Embedding binary sequences into Bernoulli site percolation on $\mathbb{Z}^3$

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Abstract

We investigate the problem of embedding infinite binary sequences into Bernoulli site percolation on $\mathbb{Z}^d$ with parameter $p$, known also as percolation of words. In 1995, I. Benjamini and H. Kesten proved that, for $d \geq 10$ and $p = 1/2$, all sequences can be embedded, almost surely. They conjectured that the same should hold for $d \geq 3$. In this paper we consider $d \geq 3$ and $p \in (p_c(d), 1 - p_c(d))$, where $p_c(d) < 1/2$ is the critical threshold for site percolation on $\mathbb{Z}^d$. We show that there exists an integer $M = M(p)$, such that, a.s., every binary sequence, for which every run of consecutive 0s or 1s contains at least $M$ digits, can be embedded.

1 Introduction

1.1. Statement of the result. Fix $d \geq 3$, and consider Bernoulli site percolation on $\mathbb{Z}^d$ with parameter $p \in (0, 1)$, i.e. take $(\Omega, \mathcal{A}, P_p)$, where $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, $\mathcal{A}$ is the canonical product $\sigma$-algebra, and $P_p = \bigotimes_{v \in \mathbb{Z}^d} P^v_p$, where $P^v_p(\omega_v = 1) = p = 1 - P^v_p(\omega_v = 0)$. An element $\omega \in \Omega = \{0, 1\}^{\mathbb{Z}^d}$ is called a percolation configuration.

Let $\xi \in \Xi := \{0, 1\}^\mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$. For a given $v \in \mathbb{Z}^d$ and $\omega \in \Omega$, we say that $\xi$ can be embedded in $\omega$ starting from $v$, if there exists an infinite nearest-neighbor vertex self-avoiding path $v_0 = v, v_1, v_2, \ldots$ such that $\omega_{v_i} = \xi_i$ for all $i \geq 1$. We say that $\xi$ can be embedded in $\omega$, if there exists $v = v(\omega) \in \mathbb{Z}^d$ such that $\xi$ can be embedded in $\omega$ starting from $v$.

Given $v \in \mathbb{Z}^d$ and $\omega \in \Omega$, define

$$S_v(\omega) := \{\xi \in \Xi : \xi \text{ can be embedded in } \omega \text{ starting from } v\}$$

and

$$S_\infty(\omega) := \cup_{v \in \mathbb{Z}^d} S_v(\omega),$$

which are measurable [3 Prop. 2].

Among other results, Benjamini and Kesten proved in [3] that for $d \geq 10$ and $p = 1/2$, all sequences can be embedded, almost surely. More precisely,
Theorem 1. Consider $\mathbb{Z}_+^d$ with all edges oriented in the positive direction and $p = 1/2$. Then for $d \geq 10$,

$$\mathbb{P}_{1/2}\left(S_\infty = \Xi\right) = 1,$$

and for $d \geq 40$,

$$\mathbb{P}_{1/2}\left(S(v) = \Xi \text{ for some } v\right) = 1.$$ (2)

They also conjectured [3, Open problem 2, p. 1029] that (1) should hold for all $d \geq 3$.

The main result of this work makes a step forward in the search for an affirmative answer to the question of Benjamini and Kesten.

For $\xi = (\xi_1, \xi_2, \ldots) \in \Xi$, we define $I(\xi) := \{i \geq 1 : \xi_i \neq \xi_{i+1}\}$. Set $i_0(\xi) := 0$, and $i_{j+1}(\xi) := \inf\{I(\xi) \setminus \{i_0(\xi), \ldots, i_j(\xi)\}\}$. Let

$$r_j(\xi) := i_{j+1}(\xi) - i_j(\xi).$$

This is well-defined as soon as $|I(\xi)| > j$. If $I(\xi) = \{i_1(\xi), \ldots, i_j(\xi)\}$, for some $j < +\infty$, we set $r_{j+k}(\xi) := +\infty$ for all $k \geq 1$. When $|I(\xi)| < +\infty$, we say that $\xi$ is ultimately monochromatic, and denote $\Xi_{um} := \{\xi : \xi$ is ultimately monochromatic$\}.$

Definition 1. If there exists $M \geq 1$ such that $r_j(\xi) \geq M$ for all $j \geq 1$, we say that $\xi$ is $M$-stretched. For a given $M$, we denote $\Xi_M := \{\xi : \xi$ is $M$-stretched$\}.$

Let $p_c(d)$ be the critical threshold for site percolation on $\mathbb{Z}_+^d$, which for $d \geq 3$ is strictly smaller than $1/2$ (see [5]). We are ready to state our main result:

Theorem 2. Let $d \geq 3$. Consider site percolation on $\mathbb{Z}_+^d$ with parameter $p \in (p_c(d), 1 - p_c(d))$. There exists $M(p)$ for which:

$$\mathbb{P}_p(\Xi_M(p) \subset S_\infty) = 1.$$ (3)

Remark 1. Since $p \in (p_c(d), 1 - p_c(d))$, it implies that all ultimately monochromatic sequences can be embedded almost surely, thus

$$\mathbb{P}_p((\Xi_M(p) \cup \Xi_{um}) \subset S_\infty) = 1.$$ (4)

Remark 2. The following construction shows that for site percolation on $\mathbb{Z}_+^d$ with parameter $p = 1/2$, the value $M = 2$ can be achieved as soon as $p_c(d - 1) < 1/4$. For that, consider the subset $\mathbb{Z}^{d-1} \times \{0, 1\}$ of $\mathbb{Z}_+^d$, and call a vertex $v = (v_1, \ldots, v_{d-1}) \in \mathbb{Z}^{d-1}$ good if $v' := (v_1, \ldots, v_{d-1}, 0)$ and $v'' := (v_1, \ldots, v_{d-1}, 1)$ satisfy $\omega_{v'} = 0$ and $\omega_{v''} = 1$. Hence, vertices of
$Z^{d-1}$ are good with probability $1/4$ each, independently of each other. If $p_c(d-1) < 1/4$, then there exists a.s. an infinite self-avoiding path $\gamma$ of good vertices in $Z^{d-1}$. Then, in the subset of $Z^{d-1} \times \{0,1\}$ with projection $\gamma$ on $Z^{d-1}$, one can embed any 2-stretched binary sequence. Numerical simulations in [7] suggest that this happens for $d \geq 5$ (they provide $p_c(4) = 0.19688\ldots$).

1.2. Comments and conjectures. It seems that Dekking [9] was the first to consider the question of whether $S(v)$ is equal to $\Xi$ in the context of percolation on regular trees. Benjamini and Kesten [3] investigated this problem in a general setup under the name percolation of words, and considered the case $p = 1/2$. For motivation, historical account and some related works, see their paper and the references therein. Besides the question which motivated our present work, another question discussed in [3] was: what happens for low-dimensional graphs? In particular, an interesting case is when the value $1/2$ is the critical parameter for site percolation as, for example, on the triangular lattice $T$. Since in this case, a.s. neither open nor closed infinite clusters exist, some sequences cannot be embedded, and therefore $P_p(S_\infty = \Xi) = 0$. Thus, one may ask how rich the set of binary sequences which can be embedded is. Even if one cannot embed all the sequences, it is possible that $S_\infty$ consists of “almost all” sequences in the following sense: let $\nu_\mu = \otimes_{i=1}^\infty \nu_\mu^i$ be the Bernoulli product measure with parameter $\mu$ on the set of binary sequences $\Xi$, i.e. $\nu_\mu^i(\xi_j = 1) = \mu, j = 1, 2, \ldots$. For a rather general class of graphs, and in particular on $Z^d$, for each $\xi \in \Xi$,

$$\rho(\xi) := P_p(\xi \text{ can be embedded from some } v) = 0 \text{ or } 1.$$  

We will say that $\xi$ percolates if $\rho(\xi) = 1$. Moreover, see [3, (1.12)],

$$\nu_\mu(\{\xi : \rho(\xi) = 1\}) = 0 \text{ or } 1.$$  

In the former (latter) case we say that almost no sequence (almost all sequences, respectively) can be embedded. In [10] it was shown that in the case of the triangular lattice $T$ and $p = 1/2$, almost all sequences can be embedded regardless of the value $0 < \mu < 1$. Returning to our original question of embeddings on $Z^d$, observe that the monochromatic sequences $0 := (0,0,\ldots)$ and $1 := (1,1,\ldots)$ are the least likely to percolate, in the sense that for any $\xi \in \Xi$ and any $v$,

$$P_p(\xi \text{ can be embedded in } \omega \text{ starting from } v) \geq \min_{\zeta \in \{0,1\}} P_p(\zeta \text{ can be embedded in } \omega \text{ starting from } v),$$  

which follows from [16, Prop. 3.1], see also [13, Lemma 2]. Inequality (5) immediately implies that on $Z^d$, $d \geq 3$, for $p \in (p_c(d), 1 - p_c(d))$, almost all
binary sequences can be embedded almost surely. Though, the a.s. simultaneous occurrence of $0 := (0, 0, \ldots)$ and $1 := (1, 1, \ldots)$ strongly supports the idea that all binary sequences can be embedded, it still remains far from being understood and settled. Besides Theorem 1 in [3] mentioned above, for $\mathbb{Z}^d, d \geq 10$, the only low-dimensional result was obtained in [11], where it was shown that $\mathbb{P}_{p}(S_\infty = \Xi) = 1$ if $p \in (p_c(d), 1 - p_c(d))$ for $\mathbb{Z}^2_{cp}$ – the close-packed graph of $\mathbb{Z}^2$, that is, the graph obtained by adding to each face of $\mathbb{Z}^2$ the two diagonal edges.

**Conjecture and open problems.** The following classification conjecture was stated by two of the authors.$^1$

Let $G$ be an infinite graph, with uniformly bounded degree, and $p_G^c$ denote its critical threshold for site percolation.

I. If $p_G^c > 1/2$, then

a) For $p \in (0, 1 - p_G^c) \cup [p_G^c, 1)$, there exists $0 < \mu_c(p) < 1$, such that for $p \leq 1 - p_G^c$ almost all binary sequences can be embedded if $\mu \leq \mu_c(p)$ and almost no binary sequences can be embedded if $\mu_c(p) < \mu$. Similar holds for $p_G^c \leq p$: almost all sequences can be embedded if $\mu_c(p) \leq \mu$, and almost no sequences can be embedded if $\mu < \mu_c(p)$.

b) If $p \in (1 - p_G^c, p_G^c)$, then for any $\mu$, almost no sequences can be embedded.

II. If $p_G^c \leq 1/2$, then

a) For $p \in (0, p_G^c) \cup (1 - p_G^c, 1)$, there exists $0 < \mu_c(p) < 1$, such that almost all binary sequences can be embedded for $\mu \leq \mu_c(p)$ if $p \leq p_G^c$, and for $\mu_c(p) \leq \mu$ if $1 - p_G^c \leq p$. Respectively, almost no sequences can be embedded if $\mu > \mu_c(p)$ or $\mu < \mu_c(p)$.

b) If $p = p_G^c$ or $p = 1 - p_G^c$, then almost all sequences can be embedded for all $0 < \mu < 1$.

c) If $1 - p_G^c < p < p_G^c$, then almost all sequences can be embedded.

Cases a) of I and II are similar. For the case $p_G^c \leq p$ of I a) or $1 - p_G^c \leq p$ of II a), a Peierls’ type argument shows that $0 < \mu_c$. To obtain $\mu_c < 1$ is more difficult due to the multi-scale nature of the problem, and requires elaborated tools. It is a corollary of the main Theorem 1 of [9]. Problems of similar nature are treated in [11] and [8].

**Open problem 1.** Does I b) hold under these general hypotheses on $G$, or are there some additional assumptions required?

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$^1$B.N.B.L. and V.S.
Open problem 2. Establish II b). In [10], it was established for the triangular lattice. However, their proof heavily uses geometric properties of $T$. It would be interesting to establish II b) for percolation on the Voronoi tessellation, or for site percolation on a “random” triangular lattice – the graph obtained from $\mathbb{Z}^2$ by adding to its faces only one, randomly chosen diagonal (with angle $\pm \pi/4$). It is known ([3] and [15] respectively) that for these graphs the site percolation threshold equals $1/2$. The methods of [10] do not apply for these models.

2 Proof of Theorem 2

2.1 Preliminaries

From now on, we restrict ourselves to the case where $d = 3$, but the same proofs apply to higher dimensions, up to minor modifications. The main idea of the proof is to construct a pair of self-avoiding 0- and 1-paths that approach each other with a certain regularity in some particular structures, that we call outlets. This pair of paths is represented schematically in Figure 1.

Let us now start the construction. Considering the box $R = [0, l^{(1)}] \times [0, l^{(2)}] \times [0, l^{(3)}]$, we define

$$C(R) := \{\text{there exists a 1-path inside } R, \text{ connecting its left side } [0, l^{(1)}] \times \{0\} \times [0, l^{(3)}] \text{ to its right side } [0, l^{(1)}] \times \{l^{(2)}\} \times [0, l^{(3)}]\},$$

and also

$$V_1^1(R) := U_1^1(R)^c,$$
where
\[ U_1^t(R) := \{ \text{there exist two disjoint connected 1-subsets of } R, \text{ both} \]
\[ \text{with diameter at least } t, \text{ which are not connected} \]
\[ \text{by a 1-path lying entirely in } R \}. \] (8)
We will also use the notation \( V_0^t \) for the event obtained by replacing 1-vertices by 0-vertices in the definition of \( V_1^t \).

The following lemma will be used repeatedly.

**Lemma 1.** There exist constants \( c_1 = c_1(p) > 0 \) and \( c_2 = c_2(p) > 0 \) such that, for all \( N \geq 1 \) and all \( l^{(j)}_N \in [N, 10N] \) \((j = 1, 2, 3)\), we have:
\[ P_p(\mathcal{C}(R)) \geq 1 - c_1 e^{-c_2 N}, \] (9)
and
\[ P_p(V_k^t(R)) \geq 1 - c_1 e^{-c_2 N} \quad (k = 0, 1), \] (10)
where \( R = [0, l^{(1)}_N] \times [0, l^{(2)}_N] \times [0, l^{(3)}_N] \).

These two properties follow from Theorem 5 in \cite{14}. Strictly speaking, they are stated for hypercubes in that paper, but they can be adapted to the case of more general boxes, of the form considered here, by standard gluing arguments.

### 2.2 Construction of outlets

In order to make our arguments more symmetric, we also consider the shifted lattice \( \mathbb{Z}^3_\star = \mathbb{Z}^3 + (0, 1/2, 1/2) \). We further denote by \( \omega \in \{0, 1\}^{\mathbb{Z}^3_\star} \) a generic 0 and 1 site percolation configuration. We start with a lemma.

Denote by \( \Gamma(R_L) \) the event that there exists a 1-path \( \gamma \) that stays inside \( R_L = [(-L, L) \times (0, 8L) \times (0, 2L)] \cap \mathbb{Z}^3_\star \) and connects \((0, 1/2, 1/2)\) to the right side of \( R_L \) (see Figure 2).

**Lemma 2.** There exist \( \delta_0 > 0 \) and \( L_0 \geq 1 \), depending only on \( p \), such that for all \( L \geq L_0 \),
\[ P_p(\Gamma(R_L)) \geq \delta_0. \]

**Proof.** We know from \cite{2} that the critical threshold in a quarter space \( \mathbb{Q} := \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{Z}_+ \) coincides with \( p_c(\mathbb{Q}) \), which implies that with probability at least \( \delta_0 = \theta(\mathbb{Q}) > 0 \), the vertex \((0, 1/2, 1/2)\) is connected to a site \( v \in \mathbb{Z}^3_\star \) with \( \|v\|_\infty \geq L - 1 \) by a 1-path \( \gamma \) lying inside \( R_L \). On the other hand, it follows from \cite{9} that there exists a left-right crossing \( \tilde{\gamma} \) of \( R_L \) with probability at least \( 1/2 \) (for \( L \) large enough). The FKG inequality implies that with probability at least \( \delta_0/2 \), both paths exist, and these two connected 1-sets can be combined with the help of \cite{10}: they are connected by a 1-path staying in \( R_L \) with probability arbitrarily close to one (by taking \( L \) large enough), which creates a path from \((0, 1/2, 1/2)\) to the right side of \( R_L \). \( \square \)
Our main construction will be based on “outlets”, that we define now (see Figure 3):

**Definition 2.** Given a configuration \( \omega \in \{0,1\}^Z \), we say that the origin \( 0 \) in \( \mathbb{Z}^3 \) is an elementary outlet if \( \omega_v = 1 \) for \( v = (0, \pm 1/2, 1/2), (0, \pm 1/2, 3/2) \), and \( \omega_v = 0 \) for \( v = (0, \pm 1/2, -1/2), (0, \pm 1/2, -3/2) \).

We will later need to refer to the vertices in such an outlet: we denote the 1-vertices by \( b_{\pm\pm} = (0,0,1) + (0,\pm 1/2,\pm 1/2) \), and the 0-vertices by \( w_{\pm\pm} = (0,0,-1) + (0,\pm 1/2,\pm 1/2) \).

**Definition 3.** Given a configuration \( \omega \in \mathbb{Z}_+^3 \), we say that the origin \( 0 \in \mathbb{Z}^3 \) is an L-outlet if:

(i) 0 is an elementary outlet,

(ii) and there are four connecting paths \( (\gamma_i)_{1 \leq i \leq 4} \), as depicted on Figure 4:

- \( \gamma_1 \) (resp. \( \gamma_2 \)) is a 1-path staying inside \( R_L^{(1)} = [(-L,L) \times (0,8L) \times (1,2L+1)] \cap \mathbb{Z}_+^3 \) (resp. \( R_L^{(2)} = [(-L,L) \times (-8L,0) \times (1,2L+1)] \cap \mathbb{Z}_+^3 \) )

Figure 2: The connecting path from Lemma 2

Figure 3: An elementary outlet (contained in the plane \( x = 0 \)).
Figure 4: An L-outlet consists of an elementary outlet, together with four connecting paths as depicted.

and connecting the vertex \((0, 1/2, 3/2)\) (resp. \((0, -1/2, 3/2)\)) to the right side of \(R_L^{(1)}\) (resp. the left side of \(R_L^{(2)}\)),

- \(\gamma_3\) (resp. \(\gamma_4\)) is a 0-path staying inside \(R_L^{(3)} = [(-L, L) \times (-8L, 0) \times (-2L - 1, -1)] \cap \mathbb{Z}_3^3\) (resp. \(R_L^{(4)} = [(-L, L) \times (0, 8L) \times (-2L - 1, -1)] \cap \mathbb{Z}_3^3\)) and connecting the vertex \((0, -1/2, -3/2)\) to the left side of \(R_L^{(3)}\) (resp. the right side of \(R_L^{(4)}\)).

Given a configuration \(\omega \in \{0, 1\}^\mathbb{Z}_3\), we say that the vertex \(v \in \mathbb{Z}_3^3\) is an L-outlet if 0 is an L-outlet for the configuration \(\tau_v(\omega)\), where \((\tau_v(\omega))_u = \omega_{u+v}\).

Lemma 3. There exist \(\delta_1 > 0\) and \(L_1 \geq 1\), depending only on \(p\), such that for all \(L \geq L_1\),

\[
P_p(0 \text{ is an L-outlet}) \geq \delta_1. \tag{11}\]

Proof. Applying Lemma 2 with \(p\) or \(1-p\) to each of the four disjoint boxes \(R_L^{(1)}, \ldots, R_L^{(4)}\), we get that with a probability at least \(\delta_0(p^2\delta_0(1-p)^2)\), the four paths \(\gamma_1, \ldots, \gamma_4\) all exist. Finally, with probability \(p^2(1-p)^2\), each of the four vertices \((0, \pm 1/2, \pm 1/2)\) has the right 0-1 value. This completes the proof, with \(\delta_1 = p^2(1-p)^2\delta_0(p^2\delta_0(1-p)^2)\).

2.3 Block argument

Definition 4. We say that the box

\[
B_L = ((-2L, 2L) \times (-8L, 8L) \times (-2L - 1, 2L + 1)) \cap \mathbb{Z}_3^3 \tag{12}
\]

is good if the following three properties are satisfied:
(i) there exists $k \in (-L,L)$ such that $(k,0,0)$ is an $L$-outlet,
(ii) the event $V_L^1((-6L,6L) \times (4L,8L) \times (1,2L+1))$ occurs,
(iii) the event $V_L^2((-6L,6L) \times (4L,8L) \times (-2L-1,-1))$ occurs.

Note that property (i) only depends on the state of the vertices inside $B_L$ (since $B_L$ contains the four boxes in the definition of $(k,0,0)$ being an $L$-outlet).

**Lemma 4.** One has:

$$\mathbb{P}_p(B_L \text{ is good}) \longrightarrow 1 \text{ as } L \rightarrow \infty. \tag{13}$$

**Proof.** It follows from Lemma 3 and ergodicity of the measure $\mathbb{P}_p$ under lattice translations, that

$$\mathbb{P}_p(\exists k \in (-L,L) \text{ s.t. } (k,0,0) \text{ is an } L\text{-outlet}) \longrightarrow 1 \text{ as } L \rightarrow \infty.$$ 

By (10), we also have that the probabilities of the events $V_L^1((-6L,6L) \times (4L,8L) \times (1,2L+1))$ and $V_L^2((-6L,6L) \times (4L,8L) \times (-2L-1,-1))$ tend to 1 as $L \rightarrow \infty$, so the result follows. 

We now describe the block argument that will be used in order to prove Theorem 2. For each pair $(i,j) \in \mathbb{Z}^2$, we first introduce $v_L(i,j) = (4iL,12jL,0)$. We then define the lattice $\mathbb{Z}^2_L = (V_L,E_L)$ having vertex set

$$V_L = \{v_L(i,j) : i + j \text{ is even}\},$$

and edge set $E_L$ given by

$$\langle v_L(i,j), v_L(i',j') \rangle \in E_L \text{ if, and only if, } |i - i'| = 1 \text{ and } |j - j'| = 1.$$ 

We get in this way an isomorphic copy of $\mathbb{Z}^2$ (see Figure 5). An infinite oriented path in $\mathbb{Z}^2_L$ is a sequence of vertices $v_L(i_0,j_0), v_L(i_1,j_1), v_L(i_2,j_2) \ldots$ such that for all $k \geq 0$, $\langle v_L(i_k,j_k), v_L(i_{k+1},j_{k+1}) \rangle \in E_L$, and also $j_{k+1} = j_k + 1$. The vertex $v_L(i,j) \in V_L$ is said to be occupied if the associated box $R_L + v_L(i,j)$ is good.

**Lemma 5.** There exists $L_2$ such that: for all $L \geq L_2$, there exists almost surely an infinite oriented path in $\mathbb{Z}^2_L$ of occupied vertices.

**Proof.** We know from Lemma 2 that for any pair $(i,j)$,

$$\mathbb{P}_p(v_L(i,j) \text{ is occupied}) = \mathbb{P}_p(v_L(0,0) \text{ is occupied}) \rightarrow 1 \text{ as } L \rightarrow \infty.$$ 

Moreover, the event that $v_L(i,j)$ is occupied depends only on the state of the vertices of $V_L$ within a graph distance at most two, so that we have a 2-dependent percolation process. We can thus use a domination by independent percolation [12], which completes the proof. 

2.4 End of the proof

We are now in a position to complete the proof. Consider $L_2$ provided by the previous lemma. First, it is an easy observation that from an oriented path as before, one gets a sequence of elementary outlets, where two successive outlets are connected as on Figure 1 by 1- and 0-paths of length at most

$$\ell = \ell(L_2) = (12L_2 - 1) \times 12L_2 \times (2L_2 + 1).$$

Let us be a bit more precise. If we denote by $(b_i^{+\pm})_{i \geq 0}$ and $(w_i^{+\pm})_{i \geq 0}$ the vertices that compose the successive outlets, we have constructed two sequences of paths $(\gamma_b^i)_{i \geq 0}$ and $(\gamma_w^i)_{i \geq 0}$, all disjoint of each other, such that for all $i \geq 0$,

- $\gamma_b^i$ (resp. $\gamma_w^i$) starts at $b_i^{++}$ (resp. $w_i^{++}$) and ends at $b_{i+1}^{++}$ (resp. $w_{i+1}^{++}$),
- and $\gamma_b^i$ (resp. $\gamma_w^i$) has a number $\lambda_b^i$ (resp. $\lambda_w^i$) of vertices (including the extremities) which is at most $\ell$.

Let us now take $M_0 = \ell^2$, and explain how to use these paths to embed any $M_0$-stretched sequence $\xi$. Recall that we denote by $(l_i^\xi)_{i \geq 1}$ the lengths of the successive runs of 0s and 1s. Let us first assume that $\xi$ is not ultimately monochromatic, which means that $\ell^2 \leq l_i^\xi < \infty$ for all $i \geq 1$. We show by induction that the first $j$ runs can be embedded starting from the first outlet, and ending in the $k_j$-th outlet (for some $k_j$), on one of the four center vertices $b_{k_j}^{++}$ or $w_{k_j}^{++}$, that we denote by $v_j$.

We just need to explain how to embed the $(j+1)$th run, of length $l_{j+1}^\xi \geq M_0 = \ell^2$. We start to embed it at $v_j'$, the vertex in the $k_j$-th outlet which is adjacent to $v_j$ and has opposite 0-1 value. We may assume without
Let us introduce \( I = \sup \{ i \geq 1 : \lambda^{k_j}_b + \ldots + \lambda^{k_j+I-1}_b \leq \ell^\xi_{j+1} - 4 \} \), then

- from \( \lambda^{k_j}_b + \ldots + \lambda^{k_j+I-1}_b \geq \ell^\xi_{j+1} - 3 \) and the definition of \( I \), we have
  \[
  \ell^\xi_{j+1} - \ell - 3 \leq \lambda^{k_j}_b + \ldots + \lambda^{k_j+I-1}_b \leq \ell^\xi_{j+1} - 4,
  \] (14)

- and using that \( \lambda^{k_j}_b + \ldots + \lambda^{k_j+I-1}_b \leq (I + 1) \times \ell \), we get that
  \[
  I \geq \frac{\ell^\xi_{j+1} - 3}{\ell} - 1 \geq \frac{\ell^2 - 3}{\ell} - 1 \geq \ell - 2.
  \] (15)

Let us now consider the path \( \rho_{j+1} \) obtained by starting from \( v'_j \), following \( \gamma^{k_j}_b, \ldots, \gamma^{k_j+I-1}_b \) (if \( v'_j \) is \( b^{k_j}_j \), then we use \( b^{k_j}_j \) before following \( \gamma^{k_j}_b \)), and ending with one extra vertex at \( b^{k_j+I}_j \). This path has a length \( L \in L^{k_j}_b + \ldots + \lambda^{k_j+I-1}_b + \{2, 3\} \), which satisfies

\[
\ell^\xi_{j+1} - \ell - 1 \leq L \leq \ell^\xi_{j+1} - 1
\]

(using [14]). First, we can make sure that \( L \) and \( \ell^\xi_{j+1} \) have the same parity: if they have different parity, we add \( b^{k_j+I}_j \) to the end of \( \rho_{j+1} \). We thus get a path \( \rho'_{j+1} \), with a length \( L' \) satisfying

\[
\ell^\xi_{j+1} - \ell - 1 \leq L' \leq \ell^\xi_{j+1}.
\]

Now, to reach a path of length \( \ell^\xi_{j+1} \) exactly, we just need to play with the \((I-1)\) intermediary outlets, with indices from \( k_j + 1 \) to \( k_j + I - 1 \). Indeed, each of these outlets allows one to add two 1-vertices to \( \rho'_{j+1} \) (by making a detour via the two center vertices), and we need to add at most \((\ell + 1)\) 1-vertices: this can be done since

\[
I - 1 \geq \ell - 3 \geq \frac{\ell + 1}{2},
\]

which finally shows that the \((j+1)\)th run can be seen, as desired. The result then follows by induction.

Clearly, the case when \( \xi \) is ultimately monochromatic can be handled in the same way, except that the procedure ends after a finite number of steps. This completes the proof of Theorem 2.
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