Equivariant Singular Riemann-Roch Theorem

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1 Introduction

The Riemann-Roch theorem [1], [2] has played such a key role in the study of algebraic geometry, algebraic topology or even number theory, that many attempts have been made to generalize it to the equivariant case (for example [3], [4]). Here we consider the case of the equivariant singular Riemann-Roch theorem, following the framework of P. Baum, W. Fulton and R. MacPherson.

2 Totaro’s approximation of $EG$

The purpose of this section is to describe a construction, due to Totaro [5], of an algebro-geometric substitute for the classifying space of topological group.

For topological group $G$, the spaces $EG$ and $BG$ are infinite dimensional, it is hard to control in general. What Totaro produces for a reductive algebraic group is a directed system of $G$-bundles $E_n \to B_n$ with the following property: for any algebraic principal $G$-bundle $E \to X$, there is a map $X' \to X$ with the fiber isomorphic to affine space $\mathbb{A}^m$, such that the pullback bundle $E' \to X'$ is pulled back from one of the bundles in the directed system by a map $X' \to B_n$.

For a complex linear reductive algebraic group $G$, Totaro constructs the directed system as follows:

Every object in the directed system is a pair $(V, V')$, such that $V$ is a representation of $G$, $V'$ is a non-empty open set of $V$, $G$ acts freely on $V'$, $V' \to V'/G$ is a principal $G$-bundle. $(V, V') < (W, W')$ if $V$ is a subspace of $W$, $V' \subset W'$, $\text{codim}_V(V - V') < \text{codim}_W(W - W')$. The morphisms are just inclusions.
For some groups, there is a convenient choice of the directed system. For example, if \( G = T \) is a split torus of rank \( n \), then \( \{ (V^n, (V - \{0\})^n) | \dim V = l \} \) is the directed system.

### 3 Equivariant cohomology

In general [1], we define

\[ H^*_G(X) = H^*(X \times^G EG), \]

Using the directed system above,

**Lemma 3.1** For a complex Lie group \( G \) and for \( X \) a complex algebraic variety on which \( G \) operates algebraically, we have

\[ H^*_G(X) = \lim_{\leftarrow} H^*(X \times^G V') \]

**Proof.** See [3] ■

For every \( G \)-vector bundle \( E \) over \( X \), \( E \times^G V' \rightarrow X \times^G V' \) and \( E \times^G EG \rightarrow X \times^G EG \) are all \( G \)-vector bundles, so we can define the Chern character \( ch^G(E) \in H^*_G(X) \) and \( ch^V(E) \in H^*(X \times^G V') \) by

\[ ch^G(E) = ch(E \times^G EG) \]

and

\[ ch^V(E) = ch(E \times^G V') \]

We know that under the map \( H^*_G(X) \rightarrow H^*(X \times^G V') \), \( ch^G(E) \) goes to \( ch^V(E) \).

When \( X \) is smooth, then for any \((V, V')\) in the directed system, \( X \times^G V' \) is smooth. So we can define the cohomological equivariant Todd class \( Td^G(X) \) to be the element in \( H^*_G(X) \) which maps to \( Td(X \times^G V') \) for every \((V, V')\) in the directed system.

### 4 Equivariant homology

Now for a complex linear reductive algebraic group \( G \), by using the constructed directed system, we can define the equivariant homology [5] for a \( G \)-variety \( X \), suppose \( \dim_{\mathbb{C}} X = n \).
\[ H^G_i(X) = \lim_{\leftarrow (V, V')} H^{BM}_{i+2l-2g}(X \times^G V') \]

where \( l = \dim C_V, \, g = \dim C_G \).

**Remark** It is possible that \( H^G_i(X) \neq 0 \) for negative \( i \).

**Proposition 4.1** \( H^G_*(X) \) is independent of \( (V, V') \) if codim\( _V(V - V') \) big enough.

\( H^G_*(X) \) is a module over \( H^G_*(X) \). In other words there is a cap product:
\[
\cap : H^G_*(X) \otimes H^G_*(X) \rightarrow H^G_*(X) \text{ defined in the following way:}
\]

If \( \alpha \in H^G_*(X) \), then for any \( (V, V') \), the image of \( \alpha \) under the map \( H^G_*(X) \rightarrow H^*(X \times^G V') \) defines a map by cap product: \( H^{BM}(X \times^G V') \rightarrow H^G_*(X) \). Go over to the direct limit, these maps induce a map from \( H^G_*(X) \) to \( H^G_*(X) \). This gives us the module structure.

For a non-singular variety \( X \), we can define the orientation class \([X]_G \in H^{2n}_{2n}(X)\) as follows:

For any object \( (V, V') \) in the directed system, \( X \times^G V' \) is non-singular, so we have a class \([X \times^G V'] \in H^{BM}_{2n+2l-2g}(X \times^G V') \), \{\([X \times^G V']\)\} gives the orientation class \([X]_G \).

### 5 Equivariant Riemann-Roch Theorem

Now we can define the equivariant Todd class \( \tau^G \) as follows,

For any equivariant coherent \( G \)-sheaf \( \mathcal{F} \) over \( X \) (\( X \) is a complex algebraic variety), if \( p : X \times V' \rightarrow X \) is the projection, then the pullback sheaf \( p^* \mathcal{F} \) is a sheaf over \( X \times V' \), it descends to a coherent sheaf \( \mathcal{F}_V \) over \( X \times^G V' \). \( \tau(\mathcal{F}_V) \) is in \( H^{BM}(X \times^G V') \), so we get a map \( K^G_0(X) \rightarrow H^{BM}_*(X \times^G V') \). Obviously, it is compatible with the transition maps, so we get a map \( K^G_0(X) \rightarrow H^G_*(X) \), that is \( \tau^G \).

**Theorem 5.1** (Equivariant Riemann-Roch Theorem) \( \tau^G \) is the unique natural transformation (with respect to \( G \)-equivariant proper maps) from \( K^G_0(X) \) to \( H^G_*(X) \), such that for any \( G \)-variety \( X \), we have the following commutative diagram:

\[
\begin{array}{ccc}
K^G_0(X) & \otimes & K^G_0(X) \\
\downarrow & \nearrow \otimes \tau^G & \downarrow \tau^G \\
H^G_*(X) & \otimes & H^G_*(X) \\
\end{array}
\]

\[
\begin{array}{ccc}
K^G_0(X) & \otimes & K^G_0(X) \\
\downarrow & \nearrow \otimes \tau^G & \downarrow \tau^G \\
H^G_*(X) & \otimes & H^G_*(X) \\
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\begin{array}{ccc}
K^G_0(X) & \otimes & K^G_0(X) \\
\downarrow & \nearrow \otimes \tau^G & \downarrow \tau^G \\
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\end{array}
\]

\[
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K^G_0(X) & \otimes & K^G_0(X) \\
\downarrow & \nearrow \otimes \tau^G & \downarrow \tau^G \\
H^G_*(X) & \otimes & H^G_*(X) \\
\end{array}
\]

\[
\begin{array}{ccc}
K^G_0(X) & \otimes & K^G_0(X) \\
\downarrow & \nearrow \otimes \tau^G & \downarrow \tau^G \\
H^G_*(X) & \otimes & H^G_*(X) \\
\end{array}
\]

\[
\begin{array}{ccc}
K^G_0(X) & \otimes & K^G_0(X) \\
\downarrow & \nearrow \otimes \tau^G & \downarrow \tau^G \\
H^G_*(X) & \otimes & H^G_*(X) \\
\end{array}
\]
and if \( f : X \rightarrow Y \) is a proper \( G \)-equivariant algebraic map, then

\[
\begin{array}{ccc}
K_0^G(X) & \xrightarrow{f_*} & K_0^G(Y) \\
\downarrow \tau^G & & \downarrow \tau^G \\
H_*^G(X) & \xrightarrow{f_*} & H_*^G(X)
\end{array}
\]

Furthermore, if \( X \) is non-singular, and \( \mathcal{O}_X \) is the structure sheaf, then

\[
\tau^G(\mathcal{O}_X) = Td^G(X) \cap [X]_G
\]

**Proof.** In fact, the proof is already encoded in the definition. ■

From the discussion, we have the following commutative diagram:

\[
\begin{array}{ccc}
K_0^G(X) & \xrightarrow{\tau^G} & H_*^G(X) \\
\downarrow & & \downarrow \\
K_0(X) & \xrightarrow{\tau} & H_*(X)
\end{array}
\]

where both downarrows are induced by forgetting functors.

### 6 Equivariant Riemann-Roch theorem with value in equivariant Chow group

For linear algebraic group \( G \), by using the same directed system, we can define \[\]

\[
A_i^G(X) = \lim_{\rightarrow} A_{i+2l-2g}^G(X \times^G V')
\]

where \( l = \dim_c V, \ g = \dim_c G \).

For the equivariant operational Chow group \( A_*^G(X) \), we define it as

\[
A_*^G(X) = \lim_{\leftarrow} A^i(X \times^G V')
\]

This definition concises with the definition in [6].

As in the previous section, we can define the Chern character, the Todd class and get a theorem similar to Theorem 5.1, except we use \( A_*^G(X) \otimes \mathbb{Q} \) and \( A_*^G(X) \otimes \mathbb{Q} \) to replace \( H^*_G(X) \) and \( H^*_*(X) \).
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