ON PRODUCT OF DIFFERENCE SETS FOR SETS OF
POSITIVE DENSITY

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Abstract. In this paper we prove that given two sets \(E_1, E_2 \subseteq \mathbb{Z}\) of positive density, there exists \(k \geq 1\) which is bounded by a number depending only on the densities of \(E_1\) and \(E_2\) such that \(k\mathbb{Z} \subseteq (E_1 - E_1) \cdot (E_2 - E_2)\). As a corollary of the main theorem we deduce that if \(\alpha, \beta > 0\) then there exist \(N_0\) and \(d_0\) which depend only on \(\alpha\) and \(\beta\) such that for every \(N \geq N_0\) and \(E_1, E_2 \subseteq \mathbb{Z}_N\) with \(|E_1| \geq \alpha N, |E_2| \geq \beta N\) there exists \(d \leq d_0\) a divisor of \(N\) satisfying \(d\mathbb{Z}_N \subseteq (E_1 - E_1) \cdot (E_2 - E_2)\).

1. Introduction

One of the main themes of additive combinatorics is sum-product estimates. It goes back to Erdős and Szemerédi [3] who conjectured that for any finite set \(A \subseteq \mathbb{Z}\) (or in \(\mathbb{R}\), for every \(\varepsilon > 0\) we have

\[ |A + A| + |A \cdot A| \gg |A|^{2-\varepsilon}, \]

where the \(A + A = \{a + b | a, b \in A\}\), and \(A \cdot A = \{ab | a, b \in A\}\). Currently the best known estimate is due to Konyagin-Shkredov [6] and it is based on the beautiful previous breakthrough work by Solymosi [7]:

\[ |A + A| + |A \cdot A| \gg |A|^{4/3+c}, \]

for any \(c < 5/9813\).

In this paper we study a slightly twisted, but nevertheless related, sum-product phenomenon. Namely, we address the following

Question 1. For a given infinite set \(E \subseteq \mathbb{Z}\), how much structure does possess the set \((E - E) \cdot (E - E)\) ?

We will restrict our attention to sets having positive density, see the definition below.

Furstenberg [5] noticed a intimate connection between difference sets for sets of positive density, and the sets of return times of a set of positive measure in measure-preserving systems. In this paper we will establish an arithmetic richness of a set of return times of a set of a positive measure to itself within a measure-preserving system. Recall that a triple \((X, \mu, T)\) is a measure-preserving system if \(X\) is a compact metric space, \(\mu\) is a probability
measure on the Borel σ-algebra of \( X \), and \( T : X \to X \) is a bi-measurable map which preserves \( \mu \). For a measurable set \( A \subset X \) with \( \mu(A) > 0 \) the set of return times from \( A \) to itself is:

\[
R(A) = \{ n \in \mathbb{Z} \mid \mu(A \cap T^n A) > 0 \}.
\]

We will denote by \( E^2 = \{ e^2 \mid e \in E \} \) the set of squares of \( E \subset \mathbb{Z} \). It has been proved by Björklund and the author [2] that for any three sets of positive measure \( A, B, \) and \( C \) in measure-preserving systems there exists \( k \geq 1 \) (depending on the sets \( A, B, \) and \( C \)) such that \( k \mathbb{Z} \subset R(A) \cdot R(B) - R(C)^2 \). One of the motivations for this work was to show that \( k \) in the latter statement depends only on the measures of the sets \( A, B, \) and \( C \). We prove the latter, and even more surprisingly, we show that \( R(C) \) can be omitted. We have

**Theorem 1.1.** Let \( (X, \mu, T) \) and \( (Y, \nu, S) \) be measure-preserving systems, and let \( A \subset X, B \subset Y \) be measurable sets with \( \mu(A) > 0 \), and \( \nu(B) > 0 \). Then there exist \( k_0 \) depending only on \( \mu(A) \) and \( \nu(B) \), and \( k \leq k_0 \) such that \( k \mathbb{Z} \subset R(A) \cdot R(B) \).

This result has a few combinatorial consequences. To state the first application, we recall that the upper Banach density of a set \( E \subset \mathbb{Z} \) is defined by

\[
d^*(E) = \limsup_{N \to \infty} \sup_{a \in \mathbb{Z}} \frac{|E \cap \{a, a+1, \ldots, a+(N-1)\}|}{N}.
\]

Through Furstenberg’s correspondence principle [5], we obtain

**Corollary 1.1.** Let \( E_1, E_2 \subset \mathbb{Z} \) be sets of positive upper Banach density. Then there exist \( k_0 \) which depends only on the densities of \( E_1 \) and \( E_2 \) and \( k \leq k_0 \) such that

\[
k \mathbb{Z} \subset (E_1 - E_1) \cdot (E_2 - E_2).
\]

Another application of Theorem 1.1 is the following result.

**Corollary 1.2.** For any \( \alpha, \beta > 0 \) there exist \( N_0 \) and \( d_0 \), depending only on \( \alpha \) and \( \beta \), such that for every \( N \geq N_0 \) and \( E_1, E_2 \subset \mathbb{Z}_N \) with \( |E_1| \geq \alpha N, |E_2| \geq \beta N \) there exists \( d \leq d_0 \) which is a divisor of \( N \) and \( d \mathbb{Z}_N \subset (E_1 - E_1) \cdot (E_2 - E_2) \).

Corollary 1.2 implies also that if \( p \) is a large enough prime and \( E_1, E_2 \subset \mathbb{Z}_p \) satisfy \( |E_1| \geq \alpha p, |E_2| \geq \beta p \), then \( (E_1 - E_1) \cdot (E_2 - E_2) = \mathbb{Z}_p \). This also follows from a result by Hart-Iosevich-Solymosi [4] who proved that if \( E \subset \mathbb{F}_q \) (where \( \mathbb{F}_q \) is a field with \( q \) elements) with \( |E| \geq q^{3/4+\varepsilon} \) then for \( q \) large enough \( (E - E) \cdot (E - E) = \mathbb{F}_q \).

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2. Proof of Theorem 1.1

Let us assume that \((X, \mu, T)\) is a measure-preserving system, and let \(A \subset X\) be a measurable set with \(\mu(A) > 0\). Recall that the set of return times of \(A\) is defined by
\[
R(A) = \{n \in \mathbb{Z} \mid \mu(A \cap T^n A) > 0\}.
\]

The theorem will follow from the following statement.

Lemma 2.1. For every \(L \geq 1\) and every \(b \in \mathbb{Z} \setminus \{0\}\) there exists \(m \leq \lfloor \frac{1}{\mu(A)} \rfloor + 1\) such that
\[
\{mb, 2mb, \ldots, Lmb\} \subset R(A).
\]

Indeed, let \(R(A)\) and \(R(B)\) be sets of return times for measurable sets \(A\) and \(B\) of positive measures. Then choose \(N = \lfloor \frac{1}{\nu(B)} \rfloor + 1\). Then for every \(b \in \mathbb{Z} \setminus \{0\}\) there exist \(1 \leq i < j \leq N\) such that \(\nu((S^b)^i B \cap (S^b)^j B) > 0\). Then by \(S\)-invariance of \(\nu\) it follows that there exists \(1 \leq m \leq N\) \((m = j - i)\) such that \(mb \in R(B)\).

Let us define \(L = N!\). By Lemma 2.1 there exists \(n = n(L, \mu(A))\) such that for every \(b \in \mathbb{Z} \setminus \{0\}\) there exists \(m \leq n\) with \(\{mb, 2mb, \ldots, Lmb\} \in R(A)\).

Let us define \(k = L \cdot n!\). Take any \(b \in \mathbb{Z} \setminus \{0\}\). By the choice of \(n\), there exists \(m \leq n\) such that \(\{mb, 2mb, \ldots, Lmb\} \in R(A)\). By the choice of \(N\) it follows that there exists \(1 \leq j \leq N\) such that \(j \cdot \frac{k}{Lm} \in R(B)\). Also, \(\frac{Lm}{j}\) is an integer less or equal than \(Lm\), therefore \(\frac{Lm}{j} b \in R(A)\). Thus \(kb = \frac{Lm}{j} b \cdot j \cdot \frac{k}{Lm} \in R(A) \cdot R(B)\). This finishes the proof of Theorem 1.1.

Proof of Lemma 2.1. Let \((X, \mu, T)\) be a measure-preserving system, and let \(A \subset X\) be a measurable set, and let \(b \in \mathbb{Z} \setminus \{0\}\). We introduce a new product system \(Z = \prod_{i=1}^{L} X\) with the transformation \(S = \prod_{i=1}^{L} T^{ib}\) and the product measure \(\nu = \prod_{i=1}^{L} \mu\). Then \((Z, \nu, S)\) is a measure-preserving system, and the set \(\tilde{A} = \prod_{i=1}^{L} A\) has measure
\[
\nu(\tilde{A}) = \mu(A)^L > 0.
\]

Then by Poincaré lemma there exists \(m \leq \lfloor \frac{1}{\mu(A)} \rfloor + 1\) such that
\[
\nu(\tilde{A} \cap S^m \tilde{A}) > 0.
\]

The latter means that for every \(1 \leq i \leq L\) we have
\[
\mu(A \cap T^{ibm} A) > 0.
\]

This proof of the lemma has been proposed to the author by I. Shkredov.
Therefore, we have \( \{bm, 2bm, \ldots, Lbm\} \in R(A) \) for \( m \leq \left\lfloor \frac{1}{\mu(A)} \right\rfloor + 1 \).

3. Proofs of Corollaries 1.1 and 1.2

Furstenberg [5] in his seminal work on Szemerédi’s theorem showed:

**Correspondence Principle.** Given a set \( E \subset \mathbb{Z} \) there exists a measure-preserving system \((X, \mu, T)\) and a measurable set \( A \subset X \) such that for all \( n \in \mathbb{Z} \) we have

\[
d^*(E \cap (E + n)) \geq \mu(A \cap T^n A),
\]

and

\[
d^*(E) = \mu(A).
\]

**Proof of Corollary 1.1.** Let \( E_1, E_2 \subset \mathbb{Z} \) be sets of positive densities. Then by Furstenberg’s correspondence principle there exist measure-preserving systems \((X, \mu, T)\) and \((Y, \nu, S)\) and measurable sets \( A \subset X, B \subset Y \) that satisfy

\[
\mu(A) = d^*(E_1), \quad \nu(B) = d^*(E_2)
\]

and

\[
R(A) \subset E_1 - E_1, \quad R(B) \subset E_2 - E_2.
\]

By Theorem 1.1 there exist \( k(\mu(A), \nu(B)) \) and \( k \leq k(\mu(A), \nu(B)) \) such that \( k \mathbb{Z} \subset R(A) \cdot R(B) \). The latter statement implies the conclusion of the corollary.

**Proof of Corollary 1.2.** Let \( \alpha > 0, \beta > 0 \) and let \( E_1, E_2 \subset \mathbb{Z}_N \) with \( |E_1| \geq \alpha N \), and \( |E_2| \geq \beta N \). It is clear that \( X = \mathbb{Z}_N \) with the shift map \( T x = x + 1 \) (mod \( N \)) and the uniform measure \( \mu \) on \( X \) defined by \( \mu(E) = \frac{|E|}{N} \) for any \( E \subset X \) is a measure-preserving system. It is also clear that for \((X, \mu, T)\) and the sets \( E_1, E_2 \subset X \) we have \( R(E_1) = (E_1 - E_1) + N \mathbb{Z} \) and \( R(E_2) = (E_2 - E_2) + N \mathbb{Z} \). Then by Theorem 1.1, it follows that if \( N \geq N_0 \), where \( N_0 \) depends only on \( \alpha \) and \( \beta \), then there exist \( k(\alpha, \beta) \) and \( k \leq k(\alpha, \beta) \) such that \( k \mathbb{Z} \subset R(E_1) \cdot R(E_2) \). Then by the Chinese Remainder theorem for \( d = \gcd(k, N) \leq k \) we have \( d \mathbb{Z} \subset (E_1 - E_1) \cdot (E_2 - E_2) + N \mathbb{Z} \), which implies the statement of the corollary.

\[\]
4. Further problems

To formulate the first problem, we mention a recent result by Björklund-Bulinski [1], who proved, in particular, that for any \( E \subset \mathbb{Z}^3 \) of positive density there exists \( k \geq 1 \), depending on the set \( E \) and not only on its density, such that

\[
k \mathbb{Z} \subset \{ x^2 - y^2 - z^2 \mid (x, y, z) \in E - E \}.
\]

Recall, the definition of the upper Banach density of a set \( E \subset \mathbb{Z}^2 \):

\[
d^*(E) = \limsup_{b-a \to \infty, d-c \to \infty} \frac{|E \cap [a, b) \times [c, d)|}{(b-a)(d-c)}.
\]

**Problem 1.** Is it true that given \( E_1, E_2 \subset \mathbb{Z} \) of positive density there exist \( k_0 \), which depends only on \( d^*(E_1) \) and \( d^*(E_2) \), and \( k \leq k_0 \) such that \( k \mathbb{Z} \subset \{ x^2 - y^2 \mid (x, y) \in E - E \} \)?

The next two problems arise naturally by Theorem 1.1 and the following result proved by Björklund and the author in [2]:

**Theorem 4.1.** Let \( E \subset \text{Mat}_d^0(\mathbb{Z}) = \{(a_{ij}) \in \mathbb{Z}^{d \times d} \mid \text{tr}(a_{ij}) = 0\} \) be a set of positive density. Then there exists \( k \geq 1 \) (which a priori depends on the set \( E \) and not only on its density) such that for any matrix \( A \in k \cdot \text{Mat}_d^0(\mathbb{Z}) \) there exists \( B \in E - E \) such that the characteristic polynomial of \( B \) coincides with the characteristic polynomial of \( A \).

**Problem 2.** Is it true that given \( E \subset \mathbb{Z}^2 \) of positive upper Banach density, there exist \( k_0 \) that depends only on \( d^*(E) \) and \( k \leq k_0 \) such that

\[
k \mathbb{Z} \subset \{ xy \mid (x, y) \in E - E \}.
\]

We also would like to establish the quantitative version of Theorem 4.1:

**Problem 3.** Is it true that the parameter \( k \) in Theorem 4.1 depends only on the density of the set \( E \subset \text{Mat}_d^0(\mathbb{Z}) \)?

In view of Corollary 1.2 we believe that a similar statement holds true for any finite commutative ring.

**Conjecture 1.** Let \( \alpha > 0 \). Then there exist \( N \) and \( k \) depending only on \( \alpha \) such that for any finite commutative ring \( R \) with \( |R| \geq N \) and any set \( E \subset R \) satisfying \( |E| \geq \alpha |R| \) the set \( (E - E) \cdot (E - E) \) contains a subring \( R_0 \) such that \( |R|/|R_0| \leq k \).
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