Anomalous Transport Behavior in Quantum Magnets

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Abstract: Transport behavior characterized by a low-temperature electrical resistivity that displays a power-law behavior \( \rho(T \to 0) \propto T^s \), with an exponent \( s < 2 \), is commonly observed in magnetic materials in both the magnetic and nonmagnetic phases. We give a pedagogical overview of this phenomenon that summarizes both the experimental situation and the state of its theoretical understanding. We also put it in context by drawing parallels with unusual power-law transport behavior in other systems.

Keywords: Strongly correlated electrons, quantum magnets, non-Fermi-liquid transport behavior

1. Introduction

Simple metals are characterized, \textit{inter alia}, by a low-temperature \((T)\) behavior of the electrical resistivity \( \rho \) given by a power-law \( \delta \rho(T \to 0) \propto T^2 \) [1,2], with \( \delta \rho = \rho - \rho_0 \) the temperature-dependent part of the resistivity and \( \rho_0 \) the residual resistivity. This is often considered one of the hallmarks of a Fermi liquid, and a stronger \( T \)-dependence of the form

\[ \delta \rho(T \to 0) = A_s T^s \]

with an exponent \( s < 2 \) is often referred to as “non-Fermi-liquid” (NFL) (transport) behavior, although this designation requires some qualification, as we will discuss in Sec. 5. A prominent example is the linear \( T \)-dependence of the resistivity in the normal phase of hole-doped high-\( T_c \) superconductors near optimal doping [3,4]. Examples in other systems are provided by various ferromagnets with a low Curie temperature. Sato observed a behavior given by Eq. (1) with \( s \approx 1.50 \) to 1.65 in Pd-doped Ni₃Al [5]. Very similar behavior was found in pressure-tuned Ni₃Al [6], as well as in pressure-tuned ZrZn₂ [7]. Another example is provided by the helical magnet MnSi, which shows a very clean \( s = 3/2 \) behavior in a temperature range from a few mK to several K [8]. The measured resistivities of ZrZn₂ and MnSi are shown in Fig. 1 as representative examples. We note that the anomalous transport behavior in ZrZn₂ is observed in both the ordered and the disordered phases, whereas in MnSi it shows only in the disordered phase. In the helically ordered phase of MnSi one observes \( \delta \rho \propto T^2 \), albeit with a large prefactor \( A_2 \), an observation we will come back to.

Surprisingly, this anomalous transport behavior is far from being completely understood, despite having been observed for many decades in many different materials. In this paper we provide a pedagogical overview of this problem and the solutions that have been proposed.

2. Soft modes as the origin of power-law relaxation rates

It is intuitively plausible that any power-law behavior of relaxation rates, including those that determine transport coefficients, requires the scattering of conduction electrons by soft or massless excitations, i.e., excitations whose characteristic frequency vanishes in the limit of vanishing wave number: A gapped excitation, whose frequency remains nonzero in this limit, will get frozen out at temperatures small compared to the gap and produce an exponentially small relaxation rate. This can be demonstrated explicitly by means of some very simple and general arguments.
Figure 1. Left panel: Observed temperature-pressure phase diagram of ZrZn$_2$, with the false colors indicating the value of the resistivity exponent $s$. The white lines represent lines of second order (solid) and first order (dashed) transitions between the ferromagnetic (FM) and paramagnetic (PM) phases, and TCP denotes the tricritical point where the order of the transition changes. After Fig. 2 in Ref. [7]; this version taken from Ref. [9]. Right panel: Measured resistivity of MnSi in the nonmagnetic phase. After Ref. [8].

As a very simple schematic example, consider noninteracting electrons described by an action $S_0[\bar{\psi}, \psi]$. $\bar{\psi}(x)$ and $\psi(x)$ are fermionic fields, $x \equiv (\vec{x}, \tau)$ comprises the real-space position $\vec{x}$ and the imaginary-time variable $\tau$, and we suppress discrete degrees of freedom such as spin, band indices, etc., in our notation. Let the single-electron energy-momentum relation be $\epsilon_{\vec{k}}$, and denote the chemical potential by $\mu$. The Fermi surface is then characterized by $\xi_{\vec{k}} \equiv \epsilon_{\vec{k}} - \mu = 0$. Consider a generalized electron density $n(x) = \bar{\psi}(x)\psi(x)$, its fluctuations $\delta n(x) = n(x) - \langle n(x) \rangle$, and denote its Fourier transform by $n(k)$, with $k \equiv (\vec{k}, \omega_n)$ a 4-vector that comprises a wave vector $\vec{k}$ and a fermionic Matsubara frequency $\omega_n$. Examples of $n(x)$ are the number density, the spin density, or any other moment of a general phase-space density. In addition, let $\delta N(x)$ be a non-electronic density fluctuation that is governed by a Gaussian action

$$S_{\text{fluct}}[\delta N] = -\frac{1}{2} \int dx \, dy \, \delta N(x) \, \chi^{-1}(x-y) \, \delta N(y)$$

with $\chi$ the physical susceptibility appropriate for the fluctuations $\delta N$, and couples to the electronic density via a short-ranged interaction potential $v:

$$S_{\text{coup}} = \int dx \, dy \, \delta N(x) \, \delta(\tau_x - \tau_y) \, v(\vec{x} - \vec{y}) \, \delta n(y) .$$

An example of $\delta N$ are ionic density fluctuations, in which case $n$ is the electronic number density, $v$ is the screened Coulomb interaction, and $S_{\text{coup}}$ describes the electron-phonon coupling. If we integrate out the fluctuations $\delta N$ we obtain an effective electronic action

$$S_{\text{eff}}[\bar{\psi}, \psi] = S_0[\bar{\psi}, \psi] + \frac{1}{2} \int k \, V(k) \, \delta n(k) \, \delta n(-k)$$

with an effective potential $V(k) = \left( v(\vec{k}) \right)^2 \chi(k)$. Since the potential $v$ is short ranged we can, for the purpose of studying long-wavelength effects, replace this expression by

$$V(k) = C \chi(k)$$

with $C = v^2(\vec{k} = 0)$ a coupling constant. The integration measures in Eqs. (2,3) and (4a), respectively, are $\int dx \equiv \int_V dx \int_0^{1/T} d\tau$ and $\int k = (1/V) \sum_{k} T \sum_{\omega_n}$, with $V$ the system volume. We use units such that the Boltzmann constant $k_B = 1$. 
The effective electron-electron interaction described by the potential $V$ can be interpreted as an exchange of $\delta N$ fluctuations by the electrons. It leads to an electron-self energy that is given, in Hartree-Fock approximation, by

$$\Sigma(p) = \int k V(k) G(p - k)$$

The single-particle relaxation rate $1/\tau_{sp}$, i.e., the inverse quasiparticle lifetime due to the effective interaction, averaged over the Fermi surface, is given by

$$\frac{1}{2\tau_{sp}} = -\frac{1}{N_F V} \sum_p \delta(\xi_p) \Sigma''(\bar{p}, 0)$$

$$= 2N_F \int_{\pm\infty} du \bar{V}''(u) \frac{1}{\sinh(u/T)}$$

(6)

Here $N_F$ is the electronic density of states at the Fermi surface, $\Sigma''(\bar{p}, \omega) = \text{Im} \Sigma(\bar{p}, i\omega_n \to \omega + i0)$ is the spectrum of the self energy, and

$$\bar{V}''(u) = \frac{1}{(N_F V)^2} \sum_{\bar{k}, \bar{p}} \delta(\xi_{\bar{k}}) \delta(\xi_{\bar{p}}) V''(\bar{k} - \bar{p}, u)$$

is the spectrum of the effective potential averaged over the Fermi surface. For simplicity, we ignore the splitting of the Fermi surface in magnets for the time being. We will add this feature, and several important others, in Sec. 4.

2.1. Power-law relaxation rates from exchange of particles

To specify the effective potential $V$, consider a particle-like excitation with a wave-number dependent resonance frequency $\omega_0(\bar{k} \to 0) = c |\bar{k}|^n$, in which case the spectrum of the susceptibility $\chi$ has the form

$$\chi''(\bar{k}, u) \propto |u|^m \text{sgn} (u) \delta(u^2 - \omega_0^2(\bar{k}))$$

(8)

Here $c$ is a stiffness coefficient, we ignore a prefactor that we absorb into the coupling constant $C$, and we neglect any damping of the excitation. The values of the exponents $n$ and $m$ depend on the nature of the particles, we will see examples below.

Performing the wave-number integrals in Eq. (7) we find

$$\bar{V}''(u) \propto |u|^{m+(d-1-2n)/n} \text{sgn} (u)$$

(9)

Via Eq. (6) this leads to

$$1/\tau_{sp} \propto T^{m+(d-1-n)/n}$$

(10)

with $d$ the spatial dimensionality.

These simple considerations illustrate a basic point: The power-law behavior of $1/\tau_{sp}$ hinges on the resonance frequency $\omega_0$ scaling as a power of $|\bar{k}|$ for $\bar{k} \to 0$. This is the defining property of a mode that is soft, or gapless, or massless.

As a well-known example, consider longitudinal phonons. In this case, $\delta n$ and $\delta N$ are the electronic and ionic number density fluctuations, respectively, and $v$ is a screened Coulomb interaction. The susceptibility $\chi$ has the same form as in a classical fluid and is characterized by $m = 2$ and $n = 1$ [10]. We thus have $1/\tau_{sp} \propto T^d$. In $d = 3$ this is the well-known $T^3$ law for the single-particle relaxation rate due to phonons [11].

We note that we have made several simplifying assumptions so far, in addition to the assumption of a single Fermi surface mentioned above. First, we have considered only the single-particle relaxation rate, rather than the more complicated transport rate $1/\tau_T$ which determines the electrical resistivity. (The single-particle rate does, however, have the same $T$-dependence as the thermal resistivity, at
least at the level of the Boltzmann equation [1].) Second, we have assumed an isotropic resonance frequency that depends only on the magnitude of the wave vector. Third, we have ignored the effects of quenched disorder, which is always present in real materials, if possibly only very weakly. Relaxing this constraint is important in order to understand the experimental results we are interested in; we will discuss this in Sec. 4.

2.2. Power-law relaxation rates from exchange of unparticles

Another possibility is the exchange of fluctuations that are described by a continuous spectrum that is scale invariant but lacks the resonance peak characteristic of particles:

\[ \chi''(\vec{k}, u) \propto u^m/|\vec{k}|^n \] (11)

Spectra of this type are familiar from condensed-matter physics; the most common example is the Lindhard function [12]. In a particle-physics context such excitations have been dubbed ‘unparticles’ [13]. The wave-vector integrals in Eq. (7) then simply lead to a prefactor, and the single-particle and transport rates are the same except for a prefactor of \( O(1) \), \( 1/\tau_{\text{sp}} \approx 1/\tau_{\text{tr}} \equiv 1/\tau \). The temperature dependence of either relaxation rate is determined by the exponent \( m \) only, and we have

\[ 1/\tau \propto T^{m+1} \] (12)

An obvious example is the case of Coulomb scattering. In this case \( \delta N \) and \( \delta n \) both represent electronic number-density fluctuations that interact via a screened Coulomb interaction. \( \chi'' \) then is the spectrum of the Lindhard function, and hence \( m = n = 1 \), which leads to \( 1/\tau \propto T^2 \). We note that at the level of quantum electrodynamics, the objects exchanged by the electrons in this example are of course particles, viz., virtual photons. However, at the level of an effective low-energy theory where the microscopic details have been integrated out, the effects of this exchange manifest themselves in the form of a continuous spectrum, viz., the dynamically screened Coulomb interaction.

For later reference we restore the prefactors, which leads to the familiar result for the Coulomb scattering rate

\[ 1/\tau = \pi T^2/2\epsilon_F \] (13)

with \( \epsilon_F \) the Fermi energy. The above derivation is similar in spirit to the one in Ref. [14]. It is remarkable that the argument of interacting density fluctuations still works if the fluctuation \( \delta N \) that interacts with the electronic fluctuation \( \delta n \) is itself an electronic density fluctuation created by all the other electrons. This aspect was stressed in Ref. [15].

3. Experimental results

For a classification of experimental results that show anomalous transport behavior it is crucial to distinguish between two different cases. In the first case, the anomalous behavior is observed only in a narrow region of the phase diagram, usually in the vicinity of a known or suspected critical point. Its observation thus requires fine tuning. An example is the \( T^{3/2} \) resistivity combined with a logarithmic \( T \)-dependence of the specific-heat coefficient observed near a probable quantum critical point in NbFe\(_2\) [16]. In the second case, the anomalous behavior is generic in the sense that it is observed in large regions of the phase diagram. This distinction is crucial, since critical points necessarily lead to critical fluctuations that can serve as the scale invariant excitations underlying the mechanism discussed in Sec. 2. Another important distinction is between clean systems that contain no or very little quenched disorder, and disordered ones. This is because quenched disorder leads to diffusive electron dynamics that can lead to anomalous transport behavior via well-known mechanisms [17,18]. The anomalous transport behavior that is hardest to understand thus occurs in systems that are clean, as evidenced by a small residual electric resistivity \( \rho_0 \) or a large mean-free path, and show generic anomalous behavior that does not require fine tuning.
Two materials that fall into the latter category are the ferromagnet \( \text{ZrZn}_2 \), and the helimagnet \( \text{MnSi} \). The cleanest samples for either systems have a \( \rho_0 \approx 0.3 \mu\Omega\text{cm} \), and the transition from the magnetic to the nonmagnetic phase can be triggered by applying hydrostatic pressure. The magnetic quantum phase transitions are well established to be first order [19,20], so critical fluctuations are not a viable candidate for explaining the observed transport anomalies. See Ref. [21] and references therein for a review of the magnetic properties of these materials.

The phase diagram of \( \text{ZrZn}_2 \) is shown in Fig. 1. The resistivity exponent \( s \), determined by the slope of a log-log plot of the electrical resistivity, is less than 2 in a large part of the phase diagram, ranging from ambient pressure to twice the critical pressure, and from the lowest temperatures achievable to about 20K. The smallest exponent \( s \approx 1.5 \) was found in a temperature region around 10K in the paramagnetic phase [7]. Data at ambient pressure have been fitted to Eq. (1) with \( s = 5/3 \) for samples with residual resistivities between 0.3\( \mu\Omega\text{cm} \) and 6.4\( \mu\Omega\text{cm} \), while a magnetic field of 9T restores a \( T^2 \) behavior [22]. The respective prefactors are \( A_{5/3} \approx 0.021 \mu\Omega\text{cm}/K^{5/3} \) and \( A_2 \approx 0.003 \mu\Omega\text{cm}/K^2 \). We will discuss interpretations of this behavior in Secs. 4 and 5, where we will show that an equally good fit of the data is obtained by a superposition of \( s = 3/2 \) and \( s = 2 \).

\( \text{MnSi} \) is a helimagnet with a rather long helical pitch wavelength of about 180Å [23]. Hydrostatic pressure destroys the helical order [20] and drives the system into a phase with no long-range magnetic order. There is, however, evidence for strong fluctuations in the nonmagnetic phase [24]. The phase diagram in the temperature-pressure plane is shown in the left panel of Fig. 2. The magnetic phase transition at low temperatures is first order, as is generically the case in clean metallic ferromagnets and long-wavelength helimagnets; for a review of the magnetic properties see Ref. [21]. Throughout the nonmagnetic phase, from the critical pressure \( p_c \approx 15\text{kbar} \) to about 50 kbar, and over a temperature range from a few mK to almost 10K, the electrical resistivity displays a \( T^{3/2} \) behavior with a prefactor ranging from \( A_{3/2} \approx 0.1 \mu\Omega\text{cm}/K^{3/2} \) at high pressure to 0.22\( \mu\Omega\text{cm}/K^{3/2} \) near the critical pressure [8,25], see Figs. 1, 2. In the helical phase the electrical resistivity shows a \( T^2 \) behavior with a prefactor \( A_2 \approx 0.03 \mu\Omega\text{cm}/K^2 \) at ambient pressure that rises, first gradually and eventually sharply, to \( A_2 \approx 0.12 \mu\Omega\text{cm}/K^2 \) as the critical pressure is approached from below [25]. We note that these prefactors
are surprisingly large. They are larger by a factor of 10 compared to their counterparts in ZrZn$_2$, and larger by many orders of magnitude compared to the Coulomb scattering contribution given by the Drude formula in conjunction with the scattering rate in Eq. (13). The $T^2$ behavior in the helical phase is thus as anomalous as the $T^{3/2}$ behavior in the disordered phase, even though the exponent value happens to coincide with the one characteristic of ordinary Fermi-liquid behavior.

In a magnetic field, MnSi has a phase known as the A-phase that consists of a skyrmionic spin texture with the cores of the skyrmions forming a hexagonal lattice of columns in the material [27], see the right panel in Fig. 2. The $T^{3/2}$ behavior of the resistivity in the paramagnetic phase persists in a nonzero field up to the crossover to the field-polarized ferromagnetic region, and neutron scattering has provided evidence for strong fluctuations in the paramagnetic phase [24].

Neither in the case of ZrZn$_2$ nor in that of MnSi is there any reason to believe that either critical fluctuations or diffusive electron dynamics lead to the observed anomalous transport behavior. The explanation thus must lie in generic excitations that are extraneous to the conduction electrons. We will discuss proposals along these lines in Sec. 4.

Another example of generic anomalous transport behavior in quantum magnets is provided by the isostructural compounds Ni$_3$Al and Ni$_3$Ga, which can be prepared with various Ni concentration around the stoichiometric value. Ni$_3$Al shows ferromagnetic order below 15-41K, depending on the exact composition, and a ferromagnetic-to-paramagnetic quantum phase transition (QPT) can be triggered by means of hydrostatic pressure, see Refs. [21,25] and references therein. The transition is suspected to be first order [6,25], and stoichiometric samples have residual resistivities $\rho_0 \approx 1\,\mu\Omega\text{cm}$. The resistivity exponent is $s \geq 1.5$ on either side of the transition, see Fig. 3, and the prefactor is $A_1 \approx 0.01\,\mu\Omega\text{cm/K}$ [25]. Similar behavior is observed in (Ni$_{1-x}$Pd$_x$)$_3$Al, which undergoes a ferromagnetic QPT at $x \approx 0.095$ [5]. This transport behavior is very similar to that observed in ZrZn$_2$. Stoichiometric Ni$_3$Ga is paramagnetic and remains so for Ni-poor compositions, but it has a ferromagnetic ground state for Ni-rich compositions. In the ferromagnetic samples $s \approx 1.5$ with a prefactor $A_3/2 \approx 0.04\,\mu\Omega\text{cm/K}^{3/2}$, whereas $s = 2$ in the paramagnetic samples with $A_2 \approx 0.001\,\mu\Omega\text{cm/K}^2$, see Ref. [25] and references therein. This situation is the reverse of the one in MnSi, where $s = 2$ in the magnetically ordered phase and $s = 1.5$ in the disordered phase.

![Figure 3. Left panel: Electrical resistivity of Ni$_3$Al plotted vs. $T^{3/2}$ for pressure values below, close to, and above the critical pressure $p_c$. After Fig. 4 in Ref. [6]. Right panel: Resistivity exponent $s$ for Ni$_3$Al with a range of Ni concentrations. Data from Ref. [29] as replotted in Ref. [25]. After Fig. 3 in Ref. [25].](image)

4. Theoretical explanations

As we have seen in Sec. 2, any explanation of the generic transport anomalies observed in various quantum magnets must involve the scattering of electrons by soft generic excitations. In magnetically ordered phases, obvious candidates are the Goldstone modes that result from the magnetic order. In phases without long-range magnetic order there are two possibilities. Either, strong fluctuations that are remnants of the long-range order may provide scattering mechanisms that can lead to generic transport anomalies. Candidates for such fluctuations have been observed in the nonmagnetic phase
of MnSi [24] and discussed as a possible origin of the observed $T^{3/2}$ behavior [30]. Or, weak quenched disorder may provide droplets of the ordered phase within the disordered one (and vice versa) [31]. This explains the widespread observation of phase separation away from the coexistence curve of a first-order of a first-order phase transition, and it also provides a way for scattering mechanisms that are germane to the magnetic phase to persist in parts of the disordered phase. We will discuss these mechanisms in more detail later in this section, and also in Sec. 5.

For ferromagnets in the ordered phase, the magnon contribution to the resistivity was considered early on and found to produce a $T^2$ behavior [32,33], in agreement with experimental results on Fe, Co, and Ni [34]. Later work confirmed this, and also considered scattering by the continuum of Stoner excitations (another example of the ‘unparticles’ mentioned in Sec. 2) [35,36]. The $T^2$ behavior is valid only above a characteristic temperature that is related to the exchange splitting [37], as we will see explicitly below. The behavior of both the electric and thermal resistivities in various temperature regimes was discussed in Ref. [15]. For helimagnets, the Goldstone modes and their contribution to the scattering rates were derived in Refs. [38–41]. Recently, electron scattering from Goldstone modes in both ferromagnets and helimagnets has been reconsidered, and several new mechanism for anomalous transport behavior have been discussed [9]. In this section we give a summary of the current state of the theory. We focus on three mechanisms that yield a resistivity exponent $s = 3/2$ at least in some temperature regime, for more complete results see Ref. [9]. For completeness, and to clarify some common misconceptions, we also briefly discuss the effects of non-generic critical fluctuations, and the extent to which they exist.

4.1. Scattering by magnetic Goldstone modes

As we mentioned in Sec. 2, the basic considerations presented there need to be generalized and refined in several ways in order to be applicable to magnetic materials. We start with a discussion of clean systems, and then consider the effects of weak disorder.

4.1.1. Clean systems

In the ordered phase of both ferromagnets and helimagnets the effective local magnetic field seen by the conduction electrons leads to an exchange splitting $\lambda$ of the Fermi surface. We thus need to distinguish between intraband scattering, where a magnon is exchanged between electrons in the same sub-band of the exchange-split Fermi surface, and interband scattering, where the exchange is between electrons in different sub-bands. At asymptotically low temperatures in clean systems the latter will always lead to exponentially small rates, as the scattering processes get frozen out for temperatures small compared to the exchange splitting. However, they can provide the leading contribution to scattering in a pre-asymptotic temperature window whose lower limit can be rather low, and thus need to be considered. Furthermore, weak quenched disorder eliminates the exponential suppression, as we will see. In ferromagnets, the magnons do not couple electrons in the same sub-band, and thus interband scattering is the only mechanism available. The effective potential for interband scattering is given by Eq. (7), but with shifted arguments of the $\delta$-functions that reflect the fact that the electrons with wave vector $\vec{k}$ live on a different Fermi surface than those with wave vector $\vec{p}$. The coupling constant $C$ is given by the square of the exchange interaction $\Gamma_t$, and the resulting expression for the single-particle interband scattering rate is

$$\frac{1}{\tau_{sp}} \propto N_F T^2 \int_{-\infty}^{\infty} du \frac{1}{\sinh(u/T)} \frac{1}{N_F^2 V^2} \sum_{\vec{k},\vec{p}} \delta(\xi_{\vec{k}+\vec{p}} + \lambda) \delta(\xi_{\vec{p}} - \lambda) \chi''(\vec{k}, u)$$

(14)
The transport rate is given by the same expression with an additional factor of $\vec{k}^2/k_F^2$ in the integrand, with $k_F$ the Fermi wave number. This is known as the backscattering factor that suppresses large-angle scattering \[1\]. For the transport interband scattering rate we thus have

$$ \frac{1}{\tau_{tr}} \propto N_F k_F^2 \int_{-\infty}^{\infty} du \frac{1}{\sinh(u/T)} \frac{1}{N_F^2 V^2} \sum_{\vec{k},\vec{\rho}} (\vec{k}^2/k_F^2) \delta(\vec{k}+\vec{\rho} - \lambda) \delta(\vec{k} + \lambda) \chi''(\vec{k}, u) \tag{15} $$

4.1.2. Systems with weak disorder

Quenched disorder, however weak, is present in all real materials and leads to a nonzero scattering rate $1/\tau_0$ even at $T = 0$ and a corresponding residual resistivity $\rho_0$. The cleanest samples of the magnetic systems discussed here have residual resistivities of a few tenth of a $\mu\Omega cm$. While very clean by the standards of these compounds, these values are large compared to the residual resistivities of many nonmagnetic metals (e.g., the residual resistivity of commercial Cu wire is less than 1 n$\Omega cm$). This motivates the consideration of disorder in the ballistic or weak-disorder regime \[42\], which in magnets is characterized by the condition $\lambda \tau_0 \gg 1$ \[41\]. A rigorous treatment requires elaborate diagrammatic calculations, but the net effect can be described by using simple heuristic arguments \[9\].

Consider the expression for the clean single-particle rate in Eq. (14). Performing the wave-number convolution integral yields

$$ \frac{1}{N_F V} \sum_{\vec{k}} \delta(\vec{k}+\vec{\rho} - \lambda) \delta(\vec{k} + \lambda) \propto \int_{-1}^{1} d\eta \delta(k v_F \eta - 2\lambda) = \frac{1}{v_F k} \Theta(\vec{k} - 2\lambda/v_F) \tag{16} $$

where $v_F$ is the Fermi velocity. The step function leads to the exponential suppression of the rates at asymptotically low temperatures mentioned above \[15,37\]. Weak disorder replaces the $\delta$-function with a Lorentzian, and in the limit $v_F |\vec{k}|/\lambda \ll 1, \lambda \tau_0 \gg 1$ the step function gets replaced by

$$ \frac{1}{v_F |\vec{k}|} \Theta(\vec{k} - 2\lambda/v_F) = \int_{-1}^{1} d\eta \delta(k v_F \eta - 2\lambda) \rightarrow \int_{-1}^{1} d\eta \frac{1/\tau_0}{(v_F |\vec{k}| \eta - 2\lambda)^2 + 1/\tau_0^2} \approx \frac{1}{\lambda^2 \tau_0} \tag{17} $$

The disorder thus eliminates the lower cutoff for the $\vec{k}$-integral and leads to an extra factor of $v_F |\vec{k}|/\lambda^2 \tau_0$ in the integrand. Since $\vec{k}$ scales as a positive power of the temperature, this implies that the power-law $T$-dependence of the single-particle rate is weaker than in the corresponding clean system, but extends to zero temperature.

In the case of the transport rate, the same arguments apply, but in addition the disorder eliminates the backscattering factor since it leads to more isotropic scattering. The effective extra factor in the integrand thus is $(\epsilon_F/\lambda^2 \tau_0)k_F/|\vec{k}|$, and the disorder strengthens the $T$-dependence of the rate, in addition to eliminating the exponential suppression at asymptotically low $T$. The single-particle rate and the transport rate thus are qualitatively the same and given by

$$ \frac{1}{\tau_{sp}} \propto \frac{1}{\tau_{tr}} \propto \frac{N_F k_F^2}{\lambda^2 \tau_0} \int_{-\infty}^{\infty} du \frac{1}{\sinh(u/T)} \frac{1}{N_F V} \sum_{\vec{k}} \chi''(\vec{k}, u) \tag{18} $$

$1/\tau_{tr}$ and $1/\tau_{sp}$ determine the electrical and thermal resistivities via the Drude formula

$$ \rho = m_e/n_e e^2 \tau \tag{19} $$

with $e$, $m_e$, and $n_e$ the electron charge, mass, and density, respectively.
4.2. Application to quantum ferromagnets

Now consider the scattering of electrons by magnons in ferromagnets. The relevant resonance frequency ($\omega_0$ in Sec. 2) is

$$\omega_{\text{FM}}(\vec{k} \to 0) = D\vec{k}^2$$

with $D$ the spin-stiffness coefficient, and the corresponding susceptibility is, apart from a numerical prefactor [10],

$$\chi_{\text{FM}}(\vec{k}, i\Omega) \propto \frac{m_0 D\vec{k}^2}{\omega_{\text{FM}}(\vec{k})^2 - (i\Omega)^2}$$

$m_0$ is the magnetization scale that determines the exchange splitting $\lambda$ via $\lambda = \Gamma t m_0$. Two other relevant energy scales are the largest energy that can be carried by a magnon (i.e., the magnetic equivalent of the Debye temperature)

$$T_1 = Dk_F^2$$

and the smallest energy that can be transferred by means of magnon exchange,

$$T_0 = Dk_0^2 \approx T_1 (\lambda/\epsilon_F)^2$$

where $k_0 = \lambda/v_F$.

For clean ferromagnets, Eqs. (14, 15) yield the results of Refs. [15,35,37], viz., a $T\ln T$ and $T^2$ behavior for $1/\tau_{\text{sp}}$ and $1/\tau_{\text{tr}}$, respectively, for $T_0 < T < T_1$, and exponentially small rates for $T < T_0$. In the presence of ballistic quenched disorder, Eq. (18) leads to both rates scaling as $T^{3/2}$, which results in a resistivity contribution

$$\delta \rho_{\text{FM}} = A_{3/2}^{\text{FM}} T^{3/2}$$

with a prefactor

$$A_{3/2}^{\text{FM}} = \gamma_1 \rho_0 / T_1 \sqrt{T_0}$$

where $\gamma_1$ is a numerical prefactor. Since the single-particle rate has the same $T$-dependence as the transport rate, this result also holds for the thermal resistivity, only the numerical prefactor is different. It is valid for $T_{\text{ball}} \ll T \lesssim T_0$, with

$$T_{\text{ball}} = T_1 / (\epsilon_F T_0)^2$$

The lower limit on the temperature window is dictated by the constraints on the ballistic disorder regime [9]. For lower temperatures, the electron dynamics are diffusive.

Parameter values appropriate for ZrZn$_2$ have been estimated in Ref. [9]. The result is a prefactor $A_{3/2}^{\text{FM}} \approx 0.01 \mu \Omega \text{cm}/K^{3/2}$, which is very close to what is observed in this material, see the discussion in Secs. 3 and 5. In order for this mechanism to explain the anomalous transport behavior on either side of the first-order QPT, the droplet formation discussed in Ref. [31] is crucial.

4.3. Application to quantum helimagnets

In helimagnets there are two different soft modes that are candidates for explaining anomalous transport behavior. One are the helimagnons, which are the Goldstone modes of the spontaneously broken symmetry that is present in the helically ordered phase [38]. The other are fluctuations of the columnar skyrmion structure that is observed, e.g., in the A-phase of MnSi, see Fig. 2 and Ref. [27]. Columnar fluctuations are familiar from the theory of liquid crystals [43] and have been studied in the context of helimagnets in Refs. [30] and [44].

4.3.1. Scattering by columnar fluctuations in skyrmionic phases

Consider a hexagonal lattice of columns in the $z$-direction that fluctuate about their equilibrium positions, as shown in Fig. 4. Such fluctuations have an anisotropic dispersion relation, with the resonance frequency scaling linearly with the wave number for wave vectors $\vec{k}_\perp$ perpendicular to the
Figure 4. Hexagonal lattice of columns, and fluctuations about this state. $a$ is the lattice constant, and $\vec{u}(\vec{x})$ is the displacement vector. From Ref. [44].

columns, and quadratic with the wave number for wave vectors in the direction of the columns [43]. If the columnar structure is due to skyrmions comprised by a superposition of three helices with pitch wave number $q$, as proposed in Ref. [27], then the resonance frequency is [9,45]

$$\omega_{\text{sky}}(\vec{k}) = \begin{cases} 
D\sqrt{k_z^4 + \vec{k}_{\perp}^2 q^2} & \text{for } Dq^4/k_F^2 \lesssim \omega_{\text{sky}} \lesssim Dq^2 \\
D(k_F^2/q^2)(k_z^2/q^2 + \vec{k}_{\perp}^2) & \text{for } \omega_{\text{sky}} \lesssim Dq^4/k_F^2
\end{cases} \quad (26)$$

For $\omega_{\text{sky}} \gtrsim Dq^2$ the behavior crosses over to the ferromagnetic one given by Eq. (20). The corresponding susceptibility is

$$\chi_{\text{sky}}(\vec{k}, i\Omega) \propto \frac{m_0}{\omega_{\text{sky}}^2(\vec{k}) - (i\Omega)^2} \begin{cases} 
Dq^2 & \text{for } \omega_{\text{sky}} \gtrsim Dq^4/k_F^2 \\
(k_F/q)^2\omega_{\text{sky}}(\vec{k}) & \text{for } \omega_{\text{sky}} \lesssim Dq^4/k_F^2
\end{cases} \quad (27)$$

In the presence of ballistic disorder, the behavior of the mode in the upper frequency range leads, in conjunction with Eq. (18), to the qualitatively same result for both the electrical and the thermal resistivity as in the ferromagnetic case,

$$\delta\rho_{\text{sky}} = A_{3/2}^{\text{sky}} \rho_0^{3/2} \quad (28a)$$

with a prefactor

$$A_{3/2}^{\text{sky}} = \gamma_2 \rho_0 / T_1 \sqrt{T_0} \quad (28b)$$

where $\gamma_2$ is another numerical factor. This is valid for $\text{Max}(T_{\text{ball}}, T q^2/k_F^2) \lesssim T \lesssim T_q$, with

$$T_q = Dq^2 \quad (29)$$

another energy scale, this one relevant for helimagnets. Since the behavior for $T \gtrsim T_q$ crosses over to the ferromagnetic one, which is the same except for the numerical prefactor, the effective upper limit of the region of validity is the greater of $T_0$ and $T_q$.

For temperatures lower than $T q^2(k_F/q)^2$ the lower frequency regime in Eqs. (26, 27) is relevant, and the $T$-dependence of the resistivity crosses over to a $T^{5/4}$ behavior. However, for helimagnets with a small $q/k_F$ this crossover temperature is extremely low and may not be larger than $T_{\text{ball}}$, so this behavior may not be observable. For instance, an estimate for MnSi yields [9] $T_q \approx 250$ mK, $T_{\text{ball}} \approx 1$ mK. With $q/k_F \approx 0.03$ [23], this yields a crossover temperature of about 0.2 mK, which is lower than $T_{\text{ball}}$ and hence not observable.

With parameter values as appropriate for MnSi, an estimate of the prefactor shows that it is within a factor of 5 within what is observed in the nonmagnetic phase of MnSi [9]. In order for this mechanism
to be operative in that phase there must exist strong columnar fluctuations. There is experimental evidence for this to be the case [24]. A theoretical analysis of the possible nature of this phase was given in Ref. [26].

4.3.2. Scattering by helimagnons

A third mechanism for a $T^{3/2}$ behavior of the electrical resistivity is provided by scattering of electrons by helimagnons, the Goldstone modes of helical order, in clean helimagnets. The dispersion relation and the susceptibility for the helimagnons are [38]

$$\omega_{HM}(\vec{k}) = D \sqrt{q^2 k_z^2 + k_\perp^4}$$

and

$$\chi_{HM}(\vec{k}, i\Omega) \propto \frac{m_0 Dq^2}{\omega_{HM}(\vec{k}) - (i\Omega)^2}$$

Here $q$ is the modulus of the helical pitch wave vector, which we again take to point in the $z$-direction. This is valid for $\omega_{HM} \lesssim Dq^2$, for larger wave numbers the behavior crosses over to the ferromagnetic one. We note that the numerator of the susceptibility is independent of the wave number, whereas in the ferromagnet it is proportional to $k^2$. As a result, the helimagnon susceptibility is softer than the ferromagnon one, even though the Goldstone mode is stiffer in the helimagnet than in the ferromagnet.

Equation (15) now yields a contribution to the electrical resistivity that is given by [9]

$$\delta \rho_{HM} = A_{3/2}^{HM} \frac{T^{3/2}}{T}$$

with

$$A_{3/2}^{HM} = \rho_\lambda \gamma_3 q / k_F T_{1}^{3/2}$$

Here $\rho_\lambda = \lambda m_e / n_e e^2$ is a resistivity scale, and $\gamma_3$ is another numerical factor. This result is valid for $T_0 \lesssim T \lesssim T_q$, provided this window exists; for $T \lesssim T_0$ the rate is exponentially suppressed. In MnSi the window does not exist since $T_q < T_0$ [9].

In the presence of ballistic disorder, Eqs. (30, 31) in conjunction with Eq. (18) yields a stronger $T$-dependence, viz., $\delta \rho_{HM} \propto T \ln T$. At sufficiently low temperatures the observed $T^2$ behavior of unknown origin is predicted to cross over to this behavior, see Ref. [9] and the discussion in Sec. 5 below.

4.4. Scattering by critical fluctuations

By now it is well established, both theoretically and experimentally, that the QPT in clean metallic ferromagnets is generically first order [21]. However, historically it was believed that the transition is second order and hence is accompanied by critical fluctuations [47]. We briefly discuss the influence of these fluctuations on the resistivity, for two reasons: (1) The transition can be weakly first order in some materials, and critical fluctuations may be observable in a pre-asymptotic regime, and (2) early work concerning the critical behavior has influenced the analysis of experiments even in cases where later studies found a clear first-order transition.

Mathon [46] found a resistivity exponent $s = 5/3$ due to ferromagnetic quantum critical fluctuations even before Hertz [47] developed a renormalization-group treatment of QPTs. The same result was obtained by Millis [48] with renormalization-group techniques; for a discussion of how this fits into a general scaling description of QPTs see Ref. [49]. An exponent $5/3 \approx 1.67$ can be experimentally hard to distinguish from $3/2$, especially if there are various competing power-law contributions to the resistivity that hold only in temperature windows of limited sizes, see Fig. 5 and the related discussion. Furthermore, the critical fluctuations, if any, will be present only in a rather limited region of the phase diagram and cannot explain observations of anomalous transport behavior
far from any phase transition. Still, they may well contribute in parts of the phase diagram in materials
where the QPT is weakly first order.

Sufficiently strong quenched disorder (strong enough to lead to diffusive electron dynamics)
changes the nature of the ferromagnetic QPT from first to second order. This was predicted theoretically
[50–52] and recently observed in Fe-doped MnSi [53]. The asymptotic critical behavior at this quantum
critical point is unusual and very hard to observe [51], but in a pre-asymptotic region an effective
power-law behavior with an exponent $s = 3/11$ has been predicted [49,52].

5. Discussion

We conclude by discussing various additional points and open problems.

5.1. Fermi liquids and non-Fermi liquids

The anomalous transport behavior we have discussed in this paper is often referred to as
non-Fermi-liquid (NFL) behavior. However, this term has multiple meanings. Originally devised
to describe the low-temperature behavior of fermions with a short-ranged interaction, such as He$^3$
[54–56], Landau’s Fermi-liquid theory was generalized to electrons with a long-ranged Coulomb
interaction by Silin [57]. The chief concept of Fermi-liquid theory is the existence of quasiparticles
that are continuously related to the single-particle excitations in a Fermi gas. Accordingly, the term
NFL is often used to refer to systems where the interactions are so strong that they destroy the Landau
quasiparticles. Examples are the Luttinger liquid [58], and the marginal Fermi liquid [59] where the
destruction is only logarithmic. A more readily observable feature of a Fermi liquid is an electrical
(and thermal) resistivity that has a $T^2$ temperature dependence for $T \to 0$ due to Coulomb scattering,
see Sec. 2.2. (We have, however, glossed over various complications, see Ref. [60].) Systems where
this is not the case are also often referred to as NFLs. However, it is important to keep in mind
that NFL transport behavior in this sense does not imply that no Landau quasiparticles exist, it may
merely mean that there are soft excitations that scatter the conduction electrons more strongly than
the Coulomb interaction does. We have discussed three examples of such excitations that are generic,
namely, ferromagnons, columnar fluctuations, and helimagnons, and one that requires fine tuning,
namely, ferromagnetic critical fluctuations.

5.2. Mechanism for generic scale invariance

There are a limited number of mechanisms that lead to generic soft modes, and generic scale
invariance, in many-particle systems. Three common ones are: (1) Spontaneously broken continuous
symmetries that lead to Goldstone modes, (2) conservation laws, and (3) gauge symmetries, see
Ref. [61] for a comprehensive discussion. The three examples we have discussed all belong to the
first category. They all are two-particle excitations, i.e., correlation functions of four fermion fields.
In clean fermion systems the single-particle excitations described by the Green function are also
soft. References [21,61] also discussed how rare regions in systems with quenched disorder fit into
the classification scheme of generic scale invariance. This scarcity of generic soft modes, especially
ones that can lead to a linear $T$-dependence of the electrical resistivity, is part of the motivation for
suggestions that a hidden quantum critical point underlies the “strange-metal” normal state of high-$T_c$
superconductors (for a discussion see, e.g., [62]). There is currently no consensus on the origin of this
behavior. The phenomenological marginal-Fermi-liquid description of Ref. [59] is a generic mechanism
that yields a resistivity exponent $s = 1$, but the microscopic origin of the marginal Fermi liquid is not
clear.

5.3. Uniqueness, or lack thereof, of the resistivity exponent

It is important to note that generically there are many competing contributions to the
$T$-dependence of the resistivity, and usually more than one are of comparable strength in any given
temperature regime. Examples of a well-defined exponent $s$ over a sizable temperature range, such as
Figure 5. Resistivity data (blue dots) of ZrZn$_2$ vs. $T^{3/2}$ at ambient pressure. Data (blue dots) from Fig. 4 of Ref. [7]. The solid red line is a fit using Eq. (33) with $s = 3/2$, $A_{3/2} = 0.021 \, \mu\Omega cm/K^{3/2}$ and $A_2 = 0.0033 \, \mu\Omega cm/K^2$. The dashed green line is a pure $T^{3/2}$ fit with $A_{3/2} = 0.027 \, \mu\Omega cm/K^{3/2}$. The inset shows a pure $T^{5/3}$ fit with $A_{5/3} = 0.0215 \, \mu\Omega cm/K^{5/3}$. On the scale of the figure, this fit is indistinguishable from the red line in the main figure.

$s = 3/2$ in MnSi, or $s = 1$ in high-$T_c$ superconductors, are rare and suggest one strongly dominant scattering mechanism in these systems. More commonly, the value of $s$ is less well defined and changes as a function of $T$, see the experimental data for ZrZn$_2$ in Fig. 1 and Ni$_3$Al in Fig. 3. Qualitatively, this is easy to understand from a slight extension of the discussion we have given in Sec. 4. We have focussed on scattering mechanisms that result in $s = 3/2$, however, a more complete analysis shows that there are various mechanisms in various temperature windows that lead to values of $s$ between $s = 1$ and $s = 2$, see Table I in Ref. [9].

To illustrate this point, let us discuss the behavior of ZrZn$_2$ at ambient pressure in more detail. References [7] and [22] found that the behavior of the electrical resistivity between 1K and about 15K is well described by Eq. (1) with $s = 5/3$ and $A_{5/3} \approx 0.02 \, \mu\Omega cm/K^{5/3}$. However, in general one would always expect a $T^2$ contribution (of Fermi-liquid origin or otherwise) that is additive to the leading contribution with $s < 2$. One should thus write

$$\delta \rho(T \to 0) = A_s T^s + A_2 T^2 + o(T^2)$$

References. [7,22] used $s$ and $A_s$ as given above, and $A_2 = 0$, and obtained a good fit. In Fig. 5 we reproduce this fit (in the inset) and compare it with a fit that uses Eq. (33) with $s = 3/2$, $A_{3/2} = 0.021 \, \mu\Omega cm/K^{3/2}$, and $A_2 = 0.0033 \, \mu\Omega cm/K^2$ (solid red line in the main figure). There are at least two physical motivations for this: (1) In Sec. 4 we have identified several scattering mechanisms that lead to $s = 3/2$. (2) There is no reason to believe that $A_2 = 0$. Indeed, Ref. [22] found a $T^2$ behavior in a magnetic field of 9T with a prefactor that is very close to the one used for the fit in Fig. 5. An obvious explanation is that the magnetic field gaps out the magnons, which eliminates the scattering mechanism that produces $s = 3/2$, and leaves a $T^2$ mechanism of unknown origin behind. It then is natural to assume that this $T^2$ mechanism is also present in zero field and needs to be taken into account. It is very interesting that the resulting fit, the solid red line in the main figure, is of equal quality as the pure $T^{5/3}$ fit shown in the inset. Indeed, if plotted on top of one another the two fits are indistinguishable on the scale of the figure. We conclude that the data by themselves cannot distinguish between a pure $T^{5/3}$ behavior and a $T^{3/2}$ behavior with a $T^2$ correction. A pure $T^{3/2}$ behavior, on the
other hand, gives a good fit only in a much more limited temperature regime, see the green dashed line in the figure.

In this context of multiple scattering mechanisms we also stress again that a resistivity exponent $s = 2$ does not necessarily imply that the transport behavior is conventional. For instance, the very large value of the prefactor $A_2$ observed in the helically ordered phase of MnSi cannot be explained by any known scattering mechanism. A related point is that a scattering mechanism leading to a smaller value of $s$ may not dominate over one leading to a larger value unless one goes to very low temperatures, as the crossover temperature obvious depends on the ratio of the prefactors. In the helical phase of MnSi, the helimagnons (in the form of the $T \ln T$ contribution mentioned at the end of Sec. 4.3.2) must manifest themselves at sufficiently low temperatures, but an estimate in Ref. [9] suggests that they will dominate over the unknown scattering mechanism leading to $s = 2$ only for temperatures below about 30 mK. Transport measurements in the ordered phase of MnSi to check this prediction would be very interesting.

5.4. Quenched disorder

An important component of the discussion of systems with weak quenched disorder in Sec. 4.1.2 was that the disorder suppresses the backscattering factor in the expression for the transport relaxation rate. This is a qualitative argument, and a more detailed theoretical analysis of the disorder dependence of the backscattering factor, and the related crossover in the $T$-dependence of the electrical resistivity, is desirable. The same is true for the crossover from the ballistic or weak-disorder regime to the strong-disorder regime that is characterized by diffusive electron dynamics. On the experimental side, a more detailed characterization of disorder, and how to quantify its presence, would be desirable. The residual resistivity may be a rather crude measure of disorder. For instance, there is experimental evidence for inhomogeneities in pressure-tuned systems that are not necessarily reflected in transport experiments and thus can be present even in systems with a rather small residual resistivity [63].

Author Contributions: Both authors have contributed equally to all aspects of this study.

Funding: This work was supported by the NSF under grant numbers DMR-1401449 and DMR-1401410. Part of this work was performed at the Aspen Center for Physics, supported by the NSF under Grant No. PHYS-1066293.

Acknowledgments: We thank Achim Rosch, Ronojoy Saha, and Sripoorna Bharadwaj for collaborations, and Andrey Chubukov and Arnulf Möbius for discussions.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations
The following abbreviations are used in this manuscript:

| Acronym | Definition                        |
|---------|----------------------------------|
| NFL     | Non-Fermi Liquid                 |
| QPT     | Quantum Phase Transition         |

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