RAMSEY NUMBERS AND MONOTONE COLORINGS

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Abstract. For positive integers \( N \) and \( r \geq 2 \), an \( r \)-monotone coloring of \( \binom{[1, \ldots, N]}{r} \) is a 2-coloring by \(-1\) and \(+1\) that is monotone on the lexicographically ordered sequence of \( r \)-tuples of every \((r+1)\)-tuple from \( \binom{[1, \ldots, N]}{r+1} \). Let \( R_{\text{mon}}(n; r) \) be the minimum \( N \) such that every \( r \)-monotone coloring of \( \binom{[1, \ldots, N]}{r} \) contains a monochromatic copy of \( \binom{[1, \ldots, n]}{r} \).

For every \( r \geq 3 \), it is known that \( R_{\text{mon}}(n; r) \leq \text{tow}_{r-1}(O(n)) \), where \( \text{tow}_h(x) \) is the tower function of height \( h-1 \) defined as \( \text{tow}_1(x) = x \) and \( \text{tow}_h(x) = 2^{\text{tow}_{h-1}(x)} \) for \( h \geq 2 \). The Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem imply \( R_{\text{mon}}(n; 2) = (n-1)^2 + 1 \) and \( R_{\text{mon}}(n; 3) = \left(\frac{2n-4}{3}\right) + 1 \), respectively. It follows from a result of Eliáš and Matoušek and Moshkovitz and Shapira. Using two geometric interpretations of monotone colorings, we show connections between estimating \( R_{\text{mon}}(n; r) \) and two Ramsey-type problems that have been recently considered by several researchers. Namely, we show connections with higher-order Erdős–Szekeres theorems and with Ramsey-type problems for order-type homogeneous sequences of points.

We also prove that the number of \( r \)-monotone colorings of \( \binom{[1, \ldots, N]}{r} \) is \( 2^{N^{r-1}/r!} \) for \( N \geq r \geq 3 \), which generalizes the well-known fact that the number of simple arrangements of \( N \) pseudolines is \( 2^{\Theta(N^2)} \).

1. Introduction

Let \( r \geq 2 \) be an integer. An ordered \( r \)-uniform hypergraph is a pair \( \mathcal{H} = (H, \prec) \) consisting of an \( r \)-uniform hypergraph \( H \) and a total ordering \( \prec \) of the vertices of \( H \). Let \( \mathcal{H}_1 = (H_1, \prec_1) \) and \( \mathcal{H}_2 = (H_2, \prec_2) \) be two ordered \( r \)-uniform hypergraphs. We say that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are isomorphic if there is an isomorphism between \( H_1 \) and \( H_2 \) that preserves the orders \( \prec_1 \) and \( \prec_2 \). The ordered hypergraph \( \mathcal{H}_1 \) is an ordered sub-hypergraph of \( \mathcal{H}_2 \) if \( \mathcal{H}_1 \) is a sub-hypergraph of \( \mathcal{H}_2 \) and \( \prec_1 \) is a suborder of \( \prec_2 \).

For a positive integer \( n \), we let \( K^r_n \) be the ordered complete \( r \)-uniform hypergraph on \( n \) vertices. That is, the edge set of \( K^r_n \) consists of all \( r \)-element subsets of the vertex set. We also use \( P^r_n \) to denote the monotone \( r \)-uniform path on \( n \) vertices. That is, \( P^r_n = (P^r_n, \prec) \) is an ordered \( r \)-uniform \( n \)-vertex hypergraph with edges formed by \( r \)-tuples of consecutive vertices in \( \prec \).

A coloring \( c \) of an ordered \( r \)-uniform hypergraph \( \mathcal{H} \) is a function that assigns some element from a finite set \( C \) to each edge of \( \mathcal{H} \). We say that \( \mathcal{H} \) is monochromatic in \( c \) if all edges of \( \mathcal{H} \) receive the same color via \( c \). If \( |C| = k \), then we call \( c \) a \( k \)-coloring of \( \mathcal{H} \).

The ordered Ramsey number \( R(\mathcal{H}) \) of an ordered \( r \)-uniform hypergraph \( \mathcal{H} \) is the minimum positive integer \( N \) such that for every 2-coloring \( c \) of \( K^r_N \) there is a sub-hypergraph of \( K^r_N \) that is monochromatic in \( c \) and isomorphic to \( \mathcal{H} \). It follows from Ramsey’s theorem that ordered Ramsey numbers always exist and are finite. There are examples of ordered graphs...
\( G = (G, \prec) \), for which ordered Ramsey numbers \( \overline{R}(G) \) differ significantly from the standard Ramsey numbers \( R(G) \). For example, there are ordered matchings \( M = (M, \prec) \) on \( n \) vertices for which \( R(M) \) is only linear in \( n \), while \( \overline{R}(M) \) grows superpolynomially in \( n \) \cite{1, 4}.

The motivation for studying the growth rate of the ordered Ramsey numbers \( \overline{R}(P^r_n) \) of monotone \( r \)-uniform paths comes from the classical paper by Erdős and Szekeres \cite{9}. In this paper, which was one of the starting points of both Ramsey theory and discrete geometry, Erdős and Szekeres independently reproved Ramsey’s Theorem and also proved two other important results in Ramsey theory, the Erdős–Szekeres Theorem about point sets in convex position and the Erdős–Szekeres Lemma on monotone subsequences. The latter results states that for every \( n \in \mathbb{N} \) there is a positive integer \( N(n) = (n - 1)^2 + 1 \) such that every sequence of \( N(n) \) numbers contains a nondecreasing or a nonincreasing subsequence of length \( n \). Moreover, the number \( N(n) \) is minimum possible, as there are sequences of \( (n - 1)^2 \) numbers without a monotone subsequence of length \( n \). It is easy to show that \( N(n) \leq \overline{R}(P^2_n) \). In fact, \( N(n) = \overline{R}(P^2_n) = (n - 1)^2 + 1 \) \cite{17}. The Erdős–Szekeres Theorem states that for every \( n \in \mathbb{N} \) there is a positive integer \( ES(n) \) such that every set of \( ES(n) \) points in the plane with no three collinear points contains \( n \) points that are vertices of a convex \( n \)-gon. This result is closely connected to the problem of estimating \( \overline{R}(P^3_n) \). Erdős and Szekeres \cite{9} showed \( ES(n) \leq \binom{2n-4}{n-2} + 1 \). We can again rather easily show that \( ES(n) \leq \overline{R}(P^3_n) \). The bound of Erdős and Szekeres then follows from the fact \( \overline{R}(P^3_n) = \binom{2n-4}{n-2} + 1 \) for every \( n \geq 2 \) \cite{13, 19}. Moreover, several other interesting geometric applications of estimates on \( \overline{R}(P^r_n) \) for \( r \geq 3 \) appeared, for example, variants of the Erdős–Szekeres Theorem for convex bodies \cite{13} or the higher-order Erdős–Szekeres theorems \cite{6}.

Given this motivation, the ordered Ramsey numbers \( \overline{R}(P^r_n) \) have been recently quite intensively studied \cite{6, 13, 17, 19} and their growth rate is nowadays well understood. For positive integers \( n \) and \( h \), let \( \text{tow}_h(n) \) be the tower function of height \( h - 1 \). That is, \( \text{tow}_1(n) = n \) and \( \text{tow}_h(n) = 2^{\text{tow}_{h-1}(n)} \) for every \( h \geq 2 \). Moshkovitz and Shapira \cite{19} showed that, for all positive integers \( n \) and \( r \) with \( r \geq 3 \),

\[
\overline{R}(P^r_{n+r-1}) = \text{tow}_{r-1}((2 - o(1))n).
\]

In fact, Moshkovitz and Shapira \cite{19} proved \( \overline{R}(P^r_{n+r-1}) = \rho_r(n) + 1 \), where \( \rho_r(n) \) is the number of line partitions of \( n \) of order \( r \) (see \cite{19} for definitions). For \( r = 3 \), this gives the exact formula \( \overline{R}(P^3_n) = \binom{2n-4}{n-2} + 1 \) and yields a new proof of the Erdős–Szekeres Theorem \cite{9}. Their coloring of \( K^3_n \) \( = (K^3_n, \prec) \) that gives \( \overline{R}(P^3_n) > \binom{2n-4}{n-2} \) satisfies the following transitivity property: if \( v_1 \prec v_2 \prec v_3 \prec v_4 \) are vertices of \( K^3_n \) such that \( c\{v_1, v_2, v_3\} = c\{v_2, v_3, v_4\} \), then all triples from \( \{v_1, v_2, v_3, v_4\} \) have the same color in \( c \).

More generally, for an integer \( r \geq 2 \), a 2-coloring \( c \) of \( K^r_n \) \( = (K^r_n, \prec) \) is called transitive if for every \( (r + 1) \)-tuple of vertices \( \{v_1, \ldots , v_{r+1}\} \) that satisfies \( v_1 < \cdots < v_{r+1} \) and \( c\{v_1, \ldots , v_r\} = c\{v_2, \ldots , v_{r+1}\} \) it holds that all \( r \)-tuples from \( \{v_1, \ldots , v_{r+1}\} \) have the same color in \( c \). For an ordered hypergraph \( H \), let \( \overline{R}(H) \) be the number \( \overline{R}(H) \) restricted to transitive 2-colorings. That is, \( \overline{R}(H) \) is the minimum positive integer \( N \) such that for every transitive 2-coloring \( c \) of \( K^r_N \), there is an ordered sub-hypergraph of \( K^r_N \) that is monochromatic in \( c \) and isomorphic to \( H \).

Note that \( \overline{R}(P^r_n) = \overline{R}(K^r_n) \) for all positive integers \( n \) and \( r \geq 2 \). We also remark that \( \overline{R}(P^r_n) < \overline{R}(K^r_n) \) for every \( r \geq 2 \) and every sufficiently large \( n \). For example, \( \overline{R}(P^3_n) = (n - 1)^2 + 1 \) \cite{17}, while \( \overline{R}(K^3_n) \) equals the standard Ramsey number \( R(K^3_n) \) of the complete \( r \)-uniform hypergraph on \( n \) vertices and thus \( \overline{R}(K^3_n) \) grows exponentially in \( n \) \cite{8}.

Perhaps surprisingly, the colorings of \( K^r_N \), which were found by Moshkovitz and Shapira \cite{19} and which give \( \overline{R}(P^r_{n+r-1}) > \rho_r(n) \), are not transitive for \( r > 3 \). Thus it is natural to ask the following question.
**Theorem 2.** All known lower bounds on $R(P_n^r)$ hold also for $R(P_n^r)$. They also mention a problem of deciding whether $R(P_n^r) = R(P_n^r)$ for all $n$ and $r$.

Clearly, $R(S(P_n^r)) \leq R(P_n^r)$ and, by [1], $R(S(P_n^r))$ grows at most as a tower of height $r-2$. This was also shown by Eliáš and Matoušek [6], who also proved that transitive colorings, as they admit various geometric interpretations; see Subsections 2.1 and 2.2 for examples.

**Problem 1.** [6] [19] What is the growth rate of $R(P_n^r)$?

Problem [1] was considered by Eliáš and Matoušek [6], who asked for better lower bounds on $R(P_n^r)$. Moshkovitz and Shapira [19] note that it might be very well possible that bounds comparable with the bounds for $R(P_n^r)$ hold also for $R(P_n^r)$. They also mention a problem of deciding whether $R(P_n^r) = R(P_n^r)$ for all $n$ and $r$.

In this paper, we settle Problem 1 by constructing, for all $n$ and $r$ with $r \geq 3$, transitive colorings $c_r$ of $K_{2n+1}$ with no monochromatic copy of $P_{2n+1}$, where $N \geq \text{towr}_{r-1}(1-o(1))$. In fact, we show that the colorings $c_r$ satisfy so-called monotonicity property, which is much more restrictive than the transitivity property and which admits several geometric interpretations.

### 1.1. Monotone colorings.

For a positive integer $n$, we write $[n]$ to denote the set $\{1, \ldots, n\}$. Let $S$ be a sequence of $n$ elements from some set. For a subset $\{i_1, \ldots, i_k\}$ of $[n]$, we use $S(i_1, \ldots, i_k)$ to denote the subsequence of $S$ obtained by deleting all elements from $S$ that are at position $i$ for some $j \in [k]$.

Let $r \geq 2$ be an integer. A 2-coloring $c$ of $K_{2n}^r = (K_{2n}^r, \prec)$ is called an $r$-monotone coloring of $K_{2n}^r$ if it assigns $-1$ or $+1$ to every edge of $K_{2n}^r$ such that the following monotonicity property is satisfied: for every sequence $S$ of $r+1$ vertices of $K_{2n}^r$ ordered by $\prec$ and all integers $i, j, k$ with $1 \leq i < j < k \leq r+1$, we have $c(S^{(i)}) \leq c(S^{(j)}) \leq c(S^{(k)})$ or $c(S^{(k)}) \geq c(S^{(j)}) \geq c(S^{(i)})$.

In other words, the monotonicity condition says that there is at most one change of a sign in the sequence $(c(S^{(r+1)}), \ldots, c(S^{(1)}))$. When referring to a 2-coloring that is $r$- monotone for some $r \geq 2$, we sometimes use the term monotone. We also abbreviate $-1$ and $+1$ by $-1$ and $+1$.

Note that every $r$-monotone coloring of $K_{2n}^r$ is a transitive 2-coloring of $K_{2n}^r$. For $r = 2$, transitive and 2-monotone colorings coincide. However, for $r \geq 3$, the monotonicity property is much more restrictive than the transitivity property, as $K_{2n+1}^r$ admits $2^r + 2$ transitive and only $2r + 2$ $r$-monotone colorings. An example of a transitive 2-coloring of $K_4^3$, which is not 3-monotone, is a function $c$ with $(c(\{1, 2, 3\}), c(\{1, 2, 4\}), c(\{1, 3, 4\}), c(\{2, 3, 4\})) = (-, +, +, -)$.

The notion of monotone colorings has been considered by several researchers [12] [18] [21] under different names. In some sense, monotone colorings can be viewed as more natural than transitive colorings, as they admit various geometric interpretations; see Subsections 2.1 and 2.2 for examples.

### 2. Our results

A monotone Ramsey number $R_{\text{mon}}(\mathcal{H})$ of an ordered $r$-uniform hypergraph $\mathcal{H}$ is the minimum positive integer $N$ such that for every $r$-monotone coloring $c$ of $K_{2n}^r$, there is a sub-hypergraph of $K_{2n}^r$ that is monochromatic in $c$ and isomorphic to $\mathcal{H}$.

Since every monotone coloring is transitive, we get $R_{\text{mon}}(\mathcal{P}_n^r) \leq R_{\text{trans}}(\mathcal{P}_n^r)$ and also $R_{\text{mon}}(\mathcal{P}_n^r) = R_{\text{mon}}(K_{2n}^r)$ for all $n$ and $r \geq 2$. It follows from [1] that $R_{\text{mon}}(\mathcal{P}_n^r) \leq \text{towr}_{r-1}(O(n))$. All known lower bounds on $R_{\text{mon}}(\mathcal{P}_n^r)$ are also true for $R_{\text{mon}}(\mathcal{P}_n^r)$. That is, we have $R_{\text{mon}}(\mathcal{P}_n^r) = R_{\text{trans}}(\mathcal{P}_n^r) = R(\mathcal{P}_n^r) = (n-1)^2 + 1$ [9], $R_{\text{mon}}(\mathcal{P}_n^3) = R_{\text{trans}}(\mathcal{P}_n^3) = R(\mathcal{P}_n^3) = (n-2)^2 + 1$ [9], and $R_{\text{mon}}(\mathcal{P}_n^4) = \text{towr}_3(\Theta(n))$ [22] for every $n \in \mathbb{N}$, as all the constructed transitive colorings in these results are actually monotone.

As our first main result, we prove an asymptotically tight lower bound on $R_{\text{mon}}(\mathcal{P}_n^r)$ for $r \geq 3$. Since $R_{\text{mon}}(\mathcal{P}_n^r) \leq R_{\text{trans}}(\mathcal{P}_n^r)$, this settles Problem 1.

**Theorem 2.** For positive integers $r$ and $n$ with $r \geq 3$, we have

$$R_{\text{mon}}(\mathcal{P}_{2n+r-1}^r) \geq \text{towr}_{r-1}(1-o(1))n.$$
For \( r \in \{3, 4\} \), the lower bounds from Theorem 2 asymptotically match the lower bounds obtained from results of Erdős and Szekeres [9] and Eliáš and Matoušek [6], respectively. Our construction is closer to the construction of Moshkovitz and Shapira [19], which they used to show the tight bound \( \overline{R}(P^r_{n+r-1}) \geq r! (n) + 1 \).

Our bounds on \( \overline{R}_{\text{trans}}(P^r_n) \) do not match the upper bounds on \( \overline{R}(P^r_n) \) exactly and thus deciding whether \( \overline{R}_{\text{trans}}(P^r_n) = \overline{R}(P^r_n) \) for all \( r \) and \( n \) remains an interesting open problem. It is even possible that \( \overline{R}_{\text{trans}}(P^r_n) = \overline{R}(P^r_n) \) for all \( r \) and \( n \).

Despite having several natural geometric interpretations, the monotone colorings seem to be quite unexplored. For example, we are not aware of any non-trivial estimate on the number of \( r \)-monotone colorings of \( K^r_n \) for \( r > 3 \). Here, we derive both upper and lower bounds for this number. Note that the bounds are reasonably close together, even with respect to \( r \).

**Theorem 3.** For integers \( r \geq 3 \) and \( n \geq r \), the number \( S_r(n) \) of \( r \)-monotone colorings of \( K^r_n \) satisfies

\[
2^{n-1/r^r} \leq S_r(n) \leq 2^{2^{r-2}n^{r-1}/(r-1)!}.
\]

As we will see in Subsection 2.2, Theorem 3 is a generalization of the well-known fact that the number of simple arrangements of \( n \) pseudolines is \( 2^9(n^2) \). This fact follows from Theorem 3 by setting \( r = 2 \). However, the constants in the exponents in the bounds from Theorem 3 are not the best known. Felsner and Valtr [11] showed that the number of simple arrangements of \( n \) pseudolines is at most \( 2^{9.657n^2} \), improving the previous bounds \( 20.792n^2 \) by Knuth [15] and \( 20.697n^2 \) by Felsner [10]. Felsner and Valtr [11] also proved the lower bound \( 2^{6.188n^2} \). All these bounds apply also to \( S_2(n) \).

In the rest of this section we use two geometric interpretations of monotone colorings to show connections between the problem of estimating \( \overline{R}_{\text{mon}}(P^r_n) \) and some geometric Ramsey-type problems that have been recently studied.

We note that besides the following two geometric interpretations of monotone colorings, there is also a third one, which was discovered by Ziegler [21]. He showed that monotone colorings can be interpreted as extensions of the cyclic arrangement of hyperplanes with a pseudohyperplane.

### 2.1. Higher-order Erdős–Szekeres theorems

Very recently, Miyata [18] introduced a new geometric interpretation of \((k + 2)\)-monotone colorings for \( k \in \mathbb{N} \), which are called *degree-\( k \) oriented matroids* in [15]. This interpretation concerns \( k \)-intersecting pseudoconfigurations of points (or \( k \)-pseudoconfigurations, for short), which are formed by a pair \((P, L)\) satisfying the following conditions. The set \( P = \{p_1, \ldots, p_n\} \) contains \( n \) points in the Euclidean plane ordered by their increasing \( x \)-coordinates and the set \( L \) is a collection of \( x \)-monotone Jordan arcs such that:

(i) for every \( l \in L \), there are at least \( k + 1 \) points of \( P \) lying on \( l \),

(ii) for every \((k + 1)\)-tuple of distinct points of \( P \), there is a unique curve \( l \) from \( P \) passing through each point of this \((k + 1)\)-tuple,

(iii) any two distinct curves from \( L \) cross at most \( k \) times\(^1\).

This notion naturally generalizes the concept of *generalized point sets* [14] (sometimes called *abstract order types*), which correspond to \( 1 \)-pseudoconfigurations. It also captures the essential combinatorial properties of configurations of points and graphs of polynomial functions, which is a setting considered by Eliáš and Matoušek [6] in their study of higher-order Erdős–Szekeres theorems.

A \( k \)-pseudoconfiguration \((P, L)\) of points is *simple* if each curve from \( L \) passes through exactly \( k + 1 \) points of \( P \); see Figure 1. If \((P, L)\) is simple, we let \( l_{i_1, \ldots, i_{k+1}} \) be the curve

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\(^1\)We count all crossings, not only those contained in \( P \).
Theorem 4. For $k, n \in \mathbb{N}$, there is a one-to-one correspondence between sign functions of simple $k$-pseudoconfigurations of $n$ points and $(k + 2)$-monotone colorings of $K_n^{k+2}$. The monotone coloring corresponding to a $k$-pseudoconfiguration $P$ is the sign function of $P$.

A subset $S$ of $P$ is $(k + 1)$st order monotone if the sign function of $(P, L)$ attains only or only + value on all of $(k + 2)$-tuples of $S$. Theorem 4 immediately gives the following corollary.

Corollary 5. For all positive integers $k$ and $n$, the number $\overline{\text{ES}}_{\text{mon}}(\mathcal{P}_n^{k+2})$ is the minimum positive integer $N$ such that every simple $k$-pseudoconfiguration of $N$ points contains a $(k + 1)$st order monotone subset of size $n$.

Generalizing the Erdős–Szekeres Theorem [9] to higher orders, Eliáš and Matoušek [5] introduced the following more restrictive setting in which, for every $l \in L$, $f_l$ is a function whose $(k + 1)$st derivative is everywhere non-positive or everywhere non-negative. A planar point set $P$ is in $(k + 1)$-general position if no $k + 2$ points of $P$ lie on the graph of a polynomial of degree at most $k$. By Newton’s interpolation, every $(k + 1)$-tuple of points from $P$ determines a simple $k$-pseudoconfiguration. Thus, in this setting, we can consider $(k + 1)$st order monotonicity with respect to the graphs of the polynomials of degree at most $k$. Let $\text{ES}_{k+1}(n)$ be the smallest positive integer $N$ such that every set of $N$ points in $(k + 1)$-general position contains a $(k + 1)$st order monotone subset of size $n$.

By Corollary 5, we have $\text{ES}_{k+1}(n) \leq \overline{\text{ES}}_{\text{mon}}(\mathcal{P}_n^{k+2})$ for all positive integers $k$ and $n$. It is known that this inequality is tight for $k = 1$ [9]. Eliáš and Matoušek [5] showed that $\text{ES}_2(n) = \omega_3(\Theta(n))$ and thus $\text{ES}_3(n)$ and $\overline{\text{ES}}_{\text{mon}}(\mathcal{P}_n^4)$ have asymptotically the same growth rate. They also asked about the growth rate of $\text{ES}_{k+1}(n)$ for $k > 2$. A related interesting open question is whether $\text{ES}_{k+1}(n)$ and $\overline{\text{ES}}_{\text{mon}}(\mathcal{P}_n^{k+2})$ are the same, at least asymptotically.

By Corollary 5, it suffices to show that the extremal configurations for $\overline{\text{ES}}_{\text{mon}}(\mathcal{P}_n^{k+2})$ can be ‘realized’ by graphs of polynomial functions of degree at most $k$. It is possible that the configurations obtained in the proof of Theorem 2 admit such realizations, which would solve the open problem of Eliáš and Matoušek about the growth rate of $\text{ES}_{k+1}(n)$. We hope to discuss this direction in future work.

2.2. Arrangements of pseudohyperplanes and order-type homogeneous point sets. Felsner and Weil [12] showed that, for every $r \geq 3$, there is a one-to-one correspondence
between r-monotone colorings of $K_n^r$, which they call $r$-signotopes, and arrangements of $n$ pseudohyperplanes in $\mathbb{R}^{d-1}$ that admit ‘sweeping’.

For an integer $d \geq 2$, a pseudohyperplane $H$ in $\mathbb{R}^d$ is a homeomorph of a hyperplane in $\mathbb{R}^d$ such that the two connected components of $\mathbb{R}^d \setminus H$ are homeomorphic to an open $d$-dimensional ball. Two pseudohyperplanes $H_1$ and $H_2$ cross, if $H_i$ intersects both components of $\mathbb{R}^d \setminus H_{i-1}$ for every $i \in \{1, 2\}$. An arrangement of pseudohyperplanes in $\mathbb{R}^d$ (or $d$-arrangement, for short) consists of pseudohyperplanes $H_1, \ldots, H_n$ in $\mathbb{R}^d$ such that any two pseudohyperplanes $H_i$ and $H_j$ intersect in a pseudohyperplane in $H_i \cong H_j \cong \mathbb{R}^{d-1}$ and they cross at their intersection. Moreover, for every $j \in [n]$, the intersections $H_i \cap H_j$, where $i \in [n] \setminus \{j\}$, form an arrangement of pseudohyperplanes in $H_j \cong \mathbb{R}^{d-1}$. A $d$-arrangement $A$ is simple if any $d+1$ pseudohyperplanes from $A$ have an empty intersection.

We assume that every $d$-arrangement $A$ of pseudohyperplanes $H_1, \ldots, H_n$ is normal, that is, $A$ is simple and is embedded in $\mathbb{R}^d$ in the following normalized way. Assume that $A$ is embedded in the hypercube $[0, 1]^d$ and, for $i \in [d-1]$, let $I_i$ be the $(d-i)$-dimensional subspace of $\mathbb{R}^d$ that contains the side of $[0, 1]^d$, which is obtained by setting the last $i$ coordinates to 0. We demand that $A \cap I_i$ is a $(d-i)$-arrangement of $n$ pseudohyperplanes. Moreover, the pseudohyperplanes in $A$ are labeled by increasing first coordinate at their intersection with $I_{d-1}$. The assumption that $A$ is embedded in $[0, 1]^d$ is only for convenience so that all intersections of $d$ pseudohyperplanes from $A$ are contained in $[0, 1]^d$. The reader may consider “spaces at infinity” instead by defining $I_i$ to be the $(d-i)$-dimensional affine subspace obtained by setting the last $i$ coordinates to some sufficiently small number.

A sign function of a normal $d$-arrangement $A$ of $n$ pseudohyperplanes $H_1, \ldots, H_n$ is a function $f: \binom{[n]}{d+1} \rightarrow \{-, +\}$ such that, for given $i_1 < \cdots < i_{d+1}$, $f(i_1, \ldots, i_{d+1}) = -$ if and only if the pseudoline $H_{i_1} \cap \cdots \cap H_{i_{d+1}}$, which is oriented away from $I_1$, intersects $H_{i_2}$ before $H_{i_2}$.

A normal $d$-arrangement $A$ is called a $C_d$-arrangement if the normal $(d-1)$-arrangement formed by $H \cap I_1$ for $H \in A$ has no + sign in its sign function. We note that every normal arrangement of pseudolines (that is, pseudohyperplanes in $\mathbb{R}^2$) is a $C_2$-arrangement, but this is no longer true for $C_d$-arrangements with $d \geq 3$. This is because, for $d \geq 3$, the arrangement induced by $A$ is not uniquely determined, while for $C_2$-arrangements this arrangement must be the “minimal one with respect to the sign function”. An example of a $C_2$-arrangement can be found in Figure 2.

![Figure 2](image)

**Figure 2.** A $C_2$-arrangement of four pseudolines. Here, the sign function assigns $-$ to the triple $\{1, 2, 3\}$ and $+$ to the triple $\{2, 3, 4\}$.

**Theorem 6.** If $d \geq 2$ and $n \in \mathbb{N}$, there is a one-to-one correspondence between sign functions of $C_d$-arrangements of $n$ pseudohyperplanes in $\mathbb{R}^d$ and $(d+1)$-monotone colorings of $K_n^{d+1}$. The monotone coloring corresponding to an arrangement $A$ is the sign function of $A$.

A subset $S$ of $A$ is order-type homogeneous if the sign function of $A$ attains only $-$ or only $+$ values on all of $(d+1)$-tuples of pseudohyperplanes from $S$. Theorem 6 gives the following corollary.
Corollary 7. For all positive integers \( d \geq 2 \) and \( n \), the number \( \Pi_{\text{mon}}(P_{d+1}^n) \) is the minimum positive integer \( N \) such that every \( C_d \)-arrangement of \( N \) pseudohyperplanes contains an order-type homogeneous subset of size \( n \).

An orientation of a \((d+1)\)-tuple of points \((p_1, \ldots, p_{d+1})\) with \( p_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{R}^d \) is defined as

\[
\text{sgn} \det \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & a_{2,1} & \cdots & a_{d+1,1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,d} & a_{2,d} & \cdots & a_{d+1,d}
\end{pmatrix}.
\]

A sequence of points from \( \mathbb{R}^d \), \( d \geq 2 \), is order-type homogeneous if all \((d+1)\)-tuples of points from this sequence have the same orientation. For positive integers \( n \) and \( d \geq 2 \), let \( \text{OT}_d(n) \) be the minimum positive integer \( N \) such that every sequence of \( N \) points from \( \mathbb{R}^d \) contains an order-type homogeneous subsequence with \( n \) points. Using geometric duality, the notion of order-type homogeneous sequence of points from \( \mathbb{R}^d \) transcribes to sequences of hyperplanes in \( \mathbb{R}^d \). Thus \( \text{OT}_d(n) \) is also the minimum positive integer \( N \) such that every sequence of \( N \) hyperplanes in \( \mathbb{R}^d \) contains an order-type homogeneous subsequence of size \( n \).

The function \( \text{OT}_d(n) \) was considered by many researchers [2][5][7][20]. Suk [20] showed that \( \text{OT}_d(n) \leq \text{tow}_d(O(n)) \). The results of Bárány, Matoušek, and Pór [2] and Eliáš, Matoušek, Roldán-Pensado, and Safernová [7] give an asymptotically matching lower bound \( \text{OT}_d(n) \geq \text{tow}_d(\Omega(n)) \). For \( d \geq 3 \), the arrangements obtained from their lower bound on \( \text{OT}_d(n) \) are not \( C_d \)-arrangements. A natural problem is to decide whether one can obtain similar lower bounds on \( \text{OT}_d(n) \) when restricted to \( C_d \)-arrangements of hyperplanes. Corollary 7 combined with Theorem 2 suggests that this might be true, as we obtain such bounds for \( C_d \)-arrangements of pseudohyperplanes for every \( d \geq 2 \).

3. Proof of Theorem 2

Here, for positive integers \( n \) and \( r \) with \( r \geq 3 \), we construct an \( r \)-monotone coloring \( c_r \) of \( K_n \) with no monochromatic copy of \( P_{2n+r-1}^r \) and with \( N \geq \text{tow}_{r-1}(1 - o(1))n \). First, we describe the construction of \( c_r \) and show that \( c_r \) contains no long monochromatic monotone \( r \)-uniform paths. Then we prove that the coloring \( c_r \) satisfies the monotonicity property.

Let us start with a brief overview of the construction of the coloring \( c_r \). It is carried out iteratively with respect to \( r \). For every positive integer \( n \), we will construct sets \( F_r(n) \) with \( r \geq 1 \) such that \( |F_1(n)| = 2 \), \( |F_2(n)| = 2n \), and \( |F_r(n)| = 2^{F_{r-1}(n)/2} \) for \( r \geq 3 \). The 2-coloring \( c_r \) will have \( F_r(n) \) as its vertex set. We will have a partition of \( F_r(n) \) into sets \( F^-_r(n) \), \( F^+_r(n) \), and a bijection \( \sigma_r : F^-_r(n) \rightarrow F^+_r(n) \). Elements \( A, B \in F_r(n) \) will be called equivalent, written \( A \equiv_r B \), if \( A = B \), \( A = \sigma_r(B) \), or \( B = \sigma_r(A) \). We say that elements from \( F^-_r(n) \) and \( F^+_r(n) \) have type \(-\) and \(+\), respectively. We will also define two orders \( <_r \) and \( \prec_r \); \( <_r \) will be a linear order on \( F_r(n) \) and \( \prec_r \) will be a linear order on equivalence classes under the equivalence relation \( \equiv_r \). In \( <_r \), all elements of \( F^-_r(n) \) will precede all elements of \( F^+_r(n) \), and the bijection \( \sigma_r \) will be order-reversing. Moreover, if we regard \( \prec_r \) as an ordering on \( F^-_r(n) \) and on \( F^+_r(n) \), we will have \( (F^-_r(n), \prec_r) = (F^+_r(n), \prec_r) \), and hence \( (F^+_r(n), \prec_r) = (F^-_r(n), \succ_r) \). The color of an edge \( e = \{A_1, \ldots, A_r\} \) in \( c_r \), where \( A_i \in F_r(n) \) and \( A_1 <_r \cdots <_r A_r \), is then defined using an iterative application of a function \( \gamma \) on consecutive vertices \( A_i \) and \( A_{i+1} \), where \( \gamma(A, B) \) is the first element of \( B \) in \( \preceq_{r-1} \) on which \( A \) and \( B \) differ. We apply \( \gamma \) on \( e \) until we reach a unique element of \( F_1(n) \), which is set to be the color of \( e \).

Now, we proceed by describing the construction of \( c_r \) in full detail. Let \( F_2(n) := \{(2n - i + 1, i) : i \in [n]\} \subseteq [2n]^2 \) and \( F_2^+(n) := \{(i, 2n - i + 1) : i \in [n]\} \subseteq [2n]^2 \). We define a linear ordering \( <_2 \) on the disjoint union \( F_2(n) := F_2^-(n) \cup F_2^+(n) \) by letting \((2n, 1) <_2 (2n - 1, 2) <_2 \)}.
Note that $N_2 := |F_2(n)| = 2n$. For convenience, we define $F_1^-(n) := \{ - \}$, $F_1^+(n) := \{ + \}$, $F_1(n) := \{ - , + \}$, and $- <_1 +$.

Let $\sigma_2: F_2^-(n) \to F_2^+(n)$ be the one-to-one correspondence that maps $(2n - i + 1, i)$ to $(i, 2n - i + 1)$. Two elements $A$ and $B$ from $F_2(n)$ are equivalent, written $A \equiv_2 B$, if $A = B$, $A = \sigma_2(B)$, or $B = \sigma_2(A)$. We order the equivalence classes of $F_2(n)$ under $\equiv$ by a linear order $\preceq_2$ by identifying each $A$ from $F_2^-(n)$ with $\sigma_2(A)$ and by letting $\prec_2$ be the ordering $<_2$ on $F_2^-(n)$. Slightly abusing the notation, we sometimes consider $\preceq_2$ as a linear order on $F_2(n)$. Then two equivalent elements of $F_2(n)$ are considered equal in $\preceq_2$. For $r = 1$, we let $\sigma_1(-) = +$ and $- \equiv_1 +$.

Let $r \geq 3$ be a positive integer and assume we have constructed $F_{r-1}(n)$. Let $F_r(n)$ be the collection of sets $X$ such that $X$ contains exactly one set from each equivalence class of $\equiv_{r-1}$ on $F_{r-1}(n)$. Observe that $N_r := |F_r(n)| = 2^{N_{r-1} - 1}$ and that no two sets from $F_r(n)$ are comparable in $\preceq$. Also note that the minimum and the maximum element of $F_{r-1}(n)$ in $<_{r-1}$ are equivalent and thus $X$ contains exactly one of them.

We let $F_r^-(n)$ and $F_r^+(n)$ be the subsets of $F_r(n)$ consisting of sets that contain the minimum and the maximum element of $F_{r-1}(n)$ in $<_{r-1}$, respectively. Since every element of $F_r(n)$ contains either the minimal or the maximal element of $F_{r-1}(n)$ in $<_{r-1}$, the sets $F_r^-(n)$ and $F_r^+(n)$ partition $F_r(n)$. We say that sets from $F_r^-(n)$ and $F_r^+(n)$ have type $-$ and $+$, respectively. An example for $r = 3 = n$ can be found in Figure 3.

Let $A$ and $B$ be distinct sets from $F_r(n)$ for $r \geq 3$. We let $\gamma(A, B)$ be the element from $B \cap E$, where $E$ is the first equivalence class of $(F_{r-1}(n))_{\equiv_{r-1}}$ in $<_{r-1}$ on which $A$ and $B$ differ. We define the total order $<_r$ on $F_r(n)$ by letting $A <_r B$ if $\gamma(A, B) \in F_r^+(n)$. Observe that $\gamma(A, B) \in F_r^-(n)$ if and only if $\gamma(B, A) \in F_r^+(n)$ and thus $<_r$ is indeed a total order. For $r = 2$, if $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are distinct elements from $F_2(n)$, then we let $\gamma(A, B) = -$ if $a_1 < b_1$ and, similarly, $\gamma(A, B) = +$ if $a_1 > b_1$, where $<$ is the standard ordering of $\mathbb{R}$. Note that, for $A, B \in F_2(n)$, $\gamma(A, B) = -$ if and only if $A \approx_2 B$.

We define the mapping $\sigma_r: F_r^-(n) \to F_r^+(n)$ by letting

$$\sigma_r(\{ A_1, \ldots, A_{N_{r-1} - 1/2} \}) := \{ \sigma_{r-1}(A_1), \ldots, \sigma_{r-1}(A_{N_{r-1} - 1/2}) \}.$$ 

Note that $\sigma_r$ is a one-to-one correspondence. Two elements $A$ and $B$ from $F_r(n)$ are equivalent, written $A \equiv_r B$, if $A = B$, $A = \sigma_r(B)$, or $B = \sigma_r(A)$. We again order the equivalence classes of $F_r(n)$ under $\equiv_r$ by a linear order $\preceq_r$ that is obtained by identifying each $A$ from $F_r^-(n)$ with $\sigma_r(A)$ and by letting $\prec_r$ be the ordering $<_r$ on $F_r^-(n)$. Again, slightly abusing the notation, we sometimes consider $\preceq_r$ as a linear order on the set $F_r(n)$. Thus two equivalent elements from $F_r(n)$ are the same in $\preceq_r$, $(F_r^-(n), \prec_r) = (F_r^+(n), \prec_r)$, and $(F_r^+(n), \prec_r) = (F_r^+(n), \prec_r)$.

For integers $k, r \geq 2$ and a sequence $(B_1, \ldots, B_k)$ of sets from $F_1(n)$ in which any two consecutive terms are distinct, we use $\Gamma(B_1, \ldots, B_k)$ to denote the sequence $(\gamma(B_1, B_2), \ldots, \gamma(B_{k-1}, B_k))$ of $k - 1$ sets from $F_{r-1}(n)$. Observe that, if $r \geq 3$, the definition of $\gamma$ guarantees that any two consecutive terms of $\Gamma(B_1, \ldots, B_k)$ are distinct and thus we can apply the function $\Gamma$ on $F_{r-1}(n)$.

Applying $\Gamma$ to $(B_1, \ldots, B_k)$ iteratively $i$ times, for some $i$ with $1 \leq i \leq \min\{k - 1, r - 1\}$, results in a sequence $\Gamma^i(B_1, \ldots, B_k) := \Gamma(\Gamma(\cdots \Gamma(B_1, \ldots, B_k) \cdots))$ of $k - i$ elements from $F_{r-1}(n)$. For convenience, we set $\Gamma^0(B_1, \ldots, B_k) := (B_1, \ldots, B_k)$.

Letting $K_{N_r}$ be the ordered complete $r$-uniform hypergraph with the vertex set $F_r(n)$ ordered by $<_r$, we color $K_{N_r}$ with a 2-coloring $c_r$ by letting $c_r(\{ A_1, \ldots, A_r \}) := \Gamma^{r-1}(A_1, \ldots, A_r)$ for every edge $\{ A_1, \ldots, A_r \}$ of $K_{N_r}$, with $A_1 <_r \cdots <_r A_r$.

**Lemma 8.** For all positive integers $n$ and $r$ with $r \geq 3$, there is no monochromatic copy of $\mathcal{P}_{n+1}^r$ in $K_{N_r}$ colored with $c_r$. 

\footnote{Alternatively, one might define $F_2(n) = [2n]$. However, we use this definition as it is more similar to the approach of Moshkovitz and Shapira.}
Proof. Let $\mathcal{P}$ be a monochromatic copy of $\mathcal{P}_k^r$ in $c_r$ for some integer $k \geq r$. Let $A_1 \prec \cdots \prec A_k$ be vertices of $\mathcal{P}$. Let $a_1, \ldots, a_{k-r+2}$ be the elements of $F_2(n)$ obtained by applying the function $\Gamma^{r-2}$ to sequences $(A_1, \ldots, A_{r-1}), (A_2, \ldots, A_r), \ldots, (A_k, \ldots, A_{k-r+2}, \ldots, A_k)$, respectively. The color $c_r([A_1, \ldots, A_{i+r-1}])$ of each edge $[A_1, \ldots, A_{i+r-1}]$ of $\mathcal{P}$ then equals $\gamma(a_i, a_{i+1})$. Thus if all edges of $\mathcal{P}$ have color $\prec$ in $c_r$, we obtain $a_1 >_2 \cdots >_2 a_{k-r+2}$. That is, the first coordinates of $a_1, \ldots, a_{k-r+2}$ increase and we get $k \leq 2n + r - 2$, as $a_1, \ldots, a_{k-r+2} \in F_2(n) \subseteq [2n]^2$. Similarly, if all edges of $\mathcal{P}$ have color $\succ$, then $a_1 \prec_2 \cdots \prec_2 a_{k-r+2}$ and the second coordinates of $a_1, \ldots, a_{k-r+2}$ increase, which again implies $k \leq 2n + r - 2$. □

Note that if $r = 3$, then $a_1, \ldots, a_{k-1}$ all have type $\prec$, as $A_1 \prec_3 \cdots \prec_3 A_k$. Using this fact, we could eventually obtain a better estimate $R_{\text{mon}}(\mathcal{P}_n^3) \geq 2^{n3}$. However, this is not optimal anyway, as we know that $R_{\text{mon}}(\mathcal{P}_n^3) = \left(\frac{2n^3}{3^3}\right) + 1$.

It remains to show that the coloring $c_r$ satisfies the monotonicity property. In other words, we want to show that there is at most one change of a sign in $(c_r(S^{r+1}), \ldots, c_r(S^{3}))$ for every sequence $S = (A_1, \ldots, A_{r+1})$ of sets from $F_r(n)$ with $A_1 \prec_3 \cdots \prec_3 A_{r+1}$. We first prove two auxiliary results that hold for every $r \geq 2$.

**Lemma 9.** For positive integers $n$ and $r$ with $r \geq 2$, let $(A, B, C)$ be a sequence of distinct sets from $F_r(n)$. For $r \geq 3$, $\gamma(A, C) = \min_{\prec_{r-1}} \{\gamma(A, B), \gamma(B, C)\}$ if $\gamma(A, B) \neq \gamma(B, C)$ and $\gamma(A, B), \gamma(B, C) \prec_{r-1} \gamma(A, C)$ otherwise. For $r = 2$, $\gamma(A, C) \in \{\gamma(A, B), \gamma(B, C)\}$ if $\gamma(A, B) \neq \gamma(B, C)$ and $\gamma(A, C) = \gamma(A, B) = \gamma(B, C)$ otherwise.

**Proof.** First, we assume $r \geq 3$. One of the following three cases occurs: $\gamma(A, B) \prec_{r-1} \gamma(B, C)$, $\gamma(B, C) \prec_{r-1} \gamma(A, B)$, or $\gamma(A, B) \equiv_{r-1} \gamma(B, C)$. In the first case, the sets in $B$ are the same as the sets in $C$ up to $\gamma(B, C)$ in $\prec_{r-1}$ while the sets in $A$ and $B$ differ already on $\gamma(A, B) \prec_{r-1} \gamma(B, C)$ in $\prec_{r-1}$. Thus $\gamma(A, C) = \gamma(A, B)$. Similarly, we obtain $\gamma(A, C) = \gamma(B, C)$ in the second case.

If $\gamma(A, B) \equiv_{r-1} \gamma(B, C)$, then it follows from $\gamma(A, B) \neq \gamma(B, C)$ that either $\gamma(A, B) = \sigma_r(\gamma(B, C))$ or $\gamma(\gamma(A, B)) = \gamma(B, C)$. In particular, $\gamma(B, A) = \gamma(B, C)$. The sets in $A$ and $C$ thus differ for the first time on a set that is larger then both $\gamma(A, B)$ and $\gamma(B, C)$ in $\prec_{r-1}$.
Proof. For positive integers $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$. If $\gamma(A, B) = + = \gamma(B, C)$, then $a_1 > b_1$ and $b_1 > c_1$. In particular, $a_1 > c_1$ and $\gamma(A, C) = +$. Analogously, if $\gamma(A, B) = - = \gamma(B, C)$, then $\gamma(A, C) = -$. If $\gamma(A, B) \neq \gamma(B, C)$, then $\gamma(A, C) \in \{\gamma(A, B), \gamma(B, C)\}$, as $F_1(n)$ contains only the values $-$ and $+$. \hfill $\square$

Note that if $A <_r B <_r C$ or $A >_r B >_r C$, then $\gamma(A, B)$ and $\gamma(B, C)$ have the same type and thus $\gamma(A, B) \neq \gamma(B, C)$ have different types and thus $\gamma(A, C)$ lies between them in $\prec_{r-1}$. For $r = 2$, it follows that $(\gamma(A, B), \gamma(B, C))$ has at most one change of a sign. Thus, for any distinct $A, B, C$ from $F_r(n)$ with $r \geq 2$, the sequence $(\gamma(A, B), \gamma(A, C), \gamma(B, C))$ is monotone in $\leq_{r-1}$. \hfill $\square$

**Lemma 10.** For positive integers $n$ and $r$ with $r \geq 2$, let $A, B, A', B'$ be sets from $F_r(n)$ such that $A \neq B$.

(i) Assume $A' \neq B$. If $A \leq_r A'$, then $\gamma(A, B) \geq_{r-1} \gamma(A', B)$.

(ii) Assume $A \neq B'$. If $B \leq_r B'$, then $\gamma(A, B) \leq_{r-1} \gamma(A, B')$.

Proof. We prove only part [ii], as the proof of part [i] is analogous. It is easy to verify the statement for $r = 2$ and thus we consider $r \geq 3$. We can assume $A \neq A'$, as otherwise the statement is trivial. There are three possibilities where to place $B$ with respect to $A$ and $A'$ in $<_r$. If $A <_r A' <_r B$, then Lemma 6 implies $\gamma(A, B) \leq_{r-1} \gamma(A', B)$ and, since $\gamma(A, B), \gamma(A', B) \in F_{r-1}^+(n)$, we have $\gamma(A, B) \geq_{r-1} \gamma(A', B)$. If $A <_r B <_r A'$, then $\gamma(A, B) \in F_{r-1}^+(n)$ and $\gamma(A', B) \in F_{r-1}^-(n)$ and we obtain $\gamma(A, B) \geq_{r-1} \gamma(A', B)$ immediately. Finally, if $B <_r A <_r A'$, then Lemma 6 implies $\gamma(B, A') \leq_{r-1} \gamma(B, A)$. Since $\gamma(B, A)$ and $\gamma(A, B)$ are equivalent and have distinct type, and the same is true for $\gamma(B, A')$ and $\gamma(A', B)$, we have $\gamma(A', B) \leq_{r-1} \gamma(A, B)$.

Before stating the last auxiliary result, we first introduce some definitions. For two sequences $S_1$ and $S_2$, we use $S_1 \cdot S_2$ to denote the concatenation of $S_1$ and $S_2$. A **profile** is a sequence of symbols $\leq$, $\geq$, and $\prec$, containing at least one of the symbols $\leq$ and $\geq$. Let $O_l := (\leq, \leq, \leq, \ldots)$ and $E_l := (\geq, \geq, \geq, \ldots)$ be two profiles of length $l \in \mathbb{N}$. We say that a profile $P$ of length $l$ is **odd** or **even** if it can be obtained from $O_l$ or $E_l$, respectively, by changing some occurrences of $\leq$ and $\geq$ to $\prec$. For two profiles $P_1$ and $P_2$ such that each is odd or even, if $P_1$ is odd and $P_2$ is even, then $P_1$ and $P_2$ have distinct parity. Otherwise we say that $P_1$ and $P_2$ have the same parity. The **opposite profile** $\overline{P}$ of a profile $P$ is the profile that is obtained from $P$ by replacing each term $\leq$ with $\geq$ and each term $\geq$ with $\leq$.

For positive integers $n, r$, and $s \geq 2$, let $R = (B_1, \ldots, B_s)$ be a sequence of $s$ sets from $F_r(n)$ and let $P$ be a profile of length $s - 1$. We say that $P$ is a **profile of $R$** if whenever $B_j <_r B_{j+1}$ or $B_j >_r B_{j+1}$, then the $j$th term of $P$ is $\leq$ or $\geq$, respectively, for every $j \in [s-1]$. \hfill $\square$

**Lemma 11.** For positive integers $n, r$, and $s$ with $r \geq 3$ and $3 \leq s \leq r + 1$, let $S := (A_1, \ldots, A_s)$ be a sequence of $s$ sets from $F_r(n)$ with $A_1 <_r \cdots <_r A_s$. Then the sequence $H := (\Gamma^{s-2}(S(n)), \ldots, \Gamma^{r-2}(S(1)))$ has either odd or even profile.

Proof. We recall that, for a sequence $S$ and a subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, |S|\}$, we use $S_[i_1, \ldots, i_k]$ to denote the subsequence of $S$ obtained by deleting all elements from $S$ that are at position $i_j$ for some $j \in [k]$. Also note that every sequence $(A_1, \ldots, A_k)$ of elements from $F_r(n)$ satisfies $\Gamma^{k-1}(A_1, \ldots, A_k) = \gamma(\Gamma^{k-2}(A_1, \ldots, A_k-1), \Gamma^{k-2}(A_2, \ldots, A_k))$. In particular, we have
\( \Gamma^{s-2}(S^{(s)}) = \gamma(\Gamma^{s-3}(S^{s-1,s}), \Gamma^{s-3}(S^{(1,s)})), \Gamma^{s-2}(S^{(1)}) = \gamma(\Gamma^{s-3}(S^{(1,s)}), \Gamma^{s-3}(S^{(1,2)})), \) and 
\( \Gamma^{s-2}(S^{(i)}) = \gamma(\Gamma^{s-3}(S^{i,s}), \Gamma^{s-3}(S^{(i,1)})) \) for every \( i \) with \( 2 \leq i \leq s - 1 \).

We use \( H_1 \) to denote the sequence \((\Gamma^{s-3}(S^{s-1,s}), \ldots, \Gamma^{s-3}(S^{(1,s)})))\) and \( H_2 \) to denote \((\Gamma^{s-3}(S^{(1,s)}), \ldots, \Gamma^{s-3}(S^{(1,2)})))\). Let \( G_1 := (\Gamma^{s-3}(S^{s-1,s})), H_1\) and \( G_2 := H_2 \cdot (\Gamma^{s-3}(S^{(1,2)}))\). That is, \( G_1 \) is the sequence obtained from \( H_1 \) by doubling the first term and \( G_2 \) is the sequence obtained from \( H_2 \) by doubling the last term. By the definition of the function \( \gamma \), for every \( i \in [s] \), the \( i \)th term of \( H \) equals \( \gamma(X,Y) \), where \( X \) is the \( i \)th term of \( G_1 \) and \( Y \) is the \( i \)th term of \( G_2 \).

We proceed by induction on \( s \geq 3 \) and, in each step of the induction, we construct a profile \( p(H) \) such that \( p(H) \) is a profile of \( H \) and \( p(H) \) is odd or even. We start with the base case \( s = 3 \). We have \( H = (\gamma(A_1, A_2), \gamma(A_1, A_3), \gamma(A_2, A_3)) \), \( H_1 = (A_1, A_2, A_1) \), \( G_1 = (A_1, A_1, A_2) \), \( A_2 = (A_2, A_3) \), and \( G_2 = (A_2, A_3, A_2) \). Since \( A_1 <_r A_2 <_r A_3 \), it follows from Lemma 9 that \( \gamma(A_1, A_2) = \gamma(A_1, A_3) >_{r-1} \gamma(A_2, A_3) \) if \( \Gamma(S^{(3)}) = \Gamma(S^{(1)}) \) or \( \gamma(A_1, A_3) = \gamma(A_2, A_3) \) if \( \Gamma(S^{(1)}) = \Gamma(S^{(3)}) \). We thus choose \( p(H) \) to be the even profile \((=,=)\) or the odd profile \((<,>)\), respectively. We also set \( p(H_1) := (\leq,=) \), \( p(H_2) := (<,=) \), \( p(G_1) := (=,\leq,=) \), and \( p(G_2) := (\leq,=) \). Observe that if \( \Gamma(S^{(3)}) = \Gamma(S^{(1)}) \), then \( p(H) \) is the profile \( p(G_1) \) and if \( \Gamma(S^{(1)}) = \Gamma(S^{(3)}) \), then \( p(H) \) is the profile \( p(G_2) \).

Let \( R \) be a sequence of length \( k \) with the profile \( p(R) \) assigned. We recall that the length of \( p(R) \) is \( k - 1 \). We let \( i_1(R) \) be the largest \( i \in [k] \) such that the first \( i - 1 \) terms of \( p(R) \) are all \( = \). Similarly, we let \( i_2(R) \) be the smallest \( i \in [k] \) such that the last \( k - j \) terms of \( p(R) \) are all \( = \). In other words, \( i_1(R) \) is the smallest \( i \) with \( 1 \leq i \leq k \) such that the \( i \)th term of \( p(R) \) is not \( = \), and \( i_2(R) \) is the smallest \( i \) with \( 1 \leq i \leq k \) such that for every \( j \) with \( i \leq j \leq k - 1 \), the \( j \)th term of \( p(R) \) is \( = \). Note that \( i_2(R) \geq i_1(R) + 1 \). In the case \( s = 3 \), it is easy to check that \( p(H_1) \) and \( p(H_2) \) have the same parity and \( i_1(G_1) = i_2(G_2) \).

For the induction step, we assume that \( s \geq 4 \). We first express each of the sequences \( H_1 \) and \( H_2 \) as a result of applying \( \gamma \) to two sequences, similarly as we have expressed \( H \) using \( G_1 \) and \( G_2 \). Let \( H_{1,1} := (\Gamma^{s-4}(S^{s-2,s-1,s}), \ldots, \Gamma^{s-4}(S^{(1,s-1,s)})) \) and \( H_{1,2} := (\Gamma^{s-4}(S^{(1,s-1,s)}), \ldots, \Gamma^{s-4}(S^{(1,2,s)})) \). By setting \( G_{1,1} := (\Gamma^{s-4}(S^{s-2,s-1,s})), H_{1,1} \) and \( G_{1,2} := H_{1,2} \cdot (\Gamma^{s-4}(S^{(1,2,s)})) \), we obtain that the \( i \)th term of \( H_1 \) is \( \gamma(X,Y) \), where \( X \) and \( Y \) are the \( i \)th terms of \( G_{1,1} \) and \( G_{1,2} \), respectively. We similarly proceed with \( H_2 \) and we let \( H_{2,1} := H_{1,2} \) and \( H_{2,2} := (\Gamma^{s-4}(S^{(1,2,s)}), \ldots, \Gamma^{s-4}(S^{(1,2,s)})) \). Setting \( G_{2,1} := (\Gamma^{s-4}(S^{(1,s-1,s)})), H_{2,1} \) and \( G_{2,2} := H_{2,2} \cdot (\Gamma^{s-4}(S^{(1,2,s)})) \), we get that the \( i \)th term of \( H_2 \) is \( \gamma(X,Y) \), where \( X \) and \( Y \) are the \( i \)th terms of \( G_{2,1} \) and \( G_{2,2} \), respectively; see Figure 4 for an example.

![Figure 4](example.png)

**Figure 4.** Example of the sequences used in the induction step for \( s = 4 \).

Here, we have profiles \( p(G_1) = (=,\leq,=) \) and \( p(G_2) = (=,\geq,=) \). We set \( p(H) = (=,\geq,=) \).
We now define a profile $p(H)$ and, as our induction step, we later prove that it is a profile of $H$. In fact, we prove a stronger statement by additionally showing that if $p(H_1)$ and $p(H_2)$ have the same parity then either $p(H) = p(G_1)$ or $p(H) = p(G_2)$ and also $i_1(G_1) \geq i_2(G_2)$, while if $p(H_1)$ and $p(H_2)$ have distinct parity then $i_1(G_1) \geq i_1(G_2)$ and $i_2(G_1) \geq i_2(G_2)$; see Figure 5.

(a) \hspace{1cm} i_1(G_1) \hspace{1cm} i_2(G_1)
(b) \hspace{1cm} i_1(G_1) \hspace{1cm} i_2(G_1)

\[ G_1: = \cdots = = = = \leq = \cdots = = \leq \cdots = \]

\[ G_2: = \cdots = \leq = \cdots = = \cdots = \]

\[ i_1(G_2) \hspace{1cm} i_2(G_2) \hspace{1cm} i_1(G_2) \hspace{1cm} i_2(G_2) \]

**Figure 5.** Example of the inequalities $i_1(G_1) \geq i_2(G_2)$ in the case of the same parity of the profiles $p(H_1)$ and $p(H_2)$ (part (a)) and $i_1(G_1) \geq i_1(G_2)$ and $i_2(G_1) \geq i_2(G_2)$ in the case when $p(H_1)$ and $p(H_2)$ have distinct parity (part (b)).

For every $j \in [s - 1]$, we let the $j$th term of a profile $\overline{p(G_1)} \circ p(G_2)$ be $= i_1(G_1)$ if the $j$th terms of both $\overline{p(G_1)}$ and $p(G_2)$ are equalities and we let the $j$th term of $\overline{p(G_1)} \circ p(G_2)$ be $\leq i_1(G_1)$ or $\geq i_1(G_1)$ be $\leq i_2(G_1)$ if the $j$th terms of $\overline{p(G_1)}$ or $p(G_2)$ is $\leq$. Similarly, we let the $j$th term of $\overline{p(G_1)} \circ p(G_2)$ be $\geq i_1(G_1)$ if the $j$th term of $\overline{p(G_1)}$ or $p(G_2)$ is $\geq$. Observe that if each of the profiles $p(H_1)$ and $p(H_2)$ is odd or even, then there is no $i$ with $1 \leq i \leq s - 1$ such that the $i$th term of $\overline{p(G_1)}$ is $\leq$ while the $i$th term of $p(G_2)$ is $\geq$, or vice versa. Thus $\overline{p(G_1)} \circ p(G_2)$ is correctly defined under this assumption. If $p(H_1)$ and $p(H_2)$ have distinct parity, we let $p(H)$ be the profile $\overline{p(G_1)} \circ p(G_2)$. If $p(H_1)$ and $p(H_2)$ have the same parity, we let $p(H)$ be the profile $\overline{p(G_1)}$ if $\Gamma^{s-2}(S^{(s)}) \prec_r S^{(s)}$ and the profile $p(G_2)$ if $\Gamma^{s-2}(S^{(s)}) \succ_r S^{(s)}$.

Recall that, as our induction step, we prove that $p(H)$ is a profile of $H$ and that $i_1(G_1) \geq i_1(G_2)$ and $i_2(G_1) \geq i_2(G_2)$ if $p(H_1)$ and $p(H_2)$ have distinct parity and $i_1(G_1) \geq i_1(G_2)$ if $p(H_1)$ and $p(H_2)$ have the same parity. We already observed that this statement is true for $s = 3$. Note that it follows from the induction hypothesis that the parity of $p(H)$ is the same as the parity of $\overline{p(G_1)}$ or of $p(G_2)$. In particular, the profile $p(H)$ of $H$ is odd or even, which gives the statement of the lemma.

By the induction hypothesis, for every $i \in \{1, 2\}$, the profile $p(H_i)$ is a profile of $H_i$ and $i_1(G_{i,1}) \geq i_1(G_{i,2})$ and $i_2(G_{i,1}) \geq i_2(G_{i,2})$ if $p(H_{i,1})$ and $p(H_{i,2})$ have distinct parity and $i_1(G_{i,1}) \geq i_2(G_{i,2})$ if $p(H_{i,1})$ and $p(H_{i,2})$ have the same parity. In the latter case, we also know that $p(H_i) \in \{p(G_{i,1}), p(G_{i,2})\}$.

Assume first that $p(H_1)$ and $p(H_2)$ have the same parity. We show that $i_1(G_1) \geq i_2(G_2)$ by distinguishing some cases. First, we consider the case when both $p(H_1)$ and $p(H_2)$ are odd, the other one will be symmetric. Using the definition of $G_{1,2}$ and $G_{2,1}$, the fact that $H_{1,2} = H_{2,1}$, and the fact that $p(H_{1,2}) = p(H_{2,1})$ contain at least one term which is not $=$, we obtain $i_j(G_{1,2}) = i_j(G_{2,1}) - 1$ for every $j \in \{1, 2\}$.

1. We start with the cases when at least one of the following situations occurs, either $p(H_1) \notin \{p(G_{1,1}), p(G_{1,2})\}$ or $p(H_2) \notin \{p(G_{2,1}), p(G_{2,2})\}$. Note that, by the definition of $p(H_i)$ for $i \in \{1, 2\}$, if $p(H_1) \notin \{p(G_{1,1}), p(G_{1,2})\}$, then $p(H_1) = \overline{p(G_{1,1})} \circ p(G_{1,2})$ and the profiles $p(H_{1,1})$ and $p(H_{1,2})$ have distinct parity.

   (a) If $p(H_1) \notin \{p(G_{1,1}), p(G_{1,2})\}$ and $p(H_2) \in \{p(G_{2,1}), p(G_{2,2})\}$, then $p(H_{1,1})$ is even and $p(H_{1,2}) = p(H_{2,1})$ and $p(H_{2,2})$ are odd. Since $p(H_{1,1})$ and $p(H_{1,2})$ have distinct parity, we get $i_1(G_{1,1}) \geq i_1(G_{1,2})$ and $i_2(G_{1,1}) \geq i_2(G_{1,2})$. Since
(a) If $p(H_1) = \overline{p(G_1)} \circ p(G_1)$ and $i_1(G_1) \geq i_2(G_1)$, we get $i_1(G_1) = i_1(G_1) + 1$ by the definition of $G_1$. Since $p(H_2)$ is odd, we have $p(H_2) = p(G_2)$. Thus $i_2(G_2) = i_2(G_2)$. Altogether, it follows from $i_1(G_1) \geq i_2(G_2)$ and $i_1(G_1) = i_1(G_1) + 1$ that

$$i_1(G_1) = i_1(G_1) + 1 = i_1(G_2) \geq i_2(G_2) = i_2(G_2).$$

(b) If $p(H_1) \in \{\overline{p(G_1)}\} \circ p(G_1)$ and $p(H_2) \notin \{\overline{p(G_2)}\} \circ p(G_2)$, then $p(H_1)$ and $p(H_2)$ are even and $p(H_2)$ is odd. Since $p(H_1) \circ p(H_2)$ have the same parity, we get $i_1(G_1) \geq i_2(G_2)$.

(c) If $p(H_2) \notin \{\overline{p(G_1)}\} \circ p(G_1)$ and $p(H_2) \notin \{\overline{p(G_2)}\} \circ p(G_2)$, then $p(H_1)$ and $p(H_2)$ have distinct parity and also $p(H_2)$ and $p(H_2)$ have distinct parity. This, however, implies that either $p(H_1)$ or $p(H_2)$ is even, which is impossible.

(2) Thus now we are left with the cases $p(H_1) \in \{\overline{p(G_1)}\} \circ p(G_1)$ and $p(H_2) \in \{\overline{p(G_2)}\} \circ p(G_2)$. We deal with all four cases.

(a) If $p(H_1) = \overline{p(G_1)}$ and $p(H_2) = p(G_2)$, then $p(H_1)$ and $p(H_2)$ are odd and we have $i_1(G_1) = i_1(G_1) + 1$. If the parity of $p(H_1)$ is $p(H_2)$ is even, then $p(H_1)$ and $p(H_2)$ have distinct parity and $p(H_2)$ and $p(H_2)$ have the same parity. It follows that $i_1(G_1) \geq i_1(G_1)$ and $i_1(G_1) \geq i_2(G_2)$. Using $i_1(G_1) = i_1(G_1) + 1$, we derive

$$i_1(G_1) + 1 = i_1(G_2) + 1 = i_1(G_2) \geq i_2(G_2) = i_2(G_2).$$

If the parity of $p(H_2)$ is even, then $p(H_1)$ and $p(H_2)$ have the same parity, while $p(H_2)$ and $p(H_2)$ have distinct parity. This implies $i_1(G_1) \geq i_2(G_2)$ and $i_2(G_2) \geq i_2(G_2)$ and we derive

$$i_1(G_1) + 1 = i_2(G_2) + 1 = i_2(G_2) \geq i_2(G_2) = i_2(G_2).$$

(b) Assume that $p(H_1) = \overline{p(G_1)}$ and $p(H_2) = p(G_2)$. Then $p(H_1)$ and $p(H_2)$ are both even. It also follows that $i_1(G_1) = i_1(G_1) + 1$ and $i_2(G_2) = i_2(G_2)$. Since the profiles $p(H_1)$ and $p(H_2)$ have the same parity, we have $i_1(G_1) \geq i_2(G_2)$, which gives

$$i_1(G_1) = i_1(G_1) + 1 = i_2(G_2) + 1 = i_2(G_2) = i_2(G_2).$$

(c) If $p(H_1) = p(G_1)$ and $p(H_2) = p(G_2)$, then both $p(H_2)$ and $p(H_2)$ are odd. We also have $i_1(G_1) = i_1(G_1) + 1$ and $i_2(G_2) = i_2(G_2)$. It follows that $p(H_2)$ and $p(H_2)$ have the same parity, which gives $i_1(G_1) \geq i_2(G_2)$. This implies

$$i_1(G_1) = i_1(G_1) + 1 = i_1(G_2) \geq i_2(G_2) = i_2(G_2).$$

(d) We cannot have $p(H_1) = p(G_1)$ and $p(H_2) = p(G_2)$, as otherwise $p(H_2)$ and $p(H_2)$ have distinct parity, which is impossible, as $p(H_2) = p(H_2)$. 


Altogether, if \( p(H_1) \) and \( p(H_2) \) are odd, we have \( i_1(G_1) \geq i_2(G_2) \). Note that in the above case analysis, we only rely on the facts that the parity of two profiles is the same or different, we do not use the actual parity. Thus, by symmetry, the inequality \( i_1(G_1) \geq i_2(G_2) \) holds if both \( p(H_1) \) and \( p(H_2) \) are even.

Now, we use the fact \( i_1(G_1) \geq i_2(G_2) \) to show that \( p(H) \) is a profile of \( H \) if \( p(H_1) = p(G_1) \) if \( \Gamma^{s-2}(S(i)) \sim_{r-s+2} \Gamma^{s-2}(S(1)) \) or \( p(H_2) = p(G_2) \) if \( \Gamma^{s-2}(S(i)) \sim_{r-s+2} \Gamma^{s-2}(S(1)) \). Recall that we assume that \( p(H_1) \) and \( p(H_2) \) have the same parity. Thus \( \Gamma^{s-3}(S^{(s-1,i)}) \leq_{r-s+3} \Gamma^{s-3}(S^{(s-1,i)}) \) or \( \Gamma^{s-3}(S^{(s-1,i)}) \geq_{r-s+3} \Gamma^{s-3}(S^{(s-1,i)}) \). This implies that the first term \( \Gamma^{s-2}(S(i)) = \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) \) of \( H \) and the last term \( \Gamma^{s-2}(S(1)) = \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) \) of \( H \) have the same type and, assuming \( s \leq r \), they are not equivalent. Thus we either have \( \Gamma^{s-2}(S(i)) \sim_{r-s+2} \Gamma^{s-2}(S(1)) \) or \( \Gamma^{s-2}(S(i)) \leq_{r-s+2} \Gamma^{s-2}(S(1)) \). In the first case, Lemma 9 with the parameters \( A := \Gamma^{s-3}(S^{(s-1,i)}), B := \Gamma^{s-3}(S^{(s-1,i)}), C := \Gamma^{s-3}(S^{(s-1,i)}) \) implies \( \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) = \Gamma^{s-2}(S(1)) \) and in the second case, the lemma with the same parameters gives \( \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) \). For \( s = r + 1 \), the terms of \( H \) lie in \( F_1(n) \) and Lemma 9 gives \( \Gamma^{s-2}(S(1)) = \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) = \Gamma^{s-2}(S(1)) \) immediately.

We know that the term \( \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) \) equals the first term \( \Gamma^{s-2}(S(i)) \) of \( H \) if \( \Gamma^{s-2}(S(i)) \sim_{r-s+2} \Gamma^{s-2}(S(1)) \) and to the last term \( \Gamma^{s-2}(S(1)) \) of \( H \) otherwise. We assume without loss of generality that \( \Gamma^{s-2}(S(i)) \leq_{r-s+2} \Gamma^{s-2}(S(1)) \), as the other case is symmetric. For \( j = i_1(G_1) \), the inequality \( i_1(G_1) \geq i_2(G_2) \) implies that \( j \)th term of \( H \) equals \( \gamma(\Gamma^{s-3}(S^{(s-1,i)}), \Gamma^{s-3}(S^{(s-1,i)})) \). For every \( i \) with \( i \leq j \), the \( i \)th term of \( H \) is obtained by applying \( \gamma \) to the first term of \( G_1 \) and the \( i \)th term of \( G_2 \). Since \( G_2 \) is either non-decreasing or non-increasing in \( \leq_{r-s+3} \), Lemma 10 implies that all the \( j \)th terms of \( H \) are monotone in \( \leq_{r-s+2} \). Thus, since the first term of \( H \) and the \( j \)th term of \( H \) are both equal to \( \Gamma^{s-2}(S(i)) \), we get that all the first \( j \) terms of \( H \) are equal. Since \( j = i_1(G_1) \leq i_2(G_2) \), for every \( i \) with \( i \geq j \), the \( i \)th term of \( H \) is obtained by applying \( \gamma \) to the \( i \)th term of \( G_1 \) and the last term of \( G_2 \). Together with the previous fact, Lemma 10 implies that \( p(H) = p(G_1) \) and it is a profile of \( H \).

For the rest of the proof we assume that the profiles \( p(H_1) \) and \( p(H_2) \) have distinct parity. For \( i \in [s-1] \), let \( (p_i,q_i) \) be the pair consisting of the \( i \)th term \( p_i \) of \( G_1 \) and the \( i \)th term \( q_i \) of \( G_2 \). It follows from the definition of \( G_1 \) and \( G_2 \) that \( (p_i,q_i) \in \{(=,-),(-,-),(-,\geq),(\leq,\geq)\} \) if \( p(H_1) \) is odd and \( p(H_2) \) is even and that \( (p_i,q_i) \in \{(=,+),(+,-),(+,\leq),(\geq,\leq)\} \) if \( p(H_1) \) is even and \( p(H_2) \) is odd. Thus, by Lemma 10 and by the fact that the \( i \)th term of \( H \) is obtained by applying \( \gamma \) to the \( i \)th terms of \( G_1 \) and \( G_2 \) for each \( i \in [s] \), the profile \( p(H) \) is odd or even and it is a profile of \( H \). For example, in the case \( (p_i,q_i) \in \{(=,\leq),\leq,\leq\} \), we apply part (i) of Lemma 10 with \( A := \text{ith term of } G_1, A' := (i+1)\text{st term of } G_1, \) and \( B := \text{ith term of } G_2, \) which is also the \( (i+1)\text{st term of } G_2 \). Note that in the cases \( (p_i,q_i) \in \{(\geq,\geq),\geq,\geq\} \), we apply Lemma 10 twice.

It remains to show that \( i_1(G_1) \geq i_1(G_2) \) and \( i_2(G_1) \geq i_2(G_2) \). Let \( j \in \{1,2\} \). Since \( p(H_1) \in \{p(G_1),p(G_2),p(G_1) \circ p(G_1)\} \), we have \( i_j(H_1) \in \{i_j(G_1),i_j(G_2)\} \). Similarly, \( p(H_2) \in \{p(G_2),p(G_2),p(G_2) \circ p(G_2)\} \) and thus \( i_j(H_2) \in \{i_j(G_1),i_j(G_2)\} \). Since \( p(H_1) = p(H_2) \), it follows from the definition of \( G_1 \) and \( G_2 \) that \( i_j(G_1) + 1 = i_j(G_2) \). We recall that \( i_j(G_{k,l}) \geq i_j(G_{k,l}) \) for all \( k,l \in \{1,2\} \). Thus the induction hypothesis gives \( i_j(G_{1,1}) \geq i_j(G_{2,1}) \) and \( i_j(G_{2,1}) \geq i_j(G_{2,2}) \). Altogether, we obtain \( i_j(H_1) \geq i_j(G_{2,1}) - 1 \) and \( i_j(H_2) \leq i_j(G_{2,1}) \). It follows from the definition of \( G_1 \) and \( G_2 \) that \( i_j(G_1) = i_j(H_1) + 1 \) and \( i_j(G_2) = i_j(H_2) \). This implies \( i_j(G_1) \geq i_j(G_2) \).
Lemma 11 is sufficient to guarantee the monotonicity property for $c_r$.

**Corollary 12.** For every integer $r$ with $r \geq 3$, the coloring $c_r$ is $r$-monotone.

**Proof.** For a sequence $S := (A_1, \ldots, A_{r+1})$ of sets from $F_r(n)$ with $A_1 \prec_r \cdots \prec_r A_{r+1}$, we show that there is at most one change of a sign in the sequence $(c_r(S^{(r+1)}), \ldots, c_r(S^{(1)}))$. By Lemma 11 applied for $s := r + 1$, the sequence $(\Gamma^{r-1}(S^{(r+1)}), \ldots, \Gamma^{r-1}(S^{(1)}))$ has odd or even profile and, in particular, this sequence is monotone in $\preceq_1$. The rest follows from the fact that $c_r(S^{(i)}) = \Gamma^{r-1}(S^{(i)})$ for every $i \in [r+1]$. □

Lemma 8 and Corollary 12 together give the statement of Theorem 2.

Comparison with the construction by Moshkovitz and Shapira. For positive integers $c$ and $r$ (for every $r \geq 4$, similarly, we say that $\sigma$ is positive, and $\sigma$ is negative, if it satisfies one of the following three conditions: the reduction of $\sigma$ is positive, $\sigma = (1, \ldots, 1, 2)$ and $r$ is even, or $\sigma = (r - 1, 1)$ and $r$ is even. Similarly, we say that $\sigma$ is negative if it satisfies one of the following three conditions: the reduction of $\sigma$ is positive, $\sigma = (1, \ldots, 1, 2)$ and $r$ is even, or $\sigma = (r - 1, 1)$ and $r$ is odd. The notion of negative and positive integer compositions is illustrated in Figure 6. Note that, for every $r \geq 3$, the only two compositions of $c$ that are not negative nor positive are $(1, \ldots, 1)$ and $(r)$.

4. **Proof of Theorem 3**

In this section, we prove Theorem 3 by showing that the number of $r$-monotone colorings of $K_r^n$ is of order $2^{n-1}/r^{o(r)}$ for $r \geq 3$ and $n \geq r$. We first derive the lower bound in Subsection 4.1 and then, in Subsection 4.2, we prove the upper bound.

4.1. **A lower bound on the number of monotone colorings.** Here we provide a lower bound $2^{n-1}/r^{o(r)}$ on the number of $r$-monotone colorings of $K_r^n$ with $r \geq 3$ and $n \geq r$. The construction is inspired by the method used by Matoušek 16 to show that there are $2^{O(n^2)}$ simple arrangements of $n$ pseudolines.

First, we introduce some definitions. A composition of a positive integer $m$ into $k$ parts, $k \in \mathbb{N}$, is an ordered $k$-tuple $(p_1, \ldots, p_k)$ of positive integers with $p_1 + \cdots + p_k = m$. It is well-known and easy to show that the number of compositions of $m$ into $k$ parts is exactly $\binom{m-1}{k-1}$. In particular, the total number of compositions of $m$ is $\sum_{i=1}^{m} \binom{m-1}{i-1} = 2^{m-1}$.

Let $r$ and $k$ be integers with $r \geq 3$ and $1 \leq k \leq r$. Let $\sigma = (p_1, \ldots, p_k)$ be a composition of $r$ into $k$ parts. The reduction step on $\sigma$ maps $\sigma$ to the composition $(p_1, \ldots, p_k - 1)$ if $p_k > 1$ or to the composition $(p_1, \ldots, p_{k-1})$ if $p_k = 1$. We say that a composition $\sigma'$ is the reduction of $\sigma$ if $\sigma'$ is a composition of one of the forms $(1, \ldots, 1, 2)$ or $(p, 1)$, for some $p > 1$, and is obtained from $\sigma$ by a sequence of reduction steps. Note that $\sigma$ has a reduction if and only if $\sigma \neq (1, \ldots, 1)$ and $\sigma \neq (r)$. Moreover, the reduction, if it exists, is unique.

We now recursively define the sign of a composition $\sigma$ of $r$ using the sign of its reduction. This is carried out by induction on $r$. If $r = 3$, then $\sigma$ is negative if $\sigma = (1, 2)$ and $\sigma$ is positive if $\sigma = (2, 1)$. For $r > 3$, we say that $\sigma$ is negative if it satisfies one of the following three conditions: the reduction of $\sigma$ is negative, $\sigma = (1, \ldots, 1, 2)$ and $r$ is odd, or $\sigma = (r - 1, 1)$ and $r$ is even. Similarly, we say that $\sigma$ is positive if it satisfies one of the following three conditions: the reduction of $\sigma$ is positive, $\sigma = (1, \ldots, 1, 2)$ and $r$ is even, or $\sigma = (r - 1, 1)$ and $r$ is odd. Note that, for every $r \geq 3$, the only two compositions of $c$ that are not negative nor positive are $(1, \ldots, 1)$ and $(r)$.
Let $r$ and $h$ be positive integers with $r \geq 3$. We set $n := r^h$ and $m := n/r = r^{h-1}$. We now present a construction of a 3-coloring $c_{r,h}$ of $K^r_n$ with colors $\{-0, +\}$ such that every 2-coloring that is obtained by replacing each occurrence of the color 0 with either $-$ or $+$ is $r$-monotone. The construction is carried out recursively starting with the case $h = 1$, in which $n = r$ and $c_{r,1}$ is the coloring that assigns the color 0 to the only edge $[r]$ of $K^r_n$.

For $h \geq 2$, we let $V_i := \{(i-1)m + 1, \ldots, im\}$ for every $i \in [r]$ and we let $[n]$ be the vertex set of $K^r_n$. Note that the sets $V_1, \ldots, V_r$ partition $[n]$ and form consecutive intervals of size $m$ in the ordering $<$ on $[n]$.

We define the 3-coloring $c_{r,h}$ of $K^r_n$ on $[n]$ as follows. Let $e = \{v_1, \ldots, v_r\} \in \binom{[n]}{r}$ be an edge of $K^r_n$. The sets $V_1, \ldots, V_r$ partition $e$ into nonempty sets $e_{1}, \ldots, e_{k}$, for some $k \in [r]$, that are consecutive in $<$. We let $p_i$ be the size of $e_i$ for every $i \in [k]$ and we use $\sigma$ to denote the composition $(p_1, \ldots, p_k)$ of $r$. We choose $c_{r,h}(e) := -$ if $\sigma$ is negative and $c_{r,h}(e) := +$ if $\sigma$ is positive. It remains to assign the color $c_{r,h}(e)$ to edges $e$ for which $\sigma$ is not negative nor positive, that is, to edges $e$ for which either $\sigma = (r)$ or $\sigma = (1, \ldots, 1)$. If $\sigma = (r)$, then $e \subseteq V_i$ for some $i \in [r]$ and, in particular, $\{v_1 - (i-1)m, \ldots, v_r - (i-1)m\} \subseteq [m]$. We then use the coloring $c_{r,h-1}$ from the previous step of the construction and we let $c_{r,h}(e) := c_{r,h-1}(\{v_1 - (i-1)m, \ldots, v_r - (i-1)m\})$. If $\sigma = (1, \ldots, 1)$, then each vertex $v_i$ lies in the set $V_i$. In this case, we use $v'_i$ to denote the integer $v_i - (i-1)m$ from $[m]$ and we let

$$c_{r,h}(e) := \begin{cases} - \text{ if } \sum_{i \in [r]} v'_i < \sum_{i \in [r]} v'_i, & \text{if } i \text{ even} \\ 0 \text{ if } \sum_{i \in [r]} v'_i = \sum_{i \in [r]} v'_i, & \text{if } i \text{ odd} \\ + \text{ if } \sum_{i \in [r]} v'_i > \sum_{i \in [r]} v'_i, & \text{if } i \text{ even} \end{cases}$$

This finishes the construction of $c_{r,h}$. We show that no matter how we replace zeros with $-$ or $+$ signs in $c_{r,h}$, the resulting coloring is $r$-monotone.

**Lemma 13.** For $h \geq 1$ and $r \geq 3$, let $c$ be an arbitrary 2-coloring of $K^r_n$ that is obtained from $c_{r,h}$ by replacing each occurrence of 0 with $-$ or $+$. Then $c$ is an $r$-monotone coloring of $K^r_n$.

**Proof.** We prove the statement by induction on $h$. For $h = 1$, the statement is trivial as $n = r$ and there is only a single edge in $K^r_n$.

![Figure 6. Examples of negative and positive compositions of $r \in \{3, 4, 5\}$.](image)

The compositions of $r$ are illustrated as elements of the partially ordered set $(2^{[r-1]}, \subseteq)$ and the sign $-$ or $+$ next to a composition $\sigma$ denotes whether $\sigma$ is negative or positive, respectively.
Now, assume that \( h \geq 2 \). We further assume that the statement is true for \( h - 1 \). Let \( F = \{v_1, \ldots, v_{r+1}\} \subseteq [n] \) be an \((r+1)\)-tuple of vertices of \( K_n \), with \( v_1 < \cdots < v_{r+1} \) and let \( j_1 < \cdots < j_r \) be indices with \( F \cap V_{j_i} \neq \emptyset \). We let \( \sigma = (p_1, \ldots, p_k) \), \( k \in [r] \), be the composition of \( r+1 \), where \( p_i = |F \cap V_{j_i}| \) for every \( i \in [k] \). For every \( i \in [r+1] \), we let \( e_i \) be the edge \( F \setminus \{v_i\} \). Similarly as before, for every \( i \in [r+1] \), the partitioning of each edge \( e_i \) by \( V_1, \ldots, V_r \) determines a composition \( \sigma_i \) of \( r \). Note that each \( \sigma_i \) can be obtained from \( \sigma \) by decreasing \( p_j \) by 1 if \( p_j > 1 \) or by removing \( p_j \) if \( p_j = 1 \), where \( j \) is a number from \( [k] \) such that \( \sum_{j=1}^{r+1} p_j < i \) and \( \sum_{j=1}^{r+1} p_j \geq i \).

We show that \( c \) is \( r \)-monotone by proving that there is at most one change of a sign in the sequence \( S_F := (c(e_1), \ldots, c(e_{r+1})) \). Since there are only \( r \) sets \( V_1, \ldots, V_r \) in the partition of \([n] \), we cannot have \( \sigma = (1, \ldots, 1) \). If \( \sigma = (r+1) \), then \( F \subseteq V_i \) for some \( i \in [r] \) and the statement follows from the induction hypothesis for \( h - 1 \). Thus we can assume that \( \sigma \) is positive or negative.

We first deal with the case \( \sigma = (1, \ldots, 1, 2, 1, \ldots, 1) \), that is, \( p_j = 2 \) for some \( j \in [r] \) and \( p_i = 1 \) for every \( i \in [r] \setminus \{j\} \). For such a \( \sigma \), we have \( \sigma_j = (1, \ldots, 1) = \sigma_{j+1} \), every \( \sigma_i \) with \( i > j \) has the \( j \)th coordinate 2 and all other \( 1 \), and \( \sigma_i \) with \( i < j \) has the \((j-1)\)st coordinate 2 and all other \( 1 \). We show that if \( \sigma_i \) has the value 2 on an odd coordinate, then \( c_{r,h}(e_i) = + \). This is because we can perform reduction steps until we reach the reduction (1, \ldots, 1, 2) of \( \sigma_i \). This reduction has an odd number of parts, which implies that it is a composition of an even number and thus the reduction of \( \sigma_i \) is positive. By the definition of \( c_{r,h} \), we obtain \( c_{r,h}(e_i) = + \). Similarly, if the value 2 is on an even coordinate of \( \sigma_i \), then \( c_{r,h}(e_i) = - \).

 Altogether, we see that there are \( \xi, \xi', \xi'' \in \{-, +\} \) such that \( S_F = (\xi, \ldots, \xi, \xi', \xi'', -\xi, \ldots, -\xi) \), where \( \xi' \) and \( \xi'' \) are on the \( j \)th and the \((j+1)\)st coordinate, respectively. Moreover, \( \xi = + \) if \( j \) is even and \( \xi = - \) if \( j \) is odd. Since \( v_j, v_{j+1} \in V_{j'} = V_j \) and \( v_j < v_{j+1} \), we have \( v_j' < v_{j+1}' \). Moreover, since \( e_j = F \setminus \{v_j\} \), \( e_{j+1} = F \setminus \{v_{j+1}\} \), the definition of \( c_{r,h} \) implies that \( \xi' \leq \xi'' \) if \( j \) is odd and \( \xi' \geq \xi'' \) if \( j \) is even and either \( c_{r,h}(e_j) \) or \( c_{r,h}(e_{j+1}) \) is not 0. Thus there is at most one change of a sign in \( S_F \).

In the rest of the proof, we assume that \( \sigma \) is a negative or a positive composition of \( r+1 \) that is not of the form (1, \ldots, 1, 2, 1, \ldots, 1). Let \( \sigma' \) be the reduction of \( \sigma \). We know that \( \sigma' \) is a composition of some integer \( r' \) with \( 3 \leq r' \leq r+1 \) and \( \sigma' = (r' - 1, 1) \) or \( \sigma' = (1, \ldots, 1, 2) \).

First, we consider the case where \( \sigma' \) is of the form \((r' - 1, 1)\). For every \( i \in [k] \) with \( i > r' \), the composition \( \sigma_i \) has the same reduction as \( \sigma \) and thus all the edges \( e_i \) with \( i > r' \) have the same color \( \xi \in \{-, +\} \) in \( c_{r,h} \). Assume that \( r' > 3 \). Then every composition \( \sigma_i \) with \( i < r' \) has the reduction \((r' - 2, 1)\) and thus every edge \( e_i \) with \( i < r' \) has the color \(-\xi \) in \( c_{r,h} \). It follows that \( c \) is \( r \)-monotone, as \( S_F = (\xi, \ldots, \xi', \xi', \xi'', \ldots, \xi', \xi'' \ldots, \xi) \) for some \( \xi' \in \{-, +\} \). Now, assume \( r' = 3 \). Since \( \sigma \neq (2, 1, \ldots, 1) \), there is an entry in \( \sigma \) of size larger than 1 not lying on the first position and thus \( \sigma_1 \) and \( \sigma_2 \) have the same reduction of the form \((1, 1, 2) \). Since \( r + 1 \geq 4 \) and \( r' = 3 \), there is at least one entry in \( \sigma_3 \) besides the first entry \( r' - 1 = 2 \) and thus \( \sigma_3 \) has the same reduction \((r'-1, 1) = (2, 1)\) as any \( \sigma_i \) with \( i > r' = 3 \). It follows that \( S_F = (\xi', \xi', \xi', \xi', \xi', \xi', \xi') \) for some \( \xi' \in \{-, +\} \).

Now, we consider the case \( \sigma' = (1, \ldots, 1, 2) \). The composition \( \sigma' \) is the reduction of \( \sigma \) for every \( i \in [k] \) with \( i > r' \) and thus all the edges \( e_i \) with \( i > r' \) have the same color \( \xi \in \{-, +\} \) in \( c_{r,h} \). Since \( \sigma \neq (1, \ldots, 1, 2, 1, \ldots, 1) \), the compositions \( \sigma_{r-1} \) and \( \sigma_r \) have the same reduction. Assume \( r' > 3 \). Then every \( \sigma_i \) with \( i \leq r'-2 \) has the reduction \((1, \ldots, 1, 2) \), which is a composition of \( r'-1 \). Consequently, for every \( i \leq r'-2 \), the edge \( e_i \) has color \(-\xi \) in \( c_{r,h} \). Thus \( S_F = (-\xi, \ldots, -\xi, \xi', \xi', \ldots, \xi') \) for some \( \xi' \in \{-, +\} \). If \( r' = 3 \), then \( \sigma' = (1, 2) \) and the reduction of \( \sigma_i \) is \((p_2, 1) \). If \( p_2 \geq 3 \), then the compositions \( \sigma_2, \ldots, \sigma_{r+1} \) have the same reduction and \( S_F = (\xi', \xi', \ldots, \xi) \) for some \( \xi' \in \{-, +\} \). If \( p_2 = 2 \), then the reduction of \( \sigma_1 \) is \((2, 1) \) and, since \( (2, 1) \) is positive and \( (1, 2) \) is negative, we obtain \( S_F = (+, \xi', \xi', \ldots, -) \).
for some $\xi' \in \{-, +\}$. In any case, there is at most one change of a sign in $S_F$ and $c$ is $r$-monotone.

By Lemma 13, every coloring obtained from $c_{r,h}$ is $r$-monotone. Thus, to finish the proof of the lower bound in Theorem 3, it suffices to estimate the number of such colorings from below.

**Lemma 14.** For positive integers $h$ and $r$ with $r \geq 3$, there are at least

$$2^{r(r-1)(h-1)-2r}$$

colorings that can be obtained from $c_{r,h}$ by replacing each occurrence of 0 with $-$ or $+$.

**Proof.** Let $f_s(h)$ be the number of 2-colorings that can be obtained from $c_{r,h}$ by replacing each occurrence of color 0 with either $-$ or $+$. Clearly, we have $f_s(1) = 2$. For $h \geq 2$, we have $f_s(h) \geq 2^x$, where $x$ is the number of edges of color 0 in $c_{r,h}$ that are not contained in any $V_i$. We recall that $m = e^{h-1} \geq r$.

We estimate the number $x$ as follows. Consider an arbitrary $(r-1)$-tuple $T = (t_1, \ldots, t_{r-1})$ of numbers from $[\lfloor m/2 \rfloor]$ such that not all terms of $T$ are equal. Clearly, there are $[m/2]^{r-1} - [m/2]$ such $(r-1)$-tuples. Let $I$ and $J$ be two sets of sizes $[\lfloor r-1/2 \rfloor]$ and $[(r-1)/2]$, respectively, whose union is a partition of $[r-1]$ such that $d := \sum_{i \in I} t_i - \sum_{j \in J} t_j$ is minimum and positive. Such a partition exists, as not all terms of $T$ are equal and $|I| \geq |J|$. We claim that $d \leq m$.

Suppose for contradiction that $d > m$. Let $t_a$ be the largest element from $(t_i : i \in I)$ and let $t_b$ be the smallest element from $(t_j : j \in J)$. Note that $t_a > t_b$, as $d > m$ and every element from $(t_i : i \in I)$ is at most $\lfloor m/2 \rfloor \leq m$. Let $I' := (I \setminus \{a\}) \cup \{b\}$ and $J' := (J \setminus \{b\}) \cup \{a\}$. The value $\sum_{i \in I'} t_i - \sum_{j \in J'} t_j$ decreases by $2(t_a - t_b)$ when compared to $\sum_{i \in I} t_i - \sum_{j \in J} t_j$. Since $1 \leq 2(t_a - t_b) \leq 2 \lfloor m/2 \rfloor - 2 \leq m$, we have $0 < \sum_{i \in I'} t_i - \sum_{j \in J'} t_j < d$, which contradicts the choice of $I$ and $J$.

We let $E := (t_i : i \in I)$ and $O$ be the sequence that is obtained from $(t_j : j \in J)$ by adding the element $d \in [m]$. Then $|O| = (\lfloor r-1/2 \rfloor) + 1 = \lfloor r/2 \rfloor$, $|E| = (\lfloor r-1/2 \rfloor) - \lfloor r/2 \rfloor$, and $\sum_{o \in O} t_o = \sum_{e \in E} t_e$. We choose $v'_{2i-1}$ to be the $i$th element of $O$ for every $i \in [\lfloor r/2 \rfloor]$ and $v'_{2j}$ to be the $j$th element of $E$ for every $j \in [\lfloor r/2 \rfloor]$. Then we set $v_i := v'_i + (i - 1)m$ for each $i \in [r]$ and obtain $c_r(h)(v_1, \ldots, v_r) = 0$. One sequence $(v'_1, \ldots, v'_r)$ is obtained from at most $r! (r-1)$-tuples $T$ with $d$ added. Thus $x \geq (\lfloor m/2 \rfloor)^{r-1} - \lfloor m/2 \rfloor/r!$. Altogether, we have an estimate $f_s(h) \geq 2^{\lfloor m/2 \rfloor^{r-1} - \lfloor m/2 \rfloor/r!}$, which is at least $2^{r(r-1)(h-1)-2r}$, as $m = e^{h-1}$. \qed

If $n = r^h$, then the bound from Lemma 14 gives the lower bound $2^{n^{r-1}/r^{3r}}$ on the number of $r$-monotone colorings of $K_n^r$. For $n$ that is not a power of $r$, we have $r^{h-1} < n < r^h$ for some $h \in \mathbb{N}$ and we can use the estimate $2^{n^{r-1}/r^{3r}}$.

### 4.2 An upper bound on the number of monotone colorings

Here, using a result of Felsner and Valtr 11, we show that, for integers $r \geq 3$ and $n \geq r$, the number of $r$-monotone colorings of $K_n^r$ is at most $2^{(r-2)n^{r-1}/(r-1)}$.

We proceed by induction on $r$. For $r = 3$, Felsner and Valtr 11 showed that the number of sign functions of simple arrangements of $n$ pseudolines is at most $20.657n^2 \leq 2^{n^2}$. By Theorem 6, sign functions of simple arrangements of $n$ pseudolines correspond to 3-monotone colorings of $K_n^3$ and thus the number of such monotone colorings is also at most $2^{n^2}$. This constitutes the base case.

For the induction step, we assume $r \geq 4$. Let $c$ be an $r$-monotone coloring of $K_n^r$ with vertex set $[n]$. For $i \in \{r, \ldots, n\}$, the $i$th projection of $K_n^r$ is the function $p_i$ that maps an edge $\{v_1, \ldots, v_{r-1}, i\}$ of $K_n^r$ with $v_1 < \cdots < v_{r-1} < i$ to $\{v_1, \ldots, v_{r-1}\}$. The image of $K_n^r$ via $p_i$ is the ordered complete $(r-1)$-uniform hypergraph $K_{r-1}^{r-1}$. Note that for every edge $e$ of $K_{r-1}^{r-1}$, there is a unique edge $e' = p_i^{-1}(e)$ of $K_n^r$ with $p_i(e') = e$. If $c$ is an $r$-monotone coloring of $K_n^r$,
then we use $p_i(c)$ to denote the 2-coloring of $K_{r-1}^r$ obtained by coloring an edge $e$ of $K_{r-1}^r$ with the color $c(p_i^{-1}(e))$.

We show that every $p_i(c)$ is an $(r - 1)$-monotone coloring of $K_{r-1}^r$. Suppose for contradiction that there is an $i \in \{r, \ldots, n\}$ such that $p_i(c)$ is not an $(r-1)$-monotone coloring of $K_{r-1}^r$. Then there is an $r$-tuple $R$ of vertices from $[i-1]$ such that the sequence $S_R = (p_i(c)(R(1)), \ldots, p_i(c)(R(r)))$ has at least two changes of a sign. It follows from the definition of $p_i$ that, for the $(r+1)$-tuple $T = R \cup \{i\}$, we have $c(T(j)) = p_i(c)(R(j))$ for every $j \in [r]$. Thus the sequence $S_T = (c(T(r+1)), \ldots, c(T(1)))$ equals to the sequence that is obtained from $S_R$ by adding the first coordinate $c(T(r+1)) = c(R)$.

Then, however, there are at least two changes of a sign in $S_T$, which contradicts the assumption that $c$ is $r$-monotone.

Every $r$-monotone coloring $c$ of $K_n^r$ thus yields a sequence $S_c = (p_1(c), \ldots, p_n(c))$ of $(r - 1)$-monotone colorings. Moreover, the mapping $c \mapsto S_c$ is injective. For every $i \in \{r, \ldots, n\}$, the number of choices for $p_i(c)$ is at most $2^{2^{(r-2)(i-1)r^2-(r-2)i}}$ by the induction hypothesis. Altogether, the number of sequences $S_c$, and thus also the number of $r$-monotone colorings of $K_n^r$, is at most

$$\prod_{i=r}^{n} 2^{2^{r-3}(i-1)r^2-(r-2)i!} \leq 2^{(2^{r-3}r^2-(r-2)i!)} \sum_{i=1}^{n} i^{r-2} \leq 2^{2^{r-3}r^{r-1}((r-1)!)}.$$

To derive the last inequality, we used the estimate $\sum_{i=1}^{n} i^{r-2} \leq \frac{n^{r-1}}{r-1} + n^{r-2} \leq 2n^{r-1}/(r-1)$ for the power sum [9]. This finishes the proof of the upper bound in Theorem 3.

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