New Simpson’s Type Estimates for Two Newly Defined Quantum Integrals

Muhammad Raees 1, Matloob Anwar 1, Miguel Vivas-Cortez 2, Artion Kashuri 3, Muhammad Samraiz 4 and Gauhar Rahman 5

1 School of Natural Sciences, National University of Sciences and Technology, H-12, Islamabad 44000, Pakistan; muhammad.raees@sns.nust.edu.pk or raeesquai@gmail.com (M.R.); manwar@sns.nust.edu.pk (M.A.)
2 Escuela de Ciencias Físicas y Matemáticas, Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076, Apartado, Quito 17-01-2184, Ecuador
3 Department of Mathematics, Faculty of Technical Science, University “Ismail Qemali”, 9400 Vlora, Albania; artion.kashuri@univlora.edu.al or artionkashuri@gmail.com
4 Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan; muhammad.samraiz@uos.edu.pk or msamraizuos@gmail.com
5 Department of Mathematics and Statistics, Hazara University, Mansehra 21300, Pakistan; gauhar55uom@gmail.com or drgauhar.rahman@hu.edu.pk

* Correspondence: MJVIVAS@puce.edu.ec

Abstract: In this paper, we give some correct quantum type Simpson’s inequalities via the application of $q$-Hölder’s inequality. The inequalities of this study are compatible with famous Simpson’s 1/8 and 3/8 quadrature rules for four and six panels, respectively. Several special cases from our results are discussed in detail. A counter example is presented to explain the limitation of Hölder’s inequality in the quantum framework.

Keywords: Simpson’s inequality; convex function; $q$-Hölder’s inequality; quantum derivatives; quantum integrals

MSC: 26B25; 26D10; 26D15

1. Introduction

The numerical integration and the numerical estimations of definite integrals is a vital piece of applied sciences. Simpson’s rules are momentous among the numerical techniques. The procedure is credited to Thomas Simpson (1710–1761). Johannes Kepler worked on a similar estimation technique about a century ago, so the algorithm is sometimes referred to as Kepler’s formula. Simpson’s formula uses the three-step Newton–Cotes quadrature rule, similar estimation technique about a century ago, so the algorithm is sometimes referred to as Newton type results. The following are the rules devised by Simpson.

1. Simpson’s 1/3 rule:

$$\int_{\lambda_1}^{\lambda_2} \omega(q) dq \leq \frac{h}{3} \left[ \omega_1 + 4 \omega_2 + 2 \omega_3 + 4 \omega_4 + 2 \omega_5 + \ldots + 4 \omega_n + \omega_{n+1} \right] - \frac{(\lambda_2 - \lambda_1)}{180} h^4 \omega^{(4)}(\Lambda), \quad (1)$$

where $\lambda_1 \leq \Lambda \leq \lambda_2$ and the number of panels are even.

2. Simpson’s 3/8 rule:

$$\int_{\lambda_1}^{\lambda_2} \omega(q) dq = \frac{3h}{8} \left[ \omega_1 + 3 \omega_2 + 3 \omega_3 + 2 \omega_4 + 3 \omega_5 + 3 \omega_6 + \ldots + 3 \omega_n + \omega_{n+1} \right] - \frac{(\lambda_2 - \lambda_1)}{80} h^4 \omega^{(4)}(\Lambda), \quad (2)$$

where $\lambda_1 \leq \Lambda \leq \lambda_2$ and the number of panels are integral multiples of 3.
An exceptionally popular estimation relating to the above rules is called Simpson’s inequality and is presented as follows:

**Theorem 1.** Let \( \varphi : [\Lambda_1, \Lambda_2] \rightarrow \mathbb{R} \) be a fourth-order differentiable function on \((\Lambda_1, \Lambda_2)\), where
\[
\|\varphi^{(4)}\|_\infty := \sup_{\varphi \in (\Lambda_1, \Lambda_2)} |\varphi^{(4)}(\xi)| < \infty,
\]
then
\[
\left| \frac{1}{6} \left[ \varphi(\Lambda_1) + 4\varphi \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + \varphi(\Lambda_2) \right] - \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \varphi(\xi) d\xi \right| \leq \frac{1}{2880} \|\varphi^{(4)}\|_\infty (\Lambda_2 - \Lambda_1)^2. \tag{3}
\]

Lately, many researchers have zeroed in on Simpson’s type inequalities for differentiable classes of convex functions. In particular, a few mathematicians have chipped away at Simpson and Newton type results for convex mappings, as convexity hypotheses are powerful and a solid strategy for tackling incredible number of issues which emerge in numerous parts of applied sciences. For example, Dragomir et al. [1] introduced Simpson type inequalities alongside their applications to quadrature formula in numerical integration. After this beginning, Simpson type inequalities via \( s \)-convex functions were developed by Alomari et al. [2]. Sarikaya et al. [3] proved the variants of Simpson type inequalities dependent upon \( s \)-convexity. Noor et al. [4,5] established some Newton type inequalities for harmonic convex and \( p \)-harmonic convex functions. Alongside these, some new Newton type inequalities for functions whose local fractional derivatives are generalized convex are proven by Iftikhar et al. [6].

In recent decades, several attempts have been made by researchers to obtain the variants and applications of integral inequalities. In the last couple of years, an assortment of novel methodologies has been used by scientists in summing up the traditional results about integral inequalities. One of those approaches is using the ideas of quantum calculus. It is very notable to everybody that quantum math is calculus without limits. Generally, the subject of quantum calculus (shortly, \( q \)-calculus) can be followed back to Euler and Jacobi, yet in the many years hence, it has encountered a quick development [7]. The theory gained popularity during the 20th century after the work of Jackson (1910) on defining an integral later known as the \( q \)-Jackson integral (see previous studies [7–11]). In \( q \)-calculus, the classical derivative is replaced by the \( q \)-difference operator in order to deal with non-differentiable functions (see Almeida and Torres and Cresson et al. [12,13]). Applications of \( q \)-calculus can be found in various branches of mathematics and physics, and the interested readers should consider [14–22] for valuable information. The development of quantum calculus enhanced the interest of researchers to add new ideas in the ongoing theory. Tariboon and Ntouyas [23] presented the idea of quantum integral over the finite interval, acquired a few \( q \)-analogues of traditional mathematical objects and opened a new setting of exploration. For example, the \( q \)-analogue of Hölder’s inequality, Hermite–Hadamard inequality, Ostrowski inequality, Cauchy–Bunyakovsky-Schwarz, Grüss, Grüss–Cebyshev, and other integral inequalities have been developed. Sudsutad et al. [24] and Noor et al. [25] acquired some extensions in the \( q \)-trapezoid type inequalities via first time \( q \)-differentiability. Liu and Zhuang [26] determined some \( q \)-analogues of trapezoid-like inequalities for twice quantum differentiable functions. Zhuang et al. [27] proved some more general \( q \)-analogues of trapezoid-like inequalities for first order quantum differentiable functions. Budak et al. [28] established some new simpson’s inequalities for \( q \)-integrals. For some more nitty gritty review with the application point of perspective, we allude to (see [24–45] and the references therein).

Adding motivation to these outcomes, particularly the outcomes created in [28], we notice that it is feasible to treat quantum integral operators introduced in [23,41] to jointly create some new Simpson type inequalities as in [28]. For this reason, we aim to achieve the following objectives.

1. To obtain a new Simpson’s type inequality depending upon the two newly defined quantum integrals given in Definitions 3 and 5 which is analogous to 1/3 quadrature formula given by (1) for four panels.
2. To extend Simpson’s 3/8 quadrature Formula (2) for six panels in the quantum calculus via indicated quantum integrals.

3. To present a counter example which explains the limiting nature of Hölder’s inequality in the quantum framework of calculus.

4. To re-capture the classical results involving the classical Hölder’s inequality and making comparison with the results due to \(q\)-Hölder’s inequality.

2. Preliminaries

Throughout the paper, let \(W := [\Lambda_1, \Lambda_2] \subset \mathbb{R}\) with \(0 \leq \Lambda_1 < \Lambda_2\) be an interval and \(W^c\) be the interior of \(W\). Assume further that \(0 < q_r < 1\) be a constant.

This section is devoted to the basic and fundamental results in the \(q\)-calculus. We start by collating foundational results and definitions suitable for ongoing study.

**Definition 1.** A function \(\omega : W \rightarrow \mathbb{R}\) is called convex, if the inequality

\[
\omega(q_1 \beta_1 + (1 - q) \beta_2) \leq q \omega(\beta_1) + (1 - q) \omega(\beta_2),
\]

is satisfied for all \(\beta_1, \beta_2 \in W\) and \(q \in [0, 1]\).

Recall that the all-time famous Jackson integral [11] from 0 to an arbitrary real number \(\Lambda\) characterized as follows:

\[
\int_{0}^{\Lambda} \omega(\zeta) d_{q_r} \zeta = (1 - q_r) \Lambda \sum_{j=0}^{\infty} q_r^j \omega(\Lambda q_r^j),
\]

provided the series on the right side converges absolutely. Moreover, he gave the integral for an arbitrary finite interval \([\Lambda_1, \Lambda_2]\) as

\[
\int_{\Lambda_1}^{\Lambda_2} \omega(\zeta) d_{q_r} \zeta = \int_{0}^{\Lambda_2} \omega(\zeta) d_{q_r} \zeta - \int_{0}^{\Lambda_1} \omega(\zeta) d_{q_r} \zeta.
\]

In [23], the authors, while developing some classical inequalities in the quantum framework, studied the concept of \(q_r\)-differentiation and \(q_r\)-integration over the finite interval.

**Definition 2.** For a continuous function \(\omega : W \rightarrow \mathbb{R}\) and \(0 < q_r < 1\), then \(q_r \Lambda_1\)-derivative of \(\omega\) at \(\Lambda \in W\) is expressed by the quotient:

\[
\Lambda_1 D_{q_r} \omega(\Lambda) = \frac{\omega(\Lambda) - \omega(q_r \Lambda + (1 - q_r) \Lambda_1)}{(1 - q_r)(\Lambda - \Lambda_1)}, \quad \Lambda \neq \Lambda_1.
\]

The function \(\omega\) is called \(q_r \Lambda_1\)-differentiable on \(W\), if \(\Lambda_1 D_{q_r} \omega(u)\) exists for all \(u \in W\). It is evident that

\[
\Lambda_1 D_{q_r} \omega(\Lambda_1) = \lim_{\Lambda \to \Lambda_1} \Lambda_1 D_{q_r} \omega(\Lambda).
\]

If \(\Lambda_1 = 0\), then the \(q_r\)-derivative in classical sense [7] is obtained:

\[
D_{q_r} \omega(\Lambda) = \frac{\omega(\Lambda) - \omega(q_r \Lambda)}{\Lambda - q_r \Lambda}.
\]

**Definition 3.** Let \(\omega : W \rightarrow \mathbb{R}\) be a continuous function and \(0 < q_r < 1\). The definite \(q_r \Lambda_1\)-integral of the function \(\omega\) is characterized by the expression

\[
\int_{\Lambda_1}^{\Lambda} \omega(\zeta) d_{q_r} \zeta = (1 - q_r)(\alpha - \Lambda_1) \sum_{j=0}^{\infty} q_r^j \omega(q_r^j \alpha + (1 - q_r^j) \Lambda_1), \quad \alpha \in W.
\]
In the same paper, the authors also proved the following $q_r$-Hölder inequality.

**Theorem 2.** Let $\omega_1, \omega_2 : W \to \mathbb{R}$ be two continuous functions. Then, the inequality

\[
\int_{\Lambda_1} y^{\prime} \left| \omega_1(e) \omega_2(e) \right| \Lambda_1 d_{q_r} e \leq \left( \int_{\Lambda_1} \left| \omega_1(e) \right|^{k_1} \Lambda_1 d_{q_r} e \right)^{\frac{1}{k_1}} \left( \int_{\Lambda_1} \left| \omega_2(e) \right|^{k_2} \Lambda_1 d_{q_r} e \right)^{\frac{1}{k_2}},
\]

holds for all $y \in W$ and $k_1, k_2 > 1$ with $k_1^{-1} + k_2^{-1} = 1$.

In [41], the authors presented an analogous notion of $q_r$-derivatives and $q_r$-integrals by introducing the $q_r^{\Lambda_2}$-derivative and $q_r^{\Lambda_2}$-integrals over the finite real interval $W$.

**Definition 4.** For a continuous function $\omega : W \to \mathbb{R}$ and $0 < q_r < 1$, then $q_r^{\Lambda_2}$-derivative of $\omega$ at $\Lambda \in W$ is defined by the quotient:

\[
^{\Lambda_2} D_{q_r} \omega(\Lambda) = \frac{\omega(\Lambda) - \omega(q_r \Lambda + (1 - q_r) \Lambda_2)}{(1 - q_r)(\Lambda - \Lambda_2)} , \quad \Lambda \neq \Lambda_2,
\]

**Definition 5.** Let $\omega : W \to \mathbb{R}$ be a continuous function and $0 < q_r < 1$. The definite $q_r^{\Lambda_2}$-integral of the function $\omega$ is characterized by the expression

\[
\int_{\beta}^{\Lambda_2} \omega(\xi) \Lambda_2 d_{q_r} \begin{pmatrix} \xi \\ \beta \end{pmatrix} = \frac{1}{\beta - \Lambda} \sum_{\delta = 0}^{\omega} q_r^\delta \omega(q_r^\delta \beta + (1 - q_r^\delta) \Lambda_2), \quad \beta \in W.
\]

**Remark 1.** It is worth mentioning that

(i) the left $q_r \Lambda_1$-derivatives and right $q_r^{\Lambda_2}$-derivatives are not same for general functions defined over the finite real interval $[\Lambda_1, \Lambda_2]$. Indeed, if $\omega(\Lambda) = \Lambda^2$, then

\[
^{\Lambda_2} D_{q_r} \omega(\Lambda) = (1 + q_r)\Lambda + (1 - q_r)\Lambda_2 \neq (1 + q_r)\Lambda + (1 - q_r)\Lambda_1 = \Lambda_1^{\Lambda_2} D_{q_r} \omega(\Lambda).
\]

However,

\[
^{\Lambda_2} D_{q_r} \omega(\Lambda) = \omega(\Lambda) = \Lambda_1^{\Lambda_2} D_{q_r} \omega(\Lambda)
\]

provided that $q_r \to 1^-$.

(ii) The $q_r$-integrals $\int_{\Lambda_1}^{\Lambda_2} \omega(\Lambda) \Lambda_2 d_{q_r} \Lambda$ and $\int_{\Lambda_1}^{\Lambda_2} \omega(\Lambda) \Lambda d_{q_r} \Lambda$ are different for general functions.

For instance,

\[
\int_{\Lambda_1}^{\Lambda_2} \Lambda \Lambda_2 d_{q_r} \Lambda = \frac{\Lambda_2 - k_1}{1 + q_r} [k_1 + q_r \Lambda_2] \neq \frac{\Lambda_2 - k_1}{1 + q_r} [k_1 q_r + \Lambda_2] = \int_{\Lambda_1}^{\Lambda_2} \Lambda k_1 d_{q_r} \Lambda.
\]

Furthermore,

\[
\int_{\Lambda_1}^{\Lambda_2} \Lambda \Lambda_2 d_{q_r} \Lambda = \frac{\Lambda_2^2 - k_1^2}{2} = \int_{\Lambda_1}^{\Lambda_2} \Lambda k_1 d_{q_r} \Lambda,
\]

subject to the condition that $q_r \to 1^-$.

It is also important to notice that, for an integer $n$, the quantum analogue is

\[
[n]_{q_r} = \frac{1 - q_r^n}{1 - q_r} = 1 + q_r + q_r^2 + \ldots + q_r^{n-1}, \quad 1 \neq q_r.
\]
Clearly, the limiting value is \( n \) for \( q_r \to 1^- \). For a detailed survey about the quantum analogues of integers and polynomials we refer to [7].

In [28], the authors presented the following Simpson’s type inequalities.

**Theorem 3.** Let \( \omega : \mathcal{W} \to \mathbb{R} \) be a continuous and \( q_r^{\Lambda_2} \)-differentiable function on \( \mathcal{W}^n \) and \( 0 < q_r < 1 \). If \( |\Lambda_2D_{q_r}\omega| \) is convex and integrable on \( \mathcal{W} \), then

\[
\left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\wp) \Lambda_2d_{q_r}\wp - \frac{1}{6} \left[ \omega(\Lambda_1) + 4\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + \omega(\Lambda_2) \right] \right| \leq (\Lambda_2 - \Lambda_1) \left\{ \left| [M_1(q_r) + M_2(q_r)]\Lambda_2D_{q_r}\omega(\Lambda_1) \right| + \left| [M_3(q_r) + M_4(q_r)]\Lambda_2D_{q_r}\omega(\Lambda_2) \right| \right\},
\]

where

\[
M_1(q_r) := \int_0^1 q_r\wp - \frac{1}{6} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{1 - 2q_r - 2q_r^2}{24(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{1}{3}, \\
\frac{16q_r^3}{216(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{1}{3} \leq q_r < 1, \\
\end{cases}
\]

\[
M_2(q_r) := \int_0^1 q_r\wp - \frac{1}{6} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{1 - 4q_r^3}{24(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{1}{3}, \\
\frac{36q_r + 12q_r^2 + 12q_r + 1}{216(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{1}{3} \leq q_r < 1, \\
\end{cases}
\]

\[
M_3(q_r) := \int_0^1 q_r\wp - \frac{5}{6} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{15 - 6q_r - 6q_r^2}{24(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{5}{6}, \\
\frac{18q_r^2}{216(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{5}{6} \leq q_r < 1, \\
\end{cases}
\]

and

\[
M_4(q_r) := \int_0^1 q_r\wp - \frac{5}{6} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{5 - 8q_r + 8q_r^2 - 8q_r^3}{24(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{5}{6}, \\
\frac{12q_r^2 + 12q_r + 5}{216(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{5}{6} \leq q_r < 1, \\
\end{cases}
\]

**Theorem 4.** Let \( \omega : \mathcal{W} \to \mathbb{R} \) be a continuous and \( q_r^{\Lambda_2} \)-differentiable function on \( \mathcal{W}^n \) and \( 0 < q_r < 1 \). If \( |\Lambda_2D_{q_r}\omega| \) is convex and integrable on \( \mathcal{W} \), then

\[
\left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\wp) \Lambda_2d_{q_r}\wp - \frac{1}{6} \left[ \omega(\Lambda_1) + 3\omega \left( \frac{2\Lambda_1 + \Lambda_2}{3} \right) + \omega(\Lambda_2) \right] \left[ \frac{2\Lambda_1 + \Lambda_2}{3} \right] \right| \leq (\Lambda_2 - \Lambda_1) \left\{ \left| [P_1(q_r) + P_3(q_r) + P_5(q_r)]\Lambda_2D_{q_r}\omega(\Lambda_1) \right| + \left| [P_2(q_r) + P_4(q_r) + P_6(q_r)]\Lambda_2D_{q_r}\omega(\Lambda_2) \right| \right\},
\]

where

\[
P_1(q_r) := \int_0^1 q_r\wp - \frac{1}{8} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{3 - 5q_r - 5q_r^2}{216(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{16q_r^2 + 16q_r - 6q_r^3}{6912(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{3}{8} \leq q_r < 1, \\
\end{cases}
\]

\[
P_2(q_r) := \int_0^1 q_r\wp - \frac{1}{8} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{6 - q_r^2 - 15q_r^3}{216(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{48q_r^2 + 48q_r^2 + 48q_r - 3}{6912(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{3}{8} \leq q_r < 1, \\
\end{cases}
\]

\[
P_3(q_r) := \int_0^1 q_r\wp - \frac{1}{2} (1 - \wp) d_{q_r}\wp = \begin{cases} \frac{9 - 5q_r - 5q_r^2}{54(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } 0 < q_r < \frac{3}{4}, \\
\frac{6q_r^2 + 6q_r - 3}{108(1 + q_r)(1 + q_r + q_r^2)}, & \text{if } \frac{3}{4} \leq q_r < 1, \\
\end{cases}
\]
\[ P_3(q_r) := \int_{\frac{3}{8}}^{1} \frac{1}{3} (1 - \varphi) d_\varphi \varphi = \begin{cases} \frac{5q_r + 5q_r^3 - 9q_r}{54(1+q_r)(1+q_r+q_r^3)}, & \text{if } 0 < q_r < \frac{3}{4}, \\ \frac{6q_r^3 + 3}{108(1+q_r)(1+q_r+q_r^3)}, & \text{if } \frac{3}{4} \leq q_r < 1, \end{cases} \]  

\[ P_4(q_r) := \int_{\frac{1}{3}}^{1} \frac{1}{3} (1 - \varphi) d_\varphi \varphi = \begin{cases} \frac{5q_r + 5q_r^3 - 9q_r}{54(1+q_r)(1+q_r+q_r^3)}, & \text{if } 0 < q_r < \frac{3}{4}, \\ \frac{6q_r^3 + 3}{108(1+q_r)(1+q_r+q_r^3)}, & \text{if } \frac{3}{4} \leq q_r < 1, \end{cases} \]  

and

\[ P_5(q_r) := \int_{\frac{1}{3}}^{1} \frac{1}{3} (1 - \varphi) d_\varphi \varphi = \begin{cases} \frac{5q_r + 5q_r^3 - 9q_r}{54(1+q_r)(1+q_r+q_r^3)}, & \text{if } 0 < q_r < \frac{3}{4}, \\ \frac{6q_r^3 + 3}{108(1+q_r)(1+q_r+q_r^3)}, & \text{if } \frac{3}{4} \leq q_r < 1. \end{cases} \]

Remark 2. If \( q_r \rightarrow 1^- \) in inequality (14), then

\[ \left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) d\varphi - \frac{1}{6} \left[ \omega(\Lambda_1) + 4\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + \omega(\Lambda_2) \right] \right| \leq \frac{5(\Lambda_2 - \Lambda_1)}{72} \{ |\omega'(\Lambda_1)| + |\omega'(\Lambda_2)| \}. \]  

Remark 3. If \( q_r \rightarrow 1^- \) in inequality (19), then

\[ \left| \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) d\varphi - \frac{1}{6} \left[ \omega(\Lambda_1) + 3\omega \left( \frac{2\Lambda_1 + \Lambda_2}{3} \right) + 3\omega \left( \frac{\Lambda_1 + 2\Lambda_2}{2} \right) + \omega(\Lambda_2) \right] \right| \leq \frac{25(\Lambda_2 - \Lambda_1)}{576} \{ |\omega'(\Lambda_1)| + |\omega'(\Lambda_2)| \}. \]

3. Auxiliary Results

We are ready to prove our main results. At start, we present two multi-parameter identities for \( q_r\Lambda_1 \) and \( q_r^{\Lambda_2} \)-differentiable functions which provide some useful inequalities of Simpson’s type.

Lemma 1. Let \( \omega : \mathcal{W} \rightarrow \mathbb{R} \) be a \( q_r\Lambda_1 \) and \( q_r^{\Lambda_2} \)-differentiable function on \( \mathcal{W}^q \) with \( 0 < q_r < 1 \). If \( \Lambda_1 D_q \omega \) and \( \Lambda_2 D_q \omega \) are continuous and \( q_r\Lambda_1 \), \( q_r^{\Lambda_2} \)-integrable functions on \( \mathcal{W} \), then the following identity holds:

\[ \Omega_1(\Lambda_1, \Lambda_2; q_r) = \frac{\Lambda_2 - \Lambda_1}{4} \left[ \int_0^{q_r} R_1(q_r, \varphi) \left( \Lambda_1 D_q \omega \left( \frac{\varphi}{2} \Lambda_2 + \left( 1 - \frac{\varphi}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \omega \left( \frac{\varphi}{2} \Lambda_2 + \left( 1 - \frac{\varphi}{2} \right) \Lambda_1 \right) \right) d_q \varphi \right], \]  

where

\[ R_1(q_r, \varphi) := \begin{cases} q_r \varphi - \frac{1}{6}, & \text{if } 0 \leq \varphi < \frac{1}{2}, \\ q_r \varphi - \frac{3}{6}, & \text{if } \frac{1}{2} \leq \varphi \leq 1, \end{cases} \]

and

\[ \Omega_1(\Lambda_1, \Lambda_2; q_r) := \frac{1}{12} \left[ \omega(\Lambda_1) + 4\omega \left( \frac{3\Lambda_1 + \Lambda_2}{4} \right) + 2\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + 4\omega \left( \frac{\Lambda_1 + 3\Lambda_2}{4} \right) + \omega(\Lambda_2) \right] \]

\[ - \frac{1}{\Lambda_2 - \Lambda_1} \left[ \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) \Lambda_1 D_q \varphi + \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) \Lambda_2 D_q \varphi \right]. \]
**Proof.** By utilizing the property of \( q_r \)-integrals given in Equation (6), we have

\[
\int_0^1 R_1(q_r, \varrho) \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho
\]

\[
= \int_0^1 q_r \varphi \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho
\]

\[
+ \frac{2}{\Lambda_2 - \Lambda_1} \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho
\]

\[
= \frac{\Lambda_1}{6} \int_0^1 \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho. \quad (31)
\]

By the application of Definitions 2 and 4 of \( q_r \)-derivatives, we find

\[
\Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right)
\]

\[
= \left[ \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right] \frac{(\Lambda_2 - \Lambda_1) q_r \varrho}{2}.
\]

Now, incorporating the property of Jackson’s integral given by (5), we have

\[
\int_0^1 \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho
\]

\[
= \frac{2}{\Lambda_2 - \Lambda_1} \left[ \sum_{\beta = 0}^{\infty} \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \sum_{\beta = 0}^{\infty} \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right]
\]

\[
+ \sum_{\beta = 0}^{\infty} \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \sum_{\beta = 0}^{\infty} \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right)
\]

\[
= \frac{2}{\Lambda_2 - \Lambda_1} \left[ \varphi \left( \frac{3 \Lambda_1 + \Lambda_2}{4} \right) \right] - \varphi \left( \Lambda_1 \right) - \varphi \left( \Lambda_2 \right). \quad (32)
\]

Similarly,

\[
\int_0^1 \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho
\]

\[
= \frac{2}{\Lambda_2 - \Lambda_1} \left[ 2 \varphi \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) \right] - \varphi \left( \Lambda_1 \right) - \varphi \left( \Lambda_2 \right). \quad (33)
\]

Finally, we have

\[
\int_0^1 q_r \varphi \left( \Lambda_1 D_q \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right) - \Lambda_2 D_q \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right) \right) dq_r \varrho
\]

\[
= \frac{2}{\Lambda_2 - \Lambda_1} \left[ 2 \varphi \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) \right] - (1 - q_r) \sum_{\beta = 0}^{\infty} \varphi \left( \frac{\varrho}{2} \Lambda_2 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_1 \right)
\]

\[
+ (1 - q_r) \sum_{\beta = 0}^{\infty} \varphi \left( \frac{\varrho}{2} \Lambda_1 + \left( 1 - \frac{\varrho}{2} \right) \Lambda_2 \right)
\]

\[
= \frac{4}{\Lambda_2 - \Lambda_1} \varphi \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) - \frac{4}{(\Lambda_2 - \Lambda_1)^2} \left( \int_{\Lambda_1}^{\Lambda_2} \varphi (\tau) \Lambda_1 d\tau, \tau + \int_{\Lambda_2}^{\Lambda_1} \varphi (\tau) \Lambda_2 d\tau, \tau \right). \quad (34)
\]
The desired equality (28) is obtained by utilizing the Equations (32)-(34) in (31) and multiplying the outcome with $\frac{\Delta u - \Delta v}{4}$. 

**Lemma 2.** Let $\omega : W \to \mathbb{R}$ be a $q_1 v_1$- and $q_2 v_2$-differentiable function on $W^0$ with $0 < q_r < 1$. If $\Lambda_1 D_{q_1} \omega$ and $\Lambda_2 D_{q_2} \omega$ are continuous and $(q_1, q_2)$-integrable functions on $W$, then the following identity holds:

$$
\Omega_2(\Lambda_1, \Lambda_2; q_r) = \frac{\Lambda_2 - \Lambda_1}{4} \left[ \int_0^1 R_2(q_r, e) \left( \Lambda_1 D_{q_1} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_1 \right) \right. \right.
$$

- $\Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e, = \Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e,

where

$$
R_2(q_r, e) := \begin{cases} 
q_r e - \frac{1}{8}, & \text{if } 0 \leq e < \frac{1}{4}, \\
q_r e - \frac{1}{2}, & \text{if } \frac{1}{4} \leq e < \frac{3}{4}, \\
q_r e - \frac{3}{8}, & \text{if } \frac{3}{4} \leq e \leq 1,
\end{cases}
$$

and

$$
\Omega_2(\Lambda_1, \Lambda_2; q_r) := \frac{1}{16} \left[ \omega(\Lambda_1) + 3\omega \left( \frac{5\Lambda_1 + \Lambda_2}{6} \right) + 3\omega \left( \frac{2\Lambda_1 + \Lambda_2}{3} \right) + 2\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) 
$$

+ $3\omega \left( \frac{\Lambda_1 + 2\Lambda_2}{3} \right) + 3\omega \left( \frac{\Lambda_1 + 5\Lambda_2}{6} \right) + \omega(\Lambda_2) \right]$

- $\frac{1}{\Lambda_2 - \Lambda_1} \left[ \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) \Lambda_1 d_q, \varphi + \int_{\frac{\Lambda_1 + \Lambda_2}{2}}^{\Lambda_2} \omega(\varphi) \Lambda_1 d_q, \varphi \right].$

**Proof.** By applying the property of $q_r$-integrals given in equation (6), we have

$$
\int_0^1 R_2(q_r, e) \left( \Lambda_1 D_{q_1} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_1 \right) \right. \right.
$$

- $\Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e, = \Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e,

$$
= \frac{3}{8} \left[ \int_0^1 \left( \Lambda_1 D_{q_1} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_1 \right) \right. \right.
$$

- $\Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e, + \int_0^{\frac{\Lambda_1 + \Lambda_2}{2}} \left( \Lambda_1 D_{q_1} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_1 \right) \right. \right.
$$

- $\Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e, + \int_{\frac{\Lambda_1 + \Lambda_2}{2}}^{\Lambda_2} \left( \Lambda_1 D_{q_1} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_1 \right) \right. \right.
$$

- $\Lambda_2 D_{q_2} \omega \left( \frac{e}{2} \Lambda_1 + \left(1 - \frac{e}{2}\right) \Lambda_2 \right) d_q, e, \right].$

The rest of the proof follows the same approach as in Lemma 1. 

**4. Simpson’s Type Inequalities Related to Simpson’s 1/3 Quadrature Rule Via Four Panels**

In this section, we give Simpson’s type inequalities related to Simpson’s 1/3 quadrature rule via four panels. We start with the following main result.
Theorem 5. Let $\omega : \mathcal{W} \to \mathbb{R}$ satisfy the assumptions of Lemma 1. In addition, if $|\Lambda_1 D_q \omega|$ and $|\Lambda_2 D_q \omega|$ are convex functions, then

$$|\Omega_1 (\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \left( |\Lambda_1 D_q \omega (\Lambda_2)| + |\Lambda_2 D_q \omega (\Lambda_1)| \right) I_1(q_r) + \left( |\Lambda_1 D_q \omega (\Lambda_1)| + |\Lambda_2 D_q \omega (\Lambda_2)| \right) I_2(q_r) \right\},$$

(37)

where

$$I_1(q_r) := \int_0^\frac{1}{2} q_r e - 1 \frac{q^2 + q - 2}{6} d_q e + \int_\frac{1}{2}^1 q_r e - \frac{5}{6} \frac{q^2}{2} d_q e$$

$$= \begin{cases} -\frac{q^2 + q - 2}{6(q^2 + 2q + 2q + 1)} & \text{if } 0 < q_r < \frac{1}{3}, \\ \frac{q^2 - 2q - 32}{9q^2 + 9q - 32} & \text{if } \frac{1}{3} < q_r < \frac{5}{6}, \\ \frac{2q^2 + 2q - 1}{2q^2 + 2q + 1} & \text{if } \frac{5}{6} < q_r < 1. \end{cases}$$

(38)

and

$$I_2(q_r) := \int_0^\frac{1}{2} q_r e - 1 \frac{q}{2} \left( 1 - \frac{q}{2} \right) d_q e + \int_\frac{1}{2}^1 q_r e - \frac{5}{6} \left( 1 - \frac{q}{2} \right) d_q e$$

$$= \begin{cases} -\frac{3q^2 + q + 1}{6(q^2 + 2q + 2q + 1)} & \text{if } 0 < q_r < \frac{1}{3}, \\ \frac{-18q^2 + 33q^2 + 33q + 10}{12q^2 + 14q^2 + 14q + 5} & \text{if } \frac{1}{3} < q_r < \frac{5}{6}, \\ \frac{7q^2 + 2q + 1}{2q^2 + 2q + 1} & \text{if } \frac{5}{6} < q_r < 1. \end{cases}$$

(39)

Proof. Consider Lemma 1. By taking modulus on both sides of the identity (28), we obtain

$$|\Omega_1 (\Lambda_1, \Lambda_2; q_r)|$$

\begin{align*}
\leq & \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \left( |\Lambda_1 D_q \omega (\frac{q}{2} \Lambda_2 + (1 - \frac{q}{2}) \Lambda_1)| + |\Lambda_2 D_q \omega (\frac{q}{2} \Lambda_1 + (1 - \frac{q}{2}) \Lambda_2)| \right) d_q e \\
& + \left( |\Lambda_1 D_q \omega (\frac{q}{2} \Lambda_2 + (1 - \frac{q}{2}) \Lambda_1)| + |\Lambda_2 D_q \omega (\frac{q}{2} \Lambda_1 + (1 - \frac{q}{2}) \Lambda_2)| \right) d_q e \right\}.
\end{align*}

(40)

As $|\Lambda_1 D_q \omega|$ and $|\Lambda_2 D_q \omega|$ are convex functions, therefore

$$\int_0^\frac{1}{2} q_r e - 1 \frac{q}{6} \left( |\Lambda_1 D_q \omega (\frac{q}{2} \Lambda_2 + (1 - \frac{q}{2}) \Lambda_1)| + |\Lambda_2 D_q \omega (\frac{q}{2} \Lambda_1 + (1 - \frac{q}{2}) \Lambda_2)| \right) d_q e$$

\begin{align*}
\leq & \left( |\Lambda_1 D_q \omega (\Lambda_2)| + |\Lambda_2 D_q \omega (\Lambda_1)| \right) \int_0^\frac{1}{2} q_r e - 1 \frac{q}{6} d_q e \\
& + \left( |\Lambda_1 D_q \omega (\Lambda_1)| + |\Lambda_2 D_q \omega (\Lambda_2)| \right) \int_0^\frac{1}{2} q_r e - 1 \frac{q}{6} \left( 1 - \frac{q}{2} \right) d_q e.
\end{align*}

(41)
Similarly,
\[
\int_{\frac{1}{2}}^{1} q_{r} \frac{5}{6} \left[ \left| \Lambda_{1} D_{q_{r}} \omega \left( \frac{\mu}{2} \Lambda_{1} + \left( 1 - \frac{\mu}{2} \right) \Lambda_{2} \right) \right| + \left| \Lambda_{2} D_{q_{r}} \omega \left( \frac{\mu}{2} \Lambda_{1} + \left( 1 - \frac{\mu}{2} \right) \Lambda_{2} \right) \right| \right] d_{q_{r}} \leq \left( \left| \Lambda_{1} D_{q_{r}} \omega (\Lambda_{2}) \right| + \left| \Lambda_{2} D_{q_{r}} \omega (\Lambda_{1}) \right| \right) \int_{\frac{1}{2}}^{1} |q_{r}| - \frac{5}{6} \left| d_{q_{r}} \right|, \tag{42}
\]
and
\[
\frac{1}{2} \left[ 1 \right] q_{r} \frac{5}{6} \left[ \left| \Lambda_{1} D_{q_{r}} \omega (\Lambda_{2}) \right| + \left| \Lambda_{2} D_{q_{r}} \omega (\Lambda_{1}) \right| \right] \int_{\frac{1}{2}}^{1} q_{r} - \frac{5}{6} \left( 1 - \frac{\mu}{2} \right) d_{q_{r}} \omega .
\]
Using inequality (41) and (42) in (40), we have the desired inequality (37). □

Before proving the next main result, we give a counter example that \( q_{r} \)-Hölder’s inequality has some limitations.

**Example 1.** Let \( \omega_{1}(x) = \sqrt{x} \) and \( \omega_{2}(x) = \sqrt{x^3} \). Suppose further that \( \kappa_{1} = 2 = \kappa_{2} \). Now,
\[
\int_{\frac{1}{2}}^{1} \sqrt{x} \sqrt{x^3} d_{q_{r}} x = \int_{\frac{1}{2}}^{1} x^2 d_{q_{r}} x = \int_{0}^{1} x^2 d_{q_{r}} x - \int_{0}^{\frac{1}{2}} x^2 d_{q_{r}} x = \frac{7}{8(q_{r}^2 + q_{r} + 1)}. \tag{43}
\]
Also,
\[
\int_{\frac{1}{2}}^{1} x d_{q_{r}} x = \frac{3}{4(q_{r} + 1)} \text{ and } \int_{\frac{1}{2}}^{1} x^2 d_{q_{r}} x = \frac{15}{16(q_{r}^3 + q_{r}^2 + q_{r} + 1)}. \tag{44}
\]
Finally, we check the inequality for specific values for \( q_{r} \).
**L.H.S** for \( q_{r} = \frac{15}{100} < 1 \),
\[
\int_{\frac{1}{2}}^{1} \sqrt{\sqrt{x} \sqrt{x^3}} d_{q_{r}} x = \frac{7}{8 \left( \left( \frac{15}{100} \right)^{2} + \frac{15}{100} + 1 \right)} = 0.74627. \tag{45}
\]
**R.H.S** for \( q_{r} = \frac{15}{100} < 1 \),
\[
\left( \int_{\frac{1}{2}}^{1} x d_{q_{r}} x \right)^{\frac{1}{2}} \left( \int_{\frac{1}{2}}^{1} x^2 d_{q_{r}} x \right)^{\frac{1}{2}} = \left( \frac{3}{4(q_{r} + 1)} \right)^{\frac{1}{2}} \left( \frac{15}{16(q_{r}^3 + q_{r}^2 + q_{r} + 1)} \right)^{\frac{1}{2}} = 0.72109, \tag{46}
\]
which justifies our claim.

The example suggests that the inequality
\[
\int_{\mu}^{1} \left| \omega_{1}(e) \omega_{2}(e) \right| \Lambda_{1} d_{q_{r}} e \leq \left( \int_{\mu}^{1} \left| \omega_{1}(e) \right|^{\frac{q_{r}}{1}} \Lambda_{1} d_{q_{r}} e \right)^{\frac{1}{q_{r}}} \left( \int_{\mu}^{1} \left| \omega_{2}(e) \right|^{\frac{q_{r}}{1}} \Lambda_{1} d_{q_{r}} e \right)^{\frac{1}{q_{r}}} \tag{47}
\]
generally not true for \( \mu < \Lambda_{1} \). In other words, the \( q_{r} \)-Hölder’s inequality is true if \( \mu = \Lambda_{1} \).
Theorem 6. Let \( \omega : W \to \mathbb{R} \) satisfies the assumptions of Lemma 1. In addition, if \( |\Lambda_1 D_{\eta r} \omega|^q_2 \) and \( |\Lambda^2 D_{\eta r} \omega|^q_2 (\kappa_2 > 1) \) are convex functions on \( W \), then

\[
|\Omega_1 (\Lambda_1, \Lambda_2; \eta r)| \leq \frac{\Lambda_2 - \Lambda_1}{12 q' \nu (1 + \eta r)} \left\{ \left( |\Lambda_1 D_{\eta r} \omega (\Lambda_2)|^{q_2} + (3 + 4q_2) |\Lambda_1 D_{\eta r} \omega (\Lambda_1)|^{q_2} \right)^{\frac{1}{q_2}} + \right. \\
+ \left. \left( |\Lambda^2 D_{\eta r} \omega (\Lambda_1)|^{q_2} + (3 + 4q_2) |\Lambda^2 D_{\eta r} \omega (\Lambda_2)|^{q_2} \right)^{\frac{1}{q_2}} \right\} + 3(I_{\eta_2}(\eta r))^{\frac{1}{2}} \sum_{\delta=0}^\infty q_2' \left( q_2' (1 - q_2') \right)^{q_1} (1 - q_2') \frac{\kappa_2 - 1}{\kappa_2},
\]

where \( \kappa_1^{-1} + \kappa_2^{-1} = 1 \), and

\[
I_{\eta_2}(\eta r) := \int_0^{\eta r} q_2 \left( \frac{1}{2} \right)^{q_1} |D_{\eta r} \omega (\Lambda_2) - \Lambda_1 D_{\eta r} \omega (\Lambda_2) - \Lambda_1 D_{\eta r} \omega (\Lambda_1)|^{q_2} d_q e = \int_0^{\eta r} \left( \sum_{\delta=0}^\infty q_2' \left( q_2' (1 - q_2') \right)^{q_1} (1 - q_2') \frac{\kappa_2 - 1}{\kappa_2} \right) d_q e.
\]

Proof. By utilizing the properties of \( q_r \)-integrals, we also have identically

\[
\Omega_1 (\Lambda_1, \Lambda_2; \eta r) = \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \int_0^{\eta r} \left( \frac{1}{2} \right)^{q_1} \left( \Lambda_1 D_{\eta r} \omega (\frac{\eta_2}{2} \Lambda_2 + (1 - \frac{\eta_2}{2}) \Lambda_1) - \Lambda_1 D_{\eta r} \omega \left( \frac{\eta_2}{2} \Lambda_1 + (1 - \frac{\eta_2}{2}) \Lambda_2 \right) \right) d_q e \\
+ \left( \frac{1}{2} \right)^{q_1} \int_0^{\eta r} \left( \Lambda_1 D_{\eta r} \omega (\frac{\eta_2}{2} \Lambda_2 + (1 - \frac{\eta_2}{2}) \Lambda_1) - \Lambda_1 D_{\eta r} \omega \left( \frac{\eta_2}{2} \Lambda_1 + (1 - \frac{\eta_2}{2}) \Lambda_2 \right) \right) d_q e \right\}. \tag{50}
\]

By the applications of modulus, \( q_r \)-Hölder’s inequality and the convexity of \( |\Lambda_1 D_{\eta r} \omega|^q_2 \) and \( |\Lambda^2 D_{\eta r} \omega|^q_2 \), we have

\[
\int_0^{\eta r} q_2 \left( \frac{1}{2} \right)^{q_1} |D_{\eta r} \omega (\frac{\eta_2}{2} \Lambda_2 + (1 - \frac{\eta_2}{2}) \Lambda_1)| d_q e \\
+ \left( \frac{1}{2} \right)^{q_1} \int_0^{\eta r} \left( \Lambda_1 D_{\eta r} \omega (\frac{\eta_2}{2} \Lambda_2 + (1 - \frac{\eta_2}{2}) \Lambda_1) - \Lambda_1 D_{\eta r} \omega \left( \frac{\eta_2}{2} \Lambda_1 + (1 - \frac{\eta_2}{2}) \Lambda_2 \right) \right) d_q e \leq \left( \int_0^{\eta r} q_2 \left( \frac{1}{2} \right)^{q_1} \right)^{\frac{1}{q_2}} \left\{ \left( \int_0^{\eta r} \frac{1}{2} |\Lambda_1 D_{\eta r} \omega (\Lambda_2)|^{q_2} d_q e + \int_0^{\eta r} \frac{1}{2} |\Lambda_1 D_{\eta r} \omega (\Lambda_1)|^{q_2} d_q e \right)^{\frac{1}{q_2}} \\
+ \left( \int_0^{\eta r} \frac{1}{2} |\Lambda^2 D_{\eta r} \omega (\Lambda_1)|^{q_2} d_q e + \int_0^{\eta r} \frac{1}{2} |\Lambda^2 D_{\eta r} \omega (\Lambda_2)|^{q_2} d_q e \right)^{\frac{1}{q_2}} \right\} \\
+ \left( \frac{1}{2} \right)^{q_1} \int_0^{\eta r} \left( \frac{1}{2} |\Lambda_1 D_{\eta r} \omega (\Lambda_2)|^{q_2} + \frac{1}{2} |\Lambda_1 D_{\eta r} \omega (\Lambda_1)|^{q_2} \right) d_q e \right\} \\
+ \left( \frac{1}{2} \right)^{q_1} \int_0^{\eta r} \left( \frac{1}{2} |\Lambda^2 D_{\eta r} \omega (\Lambda_1)|^{q_2} + \frac{1}{2} |\Lambda^2 D_{\eta r} \omega (\Lambda_2)|^{q_2} \right) d_q e \right\} \leq \left( \frac{2 |\Lambda_1 D_{\eta r} \omega (\Lambda_2)|^{q_2} + 2 (1 + 2q_2) |\Lambda_1 D_{\eta r} \omega (\Lambda_1)|^{q_2}}{4 (1 + \eta r)} \right)^{\frac{1}{2}} \\
+ \left( \frac{2 |\Lambda^2 D_{\eta r} \omega (\Lambda_1)|^{q_2} + 2 (1 + 2q_2) |\Lambda^2 D_{\eta r} \omega (\Lambda_2)|^{q_2}}{4 (1 + \eta r)} \right)^{\frac{1}{2}}. \tag{51}
\]
In a similar way, we find

\[ \begin{align*}
&\int_0^1 |\Lambda_1 D_{q_r} \omega \left( \frac{e}{2} \Lambda_2 + \left( 1 - \frac{e}{2} \right) \Lambda_1 \right) | \, d_{q_r} \omega \\
&+ \int_0^1 |\Lambda_2 D_{q_r} \omega \left( \frac{e}{2} \Lambda_1 + \left( 1 - \frac{e}{2} \right) \Lambda_2 \right) | \, d_{q_r} \omega \\
&\leq \left( \int_0^1 1 \, d_{q_r} \omega \right)^{\frac{1}{2}} \left[ \left( \int_0^1 |\Lambda_1 D_{q_r} \omega \left( \frac{e}{2} \Lambda_2 + \left( 1 - \frac{e}{2} \right) \Lambda_1 \right) |^{\kappa_2} \, d_{q_r} \omega \right)^{\frac{1}{\kappa_2}} \\
&+ \left( \int_0^1 |\Lambda_2 D_{q_r} \omega \left( \frac{e}{2} \Lambda_1 + \left( 1 - \frac{e}{2} \right) \Lambda_2 \right) |^{\kappa_2} \, d_{q_r} \omega \right)^{\frac{1}{\kappa_2}} \right]^{\frac{1}{2}} \\
&\leq \left( \frac{1}{2} \right)^{\frac{1}{\kappa_2}} \left[ \left( |\Lambda_1 D_{q_r} \omega(\Lambda_2)|^{\kappa_2} + (3 + 4q_r) |\Lambda_1 D_{q_r} \omega(\Lambda_1)|^{\kappa_2} \right) \left( 8(1 + q_r) \right) \right]^{\frac{1}{\kappa_2}} \\
&+ \left( |\Lambda_2 D_{q_r} \omega(\Lambda_1)|^{\kappa_2} + (3 + 4q_r) |\Lambda_2 D_{q_r} \omega(\Lambda_2)|^{\kappa_2} \right) \left( 8(1 + q_r) \right) \right]^{\frac{1}{\kappa_2}}. \tag{52}
\end{align*} \]

The required inequality (48) is thus obtained by utilizing inequalities (51) and (52) in (50).

**Theorem 7.** Let \( \omega : W \to \mathbb{R} \) satisfies the assumptions of Lemma 1. In addition, if \(|\Lambda_1 D_{q_r} \omega|^{\kappa_2}\) and \(|\Lambda_2 D_{q_r} \omega|^{\kappa_2}\) are convex functions on \( W \) with \( \kappa_2 \geq 1 \), then

\[ \begin{align*}
|\Omega_1(\Lambda_1, \Lambda_2; q)| \leq \frac{\Lambda_2 - \Lambda_1}{12} \left( 4(1 + q_r) \right) \left\{ \left( |\Lambda_1 D_{q_r} \omega(\Lambda_2)|^{\kappa_2} + (3 + 4q_r) |\Lambda_1 D_{q_r} \omega(\Lambda_1)|^{\kappa_2} \right) \left( 8(1 + q_r) \right) \right\}^{\frac{1}{\kappa_2}} \\
+ \left( |\Lambda_2 D_{q_r} \omega(\Lambda_1)|^{\kappa_2} + (3 + 4q_r) |\Lambda_2 D_{q_r} \omega(\Lambda_2)|^{\kappa_2} \right) \left( 8(1 + q_r) \right) \right\}^{\frac{1}{\kappa_2}} \\
+ 3 \sqrt[\kappa_2]{4(1 + q_r)(\mathcal{I}_{\Lambda_1}(q_r))^{\frac{1}{\kappa_2}}} \left\{ \left( |\Lambda_1 D_{q_r} \omega(\Lambda_2)|^{\kappa_2} \mathcal{I}_c(q_r) + |\Lambda_1 D_{q_r} \omega(\Lambda_1)|^{\kappa_2} \mathcal{I}_d(q_r) \right) \left( 8(1 + q_r) \right) \right\}^{\frac{1}{\kappa_2}} \\
+ \left( |\Lambda_2 D_{q_r} \omega(\Lambda_1)|^{\kappa_2} \mathcal{I}_c(q_r) + |\Lambda_2 D_{q_r} \omega(\Lambda_2)|^{\kappa_2} \mathcal{I}_d(q_r) \right) \left( 8(1 + q_r) \right) \right\}^{\frac{1}{\kappa_2}}, \tag{53}
\end{align*} \]

where

\[ \begin{align*}
\mathcal{I}_c(q_r) := \int_0^1 |q_r e - \frac{3}{2}| \, d_{q_r} e = \begin{cases} 
-\frac{a_0^2 + q_r^2 - 5}{12(a_0^2 + 2q_r^2 + 2q_r + 2q_r + 1)}, & \text{if } 0 < q_r < \frac{5}{6}, \\
\frac{18a_0^2 + 16q_r + 35}{216(a_0^2 + 2q_r^2 + 2q_r + 1)}, & \text{if } \frac{5}{6} \leq q_r < 1,
\end{cases} \tag{54}
\end{align*} \]

and

\[ \begin{align*}
\mathcal{I}_d(q_r) := \int_0^1 |q_r e - \frac{3}{2}| \, d_{q_r} e = \begin{cases} 
-\frac{2a_0^2 + 3q_r^2 + 9q_r + 5}{12(a_0^2 + 2q_r^2 + 2q_r + 1)}, & \text{if } 0 < q_r < \frac{5}{6}, \\
\frac{36a_0^2 + 138q_r^2 + 85}{216(a_0^2 + 2q_r^2 + 2q_r + 1)}, & \text{if } \frac{5}{6} \leq q_r < 1,
\end{cases} \tag{55}
\end{align*} \]

and \( \mathcal{I}_{\Lambda_1}(q_r) \) is obtained from \( \mathcal{I}_{\Lambda_1 \kappa_2}(q_r) \), which is same as given in Theorem 6.
Proof. Consider again Lemma 1. The property of modulus, power mean’s inequality, and the convexity of \(|\Lambda_1 D_{q_r}\omega|^{q_2}\) and \(|\Lambda_2 D_{q_r}\omega|^{q_2}\) leads to

\[
\begin{align*}
\left(\int_0^1 \left | q_r - \frac{5}{6} \right | \frac{1}{q_2} & \left | \Lambda_1 D_{q_r}\omega \left( \frac{6}{2} \Lambda_2 + \left(1 - \frac{6}{2}\right) \Lambda_1 \right) \right | \, d_{q_r} \right)^{\frac{1}{q_2}} \\
+ \left(\int_0^1 \left | q_r - \frac{5}{6} \right | \frac{1}{q_2} & \left | \Lambda_2 D_{q_r}\omega \left( \frac{6}{2} \Lambda_2 + \left(1 - \frac{6}{2}\right) \Lambda_1 \right) \right | \, d_{q_r} \right)^{\frac{1}{q_2}} \\
\leq \left(\int_0^1 \left | q_r - \frac{5}{6} \right | \frac{1}{q_2} & \left | \Lambda_1 D_{q_r}\omega \left( \frac{6}{2} \Lambda_2 + \left(1 - \frac{6}{2}\right) \Lambda_1 \right) \right | \, d_{q_r} \right)^{\frac{1}{q_2}} \\
+ \left(\int_0^1 \left | q_r - \frac{5}{6} \right | \frac{1}{q_2} & \left | \Lambda_2 D_{q_r}\omega \left( \frac{6}{2} \Lambda_2 + \left(1 - \frac{6}{2}\right) \Lambda_1 \right) \right | \, d_{q_r} \right)^{\frac{1}{q_2}} \\
\leq (I_{h_{\Lambda_1}}(q_r))^{\frac{1}{q_2}} & \left[ (|\Lambda_1 D_{q_r}\omega(\Lambda_2)|^{q_2} I_d(q_r) + |\Lambda_1 D_{q_r}\omega(\Lambda_1)|^{q_2} I_d(q_r))^{\frac{1}{q_2}} \\
+ (|\Lambda_2 D_{q_r}\omega(\Lambda_1)|^{q_2} I_d(q_r) + |\Lambda_2 D_{q_r}\omega(\Lambda_2)|^{q_2} I_d(q_r))^{\frac{1}{q_2}} \right].
\end{align*}
\]

(56)

By some parallel calculations, we obtain

\[
\begin{align*}
\left(\int_0^1 \left | \Lambda_1 D_{q_r}\omega \left( \frac{6}{2} \Lambda_2 + \left(1 - \frac{6}{2}\right) \Lambda_1 \right) \right | \, d_{q_r} \right)^{\frac{1}{q_2}} \\
+ \left(\int_0^1 \left | \Lambda_2 D_{q_r}\omega \left( \frac{6}{2} \Lambda_2 + \left(1 - \frac{6}{2}\right) \Lambda_1 \right) \right | \, d_{q_r} \right)^{\frac{1}{q_2}} \\
\leq \left(\frac{1}{2} \right)^{1 - \frac{1}{q_2}} & \left[ \left( |\Lambda_1 D_{q_r}\omega(\Lambda_2)|^{q_2} + (3 + 4q_r) |\Lambda_1 D_{q_r}\omega(\Lambda_1)|^{q_2} \right) \frac{8}{8(1 + q_r)} \right]^{\frac{1}{q_2}} \\
+ \left( |\Lambda_2 D_{q_r}\omega(\Lambda_1)|^{q_2} + (3 + 4q_r) |\Lambda_2 D_{q_r}\omega(\Lambda_2)|^{q_2} \right) \frac{8}{8(1 + q_r)} \right]^{\frac{1}{q_2}}.
\end{align*}
\]

(57)

The required inequality (53) is thus obtained by utilizing (56) and (57) in (50).

\[\square\]

5. Quantum Analogues of Simpson’s Inequalities Related to Simpson’s 3/8 Rule

This section is devoted to the extension of Simpson’s 3/8 rule. Here we present our results utilizing six panels.

Theorem 8. Let \(\omega : \mathcal{W} \rightarrow \mathbb{R}\) satisfies the assumptions of Lemma 2. In addition, if \(|\Lambda_1 D_{q_r}\omega|\) and \(|\Lambda_2 D_{q_r}\omega|\) are convex functions on \(\mathcal{W}\), then

\[
|\Omega_2(\Lambda_1, \Lambda_2; q_r)| \leq \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \left( |\Lambda_1 D_{q_r}\omega(\Lambda_2) + |\Lambda_2 D_{q_r}\omega(\Lambda_1) | \right) I_M(q_r) \\
+ \left( |\Lambda_1 D_{q_r}\omega(\Lambda_1) + |\Lambda_2 D_{q_r}\omega(\Lambda_2) | \right) I_M(q_r) \right\},
\]

(58)
where

\[
I_M(q_r) := \sum_{\gamma=3}^{5} I_\gamma(q_r) = \begin{cases} 
-\frac{q_r^2}{6}(q_r^2 + 2q_r + 1) & \text{if } 0 < q_r < \frac{3}{5}, \\
\frac{q_r^2}{1984}(q_r^2 + 2q_r + 1) & \text{if } \frac{3}{5} < q_r < \frac{3}{4}, \\
n\frac{q_r^2}{13824}(q_r^2 + 2q_r + 1) & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
n\frac{q_r^2}{320q_r^2 + 320q_r + 11} & \text{if } \frac{7}{8} < q_r < 1,
\end{cases}
\]

(59)

and

\[
I_N(q_r) := \sum_{\nu=6}^{8} I_\nu(q_r) = \begin{cases} 
-\frac{3q_r^2 + q_r + 1}{6}(q_r^2 + 2q_r + 1) & \text{if } 0 < q_r < \frac{3}{5}, \\
\frac{3q_r^2 + q_r + 1}{4992(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{5} < q_r < \frac{3}{4}, \\
n\frac{3q_r^2 + q_r + 1}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
n\frac{3q_r^2 + q_r + 1}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1,
\end{cases}
\]

(60)

\[
I_3(q_r) := \int \frac{q_r e - \frac{1}{8}}{2} d_q e = \begin{cases} 
-\frac{5q_r^2 - 5q_r + 3}{432(q_r^2 + 2q_r + 1)} & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{5q_r^2 - 5q_r + 3}{4608(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{8} < q_r < \frac{1}{2}, \\
\frac{5q_r^2 - 5q_r + 3}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{1}{2} < q_r < \frac{3}{4}, \\
\frac{5q_r^2 - 5q_r + 3}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
\frac{5q_r^2 - 5q_r + 3}{1984(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]

(61)

\[
I_4(q_r) := \int \frac{q_r e - \frac{1}{2}}{2} d_q e = \begin{cases} 
-\frac{47q_r^2 - 47q_r + 105}{432(q_r^2 + 2q_r + 1)} & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{47q_r^2 - 47q_r + 105}{224q_r^2 + 224q_r + 525} & \text{if } \frac{3}{8} < q_r < \frac{1}{2}, \\
\frac{47q_r^2 - 47q_r + 105}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{1}{2} < q_r < \frac{3}{4}, \\
\frac{47q_r^2 - 47q_r + 105}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
\frac{47q_r^2 - 47q_r + 105}{1984(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]

(62)

\[
I_5(q_r) := \int \frac{q_r e - \frac{7}{8}}{2} d_q e = \begin{cases} 
-\frac{30q_r^2 - 7q_r - 15}{432(q_r^2 + 2q_r + 1)} & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{30q_r^2 - 7q_r - 15}{960q_r^2 + 656q_r + 75} & \text{if } \frac{3}{8} < q_r < \frac{1}{2}, \\
\frac{30q_r^2 - 7q_r - 15}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{1}{2} < q_r < \frac{3}{4}, \\
\frac{30q_r^2 - 7q_r - 15}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
\frac{30q_r^2 - 7q_r - 15}{1984(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]

(63)

\[
I_6(q_r) := \int \frac{q_r e - \frac{1}{8}}{2} d_q e = \begin{cases} 
-\frac{18q_r^2 + 5q_r + 9}{108(q_r^2 + 2q_r + 1)} & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{18q_r^2 + 5q_r + 9}{432(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{8} < q_r < \frac{1}{2}, \\
\frac{18q_r^2 + 5q_r + 9}{224q_r^2 + 224q_r + 525} & \text{if } \frac{1}{2} < q_r < \frac{3}{4}, \\
\frac{18q_r^2 + 5q_r + 9}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
\frac{18q_r^2 + 5q_r + 9}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]

(64)

\[
I_7(q_r) := \int \frac{q_r e - \frac{1}{2}}{2} d_q e = \begin{cases} 
-\frac{114q_r^2 + 59q_r + 21}{432(q_r^2 + 2q_r + 1)} & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{114q_r^2 + 59q_r + 21}{192q_r^2 + 592q_r + 483} & \text{if } \frac{3}{8} < q_r < \frac{1}{2}, \\
\frac{114q_r^2 + 59q_r + 21}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{1}{2} < q_r < \frac{3}{4}, \\
\frac{114q_r^2 + 59q_r + 21}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
\frac{114q_r^2 + 59q_r + 21}{1984(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]

(65)

\[
I_8(q_r) := \int \frac{q_r e - \frac{7}{8}}{2} d_q e = \begin{cases} 
-\frac{33q_r^2 + 33q_r + 11}{432(q_r^2 + 2q_r + 1)} & \text{if } 0 < q_r < \frac{3}{8}, \\
\frac{33q_r^2 + 33q_r + 11}{192q_r^2 + 592q_r + 483} & \text{if } \frac{3}{8} < q_r < \frac{1}{2}, \\
\frac{33q_r^2 + 33q_r + 11}{13824(q_r^2 + 2q_r + 1)} & \text{if } \frac{1}{2} < q_r < \frac{3}{4}, \\
\frac{33q_r^2 + 33q_r + 11}{576(q_r^2 + 2q_r + 1)} & \text{if } \frac{3}{4} < q_r < \frac{7}{8}, \\
\frac{33q_r^2 + 33q_r + 11}{1984(q_r^2 + 2q_r + 1)} & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]

(66)

Proof. The proof is skipped as it follows the same lines as used in the previous Theorem 5 by utilizing the identity given in Lemma 2. □
Theorem 9. Assume that \( \omega : \mathcal{W} \to \mathbb{R} \) satisfies the assumptions of Lemma 2. In addition, if \( |\Lambda_1 D_q \omega|^{k_2} \) and \( |\Lambda_2 D_q \omega|^{k_2} \) are convex functions on \( \mathcal{W} \), then

\[
|\Omega_2(\Lambda_1, \Lambda_2; \varphi_r)| \leq \frac{\Delta_2 - \Delta_1}{32 \sqrt{q} \beta(1 + q)} \left\{ \left( |\Lambda_1 D_q \omega(\Lambda_2)|^{k_2} + (5 + 6q_r) |\Lambda_1 D_q \omega(\Lambda_1)|^{k_2} \right)^\frac{1}{2} + \left( |\Lambda_2 D_q \omega(\Lambda_1)|^{k_2} + (5 + 6q_r) |\Lambda_2 D_q \omega(\Lambda_2)|^{k_2} \right)^\frac{1}{2} \right. \\
+ 2 \left( |\Lambda_1 D_q \omega(\Lambda_2)|^{k_2} + (4 + 6q_r) |\Lambda_1 D_q \omega(\Lambda_1)|^{k_2} \right)^\frac{1}{2} + 2 \left( |\Lambda_2 D_q \omega(\Lambda_1)|^{k_2} + (4 + 6q_r) |\Lambda_2 D_q \omega(\Lambda_2)|^{k_2} \right)^\frac{1}{2} \right. \\
+ 8 (I_{r,1}(q_r))^\frac{1}{2} \left[ \left( |\Lambda_1 D_q \omega(\Lambda_2)|^{k_2} + (3 + 6q_r) |\Lambda_1 D_q \omega(\Lambda_1)|^{k_2} \right)^\frac{1}{2} \right. \\
+ \left( |\Lambda_2 D_q \omega(\Lambda_1)|^{k_2} + (3 + 6q_r) |\Lambda_2 D_q \omega(\Lambda_2)|^{k_2} \right)^\frac{1}{2} \right\},
\]  

(67)

where \( \kappa_1^{-1} + \kappa_2^{-1} = 1 \), and

\[
I_{r,1}(q_r) := \left\{ \int_0^1 |q_r e - \frac{7}{8} - \frac{5}{8} \right\} d_q e = \begin{cases} 
(1 - q_r) \sum_{\delta=0}^{\infty} q_r^\delta (q_r q_r + 1)^{\delta - 1}, & \text{if } 0 < q_r < \frac{7}{8}, \\
(1 - q_r) \sum_{\delta=0}^{\infty} q_r^\delta (q_r q_r - 1)^{\delta - 1}, & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]  

(68)

Proof. The required inequality (67) can be achieved if we consider Lemma 2 and follow the approach used in the proof of Theorem 6. \( \square \)

Theorem 10. Assume that \( \omega : \mathcal{W} \to \mathbb{R} \) satisfies the assumptions of Lemma 2. In addition, if \( |\Lambda_1 D_q \omega|^{k_2} \) and \( |\Lambda_2 D_q \omega|^{k_2} \) are convex functions on \( \mathcal{W} \) with \( \kappa_2 \geq 1 \), then

\[
|\Omega(\Lambda_1, \Lambda_2; \varphi_r)| \leq \frac{\Delta_2 - \Delta_1}{32 \sqrt{q} \beta(1 + q)} \left\{ \left( |\Lambda_1 D_q \omega(\Lambda_2)|^{k_2} + (5 + 6q_r) |\Lambda_1 D_q \omega(\Lambda_1)|^{k_2} \right)^\frac{1}{2} + \left( |\Lambda_2 D_q \omega(\Lambda_1)|^{k_2} + (5 + 6q_r) |\Lambda_2 D_q \omega(\Lambda_2)|^{k_2} \right)^\frac{1}{2} \right. \\
+ 2 \left( |\Lambda_1 D_q \omega(\Lambda_2)|^{k_2} + (2 + 3q_r) |\Lambda_1 D_q \omega(\Lambda_1)|^{k_2} \right)^\frac{1}{2} + 2 \left( |\Lambda_2 D_q \omega(\Lambda_1)|^{k_2} + (2 + 3q_r) |\Lambda_2 D_q \omega(\Lambda_2)|^{k_2} \right)^\frac{1}{2} \right. \\
+ 8 \sqrt{q} \theta_1(q_r) \left( I_{r,1}(q_r) \right)^{1/2} \left[ \left( |\Lambda_1 D_q \omega(\Lambda_2)|^{k_2} I_d(q_r) + |\Lambda_1 D_q \omega(\Lambda_1)|^{k_2} I_b(q_r) \right)^\frac{1}{2} \right. \\
+ \left. \left( |\Lambda_2 D_q \omega(\Lambda_1)|^{k_2} I_d(q_r) + |\Lambda_2 D_q \omega(\Lambda_2)|^{k_2} I_b(q_r) \right)^\frac{1}{2} \right\},
\]  

(69)

where

\[
I_d(q_r) := \left\{ \int_0^1 |q_r e - \frac{7}{8} - \frac{5}{2} \right\} d_q e = \begin{cases} 
\frac{-q_r^2 - 2q_r + \frac{7}{2}}{16(q_r^2 + 2q_r + 2q_r + 1)}, & \text{if } 0 < q_r < \frac{7}{8}, \\
\frac{q_r^2 - 2q_r - 13q_r + 7}{512(q_r^2 + 2q_r + 2q_r + 1)}, & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]  

(70)

\[
I_b(q_r) := \left\{ \int_0^1 |q_r e - \frac{7}{8} - \frac{5}{2} \right\} d_q e = \begin{cases} 
\frac{-2q_r^2 + 13q_r^2 + 13q_r + 7}{16(q_r^2 + 2q_r + 2q_r + 1)}, & \text{if } 0 < q_r < \frac{7}{8}, \\
\frac{2q_r^2 + 36q_r^2 + 30q_r + 217}{512(q_r^2 + 2q_r + 2q_r + 1)}, & \text{if } \frac{7}{8} < q_r < 1.
\end{cases}
\]  

(71)
and \( I_{c_1}(q_r) \) is achieved from \( I_{c_\lambda_1}(q_r) \), which is given in Theorem 9.

**Proof.** The proof is skipped as it is similar to the Theorem 7. □

### 6. Simpson’s Type Inequalities Associated with Classical Integrals

This section is devoted to some classical versions of the inequalities developed for \( q_r \)-integrals. Some results are deduced from the previous section. The rest of the results are proved to examine the variation of two theories.

**Corollary 1.** Let \( \omega : \mathcal{W} \to \mathbb{R} \) be a differentiable function on \( \mathcal{W}^0 \). If \( \omega' \) is integrable function on \( \mathcal{W} \), then the following identity holds:

\[
\frac{1}{12} \left[ \omega(\Lambda_1) + 4\omega' \left( \frac{3\Lambda_2 + \Lambda_1}{4} \right) + 2\omega' \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + 4\omega' \left( \frac{\Lambda_2 + 3\Lambda_1}{4} \right) + \omega(\Lambda_2) \right]
- \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega'(\varphi) \, d\varphi.
\]

\[
= \frac{\Lambda_2 - \Lambda_1}{4} \left[ \right. \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega'(\varphi) \, d\varphi \left. \right],
\]

where

\[
R_1(\varphi) := \left\{ \begin{array}{ll}
\varphi - \frac{1}{6}, & \text{if } 0 \leq \varphi < \frac{1}{2}, \\
\varphi - \frac{5}{6}, & \text{if } \frac{1}{2} \leq \varphi \leq 1.
\end{array} \right.
\]

**Proof.** Consider Lemma 1. If \( q_r \to 1^- \), then we find the desired identity. □

**Corollary 2.** Let \( \omega : \mathcal{W} \to \mathbb{R} \) satisfies the assumptions of Corollary 1. In addition, if \( |\omega'| \) is convex function, then

\[
\frac{1}{12} \left[ \omega(\Lambda_1) + 4\omega' \left( \frac{3\Lambda_2 + \Lambda_1}{4} \right) + 2\omega' \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + 4\omega' \left( \frac{\Lambda_2 + 3\Lambda_1}{4} \right) + \omega(\Lambda_2) \right]
- \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega'(\varphi) \, d\varphi
\leq \frac{5(\Lambda_2 - \Lambda_1)}{144} \left[ |\omega'(\Lambda_1)| + |\omega'(\Lambda_2)| \right].
\]

**Proof.** Consider Theorem 5. If we let \( q_r \to 1^- \), then the above inequality is obtained. □

**Corollary 3.** Let \( \omega : \mathcal{W} \to \mathbb{R} \) satisfies the assumptions of Corollary 1. In addition, if \( |\omega'|^{k_2} \) (\( k_2 > 1 \)), is convex function, then

\[
\frac{1}{12} \left[ \omega(\Lambda_1) + 4\omega' \left( \frac{3\Lambda_2 + \Lambda_1}{4} \right) + 2\omega' \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + 4\omega' \left( \frac{\Lambda_2 + 3\Lambda_1}{4} \right) + \omega(\Lambda_2) \right]
- \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega'(\varphi) \, d\varphi
\leq \frac{\Lambda_2 - \Lambda_1}{4} \left[ \right. \left( \frac{1 + 2^{k_2+1}}{6^{k_2+1}(k_2 + 1)} \right)^{\frac{1}{k_2+1}} \left\{ \left[ |\omega'(\Lambda_2)|^{k_2} + 7|\omega'(\Lambda_1)|^{k_2} \right]^{\frac{1}{k_2}} + \left[ |\omega'(\Lambda_1)|^{k_2} + 7|\omega'(\Lambda_2)|^{k_2} \right]^{\frac{1}{k_2}} + \left[ 3|\omega'(\Lambda_1)|^{k_2} + 5|\omega'(\Lambda_2)|^{k_2} \right]^{\frac{1}{k_2}} \left. \right].
\]
where \( \kappa_1^{-1} + \kappa_2^{-1} = 1 \).

**Proof.** Firstly, we note that

\[
\int_0^{\frac{1}{2}} |e - \frac{1}{6}|^{\kappa_1} \, de = \frac{1 + 2^{\kappa_1+1}}{6^{\kappa_1+1}(\kappa_1+1)} = \int_0^{\frac{1}{2}} |e - \frac{5}{6}|^{\kappa_1} \, de. \tag{76}
\]

By the use of modulus on both sides of the identity (72), we have

\[
\frac{1}{12} \left[ a(\Lambda_1) + 4a \left( \frac{3\Lambda_1 + \Lambda_2}{4} \right) + 2a \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) + 4a \left( \frac{3\Lambda_1 + \Lambda_2}{4} \right) + a(\Lambda_2) \right]
- \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \int_0^{\frac{1}{2}} |e - \frac{1}{6}| |a'(\frac{e}{2} \Lambda_2 + (1 - \frac{e}{2}) \Lambda_1)| + |a'(\frac{e}{2} \Lambda_1 + (1 - \frac{e}{2}) \Lambda_2)| \right\} \, de
\leq \frac{\Lambda_2 - \Lambda_1}{4} \left\{ \int_0^{\frac{1}{2}} |e - \frac{5}{6}| |a'(\frac{e}{2} \Lambda_2 + (1 - \frac{e}{2}) \Lambda_1)| + |a'(\frac{e}{2} \Lambda_1 + (1 - \frac{e}{2}) \Lambda_2)| \right\} \, de \tag{77}
\]

If we apply the H"older’s inequality and utilize the convexity of \(|a'|^{\kappa_2}\), we find

\[
\int_0^{\frac{1}{2}} |e - \frac{1}{6}| |a'(\frac{e}{2} \Lambda_2 + (1 - \frac{e}{2}) \Lambda_1)| + |a'(\frac{e}{2} \Lambda_1 + (1 - \frac{e}{2}) \Lambda_2)| \, de
\leq \left( \int_0^{\frac{1}{2}} |e - \frac{1}{6}|^{\kappa_1} \, de \right)^{\frac{1}{\kappa_1}} \left[ |a'(\Lambda_2)|^{\kappa_2} \int_0^{\frac{1}{2}} \frac{e}{2} \, de + |a'(\Lambda_1)|^{\kappa_2} \int_0^{\frac{1}{2}} \left(1 - \frac{e}{2}\right) \, de \right]^{\frac{1}{\kappa_2}}
+ \left( |a'(\Lambda_1)|^{\kappa_2} \int_0^{\frac{1}{2}} \frac{e}{2} \, de + |a'(\Lambda_2)|^{\kappa_2} \int_0^{\frac{1}{2}} \left(1 - \frac{e}{2}\right) \, de \right)^{\frac{1}{\kappa_2}} \tag{78}
\]

Similarly,

\[
\int_0^{\frac{1}{2}} |e - \frac{5}{6}| |a'(\frac{e}{2} \Lambda_2 + (1 - \frac{e}{2}) \Lambda_1)| + |a'(\frac{e}{2} \Lambda_1 + (1 - \frac{e}{2}) \Lambda_2)| \, de
\leq \left( \int_0^{\frac{1}{2}} |e - \frac{5}{6}|^{\kappa_1} \, de \right)^{\frac{1}{\kappa_1}} \left[ |a'(\Lambda_2)|^{\kappa_2} \int_0^{\frac{1}{2}} \frac{e}{2} \, de + |a'(\Lambda_1)|^{\kappa_2} \int_0^{\frac{1}{2}} \left(1 - \frac{e}{2}\right) \, de \right]^{\frac{1}{\kappa_2}}
+ \left( |a'(\Lambda_1)|^{\kappa_2} \int_0^{\frac{1}{2}} \frac{e}{2} \, de + |a'(\Lambda_2)|^{\kappa_2} \int_0^{\frac{1}{2}} \left(1 - \frac{e}{2}\right) \, de \right)^{\frac{1}{\kappa_2}} \tag{79}
\]

The desired inequality is thus obtained by utilizing inequalities (76), (78), and (79) in (77). \(\Box\)
Corollary 4. Let \( \omega : W \to \mathbb{R} \) be a differentiable function on \( W^o \). If \( \omega' \) is integrable function on \( W \), then the following identity holds:

\[
\begin{align*}
\frac{1}{16} \left[ \begin{array}{c}
\omega(\Lambda_1) + 3\omega \left( \frac{5\Lambda_1 + \Lambda_2}{6} \right) + 3\omega \left( \frac{2\Lambda_1 + \Lambda_2}{3} \right) + 2\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) \\
+ 3\omega \left( \frac{\Lambda_1 + 2\Lambda_2}{3} \right) + 3\omega \left( \frac{\Lambda_1 + 5\Lambda_2}{6} \right) + \omega(\Lambda_2) \\
\end{array} \right] - \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) d\varphi, \\
\end{align*}
\]

\( (80) \)

Proof. If we assume \( q_r \to 1^- \) in Lemma 2, then we find the identity (80) as a consequence.

Corollary 5. Assume that \( \omega : W \to \mathbb{R} \) satisfies the assumptions of Corollary 4. In addition, if \( \omega' \) is convex function, then

\[
\begin{align*}
\frac{1}{16} \left[ \begin{array}{c}
\omega(\Lambda_1) + 3\omega \left( \frac{5\Lambda_1 + \Lambda_2}{6} \right) + 3\omega \left( \frac{2\Lambda_1 + \Lambda_2}{3} \right) + 2\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) \\
+ 3\omega \left( \frac{\Lambda_1 + 2\Lambda_2}{3} \right) + 3\omega \left( \frac{\Lambda_1 + 5\Lambda_2}{6} \right) + \omega(\Lambda_2) \\
\end{array} \right] - \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) d\varphi \\
\leq \frac{25(\Lambda_2 - \Lambda_1)}{1152} |\omega'(\Lambda_1)| + |\omega'(\Lambda_2)|. \\
\end{align*}
\]

\( (82) \)

Proof. Consider Theorem 8. If we let \( q_r \to 1^- \), then as a consequence, we obtain the desired inequality.

Corollary 6. Assume that \( \omega : W \to \mathbb{R} \) satisfies the assumptions of Corollary 4. In addition, if \( |\omega'|^{\kappa_2} (\kappa_2 > 1) \), is convex function, then

\[
\begin{align*}
\frac{1}{16} \left[ \begin{array}{c}
\omega(\Lambda_1) + 3\omega \left( \frac{5\Lambda_1 + \Lambda_2}{6} \right) + 3\omega \left( \frac{2\Lambda_1 + \Lambda_2}{3} \right) + 2\omega \left( \frac{\Lambda_1 + \Lambda_2}{2} \right) \\
+ 3\omega \left( \frac{\Lambda_1 + 2\Lambda_2}{3} \right) + 3\omega \left( \frac{\Lambda_1 + 5\Lambda_2}{6} \right) + \omega(\Lambda_2) \\
\end{array} \right] - \frac{1}{\Lambda_2 - \Lambda_1} \int_{\Lambda_1}^{\Lambda_2} \omega(\varphi) d\varphi \\
\leq \frac{\Lambda_2 - \Lambda_1}{1152} \sqrt{2(\kappa_1 + 1)^{-3}} \left[ \begin{array}{c}
\omega' \left( \frac{\Lambda_2}{2} \right) + 3[\omega' \left( \frac{\Lambda_1 + \Lambda_2}{2} \right)]^{1/2} \\
+ \omega'(\Lambda_2) \\
\end{array} \right] \\
+ \left( |\omega'(\Lambda_1)|^{1/2} + 3|\omega'(\Lambda_2)|^{1/2} \right)^{1/2} + \sqrt{\frac{3}{8}} \left( 3|\omega'(\Lambda_2)|^{1/2} + 9|\omega'(\Lambda_1)|^{1/2} \right)^{1/2} \\
+ \left( 3|\omega'(\Lambda_1)|^{1/2} + 9|\omega'(\Lambda_2)|^{1/2} \right)^{1/2} + \sqrt{\frac{3}{8}} \left( 3|\omega'(\Lambda_2)|^{1/2} + 9|\omega'(\Lambda_1)|^{1/2} \right)^{1/2} \\
+ \left( 3|\omega'(\Lambda_1)|^{1/2} + 9|\omega'(\Lambda_2)|^{1/2} \right)^{1/2}, \\
\end{align*}
\]

\( (83) \)

where \( \kappa_1^{-1} + \kappa_2^{-1} = 1 \).

Proof. The proof is skipped as it is on the same lines which are followed in the proof of Corollary 3.
7. Conclusions

The current study discusses the Simpson’s type quantum inequalities. The study brings into the spotlight some new estimates for Simpson’s type quadrature rules keeping four and six panels. The inequalities due to this study proves to be analogous to the classical rules. Our inequality (37) is analogous to the Simpson’s 1/3 rule (1) for four panels while our inequality (58) given in Theorem 8 is a quantum version of Simpson’s 3/8 rule (2) for six panels.

We also notice that Corollary 1 is a special case of Lemma 1 while Corollary 2 is a special case of Theorem 5. Corollary 4 gives an identity for six panels for classical integrals and is a special case of identity given in Lemma 2. Corollary 5 is obtained as a special case from Theorem 8.

It is worth mentioning that the theory of quantum differentiable and integrable functions is not completely parallel to the classical calculus. Indeed, the $q_r$-Hölder’s inequality is weak compared to classical cases. Our results in Corollarys 3 and 6 are not special cases of Theorems 6 and 9, respectively. We have presented a counter example to explain the limiting nature of Hölder’s inequality in quantum framework of calculus. In some recent papers, the researchers have used the $q_r$-Hölder’s inequality in the strong form which should be carefully re-examined. Finally, we found that the application of $q_r$-Hölder’s inequality gives some new variants of the classical results. Now, due to this varying nature, researchers should not only address the special feature.

Finally, we remark that the result in Corollarys 2 and 5 reveals, in comparison with the established inequalities given in Remarks 2 and 3, that if the number of panels are doubled, then the error reduces half of the previous.

We feel that this study will inspire the researchers working in the area of quantum calculus.

Author Contributions: Conceptualization, M.R. and M.A.; methodology, M.R., M.A. and M.S.; software, M.R.; validation, M.R., M.A. and M.V.-C.; formal analysis, M.A., M.S. and G.R.; investigation, M.A. and A.K.; resources, M.R. and A.K.; data curation, M.R.; writing—original draft preparation, M.R., M.A. and M.S.; writing—review and editing, M.V.-C., A.K. and G.R.; visualization, M.V.-C. and A.K.; supervision, M.A.; project administration, M.V.-C.; funding acquisition, M.V.-C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References
1. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On Simpson’s inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [CrossRef]
2. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of Simpson’s type for s-convex functions with applications. *RGMIA Res. Rep. Coll.* **2009**, *4*, 1–18.
3. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson’s type for convex functions. *RGMIA Res. Rep. Coll.* **2010**, *13*, 2.
4. Noor, M.A.; Noor, K.I.; Iftikhar, S. Some Newton’s type inequalities for harmonic convex functions. *J. Adv. Math. Stud.* **2016**, *9*, 7–16.
5. Noor, M.A.; Noor, K.I.; Iftikhar, S. Newton inequalities for p-harmonic convex functions. *Honam. Math. J.* **2018**, *40*, 239–250.
6. Iftikhar, S.; Komam, P.; Erden, S. Newton’s type integral inequalities via local fractional integrals. *Fractals* **2020**, 2020. [CrossRef]
7. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2001.
8. Brito, A.M.C.; Cruz, D.A. Symmetric Quantum Calculus. Ph.D. Thesis, Aveiro University, Aveiro, Portugal, 2012.
9. Ernst, T. *The History of Q-Calculus and New Method*; Department of Mathematics Uppsala University: Uppsala, Sweden, 2000.
10. Jackson, F.H. On $q$-functions and a certain difference operator. *Trans. Roy. Soc. Edin.* **1908**, *46*, 253–281. [CrossRef]
11. Jackson, F.H. On $q$-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
12. Almeida, R.; Torres, D.F.M. Non-differentiable variational principles in terms of a quantum operator. *Math. Methods Appl. Sci.* **2011**, *34*, 2231–2241.
13. Cresson, J.; Frederico, G.S.F.; Torres, D.F.M. Constants of motion for non-differentiable quantum variational problems. *Topol. Methods Nonlinear. Anal.* **2009**, *33*, 217–231. [CrossRef]
14. Askey, R.; Wilson, J. Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomials. *Mem. Am. Math. Soc.* 1985, *54*, 1–35.
15. Ismail, M.E.H.; Simeonov, P. $q$-Difference operators for orthogonal polynomials. *J. Comput. Appl. Math.* **2009**, *233*, 749–761. [CrossRef]
16. Sofonea, D.F. Numerical analysis and $q$-calculus. *I. Octogon.* **2003**, *11*, 151–156.
17. Srivastava, H.M. Some generalizations and basic (or $q$-) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inf. Sci.* 2011, *5*, 390–444.
18. Srivastava, H.M.; Choi, J. *Zeta and q-Zeta Functions and Associated Series and Integrals*; Elsevier: Amsterdam, The Netherlands, 2012.
19. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [CrossRef]
20. Noor, M.A.; Cristescu, G.; Awan, M.U. Bounds having Riemann type quantum integrals via strongly convex functions. *Adv. Differ. Equ.* **2019**, *2019*, 1–13. [CrossRef]
21. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum integral inequalities for co-ordinated convex functions. *Appl. Math. Comput.* **2015**, *269*, 242–251. [CrossRef]
22. Liu, W.J.; Zhuang, H.F. Some quantum estimates of Hermite–Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2017**, *7*, 501–522.
23. Zhang, Y.; Du, T.S.; Wang, H.; Shen, Y.J. Different types of quantum integral inequalities via $(a, m)$-convexity. *J. Inequal. Appl.* **2018**, *2018*, 264. [CrossRef] [PubMed]
24. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* **2020**, *1–13*. [CrossRef]
25. Bin-Mohsin, B.; Awan, M.U.; Noor, M.A.; Riahi, L.; Noor, K.I.; Almutairi, B. New quantum Hermite–Hadamard inequalities utilizing harmonic convexity of the functions. *IEEE Access* **2019**, *7*, 20479–20483. [CrossRef]
26. Du, T.; Luo, C.; Yu, B. Certain Quantum estimates on the parametrized integral inequalities and their applications. *J. Math. Inequal.* **2021**, *15*, 201–228. [CrossRef]
27. Erḋen, S.; Iftikhar, S.; Delavar, R.M.; Kumam, P.; Thounthong, P.; Kumam, W. On generalizations of some inequalities for convex functions via quantum integrals. *RACSAM* **2020**, *114*, 1–15. [CrossRef]
28. Gauchman, H. Integral inequalities in $q$-calculus. *Comput. Math. Appl.* **2004**, *47*, 281–300. [CrossRef]
29. Jhanthanam, S.; Jessada, T.; Sotiris, N.; Kamsing, N. On $q$-Hermite–Hadamard inequalities for differentiable convex functions. *Mathematics* **2019**, *7*, 632. [CrossRef]
30. Khan, M.A.; Noor, M.; Nwaeeze, E.R.; Chu, Y.-M. Quantum Hermite—Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 1–20. [CrossRef]
31. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite—Hadamard inequalities via Riemann type quantum integrals. *Adv. Differ. Equ.* **2020**, *2020*, 1–15. [CrossRef]
32. Noor, M.A.; Cristescu, G.; Awan, M.U. Bounds having Riemann type quantum integrals via strongly convex functions. *Stud. Sci. Math. Hungar.* **2017**, *54*, 221–240. [CrossRef]
33. Nwaeeze, E.R.; Tameru, A.M. New parameterized quantum integral inequalities via $η$-quasiconvexity. *Adv. Differ. Equ.* **2019**, *2019*, 425. [CrossRef]
34. Riahi, L.; Awan, M.U.; Noor, M.A. Some complementary $q$-bounds via different classes of convex functions. *Politehn. Univ. Buchar. Sci. Bull. Ser. A Appl. Math. Phys.* **2017**, *79*, 171–182.
35. Sudsutad, W.; Ntouyas, S.K.; Tariboon, J. Integral inequalities via fractional quantum calculus. *J. Inequal. Appl.* **2016**, *2016*, 81. [CrossRef]
36. Zhuang, H.; Liu, W.; Park, J. Some quantum estimates of Hermite–Hadamard inequalities for quasi-convex functions. *Mathematics* **2019**, *7*, 152. [CrossRef]
37. Bermudo, S.; Körüs, P.; Valdés, J.E.N. On $q$-Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hungar.* **2020**, *162*, 364–374. [CrossRef]
38. Vivas-Cortez, M.J.; Liko, R.; Kashuri, A.; Hernández, J.E.H. New quantum estimates of trapezium–type inequalities for generalized $φ$-convex functions. *Mathematics* **2019**, *7*, 19. [CrossRef]
39. Vivas-Cortez, M.J.; Kashuri, A.; Liko, R.; Hernández, J.E.H. Quantum estimates of Ostrowski inequalities for generalized $φ$-convex functions. *Symmetry* **2019**, *11*, 16. [CrossRef]
40. Vivas-Cortez, M.J.; Liko, R.; Hernández, J.E.H. Some inequalities using generalized convex functions in quantum analysis. *Symmetry* **2019**, *11*, 14. [CrossRef]
41. Vivas-Cortez, M.J.; Ali, M.A.; Kashuri, A.; Sial, I.B.; Zhang, Z. Some new Newton’s ‘type integral inequalities for co-ordinated convex functions in quantum calculus. *Symmetry* **2020**, *12*, 1476. [CrossRef]