NON-NORMAL AFFINE MONOIDS

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Abstract. We give a geometric description of the set of holes in a non-normal affine monoid $Q$. The set of holes turns out to be related to the non-trivial graded components of the local cohomology of $\mathbb{K}[Q]$. From this, we see how various properties of $\mathbb{K}[Q]$ like local normality and Serre’s conditions $(R_1)$ and $(S_2)$ are encoded in the geometry of the holes. A combinatorial upper bound for the depth the monoid algebra $\mathbb{K}[Q]$ is obtained and some special cases where equality holds are identified. We apply this results to simplicial and to seminormal affine monoids. In particular, we prove a special case of the Eisenbud-Goto conjecture.

Moreover, we consider the dependence of the local cohomology of $\mathbb{K}[Q]$ on the characteristic of the field. In particular, we show that Serre’s condition $(S_3)$ does not depend on $\mathbb{K}$. On the other hand, we construct two classes of examples of affine monoids with additional properties whose depth does depend on $\mathbb{K}$.

1. Introduction

Let $Q$ be an affine monoid, i.e. a finitely generated submonoid of $\mathbb{Z}^N$ for some $N \in \mathbb{N}$. Further, let $\overline{Q}$ denote the normalization of $Q$. In this paper, we give a geometric description of the set of holes $\overline{Q} \setminus Q$ in $Q$ and relate it to properties of $Q$. Our first main result is the following.

**Theorem (Theorem 3.8).** Let $Q$ be an affine monoid. There exists a (not necessarily disjoint) decomposition

$$\overline{Q} \setminus Q = \bigcup_{i=1}^l (q_i + \mathbb{Z}F_i) \cap \mathbb{R}_+ Q$$

with $q_i \in \overline{Q}$ and faces $F_i$ of $Q$. If the union is irredundant (i.e. no $q_i + \mathbb{Z}F_i$ can be omitted), then the decomposition is unique.

We call a set $q_i + \mathbb{Z}F_i$ from (1) a $j$-*dimensional family of holes*, where $j$ is the *dimension of $F$. (See Section 2 for the definition of the *dimension). There is an algebraic interpretation of the sets appearing in (1). Let $\mathbb{K}$ be a field and $\mathbb{K}[Q]$ be the monoid algebra of $Q$. Then the faces in (1) correspond to the associated primes of the quotient $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$. The same face may appear several times in (1), in fact, the number of times a face appears equals the multiplicity of the corresponding prime.
In [2, Prop. 2.35] a different decomposition of the holes is considered. It is shown in [2] that one can always find a decomposition of \( \overline{Q} \setminus Q \) into a disjoint union of translates of faces of \( Q \):

\[
\overline{Q} \setminus Q = \bigcup_{i=1}^{l} q_i + F_i
\]

In fact, this statement and its proof have been the motivation for proving Theorem 3.8. Figure 1 shows an example of both kinds of decompositions. The decomposition given in (2) is disjoint, but far from being unique. On the other hand, our decomposition is (in general) not disjoint, but it is unique in the sense that the sets \( q_i + \mathbb{Z}F_i \) are uniquely determined up to reordering. Moreover, we show in Proposition 3.10 that it behaves nicely under localization.

In the third section of this paper, we consider the local cohomology of \( K[Q] \) with support on the *maximal ideal \( m \). There is a close relation to the families of holes that is summarized in the next result.

**Theorem** (Corollary 4.4 and Corollary 4.6). Let \( q \in \mathbb{Z}Q \) such that \( q \notin \text{int} \overline{Q} \). If \( H_{m}^{i+1}(K[Q])_q \neq 0 \) for some \( i \), then \( q \) is contained in a family of holes of *dimension at least \( i \). On the other hand, every \( i \)-*dimensional family of holes contains an element \( q \in \mathbb{Z}Q \) such that \( H_{m}^{i+1}(K[Q])_q \neq 0 \).

The hypothesis that \( q \notin \text{int} \overline{Q} \) is not an essential restriction, as we will see in the discussion leading to the result above. Several ring-theoretic properties of \( K[Q] \) can be described in terms of the families of holes.

**Theorem** (Theorem 5.2). Let \( Q \) be an affine monoid of *dimension \( d \). The following holds:

- If \( d \geq 2 \), then *depth \( K[Q] = 1 \) if and only if there is a 0-*dimensional family of holes.
- \( Q \) is locally normal if and only if there is no family of holes of positive *dimension.
• $K[Q]$ satisfies Serre’s condition $(R_1)$ if and only if there is no family of holes of dimension $d - 1$.

• $K[Q]$ satisfies Serre’s condition $(S_2)$ if and only if every family of holes has dimension $d - 1$.

Note, this implies that if $Q$ is locally normal, but not normal, then $\ast\text{depth } K[Q] = 1$. The first item of the preceding theorem generalizes to an upper bound on the $\ast$depth of $K[Q]$:

**Theorem (Theorem 5.3).** If $Q$ has an $i$-dimensional family of holes, then the $\ast$depth of $K[Q]$ is at most $i + 1$.

This theorem states that a non-normal affine monoid with “few” holes has a low $\ast$depth. This is somewhat counterintuitive, because Hochster’s Theorem ([2, Theorem 6.10]) states that the absence of holes, i.e., normality, implies maximal $\ast$depth. In small examples, it is often not too difficult to determine the bound given by this theorem geometrically. This can be easier than to compute the actual $\ast$depth algebraically. In general, the $\ast$depth may be strictly smaller than the bound given by Theorem 5.3. However, in Proposition 5.4, Proposition 5.8 and Proposition 5.19 we identify some special cases where equality holds. See Example 5.6 for an application.

In the further parts of Section 5, we apply our results to simplicial and semi-normal affine monoids. For simplicial affine monoids, the characterization of the Cohen-Macaulay property given in [7] is extended to the non-positive case. For seminormal affine monoids, we give a new proof of the cohomological characterization of seminormality of [4]. While our proof is not actually simpler than the original one, we believe that it offers a new, more geometric perspective. Moreover, we extend this and some other results of [4] to the non-positive case.

In Section 6, we consider the dependence of local cohomology on the characteristic of $K$ in the case of affine monoid algebras. It is known that in general $\ast$depth $K[Q]$ can depend on the characteristic [22]. However, we will see in Proposition 6.1 that certain parts of the local cohomology do not depend on the field. In particular, it turns out that Serre’s condition $(S_3)$ is independent of the characteristic. Moreover, if $\ast\text{dim } Q \leq 5$ then the support of $H^n_m(K[Q])$ also does not depend on char $K$. These results are best possible in the sense that the statements are wrong for $(S_4)$ and for $\ast\text{dim } Q = 6$. We see in Section 5 that if $Q$ is either simplicial or seminormal, locally simplicial and positive, then the Cohen-Macaulayness of $K[Q]$ is independent of $K$. Somewhat surprisingly, this does not hold for the $\ast$depth. We construct a simplicial seminormal affine monoid whose $\ast$depth varies with $K$ in Proposition 6.4. Moreover, we construct a locally simplicial seminormal affine monoid satisfying Serre’s condition $(S_2)$ whose $\ast$depth depends on the characteristic in Proposition 6.5. This was motivated by [4, Corollary 5.6]. That result states that in this situation $K[Q]$ is Cohen-Macaulay under the additional assumption that $Q$ is positive. See also Proposition 5.18 for a slight generalization of this result.

In the last section, some additional results are listed. We give a direct proof that our criterion for Serre’s condition $(S_2)$ is equivalent to the one given in [14, 22]. Further, an interpretation of the greatest $\ast$dimension of a family of holes is given. We also obtain a curious characterization of normal affine monoids. Finally, we give a bound on the Castelnuovo-Mumford regularity of seminormal homogeneous affine monoids and prove a special case of the Eisenbud-Goto conjecture (Theorem 7.7).
2. Preliminaries and notation

An affine monoid $Q$ is a finitely generated submonoid of the additive monoid $\mathbb{Z}^N$ for an $N \in \mathbb{N}$. Equivalently, $Q$ is a commutative, finitely generated, cancellative and torsionfree monoid. For general information about affine monoids see [2] or [16]. We denote the group generated by $Q$ by $\mathbb{Z}Q$, the convex cone generated by $Q$ by $\mathbb{R}_+ Q \subseteq \mathbb{R}^N$ and the normalization by $\overline{Q} = \mathbb{Z}Q \cap \mathbb{R}_+ Q$. Recall that an element $q \in \mathbb{Z}Q$ is contained in $\overline{Q}$ if and only if a multiple of $q$ lies in $Q$. A face $F \subseteq Q$ of $Q$ is a subset such that for $a, b \in Q$ the following holds:

$$a + b \in F \iff a, b \in F$$

A unit is an element in $u \in Q$, such that $-u \in Q$. The set of units forms a face $F_0$ that is contained in every other face of $Q$. We call $Q$ positive if $0$ is the only unit in $Q$. For every element $q \in Q$, there exists a unique minimal face $F$ containing $q$. We say that $q$ is an interior point of $F$ and write $\text{int} F$ for the set of interior points of $F$. Note that by definition $0 \in \text{int} F_0$. The dimension of a face $F$ is the rank of the free abelian group $\mathbb{Z}F$ generated by $F$. Since we are working with not-necessarily positive affine monoids, it is more convenient to consider a normalized version of the dimension. So we define the *dimension* as $\dim Q := \dim Q - \dim F_0$, and $\dim F := \dim F - \dim F_0$ for every face $F$ of $Q$. For a field $\mathbb{K}$, we write $\mathbb{K}[Q]$ for the monoid algebra of $Q$. Further, for an element $q \in Q$, we write $x^q \in \mathbb{K}[Q]$ for the corresponding monomial. For a face $F$ we define $p_F \subseteq \mathbb{K}[Q]$ to be the vector space generated by those monomials $x^q$ such that $q \in Q \setminus F$. Then $p_F$ is a monomial prime ideal of $\mathbb{K}[Q]$ and all monomial prime ideals are of this type. Moreover, $\mathbb{K}[Q]/p_F \cong \mathbb{K}[F]$. $\mathbb{K}[Q]$ carries a natural $\mathbb{Z}Q$-grading. With respect to this grading, the homogeneous ideals $\mathbb{K}[Q]$ are exactly the monomial ideals. Thus the ideal $p_F$ associated to the minimal face is the unique maximal graded ideal of $\mathbb{K}[Q]$. We will sometimes write $m$ for this ideal. Its height equals the maximal length of a descending chain of faces of $Q$, so $(\mathbb{K}[Q], m)$ is a *local ring of *dimension $\dim Q$. More general, the height of $p_F$ equals $\dim F$ for every face $F$. The *depth* of $\mathbb{K}[Q]$ is the maximal length of a regular sequence in $m$. Equivalently, it is the depth of the (inhomogeneous) localization $\mathbb{K}[Q]_m$. For a face $F$ of $Q$, we denote by

$$Q_F := \{ q - f \mid q \in Q, f \in F \}$$

the localization of $Q$ at $F$. It holds that $\mathbb{K}[Q_F] = \mathbb{K}[Q]_{(p_F)}$, where the later is the homogeneous localization of $\mathbb{K}[Q]$ at $p_F$. If $q$ is an interior point in $F$, then it is enough to adjoin an inverse of $q$ to $Q$ to obtain $Q_F$. Note that localizations are almost never positive. The faces of $Q$ are in bijection with the faces of $\overline{Q}$, but as sets they may be different. Therefore, for a face $F$ of $Q$, we write $F\overline{\mid} := \{ q \in \overline{Q} \mid \exists n \in \mathbb{N} : nq \in F \}$ for the corresponding face of $\overline{Q}$.

Lemma 2.1. Normalization and localization commute. More precisely, if $F \subseteq Q$ is a face, then it holds that $(Q_F) = (\overline{Q})_F$. Moreover, it makes no difference if we localize $\overline{Q}$ as a $Q$-module or as an affine monoid on its own: $(\overline{Q})_F = (\overline{Q})_{\overline{F}}$.

Proof. The equality $(Q_F) = (\overline{Q})_F$ follows from the corresponding algebraic statement, see [5, Prop. 4.13]. Further, $(\overline{Q})_F = (\overline{Q})_{\overline{F}}$, because $F$ contains an interior point of $F\overline{\mid}$. $\square$

A set $M$ is called a $Q$-module if there is an operation $Q \times M \rightarrow M$ (additively written) of $Q$ on $M$, such that $(q + p) + m = q + (p + m)$ and $0 + m = m$ for
We call this polytope cone and also consider the localization for general modules, but we only need this special case. Note that if \( U \subseteq M \subseteq \mathbb{Z}Q \) are modules, then \( \mathbb{K}\{M_F\}/\mathbb{K}\{U_F\} = \mathbb{K}\{M\}/\mathbb{K}\{U\}(p_F) \). Moreover, note that \( \mathbb{K}\{Q\} \) and \( \mathbb{K}\{\overline{Q}\} \) are isomorphic as vector spaces, but the former is considered as a \( \mathbb{K}\{Q\} \)-module, while the latter is the monoid algebra of \( \overline{Q} \).

For a \( \mathbb{Z}Q \)-graded \( \mathbb{K}[Q] \)-module \( N \) (in the algebraic sense), the support of \( N \), \( \text{Supp} N \), is defined to be the set of those \( q \in \mathbb{Z}Q \), such that there exists an element of degree \( q \) in \( N \). If \( M \) is a \( Q \)-module and \( U \subseteq M \) a submodule, then \( \text{Supp} \mathbb{K}\{M\}/\mathbb{K}\{U\} = M \setminus U \).

**Definition 2.2.** Let \( Q \) be an affine monoid. We call \( Q \) locally normal if every localization \( Q_F \) at a face \( F \neq F_0 \) is normal.

Since localizations of normal affine monoids are again normal, it is enough to consider faces of dimension 1. For a polytope \( P \subseteq \mathbb{R}^{N-1} \) with integer vertices, one often considers the topological affine monoid \( Q(P) \subseteq \mathbb{Z}^N \) generated by the set \( \{(p,1) \mid p \in P \cap \mathbb{Z}^{N-1}\} \). In this case, \( Q(P) \) is locally normal if and only if the polytope \( P \) is very ample. To see this, note that the localization of \( Q(P) \) at a vertex \((v,1)\) splits into a direct sum of the corner cone on \( v \) and a copy of \( \mathbb{Z} \). Thus, \( Q(P) \) is locally normal if and only if all the corner cones of \( P \) are normal, which is the definition of very ampleness. It is known that \( P \) is very ample if and only if there are only finitely many holes in \( Q(P) \). The corresponding statement in the general case is the following:

**Proposition 2.3.** An affine monoid is locally normal if and only if

\[
\text{rank}_{\mathbb{K}[F_0]} \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] < \infty
\]

for any field \( \mathbb{K} \). This rank does not depend on the field.

Recall that the face \( F_0 \) consists of the units of \( Q \), so this lemma amounts to saying that there are only finitely many holes up to units. Every graded module over \( \mathbb{K}[F_0] \) is free (cf. [8, Theorem 1.1.4]), so the rank is well-defined. We will not use this proposition in the sequel, so we omit the proof.

Finally, we need the definition cross section polytope of a polyhedral cone \( C \subseteq \mathbb{R}^d \). If \( C \subseteq \mathbb{R}^d \) is a cone, then we call the set of elements \( c \in C \) such that \(-c \in C\) the lineality space of \( C \) and denote it by \( \text{lin}(C) \). A cone \( C \) is pointed if \( \text{lin}(C) = \{0\} \). Every cone can be decomposed as \( C = C' + \text{lin}(C) \), where \( C' \) is a pointed cone and \( C' \cap \text{lin}(C) = \{0\} \). The cone over a polytope \( P \subseteq \mathbb{R}^{d-1} \) is defined as \( C(P) := \mathbb{R}_+ \{ (p,1) \in \mathbb{R}^d \mid p \in P \} \). This is a pointed cone, and every pointed cone can be written as cone over a polytope. For a given pointed cone \( C \), a polytope \( P \) such that \( C = C(P) \) can be obtained by intersecting \( C \) with a suitable hyperplane.

We call this polytope \( P \) a cross section polytope of \( C \), it is uniquely determined up to affine transformations. If \( Q \) is an affine monoid, then the face lattice of a cross section polytope of \( \mathbb{R}_+ Q \) is isomorphic to the face lattice of \( Q \).

3. The structure of the set of holes

In this section, we describe the structure of the set of holes \( \overline{Q} \setminus Q \). Following an idea from [2, p. 139], we consider a more general situation. Let \( M \) be a finitely generated \( Q \)-submodule of \( \mathbb{Z}Q \) and let \( U \subseteq M \) be a submodule of \( M \). We are
interested in the structure of the difference $M \setminus U$. Clearly, in the case $M = \overline{Q}$ and $U = Q$ this corresponds to the holes $\overline{Q} \setminus Q$. While for our purpose it would actually suffice to consider this case, we believe that the additional generality makes the exposition more clear. Another case of potential interest is $N = Q$ and $U \subset Q$ a submodule. This corresponds to a monomial ideal in $\mathbb{K}[Q]$. As noted above, the set $M \setminus U$ can be encoded as the support of the quotient $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. The following simple observation is the key idea: Consider an $m \in M \setminus U$ and a $q \in Q$. Let $x^m$ and $x^q$ denote the corresponding monomials in $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ resp. in $\mathbb{K}[Q]$. Then
\[
q + m \in U \iff x^q x^m = 0.
\]
Now let $F$ be a face of $Q$. It holds that $m \in M_F$, because $M \subset M_F$. However, $m \in U_F$ if and only if $x^m$ goes to zero when localizing the quotient $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ at $p_F$. This is the case if and only if the annihilator of $x^m$ is not contained in $p_F$, i.e. if there is a $q \in F$ such that $q + m \in U$. Consider the case that $m \notin U_F$ and $F$ is maximal with this property. By this we mean that $m \in U_G$ for all faces $G \supseteq F$. Because $p_G \subset p_F$, this is equivalent to $p_F$ being a minimal prime over the annihilator of $x^m$. We summarize what we have proven:

**Lemma 3.1.** Let $F$ be a face of $Q$, $m \in M \setminus U$ and $x^m$ be the corresponding monomial in $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. Then $F + m \subseteq M \setminus U$ if and only if $p_F$ contains the annihilator of $x^m$. Moreover, $F$ is a maximal face with this property if and only if $p_F$ is a minimal prime of the annihilator of $x^m$.

In view of our objective to find an irredundant decomposition of the set $M \setminus U$, it seems natural to take the largest possible pieces. Therefore, we consider the family of sets
\[
\mathcal{F}(M) = \{ \mathbb{Z}F + m \subset \mathbb{Z}Q \mid m \in M \setminus U, p_F \text{ is a minimal prime over } \text{Ann} x^m \}.
\]
These sets will yield the desired decomposition. Note that for $m, n \in M \setminus U$ with $m - n \in \mathbb{Z}F$, it holds that $p_F$ is a minimal prime over $\text{Ann} x^m$ if and only if it is minimal over $\text{Ann} x^n$. So we are free in choosing representatives of the sets in $\mathcal{F}(M)$. We first show that their union comprises all of $M \setminus U$:

**Lemma 3.2.** It holds that
\[
M \setminus U = \bigcup_{S \in \mathcal{F}(M)} S \cap M
\]

**Proof.** Every monomial $x^m \in \mathbb{K}\{M\}/\mathbb{K}\{U\}$ has at least one minimal prime over its annihilator, so the left-hand side of (3) is clearly contained in the right-hand side. On the other hand, consider $n \in (\mathbb{Z}F + m) \cap M$ for $\mathbb{Z}F + m \in \mathcal{F}(M)$. There exist $f_1, f_2$ in $F$ such that $f_1 + m = f_2 + n$. It follows that $n \notin U$, because $F + m \subset M \setminus U$ by Lemma 3.1, and hence $n \in M \setminus U$. \qed

Next, we consider the behaviour of $\mathcal{F}(M)$ under localization.

**Lemma 3.3.** Let $F \subseteq G$ be faces of $Q$ and let $m \in \mathbb{Z}Q$. Then $\mathbb{Z}G + m \in \mathcal{F}(M)$ if and only if $\mathbb{Z}G + m \in \mathcal{F}(M_F)$.

**Proof.** If $m \in M_F \setminus U_F$, then there exists an $f \in F$ such that $f + m \in M \setminus U$ and, since $F \subset G$, it holds that $\mathbb{Z}G + m = \mathbb{Z}G + f + m$. So we can assume that $m \in M \setminus U$. In this case, $\mathbb{Z}G + m \in \mathcal{F}(M)$ if and only if $p_G$ is minimal over the annihilator of $x^m \in \mathbb{K}\{M\}/\mathbb{K}\{U\}$. But this property is preserved under localization with $p_F \supseteq p_G$. Hence the claim follows. \qed
Recall that a graded module $N$ over a graded ring $R$ is called \textit{*simple} if it has no nontrivial graded submodules. A \textit{*composition series} of $N$ is a chain $0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r = N$ of graded modules, such that each quotient $N_i/N_{i-1}$ is \textit{*simple}. If $N$ has a \textit{*composition series of length} $r$, then we call $r$ the \textit{*length of} $N$. As in the ungraded case, one proves that the \textit{*length does not depend on the choice of the \textit{*composition series}. It is not difficult to see that if $(R, \mathfrak{m})$ is \textit{*local}, then $N$ is \textit{*simple} if and only if $N_{\mathfrak{m}}$ is simple. Hence, the \textit{*length of} $N$ equals the length of $N_{\mathfrak{m}}$. We now prove the finiteness of our decomposition.

\begin{lemma}
For every face $F$ of $Q$, the number of sets of the form $\mathbb{Z}F + m \in \mathcal{F}(M)$ equals the multiplicity of $p_F$ on $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. In particular, $\mathcal{F}(M)$ is finite. Moreover, a face $F$ appears in $\mathcal{F}(M)$ if and only if $p_F$ is an associated prime of $\mathbb{K}\{M\}/\mathbb{K}\{U\}$.
\end{lemma}

\begin{proof}
Consider the $\mathbb{K}[Q]$-module 

$$N := H^0_{p_F}(\mathbb{K}\{M\}/\mathbb{K}\{U\}) = \{ x \in \mathbb{K}\{M\}/\mathbb{K}\{U\} \mid p_F^n x = 0 \text{ for } n \gg 0 \} .$$

The multiplicity of $p_F$ in $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ is defined as the length of the localization $N_{p_F}$, see [5, p. 102]. Note that $N = \bigcup_{n>0} (\mathbb{K}\{M\}/\mathbb{K}\{U\}) p_F^n$ is $\mathbb{Z}Q$-graded. Therefore, by the discussion above the length of $N_{p_F}$ equals the \textit{*length} of 

$$N_{(p_F)} = H^0_{p_F}((\mathbb{K}\{M\}/\mathbb{K}\{U\})_{(p_F)}) = H^0_{p_F}((\mathbb{K}\{M_F\}/\mathbb{K}\{U_F\}))$$

Here, the first equality is a standard result about the localization of local cohomology, cf. [15, Prop 7.15]. Therefore the multiplicity is invariant under localiza-

\end{proof}

Consider a \textit{*composition series} $0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r = N$. Every \textit{*simple module} is of the form $\mathbb{K}[Q]/p_F(m) = \mathbb{K}[F](m)$, where $(\cdot)(m)$ indicates a shift in the grading by $m \in \mathbb{Z}Q$. Hence Supp $N = \bigsqcup_i \text{Supp } N_i/N_{i+1} = \bigcup_i \mathbb{Z}F + m_i$. Moreover, since every graded component of $N$ has vector space dimension 1 over $\mathbb{K}$, this union is disjoint.

On the other hand, we claim that Supp $N$ is the union of the sets $\mathbb{Z}F + m \in \mathcal{F}(M)$ (for our fixed $F$). Indeed, $m \in M \setminus U$ is contained in Supp $N$ if and only $x^m$ is annihilated by some power of $p_F$. This is equivalent to saying that $p_F$ is a minimal prime over the annihilator of $x^m$, because $p_F$ is the \textit{*maximal ideal of} $\mathbb{K}[Q]$\footnote{This is a graded version of [5, Cor. 2.19]. It can be proven analogously to the ungraded case.}. Thus, the number of sets of the form $\mathbb{Z}F + m$ in $\mathcal{F}(M)$ equals the length $r$ of the \textit{*composition series}.

We now turn to proving the irredundancy and the uniqueness of (3). This has a more geometric flavour than the preceding algebraic arguments. First, we give a variant of the well-known fact that a vector space over an infinite field cannot be written as a union of finitely many subspaces.

\begin{lemma}
Let $V$ be a vector space over $Q$ and $C \subseteq V$ be a convex cone (i.e. a subset such that for $v, w \in C$ and $\lambda, \mu \geq 0$ it follows $\lambda v + \mu w \in C$). If $C$ contains a generating set of $V$, then it is not contained in any finite union of proper subspaces of $V$.
\end{lemma}
\textbf{Proof.} Assume on the contrary that the cone $C$ is contained in the union of finitely many proper subspaces $V_1, \ldots, V_l$ of $V$. We may further assume that none of the subspaces is contained in the union of the others and that every $V_i$ has a non-empty intersection with $C$. We have certainly at least two subspaces, because $C$ contains a generating set of $\mathbb{Q}$. 

Indeed, it holds that $Q \subseteq G_i$ for every $i$. As a notation, for a subset $S \subseteq \mathbb{Q}^N$, we write $Q S$ for the $Q$-subspace generated by $S$. Then, $Q(ZF \cap G_i) \subseteq QF$ for every $i$. Indeed, it holds that $Q(ZF \cap G_i) \subseteq QF \cap QG_i \subseteq QF$. The second inclusion is strict except in the case that $QF \subseteq QG_i$. But this would imply that $F \subseteq G_i$, because $F = QF \cap Q$ and $G_i = QG_i \cap Q$. Here we use that $F$ and $G_i$ are faces of a common affine monoid.

By Lemma 3.5, we can find an element $\hat{p}$ in the cone generated by $F$ that is not contained in any $Q(ZF \cap G_i)$. By multiplication with a positive scalar, we can assume $\hat{p} \in F$. For every non-negative integer $\lambda$, it holds that $\lambda \hat{p} + q \in F$ for $q \in \bigcup_i ZG_i + p_i$. Since the union is finite, there exists an index $i$ and two different integers $\lambda, \lambda' \in \mathbb{Z}$ such that $\lambda \hat{p} + q, \lambda' \hat{p} + q \in ZG_i + p_i$. But now it follows that $(\lambda - \lambda')\hat{p} \in ZF \cap ZG_i$ and thus $\hat{p} \in Q(ZF \cap ZG_i)$, a contradiction to our choice of $\hat{p}$. \hfill \square

Now we are ready to prove that our decomposition is in fact irredundant and unique.

\textbf{Lemma 3.6.} Let $q, p_1, \ldots, p_l \in \mathbb{Z}Q$ be lattice points and let $F, G_1, \ldots, G_l$ be (not necessarily distinct) faces of $Q$. If $F + q$ is contained in the union $\bigcup_i ZG_i + p_i$, then it is already contained in one of the sets $ZG_i + p_i$.

Note that this Lemma does not hold for arbitrary subgroups of $\mathbb{Z}^N$, for example $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z} + 1)$.

\textbf{Proof.} We may assume that $F + q$ has a non-empty intersection with every $ZG_i + p_i$ for $1 \leq i \leq l$. If $F \subseteq G_i$ for any $i$, then $F + q \subseteq ZF + q' \subseteq ZG_i + q' = ZG_i + p_i$ for $q' \in F + q \cap ZG_i + p_i$. Thus in this case our claim holds. We will show that there exists always an $i$ such that $F \subseteq G_i$.

Assume that $F \not\subseteq G_i$ for every $i$. As a notation, for a subset $S \subseteq \mathbb{Q}^N$, we write $Q S$ for the $Q$-subspace generated by $S$. Then, $Q(ZF \cap G_i) \subseteq QF$ for every $i$. Indeed, it holds that $Q(ZF \cap G_i) \subseteq QF \cap QG_i \subseteq QF$. The second inclusion is strict except in the case that $QF \subseteq QG_i$. But this would imply that $F \subseteq G_i$, because $F = QF \cap Q$ and $G_i = QG_i \cap Q$. Here we use that $F$ and $G_i$ are faces of a common affine monoid.

By Lemma 3.5, we can find an element $\hat{p}$ in the cone generated by $F$ that is not contained in any $Q(ZF \cap G_i)$. By multiplication with a positive scalar, we can assume $\hat{p} \in F$. For every non-negative integer $\lambda$, it holds that $\lambda \hat{p} + q \in F$ for $q \in \bigcup_i ZG_i + p_i$. Since the union is finite, there exists an index $i$ and two different integers $\lambda, \lambda' \in \mathbb{Z}$ such that $\lambda \hat{p} + q, \lambda' \hat{p} + q \in ZG_i + p_i$. But now it follows that $(\lambda - \lambda')\hat{p} \in ZF \cap ZG_i$ and thus $\hat{p} \in Q(ZF \cap ZG_i)$, a contradiction to our choice of $\hat{p}$. \hfill \square

Now we are ready to prove that our decomposition is in fact irredundant and unique. 

\textbf{Lemma 3.7.} Consider a finite decomposition

\[ M \setminus U = \bigcup_i (ZG_i + m_i) \cap M \]

of $M \setminus U$ with $m_i \in M \setminus U$ and faces $G_i$ of $Q$. Then every set in $F(M)$ appears in this decomposition. Thus, (3) defines the unique irredundant finite decomposition of $M \setminus U$.

\textbf{Proof.} Let $m \in M \setminus U$ and $F$ a face of $Q$ such that $ZF + m \in F(M)$. By Lemma 3.6, there exists an index $i$ such that $F + m \subseteq ZG_i + m_i$. In particular, $ZG_i + m = ZG_i + m_i$. Hence, $F \subseteq ZG_i$ and therefore $F \subseteq ZG_i \cap Q = G_i$. On the other
hand, by Lemma 3.1, $F$ is a maximal face of $Q$ such that $F + m \subseteq M \setminus U$. But $G_i + m \subseteq \mathbb{Z}G_i + m \cap M \subseteq M \setminus U$, so we conclude that $G_i = F$. Whence $\mathbb{Z}F + m = \mathbb{Z}G_i + m_i$. □

This completes the proof of our theorem:

**Theorem 3.8.** Let $Q$ be an affine monoid, let $M \subseteq \mathbb{Z}Q$ be a module and let $U \subseteq M$ be a submodule. Then there exists a unique irredundant finite (non-disjoint) decomposition

$$ M \setminus U = \bigcup_i (ZF_i + m_i) \cap M. $$

The number of times $F$ appears in (4) equals the multiplicity of $p_F$ on $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. In particular, a face $F$ appears in (4) if and only if $p_F$ is an associated prime of $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. Geometrically, a set $ZF_i + m_i$ appears in (4) if and only if $F_i$ is a maximal face such that $F_i + m_i \subseteq M \setminus U$.

From now on, we specialize to the case $M = \overline{Q}$ and $U = Q$. For the ease of reference, call a face $F$ associated to $Q$ if it appears in (4). The following is immediate:

**Corollary 3.9.** $Q$ has a $j$-dimensional family of holes if and only if there is a $j$-dimensional associated face of $Q$.

We get a description of the holes of the localization $Q_F$ from Lemma 3.3:

**Proposition 3.10.** Let $F$ be a face of $Q$. The families of holes of $Q_F$ are exactly those families of holes $\mathbb{Z}G + q$ of $Q$ which satisfy $F \subseteq G$. In particular, $Q_F$ is normal if and only if no associated face contains $F$.

We would like to point our another special case of Theorem 3.8. Set $Q = M = \mathbb{N}^n$ for some $n \in \mathbb{N}$ and let $U \subseteq \mathbb{N}^n$ be a module generated by vectors $v_1, \ldots, v_r \in \mathbb{N}^n$, such that every entry of $v_i$ is either 0 or 1 for every $i$. Then $\mathbb{K}\{U\}$ is a squarefree monomial ideal in the polynomial ring $\mathbb{K}\{Q\} = \mathbb{K}\{x_1, \ldots, x_n\}$ and thus the Stanley-Reisner ideal of some simplicial complex $\Delta \subseteq 2^{[n]}$. Then (4) corresponds to the well-known primary decomposition of Stanley-Reisner ideals, see [16, Theorem 1.7]. In particular, the faces appearing in (4) correspond to the facets of $\Delta$. Moreover, the dimension of $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ equals the maximal dimension of the faces in (4), and thus one plus the dimension of $\Delta$.

4. **Local cohomology and holes**

In this section, we consider the local cohomology of the monoid algebra $\mathbb{K}\{Q\}$ with support on the maximal graded ideal $m := p_{F_0}$. Recall that the local cohomology can be computed by the Ishida complex [14] as follows: Consider the $\mathbb{Z}Q$-graded complex

$$ 0 \rightarrow \mathbb{K}\{Q\} \rightarrow \bigoplus_{F \in F_1} \mathbb{K}\{Q_F\} \rightarrow \cdots \rightarrow \bigoplus_{F \in F_{d-1}} \mathbb{K}\{Q_F\} \rightarrow \mathbb{K}\{Q\} \rightarrow 0 $$

where $F_i$ denotes the set of $i$-dimensional faces of $Q$. The maps are given by $\delta_i : \mathbb{K}\{Q_F\} \ni x^F \mapsto \sum_{G \supseteq F} \epsilon(F, G)x^G$ via the canonical inclusion $\mathbb{K}\{Q_F\} \rightarrow \mathbb{K}\{Q_G\}$ for $F \subseteq G$, and $\epsilon(F, G)$ is an appropriate sign function. See [16, Section 13.3] for the exact definition. The cohomological degrees are chosen such that the modules $\mathbb{K}\{Q\}$ and $\mathbb{K}\{\mathbb{Z}Q\}$ sit in degree 0 respectively $d$. 
Theorem 4.1 (Thm. 13.24, [16]). The local cohomology of any \( \mathbb{K}[Q] \)-module \( M \) supported on \( \mathfrak{m} \) is the cohomology of the Ishida complex tensored with \( M \):

\[
H^i_m(M) \cong H^i(M \otimes \mathcal{U}_Q)
\]

The isomorphism respects the \( \mathbb{Z}Q \)-grading.

We use the Ishida complex to relate the local cohomology of \( \mathbb{K}[Q] \) to the local cohomology of \( \mathbb{K}\{Q\}/\mathbb{K}[Q] \).

Theorem 4.2. Let \( Q \) be an affine monoid of *dimension* \( d \), \( i \leq d \) and integer and let \( q \in \mathbb{Z}Q \). Then the following holds:

1. If \( i < d \), then
   \[
   H^i_m(\mathbb{K}[Q]) \cong H^{i-1}_m(\mathbb{K}\{Q\}/\mathbb{K}[Q]).
   \]

2. If \( i = d \) and \( q \notin -\text{int} \overline{Q} \), then
   \[
   H^i_m(\mathbb{K}[Q])_q \cong H^{i-1}_m(\mathbb{K}\{Q\}/\mathbb{K}[Q])_q
   \]
   a \( \mathbb{K} \)-vector spaces.

3. If \( q \in -\text{int} \overline{Q} \), then
   \[
   H^i_m(\mathbb{K}[Q])_q = \begin{cases} 
   \mathbb{K} & \text{if } i = d, \\
   0 & \text{otherwise}.
   \end{cases}
   \]

Proof. First, we compute the local cohomology of \( \mathbb{K}\{Q\} \). For this we compare the Ishida complex of \( \mathbb{K}\{Q\} \) with the complex \( \mathbb{K}\{Q\} \otimes \mathcal{U}_Q \) over \( \mathbb{K}[Q] \). It holds that \( (\overline{Q})_F = (\overline{Q})_F^\pi \) (Lemma 2.1) for every face \( F \) of \( Q \). Therefore, for every \( q \in \mathbb{Z}Q \), the degree \( q \) part of \( \mathcal{U}_Q \) coincides with the degree \( q \) part of \( \mathbb{K}\{Q\} \otimes \mathcal{U}_Q \) (as a complex of \( \mathbb{K} \)-vector spaces). Hence, the support of the local cohomology of \( \mathbb{K}\{Q\} \) (as a ring) equals the support of the local cohomology of \( \mathbb{K}\{Q\} \) (as a \( \mathbb{K}[Q] \)-module). In particular, \( H^i_m(\mathbb{K}\{Q\})_q = 0 \) for \( i < d \), because \( \mathbb{K}\{Q\} \) is Cohen-Macaulay (as a ring). Moreover, \( H^d_m(\mathbb{K}\{Q\})_q = 0 \) for \( q \notin -\text{int} \overline{Q} \). For this, note that \( H^d_m(\mathbb{K}[Q])_q \neq 0 \) if and only if \( q \) is not in the image of the map \( \delta_{d-1} \) in \( \mathcal{U}_Q \). Since \( \overline{Q} \) is normal, this is equivalent to \( \sigma_F(q) < 0 \) for every facet, hence \( q \in -\text{int} \overline{Q} \).

Next, we consider the short exact sequence

\[
0 \to \mathbb{K}[Q] \to \mathbb{K}\{Q\} \to \mathbb{K}\{Q\}/\mathbb{K}[Q] \to 0
\]

The corresponding long exact sequence in cohomology gives immediately that \( H^d_m(\mathbb{K}[Q]) \cong H^{d-1}_m(\mathbb{K}\{Q\}/\mathbb{K}[Q]) \) for \( i < d \).

For \( i = d \), we make a case distinction. If \( q \notin -\text{int} \overline{Q} \), then \( H^d_m(\mathbb{K}\{Q\})_q = 0 \) by the discussion above. Hence one can read off from the long exact sequence that \( H^d_m(\mathbb{K}[Q])_q \cong H^{d-1}_m(\mathbb{K}\{Q\}/\mathbb{K}[Q])_q \), because the maps are homogeneous. On the other hand, if \( q \in -\text{int} \overline{Q} \), then \( q \notin Q_F \) for any face \( F \) of \( Q \). So the degree \( q \) part of \( \mathcal{U}_Q \) is just \( 0 \to \mathbb{K} \to 0 \) with the \( \mathbb{K} \) in cohomological degree \( d \).

Corollary 4.3. If \( Q \) is not normal, then *depth* \( \mathbb{K}[Q] = *depth* \mathbb{K}\{Q\}/\mathbb{K}[Q] + 1 \).

Proof. Recall that the *depth* of a *local* ring \((R, \mathfrak{m})\) equals the minimal degree \( i \) such that \( H^i_m(R) \neq 0 \). Hence the claim is immediate from Theorem 4.2. □

\(^2\)For local rings, this is a theorem by Grothendieck, cf. [3, Theorem 3.5.7]. The *local* case can be reduced to the local case by localizing at the *maximal* ideal.
We give a condition on which graded components of the local cohomology of $\mathbb{K}[Q]$ can be nonzero:

**Corollary 4.4.** Let $q \in \mathbb{Z}Q$ such that $q \notin -\text{int} \mathcal{Q}$. If $H^{i+1}_m(\mathbb{K}[Q])_q \neq 0$ for some $i$, then $q$ is contained in a family of holes of $+\text{dimension}$ at least $i$.

**Proof.** By Theorem 4.2, we have $H^{i+1}_m(\mathbb{K}[Q])_q \cong H^i_m(\mathbb{K}[\mathcal{Q}]/\mathbb{K}[Q])_q$. We consider the $i^{\text{th}}$ module in $\mathcal{U}_Q \otimes \mathbb{K}[\mathcal{Q}]/\mathbb{K}[Q]$. It is

$$\bigoplus_{F \in \mathcal{F}_i} \mathbb{K}[Q_F] \otimes \mathbb{K}[\mathcal{Q}]/\mathbb{K}[Q] = \bigoplus_{F \in \mathcal{F}_i} \mathbb{K}[\mathcal{Q}_F]/\mathbb{K}[Q_F]$$

If $H^{i+1}_m(\mathbb{K}[Q])_q \neq 0$, then there is an element of degree $q$ in this module. Hence there is a face $F$ of $+\text{dimension}$ such that $q \in \mathcal{Q}_F \setminus Q_F$. Now our description of the holes in the localization $Q_F$ (cf. Proposition 3.10) implies that $q$ is contained in a family of holes $\mathbb{Z}G + p$ of $Q$ with $F \subset G$. In particular, $\dim G \geq \dim F = i$. □

Our next goal is a partial converse to Corollary 4.4. For this, we take a closer look at the Ishida complex. In this we follow [16, Section 12.2]. Fix an element $q \in \mathbb{Z}Q$. The part of $\mathcal{U}_Q$ in degree $q$ is determined by the faces $F \subset Q$ such that $q \in Q_F$. Therefore, we consider the set $\nabla(q) := \{ F \subset Q \mid q \in Q_F, F \text{ a face} \}$. This set is clearly closed under going up in the face lattice of $Q$. Now let $P$ be a cross-section polytope of $\mathbb{R}_+Q$ and let $P^\vee$ be the polar polytope of $P$. Then the face lattice of $P^\vee$ equals the order dual of the face lattice of $Q$ (i.e. the face lattice of $P$ turned upside down). Hence the images $\nabla(q)^\vee$ of the faces in $\nabla(q)$ in the face lattice of $P^\vee$ form a set that is closed under going down. In other words, $\nabla(q)^\vee$ is a polyhedral subcomplex of the boundary complex of $P^\vee$. Because $\nabla(q)$ corresponds to the part of $\mathcal{U}_Q$ in degree $q$, we can reinterpret this part as an (augmented) polyhedral chain complex for $\nabla(q)^\vee$, while reversing the cohomological degrees. So the reduced homology of the polyhedral cell complex $\nabla(q)^\vee$ gives us the local cohomology of $\mathbb{K}[Q]$ in degree $q$ ([16, p. 258]):

$$(5) \quad H^i_m(\mathbb{K}[Q])_q = \tilde{H}_{d-i}(\nabla(q)^\vee, \mathbb{K})$$

Here, $d = \dim Q = \dim P + 1$. Using this formula, we can explicitly compute part of the local cohomology of $\mathbb{K}[Q]$:

**Proposition 4.5.** Let $Q$ be an affine monoid and let $\mathbb{Z}F + p$ be a $j$-*dimensional family of holes. Let $q \in \mathbb{Z}F + p$ an element that lies beyond every facet $G$ not containing $F$. By this we mean that $\sigma_G(q) < 0$, where $\sigma_G$ is the supporting linear form of $G$. Then

$$H^i_m(\mathbb{K}[Q])_q = \begin{cases} \mathbb{K} & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** We prove that $\nabla(q) = \{ G \mid G \supseteq F \}$. Thus $\nabla(q)^\vee$ is the boundary complex of the face $F$ corresponding to $F$ in the polar polytope $P^\vee$. This is a sphere of dimension $\dim F - 1 = \dim P - 1 - (\dim F - 1) - 1 = d - 2 - j$. So by (5) it follows

$$H^i_m(\mathbb{K}[Q])_q = \tilde{H}_{d-i}(S^{d-2-j}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

To compute $\nabla(q)$, we first consider a face $G$ that does not contain $F$. For such a $G$ we can find a facet $G' \supset G$ that does not contain $F$. By our assumption, $q$ lies beyond $G'$ and hence $q \notin Q_{G'}$. Thus $q \notin Q_{G'}$ and therefore $G \notin \nabla(q)$. Next, by our
choice of \( q \), it holds that \( q \in \overline{Q} \setminus Q \). In particular \( F \notin \nabla(q) \). Moreover, \( q \in \overline{Q} \) for every \( G \supseteq F \), because \( \overline{Q} \supseteq \overline{Q} \). It remains to show that \( q \in Q \) for every \( G \supseteq F \). So assume on the contrary that \( q \in \overline{Q} \setminus Q \) for such a \( G \). There exists an element \( f \) from the interior of \( F \) to get \( q + f \in \overline{Q} \setminus Q \). But this implies \( G + q + f \subset \overline{Q} \), which contradicts our choice \( q \in \mathbb{Z}F + p \), by Theorem 3.8. □

This gives a partial converse to Corollary 4.4:

**Corollary 4.6.** Every \( i \)-dimensional family of holes contains an element \( q \in \mathbb{Z}Q \), such that \( H_{m+1}^i(\mathbb{K}[Q])q \cong \mathbb{K} \). If \( i > 0 \), then there are in fact infinitely many such elements (even up to units).

**Proof.** Let \( \mathbb{Z}F + p \) be a family of holes and let \( q' \in \text{int} \ F \). For every facet \( G \not
supseteq F \) it holds that \( \sigma_G(q') > 0 \). Hence, \( p - mq' \) satisfies the hypothesis of Proposition 4.5 for every sufficiently large \( m \in \mathbb{N} \). This yields infinitely many non-vanishing graded components of \( H_{m+1}^i(\mathbb{K}[Q]) \). If \( i > 0 \), then \( F \neq F_0 \) and thus \( q' \notin F_0 \). So these components are \( \mathbb{K}[F_0] \)-linearly independent. □

The preceding proof shows in particular that \( H_{m+1}^i(\mathbb{K}[Q]) \cong H_m^i(\mathbb{K}[Q]/\mathbb{K}[Q]) \) is not finitely generated if \( \mathbb{K}[Q]/\mathbb{K}[Q] \) has an associated prime of dimension \( i > 0 \). This is true for general finitely generated modules over an (ungraded) local ring which is a homomorphic image of a local Gorenstein ring, see [1, Corollary 11.3.3]. For later use, we give a criterion for the vanishing of certain parts of the local cohomology.

**Lemma 4.7.** Let \( q_1, q_2, \ldots \) be a sequence of elements in \( \mathbb{Z}Q \) such that \( H_m^i(\mathbb{K}[Q])q_j \cong H_m^i(\mathbb{K}[Q])q_j \) for every \( j \). Assume further that there is a facet \( F \) of \( Q \) such that \( \sigma_F(q_j) < \sigma_F(q_{j+1}) \) for every \( j \). Then \( H_m^i(\mathbb{K}[Q])q_j = 0 \) for every \( j \).

**Proof.** Assume to the contrary that \( H_m^i(\mathbb{K}[Q])q_j \neq 0 \). Consider the submodules \( H_l \subseteq H_m^i(\mathbb{K}[Q]) \) generated by \( \{ H_m^i(\mathbb{K}[Q])q_j \mid j \geq l \} \). Clearly \( H_{l+1} \subseteq H_l \). By our hypothesis, \( \sigma_F(q_j) < \sigma_F(q_{j+1}) \) for every \( j > l \). Therefore, \( q_j \) is not contained in the \( Q \)-submodule of \( \mathbb{Z}Q \) generated by the \( q_j \) for \( j > l \). This implies that \( H_{l+1} \not\subseteq H_l \), so we get an infinite descending chain of submodules. This contradicts the fact that \( H_m^i(\mathbb{K}[Q]) \) is *Artinian* 3. □

We give an example to demonstrate the geometric meaning of the results in this section.

**Example 4.8.** Consider the affine monoid \( Q \subset \mathbb{Z}^3 \) generated by \((0, 0, 1), (1, 0, 1), (0, 2, 1), (1, 2, 1), (0, 3, 1) \) and \((1, 3, 1) \). It is shown in the left part of Figure 2. This example is taken from [22]. The holes of \( Q \) form a “wall” parallel to the \( xz \)-plane. Hence, nontrivial local cohomology of \( \mathbb{K}[Q] \) can only appear in the degrees of this wall. The right part of Figure 2 shows this wall and the intersections with the facet defining hyperplanes. In each region, \( \nabla(\cdot) \) and thus \( H_m^i(\mathbb{K}[Q]) \) is constant. In the shaded unbounded region pointing downwards, we have \( H_m^2(\mathbb{K}[Q]) \neq 0 \) by Proposition 4.5. All other unbounded regions do not support local cohomology by Lemma 4.7 (just take the points on any ray reaching outwards). So the only part of the local cohomology that is not classified so far is the lattice point \( q \) in the small shaded triangle. In fact, one may compute directly that \( \dim_k H_m^2(\mathbb{K}[Q])q = 1 \).

3This seems to be a folklore result, but I was not able to find a proof in the literature. It can be obtained from the local version (cf. [1, Theorem 7.1.3]) by localization at the *maximal ideal.
5. Applications

5.1. Special configurations of holes. In this section, we show that various ring-theoretical properties of $K[Q]$ correspond to special configurations of the holes in $Q$. For positive $Q$, the next proposition appeared as Corollary 5.3 in [19].

Proposition 5.1. Let $Q$ be an affine monoid with $\dim Q \geq 2$. Then $H^1_m(K[Q])_q \neq 0$ if and only if $q$ is contained in a zero-*dimensional family of holes. In this case, $H^1_m(K[Q])_q = K$ and $H^i_m(K[Q])_q = 0$ for $i \neq 1$.

Proof. By Theorem 4.2, we have $H^1_m(K[Q]) \cong H^0_m(K\{Q\}/K[Q])$. In the proof of Lemma 3.4, we have already verified that the support of $H^0_m(K\{Q\}/K[Q])$ is the union of the zero-*dimensional families of holes.

For the second claim, note that $H^i_m(K\{Q\}/K[Q])$ is a submodule of $K\{Q\}/K[Q]$ and every nontrivial homogeneous component of $K\{Q\}/K[Q]$ has dimension 1 over $K$. Moreover, if $H^i_m(K[Q])_q \neq 0$ for some $i > 1$, then by Corollary 4.4 $q$ has to be contained in a family of holes $ZG + p$ of *dimension $i - 1 > 0$. But then the zero-*dimensional family of holes containing $q$ is also contained in $ZG + p$ (because the minimal face $F_0$ is contained in $G$), which is absurd. \hfill \Box

Theorem 5.2. Let $Q$ be an affine monoid of *dimension $d$. The following holds:

- If $d \geq 2$, then *depth $K[Q] = 1$ if and only if there is a 0-*dimensional family of holes.
- $Q$ is locally normal if and only if there is no family of holes of positive *dimension.
- $K[Q]$ satisfies Serre’s condition $(R_1)$ if and only if there is no family of holes of *dimension $d - 1$.
- $K[Q]$ satisfies Serre’s condition $(S_2)$ if and only if every family of holes has *dimension $d - 1$.  

\begin{figure}[h]
  \centering
  \includegraphics[width=0.8\textwidth]{example.png}
  \caption{The example of Trung and Hoa}
\end{figure}
Proposition 5.1

Next, we prove the criterion for local normality. Recall that $\mathbb{K}[Q]$ is locally normal if its localizations at all $1$-dimensional faces are normal. By Proposition 3.10, this is clearly equivalent to the statement that there are no families of holes of positive dimension.

To prove the criterion for Serre’s condition $(R_1)$, recall $\mathbb{K}[Q]$ satisfies $R_1$ if and only if $\mathbb{K}[Q_F]$ is regular for every facet $F$. On the other hand, by Proposition 3.10, $Q_F$ is normal for every facet if and only if there is no $(d-1)$-dimensional family of holes. Every $1$-dimensional normal affine monoid is isomorphic to $\mathbb{Z}^m \oplus \mathbb{N}$ for some $m \in \mathbb{N}$. But these are exactly the $1$-dimensional regular affine monoids, cf. [2, Prop. 4.45].

Finally, we prove the criterion for Serre’s condition $(S_2)$. By above discussion and Proposition 3.10, it holds for any face $F$ of $Q$ that $\text{depth} \mathbb{K}[Q_F] = 1$ if and only if $F$ is associated to $Q$. On the other hand, it holds that $\text{dim} \mathbb{K}[Q_F] = 1$ if and only if $F$ is a facet of $Q$. Therefore, Serre’s condition $(S_2)$ is satisfied if and only if every associated face of $Q$ is a facet.

Note that the preceding theorem implies that $Q$ is normal if and only if it satisfies $(R_1)$ and $(S_2)$. Hence as a corollary we obtain Serre’s Theorem (cf. [3, Theorem 2.2.22]) for affine monoid algebras. Further, if $\text{dim} Q \geq 2$, then $Q$ is normal if and only if it is locally normal and $\text{depth} \mathbb{K}[Q] \geq 2$. It follows from Serre’s Theorem that this statement also holds for general Noetherian rings. The first part of the Theorem 5.2 can be generalized to an upper bound on the *depth:

**Theorem 5.3.** If $Q$ has an $i$-*dimensional family of holes, then the *depth of $\mathbb{K}[Q]$ is at most $i + 1$.

**Proof.** If $Q$ has an $i$-*dimensional family of holes, then $H_{m+1}(\mathbb{K}[Q]) \neq 0$ by Corollary 4.6. Hence $\text{depth} \mathbb{K}[Q] \leq i + 1$. Alternatively, by Corollary 4.3 we can consider the depth of $\mathbb{K}[Q]/\mathbb{K}[Q]$. The families of holes of $Q$ correspond to the associated primes of $\mathbb{K}[Q]/\mathbb{K}[Q]$ (cf. Theorem 3.8), so the claim follows from the general fact that the *depth of a module is bounded above by the *dimensions of its associated primes$^4$.

We identify a special case where equality holds:

**Proposition 5.4.** If $\overline{Q} \setminus Q = q + F$ for an element $q \in \overline{Q} \setminus Q$ and a face $F$ of $Q$, then $\text{depth} \mathbb{K}[Q] = 1 + \text{dim} F$.

Note that it is not sufficient to require that there is only one family of holes, see Example 4.8. Before we can prove Proposition 5.4, we need another lemma:

**Lemma 5.5.** Let $q \in Q$ and $F$ a face of $Q$, such that $F + q \subseteq \overline{Q} \setminus Q$. Then $F + q \subseteq \overline{Q} \setminus Q$.

**Proof.** Assume the contrary. Then there exists an element $f \in F$ such that $f + q \in Q$. We can write $f = f_1 - f_2$ with $f_1, f_2 \in F$. But $f + f_2 + q = f_1 + q \in \overline{Q} \setminus Q$ by assumption, a contradiction.

$^4$Again, I only found a local version of this result in the literature, cf. [3, Prop 1.2.13]. But Goto and Watanabe have shown in [5] that the *depth and *dimension are preserved under localization.

So the *local case follows from the local one by localization.
Proof of Proposition 5.4. By Lemma 3.1, our hypothesis is equivalent to the statement that the module \( \mathbb{K}(Q)/\mathbb{K}[Q] \) is cyclic with annihilator \( p_F \). Hence \( \mathbb{K}(Q)/\mathbb{K}[Q] \cong \mathbb{K}[Q]/p_F = \mathbb{K}[F] \) (The isomorphism shifts the grading). Next, recall that the *depth of a module \( M \) over a ring \( R \) equals its *depth over \( R/ \text{Ann} M \). Together with Corollary 4.3 this yields
\[
\text{depth}_{\mathbb{K}[Q]} \mathbb{K}[Q] = 1 + \text{depth}_{\mathbb{K}[F]} \mathbb{K}[F].
\]
Now we use Lemma 5.5 to conclude that \( \overline{Q} \setminus Q = F + q \subseteq \overline{F} + q \subseteq \overline{Q} \setminus Q \), so \( F = \overline{F} \). Hence \( \mathbb{K}[F] \) is normal and the result follows from Hochster’s Theorem. \( \square \)

We give an example how one can effectively compute the *depth using Proposition 5.4.

**Example 5.6.** Let \( G \) be a graph with vertex set \( V \) and edge set \( E \). We associate an affine monoid to \( G \), the toric edge ring \( k[G] \), introduced in [18]. This is the monoid algebra to the monoid generated by the vectors \( e_i + e_j \in \mathbb{Z}^{|V|} \), where \( \{ i, j \} \) is an edge of \( G \) and \( e_i, e_j \) denotes unit vectors indexed by the vertices of \( G \). For positive \( k \in \mathbb{N} \) consider the graph \( G_{k+6} \) in Figure 3. In [12] the *depth of the toric edge ring of this family of graphs is computed. We will show that these edge rings satisfy the assumption of Proposition 5.4, and thus give an alternative computation of the *depth.

First, it is known that \( \mathbb{K}[Q] \) is generated as a \( \mathbb{K}[Q] \)-module by \( x_1x_2x_4x_5x_6 \), i.e. the monomial corresponding to the vector \( q \in \overline{Q} \subseteq \mathbb{R}^{k+6} \) which assigns 1 to the vertices \( 1, \ldots, 6 \). If we add one of the “middle” edges, e.g. \( \{ 3, 8 \} \), to \( q \), then it is easy to see that the result lies in \( Q \). On the other hand, if we add any combination of edges from the triangles to \( q \), then the result will always be in \( \overline{Q} \setminus Q \). To see this, note that the sum over the vertices of each triangle is always odd. This implies that \( \overline{Q} \setminus Q = F + q \), where \( F \) is the face spanned by the six edges in the triangles. The dimension of \( F \) is 6, so by Proposition 5.4 it follows that depth \( \mathbb{K}[Q] = 1 + 6 = 7 \).

We generalize this computation to show that every toric edge ring can be realized as a combinatorial pure subring of a toric edge ring of *depth at most 7. The construction is as follows: To a given graph \( G \), add two triangles on six (in total) new vertices. Then connect every vertex of \( G \) with every new vertex. Obviously, the toric edge ring of \( G \) is a combinatorial pure subring of the edge ring of this bigger graph, because \( G \) is an induced subgraph of the later. Then it is not difficult to see that the face spanned by the six edges in the triangles is associated. Its dimension is six, so the *depth of the ring is at most seven. In [12] it was conjectured that

![Figure 3. The graph \( G_{k+6} \)](image)
every toric edge ring has a *depth of at least seven. So we consider it as likely that the toric edge ring we constructed has a *depth of exactly seven.

5.2. Simplicial affine monoids. An affine monoid $Q$ is called simplicial if its cross section polytope is a simplex. Equivalently, $Q$ is simplicial if its face lattice is a boolean lattice. Some authors require simplicial affine monoids to be positive, but we allow non-positive simplicial affine monoids. A well-known result by Goto, Suzuki and Watanabe [7] states that if $Q$ is simplicial, positive and satisfies Serre’s condition $(S_2)$, then $\mathbb{K}[Q]$ is Cohen-Macaulay. We give a proof of this result without the positivity assumption using our description of Serre’s condition $(S_2)$.

Proposition 5.7. Let $Q$ be a simplicial affine monoid. If $Q$ satisfies Serre’s condition $(S_2)$, then $\mathbb{K}[Q]$ is Cohen-Macaulay.

We also identify another case where the upper bound on the *depth given in Corollary 4.3 is tight.

Proposition 5.8. Let $Q$ be a simplicial affine monoid. If the families of holes of $Q$ are pairwise disjoint, then the *depth of $\mathbb{K}[Q]$ equals one plus the smallest *dimension of a family of holes.

Both results depend on the following lemma. Recall that we defined $\nabla(q)$ to be the set of faces of $Q$ such that $q \in F_F$ for $q \in \mathbb{Z}Q$. The complex $\nabla(q)^\vee$ is defined by turning the face poset of $\nabla(q)$ upside down. It is a subcomplex of the polytope $P^\vee$ polar to the cross-section polytope $P$ of $\mathbb{R}_+Q$. Since $P$ is a simplex, the same holds for $P^\vee$. So $\nabla(q)^\vee$ is a simplicial complex whose vertices correspond to the facets of $Q$. A minimal non-face of a simplicial complex $\Delta$ is a minimal face of the ambient simplex that is not a face of $\Delta$.

Lemma 5.9. Let $Q$ be a simplicial affine monoid, $q \in \mathbb{Z}Q$ and $i \geq 2$. The $(i - 1)$-dimensional minimal non-faces of $\nabla(q)^\vee$ correspond to the $(d - i)$-*dimensional families of holes containing $q$.

Proof. For $q \in \mathbb{Z}Q$ let $\nabla(q)$ denote the set of faces $F$ of $Q$ such that $q \in \overline{Q_F}$. Obviously it holds that $\nabla(q) \subset \nabla(q)$. Hence $\nabla(q)^\vee$ is a simplicial subcomplex of the simplicial complex $\nabla(q)^\vee$ contained in the boundary complex of $P^\vee$. Every minimal non-face of $\nabla(q)^\vee$ that is not contained in $\nabla(q)^\vee$ is also a minimal non-face of the latter.

So we start by computing the minimal non-faces of $\nabla(q)^\vee$. For this we claim that $\nabla(q)$ has a unique minimal element. We write $F_\geq$ for the set of facets $F$ of $Q$ such that $\sigma_F(q) \geq 0$ and we write $F_<$ for the set of facets $F$ such that $\sigma_F(q) < 0$. Our candidate for the unique minimal element is the intersection $G$ of the facets in $F_\geq$. This is indeed a face because $Q$ is simplicial. If $p \in \text{int } G$ is an interior element, then by construction $\sigma_F(p) > 0$ for all $F$ in $F_<$. Therefore, $q + mp \in \overline{Q_F}$ for $m \gg 0$ and hence $q \in \overline{Q_G}$. On the other hand, let $G'$ be a face such that $q \in \overline{Q_{G'}}$. Then there exists an element $g \in G'$ such that $q + g \in \overline{Q}$. It follows that $\sigma_F(g) > 0$ for all $F \in F_<$. Hence $G$ is not contained in any facet in $F_<$ and can therefore be written as an intersection of facets in $F_\geq$. It follows that $F \subset G$. We now return the face lattice of $\nabla(q)^\vee$. Because $\nabla(q)$ has a unique minimal element, $\nabla(q)^\vee$ is isomorphic to the complex of faces of a simplex. In particular, the minimal non-faces of $\nabla(q)^\vee$ are only vertices.

Next, we consider the minimal non-faces of $\nabla(q)^\vee$ that are contained in $\nabla(q)^\vee$. This minimal non-faces correspond to the maximal faces $F$ such that $q \in \overline{Q_F} \setminus Q_F$. **
But these are exactly the families of holes containing $q$. So the minimal non-faces of $\nabla(q)^{\vee}$ correspond either to the families of holes containing $q$ or they are vertices (these come from the minimal non-faces of $\nabla(q)^{\vee}$).

Proof of Proposition 5.7. By Theorem 5.2, Serre’s condition $(S_2)$ implies that all families of holes have dimension $d - 1$. So $\nabla(q)^{\vee}$ is a simplicial complex with only 0-dimensional minimal non-faces for every $q \in \mathbb{Z}Q$. In other words, $\nabla(q)^{\vee}$ is either a simplex or empty. So the only possible nontrivial (reduced) homology lies in degree $-1$. By (5), this amounts to saying that $H^i_m(\mathbb{K}[Q]) = 0$ for $i < d$, so $\mathbb{K}[Q]$ is Cohen-Macaulay.

Proof of Proposition 5.8. Every $q \in \mathbb{Z}Q$ is contained in at most one family of holes. Hence $\nabla(q)^{\vee}$ is a simplicial complex with only one minimal non-face of positive dimension. This is either a ball of a sphere. Evaluating (5) then yields the result.

Since we allow non-positive affine monoids in Proposition 5.7 we immediately obtain a local version.

Corollary 5.10. Let $Q$ be a locally simplicial affine monoid. If $Q$ satisfies Serre’s condition $(S_2)$, then $\mathbb{K}[Q]$ is locally Cohen-Macaulay for every field $\mathbb{K}$.

5.3. Seminormal affine monoids. In this subsection, we apply our results to seminormal affine monoids. This way we reprove and extend some results of [2]. Recall that an affine monoid $Q$ is called seminormal if $2q, 3q \in Q$ implies $q \in Q$ for $q \in \mathbb{Z}Q$. Equivalently, for every $q \in \overline{Q} \setminus Q$, the set $\{ m \in \mathbb{N} \mid mq \in Q \}$ is contained in a proper subgroup of $\mathbb{Z}$. First, we give a geometric characterization of seminormality that is similar in spirit to the characterizations given in [2, p. 66f].

Proposition 5.11. Let $Q$ be an affine monoid. $Q$ is seminormal if and only if for every family of holes $ZF + q$ it holds that $q \in QF$.

Here $QF$ denotes the $\mathbb{Q}$-subspace of $QQ$ generated by $F$.

Proof. First, assume that the condition in the statement is satisfied. Consider a family of holes $ZF + q$. Since $q \in \overline{Q} \setminus Q$, there exists an $m \in \mathbb{N}$ such that $mq \in Q$. By our assumption, it holds that $mq \in F$ and therefore $jmq + q \in ZF + q \cap \overline{Q} \subset \overline{Q} \setminus Q$ for every $j \in \mathbb{N}$. It follows that either $2q \notin Q$ or $3q \notin Q$. Thus, $Q$ is seminormal.

On the other hand, assume there is a family of holes $ZF + q$ such that $q \notin QF$. Then there exists an element $p \in ZF + q$ such that $p \in \text{int} \overline{G}$ and $p \notin G$ for some face $G \supset F$. Thus $Q$ is not seminormal by [2, Proposition 2.40].

Corollary 5.12. Localizations of seminormal affine monoids are again seminormal.

Proof. This follows from Proposition 5.11 and the description of the families of holes of a localization given in Proposition 3.10.

Corollary 5.13 (Corollary 5.4.[4]). Let $Q$ be a seminormal positive affine monoid of dimension at least 2. Then depth $\mathbb{K}[Q] \geq 2$.

Proof. If $Q$ is positive, then the minimal face $F_0$ contains only the origin $0 \in \mathbb{Z}Q$. By Proposition 5.11 every 0-dimensional family of holes would be contained in $QF_0 = \{0\} \subset Q$, so there is no 0-dimensional family of holes. Hence the claim follows from Theorem 5.2.
This result is not valid if one omits the requirement that $Q$ is positive. For example, consider $Q \subset \mathbb{Z}^3$ defined by

$$Q = \{ (x, y, z) \in \mathbb{Z}^3 \mid x, y \geq 0, z \text{ even or } x > 0 \text{ or } y > 0 \}$$

This monoid is seminormal, has dimension 2 and has a 0-dimensional family of holes, namely the odd points on the z-axis. So it has depth $\mathbb{K}[Q] = 1$ by Theorem 5.2.

Next we give a preliminary characterization of seminormality. Geometrically, we show that the graded components of the local cohomology of a seminormal affine monoid are, in a certain sense, constant on rays from the origin.

**Lemma 5.14.** An affine monoid $Q$ is seminormal if and only if it satisfies the following condition: For every $q \in \mathbb{Z}Q$ there exists a positive $m \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ and every $i \in \mathbb{N}$ it holds that $H^i_m(\mathbb{K}[Q])_q \cong H^i_m(\mathbb{K}[Q])_{(1+mq)}q$ (as $\mathbb{K}$-vector space).

**Proof.** Assume that $Q$ is seminormal and fix an element $q \in \mathbb{Z}Q$. We will find an $m \in \mathbb{N}$ such that $V(q) = V((1 + mj)q)$ for every $j \in \mathbb{N}$. This implies our claim by (5). First, note that $q \notin Q_F$ implies $mq \notin Q_F$ for every $m \in \mathbb{N}$. Similarly, $q \notin \overline{Q_F}$ implies $mq \notin \overline{Q_F}$ for every $m \in \mathbb{N}$. So it remains to show the following: There exists an $m \in \mathbb{N}$, such that for every face $F$ with $q \in \overline{Q_F} \setminus Q_F$ and every $j \in \mathbb{N}$ it holds that $(1 + jmq)q \in \overline{Q_F} \setminus Q_F$. For this, consider a face $F$ of $Q$ such that $q \in \overline{Q_F} \setminus Q_F$. By Corollary 5.12, the localization $Q_F$ is seminormal and thus the set \{ $m \in \mathbb{N} \mid mq \in Q_F$ \} is contained in a proper subgroup of $\mathbb{Z}$. Since there are only finitely many such faces, we can choose an $m$ in the intersection of these subgroups (for example, the product of the generators). Then $1 + jm$ is not contained in any of these subgroups for every $j \in \mathbb{N}$. Whence our claim follows.

For the converse, assume that $Q$ is not seminormal. Let $ZF + q$ be a family of holes such that $q \notin QF$. Then there exists a facet $G \supset F$ of $Q$ such that $\sigma_G(q) > 0$. By Proposition 4.5, we can find an $p \in ZF + q$ such that $H^i_m(\mathbb{K}[Q])_p \neq 0$ for $i = \dim F + 1$. Now the sequence $p_j := (1 + mj)p$ for $j = 0, 1, \ldots$ satisfies the hypothesis of Lemma 4.7, so we conclude that $H^i_m(\mathbb{K}[Q])_p = 0$, a contradiction. \(\square\)

With a little more care, one can show that the $m$ in the preceding lemma can be chosen independently of $q$. We extend the characterization of seminormality given in Theorem 4.7 of [4].

**Theorem 5.15.** Let $Q$ be an affine monoid. The following statements are equivalent:

1) $Q$ is seminormal.
2) $H^i_m(\mathbb{K}[Q])_q = 0$ for all $q \in \mathbb{Z}Q$ such that $q \notin -\overline{Q}$ and all $i$.
3) $H^i_m(\mathbb{K}[Q])_q = 0$ for all $q \in \mathbb{Z}Q$ such that $q \notin -\overline{Q}$ and all $i$ such that $Q$ has an $(i + 1)$-dimensional family of holes.

Note that the third condition generalizes Theorem 4.9 in [4].

**Proof.** 1) $\Rightarrow$ 2) Let $Q$ be seminormal and let $q \in \mathbb{Z}Q$. By Lemma 5.14, there exists a positive integer $m$ such that $H^i_m(\mathbb{K}[Q])_q \cong H^i_m(\mathbb{K}[Q])_{(1+mq)}q$ for every $i$ and every $j \in \mathbb{N}$. If $q \notin -\overline{Q}$, then the sequence $q_j := (1 + mj)q$ satisfies the condition of Lemma 4.7, so we conclude that $H^i_m(\mathbb{K}[Q])_q = 0$.

2) $\Rightarrow$ 3) This is obvious.
3) ⇒ 1) Assume that $Q$ is not seminormal. Then, by Proposition 5.11, there is a family of holes $ZF + q$ of $Q$ such that $q \notin QF$. There exists a facet $G$ containing $F$ such that $\sigma_G(q) > 0$. By Corollary 4.6, there exists an element $p \in ZF + q$ such that $H^i_m(\mathbb{K}[Q])_p \neq 0$ where $i = \dim F + 1$. The linear form $\sigma_G$ is constant on $ZF + q$, so $\sigma_G(p) > 0$ and hence $p \notin \overline{Q}$.

Our next results extend Proposition 4.15 of [4]. In the non-positive case we may have nontrivial local cohomology in the degrees $F^Q_0 := \overline{Q} \cap (-\overline{Q}) = QF_0 \cap ZQ$. Note that if $Q$ is positive, then $F^Q_0 = \{0\} \subset Q$, so there can be no local cohomology supported in $F^Q_0$. Moreover, if $H^i_m(\mathbb{K}[Q])_q \neq 0$ for some $q \in ZQ$ then $q \in -\overline{Q}$ by the preceding theorem. So in this case, $q \notin F^Q_0$ if and only if $q \notin \overline{Q}$. We will see in Proposition 6.5 below that it is indeed necessary to consider this graded components separately.

**Proposition 5.16** (Prop. 4.14, [4]). Let $Q$ be a seminormal affine monoid. If $H^i_m(\mathbb{K}[Q])_q \neq 0$ for some $q \in ZQ$, $q \notin F^Q_0$, then $H^i_m(\mathbb{K}[Q])$ is not finitely generated.

**Proof.** Assume to the contrary that $H^i_m(\mathbb{K}[Q])$ is finitely generated, say, in degrees $p_1, \ldots, p_s \in ZQ$. We assumed that $q \notin \overline{Q}$, so there exists a facet $F$ such that $\sigma_F(q) < 0$. By Lemma 5.14, there is an $m \in \mathbb{N}$ such that $H^i_m(\mathbb{K}[Q])_{(1 + mj)q} \neq 0$ for every $j \in \mathbb{N}$. For sufficiently large $j \in \mathbb{N}$, it holds that $\sigma_F((1 + mj)q) < \sigma_F(p_k)$ for every $k$, so the graded component in this degree cannot be generated by our supposed set of generators, a contradiction.

**Corollary 5.17.** Let $Q$ be a seminormal affine monoid such that $H^i_m(\mathbb{K}[Q])_q = 0$ for every $q \in F^Q_0$ and every $i < \dim Q$. Then $$\text{^\text{\textdagger}depth } \mathbb{K}[Q] = \min \{ \text{^\textdaggerdepth } \mathbb{K}[Q_F] + 1 \mid F \text{ a face, } \dim F = 1 \}.$$ In particular, $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if it is locally Cohen-Macaulay.

**Proof.** Our hypothesis implies that all the non-vanishing local cohomology modules of $\mathbb{K}[Q]$ are not finitely generated. Therefore, the claim is a consequence of graded version of Grothendieck’s Finiteness Theorem, cf. [1, Theorem 13.1.17] and Exercise 9.5.4 (i) in the same book\(^5\).

There are analogues of Proposition 5.7 and Proposition 5.8 for the seminormal case. The first of the following results appeared in [4, Corollary 5.6] for positive $Q$. We call an affine monoid **locally simplicial** if every localization is simplicial. Equivalently, $Q$ is locally simplicial if its cross section polytope is simple. To see that this in fact equivalent, recall that a polytope is simple if and only if every vertex figure is a simplex, cf. [23, prop. 2.16].

**Proposition 5.18.** Let $Q$ be a seminormal locally simplicial affine monoid such that $H^i_m(\mathbb{K}[Q])_q = 0$ for every $q \in F^Q_0$ and every $i < \dim Q$. Then $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if $Q$ satisfies Serre’s condition $(S_2)$.

We will see below in Proposition 6.5 that the hypothesis on the local cohomology cannot be dropped.

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\(^5\)Although [1] treats the $\mathbb{Z}$-graded case only, the proof given there generalizes to the multigraded case using the results in [6].
Proposition 5.19. Let $Q$ be a seminormal affine monoid. If its the families of holes are pairwise disjoint, then the *depth of $\mathbb{K}[Q]$ equals one plus the smallest *dimension of a family of holes.

The proofs proceed similarly to the simplicial case, once we have shown the following fact:

Lemma 5.20. Let $Q$ be a seminormal affine monoid, $q \in -\overline{Q}$. Let $\nabla(q)$ denote the set of faces of $Q$ such that $q \in \overline{Q}_F$. Then $\nabla(q)$ has a unique minimal element.

Proof. The claim is a statement about $\overline{Q}$, so we may assume that $Q = \overline{Q}$. There exists a unique face $F$ such that $q \in -\text{int } F$. Evidently $q \in \overline{Q}_F$. We show that this $F$ is the unique minimal element. So let $G$ be a facet of $Q$ that does not contain $F$. Then $\sigma_F(q) < 0$ and hence $q \notin Q_G$. The same holds then for every face contained in $G$. It follows that every face $G \in \nabla(q)$ is contained only in those facets that contain $F$. Hence, $F$ is the unique minimal element of $\nabla(q)$. □

Proof of Proposition 5.18. $Q$ is locally simplicial, so the cross-section polytope $\mathcal{P}$ of $\mathbb{R}_+Q$ is simple. Hence its polar $\mathcal{P}^\vee$ is a simplicial polytope.

Let $q \in -\overline{Q}$. By the preceding lemma, $\nabla(q)^\vee$ is a face of $\mathcal{P}^\vee$. We only need to consider those $q$ such that $H^i_m(\mathbb{K}[Q])_q \neq 0$ for some $i < \dim Q$. In this case, by assumption, it holds that $q \notin \overline{Q}$. Equivalently, $\nabla(q)^\vee$ is a proper face of $\mathcal{P}^\vee$ and thus a simplex. Now the proof is analogous to the proof of Proposition 5.7. □

We would like to point out an alternative way to derive Proposition 5.18 from Proposition 5.7. Assume that $Q$ is locally simplicial, seminormal, satisfies $(S_2)$ and $H^i_m(\mathbb{K}[Q])_q = 0$ for every $q \in F_0^{\overline{Q}}$. Then $\mathbb{K}[Q]$ is locally Cohen-Macaulay by Corollary 5.10 and hence Cohen-Macaulay by Corollary 5.17.

Proof of Proposition 5.19. Let $q \in -\overline{Q}$. We have seen in the preceding proof that $\nabla(q)^\vee$ is a face of the polytope $\mathcal{P}^\vee$, though now $\mathcal{P}^\vee$ is not assumed to be simplicial and $\nabla(q)^\vee$ might not be a proper face. However, by our hypothesis $\nabla(q)^\vee$ is obtained from $\nabla(q)^\vee$ by removing just one face. Hence, as in the proof of Proposition 5.8, $\nabla(q)^\vee$ is either a ball or a sphere and the claim follows from (5). □

6. Dependence on the characteristic

In this section, we show that the local cohomology of low-dimensional affine monoid algebras does not depend on the field. Moreover, we give two constructions of affine monoid algebras with certain properties whose *depth does depend on $\mathbb{K}$.

In the following proposition, the case $i = 0$ is actually trivial and the cases $i = 1$ and $i = d$ follow from the description of $H^1_m(\mathbb{K}[Q])$ and $H^d_m(\mathbb{K}[Q])$ given in [19], at least for positive $Q$.

Proposition 6.1. Let $Q$ be an affine monoid of *dimension $d$ and let $q \in \mathbb{Z}Q$. The vector space dimension $\dim_\mathbb{K} H^i_m(\mathbb{K}[Q])_q$ of the local cohomology modules does not depend on the characteristic of the field $\mathbb{K}$ for $i \in \{0, 1, 2, d-1, d\}$.

Proof. We use (5) to compute $\dim_\mathbb{K} H^i_m(\mathbb{K}[Q])_q$. If $i$ equals $d$ or $d-1$, then this amounts to the $-1^{st}$ and $0^{th}$ Betti number of the polyhedral cell complex $\nabla(q)^\vee$, and these numbers do not depend on the characteristic.

For the other values of $i$, we use Alexander duality. Recall that $\nabla(q)^\vee$ is a subcomplex of the boundary complex of the polytope $\mathcal{P}^\vee$, which is a $(d-2)$-sphere.
Indeed, if $\nabla(q)^\vee$ contains the interior of $P^r$, then $q \in Q$, so the local cohomology vanished in degree $q$ by Corollary 4.4. By Alexander duality (cf. [9, Theorem 3.44]), it holds that

$$\dim_K H^i_m(\mathbb{K}[Q])_q = \dim_K \tilde{H}_{d-i-1}(\nabla(q)^\vee, \mathbb{K}) = \dim_K H^{i-2}(S^{d-2} \setminus \nabla(q)^\vee, \mathbb{K}).$$

As in the case above, this number does not depend on $K$ for $i \leq 2$, because the $-1$st and the $0$th Betti numbers of $S^{d-2} \setminus \nabla(q)^\vee$ are independent of $K$. □

**Corollary 6.2.** If $\dim Q \leq 5$, then $\dim K H^i_m(\mathbb{K}[Q])_q$ is independent of $K$ for any $i$ and any $q \in \mathbb{Z}Q$.

**Proof.** If $\dim Q \leq 4$, this follows at once from Proposition 6.1. For $\dim Q = 5$ we only need to consider $H^2_m(\mathbb{K}[Q])_q = \tilde{H}_1(\nabla(q)^\vee, \mathbb{K})$. Again, we use that $\nabla(q)^\vee$ is a subcomplex of a 3-sphere, so by [9, Cor. 4.45], $\tilde{H}_1(\nabla(q)^\vee, \mathbb{Z})$ is torsionfree. On the other hand, the $0$th reduced homology of a complex is always torsionfree, so by the universal coefficient theorem (cf. [9, Cor. 3A.6]) the dimension $\dim_K \tilde{H}_1(\nabla(q)^\vee, \mathbb{K})$ does not depend on $K$. □

**Corollary 6.3.** Serre’s condition $(S_3)$ does not depend on the characteristic of the field.

**Proof.** By Proposition 6.1, the property of $\mathbb{K}[Q]$ having *depth at least 3 does not depend on the characteristic, so the same holds for $(S_3)$. □

There are 6-*dimensional affine monoids $Q$ such that *depth $\mathbb{K}[Q]$ does depend on the characteristic of the field. An example is given in [22, p.165]. The affine monoid $Q$ constructed in that paper has *dimension 6 and $H^3_m(\mathbb{K}[Q])$ vanishes if and only if $\text{char } \mathbb{K} \neq 2$. This shows that the results above cannot be improved.

We have seen in that if $Q$ is either simplicial or seminormal, locally simplicial and positive, then the Cohen-Macaulayness of $\mathbb{K}[Q]$ does not depend on the field $K$. In the second part of this subsection, we give two examples showing how these results cannot be extended. First, we construct a seminormal, simplicial and positive affine monoid whose *depth does depend on the characteristic. Secondly, we construct a seminormal, locally simplicial, non-positive affine monoid whose Cohen-Macaulayness depends on $K$. This shows that the assumption on the local cohomology in Corollary 5.17 and Proposition 5.18 is necessary. As a notation, for $q = (q_1, \ldots, q_d) \in \mathbb{N}^d$, we write $\text{Supp } q = \{ i \in [d] \mid q_i \neq 0 \}$ and $\deg q = \sum_i q_i$. For background information on Stanley-Reisner rings, see Chapter 1 of [16].

**Proposition 6.4.** Let $\Delta$ be an simplicial complex on the vertex set $[d]$ with Stanley-Reisner ring $\mathbb{K}[\Delta]$. Then there exists a seminormal simplicial positive affine monoid $Q = Q(\Delta)$ of dimension $d$ such that

$$H^i_m(\mathbb{K}[Q])_q = \begin{cases} H^{i-1}_m(\mathbb{K}[\Delta])_q & \text{if } \deg q \text{ is odd and } \text{Supp } q \in \Delta; \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq d - 1$. If $\Delta$ is acyclic, then $\text{depth } \mathbb{K}[Q] = \text{depth } \mathbb{K}[\Delta] + 1$.

If $\Delta$ is the cone over the triangulated projective plane, then this yields an example of a seminormal simplicial affine monoid whose *depth depends on the characteristic.
We construct $Q$ as a submonoid of $\mathbb{N}^d$. Let $Q$ be the set of all elements $q \in \mathbb{N}^d$ such that either $\text{Supp} q \notin \Delta$ or $\deg q$ is even. One can verify directly that this is a seminormal simplicial positive affine monoid. We identify the faces of $Q$ with the subsets of $[d]$. The families of holes then correspond to the facets of $\Delta$.

We compute the local cohomology. Let $q \in \mathbb{Z}Q$ such that $H^i_m(\mathbb{K}[Q])_q \neq 0$. By Corollary 4.4 it follows that $\deg q$ is odd and $\text{Supp} q \in \Delta$. In particular, $q \in -\text{int} F$ for a unique face $F \in \Delta$. The set $\nabla(q)$ contains the faces containing $F$ that are not in $\Delta$, in other words $\nabla(q) = \{ G \subset [d] \mid F \subset G, G \setminus F \notin \text{lk}_\Delta F \}$. Here $\text{lk}_\Delta F = \{ G \subset [d] \mid G \cap F = \emptyset, F \cup G \in \Delta \}$ denotes the link of $F$ in $\Delta$. It follows that $\nabla(q)^\vee$ equals the Alexander dual of the link of $F$. Using (5), Alexander duality ([16, Theorem 5.6]) and Hochster’s Formula ([10, Theorem A.7.3]), we compute

$$H^i_m(\mathbb{K}[Q])_q = \tilde{H}_{d-1-i}(\nabla(q)^\vee, \mathbb{K}) = \tilde{H}_{d-|F|-i-2}(\text{lk}_\Delta F, \mathbb{K}) = H^{i-1}_m(\mathbb{K}[\Delta])_q.$$  

Here, $|F|$ denotes the number of vertices of $F$, so $d - |F|$ is the cardinality of the ground set of $\text{lk}_\Delta F$. In particular, it follows that depth $\mathbb{K}[Q] \geq 1 + \text{depth} \mathbb{K}[\Delta]$.

Now assume that $\Delta$ is acyclic. If $H^{i-1}_m(\mathbb{K}[\Delta]) \neq 0$, then by Hochster’s formula there exists a face $F \subset [d]$, such that $H^{i-|F|-2}_m(\text{lk}_\Delta F, \mathbb{K}) \neq 0$. Because $\Delta$ is acyclic, it holds that $F \neq \emptyset$. So we can find an element $q \in -\text{int} F$ with odd degree. By above computation, it holds that $H^i_m(\mathbb{K}[Q])_q = H^{i-|F|-2}_m(\text{lk}_\Delta F, \mathbb{K}) \neq 0$ and thus depth $\mathbb{K}[Q] \leq 1 + \text{depth} \mathbb{K}[\Delta]$.

**Proposition 6.5.** There exists a seminormal locally simplicial affine monoid $Q$ satisfying $(S_2)$, such that $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if char $\mathbb{K} \neq 2$.

**Proof.** Let $\Delta$ be a simplicial complex on $d$ vertices. We consider $\Delta$ as a subcomplex of the full simplex $\Gamma$ on the $d$ vertices. Assume that $\Delta \subseteq \Gamma$, so $\Delta$ is in fact a subcomplex of the boundary complex of $\Gamma$. Next, we pass to the barycentric subdivision of $\Delta$ and $\Gamma$. This way, we obtain an inclusion $\text{sd}(\Delta) \subset \text{sd}(\Gamma)$, where $\text{sd}(\Delta)$ is homeomorphic to $\Delta$ and $\text{sd}(\Delta)$ is a *vertex-induced subcomplex* of $\text{sd}(\Gamma)$. This means that there is a subset $V$ of the vertices of $\text{sd}(\Gamma)$, such that $\text{sd}(\Delta)$ is the restriction of $\text{sd}(\Gamma)$ to $V$. We now consider the dual $\text{sd}(\Gamma)^\vee$ of $\text{sd}(\Gamma)$. We can realize $\text{sd}(\Gamma)^\vee$ as the boundary complex if a (necessarily) simple polytope $P$. Indeed, $P$ is just the polar of the barycentric subdivision of a simplex.

Let $P'$ be the cone over $P$ with apex $v$. Note that dim $P' = d$. We embed $P'$ as a lattice polytope into $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$ and we write $C \subset \mathbb{R}^{n+1}$ for the cone generated by $P'$ and $Q_1$ for the affine monoid generated by the lattice points in $C$. Note that we have a bijective correspondence between the vertices of $\text{sd}(\Gamma)$ and the facets of $Q_1$ that contain $v$. We define $Q_2 \subset Q_1$ as the subset of all elements of even degree, and those elements $q$ of odd degree, such that every facet containing $q$ and $v$ corresponds to a vertex in the set $V$ defined above. $Q_2$ is seminormal, because we restricted $Q_1$ to subgroups in faces, and it satisfies $(S_2)$, because we did the restriction facetwise. Let $F = Q_2 \cap \mathbb{R}_+v$ be the one-dimensional face of $Q_2$ corresponding to the ray through $v$.

Finally, we set $Q := Q_2, v$. Then $Q$ inherits seminormality and $(S_2)$ from $Q_2$. Moreover, $Q$ is simple, because its faces are in bijection with the faces of the simple polytope $P$ and $\dim Q = d$. Let us compute the local cohomology of $Q$. Fix a $q \in -\overline{Q}$. If $q \notin \overline{Q}$, then by the proof of Proposition 5.18 if holds that $H^i_m(\mathbb{K}[Q])_q = 0$.
for all \( i < \dim Q \). On the other hand, if \( q \in \overline{Q} \cap (-Q) \) and \( q \notin Q \), then by construction \( \nabla(q)^* = \text{sd}(\Delta) \). Hence in this case \( H_{m}((\mathbb{K}[Q])_{q}) = \tilde{H}_{d-1-i}(\Delta, \mathbb{K}) \). So if we choose \( \Delta \) to be triangulated projective plane, then \( \mathbb{K}[Q] \) is Cohen-Macaulay if and only if \( \text{char} \mathbb{K} \neq 2 \).

We would like to point out that instead of the construction in the preceding proof one can also consider the affine monoid \( M \) constructed in [4, Theorem 7.4]. The localization of \( M \) at the vertex \( v \) (in the notation of [4]) yields another affine monoid satisfying the claim of Proposition 6.5. We chose to present the new construction because we consider it as simpler than the one given in [4].

Finally, note that the constructions given in Proposition 6.4 and Proposition 6.5 are optimal in the following sense. Every example of a monoid algebra depending on the characteristic needs to have at least \(*\)dimension 6. Since the triangulated projective plane has 6 vertices, the construction of Proposition 6.5 has the minimal possible \(*\)dimension. The cone of the triangulated projective plane has 7 vertices, so the \(*\)dimension of the affine monoid constructed in Proposition 6.4 has \(*\)dimension 7. But by Corollary 5.17, the \(*\)depth of \( \mathbb{K}[Q] \) is determined by its 6-\(*\)dimensional localizations, so again the \(*\)dimension is minimal.

### 7. Additional results

#### 7.1. Intersection of localizations

In [19] and independently in [14] a combinatorial criterion for Serre’s condition \((S_2)\) is given. Namely, the monoid algebra \( \mathbb{K}[Q] \) satisfies Serre’s condition \((S_2)\) if and only if
\[
Q = \bigcap_{F \text{ facet of } Q} Q_F.
\]
We give a direct proof that this condition is equivalent to our criterion for \((S_2)\) in Theorem 5.2.

**Proposition 7.1.** Let \( Q \) be an affine monoid of \(*\)dimension \( d \). Then \( Q \) satisfies (6) if and only if every family of holes of \( Q \) has \(*\)dimension \( d-1 \).

This follows from the more general

**Proposition 7.2.** For \( 0 \leq i \leq d-2 \) it holds that
\[
\bigcap_{\dim G = i} Q_G \subseteq \bigcap_{\dim G = i+1} Q_G,
\]
where the intersection runs over the faces of \( Q \) of the indicated \(*\)dimension. The inclusion is strict if and only if there exists a family of holes of \(*\)dimension \( i \).

**Proof.** The inclusion is obvious, because \( F \subset G \) implies \( Q_F \subset Q_G \). So we need only to prove the case of equality.

If the inclusion is strict, then we can choose an element \( q \) from the difference of the right and left hand side of (7). For this \( q \), there exists a \( i \)-\(*\)dimensional face \( F \) with \( q \notin Q_F \), but \( q \in Q_G \) for every face \( G \supseteq F \). By Lemma 3.1, this implies that \( p_F \) is a minimal prime over the annihilator of \( x^q \). Thus is associated by Theorem 3.8.

On the other hand, assume there is an \( i \)-\(*\)dimensional associated face \( F \). Then there exists a monomial \( x^q \in \mathbb{K}[Q]/\mathbb{K}[Q] \) with annihilator \( p_F \), in particular \( q \notin Q_F \). Since \( p_F \nsubseteq p_G \) for all faces \( G \) of \(*\)dimension \( i+1 \), it follows from Lemma 3.1 that \( G + q \nsubseteq Q \) and thus \( q \in Q_G \) for all \((i+1)\)-\(*\)dimensional faces \( G \). Hence, \( q \) is contained in the right-hand side, but not in the left-hand side of (7).
Note that \( Q = Q_{F_0} = \bigcap_{\dim F = 0} Q_F \). Therefore, \( Q \) satisfies (6) if and only if all associated faces are facets. Let us add some remarks here. There is a chain of inclusions
\[
Q \subseteq \bigcap_{F \in F_1} Q_F \subseteq \bigcap_{F \in F_2} Q_F \subseteq \ldots \subseteq \bigcap_{F \in F_{d-1}} Q_F \subseteq \overline{Q}.
\]
This chain of inclusions gives rise to a similar chain on the algebra \( \mathbb{K}[Q] \) and also on the quotients modulo \( \mathbb{K}[Q] \). It yields a filtration of \( \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] \) that turns out to be the dimension filtration, as defined in [20]. It follows that if the \( \mathbb{K}[Q] \)-module \( \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] \) is sequentially Cohen-Macaulay, then *depth of \( \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] \) equals the smallest non-zero component in the filtration. In view of Corollary 4.3, this means that the *depth of \( \mathbb{K}[Q] \) is one more than the smallest *dimension of a family of holes.

7.2. The biggest family of holes. We have seen that the smallest *dimension of a family of holes gives an upper bound for the *depth. Moreover, in some cases, equality holds. As a supplement to this, we give an interpretation of the greatest *dimension of a family of holes.

**Proposition 7.3.** Let \( Q \) be a non-normal affine monoid. The following numbers are equal:

1. The maximal *dimension of a family of holes of \( Q \).
2. The minimal *dimension \( j \) such that all localizations \( Q_F \) at faces of *dimension strictly greater than \( j \) are normal.

If \( Q \) is homogeneous, then this number equals the degree of the difference between the Hilbert polynomials of \( \mathbb{K}\{\overline{Q}\} \) and \( \mathbb{K}[Q] \). If \( Q \) satisfies (\( R_1 \)), then this number equals the maximal \( i < d - 1 \) such that \( H^{i+1}_m(\mathbb{K}[Q]) \neq 0 \).

See [21, Theorem 13.12] for a variant of this result (stated without proof).

**Proof.** The first claim is immediate from Proposition 3.10. For the statement about the Hilbert polynomials, note that the mentioned difference is just the Hilbert polynomial of \( \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] \). The statement about the local cohomology follows from Corollary 4.4 and Proposition 4.5, or more directly from the exact sequence in the proof of Corollary 4.3.

7.3. A criterion for normality. From our proof of Theorem 4.2 we can extract a curious characterization of normal affine monoids.

**Proposition 7.4.** An affine monoid \( Q \) is normal if and only if \( \mathbb{K}\{\overline{Q}\} \) has finite projective dimension over \( \mathbb{K}[Q] \).

**Proof.** If \( Q \) is normal, then \( \mathbb{K}[Q] = \mathbb{K}\{\overline{Q}\} \) is free and thus has projective dimension 0. On the other hand, assume that \( \mathbb{K}\{\overline{Q}\} \) has finite projective dimension. The proof of Theorem 4.2 shows in particular that \( \mathbb{K}\{\overline{Q}\} \) is a Cohen-Macaulay \( \mathbb{K}[Q] \)-module. Therefore, it holds that *depth of \( \mathbb{K}\{\overline{Q}\} \) = *dim \( \mathbb{K}\{\overline{Q}\} \) = *dim \( \mathbb{K}[Q] \). Hence, by the graded Auslander-Buchsbaum Formula\(^6\) we have *depth of \( \mathbb{K}\{\overline{Q}\} \) + pd \( \mathbb{K}\{\overline{Q}\} \) = *depth of \( \mathbb{K}[Q] \) ≤ *dim \( \mathbb{K}[Q] \) = *depth of \( \mathbb{K}\{\overline{Q}\} \). It follows that the projective dimension of \( \mathbb{K}\{\overline{Q}\} \) over \( \mathbb{K}[Q] \) is zero, thus \( \mathbb{K}\{\overline{Q}\} \) is a projective \( \mathbb{K}[Q] \)-module. But \( \mathbb{K}[Q] \) is a *local ring, so graded projective modules are free.

\(^6\) Again, the graded Auslander-Buchsbaum Formula can be derived from its local version (cf. [3, Theorem 1.3-3]) by localization at the *maximal ideal.

It remains to show that \( \mathbb{K}\{Q\} \) has rank 1 over \( \mathbb{K}\{Q\} \). Assume the contrary. Then we can find two \( \mathbb{K}\{Q\} \)-linearly independent elements \( x^{p_1}, x^{p_2} \in \mathbb{K}\{Q\} \). There exist an element \( q \in Q \) such that \( p_1 + q, p_2 + q \in Q \), cf. [2, Prop 2.33]. Thus \( x^{p_1}q x^{p_2} - x^{p_2}q x^{p_1} = 0 \) with \( x^{p_1}q, x^{p_2}q \in \mathbb{K}\{Q\} \), a contradiction to our assumption. \( \square \)

7.4. Regularity of seminormal affine monoids. Let \( Q \) be an affine monoid which is homogeneous, i.e. it admits a generating set such that all generators are contained in a common affine hyperplane. In this case \( \mathbb{K}\{Q\} \) carries a natural \( \mathbb{Z} \)-grading such that all generators are of degree 1. The Castelnuovo-Mumford regularity is defined as

\[
\text{reg} \mathbb{K}\{Q\} := \max \{ i + j \mid H^i_m(\mathbb{K}\{Q\})_j \neq 0 \}
\]

If \( q \in Q \) is an element in the interior of \( Q \), then it is known that \( H^d_m(\mathbb{K}\{Q\})_{−q} \neq 0 \). This gives us a lower bound on the regularity: If \( m \in \mathbb{N} \) denotes the smallest degree of an interior element of \( Q \), then \( \text{reg} \mathbb{K}\{Q\} \geq d − m \).

In the case that \( Q \) is seminormal, it was already noted in [13, Remark 5.34] that the local cohomology vanishes in positive degrees. This implies that \( \text{reg} \mathbb{K}\{Q\} \leq d \).

We get a slightly stronger bound from Theorem 4.7 of [4] (resp. Theorem 5.15), namely the local cohomology vanishes in all non-negative degrees. Thus we get the following bound on the regularity:

**Proposition 7.5.** Let \( Q \) be a homogeneous seminormal affine monoid of dimension \( d \). Then \( \text{reg} \mathbb{K}\{Q\} \leq d − 1 \). If \( Q \) contains an element of degree 1 in its interior, then equality holds.

This generalizes the bound for the normal case in [21, Theorem 13.14] and the bound for the seminormal simplicial case in [17, Theorem 3.14]. A famous open conjecture in commutative algebra is the Eisenbud-Goto conjecture:

**Conjecture 7.6** ([6]). Let \( S \) be the polynomial ring with the standard grading and let \( I \subseteq S \) be a homogeneous prime ideal. Then

\[
\text{reg} S/I \leq \text{mult} S/I - \text{codim} S/I
\]

where \( \text{mult} \) is the multiplicity and \( \text{codim} \) is the codimension.

**Theorem 7.7.** Let \( Q \) be a homogeneous affine monoid. If \( Q \) is seminormal and contains an inner point in degree 1, then Conjecture 7.6 holds for \( \mathbb{K}\{Q\} \).

**Proof.** Let \( d \) be the dimension of \( Q \). By the discussion above, we know that the regularity of \( \mathbb{K}\{Q\} \) is \( d − 1 \). We may assume that \( Q \subseteq \mathbb{Z}^d \) and \( \mathbb{Z}Q = \mathbb{Z}^d \). Let \( \mathcal{P} \) be the convex hull of the elements of degree 1 of \( Q \). So \( \mathcal{P} \) is a \((d−1)\)-dimensional convex polytope. The multiplicity of \( \mathbb{K}\{Q\} \) can be computed as the normalized volume of \( \mathcal{P} \) (cf. [2, Theorem 6.54]). Finally, the codimension of \( \mathbb{K}\{Q\} \) is \( n − d \), where \( n \) is the number of generators of \( Q \). Since every generator of \( Q \) has degree 1, \( n \) is bounded above by the number of lattice points in \( \mathcal{P} \). So the claim follows from the following geometric proposition. \( \square \)

**Proposition 7.8.** Let \( \mathcal{P} \subseteq \mathbb{R}^{d−1} \) be a polytope with integral vertices that has a lattice point in its interior. Let \( N \) be the number of all lattice points in \( \mathcal{P} \). Then the normalized volume of \( \mathcal{P} \) is at least \( N − 1 \).
Proof. Let $p$ be an inner lattice point in $\mathcal{P}$. By Carathéodory’s Theorem, $p$ lies in the convex hull of $d$ other lattice points of $\mathcal{P}$. Every $(d-1)$-subset of these lattice points together with $p$ forms an lattice simplex. Since every lattice simplex has normalized volume of at least 1, the convex hull of the $d$ lattice points has normalized volume of at least $d$. Now we add the other lattice points of $\mathcal{P}$, one after the other. Every time, we get at least one new simplex in the convex hull, to the normalized volume increases by 1. If the number of lattice points in $\mathcal{P}$ is $N$, then the normalized volume is at least $d + (N - d - 1) \cdot 1 = N - 1$. □

There is another proof of Proposition 7.8 using the $\delta$-vector (sometimes called $h^*$-vector) $\delta = (\delta_0, \ldots, \delta_{d-1})$ of the polytope, see [11, p.101]. This is a vector with $d$ non-negative integer entries which sum up to the normalized volume of $\mathcal{P}$. The last entry $\delta_{d-1}$ counts the number of interior lattice points of $\mathcal{P}$ and $\delta_1 + d$ equals the total number of lattice points in $\mathcal{P}$. If $\mathcal{P}$ has an interior lattice point, then $1 \leq \delta_1 \leq \delta_i$ for $2 \leq i \leq d - 2$ (cf. [11, Theorem 36.1]). From this, we compute that

$$\sum_{i=0}^{d-1} \delta_i - (\delta_1 + d - 1) \geq d - 1 + \delta_1 - (\delta_1 + d - 1) \geq 0.$$ 

But this is exactly the claim of Proposition 7.8.

The conclusion of Proposition 7.8 does not hold without the assumption on the existence of an inner point. For example, the triangle with vertices $(0, 0), (1, 0), (0, 2)$ in the plane has four lattice points, but the normalized volume is only 2. So this approach cannot be used to prove Conjecture 7.6 for more general seminormal affine monoids.

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REFERENCES

[1] M.P. Brodmann and R. Y. Sharp. Local cohomology. Cambridge Univ Press, 1998.
[2] W. Bruns and J. Gubeladze. Polytopes, rings, and K-theory. Springer, 2009.
[3] W. Bruns and J. Herzog. Cohen-Macaulay rings. Cambridge Univ Press, 1998.
[4] W. Bruns, P. Li, and T. Römer. On seminormal monoid rings. J. Algebra, 302(1), 2006.
[5] D. Eisenbud. Commutative algebra with a view toward algebraic geometry. Springer, 1995.
[6] D. Eisenbud and S. Goto. Linear free resolutions and minimal multiplicity. J. Algebra, 88(1), 1984.
[7] S. Goto, N. Suzuki, and K. Watanabe. On affine semigroup rings. Japan J. Math, 2(1), 1976.
[8] S. Goto and K. Watanabe. On graded rings. II. Tokyo Journal of Mathematics, 1, 1978.
[9] A. Hatcher. Algebraic topology. Cambridge Univ Press, 2002.
[10] J. Herzog and T. Hibi. Monomial Ideals. Springer, 2011.
[11] T. Hibi. Algebraic combinatorics on convex polytopes. Carslaw publications, 1992.
[12] T. Hibi, A. Higashitani, K. Kimura, and A.B. O’Keefe. Depth of edge rings arising from finite graphs. Proc. Amer. Math. Soc, 139, 2011.
[13] M. Hochster and J. L. Roberts. The purity of the Frobenius and local cohomology. Adv. Math., 21, 1976.
[14] M.-N. Ishida. The local cohomology groups of an affine semigroup ring. In Algebraic geometry and commutative algebra, in honor of M. Nagata, volume 1. Kinokuniya, 1988.
[15] S.B. Iyengar, G.J. Leuschke, A. Leykin, C. Miller, E. Miller, A.K. Singh, and U. Walther. Twenty-four hours of local cohomology. Amer. Math. Soc, 2007.
[16] E. Miller and B. Sturmfels. Combinatorial commutative algebra. Springer, 2005.
[17] M. J. Nitsche. Castelnuovo-Mumford regularity of seminormal simplicial affine semigroup rings. J. Algebra, 368, 2012.
[18] H. Ohsugi and T. Hibi. Normal polytopes arising from finite graphs. J. Algebra, 207(2), 1998.
[19] U. Schäfer and P. Schenzel. Dualizing complexes of affine semigroup rings. Trans. Amer. Math. Soc, 322(2), 1990.
[20] P. Schenzel. On the dimension filtration and Cohen-Macaulay filtered modules. In Commutative algebra and algebraic geometry, volume 206 of Lect. Notes in Pure and Appl. Math. Dekker, 1999.
[21] B. Sturmfels. Gröbner bases and convex polytopes. Amer. Math. Soc, 1996.
[22] N.V. Trung and L.T. Hoa. Affine semigroups and Cohen-Macaulay rings generated by monomials. Trans. Amer. Math. Soc, 1986.
[23] G.M. Ziegler. Lectures on polytopes. Springer, 1995.

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