Critical behavior of random spin systems

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Abstract

We provide a strategy to find in few elementary calculations the critical exponents of the overlaps for dilute spin glasses, in absence of external field. Such a strategy is based on the expansion of a suitably perturbed average of the overlaps, which is used in the formulation of the free energy as the difference between a cavity part and the derivative of the free energy itself, considered as a function of the connectivity of the model. We assume the validity of certain reasonable approximations, e.g. that higher powers of overlap monomials are of smaller magnitude near the critical point, of which we do not provide a rigorous proof.
1 Introduction

Dilute spin glasses are important because of two reasons at least. Despite their mean field nature, they share with finite-dimensional models the fact that each spin interact with a finite number of other spins. Secondly, they are mathematically equivalent to some random optimization problems. The stereotypical model of dilute spin glasses is the Viana-Bray model [9], which is equivalent to the Random X-OR-SAT optimization problem in computer science, and the model we use as a guiding example here. In the original paper [9] the equilibrium of the model was studied, even in the presence of an external field, but the critical behavior was not investigated. In the case of fully connected Gaussian models, the critical exponents were computed in a recent mathematical study [1]. Here we use the techniques developed in [3] for finite connectivity spin glasses to extend the methodology of [1] to the case of dilute spin glasses. We compute the critical exponents of the overlaps among several replicas (whose distributions constitute the order parameter of the model [5]).

2 Model, notations, previous results

Given \( N \) points and families \( \{i_\nu, j_\nu, k_\nu\} \) of i.i.d random variables uniformly distributed on these points, the (random) Hamiltonian of the Viana-Bray model is defined on Ising \( N \)-spin configurations \( \sigma = (\sigma_1, \ldots, \sigma_N) \) through

\[
H_N(\sigma, \alpha) = -\sum_{\nu=1}^{P_\zeta} J_\nu \sigma_{i_\nu} \sigma_{j_\nu},
\]

where \( P_\zeta \) is a Poisson random variable with mean \( \zeta \), \( \{J_\nu = \pm 1\} \) are i.i.d. symmetric random variables and \( \alpha > 1/2 \) is the connectivity. The expec-
tation with respect to all the \((quenched)\) random variables defined so far will be denoted by \(E\), while the Gibbs expectation at inverse temperature \(\beta\) with respect to this Hamiltonian will be denoted by \(\Omega\), and depends clearly on \(\alpha\) and \(\beta\). We also define \(\langle \cdot \rangle = E\Omega(\cdot)\). The pressure, i.e. minus \(\beta\) times the free energy, is by definition

\[
A_N(\alpha) = \frac{1}{N} E \ln \sum_{\sigma} \exp(-\beta H_N(\sigma, \alpha)) .
\]

When we omit the dependence on \(N\) we mean to have taken the thermodynamic limit. The quantities encoding the thermodynamic properties of the model are the overlaps, which are defined on several configurations \((replicas)\) \(\sigma^{(1)}, \ldots, \sigma^{(n)}\) by

\[
q_1 \cdots q_n = \frac{1}{N} \sum_{i=1}^{N} \sigma^{(1)}_i \cdots \sigma^{(n)}_i .
\]

When dealing with several replicas, the Gibbs measure is simply the product measure, with the same realization of the quenched variables, but the expectation \(E\) destroys the factorization. We define \(\beta_c\) as the inverse temperature such that \(2\alpha \tanh^2 \beta_c = 1\).

We are going to need the \emph{cavity} function given by

\[
\psi_N(\alpha', \alpha) = E \ln \Omega \exp \beta \sum_{\nu=1}^{P_{2\alpha'}} J'_\nu \sigma_{k_\nu}
\]

where the quenched variables appearing explicitly in this expression are independent copies of those in \(\Omega\). When the perturbation \(\sum_{\nu=1}^{P_{2\alpha'}} J'_\nu \sigma_{k_\nu}\) is added to the Hamiltonian, the corresponding Boltzmann factor will give place to Gibbs and quenched expectations denoted by \(\Omega'_t(\cdot), \langle \cdot \rangle'_t\), and the subindex \(t\) is simply omitted when \(t = 1\). This perturbation encoded in \(\psi\), when \(\alpha' = \alpha\), is equivalent to the addition of a new spin to the
system (which can be interpreted as a gauging or spin-flip variable). As a consequence [3] gauge (or simply spin-flip in our case) invariant overlap monomials are those such that each replica appears an even number of times in them, and are stochastically stable: their average does not depend on the perturbation in the thermodynamic limit. The other overlap monomials are not invariant nor stochastically stable (the two concepts are equivalent), but their perturbed average can be expressed in terms of a power series in $t$, with ($t$-independent) stochastically stable (or invariant) averaged overlap polynomials as coefficients, in the thermodynamic limit. This is done by an iterative use of the following proposition, proven in [3].

**Proposition 1** Let $\Phi$ be a function of $s$ replicas. Then the following cavity streaming equation holds

$$ \frac{d\langle \Phi \rangle}{dt} = -2\alpha' \langle \Phi \rangle' t + 2\alpha' E \omega^{\frac{1}{\beta}} q_1 \cdot \cdot \cdot q_n \sum_{a<b} \sigma_i^{(a)} \sigma_j^{(b)} + \sum_{a<b<c} \sigma_i^{(a)} \sigma_j^{(b)} \sigma_k^{(c)} \theta^3 + \cdots \} \{ 1 - sJ \theta \omega \\
+ \frac{s(s+1)}{2!} \theta^2 \omega^2 - \frac{s(s+1)(s+2)}{3!} J \theta^3 \omega^3 + \cdots \} \} (1) $$

where $\omega = \Omega'_t(\sigma_i)$, $\theta = \tanh \beta$.

Consider for simplicity the case of $\Phi = q_1 \cdot \cdot \cdot q_n$. In the right hand side above, consisting of the product of two factors in which each term brings a new overlap multiplying $\Phi$, there is only one spin-flip invariant overlap: $q_1^2 \cdot \cdot \cdot q_n^2$. But for the other terms we can use again the streaming equation, and each non-invariant overlap will be multiplied by a suitable overlap so that the number of replicas appearing an odd number of times decreases
(by two). Notice though that each time we use the streaming equation the corresponding exponent of $\alpha'$ (eventually taken equal to $\alpha$) increase by one and so does the order of the monomial. Let us be more explicit in the case of interest, and we will see that we do not need any explicit calculation, we only need to observe that monomial of order three or higher are multiplied by $t^2$ or higher powers of $t$.

3 The expansion

In the case of $\Phi = q_{12}, q_{1234}, \ldots$, the previous proposition yields, integrating back in $dt$ once the thermodynamic limit is taken

$$\langle q_{12} \rangle_t' = \tau' t \langle q_{12}^2 \rangle - 2\tau'^2 t^2 \langle q_{12} q_{23} q_{31} \rangle + O(q^4)$$  \hspace{1cm} (2)

$$\vdots$$

$$\langle q_{1\ldots2n} \rangle_t' = \tau' \theta^{2n-2} t \langle q_{1\ldots2n}^2 \rangle + t^2 O(q^3) + \cdots$$  \hspace{1cm} (3)

where $\tau' = 2\alpha' \theta^2$ and we neglected monomials with the products of at least four overlaps. As an example, we gave the explicit form of the monomial of order three for $n = 2$.

These expansions will be used to expand $\psi$ in terms of averaged stable overlap monomials.

If we take $t = 1$ and let $\beta$ be very close to $\beta_c$, we know [3] that we can replace $\langle q_{12} \rangle'$ by $\langle q_{12}^2 \rangle$, in the left hand side of (2). This provides a relation, valid at least sufficiently close to the critical temperature, between $\langle q_{12}^2 \rangle$ and $\langle q_{12} q_{23} q_{31} \rangle$, as we neglect the higher order monomials in (2):

$$(\tau - 1) \langle q_{12}^2 \rangle = 2 \langle q_{12} q_{23} q_{31} \rangle$$  \hspace{1cm} (4)
Notice incidentally that this relation is compatible with the well known fact \cite{7} that the fluctuations of the rescaled overlap $Nq_{12}^2$ diverge only when $\tau \to 1$ (and not at higher temperatures), being $N\langle q_{12}q_{23}q_{31} \rangle$ small (due to the central limit theorem) as it is the sum of $N^3$ bounded variables divided by $N^2$ instead of $N^{3/2}$.

4 Orders of magnitude

In the expansions of the previous section, we need to understand which terms are small near the critical point. We know that above the critical temperature all the overlaps are zero, and that those which are not zero by symmetry become non-zero below the critical temperature; therefore we assume that slightly below such a temperature the overlaps are very small. More precisely, we know that for instance

$$\langle q_{12}^2 \rangle = E\Omega^2(\sigma_i \sigma_{i2})$$

is very small, and so is therefore $\Omega^2(\sigma_i \sigma_{i2})$. This means that for temperatures sufficiently close to the critical one $\Omega^4(\sigma_i \sigma_{i2})$ is negligible as compared to $\Omega^2(\sigma_i \sigma_{i2})$. In other words $\langle q_{1234}^2 \rangle$ is assumed to be of a smaller order of magnitude than $\langle q_{12}^2 \rangle$. Furthermore, if $q_{12}$ is small $q_{12}^4$ has to be of an even smaller order of magnitude. In fact we reasonably assume that

$$\langle q_{12}^4 \rangle = E\Omega^2(\sigma_i \sigma_{i2} \sigma_{i3} \sigma_{i4})$$

which is of order two in $\Omega$, is of a smaller order than $\langle q_{12}^2 \rangle$, which is also of order two in $\Omega$. An explanation comes from the self-averaging discussed in \cite{6}, which tells us that $E\Omega(\sigma_i \sigma_{i2} \sigma_{i3} \sigma_{i4})$ is of the same order as $E\Omega(\sigma_i \sigma_{i2})\Omega(\sigma_{i3} \sigma_{i4})$, which is of order two in $\Omega$, and hence increasing the number of spins in the expectation $\Omega$ is basically equivalent.
to increasing the order in $\Omega$. This is actually proven in a perturbed system [6], but it is reasonable to assume that the consequences of self-averaging (not the self-averaging itself) on the orders of magnitude of the considered quantities is not lost when the perturbation is removed, and the monomials we have are the result of the streaming equation, in which the measure is perturbed. Consistently, (4) implies that near the critical point $\langle q_{12}q_{23}q_{31} \rangle$ is smaller than $\langle q_{12}^2 \rangle$, and the two critical exponents differ by one. All these observations lead to the following criterion. We define the degree of an averaged overlap monomial as the sum of the degrees of each overlap in it, where the degree of an overlap is its exponent times its number of replicas. For instance $\langle q_{12}^2 q_{12}^2 q_{34}^2 \rangle$ is of order $4 \times 2 + 2 \times 2 + 2 \times 2 = 16$. The definition we just gave coincides with the one that can be given in terms of $\Omega$ expectations, provided one multiplies the exponent of each $\Omega$-expectation by the number of randomly chosen spins appearing in it. For example $\langle q_{1234}^2 q_{12}^2 q_{34}^2 \rangle = \mathbb{E}\Omega^2(\sigma_i, \sigma_i, \sigma_i, \sigma_i)\Omega^2(\sigma_i, \sigma_i, \sigma_i, \sigma_i)\Omega^2$ is of order $2 \times 4 + 2 \times 4 = 16$. Given an integer $m$, a monomial of order $2m + 2$ will be considered negligible, near the critical point - where all overlaps are very small, with respect to a monomial of order $2m$.

5 The transition

It is well known that all the overlaps are zero above the critical temperature $1/\beta_c$ where the replica symmetric solution holds, and that below this temperature the overlap between two replicas fluctuates and its square become non-zero [7]. As pointed out in [9], the use of the replica trick within a quadratic approximation can only provide the correct transition for the
overlap between two replicas, while overlaps of more replicas would seem to be zero down to lower temperatures before starting fluctuating. Moreover, within that method no information about the critical exponents was found. Our method allows to gain information about the critical exponents of all overlap monomials. Let us start by showing that there is only one critical point for all overlap monomials. By convexity, we have

\[ \langle q_{1...2n}^2 \rangle = \mathbb{E} \Omega^{2n}(\sigma_i, \sigma_{i+}) \geq (\mathbb{E} \Omega^2(\sigma_i, \sigma_{i+}))^n = \langle q_{12}^2 \rangle^n \]

so that all overlaps are non-zero whenever \( \langle q_{12}^2 \rangle \) is, i.e. below the critical temperature \( 1/\beta_c \). As a further example, a slightly more accurate use of convexity yields immediately

\[ \langle q_{1234}^2 \rangle \geq \langle q_{12}^2 q_{34}^2 \rangle \geq \langle q_{12}^2 \rangle \]

This means that the critical exponents of \( q_{1234}^2 \) and \( q_{12}^2 q_{34}^2 \) cannot be larger than twice the critical exponent of \( q_{12}^2 \), but cannot be smaller than this critical exponent itself either, as \( \langle q_{1234}^2 \rangle \leq \langle q_{12}^2 \rangle \).

6 Critical exponents

We will now relate the free energy to its derivative and to the cavity function. The following theorem follows easily from the results of [3], and here we only sketch the proof, based on standard convexity arguments.

**Theorem 1** In the thermodynamic limit, we have

\[ A(\alpha) = \ln 2 + \psi(\alpha, \alpha) - \alpha A'(\alpha) \]

for all values of \( \alpha, \beta \), where \( A' \) is the derivative of \( A \).
Sketched Proof. It was proven in \[5\] that
\[
A(\alpha) = \lim_{N} \mathbb{E} \ln \Omega \left( \sum_{\sigma_{N+1}} \exp(\beta \sum_{\nu=1}^{P_{2n}} J_{\nu} \sigma_{\nu} \sigma_{N+1}) \right) - \mathbb{E} \ln \Omega(\exp(-\beta(H'_{N}(\alpha/N)))) \tag{5}
\]
where the quenched variables in \(H'\) are independent of those in \(\Omega\), just like for the first term in the right hand side. The second term in the right hand side is easy to compute, at least in principle \[5\], and it is the derivative of \(A\) multiplied by \(\alpha\), because
\[
\mathbb{E} \ln \Omega(\exp(-\beta(H'_{N}(\alpha/N)))) = NA(\alpha(1 + 1/N) - NA(\alpha) .
\]
This leads to the result to prove, as the gauge invariance of \(\Omega\) allows to take out the sum over \(\sigma_{N+1}\) as \(\ln 2\), and therefore the first term in the right hand side of (5) is precisely \(\psi\).

It is easy to see that \[5\]
\[
\partial_{1} \psi(\alpha', \alpha) = 2 \sum_{n} \frac{g_{2n}^{2n}}{2n}(1 - \langle q_{1\ldots2n}' \rangle) \tag{6}
\]
\[
A'(\alpha) = \sum_{n} \frac{g_{2n}^{2n}}{2n}(1 - \langle q_{1\ldots2n}' \rangle) . \tag{7}
\]
From the theorem we have then
\[
A'(\alpha) = \partial_{1} \psi(\alpha, \alpha) + \partial_{2} \psi(\alpha, \alpha) - A'(\alpha) - \alpha A''(\alpha) .
\]
But we know \[3\] that near the critical point saturation \(\langle g_{2n}' \rangle \rightarrow \langle \xi_{2n}^{2} \rangle \) occurs in the thermodynamic limit, so that \(\partial_{1} \psi(\alpha, \alpha) \rightarrow A'(\alpha)\) and therefore we have just proven the next

**Proposition 2** In the thermodynamic limit
\[
\partial_{2} \psi(\alpha, \alpha) - \alpha A''(\alpha) = 0 . \tag{8}
\]
Notice that if in the statement of Theorem 1 we assumed saturation \( \langle q_1 \ldots 2n \rangle_t \to \langle q_1^2 \ldots 2n \rangle \) not just for \( t = 1 \) but for all \( t \) (once \( \psi(t\alpha,\alpha) \) is written using (6) as the integral of its derivative with respect to \( t \)), we would obtain \( \psi = 2A' \) and

\[
A(\alpha) = \alpha A'(\alpha) + \ln 2 ,
\]

which, as the initial condition is easily checked to be \( A'(0) = \ln \cosh \beta \), gives the well known replica symmetric solution \( A(\alpha) = \ln 2 + \alpha \ln \cosh \beta \).

This means that stability and saturation of the overlaps are equivalent to replica symmetry.

Now let us analyze (8). We consider \( \psi(\alpha',\alpha) \) as the integral of its derivative with respect to its first argument. The derivative, given in (4), contains the perturbed averaged overlaps, which we expand using (2)–(3) etc.. In this expansions the variable \( \alpha' \) appears only explicitly in front of the averaged overlap monomials, which do not depend on \( \alpha' \), they only depend on \( \alpha \). Therefore we can perform explicitly the integration of these simple power series in \( \alpha' \). The dependence on \( \alpha \) of \( \psi(\alpha',\alpha) \) is hence only in the averaged overlap monomials, and the same holds for \( A'(\alpha) \), because of (7). Therefore the derivatives of \( \psi(\alpha',\alpha) \) and of \( A'(\alpha) \) with respect to \( \alpha \) in (8) involve only the averaged overlap monomials. In other words if we define \( \tilde{A}(\alpha',\alpha) = \ln 2 + \psi(\alpha',\alpha) - \alpha' A'(\alpha) \), so that \( A(\alpha) = \tilde{A}(\alpha,\alpha) \) thanks to Theorem 1 equation (8) amounts to say that \( \partial_2 \tilde{A}(\alpha,\alpha) = 0 \). But since the second argument appears only in the averaged overlap monomials, we can consider \( A(\alpha) = \tilde{A}(\alpha,\alpha) \equiv \tilde{A}(\alpha,p_1(\alpha),p_2(\alpha),\ldots) \) a function of the averaged overlap monomials, here called \( p_1(\alpha),p_2(\alpha),\ldots \), such that

\[
\partial_2 \tilde{A} = \sum_m \frac{\partial \tilde{A}}{\partial p_m} \frac{dp_m}{d\alpha} = 0 . \tag{9}
\]
We can now use (2)-(3) etc. to have an explicit expansion of $A(\alpha)$ and deal with the differential equation (9). The result is easy to obtain and reads

$$A(\alpha) = \ln 2 + \frac{\tau}{2} - \frac{\tau}{4}(\tau - 1)(q_{12}^2) + \frac{\tau^3}{3}(q_{12}q_{23}q_{13}) + O(q^4)$$

$$+ \frac{\theta^2}{4} - \frac{\tau^2}{8}(\tau^2 - 1)(q_{12}^2) - \frac{3\tau^3}{4}(q_{12}q_{12}q_{12}q_{13}) + O(q^4))$$

$$+ O(\theta^4) \ . \ (10)$$

Notice that this expansion extends the one found in [9]. As a first approximation we may consider

$$A(\alpha) \sim \ln 2 + \frac{\tau}{2} - \frac{\tau}{4}(\tau - 1)(q_{12}^2) + \frac{\tau^3}{3}(q_{12}q_{23}q_{13})$$

and (9) becomes

$$- \frac{1}{4}(\tau - 1)\frac{d(q_{12}^2)}{d\alpha} + \frac{1}{3}\frac{d(q_{12}q_{23}q_{31})}{d\alpha} = 0 \ \ (11)$$

because

$$\frac{\hat{A}}{\partial(q_{12}^2)} = - \frac{\tau}{4}(\tau - 1) \sim - \frac{1}{4}(\tau - 1) \ ,$$

$$\frac{\hat{A}}{\partial(q_{12}q_{23}q_{31})} = \frac{\tau^3}{3} \sim \frac{1}{3} \ .$$

But now the use of (11) in (11) offers

$$- \frac{1}{4}(\tau - 1)\frac{d(q_{12}^2)}{d\alpha} + \frac{1}{3}\frac{d(\tau - 1)(q_{12}^2)}{d\alpha} = 0$$

from which, after a couple of elementary steps

$$(\tau - 1)\frac{d(q_{12}^2)}{d(\tau - 1)} - 2q_{12}^2 = 0 \ .$$

This equation is as accurate as close the temperature is to the critical one, and the solution is easy to find:

$$\langle q_{12}^2 \rangle = (\tau - 1)^2 \ ,$$

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describing the critical behavior of the overlap slightly below the critical temperature. The critical exponent is hence two.

Notice that (4) imply that $\langle q_{12} q_{23} q_{31} \rangle$ is zero above the temperature $1/\beta_2$ and positive slightly below. Moreover, (4) gives the critical exponent for $\langle q_{12} q_{23} q_{31} \rangle$: three.

From our analysis in the previous sections, we conclude that the critical exponent of $q_{1234}^2$ is strictly larger than three, but no larger than four. The criterion explained in the section on the order of magnitudes, together with (4) and the critical exponent of $q_{12}^2$, provides a relation between the degree of an overlap monomial and its critical exponent: degree $2m$ corresponds to critical exponent $m$. So for instance the critical exponent of $q_{1\ldots2n}^2$, which is of order $4n$, is $2n$.

In the infinite connectivity limit we recover the all the critical exponents for the fully connected Gaussian SK model [1].

Remark. If we extended the use of $\langle q_{1\ldots2n} \rangle^\prime \rightarrow \langle q_{1}^2 q_{2\ldots2n} \rangle$ to lower temperatures, such that $2 \alpha \theta^{2n} \equiv \tau_{2n} \sim 1$, we would obtain for $q_{1\ldots2n}^2$, for all $n$, the same identical differential equation we got for $q_{12}^2$. We would then get the same approximated behavior one gets using the replica method in a quadratic approximation [9]: $q_{2n}^2$ would be zero above the temperature such that $\tau_{2n} = 1$, then it starts fluctuating, with critical exponent two. In this sense the replica method with quadratic approximation is equivalent to extending stochastic stability below the critical point.
7 Summary and conclusions

Our strategy required the expansion of the averaged overlaps in powers of a perturbing parameter with stochastically stable overlap monomials as coefficient (similarly to the expansion exhibited in [2] for Gaussian models). This allowed to write the free energy in terms of overlap fluctuations and to discover that it does not depend on a certain family of these monomials. As a consequence, we obtained a differential equation whose solution, once all small terms are neglected, gave the critical behavior of the overlaps.

Our method is ultimately based on stochastic stability, but such a stability is proven or at least believed to hold in several contexts, therefore generalizations of our method to finite dimensional spin glasses, to the traveling salesman problem, to the K-SAT problem, to neural networks and to other cases are not to be excluded and are being studied. We plan on reporting soon on these topics [4].

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