SQUARE ROOTS OF NEARLY PLANAR GRAPHS

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Abstract. We prove that it is NP-hard to decide whether a graph is the square of a 6-apex graph. This shows that the square root problem is not tractable for squares of sparse graphs (or even graphs from proper minor-closed classes).

1. Introduction

Given a graph $G$, the square of $G$, denoted $G^2$, is the graph where $V(G^2) = V(G)$, and $uv \in E(G^2)$ if and only if the distance between $u$ and $v$ in $G$ is at most 2. For a graph $G$, a square root of $G$ is a graph $H$ such that $H^2 = G$. Note that a graph may have many possible square roots, for example, $K_5$ has both $C_5$ and $K_{1,4}$ as square roots.

Let $H$ be a fixed class of graphs. The problem we are interested in is:

$H$-square-root

Instance: A graph $G$.

Question: Is there a square root $H$ of $G$ such that $H \in H$?

There are some classes for which $H$-square-root is known to be in P. For instance, if $H$ is the class of bipartite graphs [6], outerplanar graphs [5], proper interval graphs [7] or the class of graphs with girth at least 6 [3], then $H$-square-root is in P. Further, it is possible to compute some cut vertices of square roots given the square, [1], so the $H$ problem is in P for classes of graphs such as trees, cacti, and block graphs. On the other hand, the problem is known to be NP-complete when $H$ is the class of all graphs [8] (despite a characterization of when a graph has a square root [9]), graphs with girth at least 5 [2], chordal graphs [7] and split graphs [10].

Observing that the hard graph classes above are relatively dense, it has been asked multiple times if the $H$-square-root problem is in P for sparse graph classes (for instance, see [5]). We show here that the answer is no. Recall that a graph $G$ is $k$-apex if there exists a set of at most $k$ vertices such that the deletion of these vertices results in a planar graph. Using a mild tweak on the reduction in [7] (showing the hardness of finding chordal square roots), we prove the following theorem:

Theorem 1.1. The $H$-square-root problem is NP-complete for the class $H$ of 6-apex graphs.

Actually, we prove the following slightly stronger result, showing the hardness of the following version of the problem.

Promise-$H$ square-root

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Instance: Graph $G$ that either has no square root, or has a square root belonging to $\mathcal{H}$.

Question: Does $G$ have a square root?

The difference here is that we are allowed not to deal with the graphs that have a square root, but no square root belonging to $\mathcal{H}$ (whereas $\mathcal{H}$-square-root has to answer NO for such instances).

**Theorem 1.2.** The promise-$\mathcal{H}$ square-root problem is NP-complete for the class $\mathcal{H}$ of 6-apex graphs.

Clearly, Theorem 1.2 implies Theorem 1.1. Let us remark that the class of 6-apex graphs is minor-closed. As such, it lies at the bottom of the bounded-expansion hierarchy of sparse graph classes [10]. Still, it is natural to ask whether the apex vertices are indeed necessary, or whether already the planar square-root problem is hard. We believe the latter is the case.

**Conjecture 1.3.** The $\mathcal{H}$-square-root problem is NP-complete for the class $\mathcal{H}$ of planar graphs.

2. **Set Splitting Preliminaries**

Before proving Theorem 1.1, we first show a variant of set splitting is NP-complete. Let $S$ be a set and $\mathcal{C}$ a collection of subsets of $S$. The *incidence graph* of $S$ and $\mathcal{C}$ is the bipartite graph $I$ with bipartition $(A,B)$ such that for every element $e \in S$, we have a vertex $x_e \in A$, and for every set $c \in \mathcal{C}$, we have a vertex $v_c \in B$, and $x_e v_c \in E(I)$ if and only if $e \in c$. We are interested in the following problem:

**Three parts planar set splitting**

**Instance:** A set $S$, a collection $\mathcal{C}$ of subsets of $S$ where all subsets have at least three elements, and such that the incidence graph of $S$ and $\mathcal{C}$ is planar.

**Question:** Is there a partition of $S$ into three sets $S_1, S_2, S_3$ such that for all $c \in \mathcal{C}$ there is an element of $c$ in $S_i$ for each $i \in \{1, 2, 3\}$?

We first prove this problem is NP-complete by a reduction from reducing planar 3-colouring, which is well-known to be NP-complete [4].

**Observation 2.1.** The three parts planar set splitting problem is NP-complete.

**Proof.** Let $G$ be an instance of planar 3-colouring. Let $G'$ be the graph obtained from $G$ by taking every edge $e = xy$ and adding a new vertex $z_{xy}$ adjacent to both $x$ and $y$. Observe that $G'$ is planar, and that $G'$ is 3-colourable if and only if $G$ is 3-colourable. Further note that for any new vertex $z_{xy}$, in any 3-colouring of $G'$, all three colours appear on the triangle $x, y, z_{xy}$.

Now let $S$ be the set of vertices of $G'$, and let $\mathcal{C}$ be the collection of triangles which contain a vertex in $V(G') - V(G)$. Notice that the incidence graph of $S$ and $\mathcal{C}$ is planar, since for any vertex in $V(G') - V(G)$, this vertex corresponds to a vertex of degree one in the incidence graph, and after removing all such vertices, we simply end up with a graph that is a subdivision of $G$. We claim that $G'$ (and hence $G$) has a 3-colouring if and only
if there is a partition of $S$ into three sets $S_1, S_2, S_3$ such that for each set $c \in \mathcal{C}$, there is an element of $c$ which belongs to $S_i$ for $i \in \{1, 2, 3\}$.

To see this, suppose $G'$ has a 3-colouring $f$. Let $S_i$ for $i \in \{1, 2, 3\}$ be the set of vertices which get colour $i$ in $f$. This is a partition of $S$, and for any triangle $T$ containing a vertex in $V(G') - V(G)$, $f$ colours the triangle with three distinct colours, hence we get a solution to the three parts planar set splitting problem. Conversely, if $S_1, S_2, S_3$ is a partition of $S$ such that for any set $c \in \mathcal{C}$, there is an element of $c$ in each of $S_1, S_2, S_3$, then simply colour the vertices of $G'$ with colour $i$ if the vertex lies in $S_i$. Since every edge of $G'$ is contained in a triangle belonging to $\mathcal{C}$, this is a proper 3-colouring of $G'$, completing the reduction. □

3. The reduction

Before jumping into the reduction, we first recall the “tail” structure which is frequently used in square root complexity results (see, for example, [2, 3, 5]).

Let $G$ be any graph. Let $N_G(v)$ denote the set of neighbours a vertex $v$ in $G$. An $X$-tail at a vertex $v \in V(G)$ consists of three vertices $v_1, v_2, v_3 \in V(G)$ such that $N_G(v) \neq \{v_2, v_3\}$, $N_G(v_1) = \{v_2, v_3\}$, $N_G(v_2) = \{v_1, v_3, v\}$, and $N_G(v_3) = \{v_1, v_2, v\} \cup X$, where $X$ is a subset of $N_G(v) \setminus \{v_2, v_3\}$. The following observation is well known and illustrates why tails are useful:

**Observation 3.1** ([3]). If $G$ is a graph and $v_1, v_2, v_3$ is an $X$-tail at $v$, then for any square root $H$ of $G$, we have that $N_H(v_1) = \{v_2\}$, $N_H(v_2) = \{v_1, v_3\}$, $N_H(v_3) = \{v, v_2\}$, and $N_H(v) = \{v_3\} \cup X$.

Now we give the reduction from the three part planar set splitting problem, which we have shown to be NP-complete in the previous section.

Let $S, \mathcal{C}$ be an instance of the three part planar set splitting problem, with $\mathcal{C} \neq \emptyset$. We construct a graph $G$ with the following vertex set:

- A vertex $x_s$ for each element $s \in S$,
- For each $c \in \mathcal{C}$, we add a vertex $x_c$, as well as three additional vertices $x^1_c, x^2_c, x^3_c$,
- Vertices $a_i, b_i, b^1_i, b^2_i, b^3_i$ for $i \in \{1, 2, 3\}$.

And edge set:

(i) For any two distinct elements $s, s' \in S$, we add the edge $x_s x_{s'}$.

(ii) For $i \in \{1, 2, 3\}$, and any element $s \in S$, we add the edges $a_i x_s$ and $b_i x_s$.

(iii) For $i \in \{1, 2, 3\}$ and for any set $c \in \mathcal{C}$, we add the edges $a_i x_c$ and $b_i x_c$.

(iv) For any two distinct sets $c_1, c_2 \in \mathcal{C}$, if $c_1 \cap c_2 \neq \emptyset$ then we add the edge $x_{c_1} x_{c_2}$.

(v) For each $c \in \mathcal{C}$, add edges $x^1_c x^2_c x^3_c, x^2_c x^3_c x^1_c, x^3_c x^1_c x^2_c$. Further, for each element $s \in c$, add the edges $x^3_c x_s$ and $x_s x_c$. Hence, $x^1_c, x^2_c, x^3_c$ is an $\{x_s : s \in c\}$-tail at $x_c$.

(vi) For $i, j \in \{1, 2, 3\}$, we add the edges $a_i b_j$, and the edges $b_i b_j$ if $i < j$.

(vii) For $i \in \{1, 2, 3\}$, we add the edges $b^1_i b^2_i, b^1_i b^3_i, b^2_i b^3_i, b^3_i b_i$, for every element $s \in S$, add the edge $b^i_s x_s$, and add the edge $b^i_s a_i$. Hence, $b^1_i, b^2_i, b^3_i$ is an $(\{x_s : s \in S\} \cup \{a_i\})$-tail at $b_i$. 
Proof. Because of the tail at $x_c$, Observation 3.1 implies that $N_H(x_c) = \{x_c^3\} \cup \{x_s : s \in c\}$ and that $x_c^3$ has no neighbors outside of the tail. Since $G$ has the edge $x_c a_i$, $x_c$ must be at distance two from $a_i$, and this implies that there is an edge $x_c a_i$ for some $s \in c$.

**Observation 3.3.** Let $H$ be a square root of $G$. For any element $s \in S$, the vertex $x_s$ is adjacent to at most one of $a_i$ for $i \in \{1, 2, 3\}$ in $H$.

**Proof.** Since $\{a_1, a_2, a_3\}$ is an independent set in $G$, no two of these vertices can have a common neighbor in $H$.

Combining the previous two observations, we obtain one implication of the reduction.

**Corollary 3.4.** If $G$ has a square root, then $S, C$ is a YES-instance of the three part planar set splitting problem.

**Proof.** Suppose that $H$ is a square root of $G$. For $i \in \{1, 2, 3\}$, let $S'_i = \{s \in S : x_s a_i \in E(H)\}$. By Observation 3.3 these sets are pairwise disjoint. Hence, $S_1 = S'_1$, $S_2 = S'_2$, and $S_3 = S \setminus (S'_1 \cup S'_2)$ is a partition of $S$ with $S'_3 \subseteq S_3$. Moreover, by Observation 3.2, for each $c \in C$ and $i \in \{1, 2, 3\}$, we have $c \cap S_i \neq \emptyset$.

We now need to show the converse.

**Observation 3.5.** If $S, C$ is a YES-instance of the three planar set splitting problem, then $G$ has a 6-apex square root.

**Proof.** Let $S_1, S_2, S_3$ be a partition of $S$ such that for each $c \in C$ and $i \in \{1, 2, 3\}$, we have that there is an element $s_{c,i} \in c \cap S_i$. Let us also choose $s_{c,4} \in c$ arbitrarily. Let $H$ be the graph with $V(H) = V(G)$ and with the following edges:

- The edges forced by the tails, that is, for each $c \in C$, the edges $x_c^1 x_c^2, x_c^2 x_c^3, x_c^3 x_c$, and $x_c x_s$ for each $s \in c$; and for $i \in \{1, 2, 3\}$, the edges $b_i^1 b_i^2, b_i^2 b_i^3, b_i^3 b_i$, and $b_i x_s$ for each $s \in S_i$.
- The edges $x_s a_i$ for each $i \in \{1, 2, 3\}$ and $s \in S_i$.

Note that $H - \{a_1, a_2, a_3, b_1, b_2, b_3\}$ is planar, as it is obtained from the incidence graph of $S, C$ by adding pendant paths corresponding to the tails. Hence, $H$ is 6-apex.

Now, $H^2$ contains the edges (i), since $x_s$ and $x_{s'}$ are both adjacent to $b_1$ in $H$. It contains the edges (ii), since $x_s b_i, a_s b_i \in E(H)$. It contains the edges (iii), since $a_s x_{c_{s,i}}, b_s x_{c_{s,i}}, x_{c_{s,i}} \in E(H)$. It contains the edges (iv), since if $s \in c_1 \cap c_2$, then $x_{c_1} x_s, x_{c_2} x_s \in E(H)$. It contains the edges (v), since these follow from the path $x_c^1 x_c^2 x_c^3$ and the edges $x_c x_s$ for $s \in c$. It contains the edges (vi), since for any $c \in C$, $a_x x_{s_{c,i}}, b_x x_{s_{c,i}}, b_x x_{s_{c,i}} \in E(H)$. Finally, it contains the edges (vii), since these follow from the path $b_i^1 b_i^2 b_i^3 b_i$ and the edges $b_i a_i$ and $b_i x_s$ for $s \in S$. Therefore, $G \subseteq H^2$. 

We claim that $G$ has a square root $H$ if and only if $S, C$ is a YES-instance of the three part planar set splitting problem, and if it has one, $H$ is 6-apex. We collect some basic observations about any square root of $G$ (if it exists).
To see that $G = H^2$, it suffices to show that the neighborhood of each vertex of $H$ induces a clique in $G$. For $i \in \{1, 2, 3\}$,

- $N_H(a_i) = S_i \cup \{b_i\}$ is a clique in $G$ covered by the edges (i) and (ii);
- $N_H(b_i) = S \cup \{a_i, b_i^3\}$ is a clique in $G$ covered by the edges (i), (ii), and (vii);
- for $j \in \{1, 2, 3\}$, $N_H(b_j^i)$ is a clique in $G$ of size at most two covered by the edges (vii);
- for $s \in S_i$, $N_H(x_s) = \{x_{c} : c \in \mathcal{C}, s \in c\} \cup \{a_i, b_1, b_2, b_3\}$ is a clique in $G$ covered by the edges (iii), (iv), and (vi).

Moreover, for $c \in \mathcal{C}$, $N_H(x_c) = \{x_s : s \in c\} \cup \{x_c^3\}$ is a clique in $G$ covered by the edges (i) and (v), and for $j \in \{1, 2, 3\}$, $N_H(x_c^j)$ is clique in $G$ of size at most two covered by the edges (v).

We can now prove our main result.

**Proof of Theorem 1.2.** Clearly, promise-$\mathcal{H}$ square-root problem is in NP, since we can guess a square root $H$ and easily check if $H^2 = G$.

Given an instance $S, \mathcal{C}$ of the three planar set splitting problem, we create the graph $G$ as described at the beginning of the section; clearly, this can be done in polynomial time. By Corollary 3.4 if $S, \mathcal{C}$ is a NO-instance, then $G$ has no square root; and by Observation 3.5 if it is a YES-instance, then $G$ has a 6-apex square root. Hence, this is a polynomial-time reduction from the the three planar set splitting problem to the promise-$\mathcal{H}$ square-root problem, showing the NP-completeness of the latter. □

References

[1] Guillaume Ducoffe. Finding cut-vertices in the square roots of a graph. In Hans L. Bodlaender and Gerhard J. Woeginger, editors, *Graph-Theoretic Concepts in Computer Science*, pages 234–248, Cham, 2017. Springer International Publishing.

[2] Babak Farzad and Majid Karimi. Square-root finding problem in graphs, A complete dichotomy theorem. *CoRR*, abs/1210.7684, 2012.

[3] Babak Farzad, Lap-chi Lau, Bang Van Le, and Ngoc Nguyen Tuy. Complexity of finding graph roots with girth conditions. *Algorithmica*, 62:38–53, 2012.

[4] Michael Garey and David Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. WH Freeman & Co. New York, NY, USA, 1979.

[5] Petr A. Golovach, Pinar Heggernes, Dieter Kratsch, Paloma T. Lima, and Daniël Paulusma. Algorithms for outerplanar graph roots and graph roots of pathwidth at most 2. *Algorithmica*, 81(7):2795–2828, jul 2019.

[6] Lap Chi Lau. Bipartite roots of graphs. *ACM Trans. Algorithms*, 2(2):178–208, apr 2006.

[7] Lap Chi Lau and Derek G. Corneil. Recognizing powers of proper interval, split, and chordal graphs. *SIAM Journal on Discrete Mathematics*, 18(1):83–102, 2004.

[8] Rajeev Motwani and Madhu Sudan. Computing roots of graphs is hard. *Discrete Applied Mathematics*, 54(1):81–88, 1994.
[9] A. Mukhopadhyay. The square root of a graph. *Journal of Combinatorial Theory, 2*:290–295, 1967.

[10] J. Nešetřil and P. Ossona de Mendez. *Sparsity (Graphs, Structures, and Algorithms)*, volume 28 of *Algorithms and Combinatorics*. Springer, 2012.

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