ON THE LEADING CONSTANT IN THE MANIN-TYPE CONJECTURE FOR CAMPANA POINTS

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Abstract. We compare the Manin-type conjecture for Campana points recently formulated by Pieropan, Smeets, Tanimoto and Várilly-Alvarado with an alternative prediction of Browning and Van Valkenborgh in the special case of the orbifold \((\mathbb{P}^1, D)\), where \(D = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]\). We find that the two predicted leading constants do not agree, and we discuss whether thin sets could explain this discrepancy. Motivated by this, we provide a counterexample to the Manin-type conjecture for Campana points, by considering orbifolds corresponding to squareful values of binary quadratic forms.

1. Introduction

The study of Campana points is an emerging area of interest in arithmetic geometry as a way to interpolate between rational and integral points. Campana orbifolds, first introduced in [4] and [5], consist of a variety \(X\) and a weighted boundary divisor \(D\) of \(X\). The Campana points associated to the orbifold \((X, D)\) can be viewed as rational points of \(X\) that are integral with respect to \(D\). In the recent paper [16], Pieropan, Smeets, Tanimoto and Várilly-Alvarado formulate a Manin-type conjecture for the quantitative study of Campana points on Fano Campana orbifolds, which henceforth we shall refer to as the *PSTV-A conjecture*. The authors establish their conjecture in the special case of vector group compactifications, using the height zeta function method developed by Chambert-Loir and Tschinkel [7], [8].

The arithmetic study of Campana points is still in its early stages. Initial results in [2], [22] and [3], which predate the formulation of the PSTV-A conjecture, concern squareful and \(m\)-full values of hyperplanes of \(\mathbb{P}^{n+1}\). (We recall that a nonzero integer \(z\) is \(m\)-full if for any prime \(p\) dividing \(z\), we have \(p^m \mid z\), and squareful if it is 2-full.) Following discussions in the Spring 2006...
MSRI program on rational and integral points on higher dimensional varieties, Poonen [17] posed the problem of finding the number of coprime integers $z_0, z_1$ such that $z_0, z_1$ and $z_0 + z_1$ are all squareful and bounded by $B$. In the language of the PSTV-A conjecture, this corresponds to counting Campana points on the orbifold $(\mathbb{P}^1, D)$, where $D$ is the divisor $\frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$. Upper and lower bounds for this problem were obtained by Browning and Van Valckenborgh [2], but finding an asymptotic formula remains wide open. Van Valckenborgh [22] considers a higher-dimensional analogue of this problem by defining a Campana orbifold $(\mathbb{P}^n, D)$, where

$$D_i = \begin{cases} \{z_i = 0\}, & \text{if } 0 \leq i \leq n, \\ \{z_0 + \cdots + z_n = 0\}, & \text{if } i = n + 1. \end{cases}$$

Choosing the height $H$ on $\mathbb{P}^n(\mathbb{Q})$ defined by

$$H(z) = \max(|z_0|, \ldots, |z_n|, |z_0 + \cdots + z_n|),$$

for a representative $(z_0, \ldots, z_n) \in \mathbb{Z}_{\text{prim}}^{n+1}$, this leads to the counting problem

$$(1.2) \quad N_n(B) := \frac{1}{2^#} \# \left\{ (z_0, \ldots, z_{n+1}) \in (\mathbb{Z}_{\neq 0})_{\text{prim}}^{n+2} : \begin{array}{c} z_0 + \cdots + z_n = z_{n+1}, \\ \sum_{0 \leq i \leq n+1} m_i (m_i + 1) \geq 1. \end{array} \right\}.$$ 

Van Valckenborgh [22, Theorem 1.1] proves that for any $n \geq 3$, we have $N_n(B) \sim cB^{n/2}$ as $B \to \infty$, for an explicit constant $c > 0$. In [19], we extend the treatment to handle the case $n = 2$. Work of Browning and Yamagishi [3] considers a more general orbifold $(\mathbb{P}^n, D)$, where the $D_i$ are as above, and

$$D = \sum_{i=0}^{n+1} (1 - \frac{1}{m_i})D_i$$

for integers $m_0, \ldots, m_{n+1} \geq 2$. Their main result is an asymptotic formula for the number of Campana points on this orbifold (with the same height as in (1.1)), under the assumption that there exists some $j \in \{0, \ldots, n+1\}$ such that

$$\sum_{0 \leq i \leq n+1} \frac{1}{m_i(m_i + 1)} \geq 1.$$ 

Following the formulation of the PSTV-A conjecture, several further cases have been treated. Pieropan and Schindler [15] establish the PSTV-A conjecture for complete smooth split toric varieties satisfying an additional technical assumption, by developing a very general version of the hyperbola method. Xiao [23] treats the case of biequivariant compactifications of the Heisenberg
group over $\mathbb{Q}$, using the height zeta function method. Finally, Streeter [21] studies $m$-full values of norm forms by counting Campana points on the orbifold $(\mathbb{P}^{d-1}_K, (1-\frac{1}{m})V(N_{E/K}))$, where $K$ is a number field, $V(N_{E/K})$ is the divisor cut out by a norm form associated to a degree-$d$ Galois extension $E/K$, and $m \geq 2$ is an integer which is coprime to $d$ if $d$ is not prime.

In [16], [15] and [23], the leading constants for the counting problems considered were reconciled with the prediction from the PSTV-A conjecture. In the case of Campana points for norm forms, Streeter [21, Section 7.3] provides an example where the leading constant in [21, Theorem 1.4] differs from the constant defined in the PSTV-A conjecture. It remains unclear whether this could be explained by the removal of a thin set. For the papers [2], [22] and [3], however, no subsequent attempts to compare the leading constants have been made. In this paper, we attempt to remedy this by making a detailed study of the leading constant from [2] in the context of the PSTV-A conjecture.

We now summarise the approach employed by Van Valckenborgh in the proof of [22, Theorem 1.1]. We can write each nonzero squareful number $z_i$ uniquely in the form $x_i^2y_i^3$ for a positive integer $x_i$ and a squarefree integer $y_i$. For a fixed choice of $y = (y_0, \ldots, y_{n+1}) \in (\mathbb{Z}_{\neq 0})_{\text{prim}}^{n+2}$, the equation $z_0 + \cdots + z_n = z_{n+1}$ can be viewed as a quadric $Q_y$ in $\mathbb{P}^{n+1}$ defined by the equation

$$y_0^3x_0^2 + \cdots + y_n^3x_n^2 = y_{n+1}^3x_{n+1}^2.$$

Using the circle method, one can estimate the number $N_y^+(B)$ of rational points $[x_0 : \cdots : x_{n+1}]$ on $Q_y$ with $(x_0, \ldots, x_{n+1}) \in (\mathbb{Z}_{\neq 0})_{\text{prim}}^{n+2}$, satisfying the conditions $\gcd(x_0y_0, \ldots, x_{n+1}y_{n+1}) = 1$ and $|x_i^2y_i^3| \leq B$ for all $i \in \{0, \ldots, n+1\}$. Now

$$N_n(B) = \frac{1}{2n+2} \sum_{y \in (\mathbb{Z}_{\neq 0})^{n+2}} \mu^2(y_0) \cdots \mu^2(y_{n+1}) N_y^+(B),$$

where $\mu$ denotes the Möbius function. The factor $\frac{1}{2n+2}$ is obtained from the factor $1/2$ in (1.2), together with the fact that for each $(z_0, \ldots, z_{n+1}) \in (\mathbb{Z}_{\neq 0})_{\text{prim}}^{n+2}$, there are $2^{n+1}$ corresponding points $[x_0 : \cdots : x_{n+1}]$ enumerated by $N_y^+(B)$, differing only by changes of signs of $x_0, \ldots, x_{n+1}$. One seeks to obtain an asymptotic formula for $N_n(B)$ by getting enough uniformity in the asymptotic formulas for $N_y^+(B)$.

With this approach, the leading constant for $N_n(B)$ is expressed as an infinite sum of constants $c_y$ arising from Manin’s conjecture applied to $N_y^+(B)$.
This is the point of view taken in [22, Section 5] for $n \geq 3$, and it is also how we express the leading constant in [19] for the case $n = 2$. When $n = 1$, it leads to the following prediction [2, Conjecture 1.1].

**Conjecture 1.1** (Browning, Van Valckenborgh, 2012). We have

$$N_1(B) \sim 3c_{BV}B^{1/2},$$

where the constant $c_{BV}$ is given explicitly in [2, (2–12)] (and also in (4.12)), and is expressed as a sum over $(y_0, y_1, y_2)$ of constants arising from Manin's conjecture applied to the conics $x_0^2y_0^3 + x_1^2y_1^3 = x_2^2y_2^3$.

The reason for the factor 3 in Conjecture 1.1 is explained in Lemma 4.1, and is due to the counting problem considered in [2] being over $\mathbb{N}_3 \text{prim}$ rather than $(\mathbb{Z}_\neq 0)^3 \text{prim}$.

By focusing on the contribution to $N^+_y(B)$ from the range $|y| \leq B^\theta$, for a small absolute constant $\theta > 0$, it is possible to prove the lower bound

$$N_1(B) \geq 3c_{BV}B^{1/2}(1 + o(1)),$$

where $c_{BV}$ is as defined in Conjecture 1.1. This is achieved in [2, Theorem 1.2], where it is also established that $c_{BV}$ takes the numerical value 2.68... correct to two decimal digits.

Throughout this paper, we let $c_{PSTV-A}$ denote the leading constant predicted by the PSTV-A conjecture [16]. For the orbifold $(\mathbb{P}^1, \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty])$ corresponding to the counting problem $N_1(B)$, there does not appear to be any obvious thin set to remove. Therefore, we might naturally expect that $c_{PSTV-A}$ is the leading constant for $N_1(B)$ itself, and consequently, in view of the lower bound in (1.3), that $c_{PSTV-A} \geq 3c_{BV}$. In Section 3 we shall prove the following result, which shows that in fact, $c_{PSTV-A} < 3c_{BV}$.

**Theorem 1.2.** For the orbifold corresponding to the counting problem $N_1(B)$, the leading constant predicted by the PSTV-A conjecture is

$$c_{PSTV-A} = \frac{9}{2\pi} \prod_p \left(1 + \frac{3p^{-3/2}}{1 + p^{-1}}\right).$$

Moreover, $c_{PSTV-A}/3 = 2.56785632...$, accurate up to eight digits.
We define
\[ C = \left\{ [z_0 : z_1] \in \mathbb{P}^1(\mathbb{Q}) : (z_0, z_1) \in \mathbb{Z}^2_{\text{prim}}, z_0, z_1, z_0 + z_1 \text{ squareful and nonzero} \right\} \]
to be the set of Campana points under consideration. If the PSTV-A conjecture is correct, the discrepancy between \( c_{\text{PSTV-A}} \) and \( 3c_{\text{BV}} \) could be explained by thin sets in one of the following two ways:

1. The set of Campana points \( C \) is itself thin, a situation which is explicitly excluded in the statement [16, Conjecture 1.1] of the PSTV-A conjecture.

2. There is a thin set \( T \subset C \) of Campana points such that the removal of \( T \) from the count \( N_1(B) \) reduces the leading constant to \( c_{\text{PSTV-A}} \).

Recent work of Nakahara and Streeter [12] tackles the question of when the set of Campana points corresponding to a log Fano orbifold \((\mathbb{P}^n, D)\) can be a thin set. The authors establish a connection between thin sets of Campana points and weak approximation, in the spirit of Serre’s arguments in [18, Theorem 3.5.7]. Together with [12, Corollary 1.4], this implies that \( C \) is not itself thin. It remains to consider whether the second explanation above could hold. In Section 5, we prove the following result.

**Theorem 1.3.** Suppose that Conjecture 1.1 holds. Let the height function \( H \) be as defined in (1.1) for \( n = 1 \). Then for any real number \( \lambda \in (0, 3c_{\text{BV}}] \), there is a Campana thin subset \( T \subset C \), as defined in [16, Definition 3.7], such that
\[ \# \{ z \in C \setminus T : H(z) \leq B \} \sim \lambda B^{1/2}. \]

Theorem 1.3 demonstrates that if Conjecture 1.1 holds, we can obtain any leading constant in \((0, 3c_{\text{BV}}]\), including the constant \( c_{\text{PSTV-A}} \), by the removal of an appropriate thin set. From this point of view, the PSTV-A conjecture as stated in [16] seems somewhat unsatisfactory, in that all points can lie on accumulating thin subsets. However, there does not appear to be any thin set with a clear geometric meaning which we can remove in order to obtain the constant \( c_{\text{PSTV-A}} \), and so currently \( 3c_{\text{BV}} \) seems the most natural prediction for the leading constant in this example.

**Remark 1.4.** We have considered \( N_1(B) \) for simplicity, but it seems likely that similar statements hold for \( N_n(B) \) with \( n \geq 2 \). In these cases, as mentioned
above, we know that the analogue of Conjecture 1.1 holds, and so we should be able to obtain unconditional analogues of Theorem 1.3 for any $n \geq 2$.

Motivated by the above example, in Section 6 we carry out a similar comparison in the case of squareful values of a binary quadratic form. For fixed positive, squarefree and coprime integers $a, b$ satisfying $a, b \equiv 1 \pmod{4}$, we consider the counting problem

$$N(B) = \frac{1}{2} \# \{(x, y) \in \mathbb{Z}_{\text{prim}}^2 : |x|, |y| \leq B, ax^2 + by^2 \text{ squareful}\}.$$  

This corresponds to the Campana orbifold $(X, D) = (\mathbb{P}^1, \frac{1}{2}V(ax^2 + by^2))$ over $\mathbb{Q}$, together with the obvious $\mathbb{Z}$-model $(\mathcal{X}, \mathcal{D})$, and the height $H$ on $\mathbb{P}^1(\mathbb{Q})$ given by $H([x : y]) = \max(|x|, |y|)$ for $(x, y) \in \mathbb{Z}_{\text{prim}}^2$. By [12, Theorem 1.1] and [12, Proposition 3.15], the set of Campana points in this example is not itself thin. In Theorem 6.1 we compute the constant $c_{\text{PSTV-A}}$ for this example.

In Section 6, we also prove the following theorem, which can be thought of as an unconditional analogue of Conjecture 1.1 for the counting problem $N(B)$.

**Theorem 1.5.** For any $\epsilon > 0$, we have $N(B) = cB + O(B^{85/86+\epsilon})$, where the implied constant depends only on $a, b$ and $\epsilon$. The leading constant $c$ is given explicitly in (6.19) as a sum over $v$ of constants arising from Manin’s conjecture applied to the conics $ax^2 + by^2 = u^2v^3$.

**Remark 1.6.** When $a = 1$, $N(B)$ counts squareful values of the norm form $x^2 + by^2$. This is a very special case of a result by Streeter [21, Theorem 1.4]. The constant from [21, Theorem 1.4] and the constant $c$ from Theorem 1.5 must therefore agree. However, the proof of [21, Theorem 1.4] proceeds via very different methods, using height zeta functions and Fourier analysis, leading to a constant that involves a sum of limits of global Fourier transforms of 2-torsion toric characters.

The constants $c$ and $c_{\text{PSTV-A}}$ are often not equal. In the norm form case $a = 1$, we show that $c_{\text{PSTV-A}} < c$ whenever $b > 1$. Analogously to Theorem 1.3, any constant in $(0, c]$ could be obtained by the removal of an appropriate thin set. When $a, b > 1$, however, we shall show that sometimes $c < c_{\text{PSTV-A}}$. 
The significance of this is that thin sets cannot explain the discrepancy between the constants. Thus Theorem 1.5 provides the basis for the following counterexample to the leading constant predicted by the PSTV-A conjecture.

**Corollary 1.7.** Let $a = 37$ and $b = 109$. Then the PSTV-A conjecture does not hold for the orbifold $(\mathcal{X}, \mathcal{D})$ and the height $H$ defined above.

2. The Manin-type conjecture for Campana points

In this section, we recall from [16] the definition of Campana points and the statement of the PSTV-A conjecture. Throughout this section, we work over a number field $K$.

**Definition 2.1.** A *Campana orbifold* is a pair $(X, D)$, where $X$ is a smooth variety over $K$ and

$$D = \sum_{\alpha \in \mathcal{A}} \epsilon_{\alpha} D_{\alpha}$$

is an effective Weil $\mathbb{Q}$-divisor of $X$ over $K$ (where the $D_{\alpha}$ are prime divisors) such that

1. For all $\alpha \in \mathcal{A}$, either $\epsilon_{\alpha} = 1$ or $\epsilon_{\alpha}$ takes the form $1 - 1/m_{\alpha}$ for some $m_{\alpha} \in \mathbb{Z}_{\geq 2}$.
2. The support $D_{\text{red}} = \sum_{\alpha \in \mathcal{A}} D_{\alpha}$ of $D$ has strict normal crossings on $X$.

We say that a Campana orbifold is *klt* if $\epsilon_{\alpha} \neq 1$ for all $\alpha \in \mathcal{A}$.

Let $(X, D)$ be a Campana orbifold. Campana points will be defined as points $P \in X(K)$ satisfying certain conditions. These conditions are dependent on a finite set $S$ of places of $K$ containing all archimedean places, and a choice of *good integral model* of $(X, D)$ over $\mathcal{O}_{K,S}$. This model is defined to be a pair $(\mathcal{X}, \mathcal{D})$, where $\mathcal{X}$ is a flat, proper model of $X$ over $\mathcal{O}_{K,S}$, with $\mathcal{X}$ regular, and

$$\mathcal{D} = \sum_{\alpha \in \mathcal{A}} \epsilon_{\alpha} \mathcal{D}_{\alpha},$$

where $\mathcal{D}_{\alpha}$ denotes the Zariski closure of $D_{\alpha}$ in $\mathcal{X}$.

**Definition 2.2.** Let $P \in (X \setminus D_{\text{red}})(K)$. For a place $v \notin S$, let $\mathcal{P}_v$ denote the induced point in $\mathcal{X}(\mathcal{O}_v)$ obtained via the valuative criterion for properness, as stated in [10] Thm. II.4.7. For $\alpha \in \mathcal{A}$, we define the *intersection multiplicity*
n_v(\mathcal{D}_\alpha, P) of \mathcal{D}_\alpha and P at v to be the colength of the ideal \mathcal{P}_v \mathcal{D}_\alpha in \mathcal{O}_v. The intersection number of P and \mathcal{D} at v is defined to be

\[ n_v(\mathcal{D}, P) = \sum_{\alpha \in \mathcal{A}} \epsilon_{\alpha} n_v(\mathcal{D}_\alpha, P). \]

**Definition 2.3.** Let \((X, D)\) be a Campana orbifold with a good integral model \((\mathcal{X}, \mathcal{D})\) over \(\mathcal{O}_{K,S}\). A point \(P \in (X \setminus \red D)(K)\) is a Campana \(\mathcal{O}_{K,S}\)-point of \((\mathcal{X}, \mathcal{D})\) if for all \(v \notin S\) and all \(\alpha \in \mathcal{A}\), we have

1. If \(\epsilon_{\alpha} = 1\), then \(n_v(\mathcal{D}_\alpha, P) = 0\).
2. If \(\epsilon_{\alpha} \neq 1\), so that \(\epsilon_{\alpha} = 1 - 1/m_{\alpha}\) for some \(m_{\alpha} \in \mathbb{Z}_{\geq 2}\), then either \(n_v(\mathcal{D}_\alpha, P) = 0\) or \(n_v(\mathcal{D}_\alpha, P) \geq m_{\alpha}\).

We denote the set of Campana \(\mathcal{O}_{K,S}\)-points of \((\mathcal{X}, \mathcal{D})\) by \((\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})\).

**Example 2.4.** When \(K = \mathbb{Q}\), Campana points are related to \(m\)-full values of polynomials. We consider projective space \(X = \mathbb{P}^n\), and a strict normal crossings divisor

\[ D = \sum_{i=0}^{k} \left(1 - \frac{1}{m_i}\right) D_i, \]

where \(m_i \geq 2\) are integers, and \(D_i\) are prime divisors on \(X\) defined by irreducible polynomials \(f_i\) with integral coefficients. Choosing the obvious good integral model \((\mathcal{X}, \mathcal{D})\), a rational point \(z \in (X \setminus \bigcup_{i=0}^{k} D_i)(\mathbb{Q})\), represented by \((z_0, \ldots, z_n) \in \mathbb{Z}_{\text{prim}}^{n+1}\), is a Campana \(\mathbb{Z}\)-point of \((\mathcal{X}, \mathcal{D})\) if and only if \(f_i(z_0, \ldots, z_n)\) is \(m_i\)-full for all \(i \in \{0, \ldots, k\}\). In particular, the Campana points \(C\) defined in (1.5) fit into this context, by making the choices \(X = \mathbb{P}^1, k = 2, m_0 = m_1 = m_2 = 2\), and \(f_0 = z_0, f_1 = z_1, f_2 = z_0 + z_1\).

**Definition 2.5.** We recall that for an irreducible variety \(X\) over \(K\), a subset \(A \subset X(K)\) is type I if \(A = Z(K)\), where \(Z\) is a proper closed subvariety of \(X\), and type II if \(A = \varphi(V(K))\), where \(V\) is an integral projective variety with \(\dim(V) = \dim(X)\) and \(\varphi: V \to X\) is a dominant morphism of degree at least 2. A thin set of \(X(K)\) is a subset of \(X(K)\) which is contained in a finite union of type I and type II sets. In [13, Definition 3.7], a thin set of Campana \(\mathcal{O}_{K,S}\)-points is defined to be the intersection of a thin set of \(X(K)\) with the set of Campana points \((\mathcal{X}, \mathcal{D})(\mathcal{O}_{K,S})\).
We now come to the statement of the PSTV-\(A\) conjecture given in [16, Conjecture 1.1]. Let \((X, D)\) be a Campana orbifold over \(K\) with a good integral model \((\mathcal{X}, \mathcal{D})\) over \(\mathcal{O}_{K, S}\). Let \((\mathcal{L}, \| \cdot \|)\) be an adelically metrized big and nef line bundle on \(X\) with associated divisor class \([L]\). Let \(H_L: X(K) \to \mathbb{R}_{\geq 0}\) denote the corresponding height function, as defined in [13, Section 1]. We recall that the effective cone \(\Lambda_{\text{eff}}\) of a variety \(X\) is defined as
\[
\Lambda_{\text{eff}} = \{ [D] \in \text{Pic}(X) \otimes \mathbb{R} : [D] \geq 0 \}.
\]

**Definition 2.6.** Let \([K_X]\) denote the canonical divisor class. Given the above data, we define
\[
a = \inf \{ t \in \mathbb{R} : t[L] + [K_X] + [D] \in \Lambda_{\text{eff}} \},
\]
and we define \(b\) to be the codimension of the minimal supported face of \(\Lambda_{\text{eff}}\) which contains \(a[L] + [K_X] + [D]\).

**Conjecture 2.7** (Pieropan, Smeets, Tanimoto, Várilly-Alvarado). Let \((X, D)\) be a klt Campana orbifold, and suppose that \(-(K_X + D)\) is ample (in this case we say that the orbifold is *Fano*). Assume that the set of Campana points \((\mathcal{X}, \mathcal{D})(\mathcal{O}_{K, S})\) is not itself thin. Then there is a thin set \(\mathcal{T}\) of Campana \(\mathcal{O}_{K, S}\)-points such that
\[
\# \{ P \in (\mathcal{X}, \mathcal{D})(\mathcal{O}_{K, S}) \setminus \mathcal{T} : H_{\mathcal{X}}(P) \leq B \} \sim c_{\text{PSTV-A}} B^a (\log B)^{b-1},
\]
as \(B \to \infty\), where \(a, b\) are as in Definition 2.6 and \(c_{\text{PSTV-A}} > 0\) is an explicit constant described in [16, Section 3.3].

### 3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We keep the notation from the previous section. We recall from Example 2.4 that the Campana orbifold corresponding to \(N_1(B)\) is the orbifold \((\mathbb{P}^1, D)\) defined over \(\mathbb{Q}\), where \(D\) is the divisor \(\sum_{i=0}^{2} \frac{1}{2} D_i\) and
\[
D_0 = \{ z_0 = 0 \}, \quad D_1 = \{ z_1 = 0 \}, \quad D_2 = \{ z_0 + z_1 = 0 \}.
\]
We choose \(S = \{ \infty \}\) for the set of bad places, and fix the obvious model of \((\mathbb{P}^1, D)\) over \(\mathbb{Z}\). We shall work with the height
\[
H(z) = \max(|z_0|, |z_1|, |z_0 + z_1|)
\]
for \((z_0, z_1) \in \mathbb{Z}_{\text{prim}}^2\) representing \(z\). This choice of height corresponds to the ample line bundle \(\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)\), metrized by the generating set \(\{z_0, z_1, z_0 + z_1\}\) for the global sections of \(\mathcal{L}\).

The degree function gives an isomorphism \(\text{Pic}(\mathbb{P}^1) \cong \mathbb{Z}\). Under this isomorphism, the line bundle \(\mathcal{L}\) maps to 1 and \(\Lambda_{\text{eff}}\) is identified with \(\mathbb{R}_{\geq 0}\). Since \(\deg D = 3/2\) and \(\deg[K_{\mathbb{P}^1}] = -2\), we have
\[
a = \inf \left\{ t \in \mathbb{R} : t - 2 + \frac{3}{2} \geq 0 \right\} = \frac{1}{2}.
\]
The minimal supported face of \(\Lambda_{\text{eff}}\) which contains \(a[L] + [K_{\mathbb{P}^1}] + [D] = 0\) is \(\{0\}\), which has codimension 1 in \(\Lambda_{\text{eff}}\), and so \(b = 1\). These values of \(a\) and \(b\) are compatible with Conjecture 1.1.

We now turn our attention to the definition of the leading constant \(c_{\text{PSTV-A}}\), and its computation for the orbifold and height function corresponding to \(N_1(B)\). For a description of \(c_{\text{PSTV-A}}\) in full generality, we refer the reader to [16, Section 3.3]. Here, for simplicity, we define \(c_{\text{PSTV-A}}\) in the case when \(X\) is a smooth projective variety over \(\mathbb{Q}\) satisfying \(a[L] + [K_X] + [D] = 0\) (this latter hypothesis in particular holds when \(\text{Pic}(X) \cong \mathbb{Z}\)). These assumptions certainly hold in the setting of Theorem 1.2, where we take \(X = \mathbb{P}^1\). The constant \(c_{\text{PSTV-A}}\) is given by the formula
\[
(3.2) \quad c_{\text{PSTV-A}} = \frac{\alpha \beta \tau}{a(b-1)!},
\]
and we proceed to discuss each of the factors \(\alpha, \beta, \tau\) in turn.

Let \(\varrho\) denote the rank of \(\text{Pic}(X)\). The dual effective cone \(\Lambda_{\text{eff}}^*\) is defined as
\[
\Lambda_{\text{eff}}^* = \{ y \in (\text{Pic}(X) \otimes \mathbb{Z})^* : \langle y, r \rangle \geq 0 \text{ for all } r \in \Lambda_{\text{eff}} \}.
\]
Here \(\Lambda_{\text{eff}}^* \cong (\mathbb{R}^\varrho)^* \cong \text{Hom}_R(\mathbb{R}^\varrho, \mathbb{R})\) is the usual vector space dual, and \(\langle \cdot, \cdot \rangle\) is the tautological pairing defined by \(\langle y, r \rangle = y(r)\).

The definition of \(\alpha\) from [16, Section 3.3] is closely related to the \(\alpha\)-constant from the classical Manin conjecture. In general, the definition involves a rigid effective divisor \(E\) which is \(\mathbb{Q}\)-linearly equivalent to \(aL + K_X + D\). However, if \((X, D)\) is any Campana orbifold with \(E = 0\) and we write \(D = \sum_{i=0}^k \epsilon_i D_i\) for prime divisors \(D_i\), then the definition of \(\alpha\) simplifies to
\[
(3.3) \quad \alpha = \prod_{i=0}^k (1 - \epsilon_i) \int_{\Lambda_{\text{eff}}^*} e^{-\langle[L], x \rangle} dx.
\]
In our example, \( k = 2 \) and \( \epsilon_i = 1/2 \) for \( 0 \leq i \leq 2 \). Since \( \deg L = 1 \), we have \( \langle [L], x \rangle = x \). Therefore
\[
\alpha = \left( \frac{1}{2} \right)^3 \int_0^{\infty} e^{-x} \, dx = \frac{1}{8}.
\]

When \( a[L] + [K_X] + [D] = 0 \), the constant \( \beta \) from [16, Section 3.3] agrees with the definition of \( \beta \) in Manin’s conjecture. The \( \beta \)-constant plays no rôle in our example, since \( \beta = 1 \) whenever \( \text{Pic}(X) \cong \mathbb{Z} \) (see for example [11, Definition 5.12, Remark 5.13]).

Substituting \( a = \frac{1}{2}, b = 1, \alpha = \frac{1}{8} \) and \( \beta = 1 \) into (3.2), in our example, we conclude that
\[
(3.4) \quad c_{\text{PSTV-A}} = \tau_4.
\]

We now describe the Tamagawa number \( \tau \). Again, we do not give the definition in full generality, but assume for simplicity that \( a[L] + [K_X] + [D] = 0 \). It follows from [16, Section 3.3] that
\[
(3.5) \quad \tau = \int_{\mathcal{U}(\mathbb{A}_Q)} H(x, 0)^{-1} \, d\tau_{X,D}.
\]

We explain the notation used in this equation. In the integrand, \( H(x, 0) \) denotes the height of \( x \) with respect to the zero line bundle, and so this is identically 1. In [16, Section 3.3], two alternative definitions of \( \mathcal{U}(\mathbb{A}_Q) \) are given. The first is as a topological closure of the Campana \( \mathcal{O}_{K,S} \)-points of \( (X, \mathcal{D}) \) in \( X(\mathbb{A}_Q) \), and the second is in terms of the Brauer–Manin pairing. In general, it is not known whether the two definitions coincide, but in our situation the definitions do agree since there is no Brauer–Manin obstruction. Finally, the measure \( \tau_{X,D} \) is defined to be \( H_D \tau_X \), where \( \tau_X \) is the usual Tamagawa measure from Manin’s conjecture as defined in [13, Section 2], and \( H_D \) is a height function associated to \( D \) defined as follows. We write \( D = \sum_{i=0}^{k} \epsilon_i D_i \) for prime divisors \( D_i \). We fix an adelic metrization on the line bundles \( \mathcal{O}_X(D_i) \) associated to each of the divisors \( D_i \). This induces a height \( H_{D_i} \) as described in [13, Définition 1.2]. We then define
\[
H_D = \prod_{i=1}^{k} H_{D_i}^{\epsilon_i}.
\]
Below, we let \( p \) denote any prime (or any non-archimedean place of \( \mathbb{Q} \)), and we let \( v \) denote any place of \( \mathbb{Q} \), including the archimedean place \( v = \infty \). We let \( dx_{i,p} \) denote the usual \( p \)-adic measure with respect to \( x_i \), and \( dx_{i,\infty} \) denote the usual Lebesgue measure. We denote by \( \mathcal{K}_X \) the canonical line bundle of \( X \). In the notation of \([13, \text{Section 2}]\), we obtain

\[
\tau_{X,D} = H_D \omega_\infty \prod_p \det(1 - p^{-1} \text{Frob}_p | \text{Pic}(\mathcal{X}^{[p]})) \omega_p,
\]

where

\[
(3.6) \quad \omega_v = \frac{|dz_1, v \wedge \cdots \wedge dz_n, v|}{\|dz_1 \wedge \cdots \wedge dz_n\|_{\mathcal{K}_X,v}}.
\]

We now use the definitions above to compute \( \tau \) in our example. We have

\[
\det(1 - p^{-1} \text{Frob}_p | \text{Pic}(\mathcal{X}^{[p]})) = 1 - p^{-1}
\]

for all primes \( p \). In fact, this is true whenever \( \text{Pic}(X_{\mathbb{Q}}) \cong \mathbb{Z} \) \([11, \text{Remark 6.10}]\). Fixing \( i \in \{0, 1, 2\} \) and writing \( z_2 = z_0 + z_1 \), we define sections \( s_{D_i} = z_i \). We take the metrization on \( \mathcal{O}_{\mathbb{P}^1}(D_i) \) obtained from pulling back the metrization on \( \mathcal{O}_{\mathbb{P}^1}(1) \) via the obvious isomorphism \( \mathcal{O}_{\mathbb{P}^1}(D_i) \cong \mathcal{O}_{\mathbb{P}^1}(1) \). Since we are using the metrization on \( \mathcal{O}_{\mathbb{P}^1}(1) \) arising from the generating set \( \{z_0, z_1, z_2\} \), we obtain

\[
H_{D_i}(z) = \prod_v \|s_{D_i}(z)\|_v^{-1} = \frac{\max(|z_0|, |z_1|, |z_2|)}{|z_i|}
\]

on \( (\mathbb{P}^1 \setminus D_i)(\mathbb{Q}) \), for \( (z_0, z_1) \in \mathbb{Z}_{\text{prim}}^2 \) representing \( z \). Therefore,

\[
(3.7) \quad H_D(z) = \frac{\max(|z_0|, |z_1|, |z_2|)^{3/2}}{|z_0 z_1 z_2|^{1/2}}
\]

on the open set \( (\mathbb{P}^1 \setminus \text{supp}(D))(\mathbb{Q}) \), where \( \text{supp}(D) = D_0 \cup D_1 \cup D_2 \).

The property that \( z \in (X \setminus \text{supp}(D))(\mathbb{Q}) \) is a Campana point is a local condition. In our example, it is the condition that for all primes \( p \), we have

\[
\nu_p(z_0), \nu_p(z_1), \nu_p(z_0 + z_1) \neq 1
\]

for \( (z_0, z_1) \in \mathbb{Z}_{\text{prim}}^2 \) representing \( z \), where \( \nu_p \) denotes the \( p \)-adic valuation. Let \( \Omega_p \) denote the subset of \( \mathbb{P}^1(\mathbb{Q}_p) \) cut out by this local condition, and define \( \Omega_\infty = \mathbb{P}^1(\mathbb{R}) \). The expression \((3.5)\) becomes

\[
(3.8) \quad \tau = \sigma_\infty \prod_p (1 - p^{-1}) \sigma_p,
\]
where
\[ \sigma_v = \int_{\Omega_v} \frac{\max(|z_0|_v, |z_1|_v, |z_0 + z_1|_v)^{3/2}}{|z_0z_1(z_0 + z_1)|^{1/2}} \mathrm{d}\omega_v. \]

To compute \( \sigma_v \), we use the chart \( U_v = \{ [t : 1] : t \in \mathbb{Q}_v \} \), equipped with the natural maps \( f_v : U_v \to \mathbb{Q}_v \) given by \([t : 1] \mapsto t\). The only point on \( \Omega_v \) not in \( U_v \) is \([1 : 0]\), which has measure zero, and so we may replace the range of integration with \( \Omega_v \cap U_v \). A point \([z_0 : z_1]\) on \( U_v \) satisfies \( t = z_0 / z_1 \). Let \( dt \) denote the usual \( p \)-adic measure or the Lebesgue measure as appropriate. We recall that there is an isomorphism \( \mathcal{X}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \), which on the chart \( U_v \) is given by mapping \( dt \) to \( z_1^{-2} \). Therefore, in the notation of [13 Section 2], we have
\[ \|dt\|_{\mathcal{X}_{\mathbb{P}^1},v} = \|z_1^{-2}\|_{\mathcal{O}_{\mathbb{P}^1}(-2),v} = \frac{|z_1|^{-2}}{\max(|z_0|_v, |z_1|_v, |z_0 + z_1|_v)^{-2}}. \]
Recalling (3.6), we obtain
\[ H_{D,v}\omega_v = \frac{\max(|z_0|_v, |z_1|_v, |z_0 + z_1|_v)^{3/2}|z_1|_v^2}{|z_0z_1(z_0 + z_1)|^{1/2} \max(|z_0|_v, |z_1|_v, |z_0 + z_1|_v)^2} \int dt \\
= \frac{dt}{|t(1 + t)|^{1/2} \max(|t|_v, 1, |1 + t|_v)^{1/2}}. \]
(3.9)
When \( v = \infty \), we have \( f_v(\Omega_v \cap U_v) = \mathbb{R} \). Therefore,
\[ \sigma_\infty = \int_{\mathbb{R}} \frac{dt}{|t(1 + t)|^{1/2} \max(|t|, 1, |1 + t|)^{1/2}} \\
= \int_{-\infty}^{-1} \frac{dt}{|1 + t|^{1/2}|t|} + \int_{-1}^{0} \frac{dt}{|t(1 + t)|^{1/2}} + \int_{0}^{\infty} \frac{dt}{t^{1/2}(1 + t)}. \]
Each of these integrals is equal to \( \pi \), and so we conclude that \( \sigma_\infty = 3\pi \). In the following lemma, we compute \( \sigma_v \) when \( v < \infty \).

**Lemma 3.1.** We have \( \sigma_p = 1 + p^{-1} + 3p^{-3/2} \).

**Proof.** We recall that \( \Omega_p \) consists of the points \([z_0 : z_1] \in \mathbb{P}^1(\mathbb{Q}_p) \) such that \( \min(\nu_p(z_0), \nu_p(z_1)) = 0 \) and \( \nu_p(z_0), \nu_p(z_1), \nu_p(z_0 + z_1) \neq 1 \). From this, we see that \( f_p(\Omega_p \cap U_p) \) is the set of all \( t \in \mathbb{Q}_p \) which satisfy the conditions \( t, t + 1 \neq 0 \) and \( \nu_p(t), \nu_p(1 + t) \neq \pm 1 \). Therefore,
\[ \sigma_p = \int_{\nu_p(t), \nu_p(1 + t) \neq \pm 1} \frac{dt}{|t(1 + t)|^{1/2} \max(|t|_p, 1, |1 + t|_p)^{1/2}}. \]
(3.10)
By the ultrametric triangle inequality, \( \max(1, |t|_p, |1 + t|_p) = \max(1, |t|_p) \).
We now consider separately the contribution to the integral from the regions \( R_1, R_2, R_3 \) defined respectively by the conditions

1. \( \nu_p(t) \geq 2 \),
2. \( \nu_p(t) = 0 \),
3. \( \nu_p(t) < -2 \).

In the region \( R_1 \), we have \( |1 + t|_p = 1 \) and \( \max(|t|_p, 1) = 1 \). We recall also that for any \( j \in \mathbb{Z} \), the \( p \)-adic measure of the set of \( t \in \mathbb{Q}_p \) with \( \nu_p(t) = j \) is \( (1 - p^{-1})p^{-j} \). Hence the contribution to (3.10) from \( R_1 \) is

\[
\int_{t \in \mathbb{Q}_p \atop \nu_p(t) \geq 2} \frac{dt}{|t|_p^{1/2}} = \sum_{j=2}^{\infty} (1 - p^{-1})p^{-j/2} = p^{-1} + p^{-3/2}.
\]

In the region \( R_2 \), we have \( \max(1, |t|_p) = 1 \). We further subdivide this region according to the value of \( \nu_p(1 + t) \), remembering that the case \( \nu_p(1 + t) = 1 \) must be excluded. We define

\[
S_j = \{ t \in \mathbb{Z}_p^\times : \nu_p(1 + t) = j \}.
\]

When \( j < 0 \), we have \( S_j = \emptyset \). When \( j = 0 \), the measure of \( S_j \) is \( 1 - 2p^{-1} \), because \( t \in S_0 \) if and only if the reduction of \( t \) modulo \( p \) is not 0 or \(-1\). (In the case \( p = 2 \), we have \( 1 - 2p^{-1} = 0 \), which is consistent with the fact that it is not possible for \( t \) and \( 1 + t \) to both be in \( \mathbb{Z}_2^\times \).) When \( j \geq 2 \), elements \( t \in S_j \) are precisely elements of the form \( t = -1 + s \) for some \( s \in \mathbb{Q}_p \) with \( \nu_p(s) = j \), and so \( S_j \) has measure \( p^{-j}(1 - p^{-1}) \). We conclude that the contribution to (3.10) from the region \( R_2 \) is

\[
\int_{t \in \mathbb{Z}_p^\times \atop \nu_p(1 + t) \neq \pm 1} \frac{dt}{|1 + t|_p^{1/2}} = 1 - 2p^{-1} + \sum_{j=2}^{\infty} (1 - p^{-1})p^{-j/2} = 1 - p^{-1} + p^{-3/2}.
\]

Finally, in the region \( R_3 \), we have \( |1 + t|_p = 1 \) and \( \max(1, |t|_p) = |t|_p \), and so we obtain a contribution from \( R_3 \) of

\[
\int_{t \in \mathbb{Q}_p \atop \nu_p(t) \leq -2} \frac{dt}{|t|_p^{3/2}} = \sum_{j=2}^{\infty} (1 - p^{-1})p^{-j/2} = p^{-1} + p^{-3/2}.
\]

Combining the three regions, we conclude that

\[
\sigma_p = (p^{-1} + p^{-3/2}) + (1 - p^{-1} + p^{-3/2}) + (p^{-1} + p^{-3/2}) = 1 + p^{-1} + 3p^{-3/2},
\]
as required. □
We now complete the proof of Theorem 1.2. We recall that \( c_{\text{PSTV-A}} = \frac{\tau}{4} \), and \( \sigma_{\infty} = 3\pi \). Together with Lemma 3.1 and (3.8), this implies that

\[
c_{\text{PSTV-A}} = \frac{1}{4} \sigma_{\infty} \prod_{p} (1 - p^{-1}) \sigma_{p}
\]

\[
= \frac{1}{4} \cdot 3\pi \prod_{p} \left( 1 + 3p^{-3/2} - p^{-2} - 3p^{-5/2} \right)
\]

\[
= \frac{1}{4} \cdot 3\pi \prod_{p} \left( 1 + \frac{3p^{-3/2}}{1 + p^{-1}} \right) (1 - p^{-2}).
\]

Since \( \prod_{p}(1 - p^{-2}) = 1/\zeta(2) = 6/\pi^2 \), we obtain the expression for \( c_{\text{PSTV-A}} \) claimed in (1.4).

In order to estimate the numerical value of \( c_{\text{PSTV-A}} \), we evaluate the Euler product \( \prod_{p}(1 - p^{-1}) \sigma_{p} \) by removing convergence factors. Using (3.11) we have

\[
\prod_{p}(1 - p^{-1}) \sigma_{p} = \prod_{p} \left( 1 + \frac{3p^{-3/2}}{1 + p^{-1}} \right) (1 - p^{-2})
\]

\[
= \zeta(3/2)^3 \cdot \frac{\zeta(4)}{\zeta(2)} \cdot \left( \frac{\zeta(5)}{\zeta(5/2)} \right)^3 \prod_{p} f(p),
\]

where \( f(p) = 1 + O(p^{-3}) \) is an explicit polynomial in \( p^{-1} \). The resulting Euler product now converges quickly enough to obtain an approximation for \( c_{\text{PSTV-A}} \) accurate to eight decimal digits by taking the product over the first 1000 primes.

4. MANIN’S CONJECTURE FOR THE FAMILY OF CONICS

In this section, we describe the alternative approach of Browning and Van Valckenborgh [2, Section 2] for predicting the leading constant for the counting problem \( N_1(B) \) from (1.2). The counting function considered in [2] is given by

\[
\widetilde{N}_1(B) = \# \left\{ (z_0, z_1, z_2) \in \mathbb{N}_{\text{prim}}^3 : z_0 + z_1 = z_2, \ z_0, z_1, z_2 \leq B, \ z_0, z_1, z_2 \text{ squareful} \right\}.
\]

This is very similar to \( N_1(B) \), the only differences being the presence of the factor \( 1/2 \) in (1.2), and that in \( \widetilde{N}_1(B) \) we require \( (z_0, z_1, z_2) \in \mathbb{N}_{\text{prim}}^3 \), whilst in \( N_1(B) \) we only require \( (z_0, z_1, z_2) \in (\mathbb{Z}_{\neq 0})^3_{\text{prim}} \). The following lemma compares \( N_1(B) \) with \( \widetilde{N}_1(B) \).

**Lemma 4.1.** We have \( N_1(B) = 3\widetilde{N}_1(B) \).
Proof. For convenience, we use the notation $\tilde{S}_1(B)$ to mean the set which $\tilde{N}_1(B)$ enumerates. For $\epsilon \in \{\pm 1\}^3$, we define

$$S_\epsilon(B) = \left\{(z_0, z_1, z_2) \in (\mathbb{Z}_{\neq 0})^3_{\text{prim}} : \begin{array}{l} z_0 + z_1 = z_2, |z_i| \leq B \text{ for all } i \\ z_i \text{ squareful}, \text{sgn}(z_i) = \epsilon_i \text{ for all } i \end{array} \right\},$$

and $N_\epsilon(B) = \#S_\epsilon(B)$. Then

$$2N_1(B) = \sum_{\epsilon \in \{\pm 1\}^3} N_\epsilon(B). \quad (4.1)$$

For $\epsilon = (1, 1, -1)$ or $\epsilon = (-1, -1, 1)$, we have $N_\epsilon(B) = 0$. For $\epsilon = (1, 1, 1)$ or $\epsilon = (-1, -1, -1)$, we have $N_\epsilon(B) = \tilde{N}_1(B)$, and so these choices of $\epsilon$ contribute $2\tilde{N}_1(B)$ to the sum in (4.1).

For the remaining four choices of $\epsilon$, it can be checked that there is a permutation $\sigma \in S_3$ such that the map

$$S_\epsilon(B) \to \tilde{S}_1(B)$$

$$(z_0, z_1, z_2) \mapsto \sigma(|z_0|, |z_1|, |z_2|)$$

is a bijection. Therefore $N_\epsilon(B) = \tilde{N}_1(B)$, and these choices of $\epsilon$ contribute $4\tilde{N}_1(B)$ to the sum in (4.1). \qed

In the remainder of this section we record the explicit description of $c_{BV}$ from [2], and define some notation which will be useful later.

Recalling the discussion in the introduction, for a fixed $y = (y_0, y_1, y_2)$ in $(\mathbb{Z}_{\neq 0})^3$, we consider the conic $C_y$ defined by the polynomial

$$F_y(x_0, x_1, x_2) = y_0^3 x_0^2 + y_1^2 x_1^2 - y_2^3 x_2^2. \quad (4.2)$$

We define an anticanonical height $H_y$ on $C_y$ given by

$$H_y(x) = \max \left(\{|y_0^2 x_0^2|, |y_1^3 x_1^2|, |y_2^3 x_2^2|\} \right)^{1/2}, \quad (4.3)$$

where $(x_0, x_1, x_2) \in (\mathbb{Z}_{\neq 0})^3_{\text{prim}}$ represents the point $x \in C_y(\mathbb{Q})$. We define

$$N_{C_y, H_y}(B^{1/2}) = \#\{x \in C_y(\mathbb{Q}) : H_y(x) \leq B^{1/2}\},$$

and $N_{{C_y}_y, H_y}(B^{1/2})$ in the same way, but with the additional coprimality condition $\gcd(x_0 y_0, x_1 y_1, x_2 y_2) = 1$. Then

$$\tilde{N}_1(B) = \frac{1}{4} \sum_{y \in \mathbb{N}^3} \mu^2(y_0 y_1 y_2) N_{{C_y}_y, H_y}(B). \quad (4.4)$$
The presence of the factor $1/4$ in (4.4) is due to the fact that in $N^+_{C_y,H_y}(B^{1/2})$ the points $x$ we count lie in $\mathbb{P}^2(\mathbb{Q})$, which allows for four choices of sign for the coordinates of $x$ corresponding to each point $(z_0, z_1, z_2)$ enumerated by $\hat{N}_1(B)$.

As mentioned in [2, Section 3], it is easy to show that there is an absolute constant $\delta > 0$ and an explicit constant $c_{H_y}(C_y(\mathbb{A}_Q)^+)$ depending on $y$ such that

\begin{equation}
N^+_{C_y,H_y}(B^{1/2}) = c_{H_y}(C_y(\mathbb{A}_Q)^+)B^{1/2}(1 + O_y(B^{-\delta})),
\end{equation}

where the error term has at worst polynomial dependence on $y$. The constant $c_{H_y}(C_y(\mathbb{A}_Q)^+)$ is a special case of the constant conjecturally formulated by Peyre [13, Définition 2.5]. Here, $C_y(\mathbb{A}_Q)^+$ denotes the open subset of $C_y(\mathbb{A}_Q)$ given by the conditions $\min_{0 \leq i \leq 2}(\nu_p(x_iy_i)) = 0$ for all primes $p$, and is intended to reflect the coprimality condition $\gcd(x_0y_0, x_1y_1, x_2y_2) = 1$ imposed on $N^+_{C_y,H_y}(B^{1/2})$ in (4.4). The computation of $c_{H_y}(C_y(\mathbb{A}_Q)^+)$ then involves the Tamagawa measure of $C_y(\mathbb{A}_Q)^+$ in place of the full adelic space $C_y(\mathbb{A}_Q)$. In the light of (4.4), it is natural to predict that

\begin{equation}
\hat{N}_1(B) \sim c_{BV}B^{1/2},
\end{equation}

with

\begin{equation}
c_{BV} = \frac{1}{4} \sum_{y \in \mathbb{N}^3} \mu^2(y_0y_1y_2)c_{H_y}(C_y(\mathbb{A}_Q)^+).
\end{equation}

In what follows, we shall use for brevity the notation

\begin{equation}
\gamma(d) := \prod_{\substack{p|d \\ p > 2}} \left(1 + \frac{1}{p}\right)^{-1}.
\end{equation}

In [2, Section 2], it is established that

\begin{equation}
c_{H_y}(C_y(\mathbb{A}_Q)^+) = \frac{4}{\pi} \cdot \frac{\mu^2(y_0y_1y_2)\gamma(y_0y_1y_2)}{(y_0y_1y_2)^{3/2}} \sigma_{2,y}(\mathbf{y}),
\end{equation}
where
\[(4.10)\]
\[g(y) = \prod_{p | y_0 \atop p > 2} \left( 1 + \left( \frac{y_1 y_2}{p} \right) \right) \prod_{p | y_1 \atop p > 2} \left( 1 + \left( \frac{y_0 y_2}{p} \right) \right) \prod_{p | y_2 \atop p > 2} \left( 1 + \left( \frac{-y_0 y_1}{p} \right) \right),\]
\[(4.11)\]
\[\sigma_{2,y} = \lim_{r \to \infty} 2^{-2r} \# \left\{ x \in \left( \mathbb{Z}/2^r \mathbb{Z} \right)^3 : \begin{array}{c} y_0^2 x_0^2 + y_1^2 x_1^2 \equiv y_2^2 x_2^2 \pmod{2^r}, \\ \min_{0 \leq i \leq 2} \nu_2(x_i y_i) = 0 \end{array} \right\}.\]

Combining with (4.7), we conclude that
\[(4.12)\]
\[c_{BV} = \frac{1}{\pi} \sum_{y \in \mathbb{N}^3} \frac{\mu^2(y_0 y_1 y_2) y(y_0 y_1 y_2)}{(y_0 y_1 y_2)^{3/2}} \sigma_{2,y} g(y).\]

From [2, Lemma 2.2], we have the calculation
\[(4.13)\]
\[\sigma_{2,y} = \begin{cases} 1, & \text{if } 2 \nmid y_0 y_1 y_2 \text{ and } \neg \{y_0 \equiv y_1 \equiv -y_2 \pmod{4}\}, \\ 2, & \text{if } 2 \nmid y_0 \text{ and } y_1 \equiv y_2 \pmod{8}, \\ 2, & \text{if } 2 \mid y_1 \text{ and } y_0 \equiv y_2 \pmod{8}, \\ 2, & \text{if } 2 \mid y_2 \text{ and } y_0 \equiv -y_1 \pmod{8}, \\ 0, & \text{otherwise}. \end{cases}\]

As a consequence of quadratic reciprocity, it can be shown that the condition \(\neg \{y_0 \equiv y_1 \equiv -y_2 \pmod{4}\}\) is automatically satisfied whenever \(g(y) \neq 0\).

**Remark 4.2.** The expression for \(c_{BV}\) given in (4.12) is a sum of products of local densities arising from Manin’s conjecture, but it is not multiplicative in \(y\), and it does not appear possible to express \(c_{BV}\) as a single Euler product. This is in contrast to the constant \(c_{PSTV-A}\), which is defined as a product of local densities.

5. Thin sets

In this section, we prove Theorem 1.3. We recall the definition of the set of Campana points \(\mathcal{C}\) from (1.3) and the corresponding counting problem \(N_1(B)\) from (1.2), with the height \(H\) as defined in (3.1). From Definition 2.5, the Campana thin subsets of \(\mathcal{C}\) take the form \(T = T \cap \mathcal{C}\), where \(T\) is a thin subset of \(\mathbb{P}^1(\mathbb{Q})\). For a set \(S \subseteq \mathbb{P}^1(\mathbb{Q})\), we define \(N_1(S, B) = \#\{z \in S : H(z) \leq B\}\). In particular, we have \(N_1(\mathcal{C}, B) = N_1(B)\).
For fixed integers $y_0, y_1, y_2$ satisfying $\mu^2(y_0y_1y_2) = 1$, we recall that $C_y$ denotes the conic $y_0^3x_0^2 + y_1^3x_1^2 = y_2^3x_2^2$. Consider the morphism

$$\varphi_y : C_y \to \mathbb{P}^1,$$

$$[x_0 : x_1 : x_2] \mapsto [y_0^3x_0^2 : y_1^3x_1^2].$$

The image $T_y := \varphi_y(C_y)$ is a thin subset of $\mathbb{P}^1(\mathbb{Q})$. Therefore $T_y \cap \mathcal{C}$ is a thin set of Campana points. Explicitly, $T_y \cap \mathcal{C}$ is described by the set

$$\left\{ [z_0 : z_1] \in \mathbb{P}^1(\mathbb{Q}) : (z_0, z_1) \in \mathbb{Z}^2_{\text{prim}}, z_0, z_1, z_0 + z_1 \neq 0, (z_0, z_1, z_0 + z_1) = (y_0^3x_0^2, y_1^3x_1^2, y_2^3x_2^2) \text{ for } x_0, x_1, x_2 \in \mathbb{Z} \right\}.$$  

Since $\gcd(z_0, z_1) = 1$ if and only if $\gcd(z_0, z_1, z_0 + z_1) = 1$, we may replace the condition $\gcd(z_0, z_1) = 1$ with $\gcd(x_0y_0, x_1y_1, x_2y_2) = 1$. Hence if $y \in \mathbb{N}^3$, then $N_1(T_y \cap \mathcal{C}, B)$ is just the quantity $\frac{1}{4}N_{C_y, H_y}(B^{1/2})$ considered in Section 4. For $y \in \mathbb{N}^3$ satisfying $\mu^2(y_0y_1y_2) = 1$, we define thin sets

$$T_y' = \bigcup_{w \in (\mathbb{Z}_+)^3} T_w,$$

where from (4.5) and (4.9), each set appearing in this union satisfies

$$N_1(T_y' \cap \mathcal{C}, B) = 3N_1(T_y \cap \mathcal{C}, B) = 3\mathcal{N}_1(T_y \cap \mathcal{C}, B).$$

To summarise, we have a disjoint union

$$\mathcal{C} = \bigcup_{y \in \mathbb{N}^3, \mu^2(y_0y_1y_2) = 1} (T_y' \cap \mathcal{C}),$$

where from (4.5) and (4.9), each set appearing in this union satisfies

$$N_1(T_y' \cap \mathcal{C}, B) = \frac{3}{4}N_{C_y, H_y}(B^{1/2}) \sim \frac{3}{\pi} \left( \frac{\gamma(y_0y_1y_2)}{(y_0y_1y_2)^{3/2}} \sigma_{2, y}^2(y) \right) B^{1/2}.$$

For a large integer $M$, we define

$$\mathcal{T}_M = \bigcup_{y \in \mathbb{N}^3, \mu^2(y_0y_1y_2) = 1} (T_y' \cap \mathcal{C}).$$

This is a thin set of Campana points, because it is a finite union of the thin sets $T_y' \cap \mathcal{C}$. We now assume Conjecture 1.1 holds, namely that $N_1(B) \sim 3c_{BV}B^{1/2}$. 


We deduce that
\[
N_1(C \setminus \mathcal{T}_M, B) = N_1(B) - N_1(\mathcal{T}_M, B) \sim 3c_{BV} - \frac{3}{\pi} \sum_{y \in \mathbb{N}^3 \atop y_0, y_1, y_2 \leq M} \frac{\mu^2 y_0 y_1 y_2 \gamma(y_0 y_1 y_2)}{(y_0 y_1 y_2)^{3/2}} \sigma_{2, \vartheta}(y).
\]

Since the sum is convergent, this quantity tends to zero as \(M \to \infty\). Therefore, we have shown that we can obtain an arbitrarily small positive constant by removing a thin set. We can now complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We fix \(\lambda \in (0, 3c_{BV}]\). For a subset \(S \subset C\), we define
\[
S(B) = \{z \in S : H(z) \leq B\},
\]
so that \#S(B) = \(N_1(S, B)\) in our earlier notation. We require a Campana thin subset \(\mathcal{T} \subseteq C\) with \(#\mathcal{T}(B) \sim (3c_{BV} - \lambda)B^{1/2}\).

For an appropriate choice of \(M\), the thin set \(\mathcal{T}_M\) defined in \((5.2)\) satisfies \(#\mathcal{T}_M(B) \sim (3c_{BV} - \lambda_0)B^{1/2}\) for some \(\lambda_0 \leq \lambda\). By definition, any subset of \(\mathcal{T}_M\) is also thin. Therefore, it suffices to find a subset \(\mathcal{T} \subseteq \mathcal{T}_M\) such that
\[
\frac{\#\mathcal{T}_M(B)}{\#\mathcal{T}(B)} \sim \frac{3c_{BV} - \lambda_0}{3c_{BV} - \lambda}.
\]

To achieve this, we take any subset \(A \subseteq \mathbb{N}\) of the desired asymptotic density
\[
\frac{A \cap [1, B]}{B} \sim \frac{3c_{BV} - \lambda_0}{3c_{BV} - \lambda}.
\]

We enumerate the elements of \(\mathcal{T}_M(B)\) by writing \(\mathcal{T}_M(B) = \{t_1, t_2, \ldots, t_R\}\), with \(H(t_i) \leq H(t_j)\) whenever \(i \leq j\). Then the set
\[
\mathcal{T} = \{t_i \in \mathcal{T}_M : i \in A\}
\]
is thin and satisfies \((5.3)\), as required. \(\square\)

6. **Squareful values of binary quadratic forms**

In this section, we study the constant \(c_{PSTV-A}\) for an orbifold corresponding to squareful values of the binary quadratic form \(ax^2 + by^2\). We recall the setup from the introduction. Throughout this section, \(a\) and \(b\) denote positive
integers satisfying $\mu^2(ab) = 1$ and $a, b \equiv 1 \pmod{4}$. We consider the Campana orbifold $(X, D)$ over $\mathbb{Q}$, where $X = \mathbb{P}^1$ and $D$ is the divisor $V(ax^2 + by^2)$, with the obvious good integral model $(\mathcal{X}, \mathcal{D})$. The set of Campana points in this example is not itself thin, as can be seen by combining [12, Theorem 1.1] and [12, Proposition 3.15]. Hence the PSTV-A conjecture applies to this orbifold.

We take the naive height $H$ on $\mathbb{P}^1$, which is given by $H([x : y]) = \max(|x|, |y|)$ for $(x, y) \in \mathbb{Z}_{\text{prim}}^2$. The resulting counting problem is

$$N(B) = \frac{1}{2} \# \{(x, y) \in \mathbb{Z}_{\text{prim}}^2 : |x|, |y| \leq B, ax^2 + by^2 \text{ squareful}\}.$$ 

This section is organized as follows. In Section 6.1, we compute $c_{\text{PSTV-A}}$ for the orbifold $(\mathcal{X}, \mathcal{D})$ and the height $H$. In Section 6.2, we prove the asymptotic formula for $N(B)$ given in Theorem 1.5. Finally, in Section 6.3, we prove Corollary 1.7 by comparing the constants obtained in Sections 6.1 and 6.2.

6.1. The constant from the PSTV-A conjecture. The aim of this section is to prove the following theorem. We recall the notation $\gamma(n)$ from (4.8).

**Theorem 6.1.** For the orbifold and the height function defined above, the constant $c_{\text{PSTV-A}}$ is equal to

$$\frac{4\gamma(ab)}{\pi^2} \left( \frac{\sinh^{-1}\left(\frac{\sqrt{a/b}}{\sqrt{a}}\right)}{\sqrt{a}} + \frac{\sinh^{-1}\left(\frac{\sqrt{b/a}}{\sqrt{b}}\right)}{\sqrt{b}} \right) \prod_{p \not\mid ab} \left( 1 + \frac{1 + \left(\frac{-ab}{p}\right)}{(1 + p^{-1})p^{3/2}} \right).$$

To prove Theorem 6.1, we follow the framework from Section 3. We keep the convention from Section 3 that $p$ ranges over all primes, and $v$ ranges over all primes and $v = \infty$. We have $\alpha = 1/2$ and $\beta = 1$, and so $c_{\text{PSTV-A}} = \tau/2$. The divisor $V(ax^2 + by^2)$ on $\mathbb{P}^1$ has degree 2, and corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$. With the usual metrization, this line bundle determines the height function $\max(|x|^2, |y|^2)$ for $(x, y) \in \mathbb{Z}_{\text{prim}}^2$. Choosing the section $ax^2 + by^2$, we obtain

$$H_D = \prod_v H_{D,v},$$

where

$$H_{D,v} = \frac{\max(|x|^v, |y|^v)}{|ax^2 + by^2|^v_{1/2}}.$$ 

We use the chart $y \neq 0$, and take $z = x/y$. Then for any prime $p$, we have $\nu_p(ax^2 + b) = \nu_p(ax^2 + by^2) - 2\nu_p(y)$. Consequently, the local Campana condition
that $\nu_p(ax^2 + by^2) - 2\min(\nu_p(x), \nu_p(y)) \neq 1$ is equivalent to the condition that $\nu_p(ax^2 + b)$ is not equal to 1 or a negative odd integer. Below, we denote by $\Omega_p$ the set of elements $z \in \mathbb{Q}_p$ satisfying this local Campana condition, and we set $\Omega_\infty = \mathbb{R}$. We let $dz$ denote the usual $p$-adic measure or the Lebesgue measure, as appropriate. We obtain

$$c_{\text{PSTV-A}} = \frac{1}{2} \sigma_\infty \prod_p (1 - p^{-1}) \sigma_p,$$

where

$$\sigma_v = \int_{\Omega_v} \frac{dz}{\max(|z|_v, 1)|az^2 + b|^{1/2}_v}.$$

To compute $\sigma_\infty$, we divide into regions $|z| \leq 1$ and $|z| > 1$. This yields

$$\sigma_\infty = \int_{|z| \leq 1} \frac{dz}{(az^2 + b)^{1/2}_v} + \int_{|z| > 1} \frac{dz}{|z|(az^2 + b)^{1/2}_v},$$

$$(6.3) \quad = 2 \left( \frac{\sinh^{-1} \left( \sqrt{a/b} \right)}{\sqrt{a}} + \frac{\sinh^{-1} \left( \sqrt{b/a} \right)}{\sqrt{b}} \right).$$

**Lemma 6.2.** We have

$$\sigma_p = \begin{cases} 1 + p^{-1} + \left( 1 + \left( \frac{ab}{p} \right) \right) p^{-3/2}, & \text{if } p \nmid 2ab \\ 1, & \text{if } p \mid 2ab. \end{cases}$$

**Proof.** We split $\Omega_p$ into three regions $R_1, R_2, R_3$, defined respectively by the conditions

- (1) $\nu_p(z) \geq 1$,
- (2) $\nu_p(z) < 0$,
- (3) $\nu_p(z) = 0$.

We also divide into four cases $p \nmid 2ab, p|a, p \mid b$, and $p = 2$. We let $\mu_p$ denote the usual $p$-adic measure.

**Case 1.** $p \nmid 2ab$: On $R_1$, we have $|az^2 + b|_p = 1$ and $\max(|z|_p, 1) = 1$, so

$$\int_{R_1} \frac{dz}{\max(|z|_p, 1)|az^2 + b|^{1/2}_p} = \int_{z \in \mathbb{Q}_p, \nu_p(z) \geq 1} 1 \, dz = p^{-1}. $$
On $R_2$, we have $|az^2 + b|_p = |z|_p^2$ and $\max(|z|_p, 1) = |z|_p$, so we obtain a contribution of

$$\int_{z \in \mathbb{Q}_p, \nu_p(z) < 0} \frac{dz}{|z|_p^2} = \sum_{j=-\infty}^{-1} p^{2j} \mu_p(\{z \in \mathbb{Q}_p : \nu_p(z) = j\})$$

$$= \sum_{j=1}^{\infty} (1 - p^{-1})p^{-j}$$

$$= p^{-1}.$$

On $R_3$, we have $|az^2 + b|_p \leq 1$ and $\max(|z|_p, 1) = 1$. For $j \geq 0$, we define

$$f(j) = \mu_p(\{z \in \mathbb{Z}_p^\times : \nu_p(az^2 + b) = j\})$$

$$g(j) = \mu_p(\{z \in \mathbb{Z}_p^\times : \nu_p(az^2 + b) \geq j\}).$$

We have

$$(6.4) \quad \int_{R_3} \frac{dz}{\max(|z|_p, 1)|az^2 + b|_p^{1/2}} = \int_{z \in \mathbb{Z}_p^\times \cap \Omega_p} \frac{dz}{|az^2 + b|_p^{1/2}} = \sum_{j \geq 0, j \neq 1} p^{j/2} f(j).$$

Clearly $f(j) = g(j) - g(j+1)$ for any $j \geq 0$. We now compute $g(j)$. We have $g(0) = \mu_p(\mathbb{Z}_p^\times) = 1 - p^{-1}$. By Hensel’s Lemma, for $j \geq 1$, we have

$$g(j) = p^{-j} \#\{z \pmod{p^j} : az^2 \equiv -b \pmod{p^j}\}$$

$$= p^{-j} \left(1 + \left(-\frac{ab}{p}\right)\right).$$

Therefore, the right hand side of (6.4) equals

$$1 - p^{-1} - p^{-1} \left(1 + \left(-\frac{ab}{p}\right)\right) + \sum_{j \geq 2} (1 - p^{-1})p^{-j/2} \left(1 + \left(-\frac{ab}{p}\right)\right)$$

$$= 1 - p^{-1} + \left(1 + \left(-\frac{ab}{p}\right)\right)p^{-3/2}.$$

Combining the three regions, we have completed the proof for primes $p \nmid 2ab$.

Case 2. $p \mid b$: This time, the region $R_1$ contributes zero, because if $p \mid b$ and $\nu_p(z) \geq 1$ then $\nu_p(az^2 + b) = 1$ (by the assumption that $b$ is squarefree), and so $z \notin \Omega_p$. The region $R_2$ contributes $p^{-1}$ to the integral in (6.2) by the same
calculation as in Case 1. On the region $R_3$ we have $\nu_p(a z^2 + b) = 0$, and so
\[
\int_{R_3} \frac{dz}{\max(|z|_p, 1)|az^2 + b|^{1/2}_p} = \int_{z \in \mathbb{Z}_p^\times} 1\ dz = 1 - p^{-1}.
\]
Hence $\sigma_p = p^{-1} + 1 - p^{-1} = 1$.

**Case 3.** $p \mid a$: The region $R_1$ contributes $p^{-1}$ by the same calculation as in Case 1. On $R_2$, we have $\nu_p(a z^2 + b) = 2\nu_p(z) + 1$, which is an odd negative integer, and so the contribution is zero. On $R_3$, we have $\nu_p(a z^2 + b) = 0$ (since $p \nmid b$ by the assumptions $p \mid a$ and $\mu^2(ab) = 1$), and so we obtain a contribution of $1 - p^{-1}$ as in Case 2. Combining, we have $\sigma_p = p^{-1} + 1 - p^{-1} = 1$.

**Case 4.** $p = 2$: Regions $R_1$ and $R_2$ contribute $p^{-1}$ to the integral in (6.2) as in Case 1. The region $R_3$ contributes zero. To see this, we note that if $z \in \mathbb{Z}_2^\times$, then $z^2 \equiv 1 (\text{mod } 4)$. However, since $a, b \equiv 1 (\text{mod } 4)$, we have $az^2 + b \equiv 2 (\text{mod } 4)$, and hence $\nu_p(a z^2 + b) = 1$. Hence $\sigma_2 = 2^{-1} + 2^{-1} = 1$. \(\square\)

Let $\sigma_\infty$ be as given in (6.3). We conclude from (6.1) and Lemma 6.2 that
\[
c_{\text{PSTV-A}} = \frac{\sigma_\infty}{2} \prod_{p \nmid 2ab} \left( 1 - p^{-1} \right) \left( 1 + p^{-1} + \left( 1 + \left( \frac{-ab}{p} \right) \right) p^{-3/2} \right) \prod_{p \mid 2ab} (1 - p^{-1})
\] 
\[
= \frac{\sigma_\infty}{2} \cdot \frac{6}{\pi^2} \prod_{p \nmid 2ab} \left( 1 + \frac{1 + \left( \frac{-ab}{p} \right)}{(1 + p^{-1})p^{3/2}} \right) \prod_{p \mid 2ab} \frac{1}{1 + p^{-1}}
\] 
\[
= \frac{2\sigma_\infty \gamma(ab)}{\pi^2} \prod_{p \nmid 2ab} \left( 1 + 1 + \left( \frac{-ab}{p} \right) \right) \left( 1 + \frac{1 + \left( \frac{-ab}{p} \right)}{(1 + p^{-1})p^{3/2}} \right).
\]

This completes the proof of Theorem 6.1.

6.2. **The asymptotic formula for** $N(B)$. In this section, we prove Theorem 1.5. We write $ax^2 + by^2 = u^2 v^3$ for $v \in \mathbb{Z}_{\neq 0}$ squarefree and $u \in \mathbb{N}$. If $\gcd(x, y) = 1$, then $\gcd(a, v) = 1$ and $\gcd(b, v) = 1$. This is because if $p \mid \gcd(a, v)$ then $p \mid by^2$, and since $\gcd(a, b) = 1$, this implies that $p \mid y$. But then $p^2 \mid ax^2$, and since $a$ is squarefree, we have $p \mid x$. This contradicts the assumption $\gcd(x, y) = 1$. The argument to show that $\gcd(b, v) = 1$ is the same by symmetry. Hence $a, b$ and $v$ are squarefree and pairwise coprime, in other words $\mu^2(abv) = 1$. 
Moreover, the assumptions $a, b > 0$ imply that $v > 0$. Therefore, we have

$$N(B) = \frac{1}{2} \sum_{v=1}^{\infty} \mu^2(abv)N_v(B),$$

where

$$N_v(B) = \frac{1}{2} \# \{ (x, y, u) \in \mathbb{Z}^3 : \gcd(x, y) = 1, |x|, |y| \leq B, ax^2 + by^2 = u^2v^3 \}.$$

The factor $1/2$ comes from the fact that there are two choices for the sign of $u$ in $[x : y : u]$ corresponding to each point $[x : y]$ enumerated by $N(B)$.

Throughout this section, all implied constants depend only on $a, b$ and $\epsilon$. We split the sum over $v$ into ranges $v < B^\delta$ and $v \geq B^\delta$, for a fixed $\delta > 0$. To deal with the range $v \geq B^\delta$, we note that $ax^2 + by^2 = u^2v^3$ and $|x|, |y| \leq B$ together imply that $u^2v^3 \ll B^2$, so $u \ll Bv^{-3/2}$. Therefore, there are $O(Bv^{-3/2})$ choices for $u$. Applying a result of Browning and Gorodnik [1, Theorem 1.11], for any fixed $u, v$, we have

$$\# \{ (x, y) \in \mathbb{Z}_{\text{prim}}^2 : ax^2 + by^2 = u^2v^3 \} = O(B^\epsilon).$$

Hence $N_v(B) \ll B^{1+\epsilon}v^{-3/2}$. Taking a sum over $v \geq B^\delta$, we obtain

$$\sum_{v \geq B^\delta} \mu^2(abv)N_v(B) \ll B^{1+\epsilon-\delta/2},$$

and so the contribution from the range $v \geq B^\delta$ is negligible.

For the range $v < B^\delta$, we view the equation $ax^2 + by^2 = u^2v^3$ as a conic, with $a, b$ and $v$ fixed. Sofos [20] counts rational points on isotropic conics by using a birational map from the conic to $\mathbb{P}^1$ in order to parameterise the solutions as lattice points. Unfortunately, we cannot apply [20, Theorem 1.1] directly, since the coprimality condition $\gcd(x, y, u) = 1$ is used instead of $\gcd(x, y) = 1$. However, the argument can be adapted to deal with this alternative coprimality condition. We summarise the main alterations required.

Let $Q$ be a non-singular quadratic form in 3 variables with integer coefficients. Let $\Delta_Q$ denote the discriminant of $Q$, and $\langle Q \rangle$ the maximum modulus of the coefficients of $Q$. Suppose that $\| \cdot \|$ is a norm isometric to the supremum norm. For convenience, below we use variables $x = (x_1, x_2, x_3)$ in place of $(x, y, u)$. We define

$$N_{\| \cdot \|}(Q, B) = \# \{ x \in \mathbb{Z}^3 : \gcd(x_1, x_2) = 1, Q(x) = 0, \|x\| \leq B \}.$$
This is the same as the counting function from [20], but with the condition $\gcd(x_1, x_2) = 1$ instead of $\gcd(x_1, x_2, x_3) = 1$. We let $Q_v(x) = ax_1^2 + bx_2^2 - v^3x_3^2$, and define a norm $\| \cdot \|$ by $\|x\| = \max(|x_1|, |x_2|)$. There is a constant $C$, depending only on $a$ and $b$, such that $\|x\| = \max(|x_1|, |x_2|, C\varepsilon^{3/2}|x_3|)$, and so $\| \cdot \|$ is isometric to the supremum norm. In our earlier notation, we have $N_v(B) = N_{\| \cdot \|}(Q_v, B)$.

As in [20, Section 6], the first stage is to apply a linear change of variables in order to transform $Q_v$ into a quadratic form $Q$ satisfying $Q(0, 1, 0) = 0$. We assume that $N_v(B) > 0$, so that there exists $(t_{12}, t_{22}, t_{32}) \in \mathbb{Z}^3$ with $Q_v(t_{12}, t_{22}, t_{32}) = 0$ and $\gcd(t_{12}, t_{22}) = 1$; we shall choose the smallest such solution. Then we can find integers $t_{11}, t_{21}$ such that $t_{11}t_{22} - t_{21}t_{12} = 1$ and $|t_{11}|, |t_{21}| \leq \max(|t_{12}|, |t_{22}|)$. Let

$$M = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define $Q(x) = Q_v(Mx)$ and $\|x\|' = \|Mx\|$. Since the first $2 \times 2$ minor of $M$ is an element of $SL_2(\mathbb{Z})$, the coprimality condition $\gcd(x_1, x_2) = 1$ is preserved under this transformation. Therefore $N_{\| \cdot \|}(Q_v, B) = N_{\| \cdot \|}(Q, B)$, which we shall abbreviate to $N(Q, B)$.

The forms $L(s, t)$ and $g(s, t)$ defined in [20, Equation (2.3)] can be written explicitly as

$$L(s, t) = (2at_{11}t_{12} + 2bt_{21}t_{22})s - 2v^3t_{32}t,$$

$$g(s, t) = (at_{11}^2 + bt_{21}^2)s^2 - v^3t^2.$$

As in [20, Equation (2.4)], we let $q = (q_1, q_2, q_3) = (q_1(s, t), q_2(s, t), q_3(s, t))$, where

$$q_1(s, t) = sL(s, t), \quad q_2(s, t) = -g(s, t), \quad q_3(s, t) = tL(s, t).$$

By applying the parameterisation argument from [20, Lemma 3.1], we find that $N(Q, B) = \mathcal{N}(Q, B) + O(1)$, where

$$\mathcal{N}(Q, B) = \# \left\{ (s, t) \in \mathbb{Z}_\text{prim}^2 : t > 0, \|q\|' \leq \lambda B, \gcd\left(\frac{q_1}{\lambda}, \frac{q_2}{\lambda}\right) = 1 \right\}$$

and $\lambda = \gcd(q_1, q_2, q_3)$.

We now take a sum over the possible values of $\lambda$. Due to our alternative coprimality condition, in (6.9) we have the stronger condition $\gcd\left(\frac{q_1}{\lambda}, \frac{q_2}{\lambda}\right) = 1$.
in place of \( \gcd(\frac{q_1}{\lambda}, \frac{q_2}{\lambda}, \frac{q_3}{\lambda}) = 1 \), and so when applying M"obius inversion we take a sum over a variable \( r \) with \( r \mid (\frac{q_1}{\lambda}, \frac{q_2}{\lambda}, \frac{q_3}{\lambda}) \) in place of Sofos’ sum over \( k \mid (\frac{q_1}{\lambda}, \frac{q_2}{\lambda}, \frac{q_3}{\lambda}) \).

As in [20, Equation (3.2)], we define
\[
M^*_{\sigma,\tau}(T, n) = \#\{(s, t) \in \mathbb{Z}^2_{\text{prim}} : (s, t) \equiv (\sigma, \tau) \pmod{n}, t > 0, \|q\|' \leq T\}.
\]

Then similarly to [20, Lemma 3.2], we obtain
\[
N(Q, B) = \sum_{\lambda | \Delta_Q} \sum_{r} \mu(r) \sum_{\sigma, \tau} M^*_{\sigma,\tau}(B\lambda, r\lambda),
\]
where \( \sum^+_r \) denotes a sum over residues \( \sigma, \tau \) modulo \( r\lambda \) such that \( \lambda \mid q(\sigma, \tau), r\lambda \mid (q_1(\sigma, \tau), q_2(\sigma, \tau)) \) and \( \gcd(\sigma, \tau, r\lambda) = 1 \).

We now explain why, with our choice of \( Q \), we may restrict the \( r \)-sum in (6.10) to divisors of \( \lambda \). Since \( r \) is squarefree, it suffices to show that for any prime \( p \mid (q_1, q_2) \), we also have \( p \mid q_3 \). (In general, \( \gcd(q_1, q_2) \) can still be larger than \( \lambda \) since its prime factors can occur with higher multiplicity.) Suppose that \( p \mid (q_1, q_2) \). We immediately deduce that \( p \mid q_3 \) if \( p \mid L(s, t) \), and so using \( p \mid q_1 \) we may assume that \( p \mid s \). Since \( \gcd(s,t) = 1 \) and \( p \mid q_2 \), we see from (6.8) that \( p \mid v \). However, then from (6.7) we have that \( p \mid L(s,t) \) after all, and so \( p \mid q_3 \), as desired.

An asymptotic formula for \( N(Q, B) \) can now be deduced by applying the lattice counting results from [20, Section 4] to estimate \( M^*_{\sigma,\tau}(B\lambda, r\lambda) \). We maintain control over the resulting error terms after performing the summations in (6.10) thanks to the restriction on the \( r \)-sum. Similarly to [20, Proposition 2.1], we obtain
\[
N(Q, B) = c_v B + O((BK)^{1/2+\epsilon}(|\Delta_Q| + \langle Q \rangle)^{1+\epsilon})
\]
for some constant \( c_v > 0 \), where
\[
K = \sup_{x \neq 0} \left( 1 + \frac{\|x\|_{\infty}}{\|x\|'} \right)
\]
and \( \|x\|_{\infty} = \max(|x_1|, |x_2|, |x_3|) \) denotes the supremum norm of \( x \).

We have \( \Delta_Q = \Delta_{Q_v} = abv^3 \ll v^3 \). Let \( \|M\|_{\infty} \) denote the maximum modulus of the entries of \( M \). Then \( \langle Q \rangle \ll \|M\|^2_{\infty} \). Moreover, making a change of variables from \( x \) to \( M^{-1}x \) in the definition of \( K \), we have
\[
K = \sup_{x \neq 0} \left( 1 + \frac{\|M^{-1}x\|_{\infty}}{\|x\|'} \right) \ll \|M^{-1}\|_{\infty} \sup_{x \neq 0} \left( 1 + \frac{\|x\|_{\infty}}{\|x\|'} \right) \ll \|M^{-1}\|_{\infty}.
\]
Using the bound $\|M^{-1}\|_{\infty} \ll \|M\|_{\infty}^2$, we conclude that
\begin{equation}
N(Q, B) = c_v B + O((B\|M\|_{\infty}^2)^{1/2+\epsilon}(v^3 + \|M\|_{\infty}^2)^{1+\epsilon}).
\end{equation}

We recall that $\|M\|_{\infty} = \max(|t_{12}|, |t_{22}|, |t_{32}|)$ is the size of the least integral solution to $Q_v(x) = 0$ with $\gcd(x_1, x_2) = 1$. Cassels [6] establishes an upper bound for the smallest integral solution to a quadratic form. In the following lemma, we find a bound for the least solution satisfying our additional coprimality condition.

**Lemma 6.3.** Suppose that $a, b, v$ are integers with $\mu^2(abv) = 1$. Let $Q_v$ denote the quadratic form $ax_1^2 + bx_2^2 - v^3x_3^2$. Then if the system
\begin{equation}
\begin{aligned}
Q_v(x) &= 0, \\
x &\in \mathbb{Z}^3, \ \gcd(x_1, x_2) = 1
\end{aligned}
\end{equation}
has a nontrivial solution, it has a solution satisfying $\|x\|_{\infty} \ll |v|^7$.

We deduce Lemma 6.3 from the following result of Dietmann, which generalises Cassel’s argument by imposing congruence conditions on the variables.

**Lemma 6.4.** [9, Proposition 1] Let $Q$ be a non-degenerate quadratic form in 3 variables with integral coefficients. Let $\xi \in \mathbb{Z}^3$ and $\eta \in \mathbb{N}$. Suppose that there exists an integral solution to the system
\begin{equation}
\begin{aligned}
Q(x) &= 0, \\
x &\equiv \xi \ (\text{mod } \eta).
\end{aligned}
\end{equation}
Then there exists an integral solution to this system satisfying
$$\|x\|_{\infty} \ll \max\{\eta^2|\Delta_Q|^2(\langle Q \rangle)^2, \eta^3|\Delta_Q|^3\}.$$  

**Proof of Lemma 6.3.** Suppose that $y = (y_1, y_2, y_3)$ is a solution to (6.13). Let $Q_0$ denote the quadratic form $ax_1^2 + bx_2^2 - vx_3^2$. Then clearly $Q_0(y_1, y_2, vy_3) = 0$. Let $\eta = |v|$ and let $\xi = (\xi_1, \xi_2, 0)$ denote the residues of $(y_1, y_2, vy_3)$ modulo $\eta$. We have $\gcd(y_1, v) = 1$, because if $p \mid (y_1, v)$ then $p \mid by_2^2$, but since $\mu^2(abv) = 1$ this implies $p \mid y_2$, contradicting the assumption $\gcd(y_1, y_2) = 1$. Consequently, $\xi_1$ is invertible modulo $\eta$.

Since $\Delta_{Q_0} \ll |v|$ and $\langle Q_0 \rangle \ll |v|$, we find from Lemma 6.4 an integral solution $z = (z_1, z_2, z_3)$ to (6.14) with the above choice of $Q_0, \eta, \xi$, and with $\|z\|_{\infty} \ll |v|^7$. Choose $x = (z_1, z_2, z_3/v)/\lambda$, where $\lambda = \gcd(z_1, z_2, z_3/v)$. This
is an integral solution to \( Q_v = 0 \) because \( z_3 \equiv 0 \pmod{v} \). Additionally, the bound \( \|z\|_\infty < |v|^7 \) implies that \( \|x\|_\infty < |v|^7 \). To complete the proof, it suffices to show that \( \gcd(x_1, x_2) = 1 \), or equivalently that \( \gcd(z_1, z_2) = \lambda \). Clearly \( \lambda | \gcd(z_1, z_2) \). Conversely, suppose that \( h | (z_1, z_2) \). From \( Q_0(z) = 0 \), we see that \( h | vz_3 \). However, since \( z_1 \equiv \xi_1 \pmod{\eta} \) and \( \xi_1 \) is invertible modulo \( \eta \), we have \( \gcd(h, v) = 1 \). Therefore, \( h | z_3/v \), and so \( h | \lambda \), as required. □

Substituting the bound \( \|M\|_\infty \ll v^7 \) from Lemma 6.3 into (6.12) we conclude that

\[
N_v(B) = c_v B + O(B^{1/2+\epsilon_v}).
\]

The leading constant \( c_v \) could be computed explicitly from the above method. However, we note that by [14, Example 3.2], equidistribution holds for smooth isotropic conics, and so \( c_v \) is known to be the constant predicted in Manin’s conjecture. More precisely, we have

\[
c_v = \frac{1}{2} \sigma_{\infty,v} \prod_p \sigma_{p,v},
\]

where \( \sigma_{\infty,v} \) is the real density from Manin’s conjecture applied to \( N_v(B) \), and

\[
\sigma_{p,v} = \lim_{n \to \infty} \frac{M_v(p^n)}{p^{2n}},
\]

\[
M_v(p^n) = \# \{(x, y) \pmod{p^n} : p \nmid \gcd(x, y), ax^2 + by^2 \equiv u^2 v^3\}.
\]

Combining with (6.6) and choosing \( \delta = 1/43 \), we obtain

\[
(6.15) \quad N(B) = \frac{1}{2} \sum_{v \leq B^\delta} \mu^2(abv)c_v B + O(B^{85/86+\epsilon}).
\]

We are now in a very similar situation to the one encountered in Section 4. Let \( C_{(a, b, v)}, H_{(a, b, v)} \) and \( \sigma_{2,(a, b, v)} \) be as defined in (4.2), (4.3) and (4.9) respectively, but with \( (y_0^3, y_1^3, y_2^3) \) replaced by \( (a, b, v^3) \). Then, analogously to (4.7), we have

\[
c_{H_{(a, b, v)}}(C_{(a, b, v)}(\mathbb{A}_Q)^+) = \frac{4}{\pi} \cdot \frac{\mu^2(abv)\gamma(abv)}{(abv^3)^{1/2}} \sigma_{2,(a, b, v)} \psi(a, b, v).
\]

The only difference between \( c_v \) and \( c_{H_{(a, b, v)}}(C_{(a, b, v)}(\mathbb{A}_Q)^+) \) lies in the computation of the density at the real place, since we are using a different height to \( H_{(a, b, v)} \). Replacing the real density \( \pi/(abv^3)^{1/2} \) appearing in [2, Section 2.3]...
with the appropriate real density \( \sigma_{\infty,v} \) for our setup, we have

\[
(6.16) \quad c_v = \frac{(abv^3)^{1/2}}{\pi} \sigma_{\infty,v} c_{H(a,b,v)} (C_{(a,b,v)}(A_{\mathbb{Q}})^+).
\]

To compute \( \sigma_{\infty,v} \), we use the Leray form as in [2, Section 2.3] to obtain

\[
\sigma_{\infty,v} = \frac{1}{2} \frac{v^3}{2} \int_{[-1,1]^2} \frac{dx \, dy}{\sqrt{ax^2 + by^2}}
= \frac{1}{2} \frac{v^3}{2} \int_{-1}^{1} \frac{2}{\sqrt{a}} \sinh^{-1} \left( \frac{1}{y} \sqrt{\frac{a}{b}} \right) dy
= \frac{2}{v^3/2 \, a^{1/2}} \left( \sinh^{-1} \left( \sqrt{\frac{a}{b}} \right) + \frac{\sinh^{-1} \left( \sqrt{\frac{b}{a}} \right)}{\sqrt{\frac{b}{a}}} \right)
\]

\[
(6.17) \quad = \frac{2}{v^3/2} \left( \sinh^{-1} \left( \sqrt{\frac{a}{b}} \right) + \sinh^{-1} \left( \sqrt{\frac{b}{a}} \right) \right).
\]

Hence \( \sigma_{\infty,v} = \sigma_{\infty} v^{-3/2} \), where \( \sigma_{\infty} \) is the real density from the PSTV-A conjecture, as computed in (6.3). Due to the assumptions \( a \equiv b \equiv 1 \pmod{4} \), the density at the prime 2 from (4.13) simplifies to

\[
\sigma_{2,(a,b,v)} = \begin{cases} 
1, & \text{if } v \equiv 1 \pmod{4}, \\
0, & \text{otherwise}.
\end{cases}
\]

Combining this with (6.16) and (4.9), we conclude that

\[
(6.18) \quad c_v = \begin{cases} 
\frac{4 \sigma_{\infty} \gamma(abv) \varrho(abv)}{\pi^3 v^{3/2}}, & \text{if } v \equiv 1 \pmod{4}, \\
0, & \text{otherwise}.
\end{cases}
\]

In particular, \( c_v \ll v^{-3/2+\epsilon} \). This allows us to extend the sum in (6.15) to an infinite sum over \( v \), with the same error term \( O(B^{1+\epsilon-\delta/2}) \) that is already present in (6.15). We conclude that \( N(B) = cB + O(B^{85/86+\epsilon}) \), where

\[
(6.19) \quad c = \frac{2 \sigma_{\infty}}{\pi^2} \sum_{v \equiv 1 \pmod{4}} \frac{\mu^2(abv) \gamma(abv) \varrho(a,b,v)}{v^{3/2}}.
\]

This completes the proof of Theorem 1.5.
6.3. **Comparison of \(c\) and \(c_{\text{PSTV-A}}\).** Continuing from (6.19), we pull out a factor \(\gamma(ab)\), and replace \(\mu^2(abv)\) with \(\mu^2(v)\) and the condition \(\gcd(v, ab) = 1\). This allows us to rewrite \(c\) as

\[
(6.20) \quad c = R \sum_{v \equiv 1 \pmod{4}, \gcd(v, ab) = 1} \frac{\mu^2(v)\gamma(v)\varrho(a, b, v)}{v^{3/2}},
\]

where \(R := 2\sigma_{\infty}(\gamma(ab))/\pi^2\) is the same factor that appears in (6.5). It remains to compare the sum in (6.20) with the Euler product from (6.5).

Using quadratic reciprocity, it can be shown that if \(\varrho(a, b, v) \neq 0\) and \(v\) is odd, then \(v \equiv 1 \pmod{4}\). We define

\[
f(v) = \frac{\mu^2(v)\gamma(v)}{v^{3/2}} \prod_{p | v} \left(1 + \left(\frac{-ab}{v}\right)\right).
\]

The function \(f\) is multiplicative in \(v\). Therefore, from (6.20), we have

\[
\frac{c}{R} = \sum_{\gcd(v, 2ab) = 1} f(v) \prod_{p | a} \left(1 + \left(\frac{bv}{p}\right)\right) \prod_{p | b} \left(1 + \left(\frac{av}{p}\right)\right)
\]

\[
= \sum_{k | a} \sum_{l | b} \sum_{\gcd(v, 2ab) = 1} f(v) \left(\frac{bv}{k}\right) \left(\frac{av}{l}\right),
\]

where the summand is multiplicative in \(v\). We conclude that

\[
(6.21) \quad \frac{c}{R} = \sum_{k | a} \sum_{l | b} \prod_{p | 2ab} \left(\frac{b}{k} \left(\frac{a}{l}\right) + \left(\frac{bp}{k}\right)\left(\frac{ap}{l}\right)\left(1 + \left(\frac{-ab}{p}\right)\right)\right).
\]

We recognise the contribution to (6.21) from \(k = l = 1\) as precisely the Euler product \(c_{\text{PSTV-A}}/R\) from (6.5). If \(a = 1\) and \(b \neq 1\), which is a special case of the norm forms considered in [21], then (6.21) simplifies to

\[
\frac{c}{R} = \sum_{l | b} \prod_{p | 2b} \left(1 + \left(\frac{b}{p}\right)\left(1 + \left(\frac{-b}{p}\right)\right)\right)\left(\frac{1 + \left(\frac{-b}{p}\right)}{1 + p^{-1}}\right)^{p^{3/2}}.
\]

The contribution from each divisor \(l\) is positive. We summarize as follows.

**Lemma 6.5.** Suppose that \(a = 1\), \(\mu^2(b) = 1\) and \(b \equiv 1 \pmod{4}\). Then

i) \(c_{\text{PSTV-A}} = c\) if \(b = 1\),

ii) \(c_{\text{PSTV-A}} < c\) if \(b > 1\).
Similarly to the situation from Section 5, we can obtain any constant in
\((0, c]\), including \(c_{\text{PSTV-A}}\) itself, by the removal of an appropriate thin set.

Finally, we show that when \(a, b > 1\), it is possible that \(c < c_{\text{PSTV-A}}\). Since
the removal of thin sets can only reduce the constant \(c\), this provides a coun-
terexample to the leading constant predicted by the PSTV-A conjecture.

Proof of Corollary 1.7. We take \(a, b > 7\) to be distinct primes satisfying the
following conditions.

(1) \(a, b \equiv 1 \pmod{4}\).
(2) \(\left(\frac{a}{p}\right) = -1\).
(3) \(\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right) = 1\) for \(p \in \{3, 7\}\).
(4) \(\left(\frac{a}{5}\right) = -1\) and \(\left(\frac{b}{5}\right) = 1\).

The pair \(a = 37, b = 109\) satisfies conditions (1)–(4). In fact, (1)–(4) are
equivalent to \(a, b\) lying in certain congruence classes, and so by Dirichlet’s
theorem on primes in arithmetic progressions, these conditions are satisfied by
infinitely many pairs of distinct primes \(a, b\).

Using conditions (1) and (2), the right hand side of (6.21) simplifies to
(6.22) \[ \frac{c}{R} = S(\chi_0) - S(\chi_1) - S(\chi_2) + S(\chi_3), \]
where

\[ S(\chi_0) = \prod_{p \nmid 2ab} \left(1 + \frac{1+\left(-\frac{ab}{p}\right)}{(1+p^{-1})p^{3/2}}\right), \quad S(\chi_1) = \prod_{p \nmid 2ab} \left(1 + \frac{\left(\frac{a}{p}\right)(1+\left(-\frac{ab}{p}\right))}{(1+p^{-1})p^{3/2}}\right), \]
\[ S(\chi_2) = \prod_{p \nmid 2ab} \left(1 + \frac{\left(\frac{b}{p}\right)(1+\left(-\frac{ab}{p}\right))}{(1+p^{-1})p^{3/2}}\right), \quad S(\chi_3) = \prod_{p \nmid 2ab} \left(1 + \frac{\left(\frac{ab}{p}\right)(1+\left(-\frac{ab}{p}\right))}{(1+p^{-1})p^{3/2}}\right). \]

Since \(S(\chi_0) = c_{\text{PSTV-A}}/R\), it suffices to show that \(S(\chi_3) - S(\chi_1) - S(\chi_2) < 0\).
From conditions (3) and (4), we have that all the Euler factors for \(S(\chi_1), S(\chi_2)\)
and \(S(\chi_3)\) are equal to 1 for \(p \leq 7\). For \(p > 7\), we estimate the Euler factors
trivially to obtain

\[ S(\chi_3) - S(\chi_1) - S(\chi_2) \leq \prod_{p > 7} \left(1 + \frac{2}{(1+p^{-1})p^{3/2}}\right) - 2 \prod_{p > 7} \left(1 - \frac{2}{(1+p^{-1})p^{3/2}}\right). \]

Similarly to the end of Section 3, we can use convergence factors to compute
numerically that

\[ \prod_p \left(1 + \frac{2}{(1+p^{-1})p^{3/2}}\right) \left(1 - \frac{2}{(1+p^{-1})p^{3/2}}\right)^{-1} = 15.206698... < 16. \]
On the other hand, it can be computed that
\[
\prod_{p \leq 7} \left(1 + \frac{2}{(1 + p^{-1})p^{3/2}}\right) \left(1 - \frac{2}{(1 + p^{-1})p^{3/2}}\right)^{-1} = 8.231089... > \frac{16}{2}.
\]
It follows that \(S(\chi_3) - S(\chi_1) - S(\chi_2) < 0\), as required. □

**Remark 6.6.** In the examples considered above, the divisor \(D\) does not have strict normal crossings at the primes dividing \(ab\). From this point of view, it seems natural to ask whether counting Campana \(\mathbb{Z}[1/ab]\)-points instead of Campana \(\mathbb{Z}\)-points reconciles the two leading constants \(c\) and \(c_{PSTV-A}\). However, it can be checked that in this setup, by a similar argument to the proof of Corollary 1.7, there are still values of \(a, b\) which provide a counterexample to the PSTV-A conjecture.

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