Scalar hairs and exact vortex solutions in 3D AdS gravity

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Abstract

We investigate three-dimensional (3D) Anti-de Sitter (AdS) gravity coupled to a complex scalar field $\phi$ with self-interaction potential $V(|\phi|)$. We show that the mass of static, rotationally symmetric, AdS black hole with scalar hairs is determined algebraically by the scalar charges. We recast the field equations as a linear system of first order differential equations. Exact solutions, describing 3D AdS black holes with real spherical scalar hairs and vortex-black hole solutions are derived in closed form for the case of a scalar field saturating the Breitenlohner-Freedman (BF) bound and for a scalar field with asymptotic zero mass. The physical properties of these solutions are discussed. In particular, we show that the vortex solution interpolates between two different AdS$_3$ vacua, corresponding respectively to a $U(1)$-symmetry-preserving maximum and to a symmetry-breaking minimum of the potential $V$.

1 Introduction

One of the most striking features of black hole solutions in any space-time dimension is their uniqueness. Typically, they are characterized by a bunch of parameters, which are asymptotic charges associated with global symmetries of the solution. One question that has been debated since a long time is the uniqueness of black hole solutions in the presence of scalar fields. The issue is rather involved and a precise formulation of the no-hair conjecture, i.e the absence of nontrivial scalar field configurations in a black hole background, has not yet been given [1, 2, 3]. In this context the relevant question concerns not only the presence of scalar hairs but also the possibility of having scalar charges independent from the black hole mass.

In recent years the investigation on black hole solutions with scalar hairs has been mainly focused on asymptotically AdS solutions [4]. The obvious reason behind such interest is the anti-de Sitter/conformal field theory (AdS$_d$/CFT$_{d-1}$) correspondence. The dynamics of scalar fields in the AdS$_d$ background gives crucial information about the dynamics of the dual $(d - 1)$-dimensional field theory. For instance, scalar hair configuration may interpolate between two different AdS geometries (one

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asymptotic and the other near-horizon) similarly to what happens for charged AdS black holes. An other important example is given by the recently discovered holographic superconductors. It is exactly a nontrivial scalar hair black hole solution of four-dimensional (4D) AdS gravity that breaks the $U(1)$ symmetry and is responsible for the superconducting phase transition in the dual three-dimensional (3D) field theory.

No-hair theorems for asymptotically AdS black hole have been discussed in Ref. for the case $d > 3$. In that paper Hertog has shown that AdS black hole solutions with spherical scalar hairs can only exist if the positive energy theorem is violated. He was also able to construct numerical black hole solution with scalar hairs. This result has played an important role for the development of holographic superconductors. In fact, the existence of 4D black holes dressed with a scalar hair is a necessary condition for having a phase transition in the dual theory.

In this paper we will consider the $d = 3$ case. We will investigate the existence of black hole solutions with spherical scalar hairs in 3D AdS spacetime. Because three spacetime dimensions may allow for topologically nontrivial global vortex configuration for the scalar field, we consider the case of a complex scalar field. The interest for global vortex solutions is not only motivated by the search for nontrivial scalar hairs. The study of black holes formed by global vortex is interesting by itself, as a particular case of black hole solutions supported by solitons. This kind of solutions may have some relevance in cosmology (as viable cosmic string candidates), in the context of the AdS/CFT correspondence as nonperturbative 3D bulk solutions interpolating between different AdS vacua and also as possible gravitational duals of condensed matter systems.

As a first step we analyze the dynamics of a complex scalar field $\phi$ with an arbitrary potential $V(|\phi|)$ in AdS. We demonstrate a theorem stating that for static, rotationally symmetric black hole solutions with scalar hairs the black hole mass is determined algebraically by the scalar charges. As a second step we show that the field equations can be recast in the form of a system of first order linear differential equations for the scalar potential $V$ and the spacetime metric. In principle, this allows us to find exact solutions of the field equations once the form of the scalar field is fixed. We use this method to find exact black hole solutions with spherical scalar hairs (both real and vortex-like) in the case of a scalar field saturating the BF bound and for a scalar field with zero mass. These dressed black hole solutions are completely characterized by a scalar charge $c$ and a vortex winding number $n$, the black hole mass being determined by $c$ and $n$.

In the case of a scalar field saturating the BF bound, the potential $V(|\phi|)$ has a $W$, Higgs-like, form and a global $U(1)$ symmetry. We find that the black hole-vortex solutions interpolates between two AdS$_3$ geometries with different AdS lengths. Choosing appropriately the parameters of the potential these AdS$_3$ geometries correspond respectively to the $U(1)$-symmetry-preserving maximum and to the symmetry-breaking minimum of the Higgs potential.

The structure of the paper is as follows. In Sect. 2, we discuss the dynamics of a complex scalar field in 3D AdS spacetime. In Sect. 3, we discuss no-hair theorems and prove the theorem stating that the mass of AdS$_3$ black holes with spherical scalar hairs is determined by $c$ and $n$. In Sect. 4 we recast the field equations in linear form. In Sect. 5 we derive and discuss exact solutions of our system describing AdS$_3$ black holes endowed with a real scalar hair saturating the BF bound. We also investigate their thermodynamics and show that their free energy is always higher.
then the free energy of the Banados-Teitelboim-Zanelli (BTZ) black hole \[15\]. In Sect. 6 we derive and discuss exact black hole-vortex solutions obtained for two different choices of the scalar field profile (\(|\phi| = c/r\) and \(|\phi| = c/r^2\)) and discuss the linear stability of the BTZ background. Finally in Sect. 7 we present our concluding remarks.

2 Dynamics of a complex scalar field in 3D AdS spacetime

Let us consider 3D Einstein gravity coupled to a complex scalar field \(\phi\):

\[
I = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left[ R - V(|\phi|) - \frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \phi \right],
\]

where \(V(|\phi|)\) is the potential for the scalar and for convenience we set the 3D gravitational constant \(G\) equal to \(\frac{1}{8}\). The action (1) is invariant under global \(U(1)\) transformations acting on the scalar field as \(\phi \to e^{i\beta} \phi\). We are only interested in solutions for the 3D metric that are spherically symmetric and asymptotically AdS. Therefore, we require that \(V\) has a local extremum at \(\phi = 0\) (corresponding to \(r = \infty\)), such that

\[
V(0) = -\frac{2}{L^2},
\]

where \(L\) is the AdS length. For a scalar field in the AdS-d spacetimes stability of the asymptotic solution does not necessarily requires \(\phi = 0\) to be a minimum. Only the weaker BF bound, \(m^2 L^2 \geq -(d-1)^2/4\), involving the squared mass \(m^2 = V''(0)\) of the scalar (the prime denotes derivation with respect to \(|\phi|\)), must be satisfied.

The scalar field must behave asymptotically as

\[
\varphi = \frac{c_-}{r^{\lambda_-}} + \frac{c_+}{r^{\lambda_+}} + ..., \tag{3}
\]

where \(\lambda_{\pm} = \frac{1}{2} \left( d - 1 \pm \sqrt{(d-1)^2 + 4m^2 L^2} \right)\).

The field equations following from the action (1) are,

\[
R_{\mu\nu} - g_{\mu\nu} V(|\phi|) - \frac{1}{4} \partial_\mu \bar{\phi} \partial^\nu \phi - \frac{1}{4} \partial_\mu \phi \partial^\nu \bar{\phi} = 0,
\]

\[
\frac{1}{2\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) - \frac{\partial V(|\phi|)}{\partial \phi} = 0,
\]

\[
\frac{1}{2\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \bar{\phi}) - \frac{\partial V(|\phi|)}{\partial \bar{\phi}} = 0. \tag{4}
\]

We are interested in static, spherically symmetric solutions of these equations, therefore we choose the following ansatz for the metric:

\[
ds^2 = -e^{2f(r)} dt^2 + e^{2h(r)} dr^2 + r^2 d\theta^2. \tag{5}
\]

The most general form of the complex scalar compatible with spherical symmetry of the metric is given in terms of a real modulus field \(\varphi(r)\) depending only on \(r\) and a

\[\text{[1]}\text{When the BF bound is saturated the asymptotic behavior of the scalar field may involve a logarithmic branch, whose back reaction also modifies the asymptotic behavior of the metric \[14\]. However, throughout this paper we will not consider this logarithmic branch.}\]
phase depending linearly on $\theta$. This is a consequence of the global $U(1)$ symmetry of the action (1). For topologically nontrivial spacetimes the phase becomes physically relevant and the proportionality factor is quantized. We will therefore use for $\phi$ the ansatz for static global vortices:

\[ \phi(r, \theta) = \varphi(r)e^{in\theta}, \]  

where $n$ is the winding number of the vortex.

Using Eqs. (5) and (6) in the field equations (4) we obtain:

\[ V + \frac{1}{r}e^{-2h}\dot{f} + e^{-2h}\dot{f}^2 - e^{-2h}\ddot{f} + e^{-2h}\dddot{f} = 0 \]

\[ -V + \frac{1}{r}e^{-2h}\dot{h} - e^{-2h}\dot{f}^2 + e^{-2h}\ddot{h} - e^{-2h}\dddot{f} - \frac{1}{2}e^{-2h}\dot{\varphi}^2 = 0 \]

\[ -V - \frac{1}{r}e^{-2h}\dot{f} + \frac{1}{r}e^{-2h}\dot{h} - \frac{1}{2}\frac{n^2}{r^2}\varphi^2 = 0 \]

\[ -V' + e^{-2h}\left(\frac{1}{r} + \dot{f} - \dot{h}\right)\dot{\varphi} + e^{-2h}\ddot{\varphi} - \frac{n^2}{r^2}\varphi = 0, \]  

(7)

where the dot indicates the derivative with respect to the radial coordinate $r$. Only three of the four Eqs. in (7) are independent. For $\phi \neq \text{const.}$ the system can be drastically simplified by combining the first and second equation and multiplying the fourth by $\dot{\varphi}$,

\[ \ddot{h} + \dot{f} = \frac{1}{2}r\dot{\varphi}^2 \]

\[ \dot{h} - \dot{f} = r \left(V + \frac{1}{2} \frac{n^2}{r^2}\varphi^2\right)e^{2h} \]

\[ e^{-2h}\left[\left(\frac{1}{r} + \dot{f} - \dot{h}\right)\dot{\varphi}^2 + \dot{\varphi}\dot{\varphi}\right] - \frac{1}{2}\frac{n^2}{r^2}\varphi^2 - V = 0 \]  

(8)

### 3 Scalar hairs

In this section we will discuss no-hair theorems for the black solutions of 3D AdS gravity coupled with a scalar field. The field equations (7) admit as solution the BTZ black hole. This is obtained by setting $\varphi = 0$. Requiring that $V'(0) = 0$ and using Eq. (2), one easily finds:

\[ e^{2f} = e^{-2h} = \frac{r^2}{L^2} - M, \]  

where $M$ is the black hole mass. One important question one can ask concerns the existence of 3D AdS black holes with scalar hairs, i.e. solutions of the system (8) with a non-constant scalar, $\varphi \neq \text{const.}$ Furthermore, assuming such black holes with scalar hairs to exist, one would also like to know whether the scalar charges $c_\pm$ (see Eq. (3)) are independent from the black hole mass $M$.

In the 4D case it has been shown that the existence of AdS black holes with scalar hairs is crucially related with the violation of positive energy theorem [4]. Generically, this will be also true for the 3D case. Therefore, we expect black holes with scalar hairs to exist when the scalar mass squared $m^2$ becomes negative. In
the next sections we will derive in closed form, exact black hole solutions with scalar hairs in the case of $m^2$ saturating the BF bound and for $m^2 = 0$. In this section we will be concerned with the second part of the question above. We will show that the following theorem holds:

If we couple 3D AdS gravity with a complex scalar field, the mass of static, rotationally symmetric solutions of the theory is determined algebraically by the scalar charges $c_\pm$ and by the winding number $n$.

In order to demonstrate this theorem we first solve for $e^{2f}$ the system (8). One easily finds,

$$e^{2f} = a_0 e^{\int dr \dot{\varphi}^2} \left[ \dot{V} + r \left( V + \frac{1}{2} n^2 \varphi^2 \right) \dot{\varphi}^2 + \frac{1}{2} \frac{n^2}{r^2} (\dot{\varphi}^2) \right] \left( \frac{1}{r} \frac{d}{dr} \dot{\varphi} + \ddot{\varphi} \right)^{-1}, \quad (10)$$

where $a_0$ is an integration constant, which can be scaled away by a rescaling of the time coordinate $t$. Because the $g_{tt}$ component of the metric is a function of $V, \varphi, n$ the mass of the solution is determined by $c_\pm, n$. This can be explicitly verified using in the previous equation the asymptotic expansion for the scalar field (3) and that for $g_{tt}$,

$$e^{2f} = \frac{r^2}{L^2} - M + O(\frac{1}{r}), \quad (11)$$

one immediately finds that $M$ is determined, algebraically, by $c_\pm$ and $n$. If only a single fall-off mode for the scalar (only one independent scalar charge $c_\pm$) is present in Eq. (3), the black hole mass $M$ is completely determined by the scalar charge $c_\pm$ and by the winding number $n$.

4 Linear form of the field equations

In order to find exact solutions of the system (8) one has to choose a form of the potential $V(\varphi)$ for the scalar field. Alternatively, one can choose a form for the scalar field $\varphi(r)$ and solve (8) in terms of $h(r), f(r), V(r)$. This method has two main advantages. It allows to fix from the beginning the asymptotic behavior of the field $\varphi$. The field equations (8) can be rewritten in the form of a system of first order linear differential equations. This can be achieved introducing in Eqs. (8) the new variables $S, Z$

$$S = \alpha e^{-2h}, \quad Z = \alpha (V + \frac{1}{2} \frac{n^2}{r^2} \varphi^2 - Y_0), \quad (12)$$

where $\alpha(r) = e^{\int dr \dot{\varphi}^2}$ and $Y_0$ is a solution of the equation $\dot{Y}_0 + r \varphi^2 Y_0 + \frac{n^2}{r} \varphi^2 = 0$. After eliminating $f$ and using Eqs. (12), the second and third equations in (8) can be written in the form

$$\dot{S} - p(r)S + rZ + q(r) = 0, \quad \dot{Z} - g(r)S = 0, \quad (13)$$

where $p(r) = \frac{1}{2} r (\dot{\varphi}^2), q(r) = r \alpha Y_0, g(r) = \frac{1}{2 \varphi^2} (d/dr)(r^2 \dot{\varphi}^2)$. This is a first order system of linear differential equations, which allows to determine the potential $V$ and the metric function $h$ once $\varphi(r)$ is given. The metric function $f$ then follows just by integrating the first equation in (8).

Although this method for solving Eq. (8) can be used for a generic $\varphi(r)$, in this paper we will mainly apply it to the case of a scalar field that saturates the BF
bound in 3D, this implies \( \lambda_+ = \lambda_- = 1 \), \( m^2 = -1/L^2 = m_{B,F}^2 \), i.e.\(^2\)

\[ \varphi = \frac{c}{r}, \quad V'(0) = \frac{1}{2}V(0). \]  

(14)

In the next sections we will derive in closed form exact black hole and vortex solutions corresponding a scalar field given by Eq. (14). We will also briefly consider solutions of Eqs. (8) in the case of scalar field with \( m^2 = 0 \), i.e for \( \varphi = c/r^2 \).

5 Black hole solutions with real scalar hairs

For a real scalar field, i.e. for \( n = 0 \), and \( \varphi \) given by Eq. (14) the method described in the previous section allows us to find the following solution of the system (8),

\[ V(\varphi) = -2\lambda_1 e^{\frac{\varphi^2}{2}} + \frac{\lambda_1}{2} \varphi^2 e^{\frac{\varphi^2}{2}} - 2\lambda_2 e^{\frac{\varphi^2}{2}}, \]

\[ e^{-2h} = r^2 e^{\frac{\varphi^2}{4}} \left( \lambda_1 e^{\frac{\varphi^2}{4}} + \lambda_2 \right), \]

\[ e^{2f} = e^{-\frac{\varphi^2}{2r^2}} e^{-2h}. \]  

(15)

where \( \lambda_{1,2} \) are parameters entering in the potential. They define the cosmological constant of the AdS spacetime. In fact using Eq. (2) one finds

\[ L^{-2} = \lambda_1 + \lambda_2. \]  

(16)

Thus, AdS solution exist only for \( \lambda_1 > -\lambda_2 \). For \( \lambda_1 > 0 \) the potential has the \( W \) form typical of Higgs potentials. The maximum of the potential, corresponds to an AdS3 vacuum with \( \varphi = 0 \) and \( V(0) = -2/L^2 \). It preserves the global \( U(1) \) symmetry of the action (1). Conversely, the minimum of the potential corresponds to an other AdS3 vacuum with \( \varphi = \varphi_m \) and a different AdS length \( l \), \( -2l^{-2} = V(\varphi_m) \). This vacuum breaks spontaneously the \( U(1) \) symmetry of the action (1). The shape of the potential for \( \lambda_1 > 0 \) is shown in figure 1 (left).

For \( \lambda_1 < 0 \) the potential has only the symmetry preserving maximum at \( \varphi = 0 \) and no symmetry breaking minima. The shape of the potential for \( \lambda_1 < 0, \lambda_2 > 0, \lambda_2 > |\lambda_1| \) is shown in figure 1 (right).

Let us now consider the metric part of the solution (15). The solution describes a black hole only for \( \lambda_1 < 0, \lambda_2 > 0 \), which in view of Eq. (16) requires \( \lambda_2 > |\lambda_1| \). As expected, the black hole mass \( M \) is determined in terms of the scalar charge \( c \) by the asymptotic expansion (11) of \( e^{2f} \),

\[ M = \frac{\lambda_2 c^2}{4}. \]  

(17)

The event horizon is located at

\[ r_h = \gamma \sqrt{M}, \quad \gamma = \left[ \lambda_2 \ln(-\frac{\lambda_2}{\lambda_1}) \right]^{-\frac{1}{2}}. \]  

(18)

\( r = 0 \) is a curvature singularity with the scalar curvature behaving as \( R \sim \exp(2M/(r^2\lambda_2))/r^4 \).

\(^2\) We do not consider the logarithmic branch (see footnote 1)
Figure 1: Left. The form of the potential $V(\phi)$ for the case $\lambda_1, \lambda_2 > 0$. The picture shows the potential for $\lambda_1 = \lambda_2 = 1$. Right. The form of the potential $V(\phi)$ for the case $\lambda_1 < 0, \lambda_2 > 0, \lambda_2 > |\lambda_1|$. The picture shows the potential for $\lambda_1 = -1, \lambda_2 = 2$.

For $\lambda_1 > 0$ the solution (15) describes a naked singularity. Because black hole solutions exist only in the range of parameters for which the potential $V$ has no minima, it follows that there is no $n = 0$ black hole solution interpolating between the $U(1)$-symmetry preserving vacuum at $\varphi = 0$ and the symmetry breaking vacuum at $\varphi = \varphi_m$. The symmetry-preserving and the symmetry-breaking vacua seem therefore gravitationally disconnected. Black hole solutions with scalar hairs exist only for the potential shown in right hand side of figure 1.

It is also interesting to have a short look at the thermodynamics of the black hole described by the solution (15). Temperature and entropy are given by

$$T = \frac{1}{2 \gamma \pi} \sqrt{M}, \quad S = 4 \pi \gamma \sqrt{M},$$

(19)

where $\gamma$ has been defined in Eq. (18). One can easily show that $M, T, S$ satisfy the first principle, $dM = TdS$. The thermodynamical behavior of our black hole follows closely that of the BTZ black hole, which is given by the same equations (19) with $\gamma = L$. If we compute the free energy of the BTZ black hole and that of the scalar-hair black hole (15) one finds respectively:

$$F_{BTZ} = -4\pi L^2 T^2, \quad F_{SH} = -4\pi \gamma^2 T^2.$$

(20)

Because in the range of existence of the black hole solutions ($\lambda_2 > |\lambda_1|$), we have always $\gamma < L$, it follows that $F_{SH} > F_{BTZ}$. This is consistent with the stability the BTZ black hole in the presence of a real scalar field with mass satisfying the BF bound and with the interpretation of the scalar hair solution (15) as an excitation of AdS$_3$.

### 6 Vortex solutions

Before considering vortex-like, black hole solutions, let us first consider the vacuum, $n = 0$, solutions of a model with a potential for the scalar of the form:

$$V(\varphi) = -4\Lambda^2 - \Lambda^2 \varphi^2 - \frac{\Lambda^2}{2} \varphi^4 - 2\lambda_1 e^{\varphi^2} + \frac{\lambda_1}{2} \varphi^2 e^{\varphi^2} - 2\lambda_2 e^{\varphi^2},$$

(21)
where $\Lambda, \lambda_1 > 0, \lambda_2 < 0$ are parameters. Also in this case the maximum of the potential, corresponds to a AdS$_3$, $U(1)$-symmetry preserving vacuum, with $\varphi = 0$ and

$$V(0) = -\frac{2}{L_1^2} = -4\Lambda^2 - 2\lambda_1 - 2\lambda_2,$$

where as usual $L_1$ is the AdS length. Existence of this AdS$_3$ vacuum obviously requires $2\Lambda^2 + \lambda_1 + \lambda_2 > 0$.

The mass of the scalar excitation near $\varphi = 0$ saturates the BF bound, in fact we have $m^2 = V''(0) = (1/2)V(0) = -1/L_1^2 = m^2_{BF}$. We have also a minimum of the potential describing a $U(1)$-symmetry breaking AdS$_3$ vacuum at $\varphi = \hat{\varphi}_m$ with AdS length $l_1$ given by $-2l_1^{-2} = V(\hat{\varphi}_m)$.

Using the method described in Sect. 4, a $n \neq 0$ vortex-like, black hole solution of the field equations (8) can be obtained for a profile of the complex scalar field given by

$$\phi = \frac{c}{r} e^{in\theta}, \quad \Lambda = \frac{n}{c}$$

when the parameter $\Lambda$ of the potential is fixed in terms of $c$ and $n$: $\Lambda = n/c$ and $\Lambda^2 \leq \frac{\lambda_1^2}{3\lambda_1}$.

The potential (21) with $\Lambda = n/c$ has the physical meaning of the effective potential seen by a vortex with scalar charge $c$ and winding number $n$. The metric part of the black hole solution turns out to be,

$$e^{-2h} = \lambda_1 r^2 e^{2\frac{\varphi^2}{c^2}} + \lambda_2 r^2 e^{\frac{\varphi^2}{c^2}} + \frac{2n^2}{c^2} r^2, \quad e^{2f} = e^{-\frac{e^2}{2r^2}} e^{-2h}.$$

The black hole mass is given in terms of the scalar charge $c$ and the winding number $n$:

$$M = \frac{\lambda_2 c^2}{4} + n^2.$$

The black hole solution (24) has an outer and inner horizon located respectively at

$$r_{\pm} = \frac{c}{2} \left( \ln \left\{ \frac{1}{2\lambda_1} \left[ \frac{1}{2\lambda_2} - \sqrt{\frac{\lambda_2^2}{c^2} - \frac{8\lambda_1 n^2}{c^2}} \right] \right\} \right)^{-\frac{1}{2}}.$$

The black hole solution exists for

$$\frac{M}{n^2} \leq 1 - 2 \frac{\lambda_1}{|\lambda_2|}, \quad |\lambda_2| > 2 \lambda_1.$$

The black hole becomes extremal (single event horizon) when the bound (27) is saturated. In general the radial coordinate $r_m$ corresponding to the minima of the potential (21) can lie either outside or inside the event horizon. The requirement $r_m \geq r_h$ implies $M/n^2 \geq 1 - (9/4)(\lambda_1/|\lambda_2|)$.

Thus the vortex black hole solution (24) interpolates between a symmetry-preserving AdS$_3$ vacuum at $\varphi = 0$ and a symmetry breaking AdS$_3$ vacuum at $\varphi = \varphi_m$ for

$$1 - \frac{9\lambda_1}{4|\lambda_2|} \leq \frac{M}{n^2} \leq 1 - 2 \frac{\lambda_1}{|\lambda_2|}.$$

Notice that differently from the usual case the vortex connects a maximum of the potential in the $r \to \infty$ asymptotic region with a minimum of the potential in the
interior region. This is related with the fact that the energy of the vortex although finite is always negative. The energy $E$ of the vortex is given by

$$E = \int_{r_+}^\infty r dr d\theta \langle T^t_t \rangle - E_{\text{vacuum}},$$

where $T^t_t$ is the stress energy tensor of the complex scalar field $\phi$ and we have subtracted the contribution of the $\phi = 0$ vacuum. We find after a little algebra,

$$E = \frac{c^2}{4} (\lambda_2 + 2\lambda_1) = (M - n^2)(1 - 2\frac{\lambda_1}{|\lambda_2|}).$$

(30)

The energy of the vortex measured with respect to the vacuum is finite and always negative.

### 6.1 Stability of the solutions

Let us first consider the solutions with a real scalar field. In this case the potential $V(\phi)$, given in Eq. (15) with $\lambda_1 < 0$, admits as solution both the BTZ black hole and a dressed black hole with a real scalar field $\phi = c/r$. The dressed black hole always has a free energy which is larger than that of the BTZ black hole. Correspondingly, the dressed black hole is always unstable and it will decay to the stable BTZ black hole, losing its real scalar hair. This is confirmed by a linear stability analysis. In the BTZ background scalar perturbations, $\delta \phi$, decouple from the metric. The equation of motion for the scalar perturbation is given by the Klein-Gordon equation for a scalar field propagating in AdS with a mass $m^2 = -1/L^2$ which saturates the BF bound of AdS$_3$. Thus the BTZ solution is (at least linearly) stable.

The story does not change so much if we consider solutions with a complex scalar field. In this case the potential $V(\phi)$, supporting vortex solutions, $\phi = (c/r)e^{i\theta}$ in given in Eq. (21). We can consider scalar perturbations around the BTZ black hole, which is again solution of the equations of motions. Because the potential only depends on $\phi = |\phi|$, at the linear level the equation for scalar perturbations near extrema of the potential always decouple in a holomorphic and antiholomorphic part:

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \partial^\mu \delta \phi) + \frac{1}{L^2} \delta \phi = 0, \quad \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \partial^\mu \bar{\delta \phi}) + \frac{1}{L^2} \bar{\delta \phi} = 0,$$

i.e. we have two independent real scalar perturbations, propagating in the BTZ background with the same mass saturating the BF bound, $m^2 = -1/L^2 = m^2_{BF}$. This again ensures linear stability of the BTZ black hole.

### 6.2 Other Vortex solutions

Let us now briefly consider a scalar field with $m^2 = V''(0) = 0$. From equation (3) it follows that $\phi$ must have the form $\phi = \frac{c}{r^2}$. We will therefore look for a complex scalar field of the form:

$$\phi = \frac{c}{r^2} e^{i\theta}.$$

(31)
In this case the method described in Sect. 4 gives the following solution for the potential \( V(\varphi) \),
\[
V(\varphi) = -2\lambda_1 e^{\varphi^2} - \frac{n^2}{2c} \varphi + 2\sqrt{\frac{2}{\pi}} \lambda_2 \varphi e^{\varphi^2} + 2\lambda_1 \varphi^2 e^{\varphi^2} + \\
\frac{n^2\sqrt{\pi}}{4c} \varphi^2 e^{\varphi^2} \left[ 1 - \text{erf}(\varphi) \right] - \frac{n^2\sqrt{\pi}}{4c} \varphi^2 \left[ 1 - \text{erf}(\varphi) \right] \\
-2\lambda_2 e^{\varphi^2} \text{erf} \left( \frac{\varphi}{\sqrt{2}} \right) + 2\lambda_2 \varphi^2 e^{\varphi^2} \text{erf} \left( \frac{\varphi}{\sqrt{2}} \right) - \frac{n^2}{2c} \varphi^3,
\] (32)
where \( \lambda_1, \lambda_2 \) are constants.

The metric part of the solution is given by
\[
e^{2f} = r^2 \left\{ \lambda_1 + \lambda_2 \text{erf} \left( \frac{c}{\sqrt{2r^2}} \right) + \frac{n^2\sqrt{\pi}}{8c} \left[ 1 - \text{erf} \left( \frac{c}{r^2} \right) \right] \right\}, \quad e^{-2h} = e^{\frac{2}{r^2}} e^{2f},
\] (33)
whereas for the AdS length we have,
\[
V(0) = -\frac{2}{L^2} = -2\lambda_1 - \frac{n^2\sqrt{\pi}}{4c}.
\] (34)

Hence, the solutions are asymptotically AdS for \( \lambda_1 > -\frac{n^2\sqrt{\pi}}{8c} \).

The mass \( M \) of the solution can be easily computed using the asymptotic expansion of \( \exp(2f) \). One has,
\[
M = -\sqrt{\frac{2\pi}{\lambda_2}} c + \frac{n^2}{4}.
\] (35)

The question about the presence of event horizon in the generic, \( n \neq 0 \), solution given by Eq. (33) is rather involved. We will not address this problem here, but we will just consider the \( n = 0 \) solutions. The solutions are asymptotically AdS for \( \lambda_1 > 0 \). They describe a black hole for \( \lambda_2 < 0 \), \( |\lambda_2| \geq \lambda_1 \). The position of event horizon \( r_h \) is the solution of the equation
\[
\text{erf} \left( \frac{c}{\sqrt{2r_h^2}} \right) = \frac{\lambda_1}{|\lambda_2|}.
\]

7 Conclusions

In this paper we have derived and discussed exact, spherically symmetric, solutions of 3D AdS gravity coupled with a complex scalar field \( \phi \). Our method allows to determine the potential \( V(|\phi|) \) once the form of \( |\phi(r)| \) is fixed. A scalar field profile corresponding to the saturation of the BF bound requires a Higgs-like potential \( V \), which allows both for BTZ black holes solutions - corresponding to a constant scalar field- and for black hole solutions with scalar hairs.

A generic feature of 3D black holes with scalar hairs is that the black hole mass is determined algebraically by the scalar charges and by the winding number of the vortex. The main difference between 3D black hole solutions with real scalar hairs and the BTZ black hole is the presence of a curvature singularity at \( r = 0 \), where the scalar curvature behaves as \( R \sim e^{2M/(r^2\lambda_2)}/r^4 \). On the other hand, the thermodynamical behavior of these dressed black hole solutions is very similar to that of the BTZ black hole. They share the same mass/temperature relation...
$M \sim T^2$ and we were able to show that the BTZ black hole remains always stable, being the free energy of the dressed solution always bigger than that of the BTZ black hole.

Conversely, the black hole-vortex solutions we have derived in this paper share many features with both rotating [5] and electrically charged [6] 3D black hole solutions. They are characterized by the presence of an inner and outer horizon and the vortex interpolates between two AdS$_3$ geometries with different AdS lengths. For some choice of the parameters, these AdS$_3$ geometries correspond respectively to the $U(1)$-symmetry-preserving maximum and to the symmetry-breaking minimum of the Higgs potential. In the AdS/CFT language this means that the vortex interpolates between two 2D CFTs with different central charges. It is interesting to notice that these interpolating features of the vortex solution are more similar to those of the rotating solution of Ref. [5] (both interpolated geometries are AdS$_3$) than to those of the electrically charged solutions of Ref. [6] (the solution interpolates between AdS$_3$ and AdS$_2 \times S^1$).

Also the non trivial solution for the scalar field could have an interesting holographic interpretation. The scalar charge $c$ has to be thought of as a nonvanishing VEV for some boundary operator. This could be a signal of a phase transition in the dual 2D boundary field theory. However, this is a rather involved issue because of the well-known results about phase transitions in statistical systems with only one spacelike dimension [10].

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