Modified projective synchronization between fractional order complex chaotic systems

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Abstract. In this paper, the modified projective synchronization between two fractional order complex chaotic systems is deeply studied. The controller is designed on the basis of stability theory of fractional order complex chaotic systems to synchronize the considered systems. Finally, the numerical solution is investigated to support it.

1. Introduction
In mathematics, the concept of fractional order calculus was discussed nearly from the time of origination of differential calculus. But the lack of literature on the fractional calculus deficits the development of the field. The idea of fractional order derivatives is first coined by Leibnitz in 1695[1]. Though there was some development in the topic, it was almost theoretical. Only after the works of Liouville, Grunwald, Letnikov and Riemann, in 19th century, it was understood that the real order differentials and integrals are of great importance than the integral order in solving many real world problems. The uses of fractional order calculus[2] are experienced in the areas of science, engineering and technology. Fractional derivatives play a key role in describing the mechanical and electrical properties of many real models. It also contributed a lot in the development of the theory of fractals and also in the theory of modelling and controlling various dynamical systems. Some of the perspectives of fractional order derivatives and integrals that are studied with visco-elasticity, diffusion mechanics, fluid mechanics, wave propagation, electric transmission, image processing, signal processing, economics show the richness of dynamical behaviors. In the past, only problems on real systems are considered for study. But recently, a lot of study is going on with complex chaotic systems and is proved to be useful in solving many practical problems.

This turns our attention towards fractional order complex chaotic systems and its synchronization. The discussions on the synchronization of chaotic systems has been popular among researchers only after the work of Ott et.al.[3], Pecora and Carroll[4]. The drive and response system, either they are identical chaotic systems or not, even when they start at different values tend to synchronize at some stage, provided the Lyapunov exponent of the systems are negative. Most of this kind of problems are identified in neural networks, data processing, signal processing, circuit analysis, ecological systems and oscillatory systems. Those systems are synchronized by applying some control parameter. Various types of chaotic synchronization has been investigated over the years, of which projective synchronization [5, 6, 7], time scale synchronization[8] are the recent one.
Several methods are being used to synchronize the systems. In 2013, Lei Wang et.al.[9] used modified projective synchronization to control two real chaotic systems. In 2014, Sun et.al.,[10] used modified projective synchronization, stability theory to control real and complex chaotic systems. In 2016, Ajit et.al.,[11] synchronized two different real chaotic systems using active control method. Velmurugan and Rakkiyappan[12] used stability theory, Caputo fractional derivative, hybrid synchronization scheme to synchronize fractional order chaotic complex nonlinear systems with time delays. All these synchronization schemes are investigated on the ground of Lyapunov stability theory. A scaling factor that is designed for the control is a desired constant value to reach synchronization. For this advantage, modified projective synchronization is preferred over the other in solving the fractional order complex chaotic systems.

The rest of the paper is organized as follows. Section 2 lists some of the prerequisite definitions, theorems for the investigation. In Section 3, we give the details of study scheme on two fractional order complex chaotic systems. After this, numerical simulations are given in Section 4 to show the effectiveness of the obtained results. We conclude the paper with some general remarks in Section 5.

2. Preliminaries
The basic definitions, lemma and properties required for the synchronization are given below.

**Definition 2.1** Fractional Integral[1] For any function \( x(t) \), the fractional integral of the order \( \mu \) is defined as

\[
I^\mu x(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} (t - \zeta)^{\mu - 1} x(\zeta) d\zeta
\]

where \( t \geq t_0, \mu > 0 \).

**Definition 2.2** Caputo’s Fractional Derivative[1] For any function \( x(t) \in C^n([t_0, \infty), R) \), the Caputo’s fractional derivative of order \( \mu \) is defined by

\[
D^\mu x(t) = \frac{1}{\Gamma(a - \mu)} \int_{t_0}^{t} x^{(a)}(\zeta) (t - \zeta)^{a-\mu+1} d\zeta
\]

where \( t > t_0 \) and \( a \) is a positive integer such that \( a - 1 < \mu < a \).

Moreover, when \( 0 < \mu < 1 \)

\[
D^\mu x(t) = \frac{1}{\Gamma(1 - \mu)} \int_{t_0}^{t} x'(\zeta) (t - \zeta)^{\mu} d\zeta.
\]

**Lemma 2.3** Barbalat’s Lemma[13] Suppose \( f(t) \in R \) and \( \lim_{t \to \infty} f(t) = \delta \) where \( \delta < \infty \). If \( f' \) is uniformly continuous, then \( \lim_{t \to \infty} f'(t) = 0 \).

**Definition 2.4** Modified Projective Synchronization[10] For the complex drive system (1) and complex response system (2), there exists a scaling matrix \( \beta \) that satisfies the following condition

\[
\lim_{t \to \infty} ||\hat{e}(t)|| = 0,
\]

then the systems achieve modified projective synchronization.
3. The Synchronization Scheme

The scheme for the modified projective synchronization of complex drive and response system is explained in this section.

Consider any two fractional order complex chaotic system as follows

\[ D^\mu x = f(x) \] (drive system) \tag{1}

\[ D^\mu y = g(y) + \eta \] (response system) \tag{2}

where \( x = (x_1, x_2, x_3, \ldots, x_n)^T \) is the state complex vector of the drive system (1) with \( x_i = w_i + jw'_i \) and \( f(x) = (f_1(x), f_2(x), f_3(x), \ldots, f_n(x))^T \) is the complex continuous nonlinear function vector, \( y = (y_1, y_2, y_3, \ldots, y_n)^T \) is the state complex vector of the response system (2) with \( y_i = u_i + jv'_i \), \( g(y) = (g_1(y), g_2(y), g_3(y), \ldots, g_n(y))^T \) is the complex continuous nonlinear function vector. The designed complex controller is defined as \( \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n)^T \) where \( \eta_i = v_i + jv'_i \).

Let the error function be defined as

\[ \dot{e}(t) = x(t) - \beta y(t) \] \tag{3}

where \( \dot{e}(t) = (\dot{e}_1(t), \dot{e}_2(t), \dot{e}_3(t), \ldots, \dot{e}_n(t))^T \), \( \beta = diag(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \) is an \( n \)-order diagonal matrix with each \( \beta_i \neq 0 \) (i = 1, 2, ..., n) for all \( t \).

**Theorem 3.1** For any invertible diagonal matrix \( \beta \), if the controller

\[ \eta = \beta^{-1}f(x) - g(y) + \kappa \beta^{-1}\dot{e}(t) \] \tag{4}

then the systems lead to modified projective synchronization. Here \( \kappa > 0 \) is a constant.

**Proof.** Depending on the drive system (1) and the response system (2), the error function (3) is rewritten as

\[ D^\mu \dot{e}(t) = D^\mu x(t) - \beta D^\mu y(t) = -\kappa \dot{e}(t) \] \tag{5}

To check the stability, select a positive definite function \( V(t) = \frac{1}{2} \dot{e}^T \dot{e} \). The derivative of \( V \) with respect to the time \( t \) along the trajectory of the error function is

\[ D^\mu V(t) = D^\mu \left[ \frac{1}{2} \dot{e}^T \dot{e} \right] = -\kappa \dot{e}^T \dot{e} \leq 0. \text{ (by (5))} \]

Since the function \( V \) is positive definite and \( \dot{V} \) is negative semi-definite, the asymptotic stability of the error system is not attained. But by (6), we get \( \dot{e}_1, \dot{e}_2, \dot{e}_3, \ldots, \dot{e}_n \in L_\infty \). And from the definition,

\[ \int_0^t ||\dot{e}(t)||dt = \int_0^t \dot{e}^T \dot{e}dt \leq \frac{V(0)}{\kappa} \]

and thus \( \dot{e}_1, \dot{e}_2, \dot{e}_3, \ldots, \dot{e}_n \in L_2 \).

From (3), we see that \( D^\mu \dot{e}_1, D^\mu \dot{e}_2, D^\mu \dot{e}_3, \ldots, D^\mu \dot{e}_n \in L_\infty \).

By Barbalat's lemma, we get \( \dot{e}_1, \dot{e}_2, \dot{e}_3, \ldots, \dot{e}_n \to 0 \), when \( t \to \infty \). Thus, the modified projective synchronization of the systems is achieved.

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4. Numerical Simulations

4.1. Modified Projective Synchronization of complex Lorenz system and complex Rossler system

Let us consider the complex Lorenz system

\[ D^\mu x_1 = a(x_2 - x_1) \]
\[ D^\mu x_2 = bx_1 - x_2 - x_1x_3 \]
\[ D^\mu x_3 = -cx_3 + \frac{1}{2}(x_1x_2 + x_3x_2) \]  

(7)

as the drive where \( a = 10, \ b = 28, \ c = 8/3 \) and the initial value of \((x_1, x_2, x_3) = (3 + i, 2 + i, 1)\).

Let the complex Rossler system

\[ D^\mu y_1 = -y_2 - y_3 \]
\[ D^\mu y_2 = y_1 + dy_2 \]
\[ D^\mu y_3 = e - fy_3 + \frac{1}{2}(y_1y_3 + y_1y_3) \]  

(8)

be the response system with \( d = 0.2, \ e = 0.2, \ f = 5.7 \) and the initial value of \((y_1, y_2, y_3) = (1 + i, 1 + i, 0)\).
Let the controller be defined as

\begin{align*}
v_1 &= \Re \left\{ \frac{a}{\beta_1} (x_2 - x_1) - (-y_2 - y_3) + \frac{\kappa}{\beta_1} \hat{e}_1 \right\} \\
v_2 &= \Im \left\{ \frac{a}{\beta_1} (x_2 - x_1) - (-y_2 - y_3) + \frac{\kappa}{\beta_1} \hat{e}_1 \right\} \\
v_3 &= \Re \left\{ \frac{1}{\beta_2} (bx_1 - x_2 - x_1 x_3) - (y_1 + dy_2) + \frac{\kappa}{\beta_2} \hat{e}_2 \right\} \\
v_4 &= \Im \left\{ \frac{1}{\beta_2} (bx_1 - x_2 - x_1 x_3) - (y_1 + dy_2) + \frac{\kappa}{\beta_2} \hat{e}_2 \right\} \\
v_5 &= \Re \left\{ \frac{1}{\beta_3} \left( -cx_3 + \frac{1}{2} (\bar{\tau}_1 x_2 + x_1 \bar{\tau}_2) \right) \\
&\quad - \left( e - fy_3 + \frac{1}{2} (\bar{\tau}_1 y_3 + y_1 \bar{\tau}_3) \right) + \frac{\kappa}{\beta_3} \hat{e}_3 \right\} \\
v_6 &= \Im \left\{ \frac{1}{\beta_3} \left( -cx_3 + \frac{1}{2} (\bar{\tau}_1 x_2 + x_1 \bar{\tau}_2) \right) \\
&\quad - \left( e - fy_3 + \frac{1}{2} (\bar{\tau}_1 y_3 + y_1 \bar{\tau}_3) \right) + \frac{\kappa}{\beta_3} \hat{e}_3 \right\}
\end{align*}

For the numerical simulation, Adam’s Bashforth Moulton method is used with a time interval of 0.01 and projection constant \( \kappa = 0.6 \). The simulation results are studied and thus the

**Figure 2.** Controlled Response system

![Controlled Response system](image)
Figure 3. Error function

Figure 4. Drive vs Response
Figure 5. Drive vs Response (contd.)

Figure 6. Controller function
feasibility and effectiveness of the scheme is verified. It is also found that the change in the fractional order of the system does not affect the synchronization much.

5. Conclusions
The indeterministic behavior of the two different fractional order complex dynamical systems has been addressed and are synchronized using modified projective synchronization. Based on the theory of stability, a complex controller has been applied to attain the synchronization, in which, some positions in the response system have been controlled. Considering the case of asymptotic stability of the error between the systems, synchronization is reached by introducing Lyapunov function or functionals. In the end, a numerically solved example is provided to demonstrate the effectiveness and occurrence of the synchronization of two different complex systems. In our view, the proposed control scheme is more useful in the process of synchronizing complex networks with large number of nodes at a minimum time consumption and low installation cost, digital processing and data handling. This solution may also be useful in diagnosing cancer affected areas. However the scheme needs to be experimented in field.

References
[1] Podlubny I 1999 Fractional Differential Equations Academic Press, New York.
[2] Oldham K and Spanier J 1974 The fractional calculus-Theory and applications of differentiation and integration to arbitrary order Academic Press, New York.
[3] Ott E, Grebogi C and Yorke J A 1990 Controlling chaos Physical Review Letter 64 pp 1196–1199.
[4] Pecora L M and Carroll T L 1990 Synchronization in Chaotic Systems Phys. Rev. Lett. 64(8) pp 821–824.
[5] Mainieri R and Rehacek J 1999 Projective synchronization in three-dimensional chaotic systems Phys. Rev. Lett. 82(15) pp 3042–3045.
[6] Chang C M and Chen H K 2010 Chaos and hybrid projective synchronization of commensurate and incommensurate fractional-order Chen-Lee systems Nonlinear Dyn. 62(4) pp 851–858.
[7] Wang Z L 2010 Projective synchronization of hyperchaotic Lü system and Liu system Nonlinear Dyn. 59(3) pp 455–462.
[8] Hramov A E and Koronovskii A A 2005 Time scale synchronization of chaotic oscillators Phys. D. 206(3) pp 252–264.
[9] Wang L, Zhen B and Xu J 2013 A Simple Approach to Achieve Modified Projective Synchronization between Two Different Chaotic Systems The Scientific World Journal pp 1–7.
[10] Sun J, Shen Y and Zhang X 2014 Modified projective and modified function projective synchronization of a class of real nonlinear systems and a class of complex nonlinear systems Nonlinear Dyn. DOI 10.1007/s11071-014-1558-z.
[11] Ajit K S, Vijay K Y and Das S 2016 Synchronization between fractional order complex chaotic systems, Int. J Dynam. Control DOI 10.1007/s40435-016-0226-1.
[12] Velmurugan G and Rakkiyappan R 2015 Hybrid Projective Synchronization of Fractional-Order Chaotic Complex Nonlinear Systems With Time Delays J. Comput. Nonlinear Dynam. 11(3) pp 031016-1-031016-7.
[13] Slotine J J E and Li W 1991 Applied Nonlinear Control Prentice Hall, New Jersey pp 123–125.