AUSLANDER CORRESPONDENCE FOR KAWADA RINGS

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Abstract. We study the Auslander ring of a basic left K"{o}the ring \( \Lambda \) and give a characterization of basic left K"{o}the rings in terms of their Auslander rings. We also study the functor category \( \text{Mod}((\Lambda\text{-mod})^{\text{op}}) \) and characterize basic left K"{o}the rings \( \Lambda \) by using functor categories \( \text{Mod}((\Lambda\text{-mod})^{\text{op}}) \). As a consequence we show that there exists a bijection between the Morita equivalence classes of left Kawada rings and the Morita equivalence classes of Auslander generalized right QF-2 rings.

1. Introduction

A ring \( \Lambda \) is said to be of finite representation type or representation-finite if it is left artinian and there are only finitely many finitely generated indecomposable left \( \Lambda \)-modules up to isomorphism. The class of representation-finite rings is one of the most important class of rings. Auslander in his famous theorem, which is called Auslander correspondence, provided a bijection between the set of Morita equivalence classes of representation-finite rings and that of rings with nice homological properties. Let \( \Gamma \) be an artinian ring. The global dimension of \( \Gamma \), which is denoted by \( \text{gl.dim}(\Gamma) \), is at most \( n \) if the projective dimension of each left \( \Gamma \)-module is less than or equal \( n \). The dominant dimension of \( \Gamma \), which is denoted by \( \text{dom.dim}(\Gamma) \), is at least \( n \) if the first \( n \) terms in the minimal injective resolution of \( 1 \Gamma \) are projectives. An artinian ring \( \Gamma \) is called an Auslander ring if \( \text{gl.dim}(\Gamma) \leq 2 \) and \( \text{dom.dim}(\Gamma) \geq 2 \). According to the Auslander correspondence [5, Corollary 4.7], there exists a bijection between the set of Morita equivalence classes of representation-finite rings and that of Auslander rings \( \Gamma \). It is given by \( \Lambda \to \Gamma := \text{End}_{\Lambda}(M) \), where \( M = \bigoplus_{i=1}^{n} M_i \) and \( \{M_1, \cdots, M_n\} \) is a complete set of representative of the isomorphic classes of finitely generated indecomposable left \( \Lambda \)-modules. \( M \) is called the Auslander generator of \( \Lambda\text{-mod} \). Auslander correspondence gives a very nice connection between concept of representation-finiteness which is a representation theoretic property and the concept of Auslander ring which is a homological property. On the other hand, the category \( \text{Mod-}\Gamma \) is equivalent to the functor category \( \text{Mod}((\Lambda\text{-mod})^{\text{op}}) \), where \( \text{Mod}((\Lambda\text{-mod})^{\text{op}}) \) is the category of all additive covariant functors from \( \Lambda\text{-mod} \) to the category of abelian groups. This equivalence propose to use the functor categories for studying the representation-finite
rings which leads to functorial approach in representation theory (see [3, 5]). A functor $F \in \text{Mod}((\Lambda\text{-mod})^{op})$ is called noetherian (resp., artinian) if it satisfies the ascending (resp., descending) chain condition on subfunctors. $F$ is called finite if it is both noetherian and artinian and $F$ is called locally finite if every finitely generated subfunctor of $F$ is finite. The category $	ext{Mod}((\Lambda\text{-mod})^{op})$ is said to be locally finite if every $F \in \text{Mod}((\Lambda\text{-mod})^{op})$ is locally finite. Auslander in Theorem 3.1 of [5] proved that a ring $\Lambda$ is of finite representation type if and only if $\text{Mod}((\Lambda\text{-mod})^{op})$ is locally finite if and only if every representable functor of $\text{Mod}((\Lambda\text{-mod})^{op})$ is both artinian and noetherian (see also [4]).

It is known that every finitely generated $\mathbb{Z}$-module decompose to the finite direct sum of cyclic modules. Artinian rings with this property are representation-finite rings. These rings are precisely rings over which every module is a direct sum of cyclic modules. A ring $\Lambda$ is called left Köthe if every left $\Lambda$-module is a direct sum of cyclic modules. Köthe studied this class of rings which is an important subclass of representation-finite rings. He showed that commutative artinian rings which have this property are serial. Also he posed the question to classify the non-commutative rings with this property [15] (see also [11] and references therein). Kawada completely solved the Köthe’s problem for the basic finite dimensional $K$-algebras. Kawada’s papers contain a set of 19 conditions which characterize Kawada algebras, as well as, the list of all possible finitely generated indecomposable modules [12, 13, 14]. A ring $\Lambda$ is called left Kawada if any ring Morita equivalent to $\Lambda$ is a left Köthe ring [18]. Ringel showed that any finite dimensional $K$-algebra of finite representation type is Morita equivalent to a Köthe algebra. By using of the multiplicity-free of top and soc of finitely generated indecomposable modules, he also gave a characterization of Kawada algebras [18]. In [7] the authors proved the Ringel’s result for artinian rings.

Since any left Köthe ring is representation-finite, the above results of Auslander propose to use the Auslander ring and the functor category $\text{Mod}((\Lambda\text{-mod})^{op})$ for study left Köthe rings. In this paper we study left Köthe rings by using their Auslander rings and their functor categories. We say that a semiperfect ring with enough idempotents $R$ is generalized left (resp., right) QF-2 if every indecomposable projective unitary left (resp., right) $R$-module $P$ has multiplicity-free socle (i.e. the composition factors of $\text{soc}(P)$ are pairwise non-isomorphic). In the following theorem, by using the notion of generalized right QF-2 rings, we give a characterization of basic left Köthe rings.

**Theorem A.** (See Theorem 3.2) Let $\Lambda$ be a basic ring. Then the following conditions are equivalent.

(a) $\Lambda$ is a left Köthe ring.
(b) \( \Lambda \) is a ring of finite representation type and the Auslander ring \( T \) of \( \Lambda \) is a generalized right QF-2 ring.

(c) \( \Lambda \) is a left artinian ring and \( T = \widehat{\text{End}}_\Lambda(V) \) is a left locally finite generalized right QF-2 ring, where \( V \) is a direct sum of modules in a complete set of representative of the isomorphic classes of finitely generated indecomposable left \( \Lambda \)-modules.

(d) \( \Lambda \) is a left artinian ring and for each indecomposable left \( \Lambda \)-module \( M \), \( \text{Hom}_\Lambda(M, -) \simeq \varphi(-) \) for some positive-primitive formula \( \varphi(x) \) over \( \Lambda \).

(e) \( \Lambda \) is a left artinian ring and every indecomposable projective object in \( \text{Mod}((\Lambda\text{-mod})^{\text{op}}) \) has finitely generated essential multiplicity-free socle.

Using the above theorem, we prove the following theorem, which gives a characterization of left Kawada rings.

**Theorem B.** (See Theorem [3.3]) There exists a bijection between the Morita equivalence classes of left Kawada rings and the Morita equivalence classes of Auslander generalized right QF-2 rings.

Before proving our main results in Section 3 we prove some preliminary results in the following section.

### 1.1. Notation.

Throughout this paper all rings are associative with unit unless otherwise stated. Let \( R \) be a ring (not necessary with unit). We write all homomorphisms of left (resp., right) \( R \)-modules on the right (resp., left), so \( fg \) (resp., \( g \circ f \)) means “first \( f \) then \( g \)”. We denote by \( R\text{-Mod} \) (resp., \( \text{Mod-R} \)) the category of all left (resp., right) \( R \)-modules and by \( J(R) \) the Jacobson radical of \( R \). Also we denote by \( R\text{-mod} \) (resp., \( \text{mod-R} \)) the category of all finitely generated left (resp., right) \( R \)-modules. A left (resp., right) \( R \)-module \( M \) is called unitary if \( RM = M \) (resp., \( MR = M \)).

We denote by \( R\text{Mod} \) (resp., \( \text{ModR} \)) the category of all unitary left (resp., right) \( R \)-modules. We denote by \( \text{Mod}((R\text{-mod})^{\text{op}}) \) the category of all additive covariant functors from \( R\text{-mod} \) to the category of abelian groups, which is an abelian category. We write \( \bigoplus_A M \) and \( \prod_A M \) for the direct sum of \( \text{card}(A) \) copies of an \( R \)-module \( M \) and the direct product of \( \text{card}(A) \) copies of an \( R \)-module \( M \), respectively. For a left \( R \)-module \( V \) we denoted by \( \text{Add}(V) \) the full subcategory of \( R\text{-Mod} \) whose objects are all left \( R \)-modules that are isomorphic to direct summands of \( \bigoplus_A V \) for any set \( A \). We denoted by \( \text{Proj}(R) \) (resp., \( \text{Proj}(R^{\text{op}}) \)) the full subcategory of \( R\text{Mod} \) (resp., \( \text{ModR} \)) whose objects are projective left (resp., right) \( R \)-modules. Let \( N \) be a unitary left \( R \)-module. We denote by \( E(N) \), \( \text{rad}(N) \), \( \text{top}(N) \) and \( \text{soc}(N) \) the injective hull of \( N \) in \( R\text{Mod} \), radical of \( N \), top of \( N \) and socle of \( N \), respectively. Let \( f : X \to Y \) be an \( R \)-module homomorphism and assume that \( M \) is an \( R \)-submodule of \( X \). We denote by \( f |_M \) the restriction of \( f \) to \( M \). Let \( L \) and \( N \) be
two left $R$-modules. The submodule $\text{Re}(L,N) = \bigcap \{ \text{Ker} f \mid f \in \text{Hom}_R(L,N) \}$ is called reject of $N$ in $L$.

2. Preliminaries

Let $\mathcal{U}$ be a non-empty set of left $\Lambda$-modules. A left $\Lambda$-module $M$ is said to be generated by $\mathcal{U}$ if for every pair of distinct morphisms $f,g : M \to B$ in $\Lambda$-Mod there exists a morphism $h : U \to M$ with $U \in \mathcal{U}$ such that $hf \neq hg$. Also, $\mathcal{U}$ is called a generating set for $\Lambda$-Mod if every left $\Lambda$-module generated by $\mathcal{U}$. A left $\Lambda$-module $U$ is called generator in $\Lambda$-Mod if set $\mathcal{U} = \{U\}$ is a generating set for $\Lambda$-Mod (see [16, Ch. V, Sect. 7]). Let $\Lambda$ be a ring of finite representation type and $\{X_1, \cdots , X_n\}$ be a complete set of representative of the isomorphic classes of finitely generated indecomposable left $\Lambda$-modules. A left $\Lambda$-module $X$ is called an Auslander generator of $\Lambda$-mod. Also the endomorphism ring of $X$ is called Auslander ring of $\Lambda$. Note that by [5, Proposition 3.6], the Auslander ring of $\Lambda$ is an artinian ring.

Let $\Lambda$ be a ring and $\{V_\alpha \mid \alpha \in J\}$ be a family of finitely generated left $\Lambda$-modules. Set $V = \bigoplus_{\alpha \in J} V_\alpha$ and for each $\alpha \in J$, let $e_\alpha = \pi_\alpha e_\alpha$, where $\pi_\alpha : V \to V_\alpha$ is the canonical projection and $e_\alpha : V_\alpha \to V$ is the canonical injection. For each left $\Lambda$-module $X$, we define as in [9, Page 40], $\text{Hom}_\Lambda(V,X) = \{ f \in \text{Hom}_\Lambda(V,X) \mid (V)e_\alpha f = 0 \text{ for almost all } \alpha \in J\}$. For $X = V$, we write $\widehat{\text{Hom}}_\Lambda(V,V) = \widehat{\text{End}}_\Lambda(V)$. Let $R$ be a ring (not necessary with unit). $R$ is called a ring with enough idempotents if there exists a family $\{q_\alpha \mid \alpha \in I\}$ of pairwise orthogonal idempotents of $R$ with $R = \bigoplus_{\alpha \in I} Rq_\alpha = \bigoplus_{\alpha \in I} q_\alpha R$ (see [9, Page 39]). $T = \widehat{\text{End}}_\Lambda(V)$ is a ring with enough idempotents because of $T = \bigoplus_{\alpha \in J} Te_\alpha = \bigoplus_{\alpha \in J} e_\alpha T$. Fuller in [9, Page 40] defined a covariant functor $\widehat{\text{Hom}}_\Lambda(V,-) : \Lambda$-Mod $\to \text{TMod}$ as follows. For any morphism $f : X \to Y$ in $\Lambda$-Mod, he defined $\widehat{\text{Hom}}_\Lambda(V,f) : \widehat{\text{Hom}}_\Lambda(V,X) \to \widehat{\text{Hom}}_\Lambda(V,Y)$ via $g \to gf$. From [9, Pages 40-41] we observe that the covariant functor $\widehat{\text{Hom}}_\Lambda(V,-)$ is a left exact functor and also preserves direct sums. Moreover it is an additive equivalence between the full subcategory $\text{Add}(V)$ of $\Lambda$-Mod and the full subcategory $\text{Proj}(T)$ of $\text{TMod}$ with the inverse equivalence $V \otimes_T -$ . Note that the covariant functor $\widehat{\text{Hom}}_\Lambda(V,-) : \Lambda$-Mod $\to \text{TMod}$ preserves indecomposable modules, injective modules and essential extensions when $V$ is a generator in $\Lambda$-Mod (see [19, Proposition 51.7]).

Also it is easily seen that $V$ is a generator in $\Lambda$-Mod when $\Lambda$ is left artinian and $\{V_\alpha \mid \alpha \in J\}$ is a complete set of representative of the isomorphic classes of finitely generated indecomposable left $\Lambda$-modules. By using [5, Proposition 4.2 and Corollary 4.8] and [19, Proposition 46.7] we have the following result.

**Lemma 2.1.** Let $\Lambda$ be a ring of finite representation type, $X$ be an Auslander generator and $T = \text{End}_\Lambda(X)$. Then
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(a) The functor $\text{Hom}_\Lambda(X, -) : \Lambda\text{-Mod} \to \text{Proj}(T)$ is an equivalence which preserves and reflects finitely generated modules.

(b) The functor $\text{Hom}_\Lambda(X, -)$ preserves essential extension.

(c) If $E$ is an injective left $\Lambda$-module, then $\text{Hom}_\Lambda(X, E)$ is an injective left $T$-module.

A ring $\Lambda$ is called semiperfect if $\Lambda / J(\Lambda)$ is a semisimple ring and idempotents in $\Lambda / J(\Lambda)$ can be lifted to $\Lambda$ (see [2, Page 303]). It is well-known that a ring with enough idempotents $R$ is semiperfect if and only if every finitely generated unitary left $R$-module has a projective cover in $R\text{Mod}$ and only if every finitely generated projective unitary left $R$-module is a direct sum of local modules (see [10, Page 95] and [19, Proposition 49.10]).

A semiperfect ring $\Lambda$ is called left (resp., right) $QF$-2 if every indecomposable projective left (resp., right) $\Lambda$-module has a simple essential socle (see [10, Sect. 4]). A left $\Lambda$-module $M$ has multiplicity-free socle if composition factors of $\text{soc}(M)$ are pairwise non-isomorphic (see [18, Sect. 1]). We say that a semiperfect ring with enough idempotents $R$ is generalized left (resp., right) $QF$-2 if every indecomposable projective unitary left (resp., right) $R$-module has multiplicity-free socle.

**Proposition 2.2.** Let $\Lambda$ be a ring of finite representation type. Then the following conditions are equivalent.

(a) The Auslander ring of $\Lambda$ is a generalized left $QF$-2 ring.

(b) Every indecomposable left $\Lambda$-module has multiplicity-free socle.

**Proof.** Let $X$ be an Auslander generator of $\Lambda\text{-mod}$ and $T = \text{End}_\Lambda(X)$. Assume that $M$ is a finitely generated left $\Lambda$-module. Since $\Lambda$ is left artinian, by [19, Propositions 21.3 and 31.4], $\text{soc}(M)$ is a finitely generated and essential submodule of $M$. It is easy to see that $\text{soc}(M) = \bigoplus_{i=1}^n T_i$, where $n \in \mathbb{N}$ and each $T_i$ is a simple left $\Lambda$-module. Since $\text{soc}(M)$ is an essential submodule of $M$, we get $E(M) \cong \bigoplus_{i=1}^n E(T_i)$ as $\Lambda$-modules. On the other hand, by Lemma 2.1(b), there exists an essential monomorphism $\text{Hom}_\Lambda(X, M) \to \text{Hom}_\Lambda(X, E(M))$ as $T$-modules. Also by Lemma 2.1(c), we see that $\text{Hom}_\Lambda(X, E(M))$ is an injective left $T$-module. Therefore

$$E(\text{Hom}_\Lambda(X, M)) \cong \text{Hom}_\Lambda(X, E(M)) \cong \bigoplus_{i=1}^n \text{Hom}_\Lambda(X, E(T_i))$$

as $T$-modules. $\text{End}_T(\text{Hom}_\Lambda(X, E(T_i)))$ is a local ring because the functor

$$\text{Hom}_\Lambda(X, -) : \Lambda\text{-Mod} \to \text{Proj}(T)$$
is an equivalence and \( \text{End}_{\Lambda}(E(T_i)) \) is a local ring.

\((a) \Rightarrow (b)\). Let \( M \) be an indecomposable left \( \Lambda \)-module. Since \( \Lambda \) is a ring of finite representation type, by [5, Corollary 4.8], \( M \) is finitely generated. Hence \( \text{soc}(M) = \bigoplus_{i=1}^{n} T_i \), where \( n \in \mathbb{N} \) and each \( T_i \) is a simple left \( \Lambda \)-module. We show that \( T_i \not\cong T_j \) when \( i \neq j \). By Lemma 2.1(a), \( \text{Hom}_{\Lambda}(X,M) \) is a finitely generated indecomposable projective left \( T \)-module. Since \( T \) is an artinian ring, \( \text{soc}(\text{Hom}_{\Lambda}(X,M)) = \bigoplus_{i=1}^{r} S_i \), where \( r \in \mathbb{N} \), each \( S_i \) is a simple left \( T \)-module. Moreover \( E(\text{Hom}_{\Lambda}(X,M)) = \bigoplus_{i=1}^{n} E(S_i) \) as \( T \)-modules. By assumption, \( S_i \not\cong S_j \) when \( i \neq j \). On the other hand, by above statement \( E(\text{Hom}_{\Lambda}(X,M)) = \bigoplus_{i=1}^{n} \text{Hom}_{\Lambda}(X,E(T_i)) \) as \( T \)-modules, where \( n \in \mathbb{N} \), each \( T_i \) is a simple left \( \Lambda \)-module and \( \text{soc}(M) = \bigoplus_{i=1}^{n} T_i \). Also \( \text{End}_T(\text{Hom}_{\Lambda}(X,E(T_i))) \) is a local ring for each \( 1 \leq i \leq n \). Therefore \( t = n \) and for each \( 1 \leq i \leq n \), \( \text{E}(S_i) \cong \text{Hom}_{\Lambda}(X,E(T_i)) \) as \( T \)-modules. We can certainly conclude that \( T_i \not\cong T_j \) when \( i \neq j \) since otherwise \( E(T_i) \cong E(T_j) \) as \( \Lambda \)-modules for some \( i \neq j \). This implies that \( \text{Hom}_{\Lambda}(X,E(T_i)) \cong \text{Hom}_{\Lambda}(X,E(T_j)) \) as \( T \)-modules. \( E(S_i) \cong E(S_j) \) yields \( S_i \cong S_j \) which is a contradiction. Therefore every indecomposable left \( \Lambda \)-module has multiplicity-free socle.

\((b) \Rightarrow (a)\). Let \( P \) be an indecomposable projective left \( T \)-module. Then \( P \) is a finitely generated left \( T \)-module because \( T \) is an artinian ring. Hence by Lemma 2.1(a), \( P \cong \text{Hom}_{\Lambda}(X,M) \) for some a finitely generated indecomposable left \( \Lambda \)-module \( M \). Then \( E(P) \cong E(\text{Hom}_{\Lambda}(X,M)) = \bigoplus_{i=1}^{n} \text{Hom}_{\Lambda}(X,E(T_i)) \) and \( \text{End}_T(\text{Hom}_{\Lambda}(X,E(T_i))) \) is a local ring for each \( 1 \leq i \leq n \), where \( n \in \mathbb{N} \), \( \text{soc}(M) = \bigoplus_{i=1}^{n} T_i \) and each \( T_i \) is a simple left \( \Lambda \)-module. On the other hand, \( \text{soc}(P) = \bigoplus_{i=1}^{s} S_i \) and \( E(P) \cong \bigoplus_{i=1}^{s} E(S_i) \), where \( s \in \mathbb{N} \) and each \( S_i \) is a simple left \( T \)-module. Therefore \( s = n \) and \( E(S_i) \cong \text{Hom}_{\Lambda}(X,E(T_i)) \) for each \( 1 \leq i \leq n \). If \( S_i \not\cong S_j \) as \( T \)-modules for some \( i \neq j \), then \( E(S_i) \cong E(S_j) \) as \( T \)-modules and hence \( \text{Hom}_{\Lambda}(X,E(T_i)) \cong \text{Hom}_{\Lambda}(X,E(T_j)) \) as \( T \)-modules. So by Lemma 2.1(a), we have \( E(T_i) \cong E(T_j) \) as \( \Lambda \)-modules. It follows that \( T_i \cong T_j \) as \( \Lambda \)-modules which is a contradiction. Consequently \( T \) is a generalized left QF-2 ring and the result follows.

Remark 2.3 and Lemma 2.4 are well-known to experts, but since we could not find a suitable reference in the literature, we include these results here.

**Remark 2.3.** Let \( \Lambda \) be a left artinian ring and \( \{V_{\alpha} \mid \alpha \in J\} \) be a family of finitely generated indecomposable left \( \Lambda \)-modules. Set \( V = \bigoplus_{\alpha \in J} V_{\alpha} \). Let \( \pi_{\alpha} : V \to V_{\alpha} \) be the canonical projection and \( e_{\alpha} : V_{\alpha} \to V \) be the canonical injection. Set \( e_{\alpha} = \pi_{\alpha} e_{\alpha} \) for each \( \alpha \in J \). Then \( T = \text{End}_{\Lambda}(V) = \bigoplus_{\alpha \in J} T e_{\alpha} = \bigoplus_{\alpha \in J} e_{\alpha} T \). Since \( \text{Hom}_{\Lambda}(V,-) : \text{Add}(V) \to \text{Proj}(T) \) is an equivalence of categories and \( T e_{\alpha} \cong \text{Hom}_{\Lambda}(V,V_{\alpha}) \) as \( T \)-modules we have for each \( \alpha \in J \), \( e_{\alpha} T e_{\alpha} \cong \text{End}_{\Lambda}(V_{\alpha}) \) as rings. \( V_{\alpha} \) has finite length and also it is indecomposable, then by [19, Proposition 32.4(3)], \( e_{\alpha} T e_{\alpha} \) is a local ring. It follows that every finitely generated projective unitary left \( T \)-module is a direct sum of local modules. Consequently \( T \) is a semiprfect ring.
Lemma 2.4. Let $\Lambda$ be a ring and assume that $\{V_\alpha \mid \alpha \in I\}$ is a family of non-isomorphic finitely generated indecomposable left $\Lambda$-modules. Set $T = \widehat{\text{End}}_\Lambda(V)$, where $V = \bigoplus_{\alpha \in I} V_\alpha$. If $\Lambda$ is a left artinian ring, then $\{Te_\alpha/rad(Te_\alpha) \mid \alpha \in I\}$ is a complete set of representative of the isomorphic classes of simple unitary left $T$-modules, where $e_\alpha = \pi_\alpha e_\alpha$. $\pi_\alpha : V \rightarrow V_\alpha$ is the canonical projection and $e_\alpha : V_\alpha \rightarrow V$ is the canonical injection.

Proof. By Remark 2.3, $\text{End}_T(\alpha)$ is a local ring. $Te_\alpha$ is a projective unitary left $T$-module and hence $\text{rad}(Te_\alpha)$ is a superfluous maximal submodule of $Te_\alpha$. Then $Te_\alpha/\text{rad}(Te_\alpha)$ is a simple unitary left $T$-module. Also the canonical projection $Te_\alpha \rightarrow Te_\alpha/\text{rad}(Te_\alpha)$ is a projective cover of $Te_\alpha/\text{rad}(Te_\alpha)$ in $T\text{Mod}$. If $Te_\alpha/\text{rad}(Te_\alpha) \cong Te_\beta/\text{rad}(Te_\beta)$ as $T$-modules, then $Te_\alpha \cong Te_\beta$. It is easy to see that $V_\alpha \cong V_\beta$ as $\Lambda$-modules and hence $\alpha = \beta$. Therefore $\{Te_\alpha/\text{rad}(Te_\alpha) \mid \alpha \in I\}$ is a set of non-isomorphic simple unitary left $T$-modules. It remains to prove that every simple unitary left $T$-module $S$ is isomorphism to $Te_\alpha/\text{rad}(Te_\alpha)$ for some $\alpha \in I$. Let $S$ be a simple unitary left $T$-module. By Remark 2.3 $T$ is a semiperfect ring and hence $S$ has a projective cover in $T\text{Mod}$. Let $f : P \rightarrow S$ be a projective cover of $S$ in $T\text{Mod}$. Since $\text{Ker} f$ is a maximal superfluous submodule of $P$, $P$ is an indecomposable unitary left $T$-module. It follows that $P \cong Te_\alpha$ as $T$-modules for some $\alpha \in I$ and hence $S \cong Te_\alpha/\text{rad}(Te_\alpha)$ as $T$-modules and the result follows.

Let $\Lambda$ be a ring and $\{V_\alpha \mid \alpha \in J\}$ be a family of finitely generated left $\Lambda$-modules. Set $V = \bigoplus_{\alpha \in J} V_\alpha$ and $R = \widehat{\text{End}}_\Lambda(V)$. As in [9, Page 40], we call $R$ the left functor ring of $\Lambda$ when $\{V_\alpha \mid \alpha \in J\}$ is a complete set of representative of the isomorphic classes of finitely generated left $\Lambda$-modules. Two rings with enough idempotents $R$ and $S$ are said to be Morita equivalent if there exists an additive covariant equivalence between the category of unitary left $R$-modules and the category of unitary left $S$-modules (see [1, Sect. 3]).

Proposition 2.5. Let $\Lambda$ be a ring and $\{V_\alpha \mid \alpha \in J\}$ be a complete set of representative of the isomorphic classes of finitely generated indecomposable left $\Lambda$-modules. Set $V = \bigoplus_{\alpha \in J} V_\alpha$ and $T = \widehat{\text{End}}_\Lambda(V)$. If every left $\Lambda$-module is a direct sum of finitely generated modules, then $T$ is Morita equivalent to the left functor ring $R$ of $\Lambda$.

Proof. First we show that $\text{Proj}(R)$ and $\text{Proj}(T)$ are equivalent. Let $\{U_\alpha \mid \alpha \in I\}$ be a complete set of representative of the isomorphic classes of finitely generated left $\Lambda$-modules such that $R = \widehat{\text{End}}_\Lambda(U)$, where $U = \bigoplus_{\alpha \in I} U_\alpha$. By assumption $\text{Add}(U) = \Lambda\text{-Mod}$ and also by [6, Theorem 4.4], we see that $\Lambda$ is a left artinian ring. Since $\text{Add}(U) = \Lambda\text{-Mod}$, the functor $\text{Hom}_\Lambda(U, -) : \Lambda\text{-Mod} \rightarrow \text{Proj}(R)$ is an additive equivalence with the inverse equivalence $U \otimes_R -$. On the
other hand, every finitely generated left \( \Lambda \)-module is a direct sum of indecomposable modules because \( \Lambda \) is left artinian. Then \( \text{Add}(V) = \Lambda\text{-Mod} \) and we have an additive equivalence \( \widehat{\text{Hom}}_{\Lambda}(V, -) : \Lambda\text{-Mod} \to \text{Proj}(T) \) with the inverse equivalence \( V \otimes_T - \). Therefore we obtain an additive equivalence \( \widehat{\text{Hom}}_{\Lambda}(V, U \otimes_R -) : \text{Proj}(R) \to \text{Proj}(T) \) with the inverse equivalence \( \widehat{\text{Hom}}_{\Lambda}(U, V \otimes_T -) : \text{Proj}(T) \to \text{Proj}(R) \). Now we show that this equivalence preserves finitely generated modules. Let \( Y \) be a finitely generated projective unitary left \( R \)-module. Set \( X = U \otimes_R Y \). Since \( \widehat{\text{Hom}}_{\Lambda}(U, X) \cong Y \) and \( \widehat{\text{Hom}}_{\Lambda}(U, -) \) preserves direct sums, \( X \) is a finitely generated left \( \Lambda \)-module and hence \( X \) is a finite direct sum of finitely generated indecomposable left \( \Lambda \)-modules. Then \( \widehat{\text{Hom}}_{\Lambda}(V, X) \) is a finitely generated projective unitary left \( T \)-module. Consequently the equivalence \( \widehat{\text{Hom}}_{\Lambda}(V, U \otimes_R -) : \text{Proj}(R) \to \text{Proj}(T) \) preserves finitely generated projective modules. By the similar argument, we can see that it also reflects finitely generated projective modules. Therefore the result follows from \([8] \) Theorem 3.10]. \( \square \)

Let \( \Lambda \) be a ring and \( \mathcal{U} \) be a non-empty set of left \( \Lambda \)-modules. A left \( \Lambda \)-module \( M \) is said to be\textit{cogenerated by} \( \mathcal{U} \) if for every pair of distinct morphisms \( f, g : B \to M \) in \( \Lambda\text{-Mod} \), there exists a morphism \( h : M \to U \) with \( U \in \mathcal{U} \) such that \( fh \neq gh \). Also, \( \mathcal{U} \) is called a \textit{cogenerating set} for \( \Lambda\text{-Mod} \) if every left \( \Lambda \)-module cogenerated by \( \mathcal{U} \). A left \( \Lambda \)-module \( U \) is called \textit{cogenerator} if the set \( \mathcal{U} = \{ U \} \) is a cogenerating set for \( \Lambda\text{-Mod} \) (see \([16] \) Ch. V, Sect. 7). A ring \( \Lambda \) is called \textit{left Morita} if there is an injective cogenerator left \( \Lambda \)-module \( Q \) such that it is an injective cogenerator right \( \Delta \)-module and \( \Delta \cong \text{End}_{\Lambda}(Q) \), where \( \Delta = \text{End}_{\Lambda}(Q) \) (see \([19] \) Ch. 9, Sect. 47)). In this case we say that \( \Lambda \) is left Morita to \( \Delta \).

**Proposition 2.6.** Let \( \Lambda \) be a ring and \( R \) be the left functor ring of \( \Lambda \). If \( \Lambda \) is a left artinian ring and every finitely generated indecomposable projective unitary right \( R \)-module has finitely generated essential socle, then \( \Lambda \) is a left Morita ring.

**Proof.** By \([10] \) Theorem 3.3] it is enough to show that \( \text{soc}(R_R) \) is essential in \( R_R \) and also \( R_R \) has only finitely many non-isomorphic simple right \( R \)-modules. Suppose that \( \{ U_{\alpha} \mid \alpha \in I \} \) is a complete set of representative of the isomorphic classes of finitely generated left \( \Lambda \)-modules and \( R = \widehat{\text{End}}_{\Lambda}(U) \), where \( U = \bigoplus_{\alpha \in I} U_{\alpha} \). \( U \) is a generator in \( \Lambda\text{-Mod} \). Since \( \Lambda \) is left artinian, by \([19] \) Proposition 51.7(9)], \( R \) is a semiperfect ring. Therefore every finitely generated projective unitary right \( R \)-module is a direct sum of indecomposable modules. Hence \( \text{soc}(R_R) \) is an essential submodule of \( R_R \). Now we show that \( R_R \) has only finitely many non-isomorphic simple right \( R \)-modules. \( \Lambda \Lambda \cong U_{\alpha} \) for some \( \alpha \in I \) and so \( U \cong e_{\alpha}R \) as \( R \)-modules, where \( e_{\alpha} = \pi_{\alpha} \varepsilon_{\alpha} \), \( \pi_{\alpha} : U \to U_{\alpha} \) is the canonical projection and \( \varepsilon_{\alpha} : U_{\alpha} \to U \) is the canonical injection. Moreover \( e_{\alpha}R = \bigoplus_{i=1}^{\alpha} P_{i} \), where each \( P_{i} \) is an indecomposable right \( R \)-module and hence \( \text{soc}(e_{\alpha}R) \cong \bigoplus_{i=1}^{\alpha} \text{soc}(P_{i}) \) as \( R \)-modules. Since every finitely generated indecomposable
projective unitary right \( R \)-module has finitely generated socle, \( \text{soc}(e_\alpha R) \) is finitely generated. In order to complete the proof, it is sufficient to show that for each simple right \( R \)-submodule \( L \) of \( R \), there is an \( R \)-module monomorphism \( L \to e_\alpha R \). Since \( e_\alpha R \) is faithful, \( \text{Re}(R, e_\alpha R) = 0 \). It follows that by [19, Proposition 14.3], there exists an \( R \)-module monomorphism \( \varphi : R_R \to \prod_\Lambda e_\alpha R \). Let \( L \) be a simple submodule of \( R_R \). Then \( L = aR \) for some \( 0 \neq a \in R \). Put \( \varphi(a) = (y_i) \), where each \( y_i \in e_\alpha R \). Since \( \varphi \) is a monomorphism, \( y_i \neq 0 \) for some \( i \). Without loss of generality we can assume that \( y_1 \neq 0 \). Since \( R \) is a ring with enough idempotents, there exists an idempotent \( e \in R \) such that \( ae = a \) and \( y_1 e = y_1 \). Therefore the \( R \)-module homomorphism \( \pi_1 \circ \overline{\varphi} : L \to e_\alpha R \) is a non-zero morphism, where \( \overline{\varphi} := \varphi|_L \) and \( \pi_1 : \prod_\Lambda e_\alpha R \to e_\alpha R \) is the canonical projection and the result follows. \( \square \)

A left \( \Lambda \)-module \( W \) is called minimal cogenerator if \( W \cong \bigoplus_{i \in I} E(S_i) \), where \( \{S_i \mid i \in I\} \) is a complete set of representative of the isomorphic classes of simple left \( \Lambda \)-modules (see [19, Ch. 3, Sect. 17]). Let \( \Lambda \) and \( \Delta \) be two rings. An additive contravariant functor \( F : \Lambda \text{-mod} \to \text{mod-\Delta} \) is called duality if it is an equivalence of categories (see [19, Ch. 9, Sect. 47]).

**Remark 2.7.** Let \( \Lambda \) be a ring and assume that \( W \) is a minimal cogenerator left \( \Lambda \)-module. Assume that \( \Lambda \) is left artinian and \( W \) is a finitely generated left \( \Lambda \)-module. Then we derive from [19, Proposition 47.15] that \( \Delta = \text{End}_\Lambda(W) \) is a right artinian ring, \( W \) is an injective cogenerator in \( \text{Mod-\Delta} \) and \( \Lambda \cong \text{End}_\Delta(W) \) as rings. Therefore \( \Lambda \) is a left Morita to \( \Delta \). Also we conclude from [19, Proposition 47.3] that the functor \( \text{Hom}_\Lambda(-, W) : \Lambda \text{-mod} \to \text{mod-\Delta} \) is a duality with the inverse duality \( \text{Hom}_\Lambda(-, W) : \text{mod-\Delta} \to \Lambda \text{-mod} \). By [2, Proposition 24.5] for each finitely generated left \( \Lambda \)-module \( X \), the lattice of all \( \Lambda \)-submodules of \( M \) and the lattice of all \( \Delta \)-submodules of \( \text{Hom}_\Lambda(M, W) \) are anti-isomorphic. Hence for each finitely generated left \( \Lambda \)-module \( X \), \( \text{soc}(\text{Hom}_\Lambda(X, W)) \cong \text{Hom}_\Lambda(\text{top}(X), W) \) as \( \Delta \)-modules.

### 3. Main results

Let \( A \) and \( B \) be two \( n \times 1 \) and \( n \times m \) matrices over \( \Lambda \), respectively. A formula \( \varphi(x) \) is called left positive-primitive formula over \( \Lambda \) in the free variable \( x \) if it has the form \( \exists y \ (A \ B)^t (x, y) = 0 \), where \( y = (c_1, \ldots, c_m) \) with entries in \( \Lambda \) (see [17]). A left positive-primitive formula is completely determined by the matrices \( A \) and \( B \). For a left \( \Lambda \)-module \( M \), the above left positive-primitive formula define a subgroup

\[
\varphi(M) = \{ a \in M \mid \exists c_1, \ldots, c_m \text{ in } M \text{ with } (A \ B)(a \ c_1 \ \cdots \ c_m)^t = 0 \}.
\]

If \( f : M \to N \) is a \( \Lambda \)-module homomorphism, then \( (a)f \in \varphi(N) \) when \( a \in \varphi(M) \). This yields a covariant functor \( \varphi(-) : \Lambda\text{-mod} \to \text{Ab} \) which is a subfunctor of the forgetful functor \( \text{Hom}_\Lambda(\Lambda\Lambda, -) \).
Note that a subfunctors $F$ of the forgetful functor $\text{Hom}_\Lambda(\_\Lambda, -)$ is finitely generated if and only if $F \simeq \varphi(-)$ for some positive-primitive formula $\varphi(x)$ over $\Lambda$ (see [17, Corollary 12.4]).

**Remark 3.1.** Let $\Lambda$ be a ring and suppose that $R = \widehat{\text{End}}_\Lambda(U)$, where $U = \bigoplus_{\alpha \in I} U_\alpha$ and $\{U_\alpha \mid \alpha \in I\}$ is a complete set of representative of the isomorphic classes of finitely generated left $\Lambda$-modules. We define as in [19] a functor

$$F : \text{Mod}R \rightarrow \text{Mod}((\Lambda\text{-mod})^{\text{op}})$$

as follows. For each $M \in \text{Mod}R$, we set $F(M) = \text{Hom}_R(\text{Hom}_\Lambda(-, U), M)$. It is easy to check that $\text{Hom}_R(\text{Hom}_\Lambda(-, U), M) : \Lambda\text{-mod} \rightarrow \text{Ab}$ is an additive covariant functor. If $f : M \rightarrow N$ is a morphism in $\text{Mod}R$, then $F(f) = \text{Hom}_R(\text{Hom}_\Lambda(-, U), f)$ which is a natural transformation from $\text{Hom}_R(\text{Hom}_\Lambda(-, U), M)$ to $\text{Hom}_R(\text{Hom}_\Lambda(-, U), N)$. By [19, Proposition 52.5], $F$ is an equivalence of categories. It is not difficult to see that the functor $F$ preserves and reflects simple objects, projective objects, monomorphisms, epimorphisms, essential monomorphisms and direct sums. Let $F$ be an object in $\text{Mod}((\Lambda\text{-mod})^{\text{op}})$. There exists a subfunctor $G$ of $F$ such that the following diagram is commutative

$$\begin{array}{ccc}
F(\text{soc}(X)) & \xrightarrow{F(\ell)} & F(X) \\
\downarrow{h} & & \downarrow{\phi} \\
G & & F
\end{array}$$

, where $\phi$ and $h$ are isomorphisms and $\ell : \text{soc}(X) \rightarrow X$ is the canonical injection in $\text{Mod}R$. Note that the functor $G$ is independent of choice $X$. The subfunctor $G$ of $F$ is called socle of $F$ and we denote it by $\text{soc}(F)$. The indecomposable projective objects with finitely generated essential multiplicity-free socle in $\text{Mod}((\Lambda\text{-mod})^{\text{op}})$ correspond to the indecomposable projective unitary right $R$-modules with finitely generated essential multiplicity-free socle under the equivalence $F$. From now on, we fix the functor $F$.

A semiperfect ring $\Lambda$ is called basic if $\Lambda/J(\Lambda)$ is a direct sum of division rings [2]. By [2, Proposition 27.14], any semiperfect ring is Morita equivalent to a basic ring. A left $\Lambda$-module $M$ has *multiplicity-free top* if composition factors of $\text{top}(M)$ are pairwise non-isomorphic (see [18, Sect. 1]). Let $R$ be a ring with enough idempotents. We recall that $R$ is called *left locally noetherian* (resp., *left locally artinian*) if every finitely generated unitary left $R$-module is noetherian (resp., artinian) (see [19, Ch. 5, Sect. 27]). Also $R$ is called *left locally finite* if it is both left locally artinian and left locally noetherian (see [19, Ch. 6, Sect. 32]).

Note that $R$ being left (resp., right) artinian generalized left (resp., right) QF-2 is a Morita invariant property. We are now in a position to prove our main result.
Theorem 3.2. Let $\Lambda$ be a basic ring. Then the following conditions are equivalent.

(a) $\Lambda$ is a left K"{o}the ring.
(b) $\Lambda$ is a ring of finite representation type and the Auslander ring $T$ of $\Lambda$ is a generalized right QF-2 ring.
(c) $\Lambda$ is a left artinian ring and $T = \text{End}_\Lambda(V)$ is a left locally finite generalized right QF-2 ring, where $V$ is a direct sum of modules in a complete set of representative of the isomorphic classes of finitely generated indecomposable left $\Lambda$-modules.
(d) $\Lambda$ is a left artinian ring and for each indecomposable left $\Lambda$-module $M$, $\text{Hom}_\Lambda(M, -) \cong \varphi(-)$ for some positive-primitive formula $\varphi(x)$ over $\Lambda$.
(e) $\Lambda$ is a left artinian ring and every indecomposable projective object in $\text{Mod}((\Lambda\text{-mod})^{op})$ has finitely generated essential multiplicity-free socle.

Proof. (a) $\Rightarrow$ (b). Let $\Lambda$ be a left K"{o}the ring. By [7, Corollary 3.2], $\Lambda$ is of finite representation type. Let $\{X_1, \cdots, X_m\}$ be a complete set of representative of the isomorphic classes of finitely generated indecomposable left $\Lambda$-module. $T = \text{End}_\Lambda(X)$ is an Auslander ring of $\Lambda$, where $X = \bigoplus_{i=1}^m X_i$. We show that $T$ is a generalized right QF-2 ring. Let $W$ be a minimal cogenerator left $\Lambda$-module and $\Delta = \text{End}_\Lambda(W)$. Since $\Lambda$ is of finite representation type, $W$ is a finitely generated left $\Lambda$-module. According to Remark 2.7, $\text{Hom}_\Lambda(-, W): \Lambda\text{-mod} \to \text{mod-}\Delta$ is a duality and $\Delta$ is a right artinian ring. It is easily seen that $\{\text{Hom}_\Lambda(X_1, W), \cdots, \text{Hom}_\Lambda(X_m, W)\}$ is a complete set of representative of the isomorphic classes of finitely generated indecomposable right $\Delta$-modules. Then $\Delta$ is a ring of finite representation type and $\Theta = \text{End}_\Delta(\bigoplus_{i=1}^m \text{Hom}_\Lambda(X_i, W))$ is the Auslander ring of $\Delta$. We conclude that the Auslander ring $T$ of $\Lambda$ is isomorphic to the Auslander ring $\Theta$ of $\Delta$ because $\text{Hom}_\Lambda(X, W) \cong \bigoplus_{i=1}^m \text{Hom}_\Lambda(X_i, W)$ as right $\Delta$-modules. Since the property "artinian generalized right QF-2" is Morita invariant, it is enough to show that $\Theta$ is a generalized right QF-2 ring. According to Proposition 2.2 it is enough to show that each $\text{Hom}_\Lambda(X_i, W)$ has multiplicity-free socle. Let $1 \leq l \leq m$. $\text{top}(X_i) = \bigoplus_{i=1}^t S_i$, where $t \in \mathbb{N}$ and each $S_i$ is a simple left $\Lambda$-module. By [7, Corollary 3.3], $X_i$ has multiplicity-free top and so $S_i \ncong S_j$ when $i \neq j$. It follows that $\text{Hom}_\Lambda(S_i, W) \ncong \text{Hom}_\Lambda(S_j, W)$ when $i \neq j$. On the other hand, by Remark 2.7 $\text{soc}(\text{Hom}_\Lambda(X_i, W)) \cong \text{Hom}_\Lambda(\text{top}(X_i), W)$ as $\Delta$-modules and hence $\text{soc}(\text{Hom}_\Lambda(X_i, W)) \cong \bigoplus_{i=1}^t \text{Hom}_\Lambda(S_i, W)$ as $\Delta$-modules. Therefore $\text{Hom}_\Lambda(X_i, W)$ has multiplicity-free socle and the result follows.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (d). First we show that $\Lambda$ is a ring of finite representation type. Since $\Lambda$ is left artinian, there are only finitely many non-isomorphic simple left $\Lambda$-modules. Let $\{S_1, \cdots, S_n\}$ be a complete set of representative of the isomorphic classes of simple left $\Lambda$-modules. Let
\{V_\alpha \mid \alpha \in I\} be a complete set of representative of the isomorphic classes of finitely generated indecomposable left \(\Lambda\)-modules and \(T = \text{End}_\Lambda(V)\), where \(V = \bigoplus_{\alpha \in I} V_\alpha\). Let \(\alpha \in I\). Since \(V_\alpha\) is a finitely generated left \(\Lambda\)-module, there exists a non-zero \(\Lambda\)-module epimorphism \(h : V_\alpha \to S_i\) for some \(1 \leq i \leq n\). We know that \(V\) is a generator in \(\Lambda\)-Mod. Hence by [19 Proposition 51.5(1)], \(\text{Hom}_\Lambda(V, h) : \text{Hom}_\Lambda(V, V_\alpha) \to \text{Hom}_\Lambda(V, S_i)\) is non-zero. Consequently there is a non-zero \(T\)-module epimorphism \(\varphi : T_{\epsilon_\alpha} \to M\) where \(M\) is a \(T\)-submodule of \(\text{Hom}_\Lambda(V, S_i)\).

Since by the proof of Lemma 2.4, \(\text{rad}((\_ \oplus V)^\alpha)\) is a maximal superfluous submodule of \(T_{\epsilon_\alpha}\), we can see that \(\varphi(\text{rad}(T_{\epsilon_\alpha})) = \text{rad}(M)\). On the other hand, since \(T\) is a left locally noetherian ring, \(\text{Hom}_\Lambda(V, S_i)\) is a noetherian left \(T\)-module. It follows that \(M\) is a finitely generated left \(T\)-module and so it has a maximal submodule. This means that \(\text{rad}(M)\) is a proper submodule of \(M\). Therefore the \(T\)-module epimorphism \(\overline{\varphi} : T_{\epsilon_\alpha}/\text{rad}(T_{\epsilon_\alpha}) \to M/\text{rad}(M)\) defined by \(x + \text{rad}(T_{\epsilon_\alpha}) \mapsto (x)\varphi + \text{rad}(M)\) is non-zero. Since \(\text{rad}(T_{\epsilon_\alpha})\) is a maximal submodule of \(T_{\epsilon_\alpha}\), \(T_{\epsilon_\alpha}/\text{rad}(T_{\epsilon_\alpha}) \cong M/\text{rad}(M)\) as \(T\)-modules. Since \(T\) is a left locally finite ring, \(\text{Hom}_\Lambda(V, S_i)\) has finite length. This implies that \(M/\text{rad}(M)\) is a composition factor of \(\text{Hom}_\Lambda(V, S_i)\). By Lemma 2.4, \(\{T_{\epsilon_\alpha}/\text{rad}(T_{\epsilon_\alpha}) \mid \alpha \in I\}\) is a complete set of representative of the isomorphic classes of simple unitary left \(T\)-modules. Therefore this set is finite. Consequently \(\Lambda\) is a ring of finite representation type. Next we show that every finitely generated indecomposable left \(\Lambda\)-module has multiplicity-free top. Let \(W\) be the minimal cogenerator in \(\Lambda\)-Mod. \(W\) is finitely generated and by Remark 2.7 the functor \(\text{Hom}_\Lambda(\_ , W) : \Lambda\text{-mod} \to \text{mod-}\Delta\) is a duality with the inverse duality \(\text{Hom}_\Delta(\_ , W) : \text{mod-}\Delta \to \Lambda\text{-mod}\), where \(\Delta = \text{End}_\Lambda(W)\). Moreover for each finitely generated left \(\Lambda\)-module \(X\), we have \(\text{soc}(\text{Hom}_\Lambda(X, W)) \cong \text{Hom}_\Lambda(\text{top}(X), W)\) as \(\Delta\)-modules. Let \(X\) be a finitely generated indecomposable left \(\Lambda\)-module. Then \(\text{top}(X) = \bigoplus_{j=1}^r S_j\), where each \(S_j\) is a simple left \(\Lambda\)-module. It follows that \(\text{soc}(\text{Hom}_\Lambda(X, W)) \cong \bigoplus_{j=1}^r \text{Hom}_\Lambda(S_j, W)\) as \(\Delta\)-modules. We observe that each \(\text{Hom}_\Lambda(S_j, W)\) is a simple right \(\Delta\)-module. Since \(T\) is generalized right \(\text{QF}-2\), \(\text{Hom}_\Lambda(S_j, W) \cong \text{Hom}_\Lambda(S_k, W)\) when \(j \neq k\). We conclude that \(X\) has multiplicity-free top.

Then by [7, Corollary 3.2], every indecomposable left \(\Lambda\)-module is cyclic. Let \(M\) be an indecomposable left \(\Lambda\)-module. Then there exists a \(\Lambda\)-module epimorphism \(f : \_\Lambda \to M\). It follows that there is a monomorphism \(\text{Hom}_\Lambda(M, -) \to \text{Hom}_\Lambda(\Lambda\Lambda, -)\) in \(\text{Mod}((\Lambda\text{-mod})^{\text{op}})\). Therefore \(\text{Hom}_\Lambda(M, -) \cong \varphi(-)\) for some positive-primitive formula \(\varphi(x)\) over \(\Lambda\).

\((d) \Rightarrow (a)\). Let \(M\) be a finitely generated indecomposable left \(\Lambda\)-module and assume that there exists a monomorphism \(\eta : \text{Hom}_\Lambda(M, -) \to \text{Hom}_\Lambda(\Lambda\Lambda, -)\) in \(\text{Mod}((\Lambda\text{-mod})^{\text{op}})\). Yoneda’s Lemma implies that \(\eta = \text{Hom}_\Lambda(f, -)\) for some \(\Lambda\)-module homomorphism \(f : \_\Lambda \to M\). It is easy to see that \(f\) is an epimorphism. This yields that there is a finite upper bound for the lengths of finitely generated indecomposable left \(\Lambda\)-modules and hence by [19 Proposition 54.3], \(\Lambda\) is of finite representation type. Therefore by [5, Corollary 4.8], \(\Lambda\) is a left Köthe ring.
and the result follows.

$(c) \Rightarrow (e)$. By using of the proof of $(c) \Rightarrow (d)$, we can see that $\Lambda$ is of finite representation type and hence $T$ is the Auslander ring of $\Lambda$. Also by $[5]$ Corollary 4.8, every left $\Lambda$-module is a direct sum of finitely generated indecomposable left $\Lambda$-modules. By Proposition 2.5 there exists an additive covariant equivalence $H : \text{Mod-}T \to \text{Mod}R$. It is easy to see that $H$ preserves essential monomorphisms and simple modules. By Remark 3.1 the functor $F \circ H : \text{Mod-}T \to \text{Mod}((\Lambda\text{-mod})^{\text{op}})$ is an additive equivalence of categories. Let $G$ be an indecomposable projective object in $\text{Mod}((\Lambda\text{-mod})^{\text{op}})$. By Remark 3.1 there is an indecomposable projective unitary right $R$-module $X$ such that $F(X) \cong G$. Since by the proof of $[8]$ Theorem 3.10, the functor $H$ reflects indecomposable projective objects, $H(Q) \cong X$ for some indecomposable projective right $T$-module $Q$. Since $T$ is an artinian ring, $Q$ is finitely generated and by $[19]$ Propositions 21.3 and 31.4, $\text{soc}(Q)$ is finitely generated and essential in $Q$. Then $\text{soc}(Q) = \bigoplus_{i=1}^{n} S_i$, where $n \in \mathbb{N}$ and each $S_i$ is a simple right $T$-module. Since $T$ is a generalized right QF-2 ring, $S_i \not\cong S_j$ when $i \neq j$. On the other hand there is an essential monomorphism $h : H(\text{soc}(Q)) \to H(Q)$. We know that $H(\text{soc}(Q)) \cong \bigoplus_{i=1}^{n} H(S_i)$ as $R$-modules. Therefore $\text{soc}(H(Q)) = \bigoplus_{i=1}^{n} T_i$ is an essential submodule of $H(Q)$, where each $T_i \cong H(S_i)$ as $R$-modules. Moreover $T_i \not\cong T_j$ when $i \neq j$. Now we have an essential monomorphism $\eta : F(\text{soc}(H(Q))) \to F(H(Q))$ in $\text{Mod}((\Lambda\text{-mod})^{\text{op}})$. Since $F(H(Q)) \cong G$, there is an essential monomorphism $\tau : F(\text{soc}(H(Q))) \to G$ in $\text{Mod}((\Lambda\text{-mod})^{\text{op}})$. So $\text{Im}(\tau)$ is an essential subfunctor of $G$ which is isomorphism to $F(\text{soc}(H(Q)))$. It is easy to see that $F(\text{soc}(H(Q))) \cong \bigoplus_{i=1}^{n} F(T_i)$ and also $F(T_i) \not\cong F(T_j)$ when $i \neq j$. Moreover each $F(T_i)$ is a simple object in $\text{Mod}((\Lambda\text{-mod})^{\text{op}})$. Therefore $G$ has finitely generated essential multiplicity-free socle and the result follows.

$(e) \Rightarrow (a)$. Let $R$ be the left functor ring of $\Lambda$. Then every indecomposable projective unitary right $R$-module has finitely generated essential multiplicity-free socle. By Proposition 2.6 $\Lambda$ is a left Morita ring. Choose $W$ the minimal cogenerator left $\Lambda$-module. By $[19]$ Proposition 47.15(2), $W$ is a finitely generated left $\Lambda$-module and $\Delta = \text{End}_\Lambda(W)$ is a right artinian ring. We show that every finitely generated indecomposable right $\Delta$-module has multiplicity-free socle. Let $N$ be a finitely generated indecomposable right $\Delta$-module. Since $\Delta$ is right artinian, by $[19]$ Propositions 21.3 and 31.4, $\text{soc}(N)$ is finitely generated and essential in $N$. It follows that $\text{soc}(N) = \bigoplus_{i=1}^{s} T_i$, where $s \in \mathbb{N}$ and each $T_i$ is a simple right $\Delta$-module. Also $E(N) \cong \bigoplus_{i=1}^{s} E(T_i)$ as $\Delta$-modules. We show that $T_i \not\cong T_j$ when $i \neq j$. Let $\{U_\alpha \mid \alpha \in I\}$ be a complete set of representative of the isomorphic classes of finitely generated left $\Lambda$-modules such that $R = \widehat{\text{End}_\Lambda(U)}$, where $U = \bigoplus_{\alpha \in I} U_\alpha$. By Remark 2.7 the functor $\text{Hom}_\Lambda(-, W) : \text{\Lambda-mod} \to \text{mod-}\Delta$ is a duality and hence $\{\text{Hom}_\Lambda(U_\alpha, W) \mid \alpha \in I\}$ is a complete set of representative of the isomorphic classes.
of finitely generated right $\Delta$-modules. Then $T = \widehat{\text{End}}_\Delta(V)$ is a right functor ring of $\Delta$, where $V = \bigoplus_{\alpha \in \mathcal{J}} \text{Hom}_\Delta(U_\alpha, W)$. Since $V$ is a generator in $\text{Mod-}\Delta$ we can see that $\widehat{\text{Hom}}_\Delta(V, E(N))$ is injective in $\text{Mod} T$. Also there is an essential monomorphism $\text{Hom}_\Delta(V, N) \to \text{Hom}_\Delta(V, E(N))$ in $\text{Mod} T$. Thus $E(\widehat{\text{Hom}}_\Delta(V, N)) \cong \bigoplus_{t=1}^s \widehat{\text{Hom}}_\Delta(V, E(T_t))$ as right $T$-modules. Since by Remark 2.7, $W$ is an injective cogenerator right $\Delta$-module, each $E(T_t)$ is isomorphic to a direct summand of $W$. It follows that each $E(T_t)$ is finitely generated right $\Delta$-module. Since $\text{Hom}_\Delta(V, -) : \text{Add}(V) \to \text{Proj}(T^{\text{op}})$ is an equivalence of categories, $\widehat{\text{Hom}}_\Delta(V, N)$ and each $\widehat{\text{Hom}}_\Delta(V, E(T_t))$ are projective unitary right $T$-modules. Moreover $\text{End}_T(\widehat{\text{Hom}}_\Delta(V, E(T_t))) \cong \text{End}_\Delta(E(T_t))$ as rings. By [19, Proposition 19.9], $\text{End}_T(\widehat{\text{Hom}}_\Delta(V, E(T_t)))$ is a local ring. On the other hand, by Remark 2.7, $\Lambda$ is left Morita to $\Delta$. So we deduce from [10, Lemma 3.1] that every indecomposable projective unitary right $T$-module has finitely generated essential multiplicity-free socle. This yields that $\text{soc}(\widehat{\text{Hom}}_\Delta(V, N)) = \bigoplus_{t=1}^s T'_t$ with $T'_t \cong T'_j$ when $i \neq j$, where $m \in \mathbb{N}$ and each $T'_t$ is a simple unitary right $T$-module. In addition $E(\text{Hom}_\Delta(V, N)) \cong \bigoplus_{t=1}^s E(T'_t)$ as $T$-modules. Therefore $\bigoplus_{t=1}^s \widehat{\text{Hom}}_\Delta(V, E(T_t)) \cong \bigoplus_{t=1}^s E(T'_t)$. Since $\text{End}_T(E(T'_t))$ is a local ring, $m = s$ and for each $i$, $E(T'_i) \cong \text{Hom}_\Delta(V, E(T_i))$ as $T$-modules. If $T_i \cong T_j$, then $E(T_i) \cong E(T_j)$. Thus $\widehat{\text{Hom}}_\Delta(V, E(T_i)) \cong \widehat{\text{Hom}}_\Delta(V, E(T_j))$. It follows that $E(T'_i) \cong E(T'_j)$ and hence $i = j$. Consequently every finitely generated indecomposable right $\Delta$-module has multiplicity-free socle. Let $M$ be a finitely generated indecomposable left $\Lambda$-module. Since $\Lambda$ is left artinian, $\text{top}(M) = \bigoplus_{t=1}^s S_t$, where $t \in \mathbb{N}$ and each $S_t$ is a simple left $\Lambda$-module. By Remark 2.7, $\text{soc}(\text{Hom}_\Lambda(M, W)) \cong \bigoplus_{t=1}^s \text{Hom}_\Lambda(S_t, W)$ as $\Delta$-modules and $\text{Hom}_\Lambda(M, W)$ is a finitely generated indecomposable right $\Delta$-module. Hence $\text{Hom}_\Lambda(S_i, W) \cong \text{Hom}_\Lambda(S_j, W)$ when $i \neq j$. This implies that $S_i \cong S_j$ when $i \neq j$. Therefore every finitely generated indecomposable left $\Lambda$-module has multiplicity-free top. Consequently by [7, Corollary 3.2], $\Lambda$ is a left K"{o}the ring and the result follows.}

As a consequence, we obtain the Auslander correspondence for left Kawada rings.

**Theorem 3.3.** There exists a bijection between the Morita equivalence classes of left Kawada rings and the Morita equivalence classes of Auslander generalized right QF-2 rings.

**Proof.** Let $\mathfrak{A}$ be the set of Morita equivalence classes of rings of finite representation type and $\mathfrak{B}$ be the set of Morita equivalence classes of Auslander rings. By [5, Corollary 4.7], the mapping $\varphi : \mathfrak{A} \to \mathfrak{B}$ via $\Lambda \mapsto \text{End}_\Lambda(U)$ is a bijection, where $U$ is a direct sum of all modules in a complete set of representatives of the isomorphic classes of finitely generated indecomposable left $\Lambda$-modules. Let $\Lambda$ be a left Kawada ring. Since $\Lambda$ is a semiperfect ring, $\Lambda$ is Morita equivalent to $\Lambda_0$ for some basic ring $\Lambda_0$. $\Lambda_0$ is a left K"{o}the ring and so by Theorem 3.2, $\varphi(\Lambda_0)$ is a generalized right QF-2 ring. Since $\varphi(\Lambda)$ is Morita equivalent to $\varphi(\Lambda_0)$, we can see that the
Auslander ring $\phi(\Lambda)$ is a generalized right QF-2 ring. Let $R$ be an Auslander generalized right QF-2 ring. Then there exists a representation-finite ring $\Delta$ such that $\phi(\Delta)$ is Morita equivalent to $R$. Since $\Delta$ is semiperfect, $\Delta$ is Morita equivalent to a basic ring $\Delta_0$. It follows that $\phi(\Delta_0)$ is Morita equivalent to $\phi(\Delta)$. This leads that $R$ is Morita equivalent to $\phi(\Delta_0)$. Consequently $\phi(\Delta_0)$ is a generalized right QF-2 ring. So we deduce from Theorem 3.2 that $\Delta_0$ is a left Köthe ring. Hence by [7, Corollary 4.2], $\Delta$ is a left Kawada ring. Therefore the mapping $\phi$ induces a bijection between the Morita equivalence classes of left Kawada rings and the Morita equivalence classes of Auslander generalized right QF-2 rings.

□

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