Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering:

Random Matrix approach for systems with broken time-reversal invariance.

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Assuming the validity of random matrices for describing the statistics of a closed chaotic quantum system, we study analytically some statistical properties of the S-matrix characterizing scattering in its open counterpart. In the first part of the paper we attempt to expose systematically ideas underlying the so-called stochastic (Heidelberg) approach to chaotic quantum scattering. Then we concentrate on systems with broken time-reversal invariance coupled to continua via M open channels; \( a = 1, 2, \ldots, M \). A physical realization of this case corresponds to the chaotic scattering in ballistic microstructures pierced by a strong enough magnetic flux. By using the supersymmetry method we derive an explicit expression for the density of S-matrix poles (resonances) in the complex energy plane. When all scattering channels are considered to be equivalent our expression describes a crossover from the \( \chi^2 \) distribution of resonance widths (regime of isolated resonances) to a broad power-like distribution typical for the regime of overlapping resonances. The first moment is found to reproduce exactly the Moldauer-Simonius relation between the mean resonance width and the transmission coefficient. Under the same assumptions we derive an explicit expression for the parametric correlation function of densities of eigenphases \( \theta_a \) of the S-matrix (taken modulo \( 2\pi \)). We use it to find the distribution of derivatives \( \tau_a = \partial \theta_a / \partial E \) of these eigenphases with respect to the energy ("partial delay times") as well as with respect to an arbitrary external parameter. We also find the parametric correlations of the Wigner-Smith time delay \( \tau_w(E) = \frac{1}{M} \sum_a \partial \theta_a / \partial E \) at two different energies \( E - \Omega / 2 \) and \( E + \Omega / 2 \) as well as at two different values of the external parameter. The relation between our results and those following from the semiclassical approach as well as the relevance to experiments are briefly discussed.

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I. INTRODUCTION.

Chaotic scattering has been a subject of a rather intensive research activity during the last decade, both theoretically (see reviews [1][2]) and experimentally [3][4]. This phenomenon is encountered in a variety of physical systems ranging from atomic nuclei [5][6][7], atoms [8][9][10] and molecules [11][12][13] to mesoscopic ballistic devices [14][15] and microwave cavities [16][17]. The most fundamental object characterizing the process of quantum scattering is the unitary $S-$matrix relating the amplitudes of waves incoming onto the system and the amplitudes of scattered (outgoing) waves. Because of the chaotic nature of the underlying scattering dynamics the $S-$matrix characteristics behave in an irregular way when parameters of either incoming waves (e.g. energy) or of the scattering region (e.g. the form or strength of the scattering potential, the strength of the magnetic field through the ballistic microstructure, etc.) are slightly changed. Because of this fact it seems to be most adequate to describe such a behavior in terms of some statistical measures: distributions and correlation functions.

At present, there are two complementary theoretical tools employed to calculate statistical characteristics of open quantum systems exhibiting the phenomenon of chaotic scattering. These are the semiclassical [18][19][20][21][22] and the stochastic approaches [23][24], the relation between both methods being in some detail discussed in [25]. The semiclassical approach operates with the genuine microscopic Hamiltonians and allows for treating particular systems with full account of their specific features. The starting point for such an approach is a representation of $S-$matrix elements in terms of a sum over the classical periodic orbits, the method going back to works by Gutzwiller [26] and Balian and Bloch [27]. The statistical characteristics are sampled usually over some range of energies or changing the system parameters.

It is however known, that the majority of (both closed and open) quantum chaotic systems of quite different microscopic nature shows a great degree of universality in their properties on the appropriate energy scale. More precisely, the statistical characteristics of closed systems turn out to be independent of the microscopic details when sampled on the energy intervals $\delta E$ large in comparison with a mean separation between two adjacent levels $\Delta$, but smaller then the energy scale $E_c = \hbar/t_c$, with $t_c$ standing for the relaxation time necessary for the classically chaotic system to cover uniformly the constant energy shell [28]. Because of this universality one achieves the correct description of the properties of such systems [29] by exploiting the similarity with ensembles of large Gaussian random matrices $\hat{H}$ of the size $N \times N$ characterized by the following probability distribution:

$$
\mathcal{P}_\beta \propto \exp \left( -\frac{\beta N}{4} \text{Tr} \hat{H}^2 \right)
$$

(1)

where the matrices are considered to be real symmetric ($\beta = 1$, Gaussian Orthogonal Ensemble: GOE), Hermitian ($\beta = 2$, Gaussian Unitary Ensemble: GUE) or consisting of real quaternions ($\beta = 4$, Gaussian Symplectic Ensemble: GSE). Ensembles with $\beta = 1$ ($\beta = 2$) serve to describe spectra of closed quantum chaotic systems systems with preserved (broken, e.g by applied magnetic field or by Aharonov-Bohm magnetic flux) time-reversal invariance, correspondingly. At last, the ensemble corresponding to $\beta = 4$ describes systems with preserved time-reversal invariance displaying strong spin-orbit scattering which should be taken into account.

Properties of all these ensembles were studied long ago [30][31]. The mean level density $\nu_{sc}(E) = \langle \nu(E) \rangle$ in the limit $N \to \infty$ is given by the so-called Wigner semicircle law:

$$
\nu_{sc}(E) = \frac{1}{N} \text{Tr} \delta(E - \hat{H}) = \frac{1}{2\pi} \sqrt{4 - E^2}
$$

(2)

where the angular brackets stand for the ensemble averaging. The radius of the semicircle is equal (in chosen normalization) to $E_{sc} = 2$, so that the average spacing between eigenvalues is $4/N$ while the local spacing around the point $E$ is $\Delta(E) = (N\nu_{sc})^{-1}$.

The mean level density is the simplest quantity characterizing the spectrum of any system. It is not very informative from a physical point of view since it is insensitive to the fine structure of the spectrum. It is also the same for all universality classes. Actually, no real physical system is known where the mean level density follows the semicircle law, Eq. (2).

In contrast, the two-point spectral correlation function:

$$
R_2(\omega) = \Delta^2(E) \langle \nu(E - \Omega) \nu(E + \Omega) \rangle - 1
$$

(3)

where $\omega = 2\pi\Omega/\Delta(E)$ is known to be universal, i.e. independent of the microscopic details and has the same form both for generic chaotic systems and for the Gaussian Ensembles of definite symmetry. The same universality extends to other spectral properties, such as the nearest neighbors spacing distribution, etc. [29].
Despite the apparent success in the exploitation of random matrix results for describing spectra of quantum chaotic systems of different nature [23,25] it was a long standing problem to justify the validity of such an approach microscopically. Some insight was achieved within the semiclassical approach long ago by Berry [2]. Very recently Muzykantsky and Khmelnitsky [3] and especially Andreev et al. [4] managed to find a way to prove this conjecture by a nontrivial combination of field-theoretical and semiclassical ideas. In parallel, traditional semiclassical methods were also significantly improved [35]. These results put applications of random matrices for the description of universal features of closed chaotic systems on a firm ground.

Provided the properties of a Hamiltonian $H_{in}$ for a closed chaotic system are specified, one can consider its open counterpart and work out the $S$-matrix by standard methods in the theory of quantum scattering [36–41]. As the result, the scattering matrix is expressed in terms of both the Hamiltonian $H_{in}$ and matrix elements describing the coupling of the internal motion to “open channels” i.e. the states of the system asymptotically far from the chaotic region. Correspondingly, one can try to extract the statistics of $S$-matrix inherited from the mentioned universal “random matrix” properties of $H_{in}$ [42].

In principle, it is far from being obvious that the coupling to continua does not wash out the universal features. The key observation (made long ago in the context of nuclear physics, see e.g. [43]) is that typically there are two well-separated time stages associated with the scattering process: an immediate “prompt” response (so-called direct processes) and a delayed, or equilibrated response associated with the formation of long-living states, or resonances. In the energy domain direct processes are described by smooth $S$-matrix characteristics averaged over a large energy interval. Such characteristics must be, of course, highly non-universal and are determined mainly by system-specific boundary conditions on the boundary of the scattering region. At the same time, resonance response happening on much shorter energy scale manifests itself in a form of a random signal on top of the smooth averaged characteristics. Formation of the long-living resonances is intimately related to the internal dynamics inside the scattering region. It is natural to expect that the universal features of the chaotic quantum dynamics will be reflected in the universal statistical characteristics of the $S$-matrix on the “resonance” energy scale, as long as the characteristic times (e.g. measured by inverse widths of the resonances) will be much longer than the time scale of the direct response.

To find some adequate description of these universal features one can substitute the Hamiltonian $H_{in}$ by the Gaussian random matrix, Eq. (1). This way was pioneered by Verbaarschot, Weidenmuller and Zirnbauer [42] who calculated the correlation function of $S$-matrix elements at two different energies for arbitrary number $M$ of open channels satisfying $M \ll N$, with $N$ being the total number of resonances. It was indeed found that the $N \times M$ matrix elements describing the coupling of the internal region to open channels enter the final result in the form of only $M$ simple combinations, the so-called "transmission coefficients". In full agreement with the "two distinct time scale" picture, these coefficients just measure the portion of the flux in a given channel which is not reflected back immediately, but penetrates the interaction region and participates in the formation of the long-living resonances [23,25]. The approach developed in [12] (following [24] we will call it "Heidelberg approach" henceforth) turned out to be very fruitful and serves as a case study for all further development in the field.

One can also try to make use of the expected universality directly on the level of $S$-matrix without any reference to the system Hamiltonian. Such a method was developed in great detail in a series of papers by Mello and co-workers [12,13,24]. The probability density for the whole $S$-matrix can be obtained if one makes the assumption of minimal information content of such a distribution respecting the requirements of $S$-matrix unitarity, analyticity and constraints imposed by absence or presence of the time-reversal invariance. Provided all the system-specific relevant information is encoded in the value of the average $S$-matrix $\langle S \rangle$ the joint probability $P(\vec{S})d\mu(\vec{S})$ is given by:

$$P(\vec{S})d\mu(\vec{S}) \propto \frac{\det \left( \mathbb{I} - \langle S \rangle \langle S' \rangle \right) \left( \beta M + 2 - \beta \right)/2}{\det \left( \mathbb{I} - S \langle S' \rangle \right) \left( \beta M + 2 - \beta \right)/2} d\mu_\beta(\vec{S})$$

with the following measure $d\mu_\beta(\vec{S})$:

$$d\mu_\beta(\vec{S}) \propto \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^\beta \prod_{a=1}^M d\theta_a dU$$

where $\theta_a; \ a = 1,...,M$ are eigenphases of the $S$-matrix, the volume element $dU$ is generated by the corresponding eigenvectors, $\beta = 1,2,4$ as before and $M$ is the dimension of the $S$-matrix equal to the number of open channels. The distribution $P(\vec{S})$ is called the Poisson’s kernel.

For the particular case $\langle S \rangle = 0$ the distribution Eq. (4) just coincides with the measure $d\mu_\beta(\vec{S})$. Such distributions were considered long ago by Dyson and known as the Dyson Circular Ensembles [30]. They were found to describe very satisfactorily the $S$-matrix statistics for some realistic models of the chaotic scattering [15]. The general Poisson’s
kernel, Eq.(1) was verified as well \[20,24\] and proved to be a very useful tool to predict fluctuations of transmissions through ballistic microstructures. It is natural to expect that the same distribution can be actually derived from the Heidelberg approach. It turned out that the problem is quite involved technically, however. In his insightful paper \[10\] Brouwer succeeded to derive the Poisson’s kernel distribution assuming that the Hamiltonian $H_{in}$ is taken from a quite specific Lorentzian ensemble of random matrices. Since the spectral properties of the latter ensemble and those of Gaussian matrices, Eq.(1), are identical as long as the matrix size $N \to \infty$, one expects that the equivalence of the two approaches can be shown for this generic case as well.

If one wishes to study the dependence of the $S$–matrix on external parameters without explicitly considering the system Hamiltonian, one should make some additional statistical assumptions beyond the minimum information approach. One possible way is to simulate such a dependence by a kind of “Brownian motion” in the corresponding $S$–matrix space \[17\]. It turns out, however, that the Brownian motion picture is in disagreement with the results obtained starting from the Heidelberg formalism. Therefore, the Heidelberg approach seems to be the only consistent stochastic method when we are interested in the parametric variations of the $S$–matrix characteristics. An example of such kind of calculation can be found in \[18\]. Another important advantage of the Heidelberg approach as compared with that by Mello and collaborators is that it operates with an energy-dependent $S$–matrix $S(E)$. As such, it allows to study not only spectral correlations of different physical quantities but, in principle, also contains information about such features of the $S$–matrix as resonances in the complex energy plane $E = E + iY$. The notion of resonances, representing long-lived intermediate states of open system to which bound states of its closed counterpart are converted due to coupling to continua, is one of the most fundamental concepts in the domain of quantum scattering \[19\]. On a formal level resonances show up as poles of the scattering matrix occurring at complex energies $E_k = E_k - i\Gamma_k$, where $E_k$ and $\Gamma_k$ are called position and widths of the resonance, correspondingly.

The general problem of determining the domain of concentration and the distribution of poles of the $S$–matrix in the complex plane is of fundamental interest in the general theory of scattering \[50\]. Powerful numerical methods are available (e.g. the method of complex scaling \[21\]) allowing one to extract resonance parameters for models in atomic and molecular physics. Whereas the issue of energy level statistics in closed chaotic systems was addressed in the complex plane is of fundamental interest in the general theory of scattering \[50\]. Powerful numerical methods occurs:\[M\]

where $\rho(\nu) = \frac{(\nu/2)^{(\nu/2)}}{\Gamma(\nu/2)} y_s^{\nu/2-1} e^{-\frac{\nu}{2} y_s}$.\[\text{(6)}\]

where $y_s$ stands for the resonance widths normalized to its mean value: $y_s = \Gamma/\langle \Gamma \rangle$, the parameter $\nu = M$ ($\nu = 2M$) for systems with preserved (broken) time reversal invariance, and $\Gamma(z)$ in Eq.(8) stands for the Euler Gamma-function. The case $\nu = 1$ is known as the Porter-Thomas distribution \[67\]. It was shown to be in agreement with experimental data in neutron-nuclei resonances \[67\], the fluorescence excitation spectrum of the NO$_2$ molecule \[68\], resonance dissociation of HO$_2$ molecule \[69\], the diamagnetic Rydberg spectrum in lithium atom \[10\] and in microwave cavities \[12\]. Indirectly that distribution manifests itself in fluctuations of tunneling conductance through ballistic quantum dots \[69,70\].

When the coupling to continua increases resonances start to overlap and the simple perturbative result Eq.(8) loses its validity. Finally, when the coupling to continua exceeds some critical value, the so-called ”trapping phenomenon” occurs: $M$ very unstable states (broad resonances) are formed, whereas the rest $N - M$ resonances go back to the real axis, i.e. become more and more narrow with increasing coupling, see \[4,5,10\] and the end of Sec.III for a more detailed discussion. Such a ”reorganization” of the spectrum is the most pronounced when the number of channels $M$ is of the same order as the (large) number of resonances $N$. This range of parameters $M \propto N \gg 1$ always corresponds to the condition $\langle \Gamma \rangle \gg \Delta$ which is just the opposite limiting case as compared with the domain of validity of the $\chi^2$ distribution. Under this condition one can calculate the density of resonance poles analytically \[55,65\]. However, frequently one encounters the case of few open channels and moderately overlapping resonances $\Gamma \sim \Delta \sim \nu/2$. In this situation, which is in some sense generic, one can neither rely upon the distribution Eq.(8), nor use the results of \[55,65\]. The general distribution of resonance widths describing a crossover from isolated to overlapping resonances was found recently by the present authors for the particular case of an open chaotic system with broken time-reversal invariance coupled to continua via $M \ll N$ equivalent channels \[7\]. One of the main goals of the present paper is to give a detailed derivation and subsequent analysis of that distribution, also for the case of non-equivalent channels.

Apart from the $S$–matrix elements and $S$–matrix poles, the set of scattering phase shifts $\theta_n$ (defined via the $S$–
matrix eigenvalues \( \exp i\theta_a; \quad a = 1, 2, \ldots, M \) are intensively used to characterize the chaotic scattering, see \[1\]-[4].

Quite recently, their statistical characteristics were studied numerically in some detail for chaotic \[7\]-[10] as well as for disordered \[4\]-[6] systems. The derivatives of phase shifts over the energy \( \tau_a = \partial \theta_a / \partial E \) (we propose to call them "partial delay times") are particularly interesting and related to the mean time spent by a quantum particle in the interaction domain.

The issue of the time scales associated with different stages of the quantum scattering process (e.g. tunneling, reflection and transmission) is quite a controversial subject which is under an intensive discussion for a long time, see [77-80] and references therein. In particular, ambiguities arise because there is no a self-adjoint time operator in Hilbert space, analogous to the position operator; instead, the wave function depends on time as a parameter.

Relegating all the essential details and derivations to section II, we just mention here that if \( \psi(x, t) \) denotes a wave packet at time \( t \) for a quantum particle moving in a potential \( U(x) \) (as such satisfying the Schrödinger equation \( i\hbar \partial \psi / \partial t = [-\hbar^2/2m + U(x)] \psi \) then the real number:

\[
\tau_r[\psi] = \int_{-\infty}^{\infty} dt \int_{|x| \leq r} |\psi(x, t)|^2 d^3x
\]

may be interpreted as the total time spent by this state during its evolution inside the ball of the radius \( r \) centered at the origin (we assume \( \psi(x, t = 0) \) normalized to unity). If \( \psi_f(x, t) \) denotes a freely evolving wave packet (i.e. solution of the same Schrödinger equation with \( U(x) = 0 \) and condition: \( \psi_f(t = -\infty) \sim \psi(t = -\infty) \)), the difference \( \tau(r) = \tau_r[\psi] - \tau_r[\psi_f] \) corresponds to the time delay inside the same ball due to scattering by the potential \( U(x) \). The global time delay is defined as \( \tau_d = \tau(r \to \infty) \) and under quite general conditions (see e.g. [79]) can be shown to be equal to the time-independent expectation value

\[
\tau_d = \int dx_1 \int dx_2 \psi^*(x_1, t)T_d(x_1, x_2)\psi(x_2, t)
\]

where \( T_d(x_1, x_2) \) are matrix elements of a Hermitian time delay operator \( \hat{T}_d \) in the position representation. This operator turns out to be commuting with the Hamiltonian and intimately related to the so called Wigner-Smith time delay matrix \[1\] defined in terms of the \( S \)-matrix as:

\[
\hat{\tau}_w(E) = i\hbar \frac{\partial \hat{S}^\dagger}{\partial E} \hat{S}
\]

In particular, following the papers [79]-[82] we show in section II that the eigenvalues of the operator \( \hat{T}_d \) just coincide with the eigenvalues of \( \hat{\tau}_w(E) \). The quantum mechanical expectation value of the time delay averaged over different channels turns out to be equal to [82]-[85]:

\[
\tau(E) = \frac{i\hbar}{M} \text{Tr} \left[ \frac{\partial \hat{S}^\dagger}{\partial E} \hat{S} \right] = -\frac{i\hbar}{M} \frac{\partial}{\partial E} \ln \text{Det} S(E) = \hbar \frac{1}{M} \sum_{a=1}^{M} \frac{\partial \theta_a}{\partial E}
\]

where the bar denotes the averaging over the energy spectrum of the packet. This shows the relation between the phase shift derivatives and mean time delay mentioned above.

A quite detailed analysis of the time-delay problem was given in the context of nuclear physics by Lyuboshits [83,84] and other authors [77]. In particular, for a wave packet of arbitrary form Lyuboshits [84] suggested a concept of the probability distribution of its time delay. His definition is based on the interpretation of the quantity \( P(t) = \int_{x \in V} |\psi(x, t)|^2 d^3x \) as the quantum mechanical probability to be found within the volume \( V \) at instant \( t \). Then the time derivative \( \partial P / \partial t \) can be used to define the distribution of times spent inside the volume \( V \). A general and illuminating discussion of the time evolution properties of wave packets in a generic chaotic systems can be found in [82].

On the other hand, the existence of the Hermitian time-delay operator \( \hat{T}_d \) in Hilbert space suggests an alternative definition of the time delay statistics by the natural requirement that \( \overline{\tau \Psi} = \langle \Psi(t) | \hat{T}_d | \Psi(t) \rangle \) for any wave packet \( \Psi(t) \). Then the problem is reduced basically to study the statistical properties of the Wigner-Smith time delay matrix \( \hat{\tau}_w(E) \).

The chaotic scattering makes the Wigner-Smith time delay matrix (in particular, the quantity \( \frac{1}{M} \text{Tr} \hat{\tau}_w \)) which is called just Wigner time delay) to be a strongly fluctuating function of the energy \( E \) as well as of any external parameter \( X \). From this point of view we can speak about distributions and correlation functions of these quantities. Similarly, the distribution of partial delay times can be used to characterize variations of time scales associated with the chaotic scattering process. Various statistical aspects of time evolution of the chaotic quantum systems were studied earlier in some details in [2],[86],[87].
Being an important characteristic of the scattering process, the statistics of phase shifts and their derivatives deserve to be investigated in more detail. Additional interest to the problem attaches the fact of relevance of Wigner time delay in condensed matter physics. Indeed, in a series of papers by Büttiker and collaborators the Wigner time delay was related to the frequency-dependent response of mesoscopic capacitors [88–90]. A more detailed discussion of this issue can be found in the last section of the present paper.

More general parametric derivatives of the scattering phase shifts can also be related to some observable quantities. As a particular example we mention the relation between the persistent currents and the derivative of the total phase shift over the magnetic flux derived by Akkermans et al. [91]. These authors considered "open mesoscopic networks": two-dimensional systems of conducting loops coupled to infinitely long ideal leads (waveguides). The loops can encircle a flux tube with flux \( \phi \). The expectation value of the persistent current around the flux tube in the state \( |\Psi\rangle \) is \( I(\Psi, \phi) = -\langle \Psi | d\mathcal{H}/d\phi | \Psi \rangle \). For the case of a closed system (i.e. when loops are disconnected from the leads) each discrete level \( E_n(\phi) \) carries the current \(-dE_n/d\phi \). When the system is open, it turns out that the differential contribution of the scattering states at energy \( E \) to the persistent current can be expressed in terms of phase shift derivatives as [91]:

\[
dI(E, \phi) = \frac{1}{2\pi i} \frac{\partial}{\partial \phi} \left[ \ln \det \hat{S}(E, \phi) \right] dE = \frac{1}{2\pi} \frac{\partial}{\partial \phi} \left[ \sum_a \theta_a(E, \phi) \right] dE
\]

(10)

In the present paper we give quite a detailed analysis of the statistical properties of scattering phase shifts and their derivatives for generic chaotic scattering in a system with broken time-reversal invariance. The extension of our results to other symmetry classes as well as to the crossover regimes will be published elsewhere [92]. We find it to be most informative to concentrate our attention on the so-called \( K \)-matrix related to the scattering matrix as:

\[
\hat{S} = \frac{i - iK}{i + iK}
\]

(11)

This equation shows that eigenphases \( \theta_a(E, X) \), \( a = 1, \ldots, M \) considered modulo \( 2\pi \) are determined in a unique way by the eigenvalues \( z_a(E, X) \) of the \( K \)-matrix, where we have indicated explicitly both the energy dependence and dependence on an external parameter \( X \). First of all, we calculate explicitly the correlation function of densities of the eigenvalues of the \( K \)-matrix at two different energies \( E \pm \Omega \) and parameter values \( X \pm \delta X \). When \( \Omega = \delta X = 0 \) this correlation function turns out to be the same as the pair correlation function following from the Poisson’s kernel. This fact confirms the expected equivalence of the minimum information approach and Hamiltonian approach for the case of fixed energy, and as such extends the earlier studies on that subject [40]. From that moment we concentrate on the statistics of delay times and parametric derivatives of phase shifts. First, we derive and analyze the general expression for the distribution of "partial delay times" \( \tau_a = \partial \theta_a/\partial E \) (here and henceforth we frequently put \( \hbar = 1 \)) as well as derivatives \( \partial \theta_a/\partial X \). This distribution, being an interesting characteristic of the chaotic scattering by itself, also allows us to detect the qualitative features of the Wigner time delay distribution. In particular, for the one-channel system the partial delay time is exactly the same as the Wigner time delay. After that, we derive the parametric correlation function of the Wigner time delay and show some interesting correspondences with the results of the semiclassical approach. A short account of our results was published earlier [72,93].

The organization of the paper is as follows. Section II is meant to be a kind of introduction to the random matrix method of the description of an open chaotic quantum system. It is based mainly on the original papers by other authors [72,93,71,94,95]. Considering a particular generic example we discuss the main ingredients of the model and present a quite detailed discussion of the time-delay operator and other quantities characterizing time evolution in such systems.

In Section III we derive and analyze the density of the S-matrix poles in the complex plane. Section IV is devoted to the statistics of eigenvalues of the \( K \)-matrix, phase shifts and their derivatives and analyses different statistical aspects of the Wigner time delay. It contains also a kind of semiclassical analysis of the parametric correlations of Wigner time delays. Concluding remarks and a discussion of the potential experimental relevance of the obtained results can be found in the final Section V.

II. SCATTERING PROBLEM FOR RANDOM-MATRIX HAMILTONIANS

A. General description of the model.

A model which is most appropriate for incorporating random matrix ideas for describing the phenomenon of quantum chaotic scattering was discussed in great details in the works by Verbaarschot, Weidenm"uller and Zirnbauer.
A general construction actually goes back to works by Feshbach [43] and Lewenkopf and Weidenmüller [22]. A particular generic example of a scattering system depicted schematically in Fig.1. is a two-dimensional cavity of irregular shape with impenetrable walls coupled to an infinite waveguide (lead) of width $d$. The model is simplified by neglecting any direct coupling between different channels; hence the corresponding term in the Eq. (12) is diagonal in $a$. The first term describes the Hamiltonian $H_{in}$ of the "closed" chaotic system possessing $N \gg 1$ bound states, which are eigenstates of $H_{in}$. In the spirit of the Random Matrix universality conjecture, we simulate this part of the Hamiltonian by taking $H_{in}$ to be a Gaussian random $N \times N$ matrix. The number $N$ is considered to be large: $N \gg 1$. To describe the interaction between channels and bound states(converting bound states into resonances) one adds to the Hamiltonian Eq. (12) the "interaction term":

$$\tilde{V} = \sum_{l,a} \left( | l \rangle \int dE W_{la} \langle a, E | + \text{herm. conj.} \right) .$$

In any practical implementation of such a procedure one should make sure that the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \tilde{V}$ is self-adjoint. This point is not at all trivial (see the example below). The general way of self-adjoint matching of Hamiltonians with internal structure and those describing motion in external scattering channels was suggested by Pavlov and collaborators [40]. A particularly convenient formulation of the Pavlov's method suggested by Makarov [11] was applied to the problem of chaotic scattering in the recent papers [22].

After the self-adjoint matching is done one can employ standard methods in scattering theory (see the mentioned books [36,37]) in order to write down the Lipmann-Schwinger equation for the in-and outgoing scattered waves and find an explicit expression for the scattering matrix.

### B. From Random Matrix Hamiltonian to Scattering Matrix.

Instead of demonstrating such a formal derivation within a quite abstract "projection formalism" (see e.g. [22,82]) we find it to be more illuminating to show how to derive the $S$–matrix in an alternative way [44,95]. To do this let us confine ourself to a particular generic example of a scattering system depicted schematically in Fig.1.: a two dimensional cavity of irregular shape with impenetrable walls coupled to an infinite waveguide (lead) of width $d$. Let us mention, that it is one of the favorite models for the study of generic features of chaotic scattering, both theoretically (see e.g. the "frying pan" model in [12]) and experimentally [4]. The propagation of a quantum particle inside the lead is described by the Schrödinger equation:
where \( \hat{\Psi}(x, y) = \psi_a(x) \phi_\alpha(y) \), where

\[
\psi_a(x) = \frac{1}{(2\pi\hbar^2k_a/m)^{1/2}} [A_a e^{-ik_a x} + B_a e^{ik_a x}] ; \quad \phi_\alpha(y) = (2/d)^{1/2} \sin \left[ \left( \frac{a\pi}{d} \right) (y + d/2) \right]
\]

for \( x \geq 0; \ |y| \leq d/2; \ a = 1, 2, \ldots, M, \) with the number \( M \) of open channels at the energy \( E = \frac{\hbar^2k^2}{2m} \) being equal to the largest integer less or equal to \( \frac{k_d}{\pi} \) and the wave vector \( k_a \) being equal to \( k_a = \left( k^2 - \left( \frac{a\pi}{d} \right)^2 \right)^{1/2} \), so that \( \frac{d^2}{d^2x^2} \psi_a = k_a^2 \psi_a \).

The running waves are properly normalized to energy \( \delta \)-functions; unitarity of the \( S \)-matrix to be introduced later is related to the conservation of the probability flux.

The situation of the waveguide disconnected from the cavity we describe by the boundary conditions: \( \partial \psi_a/\partial x |_{x=0} = 0 \). This means that the particle in each channel is just reflected back: \( A_a = B_a \). The corresponding \( S \)-matrix relating the vectors of incoming \( A = (A_1, \ldots, A_M)^T \) and outgoing \( B = (B_1, \ldots, B_M)^T \) amplitudes: \( B = SA \) is just the unity \( M \times M \) matrix: \( S = I \). The role of the vector \( |a, E \rangle \) of general construction (see Eqs. (12,13)) is played by the vector \( \Psi \) corresponding to the particular choice of amplitudes of incoming waves: \( A_a = 1; B_a = 0 \). The Hamiltonian of the particle motion inside the cavity is simulated by the \( N \times N \) random Hermitian matrix \( \hat{H}_{in} \). Correspondingly, the "internal" wave function is represented by the \( N \)-component vector \( u = (u_1, \ldots, u_N)^T \). The wave functions of the scattering system as a whole ("cavity attached to the lead") are therefore vectors \( \Phi = \left( \begin{array}{c} u \\ \Psi \end{array} \right) \) from the Hilbert space \( L^2(R^+, CM) \oplus C^N \) supplied with the scalar product:

\[
(\Phi_1, \Phi_2) = u_1^\dagger u_2 + (\Psi_1, \Psi_2); \quad \text{where} \quad (\Psi_1, \Psi_2) = \int_{-d/2}^{d/2} dy \int_0^\infty dx \Psi_{1}^\dagger \Psi_2
\]

Let us define the Hamiltonian operator \( \hat{H} \) of the system as a whole acting in that Hilbert space as:

\[
\hat{H} \left( \begin{array}{c} u \\ \Psi \end{array} \right) = \left( \begin{array}{c} \hat{H}_{in} u + \int_{-d/2}^{d/2} dy \int_0^\infty dx W(x, y) \Psi_1(x, y) \\ \mathcal{V}(x, y) u + \hat{H}_{ch} \Psi \end{array} \right)
\]

where \( \hat{H}_{ch} \) is the operator diagonal in the channel space:

\[
\hat{H}_{ch} = \text{diag} \left( \frac{-\hbar^2}{2m} (\partial_x^2 + \partial_y^2), \ldots, \frac{-\hbar^2}{2m} (\partial_x^2 + \partial_y^2) \right)
\]

and \( W(x, y) \) and \( \mathcal{V}(x, y) \) are \( N \times M \) and \( M \times N \) rectangular matrices describing a coupling between two parts of the Hilbert space. Let us assume for simplicity that the coupling is local along the waveguide. \( W(x, y) = \delta(x) W(y), \) so that \( \int_{-d/2}^{d/2} dy \int_0^\infty dx W(x, y) \Psi_1(x, y) = \int_{-d/2}^{d/2} dy W(y) \Psi_1(x, y) |_{x=0} = w_{i\alpha} \psi_i(x = 0), \) where \( w_{i\alpha} = \int_{-d/2}^{d/2} dy W_{i\alpha}(y) \phi_\alpha(y) \) \( i = 1, \ldots, N, \ a = 1, \ldots, M \) and \( \psi_i(x) = (\psi_1(x), \ldots, \psi_M(x))^T \). On the other hand, we have to put \( \mathcal{V}(x, y) \equiv 0 \) in order to be consistent with the locality of the coupling and to stay in the space spanned by the vectors \( \Psi \).

The operator \( \hat{H} \) defined in such a way is not, in general, a self-adjoint one. Indeed,

\[
(\hat{H}\Phi_1, \Phi_2) = u_1^\dagger \hat{H}_{in} u_2 + (\hat{H}_{ch} \Psi_1, \Psi_2) + \psi_1^\dagger(x = 0) \hat{w} u_2
\]

and

\[
(\Phi_1, \hat{H}\Phi_2) = u_1^\dagger \hat{H}_{in} u_2 + (\Psi_1, \hat{H}_{ch} \Psi_2) + u_1^\dagger \hat{w} \psi_2(x = 0)
\]

From the definition of the operator \( \hat{H}_{ch} \) and that of the scalar product \( (\Psi_1, \Psi_2) \) one can easily find after using partial integration and the fact \( \hat{H}_{in} = \hat{H}_{in} \) that:

\[
(\hat{H}\Phi_1, \Phi_2) - (\Phi_1, \hat{H}\Phi_2) = \frac{\hbar^2}{2m} \left\{ \left( \frac{\partial^2}{\partial x^2} \right) \psi_2 - \psi_1^\dagger \left( \frac{\partial}{\partial x} \psi_2 \right) \right\} |_{x=0} + \psi_1^\dagger(x = 0) \hat{w} u_2 - u_1^\dagger \hat{w} \psi_2(x = 0)
\]
In order to have a self-adjoint Hamiltonian operator $\hat{H}$ one has to impose some appropriate boundary conditions at the point $x = 0$ ensuring that the expression above is vanishing \[96\]. The most obvious (however, not the most general) choice is:

$$\hat{w}^\dagger u = \frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x} \psi \right) |_{x=0} \tag{19}$$

On the other hand, the solution of the Schrödinger equation for the whole system: $\hat{H} \Phi = E \Phi$; $E = \frac{\hbar^2 k^2}{2m}$ (we call these solutions the "scattering states" and denote them $\Phi_E$ henceforth) leads immediately to the relation:

$$u = \left( E - \hat{H}_{in} \right)^{-1} \hat{w} \psi(x = 0) \tag{20}$$

which together with Eq.\((13)\) yields the following equation for the vector $\psi(x)$:

$$\hat{w}^\dagger \left( E - \hat{H}_{in} \right)^{-1} \hat{w} \psi(x = 0) = \frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x} \psi \right) |_{x=0}; \quad \psi(x) = \left( \frac{m}{2\pi \hbar^2} \right)^{1/2} \frac{1}{\sqrt{k_m}} \left[ A_1 e^{-i k_1 x} + B_1 e^{i k_1 x} \right]$$

This equation allows us to find easily the unitary scattering matrix:

$$\hat{S} = \left( i - i \hat{K} \right) \times \left( i + i \hat{K} \right)^{-1}; \quad \hat{K} = \pi \hat{W}^\dagger \frac{1}{E - \hat{H}_{in}} \hat{W} \tag{22}$$

where $\hat{W} = \sqrt{2m/\pi \hbar^2} \hat{w} \text{diag}(k_1^{-1/2}, ..., k_M^{-1/2})$, $\hat{I}$ is the unity matrix of the corresponding dimension. Often we treat diagonal matrices proportional to $\hat{I}$ simply as numbers. Sometimes, we indicate the dimension as index, here it is $\hat{I} = \hat{I}_M$.

The expression Eq.\((22)\) can be also rewritten in another form, frequently used in applications. To this end we write:

$$\left[ i - i \hat{K} \right] \times \left[ i + i \hat{K} \right]^{-1} = \left[ \left( i + i \hat{K} \right) - 2i \hat{K} \right] \times \left[ i + i \hat{K} \right]^{-1} = 1 - 2i \left( i + i \hat{K} \right)^{-1} \hat{K}$$

and use the identity

$$\hat{W}^\dagger \left[ i + i \pi \hat{W}^\dagger \frac{1}{E - \hat{H}_{in}} \hat{W} \right]^{-1} \hat{W}^\dagger \left[ i + i \pi \hat{W}^\dagger \frac{1}{E - \hat{H}_{in}} \hat{W} \right]^{-1} = \sum_{k=0}^{\infty} (-ix)^k \left[ \hat{W}^\dagger \frac{1}{E - \hat{H}_{in}} \hat{W} \right]^{k+1}$$

which means that the scattering matrix can be written in the form:

$$\hat{S} = \hat{I} - 2i \pi \hat{W}^\dagger \frac{1}{E - \hat{H}_{in}} \hat{W} \tag{24}$$

where the non Hermitian effective Hamiltonian $\hat{H}_{eff}$ is given by $\hat{H}_{eff} = \hat{H}_{in} - i \hat{\Gamma}$ and $\hat{\Gamma} = \pi \hat{W} \hat{W}^\dagger$.

The expression for the scattering matrix of the form Eq.\((24)\) appears generally when one describes an open quantum system decaying into several open channels, see e.g. \[89\]. In particular, it can be derived from the general Hamiltonian Eq.\((12,13)\) under the assumption that the elements of the matrix $\hat{W}$ are energy-independent. This was just a starting point in the approach by Weidenmüller and collaborators \[12\], \[13\]. In the derivation above the matrix $\hat{W}$ does depend on the energy $E$ via the parameters $k_n = \left[ k^2 - \left( \frac{\alpha_n}{\hbar} \right)^2 \right]^{1/2}$. We, however, will be mostly interested in the fluctuation properties of the energy-dependent $S$-matrix. The typical energy scale of such fluctuations is given by the mean level spacing $\Delta$ - a typical separation between the adjacent eigenvalues of the matrix $\hat{H}_{in}$. Far from the thresholds, as long as $\Delta$ is negligible in comparison with the difference between the adjacent threshold energies $\frac{\hbar^2}{2m}(k_M^2 - k_{M-1}^2) = \frac{2M-1}{2m} \left( \frac{\alpha}{\hbar} \right)^2$ we can safely neglect the energy dependence of the matrix $\hat{W}$. In view of $\Delta \propto 1/N$ the latter requirement is always satisfied in the limit $N \gg 1$ which is the only case studied in the present paper.
C. Time Evolution of the Wave Packets: Staying Probability and Time Delay Operator for Open Quantum Chaotic Systems

Before describing the time evolution of wave packets let us note that any particular scattering state \( \Phi_E = (u(E), \Psi(E))^T \) is uniquely specified by the set of incoming amplitudes \( A = (A_1(E), ..., A_M(E))^T \). Being the eigenfunctions of the Hermitian Hamiltonian \( \mathcal{H} \) the scattering states must be orthogonal. Below we verify by a direct calculation the validity of the orthogonality condition:

\[
(\Phi_{E_1}, \Phi_{E_2}) = u^\dagger(E_1) u(E_2) + (\Psi_{E_1}, \Psi_{E_2}) = \delta(E_1 - E_2) A_1^\dagger A_2
\]  
(25)

Such a calculation allows one to derive some helpful relations that are used later on.

By using the identities

\[
\frac{1}{2\pi} \int_0^\infty dx e^{-ix} = \frac{1}{2} \delta(u) + \frac{m}{\hbar^2} \frac{1}{k_a(E_1)k_a(E_2)} \delta(k_a(E_1) - k_a(E_2)) = \delta(E_1 - E_2)
\]

and exploiting the definition of the \( S \)-matrix: \( B(E) = \hat{S}(E)A(E) \) and its unitarity one easily finds:

\[
(\Psi_{E_1}, \Psi_{E_2}) = \delta(E_1 - E_2) \left( A_1^\dagger A_2 + \frac{1}{2\pi i(E_1 - E_2)} A_1^\dagger \left( \hat{S}(E_1)\hat{S}(E_2) - \hat{I} \right) A_2 \right)
\]

(26)

Here we assumed that both energies \( E_1, E_2 \) are far from thresholds and close to one another, so that effectively we can put \( k_a(E_1) = k_a(E_2) \) in the expression above everywhere (this is consistent with neglecting the energy dependence of \( W \) as discussed earlier), except in the denominator, where \( (E_1 - E_2) \propto (k_a^2(E_1) - k_a^2(E_2)) \).

Let us now use the relation Eq. (22) rewritten in the following form:

\[
u_E = \frac{1}{2} \frac{1}{E - H_m} \hat{W} \left( \hat{I} + \hat{S}(E) \right) A
\]

(27)

Hence:

\[
u_{E_1}^\dagger u_{E_2} = \frac{1}{4} A_1^\dagger \left( \hat{I} + \hat{S}^\dagger(E_1) \right) \hat{W}^\dagger \left[ \frac{1}{E_1 - H_m} \right] = \frac{1}{4} \left[ \frac{1}{E_1 - H_m} \right] \hat{W} = \frac{1}{\pi(E_2 - E_1)} \left( \hat{K}(E_1) - \hat{K}(E_2) \right)
\]

(28)

Now we use the identity:

\[
\hat{W}^\dagger \frac{1}{E_1 - H_m} \frac{1}{E_2 - H_m} \hat{W} = \frac{1}{E_2 - E_1} \hat{W}^\dagger \left[ \frac{1}{E_1 - H_m} - \frac{1}{E_2 - H_m} \right] \hat{W} = \frac{1}{\pi(E_2 - E_1)} \left( \hat{K}(E_1) - \hat{K}(E_2) \right)
\]

(29)

where we exploited the definition of the \( \hat{K} \) matrix, see Eq. (22) neglecting the energy dependence of the matrix \( \hat{W} \).

Relation Eq. (24) between \( S \)-matrix and \( K \)-matrix can be written also as \( \hat{K}(E) = -i \left( \hat{I} - \hat{S}(E) \right) \left( \hat{I} + \hat{S}(E) \right)^{-1} = \hat{K}^\dagger(E) \). Substituting this relation into Eq. (24) we use it to reduce Eq. (28) to the following final form:

\[
u_{E_1}^\dagger u_{E_2} = \frac{1}{2\pi i(E_1 - E_2)} A_1^\dagger \left( \hat{S}^\dagger(E_1)\hat{S}(E_2) - \hat{I} \right) A_2
\]

(30)

We see that when combined together the Eqs. (24) and (30) produce exactly the orthogonality condition Eq. (25).

In particular, this orthogonality condition allows us to use the scattering states \( \Phi_E \) corresponding to the choice of incoming amplitudes \( A_a = 1, A_{b \neq a} = 0 \) as a convenient basis in the full Hilbert space. Denoting \( \Phi_E^{(a)} \equiv \Phi_E^{(a)} \) one can write down the total Hamiltonian \( \mathcal{H} \) as:

\[
\mathcal{H} = \int dE \sum_{a=1}^M | \Phi_E^{(a)} \rangle E \langle \Phi_E^{(a)} |
\]

(31)

Now we are prepared to answer the following question: given a wave packet \( | \Phi(t) \rangle = (u(t), \Psi(t))^T \) which evolves according the Schrödinger equation \( i\hbar(\partial/\partial t) | \Phi(t) \rangle = \mathcal{H} | \Phi(t) \rangle \), how to express in terms of the scattering matrix the probability for the corresponding particle to be found inside the ”chaotic” domain \( x < 0 \) at instant \( t \).
According to the rules of quantum mechanics this probability is just given by $P(t) = \mathbf{u}^\dagger(t)\mathbf{u}(t)$. Let us expand the wave packet over the scattering states $|\Phi_E\rangle = \sum_a^{M} A_a |\Phi_E^{(a)}\rangle$ as:

$$|\Phi(t)\rangle = \int dE f(E) |\Phi_E\rangle \exp -\frac{it}{\hbar} E; \quad \int dE |f(E)|^2 = 1 \quad (32)$$

where the coefficients $f(E)$ determine the initial form of the wave packet and we assume: $\mathbf{A}^\dagger \mathbf{A} = 1$ so that $\langle \Phi(t) | \Phi(t) \rangle = 1$. This immediately gives us the desired expression:

$$P(t) = \int dE_1 dE_2 f^\dagger(E_1) f(E_2) \mathbf{u}_{E_1}^\dagger \mathbf{u}_{E_2} \exp \frac{-it}{\hbar} (E_1 - E_2) = \int \frac{dE_1 dE_2}{2\pi i} f^\dagger(E_1) f(E_2) c - \Phi(E_1 - E_2) \sum_{a,b} A_a^* A_b \left[ (\hat{S}^\dagger(E_1)\hat{S}(E_2))_{ab} - \delta_{ab} \right] \quad (33)$$

where we made use of the Eq.(30). The mean time spent in the interaction region (i.e. the mean time delay $\tau_d$) can be found integrating this expression over the time. This operation produces a $\delta-$functional factor $\delta(E_1 - E_2)$ in the integrand which finally gives:

$$\tau_d = \int_{-\infty}^{\infty} P(t) dt = \sum_{a,b} A_a^* A_b \bar{\tau}_{ab}(E) \quad (34)$$

where

$$\bar{\tau}(E) = i\hbar \frac{\partial \hat{S}^\dagger(E)}{\partial E} \hat{S}(E) = -i\hbar \hat{S}^\dagger(E) \frac{\partial \hat{S}}{\partial E} \quad (35)$$

is the Wigner-Smith time delay matrix $[7]$ and the bar stands for the energy averaging determined by the wave packet spectrum: $\langle ... \rangle = \int dE |f(E)|^2$. If the particle comes only via a particular channel $a$ the scattering states $|\Phi_E\rangle$ coincide with the basis states $|\Phi_E^{(a)}\rangle$ and the corresponding time $\tau_d^{(a)}$ (which is natural to call the "delay time for the channel $a$") is given by the spectral average of the diagonal element $\bar{\tau}_{aa}(E)$. Then the delay time averaged over all channels is given by the spectral average of the Wigner-Smith time delay: $\overline{\tau(t)} = \int \tau(E) dE$.

The time derivative $-dP/dt$ is a current out of the chaotic region. Assuming that the spectral function $f(E)$ varies with $E$ on a much larger scale than the mean level spacing $\Delta$ (the latter scale is typical for variations of $S$-matrix elements) we can put $f(E_1) \approx f(E_2)$ in the expression Eq.(33). This results in the following expression:

$$\frac{d}{dt} P(t) = \delta(t) - p(t); \quad p(t) = \frac{1}{2\pi} \int d\epsilon e^{iEt/\hbar} \sum_{a,b} A_a^* A_b \langle \left( \hat{S}^\dagger(E + \epsilon)\hat{S}(E) \right)_{ab} \rangle \quad (36)$$

This expression can be interpreted as follows $[8,9]$. In our approximation the part of the Hilbert space corresponding to the chaotic region is not populated at $t < 0$. At $t = 0$ the wave packet reaches the chaotic region and populates its states instantly. This fact is described by the $\delta(t)$ term in the expression Eq.(33). Then the function $p(t)$ has a meaning of the distribution of duration of stay inside the chaotic region. This was just the reason to call $p(t)$ the distribution of time delays $[8,9]$. On the other hand, according to conventional rules of quantum mechanics in order to speak about the probability distribution of some observable one should be able to find a Hermitian operator in Hilbert space generating all the moments of that observable as expectation values of the integer powers of this operator. Some important insights in the issue of constructing such an operator can be found in $[79]$. To this end let us consider the following time delay operator constructed in terms of Wigner-Smith time-delay matrix Eq.(33) as:

$$\hat{T}_W = \int dE \sum_{ab} |\Phi_E^{(a)}\rangle (\bar{\tau})_{ab}(E) |\Phi_E^{(b)}\rangle \quad (37)$$

The Hermiticity of this operator follows from that of the Wigner-Smith matrix. It commutes with the Hamiltonian, Eq.(31), but is not at the same time diagonal due to the degeneracy of $\mathcal{H}$. It is evident that for any wave packet $|\Phi(t)\rangle$ the mean time delay given in Eq.(34) is just the time independent expectation value:

$$\langle \tau \rangle \equiv \tau_d = \langle \Phi(t) | \hat{T}_W | \Phi(t) \rangle \quad (38)$$

Then it is natural to define the higher moments of the time delay as:
\[ \langle \psi^p \rangle = \langle \Phi(t) \mid (\hat{T}_W)^p \mid \Phi(t) \rangle = \sum_{a,b} \hat{A}_a^* \hat{A}_b (\hat{T}^p)_{ab} \] (39)

This should be contrasted with the moments of the distribution function \( p(t) \):

\[ \int dt d^p p(t) = (i/\hbar)^p \sum_{a,b} \hat{A}_a^* \hat{A}_b \left( \frac{\partial^p \hat{S}(E)}{\partial E^p} \hat{S}(E) \right)_{ab} \] (40)

We see that only the first moment of this distribution coincides with that given by Eq. (33), all other being different. This particular example shows certain ambiguity in definition of delay time statistics. In the present paper we concentrate on statistics of Wigner-Smith time delays and related quantities: energy derivatives of \( S \)-matrix eigenphases.

D. S-matrix characteristics: poles, eigenphases and delay times

The expression (24) forms the basis for extracting the statistics of scattering poles (resonances), which are merely the complex eigenvalues of the non-Hermitian effective Hamiltonian \( \hat{H}_{eff} = \hat{H}_{in} - i\omega \hat{W} \hat{W}^\dagger \). At the same time, the expression (22) turns out to be a more convenient starting point for studying statistics of scattering phase shifts and delay times. Indeed, it is evident that scattering phase shifts \( \theta_a, a = 1, 2, ..., M \) [defined via the S-matrix eigenvalues \( \exp \{i\theta_a\} \)] are determined by the eigenvalues \( z_a(E, X) \) of the matrix \( \pi \hat{K} \) in view of the relation: \( \theta_a = -2 \arctan z_a \). Here we indicated explicitly the dependence of the eigenvalues \( z_a \) on the energy \( E \) and an external parameter \( X \) originating from the corresponding dependence of the Hamiltonian on the parameter: \( \hat{H}_{in} = \hat{H}_{in}(X) \). It is therefore convenient to characterize the statistics of phase shifts via the spectral density:

\[ \rho_{E,X}(z) = \frac{1}{M} \sum_{a=1}^{M} \delta(z - z_a(E, X)) \] (41)

Actually, the relation \( \theta_a = -2 \arctan z_a \) determines the phase shifts modulo \( 2\pi \) only. It is easy to understand that every time the energy \( E \) coincides with one of the eigenvalues \( E_n(X), n = 1, 2, ..., N \) of the Hermitian Hamiltonian \( \hat{H}_{in}(X) \) one (and only one) of the scattering phase shifts crosses the value \( 2\pi \times \text{integer} \). Indeed, assuming that the eigenvalues \( E_n \) are generically not degenerate one can write the matrix elements of the matrix \( \hat{K} \) in the vicinity of \( E = E_n \) as \( \hat{K}(E \to E_n) = \frac{1}{E - E_n} \hat{W}^* \hat{W}_{ab} \), where \( \hat{W}_{ab} \) are matrix elements of the coupling matrix \( \hat{W} \) in the basis of eigenstates \( \mid n \rangle \) of the Hamiltonian \( \hat{H}_{in} \) and \( \langle \hat{W} \rangle_{ab} = \hat{W}^*_{ab} \). We see immediately that \( \hat{K}(E \to E_n) \) has only one eigenvalue divergent at \( E = E_n \) which is given by \( z(E) = \frac{i}{\pi} n_{E_n} \sum_{a} \mid W_{na} \mid^2 \), the corresponding eigenvector being \( \nu = (W_{n1}^*, W_{n2}^*, ..., W_{nM}^*)^\dagger \). The phase shift corresponding to the infinite value of \( z(E) \) must be an integer of \( 2\pi \). At the same time all other eigenvalues of \( \hat{K}(E \to E_n) \) are exactly zero in that approximation with corresponding eigenvectors belonging to the \( M - 1 \) dimensional space orthogonal to \( \nu \). This fact just means that \( M - 1 \) corresponding eigenvalues of the exact matrix \( \hat{K}(E) \) stay finite in the vicinity of \( E_n \). Introducing the exact density of states for the closed chaotic system: \( \nu_X(E) = \frac{1}{2} \sum_n \delta(E - E_n(X)) \) and fixing the phase shift value at \( E = -\infty \) to be zero, we conclude that:

\[ \sum_{a=1}^{M} \theta_a = 2\pi N \int_{-\infty}^{E} du \nu_X(u) - 2 \sum_a \arctan z_a \] (42)

Here \( \arctan z_a \) means the principal branch: \( \mid \arctan z_a \mid < \pi/2 \). As function of energy \( E \) this expression is continuous and monotonically increasing with \( E \). The first term is proportional to the level staircase and we can forget it modulo \( 2\pi \).

We will use the relation (24) later on in order to determine the correlations of the Wigner delay times \( \tau_w(E) = (\partial/\partial E) \frac{1}{M^2} \sum a \theta_a = -(i/M)(\partial/\partial E) \ln \text{Det} \hat{S}(E) \), which , of course, are positive. For the latter quantity we also can find an independent representation by noticing that:

\[ \ln \text{Det} \hat{S}(E) = \ln \frac{\text{Det}(\hat{I} - i\hat{K})}{\text{Det}(\hat{I} + i\hat{K})} = \ln \frac{\text{Det}(\hat{I} - i\pi E_{\text{fin}}^{-1/2} \hat{W} \hat{W}^\dagger)}{\text{Det}(\hat{I} + i\pi E_{\text{fin}}^{-1/2} \hat{W} \hat{W}^\dagger)} \] (43)
where we have made use of the identity
\[
\text{Det}(\hat{I} - \hat{U}\hat{V}) = \text{Det}(\hat{I} - \hat{V}\hat{U})
\] (44)
valid for arbitrary (also rectangular) matrices $\hat{U}, \hat{V}$. From Eq. (43) we immediately obtain the simple relation:
\[
\tau_w(E) = -(2/M)\text{Im \ Tr} \left( E - \hat{H}_{in} + i\pi\hat{W}\hat{W}^+ \right)^{-1} = \frac{2}{M} \sum_{n=1}^{N} \frac{\Gamma_n/2}{(E - E_n)^2 + \Gamma_n^2/4}
\] (45)
which, in particular, shows an intimate relation between the statistics of Wigner time delay and that of $S$-matrix poles.

To study $S$-matrix characteristics within the framework of the stochastic approach, one should specify the properties of the amplitudes $W_{ia}$, which couple the internal chaotic motion to $M$ open channels. For the sake of simplicity one can restrict the consideration to the case when the $S$-matrix is diagonal after averaging: $\langle S_{ab} \rangle = \delta_{ab}\langle S_{aa} \rangle$. Such a choice is related with the mentioned absence of direct coupling between the channels [42,25] and can be ensured if one considers fixed amplitudes $W_{ia}$ satisfying the so-called orthogonality relations [42]:
\[
\sum_i W^*_{ia} W_{ib} = \frac{1}{N} \delta_{ab}
\] (46)
An alternative way to ensure the diagonality of the average $S$-matrix is to consider the amplitudes $W_{ia}$ to be independent Gaussian random variables [52,55,87]:
\[
\langle W_{ia} \rangle = 0; \quad \langle W^*_{ia} W_{jb} \rangle = \frac{\gamma_a}{N} \delta_{ab}\delta_{ij}
\] (47)
One can show, following the papers [57,58] that both choices lead to the same results as long as the number $M$ of open channels is negligible in comparison with the number of bound states $N$. Since this case is the only considered in the present paper, we restrict ourselves to the condition Eq. (46) henceforth. Provided the orthogonality condition [48] is fulfilled, one can show that the diagonal elements of the $S$-matrix are given by the following expression:
\[
\langle S_{aa} \rangle = \frac{1 - \gamma_a g(E)}{1 + \gamma_a g(E)}
\] (48)
where $g(E) = iE/2 + (1 - E^2/4)^{1/2}$. We do not give here the derivation of Eq. (48) referring the interested reader to the paper [42] for more details. However, we mention that for the one-channel case $M = 1$ relation Eq. (48) follows directly from the distribution of the phase shift $\theta$ to be found in Sec.IV. The strength of coupling to continua is convenient to be characterized via the transmission coefficients $T_a = 1 - |\langle S_{aa} \rangle|^2$ that are given for the present case by the following expression:
\[
T_a^{-1} = \frac{1}{2} \left[ 1 + \frac{1}{2\text{Re } g(E)}(\gamma_a + \gamma_a^{-1}) \right]
\] (49)
The quantity $T_a$ measures the part of the flux in channel $a$ that spends substantial part of the time in the interaction region [42,25]. This interpretation follows from the fact that the energy averaged $S$-matrix (equal to the ensemble average $\langle S \rangle$ by ergodicity requirement) describes a short time scattering ("direct response", see the Introduction). Let us also note that frequently we find it to be more convenient to use the parameters
\[
g_a = \frac{2}{T_a} - 1
\] (50)
rather than the "transmission coefficients" $T_a$.

Naively, one could suspect that the larger is the parameter $\gamma_a$, the larger is the part of the flux effectively penetrating the chaotic region. However, we see that this is not the case: both limits $\gamma_a \to 0$ and $\gamma_a \to \infty$ equally correspond to the weak effective coupling regime $T_a \ll 1$ whereas the strongest coupling (at fixed energy $E$ ) corresponds to the value $\gamma_a = 1$. The maximal possible coupling corresponding to the upper bound $T_a = 1$ is achieved in the present model for an energy interval in the vicinity of the center $E = 0$.

This feature is a purely quantum effect and is not surprising any longer, if one remembers the simplest textbook example of a quantum particle scattered on a one-dimensional "potential step": $V(x) = 0$ for $x < 0$ and $V(x) = V$ for $x > 0$. The transmission coefficient for such a problem is given by:
\[
T = 1 - |S|^2 = 4kK/(K + k)^2,
\] where
\[ k = (2mE)^{1/2}/h; \quad K = [2m(E - V)]^{1/2}/h \] and \( E > 0 \) stands for the energy of incoming particles. Similarly to the case above the transmission is very small both for the system "almost closed" classically: \( E - V \ll E \) when \( \gamma_0 \equiv (K/k)^{1/2} \ll 1 \) as well as for systems "very open classically" \( V < 0, |V| \gg E \) when \( \gamma_0 \gg 1 \), the "perfect" transmission \( T = 1 \) being possible for the only case \( V = 0 \) when \( \gamma_0 = 1 \). This simple example is of course just to remind us of the effect known in radiophysics as "impedance mismatch": the wave is always reflected back at the point of contact of two different waveguides, unless special boundary conditions are ensured.

### III. SCATTERING POLES IN COMPLEX PLANE: DISTRIBUTION OF RESONANCE WIDTHS.

#### A. Resonances as eigenvalues of a non-Hermitian Hamiltonian.

We are interested in determining the average two-dimensional density

\[ \rho(E, Y) = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta^{(2)}(E - E_j) \right\rangle = \left\langle \frac{1}{N} \sum_{j=1}^{N} \delta(E - E_j) \delta(Y - Y_j) \right\rangle \]

of complex eigenvalues \( E_j = E_j + iY_j, \quad j = 1, 2, ..., N \) of a non-Hermitian effective Hamiltonian \( \mathcal{H}_{ef} = \hat{H}_{in} - i\hat{\Gamma} \), see Eq.(4). According to the general discussion presented in the Introduction we use a \( N \times N \) random matrix \( \hat{H}_{in} \) from the Gaussian Unitary Ensemble to model the Hamiltonian of a closed chaotic system with broken time-reversal symmetry. The entries \( \Gamma_{ij} \) of the matrix \( \hat{\Gamma} \) are expressed in terms of the channel amplitudes \( W_{ia} \), \( a = 1, 2, ..., M \) as \( \Gamma_{ij} = \pi \sum_{a} W_{ia}W_{ja}^{*} \). Before presenting a general theory it is instructive to consider the important limiting case of an extremely weak coupling when we expect that the resonances are so narrow that their widths are much smaller in comparison with the mean separation between the unperturbed levels. Under these conditions a simple first order perturbation theory is adequate. Using the notation \( |n\rangle \) for the eigenvector of \( \hat{H}_{in} \) corresponding to the real eigenvalue \( E_n \) of the closed system: \( \hat{H}_{in} | n \rangle = E_n | n \rangle \), one can estimate the shift of the eigenvalues into the complex plane as

\[ -Y_n = \langle n | \hat{\Gamma} | n \rangle = \sum_{k,l=1}^{N} \langle \alpha_l | \hat{\Gamma} | \alpha_k \rangle \langle n | \alpha_l \rangle \langle \alpha_k | n \rangle \]

where \( | \alpha_k \rangle: \quad k = 1, 2, ..., N \) is an arbitrary chosen basis of orthonormal vectors. The matrix \( \hat{\Gamma} \) can be easily diagonalized and shown to have exactly \( M \) non-zero eigenvalues \( \gamma_a = \pi \sum_{i} W_{ia}^{*}W_{ia} \). Choosing \( | \alpha_k \rangle \) to be eigenbasis of the matrix \( \hat{\Gamma} \) we therefore have: \( Y_n = -\sum_{a=1}^{M} \gamma_a \langle n | \alpha_a \rangle \langle \alpha_a | n \rangle \). Now we use the well-known fact that different components \( \langle n | \alpha_a \rangle = u_a + i\nu_a \) of eigenvectors of the GUE matrices in an arbitrary basis can be treated as independent complex variables, their real and imaginary parts being independently distributed according to the Gaussian law with the variances \( u_a^{2} = \nu_a^{2} = \frac{1}{2N} \). This fact allows one to calculate the distribution of \( Y_n \) easily. Considering for simplicity the case of all equivalent channels: \( \gamma_a = \gamma \) for any channel, we get:

\[ \mathcal{P}(Y) = \delta(Y - Y_n) = \int_{-\infty}^{\infty} e^{ikY} \frac{dk}{2\pi} \left[ \int du dv \frac{N}{\pi} \exp(-(N - i\gamma k)(u^{2} + v^{2})) \right]^{M} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{(1 - ik\gamma/N)^{M}} = \frac{N^{M}}{\gamma^{M} \Gamma(M)} |Y|^{M-1} \exp \left[ - \frac{N}{\gamma} |Y| \right]. \]

for \( Y < 0 \) and zero otherwise.

We arrive at the well-known result: the widths of resonances for a slightly open chaotic system is given by the so-called \( \chi^{2} \) distribution \([31]\). Actually, the same form is applicable also for \( M \)-channel open systems with preserved time-reversal symmetry, provided one changes \( M \to M/2 \). The latter distribution for \( M = 1 \) is known as the Porter-Thomas distribution.

When the coupling to continua increases some resonances start to overlap and the simple perturbation theory loses its validity.

A general method for calculating the eigenvalue density for non-Hermitian random matrices was proposed by Sommers et al. \([18]\). These authors suggested to recover the density \( \rho(E, Y) \) from the "potential function"

\[ -\Phi(E, Y) = \frac{1}{N} \langle \ln \det (\mathcal{E} - \mathcal{H}_{ef})^{\dagger} (\mathcal{E} - \mathcal{H}_{ef}) \rangle = \left\langle \frac{1}{N} \sum_{j} \ln |\mathcal{E} - \mathcal{E}_j|^2 \right\rangle \equiv \left\langle \frac{1}{N} \sum_{j} \ln \left[ (E - E_j)^{2} + (Y - Y_j)^{2} \right] \right\rangle \]

(54)
To show that this is indeed possible it is convenient to regularize the logarithm in eq.(55) first and consider:

$$ - \Phi(E,Y)_\epsilon = \left\langle \frac{1}{N} \sum_j \ln \left[ \frac{(E-E_j)^2 + (Y-Y_j)^2 + \epsilon^2}{ \sqrt{N} \sum_j (E-E_j)^2 + (Y-Y_j)^2 + \epsilon^2} \right] \right\rangle $$ \hspace{1cm} (56)

Then one notices that the function:

$$ \rho(E,Y)_\epsilon = -\frac{1}{4\pi} (\partial_E^2 + \partial_Y^2) \Phi(E,Y)_\epsilon = \frac{1}{N\pi} \sum_j \frac{\epsilon^2}{[(E-E_j)^2 + (Y-Y_j)^2 + \epsilon^2]^2} $$ \hspace{1cm} (57)

produces the required two-dimensional density, Eq.(53), when $\epsilon \to 0$. Indeed, for an arbitrary continuous function $f(E,Y)$ one has:

$$ \lim_{\epsilon \to 0} \int_0^\infty \int_0^\infty dE dY f(E,Y) \frac{\epsilon^2}{[(E-E_j)^2 + (Y-Y_j)^2 + \epsilon^2]^2} = \lim_{\epsilon \to 0} \int_0^\infty \int_0^\infty \frac{dudv}{\pi} \frac{f(\epsilon u + E_j, \epsilon v + Y_j)}{[u^2 + v^2 + 1]^2} $$ \hspace{1cm} (58)

in agreement with the $\delta$-functional property.

In fact, the expressions Eqs.(59,57) show that $\rho(E,Y)$ can be considered as a two-dimensional density of fictitious pointlike "electric charges" $1/N$, the function $\Phi(E,Y)$ playing the role of the electrostatic potential for such a system and eq.(57) being the corresponding Poisson equation.

Actually, it turns out to be more convenient to use a slightly different regularization, as it has been actually done in [55,98], performing the limiting procedure $\kappa \to 0$ at the very end. For the so-called normal matrices (whose Hermitian conjugate $H^\dagger$ commutes with $H$) the regularized potential Eq.(59) coincides with that defined in the Eq.(56). It is a less trivial fact that one recovers the two-dimensional density of complex eigenvalues from the potential $\Phi(E,Y;\kappa)$ also in a general case of nonnormal matrices $H$. We show in the Appendix A that it is indeed the case: the density obtained from Eq.(57) by Poisson’s equation is positive and goes to a sum of $\delta$-functions with weight $1/N$ near the eigenvalues of $H_{e,f}$.

The main technical problem is to perform the averaging of the logarithmic potential $\Phi(E,Y;\kappa)$ over the random matrices from the corresponding Gaussian ensemble, eq.(1). To perform such an averaging the authors of the papers [55,98] employed the famous, but somewhat problematic "replica trick". This procedure amounts to averaging the corresponding determinant raised to an arbitrary positive integer power $n$, the average logarithm being recovered as a result of the limiting procedure $n \to 0$. In general, however, the analytical continuation $n \to 0$ is not unique. In particular, it is known that the replica trick fails to reproduce correctly the correlation function of densities of real eigenvalues of large Hermitian matrices at two points $E \pm \Omega$ of the spectrum [23]. Rather, it succeeds in giving the correct behavior of that correlation function at the scale $\Omega$ large in comparison with the typical separation between neighboring eigenvalues, known as the mean level spacing $\Delta$. As is shown below, formally the calculation of the mean eigenvalue density in the complex plane is very similar to the calculation of a correlation function of eigenvalue densities on the real axis, with the role of $\Omega$ played by the variable $iY$ measuring the distance from the real axis. We immediately conclude, that the replica trick must fail when we are interested in eigenvalues situated sufficiently close to the real axis: $Y \sim \Delta$.

To this end it is necessary to mention, that the non-Hermiticity of the matrices considered in the papers [55,98] was, in a sense, quite strong: the probability for an eigenvalue to be situated at the distance $Y \sim \Delta \propto 1/N$ vanished in the limit $N \to \infty$. Under these conditions it is not surprising that the replica trick succeeded in producing the correct result, the fact verified both by independent methods: by a variant of the supersymmetry method [58] and by direct numerical computations [55,98].

The situation described above is drastically different from that we expect to happen at the model under the present consideration. Indeed, it is known, that when the number $M$ of open channels is small in comparison with the number $N$ of relevant resonances, the majority $N-M$ of resonances are rather "narrow" and the corresponding poles are situated close to the real axis [55,59,80]. Under such a situation one has to discard the replica trick and to seek for a more reliable procedure. Fortunately, the authors of the mentioned paper [99] showed how to calculate the
two-point correlation function for real eigenvalues correctly by exploiting the method pioneered by Efetov \[100\] in the theory of disordered solids. A pedagogical introduction to the method can be found in \[101\]. In the present paper we adjust this procedure for finding the density of scattering poles in the complex plane for the few-channel case \[71\] (for the many-channel case \(M \propto N\) this density has already been calculated by N.Lehmann et al. \[58\]).

Instead of working directly with the regularized potential \(\Phi(E,Y,\kappa)\), see Eq.(50), in terms of which the two-dimensional density \(\rho(E,Y)\) is expressed as:

\[
\rho(E,Y) = -\lim_{\kappa \to 0} \frac{1}{4\pi}(\partial^2_{E} + \partial^2_{Y})\Phi(E,Y,\kappa)
\]

we prefer to consider the related function:

\[
\Phi(E,-i\Omega;\kappa) = -\frac{1}{N} \ln \det \left[ \left( E + \Omega - \hat{H}_{in} + i\Gamma \right) \left( E - \Omega - \hat{H}_{in} - i\Gamma \right) + \kappa^2 \hat{I}_N \right]
\]

(61)

It is evident, that the potential \(\Phi(E,Y,\kappa)\) can be obtained from the function \(\Phi(E,-i\Omega;\kappa)\) by analytical continuation \(-i\Omega \to Y\). As long as \(\kappa\) is finite, there is a region extending from positive to negative \(Y\) where the function \(\Phi\) is analytic in \(Y = -i\Omega\). Actually, this continuation is more convenient to perform directly on the level of densities, i.e. first to calculate the auxiliary function \[102\]:

\[
\rho(E,-i\Omega) = -\frac{1}{4\pi}(\partial^2_{E} - \partial^2_{Y})\Phi(E,-i\Omega,\kappa)
\]

(62)

and to restore the true two-dimensional density \(\rho(E,Y)\) letting \(-i\Omega \to Y\) first and then \(\kappa \to 0\):

\[
\rho(E,Y) = \lim_{\kappa \to 0} \rho(E,-i\Omega = Y)\kappa.
\]

(63)

To this end let us consider the generating function:

\[
Z(E,\Omega;E_b,\Omega_b;\kappa) = \left[ \frac{\det \left[ \left( E + \Omega - \hat{H}_{in} + i\Gamma \right) \left( E - \Omega - \hat{H}_{in} - i\Gamma \right) + \kappa^2 \hat{I}_N \right]}{\det \left[ \left( E_b + \Omega_b - \hat{H}_{in} + i\Gamma \right) \left( E_b - \Omega_b - \hat{H}_{in} - i\Gamma \right) + \kappa^2 \hat{I}_N \right]} \right]
\]

(64)

in terms of which the function \(\rho(E,-i\Omega)\kappa\) is expressed as follows:

\[
\rho(E,-i\Omega) = \frac{1}{4\pi} \left[ \left( \partial_{E} \lim_{(E_b,\Omega_b) \to (E,\Omega)} \partial_{E} \right) - \left( \partial_{\Omega} \lim_{(E_b,\Omega_b) \to (E,\Omega)} \partial_{\Omega} \right) \right] \langle Z(E,E_b,\Omega_b;\kappa) \rangle,
\]

(65)

The determinant in the denominator of expression Eq.[64] can be represented in a form of a conventional Gaussian integral over the components of a complex \(2N\)-component vector \(S = (S_1, S_2)^T\), \(dS = \prod_{j=1}^{N} \frac{dReS_j dImS_j}{\pi}\).

\[
\int [dS_1][dS_2] \exp \left\{ -S^T \left[ \begin{array}{cc}
-\kappa \hat{I} & \kappa \hat{I} \\
\kappa \hat{I} & -\kappa \hat{I}
\end{array} \right] \right\} \equiv \int [dS_1][dS_2] \exp \left\{ \kappa(S_1^T S_2 - S_2^T S_1) + iE_b(S_1^T S_1 - S_2^T S_2) + i(S_1^T, S_2^T) \begin{pmatrix}
-H_{in} & 0 \\
0 & H_{in}
\end{pmatrix} \begin{pmatrix}S_1 \\ S_2\end{pmatrix} \right\} \times
\]

\[
\exp \left\{ i\Omega_b(S_1^T S_1 + S_2^T S_2) - (S_1^T, S_2^T) \begin{pmatrix}
\hat{I} & 0 \\
0 & \hat{I}
\end{pmatrix} \begin{pmatrix}S_1 \\ S_2\end{pmatrix} \right\}
\]

(66)

At this point it is worth mentioning that all eigenvalues of \(\mathcal{H}_{ef} = \hat{H}_{in} - i\hat{\Gamma}\) (scattering poles) must be situated in the lower half of the complex plane \(\text{Im} \mathcal{E} \leq 0\). Formally it is ensured by eigenvalues of the matrix \(\hat{\Gamma}\) being real non-negative. We see that it is due to this fact that the Gaussian integral above is convergent \[103\] (for \(\Omega\) real other terms in the exponent are purely imaginary and do not spoil the convergency; at the end we may continue analytically).
The following comment is appropriate here. In principle, one can deal directly with the potential \( \Phi(E,Y,\kappa) \), Eq.(69) and succeed in finding the convergent Gaussian representation for the generating function everywhere in the complex plane \( E+iY \) (see \cite{[7], [104]}). However, the evaluation of the averaged generating function and subsequent restoration of the eigenvalue density turns out to be quite a daunting job. This is the reason why we decided to deal in our particular case with a less general, but more tractable representation Eq.(70) allowing to evaluate the generating function for two real parameters \( E, \Omega \) and then to continue analytically \( \Omega \rightarrow iY \) as explained above.

Returning to our problem we represent the determinant in the numerator of the generating function Eq.(64) in the form of a Gaussian integral over a \( 2N \)-dimensional vector \( \chi = \left( \frac{\chi_1}{\chi_2} \right) \) whose elements \( \chi_j^{(p)} \), \( j = 1, 2, ..., N; p = 1, 2 \) are anticommuting (Grassmannian) variables (see reviews \cite{[100],[101]} for more details):

\[
\text{Det}^{-1} \left[ \left( E - \Omega - \hat{H}_{in} - i \hat{\Gamma} \right) \left( E + \Omega - \hat{H}_{in} + i \hat{\Gamma} \right) + \kappa^2 \hat{I} \right] = (-1)^N \text{Det} \left[ \begin{array}{cc}
-i \left( E + \Omega - \hat{H}_{in} + i \hat{\Gamma} \right) & \kappa \hat{I} \\
\kappa \hat{I} & -i \left( E - \Omega - \hat{H}_{in} - i \hat{\Gamma} \right)
\end{array} \right] =
\]

\[
(-1)^N \int [d\chi_1][d\chi_2] \exp \left\{ -\kappa \left( \chi_1^\dagger \chi_2 + \chi_2^\dagger \chi_1 \right) + i \chi_1^\dagger \left( E - \Omega - \hat{H}_{in} - i \hat{\Gamma} \right) \chi_2 + i \chi_2^\dagger \left( E + \Omega - \hat{H}_{in} + i \hat{\Gamma} \right) \chi_1 \right\}
\]

where \([d\chi] = \prod_{k=1}^N d\chi_k^2 d\chi_k^\dagger\). In contrast to the discussion above, the integration over Grassmann variables is always well defined and one does not encounter the convergency problem.

Obviously, our generating function is the product of two Gaussian integrals defined in Eqs.(66,67). It is convenient to introduce the notion of a supervector

\[
\Psi = \left( \begin{array}{c}
\Psi_1 \\
\Psi_2
\end{array} \right) \quad \text{where} \quad \Psi_p = \left( \begin{array}{c}
S_p \\
\chi_p
\end{array} \right); \quad p = 1, 2.
\]

Then one can write the generating function in the following "supersymmetric" form:

\[
Z(E,\Omega; E_b, \Omega_b; \kappa) = (-1)^N \int [d\Psi] \exp(-(S_b[\Psi] + S_{E f}[\Psi]))
\]

where

\[
S_{E f}[\Psi] = -i \Omega \Psi^\dagger \hat{\Lambda} \hat{L} \Psi - i E \Psi^\dagger \hat{L} \Psi + i \Psi^\dagger \left( \hat{H}_{in} \otimes \hat{\Gamma} \right) \Psi
\]

and

\[
S_b[\Psi] = \kappa \Psi^\dagger \hat{\Sigma} \Psi + i (\Omega - \Omega_b) \Psi^\dagger \hat{K}_b \Psi + i (E - E_b) \Psi^\dagger \hat{L} \hat{K}_b \Psi + \Psi^\dagger \left( \hat{\Gamma} \otimes \hat{\Lambda} \hat{L} \right) \Psi
\]

Before presenting the explicit expressions for the supermatrices \( \hat{L}, \hat{\Lambda}, \hat{K}_b \) and \( \hat{\Sigma} \) we would like to make a notational convention on arranging elements of supermatrices. All these (and subsequently appearing) supermatrices are assumed to act in the space of supervectors \( \Psi \) whose element arrangement is defined in Eq.(68). Correspondingly, we subdivide each \( 4 \times 4 \) supermatrix \( \hat{Q} \) into four blocks \( \hat{Q} = \left( \begin{array}{cc}
\hat{Q}_{11} & \hat{Q}_{12} \\
\hat{Q}_{21} & \hat{Q}_{22}
\end{array} \right) \) in such a way that

\[
\Psi^\dagger \hat{Q} \Psi = \sum_{m,n=1}^2 \Psi_m^\dagger \hat{Q}_{mn} \Psi_n, \quad \text{each of these } \hat{Q}_{mn} \text{ blocks is in turn a } 2 \times 2 \text{ supermatrix } \hat{Q}_{mn} = \left( \begin{array}{cc}
Q_{bb}^{(mn)} & Q_{bf}^{(mn)} \\
Q_{fb}^{(mn)} & Q_{ff}^{(mn)}
\end{array} \right)
\]

such that \( \Psi_m^\dagger \hat{Q}_{mn} \Psi_n = S_m^\dagger Q_{bb}^{(mn)} S_n + S_m^\dagger Q_{bf}^{(mn)} \chi_n + \chi_m^\dagger Q_{fb}^{(mn)} S_n + \chi_m^\dagger Q_{ff}^{(mn)} \chi_n \). The indices \( b,f \) remind us of "bosonic"/"fermionic" nature of the commuting/Grassmannian components of supervectors, respectively.

It is necessary to note that in the present paper we use the same convention for Hermitian conjugation of \( 2 \times 2 \) supermatrices as in the paper \cite{[101]}: \( \left( \begin{array}{c}
Q_{bb} \quad Q_{bf} \\
Q_{fb} \quad Q_{ff}
\end{array} \right)^\dagger = \left( \begin{array}{c}
Q_{bb} \quad Q_{fb} \\
-Q_{fb} \quad Q_{ff}
\end{array} \right) \). This is different from the convention used in \cite{[101],[105]} and results in some subsequent differences in parametrizations.

With these conventions the \( 4 \times 4 \) supermatrices appearing in Eq.(71) are given by the following expressions:

\[
\hat{L} = \text{diag}(1,1,-1,1); \quad \hat{\Lambda} = \text{diag}(1,1,-1,-1); \quad \hat{K}_b = \text{diag}(1,0,1,0);
\]

and \( \hat{\Sigma} = \left( \begin{array}{cc}
0 & I_2 \\
-k & 0
\end{array} \right) \), where \( I_2 = \text{diag}(1,1); \quad k = \text{diag}(-1,1) \).
B. Ensemble-averaged generating function.

A quick inspection of Eqs. (69, 70, 72) makes it clear that the superintegral in hand is very similar to that emerging in the problem of calculation of the pair correlation function of eigenvalues of Hermitian random matrices at points $E \pm \Omega$, see e.g. [101]. In fact, the two expressions coincide if one neglects the exponent $S_6$. The neglected exponent $S_6$ does not contain random variables and can not prevent us from using successfully the main steps of Efetov’s standard procedure when evaluating the average value of the generating function. Below we give a short description of the main steps of the method; all further details can be found in the review [101] and references therein.

- Ensemble averaging. One can easily perform the averaging over the Gaussian-distributed matrix elements of $H_{in}$ by exploiting the identity

$$\left\langle \exp \pm i \sum_{ij} (H_{in})_{ij} \hat{U}_{ij} \right\rangle = \exp - \frac{1}{2N} \sum_{ij} \hat{U}_{ij} \hat{U}_{ij}$$

(73)

In order to write down the result of the ensemble averaging in a convenient form it is useful to introduce the supermatrix $A$ with elements

$$A_{pq}^{(mn)} = \left( \hat{L}^{1/2} \right)_{pp}^{(mm)} \sum_{i=1}^{N} (\Psi_i^p \psi_i^q) \left( \hat{L}^{1/2} \right)_{qq}^{(nn)}$$

(74)

where indices $p$ and $q$ are equal to $b$ or $f$ and we assumed the convention: $\Psi_i^b \equiv S_i$; $\psi_i^f \equiv \chi_i$. Now the ensemble-averaged value of the corresponding exponent in eq.(63) can be written as:

$$\left\langle -i \Psi^I H_{in} \hat{L} \right\rangle = \exp - \frac{1}{2N} \text{Str} \hat{A}^2$$

(75)

where the symbol Str stands for the graded trace $\text{Str} \hat{Q} = \text{Tr} \hat{Q}_{bb} - \text{Tr} \hat{Q}_{ff}$. It is also useful to notice that $\text{Str} \hat{A} \hat{Q} = \Psi^I \hat{L}^{1/2} \hat{Q} \hat{L}^{1/2} \Psi$ for an arbitrary supermatrix $\hat{Q}$. In particular, $\text{Str} \hat{A} \hat{A} = \Psi^I \hat{A} \hat{L} \Psi$.

- Hubbard-Stratonovich transformation.

As a result of ensemble averaging the superintegral representing the generating function ceased to be a Gaussian one. The further progress is based on the following identity:

$$\exp \left[ - \frac{1}{2N} \text{Str} \hat{A}^2 + i\Omega \text{Str} \hat{A} \hat{A} \right] =$$

$$\int [d\hat{R}] \exp \left\{- \frac{N}{2} \text{Str} \hat{R}^2 + i \text{Str} \hat{R} \hat{A} + N \Omega \text{Str} \hat{R} \hat{A} \right\}$$

(76)

known as the Hubbard-Stratonovich (HS) identity.

Now we can substitute this relation back into the averaged generating function, to change the order of integrations over the supervector $\Psi$ and the supermatrices $\hat{R}$, and to calculate the corresponding (Gaussian) integral over $\Psi$ exactly using the identity:

$$\int [d\Psi] \exp (-\Psi^I \hat{F} \Psi) = S\text{det}^{-1} \hat{F},$$

(77)

where the notation $S\text{det}$ stands for the graded determinant: $S\text{det} \hat{Q} = \exp \text{Str} \ln \hat{Q}$. It turns out, however, that in order to have both $\int [d\Psi]...$ and $\int [d\hat{R}]...$ convergent, one has to parametrize the set of supermatrices $\hat{R}$ in the following non-trivial fashion suggested in [100,99] (see detailed discussion in [101]):

$$\hat{R} = \hat{T}^{-1} \hat{P} \hat{T}; \quad \hat{P} = \begin{pmatrix} \hat{P}_1 - i\delta \hat{I}_2 & 0 \\ 0 & \hat{P}_2 + i\delta \hat{I}_2 \end{pmatrix}; \quad \hat{P}_m = \begin{pmatrix} p_m & \eta^*_m \\ \eta_m & i\eta_m \end{pmatrix}$$

(78)
• Saddle-point calculation.

Performing the program specified above one gets the following representation for the average generating function:

$$\langle Z(E, \Omega; E_b, \Omega_b; \kappa) \rangle = \int [d\hat{R}] \exp \left\{ -\frac{N}{2} \text{Str} \hat{R}^2 + N \Omega \text{Str} \hat{R} \hat{A} - \text{Str} \ln \hat{G} \right\}$$  \hspace{1cm} (79)

where

$$\hat{G} = \hat{G}_1 \otimes \hat{I}_N - i \hat{\Gamma} \otimes \hat{A}; \quad \hat{G}_1 = -i \kappa \hat{\Sigma}_L - (E_b - E) \hat{K}_b + (\Omega - \Omega_b) \hat{K}_b \hat{L} - E \hat{I}_4 - \hat{R}$$  \hspace{1cm} (80)

and $\hat{\Sigma}_L = \hat{L}^{-1/2} \hat{\Sigma} \hat{L}^{-1/2}$. Now one can write:

$$\text{Str} \ln \hat{G} = N \text{Str} \ln \hat{G}_1 + \text{Str} \ln \left[ \hat{I}_N - i \hat{\Gamma} \otimes \langle \Lambda \hat{G}_1^{-1} \rangle \right]$$

The second term in this expression can be rewritten as:

$$\text{Str} \ln \left[ \hat{I}_N - i \hat{\Gamma} \otimes \langle \Lambda \hat{G}_1^{-1} \rangle \right] = \sum_{a=1}^{M} \text{Str} \ln \{ \hat{I}_4 - i \gamma_a (\Lambda \hat{G}_1^{-1}) \}$$  \hspace{1cm} (81)

which easily can be verified by expanding the logarithm into the series, exploiting the orthogonality condition, Eq.(10), in each term of that expansion and resuming the whole series back \[10\]. Up to the present point we did not make use of any approximation and our calculation was essentially exact. However, we are particularly interested in the limiting case of many resonances $N \gg 1$ coupled with few open channels $M \ll N$. In this limit we expect that the resonance widths are of the same order as the mean separation between adjacent resonances $\Delta \propto 1/N$. Therefore, we can restrict our attention to the case $Y \sim 1/N$, and, correspondingly, consider $\Omega \sim 1/N$. The second fact that should be taken into account is that we are actually interested in the limit $\kappa, E_b - E, \Omega_b - \Omega \rightarrow 0$ when calculating the generating function. These facts taken together make it clear that it is sufficient to expand the logarithm $\text{Str} \ln \hat{G}_1$ in the exponent of Eq.(79) with respect to $\kappa, E_b - E, \Omega_b - \Omega$ and retain (apart from the leading terms) only terms linear in these variables. At the same time we can just neglect all these variables in the term $\sum_{a=1}^{M} \text{Str} \ln \{ \hat{I}_4 - i \gamma_a (\Lambda \hat{G}_1^{-1}) \}$ because of the condition $M \ll N$. As the result, we have:

$$\langle Z(E, \Omega; E_b, \Omega_b; \kappa) \rangle = \int [d\hat{R}] \exp \left[ -N \mathcal{L}[\hat{R}] + \delta \mathcal{L} \right]$$  \hspace{1cm} (82)

where

$$\mathcal{L}[\hat{R}] = \frac{1}{2} \text{Str} \hat{R}^2 + \text{Str} \ln (-E \hat{I} - \hat{R})$$  \hspace{1cm} (83)

$$\delta \mathcal{L} = N \Omega \text{Str} \hat{R} \hat{A} + N \text{Str} \left[ i \kappa \hat{\Sigma}_L + (E_b - E) \hat{K}_b + (\Omega_b - \Omega) \hat{K}_b \hat{L} \right] \left( -E \hat{I} - \hat{R} \right)^{-1}$$  \hspace{1cm} (84)

The form of the integrand in Eq.(82) suggests that it can be effectively calculated by the saddle-point method exploiting the large parameter $N \gg 1$. The saddle point equation is determined by stationarity of the "action" $\mathcal{L}[\hat{R}]$ and has the form $\hat{R} = (-E \hat{I} - \hat{R})^{-1}$. At the same time the discussion above makes it clear that the terms entering $\delta \mathcal{L}$ are of the order of unity when $N \rightarrow \infty$ and should be disregarded when seeking for the saddle-point solution.

Actually, it turns out that there is a whole continuous manifold of the saddle point solutions $\hat{R}_s$ equally contributing to the integral Eq.(82) in the limit $N \rightarrow \infty; \hat{R}_s = -E/2 + i \pi \nu sc \hat{T}^{-1} \hat{\Lambda} T$, where $\nu sc = \nu sc(E)$ stands for the semicircular density, Eq. (2). Introducing a new set of matrices $\hat{Q} = -i \hat{T}^{-1} \hat{\Lambda} \hat{T}$ satisfying the conditions: $\hat{Q}^2 = -\hat{I}_4; \quad \text{Str} \hat{Q} = 0$ we finally write down the averaged generating function as:

$$\langle Z(E, \Omega; E_b, \Omega_b; \kappa) \rangle = \int [d\hat{Q}] \prod_{a=1}^{M} \text{Sdet}^{-1} \left[ I + i \frac{1}{2} \gamma_a E \hat{A} + i \pi \nu sc \gamma_a \hat{Q} \hat{A} \right] \times \exp \left\{ -N \pi \nu sc \Omega \text{Str} \hat{Q} (\hat{A} - \hat{K}_b \hat{L}) - N \pi \nu sc \Omega_b \text{Str} \hat{Q} \hat{K}_b \hat{L} - i \pi \rho N \kappa \text{Str} \hat{Q} \hat{\Sigma}_L + N (E_b - E) (E + \pi \nu sc \text{Str} \hat{Q} \hat{K}_b) \right\}$$  \hspace{1cm} (85)
To obtain the required function \( \rho(E, -i\Omega) \), we should substitute this expression into the relation Eq. (82). Upon doing this one immediately notices that the first term \( \frac{\partial}{\partial \Omega} \lim_{(E, \Omega) \to (E, 0)} \frac{\rho(E, -i\Omega)}{\nu_{sc}(E)} \) produces a contribution which is of the order of \( N \), whereas the term \( \frac{\partial}{\partial \Omega} \lim_{(E, \Omega) \to (E, 0)} \frac{\rho(E, -i\Omega)}{\nu_{sc}(E)} \frac{\hat{\Lambda}}{\hat{\Sigma}}(Z) \) produces a much larger contribution of the order of \( N^2 \). Retaining only leading terms as long as \( N \to \infty \) we arrive at the following expression:

\[
\frac{\rho(E, -i\omega)}{\nu_{sc}(E)} = -\frac{1}{2} \int [d\hat{Q}] \text{Str}(\hat{Q}\hat{\Lambda}) \text{Str} \left[ \hat{Q}(\hat{\Lambda} - \hat{K}_b \hat{L}) \right] \times 
\prod_{a=1}^{M} \text{Sdet}^{-1} \left[ I + i \frac{1}{2} \gamma_a E \hat{\Lambda} + i \pi \nu_{sc} \gamma_a \hat{Q} \hat{\Lambda} \right] \exp \left[ -\frac{\omega}{2} \text{Str} \hat{Q} \hat{\Lambda} - \frac{i\epsilon}{2} \text{Str} \hat{Q} \hat{\Sigma}_L \right]
\]

where we introduced the "scaled" variables \( \omega = 2\pi \nu_{sc}(E) \Omega \) and \( \epsilon = 2\pi \nu_{sc}(E) \Omega \kappa \) and the correspondingly rescaled \( \rho(E, -i\Omega) \to \rho(E, -i\omega) \). To make the presentation self-contained we present such a parametrization in Appendix B. In the same Appendix we evaluate also some supertraces, superdeterminants and combinations of the matrix elements entering Eq. (86) as well as other superintegrals we use.

C. Distribution of resonance widths: general expression.

To perform the explicit evaluation of the superintegral on the right-hand side of Eq. (86) one has to employ the parametrization of the manifold of the \( Q \)-matrices suggested by Efetov [100]. To make the presentation self-contained we present such a parametrization in Appendix B. In the same Appendix we evaluate also some supertraces, superdeterminants and combinations of the matrix elements entering Eq. (86) as well as other superintegrals we use later on. Upon substitution of these expressions into Eq. (86) one can perform the Grassmannian integration trivially and obtains:

\[
\frac{\partial}{\partial \Omega} \rho(E, -i\omega) \nu_{sc}(E) = \frac{i}{2} \int^{\infty}_{-\infty} d\lambda_1 \int^{1}_{-1} d\lambda_2 \int^{2\pi}_{0} d\phi_1 d\phi_2 \frac{\partial}{\partial \omega} \left[ \Phi(E, \lambda_1, \phi_1, \lambda_2, \phi_2) \right]
\]

where

\[
\Phi(E, \lambda_1, \phi_1, \lambda_2, \phi_2) = \text{exp} \left[ i \lambda_1 - \lambda_2 \right] \left| \mu_1 \right| \left| \mu_2 \right| \left| \cos \phi_2 \right| \left\{ \frac{\epsilon}{2} \left( \left| \mu_1 \right| \left| \mu_2 \right| \left| \cos \phi_2 \right| + i \epsilon \right) \right\}
\]

with \( g_a = 2/T_a - 1; \quad \lambda_1^2 = 1 + |\mu_1|^2; \quad \lambda_2^2 = 1 - |\mu_2|^2 \).

Having in mind that actually we have to perform the analytic continuation \(-i\omega \to y\) in the right-hand side of the Eq. (87) we introduce two functions \( F_{1,2}(\epsilon, -i\omega) \) according to the following definitions:

\[
F_1(\epsilon, -i\omega) = \frac{1}{2\pi} \int^{\infty}_{1} d\lambda_1 \int^{2\pi}_{0} d\phi_1 \text{exp} \left[ i \omega \lambda_1 + i \epsilon \left| \mu_1 \right| \left| \sin \phi_1 \right| \right] \prod_{a=1}^{M} (g_a + \lambda_1)^{-1} =
\]

\[
\int^{\infty}_{0} \prod_{a=1}^{M} (dS_a \exp \left[-S_a g_a\right]) \int^{1}_{0} d\lambda_1 J_0(\epsilon \sqrt{\lambda_1^2 - 1}) \exp \left[-\lambda_1 (-i\omega + \sum \lambda_a)\right] =
\]

\[
\int^{\infty}_{0} \prod_{a=1}^{M} (dS_a \exp \left[-S_a g_a\right]) \Phi_1(\epsilon, -i\omega + \sum \lambda_a); \quad \Phi_1(\epsilon, y) = \frac{\exp -\sqrt{\epsilon^2 + y^2}}{\sqrt{\epsilon^2 + y^2}}
\]

and, similarly

\[
F_2(\epsilon, -i\omega) = \frac{1}{4\pi} \int^{1}_{-1} d\lambda_2 \int^{2\pi}_{0} d\phi_2 \text{exp} \left[-i \omega \lambda_2 - \epsilon \left| \mu_2 \right| \left| \cos \phi_2 \right| \right] \prod_{a=1}^{M} (g_a + \lambda_2) =
\]

\[
\sum_{k=0}^{M} D^{(M)}_{k} (g) \frac{1}{2} \int^{\frac{1}{2}}_{-\frac{1}{2}} d\lambda_2 \lambda_2^2 J_0(\epsilon \left| \mu_2 \right|) \exp \left(-i \omega \lambda_2\right) =
\]

\[
\sum_{k=0}^{M} D^{(M)}_{k} (g) \frac{\partial^k}{\partial (-i\omega)^k} \Phi_2(\epsilon, -i\omega); \quad \Phi_2(\epsilon, y) = \frac{\sinh \sqrt{\epsilon^2 + y^2}}{\sqrt{\epsilon^2 + y^2}}
\]
where

\[ D_0^{(M)} \{ g \} = \prod_{a=1}^{M} g_a; \quad D_1^{(M)} \{ g \} = \sum_{a=1}^{M} \prod_{b \neq a} g_b; \quad D_2^{(M)} \{ g \} = \sum_{a,b=1}^{M} \prod_{c \neq a,b} g_c, \text{etc.} \tag{91} \]

and \( J_0(z) \) and \( I_0(z) = J_0(iz) \) stand for the Bessel functions.

One can easily satisfy oneself that the right-hand side of Eq. (97) can be expressed in terms of the functions \( F_1(\epsilon, -i\omega) \) and \( F_2(\epsilon, -i\omega) \) and their derivatives. This gives us the possibility to perform the required analytic continuation easily and to restore the two-dimensional density \( \rho(E, y) = 2\pi \nu_{\text{sc}}(E)N, \text{see Eq. (63)} \) in the form:

\[ \frac{\partial}{\partial y} \left( \frac{\rho(E, y)}{\nu_{\text{sc}}(E)} \right) = -\lim_{\epsilon \to 0} \left\{ \frac{\partial}{\partial y} \left( F_2 \frac{\partial F_1}{\partial \epsilon} - F_1 \frac{\partial F_2}{\partial \epsilon} \right) - \epsilon \frac{\partial}{\partial \epsilon} \left[ F_1 \frac{\partial F_2}{\partial y} \right] \right\} - \frac{\epsilon^2}{2} \left\{ \frac{\partial^2 F_1}{\partial \epsilon^2} \right\} \frac{\partial F_1}{\partial y} + F_1 \frac{\partial^2 F_2}{\partial \epsilon^2} \frac{\partial F_1}{\partial y} - 4 \frac{\partial F_1}{\partial \epsilon} \frac{\partial F_2}{\partial \epsilon} \frac{\partial F_1}{\partial y} + \frac{\partial^2 F_1}{\partial \epsilon^2} \frac{\partial F_2}{\partial \epsilon} \frac{\partial F_1}{\partial y} - F_1 \frac{\partial^2 F_2}{\partial \epsilon^2} \frac{\partial F_1}{\partial y} + 2 \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial y} \right\} \tag{92} \]

The limiting transition \( \epsilon \to 0 \) is performed with the help of the identities:

\[ -\lim_{\epsilon \to 0} \left\{ \frac{\partial}{\partial \epsilon} \Phi_1(\epsilon, y) \right\} = \lim_{\epsilon \to 0} \epsilon^2 \frac{\partial^2}{\partial \epsilon^2} \Phi_1(\epsilon, y) = 2\delta(y) \tag{93} \]

which, in turn, are consequences of the following representations for the \( \delta^- \) function valid for an arbitrary integer \( k \):

\[ \delta(y) = C_k \left( \lim_{\epsilon \to 0} \frac{\epsilon^2}{(\epsilon^2 + y^2)^{2k+1/2}} \right); \quad C_1 = 1/2; \quad C_2 = 4/3 \text{ etc.} \tag{94} \]

It is useful also to note that if we substitute \( \Phi_2(\epsilon, y) \) for \( \Phi_1(\epsilon, y) \) in Eq. (93), this will produce zero instead of the \( \delta^- \)-functions on the right-hand side. The same is true also for terms like \( \lim_{\epsilon \to 0} \epsilon^2 \frac{\partial^2}{\partial \epsilon^2} \Phi_1(\epsilon, y) \). Using these observations we easily pick up all nonvanishing contributions to Eq. (24). Summing them up we get:

\[ \frac{\partial}{\partial y} \frac{\rho(E, y)}{\nu_{\text{sc}}(E)} = \frac{\partial}{\partial y} \frac{\lim}{\epsilon \to 0} \left[ F_2 \frac{\partial F_1}{\partial \epsilon} \right] \tag{95} \]

that immediately allows us to restore the density \( \rho(E, y) \) in the form:

\[ \frac{\rho(E, y)}{\nu_{\text{sc}}(E)} = F_2(\epsilon = 0, y) \int_0^\infty \left( \prod_{a=1}^{M} dS_a \right) \delta(y + \sum a S_a) \exp - \sum a g_a S_a \tag{96} \]

It is clear, however, that for any positive \( y \) the \( \delta^- \)-functional constraint in Eq. (96) is never satisfied and the right-hand side is identically zero. This result is of course just a consequence of the fact of absence of scattering poles in the upper half plane of complex energies.

We therefore consider \( y < 0 \) from now on, make the substitution \( y \to -y \) to the previous equation and consider \( y > 0 \) after that. The M-fold integral can be further simplified upon using the integral representation: \( \delta(u) = \frac{1}{2\pi} \int_{-\infty}^\infty dk \exp iku \). Finally we arrive at the following expression:

\[ \frac{\rho(E, y)}{\nu_{\text{sc}}(E)} = F_1 \{ \{ g \}; y \} F_2 \{ \{ g \}; y \} \tag{97} \]

where

\[ F_1 \{ \{ g \}; y \} = -\frac{1}{2\pi} \int_{-\infty}^\infty dk e^{-iky} \prod_{a=1}^{M} \frac{1}{g_a - ik} = (-1)^M \sum_{a=1}^{M} e^{-yg_a} \prod_{b \neq a}^{M} \frac{1}{g_a - g_b} \tag{98} \]

and

\[ F_2 \{ \{ g \}; y \} = \frac{1}{2} \int_1^1 dx e^{-y\lambda} \prod_{a=1}^{M} (g_a + \lambda) = \sum_{k=0}^{M} (-1)^k D_k^{(M)} \{ g \} \frac{d^k}{dy^k} \sinh y \tag{99} \]
and the functions $D_k^{(M)} \{g\}$ are defined in the Eq. [2].

It is clear that the function $\rho_E(y) = \frac{\rho(E_y)}{\nu_{sc}(E)}$ has the meaning of a distribution of (scaled) resonance widths for those resonances $E_j = E_j + iY_j$ whose positions $E_j$ fall into a narrow window $\delta E$ around the point $E$ of the spectrum:

$$\rho_E(y) = \frac{1}{N_E} \sum_{j=1}^{N_E} \delta(y - 2\pi \nu_{sc}(E)NY_j).$$

Such a window should contain a lot of individual resonances: $N_E \sim \frac{\delta E}{\Delta(E)} \gg 1$ in order to make the statistics representative. On the other hand, it should be small in comparison with the total width of the spectrum (in our model given by the widths of the semicircle) to ensure that the local mean level spacing $\Delta(E)$ is constant across this window.

The expressions Eqs. ([2],[3]) provide us with the explicit formula for the local-in spectrum density of scattering poles for a generic open chaotic system with broken time-reversal invariance and constitutes the main result of the present section.

D. Properties of the resonance width distribution.

- Weak coupling versus strong coupling: from $\chi^2$ distribution to power-law behavior.

When all scattering channels are equivalent, i.e. have equal transmission coefficients $T_a = T$ (hence, equal $g_a = g$) the distribution Eq. ([17]) can be represented in a quite simple and elegant form. Indeed, for this case we have:

$$\mathcal{F}_1(g, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-iky} \frac{1}{(g - ik)^M} = -\frac{1}{\Gamma(M)} y^{M-1} e^{-gy}$$

where $\Gamma(M) = (M-1)!$ stands for the Euler Gamma-function. We also can write the function $\mathcal{F}_2(g, y)$ in this case as

$$\mathcal{F}_2(g, y) = \sum_{k=0}^{M} (-1)^k \binom{M}{k} y^{M-k} \frac{dk}{dy^k} \left( \frac{\sinh y}{y} \right)$$

(101)

where $\binom{M}{k}$ stands for the binomial coefficient. One immediately sees that the expression for the density $\rho_E(y)$ can be written as:

$$\rho_E(y) = \frac{(-1)^M}{\Gamma(M)} \frac{d^M}{dy^M} \left( e^{-gy} \frac{\sinh y}{y} \right)$$

(102)

Remembering, that the "weak coupling" limit corresponds to large values of the parameter $g_a \gg 1$, one immediately notices that the distribution of resonance widths is exponentially cut at $y \gtrsim y_{max} = \max_a \{g_a^{-1}\}$. Thus, in the weak coupling limit $y_{max} \ll 1$ and we can put $\frac{\sinh y}{y} \approx 1$ everywhere. This procedure immediately results in the well-known $\chi^2$ distribution for the case of equivalent channels, see Eq. ([102]). The condition $y \ll 1$ just means that the resonances are too narrow to overlap with each other. It is therefore natural that the $\chi^2$ distribution simply follows from a first order perturbation theory, see the discussion in the beginning of the present section.

As long as the coupling to continuum becomes stronger, the parameters $g_a$ decrease towards unity. When one or more transmission coefficients $T_a$ attain their maximal value $T_a = 1$ a drastic modification of the resonance width distribution occurs. Indeed, it is easy to see that when $T_a$ (and hence the corresponding $g_a$) tends to unity, the function $\mathcal{F}_1 \{g\}, y$ behaves proportionally to $exp (-y)$ at large enough $y$. This decay exactly cancels the growing exponent $exp (y)$ originating asymptotically from $\mathcal{F}_2 \{g\}, y$. As the result, the distribution function $\rho_E(y)$ must show a pure powerlaw decay in its tail, see Fig.2.

To determine this power explicitly one should make more accurate estimates of the asymptotic behavior of both $\mathcal{F}_1$ and $\mathcal{F}_2$. Substituting $\sinh y \approx \frac{e^y}{2}$ in the definition of $\mathcal{F}_2$ one can write:

$$\mathcal{F}_2 \{g\}, y \rightarrow \infty = \frac{1}{2} e^y \sum_{p=0}^{M} \frac{(-1)^p}{y^{p+1}} U_p; \quad U_p = \sum_{k=0}^{M} (-1)^k \frac{k!}{(k-l)!} D_k^{(M)}$$

(103)
Using the definitions of the coefficients \( D_k^{(M)} \), see Eq. (11) one finds that
\[
U_0 = \prod_{a=1}^{M} (g_a - 1); \quad U_1 = -\sum_{a=1}^{M} \prod_{b \neq a} (g_b - 1); \quad \ldots \quad U_p = (-1)^p p! \sum_{a_1, \ldots, a_p = 1 \atop a \neq a_1, \ldots, a_p} \prod_{c=1}^{M} (g_c - 1)
\]

We see that the leading power of \( y \) in the asymptotic behavior of the function \( F_2 \{\{g\}, y\} \) is essentially determined by the number of parameters \( g_a \) which are simultaneously equal to unity. Let us suppose, for definiteness, that exactly \( l \) quantities \( g_1, g_2, \ldots, g_l \) are equal to unity, whereas all other \( M - l \) parameters \( g_{l+1} \leq M \) are larger (and, for simplicity, are all different). Then \( U_0 = U_1 = \ldots = U_{l-1} = 0 \) and the leading behavior of the function \( F_2 \{\{g\}, y\} \) is given by:
\[
F_2 \{\{g\}, y \to \infty\} = (-1)^l l! \frac{e^y}{2y^{l+1}} \prod_{a=l+1}^{M} (g_a - 1)
\]
(104)

Under the same conditions one can determine the leading asymptotic behavior of the function \( F_1 \{\{g\}, y\} \) by calculating the integral in eq.(103). The contribution of the \( l \)-fold pole at \( k = -i \) gives:
\[
F_1 \{\{g\}, y \to \infty\} = \frac{(-1)^l l!}{(l-1)!} y^{l-1} e^{-y} \prod_{a=l+1}^{M} \frac{1}{(g_a - 1)}
\]
(105)

This finally results in the desirable asymptotic decay law for the distribution of resonance widths:
\[
\rho_E(y \to \infty) = \frac{l}{2y^2}
\]
(106)

The physical origin of such a tail in the width distribution is discussed below. It is interesting to note that such a behavior means that the positive moments of the width distribution \( \int_0^{\infty} dy y^k \rho_E(y) \) are apparently divergent as long as \( k \geq 1 \).

It is also instructive to consider briefly the particular case of very many equivalent channels: \( M \gg 1 \); \( g_a = g \) for any channel. We find that it is most convenient to rewrite the expression Eq.(102) in an equivalent form:
\[
\rho_E(y) = \frac{1}{2\Gamma(M)y^2} \int_{y(g-1)}^{y(g+1)} dt \exp(-t - M \ln t)
\]
(107)

and to evaluate the integral by the saddle-point method. The exponent is maximal in the vicinity of \( t_s = M \). When this point is inside the integration region, i.e. \( y(g-1) < t_s < y(g+1) \) we have a nonvanishing contribution to the integral. In the opposite situation the density of resonances vanishes exponentially when \( M \gg 1 \). Picking up the nonvanishing contribution we obtain:
\[
\rho_E(y) \bigg|_{M \gg 1} = \begin{cases} \frac{M^2}{2y^2}, & y < \frac{M}{g+1} \\ 0, & y \geq \frac{M}{g+1} \end{cases}
\]
(108)

Two conclusions can be drawn from this expression:

1. In the limit of large number of channels \( M \gg 1 \) the distribution of resonance widths shows a gap: there are no resonances with widths smaller than \( y_m = \frac{M}{g+1} \).

2. A region of power-law behavior \( \rho_E(y) \propto M/y^2 \) exists not only for the critical coupling \( g = 1 \), but also in the vicinity of the critical point: \( g - 1 \ll g \). However, only for \( g = 1 \) the power-law domain extends to infinity.

The formation of a gap (a strip in the complex energy plane free of resonances) was first noticed in the numerical experiments by Moldauer \[107\] long ago and discussed in much detail by Sokolov and Zelevinsky \[53\] later on. Gaspard and collaborators (see references in \[54\]) observed such a gap in their studies of chaotic scattering in the so-called three-disk systems. Semiclassically, the number of open channels was very large and comparable with the number of resonances. In the limit \( M, N \to \infty \) but \( m = M/N \) finite for the expression for the resonance width distribution was obtained and analyzed in the papers \[53\]. Our expression Eq.(108) obtained under the conditions \( 1 \ll M \ll N \) perfectly matches the \( m \ll 1 \) limiting case of their expression.
• Mean resonance width: the Moldauer-Simonius relation.

Having at our disposal the explicit formula Eq.(7), we can, in particular, easily calculate the mean resonance width:

\[
\langle y \rangle = \int_0^\infty dy y \mathcal{F}_1 \mathcal{F}_2 = -\frac{1}{4\pi} \int_{-1}^1 d\lambda \prod_a (g_a + \lambda) \int_{-\infty}^\infty dk \prod_a \frac{1}{g_a - ik} \int_0^\infty dy e^{-y/\lambda+ik} = \\
\frac{1}{4\pi} \int_{-1}^1 d\lambda \prod_a (g_a + \lambda) \frac{\partial}{\partial\lambda} \int_{-\infty}^\infty dk \frac{1}{\lambda+ik} \prod_a \frac{1}{g_a - ik} = \\
-\frac{1}{2} \int_{-1}^1 d\lambda \prod_a (g_a + \lambda) \frac{\partial}{\partial\lambda} \prod_a \frac{1}{g_a + \lambda} = \frac{1}{2} \sum_{a=1}^M \ln \frac{g_a + 1}{g_a - 1}
\]

(109)

Remembering the relation between \( g_a \) and the transmission coefficients \( T_a \), see Eq.(3), and using the definition of the scaled level width \( y = \pi \Gamma/\Delta \) we can represent the last result in the form of a relation between the mean resonance widths \( \langle \Gamma \rangle \) and the transmission coefficients \( T_a \):

\[
\langle \Gamma \rangle = -\frac{\Delta}{2\pi} \sum_{a=1}^M \ln (1 - T_a)
\]

(110)

which can be also rewritten as:

\[
| \prod_{a=1}^M \langle S_a \rangle | = \exp -\frac{\pi \langle \Gamma \rangle}{\Delta}
\]

(111)

in view of the definition \( T_a = 1 - | \langle S_a \rangle |^2 \).

The latter formula is well-known for a long time in nuclear physics as the Moldauer-Simonius relation [108]. It was derived for systems with unbroken time-reversal symmetry by averaging the \( S \)-matrix over the energy spectrum and using the unitarity condition. The fact that we recovered this relation by ensemble averaging is in good agreement with the well-known ergodicity of the Gaussian ensembles [109]. The logarithmic divergence of \( \langle \Gamma \rangle \) at the critical coupling \( T_a = 1 \) is a direct consequence of the \( 1/y^2 \) decrease of the probability distribution, see Eq.(106).

• Strong chaos as the origin of the power-law tail of the resonance width distribution. The results presented above suggest that the powerlaw decrease \( 1/y^2 \) should be typical for chaotic systems strongly coupled to continua and is one of the clear manifestations of the strong overlap between individual resonances. It is therefore natural to try to understand the origin of such a tail qualitatively in terms of the underlying chaotic dynamics.

To this end it is interesting to mention that a little different powerlaw distribution of resonance widths, that of the form \( \rho(y) \propto y^{-3/2} \), was observed in numerical studies of a quantum chaotic system with (quasi) one-dimensional "diffusive" dynamics in the case of strong coupling to continua [110]. The authors suggested a transparent qualitative explanation of this effect based on the fact that the resonance width is proportional to the inverse time-life for a wavepacket injected into the system. The latter is determined by the time of the classical diffusion: \( t_{diff} = D/L^2 \), where \( L \) is the distance from a given point to the closest (strongly absorbing) boundary and \( D \) is a classical diffusion constant. For a semi-infinite sample this reasoning immediately gives a powerlaw width distribution \( \rho(\Gamma) \propto \Gamma^{-3/2} \).

Let us remind that the present model is based on the use of the Gaussian random matrices. Physically, it corresponds to the case of strongly chaotic classical dynamics for the closed system [23][33]. For such systems there is a typical time scale \( \delta t \) determined by an inverse Lyapunov exponent \( \lambda^{-1} \) after which the system effectively loses a memory about its initial conditions and can be found in any part of the available phase space on the energy shell with equal probability. Such systems are known as the ergodic ones. Let us show that it is just that type of classical dynamics which is responsible for the powerlaw tail \( 1/y^2 \) of the width distribution.

To understand this fact let us consider as a particular, but generic example: a particle moving with a velocity \( v \) inside an irregular-shaped two-dimensional cavity of area \( A \) and circumference \( C \propto A^{1/2} \). The chaoticity is considered to be so strong that complete "loss of memory" occurs after few collisions with walls so that \( \delta t \) can
be estimated as $\delta t \propto C/v \propto A^{1/2}/v$. Let us make a small opening of the width $d << C$ in the walls so that the particle can escape from the cavity whenever it hits the opening. Subdividing the observation time in intervals of the order of $\delta t$ we conclude that the probability to escape during one interval is just $p_0 = d/C \propto dA^{-1/2} << 1$ in view of the ergodicity and escape events during the subsequent intervals can be treated as independent (memory loss). Then the probability to stay inside the cavity for a large time $t_e$ and then to leave within the interval $[t_e, t_e + \delta t]$ can be estimated as $P(t_e)dt = \frac{p_0\delta t}{2!} \exp -p_0 t_e/\delta t$.

Considering our system semiclassically, we associate the wavelength $\lambda_d = \frac{2\pi}{k}$ with our particle and can estimate the number of the quantum mechanical states available inside the closed cavity as $N \propto \frac{A^{2/3}}{\lambda_d^2}$. Since the energy $E = mv^2/2$, the corresponding mean level spacing being of the order of $\Delta = E/N$ is proportional to $mv^2\lambda_d^3/A$. Interpreting the inverse escape time $h\gamma^{-1}$ as the resonance width $\Gamma$ and measuring it in units of $\Delta$: $y = \Gamma/\Delta$, one can find the distribution of $y$ to be given by: $P(y) = \frac{2\gamma}{\gamma^2} \exp -\frac{\gamma y}{2} \approx \frac{\gamma y}{2}$ for $y \gg M_{sc}$, where $M_{sc} = \hbar P_0/(\Delta\delta t) \propto \frac{\hbar d}{A^{2/3}mv^2\lambda_d^3} = d/\lambda_d$, which coincides with the quasiclassical estimate for the number of open channels for the present problem.

We conclude that the semiclassical arguments faithfully reproduce the same powerlaw tail of the resonance width distribution as that obtained from our random matrix model. Therefore, we expect such a tail to be a universal characteristic of the chaotic quantum scattering problem independent on the specific details of the underlying classical dynamics being sufficiently chaotic to ensure an exponential escape from the compact scattering region. The following comment is appropriate here. Dealing with realistic models of open chaotic systems containing no random parameters one always performs statistics over an interval of energies $\delta E$ on a real axis containing many resonances: $\delta E \gg \Delta$, but being small enough for a systematic variation of the smoothed level density $\nu(E)$ to be neglected: $\delta E \ll \gamma/\delta E$ (c.f definition of the quantity $p_E(y)$). One may expect that universal features of such statistics are adequately reproduced within the framework of the stochastic approach, but only on the level of "local-in-spectrum" characteristics calculated at fixed value of $E$. Indeed, any spectral averaging in the stochastic model performed on a scale comparable with the radius of the semicircle unavoidably mixes up data corresponding to very different values of the transmission coefficients, the procedure washing out any relevant physical information. In particular, it seems quite meaningless to consider quantities like the "globally" averaged resonance width $\Gamma_{gl} = \frac{1}{N} \sum_{k=1}^{N} \Gamma_k$, where the summation goes over all $N$ resonances. In our model this quantity can be trivially found from the sum rule: $N\Gamma_{gl} = -2Tr \text{Im} H_{ef} = 2\sum a \gamma_a$, and can not be related to any particular transmission coefficient. This fact, however, should not be misinterpreted as impossibility to have universal statistics of $S$-matrix poles within the stochastic approach as discussed in [25]. Rather, the quantity $\Gamma_{gl}$ can be found via the direct integration of the universal local expression $\langle \Gamma \rangle$, Eq.(114) over the energy $E$, upon substituting there the energydependent values $T_a(E), \Delta(E)$ from Eq.(49). Indeed, the following integral can be easily evaluated [11]:

$$
\lim_{N \to \infty} \sum_k \Gamma_k = N \int_{-2}^{2} \langle \Gamma(E) \rangle \nu(E)dE = \int_{-2}^{2} dE \ln \left( \frac{1 - 2\gamma_a \sqrt{1 - E^2/4} + \gamma_a^2}{1 + 2\gamma_a \sqrt{1 - E^2/4} + \gamma_a^2} \right) = \sum_{a=1}^{M} \left[ \gamma_a + \gamma_a^{-1} - |\gamma_a - \gamma_a^{-1}| \right],
$$

resulting in the expected expression $2 \sum a \gamma_a$ as long as all $\gamma_a \leq 1$.

We see that the result of the integration is always finite for any $\gamma_a$, thus concealing a specific role of the critical coupling $\gamma_a = 1$ when resonances with divergent local mean width occur sufficiently close to the center of the spectrum.

Of course, taken literally this divergence has sense only in the limit of infinite number of resonances $N \to \infty$. For any finite $N$ all resonance widths are finite and in any case can not exceed the upper bound $2 \sum a \gamma_a$. Basically, it is related to the fact that the distributions Eqs.(7) cease to be valid for the domain of very broad resonances having widths $\Gamma \sim 1$ (correspondingly, $y \sim N$). Alternatively, one may say that for large, but finite $N$ the Moldauer-Simonius relation is to be modified in a narrow domain $\delta E \propto 1/N$ in the vicinity of the energy $E = 0$, see [86].

Expression Eq.(112) can be also used to describe an interesting phenomenon happening when some coupling constants $\gamma_a$ (e.g. for the channels $a = 1, 2, ..., M_1 \leq M$) exceed the critical value $\gamma = 1$. Under this condition the result of integration in Eq.(112) is less than the exact sum rule value $2 \sum a \gamma_a$ by the quantity $\delta \Gamma = 2 \sum_{a=M_1}^{M} (\gamma_a - \gamma_a^{-1}) > 0$. This deficit reflects the existence of $M_1”$ broad resonances” of the widths.
In Sec. II (see Eq. 41), it is convenient to characterize the phase shift properties via the spectral density: knowledge of which, in particular, allows one to study statistics of "partial delay times" \( \tau_a \). As we already mentioned in Sec. II (see Eq. 11), it is convenient to characterize the phase shift properties via the spectral density: \( \rho_{E,X}(z) = \frac{1}{M} \sum_{a=1}^{M} \delta(z - z_a(E,X)) \) of eigenvalues of the matrix \( \hat{K}_X(E) = \pi W^\dagger \left( E - \hat{H}_{in}(X) \right)^{-1} W \). Our object of primary interest is the correlation function

\[
K_{E,X}(z_1, z_2) = \langle \rho_{E-\Omega/2,-X/2}(z_1) \rho_{E+\Omega/2,X/2}(z_2) \rangle - \langle \rho_{E-\Omega/2,-X/2}(z_1) \rangle \langle \rho_{E+\Omega/2,X/2}(z_2) \rangle
\]

knowledge of which, in particular, allows one to study statistics of "partial delay times" \( \tau_a = \frac{\partial \theta_a}{\partial x} \) and also the corresponding parametric derivatives.

Before addressing the issue of the spectral density correlations, it is instructive to consider in some detail the calculation of the average spectral density \( \langle \rho_{E,0}(z) \rangle \) for the few channel case. This quantity is less informative than the correlation function Eq. (113), but that simple calculation serves as a reference point for more interesting cases. Let us mention, that in the limit \( M \propto N \gg 1 \) the phase shift density was found earlier by Lehmann and Sommers [112].

### A. Averaged spectral density of \( K \)-matrix

The averaged density can be easily found provided the following functions are known

\[
f_{E,X}^{\pm}(z) = \left\langle \text{Tr} \frac{1}{z \pm i \epsilon - \hat{K}_X(E)} \right\rangle
\]

in view of the obvious relation: \( \rho_{E,X}(z) = \frac{1}{\pi M} \lim_{\epsilon \to 0} \text{Im} f_{E,X}^{-}(z) \). We restrict our attention in the present context by \( f_{E,X=0}^{-}(z) \), omitting all the indices \( \pm, E, X \) for the sake of brevity.

The function \( f(z) \) can be formally written as:

\[
f(z) = \lim_{J \to 0} \frac{\partial}{\partial J} \ln Z(J); \quad Z(J) = \frac{\text{Det} \left( (z + J)\hat{I}_M - \hat{K} \right)}{\text{Det} \left( z\hat{I}_M - \hat{K} \right)}
\]

Here and below we imply \( z = z - i \epsilon \) for the sake of brevity, implying \( f(z) \) to be analytic in the lower \( z \)-half-plane.

Due to the normalisation condition \( Z(\hat{J} = 0) = 1 \) one can write \( f(z) = \lim_{J \to 0} \frac{\partial}{\partial J} \ln Z(J) \). In order to perform the ensemble average of the generating function \( Z(J) \) in a standard way one should first get rid of the following unpleasant feature: the random matrix \( \hat{H}_{in} \) enters the expression for the generating function only via the matrix \( \hat{K} \). To this end, we can use the identity Eq. (44) and write down the determinant in the denominator of the generating function as:

\[
z^M \text{Det} \left( \hat{I}_M - z^{-1} \pi W^\dagger (E - \hat{H}_{in})^{-1} W \right) = z^M \text{Det} \left( \hat{I}_N - z^{-1} \pi (E - \hat{H}_{in})^{-1} W W^\dagger \right) = z^M \text{Det}^{-1} \left( E - \hat{H}_{in} \right) \text{Det} \left( E - \hat{H}_{in} - \pi z^{-1} W W^\dagger \right)
\]

After performing a similar manipulation with the numerator of the generating function we can write:

IV. STATISTICS OF SCATTERING PHASE SHIFTS AND TIME DELAYS

In the present section we are going to study in much detail the statistics of individual scattering phase shifts \( \theta_a \) and their derivatives, both over the energy \( E \) and over an arbitrary external parameter \( X \). As we already mentioned in Sec. II (see Eq. 11), it is convenient to characterize the phase shift properties via the spectral density: \( \rho_{E,X}(z) = \frac{1}{M} \sum_{a=1}^{M} \delta(z - z_a(E,X)) \) of eigenvalues of the matrix \( \hat{K}_X(E) = \pi W^\dagger \left( E - \hat{H}_{in}(X) \right)^{-1} W \). Our object of primary interest is the correlation function

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knowledge of which, in particular, allows one to study statistics of "partial delay times" \( \tau_a = \frac{\partial \theta_a}{\partial x} \) and also the corresponding parametric derivatives.

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\]

Here and below we imply \( z = z - i \epsilon \) for the sake of brevity, implying \( f(z) \) to be analytic in the lower \( z \)-half-plane.

Due to the normalisation condition \( Z(\hat{J} = 0) = 1 \) one can write \( f(z) = \lim_{J \to 0} \frac{\partial}{\partial J} \ln Z(J) \). In order to perform the ensemble average of the generating function \( Z(J) \) in a standard way one should first get rid of the following unpleasant feature: the random matrix \( \hat{H}_{in} \) enters the expression for the generating function only via the matrix \( \hat{K} \). To this end, we can use the identity Eq. (44) and write down the determinant in the denominator of the generating function as:

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\]

After performing a similar manipulation with the numerator of the generating function we can write:
\[
f(z) = \lim_{J \to 0} \frac{\partial}{\partial J} \left[ \left( \frac{z + J}{z} \right)^M \mathcal{F}(J) \right]: \quad \mathcal{F}(J) = \left\langle \frac{\text{Det}[E - H_{\text{eff}}(z + J)]}{\text{Det}[E - H_{\text{eff}}(z)]]} \right\rangle
\]

where we introduced the notation \( H_{\text{eff}}(z) = \hat{H}_m + \hat{W} \hat{W}^\dagger \).

Now we can use a standard procedure and represent the determinants in the denominator/numerator of the preceding equation by Gaussian integrals over \( N \) commuting/anticommuting variables. After introducing \( 2N \)-component supervector \( \Psi = \left( \begin{array}{c} S \\ \chi \end{array} \right) \), we have:

\[
\frac{\text{Det}[E - H_{\text{eff}}(z + J)]}{\text{Det}[E - H_{\text{eff}}(z)]]} = \int [d\Psi] \exp \left\{ -i \Psi \left[ (E - \hat{H}_m) - \hat{\Gamma} \otimes \hat{U} \right] \Psi \right\}
\]

where the supermatrix \( \hat{U} = \text{diag}(z^{-1}, (z + J)^{-1}) \) and \( \hat{\Gamma} = \pi \hat{W} \hat{W}^\dagger \), as before.

Now one trivially performs the averaging over the ensemble, see eq.(118) and decouples the emerging "quartic term" in the exponent with help of the Hubbard-Stratonovich transformation. In the present simple case such a decoupling is possible when one uses the set of \( 2 \times 2 \) in the exponent with help of the Hubbard-Stratonovich transform. In the present simple case such a decoupling is possible when one uses the set of \( 2 \times 2 \) matrices \( \hat{P} \) defined like in Eq.(83) as the integration manifold. Changing the order of integrations, performing the integration over \( \Psi \) explicitly, and copying steps used to derive Eq.(83), one obtains:

\[
\left\langle \frac{\text{Det}[E - H_{\text{eff}}(z + J)]}{\text{Det}[E - H_{\text{eff}}(z)]]} \right\rangle = \int d\hat{P} e^{-N \mathcal{L}(\hat{P})} \prod_{a=1}^M \text{Sdet}^{-1} \left[ \hat{L}_2 - \gamma_a (E - \hat{\Gamma} \hat{U})^{-1} \right]
\]

where

\[
\mathcal{L}(\hat{P}) = \frac{1}{2} \text{Str} \hat{P}^2 + \text{Str} \ln (E - \hat{P})
\]

Now it is convenient to use that: \( \frac{(z + J)^M}{z^n} = \text{Sdet}^M \hat{U} \). When combined with the preceding expression it gives:

\[
f(z) = \lim_{J \to 0} \frac{\partial}{\partial J} \int d\hat{P} e^{-N \mathcal{L}(\hat{P})} \prod_{a=1}^M \text{Sdet}^{-1} \left[ \hat{U}^{-1} - \gamma_a (E - \hat{P})^{-1} \right]
\]

The integral over \( \hat{P} \) is calculated in the limit \( N \gg M \) by the saddle-point method, with unique diagonal saddle-point \( \hat{P} = (E/2 + i \nu_{sc}) \hat{I}_2 \) accessible by allowed contour deformation for \( |E| \leq 2 \). This immediately yields:

\[
f(z) = \lim_{J \to 0} \frac{\partial}{\partial J} \prod_{a=1}^M \left( \frac{(z + J) - \gamma_a (E/2 + i \nu_{sc})}{z - \gamma_a (E/2 + i \nu_{sc})} \right) = \sum_{a=1}^M \frac{1}{z - \gamma_a (E/2 + i \nu_{sc})}
\]

analytic in the lower half-plane, from where we find that the mean density of eigenvalues for the matrix \( \hat{K} \) is given by a sum of Lorentzians (with \( z \) real):

\[
\rho_E(z) = \frac{1}{M} \sum_{a=1}^M \frac{\nu_{sc} \gamma_a}{(\pi \nu_{sc} \gamma_a)^2 + \left( z - \gamma_a (E/2) \right)^2}
\]

For the particular case of one open channel \( M = 1 \) the Lorentzian form of the average spectral density was first found by Mello [24]. Actually, in that particular case one can check the expression Eq.(18) for the averaged \( S \)-matrix using Eq.(122). Indeed, for \( M = 1 \) the \( S \)-matrix is reduced to the only number \( S = \exp(-2i \arctan z) \equiv 1 - 2iz/(1 + iz) \). Therefore

\[
\langle S \rangle = 1 - 2i \int_{-\infty}^{\infty} dz \rho_E(z) \frac{z}{1 + iz}
\]

The integrand has the only pole \( z_- = E \gamma/2 - i \pi \nu_{sc} \gamma \) in the lower half plane \( \text{Im} z < 0 \), and the corresponding residue immediately gives:

\[
\langle S \rangle = 1 - 2i(-2\pi i) \frac{\nu_{sc} \gamma}{z_- - E \gamma/2 - i \nu_{sc} \gamma} \frac{z_-}{1 + iz_-} = 1 - \frac{\gamma}{2} (iE + \sqrt{4 - E^2})
\]

in complete agreement with Eq.(18).
V. PAIR CORRELATION FUNCTION OF K-MATRIX SPECTRAL DENSITIES

Let us now turn our attention to the calculation of the pair correlation function \( K_{E,\Omega,\chi}(z_1, z_2) \), Eq. (113). To this end let us introduce the function

\[
f(z_1, z_2) = \frac{1}{\text{Tr}_{z_1-i\epsilon-K_{\chi/2}(E-\Omega/2)}} \frac{1}{\text{Tr}_{z_2+i\epsilon-K_{\chi/2}(E+\Omega/2)}}
\]  

(125)

related to the correlation function in Eq. (113) as:

\[
K_{E,\Omega,\chi}(z_1, z_2) = \frac{1}{2\pi M^2} \text{Re} f_c(z_1, z_2); \quad f_c(z_1, z_2) = f(z_1, z_2) - f^-(z_1)f^+(z_2)
\]  

(126)

Performing with each of the two traces of resolvents in the Eq. (125) the same manipulations as presented in Eqs. (113-117) one obviously obtains the following representation:

\[
f(z_1, z_2) = \frac{\partial^2}{\partial J_1 \partial J_2} \left[ \left( \frac{Z_j^{(1)} Z_j^{(2)}}{Z_j^{(1)} Z_j^{(2)}} \right)^M \mathcal{F}(J_1, J_2) \right] |_{J_1=J_2=0}
\]  

(127)

\[
\mathcal{F}(J_1, J_2) = \frac{\text{Det}[E - \Omega/2 - H_{\text{eff}}(-X/2, Z_j^{(1)})] \text{Det}[E + \Omega/2 - H_{\text{eff}}(X/2, Z_j^{(2)})]}{\text{Det}[E - \Omega/2 - H_{\text{eff}}(-X/2, Z_j^{(2)})] \text{Det}[E + \Omega/2 - H_{\text{eff}}(X/2, Z_j^{(2)})]}
\]

where we introduced the notations: \( Z_j^{(p)} = z_p + i(-1)^p \epsilon + J_p \); \( p = 1, 2 \) and \( H_{\text{eff}}(X; Z_j^{(p)}) = \hat{H}_m + \frac{X}{N} \hat{H}_m^{(1)} + \frac{\pi}{Z_j^{(p)}} \text{WW}^\dagger \).

This expression is quite close in its form to the generating function Eq. (64) appearing in the calculation of resonance widths distributions and we can use a similar representation for it in terms of the Gaussian(super) integrals (cf. Eq. (63)):

\[
\mathcal{F}(J_1, J_2) = (-1)^N \int [d\Psi] \exp \left\{ -iE \Psi^\dagger \hat{L}\Psi - i\frac{\Omega}{2} \Psi^\dagger \hat{\Lambda}\Psi + i\Psi^\dagger \hat{\Gamma} \otimes \left( \hat{L}\hat{U} \right) \Psi \right\} \times
\]  

(128)

\[
\left\langle \exp \left\{ i\Psi^\dagger \left( \hat{H}_m \otimes \hat{L} \right) \Psi - i\frac{X}{2N^{1/2}} \Psi^\dagger \left( \hat{H}_m^{(1)} \otimes \hat{L} \right) \Psi \right\} \right\rangle
\]

where \( \hat{U}^{-1} = \text{diag}(Z_j^{(1)} Z_j^{(1)} Z_j^{(2)} Z_j^{(2)}) = \frac{1}{2} (I_4 + \hat{\Lambda}) + \frac{1}{2} (I_4 - \hat{\Lambda}) + \text{diag}(0, J_1, 0, J_2) \) and notations for the supermatrices \( \hat{\Lambda} \) and the supervector \( \Psi \) are the same as in Eqs. (131-132).

The subsequent procedure of dealing with the ensemble average in Eq. (128) is exactly the same as that presented in details in Sec. III. The only difference is that the average is performed only over the GUE matrix \( \hat{H}_m \), whereas the matrix \( \hat{H}_m^{(1)} \) is considered to be arbitrary, but fixed from the same ensemble, see [29]. As a result, one has (cf. Eq. (76)):

\[
\mathcal{F}(J_1, J_2) = \int [dR] \exp \left\{ -\frac{N}{2} \left[ \text{Str} \hat{R}^2 - \Omega \text{Str} \hat{R}\hat{\Lambda} \right] - \text{Str} \ln \hat{G}^F \right\}
\]  

(129)

where

\[
\hat{G}^F = \hat{G}_1^F - \hat{\Gamma} \otimes \hat{U} ; \quad \hat{G}_1^F = \left[ (E \hat{I}_4 - \hat{R}) \otimes \hat{I}_N + \frac{X}{2N^{1/2}} \left( \hat{H}_m^{(1)} \otimes \hat{\Lambda} \right) \right]
\]  

(130)

so that

\[
\text{Str} \ln \hat{G}^F = \text{Str} \ln \hat{G}_1^F + \text{Str} \ln \left\{ \hat{I}_N - \left( \Gamma \otimes \hat{U} \right) \left[ \hat{G}_1^F \right]^{-1} \right\}
\]  

(131)

In turn, one can expand \( \hat{G}_1^F \) in a series with respect to \( X \):

\[
\text{Str} \ln \hat{G}_1^F = N \text{Str} \ln \left( E \hat{I}_4 - \hat{R} \right) - \sum_{l=1}^{\infty} \frac{(-X/2N^{1/2})^l}{l} \text{Tr} \left[ \hat{H}_m^{(1)} \right]^l \text{Str} \left[ \hat{\Lambda}(E \hat{I}_4 - \hat{R})^{-1} \right]^l
\]  

(132)
Now we use the fact that for any typical GUE matrix holds: \( \text{Tr} \left[ H_{\text{in}}^{(1)} \right]^{2p} = O(N) \); \( \text{Tr} \left[ H_{\text{in}}^{(1)} \right]^{2p+1} = O(1) \); where \( p \geq 0 \) is an integer. It is therefore evident, that in the limit \( N \to \infty \) the only nonvanishing term in the expansion above is that with \( l = 2 \). We also can put effectively \( G_{\text{P}} = (E \hat{I}_4 - \hat{R}) \otimes I_N \) in the second term in Eq.\([131]\) and represent it in a form of a sum over channels, see Eq.\([81]\).

Collecting all the relevant terms, we obtain:

\[
F(J_1, J_2) = \int [dR] \exp \left\{ -N \left[ \frac{1}{2} \text{Str} \hat{R}^2 + \text{Str} \ln (E \hat{I}_4 - \hat{R}) \right] \right\} \times
\exp \left\{ \frac{N\Omega}{2} \text{Str} \hat{R} \hat{\Lambda} + \frac{x^2}{8} \text{Str} \left[ \hat{\Lambda} (E \hat{I}_4 - \hat{R})^{-1} \right]^2 \right\} \prod_{a=1}^M \text{Sdet}^{-1} \left[ \hat{I}_4 - \gamma_a \hat{U} (E \hat{I}_4 - \hat{R})^{-1} \right]
\]

This integral can be evaluated in the usual manner by saddle point method in the limit \( N \gg 1 \). The saddle point manifold is parametrized again by the matrices \( \hat{R} = \frac{E}{2} \hat{I}_4 - \pi \nu_{sc} \hat{Q} \equiv \left[ E \hat{I}_4 - \hat{R} \right]^{-1} \). Remembering also that \( \text{Sdet} \hat{U} = \left( \frac{z_1(1) z_2(2)}{z_{j=1}^1 z_{j=0}^2} \right) \), we obtain the following representation for the correlation function \( f(z_1, z_2) \) in terms of the integral over the graded coset space:

\[
f(z_1, z_2) = \lim_{J_1 \to 0, J_2 \to 0} \frac{\partial^2}{\partial J_1 \partial J_2} \int [d\hat{Q}] \prod_{a=1}^M \text{Sdet}^{-1} \left[ \hat{U}^{-1} - \gamma_a \left( \frac{E}{2} \hat{I}_4 - \pi \nu_{sc} \hat{Q} \right) \right] \times
\exp \left\{ -\frac{\omega}{2} \text{Str} \hat{Q} \hat{\Lambda} + \frac{x^2}{8} \text{Str} \hat{Q} \hat{\Lambda} \hat{\Lambda} \right\}
\]

where we introduced scaled variables: \( \omega = \pi \nu_{sc} N\Omega; \quad x = \pi \nu_{sc} X \).

Remembering the definition of the supermatrix \( \hat{U}^{-1} \) and performing the expansion of the superdeterminants up to the second order with respect to \( J_1, J_2 \) one finds:

\[
\exp \left\{ \sum_{a=1}^M \text{Str} \left( \hat{B}_a^{-1} \hat{C}_1 \hat{B}_a^{-1} \hat{C}_1 \right) + \sum_{a,b=1}^M \text{Str} \left( \hat{B}_a^{-1} \hat{C}_1 \right) \text{Str} \left( \hat{B}_b^{-1} \hat{C}_2 \right) \right\}
\]

where

\[
\hat{B}_a = \frac{z_1}{2} (\hat{I}_4 + \hat{\Lambda}) + \frac{z_2}{2} (\hat{I}_4 - \hat{\Lambda}) - \gamma_a \left( \frac{E}{2} \hat{I}_4 - \pi \nu_{sc} \hat{Q} \right) \quad ; \quad \hat{C}_1 = \text{diag}(0, 1, 0, 0) \quad ; \quad \hat{C}_2 = \text{diag}(0, 0, 0, 1)
\]

In order to evaluate the integral over the coset space explicitly we substitute the corresponding expressions (see Appendix B, Eqs.\([312, 313]\)) in Eq.\([135]\) and perform the Grassmannian integrations remembering that in the chosen parametrization , see Appendix B, a nonvanishing contribution comes (apart from the terms proportional to the combination \( \alpha^* \alpha \beta^* \beta \)) also from terms in the integrand containing no Grassmannians at all (the so-called Parisi-Sourlas-Efetov-Wegner (PSEW) theorem, see \([105, 101]\)). We therefore find:

\[
f(z_1, z_2) = \sum_{a=1}^M \frac{1}{z_2(1) + i \tilde{\gamma}_a} \sum_{b=1}^M \frac{1}{z_1(b) - i \tilde{\gamma}_b} \int_1^\infty d\lambda_1 d\lambda_2 (\lambda_1 - \lambda_2)^2 \exp \left\{ -i \omega (\lambda_1 - \lambda_2) - \frac{x^2}{2} (\lambda_1^2 - \lambda_2^2) \right\} \prod_{a=1}^M \frac{D_{J_1}(a)}{D_{J_2}(a)} \times
\]

\[
\left\{ \sum_{a=1}^M \left( \frac{z_2(1) + i \tilde{\gamma}_a \lambda_1}{D_{b}(a)} - \frac{z_2(a) + i \tilde{\gamma}_a \lambda_2}{D_{b}(a)} \right) \right\} \sum_{b=1}^M \left( \frac{z_1(b) - i \tilde{\gamma}_b \lambda_1}{D_{b}(b)} - \frac{z_1(1) - i \tilde{\gamma}_b \lambda_2}{D_{b}(b)} \right) +
\sum_{a=1}^M \left\{ \frac{\tilde{\gamma}_a^2 | \mu_1 |^2}{D_{b}(a)} + \frac{\tilde{\gamma}_a^2 | \mu_2 |^2}{D_{b}(a)} \right\}
\]

where the notations \( D_{J_1, J_2}(a), \tilde{\gamma}_a, z_{1,2}(a) \) are explained in Appendix B.
It is easy to see that the expression in the first line of Eq. (137) is just the so-called “disconnected” part \( f^- (z_1) f^+ (z_2) \) of the corresponding correlation function, which is given by:

\[
\sum_{a=1}^{M} \frac{1}{z_2 (a) + i \gamma_a} \sum_{b=1}^{M} \frac{1}{z_1 (b) - i \gamma_b} = \left\langle \text{Tr} \frac{1}{z_1 - i \epsilon - K_{-X/2}(E - \Omega/2)} \right\rangle \left\langle \text{Tr} \frac{1}{z_2 + i \epsilon - K_{X/2}(E + \Omega/2)} \right\rangle
\]

The “connected” part of the correlation function \( f_c (z_1, z_2) = f (z_1, z_2) - f^- (z_1) f^+ (z_2) \) can be written in the most elegant form by noticing that:

\[
\frac{z_2 (a) + i \gamma_a \lambda_1}{\mathcal{D}_b (a)} - \frac{z_2 (a) + i \gamma_a \lambda_2}{\mathcal{D}_b (a)} = - \frac{\partial}{\partial z_1} \ln \frac{\mathcal{D}_b (a)}{\mathcal{D}_b (a)}
\]

\[
\frac{z_1 (a) - i \gamma_a \lambda_1}{\mathcal{D}_b (a)} - \frac{z_1 (a) - i \gamma_a \lambda_2}{\mathcal{D}_b (a)} = - \frac{\partial}{\partial z_2} \ln \frac{\mathcal{D}_b (a)}{\mathcal{D}_b (a)}
\]

and

\[
\frac{\hat{z}_a^2 | \mu_1 |^2}{\mathcal{D}_b (a)} + \frac{\hat{z}_a^2 | \mu_2 |^2}{\mathcal{D}_b (a)} = \frac{\partial^2}{\partial z_1 \partial z_2} \ln \frac{\mathcal{D}_b (a)}{\mathcal{D}_b (a)}
\]

Taking these relations into account one finally obtains the following compact expression:

\[
f_c (z_1, z_2) = \int_1^\infty d\lambda_1 d\lambda_2 \exp \left[ -i \omega (\lambda_1 - \lambda_2) - \frac{x^2}{2} (\lambda_1^2 - \lambda_2^2) \right] \times \]

\[
\frac{\partial^2}{\partial z_1 \partial z_2} \prod_{a=1}^{M} \left[ \frac{z_1 (a) z_2 (a) + i \gamma_a \lambda_2 (z_1 (a) - z_2 (a)) + \gamma_a^2}{\mathcal{D}_b (a)} \right] \]

with \( \gamma_a = \pi \nu_a \gamma / \gamma_a \) and \( z_p (a) = z_p - \gamma_a E / 2 \). This expression constitutes one of the central results of the present paper. In the rest of the present section we are going to use this relation intensively for extracting statistical properties of scattering phase shifts and their derivatives.

### A. Correlations of phase shift densities at fixed values of energy \( E \) and external parameter \( X \)

The general expression eq. (140) can be further simplified in the particular case \( \omega = x = 0 \). Physically this means that we are interested in studying correlation of phase shift densities at two points \( \theta_1 \) and \( \theta_2 \), but at fixed values of the energy \( E \) and the external parameter \( X \). Let us further assume that all channels are equivalent \( \gamma_a = \gamma_a \) \( a = 1, 2, ..., M \) for the sake of simplicity. Introducing notations: \( A = \hat{z}_1 \hat{z}_2 + \hat{\gamma}_a^2, \quad B = \hat{\gamma}_a (z_1 - z_2) \) where \( \hat{z}_1, \hat{z}_2 = z_{1,2} - E \gamma / 2 \) we can write:

\[
f_c^{\omega=x=0} (z_1, z_2) = \int_1^\infty d\lambda_1 \frac{\partial^2}{\partial z_1 \partial z_2} \frac{1}{\mathcal{D}_b (a + i B \lambda_2)^M} \frac{1}{\mathcal{D}_b (a + i B \lambda_1)^M} \int_1^\infty d\lambda_2 (A + i B \lambda_2)^M
\]

The integration over \( \lambda_2 \) can be easily performed yielding:

\[
f_c^{\omega=x=0} (z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \sum_{l=1}^{M} \left( \frac{M}{l} \right) (-i B)^{l-1} \int_1^\infty \frac{d\lambda_1}{[A + i B \lambda_1]^l} \{ (\lambda_1 + 1)^{l-1} - (\lambda_1 - 1)^{l-1} \} = \]

\[
\frac{\partial^2}{\partial z_1 \partial z_2} \sum_{l=1}^{M} \left( \frac{M}{l} \right) (-1)^{l-1} \sum_{q=0}^{l-1} \left( \begin{array}{c} l-1 \\ q \end{array} \right) \left( \begin{array}{c} q \\ q \end{array} \right) (A - i B)^q - 1
\]

where in the second line we expanded the brackets \( (\lambda_1 + 1)^l \) and performed the remaining integration over \( \lambda_1 \) explicitly in each term.

After differentiation over \( z_1, z_2 \) with help of the relations

\[
A \frac{\partial B}{\partial z_1,2} - B \frac{\partial A}{\partial z_1,2} = \hat{\gamma} (\hat{z}^2 + \hat{\gamma}_a^2); \quad A^2 + B^2 = (\hat{\gamma}^2 + \hat{\gamma}_a^2)(\hat{z}^2 + \hat{\gamma}_a^2)
\]

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the sums over \( l, q \) can be performed explicitly as well. As the result, we obtain:

\[
f_{\omega=x=0}(z_1, z_2) = -\frac{\gamma^2}{B^2} \left( \frac{A - iB}{A + iB} \right)^M - 1 \tag{143}
\]

In this point it is convenient to pass from the variables \( \tilde{z}_1, \tilde{z}_2 \) to new "angular" variables \( \tilde{\theta}_1, \tilde{\theta}_2 \) defined as:

\[
\tilde{z}_1 = \gamma \tan \tilde{\theta}_1, \quad \tilde{z}_2 = \gamma \tan \tilde{\theta}_2 \tag{144}
\]

We obviously have:

\[
B = \frac{\gamma^2}{\cos \tilde{\theta}_1 \cos \tilde{\theta}_2} \sin (\tilde{\theta}_1 - \tilde{\theta}_2); \quad \frac{A - iB}{A + iB} = \exp -2i(\tilde{\theta}_1 - \tilde{\theta}_2)
\]

Remembering the relation Eq.(126) between the spectral correlation function \( K_{E, \Omega, \chi}(z_1, z_2) \) and \( f(z_1, z_2) \) we notice that the pair spectral correlation function of the densities of the "angles" \( \tilde{\theta} \) defined as:

\[
K(\tilde{\theta}_1, \tilde{\theta}_2) = \langle \rho_E(\tilde{\theta}_1) \rho_E(\tilde{\theta}_2) \rangle - \langle \rho_E(\tilde{\theta}_1) \rangle \langle \rho_E(\tilde{\theta}_2) \rangle \tag{145}
\]

\[
\rho_E(\tilde{\theta}) = \frac{1}{M} \sum_{a=1}^{M} \delta(\tilde{\theta} - \arctan \frac{1}{\pi \nu_{sc}} \left[ \frac{z_a(E)}{\gamma} - E/2 \right])
\]

can be written in a very simple form:

\[
K(\tilde{\theta}_1, \tilde{\theta}_2) |_{\tilde{\theta}_1 \neq \tilde{\theta}_2} = -\left( \frac{\sin M(\tilde{\theta}_1 - \tilde{\theta}_2)}{\pi M \sin (\tilde{\theta}_1 - \tilde{\theta}_2)} \right)^2 \tag{146}
\]

One immediately recognizes in Eq.(146) the pair correlation function of the Dyson Circular Unitary Ensemble (see the book [30] where this object is called "two-level cluster function"). It corresponds to the following joint probability density of \( M \) variables \( \tilde{\theta}_a = \arctan \frac{1}{\pi \nu_{sc}} \left[ \frac{z_a(E)}{\gamma} - E/2 \right] \) \( a = 1, 2, ..., M \):

\[
P_M(\tilde{\theta}_1, ..., \tilde{\theta}_M) = \text{const} \times \prod_{a<b} | e^{2i\tilde{\theta}_a} - e^{2i\tilde{\theta}_b} |^2 \propto \prod_{a<b} \sin^2 \left( \tilde{\theta}_a - \tilde{\theta}_b \right) \tag{147}
\]

with \( -\pi/2 \leq \tilde{\theta}_a < \pi/2 \). Assuming this probability density being proven, the joint probability density of phase shifts \( \theta_a \) related to "angles" \( \tilde{\theta}_a \) as \( \tan \theta_a/2 = -\gamma \left( \tan \tilde{\theta}_a + \hat{E}/2 \right) \), where \( \hat{E} = E/(\pi \nu_{sc}) \) is given by:

\[
P_M(\theta_1, ..., \theta_M) = P_M(\tilde{\theta}_1, ..., \tilde{\theta}_M) \prod_{a=1}^{M} \left| \frac{d\tilde{\theta}_a}{d\theta_a} \right| = \frac{1}{2\gamma} \cos^2 \frac{\tilde{\theta}_a}{2} \tag{148}
\]

On the other hand one can write:

\[
\sin^2 (\tilde{\theta}_a - \tilde{\theta}_b) = \cos^2 \tilde{\theta}_a \cos^2 \tilde{\theta}_b (\tan \tilde{\theta}_a - \tan \tilde{\theta}_b) = \frac{\cos^2 \tilde{\theta}_a}{\gamma \cos^2 \theta_a/2} \frac{\cos^2 \tilde{\theta}_b}{\gamma \cos^2 \theta_b/2} \sin^2 (\theta_a/2 - \theta_b/2)
\]

so that using the identity: \( \prod_{a<b} u_a u_b = (\prod_a u_a)^{n-1} \) one obtains:

\[
P_M(\theta_1, ..., \theta_M) \propto \frac{1}{\gamma^M} \prod_{a<b} \sin^2 (\theta_a/2 - \theta_b/2) \left( \prod_c \frac{\cos^2 \tilde{\theta}_c}{\cos^2 \theta_c/2} \right)^M \tag{149}
\]

Using the relation between \( \theta_c \) and \( \tilde{\theta}_c \) and definitions of the quantities \( \hat{E}, \gamma \) one finds after a simple algebra that:

\[
\frac{\cos^2 \tilde{\theta}_c}{\cos^2 \theta_c/2} = \left( 1 + \frac{\hat{E}^2}{4} + \frac{1}{\gamma^2} \right) + e^{i\theta_c} \left( \frac{\hat{E}}{2\gamma} + \frac{1}{2} \left( 1 + \frac{\hat{E}^2}{4} - \frac{1}{\gamma^2} \right) \right) + e^{-i\theta_c} \left( -\frac{\hat{E}}{2\gamma} + \frac{1}{2} \left( 1 + \frac{\hat{E}^2}{4} - \frac{1}{\gamma^2} \right) \right) = \tag{150}
\]

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The distribution $P$ correlation function

where we made use of Eq.(124). Thus, we arrive finally at the following expression:

$$P \propto \prod_{a \neq b} | e^{i \theta_a - i \theta_b} |^{2 \sum_{c=1}^{M} | - \langle S \rangle^* e^{i \theta_c} |^{-2M}$$

(150)

which is nothing other but the Poisson's kernel distribution, Eq.(4). Here the phase shifts $\theta_a$ may be restricted to an interval $0 \leq \theta_a < 2\pi$. Inverting the argumentation, we prove, that our correlation function, Eq.(146) follows from the Poisson's kernel, Eq.(150); this we have shown in the case of equivalent channels.

**B. Distribution of partial delay times and parametric derivatives of phase shifts**

The knowledge of the spectral correlation function Eq.(113) allows one to determine the distribution $\mathcal{P}_\tau(\tau)$ of partial delay times

$$\tau_a(E) = \frac{\partial \theta_a(E)}{\partial E} = - \frac{2z_a(E)}{1 + z_a(E)^2} \frac{\partial z_a(E)}{\partial E}; \quad a = 1, ..., M$$

(151)

The distribution $\mathcal{P}_\tau(\tau)$ can be easily found if one knows the joint probability density $\mathcal{P}_E(z, v)$ defined as:

$$\mathcal{P}_E(z, v) = \frac{1}{M} \left\{ \sum_{a=1}^{M} \delta(z - z_a) \delta\left(v - \frac{\partial z_a(E)}{\partial E}\right) \right\}$$

(152)

because of the relation:

$$\mathcal{P}_\tau(\tau) = \frac{1}{M} \left\{ \sum_{a=1}^{M} \delta(\tau - \tau_a) \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz dv \mathcal{P}_E(z, v) \delta\left(\tau + \frac{2v}{1 + z^2}\right)$$

(153)

where angular brackets stand for the ensemble average as before [113].

To determine the joint probability density $\mathcal{P}_E(z, v)$ we use its relation to the spectral correlation function $K_{E, \Omega, X}(z_1, z_2)$ defined in Eq.(113):

$$\mathcal{P}_E(z, v) = M \lim_{\Omega \to +0} \Omega K_{E, \Omega, X}(z_1 = z - v\Omega/2, z_2 = z + v\Omega/2)$$

(154)

relations of this kind were first used in [114] and later on in [115-117]. To understand its origin we write for small positive $\Omega$:

$$\Omega K_{E, \Omega, X=0}(z_1 = z - v\Omega/2, z_2 = z + v\Omega/2) = \frac{1}{M^2} \left\{ \sum_{a, b=1}^{M} \delta[z - v\Omega/2 - z_a(E - \Omega/2)] \delta[z + v\Omega/2 - z_b(E + \Omega/2)] \right\} =$$

(155)

$$\Omega \frac{1}{M^2} \left\{ \sum_{a, b=1}^{M} \delta[z - v\Omega/2 + z_a(E - \Omega/2)] \delta[v\Omega + z_a(E - \Omega/2) - z_b(E + \Omega/2)] \right\}

Expanding $z_a(E - \Omega/2) - z_b(E + \Omega/2)$ at small $\Omega$ as $z_a(E) - z_b(E) - \Omega E \left( \frac{\partial z_a(E)}{\partial E} + \frac{\partial z_b(E)}{\partial E} \right) + ...$ we immediately see that in the limit $\Omega \to +0$ a nonvanishing contribution to Eq.(155) comes from the terms with equal channel indices $a = b$ and the resulting expression is equivalent to Eq.(153).

To perform the limit $\Omega \to +0$ in the most economic manner we pass from the variables $z_1, z_2$ to $z = (z_1 + z_2)/2$; $v_\alpha = (z_2 - z_1)/2\omega$, with $v_\alpha, \omega$ being the scaled variables $v_\alpha = v\Delta/2\pi$; $\omega = \pi\Omega/\Delta$. Correspondingly, the correlation function $f_c(z_1, z_2)$ acquires the form:

$$\left(\gamma^2(4 - E^2)/ \left[ 1 + \gamma^2 + \gamma\sqrt{4 - E^2} \right]\right) / \left(1 - \langle S \rangle^* e^{i \theta} \right) \left(1 - \langle S \rangle e^{-i \theta} \right)$$
\[ f(z, v_s) = \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \exp(-i\omega(\lambda_1 - \lambda_2)) \frac{1}{(\lambda_1 - \lambda_2)^2} \times \]

\[ \left[ \frac{\partial^2}{\partial z^2} - \frac{1}{\omega^2} \frac{\partial^2}{\partial v_s^2} \right] \prod_{a=1}^{M} \frac{\exp\left[ \frac{z^2 - \gamma_a E z + \gamma_a^2}{2\pi v_s \gamma_a} \right]}{1 - itv_s r_a} = \frac{1}{2M\pi^2 v_s^3} \int_{-\infty}^{\infty} dt e^{itv_s^{-1}} \prod_{a=1}^{M} \frac{1}{1 - itr_a^{-1}} \]  

(156)

According to the general discussion presented above we should take the real part of this expression and look for the term proportional to $1/\omega$ at $\omega \to +0$. It is easy to understand that such a singularity comes from that part of the integration region over the "non-compact" variable $\lambda_1$, where $\lambda_1 \propto \omega^{-1}$. After a natural rescaling $\lambda_1 = t/\omega$ one can easily extract the corresponding singular term, which turns out to come from the term with the second derivative $\frac{1}{\omega^2}\frac{\partial^2}{\partial v_s^2}$ in the expression above. Performing the calculation explicitly, we find the expression for the joint probability density of variables $z$ and $v_s$

\[ \mathcal{P}_E(z, v_s) = -\frac{1}{2M^2\pi^2} \int_{-\infty}^{\infty} dt e^{-it} \frac{\partial^2}{\partial v_s^2} \prod_{a=1}^{M} \frac{1}{1 - itv_s r_a^{-1}} = -\frac{1}{2M^2\pi^2 v_s^3} \int_{-\infty}^{\infty} dt e^{itv_s^{-1}} \prod_{a=1}^{M} \frac{1}{1 - itr_a^{-1}} \]

(157)

where $r_a = \left[ z^2 - \gamma_a E z + \gamma_a^2 \right]/2\pi v_s \gamma_a$ and we made use of the identity: $\text{Re} \int_{0}^{\infty} dt f(it) = \frac{1}{2} \int_{-\infty}^{\infty} dt f(it)$.

Performing the integration in the expression above (cf. Eq. (158)) one obtains:

\[ \mathcal{P}_E(z, v_s) = \frac{(-1)^{M-1}}{\pi M v_s^3} \prod_{a=1}^{M} \left[ \frac{z^2 - \gamma_a E z + \gamma_a^2}{2\pi v_s \gamma_a} \right] \theta(-v_s) \times \]

\[ \sum_{b=1}^{M} \exp \left[ \left( \frac{z^2 - \gamma_b E z + \gamma_b^2}{2\pi v_s \gamma_b} \right) \prod_{c \neq b} \left( \frac{z^2 - \gamma_c E z + \gamma_c^2}{2\pi v_s \gamma_c} \right) \right]^{-1} \]

where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ otherwise. We took into account that $z^2 - \gamma_a E z + \gamma_a^2 \geq 0$ as long as $|E| \leq 2$, which is just the case we are interested in (we remind that the semicircle density $\nu_{sc}(E)$ is non-vanishing for $|E| < 2$).

Substituting the expression Eq. (158) into Eq. (153) one trivially performs the integration over $v$ because of the $\delta$-function and obtains for the distribution function of scaled partial delay times $\tau_s = \frac{\tau}{2\pi}$ the following expression:

\[ \mathcal{P}_\tau(\tau_s) = \frac{(-1)^{M-1}}{2\pi M \tau_s^3} \int_{-\infty}^{\infty} \frac{dz}{(1 + z^2)} \prod_{a=1}^{M} \left[ \frac{z^2 - \gamma_a E z + \gamma_a^2}{2\pi v_s \gamma_a} \right] \sum_{b=1}^{M} \exp \left[ \left( \frac{z^2 - \gamma_b E z + \gamma_b^2}{2\pi v_s \gamma_b} \right) \right] \times \prod_{c \neq b} \left[ \frac{z^2 - \gamma_c E z + \gamma_c^2}{2\pi v_s \gamma_c} \right]^{-1} \]

\[ \left( \frac{(-1)^{M-1}}{2\pi M \tau_s^3} \int_{-\pi}^{\pi} d\phi \left[ \prod_{a=1}^{M} R_a(\phi) \right] \sum_{b=1}^{M} \exp \left[ \frac{R_b(\phi)}{\tau_s} \prod_{c \neq b} \left( R_c(\phi) - R_b(\phi) \right) \right]^{-1} \right] \]

Here

\[ R_a(\phi) = \frac{(1 + \gamma_a^2) - \gamma_a E \sin \phi + (\gamma_a^2 - 1) \cos \phi}{2\pi v_s \gamma_a} \]

and we changed the integration variable: $z = \tan(\phi/2)$.

The expression Eq. (159) provides the distribution of partial delay times for a general case of non-equivalent channels and thus constitutes one of the most important results of the present subsection.

Further simplifications are possible if we restrict our attention to the particular case of equivalent channels $\gamma_a = \gamma; \; a = 1, 2, ..., M$:

\[ \mathcal{P}_\tau(\tau_s) = \frac{1}{2\pi M \tau_s^{M-1}} \int_{-\pi}^{\pi} d\phi \left[ \frac{(1 + \gamma^2) - \gamma E \sin \phi + (\gamma^2 - 1) \cos \phi}{2\pi v_s \gamma} \right]^M \times \]

\[ \exp \left[ \frac{-1}{2\pi v_s \gamma} \tau_s \left[ (1 + \gamma^2) - \gamma E \sin \phi + (\gamma^2 - 1) \cos \phi \right] \right] \]

(160)

The last expression can be put in a more elegant form upon using the identity:

\[ \int_{-\pi}^{\pi} f(p \cos \phi + q \sin \phi) d\phi = 2 \int_{0}^{\pi} f(\sqrt{p^2 + q^2} \cos \phi) d\phi \]
and introducing the quantity $g = (\gamma + \gamma^{-1})/2\pi\nu_e$ (related to the transmission coefficient as $g = 2T^{-1} - 1$, see Eq.(50)), so that $\frac{1}{2\pi\gamma\sqrt{(\gamma^2 - 1)^2 + (\gamma E)^2}} = \sqrt{g^2 - 1}$. This finally gives:

$$P_\tau(\tau_s) = \frac{1}{\pi M!\tau_s^M} \int_0^\pi \!\! d\phi \left[ g + \sqrt{g^2 - 1} \cos \phi \right]^M \exp \left\{ -\frac{1}{\tau_s} \left[ g + \sqrt{g^2 - 1} \cos \phi \right] \right\} =$$

$$\frac{(-1)^M}{M!\tau_s^{M+1}} \frac{\partial^M}{\partial (\tau_s^{-1})^M} \left[ e^{-g\tau_s^{-1}} I_0 \left( \tau_s^{-1} \sqrt{g^2 - 1} \right) \right]$$

where $I_0(z)$ stands for the modified Bessel function. This expression provides us with the explicit form of the distribution of (scaled) partial delay times for the case of equivalent channels. Below we briefly analyze its most important features.

First of all, we notice that the distribution above assumes the simplest form for the "critical" coupling $T = 1$ (i.e. $g = 1$) corresponding to the most strong overlap of individual resonances allowed for the few-channel scattering, see preceding sections. Under this condition one finds the following distribution of scaled partial delay times:

$$P_\tau(\tau_s) = \frac{1}{M!} \tau_s^{-M-2} e^{-1/\tau_s}$$

The powerlaw tail $\tau_s^{-M-2}$ at $\tau_s \gg g$ which is evident from the expression above for $g = 1$ is actually a typical feature of the time delay distribution for any values of the parameters $g_a; a = 1, ..., M$ (see the discussion of Wigner-Smith time delay in the next subsection). For the equivalent channels we obviously have:

$$P_\tau(\tau_s \gg g) = \frac{1}{M!} \tau_s^{-M-2} P_M(g)$$

where $P_M(g)$ stands for the Legendre polynomial \[11\]. For the case of non-equivalent channels the asymptotic behavior $\tau_s^{-3M-2}$ can be inferred from Eq.(55) upon noticing that: $A_k = \sum_{b=1}^M R_b \prod_{c \neq b} (R_c - R_b)^{-1} = 0$ for $k = 0, 1, ..., M - 2$ and $A_k \neq 0$ for $k \geq M - 1$, so that the integrand is proportional to $1/\tau_s^{M-1}$. Combined with the factor $\tau_s^{-3}$ in front of the integral it gives the desired behavior.

In Fig.3 we plotted the distribution Eq.(161) for the opposite case of weakly open systems (the regime of isolated resonances: $T \ll 1$, i.e. $g \gg 1$). Under this condition when $g \gg \tau_s$ the modified Bessel function can be replaced by its asymptotic expression valid for large arguments. Taking into account also that $g - \sqrt{g^2 - 1} \approx \frac{1}{2g}$ at $g \gg 1$, we find that the distribution function Eq.(161) simplifies to the following form:

$$P_\tau(\tau_s) = \frac{(-1)^M}{M!} \tau_s^{-M-2} \frac{\partial^M}{\partial (\tau_s^{-1})^M} \left[ e^{-(2g\tau_s)^{-1}} \frac{1}{\sqrt{2\pi g\tau_s}} \right]$$

This expression is correctly normalized and plays the same role for the distribution of partial delay times as that played by the $\chi^2$ distribution in the issue of the resonance width distribution. It is necessary to mention that Eq.(164) is valid as long as $1, \tau_s \ll g$. At larger values of $\tau_s$ the behavior changes to that given by Eq.(163). It is interesting to note that in the parametrically large region $(2g)^{-1} \ll 1, \tau_s \ll g$ one can neglect the exponential term in Eq.(164) and reduce this distribution to the following form:

$$P_\tau(\tau_s) = \frac{(2M - 1)!!}{2^M M!} \frac{1}{\sqrt{2\pi g}} \tau_s^{-3/2}$$

This $\tau_s^{-3/2}$ behavior taking place irrespectively of the number of open channels is therefore the most typical feature of the partial delay times distribution for the regime of isolated resonances. The origin of such a behavior can be understood analysing the general expression for the Wigner-Smith delay times, Eq.(45) (see a more detailed discussion after the equation Eq.(193)).

At $\tau_s \sim (2g)^{-1}$ the distribution shows a maximum at a value $P(\tau_s) \sim g$ and then is cut off exponentially at smaller $\tau_s$:

$$P_\tau(\tau_s \ll (2g)^{-1}) = \frac{2g}{\pi^{1/2} M! (2g\tau_s)^{(M+3/2)}} \exp \left[ -\frac{1}{2g\tau_s} \right]$$

All these features are evident from Fig.3.
Having at our disposal the exact distribution eq. (161) it is instructive to calculate the mean value and the variance of the partial delay times. One finds:

$$
\langle \tau \rangle = \frac{2\pi}{M\Delta}; \quad \frac{\langle \tau^2 \rangle - \langle \tau \rangle^2}{\langle \tau \rangle^2} = \frac{2M(T^{-1} - 1) + 1}{M - 1}
$$

(167)

The first of these relations is quite well known [84, 88, 83, 84, 87]. It shows that the mean delay time $\langle \tau \rangle$ is determined by the mean level spacing $\Delta$ of the closed system and the number $M$ of open channels. On the other hand the magnitude of delay time fluctuations measured by the relative variance of the partial delay times distribution, see eq. (167), is determined both by $M$ and $T$. Generically, the fluctuations are the weaker the larger is the number of open channels $M$ and the stronger is the coupling to continua: $1 - T \ll T$. Let us also mention as an interesting feature the divergency of the time-delay variance at $M = 1$, which is a consequence of the powerlaw tail $\tau_s^{-M-2}$ typical for the distribution $P_s(\tau_s)$.

Here it is appropriate to mention that recently two other groups of authors [89, 118] addressed the question of delay time distribution by different approaches. Gopar, Mello and Büttiker [89] verified numerically an old conjecture by Wigner [119] concerning invariance of poles and residues of the $K$–matrix under a certain set of transformations, provided there is only one perfectly open channel: $M = 1; T = 1$ case in our notations. When combined with the $\chi^2$ distribution of residues, this conjecture was shown to produce the time delay distribution for all three symmetry classes (orthogonal, unitary and symplectic), the result for unitary class just coinciding with Eq. (162) for the coupling constant $\gamma$ following general expression for the scaled partial delay time distribution claimed to be valid for arbitrary value of $\gamma$.

Proceeding in the same manner as in Eqs. (152-155) one obtains:

$$
P_{zz}(\tau) = \frac{e^{-\gamma/\tau}}{\gamma M - 1} _1F_1[M, M + 1, (\gamma - \gamma^{-1})/\tau]
$$

(168)

where $_1F_1[M, M + 1, z]$ is the confluent hypergeometric function.

This expression is quite different from ours given in Eq. (161), the two formulas coinciding in the limit $\gamma = 1$ only. However, one can check that the expression Eq. (168) fails to fulfill the following important condition: the mean delay time must be independent of the degree of coupling to continua, measured by $\gamma$. Instead, it should be determined by the mean level spacing $\Delta$. This requirement is satisfied by our distribution Eq. (161), see Eq. (167) and is known for a long time [84, 88, 83, 84, 87]. It follows from the basic formula (15):

$$
\langle \tau \rangle = \frac{2N}{M} \int d\Gamma \int d\omega \rho(E - \omega, \Gamma) \frac{\Gamma/2}{\omega^2 + \Gamma^2/4}
$$

(169)

where $\rho(E, \Gamma) = \frac{1}{\pi} \sum_{n=1}^{N} \delta(E - E_n) \delta(\Gamma - \Gamma_n)$ is the density of the $S$–matrix poles. For few-channel case $M \ll N$ the typical scale of the width $\Gamma$ is the mean level spacing $\Delta = 1/(N \rho(E))$, see Sec.III, so that the Lorentzian factor in the integrand of Eq. (169) can be replaced by $2\pi \delta(\omega)$ when evaluating the integral. This gives $\langle \tau \rangle = 2\pi N \rho(E)/M$ in full agreement with Eq. (167). At the same time, for the particular case $M = 1$ one can find that the first moment corresponding to the distribution Eq. (168) is given by: $\Delta \langle \tau \rangle_{zz} = 2 \ln \gamma / (\gamma - \gamma^{-1})$ in contradiction with the general discussion above. This failure rules out the distribution Eq. (168) as the correct one and shows that the assumptions made in [118] are justified only as long as $T = 1$.

The distribution $P_w(w)$ of parametric derivatives of phase shifts $w_a = \frac{\partial \theta}{\partial X}$ can be found in a very similar way. Proceeding in the same manner as in Eqs. (152-155) one obtains:

$$
P_w(w) = \frac{1}{M} \left\langle \sum_{a=1}^{M} \delta(w - w_a) \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz du P_X(z, u) \delta \left( w + \frac{2u}{1 + z^2} \right)
$$

(170)

where

$$
P_X(z, u) = \frac{1}{M} \left\langle \sum_{a=1}^{M} \delta(z - z_a) \delta \left( u - \frac{\partial z_a(E, X)}{\partial X} \right) \right\rangle = \lim_{X \to 0} \frac{X K_{E, \Omega = 0, X}(z_1 = z - uX/2, z_2 = z + uX/2)}{X}
$$

(171)

Performing the limiting procedure $X \to 0$ in the same way as $\Omega \to 0$ in Eqs. (153-156) we arrive at the expression:
\[ \mathcal{P}_X(z, u_s) = -\frac{1}{2M\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t^2} e^{-t^2/2} \prod_{a=1}^{M} \left( \frac{1}{1 - itu_s r_a^{-1}} \right) = \] 

\[ \int_{-\infty}^{\infty} \frac{d\xi}{(2\pi)^{1/2}} \exp -\xi^2/2 \left[ -\frac{1}{2M\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t^2} e^{-it\xi} \prod_{a=1}^{M} \left( \frac{1}{1 - itu_s r_a^{-1}} \right) \right] \]

which can be written as:

\[ \mathcal{P}_X(z, u_s) = \int_{-\infty}^{\infty} \frac{d\xi}{(2\pi)^{1/2}} e^{-\xi^2/2} \mathcal{P}_E(z, u_s/\xi) \]  

(173)

where \( u_s = u/\pi \nu_{sc} \) and \( \mathcal{P}_E(z, u_s) \) is the joint probability density of \( z \) and its derivative over the energy studied earlier in this subsection. This fact means that a similar relation Eq.(173) holds for the distribution function \( \mathcal{P}_w(w_s) \) of scaled parametric derivatives \( w_s = \frac{1}{\pi \nu_{sc}} \frac{\partial}{\partial u} \) and that of the scaled partial delay times \( \mathcal{P}_\tau(r_s) \):

\[ \mathcal{P}_w(w_s) = \int_{0}^{\infty} \frac{d\xi}{(2\pi)^{1/2}} e^{-\xi^2/2} \mathcal{P}_\tau(|w_s|/\xi) \]  

(174)

The same relation was obtained in the paper [118] on a basis of some plausible assumptions concerning parametric derivatives of phase shifts.

C. Parametric correlations of Wigner-Smith time delays

The expression for the correlation function Eq.(140) can be used to calculate the parametric correlations of Wigner-Smith time delays \( \tau_w(E, X) \) defined as

\[ \tau_w(E, X) = -\frac{i}{M} \frac{\partial}{\partial E} \ln \det S(E, X) = \frac{1}{M} \frac{\partial}{\partial E} \sum_a \theta_a(E, X). \]  

(175)

To show this, we should remember the relation Eq.(12) between the total phase shift \( \theta = \sum_a \theta_a \), the exact density of states for the closed chaotic system \( \nu_X(E) \) and the eigenvalues \( z_a \) of the \( K \)-matrix. We see that

\[ \tau_w(E, X) = \frac{2\pi N}{M} \nu_X(E) + \tau_z(E, X); \quad \tau_z(E, X) = -2 \frac{\partial}{\partial E} \frac{1}{M} \sum_{a=1}^{M} \arctan z_a(E, X) \]  

(176)

Introducing the correlation function:

\[ C_W(\Omega, X) = \langle \tau_w(E - \Omega/2, -X/2) \tau_w(E + \Omega/2, X/2) \rangle \]

we see that it consists of three essentially different contributions:

\[ C_W(\Omega, X) = C_{\tau\tau}(\Omega, X) + \frac{2\pi N}{M} \left( C_{\nu\tau}(-\Omega, -X) + C_{\nu\tau}(\Omega, X) \right) + (2\pi N/M)^2 C_{\nu\nu}(\Omega, X) \]  

(177)

where

\[ C_{\tau\tau}(\Omega, X) = \langle \tau_z(E - \Omega/2, -X/2) \tau_z(E + \Omega/2, X/2) \rangle \]  

(178)

\[ C_{\nu\tau}(\Omega, X) = \langle \nu(E - \Omega/2, -X/2) \tau_z(E + \Omega/2, X/2) \rangle \]  

(179)

\[ C_{\nu\nu}(\Omega, X) = \langle \nu(E - \Omega/2, -X/2) \nu(E + \Omega/2, X/2) \rangle \]  

(180)

In what follows we are interested, as usual, in finding the “connected” part of all these correlation functions. This will be implicitly assumed below. The correlation function \( C_{\tau\tau}(\Omega, X) \) can be easily related to that given by Eq.(140) because of the relation:

\[ \tau_z(E, X) = -2 \frac{\partial}{\partial E} \int_{-\infty}^{\infty} dz \arctan(z) \rho_{E,X}(z) \]
where $\rho_{E,X}(z)$ is the density of $K$-matrix eigenvalues defined in the beginning of this section. As a result we have:

$$C_{\tau\tau}(\Omega, X) = \left[ \frac{\partial^2}{\partial E^2} - \frac{4\pi^2}{\Delta^2} \frac{\partial^2}{\partial \omega^2} \right] \frac{1}{2\pi^2M^2} \text{Re} \int_{-\infty}^{\infty} d_1 \arctan z_1 \int_{-\infty}^{\infty} d_2 \arctan z_2 f_c(z_1, z_2)$$ (181)

where we used the relation Eq.(126) between the correlation function $K_{\tau\omega}(z_1, z_2)$ and that given by the Eq.(140) and we used the scaled variable $\omega = \pi \Omega / \Delta$. After such a rescaling it is obvious that the term containing the second derivative $\partial^2 / \partial z_1 \partial z_2$ in the function $f_c(z_1, z_2)$ can be neglected in comparison with the second one because of the large factor $\Delta^{-2}$. Substituting now the expression Eq.(140) into eq.(181) one can easily perform the integration over $z_1, z_2$ by exploiting the presence of the second derivative $\partial^2 / \partial z_1 \partial z_2$ in the function $f_c(z_1, z_2)$ (this allows to convert factors $\arctan z_1, z_2$ into $(1 + z_1^2)^{-1}$ by partial integrations) and noticing that all poles of the expression

$$\prod_{a=1}^{M} \left[ \frac{z_1(a)z_2(a) + i\gamma_\alpha z_2(z_1(a) - z_2(a)) + z_\alpha}{z_1(a)z_2(a) + i\gamma_\alpha z_1(z_1(a) - z_2(a)) + z_\alpha} \right]$$

lie in the upper half plane $\text{Im} z_1 > 0$ with respect to the variable $z_1$ and in the lower half plane $\text{Im} z_2 < 0$ with respect to the variable $z_2$. As a result, the integration can be performed trivially by closing the integration contour over $z_1(z_2)$ in the lower (upper) half plane, correspondingly, and amounts to replacing $z_1 = -\iota; z_2 = \iota$ in the integrand. This gives:

$$C_{\tau\tau}(\omega, X) = 2 \left( \frac{\pi}{M \Delta} \right)^2 \text{Re} \int_{-\infty}^{\infty} d\lambda_2 \int_{-\infty}^{\infty} d\lambda_1 \exp\{i\omega(\lambda_1 - \lambda_2) - \frac{x^2}{2}(\lambda_1^2 - \lambda_2^2)\} I(\lambda_1, \lambda_2)$$ (182)

where

$$I(\lambda_1, \lambda_2) = 1 - \prod_{a=1}^{M} \left[ \frac{1 + \gamma_\alpha \lambda_2 - i\gamma_\alpha E/2}{1 + \gamma_\alpha \lambda_1 - i\gamma_\alpha E/2} \right] + \prod_{a=1}^{M} \left[ \frac{1 + \gamma_\alpha \lambda_2 + i\gamma_\alpha E/2}{1 + \gamma_\alpha \lambda_1 + i\gamma_\alpha E/2} \right] + \prod_{a=1}^{M} \left[ \frac{1 + 2\gamma_\alpha \lambda_2 + \gamma_\alpha}{1 + 2\gamma_\alpha \lambda_1 + \gamma_\alpha} \right]$$ (183)

The first three terms are boundary contributions due to the partial $z_1, z_2$ integrations.

Let us now show how to calculate the correlation function $C_{\nu\tau}(\Omega, X)$ which is expressed to the leading order in $N \gg 1$ as follows:

$$C_{\nu\tau}(\Omega, X) = \frac{\partial}{\partial \Omega} \int_{-\infty}^{\infty} dz \arctan z \langle \nu_{-X/2}(E - \Omega/2) \rho_{E+\Omega/2, X/2}(z) \rangle$$ (184)

To this end we represent the density $\nu_X(E)$ in terms of the corresponding resolvent as $\nu_X(E) = \frac{1}{N} \text{Im} \text{Tr}(E - i0^+ - \hat{H}_m)^{-1}$ so that

$$K_{\nu\tau}(\Omega, X; z) \equiv \langle \nu_{-X/2}(E - \Omega/2) \rho_{E+\Omega/2, X/2}(z) \rangle = \frac{1}{2\pi^2 MN} \text{Re} \left\langle \text{Tr} \left[ \frac{1}{E - \Omega/2 - \hat{K}(X/2) - i0^+} \frac{1}{z - \hat{K}(\Omega, X/2) + i0^+} \right] \right\rangle$$ (185)

The calculation of the connected part of the averaged product of two traces of resolvents in the preceding equation (we denote this quantity henceforth as $f_c^\nu(z)$ omitting an explicit mentioning of the parameters $\Omega, X$) goes along exactly the same lines as the calculation of the correlation function $f_c(z_1, z_2)$, see Eq.(127). Namely, one writes this function as:

$$f_c^\nu(z) = \frac{\partial^2}{\partial J_1 \partial J_2} \left[ \left( \frac{Z_j^{(2)}}{Z_j^{(2)}_{j=0}} \right)^M \mathcal{F}_\nu(J_1, J_2) \right] \bigg|_{J_1 = J_2 = 0}$$ (186)

$$\mathcal{F}_\nu(J_1, J_2) = \left\langle \text{Det}[E - i0^+ - \Omega/2 - \hat{H}(-X/2) + J_1] \text{Det}[E + \Omega/2 - \hat{H}_\nu(X/2; Z_j^{(2)})] \middle/ \text{Det}[E - i0^+ - \Omega/2 - \hat{H}(-X/2)] \text{Det}[E + \Omega/2 - \hat{H}_\nu(X/2; Z_j^{(2)}_{j=0})] \right\rangle$$

where we introduced the notations: $Z_j^{(2)} = z + i0^+ + J_2$ and $H_\nu(X; Z_j^{(2)}) = \hat{H}_\nu + \frac{X}{N}\sum_{\Omega} \hat{H}_m^{(1)} + \frac{X}{Z_j^{(2)}}WW^\dagger$. The generating function $\mathcal{F}_\nu(J_1, J_2)$ is expressed in a standard way as a Gaussian superintegral. Finally, the function $f_c^\nu(z)$
is reduced (after exactly the same manipulations as before, see Eqs. (128-134)) to the following representation in terms of the nonlinear $\sigma$-model:

$$f^\nu_\nu(z) = \frac{\pi}{\Delta} \int [d\Phi] \left[ E/2\hat{I}_4 - \pi\nu\tilde{Q} \right]^{(11)} \left( \prod_{a=1}^{M} \text{Sdet}^{-1} \hat{B}_\nu(a) \right) \sum_{a=1}^{M} (B_\nu^{-1}(a))^{(22)} \times$$

$$\exp \left\{ -\frac{\omega}{2} \text{Str}\tilde{Q}\hat{\lambda} + \frac{x^2}{8} \text{Str}\tilde{Q}\hat{\lambda}\tilde{Q}\hat{\lambda} \right\}$$

(187)

where the supermatrix $\hat{B}_\nu(a)$ is given by

$$\hat{B}_\nu(a) = \frac{1 + \hat{A}}{2} + \frac{1 - \hat{A}}{2} \left( z(a)\hat{I}_4 + \tilde{\gamma}_a\hat{Q} \right)$$

and we used the conventions of Appendix B for matrix elements as well as the notations: $z(a) = z - \gamma_a E/2; \tilde{\gamma}_a = \pi\nu_sc\gamma_a$. The explicit expressions for all matrix elements are given in Appendix B. Substituting them into the super-integral Eq. (187), we find again, that the term containing no Grassmannians at all gives the contribution

$$N(E/2 + i\pi\nu) \sum_{a=1}^{M} \frac{1}{z(a) + i\tilde{\gamma}_a}$$

which is exactly the "disconnected part" of the corresponding correlation function. The connected part is given by:

$$f^\nu_\nu(z) = -\frac{\pi^2}{\Delta^2} \int_{-\infty}^{1} d\lambda_2 \int_{1}^{\infty} d\lambda_1 \frac{1}{\lambda_1 - \lambda_2} \exp\left\{ i\omega(\lambda_1 - \lambda_2) - \frac{x^2}{2}(\lambda_1^2 - \lambda_2^2) \right\} \frac{\partial}{\partial z} \prod_{a=1}^{M} \frac{1}{z(a) + i\tilde{\gamma}_a} \lambda_1$$

(188)

Substituting this expression into Eq. (184) one can again trivially perform the integration over the variable $z$. As a result one obtains:

$$C_\nu\nu(\omega, X) = -\frac{1}{2\pi MN} \left( \frac{\pi}{\Delta} \right)^2 \text{Re} \int_{-1}^{1} d\lambda_2 \int_{1}^{\infty} d\lambda_1 \exp\left\{ i\omega(\lambda_1 - \lambda_2) - \frac{x^2}{2}(\lambda_1^2 - \lambda_2^2) \right\} \times$$

$$\left[ 1 - \prod_{a=1}^{M} \frac{1}{1 + \gamma_a\lambda_2 + i\gamma_a E/2} \right]$$

(189)

Finally, the parametric correlation function $C_\nu\nu(\Omega, X)$ of the densities of states for a closed chaotic system with broken time-reversal symmetry was found some time ago by Simons and Altshuler [28].

$$C_\nu\nu(\Omega, X) = \frac{1}{2\pi^2} \left( \frac{\pi}{\Delta} \right)^2 \text{Re} \int_{-1}^{1} d\lambda_2 \int_{1}^{\infty} d\lambda_1 \exp\left\{ i\omega(\lambda_1 - \lambda_2) - \frac{x^2}{2}(\lambda_1^2 - \lambda_2^2) \right\}$$

(190)

Summing up all the contributions we find the desired expression for the parametric correlation function of scaled Wigner-Smith time delays $\tilde{\tau}_s = \frac{\Delta}{2\pi}\tilde{\tau}_w$:

$$C_W(\omega, x) \equiv (\delta\tilde{\tau}_W(E - \Omega/2, -X/2)\delta\tilde{\tau}_W(E + \Omega/2, X/2)) =$$

$$\frac{1}{2M^2} \int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_1 \cos[\omega(\lambda_1 - \lambda)] \exp\left\{ -\frac{x^2}{2}(\lambda_1^2 - \lambda^2) \right\} \prod_{a=1}^{M} \frac{1 + \lambda g_a^{-1}}{1 + \lambda g_a^{-1}}$$

(191)

where we used the parameter $g_a = 2T_a^{-1} - 1$ introduced earlier.

It is interesting to mention that there exists an alternative way to derive the pair correlation function of Wigner-Smith time delays given in Eq. (191). The starting point in that case is Eq. (13). Then the calculation of the correlation of fluctuations of Wigner-Smithtime delay $\delta\tilde{\tau}_W(E, X) = \tau_w - \tilde{\tau}_w$ amounts to evaluating the average product of the resolvents of the non-Hermitian effective Hamiltonians $\hat{H} \pm i\pi W W^\dagger$. This can be done by exactly the same method as we use elsewhere in the present paper. For the case of chaotic systems with preserved TRS and no external parameter $X$ such a calculation was done earlier in [87].

Let us analyse the correlation function $C_W(x, \omega)$ in more detail. For this purpose we find it convenient to rewrite Eq. (191) in a slightly different form:
\[ C_W(\omega, x) = \frac{1}{2} \left( R_M^{(1,c)}(\omega, x) R_M^{(2,c)}(\omega, x) - R_M^{(1,s)}(\omega, x) R_M^{(2,s)}(\omega, x) \right) \]  
\[ \text{where} \]
\[ R_M^{(1,c)}(\omega, x) = \int_0^\infty dt_1 \cos(\omega t_1) \exp \left[ -x^2 t_1 - \frac{x^2 t_1^2}{2} \right] \prod_{a=1}^M \frac{1}{1 + T_a t_1/2} \]
\[ R_M^{(2,c)}(\omega, x) = \int_0^2 dt_2 \cos(\omega t_2) \exp \left[ -x^2 t_2 + \frac{x^2 t_2^2}{2} \right] \prod_{a=1}^M (1 - T_a t_1/2) \]

and the functions \( R_M^{(p,s)}(\omega, x) ; \ p = 1, 2 \) are obtained from the expressions for \( R_M^{(p,c)}(\omega, x) ; \ p = 1, 2 \) by replacing \( \cos(\omega t_p) \) by \( \sin(\omega t_p) \).

First of all, the correlation function Eq. (192) taken at \( \omega = x = 0 \) gives the variance of the Wigner-Smith time delay distribution. In principle, the corresponding integration can be performed for an arbitrary set of transmission coefficients \( T_a \). The resulting expressions turn out to be quite cumbersome. They simplify in the case of all equivalent channels \( T_a = T; \ a = 1, \ldots, M \) when we find:
\[ \frac{\langle \tau_w^2 \rangle - \langle \tau_w \rangle^2}{\langle \tau_w \rangle^2} = \frac{2}{T^2(M^2 - 1)} [1 - (1 - T)^{M+1}] \]

This expression shows the same qualitative features (divergencies at \( M = 1 \) or \( T \to 0 \)) as those following from Eq. (167).

For chaotic systems with only one open channel the Wigner-Smith time delay \( \tilde{\tau}_W \) just coincides with the partial delay time \( \tau_w \) and the corresponding distribution is given by that in Eq. (167). Unfortunately, our methods give us no possibility to find explicitly the distribution \( P_W(\tilde{\tau}_W) \) of Wigner-Smith time delay for an arbitrary number of open channels \( M > 1 \). It is natural to put forward a conjecture, that the divergence of the variance of Wigner-Smith time delay as \( M \to 1 \) indicates that a (unknown) distribution \( P_W(\tau_w) \) possess the same powerlaw tail \( P_W(\tau_w) \propto \tau_w^{-M-2} \) at large \( \tilde{\tau}_W \) as that typical for the distribution of partial phase shift times. Another argument supporting this conjecture comes from the general formula Eq. (192). Taking the value \( E \) at random it is evident that anomalously large time delay \( \tau_w(E) \sim \Gamma_n^{-1} \) corresponds to the event when \( E \) happens to be sufficiently close (at the distance \( \delta E \lesssim \Gamma_n \)) to a position \( E_n \) of an anomalously narrow resonance \( \Gamma_n \ll \Delta \). The probability of such an event can be estimated as \( \mathcal{P}_T \propto (\Gamma/\Delta)\rho(\Gamma/\Delta) \ll 1 \propto (\Gamma/\Delta)^M \), where we used the small width asymptotic \( \rho(\Delta \ll 1) \propto y^{M-1} \) of the resonance widths distribution Eq. (102). Then the asymptotic tail of the probability distribution of the time delay can be estimated as \( \mathcal{P}(\tau_w) \propto \int d\delta \mathcal{P}(\tau_w - \Gamma/n) \mathcal{P}_T \propto \tau_w^{-M-2} \) in agreement with our conjecture. Here it is appropriate to mention that the same asymptotic behavior is typical for the staying probability function \( p(t) \), see Eq. (20). The long-time asymptotic for \( p(t) \) was found for the systems with preserved time-reversal symmetry in the papers [23, 52, 80].

It is trivial to adjust the corresponding argumentation to the present case and to recover the \( \tau^{-M+2} \) behavior.

The expression Eq. (193) allows one also to show that for weakly opened systems the distribution of the scaled delay times should demonstrate the universal behavior \( \mathcal{P}(\tau) \propto \tau^{-3/2} \) in the parametrically large domain \( g^{-1} \ll \tau \ll g \), cf. Eq. (165). Indeed, for \( g \gg 1 \) the resonances do not overlap: \( \Gamma_n \ll \Delta \), and their widths \( \Gamma_n \) follow the \( \chi^2 \) distribution. It is therefore clear, that for any particular value of the energy \( E \) the sum in Eq. (193) is dominated by a single resonance whose position \( E_n \) is the closest to \( E \), that is \( \tilde{\tau}_W \approx \frac{2}{M \Gamma_n} \frac{y_n}{y_n^2 + u_n^2} \), where \( y_n = \pi \Gamma_n/\Delta \) and \( u_n = \frac{\pi}{2} (E - E_n) \). We can estimate the value of the contribution coming from all neglected terms assuming that all resonances have the same widths \( \Gamma \) and are spaced equally with the mean spacing \( \Delta \). This immediately gives the correction to be of the order of \( \delta \tilde{\tau}_W \propto (\Gamma/\Delta)^{M} \approx g^{-1} \), where we used the formula Eq. (110) in the limit \( T \ll 1 \). We conclude that the distribution of the scaled time delay is correctly reproduced by the distribution of the \( \chi^2 \) closest resonance term* as long as we are interested in the region \( \tilde{\tau}_W \gg g^{-1} \). Assuming that the variable \( u_n \) is uniformly distributed in the interval \([ -\pi, \pi ] \) we find:
\[ \mathcal{P}(\tilde{\tau}_W) = \int_0^\infty dy \mathcal{P}_x(y) \int_{-\pi}^{\pi} du \left( \tilde{\tau}_W - \frac{2}{M \Gamma_n} \frac{y}{y^2 + u^2} \right) = M^{-1/2} \frac{y^{1/2}}{(2 - M\pi y)^{1/2}} \mathcal{P}_x(y) \]

where \( y_m(\tilde{\tau}_W) = \min \left[ \frac{\tilde{\tau}_W}{M \Gamma_n}; \pi^2 M \tilde{\tau}_W \right] \). Taking into account that the \( \chi^2 \) distribution \( \mathcal{P}_x(y) \) is cut exponentially at \( y > 1/g \) we can safely neglect the term \( M\pi y \ll 1 \) and set the upper limit \( y_m = \infty \) as long as \( \tilde{\tau}_W \ll g^{-1} \). This immediately results in the anticipated \( \tilde{\tau}_W^{-3/2} \) behavior. On the other hand, we can put \( \mathcal{P}_x(y) \propto g^M y^{M-1} \) and
\[ y_m = 2/(M\tau_w) \] in the domain of extremely large time delays \( \tau_w \gg g \), which results in the \( g^M\tau_w^{-(M+2)} \) tail in full agreement with the general discussion presented above.

The behavior of the delay time distribution in the domain of extremely small time delays \( \tau_w < g^{-1} \) is determined by contribution of many resonant terms in the expression Eq. (13). However, one can argue that the distribution should be exponentially cut: \( P(\tau_w \propto \exp -[\text{const}(\tau g)^{-1}] \), as is indeed seen from the expression Eq. (16). This behavior is a typical one for a sum of random variables of the form \( \sum_n y_n/u_n^2 \), with \( \langle y_n \rangle \sim g^{-1} \) (the so-called stable Levy distribution, see similar arguments in [66]).

The correlation function Eq. (192) acquires quite a simple form for the case of many weakly open channels: \( T_a \ll 1, a = 1, 2, ..., M \) but \( \Gamma = \sum_a T_a \gg 1 \). Then we can put effectively: \( \prod_a (1 - t_2 T_a/2) \approx \exp -\Gamma t_2/2 \); \( \prod_a (1 + t_2 T_a/2)^{-1} \approx \exp -\Gamma t_2/2 \) and also neglect the terms \( \pm x^4 t_2^2 \) in the exponents of the integrands in Eqs (193,194). The corresponding integrals can be calculated exactly giving:

\[
C_W(\omega, x) = \frac{(\Gamma^2_X - \omega^2) [1 - e^{-2\Gamma X \cos 2\omega}] + 2\Gamma X \omega e^{-2\Gamma X \sin 2\omega}}{(\omega^2 + \Gamma^2_X)^2} \tag{197}
\]

where \( \Gamma_X = \Gamma + x^2 \). Let us note that for \( x = 0 \) this expression is actually valid for arbitrary \( \Gamma \). For arbitrary value of \( x \) the condition of validity is \( \Gamma_X \gg 1 \). Neglecting the exponentially small terms we arrive at the simple expression:

\[
C_W(\omega, x) = \frac{(\Gamma^2_X - \omega^2)}{(\omega^2 + \Gamma^2_X)^2} \tag{198}
\]

As will be shown in the next subsection, this expression is nothing other but the semiclassical formula for parametric time delay correlations in systems with broken time-reversal invariance. For the case of preserved time-reversal symmetry and no external parameters the Eq. (198) was derived in [7].

D. Semiclassical theory for parametric correlations of time delays.

A general semiclassical expression for the Wigner-Smith time delay in terms of a periodic orbit expansion has been given by Balian and Bloch [27]. It is formally identical to Gutzwiller’s trace formula. The corresponding expression for the pair correlation function of time delay (without taking into account a parametric dependence) for chaotic scattering was derived by Eckhardt [22]. In parallel, Berry and Keating [124] developed a method allowing to take parametric correlations into account for the case of a closed chaotic system pierced by a magnetic flux serving to break down the time-reversal symmetry. Below we show briefly how to combine both approaches to arrive at the semiclassical expression for the parametric correlation function of time delay in that case; see also related discussion in the papers [11,21,22]. The semiclassical periodic orbit expansion for the “fluctuating part” of a time delay of a quantum particle with an energy \( E + \Omega/2 \): \( \Omega \ll E \) moving in a systempierced by a magnetic flux line with flux \( \phi \) (measured in units of flux quanta \( \phi_0 = 2\pi e/c \)) is:

\[
\delta \tau_w(E + \Omega/2, \phi) = \sum_j A_j e^{i\overline{\hat{\phi}}[S_j(E) + \Omega T_j]} e^{2\pi i w_j \phi} \tag{199}
\]

where the summation goes over all periodic orbits with period \( T_j = \partial S_j/\partial E \), with \( S_j(E) \) being the corresponding action, \( A_j = \frac{e^{i\mu_j T_j}}{2\sqrt{\text{det}(M_j-1)}} \) being the amplitude and \( \mu_j, M_j \) being the Maslov phase and stability matrix corresponding to the given periodic orbit. The winding number \( w_j \) counts the number of times the orbit winds around the flux line.

Thus, for the parametric correlation function one finds:

\[
C_W(\Omega, X) \equiv \langle \delta \tau_W(E - \Omega/2, \phi - X/2) \delta \tau_W(E + \Omega/2, \phi + X/2) \rangle = \left\langle \sum_{j,k} |A_j A_k|^2 \exp \left\{ i \frac{\Omega}{\hbar} [S_j(E) - S_k(E)] + \frac{i\Omega}{2\hbar} (T_j + T_k) + 2\pi i (w_j - w_k) \phi + \pi i X (w_j + w_k) \right\} \right\rangle \tag{200}
\]

where the averaging goes over the energy spectrum. According to standard argumentation [22,120] one can restrict oneself to the so-called ”diagonal approximation” taking into account only contributions with coinciding indices \( j = k \):

\[
C_W^{\text{diag}}(\Omega, X) = \left\langle \sum_j |A_j|^2 \exp \left[ i \frac{\Omega}{\hbar} T_j + 2\pi i X w_j \right] \right\rangle \tag{201}
\]
The next important step uses the fact that winding numbers for orbits in any narrow window of periods are essentially irregular and Gaussian distributed [123,120], see also discussion in [121,122]:

$$P(w_j) = \frac{1}{2\pi\sigma(T_j)}\exp\frac{-w_j^2}{2\sigma(T_j)^2}$$

where the variance $\sigma(T_j)$ increases linearly with period: $\sigma(T) = \beta T$. The constant $\beta$ is system dependent; for a particle with mass $m$ moving in a billiard of the area $A$ it is proportional to $(2E/mA)^{1/2}$ [123,121]. Taking the discrete nature of the winding numbers into account one can write:

$$\exp 2\pi i x w_j = \sum_{k=-\infty}^{\infty} e^{-(2\pi)^2\sigma(T_j)(X-k)^2} \approx e^{-4\pi^2\beta X^2 T_j}$$

where we used $X \ll 1$ (i.e. change of the magnetic flux is much smaller than $\phi_0$) and neglected exponentially small terms $O\left(\exp -4\pi^2\sigma(T)\right)$. Substituting this average in the correlation function of time delay it is convenient to consider its Fourier transform $C(t, X) = \int d\Omega e^{-i\Omega/\hbar}\overline{C_W}(\Omega, X)$. We have:

$$C_W(t, X) = \sum_j |A_j|^2 \delta(t - T_j)e^{-4\pi^2\beta X^2 T_j} = e^{-\tilde{x}^2 t} \sum_j |A_j|^2 \delta(t - T_j)$$

where we denoted $\tilde{x} = 2\pi\beta^{1/2} X$.

For closed chaotic systems the sum in the preceding equation is known to be proportional to the time $t$ (this is the famous Hannay-Ozorio de Almeida sum rule [124]). For open chaotic systems Eckhardt [22] gave some arguments in favor of replacing this sum by $te^{-\Gamma_{cl} t}$, with $\Gamma_{cl}$ being the classical escape rate from the chaotic region. Using this fact we see that (under the assumptions we made) the semiclassical expression for the Fourier transform of the correlation function of time delays is given by:

$$C_W(t, X) = te^{-(\Gamma_{cl} + \tilde{x}^2)t}$$

After Fourier-transforming this expression back we see that the result turns out to be identical to that given in Eq.(198) upon identification $\Gamma_{cl} \rightarrow \Gamma, \tilde{x} \rightarrow x$.

Another interesting point to be mentioned is that the form of the time delay correlation function given in the eq.(107) (which contains Eq.(198) as a limiting case) was obtained by Shushin and Wardlaw [123] in the model of chaotic scattering on a leaky surface of constant negative curvature. At the first glance such a correspondence is quite a surprising fact since the model considered in [123] corresponds formally to one-channel scattering, but the result Eq.(197) was derived under the assumption of many weak channels. In order to understand that fact one should remember that the model considered in [123] possesses quite a peculiar property: all its resonance poles turned out to have exactly the same widths. It is at variance with the known form of the resonance widths distribution for one-channel scattering in a generic chaotic system, see eq.(102), where resonance widths fluctuate strongly. At the same time, if we consider the limiting case of many weak channels: $M \gg 1, g \gg 1$ and $M/g = \Gamma_{ef}$ fixed, the distribution of resonance widths tends to the delta-functional one $\rho(\Gamma) = \delta(\Gamma - \Gamma_{ef})$. This fact can be easily inferred from the Eq.(108). We see that effectively it is just the limiting case of many weak channels that corresponds to non-fluctuating resonance widths. Under these conditions the correlations of the time delays are determined by the statistics of the positions of resonances. For the model of scattering on a leaky surface of negative curvature the positions of resonances are given by the zeroes of Riemann zeta-function on the so-called critical line in the complex plane. According to the celebrated Montgomery conjecture (verified numerically [120] and supported by sound analytical results [127]) statistical properties of these zeroes are identical to those of eigenvalues of large random GUE matrices. All these facts taken into account it is no more a surprise that the correlations of time delays for both models coincide in the considered region of parameters.

VI. SUMMARY AND CONCLUSIONS

In the present paper we analyzed in much detail the universal features of statistics of resonances, phase shifts and delay times for a generic open chaotic quantum system with broken time-reversal invariance. This was achieved by replacing the Hamiltonian of the chaotic region by a large Random Matrix taken from the Gaussian Unitary Ensemble. Employing the well-developed method of mapping the problem to the so-called supersymmetric nonlinear
σ-model we succeeded in deriving explicit analytical expressions for various distributions and correlations functions, see Eqs. (97-99), 103, 114, 159, 161, 174 and 191, characterizing the above mentioned quantities for arbitrary finite number of open channels and arbitrary strength of coupling to continua.

The best candidates for checking the validity of the expressions obtained are realistic models of mesoscopic ballistic devices subject to applied magnetic field. Closed [128] as well as open [83,122,124] systems of this kind were intensively investigated recently and the statistics of S-matrix elements and related quantities was available among other characteristics. Very recently, the issue of dwell times inside the chaotic region attracted some research interest as well [130]. All these facts allow us to expect that our results can be verified independently in the numerical experiments.

1 It is important to mention that our results are also of potential experimental relevance. Indeed, the issue of time-delay fluctuations turns out to be intimately related to the statistical properties of mesoscopic capacitors [88,89]. Considering the case of a mesoscopic cavity coupled by a M-channel lead to one electronic reservoir and capacitively to another reservoir Gopar, Büttiker and Mello [89] suggested the following expression for the low frequency AC admittance of such a structure:

\[ G^I(\omega) = -i\omega C_e\alpha \quad ; \quad \alpha = \frac{\tau_w}{\eta + \tau_w} \]  

(205)

where \(C_e\) stands for a geometric capacitance relating the charge \(Q\) on the plate to the voltage \(U\) across the capacitor, \(\tau_w\) stands for the dimensionless Wigner time delay, and \(\eta = \frac{C_e}{M\epsilon\sqrt{\Delta}}\), with \(\Delta\) being the mean level spacing for the cavity.

For macroscopic cavities \(\eta \to 0\) and the dimensionless capacitance \(\alpha\) is equal to unity resulting in the classical expression for the capacitive response: \(G^I(\omega) = -i\omega C_e\). In contrast, for small enough cavities \(\eta\) has to be taken into account and the fluctuating delay time \(\tau_w\) results in a fluctuating admittance.

As it was discussed above, for one open channel \(M = 1\) the distribution of Wigner time delay is identical to the distribution of partial delay times \(P_\tau(\tau)\) and is given by Eq. (161). This fact immediately allows one to write down the distribution of the dimensionless capacitance \(\alpha\) as:

\[ P_\alpha(\alpha) = \frac{\eta}{(1-\alpha)^2} P_\tau \left[ \tau = \frac{\eta\alpha}{1-\alpha} \right] \]  

(206)

For the perfect coupling case \(T = 1\) the corresponding distribution was analyzed in [80]. In the opposite limiting case of weak coupling \(T \ll 1\) our expression just says that the variance of time delay is of the order \(1/M^2 \ll 1\) as compared with the squared mean value \(\langle \tau_w \rangle^2\). Thus, we can represent the fluctuating time delay in a form \(\tau_w = \langle \tau \rangle + \delta\tau_w\), where typical scale of the fluctuating part is of the order of \(\delta\tau_w \sim \langle \tau_w \rangle/M\). Substituting this expression to Eq. (203) and expanding with respect to the fluctuating part \(\delta\tau_w\) one obtains to the first nontrivial order:

\[ G^I(\omega) = \langle G^I(\omega) \rangle \left( 1 + \delta\tau_w \frac{\eta}{\langle \tau_w \rangle (\eta + \langle \tau_w \rangle)} + \ldots \right) \]

and immediately extracts the variance of the admittance:

\[ \frac{\langle (G^I(\omega))^2 \rangle - \langle G^I(\omega) \rangle^2}{\langle G^I(\omega) \rangle^2} = \frac{2}{T^2 M^2} \frac{\tau_{RC}^2}{\langle \tau_w \rangle^2} \]

where \(\tau_{RC}^{-1} = \langle \tau_w \rangle^{-1} + \eta^{-1}\) is the so-called RC-time and we substituted the expression Eq.(193) for the time delay variance taking into account that \(M \gg 1\). Such an expression for the particular case \(T = 1\) was very recently derived by Brouwer and Büttiker by a different method [90].
The majority of the numerical data concerning various statistical properties of the scattering matrix for open chaotic systems corresponds to the case of preserved time-reversal invariance see e.g. [133]. This case is not only simpler from numerical point of view (opposite to the situation with the analytical calculations), but also the most relevant experimentally. As a consequence, numerical studies on resonance width statistics [114,52,93,2,12] as well as on properties of scattering phase shifts and their derivatives [2,7,4,76] were restricted to the systems of that symmetry class. It is necessary to note that some analytical results for poles and time delays for time-reversal invariant scattering are already available in the literature for some time. In particular, for only one open channel the joint probability distribution of all N complex resonance poles is known [2] (however, not the density of these poles in complex plane) as well as the distribution of Wigner time delay for perfect coupling to continuum [39]. Essential progress was achieved by Lehmann et al. [7] who calculated the correlation function of time delays for two different values of energy and any number of open channels. Actually, the calculation similar to that done in the present paper can be successfully carried out for the whole crossover region between the orthogonal and unitary symmetry classes. The results will be published elsewhere [92].

Let us also mention that recent numerical results [2] show that the resonance widths distribution derived in the present paper can be applied for the systems with preserved time reversal invariance quite satisfactorily after replacing the number of channels M by M/2. This fact is not so surprising, taking into account that such feature as the powerlaw tail 1/y^2 of that distribution is actually a generic property following from the chaotic classical dynamics only, see the discussion in Sec.III.

For a majority of models in atomic and molecular physics parameters of all the resonances can be determined even without expensive calculations of S−matrix elements. The most effective method is the so-called complex scaling (or complex rotation) method [2] successfully used for the systems exhibiting chaotic behavior [16,17,52]. It is interesting to mention that a crossover from isolated to overlapping resonance regime was detected recently for the dissociation reaction \( \text{HO}_2 \rightarrow \text{H} + \text{O}_2 \) in one open channel case [16]. One can hope that applying the complex rotation method to this system one could extract the widths of resonances with sufficient accuracy and to observe a transition from the \( \chi^2 \) distribution towards that with the 1/y^2 tail. We would like to point out that the whole S−matrix as a function of energy of incoming waves was measured in real experiments [103], and even used to calculate the average time delay [10]. In principle, the positions of resonances in the complex plane can be extracted if one knows \( \hat{S}(E) \) with sufficient accuracy. For example, one can use that fact that for any number of open channels the determinant \( \det \hat{S}(E) \) as a function of energy has its singularities (which are just resonance poles \( E_n - i \Gamma_n/2 \)) only in the lower half plane \( \text{Im}E < 0 \). As a result it can be written as:

\[
\det \hat{S}(E) = e^{i \delta} \prod_n \frac{E - E_n - i \Gamma_n/2}{E - E_n + i \Gamma_n/2}
\]  

(208)

where \( \delta \) is the phase of potential scattering irrelevant for our discussion. Provided the values of \( \det \hat{S}(E) \) for real \( E \) are known, one can restore the determinant of S−matrix in the upper half-plane \( \text{Im}E > 0 \) by the relation:

\[
\det \hat{S}(E + i) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{(\omega - E)^2 + \Gamma^2} \det \hat{S}(E)
\]  

(209)

It is easy to see that the two relations (208) and (209) allow to determine all the resonance parameters \( E_n, \Gamma_n \) from zeroes of the S−matrix determinant in the upper half-plane \( \text{Im}E > 0 \). Of course, the practical implementation of this procedure requires highly accurate data for \( \hat{S}(E) \) which is not the case in the mentioned experiments due to noise and damping in resonator walls. However, one can hope that the progress in the experimental set-up could make such a measurement feasible in future.

Finally, as an interesting perspective for future research we would like to mention the issue of S−matrix statistics for systems exhibiting the Anderson localization phenomenon. This issue attracts research attention for some period [133] and increasing amount of numerical results are already available [7,110,76,134] requiring a systematic analytical insight into the problem.

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APPENDIX A: REGULARIZATION OF THE EIGENVALUE DENSITY FOR NONNORMAL RANDOM MATRICES

Let us consider a random, but fixed non-Hermitian $N \times N$ random matrix $\mathcal{H}$, for which we only assume that generically its complex eigenvalues are non-degenerate. Taking the second derivative of the potential Eq. (59) (apart from the factor $1/N$) with respect to the energy $\mathcal{E}$ and its complex conjugate $\mathcal{E}^\ast$, we obtain Poisson’s equation [35]:

$$-\frac{\partial^2}{\partial \mathcal{E} \partial \mathcal{E}^\ast} \Phi = \text{Tr} \left[ \frac{1}{(\mathcal{H} - \mathcal{E})^\dagger(\mathcal{H} - \mathcal{E}) + \kappa^2} \frac{1}{(\mathcal{H} - \mathcal{E})(\mathcal{H} - \mathcal{E})^\dagger + \kappa^2} \right] = \pi \rho_\kappa(\mathcal{E}, Y)$$  \hspace{1cm} (A1)

with a density $\rho_\kappa$ which is always positive, because the operators appearing are both positive. We will show below that $\rho_\kappa$ goes to a sum of two-dimensional $\delta$–functions in the complex energy plane:

$$\lim_{\kappa \to 0} \rho_\kappa(\mathcal{E}, Y) = \sum_{j=1}^{N} \delta^2(\mathcal{E} - \mathcal{E}_j)$$  \hspace{1cm} (A2)

where $\mathcal{E}_j$ are the eigenvalues of $\mathcal{H}$. The weight for each $\delta$–function is one. Here the integral over $\rho(\mathcal{E}, Y)d\mathcal{E}dY$ is normalized to $N$: $\int \rho_\kappa(\mathcal{E}, Y)d\mathcal{E}dY = N$, which can be kept finite. This is true for any $\kappa > 0$ and can simply be shown by Stokes theorem.

Now let us consider the Hermitian eigenvalue problem

$$[(\mathcal{H} - \mathcal{E})^\dagger(\mathcal{H} - \mathcal{E}) + \kappa^2] \psi_i = \lambda_i \psi_i$$  \hspace{1cm} (A3)

The eigenvalues $\lambda_i$ we again assume to be generically non-degenerate, and the $\psi_i$ form a complete orthonormalised set. It follows:

$$[(\mathcal{H} - \mathcal{E})(\mathcal{H} - \mathcal{E})^\dagger + \kappa^2] (\mathcal{H} - \mathcal{E}) \psi_i = \lambda_i (\mathcal{H} - \mathcal{E}) \psi_i$$  \hspace{1cm} (A4)

so that $\phi_i = (\mathcal{H} - \mathcal{E})\psi_i/\sqrt{\lambda_i - \kappa^2}$ is a normalised eigenvector of a second Hermitian eigenvalue problem for the operator $[(\mathcal{H} - \mathcal{E})(\mathcal{H} - \mathcal{E})^\dagger + \kappa^2]$. Such an eigenvector corresponds to the same eigenvalue $\lambda_i$, provided $\lambda_i \neq \kappa^2$, and $\mathcal{E}$ is not an eigenvalue of $\mathcal{H}$. If $(\mathcal{H} - \mathcal{E})\psi_0 = 0$, we nevertheless can find a normalised eigenvector $\phi_0$, orthogonal to all $\phi_i$: $i \neq 0$, with $(\mathcal{H} - \mathcal{E})(\mathcal{H} - \mathcal{E})^\dagger \phi_0 = 0$, i.e. $\phi_0$ is an eigenvector of $\mathcal{H}^\dagger$ with the eigenvalue $\mathcal{E}^\ast$.

Now we may expand $\rho_\kappa$ in terms of these eigenfunctions:

$$\rho_\kappa(\mathcal{E}, Y) = \frac{\kappa^2}{N} \sum_{ik} \frac{1}{\lambda_i} \mid (\psi_i, \phi_k) \mid^2 \frac{1}{\lambda_k}$$  \hspace{1cm} (A5)

from which one sees explicitly that $\rho_\kappa$ is positive. Here $\langle \psi_i, \phi_k \rangle$ stands for the complex scalar product in Hilbert space.

If $\mathcal{E}$ is not an eigenvalue of $\mathcal{H}$ (and $\mathcal{E}^\ast$ is not one of $\mathcal{H}^\dagger$), then $\rho_\kappa$ goes to zero proportionally to $\kappa^2$ for $\kappa \to 0$. Therefore the weight to the normalization of $\rho_\kappa$ comes only from the neighborhoods of the eigenvalues $\mathcal{E}_j$ of $\mathcal{H}$. If $\mathcal{E}_0$ is exactly an eigenvalue of $\mathcal{H}$, then we know the lowest eigenvalue $\lambda_0$ of $(\mathcal{H} - \mathcal{E}_0)^\dagger(\mathcal{H} - \mathcal{E}_0) + \kappa^2$ (which is obviously $\lambda_0 = \kappa^2$) and all other eigenvalues are higher by amounts independent of $\kappa$. That means:

$$\rho_\kappa(\mathcal{E}, Y) \approx \frac{1}{\pi \kappa^2} \mid (\psi_0, \phi_0) \mid^2$$  \hspace{1cm} (A6)

for $\mathcal{E} = \mathcal{E}_0$ and $\kappa \to 0$, which diverges as it must for a $\delta$–function at an eigenvalue $\mathcal{E}_0$.

In order to see how $\rho_\kappa$ varies with energy near an eigenvalue $\mathcal{E}_0$, we may set $\mathcal{E} = \mathcal{E}_0 + \delta \mathcal{E}$ and calculate $\lambda_0$ by perturbation theory. Only the second order perturbation contributes and the surprisingly simple result is:

$$\lambda_0 \approx \kappa^2 + \mid \delta \mathcal{E} \mid^2 \mid (\phi_0, \psi_0) \mid^2$$  \hspace{1cm} (A7)
This means that in the neighborhood of an eigenvalue $\mathcal{E}_0$ the function $\rho_\kappa$ has the form:

$$\rho_\kappa(E, Y) \approx \frac{\kappa^2 |(\phi_0, \psi_0)|^2}{\pi (\kappa^2 + |E|^2)(|\phi_0, \psi_0|^2)^2}$$  \hspace{1cm} (A8)$$

We will not consider rare cases, in which the vector $\psi_0$ is occasionally orthogonal to $\phi_0$. Then we see that $\rho_\kappa(E, Y)$ has in the limit $\kappa \to 0$ the form of a two-dimensional $\delta-$ function with weight one. Its width goes to zero like $\kappa/| (\phi_0, \psi_0)|$. This is valid in the neighborhood of one isolated eigenvalue, which is however arbitrary. This proves that the Eq. (A2) is indeed correct.

**APPENDIX B: THE PARAMETRIZATION OF THE MATRICES $\hat{Q}$**

The supermatrices $\hat{Q}$ belonging to the graded coset space $U(1,1/2)/U(1,1) \times U(1,1)$ can be parametrized as $\hat{Q} = \hat{U}^{-1} \hat{M} \hat{U}$ where

$$\hat{U} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} -iM_1 & M_{12} \\ M_{21} & iM_1 \end{pmatrix}, \quad \hat{u} = \hat{u}^{-1}, \quad \hat{v} = \hat{k}^\dagger \hat{v}^\dagger \hat{k}$$

$$M_{12} = \begin{pmatrix} \mu_1 | e^{i\phi_1} \\ 0 \end{pmatrix}, \quad M_{21} = \begin{pmatrix} \mu_1 | e^{-i\phi_1} \\ 0 \end{pmatrix},$$

where $\alpha, \alpha^*, \beta, \beta^*$ are Grassmann variables, $\lambda_1 \in (1, \infty); \lambda_2 \in (-1, 1); \phi_1, \phi_2 \in (0, 2\pi)$ and $\lambda_{1,2}$ are related to $|\mu_{1,2}|$ via $\lambda_1^2 - |\mu_1|^2 = 1; \quad \lambda_2^2 + |\mu_2|^2 = 1$.

It is convenient to have also the explicit expressions for the matrix elements of $\hat{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$. We have

$$Q_{11} = -iu^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} u \equiv \begin{pmatrix} -i [\lambda_1 - \alpha^* \alpha (\lambda_1 - \lambda_2)] & i\alpha (\lambda_1 - \lambda_2) \\ i\alpha (\lambda_1 - \lambda_2) & -i [\lambda_2 - \alpha^* \alpha (\lambda_2 - \lambda_1)] \end{pmatrix}$$  \hspace{1cm} (B2)$$

$$Q_{22} = iv^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} v \equiv \begin{pmatrix} i [\lambda_1 + \beta^* \beta (\lambda_1 - \lambda_2)] & \beta^* (\lambda_1 - \lambda_2) \\ \beta (\lambda_1 - \lambda_2) & i [\lambda_2 + \beta^* \beta (\lambda_2 - \lambda_1)] \end{pmatrix}$$  \hspace{1cm} (B3)$$

$$Q_{12} = u^{-1} \begin{pmatrix} \mu_1 & 0 \\ 0 & i\mu_2 \end{pmatrix} v \equiv \begin{pmatrix} \mu_1 (1 - \alpha^* \alpha/2)(1 + \beta^* \beta/2) - \alpha^* \beta \mu_2^* & -i\beta^* (1 - \alpha^* \alpha/2) \mu_1 + i\alpha^* (1 - \beta^* \beta/2) \mu_2^* \\ -\alpha (1 + \beta^* \beta/2) \mu_1 - \beta (1 + \alpha^* \alpha/2) \mu_2^* & i\alpha \beta^* \mu_1 + i(1 + \alpha^* \alpha/2)(1 - \beta^* \beta/2) \mu_2^* \end{pmatrix}$$

$$Q_{21} = v^{-1} \begin{pmatrix} \mu_2^* & 0 \\ 0 & i\mu_2 \end{pmatrix} u \equiv \begin{pmatrix} \mu_2^* (1 - \alpha^* \alpha/2)(1 + \beta^* \beta/2) + \alpha^* \beta \mu_2 & \beta^* (1 - \alpha^* \alpha/2) \mu_2 - \alpha (1 + \beta^* \beta/2) \mu_1^* \\ +i\alpha (1 - \beta^* \beta/2) \mu_2 - i\beta (1 + \alpha^* \alpha/2) \mu_1^* & -i\alpha \beta^* \mu_1^* + i(1 + \alpha^* \alpha/2)(1 - \beta^* \beta/2) \mu_2 \end{pmatrix}$$  \hspace{1cm} (B5)$$

where we introduced the notations: $\mu_1 = |\mu_1| \ e^{i\phi_1}; \quad \mu_2 = |\mu_2| \ e^{i\phi_2}$. The expressions above are frequently referred to as the "Efetov parametrization" for the matrices $\hat{Q}$.

We denote the corresponding measure as $d\hat{Q}$. Straightforward, but lengthy calculation gives:

$$d\hat{Q} = \frac{d\lambda_1 d\lambda_2 \ d\phi_1 d\phi_2}{(\lambda_1 - \lambda_2)^2 (2\pi)^2} d\alpha^* d\beta^* \ d\alpha \ d\beta$$  \hspace{1cm} (B6)$$

In the rest of this Appendix we present the explicit expressions for supertraces, superdeterminants, and matrix elements, entering different expressions in the main text, see the Eqs. 86, 135, 187.
1. For resonance widths calculation.
   We have:
   \[
   \text{Str} \hat{Q} \hat{\lambda} = \left( Q_{bb}^{(11)} - Q_{bb}^{(22)} \right) - \left( Q_{ff}^{(11)} - Q_{ff}^{(22)} \right) = -2i(\lambda_1 - \lambda_2)
   \]

   \[
   \text{Str} \hat{Q}(\hat{\lambda} - K_b \hat{L}) = Q_{ff}^{(22)} - Q_{ff}^{(11)} = 2i\lambda_2 - i(\lambda_1 - \lambda_2)(\alpha^* \alpha - \beta^* \beta)
   \]

   \[
   \text{Str} \hat{Q} \Sigma_L = i \left( Q_{bb}^{(12)} - Q_{bb}^{(21)} \right) - \left( Q_{ff}^{(12)} + Q_{ff}^{(21)} \right) = \\
   = i(\mu_1 - \mu_1^* - \mu_2 - \mu_2^*) - i\frac{\alpha^* \alpha - \beta^* \beta}{2}(\mu_1 - \mu_1^* + \mu_2 + \mu_2^*) - i\alpha^* \beta(\mu_2^* - \mu_1^*) - \\
   i\alpha^* \beta^*(\mu_1 + \mu_2) - \frac{i}{4}\alpha^* \beta^* \beta(\mu_1 - \mu_1^* - \mu_2 - \mu_2^*)
   \]

   The superdeterminant \( \text{Sdet}^{-1} \left[ I + i\frac{1}{2} \gamma_a E \hat{\lambda} + i\pi \nu_{sc} \gamma_a \hat{Q} \hat{\lambda} \right] \) can be easily evaluated because the supermatrix \( \hat{U} \), Eq.(B1), commutes with \( \hat{\lambda} \) and therefore can be omitted under the sign of the superdeterminant:

   \[
   \text{Sdet}^{-1} \left[ I + i\frac{1}{2} \gamma_a E \hat{\lambda} + i\pi \nu_{sc} \gamma_a \hat{Q} \hat{\lambda} \right] = \text{Sdet}^{-1} \left[ I + i\frac{1}{2} \gamma_a E \hat{\lambda} + i\pi \nu_{sc} \gamma_a M \hat{\lambda} \right] 
   \]

   The supermatrix \( \hat{A} = \left[ I + i\frac{1}{2} \gamma_a E \hat{\lambda} + i\pi \nu_{sc} \gamma_a \hat{Q} \hat{\lambda} \right] \) is however block-diagonal in the fermion-boson arrangement: \( \hat{A}_{bf} = \hat{A}_{fb} = 0 \), see Eq.(B1), and therefore \( \text{Sdet}^{-1} \hat{A} = \text{Det} \hat{A}_{ff} / \text{Det} \hat{A}_{bb} \). Trivial calculation gives:

   \[
   \text{Sdet}^{-1} \left[ I + i\frac{1}{2} \gamma_a E \hat{\lambda} + i\pi \nu_{sc} \gamma_a \hat{Q} \hat{\lambda} \right] = \frac{1 + 2\pi \nu_{sc}(E) \gamma_a \lambda_2 + \gamma_a^2}{1 + 2\pi \nu_{sc}(E) \gamma_a \lambda_1 + \gamma_a^2}
   \]

   which is reduced to the form used in the text of the paper, see e.g. Eq.(S6) upon introducing the transmission coefficients \( T_a \), Eq.(49) and the parameters \( g_a = 2/T_a - 1 \).

2. For scattering phase shifts statistics.
   First of all, using \( U \hat{\lambda} = \hat{\lambda} U \) one has

   \[
   \text{Str}(\hat{Q} \hat{\lambda} \hat{Q} \hat{\lambda}) = 2\text{Str}(M_{11}^2 - M_{12} M_{21}) = -4(\lambda_1^2 - \lambda_2^2)
   \]

   The main object entering the calculation of eigenphases correlation function is the supermatrix \( \left( \hat{B}_a \right)^{-1} \), where:

   \[
   \hat{B}_a = \frac{1}{2} z_1 (\hat{I}_4 + \hat{\lambda}) + \frac{1}{2} z_2 (\hat{I}_4 - \hat{\lambda}) - \gamma_a \left( \frac{E}{2} \hat{I}_4 - \pi \nu_{sc} \hat{Q} \right) \equiv \hat{U}^{-1} \hat{b}(a) \hat{U}
   \]

   \[
   \hat{b}(a) = \text{diag} \left( z_1(a) \hat{I}_2, z_2(a) \hat{I}_2 \right) + \tilde{\gamma}_a \hat{M}
   \]

   where we used the notations: \( z_p(a) = z_p - \gamma_a \frac{E}{2} ; \) \( p = 1, 2 \) and \( \tilde{\gamma}_a = \pi \nu_{sc} \gamma_a \), the supermatrix \( \hat{M} \) being defined in Eq.(B2). One can invert \( \hat{b}(a) \) easily noticing that in the \textit{boson-fermion} arrangement this matrix is block-diagonal:

   \[
   \hat{b}(a) = \text{diag} \left( \hat{b}_{bb}(a), \hat{b}_{ff}(a) \right), \text{ where}
   \]

   \[
   \hat{b}_{bb}(a) = \begin{pmatrix} z_1(a) - i\tilde{\gamma}_a \lambda_1 \\ \mu_1 \tilde{\gamma}_a z_2(a) + i\tilde{\gamma}_a \lambda_1 \end{pmatrix} \quad \text{and} \quad \hat{b}_{ff}(a) = \begin{pmatrix} z_1(a) - i\tilde{\gamma}_a \lambda_2 \\ \mu_2 \tilde{\gamma}_a z_2(a) + i\tilde{\gamma}_a \lambda_2 \end{pmatrix}
   \]

   so that \( \hat{b}^{-1}(a) = \text{diag} \left( \hat{b}^{-1}(a)_{bb}, \hat{b}^{-1}(a)_{ff} \right) \), where
\[
[b(a)]_{bb}^{-1} = \frac{1}{D_b(a)} \begin{pmatrix}
2(a) + i\gamma_a \lambda_1 & -\mu_1 \gamma_a \\
-\mu_1 \gamma_a & z_1(a) + i\gamma_a \lambda_1
\end{pmatrix}
\] (B9)

\[
[b(a)]_{ff}^{-1} = \frac{1}{D_f(a)} \begin{pmatrix}
2(a) + i\gamma_a \lambda_2 & -i\mu_2 \gamma_a \\
-i\mu_2 \gamma_a & z_1(a) - i\gamma_a \lambda_2
\end{pmatrix}
\] (B10)

and we introduced notations:

\[
D_b(a) \equiv \det \hat{b}_{bb}(a) = z_1(a)z_2(a) + i\gamma_a (z_1(a) - z_2(a)) \lambda_1 + \gamma_a^2
\]

\[
D_f(a) \equiv \det \hat{b}_{ff}(a) = z_1(a)z_2(a) + i\gamma_a (z_1(a) - z_2(a)) \lambda_2 + \gamma_a^2
\] (B11)

so that \( \text{Sdet} \left( \hat{B}_a^{-1} \right) = \text{Sdet} \left( \hat{b}^{-1}(a) \right) = \frac{D_f(a)}{D_b(a)} \).

Rearranging the supermatrix \( \hat{b}^{-1}(a) \) in advanced-retarded order we can find easily the supermatrix \( \hat{B}_a^{-1} = U^{-1}\hat{b}^{-1}(a)U \). Actually, we need only its elements in the fermion-fermion block (see Eq.(135)):

\[
\text{Str} \left( \hat{B}_a^{-1} \hat{C}_1 \right) = - (B_a^{-1})^{11}_{ff}; \quad \text{Str} \left( \hat{B}_a^{-1} \hat{C}_2 \right) = - (B_a^{-1})^{22}_{ff}
\]

and \( \text{Str} \left( \hat{B}_a^{-1} \hat{C}_1 \hat{B}_a^{-1} \hat{C}_2 \right) = - (B_a^{-1})^{21}_{ff} (B_a^{-1})^{12}_{ff} \).

We find, correspondingly:

\[
(B_a^{-1})^{11}_{ff} = \frac{2(a) + i\gamma_a \lambda_2}{D_f(a)} - \alpha^* \alpha \left[ \frac{2(a) + i\gamma_a \lambda_1}{D_b(a)} - \frac{2(a) + i\gamma_a \lambda_2}{D_f(a)} \right]
\] (B12)

\[
(B_a^{-1})^{22}_{ff} = \frac{z_1(a) - i\gamma_a \lambda_2}{D_f(a)} + \beta^* \beta \left[ \frac{z_1(a) - i\gamma_a \lambda_1}{D_b(a)} - \frac{z_1(a) - i\gamma_a \lambda_2}{D_f(a)} \right]
\] (B13)

\[
(B_a^{-1})^{12}_{ff} = - \frac{i\gamma_a \mu_1}{D_b(a)} \alpha \beta^* + \frac{i\gamma_a \mu_2}{D_f(a)} (1 + \alpha^* \alpha / 2) (1 - \beta^* \beta / 2)
\] (B14)

\[
(B_a^{-1})^{21}_{ff} = \frac{i\gamma_a \mu_1}{D_b(a)} \alpha \beta - \frac{i\gamma_a \mu_2}{D_f(a)} (1 + \alpha^* \alpha / 2) (1 - \beta^* \beta / 2)
\] (B15)

3. For time-delay correlations.

The main new object here is the supermatrix (see Eq.(187)):

\[
\hat{B}_\nu(a) = \frac{1 + \hat{A}}{2} + \frac{1 - \hat{A}}{2} \left[ z(a)\hat{I}_4 + \hat{\gamma}_a \hat{Q} \right] = \hat{U} \left( \begin{array}{cc}
\hat{I}_2 & 0 \\
\hat{\gamma}_a \hat{M}_{21} & z(a)\hat{I}_2 + i\hat{\gamma}_a \hat{M}_{11}
\end{array} \right) \hat{U}^{-1}
\] (B16)

where we again used that matrices \( \hat{U} \) and \( \hat{A} \) commute. The matrix \( \hat{B}_\nu(a) \) is simple to invert. Performing the calculation we find:

\[
(B_\nu(a)^{-1})^{22}_{ff} = \frac{1}{z(a) + i\gamma_a \lambda_2} - \beta^* \beta \left( \frac{1}{z(a) + i\gamma_a \lambda_1} - \frac{1}{z(a) + i\gamma_a \lambda_2} \right)
\] (B17)

and the corresponding superdeterminant is given by:

\[
\text{Sdet}^{-1} \hat{B}_\nu(a) = \frac{z(a) + i\gamma_a \lambda_2}{z(a) + i\gamma_a \lambda_1}
\] (B18)
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Note that this auxiliary function does not in general have the property of positivity.

Strictly speaking, absolute convergence is obtained only for $\Gamma$ being positive. This can be achieved formally by adding further very weakly coupled $N-M$ channels, the couplings of which can be put to zero after the calculation is done. The corresponding coupling constants should be chosen so small that they do not modify our saddle-point considerations, see below. This is always possible at any, whatever large, but fixed, value of $N$. Having such regularization in mind, the whole procedure is well-defined.

Already here one sees that contributions of $N-M$ additional very weakly coupled channels may be neglected if the corresponding couplings $\gamma_a$ approach zero faster than $1/N$.

The distribution $P_\tau(\tau)$ is actually a density of partial delay times and should not be confused with the joint probability density of all $M$ partial delay times. In general, it is also different from the distribution of Wigner-Smith time delay, being equivalent to the latter only for one open channel $M=1$.

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APPENDIX: FIGURE CAPTION

• Fig 1. A generic model for chaotic scattering: an irregular shaped cavity attached to the perfect lead. The Hamiltonian associated with the cavity region is simulated by a random matrix $\hat{H}_{in}$.

• Fig. 2 The distribution of scaled resonance widths $\rho(y)$ for $M = 1$ (solid) $M = 2$ (dash-dotted) and $M = 3$ (dotted line) equivalent open channels. The effective coupling is maximal: $g = 1$. As the result, the distributions demonstrate $M/(2y^2)$ asymptotic behavior at large $y$.

• Fig. 3 The distribution of scaled partial delay times $\mathcal{P}(\tau_s)$ for $M = 1$ (solid) $M = 2$ (dash-dotted) and $M = 3$ (dotted line) equivalent open channels. The effective coupling $g = 10$ corresponds to weakly open systems.