A COMBINATORIAL PROOF OF THE DENSE HINDMAN THEOREM

HENRY TOWSNER

ABSTRACT. The Dense Hindman’s Theorem states that, in any finite coloring of the integers, one may find a single color and a “dense” set $B_1$, for each $b_1 \in B_1$ a “dense” set $B_2^{b_1}$ (depending on $b_1$), for each $b_2 \in B_2^{b_1}$ a “dense” set $B_3^{b_1,b_2}$ (depending on $b_1, b_2$), and so on, such that for any such sequence of $b_i$, all finite sums belong to the chosen color. (Here density is often taken to be “piecewise syndetic”, but the proof is unchanged for any notion of density satisfying certain properties.) This theorem is an example of a combinatorial statement for which the only known proof requires the use of ultrafilters or a similar infinitary formalism. Here we give a direct combinatorial proof of the theorem.

1. INTRODUCTION

Hindman’s Theorem states that, in any finite coloring of the integers, some color contains an infinite set and all its finite sums. Hindman’s original proof [5] is quite complicated; fortunately, there are both simpler combinatorial arguments [1, 9] and an elegant proof based on the topology of ultrafilters (see, for instance, [3]).

Strikingly, the ultrafilter argument gives, sometimes with little additional work, various strengthenings of the theorem for which combinatorial proofs are either much harder, or not known to exist. ([6] gives a thorough exploration of many uses of ultrafilters in this context.) One such strengthening promises, not a single monochromatic infinite set and its finite sums, but a tree of such sets in which there are “many” possible first elements, for each such first element “many” possible second elements, and so on.

In this paper we give the first combinatorial proof of this theorem, modeled on Baumgartner’s proof of the ordinary Hindman’s Theorem. The key idea is the use of approximate ultrafilters, as introduced by Hirst [7]—countable collections of sets of integers which nonetheless contain enough information to complete the proof. The proof here is modeled on our related proof of the ordinary Hindman’s Theorem [8].

The reverse mathematical strength of even the ordinary Hindman’s Theorem is open; bounds are given in [2], and the gap between the lower and upper bounds on reverse mathematical strength there have not been improved. The proof given here is entirely within the bounds of second order arithmetic, but well above their upper bounds; no lower bound for the Dense Hindman’s Theorem is known besides the obvious one, that any lower bound for the ordinary Hindman’s Theorem must also bound the Dense Hindman’s Theorem.

We thank Mathias Beiglböck for bringing this question to our attention, and for many discussions about the mathematics around Hindman’s Theorem.

\footnote{Indeed, we originally found a proof quite similar to that one, and only subsequently found proof in the style of Baumgartner which we present here. This proof is slightly more elegant, and we hope that it will shed some light on the relationship between Baumgartner’s proof of the ordinary Hindman’s Theorem and the ultrafilter proof.}
2. General Definitions

Throughout this paper, variables denoted by lowercase letters will typically be natural numbers, except for \( b \), which will always belong to \( \{-1, 1\} \). Variables denoted by uppercase letters will be sets of natural numbers, and variables denoted by caligraphic letters (\( U \), \( F \), etc.) will be sets of sets of natural numbers. In fact, it would cause no harm to assume that all sets of sets of natural numbers appearing in this paper are countable, and therefore to code them using sets of natural numbers. The one exception is the property \( P \), which represents the set of sets of natural numbers satisfying some shift-invariant divisible” property, such as the infinite sets, the piecewise syndetic sets, or the sets of positive upper Banach density. (The notion of divisibility is introduced in [4].)

We adopt the notation that if \( X \) is a set of integers, \( 1 \cdot X = X \) and \( -1 \cdot X = \mathbb{N} \setminus X \). We write \( X - n \) for \( \{ m \mid m + n \in X \} \).

Definition 2.1. Let \( \mathcal{P} \) be a collection of sets of integers such that\(^2\)

\[
\begin{align*}
\text{• } &\mathbb{N} \in \mathcal{P} \\
\text{• } &\emptyset \notin \mathcal{P} \\
\text{• } &\text{If } X \subseteq Y \text{ and } X \in \mathcal{P} \text{ then } Y \in \mathcal{P} \\
\text{• } &\text{If } X_0 \cup X_1 = X \text{ and } X \in \mathcal{P} \text{ then either } X_0 \in \mathcal{P} \text{ or } X_1 \in \mathcal{P} \\
\text{• } &\text{For any } X \text{ and any } n, X \in \mathcal{P} \text{ iff } X - n \in \mathcal{P} 
\end{align*}
\]

We say a collection \( \mathcal{U} \) of sets of integers has the \( \mathcal{P} \)-finite intersection property (\( \mathcal{P} \)-fip) if

\[
\bigcap_{S \in \mathcal{F}} S \in \mathcal{P}.
\]

Natural examples of such properties \( \mathcal{P} \) include:

\[
\begin{align*}
\text{• } &\mathcal{P} \text{ is the collection of infinite sets} \\
\text{• } &\mathcal{P} \text{ is the collection of sets with positive upper Banach density} \\
\text{• } &\mathcal{P} \text{ is the collection of piecewise syndetic sets} \\
\text{• } &\mathcal{P} \text{ is the collection of sets } X \text{ such that } \sum_{x \in X} 1/x = \infty
\end{align*}
\]

Properties satisfying all but the final condition, that \( X \in \mathcal{P} \text{ iff } X - n \in \mathcal{P} \), are called divisible \(^4\).

Definition 2.2. Let \( \mathcal{U} \) be a countable collection of sets of integers\(^3\). We write \( X \in \mathcal{U} \) if there is a finite \( \mathcal{F} \subseteq \mathcal{U} \) such that \( \bigcap_{S \in \mathcal{F}} S \subseteq X \).

We say \( \mathcal{U} \) is a \( \mathcal{P} \)-semigroup if \( \mathcal{U} \) satisfies \( \mathcal{P} \)-fip, \( \mathcal{U} \) is closed under finite intersections, and whenever \( X \in \mathcal{U} \), there is a \( Y \in \mathcal{U} \) such that \( X - n \in \mathcal{U} \) for each \( n \in Y \).

There are two useful ways to view \( \mathcal{P} \)-semigroups. The first is to observe that every \( \mathcal{P} \)-semigroup represents a closed semigroup in the space of ultrafilters on the integers (namely, the collection of ultrafilters extending \( \mathcal{U} \)).

The second is to recall that an IP set is a set \( S \) such that there is an infinite \( T \subseteq S \) all of whose finite sums also belong to \( S \). Then the collection \( \{ S - n \mid n \in FS(T) \} \) (where \( FS(T) \) is the finite sums from \( T \)) is a canonical example of a \( \mathcal{P} \)-semigroup where \( \mathcal{P} \) is the collection of infinite sets. The notion of a \( \mathcal{P} \)-semigroup generalizes an IP set in two directions: first, it allows for arbitrary \( \mathcal{P} \). (Note that the appropriate requirement is not that

---

\(^2\)To keep our promise that the proof go through in second order arithmetic, we should insist that \( \mathcal{P} \) be given by some arithmetic formula; this includes all the examples given.

\(^3\)None of our arguments would change if uncountable collections—say, true ultrafilters—are allowed. However we wish to emphasize that none of our arguments will require more than countable collections.
$T \in \mathcal{P}$; rather, the requirement is that there are many possible choices for the infinite set $T$—indeed, the an infinite tree of such sets, with the number of possible branches at each level belonging to $\mathcal{P}$. This discussion will be made precise when we state the main theorem below.) Second, if we have an infinite descending sequence of IP sets $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$, their intersection may well be 0. However the union of the corresponding $\mathcal{P}$-semigroups is still a $\mathcal{P}$-semigroup. So $\mathcal{P}$-semigroups also generalize IP sets by accommodating the result of infinitely many successive refinements of an IP set.

The argument here is very similar to a proof using ultrafilters, but we emphasize that the $\mathcal{P}$-semigroups appearing in the proof are much simpler objects: they are countable collections, built with no use of the axiom of choice.

The following lemma is our essential building block:

**Lemma 2.3.** Let $\mathcal{U}$ satisfy $\mathcal{P}$-fip, and let $A$ be a set of integers. Then either $\mathcal{U} \cup \{A\}$ or $\mathcal{U} \cup \{-1 \cdot A\}$ satisfies $\mathcal{P}$-fip.

**Proof.** Suppose neither collection satisfies $\mathcal{P}$-fip. Then choose finite sets $\mathcal{F}, \mathcal{F}' \subseteq \mathcal{U}$ such that $\bigcup_{S \in \mathcal{F}} S \cap A \notin \mathcal{P}$ and $\bigcup_{S \in \mathcal{F}'} S \cap (-1 \cdot X) \notin \mathcal{P}$. Then

$$\bigcap_{S \in \mathcal{F} \cup \mathcal{F}'} S \subseteq \left( \bigcup_{S \in \mathcal{F}} S \cap X \right) \cup \left( \bigcup_{S \in \mathcal{F}'} S \cap (-1 \cdot X) \right).$$

But this is impossible, since $\bigcap_{S \in \mathcal{F} \cup \mathcal{F}'} S \in \mathcal{P}$ must hold. \qed

3. Dense Hindman’s Theorem

**Lemma 3.1.** If $\mathcal{U}$ is a $\mathcal{P}$-semigroup, $A$ is a set, $S$ is a set with the property that

$$\mathcal{U} \cup \{-1 \cdot (A - n) \mid n \notin S\}$$

satisfies $\mathcal{P}$-fip, and $\mathcal{U} \cup \{S - n \mid n \in S\}$ does not satisfy $\mathcal{P}$-fip, then there is a finite $F \subseteq S$ and a $Y \in \mathcal{U}$ such that $0 \notin Y$ and $\mathcal{U} \cup \\{\cup_{n \in F} -1 \cdot (A - n - m) \mid m \in Y\}$ satisfies $\mathcal{P}$-fip.

**Proof.** Since $\mathcal{U} \cup \{S - n \mid n \in S\}$ does not satisfy $\mathcal{P}$-fip, let $X \in \mathcal{U}$ and $F \subseteq S$ be such that $X \cap \bigcap_{n \in F} S - n \notin \mathcal{P}$. Let $Y$ be the set of $n$ such that $X - n \in \mathcal{U}$; clearly $0 \notin Y$ since $X \in \mathcal{U}$.

We claim that $\mathcal{U} \cup \\{\cup_{n \in F} -1 \cdot (A - n - m) \mid m \in Y\}$ satisfies $\mathcal{P}$-fip. Let $Z \in \mathcal{U}$ and $G \subseteq Y$ be finite. Let $Z' \in \mathcal{U}$ be such that $Z - n \in \mathcal{U}$ for each $n \in Z'$; then $Z' \cap \bigcap_{m \in G} \bigcup_{n \in F} -1 \cdot (S - n - m) \in \mathcal{U}$. If we choose any $k$ in this set, we have $Z - k \cap \bigcap_{m \in G} \bigcup_{n \in F} -1 \cdot (A - k - n - m) \in \mathcal{P}$, and therefore $Z \cap \bigcap_{m \in G} \bigcup_{n \in F} -1 \cdot (A - n - m) \in \mathcal{P}$. \qed

In particular, observe that if

$$\mathcal{U} \cup \{A - n \mid n \in S\}$$

satisfies $\mathcal{P}$-fip but whenever $m \notin S$,

$$\mathcal{U} \cup \{A - n \mid n \in S\} \cup \{A - m\}$$

does not satisfy $\mathcal{P}$-fip then $S$ satisfies the condition of this lemma.

**Definition 3.2.** We say $A$ is large relative to a $\mathcal{P}$-semigroup $\mathcal{U}$ if whenever $V$ is a $\mathcal{P}$-semigroup extending $\mathcal{U}$, $V \cup \{A\}$ satisfies $\mathcal{P}$-fip.

**Lemma 3.3.** If $C_1 \cup \cdots \cup C_n$ and $C$ is large relative to $\mathcal{U}$, there is a $V$ extending $\mathcal{U}$ and an $i$ such that $C_i$ is large for $V$. 

Proof. Applying induction, it suffices to consider the case \( n = 2 \). If \( C_1 \) is large relative to \( U \) then 1 and \( U \) suffice. Otherwise there is a \( V \) extending \( U \) such that \( V \cup \{ C_1 \} \) fails to satisfy \( \mathfrak{P} \)-fip, and therefore \( C_2 \) is large relative to \( V \).

\( \square \)

Theorem 3.4. Let \( U \) be a \( \mathfrak{P} \)-semigroup and \( A \) a set. If \( A \) is large for \( U \) then there is a \( V \) extending \( U \) such that \( A \subseteq V \) belongs to \( \mathfrak{P} \).

Proof. Given \( U \), expand to some \( \mathfrak{P} \)-semigroup \( V \) such that for every finite set \( F \), either \( \bigcap_{n \in F} A - n \) is large for \( V \) or \( \bigcup_{n \in F} -1 \cdot (A - n) \in \mathcal{V} \).

Let \( S \) be a maximal set of \( n \) so that \( 0 \in S \), \( F \subseteq S \) implies \( \bigcap_{n \in F} A - n \) is large for \( V \), and \( m \notin S \) implies that there is an \( F \subseteq S \) so that \( \bigcap_{n \in F \cup \{ m \}} A - n \) is not large for \( V \).

Then if \( F \subseteq -1 \cdot S \) is finite, we may choose a \( G \subseteq S \) so that \( \bigcap_{n \in G \cup \{ m \}} A - n \) is not large for \( V \) for any \( m \in F \). Then \( \bigcup_{n \in G \cup \{ m \}} -1 \cdot (A - n) \in \mathcal{V} \) for each \( m \in F \), and since \( V \cup \{ \bigcap_{n \in G} A - n \} \) satisfies \( \mathfrak{P} \)-fip, it follows that \( V \cup \{ -1 \cdot (A - m) \mid m \in F \} \) satisfies \( \mathfrak{P} \)-fip. This holds for every finite \( F \subseteq -1 \cdot S \), so \( V \cup \{ -1 \cdot (A - m) \mid m \notin S \} \) satisfies \( \mathfrak{P} \)-fip.

Therefore, by Lemma 3.1, if \( V \cup \{ S - n \mid n \in S \} \) does not satisfy \( \mathfrak{P} \)-fip then there is a finite set \( F \subseteq S \) and a \( V \) extending \( U \) such that \( \bigcup_{n \in F} -1 \cdot (A - n) \notin \mathcal{V} \). But this would contradict the fact that \( \bigcap_{n \in F} A - n \) is large for \( V \). So \( V \cup \{ S - n \mid n \in S \} \) satisfies \( \mathfrak{P} \)-fip, and is therefore a \( \mathfrak{P} \)-semigroup.

Since \( A \) is large for \( V \), also \( V \cup \{ S - n \mid n \in S \} \cup \{ A \} \) satisfies \( \mathfrak{P} \)-fip, so in particular, \( S \cap A \in \mathfrak{P} \). Since for each \( n \in S \), \( A \cap A - n \) is large for \( V \), the claim is proven.

\( \square \)

Definition 3.5. For any set of integers \( S \), let \( FS(S) \) be the collection of finite sums from \( S \):

\[ FS(S) = \{ \sum_{i \in T} i \mid T \subseteq S, T \text{ finite} \}. \]

Theorem 3.6. Let \( \mathbb{N} = A_1 \cup \cdots \cup A_r \). There is some \( i \leq r \) and a tree \( T \) of finite sets of integers such that:

- \( \emptyset \in T \)
- If \( F \in T \), \( \{ n \mid F \cup \{ n \} \in T \} \) belongs to \( \mathfrak{P} \)
- If \( F \in T \) then \( FS(F) \setminus \{ 0 \} \subseteq A_i \)

Proof. Since \( \mathbb{N} \) is large for the trivial \( \mathfrak{P} \)-semigroup \( \{ \mathbb{N} \} \), we may choose a \( \mathfrak{P} \)-semigroup \( U \) and an \( i \) so that \( A_i \) is large for \( U \). By the preceding theorem, we obtain a \( \mathfrak{P} \)-semigroup \( U \) extending \( U \) so that \( \{ n \in A_i \mid A_i \cap A_i - n \text{ is large for } U' \} \) belongs to \( \mathfrak{P} \). We place \( \{ n \} \) in \( T \) if \( A_i \cap A_i - n \) is large for \( U' \).

Suppose we have \( F \subseteq T \) and have a \( V \) so that \( A' := \bigcap_{n \in FS(F) \setminus \{ 0 \}} A_i - n \) is large for \( V \). Again by the preceding theorem, we may find a \( V' \) such that \( \{ m \in A' \mid A' \cap A' - m \text{ is large for } V' \} \) belongs to \( \mathfrak{P} \); then we place \( F \cup \{ m \} \) in \( T \) for each such \( m \), thereby maintaining the inductive assumption.

\( \square \)

References

[1] James E. Baumgartner. A short proof of Hindman’s theorem. J. Combinatorial Theory Ser. A, 17:384–386, 1974.
[2] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson. Logical analysis of some theorems of combinatorics and topological dynamics. In Logic and combinatorics (Arcata, Calif., 1985), volume 65 of Contemp. Math., pages 125–156. Amer. Math. Soc., Providence, RI, 1987.
[3] W. W. Comfort. Ultrafilters: some old and some new results. Bull. Amer. Math. Soc., 83(4):417–455, 1977.
[4] S. Glasner. Divisible properties and the Stone-Čech compactification. Canad. J. Math., 32(4):993–1007, 1980.
[5] Neil Hindman. Finite sums from sequences within cells of a partition of *N. J. Combinatorial Theory Ser. A*, 17:1–11, 1974.

[6] Neil Hindman and Dona Strauss. *Algebra in the Stone-Čech compactification*, volume 27 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1998. Theory and applications.

[7] Jeffry L. Hirst. Hindman’s theorem, ultrafilters, and reverse mathematics. *J. Symbolic Logic*, 69(1):65–72, 2004.

[8] Henry Towsner. Hindman’s theorem: An ultrafilter argument in second order arithmetic. http://arxiv.org/abs/0906.3882.

[9] Henry Towsner. A simple proof of Hindman’s theorem. http://arxiv.org/abs/0906.3885.