KHOVANOV-JACOBSSON NUMBERS AND INVARIANTS OF SURFACE-KNOTS DERIVED FROM BAR-NATAN’S THEORY

KOKORO TANAKA

Abstract. Khovanov introduced a cohomology theory for oriented classical links whose graded Euler characteristic is the Jones polynomial. Since Khovanov’s theory is functorial for link cobordisms between classical links, we obtain an invariant of a surface-knot, called the Khovanov-Jacobsson number, by considering the surface-knot as a link cobordism between empty links. In this paper, we define an invariant of a surface-knot which is a generalization of the Khovanov-Jacobsson number by using Bar-Natan’s theory, and prove that any $T^2$-knot has the trivial Khovanov-Jacobsson number.

1. Introduction

Khovanov [7] introduced a cohomology theory for oriented classical links which values in graded $\mathbb{Z}$-modules and whose graded Euler characteristic is the Jones polynomial. We denote Khovanov’s cohomology groups of an oriented link $L$ by $H(L, F) = \bigoplus H^i(L, F)$. His theory is powerful for classical links; for example, Bar-Natan [1] and Wehrli [13] showed that Khovanov’s cohomology is stronger than the Jones polynomial, and Rasmussen [11] gave a combinatorial proof of the Milnor conjecture by using a variant of Khovanov’s theory defined by Lee [10].

Jacobsson [6] and Khovanov [8] proved that Khovanov’s theory is functorial for link cobordisms in the following sense: a link cobordism $S \subset \mathbb{R}^3 \times [0,1]$ between classical links $L_0$ and $L_1$ induces a homomorphism $\phi_S : H(L_0, F) \rightarrow H(L_1, F)$, well-defined up to overall minus sign, under ambient isotopy of $S$ rel $\partial S$. A surface-knot $F$ is a closed connected oriented surface embedded locally flatly in $\mathbb{R}^4$, and can be considered as a link cobordism between empty links. Then the induced map $\phi_F : H(\emptyset, F) \rightarrow H(\emptyset, F)$, up to overall minus sign, gives an invariant of the surface-knot $F$. Since the cohomology group $H(\emptyset, F)$ of empty link $\emptyset$ is $\mathbb{Z}$, the map $\phi_F$ is an endomorphism of $\mathbb{Z}$. Hence we obtain an invariant of the surface-knot $F$ defined as $|\phi_F(1)| \in \mathbb{Z}$, and denote it by $KJ(F)$. This invariant is called the Khovanov-Jacobsson number in [4].

As far as the author knows, there are a few result on the computation of the Khovanov-Jacobsson numbers of surface-knots; see [4], for example. It follows from a simple observation that $KJ(F) = 0$ for any surface-knot $F$ with $\chi(F) \neq 0$ and that $KJ(F) = 2$ for a trivial $T^2$-knot $F$ (a surface-knot with $\chi(F) = 0$). It seems to be hard to compute the Khovanov-Jacobsson number in general, but Carter, Saito and Satoh [4] proved that $KJ(F) = 2$ for any $T^2$-knot $F$ obtained...
from a spun/twist-spun $S^2$-knot by attaching a 1-handle. They also proved that $KJ(F) = 2$ for any pseudo-ribbon $T^2$-knot $F$.

In this paper, we define an invariant $BN(F) \in \mathbb{Z}[t]$ of a surface-knot $F$ by using a variant of Khovanov’s theory defined by Bar-Natan [2]. This invariant is a generalization of the Khovanov-Jacobsson number such that

$$BN(F)|_{t=0} = KJ(F).$$

The main result of this paper is that the invariant $BN(F)$ is trivial for any surface-knot $F$, and hence it turns out that the Khovanov-Jacobsson number is trivial for any $T^2$-knot:

**Theorem 1.1.** For any surface-knot $F$ of genus $g$ ($g \geq 0$), we have the following.

(i) If $g$ is an even integer, then we have $BN(F) = 0$.

(ii) If $g$ is an odd integer, then we have $BN(F) = 2^{g-1}$.

**Corollary 1.2.** For any $T^2$-knot, we have $KJ(F) = 2$.

This paper is organized as follows. In Section 2 and 3, we review Khovanov’s cohomology theory for oriented classical links and the Khovanov-Jacobsson numbers of surface-knots respectively. In Section 4, we define the surface-knot invariant $BN(F)$. Section 5 is devoted to the proof of Theorem 1.1.

2. Khovanov’s cohomology theory

In this section, we briefly recall Khovanov’s cohomology theory for oriented classical links [7]. See also [1], [2], [10, Section 2], [11, Section 2], for example.

2.1. Graded $\mathbb{Z}$-module $V$ and TQFT $F$. Let $V$ be a free graded $\mathbb{Z}$-module of rank two generated by $v_+$ and $v_-$ with

$$\deg(v_+) = 1 \text{ and } \deg(v_-) = -1.$$ 

We give the graded $\mathbb{Z}$-module $V$ a Frobenius algebra structure with a multiplication $m$, a comultiplication $\Delta$, a unit $\iota$, and a counit $\epsilon$ defined by

$$m(v_+ \otimes v_+) = v_+,$$

$$m(v_+ \otimes v_-) = m(v_- \otimes v_+) = v_-,$$

$$\epsilon(v_+) = 0,$$

$$\epsilon(v_-) = 1.$$ 

The structure maps $m$, $\Delta$, $\iota$ and $\epsilon$ are graded maps of degree $-1$, $-1$, $1$ and $1$ respectively.

Khovanov’s cohomology theory is based on a $(1+1)$-dimensional TQFT $F$, a monoidal functor from oriented $(1+1)$-cobordisms to graded $\mathbb{Z}$-modules, associated to $V$. The Frobenius algebra $V$ defines $F$ by assigning $\mathbb{Z}$ to an empty 1-manifold, $V$ to a single circle, $V \otimes V$ to a disjoint union of two circles, and so on. The structure maps are assigned to elementary cobordisms such that

$$F \left( \begin{array}{c} \mathbb{Z} \times \mathbb{Z} \\ \circ \times \end{array} \right) = m : V \otimes V \rightarrow V, \quad F \left( \begin{array}{c} \mathbb{Z} \\ \circ \end{array} \right) = \iota : \mathbb{Z} \rightarrow V,$$

$$F \left( \begin{array}{c} \mathbb{Z} \times \mathbb{Z} \\ \times \times \end{array} \right) = \Delta : V \rightarrow V \otimes V, \quad F \left( \begin{array}{c} \mathbb{Z} \times \mathbb{Z} \\ \times \end{array} \right) = \epsilon : V \rightarrow \mathbb{Z}.$$ 

*The author has subsequently learned that Jacob Rasmussen [12] has a different proof of Corollary 1.2 using Lee’s theory.*
2.2. Cube of resolutions. Let $L$ be an oriented link, and $D$ an oriented link diagram of $L$ with $n$ crossings labeled by 1, 2, ..., $n$. A double point of $D$ can be resolved in two ways: one is the 0-smoothing $\mathbf{X} \sim \mathbf{X}$, and the other is the 1-smoothing $\mathbf{X} \sim \mathbf{X}$. We construct an $n$-dimensional cube $[0,1]^n$, called the cube of resolutions, from $D$ such that each vertex $v$ is decorated by a complete resolution of $D$ and each edge $e$ is decorated by a cobordism.

To each vertex $v$ of the cube, we associate the complete resolution $D_v$ as follows: the $i$th crossing is resolved by the 0-smoothing if the $i$th coordinate of $v$ is 0, and by the 1-smoothing if it is 1. Then $D_v$ is a collection of simple closed curves.

Each edge $e$ of the cube can be represented by sequences in $\{0,1,*\}^n$ with just one $*$, and has the two end vertices $v_e(0)$ and $v_e(1)$: the vertex $v_e(0)$ is obtained by substituting 0 to $*$, and the vertex $v_e(1)$ is obtained by substituting 1 to $*$. To an edge $e$ for which the $j$th coordinate is $*$, we associate the cobordism $S_e : D_{v_e(0)} \rightarrow D_{v_e(1)}$ as follows: we remove a neighborhood of the $j$th crossing, assign a product cobordism, and fill the saddle cobordism between the 0- and 1-smoothers around the $j$th crossing. The cobordism $S_e$ is either of the following two types: (i) two circles of $D_{v_e(0)}$ merge into one circle of $D_{v_e(1)}$, or (ii) one circle of $D_{v_e(0)}$ splits into two circles of $D_{v_e(1)}$.

2.3. Cube of modules. Applying $\mathcal{F}$ to the cube of resolutions, we construct another $n$-dimensional cube $[0,1]^n$, called the cube of modules. Each vertex $v$ is replaced by a graded $\mathbb{Z}$-module $\mathcal{F}(D_v)$ and each edge $e$ is replaced by a homomorphism

$$\mathcal{F}(S_e) : \mathcal{F}(D_{v_e(0)}) \rightarrow \mathcal{F}(D_{v_e(1)}).$$

The homomorphism $\mathcal{F}(S_e)$ is induced by a map $m : V \otimes V \rightarrow V$ if the cobordism $S_e$ is of type (i), and is induced by a map $\Delta : V \rightarrow V \otimes V$ if it is of type (ii).

2.4. Khovanov’s cohomology groups. For a graded $\mathbb{Z}$-module $M$ and an integer $k$, let $M\{k\}$ denote the graded $\mathbb{Z}$-module of which the $j$th graded component is the $(j-k)$th graded component of $M$. We note that $M\{k\}$ is identical to $M$ as $\mathbb{Z}$-modules. Khovanov’s cochain complex $C(D,\mathcal{F})$ of graded $\mathbb{Z}$-modules is obtained from the cube of modules as follows. The underlying group of $C(D,\mathcal{F}) (= \oplus C^i(D,\mathcal{F}))$ is defined by

$$C^i(D,\mathcal{F}) = \bigoplus_{v : |v| = n_-} \mathcal{F}(D_v)\{(|v| - n_-) + (n_+ - n_-)\},$$

where $|v|$ is the sum of all coordinates of $v$ and $n_+$ (resp. $n_-$) is the number of positive (resp. negative) crossings of $D$. The coboundary map $d$ for an element $x$ of $\mathcal{F}(D_v)$ is defined by

$$d(x) = \sum_{e_v : v_e(0) = v} (-1)^{w(e_v)} \mathcal{F}(S_{e_v})(x),$$

where $w(e)$ is the sum of all coordinates of an edge $e$ after $*$. We give some remarks about the coboundary map $d$ in Section 2.4.

The cohomology groups $H^i(D,\mathcal{F})$ of the cochain complex $C(D,\mathcal{F})$ are graded $\mathbb{Z}$-modules. Khovanov [7] proved that $C(D,\mathcal{F})$ and $C(D',\mathcal{F})$ are cochain homotopic to each other for any other diagram $D'$ of $L$. To prove it, he constructed cochain
maps

\[ f_1 : C\left( \bigwedge^i \mathcal{F} \right) \to C\left( \bigwedge^{i-1} \mathcal{F} \right), \quad g_1 : C\left( \bigwedge^{i-1} \mathcal{F} \right) \to C\left( \bigwedge^i \mathcal{F} \right), \]

\[ f_2 : C\left( \bigwedge^i \mathcal{F} \right) \to C\left( \bigwedge^{i+1} \mathcal{F} \right), \quad g_2 : C\left( \bigwedge^{i+1} \mathcal{F} \right) \to C\left( \bigwedge^i \mathcal{F} \right), \]

\[ f_3 : C\left( \bigwedge^i \mathcal{F} \right) \to C\left( \bigwedge^{i-1} \mathcal{F} \right), \quad g_3 : C\left( \bigwedge^{i-1} \mathcal{F} \right) \to C\left( \bigwedge^i \mathcal{F} \right), \]

for Reidemeister moves such that the maps \( g_i \circ f_i \) and \( f_i \circ g_i \) are cochain homotopic to the identities for each \( i = 1, 2, 3 \), but we omit the precise definition of these maps. (Refer to [7, 11, 2] for more details.) This implies that the isomorphism classes of the cohomology groups \( H^i(D, \mathcal{F}) \) are invariants of \( L \). Then we denote cohomology groups of an oriented link \( L \) by \( H(L, \mathcal{F}) = \bigoplus H^i(L, \mathcal{F}) \).

2.5. Remarks. We give some remarks about Khovanov’s cochain complex.

(I) The Frobenius algebra \( V \) has the following properties: the map \( m \) is associative, the map \( \Delta \) is coassociative, and \( \Delta \circ m = (m \otimes 1) \circ (1 \otimes \Delta) \). These properties can be interpreted as the commutativity of saddle point moves in oriented \((1 + 1)\)-cobordisms.

(II) It follows from (I) that each two dimensional face of the cube of modules is commutative. If we replace the map \( F \) by the map \((-1)^{w(e)} F(S_e)\) for each edge \( e \), then each two dimensional face becomes anti-commutative.

(III) The above (anti-)commutativity of the cube of modules ensures that the map \( d \) satisfies \( d \circ d = 0 \).

3. Khovanov-Jacobsson numbers of surface-knots

A link cobordism \( S \) between classical links \( L_0 \) and \( L_1 \) is a compact oriented surface embedded properly and locally flatly in \( \mathbb{R}^3 \times [0, 1] \) such that \( S \cap (\mathbb{R}^3 \times \{i\}) = L_i \) for each \( i = 0, 1 \). In this section, we briefly recall the map \( \phi_S : H(L_0, \mathcal{F}) \to H(L_1, \mathcal{F}) \) induced by the link cobordism \( S \) and the Khovanov-Jacobsson numbers of surface-knots (cf. [8, 8, 2]).

3.1. Movie presentations. For a fixed projection \( \pi : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^2 \times [0, 1] \), each intersection \( \pi(S) \cap (\mathbb{R}^2 \times \{t\}) \) of a link cobordism \( S \) is called a still and denoted by \( D_t \) for each \( t \in [0, 1] \). By perturbing \( S \) if necessary, we may assume that each still is a classical link diagram for all but finitely many critical values and contains at most one singular point. When \( t \) passes through a critical value, we see one of the following changes, called elementary string interactions (or ESIs, for short): Reidemeister moves (R1, R2 and R3) and Morse moves (birth, death and saddle). The collection \( \{D_t\}_{t \in [0, 1]} \) of stills is called a movie of \( S \). Refer to [3] for more details.

3.2. The maps induced by ESIs. We construct a map \( \phi_S \) for a link cobordism \( S \) represented by a single elementary string interaction. For the \( i \)-th Reidemeister move, the map \( \phi_S \) is defined to be \( f_i \) or \( g_i \) for each \( i = 1, 2, 3 \). For a birth (resp. death) move, the map \( \phi_S \) is induced by \( \iota \) (resp. \( \epsilon \)). For a saddle move, the map \( \phi_S \) is induced by either \( m \) or \( \Delta \) on each component of the cochain complex, depending on whether the saddle move merges two circle or splits one circle into two circles at the cube of resolutions level. We note that the map \( \phi_S \) is a graded map of degree 0 for a Reidemeister move, degree 1 for a birth/death move, and degree \(-1\) for a saddle move.
3.3. The maps induced by link cobordisms. For a link cobordism $S$, take a movie presentation of $S$. As mentioned above, the movie presentation of $S$ is represented as a collection of elementary string interactions: $S = S_1 \cup \cdots \cup S_k$. The map $\phi_S$ is defined to be the composite $\phi_{S_k} \circ \cdots \circ \phi_{S_1}$ of the maps induced by elementary cobordisms $S_1, \ldots, S_k$. Then the map $\phi_S$ is a graded map of degree $\chi(S)$, where $\chi(S)$ is the Euler characteristic of $S$. Jacobsson [6] and Khovanov [8] proved that a link cobordism $S$ induces a homomorphism $H(S)$, well-defined up to overall minus sign, under ambient isotopy of $S$ rel $\partial S$. (Jacobsson [6] pointed out that if we do not assume "rel $\partial S"$, then the statement does not hold in general.)

3.4. Khovanov-Jacobsson numbers. A surface-knot $F$ is a closed connected oriented surface embedded locally flatly in $\mathbb{R}^4$. When we consider the surface-knot $F$ as an oriented cobordism between empty links, the induced map $\phi_F : H(\emptyset, F) \to H(\emptyset, \emptyset)$, up to overall minus sign, gives an invariant of $F$. Since the cohomology group $H(\emptyset, F)$ of empty link $\emptyset$ is $\mathbb{Z}$, the map $\phi_F$ is an endomorphism of $\mathbb{Z}$. Hence we obtain an invariant of the surface-knot $F$ defined as $|\phi_F(1)| \in \mathbb{Z}$, and denote it by $KJ(F)$. This invariant is called the Khovanov-Jacobsson number in [4]. It is easy to see that $KJ(F) = 0$ for any surface-knot $F$ with $\chi(F) \neq 0$, since the map $\phi_F$ is a graded map of degree $\chi(F)$.

4. The surface-knot invariant derived from Bar-Natan’s theory

Bar-Natan [2] defined several variants of Khovanov’s theory. Let $V'$ be a free graded $\mathbb{Z}[t]$-module of rank two generated by $v_+$ and $v_-$ with

$$\deg(t) = -4, \quad \deg(v_+) = 1 \quad \text{and} \quad \deg(v_-) = -1.$$  

We give $V'$ a Frobenius algebra structure with a multiplication $m'$, a comultiplication $\Delta'$, a unit $\iota'$, and a counit $\epsilon'$ defined by

$$m'(v_+ \otimes v_+) = v_+, \quad \Delta'(v_+) = v_+ \otimes v_+ + v_- \otimes v_+$$

$$m'(v_+ \otimes v_-) = m'(v_- \otimes v_+) = v_- \quad \Delta'(v_-) = v_- \otimes v_- + tv_+ \otimes v_+$$

$$m'(v_- \otimes v_-) = tv_+, \quad \iota'(1) = v_+ \quad \epsilon'(v_+) = 0 \quad \epsilon'(v_-) = 1.$$  

One of his cohomology theories, implicitly defined in [2 Section 9.2], is based on a $(1+1)$-dimensional TQFT $F'$ associated to a Frobenius algebra $V'$, and we denote cohomology groups of an oriented link $L$ by $H(L, F') = \bigoplus H'(L, F')$. Here the cohomology groups $H'(D, F)$ are graded $\mathbb{Z}[t]$-modules. We note that the cohomology theory associated to $F'$ is essentially the same as that associated to the Frobenius system $F_3$ in [9], but the notational conventions are slightly different.

He proved that the cohomology theory associated to $F'$ is also functorial for link cobordisms up to sign indeterminacy (cf. [2, Proposition 6]). Given a surface-knot $F$, the induced map $\psi_F : H(\emptyset, F') \to H(\emptyset, \emptyset)$ becomes an endomorphism of $\mathbb{Z}[t]$. Hence we obtain an invariant of the surface-knot $F$ defined as

$$|\psi_F(1)| \in \mathbb{Z}[t],$$

and denote it by $BN(F)$. It follows from the definition of $V$ and $V'$ that Bar-Natan’s theory recovers Khovanov’s theory by adding the relation $t = 0$, and hence we have

$$BN(F)|_{t=0} = KJ(F).$$
We remark here that Bar-Natan’s theory also recovers Lee’s theory [10] by adding the relation $t = 1$.

5. Proof

For a surface-knot $F$, taking an arbitrary point $p$ of $F$ and cutting off a small neighborhood of $p$ which is homeomorphic to the standard disk pair $(D^4, D^2)$, we obtain a link cobordism between an empty link and a trivial knot. Then we can define the following two maps

$$
\psi_F^{(\emptyset \ra \emptyset)} : H(\emptyset, \mathcal{F}') \ra H(\emptyset, \mathcal{F}') \quad \text{and} \quad \psi_F^{(\emptyset \ra \emptyset)} : H(\emptyset, \mathcal{F}') \ra H(\emptyset, \mathcal{F}') ,
$$

where $\emptyset$ stands for a trivial knot and the cohomology group $H(\emptyset, \mathcal{F}')$ of a trivial knot is $V'$, and these two maps satisfy

$$
\psi_F = \psi_F^{(\emptyset \ra \emptyset)} \circ \iota' = \iota' \circ \psi_F^{(\emptyset \ra \emptyset)}. \quad \text{for a nonnegative integer} \ a
$$

For the connected sum $F_1 \# F_2$ of two surface-knots $F_1$ and $F_2$, the map $\psi_{F_1 \# F_2}$ can be decomposed into the composite of two maps such that

$$
\psi_{F_1 \# F_2} = \psi_{F_2}^{(\emptyset \ra \emptyset)} \circ \psi_{F_1}^{(\emptyset \ra \emptyset)} .
$$

The following two lemmas are direct consequences of the fact that

$$(m' \circ \Delta') (v_+ ) = 2v_- \quad \text{and} \quad (m' \circ \Delta') (v_- ) = 2t v_+ .$$

We note that the map $m' \circ \Delta'$ corresponds to a link cobordism between trivial knots induced by a trivial $T^2$-knot with two holes.

Lemma 5.1. If the surface-knot $F$ of genus $2m + 1$ $(m \geq 0)$ is trivial, then we have $BN(F) = 2(4t)^m$.

Lemma 5.2. If the surface-knot $F$ of genus $2m$ $(m \geq 0)$ is trivial, then we have

$$
\psi_F^{(\emptyset \ra \emptyset)}(v_- ) = \pm (4t)^m .
$$

Proof of Theorem 1.4. Since the map $\psi_F$ induced by a surface-knot $F$ is a graded map of degree $\chi(F)$ and the degree of $t$ is $-4$, it is easy to see the following:

- If the genus of a surface-knot $F$ is $2m$ $(m \geq 0)$, then we have $BN(F) = 0$.
- If the genus of a surface-knot $F$ is $2m + 1$ $(m \geq 0)$, then there exists some nonnegative integer $a$ such that $BN(F) = at^m$.

It is sufficient to prove that the above integer $a$ is equal to $2^{2m + 1}$ for any surface-knot $F$ of genus $2m + 1$.

It follows from $BN(F) = at^m$ that

$$
\psi_F^{(\emptyset \ra \emptyset)}(1) = \pm at^m v_- .
$$

Let $\Sigma_g$ denote a trivial surface-knot of genus $g$. We consider the connected sum $F \# \Sigma_{2m'}$ of $F$ and $\Sigma_{2m'}$ for a nonnegative integer $m'$. By lemma 5.2 we have

$$
\psi_{F \# \Sigma_{2m'}}(1) = (\psi_{\Sigma_{2m'}} \circ \psi_F^{(\emptyset \ra \emptyset)})(1) = \pm at^m (4t)^m' ,
$$

and hence we have $BN(F \# \Sigma_{2m'}) = at^m (4t)^m'$.

If we take the integer $m'$ such that $2m'$ is greater than the unknotting number $[11]$ of $F$, then the surface-knot $F \# \Sigma_{2m'}$ is ribbon-move equivalent to $\Sigma_{2(m + m')}$. When two surface-knots are related by ribbon-moves, it is known that the induced maps on the cohomology groups are the same (cf. [11]). Hence we have $BN(F \# \Sigma_{2m'}) = 2(4t)^{m + m'}$ by Lemma 5.1. This implies $a = 2^{2m + 1}$.

}\end{proof}
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Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro, Tokyo 153-8914, Japan
E-mail address: k-tanaka@ms.u-tokyo.ac.jp