MATRIX STRINGS, COMPACTIFICATION SCALES
AND HAGEDORN TRANSITION

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Abstract

In this work we use the Matrix Model of Strings in order to extract some non-perturbative information on how the Hagedorn critical temperature arises from eleven-dimensional physics. We study the thermal behavior of M and Matrix theories on the compactification backgrounds that correspond to string models. We obtain some information that allows us to state that the Hagedorn temperature is not unique for all Matrix String models and we are also able to sketch how the $S$-duality transformation works in this framework.

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1 Introduction

The Matrix Model of M-Theory \cite{1,2,3} has been shown to include the second quantized string spectrum \cite{20,23,24,28,31,32,30,32,27}. In particular it is possible to see that the light-cone description of Type IIA and Heterotic $E_8 \times E_8$ string theories can be obtained through the compactification of the Matrix Model on a compact manifold, which corresponds to a torus, $T^2 = S^1 \times S^1$, and the orbifold $S^1 \times S^1/\mathbb{Z}_2$ respectively. We can parameterize these backgrounds with the radii $R$ and $R_9$ of the compact dimensions. One of them is taken to be the eleven-dimensional quantum length, related to strong coupling dynamics of string theory as in \cite{16,19}. Type I string theories are also contained as an orientifold of the type II ones \cite{17,18}.

The idea that allows to obtain string theories from the Matrix Model consist in taking a membrane configuration of $N$ D-particles of the Matrix Model in a concrete compactification background. Different limits on the radii that define the classical moduli lead to different string theories. It is necessary to make the usually called ‘FLIP’. It simply consists in the interchange of the direction we consider the quantum one in such a way that the light-cone description of string models is obtained.

This new framework allows us to attempt to understand an old problem of perturbative string theory. When we study the thermal behavior of a gas of fundamental strings we arrive at the so-called Hagedorn problem (see \cite{3,8,6,7,10,11} and references therein). It consists of the presence of a critical temperature in which the canonical description of the gas breaks down. The origin of this singularity is the exponential growth of the number of states at each mass level of the strings. After an ample discussion along the last ten years one can conclude that the Hagedorn temperature is a maximum one and in the case of open universes it is reached after a phase transition \cite{10,11}. What is still lacking is a deeper understanding of what the nature of the phase transition is. We know that, at least from a perturbative point of view, it seems that at temperatures near the critical one a redistribution of the degrees of freedom of the theory is needed.

One could also wonder if non-perturbative effects would change the picture. Some works in this direction have been carried out concluding that these effects do not affect the critical behavior \cite{12,13}. We think that it is due to the fact that non-perturbative effects in string theory are relevant at scales smaller than the string one \cite{14,13}, and so they could not affect a feature that naturally arises at $\alpha'$ scales.

From the moment we know that string theories are corners of the moduli space of an eleven-dimensional fundamental theory, it seems mandatory to try to understand what the Hagedorn critical temperature means from an M-theoretical perspective.

In the limit in which the spatial compactification radius, $R_9$, goes to zero the membranes of Matrix theory become strings, and then the Hagedorn temperature does appear. Some physical properties of this phenomenon have been analyzed in \cite{33}. An interesting feature of the Hagedorn transition in String Theory is that it takes different values depending on the model we analyze. More concretely, the gas of Heterotic and Type II strings do not present the phase transition at the same temperature. The idea we will develop in next sections is that we can extract some information from the Matrix model compactifications that will allow us to draft some properties of the thermal behavior of Matrix String Theories, and clarify some aspects of the Hagedorn phenomenon from a
more fundamental perspective.

Let us sketch the idea. We will argue that the critical thermal properties of string theories are functions of a ten-dimensional parameter $\lambda$ which could be expressed in terms of eleven-dimensional variables as $\lambda = \beta R_{9}^{1/2}$. If we assume the Hagedorn temperature to be unique from the Matrix Model point of view, and $R_{9}$ to be the order parameter of the transition [15], it would be possible to establish a relation between the different Hagedorn temperatures of string theories. If we call the transition points of the perturbative string gas $\lambda_{H}$, then the relationship between different Hagedorn temperatures could be obtained by

$$\frac{\lambda_{H}^{\text{IIA}}}{\lambda_{H}^{\text{Het}}} = \left(\frac{\beta_{H} \sqrt{R_{9}^{\text{IIA}}}}{\beta_{H} \sqrt{R_{9}^{\text{Het}}}}\right)^{2} \left(\frac{R_{9}^{\text{IIA}}}{R_{9}^{\text{Het}}}\right)^{1/2}. (1)$$

We will use this approach to elucidate whether the Hagedorn temperature is unique in the M-theory framework.

Some words about the open string case are in need. We know type I string theories have the same Hagedorn temperatures as type II ones [6, 8]. This property is easily understood remembering that type I string theories may be obtained as a $Z_{2}$ orientifold of closed strings. In perturbative string theory the $Z_{2}$ projection does not change the critical point [9]. We will argue that from an eleven-dimensional point of view this is to be expected, because of the fact that the compact manifold that defines the string scale, in this case a circle, is the same. That is the properties of the quantum dimension determines the value of the critical temperature.

Another approach that could give some information on how the compactification background enters into the thermal properties of string theories is based on the study of the thermalized maximal SUGRA in eleven dimensions in those backgrounds. In next section we will show that the thermal properties of the effective supergravities, corresponding to low energy limits of string theories, are functions of the eleven-dimensional temperature and the compactification scale.

## 2 M-Theory origin of string theories

Historically M-theory appeared as the theory that would fill the lack of an underlying framework in which to understand the map of string dualities. In [16, 19] the eleven-dimensional origin of effective string theories was described. As argued in the introduction we think that we can obtain some information relating the thermal behavior of string theories and the eleven-dimensional compactification scale, by studying the M-theoretical origin of effective supergravities. This is the scope of this section.

### 2.1 M-theory on a torus

In [16] was shown that the strong coupling limit of type IIA String Theory at low energies is eleven-dimensional SUGRA on $R^{10} \times S^{1}$, where the radius of the eleven-dimensional circle is related to the string coupling constant by $R = l_{11}g_{s}^{2/3}$.

This relationship between M and IIA theories can be generalized to an arbitrary manifold in such a way that we can make the following statement: Type IIA String Theory on
a manifold $\mathcal{M}$ at strong coupling and at low energies is described by M-theory on $\mathcal{M} \times S^1$.

We will use this property to study some interesting background manifolds. An example is type IIB case. We know that by $T$-duality we can obtain Type IIB string as the dual of IIA one. More explicitly, taking $\mathcal{M} = R^9 \times S^1$ and acting with the duality on this extra $S^1$ we recover type IIB String Theory from M-theory. Another example could be $\mathcal{M} = R^9 \times S^1/\mathbb{Z}_2$ from which we will obtain type I string theory.

In order to study the relation of M-theory and the closed string theories we will analyze the particular compactifications of M-theory on $S^1$ corresponding to type IIA strings, and $S^1 \times S^1$ that recovers, after using $T$-duality, the type IIB case. Open strings and the Heterotic case will be studied later.

In the IIA case the theory is defined by the radius of the $S^1$, that we will call $R$. The free energy reduces to

$$\Lambda_{R^{10} \times S^1}(\beta) \sim -\frac{V_{11}}{l_{11}^{10}} \int \frac{ds}{s^6} \left[ \theta_3 \left( 0, \frac{2 \pi l_{11}^2}{i \beta} \right) - \theta_4 \left( 0, \frac{2 \pi l_{11}^2}{i \beta} \right) \right]$$

where $l_{11}$ is the Planck length in eleven dimensions. We want to compare this magnitude to the Helmholtz free energy of the massless spectrum of type IIA String Theory and finally check that

$$\Lambda_{R^{10} \times S^1}(\beta) = \Lambda_{IIA}(\beta_{IIA}).$$

In the type IIA String Theory the 0-brane partons have masses given by $\tilde{M}_n = n/\sqrt{\alpha'} g_s$, the tilde indicates we are measuring it in ten-dimensional units. We can include it in the free energy simply by putting this mass into string propagator. The free energy of the IIA string theory with D-partons is given by

$$\Lambda_{IIA}(\beta_{IIA}) \sim -\frac{V_{10}}{l_s^{10}} \sum_{n \in \mathbb{Z}} \int \frac{ds}{s^6} \left[ \theta_3 \left( 0, \frac{2 \pi \alpha'}{s \sqrt{\alpha'}} \right) - \theta_4 \left( 0, \frac{2 \pi \alpha'}{s \sqrt{\alpha'}} \right) \right] e^{-\alpha' \tilde{M}_n^2 s} Z(s).$$

where $Z(s)$ is the internal partition function of the theory.

To establish the relationship between the theories we must measure the ten dimensional temperature $\beta_{IIA}$ in M-theory units. Using standard Kaluza-Klein reduction it is easy to write the metrics $G_{D=10}$ in terms of $G_{D=11}$. In the case we are describing we know that the metrics are also related with the string coupling constant as $G_{10} = g_s^{2/3} G_{11}$. Then by using the relation between the radius $R$ and the string coupling $g_s$, the reduction of metrics leads to the following relation between lengths$^2$

$$\frac{L_{10}}{l_s} = \left( \frac{R}{l_{11}} \right)^{1/2} \frac{L_{11}}{l_{11}}$$

Using this, and thinking about the inverse temperature as a length, we can express the function in (3) in terms of $\beta$ and $R$, where $\beta$ is the M-theory temperature. On the other hand we have to keep in mind that the string scale is expressed as $\sqrt{\alpha'} = l_s = g_s^{-1/3} l_{11}$ in terms of eleven-dimensional parameters$^3$, $^4$, and finally we have

$$\Lambda_{IIA}(\beta) \sim -\frac{V_{10}}{\alpha'^{3/2}} \sum_{n \in \mathbb{Z}} \int \frac{ds}{s^6} \left[ \theta_3 \left( 0, \frac{2 \pi \alpha'}{s l_{11}^2} \frac{R}{l_{11}} \right) - \theta_4 \left( 0, \frac{2 \pi \alpha'}{s l_{11}^2} \frac{R}{l_{11}} \right) \right] e^{-\frac{l_{11}^3 M_s^2}{\alpha'} Z(s)}.$$
where we have let the multiplicative factors in a ten-dimensional fashion.

In order to relate this expression to M-theory free energy we have to make the dimensional reduction of the latter from eleven to ten dimensions. Before doing so, it is necessary to make an appropriate rescaling of the proper time in the previous expression

$$s = \frac{l_{11}^2}{l_s^2} t.$$  \hspace{1cm} (7)

In this way, and using the relations above, we find that (6) reads

$$A_{IIA}(\beta) \sim -\frac{V_{IIA}}{l_{11}^6} \sum_{n \in \mathbb{Z}} \int \frac{dt}{t^6} \left[ \theta_3 \left(0, \frac{i\beta^2}{2\pi tl_{11}^2} \right) - \theta_4 \left(0, \frac{i\beta^2}{2\pi tl_{11}^2} \right) \right] e^{-l_{11}^2 M_n^2 t} Z \left(\frac{l_{11}^2}{l_s^2} t \right).$$  \hspace{1cm} (8)

Now the D-particle masses are measured in eleven-dimensional units, so we have $M_n = n/R$. The connection to the free energy of M-theory on the circle is simply obtained by decoupling massive excitations of the string. It is possible to do it without integrating out the D-branes, by taking $g_s$ to be finite in such a way $l_s^{-1} \gg (l_s g_s)^{-1}$. In this limit $Z(t)$ tends to the unit and we exactly recover the free energy in (2). As expected, we have recovered the mechanism that allows the relation between strong coupling dynamics of String Theory at low energies and eleven-dimensional supergravity \[16\]. The whole KK spectrum in M-theory corresponds to bound states at threshold of D-particles of IIA theory.

The perturbative string physics is reached by taking $R \to 0$ in (2). In this region the contribution corresponding to the KK tower of momentum states, tends to one (one could see the same in the D-particle picture by simply taking a vanishing coupling constant limit) and we obtain the free energy of the ten-dimensional IIA supergravity.

Let us see how the standard Hagedorn condition of String Theory is expressed in terms of eleven-dimensional parameters. The usual method that allows the computation of the Hagedorn temperatures gives, for the free energy in (8),

$$\beta_H R^{1/2} \propto l_{11}^{3/2}.$$  \hspace{1cm} (9)

We finally see that, as guessed, the critical point of the string gas is given in terms of $\lambda = \beta R^{1/2}$.

Some words about type IIB String Theory are in need. We know that this theory is related to IIA one by $T$-duality. It seems then that by analyzing M-theory on $\mathcal{M} = \mathbb{R}^9 \times S^1 \times S^1$ and using the duality on one of the circles we will recover IIB strings from M-theory physics \[16, 14, 20, 21\]. The thermal properties of this theory are not affected by the $T$-duality because we do not change the ‘quantum’ circle of M-theory.

Another interesting related result that comes from the study of M-theory on the torus is related to the FLIP in the Matrix Model of strings \[24, 26, 25\]. From a purely M-theoretic point of view there is nothing special in taking $R$ or $R_9$ as the fundamental parameter which finally fixes the string scales. The system itself will choose the states of membranes wrapped around the smallest circle as the ‘perturbative’ ones. We will see that it is related to $S$-duality transformation. This fact allows us to say that after flipping, the thermal properties of matrix String Theory will depend on $\beta R_9^{1/2}$, as we guessed in the introduction.
2.2 M-theory on orbifolds

We can now review how M-theory contains the low energy regime of string theories with one space-time supersymmetry \[19\], that is, Heterotic and open string theories. Type I and type I' string theories can be obtained as an orientifold of type IIB and type IIA closed string, both are related by T-duality \[17, 18\]. This suggests that there should be an M-theory background that describes type I theories. In fact it is possible to recover open strings by taking \[M = R^9 \times S^1 \times S^1\], and orbifolding one of the circles \[19\]. In this way type I String Theory is obtained from eleven-dimensional physics by considering M-theory on \[M = R^9 \times S^1 / \mathbb{Z}_2 \times S^1\]. The whole map of open string theories is

\[
\text{IIA } [(\mathcal{M} = R^9 \times S^1) \times S^1] \overset{T}{\leftrightarrow} \text{IIB } [(\mathcal{M} = R^9 \times \tilde{S}^1) \times S^1] \\
\mathbb{Z}_2 \rightarrow \text{IB } [(\mathcal{M} = R^9 \times \tilde{S}^1 / \mathbb{Z}_2) \times S^1] \overset{T}{\leftrightarrow} \text{IA } [(\mathcal{M} = R^9 \times S^1 / \mathbb{Z}_2) \times S^1]
\]

(10)

where \(T\) stands for T-duality. In the previous diagram we have recovered the connected string theories without touching the eleven-dimensional \(S^1\). We know that the four theories above present the same Hagedorn temperature \[1, 8\]. In our philosophy this is due to fact that the scale of the ten-dimensional theory is governed by the common circle.

By using the symmetry that interchanges the orbifolded \(S^1\) in M-theory it is possible to recover the \(E_8 \times E_8\) Heterotic string as the M-theory on \(R^9 \times S^1 \times S^1 / \mathbb{Z}_2\). T-duality allows the relation of this theories to the \(SO(32)\) Heterotic string. We can write another relation

\[
E_8 \times E_8 [(\mathcal{M} = R^9 \times S^1) \times S^1 / \mathbb{Z}_2] \overset{T}{\leftrightarrow} SO(32) [(\mathcal{M} = R^9 \times S^1) \times \tilde{S}^1 / \mathbb{Z}_2].
\]

(11)

In these theories the scale is defined by \(S^1 / \mathbb{Z}_2\), and they have a different critical temperature as those in (10). On the other hand, it is also known that both maps, (10) and (11), are related by \(S\)-duality \[14, 21, 22, 20, 19\] (that is the flip)

\[
SO(32) [(\mathcal{M} = R^9 \times S^1) \times \tilde{S}^1 / \mathbb{Z}_2] \overset{S}{\leftrightarrow} \text{IB } [(\mathcal{M} = R^9 \times \tilde{S}^1 / \mathbb{Z}_2) \times S^1].
\]

(12)

These two theories include low energy excitations of string theories that have different Hagedorn temperatures. From our approach it is due to the type of eleven-dimensional compact dimension they are compactified in. We will argue later that this property comes from the shrinking of one of the dimensions of the orbifolded torus, and consequently from which are the membrane degrees of freedom we freeze.

From an M-theoretic point of view the change of the critical temperature by the \(S\)-duality in (12) occurs in two limits of the same compactification background. By keeping both radii to be finite it is possible to see how the Hagedorn critical point varies along a continuous path from one theory to the other. More on this subject will be said in the final discussion.

3 Hagedorn transition from M(atrix)-Theory

Before analyzing the Hagedorn problem in the Matrix String Model let us briefly review how the Matrix Model of M-theory recovers string theories from a non-perturbative perspective, and how it behaves on the corresponding backgrounds at finite temperature.
3.1 Thermodynamics of Matrix Theory on string backgrounds

For simplicity we can start with the model that describes the type IIA String Theory from Matrix Membranes [25, 24, 26]. It consists of a system of $N$ D-particles on $\mathbb{R}^9 \times S^1(R) \times S^1(R_9)$. Strings are related to those configurations that correspond to two-dimensional membranes wrapped on the torus. The radius $R$ is taken to be the light-cone one while $R_9$ is the radius of the transverse spatial circle. On this background $N$ D0-branes, composing the wrapped membrane, are described by the dimensional reduction down to $1 + 1$ dimension of SYM initially in $D = 9 + 1$ with $\mathcal{N} = 1$ and gauge group $U(N)$. The spatial dimension of the SYM theory is compactified on the $T$-dual circle $S^1(R_9)$ [23].

At this moment we have two parameters that define, at least classically, the theory. These parameters are the compactification radii of the torus $T^2$. The relevant physics here comes from the possible limits we can take on the moduli space.

We can relate this system to ten-dimensional physics in two different ways. Following the standard knowledge of $T$-duality, the action of the theory we have is simply the D1-brane action of the type IIB String Theory where the D-strings are wrapped around the compact dimension. The ten-dimensional interpretation is simply recovered by taking the parameter $R$ to be small.

On the other hand IIA String Theory is obtained from this framework by making the mentioned 'FLIP', that is using the radius $R_9$ as the responsible for the connection to ten-dimensional physics. Geometrically it can been seen as the shrinking of one of the dimensions of the membrane we started with. In this way, for scales larger than $l_{11}$ the membranes look like ten dimensional strings. The quantity $N$ stands for the KK momentum of the strings.

Some thermal properties of the Matrix Model on the torus were studied in [5]. In that work it was obtained that the loop contributions to the partition function on the torus take the following form

$$Z_n \simeq (2\pi)^{-2n} \left( \frac{\beta R}{l_{11}^2} \right)^{3n}$$

where $R$ is the light-cone direction radius [4].

The Heterotic string has also been obtained by a compactification of the Matrix Model [33, 32, 30, 28, 31]. The background manifold in this case is $R^9 \times S^1(R_9)/\mathbb{Z}_2 \times S^1(R)$. The methodology is the same as in the type II case so $E_8 \times E_8$ is reached by permuting the role of the radii, and then the relationship to perturbative ten dimensional physics is obtained by taking the orbifold radius to be infinitely small. Type I theory is obtained by shrinking the circular dimension, $S^1(R)$.

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3In order to compare the expressions here to the results in [3] one must be careful with the normalizations. Here, we are working with those of [2]

4 We should multiply $Z_n$ by $exp(-\frac{\beta R}{2})$ that comes from the light-cone degrees of freedom [5]. This exponential term does not need to be flipped because it comes from a purely light-cone frame energy and in the case of matrix strings it is recovered by the string’s momentum modes around the compact dimension $X \sim X + 2\pi R$. When computing the string partition function, in the context of Matrix String Model it will be mandatory to include the light-cone modes that will correspond with the exponential term in the above expression.
Let us be more explicit. The Matrix Model for Heterotic string is constructed starting with the so-called Heterotic Matrix Theory \cite{29, 30, 28}, that corresponds to the Matrix Model on a transverse orbifold \( S^1/\mathbb{Z}_2 \). As usual the longitudinal direction is compactified in the eleven-dimensional circle. The D-particle system in this background may be described in two ways. We can use a SYM quantum mechanics with the appropriate orbifold projection, and the addition of extra fermions coming from the orbifold planes, or by \( T \)–dualizing the orbifold direction we arrive at the dimensional reduction of \( D = 9 + 1 \) SYM theory to \( 1 + 1 \) dimensions, where the spatial one is compactified on a circle of radius \( \Sigma \) \cite{34, 29, 30, 28}. The residual gauge group is reduced to \( O(N) \), and the parameters of the theory are related to fundamental ones by

\[
\tilde{g}^2 = \frac{R^2}{R_9 l_{11}^3}, \quad \Sigma = \frac{(2\pi)^2 l_{11}^3}{RR_9^2 (14)}
\]

where \( g \) is the coupling constant of the SYM theory. The thermal properties of this model are qualitatively the same than those of theory on the torus. The difference will only arise from the internal degrees of freedom, then we can state

\[
Z_{S^1/\mathbb{Z}_2} \simeq \left( \frac{\tilde{g}^2 \beta^3}{\Sigma} \right)^n = (2\pi)^{-2n} \left( \frac{\beta R_9}{l_{11}} \right)^{3n} \quad \text{(15)}
\]

The final step in order to connect to Matrix String models we have to make the FLIP. It will be simply done by interchanging the radius \( R \) by \( R_9 \) in the previous results. We are now able to attempt the problem of the critical behavior of the string gas. This is the scope of next sections.

### 3.2 The Hagedorn transition

The previous analysis has opened to us a possibility of studying some features of Hagedorn transitions in Matrix theory.

It has been recently proposed that Hagedorn temperatures represent a deconfining point in the D-particle picture of perturbative strings. When the temperature reaches the string scale the D-partons glued together in the membrane configuration separate and a gas of individual D-particles comes out. We know that the Matrix String Models reproduce perturbative string theories when the radius \( R_9 \), which, by flipping, is related to the string coupling constant, is taken to be zero. It is precisely in this regime where the Hagedorn transition does arise, so the scale \( R_9 \) has to be considered as the order parameter of the phase transition. This is the picture qualitatively sketched in \cite{35}.

In the previous sections we have argued that the critical properties of the string gases can be expressed in terms of the eleven-dimensional parameters, the inverse temperature \( \beta \) and the eleven-dimensional radius \( R_9 \) in the form of the ‘thermal’ parameter \( \lambda = \beta R_9^{1/2} \). Let us see how this behavior comes in the Matrix Model of strings. We know that the loop expansion at finite temperature is given by powers of \( \beta R_9/l_{11}^2 \). Relating this magnitude to ten-dimensional variables, the temperature and the scale \( l_s \), and using the Hagedorn temperature, \( \beta_H \), as an input, we arrive to the following Hagedorn condition for the eleven-dimensional temperature

\[
\frac{\beta_H R_9}{l_{11}^2} \propto \frac{g_s^{1/3}}{l_{11}^{1/2}} \Rightarrow \beta_H R^{1/2} \propto l_{11}^{3/2}, \quad \text{(16)}
\]
as we obtained in section 2.

In this way it would be possible to state that a unique value of this parameter, corresponding to the critical string point $\lambda_H$, holds for various values of Matrix Theory temperature.

4 Discussion

In the previous section we have tried to extract some information from M and Matrix theories about the thermal behavior of string effective models. Finally we can conclude that, seen from eleven dimensions, the ten-dimensional critical thermodynamics behaves as a function of $\lambda = \beta R_{10}^{1/2}$, where $R_9$ is the radius of the eleventh dimension and it has to be taken as the order parameter of the Hagedorn phase transition.

The first doubt we can elucidate is whether the Hagedorn temperature is unique in the Matrix Model or if it depends on the compactification background. In perturbative String Theory this uniqueness for each model, is supported by the fact that the singularity is originated from the worldsheet degrees of freedom. In the Matrix Model, if we suppose that the Hagedorn temperature is unique for all the string models, we arrive at the conclusion that the difference should come from compactification scales. Using the expression (1) we see that the ratio $(R_{9\text{Het}}/R_{9\text{IIA}})^{1/2}$ should be different than 1. Remembering that the string theories are obtained by taking the small $R_9$ limits, we can finally conclude that the difference between closed type II and Heterotic Hagedorn temperatures cannot come from the size of the ‘quantum’ radii. This shows that it is expectable that in Matrix theory the Hagedorn temperature depends on the compactification background. It will come from the difference between reducing the internal degrees of freedom of the membrane, shrinking the circle $S^1$, as in type II and I strings, or the orbifold $S^1/Z_2$ which corresponds to the Heterotic case.

Another important consequence of the thermal behavior we have obtained is that while the critical point in String Theory is given in terms of a single value of $\lambda$, from a M-theoretical point of view it correspond to a family of $\beta$ and $R_9$ parameters. We can see that it is possible to relate this property to $S$-duality.

We will begin by focusing on the IIB case. This theory is obtained from Matrix Theory by $T$-dualizing the compact dimension of radius $R$ of the IIA matrix strings. We know that the Hagedorn condition for type IIB closed strings, $\beta_{\text{IIB}}^H = \pi \sqrt{8\alpha'}$, leads to the following one for eleven dimensional temperature

$$\beta^H = \sqrt{8\pi} \left( \frac{\ell_3^{11}}{R_9} \right)^{1/2}. \tag{17}$$

On the other hand we can introduce the thermal properties of non-perturbative effects in the same fashion. We include the degrees of freedom of D-strings in type IIB theory by taking the $R \to 0$ limit on the Matrix Theory on $S^1(R_9) \times S^1(R)$. In this case $R$ defines the string scales so the thermal parameter is $\lambda' = \beta_D R^{1/2}$. The critical temperature, $\beta_D^H$ of this gas will be the same of (17) but with $R$ instead of $R_9$. The relation between the
critical temperatures is given by
\[ \beta_H \left( \frac{R_9}{R} \right)^{1/2} = \beta_H \left( \frac{g_\ast}{g_D} \right)^{1/3} = \beta_D \tag{18} \]

We know that in the picture we are working \( R_9 \) is taken to be smaller than \( R \) in such a way the non-perturbative effects present a critical temperature bigger than fundamental strings. This property is an explanation of a known behavior presented in [13]. We have related it to \( S \)-duality by expressing the radii \( R_9 \) and \( R \) in terms of two 'string' coupling constants. We easily see that when the fundamental string coupling is small, the gas of non perturbative strings arrive at its critical temperature after the Hagedorn transition of the perturbative ones. The contrary happens for strong coupling dynamics where the D-string gas has to be taken as the perturbative one.

In the case of the Heterotic/type I duality the picture is practically the same. The main difference comes from the Hagedorn condition of string models. We know that the Heterotic gas presents its Hagedorn temperature at \( \beta_{\text{Het}}^H = (2 + \sqrt{2})\pi \sqrt{\alpha'} \). The \( S \)-dual gas of this theory is composed of open strings whose critical point is the same as in Type II theories. The relationship in terms of coupling constant, analogous to that given for the IIB string model, is for the case of Heterotic/Type I duality

\[ \beta_{\text{Het}}^H \left( \frac{g_{\text{Het}}}{g_{\text{open}}} \right)^{1/3} = \frac{(\sqrt{2} + 1)}{2} \beta_{\text{open}}^H. \tag{19} \]

Remembering that by \( S \)-duality the coupling constant are related as \( g_{\text{open}} = 1/g_{\text{Het}} \), and equivalently \( g_\ast = 1/g_D \), we can make a general statement that relates critical temperatures of \( S \)-dual gases

\[ \beta^H g^{2/3} = a \beta_{\text{Dual}}^H \tag{20} \]

where \( a \) is different for the various cases of \( S \)-dual theories. This property seems to be an evidence for the guess in [33], in the sense that the theories with the same number of supersymmetries are continuously related while it seems that there is not a continuous connection to theories with a different amount of supersymmetry.

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