ON THE MINIMAL DIMENSION OF THE ORBITS OF A $\mathbb{R}^n$-ACTION

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Abstract. Consider a smooth action of $\mathbb{R}^n$ on a connected manifold $M$, not necessarily compact, of dimension $m$ and rank $k$. Assume that $M$ is not a cylinder. Then there exists an orbit of the action of dimension $< (m + k)/2$. As a consequence, one shows that if there is a non-zero element of the ring of Pontrjagin classes of $M$ of degree $4\ell \geq 4$, then there exists an orbit of the action of dimension $\leq m - \ell - 1$.

Key words: Action of $\mathbb{R}^n$, minimal dimension of the orbits, open manifold.

2020 Mathematics Subject Classification: Primary 37C85; Secondary 57R25.

1. Introduction

In this work manifolds (always without boundary) and their associated objects are real $C^\infty$ unless another thing is said.

The degeneracy of the orbits of an action of a Lie group on a manifold is nowadays a classical question. Here we will consider the case of $\mathbb{R}^n$ acting on open manifolds. A result by E. Lima [5], of 1964, states that any action of $\mathbb{R}^n$ on a compact surface of non-vanishing Euler characteristic has a fixed point. Later on P. Molino and the author [7] have given, for compact manifolds, an upper bound for the minimal dimension of the orbits of a $\mathbb{R}^n$-action, which includes the Lima’s theorem (in $C^\infty$ class) as a particular case. The goal of this paper is to extend this result to open manifolds.

Other results of this kind for symplectic actions can be seen in [13]. For some quite interesting results in dimension three see [2].

Recall that the span of a manifold $P$ is the maximal number of vector fields on $P$ that are linearly independent at each point. The same notion but for commuting vector fields is named the rank of $P$ (defined by J.W. Milnor at the Seattle Topology Conference of 1963 and echoed by S.P. Novikov [8]). Finally, if the vector fields commute and are complete this number is called the file of $P$ (introduced by H. Rosenberg [11]). Obviously for compact manifolds file equals rank.
A manifold diffeomorphic to $\mathbb{T}^r \times \mathbb{R}^{m-r}$ will be called a cylinder of type $r$ and dimension $m$. This is our main result.

**Theorem 1.1.** Consider an action of $\mathbb{R}^n$ on a connected manifold $M$ of dimension $m$ and rank $k$. Assume that $M$ is not a cylinder. Then there is an orbit of the action of dimension $< (m + k)/2$.

(The compact case of the foregoing theorem is just the main result of [7].)

As a consequence, we show that if there is a non-zero element of the ring of Pontrjagin classes of $M$ of degree $4\ell \geq 4$, then there exists an orbit of the action of dimension $\leq m - \ell - 1$ (Proposition 5.7).

This work consists of six sections, the first one being the introduction. Sections 2 and 3 are devoted to state some results needed later on. The proof of Theorem 1.1, the main result, is given in Section 4. In Section 5 we illustrate Theorem 1.1 with several examples and one proves Proposition 5.7. Finally in Section 6 one studies the $\mathbb{R}^n$-actions whose orbits have codimension $\leq 1$.

For general questions on Lie groups actions see [9], for those on characteristic classes see [6] and finally [3] for questions on foliations.

## 2. Preliminary results

Let $V$ be a real vector space of dimension $n$ and let $G(k)$ be the Grassmann manifold of $k$-planes of $V$ (if necessary one will write $G(V, k)$ instead of $G(k)$). Given a $k'$-plane $F$ of $V$ set

$$N_F(k, r) = \{ E \in G(k) : \dim(E \cap F) = r \}, \quad r \geq 1,$$

and $N_F(k) = \bigcup_{r \geq 1} N_F(k, r)$.

Every $N_F(k, r)$ is a regular (embedded) submanifold of $G(k)$ of codimension $r(n + r - k - k')$.

Consider a manifold $P$ and a differentiable map $f : P \to G(k)$. One will say that $f$ is transverse to $N_F(k)$ if it is transverse to each stratum $N_F(k, r), \ r \geq 1$.

**Lemma 2.1.** One has:

(i) The set of those $F \in G(k')$ such that $f$ is not transverse to $N_F(k)$ is of the first category in $G(k')$. 

(ii) Assume that $P'$ is a subset of $P$ of the first category. If $\dim P \leq r_0(n + r_0 - k - k')$, where $n, k, k', r_0 \geq 0$, then the set

$$D = \{ F \in G(k') : \text{there exists } p \in P' \text{ such that } \dim(f(p) \cap F) \geq r_0 \}$$

is of the first category in $G(k')$.

Consider a foliation $\mathcal{F}$ of dimension $k$ on a $m$-manifold $M$. A transversal $T$ to $\mathcal{F}$ is called neat if no leaf of $\mathcal{F}$ intersects $T$ two or more times. A point of $M$ is said wandering if it belongs to some neat transversal. The set $W(\mathcal{F})$ of all wandering points, that is the union of all neat transversals, is open and $\mathcal{F}$-saturated.

Now assume the existence of an action of $\mathbb{R}^n$ on $M$ whose orbits are just the leaves of $\mathcal{F}$. In this case the leaves of $\mathcal{F}$ are cylinders $C_r = T^r \times \mathbb{R}^{k-r}$, $r = 0, \ldots, k$. The number $r$ one will be called the type of the leaf. Assigning to each point the type of the leaf passing through it defines a function $\tau : M \to \mathbb{N}$, which is constant on the leaves of $\mathcal{F}$.

**Lemma 2.2.** The set of those points of $W(\mathcal{F})$ such that $\tau$ is constant around them is $\mathcal{F}$-saturated, open and dense in $W(\mathcal{F})$.

**Theorem 2.3.** Assume that on a neighborhood of a point $p \in W(\mathcal{F})$ the function $\tau$ is constant. Set $r = \tau(p)$. Then there exists an open set of $M$ identified to $D \times C_r$, where $D$ is an open disc of $\mathbb{R}^{m-k}$ centered at the origin, such that:

(a) $D \times C_r$ is $\mathcal{F}$-saturated, every $\{x\} \times C_r$ is a leaf of $\mathcal{F}$ and $\{0\} \times C_r$ is that passing through $p$.

(b) One can choose coordinates $(x, \theta, y)$ on $D \times C_r = D \times T^r \times \mathbb{R}^{k-r}$ in such a way that the fundamental vector fields of the action write

$$X = \sum_{j=1}^{r} f_j(x) \frac{\partial}{\partial \theta_j} + \sum_{\ell=1}^{k-r} g_\ell(x) \frac{\partial}{y_\ell}.$$

For the proof of the foregoing results see [13] pages 271, 272, 287 and 288.

**3. Some results on vector fields**

In this section one states some technical results needed later on.

**Lemma 3.1.** Let $N$ be a regular submanifold of a manifold $M$. Assume that $N$ is a closed subset of $M$. Consider a vector field $X$ on $N$. Then there exists a vector field $X'$ on $M$ that is complete on $M - N$ and equals $X$ on $N$. 
Proof. Consider a tubular neighborhood $\pi: E \to N$ of $N$ endowed, as vector bundle, with a Riemannian metric.

Set $S_\varepsilon = \{x \in E: \|x\| < \varepsilon\}$ and $T_\varepsilon = \{x \in E: \|x\| = \varepsilon\}$, $\varepsilon > 0$.

Let $Y$ be a vector field on $E$ tangent to each $T_\varepsilon$ and such that $\pi_* Y = X$ (its existence is obvious). Consider a family $\{A_k\}_{k \geq 1}$ of open sets of $N$ such that every $\overline{A}_k$ is a compact set included in $A_{k+1}$ and $\bigcup_{k \geq 1} A_k = N$.

Set $B_k = \pi^{-1}(A_k) \cap S_{1/k}$; every $B_k$ is an open set of $E$, so of $M$, with compact adherence. Let $B = \bigcup_{k \geq 1} B_k$.

Since manifolds are normal spaces, there is a closed set $C$ of $M$ such that $C \cap N = \emptyset$ and $M - E \subseteq \overline{C}$. Consider a function $\varphi: M \to [0,1]$ such that $\varphi(N) = 1$ and $\varphi(C \cup (M - B)) = 0$. Now set $X' = \varphi Y$ on $E$ and $X' = 0$ on $\overline{C}$. As $E \cup \overline{C} = M$ and $\varphi$ vanishes on $E \cap \overline{C}$ the vector field $X'$ is well defined; moreover $X' = X$ on $N$.

For finishing it is enough to show that $X'$ is complete on $E - N$ since it vanishes on $\overline{C}$. As $X'$ is tangent to each $T_\varepsilon$, $\varepsilon > 0$, it suffices to show that it is complete on $T_\varepsilon$. But $X'|T_\varepsilon$ has compact support because $X'$ vanishes on $T_\varepsilon - \pi^{-1}(A_k)$ for any $k$ such that $1/k < \varepsilon$. □

Corollary 3.2. Consider a regular submanifold $N$ of a manifold $M$, where $N$ is a closed subset of $M$. Let $X$ and $\Phi_t$ be a vector field on $N$ and its flow respectively. Assume that $\Phi_t$ is defined on the whole $N$. Then $M$ and $(M - N) \cup \Phi_1(N)$ are diffeomorphic.

Indeed, let $X'$ be a vector field like in Lemma 3.1 and let $\Phi'_t$ be its flow. Then $\Phi'_t: M \to (M - N) \cup \Phi_1(N)$ is a diffeomorphism.

Lemma 3.3. Given $a > 0$ there exists a vector field $X$ on $\mathbb{R}^m$ such that $\Phi_1(\mathbb{R}^m) = B_a(0)$ where $\Phi_1$ is the flow of $X$.

By taking into account the diffeomorphism $x \in \mathbb{R}^m - \{0\} \to (\|x\|/a, x/\|x\|) \in \mathbb{R}^+ \times S^{m-1}$, for proving this result it suffices to exhibit a vector field $Z = g(u)\partial/\partial u$ on $\mathbb{R}^+$ vanishing near zero such that $\Phi_1(\mathbb{R}^+) = (0,1)$, where $\Phi_t$ is the flow of $Z$. For example consider a non-positive function $g: \mathbb{R}^+ \to \mathbb{R}$ such that $g((0,1/4)) = 0$ and $g(u) = -u^2$ on $[1/3, \infty)$.

In the next result, balls in $\mathbb{R}^m$ centered at the origin and the radius $r$ are denoted $B_r$ (open) or $\overline{B}_r$ (closed) and those in $\mathbb{R}^n$ centered at the origin $B'_r$ and $\overline{B}'_r$ respectively. Similarly, $S_r$ and $S'_r$ are the spheres of radius $r$ centered at the origin. On the other hand, $(x, y)$ will be a point of $\mathbb{R}^m \times \mathbb{R}^n$. 
Lemma 3.4. Given real numbers $0 < c < a$ and $0 < d < b$ set

$$A = B_a \times B'_b \setminus \{0\} \times (B'_b - B'_d) \quad \text{and} \quad A' = A - B_c \times S'_d.$$  

Then there exist $\varepsilon > 0$ and a diffeomorphism $\varphi: A \to A'$ such that $\varphi(x, y) = (x, y)$ if $\|x\| \geq a - \varepsilon$ or $\|y\| \geq b - \varepsilon$ (or both).

Proof. First observe that there exist $0 < \varepsilon < \min\{(a - c)/2, (b - d)/2\}$ and a vector field $X$ on $B_a - \{0\}$, whose flow $\Phi$ is defined everywhere for $t = 1$, such that:

- $\Phi_1(B_a - \{0\}) = B_a - B_c$,
- $X$ vanishes on $B_a - B_{a - 2\varepsilon}$.

Indeed, identify $\mathbb{R}^m - \{0\}$ to $\mathbb{R}^+ \times S^{m-1}$ in the usual way ($x \to (\|x\|, x/\|x\|)$) and endow this last space with the product coordinates $(u, v)$. Then set $X = g(u)\partial/\partial u$ where $g: (0, a) \to \mathbb{R}$ is a non-negative function, $g((0, c]) = c$ and $g((a - 2\varepsilon, a)) = 0$.

Now consider $X$ as a vector field on $(B_a - \{0\}) \times S'_d$ tangent to the first factor. Then its flow $\Phi$ is defined everywhere for $t = 1$ and:

- $\Phi_1((B_a - \{0\}) \times S'_d) = (B_a - B_c) \times S'_d$,
- $X$ vanishes on $(B_a - B_{a - 2\varepsilon}) \times S'_d$.

Since $(B_a - \{0\}) \times S'_d$ is both a closed subset and a regular submanifold of $A$, by Lemma 3.1 the vector field $X$ extends to a vector field $X'$ on $A$, which is complete outside $(B_a - \{0\}) \times S'_d$.

Let $h: A \to [0, 1]$ be a function such that:

- $h(x, y) = 1$ if $\|x\| \leq a - 2\varepsilon$ and $\|y\| \leq b - 2\varepsilon$,
- $h(x, y) = 0$ if $\|x\| \geq a - \varepsilon$ or $\|y\| \geq b - \varepsilon$.

Set $X'' = hX'$ and let $\Phi''_t$ be its flow. Then $X'' = X$ on $(B_a - \{0\}) \times S'_d$. Moreover $X''$ is complete outside $(B_a - \{0\}) \times S'_d$, it vanishes if $\|x\| \geq a - \varepsilon$ or $\|y\| \geq b - \varepsilon$, and $\Phi''_1(A) = A'$.

Therefore for finishing the proof set $\varphi = \Phi''_1$. \qed

4. Proof of Theorem 1.1

Consider an action on the left of $\mathbb{R}^n$ on a connected manifold $M$ of dimension $m$ and rank $k < m$ (i.e. $M$ is not a cylinder). Let $V$ be the Lie algebra of $\mathbb{R}^n$ and let $X_v, v \in V$, be the fundamental vector field on $M$ associated to $v$.

For each subset $V'$ of $V$ and each point $p \in M$ set $V'(p) = \{X_v(p) : v \in V'\}$.
As usual, the infinitesimal isotropy of a point \( p \) is the set \( I(p) = \{ v \in V : X_v(p) = 0 \} \). Since \( V \) is abelian, \( I(p) \) only depends on the orbit of \( p \).

Denote by \( \Sigma_r, r = 0, \ldots, m \), the set of those points of \( M \) whose orbit has dimension \( r \); that is \( p \in \Sigma_r \) if and only if \( \dim I(p) = n - r \). Let \( h_r : \Sigma_r \to G(n - r) \) be the map given by \( h_r(p) = I(p) \).

A chart \((U, x_1, \ldots, x_m)\) is said \( \Sigma_r\)-adapted if the image of \( U \) on \( \mathbb{R}^m \) is a product of open intervals and there exist vectors \( v_1, \ldots, v_r \in V \) such that \( X_{v_j} = \partial / \partial x_j, j = 1, \ldots, r \).

It is easily seen that every point of \( \Sigma_r \) belongs to the domain of some \( \Sigma_r\)-adapted chart. Therefore \( \Sigma_r \) can be covered by a countable family of \( \Sigma_r\)-adapted charts. (Here countable includes the finite case.)

**Lemma 4.1.** For every \( r = 0, \ldots, m \) the map \( h_r : \Sigma_r \to G(n - r) \) is differentiable of rank \( \leq m - r \), i.e. it can be locally extended to a differentiable map of rank \( \leq m - r \) at every point of its domain.

**Proof.** It suffices to show that \( h_r : U \cap \Sigma_r \to G(n - r) \) is differentiable and of rank \( \leq m - r \) for any \( \Sigma_r\)-adapted chart \((U, x_1, \ldots, x_m)\). Consider vectors \( v_1, \ldots, v_r \in V \) such that \( X_{v_j} = \partial / \partial x_j, j = 1, \ldots, r \) and choose \( v_{r+1}, \ldots, v_n \in V \) in a such a way that \( \{v_1, \ldots, v_n\} \) is a basis of \( V \).

Then each \( X_{v_i} = \sum_{\ell=1}^m f_{i\ell} \partial / \partial x_\ell, i = r + 1, \ldots, n \).

As \( V \) is an abelian Lie algebra functions \( f_{i\ell} \) only depend on \((x_{r+1}, \ldots, x_m)\).

Now consider the open set \( A \subset G(n - r) \) of those \((n - r)\)-planes that have as a basis

\[
\left\{ v_{r+1} - \sum_{\ell=1}^r a_{r+1\ell} v_\ell, \ldots, v_n - \sum_{\ell=1}^r a_{n\ell} v_\ell \right\},
\]

where \( a_{i\ell}, i = r + 1, \ldots, n, \ell = 1, \ldots, r \), are real numbers. Then

\[
(A, (a_{i\ell}), i = r + 1, \ldots, n, \ell = 1, \ldots, r)
\]

is a system of coordinates of \( G(n - r) \).

Let \( \tilde{h}_r : U \to A \subset G(n - r) \) be the differentiable map defined in coordinates by \( \tilde{h}_r(x) = (f_{i\ell}(x)), i = r + 1, \ldots, n, \ell = 1, \ldots, r \). Then \( \tilde{h}_r \), which only depends on \((x_{r+1}, \ldots, x_m)\), extends \( h_r : U \cap \Sigma_r \to G(n - r) \) since if \( x \in U \cap \Sigma_r \), then \( V(x) \) is spanned by \( \partial / \partial x_1, \ldots, \partial / \partial x_r \) and

\[
\left\{ v_{r+1} - \sum_{\ell=1}^r f_{r+1\ell}(x) v_\ell, \ldots, v_n - \sum_{\ell=1}^r f_{n\ell}(x) v_\ell \right\}
\]

is a basis of \( I(x) \). \( \square \)
Proposition 4.2. There always exists some orbit of dimension $\leq (m + k)/2$.

Proof. Assume that all the orbits have dimension $> (m + k)/2$. Let $(U, x_1, \ldots, x_m)$ be a $\Sigma_r$-adapted chart, $r > (m + k)/2$, and let $\hat{h}_r : U \to G(n - r)$ be like in the proof of Lemma 4.1. Consider $F \in G(k + 1)$ such that $\hat{h}_r$ is transverse to $N_F(n - r)$. Then $\text{Im}(\hat{h}_r) \cap N_F(n - r) = \emptyset$; indeed, otherwise as the rank of $\hat{h}_r \leq m - r$ and the smallest codimension among the strata of $N_F(n - r)$ is that of $N_F(n - r, 1)$ one has

$$m - r \geq \text{codim} N_F(n - r, 1) = r - k,$$

hence $r \leq (m + k)/2$, contradiction.

Observe that from $\text{Im}(\hat{h}_r) \cap N_F(n - r) = \emptyset$ follows $I(q) \cap F = \{0\}$, that is to say $\dim F(q) = k + 1$, for every $q \in U \cap \Sigma_r$.

But $M = \bigcup_{r > (m + k)/2} \Sigma_r$ may be covered by a countable family of adapted charts, and assertion (i) of Lemma 2.1 allows us to choose $F \in G(k + 1)$ such that each map $\hat{h}_r$ associated to this countable family is transverse to $N_F(n - r)$. Therefore $\dim F(p) = k + 1$ for all $p \in M$ and $\text{rank} M \geq k + 1$, contradiction.

Proposition 4.2 is due to D. Simen [12]; see [7] as well.

Remark 4.3. In fact one has proved two things slightly stronger:

- In the proposition above the rank of $M$ can be replaced by the file.
- Proposition 4.2 still holds for local actions.

From now on and until the end of Section 4 one will assume that there is no orbit of dimension $< (m + k)/2$. We will reach a contradiction. More precisely, one will construct an open subset of $M$ of rank $\geq k + 1$ that is diffeomorphic to $M$.

Since by Proposition 4.2 there exist orbits of dimension $\leq (m + k)/2$ necessarily $m + k$ is even. Let $s = (m + k)/2$.

4.1. Existence of suitable elements in $G(k + 1)$. On the open set $\Sigma_s \subset M$ the action of $\mathbb{R}^n$ defines a foliation $\mathcal{F}$ of dimension $s$ and codimension $(m - k)/2$. Let $\mathcal{W}$ be the set of its wandering points and let $\mathcal{U}$ be the set of those points of $\mathcal{W}$ around of which the type of the leaf is constant. By Lemma 2.2 $\mathcal{U}$ is $\mathcal{F}$-saturated, open and dense in $\mathcal{W}$, so $\mathcal{W} - \mathcal{U}$ is of the first category in $\mathcal{W}$.

The aim of this subsection is to prove the following result:
Proposition 4.4. There exists $F \in G(k+1)$ such that:

1. $h_s: \Sigma_s \to G(n-s)$ is transverse to $N_F(n-s)$ and $h_s^{-1}(N_F(n-s,\ell)) = \emptyset$ for every $\ell \geq 2$. Therefore $h_s^{-1}(N_F(n-s)) = h_s^{-1}(N_F(n-s,1))$.

2. $h_s^{-1}(N_F(n-s)) = h_s^{-1}(N_F(n-s,1))$ is the singular set of $F$, i.e. the set of those points $p \in M$ such that $\dim F(p) < k+1$.
   Moreover, if $\dim F(q) < k+1$ for some $q \in M$ then $\dim F(q) = k$ and $q \in U$.

3. $h_s^{-1}(N_F(n-s)) = h_s^{-1}(N_F(n-s,1))$ is empty or, both a closed subset and a regular submanifold of $M$ of dimension $s$ which is included in $U$. Moreover $h_s^{-1}(N_F(n-s))$ is saturated for the action of $\mathbb{R}^n$.

An element of $G(k+1)$ like in the proposition above will be called *suitable*.

Consider a $\Sigma_s$-adapted chart $(U, x_1, \ldots, x_m)$ and a map $\tilde{h}_s: U \to G(n-s)$ like in the proof of Lemma 4.1. Let $C$ be a subset of $U \cap \Sigma_s$.

Lemma 4.5. Assume that:

1. $C$ is of the first category in $U$.

2. If $p \in C$ then any point $q \in U$ such that $q_\ell = p_\ell, \ell = s+1, \ldots, m$, belongs to $C$ as well.

Then the set

$$D = \{ F \in G(k+1): \text{there exists } p \in C \text{ such that } \dim(h_s(p) \cap F) \geq 1 \}$$

is of the first category in $G(k+1)$.

Proof. As $h_s$ does not depend on $(x_1, \ldots, x_s)$, the set $D$ does not change if one replaces the statement $p \in C$ by $p \in T \cap C$, where $T$ is a transversal defined by $x_1 = c_1, \ldots, x_s = c_s$ for suitable $c_1, \ldots, c_s \in \mathbb{R}$. By Assumption (2), $T \cap C$ is of the first category in $T$.

By Lemma 2.1 Part (ii) applied to $\tilde{h}_s: T \to G(n-s)$ and $T \cap C$, it suffices to check that

$$\dim T \leq (n+1 - (n-s) - (k+1)) = (m - k)/2$$

which is just the case since $\dim T = (m - k)/2$. □

By covering $\Sigma_s - \hat{\Sigma}_s$ by a countable family of $\Sigma_s$-adapted charts from Lemma 4.5, follows:

Corollary 4.6. The set

$$D_1 = \{ F \in G(k+1): \text{there exists } p \in \Sigma_s - \hat{\Sigma}_s \text{ such that } \dim(h_s(p) \cap F) \geq 1 \}$$
is of the first category in $G(k + 1)$.

By Lemma 2.2 the set $W - U$ is of the first category in $W$ and $F$-saturated. Therefore by covering $W - U$ by a countable family of $\Sigma_s$-adapted charts whose domains are included in $W$ from Lemma 4.5 follows:

**Corollary 4.7.** The set

$$D_2 = \{ F \in G(k + 1) : \text{there exists } p \in W - U \text{ such that } \dim(h_s(p) \cap F) \geq 1 \}$$

is of the first category in $G(k + 1)$.

**Lemma 4.8.** There exists a set $D_3 \subset G(k + 1)$ of the first category such that if $F \in G(k + 1) - D_3$ then $\dim F(p) = k + 1$ for every $p \in \bigcup_{r>s}\Sigma_r$.

**Proof.** Consider a $\Sigma_r$-adapted chart, $r > s$, $(U, x_1, \ldots, x_m)$ and let $\tilde{h}_r: U \to G(n - r)$ be like in the proof of Lemma 4.1. By Lemma 2.1 Part (i) the set

$$E = \{ F \in G(k + 1) : \tilde{h}_r: U \to G(n - r) \text{ is not transverse to } N_F(n - r) \}$$

is of the first category.

As the minimal codimension of the strata of $N_F(n - r)$ is $r - k$, rank $\tilde{h}_r \leq m - r$ and $m - r < r - k$ since $r > s$, if $\tilde{h}_r$ is transverse to $N_F(n - r)$ then $\tilde{h}_r(U) \cap N_F(n - r) = \emptyset$. Therefore if $F \not\in E$ then $\dim F(p) = k + 1$ for any $p \in U \cap \Sigma_r$.

Finally, since $\bigcup_{r>s}\Sigma_r$ can be covered by a countable family of adapted charts, it suffices to set $D_3$ equal to the union of the sets $E$ associated to the elements of the family. □

Let $D_4 = \{ F \in G(k + 1) : h_s: \hat{\Sigma}_s \to G(n - s) \text{ is not transverse to } N_F(n - s) \}$

We claim that any $F \in G(k + 1) - D_1 \cup D_2 \cup D_3 \cup D_4$ is suitable.

Indeed, the first part of Assertion (1) of Proposition 4.4 is obvious and for the second one it is enough to remark that the rank of $h_s \leq m - s = (m - k)/2$ while, if $\ell \geq 2$, the codimension of $N_F(n - s, \ell) \geq m - k + 2$.

Now suppose that $\dim F(p) \leq k$ for some $p \in M$. Then from Corollary 4.6 and Lemma 4.8 follows that $p \in \hat{\Sigma}_s$; therefore $p \in h_s^{-1}(N_F(n - s)) = h_s^{-1}(N_F(n - s, 1))$ and $\dim F(p) = k$.

Now it is clear that $h_s^{-1}(N_F(n - s))$ is the singular set of $F$ and, as a consequence, a closed subset of $M$. 
Since $h_s$ is transverse to $N_F(n-s,1)$ and the codimension of this last submanifold equals $s - k = m - s$, it follows that $h_s^{-1}(N_F(n-s)) = h_s^{-1}(N_F(n-s,1))$ is a regular submanifold of $M$ of dimension $s$ (or empty).

For finishing one has to show that if $p \in h_s^{-1}(N_F(n-s))$ then $p \in U$. First observe that if $T$ is a small transversal to $F$ passing through $p$ then $h_s|T$ is injective (indeed, $(h_s)_*(T_pF) = 0$ and rank $h_s = m - s$). As $h_s$ is constant along the leaves of $F$, if a leaf intersects $T$ more than one time then $h_s|T$ is not injective contradiction. Thus $T$ is neat and $p \in W$. Finally, $p \in U$ because $p \notin W - U$ by Corollary 4.7.

4.2. A construction for the suitable elements of $G(k+1)$. Consider a suitable $F \in G(k+1)$. If $h_s^{-1}(N_F(n-s)) = \emptyset$ then dim $F(p) = k + 1$ everywhere and rank $M \geq k + 1$, contradiction. Therefore assume $h_s^{-1}(N_F(n-s)) \neq \emptyset$ from now on. Let $\{P_\lambda\}, \lambda \in L$, be the family of connected component of $h_s^{-1}(N_F(n-s))$. Note that $L$ is countable and non-empty, and every $P_\lambda$ a closed subset and a regular submanifold of $M$ of dimension $s$. Moreover each $P_\lambda$ is a cylinder of type let us say $r_\lambda$.

By Theorem 2.3, for every $\lambda \in L$ one may identify an open set $A_\lambda$ of $M$ including $P_\lambda$ with $D^{r_\lambda} \times C_{r_\lambda}$, in such a way that $P_\lambda = \{0\} \times C_{r_\lambda}$ and the fundamental vector fields of the action write

$$X = \sum_{j=1}^{r_\lambda} f^\lambda_j(x) \frac{\partial}{\partial \theta_j} + \sum_{\ell=1}^{s-r_\lambda} g^\lambda_\ell(x) \frac{\partial}{y_\ell}.$$

(Here one makes use of coordinates $(x, \theta, y)$ on $\mathbb{R}^{m-s} \times T^{r_\lambda} \times \mathbb{R}^{s-r_\lambda}$ instead of $(x_\lambda, \theta_\lambda, y_\lambda)$ for avoiding an over-elaborate notation.)

Remark 4.9. In the non-compact case, given two different elements $\lambda, \mu$ of $L$ it can happen that $A_\lambda \cap A_\mu \neq \emptyset$ regardless of the size of the disks $D^\lambda$ and $D^\mu$. For instance, in $\mathbb{R}^2 - \{0\}$ consider a complete vector field $Z = \varphi(x) \partial/\partial x_2$ with no zeroes and the action of $\mathbb{R}$ associated. Then the “cylinders” $\{0\} \times \mathbb{R}^+$ and $\{0\} \times \mathbb{R}^-$ cannot be separated by any couple of disks.

The next step will be to choose a suitable compact set in each $P_\lambda$ and separate them. Consider an injective map $\alpha : L \to \mathbb{Z}$ and the continuous map

$$\beta : h_s^{-1}(N_F(n-s)) = \bigcup_{\lambda \in L} P_\lambda \to \mathbb{R}$$
that equals \( \alpha (\lambda) \) on each \( P_\lambda \). As \( M \) is a normal space \( \beta \) prolongs to a continuous map \( \tilde{\beta}: M \to \mathbb{R} \). Set \( \tilde{A}_\lambda = \tilde{\beta}^{-1}((\alpha (\lambda) - 1/4, \alpha (\lambda) + 1/4)) \). Then \( \{ \tilde{A}_\lambda \}, \lambda \in L \), is a locally finite family of disjoint open sets and \( P_\lambda \subset \tilde{A}_\lambda, \lambda \in L \).

Set \( L_0 = \{ \lambda \in L: r_\lambda < s \} \). In each \( P_\lambda \) consider the compact set \( K_\lambda = \{0\} \times \mathbb{T}^{r_\lambda} \times \{0\} \) if \( \lambda \in L_0 \) and \( K_\lambda = \{0\} \times \mathbb{T}^s \) if \( \lambda \notin L_0 \). Then \( K_\lambda \subset \tilde{A}_\lambda \) and, since \( K_\lambda \) is compact, there are two balls \( B_{a\lambda}(0) \subset \mathbb{R}^{m-s} \) and \( B_{b\lambda}'(0) \subset \mathbb{R}^{s-r_\lambda} \) such that the open set \( A'_\lambda = B_{a\lambda}(0) \times \mathbb{T}^{r_\lambda} \times B_{b\lambda}'(0) \) if \( \lambda \in L_0 \), or \( A'_\lambda = B_{a\lambda}(0) \times \mathbb{T}^s \) if \( \lambda \notin L_0 \), is includes in \( \tilde{A}_\lambda \). Therefore (thought as subsets of \( M \)) \( A'_\lambda \cap A'_\mu = \emptyset \) if \( \lambda \neq \mu \).

Finally observe that, up to homotheties centered at the origin of \( \mathbb{R}^{m-s} \) and \( \mathbb{R}^{s-r_\lambda} \) respectively, one may suppose, without loss of generality, that \( A'_\lambda = B_3(0) \times \mathbb{T}^{r_\lambda} \times B_3'(0) \) if \( \lambda \in L_0 \) and that \( A'_\lambda = B_3(0) \times \mathbb{T}^s \) if \( \lambda \notin L_0 \).

Clearly, \( \{ A'_\lambda \}, \lambda \in L \), is a locally finite family

4.3. **End of the proof of Theorem 1.1** First one will construct an open subset of \( M \) which is diffeomorphic to \( M \). For every \( \lambda \in L_0 \) let \( \tilde{X}_\lambda \) be a vector field on \( \mathbb{R}^{s-r_\lambda} \) such that \( \tilde{\Phi}^1_\lambda(\mathbb{R}^{s-r_\lambda}) = B_1'(0) \) where \( \tilde{\Phi}^1_\lambda \) is the flow of \( \tilde{X}_\lambda \) (its existence is assured by Lemma 3.3). In a natural way \( \tilde{X}_\lambda \) induces a vector field \( X_\lambda \) on \( P_\lambda = \{0\} \times \mathbb{T}^{r_\lambda} \times \mathbb{R}^{s-r_\lambda} \) such that \( \tilde{\Phi}^1_\lambda(\{0\} \times \mathbb{T}^{r_\lambda}) \times \mathbb{R}^{s-r_\lambda}) = \{0\} \times \mathbb{T}^{r_\lambda} \times B_1'(0) \) where \( \tilde{\Phi}^1_\lambda \) is the flow of \( X_\lambda \).

Let \( X \) be the vector field on \( \bigcup_{\lambda \in L_0} P_\lambda \) that equals \( X_\lambda \) on each \( P_\lambda, \lambda \in L_0 \). Observe that \( \bigcup_{\lambda \in L_0} P_\lambda \) is a closed subset and a regular submanifold of \( M \); therefore from Corollary 3.2 applied to \( X \) follows that

\[
M_1 = \left( M - \bigcup_{\lambda \in L_0} P_\lambda \right) \bigcup \left( \bigcup_{\lambda \in L_0} (\{0\} \times \mathbb{T}^{r_\lambda} \times B_1'(0)) \right),
\]

where each \( \{0\} \times \mathbb{T}^{r_\lambda} \times B_1'(0) \) has to be regarded as a subset of \( P_\lambda \), is diffeomorphic to \( M \).

Now \( A'_\lambda \cap M_1 = B_3(0) \times \mathbb{T}^{r_\lambda} \times B_3'(0) - \{0\} \times \mathbb{T}^{r_\lambda} \times (B_3'(0) - B_1'(0)) \) if \( \lambda \in L_0 \) and \( A'_\lambda \cap M_1 = B_3(0) \times \mathbb{T}^s \) if \( \lambda \notin L_0 \).

Moreover the set of singular points of \( F \) belonging to \( A'_\lambda \cap M_1 \) equals \( \{0\} \times \mathbb{T}^{r_\lambda} \times B_1'(0) \) when \( \lambda \in L_0 \) and \( \{0\} \times \mathbb{T}^s \) if \( \lambda \notin L_0 \).

By Lemma 3.2 there exist \( \varepsilon > 0 \) and a diffeomorphism between

\[
E_\lambda = B_3(0) \times B_3'(0) - \{0\} \times (B_3'(0) - B_1'(0))
\]

and \( E'_\lambda = E_\lambda - B_1'(0) \times S_1' \), \( \lambda \in L_0 \), which equals the identity if \( \|x\| \geq 3 - \varepsilon \) or if \( \|y\| \geq 3 - \varepsilon \).
Therefore there exists a diffeomorphism between
\[ \tilde{E}_\lambda = B_3(0) \times T^r \times B'_2(0) - \{0\} \times T^r \times (B'_3(0) - B'_1(0)) \]
and \( \tilde{E}'_\lambda = \tilde{E}_\lambda - B_1(0) \times T^r \times S'_1 \), which equals the identity if \( \|x\| \geq 3 - \varepsilon \) or if \( \|y\| \geq 3 - \varepsilon \). Set
\[
M_2 = \left( M_1 - \bigcup_{\lambda \in L_0} \tilde{E}_\lambda \right) \bigcup \left( \bigcup_{\lambda \in L_0} \tilde{E}'_\lambda \right)
\]
Then \( M_2 \) is an open set of \( M \) that is diffeomorphic to \( M_1 \), so to \( M \). Indeed, the diffeomorphism above between \( \tilde{E}_\lambda \) and \( \tilde{E}'_\lambda \) for every \( \lambda \in L_0 \) extends by the identity on
\[
M_1 - \bigcup_{\lambda \in L_0} \tilde{E}_\lambda = M_2 - \bigcup_{\lambda \in L_0} \tilde{E}'_\lambda
\]
to a diffeomorphism between \( M_1 \) and \( M_2 \).

Observe that now
\[
A'_\lambda \cap M_2 = B_3(0) \times T^r \times B'_2(0) - \left( \{0\} \times T^r \times (B'_3(0) - B'_1(0)) \right) \cup \left( B_1(0) \times T^r \times S'_1 \right)
\]
if \( \lambda \in L_0 \) and \( A'_\lambda \cap M_2 = B_3(0) \times T^s \) if \( \lambda \notin L_0 \).

Besides, as before, the set of singular points of \( F \) belonging to \( A'_\lambda \cap M_2 \) equals \( \{0\} \times T^r \times B'_1(0) \) when \( \lambda \in L_0 \) and \( \{0\} \times T^s \) otherwise.

Observe that \( \overline{B}_{1/2}(0) \times T^r \times B'_1(0), \lambda \in L_0 \), and \( \overline{B}_{1/2}(0) \times T^s, \lambda \notin L_0 \), are closed subsets of \( M_2 \).

The next step will be to modify the (infinitesimal) action of \( F \) in order to eliminate its singularities in \( M_2 \). Thus for every \( \lambda \in L \) we need to modify the action on \( A'_\lambda \cap M_2 \).

Note that these modifications are compatible among them if their supports are contained in \( B_{1/2}(0) \times T^r \times B'_1(0), \lambda \in L_0 \), or in \( B_{1/2}(0) \times T^s, \lambda \notin L_0 \).

First assume that \( \lambda \in L_0 \). Let \( Y_1, \ldots, Y_s \) be the vector fields on \( A'_\lambda \cap M_2 \) defined by
\[
Y_1 = \partial/\partial y_1, \ldots, Y_{r_\lambda} = \partial/\partial y_{r_\lambda}, Y_{r_\lambda+1} = \partial/\partial y_{r_\lambda+1}, \ldots, Y_s = \partial/\partial y_{s-r_\lambda}.
\]
Then on \( A'_\lambda \cap M_2 \) the fundamental vector fields of the action write \( \sum_{j=1}^s h_j(x)Y_j \).

Let \( \{e_1, \ldots, e_{k+1}\} \) be a basis of \( F \) such that \( e_{k+1} \) is a basis of \( F \cap h_s(\{0\} \times T^r \times B'_1(0)) \).
(Recall that \( h_s \) is constant on \( \{0\} \times T^r \times B'_1(0) \).) Then there exist vectors \( u_1, \ldots, u_{s-k} \in V \) such that the vector subspace spanned by \( \{e_1, \ldots, e_k, u_1, \ldots, u_{s-k}\} \) is supplementary to \( h_s(\{0\} \times T^r \times B'_1(0)) \). Therefore the vector fields \( Z_1 = X_{e_1}, \ldots, Z_k = X_{e_k}, Z_{k+1} = X_{u_1}, \ldots, Z_s = X_{u_{s-k}} \) are linearly independent at every point of \( \{0\} \times T^r \times B'_1(0) \).
On the other hand, $Z_\ell = \sum_{j=1}^{s} p_{\ell j}(x) Y_j$, $\ell = 1, \ldots, s$, where the matrix $(p_{\ell j}(0))$, $\ell, j = 1, \ldots, s$, is invertible. For each $x \in B_{1/2}(0)$ set $P(x) = (p_{\ell j}(x))$, $\ell, j = 1, \ldots, s$.

As $P(0)$ is invertible, there are $\epsilon > 0$ small enough and a family of $s \times s$ matrices $Q(x) = (q_{\ell j}(x))$, $x \in B_{1/2}(0)$, depending differentiably on $x$ such that:

1. Every $Q(x)$, $x \in B_{1/2}(0)$, is invertible.
2. $Q(x) = P(x)^{-1}$ if $\|x\| \leq \epsilon$.
3. $Q(x) = Id$ if $\|x\| \geq 2\epsilon$.

On $B_{1/2}(0) \times \mathbb{T}^r \times B'_1(0)$ one modifies the infinitesimal action of $V$ by replacing each

$$X_v = \sum_{j=1}^{s} h_j^v Y_j,$$

$v \in V$, by

$$X_v^Q = \sum_{j=1}^{s} \left( \sum_{\ell=1}^{s} h_{\ell j}^v q_{\ell j} \right) Y_j.$$

Obviously “being Abelian” is not lost because $Q$ only depends on $x$.

Since $Q = Id$ on $B_{1/2}(0) - B_{2\epsilon}(0)$, the modified action extends to the whole $M_2$. Moreover the isotropy of the modified action equals that of the original one.

Observe that on $B_\epsilon(0) \times \mathbb{T}^r \times B'_1(0)$ with the modified action the vector field corresponding to $e_j$, $j = 1, \ldots, k$, is $Y_j$. Thus one may suppose, without loss of generality, that on $B_\epsilon(0) \times \mathbb{T}^r \times B'_1(0)$ the vector fields $X_{e_1}, \ldots, X_{e_k}$ do not depend on $x$, so they commute with $\partial/\partial x_1$.

Consider a function $\varphi: B_{1/2}(0) \to \mathbb{R}$ such that $\varphi(B_{1/2}(0) - B_{\epsilon/2}(0)) = 0$ and $\varphi(0) = 1$. Then on $B_{1/2}(0) \times \mathbb{T}^r \times B'_1(0)$ one modifies the action of $F$ by associating to $e_\ell$, $\ell = 1, \ldots, k$ the vector field $X_{e_\ell}$, and to $e_{k+1}$ the vector field $X_{e_{k+1}} + \varphi \partial/\partial x_1$. Clearly:

- The new action of $F$ is Abelian and has no singularities since

$$X_{e_1}, \ldots, X_{e_k}, X_{e_{k+1}} + \varphi \partial/\partial x_1$$

commute and are linearly independent at every point of $B_{1/2}(0) \times \mathbb{T}^r \times B'_1(0)$.

- It extends to the whole $M_2$.

The case $\lambda \not\in L_0$ is very similar and the details are left to readers (it suffices to delete the third factor $B'_1(0)$).

In short, we have constructed an action of $F$ on $M_2$ with no singularities. Thus rank $M_2 \geq k + 1$, hence rank $M \geq k + 1$ contradiction. Now the proof of Theorem 1.1 is finished.
5. Examples

In the next three examples one makes use of the characteristic classes for bounding from above the minimal dimension of the orbits of a $\mathbb{R}^n$-action.

**Example 5.1.** Set $M = S \times \mathbb{R}$ where $S$ is a compact connected surface of odd characteristic. Then (the Stiefel-Whitney class) $w_2(TM) \neq 0$ and the span and the rank of $S$ equal 1. Therefore by Theorem 1.1 any action of $\mathbb{R}^n$ on $M$ has an orbit of dimension $\leq 1$.

Obviously this is the best possible result since every complete and non-singular vector field gives rise to an action of $\mathbb{R}$ on $M$ all whose orbits have dimension one.

**Example 5.2.** Let $R$ be a compact, connected and orientable surface and let $\pi: M \to R$ be a complex line bundle with odd Chern class. Then as real manifold $\dim M = 4$ and $w_2(TM) \neq 0$, hence the span and the rank of $M$ are $\leq 2$.

Therefore by Theorem 1.1 any action of $\mathbb{R}^n$ on $M$ has some orbit of dimension $\leq 2$.

**Lemma 5.3.** Consider a real $\ell$-plane bundle $\pi: P \to Q$ where $Q$ is a manifold. Then there exists an action of some $\mathbb{R}^n$ on $P$ whose orbits are the fibres of $\pi: P \to Q$.

**Proof.** First observe that a section $g: Q \to P$ gives rise to a vector field $X_g$ on $P$ tangent to the fibres and constant along them (with respect to the structure of vector space) by setting $X_g(g(q)) = g(q)$, $q \in Q$; moreover $X_g$ is complete.

On the other hand if $h: Q \to P$ is another section then $[X_g, X_h] = 0$.

Finally consider a family of sections $\{g_1, \ldots, g_n\}$ such that $\{g_1(q), \ldots, g_n(q)\}$ spans $\pi^{-1}(q)$, as real vector space, for every $q \in Q$. If $\{e_1, \ldots, e_n\}$ is a basis of the Lie algebra $V$ of $\mathbb{R}^n$, then there exists an action of $\mathbb{R}^n$ on $P$ such that the fundamental vector field associated to $e_j$ is $X_{g_j}$, $j = 1, \ldots, n$. The orbits of this $\mathbb{R}^n$ actions are the fibres of $\pi: P \to Q$. \qed

Now Lemma 5.3 applied to $\pi: M \to R$ shows the existence of actions of $\mathbb{R}^n$ on $M$ whose orbits have dimension two since they are the fibres of this fibre bundle.

Recall that $\mathbb{C}P^r$ minus one point is (diffeomorphic to) the total space of the canonical complex line bundle over $\mathbb{C}P^{r-1}$. On the other hand $\mathbb{C}^\ast$ will be regarded as an open set of $\mathbb{C}P^r$ in the usual way.

Let $M_\ell$, $\ell \geq 1$, be $\mathbb{C}P^2$ minus $\ell$ different points, and let $J$ be the canonical complex structure on $\mathbb{C}P^r$. On $\mathbb{C}P^2$ consider the projective vector field $X$ which on $\mathbb{C}^2$ is written as
$X = \partial/\partial z_1 + z_1 \partial/\partial z_2$. Then $X$ only has one singular point, let us say $p$ (corresponding to the line $z_1 = 0$ of $\mathbb{C}P^2 - \mathbb{C}^2 \cong \mathbb{C}P^1$).

Moreover from the real viewpoint $X, JX$ commute and are linearly independent everywhere but $p$. Since $M_1$ can be identify to $\mathbb{C}P^2 - \{p\}$ it follows that rank $M_1 \geq 2$ and finally, as $M_1$ fibres over $\mathbb{C}P^1$ with odd Chern class, that rank $M_1 = 2$.

For constructing an action of $\mathbb{R}^n$ on $M_1$ it suffices to consider $\{X, JX, Y, JY\}$, where $Y$ is the projective vector field that on $\mathbb{C}^2$ writes $Y = \partial/\partial z_2$, as a basis of the Lie algebra of fundamental vector fields. This action has one orbit of (real) dimension four ($\mathbb{C}^2$) and one orbit of dimension two ($\mathbb{C}P^1 - \{p\}$).

A similar construction can be done for $M_2$ and $M_3$ by considering the projective vector fields $X', Y', X'', Y''$ respectively, which on $\mathbb{C}^2$ are written as $X' = \partial/\partial z_1 + z_2 \partial/\partial z_2$, $Y' = z_2 \partial/\partial z_2$, $X'' = z_1 \partial/\partial z_1 - z_2 \partial/\partial z_2$ and $Y'' = z_1 \partial/\partial z_1 + z_2 \partial/\partial z_2$.

**Remark 5.4.**

- In fact one has showed that $M_1, M_2$ and $M_3$ have file 2.
- Any holomorphic action of $\mathbb{R}^n$ on a complex compact manifold of real dimension 4 and non-vanishing characteristic always has fixed points (see Corollary 1.8 of [4]; for the analytic actions of $\mathbb{R}^2$ on dimension 4 see [1]).

From the next lemma follows that every manifold $M_{\ell}, \ell \geq 1$, has rank 2.

**Lemma 5.5.** Consider a connected open manifold $P$ of dimension $\geq 2$ and a point $p \in P$. Then $P$ and $P - \{p\}$ have the same rank and the same span.

**Proof.** Consider a non-singular function $f : P \to \mathbb{R}$. Let $Y$ be the gradient vector field of $f$ with respect to a Riemannian metric such that $Y$ is complete. Let $\gamma : \mathbb{R} \to P$ be any integral curve of $Y$; then $\gamma$ is injective and $\gamma(\mathbb{R})$ is a closed subset and a regular submanifold of $P$.

Set $C = \gamma(\mathbb{R} - (-1, 1))$; then $P$ and $P - C$ are diffeomorphic (apply Corollary 3.2 and Lemma 3.3).

Now set $p = \gamma(2)$ and let $\ell$ be either the rank or the span depending on cases. One has:

$$\ell(P) = \ell(P - C) \geq \ell(P - \{p\}) \geq \ell(P).$$

\[\square\]
Observe that the file of $P$ and that of $P - \{p\}$ may be quite different, for instance $\text{file}(\mathbb{R}^3) = 3$ but $\text{file}(\mathbb{R}^3 - \{0\}) = 1$ (see [11]).

**Example 5.6.** In Examples 5.1 and 5.2 the main tool for bounding the minimal dimension of the orbits was the Stiefel-Whitney classes. Now we will consider the Pontrjagin classes (sometimes all the Stiefel-Whitney classes but $w_0$ vanish, for instance $T\mathbb{C}P^7$).

Let $M$ be a connected manifold of dimension $m$ and rank $k$. Consider vector fields $X_1, \ldots, X_k$ commuting among them and linearly independent everywhere. Denote by $\mathcal{F}$ the foliation defined by $X_1, \ldots, X_k$. As $T\mathcal{F}$ is parallelizable, $TM$ and the normal bundle to $T\mathcal{F}$ have the same characteristic classes.

Let $\alpha$ be a non-zero element, of degree $4\ell \geq 4$, of the ring of Pontrjagin classes of $TM$. By Bott’s theorem on characteristic classes of foliations $4\ell \leq 2m - 2k$, hence $k \leq m - 2\ell$.

Therefore by Theorem 1.1 one has:

**Proposition 5.7.** Under the hypotheses above any action of $\mathbb{R}^n$ on $M$ possesses an orbit of dimension $\leq m - \ell - 1$.

A particular case of Proposition 5.7 is as follows. Let $M$ be the total space of a complex line bundle over $T^{4r}$, $r \geq 1$, with Chern class $c_1$ such that $c_1^{2r} \neq 0$. Then as real 2-plane bundle $p_1^r \neq 0$ where $p_1$ is its first Pontrjagin class. Since the tangent bundle of $T^{4r}$ is parallelizable, it is easily seen that $p_1(TM)^r \neq 0$. Now $m = 4r + 2$ and $\ell = r$; therefore by Proposition 5.7 any action of $\mathbb{R}^n$ on $M$ has an orbit of dimension $\leq 3r + 1$.

Observe that the span of $M$ equals $4r$ (while the rank $\leq 2r + 2$). Indeed, by means of a connection the vector fields on $T^{4r}$: $\partial/\partial \theta_1, \ldots, \partial/\partial \theta_{4r}$ can be lifted to $M$, so span $M \geq 4r$. As $M$ is orientable if span $M \geq 4r + 1$ then span $M = 4r + 2$ and $M$ is parallelizable, which implies $p_1(TM) = 0$, contradiction.

Moreover if $c_1$ is even, i.e. if there is $\beta \in H^2(T^{4r}, \mathbb{Z})$ such that $c_1 = 2\beta$, then $w_j(TM) = 0$, $j = 1, \ldots, 4r + 2$.

Another case is the following one. Let $M'$ be $\mathbb{C}P^7$ minus a point. Then $w_j(TM') = 0$, $j \geq 1$, but $p_1(TM')^3 \neq 0$; therefore rank$(M') \leq 8$ and any action of $\mathbb{R}^n$ on $M'$ possesses an orbit of dimension $\leq 10$. By Lemma 5.5 and Theorem 1.1 the same result holds for $\mathbb{C}P^7$ minus a finite number (\geq 1) of points.
6. $\mathbb{R}^n$-actions whose orbits have codimension $\leq 1$

For the sake of completeness, one will prove the following result:

**Proposition 6.1.** Consider an action of $\mathbb{R}^n$ on a connected manifold $M$ of dimension $m \geq 3$. Assume that the codimension of every orbit is $\leq 1$. Then the universal covering of $M$ is diffeomorphic to $\mathbb{R}^m$.

*Proof.* It is enough to prove the result when $M$ is simply connected.

First one will construct a codimension one foliation on $M$ with no vanishing cycles. If $n \leq m$ set $C = \emptyset$; otherwise let $C$ be the set of those $F \in G(m - 1)$ such that $h_m : \Sigma_m \to G(n - m)$ is not transverse to $N_F(n - m)$ [recall that $\Sigma_m$ is an open set of $M$]. By (i) of Lemma 2.1 $C$ is of the first category in $G(m - 1)$.

Observe that if $F$ belongs to $G(m - 1) - C$ then $\dim F(p) = m - 1$ for every $p \in \Sigma_m$. Indeed, if $n \leq m$ it is obvious and if $n > m$, since rank $h_m = 0$ and the codimension of each stratum of $N_F(n - m)$ is $\geq 2$, $h_m$ is transverse to $N_F(n - m)$ if and only if $h_m(\Sigma_m) \cap N_F(n - m) = \emptyset$.

Now consider a $\Sigma_{m-1}$-adapted chart $(U, x_1, \ldots, x_m)$ and an extension $\tilde{h}_{m-1} : U \to G(n-m+1)$ of $h_{m-1}$ like in the proof of Lemma 4.1. Given $a \in U$ denoted by $S_a$ and $T_a$ the slice and the transversal in $U$ defined by $x_m = a_m$ and by $x_1 = a_1, \ldots, x_{m-1} = a_{m-1}$ respectively. Observe that the set $U \cap \partial \Sigma_{m-1}$, where $\partial \Sigma_{m-1} = \Sigma_{m-1} - \circ \Sigma_{m-1}$, is saturated for the slices of $U$ and of the first category in $U$. Therefore given a transversal $T_a$ the set $T_a \cap \partial \Sigma_{m-1}$ is of the first category in $T_a$.

Since $\dim T_a = 1$, by (ii) of Lemma 2.1 applied to $\tilde{h}_{m-1} : T_a \to G(n-m+1)$ the set $D$ of those $F \in G(m - 1)$ for which there exists $p \in T_a \cap \partial \Sigma_{m-1}$ such that $\dim(h_{m-1}(p) \cap F) \geq 1$ is of the first category in $G(m - 1)$.

Note that if $F \notin D$ then $\dim F(p) = m - 1$ for every $p \in T_a \cap \partial \Sigma_{m-1}$. But $h_{m-1}$ and $\tilde{h}_{m-1}$ are constant along the slices of $U$, so if $F \notin D$ then $\dim F(q) = m - 1$ for every $q \in U \cap \partial \Sigma_{m-1}$.

As $\Sigma_{m-1}$ can be covered by a countable family of $\Sigma_{m-1}$-adapted charts, it follows the existence of a set $D' \subset G(m - 1)$ of the first category such that $\dim F(p) = m - 1$ for any $F \in (G(m - 1) - D')$ and any $p \in \partial \Sigma_{m-1}$.

Consider $F \in (G(m - 1) - C \cup D')$; then the singular set $S(F)$ is included in $\circ \Sigma_{m-1}$. Let $\mathcal{F}$ be the codimension one foliation on $M$ defined:

- On $M - S(F)$ by $F$, that is to say $T_p \mathcal{F} = F(p)$ for every $p \in M - S(F)$. 


• On $\Sigma_{m-1}$ by the orbits of the action of $\mathbb{R}^n$

It is easily checked that the definition of $\mathcal{F}$ is coherent.

As $F$ is a subalgebra of the Lie algebra $V$ of $\mathbb{R}^n$, the foliation $\mathcal{F}$ is also defined on $M - S(F)$ by an action locally free of $\mathbb{R}^{m-1}$. Therefore $\mathcal{F}$ does not have vanishing cycles on $M - S(F)$ (see Chapter 3 of [3]).

Now consider any point $q \in S(F)$. Then there exists $F' \in G(m-1)$ such that $F'(q) = T_q F$. Thus on $\Sigma_{m-1} - S(F')$, where $S(F')$ is the singular set of $F'$, the foliation $\mathcal{F}$ is defined by a locally free of $\mathbb{R}^{m-1}$ and does not have vanishing cycles.

In short, on $M$ the foliation $\mathcal{F}$ does not possesses vanishing cycles. Thus if $L$ is any leaf of $\mathcal{F}$ its fundamental group injects in $\pi_1(M) = 0$, hence $L$ is simply connected. But by the construction of $\mathcal{F}$ its leaves are cylinders, so $L = \mathbb{R}^{m-1}$. In other words, $\mathcal{F}$ is a plane foliation of $M$ and by Corollary 3, page 110, of [10] $M$ is diffeomorphic to $\mathbb{R}^m$. \hfill \Box

**Example 6.2.** Let $\mathbb{R}_e^4$ be any exotic $\mathbb{R}^4$. Then every action of $\mathbb{R}^n$ on $\mathbb{R}_e^4$ has an orbit of dimension $\leq 2$.

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