On Singularities and Instability for Different Couplings between Scalar Field and Multidimensional Geometry

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Abstract

We consider a multidimensional model of the universe given as a $D$-dimensional geometry, represented by a Riemannian manifold $(M, g)$ with arbitrary signature of $g$, $M = \mathbb{R} \times M_1 \times \cdots \times M_n$, where the $M_i$ of dimension $d_i$ are Einstein spaces, compact for $i > 1$. For Lagrangian models $L(R, \phi)$ on $M$ which depend only on the Ricci curvature $R$ and a scalar field $\phi$, there exists a conformal equivalence with minimal coupling models. For certain nonminimal models we study classical solutions and their relation to solutions in the equivalent minimal coupling model. The domains of equivalence are separated by certain critical values of the scalar field $\phi$. Furthermore, the coupling constant $\xi$ of the coupling between $\phi$ and $R$ is critical at both, the minimal value $\xi = 0$ and the conformal value $\xi_c = \frac{D-2}{4(D-1)}$. In different noncritical regions of $\xi$ the solutions behave qualitatively different. Instability can occur only in certain ranges of $\xi$.

1 Introduction

Gravitational models of multidimensional universes receive increasing interest, since they provide a class of minisuperspace models, which is rich enough to study the relation and the imprint of internal compactified extra dimensions\textsuperscript{1,2} on the external space-time. In this paper we will consider classical multidimensional models with respect to their properties in dependence on the form of the coupling between geometry and a scalar field. The $D$-dimensional geometry is represented by a Riemannian manifold $(M, g)$ with either Lorentzian or Euclidean signature of $g$, $M = \mathbb{R} \times M_1 \times \cdots \times M_n$ with $D = 1 + \sum_{i=1}^{n} d_i$, where the $M_i$ of dimension $d_i$ are Einstein spaces, compact for $i > 1$. The first of these spaces is to be considered as our observable exterior space, while all the other factors represent internal spaces to be hidden at present time.

In Sec. 2 we compare conformal transformations of Lagrangian models to conformal coordinate transformations for a $D$-dimensional geometry and define conformal equivalence.

In Sec. 3 we examine relations between conformally equivalent Lagrangian models for $D$-dimensional geometry coupled to a spatially homogeneous scalar field. Here, the

\textsuperscript{1}This paper is dedicated to Prof. D. D. Ivanenko on the occasion of his 90th birthday.
conformal coupling constant $\xi$ plays a distinguished role. We consider as example of special interest the conformal transformation between a model with minimally coupled scalar field (MCM) and a conformally equivalent model with a conformally coupled scalar field (CCM), thus generalizing previous results from Refs. 3 and 4, obtained for $n = 1$ and $D = 4$.

Sec. 4 introduces multidimensional cosmological models.

In Sec. 5 natural time gauges for multidimensional universes given by the choices of i) the synchronous time $t_s$ of the universe $M$, ii) the conformal time $\eta_i$ of a universe with the only spacial factor $M_i$, iii) the mean conformal time $\eta$, given differentially as some scale factor weighted average of $\eta_i$ over all $i$ and iv) the harmonic time $t_h$, which will be used as specially convenient in calculations on minisuperspace, since in this gauge the minisuperspace lapse function is $N \equiv 1$.

In Sec. 6 the considerations on conformally equivalent Lagrangian models from Sec. 3 will be pursued for the analysis of multidimensional cosmological models on the level of solutions.

Sec. 7 is devoted to the distinguished role of the conformal coupling constant $\xi = \frac{D-2}{4(D-1)}$ as compared to other values of $\xi$, with special emphasis to stability considerations. Number theoretical analysis of $\xi$, gives some hint, that it might play a crucial role in the dimensional reduction process, since $\xi$ is especially simple for $D = 3, 4, 6, 10$ and $26$. This is important for the cosmic evolution, since it was e.g. shown in Ref. 5 that inflation depends critically on the number of dimensions.

Sec. 8 approaches the question of stability in dependence of the dimension $D$ for a simple model with static internal spaces instability conditions for the ground state of a scalar field. The imprint of the extra dimensions yields conditions, which explicitely contain the conformal coupling constant $\xi$.

Sec. 9 resumes the perspective of the present results.

2 Conformal Transformations

One remark ab initio: throughout the following, on a geometry $g$ on $M$, conformal transformations will actually be represented as local Weyl transformations $g \rightarrow e^{2f}g$ with $f \in C^\infty(M)$.

We consider a differentiable manifold $M$. Equipped with a Riemannian structure $g_{ij}$ and scalar fields $(\phi^1, \ldots, \phi^k)$ on $M$ we obtain a Lagrangian model by imposing a Lagrangian variation principle

$$\delta S = 0 \quad \text{with} \quad S = \int_M \sqrt{|g|} L d^Dx$$  \hfill (2.1)

given by a second order Lagrangian

$$L = L(g_{ij}, \phi^1, \ldots, \phi^k; g_{ij,\ell}, \phi_{ij}^1, \ldots, \phi_{ij}^k; g_{ij,lm}).$$  \hfill (2.2)
Generally, we have to distinguish between conformal coordinate transformations in $D$-dimensional geometry and conformal transformations of Lagrangian models for $D$-dimensional geometry.

**Conformal transformation to new coordinates:**

We fix a Lagrangian model and transform the metric tensor components conformally,

$$g'_{ij} = e^{2f(x)} g_{kl},$$

via a coordinate transform satisfying

$$dx'^i = e^{-f(x)} dx^i \quad \text{or} \quad \partial x'^i / \partial x^j = e^{-f(x)} \delta^i_j.$$ (2.4)

Due to general covariance the model is still the same, though looking different in different coordinate frames.

**Conformal transformations of Lagrangian models:**

Conformal transformation of the Lagrange model keeps $M$ fixed as a differentiable manifold, but varies its additional structures conformally

$$(g_{ij}, \phi^1, \ldots, \phi^k) \rightarrow (\hat{g}_{ij}, \hat{\phi}^1, \ldots, \hat{\phi}^k),$$ (2.5)

yielding a new variational principle by demanding

$$\sqrt{|g|} L = \sqrt{|\hat{g}|} \hat{L}$$ (2.6)

for the new Lagrangian

$$\hat{L} = \hat{L}(\hat{g}_{ij}, \hat{\phi}^1, \ldots, \hat{\phi}^k; \hat{g}_{ij,l}, \hat{\phi}^1_l, \ldots, \hat{\phi}^k_l; \hat{g}_{ij,lm}).$$ (2.7)

The action remains the same but the Lagrange density becomes a new functional. Therefore, conformal transformations of models are performed in practice on a fixed coordinate patch $x^i$ of $M$. Lagrange models related in this way by a conformal transformation are called conformally equivalent.

3 Conformally Equivalent Lagrangian Models

In this section we study transformations from a Lagrangian model with minimally coupled scalar field (MCM) to a conformally equivalent one with nonminimal coupled scalar field and vice versa.

Let us follow Ref. 7 and consider an action of the kind

$$S = \int d^D x \sqrt{|g|}(F(\phi, R) - \frac{\xi}{2}(\nabla \phi)^2),$$ (3.1)
where \( F(\phi, R) \) in general describes nonminimal coupling. With
\[
\omega := \frac{1}{D-2} \ln(2\kappa^2 \lvert \frac{\partial F}{\partial R} \rvert) + C
\]
the conformal factor
\[
e^\omega = [2\kappa^2 \lvert \frac{\partial F}{\partial R} \rvert]^{\frac{1}{D-2}} e^C
\]
yields a conformal transformation from \( g_{\mu\nu} \) to the MCM metric
\[
\hat{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}.
\]
We note that CCM quantities \( x \) correspond to MCM quantities \( \hat{x} \) and the scalar field \( \phi \) to \( \Phi \) respectively.

Especially, let us consider actions being linear in \( R \):
\[
S = \int d^Dx \sqrt{|g|} (f(\phi)R - V(\phi) - \frac{\epsilon}{2}(\nabla \phi)^2).
\]
The MCM metric is then related to the metric of the conformal coupling model (CCM) by (3.4) with
\[
\omega = \frac{1}{D-2} \ln(2\kappa^2 \lvert f(\phi) \rvert) + C
\]
The scalar field in the MCM is
\[
\Phi = \kappa^{-1} \int d\phi \left\{ \frac{\epsilon(D-2)f(\phi) + 2(D-1)(f'(\phi))^2}{2(D-2)f^2(\phi)} \right\}^{1/2} =
\]
\[
= (2\kappa)^{-1} \int d\phi \left\{ \frac{2\epsilon f(\phi) + \xi^{-1}(f'(\phi))^2}{f^2(\phi)} \right\}^{1/2},
\]
where
\[
\xi := \frac{D-2}{4(D-1)}
\]
is the conformal coupling constant.

For the following we define sign\( x \) to be \( \pm 1 \) for \( x \geq 0 \) resp. \( x < 0 \). Then, with the new minimally coupled potential
\[
U(\Phi) = (\text{sign} f(\phi)) \ [2\kappa^2 \lvert f(\phi) \rvert]^{-(D/2)-2} V(\phi)
\]
the corresponding minimal action is
\[
S = \text{sign} f \int d^Dx \sqrt{\lvert \hat{g} \rvert} \left( -\frac{1}{2} ((\hat{\nabla} \Phi)^2 - \frac{1}{\kappa^2} \hat{R}) - U(\Phi) \right).
\]
To be explicit, we will concentrate now on a model with
\[
f(\phi) = \frac{1}{2}(1 - \xi \phi^2),
\]
\[ V(\phi) = \Lambda. \quad (3.12) \]

Then the constant potential \( V \) has its minimal correspondence in a non constant \( U \), given by
\[ U(\Phi) = \pm \Lambda |\kappa^2(1 - \xi \phi^2)|^{-D/D-2} \quad (3.13) \]
respectively for \( \phi^2 < \xi^{-1} \) or \( \phi^2 > \xi^{-1} \). Let us set in the following \( \epsilon = 1 \). Then with
\[ f'(\phi) = -\xi \phi \quad (3.14) \]
we obtain
\[ \Phi = \kappa^{-1} \int d\phi \left\{ \frac{1 + c \xi \phi^2}{(1 - \xi \phi^2)^2} \right\}^{1/2}, \quad (3.15) \]
where
\[ c := \frac{\xi}{\xi_c} - 1. \quad (3.16) \]

For \( \xi = 0 \) it is \( \Phi = \kappa^{-1} \phi + A \), i.e. the coupling remains minimal. To solve this integral for \( \xi \neq 0 \), we substitute \( u := \xi \phi^2 \).

To assure a solution of (3.15) to be real, let us assume \( \xi \geq \xi_c \) yielding \( c \geq 0 \). Then we obtain
\[ \Phi = \text{sign}(\phi) \frac{\text{sign}(1 - \xi \phi^2) \phi}{2 \kappa \sqrt{\xi}} \ln \left[ \frac{2 \sqrt{1 + c \sqrt{1 + c \xi \phi^2}} \sqrt{\xi} |\phi| + (2 c + 1) \xi \phi^2 + 1)^{1+c}}{2 \sqrt{c} \sqrt{1 + c \xi \phi^2} \sqrt{\xi} |\phi| + 2 c \xi \phi^2 + 1 |1 - \xi \phi^2|^1} \right] + C_{<}, \quad (3.17) \]
The integration constants \( C_{<} \) for \( \phi^2 < \xi^{-1} \) and \( \phi^2 > \xi^{-1} \) respectively may be arbitrary functions of \( \xi \) and the dimension \( D \). The singularities of the transform \( \phi \rightarrow \Phi \) are located at \( \phi^2 = \xi^{-1} \).

If the coupling is conformal \( \xi = \xi_c \), i.e. \( c = 0 \), the expressions (3.17) simplify to
\[ \kappa \Phi = \frac{1}{\sqrt{\xi_c}}[(\text{artanh}\sqrt{\xi_c \phi}) + c_{<}] \quad (3.18) \]
for \( \phi^2 < \xi_c^{-1} \) and to
\[ \kappa \Phi = \frac{1}{\sqrt{\xi_c}}[(\text{arcoth}\sqrt{\xi_c \phi}) + c_{>}] \quad (3.19) \]
for \( \phi^2 > \xi_c^{-1} \). Then the inverse formulas expressing the conformal field \( \phi \) in terms of the minimal field \( \Phi \) are
\[ \phi = \frac{1}{\sqrt{\xi_c}} \left[ \tanh(\sqrt{\xi_c \kappa \Phi} - c_{<}) \right] \quad (3.20) \]
with \( \phi^2 < \xi_c^{-1} \) and
\[ \phi = \frac{1}{\sqrt{\xi_c}} \left[ \coth(\sqrt{\xi_c \kappa \Phi} - c_{>}) \right] \quad (3.21) \]
with $\phi^2 > \xi_c^{-1}$ respectively. This result agrees with Ref. 8. For $D = 4$ it has been obtained earlier in Refs. 3, 4 and 9. In Ref. 9 it has been shown for $D = 4$, that while the MCM shows a curvature singularity, the conformal coupling model with $\phi$ of Eq. (3.20) has no such singularity.

The conformal factor is according to Eqs. (3.6) and (3.11) given by

$$\omega = \frac{1}{D-2} \ln(\kappa^2|1 - \xi_c \phi^2|) + C.$$  (3.22)

The singularity of the conformal transformation (3.22) at $\phi^2 = \xi_c^{-1}$ separates different regions in $\phi$ where conformal equivalence between the MCM and CCM holds. Eqs. (3.20) and (3.21) illustrate the qualitatively different behavior in the two regions. In Ref. 10 qualitative differences have also been found in multidimensional solutions of the respective models.

4 Lagrangian Models for Multidimensional Cosmology

We consider a geometry described by a (Pseudo-) Riemannian manifold

$$M = \mathbb{R} \times M_1 \times \ldots \times M_n,$$

with the first fundamental form

$$g \equiv ds^2 = -e^{2\gamma} dt \otimes dt + \sum_{i=1}^{n} a_i^2 ds_i^2,$$  (4.1)

where $a_i = e^{\beta_i}$ is the scale factor of the $d_i$-dimensional space $M_i$ with the first fundamental form

$$ds_i^2 = g_{kl}^{(i)} dx_{(i)}^k \otimes dx_{(i)}^l.$$  (4.2)

If we assume within a multidimensional geometry (4.1) that the $M_i$ are Einstein spaces of constant curvature, then the Ricci scalar curvature of $M$ is

$$R = e^{-2\gamma} \left\{ \left[ \sum_{i=1}^{n} (d_i \dot{\beta}_i) \right]^2 + \sum_{i=1}^{n} d_i [(\dot{\beta}_i)^2 - 2\gamma \dot{\beta}_i^2 + 2\ddot{\beta}_i^2] \right\} + \sum_{i=1}^{n} R^{(i)} e^{-2\beta}.$$  (4.3)

Let us now consider a variation principle with the action

$$S = S_{EH} + S_{GH} + S_M,$$  (4.4)

where

$$S_{EH} = \frac{1}{2\kappa^2} \int_M \sqrt{|g|} R \, dx$$

is the Einstein-Hilbert action, $S_{GH}$ is the Gibbons-Hawking boundary term$^{11}$, and $S_M$ the action of matter.
Let us consider the matter given by a minimally coupled scalar field $\Phi$ with potential $U(\Phi)$. Then the variational principle of (4.4) is equivalent to a Lagrangian variational principle over the minisuperspace $\mathcal{M}$, which is spanned by the $\beta^i$, and the scalar field $\Phi$, \[ S = \int \mathcal{L} dt, \]

with
\[ \mathcal{L} = \frac{1}{2} \mu e^{-\gamma \sum_{i=1}^n d_i \beta^i} \left\{ \sum_{i=1}^n d_i (\dot{\beta}^i)^2 - \left[ \sum_{i=1}^n d_i \dot{\beta}^i \right]^2 + \kappa^2 \dot{\Phi}^2 \right\} - V(\beta^i, \Phi), \] (4.5)

where
\[ \mu := \kappa^{-2} \prod_{i=1}^n \sqrt{\mid \det g^{(i)} \mid}. \] (4.6)

It is a convenient procedure of cosmologists, to extend the minisuperspace $\mathcal{M}$ of pure geometry directly by an additional dimension from the scalar field $\Phi$ as further minisuperspace coordinate, yielding an enlarged minisuperspace $\mathcal{MS}$.

Let us define a metric on $\mathcal{MS}$, given in coordinates $\beta^i$, $i = 1, \ldots, n+1$ with $\beta^{n+1} := \kappa \Phi$. We set
\[ G_{n+1i} = G_{i,n+1} := \delta_{i,n+1} \quad \text{and} \quad G_{kl} := d_k \delta_{kl} - d_k d_l \] (4.7)
for $i = 1, \ldots, n+1$ and $k, l = 1, \ldots, n$, thus defining the components $G_{ij}$ of the minisuperspace metric
\[ G = G_{ij} d\beta^i \otimes d\beta^j. \] (4.8)

Then we get the Lagrangian
\[ L = \frac{\mu}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j - V(\beta^i). \] (4.9)

with the energy constraint
\[ \frac{\mu}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j + V(\beta^i) = 0. \] (4.10)

Independent global conformal transformations of the spaces $M^{(i)}$ yield just translations in the functions $\beta^i$.

5 Natural Times in Multidimensional Geometry

For this geometry let us compare different choices of time $\tau$ in Eq. (4.1). The time gauge is determined by the function $\gamma$. There exist few time gauges, natural from the physical point of view.

i) The synchronous time gauge
\[ \gamma \equiv 0, \] (5.1)
for which $t$ in Eq. (4.1) is the proper time $t_s$ of the universe. The clocks of geodesically comoved observers go synchronous to that time.

ii) The conformal time gauges on $\mathbb{R} \times M_i \subset M$

$$
\gamma \equiv \beta^i,
$$

for which $t$ in Eq. (4.1) is the conformal time $\eta_i$ of $M_i$ for some $i \in \{1, \ldots, n\}$, given by

$$
d\eta_i = e^{-\beta^i} dt_s.
$$

iii) The mean conformal time gauge on $M$:

For $n > 1$ and $\beta^2 \neq \beta^1$ on $M$ the usual concept of a conformal time does no longer apply. Looking for a generalized "conformal time" $\eta$ on $M$, we set

$$
d := D - 1 = \sum_{i=1}^n d_i
$$

and consider the gauge

$$
\gamma \equiv \frac{1}{d} \sum_{i=1}^n d_i \beta^i,
$$

which yields a time $t \equiv \eta$ given by

$$
d\eta = \left( \prod_{i=1}^n a_i^{d_i} \right)^{-1/d} dt_s.
$$

Here $\prod_{i=1}^n a_i^{d_i}$ is proportional to the volume of $d$-dimensional spacial sections in $M$ and the relative time scale factor

$$
\left( \prod_{i=1}^n a_i^{d_i} \right)^{1/d} = e^{\frac{1}{d} \sum_i d_i \beta^i}
$$

is given by a scale exponent, which is the dimensionally weighted arithmetic mean of the spacial scaling exponents of spaces $M_i$. It is

$$
(dt_s)^d = e^{\sum_i d_i \beta^i} d\eta^d.
$$

On the other hand by Eq. (5.3) we have

$$
(dt_s)^d = \otimes_{i=1}^n \left( e^{\beta^i} d\eta_i \right)^{d_i},
$$

and together with Eq. (5.8) we get

$$
(d\eta)^d = e^{-\sum_i d_i \beta^i} \otimes_{i=1}^n \left( e^{\beta^i} d\eta_i \right)^{d_i}.
$$

So the time $\eta$ is a mean conformal time, given differentially as a dimensionally scale factor weighted geometrical tensor average of the conformal times $\eta_i$. An alternative to the mean
conformal time $\eta$ is given by a similar differential averaging like Eq. (5.10), but weighted by an additional factor of $e^{(1-d)\sum_i d_i \beta^i}$. This gauge is described in the following.

iv) The harmonic time gauge

$$\gamma \equiv \gamma_h := \sum_{i=1}^n d_i \beta^i \tag{5.11}$$

yields the time $t \equiv t_h$, given by

$$dt_h = \left( \prod_{i=1}^n a_i^{d_i} \right)^{-1} dt_s = \left( \prod_{i=1}^n a_i^{d_i} \right)^{\frac{1-d}{d}} d\eta. \tag{5.12}$$

In this gauge any function $\varphi$ with $\varphi(t, y) = t$ is harmonic, i.e. $\Delta[g]\varphi = 0$, and the minisuperspace lapse function is $N \equiv 1$. This gauge is especially convenient when we work in minisuperspace.

In application to the model of Sec. 4 note that, if e.g. time is harmonic in the MCM

$$\tau \equiv t^{(m)}_h, \tag{5.13}$$

in the CCM it cannot be expected to be harmonic either, i.e. in general

$$\tau \neq t^{(c)}_h. \tag{5.14}$$

In general natural time gauges are not preserved by conformal transformations of the geometry. They have to be calculated by a coordinate transformation in each of the conformally equivalent models separately.

6 Different Couplings in Multidimensional Cosmologies

In the following we want to pursue the comparison of the MCM and the CCM on the level of their classes of solutions for a multidimensional geometrical model of cosmology. Let us specify the geometry for the MCM to be of multidimensional type (4.1), with all $M_i$, $i = 1, \ldots, n$, being Ricci flat and the minimally coupled scalar field to have zero potential $U \equiv 0$. In the harmonic time gauge (5.11) with harmonic time

$$\tau \equiv t^{(m)}_h, \tag{6.1}$$

we demand this model to be a solution for Eq. (4.9) with vanishing $R^{(1)}$ and $U(\Phi)$ with $\beta^{n+1} = \kappa \Phi$. This solution is a multidimensional Kasner like universe, given by

$$\dot{\beta}^i = b^i \tau + c^i \text{ and } \dot{\gamma} = \sum_{i=1}^n d_i \dot{\beta}^i = \left( \sum_{i=1}^n d_i b^i \right) \tau + \left( \sum_{i=1}^n d_i c^i \right), \tag{6.2}$$
with \( i = 1, \ldots, n + 1 \), where with \( V \equiv 0 \) the constraint Eq. (4.10) simply reads

\[
G_{ij}b^i b^j + (b^{n+1})^2 = 0. \quad (6.3)
\]

With Eq. (3.22) the scaling powers of the universe given by Eqs. (6.2) with \( i = 1, \ldots, n \) transform to corresponding scale factors of the CCM universe

\[
\beta_i = \hat{\beta}_i - \omega = b^i \tau + \frac{1}{2 - D} \ln |1 - \xi_c(\phi)^2| + c_i + \frac{2}{2 - D} \ln \kappa - C \quad (6.4)
\]

and

\[
\gamma = \sum_{i=1}^{n} d_i \beta_i = (\sum_i d_i b^i) \tau + \frac{1}{2 - D} \ln |1 - \xi_c(\phi)^2| + (\sum_i d_i c_i) + \frac{2}{2 - D} \ln \kappa - C. \quad (6.5)
\]

Let us take for simplicity

\[
C = \frac{2}{2 - D} \ln \kappa, \quad (6.6)
\]

which yields the lapse function

\[
e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c_i) |1 - \xi_c(\phi)^2|^{1/(2-D)}}. \quad (6.7)
\]

and for \( i = 1, \ldots, n \) the scale factors

\[
e^{\beta_i} = e^{b^i \tau + c_i |1 - \xi_c(\phi)^2|^{1/(2-D)}}. \quad (6.8)
\]

Let us further set for simplicity

\[
c_< = c_> = \sqrt{\xi_c c^{n+1}}. \quad (6.9)
\]

The transformation of the scalar field from the solution (3.22) of the MCM

\[
\kappa \Phi(\tau) = b^{n+1} \tau + c^{n+1} \quad (6.10)
\]

to the scalar field of the conformal model by Eqs. (3.20) resp. (3.21) and substitution of the latter in Eqs. (6.7) and (6.8) yields a lapse function

\[
e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c_i) \cosh^{2/(2-D)}(\sqrt{\xi_c b^{n+1} \tau})} \quad (6.11)
\]

resp.

\[
e^\gamma = e^{(\sum_i d_i b^i) \tau + (\sum_i d_i c_i) \sinh^{2/(2-D)}(\sqrt{\xi_c b^{n+1} \tau})} \quad (6.12)
\]

and, with \( i = 1, \ldots, n \), nonsingular scale factors

\[
e^{\beta_i} = e^{b^i \tau + c_i \cosh^{2/(2-D)}(\sqrt{\xi_c b^{n+1} \tau})}. \quad (6.13)
\]
resp. singular scale factors
\[ e^{\beta_i} = e^{b_i \tau + c_i} | \sinh \frac{2}{\sqrt{2}} (\sqrt{\xi_c b_i^{n+1}} \tau) | \] (6.14)
of the CCM. The scale factor singularity of the MCM for \( \tau \to -\infty \) vanishes in the CCM of Eqs. (6.11) and (6.13) for a scalar field \( \phi \) bounded according to (3.20). For \( D = 4 \) this result had already been indicated by Ref. 9.

On the other hand in the CCM of Eqs. (6.12) and (6.14), with \( \phi \) according to (3.21), though the scale factor singularity of the minimal model for \( \tau \to -\infty \) has also disappeared, instead there is another new scale factor singularity at finite (harmonic) time \( \tau = 0 \).

Let us consider a special case of the nonsingular solution with \( \phi^2 < \xi_c^{-1} \), where we assume the internal spaces to be static in the MCM, i.e. \( b_i = 0 \) for \( i = 2, \ldots, n \). Then in the CCM, the internal spaces are no longer static. Their scale factors (6.13) with \( i > 2 \) have a minimum at \( \tau = 0 \). Remind that for solution (6.2) all spaces \( M_i \), internal and external, \( i = 1, \ldots, n \) have been assumed as flat. From Eq. (6.3) with \( G_{11} = d_1 (1 - d_1) \) we find that the scalar field is given by
\[ (b_i^{n+1})^2 = d_1 (d_1 - 1)(b_1^i)^2. \] (6.15)
With real \( b_1 \) then also
\[ b_i^{n+1} = \pm \sqrt{d_1 (d_1 - 1)} b_1^i \] (6.16)
is real and by Eq. (6.13) the scale \( a_1 \) of \( M_1 \) has a minimum at
\[ \tau_0 = (\sqrt{\xi_c b_i^{n+1}})^{-1} \text{artanh} \left( \frac{(2 - D)}{2 \sqrt{\xi_c}} b_1^i / b_i^{n+1} \right), \] (6.17)
with \( \tau_0 > 0 \) for \( b_1^i < 0 \) and \( \tau_0 < 0 \) for \( b_1^i > 0 \).

The points \( \tau = \tau_0 \) and \( \tau = 0 \) are the turning points in the minimum for the factor spaces \( M_1 \) and \( M_2, \ldots, M_n \) respectively. It is interesting to explain the creation of our Lorentzian universe by a "birth from nothing"\textsuperscript{12}, i.e. quantum tunneling from an Euclidean region. Let us first consider the geometry of this tunneling as usual for the external universe \( \mathbb{R} \times M_1 \). So if we cut \( M \) along the minimal hypersurface at \( \tau_0 \) in 2 pieces, one of them, say \( M' \), contains the hypersurface \( \tau = 0 \) where the internal spaces are minimal. We set \( M'' := M \setminus M' \) to be the remaining piece. Then we can choose (eventually with time reversal \( \tau \to -\tau \)) either \( M' \) or \( M'' \) as a universe \( \tilde{M} \) that is generated at \( \tau_0 \) with initial minimal scale \( a_1(\tau_0) \). In the usual quantum tunneling interpretation, at the scale \( a_1(\tau_0) \) with \( \dot{a}_1(\tau_0) = 0 \) one glues smoothly a compact simply connected Euclidean space-time region to the Lorentzian \( \tilde{M} \), yielding a joint differentiable manifold \( \hat{M} \). Then the sum of classical paths in \( \hat{M} \) passing the boundary \( \partial \hat{M} \) from the Euclidean to the Lorentzian region can be interpreted as quantum tunneling from "nothing"\textsuperscript{12} to \( \hat{M} \).
According to Ref. 13 this interpretation has a direct topological correspondence in a projective blow up of a singularity of shape \( M_2 \times \cdots \times M_n \) (the ”nothing”) to \( S^{d_1}(a_1(\tau_0)) \times M_2 \times \cdots \times M_n \), where \( S^{d_1}(a_1(\tau_0)) \) denotes the \( d_1 \)-dimensional sphere of radius \( a_1(\tau_0) \).

For \( \tilde{M} = M' \) the internal spaces shrink for (harmonic) time from \( \tau_0 \) towards \( \tau = 0 \) and expand from \( \tau = 0 \) onwards for ever, but for \( \tilde{M} = M'' \) the internal spaces expand for (harmonic) time from \( \tau_0 \) onwards for ever. So the decomposition of \( M \) in \( M' \) and \( M'' \) is highly asymmetric w.r.t. the internal spaces. For more realistic models it might be especially useful to consider the piece of \( M \) which lies between \( \tau_0 \) and \( \tau = 0 \), since it can describe a shrinking of internal spaces while the external space is expanding.

Remarkably, the multidimensional geometries with \( \tau < \tau_0 \) and \( \tau > \tau_0 \) are \( \tau \)-asymmetric to each other. Taking one as contracting, the other as expanding w.r.t. \( M \), the two are distinguished by a qualitatively different behavior of internal spaces \( M_k \), \( k \geq 2 \).

The latter allows to choose the ”arrow of time”\(^{14} \) in a natural manner determined by intrinsic features of the solutions. Note, if there is at least one internal extra space, i.e. \( n > 1 \), then the minisuperspace w.r.t. scalefactors of geometry has Lorentzian signature \((-,+\ldots,+\)\). After diagonalization of (4.7) by a minisuperspace coordinate transformation \( \beta^i \rightarrow \alpha^i \) \( (i = 1,\ldots,n) \), there is just one new scale factor coordinate, say \( \alpha^1 \), which corresponds to the negative eigenvalue of \( G \), and hence assumes the role played by time in usual quantum mechanics. (For \( n = 1 \) there are no internal spaces, but \( G_{11} < 0 \) for \( d_1 > 1 \) still provides a negative eigenvalue that is distinguished at least against the additional positive eigenvalue from the scalar field.) This shows that, at least after diagonalization, an ”external” space is distinguished against the internal spaces, because its scale factor provides a natural ”time” coordinate.

Upto now we have considered the smooth tunneling from an Euclidean region to the external universe \( \mathbb{R} \times M_1 \), where the external spaces have been considered as purely passive spectators of the tunneling process. As we have pointed out in contrast to models with only one (external) space factor \( M_1 \), the additional internal spaces \( M_2,\ldots,M_n \) yield an asymmetry of \( M \) w.r.t. (harmonic) time \( \tau \) for \( \tau_0 \neq 0 \), which is according to Eq. (6.17) the case exactly when \( D \neq 2 \) and the external space is non static, i.e. \( b_1 \neq 0 \).

In the following we want to obtain a quantum tunneling interpretation for all of \( M \), including the internal spaces. The picture becomes more complicated, since the extremal hypersurfaces of external space and internal spaces are located at different times \( \tau = \tau_0 \) resp. \( \tau = 0 \).

Let \( M_1 \) be the external space with \( b_1 > 0 \) and hence \( \tau_0 < 0 \). Let us start with an Euclidean region of complex geometry given by scale factors

\[
a_k = e^{-ib_k\tau+i\xi}\left|\sin\left(\sqrt{\xi}b^{n+1}\tau\right)\right|^{\frac{n}{2}}. \tag{6.18}\n\]

Then we can perform an analytic continuation to the Lorentzian region with \( \tau \rightarrow i\tau + \pi/(2\sqrt{\xi}b^{n+1}) \), and we require \( c_k = \tau_k - \pi b_k/(2\sqrt{\xi}b^{n+1}) \) to be the real constant of the real
The quantum creation (via tunneling) of different factor spaces takes place at different values of $\tau$.

First the factor space $M_1$ comes into real existence and after a time interval $\Delta \tau = |\tau_0|$ the internal factor spaces $M_2, \ldots, M_n$ appear in the Lorentzian region. Since $\Delta \tau$ is arbitrarily large, there is in principle an alternative explanation of the unobservable extra dimensions, independent from concepts of compactification and shrinking to a fundamental length in symmetry breaking. Here, they may have been up to now still in the Euclidean region and hence unobservable. This view is also compatible with the interpretation\textsuperscript{13} of the internal symmetries as complex resolutions of simple singularities of Cartan series ADE.

Now let us perform a transition from Lorentzian time $\tau$ to Euclidean time $i\tau$. Then with a simultaneous transition from $b^k$ to $-ib^k$ for $k = 1, \ldots, n$ the geometry remains real, since $\hat{\beta}^k = b^k\tau + c^k$ is unchanged. But the analogue of Eq. (6.16) for the Euclidean region then becomes

$$b^{n+1} = \mp i\sqrt{d_1(d_1-1)b^1}. \quad (6.19)$$

Hence the scalar field is purely imaginary. This solution corresponds to a classical (instanton) wormhole. The sizes of the wormhole throats in the factor spaces $M_2, \ldots, M_n$ coincide with the sizes of static spaces in the minimal model, i.e. $\hat{a}_2(0), \ldots, \hat{a}_n(0)$ respectively.

With Eq. (6.16) replaced by (6.19), the Eq. (6.17) remains unchanged in the transition to the Euclidean region, and the minimum of the scale $a_1$ (unchanged geometry !) now corresponds to the throat of the wormhole.

If one wants to compare the synchronous time pictures of the MCM and the CCM solutions, one has to calculate them for both metrics. In the MCM we have

$$dt_s^{(m)} = e^{\hat{\gamma}}d\tau = e^{(\sum_i d_ib^i)\tau + (\sum_i d_ic^i)}d\tau, \quad (6.20)$$

which can be integrated to

$$t_s^{(m)} = (\sum_i d_ib^i)^{-1}e^{\hat{\gamma}} + t_0. \quad (6.21)$$

The latter can be inverted to

$$\tau = (\sum_i d_ib^i)^{-1}\left\{\ln(\sum_i d_ib^i)(t_s^{(m)} - t_0) - (\sum_i d_ic^i)\right\}. \quad (6.22)$$

Setting

$$B := \sum_{i=1}^n d_ib^i \text{ and } C := \sum_{i=1}^n d_ic^i, \quad (6.23)$$

this yields the scale factors

$$\hat{a}_s^i = (t_s^{(m)} - t_0)^{b_i/B} e^{\frac{\hat{\gamma}}{B}(\ln B - C) + c_i} \quad (6.24)$$
and the scalar field
\[ \kappa \Phi = \frac{b^{n+1}}{B} \left\{ \ln B(t^{(m)}_s - t_0) \right\} - C + c^{n+1}. \]  
(6.25)

Let us define for \( i = 1, \ldots, n + 1 \) the numbers
\[ \alpha^i := \frac{b^i}{B}. \]  
(6.26)

With (6.23) they satisfy
\[ \sum_{i=1}^{n} d_i \alpha^i = 1, \]  
(6.27)

and by Eq. (6.3) also
\[ \alpha^{n+1} = \sqrt{1 - \sum_{i=1}^{n} d_i (\alpha^i)^2}. \]  
(6.28)

Eqs. (6.24) shows, that the solution (6.2) is a generalized Kasner universe with exponents \( \alpha^i \) satisfying generalized Kasner conditions (6.27) and (6.28).

In the conformal model the synchronous time is given as
\[ t_s^{(c)} = \int e^\gamma d\tau = \int \cosh^{\frac{2}{D-2}}(\sqrt{\xi_c b^{n+1}} \tau) e^{B\tau+C} d\tau \]  
(6.29)

resp.
\[ t_s^{(c)} = \int e^\gamma d\tau = \int \sinh^{\frac{2}{D-2}}(\sqrt{\xi_c b^{n+1}} \tau) e^{B\tau+C} d\tau. \]  
(6.30)

Similarly one could also try to calculate other time gauges for both metrics.

### 7 \( \xi_c \) and Resonance of Coupling in Different Dimensions

In this section we examine the resonance of the coupling in dependence of the dimension. First we consider only conformal coupling constants \( \xi_c = \frac{D-2}{4(D-1)} \) and study their prime factorization. Table 1 lists \( \xi_c := \frac{r}{q} \) with only trivial greatest common divisor of \( r \) and \( q \), i.e. \( \gcd(r, q) = 1 \), \( p_{max} \) the maximal prime factor contained in either \( r \) or \( q \) and the least common multiple \( \text{lcm}(\xi) := \text{lcm}(r, q) \), for dimensions \( D = 3 \ldots 30. \)

| \( D \) | 3 4 5 6 7 8 9 10 11 12 13 14 15 16 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \xi_c = \frac{r}{q} \) | \( \frac{1}{8} \) | \( \frac{1}{6} \) | \( \frac{3}{16} \) | \( \frac{1}{10} \) | \( \frac{5}{16} \) | \( \frac{3}{14} \) | \( \frac{7}{16} \) | \( \frac{2}{9} \) | \( \frac{9}{16} \) | \( \frac{5}{12} \) | \( \frac{11}{16} \) | \( \frac{3}{11} \) | \( \frac{13}{16} \) |
| \( p_{max} \) | 2 3 3 5 7 7 3 5 11 11 13 13 7 13 |
| lcm | 8 6 48 5 120 42 224 18 360 110 528 39 728 210 |

| 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 15 | 17 | 17 | 19 | 19 | 5 | 21 | 11 | 33 | 13 | 55 | 6 | 13 | 27 | 72 |
| 5 | 17 | 17 | 19 | 19 | 7 | 23 | 23 | 5 | 13 | 13 | 7 | 29 |
| 960 | 68 | 1224 | 342 | 1520 | 105 | 1848 | 506 | 2208 | 150 | 2600 | 702 | 3024 | 203 |

| \( p_{max} \) and lcm of \( \xi_c \) for \( D = 3, \ldots, 30. \) |
Can we give an interpretation of this numbers in terms of higher stability of certain dimensions with higher ‘simplicity’ of $\xi$?

To answer this question we assume that a system is described by a Lagrangian

$$L(q, \dot{q}) := L_1(q, \dot{q}) + \xi_c L_2(q, \dot{q}),$$

where $q$ denotes the configuration variables. The conjugate momenta are

$$p := \frac{\partial L}{\partial \dot{q}},$$

so the Legendre transform is given by the linear differential operator

$$O_\dot{q} := \dot{q} \frac{\partial}{\partial \dot{q}} - 1$$

applied to $L$. Since the operator $O_\dot{q}$ is linear we obtain

$$H = O_\dot{q} L = O_\dot{q} L_1 + \xi_c O_\dot{q} L_2 = H_1 + \xi_c H_2.$$  

Now suppose the configuration variables to be (massless) scalar fields. Their energy has to be quantized. Thus for the ground state $|0\rangle$ we yield (in natural units with $\hbar = 1$)

$$< 0 | H | 0 > = \omega_1 + \xi_c \omega_2$$

with frequencies $\omega_{1,2} = < 0 | H_{1,2} | 0 >$ respectively. The latter are the better in resonance the smaller the lcm of $\xi_c$ and the ‘simpler’ the fraction $\xi_c$. If we consider this resonance as supporting the evolution leading to our present state, we find different stability of different dimensions, according to the ‘simplicity’ of $\xi_c$ or the smallness of the lcm($\xi_c$). From the table above we find $D = 6$ most stable, followed by $D = 4$ and $D = 3$. In these dimensions $\xi_c^{-1}$ is just an integer. Besides $D = 6, 4, 3$ the next best is $D = 10$ with $\xi_c = \frac{2}{9}$. Note that all dimensions of the form $D = 4i + 2, i \in \mathbb{N}$ are more stable than their neighbouring dimensions. If we admit for the rational composition only the first 3 prime numbers, then in the range $10 < D < 81$ the next best choice is $D = 26$. While for $D = 82 = 4 \cdot 20 + 2$ also $p_{max} = 5$, the corresponding lcm($\frac{20}{81}$) = 1620 is already more than 10 times higher than that of $D = 26$.

For general coupling constants $\xi$ this considerations become relevant for regions in which $\xi \rightarrow \xi_c$ assymptotically. There might be dynamical necessity to avoid this regions for their conformal resonances, and this even more for the stable dimensions. Even more the stability of the dimensions with high resonance might be due to an evolutionary effect the avoiding of the resonances has on the dynamics. For a better understanding of the latter is necessary to understand the conditions for symmetry breaking. A necessary condition for this is the instability of the vacuum.
8 The Negative Mass Condition for Vacuum Instability

In the following we look for dependence of the vacuum instability on the (not necessarily conformal) coupling constant in different dimensions of external space $M_1$ in the case that $M_i$ for $i > 1$ are all flat.

More specifically we consider a model where the internal spaces $M_i$ for $i > 1$ are Ricci flat and static. The dynamics then is equivalent to that of a model on $\mathbb{R} \times M_1$. We consider a scalar field on the background of the curved space-time $M$ or equivalently $\mathbb{R} \times M_1$. Note that in contrast to previous sections here the backreaction of the scalar field on the geometry will be assumed to be negligible. In the ground state through the evolution of $a_1$ the extra dimensions leave a dynamical imprint in $M_1$.

A necessary condition for vacuum instability is a negative effective mass

$$M^2 < 0. \quad (8.1)$$

For the rest of this section we choose the conformal time gauge

$$\gamma = \beta^1, \quad (8.2)$$
i.e. $t \equiv \eta_1$, the conformal time of the external space $M_1$. Correspondingly, for any $x$ we define

$$\dot{x} := \frac{\partial x}{\partial \eta_1}, \quad (8.3)$$
i.e. differently from the convention of the previous sections (where the dot denotes the derivative w.r.t. the harmonic time), now and in the following the dot denotes the partial derivative w.r.t. the conformal time of $M_1$.

The effective mass $M$ of the scalar field is given as $\dot{a} \equiv a_1$)\n
$$M^2 = m^2 a^2 + \xi d(d-1)k + d(\xi - \xi_c)(2\frac{\dot{a}}{a} - (d - 3)\frac{\dot{a}^2}{a^2}), \quad (8.4)$$

where $m$ is the bare mass. In the following we assume for simplicity vanishing bare mass, $m = 0$, and examine the condition (8.4) in different cases depending on the curvature $k$ of the exterior space and the ratio $\frac{\dot{a}}{a}$. Let us substitute

$$A_d := d(d-3)\frac{\dot{a}^2}{a^2} \quad (8.5)$$
in Eq. (8.3). Note that $A_d = 0$ for critical dimension $d = 3$ and $A_d < 0$ for $d > 3$ respectively.

With Eq. (8.5) we obtain

$$M^2 = kd(d-3)\xi + d(\xi - \xi_c)(2\frac{\dot{a}}{a} - (d - 3)\frac{\dot{a}^2}{a^2})$$

$$= [d(k(d-1) \pm 2(1 + \alpha)) - A_d]\xi + [A_d \mp 2d(1 + \alpha)]\xi_c, \quad (8.6)$$
where we set
\[
\frac{\dot{a}}{a} =: \pm (1 + \alpha)
\] respectively for expansion and contraction. We define the conformal parameter
\[
H_c := \frac{\dot{a}}{a} = H \dot{t}
\] given by the Hubble parameter \(H\) and the conformal scale factor \(a = e^\gamma = a_1\) to satisfy the differential equation
\[
\dot{H}_c + H_c^2 = \pm(1 + \alpha),
\] which for constant \(\alpha\) has the solutions
\[
H_c(t) = \sqrt{1 + \alpha} \tanh[\sqrt{1 + \alpha}(t - t_0)],
\] and
\[
H_c(t) = -\sqrt{1 + \alpha} \tan[\sqrt{1 + \alpha}(t - t_0)].
\] respectively to the sign in Eq. (8.9).

Condition (8.4) is satisfied, if and only if
\[
\xi > \xi_c \frac{A_d + 2d(1 + \alpha)}{A_d - kd(d \pm k2(1 + \alpha) - 1)},
\] respectively for
\[
(d - 3)H_c^2 > \pm 2(1 + \alpha) + k(d - 1).
\]

In the following we restrict to the case of constant \(\alpha\), represented by \(\alpha = 0\). Let us consider the following cases:

Case \(k = 1, \dot{a}/a = -1\):

In this case Eq. (8.6) reads
\[
M^2 = d(d - 3)\xi + \frac{1}{2}(d - 1) - (\xi - \xi_c)A_d.
\]

For \(d = 3\) there is no vacuum instability (and hence no symmetry breaking) \(\forall \xi\), since \(M^2 = 1\) contradicts the condition (8.4).

For \(d \neq 3\) we find
\[
M^2 = d(d - 3)[(1 - (\ddot{a}/a)^2]\xi + \frac{d - 1}{4}[(d - 3)(\dot{a}/a)^2 + 2]
\]
\[
= [d(d - 3) - A_d]\xi + \xi_c(A_d + 2d).
\]

Therefore condition (8.4) is satisfied, if and only if
\[
\xi > \xi_c \frac{(d - 3)(H_c^2 - 2)}{(d - 3)[(H_c^2 - 1)]} = \xi_c \frac{A_d + 2d}{A_d - d(d - 3)},
\]
respectively for \( H^2_c \geq 1 \) if \( d > 3 \) or \( H^2_c \leq 1 \) if \( d < 3 \).

Concerning the latter case especially for \( d = 2 \) we have \( A_2 = -2 \frac{\dot{a}^2}{a^2} \) and Eq. (8.15) becomes
\[
M^2 = 2[(\frac{\dot{a}}{a})^2 - 1]\xi + \frac{1}{4}[2 - (\frac{\dot{a}}{a})^2] = -(2 + A_2)\xi + \xi c(4 + A_2),
\]
yielding the condition
\[
\xi > \xi c \frac{H^2_c - 2}{H^2_c - 1} = \xi c \frac{A_2 + 4}{A_2 + 2},
\] (8.18)
for \( H^2_c \leq 1 \) respectively.

Case \( k = -1, \frac{\dot{a}}{a} = 1 \):
In this case Eq. (8.6) reads
\[
M^2 = 2\{[(\frac{\dot{a}}{a})^2 - 1]\xi + \frac{1}{4}[2 - (\frac{\dot{a}}{a})^2]\} = -(2 + A_2)\xi + \xi c(4 + A_2),
\]
yielding the condition
\[
\xi > \xi c \frac{H^2_c - 2}{H^2_c - 1} = \xi c \frac{A_2 + 4}{A_2 + 2},
\]
(8.18)
For \( d = 3 \) it is \( M^2 = -1 \) and the vacuum is unstable (hence symmetry breaking could occur) \( \forall \xi \).

For \( d \neq 3 \) we find
\[
M^2 = -d(d - 3)\xi - \frac{1}{2}(d - 1) - (\xi - \xi c)A_d.
\]
(8.19)
Therefore condition (8.4) is satisfied, if and only if
\[
\xi > \xi c \frac{(d - 3)H^2_c - 2}{(d - 3)[H^2_c + 1]} = \xi c \frac{A_d - 2d}{A_d + d(d - 3)},
\]
(8.21)
respectively for \( d > 3 \) and \( d < 3 \).

Concerning the latter case especially for \( d = 2 \) we have \( A_2 = -2 \frac{\dot{a}^2}{a^2} \) and Eq. (8.20) becomes
\[
M^2 = 2[(\frac{\dot{a}}{a})^2 + 1]\xi - \frac{1}{4}[2 + (\frac{\dot{a}}{a})^2] = (A_2 + 2)\xi - \xi c(A_2 + 4),
\]
(8.22)
yielding the condition
\[
\xi < \xi c \frac{H^2_c + 2}{H^2_c + 1} = \xi c \frac{A_2 + 4}{A_2 + 2}.
\]
(8.23)

9 Conclusion

We have examined conformally equivalent Lagrangian models with a scalar field coupled to geometry.

In Sec. 3 the conformal transformation of the minimally coupling model (MCM) to the conformal coupling model (CCM) has been performed in arbitrary dimensions \( D \), with the conformal factor and scalar field in agreement with the results of Ref. 8. By Eq. (3.17)
the proper generalization of the scalar field from the conformal coupling case to that of an arbitrary coupling constant $\xi$ has been found. (Note that Eq. (5) in Ref. 3 holds only for $\xi = \xi_c = \frac{1}{6}$ with $D = 4$).

In Sec. 4 the geometry has been specialized to multidimensional cosmological models.

In Sec. 5 we have considered natural time gauges in multi-dimensional universes: (i) synchronous time, (ii) conformal times of different factor spaces, (iii) mean conformal time and (iv) harmonic time.

In Sec. 6 the equivalent Lagrangian models of Sec. 3 have been compared on the level of multidimensional solutions. For the case of a massless ($U(\Phi) = 0$) minimally coupled scalar field $\Phi$ we found the multidimensional generalization of the classical Kasner solution. The conformal transformation of the Kasner solution for the MCM with flat internal spaces $M_i$ yields the nonsingular solution (6.13) for $\phi^2 < \xi_c^{-1}$ and to the singular solution (6.14) for $\phi^2 > \xi_c^{-1}$. This resolution of the scale factor singularity of the Kasner solution for a proper CCM solution (6.13) confirms for arbitrary dimension $D$ what has been indicated in Ref. 9 for $D = 4$. At $\phi^2 = \xi_c^{-1}$ there is a singularity of the conformal transformation. The conformal equivalence only holds separately in the ranges $\phi^2 < \xi_c^{-1}$ and $\phi^2 > \xi_c^{-1}$.

In the special case of static internal spaces, we find a minimal scale $a_1(\tau_0)$ at (harmonic) time $\tau_0$ where the birth of the universe $M$ is happening. Analytic continuation of this solution to the Euclidean time region (preserving real geometry) yields a purely imaginary scalar field. This solution corresponds to an (instanton) wormhole, where the scale $a_1(i\tau_0)$ now indicates the throat of the wormhole.

In Ref. 10 these solutions have also been compared to their quantum counterparts. However, conformal equivalence transformations of the classical Lagrangian models and minisuperspace conformal transformations are conceptually very different procedures\textsuperscript{15}. This is in analogy to the difference between conformal coordinate transformations and conformal transformations of Lagrangian models.

In Sec. 7 a number theoretical analysis of the conformal coupling constant $\xi_c$ indicates some kind of resonance of the coupling for different dimensions. This distinguishes the dimensions 3, 4, 6, 10 and 26 against all other dimensions $D < 82$.

In Sec. 8 it was shown how $\xi_c$ also enters crucially in a negative mass condition, necessary for a vacuum instability. The corresponding inequalities have been derived for a scalar field on the background of a multidimensional geometry with static internal spaces.

However further investigations will be required to yield a more detailed and more general understanding of the conditions for dynamical or spontaneous compactification. In Ref. 16 it is shown for the model of Eq. (4.5) how both types of compactification can be obtained in the flat case.
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