We provide a detailed description of the nonequilibrium time evolution of an interacting homogeneous Bose-Einstein condensate. We use a nonperturbative in-medium quantum field theory approach as a microscopic model for the Bose gas. The real-time dynamics of the condensate is encoded in a set of self-consistent equations which corresponds to an infinite sum of ladder Feynman diagrams. The crucial role played by the interaction between fluctuations for the instability generation is thoroughly described.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) is a remarkable phenomenon that has been present amongst the whole spectrum of research in theoretical physics since its proposal [1]. It has challenged experimentalists with its evasive nature until about five years ago, when it was finally observed in a series of elaborate experiments with weakly-interacting atomic v apours confined in magnetic traps and cooled down to the realm of temperatures of the order of fractions of microkelvins [2]. Such an astonishing environment was made possible by the combination of refined techniques such as laser cooling, magneto-optical trapping and evaporative cooling. Since this remarkable experimental accomplishment was made possible by the combination of refined techniques such as laser cooling, magneto-optical trapping and evaporative cooling. Since this remarkable experimental accomplishment, a great amount of theoretical investigation has been stimulated (see [3] for a comprehensive review). In part, this interest grew up because several experimental features can be determined by fine-tuning various interesting parameters with a high level of control and accuracy. Moreover, if one really intends to describe the actual evolution of the condensate formation, then finite-density, non-zero temperature, and nonequilibrium dynamics effects will have to be taken into consideration. This may be accomplished quite naturally by in-medium nonequilibrium quantum field theory methods. This set of characteristics makes BEC one of the most attractive and promising systems in which one can use models and approximations that could also prove useful in very different environments such as neutron stars or heavy-ion collisions [4].

Recent experiments with dilute atomic gases [5] were able to start probing quantities which are relevant to the understanding of the underlying dynamics of BEC, such as the time scales for the condensate formation and its final size. On the theoretical side, the microscopic behavior of such an environment can be appropriately described by the nonequilibrium Schwinger-Keldysh closed time-path formalism in the quantum field theory approach [6,7]. In fact, the first steps in this direction were performed by Stoof [8], and provided a qualitative idea of the various times scales involved in the BEC process. However, a microscopic approach to the condensate onset and growth which can explain the mechanism of instability generation and describe the time evolution that follows in a quantitative way is still crude and incomplete, even in the case of a homogeneous gas. The second stage in the whole process, i.e. the growth of the condensate, was considered by Gardner et al. [9] through a quantum kinetic formalism and the construction of a master equation for a density operator describing the state of the condensate, which is equivalent to a Boltzmann equation describing a quasi-equilibrium growth of the condensate. Nevertheless, as pointed out by Stoof [8], the simplest formulations based on kinetic theory do not allow for the observation of a macroscopic occupation of the one-particle ground state, and the question of the instability of the Bose gas system in the homogeneous case is a non-trivial one.

In this article, we undertake the task of providing the simplest yet sensible nonequilibrium in-medium quantum field theory description of the dynamics of condensation of an interacting homogeneous Bose-Einstein gas that is abruptly quenched from a temperature far above to a temperature far below the critical one [10]. This
approach yields a nonperturbative set of self-consistent equations which correspond to a resummation of the ladder Feynman diagrams discussed in ref. [8]. In fact, this sum can be interpreted in a simpler, non-diagrammatic way by a careful analysis of the role played by the fluctuations, which shows that their interaction are crucial to the mechanism of instability generation in a model with no instabilities at the mean field level. Moreover, as will be described below, one can separate the $k$-space into three regions that differ in their stability properties, the scale being determined by the interaction strength and the density of condensate.

The article is organized as follows: Section II presents the microscopic model Lagrangian density and an effective theory for the fluctuations. Section III contains the self-consistent integro-differential equations for the condensate evolution that follow from the nonequilibrium approach. Besides, there is a detailed discussion of the assumptions and approximations that one has to make in order to provide an analytic treatment of the problem. In Section IV, we discuss our results for the density of condensate as a function of time and present our final comments.

II. EFFECTIVE LAGRANGIAN DENSITY FOR THE FLUCTUATIONS

The interaction Hamiltonian that describes a homogeneous gas of interacting bosons of mass $m$ has the following second-quantized form (throughout this paper we use units such that $\hbar = 1$):

$$H = \frac{1}{2} \int d^3x \int d^3x' \phi^\dagger(\vec{x}, t) \phi^\dagger(\vec{x}', t) V(\vec{x} - \vec{x}') \phi(\vec{x}', t) \phi(\vec{x}, t) ,$$

where $\phi(\vec{x}, t)$ and $\phi^\dagger(\vec{x}, t)$ are the boson annihilation and creation Heisenberg field operators, and $V(\vec{x} - \vec{x}')$ is the two-body interatomic potential. Since we will consider a weakly-interacting dilute gas, we are allowed to use the fact that, in this case, only binary collisions at low energy are relevant, so that we can approximate the actual interatomic potential by the much simpler form

$$V(\vec{x} - \vec{x}') = g \delta(\vec{x} - \vec{x}') ,$$

where the coupling constant $g$ is related to the $s$-wave scattering length $a$ through

$$g = \frac{4\pi a}{m} .$$

Then, for a weakly-interacting dilute and cold gas, we end up with a hard core interaction potential. In fact, one is reasonably safe to use such approximation since typical values for the dimensionless parameter which controls the validity of the dilute-gas approximation, the number of particles in a scattering volume, i.e. $n|a|^3$, is always less than $\sim 10^{-2}$. Inspired by the discussion above, one can write a field theory model for this system. It corresponds to the simplest model for a nonrelativistic complex Bose field, and its Lagrangian density is given by

$$\mathcal{L} = \phi^\dagger \left( i \frac{d}{dt} + \frac{1}{2m} \nabla^2 \right) \phi + \mu \phi^\dagger \phi - \frac{g}{2} (\phi^\dagger \phi)^2 ,$$

where the complex scalar field $\phi(\vec{x}, t)$ represents charged spinless bosons of mass $m$, and $g$ is the coupling constant defined above. In [8] we have also explicitly introduced a chemical potential $\mu$ that guarantees a constant total density of particles

$$\langle \phi^\dagger \phi \rangle = n .$$

We also assume that the system is coupled to a heat bath environment with inverse temperature $\beta = 1/(k_BT)$. The role played by temperature is very vague at this point. It will be discussed in detail later. For the moment, it is here just to remind us that we will consider the nonequilibrium evolution of the system at finite density and nonzero temperature.

Since we are interested in describing the onset and the time evolution of the Bose-Einstein condensate, it is convenient to decompose the original fields $\phi$ and $\phi^\dagger$ into a condensate (zero-momentum mode) part given by $\varphi_0$ and $\varphi_0^\dagger$, and a fluctuation part, $\varphi$ and $\varphi^\dagger$, that accounts for the atoms outside the condensate. Assuming a homogeneous condensate, we define the following decomposition:

$$\phi(\vec{x}, t) = \varphi_0(t) + \varphi(\vec{x}, t) ,$$

$$\phi^\dagger(\vec{x}, t) = \varphi_0^\dagger(t) + \varphi^\dagger(\vec{x}, t) .$$

Note that we make a time-dependent shift, i.e., we take $\varphi_0(t)$ as an arbitrary function of time that will be determined by the dynamics of the system. This procedure is different from the usual equilibrium analysis of systems that feature spontaneous symmetry breaking, where one usually performs a constant shift. Since we intend to describe the nonequilibrium evolution of the condensate, we must keep the time dependence.

The substitution of the shifted fields [8] in the Lagrangian [8] up to second order yields the Bogoliubov approximation for quasiparticles, which neglects the interaction between fluctuations and is reasonable only at temperatures well below the critical temperature for the condensate formation. At this level of approximation, the system is absolutely stable, and this corresponds to the stationary equilibrium situation that is achieved after the whole dynamical evolution of the condensate has taken place. In order to allow for the appearance of an instability, which will be responsible for the onset and subsequent evolution of the condensate as a function of time, we must go beyond the Bogoliubov approximation. The simplest extension is to implement a mean field approximation in the interactions between the fluctuation fields. It turns out that even this simple improvement in
our approach is able to trigger the instability that will drive the condensate formation. The mean field approximation corresponds to a self-consistent Hartree approach, and amounts to the following modification in the interaction term for fluctuations:

\[
g(\varphi^* \varphi)^2 = 4g(\varphi^* \varphi)\varphi^* \varphi + g(\varphi^* \varphi^*)\varphi + g(\varphi^* \varphi^*)\varphi + \left[g(\varphi^* \varphi)^2 - 4g(\varphi^* \varphi^*)\varphi - g(\varphi^* \varphi^*)\varphi^* \varphi^* - g(\varphi^* \varphi^*)\varphi - g(\varphi^* \varphi^*)\varphi^* \varphi^*\right],
\]

where the first terms in the rhs is taken as part of the quadratic Lagrangian for fluctuations, and the terms inside the square brackets is taken as part of the interaction Lagrangian. The terms \(\langle \varphi \varphi \rangle\) and \(\langle \varphi^* \varphi^* \rangle\) are usually known as the anomalous density terms while \(\langle \varphi^* \varphi \rangle\) is the non-condensate density. In the following we work with the approximation of neglecting the anomalous density terms in \(\mathcal{L}_0\), which is equivalent to the Hartree-Fock-Bogoliubov-Popov (HFBP) approximation for dilute Bose gases, which is generally valid for \(T \ll T_c\), when the contributions from the anomalous and non-condensate densities are very small as compared to the condensate density. This approach has also shown to give a reasonable first approximation for the description of the excitation spectrum at all temperatures provided the non-condensate density is calculated self-consistently \(\mathcal{L}_0\).

From the decomposition of the fields and\(\mathcal{L}_0\) in the HFBP approximation, the quadratic part of the Lagrangian density for the fluctuations, \(\mathcal{L}_0(\varphi, \varphi^*)\), may then be written as

\[
\mathcal{L}_0(\varphi, \varphi^*) = \varphi^* \left[ \frac{i}{c} \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 \right] \varphi + \varphi^* \left( -\frac{g}{2} \varphi^2 \right) \varphi + \varphi \left( -\frac{g}{2} \varphi^* \varphi \right),
\]

(8)

Here, we have used the fact that, under the field decomposition in the condensate and out of the condensate modes, the density constraint then becomes

\[
\langle \varphi^* \varphi \rangle = \langle \varphi \varphi^* \rangle = n.
\]

(9)

Additionally, assuming that at the initial time the system is mostly composed of particles outside the condensate, \(\langle \varphi^* \varphi \rangle \approx \langle \varphi^* \varphi \rangle\) (at \(t = 0\)), simple relations involving the generating functional for the correlation functions (see, for instance, the last section of chapter 2 in \[\mathcal{L}_0\]) allow us to write the total number density \(n\) of particles in terms of the chemical potential \(\mu\), valid in the mean-field approximation for the potential, as: \(\mu = 2g\varphi\). This expression does not depend on any equilibrium property of the system, depending only on the mean-field approximation. Notice that this value for \(\mu\) coincides with the usual one obtained in the equilibrium treatments of dilute Bose gases in the HFBP approximation \([\mathcal{L}_0]\). Moreover, it satisfies the Hugenholtz-Pines relation \([\mathcal{L}_0]\) that would be obtained in the equilibrium problem. In spite of this, our derivation is valid for the out-of-equilibrium regime and should not to be confused with the equilibrium treatments. These considerations lead to the quadratic Lagrangian for the fluctuations shown above (Eq. \(\mathcal{L}_0\)).

### III. The Quench Scenario, Green Functions and Self-Consistent Equations

The scenario proposed here assumes that, for \(t < 0\), the initial state is in equilibrium at a temperature \(T_i \gg T_c\). At \(t = 0\) the system is then abruptly quenched to a much lower temperature \(T_f \ll T_c\), where \(T_f\) is the temperature of the thermal bath in which the system is immersed (the one associated with the quantity \(\beta\) that appears in our description) and, of course, it will be the equilibrium temperature which the system will reach asymptotically as \(t \to \infty\). We should remark, then, that the initial temperature itself is not important in our study and it will not appear in our equations. It is only a conceptual tool that will help us in the definition of the initial conditions for the number densities, as will be clear below. In fact, since our approach is strongly out of equilibrium, it is not possible to define a “temperature parameter” when the condensate is evolving as it is usually done in a quasi-equilibrium treatment. Technically, this means that the final equilibrium temperature, \(T_f = 1/(k_B \beta)\), will show up in the nonequilibrium propagators through boundary conditions (see below).

This kind of quench is easily attained in the experiments of Bose-Einstein condensation of atomic gases, where the typical relaxation time scales are long enough (around \(\sim 0.1\)s, depending on the temperature \(\mathcal{L}_0\)) to allow for a fast drop in the temperature of the system that evolves afterwards out of equilibrium. With this choice of initial state, it is reasonable to consider the dynamics of the building up of the condensate state, which at the initial time is \(n_{\text{cond.}}(t = 0) = |\varphi_0(t = 0)|^2 \approx 0\), and the depletion of the excited states, which at \(t = 0\) is given by \(n_{\text{exc.}}(t = 0) = \langle \varphi^* \varphi \rangle \approx n\), as essentially a two-level problem. It is clear that this approximation breaks down for temperatures close to the critical temperature, where the detailed treatment would require a thorough study of the dynamics among the many levels of excited states. Furthermore, at this point the mean field approximation would not be reliable any more.

In the scenario described above, the condensate builds up subject to the density constraint relation, which may be expressed in terms of the averages of the real and the imaginary parts of \(\varphi\) and \(\varphi^*\) (\(\varphi = \varphi_1 + i\varphi_2\) and \(\varphi^* = \varphi_1 - i\varphi_2\)). Spatial translation invariance yields:

\[
|\varphi_0(t)|^2 + n_{\text{exc.}}(t) = n,
\]

(10)

\[
n_{\text{exc.}}(t) = \langle \varphi_1(t) \varphi_1(t) \rangle_{\varphi_0} + \langle \varphi_2(t) \varphi_2(t) \rangle_{\varphi_0}.
\]

In order to express these self-consistent equations in a concrete and convenient form let us define the field averages in terms of the Green functions for the fields.
The Green functions can be defined from a generating functional for our Lagrangian model, Eq. (4), where the generating functional $Z[J]$, in terms of an external source $J$, is given by

$$Z[J] = \int_D D\phi D\phi^* \exp \{ iS[\phi, \phi^*, J] \}, \quad (11)$$

where the classical action is given by

$$S[\phi, \phi^*, J] = \int_D d^4 x \{ \mathcal{L}[\phi, \phi^*] + J(x)\phi^*(x) + J^*(x)\phi(x) \}.$$  \quad (12)

In (12), the time integration is along a contour suitable for real-time evaluations, which we choose as being the Schwinger closed-time path formalism of common use in condensed matter non-equilibrium problems, where the time path $C$ goes from $-\infty$ to $+\infty$ and then back to $-\infty$. The functional integration in (12) is over fields along this time contour. As in the Euclidean time formulation, the scalar field is still periodic in time, but now $\phi(t, \vec{x}) = \phi(t-i\beta, \vec{x})$. Temperature appears due to the boundary conditions, but now real time is explicitly present in the integration contour. Denoting by $J^+$ and $J^-$ the sources in the $-\infty$ to $+\infty$ path and $+\infty$ to $-\infty$ paths, respectively, along the time path contour, the generating functional $Z[J^+, J^-]$ can be written as

$$Z[J^+, J^-] = \int_D D\phi D\phi^* \exp \left\{ i \int d^4 x \left\{ \mathcal{L}[\phi_+, \phi_+^*, J^+] - \mathcal{L}[\phi_-, \phi_-^*, J^-] \right\} \right\}. \quad (13)$$

By performing the path integrals over the quadratic forms as usual, one obtains that the generating functional in the Schwinger’s closed-time path formalism can be written as (with $\phi$, $\phi^*$ written in terms of the real fields $\phi_1$ and $\phi_2$)

$$Z[J^+, J^-] = \exp \left\{ i \int d^4 x \left[ \mathcal{L}_{int} \left( -i \frac{\delta}{\delta J^+_j} \right) - \mathcal{L}_{int} \left( i \frac{\delta}{\delta J^-_j} \right) \right] \right\} \times \exp \left\{ \frac{i}{2} \int d^4 x d^4 y J^+_j(x) G_{jj}^{ab}(\vec{x}, \vec{y}) J^+_j(y) \right\} \quad (14)$$

with $a, b = +, -$ and $j = 1, 2$. The Green functions that enter the integrals along the closed-time path contour in $G_{jj}^{ab}$ are given by $G_{jj}^{ab}$ (in momentum space)

$$G_{jj}^{ab}(k, t, t') = G_{jj}^{ab}(k, t, t') \theta(t-t') + G_{jj}^{ab}(k, t, t') \theta(t'-t),$$

$$G_{jj}^{+-}(k, t, t') = G_{jj}^{+-}(k, t, t') \theta(t-t') + G_{jj}^{+-}(k, t, t') \theta(t'-t),$$

$$G_{jj}^{+-}(k, t, t') = -G_{jj}^{+-}(k, t, t'),$$

$$G_{jj}^{++}(k, t, t') = -G_{jj}^{++}(k, t, t'). \quad (15)$$

The functions $G^>$ and $G^<$ satisfy the property $G_{jj}^{+-}(k, t, t') = G_{jj}^{+-}(k, t - i\beta, t')$, which is recognized as the periodicity condition in real time, Kubo-Martin-Schwinger (KMS) condition. Here, $\beta$ is the inverse of the temperature of the thermal bath and appears as a consequence of the boundary conditions arising from the construction of the complex time path.

The field averages of last section can then be expressed in terms of the Green functions for $\phi_1$ and $\phi_2$ as ($j = 1, 2$)

$$\langle \phi_j(t)\phi_j(t') \rangle = \int \frac{d^4 k}{(2\pi)^4} \left[ -iG_{jj}^<(k, t, t') \right], \quad (16)$$

$G^>$ and $G^<$ are constructed from the homogeneous solutions to the operator of quadratic fluctuation modes which, using Eq. (8) expressed in terms of $\phi_1$ and $\phi_2$, are given by (in momentum space)

$$\frac{d\chi_1(k, t)}{dt} + \left( \frac{k^2}{2m} + g|\phi_0|^2 \right) \chi_1(k, t) = 0,$$

$$\frac{d\chi_2(k, t)}{dt} - \left( \frac{k^2}{2m} - g|\phi_0|^2 \right) \chi_2(k, t) = 0. \quad (17)$$

The boundary conditions for the solutions of the equations above are such that, for $t < 0$, $|\phi_0(t)|^2 = 0$ and $\chi_1(k, t)$ and $\chi_2(k, t)$ can, for example, be given by $\chi_1(k, t) = e^{i\epsilon k t}/\sqrt{2N}$ and $\chi_2(k, t) = ie^{i\epsilon k t}/\sqrt{2N}$, where $\epsilon_k = k^2/(2m)$ and the normalization factor $N$ can be fixed by imposing the constraint condition, Eq. (8), at $t = 0$ ($|\phi_0(0)|^2 = 0$). We should stress that, from our choice of initial configuration, we have a well-defined initial state, given by $n_0 \simeq 0$ and $n_{exc} \simeq n$, and a well-defined equilibrium final state, at temperature $T_f \ll T_c$, towards which the system will evolve. The final situation, then, is an equilibrium one, which allows for the definition of a final equilibrium density matrix, $\rho = Tr \ e^{-\beta H}$, where $\beta$ is the inverse of the temperature of the heat bath that was responsible for the quench ($T_f \ll T_c$). Therefore, one can consistently define the Schwinger-Keldysh closed-time path procedure in order to describe the nonequilibrium evolution that will follow. The conditions fulfilled by the nonequilibrium propagators in that formalism are analogous to the KMS conditions obtained in the imaginary time description of equilibrium problems.

By decoupling the set of equations in (17), we obtain

$$\ddot{\chi}_1 - \frac{\omega_+}{\omega_-} \dot{\chi}_1 + \omega_+ \omega_- \chi_1 = 0,$$

$$\ddot{\chi}_2 - \frac{\omega_+}{\omega_+} \dot{\chi}_2 + \omega_+ \omega_- \chi_2 = 0, \quad (18)$$

where $\omega_\pm = k^2/(2m) \pm g|\phi_0|^2$. Defining $\xi_1 = \chi_1/\sqrt{\omega_-}$ and $\xi_2 = \chi_2/\sqrt{\omega_+}$, Eq. (18) can also be written as

$$\ddot{\xi}_1 + \omega_+^2 \xi_1 = 0,$$

$$\ddot{\xi}_2 + \omega_-^2 \xi_2 = 0, \quad (19)$$

where
\[ \omega^2_{1(2)} = \frac{1}{2} \frac{\dot{\omega}_{-(-)} + \dot{\omega}_{-(-)}}{\omega_{-(-)}} - \frac{3}{4} \frac{\dot{\omega}^2_{-(-)}}{\omega_{-(-)}} + \omega \omega_{-} . \]  

We can see from these decoupled equations that those modes with \( k^2/(2m) < g|\varphi_0|^2 \) are the unstable ones and they will drive initially the excited particles towards condensation. At this point, one should recall that the instability arises as a consequence of the interaction between fluctuations. Should one discard those interactions, the mean-field description would result in an absolutely stable system. This situation is completely different from the one encountered in the relativistic approach \cite{15}. Those unstable modes, for which \( k^2/(2m) \) is smaller than \( g|\varphi_0|^2 \), correspond to the (exponential) growth of the long-wavelength fluctuations which drive the process of phase transition, or condensation in our case. This is similar to the phenomenon of spinodal decomposition in statistical mechanics, typical of second order phase transitions \cite{16}.

In terms of the mode functions \( \chi_1 \) and \( \chi_2 \), and taking into account the boundary conditions for them, the Green's functions can be expressed as \( (j = 1, 2) \)

\[ G_{\beta\beta}^j(k, t, t') = \frac{i}{1 - e^{-\beta \omega}} \left[ \chi_j(k, t) \chi_j^*(k, t') + e^{-\beta \omega} \chi_j^*(k, t) \chi_j(k, t') \right] . \]  

The Green's functions for the complex fields \( \varphi, \varphi^* \) are expressed as usual by \cite{3}:

\[ \langle \varphi(t) \varphi^*(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \left[ \lambda \varphi \varphi^* (k, t, t) \right] , \]  

\[ \langle \varphi^*(t) \varphi(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \left[ \lambda \varphi \varphi^* (k, t, t) \right] . \]  

In terms of (22) and (23), we have that the non-condensate density \( n_{\text{exc}}(t) \) can be written as

\[ n_{\text{exc}}(t) = \langle \varphi_1(t) \varphi_2(t) \rangle + \langle \varphi_2(t) \varphi_2(t) \rangle = \frac{1}{2} \left( \langle \varphi(t) \varphi^*(t) \rangle + \langle \varphi^*(t) \varphi(t) \rangle \right) , \]  

and the KMS condition can be expressed, in this case, as \( G_{\varphi \varphi}^j(k, t - i\beta, t') = [G_{\varphi \varphi}^j(k, t, t')]^* \), or \( G_{\varphi \varphi}^j(k, t - i\beta, t') = [G_{\varphi \varphi}^j(k, t, t')]^* \).

Using Eq. (24), we then obtain that \( n_{\text{exc}}(t) \) can be expressed as

\[ n_{\text{exc}}(t) = \int \frac{d^3k}{(2\pi)^3} \left[ |\chi_1(k, t)|^2 + \right. \]  

\[ \left. + |\chi_2(k, t)|^2 \right] \coth \left( \frac{\beta \epsilon_k}{2} \right) . \]  

At \( t = 0 \), from the initial conditions imposed over the system, the system is all in the non-condensate state and the condensate density \( n_c \equiv |\varphi_0|^2 = 0 \): \( \chi_1 \) and \( \chi_2 \) are purely oscillatory functions. We then obtain that

\[ n_{\text{exc}}(0) \equiv n = \langle \varphi^* \varphi \rangle |_{t=0} = \langle \varphi_1^2 \rangle |_{t=0} + \langle \varphi_2^2 \rangle |_{t=0} , \]  

which, from Eq. (25), gives

\[ n = \frac{1}{2N\pi^2} \int dk k^2 \coth \left( \frac{\beta \epsilon_k}{2} \right) . \]  

The \( T = 0 \) part of (27) is divergent and represents the zero-point energy. By subtracting the zero-point energy we obtain for the normalization \( N \)

\[ N = \frac{2}{n} \zeta(3/2) \left( \frac{m}{2\pi\beta} \right)^{3/2} . \]  

Using the expression for the density in terms of the critical temperature for condensation of an ideal Bose gas \( (1/\beta_0) \) \cite{17}, \( n = \zeta(3/2) \left( m/(2\pi\beta_0) \right)^{3/2} \), we find a simple expression for the normalization \( N \) as given by \( N = 2(\beta_0/\beta)^{3/2} \), which corresponds to the pre-factor used in Ref. \cite{18}. Using these constraint conditions together with the initial boundary conditions, from Eq. (24), we can then express the condensate density as

\[ |\varphi_0(t)|^2 = \frac{1}{2\pi^2} \left( \frac{\beta_0}{\beta_c} \right)^{3/2} \int_0^{\frac{\pi a^2|\varphi_0(t)|^2}{2}} dk k^2 \times \]  

\[ \times \left[ 1 - N \left( |\chi_1(k, t)|^2 + |\chi_2(k, t)|^2 \right) \right] n_k(\beta) , \]  

where \( n_k(\beta) = (e^{\beta \epsilon_k} - 1)^{-1} \). The ultraviolet divergences were explicitly cancelled by subtracting the \( T = 0 \) component of (27). This is fine, since the contributions from the non-condensate part of the density are negligible for weak interacting dilute Bose gas systems at \( T = 0 \). Note also that, in our out-of-equilibrium approach, there are no infrared divergences since the finite time is a natural regulator. However, for the equilibrium \( t \to \infty \), the critical temperature will be modified by the interactions as pointed out in Ref. \cite{19,20}.

Equations (17) and (18) form an integro-differential system that may be solved for \( \varphi_0(t) \) numerically, given the initial conditions for \( \varphi_0(t), \chi_1(k, t) \) and \( \chi_2(k, t) \) mentioned before. Indeed, this system of equations determines completely the time evolution of the condensate density as a function of the temperature and of the total density of the gas. At this point, one could ask about the contribution from those terms responsible for the decay of quasiparticles which are related, microscopically, to the imaginary part of the self-energies in higher order perturbation theory. These contributions are important to describe the detailed dynamics for temperatures close to the critical temperature, which is not the case we have at hand. In such range of temperatures, the typical time scales for condensation are long enough so that scattering and growth among adjacent levels, which are described by those microscopic processes, are of relevance to the dynamics in this regime. Those processes have been described in the literature through the use of quantum kinetic theory, i.e. quantum Boltzmann equations, and could as well be done within the approach.
described here when considering higher-order corrections to our field averages.

Equation (29) is the first order term in the finite-temperature quantum field perturbation expansion for \( \langle \phi^2 \rangle \). Higher-order corrections for the equal-time two-point field averages can be expressed in terms of the coincidence limit of the (causal) two-point Green functions \( G_{\phi \phi'} \) and \( G_{\phi' \phi} \), which satisfy the Dyson equation (the indices stand for \( \phi \) and \( \phi' \)):

\[
G_{ij} = G^0_{ij} + G^0_{ik} \Sigma_{kl} G_{lj} ,
\]

(30)

where \( \Sigma_{ij} \) is the (matrix) self-energy, and \( G^0_{ij} \) is the zeroth-order non-interacting Green function, satisfying the equation (in momentum space)

\[
\left( \pm i \frac{d}{dt} - \epsilon_k \right) G^0_{\phi \phi', (\phi', \phi)}(k, t, t') = \delta(t - t) .
\]

(31)

One of the advantages of expressing the Green’s functions in terms of the solutions of (17) is the possibility of obtaining, in an unambiguous way, all higher-order corrections to the two-point and many-point functions.

IV. RESULTS AND DISCUSSION

From Eq. (21) we find that initially the dynamics will be dominated by essentially two regimes in the \( k \)-space: for \( (k^2/2m) < g|\phi_0|^2 \), or spinodal regime, is the regime characterized by the exponential growth of the fluctuations, driving the system towards Bose-Einstein condensation. Following a crossover at \( (k^2/2m) \sim g|\phi_0|^2 \), the modes for which \( (k^2/2m) \gg g|\phi_0|^2 \) will be oscillatory stable modes. The energy of these modes are high enough such that they effectively sample a symmetric potential and will not contribute to the dynamics. These properties determined our choice of momentum cut-off in Eq. (21). Equations (17) and (23) form a set of integro-differential equations that can be solved numerically, given the initial conditions mentioned before. In Fig. 1 we show the results for the condensate density \( |\phi_0|^2 \), as a function of time, for different values of temperature. From these results we notice the characteristic growth curves for the condensate. For very initial times, the system is dominated by the stable modes. As the condensate develops the number of unstables modes increases. At some point they dominate and make the condensate grow exponentially fast, until the equilibrium state is reached. This state is determined by the temperature at which the system is quenched below the critical temperature, and by the total density.

It is important to point out that the evolution of the condensate is completely driven by the interaction between the microscopic fluctuations of the field around the condensate. In fact, in this non-relativistic system, the interaction between fluctuations is responsible for the appearance of the instability which allows for the onset of the condensate (see, e.g., [13] for a description of what happens in the relativistic case). One should also note that not all the excited particles condense, since there will always be a fraction (which depends on various parameters for a particular system, and on the temperature of the thermal bath) of excited modes, with high enough frequency, that remains stable.

![FIG. 1](image_url)

FIG. 1. Condensate density as a function of time for \( na^3 = 0.01 \) and \( T_1/T_0 = 0.06 \), \( T_2/T_0 = 0.08 \) and \( T_3/T_0 = 0.1 \). Here, \( \tau \equiv (\hbar/ma^2)t \) is a dimensionless time and \( \rho_0 \equiv a^3|\phi_0|^2 \) is a dimensionless density.

The system of equations we have derived is completely non-perturbative and able to describe the time evolution of BEC in the extremely far from equilibrium regime, where standard methods based on the mean-field Gross-Pitaevski equations are not applicable. In this sense, our description should complement the usual approach via Boltzmann equation. Furthermore, the system of equations obtained here, can be viewed as a resummation of the ladder Feynman diagrams mentioned by Stoof in [8].

In summary, we have developed an out-of-equilibrium non-perturbative quantum field theory description of the condensation process of an interacting homogeneous Bose-Einstein gas quenched below the critical temperature. Although we focused this article on the instability process that generates the condensate (i.e., the short time behavior), for \( t \to \infty \) our results confirm the expected behavior for this limit [8]. However, the equilibrium \( (t \to \infty) \) values of the condensate fraction are lower than the experimental results [4] and the calculations of Dalfovo et. al. [8]. This may be due to our approximation of neglecting incoherent collisional processes, which is a valid approximation in an infinite homogeneous gas.
at very low temperatures and densities, but otherwise may give an important contribution. We expect that the self-consistent inclusion of pair terms should account for most of these contributions.

In spite of the absence of non-homogeneity effects, we hope that the approach developed here may be useful in the analysis of transients in realistic Bose-Einstein condensation experiments with atomic gases. Our results should also be applicable to trapped atomic gases in the central region of wide traps. Moreover, with a suitable generalization of the formalism presented above, we could be able to develop a theoretical description of the dynamical aspects of a recently proposed experiment [21], regarding the Bose-Einstein condensation in a weakly-interacting photon gas in a nonlinear Fabry-Perot cavity.

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[1] A. Griffin in: M. Inguscio, S. Stringari and C. Wieman (eds.), Bose-Einstein Condensation in Atomic Gases (Italian Physical Society, 1999).
[2] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[3] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[4] J. A. Pons, J. A. Miralles, M. Prakash and J. M. Lattimer, astro-ph/0008389, J. A. Pons, A. W. Steiner, M. Prakash and J. M. Lattimer, astro-ph/0102015, D. Boyanovsky, H. J. de Vega and R. Holman, Phys. Rev. D 51, 734 (1995). O. Scavenius and A. Dumitru, Phys. Rev. Lett. 83, 4697 (1999). O. Scavenius, A. Dumitru, E. S. Fraga, J. T. Lenaghan and A. D. Jackson, hep-ph/0009171. A. Dumitru and R. D. Pisarski, hep-ph/0010083.
[5] H.-J. Miesner, D. M. Kurn, M. R. Andrews, D. S. Durfee, S. Inoye and W. Ketterle, Science 279, 1005 (1998).
[6] J. Schwinger, J. Math. Phys. (N.Y.) 2, 407 (1961); P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. (N.Y.) 4, 1 (1963); 4, 12 (1963); L. V. Keldysh, Zh. Eksp. Teor. Fiz. 47, 1515 (1964); K. Chou, Z. Su, B. Hao and L. Yu, Phys. Rep. 118, 1 (1985).
[7] M. Le Bellac, Thermal Field Theory (Cambridge University Press, Cambridge, 1996).
[8] H. T. C. Stoof, J. Low Temp. Phys. 114, 11 (1999). Phys. Rev. Lett. 66, 3148; Phys. Rev. A49, 3824 (1994); Phys. Rev. A45, 8398 (1992).
[9] C. W. Gardiner, P. Zoller, R. J. Ballagh and M. J. Davis, Phys. Rev. Lett. 79, 1793 (1997); C. W. Gardiner, M. D. Lee, R. J. Ballagh, M. J. Davis and P. Zoller, Phys. Rev. Lett. 81, 5266 (1998).
[10] Most of the results discussed here were presented in a much shorter version in ref. [18].
[11] A. Griffin, Phys. Rev. B53, 9341 (1996).
[12] L. P. Kadanoff and G. Baym, Quantum Statistical Mechanics, Green's Functions Methods in Equilibrium and Nonequilibrium Problems (Addison-Wesley Publ. Co., New York, 1989).
[13] N. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).
[14] M.-O. Mewes, M. R. Anderson, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend and W. Ketterle, Phys. Rev. Lett. 77, 988 (1996).
[15] D. Boyanovsky, D. Lee and A. Singh, Phys. Rev. D 48, 800 (1993). D. Boyanovsky, H. J. de Vega, R. Holman and J. Salgado, Phys. Rev. D 59, 125009 (1999).
[16] J. D. Gunton, M. San Miguel and P.S. Sahni, The Dynamics of First-Order Phase Transitions, in: Phase Transitions and Critical Phenomena, vol. 8, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1983).
[17] A. Griffin, Excitations in a Bose-condensed liquid (Cambridge University Press, Cambridge, 1993).
[18] D. G. Barci, E. S. Fraga and R. O. Ramos, Phys. Rev. Lett. 85, 479 (2000).
[19] G. Baym, J.-P. Blaizot, M. Holzman, F. Laloe and D. Vautherin, Phys. Rev. Lett. 83, 1703 (1999).
[20] F. F. S. Cruz, M. B. Pinto and R. O. Ramos, Phys. Rev. B64, 014515 (2001).
[21] R. Y. Chiao and J. Boyce, Phys. Rev. A60, 4114 (1999).
[22] A. Tanzini and S. P. Sorella, Phys. Lett. A263, 43 (1999).