CHERN CLASSES OF CRYSTALS

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Abstract. The crystalline Chern classes of the value of a locally free crystal vanish on a smooth variety defined over a perfect field. Out of this we conclude new cases of de Jong’s conjecture relating the geometric étale fundamental group of a smooth projective variety defined over a perfect field and the triviality of its category of isocrystals. We also discuss the case of the Gauss-Manin convergent isocrystal.

1. Introduction

On a smooth algebraic variety $X$ defined over the field $\mathbb{C}$ of complex numbers, a vector bundle $E$ endowed with an integrable connection $\nabla : E \to \Omega^1_X \otimes O_X$ $E$ has vanishing Chern classes $c_i^{dR}(E)$ in de Rham cohomology $H^{2i}_{dR}(X/\mathbb{C})$.

The standard way to see this is applying Chern-Weil theory: the successive traces of the iterates of the curvature $\nabla^2 \in \text{Hom}(E, \Omega^2_X \otimes E)$ are identified with the Newton classes $N_i(E)$, and the $\mathbb{Q}$-vector spaces spanned by $c_i^{dR}(E), 1 \leq i \leq n$ and the $N_i(E), 1 \leq i \leq n$ are the same in $H^{2i}_{dR}(X)$ [Che52]. In particular, the method loses torsion information, and for example, a torsion class in integral $\ell$-adic cohomology $H^2(X_{\overline{k}}, \mathbb{Z}_\ell(1))$ for some $\ell$ is the first $\ell$-adic Chern class of some line bundle which carries an integrable connection.

If $(X, E)$ is defined over a field $k$ of characteristic 0 and $E$ admits an integrable connection after base changing to $\mathbb{C}$ for a complex embedding $k \hookrightarrow \mathbb{C}$, one still has vanishing $0 = c_i^{dR}(E) \in H^{2i}_{dR}(X/k)$ for $i \geq 1$ because Chern classes in de Rham cohomology are functorial and de Rham cohomology satisfies base change property as ultimately coherent cohomology does.

The first purpose of this article is to present a similar vanishing statement where $k$ is now a perfect field of characteristic $p > 0$, $E$ is replaced by the value $E_X$ on $X$ of a $p$-torsion free crystal $E$, and de Rham cohomology is replaced by crystalline cohomology. Let $W = W(k)$ be the ring of Witt vectors on $k$. We denote by $H^i_{\text{crys}}(X/W)$ the integral crystalline cohomology of $X$ and by $c_i^{\text{crys}}(E_X)$ the crystalline Chern classes of $E_X$ in $H^{2i}_{\text{crys}}(X/W)$.

Recall (see [ES15] Section 1 for an overview of the concepts) that a crystal is a sheaf of $O_{X/W}$-modules of finite presentation on the crystalline site of $X/W$, such that the transition maps are isomorphisms, that crystals build a $W$-linear category $\text{Crys}(X/W)$, of which the $\mathbb{Q}$-linearization $\text{Crys}(X/W) \otimes \mathbb{Q} \to \text{Crys}(X/W)_{\mathbb{Q}}$ is the category of isocrystals. Then $\text{Crys}(X/W)_{\mathbb{Q}}$ is a Tannakian category over $K$, the fraction field of $W$. Any $E \in \text{Crys}(X/W)_{\mathbb{Q}}$ is of the shape $\mathbb{Q} \otimes E$ where $E$ is a lattice, that is a $p$-torsion free crystal. Yet it is an open question whether one can choose $E$ to be locally free.

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Theorem 1.1. Let $X$ be a smooth variety defined over a perfect field $k$ of characteristic $p > 0$. If $E \in \text{Crys}(X/W)$ is a locally free crystal, then $c_i^{\text{crys}}(E_X) = 0$ for $i \geq 1$.

It is proved in [ES15] Prop. 3.1 that given an isocrystal $\mathbb{Q} \otimes E \in \text{Crys}(X/W)_{\mathbb{Q}}$, $c_i^{\text{crys}}(E_X)$ does not depend on the choice of the lattice $E$. In particular, if this was true that any isocrystal carries a locally free lattice, then Theorem 1.1 would imply that $c_i^{\text{crys}}(E_X) = 0$ for $i \geq 1$ for any $p$-torsion free crystal $E \in \text{Crys}(X/W)$.

The proofs of Theorem 1.1 imply also the following variant for Chern classes in torsion crystalline cohomology: Let $W_n := W/p^nW$. Then, if $X$ is as in Theorem 1.1 and if $E$ is a locally free crystal on $X/W_n$, then $c_i^{\text{crys}}(E_X)$ is zero in the torsion crystalline cohomology group $H^{2i}_{\text{crys}}(X/W_n)$ for $i \geq 1$. See Remarks 2.1 and 3.4.

Recall that the Frobenius acts on $\text{Crys}(X/W)$ and $\text{Crys}(X/W)_{\mathbb{Q}}$. Locally $\text{Crys}(X/W)$ is equivalent to the category of quasi-nilpotent integrable connections on a formal lift of $X$ over $W$, and the action is just given by the Frobenius pull-back of a connection. The category $\text{Conv}(X/K)$ of convergent isocrystals is the largest full subcategory of $\text{Crys}(X/W)_{\mathbb{Q}}$ which is stabilized by the Frobenius action. It is proved in [ES15] Prop. 3.1 that given an isocrystal $E$ such that $\mathbb{Q} \otimes E$ is trivial and the maximal Mumford slope of $E$ is as in Theorem 1.1 and if $E$ is isomorphic to $\mathbb{Z} \otimes_{\mathbb{Z}} E$ for any lattice $E$ of a convergent isocrystal, regardless of the existence of a locally free lattice.

For a natural number $r$, set $N(r)$ to be the maximum of the lower common multiples of $a$ and $b$ for all choices $a, b \geq 1, a + b \leq r$. As in [ES15], Theorem 1.1 enables one to prove the following case of de Jong’s conjecture ([ES15, Conj. 2.1]).

Theorem 1.2. Let $X$ be a smooth projective variety over a perfect field $k$ of characteristic $p > 0$. If the étale fundamental group of $X \otimes_k \bar{k}$ is trivial and the maximal Mumford slope of the sheaf of 1-forms is bounded above by $N(r)^{-1}$, then any isocrystal $E$ which is an iterated extension of irreducible isocrystals of rank $\leq r$ having locally free lattices is isomorphic to $\mathcal{O}_{X/K}^{\oplus \text{rank}(E)}$.

Given Theorem 1.1, the proof is nearly the same as the one of the main Theorem [ES15, Thm.1.1], with some differences which we explain in Section 4.

The second main theorem is de Jong’s conjecture for convergent isocrystals coming from geometry, which generalizes [ES15, Thm. 2.14]. Recall that, for a smooth proper morphism $f : Y \to X$ of varieties over a perfect field $k$ of characteristic $p > 0$, the Gauß-Manin convergent isocrystal $R^if_*\mathcal{O}_{Y/K}$ is defined by Ogus [Ogu84].

Theorem 1.3. Let $f : Y \to X$ be a smooth proper morphism between smooth proper varieties over a perfect field $k$ of characteristic $p > 0$. If the étale fundamental group of $X \otimes_k \bar{k}$ is trivial, then the Gauß-Manin convergent isocrystal $R^if_*\mathcal{O}_{Y/K}$ is isomorphic to $\mathcal{O}_{Y/K}^{\oplus r}$, where $r$ is its rank.

Remark 1.4. For a smooth projective morphism $f : Y \to X$ between varieties defined over a perfect field $k$ of characteristic $p > 0$, Lazda [Laz15 Cor. 5.4] proved recently that the Gauß-Manin convergent isocrystal $R^if_*\mathcal{O}_{Y/K}^\dagger$ canonically lifts to an overconvergent isocrystal on $X$, which we denote by $R^if_*\mathcal{O}_{Y/K}^\dagger$ and call the Gauß-Manin overconvergent isocrystal. When $X$ is smooth, geometrically simply connected, $k$ is a finite field and $p \geq 3$, we prove that $R^if_*\mathcal{O}_{Y/K}^\dagger$ is trivial as overconvergent isocrystal on $X$. See Section 5.

We now explain the methods used in order to prove Theorem 1.1.
There are two ways to prove the vanishing of $c_i^{dR}(E) \in H_{dR}^{2i}(X/k) (i > 0)$ for a locally free sheaf $E$ equipped with an integrable connection which does not use Chern-Weil theory.

One method uses a modified splitting principle as developed in [Esn88] and [Esn00]. On the projective bundle $\mathbb{P}(E) \to X$, the integrable connection induces a differential graded algebra $(\Omega^\bullet)_{\mathbb{P}(E)}$, which is a quotient $H^i(\mathbb{P}(E), \Omega^\bullet)$ of the de Rham complex, and which cohomologically splits $H_{dR}^{2i}(X/k)$ in $H^i(\mathbb{P}(E), \Omega^\bullet)$. Then one shows that the $\Omega^1$-connection on $\pi^*E$ induced by $\nabla$ stabilizes $O_{\mathbb{P}(E)}(1)$. Hence, when the rank of $E$ is two, $\pi^*E$ is an extension of $\Omega^1$-connections of rank 1 and so one can prove the vanishing of the classes in $H^{2i}(\mathbb{P}(E), \Omega^\bullet)$, thus in $H_{dR}^{2i}(X/k)$. In the case of rank $r$, one repeats the above argument $(r - 1)$-times to obtain a filtration by $\Omega^1$-connections of rank 1.

We adapt this construction to the crystalline case as follows. (In the introduction, we assume the existence of a closed embedding $\tau : \mathbb{P}(E) \to P$ of $X$ into a smooth $p$-adic formal scheme $P$ over $W$ to ease the explanation.) One considers the projective bundle $\pi : \mathbb{P}(E_W) \to D$ of the value $E_W$ of a locally free crystal $E$ on the PD-hull $X \to D \to \mathbb{P}_W$ of the embedding $X \to P$. One shows that the connection on $E_W$ induces a quotient differential graded algebra $\Omega^\bullet_{\mathbb{P}(E_W)} \to \Omega^\bullet$ of the PD-de Rham complex $\Omega^\bullet_{\mathbb{P}(E_W)}$. Then one shows that the $\Omega^1$-connection on $\pi^*(E_W)$ respects $O_{\mathbb{P}(E_W)}(1)$. Thus we can argue as in the de Rham case and obtain the required vanishing. See Section 2.

Another way on the de Rham side is to say that local trivializations of $E$ yield a simplicial scheme $X_\bullet$ augmenting $X$ together with a morphism $e : X_\bullet \to BGL(r)$, defined by the transition functions, to the simplicial classifying scheme $BGL(r)$, where $r$ is the rank of $E$. This induces the maps $H^{2i}(BGL(r), \Omega^\bullet) \to H_{dR}^{2i}(X_\bullet/k) \cong H_{dR}^{2i}(X/k)$. If $k = \mathbb{C}$ and $E$ carries an integrable connection, we have a similar map $H^{2i}(BGL(r), \Omega^\bullet) \to H_{dR}^{2i}(X_\bullet/k) \cong H_{dR}^{2i}(X/k)$ which is identified with the previous one and factors through the cohomology $H^{2i}(BGL(r), \Omega^\bullet) = H^{2i}(BGL(r), \Omega^\bullet)$ of the discrete classifying simplicial space $BGL(r)$. Thus $c_i^{dR}(E)$ is in the image of the composite map $H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(BGL(r), \Omega^\bullet)$, which is zero for $i \geq 1$.

We adapt this construction to the crystalline case as follows. Given a closed embedding $X \subset P$ into a smooth $p$-adic formal scheme over $W$, one defines $D_\bullet$ to be the simplicial scheme such that $D_0$ is the PD-hull of the diagonal in $P^{\times n+1}$. Then the crystalline Poincaré lemma and the Čech-Alexander resolution equate $H^i_{\text{cris}}(X/W)$ both with $H^i(D_\bullet, \Omega^\bullet)$ and with $H^i(D_\bullet, \Omega^\bullet)$ (see Proposition 3.3). Thus, defining a certain simplicial version $D_\bullet$ of $D_\bullet$, to which $X_\bullet$ maps, $E$ has the free value $E_{D_\bullet}$ and this induces the map $H_{\text{cris}}^{2i}(BGL(r), \Omega^\bullet/W) = H^{2i}(BGL(r), \Omega^\bullet) \to H^{2i}(D_\bullet, \Omega^\bullet) \cong H^{2i}(D_\bullet, \Omega^\bullet)$, which is zero for $i \geq 1$. See Section 3.

We now explain the methods used in order to prove Theorem 1.3 assuming $k$ is a finite field.

Let us assume first that $f : Y \to X$ is an abelian scheme and present then an $\ell$-adic argument due to G. Faltings. The arithmetic fundamental group of $X$ acts on $R^if_*\mathbb{Q}_\ell$ via $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, thus by Tate’s theorem [Tat66], all fibers of $f$ over $\mathbb{F}_q$-points of $X$ are isogeneous, thus the Gauß-Manin convergent isocrystal $R^if_*\mathcal{O}_Y/k$ is trivial.

In general, one has to replace Tate’s motivic theorem by a result of Chiarellotto-Le Stum [CLS98] (generalizing Katz-Messing theorem [KM73]) and Abe’s Čebotarev density theorem.
which yields the triviality of the semi-simplification of $R^if_*\mathcal{O}_{Y/K}^\dual$ (see Remark \[1.34\] for notation) in the category $\text{Conv}^\dual(X/K)$ of overconvergent isocrystals on $X$. Finally one has to go from the semi-simplification to the original Gauß-Manin isocrystal by showing that there are no extensions of the trivial overconvergent isocrystal by itself on $X$ when $X$ is proper or $p \geq 3$ (Theorem \[5.1\]).

Over a non-finite field (when $X$ is proper), one reduces the proof to the case of a finite ground field by a specialization argument. See Section \[5\].

Finally, in Section \[6\] we prove a very weak form of a Lefschetz theorem for isocrystals. 

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2. Crystalline modified splitting principle

The aim of this section is to prove Theorem \[1.3\] using a crystalline modified splitting principle.

Let $X$ be a smooth variety over a perfect field $k$ of characteristic $p > 0$ and let $X_\bullet \to X$ be a simplicial scheme augmented to $X$ defined as the \v{C}ech hypercovering associated to an open covering $X = \bigcup_{i \in I} X_i$ which admits a closed embedding $X_\bullet \to D_\bullet$ into a simplicial $p$-adic formal scheme $D_\bullet$ over $W$ such that, for any $n \in \mathbb{N}$ and Zariski locally on $X_\bullet$, $X_\bullet \to D_\bullet$ is the PD-envelope of a closed immersion $X_\bullet \to Y$ of $X_\bullet$ into a smooth $p$-adic formal scheme $Y$ over $W$. (Note that, for any smooth variety $X$ over $k$, there exists such a system.)

2.1. Generalities on the first crystalline Chern class, and variants of it. One denotes by $(X/W)_{\text{crys}}, (X_\bullet/W)_{\text{crys}}$ the crystalline topos of $X/W$, $X_\bullet/W$ respectively (see \[BO78, p.5.3\]), by $(X_\bullet/W)_{\text{crys}}|_{D_\bullet}$ its restriction to $D_\bullet$ (\[BO78, p.5.22\]), by $(-)_{\text{Zar}}$ the Zariski topos. In particular the canonical morphism of topoi $X_{\bullet} \to D_{\bullet}$ is an equivalence. On $D_{\bullet}$ one defines the PD-de Rham complex $\bar{\Omega}_{D_{\bullet}}$, which is a quotient differential graded algebra of the de Rham complex $\Omega^\bullet_{D_{\bullet}}$, with the submodule $\mathcal{K} \subset \Omega^1_{D_{\bullet}}$, topologically spanned by $da^m - a^{m-1}da (a \in I = \text{Ker}(\mathcal{O}_{D_{\bullet}} \to i_*\mathcal{O}_{X_\bullet(n)}), n, m \in \mathbb{N}$) spans all relations, that is

$$\bar{\Omega}^\bullet_{D_{\bullet}} = \Omega^\bullet_{D_{\bullet}}/\mathcal{K} \wedge \Omega^{\bullet-1}_{D_{\bullet}}$$

and in addition, each $\bar{\Omega}^i_{D_{\bullet}}$ is locally free over $\mathcal{O}_{D_{\bullet}}$ with the relation

$$\bar{\Omega}^i_{D_{\bullet}} = \wedge^i_{\mathcal{O}_{D_{\bullet}}} \bar{\Omega}^1_{D_{\bullet}}$$

(\[Ill79, Prop. 3.1.6\]).
One has the following commutative diagram of topoi [BO78, p.6.12]

\[
\begin{array}{ccc}
(X_\bullet /W)_\mathrm{crys} & \xrightarrow{\varphi} & D_\bullet \mathrm{Zar} = X_\bullet \mathrm{Zar} \\
\downarrow j & & \downarrow u \\
(X_\bullet /W)_\mathrm{crys} & & \\
\end{array}
\]

The complex

\[
L(\Omega^\bullet_{D_\bullet}) := j_* \varphi^*(\Omega^\bullet_{D_\bullet})
\]

in \((X_\bullet /W)_\mathrm{crys}\) is defined in [BO78, p.6.13], and it is proved in [BO78, Thm.6.12] that the natural map

\[
\mathcal{O}_{X_\bullet/W} \to L(\Omega^\bullet_{D_\bullet})
\]

is a quasi-isomorphism in \((X_\bullet /W)_\mathrm{crys}\). As a consequence, restricting (2.5) to the crystalline units, one obtains a quasi-isomorphism in \(X_\bullet \mathrm{Zar}\)

\[
Ru_* \mathcal{O}^\times_{X_\bullet/W} \cong (\mathcal{O}^\times_{D_\bullet})^d \xrightarrow{\log} \Omega^1_{D_\bullet} \to \Omega^2_{D_\bullet} \to \ldots).
\]

The exact sequence

\[
0 \to \mathcal{I}_{X_\bullet/W} \to \mathcal{O}_{X_\bullet/W} \to \iota_* \mathcal{O}_{X_\bullet} \to 0
\]

in \((X_\bullet /W)_\mathrm{crys}\), defining \(\mathcal{I}_{X_\bullet/W}\), yields an exact sequence

\[
1 \to (1 + \mathcal{I}_{X_\bullet/W}) \to \mathcal{O}^\times_{X_\bullet/W} \to \iota_* \mathcal{O}^\times_{X_\bullet} \to 1
\]

in \((X_\bullet /W)_\mathrm{crys}\). By [BI70, 2.1] and the functoriality of the construction there, the connecting homomorphism

\[
H^1(X, \mathcal{O}^\times_X) \to H^1(X_\bullet, \mathcal{O}^\times_{X_\bullet}) \to H^2((X_\bullet /W)_\mathrm{crys}, 1 + \mathcal{I}_{X_\bullet/W}),
\]

followed by the logarithm

\[
H^2((X_\bullet /W)_\mathrm{crys}, 1 + \mathcal{I}_{X_\bullet/W}) \to H^2((X_\bullet /W)_\mathrm{crys}, \mathcal{I}_{X_\bullet/W})
\]

and the natural map

\[
H^2((X_\bullet /W)_\mathrm{crys}, \mathcal{I}_{X_\bullet/W}) \to H^2_{\mathrm{crys}}(X_\bullet /W) \cong H^2_{\mathrm{crys}}(X/W)
\]

precisely computes \(c^\mathrm{crys}_1\). Thus applying \(Ru_*\) to (2.8) and using (2.6), we conclude that the connecting homomorphism \(H^1(X, \mathcal{O}^\times_X) \to H^1(X_\bullet, \mathcal{O}^\times_{X_\bullet}) \to H^2(D_\bullet, (1 + I_\bullet) \xrightarrow{d\log} \Omega^\geq_1_{D_\bullet}(\bullet))\)

of the exact sequence in \(D_\bullet \mathrm{Zar}\)

\[
1 \to ((1 + I_\bullet) \xrightarrow{d\log} \Omega^\geq_1_{D_\bullet}(\bullet)) \to (\mathcal{O}^\times_{D_\bullet})^d \xrightarrow{\log} \Omega^\geq_1_{D_\bullet}(\bullet) \to \mathcal{O}^\times_X \to 1,
\]

followed by the logarithm \(H^2(D_\bullet, (1 + I_\bullet) \xrightarrow{d\log} \Omega^\geq_1_{D_\bullet}(\bullet)) \to H^2(D_\bullet, I \xrightarrow{d} \Omega^\geq_1_{D_\bullet}(\bullet))\) and the natural map \(H^2(D_\bullet, I \xrightarrow{d} \Omega^\geq_1_{D_\bullet}(\bullet)) \to H^2(D_\bullet, \Omega^\bullet_{D_\bullet}(\bullet)) = H^2_{\mathrm{crys}}(X_\bullet /W) = H^2_{\mathrm{crys}}(X/W)\) precisely computes \(c^\mathrm{crys}_1\). In particular, if \(E_X\) is a locally free sheaf of rank 1 on \(X\), then by (2.6), \(c^\mathrm{crys}_1(E_X) = 0\) if \(E_X\) is the value on \(X\) of a locally free crystal of rank 1 on \(X\).
More generally, let \( \tau : \tilde{\Omega}^n_{\bullet} \rightarrow A^1_{\bullet} \) be a surjection of sheaves of differential graded algebras on \( D_{\bullet}^{Zar} \), such that, for any \( n \in \mathbb{N} \), \( A^1_{(n)} \) is locally free over \( O_{D_{(n)}} \). \( A^0_{(n)} = O_{D_{(n)}} \), \( A^n_{(n)} = \wedge^n_{O_{D_{(n)}}} A^1_{(n)} \). A \( \tau \)-connection (see Esn88 (2.1)) on a locally free sheaf \( E_{D_{\bullet}} \) on \( D_{\bullet} \) is an additive map \( \nabla_\tau : E_{D_{\bullet}} \rightarrow A^1_{\bullet} \otimes_{O_{D_{\bullet}}} E_{D_{\bullet}} \), which fulfills the \( \tau \)-Leibniz rule \( \nabla_\tau(e) = \tau d(e) \otimes e + \lambda \otimes \nabla_\tau(e) \). Then \( \nabla_\tau \circ \nabla_\tau : E_{D_{\bullet}} \rightarrow A^2_{\bullet} \otimes E_{D_{\bullet}} \), where \( \nabla_\tau(e \otimes e) = (-1)^i \tau d(e) + \omega \otimes \nabla_\tau(e) \) for \( \omega \in A^1_{(n)} \), is \( O_{D_{\bullet}} \)-linear. The \( \tau \)-connection is integrable if \( \nabla_\tau \circ \nabla_\tau = 0 \). One pushes down (2.9) along \( \tau \) and obtains the exact sequence in \( D_{\bullet}^{Zar} \)

\[
1 \rightarrow ((1 + I_{\bullet}(x)) \xrightarrow{d \log} A^1_{\bullet}) \rightarrow (O_{\tilde{D}_{\bullet}} \xrightarrow{d \log} A^1_{\bullet}) \rightarrow O_X^{\bullet} \rightarrow 1.
\]

Then, for a locally free sheaf \( E_X \) of rank 1 on \( X \), \( 0 = \tau(c_{1,\text{crys}}(E_X)) \in H^2(D_{\bullet}, A^1_{\bullet}) \) if \( E_X|_{X_{\bullet}} \) is the restriction to \( X_{\bullet} \) of a line bundle \( E_{D_{\bullet}} \) on \( D_{\bullet} \) which is endowed with an integrable \( \tau \)-connection.

2.2. Modified splitting principle. Let \( \tau : \tilde{\Omega}^n_{\bullet} \rightarrow A^1_{\bullet} \) be a surjection of sheaves of differential graded algebras on \( D_{\bullet}^{Zar} \) such that, for any \( n \in \mathbb{N} \), \( A^1_{(n)} \) is locally free over \( O_{D_{(n)}} \), \( A^0_{(n)} = O_{D_{(n)}} \) and \( A^n_{(n)} = \wedge^n_{O_{D_{(n)}}} A^1_{(n)} \). Also, let \( E_X \) be a locally free sheaf on \( X \) such that its restriction \( E_X|_{X_{\bullet}} \) to \( X_{\bullet} \) extends to a locally free sheaf \( E_{D_{\bullet}} \) on \( D_{\bullet} \) endowed with an integrable \( \tau \)-connection. One defines \( X' = \mathbb{P}(E_X), X'_{\bullet} = \mathbb{P}(E_X|_{X_{\bullet}}), D'_{\bullet} = \mathbb{P}(E_{D_{\bullet}}) \), together with the augmentation \( X'_{\bullet} \rightarrow X' \) and the closed embedding \( i' : X'_{\bullet} \rightarrow D'_{\bullet} \). One has a cartesian squares of (simplicial formal) schemes (over \( W \))

\[
\begin{array}{ccc}
X' & \leftarrow & X'_{\bullet} \\
\pi \downarrow & & \downarrow \pi \\
X & \leftarrow & X_{\bullet} \rightarrow D_{\bullet}.
\end{array}
\]

The PD-structure on \( I_{\bullet} \) extends uniquely to a PD-structure on \( I'_{\bullet} = \text{Ker}(O_{D'_{\bullet}} \rightarrow O_{X'_{\bullet}}) \) as \( \pi \) is flat (Botts Prop. 3.21), and \( i' \) is again the PD-envelope of a closed immersion from \( X'_{(n)} \) to a smooth \( p \)-adic formal scheme over \( W \) Zariski locally on \( X_{(n)} \) for each \( n \). Thus one can define \( \tilde{\Omega}^n_{D'_{\bullet}} \) and one has an exact sequence

\[
0 \rightarrow \pi^* \tilde{\Omega}^1_{D'_{\bullet}} \rightarrow \tilde{\Omega}^1_{D'_{\bullet}} \rightarrow \tilde{\Omega}^1_{D'_{\bullet}}/D_{\bullet} \rightarrow 0
\]

on \( X'_{\bullet}^{\text{Zar}} \). Setting \( K_{\tau} = \text{Ker}(\tilde{\Omega}^1_{D_{\bullet}} \rightarrow A^1_{\bullet}) \), one defines \( \tilde{\Omega}^1_{D'_{\bullet},\tau} = \tilde{\Omega}^1_{D_{\bullet},\tau}/\pi^* K_{\tau} \). By definition, (2.12) pushes down to an exact sequence

\[
0 \rightarrow \pi^* A^1_{\bullet} \rightarrow \tilde{\Omega}^1_{D'_{\bullet}} \rightarrow \tilde{\Omega}^1_{D'_{\bullet}}/D_{\bullet} \rightarrow 0
\]

on \( X'_{\bullet}^{\text{Zar}} \). One defines \( \tilde{\Omega}^n_{D'_{\bullet},\tau} = \wedge^n_{O_{D'_{\bullet}}} \tilde{\Omega}^1_{D'_{\bullet},\tau} \). By Esn88 Claim p. 332, the quotient homomorphism \( r : \tilde{\Omega}^1_{D'_{\bullet}} \rightarrow \tilde{\Omega}^1_{D'_{\bullet},\tau} \) extends to a quotient

\[
\tilde{\Omega}^n_{D'_{\bullet}} \rightarrow \tilde{\Omega}^n_{D'_{\bullet},\tau}
\]
of differential graded algebras on $X'_{(\bullet)Zar}$, where the differential on $\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}$ is denoted by $rd$.

The connection $\nabla_{\tau}$ on $E_{D_{(\bullet)}}$ induces a pull-back connection $\pi^*\nabla_{\tau} : \pi^*E_{D_{(\bullet)}} \to \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/(\pi^*E_{D_{(\bullet)}})$, its restriction to $\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1)$ via the exact sequence
\begin{equation}
0 \to \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1) \to \pi^*E_{D_{(\bullet)}} \to \mathcal{O}_{D_{(\bullet)}}(1) \to 0,
\end{equation}
followed by the projection $\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1) \to \mathcal{O}_{D_{(\bullet)}}(1) \to 0$, defines a section
\begin{equation}
\sigma : \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1) \to \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1)
\end{equation}
of $-p$ ($\Omega_{D_{(\bullet)}}^{\bullet}$). Thus $\tau' = 1 + p \circ \sigma : \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/(\pi^*A_{(\bullet)}^1)$ is a section of $i$. By $\Omega_{D_{(\bullet)}}^{\bullet}$, $\tau'$ induces a surjective homomorphism
\begin{equation}
\tau' : \tilde{\Omega}_{D_{(\bullet)}}^{\bullet} \to \pi^*A_{(\bullet)}^1
\end{equation}
of differential graded algebras, where the differential on the right is induced by $\tau' \circ rd$.

The main point is then that the $\pi^*A_{(\bullet)}^1$-valued connection $\nabla' : \pi^*E_{D_{(\bullet)}} \to \pi^*A_{(\bullet)}^1 \otimes \mathcal{O}_{D_{(\bullet)}}$ is integrable and respects the flag (2.15), thus induces integrable connections
\begin{equation}
\nabla' : \mathcal{O}_{D_{(\bullet)}}(1) \to \pi^*A_{(\bullet)}^1 \otimes \mathcal{O}_{D_{(\bullet)}}(1), \quad \nabla' : \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1) \to \pi^*A_{(\bullet)}^1 \otimes \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1).
\end{equation}
Consequently we can iterate the construction, replacing $X$ by $X', X_{(\bullet)}, D_{(\bullet)}$ by $D'_{(\bullet)}$, $E_X$ by the descent of $\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}/D_{(\bullet)}(1)|X'_{(\bullet)}$ to $X', E_{D_{(\bullet)}}$ by $\tilde{\Omega}_{D'_{(\bullet)}}^{\bullet}/D'_{(\bullet)}(1)$, $\sigma$ by $\tau' \circ \tau$.

2.3. **Proof of Theorem 1.1** Further iterating, after $(r-1)$-steps, one obtains a diagram as (2.11), where now $X'$ is the complete flag bundle over $X$, with (2.17) becoming a surjective homomorphism
\begin{equation}
\tau : \tilde{\Omega}_{D'_{(\bullet)}}^{\bullet} \to \pi^*\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}
\end{equation}
of differential graded algebras, and with a filtration on $\pi^*E_X$ with graded being a sum of locally free sheaves $L_j$ of rank 1 such that the restriction $L_j|X'_{(\bullet)}$ of $L_j$ to $X'_{(\bullet)}$ extends to a locally free sheaf on $D'_{(\bullet)}$ endowed with an integrable $\pi^*\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}$-connection. In addition, from $\tilde{\Omega}_{D_{(\bullet)}}^{\bullet}$, Lemma in (1.3), (2.7), the composite
\begin{equation}
\tilde{\Omega}_{D_{(\bullet)}}^{\bullet} \to R\pi_*\tilde{\Omega}_{D'_{(\bullet)}}^{\bullet} \xrightarrow{R\pi_*\tau} R\pi_*\pi^*\tilde{\Omega}_{D_{(\bullet)}}^{\bullet} \cong \tilde{\Omega}_{D_{(\bullet)}}^{\bullet}
\end{equation}
is the identity on $X_{(\bullet)Zar}$. By the standard Whitney product formula for crystalline Chern classes [Gro85, III Thm. 1.1.1] for $i \geq 1$, $c_i^{crys}(\pi^*E_X)$ is a sum of products of $c_i^{crys}(L_j)$, which by Subsection 2.1 maps to 0 in $H^2(D_{(\bullet)}, \pi^*\tilde{\Omega}_{D_{(\bullet)}}^{\bullet})$. Thus $c_i^{crys}(E_X) \in H^{2i}(D_{(\bullet)}, \tilde{\Omega}_{D_{(\bullet)}}^{\bullet})$ maps to 0 in $H^{2i}(D_{(\bullet)}, \tilde{\Omega}_{D_{(\bullet)}}^{\bullet})$ via the composite map in (2.19), which is the identity. This shows Theorem 1.1.

**Remark 2.1.** Let $W_n := W/p^nW$. Then, by replacing $W$ by $W_n$ and $D_{(\bullet)}$ by its mod $p^n$ reduction, we see that the proof above gives the following variant of Theorem 1.1 If $X$ is as in Theorem 1.1 and if $E$ is a locally free crystal on $X/W_n$, then $c_i^{crys}(E_X)$ is zero in the torsion crystalline cohomology group $H^{2i}_{crys}(X/W_n)$ for $i \geq 1$. Because $H^{2i}_{crys}(X/W) \otimes W \to
$H^2_{\text{crys}}(X/W_n)$ is injective, it implies that the Chern classes $c^\text{crys}_i(E_X)$ in $H^2_{\text{crys}}(X/W) (i \geq 1)$ are divisible by $p^n$ in this case.

2.4. Remark on Chern-Simons theory. In [Esn92] and [Esn00], a version of the modified splitting principle which is slightly more elaborate than the one used in [2.2] was performed in order to construct classes $c_i(E, \nabla) \in H^i(X, \mathcal{K}_i^M \overset{d \log}{\longrightarrow} \Omega^1_X \overset{d}{\longrightarrow} \ldots)$ of a bundle with an integrable connection $(E, \nabla)$, depending on $\nabla$, where $\mathcal{K}_i^M$ is the Zariski sheaf of Milnor $K$-theory. Those classes lift both the Chow classes in $CH^i(X) = H^i(X, \mathcal{K}_i^M)$ via the obvious forgetful map, and the Chern-Simons classes in $H^{2i-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(i))$, if $k = \mathbb{C}$. We hope to be able to define a crystalline version of Chern-Simons theory, yielding classes lifting both Chow classes and classes in syntomic cohomology.

3. The crystalline version of the discrete classifying space $BGL(r)$

The aim of this section is to prove Theorem 1.1 using a crystalline version of the discrete classifying space $BGL(r)$.

Let $X$ be a smooth variety defined over a perfect field $k$ of characteristic $p > 0$.

3.1. Čech-Alexander resolutions of $\mathcal{O}_{X/W}$. Fix a closed embedding $X \hookrightarrow Y$ into a $p$-adic smooth formal scheme $Y$ over $W$, and one defines $D(n)$ as the PD-envelope of $X$ in the diagonal embedding $Y^{n+1}$, where $n+1$ means the product over $W$. Then one has the canonical morphism of topoi

$$j_n : (X/W)_{\text{crys}}|_{D(n)} \rightarrow (X/W)_{\text{crys}}$$

and the Čech-Alexander resolution [BO78 5.29] of the abelian sheaf $\mathcal{O}_{X/W}$

$$\mathcal{O}_{X/W} \overset{\cong}{\rightarrow} (j_0 \circ j_0^* \mathcal{O}_{X/W} \rightarrow j_1 \circ j_1^* \mathcal{O}_{X/W} \rightarrow \ldots).$$

We also use the following variant of (3.2). Fix two closed embeddings $X \hookrightarrow Y$ and $X \hookrightarrow Z$, where both $Y/W$ and $Z/W$ are formal schemes and $Y/W$ is smooth, one defines $D(n)$ as the PD-envelope of $X$ in the diagonal embedding $Y^{n+1} \times_W Z$. Then one has the canonical morphisms of topoi

$$j_n : (X/W)_{\text{crys}}|_{D(n)} \overset{\ell_n}{\rightarrow} (X/W)_{\text{crys}}|_{D(-1)} \overset{j^{-1}}{\longrightarrow} (X/W)_{\text{crys}}.$$ 

On $(X/W)_{\text{crys}}|_{D(-1)}$, one has the Čech-Alexander resolution [BO78 5.29] of the abelian sheaf $j_{-1}^* \mathcal{O}_{X/W}$

$$j_{-1}^* \mathcal{O}_{X/W} \overset{\cong}{\rightarrow} (\ell_0 \circ j_0^* \mathcal{O}_{X/W} \rightarrow \ell_{1} \circ j_1^* \mathcal{O}_{X/W} \rightarrow \ldots).$$

Hence, applying the exact functor $j_{-1*}$ ([BO78 Cor.5.27.1]), one obtains the resolution of the abelian sheaf $j_{-1*}j_{-1}^* \mathcal{O}_{X/W}$

$$j_{-1*}j_{-1}^* \mathcal{O}_{X/W} \overset{\cong}{\rightarrow} (\ell_0 \circ j_0^* \mathcal{O}_{X/W} \rightarrow j_{1*} j_1^* \mathcal{O}_{X/W} \rightarrow \ldots).$$
3.2. Various simplicial constructions to compute crystalline cohomology. Let $X = \bigcup_{i \in I} X_i$ be a finite covering of $X$ by affine open subvarieties. We assign to it the standard Mayer-Vietoris simplicial scheme, the definition of which we recall now.

We choose a total order on $I$, define the set of tuples $I^<_\leq := \{(i_0, \ldots, i_n) : i_0 \leq i_1 \leq \ldots \leq i_n\}$, and $I^<_\leq$ to be the disjoint union of the $I^<_\leq$. For $J = (i_0, \ldots, i_n) \in I^<_\leq$, one sets $X_J = \cap_{j \in J} X_j$. One upgrades $I^<_\leq$ to a category. The Hom-set $\text{Hom}_{I^<_\leq}(J, J')$, for $J = (i_0, \ldots, i_n)$ and $J' = (i'_0, \ldots, i'_n)$, consists of those non-decreasing maps $\varphi: [n] \to [n']$, where $[n] = \{0, 1, \ldots, n\}$, with the property that $i_a = i'_{\varphi(a)}$ for all $a \in \{0, \ldots, n\}$. Thus to $\varphi \in \text{Hom}_{I^<_\leq}(J, J')$, one assigns the open embedding $X_J \hookrightarrow X_{J'}$, which one denotes by $\varphi^*$.

If $\varphi \in \text{Hom}_{I^<_\leq}(J, J')$ with $J' = (i'_0, \ldots, i'_n)$, $J$ is necessarily equal to $(i'_\varphi(0), \ldots, i'_\varphi(n))$. Thus, given any non-decreasing map $\varphi: [n] \to [n']$ and $J' \in I^<_\leq$, there is one and only one $J \in I^<_\leq$ such that $\varphi \in \text{Hom}_{I^<_\leq}(J, J')$, in particular the open embedding $\varphi^* : X_J \to X_J$ is determined as well. Denoting by $X_{(n)} = \sqcup_{J \in I^<_\leq} X_J$ the disjoint union of the $X_J$ over all the $J \in I^<_\leq$, one defines the map

$$\varphi^* : X_{(n')} \to X_{(n)}$$

for a non-decreasing map $\varphi: [n] \to [n']$ as the disjoint union of the maps $X_{J'} \xrightarrow{\varphi^*} X_J \hookrightarrow X_{(n)}$ for $J' \in I^<_\leq$. Using the definition [Del74, Section 5], (3.6) defines the simplicial scheme $X_{(*)}$ which augments to $X$

$$X_{(*)} \xrightarrow{\mathcal{G}} X.$$

Remark 3.1. The simplicial scheme $X_{(*)}$ here differs from the simplicial scheme $X_{(*)}$ which appeared in Section 2.

The aim of this section is to prove the

Proposition 3.2. The augmentation map induces a quasi-isomorphism

$$e^* : R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) \to R\Gamma((X_{(*)}/W)_{\text{crys}}, \mathcal{O}_{X_{(*)}/W})$$

in the derived category $D(\text{Ab})$ of abelian groups.

Proof. The proof goes by induction on the cardinality $|I|$ of $I$. If $|I| = 1$, one sets $G = R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) \in D(\text{Ab})$. Then the right hand side of (3.8) reads

$$G \xrightarrow{\alpha_0} G \to \ldots \xrightarrow{\alpha_0} G \to \ldots$$

where $\alpha_n = \text{id}$ for $n$ even, and $\alpha_n = 0$ for $n$ odd.

For $|I| > 1$, we subdivide the simplicial construction $X_{(*)}$ as follows. Let $0$ be the minimal element of $I$. One sets $I' \sqcup I''$, with $I' = \{0\}$ and $I'' = I \setminus I'$. Then one has

$$I^<_\leq = I^<_\leq \sqcup (\sqcup_{n,m \in \mathbb{N}} I^<_\leq \times I^{\leq m}_\leq) \sqcup I^<_\leq.$$

Setting $X_{J', J''} = X_{J'} \cap X_{J''}$ for $J' \in I^<_\leq$ and $J'' \in I^{\leq m}_\leq$, one sets $X_{(n,m)} = \sqcup_{J' \in I^<_\leq, J'' \in I^{\leq m}_\leq} X_{J', J''}$. Then $X_{(*)}$ forms a bisimplicial scheme ([Fri82, p. 17]) and one has the commutative diagram

$$\begin{array}{ccc}
X'_{(*)} & \xrightarrow{e'} & X_{(*)} & \xrightarrow{e''} & X''_{(*)} \\
\downarrow & & \downarrow & & \downarrow \\
X' & & X_{(*)} & & X''
\end{array}$$
with \( X' = X_0 \) and \( X'' = \cup_{i \in I \setminus \{0\}} X_i \). By induction, Proposition \( \ref{3.2} \) applies to \( \epsilon' \) and \( \epsilon'' \). On the other hand, \( X_{(\bullet, m)} \) is the constant simplicial scheme on \( X' \cap X'' \). So by the case \( |I| = 1 \), one has

\[
R\Gamma((X' \cap X''_{(m)}/W)_{\text{crys}}, O_{X' \cap X''_{(m)}/W}) \cong R\Gamma((X_{(\bullet, m)}/W)_{\text{crys}}, O_{X_{(\bullet, m)}/W}).
\]

From this and the induction hypothesis one deduces the isomorphism

\[
\epsilon^* : R\Gamma((X' \cap X''_{(m)}/W)_{\text{crys}}, O_{X' \cap X''_{(m)}/W}) \cong R\Gamma((X' \cap X''_{(\bullet)}/W)_{\text{crys}}, O_{X' \cap X''_{(\bullet)}/W}) \cong R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, O_{X_{(\bullet)}/W}).
\]

One now extends \( \ref{3.8} \) to a diagram

\[
\begin{array}{ccc}
R\Gamma((X/W)_{\text{crys}}, O_{X/W}) & \xrightarrow{\epsilon^*} & R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, O_{X''_{(\bullet)}/W}) \\
\downarrow & & \downarrow \\
R\Gamma((X'/W)_{\text{crys}}, O_{X'/W}) \oplus R\Gamma((X''/W)_{\text{crys}}, O_{X''/W}) & \xrightarrow{\epsilon^* \oplus \epsilon''^*} & R\Gamma((X'_{(\bullet)}/W)_{\text{crys}}, O_{X''_{(\bullet)}/W}) \\
\downarrow & & \downarrow \\
R\Gamma((X' \cap X''/W)_{\text{crys}}, O_{X' \cap X''/W}) & \xrightarrow{\epsilon_1^*} & R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, O_{X''_{(\bullet)}/W})
\end{array}
\]

where the left vertical triangle is induced by the quasi-isomorphism in \( \text{[Ber]} \text{IV (3.5.4)} \) and the right one exists by construction. As \( \epsilon'^*, \epsilon''^*, \epsilon^*_1 \) are isomorphisms, so is \( \epsilon^* \).

### 3.3. Lifting the simplicial construction to PD-envelopes.

Keeping the same notations, we choose for each affine \( X_i \) a closed embedding \( X_i \hookrightarrow Y_i \) into a smooth \( p \)-adic formal scheme \( Y_i \) over \( W \). One defines the PD-envelope \( \alpha_J : X_J \hookrightarrow D_J \) of \( Y_{i_0} \times_W \cdots \times_W Y_{i_n} \) for \( J = (i_0, \ldots, i_n) \in I^n_\leq \). As in \( \ref{3.7} \) and \( \ref{3.11} \) one has the simplicial formal scheme \( D_{(\bullet)} \) and the diagram of simplicial formal schemes

\[
D'_{(\bullet)} \longleftarrow D_{(\bullet, \bullet)} \longrightarrow D''_{(\bullet)}
\]

this time without augmentation. On the other hand, for each \( J \in I_\leq \), one has as in \( \ref{2.3} \) the diagram of topoi

\[
\begin{array}{ccc}
(X_J/W)_{\text{crys}} |_{D_J} & \xrightarrow{\phi_J} & D_J_{\text{Zar}} = X_J_{\text{Zar}} \\
\downarrow j_J & & \downarrow u_J \\
(X_J/W)_{\text{crys}} & &
\end{array}
\]

to which one applies \( \ref{2.4} \) and the quasi-isomorphism \( \ref{2.5} \), which in addition is functorial. Thus, combined with \( \ref{3.8} \), it yields quasi-isomorphisms

\[
R\Gamma((X/W)_{\text{crys}}, O_{X/W}) \cong R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, O_{X_{(\bullet)}/W})) \cong
R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\Omega^\bullet_{D_{(\bullet)}})) \cong R\Gamma(D_{(\bullet)}, \Omega^\bullet_{D_{(\bullet)}}).
\]

(The last isomorphism follows from \( \text{[BO78, Cor. 5.27.2]} \).)
Proposition 3.3. The forgetful morphism

\[ R\Gamma(D_{(\bullet)}, \Omega_{D_{(\bullet)}}^\bullet) \rightarrow R\Gamma(D_{(\bullet)}, \mathcal{O}_{D_{(\bullet)}}) \]

is a quasi-isomorphism.

Proof. It is enough to show that the map

\[ R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) \rightarrow R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D_{(\bullet)}})) \]

is a quasi-isomorphism. As for Proposition 3.2, we argue by inductive reasoning. For \( |I| = 1 \), the right hand side is \( R\Gamma((X/W)_{\text{crys}}, (-)) \) of the right hand side of (3.2), thus computes \( R\Gamma((X/W)_{\text{crys}}, \mathcal{O}_{X/W}) \). For general \( I \), we argue as in the proof of Proposition 3.2 If we put \( D' := D_0 \), we have the isomorphism

\[ R\Gamma((X' \cap X''_{(m)}/W)_{\text{crys}}, L(\mathcal{O}_{D''_{(m)}})) \cong R\Gamma((X_{(\bullet, m)}/W)_{\text{crys}}, L(\mathcal{O}_{D_{(\bullet, m)}})) \]

which is \( R\Gamma((X/W)_{\text{crys}}, (-)) \) of the resolution (3.3). From this and the induction hypothesis one deduces the isomorphism

\[ \epsilon_* : R\Gamma((X' \cap X''/W)_{\text{crys}}, \mathcal{O}_{X'\cap X''/W}) \cong R\Gamma((X'_*/W)_{\text{crys}}, L(\mathcal{O}_{D'_{(\bullet)}})) \cong R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D_{(\bullet)}})). \]

Then one has the diagram as (3.14) with the triangle in the right column is replaced by the triangle

\[ R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D_{(\bullet)}})) \rightarrow R\Gamma((X'_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D'_{(\bullet)}})) \oplus R\Gamma((X''_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D''_{(\bullet)}})) \rightarrow R\Gamma((X_{(\bullet)}/W)_{\text{crys}}, L(\mathcal{O}_{D_{(\bullet)}})), \]

which exists by construction. Thus one concludes by induction.

3.4. Proof of Theorem 1.1. One defines \( G \) to be the group scheme \( GL(r) \) over \( \mathbb{Z} \), and for any scheme \( S \), one writes \( G_S \) for the induced group scheme over \( S \), or by abuse of notations \( G_R \) for \( S = \text{Spec}(R) \). So one has \( G_k, G_W \) and its \( p \)-adic completion \( \hat{G}_W \). One has the classifying (formal) simplicial schemes \( BG_k, BG_W, \hat{BG}_W \). The datum of a locally free crystal \( E \) of rank \( r \), together with a trivialization of \( E_{X|X_i} = E_{X_i} \), where \( X = \cup_{i \in I} X_i \) is a finite affine covering, and a lift of the trivialization to a trivialization of \( E_D|_{D_i} \), yields, via the transition functions, a commutative diagram

\[ X_{(\bullet)} \xrightarrow{\alpha} D_{(\bullet)} \xrightarrow{\hat{\beta}} \hat{BG}_W \xrightarrow{\iota} BG_W \]

where \( \alpha, \hat{\beta} \) and \( \iota \) are the canonical morphisms. The crystalline Chern classes come as the pull-back from the universal Chern classes via

\[ f^* : H^{2i}((BG_k/W)_{\text{crys}}, \mathcal{O}_{BG_k/W}) \rightarrow H^{2i}((X_{(\bullet)}/W)_{\text{crys}}, \mathcal{O}_{X_{(\bullet)}/W}) \]
computed via
\begin{equation}
(3.25)
\begin{aligned}
& h^* : H^2i(BG_W, \Omega_{BG_w}^\bullet) \to H^2i(BG_W, \Omega_{BG_w}^\bullet) \\
& \quad \to H^2i(D(\bullet), \Omega_{D(\bullet)/W}^\bullet) \to H^2i(D(\bullet), \Omega_{D(\bullet)/W}^\bullet).
\end{aligned}
\end{equation}
Thus $h^*$ factors through the forgetful map $H^2i(BG_W, \Omega_{BG_w}^\bullet) \to H^2i(BG_W, \Omega_{BG_w}^\bullet)$, which is zero for $i \geq 1$ because the composition of it with the surjective map $H^2i(BG_W, \Omega_{BG_w}^{\geq i}) \to H^2i(BG_W, \Omega_{BG_w}^\bullet)$, (Theorem 1.1\textsuperscript{[Ogu94 II Thm. 1.1]]) is zero. This proves Theorem 1.1. \qed

Remark 3.4. By replacing $W$ by $W_n = W/p^n W$, we see that the proof above also gives the following variant of Theorem 1.1. If $X$ is as in Theorem 1.1 and if $E$ is a locally free crystal on $X/W_n$, then $c_i^{\text{crys}}(E_X)$ is zero in the torsion crystalline cohomology group $H^2i_{\text{crys}}(X/W_n)$ for $i \geq 1$. In the final step, we need the surjectivity of the map $H^2i(BG_W, \Omega_{BG_w}^{\geq i}) \to H^2i(BG_W, \Omega_{BG_w}^\bullet)$, but it follows from the isomorphism
\begin{equation}
H^2i(BG_W, \Omega_{BG_w}^\bullet) \otimes_W W_n = H^2i(BG_W, \Omega_{BG_w}^\bullet)
\end{equation}
which is true because $H^n(BG_W, \Omega_{BG_w}^\bullet)(n \in \mathbb{N})$ are free over $W$ (Theorem 1.1\textsuperscript{[Ogu90 II Thm. 1.1]}).

4. Generalization of the main Theorem [ES15 Thm.1.1] from convergent isocrystals to isocrystals possessing a locally free lattice

The aim of this section is to prove Theorem 1.2. The proof is the same as the one of [ES15 Thm.1.1] except some points which we explain now.

First assume that $k$ is algebraically closed. We have $H^1_{\text{crys}}(X/W) = 0$ by [ES15 Prop. 2.9(2)]. (See also Theorem 5.1.) So we may assume that $\mathcal{E}$ is irreducible (of rank $s \leq r$) to prove the theorem. Also, there exists $N \in \mathbb{N}$ such that the restriction $H^1_{\text{crys}}(X/W_n) \to H^1_{\text{crys}}(X/k)$ is zero for any $n \geq N$. By using Theorem 1.1 and the remark after it) in place of [ES15 Prop. 3.1], we see that $\mathcal{E}$ admits a lattice $E$ with $E_X$ strongly $\mu$-stable as $\mathcal{O}_X$-module, by [ES15 Prop. 4.2]. Because $E_X$ has vanishing Chern classes, we can argue as in Section 3 in [ES15], and by [ES15 Cor. 3.8], we see that there exists $a \in \mathbb{N}$ such that $(F^a)^*E_N \in \text{Crys}(X/W_n)$ is trivial.

Let $D_{n,m}$ be the category of pairs $(G, \varphi)$, where $G \in \text{Crys}(X/W_{n+m})$ and $\varphi$ is an isomorphism between the restriction of $G$ to $\text{Crys}(X/W_n)$ and $\mathcal{O}_X^{\times}/W_n$. Then the same computation as in [ES15 Prop. 3.6] implies the isomorphism as pointed sets
\begin{equation}
D_{n,m} \cong M(s \times s, H^1_{\text{crys}}(X/W_m))
\end{equation}
for $1 \leq m \leq n$, which is compatible with respect to $m$.

Applying (3.1) to the pairs $(n, m) = (N, N), (N, 1)$, we conclude that $(F^a)^*E_{N+1} \in \text{Crys}(X/W_{N+1})$, which is the image of $(F^a)^*E_{2N} \in \text{Crys}(X/W_{2N})$ via the restriction
\begin{equation}
D_{N, N} \cong M(s \times s, H^1_{\text{crys}}(X/W_N)) \to M(s \times s, H^1_{\text{crys}}(X/W_1)) \cong D_{N, 1}
\end{equation}
is trivial. We continue similarly to show that $(F^a)^*E_n \in \text{Crys}(X/W_n)$ is trivial for all $n \geq N$. Hence $(F^a)^*E$ is trivial. Because the endofunctor $F^* : \text{Crys}(X/W) \to \text{Crys}(X/W)$ is fully faithful at least when restricted to locally free crystals ([Ogu94 Ex. 7.3.4]), we see that $E$ itself is trivial. Hence $\mathcal{E}$ is also trivial.

Now we prove the theorem for general $k$. If we take a locally free lattice $E$ of $\mathcal{E}$ and denote the pullback of $E$ to $(X \otimes_k \overline{W(k)})_{\text{crys}}$ by $\overline{E}$, we see by base change theorem that
\[ Q \otimes_\mathbb{Z} (W(\overline{k}) \otimes_W H^0((X/W)_{\text{crys}}, E)) \cong Q \otimes_\mathbb{Z} H^0((X \otimes_{\mathbb{Z}} \overline{k}/W(\overline{k}))_{\text{crys}}, E), \] and the latter is \( s \)-dimensional over the fraction field of \( W(\overline{k}) \). So \( Q \otimes_\mathbb{Z} H^0((X/W)_{\text{crys}}, E) \) is \( s \)-dimensional over \( K \) and hence \( E \) is trivial.

5. Gauss-Manin convergent isocrystal

The aim of this section is to prove Theorem 1.3. The proof is inspired by the discussion with G. Faltings related in the introduction.

5.1. First rigid cohomology of a smooth simply connected variety is trivial. We shall prove the following theorem which is used in order to pass from the triviality of the semi-simplification of the Gauss-Manin overconvergent isocrystal to the triviality of the Gauss-Manin overconvergent isocrystal itself.

As usual, if \( k \) is a perfect field, then one denotes by \( W = W(k) \) its ring of Witt vectors and by \( K \) the field of fractions of \( W \).

**Theorem 5.1.** Let \( X \) be a smooth connected variety defined over an algebraically closed field \( k \) of characteristic \( p \) > 0, and assume that \( X \) is proper or \( p \geq 3 \). If \( \pi_{\text{ét,ab}}^1(X) = 0 \), then \( H^1_{\text{rig}}(X/K) = 0 \).

**Proof.** If \( X \) is proper, this is [ES15, Thm. 0.1 1)]. Else, let \( \ell \) be a prime not equal to \( p \). When \( X \) is proper, the assumption \( \pi_{\text{ét,ab}}^1(X) = 0 \) implies that \( 0 = H^1_{\text{ét}}(X, \mathbb{Q}_\ell) = H^1_{\text{ét}}(\text{Pic}^0_{\text{red}}(X), \mathbb{Q}_\ell) \) and so \( \text{Pic}^0_{\text{red}}(X) = 0 \). Hence \( H^1_{\text{crys}}(X/W) = 0 \) because it is the Dieudonné crystal associated to \( \text{Pic}^0_{\text{red}}(X) \) and so \( H^1_{\text{rig}}(X/K) \) is also equal to 0.

In case \( X \) is not proper, we use the Picard 1-motive \( M := \text{Pic}^+(X) \) defined in [ABV05]. Put \( M = [L \to G] \), where \( L \) is a \( \mathbb{Z} \)-module and \( G \) is a semi-abelian variety. Because the \( \ell \)-adic realization \( V_\ell M \) of \( M \) is equal to \( H^1_{\ell}(X, \mathbb{Q}_\ell) \) by [Man13] and fits into the exact sequence

\[ 0 \to V_\ell G \to V_\ell M \to V_\ell L \to 0, \]

we have \( G = 0, L = 0 \) hence \( M = 0 \). So the crystalline realization of \( M \), which is equal to \( H^1_{\text{rig}}(X/K) \) by [ABV05] when \( p \geq 3 \), vanishes. \( \square \)

5.2. Gauss-Manin convergent isocrystal. In this subsection, we give some preliminaries on convergent \( F \)-isocrystals and then recall Ogus’ definition of Gauss-Manin convergent \( F \)-isocrystal \( R^f f_* \mathcal{O}_{Y/K} \) for a smooth proper morphism \( f : Y \to X \) with \( X \) smooth over a perfect field \( k \) (see [Ogu84, Section 3]).

Recall that objects in the convergent site on \( X \) over \( K \) are enlargements, which are the diagrams of the form \( (X \leftarrow (T \otimes_W k)_{\text{red}} \to T) \) over \( W \), where \( T \) is a \( p \)-adic formal scheme of finite type and flat over \( W \). One defines a convergent isocrystal on \( X/K \) as a crystal of \( \mathbb{Q} \otimes_\mathbb{Z} \mathcal{O}_T \)-modules on enlargements. Crystal means a sheaf of coherent \( \mathbb{Q} \otimes_\mathbb{Z} \mathcal{O}_T \)-modules with transition functions being isomorphisms. We denote the category of convergent isocrystals on \( X/K \) by \( \text{Conv}(X/K) \). The convergent site is functorial for \( X/W \), and so we can define the pullback functor \( F^* : \text{Conv}(X/K) \to \text{Conv}(X/K) \) induced by the Frobenius \((F_X, F_W)\) on \( X \) and \( W \). Then we define the category \( F\text{-Conv}(X/K) \) of convergent \( F \)-isocrystals on \( X/K \) as the category of pairs \( (E, \Phi) \), where \( E \in \text{Conv}(X/K) \) and \( \Phi \) is an isomorphism \( F^*E \to E \). \( \Phi \) is called a Frobenius structure on \( E \).

We define the \( p \)-adic convergent site on \( X \) over \( K \) as a variant of the convergent site: The objects in it are the \( p \)-adic enlargements, which are the diagrams of the form \( (X \leftarrow T \otimes_W k \to T) \) over \( W \), where \( T \) is as before. As in the case of convergent site, we can define the category
of $p$-adic convergent isocrystals and that of $p$-adic convergent $F$-isocrystals, which we denote by $p\text{-Conv}(X/K)$, $F\text{-pConv}(X/K)$ respectively.

Then we have the sequence of functors

$$F\text{-Conv}(X/K) \rightarrow F\text{-Crys}(X/W)_{\mathbb{Q}} \rightarrow F\text{-pConv}(X/K),$$

in which the first one is the inverse of the functor $M \mapsto M^{an}$ in [Ber96 Thm. 2.4.2]. For any $p$-adic enlargement $T := (X \leftarrow T \otimes_k k \leftarrow T)$, $T_n := (T \otimes_k k \leftarrow T \otimes W/p^n W)$ ($n \in \mathbb{N}$) are objects in the crystalline site $(T \otimes k/W)_{\text{cryst}}$. Then, for any $E \in \text{Crys}(X/W)$, $T \mapsto \lim_n (h^*E)_{T_n}$ defines an object in $p\text{Conv}(X/K)$ and this induces the second functor in (5.1). The functors in (5.1) are known to be equivalences by [Ber96 Thm. 2.4.2] and [Ogu84 Prop. 2.18].

We recall the definition of the Gauß-Manin convergent $F$-isocrystal

$$R^if_*\mathcal{O}_{Y/K} \in F\text{-Conv}(X/K)$$

for a smooth proper morphism $f : Y \rightarrow X$ (see [Ogu84 Section 3]). It is defined as the unique object such that, for any $p$-adic enlargement $T := (X \leftarrow T \otimes_k k \leftarrow T)$, the value $(R^if_*\mathcal{O}_{Y/K})_T$ at $T$ of $R^if_*\mathcal{O}_{Y/K}$ as an object in $F\text{-pConv}(X/K)$ is given by $\mathbb{Q} \otimes R^n(f^*\mathcal{O}_{Y/K}/T)$, where $(f^*\mathcal{O}_{Y/K}/T)$ is the morphism of topoi induced by the pull-back $f_T : Y_T \rightarrow T \otimes k$ of $f$ by $T \otimes k \rightarrow X$.

Assume now that $X$ admits a closed embedding $\iota : X \hookrightarrow P$ into a $p$-adic smooth formal scheme $P$ over $\text{Spf } W$ and give another description of convergent $F$-isocrystals and the Gauß-Manin convergent isocrystals. For $n = 0, 1, 2$, let $X \leftarrow X(n) \hookrightarrow Z(n)$ be the universal $p$-adic enlargement [Ogu84 Prop. 2.3] of $X \leftarrow P \times_W \cdots \times_W P$, and let $F\text{-Str}(X \hookrightarrow P/W)$ be the category of triples $(E, \epsilon, \Phi)$ consisting of a coherent $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{Z(0)}$-module $E$, an isomorphism $\epsilon : p_2^*E \xrightarrow{\cong} p_1^*E$ ($p_i : Z(1) \rightarrow Z(0)$ are the projections) satisfying cocycle condition, and an isomorphism $\Phi : F^*(E, \epsilon) \xrightarrow{\cong} (E, \epsilon)$ (where $F$ is the pullback by Frobenius on $X(n) \hookrightarrow Z(n)$).

Then we have an equivalence of categories [(2.11), Ogu]

$$F\text{-pConv}(X/K) \cong F\text{-Str}(X \hookrightarrow P/W).$$

Via the equivalences (5.1) and (5.2), the Gauß-Manin convergent isocrystal $R^if_*\mathcal{O}_{Y/K}$ is described as the triple $(\mathbb{Q} \otimes R^nf(0)_{\text{crys}}\mathcal{O}_{Y(0)/Z(0)}, \epsilon, \Phi) \in F\text{-Str}(X \hookrightarrow P/W)$, where $f(0)_{\text{crys}} : (Y(0)/Z(0))_{\text{crys}} \rightarrow Z(0)_{\text{Zar}}$ is the morphism of topoi induced by the pull-back $f(0) : Y(0) \hookrightarrow X(0)$ of $f$ by $X(0) \rightarrow X$ and $\epsilon, \Phi$ are defined by the functoriality of crystalline cohomology sheaves.

Let us denote the category of overconvergent $F$-isocrystals on $X/K$ [Ber96 (2.3.7)] by $F\text{-Conv}^{\dagger}(X/K)$, and the structure overconvergent isocrystal by $\mathcal{O}^\dagger_{X/K}$ to distinguish it from $\mathcal{O}_{X/K}$. There exists a natural restriction functor $F\text{-Conv}^{\dagger}(X/K) \rightarrow F\text{-Conv}(X/K)$ which is fully faithful ([Ked04]). When $f : Y \rightarrow X$ is smooth projective, Lazda [Laz15 Cor. 5.4] proved that the convergent $F$-isocrystal $R^if_*\mathcal{O}_{Y/K}$ lifts to an overconvergent $F$-isocrystal on $X$, which we denote by $R^if_*\mathcal{O}^\dagger_{Y/K}$. When $X$ is proper, the categories $F\text{-Conv}(X/K)$, $F\text{-Conv}^{\dagger}(X/K)$ are the same and so one can equate $R^if_*\mathcal{O}^\dagger_{Y/K}$ and $R^if_*\mathcal{O}_{Y/K}$.

**Remark 5.2.** In this subsection, the Frobenius structure on a (n) (over)convergent isocrystal is defined with respect to the pullback functor induced by the Frobenius $(F_X, F_W)$ on $X$ and
5.3. **Case where the ground field \( k \) is finite.** In this subsection, we prove Theorem 1.3 in the case where \( k \) is a finite field and the statement in Remark 1.4. In order to prove them, we may replace \( k \) by a finite extension \( k' \), and \( K \) by a finite possibly ramified extension of the field of fractions of the ring of Witt vectors over \( k' \), such that the following hold:

1. \( X \) has a \( k \)-rational point \( x \).
2. The eigenvalues of the action of Frobenius \( F_x \) on \( L_\ell := H^i_{\text{ét}}(Y \times_X \bar{x}, \mathbb{Q}_\ell) \) belong to a number field \( K_0 \) contained in \( K \).
3. There exists a \( K_0 \)-vector space \( L \) with linear action \( F \) and an inclusion of fields \( K_0 \hookrightarrow \mathbb{Q}_\ell \) such that \( L \otimes_{K_0} \mathbb{Q}_\ell \) is isomorphic to \( L_\ell \) as vector spaces with an action.

(Note that overconvergent isocrystals and rigid cohomologies are defined even when the base complete discrete valuation ring \( O_K \) is ramified over \( W(k) \) (\cite{Ber96}, \cite{LeS07}), and the rigid cohomology satisfies the base change property for finite extension of the base (\cite{ChT03} Cor. 11.8.2)).

Until the end of this subsection, we consider the Frobenius structure with respect to the pullback induced by \((F_d, \text{id})\), where \( d = \log_p |k| \). (See Remark 5.2.) For any closed point \( y \) in \( X \), one has the base change isomorphism

\[
y^* R^i f_* \mathcal{O}_{Y/K}^\dagger = y^* R^i f_* \mathcal{O}_{Y/K} = H^i_{\text{rig}}(Y \times_X y/K)
\]

by \cite{Ogu84} Rmk.3.7.1 and \cite{Ber97} Prop. 1.8, Prop. 1.9, and the set of Frobenius eigenvalues on it is the same as that on \( \ell \)-adic cohomology \( H^i(Y \times_X \bar{y}, \mathbb{Q}_\ell) \) by \cite{KM74} Thm.1, \cite{CLS98} Cor. 1.3. The assumption \( \pi_1^\dagger(X \otimes \overline{k}) = \{1\} \) implies that the action of Frobenius \( F_y \) on \( H^i(Y \times_X \bar{y}, \mathbb{Q}_\ell) \) is identified with the action of \( F_d \) (where \( d_y = \deg(y/k) \)) on \( L_\ell \), hence the action of \( F \otimes \text{id} \) on \( L \otimes_{K_0} \mathbb{Q}_\ell \).

Let \( \mathcal{E}_0 \) be the overconvergent \( F \)-isocrystal on \( X \) defined by \((L \otimes_{K_0} K) \otimes_{K_0} \mathcal{O}_{X/K}^\dagger, F \otimes \text{id})\). Then, by construction, \( R^i f_* \mathcal{O}_{Y/K}^\dagger \) and \( \mathcal{E}_0 \) have the same eigenvalues of Frobenius action on any closed point of \( X \). Then Abe’s Čebotarev’s density theorem \cite[A.3]{Abe13} implies that the semi-simplification of \( R^i f_* \mathcal{O}_{Y/K}^\dagger \) is the same as that of \( \mathcal{E}_0 \). Hence the semi-simplification of \( R^i f_* \mathcal{O}_{Y/K}^\dagger \) is trivial as overconvergent isocrystal on \( X \). By Theorem 5.1, extensions of \( \mathbb{Q} \otimes \mathcal{O}_{X/W}^\dagger \) by itself are trivial when \( X \) is proper or \( p \geq 3 \). This finishes the proof.

5.4. **General case.** In this subsection, we prove Theorem 1.3 by a spreading out argument allowing \( k \) to be a perfect field, but assuming \( X \) to be proper. Let \( f : Y \to X \) be as in the statement of Theorem 1.3 and let \( g : X \to \text{Spec}k \) be the structure morphism. Also, let \( X = \bigcup_{i \in I} X_i \) be an affine open covering of \( X \) and take a closed embedding \( X_i \to P_i \) into a smooth \( p \)-adic formal scheme \( P_i \) over \( W \) for each \( i \in I \). We prove that the Gauß-Manin convergent isocrystal \( \mathcal{E}_{\text{conv}} := R^i f_* \mathcal{O}_{Y/K} \in F-\text{Conv}(X/K) \) is trivial as an object in \( \text{Conv}(X/K) \). Denote by \( \mathcal{E}_{\text{crys}}, \mathcal{E}_{p-\text{crys}} \) the image of \( \mathcal{E}_{\text{conv}} \) in \( F-\text{Crys}(X/W)_Q, F-p-\text{Conv}(X/K) \) via \( \text{[5.1]} \).

We can find a connected affine scheme \( T = \text{Spec} A \) smooth of finite type over \( \mathbb{F}_p \), a \( p \)-adic formal lift \( T := \text{Spf} A \) of \( T \) which is smooth over \( \text{Spf} \mathbb{Z}_p \) and endowed with a lift of Frobenius,
and proper smooth morphisms $Y_T \xrightarrow{f_T} X_T \xrightarrow{g_T} T$ which fit into the commutative diagram

(5.4)

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y_T \\
\downarrow & & \downarrow \\
X & \xrightarrow{\alpha} & X_T \\
\downarrow & & \downarrow \\
\text{Spec } k & \xrightarrow{g_T} & \text{Spec } F_p \\
\downarrow & & \downarrow \\
\text{Spf } W & \xrightarrow{T} & \text{Spf } \mathbb{Z}_p,
\end{array}
\]

where the squares with symbol $\square$ means cartesian squares. Also, we may assume the existence of an open covering $X_T = \bigcup_{i \in I} X_{i,T}$ which induces $X = \bigcup_{i \in I} X_i$ and closed immersions $X_{i,T} \hookrightarrow P_{i,T}$ ($i \in I$) into a $p$-adic formal scheme $P_{i,T}$ smooth over $T$ with $P_{i,T} \times T \text{Spf } W = P_i$.

Let $\tilde{E}_{\text{conv}} := R^n g_{T,\text{conv}} \star O_{Y_T/\mathbb{Q}_p} \in F_{\text{-Conv}}(X_T/\mathbb{Q}_p)$ be the Gauß-Manin convergent $F$-isocrystal defined by the right column of (5.4), and let $\tilde{E}_{T,\text{crys}}^n, \tilde{E}_{T,\text{pconv}}^n$ be its images in the categories $F_{\text{-Crys}}(X_T/\mathbb{Z}_p)_\mathbb{Q}, F_{\text{-Conv}}(X_T/\mathbb{Q}_p)$. Then the morphism $\alpha$ in (5.4) induces the pullback functor

(5.5)

$F_{\text{-Crys}}(X_T/\mathbb{Z}_p)_\mathbb{Q} \to F_{\text{-Crys}}(X_T/T)_\mathbb{Q} \xrightarrow{\alpha^*} F_{\text{-Crys}}(X/\mathbb{Z}_p)_\mathbb{Q}$.

We denote this functor by $\tilde{\alpha}^*$. Then we have the map

(5.6)

$\tilde{\alpha}^* \tilde{E}_{T,\text{crys}} \to E_{\text{crys}}$

defined by functoriality.

To prove that the map (5.6) is an isomorphism, we may work locally on each $X_i$. So we may assume that $X_T$ admits a closed embedding $X_T \hookrightarrow P_T$ into a $p$-adic formal scheme $P_T$ smooth over $T$. Put $P := P_T \times T W$. Then the functor (5.5) is identified with the composite

(5.7)

$F_{\text{-Conv}}(X_T/\mathbb{Q}_p) \cong F_{\text{-Str}}(X_T \hookrightarrow P_T/\mathbb{Z}_p) \xrightarrow{\alpha^*} F_{\text{-Str}}(X \hookrightarrow P/W) \cong F_{\text{-Conv}}(X/\mathbb{Q}_p)$

via the second equivalence in (5.1). It suffices to prove that the map

(5.8)

$\tilde{\alpha}^* \tilde{E}_{T,\text{pconv}} \to E_{\text{pconv}}$

induced by (5.6) via the second equivalence in (5.1) is an isomorphism, and to see this, it suffices to prove that the base change map

$\alpha_{Z(n)}^*(\mathbb{Q} \otimes R^i g(n)_{T,\text{crys}}, O_{Y_T(n)/Z_T(n)}) \to \mathbb{Q} \otimes R^i g(n)_{\text{crys}}, O_{Y(n)/Z(n)}$

induced by diagrams

(5.9)

\[
\begin{array}{ccc}
Y(n) & \xrightarrow{g(n)} & Y_T(n) \\
\downarrow & & \downarrow \\
X(n) & \xrightarrow{\alpha_{Z(n)}} & X_T(n) \\
\downarrow & & \downarrow \\
Z(n) & \xrightarrow{\alpha_{Z(n)}} & Z_T(n)
\end{array}
\]
is an isomorphism, where \( X_T \leftarrow X_T(n) \leftarrow Z_T(n) \) is the universal \( p \)-adic enlargement of \( X_T \leftarrow P_T \times_{Z_p} \cdots \times_{Z_p} P_T \) and \( X \leftarrow X(n) \leftarrow \mathcal{Z}(n) \) is the universal \( p \)-adic enlargement of \( X \leftarrow P \times_W \cdots \times_W P \). This follows from the base change theorem of crystalline cohomology

\[
\Lambda^* \Omega_{X(n)/Z(n)} \xrightarrow{\cong} \Omega_{Y(n)/Z(n)}
\]

([BO78], Thm. 7.8), and the flatness of \( \mathbb{Q} \otimes R^i g_j(\mathcal{E}_{T(n)/Z(n)}) \) for \( j \geq 0 \). Hence the map (5.6) is an isomorphism.

The crystalline cohomology \( H^j((X_T/Z_p)_{\text{crys}}) \) is trivial (and so \( \mathcal{E}_{T,\text{crys}} \) is trivial), as required. So the proof of the theorem is finished modulo the finiteness and the base change property we used above.
Finally, we prove the finiteness and the base change property. In the following, let \( \text{Spec} \, A_1 \) be an affine regular scheme of characteristic \( p > 0 \), let \( \text{Spf} \, A \) be a \( p \)-adic formal scheme flat over \( \text{Spf} \, \mathbb{Z}_p \) such that \( A/pA = A_1 \). Let \( X \to \text{Spec} \, A_1 \) be a projective smooth morphism, and let \( \mathcal{E} \) be an isocrystal on \( (X/A)_{\text{crys}} \) which is locally free as an isocrystal. Moreover, assume that we are given the following cartesian diagram such that \( \text{Spec} \, A'_1 \) is also regular.

\[
\begin{array}{ccc}
X & \stackrel{\alpha}{\longrightarrow} & X' \\
\downarrow & & \downarrow \\
\text{Spec} \, A_1 & \longleftarrow & \text{Spec} \, A'_1 \\
\downarrow & & \downarrow \\
\text{Spf} \, A & \longleftarrow & \text{Spf} \, A'.
\end{array}
\]

Then we have the following:

**Proposition 5.3.** In the situation above, we have the following:

1. \( R\Gamma((X/A)_{\text{crys}}, \mathcal{E}) \) is a perfect complex of \( (\mathbb{Q} \otimes A) \)-modules.
2. \( A' \otimes_A^L R\Gamma((X/A)_{\text{crys}}, \mathcal{E}) \to R\Gamma((X'/A')_{\text{crys}}, \alpha^* \mathcal{E}) \) is a quasi-isomorphism.

**Remark 5.4.** Although \( \mathcal{E} \) is locally free as an isocrystal, it is not clear at all that there exists a lattice \( E \) of \( \mathcal{E} \) with \( E \) locally free as a crystal. This is the reason that the base change theorem in [BO78] is not enough for us.

**Proof.** Let \( X \to X \) be a simplicial scheme augmented to \( X \) defined as the Čech hypercovering associated to an open covering \( X = \bigcup_{i \in I} X_i \) which admits a closed embedding \( X \to Y \to \) into a simplicial \( p \)-adic formal scheme \( Y \) smooth over \( A \), and let \( D(\bullet) \) be the PD-envelope of it. For \( n \in \mathbb{N} \), put \( A_n := A/p^n A, D(\bullet)n := D(\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \mathbb{Z} \). Also, let \( E \) be a \( p \)-torsion free crystal on \( X/A \) with \( \mathcal{E} = \mathbb{Q} \otimes E \), let \( E_n \) be the restriction of \( E \) to \( (X/A_n)_{\text{crys}} \) and \( E(\bullet)n \) the value of \( E \) at \( D(\bullet)n \). Then \( R\Gamma((X/A_n)_{\text{crys}}, E_n) = R\Gamma(D(\bullet)n, E(\bullet)n \otimes \Omega_{D(\bullet)n}^\bullet) \) and this is quasi-isomorphic to a bounded complex of \( A_n \)-modules flat over \( \mathbb{Z}/p^n \mathbb{Z} \) which is compatible with respect to \( n \). Hence

\[
R\Gamma((X/A_n)_{\text{crys}}, E_n) \otimes_{\mathbb{Z}_n} A_{n-1} = R\Gamma((X/A_n)_{\text{crys}}, E_n) \otimes_{\mathbb{Z}/p^n \mathbb{Z}} \mathbb{Z}/p^{n-1} \mathbb{Z} = R\Gamma((X/A_{n-1})_{\text{crys}}, E_{n-1}).
\]

Also, \( R\Gamma((X/A)_{\text{crys}}, E_1) = R\Gamma(X, \mathcal{E}_X \otimes \Omega_X^\bullet) \) is a perfect complex of \( A_1 \)-modules, because \( A_1 \) is regular. Hence, by [BO78, B.10], \( R\Gamma((X/A)_{\text{crys}}, E) = \varinjlim_n R\Gamma((X/A_n)_{\text{crys}}, E_n) \) is a perfect complex of \( A \)-modules, and so \( R\Gamma((X/A)_{\text{crys}}, \mathcal{E}) := \mathbb{Q} \otimes R\Gamma((X/A)_{\text{crys}}, E) \) is a perfect complex of \( (\mathbb{Q} \otimes A) \)-modules. This proves (1).

Next we prove (2). the left hand side is equal to

\[
\mathbb{Q} \otimes (A' \otimes_A^L R\Gamma((X/A)_{\text{crys}}, E)) = \mathbb{Q} \otimes (\varinjlim_n (A'_n \otimes_{A_n}^L R\Gamma((X/A_n)_{\text{crys}}, E_n)))
\]

because \( R\Gamma((X/A)_{\text{crys}}, E) \) is perfect, and the right hand side is equal to

\[
\mathbb{Q} \otimes \varinjlim_n R\Gamma((X/A'_n)_{\text{crys}}, \alpha^* E_n).
\]

Let us define \( C_n \) by

\[
C_n := \text{Cone}(A'_n \otimes_{A_n}^L R\Gamma((X/A_n)_{\text{crys}}, E_n) \to R\Gamma((X/A'_n)_{\text{crys}}, \alpha^* E_n)).
\]
Then it suffices to prove that \( \mathbb{Q} \otimes R \varinjlim_n C_n = 0 \). To prove this, we may work Zariski locally on \( X \) (because of cohomological descent). So we can reduce to the case that \( X \to \text{Spec} A_1 \) is liftable to an affine smooth morphism \( X = \text{Spf} B \to \text{Spf} A \). (But we lose the projectivity of \( X \).) Let \( \mathcal{X}_n := \mathcal{X} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n \mathbb{Z} \), \( \mathcal{X}_n := \mathcal{X} \otimes_A A_n \) and let \( E_n^X \) be the value of \( E_n \) at \( \mathcal{X}_n \). Then

\[
C_n = \text{Cone}(A'_n \otimes_{A_n} \Gamma(\mathcal{X}_n, E_n^X \otimes \Omega^*_{\mathcal{X}_n/A_n}) \to \Gamma(\mathcal{X}_n', \alpha^* E_n^X \otimes \Omega^*_{\mathcal{X}_n'/A_n'})).
\]

For a finitely generated \( p \)-torsion free \( B \)-module \( M \), put \( M_n := M \otimes_A A_n \) and

\[
C_n(M) := \text{Cone}(A'_n \otimes_{A_n} M_n \to A'_n \otimes_{A_n} M_n).
\]

Then it suffices to prove that \( \mathbb{Q} \otimes R \varinjlim_n C_n(M) = 0 \) when \( \mathbb{Q} \otimes B \) is a projective \( \mathbb{Q} \otimes B \)-module. If we take a resolution \( N^* \to M \) of \( M \) by finitely generated free \( B \)-modules, \( N^*_n \to M_n \) is also a resolution because \( M \) is \( p \)-torsion free. Then

\[
(5.15) \quad \mathbb{Q} \otimes R \varinjlim_n C_n(M) = \mathbb{Q} \otimes R \varinjlim_n \text{Cone}(A'_n \otimes_{A_n} N^*_n \to A'_n \otimes_{A_n} M_n) = \text{Cone}(\mathbb{Q} \otimes A') \otimes_{Q \otimes A} \mathbb{Q} = 0.
\]

(The last equality follows from the projectivity of \( \mathbb{Q} \otimes N^* \), \( \mathbb{Q} \otimes M \)) So the proof of (2) is finished. \( \square \)

6. A very weak form of a Lefschetz theorem for isocrystals

Let \( X \) be a smooth projective variety defined over a perfect field \( k \) of characteristic \( p > 0 \). One expects that given an irreducible isocrystal \( \mathcal{E} \in \text{Crys}(X/W)_\mathbb{Q} \), there is an ample divisor \( Y \subset X \) defined over the same field such that \( \mathcal{E}|_Y \) is irreducible in \( \text{Crys}(Y/W)_\mathbb{Q} \). One could also expect the existence of such \( Y \) which is independent of \( \mathcal{E} \). One could also weaken all those variants by requesting the irreducibility only for convergent isocrystals \( \mathcal{E} \in \text{Conv}(X/K) \). Note that any subisocrystal of a convergent isocrystal is a convergent isocrystal, thus this version for convergent isocrystals is just the restriction to \( \text{Conv}(X/K) \) of version for isocrystals.

We prove here a very weak form of this expectation.

**Theorem 6.1.** Let \( X \) be a smooth projective variety defined over a perfect field \( k \) of characteristic \( p > 0 \) which lifts to \( W_2(k) \). For any ample line bundle \( \mathcal{H} \), there is an integer \( n_0 > 0 \) such that for any \( n \geq n_0 \), any isocrystal \( \mathcal{E} \in \text{Crys}(X/W)_\mathbb{Q} \) of rank \( \leq p \) which admits a lattice \( E \) with \( E_X \) \( \mu \)-stable in \( \text{Crys}(X/k) \) has the property that \( E_Y \in \text{Crys}(Y/k) \) is \( \mu \)-stable as well for any smooth divisor \( Y \subset X \) in \( |\mathcal{H}^n| \) lifting to \( W_2(k) \) on which the restriction \( E_Y \) of the crystal \( E_X \) to \( Y \) is torsion free. Moreover, the torsion freeness condition for \( E_Y \) is automatically satisfied when \( \mathcal{E} \) admits a locally free lattice (which is not necessarily equal to \( E \)).

To link Theorem 6.1 to the expectation, one needs the following theorem.

**Theorem 6.2.** Let \( X \) be a smooth projective variety defined over a perfect field \( k \) of characteristic \( p > 0 \). Let \( \mathcal{E} \in \text{Crys}(X/W)_\mathbb{Q} \) which admits a lattice \( E \) such that \( E_X \) is \( \mu \)-stable in \( \text{Crys}(X/k) \). Then \( \mathcal{E} \) is irreducible.

**Proof.** Let \( \mathcal{E}' \subset \mathcal{E} \) be an irreducible subisocrystal and \( \mathcal{E}' \) be a lattice of \( \mathcal{E}' \) which is \( \mu \)-semistable in \( \text{Crys}(X/k) \). (See [ES15, Prop. 4.1] for the existence of such lattice.) Then there is an integer \( n \) such that \( E' \subset E \subset E(nX) \subset E \). By replacing \( E' \) by \( E'(nX) \), we can assume that the inclusion \( \iota : E' \to E \) in \( \text{Crys}(X/W) \) induces a nonzero
map \( \iota_X : E'_X \rightarrow E_X \) in \( \text{Crys}(X/k) \). The locus \( \Sigma \subseteq X \) on which \( E'_X, E_X \) are possibly not locally free have codimension \( \geq 2 \). Let \( \mathcal{H} \) be the ample line bundle defining the \( \mu \)-(semi)stability, and \( C \) be a smooth complete intersection curve in a high power of the linear system \( |\mathcal{H}| \), which is disjoint of \( \Sigma \). Then the restrictions \( E'_C, E_C \) of \( E'_X, E_X \) to \( C \), which is the same as the values of \( E'_X, E_X \) to \( C \), are locally free. By Theorem \[1.1\] the degree of \( E'_C, E_C \) are 0. Thus the slopes \( \mu(E'_X), \mu(E_X) \) are 0. Then \( 0 = \mu(E'_X) \leq \mu(\text{Im} \iota_X) < \mu(E_X) = 0 \) unless \( \iota_X \) is surjective. Hence \( \iota_X \) is surjective and so is \( \iota \). Hence \( \mathcal{E}' \cong \mathcal{E} \).

**Remark 6.3.** The condition in Theorem \[6.2\] implies irreducibility, but is not equivalent to it. Let \( X \) be a smooth projective curve of genus \( \geq 2 \), with lift \( X_W \) to \( W = W(k) \), let \( a, b \in H^0(X_W, \Omega^1_{X_W/W}) \) be linearly independent differential forms. Define the connection on \( E = \mathcal{O}_{X_W} e_1 \oplus \mathcal{O}_{X_W} e_2 \) by the matrix \( A = \begin{pmatrix} 0 & pa \\ pb & 0 \end{pmatrix} \), which is the value of \( \nabla \) on the two basis vectors \( e_i \). Then \( E_X \) is a trivial connection. As a consequence, \( E \) is locally nilpotent, thus is a crystal. It is irreducible as an isocrystal, as any sub line bundle of degree 0 of \( \mathcal{O}_{X_K} e_1 \oplus \mathcal{O}_{X_K} e_2 \) is isomorphic to \( \mathcal{O}_{X_K} \), spanned by \( \lambda_1 e_1 + \lambda_2 e_2 \), \( (\lambda_1, \lambda_2) \neq (0, 0) \) where \( \lambda_i \in K \) and there is no form \( \omega \in H^0(X_K, \Omega^1_{X_K}) \) such that \( \lambda_2 pa = \lambda_1 \omega, \lambda_1 pb = \lambda_2 \omega \).

We prove now that there is no lattice \( E' \) of \( \mathcal{E} \) such that \( E'_X \) is \( \mu \)-stable in \( \text{Crys}(X/k) \). The argument is essentially the same as in \[1\] Thm. 5.2]. Assuming there is such a lattice \( E' \), after replacing \( E' \) by \( E'(nX) \) for a suitable integer \( n \), there is an inclusion \( \iota : E' \rightarrow E \) inducing the identity on \( \mathcal{E} \) such that the induced map \( \iota_X : E'_X \rightarrow E_X \) is non-zero. By Theorem \[1.1\] \( \mu(E'_X) = \mu(E_X) = 0 \). If \( \iota_X \) is not surjective, \( \mu(E'_X) \leq \mu(\text{Im} \iota_X) \leq \mu(E_X) \), a contradiction. So \( \iota_X \), and thus \( \iota \) are surjective. Then \( E' = E \) and thus \( E_X = E'_X \), which is impossible as \( E_X \) is not stable.

**Remark 6.4.** Irreducible convergent isocrystals do not necessarily satisfy the condition in Theorem \[6.2\] We now construct an example.

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( X \) be a smooth projective curve such that the \( p \)-rank of its Jacobian is \( r \geq 2 \). Let \( K, K_0 \) be the fraction field of \( W(k) \), \( W(\mathbb{F}_p) \) respectively. Let \( \rho : \pi_1(X) \rightarrow GL_2(\mathbb{Z}_p) \) be a continuous representation such that \( \rho_{K_0} : \pi_1(X) \rightarrow GL_2(\mathbb{Z}_p) \rightarrow GL_2(K_0) \) is irreducible and that \( \rho \) modulo \( p \) is trivial. Because the pro-\( p \) completion \( \pi_1(X)^p \) of the étale fundamental group \( \pi_1(X) \) is isomorphic to a free pro-\( p \) group \( \langle \gamma_1, ..., \gamma_r \rangle \) of rank \( r \), the composition

\[
\pi_1(X) \rightarrow \pi_1(X)^p \cong \langle \gamma_1, ..., \gamma_r \rangle \rightarrow GL_2(\mathbb{Z}_p)
\]

defined by

\[
\gamma_1 \mapsto \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad \gamma_2 \mapsto \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad \gamma_i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (i \geq 3)
\]

satisfies the conditions required for \( \rho \). Let \( (\mathcal{E}, \Phi), \Phi : F^* \mathcal{E} \cong \mathcal{E} \), be the unit-root convergent \( F \)-isocrystal associated to \( p\mathbb{Q}_p : \pi_1(X) \rightarrow GL_2(\mathbb{Z}_p) \rightarrow GL_2(\mathbb{Q}_p) \) by \[1\] Thm. 2.1] and let \( E \) be the lattice of \( \mathcal{E} \) associated to \( \rho \) by the construction given in \[1\] Prop. 2.3, Rmk. 2.3.2]. Namely, if we denote the finite Galois covering associated to the kernel of \( \rho \) modulo \( p^n \) by \( f_n : Y_n \rightarrow X \) and denote its Galois group by \( G_n \), \( E \) is defined to be the inverse limit of \( (f_n \text{cryst} \mathcal{O}_{Y_n/W_n})^{G_n} \ (n \in \mathbb{N}) \), where the action of \( G_n \) is induced by its action on \( Y_n \) and \( \rho \) modulo \( p^n \). Then \( E_X = (f_1 \text{cryst} \mathcal{O}_{Y_1/k})^{G_1} \) is trivial because \( \rho \) is trivial modulo \( p \).

We prove that \( \mathcal{E} \) is irreducible as a convergent isocrystal. Assume on the contrary that \( \mathcal{E} \) has a convergent subsisocrystal \( \mathcal{E}' \) of rank 1. Then \( 0 \subset \mathcal{E}' \subset \mathcal{E} \) is a Jordan-Hölder sequence
of $\mathcal{E}$, and so is $0 \subset F^{*}\mathcal{E}' \subset F^{*}\mathcal{E} \cong \mathcal{E}$. Hence $\{\mathcal{E}', \mathcal{E}/\mathcal{E}'\} = \{F^{*}\mathcal{E}', F^{*}\mathcal{E}/F^{*}\mathcal{E}'\}$ and so we have an isomorphism $(F^{*})^{2}\mathcal{E}' \cong \mathcal{E}'$, which we denote by $\alpha$. By replacing $\alpha$ by $p^{n}\alpha$ for a suitable integer $n$, we may assume that the slope of $(\mathcal{E}', \alpha)$ is 0. We denote the composite $\Phi \circ F^{*}\Phi : (F^{*})^{2}\mathcal{E} \cong \mathcal{E}$ by $\Phi_{2}$. Let $t_{1} : \mathcal{E}' \to \mathcal{E}$ be the inclusion map and let $t_{2} : \mathcal{E}' \to \mathcal{E}$ be the composite $\mathcal{E}' \overset{\alpha^{-1}}{\to} (F^{*})^{2}\mathcal{E}' \to (F^{*})^{2}\mathcal{E} \overset{\Phi_{2}}{\to} \mathcal{E}$. Both $t_{1}, t_{2}$ are maps of convergent isocrystals. If $t_{2}$ is equal to $t_{1}$ up to scalar $a \in K \setminus \{0\}$, $(\mathcal{E}', a \alpha)$ is a convergent $F$-subisocrystal of $(\mathcal{E}, \Phi_{2})$. Because $(\mathcal{E}, \Phi_{2})$, corresponding to $\rho_{K_{0}} : \pi_{1}(X) \to GL_{2}(\mathbb{Z}_{p}) \to GL_{2}(K_{0})$, has pure slope 0, $(\mathcal{E}', a \alpha)$ is necessarily of slope 0 and so it induces a nontrivial subrepresentation of $\rho_{K_{0}}$, which contradicts the irreducibility of $\rho_{K_{0}}$. Hence $t_{2}$ is not a scalar multiple of $t_{1}$ and so $t := t_{1} \oplus t_{2}$ defines an isomorphism $\mathcal{E}' \oplus \mathcal{E}' \cong \mathcal{E}$ of convergent isocrystals. If we identify $\mathcal{E}' \oplus \mathcal{E}'$ and $\mathcal{E}$ via $\iota$, the isomorphism $\Phi_{2} : (F^{*})^{2}\mathcal{E}' \oplus (F^{*})^{2}\mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}'$ is written in the form $f \circ (\alpha \oplus \alpha)$, where $f \in \text{End}(\mathcal{E}' \oplus \mathcal{E}') \cong M(2 \times 2, K)$. Because $(\mathcal{E}, \Phi_{2})$, $(\mathcal{E}', \alpha)$ have pure slope 0, the $F$-isocrystal $(K^{2}, f)$ on $K$ has pure slope 0. Hence there exists a nonzero $(x_{1}, x_{2}) \in K^{2}$ with $f(F^{2}(x_{1}), F^{2}(x_{2})) = (x_{1}, x_{2})$. Then we see that the image of $x_{1} \oplus x_{2} : \mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}' \cong \mathcal{E}$ is stable under $\Phi_{2}$ and thus defines a convergent $F$-subisocrystal of $(\mathcal{E}, \Phi_{2})$, which again leads to a contradiction. Hence $\mathcal{E}$ is irreducible as convergent isocrystal, as required.

Finally, the same argument as in Remark 6.3 shows that there is no lattice $E'$ of $\mathcal{E}$ such that $E_X$ is $\mu$-stable in $\text{Crys}(X/k)$.

Proof of Theorem 6.7. As $E_X$ is torsion free, we find a smooth complete intersection curve $C$ in $|H^m|$ such that $E_C$ as a coherent sheaf is locally free. Thus by Theorem 1.1 $E_C$ has degree 0. Thus $\mu(E_C) = 0$, and for all smooth divisors $Y \in |H^m|$ with $E_Y$ torsion free, $\mu(E_Y) = 0$. By [Lan14] Cor. 5.10 the Higgs sheaf $(V, \theta)$ associated to $E_X$ by the Ogus-Vologodsky correspondence [OV07] Thm. 2.8 is $\mu$-stable of degree 0 as well. By [Lan15] Thm. 10], $(V, \theta)|_Y$ is $\mu$-stable of degree 0, thus again by [Lan14] Cor. 5.10], $E_Y$ is $\mu$-stable of degree 0. When $\mathcal{E}$ admits a locally free lattice, Theorem 1.1 and the independence of crystalline Chern classes with respect to lattices, the normalized Hilbert polynomial of $E_X$ is the same as that of $O_X$. Thus $E_X$ is locally free by [Lan15] Thm. 11] and so $E_Y$ is automatically torsion free. 

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