AFFINE SYMMETRIES OF ORBIT POLYTOPES

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Abstract. An orbit polytope is the convex hull of an orbit under a finite group \( G \leq \text{GL}(d, \mathbb{R}) \). We develop a general theory of possible affine symmetry groups of orbit polytopes. For every group, we define an open and dense set of generic points such that the orbit polytopes of generic points have conjugated affine symmetry groups. We prove that the symmetry group of a generic orbit polytope is again \( G \) if \( G \) is itself the affine symmetry group of some orbit polytope, or if \( G \) is absolutely irreducible. On the other hand, we describe some general cases where the affine symmetry group grows.

We apply our theory to representation polytopes (the convex hull of a finite matrix group) and show that their affine symmetries can be computed effectively from a certain character. We use this to construct counterexamples to a conjecture of Baumeister et. al. on permutation polytopes [Advances in Math. 222 (2009), 431–452, Conjecture 5.4].

1. Introduction

Let \( G \leq \text{GL}(d, \mathbb{R}) \) be a finite group. An orbit polytope of \( G \) is defined as the convex hull of the orbit \( Gv \) of some point \( v \in \mathbb{R}^d \). We denote it by

\[
P(G, v) = \text{conv}\{gv \mid g \in G\}.
\]

Orbit polytopes have been studied by a number of authors [1, 32, 2, 30, 9], especially orbit polytopes of finite reflection groups, which are often called generalized permutahedra, or simply permutahedra [6, 36, 23, 14, 15]. In the language of Sanyal, Sottile and Sturmfels [33], orbit polytopes are polytopal orbitopes. (An orbitope is the convex hull of an orbit of a compact groups, not necessarily finite.)

In this paper we study the affine symmetry groups of orbit polytopes. An affine symmetry of a polytope \( P \subset \mathbb{R}^d \) is a bijection of \( P \) which is the restriction of an affine map \( \mathbb{R}^d \to \mathbb{R}^d \). We write \( \text{AGL}(P) \) for the affine symmetry group of a polytope \( P \).

Clearly, the affine symmetry group of an orbit polytope \( P(G, v) \) always contains the symmetries induced by \( G \). Depending on the group and on the point \( v \), there may be additional symmetries or not. In particular, certain symmetry groups imply additional symmetries for all orbit polytopes. In this paper we develop a general theory to explain this phenomenon. We begin by looking at some very simple examples.

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1.1. Illustrating examples. Let \( G = \langle t, s \rangle \cong D_4 \), the dihedral group\(^1\) of order 8. Here \( t \) denotes a counterclockwise rotation by a right angle, and \( s \) a reflection (in the plane). Figure 1 shows two “generic” orbit polytopes. Their affine symmetry group is only the group \( G \) itself. In contrast, the orbit polytopes in Figure 2 are atypical: The first one has a larger affine symmetry group, namely the dihedral group \( D_8 \) of order 16. The other one has affine symmetry group \( D_4 \), but it has fewer vertices than the typical orbit polytope. Of course, this happens because the stabilizer of \( v \) is nontrivial. Finally, if we take for \( v \) the fixed point of the rotation, then we get a degenerate orbit polytope of dimension zero.

In general, given a finite group \( G \leq \text{GL}(d, \mathbb{R}) \), there may be three kinds of “exceptional” points: First, there may be points such that the orbit polytope \( P(G, v) \) is not full-dimensional. Let us call a point \( v \in \mathbb{R}^d \) a generating point (for \( G \)) if \( \mathbb{R}^d = \langle gv \mid g \in G \rangle \). If there exists a generating point, then the set of non-generating points is the zero set of some non-zero polynomials, as is not difficult to see (Lemma 4.2 below). In the example with \( G = D_4 \), only the origin does not generate a full-dimensional orbit polytope.

Second, there may be points \( v \) which are stabilized by some non-identity elements of \( G \). The set of such points is a finite union of proper affine subspaces, since the fixed space of each \( g \in G \setminus \{1\} \) is a proper subspace.

\(^1\) In this paper we follow the convention of geometers and write \( D_n \) for the group of the \( n \)-gon with \( 2n \) elements. Most group theorists write \( D_{2n} \) instead.
Finally, there may be points such that the corresponding orbit polytope has more symmetries than a “generic” orbit polytope. The first aim of this paper is to make this statement more precise (see Theorem 4.3). In particular, it is not obvious in general that “almost all” orbit polytopes have the same symmetry group, and that the other ones usually have more symmetries. For example, it is known that in general orbit polytopes of the same group may have quite different face lattices, even for “generic” points [30].

In our example, the symmetry group of “almost every” orbit polytope is again $G$. This is not always the case. For a simple example, let $G = \langle t \rangle$, where $t$ is a rotation by a right angle in 2-dimensional space. Then every orbit polytope is a square, and the affine symmetry group is always isomorphic to the dihedral group $D_4$ of order 8 (Figure 4). (Again, there is the trivial exception of the orbit of the fixed point of $t$.) From the first example, we know that if we take an orbit polytope of this new symmetry group, then its affine symmetry group does no longer grow for “almost all” points $v$. This will be seen to be a general phenomenon (Corollary 5.4).

We also see that the different orbit polytopes of $G = \langle t \rangle$, as $v$ varies, have not exactly the same symmetries (we have reflections at different axes), but the resulting groups are conjugate in the group of all affine isomorphisms. Actually, more is true: If
we identify the vertices of an orbit polytope with the corresponding group elements, then the affine symmetry groups of all orbit polytopes induce the same permutations on $G$. Again, this is a general phenomenon (Theorems 4.3 and 5.3).

1.2. Affine symmetries of orbit polytopes: results. For a given finite group $G \leq \text{GL}(d, \mathbb{R})$ such that at least one orbit polytope of $G$ is full-dimensional, we define a set of generic points (Section 4). If $v$ is generic, we call $P(G, v)$ a generic orbit polytope. We prove the following:

**Theorem A.** The set of generic points is the complement of the zero set of certain polynomials. The affine symmetry groups of all the generic orbit polytopes are conjugate in $\text{GL}(d, \mathbb{R})$. Moreover, the affine symmetry group of any full-dimensional orbit polytope $P(G, v)$ contains a conjugate of the affine symmetry group of a generic orbit polytope.

In the examples above, the affine symmetry group of a generic orbit polytope has order 8 in both cases. In the case of $G = \langle t \rangle \cong C_4$, every point except the fixed point of $t$ is generic. In the case of $D_4$, the non-generic points are the union of eight lines through the origin (Figure 3).

We should also mention that the exceptional points are not necessarily a finite union of proper subspaces, as is the case in our simple examples.

In the general case, it follows that every finite group $G \leq \text{GL}(d, \mathbb{R})$ defines a unique conjugacy class of subgroups of $\text{GL}(d, \mathbb{R})$ containing the groups $\hat{G} = \text{AGL}(P(G, v))$ for $v$ generic. Clearly, $P(G, v) = P(\hat{G}, v)$, but if $|G| < |\hat{G}|$, then $v$ has nontrivial stabilizer in $\hat{G}$ and thus $v$ is not generic for $\hat{G}$. However, we have the following:

**Theorem B.** Let $\hat{G} = \text{AGL}(P(G, v))$ be the affine symmetry group of the full-dimensional orbit polytope $P(G, v)$. If $w$ is generic for $\hat{G}$, then $\text{AGL}(P(\hat{G}, w)) = \hat{G}$.

Thus we have some sort of closure operator on the conjugacy classes of finite subgroups of $\text{GL}(d, \mathbb{R})$ generating full-dimensional orbit polytopes. We call a group $G$ generically closed if $\text{AGL}(P(G, v)) = G$ for all generic $v$. Thus the symmetry group of a full-dimensional orbit polytope is generically closed.

If a group is not generically closed, every full-dimensional orbit polytope has additional affine symmetries, as in the example $G \cong C_4$ above. Naturally, this leads to the problem of characterizing generically closed groups.

More generally, we may begin with an abstract finite group $G$, and consider various representations $D: G \rightarrow \text{GL}(d, \mathbb{R})$. We will see (Theorem 7.3) that there are only finitely many similarity classes of representations such that the space contains full-dimensional orbit polytopes of $D(G)$. We may ask: for which of these (faithful) representations of the given group is the image $D(G)$ generically closed?

**Theorem C.** If $D: G \rightarrow \text{GL}(d, \mathbb{R})$ is absolutely irreducible, then a generic orbit polytope has only affine symmetry group $D(G)$.

For every group of order $\geq 3$, there are representations such that $D(G)$ is not generically closed (for example, the regular representation yields a simplex with $|G|$ vertices as orbit polytope), but there may be no representations such that $D(G)$ is
generically closed. For example, abelian groups containing elements of order greater than 2 are never generically closed (see Proposition 10.3).

Thus we may ask for which groups there is a faithful representation at all such that \( D(G) \) is generically closed. This is equivalent to a question of Babai [1], namely, which groups are isomorphic to the affine symmetry group of an orbit polytope. (Babai [1] classified groups that are isomorphic to the orthogonal symmetry group of an orbit polytope.) We have only partial answers to these questions so far, but we give some constructions of groups (or representations of groups) which are not generically closed, and we conjecture that these are in fact all such groups (cf. Conjecture 10.2).

On the positive side, it follows from the more precise Theorem E below that every elementary abelian 2-group of order \( \neq 4, 8, 16 \) is isomorphic to the affine symmetry group of one of its orbit polytopes. On the other hand, the elementary abelian groups of orders 4, 8 and 16 are not affine symmetry groups of orbit polytopes. These groups are the only groups we know of, which are isomorphic to the orthogonal symmetry group of an orbit polytope, but not to the affine symmetry group of an orbit polytope.

Studying the different possible orbit polytopes of a fixed group \( G \) is related to McMullen’s theory of realizations of abstract regular polytopes [25, 28, 26, 27]. For a given finite group \( G \) and a subgroup \( H \leq G \), McMullen studies congruence classes of orbit polytopes of \( G \) such that \( H \) fixes a vertex. The group \( G \) is usually assumed to be the automorphism group of an abstract regular polytope [29] and \( H \) a stabilizer of a vertex, and then the orbit polytopes are called realizations of the abstract regular polytope. However, most of the arguments are actually valid for an arbitrary group \( G \) and subgroup \( H \). The congruence classes of such orbit polytopes form a pointed convex cone, the realization cone. Since we consider orbit polytopes up to a certain affine equivalence (see Definition 6.1), we further identify orbit polytopes in this cone. For example, the interior of the realization cone consists of non-congruent simplices, but these are all affinely equivalent.

1.3. Representation polytopes: results. An interesting class of orbit polytopes which have additional affine symmetries are the representation polytopes. A representation polytope is defined as the convex hull of \( D(G) \), where \( D: G \to \text{GL}(d, \mathbb{R}) \) is a representation of an abstract finite group \( G \). If the image group consists of permutation matrices, the polytope is called a permutation polytope. A well known example is the celebrated Birkhoff polytope of doubly stochastic matrices (also known as assignment polytope), which is the convex hull of all permutation matrices of a fixed dimension. Permutation polytopes and some other special classes of representation polytopes have also been studied by a number of people [12, 4, 22, 13].

Here we study representation polytopes as special cases of orbit polytopes. Representation polytopes usually have a big group of affine symmetries (with the notable exception of elementary abelian 2-groups, see below). The permutations of the vertices induced by the affine symmetry group of a representation polytope can be computed from a certain character. To define this character, we use the following notation: For a representation \( D: G \to \text{GL}(d, \mathbb{R}) \), we write \( \text{Irr} \, D \) for the set of irreducible (complex) characters of \( G \) which occur in the character of \( D \). Then we have:
**Theorem D.** Let $D: G \to \text{GL}(d, \mathbb{R})$ be a representation and set

$$
\gamma = \sum_{\chi \in \text{Irr} D} \chi(1). 
$$

Let $\pi$ be a permutation of $G$. Then there is an affine symmetry of the corresponding representation polytope $P(D) = P(G, I)$ sending $D(g)$ to $D(\pi(g))$ if and only if

$$
\gamma(\pi(g)^{-1}\pi(h)) = \gamma(g^{-1}h) \quad \text{for all} \quad g, h \in G.
$$

Computing the affine symmetry group of a representation polytope can be viewed as a linear preserver problem. This is the problem of determining the set of linear transformations of $\mathbb{M}_n(\mathbb{R})$ mapping a subset $G \subseteq \mathbb{M}_n(\mathbb{R})$ to itself. This problem has already been studied, for example when $G$ is a finite irreducible reflection group [19, 20, 21].

We use Theorem D to construct counterexamples to a conjecture of Baumeister et. al. [4, Conjecture 5.4]. Namely, we have:

**Theorem E.** For every elementary abelian 2-group $G$ of order $\neq 4, 8, 16$ there is a (permutation) representation $D: G \to \text{GL}(d, \mathbb{R})$ such that the corresponding representation polytope has affine symmetry group $D(G)$.

These representation polytopes are constructed as cut polytopes of certain graphs. (It is an easy consequence of the general theory in Sections 7–8 that every orbit polytope of an elementary abelian 2-group is affinely equivalent to a permutation polytope.)

Finally, we have another amusing characterization of representation polytopes among orbit polytopes:

**Theorem F.** The orbit polytope $P(G, v)$ is affinely equivalent to a representation polytope of the same group $G$ if and only if $P(G, v)$ has an affine symmetry sending every vertex $gv$ to $g^{-1}v$.

1.4. **Outline.** The paper is organized as follows: Section 2 contains preliminary remarks. In Section 3, we review and slightly generalize a criterion of Bremner, Dutour Sikirić and Schürmann [7] which allows to effectively compute the affine symmetries of a polytope. (A corollary is that an affine group can not act transitively on the 2-subsets of the vertices of a polytope, unless the polytope is a simplex, a fact which we could not find in the literature.) We define generic points in Section 4 and then prove Theorems A, B and C in Sections 4–5. In Section 6, we begin the study of representation polytopes. Section 7 contains more technical material. This material is, however, indispensable for a deeper understanding of the different possible orbit polytopes belonging to a fixed abstract finite group, and is also needed in Section 8. This section contains different characterizations of representation polytopes (including Theorem F) and the proof of Theorem D. In Section 9 we consider orbit polytopes of elementary abelian 2-groups and construct the representation polytopes of Theorem E. Finally, in the last section we discuss some open questions and conjectures.
2. Generalities

As in the introduction, $G \leq \text{GL}(d, \mathbb{R})$ is a finite group and

$$P(G, v) = \text{conv}\{gv \mid g \in G\}$$

the orbit polytope of some $v \in \mathbb{R}^d$. We also use the notation $P(G, v)$, if $G$ is some abstract finite group together with a representation $D: G \to \text{GL}(d, \mathbb{R})$, and $v \in \mathbb{R}^d$.

Notice that every $gv$ is a vertex of $P(G, v)$: A priori, the vertices are a subset of $Gv$. Every element of $G$ induces a symmetry of $P(G, v)$ onto itself and thus maps vertices to vertices. Thus every element of $Gv$ is a vertex.

We need a straightforward generalization of an observation by Guralnik and Perkinson [12]. We use the notation $\text{Fix } G = \{v \in \mathbb{R}^d \mid gv = v \text{ for all } g \in G\}$ for the fixed space of $G$ in $\mathbb{R}^d$, and we write $\text{aff } X$ for the affine hull of a set of points $X \subseteq \mathbb{R}^d$. Recall that $\text{aff } X = \{\sum_{x \in X} \lambda_x x \mid \sum_{x \in X} \lambda_x = 1\}$.

2.1. Lemma. We have

$$\left\{ \frac{1}{|G|} \sum_{g \in G} gv \right\} = P(G, v) \cap \text{Fix } G = \text{aff}(Gv) \cap \text{Fix } G.$$

Thus the following are equivalent:

(i) $\sum_{g \in G} gv = 0$,
(ii) $0 \in P(G, v)$,
(iii) $0 \in \text{aff}(Gv) = \text{aff}\{gv \mid g \in G\}$.
(iv) $\langle gv \mid g \in G \rangle \cap \text{Fix } G = \{0\}$.

Proof. Obviously, $(1/|G|) \sum_{g} gv \in P(G, v) \cap \text{Fix } G$. Let $E_1$ be the matrix

$$E_1 = \frac{1}{|G|} \sum_{g \in G} g.$$

It is easy to see (and well known) that $w \in \text{Fix } G$ if and only if $E_1 w = w$. Let $w \in \text{aff}(Gv) \cap \text{Fix } G$ and write

$$w = \sum_{g \in G} \lambda_g gv, \quad \sum_{g \in G} \lambda_g = 1.$$

It follows

$$w = E_1 w = \sum_{g \in G} E_1 \lambda_g gv = \sum_{g \in G} \lambda_g E_1 v = E_1 v.$$

Thus $\text{aff}(Gv) \cap \text{Fix } G = \{E_1 v\}$. The same argument shows that $\langle gv \mid g \in G \rangle \cap \text{Fix } G = \langle E_1 v \rangle$.

The equivalence of the assertions follows. \qed

Note that $E_1 v$ is the barycenter of the orbit polytope $P(G, v)$, and that the translated polytope $P(G, v) - E_1 v$ is the orbit polytope of $v - E_1 v$. It is thus no loss of generality to assume that $E_1 v = 0$. The barycenter of an orbit polytope is its only point which is fixed by every element of $G$. 
Note that we could have started with a finite subgroup $G$ of the affine group $\text{AGL}(d, \mathbb{R})$. Since every finite group of affine transformations fixes a point (namely, the barycenter of an orbit), we can choose a coordinate system such that the elements of $G$ are represented by matrices. It is thus no real loss of generality to assume $G \leq \text{GL}(d, \mathbb{R})$ from the beginning.

If we want to compute the affine symmetry group of an orbit polytope $P(G, v)$, we can restrict our attention to the affine space generated by the orbit $Gv$. We can thus assume that $P(G, v)$ is full-dimensional. This already implies (by Lemma 2.1) that $P(G, v)$ is centered at the origin. The affine symmetries of $P(G, v)$ are thus realized by linear maps.

We use the following general notation: For any set $S \subset \mathbb{R}^d$, we write $\text{AGL}(S)$ for the set of affine maps $\text{aff}(S) \to \text{aff}(S)$ that permute $S$, and $\text{GL}(S)$ for the set of linear maps $\langle S \rangle \to \langle S \rangle$ that permute $S$. Thus for a polytope $P$ with vertex set $S$ we have $\text{AGL}(P) = \text{AGL}(S)$. If $P$ is centered at the origin, then $\text{AGL}(P) = \text{GL}(P) = \text{AGL}(S) = \text{GL}(S)$.

### 3. Linear Isomorphisms

We can compute the affine symmetries of a polytope using a result by Bremner, Dutour Sikirić and Schürmann [7]. Since we will need a slight generalization, we give a complete proof in this section. We will then apply this criterion to orbit polytopes.

Actually, the result we are going to reprove is a criterion about isomorphisms of vector families. Let $K$ be a field and let $(v_i | i \in I)$ and $(\tilde{v}_i | i \in I)$ be two families of vectors in $K^d$ indexed by the same finite set $I$. (In our applications, we will usually have $K = \mathbb{R}$, but we will also need the case where $K = \mathbb{R}(X_1, \ldots, X_n)$ is a function field.) Following Bremner, Dutour Sikirić and Schürmann [7], we form the $d \times d$-matrix

$$Q = \sum_{i \in I} v_i v_i^t = VV^t, \quad V = (v_i | i \in I).$$

Here $V$ is a matrix with columns indexed by $I$. Note that $Q$ is invertible if $K = \mathbb{R}$ and $K^d = \langle v_i | i \in I \rangle$, since then $Q$ is positive definite. (Over $K = \mathbb{C}$, we would have to use the conjugate transpose instead of the transpose, but we will not need this case.) Similarly, we write $\tilde{V} = (\tilde{v}_i | i \in I)$ and $\tilde{Q} = \tilde{V}\tilde{V}^t$. The next result generalizes [7, Proposition 3.1]:

**3.1. Proposition.** Let $Q$ and $\tilde{Q}$ be invertible. There is a $d \times d$-matrix $A$ such that $Av_i = \tilde{v}_i$ for all $i \in I$ if and only if $V^tQ^{-1}V = \tilde{V}^t\tilde{Q}^{-1}\tilde{V}$. In this case, we have $A = \tilde{V}V^tQ^{-1}$.

**Proof.** Since $Q$ and $\tilde{Q}$ have full rank, we must have $K^d = \langle v_i | i \in I \rangle = \langle \tilde{v}_i | i \in I \rangle$. In particular, there is at most one $A$ with $Av_i = \tilde{v}_i$.
Assume that $A$ exists. Note that $A$ is necessarily invertible since it maps a generating system to a generating system. By assumption, $AV = \tilde{V}$. It follows

$$
\tilde{V}^t \tilde{Q}^{-1} \tilde{V} = \tilde{V}^t (\tilde{V} \tilde{V}^t)^{-1} \tilde{V} = \tilde{V}^t (AVV^t A^t)^{-1} \tilde{V} = \tilde{V}^t (A^t)^{-1} (VV^t)^{-1} A^{-1} \tilde{V} = V^t Q^{-1} V.
$$

Conversely, if $V^t Q^{-1} V = \tilde{V}^t \tilde{Q}^{-1} \tilde{V}$, then

$$
\tilde{V} = \tilde{Q} \tilde{Q}^{-1} \tilde{V} = \tilde{V} V^t \tilde{Q}^{-1} \tilde{V} = \tilde{V} V^t Q^{-1} V,
$$

so we can take $A = \tilde{V} V^t Q^{-1}$.

Let $V = (v_i \ | \ i \in I)$ be a vector family in $K^d$ and $\sigma \in \text{Sym}(I)$ be a permutation of $I$. We say that $\sigma$ is a linear symmetry of $V$ if there is $A \in \text{GL}(d, K)$ with $Av_i = v_{\sigma(i)}$. We write

$$
\text{LinSym}(V) = \{ \sigma \in \text{Sym}(I) \mid \exists A \in \text{GL}(d, K) : Av_i = v_{\sigma(i)} \}
$$

and call this the linear symmetry group of $(v_i \ | \ i \in I)$. Proposition 3.1 gives, in particular, a criterion for when $\sigma \in \text{LinSym}(V)$.

3.2. Corollary. Let $\sigma \in \text{Sym}(I)$ and $V = (v_i \ | \ i \in I) \in K^{d \times I}$ be such that $Q = VV^t$ is invertible, and set $W = V^t Q^{-1} V$. Then $\sigma \in \text{LinSym}(V)$ if and only if

$$
P(\sigma)^{-1} WP(\sigma) = W
$$

where $P(\sigma) \in K^{I \times I}$ is the permutation matrix belonging to $\sigma$. In this case, for $A(\sigma) = VP(\sigma)V^t Q^{-1}$ we have $A(\sigma)v_i = v_{\sigma(i)}$ for all $i \in I$.

**Proof.** Write $\tilde{V} = VP(\sigma)$, so that $\tilde{V}$ has column $v_{\sigma(i)}$ at place $i$. Then $\tilde{V} \tilde{V}^t = VV^t$ since $P(\sigma)^t = P(\sigma)^{-1}$. The result follows from Proposition 3.1.

If $Q$ is invertible, write $W = V^t Q^{-1} V = (w_{ij})$, so $w_{ij} = v_i^t Q^{-1} v_j$. Let $G(V)$ be the complete graph with vertex set $I$, vertex colors $w_{ii}$ and edge colors $w_{ij}$. The last corollary tells us that the linear symmetries of $(v_i \ | \ i \in I)$ yield isomorphisms of the edge colored graph $G(V)$ and vice versa. This means that in practice one can compute the linear symmetries by computing graph automorphisms, using software like **nauty** [24].

The map $\sigma \mapsto A(\sigma)$ is a group homomorphism from $\text{LinSym}(V)$ onto $\text{GL}(\{v_i \ | \ i \in I\})$. (Recall that we write $\text{GL}(S)$ for the set of matrices $A \in \text{GL}(d, K)$ mapping a set $S \subseteq K^d$ onto itself. Under the assumptions of Corollary 3.2, $S = \{v_i \ | \ i \in V\}$ is finite and generates $\mathbb{R}^d$, so $\text{GL}(S)$ is finite and isomorphic to a permutation group on $S$.) Notice that we do not exclude the possibility that $i \mapsto v_i$ is not injective. In that case, $\text{LinSym}(V) \to \text{GL}(\{v_i \ | \ i \in I\})$ has a nontrivial kernel, namely the permutations preserving the fibers of $i \mapsto v_i$. If $i \mapsto v_i$ is injective, then $\text{LinSym}(V) \cong \text{GL}(\{v_i \ | \ i \in I\})$.

Corollary 3.2 has the following amusing consequence. (One can also prove this using the representation theory of finite groups, in particular, the decomposition of a permutation representation into irreducible representations over $\mathbb{R}$.)
3.3. Corollary. If the affine symmetry group of a polytope $P$ acts transitively on the $2$-subsets of its vertices, then $P$ is affinely equivalent to a simplex.

Proof. Without loss of generality, we may embed $P$ in $\mathbb{R}^d$ such that $P$ is full-dimensional and centered at the origin. We can thus assume that the affine symmetries of $P$ are linear. Let $v_1, \ldots, v_n$ be the vertices of $P$ and let $W = V'Q^{-1}V = (w_{ij})$ be the corresponding vertex and edge color matrix. Let $i \neq j \in \{1, \ldots, n\}$. Then there is a linear symmetry of $P$ mapping the vertices $\{v_1, v_2\}$ to $\{v_i, v_j\}$. It follows from Corollary 3.2 that $w_{ij} = w_{12}$ or $w_{ij} = w_{21}$. Since $W$ is symmetric anyway, this means that $w_{ij} = w_{12}$ for all $i \neq j$. So all entries off the diagonal of $W$ are equal.

A permutation group which acts transitively on the $2$-subsets of a set with $n \neq 2$ elements is also transitive on the set itself. It follows $w_{11} = w_{22} = \cdots = w_{nn}$ for $n \neq 2$. (For $n = 2$, the corollary is trivially true anyway.)

Again by Corollary 3.2 it follows that every permutation of the vertices is induced by a linear map. It follows easily that $P$ is a simplex: Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a linear dependence of the vertices, i. e., $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. Every permutation of the coordinates of $\lambda$ yields also a linear dependence. By applying the transposition $(i, j)$ and subtracting dependencies, we see that $(\lambda_i - \lambda_j)v_i + (\lambda_j - \lambda_i)v_j = 0$. Since $v_i \neq v_j$, it follows that $\lambda_i = \lambda_j$ for all $i \neq j$. Therefore, there is, up to scalars, at most one linear dependence, and thus the affine hull of the vertices has dimension $n - 1$. It follows that $P$ is a simplex. \hfill \Box

Now let $G$ be a finite group acting by linear transformations on the vector space $V = \mathbb{R}^d$, and let $P = P(G, v)$ be an orbit polytope. We assume that $P$ is centered at the origin. Then $AGL(P) = GL(P) = GL(Gv)$. We view the orbit $Gv$ as a vector family indexed by elements of $G$. (If $v$ has a non-trivial stabilizer in $G$, then different group elements are mapped to the same vector.) We consider the matrix

$$Q = \sum_{g \in G} (gv)(gv)^t = \sum_{g \in G} g(vv^t)g^t.$$  

For any $g \in G$, we have

$$g^{-1}Q = \sum_{x \in G} g^{-1}x(vv^t)x^t = \sum_{y \in G} y(vv^t)(gy)^t = Qg^t.$$  

Thus $g^tQ^{-1} = Q^{-1}g^{-1}$ and

$$(gv)^tQ^{-1}hv = v^tg^tQ^{-1}hv = v^tQ^{-1}(g^{-1}hv).$$

By Corollary 3.2, the linear symmetries of $P(G, v)$ come from the graph isomorphisms of the vertex and edge colored graph with vertices $g \in G$ and colors $w_{g,h} = (gv)^tQ^{-1}(hv) = v^tQ^{-1}(g^{-1}hv)$. Let $f: G \to \mathbb{R}$ be defined by

$$f(g) = w_{1,g} = v^tQ^{-1}(gv) = (g^{-1}v)^tQ^{-1}v.$$  

Thus $w_{g,h} = f(g^{-1}h)$. Corollary 3.2 yields the following result.

3.4. Corollary. A permutation $\pi \in \text{Sym}(G)$ defines a linear symmetry of the orbit polytope $P(G, v)$ if and only if $f(g^{-1}h) = f(\pi(g)^{-1}\pi(h))$ for all $g, h \in G$, where $f(g) = v^tQ^{-1}gv$ and $Q = \sum_{g \in G}(gv)(gv)^t$. 

4. Generic points

Let $G \leq \text{GL}(d, \mathbb{R})$ be a finite group. We now work to define “generic” points in $\mathbb{R}^d$ with respect to $G$ and affine symmetries of orbit polytopes. We will see that the non-generic points are the zero set of some nonzero polynomials. Thus they form a proper algebraic subset. We begin by considering different sets of points which are not “generic”.

4.1. Lemma. The set of points $v$ such that

$$G_v := \{ g \in G \mid g v = v \} > \{1\}$$

is a finite union of proper subspaces of $V = \mathbb{R}^d$.

Proof. For every $g \neq 1$, the fixed space $\{ v \in V \mid g v = v \}$ is a proper subspace of $V$. □

Points $v$ with trivial stabilizer $G_v$ are called “in general position” by Ellis, Harris and Sköldberg [9]. However, these points are not general enough for our purposes, so we do not adopt this terminology. More important for what follows is the next exception that can occur.

4.2. Lemma. Let $m := \max \{ \dim P(G, v) \mid v \in \mathbb{R}^d \}$. Then

$$\{ v \in \mathbb{R}^d \mid \dim P(G, v) < m \}$$

is the zero set of a nonzero ideal of the polynomial ring $\mathbb{R}[X_1, \ldots, X_d]$.

Proof. Enumerate $G = \{ g_1, \ldots, g_n \}$. For each $v \in \mathbb{R}^d$, we can form the $(d + 1) \times n$-matrix $V$ with columns $(g_i v)_1 \in \mathbb{R}^{d+1}$. The rank $\text{rk}(V)$ of $V$ equals the dimension of the affine hull of $Gv$, so $\text{rk}(V) = \dim P(G, v)$. We have $\text{rk}(V) < m$ if and only if every $m \times m$ subdeterminant vanishes. If we regard the entries of $v$ as indeterminates $X_1, \ldots, X_d$, these subdeterminants define a number of polynomials. Since there is a vector $v_0 \in \mathbb{R}^d$ such that $\dim P(G, v_0) = m$, these polynomials generate a non-zero ideal. □

Assume that $\mathbb{R}^d = \text{aff} \{ g v_0 \mid g \in G \}$ for at least one $v_0 \in \mathbb{R}^d$. Then the vectors $v$, such that the orbit polytope $P(G, v)$ is not full-dimensional, form a proper algebraic subset of $\mathbb{R}^d$. Let us call a vector $v \in \mathbb{R}^d$ a generating point (for $G$), if $\text{aff} \{ g v \mid g \in G \} = \mathbb{R}^d$. (The terminology is justified by the fact that such a vector generates $\mathbb{R}^d$ as a module over the group algebra $\mathbb{R}G$, cf. Section 7.) The generating points, if there are any at all, form an open, dense subset of $\mathbb{R}^d$. Notice that if a generating point exists, then Lemma 2.1 yields that $G$ fixes no non-zero element of $\mathbb{R}^d$, and the affine and the linear space generated by any $G$-orbit coincide. Also affine symmetries of orbit polytopes are then restrictions of linear maps to the polytope.

Let $X = (X_1, \ldots, X_d)^t$ be a vector of indeterminates. This is an element of $\mathbb{R}[X]^d \subseteq \mathbb{R}(X)^d$, where $\mathbb{R}(X)$ is the field of rational functions in $d$ indeterminates $X_1, \ldots, X_d$. Since $\mathbb{R} \subseteq \mathbb{R}(X) =: K$, we may view $G$ as a subgroup of $\text{GL}(d, K)$ and $(gX)_{g \in G}$ as a vector family in $K^d$. We define the generic orbit permutation group of
$G \leq \text{GL}(d, \mathbb{R})$ to be the group of linear symmetries of the vector family $(gX)_{g \in G}$ in $K^d$, which is a subgroup of the group of all permutations of $G$, namely

$$\text{LinSym}((gX)_{g \in G}) = \{ \sigma \in \text{Sym}(G) \mid \exists A \in \text{GL}(d, K) : \forall g \in G : AgX = \sigma(g)X \}.$$  

\text{LinSym}((gX)_{g \in G}) always contains the subgroup isomorphic to $G$ via left action of $G$ on itself.

Now assume that generating points exist. It follows that $K^d = \langle gX \mid g \in G \rangle$, where the linear span is taken over $K = \mathbb{R}(X)$. To every $\sigma \in \text{LinSym}((gX)_g)$ corresponds a unique matrix

$$A_\sigma = A_\sigma(X) = A_\sigma(X_1, \ldots, X_d) \in \text{GL}(d, K)$$

such that $A_\sigma gX = \sigma(g)X$. The map $\sigma \mapsto A_\sigma$ is a group homomorphism, its image is $\text{GL}(GX) = \text{GL}((gX \mid g \in G))$. We have

$$\text{LinSym}((gX)_g) \cong \text{GL}(GX) \text{ via } \sigma \mapsto A_\sigma.$$

Notice that $G \leq \text{GL}(GX)$. If $G < \text{GL}(GX)$, then the matrices in $\text{GL}(GX) \setminus G$ can have non-constant entries.

4.3. Theorem. Let $G \leq \text{GL}(d, \mathbb{R})$ be a finite group for which generating points exist. For every generating point $v$ we have

$$\text{LinSym}((gX)_g) \leq \text{LinSym}((gv)_g).$$

The set of generating points $v$ such that $\text{LinSym}((gX)_g) < \text{LinSym}((gv)_g)$ is a proper algebraic subset of the set of all generating points.

Proof. Let $V(X)$ be the $(d \times G)$-matrix with columns $gX$ for $g \in G$. We form the matrix

$$Q(X) = Q(X_1, \ldots, X_d) := \sum_{g \in G} (gX)(gX)^t = V(X)V(X)^t$$

as in Section 3. Note that we have $\langle gv \mid g \in G \rangle = \mathbb{R}^d$ if and only if det $Q(v) \neq 0$. Since there are generating points, det $Q(X)$ is not the zero polynomial. Thus $Q(X)$ is invertible as a matrix over the function field $\mathbb{R}(X) = \mathbb{R}(X_1, \ldots, X_d)$. Therefore, Corollary 3.2 applies over $K = \mathbb{R}(X)$: For $\sigma \in \text{Sym}(G)$, we have $\sigma \in \text{LinSym}((gX)_g)$ if and only if $\sigma$ leaves the matrix $W(X) = V(X)^tQ(X)^{-1}V(X) \text{ fixed}$. If we replace $X$ by a $v$ such that $Q(v)$ is invertible, we get by evaluation the matrix $W(v)$ characterizing $\text{LinSym}((gv)_g)$. It follows that

$$\text{LinSym}((gX)_g) \leq \text{LinSym}((gv)_g).$$

Moreover, for every $\sigma \in \text{Sym}(G) \setminus \text{LinSym}((gX)_g)$, we have

$$P(\sigma)^{-1}W(X)P(\sigma) \neq W(X).$$

Thus the generating points $v$ such that $P(\sigma)^{-1}W(v)P(\sigma) = W(v)$ are zeros of some nonzero polynomials. This shows the last assertion.

4.4. Definition. Let $G \leq \text{GL}(d, \mathbb{R})$ be a finite group such that at least one orbit polytope of $G$ is full-dimensional. A point $v \in \mathbb{R}^d$ is called generic (for $G$), if $P(G, v)$ is full-dimensional, if $v$ has trivial stabilizer and if $\text{LinSym}((gX)_g) = \text{LinSym}((gv)_g)$. 

□
Now we have the first statement of Theorem A from the introduction:

4.5. Corollary. Let \( G \leq \GL(d, \mathbb{R}) \) be a finite group for which generating points exist. The set of non-generic points for \( G \) is a proper algebraic subset of \( \mathbb{R}^d \) (that is, the set of common zeros of a non-empty set of non-zero polynomials in \( \mathbb{R}[X_1, \ldots, X_d] \)).

Proof. The non-generic points are points in the union of the finitely many proper subvarieties defined in Lemmas 4.1 and 4.2 and in Theorem 4.3. □

Thus almost all points are generic for a given group, and the generic points form an open, dense subset of \( \mathbb{R}^d \). All generic points behave in the same way with respect to affine symmetries of the \( G \)-orbit polytope. This shows that the above definition is “the right one”, at least for the purposes of this paper. However, the orbit polytopes of two generic points are not necessarily combinatorially equivalent, as an example by Onn [30] shows. (The points called generic by Onn are generic in our sense, but not conversely.) The orbit polytopes in Onn’s example have dimension 5.

5. The generic symmetry group

We keep the notation of the last section: \( G \leq \GL(d, \mathbb{R}) \) is a finite group for which full-dimensional orbit polytopes exist. We call the orbit polytope \( P(G, v) \) of a generic point a \textbf{generic orbit polytope}. In this section we prove Theorems A, B and C from the introduction.

We begin with a technical result. Here, as in the last section, \( X = (X_1, \ldots, X_d)^t \) is a vector of indeterminates.

5.1. Proposition. For every \( v \in \mathbb{R}^d \) with \( \mathbb{R}^d = \langle gv \mid g \in G \rangle \), evaluation at \( v \) defines a group homomorphism

\[
\text{eval}_v : \GL(GX) \to \GL(Gv)
\]

such that the diagram

\[
\begin{array}{ccc}
\text{LinSym(}(gX)_{g}\text{)} & \longrightarrow & \text{LinSym(}(gv)_{g}\text{)} \\
\downarrow & & \downarrow \\
\GL(GX) & \longrightarrow & \GL(Gv) \\
& \text{eval}_v & \\
\end{array}
\]

commutes. If \( v \) is generic, all maps in the diagram are isomorphisms.

Proof. Let \( \sigma \in \text{LinSym(}(gX)_{g}\text{)} \) and \( A_\sigma(X) \in \GL(GX) \) be the corresponding matrix. Recall from Corollary 3.2 that

\[
A_\sigma(X) = V(X)P(\sigma)V(X)^tQ(X)^{-1}.
\]

It follows that the entries of \((\det Q(X)) \cdot A_\sigma(X)\) are polynomials. In particular, if \( \det Q(v) \neq 0 \), then evaluation at \( v \) is well defined for the entries of \( A_\sigma(X) \). This shows the existence of the homomorphism \( \GL(GX) \to \GL(Gv) \).

The commutativity of the diagram is clear. The right vertical map is always onto, and is injective if \( v \) has trivial stabilizer. In particular, if \( v \) is generic, all maps in the diagram are isomorphisms. □
5.2. Remark. The map eval\_v is always injective. This is clear if v has trivial stabilizer. However, for the proof of Corollary 5.4 below, the case where v has a nontrivial stabilizer is essential. A proof of injectivity in the general case using a continuity argument was communicated to us by Jan-Christoph Schlage-Puchta. We give here a variant of his proof, although the statement follows also from Theorem 5.3 below.

Suppose that k = k(X) ∈ GL(GX) is a matrix in the kernel of eval\_v. Write \( Gv = \{ v = v_1, v_2, \ldots, v_n \} \) with distinct v\_i’s. Evaluation at v maps GX onto Gv, with fibers \( \Omega_i = \{ gX \mid gv = v_i \} \). Since k(v) is the identity, k(X) maps each fiber \( \Omega_i \) onto itself. It follows that k(X) fixes the barycenter

\[ s_i = s_i(X) = \frac{1}{|\Omega_i|} \sum_{gX \in \Omega_i} gX \in \mathbb{R}[X]^d \]

of each \( \Omega_i \). Evaluation at v maps \( s_i \) to \( s_i(v) = v_i \). But \( Gv = \{ v_1, \ldots, v_n \} \) contains an \( \mathbb{R} \)-basis of \( \mathbb{R}^d \). It follows that the corresponding d-subset of \( \{ s_1(X), \ldots, s_n(X) \} \) is linearly independent over \( \mathbb{R}[X] \) and thus a basis of \( \mathbb{R}(X)^d \) over \( \mathbb{R}(X) \). But \( k(X)s_i(X) = s_i(X) \) for each \( i \) and thus \( k(X) \) fixes a basis. It follows \( k(X) = I \) as was to be shown.

Let \( K = \mathbb{R}(X) \). We may view the map

\[ D_X : \text{LinSym}((gX)_g) \to \text{GL}(GX) \subseteq \text{GL}(d,K), \]

\[ \sigma \mapsto D_X(\sigma) = A_\sigma(X), \]

as a representation of the abstract group \( \text{LinSym}((gX)_g) \) over the field \( K \). Similarly, we can view the composed map \( D_v \) in

\[ \begin{array}{c}
\text{LinSym}((gX)_g) \\
\downarrow D_X
\end{array} \quad \begin{array}{c}
D_v \\
\text{eval}_v
\end{array} \quad \begin{array}{c}
\text{GL}(GX) \\
\text{GL}(Gv)
\end{array} \quad \begin{array}{c}
\text{GL}(d,K)
\end{array} \]

as a representation of \( \text{LinSym}((gX)_g) \) with image in \( \text{GL}(d,\mathbb{R}) \) or in \( \text{GL}(d,K) \).

5.3. Theorem. Let \( v \) and \( w \) be generating points. Then the representations \( D_X \) and \( D_v \) are similar over \( K \), and the representations \( D_v \) and \( D_w \) are similar over \( \mathbb{R} \), that is, there exist \( S \in \text{GL}(d,K) \) and \( T \in \text{GL}(d,\mathbb{R}) \) such that

\[ D_v(\sigma) = S^{-1}D_X(\sigma)S = T^{-1}D_w(\sigma)T \]

for all \( \sigma \in \text{LinSym}((gX)_g) \). In particular, \( \text{eval}_v : \text{GL}(GX) \to \text{GL}(Gv) \) is injective for every generating point \( v \) (generic or not).

Proof. Representations over fields of characteristic zero are similar if and only if they have the same character [18, Ch. XVIII, Thm. 3]. Thus it suffices to show that \( D_X \) and \( D_v \) have the same character for all generating points \( v \). Let \( \chi_X \) and \( \chi_v \) be the characters of \( D_X \) and \( D_v \), and let \( \sigma \in \text{LinSym}((gX)_g) \). Since \( D_X(\sigma) \) is a matrix with entries in \( K = \mathbb{R}(X) \), we have \( \chi_X(\sigma) \in \mathbb{R}(X) \). But the values of a character of a finite group are always algebraic integers (in fact, the values are sums of roots of unity [17, Lemma 2.15]). In particular, \( \chi_X(\sigma) \in \mathbb{R} \). Since we get \( D_v(\sigma) \) by evaluating \( D_X(\sigma) \) at
Theorem 9.2. The following is Theorem C from the introduction.

\[ \chi_X(\sigma) = \chi_X(\sigma) \] for all generating points \( \sigma \), and thus the representations \( D_X \) and \( D_v \) are similar.

Notice that Theorem A from the introduction follows from Theorem 5.3, together with the results from Section 4. In particular, all groups of the form \( \text{GL}(Gv) \) with \( v \) generic for \( G \) in \( \mathbb{R}^d \) are conjugate in \( \text{GL}(d, \mathbb{R}) \). Thus every finite group \( G \) for which generic vectors exist, defines a conjugacy class of finite subgroups of \( \text{GL}(d, \mathbb{R}) \). Moreover, conjugate subgroups define the same conjugacy class. We call a subgroup \( G \subseteq \text{GL}(d, \mathbb{R}) \) generically closed if \( G = \text{GL}(Gv) \) for at least one generating point \( v \). Of course, by the results so far, this is then true for all generic \( v \).

The next result shows that the symmetry group \( \text{GL}(Gv) \) of a (full-dimensional) orbit \( Gv \) is generically closed (Theorem B from the introduction).

5.4. Corollary. Let \( v \) be a generating point for \( G \), write \( \hat{G} = \text{GL}(Gv) \) and let \( w \) be generic for \( \hat{G} \). Then \( \text{GL}(\hat{G}w) = \hat{G} \).

Proof. By assumption, \( v \) is a generating point for \( G \), and thus for \( \hat{G} \). By Proposition 5.1 applied to \( \hat{G} \), it follows that \( \text{GL}(\hat{G}w) \cong \text{GL}(\hat{G}X) \). Theorem 5.3 yields that the evaluation at \( v \) maps \( \text{GL}(\hat{G}X) \) injectively into \( \text{GL}(\hat{G}v) \). But \( \hat{G}v = Gv \) and \( \text{GL}(Gv) = \hat{G} \). Thus \( \text{GL}(\hat{G}w) \cong \text{GL}(\hat{G}X) \) is isomorphic to a subgroup of \( \hat{G} \). Since \( \hat{G} \subseteq \text{GL}(\hat{G}w) \), the result follows.

We have one other sufficient criterion for a group to be generically closed. Recall that a group \( G \subseteq \text{GL}(d, \mathbb{R}) \) (or a representation \( D: G \to \text{GL}(d, \mathbb{R}) \) of an abstract group \( G \)) is called absolutely irreducible, if for every field \( K \supseteq \mathbb{R} \), the space \( K^d \) has no \( G \)-invariant subspaces besides \( \{0\} \) and \( K^d \). A group \( G \) is absolutely irreducible if and only if the centralizer of \( G \) in \( \text{GL}(d, \mathbb{R}) \) consists only of the scalar matrices [17, Theorem 9.2]. The following is Theorem C from the introduction.

5.5. Theorem. Suppose that \( G \subseteq \text{GL}(d, \mathbb{R}) \) is absolutely irreducible. If \( v \) is generic for \( G \), then \( \text{GL}(Gv) = G \).

Proof. Let \( v \) and \( w \) be generating points. By Theorem 5.3, there is a matrix \( S \) such that \( D_v(\sigma) = S^{-1}D_w(\sigma)S \) for all \( \sigma \in \text{LinSym}(gX)_g \). For \( g \in G \), the group \( \text{LinSym}(gX)_g \) contains the permutation \( \lambda_g \) that maps \( x \in G \) to \( gx \), and we have \( D_v(\lambda_g) = g = D_w(\lambda_g) \). It follows that \( S^{-1}gS = g \) for all \( g \in G \). Since \( G \) is absolutely irreducible, this yields \( S \in \mathbb{R} \) and thus \( D_v(\sigma) = D_w(\sigma) \) for all \( \sigma \). It follows that \( \hat{G} := D_v(\text{LinSym}(gX)_g) \) is independent of \( v \), and thus \( \hat{G} = \text{GL}(Gv) = \text{GL}(Gw) \) for all generic points \( v \) and \( w \).

Now pick a point \( v \) that is generic for both \( G \) and \( \hat{G} \). Then \( \hat{G}v = Gv \) since \( \hat{G} = \text{GL}(Gv) \). Since \( v \) has trivial stabilizer in both groups, it follows that \( \hat{G} = G \).

6. REPRESENTATION POLYTOPES

Let \( G \) be a finite group and \( D: G \to \text{GL}(d, \mathbb{R}) \) a real representation. The associated representation polytope \( P(D) \) is the convex hull of the matrices \( D(g) \) in the space
of all $d \times d$-matrices $[12]$. Of course, a representation polytope is a very special orbit polytope, namely

$$P(D) = \text{conv}\{D(g) \mid g \in G\} = P(G, I),$$

where $I$ is the identity matrix and $g \in G$ acts on the vector space of matrices by left multiplication with $D(g)$. However, in this section we show that representation polytopes are in fact generic orbit polytopes in a suitable space. We will see that their affine symmetry group is strictly bigger than $G$, except perhaps when $G$ is an elementary 2-group.

We need the following technical notion of equivalence between orbit polytopes of the same group $G$.

**6.1. Definition.** Let $G$ be an (abstract) finite group acting affinely on two spaces $V$ and $W$, and let $v \in V$ and $w \in W$. We say that the orbit polytopes $P(G, v)$ and $P(G, w)$ are **affinely $G$-equivalent** if there is an affine isomorphism $\alpha : P(G, v) \rightarrow P(G, w)$ such that $\alpha(gx) = g\alpha(x)$ for all $x \in P(G, v)$ and $g \in G$.

This is stronger than mere affine equivalence. For example, if $G = D_4 = \langle t, s \mid s^2 = t^4 = 1, sts = t^{-1} \rangle$, the orbit polytope of a point $v$ with $sv = v$ and the orbit polytope of a point $w$ with $stw = w$ are affinely equivalent (both are squares), but not as $G$-sets. This follows from the fact that $s$ fixes vertices of $P(G, v)$, but not of $P(G, w)$. Of course, in this case, there is an automorphism $\varphi$ of the group mapping $s$ to $st$, and so we can find an affine isomorphism $\alpha : P(G, v) \rightarrow P(G, w)$ with $\alpha(gx) = \varphi(g)\alpha(x)$. This leads to a weaker notion of equivalence (cf. [3]), but we will not need this here.

For another example, let $G = C_4 \times V_4$ be the direct product of $C_4$, a cyclic group of order 4, and the Klein four group $V_4$. Both a square and a 3-simplex are orbit polytopes of $C_4$ and $V_4$, and thus we get the direct product of the square and the 3-simplex as an orbit polytope of $G$ in two different ways. These are not affinely $G$-equivalent, not even in a weaker sense as in the last example.

**6.2. Lemma.** Let $D : G \rightarrow \text{GL}(d, \mathbb{R})$ be a representation and $A \in \text{GL}(d, \mathbb{R})$. Then $P(G, A)$ and $P(D) = P(G, I)$ are affinely $G$-equivalent.

**Proof.** Multiplication from the right with $A$ yields an affine map from $P(D) = P(G, I)$ to $P(G, A)$ commuting with left action of $G$, and multiplication with $A^{-1}$ yields the inverse. \qed

Since representation polytopes are special cases of orbit polytopes, the notions of the last sections apply. Of course, the subspace $X = \langle D(g) \mid g \in G \rangle \subseteq \mathbb{R}^{d \times d}$ generated by the image of a representation is in general (much) smaller than the space of all matrices. (We have $X = \mathbb{R}^{d \times d}$ if and only if $D$ is absolutely irreducible [17, Theorem 9.2].) Recall that $A \in X$ is **generic**, if $X = \langle D(g)A \mid g \in G \rangle$, if $A$ has trivial stabilizer in $G$ and if the linear symmetry group $\text{LinSym}(\langle D(g)A \mid g \rangle)$ contains only the generic permutations. The stabilizer of any $A$ contains at least the kernel of $D$ in $G$, that is, the normal subgroup of elements $n \in G$ such that $D(n) = I$. Therefore, we assume now that $D$ is **faithful**, that is, $\ker D = \{1_G\}$.

**6.3. Proposition.** Let $D : G \rightarrow \text{GL}(d, \mathbb{R})$ be a faithful representation and let $X = \langle D(G) \rangle \subseteq \mathbb{R}^{d \times d}$ be the subspace generated by the image of $G$. If $A \in X$ is a generating
point for $G$, then $P(G, A)$ and $P(D)$ are affinely $G$-equivalent. In particular, all generating points are generic.

Proof. From $I \in X = \langle D(g)A \mid g \in G \rangle$ it follows that $I = \sum_r r_d D(g)A$ for some $r_g \in \mathbb{R}$. But then $A$ is invertible with inverse $\sum r_g D(g)$. The first claim follows from Lemma 6.2. Since all full-dimensional orbit polytopes in $X$ are affinely $G$-isomorphic, their affine symmetry groups are conjugate, and its vertices have trivial stabilizer. Thus all generating points are generic. \qed

We mention in passing that $A \in X$ is a generating point in $X$ for $G$ if and only if it is invertible. This follows since $X$ is a subalgebra of $\mathbb{R}^{d \times d}$.

In particular, the representation polytope $P(D)$ itself is generic in its space. The affine symmetry group of a representation polytope is always bigger than $D(G)$, except perhaps when $G$ is an elementary abelian 2-group:

6.4. Proposition. Let $D : G \to \text{GL}(d, \mathbb{R})$ be a faithful representation. Then the affine symmetry group $\text{AGL}(P(D))$ contains the following maps:

(i) for every $h \in G$, the map sending $D(g)$ to $D(h)D(g)$,
(ii) for every $h \in G$, the map sending $D(g)$ to $D(g)D(h)$,
(iii) the map sending $D(g)$ to $D(g^{-1})$.

We have $|\text{AGL}(P(D))| \geq 2|G||G : Z(G)|$, or $G$ is an elementary abelian 2-group.

Proof. Left and right multiplication by $D(h)$ is a linear map on $\mathbb{R}^{d \times d}$ and permutes the vertices $D(g)$, thus (i) and (ii).

To see (iii), assume first that $D(g)$ is orthogonal for all $g \in G$. Then the linear map sending a matrix $A$ to its transposed matrix $A^t$ sends $D(g)$ to $D(g)^t = D(g^{-1})$ and thus maps $P(D)$ onto itself.

In general, the representation $D$ is similar to an orthogonal one [17, Theorem 4.17], so there is a non-singular matrix $S$ such that $S^{-1}D(g)S$ is orthogonal for all $g \in G$. Then the linear map $A \mapsto S(S^{-1}AS)^tS^{-1}$ sends $D(g)$ to $S(S^{-1}D(g)S)^tS^{-1} = S(S^{-1}D(g)S)^{-1}S^{-1} = D(g^{-1})$.

To estimate the order of the subgroup of $\text{AGL}(P(D))$ generated by the maps described in (i), (ii) and (iii), we identify it with a subgroup of $\text{Sym}(G)$. For every $g \in G$, let $l(g) \in \text{Sym}(G)$ be left multiplication with $g$, and $r(g) \in \text{Sym}(G)$ right multiplication with $g$. Every $l(g)$ commutes with every $r(h)$. We have $l(g)r(h) = \text{Id}_G$ if and only if $gxh = x$ for all $x \in G$, which is the case if and only if $g = h^{-1}$ and $g \in Z(G)$. Thus $|l(g)r(G)| = |G||G : Z(G)|$.

Finally, the map $\varepsilon$ sending $x$ to $x^{-1}$ is in $l(G)r(G)$ if and only if there are $g$ and $h \in G$ such that $x^{-1} = gxh$ for all $x \in G$. The case $x = 1$ yields then $g = h^{-1}$, and we have $(xy)^{-1} = (xy)^h = x^{-1}y^{-1}$ for all $x, y \in G$. Thus $G$ is abelian and every element has order 2. Thus $G$ is an elementary abelian 2-group. In every other case, we have $|\langle \varepsilon, l(G), r(G) \rangle| \geq 2|G||G : Z(G)|$. \qed

6.5. Remark. The map $\varepsilon$ above normalizes $l(G)r(G)$. Thus $\langle \varepsilon, l(G), r(G) \rangle$ has order $2|G||G : Z(G)|$, except when $G$ is an elementary abelian 2-group.
Later, when we have shown how to compute the affine symmetries of representation polytopes from a certain character, we will construct representation polytopes of elementary abelian $2$-groups that have no additional affine symmetries.

7. ORBIT POLYTOPES AS SUBSETS OF THE GROUP ALGEBRA

Let $G$ be an (abstract) finite group. For each representation $D : G \to \text{GL}(d, \mathbb{R})$ and for each $v \in \mathbb{R}^d$ we get an orbit polytope $P(G, v) := P(D(G), v)$. We may ask, for example, whether there is a representation of $G$ and an orbit polytope $P(G, v)$ such that the affine symmetry group of $P(G, v)$ is isomorphic to $G$. The present section provides some basic results for dealing with such questions.

We will use the module theoretic view of representation theory and the basic structure theory of semisimple rings [17, 18]. Recall that any representation $D : G \to \text{GL}(V)$ endows $V$ with the structure of a left module over the group algebra $\mathbb{R}G$, which is by definition the set of formal sums $\sum_{g \in G} r_g g$, $r_g \in \mathbb{R}$, together with component-wise addition and multiplication extended distributively from multiplication in the group. Conversely, any left $\mathbb{R}G$ module $V$ defines a representation $D : G \to \text{GL}(V)$, where $D(g) : V \to V$ is the map $v \mapsto gv$. Similar representations correspond to isomorphic $\mathbb{R}G$-modules and conversely.

The group algebra has a canonical inner product defined by $\langle g, h \rangle = \delta_{gh}$ for $g, h \in G$. This inner product can be used to show that any left ideal of $\mathbb{R}G$ has a left ideal complement (Maschke’s theorem for $\mathbb{R}G$): namely, the orthogonal complement of a (left) ideal is again a (left) ideal.

Let $V$ be a left $\mathbb{R}G$-module and $v \in V$. The $\mathbb{R}$-subspace $\langle Gv \rangle = \langle gv \mid g \in G \rangle$ generated by the $G$-orbit of $v$ equals $\langle Gv \rangle = \{ \sum_{g \in G} r_g gv \mid r_g \in \mathbb{R} \} = \{ av \mid a \in \mathbb{R}G \} = \mathbb{R}Gv$. This is the cyclic $\mathbb{R}G$-module generated by $v$. The orbit polytope $P(G, v)$ lives in this submodule of $V$. Notice that when $\alpha : V \to W$ is an isomorphism between two $\mathbb{R}G$-modules, then $\alpha$ maps an orbit polytope $P(G, v)$ to the orbit polytope $P(G, \alpha(v))$, which is affinely $G$-equivalent to $P(G, v)$ in the sense of Definition 6.1. Conversely, assume that $P(G, v)$ and $P(G, w)$ are affinely $G$-equivalent. The affine isomorphism $\alpha : P(G, v) \to P(G, w)$ extends to an isomorphism of affine hulls. If both polytopes are centered at the origin, then this isomorphism is linear and an isomorphism of $\mathbb{R}G$-modules. In the other cases, we can first translate the polytopes into polytopes centered at the origin. It follows that the orbit polytopes $P(G, v)$ and $P(G, w)$ are affinely $G$-equivalent if and only if the modules $\mathbb{R}Gv$ and $\mathbb{R}Gw$ are isomorphic, up to a trivial module summand.

It is a consequence of the general theory of semisimple rings that a cyclic module is isomorphic to a left ideal in the group algebra, and that this left ideal is generated by an idempotent $f$ (that is, $f^2 = f$). Thus every orbit polytope is affinely $G$-equivalent
to an orbit polytope \( P(G, f) \) contained in the group algebra. We now reprove this, giving a concrete formula for \( f \).

7.1. **Theorem.** Suppose \( G \) acts linearly on \( V = \mathbb{R}^d \) and \( v \in \mathbb{R}^d \) is such that \( V = R G v \). Set

\[
Q := \sum_{g \in G} (g v) (g v)^t \in \mathbb{R}^{d \times d} \quad \text{and} \quad f := \sum_{g \in G} \left( (g v)^t Q^{-1} v \right) \cdot g \in RG.
\]

Then \( P(G, v) \) is affinely \( G \)-equivalent to \( P(G, f) \), and \( f \) is an idempotent with \( f v = v \), \( R G f \cong V \) and \( \langle 1 - f, f \rangle = 0 \).

The last equation means that \( R G = R G f \oplus R G (1 - f) \) is an orthogonal direct sum. Notice that \( Q \) is defined as in Section 3.

**Proof of Theorem 7.1.** Define a map \( \mu : V \to R G \) by

\[
\mu(x) = \sum_{g \in G} \left( (g v)^t Q^{-1} x \right) \cdot g, \quad x \in V.
\]

Notice that \( f = \mu(v) \). First we show that \( \mu \) is an homomorphism of \( R G \)-modules: For \( h \in G \) and \( x \in V \), we have

\[
\mu(hx) = \sum_{g \in G} \left( (g v)^t Q^{-1} h x \right) g = \sum_{g \in G} \left( (g v)^t (h^{-1})^t Q^{-1} x \right) g
\]

\[
= \sum_{g \in G} \left( (h^{-1} g v)^t Q^{-1} x \right) g = \sum_{g \in G} \left( (g v)^t Q^{-1} x \right) \tilde{h} g = h \mu(x).
\]

(The second equality uses a property established before Corollary 3.4.)

Next we show that \( \mu(x) v = x \) for all \( x \in V \):

\[
\mu(x) v = \sum_{g \in G} \left( (g v)^t Q^{-1} x \right) g v
\]

\[
= \sum_{g \in G} (g v) \cdot (g v)^t Q^{-1} x = QQ^{-1} x = x.
\]

In particular, \( f v = \mu(v) v = v \), and \( f^2 = f \mu(v) = \mu(f v) = \mu(v) = f \).

Moreover, it follows that \( \mu \) is injective, and is an isomorphism from \( V \) onto

\[
\mu(V) = \mu(R G v) = R G \mu(v) = R G f.
\]

The restriction of \( \mu \) to \( P(G, v) \) is an affine \( G \)-equivalence from \( P(G, v) \) onto \( P(G, f) \).

Finally, for any \( a = \sum_g a_g g \in R G \) such that \( a v = 0 \) and \( x \in V \), we have

\[
\left\langle \sum_g a_g g, \mu(x) \right\rangle = \sum_g a_g \left( (g v)^t Q^{-1} x \right)
\]

\[
= \left( \sum_g a_g g v \right)^t Q^{-1} x = 0^t Q^{-1} x = 0.
\]

With \( a = 1 - f \) and \( x = v \), we get \( \langle 1 - f, f \rangle = 0 \) as claimed. \( \square \)

The map \( \mu \) of the last proof is a splitting of the left module homomorphism \( \kappa : R G \to V \) defined by \( \kappa(a) = a v \), since we have seen that \( \mu(x) v = x \) for all \( x \in V \). Moreover, \( a \mapsto \mu(\kappa(a)) = a f \) is the orthogonal projection from \( R G \) onto \( R G f \).

Let \( \pi \) be a permutation of the group \( G \). We extend \( \pi \) to a linear map \( R G \to R G \), which we still denote by \( \pi \). Corollary 3.4 yields that \( \pi \in \text{LinSym}(G v) \) if and only if
\[ \pi(gf) = \pi(g)f \] for all \( g \in G \). This can easily be verified directly, using that \( \mathbb{R}G \) is the orthogonal sum of \( \mathbb{R}Gf \) and \( \mathbb{R}G(1-f) \). A consequence is the following:

7.2. Corollary. Assume that \( f^2 = f \) and \( \langle 1-f, f \rangle = 0 \). Then \( \text{LinSym}(Gf) = \text{LinSym}(G(1-f)) \).

Note that the vector configuration \( \{ g(1-f) \mid g \in G \} \) is just the dual one to (the Gale diagram of) \( \{ gf \mid g \in G \} \) [35, Chapter 6]. Thus the last corollary is nothing new. One should notice, however, that it is possible that \( f \) has nontrivial stabilizer \( H > 1 \), while the stabilizer of \( (1-f) \) is trivial. Indeed, if \( gf = f \) and \( g(1-f) = 1-f \), then \( g = g \cdot 1 = gf + g(1-f) = f + (1-f) = 1 \). Thus the intersection of the two stabilizers is trivial.

If \( H \) is the stabilizer of \( f \), then every permutation of \( G \) which maps each left coset of \( H \) to itself is in \( \text{LinSym}(Gf) = \text{LinSym}(G(1-f)) \). Such a permutation induces the identity on \( P(G, f) \), but in general induces a non-identity symmetry on \( P(G, 1-f) \). For example, we may view a tetrahedron as an orbit polytope of the symmetric group \( S_3 \), so that \( S_3 \) stabilizes a vertex. The dual of this polytope has dimension \( 24 - 1 - 3 = 20 \), has 24 vertices and affine symmetry group of order \( 24 \cdot 6^4 \).

In the rest of this section, we discuss some consequences of the general structure theory of semisimple rings for orbit polytopes. (By Maschkes theorem, \( \mathbb{R}G \) is semisimple.)

There are only a finite number of non-isomorphic simple left \( \mathbb{R}G \)-modules, say \( S_1, \ldots, S_r \) [18, Ch. XVII, § 4]. Every \( \mathbb{R}G \)-module \( V \) of finite dimension over \( \mathbb{R} \) is isomorphic to a direct sum \( m_1S_1 \oplus \cdots \oplus m_rS_r \), where the multiplicities \( m_i \in \mathbb{N} \) are uniquely determined by the isomorphism type of \( V \). If \( W \cong m_1S_1 \oplus \cdots \oplus m_rS_r \) is another left \( \mathbb{R}G \)-module, then \( V \) is isomorphic to a submodule of \( W \) if and only if \( m_i \leq n_i \) for all \( i \).

In particular, we can write \( \mathbb{R}G \cong d_1S_1 \oplus \cdots \oplus d_rS_r \) with \( d_i \in \mathbb{N} \). We have seen in Theorem 7.1 that if a module \( V \) has the form \( V = \mathbb{R}Gv \), then it is isomorphic to a submodule (that is, a left ideal) of the regular module \( \mathbb{R}G \). Conversely, each left ideal \( L \leq \mathbb{R}G \) is generated by an idempotent \( f \). (Choose a complement \( A \) of \( L \) and a decomposition \( 1 = f + e \) with \( f \in L \) and \( e \in A \).)

Thus \( V = m_1S_1 \oplus \cdots \oplus m_rS_r \) is cyclic as \( \mathbb{R}G \)-module if and only if \( m_i \leq d_i \) for all \( i \). In particular, there are only finitely many isomorphism classes of cyclic \( \mathbb{R}G \)-modules, and every possible orbit polytope of \( G \) under some representation is contained in one of these cyclic modules, up to affine \( G \)-equivalence.

By Lemma 2.1 we may assume that an orbit polytope is centered at the origin. This means that the corresponding cyclic \( \mathbb{R}G \)-module does not contain the trivial module as constituent. Conversely, if an orbit polytope \( P(G, v) \) is full-dimensional in \( V \), which means that \( V = \text{aff}(P(G, v)) \), then \( P(G, v) \) is centered at the origin by Lemma 2.1, and the trivial module is not a constituent of \( V \). Thus we have proved the following result:

7.3. Theorem. Let \( S_1 = \mathbb{R} \) (the trivial module), \( S_2, \ldots, S_r \) be a set of representatives of the different isomorphism classes of simple left \( \mathbb{R}G \)-modules, and let \( V \) be an
affine left $\mathbb{R}G$-module. Write
\[ V \cong m_1 S_1 \oplus \cdots \oplus m_r S_r \quad \text{and} \quad \mathbb{R}G \cong d_1 S_1 \oplus \cdots \oplus d_r S_r. \]
Then $V$ contains full-dimensional orbit polytopes $P(G,v)$ if and only if $m_1 = 0$ and $m_i \leq d_i$ for all $i$.

We should mention that in practice, one can determine the multiplicities $m_i$ by just looking at the character of the module $V$, using the orthogonality relations of character theory. Of course, we have to know the characters of the modules $S_i$, which can be derived from the irreducible complex characters. (See any reference on character theory of finite groups, for example [17], [34].)

A possible application of Theorem 7.3 is as follows: Suppose we are given a finite group $G$, and we want to know whether there is an orbit polytope $P(G,v)$ such that $\text{AGL}(P(G,v)) \cong G$. Then there are only finitely many representations of $G$ we have to check, namely the subrepresentations of the regular representation. Using Corollary 7.2, we only have to check half of these representations.

We discuss one further topic in this section. Let $V$ be a module not necessarily containing full-dimensional orbit polytopes. In Lemma 4.2 we showed that for “almost all” vectors $v \in V$, the subspace $\langle gv \mid g \in G \rangle = \mathbb{R}Gv$ has the maximal possible dimension. The general structure theory of semisimple rings yields also that all cyclic submodules of maximal dimension are isomorphic:

7.4. Proposition. Let $V$ be a finite dimensional $\mathbb{R}G$-module and set
\[ m := \max \{ \dim_{\mathbb{R}}(\mathbb{R}Gv) \mid v \in V \}. \]
If $\dim_{\mathbb{R}}(\mathbb{R}Gv_1) = \dim_{\mathbb{R}}(\mathbb{R}Gv_2) = m$, then $\mathbb{R}Gv_1 \cong \mathbb{R}Gv_2$ as $\mathbb{R}G$-modules.

Proof. Let $m_i$ and $d_i$ be as before and set $e_i := \min \{ m_i, d_i \}$. The multiplicity of $S_i$ in any cyclic submodule $\mathbb{R}Gv \leq V$ is bounded above by $e_i$. Thus the dimension of such a submodule over $\mathbb{R}$ is bounded above by $e_1 \dim_{\mathbb{R}} S_1 + \cdots + e_r \dim_{\mathbb{R}} S_r$.

Since $e_i \leq m_i$, the module $V$ has a submodule $W \cong e_1 S_1 \oplus \cdots \oplus e_r S_r$, which is also isomorphic to a submodule of $\mathbb{R}G$. Then there is $v \in W \leq V$ such that $W = \mathbb{R}Gv = \langle gv \mid g \in G \rangle$. This shows that
\[ e_1 \dim_{\mathbb{R}} S_1 + \cdots + e_r \dim_{\mathbb{R}} S_r = m, \]
and if $m = \dim_{\mathbb{R}}(\mathbb{R}Gv)$, then $\mathbb{R}Gv \cong e_1 S_1 \oplus \cdots \oplus e_r S_r$. \hfill \Box

As a consequence, we can define generic points in arbitrary $\mathbb{R}G$-modules as points generating a submodule of the maximal possible dimension, and being generic in this submodule. Then all generic orbit polytopes have essentially “the same” affine symmetry group.

8. Representation polytopes as subsets of the group algebra

In this section we characterize representation polytopes among orbit polytopes, and we show how to compute their affine symmetries from a certain character (Theorem D from the introduction).
8.1. Theorem. Let $f \in RG$ be an idempotent. Then $f \in Z(RG)$ if and only if $P(G, f)$ is affinely $G$-equivalent to a representation polytope $P(D)$ (where $D$ is a representation of the same group $G$). Moreover, we can choose $f$ such that $\ker D = RG(1 - f)$.

We need the following simple property of semisimple rings:

8.2. Lemma. Let $e$ be an idempotent in a semisimple ring $A$. Then $Ae$ is an ideal (i.e. two-sided) if and only if $e \in Z(A)$.

Proof. By Wedderburn-Artin structure theory, $A \cong A_1 \times \cdots \times A_k$ is a direct product of simple rings $A_i$. Let $e_i$ be the projection of $e$ to $A_i$. If $Ae$ is a two-sided ideal of $A$, then $A_ie_i$ is a two-sided ideal of $A_i$. Thus either $A_ie_i = \{0\}$ and $e_i = 0$, or $A_ie_i = A_i$, which yields $e_i = 1_{A_i}$, since $e_i$ is invertible and an idempotent. In any case, $e_i \in Z(A_i)$ and so $e \in Z(A)$. The converse is trivial. □

Proof of Theorem 8.1. If $f \in Z(RG)$, then $RG(1 - f)$ is an ideal of $RG$ and there is a representation $D$ such that $D$ as algebra homomorphism $RG \to M_n(\mathbb{R})$ has kernel $RG(1 - f)$. (For example, we can take the representation corresponding to the action of $G$ on $RGf$.) Then $D$ yields an affine isomorphism of $G$-sets from $P(G, f)$ onto $P(D)$.

Conversely, assume that $D$ is a representation and $\alpha : P(G, f) \to P(D)$ is an affine isomorphism such that $\alpha(gf) = D(g)$ for all $g \in G$. First we show that we can assume that $\alpha$ is the restriction of an injective linear map $RGf \to M_n(\mathbb{R})$. Let $e_1 = (1/|G|) \sum_g g \in Z(RG)$. The barycenter of $P(G, f)$ is the idempotent $e_1f$, which is either $e_1$ or 0, and the barycenter of $P(D)$ is $D(e_1)$. We are done if both centers are zero, or both are non-zero. If $e_1f = 0$, but $D(e_1) \neq 0$, then we can replace $f$ by $f + e_1$, since $P(G, f)$ and $P(G, f + e_1)$ are affinely equivalent, and assume that $e_1f = e_1 \neq 0$. If $e_1f = 0$, but $D(e_1) = 0$, then we replace $f$ by $f - e_1$.

So assume that $\alpha : RGf \to M_n(\mathbb{R})$ is linear and injective, and sends $gf$ to $D(g)$. Then

$$\alpha \left( \sum_{g \in G} a_g gf \right) = \sum_{g \in G} a_g \alpha(gf) = \sum_{g \in G} a_g D(g) = D \left( \sum_{g \in G} a_g g \right).$$

For the rest of this proof, write $e = (1 - f)$ and $A = RG$. We have $D(f) = \alpha(f \cdot f) = \alpha(f) = I$ and $D(e) = \alpha(ef) = \alpha(0) = 0$. For $a \in A$, we have $\alpha(eaf) = D(ea) = 0$ and thus $eaf = 0$ since $\alpha$ is injective. It follows that $ea \subseteq Ae$ and thus $Ae$ is a two-sided ideal of $A$. The lemma yields $e \in Z(A)$ and thus $f = 1 - e \in Z(A)$. □

8.3. Remark. Let $V = RGf$. In the notation of Theorem 7.3, $f \in Z(RG)$ if and only if each multiplicity $m_i$ of the simple module $S_i$ in $V$ is either 0 or $d_i$ (the multiplicity of $S_i$ in $RG$).

8.4. Corollary. Let $G$ be a finite group. Every orbit polytope is affinely $G$-equivalent to a representation polytope, if and only if $RG$ is a direct product of division rings, if and only if $G$ is an abelian group or a direct product of the quaternion group of order 8 with an elementary abelian 2-group.

Proof. A semisimple ring in general is (by Wedderburn-Artin) a direct product of matrix rings over division rings, and is thus a direct product of division rings if and
only if all idempotents are central. Thus Theorem 8.1, together with Theorem 7.1, yields the first equivalence.

If $G$ is abelian, all idempotents are central, since $\mathbb{R}G$ is commutative. If $G$ is a direct product of the quaternion group of order 8 and an elementary abelian 2-group, then $\mathbb{R}G$ is a direct product of copies of $\mathbb{R}$ and Hamilton’s division ring of quaternions.

Conversely, let $G$ be a group such that all idempotents of the group algebra $\mathbb{R}G$ are contained in the center $Z(\mathbb{R}G)$. Let $H \leq G$ be a subgroup. Then $e_H := (1/|H|) \sum_{h \in H} h$ is an idempotent in $\mathbb{R}G$. Thus $g^{-1} e_H g = e_H$ for all $g \in G$, so $H$ is a normal subgroup. It follows that every subgroup of $G$ is normal. Such groups have been classified by Dedekind [16, Satz III.7.12 on p. 308]: Either $G$ is abelian or $G$ is a direct product $Q_8 \times E \times A$, where $Q_8$ is the quaternion group with 8 elements, $E$ is an elementary abelian 2-group and $A$ is abelian of odd order. But if $A > 1$ in the second case, then $\mathbb{R}A$ has a summand isomorphic to the complex numbers $\mathbb{C}$, and thus $\mathbb{R}G \cong \mathbb{R}[Q_8 \times E] \otimes \mathbb{R} \mathbb{R}A$ has a summand $\mathbb{H} \otimes \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$, where $\mathbb{H}$ is the division ring of the quaternions. Thus $A = 1$. □

The central idempotents of the group algebra can be described using the irreducible characters. We first recall the description of the central idempotents in the complex group algebra $\mathbb{C}G$. As usual, we write $\text{Irr}(G)$ for the set of complex irreducible characters of a group $G$. To every $\chi \in \text{Irr}(G)$ corresponds the central idempotent [17, Theorem 2.12]

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

An arbitrary idempotent in $Z(\mathbb{C}G)$ is the sum of some of these. Thus each idempotent $f$ in $Z(\mathbb{C}G)$ has the form

$$f = \frac{1}{|G|} \sum_{g \in G} \gamma(g^{-1}) g \quad \text{with} \quad \gamma = \sum_{\chi \in S} \chi(1) \chi \quad \text{for some} \quad S \subseteq \text{Irr}(G).$$

(This $\gamma$ is actually the character of the ideal $\mathbb{C}Gf$ as left $\mathbb{C}G$-module.)

For $f \in Z(\mathbb{R}G)$, we get the same conclusion, with the additional requirement that $\chi$ and its complex conjugate $\overline{\chi}$ are either both in $S$ or both not.

Given a representation $D$, we write $\text{Irr}(D)$ for the set of (complex) irreducible constituents of the character of $D$. Then the kernel of $D$, viewed as algebra homomorphism $\mathbb{R}G \to \mathbb{M}_n(\mathbb{R})$, is $\mathbb{R}G(1 - f)$, where $f$ is the sum of those $e_\chi$ such that $\chi \in \text{Irr}(D)$. Now we can prove Theorem D from the introduction, which we restate here for the readers convenience:

8.5. Theorem. Let $D : G \to \mathbb{M}_n(\mathbb{R})$ be a representation and set

$$\gamma = \sum_{\chi \in \text{Irr}(D)} \chi(1).$$

Then the permutation $\pi : G \to G$ is in $\text{AGL}(P(D))$ if and only if

$$\gamma(\pi(g)^{-1} \pi(h)) = \gamma(g^{-1} h) \quad \text{for all} \quad g, h \in G.$$

(For example, this holds if $\pi$ is a group automorphism of $G$ fixing $\gamma$.)
Proof. The representation polytope $P(D)$ is isomorphic to $P(G, f)$ with $f \in \mathbb{Z}(\mathbb{R}G)$ and $\ker D = \mathbb{R}G(1 - f)$. Then

$$f = \frac{1}{|G|} \sum_{g \in G} \gamma(g^{-1})g$$

by the remarks above. On the other hand, we may view $P(D)$ as a full-dimensional orbit polytope $P(G, v)$ in some space $\mathbb{R}^d$, and we may construct $f$ as in Theorem 7.1. It follows that

$$(1/|G|)\gamma(g) = (g^{-1}v)^tQ^{-1}v = v^tQ^{-1}(gv).$$

The result now follows from Corollary 3.4. \hfill \Box

Notice that the character $\gamma$ is in general not the character of the representation. Two representations yield affinely $G$-equivalent representation polytopes if and only if they have the same non-trivial constituents. For all these representations, we have to use the same character $\gamma$ to compute the affine symmetries.

We close this section with the following surprising characterization of representation polytopes among orbit polytopes, which is Theorem F from the introduction:

8.6. Theorem. Let $P(G, v)$ be an orbit polytope of a finite group $G$. Then $P(G, v)$ is affinely $G$-equivalent to a representation polytope of $G$ if and only if there is an $\alpha \in AGL(P(G, v))$ such that $\alpha(gv) = g^{-1}v$ for all $g \in G$.

Proof. We have seen in Proposition 6.4(iii) that a representation polytope $P(D)$ has an affine symmetry mapping $D(g)$ to $D(g^{-1})$.

Conversely, assume that there is such $\alpha$. Write $f(g) = v^tQ^{-1}(gv)$ as in Section 3. By Corollary 3.4, we have that $f(gh^{-1}) = f(g^{-1}h)$ for all $g, h \in G$. But we also have

$$f(g^{-1}) = v^tQ^{-1}(g^{-1}v) = \left(v^tQ^{-1}(g^{-1}v)\right)^t = v^t(g^{-1})^tQ^{-1}v = v^tQ^{-1}gv = f(g).$$

Combining both properties, we get

$$f(hg^{-1}) = f((hg^{-1})^{-1}) = f(h^{-1}g^{-1}) = f(g^{-1}h).$$

It follows that

$$f = \sum_{g \in G} f(g^{-1})g \in \mathbb{Z}(\mathbb{R}G).$$

By Theorem 7.1, we have $P(G, v) \cong P(G, f)$, and Theorem 8.1 yields that $P(G, f) \cong P(D)$ for some representation $D$. \hfill \Box

9. Some orbit polytopes of elementary abelian 2-groups

In this section we show that every elementary abelian 2-groups of order $2^n$ with $n \geq 5$ is the affine symmetry group of one of its orbit polytopes. To do this, we show that cut polytopes of graphs are orbit polytopes of elementary abelian 2-groups, and then exhibit a class of graphs such that the corresponding orbit polytope has no additional affine symmetries. At the end of the section, we also explain why these orbit polytopes yield counterexamples to a conjecture of Baumeister et. al. [4, Conjecture 5.4].
We begin with some general remarks. Recall that an elementary abelian 2-group $G$ of order $2^n$ is isomorphic to the additive group $\mathbb{F}_2^n$ and can be viewed as a vector space over $\mathbb{F}_2$. Every representation $G \to \text{GL}(d, \mathbb{R})$ is similar to a representation $D$ of the form

$$g \mapsto D(g) = \begin{pmatrix} \lambda_1(g) \\ \lambda_2(g) \\ \vdots \\ \lambda_d(g) \end{pmatrix},$$

where each $\lambda_i : G \to \{\pm 1\}$ is a linear character which is a constituent of $D$. Every simple $\mathbb{R}G$-module is one-dimensional and corresponds to a unique linear character of $G$. We have $\mathbb{R}G \cong \mathbb{R}[G]$ (as $\mathbb{R}$-algebras). By Theorem 7.3, $\mathbb{R}^d$ contains full-dimensional orbit polytopes of $G$ if and only if all $\lambda_i$'s are different and the trivial character is not among them.

It follows that every representation $D : \mathbb{F}_2^n \to \text{GL}(d, \mathbb{R})$ is similar to one arising from the following construction: Let $C$ be a $d \times n$-matrix over $\mathbb{F}_2$. For a vector $y = (y_1, \ldots, y_d)^t \in \mathbb{F}_2^d$, we write $(-1)^y = ((-1)^{y_1}, \ldots, (-1)^{y_d})^t \in \mathbb{R}^d$. Then define a representation $D$ by $D(x) = \text{diag}((-1)^{Cx})$ for $x \in \mathbb{F}_2^n$. The representation is faithful if and only if $C$ has rank $n$. Every orbit polytope is affinely $G$-equivalent to a representation polytope (by Corollary 8.4, but it is easy to see this directly here).

Notice that the character of such a representation is given by $\gamma(x) = d - 2w(Cx)$, where $w(y)$ denotes the Hamming weight of $y \in \mathbb{F}_2^d$. The rows of $C$ correspond to the irreducible constituents of $D$. The vector space $\mathbb{R}^d$ contains full-dimensional orbit polytopes if all rows of $C$ are different, and $C$ has no zero row. Equivalently, we have $[\gamma, \lambda] \in \{0, 1\}$ for all $\lambda \in \text{Irr} G$ and $[\gamma, 1_G] = 0$. For convenience, let us call such a character an ideal character.

If $\gamma$ is an ideal character, then a permutation $\pi$ of $G = \mathbb{F}_2^n$ yields an affine symmetry of $P(D)$ if and only if $\gamma(\pi(y) - \pi(x)) = \gamma(y - x)$ for all $x, y \in \mathbb{F}_2^n$. (This is Theorem 8.5 with additive notation for the group $G$.) In particular, every automorphism $A$ of $G = \mathbb{F}_2^n$ that fixes $\gamma$ induces an affine symmetry of the representation polytope which maps 0 to 0. If there is such an automorphism, then $\text{AGL}(P(D)) > D(G)$.

The group $\text{Aut}(G) = \text{GL}(n, 2)$ acts on the set of ideal characters of degree $d$ by $(\gamma, A) \mapsto \gamma \circ A$. There are $\binom{2^n - 1}{d}$ ideal characters of degree $d$. It follows that if $\binom{2^n - 1}{d} < |\text{GL}(n, 2)|$, then every ideal character of degree $d$ has non-trivial stabilizer in $\text{GL}(n, 2)$ and thus every $d$-dimensional orbit polytope of a group $G \cong \mathbb{F}_2^n$ has additional affine symmetries. For $n = 2, 3$ and 4 this is the case for all $d$. (E. g., for $n = 4, d = 7$ we have $15 \choose 7 = 6435 < |\text{GL}(4, 2)| = 20160$.) Thus orbit polytopes of the elementary abelian 2-groups of orders 4, 8 and 16 have additional affine symmetries.

9.1. Example. Consider the following $12 \times 5$-matrix over $\mathbb{F}_2$:

$$ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}^t. $$
The representation $D: \mathbb{F}_2^5 \to \text{GL}(12, \mathbb{R})$ sending $x \mapsto D(x) = \text{diag}((-1)^{C_x})$ yields a subgroup of $\text{GL}(12, \mathbb{R})$ generated by diagonal matrices corresponding to the rows of the matrix above. We computed (using Theorem 8.5 and the computer algebra system GAP [10]) that the affine symmetry group of the corresponding representation polytope contains no additional elements. (This representation polytope is isomorphic to a permutation polytope of the same group, see Lemma 9.8 below.)

Computational experiments suggest that if $n < d < 2^n - 1$ and $d$ “sufficiently far” from both $n$ and $2^n - 1$, then most possible choices of $C$ yield a representation polytope $P(D)$ with no additional affine symmetries.

In the remainder of this section, we construct orbit polytopes of $G \cong \mathbb{F}_2^n$ without additional symmetries for $n \geq 6$. For this purpose we consider a restricted class of ideal characters coming from graphs.

Let $\Gamma = (V,E)$ be a finite simple graph with vertex set $V$ and edge set $E \subseteq \binom{V}{2}$. We consider the power sets $2^V$ and $2^E$ as vector spaces over $\mathbb{F}_2$, where the vector addition is given by symmetric difference, i.e. $A + B = (A \setminus B) \cup (B \setminus A)$. Define

$$C: 2^V \to 2^E, \quad A \mapsto \{e \in E \mid |e \cap A| = 1\},$$

i.e., a vertex set $A$ is mapped to the set of edges which connect an element of $A$ with an element of $V \setminus A$. This is a linear map, as is checked easily. The matrix of $C$ with respect to the standard bases of $2^V$ and $2^E$ is the incidence matrix of the graph $\Gamma$, that is, the $E \times V$-matrix $C = (c_{ev})_{e \in E, v \in V}$ with entry $c_{ev} = 1$ if $v \in e$ and $c_{ev} = 0$ otherwise. We call the image of $C$ the cut space of $\Gamma$ and denote it by $CT$. The elements of $CT$ are called cut sets.

We collect some easy facts about the cut space.

9.2. Lemma.

(i) The kernel of $C$ is generated by the vertex sets of the connected components of $\Gamma$. Therefore, $CT$ is a $(|V| - t)$-dimensional subspace of $2^E$, where $t$ is the number of those components. In particular, $CT$ has dimension $|V| - 1$ if $\Gamma$ is connected.

(ii) As a subgraph of $\Gamma$, any cut set is bipartite. In particular all circles in a cut set are of even length.

Let $D: 2^V \to \text{GL}(E, \mathbb{R})$ be the representation defined (as above) by $D(A) = \text{diag}((-1)^{C(A)})$, so that $A$ is sent to the diagonal $E \times E$-matrix with entry $d_{ee} = -1$ if $e \in C(A)$ and $d_{ee} = 1$ if $e \not\in C(A)$, and let $\chi$ be the character of $D$. Then $\chi(A) = |E| - 2|C(A)|$ and $\ker \chi = \ker C$, and $\chi$ is an ideal character. Thus we know that full-dimensional orbit polytopes of $2^V / \ker C$ in $\mathbb{R}^{|E|}$ exist, and that they are centered at the origin and affinely equivalent to each other. Actually, they are affinely equivalent to the so called cut polytope of $\Gamma$.

Since $D$ and $\chi$ are not faithful as representation and character of $2^V$, it is more convenient to view them as representation and character of the elementary abelian group $CT$. Thus $\chi(S) = |E| - 2|S|$ for $S \subseteq E$, $S \in CT$. The vertices of the cut polytope correspond to the cut sets. The affine symmetries of the cut polytope are induced by those permutations $\sigma$ of $CT$ which satisfy $\chi(S^\sigma + T^\sigma) = \chi(S + T)$ for
all $S, T \in C\Gamma$. The group $G = C\Gamma$ itself induces a subgroup of the affine symmetry group of its orbit polytope, which acts regularly on the vertices of the orbit polytope. Since we want to know whether or not the affine symmetry group is strictly larger than $G$, it suffices to study the stabilizer of an element in the affine symmetry group. We call a permutation $\sigma$ of the elements of $C\Gamma$ admissible if it satisfies $\emptyset^\sigma = \emptyset$ and $|S^\sigma + T^\sigma| = |S + T|$ for all $S, T \in C\Gamma$. The admissible permutations are exactly the permutations induced by affine symmetries of the cut polytope stabilizing the identity. We now list some useful properties of admissible permutations.

9.3. Lemma. For all $S, T \in C\Gamma$ and all admissible permutations $\sigma$ we have

(i) $|S^\sigma| = |S|,$

(ii) $|S^\sigma \cap T^\sigma| = |S \cap T|.$

Proof. The first equation follows directly from the definition of admissible maps by setting $T = \emptyset.$ The second equation follows from the first one, and from $|S \cap T| = |S| + |T| - |S + T|.$

Let $\pi$ be a graph automorphism of $\Gamma$. Then $\pi$ induces an admissible permutation of $C\Gamma$ in a natural way. In general, not every admissible permutation comes from an automorphism, e.g. if $\Gamma$ is a forest, then any singleton set $\{e\}$ is an element of $C\Gamma$, and therefore each permutation of $E$ leads to an admissible permutation. However, it is clear that not each of these permutations is induced by a graph automorphism unless all connected components of $\Gamma$ contain at most one edge.

For an admissible map $\sigma$ to be induced by a graph automorphism, it is clearly necessary that $\sigma$ maps principal cut sets $C(\{v\})$ (or simply $C(v)$) to principal cut sets again. We show that for certain graphs this condition is already sufficient.

9.4. Lemma. Let $\Gamma$ be a graph such that no cut set is a cycle of length 4, and let $\sigma$ be an admissible map which permutes principal cut sets of $\Gamma$. Then $\sigma$ is induced by a graph automorphism.

Proof. Let $\pi$ be the permutation of the vertices with $C(u)^\sigma = C(u^\pi)$ for all $u \in V$. Then $\pi$ is a graph automorphism: Two vertices $v, w$ are adjacent if and only if $|C(v) \cap C(w)| = 1$, if and only if $|C(v)^\sigma \cap C(w)^\sigma| = 1$, if and only if $v^\sigma$ and $w^\sigma$ are adjacent.

Since any graph automorphism induces an admissible permutation, we may replace $\sigma$ by the composition $\sigma \circ \pi^{-1}$. Thereby, we reduced the problem to show that any admissible permutation $\sigma$ which fixes all principal cut sets must be the identity.

For this purpose we will first show that also $S := C(u) + C(v)$ is fixed by $\sigma$ for any pair of adjacent vertices $u, v \in V$. Let $e = uv$. We have $C(u) \cap S = C(u) \setminus \{e\}$ and $|C(u) \cap S^\sigma| = |C(u) \cap S| = |C(u)| - 1$. If $e \notin S^\sigma$, then $C(u) \cap S^\sigma = C(u) \cap S$, and similarly $C(v) \cap S^\sigma = C(v) \cap S$, and thus $S = S^\sigma$.

So if $S^\sigma \neq S$, then $e \in S^\sigma$. Then there must be an edge $f \in C(u) \setminus S^\sigma$, and an edge $g \in C(v) \setminus S^\sigma$, with $f = ux$ and $g = vy$ for some vertices $x \neq v$ and $y \neq u$. Thus $S^\sigma$ contains all edges of $S$ but $f$ and $g$, and contains $e$ which is not in $S$. Since $|S^\sigma| = |S|$, there is exactly one further edge $h$ in $S^\sigma \setminus S$. Since $|C(x) \cap S^\sigma| = |C(x) \cap S|$ and $f \in C(x) \cap S$, but $f \notin C(x) \cap S^\sigma$, and $e \notin C(x)$, we conclude that $h \in C(x) \cap S^\sigma$. 

The same argument with \( y \) and \( g \) instead of \( x \) and \( f \) shows that \( h \in C(y) \cap S^\sigma \). Thus we have \( h \in C(x) \cap C(y) \). So either \( x = y \) or \( h = xy \).

If \( x = y \), then the cut set \( S^\sigma + C(x) \) contains the circle \( uvx \) of odd length 3 which contradicts Lemma 9.2(ii). If \( h = xy \), then \( S^\sigma + S = \{ f, g, e, h \} \) is a cut set of \( \Gamma \) which clearly forms a cycle of length 4 in contradiction to our assumption. Hence we have shown \( S^\sigma = S \).

To finish the proof we notice that for any edge \( e = uv \) and any cut set \( S \) we have \( e \in S \) if and only if \( |S \cap (C(u) + C(v))| < |S \cap C(u)| + |S \cap C(v)| \). By the previous steps and Lemma 9.3 the latter condition is clearly invariant under \( \sigma \), i.e. we have \( e \in S \iff e \in S^\sigma \), and hence we can conclude \( S^\sigma = S \) for all cut sets \( S \).

We now introduce a class of graphs where each admissible map is induced by a graph automorphism. These graphs will be complements of certain trees. Recall that every edge of \( \Gamma \) is adjacent to at least one vertex in \( A \). Write \( \tau(\Gamma) \) for the smallest possible size of a vertex cover of \( \Gamma \). Let

\[
\mathcal{T} := \{ \Gamma = (V, E) \mid |V| \geq 7, \Gamma \text{ is a tree}, \tau(\Gamma) > 3 \},
\]

where \( \Gamma \) denotes the complement graph of \( \Gamma \). We will use a simple estimation of the sizes of cut sets and Lemma 9.4 to obtain the following result.

9.5. Proposition. Let \( \Gamma \in \mathcal{T} \), then each admissible permutation of \( \Gamma \) is induced by a graph automorphism.

Proof. Let \( n := |V| \), where \( V \) is the vertex set of \( \Gamma \), and let \( \sigma \) be any admissible permutation of \( \Gamma \). We will compare the sizes of principal cut sets to those of non-principal ones. Let \( v \in V \) be an arbitrary vertex. Then \( v \) has at most \( n - 2 \) neighbors in \( \Gamma \), so \( |C(v)| \leq n - 2 \). Let \( v, w \in V \) be two different vertices and set \( A = \{ v, w \} \). In \( \Gamma \) there are at least two edges which are not incident with \( v \) or \( w \), because otherwise we could cover all edges of \( \Gamma \) with three or less vertices. Hence in \( \Gamma \) there are at most \( n - 3 \) edges between \( A \) and \( A^c \), so in \( \Gamma \) we have \( |C(A)| \geq |A| \cdot |A^c| - (n - 3) = 2 \cdot (n - 2) - (n - 3) = n - 1 \). Finally, let \( A \subseteq V \) be any subset with \( 3 \leq |A| \leq \frac{n}{2} \), so that \( C(A) \) is any cut set not considered yet. In \( \Gamma \) there are at most \( n - 1 \) edges between \( A \) and \( A^c \), because there are \( n - 1 \) edges in total. Hence, in \( \Gamma \) we have the inequality \( |C(A)| \geq |A| \cdot (n - |A|) - (n - 1) \). Now since the real function \( x \mapsto x \cdot (n - x) - (n - 1) \) is increasing over the interval \( [3, \frac{n}{2}] \), the right hand side of the inequality attains its global minimum at \( |A| = 3 \). Hence, \( |C(A)| \geq 3 \cdot (n - 3) - (n - 1) = 2n - 8 \geq n - 1 \), where the last inequality holds because of \( n \geq 7 \).

So far, we showed that any principal cut set has at most \( n - 2 \) elements whereas any non-principal cut set has at least \( n - 1 \) elements. In particular, no cut set is a cycle of length 4, since principal cut sets are acyclic and non-principal cut-sets have more than 4 elements. This shows that any admissible permutation \( \sigma \) maps principal cut sets to principal cut sets, and in combination with Lemma 9.4 we see that \( \sigma \) must be induced by a graph automorphism.

The previous explanations and Proposition 9.5 show that any graph \( \Gamma = (V, E) \) of the class \( \mathcal{T} \) leads to a \(|E|\)-dimensional faithful real representation of the elementary
abelian group $C\Gamma$, where the stabilizer of $AGL(P)$ of any full-dimensional orbit polytope $P$ at any vertex is isomorphic to $Aut(\Gamma)$. In particular, we see that $C\Gamma$ is generically closed with respect to this representation if and only if $Aut(\Gamma) = 1$, i.e. if $\Gamma$ is asymmetric. The following lemma shows that there are “enough” asymmetric graphs in $\mathcal{T}$.

9.6. Lemma. For all natural numbers $n \geq 8$ there is a connected asymmetric graph $\Gamma \in \mathcal{T}$ with $|V(\Gamma)| = n$.

Proof. It is obvious that the tree shown in Figure 5 is asymmetric and that its edges cannot be covered by 3 or less vertices, if the tree has 8 or more vertices. Hence its complement, which must also be asymmetric, lies in $\mathcal{T}$.

The only admissible permutation of the graph in Figure 5 with 7 vertices is the identity, too, as one can easily check with a computer. Unfortunately, its complement’s edges can be covered by 3 vertices, hence the graph does not belong to $\mathcal{T}$ and we cannot apply Proposition 9.5.

Now, for any connected graph $\Gamma = (V,E)$ with $n+1$ vertices the cut space $C\Gamma$ is an elementary abelian $2$-group of order $2^n$ (by Lemma 9.2 (i)), and the cut polytope of $\Gamma$ has dimension $|E|$ and can be viewed as an orbit polytope of this group. In particular, for each $n \geq 6$ our construction yields an orbit polytope of an elementary abelian $2$-group of order $2^n$, which is also the affine symmetry group of the polytope. The polytope has dimension $|E| = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$ for the graphs of the last lemma. Together with Example 9.1 and the remark before that example, we get the following result.

9.7. Theorem. The elementary abelian $2$-group of order $2^n$ is the affine symmetry group of one of its orbit polytopes if and only if $n \notin \{2, 3, 4\}$.

Finally, we consider permutation polytopes. Let $G \leq Sym_d$ be a permutation group and let $D: G \to GL(d, \mathbb{R})$ be the corresponding representation as a group of permutation matrices. Then $P(D)$ is called a permutation polytope, also written as $P(G)$. In their paper “On permutation polytopes” [4], Baumeister et. al. point out that left and right multiplication with elements of $D(G)$ induce affine automorphisms of $P(G)$ and that thus the affine automorphism group of $P(G)$ is bigger than $G$ for non-abelian groups. They conjecture this also to be true for abelian groups $G$ of order $|G| > 2$. Now if $G$ contains elements $g$ with $g^2 \neq 1$, then transposition of matrices...
yields an additional affine symmetry of $P(G)$, thereby verifying the conjecture for these groups.\footnote{In fact, Thomas Rehn \cite[Theorems A.2 and A.4]{rehn2009} has shown that for abelian groups of exponent greater than 2, the affine symmetry group is usually much larger than $|G|$. But notice that the proof of Lemma A.7 and thus of Theorem A.2 for elementary abelian 2-groups is wrong.}

However, for elementary abelian 2-groups of order $|G| \geq 2^5$, the conjecture is false. This follows from Theorem 9.7 and the following simple observation (which completes the proof of Theorem E from the introduction):

9.8. **Lemma.** Let $G$ be an elementary abelian 2-group. Then every orbit polytope of $G$ is affinely $G$-equivalent to a permutation polytope.

**Proof.** An orbit polytope of an abelian group is affinely $G$-equivalent to a representation polytope (Corollary 8.4). Let $D$ be a representation of $G$. The abelian group $D(G)$ is simultaneously diagonalizable. Let $\{b_1, \ldots, b_d\}$ be a basis of eigenvectors. Then $D(G)$ permutes the set $\{\pm b_1, \ldots, \pm b_d\}$, with $d$ orbits of length 2. The corresponding permutation representation $D_1$ of $G$ is similar to
\[
\begin{pmatrix}
I & 0 \\
0 & D
\end{pmatrix},
\]
where $I$ is the $d \times d$ identity matrix. It follows that the representation polytopes of $D_1$ and $D$ are affinely equivalent as $G$-sets. \qed

10. **Open questions and conjectures**

10.1. **Question.** Fix a finite group $G$. For which $\mathbb{R}G$-modules $V$ is the image of $G$ in $\text{GL}(V)$ generically closed? That is, when is it true that a generic orbit polytope in $V$ has no additional affine symmetries?

We have seen in Theorem 5.5 that if $V$ is absolutely irreducible (that is, $\text{End}_{\mathbb{R}G}(V) = \mathbb{R}$), then the image of $G$ in $\text{GL}(V)$ is generically closed. If $V = \mathbb{R}G$ is the regular module (or the regular module minus the trivial module), then the generic orbit polytope is a simplex and all permutations of the vertices come from affine symmetries. From the results of Section 8 it follows that if $m_i \in \{0, d_i\}$ for all $i$, where $m_i$ and $d_i$ are the multiplicities of the simple module $S_i$ in $V$ and in $\mathbb{R}G$, respectively, then the full-dimensional orbit polytopes in $V$ are in fact representation polytopes. We have already seen that these polytopes have a big group of affine symmetries, except for elementary abelian 2-groups.

A broader class of modules such that the affine symmetry group of a generic orbit polytope “grows” can be constructed as follows. Let $V$ be a cyclic $\mathbb{R}G$-module, so that $V$ contains full-dimensional orbit polytopes. We decompose $V$ into an “ideal component” and a “non-ideal component” as follows: We may write
\[
V \cong m_1 S_1 \oplus \cdots \oplus m_r S_r \leq d_1 S_1 \oplus \cdots \oplus d_r S_r \cong \mathbb{R}G,
\]
where the $S_i$ are the different simple $\mathbb{R}G$-modules up to isomorphism, and $0 \leq m_i \leq d_i$. Then the ideal component $I$ of $V$ is the sum of those $m_i S_i$ such that $m_i = d_i$, and the non-ideal component $L$ is the sum of the $m_i S_i$ with $m_i < d_i$. Thus $V \cong I \oplus L$,
where $I$ is an ideal of $RG$, and $L$ is a left ideal where each simple constituent occurs with strictly smaller multiplicity than in the group algebra.

Let $N$ be the kernel of $G$ acting on $L$. Let $\alpha$ be a group automorphism of $G$ such that it maps each coset of $N$ in $G$ onto itself, and also the ideal $I$. For example, conjugation with any $n \in N$ has this property, and thus if $N \not\subseteq Z(G)$, then there is such an $\alpha$ which is not the identity. Then $\alpha$ as an algebra automorphism of $RG$ maps $I \oplus L$ onto itself, and leaves the elements of $L$ fixed. A generic orbit polytope $P(G,v) \subseteq V$ has thus an affine symmetry sending $gv$ to $\alpha(g)v$.

The following, if true, would provide an (almost complete) answer to Question 10.1:

**10.2. Conjecture.** Let $V = I \oplus L$ be a faithful $RG$-module, where $I$ is the ideal component of $V$. If $G$ acts faithfully on $L$ then $|\text{AGL}(P(G,v))| = |G|$ for every generic orbit polytope in $P(G,v)$ in $V$.

For a concrete example, let $G = \text{Sym}(4)$, the symmetric group on four letters. There are four nontrivial simple $RG$-modules $S_1, \ldots, S_4$ of dimensions 1, 2, 3, 3 respectively. Of these, $S_1$ (the signum representation) has kernel $A_4$, and $S_2$ has kernel $V_4$, while $S_3$ and $S_4$ have trivial kernel. We have

$$RG \cong \mathbb{R} \oplus S_1 \oplus 2S_2 \oplus 3S_3 \oplus 3S_4.$$  

A faithful $RG$-module containing full-dimensional orbit polytopes has the form

$$m_1S_1 \oplus m_2S_2 \oplus m_3S_3 \oplus m_4S_4,$$

where $m_i \leq \dim S_i$ and $m_3 + m_4 > 0$. There are $2 \cdot 3 \cdot 4 \cdot 4 - 2 \cdot 3 = 90$ such modules. Of these, $2^4 - 2^2 = 12$ are ideals of the group algebra, and another 6 modules have the form $S_2 \oplus I$, where $I$ is an ideal of $RG$ and at least one of $S_3$ and $S_4$ occurs in $I$. The generic orbit polytopes in these modules have additional affine symmetries. The generic orbit polytopes in the remaining 72 modules have only affine symmetries induced by $G$, so Conjecture 10.2 holds for $G = \text{Sym}(4)$. We have verified Conjecture 10.2 for all groups of order $\leq 47$, using GAP’s library of small groups [10]. We have also checked some few bigger groups with “few” conjugacy classes, e.g. $A_5$.

Babai [1] classified the finite groups which are isomorphic to the orthogonal symmetry group of an orbit polytope, and asked the question which abstract finite groups occur as affine symmetry group of an orbit polytope. Following Babai, we call a finite group $G$ **generalized dicyclic**, if it has an abelian subgroup $A$ of index 2 and an element $g \in G \setminus A$ of order 4 such that $g^{-1}ag = a^{-1}$ for all $a \in A$.

**10.3. Proposition.** The following groups are not isomorphic to the affine symmetry group of an orbit polytope: abelian groups of exponent greater than 2, and generalized dicyclic groups.

**Proof.** Babai [1, Theorem 1] showed that the groups in the statement of the proposition are not the orthogonal symmetry groups of orbit polytopes. (By the orthogonal symmetry group of a polytope $P \subseteq \mathbb{R}^d$, we mean the group of orthogonal transformations of $\mathbb{R}^d$ mapping $P$ onto itself.)

Now if a finite group $G$ (say) is the affine symmetry group of a polytope $P$, then there is an affine automorphism $\sigma$ such that $\sigma G \sigma^{-1}$ is orthogonal. Since $\sigma G \sigma^{-1}$ is
the affine symmetry group of the polytope \( \sigma(P) \), it is also the orthogonal symmetry group of the polytope \( \sigma(P) \). The proposition is thus an immediate corollary of Babai’s result.

Of course, the groups of Proposition 10.3 contain the groups of Corollary 8.4. (That generalized dicyclic groups do not occur can also be shown using the methods of this paper and some character theory: One can show that \( g^2 \) acts trivial on the non-ideal component of any module, where \( g \in G \setminus A \). Assuming this, one can show that the permutation sending \( x \in G \setminus A \) to \( xg^2 \) and \( a \in A \) to itself induces an affine symmetry of every orbit polytope of \( G \).) We propose the following conjecture:

10.4. **Conjecture.** The only finite groups that occur not as affine symmetry group of an orbit polytope are those from Proposition 10.3 and the elementary abelian groups of order 4, 8 and 16.

That the last three groups do not occur has been shown in Section 9. By the results of this paper, we know that the following groups occur as affine symmetry groups of an orbit polytope:

- Elementary abelian 2-groups of order \( 2^n \neq 4, 8, 16 \).
- Finite groups which have a faithful absolutely irreducible representation over the real numbers.

The last item includes all non-abelian finite simple groups \( G \), as is not difficult to see using the Feit-Thompson theorem and the theory of the Frobenius-Schur indicator [17, Chapter 4]. Another example are the finite reflection groups. Many other groups have a faithful irreducible representation over the reals, but for example no group of odd order can have such a representation, and no group with center of size > 2.

Of course, Conjecture 10.2 is also relevant for Conjecture 10.4, but there are some groups \( G \) such that \( G \) does not act faithfully on any non-ideal component, but the group is realizable as an affine symmetry group of an orbit polytope. An example is the group \( G = Q_8 \times Q_8 \). It has exactly one irreducible module over the reals which occurs more than once as constituent of the group algebra. This module is not faithful, the kernel has order 2. All other irreducible modules are isomorphic to (two-sided) ideals of the group algebra \( \mathbb{R}G \). Nonetheless, there are faithful representations \( D \) of \( G \) such that \( D(G) \) is generically closed, and thus \( G \) is isomorphic to the affine symmetry group of an orbit polytope.

We have said nothing in this paper about the combinatorial symmetry group of orbit polytopes. A combinatorial symmetry of a polytope \( P \) is a permutation of its vertices which maps faces of \( P \) to faces of \( P \). Already the example of the dihedral group \( D_4 \) (or \( D_n \)) shows that the combinatorial symmetry group of an orbit polytope is usual bigger than the affine symmetry group. The generic orbit polytope of \( D_4 \) is combinatorially an 8-gone. There are, however, special points such that the orbit polytope is a regular 8-gone, and for these points, the combinatorial and the affine symmetry groups agree. We conjecture that this is a general phenomenon:

10.5. **Conjecture.** Let \( G \leq \text{GL}(d, \mathbb{R}) \) be finite and \( P(G,v) \) a full-dimensional orbit polytope. Then there is a point \( v_0 \) such that \( P(G,v) \) and \( P(G,v_0) \) are combinatorially...
equivalent and such that all combinatorial symmetries of $P(G, v_0)$ are affine symmetries of $P(G, v_0)$.

An interesting example in case is the rotation group $T$ of the tetrahedron in dimension 3. This group is isomorphic to the alternating group $A_4$ and has order 12. The generic orbit polytope is an icosahedron, but of course a skew icosahedron having only $T$ as affine symmetry group. However, for special points the orbit polytope is a regular icosahedron with symmetry group of order 120. This is also the combinatorial symmetry group of the icosahedron. If the tetrahedron we begin with has rational coordinates, then the points such that the orbit polytope is a regular icosahedron all have irrational coordinates, because the 3-dimensional representation of the icosahedron group is not realizable over the rational numbers.

Bokowski, Ewald and Kleinschmidt [5] constructed the first example of a polytope such that its combinatorial symmetry group is bigger than the affine symmetry group of all possible realizations. Other examples have been constructed since then, but none of them, to the best of our knowledge, is an orbit polytope.

If $P(G)$ is a representation polytope belonging to the group $G \leq \text{GL}(d, \mathbb{R})$, then $P(G)$ is affinely equivalent to every other orbit polytope $P(G, A)$ which generates the same subspace of the matrix space as $G$. Thus the following conjecture would follow from the last one:

10.6. **Conjecture.** The combinatorial and the affine symmetry group of representation polytopes agree.

We have verified this for all rational representations of groups of order $\leq 31$ using GAP [10] and, in particular, Dutour Sikirić’s GAP-package polyhedral [8]. (Both polymake [11] and polyhedral can compute only with polytopes with rational vertices or vertices in quadratic extension fields.) Another example is the Birkhoff polytope, the representation polytope of the natural representation of the symmetric group $S_n$. Using the known facet structure of the Birkhoff polytope, it is not too difficult to show that its combinatorial symmetry group only contains the symmetries described in Proposition 6.4, which are of course affine.

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