2-VECTOR BUNDLES, D-BRANES AND FROBENIUS MANIFOLDS

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ABSTRACT. We show that if $M$ is a Frobenius manifold of dimension $n$ such that $T_x M$ is semisimple for every $x \in M$, then there exists a canonical 2-vector bundle $B$ over $M$ of rank $n$. This 2-vector bundle encodes the information about the maximal category of $D$-branes associated to the open closed topological field theories defined by the Frobenius algebras $T_x M$. In particular this construction answers a conjecture of Graeme Segal in [Seg07]. We also explain the relation of the labels of the $D$-branes to Azumaya algebras and twisted vector bundles on the spectral cover $S$ of $M$.

1. INTRODUCTION

The aim of the present work is to give a positive answer to a remark of G. Segal, in [Seg07], about a possible relation between 2-vector bundles and the moduli space of topological field theories.

The geometric objects involved in this remark – 2-vector bundles – are a topological generalisation of the algebraic notion of 2-vector spaces. The notion of 2-vector spaces was introduced by M. Kapranov and V. Voedvodski in [KV94]. This notion is a categorification of the concept of vector space. The idea of 2-vector bundles was proposed as a geometric model for elliptic cohomology. Constructions and definitions of 2-vector bundles were proposed by J.L. Brylinsky [Bry98] and N. Baas, B. Dundas and J Rognes [BDR04a].

The ideas behind these constructions are related to physics, in particular string theory. At the beginning of the 90’s it was suggested, see for example [Fre94], that some form of 2-vector spaces should be attached to the endpoints of an open string. In [Seg07] Graeme Segal suggested that there might be a relation between 2-vector bundles and the moduli space of topological field theories.

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Two dimensional topological field theories can be algebraically described in terms of commutative Frobenius algebras. In the case of commutative semisimple Frobenius algebras G. Moore and G. Segal [MS06, ea09] found a geometric description of these algebras as the algebras of functions on finite sets equipped with a measure. These finite sets play the role of spacetimes in the theory. It is then natural to think that a smoothly varying family of 2d-topological field theories is a pair \((\pi : S \to M, f)\) formed by a smooth manifold \(M\) with a fixed finite sheeted covering space \(S \to M\) and a function \(f : S \to \mathbb{R}\). The points \(x\) of \(M\) parametrise the topological field theories of the family defined by the fibres \(\pi^{-1}(x)\) with the measure induced by \(f\). This type of structure appeared in the work of Saito about unfoldings of singularities. The structure reappeared in the notion of a Frobenius manifold defined by B. Dubrovin [Dub95]. A Frobenius manifold is basically a manifold \(M\) with the property that the tangent spaces \(T_xM\) have a structure of a Frobenius algebra \(\forall x \in M\). Frobenius manifolds define a geometric model for the solutions of WDVV equations. These equations capture the deformations of topological conformal field theories. When \(T_xM\) is semisimple for every \(x \in M\), then \(M\) has a canonically associated covering space \(S \to M\) called the spectral cover. This covering space has a natural function on it which provides the measure. This fact provides the connection between the viewpoints of Moore and Segal and the definition given by Dubrovin.

The plan of this paper is the following. In Section 2 we recall some basic facts and definitions about open and closed 2d-topological field theories, Frobenius algebras, Calabi-Yau categories and Frobenius manifolds, we also define Cardy categories. In the next section we describe 2-vector spaces, 2-vector bundles and twisted vector bundles. We shall give a summary of Moore and Segal description of the maximal category of D-branes in an open and closed topological field theory. In section 4 we introduce the notion of Cardy fibrations over Frobenius manifolds. Then in Section 5 we give a local characterisation of maximal Cardy fibrations and we show that maximal Cardy fibrations define two vector bundles. In the next section, Section 6 we show that some sectors of a maximal Cardy fibration are related to twisted vector bundles and Azumaya algebras over the spectral cover of a Frobenius manifold.

2. TOPOLOGICAL D-BRANES AND FROBENIUS MANIFOLDS
2.1. **Open and closed 2d topological field theories.** Let us roughly describe the basic aspects of 2d topological field theories (TFT). Precise definitions for closed theories can be found, for example, in the article of L. Abrams [Abr96]; the references for the relevant definitions for open and closed topological field theories are the article of G. Moore and G. Segal [MS06] and chapter 2 of [ea09].

The description of a closed TFT is as follows: Let Cob denote the (pseudo) category whose objects are closed, oriented, 1-dimensional manifolds. We shall consider the empty set as an object of Cob. If $N$ and $M$ are objects of Cob, then a morphism $N \to M$ is an ordered pair $(\Sigma, \phi)$ formed by a compact two dimensional oriented manifold $\Sigma$ and an orientation preserving diffeomorphism $\phi : \partial \Sigma \to N \sqcup -M$, where $-M$ denotes the manifold $M$ with the opposite orientation. A convenient way to describe a cobordism is to use the following diagram $N \to \Sigma \leftarrow -M$. Two cobordisms $(\Sigma, \phi)$ and $(\Sigma_1, \psi)$ are to be identified if there is an orientation preserving diffeomorphism $\alpha : \Sigma \to \Sigma_1$ such that the diagram

![Diagram of cobordisms](attachment:diagram.png)

commutes. Any object of Cob is diffeomorphic to a disjoint union of copies of the standard circle $S^1$ and the empty set. If $C_1$ and $C_2$ are objects of Cob and $\Sigma$ is a cobordism between them, the circles in $C_1$ are called *ingoing* and the circles in $C_2$ are called *outgoing*. The morphisms in Cob are generated by the following figures, where the outgoing circles are written to the right

![Figure 1. Generators of morphisms in Cob.](attachment:figure1.png)

The category Cob has a monoidal structure induced by the disjoint union of manifolds.
Definition 2.1. Let Vect denote the category of vector spaces over $\mathbb{C}$. A 2d-closed topological field theory (or TFT) is a functor $F : \text{Cob} \to \text{Vect}$ such that

$$F(C_1 \sqcup C_2) = F(C_1) \otimes F(C_2). \quad (1)$$

A closed topological field theory is determined by two vector spaces: The vector space $A = F(S^1)$ and the vector space $F(\emptyset)$. The multiplicative condition, given by equation (1), implies that $F(\emptyset)$ is the ground field $\mathbb{C}$. On the other hand the generators of the morphisms in Cob induce on $A$ the following structure

$$
\begin{align*}
\begin{matrix}
\text{C} & \xrightarrow{i} & \text{A}, \\
\text{A} \otimes \text{A} & \xrightarrow{\mu} & \text{A}, \\
\text{A} & \xrightarrow{\theta} & \mathbb{C}.
\end{matrix}
\end{align*}
$$

Figure 2. Structure of $A$.

We shall assume that the homomorphism associated to the cylinder is the identity. The morphisms in diagram must satisfy compatibility conditions. The algebraic structure induced on $A$ is the structure of a Frobenius algebra – the precise definition is in Section 2.2.

Categories of branes are obtained when one considers a bigger cobordism category, namely open and closed cobordism. The (pseudo) category $\text{Ocob}$ is the category where the objects are one dimensional, compact, oriented manifolds with (possibly empty) boundary. If the boundary is non-empty we shall suppose that each connected component of the boundary is labelled by an element of a fixed set $B$. The set $B$ is called the set of boundary conditions. Any element of $\text{Ocob}$ is diffeomorphic to a disjoint union of elements of the form given in figure 3.

A cobordism $\Sigma$ between two objects $C_0$ and $C_1$ of $\text{Ocob}$ is an oriented surface whose boundary consists of three parts $\partial \Sigma = C_0 \cup C_1 \cup \text{cstr}$. The part $C_{\text{cstr}}$ is called the constrained boundary and is a cobordism from $\partial C_0$ to $\partial C_1$. The components of $C_{\text{cstr}}$ are labelled in a way compatible with the labelling of $\partial C_0$ and $\partial C_1$. See figure 4.
Definition 2.2. An open and closed topological field theory is a functor $F : \text{Ocob} \to \text{Vect}$ satisfying the multiplicative axiom (1).

An open and closed topological field theory is algebraically described by a certain class of “self-dual” categories which we shall call Cardy categories, see Section 2.3. We shall write $E_{ab}$ for the image of the interval with labels $a$ and $b$. See figure 3.

2.2. Frobenius algebras. We will recall here some basic facts about Frobenius algebras. A general reference about Frobenius algebras over a field is \cite{Abr96}. We shall need also some basic definitions about Frobenius algebras over rings – see for example \cite{EN55}. Let $R$ denote a commutative ring. If $A$ is an $R$-module we shall write $A^* = \text{Hom}_R(A, R)$ for the dual module. If $A$ is an $R$-algebra, then $A^*$ has a natural structure of a left $A$-module given by

$$(a \varphi)(b) = \varphi(ab),$$

(2)
for $a, b$ in $A$ and $\varphi$ in $A^*$. We shall write $(, )$ for the evaluation map $A^* \times A \to R$.

**Definition 2.3.** Let $R$ denote a commutative ring. A $R$-Frobenius algebra is a quadruple $(A, e, \mu, \Phi)$ consisting of a finitely generated, projective, associative $R$-algebra $A$ with unit $e$, multiplication map $\mu : A \times A \to A$ and a left $A$-module isomorphism

$$\Phi : A \to A^*.$$  

In the case when $R = \mathbb{C}$ we obtain the following equivalent definition.

**Definition 2.4.** A Frobenius algebra is a quadruple $(A, e, \mu, \theta)$ consisting of a finitely generated, associative $\mathbb{C}$-algebra $A$ with unit $e$, multiplication map $\mu : A \times A \to A$ and a linear form $\theta : A \to \mathbb{C}$, called the trace, such that the bilinear form $g : A \times A \to \mathbb{C}$ given by:

$$g(x, y) = \theta(\mu(x, y))$$

is non-degenerate.

The trace $\theta$ is related to $\Phi$ via the identity $\theta(a) = \langle \Phi(a), e \rangle$.

**Remark 2.5.** The form $g$ will be called the metric. The trace $\theta$ can be recovered from $g$ and the unit element $e$ by $\theta(x) = g(x, e)$.

We shall usually write $(x, y) \to xy$ for the multiplication map $\mu(x, y)$. An important piece of information associated to a Frobenius algebras is the trilinear map

$$c : V \times V \times V \to \mathbb{C},$$

given by $c(x, y, z) = \theta(xyz).$  

**Definition 2.6.** A $\mathbb{C}$-Frobenius algebra $A$ is semisimple if it has no nilpotents.

An important result in the theory of semisimple Frobenius algebras over $\mathbb{C}$ is given by the following proposition, see [Hit97, Prop 2.2] for a proof.

**Proposition 2.7.** If $A$ is a semisimple commutative $\mathbb{C}$-Frobenius algebra of dimension $n$, then there exists a basis $e_1, \ldots, e_n$ of $A$ such that:
2.3. Calabi-Yau and Cardy categories. Let $R$ be a commutative ring.

**Definition 2.8.** A Calabi-Yau category over $R$ is a category $\mathcal{C}$ satisfying:

1. For any pair of objects $a, b$ of $\mathcal{C}$ the space of homomorphisms
   
   $E_{ab} := \text{Hom}_{\mathcal{C}}(a, b)$

   is a finitely generated, projective $R$ module.

2. The composition
   
   $E_{ab} \times E_{bc} \to E_{ac}$

   is an $R$-bilinear map.

3. For each $a \in \text{Obj}(\mathcal{C})$ there exists a homomorphism of $R$-modules
   
   $\theta_a : E_{aa} \to R$, \hspace{1cm} (7)

   that induces a left $E_{aa}$-modules isomorphism

   $E_{aa} \to E_{aa}^*$ \hspace{1cm} (8)

   where $E_{aa}^*$ denotes the dual $R$-module. This condition implies that $E_{aa}$ is a Frobenius algebra over $R$.

4. The pairings

   $E_{ab} \otimes E_{ba} \to E_{aa} \xrightarrow{\theta_a} R$ \hspace{1cm} (9)

   $E_{ba} \otimes E_{ab} \to E_{bb} \xrightarrow{\theta_b} R$ \hspace{1cm} (10)

   induce isomorphisms $E_{ab} \simeq E_{ba}^*$. If $\varphi \in E_{ab}$, and $\psi \in E_{ba}$, then

   $\theta_a(\varphi \cdot \psi) = \theta_b(\psi \cdot \varphi)$ \hspace{1cm} (11)
**Remark 2.9.** This definition is an extension of the usual definition of Calabi-Yau category [Cos07]. The main change is that we replace $\mathbb{C}$-vector spaces by finitely generated, projective $R$ modules. This is essentially the same change from the definition of Frobenius algebra over a field to Frobenius algebras over a ring.

**Definition 2.10.** Let $A$ be a commutative Frobenius algebra over $R$. A Calabi-Yau category over $A$ is a category $\mathcal{C}$ satisfying:

1. For each object $a$ there exists a pair of $R$-linear morphisms
   
   
   $\iota_a : A \to E_{aa}$, and $\iota^a : E_{aa} \to A$.

   such that
   
   (a) $\iota_a$ is a homomorphism of $R$-algebras.
   
   (b) For $r \in A$ and $\psi \in E_{ab}$ it holds that

   $$\iota_a(r)\psi = \psi\iota_b(r).$$

   (c) The morphisms $\iota_a$ and $\iota^a$ are adjoints in the sense that

   $$\theta^{-1}(\iota^a(\psi)) = \theta^{-1}(\psi\iota_a(\phi)),$$

   for all $\psi \in E_{aa}$, $\phi \in A$.

   Since $E_{ab}$ and $E_{ba}$ are in duality, if $E_{ab}$ is a free $R$-module and $\psi_\nu$ is a basis of $E_{ab}$ let $\psi^\nu$ be the dual basis of $E_{ba}$. Define $\pi_a^b : E_{aa} \to E_{bb}$ by

   $$\pi_a^b(\psi) := \sum_\nu \psi_\nu \psi^\nu.$$

   **Proposition 2.11.** If $E_{ab}$ (and $E_{ba}$) are free $R$-modules, then the homomorphism $\pi_a^b$ is independent of the choice of basis $\psi_\nu$ and $\psi^\nu$.

   **Proof.** Let $\psi_\alpha$ and $\psi^\alpha$ be another pair of dual basis of $E_{ab}$ and $E_{ba}$. Then

   $$\psi_\alpha = \sum_\nu a^\nu_\alpha \psi_\nu,$$

   and

   $$\psi^\alpha = \sum_\mu b_\mu^\alpha \psi^\mu,$$
for certain matrices \([a^\nu_\alpha]\) and \([b^\beta_\mu]\). The conditions \(\delta^\beta_\alpha = \varphi_\alpha \varphi^\beta\) and \(\delta^\mu_\nu = \psi_\nu \psi^\mu\) imply that

\[
\delta^\beta_\alpha = \varphi_\alpha \varphi^\beta = \left(\sum_\nu a^\nu_\alpha \psi^\nu\right) \left(\sum_\mu b^\mu_\beta \psi^\mu\right) = \sum_\nu \sum_\mu a^\nu_\alpha b^\mu_\beta \psi^\nu \psi^\mu = \sum_\nu \sum_\mu a^\nu_\alpha b^\mu_\beta \delta^\nu_\mu
\]

\[
= \sum_\nu a^\nu_\alpha b^\beta_\mu.
\]

Hence the matrix \([a^\nu_\alpha]\) is the inverse of the matrix \([b^\beta_\mu]\). Therefore if \(\psi \in E_{aa}\)

\[
\sum_\alpha \varphi_\alpha \psi^\alpha = \sum_\alpha \sum_\nu \sum_\mu a^\nu_\alpha b^\mu_\alpha \psi^\nu \psi^\mu = \sum_\nu \sum_\mu \delta^\nu_\nu \psi^\nu \psi^\nu = \sum_\nu \psi^\nu \psi^\nu.
\]

Q.E.D.

We want to extend the definition of \(\pi^a_b\) to finitely generated, projective \(R\)-modules. Let \(X = \text{spec}(R)\). Then the modules \(E_{ab}\) and \(E_{ba}\) define locally free sheaves \(\tilde{E}_{ab}\) and \(\tilde{E}_{ba}\) over \(X\). Let \(U_i, i \in I\) be a covering of \(X\) such that \(\tilde{E}_{ab}(U_i)\) and \(\tilde{E}_{ba}(U_i)\) are \(O_X(U_i)\)-free modules. Then for each pair of indices \(i, j \in I\) also the modules \(\tilde{E}_{ab}(U_i \cap U_j)\) and \(\tilde{E}_{ba}(U_i \cap U_j)\) are free \(O_X(U_i \cap U_j)\) modules.

Hence we have homomorphisms

\[
\pi^a_b(U_i) : \tilde{E}_{aa}(U_i) \to \tilde{E}_{bb}(U_i),
\]

\[
\pi^a_b(U_j) : \tilde{E}_{aa}(U_j) \to \tilde{E}_{bb}(U_j),
\]

\[
\pi^a_b(U_i \cap U_j) : \tilde{E}_{aa}(U_i \cap U_j) \to \tilde{E}_{bb}(U_i \cap U_j).
\]

By Proposition 2.11 the restriction of \(\pi^a_b(U_i)\) and \(\pi^a_b(U_j)\) to \(U_i \cap U_j\) coincide with \(\pi^a_b(U_i \cap U_j)\).

Hence there is a globally well defined homomorphism of sheaves \(\pi^a_b : \tilde{E}_{aa}(X) \to \tilde{E}_{bb}(X)\) which is the same as a module homomorphism

\[
\pi^a_b : E_{aa} \to E_{bb}.
\]

**Definition 2.12.** A Cardy category is a Calabi-Yau category over a \(R\)-Frobenius algebra \(A\) that satisfy the following condition, called the Cardy condition,

\[
\pi^a_b = \iota_b \circ \iota^a.
\]

(15)
for any pair of labels $a, b$.

2.4. **The maximal category of branes.** In this section we will discuss some results of G. Moore and G. Segal [MS06] regarding the structure of the algebras $E_{ab}$ corresponding to the open sector of an open and closed topological field theory. We will only consider the case for which the Frobenius algebra $A$ of the closed sector is semisimple which is the hypothesis used by Moore and Segal.

Let $A$ be an associative, commutative, semisimple Frobenius algebra over $\mathbb{C}$, and suppose $\text{dim}_\mathbb{C} A = n$. We then have a system of orthogonal idempotents $e_1, \ldots, e_n$ which determine the simple components; i.e. $A \cong \bigoplus_i \mathbb{C} e_i$, and each summand $\mathbb{C} e_i$ is isomorphic to $\mathbb{C}$. The prime ideals of $A$ can be identified with the set $X = \{e_1, \ldots, e_n\}$. This set plays the role of space-time. The algebra $A$ is the algebra of observables. The Frobenius structure on $A$ induces a measure $\mu$ on $X$ by $\mu(e_i) = \theta(e_i)$.

**Theorem 2.13** ([MS06], Theorem 2). For each object $a \in \mathcal{B}$, the algebra $E_{aa}$ is semisimple.

**Remark 2.14.** By the previous result the algebra $E_{aa}$ can be regarded as a sum $\bigoplus_i M(a, i)$ of matrix algebras $M(a, i) := M_{d(a, i)}(\mathbb{C})$. In other words, it is possible to find complex vector spaces $V_{a,i}$ such that

$$E_{aa} \cong \bigoplus_{i=1}^n \text{End}(V_{a,i}),$$

(16)

where $\text{dim} V_{a,i} = d(a, i)$. Moreover, the matrix algebra $M(a, i) = \text{End}(V_{a,i})$ corresponds under the isomorphism (16) with the subalgebra $i_a(e_i)E_{aa}$. Elements of $E_{aa}$ will be denoted by a tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$, where $\sigma_i \in M(a, i)$. If $\epsilon_i \in E_{aa}$ denotes the tuple consisting of the identity matrix $1_{a,i} \in M(a, i)$ in the $i$-th coordinate and all others equals to zero, then $i_a(\epsilon_i) = \epsilon_i$ or is equal to zero.

We can give an explicit characterization for the morphisms $\theta_{a,i} : \mathcal{I}^a$ and $\pi^a_{\mathcal{I}}$. For $\sigma = (\sigma_1, \ldots, \sigma_n) \in E_{aa}$, the equality $\theta_a(\sigma \tau) = \theta_a(\tau \sigma)$ implies that

$$\theta_a(\sigma) = \sum_i \lambda_i \text{tr}(\sigma_i)$$

for some constants $\lambda_i \in \mathbb{C}$. 
Fixing a square root $\lambda_i = \sqrt{\theta(e_i)}$ for each $i$, we arrive at the following expressions

$$\theta_a(\sigma) = \sum_i \sqrt{\theta(e_i)} \text{tr}(\sigma_i),$$

$$i^a(\sigma) = \sum_i \frac{\text{tr}(\sigma_i)}{\sqrt{\theta(e_i)}} e_i,$$

$$\pi_i^a(\sigma) = \sum_i \frac{\text{tr}(\sigma_i)}{\sqrt{\theta(e_i)}} i_b(e_i),$$

where in the last equality, the trace $\text{tr}$ is the one corresponding to $E_{aa}$.

A characterization like the one provided in theorem 2.13 holds for the spaces $E_{ab}$.

Lemma 2.15 ([MS06]). If $C$ is semisimple, then for each pair $a, b \in \mathcal{B}$ we have an isomorphism

$$E_{ab} \cong \bigoplus_{i=1}^n \text{Hom}_C(V_{a,i}, V_{b,i}),$$

for some finite-dimensional complex vector spaces $V_{a,i}, V_{b,i}$.

Note that the vector spaces in the right hand side of equation (17) are the ones appearing in the decompositions of $E_{aa}$ and $E_{bb}$; see remark 2.14.

2.5. Frobenius manifolds. In this section we shall briefly review the definition of a Frobenius Manifold in the sense of Manin [Man99, Definition 1.1.1]. We shall write $M$ or $(M, \mathcal{O}_M)$ for a manifold, where the word manifold means a $C^\infty$, real analytic or complex analytic manifold, and $\mathcal{O}_M$ denotes the (complexified) structure sheaf of $M$– we shall assume that $\mathcal{O}_M$ is a sheaf of $C$ algebras and that for every $x \in M$ the reduced field $k_x$ is $C$. We shall write $\mathcal{T} M$ for the complexified tangent sheaf and $\mathcal{T}^* M$ for the complexified cotangent sheaf. The sections of $\mathcal{T} M$ acts as derivations on $\mathcal{O}_M$. If $(x^1, \ldots, x^n)$ is a system of coordinates, then the $x^i$ determine vector fields $\partial_i$ such that

$$df = \sum_{i=1}^{m+n} dx^i \partial_i f.$$ 

The vector fields $\partial_i$ locally generate $\mathcal{T} M$. The one forms $dx^i$ locally generate the cotangent sheaf.

Definition 2.16. A manifold $M$ with multiplication on the tangent sheaf is a triple $(\mathcal{T} M, \mu, e)$, where

$$\mu : \mathcal{T} M \otimes \mathcal{T} M \to \mathcal{T} M$$

is an associative $\mathcal{O}_M$-bilinear map of sheaves and $e$ is a global vector field which is the unit for $\mu$. 
If a manifold $M$ has a multiplication on the tangent sheaf, then the deviation from a Poisson algebra structure on $\mathcal{T}M$ is given by the following expression
\[
P_x(z, w) := [x, \mu(z, w)] - \mu([x, z], w) - \mu(z, [x, w]).
\] (19)

**Definition 2.17.** A manifold $M$ with multiplication on the tangent sheaf is an $F$-manifold, see for example [HM99], if the multiplication $\mu$ satisfies
\[
P_{\mu(x, y)}(z, w) = \mu(x, P_y(z, w)) + \mu(y, P_x(z, w)).
\] (20)

**Definition 2.18.** An affine flat structure on a manifold $M$ is a subsheaf $\mathcal{T}_F M$ of the tangent sheaf $\mathcal{T}M$ of linear spaces such that $\mathcal{T}M = \mathcal{T}_F M \otimes_k \mathcal{O}_M$ and the tangent bracket of pairs of its sections vanish. The elements of $\mathcal{T}_F M$ are called flat vector fields.

**Definition 2.19.** A metric $g$ on a manifold $M$ is compatible with an affine flat structure if $g(x, y)$ is constant for flat vector fields $x, y$.

The next condition defines the compatibility between the metric and the multiplication on the tangent sheaf.

**Definition 2.20.** A metric on a manifold with multiplication on the tangent sheaf is invariant if
\[
g(\mu(x, y), z) = g(x, \mu(y, z)).
\] (21)

**Remark 2.21.** An invariant metric determines a tensor $c$ given by $c(x, y, z) = g(\mu(x, y), z)$ and a morphism $\theta_M : \mathcal{T}M \rightarrow \mathcal{O}_M$ given by $\theta(x) = g(e, x)$ – see 2.5.

**Definition 2.22.** Let $M$ be a manifold. A pre-Frobenius structure on $M$ is a triple $(\mathcal{T}_F M, g, c)$ formed by an affine flat structure $\mathcal{T}_F M$ on $M$, a compatible metric $g$, and an even symmetric tensor
\[
c : S^3(\mathcal{T}M) \rightarrow \mathcal{O}_M.
\]
A manifold with a pre-Frobenius structure will be called a pre-Frobenius manifold.

A pre-Frobenius manifold has a multiplication $\mu$ on the tangent sheaf given by
\[
\mathcal{T}M \otimes \mathcal{T}M \xrightarrow{\zeta} \mathcal{T}^*M \xrightarrow{\bar{\zeta}} \mathcal{T}M.
\]
The metric in this case is invariant under the multiplication. A local potential $\Phi$ for $\mathcal{F}M$ is an even function such that for any flat local tangent fields $x, y, z$,

$$c(x, y, z) = (xyz)\Phi$$  \hfill (22)

A pre-Frobenius manifold is called potential if $c$ admits everywhere a local potential.

**Definition 2.23.** A Frobenius manifold is an associative, potential pre-Frobenius manifold.

**Definition 2.24.** A Frobenius manifold is called semisimple, if there is everywhere a local isomorphism of sheaves of algebras $\mathcal{F}M \simeq \mathcal{O}_M^n$.

2.5.1. The spectral cover $S_M$. Let $M$ be an $n$-dimensional F-manifold in an analytic category. Let $S_M$ be the relative affine spectrum of the $\mathcal{O}_M$-algebra $\mathcal{F}M$. The space $S_M$ is a manifold in the same class that $M$ and it is endowed with two structure maps $\bar{\pi} : S_M \to M$ and $s : \mathcal{F}M \to \bar{\pi}_*(\mathcal{O}_{S_M})$. The morphism $s$ is an isomorphism of sheaves. If $M$ is a semisimple manifold, then $\bar{\pi}$ is étale [Man99, section 8.1]. When $M$ is fixed we shall write $S$ for $S_M$.

If $M$ is a Frobenius manifold, then there is a natural trace $\theta_M : \mathcal{O}_S \to \mathcal{O}_M$ given by the composition

$$\mathcal{O}_S \to \pi_*\mathcal{O}_S \xrightarrow{s^{-1}} \mathcal{F}M \xrightarrow{\theta_M} \mathcal{O}_M,$$

where $\theta_M$ is the trace defined in Remark 2.21. With this structure the Frobenius algebras obtained from $(\pi_*\mathcal{O}_M, \theta_M)$ are isomorphic, via $s$, to the Frobenius algebras obtained from $(\mathcal{F}M, \theta_M)$.

The construction of the spectral cover is part of a more general framework, namely that of the analytic spectrum, introduced by C. Houzel [Hou61] to study finite morphism of analytic spaces. He defines the analytic spectrum for algebras of finite presentation over an analytic space, which include finite algebras (those algebras which are coherent modules): let $\Gamma$ be a finite presentation $\mathcal{O}_M$-algebra and $f : N \to M$ a space over $M$ (in particular, if $E$ is a vector bundle, then its sheaf of sections is coherent and thus of finite presentation). Define a contravariant functor $S_\Gamma$ from spaces over $M$ to the category of sets by

$$S_\Gamma(N, f) = \text{Hom}_{\mathcal{O}_N\text{-alg}}(f^*\Gamma, \mathcal{O}_N)$$
This functor is then representable, and we have a bijection between $S_T(N, f)$ and holomorphic maps $N \to \text{Specan} \Gamma$, where $\text{Specan} \Gamma$ is the analytic spectrum. Even with these nice algebras, the space $\text{Specan} \Gamma$ may have singularities. For detailed descriptions we refer the reader to [Hou61]; check also [Fis76]. The case in which we are interested deals with a bundle of algebras $E$ such that $E_x$ is semisimple for each $x$ (see below). If $M = N$ and $f : M \to M$, then the construction of the analytic spectrum provides a bijection between the subspace of the dual bundle $(f^*E)^*$ consisting of morphisms of algebras and maps $M \to \text{Specan} \Gamma_E$. For $f = \text{id}_M$, this is just expressing that every morphism of algebras $\varphi : E \to C$ is determined by a map $M \to \text{Specan} \Gamma_E$ (for each $x$ this is just choosing the kernel of the restriction $\varphi_x : E_x \to C$).

**Proposition 2.25.** For a bundle of algebras $E$ over $M$ there exists an isomorphism of $\mathcal{O}_M$-algebras

$$\pi_*\mathcal{O}_{S_E} \cong \Gamma_E,$$

(23)

**Proof.** consider the sequence of maps

$$\Gamma_E \longrightarrow p_*\mathcal{O}_{E^*} \longrightarrow \pi_*\mathcal{O}_{S_E},$$

$$X \mapsto \tilde{X} \mapsto \tilde{X}|_S$$

where $p : E^* \to M$ is the canonical projection (we are considering $S_E$ as a subspace of $E^*$; then $\pi$ is just the restriction of $p$ to $S_E$), and $\tilde{X} : p^{-1}(U) = E^*|_U \to C$ is the map given by

$$\tilde{X}(x, \varphi) = \varphi(X(x)).$$

The composite map

$$\Gamma_E \longrightarrow \pi_*\mathcal{O}_{S_E}$$

(24)

is then easily seen to be an isomorphism of $\mathcal{O}_M$-algebras (recall that $(x, \varphi) \in S_E$ if and only if $\varphi$ is an algebra homomorphism).

The inverse can be described easily: Given a map $\bar{f} : \pi^{-1}(U) \to C$, let $X_{\bar{f}} \in \Gamma_E(U)$ be the local section defined as follows: pick an $x \in U$ an assume that $U$ is semisimple (if it is not, we can choose a smaller open neighborhood around $x$); let $\{e_i\}$ be a local frame of idempotent sections for $E|_U$. Then

$$X_{\bar{f}}(x) = \sum_i \bar{f}(x, \varphi_i)e_i(x),$$
where $\varphi_i : E_x \to \mathbb{C}$ is the algebra homomorphism which verifies $\varphi_i(e_i(x)) \neq 0$ (in fact, $\varphi_i(e_i(x)) = 1$ as $\varphi_i(1) = 1$). The assignment $\tilde{f} \mapsto X_{\tilde{f}}$ is then the inverse of (24). Q.E.D.

Moreover, each summand $O_{S_N} \otimes O_{x_0} \mathbb{C}$ is invariant under the action of any multiplication operator, and thus it is the space of generalized eigenvectors.

It holds the following result, which is in fact Housel’s definition of the spectral cover.

**Proposition 2.26.** Let $E \to M$ be a bundle of associative and commutative algebras. Then

1. The analytic spectrum $S_E$ represents the functor (which we denote with the same symbol) $S_E(N, f) = \text{Hom}_{\mathcal{O}_N - \text{alg}}(f^*E, \mathbb{C})$ from spaces over $M$ to the category of sets (here $\mathbb{C}$ means the trivial line bundle $N \times \mathbb{C}$).

2. If $E_x$ is semisimple for each $x$, then $\pi : S_E \to M$ is a covering space.

### 3. 2-VECTOR SPACES AND 2-VектOR BUNDLES

#### 3.1. 2-VECTOR SPACES

We will now give an overview of the categorical analogues of vector spaces and vector bundles. There are several definitions of 2-vector space in the literature due, among others, to Kapranov-Voevodsky [KV94], Baez-Crans [BC04] and Elgueta [Elg06]. We will adopt the definition of 2-vector spaces of Kapranov and Voevodsky. The references for our treatment of monoidal categories are [Mac71, Kel82].

**Definition 3.1.** A rig category is a category $\mathcal{R}$ with two symmetric monoidal structures $(\mathcal{R}, \oplus, 0)$ and $(\mathcal{R}, \otimes, 1)$ together with distributivity natural isomorphisms $X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z)$ and $(X \oplus Y) \otimes Z \to (X \otimes Z) \oplus (Y \otimes Z)$ verifying some coherence axioms which are detailed in [Lap72, Kel74].

An important example, which will be extensively used in what follows, is the category Vect of finite dimensional vector spaces over $\mathbb{C}$. The operations are given by direct sum (with $0 = \{0\}$, the trivial vector space) and tensor product (with $1 \cong \mathbb{C}$).

**Definition 3.2.** Let $\mathcal{R}$ be a rig category. A left module category over $\mathcal{R}$ is a monoidal category $(\mathcal{M}, \oplus, 0)$ together with an action (bifunctor)

$$\otimes : \mathcal{R} \times \mathcal{M} \to \mathcal{M}$$
and natural isomorphisms

\[ A \otimes (B \otimes X) \longrightarrow (A \otimes B) \otimes X \]

\[ (A \oplus B) \otimes X \longrightarrow (A \otimes X) \oplus (B \otimes X) \]

\[ A \otimes (X \oplus Y) \longrightarrow (A \otimes X) \oplus (A \otimes Y) \]

\[ \tau_X = \tau : 1 \otimes X \longrightarrow X \quad \rho_A = \rho : A \otimes 0 \longrightarrow 0 \quad \lambda_X = \lambda : 0 \otimes X \longrightarrow 0 \]

for any given objects \( A, B \in R \) and \( X, Y \in M \), which are required to satisfy coherence conditions analogous to the ones for a rig category. Right module categories are defined analogously.

An \( R \)-module functor between \( R \)-modules \( M \) and \( N \) is a functor \( F : M \rightarrow N \) such that

\[ F(X \oplus Y) \cong F(X) \oplus F(Y) \]

\[ F(A \otimes X) \cong A \otimes F(X). \]

The isomorphisms should be natural in \( X \) and \( Y \) in the first case and natural in \( A \) and \( X \) in the second case.

Given \( n \in \mathbb{N} \), consider now the product category \( \text{Vect}^n \); its objects and morphisms are \( n \)-tuples of vector spaces and linear transformations respectively. The \( \text{Vect} \) module structure is provided by the operations

\[ (V_1, \ldots, V_n) \oplus (W_1, \ldots, W_n) = (V_1 \oplus W_1, \ldots, V_n \oplus W_n), \]

\[ V \otimes (V_1, \ldots, V_n) = (V \otimes V_1, \ldots, V \otimes V_n). \]

Any object \((V_1, \ldots, V_n)\) can be decomposed, just like vectors in euclidean \( n \)-space, in the following way

\[ (V_1, \ldots, V_n) = (V_1 \otimes C_1) \oplus \cdots \oplus (V_n \otimes C_n), \]

where \( C_i \) is the tuple whose \( i \)-th entry is equal to \( C \) and all others equal to the trivial vector space. Using this decomposition any \( \text{Vect} \)-module functor can be determined on objects by its values in each \( C_i \),

\[ F(V_1, \ldots, V_n) \cong (V_1 \otimes F(C_1)) \oplus \cdots \oplus (V_n \otimes F(C_n)). \]  

(25)

We can define some extra structure in the category of \( R \)-modules by introducing morphisms between morphisms or 2-arrows. Given two \( R \)-modules \( M \) and \( N \) and module functors \( F, G : \)
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M → N, we define a 2-morphism θ : F → G as a natural transformation. This provides the category of R-modules with a structure of 2-category.

**Definition 3.3.** A Vect-module category V is called a 2-vector space if it is Vect-module equivalent to the product Vect^n for some natural number n. In other words, V is a 2-vector space if and only if there exists a natural number n and a Vect-module functor V → Vect^n which is also an equivalence of categories.

The proof of the following theorem can be found in [KV94].

**Theorem 3.4.** If F : Vect^n → Vect^m is an equivalence, then n = m.

By the previous result, the number n in definition 3.3 is well defined and it is called the rank of the 2-vector space V. The 2-vector space Vect^n plays, in this categorical setting, the same role that the space C^n plays in linear algebra. We will denote by 2Vect the (2-)category of 2-vector spaces of finite rank. Morphisms between 2-vector spaces can be characterised in a similar way as linear maps between vector spaces. To see this, consider first an m × n matrix

\[
A = \begin{pmatrix}
V_{11} & \cdots & V_{1n} \\
\vdots & \ddots & \vdots \\
V_{m1} & \cdots & V_{mn}
\end{pmatrix}
\]

where the entries V_{ij} are C vector spaces of finite dimension. If V := (V_1, ..., V_n) ∈ Vect^n, then the product

\[
AV = \left( \sum_j V_{ij} \otimes V_j, \ldots, \sum_j V_{mj} \otimes V_j \right)
\]

is a well defined object of the category Vect^m; given now a map f := (f_1, ..., f_n) : V → W, where W := (W_1, ..., W_n), there exists an induced map Af : AV → AW given by

\[
Af = \left( \sum_j \text{id}_{ij} \otimes f_j, \ldots, \sum_j \text{id}_{mj} \otimes f_j \right),
\]

where id_{ij} : V_{ij} → V_{ij} is the identity map. Moreover, the correspondence

\[
V \mapsto AV \\
f \mapsto Af
\]
is a Vect-module functor $\text{Vect}^n \to \text{Vect}^m$. Composition of such morphisms is given by usual multiplication of matrices, and two matrices $A = (V_{ij})$ and $B = (W_{ij})$ of the same size are naturally isomorphic if and only if $V_{ij}$ is isomorphic to $W_{ij}$ for each $i, j$.

Note that equation (25) readily implies that a morphism $F : \text{Vect}^n \to \text{Vect}^m$ is naturally isomorphic to the $m \times n$ matrix with columns given by $F(C_1), \ldots, F(C_n)$. For a morphism $F : V \to W$ between 2-vector spaces, if $u : V \to \text{Vect}^n$ and $v : W \to \text{Vect}^m$ are equivalences with inverses $\tilde{u}$ and $\tilde{v}$ respectively, then $vF\tilde{u}$ is naturally isomorphic to a matrix $A$, and hence $F$ can be represented as $\tilde{v}Au$ for some matrix $A$.

Let now $A = (V_{ij})$ be an $n \times n$ matrix which is an equivalence $\text{Vect}^n \to \text{Vect}^n$, and let $B = (W_{ij})$ represent the inverse, up to equivalence. As the identity morphism of $\text{Vect}^n$ can be represented by the “scalar” matrix $C \text{Id}$, we have natural isomorphisms $AB \cong C \text{Id} \cong BA$. Taking dimensions coordinatewise we can form the dimension matrices $d(A) := (\dim V_{ij})$ and $d(B) = (\dim W_{ij})$. Then, as the dimension matrices has natural entries, necessarily $\det d(A) = \pm 1$. But not every matrix satisfying this property is in fact an equivalence, and this is the main problem behind the short supply of equivalences $\text{Vect}^n \to \text{Vect}^n$. For example, take $n = 2$ and consider the morphisms given by the matrices

$$A_k = \begin{pmatrix} C & C \\ C^{k-1} & C^k \end{pmatrix}.$$ 

Then $d(A_k) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & k \end{smallmatrix}\right)$ and $\det d(A_k) = 1$. But, no matter which $k \in \mathbb{N}$ we choose, there is no inverse for $A_k$, and hence it is not an equivalence of 2-vector spaces. The example below explicitly shows the scarcity of equivalences for $n = 2$.

**Example 3.5.** Let $A = (V_{ij})$ be an autoequivalence of $\text{Vect}^2$ and $B = (W_{ij})$ and inverse. Let $a_{ij} := \dim V_{ij}$, $b_{ij} := \dim W_{ij}$ and then $d(A) = (a_{ij})$ and $d(B) = (b_{ij})$. From the natural isomorphisms $AB \cong C \text{Id} \cong BA$ we deduce that the following equations must hold

$$a_1b_{1j} + a_2b_{2j} = \delta_{ij}, \quad (26)$$ 

for $i, j = 1, 2$. In particular, the matrix $d(B)$ is the inverse of the matrix $d(A)$; hence

$$d(B) = \epsilon \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix},$$
where $\varepsilon = \pm 1$ is the determinant of $d(A)$. If $\varepsilon = 1$, then necessarily $a_{12} = a_{21} = 0$; this fact together with equation (26) yields

$$a_{ii}b_{ii} = 1$$

for $i = 1, 2$, and then $a_{11} = a_{22} = 1$. For $\varepsilon = -1$ we obtain $a_{ii} = 0$ for $i = 1, 2$ and $a_{12} = a_{21} = 1$. Thus, the only equivalences $\text{Vect}^2 \rightarrow \text{Vect}^2$ (up to isomorphism) have the form

$$ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}. $$

### 3.1.1. 2-Vector Bundles

The notion of 2-vector bundle (of rank 1) was introduced by Brylinski in [Bry98] as a way of describing some cohomology classes associated to symplectic manifolds in terms of 2-vector spaces (as an alternative to gerbes). His definition resembles the definition of the sheaf of sections of a vector bundle. Another notion of 2-vector bundle was proposed by Baas, Dundas and Rognes (BDR) in [BDR04b]. Their definition resembles the cocycles for a vector bundles.

We shall give here another definition of 2-vector bundles which is a generalisation to higher ranks of Brylinski’s definition. We shall define some geometric 2-vector bundles naturally associated to a Frobenius manifold. In section 6 we shall show that the 2-vector bundles constructed in this way are 2-vector bundles in the sense of Baas, Dundas and Rognes.

If $R \rightarrow B \leftarrow M$ are fibred categories or stacks over $B$, then an action of $R$ on $M$ is a morphism of fibred categories $R \times_B M \rightarrow M$. If $M$, has some extra structure we shall ask the action to preserve such structure. For instance, if the category $M$ is additive, then we should have a natural distributivity isomorphism $A \cdot (X \oplus Y) \cong A \cdot X \oplus A \cdot Y$, plus other properties involving $1$ and $0$.

We shall write $[\text{Vect}, M]$ for the fibred category that associated to each open set $U \subset M$ the category of vector bundles over $U$. The definition of 2-vector bundle given by Brylinski in [Bry98] reads as follows.

**Definition 3.6.** Let $M$ be a manifold and let $\text{Op}(M)$ denote the category of open sets of $M$. A fibred category $M \rightarrow \text{Op}(M)$ is said to be a 2-vector bundle of rank 1 over $M$ if the following conditions hold:

1. For each open subset $U \subset M$, the fibre $M(U)$ is an additive category.
(2) There exists an action $(E, X) \mapsto E \cdot X$ of the (fibred) category $\text{[Vect, } M\text{]}$ on $M$.

(3) Given any $x \in M$, there exists an open neighbourhood $U$ of $x$ and an object $X_U \in M(U)$ (called a local generator) such that the functor $\text{Vect}(U) \to M(U)$ given by $E \mapsto E \cdot X_U$ is an equivalence of categories, where $\cdot$ denotes the action.

(4) $M \to \text{Op}(M)$ is a stack.

We now extend the definition to higher ranks. For some technical reasons instead of the (fibred) category of vector bundles, we shall consider the (fibred) category of locally-free sheaves over $M$. We shall write $\text{LF}_O^M$ for the (fibred) category of locally free $O_M$-modules on $M$.

**Definition 3.7.** A fibred category $M \to \text{Op}(M)$ is said to be a $2$-vector bundle of rank $n$ over $M$ if and only if the following conditions hold:

1. For each open subset $U \subset M$, the fibre $M(U)$ is an additive category.
2. There exists an action $(\mathcal{M}, X) \mapsto \mathcal{M} \cdot X$ of $\text{LF}_O^M$ on $M$.
3. Given any $x \in M$, there exists an open neighbourhood $U \ni x$ and objects $X_1, \ldots, X_n$ in $M(U)$ (called local generators) such that the functor $\text{LF}_O^U \to M(U)$ given by
   
   $$(\mathcal{M}_1, \ldots, \mathcal{M}_n) \mapsto \mathcal{M}_1 \cdot X_1 \oplus \cdots \oplus \mathcal{M}_n \cdot X_n$$
   
   is an equivalence of categories.
4. $M \to \text{Op}(M)$ is a stack.

** Remark 3.8.** Note that the local equivalence of the previous definition preserves both the action and the additive structure; that is, if $\Phi$ is such an equivalence, $\mathcal{L} \in \text{LF}_O^U$ and $\mathcal{M}, \mathcal{N} \in \text{LF}_O^n$, then

$$\Phi ((\mathcal{L} \otimes \mathcal{M}) \oplus \mathcal{N}) \cong (\mathcal{L} \otimes \Phi(\mathcal{M})) \oplus \Phi(\mathcal{N}).$$

** Example 3.9.** Let $M = \{x\}$ be a one-point space. A 2-vector bundle of rank $n$ over $M$ is then an additive category $M$ equivalent to the category $\text{LF}_O^n$. As $O(M) \cong \mathbb{C}$, then $M$ is equivalent to the $n$-fold product of the category of $\mathbb{C}$-modules; that is, it is a 2-vector space (of rank $n$).

The following result shall be useful later.
Proposition 3.10. Let $\Phi : LF^n_{\mathcal{O}_M} \to LF^n_{\mathcal{O}_M}$ be a functor which preserves the action and the additive structure. Then there exists an $m \times n$ matrix $A := (M_{ij})$ of $\mathcal{O}_M$-modules such that $\Phi$ is naturally isomorphic to multiplication by $A$.

The proof is completely analogous to the one for 2-vector spaces. Moreover, this kind of morphisms share with 2-vector spaces the same shortage of equivalences.

We shall now introduce Baas-Dundas-Rognes (BDR) 2-vector bundles – see [BDR04b] for more details.

Definition 3.11. Let $A$ be a poset and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an ordered open cover of a topological space $M$ by open subsets. A Bass-Dundas-Rognes 2-vector bundle (BDR 2-vector bundle for short) is a law that assigns to each pair $\alpha, \beta \in A$ a matrix $E^{\alpha\beta} := (E^{\alpha\beta}_{ij})$ of (constant rank) vector bundles over $U_\alpha \cap U_\beta = U_{\alpha\beta}$ (for each $\alpha < \beta$) subject to the following conditions:

1. $\det \left( \text{rk } E^{\alpha\beta}_{ij} \right) = \pm 1$.
2. For $\alpha < \beta < \gamma$ in $A$ and $U_{\alpha\beta\gamma} \neq \emptyset$, we have isomorphisms

$$
\phi^{\alpha\beta\gamma}_{ik} : \bigoplus_j E^{\alpha\beta}_{ij} \otimes E^{\beta\gamma}_{jk} \xrightarrow{\cong} E^{\alpha\gamma}_{ik}.
$$

As for morphisms of 2-vector spaces, this condition can also be expressed in matrix form $\phi^{\alpha\beta\gamma} : E^{\alpha\beta} E^{\beta\gamma} \cong E^{\alpha\gamma}$.

3. For $\alpha < \beta < \gamma < \delta$ with $U_{\alpha\beta\gamma\delta} \neq \emptyset$, the following diagram of bundles over $U_{\alpha\beta\gamma\delta}$ should commute

$$
\begin{array}{ccc}
E^{\alpha\beta} \otimes (E^{\beta\gamma} \otimes E^{\gamma\delta}) & \rightarrow & (E^{\alpha\beta} \otimes E^{\beta\gamma}) \otimes E^{\gamma\delta} \\
\downarrow & & \downarrow \\
E^{\alpha\beta} \otimes E^{\beta\delta} & \rightarrow & E^{\alpha\delta} \leftarrow E^{\alpha\gamma} \otimes E^{\gamma\delta},
\end{array}
$$

where the top arrow is the associativity isomorphism derived from the associativity of the tensor product of vector bundles and the other arrows are defined from the isomorphisms of the previous item.
3.2. Azumaya algebras and twisted vector bundles. In this section we will introduce some basic material regarding Azumaya algebras, as well as an introduction to twisted vector bundles. The former are strongly related to the latter, and this relationship will also appear later in section 6. The treatment of twisted bundles is mainly based on [Kar10].

3.2.1. Azumaya Algebras. If $F$ is a field (which we assume to have characteristic equal to zero), a central simple algebra over $F$ is a simple (associative) algebra with center equal to $F$. Replacing $F$ with a commutative local ring $R$ leads to the notion of Azumaya algebra; that is, an associative $R$-algebra $A$ is an Azumaya algebra if and only if there exists some $k \in \mathbb{N}$ such that $A \cong R^k$ as $R$-modules and also the algebra homomorphism $\varphi : A \otimes_R A^\circ \to \text{End}_R(A) \cong M_k(A)$ given by $\varphi(x \otimes y)(z) = xyz$ is an isomorphism, where $A^\circ$ is the algebra with underlying set $A$ and operation given by $x \cdot y = yx$ (the right hand side is multiplication in $A$). Auslander and Goldman [AG60] generalized this definition to include any commutative (not necessarily local) base ring.

**Definition 3.12.** An Azumaya algebra over $(M, \mathcal{O}_M)$ is a coherent sheaf of $\mathcal{O}_M$-algebras locally isomorphic to the sheaf $M_k(\mathcal{O}_M)$.

**Remark 3.13.** By Proposition 2.1 (b) of [Mil80], see also section 1 of [Gro68], an Azumaya algebra over $(M, \mathcal{O}_M)$ is a locally free sheaf of algebras such that the fibres are $M_k(\mathbb{C})$.

If $E \to M$ is a vector bundle, the sheaf of sections of $\text{End}(E)$ is an Azumaya algebra. Not every Azumaya algebra has this form. However the situation is different if one considers twisted vector bundles.

3.2.2. Twisted vector bundles. Twisted vector bundles can be thought of as a model for twisted K-theory [AS05], just as vector bundles are models of topological K-theory.

**Definition 3.14.** A twisted vector bundle $E$ over $M$ is a tuple

$$ E = (\mathcal{U}, U_i \times V, g_{ij}, \lambda_{ijk}) $$

consisting of the following data:

1. An open cover $\mathcal{U} = \{U_i\}$ of $M$. 
(2) A (trivial) vector bundle $U_i \times V$ over each $U_i \in \mathcal{U}$, where $V$ is a finite dimensional complex vector space (which shall usually be taken to be complex $n$-space).

(3) Two families of maps $g_{ij} : U_{ij} \to \text{GL}(V)$ and $\lambda_{ijk} \in \mathcal{O}(U_{ijk})$ such that $\lambda := (\lambda_{ijk})$ is a Čech 2-cocycle, each map $\lambda_{ijk}$ takes values in $\mathbb{C} \times$ and

$$g_{ii} = 1 , \quad g_{ji} = g_{ij}^{-1} , \quad g_{ij} g_{jk} = \lambda_{ijk} g_{ik}$$

over $U_{ijk}$.

Two twisted bundles $E = (\mathcal{U}, U_i \times V, g_{ij}, \lambda_{ijk})$ and $F = (\mathcal{V}, V_r \times V, f_{rs}, \mu_{rst})$ will be regarded as equal if the cocycles of $E$ and $F$ are equal over members of the refinement $\mathcal{U} \cap \mathcal{V}$.

Twisted vector bundles admit the same operations as ordinary bundles.

**Definition 3.15.** Let $E = (\mathcal{U}, U_i \times V, g_{ij}, \lambda_{ijk})$ and $F = (\mathcal{V}, V_r \times V, f_{ij}, \mu_{ijk})$ be twisted vector bundles over $M$. A morphism $\phi : E \to F$ is a family of bundle morphisms

$$\phi_i : U_i \times V \to U_i \times W$$

such that the following square

$$\begin{array}{ccc}
U_{ij} \times V & \xrightarrow{\phi_j} & U_{ij} \times W \\
1 \times g_{ij} & \downarrow & 1 \times f_{ij} \\
U_{ij} \times V & \xrightarrow{\phi_i} & U_{ij} \times W
\end{array}$$

(27)

commutes.

**Lemma 3.16.** Two twisted bundles $E = (\mathcal{U}, U_i \times V, g_{ij}, \lambda_{ijk})$ and $F = (\mathcal{V}, U_i \times W, f_{ij}, \mu_{ijk})$ are isomorphic if and only if there exists a family of maps $\{u_i : U_i \to \text{Iso}(V, W)\}$ such that

$$f_{ij} = u_i g_{ij} u_j^{-1}.$$

Further properties are given in the following

**Lemma 3.17.** Let $E$ and $F$ be twisted bundles. Then

(1) $E \otimes F \cong E$ if and only if $F$ is an ordinary line bundle.
(2) The dual bundle $E^*$ has twisting $\lambda^{-1}$.

(3) If $E$ has twisting $\lambda$ and $F$ has twisting $\lambda^{-1}$, then $E \otimes F$ is an ordinary vector bundle. In particular, $E^* \otimes F$ is also a vector bundle if $E$ and $F$ have the same twisting.

(4) If $E$ is defined over the trivial open cover $\Omega = \{ M \}$, then $E$ is a trivial vector bundle, and conversely.

Of particular interest is the twisted vector bundle $\text{Hom}(E, F)$, which is defined by

$$\text{Hom}(E, F) = (\Omega, U_i \times \text{Hom}_C(V, W), h_{ij}, \lambda_{ijk}^{-1} h_{ijk}),$$

where $h_{ij} : U_{ij} \rightarrow \text{GL}(\text{Hom}_C(V, W))$ is given by $h_{ij}(x)(u) = f_{ij}(x)u g_{ij}(x)^{-1}$. If $F$ is also a $\lambda$-twisted bundle (i.e. $\mu = \lambda$), then the data defining $\text{Hom}(E, F)$ in fact defines an ordinary vector bundle (there is no twisting!), which is denoted by $\text{HOM}(E, F)$. If $E = F$, then $\text{HOM}(E, F)$ will be denoted $\text{END}(E)$.

**Lemma 3.18** ([Kar10], Proposition 3.1). The vector space $\text{Hom}_{TVB}(M)(E, F)$ can be canonically identified with the space of sections of the bundle $\text{HOM}(E, F)$.

**Theorem 3.19** ([Kar10], Theorem 3.2). Assume $A$ is an Azumaya algebra over $M$. Then, there exists a twisted bundle $E$ such that

$$A \cong \text{END}(E).$$

### 3.2.3. The Twisted Picard Group.

For the following discussion it will be useful to recall the definition of the Picard group of a manifold $M$; consider the set of isomorphism classes of (ordinary) line bundles over $M$. If $L, K$ are line bundles, then $[L] \cdot [K] := [L \otimes K]$ provides the set of isomorphism classes of line bundles with a structure of abelian group. This group is called the *Picard group of $M$* and is denoted by $\text{Pic}(M)$.

Analogously, twisted line bundles also enjoy some remarkable properties, like line bundles do. Given a twisted bundle $E$, we shall denote by $[E]$ its isomorphism class. Let us restrict ourselves to considering isomorphism classes of twisted line bundles over a manifold $M$. We define a product in the following way:

$$[L] \cdot [K] := [L \otimes K],$$

extending the one for line bundles.
Theorem 3.20. The set of isomorphism classes of twisted line bundles together with the operation (28) is an abelian group which contains Pic(M) as a subgroup.

Proof. Associativity and commutativity of the operation follow from the ones of the tensor product, as stated in 3.17.

Let L be a twisted line bundle; if $e^1$ denotes the trivial line bundle over M, then $L \otimes e^1 \cong L$; to see this, consider the family of maps

$$\phi_i : U_i \times (\mathbb{C} \otimes \mathbb{C}) \rightarrow U_i \times \mathbb{C}$$

given by $\phi_i(x, z \otimes w) = (x, zw)$. These maps define a morphism of twisted bundles

$$\phi : L \otimes e^1 \rightarrow L,$$

with inverse given by the family $\phi^{-1}_i(x, z) = (x, z \otimes 1)$. Hence, $[e^1] = 1$, the unit of the group.

Let now $[L]$ be an arbitrary class. Then, $L \otimes L^*$ is an ordinary line bundle; denoting this bundle by $L$, we have that

$$[L]^{-1} = [L^* \otimes L^*].$$

The inclusion of Pic(M) as a subgroup is clear from the previous discussion. Q.E.D.

The group introduced in the previous theorem will be called the twisted Picard group of M and denoted by TPic(M).

Assume now that TVB(M) and Vect(M) are sets consisting of twisted bundles (with arbitrary twisting) over M and vector bundles over M, respectively, and consider the equivalence relations $E \sim E \otimes L$ and $E \sim E \otimes L$, where L is a twisted line bundle and L is a line bundle. In the following result, $[E]$ will denote the class of E according to the relation $E \sim L \otimes E$; the same notation will be used for ordinary vector bundles.

Theorem 3.21. There exists a non-canonical bijection

$$\Psi : TVB(M) / E \sim L \otimes E \rightarrow Vect(M) / E \sim L \otimes E.$$ 

Proof. For each twisting $\lambda$, let us fix a twisted line bundle $L_\lambda$ with that twisting. Now consider the map

$$\Psi[E] = [E \otimes L_\lambda^{-1}],$$
where $E$ has twisting $\lambda$.

We check that this correspondence is well-defined: first note that the twisting of $E \otimes L_{\lambda^{-1}}$ is $\lambda \lambda^{-1} = 1$, and hence it is an ordinary line bundle. Now suppose that $[E] = [F]$, where $E$ has twisting $\lambda$ and $F$ twisting $\mu$; this implies the existence of a twisted line bundle $L$ such that $F \cong L \otimes E$. In particular, if $L$ has twisting cocycle equal to $\epsilon$, then $\mu = \epsilon \lambda$. We now have to check that $[E \otimes L_{\lambda^{-1}}] = [F \otimes L_{\mu^{-1}}];$ in other words, we should find a line bundle $L$ such that $E \otimes L_{\lambda^{-1}} \cong L \otimes E \otimes L_{\mu^{-1}} \otimes L$. Take now

$$L := L_{\mu} \otimes L^* \otimes L_{\lambda^{-1}};$$

then $L$ is an ordinary line bundle, as the twisting of the product of the right hand side is precisely $\mu \epsilon^{-1} \lambda^{-1} = \epsilon \lambda \epsilon^{-1} \lambda^{-1} = 1$. We then have

$$L \otimes E \otimes L_{\mu^{-1}} \otimes L \cong L \otimes E \otimes L_{\mu^{-1}} \otimes L_{\mu} \otimes L^* \otimes L_{\lambda^{-1}} \cong E \otimes L_{\lambda^{-1}},$$

as desired.

Assume now that $E$ and $F$ are twisted bundles with twistings $\lambda$ and $\mu$ respectively such that there exists a line bundle $L_0$ with $F \otimes L_{\mu^{-1}} \cong L_0 \otimes E \otimes L_{\lambda^{-1}}$. Multiplying by $L_{\mu}$ at both sides, we obtain

$$F \otimes L_1 \cong L_0 \otimes E \otimes L_{\lambda^{-1}} \otimes L_{\mu},$$

where $L_1 = L_{\mu} \otimes L_{\mu^{-1}}$. Multiplying now by the dual line bundle $L_1^*$ yields

$$F \cong E \otimes L_{\lambda^{-1}} \otimes L_{\mu} \otimes L_0 \otimes L_1^*.$$

As $L_{\lambda^{-1}} \otimes L_{\mu} \otimes L_0 \otimes L_1^*$ is a twisted line bundle (with twisting $\mu \lambda^{-1}$), then $[F] = [E]$ and hence $\Psi$ is injective.

Let now $E$ be an arbitrary bundle. Then $E \otimes L_{\lambda}$ is a $\lambda$-twisted vector bundle and then

$$\Psi [E \otimes L_{\lambda}] = [E \otimes L_{\lambda} \otimes L_{\lambda^{-1}}] = [E].$$

Q.E.D.
4. Cardy Fibrations

We shall now define an extension of the notion of Calabi-Yau category.

**Definition 4.1.** Let $R$ be a commutative ring with unit. An $R$-linear category $C$ is a Calabi-Yau category if for each pair of objects $a, b$ in $C$, the set of arrows $\text{Hom}_C(a, b)$ is a finitely generated projective $R$-module and for each element $a$ in $C$ there exists a linear form

$$\theta_a : \text{Hom}_C(a, a) \rightarrow R$$

such that

1. the induced pairing

$$\text{Hom}_C(a, b) \otimes_R \text{Hom}_C(b, a) \rightarrow \text{Hom}_C(a, a) \xrightarrow{\theta_a} R$$

is a perfect pairing

2. given arbitrary arrows $\sigma : a \rightarrow b$ and $\tau : b \rightarrow a$, the equality $\theta_a(\tau \sigma) = \theta_b(\sigma \tau)$ holds.

**Definition 4.2.** Let $M$ be a semisimple manifold with multiplication $M$ a CY-category $\mathcal{B}$ over $M$ is a fibred category over the category of open sets of $M$ such that for each open set $U \subset M$ the category $\mathcal{B}(U)$ is an $\mathcal{O}(U)$-CY category.

Let us fix a semisimple manifold with multiplication $M$, with structure sheaf $\mathcal{O} = \mathcal{O}_M$ and let $\mathcal{B}$ be an $\mathcal{O}$-linear CY category over $M$. For objects $a, b \in \mathcal{B}(U)$, let us denote by $\Gamma_{ab}$ the presheaf $\text{Hom}_{\mathcal{B}}(a, b)$ over $U$ given by

$$V \mapsto \text{Hom}_{\mathcal{B}(V)}(a|_V, b|_V).$$

By definition of CY category, we have that $\Gamma_{aa}$ is a Frobenius $\mathcal{O}_U$-algebra for each $a \in \mathcal{B}(U)$.

**Notation 4.3.** Recall that if the base manifold is clear, we shall suppress the subscript of the structure sheaf when taking local sections; e.g. instead of using the notation $\mathcal{O}_M(U)$ for $U \subset M$, we will only write $\mathcal{O}(U)$; and the restriction $\mathcal{O}_M|_U$ shall be denoted $\mathcal{O}_U$. The same considerations are applied to the tangent sheaf $\mathcal{T}_M$ of a manifold $M$.

We now turn to the relevant definitions.
Definition 4.4. A Calabi-Yau (CY) fibration over a semisimple manifold $M$ is a pair $(\mathcal{B}, \mathcal{U})$ (the open cover shall be omitted form the notation), where $\mathcal{B}$ is a CY category over $M$ and $\mathcal{U} = \{U_a\}$ is an open cover of $M$, subject to the following conditions:

1. Each $U_a \in \mathcal{U}$ is semisimple.
2. $\mathcal{B}$ is a stack.
3. Given any $U_a \in \mathcal{U}$ and objects $a, b \in \mathcal{B}(U_a)$, the sheaf $\Gamma_{ab}$ is a locally-free locally finitely generated $\mathcal{O}_{U_a}$-module. Objects of $\mathcal{B}(U)$ are called labels, boundary conditions or D-branes over $U$.
4. For each $U_a \in \mathcal{U}$ and each object $a \in \mathcal{B}(U_a)$, we have transition (sheaf) homomorphisms

$$t_a : \mathcal{T}_U \rightarrow \Gamma_{aa}, \quad t^a : \Gamma_{aa} \rightarrow \mathcal{T}_U.$$ 

The previous data is subject to the following conditions:

(a) $t_a$ is a morphism of $\mathcal{O}_{U_a}$-algebras (preserves multiplication and unit) and $t^a$ is an $\mathcal{O}_{U_a}$-linear map. In particular, $t_a$ provides $\Gamma_{aa}$ with a $\mathcal{T}_{U_a}$-algebra structure.

(b) $t_a$ is central: given $X \in \mathcal{T}(V)$ and $\sigma \in \Gamma_{ab}(V)$, we have

$$\sigma t_a(X) = t_b(X)\sigma$$

in $\Gamma_{ab}(V)$, for each $V \subset U_a$.

(c) There is an adjoint relation between $t_a$ and $t^a$ given by

$$\theta(t^a(\sigma)X) = \theta_a(\sigma t_a(X)),$$

for each $X \in \mathcal{T}_{U_a}$ and $\sigma \in \Gamma_{ab}$.

Remark 4.5. For some technical considerations, we will assume that our CY fibrations $\mathcal{B}$ verify that for each open subset $U \subset M$, the skeleton $\text{sk} \mathcal{B}(U)$ of the category $\mathcal{B}(U)$ is a set.

4.1. Cardy Fibrations. For $U_a \in \mathcal{U}$ open and $a, b \in \mathcal{B}(U_a)$, if $\Gamma_{ab}$ restricted to $U$ is trivial pick a local basis $\{\sigma_i\}$ of $\Gamma_{ab}$ and let $\{\sigma^i\}$ be a basis of $\Gamma^*_{ab}$ dual to $\{\sigma_i\}$. Define the map $\pi^a_b : \Gamma_{aa} \rightarrow \Gamma_{bb}$ by

$$\pi^a_b(\sigma) = \sum_i \sigma_i \sigma^i.$$ 

\footnote{In particular, the presheaf $\mathcal{O}$ is a sheaf.}
Some comments are in place: the sequence of maps

$$\Gamma_{ba} \otimes \Gamma_{ab} \longrightarrow \Gamma_{bb} \xrightarrow{\theta_a} \mathcal{O}_U \quad (33)$$

induces a duality isomorphism $\Gamma_{ba} \xrightarrow{\cong} \Gamma_{ab}^*$. The dual basis in the definition of $\pi_b^a$ is in fact the preimage of the dual basis of $\{\sigma_i\}$ under this isomorphism. Another key observation is stated in the following

**Proposition 4.6.** The map $\pi_b^a$ does not depend on the chosen (local) basis.

**Proof.** As $\Gamma_{aa}$, $\Gamma_{bb}$ and $\Gamma_{ba}$ are locally-free, we can pick an open cover $U_a$ of $U_b$ such that $\Gamma_{aa}|_V \cong \mathcal{O}^{n_a}$, $\Gamma_{ba}|_V \cong \mathcal{O}^{n_{ba}}$, etc. for each $V \in U_a$. Pick then a basis $B = \{e_1, \ldots, e_{n_{ba}}\}$ for $\Gamma_{ba}|_V$. Let $B' = \{e_1', \ldots, e_{n_{ba}}'\}$ be the corresponding dual basis for $\Gamma_{ba}^*$. Then, in terms of this basis we have $\pi_b^a(\sigma) = \sum_i e_i \sigma e_i$. Let $D = \{f_1, \ldots, f_{n_{ba}}\}$ be another basis over $V$ with dual basis $D'$. We then have

$$f_i = \sum_j \lambda_{ij} e_j \quad \text{and} \quad f^i = \sum_j \mu_{ij} e^j.$$

Replacing these linear combinations in the equality $\delta_{ij} = f^i(f_j)$ we obtain

$$\delta_{ij} = \sum_k \mu^{ik} \lambda_{jk}.$$

If $A := (\lambda_{ij})$ and $B := (\mu^{ij})$ then the previous equality implies that $AB^t = I$ or, equivalently, $A^tB = I$, which in terms of the coefficients is expressed by $\delta_{ij} = \sum_k \lambda_{ki} \mu^{kj}$. We now compute

$$\sum_i f_i \sigma f^i = \sum_i \left( \sum_j \lambda_{ij} e_j \right) \sigma \left( \sum_k \mu^{ik} e^k \right) = \sum_{j,k} \lambda_{ij} \mu^{jk} e_j \sigma e^k = \sum_{j,k} \delta_{jk} e_j \sigma e^k = \sum_j e_j \sigma e^j,$$

as desired. Q.E.D.

---

2By a *basis* we mean a system of linearly independent generators $e_1, \ldots, e_{n_{ba}} \in \Gamma_{ba}(V)$ such that $\{e_1|_W, \ldots, e_{n_{ba}}|_W\}$ is also linearly independent and generates $\Gamma_{ba}(W)$ for each $W \subset V$. For instance, let $u_{1}, \ldots, u_{n_{ba}} \in \mathcal{O}(V)$ be units; then, if $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the sections $u_1 e_1, \ldots, u_{n_{ba}} e_{n_{ba}}$ form a basis.
Then, when defining \( \pi^a_b \) locally on each \( V \), we have that, by the previous computation, these expressions coincide over non-empty overlaps, and thus can be glued together to obtain a morphism over \( U_a \in \mathcal{U} \)

\[
\pi^a_b : \Gamma_{aa} \longrightarrow \Gamma_{bb}.
\]

This final layer of structure is included in the following

**Definition 4.7.** A Calabi-Yau fibration \( \mathcal{B} \) is called a **Cardy fibration** if and only if the following condition, called the **Cardy condition**, holds for each open subset \( U_a \in \mathcal{U} \): For \( a, b \in \mathcal{B}(U_a) \),

\[
\pi^a_b = \iota_b \iota^a.
\]

In other words, the following triangle

\[
\begin{array}{ccc}
\Gamma_{aa} & \xrightarrow{\pi^a_b} & \Gamma_{bb} \\
\downarrow \iota^a & & \downarrow \iota_b \\
\mathcal{T}_U & & \\
\end{array}
\]

should commute.

We shall deal with Cardy fibrations all along.

**Definition 4.8.** A Cardy fibration \( \mathcal{B} \) is said to be **trivializable** if conditions (3), (4)a-c in definition 4.4 and the Cardy condition hold also for any open subset of each \( U_a \in \mathcal{U} \).

5. **Algebraic Properties of Maximal Cardy Fibrations**

This section will be devoted to describing in detail the stack of boundary conditions \( \mathcal{B} \). We will first deal with morphisms and later with the whole category.
5.1. **Local characterization of categories of branes.** The main idea now is to pick a point \( x \in M \) and prove that all the fibres over \( x \) of the sheaves involved in this discussion define a brane category in the sense of Moore and Segal. This approach will let us generalize all the results of \([MS06]\) to Cardy fibrations.

Let us fix a point \( x \in M \), and assume that \( x \in U_\alpha \), where \( U_\alpha \) is semisimple. Given arbitrary labels \( a, b \in \mathcal{B}(U_\alpha) \), let us denote by \( E_{ab} \) the fibre over \( x \) for the sheaf \( \Gamma_{ab} \). We need to show that the vector spaces \( T_x M \) and \( E_{ab} \), together with the appropriate morphisms, form a CY category in the sense of Moore and Segal.

Let us denote by \( p_{ab} \) (or just \( p \) if the labels are clear) the sequence of projections

\[
\Gamma_{ab}(U_\alpha) \rightarrow \Gamma_{ab,x} \rightarrow E_{ab},
\]

where \( \Gamma_{ab,x} \) is the stalk over \( x \) of the sheaf \( \Gamma_{ab} \). Let \( 1_a \in \Gamma_{aa}(U_\alpha) \); let us identify a label \( a \in \mathcal{B}(U_\alpha) \) with \( 1_a \), and denote \( p_{aa}(1_a) \) by \( \bar{a} \).

We now define the category of boundary conditions \( \mathcal{B}_x \); its objects are given by

\[
\text{Obj } \mathcal{B}_x = \{ \bar{a} = p_{aa}(1_a) \mid a \in \mathcal{B}(U_\alpha) \}.
\]

If \( \bar{a}, \bar{b} \in \mathcal{B}_x \), consider the corresponding units \( 1_a \in \Gamma_{aa}(U_\alpha) \) and \( 1_b \in \Gamma_{bb}(U_\alpha) \). Then we define

\[
\text{Hom}_{\mathcal{B}_x}(\bar{a}, \bar{b}) = E_{ab}.
\]

With this definition, \( \text{Hom}_{\mathcal{B}_x}(\bar{a}, \bar{b}) \) is a \( \mathbb{C} \)-vector space, with dimension equal to the rank of \( \Gamma_{ab} \). We shall denote this vector space by \( O_{\bar{a}, \bar{b}} \).

We also have the linear forms \( \theta : \mathcal{F}_M \rightarrow \mathcal{O} \) and \( \theta_a : \Gamma_{aa} \rightarrow \mathcal{O} \) which induce linear maps on the fibres

\[
\bar{\theta}_x : T_x M \rightarrow \mathbb{C}
\]

\[
\theta_{\bar{a}} : O_{\bar{a}} \rightarrow \mathbb{C}
\]

which provide \( T_x M \) and \( O_{\bar{a}} \) with a Frobenius \( \mathbb{C} \)-algebra structure.

In the same fashion, the transition morphisms \( t_a \) and \( t^a \) induce maps

\[
T_x M \xleftarrow{t_a} O_{\bar{a}} \xrightarrow{t^a} T_x M.
\]

**Lemma 5.1.** Let \( x_0, x_1 \in U_\alpha \). If \( U_\alpha \) is connected, then the categories \( \mathcal{B}_{x_0} \) and \( \mathcal{B}_{x_1} \) are isomorphic.
Proof. Let us consider two labels \(a, b \in B(U_\alpha)\); to distinguish between the two fibres, let \(F_x(M)\) be the fibre over \(x\) of the locally free module \(M\); likewise, let us denote by \(p^0_{aa}\) (for \(x_0\)) or \(p^1_{aa}\) (for \(x_1\)) the projection. By connectivity assumptions, the ranks of \(\Gamma_{aa}\) and \(\Gamma_{ab}\) are constant and we can therefore fix isomorphisms

\[
\phi_{aa} : F_{x_0}(\Gamma_{aa}) \cong F_{x_1}(\Gamma_{aa}) \quad \text{and} \quad \phi_{ab} : F_{x_0}(\Gamma_{ab}) \cong F_{x_1}(\Gamma_{ab})
\]

such that the diagrams

\[
\begin{array}{ccc}
\Gamma_{aa}(U_\alpha) & \xrightarrow{\phi_{aa}} & \Gamma_{ab}(U_\alpha) \\
\downarrow & & \downarrow \\
F_{x_0}(\Gamma_{aa}) & \xrightarrow{\phi_{ab}} & F_{x_1}(\Gamma_{ab})
\end{array}
\]

commute, where the unlabelled arrows are canonical projections. In particular, this commutativity implies that, for example, \(p^0_{aa}(1_a) \in F_{x_0}(\Gamma_{aa})\) is mapped onto \(p^1_{aa}(1_a)\).

We now define a functor \(F : \mathcal{B}_{x_0} \rightarrow \mathcal{B}_{x_1}\); on objects, if \(a_0 := p^0_{aa}(1_a)\), then

\[
F(a_0) = \phi_{aa}(a_0).
\]

Let now \(\sigma : a_0 \rightarrow b_0\) be an arrow in \(\mathcal{B}_{x_0}\). That is, \(\sigma\) is an element of \(F_{x_0}(\Gamma_{ab})\). Then we define

\[
F(\sigma) = \phi_{ab}(\sigma).
\]

The inverse of this functor is constructed in the same way, by considering \(\phi_{aa}^{-1}\) and \(\phi_{ab}^{-1}\). Q.E.D.

**Theorem 5.2.** The category \(\mathcal{B}_x\), together with the Frobenius algebra \(T_xM\) and the structure maps \(\bar{\theta}_x\), \(\theta_\pi\), \(\iota_\pi\) and \(\mathfrak{g} (\bar{a} \in \mathcal{B}_x)\) defines a brane category in the sense of Moore and Segal.

From theorem 5.2, we can deduce the following

**Theorem 5.3.** Let \(a \in B(U_\alpha)\) with \(U_\alpha\) connected. Then, the sheaf \(\Gamma_{aa}\) is locally isomorphic to a sum \(\bigoplus_i M_{d(i,j)}(\mathcal{O}_{U_\alpha})\) of matrix algebras.
**Proof.** Fix \(x_0 \in U_a\) and let \(\{e_1, \ldots, e_n\}\) be a frame of orthogonal, idempotent sections in \(\mathcal{F}(U_a)\). Then, for the category \(\mathcal{F}_X\), we have Moore and Segal’s Theorem 2 \([2.13]\) at our disposal. We have that \(O_{\pi} = \bigoplus_i \iota_{\pi}(e_i(x_0))O_{\pi}\); by \([2.13]\)

\[
O_{\pi} = \text{Hom}_{\mathcal{F}_{X \times 0}}(\pi, \pi) \cong \bigoplus_{i=1}^n M_{d(x_0, \pi, i)}(\mathbb{C});
\]

moreover, the matrix algebra \(M_{d(x_0, \pi, i)}(\mathbb{C})\) corresponds to the summand \(\iota_{\pi}(e_i(x_0))O_{\pi}\). On the other hand, we have that, locally around \(x_0\), the sheaf \(\Gamma_{aa}\) is isomorphic to \(O_{U_a}^{n_a}\) for some integer \(n_a\). We should link this isomorphism with the pointwise decomposition given in equation \([35]\).

It is sufficient to work with only one idempotent; we thus consider the algebra \(\iota_a(e_i)\Gamma_{aa}\), which is a locally free module, being the image of the idempotent map \(L_i : \Gamma_{aa} \to \Gamma_{aa}\) given by \(L_i(\sigma) = \iota_a(e_i)\sigma\). Assume that \(x_0 \in V\), where \(V\) is a neighborhood such that \(\iota_a(e_i)|_{\Gamma_{aa}}|_V \cong O_V^{n_a(i)}\). The fibre over \(x_0\) of \(\iota_a(e_i)\Gamma_{aa}\) is precisely \(\iota_{\pi}(e_i(x_0))O_{\pi}\), which is isomorphic to \(M_{d(x_0, \pi, i)}(\mathbb{C})\). If \(x_1 \in V\) is another point, then \(\iota_{\pi}(e_i(x_1))O_{\pi}\) is isomorphic to \(M_{d(x_1, \pi, i)}(\mathbb{C})\). But the local triviality of \(\iota_a(e_i)\Gamma_{aa}\) implies that

\[
d(x_1, \pi, i) = n_a(i) = d(x_0, \pi, i).
\]

Hence, by remark \([3.13]\) the decomposition \([35]\) extends to a neighborhood of \(x_0\), as we wanted to prove. Q.E.D.

**Remark 5.4.** From the previous result we can also deduce that the matrix algebra \(M_{d(a,i)}(O_V)\) corresponds (locally) to the subalgebra \(\iota_a(e_i)\Gamma_{aa}\).

For \(a, b \in \mathcal{B}(U_a)\), and again by the CY structure of \(\mathcal{F}_X\), we have an isomorphism

\[
O_{\pi} = \text{Hom}_{\mathcal{F}_{X \times 0}}(\pi, \pi) \cong \bigoplus_{i=1}^n \text{Hom}_{\mathcal{C}}\left(C^{d(\pi, i)}, C^{d(\pi, i)}\right),
\]

and thus the following result, which is proved following the same procedure of the previous theorem (note that in this case we have the idempotent morphism \(L_i : \Gamma_{ab} \to \Gamma_{ab}\), \(L_i(\sigma) = \iota_b(e_i)\sigma\) which, by the centrality condition \([31]\), coincides with the morphism \(\Gamma_{ab} \to \Gamma_{ab}\) given by \(\sigma \to \sigma\iota_a(e_i))\).

**Theorem 5.5.** In the situation of theorem \([5.3]\) for \(a, b \in \mathcal{B}(U_a)\) we have a local isomorphism between \(\Gamma_{ab}\) and \(\bigoplus_{i=1}^n \text{Hom}_{\mathcal{C}}\left(O_{U}^{d(a,i)}, O_{U}^{d(b,i)}\right)\).
Likewise, an arrow \( c \) additive category for each of maps is then given by multiplying matrices. As a consequence, we obtain thus a structure of a morphism of additive Cardy categories. Let 

\[ \theta \, : \, a \rightarrow b, \quad i^a \, : \, \text{a morphism over } U, \quad \pi^n_b \] 

Remark 5.6. Observe that the dimensions \( d(a,i) \) in theorem 5.5 are the same as the ones in 5.3; this is deduced from the proof of Moore and Segal’s theorem 2 in [MS06]. And also in this case, the summand \( \text{Hom}_{\mathcal{C}}(\mathcal{C}^{d(a,i)}_{V}, \mathcal{C}^{d(b,i)}_{V}) \) corresponds to the submodule \( \pi^a_{b} \Gamma_{ab}|V = \Gamma_{ab}|_{\nu_{a}}(e_{i}) \).

From these last results, and following the same procedures done in section 2.4, we can derive local expressions for the morphisms \( \theta_{a}, i^{a} \) and \( \pi^{a}_{b} \). Let \( a, b \in \mathcal{M}(U_{a}) \) and let \( x \in U_{a} \). Assume that \( U \ni x \) is a neighborhood such that \( \Gamma_{ab}|U \) is isomorphic to a sum \( \bigoplus_{i}\mathbb{M}_{d(a,i)}(\mathcal{C}_{U}) \) (in that case an element \( \sigma \in \Gamma_{ab}|U \) can be represented as a tuple \( (\sigma_{i}) \), where \( \sigma_{i} \in \mathbb{M}_{d(a,i)}(\mathcal{C}_{U}) \)). If \( \{ e_{1}, \ldots, e_{n} \} \) is a frame of orthogonal, idempotent sections for \( \mathcal{M} \) over \( U_{a} \), then we have the following expressions for \( \theta_{a}, i^{a} \) and \( \pi^{a}_{b} \) over \( U \):

\[
\theta_{a}(\sigma) = \sum_{i} \sqrt{\theta(e_{i})} \, \text{tr}(\sigma_{i}),
\]

\[
i^{a}(\sigma) = \sum_{i} \frac{\text{tr}(\sigma_{i})}{\sqrt{\theta(e_{i})}} \, e_{i},
\]

\[
\pi^{a}_{b}(\sigma) = \sum_{i} \frac{\text{tr}(\sigma_{i})}{\sqrt{\theta(e_{i})}} \, i_{b}(e_{i}).
\]

5.2. Enlarging the categories of branes. Let \( \mathcal{B} \) be a Cardy fibration over a manifold \( M \).

5.2.1. Additive structure. We shall show that \( \mathcal{B} \) can be embedded in a canonical way into a fibration of additive Cardy categories. Let \( U \subset M \) be any open subset and \( a, b, c \in \mathcal{B}(U) \); based on properties of modules, we shall define a new label \( a \oplus b \); we put

\[
\Gamma_{(a \oplus b),c} := \Gamma_{ac} \oplus \Gamma_{bc},
\]

\[
\Gamma_{c,(a \oplus b)} := \Gamma_{ca} \oplus \Gamma_{cb}.
\]

A morphism \( a \oplus b \rightarrow c \) shall be represented as a row matrix \( (\sigma \, \tau) \), where \( \sigma : a \rightarrow c, \tau : b \rightarrow c \). Likewise, an arrow \( c \rightarrow a \oplus b \) is a column matrix \( (\tau \, \sigma) \), for \( \sigma : c \rightarrow a, \tau : c \rightarrow b \). Thus, a map \( a_{1} \oplus a_{2} \rightarrow b_{1} \oplus b_{2} \) can be represented as a matrix \( (\sigma_{ij} \, \tau_{ij}) \), where \( \sigma_{ij} : a_{i} \rightarrow b_{j} \). Composition of maps is then given by multiplying matrices. As a consequence, we obtain thus a structure of additive category for each \( \mathcal{B}(U) \). For a new object \( a \oplus b \) we define \( \theta_{a \oplus b} : \Gamma_{(a \oplus b), a \oplus b} \rightarrow \mathcal{C}_{U} \) by

\[
\theta_{a \oplus b} (\sigma_{11}, \sigma_{21}) = \theta_{a}(\sigma_{11}) + \theta_{b}(\sigma_{21}).
\]

Regarding nondegeneracy of the linear forms we have the following
Proposition 5.7. The diagram
\[
\begin{array}{c}
\Gamma_{(a \oplus b)c} \otimes \Gamma_{c(a \oplus b)} \rightarrow \Gamma_{(a \oplus b)(a \oplus b)} \rightarrow \mathcal{O}_U \\
\downarrow \quad \quad \downarrow \\
\Gamma_{c(a \oplus b)} \otimes \Gamma_{(a \oplus b)c} \rightarrow \Gamma_c \otimes \Gamma_{(a \oplus b)c} \rightarrow \mathcal{O}_U
\end{array}
\]

is commutative, and the top and bottom composite bilinear maps are non-degenerate pairings (the vertical arrow on the left is the twisting map).

We now define the transition morphisms \( \iota_{(a \oplus b)} : \mathcal{T}_{Ua} \rightarrow \Gamma_{(a \oplus b)(a \oplus b)} \) and \( \iota^{(a \oplus b)} : \Gamma_{(a \oplus b)(a \oplus b)} \rightarrow \mathcal{T}_{Ua} \) by the equations
\[
\iota_{(a \oplus b)}(X) = \begin{pmatrix} \iota_a(X) & 0 \\ 0 & \iota_b(X) \end{pmatrix},
\]
\[
\iota^{(a \oplus b)}(c_{11} c_{21} c_{12} c_{22}) = \iota^a(c_{11}) + \iota^b(c_{22}).
\] (38)

In particular, note that both \( \iota_{a \oplus b} \) and \( \iota^{a \oplus b} \) are \( \mathcal{O}_{Ua} \)-linear, and \( \iota_{a \oplus b} \) is an algebra homomorphism which preserves the unit.

The following result shall be useful to prove the Cardy condition.

Lemma 5.8. For the maps \( \pi_{c \oplus b}^a \) and \( \pi_{b \oplus c}^a \) the following equalities hold
\[
\pi_{c \oplus b}^a = \pi_{c}^a + \pi_{b}^a,
\]
\[
\pi_{b \oplus c}^a = \begin{pmatrix} \pi_{a}^b & 0 \\ 0 & \pi_{a}^c \end{pmatrix}.
\]

Theorem 5.9. Given \( a, b \in \mathcal{B}(U_a) \), the maps \( \theta_{a \oplus b}, \iota_{(a \oplus b)} \) and \( \iota^{(a \oplus b)} \) verify the centrality, adjoint and Cardy conditions.

Proof. For the centrality condition, take \( \sigma : a \oplus b \rightarrow c \), which can be represented by a matrix \( (c_{11} c_{21}) \). Then
\[
\sigma \iota_{a \oplus b}(X) = (c_{11} c_{21}) \begin{pmatrix} \iota_a(X) & 0 \\ 0 & \iota_b(X) \end{pmatrix} = (c_{11} \iota_a(X) c_{21} \iota_b(X)).
\]

The equality \( \sigma \iota_{a \oplus b}(X) = \iota_c(X) \sigma \) now follows from the centrality condition for the morphisms \( \iota_a, \iota_c \) and \( \iota_b, \iota_c \).
We now verify the adjoint relation $\theta_{a \oplus b} (\sigma_{a \oplus b} (X)) = \theta \left( i^a \oplus i^b (\sigma) X \right)$; so let $\sigma : a \oplus b \rightarrow a \oplus b$ be given by $(\sigma_i)^t$. Then the adjoint relation between $i^a, i^a$ and the one between $i^b, i^b$ let us write

$$\theta_{a \oplus b} (\sigma_{a \oplus b} (X)) = \theta \left( i^a \oplus i^b (\sigma) X \right),$$

as desired.

For the Cardy condition, we now check that $\pi_{c \oplus d} = \iota_{c \oplus d} i^{a \oplus b}$. The right hand side is

$$\iota_{c \oplus d} i^{a \oplus b} = \left( \begin{array}{cc} i^a (\sigma_{11}) + i^b (\sigma_{22}) & 0 \\ 0 & i_d \left( i^a (\sigma_{11}) + i^b (\sigma_{22}) \right) \end{array} \right),$$

where in the last equality we used the Cardy condition. The rest now follows from lemma 5.8.

Q.E.D.

**Corollary 5.10.** Any maximal Cardy fibration is additive.

5.2.2. The Action of the Category of Locally Free Sheaves. In this section we shall prove that another enlargement of the category $\mathcal{B}$ can be made, by considering a label of the form $\mathcal{M} \otimes a$, where $\mathcal{M}$ is a locally free $\mathcal{O}_U$-module and $a \in \mathcal{B}(U)$. A consequence of this construction is that every maximal fibration enjoys, besides an additive structure, an action of the (fibred) category of locally free sheaves, which is compatible with the additive structure.

So let the locally free $\mathcal{O}_U$-module $\mathcal{M}$ be given, as well as a brane $a \in \mathcal{B}(U)$ over $U$. The new product brane $\mathcal{M} \otimes a$ is defined by

$$\Gamma_{(\mathcal{M} \otimes a)b} = \mathcal{M}^* \otimes \Gamma_{ab},$$

$$\Gamma_{b(\mathcal{M} \otimes a)} = \mathcal{M} \otimes \Gamma_{ba},$$

(39)
where the tensor product is taken over \( O_U \). In particular, we also have that
\[
\Gamma((\mathcal{M} \otimes a)(\mathcal{N} \otimes b)) = \text{Hom}(\mathcal{M}, \mathcal{N}) \otimes \Gamma_{ab},
\]
by the canonical identification between \( \mathcal{M} \otimes \mathcal{N} \) and \( \text{Hom}(\mathcal{M}, \mathcal{N}) \) (so an object of the form \( \varphi \otimes x \) shall be regarded as a homomorphism \( \mathcal{M} \to \mathcal{N} \)). Note that this definition let us also define a restriction \( (\mathcal{M} \otimes a)|_V := \mathcal{M}|_V \otimes a|_V \). Moreover, if we work on a semisimple subset \( U_\alpha \subset U \), then \( \Gamma((\mathcal{M} \otimes a)|_V) \) and \( \Gamma_{b,(\mathcal{M} \otimes a)} \) are locally free.

The composition pairing
\[
\Gamma((\mathcal{M} \otimes a)(\mathcal{N} \otimes b)) \otimes \Gamma((\mathcal{N} \otimes b)(\mathcal{P} \otimes c)) \longrightarrow \Gamma((\mathcal{M} \otimes a)(\mathcal{P} \otimes c))
\]
(40)
can be also written as
\[
\mathcal{M}^* \otimes \mathcal{N} \otimes \mathcal{N}^* \otimes \mathcal{P} \otimes \Gamma_{ab} \otimes \Gamma_{bc} \longrightarrow \mathcal{M}^* \otimes \mathcal{P} \otimes \Gamma_{ac};
\]
hence, the map (40) is built from two composition pairings, the one corresponding to composition of module homomorphisms, namely \( \mathcal{M}^* \otimes \mathcal{N} \otimes \mathcal{N}^* \otimes \mathcal{P} \to \mathcal{M}^* \otimes \mathcal{P} \), and the one corresponding to composition of maps of branes, \( \Gamma_{ab} \otimes \Gamma_{bc} \to \Gamma_{ac} \).

**Lemma 5.11.** We have a duality isomorphism \( \Gamma((\mathcal{M} \otimes a)|_V) \cong \Gamma_{b,(\mathcal{M} \otimes a)}^* \).

**Proposition 5.12.** The correspondence \( (\mathcal{M}, a) \mapsto \mathcal{M} \otimes a \) defines an action
\[
\text{LF}_{\mathcal{O}_U} \times \mathcal{B}(U) \longrightarrow \mathcal{B}(U)
\]
which is compatible with the additive structure.

**Theorem 5.13.** With the previous definitions, the action \( \text{LF}_{\mathcal{O}_U} \times U \mathcal{B}|_U \to \mathcal{B}|_U \) is compatible with all the structures in a Cardy fibration.

We thus obtain the following

**Corollary 5.14.** Any maximal CY category \( \mathcal{B} \) over \( M \) comes equipped with a linear action \( \text{LF}_{\mathcal{O}_M} \times \mathcal{B} \to \mathcal{B} \).
5.2.3. Pseudo-Abelian Structure. We shall now show that besides the additive structure and the action of the category of locally free sheaves, any maximal Cardy fibration should be pseudo-abelian. That is to say, given \( a \in \mathcal{B}(U) \) and an arrow \( \sigma_0 : a \to a \) such that \( \sigma_0^2 = \sigma_0 \), we shall assume that there exists branes \( K_0 := \text{Ker} \sigma_0 \) and \( I_0 := \text{Im} \sigma_0 \) (which can also be taken as \( \text{Ker}(1_a - \sigma_0) \)) such that

- The brane \( a \) decomposes as a sum \( a \cong K_0 \oplus I_0 \) and
- using matrix notation, the map \( \sigma_0 \) is given by \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

As was done for the additive structure and the action of the category of locally free modules, the enlargement of the category of branes by adding kernels should be done by defining all the structure maps for this new object \( K_0 \), namely \( \theta_{K_0}, \iota_{K_0}, \iota_{K_0}^* \), along with the verification of their properties. In particular, it should be noted that this definitions should agree with the additive structure.

First note that an arrow \( K_0 \to K_0 \) is a composite of the form

\[
K_0 \xrightarrow{i_1} K_0 \oplus I_0 \xrightarrow{\sigma} K_0 \oplus I_0 \xrightarrow{\text{pr}_1} K_0
\]

for some arrow \( \sigma : a \to a \), and hence \( \Gamma_{K_0 K_0} \subset \Gamma_{aa} \) is a submodule. In fact, we have that

\[
\Gamma_{aa} = \Gamma_{K_0 K_0} \oplus \Gamma_{K_0 I_0} \oplus \Gamma_{I_0 K_0} \oplus \Gamma_{I_0 I_0}.
\]

For \( a \in \mathcal{B}(U_a) \), consider the homomorphism \( \rho : \Gamma_{aa} \to \Gamma_{aa} \) given by

\[
\rho \left( \begin{array}{cc} c_1 & c_2 \\ c_1 & c_2 \end{array} \right) = \left( \begin{array}{cc} 0 & c_2 \\ 0 & c_2 \end{array} \right).
\]

Then \( \rho \) is clearly a projection with kernel \( \Gamma_{K_0 K_0} \) which is then locally-free. A similar argument can be used to prove that for any label \( b \in \mathcal{B}(U_a) \), \( \Gamma_{K_0 b} \) is also locally-free; consider \( \Gamma_{ab} = \Gamma_{K_0 b} \oplus \Gamma_{I_0 b} \) and the map \( \eta : \Gamma_{ab} \to \Gamma_{ab} \) which projects to \( \Gamma_{I_0 b} \). Proposition 5.15 shows that also \( \Gamma_{bK_0} \cong \Gamma_{K_0 b}^* \) is locally free.

We now turn to the structure maps. If \( a \cong K_0 \oplus I_0 \), the fact that

\[
\theta_a \left( \begin{array}{cc} 0 & c_1 \\ 0 & c_1 \end{array} \right) = \theta_a \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) = 0
\]

suggests the definition of the linear form \( \theta_{K_0} : \Gamma_{K_0 K_0} \to \mathcal{O}_U \) by

\[
\theta_{K_0}(\sigma) = \theta_a \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right).
\]
Proposition 5.15. The diagram

\[
\begin{array}{ccc}
\Gamma_{K_0} \otimes \Gamma_{K_0} & \overset{\theta_{K_0}}{\longrightarrow} & \mathcal{O}_U \\
\Downarrow & & \Downarrow \\
\Gamma_{K_0} \otimes \Gamma_{K_0} & \overset{\theta_{K_0}}{\longrightarrow} & \mathcal{O}_U \\
\end{array}
\]

is commutative, and the top and bottom composite bilinear maps are non-degenerate pairings (the vertical arrow on the left is the twisting map).

Lemma 5.16. We have \( \varphi_{12} = \varphi_{21} = 0 \).

Theorem 5.17. The maps \( \theta_{K_0}, \iota_{K_0} \) and \( \iota_{K_0} \) satisfy the centrality, adjoint and Cardy conditions.

Hence, we obtain the following

Corollary 5.18. Any maximal CY category \( \mathcal{B} \) over \( M \) is pseudo-abelian.

5.2.4. Local Structure of a maximal Cardy Fibration. There is a further assumption to be made about maximal categories in order to obtain a full description.

Definition 5.19. Let \( U \subset M \) be a semisimple open subset. We shall say that a label \( a \in \mathcal{B}(U) \) is supported on an index \( i_0 \) if

\[ \iota_a(e_{i_0}) = 1_a. \]

Equivalently, \( \iota_a(e_j) = 0 \) for each \( j \neq i_0 \).

Lemma 5.20. Let \( i \neq j \) be two indices, \( 1 \leq i, j \leq n \) and let \( a, b \) be labels over a semisimple open subset of \( M \). If \( a \) and \( b \) are supported on \( i \) and \( j \) respectively, then \( \Gamma_{ab} = 0 \).

Proof. Pick an arrow \( \sigma \in \Gamma_{ab} \). Then \( \sigma = \sigma 1_a = \sigma \iota_a(e_i) = \iota_b(e_i) \sigma = 0 \), as claimed. Q.E.D.

Lemma 5.21. Let \( \mathcal{B} \) be a maximal category of branes and \( U \) a semisimple open subset. For each index \( i, 1 \leq i \leq n \), there exists a label \( \xi_i \) supported on \( i \).
Proof. Assume that this statement is false. We shall see that the maximality of \( \mathcal{B} \) will not allow this to happen.

So we first assume that \( i_a(e_j) = 0 \) for each index \( j \) and each \( a \in \mathcal{B}(U) \). We define a new category \( \mathcal{C} \): the objects of \( \mathcal{C}(U) \) are objects of \( \mathcal{B}(U) \) plus one label, which we denote by \( \xi_i \). We also define

- \( \Gamma_{\xi_i\xi_i} = \mathcal{O}_U \).
- \( \Gamma_{\xi_i\xi_i} = \Gamma_{a\xi_i} = 0 \); this definition is motivated by lemma \( 5.20 \).
- \( \theta_{\xi_i} : \Gamma_{\xi_i\xi_i} = \mathcal{O}_U \rightarrow \mathcal{O}_U \) is the identity.
- \( \eta \in \mathcal{B}(U) \) be a local vector field. Then \( \iota_{\xi_i} : \mathcal{T}_U \rightarrow \Gamma_{\xi_i\xi_i} \) and \( \iota_{\xi_i} : \mathcal{T}_U \rightarrow \mathcal{T}_U \) are given by

\[
\iota_{\xi_i}(X) = \lambda_i \quad \text{and} \quad \iota_{\xi_i}(\lambda) = \lambda e_i.
\]

These definitions make \( \mathcal{C} \) a Cardy fibration, contradicting the maximality of \( \mathcal{B} \). Q.E.D.

Proposition 5.22. Let \( U \) be a semisimple neighborhood. For each index \( i = 1, \ldots, n \), there exists a label \( \xi_i \in \mathcal{B}(U) \) supported on \( i \) such that \( \Gamma_{\xi_i\xi_i} = \mathcal{O}_U \).

Proof. Let \( i \) be an index, \( 1 \leq i \leq n \). By lemma \( 5.21 \) we can pick a label \( a_i \) supported in \( i \). If \( \Gamma_{a_i a_i} \cong \mathcal{O}_U \), then \( \xi_i := a_i \) is the label we are looking for. If not, we have that \( \Gamma_{a_i a_i} \) can be taken to be a matrix algebra \( \mathbb{M}_{d_i}(\mathcal{O}_U) \) (the construction of such a label is assured by maximality of the category of branes, and can be proved by following exactly the same procedure used in the proof of lemma \( 5.21 \)). Let then \( \sigma \in \Gamma_{a_i a_i} \) be an idempotent matrix, which can be regarded as a morphism \( \sigma : \mathcal{O}_U \rightarrow \mathbb{M}_{d_i}(\mathcal{O}_U) \). Moreover, assume that \( \sigma \) is the projection

\[
\sigma(\lambda_1, \ldots, \lambda_n) = (\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n).
\]

Then, as the category of branes is pseudo-abelian, we have that \( \text{Ker} \sigma \cong \mathcal{O}_U \in \mathcal{B}(U) \). As \( \mathcal{O}_U \) is indecomposable, we should have \( \Gamma_{\text{Ker} \sigma \mathcal{O}_U} \cong \mathcal{O}_U \), and hence \( \xi_i := \text{Ker} \sigma \) is the object we were looking for. Q.E.D.

Lemma 5.23. \( \Gamma_{\xi_i\xi_j} = 0 \) for \( i \neq j \).

Proof. This is an immediate consequence of lemma \( 5.20 \). Q.E.D.

We shall need the following decomposition for \( \Gamma_{ab} \).
Proposition 5.24. For labels \( a, b \in \mathcal{B}(U) \), with \( U \) a semisimple neighborhood, we have an isomorphism

\[
\Gamma_{ab} \cong \bigoplus_i \Gamma_{a \xi^i} \otimes \Gamma_{\xi^i b}.
\]

Proof. Define the map \( \phi : \bigoplus_i \Gamma_{a \xi^i} \otimes \Gamma_{\xi^i b} \rightarrow \Gamma_{ab} \) by

\[
\phi(\sigma_1 \otimes \tau_1, \ldots, \sigma_n \otimes \tau_n) = \sum_i \tau_i \sigma_i.
\]  
(41)

Using the characterization given in 5.5, we have a local isomorphism

\[
\bigoplus_i \Gamma_{a \xi^i} \otimes \Gamma_{\xi^i b} \cong \bigoplus_i \bigg( \bigoplus_j \text{Hom}_{\mathcal{O}_U}(\mathcal{O}^d_{\mathcal{O}_U}(a, j), \mathcal{O}^d_{\mathcal{O}_U}(\xi^i, j)) \bigg) \otimes \bigg( \bigoplus_k \text{Hom}_{\mathcal{O}_U}(\mathcal{O}^d_{\mathcal{O}_U}(\xi^i, k), \mathcal{O}^d_{\mathcal{O}_U}(b, k)) \bigg).
\]

By 5.23, we have that

\[
d(\xi^i, k) = \delta_{ik},
\]

and thus

\[
\bigoplus_i \Gamma_{a \xi^i} \otimes \Gamma_{\xi^i b} \cong \bigoplus_i \text{Hom}_{\mathcal{O}_U}(\mathcal{O}^d_{\mathcal{O}_U}(a, i), \mathcal{O}^d_{\mathcal{O}_U}(b, i)).
\]

On the other hand, by 5.3, we also have that, locally, \( \Gamma_{ab} \cong \bigoplus_i \text{Hom}_{\mathcal{O}_U}(\mathcal{O}^d_{\mathcal{O}_U}(a, i), \mathcal{O}^d_{\mathcal{O}_U}(b, i)) \). Combining these facts with (41) we conclude that the stalk maps \( \phi_x \) are in fact bijections for each \( x \in U \).

Q.E.D.

A useful consequence of 5.24 is the following

Corollary 5.25. For each label \( b \) over \( U \), we have an isomorphism \( b \cong \bigoplus_i \Gamma_{\xi^i b} \otimes \xi^i \).

Proof. Take any label \( c \). By equations (39) and duality we have

\[
\text{Hom}_{\mathcal{O}_U}\left(\bigoplus_i \Gamma_{\xi^i b} \otimes \xi^i, c\right) \cong \bigoplus_i \Gamma_{b \xi^i} \otimes \text{Hom}_{\mathcal{O}_U}(\xi^i, c)
\]

\[
\cong \bigoplus_i \Gamma_{b \xi^i} \otimes \Gamma_{\xi^i c} \cong \Gamma_{bc}.
\]

As \( c \) is arbitrary, the result follows. Q.E.D.

Note that the coefficient modules in the previous result are unique, up to isomorphism: if \( b \cong \bigoplus_i \mathcal{M}_i \otimes \xi^i \), then

\[
\Gamma_{\xi^i b} \cong \bigoplus_i \mathcal{M}_i \otimes \Gamma_{\xi^i \xi^i} \cong \mathcal{M}_i.
\]

The next result addresses some uniqueness issues.

Proposition 5.26. Let \( \xi^i \in \mathcal{B}(U) \) be as in 5.22 where \( U \) is semisimple.
(1) Let $\eta_i$ be a label with the same properties as $\xi_i$. Then, there exists an invertible sheaf $L$ over $U$ such that $\eta_i \cong L \otimes \xi_i$. The converse statement also holds.

(2) If $M$ is a locally-free module such that $M \otimes \xi_i \cong \xi_i$, then $M \cong O_U$.

Proof. For the first item, by 5.20 and 5.25, we have that $\eta_i \cong \bigoplus_j \Gamma_{\xi_i} \otimes \xi_j \cong \Gamma_{\xi_i} \otimes \xi_i$. Define now $M_i = \Gamma_{\xi_i} \eta_i$. Then,

$$O_U \cong \Gamma_{\eta_i} \cong \Gamma_{(M \otimes \xi_i)(\xi_i \otimes \xi_i)} \cong M_i \otimes \Gamma_{\xi_i} \eta_i \cong \Gamma_{\xi_i} \eta_i \otimes \Gamma_{\xi_i} \eta_i.$$ 

The converse is immediate by properties of the action $L \otimes \xi_i$.

For (2), as $M \otimes \xi_i \cong \xi_i$, the modules $\Gamma_{\xi_i} \eta_i$ and $\Gamma_{\xi_i}(M \otimes \xi_i)$ are isomorphic. Hence,

$$O_U \cong \Gamma_{\xi_i}(M \otimes \xi_i) \cong M \otimes \Gamma_{\xi_i} \eta_i \cong M,$$

as desired. Q.E.D.

Theorem 5.27. There exists an open cover $U$ of $M$ and an equivalence of categories

$$\mathcal{B}(U) \simeq LF^p_{O_U}$$

for each $U \in \mathcal{U}$, where $LF^p_{O_U}$ denotes the $n$-fold fibred product of $LF_{O_U}$.

Proof. Let $\mathcal{U} = \{U_a\}$ be an open cover of $M$, where each $U_a$ is semisimple. Define a functor $F_a : \mathcal{B}(U_a) \rightarrow LF^p_{O_{U_a}}$ on objects by

$$F_a(a) = (\Gamma_{\xi_1 a}, \ldots, \Gamma_{\xi_n a}),$$

where the objects $\xi_i$ are the ones of proposition 5.22 and on arrows by $F_a(\sigma) = \sigma_i$; that is, if $\sigma : a \rightarrow b$, then $F_a(\sigma)(\tau_1, \ldots, \tau_n) = (\sigma \tau_1, \ldots, \sigma \tau_n)$. We now define $G_a : LF^p_{O_{U_a}} \rightarrow \mathcal{B}(U_a)$ on objects by

$$G_a(M_1, \ldots, M_n) = \bigoplus_i M_i \otimes \xi_i$$

and on arrows by

$$G_a(f_1, \ldots, f_n) = (f_1 \otimes id_{\xi_1}, \ldots, f_n \otimes id_{\xi_n}),$$

where $f_i : M_i \rightarrow N_i$. 


We then have that $F_\alpha G_\alpha (\mathcal{M}_1, \ldots, \mathcal{M}_n) = (\Gamma_{\xi,\pi}, \ldots, \Gamma_{\xi,\pi})$, where $\mathcal{A} := \bigoplus_j \mathcal{M}_j \otimes \xi_j$. Now,

$$\Gamma_{\xi,\pi} \cong \bigoplus_j \text{Hom}_U(\xi_j, \mathcal{M}_j \otimes \xi_j) \cong \bigoplus_j \mathcal{M}_j \otimes \text{Hom}_U(\xi_j, \xi_j) \cong \mathcal{M}_i$$

by (39) and 5.23.

The other way, we have $G_\alpha F_\alpha (a) = \bigoplus_i \Gamma_{\xi,\pi} \otimes \xi_i$, which is isomorphic to $a$ by 5.25. Q.E.D.

In terms of the spectral cover, over each semisimple $U \subset M$ we have $\pi^{-1}(U) = \bigsqcup_{i=1}^n U_i$, where each $U_i$ is homeomorphic to $U$ by the projection $\pi: S \to M$, and thus we can write the $n$-fold product $LF_{\mathcal{O}_U}$ as the pushout $(\pi_* LF_{\mathcal{O}_S})(U) = LF_{\mathcal{O}_{\pi^{-1}(U)}}$. But $\mathcal{O}_{\pi^{-1}(U)}$ is the sheaf $(\pi_* \mathcal{O}_S)|_U$, which is in turn isomorphic to the tangent sheaf $\mathcal{T}_U$ by proposition 2.25. Moreover, if $f: M \to N$ is a continuous map, then, by definition, the fibred categories $f_* LF_{\mathcal{O}_M}$ and $LF f_* \mathcal{O}_M$ are equal. Thus, combining all these facts we can deduce that

$$\pi_* LF_{\mathcal{O}_S} = LF_{\pi_* \mathcal{O}_S} \cong LF_{\mathcal{T}_M}.$$ 

**Corollary 5.28.** Given a maximal Cardy fibration $\mathcal{B}$ over a massive manifold $M$, there exists an open cover $\mathcal{U}$ of $M$ such that the category $\mathcal{B}(U)$ is equivalent to the category $LF_{\mathcal{T}_U}$ of locally free $\mathcal{T}_U$-modules.

Before stating the next result, we give a preliminary definition. Given a vector bundle $E$ we can construct the exterior powers $\bigwedge^k E$ which for a point $x \in M$ have fibre $\bigwedge^k E_x$. Given now a bundle map $\phi: E \to F$, we have that $\phi^\wedge_k: \bigwedge^k E \to \bigwedge^k F$ is given by

$$\phi^\wedge_k(e_1 \wedge \cdots \wedge e_n) = \phi(e_1) \wedge \cdots \wedge \phi(e_n).$$

After this brief comment about exterior powers, we can now give the definition we need (see [Aud98] and references cited therein). A Higgs pair for a manifold $M$ is a pair $(E, \phi)$, where $E$ is a vector bundle and $\phi: TM \to \text{End}(E)$ is a morphism such that $\phi \wedge \phi = 0$. This last condition is expressing that for each $x \in M$, the endomorphisms $\phi_x(v) \in \text{End}(E_x)$ (for $v \in T_x M$) commute.

**Corollary 5.29.** Given $a \in \mathcal{B}(U_a)$, the transition homomorphism $\iota_a$ consists of $n$ Higgs pairs for $U_a$. 

The meaning of “consists of n Higgs pairs” is explained in the following proof.

Proof. From theorem 5.27 we have an equivalence \( F_\alpha : \mathcal{B}(U_\alpha) \to LF^\alpha_{U_\alpha} \); in particular, given a label \( a \in \mathcal{B}(U_\alpha) \), we have a bijection

\[
\text{Hom}_{\mathcal{B}(U_\alpha)}(a, a) \longrightarrow \text{Hom}_{LF^\alpha_{U_\alpha}}(F_\alpha(a), F_\alpha(a)),
\]

which is in fact an isomorphism of algebras

\[
\Gamma_{\alpha a} \longrightarrow \bigoplus_k \text{End}_{LF^\alpha_{U_\alpha}}(\Gamma_{\xi_k a}).
\]

We can then assume that the transition homomorphism \( \iota_a : \mathcal{T}_{U_\alpha} \to \Gamma_{\alpha a} \) is in fact a morphism

\[
iota_a : \mathcal{T}_{U_\alpha} \longrightarrow \bigoplus_k \text{End}_{LF^\alpha_{U_\alpha}}(\Gamma_{\xi_k a});
\]

in other words, the map \( \iota_a \) consists of \( n \) morphisms

\[
iota^k_a : \mathcal{T}_{U_\alpha} \longrightarrow \text{End}_{LF^\alpha_{U_\alpha}}(\Gamma_{\xi_k a}).
\]

In our case, we have that the morphism \( \iota_a \) is central; this condition can be also expressed by saying that the morphisms \( \iota^k_a \) are central \((k = 1, \ldots, n)\). Hence, for each \( k = 1, \ldots, n \), \((\Gamma_{\xi_k a}, \iota^k_a)\) is a Higgs pair for \( U_\alpha \). Q.E.D.

We shall now describe the BDR 2-vector bundle structure for the stack \( \mathcal{B} \) (check definition 3.11 for details).

We first point out that, being \( M \) paracompact, the open cover by semisimple open subsets \( U = \{ U_\alpha \} \) can be taken to be indexed by a poset (which we shall not include in our notation). For each index \( i = 1, \ldots, n \), let \( \xi^\alpha_i \in \mathcal{B}(U_\alpha) \) be a label as in proposition 5.22. Let \( U_\beta \) be another semisimple subset such that \( U_{\alpha \beta} \neq \emptyset \) and let \( \{ e^\alpha_i \} \) and \( \{ e^\beta_i \} \) be frames of simple idempotent sections over \( U_\alpha \) and \( U_\beta \) respectively. We then have a permutation \( u = u_{\alpha \beta} : \{ 1, \ldots, n \} \to \{ 1, \ldots, n \} \) such that, over \( U_{\alpha \beta} \),

\[
e^\alpha_i = e^\beta_{u(i)}.
\]

By proposition 5.26 the previous equation is equivalent to the existence of invertible sheaves \( \mathcal{L}^{\alpha \beta}_i \) such that, over \( U_{\alpha \beta} \),

\[
\xi^\alpha_i \cong \mathcal{L}^{\alpha \beta}_{u(i)} \otimes \xi^\beta_{u(i)}.
\]
Write $\xi^\alpha := (\xi_1^\alpha, \ldots, \xi_n^\alpha)^t$. Then, we can write the previous equation in matrix form

$$\xi^\alpha \sim A_u^{\alpha \beta} \xi^\beta,$$

where $A_u^{\alpha \beta}$ is a matrix obtained from the diagonal matrix

$$\text{diag} \left( L_1^{\alpha \beta}, \ldots, L_n^{\alpha \beta} \right)$$

by applying the permutation $u$ to its columns. Let now $\gamma$ be such that $U_{\alpha \beta \gamma} \neq \emptyset$ and suppose that the idempotents are permuted according to $v$ over $U_{\beta \gamma}$ and $w$ over $U_{\alpha \gamma}$.

**Lemma 5.30.** We have an isomorphism $A_u^{\alpha \beta} A_v^{\beta \gamma} \cong A_w^{\alpha \gamma}$ (i.e. the corresponding matrix entries on each side have isomorphic bundles).

**Proof.** Assume that the idempotents are permuted according to

- $u$ over $U_{\alpha \beta}$,
- $v$ over $U_{\beta \gamma}$ and
- $w$ over $U_{\alpha \gamma}$.

Then, by uniqueness, we should have $vu = w$. Now pick a vector $\xi^\gamma$. Then, the $i$-th coordinate of $A_u^{\alpha \beta} A_v^{\beta \gamma} \xi^\gamma$ is given by $L_i^{\alpha \beta} \otimes L_{u(i)}^{\beta \gamma} \otimes \xi^\gamma_{v(u(i))}$ and the one corresponding to the product $A_w^{\alpha \gamma} \xi^\gamma$ is $L_i^{\alpha \gamma} \otimes \xi^\gamma_{w(i)}$. As both objects are isomorphic to $\xi_i^\alpha$, they are both isomorphic, and hence by 5.26

$$L_i^{\alpha \beta} \otimes L_{u(i)}^{\beta \gamma} \cong L_i^{\alpha \gamma},$$

as desired.

Q.E.D.

If $A = (E_{ij})$ is an $n \times n$ matrix of vector bundles, we denote by $\text{rk} A \in \text{M}_n(\mathbb{N}_0)$ the matrix which $(i, j)$ entry is $\text{rk} E_{ij}$. Then, by definition,

$$\det \left( \text{rk} A_u^{\alpha \beta} \right) = \pm 1.$$

Moreover, associativity of the tensor product renders the following diagram

$$
\begin{array}{c}
A^{\alpha \beta} (A^{\beta \gamma} A^{\gamma \delta}) \xrightarrow{\sim} (A^{\alpha \beta} A^{\beta \gamma}) A^{\gamma \delta} \\
\downarrow \quad \downarrow \\
A^{\alpha \beta} A^{\beta \delta} \xrightarrow{\sim} A^{\alpha \delta} \xleftarrow{\sim} A^{\alpha \gamma} A^{\gamma \delta},
\end{array}
$$
commutative (see definition 3.11). We can then state the following

**Theorem 5.31.** Let \( M \) be a semisimple \( F \)-manifold of dimension \( n \). Then, any maximal Cardy fibration \( B \) over \( M \) has a canonical BDR 2-vector bundle of rank \( n \) attached to it.

### 6. BRANES AND TWISTED BUNDLES

Let now \( \mathcal{A} \) be an algebra over \( M \), i.e. a sheaf of (non necessarily commutative) \( \mathcal{O}_M \)-algebras, and assume also that \( \mathcal{A} \) is locally-free as an \( \mathcal{O}_M \)-module. Let \( \iota : \mathcal{T}_M \to \mathcal{A} \) be a central morphism; this map provides \( \mathcal{A} \) with a structure of \( \mathcal{T}_M \)-algebra.

**Lemma 6.1.** If \( S \) is the spectral cover of \( M \) with projection \( \pi : S \to M \), the topological inverse image \( \pi^{-1}\mathcal{T} \) is a sheaf of rings (and of \( \pi^{-1}\mathcal{O}_M \)-modules) and \( \pi^{-1}\mathcal{A} \) is a \( \pi^{-1}\mathcal{T} \)-algebra by means of the central morphism \( \pi^{-1}\iota : \pi^{-1}\mathcal{T} \to \pi^{-1}\mathcal{A} \) which is given by \( \pi^{-1}\iota(\sigma) = \iota_{\pi(y)}(\sigma(y)) \).

**Proof.** Recall that, for a sheaf over \( S \) over \( M \), \( \pi^{-1}S \) is the sheaf given by \( \pi^{-1}S(\tilde{U}) = S(\pi(\tilde{U})) \).

From this definition, the statement of the lemma readily follows. Q.E.D.

In the following we shall consider the ringed space \((S, \mathcal{O}_S)\) and also \( M \) with two different ringed structures: one given by \( \mathcal{O}_M \) and the other by the sheaf of algebras \( \mathcal{T} \). By proposition 2.25 we have distinguished maps \( u_1 : \mathcal{O}_M \to \pi_*\mathcal{O}_S \) and \( u_2 : \mathcal{T} \to \pi_*\mathcal{O}_S \), which can be regarded as the inclusion \( f \mapsto f1 \) and the identity, respectively. This maps define two morphisms of ringed spaces \((\pi, u_1) : (S, \mathcal{O}_S) \to (M, \mathcal{O}_M) \) and \((\pi, u_2) : (S, \mathcal{O}_S) \to (M, \mathcal{T}) \). By the adjunction between \( \pi_* \) and \( \pi^{-1} \) we have change-of-ring morphisms

\[
\pi^{-1}\mathcal{O}_M \to \mathcal{O}_S \quad \text{and} \quad \pi^{-1}\mathcal{T} \to \mathcal{O}_S,
\]

and the inverse images

\[
\pi^*\mathcal{T} = \mathcal{O}_S \otimes_{\pi^{-1}\mathcal{O}_M} \pi^{-1}\mathcal{T} \quad \text{and} \quad \pi^*\mathcal{A} = \mathcal{O}_S \otimes_{\pi^{-1}\mathcal{T}} \pi^{-1}\mathcal{A}
\]

are \( \mathcal{O}_S \)-algebras. By considering the morphism \( \pi^*\mathcal{T} \xrightarrow{1 \otimes \pi^{-1}\iota} \pi^*\mathcal{A} \), the sheaf \( \pi^*\mathcal{A} \) turns out to be a \( \pi^*\mathcal{T} \)-algebra. The actions that provide these algebra structures will be described explicitly after introducing some other tools that we need.
Lemma 6.2. Let $\mathcal{A}$ be a sheaf of commutative $\mathcal{R}$-algebras over $S$, where $\mathcal{R}$ is a sheaf of commutative rings. Then $\pi_*\mathcal{A}$ is a sheaf of $\pi_*\mathcal{R}$-algebras.

In what follows, we regard $S$ as being a submanifold of $T^*M$; i.e. points of $S$ are multiplicative linear maps $\phi : T_xM \to \mathbb{C}$, where $x = \pi(\phi)$. We now define a global section $\sigma_0 \in \Gamma(S; \pi^{-1}\mathcal{F})$ in the following way: we let $\sigma_0 : S \to \bigcup_{\phi \in S} \mathcal{F}_{\pi(\phi)}$ be given by

$$\sigma_0(\phi) := (\phi, e^\phi_x),$$

where $x = \pi(\phi)$ and $e^\phi_x$ is the germ at $x$ of the unique idempotent local section $e^\phi : U \to TM$ which verifies $\phi(e^\phi(x)) = 1$. Note that $\sigma_0$ induces a section $1 \otimes \sigma_0 \in \Gamma(S; \pi^*\mathcal{F})$ and, moreover, $\sigma_0$ as well as $1 \otimes \sigma_0$ are idempotent. Likewise, $\sigma_0$ also induces (global) idempotent sections on $\pi^{-1}\mathcal{A}$ and $\pi^*\mathcal{A}$ given by $\pi^{-1}i(\sigma_0)$ and $1 \otimes \pi^{-1}i(\sigma_0)$, respectively. To be more explicit, we have

$$1 \otimes \sigma_0 \in \Gamma(S; \pi^*\mathcal{F}) \quad , \quad 1 \otimes \sigma_0 : S \to \bigcup_{\phi \in S} \mathcal{O}_{S,\phi} \otimes \mathcal{O}_{M,\pi(\phi)} \mathcal{F}_{\pi(\phi)},$$

$$\pi^{-1}i(\sigma_0) \in \Gamma(S; \pi^{-1}\mathcal{A}) \quad , \quad \pi^{-1}i(\sigma_0) : S \to \bigcup_{\phi \in S} \mathcal{A}_{\pi(\phi)},$$

$$1 \otimes \pi^{-1}i(\sigma_0) \in \Gamma(S; \pi^*\mathcal{A}) \quad , \quad 1 \otimes \pi^{-1}i(\sigma_0) : S \to \bigcup_{\phi \in S} \mathcal{O}_{S,\phi} \otimes \mathcal{F}_{\pi(\phi)} \mathcal{A}_{\pi(\phi)},$$

given by the following expressions:

$$(1 \otimes \sigma_0)_\phi = 1 \otimes e^\phi_x,$$

$$\pi^{-1}i(\sigma_0)_\phi = i_x(e^\phi_x),$$

$$(1 \otimes \pi^{-1}i(\sigma_0))_\phi = 1 \otimes i_x(e^\phi_x),$$

where $x = \pi(\phi)$.

Proposition 6.3. Let $\mathcal{A}$ be an algebra over a space $M$ and let $e \in \mathcal{A}(M)$ be a global idempotent section. Then the assignment

$$U \mapsto e\mathcal{A}(U) = \{e\sigma \mid \sigma \in \mathcal{A}(U)\}$$

is a sheaf of ideals.

Proof. Let $\{U_i\}$ be an open cover of an open subset $U \subset M$; for each index $i$, let $\sigma_i \in e\mathcal{A}(U_i)$ such that $\sigma_i = \sigma_j$ over $U_{ij}$. Then we have:
(1) for each \( i \), there exists a section \( \tau_i \in \mathcal{A}(U_i) \) such that \( \sigma_i = e \tau_i \) and

(2) as \( \mathcal{A} \) is a sheaf, there exists a unique section \( \sigma \in \mathcal{A}(U) \) with \( \sigma|_{U_i} = \sigma_i \) for each \( i \).

Consider now the section \( e \sigma \in e\mathcal{A}(U) \). Then, over \( U_i \) we have

\[
(e \sigma)|_{U_i} = e \sigma_i = e(e \tau_i) = e \tau_i = \sigma_i,
\]

and thus, by uniqueness, \( \sigma = e \sigma \in \mathcal{A}(U) \). Q.E.D.

Notation 6.4. The sheaves \((1 \otimes \sigma_0)\pi^* \mathcal{T}_M\) and \((1 \otimes \pi^{-1}(e_0))\pi^* \mathcal{A}\), will be denoted by \( \mathcal{T}_0^* \) and \( \mathcal{A}_0^* \) respectively. The notation \( e_x^\varphi \) will be adopted for the germ \( \iota_x(e_\varphi) \).

By the previous result, the sheaves \( \mathcal{T}_0^* \) and \( \mathcal{A}_0^* \) are \( \mathcal{O}_S \)-algebras and their stalks are given by the expressions

\[
\mathcal{T}_{0,x}^* = \mathcal{O}_{S,x} \otimes_{\mathcal{O}_M} e_x^\varphi \mathcal{T}_x,
\]

\[
\mathcal{A}_{0,x}^* = \mathcal{O}_{S,x} \otimes_{\mathcal{O}_M} e_x^\varphi \mathcal{A}_x,
\]

where \( x = \pi(\varphi) \).

Notation 6.5. From now on, we will suppress the coefficient rings in the notation of the tensor product.

Proposition 6.6. There exists a canonical isomorphism of \( \mathcal{O}_S \)-algebras \( \mathcal{T}_0^* \cong \mathcal{O}_S \).

Proof. The correspondence \( \mathcal{O}_S \rightarrow \mathcal{T}_0^* \) given by \( f \mapsto f \otimes \sigma_0 \). provides the desired isomorphism. Q.E.D.

Combining 2.25 and 6.6 we have the following

Corollary 6.7. There exists a canonical isomorphism of \( \mathcal{O}_M \)-algebras

\[
\pi_* \mathcal{T}_0^* \cong \mathcal{T}.
\]

Lemma 6.8. If \( U \subset M \) is a semisimple neighborhood with basis \( \{e_1, \ldots, e_n\} \), there exists an isomorphism

\[
\mathcal{A}|_U \cong \bigoplus_i \iota(e_i) \mathcal{A}|_U.
\]
Proof. Define \( \phi : \mathcal{A} |_U \to \bigoplus_i \iota(e_i) \mathcal{A} |_U \) by

\[
\phi(\sigma) = \sum_i \iota(e_i)\sigma.
\]

Recalling that the stalk \( \left( \bigoplus_i \iota(e_i) \mathcal{A} |_U \right)_x \) is given by \( \bigoplus \iota_x^\phi e_x^\phi \mathcal{A}_x \), the statement of the lemma follows.

Q.E.D.

Theorem 6.9. The assignment \( \sigma \mapsto \sigma \) defines an isomorphism of \( \mathcal{T} \)-algebras

\[
\mathcal{A} \longrightarrow \pi_* \mathcal{A}^*_0.
\]

Proof. The equalities \( \tilde{1} = 1 \) and \( \tilde{\sigma + \tau} = \tilde{\sigma} + \tilde{\tau} \) are straightforward to verify. Let us now check that \( \tilde{\sigma \tau} = \tilde{\sigma} \tilde{\tau} \) holds. We have

\[
(\tilde{\sigma \tau})_x = \sum_{\phi \in \pi^{-1}(x)} 1 \otimes e_x^\phi \sigma_x \tau_x
\]

\[
= \sum_{\phi \in \pi^{-1}(x)} 1 \otimes e_x^\phi \sigma_x e_x^\phi \tau_x
\]

\[
= \left( \sum_{\phi \in \pi^{-1}(x)} 1 \otimes e_x^\phi \sigma_x \right) \left( \sum_{\phi \in \pi^{-1}(x)} 1 \otimes e_x^\phi \tau_x \right) = \sigma_x \tau_x.
\]

Let \( X \) be a vector field on \( M \) with local representation \( X = \sum_{\phi \in \pi^{-1}(x)} \lambda_x^\phi e_x^\phi \). We will now check that \( \tilde{X} \cdot \tilde{\sigma} = \tilde{X} \cdot \tilde{\sigma} \), which is almost a tautology. The left hand side is

\[
(\tilde{X} \cdot \tilde{\sigma})_x = \sum_{\phi \in \pi^{-1}(x)} 1 \otimes \lambda_x^\phi e_x^\phi \sigma_x.
\]

\[
= \sum_{\phi \in \pi^{-1}(x)} \tilde{\lambda}_x \otimes e_x^\phi \sigma_x,
\]

where \( \tilde{\lambda} \) is the map on \( \pi^{-1}(U) \) defined by \( \tilde{\lambda}(\phi) = \lambda(\pi(\phi)) \). But the right hand side is precisely \( (X \cdot \sigma)_x \).

We will now prove that the assignment \( \sigma \mapsto \tilde{\sigma} \) is a sheaf isomorphism, so we will check that at the level of stalks, the maps \( \mathcal{A}_x \to (\pi_* \mathcal{A}^*_0)_x \) are bijections.

Let \( \tau_x \in (\pi_* \mathcal{A}^*_0)_x \) be given by \( \tau_x = \sum_{\phi \in \pi^{-1}(x)} f_x^\phi \otimes e_x^\phi \sigma_x \). Assume also that \( f_x^\phi \) is the germ of a function, which, abusing, we denote again by \( f_x^\phi \), defined in a neighborhood \( \tilde{U}_\phi \) of \( \phi \) such that...
\[ \pi|_{\bar{U}_\varphi} \text{ is a homeomorphism. If we define} \]
\[ \sigma_x = \sum_{\varphi \in \pi^{-1}(x)} (f_\varphi \pi^{-1})_x \varepsilon_{\varphi, x} \varepsilon_{\varphi, x} \in \mathcal{A}_x, \]
then \( \sigma_x \mapsto \tau_x \).

Suppose now that \( \sigma_x = \sum_{\varphi \in \pi^{-1}(x)} 1 \otimes e_\varphi \sigma_x = 0 \). As all the modules (stalks) involved are free, this equality implies immediately that \( e_\varphi \sigma_x = 0 \) for each \( \varphi \in \pi^{-1}(x) \), and thus \( \sigma_x = 0 \). This finishes the proof. Q.E.D.

Recall now that a functor \( F : X \to Y \) is said to be essentially surjective if for each object \( Y \in Y \) there exists an object \( X \in X \) such that \( F(X) \) is isomorphic to \( Y \). For a sheaf of rings or algebras \( \mathcal{R} \), we let \( \text{Mod}_{\mathcal{R}} \) denote the category of \( \mathcal{R} \)-modules. The previous results can then be summarized in the following

**Theorem 6.10.** The functor \( \pi_* : \text{Mod}_{\mathcal{O}_S} \to \text{Mod}_{\mathcal{O}} \) is essentially surjective.

### 6.1. A Correspondence Between Branes and Twisted Vector Bundles

Consider now a global label \( a \in \mathcal{B}(M) \); we can then apply the machinery of the previous sections to the \( \mathcal{O} \)-algebra \( \Gamma_{aa} \).

Hence, by 6.10, there exists an \( \mathcal{O}_S \)-algebra \( \tilde{\Gamma}_{aa} \) such that \( \pi_* \tilde{\Gamma}_{aa} \cong \Gamma_{aa} \).

**Theorem 6.11.** \( \tilde{\Gamma}_{aa} \) is an Azumaya algebra over \( S \).

**Proof.** Let \( x \in M \) and let \( U \) be a semisimple neighborhood of \( x \), with \( \pi^{-1}(U) = \bigsqcup_i \bar{U}_i \). If \( a \in \mathcal{B}(M) \) is a global label, then we can apply 5.3 to the restriction \( a|_U \). Let \( \{e_1, \ldots, e_n\} \) be a frame of simple, orthogonal idempotent sections over \( U \). Suppose now that \( e_i \) is the section corresponding to the sheet \( \bar{U}_i \). By constructions in the previous section, and also theorem 5.3 and remark 5.4, we can write
\[ \tilde{\Gamma}_{aa}|_{\bar{U}_i} = i_a(e_i) \Gamma_{aa}|_{\pi(\bar{U}_i)} \cong i_a(e_i) \Gamma_{aa}|_U \cong M_{d(a,i)}(\mathcal{O}_U). \]

Q.E.D.

Note that the dimension of the matrix algebras may vary at different sheets: if \( \Gamma_{aa} \) is isomorphic over a semisimple \( U \) to \( \bigoplus_i M_{d_i}(\mathcal{O}_M) \), then, if \( \varphi \in \bar{U}, \pi(\varphi) = x \in U \) and \( \bar{U} \) is a sufficiently small neighborhood around \( \varphi \), we have that
\[ \tilde{\Gamma}_{aa}|_{\bar{U}} \cong M_{d_i}(\mathcal{O}_{\bar{U}}). \]
If the cover $S$ is connected, then this dimension is constant. In this case, we then have a twisted vector bundle $\mathbb{E}_a$ over $S$ such that

$$\text{END}(\mathbb{E}_a) \cong \tilde{\Gamma}_{aa}.$$  

From now on we shall assume that $S$ is connected.

Take now two boundary conditions $a, b \in \mathcal{B}(M)$ such that $\Gamma_{aa} \cong \Gamma_{bb}$. On a semisimple open subset $U_i$ we can represent both labels in the form

$$a|_{U_i} = \bigoplus_k \mathcal{M}_k \otimes \tilde{\zeta}_k,$$

$$b|_{U_i} = \bigoplus_k \mathcal{N}_k \otimes \tilde{\zeta}_k,$$

where $\mathcal{M}_k, \mathcal{N}_k$ are locally free modules and $\tilde{\zeta}_k$ are the objects of proposition 5.22. Then, $\Gamma_{aa}|_{U_i} \cong \bigoplus_k \text{End}_{\mathcal{O}_{U_i}}(\mathcal{M}_k)$ and $\Gamma_{bb}|_{U_i} \cong \bigoplus_k \text{End}_{\mathcal{O}_{U_i}}(\mathcal{N}_k)$. By theorem 6.11 and the connectivity of $S$ we can write

$$\Gamma_{aa}|_{U_i} \cong \text{End}_{\mathcal{O}_{U_i}}(\mathcal{M}^{(i)}),$$

$$\Gamma_{bb}|_{U_i} \cong \text{End}_{\mathcal{O}_{U_i}}(\mathcal{N}^{(i)}).$$

(45)

for some locally free modules $\mathcal{M}^{(i)}$ and $\mathcal{N}^{(i)}$ over $U_i$. As $\Gamma_{aa}$ and $\Gamma_{bb}$ are isomorphic, we can assure the existence of invertible sheaves $\mathcal{L}_i$ such that $\mathcal{N}^{(i)} \cong \mathcal{L}_i \otimes \mathcal{M}^{(i)}$. By shrinking the open subset if necessary, we can regard these invertible sheaves as free.

From equations (45) let us denote by $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ the locally free sheaves with local representation $\text{End}_{\mathcal{O}_{U_i}}(\mathcal{M}^{(i)})$ and $\text{End}_{\mathcal{O}_{U_i}}(\mathcal{N}^{(i)})$ respectively. Then

- $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ are Azumaya algebras. Hence, there exist twisted bundles $\mathbb{E}$ and $\mathbb{F}$ such that $\widehat{\mathcal{M}} \cong \Gamma_{\text{END}(\mathbb{E})}$ and $\widehat{\mathcal{N}} \cong \Gamma_{\text{END}(\mathbb{F})}$.

- As $\Gamma_{aa}$ and $\Gamma_{bb}$ are isomorphic, $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$ are also isomorphic. In particular, END($\mathbb{E}$) and END($\mathbb{F}$) are isomorphic.

**Proposition 6.12.** Let $\mathbb{E}$ and $\mathbb{F}$ be two twisted bundles over a space $M$. Then the algebra bundles $\text{END}(\mathbb{E})$ and $\text{END}(\mathbb{F})$ are isomorphic if and only if there exists a twisted line bundle $\mathbb{L}$ such that $\mathbb{F} \cong \mathbb{E} \otimes \mathbb{L}$. 
Proof. We make use of 3.16. Let \( \mathcal{E}, \mathcal{F} \) be given by

\[
\mathcal{E} = (\mathcal{U}, U_i \times \mathbb{C}^n, g_{ij}, \lambda_{ijk}),
\]

\[
\mathcal{F} = (\mathcal{U}, U_i \times \mathbb{C}^n, f_{ij}, \mu_{ijk}).
\]

For the “if” part, let \( \mathcal{L} \) be given by \((\mathcal{U}, U_i \times \mathbb{C}, \xi_{ij}, \eta_{ijk})\), where \( \xi_{ij} : U_{ij} \to \mathbb{C}^\times \). Assume that \( u_{ij} : U_{ij} \to \text{GL}(M_n(\mathbb{C})) \) are the cocycles for \( \text{END}(\mathcal{E} \otimes \mathcal{L}) \); then,

\[
u_{ij}(x)(A) = \xi_{ij}(x)g_{ij}(x)A\xi_{ij}(x)^{-1}g_{ij}(x)^{-1},
\]

which are precisely the cocycles for \( \text{END}(\mathcal{E}) \).

For the “only if” part, assume that \( \text{END}(\mathcal{E}) \cong \text{END}(\mathcal{F}) \) and let \( \{ \alpha_i : U_i \to \text{GL}(M_n(\mathbb{C})) \} \) be a family of maps as in 3.16. Then, for each \( n \times n \) matrix \( A \) we have

\[
f_{ij}(x)Af_{ij}(x)^{-1} = (\alpha_i(x)g_{ij}(x)\alpha_j(x)^{-1})A(\alpha_i(x)g_{ij}(x)\alpha_j(x)^{-1})^{-1},
\]

over \( U_{ij} \). This equality implies that there exists a map \( \tilde{\xi}_{ij} : U_{ij} \to \mathbb{C}^\times \) such that

\[
f_{ij}(x)^{-1}\alpha_i(x)g_{ij}(x)\alpha_j(x)^{-1} = \tilde{\xi}_{ij}(x)1 \tag{46}
\]

or, equivalently,

\[
f_{ij}(x) = \alpha_i(x)\tilde{\xi}_{ij}(x)^{-1}g_{ij}(x)\alpha_j(x)^{-1},
\]

where \( \alpha_i(x) \) is regarded here as an invertible matrix (by the Skolem-Noether theorem).

We now only need to show that \( \{ \tilde{\xi}_{ij} \} \) is a (twisted) cocycle. Multiplying equation (46) by the one corresponding to \( \tilde{\xi}_{jk} \) and using the twistings for \( \mathcal{E} \) and \( \mathcal{F} \) (we omit any reference to \( x \in U_{ijk} \) for simplicity) we obtain

\[
\alpha_i\lambda_{ijk}g_{ik}\alpha_k^{-1} = \tilde{\xi}_{ij}\tilde{\xi}_{jk}\mu_{ijk}f_{ik};
\]

rearranging the last equation we must have

\[
\tilde{\xi}_{ij}\tilde{\xi}_{jk} = \lambda_{ijk}\mu_{ijk}^{-1}\tilde{\xi}_{ik},
\]

as desired. Q.E.D.
Let now \( B(M)/\sim \) be the set of labels over \( M \) subject to the identification

\[
a \sim b \iff \Gamma_{aa} \cong \Gamma_{bb}
\]

and let \( TVB(S) \) be the set of twisted vector bundles over \( S \). We can then define a map

\[
\Phi : B(M)/\sim \to TVB(S)/E \sim L \otimes E
\]

by \( \Phi(a) = E_a \), where \( L \) is a twisted line bundle. The results obtained in the previous paragraphs let us conclude with the following characterization of branes in terms of twisted bundles.

**Theorem 6.13.** The map \( \Phi \) is injective.

In other words, we can regard each label (up to equivalence) over \( M \) as a twisted bundle (again, up to equivalence) over the spectral cover.

Now, by theorem 3.21 we have a bijection

\[
\Psi : TVB(S)/E \sim L \otimes E \xrightarrow{\cong} Vect(S)/E \sim L \otimes E,
\]

and then every brane \( a \in B(M) \) can in fact be taken as a vector bundle over \( S \), up to tensoring with a line bundle.

**References**

[Abr96] L. Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *Journal of Knot theory and its ramifications*, 5(5):569–588, 1996.

[AG60] M. Auslander and O. Goldman. The Brauer group of a commutative ring. *Transactions of the A.M.S.*, 97(3):367–409, 1960.

[AS05] M. Atiyah and G. Segal. Twisted k-theory. *ArXiv Mathematics e-prints*, 2005.

[Aud98] M. Audin. Symplectic geometry in Frobenius manifolds and quantum cohomology. *Journal of Geometry and Physics*, 25:183–204, 1998.

[BC04] J.C. Baez and A.S. Crans. Generalized 2-vector spaces and general linear 2-groups. *Theory and Applications of Categories*, 12(15):492–528, 2004.

[BDR04a] N. Baas, B. Dundas, and J. Rognes. 2-vector bundles and forms of elliptic cohomology. In Tillmann *[Til04]*, pages 18–46.

[BDR04b] N. Baas, B. Dundas, and J. Rognes. 2-vector bundles and forms of elliptic cohomology. In Tillmann *[Til04]*, pages 18–46.
[Bry98] J. L. Brylinski. Categories of vector bundles and yang-mills equations. In Getzler E. and M. Kapranov, editors, *Higher Category Theory: Workshop on Higher Category Theory and Physics, March 28-30 1997, Northwestern University, Evanston, IL*, volume 230 of *Contemporary Mathematics*, pages 69–112. AMS, 1998.

[Cos07] K. Costello. Topological conformal field theories and calabi-yau categories. *Advances in Mathematics*, 210(1):165–214, 2007.

[Dub95] B. Dubrovin. *Geometry of 2D Topological Field Theories*, volume 1620 of *Lecture Notes in Mathematics*. Springer-Verlag, 1995.

[ea09] Paul Aspinwall . . . [et al.]. *Dirichlet branes and mirror symmetry*, volume 4 of *Clay Mathematics Monographs*. A.M.S., 2009.

[Elg06] J. Elgueta. Generalized 2-vector spaces and general linear 2-groups. *ArXiv Mathematics e-prints*, 2006.

[EN55] S. Eilenberg and T. Nakayama. On the dimension of modules and algebras. II. Frobenius algebras and quasi-Frobenius rings. *Nagoya Math. J.*, 9:1–16, 1955.

[Fis76] G. Fischer. *Complex Analytic Geometry*, volume 538 of *Lecture Notes in Mathematics*. Springer-Verlag, 1976.

[Fre94] D. Freed. Higher algebraic structures and quantization. *Communications in Mathematical Physics*, 159(2):343–398, 1994.

[Gro68] A. Grothendieck. Le groupe de Brauer I, algèbres d’Azumaya et interprétations diverses. In *Dix Exposes sur la cohomologie des schemas*, pages 46–65, 1968.

[Hit97] N. Hitchin. Frobenius manifolds. In J Hurtubise, F. Lalonde, and G. Sabidussi, editors, *Gauge Theory and Symplectic Geometry, Proceedings of the NATO Advanced Study Institute and Seminaire de Mathematiques Superieures, Montreal, Canada, July 3-14, 1995*, pages 69–112. Kluwer, 1997.

[HM99] C Hertling and Y. Manin. Weak Frobenius manifolds. *International Mathematics Research Notices*, 6:277–286, 1999.

[Hou61] C. Houzel. Géométrie analytique locale II: Théorie des morphismes finis. In *Séminaire Henri Cartan 13e année No.18-21; 12, 22, 25, 15 p.* Secrétariat.mathematique, 1961.

[Kar10] M. Karoubi. Twisted bundles and twisted k-theory. *K-Theory Preprint Archives*, 2010.

[Kel74] G. Kelly. *Coherence Theorems for Lax Algebras and Distributive Laws*, volume 420 of *Lecture Notes in Mathematics*. Springer-Verlag, 1974.

[Kel82] M.G. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.

[KV94] M. M. Kapranov and V. A. Voevodski. 2-categories and Zamolodchikov tetrahedra equations. *Proceedings of Symposia in Pure Mathematics, A. M. S.*, 56:177–259, 1994.

[Lap72] M. Laplaza. Coherence for Distributivity, volume 281 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.

[Mac71] S. MacLane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, 1971.

[Man99] Y. Manin. *Frobenius manifolds, quantum cohomology and moduli spaces*, volume 47 of *Colloquium Publications*. A.M.S., 1999.
[Mil80] J. Milne. Étale cohomology. Princeton University Press, 1980.

[MS06] G. W. Moore and G. Segal. D-branes and K-theory in 2D topological field theory. ArXiv High Energy Physics - Theory e-prints - 0609042, September 2006.

[Seg07] G. Segal. What is an elliptic object? In H. Miller and D. Ravenel, editors, Elliptic Cohomology: Geometry, Applications, and Higher Chromatic Analogues, volume 342 of London Mathematical Society, Lecture Notes Series, pages 306–317. Cambridge University Press, 2007.

[Til04] Ulrike Tillmann, editor. 2002 Oxford Symposium in the Honour of the 60th Birthday of Graeme Segal, number 308 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

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