On Two Dimensional Orthogonal Knapsack Problem

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Abstract

In this paper, we study the following knapsack problem: Given a list of squares with profits, we are requested to pack a sublist of them into a rectangular bin (not a unit square bin) to make profits in the bin as large as possible. We first observe there is a Polynomial Time Approximation Scheme (PTAS) for the problem of packing weighted squares into rectangular bins with large resources, then apply the PTAS to the problem of packing squares with profits into a rectangular bin and get a $\frac{6}{5} + \epsilon$ approximation algorithm.

1 Introduction

In this paper we study the two-dimensional generalization for Knapsack: we are given a set of squares, each of which is associated with a profit. The goal is to pack a subset of the squares (items) into a rectangle (bin) to maximize the total profit packed. The problem is NP-hard in the strong sense even if each item is an unweighted square (i.e., its profit is equal to its area) \cite{17}. A little surprisingly, the research for approximation algorithm has started quite recently: Jansen and Zhang \cite{12}, Caprara and Monaci \cite{2}, Harren \cite{11} etc.

Related Work There are many literatures on rectangle packing and square packing. For a two dimensional knapsack problem in which a subset of a given set of rectangles are packed into a given rectangular bin to maximize the total profits in the bin. Jansen and Zhang proposed $2 + \epsilon$ approximation algorithm \cite{12}. When all items are squares and their profits are equal to their areas, Fishkin, Gerber, Jansen and Solis-Oba \cite{8} presented a PTAS, which was also obtained by Han, Iwama and Zhang independently \cite{10}. Jansen and Zhang \cite{13} proposed a PTAS for packing squares into a rectangular bin to maximize the number of squares packed in the bin \cite{13}. Harren \cite{11} proposed $\frac{5}{4} + \epsilon$ approximation algorithm for packing squares into a unit square bin. But his algorithm is not applicable to pack squares into a rectangular bin since his algorithm requires that every side of the bin must have the same length. Fishkin, Gerber and Jansen \cite{7} obtained a $(1 - \epsilon)$-approximation algorithm for packing a set of rectangles with profits into a large resource bin with width 1 and height larger than $(1/\epsilon)^4$.

Another related work is 2 dimensional bin packing problem in which all rectangles have to be packed into a unit square bin to minimize the number of bins required. When all items are squares, Ferreira et al. \cite{6} gave an approximation algorithm with asymptotic worst-case ratio bounded above by 1.988. Kohayakawa et al. \cite{16} and Seiden and van Stee \cite{19} independently obtained approximation algorithms with asymptotic worst-case ratio of at most $14/9 + \epsilon$ (for any $\epsilon > 0$). These results were recently improved by Bansal, Correa, Kenyon and Sviridenko \cite{1}. They proposed an asymptotic PTAS for packing $d$-dimensional cubes into the minimum number of unit cubes. For the online case, if the number of bins is unbounded, the best known asymptotic worst case ratio is 2.1439 \cite{9}.
There are also some research on the Multiple Knapsack Problem. Kellerer [14], first gave a PTAS for a special case of this problem in which all the knapsacks have identical capacity [14]. Chekuri and Khanna [3] obtained a PTAS for the general multiple knapsack problem. For packing rectangles into multiple identical rectangular bins, Fishkin et al. [7] gave a $2 + \varepsilon$ approximation algorithm.

**Main results and Techniques:** We first observe that the techniques used in Multiple Knapsack Problem [3] are useful for the problem of packing weighted squares into rectangular bins (the bins may have different dimensions) with large resources, where *large resource* means that the height of a bin is much larger than the width, and give a Polynomial Time Approximation Scheme (PTAS) for the above problem, then apply the PTAS to the problem of packing squares with profits into a rectangular bin and get a $\frac{6}{5} + \varepsilon$ approximation algorithm. For packing squares into a rectangular bin, we first introduce a simple algorithm by the techniques used in 2D bin packing [1] problems and show that if there are too many large squares packed or the rest area for packing squares is not small, the algorithm has a nice performance, then we focus on the case in which there are a few large squares packed and the rest area for small squares is also few. We propose a novel approach of packing a few large items such that the packing does not affect the future small items packing too much, and call it *corner* packing. For packing small squares into the rectilinear polygons which is generated after packing large squares into the bin,

- we first dissect the polygons into rectangular blocks such that the optimal value of packing small squares into the blocks is near the optimal value of packing small squares into the polygons,
- then call the method used in Multiple Knapsack Problem [3], to guess one sublist which has a feasible packing and profit at least $(1 - \varepsilon)OPT_b$, where $OPT_b$ is the optimal value for packing small squares into the blocks,
- lastly, we exploit the techniques used in strip packing [7, 15] to pack items in each block.

**Worst Case Ratio:** We adopt the standard measure *worst case ratio* to evaluate approximation algorithms. For any input list $L$, let $A(L)$ be the total profit packed by approximation algorithm $A$ and $OPT(L)$ be the optimal value. The worst case ratio of algorithm $A$ is thus defined as

$$R_A = \sup_L \frac{OPT(L)}{A(L)}.$$

$p(\cdot), w(\cdot)$: Given a square $q$, we use $p(q)$ and $w(q)$ to denote its profit and area respectively. And given a list of squares $L = (q_1, \ldots, q_n)$, we define $p(L) = \sum_{i=1}^n p(q_i)$ and $w(L) = \sum_{i=1}^n w(q_i)$.

## 2 Packing Squares into Rectangular Bins with Large Resources

**INSTANCE:** Given an input list $S$ of $n$ squares with profits and a set of rectangular bins $B = (B_1, B_2, \ldots, B_c)$ where $B_i = (w_i, h_i)$ and $max\{w_i, h_i\} \geq \epsilon^6 - 1$ for all $i, c, \epsilon$ are constants.

**OBJECTIVE:** Maximize the total profit packed in $B$.

Based on the ideas from the seminal papers [3, 15], we give a PTAS for the above problem. There are three steps in the PTAS. We first guess a subset of squares which can be packed into $B$ and whose total profit is near the optimal value through the technique of rounding the input instance into $O(\epsilon^{-2} \ln n)$ classes. Then for each bin, we guess the number of items packed in that bin from each class such that our guess is also near the optimal solution, i.e., we do not lose too much profit. After matching items into bins, we use the strip packing algorithm to pack items in each bin.
Rounding and guessing: Here, we consider square packing. Since there is a natural order relation between any two squares, the techniques used in Multiple Knapsack problem \cite{3} are useful for square packing too. We first round the instance into a well structured instance which has $O(\epsilon^{-1} \ln n)$ distinct profits, and more items in each profit class have at most $(\epsilon^{-1})$ distinct sizes (side length). Then we select a subset items which can be packed into the bins and has the profit as least $(1 - \epsilon)\text{OPT}(I)$. But, if the items are rectangles, we do not have the above result, since there is not an order relation between any two rectangles.

**Lemma 1** Given an above instance $I = (B, S)$ with $n$ items, in polynomial time $v = n^{O(1/\epsilon^3)}$, we can obtain instances $I_1, \ldots, I_v$ such that

- $I_j = (B, S_j)$ for $1 \leq j \leq v$, where $S_j$ is a sublist of list $S$.
- For $1 \leq j \leq v$, items in $S_j$ have $O(\epsilon^{-1} \ln n)$ distinct profits, and more items in each profit class have at most $(\epsilon^{-1})$ distinct sizes (side length).
- There is an index $j$, $1 \leq j \leq v$, such that $S_j$ has a feasible packing in $B$ and $p(S_j) \geq (1 - O(\epsilon))\text{OPT}(I)$.

**Proof.** We show how to construct instance $I_1, \ldots, I_v$ from $I = (B, S)$ such that one of them satisfies the conditions. There are four steps, which are basically from \cite{3}.

- Guess a value $O$ such that $(1 - \epsilon)\text{OPT} \leq O \leq \text{OPT}$.
- Round down the profits of items into $O(\epsilon^{-1} \ln n)$ classes such that $\frac{p_j}{1 + \epsilon} \leq p_j^- \leq p_j$, where $p_j$ is the $j$th item’s profit and $p_j^-$ is the one after rounding down, where $n$ is the number of items.
- Guess a set of sublists based the value $O$ and the rounded profits of items such that one of them is feasible to pack into the set of bins (blocks) $B$ and its total profit is at least $(1 - O(\epsilon))\text{OPT}$.
- Using the techniques in bin packing \cite{5}, in each distinct profit class, reduce the number of distinct sizes into $O(\epsilon^{-1})$ such that we lose the profit at most $O(\epsilon)\text{OPT}$. Hence, totally, each of sublists has $O(\epsilon^{-2} \ln n)$ distinct size values and profits.

Next, we give the details for the above four steps. First, we show how to guess $O$. Given a sufficiently small constant $\epsilon > 0$, let $p_{\text{max}}$ denote the largest value among item profits. We know the optimal solution is bounded by $n \cdot p_{\text{max}}$. So we guess $O$ from the set

$$\{p_{\text{max}}(1 + \epsilon)^i | 0 \leq i \leq 1 + \ln(1 + \epsilon) n \leq 2\epsilon^{-1} \ln n\}.$$ 

$(1 + \ln(1 + \epsilon) n \leq 2\epsilon^{-1} \ln n$ follows from $\ln(1 + \epsilon) \geq \epsilon - \epsilon^2 / 2 \geq \epsilon / 2$.) Therefore, one of the values in the above set is guaranteed to satisfy the desired property for $O$.

Given a value $O$ such that $\max\{p_{\text{max}}, (1 - \epsilon)\text{OPT}\} \leq O \leq \text{OPT}$, then we show how to massage the given instance into a more structured one has few distinct profits.

1. Discard all items with profits at most $\epsilon O/n$.
2. Consider the other items and divide all profits by $\epsilon O/n$ such that after scaling each profit is at most $n/\epsilon$.
3. Round down the profits of item to the nearest power of $(1 + \epsilon)$.
It is easily seen that only an $O(\epsilon)$ fraction of the optimal profit lost by the above transformation. Since $(1 + \epsilon)^{t} \leq n/\epsilon$, we have
\[ i \leq 2\epsilon^{-1} \ln n/\epsilon \leq 4\epsilon^{-1} \ln n. \]
The last inequality follows from $n/\epsilon \leq n^2$. Therefore, we can transform the instance into a new instance with $O(\epsilon^{-1} \ln n)$ distinct profits such that only an $O(\epsilon)$ fraction of the optimal profit is lost.

Next we show how to guess the items to pack on the instance with $O(\epsilon^{-1} \ln n)$ distinct profits. Let $h \leq 4\epsilon^{-1} \ln n + 1$ be the number of distinct profits in our new instance. We partition the input set of squares $S$ into $h$ set $S_1, ..., S_h$ with items in each set having the same profit. Let $U$ be the items chosen in some optimal solution and let $U_i = S_i \cap U$. Recall that we have an estimate $O$ of the optimal value. If $p(U_i) \leq \epsilon O/h$, we ignore the set $S_i$; no significant profit is lost. Hence we can assume that $\epsilon O/h \leq p(U_i) \leq O$ and approximately guess the value $p(U_i)$ for $1 \leq i \leq h$, where $P(U_i)$ is the total profit in $U_i$. More precisely, for each $i$ we guess a value $k_i \in [h/\epsilon^2]$ such that
\[ k_i(\epsilon^2 O/h) \leq p(U_i) \leq (k_i + 1)(\epsilon^2 O/h), \]
where $[h/\epsilon^2]$ stands for the set of integers $0, 1, ..., \lfloor h/\epsilon^2 \rfloor$.

We show how the numbers $k_i$ enable us to identify the items to pack and then show how the values $k_1, ..., k_h$ can be guessed in polynomial time. Given the value $k_i$ we order the items in $S_i$ in increasing size values (side length). Let $a_i$ denote the profit of an item in $S_i$. If $a_i \leq \epsilon O/h$, pick the largest number of item from this ordered set whose cumulative profit does not exceed $k_i(\epsilon^2 O/h)$.

If $a_i > \epsilon O/h$ we pick the smallest number of items, again in increasing order of side lengths, whose cumulative profits exceeds $k_i(\epsilon^2 O/h)$. The choice of items is thus completely determined by the choice of the $k_i$. For a tuple of values $k_1, ..., k_h$, let $U(k_1, ..., k_h)$ denote the set of items packed as described above.

From the above selection, there exists a valid tuple $(k_1, ..., k_h)$ with each $k_i \in [h/\epsilon^2]$ such that $U(k_1, ..., k_h)$ has a feasible packing in $B$ and $p(U(k_1, ..., k_h)) \geq (1 - \epsilon)O$.

Now we show that the values $k_1, ..., k_h$ can be guessed in polynomial time. Before that, we introduce a useful claim.

Claim 1 [3] Let $f$ be the number of $g$-tuples of non-negative integers such that the sum of tuple coordinates is equal to $d$. Then $f = \binom{d+g-1}{g-1}$. If $d + g \leq \alpha g$ then $f = O(\epsilon^{\alpha g})$.

By Claim [1] the number of $h$-tuples $(k_1, ..., k_h)$ with $k_i \in [h/\epsilon^2]$ and $\sum_i k_i \leq h/\epsilon^2$ is $O(n^{O(\epsilon^{-3})})$.

Next we show how to reduce the number of distinct sizes (side length) in each profit class. The basic idea is the one used in approximation schemes for bin packing [3]. Let $A$ be a set of $g$ items with identical profit. We order items in $A$ in non-decreasing sizes and divide them into $t = (1 + 1/\epsilon)$ groups $A_1, ..., A_t$ with $A_1, ..., A_{t-1}$ containing $\lfloor g/t \rfloor$ items each and $A_t$ containing $(g \mod t)$ items.

We discard the items in $A_{i-1}$ and for $i < t - 1$ we increase the size of every item in $A_i$ to the size of the smallest item in $A_{i+1}$. Since $A$ is ordered by size, no item in $A_i$ is larger than the smallest item in $A_{i+1}$ for each $1 \leq i < t$. It is easy to see that if $A$ has a feasible packing then the modified instance also has a feasible packing. We discard at most an $\epsilon$ fraction of the profit and the modified sizes have at most $2/\epsilon$ distinct values. Applying this to each profit class we obtain an instance with $O(\epsilon^{-2} \ln n)$ distinct size values.

Hence, we have this lemma. \hfill \Box

Distributing the selected items into each bin

After guessing a polynomial number of sublists, next we consider how to distribute the selected items in each sublist into bins. Easily to see, the possibilities to assign the selected items into bins is
bounded by $c^n$, which is an exponential size of $n$, where $c$ is the number of bins and $n$ is the number of items to be packed. But we can guess a subset from the selected items in a polynomial time such that the total profit in the subset is near the optimal solution.

After step 1, we have $(\varepsilon^{-2} \ln n)$ classes in the input instance. Let $k_i$ be the number of items of the $i$th class and let $l^j_i$ be the number of items of the $i$th class packed in the $j$th bins, where $1 \leq j \leq m$.

**Lemma 2** We can guess a set of numbers $h^j_i$ in polynomial time such that $(1 - \varepsilon)l^j_i \leq h^j_i \leq l^j_i$, where $1 \leq \varepsilon^{-2} \ln n$ and $1 \leq j \leq c$ and $c$ is the number of bins.

**Proof.** For the $j$th bin, we guess $h^j_i$ items from the $i$th class. If $k_i \leq \frac{c}{\varepsilon (1 + \varepsilon)}$ then we can guess a number $h^j_i$ such that $h^j_i = l^j_i$ in $O\left(\frac{c}{\varepsilon (1 + \varepsilon)}\right)$ time. Else, we guess a number $h^j_i$ from the set $\{\lfloor (1 + \varepsilon)^p \cdot \frac{c}{\varepsilon} \rfloor | x = 1, 2, \ldots \}$ such that $(1 - \varepsilon)l^j_i \leq h^j_i \leq l^j_i$. Since $h^j_i \leq k_i$, the number of guesses required to obtain a single $h^j_i$ is bounded by $g = \log_{1+\varepsilon} \frac{c}{\varepsilon} \leq O\left(\varepsilon^{-2} \ln c\right)$, for each class, the total number of guesses for all $h^j_i$ is bounded by $g^c \leq O\left(\varepsilon^{-2} c^c\right)$, where $1 \leq j \leq c$. Therefore for all the $O\left(\varepsilon^{-2} \ln n\right)$ size classes the total number of guesses for is bounded by $n^c$, which is a polynomial of $n$, where $c$ and $\varepsilon$ are constants. □

Since all the items in each size class have the same profit and by Lemma 2, we have $h^j_i \geq (1 - \varepsilon)l^j_i$, there exists one assignment which is feasible to $B$ and keeps at least $(1 - \varepsilon)$ times the profits. Next we consider how to packing items into each bin.

**Packing each sublist into each bin:** In each bin, we have the following property

$$\frac{\max\{w, h\}}{\min\{w, h\}} \geq \epsilon^{-4},$$

then the techniques used in [15, 7] are helpful to pack all squares into the bins.

First, we first give an important lemma for packing squares into a bin with large resource, called cutting technique.

**Lemma 3** Given an input list $L$ of squares with sides at most $\epsilon$ and two rectangular bins $B_1 = (1, a)$, $B_2 = (1 + 2\epsilon, a)$, then

$$(1 - 4\epsilon) \cdot OPT(L, B_2) \leq OPT(L, B_1),$$

where $OPT(L, B)$ is the optimal value for packing list $L$ into bin $B$.

**Proof.** Now we construct a packing in bin $B_1$ from an optimal packing in bin $B_2$ and prove its profit is at least $(1 - 4\epsilon) \cdot OPT(L, B_2)$.

Consider an optimal packing in bin $B_2$, we cut $B_2$ into $[\frac{1}{4\epsilon}]$ pieces of slices, say $S_1, S_2, \ldots, S_{[\frac{1}{4\epsilon}]}$ respectively, such that every slice has an exact width $4\epsilon$ (except the last one), shown as fig. 1 (Note that some squares may be cut into two parts, one part in $S_i$ and another part in $S_{i+1}$). Then we find a slice $S_i$ such that $p(S_i) \leq 4\epsilon OPT(L, B_2)$ and remove all squares completely contained in slice $S_i$ if any. Observe that after the above removal, all squares remaining in bin $B_2$ can be packed into $B_1$. Hence, $OPT(L, B_1) \geq (1 - 4\epsilon) \cdot OPT(L, B_2)$. □

![Figure 1: Cutting bin $B_2$ into slices](attachment:image.png)
Lemma 4 [15] There is an algorithm A which, given a list $L$ of $n$ square and a positive $\epsilon$, produces a packing of $L$ in a strip of width 1 and height $A(L)$ such that $A(L) \leq (1 + \epsilon)\text{Opt}(L) + O(1/\epsilon^2)$.

Lemma 5 For packing small squares into a constant number of bins, for each bin, if $\frac{\max\{w,h\}}{\min\{w,h\}} \geq \epsilon^{-4}$, then there is a polynomial time algorithm with an output at least $(1 - O(\epsilon))\text{OPT}$, where $\text{OPT}$ is the optimal value.

Proof. Given an instance $I = (B, S)$, where $S$ is the set of small squares with profits and $B = (B_1, B_2, \ldots, B_c)$ is the set of rectangular bins, by the method in Lemma 1, we guess a subset $S_j \subseteq S$ such that $S_j$ has a feasible packing in $B$ and $p(S_j) \geq (1 - O(\epsilon))\text{OPT}(I)$.

In an instance $I_p = (B, S_p)$, we first guess $h_j^i$ by Lemma 2. Then according to $h_j^i$ value we assign the items to each bin and use the APTAS in Lemma 4 to pack items in each bin, where $1 \leq j \leq c$. If in each bin $(w, h)$, the height used by the APTAS in Lemma 4 is bounded by $(1 + \epsilon)\max\{w, h\} + O(\min\{w, h\}/\epsilon^2)$, then we keep the assignment otherwise reject the assignment. Since there is a $S_p$ such that $S_p$ has a feasible packing in $B$. After all the guesses, there is at least one assignment remained. For the assignment, we apply the APTAS in Lemma 4 and the cutting techniques in Lemma 3 such that in each bin, the profit keeps at least $(1 - \epsilon)$ times the optimal value.

Hence we have this lemma. \hfill \Box

3. Previous Algorithms for Packing Squares

Based on previous techniques used for 2D packing problem [1, 11] and the greedy packing (which is given in appendix), we introduce a simple algorithm $A_1$ which is implied in [11] for packing a set of squares into a bin $(1, h)$, where $h \geq 1$. There are two steps in $A_1$: first group all squares by their sizes and guess one group which does not significantly affect the optimal packing and delete it from the input list, then pack large items by enumeration, lastly append small items in the “gap” of the bin. Next, we give the details of the two steps.

**Grouping:** For an integer $k = \lceil \frac{1}{\epsilon} \rceil$, where $\epsilon < (2h + 2h^2)^{-1}$ is sufficiently small and $h \geq 1$ is the bin height, we select $k$ points in the region $(0, 1]$, $P_1, \ldots, P_k$ as follows

$$ P_i = e^{\epsilon i} \quad \text{and} \quad 1 \leq i \leq k. $$

Then the interval $(0, 1]$ is divided into $k + 1$ intervals, $I_1, \ldots, I_{k+1}$, where $I_i = (P_i, P_{i-1}]$, $2 \leq i \leq k$, $I_1 = (P_1, 1]$ and $I_{k+1} = (0, P_k]$.

**Notation:** In the following, given a list $L$ of squares, $L_i$ denotes the list in which all square’s sides are in interval $I_i$, $w(L_i)$ denotes the total area of $L_i$, $p(L_i)$ the total profits of $L_i$, $|L_i|$ the number of squares in $L_i$, where $1 \leq i \leq k + 1$. In the following, given a list $L$ of squares, $L_i$ denotes the list in which all square’s sides are in interval $I_i$, $w(L_i)$ denotes the total area of $L_i$, $p(L_i)$ the total profits of $L_i$, $|L_i|$ the number of squares in $L_i$, where $1 \leq i \leq k + 1$.

**Packing:**

1. Guess an index $i$ such that $\text{OPT}(L - L_i) \geq (1 - \epsilon)\text{OPT}(L)$.

2. Get all feasible packing for $L_{i-1} \cup \cdots \cup L_1$, pack each of them into the bin, then partition the uncovered space into rectangular bins (blocks) in the method [1] and append $L_{i+1} \cup \cdots \cup L_{k+1}$ into these bins by the Greedy algorithm.

3. Output the one with the largest profit.

Since there are $k + 1$ sublists $L_1, \ldots, L_{k+1}$ in $L$, then the guess in step 1 of $A_1$ is always feasible, where $k = \lceil \frac{1}{\epsilon} \rceil$. After selecting an index $i$, we define all items in $L_{i-1} \cup \cdots \cup L_1$ as large items and
all items in $L_{i+1} \cup \cdots \cup L_{k+1}$ as small items. Note that if $i = 1$ then there are no large items, and if $i = k + 1$ then there are no small items.

$A_1$’s worst case ratio is related to the number of large items in $L_{opt}$ and the rest area for small items, where $L_{opt}$ is a sublist of $L$ to produce an optimal solution.

**Fact 1** \[1\] Given large items with sides larger than $\epsilon^{6^{i-1}}$ which can be packed in the bin $(1, h)$, and small items with sides at most $\epsilon^6$, where $i \geq 1$, if the total area of all the squares is at most $h - \epsilon^{4 \times 6^{i-1}-1}$, where $2h(1+h) < h^{-1}$, then all can be packed in the bin.

**Lemma 6** \[1, 11\] After packing large items, if the rest area in the bin is at least $\epsilon^{-1} \delta$, then $A_1(L) \geq (1-2\epsilon)OPT(L)$, where $\delta = \epsilon^{4 \times 6^{i-1}-1}$.

**Lemma 7** \[11\] Let $m$ be the number of large items in $L_{opt}$. i) $A_1(L) \geq (1-\epsilon)OPT(L)$ if $m = 0$; ii) else $A_1(L) \geq \frac{m}{m+1}(1-\epsilon)OPT(L)$.

**Lemma 8** \[11\] Algorithm $A_1$ is run in polynomial time of $n$.

4 **Corner packing**

To pack squares into a rectangular bin, there are a lot of approaches, the most two studies are NFDH \[1\] and BL. In this section, we first give a new approach, called Corner packing, which includes the above two approaches. Then we analyze the corner packing later and show that it is one of the key points for improving the worse case ratio.

During packing squares into the rectangular bin, the uncovered space of the bin may get into the rectilinear polygons. The corner packing (shown as in Fig. 5(b)) can be regarded as a sequence of packing. Every time when one square is packed into the bin, we obey the following rules:

- select one vertex of the current rectilinear polygons at which the interior angle is 90 degrees,
- place the square such that one of its corners coincides with the vertex we selected. After packing, we get the new rectilinear polygons.

Note that both NFDH and BL \[12\] belong to Corner packing, where BL packing is to pack squares in the bin as bottom as possible then as left as possible.

**Lemma 9** Assume $n$ squares are packed in the bin by corner packing, then
i) there are at most $4+2n$ vertices of all the rectilinear polygons,
ii) there are at most $2^n(n+1)!$ possibilities to pack these $n$ squares in the bin by corner packing.

(refer to the proof in the appendix).

5 **A Refined Algorithm for Packing into a Rectangular Bin**

Let $m$ be the number of large items in $L_{opt}$, where $L_{opt}$ is a sublist of $L$ to get the optimal solution. By Lemma 7 if $m$ is very large, then algorithm $A_1$ has a good performance. So, to improve algorithm $A_1$, we have to study the case in which $m$ is very small. Note that when the bin is a unit square, the situation becomes a relatively simple. Since when $m = 1$ we can transform the original packing into a special strip packing; when $m = 2, 3$ we can estimate there must be much more space left for small squares than the wasted area. This is the main idea in Harren’s paper\[11\]. If the bin is not longer a unit square, his algorithm does not work very well. To improve algorithm $A_1$, we are faced with two problems:
• How to pack a few large items such that the packing does not affect too much the future small items packing? (how to allocate large items in the bin.)

• How to pack small items in the gaps (rectilinear polygons) generated after packing large items?

Next, we give our solutions for the above questions and propose a refined algorithm called $A_2$ with the worst case ratio $\frac{6}{5} + \varepsilon$.

### 5.1 Packing a few large items

Recall that, given an $i > 1$, if a square's side length is at least $\varepsilon^{6^{i-1}}$ then it is called large, else its side length is at most $\varepsilon^{6^i}$ then it is called small. And there is a gap between large items and small items, which is very important for packing large items.

Next we show that corner packing is a good packing which does not significantly affect future small square packing when there are a few large items.

**Lemma 10** Let $m$ be the number of large items in $L_{opt}$. If $m \leq 4$, then $(1 - \varepsilon)OPT(L) \leq OPT(L, *)$, where $OPL(L)$ is the optimal value of packing $L$ into the bin and $OPT(L, *)$ is the optimal value of packing $L$ into the bin such that all large squares are packed by corner packing.

**Proof.** Here, we just give the proof when $m = 4$, since the proof for $m = 1, 2, 3$ is involved. Let $a, b, c, d$ be the four large squares in an optimal packing $L_{opt}$. Without loss of generality assume $a, b, c, d$ are placed in the bin as Fig. 2(1). Note that a large item has side at least $\varepsilon^{6^{i-1}}$ and a small item has side at most $\varepsilon^{6^i}$, where $i \geq 1$. We cut the bin into three parts, two rectangular blocks $I = (x_1, y_1)$, $II = (x_2, y_2)$ and a rectilinear polygon $P$ as shown as Fig. 2(2). Now we define two new rectangular blocks $I_\varepsilon = (x_1 + 2\varepsilon^{6^i}, y_1)$ and $II_\varepsilon = (x_2 + 2\varepsilon^{6^i})$. Then we have

$$OPT(L, B) \leq OPT(L, I_\varepsilon \cup II_\varepsilon \cup P),$$

where $OPT(L, B)$ is the optimal value of packing $L$ into the bin $B$ and $OPT(L, I_\varepsilon \cup II_\varepsilon \cup P)$ is the optimal value of packing $L$ into three rectilinear polygons $I_\varepsilon \cup II_\varepsilon \cup P$. This can be seen as follows, all squares packed into the bin $B$ as shown in Fig. 2(1) can be packed into three rectilinear polygons $I_\varepsilon \cup II_\varepsilon \cup P$. By Lemma 3 for any list $L$ of small squares, we have

$$(1 - 4\varepsilon^{6^i})OPT(L, I_\varepsilon) \leq OPT(L, I) \text{ and } (1 - 4\varepsilon^{6^i})OPT(L, II_\varepsilon) \leq OPT(L, II).$$

Then

$$(1 - 4\varepsilon^{6^i})OPT(L, I_\varepsilon \cup II_\varepsilon \cup P) \leq OPT(L, I \cup II \cup P).$$

Figure 2: An optimal packing vs. its corner packing
So, by (1), (2),
\[(1 - 4\epsilon^6)OPT(L, B) \leq OPT(L, I \cup II \cup P).\]
And we have \(OPT(L, I \cup II \cup P) \leq OPT(L, *a)\), where \(OPT(L, *a)\) is the optimal value of packing \(L\) into the bin with \(a\) at one corner, shown as in Fig. 2(3). Hence
\[(1 - 4\epsilon^6)OPT(L, B) \leq OPT(L, *a). \tag{3}\]
By the similar proof, we have
\[(1 - 4\epsilon^6)OPT(L, *a) \leq OPT(L, *ab)\]
and
\[(1 - 4\epsilon^6)OPT(L, *ab) \leq OPT(L, *abc), \quad (1 - 4\epsilon^6)OPT(L, *abc) \leq OPT(L, *abcd)\]
where \(OPT(L, *X)\) is the optimal value of packing a set \(X\) into the bin with all the items in \(X\) at corners of the bin. Therefore, we have
\[OPT(L, *abcd) \geq (1 - 4\epsilon^6)OPT(L, B) \geq (1 - \epsilon)OPT(L, B).\]
The last inequality follows from \(\epsilon \leq 1/2\).
Hence, this lemma holds. \(\square\)

5.2 Packing small items into rectilinear polygons

After packing large items, the uncovered space in the bin maybe a set of rectilinear polygons. Our strategy for packing small squares into the polygons are below:

- Dissect the polygons into rectangular blocks such that the optimal value of packing small squares into the blocks is at least \((1 - \epsilon)OPT_p\), where \(OPT_p\) is the optimal value for packing small squares into the polygons.

- To pack small items into blocks, we use the PTAS in Section 2 of packing squares into rectangular bins with large resources.

**Dissection:** After packing few large squares into the bin by corner packing, we dissect the rectilinear polygons into rectangular blocks, such that the dissection does not affect the optimal packing insignificantly.

**Lemma 11** If there are at most 4 large squares packed, and the total area of the large squares packed is at least \(h - \epsilon^4 \times 6^{i-1} - 2\), then there exist a dissection of the polygons (the uncovered space of the bin) into blocks such that
\[OPT_b \geq (1 - \epsilon)OPT_p,\]
where \(OPT_b\) (\(OPT_p\)) is the optimal value of packing small squares into blocks (polygons).

**Proof.** In this proof, we just give our dissection for four large squares packed, shown as in Fig. 3 and 4 (by dotted lines), since the number of large squares is less than 4, we have the similar dissection.

Except for case (d), if the rectilinear polygons are dissected into blocks as shown in Fig. 3, then we have a set of five rectangular blocks \(B = \{B_i\}\), where \(B_i = (w_i, h_i)\) and \(w_i \leq h_i\) for \(1 \leq i \leq 5\). (Otherwise we can exchange \(w_i\) and \(h_i\)) Using the same techniques in Lemma 10 we define a new set of five blocks \(B' = \{B'_i\}\), where \(B'_i = (w_i, h_i + 2\epsilon^6)\) for \(1 \leq i \leq 5\). Let \(P\) be the polygon(s) after
packing large squares in the bin, and $L$ be a list of small squares. Since each small square has side length at most $\epsilon^6$, we have

$$OPT(L, B') \geq OPT(L, P) \geq OPT(L, B),$$

where $OPT(L, X)$ is the optimal value for packing $L$ into $X$. Since each large square has side length at least $\epsilon^{6^{-1}}$ and $h_i \geq w_i$, we have $h_i \geq \epsilon^{6^{-1}}$. By Lemma 3, we have

$$OPT(L, B) \geq (1 - \epsilon)OPT(L, B').$$

Hence we have $OPT_b = OPT(L, B) \geq (1 - \epsilon)OPT(L, P) = (1 - \epsilon)OPT_p$.

Next, we study the case (d) of Fig. 3 and prove that our strategy shown in Fig. 4 still works. There are two possibilities for the case (d). We assign $S_1, S_2, S_3, S_4$ to the four large squares as shown in Fig. 4, where $S_i = (s_i, s_i)$. And the polygon is dissected into 5 blocks $B_1, \ldots, B_5$, where $B_i = (w_i, h_i)$.

From our dissections in Fig. 4 (by dotted lines), we have

$$\max\{w_i, h_i\} \geq \epsilon^{6^{-1}}, \text{ for } 1 \leq i \leq 4.$$ 

And since the total area of the large items in the bin is at least $h - \epsilon^{4 \times 6^{-1} - 2}$, i.e., the total area of the blocks is at most $\epsilon^{4 \times 6^{-1} - 2}$. We have

$$\min\{w_i, h_i\} \leq \frac{\epsilon^{4 \times 6^{-1} - 2}}{\epsilon^{6^{-1}}} \leq \epsilon^{2 \times 6^{-1}}.$$

The last inequality follows from $i \geq 2$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Possible packing}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Two possibilities for the case (d)}
\end{figure}
Let $\delta = \epsilon^{6^{i-1}}$, next we prove that $w_5 \leq \delta^2$ in the cases (d1) and (d2). It is trivial to see $w_5 \leq \delta^2$ in the case (d1), since $w_5 \leq w_1 = \min\{w_1, h_1\}$. Now, we consider the case (d2), since $s_1 + s_2 \leq 1$ and $s_1 - s_2 \leq h_4 \leq \delta^2$, we have

$$s_1 \leq \frac{1 + \delta^2}{2}.$$ 

Since $s_3 + s_4 \leq 1$ and $s_4 - s_3 \leq h_2 \leq \delta^2$, we have

$$s_4 \leq \frac{1 + \delta^2}{2}.$$ 

So,

$$w_5 = s_1 + s_4 - 1 \leq 1 + \delta^2 - 1 = \delta^2.$$ 

Therefore, we have in the cases (d1) and (d2), $w_5 \leq \delta^2 \leq \epsilon w_4$ and $h_5 \leq h_4$, i.e., to compare with block $B_4$, block $B_5$ is sufficiently small and can be ignored. So

$$OPT(S, B^-) \geq (1 - O(\epsilon))OPT(S, B),$$

where $S$ is the set of small squares, $B^- = \bigcup_{i=1}^{4} B_i$, $B = \bigcup_{i=1}^{5} B_i$. Hence when we pack small squares into blocks $\bigcup_{i=1}^{5} B_i$, we just consider $\bigcup_{i=1}^{4} B_i$. Then by the similar proof for other cases, we have $OPT_b \geq (1 - O(\epsilon))OPT_p$.

Hence, this lemma holds. \qed

5.3 Algorithm $A_2$ and its analysis

Next, we give the details of algorithm $A_2$ which is similar to $A_1$.

**Description of Algorithm $A_2$**

1. Group items and guess an index $i$ such that $OPT(L - L_i) \geq (1 - \epsilon)OPT(L)$ and divide the remaining into two classes, say large and small,

2. Get all feasible packing of $L_{i-1} \cup \cdots \cup L_1$, for each of them,

   (a) if there are at least 4 items or the total area of items is at most $h - \epsilon^4 \times 6^{i-1} - 2$, then pack large and small squares by algorithm $A_1$.

   (b) else locate large items as Fig. 3 and divide the gaps into $b$ locks as Fig. 3 and 4, lastly apply the method in Lemma 5 for small items.

3. Output the one with the largest profit.

**Theorem 1** For any input list $L$, $\frac{OPT(L)}{OPT(L)} \leq \frac{6}{5}(1 + O(\epsilon))$, where $\epsilon$ is sufficiently small.

**Proof.** To consider an optimal packing solution $L_{opt}$, if there are at least 5 large items in $L_{opt}$ or the total area of large items in $L_{opt}$ at most $h - \epsilon^4 \times 6^{i-1} - 2$, by Fact 11, Lemmas 6 and 7

$$A_2(L) \geq \frac{5}{6}(1 - 2\epsilon)OPT(L).$$

Else, the total area of the large items in the bin is at least $h - \epsilon^4 \times 6^{i-1} - 2$ and there are at most 4 packed.
By the dissection of the polygons into rectangular blocks, shown as in Fig. 3 and 4 in each block \((w, h)\), we make sure that \(\max\{w, h\} \geq \epsilon^{6^{i-1}}\). So,

\[
\frac{\max\{w, h\}}{\min\{w, h\}} \geq \frac{\max\{w, h\}}{\epsilon^{4 \times 6^{i-1} - 2}} \geq \frac{\epsilon^{2 \times 6^{i-1}}}{\epsilon^{4 \times 6^{i-1} - 2}} \geq \epsilon^{-4}.
\]

The last inequality follows from \(i \geq 2\). (remember when \(i = 1\), there is no large item.) By Lemmas 10, 5, we have

\[
A_2(L) \geq (1 - O(\epsilon))OPT(L)
\]

By Lemma 5, the time complexity of Algorithm \(A_2\) is a polynomial time of \(n\). Hence, this theorem holds.

6 Concluding remarks

Note that algorithm \(A_2\) can be expended to multi-dimensional cube packing.

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7 Appendix

7.1 The proof for Lemma 9

Proof. We use induction to prove part i). When \( n = 0 \), there are 4 vertices in the bin. When \( n = 1 \), there are at most 6 vertices in the rectilinear polygon. So, we assume that when \( n = k \), part i) holds, i.e., after packing \( k \) items in the bin, there are at most \( 4 + 2k \) vertices in the rectilinear polygons. When \( n = k + 1 \), we use one of \( 4 + 2k \) vertices and generate at most 3 vertices, hence the total number of vertices is at most

\[
4 + 2k - 1 + 3 = 4 + 2(k + 1).
\]

Then we can see there are at most \( 4 + 2(i - 1) \) ways to pack the \( i \)-th square, where \( i \geq 1 \). Hence to pack \( n \) items in the bin, there are at most

\[
\prod_{i=1}^{n} (4 + 2(i - 1)) = 2^n \prod_{i=1}^{n} (i + 1) = 2^n (n + 1)!
\]

possibilities. \( \square \)

7.2 NFDH packing

NFDH (Next Fit Decreasing Height) [18]. NFDH packing behaves as follows: First sort all squares by their heights, then pack them in the bin from the largest one level by level as shown in Fig. 5(a). In each level, pack them by Next Fit, namely, if the current level cannot accommodate the next item, then open a new with height equal to the current item’s height. We repeat this procedure, until there is no space for a new level in the bin. Here is a key property of NFDH.

![Figure 5: NFDH and Corner packing](image)

Greedy Algorithm

1. Sort the input list \( L \) such that \( \frac{p(A_1)}{w(A_1)} \geq \cdots \geq \frac{p(A_k)}{w(A_k)} \).

2. For \( i \) from 1 to \( n \) do
   - if \( (a_i \geq \epsilon) \) and \( (b_i \geq \epsilon) \) then
     - (a) Find a maximal index \( m \) such that \( (A_1, A_2, \ldots, A_m) \) can be packed into the current bin by NFDH and pack \( (A_1, A_2, \ldots, A_m) \) into \( (a_i, b_i) \).
     - (b) Then update list \( L \) and re-index \( L \). If \( L \) becomes empty then finish packing.