CORRELATION FUNCTIONS AND CORNER TRANSFER MATRIX OF THE CHIRAL POTT'S MODEL

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We use the Density Matrix Renormalization Group technique to compute correlation functions of the \(Z_3\)-chiral Potts quantum chain in the massive regimes. Chains of up to 70 sites are used and a clear oscillatory behavior is found. We also check the relation between the density matrix spectrum and the Corner Transfer Matrix spectrum pointed out by Nishino and Okunishi, using Baxter’s low-temperature formulae for the chiral Potts Corner Transfer Matrix eigenvalues. We find very good agreement.

1 Introduction

We report numerical calculations which apply the density-matrix renormalization group method (DMRG) for the first time to the chiral Potts quantum chain. Although the chiral Potts model is integrable, various interesting properties are not yet accessible to analytical calculation and one has to resort to numerical calculations in order to get more complete information. We shall concentrate on two aspects: first, we show clearly that in the high-temperature massive phase the correlation functions are oscillating. Second, we check the relation \(\rho \sim ABCD\) between the density matrix and the Corner Transfer Matrices (CTM), pointed out by Nishino and Okunishi [30]. For the CTM eigenvalues of the chiral Potts model we use Baxter’s low-temperature expansion formulae [10].

2 The integrable chiral Potts model

The \(Z_N\)-symmetrical chiral Potts model has been introduced by Ostlund [32] in order to describe incommensurate phase transitions in surface layers, e.g. Krypton monolayers on graphite. Centen et al. [13] introduced and analysed the chiral Potts quantum chain. Very soon it was realized that this class of models can be reformulated such that very interesting mathematical structures emerge: Howes, Kadanoff and den Nijs [25] modified the \(N = 3\) quantum chain

\[1\]To be published in "Quantization, Gauge Theory and Strings" dedicated to the memory of Efim Fradkin, ed. A. Semikhatov, M. Vasiliev and V. Zaikin, Moscow 2001.
to make it self-dual and discovered that in their version for a particular chiral angle the lowest gap depends linearly on the inverse temperature. Then von Gehlen and Rittenberg \[22\] constructed a whole set of $\mathbb{Z}_N$ symmetrical quantum chains which for any $N$ satisfy the Dolan-Grady \[19\] integrability conditions. These quantum chains are now usually called “superintegrable”.

Perk noticed \[33\] that the Dolan-Grady conditions are just conditions for generating the algebra which Onsager in 1944 had introduced to obtain the free energy of the Ising model \[31\].

Onsager’s 1944 technique for solving the Ising model made use of specific $\mathbb{Z}_2$-features and it has not been possible to adapt it for solving the $\mathbb{Z}_N$-model. The actual solution for the eigenvalues of the superintegrable model was achieved differently: In 1987 Au-Yang et al. \[5\] invented a two-dimensional $\mathbb{Z}_N$-symmetrical Yang-Baxter-integrable statistical model, which contains the chiral Potts chain hamiltonian as the derivative of its transfer matrix. Inversion relations and functional equations were found for this 2-dim. model and were used to obtain the ground state energy \[4\] and also analytic formulae for the excited levels \[4\] \[10\]. Hints that the chiral Potts model is related to the six-vertex model at quantum group parameter at root of unity came from Korepanov \[27\] and the connection was shown by Bazhanov and Stroganov \[14\].

We first summarize some basic notions of the 2-dimensional integrable chiral Potts model of \[5\]. Fig. 1 shows part of a diagonal square lattice and rapidity lines used to define the Boltzmann weights $W_{p,q}(\sigma_j - \sigma'_j)$ and $W_{p,q}(\sigma_j - \sigma'_{j+1})$ which depend on both rapidities via six functions $x_p, y_p, \mu_p, x_q, y_q, \mu_q$ (for brevity, we write $x(q)$ as $x_q$, etc.): $W_{p,q}(0) = W_{p,q}(0)$ = 1 and for $n = 1, \ldots, N-1$

$$W_{p,q}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q}, \quad W_{p,q}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p}$$

Figure 1: Diagonally drawn square lattice with $\mathbb{Z}_N$-spins $\sigma_j$ on the lattice sites. Dashed lines: $p$- and $q$-rapidity lines. Boltzmann weights $W_{p,q}(\sigma_j - \sigma'_j)$ act in SW-NE-direction, $W_{p,q}(\sigma_j - \sigma'_{j+1})$ in SE-NW-direction.
where $\omega = e^{2\pi i/N}$. Imposing $Z_N$-symmetry: $W_{p,q}(n + N) = W_{p,q}(n)$ (same for \(\overline{W}\)), implies that $x_q$ and $y_q$ are restricted to high genus Fermat-like curves:

$$k \left( x_q^N y_q^N + 1 \right) = x_q^N + y_q^N; \quad k x_q^N = 1 - k^2 \mu_q^{-N}; \quad k y_q^N = 1 - k^2 \mu_q^N \quad (1)$$

(the same also for $x_p$, $y_p$, $\mu_p$). $k^2 = 1 - k^2$ are temperature-like parameters, the low-temperature limit will be $k' \to 0$. For given $k$, up to discrete choices of roots, eqs. leave only two of the six functions independent, e.g. $x_q$ and $x_p$.

For $N = 2$ the chiral Potts model reduces to the Ising model and the Boltzmann weights can be written in terms of Jacobi elliptic functions

$$x_q = \sqrt{k} \text{sn}(q, k); \quad y_q = \sqrt{k} \text{cn}(q, k)/\text{dn}(q, k), \quad \mu_q = \sqrt{k'/\text{dn}(q, k)}.$$

$W$ and $\overline{W}$ satisfy star-triangle-relations which involve the $p$- and $q$-variables separately, not as usual only through the difference $p - q$:

$$\sum_{l=0}^{N-1} \overline{W}_{q,r}((l'-l)W_{p,r}((l'-l)\overline{W}_{p,q}((l-l''))=R_{pqr}W_{p,q}((l''-l')\overline{W}_{p,r}((l''-l''))W_{q,r}((l''-l'))) \quad (2)$$

$R_{pqr}$ can be written in terms of the $W$, $\overline{W}$ and their $N$-th roots $\bar{1}$, $\bar{2}$. From the Boltzmann weights, the transfer matrix is defined by

$$T_{p,q}(\{l\}, \{l'\}) = \prod_{j=1}^{L} \overline{W}_{p,q}(l_j - l'_j) W_{p,q}(l_j - l'_{j+1}). \quad (3)$$

and the chiral Potts quantum chain Hamiltonian arises as the derivative

$$\lim_{q \to p} \frac{\partial T_{p,q}}{\partial q} \sim H = -\sum_{j=1}^{L} \sum_{l=0}^{N-1} (\alpha_l Z_{j}^l + \lambda \bar{\alpha}_l X_{j}^l X_{j+1}^l), \quad (4)$$

where the operators $X_j$ and $Z_j$ act in the spaces $\mathbb{C}^N$ at sites $j$ and satisfy

$$Z_j X_j = X_j Z_j \omega^{\delta_{j,j}}; \quad Z_j^N = X_j^N = 1; \quad \omega = e^{2\pi i/N}; \quad (5)$$

and

$$\alpha_l = \frac{e^{i\phi(2l-N)/N}}{\sin \frac{\pi l}{N}}; \quad \bar{\alpha}_l = \frac{e^{i\phi(2l-N)/N}}{\sin \frac{\pi l}{N}}; \quad \lambda = 1/k'. \quad (6)$$

In right hand side of (3) we skipped an additive constant term. $H$ commutes with the $Z_N$-charge operator $\hat{Q} = \prod_{j=1}^{L} Z_j$. The eigenvalues of $\hat{Q}$ are $\omega^Q$ with $Q = 0, \ldots, N - 1$. Since $T_{p,q}$ depends not only on $p - q$, it matters, at which $p$ the limit $q \to p$ is taken. So, for given $\lambda$, the hamiltonian (3) depends on one chiral parameter $\phi$. The coefficients (5) are conveniently written in terms of two chiral angles $\varphi$ and $\phi$, which are related through

$$\cos \varphi = \lambda \cos \phi. \quad (6)$$
Using $\varphi$ as the independent variable, $\phi$ becomes complex for $\lambda < \mid \cos \varphi \mid$ and the hamiltonian (3) becomes non-hermitian. The degrees of freedom remain unchanged, and the previously parity-nonconserving $H$ now acquires a parity symmetry. Baxter mostly considers the 2-d model in this parameter range with $\varphi < \frac{\pi}{2}$, since there the Boltzmann weights are positive real. In the single point $\lambda = 1$ and $\varphi = 0$ we recover the Fateev-Zamolodchikov parafermionic model [20], see e.g. [4].

Performing a duality transformation on (3) we reach a model of the same type, which has $\phi$ and $\varphi$ interchanged and $\lambda$ replaced by $\lambda - 1$ but also different b.c. Howes et al. [25] originally considered a self-dual model with $\varphi = \phi$ instead of (6). Probably this model is not integrable except at the line $\varphi = \frac{\pi}{2}$ and the point $\varphi = 0$, $\lambda = 1$, where it coincides with (6).

The analysis of the general 2-d model for $N \geq 3$ is difficult since for general $k$ no simple hyperelliptic parametrization of the $W_{p,q}$, $W_{p,q}$ is possible [12]. Nevertheless, the free energy and the interface tension have been obtained circumventing an explicit uniformisation [1, 35]. In [29] a gap-formula for the quantum chain has been obtained, from which the phase structure can be calculated: In the massive phases the ground state of the chain Hamiltonian is translational invariant (momentum $P = 0$). For the high-temperature (small $\lambda$) massive phase the ground state is in the sector $Q = 0$. In the low-temperature phase varying $\lambda$, $\varphi$ or $L$, the ground state rotates among the $P = 0$, $Q$-sectors with many level-crossings. The incommensurate (IC) phase is reached when a $P \neq 0$-state dips below the lowest $P = 0$ state. The IC phase turns out to have a wedge shape, the tip of the wedge is at $\lambda = 1$, $\varphi = 0$ (the parafermionic point) and spreading out with increasing $\varphi$. At $\varphi = \phi = \frac{\pi}{2}$ for $N = 3$ the IC-phase covers the interval $0.9013 < \lambda < 1.1095$.

Much more detailed analytic information has been obtained for the special ”superintegrable” case $x_p = y_p$, $\mu_p = 1$, or, in terms of the chain parameters: $\varphi = \phi = \frac{\pi}{2}$. In this case $H$ simplifies to

$$H^{(s)} = - \frac{4}{N} \sum_{j=1}^{L} \sum_{l=1}^{N-1} \frac{1}{1 - \omega^{-l}} \left( Z_j^l + \lambda X_j^l X_{j+l}^{N-l} \right).$$

(7)

Splitting $H^{(s)}$ into two pieces by writing $-H^{(s)} = A_0 + \lambda A_1$, one finds [22] that $A_0$ and $A_1$ satisfy the Dolan-Grady [13] conditions

$$[A_0, [A_0, [A_0, A_1]]] = 16 [A_0, A_1]; \quad [A_1, [A_1, [A_1, A_0]]] = 16 [A_1, A_0].$$

(8)

Iff these conditions are met, starting from $A_0$, $A_1$ one can generate Onsager’s algebra $A$ [31] :

$$[A_l, A_m] = 4 G_{l-m}; \quad [G_l, A_m] = 2 A_{m+l} - 2 A_{m-l}; \quad [G_l, G_m] = 0; \quad l, m \in \mathbb{Z}.$$
\( \mathcal{A} \) implies an infinite set of \( \lambda \)-dependent commuting charges \([Q_m, Q_{m'}] = 0\):

\[
Q_m = \frac{1}{2} (A_m + A_{-m} + \lambda (A_{m+1} + A_{-m+1})) \quad \text{among which} \quad Q_0 \sim H^{(s)}. \tag{9}
\]

As a consequence of \( \mathcal{A} \) \[18\] all eigenvalues \( E \) of \( H^{(s)} \) depend on \( \lambda \) as

\[
E = a + b \lambda + \sum_{j=1}^{m_E} 4 m_j \sqrt{1 + 2 \lambda \cos \theta_j + \lambda^2} \tag{10}
\]

where \( a, b, m_E \) are integers, and one finds \( m_j = \pm 1/2 \). \( \mathcal{A} \) gives no information about which values of \( a, b \) and \( \theta_j \) appear. Only few facts are known about the representation theory of \( \mathcal{A} \) \[18, 17\].

Starting from functional relations \[1, 14, 13\], and solving them via Bethe-ansatz-like equations \[1, 36\], the complete finite \( L \) spectrum for the \( \mathbb{Z}_3 \) superintegrable quantum chain has been given in \[16\]. The general \( \mathbb{Z}_N \)-case is discussed in \[9\]. All \( \mathbb{Z}_N \) conserving boundary conditions (b.c.) \( X_{L+1} = \omega^R X_1; \ R = 0, 1, \ldots, N-1 \) can be treated. Eigenstates of \( H^{(s)} \) are not known for general \( \lambda \). There are high- and low-temperature expansion results \[24\].

### 3 Correlation functions

To get more information about the phases of the \( \mathbb{Z}_N \)-chiral Potts quantum chains we are interested in correlation functions. We assume periodic b.c. \( X_{L+1} = X_1 \) and will only consider expectation values in the lowest translational invariant state, which we denote by \( | v \rangle \). This is the ground state in the non-IC regime\[4\]. We concentrate on two-point functions of local operators \( \Xi_j \), where \( \Xi_j \) stands for either \( Z_j \) or \( X_j \), or powers of these operators. We assume

\[
\langle v | v \rangle = 1 \quad \text{and define as usual}
\]

\[
C_{\Xi}(r) = \langle v | \Xi_{j+r}^+ \Xi_j | v \rangle - \langle v | \Xi_{j+r}^+ | v \rangle \langle v | \Xi_j | v \rangle. \tag{11}
\]

Since \( | v \rangle \) is translational invariant, \( C_{\Xi}(r) \) will be independent of \( j \). We have \( C_{X}(-r) = C_{X}(r)^* \); \( C_{Z}(-r) = C_{Z}^*(r) = C_{Z}(r) \). Thus, although the chiral Potts model is not parity invariant, with periodic b.c. we can restrict ourselves to positive \( r \) with \( 0 < r < L/2 \). According to the asymptotic behavior expected in the massive phases we will fit our numerical data at large \( r \) to the form

\[
C_{\Xi}(r) \approx a_{\Xi} e^{-r/\xi_{\Xi}} e^{2\pi ir/\Lambda_{\Xi}} + (1 - a_{\Xi}) \delta_{r,0}. \tag{12}
\]

\[1\]Albertini and McCoy \[3\] have studied correlation functions in the superintegrable IC region via finite-size corrections to the energy levels and methods from conformal field theory.
A finite Λ means oscillatory correlations. In the high-temperature regime $C_Z$ is real and so it cannot show oscillations. $C_Z$ is real also in 8th-order low-temperature expansion [24]. So, we concentrate on $C_X(r)$ and Λ will always mean $Λ_X$. Honecker [24] gave a rigorous argument that $C_X$ must oscillate for $φ = π$ (this is a curve at $λ < 1$ and $φ > π/2$) because of the shifted parity symmetry present there. Then, by continuity, we expect such oscillations in the adjacent part of the high-temperature phase too. For $Z_3$ at $φ = π$ one gets $|Λ| = 6$, whereas from high-temperature expansion $\lim_{λ→0} Λ = 6π/ℜeφ$, so that on the superintegrable line for $λ → 0$ we should have $Λ → 12$ and $Λ = ∞$ at $φ = 0$. Observe that for small $λ$ a small change of $φ$ produces a strong change of $Λ$ and so there we expect a fast $φ$-dependence of $Λ$.

Information on $Λ$ for $λ → 1^+$ can be obtained from a finite-size analysis of the pattern of ground-state level crossings. This way [21] concludes that in the thermodynamic limit $Λ ∼ (λ − 1)^{-2/N}$ for $λ → 1^+$ and $φ = π/2$.

Numerical calculations of $C_X$ using Lanczos-type diagonalization of the sparse matrix $H$ have been performed in [23], but since only chain lengths $L ≤ 14$ or $r ≤ 7$ can be managed, $Λ$ (which, as we saw, should be 6...12 or larger) could only be crudely determined. Here now the DMRG-method can do much better, and we will now turn to describe our new calculations.

4 The DMRG-method

For many years, say from 1980-1992, numerical investigations of quantum chain spectra mainly used Lanczos-type diagonalisation of finite chain hamiltonians. In this method, the full space of states is considered, restricted only by possible conservation laws as e.g. momentum, $Z_N$-charge or parity. By an iterated application of the sparse hamiltonian matrix one achieves excellent convergence for typically about 10-20 lowest energy levels. For a $Z_3$-model chains up to about $L = 14$ sites (ca.$10^5 × 10^5$-matrices for periodic b.c.) can be treated without much effort. The extrapolation to the thermodynamic limit $L → ∞$ has to be based on these few $L = 2, . . . , 14$ data.

While in this method each chain length $L$ is treated separately, the DMRG method invented by White [37] starts with small chains and enlarges the chain step by step. This method has become standard in condensed matter physics and we will be brief in summarizing the main idea at the specific example of a $Z_3$-chain with free b.c.:

We start with a small chain $B_L$ of e.g. $L = 5$ sites. Then $H$ is a $m \times m$ matrix, where $m = 3^5 = 243$. Then we add one site and obtain a chain $B_{L+1}$ of $L + 1$
sites. We don’t want to keep all $3m$ states of $B_{L+1}$, but nevertheless want to get a good approximation for the ground state vector.

In order to find out, which states of $B_{L+1}$ are most important for our goal (how to renormalise the system), we double the system by adding a system of the same size as shown in the Figure:

![Diagram](image)

This $2(L + 1)$ sites system is called "superblock". We compute the $3^2 m^2 \approx 5 \cdot 10^5$-dimensional ground-state vector $|\Phi\rangle$ of this superblock by a Lanczos algorithm:

$$|\Phi\rangle = \sum_{i,j} \Phi_{i,j} |i\rangle_B \otimes |j\rangle_{\tilde{B}}$$  \hspace{1cm} (13)

and form the reduced density matrix taking the partial trace over the $\tilde{B}$-states:

$$\rho_{i,i'} = \sum_j \Phi_{i,j} \Phi_{i',j}^{*}.$$  \hspace{1cm} (14)

We diagonalise the reduced $(729 \times 729)$ density matrix $\rho_{i,i'}$ by some QL-algorithm and obtain the diagonal elements $\rho_i$. Only those about $m' \approx 80$ states with the largest $\rho_i$ are now kept to describe the renormalized matrix $B_{L+1}$. This procedure is repeated until we reach chains of about $L = 80$ sites, still described by a $m' \times m'$ renormalised hamiltonian matrix.

This method is most efficient for free b.c. If one wants to use periodic b.c., it is convenient to add a system repeated in the same orientation for not having two big adjacent blocks when joining the outer ends. We have not mentioned the decomposition into parity eigenstates, which is useful, even if parity is not conserved as it is in our chiral Potts model. For calculating matrix elements of operators, one has to keep track of the transformation of these operators into the new bases, which requires considerable effort.

The DMRG method works very well even up to $L = 80 \ldots 200$ sites keeping just about $m' \approx 80 \ldots 150$ states. The reason for this is the exponentially decreasing spectrum of the density matrix, which, as first pointed out by Nishino and Okunishi \[30\], can be understood from the close relation of the DMRG procedure to Baxter's variational method for calculating CTM eigenvalues \[3\]. We shall make use of this connection in the last section.

**Results for the correlation functions**

We have calculated $C_X(x)$ in the high-temperature phase from chains up to $L = 80$ sites, so that it is reasonable to look up to $r = 30$ sites. The so-called
Figure 2: Real- and imaginary parts of the exponentially stretched correlation function $e^{x/\xi}C_X(x)$. Superintegrable case $\varphi = \phi = \pi/2$, high-temperature phase at $\lambda = 0.75$. The fit gives $\Lambda = 29.5(5)$ and $\xi = 2.000(5)$.

"infinite-size" DMRG approach \cite{34} is used, but with periodic b.c. (although this is much more effort in DMRG) in order to reduce boundary effects. Because of space limitations we give just one typical example for the superintegrable case and $\lambda = 0.75$. Of course, $C_X$ decreases exponentially with $r$. So, in order to show the oscillation clearly, we multiply $C_X$ by the factor $e^{r/\xi}$ which can be done since our numerical precision is quite high. We show both the real and the imaginary part.

We fit also $\gamma$ in $\xi^{-1} = 2\gamma|1-\lambda|$ (in the superintegrable case the gap is linear in $\lambda$). Our fits give $\gamma = 1.0 \pm 0.1$ in lattice units. Going to smaller chiral angles (we give no Figures here, but these look similar to Fig.2) we fitted the exponent $\nu$ in $\xi^{-1} \sim |1-\lambda|^{\nu}$. $\nu$ is found to change little with $\varphi$, not surprisingly since the mass-gap exponent $\nu = 5/6$ for the parafermion case $\phi = 0$ is close to the value $\nu = 1$ at $\phi = \pi/2$. However, $\Lambda$ changes strongly: it must diverge for $\phi \to 0$. In the integrable case \cite{35} the behavior of $\Lambda$ with decreasing $\phi$ is complicated because at $\lambda < 1$ the range in $\lambda$ where the hamiltonian is hermitian, shrinks fast. However, for the $\mathbb{Z}_3$-self-dual (and probably not integrable) case $\phi = \varphi$ within the errors we find $\Lambda \sim \phi^{-1}$. 

0.0 5.0 10.0 15.0 20.0 25.0 30.0

\begin{center}
\begin{figure}
\begin{tikzpicture}
\begin{axis}[
width=\textwidth,
height=0.5\textwidth,
]  
\addplot [solid,mark=*,mark options={solid}] table [y expr={\thisrow{Y}+0.25*sin(2*\pi*x/29.5)}] {data.csv};  
\addplot [solid,mark=square*,mark options={solid}] table [y expr={\thisrow{Y}+0.25*cos(2*\pi*x/29.5)}] {data.csv};  
\addplot [dashed,mark=*,mark options={solid}] table [y expr={\thisrow{Y}+0.25*sin(2*\pi*x/29.5)}] {data.csv};  
\addplot [dashed,mark=square*,mark options={solid}] table [y expr={\thisrow{Y}+0.25*cos(2*\pi*x/29.5)}] {data.csv};  
\end{axis}
\end{tikzpicture}
\end{figure}
\end{center}
5 The Corner Transfer Matrix eigenvalues

Baxter’s corner transfer matrix (CTM) is a very efficient tool \[6\] for the calculation of single-spin expectation values (order parameters) of many integrable models. The CTM also plays also a crucial role in the Kyoto approach \[26\] to correlation functions of integrable models.

A CTM is defined as the partition function of a corner of a planar lattice (e.g. quadrant in case of a square lattice, or a sextant in case of a triangular lattice \[10\]). Fig. 3 shows a corner of a small square lattice. The empty circles indicate the spins forming the indices of the CTM, full circles are summed over, except for the lower diagonal spins, which are fixed or summed over, depending on whether we use fixed or free b.c. The CTM of the SE-corner of

\[
\begin{align*}
A_{\{s\},\{s'\}}
\end{align*}
\]

Figure 3: Corner transfer matrix \(A_{\{s\},\{s'\}}\)
a square lattice is usually called \(A\), the other anticlockwise rotated ones \(B, C\) and \(D\). The partition function is \(Z = Tr(ABCD)\). Baxter discovered that in the thermodynamic limit the spectrum of the CTM of the *eight-vertex model* takes a very simple form: let \(A_d\) be \(A\) diagonalised and normalized such that its maximum eigenvalue is unity. Then

\[
(A_d)_{r,r}(u) = \exp (-n_r x); \quad n_r \in \mathbb{Z},
\]

with \(x = \pi u/K\) and \(u\) is the rapidity parameter in the Boltzmann weights, \(K\) the elliptic half-period. For the derivation of (15) the difference property of the Yang-Baxter equations is used and also some analyticity property which allows to make use of a periodicity property. From (15) it is a short step to obtain the order parameter (magnetization).

Of course, also for the \(\mathbb{Z}_N\) chiral Potts model one is interested in the order parameters. We like to know \(\langle Z_j \rangle\) for \(j = 1, \ldots, N - 1\) where the spin with label 0 is far from the boundaries. From series expansions and \(Z\)-invariance
arguments there is the conjecture that the order parameters are independent of the chiral angle and their \( k' \)-dependence is \[\langle Z_j^0 \rangle = (1 - k'^2)^{(j(N-j))/2N^2}. \] (16)

Baxter \[10, 11\] has derived many important factorization and quasiperiodicity properties and functional relations of the chiral Potts CTM, but since the difference property is absent and analyticity seems to be restricted, as yet no exact proof of (16) was found. In the low-temperature limit Baxter did a truncated recursion calculation for the \( \mathbb{Z}_3 \)-case \[10\]. He obtains a 7th-order series approximation to the 17 largest eigenvalues \( M_i \) of \( M = ABCD \). These eigenvalues depend only on a temperature-like parameter, arising from the nome of the hyperelliptic functions which parametrize the Boltzmann weights.

In detail:

\[
\begin{align*}
M_2 &= M_3 = \tau; & M_4 = M_5 = 2\tau^3 - \frac{1}{2}\tau^4 - \frac{29}{8}\tau^5 - \frac{5}{16}\tau^6 + \frac{4147}{128}\tau^7 + \ldots \\
M_6 &= (3 \pm 2\sqrt{2})\tau^4 + (\mp 4\sqrt{2} - 5)\tau^5 - \left( \frac{5}{2} \pm \frac{49}{4\sqrt{2}} \right)\tau^6 + \left( 12 \pm \frac{377}{4\sqrt{2}} \right)\tau^7 + \ldots \\
M_7 &= M_8 = \frac{1}{2}\tau^4 + \frac{13}{8}\tau^5 + \frac{37}{16}\tau^6 - \frac{2995}{128}\tau^7 + \ldots \\
M_{10} &= M_{11} = 3\tau^5 + \frac{11}{3}\tau^6 - \frac{730}{27}\tau^7 + \ldots; & M_{13} = M_{14} = \frac{4}{3}\tau^6 + \frac{29}{54}\tau^7 + \ldots \\
M_{15} &= (4 \pm 2\sqrt{3})\tau^6 + (10 \pm 6\sqrt{3})\tau^7 + \ldots; & M_{16} = M_{17} = \frac{9}{2}\tau^7 + \ldots. \tag{17}
\end{align*}
\]

Here the largest eigenvalue is normalized to \( M_1 = 1 \) and \( M_2 \) is taken as the temperature scale \( \tau \), which is related to \( k' \) of \[11\] or rather, \( u = k'^2/27 \), by

\[
\tau = u + 14u^2 + 254u^3 + 5150u^4 + 111123u^5 + 2495269u^6 + 57608712u^7 + \ldots \tag{18}
\]

In contrast to other integrable models with difference property, here the spectrum of \( \log M \) is not equidistant. Baxter uses fixed spin zero b.c. on the CTM, so the ground state is non-degenerate. The degenerate levels are those with center spin 1 and 2, the non-degenerate ones have center spin zero. Baxter’s derivation of (17) is best justified in the region of real positive Boltzmann weights and it is not clear which details can be trusted when extending the formulae to the region of complex Boltzmann weights. Using (17), (18) one finds correctly \( \langle Z_0^0 \rangle = k'^{2/9} \) up to the order \( \tau^7 \) considered, in agreement with (16) as first conjectured from the superintegrable chain.

Looking into the apparent convergence of the expansions (17), (18), one is inclined to trust the lowest eigenvalues \( M_2 \ldots M_5 \) up to about \( k' \approx 0.7 \) within a few percent: For \( k' = 0.7 \) one finds \( M_2 = \tau \approx 0.25 \). However, approaching \( k' = 1 \) we clearly see that convergence of the series gets lost quite suddenly, although in 7th-order at \( k' = 0.9 \) we still remain at \( \tau \approx 0.06 \).
6 CTM eigenvalues from DMRG density matrices

Fig.4 illustrates that the matrix $M = ABCD$ is just the partition function of a lattice with a seam opened between $A$ and $D$. There is a direct relation

$$\begin{array}{c}
\sigma_2 \\
C \\
\sigma_1 \\
D
\end{array}$$

$$\begin{array}{c}
\sigma_2 \\
B \\
\sigma_1 \\
A
\end{array}$$

between $M$ and the reduced density matrix $\rho_{i,i'}$ of (14): Baxter [6], chapt.13, pointed out that in the thermodynamic limit the product of two CTM, say $AB$ in Fig.4, is the maximal eigenvector $|\Psi\rangle$ of the column-to-column transfer matrix (TM) (here moving left to right). Now, for integrable models the TM of the 2-dim. system is in one-to-one correspondence with the quantum chain hamiltonian $H$ as e.g. in (3) (vice-versa one uses the Trotter construction) and the maximal eigenvector of the TM is the ground state eigenvector of $H$, i.e. $|\Psi\rangle \sim |\Phi\rangle$ of eq.(13). So $M = ABCD$, which is summed over the closed seam between $B$ and $C$, is proportional to $\rho_{i,i'}$ of eq.(14). All this is correct for large systems, i.e. when the boundaries play no essential role.

We now compare numerically the $\rho_{i,i'}$ (which appeared as a technical tool for selecting the most relevant states in our DMRG calculation) to Baxter’s CTM eigenvalues (17). Baxter uses fixed b.c. at low $k'$. It is not easy to implement fixed b.c. in DMRG, so we use free b.c. for the dual model [8] at high temperatures $\lambda < 1$. We have studied several values of $L$ and $\phi, \lambda$. In Fig.5 we show that there is excellent agreement already when we use a small $L = 10$ system, which can be understood considering that for $\lambda \leq 0.75$ the
Figure 5: The largest 15 eigenvalues for several $\lambda$. Discrete points: Baxter’s CTM-calculation \footnote{eqs.(17)} together with our numerical density matrix eigenvalues for free b.c. and $L = 10$: solid lines: at the boundary of the hermitian region $\phi = 0$, $\varphi = \arccos \lambda$; dashed lines: on the superintegrable line.

correlation length $\xi$ is smaller than three lattice sites. E.g. for $\lambda = 0.5$ eqs.(17) give $M_4 = 2.44884 \cdot 10^{-6}$ whereas the diagonalisation of the density matrix gives $\rho_4 = 2.4418 \cdot 10^{-6}$ at $\phi = \frac{\pi}{2}$ and $\rho_4 = 2.4382 \cdot 10^{-6}$ at $\phi = 0$. Baxter’s formulae are not valid at $\lambda = 1$, so any agreement there is not significant. For the Ising and XXZ-models, the checks analogous to the ones given here were made earlier by Peschel \textit{et al.} \footnote{34}. In their cases the CTM-eigenvalues are known exactly and, apart from degeneracies, are strictly logarithmically equidistant.

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