Abstract

We introduce a tool, dynamical thickening, which overcomes the infamous discontinuity of the gradient flow endpoint map near non-degenerate critical points. More precisely, we interpret the stable foliations of certain Conley pairs \((N, L)\), established in [4], as a dynamical thickening of the stable manifold. As a first application and to illustrate efficiency of the concept we reprove a fundamental theorem of classical Morse theory, Milnor’s homotopical cell attachment theorem [1]. Dynamical thickening drastically shortens and conceptually simplifies the original proof.

Consider a connected smooth manifold \(M\) of finite dimension \(n\). Suppose \(f : M \to \mathbb{R}\) is a smooth function and \(x\) is a non-degenerate critical point of \(f\) of Morse index \(k\), that is \(df_x = 0\) and in local coordinates the Hessian matrix \((\partial^2 f/\partial x^i \partial x^j)_{i,j}\) at \(x\) has precisely \(k\) negative eigenvalues, counting multiplicities, and zero is not an eigenvalue. Set \(c := f(x)\) and assume for simplicity that the level set \(\{f = c\}\) carries no critical point other than \(x\).

Morse theory studies how the topology of sublevel sets \(M^a = \{ f \leq a \}\) changes when \(a\) runs through a critical value \(c\). A fundamental tool is the concept of a flow, also called a 1-parameter group of diffeomorphisms of \(M\). A common choice is the downward gradient flow \(\{\varphi_s\}_{s \in \mathbb{R}}\), namely the one generated by the initial value problems \(\frac{d}{ds} \varphi_s = -(\nabla f) \circ \varphi_s\) with \(\varphi_0 = \text{id}_M\). Existence is guaranteed, for instance, if the vector field is of compact support. Here \(\nabla f\) denotes the gradient vector field of \(f\) on \(M\). It is uniquely determined by the identity \(df(\cdot) = g(\nabla f, \cdot)\) after fixing an auxiliary Riemannian metric \(g\) on \(M\). Key properties of the downward gradient flow are that \(f\) decays along flow lines \(s \mapsto \varphi_s q\) and that \(\nabla f\) is orthogonal to level sets. Consequently sublevel sets are forward flow invariant. Since \(df_x = 0\) implies \((\nabla f)_x = 0\), any critical point \(x\) is a fixed point of the flow and non-degeneracy translates into hyperbolicity.

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By non-degeneracy of $x$ its unstable manifold $W^u$ and descending disk $W^u_\varepsilon$, 
$$W^u = \{ q \in M \mid \lim_{s \to -\infty} \varphi_s q = x \}, \quad W^u_\varepsilon = W^u \cap \{ f > c - \varepsilon \},$$
are embedded open disks in $M$ of dimension $k = \operatorname{ind}(x)$. The topological boundary of $W^u_\varepsilon$ is denoted by $S^u_\varepsilon$ and called a descending sphere. By the Morse-Lemma $W^u_\varepsilon \cup S^u_\varepsilon = \bar{W}^u_\varepsilon$ is an embedded closed disk (as a manifold with boundary) for $\varepsilon > 0$ sufficiently small. Considering instead the limit $s \to +\infty$ one gets the stable manifold $W^s$ and (closed) ascending disk $\bar{W}^s_x = W^s(x) \cap \{ f > c + \varepsilon \}$ which have analogous properties except that they are of codimension $k$.

In [4, Thm. C] we constructed\footnote{The infinite dimensional heat flow scenario on the free loop space of $M$ studied in [4] specializes to the case at hand if we consider only constant loops, pick $V = f$, and replace the Jacobi operator by the operator $A_x$ on $T_x M$ that represents the Hessian under $g$. This way the semi-groups become essentially exponentials of $n \times n$ matrices, the analysis becomes much simpler, and we end up with a new proof of the well known finite dimensional $\lambda$-Lemma [2]. The arguments that establish the foliation in [4] simplify significantly in finite dimensions. For details we refer the reader to our upcoming lecture notes [3].} a foliation of the neighborhood
$$N = N^\varepsilon_{x,T} := \{ q \in M \mid f(q) \leq c + \varepsilon, \ f(\varphi_{-T} q) > c - \varepsilon \}_{\text{connected component of } x}$$
of $x$ whenever $\varepsilon > 0$ is small and $T > 0$ is large. The leaves are codimension-$k$ disks bounded by the upper level set $\{ f = c + \varepsilon \}$ and parametrized by their unique point of intersection $\gamma_T$ with the unstable manifold.\footnote{Convention. As opposed to [4], in the present text all leaves are closed disks.} The leaf over $x$ is $\bar{W}^s_x$. Each point of a leaf $N(\gamma_T)$ reaches the lower level set $\{ f = c - \varepsilon \}$ in time $T$ under the downward gradient flow. Note that $\{ f = c - \varepsilon \}$ intersects the unstable manifold in the $(k-1)$-sphere $S^u_\varepsilon$ which encloses $W^u_\varepsilon$. Choosing a tubular neighborhood $D$ of $S^u_\varepsilon$ in $\{ f = c - \varepsilon \}$ provides a family of codimension-$k$ disks $D_\gamma$, one for each $\gamma \in S^u_\varepsilon$. By [3, 4] we get a continuous foliation
$$N = \bar{W}^s_\varepsilon \cup \bigcup_{T > \tau, \gamma \in S^u_\varepsilon} N(\gamma_T), \quad N(\gamma_T) = \varphi_{-T}(D_\gamma) \cap \{ f \leq c + \varepsilon \},$$
which is $C^1$ away from the ascending disk $\tilde{W}^u_\varepsilon$. It is a key fact that the leaves are diffeomorphic to $W^u_\varepsilon$ via $C^1$ maps $G^T_\varepsilon: W^u_\varepsilon \to N(\gamma_T)$ which converge in $C^1$ to the identity on $W^u_\varepsilon$, as $T \to \infty$. Furthermore, the foliation is flow invariant in the sense that $\varphi_\varepsilon$ maps the leaf $N(\gamma_T)$ into $N(\varphi_\varepsilon \gamma_T)$. Figure 1 illustrates the foliation and the qualitative behavior of the flow which is transversal to all leaves except the leaf over $x$ which is invariant. Conjugation by the diffeomorphism $G^T_\varepsilon$ produces a copy $\theta_\varepsilon$ of the flow $\varphi_\varepsilon$ on $W^u_\varepsilon$ on each leaf $N(\gamma_T)$.

After having introduced the general background and the key geometric tool we are ready to state and reprove the homotopical cell attachment theorem.

**Theorem** (Milnor [1, I Thm. 3.2]). Let $f: M \to \mathbb{R}$ be a smooth function, and let $x$ be a non-degenerate critical point with Morse index $k$. Setting $f(x) = c$, suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical point of $f$ other than $x$, for some $\varepsilon > 0$. Then, for all sufficiently small $\varepsilon$, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a $k$-cell attached.

**Proof.** Fix a Riemannian metric on $M$. Without loss of generality assume that $-\nabla f$ is of compact support\(^3\) in $M$, so it generates a flow $\{\varphi_\pi\}_{\pi \in \mathbb{R}}$ on $M$. Let $A: T_x M \to T_x M$ be the symmetric operator determined by the identity $\text{Hess}_x f (\cdot, \cdot) = g(A \cdot, \cdot)$. Set $\lambda > 0$ equal to half the distance of the spectrum of $A_x$ and the origin. Choose Morse coordinates about the critical point $x$ and pick constants $\varepsilon, \tau > 0$ according to [4, Hyp. 2.2] in order to meet the assumptions in [4] of Theorem C (existence of the invariant foliation $N = N^u_{\varepsilon, \tau}$) and Definition 2.11 (induced leaf semi-flow $\theta_\varepsilon$). Now there are two steps.

I. As indicated above the forward flow $\varphi_\varepsilon$ on $\tilde{W}^s_\varepsilon$ transfers to a forward flow $\theta_\varepsilon$ on each leaf $N(\gamma_T)$ via conjugation by the diffeomorphisms $G^T_\varepsilon$; see [4, Def. 2.11]. The key property $\lim_{\varepsilon \to \infty} \varphi_\varepsilon q = x$ on $\tilde{W}^s_\varepsilon$ translates to $\lim_{\varepsilon \to \infty} \theta_\varepsilon q = \gamma_T$ on $N(\gamma_T)$. Thus the leaf flow $\theta_\varepsilon$ deformation retracts $N$ onto $N \cap W^u = \varphi_{-\varepsilon} W^u_\varepsilon$.

II. Given the closed set $Z = M^{c+\varepsilon} \setminus N$ and its closed subset $A = M^{c-\varepsilon}$, consider the entrance time function $\mathcal{T}_A: Z \to [0, \infty)$ which assigns to each point $z \in Z$ the time when its flow trajectory enters $A$. To see that $\mathcal{T}_A$ is well defined note that $Z$ and $A$ are both forward flow invariant. Indeed $\partial A$ is a level set along which $-\nabla f$ is inward pointing. Moreover, the boundary of $Z$ consists of, firstly, some part of a level set along which $-\nabla f$ is inward pointing and, secondly, the hypersurface $\partial^- N = \varphi_{-\varepsilon}^{-1}(D) \cap M^{c+\varepsilon}$ which reaches $D \subset \partial A$ precisely in time $\tau$. This hypersurface $\partial^- N$ is indicated in Figure 1 by dotted lines, since it does just not belong to $N$, consequently by solid lines in Figure 2.

The function $\mathcal{T}_A$ is lower and upper semi-continuous, hence continuous, because the subset $A$ of $Z$ is closed and forward flow invariant, respectively; cf. [4, Pf. of Thm. B]. As the set $X := Z \setminus \{f < c-\varepsilon\}$ illustrated by Figure 2 is equal to the compact and critical point free set $f^{-1}[c-\varepsilon, c+\varepsilon] \setminus N$ the function $\mathcal{T}_A$ is bounded. Consequently the map $h: [0, 1] \times (Z \cup W^u_\varepsilon) \to Z \cup W^u_\varepsilon$ defined by

$$h(\lambda, z) = \begin{cases} z, & z \in A \cup W^u_\varepsilon, \\ \varphi_{\mathcal{T}_A(z)} z, & z \in X \setminus W^u_\varepsilon, \end{cases} \quad (1)$$

\(^3\) Otherwise, substitute for $-\rho \nabla f$ where $\rho: M \to \mathbb{R}$ is a smooth compactly supported cut-off function with $\rho \equiv 1$ on the compact set $K := f^{-1}[c-\varepsilon, c+\varepsilon]$. 

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Figure 2: Deforming $N$ onto its part in $W^u$ via $\theta$ and $X$ onto $\partial A$ via flow $\varphi$ is continuous. One readily verifies that $h$ is a strong deformation retraction of the after Step I left over space $Z \cup W^u_\varepsilon$ onto $M^{c-\epsilon} \cup W^u_\varepsilon$; see Figure 2.

By [4] dynamical thickening can be defined in infinite dimensional contexts, a backward flow is not even required.

Perspectives

In the history of Morse theory discontinuity of the flow trajectory end point map obstructed to carry out, in a simple fashion, various constructions suggested by geometry, for instance, to extend continuously open unstable disks towards their closure. It will be a future research project to investigate the role of local dynamical thickening in such cases. Inspired by our ongoing work to construct global backward foliations for the forward heat flow we are tempted to believe that their finite dimensional analogues might be useful tools as well.

References

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