DIVISORS ON GRAPHS, ORIENTATIONS, SYZYGIES, AND SYSTEM RELIABILITY

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Abstract. We study various ideals arising in the theory of system reliability. We use ideas from the theory of divisors, orientations and matroids on graphs to describe the minimal polyhedral cellular free resolutions of these ideals. In each case we give an explicit combinatorial description of the minimal generating set for each higher syzygy module in terms of the acyclic orientations of the graph, the $q$-reduced divisors and the bounded regions of the graphic hyperplane arrangement. The resolutions of all these ideals are closely related, and their Betti numbers are independent of the characteristic of the base field. We apply these results to compute the reliability of their associated coherent systems.

Contents

1. Introduction 1
2. Definitions and background 4
3. Partial orientations and $q$-reduced divisors 6
4. Graphic matroid ideals 10
5. Minimal free resolutions of system ideals 16
6. Algorithms for generating functions of system ideals 24
References 26

1. Introduction

This work is concerned with the development of new connections between the theory of oriented matroids, the theory of divisors on graphs, and the theory of system reliability. Inspired by the work of Diaconis and Sturmfels [DS98] applying algebraic techniques in probability and statistics, and following the work by Naiman-Wynn [NW92] and Giglio-Wynn [GW04] connecting system reliability to Hilbert functions of their associated ideals, we study reliability of (directed) networks through the lens of algebraic statistics and geometric combinatorics.

The starting point of this paper is to study the following network flow reliability problem. Let $G$ be a (directed) graph with a source vertex $q$ and other vertices with specific demands. Assume that the probability that the edge $e$ is working properly is $p_e$. A popular game in system reliability theory is to compute the probability of the union of certain events under various restrictions. A well-known example is the source-to-multiple-terminal (SMT) reliability which is the probability that there exists at least one (oriented) path from $q$ to every other vertex of $G$.

The classical method to compute the system reliability is to apply the inclusion-exclusion principle of probability theory which is computationally expensive. On the other hand, the system reliability formula is equal to the numerator of Hilbert series of a certain ideal associated to the network. Our main contribution is to apply the syzygy tool from computational algebra to distinguish the (non-cancelling) terms in the reliability formula for various system ideals.
For each such an ideal, we prove that each non-cancelling term is corresponding to a minimal generator of its proper higher syzygy module. We study the minimal free resolution of these ideals, in order to obtain a closed form of the inclusion-exclusion expression. We associate a set of combinatorial and algebraic objects to each term in the reliability formula, namely oriented divisors, acyclic orientations or generalized oriented spanning trees, bounded regions of graphic hyperplane arrangement, and the minimal generating sets of the syzygy modules of the associated ideal.

The given bijections between the generators of the higher syzygy modules, and the combinatorial objects classify all terms of the reliability formula in terms of the combinatorics of the network. The new algebraic approach, beside simplifying some results in the literature (see e.g., [SP78 and [5.1]], and providing some practical applications and new algorithms, gives a more clear insight into the structure of the system.

Developing the new connections between system reliability theory, divisor theory and discrete potential theory enables us to obtain many numerical and intrinsic information about a network by looking instead at its associated ideals (or its corresponding combinatorial objects) arising naturally in different settings.

1.1. Divisors on graphs. Let $G$ be a finite graph. Let $\text{Div}(G)$ be the free abelian group generated by $V(G)$. An element of $\text{Div}(G)$ is a formal sum of vertices with integer coefficients and is called a divisor on $G$. We denote by $\mathcal{M}(G)$ the group of integer-valued functions on the vertices. The Laplacian operator $\Delta : \mathcal{M}(G) \to \text{Div}(G)$ is defined by

$$\Delta(f) = \sum_{v \in V(G)} \sum_{(v,w) \in E(G)} (f(v) - f(w))(v).$$

The group of principal divisors is defined as the image of the Laplacian operator and is denoted by $\text{Prin}(G)$. Two divisors $D_1$ and $D_2$ are called linearly equivalent if their difference is a principal divisor. This gives an equivalence relation on the set of divisors. The set of equivalence classes forms a finitely generated abelian group which is called the Picard group of $G$. If $G$ is connected, then the finite (torsion) part of the Picard group has cardinality equal to the number of spanning trees of $G$. We recommend the recent survey article [LP10] for a short overview of the subject.

Associated to $G$ there is a canonical ideal which encodes the equivalences of divisors on $G$ introduced in [CRS02]. A certain initial ideal $M^G_q$ defined after fixing a vertex $q$ in $V(G)$, was extensively studied in [PS04, PPWT1, MS13a].

1.2. Polyhedral cellular free resolutions. Let $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field $k$ on $n$ variables with its usual $\mathbb{Z}^n$-grading, and let $I \subset R$ be an ideal generated by monomials $I = \langle m_1, m_2, \ldots, m_t \rangle$. A graded free resolution of $I$ is an exact sequence of the form

$$0 \to \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_0 \xrightarrow{\varphi_0} I \to 0$$

where all $F_i$’s are free $R$-modules and all differential maps $\varphi_i$’s are graded. The resolution is called minimal if $\varphi_{i+1}(F_{i+1}) \subseteq mF_i$ for all $i \geq 0$, where $m = \langle x_1, \ldots, x_n \rangle$. The $i$-th Betti number $\beta_i(I)$ of $I$ is the rank of $F_i$. The $i$-th graded Betti number in degree $j \in \mathbb{Z}^n$, denoted by $\beta_{i,j}(I)$, is the rank of the degree $j$ part of $F_i$. These integers encode very subtle numerical information about the ideal (e.g. its Hilbert series).

One natural way to describe a resolution of an ideal is through the construction of a polyhedral complex whose faces (vertices, edges, and higher dimensional cells) are labeled by monomials in such a way that the chain complex determining its cellular homology realizes a graded free resolution of the ideal. The study of cellular resolutions was initiated in [BS98] (to where we refer for further details). Cellular resolutions have the advantage that algebraic resolutions can in some sense be given a global description, and they also lead to combinatorially interesting geometric complexes.
1.3. Coherent systems. We quickly recall some basic notions from reliability theory and we recommend [Doh03, Sec.6] and the survey articles [ABS84] by Agrawal-Barlow, and [JJM88] by Johnson-Malek for a more detailed overview of the subject.

Let $G$ be a finite (directed) graph on the vertex set $V(G)$ and the edge set $E(G)$. We assume that every vertex of $G$ is always operational, but the edge $e$ is operational with probability $p_e$. Let $\varphi$ be a binary function from the collection of all subsets $A$ of $E(G)$ to $\{0, 1\}$ such that for each subset $A$ with $\varphi(A) = 1$ and for each $e$ in $E(G)$, we have that $\varphi(A \cup \{e\}) = 1$. The pair $S = (E(G), \varphi)$ is called a coherent system. For $A \subseteq E(G)$, $\varphi(A)$ represents the state of the system; the system is working properly if $\varphi(A) = 1$, and it is failed if $\varphi(A) = 0$. For a system $(E(G), \varphi)$, a set $P \subseteq E(G)$ with $\varphi(P) = 1$ is called a path set, and a set $C \subseteq E(G)$ with $\varphi(E(G) \setminus C) = 0$ is called a cut set. The failure set $F$ of $S$ is the set of all failure subsets of $E(G)$, and the nonfailure set or operating set is the complement of $F$ containing all subsets $A$ of $E(G)$ with $\varphi(A) = 1$.

We note that each set $A$ can be regarded as a point $(a_1, a_2, \ldots, a_m)$ in $\{0, 1\}^{E(G)}$, and we can associate a squarefree monomial to $A$ by considering $A$ as its exponent vector. We will often abuse notation and use $A$ to denote its corresponding monomial. We denote $I_{G, \varphi}$ for the ideal generated by monomials corresponding to the operating subsets of $S$. The reliability of a system $S$ denoted by $R(S)$ is the probability of the nonfailure set $F^c$. The unreliability of $S$ denoted by $U(S)$ is the probability of the failure set $F$. Giglio and Wynn in [GW04] discuss the relation between the reliability of the system $(E(G), \varphi)$ and Hilbert series of its associated ideal $I_{G, \varphi}$. In fact, one can obtain $R(S)$ by evaluating the $h$-polynomial (the numerator of the $Z^{E(G)}$-graded Hilbert series) of $I_{G, \varphi}$ in $p_e$'s, i.e. substituting each variable $x_e$ with its corresponding probability $p_e$, see [GW04]. In particular the (multigraded) Betti numbers give the exact formula for the system reliability. The special networks such as $k$-out-of-$n$ have been studied in [GW04], and the general case was stated as an open problem. For some other systems $S$, a nonminimal free resolution of $I_{G, \varphi}$ have been studied to obtain an expression for the multigraded Hilbert function which can be truncated to obtain bounds for the reliability of the system (see e.g., [GW04, SzW09]). We note that if the resolution is minimal, then the bounds obtained by truncating the Hilbert function is much sharper than the bounds given by a nonminimal free resolution.

1.4. Outline and our results. The goal of this paper is threefold:

(1) To provide an overview of the state-of-the-art on acyclic orientations and divisors on graphs. See Proposition 3.6 and Theorem 3.10.

(2) To describe the invariants of cut ideals in terms of a polyhedral complex and combinatorics of $G$ (Theorem 5.11). Parallely to compute the invariants of SMT ideals in terms of the combinatorics of $G$, the minimal generating set of the higher syzygy modules, the minimal prime decomposition, Castelnuovo-Mumford regularity (Theorem 5.7 and Corollary 5.8).

(3) To investigate the relation between SMT ideals, cut ideals, and toppling ideals. Providing a comprehensive and combinatorial description of their syzygies allows us to read the reduced divisors as algebraic invariants.

We start by studying an equivalence classes of acyclic orientations introduced by Gioan [Gio07] and show how they are intimately related to the equivalence classes of divisors. This relation can be encoded precisely through a natural subclass of acyclic orientations which turns out to be a finer description for the set of reduced divisors of graph, see Definition 3.8 and Theorem 3.10. Then we step back to study these combinatorial objects from an algebraic point of view. We would like to read each such a combinatorial object as a representative of a generator of a syzygy module of a proper ideal in $\mathbb{S}_E$. In particular, we study the cut ideal and the path ideal associated to a directed or an undirected network. We give an explicit description of a minimal generating set for each higher syzygy module of these ideals. We show that in each case the given minimal generating set can be directly read from the combinatorics of the graph. Quite surprisingly, the Betti numbers of the ideals arising from a directed graph $G$ coincide with the Betti numbers of the ideals associated to
its corresponding undirected graph if both orientations of an (undirected) edge appears in $E(G)$. However, considering the orientations on the edges enables us to obtain finer descriptions for the syzygy elements. In §6.1 we show that each element in the minimal generating set of each syzygy module has a unique multidegree which is associated to a unique combinatorial object. We remark that this situation rarely happens, and the core idea of our results is to apply this fact to obtain a nice interpretation and algorithms for the Betti numbers, Hilbert function and so the system reliability formula.

The cut ideal can be thought of as an oriented variant of the monomial ideal $M^q_G$ coming from the divisor theory. For two distinguished vertices $q, t$ of $G$ (a source at $q$ and a target at $t$), we study the ideal $C^q_G$ in §5.1 whose associated system measures the non-connectivity of $q$ and $t$. We construct the polyhedral complex resolving the minimal free resolution of each such ideal, and we show that varying the choice of the target vertex $t$, and gluing all the constructed complexes together we obtain the graphic hyperplane arrangement studied extensively in the literature, see [GZ83, NPS02]. The basic idea is that if the subideals $C^q_G$ of the ideal $O^q_G$ with an associated polyhedral complex $D^{q,t}_{G}$ (supporting its minimal free resolution) have been chosen carefully, then the construction of $D^{q,t}_{G}$ can be realized as an iterative procedure where in each step a geometric complex is removed in a certain way from the graphic hyperplane arrangement, see Theorem 5.1.

In §5.1.1 we investigate the minimal free resolutions of SMT systems that can be recovered from the acyclic orientations of graph. In particular, we are interested in a family of resolutions whose bases elements in each higher syzygy module appear with multiplicity one. This implies that associated to each multidegree there is exactly one syzygy element. This fact together with the methods established in §5.1 for directed graphs, breaks the computational problem down into several parallel tasks. More precisely, we can list all acyclic orientations of $G$, and compute the minimal free resolutions of their associated ideals. After having all in hand, each such a resolution is encoded as a subcomplex in the resolution of the SMT ideal, see Theorem 5.6, Remark 5.17 and Example 5.19.

The strength of our method is that we only read the non-cancelling terms in $R(S)$. For example, the graph depicted in Figure 8 has 8 spanning trees; using the inclusion-exclusion principle, we have to check $2^8 - 1$ terms to compute the reliability formula, but in Example 5.14 we show that only 23 of these terms are non-cancelling which are encoded in the resolution of $R(S)$. We then step back and define two subideals $P^q_G$ and $P^q_G$ of the SMT ideals for given vertex $q$. These ideals arose in system reliability theory. See §5.1.3 and Example 5.22.

One immediate corollary of the combinatorial description of the syzygies is to give bijection between the set of divisors associated to oriented $k$-spanning trees, and $q$-reduced divisors of degree $k - 1$ with $D(q) = -1$ and the minimal generating set of the $k$-syzygy module of SMT ideals (see Theorem 5.10 and Theorem 5.6).

Quite surprisingly, many ideas from discrete potential theory on graphs, mostly coming from [FGK13], fit together nicely to give algorithms for constructing the generators of our ideals in §6.

2. Definitions and background

2.1. Graphs and divisors. Throughout this paper, a graph means a finite, connected, unweighted multigraph without loops. As usual, the set of vertices and edges of a graph $G$ are denoted by $V(G)$ and $E(G)$. A pointed graph $(G, q)$ is a graph together with a choice of a distinguished vertex $q \in V(G)$. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the induced subgraph of $G$ with the vertex set $S$; the edges of $G[S]$ are exactly the edges that appear in $G$ over the set $S$.

An element of $\text{Div}(G)$ is written as $D = \sum_{v \in V(G)} a_v(v)$ and is called a divisor on $G$. The coefficient $a_v$ in $D$ is also denoted by $D(v)$. A divisor $D$ is called effective if $D(v) \geq 0$ for all $v \in V(G)$. The set of effective divisors is denoted by $\text{Div}_+(G)$. We write $D \leq E$ if $E - D \in \text{Div}_+(G)$. For $D \in \text{Div}(G)$, let $\deg(D) = \sum_{v \in V(G)} D(v)$ be the degree of $D$. 


Since principal divisors defined in \( \mathbb{N} \) have degree zero, the map \( \deg : \text{Div}(G) \to \mathbb{Z} \) descends to a well-defined map \( \deg : \text{Pic}(G) \to \mathbb{Z} \). Two divisors \( D_1 \) and \( D_2 \) are called linearly equivalent if they become equal in \( \text{Pic}(G) \). In other words:

\[
D_1 \sim D_2 \iff \text{there exists a function } f \text{ with } D_2 = D_1 - \Delta(f)
\]

In this case we write \( D_1 \sim D_2 \). The linear system \( |D| \) of \( D \) is defined as the set of effective divisors that are linearly equivalent to \( D \).

Let \( \mathcal{E}(G) \) denote the set of oriented edges of \( G \); for each edge in \( E(G) \) there are two edges \( e \) and \( \bar{e} \) in \( \mathcal{E}(G) \). So we have \(|\mathcal{E}(G)| = 2m\). An element \( e \) of \( \mathcal{E}(G) \) is called an oriented edge, and \( \bar{e} \) is called the inverse edge. We have a map

\[
\mathcal{E}(G) \to V(G) \times V(G)
\]

\[
e \mapsto (e_+, e_-)
\]

sending an oriented edge \( e \) to its head (or its terminal vertex) \( e_+ \) and its tail (or its initial vertex) \( e_- \). Note that \( e_+ = e_- \) and \( \bar{e} = e_+ \).

An orientation of \( G \) is a choice of subset \( O \subseteq \mathcal{E}(G) \) such that \( \mathcal{E}(G) \) is the disjoint union of \( O \) and \( \bar{O} = \{ \bar{e} : e \in O \} \). An orientation is called acyclic if it contains no directed cycle. A partial orientation of \( G \) is a choice of subset \( P \subseteq \mathcal{E}(G) \) that is strictly contained in an orientation \( O \) of \( G \). A partial orientation is called acyclic if the induced orientation on the graph obtained by contracting all its components is acyclic.

Let \( O \) be an orientation of \( G \). A vertex \( q \) is called a source for \( O \) if \( q = e_- \) for every \( e \in O \) which is incident to \( q \). Let \( P \) be a partial orientation of \( G \), and let \( H \) be the associated connected component containing the vertex \( q \). Then \( H \) is called a source for \( P \), if \( H \) corresponds to a source in the graph obtained by contracting all components of \( P \). We define \( D_P \) to be the divisor associated to \( P \) with \( D_P(v) = \deg_P(v) - 1 \), where \( \deg_P(v) \) denotes the number of oriented edges directed to \( v \) in \( P \). Given disjoint nonempty subsets \( A, B \) of \( V(G) \) we define

\[
\mathcal{E}(A, B) = \{ e \in \mathcal{E}(G) : e_+ \in A, e_- \in B \}.
\]

To the ordered pair \( (A, B) \) we also assign the effective divisor

\[
D(A, B) = \sum_{v \in A} |\mathcal{E}(\{v\}, B)|(v).
\]

In other words, the support of \( D(A, B) \) is a subset of \( A \) and for \( v \in A \) the coefficient of \( (v) \) in \( D(A, B) \) is the number of edges between \( v \) and \( B \). For a nonempty subset \( A \) of \( V(G) \), \( \mathcal{E}(A, A^c) \) is called a cut of \( G \), and any proper subset \( C \) of \( \mathcal{E}(A, A^c) \) is called a partial cut of \( G \). A cut \( C = \mathcal{E}(A, A^c) \) is called connected if \( G[A] \) and \( G[A^c] \) are connected.

2.1.1. Toppling ideals. Let \( k \) be a field and let \( k[x] \) be the polynomial ring in the \( n \) variables \( \{ x_v : v \in V(G) \} \). Any effective divisor \( D \) gives rise to a monomial \( x^D = \prod_{v \in V(G)} x_v^{D(v)} \). Associated to every graph \( G \) there is a canonical ideal which encodes the linear equivalences of divisors on \( G \). This ideal is implicitly defined in Dhar’s seminal paper \cite{Dhar90}. The ideal was introduced in \cite{CRS02} to address computational questions in chip-firing dynamics using Gröbner bases.

\[
I_G = \langle x^{D_1} - x^{D_2} : D_1 \sim D_2 \text{ both effective divisors} \rangle.
\]

Consider a total ordering of the set of variables \( \{ x_v : v \in V(G) \} \) compatible with the distances of vertices from \( q \) in \( G \):

\[
(2.2) \quad \text{dist}(w, q) < \text{dist}(v, q) \Rightarrow x_w < x_v .
\]

Here, the distance between two vertices in a graph is the number of edges in a shortest path connecting them. The above ordering can be thought of as an ordering on vertices induced by running the breadth-first search algorithm starting at the root vertex \( q \). Then we denote \( M_G^O \) for the initial ideal of \( I_G \) with respect to the degree reverse lexicographic ordering on \( k[x] \) induced by the total ordering on the variables given in (2.2).
2.2. Ideals arising in the theory of system reliability. We fix a pointed graph \((G,q)\) with the edge set \(E(G)\), and the oriented edge set \(\mathbb{E}(G)\). Let \(k\) be a field and let \(R = k[\mathbf{x}]\) be the polynomial ring in the \(m\) variables \(\{x_e : e \in E(G)\}\), and \(S = k[\mathbf{y}]\) be the polynomial ring in the variables \(\{y_e : e \in \mathbb{E}(G)\}\).

2.2.1. SMT ideals. The ideal corresponding to the SMT system is the spanning tree ideal of \(G\). For each spanning tree \(T\) of \(G\), let \(\mathcal{O}_T\) denote the orientation of \(T\) with a unique source at \(q\) (i.e. the orientation obtained by orienting all paths away from \(q\)), see Figure 8. Any spanning tree \(T\) of \(G\) gives rise to two monomials

\[
y^T = \prod_{e \in \mathcal{O}(T)} y_e \quad \text{and} \quad x^T = \prod_{e \in \mathcal{E}(T)} x_e .
\]

The oriented spanning tree ideal of \(G\) is defined as

\[
\mathfrak{T}_G^q = \{y^T : T \text{ is a spanning tree of } G\} \subset S ,
\]

and the spanning tree ideal of \(G\) is defined as

\[
\mathfrak{T}_G = \{x^T : T \text{ is a spanning tree of } G\} \subset R .
\]

2.2.2. Path ideals. For two distinguished vertices \(q\) and \(t\) of \(G\), denote by \(R(q,t)\) the probability of successful communication between \(q\) and \(t\) which requires the examination of all (directed) paths in \(G\) from \(q\) to \(t\). Similarly, for a subset \(T\) of \(V(G)\), we denote \(R(q,T)\) for the probability of successful communication between \(q\) and every vertex \(t\) of \(T\). To each tree \(P\) of \(G\) with \(\{q\} \cup T \subseteq V(P), E(P) \subset \mathbb{E}(G),\) and \(E(P) \subset \mathbb{E}(G)\), we associate two monomials

\[
y^P = \prod_{e \in \mathcal{O}(P)} y_e \quad \text{and} \quad x^P = \prod_{e \in \mathcal{E}(P)} x_e ,
\]

where \(\mathcal{O}(P)\) denotes the unique orientation of \(P\) with a unique source at \(q\) such that \(\text{indeg}_{\mathcal{O}(P)}(t) > 0\) for all \(t \in T\). We denote \(\mathcal{P}^q_{q,T}\) and \(\mathcal{P}^q_{q,T}\) for the ideal generated by all the monomials \(y^P\) and \(x^P\), respectively. Note that for \(T = V(G)\backslash\{q\}\), this coincide with the (oriented) spanning tree ideal studied in [2.2.1].

2.2.3. Cut ideals. Here we explain the dual of SMT system in details. Each connected cut \(C = \mathbb{E}(A,A^c)\) of \(G\) gives rise to two monomials

\[
y^C = \prod_{e \in \mathcal{C}} y_e \quad \text{and} \quad x^C = \prod_{e \in \mathcal{C}} x_e .
\]

Here, for each cut \(C = \mathbb{E}(A,A^c)\), we assume that \(q \in A^c\). The oriented cut ideal \(\mathfrak{C}^q_C\), and the cut ideal \(\mathfrak{C}_C\) of \(G\) are respectively generated by all the monomials \(y^C\) and \(x^C\). Let \(t\) be a vertex in \(V(G)\backslash\{q\}\), and \(\mathcal{S}^q_{q,t}\) be the set containing all connected cuts \(\mathbb{E}(A,A^c)\) of \(G\), with \(t \in A\) and \(q \in A^c\). Then \(\mathfrak{C}^q_{q,t}\) and \(\mathfrak{C}_{q,t}\) denote the ideals generated by the monomials \(y^C\) and \(x^C\) corresponding to the cuts in \(\mathcal{S}^q_{q,t}\). The ideals \(\mathfrak{C}_C\) and \(\mathfrak{C}_{q,t}\) are called the mincut ideal of all-terminal, and 2-terminal networks, respectively (see, e.g., [8], [11] and references therein).

3. Partial orientations and q-reduced divisors

This section, while elementary, is one of the technical parts of the paper. The reader is invited to draw graphs and flowcharts to follow the proofs. The main result of this section is Theorem 3.10 and the ingredients needed for the proof are listed in Definition 3.4 and Proposition 3.6.

From now on we fix a graph \(G\) and we let \(n = |V(G)|\). For each cut \(\mathcal{C} = \mathbb{E}(A,A^c)\) of \(G\), we define its inverse as \(\overline{\mathcal{C}} = \{e : e \in \mathcal{C}\}\). The result of applying a cut-inverse operation on a partial orientation \(\mathcal{P}\) and the cut \(\mathbb{E}(A,A^c)\), is the (partial) orientation \(\mathcal{P}'\) which only inverses the edges of \(\mathbb{E}(A,A^c)\), but preserves the other edges of \(\mathcal{P}\) unchanged. Thus the only difference between \(\mathcal{P}\) and \(\mathcal{P}'\) is on the edges of \(\mathbb{E}(A,A^c)\). In other words, \(\mathcal{P}' = \mathbb{E}(A^c,A) \cup (\mathcal{P} \backslash \mathbb{E}(A,A^c))\). Similarly one
can define the inverse of a cycle in a partial orientation \( P \). The operation inverting a cut in \( P \) is called a cut reversal, and the operation inverting a cycle is called a cycle reversal (see [Gio07] for more details).

**Definition 3.1.** Two (partial) orientations \( P \) and \( P' \) are called equivalent in cut-cycle reversal system if there exists a sequence \( P = P_1, P_2, \ldots, P_k = P' \) of orientations such that for each \( i \), \( P_{i+1} \) is obtained from \( P_i \) by inverting a cut or a cycle in \( P_i \) (see Figure 1).

**Figure 1.** \( P_1, P_2, P_3, P_4 \): where \( P_2 \) is obtained by a cut-inverse from \( P_1 \), \( P_3 \) by a cut-inverse from \( P_2 \), and \( P_4 \) by a cycle-inverse from \( P_3 \)

The following special partial orientations of \( G \) arise naturally in our setting.

**Remark 3.2.** It is easy to find two different (partial) orientations having identical associated divisors. For example, let \( e \in P \) and \( e' \notin P \) with \( e_+ = e'_- \). Then for \( P' = \{e'\} \cup P \setminus \{e\} \) we have that \( D_P = D_{P'} \). Moreover, \( P' \cap E(A^c, A) \subseteq P \cap E(A^c, A) \) for any \( A \), with \( e', e'_- \in A^c \) and \( e_+ \in A \). We write \( P \sim_1 P' \) if there is a sequence of moves, taking \( P \) to \( P' \) by exchanging pair of edges in each step, as explained.

So we slightly modify Definition 3.1 as follows:

\[
(3.1) \quad P \sim P' \iff \text{there exists a sequence } P = P_1, P_2, \ldots, P_k = P', \text{ where for each } i, P_{i+1} \text{ is a (partial) orientation such that } P_{i+1} \sim_1 P_i, \text{ or it is obtained from } P_i \text{ by inverting a cut, or a cycle.}
\]

**Example 3.3.** Let \( G \) be the following graph on the vertices \( v_1, v_2, \ldots, v_5 \). We fix \( A = \{v_4\} \). We start from \( P \), and in each step, we substitute the red edge \( e \) with the blue edge directed to \( e_+ \), to obtain a new orientation. Then, as we see, their associated divisors coincide, and \( |P_1 \cap E(A^c, A)| = 2, |P_2 \cap E(A^c, A)| = 1 \) and \( |P_3 \cap E(A^c, A)| = 0 \).

**Figure 2.** \( P, P_1, P_2, \) and \( P_3 \)

This example motivates the following definition.

**Definition 3.4.** Fix a subset \( A \subseteq V(G) \) and the orientation \( P \) of \( G \). The set of all (partial) orientations \( P' \) of \( G \) with \( D_{P'} = D_P \), will be denoted by \( S(P) \). Let \( \text{E}(A, P) \) denote the ordering on the elements of \( S(P) \), given by reverse inclusion:

\[
P'' \preceq P' \iff P' \cap E(A^c, A) \subseteq P'' \cap E(A^c, A).
\]
We would like to find such partial orientations \( \mathcal{P}' \), when \( \mathcal{P}' \cap \mathbb{E}(A^c, A) \) has the smallest possible size, and among them, we consider those with the maximum number of oriented edges from \( A^c \) to \( A \). We fix, once and for all, a total ordering extending \( \preceq_{(A, \mathcal{P})} \). By a slight abuse of notation, \( \preceq_A \) will be used to denote this total ordering extension. In particular \( \preceq_A \) will denote the associated total order. We denote the maximal element of \( S(\mathcal{P}) \) (with respect to \( \preceq_A \)) with \( \mathcal{P}_A \).

**Lemma 3.5.** Fix a subset \( A \subset V(G) \) and a partial orientation \( \mathcal{P} \) of \( G \). Then there exist a cut \( C \) and a partial orientation \( \mathcal{P}' \) such that \( C \subset \mathcal{P}' \) and \( \mathcal{P}' \sim_1 \mathcal{P}_A \). If \( |\mathbb{E}(A^c, A) \cap \mathcal{P}_A| > 0 \), then we can choose \( C \) with \( |\mathbb{E}(A^c, A) \cap C| > 0 \).

**Proof.** We consider two different cases:

(i) \( |\mathbb{E}(A^c, A) \cap \mathcal{P}_A| > 0 \): Assume that \( e' \in \mathbb{E}(A^c, A) \). Then consider the set
\[
C = \{e'\} \cup \{e_+ : e_+ \in A^c \text{ and there is a path } e_+ = e_1, e_2, \ldots, e_n \in \mathcal{P}_A \},
\]
and let \( C = \mathbb{E}(C, C^c) \). It is clear that \( |\mathbb{E}(C, C^c)| = 0 \), since otherwise the vertex \( e_+ \) corresponding to \( e \in \mathbb{E}(C, C^c) \) would be in \( C \), as well. By contrzary assume that \( C \subseteq \mathbb{E}(C, C^c) \), say \( e \in \mathbb{E}(C, C^c) \). Consider the path \( e_1, e_2, \ldots, e_k \) with \( (e_k)_+ = e_+ \). Then the partial orientation \( \mathcal{P}' = \{e\} \cup (\mathcal{P}_A \setminus \{e_k\}) \) belongs to \( S(\mathcal{P}_A) \). By continuing the same procedure, we keep moving an unoriented edge closer to \( e_1 \) so that we can unorient the edge \( e_1 \) and add another oriented edge in \( G[C^c] \). This way the associated divisor will not be changed, however for the new orientation \( \mathcal{P}' \) we have \( \mathcal{P}' \cap \mathbb{E}(A^c, A) = (\mathcal{P}_A \cap \mathbb{E}(A^c, A)) \setminus \{e_1\} \), a contradiction (to the choice of \( \mathcal{P}_A \)).

(ii) \( |\mathbb{E}(A^c, A) \cap \mathcal{P}_A| = 0 \): First, note that each edge between \( A \) and \( A^c \), either is undirected, or it is directed from \( A^c \) to \( A \). If \( \mathbb{E}(A, A^c) \subset \mathcal{P}_A \), then \( \mathbb{E}(A, A^c) \) is already a cut. Otherwise, there is an unoriented edge between \( A \) and \( A^c \). We may use this edge, and apply the same argument as proof of (i) (by moving this undirected edge inside a set, and adding a directed edge to \( \mathcal{P}_A \) to get a contradiction as in (i)).

Our main goal in the following proposition is to show that two equivalence classes \([3.1]\) and \([2.1]\) are intimately related and in fact they are equal, i.e., two (partial) orientations are equivalent in the (modified) cut-cycle reversal system in the sense of \([3.1]\), if and only if their corresponding divisors are linearly equivalent.

**Proposition 3.6.** \( \mathcal{P} \sim \mathcal{P}' \Leftrightarrow D_P \sim D_{P'} \).

**Proof.** Let \( \mathcal{P} \) be a partial orientation. Exchanging a pair of edges as Remark \([3.2]\) and also inverting a cycle in \( \mathcal{P} \), keep the associated divisor unchanged. On the other hand, by inverting a cut in \( \mathcal{P} \) we obtain an orientation whose associated divisor is linearly equivalent to \( D_P \) as in \([2.1]\). Thus \( \mathcal{P} \sim \mathcal{P}' \) implies that \( D_P \sim D_{P'} \).

Now assume that \( D_P \sim D_{P'} \). Then by \([2.1]\) there exists (an integer valued) function \( f \) with \( D_{P'} = D_P - \Delta(f) \). The proof is by induction on \( |\text{supp}(f)| \). First assume that \( D_{P'} = D_P \). Then we show that one can obtain \( \mathcal{P}' \) from \( \mathcal{P} \), only by performing cycle-inverse, and exchanging pair of oriented edges as Remark \([3.2]\). The proof is by reverse induction on \( |\mathcal{P} \cap \mathcal{P}'| \). Assume that \( e \in \mathcal{P} \setminus \mathcal{P}' \). Since \( \text{indeg}_P(e) = \text{indeg}_P(e') \), there exists an edge \( e' \in \mathcal{P} \setminus \mathcal{P}' \) with \( (e_1)_+ = e'_+ \). Now we consider two cases, and in each case we find a third orientation \( \mathcal{P}'' \) such that \( \mathcal{P} \cap \mathcal{P}' \) is a proper subset of \( \mathcal{P} \cap \mathcal{P}'' \) and \( \mathcal{P}' \cap \mathcal{P}'' \). Therefore, the result follows by induction hypothesis.

Case 1. \( \bar{e} \notin \mathcal{P}' \) or \( \bar{e}' \notin \mathcal{P} \): If \( \bar{e} \notin \mathcal{P}' \), then we let \( \mathcal{P}'' \) be the orientation obtained from \( \mathcal{P} \) by replacing \( e \) with \( e' \) as Remark \([3.2]\). If \( \bar{e}' \notin \mathcal{P} \), then we perform the similar operation (replacing \( e \) with \( e' \)) on \( \mathcal{P}' \).

Case 2. \( \bar{e} \in \mathcal{P}' \) and \( \bar{e}' \in \mathcal{P} \): Since \( \text{indeg}_P(e'_-) = \text{indeg}_P(e_-) \), there exists \( e_1 \in \mathcal{P} \setminus \mathcal{P}' \) with \( (e_1)_+ = e'_- \), such that either \( e_1 \) is undirected in \( \mathcal{P} \), or its inverse is in \( \mathcal{P} \). In the first case, we are in Case(1). Otherwise, by continuing the same argument, we keep moving along a path \( \bar{e}, e', e_1, \ldots \) in \( \mathcal{P}' \) and along its inverse in \( \mathcal{P} \) which will be terminated at some point. So this way, we can create an oriented cycle. Now inverting this cycle in \( \mathcal{P} \) we obtain \( \mathcal{P}'' \) which is either equal to \( \mathcal{P}' \), or it has more intersection with both \( \mathcal{P} \) and \( \mathcal{P}' \), as desired.

8
Now assume that $|\text{supp}(f)| > 0$ and let $A = \text{supp}(f)$. By Definition 3.3 we may assume that $\mathcal{P} = \mathcal{P}_A$. Note that $D_{\mathcal{P}} = D_{\mathcal{P}_A}$. If $|E(A^+, A) \cap \mathcal{P}| > 0$, by Lemma 3.5 there exists a cut $C = E(C, C^c) \subset \mathcal{P}$ with $|E(A^+, A) \cap C| > 0$. This implies that at least $|E(A^+, A) \cap C|$ edges are directed from $A$ to $C$, and for each $e$ in $E(A^+, A) \cap C$, the indegree of $e_+$ in $\mathcal{P}'$ is greater than its indegree in $\mathcal{P}$. Therefore $D_{\mathcal{P}'}(e_+) > D_{\mathcal{P}}(e_+)$, a contradiction. Thus we have $|E(A^+, A) \cap \mathcal{P}| = 0$. If $E(A, A^c) \subset \mathcal{P}$, i.e. $E(A, A^c)$ is a cut, then we can inverse this cut to obtain $\mathcal{P}'$. By applying the induction assumption on the support of the function taking $D_{\mathcal{P}'}$ to $D_{\mathcal{P}}$, we conclude that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ and to $\mathcal{P}'$, as desired. Otherwise, applying Lemma 3.5 we obtain a cut $C' \subset \mathcal{P}$, and we process the cut-inverse corresponding to $C'$ in order to use the induction hypothesis. \hfill \Box

**Remark 3.7.** Note that an acyclic partial orientation can not be equivalent to a partial orientation without any source. Once we know that $O$ does not have any source it follows that any vertex belongs to a directed cycle. On the other hand, even if we can find a cut $E(A, A^c) \subset O$ to perform a cut-inverse, the vertices of $A$ and $A^c$ still belong to a cycle in the obtained orientation.

The following special class of acyclic (partial) orientations of $G$ arises naturally in our setting, where $g$ denotes the genus of the graph, i.e. $g = |E(G)| - |V(G)| + 1$.

**Definition 3.8.** Fix a pointed graph $(G, q)$ with the (oriented) edge set $E(G)$. For each integer $0 \leq k \leq g$, an oriented $k$-spanning tree $\mathcal{T}$ of $(G, q)$ is a connected subgraph of $G$ on $V(G)$ with a unique source at $q$ such that

- $E(\mathcal{T}) \subset E(G)$ with $|E(\mathcal{T})| = n - 1 + k$,
- $\mathcal{T}$ is acyclic.

The set of all oriented $k$-spanning trees of $(G, q)$ will be denoted by $\mathcal{S}_k(G, q)$. The set $\mathcal{S}_0(G, q)$ corresponds to the set $\{O_T : T$ is a spanning tree of $G\}$.

**Remark 3.9.** Note that out assumption that $T$ has a unique source at $q$ implies that for each vertex $v \in V(G) \setminus \{q\}$, there exists an oriented path from $q$ to $v$ in $\mathcal{T}$.

### 3.1. Reduced divisors and oriented $k$-spanning trees

For a fixed vertex $q$, a divisor $D$ is called $q$-reduced if:

1. $D(v) \geq 0$ for all $v \in V(G) \setminus \{q\}$.
2. For every non-empty subset $A \subset V(G) \setminus \{q\}$, there exists $v \in A$ with $D(v) < |E(A, \{v\})|$. 

Note that for undirected graphs we have that $|E(A, \{v\})| = |E(A, \{v\})|$. The $q$-reduced divisors play an important role in divisor theory, because each divisor has a unique equivalent divisor among $q$-reduced divisors (see, e.g., [Dhar90, CLB03, CRS02, DN07, BS13]). Using Dhar’s burning algorithm, one can start from a source $q$, and check the availability of a vertex $v_1 \in V(G) \setminus \{q\}$ with $D(v_1) < |E(\{q, \{v_1\})|$. If there exists such a vertex, then the next step will check the same procedure to find a vertex $v_2$ with $D(v_2) < |E(\{q, v_1, \{v_2\})|$. The divisor is $q$-reduced if and only if by continuing this procedure we can enlarge the set containing $q$ to $V(G)$, (see [BS13, Alg. 4]). We may assume that $D(q) = -1$ for all $q$-reduced divisors.

We end this section by the following combinatorial result which shows how the oriented $k$-spanning trees are naturally arisen in the theory of (reduced) divisors. Later in [6.11] we study oriented $k$-spanning trees, and so the reduced divisors, from an algebraic point of view.

**Theorem 3.10.** For each $\mathcal{T} \in \mathcal{S}_k(G, q)$, its corresponding divisor $D_{\mathcal{T}}$ is $q$-reduced. Associated to every $q$-reduced divisor $D$, there exists an oriented $k$-spanning tree $\mathcal{T}$ with $D_{\mathcal{T}} \sim D$. 

1Gioan in [Gio07] defines a $q$-connected orientation as a totally orientation of $G$ in which every vertex is reachable from $q$ by a directed path. We use the notation $\mathfrak{T}_k$ and the name oriented $k$-spanning tree in order to emphasize that $\mathfrak{T}_k$ is an acyclic subgraph of $G$ containing a rooted spanning tree with $k$ extra edges.
Proof. Let $T$ be an oriented $k$-spanning tree. We show that $D_T$ is $q$-reduced. Note that $D(q) = -1$. Note that $V(G)\backslash \{q\}$ has at least one vertex, say $v_1$, with indeg$_{V(G)\backslash \{q\}}(v_1) = 0$, otherwise we have an oriented cycle in $G$. Thus indeg$_T(v_1) = |E(\{v_1\}, \{q\})| = |E(\{v_1\}, \{q\})|$ and $|E(\{q, v_1\}, \{q, v_1\})^c| = 0$. Thus indeg($v_1$) = 1 = $D(v_1) < |E(\{q\}, \{v_1\})|$. By continuing the same procedure we can enlarge $\{q, v_1\}$ to $V(G)$, as needed in Dhar’s algorithm which implies that $D_T$ is $q$-reduced.

Let $D$ be a $q$-reduced divisor of degree $k-1$ (with $D(q) = -1$). First we find a partial orientation $P$ with $D_P = D$. We may assume $D(v) < \text{deg}(v)$ for each $v$. Let $P$ be an arbitrary orientation with $D_P < D$. Assume that indeg$_P(v) < D(v)$. If there exists an unoriented edge adjacent to $v$, then we orient this edge toward $v$ to increase the indegree of $v$, and get closer to the desired orientation. Otherwise $|E(\{v\}^c, \{v\}) \cap P| > 0$. So by applying Lemma 3.5 we obtain a cut $C = E(C, C^c)$ with $v \in C$. Now by inverting this cut, we increase the indegree of $v$ and the obtained partial orientation is closer to $D$. Note that by the same argument used in proof Lemma 3.5 in case that the indegree of $v$ is greater than $D(v)$, we can use an unoriented edge in $P$, and keep exchanging pairs of edges as in Remark 3.2 to obtain an unoriented edge adjacent to $v$. Then that we can unorient this edge to decrease the indegree of $v$. By continuing the same argument, we keep decreasing the number $D(v) - D_P(v)$ for each vertex $v$ and we get the desired orientation.

Now we want to show that $P \in \mathcal{G}_k(G, q)$. Our assumption that $D(q) = -1$ implies that $q$ is a source. We may assume that $P = P(q)$ as in Definition 3.4. The idea is to start form $q$, and add all other vertices of $G$ to $\{q\}$, step-by-step, by applying Dhar’s algorithm so that all oriented edges are directed from the set containing $q$ to its complement. Since $D$ is $q$-reduced, there exists $v_1 \in V(G)\backslash \{q\}$ such that no edge is directed to $v_1$ in $V(G)\backslash \{q\}$. Therefore, $|E(\{q, v_1\}^c, \{q, v_1\})| = 0$. If $E(\{q, v_1\}, \{q, v_1\})^c \subset P$, then $E(\{q, v_1\}, \{q, v_1\})^c$ is a cut, and we perform a cut-inverse to increase the number of vertices with the property that there exists a path from $q$ to them. Note that inverting cuts will never produce a cycle. Otherwise, $E(\{q, v_1\}, \{q, v_1\})\backslash P$ contains at least an edge $e$. Our assumption on $P(q)$ implies that none of the vertices of $V(G)\backslash \{q, v_1\}$ is oriented to $v_2 = e^-$. Then we use the same argument for $\{q, v_1, v_2\}$, and by continuing the same procedure, we keep enlarging the set containing $q$ to $V(G)$.

Remark 3.11. Note that in the proof of Theorem 3.10 corresponding to each divisor $D$ with $0 \leq \text{deg}(D) < q$, we have found a (partial) orientation $P$ with $D \leq D_P$. We are mostly interested in having the equivalency $D \sim D_P$. We have shown that if $D$ is $q$-reduced, then $P$ is indeed an oriented $\text{deg}(D)$-spanning tree with $D \sim D_P$. It is also clear that $\text{deg}(D) = \text{deg}(D_P)$ implies the equality $D = D_P$. In particular the equality holds for $\text{deg}(D) = g - 1$, since $\text{deg}(D_P) \leq g - 1$ (see [ABKSE13] Thm. 4.7 for the same statement).

4. Graphic matroid ideals

In this section, we first recall some definitions related to ideals arising in matroid theory, and also related to $\mathcal{M}_G^0$ from [21]. An important feature that we want to emphasize in this section is the relation between the ideals associated to an undirected network, their corresponding oriented versions, and their Alexander duals. The main results needed in [15] and [16] are Proposition 4.3 and Corollary 4.9. Other results are used as ingredients to establish these results.

4.1. Oriented matroid ideals. Here, we quickly recall some basic notions from oriented matroid theory. Our main goal is to fix our notation. A secondary goal is to keep the paper self-contained. Most of the material here is well-known and we refer to [5] and [16] for proofs and more details.

An oriented hyperplane arrangement is a real hyperplane arrangement along with a choice of a positive side for each hyperplane. Equivalently, one may fix a set of linear forms vanishing on hyperplanes to fix the orientation. For each (oriented) hyperplane arrangement $A = \{H_1, \ldots, H_m\}$ with hyperplanes $H_j = \{v \in \mathbb{R}^{n-1} : h_j(v) = c_j\}$ living in $\mathbb{R}^{n-1}$, one can consider a central
hyperplane arrangement \( \mathcal{C} = \{ \mathcal{H}_1, \ldots, \mathcal{H}_m, \mathcal{H}_g \} \) in \( \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \) such that

\[
\mathcal{H}_j = \{ (v, w) \in \mathbb{R}^{n-1} \times \mathbb{R} : h_j(v) = c_j w \} \quad \text{and} \quad \mathcal{H}_g = \{ (v, w) \in \mathbb{R}^{n-1} \times \mathbb{R} : w = 0 \}.
\]

Now if we restrict ourselves to the positive side of \( \mathcal{H}_g \), more precisely, consider the restriction of \( \mathcal{C} \) to the hyperplane \( \{ (v, w) \in \mathbb{R}^{n-1} \times \mathbb{R} : w = 1 \} \) we obtain an affine hyperplane arrangement. The corresponding (affine) oriented matroid \( \mathcal{M} \) is a matroid on the ground set \( E = \{ 1, \ldots, m, g \} \) such that the set of its covectors \( \mathcal{L} \) is the image of the map

\[
\mathbb{R}^n \to \{-, 0, +\}^E \quad (v, w) \mapsto (\text{sign}(h_1(v) - c_1 w), \ldots, \text{sign}(h_m(v) - c_m w), \text{sign}(w)).
\]

Let \( R = k[x] \) be the polynomial ring in \( m \) variables \( \{ x_i : H_i \in \mathcal{A} \} \) and \( S = k[x, y] \) be the polynomial ring in \( 2m \) variables \( \{ x_i, y_i : H_i \in \mathcal{A} \} \). For any (affine) oriented hyperplane arrangement one can define (see \cite{NPS02}) the associated oriented matroid ideal: let \( \{ h_j \} \) be \( m \) nonzero linear forms defining the hyperplane arrangement \( A \) with hyperplanes \( \mathcal{H}_j = \{ v \in \mathbb{R}^{n-1} : h_j(v) = c_j \} \) in \( \mathbb{R}^{n-1} \). The oriented matroid ideal associated to \( A \) is the ideal in \( 2m \) variables of the form:

\[
\mathcal{O} = \langle m(v) : v \in \mathbb{R}^{n-1} \rangle \subset K[x, y],
\]

where for each \( v \in \mathbb{R}^{n-1} \)

\[
m(v) = \prod_{h_i(v) > c_i} x_i \prod_{h_i(v) < c_i} y_i,
\]

the multiplication being over all \( i = 1, \ldots, m \). Note that any two points in the relative interior of a cell will give raise to the same monomial. Moreover, the hyperplanes \( \mathcal{H}_1, \ldots, \mathcal{H}_m \) partition \( \mathbb{R}^{n-1} \) into relatively open convex polyhedra called the cells of the corresponding arrangement. The cells of dimension zero are called vertices. A cell is called a bounded cell if it is bounded as a subset of \( \mathbb{R}^{n-1} \), and we denote \( \mathcal{B}_A \) for the set consisting of all bounded cells of \( \mathcal{A} \) which is a regular CW-complex (see \cite{NPS02} for more details).

There is a canonical surjective \( k \)-algebra homomorphism \( \phi : S \to R \) defined by sending \( x_i \) and \( y_i \) to \( x_i \) for all \( i \). The kernel of this map is precisely the ideal generated by \( \{ x_1 - y_1, \ldots, x_m - y_m \} \), which we denote by \( a \). The induced isomorphism \( \phi : S/a \overset{\sim}{\to} R \) is the algebraic indegree map, and it relates the ideals \( \mathcal{O} \) to the ideal \( \mathcal{O}' = \langle m(v) : v \in \mathbb{R}^{n-1} \rangle \subset R \).

**Proposition 4.1.** Let \( V \) be the set consisting of the vertices of the bounded complex \( \mathcal{B}_A \), and for \( v \in V \), let \( \mathcal{P}_v = \{ x_i : h_i(v) \neq c_i \} \). Then the minimal prime decomposition of the Alexander dual of the ideal \( \mathcal{O} \) is \( \mathcal{O}' = \bigcap_{v \in V} \mathcal{P}_v \).

**Proof.** First note that from the Alexander duality definition, the minimal prime decomposition of \( \mathcal{O}' \) is \( \mathcal{O}' = \bigcap_{v \in V} \mathcal{P}_v \), where \( \mathcal{P}_v = \{ x_i : h_i(v) > c_i \} + \{ y_j : h_j(v) < c_j \} \). Also note that for any \( i \) and \( \mathcal{O}'_i = \mathcal{O}' \otimes S/(x_1 - y_1, \ldots, x_i - y_i) \), the minimal primes of \( \mathcal{O}'_i \) are obtained from the minimal primes of \( \mathcal{O}' \) by specializing the variable \( y_\ell \) to \( x_i \) for all \( \ell \leq i \). Let \( \mathcal{P}_v \) denote the ideal obtained from \( \mathcal{P}_v \) by identifying the variables \( x_\ell \) and \( y_\ell \) for each \( \ell \leq i \). It is easy to see that \( \mathcal{O}' \subseteq \bigcap \mathcal{P}_v \).

Assume that \( m \) is a monomial in \( \bigcap \mathcal{P}_v \). We want to show that \( m \in \mathcal{O}' \). Let \( x_i \in \text{supp}(m) \) and \( m = x_i m' \) for some \( m' \). Thus \( x_i y_i m' \in \mathcal{P}_v \), and so \( x_i m' \) or \( y_i m' \) belongs to \( \mathcal{P}_v \), since each prime ideal contains at most one of the variables \( x_i \) or \( y_\ell \). However for each \( v \) the images of the both monomials \( x_i m' \) and \( y_i m' \) in \( \mathcal{P}_v \) are equal to \( m \). Therefore \( m \in \mathcal{O}' \). If \( x_i \notin \text{supp}(m) \), then \( m \in \mathcal{P}_v \) shows that \( m \in \mathcal{P}_v \). This implies that \( m \) belongs to \( \bigcap \mathcal{P}_v \) and so to \( \mathcal{O}' \). Thus after identifying \( x_i \) and \( y_\ell \) we obtain again \( m \) (since \( x_i, y_\ell \notin \text{supp}(m) \)) which belongs to \( \mathcal{O}' \). \( \square \)

**4.2. Stanley-Reisner ideals.** Consider the polynomial ring \( k[z] \) (over the field \( k \)) in variables \( z = \{ z_1, \ldots, z_r \} \). Given an abstract simplicial complex \( \Sigma \), the squarefree monomial ideal in \( k[z] \) defined as \( I_\Sigma = \langle z^\tau : \tau \notin \Sigma \rangle \subset K[z] \) is called the *Stanley-Reisner ideal* of \( \Sigma \). The *Stanley-Reisner ring* (or *face ring*) \( k[\Sigma] \) is, by definition, \( k[z]/I_\Sigma \). In fact, this gives a bijective correspondence between squarefree monomial ideals inside \( k[z] \) and abstract simplicial complexes on the vertices \( \{ z_1, \ldots, z_r \} \) (see, e.g., \cite{MS05} Chapter 1). The simplicial complex \( \Sigma \) is called Cohen-Macaulay if \( k[\Sigma] \) is Cohen-Macaulay. A (pure) “shellable” simplicial complex is Cohen-Macaulay (see, e.g., \cite{Sta08}).
Chapter III] or [MS05, Chapter 13]). In general, \( \text{dim}(k[\Sigma]) \) is equal to the maximal cardinality of the faces of \( \Sigma \) (see, e.g., [Sta90, p.53]).

The oriented matroid ideal \( \mathcal{O} \subset S \) is a squarefree monomial ideal. Let \( \Delta_o \) denote its associated simplicial complex on \( 2m \) vertices \( \{x_1, \ldots, x_m, y_1, \ldots, y_m\} \). Similarly, let \( \Delta_o \) denote its associated simplicial complex on \( m \) vertices \( \{x_1, \ldots, x_m\} \).

**Remark 4.2.** The simplicial complexes \( \Delta_o \) and \( \Delta_o \) are shellable, and so their corresponding Stanley-Reisner rings are Cohen-Macaulay of dimension \( 2m - r \), where \( r = \text{rank}(M\setminus G) \). Moreover, the \( \mathbb{Z} \)-graded (and the multigraded) Betti numbers of \( \mathcal{O} \) and \( \mathcal{O} \) coincide (see [MS13, Thm. 10.3]).

**Proposition 4.3.**

(i) The set \( \{x_1 - y_1, \ldots, x_m - y_m\} \) forms a regular sequence for \( S/\mathcal{O}^\vee \).

(ii) The \( \mathbb{Z} \)-graded (and multigraded) Betti numbers of \( \mathcal{O}^\vee \) and \( \mathcal{O}^\vee \) coincide.

**Proof.** (i) By contrary assume that \( x_1 - y_1 \) is a zerodivisor element modulo \( \mathcal{O}^\vee \). Then there exists \( \nu \) such that \( x_1 - y_1 \), and so \( x_1, y_1 \) belongs to \( P_\nu \) which is a contradiction, since at most one of the variables \( x_1, y_1 \) belongs to \( P_\nu \).

Now we show that \( x_2 - y_2 \) is nonzerodivisor modulo \( \mathcal{O}^\vee \). Note that by Proposition 4.1 the prime components of \( \mathcal{O}^\vee \) are obtained by identification \( x_1 = y_1 \) in prime components \( \mathcal{O}^\vee \). If \( x_2 - y_2 \) is a zerodivisor element modulo \( \mathcal{O}^\vee \), then there exists \( \nu \) such that \( x_2, y_2 \) belong to \( P_\nu \). Thus \( x_2, y_2 \) belong to a prime component of \( \mathcal{O}^\vee \), and so to \( P_\nu \). This contradicts by the fact that each prime component of \( \mathcal{O}^\vee \) contains at most one of the variables \( x_2 \) or \( y_2 \).

Then continuing the same argument, we show that \( x_1 - y_i \) is a nonzerodivisor modulo \( \mathcal{O}^\vee \) for all \( i \).

(ii) The result follows by [Eis95, Lem. 3.15] (see also [BH93, Prop. 1.15]). \( \square \)

**Remark 4.4.**

(i) The ideals \( \mathcal{O} = I_{\Delta_o} \) and \( \mathcal{O} = I_{\Delta_o} \) (by Remark 1.2(i)) are both Cohen-Macaulay, so by [ER98, Thm 3] the minimal free resolutions of the ideals \( \mathcal{O}^\vee \) and \( \mathcal{O}^\vee \) are “linear”.

(ii) By Proposition 4.3(ii) \( \text{pd}(\mathcal{O}^\vee) = \text{pd}(\mathcal{O}^\vee) \) and \( \text{reg}(\mathcal{O}^\vee) = \text{reg}(\mathcal{O}^\vee) \). Thus by Auslander-Buchsbaum formula we obtain that \( \text{depth}(\mathcal{O}^\vee) = \text{depth}(\mathcal{O}^\vee) - m \).

4.3. **Graphic hyperplane arrangements.** In this section we introduce graphic hyperplane arrangements and their associated ideals. We recall some basic definitions and constructions, and we refer to the book [Sta04] and papers [GZ98, NPS02, MS13] for proofs and more details. We consider the pointed graph \((G, q)\) on the vertex set \([n]\) with the edge set \(E(G)\).

Following [GZ98], we define the **graphic hyperplane arrangement** as follows. This arrangement lives in the Euclidean space \(C^0(G, \mathbb{R})\), i.e. the vector space of all real-valued functions on \(V(G)\) endowed with the bilinear form

\[
\langle f_1, f_2 \rangle = \sum_{v \in V(G)} f_1(v)f_2(v).
\]

Let \( C^1(G, \mathbb{R}) \) be the vector space of all real-valued functions on \( E(G) \), and let \( \partial^* : C^0(G, \mathbb{R}) \to C^1(G, \mathbb{R}) \) denote the usual coboundary map.

For each edge \( e \in E(G) \) let \( \mathcal{H}_e \subset C^0(G, \mathbb{R}) \) denote the hyperplane

\[
\mathcal{H}_e = \{ f \in C^0(G, \mathbb{R}) : (\partial^* f)(e) = 0 \}.
\]

Consider the arrangement

\[
\mathcal{H} = \{ \mathcal{H}_e : e \in E(G) \}
\]

in \( C^0(G, \mathbb{R}) \). Since \( G \) is connected, we know \( \bigcap_{e \in E(G)} \mathcal{H}_e \) is the 1-dimensional space of constant functions on \( V(G) \), which is the same as the kernel of \( d \). We define the **graphic arrangement** corresponding to \( G \), denoted by \( \mathcal{H}_G \), to be the restriction of \( \mathcal{H}' \) to the hyperplane

\[
(Ker(d))^\perp = \{ f \in C^0(G, \mathbb{R}) : \sum_{v \in V(G)} f(v) = 0 \}.
\]
To define the regions of $H$ with the real number $v$ (see, e.g., [GZ83, Lem. 7.1 and Lem. 7.2]). In particular, the connected cuts of $H$ to the lowest dimensional regions of $H$. Given any function $f \in C^0(G, \mathbb{R})$ one can label each vertex $v$ with the real number $f(v)$. In this way we obtain an acyclic partial orientation of $G$ by directing $v$ toward $u$ if $f(u) < f(v)$. Recall this means we have an acyclic orientation on the graph $G/f$ obtained by contracting all unoriented edges (i.e. all edges $\{u, v\}$ with $f(u) = f(v)$).

We are mainly interested in acyclic orientations of $G$ with a unique source at $q \in V(G)$. For this purpose, we define

$$\mathcal{H}^q = \{ f \in C^0(G, \mathbb{R}) : f(q) = -1 \}.$$  

The restriction of the arrangement $\mathcal{H}_G$ to $\mathcal{H}^q$ will be denoted by $\mathcal{H}_G^q$. We denote the bounded complex (i.e. the polyhedral complex consisting of bounded cells) of $\mathcal{H}_G^q$ by $\mathcal{E}_G^q$. By 4.1, the restriction of $\mathcal{H}_G$ to $\mathcal{H}^q$ coincides with the restriction of $\mathcal{H}_G$ to

$$(\mathcal{H}^q)' = \{ f \in C^0(G, \mathbb{R}) : \sum_{v \neq q} f(v) = 1 \}.$$  

The regions of $\mathcal{E}_G^q$ are corresponding to acyclic orientations with a unique source at $q$ (see e.g., [GZ83, Theorem 7.3]). Fixing an orientation $\mathcal{O}$ of the graph $G$ will fix the linear forms $(df)(e) = f(e_+) - f(e_-)$ for $e \in \mathcal{O}$ and gives an orientation to the hyperplane arrangement $\mathcal{H}_G^q$. The oriented matroid ideal associated to this oriented hyperplane arrangement $\mathcal{H}_G^q$ will be denoted by $C_G^q$ (instead of $\mathcal{O}_G^q$) and will be called the graphic oriented matroid ideal associated to $G$ and $q$. See [NPS02] for more details and Figure 3 corresponding to the graph $G$ in Example 5.18 depicted in Figure 4.

**Notation.** From now on we let $C_G^q$, $C_G$, $\mathcal{T}_G^q$ and $\mathcal{T}_G$, respectively, denote the ideals $\mathcal{O}_G^q$, $\mathcal{O}_G^q$, $\mathcal{O}_G^q$, and $\mathcal{O}_G^q$ from 4.1.

---

**Figure 3.** $\mathcal{H}_G^q$, $C_G^q$, and the monomial labels on the vertices

---

2We use the notation $C_G^q$ in order to be coherent with the notation used in [GZ83, §2.2.3] however in [NPS02] it is denoted by $\mathcal{O}_M$, where $M$ is the associated graphic matroid.
Remark 4.5. In [MS12] the description of the generating sets and the Betti numbers of $\mathcal{M}_G^q$, $C_G^q$ and $C_G$ is in terms of the “connected flags” of $G$. Fix a vertex $q \in V(G)$ and an integer $k$. A connected $k$-flag of $G$ (based at $q$) is a strictly increasing sequence $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k = V(G)$ such that $q \in U_1$ and all induced subgraphs on vertex sets $U_i$ and $U_{i+1} \setminus U_i$ are connected. Associated to any connected $k$-flag one can assign a “partial orientation” on $G$ by orienting edges from $U_i$ to $U_{i+1} \setminus U_i$ (for all $1 \leq i \leq k-1$) and leaving all other edges unoriented. Two connected $k$-flags are considered equivalent if the associated partially oriented graphs coincide. The Betti numbers correspond to the numbers of connected flags up to this equivalence. For a complete graph all flags are connected and all distinct flags are inequivalent. So in this case the Betti numbers are simply the face numbers of the order complex of the poset of those subsets of $V(G)$ that contain $q$ (ordered by inclusion). These numbers can be described using classical Stirling numbers, see [PS04, MS13a].

4.4. Ideals of directed graphs. Since we are representing the main results of [5] for both directed and undirected graphs, we need to extend the results of [4.3] to directed graphs. Let $G$ be a directed graph with a source at $q$, and let $E(G)$ denote the set of its oriented edges. Note that for undirected graphs $|\mathcal{E}(G)| = 2|\mathcal{E}(G)|$. The ideal $C_G^q \subset S$ is a squarefree monomial ideal. Let $\Sigma_G^q$ denote its associated simplicial complex on the vertices $\{ye : e \in E(G)\}$.

Proposition 4.6.

(i) The number of facets of $\Sigma_G^q$ is the same as the number of ‘oriented spanning trees’ of $G$. For each oriented spanning tree $T$ of $G$, the corresponding facet $\tau_T$ is:

$$\tau_T = \{ye : e \in E(G)\setminus \mathcal{O}_T\}.$$ 

(ii) For each oriented spanning tree $T$ of $G$, let $P_T = \langle ye : e \in \mathcal{O}_T\rangle$. The minimal prime decomposition of $C_G^q$ is

$$C_G^q = \bigcap_T P_T,$$

the intersection being over all oriented spanning trees of $G$.

(iii) For each facet $\tau$ of $\Sigma_G^q$ we have $|\tau| = |\mathcal{E}(G)| - n + 1$. Therefore

$$\dim(\mathbb{K}[\Sigma_G^q]) = |\mathcal{E}(G)| - n + 1.$$

(iv) $\Sigma_G^q$ is Cohen-Macaulay.

Proof. The ideal $C_G^q$ is generated by monomials $\prod_{e \in E(A,A^c)} ye$, where $q \in A^c \subset V(G)$ and $E(A,A^c) \subset \mathcal{E}(G)$ denotes the set of oriented edges from $A^c$. First we show that for each oriented spanning tree $T$, the monomial $m_T := \prod_{e \in \mathcal{E}(G)\setminus \mathcal{O}_T} ye$ does not belong to $C_G^q$. Clearly $m_T \in C_G^q$ if and only if $m_T$ is divisible by one of the given generators $\prod_{e \in E(A,A^c)} ye$. But

$$\prod_{e \in E(A,A^c)} ye \mid \prod_{e \in \mathcal{E}(G)\setminus \mathcal{O}_T} ye \iff \mathcal{E}(A, A^c) \subseteq (\mathcal{E}(G)\setminus \mathcal{O}_T).$$

However, it follows from the definition of $\mathcal{O}_T$ that it must contain some element of $\mathcal{E}(A, A^c)$ for any $A$. This shows that $\tau_T = \{ye : e \in \mathcal{E}(G)\setminus \mathcal{O}_T\}$ is a face in the simplicial complex $\Sigma_G^q$.

Next we show that $\tau_T$ must be a facet; for $f \in \mathcal{O}_T$ removing $f$ from the tree gives a partition of $V(T) = V(G)$ into two connected subsets $B$ and $B^c$ with $f_- \in B^c$ and $f_+ \in B$. Then the monomial $m_T \cdot ye$ is divisible by $\prod_{e \in E(B,B^c)} ye$. It remains to show that for any monomial $m = \prod_{e \in F} ye$ that does not belong to $C_G^q$, we have $F \subseteq \mathcal{E}(G)\setminus \mathcal{O}_T$ for some oriented spanning tree $T$. To do this, we repeatedly use the fact that $m$ is not divisible by generators of the form $\prod_{e \in E(A,A^c)} ye$ for various $A$, and construct an oriented spanning tree $T$. This procedure is explained in Algorithm 1.

Note that if $\prod_{e \in F} ye$ is not divisible by $\prod_{e \in E(A,A^c)} ye$ then there exists an $e \in \mathcal{E}(A, A^c)$ such that $e \notin F$. The orientation $\mathcal{O}_T$ is also induced by Algorithm 1.

(ii) follows from (i) and [MS04, Thm. 1.7].
4.4.1. \[ h \] resolution for

... identifying the variables 

One example of this observation is that we can obtain the expressing of the 

... numerator of Hilbert series) of these ideals, in terms of the Tutte polynomial of 

\[ M \]

**Remark** by Proposition 4.3.

**Proof.**

... complex

... Corollary 4.9.

... complex of the cut ideal introduced in

\[ \{ \]

**Theorem.** Let \[ H \] be the undirected graph corresponding to \[ G \], i.e., \[ H \] is obtained from \[ G \] by removing directions from the edges of \[ G \]. Now assume that \[ T_1, \ldots, T_k \] is the \[ \text{shelling order} \] for the facets of \[ \Sigma_G^q \]. Note that this induces the shelling order \[ T_1^*, \ldots, T_k^* \] on the set of oriented spanning trees of \[ H \]. Assume that \[ E_1, \ldots, E_k \] be the facets of \[ \Sigma_G^q \]. It is easy to see that each facet \[ E_j \] corresponds to a facet \[ T_{s_j}^* \] of \[ \Sigma_G^q \], where \[ T_{s_j}^* = E_j \cup F \] for \[ F = \{ y_e, y_\bar{e} \mid e \in E(H) \} \setminus \{ y_e \mid e \in E(G) \} \].

Assume that \[ s_{p_1} < s_{p_2} < \cdots < s_{p_t} \]. Now we show that \[ E_{p_1}, \ldots, E_{p_t} \] forms a shelling order for the facets of \[ \Sigma_G^q \]. Assume that for \[ \ell < j \] and \[ e \in E_{p_\ell} \setminus E_{p_j} \] we have that \[ T_{\ell} \setminus T_{s_{p_j}} = \{ e \} \] for some \[ \ell < j \]. This shows that for each \[ e' \in T_{\ell} \] \( e' \neq e \) we also have \[ e' \in T_{s_{p_j}} \] and so \[ e' \in E(G) \]. Thus \[ T_{\ell} \setminus F \subseteq E(G) \] which completes the proof. \( \square \)

**Example 4.8.** Consider the graph \[ G \] in Figure 8. For the spanning tree with the edge set \[ \{ y_1, y_3, y_4 \} \] (and \[ P_T = \{ y_1, y_3, y_4 \} \]) we have

\[ \tau_T = \{ y_e : e \in E(G) \} \setminus \{ y_1, y_3, y_4 \} = \{ y_1, y_2, y_3, y_4, y_5 \} \] .

We are now ready to use our results in this section to give a precise description of the polyhedral complex of the cut ideal introduced in \([2,2,3]\) for a directed graph.

**Corollary 4.9.** The set \( \{ y_e - y_\bar{e} \mid e \in E(G) \} \) forms a regular sequence for \( C_G^Q \). The polyhedral cell complex \( B_G^Q \) supports a minimal graded free resolution for \( C_G^Q \) and \( C_G \). In particular, the multigraded Betti numbers of \( C_G^Q \) and \( C_G \) coincide.

**Proof.** As an immediate consequence of Lemma \([1,7]\) and \([2,3,5] \) Lem. 3.15, similar to \([NPS02] \) Cor 2.7] we have that \( \{ y_e - y_\bar{e} : e \in E(G) \} \) forms a regular sequence for \( C_G^Q \). Now the proof follows by Proposition \( 4.3 \)\( \square \)

**Remark 4.10.** Note that \( C_G^Q \) is exactly the ideal \( C_G \). The relation between the ideals \( C_G^Q \) and \( M_G^Q \) from \([2,1]\) can be understood via the regular sequence (see \([2,3,5]\)). The labeled polyhedral cell complex \( B_G^Q \) supports a graded minimal free resolution for \( C_G^Q \). We relabel this complex by identifying the variables \( y_e \) with \( x_{e+} \). The resulting labeled complex supports a minimal free resolution for \( M_G^Q \), see e.g., \([NPS02] \) \([2,3,5]\) for more details.

### 4.4.1. h-polynomials

The equality of the Betti tables of the ideals \( C_G^Q, C_G \) and \( M_G^Q \) allows us to prove many numerical facts about one ideal by looking instead at another ideal in this family. One example of this observation is that we can obtain the expressing of the \( h \)-polynomials (the numerator of Hilbert series) of these ideals, in terms of the Tutte polynomial of \( G \). So we obtain...
the following result. The $h$-vector of $S/C_G^t, R/C_G$ and $k[x_v : v \in V(G)]/M^2_G$, is equal to $T(1, y)$, where $T(x, y)$ is the Tutte polynomial of the graph (Björklund page 236).

4.4.2. Multiplicities. For a finitely generated (graded) module $M$ of dimension $d > 0$ over a polynomial ring, the multiplicity of $M$ is defined to be the leading coefficient of the Hilbert polynomial of $M$ (i.e., the polynomial defining $i \mapsto \dim(M_i)$ for $i \gg 0$). We will denote this quantity by $e(M)$. Since the Hilbert polynomial is completely determined by the Betti table (see, e.g., [MS05, Theorem 8.20 and Proposition 8.23]), the multiplicity is also determined by the Betti table.

In our situation, Proposition [6.6][iii] implies that $\dim(S/C_G^t) = |E(G)| - n + 1$. Also, for each spanning tree $T$ we have $\dim(S/P_T) = |E(G)| - n + 1$ and $e(S/P_T) = 1$. Thus by [GP08, Lemma 5.3.11] we have

$$e(S/C_G^t) = e(S/C_G) = \sum_T e(S/P_T) = \varepsilon(G),$$

where $\varepsilon(G)$ denotes the number of oriented spanning trees of $G$.

5. Minimal free resolutions of system ideals

We now study the ideals associated to coherent systems introduced in [222] in details.

5.1. Cut ideals. First we recall that given a polyhedral complex and a subset $U$ of its vertices, its induced subcomplex on $U$, is the set of all its faces whose vertices belong to $U$. The polyhedral complex $B_G^t$ associated to the cut ideal of $G$ in Corollary [4.9] naturally contains the subcomplexes induced on the vertices of top dimensional regions. We will show that the subcomplex of $B_G^t$, supports the minimal free resolution of its corresponding ideal, i.e., the ideal generated by the monomials associated to its vertices, provided that the subcomplex has been chosen nicely.

Let $S_{q,t} = \{C_1, \ldots, C_t\}$ be the set containing all connected cuts $C_i = E(A_i, A_i^c)$ of $G$, with $t \in A_i$ and $q \in A_i^c$, see [222.3]. For each $i$, let $c_i$ denote the vertex of $B_G^t$ corresponding to the cut $C_i$. We denote $D_G^{q,t}$ for the induced subcomplex of $B_G^t$ on the vertices $c_1, \ldots, c_t$. By a slight abuse of notation, $c_i$ will be used to denote its corresponding $\mathbb{N}^n$-vector, where the $r$-th entry corresponding to the vertex $v_r$, is $|\{e \in C_i : e_r = v_r\}|$. In order to have $D_G^{q,t}$ as a polyhedral complex supporting the minimal free resolution of $C_G^{q,t}$, we first label the vertices (0-dimensional faces) of $D_G^{q,t}$; we associate the monomial $y^{c_i}$ from [223] as the label of the vertex $c_i$.

**Theorem 5.1.** The labeled polyhedral cell complex $D_G^{q,t}$ gives a $\mathbb{Z}^{2m}$-graded minimal free resolution for $C_G^{q,t}$. In particular, the $\beta_d(C_G^{q,t})$ counts the $d$-dimensional bounded regions of $D_G^{q,t}$ for all $d$.

**Proof.** By [MS05, Prop. 4.5] we only need to check that $(D_G^{q,t})_{\leq \delta}$ is acyclic for any $\delta \in \mathbb{N}^n$, and the monomial labels of each pair of the cells $F_1 \subseteq F_2$ are different. The latter is clear because the labeling of the cells are corresponding to the partitions of $G$, see e.g., [GZS3, p.112]. We follow the strategy of the proof of Lemma 6.4 in [DS12]. Assume that $c_1, \ldots, c_t$ are the vertices of $(D_G^{q,t})_{\leq \delta}$, and $C_1, \ldots, C_t$ are their corresponding cuts. Since the ideal $C_G^{q,t}$ and the labels of the cells are all squarefree, we can assume that $\delta$ is squarefree as well. Let $J_1 = V(G) \setminus \{e_+ : e \in C_1 \cup \cdots \cup C_t\}$. Note that $q \in J_1$, since $q$ is a source. Let $J_2 \subset V(G) \setminus J_1$ be the connected component of $G[V(G) \setminus J_1]$ containing $q$. Now consider the subset $L = V(G) \setminus J_2$ and let $\mathbf{p}$ be the $n$-vector $\frac{1}{|L|} \epsilon_L$, where $\epsilon_L$ is 1 in entries corresponding to the vertices in $L$, and 0 in other entries. We claim that $\mathbf{p}$ is the star point of $(D_G^{q,t})_{\leq \delta}$. Note that $c_1, \ldots, c_t$ are the vertices of $B_G^t$, and the graph obtained by contracting the vertices corresponding to $\mathbf{p}$ will have a unique source. Assume that $\mathbf{p}$ belongs to some cell labeled by the monomial $a_{\mathbf{p}}$ of $S$. Note that since $J_2 \subset A_i^c$ for each $C_i = E(A_i, A_i^c)$, we have $t \notin J_2$. Thus $\mathbf{p}$ belongs to $D_G^{q,t}$.

We now show that $a_{\mathbf{p}} \subset \delta$. It is clear that $supp(c_i) \subseteq L$ for all $i$, since one of the endpoints of each edge in the cut has positive indegree. Now consider $\ell \in L$ with $a_{\ell} > 0$. Thus there exists an edge $e \in C_1 \cup \cdots \cup C_t$, and a path from $q$ to $e_+$ not having intersection with any other vertex in $V(C_1 \cup \cdots \cup C_t)$. Thus $e_-$ belongs to $V(C_1 \cup \cdots \cup C_t)$ and so $\delta_{\ell} \geq 1$. [16]
Let \( r \) be a point in \((D^q_t G) \leq \delta\). If \( r_i > r_j \) for some edge \( \{i, j\} \), then \( i \in L \) and so \( p_i = \frac{1}{|L|} \geq p_j \). Thus no hyperplane \( \mathcal{H}_e \) (strictly) separates \( r \) and \( p \). Therefore the line segment \((p, r)\) connecting \( p \) and \( r \), sits inside some cell \( R \) of \( B^q_t G \). Now we have to show that any interior point of \((p, r)\) is also in \((D^q_t G) \leq \delta\). Note that the support of the monomial associated to \( R \) is a subset of \( \delta \) in \( B^q_t G \), and so in \( D^q_t G \). By [MS13b, Lemma 6.2] the graph obtained by contracting the vertices corresponding to any interior point of \((p, r)\) has again a unique source, and so the region \( R \) containing \((p, r)\) belongs to \( D^q_t G \). □

**Remark 5.2.** The set \( \{y_e - y_{\bar{e}} : e, \bar{e} \in \mathcal{E}(G)\} \) forms a regular sequence for \( S/\mathcal{C}_q^t G \). Then by [MS13b, Thm. A.11] (see also [Eis05, Lem. 3.15]) relabeling the vertices of the complex \( D^q_t G \) with monomials \( x^C \) (instead of \( y^C \) corresponding to cuts \( C \) ), gives us a polyhedral complex supporting the minimal free resolution of \( \mathcal{C}_q^t \). Note that as usual, we extend the labeling to all faces by the least common multiple rule.

**Example 5.3.** Consider the graph \( G \) depicted in Figure 4. Let \( q \), the unique source, be the (red) vertex at the bottom, and \( t \), a sink, be the (blue) vertex at the left. Acyclic partial orientations of \( G \) associated to connected cuts \( \mathcal{E}(A, A') \) with \( q \in A' \) and \( t \in A \), are depicted in Figure 4. The corresponding mincut ideal is

\[
\mathcal{C}_{q, t} = \langle x_2 x_3 x_5, x_2 x_4 x_5, x_1 x_2 \rangle.
\]

The corresponding acyclic partial orientations (with 2 components) are denoted by \( p_4, p_5, p_6 \) in Figure 6(a), see also Figure 8. The acyclic partial orientations with 3 components (corresponding to the edges in \( B^t_G \)) and the acyclic orientation with 4 components (corresponding to the unique top dimensional region) are depicted in Figures 5. The induced subcomplex of \( D^q_t G \) on the vertices \( p_4, p_5, p_6 \) which has three vertices, three edges, and one 2-dimensional region resolves the minimal free resolution of the ideal \( \mathcal{C}_{q, t} \). Also note that the Alexander dual of the mincut ideal is the ‘path ideal’ between two vertices \( q \) and \( t \). Here \( \mathcal{P}_{q, t} = \langle x_2, x_1 x_3 x_4, x_1 x_5 \rangle \).

**Example 5.4.** Consider the complete graph \( K_4 \) on \([4]\) with the unique source at 4. The acyclic partial orientations of \( K_4 \) are encoded in the barycentric subdivision of the 2-simplex which supports the minimal free resolution of the ideals \( \mathcal{C}_{K_4}^t, \mathcal{C}_{K_4}^t \), and the ideal

\[
\mathcal{M}_{K_4}^t = \langle x_1^3, x_2^3, x_3^3, x_1^2 x_2^2, x_1^2 x_3^2, x_2^2 x_3^2, x_1 x_2 x_3 \rangle.
\]

Figure 6(b) shows the three regions corresponding to the choice of a sink \( t \) as 1, 2 or 3. See Corollary 4.13 and Remark 4.10.
Example 5.5. Consider the graph $G$ depicted in Figure 7 from [Doh03, Exam. 6.2.1]. Let the blue vertex denote the unique source at $q$ and the red vertex denote the sink at $t$. Note that

$C_{q,t} = \langle x_1x_2x_3, x_2x_3x_4x_6, x_6x_7x_8, x_1x_2x_5x_8, x_1x_4x_7x_8, x_3x_5x_6x_7, x_2x_4x_5x_6x_8, x_1x_3x_4x_5x_7 \rangle$

and the minimal free resolution of $R/C_{q,t}$ is:

$0 \rightarrow R^4 \rightarrow R^{14} \rightarrow R^{17} \rightarrow R^8 \rightarrow R \rightarrow R/C_{q,t} \rightarrow 0$.

The acyclic partial orientations of $G$ with a unique source at $q$ and a sink at $t$, can be read from the multigraded Betti numbers.

5.1.1. Syzygies and free resolutions for $T_G$ and $T^q_G$. Here we present the exact SMT reliability for directed and undirected graphs. Algebraically, this corresponds to computing the multigraded Betti numbers of SMT ideals associated to graphs. However, the results are written in the algebraic language of ideals, the proofs are based on graph theoretical arguments. We derive the implicit representation of the higher syzygy modules of these ideals in terms of the oriented $k$-spanning trees (see Definition 3.8), which also extends the results in [SP78], where Satyanarayana and Prabhakar have presented a topological formula for evaluating the exact SMT reliability for directed networks by giving a combinatorial recipe to read the (non-cancelling) terms in the reliability formula. Our main result in this section is analogous to [MS12, Thm. 5.3] for the cut ideal and the initial ideal of the toppling ideal of $G$, see Remark 4.5. We remark that the ideals associated to spanning trees arise in different contexts, see e.g. [NPS02, MS13b, KMS14].
We recall that there is a bijection between the generators of $\mathcal{I}_G$, the generators of $\mathcal{I}_G^G$ and $\mathcal{S}_0(G, q)$. In Theorem 5.6 we give a generalization of this fact.

**Notation.** Let $\mathcal{I}$ be an element of $\mathcal{S}_k(G, q)$ for $k > 0$, and $e \in \mathcal{E}(\mathcal{I})$.

1. $\mathcal{I}^e$ denotes the subgraph of $\mathcal{I}$ with the edge set $\mathcal{E}(\mathcal{I}^e) = \mathcal{E}(\mathcal{I}) \setminus \{e\}$. We set $W(\mathcal{I}) = \{ e \in \mathcal{E}(\mathcal{I}) : e \in \mathcal{S}_{k-1}(G, q) \}$.
2. Given an arbitrary ordering on the edges of the graph, say $\mathcal{E}(G) = \{e_1, \ldots, e_m\}$, we set $c(\mathcal{I}, e) = (-1)^{j}$, when $e$ is the $j$th edge among the edges with endpoint $e_+$.  

**Theorem 5.6.** Fix a pointed graph $(G, q)$. For each $k \geq 0$ there exists a natural injection 

$$\psi_k : \mathcal{S}_k(G, q) \hookrightarrow \text{syz}_k(\mathcal{I}_G^G)$$

such that the set $\text{Image}(\psi_k)$ forms a minimal generating set for $\text{syz}_k(\mathcal{I}_G^G)$.

**Proof.** For $k = 0$ the result is clear. Here $\psi_0(\mathcal{I}_G^G)$ is the generating set of $\mathcal{I}_G^G$, and 

$$\psi_0 : \mathcal{S}_0(G, q) \hookrightarrow \text{syz}_0(\mathcal{I}_G^G) = \mathcal{I}_G^G \quad \mathcal{I} \mapsto \prod_{e \in \mathcal{E}(\mathcal{I})} y_e.$$

The proof is by induction on $k \geq 1$. We define 

$$\psi_k : \mathcal{S}_k(G, q) \hookrightarrow \text{syz}_k(\mathcal{I}_G^G) \quad \mathcal{I} \mapsto \sum_{e \in W(\mathcal{I})} (-1)^{c(\mathcal{I}, e)} y_e[\psi_{k-1}(\mathcal{I}^e)].$$

**Base case.** Assume that $\mathcal{I} \in \mathcal{S}_1(G, q)$. Then $\mathcal{I}$ has a unique cycle, and there exists a unique vertex $v$ such that $\text{indeg}(v) = 2$. Therefore $W(\mathcal{I}) = \{e, e' : e_+ = e'_+\}$ and $c(\mathcal{I}, e) = c(\mathcal{I}, e') = 0$. Thus $\mathcal{I}^e, \mathcal{I}^{e'}$ belong to $\mathcal{I}_G^G$, and $y_e\psi_0(\mathcal{I}^e) - y_{e'}\psi_0(\mathcal{I}^{e'}) = 0$ which implies that 

$$\psi_1(\mathcal{I}) = y_e[\mathcal{I}^e] - y_{e'}[\mathcal{I}^{e'}]$$

is an element of $\text{syz}_1(\mathcal{I}_G^G)$.

Now assume that $\sum(-1)^{c(\mathcal{I}, e)} y_e[\mathcal{I}_i]$ is an element of the minimal generating set of $\text{syz}_1(\mathcal{I}_G^G)$, that is, $\sum(-1)^{c(\mathcal{I}, e)} y_e\psi_0(\mathcal{I}_i) = 0$. Note that here we use Remark 3(i) that the resolution of $\mathcal{I}_G^G$, as the Alexander dual of $\mathcal{C}_G^0$, is linear. Therefore for each monomial $M_i = y_{e_i} \prod_{e \in \mathcal{E}(\mathcal{I}_i)} y_e$ corresponding to an oriented tree $\mathcal{I}_i$ such that $M_i = M_j$. This implies that $\mathcal{E}(\mathcal{I}_i) \cup \{e_i\} = \mathcal{E}(\mathcal{I}_i) \cup \{e_j\}$. We claim that the induced graph $\mathcal{I} \subseteq \mathcal{I}_i$ with the edges $\mathcal{E}(\mathcal{I}) = \mathcal{E}(\mathcal{I}_i) \cup \{e_i\}$ is acyclic. Otherwise it should have a cycle $C$ including the edges $e_i$ and $e_j$. Let $v_i = e_{i+}$ and $v_j = e_{j+}$ with $v_i \neq v_j$ (the case $v_i = v_j$ is clear). On the other hand, since $e_i \notin \mathcal{I}_i$, there exists a (unique) path $P$ with $\mathcal{E}(P) \subseteq \mathcal{E}(\mathcal{I}_i)$ from $q$ to $v_i$ not going through the edge $e_i$. Similarly, there exists a (unique) path $P'$ with $\mathcal{E}(P') \subseteq \mathcal{E}(\mathcal{I}_j)$ from $q$ to $v_j$ not going through the edge $e_j$. Thus the subgraph with the edge set $\mathcal{E}(P) \cup \mathcal{E}(P') \cup \mathcal{E}(C) \setminus \{e_j\} \subseteq \mathcal{E}(\mathcal{I}_j)$ contains a cycle which is a contradiction by our assumption that $\mathcal{I}_i \in \mathcal{S}_0(G)$ and it is acyclic.

**Induction hypothesis.** Now let $k > 1$ and assume $\text{Image}(\psi_{k-1}) \subseteq \text{syz}_{k-1}(\mathcal{I}_G^G)$ forms a minimal generating set for $\text{syz}_{k-1}(\mathcal{I}_G^G)$.

Now assume that $s = \sum(-1)^{c(\mathcal{I}, e_i)} y_{e_i}[\mathcal{I}_i]$ is an element of the minimal generating set of $\text{syz}_{k-1}(\mathcal{I}_G^G)$, that is, $\sum(-1)^{c(\mathcal{I}, e_i)} y_{e_i}\psi_{k-1}(\mathcal{I}_i) = 0$ since by Remark 3(i) we know that the resolution of $\mathcal{I}_G^G$ is linear. Note that corresponding to each term $M_{i_j} = y_{e_j}y_{e_j}\psi_{k-2}(\mathcal{I}_j')$ with $e_j \in \mathcal{E}(\mathcal{I}_i)$, there exists a unique term $y_{e_j}\psi_{k-1}(\mathcal{I}_j)$ with $e_j \in \mathcal{E}(\mathcal{I}_j)$ such that $M_{i_j} = y_{e_j}y_{e_j}\psi_{k-2}(\mathcal{I}_j')$ is equal to $M_j$. In particular, we have $\mathcal{I}_j \setminus \{e_j\} = \mathcal{I}_i \setminus \{e_i\}$. Set $\mathcal{I}$ be the subgraph of $G$ with the edge set $\mathcal{E}(\mathcal{I}_i) \cup \{e_i\}$. Now we show that:

1. the subgraph $\mathcal{I}$ belongs to $\mathcal{S}_k(G, q)$.
2. all terms of $\psi_k(\mathcal{I})$ have been appeared in $s = \sum(-1)^{c(\mathcal{I}, e_i)} y_{e_i}[\mathcal{I}_i]$. 

19
Corollary 5.11. Hilbert series of 1-spanning trees induced on the edges 1, Example 5.10. of the base field acyclic orientations on the induced subgraph on edges corresponding to The Betti numbers of the ideal Corollary 5.9. Corollary 5.8. $T \subset C$ forms a regular sequence for $G$.

Thus the Betti numbers of $T = \bigcap_{i} C_i$ are given by

$$\beta_{i,n} = |\mathcal{S}_{i,n}(G,q)|,$$

where $|\mathcal{S}_{i,n}(G,q)|$ is the number of $i$-spanning trees of $G$ and $|\mathcal{S}_{i,j}(G,q)|$ is the number of possible acyclic orientations on the induced subgraph on edges corresponding to $j$.

Example 5.10. In our running example, $\beta_{1,(1,1,1,1,0)} = 3$ is corresponding to the three oriented 1-spanning trees induced on the edges 1, 2, 3, 4. See Figure 9.

As an immediate consequence of the constructed minimal free resolution, we have obtained Hilbert series of $R/\mathcal{I}_G$ in its natural grading by the groups $Z$ and $Z^m$.

Corollary 5.11. Hilbert series of $R/\mathcal{I}_G$ in the grading by the group $Z$ equals

$$1 + \sum_{i=0}^{m} (-1)^{i+1} |\mathcal{S}_{i-1}(G,q)| t^{i} - 1 = \frac{1 + \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} |\mathcal{S}_{j}(G,q)| t^{i+j}}{(1 - t)^{|\mathcal{S}_0(G,q)|}}$$

and Hilbert series of $R/\mathcal{I}_G$ in the grading by the group $Z^m$ equals

$$1 + \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} |\mathcal{S}_{i,j}(G,q)| t^{i+j} = \frac{1 + \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{i+j+k} |\mathcal{S}_{i,j,k}(G,q)| t^{i+j+k}}{(1 - t)^{|\mathcal{S}_0(G,q)|}}$$

Example 5.12. Let $G = C_n$ be the cycle on $n$ vertices. Then $|\mathcal{S}_{0,n-1}(G,q)| = n$ and $|\mathcal{S}_{1,n}(G,q)| = n$. Thus the Betti numbers of $\mathcal{I}_G$ are given by

$$\beta_0 = 1, \beta_{1,n-1} = n, \beta_{2,n} = n - 1$$

It turns out that similar statement holds for the ideals whose based graphs are cactus, i.e., connected graphs in which each edge belongs to at most one cycle. Here we have removed the proofs, and we just mention the statement for a cactus graph.
Example 5.13. Let $G$ be a cactus graph with induced cycles $C_1, \ldots, C_k$ with $|V(C_i)| = n_i$ for $i = 1, \ldots, k$. Then one can easily see, by induction on $k$, that the number of spanning trees of $G$ is $|\mathcal{S}_{0,|E(G)|-k}(G, q)| = n_1 n_2 \cdots n_k$ and for $i > 0$, $j = i + |V(G)| - 1$ we have

$$|\mathcal{S}_{i,j}(G, q)| = \sum_{\mathbf{D} = (d_{i_1}, d_{i_2}, \ldots, d_{i_k}) \mid i_1 < i_2 < \cdots < i_k \leq k} (n_{i_1} - 1)(n_{i_2} - 1) \cdots (n_{i_k} - 1) \prod_{a \in [k] \setminus D} n_a.$$ 

For example for the cactus graph with two cycles of length 3 and 4, we have

$$\beta_0 = 1, \; \beta_{1,5} = 12, \; \beta_{2,6} = 17, \; \beta_{3,7} = 6.$$ 

In particular if $n_i = n$ for all $i$, then the Betti numbers of $\mathcal{S}_G$ are given by

$$\beta_{i,j} = \begin{cases} n^k, & \text{if } i = 0, \; j = |E(G)| - k; \\ \binom{k}{i}(n - 1)^i n^{k - i}, & \text{if } i > 0, \; j = i + |V(G)| - 1; \\ 0 & \text{otherwise.} \end{cases}$$ 

For example for a cactus graph with three cycles of length 3, we have

$$\beta_0 = 1, \; \beta_{1,6} = 27, \; \beta_{2,7} = 54, \; \beta_{3,8} = 36, \; \beta_{4,9} = 8.$$ 

Example 5.14. Consider the graph $G$ depicted in Figure 8 with the fixed orientation $O$. Let $q$ be the distinguished (red) vertex at the bottom. Then

$$\mathcal{S}_G = \langle x_1 x_2 x_3, x_2 x_3 x_5, x_1 x_3 x_5, x_1 x_2 x_4, x_2 x_4 x_5, x_1 x_4 x_5, x_2 x_3 x_4, x_1 x_3 x_4 \rangle$$

and

$$\mathcal{S}_G^O = \langle y_1 y_2 y_3, y_2 y_3 y_5, y_1 y_3 y_5, y_1 y_2 y_4, y_2 y_4 y_5, y_1 y_4 y_5, y_2 y_4 y_4, y_1 y_4 y_4 \rangle.$$ 

Then the Betti numbers of $R/\mathcal{S}_G$ (and similarly $S/\mathcal{S}_G^O$) are given by

$$\beta_0 = 1, \; \beta_{1,3} = 8, \; \beta_{2,4} = 11, \; \beta_{3,5} = 4,$$

where $|\mathcal{S}_0(G, q)| = 8$, $|\mathcal{S}_1(G, q)| = 11$, and $|\mathcal{S}_2(G, q)| = 4$.

![Figure 8. Graph G with a fixed orientations, and its spanning trees](image)

![Figure 9. Oriented 1-spanning trees and 2-spanning trees $O_1, \ldots, O_4$](image)

Example 5.15. The nonzero graded Betti numbers of the SMT ideals corresponding to the four orientations of $G$ denoted by $O_1, \ldots, O_4$ in Figure 9 are as follows (all equal to 1):

$$S/\mathcal{S}_{O_1} : \; \beta_{1,245}, \; \beta_{1,213}, \; \beta_{1,253}, \; \beta_{1,124}, \; \beta_{2,1234}, \; \beta_{2,2345}, \; \beta_{2,1245}, \; \beta_{2,1325}, \; \beta_{3,12345}.$$
From this we can immediately read off the graded Betti numbers of \( T^q_G \) by keeping only one copy of each Betti number; we have 8 distinct multidegrees at the first syzygy module, 11 at the second syzygy module, and 4 at the third syzygy module.

**Remark 5.16.** In fact the constructed minimal free resolution of \( T^q_G \) arises in a more natural geometric context, and supported on a cellular complex which will be addressed in more details in the next paper. Another interesting result about the SMT ideals is that all their powers have minimal linear resolutions. Moreover, the \( \beta_{i,j}(\mathfrak{T}^q_G)^k \) is 1 or 0 for all \( j \in \mathbb{Z} | E(G) | \) and for all \( k \). Note that \( \mathfrak{T}_G \) is a matroid ideal and applying [CH03, Prop. 5.2, Thm. 5.3] we expect that all powers of \( \mathfrak{T}_G \) has linear resolutions, but the ideal \( \mathfrak{T}^q_G \) is not matroid.

**Remark 5.17.** We observe that using the results presented in §4.4, Theorem 5.6 can be naturally formulated for an arbitrary directed graph. The syzygies of \( \mathfrak{T}_G \) can be derived by finding the syzygies of the SMT ideals of the (totally) acyclic orientations of \( G \). Hence, this breaks the computational problem down into several parallel tasks. We list all acyclic orientations \( O_1, \ldots, O_r \) of \( G \) in which every edge of \( G \) is oriented, and we compute the minimal free resolutions of their associated ideals. The combinatorial bijection presented in Theorem 5.6 implies that the minimal generating set of the \( k \)-syzygy module of \( \mathfrak{T}_G \), is the union of the minimal generating sets of the \( k \)-syzygy modules of \( \mathfrak{T}_{O_i} \) for all \( i \) and all \( k \).

The following example illustrates Remark 5.17 for a directed graph.

**Example 5.18.** Let \( G \) be the following graph with the edge set [6]. We fix the red vertex as the distinguished vertex. Then

\[
\mathfrak{T}^q_G = \langle y_1 y_2 y_4 y_6, y_1 y_4 y_5 y_6, y_1 y_2 y_3 y_6, y_2 y_3 y_4 y_6 \rangle,
\]

where \( y_e \) corresponds to the oriented edge \( e \) and the variable \( y_4 \) is associated to the edge 4 directed toward the vertex (6). And \( y_4 \) for its inverse. Then the minimal free resolution of \( S/\mathfrak{T}^q_G \) is \( 0 \to S^3 \to S^4 \to S \to S/\mathfrak{T}^q_G \to 0 \).

**Figure 10.** Directed graph \( G \), and its oriented spanning trees

**Figure 11.** Oriented 1-spanning trees of \( G \)
5.1.2. Relation to reduced divisors. The $q$-reduced divisors played a prominent role in Baker-Norine’s proof of Riemann-Roch theory for finite graphs. Our approach to study the oriented $k$-spanning trees provides new combinatorial description for the rank of divisors. While this paper was being prepared, the preprint [Bac14] was posted on the arXiv by Spencer Backman who applies some similar results to provide a new proof of the Riemann-Roch theorem. However our perspective is mostly geometric combinatorics and commutative algebra, see Theorem 5.2. There are several other bijections in the literature between the maximum $G$-parking functions, the set of spanning trees of with no broken circuit and particular acyclic orientations of $G$, see e.g. [BCT10]. We would like to remark that the given bijection is different from those appeared in the literature and has not been studied before.

Corollary 5.19. For each $k$-syzygy element of $\mathfrak{T}_G$, its corresponding oriented $k$-spanning tree represents a $q$-reduced divisor of degree $k - 1$. For a $q$-reduced divisor $D$ of degree $k - 1$ with $D(q) = -1$, there exists $\mathfrak{T} \in \mathfrak{S}_k(G, q)$ and a $k$-syzygy element $\psi_k(\mathfrak{T})$ such that $D = D_\mathfrak{T}$.

5.1.3. Syzygies and free resolutions for $\mathcal{P}^O_{q,t}$ and $\mathcal{P}_{q,t}$. Following the discussion for the ideals of the SMT modules, it would be interesting to derive the implicit formulas for the syzygies of path ideals in terms of the combinatorial data provided by the graph. Theorem 5.21 presents the algebraic analogue of [SP78, Thm. 1]. A pointed graph $(G, q, t)$ is a graph together with two distinguished vertices $q, t$ of $V(G)$ such that $q$ is a unique source, and $t$ is a sink.

Definition 5.20. Fix a graph $(G, q, t)$, and let $g = \vert E(G) \vert - \vert V(G) \vert + 1$ denote the genus of $G$. For each integer $0 \leq k \leq g$, an oriented path $\mathcal{P}$ of $(G, q, t)$ is a connected subgraph of $G$ with a unique source at $q$ such that

- $V(\mathcal{P}) \subseteq V(G)$ with $q, t \in V(\mathcal{P})$ and $E(\mathcal{P}) \subseteq E(G)$,
- $\mathcal{P}$ is acyclic.

Let $\mathcal{P}'$ be the shortest oriented path with $E(\mathcal{P}') \subseteq E(\mathcal{P})$ and $\vert E(\mathcal{P}) \setminus E(\mathcal{P}') \vert = k$, then $\mathcal{P}$ is called an oriented $k$-path. The set of all oriented $k$-paths of $(G, q, t)$ will be denoted by $\mathfrak{S}_k(G, q, t)$. Note that $\mathfrak{S}_g(G, q, t)$ is the set containing all minimal (directed) paths from $q$ to $t$ with associated monomial $y^P = \prod_{e \in E(\mathcal{P})} y_e$. We recall that the ideal $\mathcal{P}^O_{q,t}$ (and $\mathcal{P}_{q,t}$) is generated by all monomials associated to the elements of $\mathfrak{S}_0(G, q, t)$, see [2.2.2].

In Theorem 5.20 if we replace $\mathfrak{S}_k(G, q)$ and $\mathfrak{T}_G$ by $\mathfrak{S}_k(G, q, t)$ and $\mathcal{P}^O_{q,t}$, then an analogous statement holds.

Theorem 5.21. Fix a pointed graph $(G, q, t)$, and let $g$ be the genus of $G$. For each $0 \leq k \leq g$ there exists a natural injection

$$\psi_k : \mathfrak{S}_k(G, q, t) \rightarrow \text{syz}_k(\mathcal{P}^O_{q,t})$$

such that the set $\text{Image}(\psi_k)$ forms a minimal generating set for $\text{syz}_k(\mathcal{P}^O_{q,t})$.

Example 5.22. Coming back to Example 5.3 we have that

$$\mathcal{P}_{q,t} = \langle x_1x_6, x_2x_4x_6, x_3x_4x_5x_6, x_2x_7, x_1x_4x_7, x_3x_5x_7, x_3x_8, x_2x_5x_8, x_1x_4x_5x_8 \rangle$$

with the minimal free resolution

$$0 \rightarrow \mathbb{R}^4 \rightarrow \mathbb{R}^{18} \rightarrow \mathbb{R}^{31} \rightarrow \mathbb{R}^{25} \rightarrow \mathbb{R}^{9} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathcal{P}_{q,t} \rightarrow 0,$$

where $\sum \beta_i(\mathcal{P}_{q,t}) = 117$ corresponds to the acyclic orientations of $G$ containing a path from $q$ to $t$. However the inclusion-exclusion expression of the reliability formula will contain $2^9 - 1 = 511$ terms (with only 117 non-cancelling terms). Note that we do not need to list all acyclic orientations of $G$. We only list the minimal paths between $q$ and $t$ which can be obtained inductively (see [6.1]), and then the command `peek betti res $\mathcal{P}_{q,t}$ in Macaulay2 [GS]`, gives us the list of all multigraded Betti numbers corresponding to desired acyclic orientations. For example (considering one variable for each edge) the multigraded Betti number in degree $(0, 1, 0, 1, 1, 1, 1, 1)$ corresponds to the (unique) acyclic orientation on the edge set $\{2, 4, 5, 6, 7, 8\}$ with properties listed in Definition 5.20 for $k = 2$. 

23
6. Algorithms for Generating Functions of System Ideals

Here we state some ideas from discrete potential theory on graphs from [FGK13] which enable us to apply our results from [5] to break down the computation of Hilbert function into several parallel tasks. We discuss some computational aspects of our results, by presenting concrete algorithms for computing the minimal generating sets of the system ideals. The generating function of a monomial ideal is the sum of the monomials in its minimal generating set.

6.1. Generating functions of path ideals. Given a subset $T$ of $V(G)\setminus\{q\}$, computing the probability of successful communication between distinguished vertices $q$ and all vertices $t$ of $T$, requires the examination of all (directed) trees containing the vertices $T\cup\{q\}$, see [2.2.2]. The exact calculation of $R(q,T)$ (even if for each edge $p_e = p$ for some constant value $p$) is not feasible for large graphs. However there are some criteria giving some approximate bounds for the reliability of the system. Combining our results presented in [5] and the algorithms given by Fomin, Grigoriev, and Koshevoy in [FGK13] (for computing generating functions of spanning trees of $G$) we provide new algebraic methods and algorithms to compute the exact value for the reliability of the system.

| Input: | A subset $T \subset V(G)\setminus\{q\}$. |
|-------|-----------------------------------|
| Output: | The generating function $\varphi_{q,T}(G)$ of $\mathcal{P}_{q,T}^O$. |
| Initialization: | $G_0 = G$, $w_0(e) = y_e$ for $e \in E(G)$, $V(G)\setminus\{q\} = \{v_1, \ldots, v_n\}$, $v_n \in T$, and $i = 1$. |
| while $i < n$ do | |
| Let $A_i = \{e \in E(K_{n+1}) : e_-, v_i, e_+ \text{ is an oriented path in } G_{i-1}\}$. |
| Let $G_i$ be a graph with $V(G_i) = V(G_{i-1})\setminus\{v_i\}$ and $E(G_i) = A_i \cup E(G_{i-1})\setminus\{e : e_+ = v_i \text{ or } e_- = v_i\}$. |
| if $e \not\in A_i$, then let $w_i(e) = w_{i-1}(e)$. |
| else $w_i(e) = w_{i-1}(e) + (\mu_{G_{i-1}}(v_i))^{-1}w_{i-1}(e_1)w_{i-1}(e_2)$, where $e_1 = (e_-, v_i)$ and $e_2 = (v_i, e_+)$. |
| $i \leftarrow i + 1$. |
| end |

Output $\varphi_T(G) = w_{n-1}(e_{q,v_n})\prod_{i=1}^{n-1} \mu_{G_{i-1}}(v_i)$, where $e_{q,v_n}$ is the edge oriented from $q$ toward $v_n$ in $G_{n-1}$.

Algorithm 2: Finding the generating function of the path ideal $\mathcal{P}_{q,T}^O(G)$.

Algorithm 2 is a slight modification of the star-mesh transformation in a directed network established in [FGK13] Lemma 7.3. The idea behind the algorithm is to compute the generating function of $G$ inductively. Having the monomial terms of the generating function of a system ideal in hands, we now can apply the algebraic results of [5.1.1] to compute the multigraded free resolution of these ideals. Note that each multigraded Betti number corresponds to a unique oriented subgraph of $G$. This way all acyclic orientations of graph can be read directly from the multigraded Betti numbers of $\mathcal{P}_{q,T}^O$. 

Notation. Given a subset $T \subset V(G)\setminus\{q\}$, we consider the ideal $\mathcal{P}_{q,T}^O$ from [2.2.2]. We denote $\varphi_{q,T}(G)$ for the generating function of $\mathcal{P}_{q,T}^O$, i.e. the sum of the monomials (minimally) generating $\mathcal{P}_{q,T}^O$. In Algorithm 2 we assume that $V(G)\setminus\{q\} = \{v_1, \ldots, v_n\} \text{ and } v_n \in T$. We define $\mu_G(v) = \sum_{\{e \in E(G) : e_+, e_- = v\}} w(e)$ if $v \in T$, and $\mu_G(v) = 1$ if $v \not\in T$, where $w(e)$ is the sum of the multidegrees associated to the edge $e$, i.e., if the graph is simple, then $w(e) = y_e$, and if $e$ is representing two edges $e_1, e_2$ with $e_1 = e_2$ and $e_{-1} = e_{-2}$, then $w(e) = w(e_1) + w(e_2)$. In fact, $w(e)$ is the vector (in $\mathbb{Z}_{2m}$) corresponding to the multidegree of the edge $e$. So in algebraic language, the summing rule for the function $w$ on the parallel edges, is nothing but the sum of the vectors corresponding to the multidegrees of their associated variables. In the following, we let $K_{n+1}$ be the complete graph on the vertex set $V(G)$ and we let $y_e y_e = 1$. 

24
Example 6.1. In Figure 12 we apply Algorithm 2 for the graph in Figure 7(b) step-by-step to obtain the generic function $\varphi_t(G) = 1.6 + 1.\bar{4}.7 + 2.4.6 + 1.\bar{4}.5.8 + 3.8 + 2.5.8 + 2.7 + 3.5.7 + 3.4.5.6$

for the ideal $\mathcal{P}_{q,t}^O$. Thus we have

$$\mathcal{P}_{q,t}^O = \langle y_1y_6, y_1y_7y_7, y_2y_4y_6, y_1y_3y_5y_8, y_3y_8, y_2y_5y_8, y_2y_7, y_3y_5y_7, y_3y_4y_5y_6 \rangle$$

6.2. Generating functions of SMT ideals. For undirected graphs, there exist other efficient techniques from the theory of electrical networks in order to compute $\varphi_t(G)$ (see e.g., [FGK13, Corollary 6.6]. The idea behind these algorithms is as follows. Think of the weight (or multidegree) $x_e$ of each edge $e$ of $G$ as its electrical conductance. Note that summing up the multidegrees as Algorithm 2 means combining the parallel edges into one edge in terms of electrical networks.

The generating function of the path ideal for $T = V(G) \setminus \{q\}$ will be denoted by $\varphi_t(G)$ (instead of $\varphi_T(G)$). The effective conductance of an edge $e$ is

$$\text{effcond}_G(e) = \frac{\varphi(G)}{\varphi(G_e)}, \quad (6.1)$$

where $G_e$ denotes the graph obtained from $G$ by contracting the edge $e$, (see e.g., Kirchhoff’s effective conductance formula [Wag05 Sec. 2] and [FGK13 Lem. 6.4]). Without loss of generality we may assume that the subgraph induced on the vertices $v_1, \ldots, v_i$ is connected to $v_{i+1}$. We define the graphs $G_i$ recursively by $G_1 = G$ and $G_{i+1} = (G_i)_{e_i}$, where $(G_i)_{e_i}$ is the graph obtained from $G_i$ by contracting the edge $e_i$ which joins the vertex $v_{i+1}$ to the vertex obtained by gluing all the vertices $v_1, \ldots, v_i$ together. Then using (6.1) inductively, we can compute the generic function of $G$ in terms of the effective conductances on the edges $e_i$ as follows:

$$\varphi(G) = \prod_{i=1}^{n-1} \text{effcond}_{G_i}(e_i), \quad (6.2)$$

\[\text{In the expression of the generic function, we use the indices } i \text{ and } \bar{i} \text{ instead of the variables } y_i \text{ and } y_{\bar{i}}.\]
6.3. Reliability of the dual systems. We first recall that the minimal cuts are the minimal subsets of \( E(G) \) whose failure ensures the failure of the system. Similarly, minimal paths are the minimal sets in the nonfailure set. For every system \( S = (E(G), \varphi) \), its dual denoted by \( S' = (E(G), \varphi') \) is defined on \( E(G) \) with \( \varphi'(A) = 1 - \varphi(A^c) \). In other words, a path set of a system \( S \) is a cut set of the dual system \( S' \). On the other hand, the ideal associated to \( S' \) is the Alexander dual \([\text{MS05 Thm. 5.20}]\) of the ideal associated to \( S \). By Alexander inversion formula \([\text{MS05 Thm. 5.14}]\) we can also compute the \( h \)-polynomial of \( \mathcal{P}_{q,t} \) (and \( \sum_G \)) from the \( h \)-polynomial of \( \mathcal{C}_{q,t} \) (and \( \mathcal{C}_G \)) as

\[
h(S/C_{q,t};y) = h(P_{q,t};1-y) \quad \text{and} \quad h(R/C_G;x) = h(\sum_G;1-x).
\]

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