A NOTE ON THE 4-GIRTH-THICKNESS OF $K_{n,n,n}$

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Abstract. The 4-girth-thickness $\theta(4, G)$ of a graph $G$ is the minimum number of planar subgraphs of girth at least four whose union is $G$. In this paper, we obtain that the 4-girth-thickness of complete tripartite graph $K_{n,n,n}$ is $\left\lceil \frac{n+1}{3} \right\rceil$ except for $\theta(4, K_{1,1,1}) = 2$. And we also show that the 4-girth-thickness of the complete graph $K_{10}$ is three which disprove the conjecture $\theta(4, K_{10}) = 4$ posed by Rubio-Montiel (Ars Math Contemp 14(2) (2018) 319).

1. Introduction

The thickness $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs whose union is $G$. It was defined by W.T.Tutte [8] in 1963. Then, the thicknesses of some graphs have been obtained when the graphs are hypercube [6], complete graph [11 2 9], complete bipartite graph [3] and some complete multipartite graphs [5 10 11].

In 2017, Rubio-Montiel [7] define the $g$-girth-thickness $\theta(g, G)$ of a graph $G$ as the minimum number of planar subgraphs whose union is $G$ with the girth of each subgraph is at least $g$. It is a generalization of the usual thickness in which the 3-girth-thickness $\theta(3, G)$ is the usual thickness $\theta(G)$. He also determined the 4-girth-thickness of the complete graph $K_n$ except $K_{10}$ and he conjecture that $\theta(4, K_{10}) = 4$. Let $K_{n,n,n}$ denote a complete tripartite graph in which each part contains $n$ ($n \geq 1$) vertices. In [11], Yang obtained $\theta(K_{n,n,n}) = \left\lceil \frac{n+1}{3} \right\rceil$ when $n \equiv 3(\mod 6)$.

In this paper, we determine $\theta(4, K_{n,n,n})$ for all values of $n$ and we also give a decomposition of $K_{10}$ with three planar subgraphs of girth at least four, which shows $\theta(4, K_{10}) = 3$.

2. The 4-girth-thickness of $K_{n,n,n}$

Lemma 1. [4] A planar graph with $n$ vertices and girth $g$ has edges at most $\frac{2}{g-2}(n-2)$.

Theorem 2. The 4-girth-thickness of $K_{n,n,n}$ is

$$\theta(4, K_{n,n,n}) = \left\lceil \frac{n+1}{2} \right\rceil$$

except for $\theta(4, K_{1,1,1}) = 2$.

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Proof. It is trivial for \( n = 1 \), \( \theta(4, K_{1,1,1}) = 2 \). When \( n > 1 \), because \( |E(K_{n,n,n})| = 3n^2, |V(K_{n,n,n})| = 3n \), from Lemma 1 we have

\[
\theta(4, K_{n,n,n}) \geq \left\lceil \frac{3n^2}{2(3n - 2)} \right\rceil = \left\lceil \frac{n}{2} + \frac{1}{3} + \frac{2}{3(3n - 2)} \right\rceil = \left\lceil \frac{n + 1}{2} \right\rceil.
\]

In the following, we give a decomposition of \( K_{n,n,n} \) into \( \lceil \frac{n+1}{2} \rceil \) planar subgraphs of girth at least four to complete the proof. Let the vertex partition of \( K_{n,n,n} \) be \((U, V, W)\), where \( U = \{u_1, \ldots, u_n\}, V = \{v_1, \ldots, v_n\} \) and \( W = \{w_1, \ldots, w_n\} \). In this proof, all the subscripts of vertices are taken modulo \( 2p \), except that of \( u_{2p+1}, v_{2p+1}, w_{2p+1} \).

Case 1. When \( n = 2p \) (\( p \geq 1 \)).

Let \( G_1, \ldots, G_{p+1} \) be the graphs whose edge set is empty and vertex set is the same as \( V(K_{2p,2p,2p}) \).

Step 1: For each \( G_i \) (\( 1 \leq i \leq p \)), arrange all the vertices \( u_1, v_{3-2i}, u_2, v_{4-2i}, u_3, v_{5-2i}, \ldots, u_{2p}, v_{2p-2i+2} \) on a circle and join \( u_j \) to \( v_{j+2-2i} \) and \( v_{j+1-2i} \), \( 1 \leq j \leq 2p \). Then we get a cycle of length \( 4p \), denote it by \( G^1_i \) (\( 1 \leq i \leq p \)).

Step 2: For each \( G^1_i \) (\( 1 \leq i \leq p \)), place the vertex \( w_{2i-1} \) inside the cycle and join it to \( u_1, \ldots, u_{2p} \), place the vertex \( w_{2i} \) outside the cycle and join it to \( v_1, \ldots, v_{2p} \). Then we get a planar graph \( G^2_i \) (\( 1 \leq i \leq p \)).

Step 3: For each \( G^2_i \) (\( 1 \leq i \leq p \)), place vertices \( w_{2j} \) for \( 1 \leq j \leq p \) and \( j \neq i \), inside of the quadrilateral \( w_{2i-1}u_{2i-1}v_{2i}w_{2i} \) and join each of them to vertices \( w_{2i-1} \) and \( u_{2i} \). Place vertices \( w_{2j-1} \), for \( 1 \leq j \leq p \) and \( j \neq i \), inside of the quadrilateral \( w_{2j}v_{2i-1}u_{k}w_{2j} \), in which \( u_k \) is some vertex from \( U \). Join each of them to vertices \( v_{2i-1} \) and \( v_{2i} \). Then we get a planar graph \( \overline{G}_i \) (\( 1 \leq i \leq p \)).

Step 4: For \( G_{p+1} \), join \( w_{2i-1} \) to both \( v_{2i-1} \) and \( v_{2i} \), join \( w_{2i} \) to both \( u_{2i-1} \) and \( u_{2i} \), for \( 1 \leq i \leq p \), then we get a planar graph \( \overline{G}_{p+1} \).

For \( \overline{G}_1 \cup \cdots \cup \overline{G}_{p+1} = K_{n,n,n} \), and the girth of \( \overline{G}_i \) (\( 1 \leq i \leq p + 1 \)) is at least four, we obtain a 4-girth planar decomposition of \( K_{2p,2p,2p} \) with \( p + 1 \) planar subgraphs. Figure 1 shows a 4-girth planar decomposition of \( K_{4,4,4} \) with three planar subgraphs.

![Graph](image-url)
Case 2. When \( n = 2p + 1 \) \((p > 1)\).

Base on the 4-girth planar decomposition \( \{ G_1, \ldots, G_{p+1} \} \) of \( K_{2p,2p,2p} \), by adding vertices and edges to each \( G_i \) \( (1 \leq i \leq p+1) \) and some other modifications on it, we will get a 4-girth planar decomposition of \( K_{2p+1,2p+1,2p+1} \) with \( p+1 \) subgraphs.

Step 1: (Add \( u_{2p+1} \) to \( G_i \), \( 1 \leq i \leq p \)) For each \( G_i \) \( (1 \leq i \leq p) \), we notice that the order of the \( p-1 \) interior vertices \( w_{2j}, 1 \leq j \leq p \), and \( j \neq i \) in the quadrilateral \( w_{2i-1}w_{2i-1}v_{1i}u_{2i} \) of \( G_i \) has no effect on the planarity of \( G_i \). We adjust the order of them, such that \( w_{2i-1}w_{2i-1}w_{2p-2i+2}u_{2i} \) is a face of a plane embedding of \( G_i \). Place the vertex \( u_{2p+1} \) in this face and join it to both \( w_{2i-1} \) and \( w_{2p-2i+2} \). We denote the planar graph we obtain by \( \tilde{G}_i \) \( (1 \leq i \leq p) \).

Step 2: (Add \( v_{2p+1} \) and \( w_{2p+1} \) to \( \tilde{G}_1 \)) Delete the edge \( v_1u_2 \) in \( \tilde{G}_1 \), put both \( v_{2p+1} \) and \( w_{2p+1} \) in the face \( w_ku_1v_1w_tv_2u_2 \) in which \( w_k \) is some vertex from \( \{ w_{2j} | 1 < j \leq p \} \) and \( w_t \) is some vertex from \( \{ w_{2j-1} | 1 < j \leq p \} \). Join \( v_{2p+1} \) to \( w_{2p+1} \), join \( v_{2p+1} \) to \( u_1, u_2 \), and join \( w_{2p+1} \) to \( v_1, v_2 \), we get a planar graph \( \tilde{G}_1 \).

Step 3: (Add \( v_{2p+1} \) and \( w_{2p+1} \) to \( \tilde{G}_i \), \( 2 \leq i \leq p \)) For each \( \tilde{G}_i \) \( (2 \leq i \leq p) \), place the vertex \( v_{2p+1} \) in the face \( w_ku_{2i-1}v_{1i}w_{2i} \) in which \( w_k \) is some vertex from \( \{ w_{2j} | 1 \leq j \leq p \) and \( j \neq i \} \), and join it to \( u_{2i-1} \) and \( u_{2i} \). Place the vertex \( w_{2p+1} \) in the face \( w_kv_{2i-1}u_{ti}v_{2i} \) in which \( w_k \) is some vertex from \( \{ w_{2j-1} | 1 \leq j \leq p \) and \( j \neq i \} \) and \( u_t \) is some vertex from \( U \). Join \( w_{2p+1} \) to both \( v_{2i-1} \) and \( v_{2i} \), we get a planar graph \( \tilde{G}_i \) \( (2 \leq i \leq p) \).

Step 4: (Add \( u_{2p+1}, v_{2p+1} \) and \( w_{2p+1} \) to \( \tilde{G}_{p+1} \)) We add \( u_{2p+1}, v_{2p+1} \) and \( w_{2p+1} \) to \( \tilde{G}_{p+1} \). For \( 1 \leq i \leq 2p \), join \( u_{2p+1} \) to each \( v_i \), join \( v_{2p+1} \) to each \( w_i \), join \( w_{2p+1} \) to each \( u_i \), join \( u_{2p+1} \) to both \( v_{2p+1} \) and \( w_{2p+1} \), and join \( v_1 \) to \( u_2 \), then we get a planar graph \( \tilde{G}_{p+1} \). Figure 2 shows a plane embedding of \( \tilde{G}_{p+1} \).

For \( \tilde{G}_1 \cup \cdots \cup \tilde{G}_{p+1} = K_{n,n,n} \), and the girth of \( \tilde{G}_i \) \( (1 \leq i \leq p + 1) \) is at least four, we obtain a 4-girth planar decomposition of \( K_{2p+1,2p+1,2p+1} \) with \( p+1 \) planar subgraphs. Figure 3 shows a 4-girth planar decomposition of \( K_{5,5,5} \) with three planar subgraphs.

Case 3. When \( n = 3 \), Figure 4 shows a 4-girth planar decomposition of \( K_{3,3,3} \) with two planar subgraphs.

Summarizing the above, the theorem is obtained. \( \square \)
In [7], the author posed the question whether \( \theta(4, K_{10}) = 3 \) or 4, and conjectured that it is four. We disprove his conjecture by showing \( \theta(4, K_{10}) = 3 \).

**Remark 3.** The 4-girth-thickness of \( K_{10} \) is three.

**Proof.** From [7], we have \( \theta(4, K_{10}) \geq 3 \). We draw a 4-girth planar decomposition of \( K_{10} \) with three planar subgraphs in Figure 3 which shows \( \theta(4, K_{10}) \leq 3 \). The remark follows.

\[ \square \]
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Figure 4. A 4-girth planar decomposition of $K_{3,3,3}$

Figure 5. A 4-girth planar decomposition of $K_{10}$

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