COMPRESSED RANDOM VARIABLES IN THE GRAPH $W^*$-PROBABILITY SPACES

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Abstract. In [16] and [17], we observed the amalgamated free probability theory on the graph $W^*$-probability space $(W^*(G), E)$ over the diagonal subalgebra $D_G$. In [18], we consider the diagonal compressed random variables in $(W^*(G), E)$ and observed the amalgamated freeness in the $W^*$-probability space $(W^*(G), E)$ over the $D_G$. In particular, for $V_N = \{v_0\} \subset V(G)$, we could construct the (scalar-valued) tracial $W^*$-probability space, so-called the vertex compressed graph $W^*$-probability space. In this paper, we will consider the off-diagonal compressed random variables in $(W^*(G), E)$. After fixing $v_1 \neq v_2$ in $V(G)$, we define the $(v_1, v_2)$-off-diagonal compressed random variable of $a \in (W^*(G), E)$, $L_{v_1}aL_{v_2}$. We will consider the free probability data on such elements in $(W^*(G), E)$. Also, we will consider the compressed random variables $PaP$ for the $D_G$-valued random variable $a \in (W^*(G), E)$, compressed by the projection $P = L_{v_1} + L_{v_2} + \ldots + L_{v_N} \in D_G$. To observe the free probability data for such compressed random variable, we use the diagonal-compression and off-diagonal compression. We can figure out that only the diagonal-compression affects the compressed free probability on $W^*(G)$. In fact, $D_G$-valued moment series and R-transform of the compressed random variable of $a$ are same as those of diagonal compressed random of it.

In [16], we constructed the graph $W^*$-probability spaces. The graph $W^*$-probability theory is one of the good example of Speicher’s combinatorial free probability theory with amalgamation. In [16], we observed how to compute the moment and cumulant of an arbitrary random variables in the graph $W^*$-probability space and the freeness on it with respect to the given conditional expectation. Also, in [17], we consider certain special random variables of the graph $W^*$-probability space, for example, semicircular elements, even elements and R-diagonal elements. This shows that the graph $W^*$-probability spaces contain the rich free probabilistic objects. Roughly speaking, graph $W^*$-algebras are $W^*$-topology closed version of free semigroupoid algebras defined and observed by Kribs and Power in [10].

Throughout this paper, let $G$ be a countable directed graph and let $F^+(G)$ be the free semigroupoid of $G$, in the sense of Kribs and Power. i.e., it is a collection of all vertices of the graph $G$ as units and all admissible finite paths, under the admissibility. As a set, the free semigroupoid $F^+(G)$ can be decomposed by

$$F^+(G) = V(G) \cup FP(G),$$

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where $V(G)$ is the vertex set of the graph $G$ and $FP(G)$ is the set of all admissible finite paths. Trivially the edge set $E(G)$ of the graph $G$ is properly contained in $FP(G)$, since all edges of the graph can be regarded as finite paths with their length 1. We define a graph $W^*$-algebra of $G$ by

$$W^*(G) \overset{\text{def}}{=} \mathbb{C}[\{L_w, L^*_w : w \in \mathbb{F}^+(G)\}]^w,$$

where $L_w$ and $L^*_w$ are creation operators and annihilation operators on the generalized Fock space $H_G = l^2(\mathbb{F}^+(G))$ induced by the given graph $G$, respectively. Notice that the creation operators induced by vertices are projections and the creation operators induced by finite paths are partial isometries. We can define the $W^*$-subalgebra $D_G$ of $W^*(G)$, which is called the diagonal subalgebra by

$$D_G \overset{\text{def}}{=} \mathbb{C}[\{L_v : v \in V(G)\}]^w.$$

Then each element $a$ in the graph $W^*$-algebra $W^*(G)$ is expressed by

$$a = \sum_{w \in \mathbb{F}^+(G ; a), u_w \in \{1, *\}} p_w L_u^w, \quad \text{for } p_w \in \mathbb{C},$$

where $\mathbb{F}^+(G : a)$ is a support of the element $a$, as a subset of the free semigroupoid $\mathbb{F}^+(G)$. The above expression of the random variable $a$ is said to be the Fourier expansion of $a$. Since $\mathbb{F}^+(G)$ is decomposed by the disjoint subsets $V(G)$ and $FP(G)$, the support $\mathbb{F}^+(G : a)$ of $a$ is also decomposed by the following disjoint subsets,

$$V(G : a) = \mathbb{F}^+(G : a) \cap V(G)$$

and

$$FP(G : a) = \mathbb{F}^+(G : a) \cap FP(G).$$

Thus the operator $a$ can be re-expressed by

$$a = \sum_{v \in V(G : a)} p_v L_v + \sum_{w \in FP(G : a), u_w \in \{1, *\}} p_w L_u^w.$$

Notice that if $V(G : a) \neq \emptyset$, then $\sum_{v \in V(G : a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Thus we have the canonical conditional expectation $E : W^*(G) \rightarrow D_G$, defined by

$$E(a) = \sum_{v \in V(G : a)} p_v L_v,$$
for all \(a = \sum_{w \in \mathbb{F}^+(G:a), u_w \in \{1, \star\}} p_w L_w^a\) in \(W^*(G)\). Then the algebraic pair \((W^*(G), E)\) is a \(W^*\)-probability space with amalgamation over \(D_G\) (See [16]). It is easy to check that the conditional expectation \(E\) is faithful in the sense that if \(E(a^*a) = 0_{D_G}\), for \(a \in W^*(G)\), then \(a = 0_{D_G}\).

For the fixed operator \(a \in W^*(G)\), the support \(\mathbb{F}^+(G : a)\) of the operator \(a\) is again decomposed by

\[
\mathbb{F}^+(G : a) = V(G : a) \cup FP_*(G : a) \cup FP_0^*(G : a),
\]

with the decomposition of \(FP(G : a)\),

\[
FP(G : a) = FP_*(G : a) \cup FP_0^*(G : a),
\]

where

\[
FP_*(G : a) = \{ w \in FP(G : a) : \text{both } L_w \text{ and } L_w^* \text{ are summands of } a \}
\]

and

\[
FP_0^*(G : a) = FP(G : a) \setminus FP_*(G : a).
\]

The above new expression plays a key role to find the \(D_G\)-valued moments of the random variable \(a\). In fact, the summands \(p_v L_v\)'s and \(p_w L_w + p_w^* L_w^*\), for \(v \in V(G : a)\) and \(w \in FP_*(G : a)\), act for the computation of \(D_G\)-valued moments of \(a\). By using the above partition of the support of a random variable, we can compute the \(D_G\)-valued moments and \(D_G\)-valued cumulants of it via the lattice path model \(LP_n\) and the lattice path model \(LP_n^*\) satisfying the \(\star\)-axis-property. At a first glance, the computations of \(D_G\)-valued moments and cumulants look so abstract and hence it looks useless. However, these computations, in particular the computation of \(D_G\)-valued cumulants, provides us how to figure out the \(D_G\)-freeness of random variables by making us compute the mixed cumulants. As applications, in the final chapter, we can compute the moment and cumulant of the operator that is the sum of \(N\)-free semicircular elements with their covariance \(2\).

Based on the \(D_G\)-cumulant computation, we can characterize the \(D_G\)-freeness of generators of \(W^*(G)\), by the so-called diagram-distinctness on the graph \(G\), i.e., the random variables \(L_{w_1}\) and \(L_{w_2}\) are free over \(D_G\) if and only if \(w_1\) and \(w_2\) are diagram-distinct the sense that \(w_1\) and \(w_2\) have different diagrams on the graph \(G\). Also, we could find the necessary condition for the \(D_G\)-freeness of two arbitrary random variables \(a\) and \(b\), i.e., if the supports \(\mathbb{F}^+(G : a)\) and \(\mathbb{F}^+(G : b)\) are diagram-distinct, in the sense that \(w_1\) and \(w_2\) are diagram distinct for all pairs \((w_1, w_2) \in \mathbb{F}^+(G : a) \times \mathbb{F}^+(G : b)\), then the random variables \(a\) and \(b\) are free over \(D_G\).

In [17], we considered some special \(D_G\)-valued random variables in a graph \(W^*\)-probability space \((W^*(G), E)\). The those random variables are the basic objects to study Free Probability Theory. We can conclude that

(i) if \(l\) is a loop, then \(L_l + L_l^*\) is \(D_G\)-semicircular.
(ii) if \( w \) is a finite path, then \( L_w + L_w^* \) is \( D_G \)-even.

(iii) if \( w \) is a finite path, then \( L_w \) and \( L_w^* \) are \( D_G \)-valued R-diagonal.

In [18], we observed the diagonal compressed random variables in the graph \( W^*(G, E) \). Let \( v_1, \ldots, v_N \in V(G) \) and let \( a \) be a \( D_G \)-valued random variable in \( (W^*(G, E)) \). Define the diagonal compressed random variable of \( a \) by

\[
C_V(a) = L_{v_1}aL_{v_1} + \ldots + L_{v_N}aL_{v_N}.
\]

Notice that if \( v \in V(G) \), then \( L_vaL_v \) is the compressed random variable by \( L_v \) and the compressed random variable has its support contained in \( \{v\} \cup \text{loop}_v(G) \), where \( \text{loop}_v(G) = \{l \in \text{loop}(G) : l = vlv\} \).

The main purpose of this paper is to show that the compressed random variable \( P_VaP_V \) of a projection \( P_V = \sum_{j=1}^{N} L_{v_j} \), where \( V = \{v_1, \ldots, v_N\} \), has the same free probabilistic information with the diagonal compressed random variable \( C_V(a) \), i.e., the compressed random variables \( P_VaP_V \) and \( C_V(a) \) have the same \( D_G \)-valued moments and cumulants.

1. Graph \( W^* \)-Probability Theory

Let \( G \) be a countable directed graph and let \( \mathbb{F}^+(G) \) be the free semigroupoid of \( G \). i.e., the set \( \mathbb{F}^+(G) \) is the collection of all vertices as units and all admissible finite paths of \( G \). Let \( w \) be a finite path with its source \( s(w) = x \) and its range \( r(w) = y \), where \( x, y \in V(G) \). Then sometimes we will denote \( w \) by \( w = xwy \) to express the source and the range of \( w \). We can define the graph Hilbert space \( H_G \) by the Hilbert space \( l^2(\mathbb{F}^+(G)) \) generated by the elements in the free semigroupoid \( \mathbb{F}^+(G) \). i.e., this Hilbert space has its Hilbert basis \( \mathcal{B} = \{\xi_w : w \in \mathbb{F}^+(G)\} \). Suppose that \( w = e_1 \ldots e_k \in FP(G) \) is a finite path with \( e_1, \ldots, e_k \in E(G) \). Then we can regard \( \xi_w \) as \( \xi_{e_1} \otimes \ldots \otimes \xi_{e_k} \). So, in [10], Kribs and Power called this graph Hilbert space the generalized Fock space. Throughout this paper, we will call \( H_G \) the graph Hilbert space to emphasize that this Hilbert space is induced by the graph.

Define the creation operator \( L_w \), for \( w \in \mathbb{F}^+(G) \), by the multiplication operator by \( \xi_w \) on \( H_G \). Then the creation operator \( L \) on \( H_G \) satisfies that

\[
(i) \quad L_w = L_{xwy} = L_x L_w L_y, \quad \text{for } w = xwy \text{ with } x, y \in V(G).
\]
(ii) $L_{w_1}L_{w_2} = \begin{cases} L_{w_1w_2} & \text{if } w_1w_2 \in \mathbb{F}^+(G) \\ 0 & \text{if } w_1w_2 \notin \mathbb{F}^+(G), \end{cases}$

for all $w_1, w_2 \in \mathbb{F}^+(G)$.

Now, define the annihilation operator $L_w^*$, for $w \in \mathbb{F}^+(G)$ by

$$L_w^* \xi_{w'} \overset{\text{def}}{=} \begin{cases} \xi_h & \text{if } w' = wh \in \mathbb{F}^+(G) \xi \\ 0 & \text{otherwise.} \end{cases}$$

The above definition is gotten by the following observation:

$$< L_w \xi_h, \xi_{w'} > = < \xi_h, \xi_{w'} > = 1 = < \xi_h, \xi_h > = < \xi_h, L_w^* \xi_{w'} >,$$

where $<,>$ is the inner product on the graph Hilbert space $H_G$. Of course, in the above formula we need the admissibility of $w$ and $h$ in $\mathbb{F}^+(G)$. However, even though $w$ and $h$ are not admissible (i.e., $wh \notin \mathbb{F}^+(G)$), by the definition of $L_w^*$, we have that

$$< L_w \xi_h, \xi_h > = < 0, \xi_h > = 0 = < \xi_h, 0 > = < \xi_h, L_w^* \xi_h > .$$

Notice that the creation operator $L$ and the annihilation operator $L^*$ satisfy that

$$L_w^* L_w = L_y \quad \text{and} \quad L_w L_w^* = L_x, \quad \text{for all } w = xwy \in \mathbb{F}^+(G),$$

under the weak topology, where $x, y \in V(G)$. Remark that if we consider the von Neumann algebra $W^*\{L_w\}$ generated by $L_w$ and $L_w^*$ in $B(H_G)$, then the projections $L_y$ and $L_x$ are Murray-von Neumann equivalent, because there exists a partial isometry $L_w$ satisfying the relation (1.1). Indeed, if $w = xwy$ in $\mathbb{F}^+(G)$, with $x, y \in V(G)$, then under the weak topology we have that

$$L_w L_w^* L_w = L_w \quad \text{and} \quad L_w^* L_w L_w^* = L_w^* .$$

So, the creation operator $L_w$ is a partial isometry in $W^*\{L_w\}$ in $B(H_G)$. Assume now that $v \in V(G)$. Then we can regard $v$ as $v = v/v$. So,

$$L_v L_v^* L_v = L_v \quad = L_v L_v^* = L_v^* .$$

This relation shows that $L_v$ is a projection in $B(H_G)$ for all $v \in V(G)$.

Define the graph $W^*$-algebra $W^*(G)$ by
\[ W^*(G) \overset{\text{def}}{=} \mathbb{C}[\{L_w, L^*_w : w \in F^+(G)\}]^w. \]

Then all generators are either partial isometries or projections, by (1.2) and (1.3). So, this graph \( W^* \)-algebra contains a rich structure, as a von Neumann algebra. (This construction can be the generalization of that of group von Neumann algebra.) Naturally, we can define a von Neumann subalgebra \( D_G \subset W^*(G) \) generated by all projections \( L_v, v \in V(G) \).

\[ D_G \overset{\text{def}}{=} W^* (\{L_v : v \in V(G)\}). \]

We call this subalgebra the **diagonal subalgebra** of \( W^*(G) \). Notice that \( D_G = \Delta|G| \subset M|G|(\mathbb{C}) \), where \( \Delta|G| \) is the subalgebra of \( M|G|(\mathbb{C}) \) generated by all diagonal matrices. Also, notice that \( 1_{D_G} = \sum_{v \in V(G)} L_v = 1_{W^*(G)}. \)

If \( a \in W^*(G) \) is an operator, then it has the following decomposition which is called the Fourier expansion of \( a \);

\[ a = \sum_{w \in F^+(G : a), u_w \in \{1, \ast\}} p_w L_u^w, \]

where \( p_w \in \mathbb{C} \) and \( F^+(G : a) \) is the support of \( a \) defined by

\[ F^+(G : a) = \{ w \in F^+(G) : p_w \neq 0 \}. \]

Remark that the free semigroupoid \( F^+(G) \) has its partition \( \{V(G), FP(G)\} \), as a set. i.e.,

\[ F^+(G) = V(G) \cup FP(G) \quad \text{and} \quad V(G) \cap FP(G) = \emptyset. \]

So, the support of \( a \) is also partitioned by

\[ F^+(G : a) = V(G : a) \cup FP(G : a), \]

where

\[ V(G : a) \overset{\text{def}}{=} V(G) \cap F^+(G : a) \]

and

\[ FP(G : a) \overset{\text{def}}{=} FP(G) \cap F^+(G : a). \]

So, the above Fourier expansion (1.4) of the random variable \( a \) can be re-expressed by

\[ a = \sum_{v \in V(G : a)} p_v L_v + \sum_{w \in FP(G : a), u_w \in \{1, \ast\}} p_w L_u^w. \]
We can easily see that if $V(G:a) \neq \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Also, if $V(G:a) = \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v = 0_{D_G}$. So, we can define the following canonical conditional expectation $E : W^*(G) \to D_G$ by

$$E(a) = E \left( \sum_{v \in V(G:a)} p_v L_v \right)^{\overset{\text{def}}{=} \sum_{v \in V(G:a)} p_v L_v},$$

for all $a \in W^*(G)$. Indeed, $E$ is a well-determined conditional expectation.

**Definition 1.1.** Let $G$ be a countable directed graph and let $W^*(G)$ be the graph $W^*$-algebra induced by $G$. Let $E : W^*(G) \to D_G$ be the conditional expectation defined above. Then we say that the algebraic pair $(W^*(G), E)$ is the graph $W^*$-probability space over the diagonal subalgebra $D_G$. By the very definition, it is one of the $W^*$-probability space with amalgamation over $D_G$. All elements in $(W^*(G), E)$ are called $D_G$-valued random variables.

We have a graph $W^*$-probability space $(W^*(G), E)$ over its diagonal subalgebra $D_G$. We will define the following free probability data of $D_G$-valued random variables.

**Definition 1.2.** Let $W^*(G)$ be the graph $W^*$-algebra induced by $G$ and let $a \in W^*(G)$. Define the $n$-th ($D_G$-valued) moment of $a$ by

$$E(d_1a d_2a ... d_na),$$

where $d_1, ..., d_n \in D_G$. Also, define the $n$-th ($D_G$-valued) cumulant of $a$ by

$$k_n(d_1a, d_2a, ..., d_na) = C^{(n)}(d_1a \otimes d_2a \otimes ... \otimes d_na),$$

for all $n \in \mathbb{N}$, and for $d_1, ..., d_n \in D_G$, where $\hat{C} = (C^{(n)})_{n=1}^{\infty} \in I^e(W^*(G), D_G)$ is the cumulant multiplicative bimodule map induced by the conditional expectation $E$, in the sense of Speicher. We define the $n$-th trivial moment of $a$ and the $n$-th trivial cumulant of $a$ by

$$E(a^n) \quad \text{and} \quad k_n \left( a, a, ..., a \atop \text{n-times} \right) = C^{(n)}(a \otimes a \otimes ... \otimes a),$$

respectively, for all $n \in \mathbb{N}$.

To compute the $D_G$-valued moments and cumulants of the $D_G$-valued random variable $a$, we need to introduce the following new definition:
Definition 1.3. Let \((W^*(G), E)\) be a graph \(W^*\)-probability space over \(D_G\) and let \(a \in (W^*(G), E)\) be a random variable. Define the subset \(FP_\ast(G : a)\) in \(FP(G : a)\) by

\[ FP_\ast(G : a) \overset{def}{=} \{ w \in \mathbb{F}^*(G : a) : \text{both} \ L_w \text{ and } L_w^* \text{ are summands of } a \} . \]

And let \(FP_\ast^c(G : a) \overset{def}{=} FP(G : a) \setminus FP_\ast(G : a)\).

We already observed that if \(a \in (W^*(G), E)\) is a \(D_G\)-valued random variable, then \(a\) has its Fourier expansion \(a = a_d + a_0\), where

\[ a_d = \sum_{v \in V(G : a)} p_v L_v \]

and

\[ a_0 = \sum_{w \in FP(G : a), u_w \in \{1, *\}} p_w L_{u_w} . \]

By the previous definition, the set \(FP(G : a)\) is partitioned by

\[ FP(G : a) = FP_\ast(G : a) \cup FP_\ast^c(G : a) , \]

for the fixed random variable \(a\) in \((W^*(G), E)\). So, the summand \(a_0\), in the Fourier expansion of \(a = a_d + a_0\), has the following decomposition ;

\[ a_0 = a_{(*)} + a_{(\text{non-*})} , \]

where

\[ a_{(*)} = \sum_{l \in FP_\ast(G : a)} (p_l L_l + p_l L_l^*) \]

and

\[ a_{(\text{non-*})} = \sum_{w \in FP_\ast^c(G : a), u_w \in \{1, *\}} p_w L_{u_w} , \]

where \(p_l\) is the coefficient of \(L_l^*\) depending on \(l \in FP_\ast(G : a)\).

1.1. \(D_G\)-Moments and \(D_G\)-Cumulants of Random Variables.

Throughout this chapter, let \(G\) be a countable directed graph and let \((W^*(G), E)\) be the graph \(W^*\)-probability space over its diagonal subalgebra \(D_G\). In this chapter, we will compute the \(D_G\)-valued moments and the \(D_G\)-valued cumulants of arbitrary random variable.
Define the lattice path $l$ where $(G, ∗)$. If $w$ be a countable directed graph and assume that $w_1, ..., w_n \in F^+(G)$, then $L_{w_1}^w ... L_{w_n}^w \in (W^*(G), E)$ be a $D_G$-valued random variable. In this section, we will define a lattice path model for the random variable $L_{w_1}^w ... L_{w_n}^w$.

Recall that if $w = e_1, ..., e_k \in FP(G)$ with $e_1, ..., e_k \in E(G)$, then we can define the length $|w|$ of $w$ by $k$, i.e., the length $|w|$ of $w$ is the cardinality $k$ of the admissible edges $e_1, ..., e_k$.

**Definition 1.4.** Let $G$ be a countable directed graph and $F^+(G)$, the free semigroupoid. If $w \in F^+(G)$, then $L_w$ is the corresponding $D_G$-valued random variable in $(W^*(G), E)$. We define the lattice path $l_w$ of $L_w$ and the lattice path $l_w^{-1}$ of $L_w^*$ by the lattice paths satisfying that:

(i) the lattice path $l_w$ starts from $* = (0, 0)$ on the $\mathbb{R}^2$-plane.

(ii) if $w \in V(G)$, then $l_w$ has its end point $(0, 1)$.

(iii) if $w \in E(G)$, then $l_w$ has its end point $(1, 1)$.

(iv) if $w \in E(G)$, then $l_w^{-1}$ has its end point $(-1, -1)$.

(v) if $w \in FP(G)$ with $|w| = k$, then $l_w$ has its end point $(k, k)$.

(vi) if $w \in FP(G)$ with $|w| = k$, then $l_w^{-1}$ has its end point $(-k, -k)$.

Assume that finite paths $w_1, ..., w_s$ in $FP(G)$ satisfy that $w_1...w_s \in FP(G)$. Define the lattice path $l_{w_1} ... l_{w_s}$ by the connected lattice path of the lattice paths $l_{w_1}, ..., l_{w_s}$, i.e., $l_{w_2}$ starts from $(k_{w_1}, k_{w_1}) \in \mathbb{R}^+$ and ends at $(k_{w_1} + k_{w_2}, k_{w_1} + k_{w_2})$, where $|w_1| = k_{w_1}$ and $|w_2| = k_{w_2}$. Similarly, we can define the lattice path $l_{w_1}^{-1} ... l_{w_s}^{-1}$ as the connected path of $i_{w_1}^{-1}, i_{w_2}^{-1}, ..., i_{w_s}^{-1}$.

**Definition 1.5.** Let $G$ be a countable directed graph and assume that $L_{w_1}, ..., L_{w_n}$ are generators of $(W^*(G), E)$. Then we have the lattice paths $l_{w_1}, ..., l_{w_n}$ of $L_{w_1}, ..., L_{w_n}$, respectively in $\mathbb{R}^2$. Suppose that $L_{w_1}^w ... L_{w_n}^w \neq 0_{D_G}$ in $(W^*(G), E)$, where $u_{w_1}, ..., u_{w_n} \in \{1, *\}$. Define the lattice path $l_{w_1}^{u_{w_1}, ..., u_{w_n}}$ of nonzero $L_{w_1}^w ... L_{w_n}^w$ by the connected lattice path of $l_{w_1}^{u_{w_1}}, ..., l_{w_n}^{u_{w_n}}$, where $t_{w_j} = 1$ if $u_{w_j} = 1$ and $t_{w_j} = -1$ if...
Let $u_{w_1} = \ast$. Assume that $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} = 0_{D_G}$. Then the empty set $\emptyset$ in $\mathbb{R}^2$ is the lattice path of it. We call it the empty lattice path. By $LP_n$, we will denote the set of all lattice paths of the $D_G$-valued random variables having their forms of $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$, including empty lattice path.

Also, we will define the following important property on the set of all lattice paths:

**Definition 1.6.** Let $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n} \neq \emptyset$ be a lattice path of $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \neq 0_{D_G}$ in $LP_n$. If the lattice path $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ starts from $\ast$ and ends on the $\ast$-axis in $\mathbb{R}^+$, then we say that the lattice path $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ has the $\ast$-axis-property. By $LP_n^{\ast}$, we will denote the set of all lattice paths having their forms of $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ which have the $\ast$-axis-property. By little abuse of notation, sometimes, we will say that the $D_G$-valued random variable $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$ satisfies the $\ast$-axis-property if the lattice path $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ of it has the $\ast$-axis-property.

The following theorem shows that finding $E \left( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \right)$ is checking the $\ast$-axis-property of $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$.

**Theorem 1.1.** (See [15]) Let $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable, where $u_{w_1}, \ldots, u_{w_n} \in \{1, \ast\}$. Then $E \left( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \right) \neq 0_{D_G}$ if and only if $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$ has the $\ast$-axis-property (i.e., the corresponding lattice path $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ of $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$ is contained in $LP_n^{\ast}$). Notice that $\emptyset \notin LP_n^{\ast}$. \hfill $\square$

By the previous theorem, we can conclude that $E \left( L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \right) = v$, for some $v \in V(G)$ if and only if the lattice path $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ has the $\ast$-axis-property (i.e., $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n} \in LP_n^{\ast}$).

### 1.1.2. $D_G$-Valued Moments and Cumulants of Random Variables

Let $w_1, \ldots, w_n \in \mathbb{F}^+(G)$, $u_{w_1}, \ldots, u_{w_n} \in \{1, \ast\}$ and let $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable. Recall that, in the previous section, we observed that the $D_G$-valued random variable $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n} = e \in (W^*(G), E)$ with $v \in V(G)$ if and only if the lattice path $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n}$ of $L_{w_1}^{u_1} \ldots L_{w_n}^{u_n}$ has the $\ast$-axis-property (equivalently, $l_{w_1}^{u_1} \ldots l_{w_n}^{u_n} \in LP_n^{\ast}$). Throughout this section, fix a $D_G$-valued random variable $a \in (W^*(G), E)$. Then the $D_G$-valued random variable $a$ has the following Fourier expansion,

$$a = \sum_{v \in V(G)} p_v L_v + \sum_{l \in FP_1(G:a)} (p_l L_l + p_l L_l) + \sum_{w \in FP_2(G:a), u_w \in \{1, \ast\}} p_w L_{w}^{u_w}.$$

Let’s observe the new $D_G$-valued random variable $d_1 a d_2 a \ldots d_n a \in (W^*(G), E)$, where $d_1, \ldots, d_n \in D_G$ and $a \in W^*(G)$ is given. Put
\[ d_j = \sum_{v_j \in V(G:d_j)} q_{v_j} L_{v_j} \in D_G, \text{ for } j = 1, \ldots, n. \]

Notice that \( V(G : d_j) = \mathbb{F}^+(G : d_j) \), since \( d_j \in D_G \rightarrow W^*(G) \).

**Proposition 1.2.** (See [16]) Let \( a \in (W^*(G), E) \) be given as above. Then the \( n \)-th moment of \( a \) is

\[
E(d_1 a \cdots d_n a) = \sum_{(u_1, \ldots, u_n) \in \Pi^n_{j=1} V(G : d_j)} \left( \prod_{j=1}^n q_{u_j} \right) \left( \prod_{j=1}^n p_{w_j} \right)
\]

\[
= \left( \prod_{j=1}^n \delta(u_j, x_j, y_j, u_{w_j}) \right) E(L_{u_1}^{w_1} \cdots L_{u_n}^{w_n}),
\]

where

\[
\delta(u_j, x_j, y_j) = \begin{cases} 
\delta_{u_j, x_j} & \text{if } u_j = 1 \\
\delta_{u_j, y_j} & \text{if } u_j = *
\end{cases}
\]

\[ \square \]

Let \( w_1, \ldots, w_n \in FP(G) \) be finite paths and \( u_1, \ldots, u_n \in \{1, *\} \). Then, by the Möbius inversion, we have

\[(1.13)\]

\[ k_n \left( L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \right) = \sum_{\pi \in NC(n)} \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) \mu(\pi, 1_n), \]

where \( \hat{E} = (E(n))_{n=1}^\infty \) is the moment multiplicative bimodule map induced by the conditional expectation \( E \) (See [16]) and where \( NC(n) \) is the collection of all noncrossing partition over \( \{1, \ldots, n\} \).

**Definition 1.7.** Let \( NC(n) \) be the set of all noncrossing partition over \( \{1, \ldots, n\} \) and let \( L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \in (W^*(G), E) \) be \( D_G \)-valued random variables, where \( u_1, \ldots, u_n \in \{1, *\} \). We say that the \( D_G \)-valued random variable \( L_{w_1}^{u_1} \cdots L_{w_n}^{u_n} \) is \( \pi \)-connected if the \( \pi \)-dependent \( D_G \)-moment of it is nonvanishing, for \( \pi \in NC(n) \). In other words, the random variable \( L_{w_1}^{u_1} \cdots L_{w_n}^{u_n} \) is \( \pi \)-connected, for \( \pi \in NC(n) \), if

\[ \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) \neq 0_{D_G}. \]

i.e., there exists a vertex \( v \in V(G) \) such that

\[ \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes \cdots \otimes L_{w_n}^{u_n} \right) = L_v. \]

For convenience, we will define the following subset of \( NC(n) \):
Definition 1.8. Let $NC(n)$ be the set of all noncrossing partitions over $\{1, \ldots, n\}$ and fix a $DG$-valued random variable $L_{u_1}^{w_1} \ldots L_{u_n}^{w_n}$ in $(W^*(G), E)$, where $u_1, \ldots, u_n \in \{1, *\}$. For the fixed $DG$-valued random variable $L_{u_1}^{w_1} \ldots L_{u_n}^{w_n}$, define

$$C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n} \overset{def}{=} \{ \pi \in NC(n) : L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \text{ is } \pi\text{-connected} \},$$

in $NC(n)$. Let $\mu$ be the Möbius function in the incidence algebra $I_2$. Define the number $\mu_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}$, for the fixed $DG$-valued random variable $L_{u_1}^{w_1} \ldots L_{u_n}^{w_n}$, by

$$\mu_{u_1, \ldots, u_n}^{w_1, \ldots, w_n} \overset{def}{=} \sum_{\pi \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}} \mu(\pi, 1_n).$$

Assume that there exists $\pi \in NC(n)$ such that $L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} = L_0$ is $\pi$-connected. Then $\pi \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}$ and there exists the maximal partition $\pi_0 \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}$ such that $L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} = L_0$ is $\pi_0$-connected. Notice that $1_n \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}$. Therefore, the maximal partition in $C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}$ is $1_n$. Hence we have that;

Lemma 1.3. (See [16]) Let $L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \in (W^*(G), E)$ be a $DG$-valued random variable having the $*$-axis-property. Then

$$E \left( L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \right) = \hat{E}(\pi) \left( L_{u_1}^{w_1} \otimes \ldots \otimes L_{u_n}^{w_n} \right),$$

for all $\pi \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}$. \(\square\)

By the previous lemmas, we have that

Theorem 1.4. (See [16]) Let $n \in 2\mathbb{N}$ and let $L_{u_1}^{w_1}, \ldots, L_{u_n}^{w_n} \in (W^*(G), E)$ be $DG$-valued random variables, where $w_1, \ldots, w_n \in FP(G)$ and $u_j \in \{1, *\}$, $j = 1, \ldots, n$. Then

$$k_n \left( L_{u_1}^{w_1} \ldots L_{u_n}^{w_n} \right) = \mu_{u_1, \ldots, u_n}^{w_1, \ldots, w_n} \cdot E(\pi_1, \ldots, L_{u_n}^{w_n}),$$

where $\mu_{u_1, \ldots, u_n}^{w_1, \ldots, w_n} = \sum_{\pi \in C_{u_1, \ldots, u_n}^{w_1, \ldots, w_n}} \mu(\pi, 1_n)$. \(\square\)
1.2. $D_G$-Freeness on $(W^* (G), E)$.

Now, we will introduce the diagram-distinctness of finite paths;

Definition 1.9. (Diagram-Distinctness) We will say that the finite paths $w_1$ and $w_2$ are diagram-distinct if $w_1$ and $w_2$ have different diagrams in the graph $G$. Let $X_1$ and $X_2$ be subsets of $FP(G)$. The subsets $X_1$ and $X_2$ are said to be diagram-distinct if $x_1$ and $x_2$ are diagram-distinct for all pairs $(x_1, x_2) \in X_1 \times X_2$.

In [16], we found the $D_G$-freeness characterization on the generator set of $W^* (G)$, as follows;

Theorem 1.5. (See [16]) Let $w_1, w_2 \in FP(G)$ be finite paths. The $D_G$-valued random variables $L_{w_1}$ and $L_{w_2}$ in $(W^* (G), E)$ are free over $D_G$ if and only if $w_1$ and $w_2$ are diagram-distinct. □

Let $a$ and $b$ be the given $D_G$-valued random variables. We can get the necessary condition for the $D_G$-freeness of $a$ and $b$, in terms of their supports. Recall that we say that the two subsets $X_1$ and $X_2$ of $FP(G)$ are said to be diagram-distinct if $x_1$ and $x_2$ are diagram-distinct, for all pairs $(x_1, x_2) \in X_1 \times X_2$.

Proposition 1.6. (See [16]) Let $a, b \in (W^* (G), E)$ be $D_G$-valued random variables with their supports $\mathbb{F}^+ (G : a)$ and $\mathbb{F}^+ (G : b)$. The $D_G$-valued random variables $a$ and $b$ are free over $D_G$ in $(W^* (G), E)$ if $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct. □

2. Diagonal Compressed Random Variables in $(W^* (G), E)$

Let $G$ be a countable directed graph and $(W^* (G), E)$, the graph $W^*$-probability space over the diagonal subalgebra $D_G$. In [18], we observed the diagonal compressed $D_G$-valued free probability on $(W^* (G), E)$. Fix a finite subset $V = \{v_1, ..., v_N\}$ of the vertex set $V(G)$ and define the diagonal compression $P_V : W^* (G) \to W^* (G)$ by

$$P_V (a) = L_{v_1} a L_{v_1} + ... + L_{v_N} a L_{v_N} \in W^* (G), \forall a \in (W^* (G), E).$$

If the given subset $V$ is a singleton set, then we will call this diagonal compression the vertex-compression. Let $a \in (W^* (G), E)$ be a $D_G$-valued random variable having its expression
and hence we can get Theorem 2.1. of Section 1.5, we can get the following theorems; the diagonal compressed random variables (See [18]). Also, by little modification over

\[ a = \sum_{v \in V(G : a)} p_v L_v + \sum_{l \in FP_*(G : a)} (p_l L_l + p_l^* L_l^*) + \sum_{w \in FP_*^c(G : a), w \in \{1, *\}} p_w L_w^{u_w}. \]

Then we have the following Fourier-like expression of the diagonal compressed random variable \( P_V(a) \) of \( a \) by \( V \);

\[ P_V(a) = \sum_{v \in V(G : P_V(a))} p_v L_v + \sum_{w \in FP_*(G : P_V(a))} (p_w L_w + p_w^* L_w^*) + \sum_{w \in FP_*^c(G : P_V(a)), w \in \{1, *\}} p_w L_w^{u_w}, \]

with

\[ V(G : P_V(a)) = V \cap V(G : a), \]

\[ FP_*(G : P_V(a)) = (\cup_{j=1}^N \text{loop}_{v_j}(G : a)) \cap FP_*(G : a) \]

and

\[ FP_*^c(G : P_V(a)) = (\cup_{j=1}^N \text{loop}_{v_j}(G : a)) \cap FP_*^c(G : a), \]

where

\[ \text{loop}_{v_j}(G : a) = \{ l \in \text{loop}(G : a) : l = v_j l v_j \}, \]

for \( j = 1, ..., N \).

Hence we can apply all amalgamated free probability information on this diagonal compressed case. In particular, we can compute the \( D_G \)-valued moments and \( D_G \)-valued cumulants of the diagonal compressed random variables like Section 1.3 and hence we can get \( D_G \)-valued moment series and \( D_G \)-valued R-transforms of the diagonal compressed random variables (See [18]). Also, by little modification of Section 1.5, we can get the following theorems;

**Theorem 2.1.** (See [18]) Let \( a, b \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( V = \{ v_1, ..., v_N \} \) be a finite subset of the vertex set \( V(G) \). Let \( P_V(a) \) and \( P_V(b) \) be the diagonal compressed random variable of \( a \) and \( b \) by \( V \) in \((W^*(G), E)\), respectively. Then

1. If \( a \) and \( b \) are free over \( D_G \) in \((W^*(G), E)\), then \( P_V(a) \) and \( P_V(b) \) are free over \( D_G \) in \((W^*(G), E)\).

2. If \( a \) and \( b \) satisfy that

\[ (V \cap V(G : a)) \cap (V \cap V(G : b)) = \emptyset \]

and

\[ \text{loop}_{v_i}(G : a) \cap \text{loop}_{v_j}(G : b) = \emptyset, \]

for all choices \((i, j) \in \{1, ..., N\}^2 \), then \( P_V(a) \) and \( P_V(b) \) are free over \( D_G \) in \((W^*(G), E)\). \( \forall \)
Theorem 2.2. (See [18]) Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable and let $V_1 = \{v_1^{(1)}, \ldots, v_{N_1}^{(1)}\}$ and $V_2 = \{v_1^{(2)}, \ldots, v_{N_2}^{(2)}\}$ be finite subsets of the vertex set $V(G)$. Suppose that

$$\text{loop}_{v_1^{(1)}}(G : a) \cap \text{loop}_{v_2^{(2)}}(G : a) = \emptyset,$$

for all choices $(i, j) \in \{1, \ldots, N_1\} \times \{1, \ldots, N_2\}$. Then the corresponding diagonal compressed random variables $P_{V_1}(a)$ and $P_{V_2}(a)$ of $a$ by $V_1$ and $V_2$ are free over $D_G$ in $(W^*(G), E)$. □

Therefore, we can again characterize the $D_G$-freeness of the diagonal compressed random variables in $(W^*(G), E)$ by the subsets of the free semigroupoid $F^+(G)$ of the graph $G$. Also, we can get the diagonal compressed R-transform calculus like in Section 1.6.

3. Off-Diagonal Random Variables in $(W^*(G), E)$

Throughout this chapter, let $G$ be a countable directed graph and $F^+(G)$, the free semigroupoid of $G$ and let $(W^*(G), E)$ be the graph $W^*$-probability space over the diagonal subalgebra $D_G$. In this chapter, we will consider the off-diagonal compressed random variables in $(W^*(G), E)$ over $D_G$. Let’s fix $v_1 \neq v_2$ in $V(G)$ and let $L_{v_1}$ and $L_{v_2}$ be the corresponding projections in $(W^*(G), E)$. Define a subset $FP_{v_1}^{v_2}(G)$ of $FP(G)$ by

$$FP_{v_1}^{v_2}(G) \overset{def}{=} \{w \in FP(G) : w = v_1 v_2\},$$

for $v_1, v_2 \in V(G)$.

Definition 3.1. Let $v_1 \neq v_2 \in V(G)$ be given. For any $D_G$-valued random variable $a \in (W^*(G), E)$, define the $(v_1, v_2)$-off-diagonal compressed random variable of a (in short $(v_1, v_2)$-compressed random variable of $a$) $v_1 a v_2$ by

$$v_1 a v_2 \overset{def}{=} L_{v_1} a L_{v_2} \in (W^*(G), E).$$

Let $a \in (W^*(G), E)$ be an arbitrary $D_G$-valued random variable having the following Fourier expansion,

$$a = \sum_{v \in V(G)} p_v L_v + \sum_{l \in FP_1(G)} (p_l L_l + p_l^* L_l^*) + \sum_{w \in FP_2^{(a)}(G), u \in \{1, \ast\}} p_w L_w^u.$$

We will denote
\[ a_d = \sum_{v \in V(G; a)} p_v L_v, \quad a_{(s)} = \sum_{l \in FP_+(G; a)} (p_l L_l + p_{l^1} L_{l^1}) \]

and

\[ a_{(non-s)} = \sum_{w \in FP_+(G; a), u \in \{1, *\}} p_w L_w^u, \]

for the given \( D_G \)-valued random variable \( a \). Thus the \((v_1, v_2)\)-compressed random variable \( v_1 a v_2 \) is determined by

\[ v_1 a v_2 = L_{v_1} a L_{v_2} = L_{v_1} a_d L_{v_2} + L_{v_1} a_{(s)} L_{v_2} + L_{v_1} a_{(non-s)} L_{v_2} = 0_D + L_{v_1} a_{(s)} L_{v_2} + L_{v_2} a_{(non-s)} L_{v_2}. \]

By definition, we have the following partition of \( FP(G : a) \), for the given random variable \( a \in (W^*(G), E) \):

\[ \{ l = v_1 l v_2 : l \in FP_+(G : a) \} = FP_+(G : a) \cap FP_{v_1}^{v_2}(G) \]

and

\[ \{ w = v_1 w v_2 : w \in FP_+(G : a) \} = FP_+(G : a) \cap FP_{v_1}^{v_2}(G). \]

So, we have that:

**Lemma 3.1.** Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( v_1 \neq v_2 \in V(G) \) be the fixed vertices. Let \( v_2 a v_1 \in (W^*(G), E) \) be the \((v_1, v_2)\)-off-diagonal compressed random variable. Then

\[ v_1 a v_2 = \sum_{w = v_1 w v_2 \in FP(G; a)} p_w L_w + \sum_{w = v_1 w v_2 \in FP(G; a)} p_w L_w^*. \]

**Proof.** By the relation that

\[ L_w = L_{vw} = L_v L_w, \quad L_w = L_{wv} = L_w L_v \]

and

\[ L_w^* = L_{vv'}^* = L_v L_{w}^*, \quad L_{w}^* = L_{w' v}^* = L_{w}^* L_{v'}, \]

under the weak topology \((v, v' \in V(G))\), we have that

\[ L_w = L_{v vv'} = L_v L_w L_{v'} \]

and

\[ L_{w}^* = L_{v vv'}^* = L_{v'} L_{w}^* L_{v}, \]

whenever \( w = vv' \) is a non-loop finite path, for \( v, v' \in V(G) \). Thus, if \( w \in FP(G : a) \), then we have that
\[ L_{v_1} L_w L_{v_2} = \begin{cases} 
L_w & \text{if } w = v_1 w v_2 \\
0_{D_G} & \text{otherwise} 
\end{cases} \]

and

\[ L_{v_1} L^*_w L_{v_2} = \begin{cases} 
L^*_w & \text{if } w = v_2 w v_1 \\
0_{D_G} & \text{otherwise}. 
\end{cases} \]

Therefore, the \((v_1, v_2)\)-compressed random variable \(v_1 a v_2\) can be

\[ v_1 a v_2 = L_{v_1} a L_{v_2} = L_{v_1} \left( \sum_{w \in FP^+(G:a), u_w \in \{1, *\}} p_w L^w u_w \right) L_{v_2} \]

\[ = \sum_{w \in FP(G:a), u_w \in \{1, *\}} p_w \left( L_{v_1} L^w u_w L_{v_2} \right) \]

\[ = \sum_{w=v_1 v_2 \in FP(G:a)} p_w L_w + \sum_{w'=v_2 w' v_1 \in FP(G:a)} p_{w'} L^*_{w'}. \]

For the convenience, we introduce the following new notation:

**Notation** Let \(v_1 \neq v_2 \in V(G)\) be given as before and let \(a \in \langle W^*(G), E \rangle\) be a \(D_G\)-valued random variable. Define

\[ FP^v_{v_1}(G : a) \overset{\text{def}}{=} FP_s(G : a) \cap FP^v_{v_1}(G) \]

and

\[ FP^v_{v_2}(G : a) \overset{\text{def}}{=} FP_s(G : a) \cap FP^v_{v_2}(G). \]

**3.1. Off-Diagonal Compressed Moments and Cumulants.**

Let \(v_1 \neq v_2 \in V(G)\) be the fixed vertices. In this section, we will consider the moments and cumulants of the \((v_1, v_2)\)-off-diagonal compressed random variables in the graph \(W^*\)-probability space \((W^*(G), E)\), over the diagonal subalgebra \(D_G\). We have that

\[ v_1 a v_2 = \sum_{w=v_1 v_2 \in FP^v_{v_1}(G:a)} p_w L_w + \sum_{w'=v_2 w' v_1 \in FP^v_{v_2}(G:a)} p_{w'} L^*_{w'}. \]
So, to compute the $D_G$-valued moments and the $D_G$-valued cumulants of $(v_1, v_2)$-off-diagonal compressed random variables in $(W^*(G), E)$ is to compute the $D_G$-valued moments and the $D_G$-valued cumulants of the $D_G$-valued random variables $x \in (W^*(G), E)$ such that

$$x = \sum_{l_1=v_1, l_2=v_2} p_{l_1} L_{l_1} + \sum_{l_2=v_2, l_1=v_1} p_{l_2} L_{l_2}^*$$

in $(W^*(G), E)$.

Suppose that $a \in (W^*(G), E)$ is a $D_G$-valued random variable and assume that $FP_*(G : a) = \{w_1, w_2, \ldots\} \subset FP(G)$.

Then, in terms of $FP_*(G : a)$, the $D_G$-valued random variable $a$ has the following summands

$$p_{w_1} L_{w_1}, \quad p_{w_2} L_{w_2}^*, \quad p_{w_3} L_{w_3}, \quad \ldots,$$

where $p_{w_i}, p_{w_i^*} \in \mathbb{C}$. By the above observation, we have the following result:

**Theorem 3.2.** Let $v_1 \neq v_2 \in V(G)$ be the fixed vertices in the graph $G$ and let $x \in (W^*(G), E)$ be a $D_G$-valued random variable with its Fourier-like expression,

$$x = \sum_{l_1=v_1, l_2=v_2} p_{l_1} L_{l_1} + \sum_{l_2=v_2, l_1=v_1} p_{l_2} L_{l_2}^*.$$

Then the $n$-th moments and $n$-th cumulants of $x$ vanish, for all $n \in \mathbb{N}$.

**Proof.** (1) Let $n = 1$. Then the first moments and the first cumulants of the $D_G$-valued random variable $x$ vanish:

$$E(x) = k_1(x) = 0_{D_G},$$

since $V(G : x) = \emptyset$.

(2) Let $n > 1$ in $\mathbb{N}$. Then the $n$-th $D_G$-valued moments vanish; By Section 1.4, we have that the $n$-th moment of the $D_G$-valued random variable $x$ is

$$E(d_1 \ldots d_n x)$$

$$= \sum_{\pi \in NC(n)} \left( \sum_{(v^{(1)}, \ldots, v^{(n)}) \in \Pi_{j=1}^{n} V(G : d_j)} (\Pi_{j=1}^{n} q_{v^{(j)}}) \right).$$
(w_1, ..., w_n) \in F P(G : x)^n, w_j = x_j w_j, y_j, u_{w_j} \in \{1, \ast\}, l^w_{w_1, \ldots, w_n} \in LP^*_n \quad (\Pi_{j=1}^n p_{w_j})

$$
\left(\Pi_{j=1}^n \delta(v^{(j)}), x_j, y_j, u_{w_j}\right) E \left(L^{u_{w_1}} \cdots L^{u_{w_n}} \right),
$$

where $x_j, y_j \in \{v_1, v_2\}$ and where $d_j = \sum_{v^{(j)} \in V(G, d_j)} q_{v^{(j)}} L_{v^{(j)}} \in D_G$ are arbitrary, $j = 1, \ldots, n$, for all $n \in \mathbb{N}$, and where $LP^*_n$ is the lattice path model satisfying the $\ast$-axis-property (See Section 1.2). But to get the nonvanishing $n$-th cumulant of $x$, we need to have at least one summand of $d_1 a \ast d_n a, L^{u_1} \cdots L^{u_n} = L_v$, for some $v \in V(G)$. Equivalently, the lattice path $l^w_{w_1, \ldots, w_n}$ should have the $\ast$-axis-property (i.e., $l^w_{w_1, \ldots, w_n} \in LP^*_n$). To do that, at least, we need to have the nonempty $FP_x(G : x)$. But

$$
V(G) \not\subseteq F^+(G : a) \text{ and } FP_x(G : x) = \emptyset.
$$

Therefore,

$$
k_1(d_1 x) = 0_{D_G} = E(d_1 a)
$$

and

$$
k_n (a, \ldots, a) = k_n (a(a), \ldots, a(a)) = 0_{D_G}.
$$

Indeed, we have that

$$
FP(G : x) \subseteq FP_{v_1}^{a_1}(G) \cup FP_{v_2}^{a_2}(G).
$$

Moreover, if $w = v_1 w v_2 \in FP(G : x) \cap FP_{v_1}^{a_1}(G)$, then the $L_w$-term of $x$ exists but the $L_{w}^{\ast}$-term does not exist in the Fourier expansion of $x$. Similarly, if $w' = v_2 w v_1 \in FP(G : x) \cap FP_{v_2}^{a_2}(G)$, then the $L_{w'}^{\ast}$-term of $x$ exists but the $L_w$-term does not exist in the Fourier expansion of $x$. Therefore, each lattice path of $L^{u_1} \cdots L^{u_n}$ does not have the $\ast$-axis-property. Since all $n$-th $D_G$-valued cumulants of $x$ vanish, all $k$-th $D_G$-valued moments of $x$ vanish, by the Möbius inversion. ■

Consider the $D_G$-valued random variable

$$
a = L_v + L_{w_1} + L_{w_1}^{\ast} + L_{w_2}^{\ast},
$$

where $v \in V(G)$ and $w_1 \neq w_2 \in FP(G)$, with $w_1 = v_1 w_1 v_2$ and $w_2 = v_2 w_2 v_1$. Then the $(v_1, v_2)$-off-diagonal compressed random variable of the $D_G$-valued random variable $a$ is

$$
v_1 a_{v_2} = L_{w_1} + L_{w_2}^{\ast}.
$$

So, we have that

$$
FP_x(G : v_1 a_{v_2}) = \emptyset
$$

and
Therefore, the $n$-th moments and the $n$-th cumulants of $v_1 a v_2$ vanish, for all $n \in \mathbb{N}$.

3.2. Off-Diagonal Compressed $D_G$-Freeness.

Let $G$ be a countable directed graph and $\mathbb{F}^+(G)$, the free semigroupoid of $G$ and let $(W^* (G), E)$ be the corresponding graph $W^*$-probability space over the diagonal subalgebra $D_G$. In this section, we will consider the $D_G$-freeness of the off-diagonal compressed random variables. Throughout this section, let $v_1, v_2, v_3$ and $v_4$ be mutually distinct vertices in $V(G)$. Since $\mathcal{F}_{P}(G : v_1 a v_2) = \emptyset$, it has vanishing $D_G$-valued $n$-th moments and $n$-th cumulants, for all $n \in \mathbb{N}$. Therefore, automatically, the $D_G$-valued moments vanish, by [12].

**Proposition 3.3.** Let $a$ and $b$ be $D_G$-valued random variables in the graph $W^*$-probability space $(W^* (G), E)$ and let $v_1 \neq v_2$ be the given vertices in $V(G)$. If $a$ and $b$ have the diagram-distinct supports, then the $(v_1, v_2)$-off-diagonal compressed random variables $v_1 a v_2 \equiv L_{v_1} a L_{v_2}$ and $v_1 b v_2 \equiv L_{v_1} b L_{v_2}$ are free over $D_G$ in $(W^* (G), E)$.

**Proof.** By the diagram-distinctness of $\mathcal{F}_{P}(G : a)$ and $\mathcal{F}_{P}(G : b)$, $\mathcal{L}_{P}(G : a)$ and $\mathcal{L}_{P}(G : b)$ are diagram-distinct, too. Note that

\[
\mathcal{F}_{P}(G : v_1 a v_2) \subset \mathcal{L}_{P}(G : a)
\]

and

\[
\mathcal{F}_{P}(G : v_1 b v_2) \subset \mathcal{L}_{P}(G : b).
\]

Therefore, $v_1 a v_2$ and $v_1 b v_2$ are free over $D_G$.

Now, we will consider the other case ;

**Proposition 3.4.** Let $v_1, v_2, v_3$ and $v_4$ be the mutually distinct vertices in $V(G)$ and let $a \in (W^* (G), E)$ be a $D_G$-valued random variable. Then the $(v_1, v_2)$-off-diagonal compressed random variable $v_1 a v_2$ and the $(v_3, v_4)$-off-diagonal compressed random variable $v_3 a v_4$ are free over $D_G$ in $(W^* (G), E)$.

**Proof.** Since $v_1, \ldots, v_4$ are mutually distinct, the $(v_1, v_2)$-off diagonal compressed random variable $v_1 a v_2$ and the $(v_3, v_4)$-off-diagonal compressed random variable $v_3 a v_4$ have the diagram-distinct supports. Therefore, they are free over $D_G$ in $(W^* (G), E)$. |
Corollary 3.5. Let $v_1, v_2, v_3$ and $v_4$ be the given vertices in $V(G)$. Define two subsets

$$W^*(G)_{v_2} = L_{v_1}W^*(G)L_{v_2} \quad \text{and} \quad W^*(G)_{v_3} = L_{v_3}W^*(G)L_{v_4},$$

in the graph $W^*$-probability space $(W^*(G), E)$. Then these subsets are free over $D_G$ in $(W^*(G), E)$. □

We can regard $W^*(G)_{v_i v_j}$ as an off-diagonal block of $W^*(G)$.

4. Compressed Free Probability on $(W^*(G), E)$

Throughout this chapter, let $G$ be a countable directed graph and $F^+(G)$, the free semigroupoid of the graph $G$ and let $(W^*(G), E)$ be the graph $W^*$-probability space over the diagonal subalgebra $D_G$. In this chapter, we will consider the compressed random variable $PaP$ of the $D_G$-valued random variable $a \in (W^*(G), E)$ by the projection $P \in W^*(G)$. Let $v_1, ..., v_N$ be vertices in $V(G)$ and define the projection

$$P = L_{v_1} + ... + L_{v_N} \in W^*(G).$$

Then it is indeed a projection in $W^*(G)$. From now, fix the finite vertices $v_1, ..., v_N \in V(G)$ and the corresponding projection $P = L_{v_1} + ... + L_{v_N}$ in $W^*(G)$. Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable. Then naturally, we can construct the compressed random variable $PaP$ of $a$ by $P$. Then this compressed random variable is again a $D_G$-valued random variable in $(W^*(G), E)$.

Notice that the diagonal-compressed random variable $P_V(a)$ by the diagonal compression $P_V : W^*(G) \to W^*(G)$, for the fixed vertex-subset $V = \{v_1, ..., v_N\}$ (See Chapter 2 and [18]) and the compressed random variable $PaP$ are totally different in $(W^*(G), E)$. For example, if $a$ is a $D_G$-valued random variable in $(W^*(G), E)$, then the diagonal compressed random variable is

$$P_V(a) \overset{df}{=} L_{v_1}aL_{v_1} + ... + L_{v_N}aL_{v_N},$$

but the compressed random variable by the projection $P$ is

$$PaP = \left( \sum_{i=1}^{N} L_{v_i} \right) a \left( \sum_{j=1}^{N} L_{v_j} \right) = \sum_{(i,j) \in \{1, ..., N\}^2} L_{v_i}aL_{v_j}. $$
Therefore, we can say that the compressed random variable \( PaP \) of \( a \) by \( P \) satisfies that

\[
PaP = P_V(a) + \sum_{(i, j) \in \{1, \ldots, N\}^2, \ i \neq j} L_{v_i} a L_{v_j}.
\]

However, in this chapter, we will observe that \( P_V(a) \) and \( PaP \) have the same free probability information.

Again, remark that the compressed random variable \( PaP \) is the sum of diagonal compressed random variable \( P_V(a) \) and the (sum of \( D_G \)-free) off-diagonal compressed random variables \( L_{v_i} a L_{v_j} \ (i \neq j \in \{1, \ldots, N\}) \). Define

\[
P_V^c(a) = \sum_{(i, j) \in \{1, \ldots, N\}^2, \ i \neq j} L_{v_i} a L_{v_j}.
\]

**Proposition 4.1.** Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( P = \sum_{j=1}^N L_{v_j} \in W^*(G) \) be a projection, where \( V = \{v_1, \ldots, v_N\} \subset V(G) \) is the finite subset of \( V(G) \). Then the compressed random variable \( PaP \) of \( a \) by \( P \) is

\[
PaP = P_V(a) + P_V^c(a),
\]

where \( P_V(a) \) is the diagonal compressed random variable of \( a \) by \( V \) and

\[
P_V^c(a) = \sum_{(i, j) \in \{1, \ldots, N\}^2, \ i \neq j} L_{v_i} a L_{v_j}.
\]

In particular, \( P_V(a) \) and \( P_V^c(a) \) are free over \( D_G \) in \( (W^*(G), E) \).

**Proof.** By the previous discussion, the compressed random variable of \( a \) by the projection \( P \) satisfies that

\[
PaP = P_V(a) + P_V^c(a),
\]

where

\[
P_V(a) = L_{v_1} a L_{v_1} + \ldots + L_{v_N} a L_{v_N}
\]

and

\[
P_V^c(a) = \sum_{(i, j) \in \{1, \ldots, N\}^2, \ i \neq j} L_{v_i} a L_{v_j}.
\]

Then, by [18] and by Chapter 3,

\[
F^+(G : P_V(a)) \subset (V \cup \text{loop}(G))
\]

and

\[
F^+(G : P_V^c(a)) \subset \text{loop'}(G).
\]

Therefore, the supports of \( P_V(a) \) and \( P_V^c(a) \) are diagram-distinct and hence they are free over \( D_G \).
4.1. Amalgamated Moments and Cumulants of Compressed Random Variables.

Remark that the compressed random variable \( PaP \) of a \( D_G \)-valued random variable \( a \in (W^*(G), E) \) by the projection \( P = \sum_{j=1}^N L_{v_j} \in W^*(G) \) has the form of

\[
PaP = PV(a) + P^cV(a),
\]

where \( V = \{v_1, ..., v_N\} \subset V(G) \) is the finite subset. Furthermore, by the previous proposition, as \( D_G \)-valued random variables in \((W^*(G), E)\), the diagonal compressed part \( PV(a) \) of \( a \) and the off-diagonal compressed part \( P^cV(a) \) are free over \( D_G \) in \((W^*(G), E)\). Therefore, we can get the following result:

**Theorem 4.2.** Let \( V = \{v_1, ..., v_N\} \) be the finite subset of the vertex set \( V(G) \) and let \( P = \sum_{j=1}^N L_{v_j} \in W^*(G) \) be the corresponding projection. Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and \( PaP \), the compressed random variable of \( a \) by \( P \). Then the \( n \)-th cumulants of \( PaP \) is

\[
k_1(d_1PaP) = \sum_{v \in V \cap (V(G):d_1) \cap V(G:a)} (q_v p_v) L_v,
\]

and

\[
k_n \left( \frac{d_1PaP, ..., d_nPaP}{n \text{-times}} \right) = \sum_{(v^{(1)}, ..., v^{(n)}) \in \prod_{k=1}^n V(G:d_k)} (\prod_{j=1}^n q_{v^{(j)}})
\]

\[
\sum_{(w_1, ..., w_n) \in \left( \cup_{k=1}^n \text{loop}_{v^{(k)}} (G:a) \cap V(G:a) \right)} \mu_{w_1, ..., w_n} \delta_{v^{(k)}, x_k} \left( L_{w_1}^{u_{w_1}} ... L_{w_n}^{u_{w_n}} \right),
\]

for all \( n > 1 \) in \( \mathbb{N} \), where \( d_k = \sum_{v^{(k)} \in V(G:d_k)} q_{v^{(k)}} L_{v^{(k)}} \in D_G \) are arbitrary for \( k = 1, ..., n \).

**Proof.** Suppose we have the compressed random variable \( PaP \) of the \( D_G \)-valued random variable \( a \). Then
where $P_V$ is the diagonal compression and $P_V^c$ is the off-diagonal compression by $V \subset V(G)$. Clearly, we have the above first cumulant of $PaP$. Also, we have that

$$k_n (d_1 PaP, \ldots, d_n PaP)$$

$$= k_n (d_1 (P_V(a) + P_V^c(a)) + \ldots + d_n (P_V(a) + P_V^c(a)))$$

$$= k_n (d_1 P_V(a), \ldots, d_n P_V(a))$$

$$+ k_n (d_1 P_V^c(a), \ldots, d_n P_V^c(a))$$

by the $D_G$-freeness of $P_V(a)$ and $P_V^c(a)$

$$= k_n (d_1 P_V(a), \ldots, d_n P_V(a))$$

$$+ \sum_{(i,j) \in \{1, \ldots, N\}^2, i \neq j} k_n (d_1(L_{v_i}aL_{v_j}), \ldots, d_n(L_{v_i}aL_{v_j}))$$

$$= k_n (d_1 P_V(a), \ldots, d_n P_V(a)) + 0_{D_G}$$

by Section 3.1

$$= k_n (d_1 P_V(a), \ldots, d_n P_V(a)),$$

for all $n \in 2\mathbb{N}$. Therefore, by [18], we can get the above result. □

**Remark 4.1.** The above theorem simply shows that

$$k_n (d_1 (PaP), \ldots, d_n (PaP)) = k_n (d_1 P_V(a), \ldots, d_n P_V(a)),$$

for all $n \in \mathbb{N}$ and for any arbitrary $d_1, \ldots, d_n \in D_G$.

This says that the off-diagonal compressed part $P_V^c(a)$ does not affect to compute the $D_G$-valued cumulants of the compressed random variable $PaP$. We can conclude that the compressed random variable $PaP$ of the $D_G$-valued random variable by $P = \sum_{j=1}^N L_{v_j}$ and the diagonal compressed random variable $P_V(a)$ of the random variable $a$ by $V = \{v_1, \ldots, v_N\}$ have the same distributions and hence they have the same $D_G$-valued R-transforms.

### 4.2. $D_G$-Freeness of Compressed Random Variables.
In this section, we will consider the $D_G$-freeness of compressed random variables. In this section, we will consider the various conditions for the $D_G$-freeness of compressed random variables.

**Theorem 4.3.** Let $a, b \in (W^*(G), E)$ be a $D_G$-valued random variable and let $V = \{v_1, ..., v_N\}$ be a finite subset of the vertex set $V(G)$ and $P = \sum_{i=1}^{N} L_{v_i}$, the corresponding projection in $W^*(G)$. Let $PaP$ and $PbP$ be the compressed random variable of $a$ and $b$ by $P$ in $(W^*(G), E)$, respectively. If $a$ and $b$ satisfy that

$$(V \cap V(G : a)) \cap (V \cap V(G : b)) = \emptyset,$$

$$\text{loop}_{v_i}(G : a) \cap \text{loop}_{v_j}(G : b) = \emptyset,$$

and

$$\text{loop}_{v_i}(G : a) \cap \text{loop}_{v_j}(G : b) = \emptyset,$$

for all choices $(i, j) \in \{1, ..., N\}^2$, then $PaP$ and $PbP$ are free over $D_G$ in $(W^*(G), E)$.

**Proof.** By the previous section, we have that

$$PaP = P_V(a) + P^c_V(a) \quad \text{and} \quad PbP = P_V(b) + P^c_V(b),$$

where $P_V, P^c_V : W^*(G) \to W^*(G)$ are the diagonal compression and off-diagonal compression by $V$, respectively. Moreover, $P_V(a)$ (resp. $P_V(b)$) and $P^c_V(a)$ (resp. $P^c_V(b)$) are free over $D_G$ in $(W^*(G), E)$. By the assumption and by [18], $P_V(a)$ and $P_V(b)$ are free over $D_G$ in $(W^*(G), E)$. By the second condition $P^c_V(a)$ and $P^c_V(b)$ are free over $D_G$ in $(W^*(G), E)$. This shows that $\{P_V(a), P^c_V(a)\}$ and $\{P_V(b), P^c_V(b)\}$ are free over $D_G$ in $(W^*(G), E)$. So, $PaP$ and $PbP$ are free over $D_G$. □

**Theorem 4.4.** Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable and let $V_1 = \{v_1^{(1)}, ..., v_{N_1}^{(1)}\}$ and $V_2 = \{v_1^{(2)}, ..., v_{N_2}^{(2)}\}$ be finite subsets of the vertex set $V(G)$. Let $P$ and $Q$ be the corresponding projections of $V_1$ and $V_2$ in $W^*(G)$, respectively. Suppose that

$$(V_1 \cap V(G : a)) \cap (V_2 \cap V(G : a)) = \emptyset,$$

and

$$\text{loop}_{v_i^{(1)}}(G : a) \cap \text{loop}_{v_j^{(2)}}(G : a) = \emptyset,$$

for all choices $(i, j) \in \{1, ..., N_1\} \times \{1, ..., N_2\}$. Then the corresponding diagonal compressed random variables $PaP$ and $QaQ$ of $a$ by $P$ and $Q$ are free over $D_G$ in $(W^*(G), E)$.

**Proof.** By hypothesis and by [18], $P_{V_1}(a)$ and $Q_{V_2}(a)$ are free over $D_G$ in $(W^*(G), E)$, where $P_{V_1}, Q_{V_2} : W^*(G) \to W^*(G)$ are diagonal compressions by $V_1$ and $V_2$, respectively. □
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