Abstract
We give a new proof of Kollár’s conjecture on the derived pushforward of the dualizing sheaf twisted by a variation of Hodge structure. This conjecture has been settled by M. Saito via the theory of mixed Hodge modules and has various applications in the investigation of Albanese maps. Our technique is $L^2$ theoretic which gives concrete constructions and proofs to the conjecture. The $L^2$ point of view allows us to generalize Kollár’s conjecture to the context of non-abelian Hodge theory.

1 Introduction
Let $f : X \to Y$ be a surjective morphism between complex projective varieties. Assume that $X$ is smooth and denote by $\omega_X$ its dualizing sheaf. In [8, 9], Kollár proves the following results which is roughly called the Kollár package in the present paper.

Torsion freeness $R^i f_* \omega_X$ is torsion free for $i \geq 0$ and $R^i f_* \omega_X = 0$ if $i > \dim X - \dim Y$.

Vanishing theorem If $L$ is an ample line bundle on $Y$, then

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{for} \quad \forall j > 0 \quad \text{and} \quad \forall i \geq 0.$$ 

Decomposition theorem $Rf_* \omega_X$ splits in $D(Y)$, i.e.,

$$Rf_* \omega_X \cong \bigoplus_q R^q f_* \omega_X[-q] \in D(Y).$$
As a consequence, the spectral sequence

\[ E_2^{pq} := H^p(Y, R^q f_* \omega_X) \Rightarrow H^{p+q}(X, \omega_X) \]

degenerates at the \( E_2 \) page.

Motivated by the proofs, Kollár [9, Sect. 5] conjectured that the Kollár package could be put into a more general framework which is closely related to variations of Hodge structure. More precisely, Kollár conjectured that there is a coherent sheaf \( S_X(\mathbb{V}) \) associated to every polarized variation of Hodge structure \( \mathbb{V} = (\mathcal{V}, \nabla, \mathcal{F}, Q) \) over some dense Zariski open subset of the regular loci \( X_{\text{reg}} \) of \( X \), such that the three results above hold when \( \omega_X \) is replaced by \( S_X(\mathbb{V}) \). This conjecture has been perfectly settled in [18] for \( \mathbb{R} \)-polarized variations of Hodge structure by using M. Saito’s theory of mixed Hodge modules [15, 17] and has applications in the investigation of Albanese maps.

The purpose of this paper has two sides.

1. We give a concrete construction of \( S_X(\mathbb{V}) \) by using certain \( L^2 \) holomorphic sections and reprove Kollár’s conjecture without using mixed Hodge modules. In addition, the \( L^2 \) method has two other advantages:

   (a) It allows us to prove Kollár’s conjecture for proper Kähler morphisms. The machinery of mixed Hodge modules does not work for this more general setting because the decomposition theorem of Saito with respect to a proper Kähler morphism is still a conjecture [16, Conjecture 0.4].

   (b) It allows us to generalize Kollár’s conjecture to coefficients twisted by a hermitian vector bundle with Nakano semi-positive curvature. This flexibility is convenient for applications.

2. We observe that, rather than the structure of the variation of Hodge structure, the validity of the Kollár package for \( S_X(\mathbb{V}) \) is a consequence of the Nakano semipositivity of the top Hodge bundle \( S(\mathbb{V}) := \mathcal{F}_{\max} \{ k | F_k \neq 0 \} \) (see Sect. 4 for the abstract Kollár package). This allows us to further generalize Kollár’s conjecture to the context of non-abelian Hodge theory. For example, we show that the Kollár package holds for \( S_X(E, h) \) (see below for its definition) when \( E \) is a subbundle of a tame harmonic bundle \( (H, \theta, h) \) with vanishing second fundamental form such that \( \overline{\theta}(E) = 0 \). A typical example is a system of Hodge bundles (equivalent to a polarized complex variation of Hodge structure via Simpson’s correspondence) in the sense of Simpson [22] on some Zariski open subset \( X^\circ \subset X_{\text{reg}} \) while \( E \) is the top nonzero Hodge summand. This is an example that supports the principle that many results for variations of Hodge structure hold for harmonic bundles. Other examples include the hard Lefschetz theorems of Simpson [24], Sabbah [14] and Mochizuki [10, 11], the vanishing theorems of Arapura [1] and Deng–Hao [6] and Mochizuki’s degeneration theory for twistor structures [10]. Notice that Kollár’s conjecture for complex variations of Hodge structure may be proved in the pattern of Saito [18] by using the theory of complex Hodge modules (see [13] or the MHM Project by Sabbah and Schnell on the homepage of Sabbah.).
Let $X$ be a complex space and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset. Let $(E, h)$ be a hermitian vector bundle on $X^o$. Denote by $j : X^o \rightarrow X$ the open immersion. The main object of the present paper is the subsheaf $S_X(E, h) \subset j_* (K_{X^o} \otimes E)$ consisting of the holomorphic forms which are locally square integrable at every point of $X$. This class of objects includes Saito’s $S$-sheaves. It has the following features:

1. $S_X(E, h)$ is a torsion free $\mathcal{O}_X$-module. It is coherent when $(E, h)$ is Nakano semi-positive and tame (Definition 2.8).
2. $S_X(E, h)$ has the functoriality property (Proposition 2.5): Let $\pi : X' \rightarrow X$ be a proper bimeromorphic map such that $\pi \circ j \cong j \circ \pi^{-1}(X^o)$ is biholomorphic. Then
   
   \[ S_X(E, h) \cong \pi_* (S_{X'}(\pi^0 E, \pi^0 h)). \]

3. Let $\mathbb{V} := (\mathcal{V}, \nabla, \mathcal{F}^*, Q)$ be an $\mathbb{R}$-polarized variation of Hodge structure with $h_Q$ the associated Hodge metric. Denote by $S(\mathbb{V}) := \mathcal{F}^{\text{max}}_{k | \mathcal{F}^* \not= \emptyset}$ the top nonzero Hodge bundle and denote $S(I_{C_X}(\mathbb{V}))$ to be Saito’s $S$-sheaf associated with the Hodge module $I_{C_X}(\mathbb{V})$. Then
   
   \[ S_X(S(\mathbb{V}), h_Q) \cong S(I_{C_X}(\mathbb{V})) \ [20, \text{Theorem 4.10}]. \]

4. Consider a tame harmonic bundle $(H, \theta, h)$ and a holomorphic subbundle $E \subset H$ with vanishing second fundamental form (equivalently, $E \subset H$ is a holomorphic direct summand). Assume that $\bar{\theta}(E) = 0$. Then the corresponding $S_X(E, h)$ is a coherent sheaf (Proposition 5.5) satisfying the Kollár package (Theorem 1.1).

All the complex spaces are assumed to be separated, reduced, paracompact, countable at infinity and of pure dimension throughout the present paper. We would like to point out that the complex spaces are allowed to be non-irreducible. The main result of the present paper is the following.

**Theorem 1.1** Let $f : X \rightarrow Y$ be a proper locally Kähler morphism (Definition 3.1) from a complex space $X$ to an irreducible complex space $Y$. Assume that every irreducible component of $X$ is mapped onto $Y$. Let $X^o \subset X_{\text{reg}}$ be a dense Zariski open subset and $(H, \theta, h)$ a tame harmonic bundle on $X^o$. Let $E \subset H$ be a holomorphic subbundle with vanishing second fundamental form. Let $\bar{\theta}$ be the adjoint of $\theta$ and assume that $\bar{\theta}(E) = 0$. Let $F$ be a Nakano semi-positive vector bundle on $X$. Then the following statements hold.

**Torsion freeness** $R^q f_* (S_X(E, h) \otimes F)$ is torsion free for every $q \geq 0$ and vanishes if $q > \dim X - \dim Y$.

**Injectivity theorem** If $L$ is a semi-positive holomorphic line bundle so that $L^{\otimes l}$ admits a nonzero holomorphic global section $s$ for some $l > 0$, then the canonical morphism

\[ R^q f_* (\times s) : R^q f_* (S_X(E, h) \otimes F \otimes L^{\otimes k}) \rightarrow R^q f_* (S_X(E, h) \otimes F \otimes L^{\otimes k+l}) \]

is injective for every $q \geq 0$ and every $k \geq 1$. 
**Vanishing theorem** If $Y$ is a projective algebraic variety and $L$ is an ample line bundle on $Y$, then

$$H^q(Y, R^p f_* (S_X(E, h) \otimes F) \otimes L) = 0, \quad \forall q > 0, \quad \forall p \geq 0.$$ 

**Decomposition theorem** Assume moreover that $X$ is a compact Kähler space. Then $Rf_*(S_X(E, h) \otimes F)$ splits in $D(Y)$, i.e.,

$$Rf_*(S_X(E, h) \otimes F) \simeq \bigoplus_q R^q f_*(S_X(E, h) \otimes F)[-q] \in D(Y).$$

As a consequence, the spectral sequence

$$E_2^{pq} : H^p(Y, R^q f_*(S_X(E, h) \otimes F)) \Rightarrow H^{p+q}(X, S_X(E, h) \otimes F)$$

degenerates at the $E_2$ page.

When $(H, \theta, h)$ is a harmonic bundle associated to an $\mathbb{R}$-polarized variation of Hodge structure $\nabla := (\nabla, F, \nabla^*, Q)$, $E = S(\nabla)$ and $F = \partial_X$, Theorem 1.1 implies Kollár’s conjecture. In this case, our construction of $S_X(\nabla)$ coincides with Saito’s.

The present paper is organized as follows. In Sect. 2, we introduce the construction of the adjoint $L^2$-extension $S_X(E, h)$ of a hermitian bundle $(E, h)$ which generalizes Saito’s $S$-sheaf. Some fundamental properties of $S_X(E, h)$ are proved and its $L^2$-Dolbeault resolution is established. After that, we introduce a harmonic representation of the derived pushforwards of $S_X(E, h)$ in Sect. 3 by generalizing Takegoshi’s work [25] to complex spaces. In Sect. 4, we prove an abstract Kollár package and illustrate the observation that: Kollár’s conjecture is a consequence of the Nakano semi-positivity of the top Hodge bundle. We prove Theorem 1.1 in Sect. 5 as an application.

**Notation:**

- Let $X$ be a complex space. A Zariski closed subset (= closed analytic subset) $Z \subset X$ is a closed subset which is locally defined as the zeros of a set of holomorphic functions. A subset $Y \subset X$ is called Zariski open if $X \setminus Y$ is Zariski closed.
- Let $\alpha$ and $\beta$ be functions, metrics or $(1, 1)$-forms. We denote $\alpha \lesssim \beta$ if $\alpha \leq C \beta$ for some $C \in \mathbb{R}_{>0}$. We say that $\alpha$ and $\beta$ are quasi-isometric if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$. We denote it by $\alpha \sim \beta$.
- A plurisubharmonic (resp. strictly plurisubharmonic) function on a complex space $X$ is a function $\lambda : X \to (-\infty, \infty)$ such that, locally at every point $x \in X$, there is a neighborhood $U$ of $x$, a closed immersion $\iota : U \to \Omega$ into a complex manifold $\Omega$ and a plurisubharmonic (resp. strictly plurisubharmonic) function $\Lambda$ on $\Omega$ such that $\iota^* \Lambda = \lambda$.
- By a $C^\infty$ form on a complex space $X$ we mean a $C^\infty$ form $\alpha$ on $X_{\text{reg}}$ so that locally at every point $x \in X$ there is an open neighborhood $U$ of $x$, a closed immersion $\iota : U \to \Omega$ into a holomorphic manifold $\Omega$ and $\beta \in C^\infty(\Omega)$ such that $\iota^* \beta = \alpha$ on $U \cap X_{\text{reg}}$.  

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2 $L^2$-extension and its $L^2$-Dolbeault resolution

2.1 $L^2$-Dolbeault cohomology and $L^2$-Dolbeault complex

Let $(Y, ds^2)$ be a hermitian manifold of dimension $n$ and $(E, h)$ a hermitian vector bundle on $Y$. Let $L_{(2)}^{p,q}(Y, E; ds^2, h)$ be the space of measurable $E$-valued $(p, q)$-forms on $Y$ which are square integrable with respect to the metrics $ds^2$ and $h$. Denote $\partial_{\text{max}}$ to be the maximal extension of the $\partial$ operator defined on the domains

$$
D_{\text{max}}^{p,q}(Y, E; ds^2, h) := \text{Dom}^{p,q}(\partial_{\text{max}}) = \{\phi \in L_{(2)}^{p,q}(Y, E; ds^2, h) | \partial \phi \in L_{(2)}^{p,q+1}(Y, E; ds^2, h)\}.
$$

Here $\partial$ is defined in the sense of distribution. The $L^2$ cohomology $H_{(2),\text{max}}^{p,\bullet}(Y, E; ds^2, h)$ is defined as the cohomology of the complex

$$
D_{\text{max}}^{p,\bullet}(Y, E; ds^2, h) := D_{\text{max}}^{p,0}(Y, E; ds^2, h) \xrightarrow{\partial_{\text{max}}} \cdots \xrightarrow{\partial_{\text{max}}} D_{\text{max}}^{p,n}(Y, E; ds^2, h).
$$

Let $X$ be a complex space and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset of the regular locus $X_{\text{reg}}$. Let $ds^2$ be a hermitian metric on $X^o$ and $(E, h)$ a hermitian vector bundle on $X^o$. Let $U \subset X$ be an open subset. Define $L_{X,ds^2}^{p,q}(E, h)(U)$ to be the space of measurable $E$-valued $(p, q)$-forms $\alpha$ on $U \cap X^o$ such that for every point $x \in U$, there is a neighborhood $V_x$ of $x$ so that

$$
\int_{V_x \cap X^o} |\alpha|_{ds^2,h}^2 \text{vol}_{ds^2} < \infty.
$$

For each $0 \leq p, q \leq n$, we define a sheaf $\mathcal{D}_{X,ds^2}^{p,q}(E, h)$ on $X$ by

$$
\mathcal{D}_{X,ds^2}^{p,q}(E, h)(U) := \{\phi \in L_{X,ds^2}^{p,q}(E, h)(U) | \partial_{\text{max}} \phi \in L_{X,ds^2}^{p,q+1}(E, h)(U)\}
$$

for every open subset $U \subset X$.

Define the $L^2$-Dolbeault complex of sheaves $\mathcal{D}_{X,ds^2}^{p,\bullet}(E, h)$ as

$$
\mathcal{D}_{X,ds^2}^{p,0}(E, h) \xrightarrow{\partial} \mathcal{D}_{X,ds^2}^{p,1}(E, h) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{D}_{X,ds^2}^{p,n}(E, h)
$$

where $\partial$ is defined in the sense of distribution.

**Definition 2.1** Let $X$ be a complex space and $ds^2$ a hermitian metric on $X_{\text{reg}}$. We say that $ds^2$ is a hermitian metric on $X$ if, for every $x \in X$ there is a neighborhood $U$ of $x$ and a holomorphic closed immersion $U \subset V$ into a complex manifold $V$ such that $ds^2|_U \sim ds^2_V|_U$ for some hermitian metric $ds^2_V$ on $V$. If $ds^2|_{X_{\text{reg}}}$ is moreover Kähler, we say that $ds^2$ is a Kähler hermitian metric on $X$. 

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Lemma 2.2 Let \( X \) be a complex space and \( X^o \subset X_{\text{reg}} \) a dense Zariski open subset. Let \( ds^2 \) be a hermitian metric on \( X^o \) and \((E, h)\) a hermitian vector bundle on \( X^o \). Suppose that for every point \( x \in X \setminus X^o \) there is a neighborhood \( U_x \) of \( x \) and a hermitian metric \( ds^2_0 \) on \( U_x \) such that \( ds^2_0|_{X^o \cap U_x} \lesssim ds^2|_{X^o \cap U_x} \). Then \( \mathcal{D}_{X, ds^2}(E, h) \) is a fine sheaf for every \( p \) and \( q \).

Proof It suffices to show that for every \( W \subset \overline{W} \subset U \subset X \) where \( W \) and \( U \) are small open subsets, there is a positive continuous function \( f \) on \( U \) such that

- \( \text{supp}(f) \subset \overline{W} \),
- \( f \) is \( C^\infty \) on \( U \cap X^o \),
- \( \bar{\partial} f \) has bounded fiberwise norm, with respect to the metric \( ds^2 \).

Choose a closed embedding \( U \subset M \) where \( M \) is a smooth complex manifold \( M \). Let \( V \subset \overline{V} \subset M \) where \( V \) is an open subset such that \( V \cap U = W \). Let \( ds^2_M \) be a hermitian metric on \( M \) so that \( ds^2_0|_{U \cap X^o} \sim ds^2_M|_{U \cap X^o} \). Let \( g \) be a positive smooth function on \( M \) whose support lies in \( \overline{V} \). Denote \( f = g|_U \), then apparently \( \text{supp}(f) \subset \overline{W} \) and \( f \) is \( C^\infty \) on \( U \cap X^o \). It suffices to show the boundedness of the fiberwise norm of \( \bar{\partial} f \). Since \( U \cap X^o \subset M \) is a submanifold, one has the orthogonal decomposition

\[
T_{M,x} = T_{U \cap X^o,x} \oplus T_{U \cap X^o,x}^\perp, \quad \forall x \in U \cap X^o.
\]

Therefore \( |\bar{\partial} f|_{ds^2} \lesssim |\bar{\partial} f|_{ds^2_0} \leq |\bar{\partial} g|_{ds^2_M} < \infty \). The lemma is proved. \( \square \)

2.2 \( S_X(E, h) \) and its basic properties

Let \( X \) be a complex space of dimension \( n \) and \( X^o \subset X_{\text{reg}} \) a dense Zariski open subset. Let \( ds^2 \) be a hermitian metric on \( X^o \) and \((E, h)\) a hermitian vector bundle on \( X^o \).

Definition 2.3 Define

\[
S_X(E, h) := \text{Ker} \left( \mathcal{D}_{X, ds^2}^{n,0}(E, h) \xrightarrow{\bar{\partial}} \mathcal{D}_{X, ds^2}^{n,1}(E, h) \right).
\]

The following proposition shows that \( S_X(E, h) \) is independent of the choice of \( ds^2 \). Hence \( ds^2 \) is omitted in the notation \( S_X(E, h) \).

Proposition 2.4 \( S_X(E, h) \) is independent of the choice of \( ds^2 \).

Proof Let \( \pi : \tilde{X} \to X \) be a desingularization of \( X \setminus X^o \) so that \( \tilde{X} \) is smooth and \( \pi \) is biholomorphic over \( X^o \). Let \( ds^2_{\tilde{X}} \) be an arbitrary hermitian metric on \( \tilde{X} \). Since \( \pi \) is a proper map, a section of \( K_{X^o} \otimes E \) is locally square integrable at \( x \in X \) if and only if it is locally square integrable near \( \pi^{-1}\{x\} \). Thus it suffices to show that

\[
\text{Ker} \left( \mathcal{D}_{\tilde{X}, \pi^* ds^2}^{n,0}(E, h) \xrightarrow{\bar{\partial}} \mathcal{D}_{\tilde{X}, \pi^* ds^2}^{n,1}(E, h) \right) = \text{Ker} \left( \mathcal{D}_{\tilde{X}, ds^2_{\tilde{X}}}^{n,0}(E, h) \xrightarrow{\bar{\partial}} \mathcal{D}_{\tilde{X}, ds^2_{\tilde{X}}}^{n,1}(E, h) \right).
\]

(2.1)

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Since the problem is local, we assume that there is an orthogonal frame of cotangent fields $\delta_1, \ldots, \delta_n$ such that
\[
\pi^* ds^2 \sim \lambda_1 \delta_1 \delta_1 + \cdots + \lambda_n \delta_n \delta_n
\] (2.2)
and
\[
dx^2 \sim \delta_1 \delta_1 + \cdots + \delta_n \delta_n.
\] (2.3)
Let $s = \delta_1 \wedge \cdots \wedge \delta_n \otimes \xi$. By (2.2) and (2.3) we obtain
\[
\int |s|^2_{\pi^* ds^2, h} \text{vol}_{\pi^* ds^2} = \int |\delta_1 \wedge \cdots \wedge \delta_n \otimes \xi|^2_{\pi^* ds^2, h} \prod_{i=1}^n \lambda_i \delta_i \wedge \delta_i
\]
\[
= \int |\xi|^2 h \prod_{i=1}^n \delta_i \wedge \delta_i
\]
\[
= \int |s|^2_{dx^2, h} \text{vol}_{dx^2}.
\]
Therefore $\int |s|^2_{\pi^* ds^2, h} \text{vol}_{\pi^* ds^2}$ is locally finite if and only if $\int |s|^2_{dx^2, h} \text{vol}_{dx^2}$ is locally finite. This proves (2.1).

**Proposition 2.5** (Functoriality property). Let $\pi : X' \to X$ be a proper holomorphic map between complex spaces which is biholomorphic over $X^\circ$. Then
\[
\pi_* S_{X'}(\pi^* E, \pi^* h) = S_X(E, h).
\]

**Proof** It follows from Proposition 2.4 that
\[
S_{X'}(\pi^* E, \pi^* h) = \text{Ker} \left( \mathcal{D}_{X', \pi^* ds^2}^{\mathbb{R}, 0}(\pi^* E, \pi^* h) \to \mathcal{D}_{X', \pi^* ds^2}^{\mathbb{R}, 1}(\pi^* E, \pi^* h) \right)
\]
and
\[
S_X(E, h) = \text{Ker} \left( \mathcal{D}_{X, ds^2}^{\mathbb{R}, 0}(E, h) \to \mathcal{D}_{X, ds^2}^{\mathbb{R}, 1}(E, h) \right).
\]
Since $\pi$ is a proper map, a section of $E$ is locally square integrable at $x \in X$ if and only if it is square integrable over some neighborhood of $\pi^{-1}\{x\}$. This proves the lemma.

The following simple lemma is convenient for applications. It allows us to shrink the domain of $(E, h)$ without changing $S_X(E, h)$. This phenomenon also appears in the class of Saito’s $S$-sheaves.
Lemma 2.6 Let $U \subset X^o$ be a dense Zariski open subset. Then

$$S_X(E, h) \simeq S_X(E\lfloor_U, h\lfloor_U).$$

Proof This is a consequence of the fact that if a locally $L^2$ function on a hermitian manifold is $\bar{\partial}$-closed away from a Zariski open subset, then it is $\bar{\partial}$-closed over the whole manifold [3, Lemma 1.3]. □

Lemma 2.7 Let $(F, h_F)$ be a hermitian vector bundle on $X$. Then

$$S_X(E, h) \otimes F \simeq S_X(E \otimes F, h \otimes h_F).$$

Proof Let $x \in X$ be a point and let $U$ be an open neighborhood of $x$ so that $F\lfloor_U \simeq O_X \oplus r_{U}^r$ and $h_F$ is quasi-isometric to the trivial metric, i.e.,

$$\sum_{i=1}^{r} a_i e_i^2_{h_F} \sim \sum_{i=1}^{r} |a_i|^2$$

where $\{e_1, \ldots, e_r\}$ is the standard frame of $O_X^{\oplus r}$ and $a_i$S are measurable functions on $U \cap X^o$. Let $ds^2$ be an arbitrary hermitian metric on $X^o$ and let $\alpha = \sum_{i=1}^{r} \alpha_i \otimes e_i$ be a measurable section of $(K_{X^o} \otimes E \otimes F)|_{U \cap X^o}$. Then

$$|\alpha|_2^2_{ds^2, h \otimes h_F} \sim \sum_{i=1}^{r} |\alpha_i|^2_{ds^2, h}$$

is finite if and only if $|\alpha_i|_{ds^2, h}$ is finite for each $i = 1, \ldots, r$. This proves the lemma. □

At the end of this subsection, we show that $S_X(E, h)$ is a coherent sheaf if the dual metric $h^*$ has at most polynomially growth at the boundary $X \setminus X^o$ and $(E, h)$ is Nakano semi-positive. Before that let us briefly recall the definition of Nakano semipositivity. Let $\Theta \in A^{1,1}(X^o, \text{End}(E))$ be a real form. Assume locally that

$$\Theta = \sqrt{-1} \sum_{i,j} \omega_{i,j} e_i \otimes e_j^*$$

where $\omega_{i,j} \in A^{1,1}_{X^o}, (e_1, \ldots, e_r) \in E$ is an orthogonal local frame of $E$ and $(e_1^*, \ldots, e_r^*) \in E^*$ is the dual frame. We say that $\Theta$ is Nakano semi-positive, denoting $\Theta \geq 0$, if the bilinear form

$$\theta(u_1, u_2) := \sum_{i,j} \omega_{i,j} (u_{1i}, \overline{u_{2j}}), \quad u_l = \sum_i u_{li} \otimes e_i \in T_{X^o} \otimes E, \quad l = 1, 2$$

is semi-positive definite. Let $\Theta_1, \Theta_2 \in A^{1,1}(X^o, \text{End}(E))$ be two real forms. We denote $\Theta_1 \geq \Theta_2$ if $\Theta_1 - \Theta_2 \geq 0$. $(E, h)$ is Nakano semi-positive if $\sqrt{-1} \Theta_h(E) \geq 0$. □

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Definition 2.8 Let $X$ be a complex space and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset. A hermitian vector bundle $(E, h)$ on $X^o$ is called tame on $X$ if, for every point $x \in X$, there is an open neighborhood $U$ of $x$, a proper bimeromorphic morphism $\pi : \tilde{U} \to U$ which is biholomorphic over $U \cap X^o$, and a hermitian vector bundle $(Q, h_Q)$ on $\tilde{U}$ such that

1. $\pi^*E|_{\pi^{-1}(X^o \cap U)} \subset Q|_{\pi^{-1}(X^o \cap U)}$ as a subsheaf.
2. There is a hermitian metric $h'_Q$ on $Q|_{\pi^{-1}(X^o \cap U)}$ so that $h'_Q|_{\pi^{-1}(X^o \cap U)} \sim \pi^*h$ on $\pi^{-1}(X^o \cap U)$ and

$$\left( \sum_{i=1}^r \|\pi^*f_i\|^2 \right)^c h_Q \lesssim h'_Q$$

for some $c \in \mathbb{R}$. Here $\{f_1, \ldots, f_r\}$ is an arbitrary set of local generators of the ideal sheaf defining $\tilde{U} \setminus \pi^{-1}(X^o) \subset \tilde{U}$.

The tameness condition (2.4) is independent of the choice of the set of local generators. In the present paper, a tame hermitian vector bundle $(E, h)$ is constructed as a subsheaf of a tame harmonic bundle (see Sect. 5.2, especially Proposition 5.5) in the sense of Simpson [23] and Mochizuki [10, 11]. In this case, Condition (2.4) comes from the tameness condition of the harmonic bundle. This is the origin of the name “tame hermitian vector bundle”.

Proposition 2.9 $S_X(E, h)$ is a coherent sheaf if $(E, h)$ is Nakano semi-positive and tame on $X$.

Proof Since the problem is local, we assume that $X$ is a germ of complex space. Let $\pi : \tilde{X} \to X$ be a desingularization so that $\pi$ is biholomorphic over $X^o$ and $D := \pi^{-1}(X \setminus X^o)$ is a simple normal crossing divisor. By abuse of notations we regard $X^o \subset \tilde{X}$ as a subset. Since $(E, h)$ is tame, we assume that there is a hermitian vector bundle $(Q, h_Q)$ on $\tilde{X}$ such that $E$ is a subsheaf of $Q|_{X^o}$ and there exists $m \in \mathbb{N}$ such that

$$|z_1 \ldots z_r|^{2m} h_Q|_E \lesssim h$$

where $z_1, \ldots, z_n$ are local coordinates on $\tilde{X}$ so that $D = \{z_1 \ldots z_r = 0\}$. There is moreover a hermitian metric $h'_Q$ on $Q|_{X^o}$ so that $h'_Q|_E \sim h$. By Proposition 2.5 there is an isomorphism

$$S_X(E, h) \simeq \pi_* \left( S_{\tilde{X}}(E, h) \right).$$

Since $\pi$ is a proper map, it suffices to show that $S_{\tilde{X}}(E, h)$ is a coherent sheaf on $\tilde{X}$. Since the problem is local and $\tilde{X}$ is smooth, we may assume that $\tilde{X} \subset \mathbb{C}^n$ is the unit ball so that $D = \{z_1 \ldots z_r = 0\}$. Without loss of generality we assume that $Q$ admits
a global holomorphic frame \( \{ e_1, \ldots, e_l \} \) which is orthonormal with respect to \( h_Q \), i.e.,

\[
(e_i, e_j)_{h_Q} = \begin{cases} 
1, & i = j \\
0, & i \neq j 
\end{cases} \quad (2.6)
\]

There is a complete Kähler metric on \( X^o \) by Lemma 2.14. Since \( Q \) is coherent\(^1\), the space \( \Gamma(\tilde{X}, S_{\tilde{X}}(E, h)) \) generates a coherent subsheaf \( \mathcal{J} \) of \( Q \) by strong Noetherian property for coherent sheaves. We have the inclusion \( \mathcal{J} \subset S_{\tilde{X}}(E, h) \) by the construction. It remains to prove the converse. By Krull’s theorem [2, Corollary 10.19], it suffices to show that

\[
\mathcal{J} \oplus S_{\tilde{X}}(E, h) \cap m_{\tilde{X},\tilde{x}}^{k+1} Q = S_{\tilde{X}}(E, h)_x, \quad \forall k \geq 0, \quad \forall x \in \tilde{X}. \quad (2.7)
\]

Let \( \alpha \in S_{\tilde{X}}(E, h)_x \) be defined in a precompact neighborhood \( V \) of \( x \). Choose a \( C^{\infty} \) cut-off function \( \lambda \) so that \( \lambda \equiv 1 \) near \( x \) and \( \text{supp} \lambda \subset V \). Denote \( |z|^2 := \sum_{i=1}^n |z_i|^2 \).

Let

\[
\psi_k(z) := 2(n + k + rm) \log |z - x| + |z|^2
\]

and \( h_{\psi_k} = e^{-\psi_k} h \). Denote \( \omega_0 := \sqrt{-1} \partial \bar{\partial} |z|^2 \). Then

\[
\sqrt{-1} \Theta_{h_{\psi_k}}(E) = \sqrt{-1} \partial \bar{\partial} \psi_k + \sqrt{-1} \Theta_h(E) \geq \omega_0.
\]

Since \( \text{supp} \lambda \alpha \subset V \) and \( \bar{\partial}(\lambda \alpha) = 0 \) near \( x \), we know that

\[
\| \bar{\partial}(\lambda \alpha) \|_{\omega_0, h_{\psi_k}} \sim \| \bar{\partial}(\lambda \alpha) \|_{\omega_0, h} \leq \| \bar{\partial} \lambda \|_{L^\infty} \| \alpha \|_{\omega_0, h} + |\lambda|^2 \| \bar{\partial} \alpha \|_{\omega_0, h} < \infty
\]

Hence [4, Theorem 5.1] gives a solution to the equation \( \bar{\partial} \beta = \bar{\partial}(\lambda \alpha) \) so that

\[
\| \beta \|_{\omega_0, h} \lesssim \int_{X^o} |\beta|_{\omega_0, h} |z - x|^{-2(n + k + rm)} \text{vol}_{\omega_0} \lesssim \| \bar{\partial}(\lambda \alpha) \|_{\omega_0, h_{\psi_k}} < \infty. \quad (2.8)
\]

Thus \( \gamma = \beta - \lambda \alpha \) is holomorphic and \( \gamma \in \Gamma(\tilde{X}, S_{\tilde{X}}(E, h)) \).

Assume that

\[
\beta = \sum_{i=1}^l f_i e_i dz_1 \wedge \cdots \wedge dz_n
\]

\(^{1}\) We do not distinguish a holomorphic vector bundle with the sheaf of its holomorphic sections.
for some holomorphic functions \( f_1, \ldots, f_l \in \mathcal{O}_\tilde{X}(X^o) \). By (2.5), (2.6) and (2.8) we obtain that

\[
\sum_{i=1}^l \int_{X^o} |f_i|^2 |z_1^m z_r^m| z - x|^{-2(n+k+rm)} \text{vol}_{\omega_0} \\
= \int_{X^o} |\beta|^2 _{\omega_0, \eta_Q} |z_1^m z_r^m| z - x|^{-2(n+k+rm)} \text{vol}_{\omega_0} \\
\lesssim \int_{X^o} |\beta|^2 _{\omega_0, \eta} |z - x|^{-2(n+k+rm)} \text{vol}_{\omega_0} < \infty.
\]

This implies that \( z_1^m z_r^m f_i \in m_{\tilde{X}, x}^{k+1+rm} \) for every \( i = 1, \ldots, l \) [5, Lemma 5.6]. Therefore \( \beta_x \in m_{\tilde{X}, x}^{k+1+Q} \) and we prove (2.7).

\[\square\]

### 2.3 \( L^2 \)-Dolbeault resolution of \( S_X(E, h) \)

In this subsection we introduce an \( L^2 \)-Dolbeault resolution of \( S_X(E, h) \). First we recall the following estimate.

**Theorem 2.10** [4, Theorem 5.1] Let \( Y \) be a complex manifold of dimension \( n \) which admits a complete Kähler metric. Let \((E, h)\) be a hermitian vector bundle such that

\[
\sqrt{-1} \Theta_h(E) \geq \omega \otimes \text{Id}_E
\]

for some (not necessarily complete) Kähler form \( \omega \) on \( Y \). Then for every \( q > 0 \) and every \( \alpha \in L^{n,q}_{(2)}(Y, E; \omega, h) \) such that \( \bar{\partial}\alpha = 0 \), there is \( \beta \in L^{r,n,q-1}_{(2)}(Y, E; \omega, h) \) such that \( \bar{\partial}\beta = \alpha \) and \( \|\beta\|^2 _{\omega, h} \leq q^{-1} \|\alpha\|^2 _{\omega, h} \).

The main theorem of this section is the following.

**Theorem 2.11** Let \( X \) be a complex space of dimension \( n \) and \( ds^2 \) a hermitian metric on a dense Zariski open subset \( X^o \subset X_{\text{reg}} \) with \( \omega \) its fundamental form. Let \((E, h)\) be a Nakano semi-positive hermitian vector bundle. Assume that, locally at every point \( x \in X \), there is a neighborhood \( U \) of \( x \), a strictly plurisubharmonic function \( \lambda \in C^2(U) \) and a bounded plurisubharmonic function \( \Phi \in C^2(U \cap X^o) \) such that

\[
\sqrt{-1} \partial\bar{\partial}\lambda|_{U \cap X^o} \lesssim \omega|_{U \cap X^o} \lesssim \sqrt{-1} \partial\bar{\partial}\Phi.
\]

Then the canonical map

\[
S_X(E, h) \to \mathcal{D}_{X, ds^2}(E, h)
\]

is a quasi-isomorphism. As a consequence, there is a canonical isomorphism

\[
H^q(X, S_X(E, h)) \simeq H^{n,q}_{(2)}(X^o, E; ds^2, h), \quad \forall q \geq 0
\]

if \( X \) is compact.
Remark 2.12 \((E, h)\) is not required to be tame in Theorem 2.11. So \(S_X(E, h)\) may not be a coherent sheaf.

Proof It suffices to show that the \(L^2\)-Dolbeault complex \(D_{X, ds^2}^{n,q}(E, h)\) is exact at \(D_{X, ds^2}^{n,q}(E, h)\) for every \(q > 0\). Since the problem is local, we consider a point \(x \in X\) and a small open neighborhood \(U\) of \(x\) so that there is \(C > 0\) and a bounded plurisubharmonic function \(\Phi \in C^2(U \cap X^o)\) such that \(C \sqrt{-1} \partial \bar{\partial} \Phi \geq \omega|_{U \cap X^o}\). Let \(h' = e^{-C \Phi} h\). Since \(\Phi\) is bounded and \((E, h)\) is Nakano semi-positive, we have \(h' \sim h\) and

\[
\sqrt{-1}\Theta_{h'}(E|_{U \cap X^o}) = C \sqrt{-1}\partial \bar{\partial} \Phi \otimes \text{Id}_E|_{U \cap X^o} + \sqrt{-1}\Theta_h(E|_{U \cap X^o}) \geq \omega \otimes \text{Id}_E|_{U \cap X^o}.
\]

By Lemma 2.14 below, we may assume that \(U \cap X^o\) admits a complete Kähler metric. By Theorem 2.10, we therefore have

\[
H_{(2)}^{n,q}(U \cap X^o, E|_{U \cap X^o}; ds^2, h) = H_{(2)}^{n,q}(U \cap X^o, E|_{U \cap X^o}; ds^2, h') = 0, \quad \forall q > 0.
\]

This proves the exactness of \(D_{X, ds^2}^{n,q}(E, h)\) at \(D_{X, ds^2}^{n,q}(E, h)\) for each \(q > 0\). Since \(\omega\) is locally bounded from below by a hermitian metric, it follows from Lemma 2.2 that \(D_{X, ds^2}^{n,q}(E, h)\) is a fine sheaf for every \(q\). The second claim therefore follows from the compactness of \(X\).

In order to apply Theorem 2.11, we introduce a type of hermitian metrics that is crucial for the present paper.

Definition 2.13 Let \(X\) be a complex space and \(X^o \subset X_{reg}\) a dense Zariski open subset. Let \(ds^2\) be a hermitian metric on \(X^o\).

1. \(ds^2\) is called to admit a \((\infty, 1)\) bounded potential locally on \(X\) if, for every point \(x \in X\) there is a neighborhood \(U\) of \(x\), a function \(\Phi \in C^\infty(U \cap X^o)\) such that \(|\Phi| + |d\Phi|_{ds^2} < \infty\) and \(ds^2|_{U \cap X^o} \sim \sqrt{-1} \partial \bar{\partial} \Phi\).
2. \(ds^2\) is called locally complete on \(X\) if, for every point \(x \in X\) there is a neighborhood \(U\) of \(x\) such that \((U \cap X^o, ds^2)\) is complete.
3. \(ds^2\) is locally bounded from below by a hermitian metric if, for every point \(x \in X\) there is a neighborhood \(U\) of \(x\) and a hermitian metric \(ds^2_0\) on \(U\) such that \(ds^2_0|_{U} \lesssim ds^2|_{U}\).

Lemma 2.14 Let \(X\) be a weakly pseudoconvex Kähler space with \(\psi\) a smooth exhausted plurisubharmonic function on \(X\). Denote \(X_c := \{x \in X| \psi(x) < c\}\) for each \(c \in \mathbb{R}\). Let \(X^o \subset X\) be a dense Zariski open subset. Then, for every \(c \in \mathbb{R}\), there exists a complete Kähler metric \(ds^2\) on \(X_c \cap X^o\) such that

1. \(ds^2_0 \lesssim ds^2\) for some hermitian metric \(ds^2_0\) on \(X_c\);
2. \(ds^2\) admits a \((\infty, 1)\) bounded potential locally on \(X_c\).
Proof Let $U$ be a neighborhood of a point $x \in X \setminus X^o$. Assume that $U \setminus X^o \subset U$ is defined by the common zeros of $f_1, \ldots, f_r \in \mathcal{O}_U(U)$. Let

$$
\varphi_U := \frac{1}{\log(- \log \sum_{i=1}^r \| f_i \|^2)} + \phi_U,
$$

where $\phi_U$ is a strictly $C^\infty$ plurisubharmonic function on $U$ so that $\varphi_U$ is strictly plurisubharmonic. Then the quasi-isometric class of $\sqrt{-1} \partial \bar{\partial} \varphi_U$ is independent of the choice of $\{ f_1, \ldots, f_r \}$ and $\phi_U$. By partition of unity the potential functions $\phi_U$ can be glued to a global function $\varphi$ on $X$ so that

$$
\sqrt{-1} \partial \bar{\partial} \varphi_U \sim \sqrt{-1} \partial \bar{\partial} \varphi
$$

near every point $x \in X \setminus X^o$ and $\varphi \equiv 0$ away from a neighborhood $V$ of $X \setminus X^o$.

Denoting $u = - \log \sum_{i=1}^r | f_i |^2$, we assume that $u > e$ on $V$ after a possible shrinking of $V$. Then

$$
\sqrt{-1} \partial \bar{\partial} \varphi|_U \sim \sqrt{-1} \frac{2 + \log u}{u^2 \log^3 u} \partial u \wedge \bar{\partial} u + \sqrt{-1} \frac{\partial \bar{\partial} u}{u \log^2 u} \sim \sqrt{-1} \frac{\partial \bar{\partial} u}{u^2 \log^2 u} + \sqrt{-1} \partial \bar{\partial} \varphi_U.
$$

Hence

$$
|\varphi| + |d \varphi| \sqrt{-1} \partial \bar{\partial} \varphi \lesssim \frac{1}{\log u} < 1.
$$

Since $\log \log u$ is a smooth exhausted function such that

$$
|d \log \log u| \sqrt{-1} \partial \bar{\partial} \varphi \leq 2,
$$

$\sqrt{-1} \partial \bar{\partial} \varphi$ is locally complete near $X \setminus X^o$ by the Hopf–Rinow theorem.

Let $c \in \mathbb{R}$ and let $\omega_0$ be a Kähler hermitian metric on $X$. By adding a constant to $\psi$, we assume $\psi \geq 0$. Then $\psi_c := \psi + \frac{1}{c - \psi}$ is a smooth exhausted plurisubharmonic function on $X_c = \{ x \in X | \psi(x) < c \}$. Hence $\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_c^2$ is a complete Kähler hermitian metric on $X_c$ [4, Theorem 1.3].

Since $X_c$ is compact,

$$
\sqrt{-1} \partial \bar{\partial} \varphi + K(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_c^2), \quad K \gg 0
$$

is positive definite and it gives the desired complete Kähler metric on $X_c \cap X^o$. \qed
3 Harmonic representation of the derived pushforwards

**Definition 3.1** [25, Definition 6.1] A morphism $f : X \to Y$ between complex spaces is called locally Kähler if $f^{-1}U$ is a Kähler space for any relatively compact open subset $U \subset Y$.

Let $f : X \to Y$ be a proper locally Kähler morphism between complex spaces and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset. Assume that every irreducible component of $X$ is mapped onto $Y$. Let $(E, h)$ be a hermitian vector bundle on $X^o$ with Nakano semi-positive curvature. The aim of this section is to introduce a harmonic representation of $R^q f_* S_X(E, h)$. The results of this section are the generalizations of [25, Sect. 4] to $L^2$-Dolbeault complexes on complex spaces.

### 3.1 Harmonic forms on weakly 1-complete spaces

Let $Z$ be a complex space of pure dimension $n$ and $Z^o \subset Z_{\text{reg}}$ a dense Zariski open subset. Let $ds^2$ be a locally complete Kähler metric on $Z^o$ which admits a $(\infty, 1)$ bounded potential locally on $Z$ (Definition 2.13). Denote $\omega$ to be the associated Kähler form. Let $(E, h)$ be a hermitian vector bundle on $Z^o$. Let $A^{p,q}(Z^o, E)$ (resp. $A^k(Z^o, E)$) be the space of $C^\infty E$-valued $(p, q)$ (resp. $k$) forms on $Z^o$ and let $A^{p,q}_{\text{cpt}}(Z^o, E) \subset A^{p,q}(Z^o, E)$ be the subspace of forms with compact support in $Z^o$. Let $*: A^{p,q}(Z^o, E) \to A^{n-q,n-p}(Z^o, E)$ be the Hodge star operator relative to $ds^2$. Since $h$ is a smooth section of $\text{Hom}(E, \bar{E}^*)$, we define an anti-isomorphism $\sharp_E : A^{p,q}(Z^o, E) \to A^{q-p,n-p}(Z^o, E^*)$ by $\sharp_E u = h(u)$. Denote by $\langle -, - \rangle_h$ the pointwise inner product on $A^{p,q}(Z^o, E)$. These operators are related by

$$\langle \alpha, \beta \rangle_h \text{vol}_{ds^2} = \alpha \wedge \sharp_E \beta. \quad (3.1)$$

Denote

$$\langle \alpha, \beta \rangle_h := \int_{Z^o} \langle \alpha, \beta \rangle_h \text{vol}_{ds^2} \quad (3.2)$$

and $\|\alpha\|_h := \sqrt{\langle \alpha, \alpha \rangle_h}$. Let $\nabla = D' + \bar{\partial}$ be the Chern connection associated to $h$. Let $\bar{\partial}_h^* = - * D'^*\text{ and } D_h^* = - * \bar{\partial}^*$ be the formal adjoints of $\bar{\partial}$ and $D'$ respectively. Denote by $\Theta_h \in A^{1,1}(Z^o, \text{End}(E))$ the Chern curvature form of $(E, h)$.

For two operators $S$ and $T$ acting on $A^\bullet(Z^o, E)$ with degree $a$ and $b$ respectively, we define the graded Lie bracket $[S, T] := S \circ T - (-1)^{ab} T \circ S$.

Denote by $L$ the Lefschetz operator with respect to $ds^2$ and denote $\Lambda$ to be the formal adjoint of $L$. Letting $\alpha$ be a form, denote $e(\alpha)$ to be the operator defined by $e(\alpha)(\beta) = \alpha \wedge \beta$. Then we have the following Kähler identities [26, Chapter V]:

$$D_h^* = -\sqrt{-1}[\bar{\partial}, \Lambda], \quad (3.3)$$

$$\bar{\partial}_h^* = \sqrt{-1}[D', \Lambda] \quad \text{and} \quad (3.4)$$
\[ e(\theta)^* = \sqrt{-1}[e(\bar{\theta}), \Lambda], \quad e(\bar{\theta})^* = -\sqrt{-1}[e(\theta), \Lambda] \]  

(3.5)

for \( \theta \in A^{1,0}(Z^o, E) \).

Denote \( \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}_h^* + \bar{\partial}_h^*\partial \) and \( \Delta_{D'} = D'D_h^* + D_h^*D' \). By (3.3), (3.4) and Jacobi’s identity

\[
[[S, T], U] + (-1)^{a(b+c)}[[T, U], S] + (-1)^{c(a+b)}[[U, S], T] = 0
\]

where \( a = \deg S, b = \deg T \) and \( c = \deg U \) respectively, we obtain that the formula

\[
\Delta_{\bar{\partial}} = \Delta_{D'} + \sqrt{-1}[e(\Theta_h), \Lambda]
\]

(3.6)

holds on \( Z^o \).

Let \( \varphi : Z^o \to \mathbb{R} \) be a \( C^\infty \) function and let \( h_\varphi := e^{-\varphi}h \). Since \( D'_{h_\varphi} = D' - e(\partial \varphi) \), we obtain the formula:

\[
\bar{\partial}_{h_\varphi}^* = \bar{\partial}_h^* + e(\bar{\partial} \varphi)^*
\]

(3.7)

and

\[
\Delta_{\bar{\partial}_{h_\varphi}} = \Delta_{D'_{h_\varphi}} + \sqrt{-1}[e(\Theta_h + \partial \bar{\partial} \varphi), \Lambda]
\]

(3.8)

where \( \Delta_{\bar{\partial}_{h_\varphi}} := [\bar{\partial}, \bar{\partial}_{h_\varphi}^*] \) and \( \Delta_{D'_{h_\varphi}} := [D'_{h_\varphi}, D'_{h_\varphi}^*] \).

By (3.3), (3.4), (3.5) and Jacobi’s identity, the Donnelly and Xavier’s formula [25, (1.9), (1.10)] can be stated as follows.

\[
[\bar{\partial}, e(\bar{\partial} \varphi)^*] + [D'_{h_\varphi}^*, e(\partial \varphi)] = \sqrt{-1}[e(\partial \bar{\partial} \varphi), \Lambda]
\]

(3.9)

and

\[
[D', e(\partial \varphi)^*] + [\bar{\partial}_{h_\varphi}^*, e(\bar{\partial} \varphi)] = \sqrt{-1}[e(\partial \bar{\partial} \varphi), \Lambda])].
\]

(3.10)

Fix a Whitney stratification of \( Z \) so that \( Z^o \) is the union of open strata. By Sard’s theorem there is a subset \( \Sigma \subset \mathbb{R} \) of measure zero so that for every \( c \in \mathbb{R} \setminus \Sigma \) and every stratum \( S \) of \( Z \), \( c \) is a regular value of \( \varphi | S \). Any \( c \in \mathbb{R} \setminus \Sigma \) is called a regular value of \( \varphi \). For any regular value \( c \) of \( \varphi \), \( \{ \varphi = c \} \) is a piecewise smooth submanifold of \( Z \). In particular, \( \{ \varphi \leq c \} \cap Z^o \) is a submanifold of \( Z^o \) with a smooth boundary \( \{ \varphi = c \} \cap Z^o \).

Denote

\[
(\alpha, \beta)_{c,h} = \int_{\{\varphi \leq c\} \cap Z^o} \langle \alpha, \beta \rangle_h \text{vol}_d s^2, \quad [\alpha, \beta]_{c,h} := \int_{\{\varphi = c\} \cap Z^o} \langle \alpha, \beta \rangle_h \text{vol}_{\{\varphi = c\}}^2
\]

and \( \|\alpha\|_{c,h} := \sqrt{(\alpha, \alpha)_{c,h}} \) for any regular value \( c \) of \( \varphi \). By Stokes’ theorem we acquire that [25, (1.1)]

\[
(\bar{\partial} \alpha, \beta)_{c,h} = (\alpha, \bar{\partial}_h^* \beta)_{c,h} + [\alpha, e(\bar{\partial} \varphi)^* \beta]_{c,h}
\]

(3.11)
and
\[
(D^\alpha, \beta)_{c,h} = (\alpha, D_h^{\ast} \beta)_{c,h} + [\alpha, e(\partial \bar{\varphi})^{\ast} \beta]_{c,h}
\] (3.12)
hold for every \( C^\infty \) forms \( \alpha \) and \( \beta \) on \( \{ \varphi \leq c \} \cap Z^o \) where either \( \alpha \) or \( \beta \) has compact support in \( \{ \varphi \leq c \} \cap Z^o \).

**Proposition 3.2** Let \((M, \omega_M)\) be a complete Kähler manifold of dimension \( n \) and let \((E, h)\) be a Nakano semi-positive holomorphic vector bundle on \( M \). If \( q \geq 1 \) and \( \alpha \in \mathcal{D}^{n,q}(M, E; \omega_M, h) \cap \text{Ker} \Delta_{\tilde{\alpha}} \), then \( \alpha \) satisfies the following equations:

1. \( \tilde{\delta} \alpha = 0, \tilde{\delta}^{\ast} \alpha = 0, D_h^{\ast} \alpha = 0 \) and \( \langle \sqrt{-1} e(\Theta_h) \Lambda \alpha, \alpha \rangle_h = 0 \) on \( M \). In particular
   \( \tilde{\delta}(\ast \alpha) = 0 \), i.e., \( \ast \alpha \in \Gamma(M, \Omega_M^{n-q}(E)) \).
2. \( e(\partial \bar{\varphi})^{\ast} \alpha = 0 \) and \( \langle \sqrt{-1} e(\partial \bar{\varphi}) \Lambda \alpha, \alpha \rangle_h = 0 \) on \( M \) for any \( C^\infty \) plurisubharmonic function \( \varphi \) on \( M \) with

\[
\sup_{x \in M} |\varphi(x)| + |d \varphi(x)|_{\omega_M} < \infty.
\]

**Proof** See [25, Theorem 3.4]. \( \square \)

Denote by \( \mathcal{P}(Z) \) the set of \( C^\infty \) plurisubharmonic functions \( \varphi : Z \to (-\infty, c_*) \) for some \( c_* \in (-\infty, \infty) \) such that \( Z_c := \{ z \in Z | \varphi(z) < c \} \) is precompact in \( Z \) for every \( c < c_* \). \( Z \) is called a **weakly 1-complete space** if \( \mathcal{P}(Z) \neq \emptyset \). For every \( \varphi \in \mathcal{P}(Z) \), denote

\[
\mathcal{H}^{p,q}(Z, E, h, \varphi) := \left\{ \alpha \in \mathcal{D}^{p,q}_{Z, d\bar{s}^2}(E, h)(Z) \mid \tilde{\delta} \alpha = \tilde{\delta}^{\ast} \alpha = 0, e(\tilde{\delta} \varphi)^{\ast} \alpha = 0 \right\}.
\]

By the regularity theorem for elliptic operators of second order, every element of \( \mathcal{H}^{p,q}(Z, E, h, \varphi) \) is \( C^\infty \) on \( Z^o \).

**Remark 3.3** The reason for the boundary condition \( e(\tilde{\delta} \varphi)^{\ast} \alpha = 0 \) lies in the Stokes’ formula (3.11). The condition \( e(\tilde{\delta} \varphi)^{\ast} \beta = 0 \) ensures that there is no boundary term in the Stokes’ formula

\[
(\tilde{\delta} \alpha, \beta)_{c,h} = (\alpha, \tilde{\delta}^{\ast} \beta)_{c,h}.
\]

This is a boundary condition on the smooth level set \( \{ \varphi = c \} \) while \( \{ \varphi \leq c \} \) exhausts the space.

**Proposition 3.4** Let \( Z \) be a weakly 1-complete space of pure dimension \( n \) and \( Z^o \subset Z_{\text{reg}} \) a dense Zariski open subset. Let \( d\bar{s}^2 \) be a Kähler metric on \( Z^o \) which is locally complete on \( Z \) and let \((E, h)\) be a hermitian vector bundle on \( Z^o \) with Nakano semi-positive curvature. Then the following assertions hold.

1. Let \( \varphi \in \mathcal{P}(Z) \). Assume that \( \alpha \in \mathcal{D}^{n,q}_{Z, d\bar{s}^2}(E, h)(Z) \) satisfies \( e(\tilde{\delta} \varphi)^{\ast} \alpha = 0 \). Then
   \( \tilde{\delta} \alpha = \tilde{\delta}^{\ast} \alpha = 0 \) if and only if \( D_h^{\ast} \alpha = 0 \) and \( \langle \sqrt{-1} e(\Theta_h + \tilde{\delta} \bar{\varphi}) \Lambda \alpha, \alpha \rangle_h = 0 \). Here
   \( \langle \sqrt{-1} e(\Theta_h + \tilde{\delta} \bar{\varphi}) \Lambda \alpha, \alpha \rangle_h = 0 \) is equivalent to
   \( \langle \sqrt{-1} e(\tilde{\delta} \bar{\varphi}) \Lambda \alpha, \alpha \rangle_h = 0 \).
(2) Let \( \varphi, \psi \in \mathcal{P}(Z) \). Then \( \mathcal{H}^{n,q}(Z, E, h, \varphi) = \mathcal{H}^{n,q}(Z, E, h, \psi) \) for every \( q \geq 0 \).

(3) For every \( q \geq 0 \) and every \( \varphi \in \mathcal{P}(Z) \), the Hodge star operator gives a well defined map

\[
* : \mathcal{H}^{n,q}(Z, E, h, \varphi) \to \text{Ker} \tilde{\partial} \left( \mathcal{D}_{Z,ds^2}^{n-q,0}(E, h)(Z) \to \mathcal{D}_{Z,ds^2}^{n-q,1}(E, h)(Z) \right).
\]

**Proof** To show (1), we suppose that \( e(\tilde{\partial} \varphi)^* \alpha = 0 \) and \( \tilde{\partial} \alpha = \tilde{\partial}_h^* \alpha = 0 \). Take any regular value \( c \) of \( \varphi \). Since \( (\varphi \leq c) \cap Z^\alpha, ds^2 \) is complete, the Hopf–Rinow theorem implies that there exists an exhaustive sequence \( \{K_v\}_{v \in \mathbb{N}} \) of compact sets of \( Z^\alpha \cap \{\varphi \leq c\} \) and functions \( \theta_v \in C^\infty(\{\varphi \leq c\} \cap Z^\alpha, \mathbb{R}) \) such that

\[
\theta_v = 1 \quad \text{on a neighborhood of } K_v, \quad \text{supp } \theta_v \subset K_v^{\alpha+1},
\]

\[
0 \leq \theta_v \leq \theta_{v+1} \leq 1 \quad \text{and } |d\theta_v|_g \leq 2^{-v}, \quad \forall v.
\]

Then \( \{\theta_v \alpha\}_{v \in \mathbb{N}} \) converges to \( \alpha \) under the norm \( \| - \| + \| \tilde{\partial} - \| + \| D' - \| + \| \tilde{\partial}_h^* - \| + \| D_h^* - \| \) for various \( h \).

Let \( h_\varphi = e^{-\varphi} h \). Since \( \tilde{\partial} \alpha = \tilde{\partial}_h^* \alpha = 0 \), by (3.6) and (3.7) we obtain that

\[
(D' D_h^* \alpha, \theta_v \alpha)_{c,h_\varphi} + (\sqrt{-1} e(\Theta_{h_\varphi}) \Lambda \alpha, \theta_v \alpha)_{c,h_\varphi} = 0.
\]

By (3.12), this is equivalent to

\[
(D_h^* \alpha, D_h^* (\theta_v \alpha))_{c,h_\varphi} + (\sqrt{-1} e(\Theta_{h_\varphi}) \Lambda \alpha, \theta_v \alpha)_{c,h_\varphi} + [e(\partial \varphi) D_h^* \alpha, \theta_v \alpha]_{c,h_\varphi} = 0.
\]

(3.13)

By the assumptions and (3.9) we have

\[
e(\partial \varphi) D_h^* \alpha = \sqrt{-1} e(\partial \tilde{\partial} \varphi) \Lambda \alpha.
\]

(3.14)

Substituting (3.14) into (3.13), we get

\[
(D_h^* \alpha, D_h^* (\theta_v \alpha))_{c,h_\varphi} + (\sqrt{-1} e(\Theta_{h_\varphi}) \Lambda \alpha, \theta_v \alpha)_{c,h_\varphi} + [\sqrt{-1} e(\partial \tilde{\partial} \varphi) \Lambda \alpha, \theta_v \alpha]_{c,h_\varphi} = 0.
\]

Letting \( v \to \infty \), we obtain that

\[
\| D_h^* \alpha \|^2_{c,h_\varphi} + (\sqrt{-1} e(\Theta_{h_\varphi}) \Lambda \alpha, \alpha)_{c,h_\varphi} + [\sqrt{-1} e(\partial \tilde{\partial} \varphi) \Lambda \alpha, \alpha]_{c,h_\varphi} = 0.
\]

(3.15)

Since \( \varphi \) is a plurisubharmonic function and \( \sqrt{-1} \Theta_{h_\varphi} \) is Nakano semi-positive, \( \sqrt{-1} \Theta_{h_\varphi} = \sqrt{-1} \Theta_h + \sqrt{-1} \partial \tilde{\partial} \varphi \) is also Nakano semi-positive. Then all the three terms in (3.15) are semi-positive. Hence they are all zero for every regular value \( c \) of \( \varphi \). This proves the necessity of (1).
To prove the sufficiency of (1), we assume that \( D_h^*\alpha = 0 \), \( \langle \sqrt{-1} e(\Theta h + \bar{\partial} \bar{\partial} \varphi)\Lambda \alpha, \alpha \rangle_h = 0 \) and \( e(\bar{\partial} \varphi)^*\alpha = 0 \). By (3.6) we have \( \Delta \tilde{\beta} \alpha = 0 \). By (3.11) we obtain

\[
(\Delta \tilde{\beta} \alpha, \theta, \alpha)_{c, h} = (\tilde{\partial}_h^*\alpha, \tilde{\partial}_h^*(\theta, \alpha))_{c, h} + (\tilde{\partial} \alpha, \bar{\partial}(\theta, \alpha))_{c, h} + [\tilde{\partial}_h^*\alpha, e(\bar{\partial} \varphi)^*(\theta, \alpha)],_{c, h} - [e(\bar{\partial} \varphi)^* \tilde{\partial} \alpha, \theta, \alpha]_{c, h} = 0. \tag{3.16}
\]

By (3.9) we acquire that

\[
\langle e(\bar{\partial} \varphi)^* \tilde{\partial} \alpha, \theta, \alpha \rangle_h = \left\langle \sqrt{-1} e(\bar{\partial} \bar{\partial} \varphi)\Lambda \alpha, \theta, \alpha \right\rangle_h = \theta \sqrt{-1} \left\langle e(\bar{\partial} \bar{\partial} \varphi)\Lambda \alpha, \alpha \right\rangle_h = 0. \tag{3.17}
\]

Combining (3.16), (3.17) and \( e(\bar{\partial} \varphi)^*(\theta, \alpha) = \theta e(\bar{\partial} \varphi)^* \alpha = 0 \), we obtain that

\[
(\tilde{\partial}_h^*\alpha, \tilde{\partial}_h^*(\theta, \alpha))_{c, h} + (\tilde{\partial} \alpha, \bar{\partial}(\theta, \alpha))_{c, h} = 0.
\]

Taking \( \nu \to \infty \) we know that

\[
\|\tilde{\partial}_h^*\alpha\|^2_{c, h} + \|\tilde{\partial} \alpha\|^2_{c, h} = 0
\]

for every regular value \( c \) of \( \varphi \). This implies that \( \tilde{\partial} \alpha = \tilde{\partial}_h^* \alpha = 0 \).

To prove (2) we set \( h_{-\psi} = e^\psi h \). By (3.9) we obtain that

\[
\tilde{\partial} e(\bar{\partial} \varphi)^*(\theta, \alpha) + e(\bar{\partial} \varphi)^* \tilde{\partial}(\theta, \alpha) + e(\bar{\partial} \varphi) D_h^* (\theta, \alpha) = \sqrt{-1} e(\bar{\partial} \bar{\partial} \varphi) \Lambda (\theta, \alpha) \tag{3.18}
\]

if \( \alpha \in H^{n, q}(Z, E, h, \varphi) \). By (3.7) and (3.11), we get that

\[
(\tilde{\partial} e(\bar{\partial} \varphi)^*(\theta, \alpha), \alpha)_{c, h_{-\psi}} = (e(\bar{\partial} \varphi)^*(\theta, \alpha), \tilde{\partial}_h^* \alpha)_{c, h_{-\psi}} + [e(\bar{\partial} \varphi)^*(\theta, \alpha), e(\bar{\partial} \varphi)^* \alpha]_{c, h_{-\psi}} = (e(\bar{\partial} \varphi)^*(\theta, \alpha), (\tilde{\partial}_h^* - e(\bar{\partial} \varphi)^* \alpha)_{c, h_{-\psi}} = -(e(\bar{\partial} \varphi)^*(\theta, \alpha), e(\bar{\partial} \varphi)^* \alpha)_{c, h_{-\psi}}. \tag{3.19}
\]

By (3.18) and (3.19) one has

\[
(\sqrt{-1} e(\bar{\partial} \bar{\partial} \varphi) \Lambda (\theta, \alpha), \alpha)_{c, h_{-\psi}} + (e(\bar{\partial} \bar{\partial} \varphi)^*(\theta, \alpha), e(\bar{\partial} \varphi)^* \alpha)_{c, h_{-\psi}} = (e(\bar{\partial} \bar{\partial} \varphi)^* \tilde{\partial}(\theta, \alpha), \alpha)_{c, h_{-\psi}} + (e(\bar{\partial} \varphi) D_h^* (\theta, \alpha), \alpha)_{c, h_{-\psi}}.
\]

Taking \( \nu \to \infty \) we acquire that

\[
(\sqrt{-1} e(\bar{\partial} \bar{\partial} \varphi) \Lambda (\alpha), \alpha)_{c, h_{-\psi}} + \|e(\bar{\partial} \varphi)^* \alpha\|^2_{c, h_{-\psi}} = 0.
\]
for every regular value $c$ of $\varphi$. Since both terms are semi-positive, we show that $e(\bar{\partial}\psi)^*\alpha = 0$. Hence $\alpha \in \mathcal{H}^{n,q}(Z, E, h, \psi)$.

It remains to show (3). Since $*$ is a bounded operator, it suffices to show that $\bar{\partial}^*\alpha = 0$ for every $\alpha \in \mathcal{H}^{n,q}(Z, E, h, \varphi)$. This follows from $-\bar{\partial}^*\alpha = D^*_h\alpha = 0$ which is proved in (1).

**Remark 3.5** There is a geometrical explanation why $\mathcal{H}^{n,q}(Z, E, h, \varphi)$ is independent of the choice of the smooth exhausted plurisubharmonic function $\varphi$. Let us consider the case when $Z$ admits a proper holomorphic map $f : Z \to S$ to a Stein space (this is the basic model of the present paper). Let $\alpha \in \mathcal{H}^{n,q}(Z, E, h, \varphi)$. Due to Proposition 3.7 (whose proof is independent of Proposition 3.4-(2)), $\alpha$ is a holomorphic $(n-q, 0)$-form which lies in $\Omega^\alpha Z_{\dim Z - \dim S - q} \otimes f^*\Omega^S_{\dim S}$ over the dense open regular loci where $f$ is a submersion. Let $\psi$ be another smooth plurisubharmonic function on $Z$. Since $f$ is a proper map, $\psi$ must be constant along each fiber of $f$. Thus $\psi = f^*\phi$ for some smooth function $\phi$ on $S$. Then $e(f^*\partial(\phi))^*\alpha$ must be zero since $\partial(\phi)$ is a $(1, 0)$-form on $S$ but $\partial^*\alpha$ is divided by a nonzero $(\dim S, 0)$-form on $S$. This shows that $\alpha \in \mathcal{H}^{n,q}(Z, E, h, \psi)$.

**Remark 3.6** By Proposition 3.4-(2), $\mathcal{H}^{n,q}(Z, E, h, \varphi)$ is independent of the choice of $\varphi \in \mathcal{P}(Z)$. Hence we simply denote $\mathcal{H}^{n,q}(Z, E, h) := \mathcal{H}^{n,q}(Z, E, h, \varphi)$ for every complex space $Z$ with $\mathcal{P}(Z) \neq \emptyset$.

### 3.2 Harmonic representation

Let us return to the relative setting. Let $f : X \to Y$ be a proper morphism from a complex space $X$ to an irreducible complex space $Y$. Denote $n := \dim_X X$ and $m := \dim_Y Y$ respectively. Assume that every irreducible component of $X$ is mapped onto $Y$. Let $X^o \subset X_{\reg}$ be a dense Zariski open subset and $(E, h)$ a hermitian vector bundle on $X^o$ with Nakano semi-positive curvature. Assume that there is a Kähler metric $ds^2$ on $X^o$ such that

1. $ds^2$ admits a $(\infty, 1)$ bounded potential locally on $X$;
2. $ds^2$ is locally complete on $X$ and is locally bounded from below by a hermitian metric (Definition 2.13).

By Lemma 2.14, such kind of metric exists locally near every fiber of $f$ and exists globally on $X^o$ when $X$ is a compact Kähler space. By Theorem 2.11 and Lemma 2.2, there is a resolution by fine sheaves

$$S_X(E, h) \to \mathcal{P}_{X, ds^2}^N(E, h).$$

(3.20)

Denote $X(T) := f^{-1}(T)$ for every subset $T \subset Y$. Denote by $L$ the Lefschetz operator with respect to $ds^2$ and by $\Lambda$ the formal adjoint of $L$.

**Proposition 3.7** Notations as above. Let $S \subset Y$ be a Stein open subset. Let $S^o \subset S_{\reg}$ and $X(S)^o \subset X(S^o) \cap X^o$ be dense Zariski open subsets so that $f : X(S)^o \to S^o$ is
a submersion. Then the Hodge star operator

\[ * : \mathcal{H}^{n,q}(X(S), E, h) \rightarrow \{ \alpha \in \mathcal{D}^{n,q,0}_{X(S), d_s^2}(E, h)(X(S)) \mid \tilde{\partial}\alpha = 0, \alpha|_{X(S)'} \in \Gamma(X(S)\circ, \Omega^n_{X,\circ} - m - q \otimes f^*\Omega^m_S) \} \]

is well defined and injective for every \( q \). As a consequence,

\begin{enumerate}
\item \( \mathcal{H}^{n,q}(X(S), E, h) = 0 \) for every \( q > n - m \);
\item For every open subset \( S' \subset S \) such that \( \mathcal{P}(S') \neq \emptyset \), the restriction map

\[ \mathcal{H}^{n,q}(X(S), E, h) \rightarrow \mathcal{H}^{n,q}(X(S'), E, h) \]

is well defined.
\end{enumerate}

**Proof** Let \( \alpha \in \mathcal{H}^{n,q}(X(S), E, h) \). By Proposition 3.4-(1), \( D^*_h \alpha = 0 \), i.e., \( \tilde{\partial} \ast \alpha = 0 \).

It remains to show that \( \ast \alpha|_{X(S)'} \in \Gamma(X(S)\circ, \Omega^n_{X,\circ} - m - q \otimes f^*\Omega^m_S) \).

Fix a closed immersion \( S \subset \mathbb{C}^N \) where \( z_1, \ldots, z_N \) are standard coordinates of \( \mathbb{C}^N \). Let \( \varphi = \sum_{i=1}^N |z_i|^2 \mathcal{S} \in \mathcal{P}(S) \). By Proposition 3.4-(1) and (3.5), one has

\[ \sum_{i=1}^N |e(\partial f^*(z_i)) \ast \alpha|_h^2 = \sum_{i=1}^N \langle e(\partial f^*(z_i)) \ast \alpha, -\sqrt{-1} e(\partial f^*(z_i)) \Lambda \alpha \rangle_h \]

\[ = \sum_{i=1}^N \langle \alpha, -\sqrt{-1} e(\partial f^*(z_i)) e(\partial f^*(z_i)) \Lambda \alpha \rangle_h \]

\[ = \langle \alpha, -\sqrt{-1} e(\partial f^* \varphi) \Lambda \alpha \rangle_h = 0. \quad (3.5) \]

Hence \( f^*(d\mathfrak{z}_i) \wedge \ast \alpha = 0 \), \( \forall i = 1, \ldots, N \). This implies that \( \alpha|_{X(S)\circ} = 0 \) if \( q > n - m \) and \( \ast \alpha|_{X(V)\cap X(S)\circ} \) can be divided by \( f^*\theta \) for any open subset \( V \subset S' \) which admits a non-vanishing holomorphic \( m \)-form \( \theta \). Therefore

\[ \ast \alpha|_{X(S)\circ} \in \Gamma(X(S)\circ, \Omega^n_{X,\circ} - m - q \otimes f^*\Omega^m_S). \]

This proves that (3.21) is well defined and \( \mathcal{H}^{n,q}(X(S), E, h) = 0 \) for every \( q > n - m \). (3.21) is injective because

\[ L^q \circ \ast = \sqrt{-1}^{(n-q)(n-q+3)} q! \text{Id} \quad (3.22) \]

holds on \( (n, q) \)-forms [26, Theorem 3.16].

It remains to show that \( \alpha|_{X(S')} \in \mathcal{H}^{n,q}(X(S'), E, h) \), i.e., \( e(\tilde{\partial} f^* \psi)^\ast \alpha|_{X(S')} = 0 \) for some \( \psi \in \mathcal{P}(S') \). Since \( \ast \alpha|_{X(S)\circ} \in \Gamma(X(S)\circ, \Omega^n_{X,\circ} - m - q \otimes f^*\Omega^m_S) \), we obtain that
Thus $e(\bar{\partial}(f^*\psi))^*\alpha|_{X(S')} = 0$ by continuity. This proves (2). \hfill \Box

By Proposition 3.7, the restriction map

\[ \mathcal{H}^{n,q}(X(V), E, h) \to \mathcal{H}^{n,q}(X(U), E, h) \]

is well defined for any pair of Stein open subsets $U \subset V \subset Y$. Hence the data

\[ U \mapsto \mathcal{H}^{n,q}(X(U), E, h), \quad U \subset Y \text{ is a Stein open subset} \]

determines a sheaf $\mathcal{H}^{n,q}_f(E, h)$ on $Y$ (after sheafification).

By (3.20) and Lemma 2.2, there is a natural morphism

\[ \mathcal{H}^{n,\bullet}_f(E, h) \to f_*(\mathcal{D}^{n,\bullet}_X, ds^2(E, h)) \cong Rf_*(S_X(E, h)). \tag{3.23} \]

This induces a canonical morphism

\[ \mathcal{H}^{n,q}_f(E, h) \to R^q f_* S_X(E, h) \]

for every $0 \leq q \leq n$. The main result of this section is

**Theorem 3.8** $\mathcal{H}^{n,q}_f(E, h)$ is a sheaf of $\mathcal{O}_Y$-modules for every $q \geq 0$. Assume that $S_X(E, h)$ is a coherent sheaf. Then the canonical morphism

\[ \mathcal{H}^{n,q}_f(E, h) \to R^q f_* S_X(E, h) \]

is an isomorphism of $\mathcal{O}_Y$-modules for every $q \geq 0$. Moreover,

\[ \mathcal{H}^{n,q}_f(E, h)(U) = \mathcal{H}^{n,q}_f(f^{-1}(U), E, h) \]

for every Stein open subset $U \subset Y$.

**Proof** Let $S \subset Y$ be a Stein open subset and $\varphi \in \mathcal{P}(X(S))$. For every $\alpha \in \mathcal{H}^{n,q}(X(S), E, h)$ and every $g \in \mathcal{O}_Y(S)$, denote $g' := f^* g$. Then Proposition 3.4-(1) implies that

\[ D^{n*}_h(g'\alpha) = -\ast \bar{\partial} \ast (g'\alpha) = g'D^{n*}_h\alpha = 0 \]

and

\[ (\sqrt{-1}e(\Theta_h + \partial\bar{\partial}\varphi)\Lambda(g'\alpha), g'\alpha)_h = |g'|^2(\sqrt{-1}e(\Theta_h + \partial\bar{\partial}\varphi)\Lambda(\alpha), \alpha)_h = 0. \]

Hence $g'\alpha \in \mathcal{H}^{n,q}(X(S), E, h, \varphi)$ by Proposition 3.4-(1). This shows that $\mathcal{H}^{n,q}_f(E, h)$ is a sheaf of $\mathcal{O}_Y$-modules for every $0 \leq q \leq n$. \hfill \copyright \ Springer
Since $S_X(E, h)$ is a coherent sheaf, to prove the remaining claims it suffices to show that the natural morphism

$$\tau_U^q : \mathcal{H}^{n,q}(X(U), E, h) \to H^q \Gamma(X(U), D^{n,q}(E, h))$$

is an isomorphism for every Stein open subset $U \subset Y$ and every $q \geq 0$. Fix a $C^\infty$ exhausted strictly plurisubharmonic function $\varphi_U$ on $U$. Denote $U_c := \{ \varphi_U < c \}$ and $\varphi := f^* \varphi_U$.

**Claim 1:** $\tau_U^q$ is injective. Assume that $\alpha \in \mathcal{H}^{n,q}(X(U), E, h)$ and $\alpha = \tilde{\beta} \varphi$ for some $\beta \in D_{X,ds^2}^{n,q-1}(E, h)(X(U))$. Take any regular value $c$ of $\varphi$. Since $\{ \varphi \leq c \} \cap Z^0, ds^2$ is complete, the Hopf–Rinow theorem implies that there exists an exhaustive sequence $\{ K_v \}_{v \in \mathbb{N}}$ of compact subsets of $Z^0 \cap \{ \varphi \leq c \}$ and functions $\theta_v \in C^\infty(\{ \varphi \leq c \} \cap Z^0, \mathbb{R})$ such that

$$\theta_v = 1 \text{ on a neighborhood of } K_v, \quad \text{supp } \theta_v \subset K_{v+1},$$

$$0 \leq \theta_v \leq \theta_{v+1} \leq 1 \quad \text{and} \quad |d\theta_v|_g \leq 2^{-v}, \quad \forall v.$$ 

Then $\{ \theta_v \alpha \}$ converges to $\alpha$ under the norm $\| - \| + \| \tilde{\partial}_\varphi (-) \|$. By (3.11), we obtain that

$$(\alpha, \theta_v \alpha)_{c,h} = (\beta, \tilde{\partial}_\varphi (\theta_v \alpha))_{c,h}$$

for every regular value $c$ of $\varphi$. Taking $v \to \infty$ one gets that

$$(\alpha, \alpha)_{c,h} = (\beta, \tilde{\partial}_\varphi \alpha)_{c,h} = 0$$

for every regular value $c$ of $\varphi$. Hence $\alpha = 0$.

**Claim 2:** $\tau_U^q$ is surjective. Let $\alpha \in \Gamma(X(U), D_{X,ds^2}^{n,q}(E, h))$ be $\tilde{\partial}$-closed.

**Step 1:** In this step we show that for every $c \in \mathbb{R}$, $\alpha|_{X(U_c)} = u_c + \tilde{\partial} \beta_c$ for some $u_c \in \mathcal{H}^{n,q}(X(U_c), E, h)$ and $\beta_c \in \Gamma(X(U_c), D_{X,ds^2}^{n,q-1}(E, h))$.

Fix $0 < c_0 < c_1$. Denote $\omega_{ds^2}$ to be the Kähler form associated to $ds^2$ and let $\omega_\lambda := \omega_{ds^2} + \sqrt{-1} \partial \bar{\partial} \lambda (\varphi - c)$ for some smooth convex function $\lambda : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\lambda(t) = 0$ if $t \leq c_0$, $\lambda(t) > 0$, $\lambda'(t) > 0$, $\lambda''(t) > 0$ if $t > c_0$ and $\int_0^{c_1} \sqrt{\lambda''(t)} dt = +\infty$. Let $h_\lambda := e^{-\lambda(\varphi)} h$. Then $\omega_\lambda \geq \omega_{ds^2}$ is a complete Kähler metric on $X(U_{c+c_1})$ and $(E, h_\lambda)$ is Nakano semi-positive. By choosing $\lambda$ sufficiently large we assume that $\alpha \in L_{2}^{n,q}(X(U_{c+c_1}), E; \omega_\lambda, h_\lambda)$. Noting that $\tilde{\partial} \alpha = 0$, there is a unique decomposition $\alpha = v_c + \gamma_c$ so that $v_c$ is a harmonic form on $U_{c+c_1}$ with respect to $\omega_\lambda$ and $h_\lambda$ while $\gamma_c$ lies in the closure of the range of $\tilde{\partial}$ in the Hilbert space $L_{2}^{n,q}(X(U_{c+c_1}), E; \omega_\lambda, h_\lambda)$. Since $\omega_\lambda|_{U_c} = \omega_{ds^2}|_{U_c}$ and $h_\lambda|_{U_c} = h|_{U_c}$, by Propositions 3.2 and 3.9 below we see that $v_c|_{U_c} \in \mathcal{H}^{n,q}(X(U_c), E, h)$ and $\gamma_c|_{U_c} = \tilde{\partial} \beta_c$ for some $\beta_c \in \Gamma(X(U_c), D_{X,ds^2}^{n,q-1}(E, h))$. 

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Step 2: By Step 1, there is a decomposition
\[ \alpha|_{U_k} = u_k + \bar{\partial}\beta_k \]
where \( u_k \in \mathcal{H}^{n,q}(X(U_k), E, h) \) and \( \beta_k \in \Gamma(X(U_k), \mathcal{S}_{X,d^2}^{n,q-1}(E, h)) \) for every \( k \in \mathbb{N} \).
Since \( u_{k+1} \) and \( u_k \) are cohomologous on \( X(U_k) \), we have \( u_{k+1}|_{X(U_k)} = u_k \) by Claim 1. Hence there is a unique \( u \in \mathcal{H}^{n,q}(X(U), E, h) \) such that \( u|_{X(U_k)} = u_k \) for every \( k \). Hence the cohomology classes \( [\alpha] \) and \( [u] \) are equal over \( X(U_k) \) for each \( k \in \mathbb{N} \).
Since \( U_k \) is a Stein space for each \( k \in \mathbb{N} \cup \{ +\infty \} \) and \( S_X(E, h) \) is a coherent sheaf, there is a canonical isomorphism
\[ \Gamma(X(U_k), \mathcal{S}_{X,d^2}^{n,q}(E, h)) \cong H^q(X(U_k), S_X(E, h)) \cong \Gamma(U_k, R^q f_!(S_X(E, h))) \]
for every \( k \in \mathbb{N} \cup \{ +\infty \} \). By regarding \( [\alpha] \) and \( [u] \) as sections of \( R^q f_!(S_X(E, h)) \) (which are equal over \( U_k \) for each \( k \in \mathbb{N} \)) we obtain that \( [\alpha] = [u] \). This proves the surjectivity of \( \tau^q_U \). \( \square \)

To complete the proof of Theorem 3.8, we go on proving the following proposition.

**Proposition 3.9** Assume that \( S_X(E, h) \) is a coherent sheaf on \( X \). Let \( V \subset X \) be an open subset and denote \( V^o := V \cap X^o \). Let
\[ \tilde{\partial} : L^{n,q-1}_{(2)}(V^o, E; ds^2, h) \to L^{n,q}_{(2)}(V^o, E; ds^2, h) \]
be the unbounded operator in the sense of distribution where \( q \geq 1 \). Suppose that \( \alpha \in \text{Im} \tilde{\partial} \). Then there is an \((n, q - 1)\)-form \( \beta \in L^{n,q-1}_{V,d^2}(E, h)(V) \) such that \( \alpha = \tilde{\partial}\beta \).

**Proof** To prove the proposition, we need the following lemma which is a modified version of Theorem 2.10.

**Lemma 3.10** Let \( Y \) be a complex manifold of dimension \( n \) which admits a complete Kähler metric. Let \( \omega = \sqrt{-1} \partial \bar{\partial} \varphi \) be a Kähler metric of \( Y \) with sup \(|\varphi| < \infty \). Let \((E, h)\) be a Nakano semi-positive hermitian vector bundle on \( Y \). Then for every \( q > 0 \) and every \( \alpha \in L^{n,q}_{(2)}(Y, E; \omega, h) \) such that \( \bar{\partial} \alpha = 0 \), there is a \( \beta \in L^{n,q-1}_{(2)}(Y, E; \omega, h) \) such that \( \bar{\partial} \beta = \alpha \) and \( \|\beta\|_{\omega, h}^2 \leq C\|\alpha\|_{\omega, h}^2 \). Here \( C \) is a constant depending only on \( \varphi \).

**Proof** Let \( h^\prime := he^{-\varphi} \). Then \( h^\prime \sim h \). Since \((E, h)\) is Nakano semi-positive, we know that
\[ \sqrt{-1} \Theta_{h^\prime}(E) = \sqrt{-1} \partial \bar{\partial} \varphi \otimes \text{Id}_E + \sqrt{-1} \Theta_h(E) \geq \omega \otimes \text{Id}_E. \]
By Theorem 2.10, there exists \( \beta \in L^{n,q-1}_{(2)}(Y, E; \omega, h^\prime) \) such that \( \bar{\partial} \beta = \alpha \) and \( \|\beta\|_{\omega, h^\prime}^2 \leq \frac{1}{2}\|\alpha\|_{\omega, h^\prime}^2 \). Since \( \varphi \) is a bounded function, we have \( \|\beta\|_{\omega, h}^2 \leq C\|\alpha\|_{\omega, h}^2 \)
where \( C \) is a constant depending only on \( \varphi \). The lemma is proved. \( \square \)
Let us return to the proof of the proposition. First we take a locally finite Stein open cover \( \{ V_j \} \) of \( V \). Denote \( V_{j_0, \ldots, j_p} := V_{j_0} \cap \cdots \cap V_{j_p} \). By Lemma 2.14, after a possible refinement we assume that there is a complete Kähler metric on \( V_{j_0, \ldots, j_p} \) which has a bounded potential for every \( j_0, \ldots, j_p \). Since \( \alpha \in \text{Im} \bar{\partial} \subset L^{n,q}_{(2)}(V^\omega, E; ds^2, h) \) and \( \ker \bar{\partial} \) is closed in \( L^{n,q}_{(2)}(V^\omega, E; ds^2, h) \), we know that \( \bar{\partial} \alpha = 0 \).

Since \( S_X(E, h) \) is a coherent sheaf on \( X \), by Theorem 2.11 and Lemma 2.2 there are isomorphisms of cohomologies

\[
H^k(\{ V_j \}, S_X(E, h)) \simeq H^k(V, S_X(E, h)) \simeq H^k(\Gamma(V, \mathcal{O}_X, E; ds^2, h)),
\]

where \( H^k(\{ V_j \}, S_X(E, h)) \) is the Čech cohomology with respect to the covering \( \{ V_j \} \).

Let us recall the construction of the corresponding Čech cocycle of \( \alpha \). Let \( \alpha_j = \alpha|_{V_j^\omega} \). By Lemma 3.10, there exists \( b_j \in L^{n,q-1}_{(2)}(V_j^\omega, E; ds^2, h) \) such that \( \alpha_j = \bar{\partial} b_j \) on \( V_j^\omega \) for each \( j \).

Suppose that \( \{ \alpha_{j_0, \ldots, j_r} \} \) and \( \{ b_{j_0, \ldots, j_r} \} \) are determined in the way as:

\[
b_{j_0, \ldots, j_r} \in L^{n,q-r-1}_{(2)}(V_{j_0, j_1, \ldots, j_p}^\omega, E; ds^2, h), \quad \alpha_{j_0, \ldots, j_r} \in L^{n,q-r}_{(2)}(V_{j_0, j_1, \ldots, j_p}^\omega, E; ds^2, h),
\]

\[
\alpha_{j_0, \ldots, j_r} = \bar{\partial} b_{j_0, \ldots, j_r} \quad \text{on} \quad V_{j_0, \ldots, j_r}^\omega \quad \text{and} \quad (\delta \alpha)_{j_0, \ldots, j_r+1} = 0.
\]

Set

\[
\alpha_{j_0, \ldots, j_r+1} := (\delta b)_{j_0, \ldots, j_r+1}
\]

which is a \( \bar{\partial} \)-closed form. It follows from Lemma 3.10 that the same statements in (3.25) also hold for \( \alpha_{j_0, \ldots, j_r+1} \). Repeating the above steps, we obtain a \( q \)-cocycle \( \{ \alpha_{j_0, \ldots, j_r} \} \in \mathcal{Z}^q(\{ V_j \}, S_X(E, h)) \) which corresponds to \( \alpha \) by (3.24).

By hypothesis there exists a sequence \( \{ \gamma_m \}_{m \in \mathbb{N}} \in L^{n,q-1}_{(2)}(V^\omega, E; ds^2, h) \) such that \( \| \alpha - \bar{\partial} \gamma_m \| \to 0 \) as \( m \to \infty \). Let \( v_m := \alpha - \bar{\partial} \gamma_m \). Since \( \bar{\partial} v_m = 0 \), by Lemma 3.10 there exists \( \{ \mu_{j,m} \} \in L^{n,q-1}_{(2)}(V_j^\omega, E; ds^2, h) \) such that \( v_m = \bar{\partial} \mu_{j,m} \) and \( \| \mu_{j,m} \| \leq C_j \| v_m \| \) for \( C_j \) depending only on \( j \). Set \( \rho_{j,m} := b_j - \mu_{j,m} - \gamma_m|_{V_j^\omega} \). Then we have \( \bar{\partial} \rho_{j,m} = 0 \), \( \bar{\partial}(\alpha_{ij} - \delta(\rho_{j,m})) = 0 \) and \( \| \alpha_{ij} - \delta(\rho_{j,m}) \| \leq C_{ij} \| v_m \| \).

Suppose that \( \{ \rho_{j_0, \ldots, j_r,m} \} \) is already determined as:

\[
\rho_{j_0, \ldots, j_r,m} \in L^{n,q-r-1}_{(2)}(V_{j_0, \ldots, j_r}^\omega, E; ds^2, h), \quad \bar{\partial}(\alpha_{j_0, \ldots, j_r+1} - (\delta \rho)_{j_0, \ldots, j_r+1,m}) = 0,
\]

\[
\bar{\partial} \rho_{j_0, \ldots, j_r,m} = 0, \quad \| \alpha_{j_0, \ldots, j_r+1} - (\delta \rho)_{j_0, \ldots, j_r+1,m} \| \leq C_{j_0, \ldots, j_r+1} \| v_m \|.
\]

We construct \( \{ \rho_{j_0, \ldots, j_r+1,m} \} \) as follows. (3.26) and Lemma 3.10 imply that there exists \( \gamma_{j_0, \ldots, j_r,m} \in L^{n,q-r-2}_{(2)}(V_{j_0, \ldots, j_r}^\omega, E; ds^2, h) \) and \( \mu_{j_0, \ldots, j_r+1,m} \in L^{n,q-r-2}_{(2)}(V_{j_0, \ldots, j_r}^\omega, E; ds^2, h) \) such that...
Throughout this section, \( f : X \to Y \) is a proper locally Kähler morphism from a complex space \( X \) to an irreducible complex space \( Y \). Assume that every irreducible component of \( X \) is mapped onto \( Y \), \( X^0 \subset X_{\text{reg}} \) is a dense Zariski open subset and \( (E, h) \) is a hermitian vector bundle on \( X^0 \) with Nakano semi-positive curvature. In this section we establish the Kollár package of \( SX \).

Theorem 4.1 (Torsion Freeness). Assume that \( S_X(E, h) \) is a coherent sheaf on \( X \). Then \( R^q f_* S_X(E, h) \) is torsion free for every \( q \geq 0 \) and vanishes if \( q > \dim X - \dim Y \).

Proof Since the problem is local, we assume that \( Y \) is Stein and there is a Kähler metric on \( X^0 \) which is locally complete, locally bounded from below by a hermitian metric and admits a \((\infty, 1)\) potential locally on \( X \) (Lemma 2.14). Define the sheaf \( \Omega_{X, ds^2, (2)}^p (E, h) \) as

\[
\Omega_{X, ds^2, (2)}^p (E, h) (U) = \left\{ \alpha \in \mathcal{D}_{X, ds^2}^p (E, h) (U) \mid \partial \alpha = 0 \right\}
\]

for every open subset \( U \subset X \).

By Proposition 3.4-(3), the Hodge star operator induces a well defined map

\[
* : \mathcal{H}_f^{n, q} (E, h) \to f_* \Omega_{X, ds^2, (2)}^{n-q} (E, h).
\]

Since the Lefschetz operator \( L \) with respect to \( ds^2 \) is bounded and \([L, \partial] = 0\), taking the finess of \( \mathcal{D}_{X, ds^2}^p (E, h) \) (Lemma 2.2) into account we get a well defined map

\[
L^q : f_* \Omega_{X, ds^2, (2)}^{n-q} (E, h) \to R^q f_* (\mathcal{D}_{X, ds^2}^{n-q} (E, h)).
\]
By Theorems 2.11 and 3.8, we get the morphisms

\[ R^q f_* S_X(E, h) \rightarrow f_* \Omega^{n-q}_{X, ds^2} (E, h) \rightarrow ^L R^q f_* S_X(E, h) \]

such that

\[ L^q \circ \ast = \sqrt{-1}^{(n-q)(n-q+3)} q! \text{Id} \quad (3.22). \]

This proves the first claim since \( f_* \Omega^{n-q}_{X, ds^2} (E, h) \) is torsion free for every \( q \geq 0 \). The vanishing result follows from Proposition 3.7-(1) and Theorem 3.8.

**Theorem 4.2** (Injectivity theorem). Assume that \( S_X(E, h) \) is a coherent sheaf on \( X \).

If \( L \) is a semi-positive holomorphic line bundle on \( X \) so that \( L \otimes l \) admits a nonzero holomorphic global section \( s \) for some \( l > 0 \), then the canonical morphism

\[ R^q f_* (\otimes s) : R^q f_* (S_X(E, h) \otimes L^k) \rightarrow R^q f_* (S_X(E, h) \otimes L^{k+l}) \]

is injective for every \( q \geq 0 \) and every \( k \geq 1 \).

**Proof** Since the problem is local, we assume that \( Y \) is Stein and there is a Kähler metric on \( X^o \) which is locally complete, locally bounded from below by a hermitian metric and admits a \((\infty, 1)\) potential locally on \( X \) (Lemma 2.14). Let \( h_L \) be the hermitian metric on \( L \) with semi-positive curvature. Then we have

\[ S_X(E, h) \otimes L^k \simeq S_X(E \otimes L^k, h \otimes h^k_L), \quad k \geq 1 \]

by Lemma 2.7. By Theorem 3.8, it is therefore sufficient to show that the canonical map

\[ \otimes s : \mathcal{H}^{n,q}(X, E \otimes L^k, h \otimes h^k_L) \rightarrow \mathcal{H}^{n,q}(X, E \otimes L^{k+l}, h \otimes h^{k+l}_L) \quad (4.1) \]

is well defined for an arbitrary Stein space \( Y \). Let \( \alpha \in \mathcal{H}^{n,q}(X, E \otimes L^k, h \otimes h^k_L, \varphi) \) for some \( \varphi \in \mathcal{P}(X) \) \((\mathcal{P}(X) \neq \emptyset \) since \( Y \) is Stein and \( f \) is proper). It follows from Proposition 3.4-(1) that

\[ D^{*s}(\alpha \otimes s) = - \ast \bar{\partial}^\ast (\alpha \otimes s) = D^{*s} \alpha \otimes s = 0 \]

and

\[
0 \leq (\sqrt{-1} e(\Theta_{h \otimes h^k_L}(E \otimes L^k) + \bar{\partial} \bar{\partial} \varphi) \Lambda(\alpha \otimes s), \alpha \otimes s)_{h \otimes h^{k+l}_L}
\]

\[
\leq \frac{k+l}{k} |s|_{h^k_L}^2 (\sqrt{-1} e(\Theta_{h_L}(E \otimes L^k) + \bar{\partial} \bar{\partial} \varphi) \Lambda(\alpha), \alpha)_{h \otimes h_L^{k+l}} = 0.
\]

Hence \( \alpha \otimes s \in \mathcal{H}^{n,q}(X, E \otimes L^{k+l}, h \otimes h^{k+l}_L, \varphi) \) by Proposition 3.4-(1). Hence (4.1) is well defined. Then (4.1) is injective because \( \{s \neq 0\} \) is dense. The proof is finished.

\[ \square \]

\[ \square \]

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Theorem 4.3 (Decomposition theorem) Assume that there exists a Kähler metric $ds^2$ on $X^o$ such that

1) $ds^2$ admits a $(\infty, 1)$ bounded potential locally on $X$;
2) $ds^2$ is locally complete on $X$ and is locally bounded from below by a hermitian metric.

Assume that $S_X(E, h)$ is a coherent sheaf on $X$. Then $Rf_*S_X(E, h)$ splits in $D(Y)$, i.e.,

$$Rf_*S_X(E, h) \simeq \bigoplus_q R^q f_*S_X(E, h)[-q] \in D(Y).$$

As a consequence, the spectral sequence

$$E_2^{pq} := H^p(Y, R^q f_*S_X(E, h)) \Rightarrow H^{p+q}(X, S_X(E, h))$$

degenerates at the $E_2$ page if $Y$ is compact.

Remark 4.4 By Lemma 2.14, such kind of metric exists if $X$ is compact.

Proof Let

$$\alpha : \bigoplus_q H^{n-q}_f(E, h)[-q] \to f_* \mathcal{G}^\bullet_{X, ds^2, h} \simeq Rf_* \mathcal{G}^\bullet_{X, ds^2, h}$$

be the inclusion map. By Theorem 3.8, $\alpha$ is a quasi-isomorphism under the hypothesis that $S_X(E, h)$ is coherent. Hence by Theorems 2.11 and 3.8 we obtain that

$$Rf_*S_X(E, h) \simeq Rf_* \mathcal{G}^\bullet_{X, ds^2, h} \simeq \bigoplus_q H^{n-q}_f(E, h)[-q] \simeq \bigoplus_q R^q f_*S_X(E, h)[-q]$$

in $D(Y)$. The degeneration of the spectral sequence follows from standard arguments. 

\[\Box\]

Theorem 4.5 (Vanishing theorem) Assume that $S_X(E, h)$ is a coherent sheaf on $X$. If $Y$ is a projective algebraic variety and $L$ is an ample line bundle on $Y$, then

$$H^q(Y, R^p f_*S_X(E, h) \otimes L) = 0, \ \forall q > 0, \ \forall p \geq 0.$$ 

Proof Since $S_X(E, h)$ is coherent, so is $R^p f_*S_X(E, h)$ for every $p \geq 0$. Then there is an integer $k$ large enough so that $H^0(Y, L^\otimes k) \neq 0$ and

$$H^q(Y, R^p f_*S_X(E, h) \otimes L^\otimes k+1) = 0, \ \forall q > 0, \ \forall p \geq 0. \ (4.2)$$

By Lemma 2.7, we acquire that

$$S_X(E, h) \otimes f^*L^\otimes l \simeq S_X(E \otimes f^*L^\otimes l, h \otimes f^*h_L^l), \ \forall l > 0 \ (4.3)$$
where $h_L$ is a hermitian metric on $L$ with positive curvature. Take $0 \neq s \in H^0(Y, L^\otimes k)$. By Theorem 4.2, the canonical map

$$\otimes f^*s : H^q(X, S_X(E \otimes f^*L, h \otimes f^*h_L)) \to H^q(X, S_X(E \otimes f^*L^\otimes k+1, h \otimes f^*h_L^{k+1})), \quad \forall q \geq 0$$

is injective. By Theorem 4.3, we know that the canonical map

$$\otimes s : H^q(Y, R^p f_* S_X(E \otimes f^*L, h \otimes f^*h_L)) \to H^q(Y, R^p f_* S_X(E \otimes f^*L^\otimes k+1, h \otimes f^*h_L^{k+1}))$$

is injective for every $p, q \geq 0$. Combining this with (4.2) and (4.3), we prove the theorem. \qed

5 Non-abelian Hodge theory and Kollár package

5.1 Harmonic bundle and variation of Hodge structure

The notion of harmonic bundles is used by Simpson [24] to establish a correspondence between local systems and semistable Higgs bundles with vanishing Chern classes over a compact Kähler manifold. A typical example of harmonic bundles comes from a polarized variation of Hodge structure (loc. cit.). A harmonic bundle produces a $\lambda$-connection structure which gives a $\square$-module on $\lambda = 1$ and a Higgs bundle on $\lambda = 0$. This is the main subject of non-abelian Hodge theory. We only review the necessary knowledge of this topic that is used in the present paper. Readers may consult [10, 11, 14, 22–24] for more details.

Let $(M, ds^2)$ be a complex manifold. A Higgs bundle $(H, \theta)$ on $M$ consists of a holomorphic vector bundle $H$ on $M$ together with an $\mathcal{O}_M$-linear map $\theta : H \to H \otimes \Omega^1_M$ such that $\theta \wedge \theta = 0$. Let $(H, \theta, h)$ be a Higgs bundle on $M$ with a hermitian metric $h$. Let $\overline{\theta}$ be the adjoint of $\theta$ and let $\partial$ be the unique $(1, 0)$-connection such that $\partial + \overline{\partial}$ is compatible with $h$.

**Definition 5.1** $(H, \theta, h)$ is called a harmonic bundle if $(\partial + \overline{\partial} + \theta + \overline{\theta})^2 = 0$. In this case, $h$ is called a harmonic metric.

Let $\nabla_h$ be the Chern connection on $H$ with respect to $h$ and let $\Theta_h(H) = \nabla_h^2$ be its Chern curvature form. Then we have the self-dual equation

$$\Theta_h(H) + \theta \wedge \overline{\theta} + \overline{\theta} \wedge \theta = 0. \quad (5.1)$$

For the purpose of the present paper, we are interested in tame harmonic bundles in the sense of Simpson [23] and Mochizuki [10, 11].

**Definition 5.2** Let $X$ be a complex space and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset.
(1) Assume that \( X \) is smooth and \( X \setminus X^o \subset X \) is a normal crossing divisor. Then the Higgs field \( \theta \) of the Higgs bundle \( (H, \theta) \) can be described as:

\[
\theta = \sum_{i=1}^r f_i \frac{d\bar{z}_i}{z_i} + \sum_{j=r+1}^n g_j dz_j \quad (5.2)
\]

under holomorphic local coordinates \( z_1, \ldots, z_n \) such that \( X \setminus X^o = \{ z_1 \ldots z_r = 0 \} \).

The harmonic bundle \( (H, \theta, h) \) is tame (with respect to \( X^o \subset X \)) if the coefficients of the characteristic polynomials \( \det(t - f_i) \) and \( \det(t - g_j) \) can be extended to the holomorphic functions on \( X \).

(2) For a general \( X \) and \( X^o \), we say that \( (H, \theta, h) \) is tame if for every point \( x \in X \) there is an open neighborhood \( U \) of \( x \), a desingularization \( \pi : \tilde{U} \to U \) such that

- \( \pi^o := \pi|_{\pi^{-1}(X^o \cap U)} : \pi^{-1}(X^o \cap U) \to X^o \cap U \) is biholomorphic and \( \pi^{-1}(U \setminus X^o) \subset \tilde{U} \) is a normal crossing divisor.
- \( (\pi^o H, \pi^o \theta, \pi^o h) \) is a tame harmonic bundle (with respect to \( \pi^{-1}(X^o \cap U) \subset \tilde{U} \)).

A typical type of tame harmonic bundles is the variation of Hodge structure.

**Definition 5.3** [22, Sect. 8] Let \( M \) be a complex manifold. Denote by \( \mathcal{A}^0_M \) the sheaf of \( C^\infty \) functions on \( M \). A polarized complex variation of Hodge structure \( \mathbb{V} := (\mathcal{V}, \nabla, \{ \mathcal{V}^{p,q} \}, Q) \) on \( M \) of weight \( k \) consists of a flat holomorphic connection \( (\nabla, \mathcal{V}) \) on \( M \) together with a decomposition \( \mathcal{V} \otimes \mathcal{O}_M \mathcal{A}^0_M = \bigoplus_{p+q=k} \mathcal{V}^{p,q} \) of \( C^\infty \) bundles, and a flat hermitian form \( Q \) on \( \mathcal{V} \) such that

1. The hermitian form \( h_Q \) which equals \((-1)^p Q \) on \( \mathcal{V}^{p,q} \) is a hermitian metric on the \( C^\infty \) complex vector bundle \( \mathcal{V} \otimes_M \mathcal{A}^0_M \).
2. The Griffiths transversality condition

\[
\nabla(\mathcal{V}^{p,q}) \subset \mathcal{A}^{0,1}(\mathcal{V}^{p+1,q-1}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{0,1}(\mathcal{V}^{p,q}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{p-1,q+1})
\]

holds for every \( p \) and \( q \). Here \( \mathcal{A}^{p,q}(\mathcal{V}^{i,j}) \) denotes the sheaf of \( C^\infty \) \((p, q)\)-forms with values in \( \mathcal{V}^{i,j} \).

Denote \( S(\mathbb{V}) := \mathcal{V}^{p_{\text{max}},k-p_{\text{max}}} \) where \( p_{\text{max}} = \max\{ p | \mathcal{V}^{p,k-p} \neq 0 \} \).

Let \( \mathbb{V} := (\mathcal{V}, \nabla, \{ \mathcal{V}^{p,q} \}, Q) \) be a polarized complex variation of Hodge structure of weight \( k \). Take the decomposition

\[
\nabla = \bar{\theta} + \bar{\partial} + \tilde{\partial} + \theta
\]

according to (5.3). The triple \( (H = \text{Ker} \tilde{\theta}, \theta, h_Q) \) is a harmonic bundle associated with \( (\mathcal{V}, \nabla, h_Q) \) via Simpson’s correspondence [22, Sect. 8]. There is moreover an orthogonal decomposition of holomorphic subbundles \( H = \bigoplus_{p+q=k} H^{p,q} \) where \( H^{p,q} = H \cap \mathcal{V}^{p,q} \) and

\[
\theta(H^{p,q}) \subset H^{p-1,q+1} \otimes \Omega_M. \quad (5.4)
\]
The following proposition is known to experts. We recall its proof in sketch for the convenience of readers.

**Proposition 5.4** Let $X$ be a complex space and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset. Let $(V, \nabla, \{\nabla^{p,q}\}, Q)$ be a polarized complex variation of Hodge structure on $X^o$. Then the associated harmonic bundle $(H = \text{Ker} \tilde{\theta}, \theta, h_Q)$ is tame.

**Proof** Let $x \in X$. Take an open neighborhood $U$ of $x$, a desingularization $\pi : \tilde{U} \to U$ such that $\pi^0 := \pi|_{\pi^{-1}(X^o \cap U)} : \pi^{-1}(X^o \cap U) \to X^o \cap U$ is biholomorphic and $\pi^{-1}(U \setminus X^o) \subset \tilde{U}$ is a normal crossing divisor. By (5.4) the Higgs field $\pi^0*\theta$ is nilpotent. Therefore $f_i$ and $g_j$ in the local expression (5.2) are nilpotent matrices for each $i = 1, \ldots, r$ and each $j = r + 1, \ldots, n$. Therefore $\det(t - f_i) = t^n$ and $\det(t - g_j) = t^n$. Thus the harmonic bundle $(H = \text{Ker} \tilde{\theta}, \theta, h_Q)$ is tame. \qed

### 5.2 Tame harmonic bundles and the Kollár package

Let $(H, h)$ be a hermitian vector bundle and $E \subset H$ a subbundle (i.e., both $E$ and $H/E$ are locally free). The second fundamental form of $E \subset H$ is defined as $b(H, E, h) = \nabla_H\big|_E - \nabla_E \in \mathfrak{A}(\text{Hom}(E, H/E))$ where $\nabla_H$ and $\nabla_E$ are the Chern connections associated with the hermitian bundles $(H, h)$ and $(E, h)$ respectively. Notice that $b(H, E, h) = 0$ is equivalent to the fact that the short exact sequence

$$0 \to E \to H \to H/E \to 0$$

splits holomorphically. Equivalently, $E$ is a holomorphic direct summand of $H$.

**Proposition 5.5** Let $X$ be a complex space and $X^o \subset X_{\text{reg}}$ a dense Zariski open subset. Let $(H, \theta, h)$ be a tame harmonic bundle on $X^o$. Let $E \subset H$ be a holomorphic subbundle with vanishing second fundamental form. Assume that $\tilde{\theta}(E) = 0$. Then $(E, h)$ is a tame hermitian vector bundle with Nakano semi-positive curvature. As a consequence, $S_X(E, h)$ is a coherent sheaf on $X$.

**Proof** To prove the tameness property (Definition 2.8), we construct $Q$ by Mochizuki’s prolongation construction. Since the problem is local, we assume that there is a desingularization $\pi : \tilde{X} \to X$ such that $\pi$ is biholomorphic over $X^o$ and $D := \pi^{-1}(X \setminus X^o)$ is a simple normal crossing divisor. By abuse of notations we identify $X^o$ and $\pi^{-1}X^o$. Since $(H, \theta, h)$ is tame, by [12, Proposition 2.53] there is a logarithmic Higgs bundle $(\tilde{H}, \tilde{\theta})$:

$$\tilde{\theta} : \tilde{H} \to \Omega_{\tilde{X}}(\log D) \otimes \tilde{H},$$

such that $(\tilde{H}, \tilde{\theta})|_{X^o}$ is holomorphically isomorphic to $(H, \theta)$. Let $D = \bigcup_{i=1}^k D_i$ be the irreducible decomposition and let $z_1, \ldots, z_n$ be holomorphic local coordinates such that $D_i = \{z_i = 0\}$ for each $i = 1, \ldots, k$. By [10, Part 3, Chapter 13], the tameness property of $(H, \theta, h)$ forces a norm estimate

$$|z_1 \ldots z_k|^{2b}|s|_{h_0} \lesssim |s|_h \quad (5.5)$$
for any local holomorphic section $s$ of $\tilde{H}$ and a constant $b > 0$ which is independent of $s$. Here $h_0$ is an arbitrary hermitian metric on $\tilde{H}$.

Let $Q := \tilde{H}$. Then $E$ is a subsheaf of $Q|_{X^o}$. By (5.5), we see that $(E, h)$ is tame.

To see that $(E, h)$ is Nakano semi-positive, we take the decomposition

$$\nabla = \partial + \bar{\partial} + \theta$$

according to (5.3). Since $E \oplus E^\perp$ is an orthogonal holomorphic decomposition of $H$ and $\tilde{\partial}(E) = 0$, we get from (5.1) that

$$\sqrt{-1} \Theta_h(E) = -\sqrt{-1} \partial \wedge \theta \geq 0.$$ 

Hence $(E, h)$ is Nakano semi-positive. By Proposition 2.9, $S_X(E, h)$ is a coherent sheaf on $X$.

**Example 5.6** (Top Hodge piece of a polarized complex variation of Hodge structure). Let $V := (V, \nabla, (V^p, q), Q)$ be a polarized complex variation of Hodge structure of weight $k$. Let

$$\nabla = \bar{\partial} + \partial + \bar{\partial} + \theta$$

be the decomposition according to (5.3). Let $(H = \text{Ker} \bar{\partial}, \theta, h_Q)$ be the tame harmonic bundle associated to $(V, \nabla, h_Q)$ [22, Sect. 8]. For the reason of degrees, $S(V)$ is a holomorphic subbundle of $H$ and $\tilde{\partial}(S(V)) = 0$. By Proposition 5.4, we acquire that $(H = \text{Ker} \bar{\partial}, \theta, h_Q)$ and $S(V)$ satisfy the conditions in Proposition 5.5. Hence $S_X(S(V), h_Q)$ is a coherent sheaf. Applying Theorem 1.1 to this case one gets a solution to Kollár’s conjecture.

When $V$ is $\mathbb{R}$-polarized, Saito [18] constructs a coherent sheaf $S(IC_X(V))$ as the top Hodge piece of the Hodge module $IC_X(V)$. The following proposition shows that $S_X(S(V), h_Q)$ coincides with Saito’s $S$-sheaf associated to $IC_X(V)$ when $V$ is $\mathbb{R}$-polarized.

**Proposition 5.7** Let $V := (V, \nabla, F^\bullet, Q)$ be an $\mathbb{R}$-polarized variation of Hodge structure. Then

$$S_X(S(V), h_Q) \simeq S(IC_X(V)).$$

**Proof** See [19] or [20, Theorem 4.10].

Now we are ready to show the Kollár package with respect to a tame harmonic bundle.

**Theorem 5.8** Let $f : X \to Y$ be a proper locally Kähler morphism from a complex space $X$ to an irreducible complex space $Y$. Assume that every irreducible component of $X$ is mapped onto $Y$. Let $X^o \subset X_{\text{reg}}$ be a dense Zariski open subset and $(H, \theta, h)$ a tame harmonic bundle on $X^o$. Let $E \subset H$ be a holomorphic subbundle with vanishing second fundamental form. Assume that $\tilde{\partial}(E) = 0$. Let $F$ be a Nakano semi-positive vector bundle on $X$. Then the following statements hold.
Torsion freeness $R^q f_*(S_X(E, h) \otimes F)$ is torsion free for every $q \geq 0$ and vanishes if $q > \dim X - \dim Y$.

Injectivity theorem If $L$ is a semi-positive holomorphic line bundle so that $L^{\otimes l}$ admits a nonzero holomorphic global section $s$ for some $l > 0$, then the canonical morphism

$$R^q f_*(\times s) : R^q f_*(S_X(E, h) \otimes F \otimes L^k) \to R^q f_*(S_X(E, h) \otimes F \otimes L^{\otimes k+1})$$

is injective for every $q \geq 0$ and every $k \geq 1$.

Vanishing theorem If $Y$ is a projective algebraic variety and $L$ is an ample line bundle on $Y$, then

$$H^q(Y, R^p f_*(S_X(E, h) \otimes F) \otimes L) = 0, \quad \forall q > 0, \quad \forall p \geq 0.$$

Decomposition theorem Assume moreover that $X$ is a compact Kähler space. Then $Rf_*(S_X(E, h) \otimes F)$ splits in $D(Y)$, i.e.,

$$Rf_*(S_X(E, h) \otimes F) \simeq \bigoplus_q R^q f_*(S_X(E, h) \otimes F)[-q] \in D(Y).$$

As a consequence, the spectral sequence

$$E_2^{pq} : H^p(Y, R^q f_*(S_X(E, h) \otimes F)) \Rightarrow H^{p+q}(X, S_X(E, h) \otimes F)$$

degenerates at the $E_2$ page.

Proof By Proposition 5.5, $S_X(E, h)$ is a coherent sheaf. Let $h_F$ be a hermitian metric on $F$ with Nakano semi-positive curvature. By Lemma 2.7, we see that

$$S_X(E, h) \otimes F \simeq S_X(E \otimes F, h \otimes h_F)$$

is a coherent sheaf on $X$. Note that $(E \otimes F, h \otimes h_F)$ is Nakano semi-positive by Proposition 5.5. It follows from the abstract Kollár package in §4 that the statements in the theorem are valid.

We end this section with remarks on two other packages of Kollár’s conjecture.

Remark 5.9 (Remark on the intersection cohomology package) In [9, Sect. 5.8], Kollár also predicts that $S(\text{IC}_X(\mathbb{V}))$ is related to the intersection complex $\text{IC}_X(\mathbb{V})$ when $\mathbb{V} = (\mathcal{V}, \nabla, F^\bullet, Q)$ is an $\mathbb{R}$-polarized variation of Hodge structure. This involves the $L^2$-representation of the intersection complex.

Theorem 5.10 Let $X$ be a compact Kähler space of pure dimension $n$ and $X^\circ \subset X_{\text{reg}}$ a dense Zariski open subset. Let $\mathbb{V} = (\mathcal{V}, \nabla, F^\bullet, Q)$ be an $\mathbb{R}$-polarized variation of
Hodge structure of weight \( r \) on \( X^o \). Then \( IH^k(X, \mathcal{V}) \) admits a pure Hodge structure of weight \( k \):

\[
IH^k(X, \mathcal{V}) = \bigoplus_{p,q \geq 0, \ p+q=k+r} IH^{p,q}(X, \mathcal{V}), \quad IH^{p,q}(X, \mathcal{V}) = IH_{q-p}(X, \mathcal{V})
\]

for every \( 0 \leq k \leq 2 \dim X \). There is moreover a morphism

\[
IC_X(\mathcal{V}) \to S(IC_X(\mathcal{V}))
\]

in the derived category of sheaves of \( \mathbb{C} \)-vector spaces which induces an isomorphism

\[
IH^{n+r,q}(X, \mathcal{V}) \to H^q(X, S(IC_X(\mathcal{V}))), \quad \forall q \geq 0.
\]

**Proof** The first statement is a consequence of [21, Theorem 1.4], which states that there is a complete Kähler metric \( ds^2 \) on \( X^o \) whose \( L^2 \)-de Rham complex \( D^{\bullet}_{X,\mathcal{V};ds^2,h} \) is quasi-isomorphic to \( IC_X(\mathcal{V}) \). As a consequence, there is a canonical isomorphism

\[
IH^k(X, \mathcal{V}) \simeq H^{2k}_{(2)}(X^o, \mathcal{V}; ds^2, h), \quad \forall k.
\]

The \((p, q)\)-decomposition of forms in \( D^{\bullet}_{X,\mathcal{V};ds^2,h} \) provides the Hodge structure on \( IH^k(X, \mathcal{V}) \). See [21, Sect. 8.3] for details. The second claim follows from the diagram

\[
\begin{array}{ccc}
D^{\bullet}_{X,\mathcal{V};ds^2,h} & \xrightarrow{\tau} & D^{n+\bullet}_{X,ds^2}(S(\mathcal{V}), h) \\
\downarrow \simeq & & \downarrow \simeq \ (\text{Theorem 2.11, Proposition 5.7}) \\
IC_X(\mathcal{V}) & & S(IC_X(\mathcal{V}))
\end{array}
\]

where \( \tau \) is taking the projection to the \( S(\mathcal{V}) \)-valued \((n, \bullet)\)-component. \( \square \)

**Remark 5.11** (Remark on the direct image package) In [9, Sect. 5.8], Kollár also predicts that

\[
R^q f_*(IC_X(\mathcal{V})) \simeq S_Y(R^q f_*IC_X(\mathcal{V})|_{Y^o})
\]

where \( Y^o \) is the Zariski open subset of \( Y \) so that \( R^q f_*IC_X(\mathcal{V})|_{Y^o} \) is a local system whose fiber at \( y \in Y^o \) is canonically isomorphic to \( H^q(X_y, IC_{X_y}(\mathcal{V}|_{X_y \cap X^o})) \). This is also a consequence of the \( L^2 \)-representation of \( IC_X(\mathcal{V}) \) [21, Theorem 1.4].

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References

1. Arapura, D.: Kodaira–Saito vanishing via Higgs bundles in positive characteristic. J. Reine Angew. Math. 755, 293–312 (2019)
2. Atiyah, M.F., Macdonald, I.G.: Introduction to Commutative Algebra, p. MR0242802. Addison-Wesley Publishing Co., Reading (1969)
3. Berndtsson, B.: An Introduction to Things $\bar{\partial}$, Analytic and Algebraic Geometry, pp. 7–76 (2010)
4. Demailly, J.-P.: Estimations $L^2$ pour l’opérateur $\bar{\partial}$ d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété kählérienne complète. Ann. Sci. École Norm. Sup. 15(4), 457–511 (1982)
5. Demailly, J.-P.: Analytic Methods in Algebraic Geometry, Surveys of Modern Mathematics, vol. 1. International Press, Somerville (2012)
6. Deng, Y., Feng, H.: Vanishing Theorem for Tame Harmonic Bundles via $l^2$-Cohomology. arXiv:1912.02586
7. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, Third, North-Holland Mathematical Library, vol. 7. North-Holland Publishing Co., Amsterdam (1990)
8. Kollár, J.: Higher direct images of dualizing sheaves. I. Ann. Math 2 123(1), 11–42 (1986)
9. Kollár, J.: Higher direct images of dualizing sheaves. II. Ann. Math. (2) 124(1), 171–202 (1986)
10. Mochizuki, T.: Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules I. Mem. Am. Math. Soc. 185(869), xii+324 (2007)
11. Mochizuki, T.: Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules II. Mem. Am. Math. Soc. 185(870), xii+565 (2007)
12. Mochizuki, T.: Kobayashi–Hitchin correspondence for tame harmonic bundles. II. Geom. Topol. 13(1), 359–455 (2009)
13. Pareschi, G., Popa, M., Schnell, C.: Hodge modules on complex tori and generic vanishing for compact Kähler manifolds.Geom. Topol. 21(4), 2419–2460 (2017)
14. Sabbah, C.: Polarizable twistor D-modules. Astérisque 300, vi+208 (2005)
15. Saito, M.: Modules de Hodge polarisables. Publ. Res. Inst. Math. Sci. 24(6), 849–995 (1989)
16. Saito, M.: Decomposition theorem for proper Kähler morphisms. Tohoku Math. J. 2(2), 127–147 (1990)
17. Saito, M.: Mixed Hodge modules. Publ. Res. Inst. Math. Sci. 26(2), 221–333 (1990)
18. Saito, M.: On Kollár’s Conjecture, Several Complex Variables and Complex Geometry, Part 2 (Santa Cruz, CA, 1989), pp. 509–517 (1991)
19. Schnell, C., Yang, R.: Hodge modules and singular hermitian metrics. arXiv:2003.09064
20. Shentu, J., Zhao, C.: $L^2$-Dolbeault resolution of the lowest Hodge piece of a Hodge module and an application to the relative Fujita conjecture. arXiv:2104.04905
21. Shentu, J., Zhao, C.: $L^2$-representation of Hodge modules. arXiv:2103.04030
22. Simpson, C.T.: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization. J. Am. Math. Soc. 1(4), 867–918 (1988)
23. Simpson, C.T.: Harmonic bundles on noncompact curves. J. Am. Math. Soc. 3(3), 713–770 (1990)
24. Simpson, C.T.: Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math. 75, 5–95 (1992)
25. Takegoshi, K.: Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms. Math. Ann. 303(3), 389–416 (1995)
26. Wells, R.O., Jr.: Differential Analysis on Complex Manifolds, Second, Graduate Texts in Mathematics, vol. 65. Springer, New York (1980)

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