ON THE FROBENIUS MANIFOLDS FOR CUSP SINGULARITIES

YUUKI SHIRAISHI AND ATSUSHI TAKAHASHI

Abstract. We show that the Frobenius manifold associated to the pair of a cusp singularity and its canonical primitive form is isomorphic to the one constructed from the Gromov-Witten theory for an orbifold projective line with at most three orbifold points. We also calculate the intersection form of the Frobenius manifold.

1. Introduction

The notion of primitive forms is introduced by K. Saito [14, 15] in his study of period mapping associated to a deformation theory of an isolated hypersurface singularity. Roughly speaking, a primitive form is a generalization of a differential of the first kind on an elliptic curve and the tools in order to define a primitive form are natural generalization of a polarized Hodge structure. The existence of primitive forms is proved for isolated hypersurface singularities [18] (for comprehensive guides, see also [6, 13]).

In the first part of the present paper, we shall investigate the canonical primitive form for a cusp singularity: Let $A = (a_1, a_2, a_3)$ be a triplet of positive integers such that $a_1 \leq a_2 \leq a_3$. Set $\mu_A = a_1 + a_2 + a_3 - 1$ and $\chi_A := 1/a_1 + 1/a_2 + 1/a_3 - 1$. We can associate to $A$ the polynomial $f_A(x) \in \mathbb{C}[x_1, x_2, x_3]$ given as

$$f_A(x) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1} \cdot x_1 x_2 x_3 \quad (1.1)$$

for some $q \in \mathbb{C}\{0\}$, which we shall call a cusp polynomial of type $A$. In particular, it defines a cusp singularity if $\chi_A < 0$. It was claimed by K. Saito [15] and is proven by M. Saito [18] that, for the universal unfolding of a cusp singularity, there exists a unique primitive form up to a constant factor with the minimal exponent $r = 1$, whose associated exponents are given by the mixed Hodge structure of the cusp singularity. First, we shall give its local expression in order to show the mirror isomorphism:

Theorem (Theorem 2.38). Assume that $\chi_A < 0$. There exists a unique primitive form $\zeta_A$ for the tuple $(\mathcal{H}_{F_A}^{(0)}, \nabla, K_{F_A})$ with the minimal exponent $r = 1$ such that

$$\zeta_A|_{s=0} = [s^{-1}_{\mu_A} dx_1 \wedge dx_2 \wedge dx_3]. \quad (1.2)$$
Once we obtain the primitive form, we can construct the Frobenius structure on the deformation space of the isolated hypersurface singularity by well-known construction \cite{17}. Next, we shall show that the Frobenius structure associated to the pair \((f_\mu, \zeta_\mu)\) given above satisfies the conditions in Theorem 3.1 of \cite{8}:

**Theorem (Theorem 1.2).** Assume that \(\chi_\mu < 0\). For the Frobenius structure of rank \(\mu_\mu\) and dimension one constructed from the pair \((f_\mu, \zeta_\mu)\), there exist flat coordinates \(t_1, t_{1,1}, \ldots, t_{3,\mu_\mu-1}, t_{\mu_\mu}\) satisfying the following conditions:

(i) The unit vector field \(e\) and the Euler vector field \(E\) are given by

\[
e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{\mu_\mu-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_\mu \frac{\partial}{\partial t_{\mu_\mu}}.
\]

(ii) The non-degenerate symmetric bilinear form \(\eta\) on \(T_M\) defined by

\[
\eta(\delta, \delta') := K^{(0)}_\mu(u \nabla_\delta \zeta_\mu, u \nabla_{\delta'} \zeta), \quad \delta, \delta' \in T_M
\]

satisfies

\[
\eta \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{\mu_\mu}} \right) = \eta \left( \frac{\partial}{\partial t_{\mu_\mu}}, \frac{\partial}{\partial t_1} \right) = 1,
\]

\[
\eta \left( \frac{\partial}{\partial t_{i,j}}, \frac{\partial}{\partial t_{i',j'}} \right) = \begin{cases} \frac{1}{a_i} & i = i_2 \text{ and } j_2 = a_i - j_1, \\ 0 & \text{otherwise.} \end{cases}
\]

(iii) The Frobenius potential \(F_{f_\mu, \zeta_\mu}\) satisfies \(E F_{f_\mu, \zeta_\mu}|_{t_1=0} = 2 F_{f_\mu, \zeta_\mu}|_{t_1=0}\):

\[
F_{f_\mu, \zeta_\mu}|_{t_1=0} \in \mathbb{C} \left[ t_{1,1}, \ldots, t_{1,\mu_\mu-1}, t_{2,1}, \ldots, t_{2,\mu_\mu-1}, t_{3,1}, \ldots, t_{3,\mu_\mu-1}, e^{\mu_\mu} \right].
\]

(iv) The restriction of the Frobenius potential \(F_{f_\mu, \zeta_\mu}\) to the submanifold \(\{t_1 = e^{\mu_\mu - 1} = 0\}\) is given as

\[
F_{f_\mu, \zeta_\mu}|_{t_1=e^{\mu_\mu}=0} = G^{(1)} + G^{(2)} + G^{(3)},
\]

where \(G^{(i)} \in \mathbb{C}[t_{1,1}, \ldots, t_{1,\mu_\mu-1}], i = 1, 2, 3\).

(v) In the frame \(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{1,1}}, \ldots, \frac{\partial}{\partial t_{3,\mu_\mu-1}}, \frac{\partial}{\partial t_{\mu_\mu}}\) of \(T_M\), the product \(\circ\) can be extended to the limit \(t_1 = t_{1,1} = \cdots = t_{3,\mu_\mu-1} = e^{\mu_\mu} = 0\). The \(\mathbb{C}\)-algebra obtained in this limit is isomorphic to

\[
\mathbb{C}[x_1, x_2, x_3] / (x_1 x_2, x_2 x_3, x_3 x_1, a_1 x_1, a_2 x_2, a_3 x_3),
\]

where \(\partial/\partial t_{1,1}, \partial/\partial t_{2,1}, \partial/\partial t_{3,1}\) are mapped to \(x_1, x_2, x_3\), respectively.

(vi) The term \(t_1 t_2 t_3 e^{\mu_\mu}\) occurs with the coefficient 1 in \(F_{f_\mu, \zeta_\mu}\).

As a consequence, we obtain the mirror isomorphism as Frobenius manifolds:
Theorem (Theorem 4.1). Assume that $\chi_A < 0$. There exists an isomorphism of Frobenius manifolds between the one constructed from the Gromov–Witten theory for $\mathbb{P}^1_A$ and the one constructed from the pair $(f_A, \zeta_A)$.

For the cases that $\chi_A > 0$ by [10, 12, 9] and the cases that $\chi_A = 0$ by [19], same statements as Theorem 4.1 are shown. Therefore, combining them with Theorem 4.1, it is shown that, for arbitrary triplet of positive integers $A$, there exists the classical mirror symmetry between the orbifold projective line with at most three orbifold points $\mathbb{P}^1_A$ and the pair of the cusp polynomial $f_A$ and the primitive form $\zeta_A$ associated to it.

In the last part of the present paper, we shall investigate the period mapping of the primitive form $\zeta_A$ for the case that $\chi_A \neq 0$. Here $\zeta_A$ is the one given in Theorem 2.38 for $\chi_A < 0$ and the one in Theorem 3.1 in [9] for $\chi_A > 0$. Let $T_A$ be the following Coxeter–Dynkin diagram:

\[
\begin{array}{ccc}
(1, a_1 - 1) & \cdots & (1, 1) \\
(2, 1) & \vdots & (3, a_3 - 1) \\
(2, a_2 - 1)
\end{array}
\]

and $h_A$ the complexified Cartan subalgebra of the Kac–Moody Lie algebra associated to $T_A$ (in particular, which is the simple Lie algebra for the case that $\chi_A > 0$). Denote by $\alpha_1, \ldots, \alpha_{(3, a_3 - 1)} \in h_A^* := \text{Hom}_C(h_A, \mathbb{C})$ simple roots corresponding the vertices in $T_A$, by $\alpha_1^\vee, \ldots, \alpha_{(3, a_3 - 1)}^\vee \in h_A$ simple coroots and by $\langle , \rangle : h_A^* \otimes_C h_A \rightarrow \mathbb{C}$ the natural pairing.

By using some isomorphisms (see Lemma 5.11, Lemma 5.12 and Lemma 5.13), we can identify $\alpha_i$, $i = 1, (1, 1), \ldots, (i, j), \ldots, (3, a_3 - 1)$ with homology classes represented by vanishing cycles in each fiber $X_{(0, s, s_{\mu_A})} \subset \mathbb{C}^3$ (see Subsection 5.1) over a general point $(0, s, s_{\mu_A})$. Under this situation, we shall consider periods of the holomorphic 2-form $\tilde{\zeta}_A$ on $X_{(0, s, s_{\mu_A})}$ induced from the primitive form $\zeta_A$ (see Subsection 5.2, (5.16)). Then we can calculate the intersection form $I_{(f_A, \zeta_A)}$ of $M := M_{(f_A, \zeta_A)}$:

Theorem (Theorem 5.14). Consider the periods

\[
x_i := \frac{1}{(2\pi \sqrt{-1})^2} \int_{\gamma_{(s, s_{\mu_A})}} \tilde{\zeta}_A, \quad i = 1, (1, 1), \ldots, (i, j), \ldots, (3, a_3 - 1), \quad (1.4)
\]
where \( \gamma_i(\mathbf{s}, \mathbf{s}_{\mu_A}) \) is the horizontal family of homology classes in \( \bigcup_{(0, \mathbf{s}, \mathbf{s}_{\mu_A}) \in \mathcal{M} \setminus \mathcal{D}} H_2(\mathcal{X}_{(0, \mathbf{s}, \mathbf{s}_{\mu_A})}, \mathbb{C}) \) identified with \( \alpha_i \), and the function

\[
x_{\mu_A} := \frac{1}{2\pi \sqrt{-1}} t_{\mu_A} = \frac{1}{2\pi \sqrt{-1}} \log s_{\mu_A}.
\]

They define flat coordinates with respect to \( I_{(f_A, \zeta_A)} \) on the monodromy covering space of \( M \setminus \mathcal{D} \). Moreover, one has

\[
I_{(f_A, \zeta_A)}(dx_i, dx_j) = -\frac{1}{(2\pi \sqrt{-1})^2} \langle \alpha_i, \alpha_j^\vee \rangle,
\]

\[
I_{(f_A, \zeta_A)}(dx_{\mu_A}, dx_i) = I_{(f_A, \zeta_A)}(dx_i, dx_{\mu_A}) = 0,
\]

\[
I_{(f_A, \zeta_A)}(dx_{\mu_A}, dx_{\mu_A}) = \frac{1}{(2\pi \sqrt{-1})^2} \chi_A.
\]

On the other hand, Dubrovin–Zhang \cite{3} constructed the Frobenius structure on a quotient space by an extended affine Weyl group \( \hat{W}_A \) for \( \hat{h}_A := h_A \times \mathbb{C} \):

**Theorem (3).** Assume that \( \chi_A > 0 \). There exists a unique Frobenius structure of rank \( \mu_A \) and dimension one on \( M_{\hat{W}_A} := \hat{h}_A / \hat{W}_A \) with flat coordinates \( t_1, t_{1,1}, \ldots, t_{i,j}, \ldots, t_{3,a_3-1}, t_{\mu_A} := (2\pi \sqrt{-1})x_{\mu_A} \) such that

\[
e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i-j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}},
\]

and the intersection form \( I_{\hat{W}_A} \) is given by

\[
I_{\hat{W}_A}(\alpha_i, \alpha_j) = -\frac{1}{(2\pi \sqrt{-1})^2} \langle \alpha_i, \alpha_j^\vee \rangle, \quad i, j = 1, (1, 1), \ldots, (3, a_3 - 1),
\]

\[
I_{\hat{W}_A}(\alpha_i, dx_{\mu_A}) = I_{\hat{W}_A}(dx_{\mu_A}, \alpha_i) = 0, \quad i = 1, (1, 1), \ldots, (3, a_3 - 1),
\]

\[
I_{\hat{W}_A}(dx_{\mu_A}, dx_{\mu_A}) = \frac{1}{(2\pi \sqrt{-1})^2} \chi_A,
\]

where we identify the cotangent space of \( M_{\hat{W}_A} \) with \( \hat{h}_A^* \oplus \mathbb{C} dx_{\mu_A} \).

The isomorphism as Frobenius manifolds between \( M_{(f_A, \zeta_A)} \) and \( M_{\hat{W}_A} \) is obtained in \cite{3} and \cite{10} for the case that \( A = (1, a_2, a_3) \) and \cite{12} for left cases such that \( \chi_A > 0 \). However, systematical proof is not found in any literature. By contrast we can give a natural proof by using Theorem 5.14 and the following theorem which might be known to experts:
Theorem (Theorem 7.1). A Frobenius manifold $M$ of rank $\mu_A$ and dimension one with the following $e$ and $E$ is uniquely determined by the intersection form $I_M$:

$$e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i-j}{a_i} t_i, \quad \frac{\partial}{\partial t_1} + \chi_A \frac{\partial}{\partial \mu_A}. \quad (1.9)$$

As a consequence, for the case that $\chi_A > 0$, we can obtain an isomorphism as Frobenius manifolds between $M_{(f_A, \zeta_A)}$ and $\hat{M}_{\hat{W}_A}$ in a natural way:

Corollary (Corollary 7.3). Assume that $\chi_A > 0$. There exists an isomorphism of Frobenius manifolds between the one constructed from the invariant theory of extended affine Weyl group $\hat{W}_A$ and the one constructed from the pair $(f_A, \zeta_A)$.

Acknowledgement

The first named author is deeply grateful to Claus Hertling and Christian Sevenheck for their valuable discussions and encouragement. He is supported by the JSPS International Training Program (ITP). The second named author is supported by JSPS KAKENHI Grant Number 24684005.

2. Notations and Terminologies

Let $A$ be a triplet $(a_1, a_2, a_3)$ of positive integers such that $a_1 \leq a_2 \leq a_3$. Set

$$\mu_A := a_1 + a_2 + a_3 - 1 \quad (2.1)$$

and

$$\chi_A := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1. \quad (2.2)$$

2.1. Universal unfolding of cusp singularities.

Definition 2.1. A polynomial $f_A(x) \in \mathbb{C}[x_1, x_2, x_3]$ given as

$$f_A(x) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1} \cdot x_1 x_2 x_3 \quad (2.3)$$

for some $q \in \mathbb{C}\{0\}$ is called the cusp polynomial of type $A$. In particular, it defines a cusp singularity if $\chi_A < 0$.

Let $f_A = f_A(x)$ be a cusp singularity of type $A$ such that $\chi_A < 0$. Then we regard $f_A$ as a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^3$.

Proposition 2.2. The germ of the hypersurface $X_0 := \{f_A(x) = 0\}$ in $(\mathbb{C}^3, 0)$ at the origin has at most an isolated singular point at the origin. In particular, we have

$$\mathcal{O}_{\mathbb{C}^3, 0} \left/ \left( \frac{\partial f_A}{\partial x_1}, \frac{\partial f_A}{\partial x_2}, \frac{\partial f_A}{\partial x_3} \right) \right. \quad (2.4)$$
is a \(\mathbb{C}\)-vector space of rank \(\mu_A\).

**Proof.** This is a well-known fact. Note that we can choose a basis of the vector space as

\[
1, \ x_i^j \ (i = 1, 2, 3, j = 1, \ldots, a_i - 1), \ x_1 x_2 x_3.
\]

\(\Box\)

We can consider the **universal unfolding** of \(f_A\).

**Definition 2.3.** A holomorphic function \(F_A(x; s, s_{\mu_A})\) defined on a neighborhood of \((0; 0, q)\) of \(\mathbb{C}^3 \times (\mathbb{C}^{\mu_A-1} \times \mathbb{C}\setminus\{0\})\) is given as follows;

\[
F_A(x; s, s_{\mu_A}) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - s_{\mu_A}^{-1} \cdot x_1 x_2 x_3 + s_1 \cdot 1 + \sum_{i=1}^3 \sum_{j=1}^{a_i-1} s_{i,j} \cdot x_i^j.
\]

(2.6)

In order to get a flat family of hypersurfaces where the fiber over \((0; 0, q)\) is isomorphic to the singularity \(X_0\) and generic fibers are smoothings of \(X_0\) containing vanishing cycles to the singularity, we have to avoid "cycles from infinity". Therefore, we shall choose suitably a domain and a range of \(F_A\) and an open set \(\mathbb{C}^{\mu_A-1} \times \mathbb{C}\setminus\{0\}\) in the following way.

Denote by

\[
p : \mathbb{C}^3 \times (\mathbb{C}^{\mu_A-1} \times \mathbb{C}) \to \mathbb{C}^{\mu_A-1} \times \mathbb{C}, \quad (x; s, s_{\mu_A}) \mapsto (s, s_{\mu_A})
\]

(2.7)

the projection map. Fix Euclidean norms \(\| \cdot \|\) on \(\mathbb{C}^3 \times (\mathbb{C}^{\mu_A-1} \times \mathbb{C}), \mathbb{C}\) and \(\mathbb{C}^{\mu_A-1} \times \mathbb{C}\). For positive real numbers \(r, \delta\) and \(\epsilon\), put

\[
\mathbb{C}_\delta := \{ w \in \mathbb{C} \mid \| w \| < \delta \}, \quad (2.8a)
\]

\[
M_\epsilon := \{ (s, s_{\mu_A}) \in \mathbb{C}^{\mu_A-1} \times \mathbb{C} \setminus \{0\} \mid \| (s, s_{\mu_A}) \| < \epsilon \}, \quad (2.8b)
\]

\[
\overline{M}_\epsilon := \{ (s, s_{\mu_A}) \in \mathbb{C}^{\mu_A-1} \times \mathbb{C} \mid \| (s, s_{\mu_A}) \| < \epsilon \}, \quad (2.8c)
\]

\[
X_{r,\delta,\epsilon} := \{ (x; s, s_{\mu_A}) \in \mathbb{C}^3 \times (\mathbb{C}^{\mu_A-1} \times \mathbb{C}\setminus\{0\}) \mid \| (x; s, s_{\mu_A}) \| < r, \| F_A(x; s, s_{\mu_A}) \| < \delta \} \cap p^{-1}(M_\epsilon), \quad (2.8d)
\]

\[
\overline{X}_{r,\delta,\epsilon} := \{ (x; s, s_{\mu_A}) \in \mathbb{C}^3 \times (\mathbb{C}^{\mu_A-1} \times \mathbb{C}) \mid \| (x; s, s_{\mu_A}) \| < r, \| F_A(x; s, s_{\mu_A}) \| < \delta \} \cap p^{-1}(\overline{M}_\epsilon). \quad (2.8e)
\]

For the choice \(1 \gg r \gg \delta \gg \epsilon > 0\) of the radius, we have

(i) \(p : X_{r,\delta,\epsilon} \to M_\epsilon\) is a smooth Stein map, which is topologically acyclic.
(ii) The holomorphic map

\[ \varphi : \mathcal{X}_{r,\delta,\epsilon} \to \mathbb{C}_\delta \times M_\epsilon, \quad (x; s, s_{\mu_A}) \mapsto (w; s, s_{\mu_A}) := (F_A(x; s, s_{\mu_A}); s, s_{\mu_A}) \]  

(2.9)

is a flat Stein map, whose fibers are smooth and transverse to \( \partial \mathcal{X}_{r,\delta,\epsilon} \) at each point of \( \partial \mathcal{X}_{r,\delta,\epsilon} \). The fiber \( \varphi^{-1}(0; 0, q) \cap \mathcal{X}_{r,\delta,\epsilon} \) over \( (0; 0, q) \) has only an isolated singular point isomorphic to \( X_0 \) and is contractible to a point.

We fix the constants \( r, \delta \) and \( \epsilon \) once for all and put \( X := X_{r,\delta,\epsilon}, X^0 := X_{r,\delta,\epsilon} \), \( M := M_\epsilon \) and \( \mathcal{M} := \mathcal{M}_\epsilon \). We also introduce the following notations for the later convenience:

**Definition 2.4.** Put

\[ X^s := \{(x; s, s_{\mu_A}) \in X \mid s = 0\}, \]  

\[ M^s := \{(s, s_{\mu_A}) \in M \mid s = 0\}, \]  

\[ F^s_A := F_A(x; 0, s_{\mu_A}). \]  

(2.10a, 2.10b, 2.10c)

**Remark 2.5.** \( M^s \) will correspond by mirror symmetry to the “small quantum cohomology subspace”, that is the reason of the letter “s”.

Set

\[ \mathcal{O}_C := \mathcal{O}_X \bigg/ \left( \frac{\partial F_A}{\partial x_1}, \frac{\partial F_A}{\partial x_2}, \frac{\partial F_A}{\partial x_3} \right). \]  

(2.11)

**Proposition 2.6.** The function \( F_A(x; s, s_{\mu_A}) \) satisfies the following conditions:

(i) \( F_A(x; 0, q) = f_A(x) \).

(ii) The \( \mathcal{O}_M \)-homomorphism \( \rho \) called the Kodaira–Spencer map defined as

\[ \rho : \mathcal{T}_M \to p_* \mathcal{O}_C, \quad \delta \mapsto \hat{\delta} F_A, \]  

(2.12)

is an isomorphism, where \( \hat{\delta} \) is a lifting on \( X \) of a vector field \( \delta \) with respect to the projection \( p \).

**Proof.** This is also a well-known fact. \( \square \)

Note that the tangent bundle \( \mathcal{T}_M \) naturally obtains an \( \mathcal{O}_M \)-algebra structure.

**Definition 2.7.** We shall denote by \( \circ \) the induced product structure on \( \mathcal{T}_M \) by the \( \mathcal{O}_M \)-isomorphism \( \hat{\delta} \). Namely, for \( \delta, \delta' \in \mathcal{T}_M \), we have

\[ \left( \hat{\delta} \circ \hat{\delta}' \right) F_A = \hat{\delta} F_A \cdot \hat{\delta}' F_A \text{ in } p_* \mathcal{O}_C. \]

(2.13)
Proof. It is almost obvious that $\iota$ follows from the following equality in (2.10). Namely, let

$$p \text{ residue classes of monomials is isomorphic to the } \mathcal{O} \text{ isomorphism of } O$$

where

$$\text{As an } \mathcal{O}_M \text{-module of rank } \infty \text{ we have the isomorphism of } O \text{ extension of } \mathcal{O}_M \text{ as a free } \mathcal{O}_M \text{-module. As an } \mathcal{O}_M \text{-algebra, it is } \mathcal{O}_M \text{-isomorphic to the } \mathcal{O}_M \text{-subalgebra of } \iota_* (p_* \mathcal{O}_C) \text{ whose } \mathcal{O}_M \text{-basis is given by the set of residue classes of monomials}$$

$$\{ 1, x_i^j (i = 1, 2, 3, j = 1, \ldots, a_i - 1), x_{i+1}^{-1} x_1 x_2 x_3 \} \text{.} \quad (2.17)$$

In particular, we have the isomorphism of $\mathcal{O}_M$-modules

$$\iota^* (p_* \mathcal{O}_C) \simeq p_* \mathcal{O}_C \text{.} \quad (2.18)$$

Proposition 2.9. As an $\mathcal{O}_M$-module, $p_* \mathcal{O}_C$ is free of rank $\mu_A$. As an $\mathcal{O}_M$-algebra, it is isomorphic to the $\mathcal{O}_M$-subalgebra of $\iota_* (p_* \mathcal{O}_C)$ whose $\mathcal{O}_M$-basis is given by the set of residue classes of monomials

$$\{ 1, x_i^j (i = 1, 2, 3, j = 1, \ldots, a_i - 1), s_{\mu_A}^{-1} x_1 x_2 x_3 \} \text{.} \quad (2.19)$$

Lemma 2.10. We have

$$p_* \mathcal{O}_C / s_{\mu_A} p_* \mathcal{O}_C \simeq \mathcal{O}_{M, \infty} [x_1, x_2, x_3] / \left( x_2 x_3, x_3 x_1, x_1 x_2, H_1(x, s), H_2(x, s) \right) \text{,} \quad (2.20)$$

is a free $\mathcal{O}_{M, \infty}$-module of rank $\mu_A$. In particular, we have

$$p_* \mathcal{O}_C / m_{(0,0)} p_* \mathcal{O}_C \simeq \mathbb{C} [x_1, x_2, x_3] / \left( x_2 x_3, x_3 x_1, x_1 x_2, a_1 x_1^{a_1} - a_2 x_2^{a_2} - a_3 x_3^{a_3} \right) \text{,} \quad (2.21)$$

where $m_{(0,0)}$ is the maximal ideal of $\mathcal{O}_M$ corresponding to the point $(0,0)$. $\square$

Proof. Some elementary calculations yield the statement.

By this lemma, we see that $p_* \mathcal{O}_C$ is free of rank $\mu_A$ as an $\mathcal{O}_M$-module. The rest follows from the following equality in $\iota_* (p_* \mathcal{O}_C)$:

$$\left[ s_{\mu_A}^{-1} x_1 x_2 x_3 \right] = \left[ a_i x_i^{a_i} + \sum_{j=1}^{a_i-1} j \cdot s_{i,j} \cdot x_i^j \right], \quad i = 1, 2, 3. \quad (2.22)$$

$\square$
Therefore, the analytic subset \( \overline{C} \) in \( \overline{X} \) defined by (2.16) is a closure of the relative critical set \( C \), which is flat and finite over \( \overline{M} \).

**Proposition 2.11.** Denote by \( \mathcal{T}_{\overline{M}}(-\log M_\infty) \) the sheaf of holomorphic vector fields on \( \overline{M} \) with logarithmic zeros along the divisor \( M_\infty \) defined as

\[
\mathcal{T}_{\overline{M}}(-\log M_\infty) := \{ \delta \in \mathcal{T}_{\overline{M}} \mid \delta s_{\mu_A} \in s_{\mu_A} \cdot \mathcal{O}_{\overline{M}} \}.
\]

(2.23)

Then, \( \mathcal{T}_{\overline{M}}(-\log M_\infty) \) is a free \( \mathcal{O}_{\overline{M}} \)-module of rank \( \mu_A \) and we have the isomorphism of \( \mathcal{O}_M \)-modules

\[
\iota^* (\mathcal{T}_{\overline{M}}(-\log M_\infty)) \simeq \mathcal{T}_M.
\]

(2.24)

Namely, \( \mathcal{T}_{\overline{M}}(-\log M_\infty) \) is an extension of \( \mathcal{T}_M \) as a free \( \mathcal{O}_M \)-module.

**Proof.** The statement follows from the fact that

\[
\mathcal{T}_{\overline{M}}(-\log M_\infty) \simeq \mathcal{O}_M \frac{\partial}{\partial s_1} \bigoplus_{1 \leq i \leq 3} \mathcal{O}_M \frac{\partial}{\partial s_{i,j}} \bigoplus_{1 \leq j \leq a_i - 1} \mathcal{O}_M s_{\mu_A} \frac{\partial}{\partial s_{\mu_A}}.
\]

(2.25)

\[\square\]

**Proposition 2.12.** The Kodaira–Spencer map \( \rho \) induces the \( \mathcal{O}_{\overline{M}} \)-isomorphism \( \overline{\rho} \)

\[
\overline{\rho}: \mathcal{T}_{\overline{M}}(-\log M_\infty) \longrightarrow p_* \mathcal{O}_C.
\]

(2.26)

**Proof.** The statement easily follows since the Kodaira–Spencer map \( \rho \) induces the \( \mathcal{O}_{\overline{M}} \)-isomorphism between \( \mathcal{T}_{\overline{M}}(-\log M_\infty) \) and the free \( \mathcal{O}_{\overline{M}} \)-submodule of \( \iota_* (p_* \mathcal{O}_C) \) spanned by the residue classes of monomials

\[
1, \ x_j^i (i = 1, 2, 3, \ j = 1, \ldots, a_i - 1), \ s_{\mu_A}^{-1} x_1 x_2 x_3.
\]

(2.27)

\[\square\]

2.2. Primitive vector field and Euler vector field.

**Definition 2.13.** The vector field \( e \) and \( E \) on \( M \) corresponding to the unit 1 and \( F \) by the \( \mathcal{O}_M \)-isomorphism (2.12) is called the primitive vector field and the Euler vector field, respectively. That is,

\[
\widehat{e} F_A = 1 \text{ and } \widehat{E} F_A = F_A \text{ in } p_* \mathcal{O}_C.
\]

(2.28)

**Proposition 2.14.** The primitive vector field \( e \) and the Euler vector field \( E \) on \( M \) are given by

\[
e = \frac{\partial}{\partial s_1}, \quad E = s_1 \frac{\partial}{\partial s_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i - 1} \frac{a_i - j}{a_i} s_{i,j} \frac{\partial}{\partial s_{i,j}} + \chi_A s_{\mu_A} \frac{\partial}{\partial s_{\mu_A}}.
\]

(2.29)

**Proof.** Easy calculation yields the statement. \[\square\]
Lemma 2.15. We have the “Euler’s identity”:

\[ F_A = EF_A + \sum_{i=1}^{3} \frac{1}{a_i} x_i \partial F_A \partial x_i. \quad (2.30) \]

Proof. Easy calculation yields the statement. \qed

Definition 2.16. An element \( g \in \mathcal{O}_M \) is of degree \( l \) for some \( k \in \mathbb{Q} \) if it satisfies the equation \( Eg = lg \) where \( E \) is the Euler vector field. If \( Eg = lg \), then \( l \) is denoted by \( \deg(g) \).

2.3. Filtered de Rham cohomology. For any non-negative integer \( i \), denote by \( \Omega^i_{X/M} \) the sheaf of relative holomorphic differential \( i \)-forms with respect to the projection \( p : X \to M \).

Definition 2.17. Define \( \Omega_{F_A} \) as

\[ \Omega_{F_A} := p_* \Omega^3_{X/M} \wedge p_* \Omega^2_{X/M}. \quad (2.31) \]

Proposition 2.18. \( \Omega_{F_A} \) is a free \( p_* \mathcal{O}_C \)-module of rank one and hence a free \( \mathcal{O}_M \)-module of rank \( \mu_A \).

Proof. A \( p_* \mathcal{O}_C \)-free base of \( \Omega_{F_A} \) can be chosen as \([\omega], \omega := dx_1 \wedge dx_2 \wedge dx_3\), and hence an \( \mathcal{O}_M \)-free basis of \( \Omega_{F_A} \) can be chosen as

\[ \{ [\omega], [x_i^j \omega] (i = 1, 2, 3, j = 1, \ldots, a_i - 1), [x_1 x_2 x_3 \omega] \}. \quad (2.32) \]

Definition 2.19. We set

\[ \mathcal{H}_{F_A} := \mathbb{R}^3 p_*(\Omega^*_{X/M} \otimes \mathcal{O}_M (\langle u \rangle)) \cup dX/M + dF_A \wedge \quad (2.33) \]

and call it the filtered de Rham cohomology group of the universal unfolding \( F_A \).

For any \( k \in \mathbb{Z} \), put

\[ \mathcal{H}_{F_A}^{(-k)} := \mathbb{R}^3 p_*(\Omega^*_{X/M} \otimes \mathcal{O}_M (\langle u \rangle)^k) \cup udX/M + dF_A \wedge \quad (2.34) \]

Obviously, one has an \( \mathcal{O}_M (\langle u \rangle) \)-isomorphism for all \( i \in \mathbb{Z} \)

\[ \mathcal{H}_{F_A}^{(0)} \simeq \mathcal{H}_{F_A}^{(-k)}, \quad \omega \mapsto u^k \omega, \quad (2.35) \]

and an \( \mathcal{O}_M (\langle u \rangle) \)-isomorphism

\[ \mathcal{H}_{F_A}^{(-k)} \otimes \mathcal{O}_M (\langle u \rangle) \simeq \mathcal{H}_{F_A}. \quad (2.36) \]
Furthermore, $\mathcal{H}^{(-k)}_{F_A}$ is naturally a submodule of $\mathcal{H}_{F_A}$ so that $\{\mathcal{H}^{(-k)}_{F_A}\}_{i \in \mathbb{Z}}$ form an increasing and exhaustive filtration of $\mathcal{H}_{F_A}$:

$$\cdots \subset \mathcal{H}^{(-k-1)}_{F_A} \subset \mathcal{H}^{(-k)}_{F_A} \subset \cdots \subset \mathcal{H}^{(0)}_{F_A} \subset \cdots \subset \mathcal{H}_{F_A},$$

such that $\mathcal{H}_{F_A}$ is complete with respect to the filtration in the following sense:

$$\bigcup_{k \in \mathbb{Z}} \mathcal{H}^{(-k)}_{F_A} = \mathcal{H}_{F_A} \text{ and } \bigcap_{k \in \mathbb{Z}} \mathcal{H}^{(-k)}_{F_A} = \{0\}.$$  

(2.37)

**Proposition 2.20.** For any $k \in \mathbb{Z}$, $\mathcal{H}^{(-k)}_{F_A}$ is an $\mathcal{O}_M[[u]]$-free module of rank $\mu_A$. In particular, we have the following short exact sequence of $\mathcal{O}_M$-modules

$$0 \rightarrow \mathcal{H}^{(-1)}_{F_A} \rightarrow \mathcal{H}^{(0)}_{F_A} \rightarrow \Omega_{F_A} \rightarrow 0.$$  

(2.39)

**Proof.** One can choose an $\mathcal{O}_M[[u]]$-free basis of $\mathcal{H}^{(0)}_{F_A}$ as

$$\{[\omega], [x_i^j \omega] (i = 1, 2, 3, j = 1, \ldots, a_i - 1), [x_1 x_2 x_3 \omega]\}$$

(2.40)

where $\omega = dx_1 \wedge dx_2 \wedge dx_3$. The rest is clear. $\Box$

**Definition 2.21.** Define an element $\zeta_A'$ of $\Gamma(M, \mathcal{H}^{(i)}_{F_A})$ as

$$\zeta_A' := [s_{1, \mu}^{-1} dx_1 \wedge dx_2 \wedge dx_3].$$

(2.41)

**Definition 2.22.** For any $k \in \mathbb{Z}$, define an $\mathcal{O}_{\overline{M}[[u]]}$-free module $\overline{\mathcal{H}}^{(-k)}_{F_A}$ of rank $\mu_A$ as

$$\overline{\mathcal{H}}^{(-k)}_{F_A} := \mathcal{O}_{\overline{M}[[u]]} \cdot u^k \zeta_A' \bigoplus_{1 \leq i \leq 3} \mathcal{O}_{\overline{M}[[u]]} \cdot u^k [x_i^j \zeta_A'] \bigoplus_{1 \leq j \leq a_i - 1} \mathcal{O}_{\overline{M}[[u]]} \cdot u^k s_{1, \mu}^{-1} x_1 x_2 x_3 \zeta_A',$$

(2.42)

where we regard $\zeta_A', x_i^j \zeta_A', s_{1, \mu}^{-1} x_1 x_2 x_3 \zeta_A'$ as elements of $\Gamma(\overline{M}, \mathcal{O}_{\overline{M}[[u]]})$.

**Proposition 2.23.** We have the isomorphism of $\mathcal{O}_{\overline{M}[[u]]}$-modules

$$ι^* \left( \overline{\mathcal{H}}^{(-k)}_{F_A} \right) \simeq \mathcal{H}^{(-k)}_{F_A}, \quad i \in \mathbb{Z}.$$  

(2.43)

Namely, $\overline{\mathcal{H}}^{(-k)}_{F_A}$ is an extension of $\mathcal{H}^{(-k)}_{F_A}$ as a free $\mathcal{O}_{\overline{M}[[u]]}$-module.

**Proof.** It is clear by Proposition 2.20. $\Box$

**Definition 2.24.** Define an $\mathcal{O}_{\overline{M}}$-free module $\overline{\Omega}_{F_A}$ of rank $\mu_A$ as

$$\overline{\Omega}_{F_A} := \mathcal{O}_{\overline{M}} \cdot r^{(0)}(\zeta_A') \bigoplus_{1 \leq i \leq 3} \mathcal{O}_{\overline{M}} \cdot r^{(0)}(x_i^j \zeta_A') \bigoplus_{1 \leq j \leq a_i - 1} \mathcal{O}_{\overline{M}} \cdot r^{(0)}(s_{1, \mu}^{-1} x_1 x_2 x_3 \zeta_A'),$$

(2.44)

where we regard $r^{(0)}(\zeta_A'), r^{(0)}(x_i^j \zeta_A'), r^{(0)}(s_{1, \mu}^{-1} x_1 x_2 x_3 \zeta_A')$ as elements of $\Gamma(\overline{M}, \mathcal{O}_{\overline{M}})$.
Proposition 2.25. We have the isomorphism of $\mathcal{O}_M$-modules

$$\iota^* (\Omega_{F_A}) \simeq \Omega_{F_A}. \quad (2.45)$$

Namely, $\Omega_{F_A}$ is an extension of $\Omega_{F_A}$ as a free $\mathcal{O}_{\overline{M}}$-module.

Proof. It is almost clear. □

Proposition 2.26. $\Omega_{F_A}$ is a free $p_*\mathcal{O}_{\overline{C}}$-module of rank one.

Proof. A $p_*\mathcal{O}_{\overline{C}}$-free base of $\Omega_{F_A}$ can be chosen as $r^{(0)}(\zeta'_A)$ in Definition 2.24. □

Proposition 2.27. We have the following short exact sequence of $\mathcal{O}_{\overline{M}}$-modules

$$0 \to \mathcal{H}_{F_A}^{(-1)} \hookrightarrow \mathcal{H}_{F_A}^{(0)} \xrightarrow{\rho^{(0)}} \Omega_{F_A} \to 0. \quad (2.46)$$

Proof. It is clear by their definitions. □

2.4. Gauß–Manin connection. We define the free $\mathcal{O}_M[[u]]$-module $\mathcal{T}_{\mathcal{C}_u \times M}$ of rank $\mu_A + 1$ as follows:

$$\mathcal{T}_{\mathcal{C}_u \times M} := \mathcal{O}_M[[u]] \frac{d}{du} \oplus \mathcal{O}_M[[u]] \otimes \mathcal{O}_M \mathcal{T}_M. \quad (2.47)$$

Definition 2.28. We define a connection, called the Gauß–Manin connection,

$$\nabla : \mathcal{T}_{\mathcal{C}_u \times M} \otimes \mathcal{O}_M \mathcal{H}_{F_A} \to \mathcal{H}_{F_A} \quad (2.48)$$

by letting; for $\delta \in \mathcal{T}_M$ and $\zeta = [\phi dx_1 \wedge \cdots \wedge dx_n] \in \mathcal{H}_{F_A}$,

$$\nabla_{\delta} \zeta := \left[ \frac{1}{u} (\delta F_A) \phi + \delta (\phi) \right] dx_1 \wedge \cdots \wedge dx_n, \quad (2.49a)$$

$$\nabla_{\frac{d}{du}} \zeta := \left[ -\frac{1}{u^2} F_A \phi + \frac{d\phi}{du} \right] dx_1 \wedge \cdots \wedge dx_n. \quad (2.49b)$$

Proposition 2.29. Gauß–Manin connection $\nabla : \mathcal{T}_{\mathcal{C}_u \times M} \otimes \mathcal{O}_M \mathcal{H}_{F_A} \to \mathcal{H}_{F_A}$ satisfies following:

(i) $\nabla$ is integrable:

$$\left[ \nabla_{\frac{d}{du}}, \nabla_{\frac{d}{du}} \right] = 0, \quad \left[ \nabla_{\frac{d}{du}}, \nabla_{\delta} \right] = 0, \quad \left[ \nabla_{\delta}, \nabla_{\delta'} \right] = \nabla_{[\delta, \delta']}, \quad \delta, \delta' \in \mathcal{T}_M. \quad (2.50)$$

(ii) $\nabla$ satisfies Griffith transversality: that is,

$$\nabla : \mathcal{T}_M \otimes \mathcal{O}_M \mathcal{H}_{F_A}^{(-k)} \longrightarrow \mathcal{H}_{F_A}^{(-k+1)}, \quad k \in \mathbb{Z}. \quad (2.51)$$

(iii) The covariant differentiation $\nabla_{\frac{d}{du}}$ satisfies

$$\nabla_{\frac{d}{du}} (\mathcal{H}_{F_A}^{(-k)}) \subset \mathcal{H}_{F_A}^{(-k+1)}, \quad k \in \mathbb{Z}. \quad (2.52)$$

Proof. See Proposition 4.5 of [17] and reference there in. □
We shall consider the extension of the Gauß–Manin connection $\nabla$ on $\mathcal{H}_F$ to the one on $\overline{\mathcal{H}}_F$. Define the free $\mathcal{O}_M[[u]]$-module $\mathcal{T}_{c\times M}(-\log M)\otimes \mathcal{O}_M$ of rank $\mu + 1$ as follows:

$$\mathcal{T}_{c\times M}(-\log M) := \mathcal{O}_M[[u]] \otimes \mathcal{O}_M(-\log M).$$

(2.53)

**Proposition 2.30.** The Gauß–Manin connection $\nabla$ on $\mathcal{H}_F$ extends to $\overline{\mathcal{H}}_F$ with logarithmic poles along $M_\infty$, namely, $\nabla$ induces the connection

$$\nabla : \mathcal{T}_{c\times M}(-\log M) \otimes \mathcal{O}_M \overline{\mathcal{H}}_F \rightarrow \overline{\mathcal{H}}_F,$$

(2.54)

satisfying the following conditions:

(i) $\nabla$ is integrable:

$$\left[ \nabla_{\frac{\partial}{\partial u}}, \nabla_{\frac{\partial}{\partial u}} \right] = 0, \quad \left[ \nabla_{\frac{\partial}{\partial A}}, \nabla_{\frac{\partial}{\partial A}} \right] = 0, \quad \left[ \nabla_{\delta}, \nabla_{\delta'} \right] = \nabla_{[\delta, \delta']}, \quad \delta, \delta' \in \mathcal{T}_M(-\log M_\infty).$$

(2.55)

(ii) $\nabla$ satisfies Griffith transversality: that is,

$$\nabla : \mathcal{T}_M(-\log M_\infty) \otimes \mathcal{O}_M \overline{\mathcal{H}}_F^{(-k)} \rightarrow \overline{\mathcal{H}}_F^{(-k+1)}, \quad k \in \mathbb{Z}.$$  

(2.56)

(iii) The covariant differentiation $\nabla_{\frac{\partial}{\partial u}}$ satisfies

$$\nabla_{\frac{\partial}{\partial u}} (\overline{\mathcal{H}}_F^{(-k)}) \subset \overline{\mathcal{H}}_F^{(-k+1)}, \quad k \in \mathbb{Z}.$$  

(2.57)

**Proof.** We shall check that

$$s_{\mu A} \nabla \frac{\partial}{\partial s_{\mu A}} (\overline{\mathcal{H}}_F) \subset \overline{\mathcal{H}}_F.$$  

(2.58a)

$$\nabla \frac{\partial}{\partial \xi_i} (\overline{\mathcal{H}}_F) \subset \overline{\mathcal{H}}_F, \quad \frac{\partial}{\partial \xi_i} (\overline{\mathcal{H}}_F) \subset \overline{\mathcal{H}}_F, \quad i = 1, 2, 3, j = 1, \ldots, a_i - 1.$$  

(2.58b)

$$u \nabla_{\frac{\partial}{\partial u}} (\overline{\mathcal{H}}_F) \subset \overline{\mathcal{H}}_F.$$  

(2.58c)

First we shall check the condition (2.58a). One has

$$s_{\mu A} \nabla \frac{\partial}{\partial s_{\mu A}} \zeta' = \frac{1}{u} s_{\mu A}^{-1} x_1 x_2 x_3 \zeta' - \zeta',$$

$$s_{\mu A} \nabla \frac{\partial}{\partial s_{\mu A}} x_i \zeta' = \frac{1}{u} s_{\mu A}^{-1} x_1 x_2 x_3 \cdot x_i \zeta' - x_i \zeta',$$

$$s_{\mu A} \nabla \frac{\partial}{\partial s_{\mu A}} s_{\mu A}^{-1} x_1 x_2 x_3 \zeta' = \frac{1}{u} (s_{\mu A}^{-1} x_1 x_2 x_3)^2 \zeta' - 2 s_{\mu A}^{-1} x_1 x_2 x_3 \zeta'.$$

The images of $s_{\mu A}^{-1} x_1 x_2 x_3 \cdot x_i \zeta'$ and $(s_{\mu A}^{-1} x_1 x_2 x_3)^2 \zeta'$ by $\iota^0(0)$ can be extended to $\overline{\Omega}_F$. By Proposition 2.20 and Proposition 2.27, we can show that $1/u \cdot s_{\mu A}^{-1} x_1 x_2 x_3 \zeta'$, $1/u \cdot (s_{\mu A}^{-1} x_1 x_2 x_3)^2 \zeta' \in \overline{\mathcal{H}}_F$. 

Next we shall check that the condition (2.58b). The assertion for $s_1$ is obvious. One has

$$\nabla \frac{\partial}{\partial x_i} \phi \cdot \zeta' = \frac{1}{u} x_i \zeta' \cdot \phi \cdot \zeta'.$$
where \( \phi \) is an element of the set \( \{ 1, \partial F_A/\partial s_{i,j} \mid i = 1, 2, 3, j = 1, \ldots, a_i - 1 \}, \partial F_A/\partial s_{\mu A} \} \).

The image of \( x_i' \cdot \phi \cdot \zeta_A \) by \( r^{(0)} \) can be extended to \( \Omega_{FA} \). By Proposition 2.26 and Proposition 2.27, we can show that \( 1/u \cdot x_i' \cdot \phi \cdot \zeta_A' \in H_{FA} \).

Finally we shall check the condition (2.58c). One has

\[
\frac{u}{\partial} \cdot \frac{\partial F_A}{\partial \phi} \cdot \zeta_A' = \frac{-1}{u} F_A \cdot \phi \cdot \zeta_A',
\]

where \( \phi \) is an element of the set \( \{ 1, \partial F_A/\partial s_{i,j} \mid i = 1, 2, 3, j = 1, \ldots, a_i - 1 \}, \partial F_A/\partial s_{\mu A} \} \).

The image of \( F_A \cdot \phi \cdot \zeta_A' \) by \( r^{(0)} \) can be extended to \( \Omega_{FA} \) since we have \( F_A = EF_A + \sum_{i=1}^{3} \frac{1}{a_i} x_i \frac{\partial F_A}{\partial x_i} \).

By Proposition 2.26 and Proposition 2.27, we can show that \( -1/u \cdot F_A \cdot \phi \cdot \zeta_A' \in H_{FA} \). The conditions (i), (ii) and (iii) follow from Proposition 2.29.

\[ \square \]

2.5. Higher residue pairing.

**Definition 2.31.** Define an \( O_M \)-bilinear form \( J_A \) on \( \Omega_{FA} \) by

\[
J_A(\omega_1, \omega_2) := -\text{Res}_{M} \left[ \begin{array}{c} \phi_1 \phi_2 dx_1 \wedge dx_2 \wedge dx_3 \\ \frac{\partial F_A}{\partial x_1} \frac{\partial F_A}{\partial x_2} \frac{\partial F_A}{\partial x_3} \end{array} \right],
\]

where \( \omega_1 = [\phi_1 dx_1 \wedge dx_2 \wedge dx_3] \) and \( \omega_2 = [\phi_2 dx_1 \wedge dx_2 \wedge dx_3] \).

**Lemma 2.32.** The \( O_M \)-bilinear form \( J_A \) on \( \Omega_{FA} \) is non-degenerate.

**Proof.** This is a well-known fact (cf. Section 10.4 of [6]). \[ \square \]

In order to define the higher residue pairing, we prepare a notation. For \( P = \sum_{i \in \mathbb{Z}} p_i u^i \in O_M((u)) \), set \( P^* := \sum_{i \in \mathbb{Z}} p_i (-u)^i \) such that \( (P^*)^* = P \).

**Definition 2.33.** An \( O_M \)-bilinear form

\[
K_{FA} : \mathcal{H}_{FA} \otimes_{O_M} \mathcal{H}_{FA} \to O_M((u))
\]

is called the higher residue pairing if it satisfies the following properties:

(i) For all \( \omega_1, \omega_2 \in \mathcal{H}_{FA} \),

\[
K_{FA}(\omega_1, \omega_2) = (-1)^3 K_{FA}(\omega_2, \omega_1)^*.
\]

(ii) For all \( P \in O_M((u)) \) and \( \omega_1, \omega_2 \in \mathcal{H}_{FA} \),

\[
P K_{FA}(\omega_1, \omega_2) = K_{FA}(P \omega_1, \omega_2) = K_{FA}(\omega_1, P^* \omega_2).
\]
(iii) For all $\omega_1, \omega_2 \in \mathcal{H}_{F_A}^{(0)}$,
\[ K_{F_A}(\omega_1, \omega_2) \in u^3 \mathcal{O}_M[[u]]. \] (K3)

(iv) The following diagram is commutative:
\[
\begin{array}{ccc}
K_{F_A} : \mathcal{H}_{F_A}^{(0)} \times \mathcal{H}_{F_A}^{(0)} & \longrightarrow & u^3 \mathcal{O}_M[[u]] \\
\downarrow & & \downarrow \text{mod } u^4 \mathcal{O}_M[[u]] \\
J_{F_A} : \Omega_{F_A} \times \Omega_{F_A} & \longrightarrow & u^3 \mathcal{O}_M.
\end{array}
\]

(v) For all $\omega_1, \omega_2 \in \mathcal{H}_F$ and $\delta \in \mathcal{T}_M$,
\[ \delta K_{F_A}(\omega_1, \omega_2) = K_{F_A}(\nabla_\delta \omega_1, \omega_2) + K_{F_A}(\omega_1, \nabla_\delta \omega_2). \] (K4)

(vi) For all $\omega_1, \omega_2 \in \mathcal{H}_F$,
\[ u \frac{d}{du} K_{F_A}(\omega_1, \omega_2) = K_{F_A}(u \nabla \frac{d}{du} \omega_1, \omega_2) + K_{F_A}(\omega_1, u \nabla \frac{d}{du} \omega_2). \] (K5)

**Definition 2.34.** Define $K_{F_A}^{(k)}$ for $k \in \mathbb{Z}$ by the coefficient of the expansion of $K_{F_A}$ in $u$
\[ K_{F_A}(\omega_1, \omega_2) := \sum_{k \in \mathbb{Z}} K_{F_A}^{(k)}(\omega_1, \omega_2) u^{k+3}, \] (2.62)
and call it the $k$-th *higher residue pairing*.

**Theorem 2.35** (K. Saito [16]). There exists a unique higher residue pairing $K_{F_A}$.

*Proof.* See Theorem of [16]. \(\Box\)

We shall consider the extension $\overline{K}_{F_A}$ of $K_{F_A}$ on $\overline{\mathcal{H}}_{F_A}$.

**Proposition 2.36.** The pairing $K_{F_A}$ on $\mathcal{H}_{F_A}^{(0)}$ induces an $\mathcal{O}_{\mathcal{T}_M}$-bilinear form
\[ \overline{K}_{F_A} : \overline{\mathcal{H}}_{F_A}^{(0)} \otimes_{\mathcal{O}_{\mathcal{T}_M}} \overline{\mathcal{H}}_{F_A}^{(0)} \longrightarrow u^3 \mathcal{O}_{\mathcal{T}_M}[[u]] \] (2.63)
whose restriction to $\iota^* \left( \overline{\mathcal{H}}_{F_A}^{(0)} \right) \simeq \mathcal{H}_{F_A}^{(0)}$ coincides with $K_{F_A}$.

*Proof.* This follows from Lemma 3.4 of [7], where $\overline{M}$, $M$, $M_\infty$, $\mathcal{H}_{F_A}^{(0)}$, $K_{F_A}$, $\overline{\mathcal{H}}_{F_A}^{(0)}$ correspond to $X$, $Y$, $D$, $\mathcal{H}$, $P$, $\mathcal{F}$ in [7], respectively. \(\Box\)

### 2.6. Primitive form.

**Definition 2.37.** An element $\zeta \in \Gamma(M, \mathcal{H}_{F_A}^{(0)})$ is called a *primitive form* for the tuple $(\mathcal{H}_{F_A}^{(0)}, \nabla, K_{F_A})$ if it satisfies following five conditions:

(i) $u \nabla \delta \zeta = \zeta$ and $\zeta$ induces $\mathcal{O}_M$-isomorphism:
\[ \mathcal{T}_M[[u]] \simeq \mathcal{H}_{F_A}^{(0)}, \quad \sum_{k=0}^\infty \delta_k u^k \mapsto \sum_{k=0}^\infty u^k (u \nabla \delta_k \zeta). \] (P1)
(ii) For all \( \delta, \delta' \in \mathcal{T}_M \),
\[
K_{F_A}(u \nabla_{\delta} \zeta, u \nabla_{\delta'} \zeta) \in \mathbb{C} \cdot u^3. \tag{P2}
\]
(iii) There exists \( r \in \mathbb{C} \) such that
\[
\nabla_{u \frac{\partial}{\partial u}} e \zeta = r \zeta. \tag{P3}
\]
(iv) There exists a connection \( \nabla : \mathcal{T}_M \times \mathcal{T}_M \rightarrow \mathcal{T}_M \) such that
\[
u u \nabla_{\delta} \nabla_{\delta'} \zeta = \nabla_{\delta \delta'} \zeta + u \nabla_{\nabla_{\delta} \delta} \zeta, \quad \delta, \delta' \in \mathcal{T}_M. \tag{P4}
\]
(v) There exists an \( \mathcal{O}_M \)-endomorphism \( N : \mathcal{T}_M \rightarrow \mathcal{T}_M \) such that
\[
u u \nabla_{\frac{\partial}{\partial u}} (u \nabla_{\delta} \zeta) = -\nabla_{E \delta} \zeta + u \nabla_{N \delta} \zeta, \quad \delta \in \mathcal{T}_M. \tag{P5}
\]
In particular, the constant \( r \) of (P3) is called the minimal exponent.

It was claimed by K. Saito \[15\] and is proven by M. Saito \[18\] that, for the universal unfolding of a cusp singularity, there exists a unique primitive form \( \zeta_A \) with the minimal exponent \( r = 1 \), whose associated exponents are given by the mixed Hodge structure of the cusp singularity. In order to prove the mirror isomorphism, we give its local expression as follows:

**Theorem 2.38** (cf. \[15, 18\]). Assume that \( \chi_A < 0 \). There exists a unique primitive form \( \zeta_A \) for the tuple \( (\mathcal{H}^{(0)}_{F_A}, \nabla, K_{F_A}) \) with the minimal exponent \( r = 1 \) such that
\[
\zeta_A|_{s=0} = [s^{-1}_{\mu A} dx_1 \wedge dx_2 \wedge dx_3]. \tag{2.64}
\]

### 3. Proof of Theorem 2.38

In this section, we shall prove Theorem 2.38. The detail of the proof is not necessary for following sections. The reader interested in the mirror isomorphism and the period mapping of the primitive form can skip this section.

Set \( \overline{M^s} := \{(s, s_{\mu A}) \in \overline{M} \mid s = 0\} \) and define elements of \( \overline{\mathcal{H}^{(0)}_{F_A}} := \mathcal{H}^{(0)}_{F_A}|_{\overline{M^s}} \) by
\[
\zeta_1 := [s^{-1}_{\mu A} dx_1 \wedge dx_2 \wedge dx_3], \tag{3.1a}
\]
\[
\zeta_{i,j} := [s^{-1}_{\mu A} x_i^j dx_1 \wedge dx_2 \wedge dx_3], \quad i = 1, 2, 3, \quad j = 1, \ldots, a_i - 1, \tag{3.1b}
\]
\[
\zeta_{\mu A} := u \nabla_{s\frac{\partial}{\partial s_{\mu A}}} \zeta_1, \tag{3.1c}
\]
and put
\[
V := \mathbb{C} \zeta_1 \bigoplus_{1 \leq i \leq 3, 1 \leq j \leq a_i - 1} \mathbb{C} \zeta_{i,j} \bigoplus \mathbb{C} \zeta_{\mu A}. \tag{3.2}
\]
It is obvious that there is an \( \mathcal{O}_{\overline{M^s}} \)-isomorphism
\[
\overline{\mathcal{H}^{(0)}_{F_A}} \simeq \mathcal{O}_{\overline{M^s}}[[u]] \otimes_{\mathbb{C}} V. \tag{3.3}
\]
From now on, we shall check that the elements above define a good section in the sense of Kyoji Saito (see Section 4 of [15]):

**Proposition 3.1.** The elements $\zeta_1$, $\zeta_{i,j}$ ($i = 1, 2, 3$, $j = 1, \ldots, a_i - 1$), $\zeta_{\mu A} \in H_{F_A}^{(0)}$ satisfy the following conditions:

(i) $K_{F_A}^s (\zeta_i, \zeta_j) \big|_{s_{\mu A} = 0} \in \mathbb{C} u^3$,

(ii) $\left( \nabla_{u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}}} \zeta_i \right) \big|_{s_{\mu A} = 0} = N_0 \left( u^{-1} \cdot \zeta_i \big|_{s_{\mu A} = 0} \right) + S_0 \left( \zeta_i \big|_{s_{\mu A} = 0} \right)$,

where $\tilde{\zeta} := t(\zeta_1, \zeta_{1,1}, \ldots, \zeta_{3,a_3-1}, \zeta_{\mu A})$, $N_0 \in M(\mu_A, \mathbb{Q})$ and $S_0 \in M(\mu_A, \mathbb{C})$ with $N_0$ nilpotent and $S_0$ diagonal. Moreover, $S_0$ has eigenvalues $\{\alpha_1, \alpha_{1,1}, \ldots, \alpha_{3,a_3-1, \alpha_{\mu A}}\}$ which coincide with the exponents of the cusp polynomial $f_A$:

$$\alpha_1 = 1, \quad \alpha_{i,j} = 1 \div \frac{j}{a_i} (i = 1, 2, 3, \quad j = 1, \ldots, a_i - 1), \quad \alpha_{\mu A} = 2.$$  \hspace{1cm} (3.4)

**Lemma 3.2.** We have

$$\nabla_{u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}}} \zeta_1 = 1 \cdot \zeta_1,$$  \hspace{1cm} (3.5a)

$$\nabla_{u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}}} \zeta_{i,j} = \left( 1 + \frac{j}{a_i} \right) \cdot \zeta_{i,j}, \quad i = 1, 2, 3, \quad j = 1, \ldots, a_i - 1,$$  \hspace{1cm} (3.5b)

$$\nabla_{u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}}} \zeta_{\mu A} = 2 \cdot \zeta_{\mu A}.$$  \hspace{1cm} (3.5c)

**Proof.** They easily follow from Lemma 3.4 of [9]. \hfill \Box

**Lemma 3.3.** We have

$$\deg \left( K_{F_A}^{(k)} (\zeta_i, \zeta_j) \right) = \deg(\zeta_i) + \deg(\zeta_j) - 3 - k, \quad k \in \mathbb{Z}.$$  \hspace{1cm} (3.6)

**Proof.** The equations (K4) and (K5) yields

$$\left( u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}} \right) K_{F_A}^s (\zeta_i, \zeta_j) = K_{F_A}^s \left( \nabla_{u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}}} \zeta_i, \zeta_j \right) + K_{F_A}^s \left( \zeta_i, \nabla_{u \frac{d}{du} + \chi_A s_{\mu A} \frac{\partial}{\partial s_{\mu A}}} \zeta_j \right).$$

The statement now easily follows from the definition of $K_{F_A}^{(k)}$ and Lemma 3.2 \hfill \Box

**Lemma 3.4.** We have

$$K_{F_A}^s (\zeta_i, \zeta_j) \in u^3 \mathcal{O}_{F_A}[[u]].$$  \hspace{1cm} (3.7)

In particular, we have

$$K_{F_A}^s (\zeta_i, \zeta_j) \big|_{s_{\mu A} = 0} \in \mathbb{C} u^3.$$  \hspace{1cm} (3.8)
Proof. By Lemma 3.3, we have \(\deg(K^{(k)}_{F_{A}^{\mu A}}(\zeta_i, \zeta_j)) < 0\) if \(k \geq 1\) except for the case \(\zeta_i = \zeta_j = \zeta_{\mu A}\) and \(k = 1\). However, \(K^{(1)}_{F_{A}^{\mu A}}(\zeta_{\mu A}, \zeta_{\mu A}) = 0\) since \(K^{(1)}_{F_{A}^{\mu A}}\) is skew-symmetric. Note that \(\deg(s_{\mu A}) = \chi_{A} < 0\). Therefore, if \(k \geq 1\), then \(K^{(k)}_{F_{A}^{\mu A}}(\zeta_i, \zeta_j) \in O_{\mathcal{M}}^{s_{\mu A}},\) which vanishes at \(s_{\mu A} = 0\). □

Lemma 3.5. The residue endomorphism \(\left(\nabla_{s_{\mu A}^\partial} s_{\mu A}^\partial \right)\) on \(H^{(1)}_{F_{A}^{\mu A}}|_{s_{\mu A}=0}\) has zero eigenvalues and \(2 \times 2\) nilpotent Jordan block, namely,

\[\nabla_{s_{\mu A}^\partial} s_{\mu A}^\partial \nabla_{s_{\mu A}^\partial} s_{\mu A}^\partial \zeta_1 = 0.\]

(3.9)

Proof. This lemma follows from Lemma 10.2 in [6]. □

Hence, combining Lemma 3.2 and Lemma 3.4 with Lemma 3.5, it turns out that the set \(\{\zeta_1, \ldots, \zeta_{\mu A}\}\) of elements in \(H^{(0)}_{F_{A}^{\mu A}}|_{s_{\mu A}=0}\) defines a good section, which gives a primitive form \(\zeta_{A}\) for the tuple \((H^{(0)}_{F_{A}^{\mu A}}, \nabla, K_{F_{A}^{\mu A}})\) with the minimal exponent \(r = 1\). The exponents defined by this primitive form \(\zeta_{A}\) is given by

\[\left\{1, 1 + \frac{1}{a_1}, \ldots, 1 + \frac{a_1 - 1}{a_1}, 1 + \frac{1}{a_2}, \ldots, 1 + \frac{a_2 - 1}{a_2}, 1 + \frac{1}{a_3}, \ldots, 1 + \frac{a_3 - 1}{a_3}, 2\right\}\]

(3.10)

and they coincide with the exponents defined by the mixed Hodge structure for \(f_{A}\). Up to a constant factor, \(\zeta_{A}\) is the primitive form associated to \(f_{A}\) obtained by the general theory developed by M. Saito (see Theorem in Section 3.6 of [18]). Therefore, we see that

\[\zeta_{A}|_{s=0} = \zeta_1 = [s_{\mu A}^{-1}dx_1 \wedge dx_2 \wedge dx_3].\]

(3.11)

This finishes the proof of Theorem 2.38.

4. Mirror Symmetry

In this section, we shall show the isomorphism of Frobenius manifolds between the one constructed from the orbifold Gromov-Witten theory of \(\mathbb{P}^1_{A}\) for the case \(\chi_{A} < 0\) and the one constructed from the pair of the cusp singularity \(f_{A}\) and the primitive form \(\zeta_{A}\) obtained in Theorem 2.38.

4.1. Mirror isomorphism. The main theorem of this section is the following:

Theorem 4.1. Assume that \(\chi_{A} < 0\). There exists an isomorphism of Frobenius manifolds between the one constructed from the Gromov–Witten theory for \(\mathbb{P}^1_{A}\) and the one constructed from the pair \((f_{A}, \zeta_{A})\).

Proof. Theorem 4.1 immediately follows from Theorem 3.1 in [8] and the following Theorem 4.2 □
In our previous paper [8], it is shown that a Frobenius structure with certain conditions can be reconstructed uniquely and the one constructed from the Gromov–Witten theory of the orbifold projective line $\mathbb{P}^1_A$ with arbitrary triplet of positive integers $A$ satisfies the conditions. We shall see the Frobenius structure constructed from the pair $(f_A, \zeta_A)$ also satisfies the conditions, i.e., the following Theorem 4.2 holds:

**Theorem 4.2.** Assume that $\chi_A < 0$. For the Frobenius structure of rank $\mu_A$ and dimension one constructed from the pair $(f_A, \zeta_A)$, there exist flat coordinates $t_1, t_{1,1}, \ldots, t_{3, a_3 - 1}$, $t_{\mu_A}$ satisfying the following conditions:

(i) The unit vector field $e$ and the Euler vector field $E$ are given by

$$e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i - 1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}.$$

(ii) The non-degenerate symmetric bilinear form $\eta$ on $T_M$ defined by

$$\eta(\delta, \delta') := K_{f_A}^{(0)}(u \nabla_\delta \zeta_A, u \nabla_\delta' \zeta), \quad \delta, \delta' \in T_M$$

satisfies

$$\eta \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{\mu_A}} \right) = \eta \left( \frac{\partial}{\partial t_{\mu_A}}, \frac{\partial}{\partial t_1} \right) = 1,$$

$$\eta \left( \frac{\partial}{\partial t_{i_1,j_1}}, \frac{\partial}{\partial t_{i_2,j_2}} \right) = \begin{cases} \frac{1}{a_i} & i_1 = i_2 \text{ and } j_2 = a_i - j_1, \\ 0 & \text{otherwise}. \end{cases}$$

(iii) The Frobenius potential $F_{f_A, \zeta_A}$ satisfies $E F_{f_A, \zeta_A} |_{t_1=0} = 2 F_{f_A, \zeta_A} |_{t_1=0}$.

$$F_{f_A, \zeta_A} |_{t_1=0} \in \mathbb{C}[t_{1,1}, \ldots, t_{1,a_1-1}, t_{1,2}, \ldots, t_{2,a_2-1}, t_{3,1}, \ldots, t_{3,a_3-1}, e^{\mu_A}].$$

(iv) The restriction of the Frobenius potential $F_{f_A, \zeta_A}$ to the submanifold $\{t_1 = e^{\mu_A} = 0\}$ is given as

$$F_{f_A, \zeta_A} |_{t_1=e^{\mu_A}=0} = G^{(1)} + G^{(2)} + G^{(3)},$$

where $G^{(i)} \in \mathbb{C}[t_{i,1}, \ldots, t_{i,a_i-1}], i = 1, 2, 3$.

(v) In the frame $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{1,1}}, \ldots, \frac{\partial}{\partial t_{3,a_3-1}}, \frac{\partial}{\partial t_{\mu_A}}$ of $T_M$, the product $\circ$ can be extended to the limit $t_1 = t_{1,1} = \cdots = t_{3,a_3-1} = e^{\mu_A} = 0$. The $\mathbb{C}$-algebra obtained in this limit is isomorphic to

$$\mathbb{C}[x_1, x_2, x_3] / (x_1x_2 - a_1x_1^{a_1} - a_2x_2^2 - a_3x_3^3),$$

where $\partial/\partial t_{1,1}, \partial/\partial t_{2,1}, \partial/\partial t_{3,1}$ are mapped to $x_1, x_2, x_3$, respectively.

(vi) The term $t_{1,1}t_{2,1}t_{3,1}e^{\mu_A}$ occurs with the coefficient one in $F_{f_A, \zeta_A}$. 

In the frame $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{1,1}}, \ldots, \frac{\partial}{\partial t_{3,a_3-1}}, \frac{\partial}{\partial t_{\mu_A}}$ of $T_M$, the product $\circ$ can be extended to the limit $t_1 = t_{1,1} = \cdots = t_{3,a_3-1} = e^{\mu_A} = 0$. The $\mathbb{C}$-algebra obtained in this limit is isomorphic to

$$\mathbb{C}[x_1, x_2, x_3] / (x_1x_2 - a_1x_1^{a_1} - a_2x_2^2 - a_3x_3^3),$$

where $\partial/\partial t_{1,1}, \partial/\partial t_{2,1}, \partial/\partial t_{3,1}$ are mapped to $x_1, x_2, x_3$, respectively.
We shall show Theorem 4.2 by checking one by one that the Frobenius structure constructed from the pair \((f_A, \zeta_A)\) satisfies the conditions in following subsections.

4.2. **Condition (i).**

**Lemma 4.3.** We have

\[
\text{Res}_{X^s/M^s} \left[ 1 \cdot dx_1 \wedge dx_2 \wedge dx_3 \right] = 0, \tag{4.1}
\]

\[
\text{Res}_{X^s/M^s} \left[ \frac{\partial F_A^s}{\partial x_1} \frac{\partial F_A^s}{\partial x_2} \frac{\partial F_A^s}{\partial x_3} \right] = 0, \quad i = 1, 2, 3, j = 1, \ldots, a_i - 1, \quad \tag{4.2}
\]

and

\[
\text{Res}_{X^s/M^s} \left[ x_1^i x_2 x_3 \cdot dx_1 \wedge dx_2 \wedge dx_3 \right] = s_{\mu A}^3. \tag{4.3}
\]

**Proof.** Some elementary calculations of residues yield the statement. \(\square\)

**Lemma 4.4.** We have

\[
K_{F_A}^{(0)}(\zeta_A, u \nabla \frac{\partial}{\partial x_i} \zeta_A) = 0, \quad K_{F_A}^{(0)}(\zeta_A, u \nabla \frac{\partial}{\partial x_{i,j}} \zeta_A) = 0, \quad i = 1, 2, 3, j = 1, \ldots, a_i - 1, \tag{4.4}
\]

and

\[
K_{F_A}^{(0)}(\zeta_A, u \nabla s_{\mu A} \frac{\partial}{\partial s_{\mu A}} \zeta_A) = 1. \tag{4.5}
\]

**Proof.** The statement follows from Lemma 4.3. \(\square\)

**Lemma 4.5.** The one form \(\theta \in \Gamma(M, \Omega^1_M)\) defined by

\[
\theta := K_{F_A}^{(0)}(\zeta_A, u \nabla \frac{\partial}{\partial x_1} \zeta_A) ds_1 + \sum_{i=1}^3 \sum_{j=1}^{a_i-1} K_{F_A}^{(0)}(\zeta_A, u \nabla \frac{\partial}{\partial x_{i,j}} \zeta_A) ds_{i,j} + K_{F_A}^{(0)}(\zeta_A, u \nabla \frac{\partial}{\partial s_{\mu A}} \zeta_A) ds_{\mu A}, \tag{4.6}
\]

is a closed form which is independent from the choice of coordinates on \(M\). Moreover, there exists a flat coordinate \(t\) such that \(\theta = dt\).

**Proof.** See section 3.3 3) of [15]. \(\square\)

Combining these two Lemmas, we have \(dt|_{s=0} = ds_{\mu A}/s_{\mu A}\). Therefore, we can choose a flat coordinate \(t_{\mu A}\) such that

\[
e^{t_{\mu A}} := s_{\mu A} \cdot t(s, e^{s_{\mu A}}), \quad \deg(t(s, e^{s_{\mu A}})) = 0, \quad t(0, e^{s_{\mu A}}) = 1, \tag{4.7a}
\]
\[ \frac{\partial t_{\mu A}}{\partial s_1} \big|_{s=0} = 0, \quad \frac{\partial t_{\mu A}}{\partial s_{i,j}} \big|_{s=0} = 0. \]  
\text{(4.7b)}

Since \( \zeta_1, \ldots, \zeta_{\mu_A} \) form a \( \mathcal{O}_M[[u]] \)-basis of \( \mathcal{H}_{F_A}^{(0)} \) at \((0, q) \in \overline{M}\), one can choose other flat coordinates \( t_1, t_{1,1}, \ldots, t_{3,\alpha_3-1} \) such that
\[ t_1|_{s=0} = t_{1,1}|_{s=0} = \cdots = t_{3,\alpha_3-1}|_{s=0} = 0 \]  
\text{(4.8)}

together with the following normalization;
\[ \frac{\partial t_1}{\partial s_1} \big|_{(s,s_{\mu_A})=(0,0)} = 1, \quad \frac{\partial t_1}{\partial s_{i,j}} \big|_{(s,s_{\mu_A})=(0,0)} = 0, \quad \frac{\partial t_1}{\partial s_{\mu_A}} \big|_{(s,s_{\mu_A})=(0,0)} = 0, \]  
\text{(4.9a)}
\[ \frac{\partial t_{i,j}}{\partial s_1} \big|_{(s,s_{\mu_A})=(0,0)} = 0, \quad \frac{\partial t_{i,j}}{\partial s_{\mu_A}} \big|_{(s,s_{\mu_A})=(0,0)} = \delta_{ii'}\delta_{jj'}, \quad \frac{\partial t_{i,j}}{\partial s^{'i,j'}} \big|_{(s,s_{\mu_A})=(0,0)} = 0, \]  
\text{(4.9b)}

where \( \delta_{ii'} \) and \( \delta_{jj'} \) are Kronecker’s deltas. In particular, flat coordinates \( t_1, t_{1,1}, \ldots, t_{3,\alpha_3-1}, t_{\mu_A} \) satisfy
\[ e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} a_i - j \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}, \]  
\text{(4.10)}

which is Condition (i).

4.3. Condition (ii).

**Lemma 4.6.** We have
\[ K_{F_A}^{(0)}(\zeta_A, u \nabla_{\partial t_1} \zeta_A) = 0, \quad K_{F_A}^{(0)}(\zeta_A, u \nabla_{\partial s_{i,j}} \zeta_A) = 0, \quad i = 1, 2, 3, j = 1, \ldots, a_i - 1, \]  
\text{(4.11)}

and
\[ K_{F_A}^{(0)}(\zeta_A, u \nabla_{\partial s_{\mu_A}} \zeta_A) = 1. \]  
\text{(4.12)}

**Proof.** By Lemma 4.5, we have
\[ dt_{\mu_A} = K_{F_A}^{(0)}(\zeta_A, u \nabla_{\partial t_1} \zeta_A) dt_1 + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} K_{F_A}^{(0)}(\zeta_A, u \nabla_{\partial s_{i,j}} \zeta_A) dt_{i,j} + K_{F_A}^{(0)}(\zeta_A, u \nabla_{\partial s_{\mu_A}} \zeta_A) dt_{\mu_A}. \]

The statement follows. \(\square\)

Note that the pairings to consider are constant since we are dealing with flat coordinates. Therefore, we can evaluate them along \( M^s \). Moreover, by the normalization \([4.9]\), we have
\[ K_{F_A}^{(0)} \left( u \nabla_{\partial s_{i,j}} \zeta_A, u \nabla_{\partial s^{'i,j'}} \zeta_A \right) \big|_{s=0} = K_{F_A}^{(0)} \left( u \nabla_{\partial t_{i,j}} \zeta_A, u \nabla_{\partial t^{'i,j'}} \zeta_A \right) \big|_{s=0}. \]  
\text{(4.13)}

The statement follows from the same argument in Subsection 4.2 in \([9]\) and Lemma 2.10.
4.4. **Condition** (iii). By $dt|_M = ds_{\mu_A}/s_{\mu_A}$, we have the $O_M$-isomorphism:

$$
T_M \left( - \log M \right)|_M \simeq O_M \frac{\partial}{\partial t_1} \bigoplus_{1 \leq i \leq 3} O_M \frac{\partial}{\partial t_{i,j}} \bigoplus_{1 \leq i \leq a_{i-1}} O_M \frac{\partial}{\partial t_{\mu_A}},
$$

where we restrict the flat coordinates in Condition (i) on $M^r$ and denote them by the same characters. Then, by Proposition 2.12, we have the $O_M$-isomorphism:

$$
p_*O_C|_M \simeq T_M \left( - \log M \right)|_M.
$$

Condition (iii) follows from (4.15) and the fact that we can take the flat coordinate $e_{\mu_A}|_M = s_{\mu_A}$.

4.5. **Condition** (iv). Recall that the ideal in the equation (2.16) restricted to $M_\infty$ is given by

$$
(x_2 x_3, x_3 x_1, x_1 x_2, H_1(x, s), H_2(x, s)),
$$

where

$$
H_i(x, s) := a_i x_i - a_{i+1} x_{i+1} + \sum_{j=1}^{a_i-1} j \cdot s_{i,j} \cdot x_i^j - \sum_{j=1}^{a_{i+1}-1} j \cdot s_{i+1,j} \cdot x_i^j, \quad i = 1, 2.
$$

In particular, we have

$$
\eta \left( \frac{\partial}{\partial s_{i,j}}, \frac{\partial}{\partial s'_{i',j'}} \right)|_{s_{\mu_A} = 0} = 0, \quad \text{if } i \neq i'.
$$

Note that the connection $\nabla$ on $T_M$ in (P4) is the Levi–Civita connection with respect to $\eta$ (see Proposition 7.9 and Proposition 7.16 of [17]) and that $t_{i,j}$ is the solution of the system

$$
\nabla^* dt_{i,j} = 0, \quad t_{i,j}|_{s = s_{\mu_A} = 0} = 0,
$$

$$
\frac{\partial t_{i,j}}{\partial s_1}|_{s = s_{\mu_A} = 0} = 0, \quad \frac{\partial t_{i,j}}{\partial s'_{i',j'}}|_{s = s_{\mu_A} = 0} = \delta_{i,i'} \delta_{j,j'}, \quad \frac{\partial t_{i,j}}{\partial s_{\mu_A}}|_{s = s_{\mu_A} = 0} = 0,
$$

where $\nabla^*$ is a connection on $\Omega^1_M$ dual to $\nabla$. Therefore, by (4.18), we have

$$
\frac{\partial t_{i,j}}{\partial s'_{i',j'}}|_{s_{\mu_A} = 0} = 0, \quad \text{if } i \neq i'.
$$

The third derivatives of the Frobenius potential with respect to flat coordinates are given by residues. For example, we have

$$
\frac{\partial^3 F_{A, \zeta_A}}{\partial t_{i_1,j_1} \partial t_{i_2,j_2} \partial t_{i_3,j_3}} = -e^{-2t_{\mu_A}} \text{Res}_{\mathbb{C}^3 \times M/M} \left[ \begin{array}{ccc} \frac{\partial F_A}{\partial x_1} & \frac{\partial F_A}{\partial x_2} & \frac{\partial F_A}{\partial x_3} \\ \frac{\partial F_A}{\partial x_1} & \frac{\partial F_A}{\partial x_2} & \frac{\partial F_A}{\partial x_3} \\ \frac{\partial F_A}{\partial x_1} & \frac{\partial F_A}{\partial x_2} & \frac{\partial F_A}{\partial x_3} \end{array} \right].
$$
Therefore, by using the above description of the ideal, the normalization (4.9) and (4.20), we can show that
\[
\lim_{\epsilon^\mu_A \to 0} \frac{\partial^n F_{fA,\zeta_A}}{\partial t_{i_1,j_1} \cdots \partial t_{i_n,j_n}} \bigg|_{t_1=t_{1,1}=- \cdots =t_{3,a_3-1}=0} \neq 0 \text{ only if } i_1 = \cdots = i_n,
\]
(4.22)
by induction on \(n\).

4.6. **Condition (v).** The condition (v) easily follows from the equation (2.16) by setting \(s_1 = s_{1,1} = \cdots = s_{3,a_3-1} = s_{\mu_A} = 0\) together with the normalization (4.9).

4.7. **Condition (vi).** Note that the coefficient of the term \(t_1 t_2 t_3 e^{t_{\mu_A}}\) is given by the limit
\[
\lim_{\epsilon^\mu_A \to 0} \left( e^{-t_{\mu_A}} \cdot \left. \frac{\partial^3 F_{fA,\zeta_A}}{\partial t_{1,1} \partial t_{2,1} \partial t_{3,1}} \right|_{t_1=t_{1,1}=- \cdots =t_{3,a_3-1}=0} \right). \tag{4.23}
\]
Note also that we have the following formula
\[
\frac{\partial^3 F_{fA,\zeta_A}}{\partial t_{1,1} \partial t_{2,1} \partial t_{3,1}} = -\text{Res}_{\mathbb{C}^3 \times M/M} \left[ \frac{\partial F_A}{\partial x_1} \frac{\partial F_A}{\partial x_2} \frac{\partial F_A}{\partial x_3} \right] \omega_0(x; s, s_{\mu_A})^{-2} dx_1 \wedge dx_2 \wedge dx_3 \right]. \tag{4.24}
\]
Then, by the normalization (4.9), the above limit is reduced to the calculation of the following limit
\[
\lim_{\epsilon^\mu_A \to 0} \left( -e^{-t_{\mu_A}} \cdot e^{-2t_{\mu_A}} \text{Res}_{\mathbb{C}^3 \times M/M} \left[ \frac{\partial F_A}{\partial x_1} \frac{\partial F_A}{\partial x_2} \frac{\partial F_A}{\partial x_3} \right] \right|_{s=0}, \tag{4.25}
\]
which is one.

5. **Periods of primitive forms**

In this section, we assume that \(\chi_A \neq 0\). For simplicity, we shall denote by \(M\) the Frobenius manifold \(M_{(f_A,\zeta_A)}\) constructed from the pair of the cusp polynomial \(f_A\) and the primitive form \(\zeta_A\), where \(\zeta_A\) is the one given in Theorem 2.38 for \(\chi_A < 0\) and the one in Theorem 3.1 in [9] for \(\chi_A > 0\). We shall systematically calculate the intersection form of \(M_{(f_A,\zeta_A)}\) by the period mappings of the primitive form.

5.1. **Period mappings and Intersection forms.** In this subsection, we shall reformulate the period mappings of primitive forms and intersection forms considered by K. Saito [15] in our situation.

Put
\[
\mathring{M} := \begin{cases} 
\mathbb{C} \times M & \text{if } \chi_A > 0, \\
\mathbb{C}_\delta \times M & \text{if } \chi_A < 0.
\end{cases} \tag{5.1}
\]
We also denote the coordinate of $\tilde{M}$ by $(w, s, s_{\mu A})$ and set $\delta_w := \partial/\partial w$.

**Definition 5.1.** For any $\kappa \in \mathbb{C}$, define the $\mathcal{D}_{\tilde{M}}$-module

\[ \mathcal{M}^{(\kappa)} := \mathcal{D}_{\tilde{M}}/\mathcal{I}^{(\kappa)}_{\tilde{M}} \]

with relations:

\[ \mathcal{I}^{(\kappa)}_{\tilde{M}} := \sum_{\partial, \partial' \in T_M} \mathcal{D}_{\tilde{M}} P(\partial, \partial') + \sum_{\partial \in T_M} \mathcal{D}_{\tilde{M}} Q(\partial), \]

\[ P(\partial, \partial') := \partial \partial' - (\partial \circ \partial') \delta_w - \nabla_{\partial' \partial}, \quad \partial, \partial' \in T_M, \]

\[ Q(\partial) := (E \circ \partial) \delta_w - (N - \kappa - 1) \partial, \quad \partial \in T_M. \]

where $\nabla$ and $N$ are the connection and the $\mathcal{O}_M$-endomorphism defined in Definition 2.37 respectively.

**Definition 5.2.** For a $\mathcal{D}_{\tilde{M}}$-module $\mathcal{N}$, we set the $\mathcal{D}_{\tilde{M}}$-module:

\[ \text{Sol}(\mathcal{N}) := \text{Hom}_{\mathcal{D}_{\tilde{M}}} (\mathcal{N}, \mathcal{O}_{\tilde{M}}). \]

**Remark 5.3.** Since $\mathcal{M}^{(\kappa)}$ has the generator 1, we can identify a solution $g \in \text{Sol}(\mathcal{M}^{(\kappa)})$ with $g(1) \in \mathcal{O}_{\tilde{M}}$.

**Definition 5.4.** We set a $\mathcal{D}_{\tilde{M}}$-module:

\[ \tilde{\mathcal{M}}^{(\kappa)} := \mathcal{D}_{\tilde{M}}/ \left( \mathcal{I}^{(\kappa)}_{\tilde{M}} + \mathcal{D}_{\tilde{M}} (E - (1 - \kappa)) \right). \]

**Remark 5.5.** Recall here that the minimal exponent $r$ of $\zeta_A$ in Theorem 2.38 and Theorem 3.1 in [9] is one. We substitute $r = 1$ to the original definition in [15].

Set $\tilde{\mathcal{D}} := \varphi(\mathcal{C})$ where $\mathcal{C}$ is the critical set of $F_A$ and $\varphi$ is the morphism defined in Section 2. Since these are also easily defined for $\chi_A > 0$ in a natural way, we omit the detail (see [9]). However, note that we have to consider everything globally for the case that $\chi_A > 0$ (for example, $\mathcal{C}$ is a submanifold in $\mathbb{C}^3 \times M$).

**Lemma 5.6.** One has

\[ \text{Sol}(\mathcal{M}^{(\kappa)}) \simeq \text{Sol}(\tilde{\mathcal{M}}^{(\kappa)}) \oplus \mathbb{C}_{\tau_{\mu A}}, \quad \text{if } \kappa = 1, \]

\[ \text{Sol}(\mathcal{M}^{(\kappa)}) \simeq \text{Sol}(\tilde{\mathcal{M}}^{(\kappa)}) \oplus \mathbb{C}, \quad \text{if } \kappa \neq 1, \]

where $\tau_{\mu A}$ is a function on $\tilde{M} \setminus \tilde{\mathcal{D}}$ such that $E \tau_{\mu A}$ is non-zero constant and $\tau_{\mu A} |_{M \setminus \tilde{D}} = t_{\mu A}$.

**Proof.** See Section 5 of [15].
Since $\text{Sol}(\mathcal{M}(\kappa))$ always contains the constant function, we have
\[d\text{Sol}(\mathcal{M}(\kappa)) \simeq \text{Sol}(\mathcal{M}(\kappa))/\mathcal{C}_M,\] (5.8)
where $d\text{Sol}(\mathcal{M}(\kappa))$ is the image of $\text{Sol}(\mathcal{M}(\kappa))$ in $\Omega^1_M$ under the differential $d$.

**Lemma 5.7.** There exist an isomorphism:
\[\Omega^1_{\hat{M},t} \simeq \mathcal{O}_{\hat{M},t} \otimes (d\text{Sol}(\mathcal{M}(\kappa))|_{w=0}), \quad t \in \hat{M}\setminus \mathcal{D},\] (5.9)
where $\mathcal{D}$ is the image of $\hat{D}$ by the natural projection to $\hat{M}$ and coincides with the support set for the kernel of the multiplication by the Euler vector field $E$.

**Proof.** See Section 5 of [15]. \□

Set
\[X_{(w,s,s_{\mu A})} := \{(x_1, x_2, x_3) \in \mathbb{C}^3| w - F_A(x, s; s_{\mu A}) = 0\}, \quad (w, s, s_{\mu A}) \in \hat{M}.\] (5.10)
We can relate an element $\omega \in \Gamma(M, \mathcal{H}_{F_A}^{(0)})$ with the Gelfand–Leray form:
\[\hat{\omega} := \text{Res}_{X_{(w,s,s_{\mu A})}} [\omega] \mathcal{F}_A(x, s; s_{\mu A}) \in \bigcup_{(w,s,s_{\mu A}) \in \hat{M}\setminus \hat{D}} H^2(X_{(w,s,s_{\mu A})}, \mathbb{C}),\] (5.11)
where $[\omega]$ is the image of $\omega$ for the Fourier–Laplace transformation: $u^{-1} \mapsto \delta_w$.

Let $\beta(w, s, s_{\mu A}) \in \bigcup_{(w,s,s_{\mu A}) \in \hat{M}\setminus \hat{D}} H_2(X_{(w,s,s_{\mu A})}, \mathbb{Z})$ be a horizontal family of homology classes defined on a simply connected domain of a covering space of $\hat{M}\setminus \hat{D}$. Then, by considering the Gelfand–Leray form of the primitive form $\zeta_A$, one can consider the period
\[\int_{\beta(w,s,s_{\mu A})} \zeta_A.\]

**Lemma 5.8.** One has
\[\int_{\beta(w,s,s_{\mu A})} \zeta_A \in \text{Sol}(\hat{\mathcal{M}}^{(1)}),\] (5.12)

**Proof.** Lemma 5.8 immediately follows from Note 2 of Section 5 in [15]. \□

Here we shall recall the intersection form of a Frobenius manifold defined by Dubrovin [4] and the intersection form considered by K. Saito [15], and compare them.

**Definition 5.9** (cf. [4]). Let $M$ be a Frobenius manifold and $E$ the Euler vector field of $M$. For 1-forms $\omega_1, \omega_2 \in \Omega^1_M$, we put
\[I_M(\omega_1, \omega_2) := i_E(\omega_1 \bullet \omega_2),\] (5.13)
where $\bullet$ is the induced operation of multiplication of tangent vectors on the Frobenius manifold $M$ and the duality between tangent and cotangent spaces established by the non-degenerate bilinear form $\eta$ of $M$, and $i_E$ is the operator of contraction of a 1-form with the Euler vector field $E$. Moreover, by using flat coordinates of $M$, one can reformulate the intersection form $I_M$ as follows:

$$I_M(\text{d}t^i, \text{d}t^j) = \sum_{k,l=1}^{\mu} \eta^{ik} \eta^{jl} E(\partial_k \partial_l F_M),$$

(5.14)

where we denote by $t^i$ the flat coordinate of $M$, by $F_M$ the Frobenius potential of $M$, and set $\partial_i := \partial/\partial t_i$ and $(\eta^{ab}) := (\eta(\partial_a, \partial_b))^{-1}$.

**Definition 5.10 (cf. [15]).** We put the $O_M$-bilinear form

$$I : \Omega^1_M \times \Omega^1_M \rightarrow O_M$$

(5.15a)

$$I(\omega_1, \omega_2) := \sum_{a,b=1}^{\mu} i_{\partial_a}(\omega_1) \cdot \eta^{ab} \cdot i_{w_{\partial_b} + E\partial_b}(\omega_2),$$

(5.15b)

where $i_{\partial_a}$ is the operator of contraction of a 1-form with the vector field $\partial_a \in T_M$, and call it the intersection form of $\tilde{M}$.

Combining Lemma 5.7 with Definition 5.9 and Definition 5.10, one can see that the $O_M$-bilinear form $I$ restricted to $w = 0$ coincides the intersection form of Frobenius manifold $I_{M(fA, \zeta_A)}$. For the simplicity, we shall denote $I_{M(fA, \zeta_A)}$ by $I(fA, \zeta_A)$.

5.2. **Calculations for the intersection form of $M(fA, \zeta_A)$**. From now on, we denote by $\tilde{\zeta}_A$ the form $\zeta_A$ restricted to $w = 0$. Namely, we set

$$\tilde{\zeta}_A := \text{Res}_{X(0,s,s_{\mu A})} \frac{[\zeta_A]}{F_A(x, s; s_{\mu A})}. \quad (5.16)$$

We shall calculate the intersection form of the Frobenius manifold $I(fA, \zeta_A)$ by considering the intersections of vanishing cycles in a fiber.

**Lemma 5.11.** The primitive form $\zeta_A$ satisfies the following equation:

$$\frac{1}{(2\pi)^2} \int_{\gamma_0(s,s_{\mu A})} \tilde{\zeta}_A = 1,$$

(5.17)

where $\gamma_0(s, s_{\mu A})$ is the horizontal family of the homology in $\bigcup_{(0,s,s_{\mu A}) \in M \setminus \mathcal{D}} H_2(X(0,s,s_{\mu A}), \mathbb{C})$ corresponding to the relative 3-cycle $\Gamma_0 = \{|x_1| = |x_2| = |x_3| = \varepsilon\} \subset \mathbb{C}^3 \times M$. In
particular, if \( \chi_A > 0 \), one has

\[
\zeta_A = [e^{-t_{\mu_A}} dx_1 \wedge dx_2 \wedge dx_3],
\]

(5.18a)

\[
\tilde{\zeta}_A = \text{Res}_{F_A = 0} \left[ \frac{e^{-t_{\mu_A}} dx_1 \wedge dx_2 \wedge dx_3}{-F_A} \right].
\]

(5.18b)

Proof. Case (i) \( \chi_A < 0 \)

We can evaluate the left hand of the equation (5.17) at \( s = 0 \). Then one has

\[
\frac{1}{(2\pi - 1)^2} \int_{\gamma_0(s, s_{\mu_A})} \tilde{\zeta}_A = \frac{1}{(2\pi - 1)^3} \int_{\Gamma_0} s_{\mu_A}^{-1} dx_1 \wedge dx_2 \wedge dx_3
\]

\[
= \frac{1}{(2\pi - 1)^3} \int_{\Gamma_0} dx_1 \wedge dx_2 \wedge dx_3 \times \left\{ \left( \sum_{n=1}^{\infty} \left( s_{\mu_A} \cdot \sum_{i=1}^{3} x_i^{a_i} \right)^n \right) \right\} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3}.
\]

Here one has

\[
x_1^{a_1} x_2^{a_2} x_3^{a_3} \neq (x_1 x_2 x_3)^n \quad \text{if} \quad e_i \geq 0, \quad 3 \sum_{i=1}^{3} e_i = n
\]

because, if the equality is attained, one has the contradictory inequation:

\[
n > \left( \sum_{i=1}^{3} \frac{n}{a_i} \right) = \sum_{i=1}^{3} e_i = n.
\]

Therefore one has

\[
\frac{1}{(2\pi - 1)^2} \int_{\gamma_0(s, s_{\mu_A})} \tilde{\zeta}_A = \frac{1}{(2\pi - 1)^3} \int_{\Gamma_0} dx_1 \wedge dx_2 \wedge dx_3 = 1.
\]

Case (ii) \( \chi_A > 0 \)

First, we shall show the equations (5.18a) and (5.18b). By Theorem 3.1 in [9], the element \( [s_{\mu_A}^{-1} dx_1 \wedge dx_2 \wedge dx_3] \in \mathcal{H}_{F_A}^{(0)} \) is a primitive form. By Lemma 4.2 in [9] and Lemma 4.5, we can choose \( t_{\mu_A} := \log s_{\mu_A} \) as a flat coordinate. Therefore we have (5.18a) and (5.18b).

Next, we shall show the equation (5.17). By equation (5.18b), one has

\[
\frac{1}{(2\pi - 1)^2} \int_{\gamma_0(s, s_{\mu_A})} \tilde{\zeta}_A = \frac{1}{(2\pi - 1)^3} \int_{\Gamma_0} e^{-t_{\mu_A}} dx_1 \wedge dx_2 \wedge dx_3
\]

\[
= \frac{1}{(2\pi - 1)^3} \int_{\Gamma_0} \left\{ \sum_{n=1}^{\infty} \left( s_{\mu_A} \cdot \sum_{i=1}^{3} x_i^{a_i} + s_1 + \sum_{i=1}^{3} \sum_{j=1}^{a_i} s_{i,j} x_i^n \right)^n \right\} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3}
\]

\[
+ \frac{1}{(2\pi - 1)^3} \int_{\Gamma_0} dx_1 \wedge dx_2 \wedge dx_3
\]

Here one has

\[
\prod_{j=1}^{a_1} x_1^{j} \prod_{j=1}^{a_2} x_2^{j} \prod_{j=1}^{a_3} x_3^{j} \neq (x_1 x_2 x_3)^n \quad \text{if} \quad e_{i,j} \geq 0, \quad 3 \sum_{i=1}^{3} \sum_{j=1}^{a_i} e_{i,j} \leq n
\]
because, if the equality is attained, one has the following contradictory inequation:

\[ n < \left( \sum_{i=1}^{3} \frac{n}{a_i} \right) = \sum_{i=1}^{3} a_i \sum_{j=1}^{n} \frac{j}{a_i} \leq \sum_{i=1}^{3} \sum_{j=1}^{n} e_{i,j} \leq n. \]

Then one has

\[ \frac{1}{(2\pi \sqrt{-1})^2} \int_{\gamma_0(s,s_{\mu A})} \tilde{\zeta}_A = \frac{1}{(2\pi \sqrt{-1})^3} \int_{\Gamma_0} dx_1 \wedge dx_2 \wedge dx_3 = 1. \]

Therefore we have Lemma 5.11. □

**Lemma 5.12.** There exists the isomorphism between lattices:

\[ (H_2(X_{(w,s,s_{\mu A})}, \mathbb{Z}), -I_{H_2}) \simeq (\tilde{h}^*_A, \langle , \rangle) \]

if \((w, s, s_{\mu A}) \in \tilde{M} \setminus \tilde{D} \), (5.19)

where \(I_{H_2} \) is the intersection form for cycles in the fiber \(X_{(w,s,s_{\mu A})} \).

**Proof.** See [5] and Section 3 of [21]. □

Here \( T_A \) is the following Coxeter–Dynkin diagram:

\[ \begin{array}{cccccccc}
(1, a_1 - 1) & \cdots & \cdots & (1, 1) & 1 & (3, 1) & (3, a_3 - 1) \\
\end{array} \]

\( \bullet \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \)

and \( \mathfrak{h}_A \) the complexified Cartan subalgebra of the Kac–Moody Lie algebra associated to \( T_A \) (in particular, which is the simple Lie algebra for the case that \( \chi_A > 0 \)). Denote by \( \alpha_1, \ldots, \alpha_{(3,a_3-1)} \in \mathfrak{h}^*_A := \text{Hom}_\mathbb{C}(\mathfrak{h}_A, \mathbb{C}) \) simple roots corresponding the vertices in \( T_A \), by \( \alpha_1^\vee, \ldots, \alpha_{(3,a_3-1)}^\vee \in \mathfrak{h}_A \) simple coroots and by \( \langle , \rangle : \mathfrak{h}^*_A \otimes \mathfrak{h}_A \to \mathbb{C} \) the natural pairing. The Weyl group \( W_A \) is a group generated by reflections

\[ r_i(h) := h - \langle \alpha_i, h \rangle \alpha_i, \quad h \in \mathfrak{h}_A, \quad i = 1, (1, 1), \ldots, (i, j), \ldots, (3, a_3 - 1), \quad (5.21) \]

where \( \langle , \rangle \) denotes the natural pairing \( \langle , \rangle : \mathfrak{h}^*_A \otimes \mathfrak{h}_A \to \mathbb{C} \). Moreover, we set \( \tilde{\mathfrak{h}}_A \) is the complexified Cartan subalgebra of the affine Lie algebra associated to \( T_A \).

**Lemma 5.13.** Denote by \( \delta \in \tilde{\mathfrak{h}}^*_A := \text{Hom}_\mathbb{C}(\tilde{\mathfrak{h}}_A, \mathbb{C}) \) the generator of the imaginary root. Then one has the isomorphism of affine spaces

\[ \mathfrak{h}_A \simeq \left\{ \tilde{h} \in \tilde{\mathfrak{h}}_A \mid \langle \tilde{h}, \delta \rangle = 1 \right\} \]

(5.22)
which is compatible with the action of the affine Weyl group $\widetilde{W}_A$ on both sides, where the action on the left hand side is defined by

$$h \mapsto w(h) + \sum_{i=1}^{a_1-1} \sum_{j=1}^{3} m_{(i,j)} \alpha_{(i,j)}^\vee, \quad m_{(i,j)} \in \mathbb{Z}$$

and the one on the right hand side is the natural one.

**Proof.** Some elementary calculations yield the statement. \qed

By Lemma 5.11, Lemma 5.12 and Lemma 5.13, we can identify $\delta$ with the 2-cycle $\gamma_0(s, s_{\mu_A})$ in the fiber. We denote by $\gamma_i(s, s_{\mu_A}) \in \bigcup_{(0,s,s_{\mu_A}) \in \tilde{M} \setminus \tilde{D}} H_2(X_{(0,s,s_{\mu_A})}, \mathbb{C})$ the image of $\alpha_i$ by the composition of the isomorphisms in Lemma 5.12 and Lemma 5.13.

Under the above notations and the natural identification between $M$ and $\tilde{M}|_{w=0}$, the following is the main theorem in this section:

**Theorem 5.14.** Consider the periods

$$x_i := \frac{1}{(2\pi \sqrt{-1})^2} \int_{\gamma_i(s, s_{\mu_A})} \zeta_A, \quad i = 1, (1,1), \ldots, (i,j), \ldots, (3, a_3 - 1),$$

where $\gamma_i(s, s_{\mu_A})$ is the horizontal family of homology classes in $\bigcup_{(0,s,s_{\mu_A}) \in \tilde{M} \setminus \tilde{D}} H_2(X_{(0,s,s_{\mu_A})}, \mathbb{C})$ identified with $\alpha_i$, and the function

$$x_{\mu_A} := \frac{1}{2\pi \sqrt{-1}} t_{\mu_A} = \frac{1}{2\pi \sqrt{-1}} \log s_{\mu_A}.$$  \hspace{1cm} (5.25)

They define the flat coordinates with respect to $I_{(f_A, \zeta_A)}$ on the monodromy covering space of $M \setminus D$. Moreover, one has

$$I_{(f_A, \zeta_A)}(dx_i, dx_j) = \frac{-1}{(2\pi \sqrt{-1})^2} \langle \alpha_i, \alpha_j^\vee \rangle, \hspace{1cm} (5.26a)$$

$$I_{(f_A, \zeta_A)}(dx_{\mu_A}, dx_i) = I_{(f_A, \zeta_A)}(dx_i, dx_{\mu_A}) = 0, \hspace{1cm} (5.26b)$$

$$I_{(f_A, \zeta_A)}(dx_{\mu_A}, dx_{\mu_A}) = \frac{1}{(2\pi \sqrt{-1})^2} \chi_A. \hspace{1cm} (5.26c)$$

**Proof.** The first assertion immediately follows from Note 2 of Section 5 in [15] and Lemma 5.6. We shall show the second assertion. First, we shall show the equations (5.26b) and
\textbf{5.26a}. By Lemma 5.7 and Definition 5.9 one has
\begin{align*}
I_{(f_A, \zeta_A)}(dt_{\mu_A}, dt_1) &= \sum_{\alpha, \beta} \eta^{\mu_A \alpha} \eta^{1 \beta} E(\partial_\alpha \partial_\beta F_{(f_A, \zeta_A)}) \\
&= 1 \cdot 1 \cdot E(\partial_1 \partial_{\mu_A} F_{(f_A, \zeta_A)}) = t_1,
\end{align*}
\begin{align*}
I_{(f_A, \zeta_A)}(dt_{\mu_A}, dt_{i,j}) &= \sum_{\alpha, \beta} \eta^{\mu_A \alpha} \eta^{(i,j) \beta} E(\partial_\alpha \partial_\beta F_{(f_A, \zeta_A)}) \\
&= 1 \cdot a_i \cdot E(\partial_i \partial_{(j, a_i-j)} F_{(f_A, \zeta_A)}) = \frac{a_i - j}{a_i} t_{i,j},
\end{align*}
\begin{align*}
I_{(f_A, \zeta_A)}(dt_{\mu_A}, dt_{\mu_A}) &= \sum_{\alpha, \beta} \eta^{\mu_A \alpha} \eta^{\mu_A \beta} E(\partial_\alpha \partial_\beta F_{(f_A, \zeta_A)}) \\
&= 1 \cdot 1 \cdot E(\partial_1 \partial_1 F_{(f_A, \zeta_A)}) = \chi_A.
\end{align*}

We shall substitute the above calculations for following equations. By Lemma 5.8 one has
\begin{align*}
I_{(f_A, \zeta_A)}(dx_{\mu_A}, dx_1) &= \sum_{\alpha, \beta} I_{(f_A, \zeta_A)}(dt_{\alpha}, dt_{\beta}) \partial_\alpha x_{\mu_A} \partial_\beta x_1 \\
&= \sum_{\beta} \frac{1}{2 \pi \sqrt{-1}} I_{(f_A, \zeta_A)}(dt_{\mu_A}, dx_{\beta}) \partial_\beta x_1 = \frac{1}{2 \pi \sqrt{-1}} E x_1 = 0,
\end{align*}
\begin{align*}
I_{(f_A, \zeta_A)}(dx_{\mu_A}, dx_{\mu_A}) &= I_{(f_A, \zeta_A)} \left( \frac{1}{2 \pi \sqrt{-1}} dt_{\mu_A}, \frac{1}{2 \pi \sqrt{-1}} dt_{\mu_A} \right) = \frac{1}{(2 \pi \sqrt{-1})^2} \chi_A.
\end{align*}

Finally we shall show the equation (5.26a):
\[ I_{(f_A, \zeta_A)}(dx_1, dx_1) = \frac{-1}{(2 \pi \sqrt{-1})^2} \langle \alpha_1, \alpha_1 \rangle. \]

The equation (5.26a) immediately follows from Lemma 5.12 and the following Lemma 5.15.

\textbf{Lemma 5.15.} Let \( \beta_i(w, s, s_{\mu_A}) \in \bigcup_{(w, s, s_{\mu_A}) \in \tilde{M} \setminus \tilde{D}} H_2(\mathcal{H}(w, s, s_{\mu_A}), \mathbb{Z}), \ i = 1, \ldots, (3, a_3 - 1) \) be a horizontal family of homology defined on a simply connected domain of a covering space of \( \tilde{M} \setminus \tilde{D} \). Then, one has
\begin{align}
-\frac{1}{(2 \pi \sqrt{-1})^2} \sum_{\alpha, \beta = 1}^{\mu_A} \partial_\alpha \left( \int_{\beta_i(w, s, s_{\mu_A})} \zeta_A \right) \cdot \eta^{ab} \cdot (E \circ \partial_b) \left( \int_{\beta_j(w, s, s_{\mu_A})} \zeta_A \right) \\
&= I_{H_2}(\beta_1(w, s, s_{\mu_A}), \beta_1(w, s, s_{\mu_A})). \tag{5.29}
\end{align}

\textbf{Proof.} See Theorem 3.4 in [15] (a factor \((-1)\) is missing in the reference).
\[ \square \]

Therefore we have Theorem 5.14.
\[ \square \]
6. Frobenius manifold $M_{\hat{W}_A}$

We shall recall the Frobenius manifold constructed from the invariant theory of an extended affine Weyl group by Dubrovin-Zhang in [3].

Under the assumption $\chi_A \neq 0$, the Cartan matrix for $T_A$ is nondegenerate. Set $\hat{h}_A := h_A \times \mathbb{C}$. The affinization $\hat{W}_A$ of $W_A$ acts on $\hat{h}_A$ by

$$(h, x_{\mu_A}) \mapsto (w(h) + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} m_{(i,j)} \alpha_i^\vee, x_{\mu_A}), \quad m_{(i,j)} \in \mathbb{Z}$$

and $\mathbb{Z}$ acts on $\hat{h}_A$ by

$$(h, x_{\mu_A}) \mapsto (h + m \omega_1^\vee, x_{\mu_A} + m), \quad m \in \mathbb{Z},$$

where $\omega_1^\vee, \omega_{(1,1)}^\vee, \ldots, \omega_{(3,a_3-1)}^\vee$ denotes the fundamental coweights, the elements of $h_A$ satisfying $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$ (where $\delta_{ij}$ is the Kronecker's delta). Then, $\hat{W}_A$ is defined as a group acting on $\hat{h}_A$ generated by $\hat{W}_A$ and $\mathbb{Z}$ with the above actions on $\hat{h}_A$. In particular, one has the following exact sequence

$$1 \to \hat{W} \to \hat{W} \to \mathbb{Z} \to 1.$$ 

By the invariant theory of $\hat{W}_A$, Dubrovink-Zhang [3] give the following:

**Theorem 6.1** ([3]). Assume that $\chi_A > 0$. There exists a unique Frobenius structure of rank $\mu_A$ and dimension one on $M_{\hat{W}_A} := \hat{h}_A/\hat{W}_A$ with flat coordinates $t_1, t_{1,1}, \ldots, t_{i,j}, \ldots, t_{3,a_3-1}, t_{\mu_A} := (2\pi\sqrt{-1})x_{\mu_A}$ such that

$$e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}},$$

and the intersection form $I_{\hat{W}_A}$ is given by

$$I_{\hat{W}_A}(\alpha_i, \alpha_j) = \frac{-1}{(2\pi\sqrt{-1})^2} \langle \alpha_i, \alpha_j^\vee \rangle, \quad i, j = 1, (1, 1), \ldots, (3, a_3 - 1),$$

$$I_{\hat{W}_A}(\alpha_i, dx_{\mu_A}) = I_{\hat{W}_A}(dx_{\mu_A}, \alpha_i) = 0, \quad i = 1, (1, 1), \ldots, (3, a_3 - 1),$$

$$I_{\hat{W}_A}(dx_{\mu_A}, dx_{\mu_A}) = \frac{1}{(2\pi\sqrt{-1})^2} \chi_A,$$

where we identify the cotangent space of $M_{\hat{W}_A}$ with $h_A^* \oplus \mathbb{C} dx_{\mu_A}$.

**Proof.** See Theorem 2.1 in [3]. □
7. ISOMORPHISM BETWEEN $M_{(f_A, \zeta_A)}$ AND $\hat{M}_{W_A}$

In this section, we shall show the isomorphism of Frobenius manifolds between the one constructed from the pair $(f_A, \zeta_A)$ in [9] and the one constructed from the invariant theory of an extended affine Weyl group $\hat{W}_A$ in [3].

7.1. Reconstruction Theorem via Intersection forms. The following Theorem 7.1 might be known to experts. However the complete proof is not found in any literature. For this reason, we shall give a proof suitable for our situation here.

**Theorem 7.1.** A Frobenius manifold $M$ of rank $\mu_A$ and dimension one with the following $e$ and $E$ is uniquely determined by the intersection form $I_M$:

$$
e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{\alpha_i-1} \frac{a_i-j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}. \tag{7.1}$$

**Proof.** We use the following relation between the product $\circ$ and the intersection form $I_M$:

**Lemma 7.2.** Denote by $\Gamma^{ij}_k$ the contravariant components of the Levi–Civita connection for the intersection form $I_M$. Then one has

$$\Gamma^{ij}_k = d_j C^{\alpha \beta}_\gamma, \tag{7.2}$$

where $d$ is the dimension of the Frobenius manifold and $d_j$ is a rational number defined by $E(t_j) = d_j t_j$, $C^{ij}_k := \sum_{a,b=1}^{\mu_A} \eta^{ai} \eta^{bj} C_{abk}$ and $C_{ijk} := \partial_i \partial_j \partial_k F_M$.

**Proof.** See Lemma 3.4 in [4] and apply $d = 1$. \hfill $\square$

One sees $C^{ij}_k$ can be reconstructed from the intersection form $I_M$ if $d_j \neq 0$. Since $d = 1, d_\beta \neq 0$ if and only if $\beta = \mu_A$. However, we have

$$C^{ij}_{\mu_A} = \sum_{a,b=1}^{\mu_A} \eta^{ai} \eta^{b\mu_A} C_{abk} = \delta^i_k,$$

where $\delta^i_k$ is Kronecker’s delta. Therefore, $C_{ijk}$ and hence the Frobenius potential $F_M$ can be reconstructed from the intersection form $I_M$ by Lemma 7.2. \hfill $\square$

7.2. Isomorphism of Frobenius manifolds.

**Corollary 7.3.** Assume that $\chi_A > 0$. There exists an isomorphism of Frobenius manifolds between the one constructed from the invariant theory of extended affine Weyl group $\hat{W}_A$ and the one constructed from the pair $(f_A, \zeta_A)$.

**Proof.** Corollary 7.3 immediately follows from Theorem 7.1, Theorem 5.14 and Theorem 6.1 \hfill $\square$
ON THE FROBENIUS MANIFOLDS FOR CUSP SINGULARITIES

REFERENCES

[1] D. Abramovich, T. Graber and A. Vistoli, Gromov–Witten theory of Deligne–Mumford stacks, Amer. J. Math. 130 (2008), no. 5, pp. 1337–1398.

[2] W. Chen and Y. Ruan, Orbifold Gromov–Witten Theory, Orbifolds in mathematics and physics (Madison, WI, 2001), pp. 25–85, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[3] B. Dubrovin and Y. Zhang, Extended Affine Weyl Groups and Frobenius Manifolds, Compositio Math. 111 (1998), pp. 167–219.

[4] B. Dubrovin, Geometry of 2d topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, (1996), pp. 120–348.

[5] W. Geigle and H. Lenzing, A class of weighted projective curves arising in representation theory of finite-dimensional algebras, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), pp. 934, Lecture Notes in Math., 1273, Springer, Berlin, (1987).

[6] C. Hertling, Frobenius manifolds and moduli spaces for singularities, Cambridge Tracts in Mathematics, Cambridge University Press, Spring (2002).

[7] C. Hertling and C. Sevenheck Limits of families of Brieskorn lattices and compactified classifying spaces, Adv. Math. 223 (2010), no. 4, pp. 1155–1224.

[8] Y. Ishibashi, Y. Shiraishi, A. Takahashi, A Uniqueness Theorem for Frobenius Manifolds and Gromov–Witten Theory for Orbifold Projective Lines, to appear in J. Reine Angew. Math., DOI: 10.1515.

[9] Y. Ishibashi, Y. Shiraishi, A. Takahashi, Primitive Forms for Affine Cusp Polynomials, arXiv:1211.1128.

[10] Todor E. Milanov, Hsian-Hua Tseng, The spaces of Laurent polynomials, $\mathbb{P}^1$-orbifolds, and integrable hierarchies, J. Reine Angew. Math, Volume (2008), Issue 622, pp189–235.

[11] Y. Manin, Frobenius manifolds, Quantum Cohomology, and Moduli Spaces, American Mathematical Soc (1999).

[12] P. Rossi, Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations, Math. Ann., 348 (2010), pp. 265–287.

[13] C. Sabbah, Frobenius manifolds: isomonodromic deformations and infinitesimal period mappings, Exposition. Math. 16 (1998), no. 1, 1–57.

[14] K. Saito, Primitive forms for a universal unfolding of a function with an isolated critical point. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1982), no. 3, pp. 775–792.

[15] K. Saito, Period mapping associated to a primitive form, Publ. RIMS, Kyoto Univ. 19 (1983) 1231–1264.

[16] K. Saito, The higher residue pairings $K_{F'}^{(k)}$ for a family of hypersurface singular points, Proceedings of Symposia in Pure Mathematics Vol. 40 (1983), part 2, pp. 441–463.
[17] K. Saito and A. Takahashi, *From Primitive Forms to Frobenius manifolds*, Proceedings of Symnposia in Pure Mathematics, 78 (2008) pp. 31–48.

[18] M. Saito, *On the structure of Brieskorn lattice*, Annales de l’institut Fourier (1989) Volume: 39, Issue: 1, Publisher: Institut Fourier, pp. 27–72.

[19] I. Satake and A. Takahashi, *Gromov–Witten invariants for mirror orbifolds of simple elliptic singularities*, Annales de l’institut Fourier 61, 2885–2907, (2011).

[20] A. Takahashi, *Weighted projective lines associated to regular systems of weights of dual type*, Adv. Stud. Pure Math. 59 (2010), pp. 371–388.

[21] A. Takahashi, *Mirror symmetry between orbifold projective lines and cusp singularities*, to appear in Adv. Stud. Pure Math. (2013).

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA OSAKA, 560-0043, JAPAN

E-mail address: sm5021sy@ecs.cmc.osaka-u.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA OSAKA, 560-0043, JAPAN

E-mail address: takahashi@math.sci.osaka-u.ac.jp