EXTENSION OF REILLY FORMULA FOR A CLASS OF
ELLiptic DIFFERENTIAL OPERATOR IN DIVERGENCE
FORM

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Abstract. In this paper, we prove the Reilly formula for the elliptic divergence type operator $L_A(u) := \text{div}(A \nabla u)$ on a compact Riemannian manifold $M$ where $A$ is a positive definite divergence free self-adjoint $(1,1)$-Codazzi tensor field on $M$ and then by assumption on extension of Ricci tensor we get some lower estimates for the first eigenvalue of $L_A$.

1. Introduction

Elliptic operators on manifolds is one of the important extensions of the Laplace operator. One knows that a second-order linear differential operator without zero order term $L : C^\infty(M) \to C^\infty(M)$ can be written as

$$Lf = \text{div}(A \nabla f) + \langle V, \nabla f \rangle,$$

where $A \in \Gamma(\text{End}(TM))$ is a self-adjoint with respect to the metric $\langle \cdot, \cdot \rangle$. So an operator of the form $L_A(f) = \text{div}(A \nabla f)$ is an important kind of elliptic operator. One of the important issue associated with the operator $L_A$ is study of the spectrum of this operator when the manifold $M$ is compact. In this regard [1, 6, 7, 13] got valuable results. One way to get a lower estimate of the first eigenvalue of the Laplace operator is Reilly formula [12]. The formula is proved by integration from the usual Bochner formula and states that for each smooth function $u$ on a manifold $M$ one has,

$$\int_M \left( (\Delta u)^2 - |\text{hess}(u)|^2 - \text{Ric} \langle \nabla u, \nabla u \rangle \right) dv_{g}$$

$$= \int_{\partial M} \left( (n-1)Hu_n^2 + II(\nabla^{\partial} u, \nabla^{\partial} u) + 2u_n\Delta^{\partial}(u) \right) dv_{\bar{g}}$$

(1.1)

where $u_n = \frac{\partial u}{\partial n}$ and $\nabla^{\partial}$, $\Delta^{\partial}$ are the gradient and Laplacian with respect to the metric of $\partial M$. This formula has many interesting consequences in geometry such as estimates of the first eigenvalue of the Laplace operator, Alexandrov’s theorem.

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and the Heintze-Karcher’s inequality. The Reilly formula \(1.1\) is similarly proved for weighted manifolds \(11\) as follows,
\[
\int_M \left( (\Delta f)u^2 - |\text{hess}(u)|^2 - \text{Ric}_f(\nabla u, \nabla u) \right) \text{dvol}_g \\
= \int_{\partial M} u_n (Hu_n - \langle \nabla H, \nabla u \rangle + \Delta (u)) \text{dvol}_{\bar{g}} \\
+ \int_{\partial M} (II(\nabla^\partial u, \nabla^\partial u) - \langle \nabla^\partial u, \nabla^\partial u_n \rangle) \text{dvol}_{\bar{g}}.
\]

Estimate of the first eigenvalue of the Laplace operator, is one of the important and long-standing problem in geometric analysis and PDE theory on manifolds. For example, it gives an upper bound for the constant in the Poincaré inequality. So it is very important to find a good lower estimate for the first eigenvalue of the Laplace operator. Another application is in the estimate of the heat kernel \(3, 4, 9, 10, 11\). Similar results have been obtained for weighted manifolds \(8\).

In this paper, we get a Reilly-type formula for the elliptic divergence type operator \( L_A u = \text{div}(A\nabla u) \), when \( A \) is a divergence free positive definite self-adjoint \((1, 1)\)-Codazzi tensor field on \( M \) and obtain some lower estimates for the first eigenvalue of this operator. Also the approach of this paper is more similar to the corresponding results for the Laplace operator.

Explicitly, the results are as follows. At first we get the Reilly formula for the elliptic operator \( L_A \), when \( A \) is a parallel tensor field. As an important consequence we get the following estimates of the lower bound of the first eigenvalue of the operator \( L_A \), when \( A \) is parallel.

**Theorem 1.1.** Let \( M \) be a closed Riemannian manifold and \( A \) be a parallel symmetric and positive semi-definite operator on \( M \) such that one of the following conditions holds,

1) \( \text{Ric}_A \geq Kg \) and \( K > 0 \) is a constant,
2) \( \text{Ric}_A(X, X) \geq K \langle AX, X \rangle \) and \( K > 0 \) is a constant.

Then, one has the following estimates for the first eigenvalue of the operator \( L_A \),

1) \( \lambda \geq \frac{\text{Trace}(A)K}{\text{Trace}(A) - \delta_1} \)

2) \( \lambda \geq \frac{\text{Trace}(A)\delta_1 K}{\text{Trace}(A) - \delta_1} \)

where \( \lambda \) is the first positive eigenvalue of the operator \( L_A \). If the equality holds then \( A \) is scalar operator, i.e. \( A = \alpha I \) for some real constant \( \alpha \) and \( M \) has constant sectional curvature \( \frac{K}{n-1} \).
As similar as the original one for the Laplace operator the estimates of the Theorem 1.1 are trivial when \( K \leq 0 \), so by adapting of the Li and Yau method we get the following results when \( K \leq 0 \).

**Theorem 1.2.** Let \( M \) be a closed Riemannian manifold, \( A \) be a parallel symmetric and positive semi-definite operator on \( M \) and \( \text{Ric}_A \geq -K \) for some \( K > 0 \), then we have the following estimate for the first eigenvalue of the operator \( L_A \),

\[
2 \left( \alpha + \sqrt{\alpha^2 + K\alpha} \right) \exp \left( -1 - \sqrt{1 + \frac{K\alpha}{\alpha}} \right) \leq \lambda,
\]

where \( \alpha = \frac{\delta_1^2}{d^2 \text{Trace}(A)} \), \( d = \text{diam}(M) \) and \( \delta_1 \) is defined in Definition 2.2.

For the Codazzi divergence free tensor fields, we get the following extended Reilly formula,

**Theorem 1.3 (Extended Reilly formula).** Let \( M \) be a Riemannian manifold with boundary \( \partial M \) and \( A \) be a \((1,1)\)-Codazzi tensor field with \( \text{div}(A) = 0 \) then,

\[
B = C,
\]

where

\[
B = \int_{\partial M} \left( (\nabla \nabla u, \nabla u, A \nhat{n}) - 2\langle \text{shape}^\partial (\nabla^\partial u), A \nhat{n} \rangle \right) dvol_{\bar{g}}
\]

\[
+ \frac{1}{2} \int_{\partial M} \langle \nabla^\partial u, (\nabla^\partial A) \nabla^\partial u \rangle dvol_{\bar{g}} + \int_{\partial M} \left( u^2_\alpha H^\partial_A - \langle A \nhat{n}, \nabla^\partial u_n \rangle \right) dvol_{\bar{g}}
\]

\[
+ \frac{1}{2} \int_{\partial M} u^2_\alpha \langle \nhat{n}, (\nabla^\partial A) \nhat{n} \rangle dvol_{\bar{g}} + \int_{\partial M} \left( \langle \nabla^\partial u, u_n \rangle \langle \nhat{n}, A \nhat{n} \rangle - u_n \Delta^\partial_A(u) \right) dvol_{\bar{g}}
\]

\[
+ \int_{\partial M} u_n \langle \nabla^\partial u, (\nabla^\partial A) \nhat{n} \rangle dvol_{\bar{g}}.
\]

and

\[
C = \int_M \left( \text{Trace} (A \circ \text{hess}^2(u)) \right) dvol_{\bar{g}} - \int_M \Delta_A(u) \Delta u dvol_{\bar{g}}
\]

\[
+ \int_M \text{Ric}_A (\nabla u, \nabla u) dvol_{\bar{g}} + \frac{1}{2} \int_M \langle \nabla u, (\Delta A) \nabla u \rangle dvol_{\bar{g}}.
\]

wherein \( H^\partial_A := \text{Trace}(A \circ \text{shape}^\partial) \) is defined as an extended mean curvature of the boundary \( \partial M \), \( \nhat{n} \) is the outward unit vector field on \( \partial M \) and \( \bar{g} \) is the restricted metric on \( \partial M \) and \( \nabla^\partial \), \( \Delta^\partial_A \) are gradient and extended Laplacian with respect to the metric \( \bar{g} \).

Similarly we get the following estimates, for the first eigenvalues.

**Theorem 1.4.** Let \( M \) be a closed Riemannian manifold and \( A \) be a positive semi-definite \((1,1)\)-Codazzi tensor on \( M \) such that \( \text{Trace}(A) \) is constant. Also for each vector field \( X \) with \(|X| = 1\) one has

\[
\text{Ric}_A (X, X) + \text{Ric}(X, AX) \geq 2K > 0,
\]
then the following estimate for the first eigenvalue of $L_A$ is obtained,

$$\lambda \geq \frac{\text{Trace}(A) K}{\text{Trace}(A) - \delta_1}.$$ 

Also, when $K \leq 0$, we have the following result.

**Theorem 1.5.** Let $M$ be a closed Riemannian manifold and $A$ be a positive semi-definite $(1,1)$-Codazzi tensor on $M$ such that $\text{Trace}(A)$ is constant. Also, let for each vector field $X$ with $|X| = 1$ one has

$$\text{Ric}(X, AX) \geq -K, \quad \delta = \max \langle X, (\nabla A)X \rangle$$

and for any vector field $X, Y, Z$ we have

$$|(\nabla_X \nabla_Y A) Z| \leq K' |X||Y||Z|$$

where $K, K', \delta \geq 0$ and $\text{diam}(M) \leq d$. Then the following estimate for the first eigenvalue of $L_A$ is obtained,

$$2 \left( \alpha + \sqrt{\alpha^2 + (K + 2K' + \delta)\alpha} \right) \exp \left( -1 - \sqrt{1 + \frac{K + 2K' + \delta}{\alpha}} \right) \leq \lambda,$$

where $\alpha = \frac{\delta^2}{\text{diam}^2(M)}$.

Finally, for more general case, we have the following result, when $K > 0$.

**Theorem 1.6.** Let $B$ be a $(1,1)$-self-adjoint tensor field and satisfies in following conditions

a) $\text{div}(B) = 0$,

b) $\nabla^2 B(X, Y)Z \leq K' |X||Y||Z|$ for any vector field $X, Y, Z$,

c) $\text{Ric}(X, BX) \geq K|X|^2$,

d) $\nabla \text{Trace}(B)$ is parallel.

Then the following estimate is obtained for the first eigenvalue of $L_B$,

$$\lambda \geq \frac{n\delta_n (K + 2nK')}{n\delta_n - \delta_1}.$$ 

2. **Preliminaries**

In this section, we summarize some preliminaries that we use in throughout paper.

**Definition 2.1.** A $(1,1)$-tensor field $A$ on a Riemannian manifold $M, \langle , \rangle)$ is self-adjoint whenever

$$\forall X, Y \in \Gamma(TM) : \langle AX, Y \rangle = \langle X, AY \rangle.$$ 

**Definition 2.2.** Let $A$ be a self-adjoint positive definite $(1,1)$-tensor field on $M$, we say $A$ is bounded if there are some constant $\alpha, \beta > 0$ such that for any vector field $X \in \Gamma(TM)$ on $M$ with $|X| = 1$, one has $\alpha < \langle X, AX \rangle < \beta$ and $\delta_1, \delta_n$ are defined as follows
a) \( \delta_1 = \min_{|X|=1} \langle X, AX \rangle \),

b) \( \delta_n = \max_{|X|=1} \langle X, AX \rangle \).

Note that when \( A \) is parallel, then \( \langle \nabla r, A \nabla r \rangle \) is constant with respect to distant function \( r(x) = \text{dist}(p, x) \), in other words \( \frac{\partial}{\partial r} \langle \nabla r, A \nabla r \rangle = 0 \). So \( \delta_1 = \min_{B(p, \varepsilon)} \langle \nabla r, A \nabla r \rangle \) and \( \delta_n = \max_{B(p, \varepsilon)} \langle \nabla r, A \nabla r \rangle \) which \( \varepsilon > 0 \) is arbitrary.

**Definition 2.3.** Let \( A \) be a self-adjoint operator on manifold \( M \) and \( \{e_i\} \) be an orthonormal basis at the computing point. We define \( L_A, \Delta_A \) and \( \text{Ric}_A \) as follow

a) \( L_A(u) := \text{div}(A \nabla u) = \sum_i \langle \nabla_{e_i} (A \nabla u), e_i \rangle \),

b) \( \Delta_A(u) := \sum_i \langle \nabla_{e_i} \nabla u, Ae_i \rangle \),

c) \( \text{Ric}_A(X, Y) := \sum_i \langle R(X, Ae_i)e_i, Y \rangle \) and we call the tensor \( \text{Ric}_A \) as an extended Ricci tensor,

where \( u \) is a smooth function and \( X, Y \) are vector fields on \( M \).

As usual for comparison results in differential geometry one needs a Bochner formula and the associated Riccati inequality. The following Theorem provided this.

**Theorem 2.4** (Extended Bochner formula). \(^7\) Let \( M \) be a smooth Riemannian manifold and \( A \) be a self-adjoint operator on \( M \), then for any smooth function \( u \) on \( M \), we have,

\[
\frac{1}{2}L_A(\nabla u^2) = \frac{1}{2} \left( \langle \nabla \nabla u, \nabla (\text{div}(A)) \rangle + \langle \nabla u, \nabla (\Delta_A u) \rangle \right) - \Delta(\nabla u, A \nabla u) + \text{Ric}_A(\nabla u, \nabla u).
\]

(2.1)

In the following proposition, we provide a generalization of Cauchy-Schwarz inequality. By this result, we can get the so-called Riccati inequality to the extended Bochner formula in Theorem 2.4 in a similar way.

**Proposition 2.5.** \(^1\) Let \( A \) be a positive semi-definite symmetric matrix, then for every matrix \( F \) we have,

\[
\text{Trace}(AF^2) \geq \frac{1}{\text{Trace}(A)}(\text{Trace}(AF))^2.
\]

and the equality holds if and only if \( A = \alpha I \) for some \( \alpha \in \mathbb{R} \).

**Definition 2.6.** Let \( (M, \langle , \rangle) \) be a Riemannian manifold and \( A \) be a \((1,1)\)-tensor field. We say that the tensor \( A \) is a Codazzi Tensor if \( (\nabla_X A)(Y, Z) = (\nabla_Y A)(X, Z) \).

**Definition 2.7.** Let \( A \) be a \((1,1)\)-tensor field on manifold \( M \), then we define \( T^A \) as follows

\[
T^A(X, Y) := (\nabla_X A)Y - (\nabla_Y A)X.
\]

Notice, \( T \) is a \((2,1)\) tensor field.
Example 2.8. If $A$ is the shape operator of the hypersurface $\Sigma^n \subset M^{n+1}$ then
\[ T^A(X, Y) = (\bar{R}(Y, X)N)^T, \]
where $\bar{R}$ is the curvature tensor on the ambient manifold $M$, and $N$ is the unit normal vector field on $\Sigma^n \subset M^{n+1}$.

We compute the second covariant derivation of the operator $A$. This lemma is useful for computation of the tensor $Ric_A$ and its relation with other geometric quantities like Laplace of the tensor $A$ and the Ricci tensor.

Lemma 2.9. Let $A$ be a self-adjoint operator on manifold $M$ and $X, Y, Z$ are vector fields on $M$, then
\[ a) \quad (\nabla^2 A)(X, Y, Z) = (\nabla^2 A)(X, Z, Y) + R(Z, Y)(AX) - A(R(Z, Y)X), \]
\[ b) \quad (\nabla^2 A)(X, Y, Z) - (\nabla^2 A)(Y, X, Z) = (\nabla_Z T^A)(X, Y). \]

Proof. For part (a) we have,
\[ \nabla^2 A(X, Y, Z) = (\nabla (\nabla A))(X, Y, Z) = (\nabla_Z (\nabla A))(X, Y). \]
\[ = \nabla_Z ((\nabla A)(X, Y)) - (\nabla A)(\nabla_Z X, Y) - (\nabla A)(X, \nabla_Z Y) \]
\[ = \nabla_Z ((\nabla Y A)X - (\nabla A)(\nabla_Z X, Y) - (\nabla_{\nabla Z Y} A)X) \]
\[ = (\nabla_Z (\nabla Y A)X + (\nabla Y A)(\nabla_Z X) - (\nabla Y A)(\nabla_Z X) - (\nabla_{\nabla Z Y} A)(X) \]
\[ = (\nabla_Z (\nabla Y A)X - (\nabla_{\nabla Z Y} A)X. \]

Similarly,
\[ \nabla^2 A(X, Z, Y) = (\nabla_Y (\nabla Z A))X - (\nabla_{\nabla Z Y} A)X. \]

Thus
\[ \nabla^2 A(X, Y, Z) - \nabla^2 A(X, Z, Y) = (\nabla_Z \nabla Y A)X - (\nabla_Y (\nabla Z A))X - (\nabla_{\nabla Z Y} A)X \]
\[ = (R(Z, Y)A)X = R(Z, Y)(AX) - A((R(Z, Y)X)). \]

For part (b), by definition of $T$, we have
\[ \nabla^2 A(X, Y, Z) = (\nabla (\nabla A))(X, Y) \]
\[ = \nabla_Z ((\nabla A)(X, Y)) - (\nabla A)(\nabla_Z X, Y) - (\nabla A)(X, \nabla_Z Y) \]
\[ = \nabla_Z ((\nabla A)(Y, X) + T^A(X, Y)) - (\nabla A)(\nabla_Z X, Y) - (\nabla A)(X, \nabla_Z Y) \]
\[ = \nabla_Z ((\nabla A)(Y, X)) + \nabla_Z (T^A(X, Y)) - (\nabla A)(\nabla_Z X, Y) - (\nabla A)(X, \nabla_Z Y) \]
\[ = (\nabla_Z (\nabla A)(Y, X)) + T^A(\nabla_Z Y, X) + T^A(Y, \nabla_Z X) \]
\[ = (\nabla_Z (\nabla A)(Y, X)) + (\nabla_{\nabla Z} T^A)(X, Y). \]

\[ \square \]
Lemma 2.10. Let $A$ be a $(1,1)$-symmetric tensor field and $\nabla^* T^A = 0$, then

$$\langle (\Delta A) X, X \rangle = \langle \nabla_X \text{div}(A), X \rangle - R_{A}(X, X) + R_{A}(X, AX),$$

where $\nabla^*$ is adjoint of $\nabla$.

Proof. For simplicity let $\{e_i\}$ be an orthonormal local frame field with $\nabla e_i, e_j = 0$ at the computing point. By computation and Lemma 2.9 we have,

$$\langle (\Delta A) X, X \rangle = \sum_i \langle \nabla^2 A(e_i, e_i, X), X \rangle = \sum_i \langle \nabla^2 A(e_i, X, e_i), X \rangle + \sum_i \langle (\nabla e_i T^A)(X, e_i), X \rangle$$

$$= \sum_i \langle \nabla^2 A(e_i, X, e_i), X \rangle - \sum_i \langle \text{div}(T^A) X, X \rangle$$

$$= \sum_i \langle \nabla^2 A(e_i, X, e_i), X \rangle.$$

So by Lemma 2.9 part (a) we have

$$\langle (\Delta A) X, X \rangle = \sum_i \langle \nabla^2 A(e_i, X, e_i), X \rangle = \sum_i \langle \nabla^2 A(e_i, X, e_i), X \rangle + \sum_i \langle (R(e_i, X)Ae_i) - A((R(e_i, X)e_i)), X \rangle$$

$$= \langle (\nabla_X \text{div}A), X \rangle - R_{A}(X, X) + R_{A}(X, AX).$$

Corollary 2.11. Let $A$ be a self-adjoint $(1,1)$-Codazzi tensor field. So $T^A = 0$ and we have,

$$\langle (\Delta A) X, X \rangle = Hess(\text{Trace} A)(X, X) - R_{A}(X, X) + R_{A}(X, AX).$$

Also, when $\Sigma^n \subset M^{n+1}$ be the hypersurface with shape operator $A$ and the ambient manifold has constant sectional curvature $c$, then

$$(\Delta A) X = n\nabla_X \nabla H - cnHX + \left(cn - |A|^2\right) AX + nHA^2 X$$

Proof. The first part is clear and for the second part refer to [2] page 333.

Corollary 2.12. In Theorem 2.4 if $A$ is parallel, then

$$\langle (\Delta A) u, u \rangle = \frac{1}{2} L_{A}(|\nabla u|^2) = \text{Trace}(Aohess^2(u)) + \langle \nabla u, \nabla (\Delta_A u) \rangle + R_{A}(\nabla u, \nabla u)$$

The extended Bochner formula (2.1) is very complicated. The complication of the formula is about the existence of parameter $\Delta_{\nabla^* A} u$. However, when $\Delta_{\nabla^* A} u \leq 0$, the Bochner formula get the simple Riccati inequality. In the following proposition, we get some parameters which seems are suitable to estimate $\Delta_{\nabla^* A} u$. These parameters show the affection of parallelness and values of eigenfunction on the value of $\Delta_{\nabla^* A} u$. The values of these parameter depend on analytic and algebraic properties of the tensor $B$. 

Proposition 2.13. Let $B$ be a $(1, 1)$-self-adjoint tensor field on the manifold $M$ then
\[
\Delta_{(\nabla_{\nabla u}, B)}u = \nabla u. \nabla u. \text{Trace}(B) - \langle \nabla u, (\Delta B) \nabla u \rangle + \sum_i \left<i, T(\nabla_{\nabla u} B)(\nabla u, e_i), e_i\right> + \sum_i e_i. \langle \nabla u, T^B(e_i, \nabla u) \rangle.
\] (2.4)

Proof. Let $B$ be a $(1, 1)$-tensor field, then
\[
\Delta_{(\nabla_{\nabla u}, B)}u = \sum_i \langle \nabla_{e_i}, \nabla u, (\nabla_{\nabla u} B) e_i \rangle
\]
\[
= \sum_i \langle \nabla_{e_i}, \nabla u, (\nabla_{\nabla u} B) \nabla u \rangle + \sum_i \langle \nabla_{e_i}, \nabla u, T^B(\nabla u, e_i) \rangle
\]
\[
= \sum_i e_i. \langle \nabla u, (\nabla_{\nabla u} B) \nabla u \rangle - \sum_i \langle \nabla u, (\nabla_{e_i} B) \nabla u \rangle - \sum_i \langle \nabla_{e_i}, \nabla u, T^B(\nabla u, e_i) \rangle
\]
\[
= \sum_i e_i. \langle \nabla u, (\nabla_{\nabla u} B) \nabla u \rangle - \langle \nabla u, (\Delta B) \nabla u \rangle - \Delta_{(\nabla_{\nabla u}, B)}u
\]
\[
+ 2 \sum_i \langle \nabla_{e_i}, \nabla u, T^B(\nabla u, e_i) \rangle
\]
\[
= \sum_i e_i. \langle \nabla u, (\nabla_{\nabla u} B) e_i \rangle + \sum_i e_i. \langle \nabla u, T^B(e_i, \nabla u) \rangle - \langle \nabla u, (\Delta B) \nabla u \rangle - \Delta_{(\nabla_{\nabla u}, B)}u
\]
\[
+ 2 \sum_i \langle \nabla_{e_i}, \nabla u, T^B(\nabla u, e_i) \rangle.
\]

In other words,
\[
\Delta_{(\nabla_{\nabla u}, B)}u = \langle \nabla u, \text{div}(\nabla_{\nabla u} B) \rangle + \sum_i e_i. \langle \nabla u, T^B(e_i, \nabla u) \rangle - \langle \nabla u, (\Delta B) \nabla u \rangle.
\]

But,
\[
\langle \nabla u, \text{div}(\nabla_{\nabla u} B) \rangle = \sum_i \langle \nabla u, \nabla_{e_i} (\nabla_{\nabla u} B) e_i \rangle = \sum_i \langle \nabla_{e_i} (\nabla_{\nabla u} B) \nabla u, e_i \rangle
\]
\[
= \sum_i \langle \nabla_{\nabla u} (\nabla_{\nabla u} B) e_i + T(\nabla_{\nabla u} B)(\nabla u, e_i), e_i \rangle
\]
\[
= \nabla u. \nabla u. \text{Trace}(B) + \sum_i \left<i, T(\nabla_{\nabla u} B)(\nabla u, e_i), e_i\right>.
\]

So,
\[
\Delta_{(\nabla_{\nabla u}, B)}u = \nabla u. \nabla u. \text{Trace}(B) - \langle \nabla u, (\Delta B) \nabla u \rangle + \sum_i \left<i, T(\nabla_{\nabla u} B)(\nabla u, e_i), e_i\right> + \sum_i e_i. \langle \nabla u, T^B(e_i, \nabla u) \rangle.
\]

We recall the following result from [5], Corollary 1.7.5. It is used for the rigidity result in Theorem 1.1.
Proposition 2.14. [5] Let $M$ be a complete Riemannian manifold, then $M$ has constant sectional curvature $H$ iff there is a non trivial smooth function $u$ on $M$ with $Hess u = -Hu g$, where $g$ is the Riemannian metric on $M$.

3. Reilly formula, when $A$ is parallel

In this section, we prove the extended Reilly formula 1.3 when $A$ is parallel.

In fact the result is valid when $\Delta \nabla u \cdot A u = 0$. As usually we integrate from the extended Bochner formula 2.12. The computation is coordinate-independent.

Theorem 3.1. (Reilly-type formula 1) Let $M$ be a complete Riemannian manifold and $A$ be a parallel self-adjoint $(1,1)$-tensor field on it, then

\begin{equation}
B = C,
\end{equation}

where

\begin{align*}
B &= \int_{\partial M} \left( \langle \nabla \nabla u, A \nabla \rangle - 2u_n \langle \text{shape} \langle \nabla u \rangle, A \nabla \rangle \right) dvol_{\tilde{g}} \\
&\quad + \int_{\partial M} \left( u_n H^A_n - u_n \langle A \nabla, \nabla \rangle \right) dvol_{\tilde{g}} \\
&\quad + \int_{\partial M} \left( \langle \nabla u, u_n \rangle \langle A \nabla, \nabla \rangle - u_n L_A^n(u) \right) dvol_{\tilde{g}},
\end{align*}

and

\begin{align*}
C &= \int_M \left( \text{Trace} \left( A \circ \text{hess}^2(u) \right) \right) dvol_g \\
&\quad - \int_M L_A(u) (\Delta u) dvol_g + \int_M \text{Ric}(u, \nabla(L_A u)) dvol_g,
\end{align*}

wherein $H^A_n := \text{Trace}(A \circ \text{shape})$ is defined as an extended mean curvature of the boundary $\partial M$, $\nabla$ is the outward unit vector field on $\partial M$ and $\tilde{g}$ is the restricted metric on $\partial M$ and $\nabla$, $L_A^n$ are gradient and extended Laplacian with respect to the metric $\tilde{g}$, respectively.

Proof. Since $A$ is parallel we have $L_A = \Delta_A$. Therefore integration from the extended Bochner formula in corollary 2.12 gives,

\begin{equation}
\frac{1}{2} \int_M L_A(|\nabla u|^2) dvol_g = \int_M \text{Trace} \left( A \circ \text{hess}^2(u) \right) dvol_g + \int_M \langle \nabla u, \nabla(L_A u) \rangle dvol_g \\
+ \int_M \text{Ric}(u, \nabla u) dvol_g.
\end{equation}

Each of the terms in the above formula are computed as follows,

\begin{equation}
\int_M L_A(|\nabla u|^2) dvol_g = \int_{\partial M} \left( \langle \nabla(|\nabla u|^2), A \nabla \rangle \right) dvol_{\tilde{g}},
\end{equation}

and

\begin{equation}
\int_M \langle \nabla u, \nabla(L_A u) \rangle dvol_g = - \int_M (L_A u)(\Delta u) dvol_g + \int_{\partial M} (L_A u)(\nabla u, \nabla) dvol_{\tilde{g}}.
\end{equation}

So, from (3.2), (3.3) and (3.4) with re-arrangement we have,

$B = C$, 

where
\begin{equation}
B = \frac{1}{2} \int_{\partial M} \left\langle \nabla(|\nabla u|^2), A \nabla \right\rangle \, d\text{vol}_g - \int_{\partial M} (L_A u) \left\langle \nabla u, A \nabla \right\rangle \, d\text{vol}_g,
\end{equation}
and
\begin{equation}
C = \int_M \text{Trace}(A \circ hess^2(u)) \, d\text{vol}_g - \int_M (L_A u)(\Delta u) d\text{vol}_g + \int_M Ric_A(\nabla u, \nabla u) d\text{vol}_g.
\end{equation}

For Dirichlet or Neumann boundary conditions we should compute \((3.5)\) on the boundary of \(M\), in other words, we need to compute the following parameters with respect to the intrinsic and extrinsic geometry of \(\partial M\),
\begin{equation}
\left\langle \nabla(|\nabla u|^2), A \nabla \right\rangle,
\end{equation}
and
\begin{equation}
(L_A u) \left\langle \nabla u, A \nabla \right\rangle.
\end{equation}

We denote the gradient and \(A\)–Laplacian with respect to the geometry of \(\partial M\) by \(\nabla^\partial\) and \(L^\partial_A\) and the gradient and \(A\)–Laplacian with respect to \(M\) by \(\nabla\) and \(L_A\). Now, we compute each item. Computation of \((3.6)\) results that
\[\frac{1}{2} \left\langle \nabla(|\nabla u|^2), A \nabla \right\rangle = \frac{1}{2} A \nabla \cdot \langle \nabla u, \nabla u \rangle = \langle \nabla_A \nabla u, \nabla u \rangle.\]

Now, for computation of \((3.7)\), let \(\{e_i\}\) be local orthonormal frame field such that \(e_n = \nabla\) be the outward unit vector field on \(\partial M\), then
\begin{align*}
L_A u &= \sum_{i<n} \left\langle \nabla_{e_i} \nabla u, Ae_i \right\rangle + \left\langle \nabla_{e_n} \nabla u, A \nabla \right\rangle \\
&= \sum_{i<n} \left\langle \nabla_{e_i} (\nabla^\partial u + (u_n \nabla), Ae_i \right\rangle + \left\langle \nabla_{e_n} \nabla u, A \nabla \right\rangle \\
&= \sum_{i<n} \left\langle \nabla_{e_i} \nabla^\partial u, Ae_i \right\rangle + \sum_{i<n} \left\langle \nabla_{e_i} ((u_n \nabla), Ae_i \right\rangle + \left\langle \nabla_{e_n} \nabla u, A \nabla \right\rangle \\
&= \sum_{i<n} \left\langle \nabla_{e_i} \nabla^\partial u, Ae_i \right\rangle + \sum_{i<n} (\partial^\partial (\nabla^\partial u, e_i), Ae_i) \\
&\quad + \sum_{i<n} (u_n \nabla_{e_i} \nabla u, Ae_i) + \left\langle \nabla_{e_n} \nabla u, A \nabla \right\rangle \\
&= L^\partial_A(u) + \sum_{i<n} \left\langle \left(\text{shap}^\partial(\nabla^\partial u), e_i \right\rangle \nabla, Ae_i \right\rangle + \left\langle \nabla, A \left(\sum_{i<n} (e_i, u_n) e_i \right) \right\rangle \\
&\quad - \sum_{i<n} u_n \left\langle \text{shap}^\partial (e_i), Ae_i \right\rangle + \left\langle \nabla \nabla u, A \nabla \right\rangle \\
&= L^\partial_A(u) + \left\langle \nabla, A \left(\sum_{i<n} \left(\text{shap}^\partial(\nabla^\partial u), e_i \right\rangle \nabla, Ae_i \right) \right\rangle + \left\langle \nabla, A \nabla^\partial u_n \right\rangle \\
&\quad - u_n \sum_{i<n} \left\langle A \circ \text{shap}^\partial (e_i), e_i \right\rangle + \left\langle \nabla \nabla u, A \nabla \right\rangle \\
&= L^\partial_A(u) + \left\langle \nabla, A \left(\text{shap}^\partial(\nabla^\partial u) \right) \right\rangle + \left\langle \nabla \nabla u_n \right\rangle - u_n H^\partial_A + Hess(u)(\nabla, A \nabla),
\end{align*}
where by definition we have \(L^\partial_A u = \sum_{i<n} \left\langle \nabla_{e_i} \nabla^\partial u, Ae_i \right\rangle\) and "shap" is the shape operator of \(\partial M\) with respect to outward unit vector field \(n\) of \(M\). We also define \(H^\partial_A := \text{Trace}(A \circ \text{shap}^\partial)\) as a generalization of the mean curvature of the boundary of \(M\).
We know that (3.5) is obtained by integration from $\langle \nabla A \nabla u, \nabla u \rangle - u_n L_A u$. So we have,

$$
\langle \nabla A \nabla u, \nabla u \rangle - u_n L_A u = \text{Hess}(u)(\nabla u, A \nabla u) - u_n L_A^0(u) - u_n \langle A \nabla u, (\text{shap}^\partial(\nabla^\theta u)) \rangle
$$

But,

$$
\text{Hess}(u)(\nabla u, A \nabla u) - \text{Hess}(u)(u_n \nabla u, A \nabla u) = \text{Hess}(u)(\nabla u, A \nabla u - u_n \nabla u_n) = \langle \nabla^\theta u, A \nabla u \rangle - u_n \langle \text{shap}^\partial(\nabla^\theta u), A \nabla u \rangle.
$$

Hence the extended Reilly formula when $A$ is parallel becomes as follows,

$$
B = C,
$$

where

$$
B = \int_{\partial M} \left( \langle \nabla \nabla \nabla u, A \nabla u \rangle - 2u_n \langle \text{shap}^\partial(\nabla^\theta u), A \nabla u \rangle \right) \text{dvol}_\bar{g}
$$

$$
+ \int_{\partial M} \left( u_n^2 H_A^\partial - u_n \langle A \nabla u, \nabla^\theta u_n \rangle \right) \text{dvol}_\bar{g}
$$

$$
+ \int_{\partial M} \left( \langle \nabla^\theta u, A \nabla u \rangle - u_n L_A^0(u) \right) \text{dvol}_\bar{g},
$$

and

$$
C = \int_M \text{Trace}(A \circ \text{hess}^2(u)) \text{dvol}_\bar{g} - \int_M (L_A u)(\Delta u) \text{dvol}_\bar{g} + \int_M \text{Ric}_A(\nabla u, \nabla u) \text{dvol}_\bar{g}.
$$

Remark 3.2. We define the second fundamental form of a hypersurface $\Sigma \subset M$ by the rule $II(X, Y) = \langle \nabla_X Y \rangle^\perp = - \langle \nabla_X N, Y \rangle$. So we call a hypersurface "convex" if the second fundamental form is negative definite.

Remark 3.3. Note that the result of the Theorem 3.1 depends on the conditions $\Delta \nabla^\gamma A u = 0$ and $\text{div}(A) = 0$, not parallelness of $A$.

4. Estimates of the First Eigenvalue of $L_A$ When $A$ is Parallel

By the Reilly-type formula, we get some estimates for the first positive eigenvalue of operator $L_A$ by some restrictions on $\text{Ric}_A$, when $A$ is parallel and the manifold is compact. For manifolds with boundary, we denote $\lambda_D$, $\lambda_N$ for the first eigenvalue with the Dirichlet and Neumann boundary conditions respectively.

Proof of the Theorem 4.1 Since $A$ is parallel, one has,

$$
\delta_1 \langle X, X \rangle \leq \langle AX, X \rangle \leq \delta_n \langle X, X \rangle,
$$
where $\delta_1, \delta_n$ are defined as Definition 2.2. Let $\lambda$ be the first positive eigenvalue of the operator $L_A(u) = \text{div}(A \nabla u)$ and $u$ be the corresponding eigenfunction, i.e., $L_A(u) + \lambda u = 0$. We can assume $u > 0$ and $\int_M u^2 dvol_g = 1$. So we have,

$$\delta_1 \int_M |\nabla u|^2 dvol_g \leq \int_M \langle A \nabla u, \nabla u \rangle dvol_g = -\int_M uL_A(u) dvol_g = \lambda \int_M u^2 dvol_g = \lambda.$$  

Similarly, one has $\delta_n \int_M |\nabla u|^2 dvol_g \geq \lambda$, so

$$\frac{\lambda}{\delta_n} \leq \int_M |\nabla u|^2 dvol_g \leq \frac{\lambda}{\delta_1}.  \tag{4.1}$$

Since $M$ has empty boundary, by Theorem 3.1 we get,

$$\int_M \text{Trace} \left( A \circ \text{hess}^2(u) \right) dvol_g - \int_M (L_Au)(\Delta u) dvol_g + \int_M \text{Ric}_A(\nabla u, \nabla u) dvol_g = 0.$$  

As $L_Au = -\lambda u$, we obtain,

$$-\int_M (L_Au)(\Delta u) dvol_g = \int_M \lambda u(\Delta u) dvol_g = -\lambda \int_M |\nabla u|^2 dvol_g.$$  

Now, we prove the two estimates of the theorem.

1) We note that $\int_M \text{Trace} \left( A \circ (\text{hess}^2(u)) \right) dvol_g \geq 0$, so

$$-\lambda \int_M |\nabla u|^2 dvol_g + \int_M \text{Ric}_A(\nabla u, \nabla u) dvol_g \leq 0,$$

and by assumption $\text{Ric}_A \geq Kg$, hence $(K - \lambda) \int_M |\nabla u|^2 dvol_g \leq 0$ or equivalently $K \leq \lambda$.

To get a better estimate for $\lambda$, we use,

$$\frac{\lambda}{\delta_n} \leq \int_M |\nabla u|^2 dvol_g \leq \frac{\lambda}{\delta_1}.  \tag{4.2}$$

By Theorem 3.1 we see that (note $\text{Trace}(A)$ is constant),

$$\frac{1}{\text{Trace}(A)} \int_M (L_Au)^2 dvol_g - \int_M (L_Au)(\Delta u) dvol_g + \int_M \text{Ric}_A(\nabla u, \nabla u) dvol_g \leq 0.$$  

So,

$$\frac{1}{\text{Trace}(A)} \lambda^2 + (K - \lambda) \int_M |\nabla u|^2 dvol_g \leq 0.$$  

And by (4.1) we get,

$$\frac{\lambda^2}{\text{Trace}(A)} + (K - \lambda) \frac{\lambda}{\delta_1} \leq 0.$$  

And finally,

$$\lambda \geq \frac{\text{Trace}(A)K}{\text{Trace}(A) - \delta_1}.$$  

2) By Theorem 3.1 and the assumption on $\text{Ric}_A$ we have,

$$\frac{1}{\text{Trace}(A)} \int_M (\lambda u)^2 dvol_g + \int_M (\lambda u)(\Delta u) dvol_g + K \int_M \langle A \nabla u, \nabla u \rangle dvol_g \leq 0.$$
We note that
\[ \int_M (A \nabla u, \nabla u) \, dvol_g = -\int_M u L_A(u) \, dvol_g = \lambda \int_M u^2 \, dvol_g = \lambda. \]

So
\[ \frac{1}{\text{Trace}(A)} \lambda^2 - \lambda \int_M |\nabla u|^2 \, dvol_g + K \lambda \leq 0. \]

And by (4.1) we get,
\[ \lambda \geq \frac{\text{Trace}(A)}{\text{Trace}(A)} \delta_1 K, \]
where \( \delta_1 \) is a constant.

For the rigidity result, let \( u \) be the corresponding eigenfunction, the equality in (1.3) and (1.4) implies equality in (4.2). By proposition 2.2 and Cauchy-Schwartz inequality one has
\[ A = \alpha I, \quad \text{Hess} u = hI, \quad \lambda = \frac{n \alpha K}{(n-1)}. \]
where \( \alpha, h \) be smooth functions. So from \( L_A u = \Delta_A u = -\frac{n \alpha K u}{(n-1)} \) we conclude \( n h \alpha = \Delta_A u = \frac{n \alpha K u}{(n-1)} \), equivalently \( h = \frac{K}{(n-1)} u \) and the result follows by proposition 2.14.

Remark 4.1. If \( A = Id_{TM} \) we get the Lichnerowicz and Obata estimate for the first eigenvalue in both cases.

**Corollary 4.2.** Let \( \Sigma^n \subset M^{n+1} \) be a complete orientable Riemannian hypersurface or be an orientable space-like hypersurface in Lorentzian manifold \( M \), which \( P_k \) is parallel and positive-definite operator and

a) \( \text{Ric}_{P_k} \geq \Lambda > 0 \),

b) \( \text{Ric}_{P_k}(X, X) \geq \Lambda |X|^2 \), where \( \Lambda > 0 \) and \( X \in \Gamma(TM) \) is arbitrary vector field.

Then one has the following estimates of the first eigenvalue \( \lambda_1 \) of the operator \( L_k \),

\[ \lambda \geq \frac{\text{Trace}(A) K}{\text{Trace}(A) - \delta_1} \geq \frac{c_k}{c_k-1} \Lambda. \]

**Corollary 4.3 (Dirichlet and Neumann boundary condition).** Let \( M \) be a compact Riemann manifold with boundary \( \partial M \) and \( A \) be a parallel symmetric and positive semi-definite operator on \( M \) such that the outward unit vector field \( \overrightarrow{n} \) is an eigenvector of \( A \) and for Dirichlet boundary condition \( \partial M \) be a convex hypersurface. Suppose that one of the following conditions holds,

1) \( \text{Ric}_A \geq Kg \) and \( K > 0 \);
2) \( \text{Ric}_A(X, X) \geq K \langle AX, X \rangle \) and \( K > 0 \).

Then we have the following estimate for the first eigenvalue of the operator \( L_A \) for Dirichlet or Neumann boundary condition,

\[ \lambda_D, \lambda_N \geq \frac{\text{Trace}(A) K}{\text{Trace}(A) - \delta_1}; \]
\[ \lambda_D, \lambda_N \geq \frac{\text{Trace}(A) \delta_1 K}{\text{Trace}(A) - \delta_1}. \]
These estimates are not useful when $K \leq 0$. So we use the Li and Yau method \cite{9} to estimate the first eigenvalue when $Ric_A \leq 0$. In principle Li-Yau’s method gets a gradient estimate on the $\lambda_1$-eigenfunction by using the Bochner formula and maximal principle.

**Proof of the Theorem 1.2.** We follow Theorem 5.3 p. 39 in \cite{9} to get the estimate. In principal, we use generalized Bochner formula in Theorem 2.4 and maximum principle similar \cite{9}. Let $u$ be an eigenfunction with $\Delta_A u = L_A u = -\lambda u$, where $\lambda > 0$. We can assume,

$$\int_M u \, dvol_g = 0, \quad 1 > \max_{x \in M} u > \min_{x \in M} u = -1.$$ 

Let us consider the function $v = \ln(a + u)$ for some constant $a > 1$. The function $v$ satisfies,

$$\nabla v = \frac{1}{a + u} \nabla u,$$

and

$$\Delta_A v = \frac{1}{a + u} \Delta_A u - \frac{1}{(a + u)^2} |\nabla u|_A^2 = -\frac{\lambda u}{a + u} - |\nabla v|_A^2,$$

where $|X|_A^2 := \langle X, AX \rangle$. Define $Q(x) := |\nabla v|^2(x)$, by the extended Bochner formula, one has,

$$\frac{1}{2} \Delta_A (Q) = \frac{1}{2} \Delta_A (|\nabla v|^2) = Trace(A \circ hess^2 v) + \nabla v. \Delta_A v + Ric_A (\nabla v, \nabla v)$$

(4.3)

$$\geq \frac{1}{Trace(A)} (\Delta_A v)^2 + \nabla v. \Delta_A v + Ric_A (\nabla v, \nabla v).$$

But,

$$\frac{1}{Trace(A)} (\Delta_A v)^2 = \frac{1}{Trace(A)} \left( -\frac{\lambda u}{a + u} - |\nabla v|_A^2 \right)^2$$

$$\geq \frac{1}{Trace(A)} \left( |\nabla v|_A^4 + \frac{2\lambda u}{a + u} |\nabla v|_A^2 \right),$$

and

$$\nabla v. \Delta_A v = \nabla v. \left( \frac{-\lambda u}{a + u} - |\nabla v|_A^2 \right) = -\lambda \frac{a}{a + u} |\nabla v|^2 - \nabla v. |\nabla v|_A^2$$

(4.4)

$$= -\lambda \frac{a}{a + u} |\nabla v|^2 - 2Hess(A\nabla v, \nabla v) - \langle \nabla v, (\nabla v, A \nabla v) \rangle.$$ 

By parallelness of $A$, we have

(4.5) $$\nabla v. \Delta_A v = -\lambda \frac{a}{a + u} |\nabla v|^2 - (A\nabla v). |\nabla v|^2.$$ 

So (4.3) is written as

(4.6)

$$\frac{1}{2} \Delta_A (Q) + (A\nabla v). |\nabla v|^2 \geq \frac{\delta^2}{Trace(A)} Q^2 + \left[ \lambda \left( \frac{2u}{(a + u) Trace(A)} - \frac{a}{a + u} \right) - K \right] Q.$$
If \( x_0 \) is the maximum point, where \( Q \) achieves its maximum, then \( \frac{1}{2} \Delta_A (Q) + (A\nabla v) \cdot |\nabla v|^2 \leq 0 \) and we have,

\[
Q(x) \leq Q(x_0) \leq \frac{\text{Trace}(A)}{\delta_1^2} \left[ \frac{a}{a-1} \lambda + K \right],
\]

for all \( x \in M \). Integrating \( \sqrt{Q} = |\nabla \ln(u + a)| \) along a minimal geodesic \( \gamma \) joining points \( u = -1 \) to \( u = \max u \geq 0 \), we have,

\[
\ln \left( \frac{a}{a-1} \right) \leq \ln \left( \frac{a + \max u}{a-1} \right) \leq \int \gamma |\nabla \ln(a + u)| \leq \frac{d \sqrt{\text{Trace}(A)}}{\delta_1} \left[ \frac{a}{a-1} \lambda + K \right]^{\frac{1}{2}}.
\]

Setting \( t = \frac{a-1}{a} \), we have

\[
t \left( \frac{\delta_1^2}{d^2 \text{Trace}(A)} (\ln t)^2 - K \right) \leq \lambda.
\]

By maximizing, we have,

\[
2 \left( \alpha + \sqrt{\alpha^2 + K \alpha} \right) \exp \left( -1 - \sqrt{1 + \frac{K}{\alpha}} \right) \leq \lambda,
\]

where \( \alpha = \frac{\delta_1^2}{d^2 \text{Trace}(A)} \).

Remark 4.4. Similar method can be applied for the manifold with boundary and \( \text{Ric}_A \geq K \), where \( K \leq 0 \).

5. Reilly formula, when \( A \) is Codazzi tensor

**Proof of the Theorem 1.3** To get the Reilly formula for general case we should integrate of the extended Bochner formula (2.4). So we should add the phrase

\[
\int_M \frac{1}{2} \left( \langle \nabla |\nabla u|^2, \text{div}(A) \rangle - \Delta_{(\nabla_{\nabla_u} A)} u \right) d\text{vol}_g,
\]

to the Reilly-type formula (3.1). By assumption, for any vector field \( X, Y \) one has \( (\nabla X) Y = (\nabla Y) X \). Let \( \{e_i\} \) be a local orthonormal frame field with \( \nabla e_i e_j (x) = 0 \). So at the point \( x \) one has

\[
\Delta_{(\nabla_{\nabla_u} A)} u = L_{(\nabla_{\nabla_u} A)} u - \langle \nabla u, \text{div}(\nabla_{\nabla_u} A) \rangle
\]

\[
= L_{(\nabla_{\nabla_u} A)} u - \left( \nabla u, \sum_i \nabla e_i \left( (\nabla_{\nabla_u} A) e_i \right) \right)
\]

\[
= L_{(\nabla_{\nabla_u} A)} u - \left( \nabla u, \sum_i \nabla e_i \left( (\nabla e_i A) \nabla u \right) \right)
\]

\[
= L_{(\nabla_{\nabla_u} A)} u - \sum_i \langle \nabla u, (\nabla e_i^2 A) \nabla u \rangle - \sum_i \langle \nabla u, (\nabla e_i A) \nabla e_i \nabla u \rangle
\]

\[
= L_{(\nabla_{\nabla_u} A)} u - \langle \nabla u, (\Delta A) \nabla u \rangle - \sum_i \langle (\nabla e_i A) \nabla u, \nabla e_i \nabla u \rangle
\]

\[
= L_{(\nabla_{\nabla_u} A)} u - \langle \nabla u, (\Delta A) \nabla u \rangle - \sum_i \langle (\nabla_{\nabla_u} A) e_i, \nabla e_i \nabla u \rangle
\]

\[
= L_{(\nabla_{\nabla_u} A)} u - \langle \nabla u, (\Delta A) \nabla u \rangle - \Delta_{(\nabla_{\nabla_u} A)} u
\]
Equivalently
\[ \Delta (\nabla_{\nabla A}u) = \frac{1}{2} \left( L(\nabla_{\nabla A}u) - \langle \nabla u, (\Delta A) \nabla u \rangle \right) , \]
so
\[ - \int_M (\Delta (\nabla_{\nabla A}u)) \, dv_{g} = - \frac{1}{2} \int_M \left( L(\nabla_{\nabla A}u) - \langle \nabla u, (\Delta A) \nabla u \rangle \right) \, dv_{g} \]
\[ = \frac{1}{2} \int_M \langle \nabla u, (\Delta A) \nabla u \rangle \, dv_{g} - \frac{1}{2} \int_M (L(\nabla_{\nabla A}u)) \, dv_{g} . \]
But \( \nabla u = \nabla^0 u + u_n \vec{n} \), then
\[ \int_M (L(\nabla_{\nabla A}u)) \, dv_{g} = \int_{\partial M} \langle (\nabla_{\nabla A}u) \nabla u, \vec{n} \rangle \, dv_{\bar{g}} = \int_{\partial M} \langle \nabla u, (\nabla_{\nabla A}u) \vec{n} \rangle \, dv_{\bar{g}} \]
\[ = \int_{\partial M} \langle \nabla^0 u, (\nabla_{\nabla A}u) \vec{n} \rangle \, dv_{\bar{g}} \]
\[ + 2 \int_{\partial M} u_n \langle \nabla^0 u, (\nabla_{\nabla A}u) \vec{n} \rangle \, dv_{\bar{g}} \]
\[ + \int_{\partial M} u_n^2 \langle \nabla^0 u, (\nabla_{\nabla A}u) \vec{n} \rangle \, dv_{\bar{g}} . \]
Also one knows
\[ \int_M \frac{1}{2} \left( \nabla |\nabla u|^2, \text{div}(A) \right) \, dv_{g} \]
\[ = \frac{1}{2} \int_M \text{div} \left( |\nabla u|^2 \text{div} A \right) \, dv_{g} - \frac{1}{2} \int_M |\nabla u|^2 \text{div} (\text{div} A) \, dv_{g} \]
\[ = \frac{1}{2} \int_{\partial M} |\nabla u|^2 \langle \text{div} A, \vec{n} \rangle \, dv_{\bar{g}} - \frac{1}{2} \int_M |\nabla u|^2 \text{div} (\text{div} A) \, dv_{g} , \]
and the formula follows. \( \square \)

By easy computation, one knows \( \text{div} A = \nabla \text{trace}(A) \), so when \( A \) is divergence free Codazzi tensor, then \( \text{trace}(A) \) is constant. In this case, we have the following results for the estimate of the first eigenvalue.

**Proof of the Theorem 1.4** By assumption one has,
\[ \delta_1 \langle X, X \rangle \leq \langle AX, X \rangle \leq \delta_n \langle X, X \rangle , \]
where \( \delta_1, \delta_n \) are defined as Definition 2.2. Let \( \lambda \) be a positive eigenvalue of the operator \( L_A(u) = \text{div}(A \nabla u) \) and \( u \) be the corresponding eigenfunction, i.e., \( L_A(u) + \lambda u = 0 \). We can assume \( u > 0 \) and \( \int_M u^2 \, dv_{g} = 1 \). So we have,
\[ \delta_1 \int_M |\nabla u|^2 \, dv_{g} \leq \int_M \langle A \nabla u, \nabla u \rangle \, dv_{g} = - \int_M u L_A(u) \, dv_{g} = \lambda \int_M u^2 \, dv_{g} = \lambda . \]
Similarly, one has \( \delta_n \int_M |\nabla u|^2 \, dv_{g} \geq \lambda \), so
\[ (5.1) \quad \frac{\lambda}{\delta_n} \leq \int_M |\nabla u|^2 \, dv_{g} \leq \frac{\lambda}{\delta_1} . \]
Since $M$ has empty boundary, by the extended Reilly formula we get,
\[ 0 = \int_M \left( \text{Trace} \left( A \circ \text{hess}^2(u) \right) \right) d\text{vol}_g - \int_M \Delta_A(u) \left( \Delta u \right) d\text{vol}_g \]
\[ + \int_M \text{Ric}_A(\nabla u, \nabla u) d\text{vol}_g + \frac{1}{2} \int_M \langle \nabla u, (\Delta A) \nabla u \rangle d\text{vol}_g \]
\[ - \frac{1}{2} \int_M |\nabla u|^2 \text{div}(\text{div} A) d\text{vol}_g. \]

By assumption $\text{Trace}(A)$ is constant and $T^A = 0$ so $\text{div} A = 0$. Also by Lemma 2.11 we have
\[ \langle \nabla u, (\Delta A) \nabla u \rangle = -\text{Ric}_A(\nabla u, \nabla u) + \text{Ric}(\nabla u, A \nabla u), \]
Hence the equation (5.2) is written as
\[ 0 = \int_M \left( \text{Trace} \left( A \circ \text{hess}^2(u) \right) \right) d\text{vol}_g - \int_M \Delta_A(u) \left( \Delta u \right) d\text{vol}_g \]
\[ + \int_M \text{Ric}_A(\nabla u, \nabla u) d\text{vol}_g + \frac{1}{2} \int_M \text{Ric}(\nabla u, A \nabla u) d\text{vol}_g. \]
By $\text{div}(A) = 0$ we have $L_A u = \Delta_A u = -\lambda u$. So by the restriction on $\text{Ric}_A(X, X) + \text{Ric}(X, AX)$ we have
\[ \frac{1}{\text{Trace}(A)} \lambda^2 + (K - \lambda) \int_M |\nabla u|^2 d\text{vol}_g \leq 0. \]
As a similar discussion in Theorem 1.1 and (5.1) we get,
\[ \frac{\lambda^2}{\text{Trace}(A)} + (K - \lambda) \frac{\lambda}{\delta_i} \leq 0. \]
And finally,
\[ \lambda \geq \frac{\text{Trace}(A) K}{\text{Trace}(A) - \delta_i}. \]
The equality condition is as Theorem 1.1. \hfill \square

**Proof of the Theorem 1.5** By proposition 2.13 and lemma 2.10 we have
\[ \Delta(\nabla \nabla_A u) = \text{Ric}_A(\nabla u, \nabla u) - \text{Ric}(\nabla u, A \nabla u) + \sum_i \left\langle T(\nabla \nabla_A e_i, e_i) \right\rangle + \sum_i \left\langle \nabla u, T^A(e_i, \nabla u) \right\rangle. \]
The tensor $A$ is Codazzi, so $\sum_i e_i \cdot \langle \nabla u, T^A(\nabla u, e_i) \rangle = 0$ and $\text{div}(A) = 0$, Thus the extended Bochner formula is become to
\[ \frac{1}{2} L_A(|\nabla u|^2) = \text{Trace} \left( A \circ \text{hess}^2(u) \right) + \langle \nabla u, \nabla(\Delta_A u) \rangle + \sum_i \left\langle T(\nabla \nabla_A u)(\nabla u, e_i), e_i \right\rangle \]
\[ + \text{Ric}(\nabla u, A \nabla u). \]
By the restrictions on $\text{Ric}(\nabla u, A \nabla u)$ and second covariant derivative of $A$, we have
\[ \frac{1}{2} L_A(|\nabla u|^2) \geq \text{Trace} \left( A \circ \text{hess}^2(u) \right) + \langle \nabla u, \nabla(\Delta_A u) \rangle - (K + 2K') |\nabla u|^2. \]
As similar computation in the proof of theorem 1.2, let \( v := \ln(a + u) \) and \( Q := |\nabla v|^2 \), then we have,

\[
\nabla v \Delta_A v = \nabla v \left( \frac{-\lambda u}{a + u} - |\nabla v|^2_A \right) = -\lambda \frac{a}{a + u} |\nabla v|^2 - \nabla v \cdot |\nabla v|^2_A.
\]

Thus,

\[
(5.3) \quad \frac{1}{2} \Delta_A Q \geq \frac{\delta_1^2 Q^2}{\text{Trace}(A)} + \left( \lambda \left( \frac{2\delta_1 u}{\text{Trace}(A)(a + u)} - \frac{a}{a + u} \right) - (K + 2K') \right) Q - \nabla v \cdot |\nabla v|^2_A.
\]

Now we need to estimate \( \nabla v \cdot |\nabla v|^2_A \). Since

\[
\nabla v \cdot |\nabla v|^2_A = 2 \text{Hess}(A \nabla v, \nabla v) + \langle \nabla v, \nabla (\nabla v) \rangle \leq A \nabla v . Q + \delta Q
\]

we can get

\[
\frac{1}{2} \Delta_A (Q) + A \nabla v . Q \geq \frac{\delta_1^2 Q^2}{\text{Trace}(A)} + \left( \lambda \left( \frac{2\delta_1 u}{\text{Trace}(A)(a + u)} - \frac{a}{a + u} \right) - (K + 2K') \right) Q.
\]

If \( x_0 \) is the maximum point, where \( Q \) achieves its maximum, then \( \Delta_A Q(x_0) \leq 0 \) and \( \nabla Q(x_0) = 0 \), so

\[
Q \leq Q(x_0) \leq \frac{\text{Trace}(A)}{\delta_1^2} \left( \frac{a\lambda}{a - 1} + K + 2K' + \delta \right).
\]

for all \( x \in M \). Integrating \( \sqrt{Q} = |\nabla \ln(u + a)| \) along a minimal geodesic \( \gamma \) joining points \( u = -1 \) to \( u = \max u \geq 0 \), we have,

\[
\ln \left( \frac{a}{a - 1} \right) \leq \ln \left( \frac{a + \max u}{a - 1} \right) \leq \int_{\gamma} |\nabla \ln(u + a)| \leq \frac{d\sqrt{\text{Trace}(A)}}{\delta_1} \left[ \frac{a\lambda}{a - 1} + K + 2K' + \delta \right].
\]

Setting \( t = \frac{a - 1}{a} \), we have

\[
\left( \ln \left( \frac{1}{t} \right) \right)^2 \leq \frac{d^2 \text{Trace}(A)}{\delta_1^2} \left[ \frac{\lambda}{t} + K + 2K' + \delta \right],
\]

in other words,

\[
t \left( \frac{\delta_1^2}{d^2 \text{Trace}(A)} (\ln(t))^2 - (K + 2K' + \delta) \right) \leq \lambda.
\]

Consequently, by maximizing, we have,

\[
2 \left( \alpha + \sqrt{\alpha^2 + (K + 2K' + \delta)\alpha} \right) \exp \left( -1 - \sqrt{1 + \frac{K + 2K' + \delta}{\alpha}} \right) \leq \lambda,
\]

where \( \alpha = \frac{\delta_1^2}{d^2 \text{Trace}(A)} \). □
6. ESTIMATE OF THE FIRST EIGENVALUE IN GENERAL CASE

Let $B$ be a $(1,1)$-self-adjoint tensor field on a closed manifold $M$. We get an estimate for the first eigenvalue of the operator $L_B$.

**Proof of Theorem 6.6** From the extended Bochner formula, we know that

\[
\frac{1}{2} L_B(\|\nabla u\|^2) = \frac{1}{2} \left( \nabla |\nabla u|^2, \text{div}(B) \right) + \text{Trace} \left( B \circ \text{hess}^2 (u) \right) + \langle \nabla u, \nabla (\Delta_B u) \rangle - \Delta (\nabla_B u) + \text{Ric}_B(\nabla u, \nabla u).
\]

Also,

\[
\langle \langle \Delta_B \nabla u, \nabla u \rangle \rangle = \langle \nabla_B \text{div}(B), \nabla u \rangle - \text{Ric}_B(\nabla u, \nabla u) + \text{Ric}(\nabla u, B \nabla u),
\]

and

\[
\Delta (\nabla_B u) = \nabla u. \nabla u. \text{Trace}(B) - \langle \nabla u, (\Delta_B) \nabla u \rangle + \sum_i \left\langle T(\nabla_B \nabla_B (u, e_i), e_i \rangle + \sum_i e_i. \langle \nabla u, T_B(e_i, \nabla u) \rangle.
\]

So from the conditions of the theorem, we have,

\[
\frac{1}{2} \Delta_B(\|\nabla u\|^2) = \text{Trace} \left( B \circ \text{hess}^2 (u) \right) + \langle \nabla u, \nabla (\Delta_B u) \rangle - \sum_i \left\langle T(\nabla_B \nabla_B (u, e_i), e_i \rangle - \sum_i e_i. \langle \nabla u, T_B(e_i, \nabla u) \rangle + \text{Ric}(\nabla u, A \nabla u).
\]

Items (b) and (c) imply,

\[
\frac{1}{2} \Delta_B(\|\nabla u\|^2) \geq \frac{(\Delta_B u)^2}{n \delta_n} + \langle \nabla u, \nabla (\Delta_B u) \rangle - \sum_i e_i. \langle \nabla u, T_B(e_i, \nabla u) \rangle + (K + 2nK')\|\nabla u\|^2.
\]

We know $L_B u = \Delta_B u = -\lambda u$ and $\partial M = \emptyset$, so by integration we have,

\[
\int_M \frac{1}{2} \Delta_A(\|\nabla u\|^2) d\text{vol}_g \geq \frac{1}{n \delta_n} \int_M (\Delta_A u)^2 d\text{vol}_g + \int_M \langle \nabla u, \nabla (\Delta_A u) \rangle d\text{vol}_g - \int_M \sum_i e_i. \langle \nabla u, T_B(e_i, \nabla u) \rangle d\text{vol}_g + \int_M (K + 2nK')\|\nabla u\|^2 d\text{vol}_g.
\]

But $\sum_i e_i. \langle \nabla u, T_B(e_i, \nabla u) \rangle$ is of divergence form, so $\int_M \sum_i e_i. \langle \nabla u, T_B(e_i, \nabla u) \rangle d\text{vol}_g = 0$, also $\frac{1}{n \delta_n} \leq \int_M \|\nabla u\|^2 d\text{vol}_g \leq \frac{\lambda^2}{\delta_1}$, so we have the following inequality,

\[
0 \geq \frac{\lambda^2}{n \delta_n} - \frac{\lambda^2}{\delta_1} + (K + 2nK') \frac{\lambda}{\delta_n}.
\]

Consequently,

\[
\lambda \geq \frac{n \delta_n (K + 2nK')}{n \delta_n - \delta_1}.
\]

□
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