Random flights in $\mathbb{R}^3$ have been introduced since the beginning of the Twentieth Century by Pearson, Kluyver and Rayleigh. In their works uniformly oriented random displacements of fixed length are considered. Random flights where changes of direction are spaced by a homogeneous Poisson process with displacements uniformly distributed have been considered in $\mathbb{R}^3$ from different viewpoints by Stadje (in $\mathbb{R}^2$ [Stadje, 1987] and $\mathbb{R}^3$ [Stadje, 1989]), Orsingher and De Gregorio (2007), Franceschetti (2007) and Garcia-Pelayo (2008).

The case of random flights with Dirichlet distributed displacements (also uniformly distributed) was considered and investigated by Le Caëñ (2010) and De Gregorio and Orsingher (2012). Possible applications of random flights are suggested by the scattering of light rays in inhomogeneous media. Recent applications to the analysis of photon propagation in the Cosmic Microwave Background (CMB) radiation have been discussed in Reimberg and Abramo (2013). Furthermore, Martens et al. (2012) have shown that the probability law of planar random motions discussed in Kolesnik and Orsingher (2005) coincides with the explicit form of the van Hove function for the run-and-tumble model in two dimensions. This work gives and interesting and strong link between explicit solutions of the Lorentz model of electron conduction and the probability theory of random flights.

Displacements have random orientation defined by the angles $(\theta_1, \theta_2, \ldots, \theta_{3-2}, \phi)$ with density

$$g(\theta_1, \ldots, \theta_{3-2}, \phi) = \frac{\Gamma \left( \frac{d}{2} \right)}{2\pi^{d/2}} \sin^{d-2} \theta_1 \sin^{d-3} \theta_2 \ldots \sin \theta_{3-2},$$

with $0 \leq \theta_j \leq \pi$, $0 \leq \phi \leq 2\pi$.

The length of displacements $\tau_j$, $j = 1, \ldots, k$ between successive changes of direction occurring at times $t_j$, $1 \leq j \leq k$, with $\tau_j = t_j - t_{j-1}$ has distribution

\begin{quote}
\textit{Date: November 4, 2013.}
\end{quote}

\begin{quote}
\textit{Key words and phrases.} Random flights, Klein–Gordon type equations, Hyper-Bessel equations, Telegraph equation.
\end{quote}
where \(0 < \tau_j < t - \sum_{n=0}^{j-1} \tau_n\), \(1 \leq j \leq k\), \(\tau_{k+1} = t - \sum_{j=1}^{k} \tau_j\), which is a Dirichlet distribution with parameters \((\delta - 1, \ldots, \delta - 1)\), with \(\delta \geq 2\). The probability density of the vector \(X_0(t) = (X_1(t), \ldots, X_0(t))\) representing the position of the moving particle at time \(t\) after \(k\) changes of direction reads

\[
p_{X_0}(x_0, t; k) = \frac{\Gamma((k+1)(\frac{\delta}{2} - 1))}{\Gamma((\frac{\delta}{2} - 1)) \Gamma((\frac{\delta}{2} - 1))} \frac{1}{t^{k(\frac{\delta}{2} - 1) - 1}} \prod_{j=1}^{k+1} \frac{\tau_j^{\delta - 2}}{\tau_j!},
\]

with \(\|x_0\| < ct\), \(\delta \geq 2\) (see Theorem 2 of De Gregorio and Orsingher (2012)). For \(\delta = 2\), formula (1.3) reduces to

\[
p_{X_0}(x_0, t; k) = \frac{k}{2\pi(t)} \left(c^2 t^2 - \|x_0\|^2\right)^{\frac{\delta}{2} - 1}, \quad k \geq 1,
\]

and was firstly obtained by Stadje (1987) in relation to finite-velocity planar random motions (see also Kolesnik and Orsingher (2005)).

In De Gregorio and Orsingher (2012) and Le Caët (2011) is shown that, if displacements \((\tau_1, \ldots, \tau_k)\) have joint Dirichlet distribution with parameters \((\frac{\delta}{2} - 1, \ldots, \frac{\delta}{2} - 1)\), that is,

\[
f_2(\tau_1, \ldots, \tau_k) = \frac{\Gamma((k+1)(\frac{\delta}{2} - 1))}{\Gamma((\frac{\delta}{2} - 1)) \Gamma((\frac{\delta}{2} - 1))} \frac{1}{t^{k(\frac{\delta}{2} - 1) - 1}} \prod_{j=1}^{k+1} \frac{\tau_j^{\delta - 2}}{\tau_j!},
\]

the density of the vector \(Y_0(t) = (Y_1(t), \ldots, Y_0(t))\) becomes

\[
p_{Y_0}(y_0, t; k) = \frac{\Gamma((k+1)(\frac{\delta}{2} - 1) + 1)}{\Gamma((\frac{\delta}{2} - 1))} \frac{1}{t^{2k(\frac{\delta}{2} - 1) - 1}} \prod_{j=1}^{k+1} \frac{(\tau_j - 1)^{\delta - 2}}{\tau_j!},
\]

with \(\|y_0\| < ct\), \(k \geq 1\), \(\delta \geq 3\).

For \(\delta = 4\) we extract from (1.6)

\[
p_{Y_4}(y_4, t; k) = \frac{k(k+1)}{2\pi^2(t)} \left(c^2 t^2 - \|y_4\|^2\right)^{\frac{\delta}{2} - 1},
\]

which coincides with formula (1.5) of Orsingher and De Gregorio (2007).

In order to obtain the unconditional distributions of \(X_0(t)\) and \(Y_0(t)\), we here take into account a randomization different from that applied in De Gregorio and Orsingher (2012). In the case of \(X_0(t), t \geq 0\), we consider the following distribution

\[
P\{\Omega_0(t) = k\} = \frac{1}{E_{\delta-1,\delta-1}(\lambda t)^{\delta-1}} \Gamma((k+1)(\delta - 1)),
\]

with \(\lambda > 0\), \(\delta \geq 2\), \(k = 0, 1, \ldots\), and

\[
E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)},
\]

represents the generalized Mittag-Leffler function (see for example Kiryakova (2000)). We observe that (1.8) includes as a special case for \(\delta = 2\) the distribution of an homogeneous Poisson process. It can be regarded as a generalization of the Poisson process in the sense that will be discussed in section 3.
In the case of $Y_0(t)$, $t \geq 0$, the randomization is performed by means of the process with the following distribution

$$
P\{N_0(t) = k\} = \frac{1}{E_{\delta-2,3-1}((\lambda t)^{\delta-2}) \Gamma((\delta - 2)k + \delta - 1)},
$$

with $\lambda > 0$, $\delta \geq 3$, $k = 0, 1, \ldots$.

By randomizing the distribution (1.3) with (1.8) and (1.6) with (1.10) we obtain the unconditional distribution of $X_3(t)$ and $Y_0(t)$, for $\delta \geq 2$ in the first case and $\delta \geq 3$ in the second one. For some values of the dimension $\delta$ these distributions have an attractive form. For example for $\delta = 3$, that is for the most interesting case for the applications, we have that

$$
P\{X_3(t) \in dx_3\} = \left(\frac{\lambda}{2c}\right)^2 \frac{1}{\pi \sinh(\lambda t)} \frac{I_1\left(\frac{\lambda}{2c} \sqrt{c^2t^2 - \|x_3\|^2}\right)}{\sqrt{c^2t^2 - \|x_3\|^2}},
$$

and

$$
P\{Y_3(t) \in dy_3\} = \frac{\lambda}{2c \pi (e^{\lambda t} - 1)} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2c}\right)^{k+1} \frac{1}{\Gamma(k+1)} \frac{E_{\frac{1}{2},\frac{1}{2}+\frac{3}{2}}\left(\frac{\lambda}{2c} \sqrt{c^2t^2 - \|y_3\|^2}\right)}{\sqrt{c^2t^2 - \|y_3\|^2}},
$$

for $x_3$ and $y_3$ belonging to $S_3^3 := \{x \in \mathbb{R}^3 : \|x_3\|^2 \leq c^2t^2\}$.

In (1.11) we have $I_1(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+1} \frac{1}{k!(k+1)!}$

while

$$
E_{\frac{1}{2},\frac{1}{2}+\frac{3}{2}}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)\Gamma(k+3)},
$$

is a multi-index Mittag-Leffler function. While the distribution (1.11) is finite near $\partial S_3^3$, the density (1.12) converges to $+\infty$ as $\|y_3\| \to ct$.

We present the partial differential equations governing the space-dependent component of the distributions $P\{X_3(t) \in dx_3\}$ and $P\{Y_3(t) \in dy_3\}$. For the case $\delta = 3$, we prove that $P\{X_3(t) \in dy_3\}$ satisfies a telegraph-type equation with time-dependent coefficients (see (3.14)), while $P\{X_3(t) \in dy_3\}$ satisfies an higher order equation involving the second-order D’Alembert operator.

For a random flight in $\mathbb{R}^3$ where changes of direction occur only at even-valued Poisson times, the explicit distribution of the current position was discussed in De Gregorio and Orsingher (2012) (see formula (4.1) below). Here we prove that its probability law satisfies the three dimensional telegraph equation

$$
\frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} - c^2 \Delta u = 0.
$$
2. Preliminaries about McBride theory

For our analysis the integer power of hyper-Bessel operators play a crucial role. For this reason we devote this section to the presentation of the basic facts of McBride theory introduced in (1975) and then extended in a series of successive papers. Our starting point is the generalized hyper-Bessel operator, considered in McBride (1975),

\[ L = x^{a_1} D x^{a_2} \cdots x^{a_n} D x^{a_{n+1}}, \]

where \( n \) is an integer number, \( a_1, \ldots, a_{n+1} \) are complex numbers and \( D = d/dx \). Hereafter we assume that the coefficients \( a_j, j = 1, \ldots, n + 1 \) are real numbers. The operator \( L \) generalizes the classical \( n \)-th order hyper-Bessel operator \( L_{B_n} = x^{-n} x^{ \frac{d}{dx} } x^{ \frac{d}{dx} } \cdots x^{ \frac{d}{dx} } \) \( n \) times.

The operator \( L \) defined in \((2.1)\) acts on the functional space

\[ F_{p, \mu} = \{ f : x^{-\mu} f(x) \in F_p \}, \]

where

\[ F_p = \{ f \in C^\infty : x^k d^k f / dx^k \in L^p, k = 0, 1, \ldots \}, \]

for \( 1 \leq p < \infty \) and for any complex number \( \mu \) (see McBride, 1975, 1982, for details). The following lemma gives an alternative representation of the operator \( L \).

**Lemma 2.1.** The operator \( L \) in \((2.1)\) can be written as

\[ Lf = m^n x^{a-n} \prod_{k=1}^{n} x^{m-b_k} D_m x^{m b_k} f, \]

where

\[ D_m := \frac{d}{dx^m} = m^{-1} x^{1-m} \frac{d}{dx}. \]

The constants appearing in \((2.4)\) are defined as

\[ a = \sum_{k=1}^{n+1} a_k, \quad m = |a - n|, \quad b_k = \frac{1}{m} \left( \sum_{i=k+1}^{n+1} a_i + k - n \right), \quad k = 1, \ldots, n. \]

For the proof see lemma 3.1, page 525 of McBride (1982).

In the analysis of the integer power (as well as the fractional power) of the operator \( L \), a key role is played by \( D_m \) appearing in \((2.4)\).

**Lemma 2.2.** Let \( r \) be a positive integer, \( a < n \), \( f \in F_{p, \mu} \) and

\[ b_k \in A_{p, \mu, m} := \{ \eta \in C : \Re(m\eta + \mu) + m \neq 1/p - ml, l = 0, 1, 2, \ldots \}, \quad k = 1, \ldots, n. \]

Then

\[ L^r f = m^n x^{-mr} \prod_{k=1}^{n} I_m^{b_k} x^{m b_k} f, \]

where, for \( \alpha > 0 \) and \( \Re(m\eta + \mu) + m > 1/p \)

\[ (I_m^{\alpha} f)(x) = \frac{x^{-m\eta-ma}}{\Gamma(\alpha)} \int_0^x (x^m - u^m)^{\alpha-1} u^{m\eta} f(u) d(u^m), \]
and for $\alpha \leq 0$

$$
(I_m^{\eta,\alpha} f)(x) = (\eta + \alpha + 1) I_m^{\eta,\alpha+1} f + \frac{1}{m} I_m^{\eta,\alpha+1} \left( \frac{d}{dx} f \right).
$$

For the proof consult [McBride (1982)], page 525. By using similar arguments, it is also possible to give the fractional generalization $L^\alpha$ of $L$. A useful result that will be used in the following section is given by the Lemma

**Lemma 2.3.** Let be $\eta + \frac{\beta}{m} + 1 > 0$, $m \in \mathbb{N}$, we have that

$$
I_m^{\eta,\alpha} x^\beta = \frac{\Gamma \left( \eta + \frac{\beta}{m} + 1 \right)}{\Gamma \left( \alpha + \eta + 1 + \frac{\beta}{m} \right)} x^\beta.
$$

### 3. Random flights governed by higher order partial differential equations

#### 3.1. The first case.

We first consider the random flights treated in [De Gregorio and Orsingher (2012)] and strictly related to the finite velocity planar random motions studied in [Kolesnik and Orsingher (2005)]. The random flights in $\mathbb{R}^\theta$ consist of the triple $(\theta, \tau, \varpi_b(t))$, where the orientation $\theta$ has distribution $\mathcal{L}(1)$, $\tau$ represents the displacements and $\varpi_b(t)$ gives the number of changes of direction recorded in $(0, t)$. The displacements, in the first model, for $\theta \geq 2$ and $\varpi_b(t) = k$, have length $\tau = (\tau_1, \ldots, \tau_k)$ with joint distribution

$$
f_1(\tau_1, \ldots, \tau_k) = \frac{\Gamma ((k + 1)(\theta - 1))}{\Gamma (\theta - 1)^{k+1} \Gamma((k+1)(\theta-1)-1)} \prod_{j=1}^{k+1} \tau_j^{\theta-2},
$$

where $0 < \tau_j < t - \sum_{n=0}^{j-1} \tau_n$, $1 \leq j \leq k$, $\tau_{k+1} = t - \sum_{j=1}^{k} \tau_j$, which is a Dirichlet distribution with parameters $(\theta - 1, \ldots, \theta - 1)$. In [De Gregorio and Orsingher (2012)], it was shown (Theorem 2) that the distribution of the moving point $X_3(t) = (X_1(t), \ldots, X_3(t))$ (with intermediate displacements possessing joint distribution $f_1$) is given by

$$
p_{X_3}(x_3; t; k) = \frac{\Gamma (k+1(\theta-1)+\frac{1}{2}) (c^2 t^2 - \|x_3\|^2)^{\frac{\theta-1}{2}}}{\Gamma (\frac{\theta}{2}(\theta-1)) \pi^{\theta/2} (ct)^{(k+1)(\theta-1)-1}},
$$

with $\|x_3\| < ct$, $\theta \geq 2$.

In the case $\theta = 2$, the density (3.2) becomes

$$
p_{X_2}(x_2; t; k) = \frac{k}{2\pi (ct)^k} (c^2 t^2 - \|x_2\|^2)^{\frac{1}{2}}, \quad \|x_2\| \leq c t^2,
$$

and coincides with formula (11) of [Kolesnik and Orsingher (2005)] for planar random motions.

The number $\varpi_b(t)$ of changes of direction is represented by an extension of the Poisson process, whose parameters depend on the dimension $\theta$, and has distribution of the following form

$$
P\{\varpi_b(t) = k\} = \frac{1}{E_{\theta-1,\theta-1} ((\lambda t)^{\theta-1})} (\lambda t)^k \Gamma((k+1)(\theta-1))
$$

$$
= \frac{1}{E_{\theta-1,\theta-1,\theta-1,\theta-1} \frac{\theta-1}{2} \frac{1}{2}} \left( \frac{\lambda t}{2} \right)^{k(\theta-1)} \Gamma\left( \frac{k+1}{\theta}(\theta-1) + \frac{1}{2} \Gamma\left( \frac{2}{\theta}(\theta-1)(k+1) \right) \right),
$$

where $E_{\theta-1,\theta-1,\theta-1,\theta-1}$ is the Erlang distribution with parameters $(\theta-1, \theta-1, \theta-1, \theta-1)$. 


with \( \lambda > 0, d \geq 2, k = 0, 1, \ldots, \) and
\[
E_{\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \frac{\delta}{2}} \left( \frac{\lambda t}{2} \right)^{(d-1)} = \sum_{k=0}^{\infty} \left( \frac{\lambda t}{2} \right)^{k(d-1)} \frac{1}{\Gamma\left(\frac{d-1}{2}(d-1) + \frac{\delta}{2}\right)\Gamma\left(\frac{d-1}{2}(k+1)\right)}
\]
is the multi-index Mittag-Leffler function. The randomization of \( \mathcal{N}_d(t) \), applied here, is different from that considered in the paper by De Gregorio and Orsingher (2012). This different randomization permits us to arrive at the PDE’s governing the distribution of \( X(t) \). The distribution (3.1) generalizes the distribution of the homogeneous Poisson process which is retrieved as a special case for \( d = 2 \). The probability generating function of the generalized Poisson process \( \mathcal{N}_d(t), t \geq 0 \), is given by
\[
(3.5) \quad G_{\theta}(u, t) = \frac{E_{\theta-1, d-1} \left( (\lambda t)^{d-1} u \right)}{E_{\theta-1, d-1} \left( (\lambda t)^{d-1} \right)}.
\]
It is simple to prove that the function
\[
(3.6) \quad f(u, t) = u^{d-2} G_{\theta}(u^{d-1}, t),
\]
satisfies the ordinary differential equation of order \( d - 1 \)
\[
(3.7) \quad \frac{d^{d-1}f}{du^{d-1}}(u, t) = (\lambda t)^{d-1} f(u, t), \quad d \geq 2.
\]
In the special case \( d = 2 \), the function (3.6) coincides with the probability generating function of the homogeneous Poisson process.

By combining (3.2) and (3.4), we obtain the probability law
\[
(3.8) \quad P\{X_0(t) \in dX_0\} = \sum_{k=1}^{\infty} P\{X_0(t) \in dX_0|\mathcal{N}_0(t) = k\} P\{\mathcal{N}_0(t) = k\}
\]
\[
= \frac{dx_0}{\pi^{d/2}(ct)^{d-2} E_{\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \frac{\delta}{2}} \left( \frac{\lambda t}{2} \right)^{(d-1)}} \times \sum_{k=1}^{\infty} \left( \frac{\lambda}{2\pi c} \right)^{k(d-1)} \frac{\Gamma\left(\frac{d}{2}(d-1)\right)\Gamma\left(\frac{d}{2}(k+1)\right)}{\Gamma\left(\frac{d}{2}(d-1)\right)\Gamma\left(\frac{d}{2}(k+1)\right)}.
\]
We remark that for \( d = 2 \), we have that
\[
(3.9) \quad P\{X_2(t) \in dx_2\} = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t+x_1^2+x_2^2}}{\sqrt{c^2t^2-x_1^2-x_2^2}}.
\]
for \( x_1^2 + x_2^2 < c^2t^2 \); which is the absolutely continuous component of the distribution of a point \((X_1(t), X_2(t))\) performing the planar random motion studied in Kolesnik and Orsingher (2005). In this paper the authors proved that the density (3.9) is the fundamental solution to the planar telegraph equation (also equation of damped waves)
\[
(3.10) \quad \frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} = c^2 \left\{ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right\} u.
\]
We are now ready to state the following

**Theorem 3.1.** The function
\[
(3.11) \quad f(x, t) = \pi^{d/2}(ct)^{d-2} E_{\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \frac{\delta}{2}} \left( \frac{\lambda t}{2} \right)^{(d-1)} \frac{P\{X_0(t) \in dx_0\}}{\prod_{j=1}^{d} dx_j}
\]
We now consider the function $$f$$ appearing in (3.13) can be considered again as a particular case of the operator (2.1) with $$k$$.

To begin with, we observe that by means of the transformation

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \quad d \geq 2.$$

Proof. To begin with, we observe that by means of the transformation

$$w = \left(c^2 t^2 - \|x_0\|^2\right)^{1/2},$$

applied in (3.12) we obtain

$$\left(\frac{d^2}{dw^2} + \frac{\partial}{w} \frac{d}{dw}\right)^{d-1} u(w) = \left(\frac{\lambda^2}{c^2}\right)^{d-1} u(w).$$

The operator appearing in (3.13) can be considered again as a particular case of the operator (2.1) with $$a_1 = -d, a_2 = d, a_3 = 0, a = 0, n = m = 2, b_1 = \frac{d-1}{2}$$ and $$b_2 = 0$$. Hence, from Lemma 2.2 we have that

$$L^{d-1} u(w) = 4^{d-1} w^{-2(d-1)} I_2^{0,1-d} I_2^{0,1-d} u(w).$$

In view of Lemma 2.2 we observe that

$$L^{d-1} w^\beta = 4^{d-1} w^{-2(d-1)} \frac{\Gamma\left(\frac{d}{2} + 1 + \frac{\beta-1}{2}\right) \Gamma\left(\frac{\beta}{2} + 1\right)}{\Gamma\left(\frac{d-1}{2} + \frac{\beta}{2} + 2 - d\right) \Gamma\left(\frac{\beta}{2} + 2 - d\right)}.$$

We now consider the function $$f(x, t)$$ in the new variable $$w$$

$$f(w) = \sum_{k=1}^{\infty} \left(\frac{\lambda}{2c}\right)^{k(d-1)} \frac{w^{k(d-1)-2}}{\Gamma\left(k\left(\frac{d-1}{2}\right)\right) \Gamma\left(\frac{d-1}{2}(k + 1)\right)}.$$

From the previous calculations we have that

$$L^{d-1} f(w) = 4^{d-1} \sum_{k=1}^{\infty} \left(\frac{\lambda}{2c}\right)^{k(d-1)} \frac{w^{k(d-1)-2-2(d-1)}}{\Gamma\left(\frac{k-1}{2}k + 1 - d\right) \Gamma\left(\frac{k}{2}(k + 1)\right) \Gamma\left(k'\left(\frac{d-1}{2}\right)\right) \Gamma\left(\frac{d}{2}(k + 1)\right)}.$$

By returning to the variables $$(x, t)$$, we finally obtain the claimed result.
We now concentrate our attention to random flights in $\mathbb{R}^3$, which is clearly relevant for applications. From (3.8), we have that the absolutely continuous component of the probability law is given by

$$P\{X_3(t) \in dx_3\} = \left(\frac{\lambda}{2c}\right)^2 \frac{1}{\pi \sinh(\lambda t)} \frac{I_1\left(\frac{\lambda}{2c} \sqrt{c^2t^2 - \|x_3\|^2}\right)}{\sqrt{c^2t^2 - \|x_3\|^2}} = p(\|x_3\|, t).$$

Since

$$\int_{S^3_{ct}} P\{X_3(t) \in dx_3\} = 1 - \frac{\lambda t}{\sinh(\lambda t)} = 1 - P\{N_3(t) = 0\},$$

the distribution of $X_3(t)$ has a singular component uniformly distributed on $\partial S^3_{ct}$, because the particle has initial uniformly distributed orientation.

**Theorem 3.2.** The probability law $p(x, t) = \frac{P\{X_3(t) \in dx_3\}}{\Pi_{j=1}^3 dx_j}$ of the random flight in $\mathbb{R}^3$, is governed by the fourth-order, homogeneous partial differential equation with time-varying coefficients

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \Delta\right)^2 p(x, t) + 2\lambda \left(\frac{\partial^2}{\partial t^2} - c^2 \Delta\right) \left(\lambda + 2b(t) \frac{\partial}{\partial t}\right) p(x, t) + 4\lambda^2 \left(\frac{\partial^2}{\partial t^2} + \lambda^2 b(t) \frac{\partial}{\partial t}\right) p(x, t) = 0,$$

where $x \in \mathbb{R}^3$ and

$$b(t) = \coth(\lambda t).$$

**Proof.** From Theorem (3.1), we have that the function

$$f(x, t) = \left(\frac{2c}{\lambda}\right)^2 \pi \sinh(\lambda t) \frac{P\{X_3(t) \in dx_3\}}{\Pi_{j=1}^3 dx_j},$$

satisfies the higher order 3-dimensional Klein-Gordon-type equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \Delta\right)^2 u(x, t) = \lambda^4 u(x, t)$$

Substituting the function (3.21) to (3.22), we obtain the governing equation for the probability law of the random flight.

We now consider the distribution $q_1(x_2, t) = q_1(x_1, x_2, t)$ of the projection of the absolutely continuous component of the probability law of $X_3(t)$ onto the plane

$$q_1(x_1, x_2, t) = \int_{\sqrt{c^2t^2 - \|x_3\|^2}} \frac{P\{X_3(t) \in dx_3\}}{\Pi_{j=1}^3 dx_j} p(\|x_3\|, t)dx_3$$

$$= \int_{\sqrt{c^2t^2 - \|x_3\|^2}} \left(\frac{\lambda}{2c}\right)^2 \frac{1}{\pi \sinh(\lambda t)} \frac{I_1\left(\frac{\lambda}{2c} \sqrt{c^2t^2 - \|x_3\|^2}\right)}{\sqrt{c^2t^2 - \|x_3\|^2}} dx_3$$

$$= \left(x_3 = \sqrt{w} \sqrt{c^2t^2 - x_1^2 - x_2^2}\right).$$
RANDOM FLIGHTS GOVERNED BY KLEIN-GORDON-TYPE PARTIAL DIFFERENTIAL EQUATIONS

\[
\begin{align*}
\lambda^2 c^2 & \sum_{k=0}^{\infty} \left( \frac{\lambda}{2c} \right)^{2k+1} \frac{1}{k!(k+1)!} \int_0^1 \left( \sqrt{1 - w^2} \right)^{2k} w^{-1/2} dw \\
& = \frac{\lambda}{2\pi c \sinh(\lambda t)} \frac{1}{\sqrt{c^2 t^2 - x_1^2 - x_2^2}} \\
& \times \cosh \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x_1^2 - x_2^2} \right) - 1,
\end{align*}
\]

for \( x_1^2 + x_2^2 \leq c^2 t^2 \). Since the projection of the singular component of the distribution of \( X_3(t) \) onto the plane \( (x_1, x_2) \) is equal to

(3.24) \[
q_2(x_1, x_2, t) = \frac{\lambda}{2\pi c \sinh(\lambda t)} \frac{1}{\sqrt{c^2 t^2 - x_1^2 - x_2^2}},
\]

we have that the projection of the distribution of \( X_3(t) \) is given by

(3.25) \[
p(x_1, x_2, t) = q_1(x_1, x_2, t) + q_2(x_1, x_2, t)
\]

\[
= \frac{\lambda}{2\pi c \sinh(\lambda t)} \frac{1}{\sqrt{c^2 t^2 - x_1^2 - x_2^2}} \cosh \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x_1^2 - x_2^2} \right)
\]

We can also consider the projection \( X_1(t) \) of the distribution \( X_3(t) \) on the line. The distribution of \( X_1(t) \) has a fine form and reads

(3.26) \[
p(x_1, t) = \frac{\lambda I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x_1^2} \right)}{2c \sinh(\lambda t)}, \quad |x| < ct.
\]

Furthermore (3.26) is a solution to the telegraph-type equation

(3.27) \[
\frac{\partial^2 p}{\partial t^2} + 2\lambda \coth(\lambda t) \frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial x^2}.
\]

This can be checked by considering that

(3.28) \[
\frac{2c}{\lambda} p(x_1, t) \cdot \sinh(\lambda t) = I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x_1^2} \right),
\]

and the function

\[
q(x_1, t) = I_0 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - x_1^2} \right),
\]

solves the equation

(3.29) \[
\frac{\partial^2 q}{\partial t^2} - \lambda^2 q = c^2 \frac{\partial^2 q}{\partial x^2}.
\]

In the same way it is simple to prove that the distribution (3.25) solves the two-dimensional telegraph-type equation

(3.30) \[
\left( \frac{\partial^2}{\partial t^2} + 2\lambda \coth(\lambda t) \frac{\partial}{\partial t} - c^2 \Delta \right) p(x_1, x_2, t) = 0.
\]

We observe that the distribution of the random flight \( X_3(t) \) satisfies a fourth-order p.d.e. while its projections on the plane and on the line are directed by second-order p.d.e.’s of the telegraph form with one time-varying coefficient.
The function

\[
f_2(\tau_1, \ldots, \tau_k) = \frac{\Gamma((k + 1)(\frac{3}{2} - 1))}{\Gamma(\frac{3}{2} - 1)^{k+1}} \frac{1}{c^{(k+1)(\frac{3}{2}-1)-1}} \prod_{j=1}^{k} \tau_j^{\frac{3}{2}-2},
\]

where \(0 < \tau_j < t - \sum_{n=0}^{k} \tau_n, 1 \leq j \leq k\), \(\tau_{k+1} = t - \sum_{j=1}^{k} \tau_j\). This kind of random flights were considered in Le Caët (2011) and De Gregorio and Orsingher (2012), where it was shown (Theorem 2) that the corresponding distribution of the moving point \(Y_\delta(t) = (Y_1(t), \ldots, Y_\delta(t))\) is given by

\[
p_{Y_\delta}(Y_\delta, t; k) = \frac{\Gamma((k + 1)(\frac{3}{2} - 1) + 1)}{\Gamma(\frac{3}{2} - 1)^{k+1}} \frac{1}{\pi^{\delta/2} (ct)^{(2k+1)(\frac{3}{2}-1)-1}},
\]

with \(|Y_\delta| < ct\). In order to obtain the unconditional distributions, we assume here that the random number of changes of direction is endowed with the following distribution (depending on the dimension \(\delta\) of the space)

\[
P(N_\delta(t) = k) = \frac{1}{E_{\delta-2,3-1}((\lambda t)^{\delta-2})} \frac{(\lambda t)^{k(\delta-2)}}{\Gamma((\delta-2)k + \delta - 1)}
\]

\[= \frac{1}{E_{\approx-1,\approx-1} (\approx)^{\approx-2}} \sum_{k=0}^{\infty} \left(\frac{\lambda t}{2}\right)^{k(\delta-2)} \frac{1}{\Gamma((k + 1)(\frac{3}{2} - 1) + 1)\Gamma(\frac{3}{2} - 1) + (\frac{3}{2} - 1)k)},
\]

with \(\lambda > 0\), \(\delta \geq 3\), \(k = 0, 1, \ldots\), and where

\[
E_{\approx-1,\approx-1} (\approx)^{\approx-2} = \sum_{k=0}^{\infty} \left(\frac{\lambda t}{2}\right)^{k(\delta-2)} \frac{1}{\Gamma((k + 1)(\frac{3}{2} - 1) + 1)\Gamma(\frac{3}{2} - 1) + (\frac{3}{2} - 1)k)}
\]

is the multi-index Mittag-Leffler function (see for example Kiryakova (2000) and references therein). By combining \((3.32)\) and \((3.33)\), we obtain the probability law

\[
P(Y_\delta(t) \in dY_\delta) = \frac{1}{\prod_{j=1}^{\delta} dy_j} \pi^{\delta/2} (ct)^{\delta-2} \sum_{k=1}^{\infty} \left(\frac{\lambda t}{2c}\right)^{k(\delta-2)} \frac{1}{\Gamma(k(\delta-2))(\delta-2))} \frac{(\lambda t)^{\delta-2}}{\Gamma((k + 1)(\frac{3}{2} - 1) + 1)\Gamma(\frac{3}{2} - 1) + (\frac{3}{2} - 1)k)}
\]

We are now ready to state the following

**Theorem 3.3.** The function

\[
f(y, t) = \pi^{\delta/2} (ct)^{\delta-2} \sum_{k=1}^{\infty} \left(\frac{\lambda t}{2c}\right)^{k(\delta-2)} \frac{1}{\Gamma(k(\delta-2))(\delta-2))} \frac{(\lambda t)^{\delta-2}}{\Gamma((k + 1)(\frac{3}{2} - 1) + 1)\Gamma(\frac{3}{2} - 1) + (\frac{3}{2} - 1)k)}
\]

solves the \(\delta\) – dimensional higher order non-homogeneous Klein-Gordon equation

\[
\left(\frac{\partial^2}{\partial t^2} - c^2 \Delta\right)^{\delta-2} u(y, t) = \lambda^{2(\delta-2)} u(y, t) + (2\lambda c)^{3-2} \frac{(c^2 t^2 - \|Y_\delta\|^2)^{\delta/2}}{\sqrt{\pi} \Gamma(1 - \frac{\delta}{2})},
\]
where \( \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial y_j^2} \).

**Proof.** The proof follows the same reasoning used in Theorem 3.1. To begin with, we observe that by means of the transformation

\[
\begin{align*}
(3.37) & \\
\left( \frac{d^2}{dw^2} + \frac{\partial}{w \, dw} \right)^{\beta-2} u(w) = \left( \frac{\lambda^2}{c^2} \right)^{\beta-2} u(w) + \left( \frac{2\lambda}{c} \right)^{\beta-2} \frac{w^{-\beta}}{\sqrt{\pi} \Gamma(1 - \frac{\beta}{2})}.
\end{align*}
\]

The operator appearing in (3.37) can be considered again as a specific case of the operator \( (3.41) \) with \( a_1 = -\delta, a_2 = 0, a_3 = 0, a = 0, n = m = 2, b_1 = \frac{d-1}{2}, b_2 = 0 \). Hence, from Lemma 2.2 we have that

\[
(3.38) \quad \left( \frac{d^2}{dw^2} + \frac{\partial}{w \, dw} \right)^{\beta-2} u(w) = L^{\beta-2} u(w) = 4^{\beta-2} w^{2(\beta-2)} I_2^{\beta-2} I_2^{\beta-2} \frac{w^{-\beta}}{L^{\beta-2}}.
\]

In view of Lemma 2.3 we observe that

\[
(3.39) \quad L^{\beta-2} w^\beta = 4^{\beta-2} w^{\beta-2(\beta-2)} \frac{\Gamma(\beta \frac{\beta}{2} + 1 + \frac{\beta-1}{2}) \Gamma(\beta \frac{\beta}{2} + 1)}{\Gamma(\beta \frac{\beta}{2} + 1 + \frac{\beta}{2} + 2 - \delta) \Gamma(\beta \frac{\beta}{2} - \delta + 3)}.
\]

We now take into account the function \( f(y,t) \) in the new variable \( w \)

\[
f(w) = \sum_{k=1}^{\infty} \left( \frac{\lambda}{2c} \right)^{k(\beta-2)} w^{k(\beta-2)} \frac{\Gamma(k(\beta \frac{\beta}{2} - 1)) \Gamma(\frac{\beta}{2} + \frac{\beta}{2} - 1)}{\Gamma(\frac{\beta}{2} + (\frac{\beta}{2} - 1)k)}
\]

From the previous calculations we have that

\[
(3.40) \quad L^{\beta-2} f(w) = 4^{\beta-2} \sum_{k=1}^{\infty} \left( \frac{\lambda}{2c} \right)^{k(\beta-2)} w^{k(\beta-2)-2(\beta-2)} \frac{\Gamma(k(\beta \frac{\beta}{2} - 1)) \Gamma(\frac{\beta}{2} + (\frac{\beta}{2} - 1)k)}{\Gamma(\beta \frac{\beta}{2} - \delta + 2) \Gamma(\beta \frac{\beta}{2} + 1 + \frac{\beta-1}{2})}
\]

\[
= 4^{\beta-2} \sum_{k'=1}^{\infty} \left( \frac{\lambda}{2c} \right)^{(k+k')(\beta-2)} w^{k'(\beta-2)} \frac{\Gamma(k'(\beta \frac{\beta}{2} - 1)) \Gamma(\frac{\beta}{2} + (\frac{\beta}{2} - 1)k')}{\Gamma(k(\beta \frac{\beta}{2} - 1)) \Gamma(\frac{\beta}{2} + (\frac{\beta}{2} - 1)k)}
\]

\[
= \left( \frac{\lambda}{c} \right)^{2(\beta-2)} f(w) + \left( \frac{\lambda}{2c} \right)^{2-\beta} \frac{w^{-\beta}}{\sqrt{\pi} \Gamma(1 - \frac{\beta}{2})}.
\]

where \( k' = k - 2 \). This means that \( f(w) \) satisfies the following equation

\[
(3.41) \quad \left( \frac{d^2}{dw^2} + \frac{\partial}{w \, dw} \right)^{\beta-2} u(w) = \left( \frac{\lambda^2}{c^2} \right)^{\beta-2} u(w) + \left( \frac{2\lambda}{c} \right)^{\beta-2} \frac{w^{-\beta}}{\sqrt{\pi} \Gamma(1 - \frac{\beta}{2})}.
\]

By going back to the variables \( (y,t) \), we arrive at the claimed result. \( \square \)

**Remark 3.4.** We observe that the inhomogeneous term in (3.36) vanishes for all even values of \( \delta \geq 4 \).

From (3.34) we have that the absolutely continuous component of the probability law of the random flight in \( \mathbb{R}^3 \) is given by

\[
(3.42) \quad P\{Y_3(t) \in dy_3\} = \frac{\lambda}{2c \pi(e^\delta - 1)} \sum_{k=0}^{\infty} \left( \frac{\lambda}{2c} \right)^{k+1} \frac{\sqrt{c^2t^2 - \|Y_3\|^2}}{\Gamma(k+\delta) \Gamma(k+\delta^2)}.
\]
The singular part of the distribution of $e^{Y_3(t)}$ is uniform on $S^3_{ct}$ and has weight equal to
\begin{equation}
(3.43) \quad \int_{S^3_{ct}} P\{Y_3(t) \in dy_3\} = 1 - \frac{\lambda t}{e^{\lambda t} - 1} = 1 - P\{N_3(t) = 0\}.
\end{equation}

**Theorem 3.5.** The probability law of the random flight in $\mathbb{R}^3$ is governed by the following non-homogeneous 3-d telegraph equation with variable coefficients
\begin{equation}
(3.44) \quad \left( \frac{\partial^2}{\partial t^2} + c_1(t) \frac{\partial}{\partial t} - c^2 \Delta \right) u(y, t) = c_2(t) u(y, t) + c_3(y, t),
\end{equation}
where $y \in \mathbb{R}^3$ and
\begin{align*}
c_1(t) &= \frac{2\lambda c^{\lambda t}}{e^{\lambda t} - 1} \\
c_2(t) &= -\frac{\lambda^2}{e^{\lambda t} - 1} \\
c_3(y, t) &= \frac{\lambda^2}{\sqrt{\pi^3(c^{\lambda t} - 1)}} \frac{(c^2 t^2 - \|y_3\|^2)^{-3/2}}{\Gamma(-\frac{1}{2})}.
\end{align*}

**Proof.** From the previous theorem, we have that the function
\begin{equation}
(3.45) \quad f(y, t) = 2\pi c t \frac{e^{\lambda t} - 1}{\lambda t} P\{Y_3(t) \in dy_3\},
\end{equation}
satisfies the inhomogeneous 3-d Klein-Gordon equation
\begin{equation}
(3.46) \quad \left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) u(y, t) = \lambda^2 u(y, t) + \left(2\lambda c\right) \frac{(c^2 t^2 - \|y_3\|^2)^{-3/2}}{\sqrt{\pi^3(-\frac{1}{2})}}
\end{equation}
By substituting the function (3.45) into (3.46), we obtain the equation governing the probability law of the random flight $Y_3(t)$.

**Remark 3.6.** The law of the projection $(Y_1(t), Y_2(t))$ on the plane of the random motion considered here $(Y_3(t))$ reads
\begin{equation}
(3.47) \quad P\{Y_1(t) \in dy_1, Y_2(t) \in dy_2\} = \frac{\lambda dy_1 dy_2}{2\pi c(e^{\lambda t} - 1)} \frac{e^{\lambda \sqrt{c^2 t^2 - (y_1^2 + y_2^2)}}}{\sqrt{c^2 t^2 - (y_1^2 + y_2^2)}},
\end{equation}
for $(y_1, y_2) \in C_{ct}$, where $C_{ct} = \{y_1, y_2 : y_1^2 + y_2^2 \leq c^2t^2\}$.
This can be checked by integrating (3.42) w.r. to $y_3$ and then by summing the contribution of the projection of the singular component of the distribution as performed in (3.25).
Furthermore, we observe that the two-dimensional distribution (3.47) satisfies the following time-varying telegraph-type equation
\begin{equation}
(3.48) \quad \left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) p + c_1(t) \frac{\partial p}{\partial t} - c_2(t) p = 0,
\end{equation}
where the functions $c_1(t)$ and $c_2(t)$ are defined in Theorem 3.5. The behavior of the distributions (3.25) and (3.47) near the edge of the circles $C_{ct}$ is similar. We note that
\begin{equation}
(3.49) \quad \lim_{y_1, y_2 \to 0} P\{Y_1(t) \in dy_1, Y_2(t) \in dy_2\} = \frac{\lambda}{2\pi c^2 t} \frac{1}{1 - e^{-\lambda t}},
\end{equation}
while

\[ \lim_{x_1,x_2 \to 0} \frac{\text{P}\{X_1(t) \in dx_1, X_2(t) \in dx_2\}}{dx_1 dx_2} = \frac{\lambda}{2\pi e^{\lambda t}} e^{\lambda t} - e^{-\lambda t}. \]

**Remark 3.7.** From (3.47) we can infer that

\[ \text{P}\{Y_1(t) \in dy_1\} = \frac{\lambda}{2c(e^{\lambda t} - 1)} \sum_{k=0}^{\infty} \left( \frac{\lambda}{2c} \sqrt{c^2 t^2 - y_1^2} \right)^k \frac{1}{[\Gamma\left(\frac{k}{2} + 1\right)]^2}. \]

This distribution is similar to (3.28) which was obtained as a projection of the planar random motion where changes of direction are paced by a homogeneous process. Both these probability distributions refer to one-dimensional random motions with random velocities (see also Stadje and Zacks (2004) on this point).

4. **Three-dimensional random flights governed by a Poisson process**

In De Gregorio and Orsingher (2012) a random motion in \( \mathbb{R}^3 \) governed by a Poisson process was introduced. In more detail, the authors studied a random motion where particles change direction only at even-valued Poisson events. They show that, in this case, the unconditional probability law is given by

\[ \text{P}\{U_3(t) \in du_3, \bigcup_{k=1}^{\infty} (N(t) = 2k + 1)\} = e^{-\lambda t} \frac{\lambda}{\pi} \left( \frac{\lambda}{2c} \sqrt{c^2 t^2 - \|u_3\|^2} \right) I_1 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - \|u_3\|^2} \right). \]

**Theorem 4.1.** The probability law (4.1) satisfies the 3-d telegraph equation

\[ \left( \frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} - c^2 \Delta \right) u(x, t) = 0, \]

where \( x \in \mathbb{R}^3 \).

**Proof.** By means of the exponential substitution

\[ u(x, t) = e^{-\lambda t} f(x,t), \]

(4.2) reduces to

\[ \left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) f(x, t) = \lambda^2 f(x, t). \]

By using the transformation

\[ w = \sqrt{c^2 t^2 - \|u_3\|^2}, \]

we convert (4.4) to the Bessel equation

\[ \frac{d^2f}{dw^2} + \frac{1}{w} \frac{df}{dw} = \frac{\lambda^2}{c^2} f. \]

We now observe that the function (4.1) can be written as

\[ \text{P}\{U_3(t) \in du_3, \bigcup_{k=1}^{\infty} (N(t) = 2k + 1)\} = e^{-\lambda t} f(x, t), \]

where

\[ f(x, t) = \frac{1}{\pi} \left( \frac{\lambda}{2c} \right)^2 \frac{1}{\sqrt{c^2 t^2 - \|u_3\|^2}} I_1 \left( \frac{\lambda}{c} \sqrt{c^2 t^2 - \|u_3\|^2} \right), \]

solves (4.2) and thus the proof of the theorem is complete. \( \Box \)
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