Theoretical modeling for quantum liquids from 1d to 2d dimensional crossover using quantum groups

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Abstract

Recent experimental and theoretical work in strongly correlated electron system necessitates a formulation to deal with 1d to 2d dimensional crossover and raises several interesting questions. A particularly interesting question is what happens to the 1d Luttinger liquid, as we go from 1d to 2d? The issue of dimensional crossover in lower dimensional systems is of great general interest, in particular the relationship between quantum liquids and dimensional crossover from 1d to 2d. Thus one may ask given a 1d Luttinger liquid, what happens if we go from 1d to 2d, does the Luttinger liquid end up as a Fermi liquid or a non-Fermi liquid [NFL] in 2d. More importantly and relevantly what are the conditions that determine which type of quantum liquid we will end up with.

Looking ahead to future applications we need to know what parameters to tune in order to engineer 2d quantum liquids from their 1d counterparts. In other words what are the parameters that determine quantum phase transitions. Can the Luttinger liquid lead to stripe formation in high temperature superconducting [HTSC] materials in a natural way? Can we classify the quantum liquids existing in 1d, 2d and 3d under symmetry classes of quantum groups in practically meaningful way? What is the nature of dimensional phase transitions and how are these [i.e. dimensional phase transitions] made possible by interactions of quantum liquids. The main point of emphasis of the present note is that the

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transition from 1d to 2d has an underlying quantum group symmetry. The relationship of 1d to 2d transition with quantum group symmetry ties in nicely with our previous proposal to model superconductivity, antiferromagnetism and related phases arising from strongly correlated electron states with quantum groups. A simple model based on two interacting chains with nearest neighbor and next to nearest neighbor interactions is suggested. It appears reasonable to assume that it is the interaction between the 1d Luttinger liquids which leads to the quantum group symmetry and the transition to 2d quantum liquid. The conditions under which this 2d quantum liquid deviates away from the 2d Fermi liquid is clearly of main interest to us. One of the prime motivations underlying our proposal is the experimental observation of stripe structure [phase] in high Tc superconductivity [HTSC] materials. In short a simple model of two coupled chains 1d chains is considered to answer an important question, how does one couple two 1d chains [Luttinger Liquid] to obtain NFL or partial NFL behavior in 2d?
The discovery of cuprate superconductors has initiated an intense interest in strongly correlated system. In particular the anomalous normal state behavior of these materials indicate strong deviations from normal Fermi Liquid behavior. A very useful and classic behavior non-Fermi liquid [NFL] is provided by one-dimensional [1d] electron gas, where the generic fixed point behavior is believed to be a Luttinger liquid [1]. Exact solutions of 1d systems are available via Bethe ansatz and its generalizations. There is a wide array of non-perturbative techniques one can use in 1d. The ‘unique’ behavior of the 1d Luttinger liquid fixed points can be traced back to or originates in special kinematics of 1d. The Fermi surface in 1d consists of just two points, the electrons interact very strongly at these two points. Moreover the energy and momentum conservation impose a single constraint on scattering processes, resulting in an enhancement in scattering phase space. Simply, the ‘limited’ phase space available in 1d for scattering is qualitatively responsible for the peculiar behavior of Luttinger liquid. It is naturally interesting to see if the peculiar behavior of 1d Luttinger liquid may carry over to 2d. Anderson [2] made an hypothesis about the form of the ground state of the 2d Hubbard model by identifying singular scattering diagrams. Thereafter several authors have tried to generalize the Luttinger liquid concepts to higher dimensions [3–6].

It is clear that the special kinematics of 1d will not be present in 2d and other higher dimensions. A straightforward generalization of Luttinger liquid to higher dimensions cannot be expected since the 1d kinematics impose a single constraint on scattering processes whereas in higher dimensions the energy and momentum [kinematics] impose several independent constraints on the scattering process. Thus one finds that these constraints lead to the stabilization of the Fermi liquid in 2d and higher dimensions [3,7]. An interesting approach to go from 1d to 2d which suggest itself would be to couple 1d Hubbard chains since the Hubbard Hamiltonian [HH] and its extensions dominate the study of strongly correlated electrons systems and the insulator metal transition. Indeed Lin et al. [8] have taken such an approach and have claimed that in 2d one ends up with Fermi liquid. So how does one avoid or bypass the Fermi liquid as one goes from 1d to 2d. One way is by the introduction of long-range or singular interactions [2,8,10]. Yet another approach is based on Anderson’s [11] ingenious suggestion that NFL behavior in higher dimensions may arise from the formation of bound anti-bound states above and below the single particle continuum. This idea has been further examined very recently by Ho and Coleman [12]. Despite these efforts a route to NFL behavior in 2d that uses strictly local interactions has not been reported.

In an attempt to gain a fundamental understanding of the relationship between antiferromagnetism and superconductivity and other quantum phases [as mentioned above] that may be present in strongly correlated electron systems we are naturally led to examine the issue of the nature of the dimensional crossover from 1d to 2d and 2d to 3d. In the present note we concentrate on the 1d to 2d case. The main point of emphasis of the present note is that the transition from 1d to 2d has a underlying quantum symmetry group. This relation of 1d to 2d transition with quantum group symmetry ties in nicely with our previous proposal to model superconductivity, antiferromagnetism and related phases arising
from strongly correlated electron states with quantum groups [14, 15]. We thus adopt the following general strategy:

- We first identify a 1d model such as 1d Hubbard Hamiltonian. At this stage we allow ourself the freedom to choose among the plethora of known 1d Hamiltonians and/or guess any reasonable 1d Hamiltonian which could form the building block of our final 2d model for HTSC and antiferromagnetism. We initially restrict our 1d Hamiltonian to be integrable for simplicity however our real goal is to relax the condition of integrability to suit a particular realistic situation of interest.

- We next couple two 1d systems and try to identify the relevant 2d ‘covering’ model.

We write the Hamiltonian for the two coupled chains interacting via the nearest neighbor and next nearest neighbor interactions as

$$H = \sum_i t [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \xi \sigma_i^z \sigma_{i+1}^z] + \sum_j t' [\tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y + \xi \tau_j^z \tau_{j+1}^z]$$

$$+ \sum_{i,j} t'' [\sigma_i^x \tau_j^x + \sigma_i^y \tau_j^y + \xi \sigma_i^z \tau_j^z] + \sum_i U \sigma_i^z \tau_i^z, \tag{1}$$

where $t$ is the nearest neighbor hopping parameter on the same chain, $t'$ is the nearest neighbor hopping parameter on different chains and $t''$ is the next to nearest neighbor hopping parameter on different chains. In Eq. 1 we have assumed $su(2)$ symmetry and written the Hamiltonian in bosonized form. $\sigma$ and $\tau$ are two independent sets of Pauli matrices. The Hamiltonian in Eq. 1 resembles that of two XXZ chains interacting with each other via nearest and next to nearest neighbor interactions. The model Hamiltonian in Eq. 1 is written in bosonized form. It is well-known in 1d bosons and fermions are closely related and that this relationship breaks down in higher dimensions. If we set $t' = t'' = \xi = 0$ in Eq. 1 we obtain the bosonized form of the $su(2)$ Hubbard Model,

$$H = \sum_i t [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y] + \sum_j t' [\tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y] + \sum_i U \sigma_i^z \tau_i^z. \tag{2}$$

The fermionic version of Eq. 2 is well-known and can be immediately written down.\(^4\)

\(^4\)We note that $\sigma$ when it appears as a subscript is a spin label in Eq. 3 and other fermionized Hamiltonians. It is not to be confused $\sigma$ used for Pauli matrices. Similar remarks apply to $\tau$. 

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\[ H = \sum_{i,\sigma} t [c_{i,\sigma}^\dagger c_{i+1,\sigma} + c_{i+1,\sigma}^\dagger c_{i,\sigma}] + \sum_i U [n_{i\uparrow} - \frac{1}{2}] [n_{i\downarrow} - \frac{1}{2}]. \]  

(3)

In Eq. 3 we have used the following common notation: The canonical Fermi operators satisfy the anticommutation relations: \( \{ c_{i,\sigma}^\dagger, c_{j,\sigma'} \} = \delta_{ij} \delta_{\sigma\sigma'} \) where \( i, j = 1, 2, \ldots, N \) and \( \sigma, \sigma' = \uparrow, \downarrow. \)

At any lattice site \( i \) there are four electronic states, namely:

\[ |0\rangle, \quad |\uparrow\rangle_i = c_{i,\uparrow}^\dagger |0\rangle, \quad |\downarrow\rangle_i = c_{i,\downarrow}^\dagger |0\rangle, \quad \text{and} \quad |\uparrow\downarrow\rangle_i = c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger |0\rangle. \]

As usual we take \( n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma} \) to denote the number operator for electrons at site \( i \) and spin \( \sigma \). It is straightforward to see that after summing over \( \sigma \) at site \( i \) one has \( n_i = n_{i\uparrow} + n_{i\downarrow} \).

The fermionic version of Eq. 1 can be written as

\[ H = \sum_{i,\sigma} t [c_{i,\sigma}^\dagger c_{i+1,\sigma} + c_{i+1,\sigma}^\dagger c_{i,\sigma}] + \xi \left( n_{i,\sigma} - \frac{1}{2} \right) \left( n_{i+1,\sigma} + \frac{1}{2} \right) \]  

\[ + \sum_{i,\sigma,j,\tau} t' [c_{i,\sigma}^\dagger d_{j,\tau} + d_{j,\tau}^\dagger c_{i,\sigma}] + \xi \left( n_{i,\sigma} - \frac{1}{2} \right) \left( n_{j,\tau} - \frac{1}{2} \right) \]  

\[ + \sum_{i,\sigma,j,\tau} t'' [c_{i,\sigma}^\dagger d_{j,\tau} + d_{j,\tau}^\dagger c_{i,\sigma}] + \xi \left( n_{i,\sigma} - \frac{1}{2} \right) \left( n_{j,\tau} - \frac{1}{2} \right) \]  

\[ + \sum_{i,\sigma,j,\tau} \left( U/2 \right) \left[ n_{i,\sigma} - \frac{1}{2} \right] \left[ n_{i,\tau} - \frac{1}{2} \right] + \left[ n_{j,\sigma} - \frac{1}{2} \right] \left[ n_{j,\tau} - \frac{1}{2} \right]. \]  

(4)

d_{j,\tau} and \( d_j^\dagger \) are canonical Fermi operators, with \( n_{j,\sigma} = d_{j,\tau}^\dagger d_{j,\tau} \).

We next want to construct a Hamiltonian which is a partially q-deformed version of the Hamiltonian in Eq. 4. To this end we define a deformation function \( D_q \) [see, for example, Ref. 21 and references therein]

\[ D_q = \exp \{-\frac{1}{2} (\eta - \sigma \gamma) n_{i,-} - \frac{1}{2} (\eta + \sigma \gamma) n_{i+1,-} \} \]  

\[ q = \exp \{ \gamma \} \]  

\[ \exp \{-\eta\} = \frac{q^{\alpha+1} - q^{-\alpha-1}}{q^{\alpha} - q^{-\alpha}} \]  

(5)

where \( \alpha \in \mathbb{C} \) is a free parameter. We note the usual definition used in quantum groups [21]

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbb{C}. \]  

(6)

The next step is to q-deform the Hamiltonian given in Eq. 4. We note that many possibilities exist if we want to partially q-deform, since we can q-deform either chain or parts which are responsible for the interaction between the chains. For simplicity we just consider the deformation of one of the chain, viz
\[ H_q = \sum_{i,\sigma} t[c^\dagger_{i,\sigma} c_{i+1,\sigma} + c^\dagger_{i+1,\sigma} c_{i,\sigma} + \xi(n_{i,\sigma} - \frac{1}{2})(n_{i+1,\sigma} - \frac{1}{2})]D_q \]
\[ \quad + \sum_{j,\tau} t[ d^\dagger_{j,\tau} d_{j+1,\tau} + d^\dagger_{j+1,\tau} d_{j,\tau} + \xi(n'_{j,\tau} - \frac{1}{2})(n'_{j+1,\tau} - \frac{1}{2})] \]
\[ \quad + \sum_{i,\sigma,j,\tau} t' [c^\dagger_{i,\sigma} d_{j,\tau} + d^\dagger_{j,\tau} c_{i,\sigma} + \xi(n_{i,\sigma} - \frac{1}{2})(n'_{j,\tau} - \frac{1}{2})] \]
\[ \quad + \sum_{i,\sigma,j,\tau} t'' [c^\dagger_{i,\sigma} d_{j+1,\tau} + d^\dagger_{j+1,\tau} c_{i,\sigma} + \xi(n_{i,\sigma} - \frac{1}{2})(n'_{j+1,\tau} - \frac{1}{2})] \]
\[ \quad + \sum_{i,\sigma,j,\tau} (U/2) \{[n_{i,\sigma} - \frac{1}{2}][n'_{j,\tau} - \frac{1}{2}] + [n_{j,\sigma} - \frac{1}{2}][n'_{j,\tau} - \frac{1}{2}]\}. \]  

(7)

There are several directions in which we can generalize the basic Hamiltonian model considered in the present note, namely

- From both physical and purely technical point of views we may consider its supersymmetric variants in particular with pair hoppings \([22,23]\).
- The q-deformed versions of the supersymmetric variants \([21,18]\).
- We may readily generalize the current model to include the effect of doping. In particular we may begin with the spin-1 version of Hamiltonian considered in this note doped with spin-1/2 carriers \([24]\).

Next we detail some background. It is well-known that the Hubbard Hamiltonian [HH] has an su(2) symmetry \([13]\), this is especially apparent if we write the HH in terms of two independent sets of Pauli matrices \(\sigma\) and \(\tau\)

\[ H^{HH} = \sum_i t[\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \tau^x_i \tau^x_{i+1} + \tau^y_i \tau^y_{i+1}] + U\sigma^z_i \tau^z_i. \]  

(8)

It is convenient to consider writing the HH in terms of some basic ‘building blocks’. It is straightforward and well-known that a building block of HH is the XX model with Hamiltonian

\[ H^{XX} = \sum_i t[\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}]. \]  

(9)

In turn the XX model can be regarded as a particular case of the XXZ model

\[ H^{XXZ} = \sum_i t[\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \xi \sigma^z_i \sigma^z_{i+1}], \]  

(10)

when \(\xi = 0\). So far we have assumed su(2) symmetry. It is well-known that exact integrability is a feature of several toy models of statistical mechanics, namely 2d Ising model,
six-vertex model, eight-vertex model, and others. A 2d classical statistical model which is a
‘covering’ model for 1d Hubbard model has been identified by Shastry \[16\]. In particular it
was shown in \[16\] that the 2d covering model of the 1d HH provides a one parameter family
transfer matrices which commute with the HH and that two transfer matrices of a family
mutually commute. Underlying the commutation relation is the Yang-Baxter factorization
condition. In turn underlying the Yang-Baxter relations and knots are quantum groups
\[19,20\]. We must be careful to distinguish between the classical Yang-Baxter relations from
their quantum counterparts. The classical Lie group underlies classical Yang-Baxter equa-
tion just as the quantum group underlies the quantum Yang-Baxter equation. Indeed this
is one way of defining quantum group from its classical Lie group counterpart. We caution
the reader that currently there is no ‘satisfactory’ general definition of a quantum group\[11\]
However it is commonly accepted \[20\] that quantum groups are certain ‘well-behaved’ Hopf
algebras and that the standard deformations of the enveloping Hopf algebras of semisimple
Lie algebras and of coordinate Hopf algebras of the corresponding Lie groups are guiding
examples. An amazing feature of quantum group theory is the unexpected connections with
apparently unrelated concepts in physics and mathematics such as Lie Groups, Lie Algebras
and their representations, special functions, knot theory, low-dimensional topology, operator
algebras, noncommutative geometry, combinatorics, quantum inverse scattering problem,
theory of integrable models, conformal and quantum field theory and perhaps others. In
summary a class of non-commutative Hopf algebra was found in the investigatons of inte-
grable systems. In turn these Hopf algebras are q-deformed function algebras of classical
groups, this structure can be taken to define a quantum group. The structure of quantum
groups further suggest that one may envision the possibility of even discarding the com-
mutativity of the algebra of coordinate functions. The new class of symmetry based on
the noncommutativity of the algebra of coordinate functions may have applications to real
physical systems [such as HTSC and related phases in strongly correlated electron systems]
other than the integrable systems. Indeed we want to go beyond integrable systems. How-
ever it is first useful to understand the exact role of quantum groups in modeling of HTSC
and other quantum phases and what happens to interacting integrable 1d models as they
evolve towards 2d. One of the prime motivations underlying our proposal is the experimen-
tal observation of stripe structure [phase] in HTSC materials. A number of experimental
techniques have recently observed that the CuO\(_2\) are rather inhomogeneous, providing evi-

\[^{\dagger}\]We note that in the present paper we have considered su(2) Lie groups as one of the basic
ingredient. The generalization to other Lie groups is an interesting issue and has been dealt by
several authors, for example the su(N) XX model and its variant has been considered by Maassarani
and Mathieu \[17\]. In context of integrable supersymmetric and q-deformed supersymmetric models
there are several works, for example see Ref. \[18\] and references therein.

\[^{\dagger\dagger}\]We mean in terms of rigorous mathematics
idence for phase separation into a two component system, i.e. carrier-rich and carrier-poor regions. In particular, extended x-ray absorption fine structure [EXAFS] demonstrated that these domains forms stripes of undistorted and distorted local structures alternating with mesoscopic length scale comparable with coherence length in HTSC. In theories based on magnetic interactions [25] for modeling of HTSC, it has been assumed that the CuO$_2$ planes in HTSC materials are microscopically homogeneous. However, a number of experimental techniques have recently observed that the CuO$_2$ are rather inhomogeneous, providing evidence for phase separation into a two component system, i.e. carrier-rich and carrier-poor regions [26]. In particular, extended x-ray absorption fine structure [EXAFS] demonstrated that these domains forms stripes of undistorted and distorted local structures alternating with mesoscopic length scale comparable with coherence length in HTSC. The neutron pair distribution function of Egami et al. [27] also provides structural evidence for two component charge carriers. Other techniques also seem to point that below a certain temperature $T^*$ the CuO$_2$ planes may have ordered stripes of carrier-rich and carrier-poor domains [27]. The emergence of experimental evidence for inhomogeneous structure has led to renewal of interest, in theories of HTSC which are based on alternative mechanism, such as phonon scattering, the lattice effect on high $T_c$ superconductivity [25,30,27]. Polarized EXAFS study of optimally doped YBa$_2$Cu$_3$O$_y$ shows in-plane lattice anomaly [26] below a characteristic temperature $T^*$ which lies above $T_c$, and close to the characteristic temperature of spin gap opening $T^*$. It is an interesting question if the in-plane lattice anomaly is related to the charge stripe or spin-phonon interaction. We note that it has been attempted in [31,32] to relate the spin gap observed in various experiments such as NMR, neutron scattering and transport properties to the short-range ordering of spin singlets.

In the context of engineering of quantum liquid behavior it is interesting to note the work by Bianconi et al., [33]. In Ref. [33] a model of stripes is constructed in the context of HTSC materials. In this model [33] a gas of free electrons with effective mass $m^*$ moves in a superlattice of quantum stripes of width $L$ separated by a periodic potential barrier. Here [33] the 2d striped phase is formed by superconducting stripes of width $L$ [U-stripes] alternated by separating stripes of width $W$ [D-stripes]. Their conclusion [33] which is of interest to us is that the maximum critical temperature is reached at the cross-over from 2d to 1d behavior. Indeed this observation is in agreement with our intuitive remark in [14], namely “superconductivity arises when two immiscible phases namely 2-D antiferromagnetic state and a 3-D metallic state, are “forced” to meet at $T_c \to \infty.”

In conclusion we have proposed a model which can lead to a theory of dimensional

$^\dagger$The following can be taken as a definition of $T^*$: $T^*$ is an onset temperature of pseudogap opening in spin or charge excitation spectra.

$^\ddagger$$T^*$ may be defined as follows: $T^*$ is an onset temperature of local phonon anomalies and $T^* < T^*$. 

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crossover from 1d to 2d. We have also presented a partially q-deformed version of the model Hamiltonian. It is expected that this model can also be used to study the underlying physics of spin ladder compounds [34].

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Note Added:-We think it is our duty scientific or otherwise to make the following remarks regarding citations and references. It is a fact nowadays that most authors do not bother to cite other works until that work belongs to a particular group or groups that they are affiliated with in one way or the other. A lot of times we tried to cite work which could be relevant to our work. But given the present climate of citations we think that is unfair to us. Moreover one simply cannot go through many works since there are so many. We have thus decided that we are not obligated just like most to consume our time going through other peoples work. In summary we can disadvantage ourself unnecessarily by going through other people work who do not care to cite work of others beside themselves and their favorite groupings. However in the present case we have followed up our usual policy of following up on references as much as possible.
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