Generalized Fractional Calculus Operators Associated with K-function

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Abstract: The aim of this paper is to study some properties of K-function introduced by Sharma. Here we establish two theorems which give the image of this K-function under the generalized fractional integral operators involving Fox’s H-function as kernel. Corresponding assertions in term of Euler, Whittaker and K-transforms are also presented. On account of general nature of H-function and K-function a number of results involving special functions can be obtained merely by giving particular values for the parameters.

Keywords: Generalized fractional integral, K-function, H-function, Euler transform, Whittaker transforms, K-transform.

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1. Introduction

K-Function: Recently, Sharma[18] introduced and studied a special function called as K-function, its relation with other special functions which is generalization of Mittag-Leffler function[9], also generalized form introduced by Prabhakar[12] and other special functions. These special functions have recently found essential applications in solving problems of physics, biology, engineering and applied sciences. The function is defined for \( \mu, \xi, \gamma \in \mathbb{C}, \Re(\mu) > 0, a_i, b_j \in \mathbb{R}(-\infty, \infty), \ a_i, b_j \neq 0; \ (i = 1, 2, \ldots, p; j = 1, 2, \ldots, q) \) as:

\[
\frac{\mu + \xi}{p} K_{q}(x) = \frac{\mu + \xi}{p} K_{q}(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{(\gamma)_n}{\Gamma(\mu n + \xi)} x^n,
\]

(1.1)

where \((a_j)_n\) and \((b_j)_n\) are Pochhammer symbols. If any numerator parameter \(a_{\mu j}\) is a negative integer or zero, then the series terminates to a polynomial in \(x\). The series (1.1) is defined when none of parameters \(a_{\mu j}, j = 1, 2, \ldots, q\) is a negative integer or zero. From the ratio test it is evident that the series is convergent for all \(x\) if \(p > q + 1\). When \(p = q + 1\) and \(|x| = 1\), the series can converge in some cases. Let \(\gamma = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\). It can be shown that when \(p = q + 1\) the series is absolutely convergent for \(|x| = 1\) if \(\Re(\gamma) < 0\), conditionally convergent for \(x = -1\) if \(0 \leq \Re(\gamma) < 1\) and divergent for \(|x| = 1\) if \(\Re(\gamma) \geq 1\).

Some important special cases of K-function are enumerated below:

1. For \(p = q = 0\), the K-function is the generalization of the Mittag-Leffler function[9] and its generalized form[12],

\[
\mu + \xi = 0 K_{0}(\ldots; x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\mu n + \xi)} x^n = E_{\mu, \xi}(x),
\]

(1.2)
2. For $\gamma = 1$ in (1.2), then K-function is the generalized Mittag-Leffler function\(^9\).

\[
\mu,1,1
0\, K_0 (\gamma; x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mu n + \xi)} = E_{\mu,1}^1 (x) = E_{\mu,1}^1 (x),
\]

(1.3)

3. For $\xi = 1$ in (1.3), we get Mittag-Leffler function\(^9\).

\[
\mu,1,1
0\, K_0 (\gamma; x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mu n + 1)} = E_{\mu,1}^1 (x) = E_{\mu,1}^1 (x) = E_{\mu,1}^1 (x).
\]

(1.4)

4. For $\mu = 1$ in (1.4), which is the exponential function\(^15\) denoted by $e^x$.

\[
\mu,1,1
0\, K_0 (\gamma; x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n + 1)} = E_{1,1}^1 (x) = E_{1,1}^1 (x) = E_{1,1}^1 (x).
\]

(1.5)

**Generalized Fractional Integral Operator:**

Now, we recall the definition of generalized fractional integral operators involving Fox’s H-function as kernel, defined by Saxena and Kumbhat\(^16\) means of the following equations:

\[
R^{\eta,\alpha}_{x,t} [f (x)] = rx^{-\eta - r\alpha - 1} \int_0^t (x - t) r^\eta \left[ kU_{p,q} \left[ \left( \frac{a_p}{b_q}, \frac{A_p}{B_q} \right) \right] f(t) dt,
\]

(1.6)

\[
K^{\delta,\alpha}_{x,t} [f (x)] = rx^{\delta} \int_1^{x} (x - t) x^\eta \left[ kV_{p,q} \left[ \left( \frac{a_p}{b_q}, \frac{A_p}{B_q} \right) \right] f(t) dt,
\]

(1.7)

where $U$ and $V$ represent the expressions

\[
\left( \frac{t^r}{x^r} \right)^U \left( \frac{1 - t^r}{x^r} \right) \quad \text{and} \quad \left( \frac{x^r}{1 - t^r} \right)^U \left( \frac{1 - x^r}{1 - t^r} \right)
\]

respectively with $\tau, \nu > 0$. The sufficient conditions of operators are given below:

(i) $1 \leq p, q < \infty, p^{-1} + q^{-1} = 1$;

(ii) $\Re(\eta + r\tau(b_j/B_j)) > -q^{-1}, \Re(\alpha + r\nu(b_j/B_j)) > -q^{-1}$;

\[
\Re(\delta + \alpha + r\tau(b_j/B_j)) > -p^{-1} (j = 1, \ldots, m);
\]

(iii) $f(x) \in L_p(0, \infty)$;

(iv) $|\arg k| < \frac{\pi}{2}$, $\lambda > 0$

where $\lambda = \sum_{j=0}^m B_j - \sum_{j=m+1}^q B_j + \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j > 0$.

For a detailed account of various properties, generalizations and applications of this function, the readers may refer to the recent work of the researchers\(^1, 2, 6, 7, 22\) and the references cited therein.

**H-Function:**

Symbol $H^{m,a}_{p,q} (x)$ stands for well known Fox H-function\(^4\), in operator (1.6) and (1.7) defined in terms of Mellin-Barnes type contour integral as:

\[
H^{m,a}_{p,q} (z) = H^{m,a}_{p,q} \left[ \left( \frac{a_p}{b_q}, \frac{A_p}{B_q} \right) \right] = \frac{1}{2\pi i} \oint_C \Gamma(s) z^s ds,
\]

(1.8)

where
\[ \chi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{i=1}^{n} \Gamma(1 - a_i + A_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i - A_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j s)}, \]

(1.9)

\( m, n, p, q \in \mathbb{N} \) with \( 1 \leq m \leq q, 0 \leq n \leq p \), \( A_i, B_i \in \mathbb{R} \), \( a_i, b_j \in \mathbb{R} \) or \( \mathbb{C} \), \( i = 1, 2, \ldots, p; j = 1, 2, \ldots, q \) such that \( A_i(b_j + k) \neq B_j(a_i - l - 1)(k, l \in \mathbb{N}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m) \).

For the conditions of analytically continuations together with the convergence conditions of H-function, one can see\(^{[10,20]}\). Throughout the present paper, we assume that these conditions are satisfied by the function.

**Euler Transform:** (Sneddon\(^{[19]}\)).

The Euler transform of a function \( f(z) \) is defined as:

\[ B\{f(z); a, b\} = \int_{0}^{1} z^{a-1}(1-z)^{b-1} f(z) dz, \quad a, b \in \mathbb{C}, \Re(a) > 0, \Re(b) > 0. \]

(1.10)

**Whittaker Transform:** (Whittaker and Watson\(^{[24]}\)).

Due to Whittaker transform, the following result holds:

\[ \int_{0}^{e} t^{\tau-1} e^{-t} W_{\tau, \omega}(t) dt = \frac{\Gamma\left(\frac{1}{2} + w + \omega\right) \Gamma\left(\frac{1}{2} - w + \omega\right)}{\Gamma(1 + w + \omega)} , \]

(1.11)

where \( \Re(w \pm \omega) > -1/2 \) and \( W_{\tau, \omega}(t) \) is the Whittaker confluent hypergeometric function

\[ W_{\omega, \tau}(z) = \frac{\Gamma(-2\omega)}{\Gamma\left(\frac{1}{2} - \tau - \omega\right)} M_{\tau, \omega}(z) + \frac{\Gamma(2\omega)}{\Gamma\left(\frac{1}{2} + \tau + \omega\right)} M_{\tau, -\omega}(z), \]

(1.12)

where \( M_{\tau, \omega}(z) \) is defined by

\[ M_{\tau, \omega}(z) = z^{1/2 + \omega} e^{-1/2 z} I_{1/2 + \omega - \tau; 2\omega + 1; z}. \]

(1.13)

**K-Transform:** (Erd\'e\'s and Watson\(^{[1]}\)).

This transform is defined by the following integral equation:

\[ \Re, \int f(x); p = g[p; u] = \int_{0}^{p} (px)^{1/2} K_{\nu}(px) f(x) dx, \]

(1.14)

where \( \Re(p) > 0; K_{\nu}(x) \) is the Bessel function of the second kind defined by \([12, \text{p.332}]\)

\[ K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0, \nu}(2z), \]

where \( W_{0, \nu}(\cdot) \) is the Whittaker function defined in equation (1.12).

The following result given in Mathai et al.\(^{[11]}\), p. 54, eq. 2.37 will be used in evaluating the integrals:

\[ \int_{0}^{\infty} t^{\rho-1} K_{\nu}(at) dt = 2^{\nu-1} a^{-\rho} \Gamma\left(\frac{\rho + \nu}{2}\right), \Re(a) > 0; \Re(\rho \pm \nu) > 0. \]

(1.15)

**2. Images of K-function under the Generalized Fractional Integral Operators**

In this section, we consider two generalized fractional integral operators involving the Fox’s H-function as the kernels and derived the following theorems:

**Theorem 2.1** Let \( \mu, \xi, \rho, \gamma \in \mathbb{C}, x > 0, \Re(\mu) > 0, \Re(\rho) > 0, f(x) \in L_{p}(0, \infty), 1 \leq p \leq 2, \)
\[ \arg k < \lambda \pi/2, \lambda > 0. \] Further, let the constants satisfy the condition \( a \in \mathbb{C} \), then the fractional integration of the product of K-function exists, under the condition

\[ p^{-1} + q^{-1} = 1, \mathfrak{R}(\eta + r \tau (b_j/B_j)) > -q^{-1}, \mathfrak{R}(\alpha + r \nu (b_j/B_j)) > -p^{-1} \]

then there hold the following formula:

\[
R_{\alpha, \beta}^{\gamma, \delta}(t^{-\alpha} p \kappa_\eta (at^\alpha)) = x^{\alpha-1} \sum_{n=0}^{\infty} \left( \frac{a_n}{b_n} \right) \frac{(\gamma)_n}{\Gamma(\mu n + \xi)n!} \left( ax^\alpha \right)^n \times H_{p+q+1}^{m+1} \left[ \left( \frac{a_n A_n}{b_n B_n} \right) \left( 1 - \frac{\eta}{r} - \frac{\beta}{r} - \frac{\gamma}{r} + \frac{\nu n}{r} - \alpha - \tau + \nu \right) \right]
\]

(2.1)

**Proof.** Let \( \ell \) be the left-hand side of (2.1), using (1.1) and (1.6), we have

\[
\ell = r x^{-\eta - \alpha - 1} \int_{0}^{\infty} t^{\eta + \beta - 1} \left( x^r - t^r \right) \left[ k \left( x \right) \right]^{(\alpha + \nu s)} \frac{1}{2\pi i} \int_{L} \mathcal{L}(s)(kU)^s d\nu \sum_{n=0}^{\infty} \left( \frac{a_n}{b_n} \right) \frac{(\gamma)_n}{\Gamma(\mu n + \xi)n!} \left( at^\alpha \right)^n dt,
\]

Changing the order of the integration under the valid condition given with the theorem, we obtain

\[
\ell = r x^{-\eta - \alpha - 1} \sum_{n=0}^{\infty} \left( \frac{a_n}{b_n} \right) \frac{(\gamma)_n}{\Gamma(\mu n + \xi)n!} \frac{1}{2\pi i} \int_{L} \mathcal{L}(s) k^s \left( x \right)^{\alpha - r s} \left( 1 - \frac{t^r}{x^r} \right)^{\alpha + \nu s} dt ds.
\]

(2.2)

Let the substitution \( t^r/x^r = w \), then \( t = xW^{1/r} \) in above term, we get

\[
x^{\alpha-1} \sum_{n=0}^{\infty} \left( \frac{a_n}{b_n} \right) \frac{(\gamma)_n}{\Gamma(\mu n + \xi)n!} \frac{1}{2\pi i} \int_{L} \mathcal{L}(s) k^s \left( W^{1/r} \right)^{\alpha + \nu s} \left( 1 - w^{\alpha + \nu s} \right) dt ds.
\]

(2.3)

Using Beta function for (2.3), the inner integral reduces to

\[
\ell = x^{\alpha-1} \sum_{n=0}^{\infty} \left( \frac{a_n}{b_n} \right) \frac{(\gamma)_n}{\Gamma(\mu n + \xi)n!} \frac{1}{2\pi i} \int_{L} \mathcal{L}(s) k^s \left[ \frac{\Gamma\left( \frac{\eta + \beta + \gamma}{\tau} + \nu s \right)}{\Gamma\left( \frac{\eta + \beta + \gamma}{\tau} + \alpha + 1 + (\tau + s) \xi \right)} \right] ds.
\]

(2.4)

Interpreting the right-hand side of (2.4), in view of the definition (1.9), we arrive at the result (2.1).

**Theorem 2.2** Let \( \mu, \xi, \eta, \gamma \in \mathbb{C}, \ x > 0, \ \mathfrak{R}(\mu) > 0, \ \mathfrak{R}(\eta) > 1, \ f(x) \in L_p(0, \infty), \ 1 \leq p \leq 2 \), \[ \arg k < \lambda \pi/2, \lambda > 0. \] Further, let the constants satisfy the condition \( a \in \mathbb{C} \), then the fractional integration of the product of K-function exists, under the condition

\[ p^{-1} + q^{-1} = 1, \mathfrak{R}(\alpha + r \nu (b_j/B_j)) > -q^{-1}, \mathfrak{R}(\delta + \alpha + r \tau (b_j/B_j)) > -p^{-1} \]

then there hold the following formula:
\[ K_{\gamma}^{\alpha,\beta}(t^{-\gamma} \mu, \xi, r_0)(at^{-\gamma}) = x^{-\gamma-1} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_n)}{(b_1, \ldots, b_q)_n} \frac{\Gamma(\gamma)}{\Gamma(\mu + \xi, n)!} \left( \frac{r}{r} \right)^{n}(at^{-\gamma})^n \]

\[
\times H_{p+2, q+1}^{m, n+2}(b_1, B_q, r_0) \left[ k \left( a_p, A_p, \left( \frac{1}{r} - \frac{\theta}{r} - \frac{v}{r} \right), (-\alpha, \nu) \right) + \left( \frac{\theta}{r} - \frac{v}{r} - \alpha, \tau + \nu \right), (b_q, B_q) \right].
\]

(2.5)

**Proof.** Let \( \varphi \) be the left-hand side of (2.5), using (1.1) and (1.7), we have

\[
\varphi = rx^{\delta} \int_{x}^{\infty} t^{-\delta - \gamma - r\alpha - 1}(t - x)^{\gamma} \frac{1}{2\pi i} \int_{L} \chi(s) (kV)^{-s} ds \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_n)}{(b_1, \ldots, b_q)_n} \frac{\Gamma(\gamma)}{\Gamma(\mu + \xi, n)!} \left( \frac{x}{t} \right)^n dt.
\]

Changing the order of the integration under the valid condition given with the theorem statement, we obtain

\[
\varphi = rx^{\delta} \int_{x}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_n)}{(b_1, \ldots, b_q)_n} \frac{\Gamma(\gamma)}{\Gamma(\mu + \xi, n)!} \left( \frac{x}{t} \right)^n \int_{x}^{\infty} \chi(s) (kV)^{-s} ds \left( t^{-\delta - \gamma - r\alpha - 1}(t - x)^{\gamma} \frac{1}{2\pi i} \int_{L} \chi(s) (kV)^{-s} ds \right). \]

Let the substitution \( x'/t' = u \), then \( t = x/u^{(1/r)} \) in above term and using beta function, we get

\[
\varphi = x^{-\gamma} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_n)}{(b_1, \ldots, b_q)_n} \frac{\Gamma(\gamma)}{\Gamma(\mu + \xi, n)!} \left( \frac{x}{t} \right)^n \int_{x}^{\infty} \chi(s) (kV)^{-s} \left( t^{-\delta - \gamma - r\alpha - 1}(t - x)^{\gamma} \frac{1}{2\pi i} \int_{L} \chi(s) (kV)^{-s} ds \right). \]

Interpreting the right-hand side of (2.7), in view of the definition (1.9), we arrive at the result (2.5).

### 3. Special Cases

(I). If we put \( p = q = 0 \), in Theorem (2.1) and Theorem (2.2), then we obtain the following interesting results on the right is known as generalized Mittag-Leffler function, introduced by Prabhakar[12] and Studied by Kilbas et al. [15] p.45.

**Corollary 3.1** For \( p = q = 0 \), equation (2.1) reduces in the following form:

\[
R_{x, r}^{\eta, \alpha}(t^{-\eta} \mu, \xi, r_0)(at^{-\eta}) = x^{-\eta+1} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_n)}{(b_1, \ldots, b_q)_n} \frac{\Gamma(\eta)}{\Gamma(\mu + \xi, n)!} \left( \frac{x}{t} \right)^n \left( t^{-\eta-\gamma} \frac{1}{2\pi i} \int_{L} \chi(s) (kV)^{-s} ds \right).
\]

(3.1)

**Corollary 3.2** For \( p = q = 0 \), equation (2.5) reduces in the following form:

\[
K_{x, r}^{\delta, \alpha}(t^{-\delta} \mu, \xi, r_0)(at^{-\delta}) = x^{-\delta} \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_n)}{(b_1, \ldots, b_q)_n} \frac{\Gamma(\delta)}{\Gamma(\mu + \xi, n)!} \left( \frac{x}{t} \right)^n \left( t^{-\delta-\gamma+r\alpha-1}(t - x)^{\gamma} \frac{1}{2\pi i} \int_{L} \chi(s) (kV)^{-s} ds \right).
\]

(3.2)

(II). For putting \( p = q = 0 \) and \( \gamma = 1 \) in Theorem (2.1) and Theorem (2.2), then we get generalized
Mittag-Leffler function [8] defined by

**Corollary 3.3** For \( \gamma = 1 \) and \( p = q = 0 \), equation (2.1) reduces in the following form:

\[
R_{\alpha,r}^{\eta,1} \left( t - \frac{1}{a} \right) K_{\alpha} \left( \frac{1}{a} \right) (t) = x^{\alpha-1} E_{\mu,1} \left( \frac{a}{x} \right) H_{p+2,q+1}^{m,n+2} \left[ \begin{array}{c} a, p, A_p \\ \frac{1 - \eta \gamma - \frac{1}{a} v n}{\eta \gamma - \frac{1}{a} v n} \end{array} \right] \left( -\alpha, \nu \right) (b_q, B_q).
\] (3.3)

**Corollary 4** For \( \gamma = 1 \) and \( p = q = 0 \), equation (2.5) reduces in the following form:

\[
K_{\alpha,r}^{\eta,1} \left( t - \frac{1}{a} \right) K_{\alpha} \left( \frac{1}{a} \right) (t) = x^{\alpha-1} E_{\mu,1} \left( \frac{a}{x} \right) H_{p+2,q+1}^{m,n+2} \left[ \begin{array}{c} a, p, A_p \\ \frac{1 - \delta \gamma - \frac{1}{a} v n}{\delta \gamma - \frac{1}{a} v n} \end{array} \right] \left( -\alpha, \nu \right) (b_q, B_q).
\] (3.4)

(III). For taking \( p = q = 0 \), \( \gamma = 1 \) and \( \xi = 1 \), Theorem (2.1) and Theorem (2.2), then we obtain Mittag-Leffler function[16] defined by

**Corollary 3.5** For \( \xi = 1 \), \( \gamma = 1 \) an \( p = q = 0 \), equation (2.1) reduces in the following form:

\[
R_{\alpha,r}^{\eta,1} \left( t - \frac{1}{a} \right) K_{\alpha} \left( \frac{1}{a} \right) (t) = x^{\alpha-1} E_{\mu,1} \left( \frac{a}{x} \right) H_{p+2,q+1}^{m,n+2} \left[ \begin{array}{c} a, p, A_p \\ \frac{1 - \eta \gamma - \frac{1}{a} v n}{\eta \gamma - \frac{1}{a} v n} \end{array} \right] \left( -\alpha, \nu \right) (b_q, B_q).
\] (3.5)

**Corollary 3.6** For \( \xi = 1 \), \( \gamma = 1 \) an \( p = q = 0 \), equation (2.5) reduces in the following form:

\[
K_{\alpha,r}^{\eta,1} \left( t - \frac{1}{a} \right) K_{\alpha} \left( \frac{1}{a} \right) (t) = x^{\alpha-1} E_{\mu,1} \left( \frac{a}{x} \right) H_{p+2,q+1}^{m,n+2} \left[ \begin{array}{c} a, p, A_p \\ \frac{1 - \delta \gamma - \frac{1}{a} v n}{\delta \gamma - \frac{1}{a} v n} \end{array} \right] \left( -\alpha, \nu \right) (b_q, B_q).
\] (3.6)

(IV). For taking \( p = q = 0 \), \( \gamma = 1 \), \( \xi = 1 \) and \( \mu = 1 \) Theorem (2.1) and Theorem (2.2), then we obtain in term of exponential function[16] denoted by \( e^x \).

**Corollary 3.7** For \( \mu = 1 \), \( \xi = 1 \), \( \gamma = 1 \) an \( p = q = 0 \), equation (2.1) reduces in the following form:

\[
R_{\alpha,r}^{\eta,1} \left( t - \frac{1}{a} \right) K_{\alpha} \left( \frac{1}{a} \right) (t) = x^{\alpha-1} E \left( \frac{a}{x} \right) H_{p+2,q+1}^{m,n+2} \left[ \begin{array}{c} a, p, A_p \\ \frac{1 - \eta \gamma - \frac{1}{a} v n}{\eta \gamma - \frac{1}{a} v n} \end{array} \right] \left( -\alpha, \nu \right) (b_q, B_q).
\] (3.7)

**Corollary 3.8** For \( p = q = 0 \), \( \gamma = 1 \), \( \xi = 1 \) an \( \mu = 1 \) equation (2.5) reduces in the following form:

\[
K_{\alpha,r}^{\eta,1} \left( t - \frac{1}{a} \right) K_{\alpha} \left( \frac{1}{a} \right) (t) = x^{\alpha-1} E \left( \frac{a}{x} \right) H_{p+2,q+1}^{m,n+2} \left[ \begin{array}{c} a, p, A_p \\ \frac{1 - \delta \gamma - \frac{1}{a} v n}{\delta \gamma - \frac{1}{a} v n} \end{array} \right] \left( -\alpha, \nu \right) (b_q, B_q).
\] (3.8)

4. Integral Transforms of Fractional Integral associated with K-Function

In this section, Euler, Whittaker and K-transforms of the results established in Theorem (2.1) and Theorem (2.2), have been obtained.

**Theorem 4.1** Assume that \( \mu, \xi, \gamma, c, d \in C, \Re(c) > 0, \Re(d) > 0, \Re(\mu) > 0, \Re(\gamma) > 0, p^{-1} + q^{-1} = 1; \)
\[ f(x) \in L_p(0, \infty), \quad 1 \leq p \leq 2, \quad |\arg k| < \lambda \pi/2, \lambda > 0, \quad \text{Re}(\eta + r \tau(b_j/B_j)) > -q^{-1}; \quad \text{Re}(\alpha + r \nu(b_j/B_j)) > -q^{-1}; \quad (j = 1, \ldots, m); \quad \text{then} \]

\[ B \left[ R_{\xi, \tau}^\eta \left( t^{\eta-1} \frac{\mu \xi \gamma}{p} K_q \left( at^\nu \right) \right); c, d \right] = \Gamma(d) \sum_{n=0}^\infty \left( \frac{(a)}{n} \right) \left( \frac{(\mu \xi \gamma)}{n} \right) \left( \frac{(\nu)}{n} \right) \frac{(\alpha)^n}{n!} \Gamma(c + \nu - 1)^n \frac{(\kappa)^n}{n!} \Gamma(c + d + \nu - 1)^n \]

\[ \times H_{p+2q+1}^{m,n+2} \left[ k \left( \begin{array}{c} a \cdot \nu \cdot \alpha \cdot \beta \\ \gamma \\ \delta \end{array} \right) \left( \begin{array}{c} 1 - \frac{n - \eta}{r} - \frac{1}{r} - \frac{vm}{r}, \tau \end{array} \right) (-\alpha, \nu) \right] \]

\[ \times H_{p+2q+1}^{m,n+2} \left[ k \left( \begin{array}{c} a \cdot \nu \cdot \alpha \cdot \beta \\ \gamma \\ \delta \end{array} \right) \left( \begin{array}{c} 1 - \frac{n - \eta}{r} - \frac{1}{r} - \frac{vm}{r}, \tau \end{array} \right) (-\alpha, \nu) \right] \]

\[ \times H_{p+2q+1}^{m,n+2} \left[ k \left( \begin{array}{c} a \cdot \nu \cdot \alpha \cdot \beta \\ \gamma \\ \delta \end{array} \right) \left( \begin{array}{c} 1 - \frac{n - \eta}{r} - \frac{1}{r} - \frac{vm}{r}, \tau \end{array} \right) (-\alpha, \nu) \right] \]

\[ \times H_{p+2q+1}^{m,n+2} \left[ k \left( \begin{array}{c} a \cdot \nu \cdot \alpha \cdot \beta \\ \gamma \\ \delta \end{array} \right) \left( \begin{array}{c} 1 - \frac{n - \eta}{r} - \frac{1}{r} - \frac{vm}{r}, \tau \end{array} \right) (-\alpha, \nu) \right] \]

**Proof.** Using (2.1) and (1.10), it gives

\[ B \left[ R_{\xi, \tau}^\eta \left( t^{\eta-1} \frac{\mu \xi \gamma}{p} K_q \left( at^\nu \right) \right); c, d \right] = \sum_{n=0}^\infty \left( \frac{(a)}{n} \right) \left( \frac{(\mu \xi \gamma)}{n} \right) \left( \frac{(\nu)}{n} \right) \frac{(\alpha)^n}{n!} \Gamma(c + \nu - 1)^n \frac{(\kappa)^n}{n!} \Gamma(c + d + \nu - 1)^n \]

\[ \times H_{p+2q+1}^{m,n+2} \left[ k \left( \begin{array}{c} a \cdot \nu \cdot \alpha \cdot \beta \\ \gamma \\ \delta \end{array} \right) \left( \begin{array}{c} 1 - \frac{n - \eta}{r} - \frac{1}{r} - \frac{vm}{r}, \tau \end{array} \right) (-\alpha, \nu) \right] \]

Further, we obtain the result (4.1). This completes the proof of the theorem.

**Theorem 4.2** Let \( \mu, \xi, \eta, \gamma, \epsilon, c \in \mathbb{C}, \quad a > 0, \quad \text{Re}(c) > 0, \quad \text{Re}(d) > 0, \quad \text{Re}(\mu) > 0, \quad \text{Re}(1 - \theta) < 1, \)

\[ p^{-1} + q^{-1} = 1; \quad f(x) \in L_p(0, \infty), \quad 1 \leq p \leq 2, \quad |\arg k| < \lambda \pi/2, \lambda > 0, \quad \text{Re}(\theta + \alpha + r \tau(b_j/B_j)) > -p^{-1}; \]

\[ \text{Re}(\alpha + r \nu(b_j/B_j)) > -q^{-1}; \quad (j = 1, \ldots, m); \quad \text{then} \]

\[ B \left[ K_{\delta, \tau}^\epsilon \left( t^{\epsilon-1} \frac{\mu \xi \gamma}{p} K_q \left( at^\nu \right) \right); c, d \right] = \Gamma(d) \sum_{n=0}^\infty \left( \frac{(a)}{n} \right) \left( \frac{(\mu \xi \gamma)}{n} \right) \left( \frac{(\nu)}{n} \right) \frac{(\alpha)^n}{n!} \Gamma(c + \nu - 1)^n \frac{(\kappa)^n}{n!} \Gamma(c + d + \nu - 1)^n \]

\[ \times H_{p+2q+1}^{m,n+2} \left[ k \left( \begin{array}{c} a \cdot \nu \cdot \alpha \cdot \beta \\ \gamma \\ \delta \end{array} \right) \left( \begin{array}{c} 1 - \frac{n - \eta}{r} - \frac{1}{r} - \frac{vm}{r}, \tau \end{array} \right) (-\alpha, \nu) \right] \]

**Proof.** In similar manner, proof of Theorem (4.1), we obtain the result (4.4).

**Theorem 4.3** Follow stated Theorem (2.1) for conditions on parameters, with \( \text{Re}(\theta \pm (\delta + \xi + \eta - 1)/2) > 1 \), then the following result holds:
\[
\Gamma_{\omega}^\infty e^{\frac{-\varphi}{2} t^{\omega-1}} W_{\omega,\varphi}(\varphi) \left\{ R_{x,w}^{a,\alpha} \left( t^{\omega-1} p K_q (a t^\varphi) \right) \right\} \, dt = \varphi^{1-\varphi-\zeta} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{1}{\Gamma(\mu n + \zeta) n!} (a \varphi^{-v})
\]

\[
\times \Gamma(\omega + \varphi + \mu - 1/2) \Gamma(\varphi + 1/2) H_{p+2,q+1}^{m,n+2} \left[ k \left( a_p, A_p \right) \left( \frac{1 - \eta}{r} - \frac{\varphi}{r} \right) \left( -\alpha, \nu \right) \right]
\]

\[
(4.5)
\]

**Proof.** Using (2.1) and (1.11), it gives

\[
\Gamma_{\omega}^\infty e^{\frac{-\varphi}{2} t^{\omega-1}} W_{\omega,\varphi}(\varphi) \left\{ R_{x,w}^{a,\alpha} \left( t^{\omega-1} p K_q (a t^\varphi) \right) \right\} \, dt = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{1}{\Gamma(\mu n + \zeta) n!} (a \varphi^{-v})
\]

\[
\times H_{p+2,q+1}^{m,n+2} \left[ k \left( a_p, A_p \right) \left( \frac{1 - \eta}{r} - \frac{\varphi}{r} \right) \left( -\alpha, \nu \right) \right]
\]

\[
\int \Gamma^\infty \left( \frac{1}{2} \left( \omega + \varphi + \mu - 1/2 \right) - \frac{\varphi}{2} \right) W_{\omega,\varphi}(\varphi) \, dt
\]

Assume that \( \varphi = k, \Rightarrow dt = dk/\varphi, \) we get

\[
= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{1}{\Gamma(\mu n + \zeta) n!} \left( a \varphi^{-v} \right)
\]

\[
\times \varphi^{1-\varphi-\zeta} \int_{0}^{\infty} k \left( \frac{1}{2} \left( \omega + \varphi + \mu - 1/2 \right) - \frac{\varphi}{2} \right) W_{\omega,\varphi}(k) \, dk
\]

(4.7)

Interpreting the right-hand side of (4.7), in view of the definition (1.11), We arrive at the result (4.5).

**Theorem 4.4** Follow stated Theorem (2.2) for conditions on parameters, with \( \Re(\pm (1/2 \pm \zeta - vn - 1)) > 1/2, \) then the following result holds:

\[
\Gamma_{\omega}^\infty e^{\frac{-\varphi}{2} t^{\omega-1}} W_{\omega,\varphi}(\varphi) \left\{ R_{x,w}^{a,\alpha} \left( t^{\omega-1} p K_q (a t^\varphi) \right) \right\} \, dt = \varphi^{1-\varphi-\zeta} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n}{(b_1)_n \ldots (b_q)_n} \frac{1}{\Gamma(\mu n + \zeta) n!} (a \varphi^{-v})
\]

\[
\times \Gamma(\omega - \varphi + \mu - 1/2) \Gamma(\varphi - \omega + \mu - 1/2) H_{p+2,q+1}^{m,n+2} \left[ k \left( a_p, A_p \right) \left( \frac{1 - \eta}{r} - \frac{\varphi}{r} \right) \left( -\alpha, \nu \right) \right]
\]

\[
(4.8)
\]

**Proof.** Proceeding like as proof of Theorem (4.3), we obtain the result (4.8).

**Theorem 4.5** Follow stated Theorem (2.1) for conditions on parameters, with \( \Re(\omega) > 0; \Re((\varphi + \omega + \mu - 1) > 0, \) then the following result holds:
\[
\int_0^\infty \frac{1}{t} K_\nu \left( \frac{1}{t} \right) dt = 2^{\rho - 3} \omega (1 - \rho - \delta) \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{\nu}{n!} \left( \frac{\nu}{\tau} \right)^n (a(\omega^2)^n)
\]

**Proof.** Using (2.1) and (1.15), it gives

\[
\int_0^\infty \frac{1}{t} K_\nu \left( \frac{1}{t} \right) dt = \int_0^\infty \frac{1}{t} K_\nu \left( \frac{1}{t} \right) dt = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{\nu}{n!} \left( \frac{\nu}{\tau} \right)^n (a(\omega^2)^n)
\]

we get

\[
\int_0^\infty \frac{1}{t} K_\nu \left( \frac{1}{t} \right) dt = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{\nu}{n!} \left( \frac{\nu}{\tau} \right)^n (a(\omega^2)^n)
\]

Interpreting the right-hand side of (4.10), we arrive at the result (4.9).

**Theorem 4.6** Follow stated Theorem (2.2) for conditions on parameters, with \( \Re(\omega) > 0 \), \( \Re(\rho - \delta - \nu) > 0 \), then the following result holds:

\[
\int_0^\infty \frac{1}{t} K_\nu \left( \frac{1}{t} \right) dt = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{\nu}{n!} \left( \frac{\nu}{\tau} \right)^n (a(\omega^2)^n)
\]

**Proof.** In parallel line of proof of Theorem (4.5), we obtain the result (4.11).

**5. Properties of Integral Operators**

Here, we established some properties of the operators as consequences of Theorem (2.1) and Theorem (2.2). These properties show compositions of power function.

**Theorem 5.1** Follow all the conditions on parameters as stated in Theorem (2.1) with \( \Re(\beta + \delta) > 0 \), then the following result holds true:
\[ x^\beta R^{\eta,\alpha}_{x,r} \left[ \frac{t^{\beta - 1} \mu_{x,y} \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x) = R^{\eta - \beta,\alpha}_{x,r} \left[ \frac{t^{\beta - 1} \mu_{x,y} \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x). \]  
(5.1)

Proof. From (2.1), the left hand side of eq. (5.1), we have
\[ x^\beta R^{\eta,\alpha}_{x,r} \left[ t^{\beta - 1} \frac{\mu_{x,y} \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x) = \sum_{n=0}^{\infty} \left( \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n a^n}{(b_1)_n \ldots (b_q)_n} \Gamma(\mu + \xi) \right) x^{\beta + \nu + m - 1} \times H^{m,n+2}_{p+2,q+1} \left[ k \left( a_p, A_p \right) \left( 1 - \frac{\eta}{r} \frac{\xi}{r} \frac{1 - v}{r} \right) \left( -\alpha, \nu \right) \right], \]  
(5.2)
again by (2.1), the right hand of (5.1) follows as
\[ R^{\eta - \beta,\alpha}_{x,r} \left[ t^{\beta - 1} \frac{\mu_{x,y} \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x) = \sum_{n=0}^{\infty} \left( \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n a^n}{(b_1)_n \ldots (b_q)_n} \Gamma(\mu + \xi) \right) x^{\beta + \nu + m - 1} \times H^{m,n+2}_{p+2,q+1} \left[ k \left( a_p, A_p \right) \left( 1 - \frac{\eta}{r} \frac{\xi}{r} \frac{1 - v}{r} \right) \left( -\alpha, \nu \right) \right], \]  
(5.3)
It seems that Theorem (5.1) readily follow due to (5.2) and (5.3).

**Theorem 5.2** Follow all the conditions on parameters as stated in Theorem (2.2) with \( \Re(1 - \beta + \nu) < 1 \), then the following result holds true:
\[ x^\beta \mathbf{K}^{\delta,\alpha}_{x,r} \left[ t^{\delta - \beta} \frac{\mu \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x) = \mathbf{K}^{\delta - \beta,\alpha}_{x,r} \left[ t^{\delta - \beta} \frac{\mu \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x). \]  
(5.4)

**Proof.** From (2.2), the left hand side of eq. (5.4), we have
\[ x^\beta \mathbf{K}^{\delta,\alpha}_{x,r} \left[ t^{\delta - \beta} \frac{\mu \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x) = \sum_{n=0}^{\infty} \left( \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n a^n}{(b_1)_n \ldots (b_q)_n} \Gamma(\mu + \xi) \right) x^{\beta - \delta - \nu} \times H^{m,n+2}_{p+2,q+1} \left[ k \left( a_p, A_p \right) \left( 1 - \frac{\delta}{r} \frac{\nu}{r} \right) \left( -\alpha, \nu \right) \right], \]  
(5.5)
again by (2.2), the right hand of (5.4) follows as
\[ \mathbf{K}^{\delta - \beta,\alpha}_{x,r} \left[ t^{\delta - \beta} \frac{\mu \xi \gamma}{p} \int K_q \left( at^\nu \right) \right](x) = \sum_{n=0}^{\infty} \left( \frac{(a_1)_n \ldots (a_p)_n (\gamma)_n a^n}{(b_1)_n \ldots (b_q)_n} \Gamma(\mu + \xi) \right) x^{\beta - \delta - \nu} \times H^{m,n+2}_{p+2,q+1} \left[ k \left( a_p, A_p \right) \left( 1 - \frac{\delta}{r} \frac{\nu}{r} \right) \left( -\alpha, \nu \right) \right]. \]  
(5.6)
It seems that Theorem (5.4) readily follow due to (5.5) and (5.6).

**6. Conclusions**

A class of generalized fractional integral operators involving Fox’s H-function as kernel applied on K-function are established. Also discussed the actions of fractional integral operators under Euler, Whittaker and K-transforms. The
majority of the results derived here are general in nature and compact forms are fairly helpful in deriving a variety of integral formulas in the theory of integral operators. Further, for various other special cases, the author refer (see, e.g., ([13,14,17,21,23])) for the interested readers.

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References

1. P. Agarwal, Q. Al-Mdallal, Y.J. Cho, and S. Jain, "Fractional differential equations for the generalized Mittag-Leffler function", Advances in Difference Equations, Vol. 2018, no. 1, 2018, Art. ID 58.
2. P. Agarwal, J.J. Nieto, M.J. Luo, "Extended Riemann-Liouville type fractional derivative operator with applications", Open Mathematics, Vol. 15, no 1, 1667-1681, 2017.
3. A. Erdelyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, "Higher Transcendental Functions", Vol.2, McGraw-Hill, New York, 1954.
4. C. Fox, "The G and H functions as symmetrical Fourier Kernels", Trans. Amer. Math. Soc., vol. 98, no. 3, pp. 395-429, 1961.
5. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, "Theory and Applications of Fractional Differential Equations", Elsevier, North Holland Math. Studies 204. Amsterdam, 2006.
6. I.O. Kymaz, A. Cetinkaya and P. Agarwal, "An extension of Caputo fractional derivative operator and its applications", Journal of Nonlinear Science and Applications, Vol. 9, no. 6, 3611-3621, 2016.
7. A.M. Khan, P. Ramani, D.L. Suthar and D. Kumar , "A note on $k_4$ fractional integral operator, Int. J. Appl. Comput. Math., Vol. 4, no. 1, 1-12, 2018.
8. G.M. Mittag-Leffler, "Sur la nuouvelle function $E_{\alpha}(x)$", C. R. Acad. Sci. Paris., vol. 137, no. 2, pp. 554-558, 1903.
9. G.M. Mittag-Leffler, "Sur la representation analytique de\'une branche uniforme une function monogene, Acta. Math., vol. 29, pp. 101-181, 1905.
10. A.M. Mathai and R.K. Saxena, "The H-functions with Applications in Statistics and other Disciplines", John Wiley and Sons, 1974.
11. A.M. Mathai, R.K. Saxena and H.J. Haubold, "The H-function Theory and Application", Springer, New York, 1954.
12. T.R. Prabhakar, "A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel", Yokohama Math. J., vol. 19, pp. 7-15, 1971.
13. S.D. Purohit, D.L. Suthar and S.L. Kalla, "Some results on fractional calculus operators associated with the M-function", Hadronic J., vol. 33, no. 3, pp. 225-236, 2010.
14. S.D. Purohit, D.L. Suthar and S.L. Kalla, "Marichev-Saigo-Maeda Fractional Integration Operators of the Bessel Functions", Mathematiche (Catania), vol. 67, no. 1, pp. 12-32, 2012.
15. E.D. Rainville, "Special Functions", Chelsea Publishing Company, Bronx, New York, 1960.
16. R.K. Saxena and R.K. Kumbhat, "Integral operators involving H-function", Indian J. Pure Appl. Math., vol. 5, pp. 1-6, 1974.
17. R.K.Saxena, J. Ram and D.L. Suthar, "Generalized fractional calculus of the generalized Mittag-Leffler functions", J. Indian Acad. Math., vol.31, no. 1, pp. 165-172, 2009.
18. K. Sharma, "Application of fractional calculus operators to related area", Gen. Math. Notes, vol. 7, no.1, pp. 33-40, 2011.
19. I.N. Sneddon, "The use of Integral Transform", New Delhi, Tata McGraw Hill, 1979.
20. H.M. Srivatava, K.C. Gupta and S.P. Goyal, "The H-Function of One and Two Variable With Applications", South Asian Publications Pvt. Ltd, New Delhi, Madras, 1982.
21. D.L. Suthar and Haile Habenom Anteneh, "Certain generalized fractional integral formulas involving the product of K-function and the general class of multivariable polynomials", Commun. Numer. Anal., vol. 2017, no. 2, pp. 101-108, 2017.
22. D.L. Suthar and H. Habenom and H. Tadesse, "Generalized fractional calculus formulas for a product of Mittag-Leffler function and multivariable polynomials," Int. J. Appl. Comput. Math., Vol. 4, no. 1, 1-12, 2018.
23. D.L. Suthar and S.D. Purohit, "Unified fractional integral formulae for the generalized Mittag-Leffler functions", J. Sci. Arts, vol. 27, no. 2, pp. 117-124, 2014.
24. E.T. Wittaker and G.N. Watson, "A course of Modern Analysis", Cambridge: Cambridge Univ. Press, 1962.