ABSTRACT. We show that the (twisted) derived category “recognizes” the three different kinds of Enriques surfaces in characteristic 2.

A. INTRODUCTION

This note concerns the study of derived categories of Enriques surfaces defined over an algebraically closed field $k$ of characteristic 2.

Derived categories were introduced by Verdier in the late 60s, with the aim to extend known theorems in algebraic geometry to a relative setting. Thanks to the work of Bondal, Orlov, Kawamata, Bridgeland, Katzarkov, Bayer-Macrì, Bayer-Macrì-Toda, and many others, it is nowadays understood that they provide a versatile and powerful tool for better understanding the geometry of algebraic varieties. However, the great majority of the techniques employed in the aforementioned studies are tailored for complex varieties (or more generally for varieties over an algebraically closed field of characteristic 0), and the research on the more algebraic setting of varieties defined over (possibly finite) fields of positive characteristic is still, in many aspects, at an embryonic stage.

In this work we concentrate on Enriques surfaces over fields of characteristic 2. By the Bombieri–Mumford classification of algebraic surfaces ([1]), an Enriques surface $X$ is a smooth projective surface with numerically trivial canonical bundle, second Betti number equal to 10 and Euler characteristic 1. Over fields of characteristic two they are split into three kinds: classical, which have a torsion canonical bundle of order 2, non-classical ordinary, whose canonical bundle is trivial and whose Frobenius morphism acts isomorphically on $H^1(X, O_X)$, and non-classical supersingular, which have again a trivial canonical bundle and are such that their Frobenius induces the zero-morphism on $H^1(X, O_X)$. In both these latter two cases we have that $H^1(X, O_X)$ is a 1-dimensional vector space. The central result of this note is the following:

**Theorem A.** Let $X$ be an Enriques surface defined over an algebraically closed field $k$ of characteristic 2. If $Y$ is a smooth $k$-scheme with the same derived category of $X$, then $Y$ is an Enriques surfaces of the same kind as $X$.

2010 Mathematics Subject Classification. Primary 14F05 Secondary 14J20, 14F05.

Key words and phrases. Derived Category, Enriques Surfaces, Positive Characteristic.
The main characters of our proof are the invariance under derived equivalence of the order of the canonical bundle and the Roquier isomorphism ([12]). Using the same ingredients, we also prove a twisted variant of Theorem A.

**Notation and Terminology.** Given a smooth projective variety $Z$, we will denote by $\mathcal{D}^b(Z)$ its bounded derived category of coherent sheaves. The symbol $T_Z$ will stand for the tangent bundle of $Z$. We say that two varieties $Z_1$ and $Z_2$ are derived equivalent or Fourier–Mukai partners if there exists an exact equivalence $\Phi : \mathcal{D}^b(Z_1) \to \mathcal{D}^b(Z_2)$.

**Acknowledgements.** I would like to thank Prof. C. Liedtke for asking me the question which this note answers at a conference in Berlin. I am also grateful to the unknown referee for many suggestions and improvements. It was him (or her) who encouraged me to pursue the variant of Theorem A proposed in the last section. Finally, I am indebted to K. Honigs for very interesting mathematical conversation and for pointing me the difficulties of using the HKR isomorphism in characteristic 2.

**B. The Proof**

We first prove that $Y$ is an Enriques surface by using a similar argument to the one in [5, Theorem 3.3]. The key ingredients are the invariance under derived equivalence of the order of the canonical bundle (see [6, Proposition 4.1]) and the isomorphisms that Fourier–Mukai transforms induce at cohomological level which, imply that derived equivalent surface have the same Betti numbers.

By standard results on derived invariants, we know that $Y$ is a surface with numerically trivial canonical bundle. In particular it is a minimal surface of Kodaira dimension 0. Denote by $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ an exact equivalence. Let $l$ be an odd prime. By [4, Lemma 3.1] the equivalence $\Phi$ descends to cohomology and yields and isomorphism

$$H^0_{\text{et}}(X, \mathbb{Q}_l) \oplus H^2_{\text{et}}(X, \mathbb{Q}_l)(1) \oplus H^4_{\text{et}}(X, \mathbb{Q}_l)(2) \simeq H^0_{\text{et}}(Y, \mathbb{Q}_l) \oplus H^2_{\text{et}}(Y, \mathbb{Q}_l)(1) \oplus H^4_{\text{et}}(Y, \mathbb{Q}_l)(2),$$

where $(i)$ denotes the Tate twist. After computing dimensions, the above isomorphism leads to the equality of Betti numbers:

$$b_2(X) = b_2(Y) = 10.$$ 

So we conclude that $Y$ is an Enriques surfaces by applying the Bombieri–Mumford classification theorem (see for example [7, Paragraph 7]).

Now we assume that $X$ is a classical Enriques surface. Then its canonical bundle has order 2. As remarked before, since derived equivalences commute with Serre functors, also the canonical bundle of $Y$ will have order 2, and therefore $Y$ is a classical Enriques surface.

Suppose conversely that the canonical bundle of $X$ (and so the one of $Y$) is trivial. The table on page 6 of [8] tells us that one can distinguish non-classical ordinary and supersingular Enriques surfaces by looking at the cohomology of their tangent bundle. Therefore, in order to conclude the proof, we need just to show that $H^0(X, T_X) \simeq H^0(Y, T_Y)$. 
This is a consequence of the Roquier isomorphism. More precisely, in [12] the author proves that the exact equivalence \( \Phi \) induces an isomorphism of algebraic groups

\[
F_{\Phi} : \text{Pic}^0(X) \rtimes \text{Aut}^0(X) \to \text{Pic}^0(Y) \rtimes \text{Aut}^0(Y),
\]

which in turns induces an isomorphism of their tangent spaces at the origin. Now, it is well known that, for any smooth projective variety \( Z \), the tangent space in \( \mathcal{O}_Z \) of \( \text{Pic}^0(Z) \) is isomorphic to \( H^1(Z, \mathcal{O}_Z) \). On the other side there is an isomorphism of the tangent space at the origin of \( \text{Aut}^0(Z) \) with \( H^0(Z, \mathcal{T}_Z) \) (cfr. [11]). Thus, since as scheme, the semi-direct product is simply the product, (1) yields an isomorphism

\[
H^1(X, \mathcal{O}_X) \oplus H^0(X, \mathcal{T}_X) \simeq H^1(Y, \mathcal{O}_Y) \oplus H^0(Y, \mathcal{T}_Y).
\]

The statement is proved by taking dimension on both sides, since both \( X \) and \( Y \) are non-classical Enriques surfaces, and so both \( H^1(X, \mathcal{O}_X) \) and \( H^1(Y, \mathcal{O}_Y) \) are one dimensional. \( \Box \)

C. Variant: Twisted Fourier–Mukai Partners of Enriques Surfaces

In this section we prove the following variant to Theorem A, involving the twisted derived categories introduced by Căldăraru in his thesis [2].

**Theorem A’.** Let \( X_1 \) and \( X_2 \) be Enriques surfaces over an algebraically closed field of characteristic 2, and consider two Brauer classes, \( \alpha_1 \) and \( \alpha_2 \), with \( \alpha_i \in \text{Br}(X_i) \). If there is an exact equivalence of the twisted bounded derived categories

\[
\Phi : \text{D}^b(X_1, \alpha_1) \to \text{D}^b(X_2, \alpha_2),
\]

then \( X_1 \) and \( X_2 \) are of the same kind.

**Proof.** We first remark that non-classical Enriques surfaces have a trivial Brauer group ([3, Proposition 5.3.5]). So if \( X_1 \) and \( X_2 \) are both non-classical then the statement can be deduced directly by Theorem A. Thus we can suppose that \( X_1 \) is classical. Then its Brauer group is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) ([3]). In order to prove the theorem we have just to show that there cannot be an exact equivalence

\[
\Phi : \text{D}^b(X_1, \alpha_1) \to \text{D}^b(X_2),
\]

where \( \alpha_1 \) is the only non-trivial class of \( \text{Br}(X_1) \) and \( X_2 \) is a non-classical Enriques surface. But, if such an equivalence should exist, by [9, Theorem 1.6.15] or [10, Theorem 22] we would have that

\[
0 \simeq H^0(X_1, \omega_{X_1}) \simeq H^0(X_2, \omega_{X_2}),
\]

which yields an obvious contradiction. \( \Box \)
References

[1] Enrico Bombieri and David Mumford, *Enriques’ classification of surfaces in char. p. III*, Inventiones mathematicae 35 (1976), no. 1, 197–232.

[2] Andrei Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, Ph.D. Thesis, 2000.

[3] François Cossec and Igor Dolgachev, *Enriques surfaces i*, Vol. 76, Springer Science & Business Media, 2012.

[4] Katrina Honigs, *Derived equivalent surfaces and abelian varieties, and their zeta functions*, Proceedings of the American Mathematical Society 143 (2015), no. 10, 4161–4166.

[5] Katrina Honigs, Luigi Lombardi, and Sofia Tirabassi, *Derived equivalences of canonical covers of hyper-elliptic and Enriques surfaces in positive characteristic*, 2016. Preprint arXiv:1606.02094.

[6] Daniel Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford University Press on Demand, 2006.

[7] Christian Liedtke, *Algebraic surfaces in positive characteristic*, Birational geometry, rational curves, and arithmetic, 2013, pp. 229–292.

[8] ———, *Arithmetic moduli and lifting of Enriques surfaces*, Journal für die reine und angewandte Mathematik (Crelles Journal) 2015 (2015), no. 706, 35–65.

[9] Hermes Martínez, *Fourier-Mukai transform for twisted sheaves*, Ph.D. Thesis, 2010.

[10] ———, *Fourier-mukai transform for twisted derived categories of surfaces*, Revista Colombiana de Matemáticas 46 (2012), no. 2, 205–228.

[11] Hideyuki Matsumura and Frans Oort, *Representability of group functors, and automorphisms of algebraic schemes*, Inventiones mathematicae 4 (1967), no. 1, 1–25.

[12] Raphaël Rouquier, *Automorphismes, graduations et catégories triangulées*, Journal of the Institute of Mathematics of Jussieu 10 (2011), no. 03, 713–751.

Department of Mathematics, University of Bergen, Allégaten 41, Bergen, Norway

E-mail address: sofia.tirabassi@uib.no