $\hat{X}$ is self-dual, as we can easily check that it leaves $B$ invariant.

Let $\hat{W}$ be the irreducible self-dual $FG$-module corresponding to $\hat{X}$. So $\hat{W}$ extends $W$. Then $S \otimes \hat{W}$ give all irreducible $FG$-modules lying over $W$, as $S$ ranges over all irreducible $FG/N$-modules. Recall that $S \otimes \hat{W} \cong S' \otimes \hat{W}$ if and only if $S \cong S'$. So $S \otimes \hat{W}$ is self-dual if and only if $S$ is self-dual. Fong’s Lemma implies that $\dim(S \otimes \hat{W})$ is an even multiple of $\dim(\hat{W})$, if $S$ is non-trivial and self-dual. So $\hat{W}$ is the unique extension of $W$ to $G$ which is self-dual. The statements about $V$ are now consequences of Lemma 6.

We will refer to $V$ as the canonical self-dual irreducible $FG$-module lying over $W$. Our Corollary, which is probably known, is an analogue of Richards’ Theorem \cite{R} for irreducible 2-Brauer characters:

**Corollary 10.** Suppose that $|G : N|$ is odd. Then

(i) If $W$ is a self-dual irreducible $FN$-module, then $W^G$ has a unique self-dual composition factor, up to isomorphism.

(ii) If $V$ is a self-dual irreducible $FG$-module then $V_N$ is a sum of distinct self-dual irreducible $FN$-modules.

In particular induction-restriction defines a natural correspondence between the self-dual irreducible $FG$-modules and the $G$-orbits of self-dual irreducible $FN$-modules.

**Proof.** We may assume that $W$ is $G$-invariant. Let $\hat{W}$ be the unique self-dual irreducible $FG$-module which extends $W$. Then all irreducible $FG$-modules lying over $W$ have the form $U \otimes W$, for some irreducible $FG/N$-module $U$. But $G/N$ has odd order. So $U$ is self-dual if and only if it is trivial. It follows that $\hat{W}$ is the unique self-dual irreducible $FG$-module lying over $W$. All composition factors of $W^G$ lie over $W$. So (i) holds.

For (ii), write $V_N = \epsilon(W_1 \oplus \cdots \oplus W_t)$, where $\epsilon,t \geq 1$ and $W_1,\ldots,W_t$ are distinct irreducible $FN$-modules. Now $V_N$ is a self-dual irreducible $FN$-module. So for each $i = 1,\ldots,t$, $W_i^* \cong W_j$, for some $j = 1,\ldots,t$. The map $i \mapsto j$ is an involutary permutation of $\{1,\ldots,t\}$. But $t$ is odd, as by Clifford Theory it divides $|G : N|$. So there exists $i$ such that $W_i^* \cong W_i$, whence all $W_1,\ldots,W_t$ are self-dual. Now by part (i), $V$ is the unique self-dual irreducible $FG$-module lying over $W_1$. It then follows from Theorem \cite{M} that $\epsilon = 1$ i.e. $V_N = W_1 \oplus \cdots \oplus W_t$.

The last statement follows from (i) and (ii). \qed
We also have a fairly obvious extension of Theorem 2 to subnormal subgroups:

**Corollary 11.** Let $S$ be a subnormal subgroup of $G$ and let $U$ be a self-dual irreducible $FS$-module. Then there is a unique self-dual irreducible $FG$-module $V$ such that $U$ occurs with odd multiplicity in $V_{↓S}$.

**Proof.** We use induction on $|G : S|$. By Theorem 2, we may assume that $S$ is not a normal subgroup of $G$. So there exists $S \subseteq N \subseteq G$ such that $S$ is subnormal in $N$ and $N$ is normal in $G$. As $|N : S| < |G : S|$, our inductive assumption implies that there is a unique self-dual irreducible $FN$-module $W$ such that $U$ has odd multiplicity in $W_{↓S}$. Let $V$ be the canonical $FG$-module over $W$.

Now $V_{↓N} = W \oplus W_2 \oplus \cdots \oplus W_t$ is the sum of the distinct $G$-conjugates of $W$. As $W$ is self-dual, all the $W_i$ are self-dual. By choice of $W$, $U$ appears with even multiplicity in $W_{↓S}$ for $i = 2, \ldots, t$. Since $V_{↓S} = W_{↓S} \oplus W_2_{↓S} \oplus \cdots \oplus W_t_{↓S}$, we deduce that $U$ appears with odd multiplicity in $V_{↓S}$.

Now let $V'$ be a self-dual irreducible $FG$-module such that $U$ occurs with odd multiplicity in $V'_{↓S}$. Write $V'_{↓N} = e(W'_1 \oplus \cdots \oplus W'_s)$, for some odd integer $e$, where $W'_1, \ldots, W'_s$ are distinct irreducible $FN$-modules. We claim that one and hence all $W'_i$ are self-dual. For suppose otherwise. As $V'_{↓N}$ is self-dual, for each $i$ there is a unique $j \neq i$ such that $W'_i^* \cong W'_j$. Then $U$, being self-dual, occurs with the same multiplicity in $W'_{↓S}$ as in $W'_{↓S}$. So $U$ occurs with even multiplicity in $e(W'_i \oplus W'_j)$, and hence with even multiplicity in $V'_{↓S}$. This contradiction proved our claim. So $e = 1$ and $V'$ is the canonical $FG$-module over $W'$. Now we may assume that $U$ appears with odd multiplicity in $W'_{↓S}$. So $W \cong W'_1$, by uniqueness of $W$ over $U$, and then $V' \cong V$, by uniqueness of $V$ over $W$. □

**Remark:** In the context of the Corollary, suppose that $S \subseteq N \subseteq G$, and let $U$ be an irreducible self-dual $FS$-module. Let $W$ be the canonical $FN$-module over $U$ and let $V$ be the canonical $FG$-module over $W$. We claim that that $U$ occurs with multiplicity 1 in $V_{↓S}$. For suppose otherwise. Then $V_{↓N}$ has an irreducible submodule $W' \neq W$ such that $U$ occurs with non-zero multiplicity in $W'_{↓S}$. As $W'$ is $G$-conjugate to $W$, it is self-dual and $\dim(W') = \dim(W)$. On the other hand, $U$ occurs with even multiplicity $e > 1$ in $W'_{↓S}$. Setting $t$ as the number of distinct $N$-conjugates of $U$, we get the contradiction

$$\dim(W') = et \dim(U) = e \dim(W) > \dim(W).$$

This proves our claim. In view of this, we ask:

**Question:** do there exist $(G, V)$ and $(S, U)$, where $G$ is a finite group, $S$ is a subnormal subgroup of $G$, $V$ is a self-dual irreducible $FG$-module and $U$ is a self-dual irreducible $FS$-module, such that $U$ occurs with multiplicity $e$ in $V_{↓S}$, where $e$ is odd but not 1?

4. **Irreducible self-dual modules of non-quadratic type**

Let $V$ be a non-trivial self-dual irreducible $FG$-module. Then by Lemma 7, $V$ affords a non-degenerate $G$-invariant alternating form $B$. Let $Q : V \to F$ be a quadratic form
which polarizes to $B$. This means that $Q(\lambda v_1) = \lambda^2 Q(v_1)$ and $Q(v_1 + v_2) = Q(v_1) + B(v_1, v_2) + Q(v_2)$, for all $\lambda \in F$ and $v_1, v_2 \in V$. However, contrary to what happens when \text{char}(F) \neq 2, Q$ is not uniquely determined by $B$. In particular $Q$ need not be $G$-invariant.

On the other hand, for many $FG$-modules each $G$-invariant quadratic form is uniquely determined by its polarization:

**Lemma 12.** Let $G$ be a finite group and let $F$ be an arbitrary field of characteristic 2. Suppose that $V$ is an $FG$-module which affords a non-degenerate $G$-invariant alternating bilinear form $B$ but $V$ has no trivial quotient. Then $V$ affords at most one $G$-invariant quadratic form which polarizes to $B$.

**Proof.** Let $Q_1$ and $Q_2$ be $G$-invariant quadratic forms on $V$ which polarize to $B$. Then $P := Q_1 + Q_2$ is a $G$-invariant quadratic form which polarizes to $2B = 0$. Thus $P(\lambda v_1) = \lambda^2 P(v_1)$, and $P(v_1 + v_2) = P(v_1) + P(v_2)$, for all $\lambda \in F$ and $v_1, v_2 \in V$.

Set $U := \{m \in V \mid P(v) = 0\}$. Then $U$ is a submodule of $V$. Let $g \in G$ and $v \in V$. Then $gv + v \in U$, as $P(gv + v) = P(gv) + P(v) = 0$. So $G$ acts trivially on $V/U$, whence $U = V$ by our hypothesis on $V$. We conclude that $P = 0$, or equivalently $Q_1 = Q_2$. \qed

Recall the notion of a canonical irreducible $FG$-module introduced after Theorem 9.

**Proposition 13.** Let $W$ be a non-trivial self-dual irreducible $FN$-module and let $V$ be the canonical self-dual irreducible $FG$-module lying over $W$. Then $V$ has quadratic type if and only if $W$ has quadratic type.

**Proof.** We adopt the notation of Theorem 9. So $W$ has a unique self-dual extension $\hat{W}$ to its stabilizer $T$ and $V = \hat{W} \uparrow^G$. Also $V \downarrow_N$ is the sum of the distinct $G$ conjugates of $W$, each occurring with multiplicity 1. So we can identify $W$ with an $F$-subspace of $V$.

Suppose first that $V$ affords a $G$-invariant quadratic form $Q$, and let $B$ be its polarization. So $B$ is a non-degenerate $G$-invariant alternating form on $V$. Let $W^\perp = \{v \in V \mid B(v, w) = 0, \text{ for all } w \in W\}$. Then $W^\perp$ is a submodule of $V \downarrow_N$ and $V \downarrow_N / W^\perp \cong W^* \cong W$ as $FN$-modules. As $W$ occurs with multiplicity 1 in $V \downarrow_N$, we deduce that $W \cap W^\perp = 0$. So the restriction $B \downarrow_W$ to $W$ is non-degenerate. Moreover the restriction $Q \downarrow_W$ of $Q$ to $W$ is an $N$-invariant quadratic form on $W$ which polarizes to $B \downarrow_W$. So $W$ is of quadratic type.

Conversely, suppose that $W$ affords a non-degenerate $N$-invariant quadratic form $q$, and let $b$ be its polarization. Then $b$ is a non-degenerate $N$-invariant alternating form on $W$. We identify $W$ and $\hat{W}$ as $F$-vector spaces. As $\hat{W}$ is self-dual and irreducible, it affords a $T$-invariant non-zero bilinear form, say $b'$. Now all $N$-invariant non-zero bilinear forms on $W$ are scalar multiples of each other, as $W$ is irreducible. So $b$ is a scalar multiple of $b'$, and in particular $b$ is $T$-invariant.

For $t \in T$, we define a quadratic form $q^t$ on $W$ by setting

$$q^t(w) := q(tw), \quad \text{for all } w \in W.$$
It is clear that $q'$ is $N$-invariant, and also that $q'$ polarizes to $b$, as $b$ is $T$-invariant. So $q' = q$, according to Lemma [12]. This establishes that $q$ is $T$-invariant, and shows that $\hat{W}$ is of quadratic type.

Next, we may decompose $V = \hat{W}^G$ as $F$-vector space
\[
V = (g_1 \otimes \hat{W}) \oplus (g_1 \otimes \hat{W}) \oplus \cdots \oplus (g_n \otimes \hat{W}),
\]
where $g_1, \ldots, g_n$ is a transversal to $T$ in $G$. By a standard procedure, we may define the induced forms $b^G$ and $q^G$ on $V$ using
\[
b^G \left( \sum_{i=1}^{n} g_i \otimes w_i, \sum_{j=1}^{n} g_j \otimes x_j \right) = \sum_{i=1}^{n} b(w_i, x_i),
\]
\[
q^G \left( \sum_{i=1}^{n} g_i \otimes w_i \right) = \sum_{i=1}^{n} q(w_i),
\]
for all $w_i, x_j \in \hat{W}$. Then $b^G$ is a $G$-invariant alternating bilinear form on $V$ and $q^G$ is a $G$-invariant quadratic form on $V$ which polarizes to $b^G$. So $V$ is of quadratic type. \[\square\]

**Proposition 14.** Let $V$ be a self-dual irreducible $FG$-module and suppose that some self-dual irreducible $FN$-module $W$ occurs with multiplicity $e > 1$ in $V \downarrow_N$. Then $e$ is even and $V$ has quadratic type.

**Proof.** We adopt the notation of Theorem [9]. So $W$ has a unique self-dual extension $\hat{W}$ to its stabilizer $T$ and $V = (S \otimes \hat{W})^G$, where $S$ is a self-dual irreducible $F(T/N)$-module. Now $W$ occurs with multiplicity $1$ in $(\hat{W}^G) \downarrow_N$. So the multiplicity $e$ of $W$ in $V \downarrow_N$ equals $\dim(S)$. As $e > 1$, we deduce that $S$ is non-trivial. But then $\dim(S)$ is even, according to Lemma [7]. Finally, since $S$ and $\hat{W}$ are both non-trivial and self-dual, $S \otimes \hat{W}$ has quadratic type, by the remark below. This in turn implies that $V = (S \otimes \hat{W})^G$ has quadratic type. \[\square\]

**Remark:** Suppose that $U$ and $V$ are $FG$-modules which afford non-degenerate $G$-invariant alternating bilinear forms $B_U$ and $B_V$, respectively. According to Sin and Willems [SW, Proposition 3.4] there is a quadratic form $Q$ on $U \otimes_F V$, which polarizes to $B_U \otimes B_V$, such that $Q(u \otimes v) = 0$, for all $u \in U$ and $v \in V$. These properties uniquely specify $Q$. For, if $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ are bases for $U$ and $V$, respectively, then for all $\lambda_{ij} \in F$
\[
Q \left( \sum \lambda_{ij} u_i \otimes v_j \right) := \sum \lambda_{ij} \lambda_{ij'} B_U(u_i, u_{i'}) B_V(v_j, v_{j'}),
\]
where $i, i'$ range over $1, \ldots, n$ and $j, j'$ over $1, \ldots, m$. Any basic tensor $u \otimes v$ can be written as $\sum \alpha_i \beta_j u_i \otimes v_j$, for scalars $\alpha_i, \beta_j$. Then in the expression for $Q(u \otimes v)$, the term indexed by $(i, j), (i', j')$ can be cancelled with the term indexed by $(i', j), (i, j')$, for $i \neq i'$. Likewise pairs of terms with $j \neq j'$ cancel. Finally, all terms with $i = i'$ or $j \neq j'$ are zero as $B_U$ and $B_V$ are alternating. It is now clear that $Q$ is $G$-invariant.
We turn our attention to those irreducible $FN$-modules that are not self-dual but are $G$-conjugate to their duals. To investigate these, we require a familiar concept.

Suppose that $W$ is an irreducible $FN$-module, with stabilizer $T$ in $G$. Then

$$T^* = \{ g \in G \mid W^g \cong W \text{ or } W^g \cong W^* \}$$

is a subgroup of $G$ containing $T$, called the extended stabilizer of $W$. If $W$ and $W^*$ are non-isomorphic and $G$-conjugate, then $|T^*: T| = 2$. Otherwise $T = T^*$.

**Proposition 15.** Let $W$ be an irreducible $FN$-module which is not self-dual. Then all self-dual irreducible $FG$-modules lying over $W$ are of quadratic type.

**Proof.** If $W$ is not $G$-conjugate to $W^*$, there are no self-dual irreducible $FG$-modules lying over $W$. So we may assume that $W$ is $G$-conjugate to $W^*$ and that $|T^*: T| = 2$.

Let $V$ be a self-dual irreducible $FG$-module lying over $W$. Then $V = U^G$, where $U$ is the unique irreducible submodule of $V_{U^G}$ lying over $W$. Likewise $V \cong V^* = (U^*)^G$. So $U^* \cong \tau U$ and preserves $B$ since $\tau$ is $T^*$-conjugate. So $X \cong X^*$.

Let $\tau \in T^* \setminus T$. Then $X_{\tau U} = U \oplus \tau U$. But $U$ and $U^*$ are non-isomorphic irreducible $G$-conjugate submodules of $X_{\tau U}$. So $X_{\tau U} = U \oplus U^*$, whence $\tau U \cong U^*$.

Let $B$ be a $G$-invariant non-degenerate alternating bilinear form on $V$, and let $X^\perp$ be the orthogonal complement of $X$ in $V_{\tau U}$. Then $X \cong X^\perp \cong V/X^\perp$. So $X \cap X^\perp = 0$, by uniqueness of $X$. This shows that the restriction $B_X$ of $B$ to $X$ is a $(T^*)$-invariant) non-degenerate alternating bilinear form on $X$. Since $U$ is irreducible but not self-dual, it is totally isotropic with respect to $B$, and likewise, so is $\tau U$. We define $Q : X \to F$ via

$$Q(u_1 + \tau u_2) = B(u_1, \tau u_2), \quad \text{for all } u_1, u_2 \in U.$$

Then $Q$ is a quadratic form which polarizes to $B_X$. As $Q$ vanishes on the subspaces $U$ and $\tau U$, it is an example of a hyperbolic form.

We now check that $Q$ is $T^*$-invariant. It is certainly $T$-invariant, as $T$ fixes $U$ and $\tau U$, and preserves $B$. Suppose that $\tau' \in T^* \setminus T$. Then $\tau' u_1 \in \tau U$ and $\tau' \tau u_2 \in U$. So

$$Q(\tau'(u_1 + \tau u_2)) = B(\tau' \tau u_2, \tau' u_1) = B(\tau u_2, u_1) = B(u_1, \tau u_2) = Q(u_1 + \tau u_2),$$

since $\tau'$ also leaves $B$ invariant. So $Q$ is indeed $T^*$-invariant.

Finally, the induced form $Q^G$ is a $G$-invariant quadratic form on $V$ which polarizes to the $G$-invariant non-degenerate alternating bilinear form $B_X^G$. So $V$ is of quadratic type, as required. \hfill $\square$

**Proof of Theorem.** Let $V$ be a self-dual irreducible $FG$-module which is not of quadratic type, such that $N$ does not act trivially on $V$. Write $V_{\downarrow N} = e(W_1 \oplus \cdots \oplus W_i)$, where
$e > 0$ and $W_1, \ldots, W_t$ is a $G$-orbit of irreducible $FN$-modules, each of which is irreducible and non-trivial.

It follows from Proposition 15 that each $W_i$ is self-dual, and then from Proposition 14 that $e = 1$. So $V$ is the canonical self-dual irreducible $FG$-module over each $W_i$. Then Proposition 13 implies that each $W_i$ has non-quadratic type.

We now show how the techniques we have developed above can be employed to obtain a criterion for all self-dual irreducible $FG$-modules to be of quadratic type.

**Corollary 16.** For each normal subgroup $N$ of a finite group $G$, all non-trivial self-dual irreducible $FG$-modules are of quadratic type if and only if the same is true for both $N$ and $G/N$.

**Proof.** Suppose first that all non-trivial self-dual irreducible $FG$-modules are of quadratic type. Then the same is obviously true for $G/N$ and we must show that it is true for $N$. Let $W$ be a non-trivial self-dual irreducible $FN$-module and let $V$ be the canonical self-dual irreducible $FG$-module over $W$. Then $V$ is irreducible and of quadratic type, from the hypothesis. So $W$ is of quadratic type, according to Proposition 13.

Conversely, suppose that all non-trivial self-dual irreducible modules of both $N$ and $G/N$ are of quadratic type. Then for the sake of contradiction, suppose that $V$ is a self-dual irreducible $FG$-module which has non-quadratic type. Theorem 3 implies that $N$ acts trivially on $V$. So $V$ is a self-dual irreducible $F(G/N)$-module of non-quadratic type, contrary to hypothesis. □

5. **Irreducible self-dual modules of non-abelian finite simple groups**

The proof of Corollary 16 shows that in an inductive approach to deciding whether a self-dual irreducible $FG$-module is of quadratic type, the main difficulty lies in solving the problem for non-abelian simple groups, and as far as we know, this is an unsolved (and difficult) question. See [W, Remark 3.4(a)].

At a simpler, but by no means straightforward, level, we can ask if every non-abelian simple group has a non-trivial irreducible module of quadratic type. The answer is no, for according to [HM], the Mathieu simple group $M_{22}$ has no such modules. We were unable to find an explicit reference to the calculations needed to verify this in the literature. So we outline a proof here, which only assumes some knowledge of the irreducible Brauer characters of certain groups.

We use the notation and decomposition matrices from [ModAtlas] and character tables from [GAP]. So $M_{22}$ has exactly two non-trivial self-dual irreducible Brauer characters $\phi_4$ and $\phi_7$, with degrees 34 and 98, respectively.

Now $M_{22}$ has two conjugacy classes of maximal subgroups isomorphic to the alternating group $A_7$. The restriction of $\phi_4$ to any $A_7$ is the sum of an irreducible character $\psi$ of degree 20 plus another of degree 14. In turn, the restriction of $\psi$ to $A_6$ is the sum of an irreducible character $\mu$ of degree 4 plus two irreducible characters of degree 8. Examining the values of $\mu$, we see that it is the Brauer character of a representation defined over $\mathbb{F}_2$. So $\mu$
Lemma 1.2] that \( Q_\lambda \) is not of quadratic type. Next we observe that \( \phi_2 \phi_3 = 2\phi_1 + \phi_7 \). Here \( \phi_2 \) and \( \phi_3 = \overline{\phi_4} \) are of degree 10, and \( \phi_1 \) is the trivial Brauer character. Now \( \phi_2 \phi_3 \) is the Brauer character of the ring \( E \) of \( F \)-endomorphisms of a module affording \( \phi_2 \). Let \( \text{Tr} : E \to F \) be the trace map and let \( W = \{ A \in E \mid \text{Tr}(A) = 0_F \} \). Then \( W \) is a submodule of \( E \) and \( E/W \) is the trivial module. So \( W \) has Brauer character \( \phi_1 + \phi_7 \). Clearly the identity map \( I \in E \) spans the unique trivial submodule of \( W \). Now for each \( A \in W \), set \( Q(A) \) as the coefficient of \( x^{n-2} \) in the \( F \)-characteristic polynomial of \( A \). Then \( Q \) is a \( G \)-invariant quadratic form, with polarization \( B(A, B) = \text{Tr}(AB) \), for all \( A, B \in W \). In particular \( I \) spans \( \text{Rad}(Q) \). As \( I \) has characteristic polynomial \( (x-1)^{10} = x^{10} + x^8 + x^2 + 1 \), we see that \( Q(I) = 1_F \). So the singular radical \( \text{Rad}_0(Q) \) is 0. Now [GW95, Theorem 1.3] implies that \( \phi_7 \) is not of quadratic type.

On the other hand, each of the remaining 25 sporadic finite simple groups does have irreducible \( FG \)-modules of quadratic type. To show this, we need the following known result. We prove it here, as we could not locate a self-contained proof in the literature:

**Lemma 17.** Let \( G \) be a finite group and let \( F \) be a perfect field of characteristic 2. Then each self-dual irreducible \( FG \)-module which is not in the principal 2-block of \( G \) has quadratic type.

**Proof.** Let \( M \) be a self-dual irreducible \( FG \)-module which is not in the principal 2-block. In particular there is no module \( W \) such that \( \text{soc}(W) \) is trivial and \( \frac{W}{\text{soc}(W)} \cong M \).

Let \( B \) be a \( G \)-invariant non-degenerate symplectic bilinear form on \( M \), and let \( Q \) be a quadratic form on \( M \) which polarizes to \( B \). For all \( g \in G \), define \( Q^g(m) := Q(gm) \). Then \( Q^g \) is a quadratic form which polarizes to \( B \), as

\[
Q^g(m_1 + m_2) = Q(gm_1) + B(gm_1, gm_2) + Q(gm_2) = Q^g(m_1) + B(m_1, m_2) + Q^g(m_2).
\]

Consider the quadratic form \( Q + Q^g \). This is additive and satisfies \( (Q + Q^g)(\lambda m) = \lambda^2 (Q + Q^g)(m) \), for all \( \lambda \in F \) and \( m \in M \). As \( F \) is perfect, there exists \( \phi_g \in M^* \) such that \( Q(gm) = Q(m) + \phi_g(m)^2 \), for all \( m \in M \). Now for \( g, h \in G \), and \( m \in M \) we have

\[
Q(m) + \phi_{gh}(m)^2 = Q(ghm) = Q(hm) + \phi_h(hm)^2 = Q(m) + \phi_h(m)^2 + h^{-1} \phi_g(m)^2.
\]

So \( \phi : G \to M^* \) satisfies the cocycle condition \( \phi_{gh} = \phi_h + h^{-1} \phi_g \), for all \( g, h \in G \).

Now take \( W \) to be the Cartesian product \( M \times F \), endowed with the obvious \( F \)-vector space structure. Define an \( FG \)-module structure on \( W \) via

\[
g(m, \lambda) = (gm, \phi_g(m) + \lambda), \quad \text{for all } m \in M, \lambda \in F \text{ and } g \in G.
\]
This is an action because for all \( g, h \in G \), we have
\[
(gh)(m, \lambda) = (ghm, \phi_{gh}(m) + \lambda)
= (ghm, \phi_g(hm) + \phi_h(m) + \lambda)
= g(hm, \phi_h(m) + \lambda)
= g(h(m, \lambda)).
\]

Clearly \( W \) has a submodule \( 0 \times F \) isomorphic to the trivial \( FG \)-module \( F \), and \( W \) modulo this submodule is isomorphic to \( M \). Our assumption on \( M \) forces \( W \cong M \oplus F \) as \( FG \)-modules. So there is \( \psi \in M^* \) such that \( m \mapsto (m, \psi(m)) \), for \( m \in M \), is an injective \( FG \)-module map \( M \to W \).

Now on the one hand \( g(m, \psi(m)) = (gm, \psi(gm)) \). On the other hand \( g(m, \psi(m)) = (gm, \phi_g(m) + \psi(m)) \). Comparing these expressions, we see that \( \phi_g(m) + \psi(gm) = \psi(m) \), for all \( g \in G \). Finally, define the quadratic form \( \hat{Q} \) on \( M \) via
\[
\hat{Q}(m) = Q(m) + \psi(m)^2, \quad \text{for all } m \in M.
\]

Then it is clear that \( \hat{Q} \) polarizes to \( B \). Furthermore, for all \( g \in G \) we have
\[
\hat{Q}(gm) = Q(gm) + \psi(gm)^2 = Q(m) + \phi_g(m)^2 + \psi(gm)^2 = Q(m) + \psi(m)^2 = \hat{Q}(m).
\]
So \( \hat{Q} \) is \( G \)-invariant. \( \square \)

Using [GAP] and [ModAtlas], the only sporadic finite simple groups which do not have a real non-principal 2-block are \( M_{11}, M_{22}, M_{23} \) and \( M_{24} \). Now \( M_{11} \) has an orthogonal irreducible \( K \)-character \( \chi_2 \), of degree 10, whose restriction to 2-regular elements is a self-dual irreducible Brauer character \( \phi_2 \). So \( \phi_2 \) has quadratic type. Similarly \( M_{24} \) has an orthogonal irreducible \( K \)-character \( \chi_7 \), of degree 252, whose restriction to 2-regular elements contains the self-dual irreducible Brauer character \( \phi_6 \) with multiplicity 1, but does not contain the trivial Brauer character. So \( \phi_6 \) has quadratic type. Finally, \( \phi_6 \) restricts to an irreducible Brauer character of a maximal subgroup \( M_{23} \). So \( M_{23} \) also has a quadratic type irreducible Brauer character.

All other simple group whose modular representations are tabulated in the modular [Atlas] have quadratic type irreducible Brauer characters, and we suspect that \( M_{22} \) may be unique among all non-abelian finite simple groups in not having such a character. We note that the automorphism group of \( M_{22} \) does have irreducible modules of quadratic type, since Proposition 15 applies to certain irreducible modules of the automorphism group that are induced from irreducible modules of \( M_{22} \) that are not self-dual.

### 6. Real weakly regular 2-blocks

We continue to assume that \( G \) is a finite group and \( N \) is a normal subgroup of \( G \). The results in this section include real refinements of [M, Theorem 4.4, Corollary 4.5]. If \( C \) is a conjugacy class of \( G \), then \( C^+ \) is the sum of its elements in \( RG \). Also \( C^o \) is the class consisting of the inverses of the elements of \( C \). Each \( z \in Z(FG) \) can be written \( z = \sum \beta(z, C)C^+ \), where \( C \) ranges over the conjugacy classes of \( G \) and \( \beta(z, C) \in F \).
We use standard notation and results on blocks. In particular, corresponding to each 2-block $B$ of $G$, there is a primitive idempotent $e_B$ of the centre $Z(FG)$ of $FG$, an $F$-algebra homomorphism $\omega_B : Z(FG) \to F$, called the central character of $B$, and a 2-subgroup $D$ of $G$ called a defect group of $B$. Then $D$ is only determined up to $G$-conjugacy, and $|D| = 2^d$, where $d \geq 0$ is called the defect of $B$. We use $\text{Irr}(B)$ and $\text{IBr}(B)$ to denote the irreducible $K$-characters and irreducible Brauer characters in $B$, respectively.

Let $\chi \in \text{Irr}(B)$, let $\psi$ be an irreducible constituent of $\chi|_N$ and let $b$ be the 2-block of $N$ containing $\psi$. Then $B$ is said to cover $b$, and the 2-blocks of $N$ covered by $B$ form a single $G$-orbit. Set $e^G_b$ as the sum of the distinct $G$-conjugates of $e_b$. Then $e^G_b$ is an idempotent in $Z(FG)$ which is the sum of the block idempotents of all blocks of $G$ which cover $b$.

Recall that $B$ is said to be weakly regular (with respect to $N$) if it has maximal defect among the set of blocks of $G$ which cover $b$. This happens if and only if $B$ has a defect group $D$ such that $DN/N$ is a Sylow 2-subgroup of the stabilizer of $b$ in $G$.

Let $\chi$ be a $K$-character or Brauer character belonging to $B$. Then $\chi(1)_2 \geq |G : D|_2$. If equality occurs, we say that $\chi$ has height 0. Recall that if $\chi$ is an irreducible $K$-character, its central character is defined by $\omega_\chi(C^+) := \chi(C^+)/\chi(1)$, for all conjugacy classes $C$ of $G$. It is classical that $\omega_\chi(C^+) \in R$, and indeed $\omega_B(C^+) = \omega_\chi(C^+)^*$ is independent of $\chi \in \text{Irr}(B)$. Suppose now that $\theta$ is an irreducible Brauer character in $B$ which has height 0. We claim that for all 2-regular conjugacy classes $C$ of $G$

\[ \frac{\theta(C^+)}{\theta(1)} \in R \quad \text{and} \quad \left( \frac{\theta(C^+)}{\theta(1)} \right)^* = \omega_B(C^+). \]  

For, it is known that there are integers $n_\chi$ such that $\theta \equiv \sum_{\chi \in \text{Irr}(B)} n_\chi \chi$ on the 2-regular elements of $G$. As $\chi(C^+)/\chi(1)$ and $\chi(1)/\theta(1)$ belong to $R$, we get

\[ \frac{\theta(C^+)}{\theta(1)} = \sum_{\chi \in \text{Irr}(B)} \left( \frac{\chi(C^+)}{\chi(1)} \right) \left( \frac{n_\chi \chi(1)}{\theta(1)} \right) \quad \text{belongs to } R. \]

Moreover \( \left( \frac{\theta(C^+)}{\theta(1)} \right)^* = \omega_B(C^+) \left( \sum_{\chi \in \text{Irr}(B)} n_\chi \chi(1) \right)^* = \omega_B(C^+). \)

Our first result includes a proof of Theorem 4(i):

**Lemma 18.** Let $b$ be a 2-block of $N$. Then the number of weakly regular 2-blocks of $G$ which cover $b$ is odd. So $G$ has a real weakly regular 2-block which covers $b$ if and only if $b$ is $G$-conjugate to $b^g$.

Let $B$ be a weakly regular 2-block of $G$ which covers $b$. Then $\beta(e_B, C) \omega_B(C^+) = \beta(e^G_b, C) \omega_b(C^+)$, for all conjugacy class $C$ of $G$ contained in $N$.

**Proof.** The first statement is proved in Lemma 5.1 of [GM], so we merely summarize the argument here. There is a defect preserving bijection between the blocks of $G$ covering $b$ and the blocks of the $G$-stabilizer of $b$ covering $b$. So we may assume that $b$ is $G$-invariant.

Let $B$ be as in the statement. In particular $e_B = e_B b$. So $1_F = \omega_B(e_b) = \omega_B(b)$. Thus there is a conjugacy class $L$ of $G$ contained in $N$ such that $\beta(e_b, L) \omega_B(L^+) \neq 0$. Now $L$ is 2-regular, as it is in the support of the block idempotent $e_b$. As $e_b$ is a sum
of block idempotents of blocks of $G$ with a defect group contained in $D$, $L$ has a defect group contained in $D$. But $\omega_L(L^+) \neq 0_F$. So $L$ has a defect group containing the defect group $D$ of $B$. We deduce that $D$ is a defect group of $L$.

Corollary 3.2 of [GM] implies that $\beta(e_B, L) = \omega_B(L^{o+})$. But $\omega_B(L^{o+}) = \omega_B^G(L^{o+})$, for each block $B'$ of $G$ which covers $b$, as $L \subseteq N$. So, again by Corollary 3.2 of [GM], $\beta(e_B, L) = \beta(e_B', L)$, if $B'$ is in addition weakly regular. On the other hand $\beta(e_B', L) = 0_F$, if $B'$ is not weakly regular. As $e_b$ is the sum of the block idempotents of all blocks of $G$ covering $b$, we see that $\beta(e_b, L) = \beta(e_B, L)\rho$, where $\rho$ is the number of weakly regular 2-blocks of $G$ covering $b$. It follows from this that $\rho$ is odd.

Suppose that there is a real weakly regular 2-block $B$ of $G$ which covers $b$. Then $B = B^o$ also covers $b^o$. So $b$ is $G$-conjugate to $b^o$. Conversely, suppose that $b$ is $G$-conjugate to $b^o$. Then taking contragredients of blocks is an involution on the set of weakly regular 2-block of $G$ covering $b$. As this set has odd size $\rho$, we deduce that there is a real weakly regular 2-block of $G$ which covers $b$.

For the last statement, let $C$ be a conjugacy class of $G$ which is contained in $N$ for which $\beta(e_B, C)\omega_B(C^+) \neq 0_F$ or $\beta(e_B, C)\omega_B(C^+) \neq 0_F$. As $\omega_B(C^+) = \omega_B(C^+)$, the argument above implies that $D$ is a defect group of $C$. But then $\beta(e_B, C) = \beta(e_B, C)\rho = \beta(e_B, C)$, as $\text{char}(F) = 2$. We conclude that $\beta(e_B, C)\omega_B(C^+) = \beta(e_b, C)\omega_B(C^+)$. □

We need one more result before proving part (ii) of Theorem 4.

**Lemma 19.** Let $b$ be a real $G$-invariant 2-block of $N$. Then $G$ has a self-dual Brauer character $\phi$ such that $\phi$ vanishes off $N$ and $\phi|_N = e(\theta_1 + \cdots + \theta_t)$ where both $e$ and $t$ are odd and $\theta_1, \ldots, \theta_t$ are distinct self-dual height 0 irreducible Brauer characters in $b$.

**Proof.** Note that we are not claiming that $\phi$ is irreducible.

Consider the $G$-set $X := \{\theta \in \text{IBr}(b) \mid \theta \text{ has height zero and } \Phi_\theta(1)_2 = |N|_2\}$. Then $X$ has odd size, according to Lemma 5. Also duality is an involution on $X$. So there is a $G$-orbit $\theta_1, \ldots, \theta_t$ in $X$, with $t$ odd and all $\theta_i$ self-dual and of height 0.

Let $T$ be the inertial group of $\theta_1$ in $G$. Then $T$ contains a Sylow 2-subgroup $S$ of $G$. As $SN/N$ is a 2-group, $\theta_1$ has a unique extension $\hat{\theta}_1$ to an irreducible Brauer character of $SN$. Notice that $\hat{\theta}_1$ vanishes off $N$, as $N$ contains all 2-regular elements of $SN$.

Set $\phi := \hat{\theta}_1 ^G$. Then $\phi$ is self-dual and $\phi|_N = \frac{1}{|SN:N|}(\theta_1 ^G)|_N = e(\theta_1 + \cdots + \theta_t)$, where $e = [T : SN]$ is odd. Finally $\phi$ vanishes off $N$ as $\theta_1$ vanishes off $N$.

We now prove the uniqueness part (ii) of Theorem 4.

**Lemma 20.** Let $b$ be a real 2-block of $N$. Then $G$ has a unique real 2-block which covers $b$ and which is weakly regular with respect to $N$.

**Proof.** We may assume that $b$ is $G$-invariant, and we let $B$ be any real weakly regular 2-block of $G$ covering $b$. Let $\phi$ be the Brauer character of $G$ defined in Lemma 19. So $\phi|_N = e(\theta_1 \cdots + \theta_t)$ where $et$ is odd and $\theta_1, \ldots, \theta_t$ are distinct self-dual height 0 irreducible Brauer characters in $b$. Write $\phi = \sum_{\mu \in \text{IBr}(G)} m_\mu \mu$, where $m_\mu$ are non-negative integers. Then $\phi_B := \sum_{\mu \in \text{IBr}(B)} m_\mu \mu$ is the $B$-part of $\phi$. 

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Let \( C \) be a 2-regular conjugacy class of \( G \) which is contained in \( N \). Then \( \theta_i(C^+) = \theta_1(C^+) \), for \( i = 1, \ldots, t \), as \( \theta_i \) is \( G \)-conjugate to \( \theta_1 \). So
\[
\left( \frac{\phi(C^+)}{\theta_1(1)} \right)^* = \left( \frac{e\theta_1(C^+)}{\theta_1(1)} \right)^* = \omega_b(C^+) = \omega_B(C^+),
\]
where we have used (5).

Next let \( \hat{e}_B \) be the unique idempotent in \( Z(RG) \) with \( \hat{e}_B^* = e_B \). Then for all \( \mu \in \text{IBr}(G) \) we have \( \mu(\hat{e}_B) = \mu(1) \) or \( 0_R \), as \( \mu \) does or does not belong to \( B \), respectively. So
\[
\left( \frac{\phi_B(1)}{\theta_1(1)} \right)^* = \left( \frac{\phi(e_B)}{\theta_1(1)} \right)^* = \sum \beta(e_B, C^+)\omega_B(C^+) = \sum \beta(e_b, C^+)\omega_b(C^+) = \omega_b(e_b) = 1_F.
\]
Here in both sums, \( C \) ranges over the conjugacy classes of \( G \) which are contained in \( N \), as \( \phi \) vanishes off \( N \). Also the middle equality arises from the last assertion in Lemma 18.

Now for each \( \mu \in \text{IBr}(B) \) with \( m_\mu \neq 0 \), we have \( \mu\downarrow_N = e_\mu(\theta_1 \cdots \theta_t) \), for some integer \( e_\mu > 0 \). Then by the previous displayed equation
\[
\frac{\phi_B(1)}{\theta_1(1)} = t \sum_{\mu \in \text{IBr}(B)} m_\mu e_\mu \quad \text{is an odd integer.}
\]
As \( m_\mu e_\mu = m_\pi e_\pi \), it follows that there is a self-dual \( \mu \in \text{IBr}(B) \) such that \( m_\mu e_\mu \) is odd. Then \( \mu \) is the canonical irreducible Brauer character of \( G \) lying over \( \theta_1 \) given by Theorem 2. As \( \theta_1 \) determines \( \mu \), which in turn determines \( B \), we conclude that \( B \) is the only real weakly regular 2-block of \( G \) which covers \( b \), as we wished to show. \( \square \)

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