ON DERIVATION DEVIATIONS
IN AN ABSTRACT PRE-OPERAD

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Abstract. We consider basic algebraic constructions associated with an abstract pre-operad, such as a \(\cdash\)-algebra, total composition \(\bullet\), pre-coboundary operator \(\delta\) and \emph{tribraces} \(\{\cdot, \cdot, \cdot\}\). A derivation deviation of the pre-coboundary operator over the tribraces is calculated in terms of the \(\cdash\)-multiplication and total composition.

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Key words. Comp(osition), (pre-)operad, Gerstenhaber theory, cup, pre-coboundary, (tri)braces, derivation deviation.

1. Introduction and outline of the paper

We consider basic algebraic constructions associated with an abstract pre-operad \(C\) (Sec. 2), such as a \(\cdash\)-algebra (Sec. 3), total composition \(\bullet\) (Sec. 4), pre-coboundary operator \(\delta\) (Sec. 5) and tribraces \(\{\cdot, \cdot, \cdot\}\) (Sec. 6). Main result of the present paper is the Main Theorem (Sec. 6). By defining (see degree notations in Sec. 2) a derivation deviation (Definition 6.2) of the pre-coboundary operator \(\delta\) over the tribraces \(\{\cdot, \cdot, \cdot\}\),

\[
\mathrm{dev}_{\{\cdot, \cdot, \cdot\}} \delta (h \otimes f \otimes g) := \delta \{h, f, g\} - \{h, f, \delta g\} - (-1)^{|g|} \{h, \delta f, g\} - (-1)^{|g|+|f|} \{\delta h, f, g\},
\]

the Main Theorem states that in a the pre-operad \(C\) one has

\[
(-1)^{|g|} \mathrm{dev}_{\{\cdot, \cdot, \cdot\}} \delta (h \otimes f \otimes g) = (h \bullet f) \cdash g + (-1)^{|h|+|f|} f \cdash (h \bullet g) - h \bullet (f \cdash g).
\]

This formula (Main Theorem) modifies and generalizes Gerstenhaber’s Theorem 5 from [2]. One can see the appearance of the extra terms with \(\delta h\), \(\delta f\) and \(\delta g\) inside the tribraces. The Gerstenhaber formula is obtained when arguments in tribraces \(\{h, f, g\}\) are the Hochshild cocycles, i.e. if \(\delta h = \delta f = \delta g = 0\). As it was stated in [2, 14], on the Hochschild cochain level the extra terms with \(\delta h\), \(\delta f\) and \(\delta g\) also appear when modifying the Gerstenhaber formula. We work out these extra terms for an abstract pre-operad and give their interpretation via a derivation deviation of \(\delta\) over the tribraces.

In this paper, we do not assume associativity constraint (see Example 5.2).

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2. Pre-operad (composition system)

Let $K$ be a unital commutative associative ring, and let $C^n (n \in \mathbb{N})$ be unital $K$-modules. For homogeneous $f \in C^n$, we refer to $n$ as the degree of $f$ and often write (when it does not cause confusion) $f$ instead of deg $f$; for example, $(-1)^f := (-1)^n$, $C^f := C^n$ and $o_f := o_n$. Also, it is convenient to use the shifted (desuspended) degree $|f| := n - 1$. Throughout this paper, we assume that $\otimes := \otimes_K$.

**Definition 2.1.** A linear (right) pre-operad (composition system) with coefficients in $K$ is a sequence $C := \{C^n\}_{n \in \mathbb{N}}$ of unital $K$-modules (an $\mathbb{N}$-graded $K$-module), such that the following conditions hold:

1. For $0 \leq i \leq m - 1$ there exist partial compositions
   
   $$o_i \in \text{Hom} (C^m \otimes C^n, C^{m+n-1}), \quad |o_i| = 0.$$

2. For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the composition relations hold,
   
   $$(h \circ_i f) \circ_j g = \begin{cases} 
   (-1)^{|f|/|g|} (h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\
   h \circ_i (f \circ_{i-1} g) & \text{if } i \leq j \leq i + |f|, \\
   (-1)^{|f|/|g|} (h \circ_{j-|f|} g) \circ_i f & \text{if } i + f \leq j \leq |h| + |f|. 
   \end{cases}$$

3. There exists a unit $I \in C^1$ such that
   
   $$I \circ_0 f = f = f \circ_1 I, \quad 0 \leq i \leq |f|.$$

In the 2nd item, the first and third parts of the defining relations turn out to be equivalent.

**Remarks 2.2.** A pre-operad is also called a comp(osition) algebra or asymmetric operad or non-symmetric operad or non-$\Sigma$ operad. The concept of (symmetric) operad was formalized by May \[9\] as a tool for the theory of iterated loop spaces. Recent studies and applications can be found in [10].

Above we modified the Gerstenhaber comp algebra defining relations \[1\] by introducing the sign $(-1)^{|f|/|g|}$ in the defining relations of the pre-operad. The modification enables us to keep track of (control) sign changes more effectively. One should also note that (up to sign) our $o_i$ is Gerstenhaber’s $o_{i+1}$ from \[1\]; we use the original (non-shifted) convention from \[2\].

**Example 2.3** (endomorphism pre-operad \[2\]). Let $A$ be a unital $K$-module and $E^n_A := \text{End}^n_A := \text{Hom} (A \otimes^n, A)$. Define the partial compositions for $f \otimes g \in E^f_A \otimes E^g_A$ as

$$f \circ_i g := (-1)^{|g|/|f|} f \circ (\text{id}_A \otimes^i g \otimes \text{id}_A \otimes (f[-i])), \quad 0 \leq i \leq |f|.$$ 

Then $E_A := \{E^n_A\}_{n \in \mathbb{N}}$ is a pre-operad (with the unit $\text{id}_A \in E^1_A$) called the endomorphism pre-operad of $A$. A few examples (without the sign factor) can be found in \[3\] as well. We use the original indexing of \[2\] for the defining formulae.

**Example 2.4** (associahedra). A geometrical example of a pre-operad (without signs $(-1)^{−}$ in the defining relations) is provided by the Stasheff associahedra, which was first constructed in \[11\]. Quite a surprising realization of the associahedra as truncated simplices was discovered and studied in \[10\], \[12\].
Notations 2.5 (scope of a pre-operad). The scope of \((h \circ_i f) \circ_j g\) is given by
\[0 \leq i \leq |h|, \quad 0 \leq j \leq |f| + |h|.
\]
It follows from the defining relations of a pre-operad that the scope is a disjoint union of
\[
\begin{align*}
B &:= \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq |h|; \ 0 \leq j \leq i - 1\}, \\
A &:= \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq i \leq |h|; \ i \leq j \leq i + |f|\}, \\
G &:= \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq i \leq |h| - 1; \ i + f \leq j \leq |f| + |h|\}.
\end{align*}
\]
Note that the triangles \(B\) and \(G\) are symmetrically situated with respect to the parallelogram \(A\) in the scope \(BAG := B \sqcup A \sqcup G\). The (recommended and impressive) picture is left for a reader as an exercise.

Recapitulation 2.6. The defining relations of a pre-operad can be easily rewritten as follows:
\[
(h \circ_i f) \circ_j g = \begin{cases} 
(-1)^{|f||g|}(h \circ_j g) \circ_{i+|g|} f & \text{if } (i, j) \in B, \\
h \circ_i (f \circ_{j-i} g) & \text{if } (i, j) \in A, \\
(-1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } (i, j) \in G.
\end{cases}
\]
The first \((B)\) and third \((G)\) parts of the relations turn out to be equivalent.

3. Cup

In this section, we recollect from \cite{1} basic facts about cup-multiplication in an abstract pre-operad.

Definition 3.1 (cf. \cite{1, 2}). In a pre-operad \(C\), let \(\mu \in C^2\). Define \(\cup := \cup_\mu : C^f \otimes C^g \to C^{f+g}\) by
\[
f \cup g := (-1)^{|f|}(\mu \circ_0 f) \circ_f g \in C^{f+g}, \quad |\cup| = 1, \quad f \otimes g \in C^f \otimes C^g.
\]
Note that \(\circ_f := \circ_{\deg f}\) and \(|\cup| = 1\) means that \(\deg \cup = 0\). The pair \(\text{Cup } C := \{C, \cup\}\) is called a \(\cup\)-algebra of \(C\).

Example 3.2. For the endomorphism pre-operad (Example 2.3) \(\mathcal{E}_A\), one has
\[
f \cup g = (-1)^{|f|g}\mu \circ (f \otimes g), \quad \mu \otimes f \otimes g \in \mathcal{E}_A^{2} \otimes \mathcal{E}_A^{f} \otimes \mathcal{E}_A^{g}.
\]

Proposition 3.3. In a pre-operad \(C\), one has
\[
\mu \circ_0 f = (-1)^{|f|}f \cup I, \quad \mu \circ_1 f = -I \cup f, \quad f \cup g = -(-1)^{|f||g|}(\mu \circ_1 g) \circ_0 f.
\]
Proof. We have
\[
(-1)^{|f|}f \cup I = (-1)^{|f|+|f|}(\mu \circ_0 f) \circ_f I = \mu \circ_0 f, \quad -I \cup f = (\mu \circ_0 I) \circ_1 f = \mu \circ_1 f.
\]
Also, calculate
\[
f \cup g = (-1)^{|f|}(\mu \circ_0 f) \circ_f g = (-1)^{|f||g|+|f|}(\mu \circ_{f-|f|} g) \circ_0 f
\]
\[
= (-1)^{|f||g|+|f|+1}(\mu \circ_1 g) \circ_0 f = -(-1)^{|f||g|}(\mu \circ_1 g) \circ_0 f,
\]
which is the required formula. \(\square\)
Lemma 3.4. In a pre-operad $C$, the following composition relations hold:

$$(f \sim g) \circ_j h = \begin{cases} 
(-1)^{g|h}(f \circ_j h) \sim g & \text{if } 0 \leq j \leq |f|, \\
(f \sim (g \circ_{j-f} h)) & \text{if } f \leq j \leq |g| + f.
\end{cases}$$

Proof. Calculate, by using the defining relations of a pre-operad:

$$(f \sim g) \circ_j h = (-1)^f[(\mu \circ_0 f) \circ_f g] \circ_j h$$

$$= \begin{cases} 
(-1)^{f+|g|h}[(\mu \circ_0 f) \circ_j h] \circ_{f+|g|h} g & \text{if } 0 \leq j \leq |f|, \\
(-1)^f(\mu \circ_0 f) \circ_f (g \circ_{j-f} h) & \text{if } f \leq j \leq |g| + f,
\end{cases}$$

$$= \begin{cases} 
(-1)^{f+|g|h}[(\mu \circ_0 (f \circ_j h)) \circ_{f+|g|h} g & \text{if } 0 \leq j \leq |f|, \\
f \sim (g \circ_{j-f} h) & \text{if } f \leq j \leq |g| + f,
\end{cases}$$

$$= \begin{cases} 
(-1)^{f+|g|h}(f \circ_j h) \sim g & \text{if } 0 \leq j \leq |f|, \\
f \sim (g \circ_{j-f} h) & \text{if } f \leq j \leq |g| + f,
\end{cases}$$

which is the required formula. □

4. Total composition

Definition 4.1 (cf. [4, 3]). In a pre-operad $C$, the total composition $\bullet : C^f \otimes C^g \to C^{f+g-1}$ is defined by

$$f \bullet g := \sum_{i=0}^{|f|} f \circ_i g \in C^{f+g-1}, \quad \bullet = 0, \quad f \otimes g \in C^f \otimes C^g.$$ 

The pair $\text{Com} C := \{C, \bullet\}$ is called the composition algebra of $C$.

Theorem 4.2. In a pre-operad $C$, one has

$$(f \sim g) \bullet h = f \sim (g \bullet h) + (-1)^{|h|g}(f \bullet h) \sim g.$$ 

Proof. Use Lemma 3.4. Note that $|f \sim g| = f + g - 1$ and calculate,

$$(f \sim g) \bullet h = \sum_{i=0}^{f+g-1} (f \sim g) \circ_i h = \sum_{i=0}^{f-1} (f \sim g) \circ_i h + \sum_{i=f}^{f+g-1} (f \sim g) \circ_i h$$

$$= (-1)^{|h|g} \sum_{i=0}^{|f|} (f \circ_i h) \sim g + \sum_{i=f}^{f+g-1} f \sim (g \circ_{i-f} h)$$

$$= (-1)^{|h|g}(f \bullet h) \sim g + \sum_{i'=0}^{|g|} f \sim (g \circ_{i'} h)$$

$$= (-1)^{|h|g}(f \bullet h) \sim g + f \sim (g \bullet h),$$

which is the required formula. □
Remark 4.3. This theorem tells us that right translations in Com $C$ are (right) derivations of the $\bowtie$-algebra. It may be anticipated from Theorem 5 of [2] that the left translations in Com $C$ are not derivations of the $\bowtie$-algebra (see the Main Theorem in Sec. 6).

5. CUP AND A PRE-COBOUNDARY OPERATOR

Definition 5.1. In a pre-operad $C$, define a pre-coboundary operator $\delta_\mu$ by

$$-\delta_\mu f := [f, \mu] := f \cdot \mu - (-1)^{|f|}\mu \cdot f, \quad \mu \otimes f \in C^2 \otimes C^f.$$

Example 5.2. In the Gerstenhaber theory [2], $C$ is an endomorphism pre-operad and $\delta_\mu$ is the Hochschild coboundary operator with the property $\delta_\mu^2 = 0$, the latter is due to the associativity $\mu \cdot \mu = 0$.

In this paper, we do not assume the associativity constraint.

Proposition 5.3. In a pre-operad $C$, one has

$$-\delta_\mu f = f \cdot I + f \cdot \mu + (-1)^{|f|} I \bowtie f, \quad \mu \otimes f \in C^2 \otimes C^f.$$

Definition 5.4. The derivation deviation of $\delta_\mu$ over $\cdot$ is defined by

$$\text{dev}_\cdot \delta_\mu (f \otimes g) := \delta_\mu (f \cdot g) - f \cdot \delta_\mu g - (-1)^{|g|}\delta_\mu f \cdot g.$$

Theorem 5.5 (cf. [2]). In a pre-operad $C$, one has

$$(1-1)^{|g|} \text{dev}_\cdot \delta_\mu (f \otimes g) = f \cdot g - (-1)^{|f|} g \cdot f, \quad \mu \otimes f \otimes g \in C^2 \otimes C^f \otimes C^g.$$

Proof. Two proofs can be found in [3].

6. MAIN THEOREM AND GERSTENHABER’S METHOD

In this section, we calculate the derivation deviation of the pre-coboundary operator over tribraces by using the Gerstenhaber auxiliary variables method. The idea of the method has been illustrated in [2].

Definition 6.1 (Gerstenhaber tribraces (cf. [2, 3])). The Gerstenhaber tribraces $\{\cdot, \cdot, \cdot\}$ are defined as a double sum over the triangle $G$ by

$$\{h, f, g\} := \sum_{(i,j) \in G} (h \circ_i f) \circ_j g \in C^{h+f+g-2}, \quad |\{\cdot, \cdot, \cdot\}| = 0, \quad h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g.$$

Definition 6.2. The derivation deviation of $\delta := \delta_\mu$ over the tribraces $\{\cdot, \cdot, \cdot\}$ is defined by

$$\text{dev}_{\{\cdot, \cdot, \cdot\}} \delta (h \otimes f \otimes g) := \delta (h, f, g) - \{h, f, \delta g\} - (-1)^{|g|}\delta (h, f, g) - (-1)^{|g|+|f|}\{\delta h, f, g\}.$$

Main Theorem (cf. [2, 3, 4]). In a pre-operad $C$, one has

$$(1-1)^{|g|} \text{dev}_{\{\cdot, \cdot, \cdot\}} \delta (h \otimes f \otimes g) = (h \cdot f) \bowtie g + (-1)^{|h|} f \bowtie (h \cdot g) - h \cdot (f \bowtie g).$$
Notations 6.3 (auxiliary variables (cf. [3])). In a pre-operad \( C \), for \((i, j) \in G\) define
\[
\Gamma_{i+1,j+1} := (-1)^{|h| + |f| + |g|} I \sim ((h \circ_i f) \circ_j g) \\
- (-1)^{|f| + |g|} \sum_{s=0}^{i-1} ((h \circ_s \mu) \circ_{i+1} f) \circ_{j+1} g + (-1)^{|f| + |g|} (h \circ_i (I \sim f)) \circ_{j+1} g,
\]
\[
\Gamma'_{i+1,j+1} := + (-1)^{|g|} (h \circ_i (f \sim I)) \circ_{j+1} g \\
- (-1)^{|f| + |g|} \sum_{s=i+1}^{j-f} ((h \circ_s \mu) \circ_i f) \circ_{j+1} g + (-1)^{|g|} (h \circ_i f) \circ_j (I \sim g),
\]
\[
\Gamma''_{i+1,j+1} := + (h \circ_i f) \circ_j (g \sim I) \\
- (-1)^{|f| + |g|} \sum_{s=j-|f|+1}^{|h|} ((h \circ_s \mu) \circ_i f) \circ_j g - ((h \circ_i f) \circ_j g) \sim I.
\]

Lemma 6.4. In a pre-operad \( C \), for \((i, j) \in G\) one has
\[
\delta((h \circ_i f) \circ_j g) - (h \circ_i f) \circ_{j-1} \delta g - (-1)^{|g|} (h \circ_i \delta f) \circ_{j+1} g = \Gamma_{i+1,j+1} + \Gamma'_{i+1,j+1} + \Gamma''_{i+1,j+1}.
\]
Proof. See Appendix A. \(\Box\)

Notations 6.5 (truncated envelope of \( G' \)). Now, define a (finite) shifted Gerstenhaber triangle:
\[
G' := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq |h|; \ i + f \leq j \leq f + |h|\}.
\]
Its envelope is the triangle
\[
G'_{\text{env}} := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq i \leq h + 1; \ i + |f| \leq j \leq f + h\} = G' \cup \partial G'_{\text{env}}.
\]
The boundary of the envelope \( G'_{\text{env}} \) is evidently \( \partial G'_{\text{env}} = G'_{\text{env}} \setminus G' \). The truncated envelope \( G'_{\text{env}} \) of \( G' \) is defined by removing its vertices,
\[
G''_{\text{env}} := G'_{\text{env}} \setminus \{((f, 0); (f + h, 0); (h + 1, f + h))\} = G' \cup \partial G''_{\text{env}}.
\]
Then, evidently, \( G''_{\text{env}} \) as having six vertices, turns out to be a (finite) hexagon.

Lemma 6.6. In a pre-operad \( C \), for \(0 \leq i \leq |h|, \ i + f \leq j \leq f + |h|\) one has
\[
(-1)^{|f| + |g|} (\delta h \circ_i f) \circ_{j-1} g = \Gamma_{ij} + \Gamma'_{i+1,j} + \Gamma''_{i+1,j+1},
\]
by definition for \( \Gamma_{0j}, \Gamma_{i+1,j+1}, \) and \( \Gamma''_{i,j+h} \) (boundary values on \( \partial G''_{\text{env}} \)).
Proof. See Appendix B. \(\Box\)

Boundary Lemma. In a pre-operad \( C \), for \((i, j) \in \partial G''_{\text{env}}\) one has
\[
\Gamma_{0j} = (-1)^{g+|h|} f \sim (h \circ_{j-1} f) g, \quad f \leq j \leq f + |h|,
\]
\[
\Gamma'_{i+1,j} = (-1)^{|g|} h \circ_{i-1} (f \sim g), \quad 1 \leq i \leq h,
\]
\[
\Gamma''_{i,j+h} = (-1)^{g} (h \circ_{i-1} f) \sim g, \quad 1 \leq i \leq h,
\]
by definition for \( \Gamma_{0f}, \Gamma'_{h,h+|f|}, \) and \( \Gamma''_{1,f+h} \) (three vertex values).
Proof. See Appendix C. \(\Box\)
Proof of the Main Theorem (Gerstenhaber’s method). First note that
\[ \{h, \delta f, g\} = \sum_{i=0}^{|h|-1} \sum_{j=i+\delta f} (h \circ_i \delta f) \circ_j g = \sum_{i=0}^{|h|-1} \sum_{j=i+f} (h \circ_i \delta f) \circ_{j+1} g. \]

By using Lemma 6.4 we have
\[
\delta \{h, f, g\} - \{h, f, \delta g\} - (-1)^{|g|} \{h, \delta f, g\} = \sum_{i=0}^{|h|-1} \sum_{j=i+f} (\Gamma_{i+1,j+1} + \Gamma'_{i+1,j} + \Gamma''_{i+1,j+1})
\]
\[
= \sum_{i=1}^{\delta h} \sum_{j=i+f} (\Gamma_{ij} + \Gamma'_{ij} + \Gamma''_{ij}) = \sum_{G'} (\Gamma_{ij} + \Gamma'_{ij} + \Gamma''_{ij}).
\]

Now use Lemma 6.6 to see that
\[
(-1)^{|f|+|g|} \{\delta h, f, g\} = \sum_{i=0}^{|\delta h|-1} \sum_{j=i+f} (-1)^{|f|+|g|} (\delta h \circ_i f) \circ_j g,
\]
\[
= \sum_{i=0}^{|h|} \sum_{j=i+f} (\Gamma_{ij} + \Gamma'_{i+1,j} + \Gamma''_{i+1,j+1})
\]
\[
= \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma_{ij} + \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma'_{ij} + \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma''_{ij}
\]
\[
= \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma_{ij} + \sum_{j=f}^{f+|h|} \Gamma_{0j} + \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma'_{ij} + \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma''_{ij}
\]
\[
+ \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma''_{ij} + \sum_{i=1}^{f+|h|} \sum_{j=i+f} \Gamma''_{i,j+f+1}
\]
\[
= \sum_{G'} (\Gamma_{ij} + \Gamma'_{ij} + \Gamma''_{ij}) + \sum_{j=f}^{f+|h|} \Gamma_{0j} + \sum_{i=1}^{h} \sum_{i+|f|} \Gamma'_{ij} + \sum_{i=1}^{h} \sum_{i+f+1} \Gamma''_{ij}.
\]

One can easily see that the resulting boxed formula is a sum over the boundary \( \partial G'_{\Gamma} \) of the truncated envelope \( G'_{\Gamma} \) of \( G \). By cancelling the sums \( \sum_{G'} \) and using the Boundary Lemma, we finally obtain
\[
\text{dev}_{i,j} \delta (f \otimes g \otimes h) = -(-1)^{|g|+|h|} \sum_{j=f}^{f+|h|} f \sim (h \circ_{j-f} g)
\]
\[
- (-1)^{|g|} \sum_{i=1}^{h} h \circ_{i-1} (f \sim g) - (-1)^{|g|} \sum_{i=1}^{h} (h \circ_{i-1} f) \sim g
\]
By using the composition relations $\delta h, \delta f$
One can see the appearance of the extra terms with $[5, 14, 13]$. algebra
based nowadays on the pioneering concept of (cf. $[2, 14]$) Theorem 6.7
which is the required formula.

Theorem 6.7 (cf. $[2, 14]$). In a pre-operad $C$, one has

$$(-1)^{|g|} \text{dev}_{\{\cdots, \}} \delta (h \otimes f \otimes g) = [h, f] \vartriangleleft g + (-1)^{|h|} f \vartriangleleft [h, g] - [h, f \vartriangleleft g].$$

Proof. Combine the Main Theorem with Theorem 4.2.

Remarks 6.8. A well known first form of this theorem, found by Gerstenhaber $[2]$ for the Hochschild cocycles, can be seen as a starting point of the modern mechanical mathematics based nowadays on the pioneering concept of (homotopy) Gerstenhaber algebra $[2, 14, 13]$.

Our Main Theorem generalizes and modifies Gerstenhaber’s Theorem 5 from $[2]$. One can see the appearance of the extra terms with $\delta h, \delta f$ and $\delta g$ inside the tribraces. The Gerstenhaber formula is obtained when arguments in tribraces $\{h, f, g\}$ are the Hochschild cocycles, i.e. if $\delta h = \delta f = \delta g = 0$. As it was stated in $[2, 14]$, on the Hochschild cochain level the extra terms with $\delta f, \delta g$ and $\delta h$ also appear when modifying the Gerstenhaber formula. Our contribution is to work out these extra terms for an abstract pre-operad and give their interpretation via a derivation deviation of $\delta$ over the tribraces.

7. Appendix A

Proof of Lemma 6.4. First note that

$$-\delta_{\mu}((h \circ_{i} f) \circ_{j} g) = (-1)^{|h|+|f|+|g|} I \vartriangleleft ((h \circ_{i} f) \circ_{j} g)$$

$$+ \sum_{s=0}^{\text{max}(|h|, |f|, |g|)} ((h \circ_{i} f) \circ_{j} g) \circ s \mu + (h \circ_{i} f) \circ_{j} g \vartriangleleft I.$$

By using the composition relations

$$((h \circ_{i} f) \circ_{j} g) \circ s \mu = \begin{cases} (-1)^{|g|} ((h \circ_{i} f) \circ s \mu) \circ_{j+1} g & \text{if } 0 \leq s \leq j - 1, \\ (h \circ_{i} f) \circ_{j} (g \circ_{s-j} \mu) & \text{if } j \leq s \leq j + |g|, \\ (-1)^{|g|} ((h \circ_{i} f) \circ_{s-|g|} \mu) \circ_{j} g & \text{if } j + g \leq s \leq |h| + |f| + |g|, \end{cases}$$

cut the above sum in three pieces,

$$-\delta((h \circ_{i} f) \circ_{j} g) = (-1)^{|h|+|f|+|g|} I \vartriangleleft ((h \circ_{i} f) \circ_{j} g)$$

$$+ (-1)^{|g|} \sum_{s=0}^{j-1} ((h \circ_{i} f) \circ s \mu) \circ_{j+1} g + \sum_{s=j}^{j+|g|} (h \circ_{i} f) \circ_{j} (g \circ_{s-j} \mu)$$

$$+ (-1)^{|g|} \sum_{s=j+g}^{\text{max}(|h|, |f|, |g|)} ((h \circ_{i} f) \circ_{s-|g|} \mu) \circ_{j} g + ((h \circ_{i} f) \circ_{j} g) \vartriangleleft I.$$
Next, cut the first sum $\sum_{s=0}^{j-1} x \circ s \mu = x \circ s \mu = x \cdot \mu = (-1)^{|x|} I \circ x - \delta \mu x - x \circ I$, for $x = f, g$,

$$
\begin{align*}
\sum_{s=j}^{j+|x|} x \circ s \mu &= \sum_{s=0}^{|x|} x \circ s \mu = x \cdot \mu = (-1)^{|x|} I \circ x - \delta \mu x - x \circ I, \\
\sum_{s=i+|f|}^{j-1} h \circ s \mu &= \sum_{s=i+1}^{j-1} h \circ s \mu, \\
\sum_{s=j+|g|}^{j+|f|+|g|} (h \circ i f) \circ s \mu &= \sum_{s=j+1}^{j+|f|} (h \circ i f) \circ s \mu = (-1)^{|f|} \sum_{s=j+1}^{j+|f|} (h \circ s \mu) \circ i f \\
&= (-1)^{|f|} \sum_{s=j-|f|+1}^{j+|f|} (h \circ s \mu) \circ i f,
\end{align*}
$$

to obtain the required formula. □

8. Appendix B

8.1. Proof of Lemma 6.6. First note that

$$
\Gamma_{ij} + \Gamma_{i+1,j} + \Gamma_{i+1,j+1} = (-1)^{|h|+|f|+|g|} I \circ ((h \circ i-1 f) \circ j-1 g)
\begin{align*}
&- (-1)^{|f|+|g|} \sum_{s=0}^{i-2} ((h \circ s \mu) \circ i f) \circ j g + (-1)^{|f|+|g|} (h \circ i-1 (I \circ f)) \circ j g \\
&+ (-1)^{|g|} (h \circ i (f \circ I)) \circ j g - (-1)^{|f|+|g|} \sum_{s=i+1}^{j-1} ((h \circ s \mu) \circ i f) \circ j g \\
&+ (-1)^{|g|} (h \circ i f) \circ j-1 (I \circ g) + (h \circ i f) \circ j (g \circ I) \\
&- (-1)^{|f|+|g|} \sum_{s=j-|f|+1}^{j+|f|} ((h \circ s \mu) \circ i f) \circ j g - ((h \circ i f) \circ j g) \circ I.
\end{align*}
$$

We must compare it term by term with

$$
- (\delta h \circ i f) \circ j g = \left( (-1)^{|h|} I \circ h + \sum_{s=0}^{|h|} h \circ s \mu + h \circ I \right) \circ i f \circ j g
\begin{align*}
&= (-1)^{|h|} ((I \circ h) \circ i f) \circ j g + \sum_{s=0}^{i-2} ((h \circ s \mu) \circ i f) \circ j g + ((h \circ i-1 \mu) \circ i f) \circ j g \\
&+ ((h \circ i \mu) \circ i f) \circ j g + \sum_{s=i+1}^{j-|f|+1} ((h \circ s \mu) \circ i f) \circ j g + ((h \circ j-|f|-1 \mu) \circ i f) \circ j g
\end{align*}
$$
+ ((h ◦_{j} f) ◦_{i} f) ◦_{j} g + \sum_{s=j-|f|+1}^{0} ((h ◦_{s} μ) ◦_{i} f) ◦_{j} g + ((h ∼ I) ◦_{i} f) ◦_{j} g.

Now, recall the sign \((-1)^{|f|+|g|}\) and use composition relations to note the \textit{ground identities}

\[
((I ∼ h) \circ_{i} f) \circ_{j} g = ((I ∼ (h \circ_{i-1} f)) \circ_{j} g = I ∼ ((h \circ_{i-1} f) \circ_{j-1} g),
\]

\[
(h \circ_{i-1} μ) \circ_{i} f = h \circ_{i-1} (μ \circ_{1} f) = -h \circ_{i-1} (I ∼ f),
\]

\[
((h \circ_{j-1} f) \circ_{i} f) \circ_{j} g = (-1)^{|f|}((h \circ_{i} f) \circ_{j-1} μ) \circ_{j} g = (-1)^{|f|}(h \circ_{i} f) \circ_{j-1} (μ \circ_{1} g)
\]

\[
= (-1)^{|f|}(h \circ_{i} f) \circ_{j-1} (I ∼ g),
\]

\[
((h \circ_{j-1} f) \circ_{i} f) \circ_{j} g = (-1)^{|f|}((h \circ_{i} f) \circ_{j} μ) \circ_{j} g = (-1)^{|f|}(h \circ_{i} f) \circ_{j} (μ \circ_{0} g)
\]

\[
= (-1)^{|f|+|g|}(h \circ_{i} f) \circ_{j} (g ∼ I),
\]

\[
((h ∼ I) \circ_{i} f) \circ_{j} g = (-1)^{|f|}((h \circ_{i} f) ∼ I) \circ_{j} g = (-1)^{|f|+|g|}((h \circ_{i} f) \circ_{j} g) ∼ I,
\]

which prove the required formula.

\[\square\]

8.2. \textbf{Proposition/recapitulation.} In a pre-operad \(C\), for \((i, j) ∈ G′\) the auxiliary variables read

\[
Γ_{ij} = -(-1)^{|h|+|f|+|g|}(I ∼ h) \circ_{i} f) \circ_{j} g - (-1)^{|f|+|g|}\sum_{s=0}^{i-1}((h \circ_{s} μ) \circ_{i} f) \circ_{j} g,
\]

\[
Γ′_{ij} = -(-1)^{|f|+|g|}\sum_{s=i-1}^{j-1}((h \circ_{s} μ) \circ_{i-1} f) \circ_{j} g,
\]

\[
Γ''_{ij} = -(-1)^{|f|+|g|}\sum_{s=j-1}^{i}((h \circ_{s} μ) \circ_{i-1} f) \circ_{j-1} g - (-1)^{|f|+|g|}((h ∼ I) \circ_{i-1} f) \circ_{j-1} g.
\]

\[\text{Proof.}\] Use the \textit{ground identities} from the previous subsection 8.1 \(\square\)

9. \textbf{Appendix C}

\textbf{Proof of the Boundary Lemma.} First prove the

\textbf{Vertex Proposition.} In a pre-operad \(C\), one has

\[
Γ_{0,f+|h|} = (-1)^{g+|h|} f ∼ (h \circ_{|h|} g),
\]

\[
Γ′_{f} = (-1)^{|g|} h \circ_{0} (f \sim g),
\]

\[
Γ''_{h,f+h} = (-1)^{|g|} (h \circ_{|h|} f) \sim g.
\]

\[\text{Proof.}\] We calculate these vertex values in a standard way, by using Lemma 5.6. First calculate \(Γ_{0,f+|h|}\). Use Lemma 5.6 and \(Γ''_{1,f+h}\) from the Boundary Lemma to note that

\[
Γ_{0,f+|h|} + Γ′_{1,f+|h|} + Γ''_{1,f+h} = Γ_{0,f+|h|} - (-1)^{|f|+|g|}\sum_{s=0}^{i}((h \circ_{s} μ) \circ_{0} f) \circ_{f+|h|} g
\]
\( + (-1)^g (h \circ_0 f) \sim g. \)

We must compare it term by term with

\[-(\delta h \circ_0 f) \circ_{f+|h|} g = \left( \left( (-1)^{|h|} (I \sim h + \sum_{s=0}^{|h|} h \circ_s \mu + h \sim I) \circ_0 f \right) \circ_{f+|h|} g \right) \]

\[= (-1)^{|h|} ((I \sim h) \circ_0 f) \circ_{f+|h|} g + \sum_{s=0}^{|h|} \left( (h \circ_s \mu) \circ_0 f \circ_{f+|h|} g \right) \]

\[+ ((h \sim I) \circ_0 f) \circ_{f+|h|} g. \]

Now, recall the sign \((-1)^{|f|+|g|}\) and use composition relations to note that

\[(-1)^{|h|} ((I \sim h) \circ_0 f) \circ_{f+|h|} g = (-1)^{|h|+|f|h} ((I \circ_0 f) \sim h) \circ_{f+|h|} g \]

\[= (-1)^{|h|+|f|h} (f \sim h) \circ_{f+|h|} g = -(-1)^{|g|+|f|} \left[ (-1)^{|h|+|f|h} f \sim (h \circ_0 g) \right]. \]

\[((h \sim I) \circ_0 f) \circ_{f+|h|} g = (-1)^{|f|h} ((h \circ_0 f) \sim I) \circ_{f+|h|} g = (-1)^{|f|h} (h \circ_0 f) \sim (I \circ_0 g) \]

\[= (-1)^{|f|h} (h \circ_0 f) \sim g, \]

which lead one to the required formula for \(\Gamma_{0,f+|h|}.\)

Next calculate \(\Gamma_{\prime f}.\) Use Lemma \([\Sigma, \Omega]\) and \(\Gamma_{0,f}\) from the Boundary Lemma to note that

\[\Gamma_{0f} + \Gamma_{\prime f} + \Gamma_{\prime \prime \prime, f+1} = (-1)^{|f|h+|g|} f \sim (h \circ_0 g) + \Gamma_{\prime f} \]

\[- (-1)^{|f|h+|g|} \sum_{s=1}^{|h|} \left( (h \circ_s \mu) \circ_0 f \circ_f g - (-1)^{|f|h+|g|} ((h \sim I) \circ_0 f) \circ_f g. \right) \]

We must compare it term by term with

\[-(\delta h \circ_0 f) \circ_f g = \left( \left( (-1)^{|h|} (I \sim h + \sum_{s=0}^{|h|} h \circ_s \mu + h \sim I) \circ_0 f \right) \circ_f g \right) \]

\[= (-1)^{|h|} ((I \sim h) \circ_0 f) \circ_f g + \sum_{s=0}^{|h|} ((h \circ_s \mu) \circ_0 f) \circ_f g \]

\[+ \sum_{s=1}^{|h|} ((h \circ_s \mu) \circ_0 f) \circ_f g + (h \sim I) \circ_0 f) \circ_f g. \]

Now, recall the sign \((-1)^{|f|h+|g|}\) and use composition relations to note that

\[((I \sim h) \circ_0 f) \circ_f g = (-1)^{|f|h} ((I \circ_0 f) \sim h) \circ_f g = (-1)^{|f|h} (f \sim h) \circ_f g \]

\[= (-1)^{|f|h} f \sim (h \circ_0 g), \]

\[((h \circ_0 \mu) \circ_0 f) \circ_f g = (h \circ_0 (\mu \circ_0 f)) \circ_f g = h \circ_0 ((\mu \circ_0 f) \circ_f g) \]

\[= -(-1)^{|f|h+|g|} \left[ (-1)^{|g|} h \circ_0 (f \sim g) \right], \]

which lead one to the required formula for \(\Gamma_{\prime f}.\)

At last calculate \(\Gamma_{\prime \prime \prime, f+1}.\) Use Lemma \([\Sigma, \Omega]\) and \(\Gamma_{\prime f, f+|h|}\) from the Boundary Lemma to note that

\[\Gamma_{|h,f+|h|} + \Gamma_{\prime f+|h|} + \Gamma_{\prime \prime \prime, f+1} = -(-1)^{|f|h+|g|+|h|} ((I \sim h) \circ_0 f) \circ_{f+|h|} g \]
\[-(\delta h \circ_h f) \circ_{f+|h|} g = \left( (-1)^{|f|} (I \circ_h + \sum_{s=0}^{|h|} h \circ_s \mu + h \circ_I) \circ_{|h|} f \right) \circ_{f+|h|} g \]

\[= (-1)^{|h|} \left( (I \circ_h f) \circ_{f+|h|} g \right) + \sum_{s=0}^{|h|-1} ((h \circ_s \mu) \circ_{|h|} f) \circ_{f+|h|} g \]

Now, recall the sign \((-1)^{|f|+|g|}\) and use composition relations to note that

\[((h \circ_{|h|} \mu) \circ_{|h|} f) \circ_{f+|h|} g = (h \circ_{|h|} (\mu \circ_{0} f)) \circ_{f+|h|} g = h \circ_{|h|} ((\mu \circ_{0} f) \circ_{f} g) \]

\[= (-1)^{|f|} h \circ_{|h|} (f \circ_{f+|h|} g), \]

\[((h \circ_{|h|} \mu) \circ_{|h|} f) \circ_{f+|h|} g = (-1)^{|f|} ((h \circ_{|h|} f) \circ_{f+|h|} g) \]

\[= (-1)^{|f|} (h \circ_{|h|} f) \circ (I \circ_{0} g) = -(-1)^{|f|+|g|} \left( (-1)^{|g|} (h \circ_{|h|} f) \circ_{f+|h|} g \right), \]

which lead one to the required formula for \(\Gamma_{h,f+|h|}'.

Now, when the vertex values found, we can accomplish the proof of the Boundary Lemma. First calculate \(\Gamma_{0,j}'). Use Lemma \[\ref{lemma_6.6}\] to note that

\[\Gamma_{0,j} + \Gamma'_{1,j} + \Gamma''_{1,j+1} = \Gamma_{0,j} - (-1)^{|f|+|g|} \sum_{s=0}^{j-f} ((h \circ_s \mu) \circ_{0} f) \circ_{j} g \]

\[-(-1)^{|f|+|g|} \sum_{s=j-|f|}^{|h|} ((h \circ_s \mu) \circ_{0} f) \circ_{j} g - (-1)^{|f|+|g|} ((h \circ_{I} f) \circ_{j} g). \]

We must compare it term by term with

\[-(\delta h \circ_{0} f) \circ_{j} g = \left( (-1)^{|h|} (I \circ_h + \sum_{s=0}^{|h|} h \circ_s \mu + h \circ_I) \circ_{|h|} f \right) \circ_{j} g \]

\[= (-1)^{|h|} ((I \circ_h g) \circ_{j} f) + \sum_{s=0}^{j-f} ((h \circ_s \mu) \circ_{0} f) \circ_{j} g \]

\[+ \sum_{s=j-|f|}^{|h|} ((h \circ_s \mu) \circ_{0} f) \circ_{j} g + ((h \circ_{I} f) \circ_{j} g). \]

Now, recall the sign \((-1)^{|f|+|g|}\) and use composition relations to note that

\[(-1)^{|h|} ((I \circ_{h} g) \circ_{j} f) = (-1)^{|h|+|f|} ((I \circ_{0} g) \circ_{j} f) \]

\[= (-1)^{|h|+|f|} (f \circ_{h} g) = -(-1)^{|f|+|g|} \left( (-1)^{|g|} (h \circ_{j-f} g) \right), \]

which leads one to the required formula for \(\Gamma_{0,j}'.\)
Next calculate \( \Gamma'_{i,i+|f|} \). Use Lemma 5.6 to note that
\[
\Gamma_{i-1,i+|f|} + \Gamma'_{i,i+|f|} + \Gamma''_{i,i+|f|} = -(-1)^{|f|+|g|+|h|}(I \circ h) \circ_{i-1} f \circ_{i+|f|} g
\]
\[
- (-1)^{|f|+|g|} \sum_{s=0}^{i-2} ((h \circ_s \mu) \circ_{i-1} f) \circ_{i+|f|} g + \Gamma'_{i,i+|f|}
\]
\[
- (-1)^{|f|+|g|} \sum_{s=i}^{|h|} ((h \circ_s \mu) \circ_{i-1} f) \circ_{i+|f|} g
\]
\[
- (-1)^{|f|+|g|} ((h \circ h) \circ_{i-1} f) \circ_{i+|f|} g.
\]
We must compare it term by term with
\[
-(\delta h \circ_{i-1} f) \circ_{i+|f|} g = \left( (-1)^{|h|} (I \circ h + \sum_{s=0}^{|h|} h \circ_s \mu + h \circ I) \circ_{i-1} f \right) \circ_{i+|f|} g
\]
\[
= (-1)^{|h|}(I \circ h) \circ_{i-1} f \circ_{i+|f|} g + \sum_{s=0}^{i-2} ((h \circ_s \mu) \circ_{i-1} f) \circ_{i+|f|} g
\]
\[
+ (h \circ_{i-1} \mu) \circ_{i-1} f \circ_{i+|f|} g + \sum_{s=i}^{|h|} ((h \circ_s \mu) \circ_{i-1} f) \circ_{i+|f|} g
\]
\[
+ ((h \circ h) \circ_{i-1} f) \circ_{i+|f|} g.
\]
Now, recall the sign \((-1)^{|f|+|g|}\) and use composition relations to note that
\[
((h \circ_{i-1} \mu) \circ_{i-1} f) \circ_{i+|f|} g = (h \circ_{i-1} (\mu \circ_0 f)) \circ_{i+|f|} g = h \circ_{i-1} ((\mu \circ_0 f) \circ f g)
\]
\[
= -(-1)^{|f|+|g|} (\sum_{s=0}^{|h|} (h \circ_s \mu) \circ_{i-1} f) \circ_{i+|f|} g + (h \circ h) \circ_{i-1} (f \circ g),
\]
which leads one to the required formula for \( \Gamma'_{i,i+|f|} \).

At last calculate \( \Gamma''_{i,f+h} \). Use Lemma 5.6 to note that
\[
\Gamma_{i-1,f+|h|} + \Gamma'_{i,f+|h|} + \Gamma''_{i,f+h} = -(-1)^{|f|+|g|+|h|}(I \circ h) \circ_{i-1} f \circ_{f+|h|} g
\]
\[
- (-1)^{|f|+|g|} \sum_{s=0}^{i-2} ((h \circ_s \mu) \circ_{i-1} f) \circ_{f+|h|} g
\]
\[
- (-1)^{|f|+|g|} \sum_{s=i-1}^{|h|} ((h \circ_s \mu) \circ_{i-1} f) \circ_{f+|h|} g + \Gamma''_{i,f+h}.
\]
We must compare it term by term with
\[
-(\delta h \circ_{i-1} f) \circ_{f+|h|} g = \left( (-1)^{|h|} (I \circ h + \sum_{s=0}^{|h|} h \circ_s \mu + h \circ I) \circ_{i-1} f \right) \circ_{f+|h|} g
\]
\[
= (-1)^{|h|}(I \circ h) \circ_{i-1} f \circ_{f+|h|} g + \sum_{s=0}^{i-2} ((h \circ_s \mu) \circ_{i-1} f) \circ_{f+|h|} g
\]
Now, recall the sign $(-1)^{|f|+|g|}$ and use composition relations to note that
\[
(h \circ_{i-1} f) \circ_{f+|h|} g = (-1)^{|f|} (h \circ_{i-1} f) \circ_{f+|h|} g
\]
\[
= (-1)^{|f|} (h \circ_{i-1} f) \circ_{f+|h|} g,
\]
which leads one to the required formula for $\Gamma''_{i,f+h}$. \hfill \square

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