Concentration points for Fuchsian groups

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March 30, 2022

Abstract: A limit point $p$ of a discrete group of Möbius transformations acting on $S^n$ is called a concentration point if for any sufficiently small connected open neighborhood $U$ of $p$, the set of translates of $U$ contains a local basis for the topology of $S^n$ at $p$. For the case of Fuchsian groups ($n = 1$), every concentration point is a conical limit point, but even for finitely generated groups not every conical limit point is a concentration point. A slightly weaker concentration condition is given which is satisfied if and only if $p$ is a conical limit point, for finitely generated Fuchsian groups. In the infinitely generated case, it implies that $p$ is a conical limit point, but not all conical limit points satisfy it. Examples are given that clarify the relations between various concentration conditions.

AMS(MOS) Subject Classification: Primary 20H10; Secondary 30F35, 30F40, 57M50

Keywords: Fuchsian group, Kleinian group, Möbius group, limit point, conical limit point, point of approximation, lamination, geodesic lamination, Schottky group, concentration, concentration point, controlled, weak, geodesic separation point
1 Introduction

The action of a Fuchsian or Kleinian group on the sphere at infinity can be examined from several viewpoints, and the resulting interplay between topology, geometry, number theory, and analysis brings richness and beauty to the subject. The topological viewpoint provides the starting point for much of the theory, in that it gives the dichotomy between the region of discontinuity and the limit set. The region of discontinuity can be regarded as the portion of the sphere at infinity with trivial or nearly trivial local dynamics. In contrast, at points in the limit set the behavior is complicated and varied.

For a limit point \( p \) a well-known type of behavior is the property of being a conical limit point. This property is often defined geometrically by saying that there is a sequence of translates of the origin (where we regard the group \( \Gamma \) as acting on the Poincaré ball \( B^m \)) that limit to \( p \) and lie within a bounded hyperbolic distance of a geodesic ray ending at \( p \). But it can also be described topologically in terms of the action of \( \Gamma \) on the sphere at infinity \( S^{m-1} \). For example, one of several such characterizations is that there exist points \( q \neq r \) in \( S^{m-1} \) and a sequence of distinct elements \( \gamma_n \in \Gamma \) such that \( p = \gamma_n(p) \to q \) and \( \gamma_n(x) \to r \) for every \( x \in S^{m-1} \setminus \{p\} \). For other topological characterizations of conical limit points, see [1, 3, 5].

Another topological aspect of the action of \( \Gamma \) on \( S^{m-1} \) is its concentration behavior. This refers to the action of \( \Gamma \) on the set of (open) neighborhoods of \( p \) in \( S^{m-1} \). The following definitions appear in [1].

**Definition:** An open set \( U \) in \( S^{m-1} \) can be concentrated at \( p \) if for every neighborhood \( V \) of \( p \), there exists an element \( \gamma \in \Gamma \) such that \( p \in \gamma(U) \) and \( \gamma(U) \subseteq V \). If in addition the element \( \gamma \) can always be selected so that \( p \in \gamma(V) \), then one says that \( U \) can be concentrated with control.

Note that \( U \) can be concentrated at \( p \) if and only if the set of translates of \( U \) contains a local basis for the topology of \( S^{m-1} \) at \( p \). Also, one can easily check from the definition that (1) there exists a neighborhood of \( p \) which can be concentrated with control if and only if there is a connected neighborhood which can be concentrated with control (take the connected component of \( U \) that contains \( p \), and require that \( \gamma^{-1}(p) \in U \cap V \), and (2) if a neighborhood of \( p \) can be concentrated with control, then every smaller neighborhood can be concentrated with control.

**Definition:** The limit point \( p \) is called a controlled concentration point for
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Γ if it has a neighborhood which can be concentrated with control at \( p \).

Concentration with control is studied in [1]. Analogously to conical limit points, \( p \) is a controlled concentration point if and only if there exist a point \( r \neq p \) in \( S^{m-1} \) and a sequence \( \gamma_n \) of distinct elements of \( \Gamma \) so that \( \gamma_n(p) \to p \) and \( \gamma_n(x) \to r \) for all \( x \in S^{m-1} - \{ p \} \). In particular, every controlled concentration point is a conical limit point. However, examples are given in [1] of conical limit points of 2-generator Schottky groups which are not controlled concentration points (see also proposition 4.1 below). For groups of divergence type, controlled concentration points have full Patterson-Sullivan measure in the limit set. There is a direct connection between controlled concentration points and the dynamics of geodesics in the hyperbolic manifold \( B^m/\Gamma \). Call a geodesic ray in \( B^m/\Gamma \) recurrent if it is the image of a geodesic ray in \( B^m \) that ends at a controlled concentration point. In an appropriate metric, the space of recurrent geodesic rays in \( B^m/\Gamma \) is a metric completion of the space of closed geodesics in \( B^m/\Gamma \) (where both spaces are topologized as subspaces of the unit tangent bundle of \( B^m/\Gamma \)).

We turn now to weaker concentration properties. It is not difficult to show (see [6]) that every limit point \( p \) has a disconnected neighborhood that can be concentrated at \( p \). So the weakest reasonable concept of concentration behavior is the following.

**Definition:** The limit point \( p \) is called a weak concentration point for \( \Gamma \) if there exists a connected open set that can be concentrated at \( p \).

Weak concentration points are studied in [3]. It turns out that for a geometrically finite group, every limit point is a weak concentration point, and for any group, all but countably many limit points are weak concentration points. A more restrictive condition is that every sufficiently small connected neighborhood can be concentrated:

**Definition:** The limit point \( p \) is called a concentration point for \( \Gamma \) if every sufficiently small connected neighborhood of \( p \) can be concentrated:

In this paper, we will study concentration behavior for Fuchsian groups. From now on, let \( \Gamma \) be Fuchsian. A slightly weaker concept than concentration point turns out to be important.

**Definition:** The limit point \( p \) is called a geodesic separation point for the Fuchsian group \( \Gamma \) if for every sufficiently small connected neighborhood \( U \) of \( p \), either \( U \) or \( S^1 - \overline{U} \) can be concentrated at \( p \).
The name of this property derives from the fact that for a geodesic separation point $p$, if $\lambda$ is any geodesic in $B^2$ whose endpoints separate $p$ from the boundary of a small neighborhood of $p$, then for any neighborhood $V$ of $p$ there exists $\gamma \in \Gamma$ so that the endpoints of $\gamma(\lambda)$ separate $p$ from the boundary of $V$. Indeed, it is easily verified that this is equivalent to the condition in the definition; this simply uses the fact that every connected neighborhood of $p$ (other than $S^1$ itself) is an interval, so corresponds to the unique geodesic in $B^2$ that runs between its endpoints.

The purpose of this paper is to investigate the general relations between these concentration properties for Fuchsian groups. We summarize them here; unless otherwise stated, references are to results that appear later in this paper. By $\Gamma$ we denote an nonelementary Fuchsian group, by $\Gamma_0$ a certain two-generator Schottky group defined in §4, and by $\Gamma_1$ a certain infinitely generated Fuchsian group defined in §5.

1. Every controlled concentration point for $\Gamma$ is a concentration point (a direct consequence of the definitions). There are uncountably many concentration points for $\Gamma_0$ that are not controlled concentration points (theorem 4.2).

2. Every concentration point for $\Gamma$ is a geodesic separation point (immediate from the definitions). There are uncountably many geodesic separation points for $\Gamma_0$ that are not concentration points (theorem 4.3 and theorem 3.2).

3. Every geodesic separation point for $\Gamma$ is a conical limit point (proposition 3.1). There are uncountably many conical limit points for $\Gamma_1$ which are not geodesic separation points (proposition 5.1). However, if $\Gamma$ is finitely generated, then every conical limit point is a geodesic separation point (theorem 3.2).

4. Every conical limit point or parabolic fixed point for $\Gamma$ is a weak concentration point (theorem 3.1 of [6]). There are uncountably many weak concentration points for $\Gamma_1$ which are neither conical limit points nor parabolic fixed points (proposition 5.1). However, if $\Gamma$ is finitely generated, then every weak concentration point is either a conical limit point or a parabolic fixed point (by theorem 2 of [3]).

5. At most countably many limit points of $\Gamma$ are not weak concentration
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points (theorem 3.6 of [§]). However, if \( \Gamma \) is finitely generated, then every limit point is a weak concentration point (corollary 2.2 of [§]).

We assume familiarity with the basic concepts of Möbius groups as exposted, for example, in [2]. We use the term Nielsen hull for the (hyperbolic) convex hull in \( B^2 \) of the limit set of a Fuchsian group \( \Gamma \), and Nielsen core for the quotient of the Nielsen hull by \( \Gamma \). Otherwise, our terminology and notation are standard. The reader may find it useful to examine the examples of §§4 and 5 before delving into §§2 and 3, whose main objective is the proof of theorem 3.2.

This manuscript is a revised version of a preprint that was circulated in 1992. The proof of theorem 3.2 is substantially rewritten, and a gap in it has been filled. Section 5 is new, and the introduction has been completely rewritten to place the results in the context of subsequent developments which have appeared in [1, 6].

2 Controlled concentration points and geodesic laminations

This section contains some sufficient conditions for a limit point of a Fuchsian group to be a controlled concentration point. In particular, if \( L \) is a compact geodesic lamination in a hyperbolic 2-manifold, then every endpoint of a leaf of the preimage of \( L \) in \( B^2 \) is a controlled concentration point.

Our first condition follows from theorem 2.3 of [1], but for simplicity we give a direct self-contained argument.

**Lemma 2.1** Let \( \Gamma \) be a torsionfree discrete group of Möbius transformations acting on the Poincaré disc \( B^m \), and let \( \pi: B^m \to B^m/\Gamma \) be the quotient map. Let \( y_0 \in B^m \) and let \( p \) be a point in \( S^{m-1} \). Let \( \alpha:[0,\infty) \to B^m \) be the geodesic ray from \( y_0 \) to \( p \), parameterized at unit speed. Suppose further that there exist numbers \( t_i \), with \( \alpha(t_i) \) limiting to \( p \), so that in the tangent bundle \( T(B^m/\Gamma) \), the images \( d\pi(\alpha'(t_i)) \) converge to \( d\pi(\alpha'(0)) \). Then \( p \) is a controlled concentration point for \( \Gamma \).

**Proof of 2.1:** Consider the hyperbolic codimension 1 hyperplane through \( y_0 \) perpendicular to \( \alpha \), and let \( U \) be the neighborhood of \( p \) in \( S^{m-1} \) which is one of the components of the complement of the boundary of the hyperplane.
For a Fuchsian group $\Gamma$ acting on the Poincaré disc $B^2$, we denote the Nielsen core of $B^2/\Gamma$ by $N(B^2/\Gamma)$. If $\Gamma$ is not elementary, then the interior of $N(B^2/\Gamma)$ is nonempty. Since the Nielsen hull is convex, it follows that if a geodesic in $B^2/\Gamma$ leaves the Nielsen core, it will never reenter. Note that such a geodesic cannot lie in a compact subset of $B^2/\Gamma$. When $\Gamma$ is finitely generated and torsionfree, its Nielsen core is a 2-manifold of finite area whose boundary is a finite collection of simple closed geodesics.

**Theorem 2.2** Let $\Gamma$ be a Fuchsian group acting on the Poincaré disc $B^2$. Suppose there exists a geodesic ray in $B^2$ which ends at $p \in S^1$, which has no transverse crossing with any of its translates, and whose image in $B^2/\Gamma$ lies in a compact subset. Then $p$ is a controlled concentration point for $\Gamma$.

**Proof of 2.2:** Let $S$ denote $B^2/\Gamma$, let $\alpha$ denote the hypothesized geodesic ray, and let $\alpha_0$ denote its image in $S$. Now $S$ is an increasing union of compact suborbifolds, and by filling in any complementary discal 2-orbifolds for these suborbifolds, we may assume that each has orbifold fundamental group which injects into $\Gamma$. Replacing $\Gamma$ by the fundamental group of one of the suborbifolds that contains $\alpha_0$, we may assume that $\Gamma$ is finitely generated. Passing to a subgroup of finite index, we may assume that $\Gamma$ is torsionfree.

We will use the elementary theory of geodesic laminations as presented in [4]. Let $L$ be the set of points $y \in S$ with the following property: there is a sequence of points $x_i$ on $\alpha$ that limits to $p$, whose images limit to $y$. Since $\alpha_0$ has no transverse self-intersections, it follows that $L$ is a nonempty compact geodesic lamination in $S$, for which each tangent vector is a limit of a sequence of tangent vectors to $\alpha_0$ at points whose preimages limit to $p$. Since $L$ is compact, it must lie in $N(S)$.
Suppose that $L$ contains a simple closed geodesic $C$. If there were no collar neighborhood of $C$ that contained the image of a subray of $\alpha$, then $\alpha_0$ would have transverse self-intersections. Therefore $\alpha_0$ must either be contained in $C$, or spiral toward $C$. In either case, $p$ is the endpoint of the axis of a hyperbolic translation in $\Gamma$, so is a controlled concentration point.

Suppose that $L$ does not contain any simple closed geodesics, and that $\alpha_0$ is contained in a leaf of $L$. Orient $\alpha$ in the direction pointing toward $p$, and let $v$ be the initial (unit) tangent vector of $\alpha_0$. Let $x_i$ be a sequence of points of $\alpha$, limiting to $p$, whose images $y_i$ limit to the starting point of $\alpha_0$, and let $v_i$ be the oriented tangent vectors to $\alpha_0$ at $y_i$. If a subsequence of the $v_i$ limits to $v$, then lemma 2.1 shows that $p$ is a controlled concentration point. So we may assume that they limit to $-v$. Let $\beta$ be the subarc of $\alpha_0$ from its initial point to some $y_i$. For $j$ sufficiently large so that $v_j$ is extremely close to $-v$, the part of $\alpha_0$ that ends at $y_j$ follows backwards along $\beta$ staying very close until after it passes $y_i$, and at a point where it passes $y_i$ it is pointing approximately in the direction of $v$ (see Figure 1). Therefore lemma 2.1 still applies.

The last case is that $L$ does not contain any simple closed geodesics, and $\alpha_0$ does not lie in a leaf of $L$. Note that $\alpha_0$ must be disjoint from $L$, since any transverse crossings with $L$ would force self-intersections of $\alpha_0$. A principal region for $L$ is a component of $N(S) - L$. Since $L$ lies in $N(S)$ and does not contain any of the boundary geodesics, we may assume (by shortening $\alpha$) that $\alpha_0$ lies in $N(S)$. Let $U$ be the principal region for $L$ that contains $\alpha_0$. Following lemma 4.4 of [4], we wish to describe the possibilities for $U$. 

![Figure 1](image-url)
In that lemma, $S$ is closed and has no cusps, and $U$ is either isometric to the interior of a finite-sided ideal polygon, or there is a compact submanifold $U_0$ of $U$, whose boundary is a union of closed geodesics, such that $U - U_0$ is isometric to the interior of a finite collection of crowns. (A crown is a complete hyperbolic surface with finite area and geodesic boundary, which is homeomorphic to $S^1 \times [0, 1] - Q$ for some finite subset $Q$ of $S^1 \times \{1\}$.) We define a cuspidal crown to be a complete hyperbolic surface with finite area and geodesic boundary, which is homeomorphic to $(B^2 \cup \partial B^2) - (\{0\} \cup Q)$, where $Q$ is a finite subset of $\partial B^2$. For later reference, we state the next observation as a lemma.

**Lemma 2.3** Let $L$ be a geodesic lamination in a hyperbolic 2-manifold $F$ of finite area, with boundary consisting of closed geodesics, and let $U$ be a component of $F - L$. Then either

1. $U$ is isometric to the interior of a finite-sided ideal polygon in $B^2$, or
2. there is a submanifold $U_0$ of $U$, whose boundary consists of closed geodesics, such that $U - U_0$ is isometric to the interior of a finite collection of crowns, or
3. $U$ is isometric to the interior of a cuspidal crown.

**Proof of 2.3:** The proof is exactly like the proof of lemma 4.4 of [4], with the case (3) arising when the element called $g$ there is parabolic.

Returning to the proof of theorem 2.2, since $\alpha_0$ is disjoint from $L$, but limits onto $L$, it must limit onto one of the noncompact boundary geodesics of a principal region of one of the forms described in lemma 2.3. It follows that there is a leaf in the preimage of the boundary leaves of $L$ that ends at $p$. Replacing $\alpha$ by a subray of that leaf ending at $p$, we are in the previous case where $\alpha_0$ was assumed to lie in a leaf, and again it follows that $p$ is a controlled concentration point.
Corollary 2.4 Let $\Gamma$ be a torsionfree Fuchsian group, and let $L$ be a compact geodesic lamination in $B^2/\Gamma$. Then the endpoints of the leaves of the preimage of $L$ in $B^2$ are controlled concentration points for $\Gamma$.

Proof of 2.4: Apply theorem 2.2 to a subray of the leaf.

3 Geodesic separation points

The first result of this section, proposition 3.1, shows that every geodesic separation point is a conical limit point. In particular, a parabolic fixed point cannot be a geodesic separation point. On the other hand, in §5 we will give an example of an infinitely generated Fuchsian group with uncountably conical limit points which are not geodesic separation points. Theorem 3.2 shows that this cannot happen in a finitely generated example, since then every limit point is either a parabolic fixed point or a geodesic separation point.

Note that proposition 3.1 implies that for Fuchsian groups, concentration points are conical limit points. Whether this holds in higher dimensions is an open question.

Proposition 3.1 Let $p$ be a geodesic separation point of a Fuchsian group $\Gamma$. Then $p$ is a conical limit point.

Proof of 3.1: Clearly, $p$ cannot be the endpoint of an interval of discontinuity with finite stabilizer. If $p$ is the endpoint of an interval of discontinuity with infinite stabilizer, then $p$ is the attracting fixed point of a hyperbolic element and hence is a conical limit point. So we may assume that every neighborhood of $p$ contains limit points of $\Gamma$ on both sides of $p$. Let $W$ be a neighborhood of $p$ such that for every connected neighborhood $U$ of $p$ with $\overline{U} \subseteq W$, either $U$ or $S^1 - \overline{U}$ can be concentrated at $p$. By the Double Density Theorem, there exists an axis $\lambda$ of a hyperbolic element of $\Gamma$ whose endpoints lie in $W$ and separate $p$ from the boundary of $W$. If $x$ is a point on this axis, then translates of $x$ lie at intervals of some fixed length $d$ along $\lambda$. Since $p$ is a geodesic separation point, there must be translates of $\lambda$ that intersect $\alpha$ arbitrarily close to $p$. On each such translate of $\lambda$, there are translates of $x$ within hyperbolic distance $d$ of $\alpha$. Therefore $p$ is a conical limit point.
Several times in the proof of theorem 3.2, we will use the observation that if a portion of a geodesic ray in $N(B^2/\Gamma)$ moves far out a cusp, but the ray does not continue all the way out the cusp, then it must behave as follows. For some time it travels almost straight out the cusp, then it starts to spiral around the cusp, finally becoming tangent to some horocycle, then it returns to the thick part of $N(B^2/\Gamma)$. (This behavior is easily seen by normalizing so that the parabolic element generating the cusp acts in the upper half-plane model as $z \mapsto z + 1$, and observing the behavior of a geodesic arc that rises to a high vertical coordinate before descending to the real line.) Note that in particular, any such ray in $N(B^2/\Gamma)$ must have self-intersections, and when it is spiraling near its tangent horocycle it must intersect any geodesic ray that travels a great deal further out the cusp at nearly right angles.

**Theorem 3.2** Let $\Gamma$ be a Fuchsian group. If $\Gamma$ is finitely generated, then every limit point of $\Gamma$ is either a parabolic fixed point or a geodesic separation point.

**Proof of 3.2:** Assume that $p$ is not a parabolic fixed point. Passing to a subgroup of finite index, we may assume that $\Gamma$ is torsionfree. If $\Gamma$ is elementary, then $p$ is the endpoint of the axis of a hyperbolic element, and hence a controlled concentration point. So we will assume that $\Gamma$ is nonelementary, and hence that its Nielsen core has nonempty interior. Since $\Gamma$ is finitely generated, its Nielsen core has finite area and has boundary a (possibly empty) finite collection of simple closed geodesics.

For later reference, we isolate the next argument as a lemma.

**Lemma 3.3** Let $\Gamma$ be a finitely generated torsionfree Fuchsian group, and let $p$ be a limit point of $\Gamma$ which is not a controlled concentration point and is not a parabolic fixed point. Let $\alpha$ be a geodesic ray ending at $p$, and lying in the interior of the Nielsen hull of $\Gamma$. Suppose $\lambda$ is a geodesic in $B^2$ that intersects $\alpha$, such that $p$ is not a limit point of the crossings of the translates of $\lambda$ with $\alpha$. Then there exists a finitely generated subgroup $\Gamma'$ of $\Gamma$ with the following properties:

(a) $p$ is a limit point of $\Gamma'$, and some subray of $\alpha$ lies in the interior of the Nielsen hull of $\Gamma'$. 


(b) Either the area of $N(B^2/\Gamma')$ is less than the area of $N(B^2/\Gamma)$, or the areas are equal and the genus of $N(B^2/\Gamma')$ is less than the genus of $N(B^2/\Gamma)$.

Proof of 3.3: Since $\alpha$ lies in the Nielsen hull of $\Gamma$, its image $\alpha_0$ in $B^2/\Gamma$ lies in $N(B^2/\Gamma)$. Let $\Gamma \lambda$ be the union of the translates of $\lambda$. Then some subray of $\alpha$ lies entirely in a component $E$ of $B^2 - \Gamma \lambda$. Note that $E$ is convex, since it is an intersection of half-spaces. Let $\Gamma'$ be the stabilizer of $E$. Since $E$ is precisely invariant under $\Gamma$, $E/\Gamma'$ maps injectively into $B^2/\Gamma$ under the covering projection $B^2/\Gamma' \to B^2/\Gamma$.

Suppose for contradiction that $p$ is not a limit point of $\Gamma'$. Then some subray of $\alpha$ injects into $E/\Gamma'$, and hence into $N(B^2/\Gamma)$. If $\alpha_0$ does not lie in a compact subset of the Nielsen core, then since $p$ is not a parabolic fixed point, $\alpha_0$ must enter and leave some cusp of $N(B^2/\Gamma)$ infinitely many times, moving farther and farther out toward the end of the cusp. This is impossible as a subray of $\alpha$ maps injectively. So $\alpha_0$ lies in a compact subset. But then, theorem 2.2 implies that $p$ is a controlled concentration point for $\Gamma$, giving the contradiction.

Now $\Gamma'$ is nonelementary, since otherwise $p$ would be a fixed point of a hyperbolic element, and hence a controlled concentration point. Therefore the interior of the Nielsen hull of $\Gamma'$ is nonempty. Since the orbit of any point of $E$ under $\Gamma'$ lies in $E$, the limit set of $\Gamma'$ lies in $\overline{E} \cap S^1$. Since $E$ is convex, this shows that the interior of the Nielsen hull of $\Gamma'$ lies in $E$. Therefore the covering map from $B^2/\Gamma'$ to $B^2/\Gamma$ restricts to a map $N(B^2/\Gamma') \to N(B^2/\Gamma)$ whose restriction to the interior of $N(B^2/\Gamma')$ is an isometric imbedding. In particular, this shows that $\Gamma'$ is finitely generated, so that $N(B^2/\Gamma')$ has boundary consisting of closed geodesics. Consequently, the topological effect of the map $N(B^2/\Gamma') \to N(B^2/\Gamma)$ must be first to include $N(B^2/\Gamma')$ into a larger (or possibly equal) surface, then (possibly) identify some pairs of boundary components. Therefore either $N(B^2/\Gamma')$ has smaller area than $N(B^2/\Gamma)$, or it has the same area and has smaller genus, or the restriction of the covering map is a homeomorphism from $N(B^2/\Gamma')$ to $N(B^2/\Gamma)$. In the latter case, $\Gamma = \Gamma'$ so their Nielsen hulls are equal. But the interior of the Nielsen hull of $\Gamma'$ is disjoint from the translates of $\lambda$, so $\alpha$ could not have intersected $\lambda$. Therefore $\Gamma' = \Gamma$ is impossible, giving assertion (b).

Finally, if no subray of $\alpha$ lies in the interior of the convex hull of $\Gamma'$, then since $\alpha$ ends at a limit point of $\Gamma'$, $\alpha_0$ must either coincide with or spiral onto a boundary geodesic for $N(B^2/\Gamma')$. But then $p$ is a fixed point of a hyperbolic
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element of $\Gamma'$, a contradiction. This completes the proof of assertion (a).

We now continue the proof of theorem 3.2. Suppose for contradiction that $p$ is not a geodesic separation point. Fix a geodesic ray $\alpha$ in $B^2$ running from a point in the interior of the Nielsen hull of $\Gamma$ to $p$, and let $\alpha_0$ denote its image in $N(B^2/\Gamma)$. Make an initial choice of connected neighborhood $W$ of $p$, small enough so that whenever the endpoints of $\lambda$ lie in $W - \{p\}$, they separate $p$ from the boundary of $W$ if and only if $\lambda$ intersects $\alpha$. Since $p$ is not a geodesic separation point, there exists a geodesic $\lambda$ with endpoints in $W - \{p\}$, so that $\lambda$ intersects $\alpha$ but for which there is a neighborhood $V$ of $p$ for which no translate of $\lambda$ separates $p$ from the boundary of $V$.

Suppose first that $p$ is not a limit point of the intersections of the translates of $\lambda$ with $\alpha$. Let $\Gamma'$ be a subgroup of $\Gamma$ obtained using lemma 3.3. By condition (a) of lemma 3.3, $p$ is a limit point of $\Gamma'$, and since $\Gamma'$ is a subgroup of $\Gamma$, $p$ is not a geodesic separation point for $\Gamma$. Replace $\Gamma$ by $\Gamma'$, shorten $\alpha$ if necessary so that it lies in the interior of the Nielsen hull of $\Gamma'$, and replace $W$ by a smaller neighborhood if necessary. By condition (b) of lemma 3.3, such a procedure can only occur finitely many times. So we may assume that $p$ is a limit point of the intersections of the translates of $\lambda$ with $\alpha$.

Let $x_i$ be a sequence of intersection points of translates $\gamma_i(\lambda)$ with $\alpha$, which limit to $p$. Since $p$ is not a geodesic separation point, the angles between $\alpha$ and $\gamma_i(\lambda)$ at the $x_i$ must limit to 0. Moreover, by passing to a subsequence we may assume that for one of the endpoints $e_1$ of $\lambda$, the sequence $\gamma_i(e_1)$ limits to $p$, while for the other endpoint $e_2$, the sequence $\gamma_i(e_2)$ limits to a point $q$ distinct from $p$ (since the $\gamma_i(e_2)$ lie in $S^1 - \nabla$).

Suppose for contradiction that no subsequence of the images of the $x_i$ in $N(B^2/\Gamma)$ converges. By passing to a subsequence, we may assume that for some cusp of $N(B^2/\Gamma)$, the images of the $x_i$ lie farther and farther out the cusp. Now $\alpha_0$ does not travel straight out the cusp, since $p$ is not a parabolic fixed point. Therefore there are portions of $\alpha_0$ that travel almost straight out the cusp for a long time, then start to spiral, becoming tangent to some horocycle, then return back to the thick part of $B^2/\Gamma$. The image $\lambda_0$ of $\lambda$ either travels straight out the cusp, or has infinitely many portions similar to those of $\alpha_0$. Consider such a portion of $\alpha_0$. At the part where it spirals near its tangent horocycle, it crosses $\lambda_0$ almost perpendicularly (either where $\lambda_0$ is traveling straight out the cusp, or on infinitely many portions
that are traveling out to horocycles much farther out the cusp). This implies that there is a sequence of nearly perpendicular crossings of $\alpha$, converging to $p$. This contradicts the choice of $\lambda$. Therefore, by taking a subsequence of the $x_i$, we may assume that the images of the $x_i$ converge to a point $s$ in $N(B^2/\Gamma)$, and moreover that the images of the (unit) tangent vectors to $\alpha$ at the $x_i$ (oriented to point toward $p$) also converge. Since the intersection angle between $\alpha$ and the $\gamma_i(\lambda)$ at the $x_i$ limit to 0, the images of the tangent vectors of the $\gamma_i(\lambda)$ at the $x_i$ (oriented to point toward the endpoint $\gamma_i(e_1)$ of $\gamma_i(\lambda)$) must also converge to the same limiting vector. Let $\mu_0$ be the geodesic in $B^2/\Gamma$ determined by this tangent vector.

Suppose for contradiction that $\mu_0$ has a transverse self-intersection. Then some segment $\sigma$ of $\mu_0$ containing $s$ has a transverse self-intersection at a point $s_0$, where it crosses itself making a positive intersection angle $\theta$. There are portions of $\alpha_0$ and $\lambda_0$ which approximate $\sigma$ arbitrarily closely. Therefore there are crossings of $\alpha_0$ with $\lambda_0$ close to $s_0$, at angles close to $\theta$. This shows that there are translates of $\lambda$ crossing $\alpha$ at angles approximately $\theta$, with the crossings limiting to $p$. This contradicts the choice of $\lambda$. We conclude that $\mu_0$ has no transverse self-intersections. By similar reasoning, $\lambda_0$ cannot intersect $\mu_0$ transversely.

Every point of $\mu_0$ must be a limit of points of $\alpha_0$, and hence $\mu_0$ lies in $N(B^2/\Gamma)$. Suppose for contradiction that it does not lie in a compact subset of $N(B^2/\Gamma)$. Since it has no self-intersections, it must travel all the way out a cusp of $N(B^2/\Gamma)$. Since $\lambda_0$ does not cross $\mu_0$ transversely, but every point of $\mu_0$ is a limit of points of $\lambda_0$, $\lambda_0$ must also travel straight out that cusp. Since $\alpha_0$ cannot travel straight out the cusp, because $p$ is not a parabolic fixed point, there must as before be infinitely many portions of $\alpha_0$ that spiral near horocycles in this cusp. This produces nearly perpendicular intersections of $\alpha_0$ with $\lambda_0$, as before contradicting the choice of $\lambda$. So we may assume that $\mu_0$ lies in a compact subset of $N(B^2/\Gamma)$. Since $\mu_0$ has no self-intersections, its closure is a geodesic lamination $L$, for which every tangent vector is a limit of tangent vectors of $\alpha_0$.

Suppose for contradiction that every subray of $\alpha_0$ has transverse crossings with $\mu_0$. Let $\mu$ be a lift of $\mu_0$ to $B^2$. Then there is a sequence of translates of $\mu$ intersecting $\alpha$ in a sequence of points $r_i$ converging to $p$. By passing to subsequences, we may assume that $r_i$ and $x_i$ alternate as one moves along $\alpha$. Since $\lambda_0$ does not cross $\mu_0$ transversely, the translates must be disjoint and must alternate as shown in Figure 2. Therefore there is a sequence of translates of $\mu$ that converges to the geodesic from $p$ to $q$. Since $L$ is closed,
this shows that the geodesic from \( p \) to \( q \) is the lift of a geodesic of \( L \). By corollary 2.4, \( p \) is a controlled concentration point for \( \Gamma \), a contradiction. So by passing to a subray of \( \alpha \), we may assume that \( \alpha_0 \) does not cross \( L \) transversely. Since \( p \) is not a controlled concentration point, corollary 2.4 shows that \( \alpha_0 \) does not lie in a leaf of \( L \), so \( \alpha_0 \) is disjoint from \( L \).

![Figure 2](image)

Cutting \( N(B^2/\Gamma) \) along \( L \), we obtain pieces as described in lemma 2.3. Now \( \alpha_0 \) lies in one of these pieces and tangent vectors of \( \alpha_0 \) lie arbitrarily close to vectors in a boundary geodesic \( \rho_0 \) of \( L \). Suppose this geodesic is not closed. Then it lies in a polygon or a crown, and some terminal segment of \( \alpha_0 \) travels out an end of the polygon or crown limiting onto \( \rho_0 \). It follows that some lift of \( \rho_0 \) ends at \( p \). By corollary 2.4, \( p \) is a controlled concentration point, a contradiction. So we may assume that \( \rho_0 \) is a simple closed loop.

We now refer to Figure 3. Since tangent vectors of \( \lambda_0 \) limit to \( \rho_0 \), there is a portion of \( \lambda_0 \) that spirals very close to \( \rho_0 \), reaches a minimum distance \( d \), then spirals away. Assuming that the portion is selected to make \( d \) sufficiently small, there is a sequence of lifts of \( \lambda_0 \) to \( B^2 \) that appear with a lift \( \rho \) of \( \rho_0 \) as shown. Similarly, there are portions of \( \alpha_0 \) that spiral in toward \( \rho_0 \), reaching a minimum distance \( d' \) from \( \rho_0 \), where \( d' \) may be selected to be much smaller than \( d \), and then spiral away. This implies there are translates \( \alpha \) as shown in Figure 3. Notice that any such lift must cross one of the translates of \( \lambda \).
§4. The Schottky Example

in Figure 3, making an angle larger than some positive lower bound $\theta_0$. A succession of translates of $\alpha$ corresponding to smaller and smaller values of $d'$ shows that there is a sequence of intersections of $\alpha$ with translates of $\lambda$, converging to $p$, at which the crossing angles are all greater than $\theta_0$. This again contradicts the choice of $\lambda$, establishing that $p$ is a geodesic separation point.

3.2

4 The Schottky example

In this section, we show that the sets of controlled concentration points, concentration points, and geodesic separation points can differ even for finitely generated Fuchsian groups. For simplicity, we will work with an explicit two-generator 2-dimensional Schottky group $\Gamma_0$, although it will be apparent that the same phenomena occur for other similar examples. The limit set of $\Gamma_0$ is a Cantor set which can be understood quite explicitly using the sequence of
crossings of a geodesic ray (ending at the limit point) with the translates of two fixed geodesics which lie in the boundary of a fundamental domain.

To define \( \Gamma_0 \), we work in the Poincaré unit disc \( B^2 \). Figure 4 shows a fundamental domain for the action of \( \Gamma_0 \) on \( B^2 \). Its frontier has two geodesics, \( a \) and \( a' \), with centers on the real axis, and two more, \( b \) and \( b' \), with centers on the imaginary axis. It is generated by two isometries, one of which preserves the real axis and carries \( a' \) to \( a \), and the other preserving the imaginary axis and carrying \( b' \) to \( b \). Fix arbitrarily a normal direction to \( a \) to call the positive direction. It determines a normal direction for each translate of \( a \); the side of the translate into which it points will be called the positive side. Similarly, we select a positive side for \( b \) and its translates. A crossing of an oriented geodesic or geodesic ray in \( B^2 \) with a translate of \( a \) or \( b \) will be called a positive crossing when it crosses from the negative side to the positive side, otherwise it will be called a negative crossing.

A few more translates of these geodesics are drawn in Figure 4; it is convenient to label all translates of \( a \) with a letter \( a \), located on the positive side, and similarly to label all translates of \( b \).

Suppose that \( \alpha \) is a geodesic segment or ray in \( B^2 \), which does not lie in
§4. The Schottky Example

a translate of $a$ or $b$. Then $\alpha$ crosses a sequence (finite or infinite, possibly of length 0) of translates of $a$ and $b$. When a geodesic segment starts or ends in a translate, or a geodesic ray starts in one, that intersection is to be counted as a crossing. To $\alpha$, we associate a sequence $S(\alpha) = x_1x_2x_3 \cdots$ of elements in the set $\{a, \overline{a}, b, \overline{b}\}$ in the following way. If the $i^{th}$ crossing of $\alpha$ with the union of the translates of $a$ and $b$ is a positive crossing with a translate of $a$, then $x_i = a$. If the $i^{th}$ crossing is a negative crossing with a translate of $a$, then $x_i = a$. For crossings with translates of $b$, the elements $b$ and $\overline{b}$ are assigned similarly. Note that $S(\alpha)$ is an infinite sequence if and only if $\alpha$ is a geodesic ray which ends at a limit point of $\Gamma_0$, and that for each sequence $S = x_1x_2x_3 \cdots$ of elements of the set $\{a, \overline{a}, b, \overline{b}\}$, there exist geodesic rays $\alpha$ with $S(\alpha) = S$.

Although we will not need it, the following fact seems worth mentioning. If $S(\alpha) = x_1x_2x_3 \cdots$ is a crossing sequence of infinite length, let $\sigma(S(\alpha))$ denote the shifted sequence $x_2x_3x_4 \cdots$. Suppose $\alpha_1$ and $\alpha_2$ are geodesic rays ending at the limit points $p_1$ and $p_2$ respectively. Then there exist $m, n \geq 0$ so that $\sigma^m(S(\alpha_1)) = \sigma^n(S(\alpha_2))$ if and only if there exists $\gamma \in \Gamma_0$ so that $\gamma(p_1) = p_2$.

Using these sequences, the controlled concentration points of $\Gamma_0$ can be characterized. The following result appears in [1], but we reproduce its proof here for the convenience of the reader.

**Proposition 4.1** Let $p$ be a limit point of $\Gamma_0$ which is the endpoint of a geodesic ray $\alpha$ with $S(\alpha) = x_1x_2x_3 \cdots$. Then $p$ is a controlled concentration point for $\Gamma_0$ if and only if $S(\alpha)$ has the following property. There exists $N$ such that for all $n \geq N$, for all positive $k$, and for all $M$, there exists $m \geq M$ such that $x_{n+i} = x_{m+i}$ for all $i$ with $0 \leq i \leq k$.

In words, past some point every finite subsequence reappears infinitely many times. This is equivalent to the condition that past some point every finite subsequence reappears at least once.

**Proof of 4.1:** Denote by $\lambda_n$ the translate of $a$ or $b$ whose crossing with $\alpha$ determines $x_n$, and by $U_n$ the neighborhood of $p$ bounded by the endpoints of $\lambda_n$. Suppose the condition in the proposition holds. By truncating $\alpha$, we may assume that every subsequence reappears infinitely often. Let $m_k$ be an integer greater than $k$ so that $x_{1+i} = x_{m_k+i}$ for $0 \leq i \leq k$. Given a neighborhood $V$ of $p$, choose $k$ so large that $\lambda_k$ has endpoints in $V$. Let $\gamma$ be the element of $\Gamma_0$ that translates $\lambda_1$ to $\lambda_{m_k}$. Note that $\gamma$ translates $\lambda_{1+i}$.
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onto $\lambda_{m+k}$ for all $0 \leq i \leq k$, hence translates $U_{1+i}$ onto $U_{m+k+i}$ for $0 \leq i \leq k$. Therefore $\gamma(U_1) = U_{m+k} \subseteq V$ and $p \in U_{m+k} = \gamma(U_{1+k}) \subseteq \gamma(V)$, showing that $U_1$ can be concentrated with control. Conversely, suppose $p$ is a controlled concentration point and choose $N$ large enough so that $U_N$, and hence every neighborhood of $p$ inside $U_N$, can be concentrated with control. For any $n, k > N$ and any $M$, there exists $\gamma$ so that $\gamma(U_n) \subseteq U_M$ and $\gamma^{-1}(p) \in U_{n+k}$. This $\gamma$ must move $\lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+k}$ onto a sequence of translates of $a$ and $b$ crossed by $\alpha$, with endpoints in $U_M$. Thus the condition of proposition [4.1] holds.

Proposition [4.1] shows immediately that not all limit points of $\Gamma_0$ are controlled concentration points. The next two theorems provide more delicate examples.

Theorem 4.2 There are uncountably many limit points of $\Gamma_0$ which are concentration points but are not controlled concentration points.

Proof of 4.2: Denote by $a_n$ a sequence of $n$ a’s, and by $\overline{a}_n$ a sequence of $n$ $\overline{a}$’s. Choose one of the uncountably many increasing sequences of positive integers $1 \leq i_1 < j_1 < i_2 < j_2 < i_3 < \cdots$, and let $p$ be a limit point which is the endpoint of a geodesic ray whose crossing sequence is

$$ba_{i_1}b\overline{a}_{j_1}ba_{i_2}b\overline{a}_{j_2}ba_{i_3}b\overline{a}_{j_3}ba_{i_4}b\overline{a}_{j_4} \cdots .$$

By proposition [4.1], $p$ is not a controlled concentration point. We will verify that it is a concentration point.

Let $W$ be the connected neighborhood of $p$ in $\partial B^2$ whose endpoints are the translate of $b$ whose intersection with $\alpha$ corresponds to the first $b$ in the crossing sequence of $\alpha$. If $\lambda$ is any geodesic whose endpoints are the endpoints of a connected neighborhood $U$ of $p$ with $U \subseteq W$, then $\alpha$ crosses $\lambda$. We will show that any such $U$ can be concentrated at $p$.

We refer to Figure 5. The geodesic labeled with $a_{i_n}$ represents the $i_n$ translates of $a$ whose positive crossings by $\alpha$ produce the block of $i_n$ a’s in the crossing sequence of $\alpha$, and similarly for the geodesic labeled with $a_{j_n}$. The geodesic $\mu$ is the unique oriented geodesic which crosses the middle one of the three translates of $b$ in Figure 5 which cross $\alpha$, and has two-ended crossing sequence $\cdots aaab\overline{a}aa \cdots$. Its endpoints are labeled $B$ and
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C, and also shown are the two translates of \( \mu \) which cross the other two translates of \( b \) that cross \( \alpha \) in Figure 5. Since the crossing sequence of \( \alpha \) contains arbitrarily long blocks of the form \( a_i \mu a_j \), it follows that there are translates of \( \alpha \) limiting onto \( \mu \) so that the images of the initial point of \( \alpha \) limit to \( B \) and the images of \( p \) limit to \( C \). This is indicated by the direction arrow on \( \mu \). Similarly, directions are labeled on the other two translates of \( \mu \).

![Figure 5](image)

We orient \( \lambda \) so that it crosses \( \alpha \) from left to right in Figure 5. Let \( E \) be the initial point of \( \lambda \). There must be an \( n \) so that \( E \) lies between the outermost and innermost of the three translates of \( b \) that cross \( \alpha \) in Figure 5. Suppose first that \( E \) is not equal to either \( B \) or \( C \). Then \( \lambda \) must cross either \( \mu \) or one of the two translates of \( \mu \) shown in Figure 5, making some nonzero angle \( \theta \) at the intersection point. Therefore it crosses almost all the translates of \( \alpha \) that limit to \( \mu \). Translating these back to \( \alpha \), we find translates of \( \lambda \) crossing \( \alpha \) from left to right at angles approximately \( \theta \), arbitrarily close to \( p \), showing that \( U \) can be concentrated at \( p \).

There remains the case where \( E \) equals either \( B \) or \( C \). For this, consider
the geodesic $\beta$ which runs from the left hand endpoint $A$ of the largest translate of $b$ in Figure 5 that crosses $\alpha$ to the left hand endpoint $D$ of the smallest one. Figure 6 shows the images of $A$, $B$, $C$, and $D$ under the element $\gamma$ of $\Gamma_0$ that moves the middle translate of $b$ in Figure 5 to a corresponding one closer to $p$. To verify that the images of $A$ and $D$ are as indicated, note that after crossing the translate of $b$ in Figure 5, $\beta$ makes $j_n$ negative crossings with $a$'s then limits onto the unlabeled side of a translate of $b$. Referring back to the fundamental domain shown in Figure 4, one sees that the latter translate of $b$ must be labeled as shown in Figure 6, hence $\gamma(D)$ is as shown. The determination of $\gamma(A)$ is similar. So $\gamma(\lambda)$ runs from $\gamma(B)$ or $\gamma(C)$ to some point lying between $\gamma(D)$ and $\gamma(A)$. This shows that $U$ can be concentrated at $p$, and completes the proof that $p$ is a concentration point.

4.2

Figure 6

Theorem 4.3 There are uncountably many limit points of $\Gamma_0$ which are not concentration points.
4. The Schottky Example

Proof of 4.3: We retain the notation of the proof of theorem 4.2. Choose one of the uncountably many increasing sequences of positive integers $1 \leq i_1 < i_2 < i_3 < \cdots$, and let $p$ be a limit point which is the endpoint of a geodesic ray whose associated sequence is

$$ba_{i_1}ba_{i_2}ba_{i_3}ba_{i_4} \cdots.$$

We will verify that $p$ is not a concentration point.

We refer to Figure 7. Let $\lambda_n$ be the geodesic which runs from the left hand endpoint of the larger translate of $b$ in Figure 7(a) to the right hand endpoint of the smaller one, and let $U_n$ be the neighborhood of $p$ whose endpoints are the endpoints of $\lambda_n$. We will show that $U_n$ cannot be concentrated at $p$. Since there are arbitrarily small such neighborhoods, this implies that $p$ is not a concentration point.

Figure 7

Call the larger translate of $b$ in Figure 7 $\mu_1$, and the smaller $\mu_2$. Suppose there is an element $\gamma \in \Gamma_0$ so that $\gamma(\lambda_n)$ crosses $\alpha$ from left to right, closer to $p$ (i.e. below the crossing of $\mu_2$ with $\alpha$). Suppose that $\gamma(\mu_1)$ crosses $\alpha$. Then since $\lambda_n$ lies on the unlabelled side of $\mu_1$, $\gamma(\mu_2)$ must lie underneath $\gamma(\mu_1)$, and as shown in Figure 7(b), $\gamma(\lambda_n)$ must lie entirely to the left of $p$. If $\gamma(\mu_2)$ crosses $\alpha$, then similar considerations show that $\gamma(\lambda_n)$ lies to the right of $p$. Suppose neither crosses $\alpha$. Observe that any translate of $b$ which lies in the
boundary of a translate of the fundamental domain that intersects \( \alpha \) near \( p \) either

(a) crosses \( \alpha \), or

(b) lies entirely to the left of \( p \) and has its labelled side underneath, or

(c) lies entirely to the right of \( p \) and has its labelled side above.

Since \( \lambda_n \) crosses only \( i_n \) translates of \( \alpha \), any translate of \( \lambda_n \) that crosses \( \alpha \) from left to right near \( p \) must either start on the unlabelled side of a translate of \( b \) or end on a labelled side of one, but neither of these is possible. Note, however, that there are translates of \( \lambda_n \) arbitrarily close to \( p \) that cross \( \alpha \) from right to left, as shown in Figure 7(c). This is consistent with the fact that, by theorem 3.2, \( p \) must be a geodesic separation point.

\section{5 The infinitely generated case}

In this section we construct an infinitely generated Fuchsian group \( \Gamma_1 \) having uncountably many conical limit points which are not geodesic separation points. This shows that in theorem 3.2 the hypothesis that \( \Gamma \) is finitely generated is necessary. Moreover, \( \Gamma_1 \) has uncountably many limit points that are weak concentration points but not conical limit points.

**Proposition 5.1** There is an infinitely generated fuchsian group \( \Gamma_1 \), containing no parabolic elements, having uncountably many weak concentration points that are not conical limit points, and uncountably many conical limit points that are not geodesic separation points.

**Proof of 5.1:** Let \( \Gamma \) be the fundamental group of the closed orientable surface \( F \) of genus 2, acting on \( B^2 \) as determined by some hyperbolic structure on \( F \). It contains no parabolic elements. Regard \( F \) as the boundary of the genus 2 handlebody \( V \), and choose elements \( a \) and \( b \) in \( \pi_1(F) \) whose images under the homomorphism \( \pi_1(F) \to \pi_1(V) \) represent free generators of \( \pi_1(V) \).

Let \( \tilde{V} \) be the infinite cyclic covering of \( V \) corresponding to the kernel of the homomorphism \( \pi_1(V) \to \mathbb{Z} \) that sends \( a \) to 1 and \( b \) to 0. This covering
§5. Another Example

can be constructed by cutting $V$ apart along a cocore disc $D_a$ for one of its 1-handles (the one corresponding to the generator $a$) and gluing infinitely many copies $\ldots, V_{-2}, V_{-1}, V_0, V_1, V_2, \ldots$, of the split-open handlebody end to end along their copies of $D_a$. For each $i$, $V_i \cap V_{i+1}$ is a lift $D^i_a$ of $D_a$. The cocore disc $D_b$ for the other 1-handle lifts to a copy $D^i_b$ in each $V_i$. Since every simple closed essential loop in $F$ is isotopic to a geodesic, we may assume that $\partial D_a$ and $\partial D_b$ are geodesics.

Let $\tilde{F}$ be the boundary of $\tilde{V}$, and let $\Gamma_1$ be the subgroup of $\Gamma$ corresponding to $\pi_1(\tilde{F})$. Denote $\tilde{F} \cap V_i$ by $F_i$. Each $F_i$ is a twice-punctured torus, with boundary $\partial D^{i-1}_a \cup \partial D^i_a$, and with a 1-handle which contains the loop $\partial D^i_b$. Fix a basepoint $x$ of $\tilde{F}$, disjoint from $\partial D_a$, and let $\tilde{x}$ be the point of the preimage of $x$ that lies in $F_0$. Choose a basepoint $\tilde{x}$ for $B^2$ that maps to $\tilde{x}$. Notice that the union over all $j \in \mathbb{Z}$ of the preimage geodesics of $\partial D^i_a \cup \partial D^i_b$ in $B^2$ forms the full preimage of $\partial D_a \cup \partial D_b$ in $B^2$. Therefore these preimage geodesics are pairwise disjoint, and for every $\epsilon > 0$, there are only finitely many with Euclidean diameter greater than $\epsilon$.

Since $\tilde{F}$ is a regular covering of $F$, $\Gamma_1$ is a normal subgroup of $\Gamma$ and hence its limit set is all of $S^1$. By corollary 3.3 of \cite{3}, every limit point of $\Gamma_1$ is a weak concentration point. We will show that uncountably many of these are not conical limit points.

For each $k$ let $c_k$ be the shortest loop based at $x$ that represents $a^k b$ in $\pi_1(F, x)$. Choose uncountably many sequences $i_1, i_2, \ldots$ of positive integers, so that no two of the sequences become equal after truncation of any initial segments. For each sequence, let $\beta$ be the ray in $F$ corresponding to the infinite product $c_{i_1} c_{i_2} \ldots$. Let $\alpha'_{0}$ be the lift of $\beta$ to $\tilde{F}$ starting at $\tilde{x}$. For each $j \geq 0$, $\alpha'_{0}$ crosses $\partial D^i_a$ exactly once. Therefore the lift of $\alpha'_{0}$ to $B^2$, starting at $\tilde{x}$, limits to a single point $p$ in $S^1$. The geodesic ray $\alpha$ from $\tilde{x}$ to $p$ also crosses the union of the preimage geodesics of $\partial D^i_a$ exactly once. Let $\alpha_0$ be its image in $\tilde{F}$. Then for each $j \geq 0$, $\alpha_0$ crosses $\partial D^i_b$ exactly once. In particular, for every compact subset $K$ of $\tilde{F}$, there is a subray of $\alpha_0$ which is disjoint from $K$. This shows that $p$ is not a conical limit point for $\Gamma_1$. The points $p$ obtained from different sequences are distinct. In fact, no two of them can even be equivalent under the action of $\Gamma_1$, for if so then some terminal segments of their defining sequences would have to be equal.

To see that uncountably many limit points of $\Gamma_1$ are conical limit points that are not geodesic separation points, modify the previous construction by letting $c_k$ be the loop in $F$ representing $a^k b a^{-k} b$. This time, $c_k$ lifts to a loop in $\tilde{F}$ that starts at $\tilde{x}$, moves into $F_k$, circles around the 1-handle in $F_k$
crossing $\partial D^k_b$, returns to $F_0$, and goes around the 1-handle in $F_0$, crossing $\partial D^0_b$ once before returning to $\tilde{x}$. Choose the $i_k$ to be positive and increasing, and proceed with the construction of $\beta$, $\alpha'_0$, $\alpha$, and $\alpha_0$ as before. The fact that $\alpha'_0$ crosses each $\partial D^i_k b$ exactly once shows that $\alpha'_0$ limits to a single point $p \in S^1$, and the geodesic ray $\alpha_0$ crosses each $\partial D^i_k b$ exactly once, since $\alpha'_0$ does. Note that the crossing angles of $\alpha_0$ with the $\partial D^i_k b$ will be bounded away from 0. This time, $p$ is a conical limit point, since $\alpha_0$ returns infinitely many times to the compact subset $\partial D^0_b$. But $p$ is not a geodesic separation point. For the crossing of $\alpha_0$ with $\partial D^i_k b$ produces a crossing of $\alpha$ with a geodesic $\lambda_k$ in the preimage of $\partial D^i_k b$. Since the crossing angles are bounded away from 0, the endpoints of the $\lambda_k$ converge to $p$. Since $\alpha_0$ crosses $\partial D^i_k b$ only once, the geodesics $\lambda_k$ show that $p$ is not a geodesic separation point.

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