A Thermodynamic Interpretation of Time for Rolling Tachyons

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Abstract

We show that the open string worldsheet description of brane decay (discussing a specific example of a rolling tachyon background) can be related to a sequence of points of thermodynamic equilibrium of a grand canonical ensemble of point charges on a circle, the Dyson gas. Subsequent instants of time are related to neighboring values of the chemical potential or the average particle number $\bar{N}$. The free energy of the system decreases in the direction of larger $\bar{N}$ or later times, thus defining a thermodynamic arrow of time. Time evolution equations are mapped to differential equations relating thermal expectation values of certain observables at different points of thermal equilibrium. This suggests some lessons concerning emergence of time from an underlying microscopic structure in which the concept of time is absent.
1 Introduction

There are a number of examples in string theory in which several dimensions of space emerge as an effective description of a system that does not contain them [1–3]. All these examples arise at their root from the duality between open and closed strings, by equivalently describing the dynamics of a system of D-branes in terms of the open string degrees of freedom that quantize them, or in terms of the closed strings that quantize the spacetime that the D-branes create. There are no similar examples of the emergence of time. A candidate setting for investigating such an emergence is provided by the unstable branes of bosonic and superstring theory [7]. In this case, open strings are localized in time [8], and one might ask how the open strings contrive to describe closed strings and spacetime at times far from the region of time where they are confined.

Previous work showed that the worldsheet correlation functions of strings in such universes can be computed at tree level from an ensemble of $U(N)$ matrices [9–14]. As the brane decays to a gas of closed strings, the rank of the matrices contributing to the ensemble increases to infinity. In examples of the emergence of space [1–3], closed

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1For recent discussions of emergence of spacetime and string theory, see [4, 5]. For a recent discussion from a condensed matter physics perspective, see [6].
string backgrounds emerge from the dynamics of $U(N)$ matrices in the large $N$ limit. This suggests that for decaying branes the geometry at late times is “emerging” from the dynamics of the large rank matrices appearing in the open string description.

Here we take a related perspective, by showing that all worldsheet correlation functions in decaying brane universes can also be computed at tree level in string theory from a simple statistical mechanical model – test charges interacting with a Dyson gas, consisting of point charges confined to a circle. Physical time maps into (an analytic continuation of) the fugacity in the gas. Thus, by a Legendre transform, time becomes related to the mean number of particles ($\bar{N}$) in the statistical system. (In turn, this is related to the rank of the matrices appearing in the matrix model description of the system [9, 12].) The free energy of the Dyson gas decreases with $\bar{N}$, giving rise to a thermodynamic arrow for time.

In classical physics, we are used to specifying the state of a system by specifying its initial conditions and then allowing the equation of motion to evolve the configuration through time. Since the Dyson gas arises here from a rewriting of worldsheet perturbation theory, the spacetime equations of motion can be derived by requiring conformal invariance (perhaps implying criticality) of the statistical system. The time evolution equations of spacetime fields then translate into equations relating the values of certain statistical observables at different points of thermodynamic equilibrium.

As a rewriting of worldsheet perturbation theory in a different, statistical language, our analysis does not describe a holographic duality in the sense of the AdS/CFT correspondence. The latter is a relationship between the spacetime theory of open strings (i.e. a quantum field theory) and a spacetime theory of closed strings. In principle, we are simply describing a dictionary between two equivalent worldsheet formulations. Nevertheless, the appearance of a thermodynamic arrow of time from our statistical formulation of the decaying brane is suggestive of some kind of emergent phenomenon. The discussion in Sec. 4, attempts to draw lessons about what it could mean for time to be “emergent”.

2 The half S-brane as a Dyson Gas

2.1 The half S-brane as an ensemble of matrix models

Bosonic string theory has a spectrum of unstable D-branes that decay by tachyon condensation. Sen [7] showed that their homogeneous decay can be described in terms of the open string theory on a D-brane by the exactly marginal boundary deformation

$$\delta S_{\text{bdry}} = \lambda \int dt \, e^{\alpha(t)} .$$  \hfill \(1\)
The parameter $\lambda$ has no meaning since it can be absorbed by redefining the origin of the target space time coordinate $X^0$. The resulting background is called the half S-brane since it represents a spacelike D-brane at $t \to -\infty$ that decays into closed strings as time passes.

Worldsheet correlation functions in the background of the decaying brane take the form

$$\bar{A}_l = \int DX^0 DX^1 \cdots DX^{25} e^{-S} \prod_{a=1}^l V_a(z_a, \bar{z}_a)$$

(2)

where the action $S$ includes the boundary deformation (1) and the $V_a$ are vertex operators constructed from $\{X^0, \bar{X}\}$ and written as functions of holomorphic coordinates on the worldsheet. String scattering amplitudes are obtained by integrating these correlation functions over all vertex operator positions (after fixing the location of one operator by using the conformal symmetry of the worldsheet).

Taking $V_a$ to be tachyon vertex operators and leaving out the trivial spatial part of the computation, the $X^0$ dependent piece of the correlator is

$$\bar{A}_l \sim \int DX^0 e^{-S} \prod_{a=1}^l e^{ik_0^a X^0(z_a, \bar{z}_a)}.$$  

(3)

At non-zero momentum it is possible to choose a gauge in which general on-shell closed string vertex operators with finite energy have the form

$$V = e^{ik_0^0 X^0} V_{sp},$$

(4)

where $V_{sp}$ is constructed entirely out of the 25 spatial fields [15]. In this gauge, and given that the boundary perturbation only depends on $X^0$, the non-trivial part of finite momentum string scattering amplitudes can be written as (3).

The worldsheet correlation function (3) can be evaluated on the disk by isolating the zero mode $x^0$ from the fluctuations as $X^0 = x^0 + X^0$, and expanding the boundary perturbation $e^{-\delta S_{bdry}}$ in a power series. The result is

$$\bar{A}_l = \int dx^0 e^{ix_0 \sum_a k_0^a} \sum_{N=0}^{\infty} \frac{(-2\pi \lambda e^{x^0})^N}{N!} \int \prod_{i=1}^N \frac{dt_i}{2\pi} (e^{X^0(t_1)} \cdots e^{X^0(t_N)} \prod_{a=1}^l e^{i\omega_a X^0(z_a, \bar{z}_a)}).$$

(5)

Now using the standard Neumann correlator of $X^0$ (see, e.g., Sec. 2.2 and 3.1 of [12]) the disk expectation value in (5) can be evaluated as

$$\bar{A}_l = \int dx^0 e^{ix_0 \sum_a \omega_a} A_l(x^0)$$

(6)

$$A_l(x^0) = \prod_{a<b} |z_a - z_b|^{-\frac{k_0^a k_0^b}{k_0^b}} \prod_{a\neq b} \prod_{1-1} |1 - z_a z_b|^{-\frac{k_0^a k_0^b}{k_0^b}} F(z_1, k_1^0; \cdots; z_l, k_l^0; x^0),$$

(7)

where the initial factors involving $z_a$ arise from contractions between the vertex operators, the integral over $x^0$ is the Fourier transformation to total energy, and $F$ isolates
all the interesting pieces in the correlator arising from the interaction between the vertex operator and the boundary perturbation (1). $A_i(x^0)$ is the amplitude written as a function of target space time. We find

$$F(\{z_a, k^0_a\}; x^0) = \sum_{N=0}^{\infty} \frac{(-2\pi \lambda e^{x^0})^N}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{i<j} [e^{it_i} - e^{it_j}]^2 \prod_{ia} |1 - z_a e^{-it_i}|^{2ik^0_a}$$

$$= \sum_{N=0}^{\infty} \frac{(-2\pi \lambda e^{x^0})^N}{N!} \int_{U(N)} dU \prod_a |\det(1 - z_a U)|^{2ik^0_a}. \quad (8)$$

The second equality arises by recognizing the measure for integration over $U(N)$.

In this way, worldsheet correlation functions for a decaying brane are expressed in terms of an ensemble of $U(N)$ matrices with varying rank $N$. As a simple example, the disk partition function is

$$Z_{\text{open}} = \langle e^{-\delta S_{\text{bdry}}} \rangle = \sum_{N=0}^{\infty} \frac{(-2\pi \lambda e^{x^0})^N}{N!} \int_{U(N)} dU$$

$$= \frac{1}{1 + 2\pi \lambda e^{x^0}}. \quad (9)$$

At early times $x^0 \to -\infty$ the first few terms in the series (9) dominate and hence only low rank $U(N)$ matrices contribute. However, as $x^0$ increases, and brane decay progresses further, matrices of larger rank become progressively more important. The series converges when $|2\pi \lambda e^{x^0}| < 1$. However, the summed expression (10) can be analytically continued to future infinity $x^0 \to \infty$. The convergence radius $x^0 \sim -\ln(2\pi \lambda)$ might be interpreted as the moment after which the tachyon has completely condensed, so that the open string worldsheet is no longer really meaningful. After this point, a closed string description is better. Indeed, the function

$$f(x^0) = \frac{1}{1 + 2\pi \lambda e^{x^0}}$$

is the coefficient of the lowest, closed string tachyon component of the boundary state of the decaying brane [7, 9].

As another example of an interesting target space observable, consider the energy-momentum tensor as a function of target space time $x^0$ [9,18]. It can be decomposed as

$$T^{MN}(x^0) = K[\eta^{MN} \mathcal{B}(x^0) + \mathcal{A}^{MN}(x^0)], \quad (12)$$

where $\mathcal{B}(x^0) = f(x^0)$ is the open string disk partition function, whereas

$$\mathcal{A}^{MN}(x^0) = 2\langle : \partial X^M(0) \bar{\partial} X^N(0) : \rangle_{\frac{1}{2} S-\text{brane}} \quad (13)$$

\footnote{For additional discussion and early references, see [11,16,17].}
is the disk one-point function of the closed string vertex operator

\[ V^{MN}(z, \bar{z}; k) =: \partial X^M(z) \bar{\partial} X^N(\bar{z}) e^{ik \cdot X(z, \bar{z})} : \]  

at zero momentum \( k = 0 \), placed at the origin \((z, \bar{z}) = 0\). (Since this is at zero momentum we cannot write this in the gauge \( \Box \).) The perturbative expansion is again expressed in terms of an ensemble of \( U(N) \) matrices with a finite convergence radius as before [9,12]. The series summation gives, for example, the \( A^{00} \) component

\[ A^{00}(x^0) = f(x^0) - 2 , \]  

which can then be analytically continued to late times.

In general, all closed string observables can be computed using the open string worldsheet theory, using similar perturbative expansions in terms of an ensemble of matrix models with a finite convergence radius, and the sum can be analytically continued to late times. For example, one-point functions of closed string on-shell vertex operators correspond to probabilities to produce closed string states in the decay [19]. The functions obtained can also be interpreted as coefficients of the associated component in the decaying brane boundary state [8,20]. The resulting closed string description is then valid at all times, even if the open string formalism is strictly valid only within the convergence radius, at suitably early times.

### 2.2 The half S-brane as a Dyson Gas: partition function

We can now show that all the S-brane disk amplitudes can also be expressed in terms of the correlation functions in a simple statistical mechanical system – the classical Coulomb gas of infinitely heavy unit charges on a unit circle, also called the Dyson gas. In particular, the S-brane disk partition function is related to the grand canonical partition function of the Dyson gas (see Appendix D of [12] for a brief discussion).

To see this, recall first that a Dyson gas [21] consists of heavy unit charges confined to live on a unit circle in a two-dimensional plane. Pairs of charges, having positions \( e^{it_i} \), interact via the two-dimensional Coulomb potential which is logarithmic. The kinetic term is zero since the particles are taken to be infinitely heavy and they do not move. The repulsive interactions give a two-body potential

\[ V(t_i, t_j) = - \ln |e^{it_i} - e^{it_j}| . \]  

The grand canonical partition function of the Dyson gas is

\[ Z_G = \sum_{N=0}^{\infty} \frac{z^N}{N!} Z_N(\beta) \]

\[ = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_0^{2\pi} \prod_{i=1}^{N} \frac{dt_i}{2\pi} \exp(-\beta \sum_{k<l} V(t_k, t_l)), \]
where $z$ is the fugacity and $Z_N(\beta)$ is the canonical partition function for $N$ particles as a function of the inverse temperature $\beta = 1/T$. Putting in the logarithmic potential (16), we see that at the special temperature $\beta^{-1} = 1/2$,

$$Z_G = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int \frac{dU}{U(N)} \frac{1}{1-z},$$

where we used the identification of the integral over $U(N)$ as in (8) and (9). Thus we see that for $\beta = 2$, the interactions between the Dyson gas particles reproduce the integrations over $U(N)$ in (8).

Comparing (10) and (20) it is tempting to identify $z = -2\pi \lambda e^{x_0}$ and $Z_G = Z_{\text{open}}$. However, this is inconvenient because fugacity is the exponential of a chemical potential, and hence this identification would require an imaginary component for the chemical potential in our Dyson gas. Rather, $Z_G$ is more naturally identified with another quantity in the decaying brane worldsheet theory, namely the expectation value of an operator $(-1)^{\hat{N}_T}$, where $\hat{N}_T$ counts the number of tachyon vertex operators from the rolling tachyon background:

$$Z'_{\text{open}} = \langle (-1)^{\hat{N}_T} \rangle_{\frac{1}{2}S-brane} = \sum_{N=0}^{\infty} (-1)^N (-2\pi \lambda e^{x_0})^N$$

$$= \frac{1}{1 - 2\pi \lambda e^{x_0}}.$$  

Identifying $Z_G \leftrightarrow Z'_{\text{open}}$ gives

$$z = e^{\beta \mu} = 2\pi \lambda e^{x_0},$$

so the chemical potential $\mu$ essentially corresponds to target space time $x_0$. Thus, considering the decaying brane at different times corresponds to considering the Dyson gas at different points of thermodynamic equilibrium, corresponding to different values of the chemical potential in the grand canonical ensemble.

The interesting physical quantities in the target space of the decaying brane are computed as expectation values of closed string vertex operators with respect to the disk partition function $Z_{\text{open}}$ rather than $Z'_{\text{open}}$. In order to obtain these observables from the Dyson gas formulation, we will need to compute the grand canonical partition function $Z_G$ as a function of a conventional fugacity and then analytically continue $z \rightarrow ze^{i\delta} \rightarrow ze^{i\pi} = -z$

$$z \rightarrow z e^{i\delta} \rightarrow ze^{i\pi} = -z$$  

(23)

to negative values of $z$, while continuing to identify time $x_0$ via the absolute value of $z$ as in (22). Recall that because the late time physics ($x_0 \rightarrow \infty$) lies beyond
the convergence radii of the series in (9) and (20), an analytic continuation in the complex fugacity plane was in any case required. The continuation (23) relating the string partition function $Z_{\text{open}}$ to the Dyson gas partition function $Z_G$ is simply an extension of this procedure.

### 2.3 The half S-brane as a Dyson Gas: correlation functions

Above we showed how the open string disk partition function is obtained from the thermodynamics of the Dyson gas. A general closed string correlation function is obtained by inserting vertex operators in the bulk of the disk. For homogenous brane decay, as described by (1), the spatial fields $X^i$, $i = 1, \ldots, 25$ in the vertex operator will simply contract amongst themselves and give a standard result on the disk. Thus, as in (3), we will leave out this trivial spatial part of the computation since we are interested in the dependence of correlators on target space time. Then, the relevant part of the general vertex operator can be written as sums of products of terms of the general form

$$: \partial^{n_1} X^0(z) \partial^{n_2} X^0(\bar{z}) e^{i k_0 X^0} : .$$

The correlation functions of such operators can be computed in the Dyson gas approach by inserting external charges at the locations of the vertex operators and computing the expectation values of suitable moments of the potential.

To illustrate, consider the stress tensor ($n_1 = n_2 = 1$ and $k^0 \to 0$). The $T^{00}$ component contains the disk partition function $Z_{\text{open}} = B(x^0)$ and the expectation value $A^{00} = 2 \langle \partial X^0 \partial X^0 \rangle$ (12). In the Dyson gas thermodynamics, the latter quantity is computed by placing an additional charge $k^0$ at $(z_0, \bar{z}_0)$ on the plane, inside the circle. The charged particles of the Dyson gas on the unit circle will produce an electrostatic potential acting on the bulk charge,

$$V^{\text{bulk} - \text{Dyson}} = -k^0 \sum_k \ln |z_0 - e^{it_k}|^2 .$$

The holomorphic and anti-holomorphic derivatives of $V^{\text{bulk} - \text{Dyson}}$ will be called holomorphic and antiholomorphic “forces”,

$$F^{\text{bulk} - \text{Dyson}} = \frac{\partial V^{\text{bulk} - \text{Dyson}}}{\partial z_0}, \quad \bar{F}^{\text{bulk} - \text{Dyson}} = \frac{\partial V^{\text{bulk} - \text{Dyson}}}{\partial \bar{z}_0} .$$

To compute $A^{00}$ in the Dyson gas, compute the grand canonical expectation value of the product of these “forces”,

$$\langle \langle F(z_0) F(\bar{z}_0) \rangle \rangle_\beta \equiv \sum_{N = 0}^{\infty} \frac{z^N}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \exp\left[ -\beta (V^{\text{Dyson}} + V^{\text{bulk} - \text{Dyson}}) \right] \cdot F^{\text{bulk} - \text{Dyson}}(z_0, \{t_i\}) \bar{F}^{\text{bulk} - \text{Dyson}}(\bar{z}_0, \{t_i\}) .$$
then move the external charge to the origin, $z_0 = \bar{z}_0 = 0$, and continue in the complex fugacity plane

$$z \rightarrow -z = -2\pi \lambda e^{\varphi},$$

and replace $k^0 \rightarrow ik^0$ to finally obtain

$$\langle \langle F(z_0) F(\bar{z}_0) \rangle \rangle_\beta \rightarrow \langle \partial X^0 \partial \bar{X}^0 : \rangle = \frac{1}{2} \cdot A^{00}(x^0).$$

This is a prescription for computing the target space stress tensor for a decaying brane in terms of expectation values in the grand canonical ensemble for a Dyson gas. Here

$$V^{\text{Dyson}} = -\sum_{k<l} \ln |e^{it_k} - e^{it_l}|$$

is the potential energy of the Dyson gas.

It is straightforward to generalize this to a comparison of more general thermal expectation values and more complicated closed and open string correlation functions. We add external charges at the locations of the vertex operators and compute the grand canonical expectation value of an appropriate moment of $V$. The potential between the external charges will produce an overall worldsheet position dependence in the amplitude (e.g., the initial factors in (7)), while the interactions with the Dyson gas particles produce the part of the amplitude that is affected by the decay of the brane (e.g., the final factor $F$ in (7)). Finally, to compute a string amplitude, the positions of all but one of the vertex operators are integrated over the disk\(^4\). This treatment can be applied to open string vertex operators too, except that here the external charges are inserted into the circle on which the Dyson gas particles are already present. The prescription is illustrated by some examples in the table below.

### 3 The flow of time

In the Dyson gas formulation, target space time has been replaced by the chemical potential $\mu$ of a statistical system. Different points of thermodynamic equilibrium characterized by different values of $\mu$ (keeping the temperature and the volume fixed) correspond to the different instants in time for the decaying brane.

#### 3.1 Time as the average particle number

It is now simple to give an interpretation of time as a measure of the average particle number in the Dyson gas. In terms of the fugacity $z$, the average particle number is:

$$\bar{N} = z \partial_z \ln Z_G = \frac{z}{1 - z}.$$  

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\(^3\)Charges must be continued to imaginary values in order to make contact with real energies.

\(^4\)We need 3 real parameters to fix the residual $SL(2, \mathbb{R})$ symmetry.
In Rolling Tachyon CFT

: $e^{i p^0 X^0(\tau)}$ :
(on the boundary)

Insert external particle with charge $p^0$ on the circle and modify the potential term by adding an interaction term for this external charged particle and Dyson gas

: $\partial^n X^0(\tau)$ :

Insert charged particle on the boundary and compute the thermal average of the $n$-th moment of the potential in grand canonical partition function of the Dyson gas

: $e^{ik^0 X^0(z,\bar{z})}$ :
(in the bulk)

Insert particle with charge $k^0$ on 2d plane inside the circle at $(z, \bar{z})$ and modify the potential by adding interaction between the bulk charge and Dyson gas particles

: $\partial^{n_1} X^0(\bar{z}) \partial^{n_2} X^0(\bar{z})$ :

Insert external charged particle inside the circle and compute thermal average of $n_1$-th holomorphic and $n_2$-th anti-holomorphic moments of the potential in the grand canonical partition function of the Dyson gas

| In Rolling Tachyon CFT | In Dyson Gas Ensemble |
|------------------------|------------------------|
| $e^{i p^0 X^0(\tau)}$ | Insert external particle with charge $p^0$ on the circle and modify the potential term by adding an interaction term for this external charged particle and Dyson gas |
| $\partial^n X^0(\tau)$ | Insert charged particle on the boundary and compute the thermal average of the $n$-th moment of the potential in grand canonical partition function of the Dyson gas |
| $e^{ik^0 X^0(z,\bar{z})}$ | Insert particle with charge $k^0$ on 2d plane inside the circle at $(z, \bar{z})$ and modify the potential by adding interaction between the bulk charge and Dyson gas particles |
| $\partial^{n_1} X^0(\bar{z}) \partial^{n_2} X^0(\bar{z})$ | Insert external charged particle inside the circle and compute thermal average of $n_1$-th holomorphic and $n_2$-th anti-holomorphic moments of the potential in the grand canonical partition function of the Dyson gas |

Table 1: Correspondence between state in CFT and thermal average in Dyson gas. Left hand side of the table is to be evaluated inside a correlator in rolling tachyon background. All the vertex operators in bosonic string theory can be obtained by taking suitable combinations of the above set.

Recall now that every positive fugacity value was in one-to-one correspondence with an instant of time $x^0$ for the decaying brane, by

$$|z| = 2\pi \lambda e^{x^0}. \quad (32)$$

So the average number of particles in the Dyson gas related to the instant $x^0$ is given by

$$\bar{N} = \frac{2\pi \lambda e^{x^0}}{1 - 2\pi \lambda e^{x^0}} \implies x^0 = \ln \left( \frac{\bar{N}}{1 + \bar{N}} \right) - \ln 2\pi \lambda. \quad (33)$$

In particular, the infinite past $x^0 = -\infty$ corresponds to vanishing fugacity and $\bar{N}$. The average particle number then increases monotonically as a function of the fugacity, corresponding to later time values $x^0$. This relation is valid up to the point $z = 1$ where $\bar{N}$ diverges. This corresponds to time value $x^0 = -\ln 2\pi \lambda$. But we have also seen that this is also the boundary of the convergence radius for typical open string worldsheet calculations, beyond which it is more appropriate to use the closed string description of the system.
The average particle number $\bar{N}$ is a continuous quantity, just like time. Interestingly, however, the underlying physical quantity in the Dyson gas is the actual number of particles $N$. This is discrete, and fluctuates around $\bar{N}$. The relative size of the fluctuations declines as $\bar{N}$ increases:

$$\delta N = \sqrt{N^2 - \bar{N}^2} = \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{2\pi N}}. \tag{34}$$

At early times (small $\bar{N}$), the relative fluctuations of $N$ are large and at later times $\bar{N}$ becomes more sharply defined. It is tempting to interpret this as “a continuous time emerging from an underlying discrete variable in the large $N$ limit”\footnote{Quantitatively, one can cut off the infinite series summations at some value $N_{\text{max}} \gg \bar{N}$ and get good approximations to all calculations. The size of the cutoff $N_{\text{max}}$ increases with time.}

### 3.2 A thermodynamic arrow of time

Consider the grand potential of the Dyson gas ensemble,

$$\Omega(\mu, T, V) = -T \ln Z_G(\mu, T, V), \tag{35}$$

where we set Boltzmann’s constant to 1. $\Omega$ satisfies the thermodynamic identity

$$\Omega = \bar{E} - TS - \mu \bar{N}, \tag{36}$$

where $\bar{E}$ is the mean energy of the system. A Legendre transformation gives the conventional Helmholtz free energy:

$$A(\bar{N}, T, V) = \Omega + \mu \bar{N} = \bar{E} - TS, \tag{37}$$

where we substitute for the chemical potential $\mu$ as a function of $T, \bar{N}$ by inverting \textcolor{red}{(31)}. $A$ measures the free energy of the system as a function of the particle number (as opposed to the chemical potential). Setting $T = 1/2$ for the Dyson gas, we find that

$$A(\bar{N}, T, V) = -\frac{1}{2} \left[(\bar{N} + 1) \ln(\bar{N} + 1) - \bar{N} \ln \bar{N}\right]. \tag{38}$$

This resembles the familiar formula for entropy as a function of the mean number of particles. Indeed, the Shannon entropy in the particle number distribution is

$$I_S = -\sum_n p(n) \ln p(n) ; \quad p(n) = \frac{z^n}{Z_G}. \tag{39}$$

where $p(n)$ is the probability that there are $n$ charged particles in the Dyson gas. Doing the sum \textcolor{red}{(39)} and using \textcolor{red}{(31)} for the mean particle number, we find

$$I_S = (\bar{N} + 1) \ln(\bar{N} + 1) - \bar{N} \ln \bar{N} \quad \Rightarrow \quad A(\bar{N}, T, V) = -T I_S. \tag{40}$$

The Shannon entropy increases with $\bar{N}$ and hence the free energy decreases with the mean particle number. Recall that the increase of $\bar{N}$ marks the passage of time. Thus the free energy decreases with the passage of time. We interpret this as a thermodynamic arrow of time for decaying brane universes.
3.3 Equations of motion: generalities

Usually, the flow of time is discussed in terms of a *time evolution* equation that determines the configuration of a system at all times, given an initial condition. Hence, it is interesting to ask how the spacetime equations of motion appear in the Dyson gas formulation of decaying branes.

In string theory the spacetime fields appear as coupling constants in the worldsheet sigma model, and their equation of motion arises by requiring vanishing of all the beta functions [22]. In the presence of an unstable brane, the worldsheet theory includes a tachyonic boundary deformation. It is more subtle to derive the beta function in the presence of such a source, there are additional corrections which can be viewed to arise from higher order string diagrams in a degenerate limit [23, 24]. The vanishing of the corresponding beta function then leads to a solution of the spacetime equations of motion containing a decaying brane. Specifically, the exactly marginal boundary deformation (1) preserves worldsheet conformal invariance and leads to a specific time dependent source term in the spacetime equations of motion. For example, the dilaton field equation in the presence of the decaying brane is

$$\nabla^2 \phi = J_{\text{dil}} = \langle V(0)e^{-\delta S_{\text{bdry}}} \rangle_{\text{disk}},$$  \hspace{1cm} (41)

where the r.h.s. is the one-point function of the dilaton vertex operator in the presence of the boundary deformation. For the homogenously rolling tachyon and a decaying $p$-brane in bosonic theory, the field equation turns out to be [25]

$$\eta^{MN} \partial_M \partial_N \phi(x^0, \vec{x}) = c \cdot \delta^{25-p}(\vec{x}) f(x^0),$$  \hspace{1cm} (42)

where $M, N = 0, \ldots, 25$, $c$ is a numerical constant and $f$ is related to the target space stress tensor. Focusing on the simplest case of a space-filling brane ($p = 25$), the equation simplifies to

$$- \partial^2_{0} \phi(x^0) = c \cdot f(x^0).$$  \hspace{1cm} (43)

Since we have essentially reformulated worldsheet perturbation theory in terms of a statistical mechanical system, it should be possible to interpret equations like (43) in terms of Dyson gas thermodynamics. Recalling that the time $x^0$ is related to the chemical potential as $x^0 \sim 2\mu$, and that $f$ is interpreted in terms of a thermodynamic average of moments of the electrostatic potential between the gas charges and an external charge at the origin, we expect an equation in the Dyson gas of the form

$$- \partial^2_{\mu} \phi(\mu) = 4c \cdot f(\mu).$$  \hspace{1cm} (44)

In other words, if we identify the quantity $\phi$ as an object in the Dyson gas, its value at all points of equilibrium (fixed by different values of $\mu$) can be determined via (44) in terms of an “initial value” $\phi(\mu_0)$ and the initial gradient $\partial_\mu \phi(\mu_0)$. 

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The identification of the spacetime fields within the Dyson gas formulation and the derivation of the equations of motion is done below and in the Appendix by translating the worldsheet beta function formulation into our statistical language. We have been unable to find a clear account in the literature of the derivation of the beta function equations for bulk fields in the presence of a source created by a tachyonic boundary deformation. While a complete calculation would clearly be an interesting basic string theory problem, the details are tangential to the discussion here. We outline the steps of the calculation in the Appendix.

### 3.4 Equations of motion: derivation

We begin by recalling how the field equations arise from requiring Weyl invariance of the decaying brane worldsheet sigma model. We start with the worldsheet action

\[ S = S_P + \delta S_{\text{bdry}} \]

\[ = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \eta^{ab} \partial_a X^M \partial_b X^N + \lambda \int ds \ e^{X^0}. \]  

(45)

Next, we compute the disk partition function, as in (10), but now the contractions between operators at the boundary will involve the curved space generalization \( d_h(s_i, s_j) \) of the flat space distance function \( d(e^{it_i}, e^{it_j}) = |e^{it_i} - e^{it_j}|. \) The correlators between bosons on the boundary become

\[ \langle X^0(s_i)X^0(s_j) \rangle = \Delta_{\text{bdry}}(s_i, s_j) = \ln d_h^2(s_i, s_j). \]  

(46)

The disk partition function then becomes

\[ Z_{\text{disk}} = Z_{sp} \times \sum_{N=0}^{\infty} \frac{(-2\pi\lambda e^{\beta 0})^N}{N!} \int \prod_i ds_i \prod_{i<j} \exp(\Delta_{\text{bdry}}(s_i, s_j)) \]  

(47)

where \( Z_{sp} \) is the trivial contribution from the 25 free spacelike bosons \( X^i \).

The corresponding statistical mechanical system (in addition to the free boson CFT) is the curved space generalization of the Dyson gas, a system where unit point charges on a (locally scaled) circle interact through a two-body Coulomb potential in curved space,

\[ V_{h}^{\text{Dyson}} = -\frac{1}{2} \sum_{i<j} \Delta_{h}^{\text{bdry}}(s_i, s_j) \]  

(48)

with the canonical partition function

\[ Z_N(\beta) = \int \prod_{i=1}^{N} \frac{ds_i}{2\pi} \exp(-\beta V_{h}^{\text{Dyson}}) \]  

(49)

The inverse temperature is again \( \beta = 2. \) In the curved case there is no obvious random matrix interpretation. As before, we must consider a grand canonical ensemble partition function to make contact with the worldsheet theory.
The next step in deriving field equations is to introduce background fields in the worldsheet sigma model, for example a general target space metric $G_{MN}(X)$ and a dilaton $\phi(X)$,

$$S[G, \Phi] = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} [h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \alpha' \tilde{R} \Phi(X)], \quad (50)$$

where $\tilde{R}$ is the worldsheet Ricci scalar. We take the target space metric to be almost flat,

$$G_{MN}(X) = \eta_{MN} + \epsilon_{MN}(X) \quad (51)$$

and choose $\epsilon_{MN}(X)$ to be a monochromatic gravitational plane wave

$$\epsilon_{MN}(X) = -4\pi g_c \epsilon_{MN} e^{ik \cdot X}, \quad (52)$$

where $g_c$ denotes the string coupling. Similarly, we take a plane wave of the dilaton:

$$\Phi(X) = -4\pi g_c \phi e^{ik \cdot X}. \quad (53)$$

Next, perform a Taylor expansion in the metric and dilaton perturbation for the disk partition function, so that it becomes

$$Z_{\text{disk}}[\epsilon, \phi] = \langle e^{-\delta S_{\text{bdry}}} \{1 + g_c : \mathcal{V} : + \frac{g_c^2}{2!} : \mathcal{V} \mathcal{V} : + \cdots \} \rangle_\eta \quad (54)$$

where

$$\mathcal{V} = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{h(\sigma)} \left[ \epsilon_{MN} h^{ab}(\sigma) \partial_a X^M \partial_b X^N e^{ik \cdot X} + \alpha' \phi \tilde{R} e^{ik \cdot X} \right] \quad (55)$$

is the graviton-dilaton vertex operator. Requiring $Z_{\text{disk}}$ to be invariant under the Weyl transformations $h_{ab} \to h_{ab}' = e^{2\omega(\sigma)} h_{ab}$ then imposes the on-shell conditions and physical polarization conditions for $\epsilon_{MN}$. Or, more precisely, in coordinate space the $\beta$-function equation gives the (linearized) Einstein equation (see e.g. [26]) in the presence of the rolling tachyon source in string frame:\footnote{The derivation of the source term is subtle. See the Appendix.}

$$R^L_{MN} + 2\nabla^L_M \nabla^L_N \phi = 8\pi T^L_{MN}. \quad (56)$$

Here the superscript $L$ means linearized. Recall that the gravitational field from which the Ricci tensor is constructed is $\tilde{R}^L_{MN}$; to first order in $\epsilon$

$$R^L_{MN} = \frac{1}{2} \left\{ \partial^2 \epsilon_{MN} - \partial_P \partial_M \epsilon^P_N - \partial_P \partial_N \epsilon^P_M + \partial_M \partial_N \epsilon^P_P \right\}. \quad (57)$$

Meanwhile, the stress tensor $T_{MN}$ appearing on the right side of (56) is equal to the disk one-point functions (12)-(13).
We now have sufficient information to identify what computation in the Dyson gas gives the thermodynamic counterpart of the Einstein equation \((56)\). Generating a weakly curved background spacetime metric corresponds to introducing additional bulk charges to the system, and considering the thermal expectation values of some (generalized) forces resulting from the point charges on the circle acting on the added test charge. Then, just like above, we should require scale invariance to derive the counterpart of the field equations. It is possible that the requirement of scale invariance is tantamount to requiring that the statistical system be at a critical point, but we have not explored this carefully. In any case, deformations of the Dyson gas which introduce additional external charges result in deformations of the spacetime.

In order to keep the exposition brief, we will ignore the 25 free spacelike bosons that play role in determining the full Einstein equation \((56)\). Thus, in what follows we will only focus on the time-time component

\[
R_{00} + 2 \partial^2_0 \phi = 8 \pi T_{00},
\]

as it will be straightforward to repeat the calculations for the other components.

Our starting point is \((54)\), which instructs us to consider the quantity

\[
Z^{\text{disk}}[\varepsilon, \phi] \equiv Z_{sp} \times (\langle \langle 1 \rangle \rangle_{\beta} + g_c \langle \langle \mathcal{O} \rangle \rangle_{\beta} + \cdots),
\]

where \(Z_{sp}\) denotes the contribution from the 25 spatial directions of \((54)\), and the rest is the non-trivial contribution from the grand canonical ensemble of charges. We can calculate that the second term in the expansion, corresponding to the second term \(\langle e^{-\delta S_{\text{bdry}}} \int dk \mathcal{V}(k) \rangle\) in \((54)\), is

\[
\langle \langle \hat{O} \rangle \rangle_{\beta} \equiv z_{\epsilon}^{k_0} \sum_{N=0}^{\infty} \frac{z_{\epsilon}^N}{N!} \langle \langle \hat{O}_N(k^0) \rangle \rangle_{\beta, N},
\]

where

\[
\langle \langle \hat{O}_N(k^0) \rangle \rangle_{\beta, N} \equiv \frac{1}{2} z_{\epsilon}^{k_0} \int d^2 \sigma \sqrt{h(\sigma)} \int \prod_{i=1}^{N} \frac{ds_i}{2\pi} \exp \left\{ -\beta \left[ : V_{h}^{\text{bulk}}(\sigma, \sigma') : |_{\sigma' \to \sigma} + V_{h}^{\text{Dyson}} + V_{h}^{\text{bulk-Dyson}} \right] \right\} \times \left[ h^{ab} : \partial_a V_{h}^{\text{bulk}} : |_{\sigma' \to \sigma} : \partial_b V_{h}^{\text{bulk}} : |_{\sigma' \to \sigma} + 2 \frac{1}{k^0} \epsilon^{ab} \partial_a V_{h}^{\text{bulk}} \partial_b V_{h}^{\text{bulk-Dyson}} + \frac{1}{(k^0)^2} h^{ab} \partial_a V_{h}^{\text{bulk-Dyson}} \partial_b V_{h}^{\text{bulk-Dyson}} - \frac{1}{2 k^0} h^{ab} : \partial_a \partial_b V_{h}^{\text{bulk}} : |_{\sigma' \to \sigma} \right].
\]

In the above, \(\sigma = (\sigma^1, \sigma^2)\) is the position of the bulk charge inside the disk, \(\epsilon^{ab}\) is the unique antisymmetric tensor in two dimensions and

\[
: V_{h}^{\text{bulk}}(\sigma, \sigma') : |_{\sigma' \to \sigma} + V_{h}^{\text{bulk-Dyson}}
\]
is a regularized net potential energy measured at $\sigma$ due to a bulk charge $k^0$ placed at $\sigma$ and Dyson gas on the boundary of the unit disk, subject to the Neumann boundary condition on the boundary. These terms can be expressed in terms of curved space generalization of bulk-bulk $\langle X^0(\sigma)X^0(\sigma') \rangle$ and bulk-boundary $\langle X^0(\sigma)X^0(s_i) \rangle$ potentials respectively:

$$V_{h}^\text{bulk-Dyson} = -k^0 \sum_i \Delta_{h}^\text{bulk-bdry}(\sigma, s_i)$$

$$V_{h}^\text{bulk}(\sigma, \sigma') = -k^0 \Delta_{h}^\text{bulk}(\sigma, \sigma'), \quad (61)$$

where both $\Delta_{h}^\text{bulk-bdry}(\sigma, s_i)$ and $\Delta_{h}^\text{bulk}(\sigma, \sigma')$ are subject to the Neumann condition on the boundary. The expression inside the square bracket of (61) is the product of the generalized forces on the bulk charge from the Dyson gas. Those expressions are regularized since these are measured at the location of the bulk charge itself. We denote this regularization scheme by $::$. Finally, to get the expression of (54) from (61), we follow these two rules:

1. Analytically continue $z \to -z = -2\pi \lambda e^{x^0}$, (2) Replace $k^0$ by $ik^0$.

In the thermodynamical system, the above quantities are thermal expectation values of particular moments of the Coulomb interaction potentials. Requiring (59) to be scale invariant yields the Einstein equation (56). (For additional details that will be involved in the derivation, see the Appendix.) Derived in this way, the Einstein equation does not refer to spacetime tensors; rather it refers to Fourier transformations of quantities in the momentum space $(k^0, \vec{k})$. This is related to the familiar Ricci tensor by the following steps. First, Fourier transform the Ricci tensor of (56) from coordinate space to momentum space:

$$R_{MN}(k) = \int dx^0 \int d^2x e^{i\vec{k} \cdot \vec{x}} R_{MN}(x)$$

$$= \frac{1}{2} \{ k^2 \epsilon_{MN}(k) - k_P k_M \epsilon^P_N - k_P k_N \epsilon^P_M + k_M k_N \epsilon^P_P \}. \quad (62)$$

Then, do the inverse Fourier transformation, but now with the chemical potential $\mu$ instead of time $x^0$. In particular, this defines

$$R_{\mu\mu}(\mu, \vec{x}) \equiv \int \frac{dk^0}{2\pi} \int d^2\vec{k} e^{-ik^0\mu + i\vec{k} \cdot \vec{x}} R_{00}(k^0, \vec{k}). \quad (63)$$

In our example, the time-time component of the Einstein equation (58) is converted to an equation for thermal expectation values

$$R_{\mu\mu} + 2\partial_\mu \phi = 8\pi T_{\mu\mu}. \quad (64)$$

---

7We subtract the usual divergent self-interaction contribution of the bulk charge but keep the finite contribution from its image charge necessitated by the Neumann boundary condition.

8Note also that the spectator CFT contained in $Z_{sp}$ is required to fully recover the answer of (54).
where \( \mu = \ln \frac{z}{\beta} \) is the chemical potential, and the stress tensor component \( T_{\mu\mu} \) is the thermal expectation value (27) in the limit \( k^0 \to 0 \). This particular component of the stress tensor turns out to be a constant, the conserved initial energy. Non-trivial chemical potential dependence would be obtained for what where initially the pressure components \( T_{ii} \) of the stress tensor.

We have sketched how the thermal ensemble counterparts of the field equations emerge from requiring scale invariance of a Dyson gas with additional test charge density included. In this way we could also derive equations of motion of other spacetime fields, such as the dilaton equation which we discussed in section 3.3.

4 Discussion

In this paper we reformulated the worldsheet description of brane decay in terms of the statistical mechanics of the Dyson gas. The progress of time was marked by an increase in the average number of particles in the gas. This gave rise to a thermodynamic arrow of time – the decrease of free energy (or, equivalently, the increase of the Shannon entropy of the particle number distribution) marked the passage of time. The spacetime equations of motion translated into relations between the expectation values of different statistical moments at different values of the chemical potential. Our analysis has essentially involved rewriting worldsheet perturbation theory in a statistical language\(^9\) and we are not precisely describing the “emergence” of time from a timeless system. Nevertheless, there are several lessons to be learned here concerning what it might mean for time to be emergent.

First, in any scenario where time is emergent, the absence of time in the underlying microscopic description will require the latter to be some kind of Euclidean or statistical system. In analogy with the AdS/CFT correspondence one might have expected some kind of relation between larger scales in the statistical system and later instants of time. In the system we have studied, the “scale” in question turned out to be the number of interacting particles in the system, and time as a continuous variable arose from this discrete quantity via a Legendre transform relating it to a chemical potential.

Second, in a scenario where time emerges from a statistical system, one might have expected a non-equilibrium flow to realize the flow in time. In fact, this is a somewhat misleading expectation. In non-equilibrium statistical mechanics, there is already a physical time and the dynamical equations describe the flow in time of statistical moments towards their equilibrium values. In our setup, we instead found that each instant of time was described by a different point of equilibrium in a statistical system.

\(^9\)The electrostatics interpretation of worldsheet Green functions as 2d Coulomb potentials between point charges is straightforward. The main issue was that here it specifically leads to a statistical mechanical system with a reinterpretation for time in target space.
The spacetime equations of motion became relations between these different points of equilibrium. Indeed, it is possible that any emergent or holographic description of time will have such a character. We are attached to the idea of a “flow in time” partly because it is conventional to construct the initial value problem in terms of data on a spacelike surface. But we could equally well have formulated it on a timelike surface of co-dimension one. Indeed, the latter is the natural way to do things in AdS space.

Most of this paper was concerned with a statistical formulation of the rolling tachyon background of Sen \(^1\). However, in order to derive the equations of motion we required Weyl invariance of deformations of the Dyson gas. These deformations, involving the addition of external charges, gave rise to new spacetime backgrounds solving the equations of motion. Thus the set of Weyl invariant deformations of the Dyson gas with \(\beta = 2\) all give statistical descriptions of spacetimes with decaying branes and different initial conditions\(^{10}\). This is reminiscent of the AdS/CFT correspondence where deformations of the CFT are dual to deformations of AdS spacetime. From this perspective, one difference is that in AdS/CFT the deformations need not be marginal – relevant deformations are dual to asymptotically AdS spacetimes that differ in the deep interior, and the RG flow of the field theory is related to the spacetime equations of motion.

A simple way of exploring the role of non-marginal deformations of the Dyson gas is to consider generalizations of Sen’s background \(^1\) that are of the form

\[
\delta S_{\text{open}} = \lambda \int dt \, e^{\alpha X^0(t)}. \tag{65}
\]

The parameter \(\alpha\) in the exponent is related to the inverse temperature of the associated Dyson gas \(\beta\) by \(\alpha = \sqrt{\beta/2}\). For \(\beta < 2\), the deformation is relevant, for \(\beta > 2\) it is irrelevant. This general system has been studied in the random matrix literature, and is known as the circular \(\beta\)-ensemble\(^{11}\). Dyson conjectured that the canonical partition function of the ensemble at fixed \(N\) is

\[
Z_N(\beta) = \frac{\Gamma(1 + \frac{\beta N}{2})}{[\Gamma(1 + \frac{\beta}{2})]^N}. \tag{66}
\]

and various proofs have been presented in the literature (see \([28]\)). Using Dyson’s formula, we can construct the grand canonical partition function

\[
Z_G(z, \beta, V) = \sum_{N=0}^{\infty} z^N \frac{N!}{N} Z_N(\beta)/N! , \tag{67}
\]

\(^{10}\)Note that although we considered deformations at the boundary of the worldsheet corresponding to the one-dimensional Dyson gas, this is just a specific example of a more general relation between scale invariant statistical mechanical systems and (time-dependent) string backgrounds. For example, it would be interesting to find a generalization for closed string tachyon condensation.

\(^{11}\)Incidentally, a longstanding problem was to find a random matrix model formulation of the general \(\beta\)-ensembles. For circular ensembles, a solution in terms of matrix models of certain sparse unitary matrices was found in \([27]\).
with $Z_N(\beta)$ given by (66). It is readily shown that the partition function diverges for $\beta > 2$ and converges for $\beta < 2$, for all values of fugacity $z$. Exactly at $\beta = 2$, the sum is convergent for $|z| < 1$, and requires analytic continuation to larger values of $z$ as described before. This suggests that $\beta = 2$ is a critical point for the Dyson gas (see [29, 30]). Furthermore, the renormalization group flow [31] of the relevant deformations ($\beta < 2$, convergent partition function) could perhaps be related to the spacetime equations of motion as in AdS/CFT and the worldsheet analyses of [32].

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A Toward deriving equations of motion using Weyl variation method

A.1 Goal

We would like to show that the equation of motions of the bulk massless fields, like graviton, dilaton etc. in presence of a decaying brane can be derived using Weyl invariance of the worldsheet non-linear sigma model. In absence of any boundary perturbation, the derivation is easy and given in Polchinski’s book [26]. In presence of a boundary perturbation, the derivation is not so straightforward.

The sigma model is described on a curved worldsheet in presence of massless closed string fields background and boundary perturbation corresponding to decaying brane. In particular, for the graviton-dilaton we would like to show that its equation of motion is given by

$$R^{L}_{MN} + 2\nabla^{L}_{M}\nabla^{L}_{N}\phi = T_{MN},$$ (68)

where $R^{L}_{MN}$ is the linearized (in $h_{MN}$) target spacetime Ricci tensor and $T_{MN}$ is the graviton-dilaton one-point function in presence of boundary perturbation, $\delta S_{\text{bdry}}$ =
\[ \lambda \int ds e^{X_0(s)} : \]
\[ T^M_M = \langle V e^{-\lambda \int ds e^{X_0(s)}} \rangle_\eta, \quad (69) \]

where \( V \) is the graviton-dilaton vertex operator defined in (55) and \( \langle O \rangle_\eta \equiv \int DXDhe^{-S_P} O \) implies that the correlation function is evaluated in the free theory using perturbative methods. Here \( S_P \) is the Polyakov action defined in the first term of (45).

### A.2 Steps

How can we derive (68)? By enforcing Weyl invariance in the full quantum theory of the worldsheet non-linear sigma model, we can achieve this. In this method we shall be able to derive the analog of (68) in momentum space. We give the steps to be carried out in this section:

1. The starting point is (54). We write \( Z_{\text{disk}}[\varepsilon, \phi] = Z_0 + Z_1 + \ldots \), where \( Z_n \) is the term with \( n \)-th power of \( V \): \( Z_n = \frac{\partial^n}{\partial \phi^n} \langle e^{-\delta S_{\text{bdry}}(\cdot \cdot \cdot)} \rangle_\eta \). Since \( \delta S_{\text{bdry}} \) involves exactly marginal boundary perturbation, \( Z_0 \) is invariant under worldsheet Weyl transformation \( \delta W h_{ab} = \delta \omega(\sigma) h_{ab} \). However, other terms \( Z_n \) \( (n > 0) \) are not. If we make sure that \( Z_1 \) is Weyl invariant, all other terms with higher powers of \( V \) will automatically be so. The Weyl variation of \( Z_1 \) is (the Polyakov action is Weyl invariant):

\[ \delta_W Z_1 = \int DXDh \ e^{-S_P} \delta_W : V : + \int DXDh \ e^{-S_P} \sum_{N=1}^{\infty} \delta_W \left( \left( -\lambda \oint ds : e^{X_0(s)} : \right)^N : V : \right). \quad (70) \]

Weyl invariance is imposed by demanding \( \delta_W Z_1 = 0 \). The notation \( : \cdot : \) around an operator implies that they are normal ordered.

2. The first term in the r.h.s. of the above equation can be derived by using the result given in Polchinski’s book. On curved worldsheet, first, we define normalized bulk operator like \( V \) as:

\[ : V := e^{D_{\text{bulk}}} V, \quad D_{\text{bulk}} = \int d^2\sigma_1 d^2\sigma_2 \Delta^\text{bulk}_h(\sigma_1, \sigma_2) \frac{\delta}{\delta X^M(\sigma_1)} \frac{\delta}{\delta X^M(\sigma_2)}. \quad (71) \]

Here \( \Delta^\text{bulk}_h(\sigma_1, \sigma_2) \) is the bulk-bulk propagator on curved worldsheet. Note that in the flat worldsheet limit (+ boundary conditions)

\[ \Delta^\text{bulk}_h(\sigma_1, \sigma_2) \rightarrow \Delta^\text{flat}_\text{flat}(z, \bar{z}; w, \bar{w}) = \frac{1}{2} \ln |z - w|^2. \quad (72) \]

Next we define the Weyl variation of the normalized bulk operator as

\[ \delta_W : V := e^{\delta_W D_{\text{bulk}}} V + \delta_W^{\exp} V, \quad (73) \]
where the last term in the above equation denotes the explicit Weyl variation. The Weyl variation of $D_{\text{bulk}}$ is

$$\delta_W D_{\text{bulk}} = \int d^2\sigma_1 d^2\sigma_2 \delta_W \Delta_h^{\text{bulk}}(\sigma_1, \sigma_2) \frac{\delta}{\delta X^M(\sigma_1)} \frac{\delta}{\delta X^M(\sigma_2)} .$$

(74)

3. Choosing a particular RG scheme\textsuperscript{12} on the worldsheet, it turns out that

$$\delta_W : \mathcal{V} := g_c \int d^2\sigma \sqrt{h} \omega \left[ \tilde{R}^L_{MN}(k) h^{ab} : \partial_a X^M \partial_b X^N e^{ik \cdot X} : + \alpha' F \tilde{R} : e^{ik \cdot X} : \right] ,$$

(75)

where

$$\tilde{R}^L_{MN}(k) = \frac{1}{2} \{ -k^2 \varepsilon_{MN} + k^N k^P \varepsilon_{MP} + k^M k^P s_{NP} - k_M k_N \varepsilon^P + 4 k_M k_N \phi \}$$

$$F = -\frac{1}{2} k^2 \phi .$$

(76)

4. The most non-trivial part for evaluating the r.h.s. of (70) is the second term.

We need to generalize the rule in (73) to the case when we have a composite operator made of bulk and boundary operators on the curved worldsheet. We would like to give a recipe for this generalized definition of normal ordering next.

5. Polchinski’s Rule Generalized : We use $\sigma_i = (\sigma_i^0, \sigma_i^1)$ for bulk and $(t_i, u_j)$ for boundary coordinates. We define the following two operators $D_{\text{bdry}}$ and $D_{\text{bulk-bdry}}$ on the boundary and bulk-boundary respectively,

$$D_{\text{bdry}} \equiv \frac{1}{2} \int du_1 \int du_2 \Delta_{\text{bdry}}(u_1, u_2) \frac{\delta}{\delta X^\rho(u_1)} \frac{\delta}{\delta X^\rho(u_2)}$$

$$D_{\text{bulk-bdry}} \equiv \frac{1}{2} \int d^2\sigma_3 \int du_3 \Delta_{\text{bulk-bdry}}(\sigma_3, u_3) \frac{\delta}{\delta X^\rho(\sigma_3)} \frac{\delta}{\delta X^\rho(u_3)} .$$

(77)

In our case the normal ordered boundary operator is : $e^{X^0(s)} := e^{D_{\text{bdry}}} e^{X^0(s)}$ which is exactly marginal. Hence its explicit Weyl variation as well as the Weyl variation of $D_{\text{bdry}}$ are vanishing. Keeping these two points in mind, we define the following generalized differential operator which does not include $D_{\text{bdry}}$:

$$D \equiv D_{\text{bulk}} + D_{\text{bulk-bdry}} .$$

(78)

Let $\mathcal{F}$ be a composite operator made of bulk and boundary operators. In our case $\mathcal{F}$ will be $\prod_i (-\lambda) e^{X^0(s_i)} \mathcal{V}$. We define the Weyl variation of the normal ordered $\mathcal{F}$ as

$$\delta_W : \mathcal{F} := (D \delta_W D)_* \mathcal{F} .$$

(79)

\textsuperscript{12}We put $\gamma = 0$ in eqns (3.6.17a-c) of [26].
The subscript $\ast$ denotes an important rule: we set terms like $(D_{\text{bulk-bdry}})^n = 0$ for $n > 1$. The reason is that there is no concept of self-contraction for computing correlators between bulk and boundary operators. The explicit Weyl variation of $\mathcal{F}$ in our case is zero since the variation for both $\prod_i e^{X_0(s_i)}$ and $\mathcal{V}$ are zero. Equation (79) will be our working definition.

A.2.1 Explicit calculations

From (77) and (78), we find that
\[(D_\delta W D)_{\ast} \mathcal{F} = (D_{\text{bulk}} \delta W D_{\text{bulk}} + D_{\text{bulk}} \delta W D_{\text{bulk-bdry}} + D_{\text{bulk-bdry}} \delta W D_{\text{bulk}}) \mathcal{F} . \quad (80)\]

In our case
\[
\mathcal{F} = \int d^2 \sigma \sqrt{h} \left[ \varepsilon_{MN} h^{ab} \partial_a X^M \partial_b X^N e^{ikX(\sigma)} + \alpha' \tilde{R} \phi e^{ikX} \right] \int \prod_k d \sigma_k \prod_j (-\lambda) e^{X_0(s_j)} \quad (81)\]

For $\mathcal{F}$ given in (81), we compute its Weyl variation $\delta_W : \mathcal{F} :$, following (79). So we need to evaluate (80). The result is:
\[
(D_\delta W D)_{\ast} \mathcal{F} = \int d^2 \sigma \prod_k d \sigma_k \sqrt{h} \varepsilon_{MN} \left[ -4k^M k^N (\partial_a \delta_W \Delta_{\text{bulk}} \partial_b \Delta_{\text{bulk}} + \partial_a \Delta_{\text{bulk}} \partial_b \delta_W \Delta_{\text{bulk}}) 
\right. \\
- 2k^2 \left\{ \eta^{MN} (\partial_a \delta_W \Delta_{\text{bulk}} + \eta^{MN} (\partial_a \partial'_b \Delta_{\text{bulk}}) \delta_W \Delta_{\text{bulk}} \\
+ ik^M (\partial_a \delta_W \Delta_{\text{bulk}} \partial_b X^N) \right\} \\
+ 2i k^0 \eta^{MN} \sum_i \partial_a \delta_W \Delta_{\text{bulk-bdry}}(\sigma, s_i) \partial_b \Delta_{\text{bulk}} \\
+ 2ik^0 \eta^{MN} \partial_a \partial_b \delta_W \Delta_{\text{bulk}} \sum_i (\partial_{\text{bulk}}(\sigma, s_i)) \\
+ 2ik^0 \eta^{MN} \partial_a \partial_b \delta_W \Delta_{\text{bulk}} \sum_i (\partial_{\text{bulk}}(\sigma, s_i)) \\
+ 2ik^0 \eta^{MN} \partial_a \partial_b \delta_W \Delta_{\text{bulk}} \sum_i (\partial_{\text{bulk}}(\sigma, s_i)) \right] e^{ikX} \prod_j (-\lambda) e^{X_0(s_j)} . \quad (82)\]

This equation actually evaluates Weyl variation of $\langle \mathcal{V} e^{-\delta S_{\text{bdry}}} \rangle_h$ where the subscript $h$ denotes curved worldsheet. This is easy to verify: first, we compute one-point function of $\mathcal{V}$ in presence of boundary perturbation on flat worldsheet, $\langle \mathcal{V} e^{-\delta S_{\text{bdry}}} \rangle_\eta$. Next, we covariantize this result for curved worldsheet to obtain $\langle \mathcal{V} e^{-\delta S_{\text{bdry}}} \rangle_h$. Finally we can compute its Weyl variation by using the rules in (73), (78) and (79).

The next step is to express the r.h.s. of (82) as $\sim \delta \omega T^M_M$. Finally we have to substitute (82), (75) and (76) in r.h.s. of (70) to obtain the momentum space version of equation of motion in (56). We leave this issue for future work.
References

[1] I. R. Klebanov, arXiv:hep-th/9108019.
[2] W. Taylor, Rev. Mod. Phys. 73, 419 (2001) arXiv:hep-th/0101126.
[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183 (2000) arXiv:hep-th/9905111.
[4] N. Seiberg, arXiv:hep-th/0601234.
[5] A. M. Polyakov, arXiv:hep-th/0602011.
[6] Z. C. Gu and X. G. Wen, arXiv:gr-qc/0606100.
[7] A. Sen, JHEP 0204, 048 (2002) arXiv:hep-th/0203211; A. Sen, JHEP 0207, 065 (2002) arXiv:hep-th/0203265; A. Sen, Int. J. Mod. Phys. A 20, 5513 (2005) arXiv:hep-th/0410103.
[8] M. Gutperle and A. Strominger, JHEP 0204 (2002) 018 arXiv:hep-th/0202210.
[9] F. Larsen, A. Naqvi and S. Terashima, JHEP 0302, 039 (2003) arXiv:hep-th/0212248; N. R. Constable and F. Larsen, JHEP 0306, 017 (2003) arXiv:hep-th/0305177.
[10] K. Okuyama, JHEP 0309 (2003) 053 arXiv:hep-th/0308172.
[11] M. Gutperle and A. Strominger, Phys. Rev. D 67, 126002 (2003) arXiv:hep-th/0301038.
[12] V. Balasubramanian, E. Keski-Vakkuri, P. Kraus and A. Naqvi, Commun. Math. Phys. 257, 363 (2005) arXiv:hep-th/0404039.
[13] J. Shelton, JHEP 0501, 037 (2005) arXiv:hep-th/0411040.
[14] N. Jokela, E. Keski-Vakkuri and J. Majumder, Phys. Rev. D 73 (2006) 046007 arXiv:hep-th/0510205.
[15] S. Hwang, Phys. Lett. B 276, 451 (1992) arXiv:hep-th/9110039; J. M. Evans, M. R. Gaberdiel and M. J. Perry, Nucl. Phys. B 535, 152 (1998) arXiv:hep-th/9806024.
[16] C. G. Callan, I. R. Klebanov, A. W. W. Ludwig and J. M. Maldacena, Nucl. Phys. B 422 (1994) 417 arXiv:hep-th/9402113.
[17] M. R. Gaberdiel, A. Recknagel and G. M. T. Watts, Nucl. Phys. B 626, 344 (2002) arXiv:hep-th/0108102.
[18] B. Chen, M. Li and F. L. Lin, JHEP 0211 (2002) 050 [arXiv:hep-th/0209222].
[19] N. Lambert, H. Liu and J. M. Maldacena, [arXiv:hep-th/0303139].
[20] T. Okuda and S. Sugimoto, Nucl. Phys. B 647 (2002) 101 [arXiv:hep-th/0208196].
[21] F. J. Dyson, J. Math. Phys. 3 (1962) 140.
[22] L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, Annals Phys. 134 (1981) 85.
[23] C. G. . Callan, C. Lovelace, C. R. Nappi and S. A. Yost, Nucl. Phys. B 288 (1987) 525.
[24] R. G. Leigh, Mod. Phys. Lett. A 4 (1989) 2767.
[25] A. Maloney, A. Strominger and X. Yin, JHEP 0310 (2003) 048 [arXiv:hep-th/0302146].
[26] J. Polchinski, String theory. Vol. 1: An introduction to the bosonic string, Chapter 3.
[27] R. Killip and I. Nenciu, Int. Math. Res. Not., 50 (2004) 2665-2701 [arXiv:math.SP/0410034].
[28] M. L. Mehta, Random Matrices, 2nd edition, Academic Press (1991).
[29] P. Fendley, F. Lesage and H. Saleur, [arXiv:hep-th/9409176].
[30] P. Fendley, H. Saleur and N. P. Warner, Nucl. Phys. B 430 (1994) 577 [arXiv:hep-th/9406125].
[31] C. L. Kane and M. P. A. Fisher, Phys. Rev. B 46 (1992) 015233.
[32] D. Z. Freedman, M. Headrick and A. Lawrence, Phys. Rev. D 73 (2006) 066015 [arXiv:hep-th/0510126].