On weak Fano manifolds with small contractions obtained by blow-ups of a product of projective spaces

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October 25, 2016

Abstract

We consider weak Fano manifolds with small contractions obtained by blowing up successively curves and subvarieties of codimension 2 in products of projective spaces. We give a classification result for a special case. In the process of proof, we describe explicitly the structure of nef cones and compute the self intersection numbers of anti-canonical divisors for such weak Fano manifolds.

Mathematics Subject Classification (2000): 14J45, 14E30

1 Introduction

A smooth projective variety is called Fano manifold if its anti-canonical divisor is ample. The classification is known up to dimension 3. However, in dimension greater than or equal to 4, there exist only partial classification results (see [2] for a recent progress).

It is essential to investigate Fano manifolds in terms of the theory of extremal contractions (see [8],[10]). Recall that a small contraction is a birational morphism whose exceptional locus has codimension greater than or equal to 2, and it does not appear as extremal contraction for smooth 3-folds. Hence, in dimension greater than or equal to 4, it is interesting to give examples of Fano manifolds having small contractions.
We can construct a smooth projective variety with a small contraction by means of successive blow-ups (see [5]): Let $Y$ be a smooth projective variety of dimension greater than or equal to 4. Let $C$ be a smooth curve on $Y$ and $S$ a smooth subvariety of $Y$ with $\text{codim}_Y S = 2$. Assume that $C$ and $S$ intersect transversally at points. Let $\pi: X \to Y$ be the blow-up along $C$ and let $S'$ be the strict transform of $S$ by $\pi$. Let $\beta: \tilde{X} \to X$ be the blow-up along $S'$. Then $\tilde{X}$ has a small contraction (see Section 2 for details). We consider the following:

**Problem.** Classify the triples $(Y, C, S)$ such that $\tilde{X}$ is a Fano manifold.

The purpose of this paper is to give a classification result in a special case for the problem expanded to the case where $\tilde{X}$ is a weak Fano manifold, i.e. a smooth projective variety with nef and big anti-canonical divisor.

Throughout the paper, we work over the field of complex numbers.

**Theorem 1.** Let $Y = \mathbb{P}^{n-1} \times \mathbb{P}^1$ with $n \geq 3$. Let $C$ be a fiber of the projection $Y \to \mathbb{P}^{n-1}$ and let $S$ be a complete intersection of two divisors of bidegrees $(a, b)$ and $(1, 1)$. Assume that $S$ is smooth and irreducible. Assume also that $S$ and $C$ intersect transversally at one point. Let $\pi: X \to Y$ be the blow-up along $C$ and let $\beta: \tilde{X} \to X$ be the blow-up along the strict transform of $S$ by $\pi$. Then $\tilde{X}$ is a weak Fano manifold if and only if $n \geq 3$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1) \text{ or } (3, 2).$$

Moreover, $\tilde{X}$ is a Fano manifold if and only if $n \geq 4$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (2, 0) \text{ or } (2, 1).$$

**Remarks:**

(1) The case $Y = \mathbb{P}^n$ seems more complicated (see Section 6).

(2) The assumption on $C$ is not so restrictive. Indeed, if $C$ is not a fiber of the projection $p : Y = \mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^{n-1}$, there exists a fiber $\Gamma$ of $p$ such that $C \cap \Gamma \neq \emptyset$. Then we have $-K_X \cdot \tilde{\Gamma} = 4 - n$, $\tilde{\Gamma}$ being the strict transform of $\Gamma$ by $\pi \circ \beta$. Hence, $-K_X$ is not nef for $n \geq 5$.

(3) Let $q : Y = \mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection. Put $y_0 := C \cap S$. Since we assume $S$ to be irreducible, $a = 0$ implies $b = 1$ and $S$ is a hyperplane in the fiber $q^{-1}(q(y_0)) \cong \mathbb{P}^{n-1}$. If $a \geq 1$, then $q|_S : S \to \mathbb{P}^1$ is surjective. The
assumption that $S$ is contained in a divisor of bidegree $(1,1)$ is natural (at least for the case where $\tilde{X}$ is a Fano manifold): Consider the open set

$$T := \{ t \in \mathbb{P}^1 \mid t \neq q(y_0) \text{ and } S \cap q^{-1}(t) \text{ is smooth} \}.$$  

If $\tilde{X}$ is a Fano manifold, so is $\tilde{X}_t := (q \circ \pi \circ \beta)^{-1}(t)$ for $t \in T$. Note that $(\pi \circ \beta)|_{\tilde{X}_t} : \tilde{X}_t \to q^{-1}(t) \simeq \mathbb{P}^{n-1}$ is the blow-up whose center consists of the point $C \cap q^{-1}(t)$ and the subvariety $S_t := S \cap q^{-1}(t)$. According to [1], there exist a hypersurface $U_t \subset q^{-1}(t) \simeq \mathbb{P}^{n-1}$ of degree $a$ ($1 \leq a \leq n$) and a hyperplane $V_t \subset q^{-1}(t) \simeq \mathbb{P}^{n-1}$ such that $S_t$ is complete intersection of $U_t$ and $V_t$. Let $V$ be the closure of the union $\bigcup_{t \in T} V_t$. Then, $V$ contains $S$ and $V$ has bidegree $(1, c)$ for some $c \geq 0$, and our theorem covers the case $c = 1$.

The present paper is organized as follows: In Section 2 we explain how to obtain a small contraction by means of blow-ups. We also fix notations which will be used constantly throughout the paper. Section 3 is devoted to determine the structure of the nef cones of $\tilde{X}$ for $(a, b) = (1, 0)$ and for any $(a, b)$ such that $a \geq 1$ and $b \geq 0$. Recently, the explicit descriptions of nef cones are of great importance in the study of Mori dream spaces (see [9]). Hence, this section is of independent interest. In Section 4 we compute $(-K_{\tilde{X}})^n$ and express it as a rational function depending on $(n, a, b)$. We will give a sufficient condition for $(-K_{\tilde{X}})^n$ to be strictly positive. Since the self intersection number of the anti-canonical divisor is an important invariant for (weak) Fano manifolds, we believe that this section is also of independent interest. In Section 5 we prove Theorem 1 using Propositions shown in Sections 3 and 4. Section 6 is a supplement in which we give several examples for the case $Y \neq \mathbb{P}^{n-1} \times \mathbb{P}^1$.

Notation. Let $(x_0 : x_1 : \cdots : x_{n-1})$ and $(s : t)$ are homogeneous coordinates of $\mathbb{P}^{n-1}$ and $\mathbb{P}^1$ respectively. Recall that a divisor $D$ on the product $\mathbb{P}^{n-1} \times \mathbb{P}^1$ is said to have bidegree $(a, b)$ if $D$ is defined by a polynomial

$$\sum c_{i_0, i_1, \ldots, i_{n-1}, j, k} x_0^{i_0} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} s^j t^k \quad (c_{i_0, \ldots, i_{n-1}, j, k} \in \mathbb{C})$$

such that $i_0 + \cdots + i_{n-1} = a$, $j + k = b$. It is equivalent to say that $D$ is a member of the linear system $|\mathcal{O}_{\mathbb{P}^{n-1} \times \mathbb{P}^1}(a, b)|$.

For a projective variety $X$, we denote by $N^1(X)$ (resp. $N_1(X)$) the set of the numerical classes of divisors (resp. 1-cycles) with real coefficients.
It is known that this is a finite dimensional vector space (see [6]), and its dimension denoted by $\rho(X)$ is called the Picard number of the variety $X$. The numerical equivalence class of a divisor $D$ (resp. a 1-cycle $C$) is denoted by $[D]$ (resp. $[C]$). We see that $N^1(X)$ and $N_1(X)$ are dual to each other via the bilinear form $N^1(X) \times N_1(X) \to \mathbb{R}$ defined by the intersection number: $([D],[C]) \mapsto D \cdot C$.

The nef cone $\text{Nef}(X)$ and the cone of curves $\text{NE}(X)$ are defined by

$$\text{Nef}(X) := \{[D] \in N^1(X) \mid D \text{ is a nef divisor}\},$$

$$\text{NE}(X) := \{ \sum a_i[C_i] \in N_1(X) \mid C_i \text{ is an irreducible curve on } X, a_i \geq 0\}.$$ 

The closure of $\text{NE}(X)$ in $N_1(X)$ is denoted by $\overline{\text{NE}}(X)$. The important fact is that the two cones $\text{Nef}(X)$ and $\overline{\text{NE}}(X)$ are dual to each other (see [7] Proposition 1.4.28).

Let $\Gamma$ be a 1-cycle on a projective variety $Y$ and let $V$ be a subvariety of $Y$. For a divisor $D$ on $V$, we denote by $(D \cdot \Gamma)_V$ the intersection number taken in $V$. Given a birational morphism $\alpha : X \to Y$, the strict transform of a subvariety $M \subset Y$ will be denoted by $\alpha^{-1}_* M$.

2 Construction of a small contraction

We follow Example (2.6) in [5]. Let $Y$ be a smooth projective variety of dimension $n \geq 3$. Let $C \subset Y$ be a smooth curve and let $S \subset Y$ be a smooth subvariety of codimension 2. Assume that $C$ and $S$ intersect transversally at one point. Put $y_0 := S \cap C$. Let $\pi : X \to Y$ be the blow-up along $C$ with the exceptional divisor $E$. Note that $\pi|_E : E \to C$ is a $\mathbb{P}^{n-2}$-bundle. Put $E_0 := \pi^{-1}(y_0)$. Let $\beta : \tilde{X} \to X$ be the blow-up along $S' := \pi^{-1}_* S$ with the exceptional divisor $F$. Let $f$ be a fiber of the $\mathbb{P}^1$-bundle $\beta|_F : F \to S'$. We put $\tilde{E} := \beta^{-1}(E)$ and $\tilde{E}_0 := \beta^{-1}_* E_0$. Note that $\tilde{E}_0$ is isomorphic to $\mathbb{P}^{n-2}$.

**Lemma 1.** There exists a birational morphism $\varphi : \tilde{X} \to X_0$, $X_0$ being a projective variety, such that $\varphi(\tilde{E}_0)$ is a point for $n \geq 4$. The same holds for $n = 3$, if we assume $-K_{\tilde{X}}$ is nef and big.

**Proof.** (See also [3] Chapter 6.) Let $\tilde{e}_0$ be a line in $\tilde{E}_0 \simeq \mathbb{P}^{n-2}$. We show that $\mathbb{R}^+[\tilde{e}_0]$ is extremal in the cone $\overline{\text{NE}}(\tilde{X})$. Assume that there exist irreducible curves $A, B \subset \tilde{X}$ such that $\tilde{e}_0 \equiv A + B$. Let $D$ be an ample divisor
on $Y$ and put $\tilde{D} := (\pi \circ \beta)^* D$. Since $\tilde{D} \cdot \tilde{c}_0 = 0$, we have $\tilde{D} \cdot A = \tilde{D} \cdot B = 0$, which implies that $A$ and $B$ are contracted by $\pi \circ \beta$. Assume $A \not\subset \tilde{E}$. Then there exists $s \in S \setminus \{y_0\}$ such that $A = (\pi \circ \beta)^{-1}(s)$. Since $(\pi \circ \beta)(B)$ is a point, $B$ is one of the following types:

1. $B = (\pi \circ \beta)^{-1}(t)$ ($t \in S \setminus \{y_0\}$)
2. $B \subset (\pi \circ \beta)^{-1}(c)$ ($c \in C \setminus \{y_0\}$)
3. $B \subset (\pi \circ \beta)^{-1}(y_0)$

In case 1, we have $\tilde{c}_0 \equiv A + B \equiv f + f = 2f$, a contradiction. In case 2, we have $F \cdot A + F \cdot B = -1 + 0 = -1$, while $F \cdot (A + B) = F \cdot \tilde{c}_0 = 1$, a contradiction. In case 3, if we put $G := F \cap \tilde{E}$ then we have $(\pi \circ \beta)^{-1}(y_0) = \tilde{E}_0 \cup G$. Assume $B \subset G$. Put $G_0 := F \cap \tilde{E}_0$. Note that $N_{G_0/G} \simeq \mathcal{O}_{\mathbb{P}^n}_{-2}(-1)$. Since $F \cdot f = -1$ and $F \cdot \tilde{c}_0 = 1$, we have $F|_G \sim -G_0$. Hence,

$$F \cdot B = F|_G \cdot B = (-G_0 \cdot B)_G.$$  

On the other hand, we have

$$F \cdot B = F \cdot \tilde{c}_0 - F \cdot A = 1 - (-1) = 2.$$  

Hence, $(G_0 \cdot B)_G = -2 < 0$ which implies that $B \subset G_0 \subset \tilde{E}_0$. Thus, $[B] \in \mathbb{R}^+[\tilde{c}_0]$, a contradiction because $\tilde{c}_0 \equiv A + B \equiv f + B$. We conclude that all the cases 1, 2, 3 do not happen. Therefore $A \subset \tilde{E}$. By a similar argument, we also have $B \subset \tilde{E}$. Now, we take intersection numbers in $\tilde{E}$:

$$-1 = (\tilde{E}_0 \cdot \tilde{c}_0)_{\tilde{E}} = (\tilde{E}_0 \cdot A)_{\tilde{E}} + (\tilde{E}_0 \cdot B)_{\tilde{E}}$$

which implies $\tilde{E}_0 \cdot A < 0$ or $\tilde{E}_0 \cdot B < 0$. Hence, $A \subset \tilde{E}_0$ or $B \subset \tilde{E}_0$. In both cases we have $[A] \in \mathbb{R}^+[\tilde{c}_0]$ and $[B] \in \mathbb{R}^+[\tilde{c}_0]$. It follows that $\mathbb{R}^+[\tilde{c}_0]$ is an extremal ray in $N_\text{E}(\tilde{X})$.

If $n \geq 4$, we have $K_{\tilde{X}} \cdot \tilde{c}_0 = 3 - n < 0$. Hence, $\mathbb{R}^+[\tilde{c}_0]$ is a $K_{\tilde{X}}$-negative extremal ray, and we are done by Contraction Theorem.

In case $n = 3$, since we assume $-K_{\tilde{X}}$ is nef and big, the linear system $|-mK_{\tilde{X}}|$ defines a morphism for a sufficiently large $m \in \mathbb{N}$ by Base Point Free Theorem. The Stein factorization gives a desired contraction because we have $-K_{\tilde{X}} \cdot \tilde{c}_0 = 0$ (note that $\tilde{E}_0 = \tilde{c}_0$ for $n = 3$).
From now on, we fix the following:

**Notation (**)**
Assume \( n \geq 3 \) and put \( Y := \mathbb{P}^{n-1} \times \mathbb{P}^1 \). Let \( p : Y \to \mathbb{P}^{n-1} \) and \( q : Y \to \mathbb{P}^1 \) be the projections. Let \( C \) be a fiber of \( p \). Put \( H := p^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) \) and \( L := q^* \mathcal{O}_{\mathbb{P}^1}(1) \).

Consider \( V \in |H + L| \) and \( U \in |aH + bL| \) where \( a \) and \( b \) are non-negative integers. Let \( S \) be the complete intersection of \( U \) and \( V \). We assume also that \( S \) is smooth and irreducible. We assume also that \( C \) and \( S \) intersect transversally at one point and put \( y_0 := C \cap S \).

Let \( h \) be a fiber of \( p \) such that \( h \neq C \) and \( h \cap S = \emptyset \) and let \( l \) be a line in \( q \) such that \( l \cap C = \emptyset \) and \( l \cap S = \emptyset \).

Let \( \pi : X \to Y \) be the blow-up along \( C \). Put \( E := \text{Exc}(\pi) \) and \( E_0 := \pi^{-1}(y_0) \). Let \( e_0 \) be a line in \( E_0 \simeq \mathbb{P}^{n-2} \) and let \( e \) be a line in a fiber different from \( E_0 \) of the \( \mathbb{P}^{n-2} \)-bundle \( \pi|_E : E \to C \). Let \( H' \) and \( L' \) be the pull backs of \( H \) and \( L \) by \( \pi \). Let \( h' \) and \( l' \) be the strict transforms of \( h \) and \( l \) by \( \pi \). Put \( S' := \pi^{-1}_* S \).

Let \( \beta : \tilde{X} \to X \) be the blow-up along \( S' \). Put \( F := \text{Exc}(\beta) \) and \( \tilde{E} := \beta^{-1}_*(E) \). Let \( \tilde{H} \) and \( \tilde{L} \) be the pull backs by \( \beta \) of \( H' \) and \( L' \). Let \( f \) be a fiber of the \( \mathbb{P}^1 \)-bundle \( \beta|_F : F \to S' \). Let \( \tilde{e}_0, \tilde{e}, \tilde{h} \) and \( \tilde{l} \) be the strict transforms by \( \beta \) of \( e_0, e, h' \) and \( l' \). Put \( V' := \pi^{-1}_* V \) and \( \tilde{V} := \beta^{-1}_* V' \).

## 3 Structure of nef cones

The following is useful to determine the structure of simplicial cones:

**Lemma 2.** Let \( (D, C) \mapsto D \cdot C \) be a bilinear form of \( \mathbb{R}^m \times (\mathbb{R}^m)^* \). Let \( V \) be a cone in \( \mathbb{R}^m \) and let \( V^* \) be its dual cone. Assume that there exist \( D_1, D_2, \ldots, D_m \in V \) and \( C_1, C_2, \ldots, C_m \in V^* \) such that \( D_i \cdot C_j = \delta_{ij} \) (Kronecker delta). Then, we have

\[
V = \mathbb{R}^+ D_1 + \mathbb{R}^+ D_2 + \cdots + \mathbb{R}^+ D_m,
\]
\[
V^* = \mathbb{R}^+ C_1 + \mathbb{R}^+ C_2 + \cdots + \mathbb{R}^+ C_m.
\]

**Proof.** Since \( D_1, \ldots, D_m \) are linearly independent, for any \( D \in V \) there exist real numbers \( a_1, \ldots, a_m \) such that \( D = a_1 D_1 + \cdots + a_m D_m \). We have

\[
a_i = (a_1 D_1 + \cdots + a_m D_m) \cdot C_i = D \cdot C_i \geq 0
\]

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Lemma 3. Let $X$ be a smooth projective variety, $V$ a prime divisor on $X$ and $D$ a divisor on $X$. If the divisors $D-V$ and $D|_V$ are nef, then $D$ is nef.

Proof. Let $\Gamma$ be a curve on $X$. If $\Gamma \not\subset V$, we have $D \cdot \Gamma = (D-V) \cdot \Gamma \geq 0$. If $\Gamma \subset V$, then we have $D \cdot \Gamma = D|_V \cdot \Gamma \geq 0$.

Now, we return to our situation (Notation (*) in Section 2).

Proposition 1. We have

$$\text{Nef}(\tilde{X}) = \mathbb{R}^+[\tilde{H}] + \mathbb{R}^+[\tilde{L}] + \mathbb{R}^+[\tilde{H} - \tilde{E}] + \mathbb{R}^+[D(a,b)],$$

where

$$D(a,b) := \begin{cases} \tilde{H} + \tilde{L} - \tilde{E} - F & \text{for } a = 0 \text{ and } b = 1, \\ 2\tilde{H} + \tilde{L} - \tilde{E} - F & \text{for } a = 1 \text{ and } b = 0, \\ 2\tilde{H} + b\tilde{L} - \tilde{E} - F & \text{for } a = 1 \text{ and } b \geq 1, \\ a\tilde{H} + \tilde{L} - \tilde{E} - F & \text{for } a \geq 2 \text{ and } b = 0, \\ a\tilde{H} + b\tilde{L} - \tilde{E} - F & \text{for } a \geq 2 \text{ and } b \geq 1. \end{cases}$$

Proof. We define 1-cycles $l(a)$ and $h(b)$ on $\tilde{X}$ by:

$$l(a) := \begin{cases} \tilde{l} - \tilde{e}_0 - f & (a = 0) \\ \tilde{l} - \tilde{e}_0 - 2f & (a = 1) \\ \tilde{l} - \tilde{e}_0 - af & (a \geq 2) \end{cases}, \quad h(b) := \begin{cases} \tilde{h} - f & (b = 0) \\ \tilde{h} - bf & (b \geq 1). \end{cases}$$

Claim. For any $a \geq 0$, we have $[l(a)] \in \text{NE}(\tilde{X})$.

Proof. Let $t_0 := q(y_0)$ and $t \in \mathbb{P}^1 \setminus \{y_0\}$. Put $y_t := C \cap q^{-1}(t)$. Put also $Y_0 := q^{-1}(t_0)$ and $Y_t := q^{-1}(t)$. We define the curve $\Gamma$ as follows: If $a = 0$, let $\Gamma$ be a line in $Y_t \simeq \mathbb{P}^{n-1}$. If $a = 1$, let $\Gamma$ be a line in $Y_t$ such that $y_t \in \Gamma$ and $S \cap \Gamma \neq \emptyset$. If $a \geq 2$, let $\Gamma$ be a line in $Y_0 \simeq \mathbb{P}^{n-1}$ such that $y_0 \in \Gamma$ and $\Gamma \subset V$. For any $a \geq 0$, $\Gamma \equiv l$ in $Y$. Put $\Gamma' := \pi_*^{-1}\Gamma$ and $\tilde{\Gamma} := \beta_*^{-1}\Gamma'$. For $a = 0$ and $a = 1$, we have $\Gamma' + e \equiv l'$. This yields $\tilde{\Gamma} + \tilde{e} \equiv \tilde{l}$ for $a = 0$ (because $\Gamma' \cap S' = \emptyset$) and $\tilde{\Gamma} + \tilde{e} + f \equiv \tilde{l}$ for $a = 1$ (because $\Gamma'$ and $S'$ intersect transversally at one point). In case $a \geq 2$, we have $\Gamma' + e_0 \equiv l'$ which yields

$$(\tilde{\Gamma} + (a-1)f) + (e_0 + f) \equiv \tilde{l}$$
because \((S' \cdot \Gamma)_{V'} = a - 1\) and \((S' \cdot e_0)_{V'} = 1\). Thus, for any \(a \geq 0\), we have 
\[l(a) = [\Gamma] \in \text{NE}(\widetilde{X}).\]

**Claim.** For any \(b \geq 0\), we have \([h(b)] \in \text{NE}(\widetilde{X})\).

**Proof.** We define the curve \(\Delta\) as follows: If \(b = 0\), let \(\Delta\) be a fiber of \(p|_U\) different from \(C\). Note that \(U\) is isomorphic to \(p(U) \times \mathbb{P}^1\) because \(U \sim aH\). If \(b \geq 1\), let \(\Delta\) be a fiber of \(p\) such that \(\Delta \subset V\) and \(\Delta \not\subset S\) \((\Delta\) is a fiber of the exceptional divisor of the blow-up \(p|_V : V \to \mathbb{P}^{n-1}\)). Since \(\Delta \equiv h\) for any \(b \geq 0\), we have

\[
(S \cdot \Delta)_U = V|_U \cdot \Delta = V \cdot h = 1 \quad \text{for } b = 0,
\]

\[
(S \cdot \Delta)_V = U|_V \cdot \Delta = U \cdot h = b \quad \text{for } b \geq 1.
\]

Put \(\tilde{\Delta} := (\pi \circ \beta)^{-1}_*\Delta\). Then, if \(b = 0\), we have \(\tilde{\Delta} + f \equiv \tilde{h}\) and if \(b \geq 1\), \(\tilde{\Delta} + bf \equiv \tilde{h}\). Thus, for any \(b \geq 0\), we get \([h(b)] = [\tilde{\Delta}] \in \text{NE}(\widetilde{X})\).\[\Box\]

**Claim.** The divisors \(\widetilde{H}, \widetilde{L}, \widetilde{H} - \widetilde{E}\) and \(D(a, b)\) are all nef.

**Proof.** We see that \(H = p^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)\) and \(L = q^*\mathcal{O}_{\mathbb{P}^1}(1)\) are nef. Hence, so are \(\widetilde{H} = (\pi \circ \beta)^*H\) and \(\widetilde{L} = (\pi \circ \beta)^*L\). Note that \(X\) is isomorphic to \(Bl_z(\mathbb{P}^{n-1}) \times \mathbb{P}^1\) where \(z\) is the point \(p(C) \in \mathbb{P}^{n-1}\). For the blow-up \(\varepsilon : Bl_z(\mathbb{P}^{n-1}) \to \mathbb{P}^{n-1}\) the divisor \(\varepsilon^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) - \text{Exc}(\varepsilon)\) is nef. Hence, so is its pull back by the projection \(X \to Bl_z(\mathbb{P}^{n-1})\), which is linearly equivalent to \(H' - E\). Therefore, \(\widetilde{H} - \widetilde{E} = \beta^*(H' - E)\) is also nef.

We show that \(D(a, b)\) is nef for \((a, b) = (0, 1)\) and for any \((a, b)\) such that \(a \geq 1\) and \(b \geq 0\).

First, we consider the case \((a, b) = (0, 1)\). Put \(H_0 := p^{-1}(p(S))\), \(H'_0 := \pi^{-1}_*H_0\) and \(\widetilde{H}_0 := \beta^{-1}_*H'_0\). Note that we have \(S = q^{-1}(q(y_0)) \cap H_0\). Note also that \(\pi|_{H'_0} : H'_0 \to H_0 \simeq \mathbb{P}^{n-2} \times \mathbb{P}^1\) is the blow-up along \(C\) and \(\beta|_{\widetilde{H}_0} : \widetilde{H}_0 \to H'_0\) is an isomorphism. Let \(L_t\) be a fiber of \(q\) such that \(y_0 \not\in L_t\). Put \(\widetilde{L}_t := (\pi \circ \beta)^{-1}_*L_t\). We see that \(\widetilde{L}_t \cap \widetilde{H}_0\) and \(F \cap \widetilde{H}_0\) are both fibers of the projection

\[
(q \circ \pi \circ \beta)|_{\widetilde{H}_0} : \widetilde{H}_0 \to \mathbb{P}^1.
\]

Hence, we have \(\widetilde{L}|_{\widetilde{H}_0} \sim \widetilde{L}_t|_{\widetilde{H}_0} \sim F|_{\widetilde{H}_0}\). Therefore,

\[
(\widetilde{H} + \widetilde{L} - \widetilde{E} - F)|_{\widetilde{H}_0} \sim (\widetilde{H} - \widetilde{E})|_{\widetilde{H}_0},
\]

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which is nef. Since $\tilde{H}_0 \sim \tilde{H} - \tilde{E} - F$, we have

$$(\tilde{H} + \tilde{L} - \tilde{E} - F) - \tilde{H}_0 \sim \tilde{L},$$

which is also nef. By Lemma 3 we conclude that $D(0, 1) = \tilde{H} + \tilde{L} - \tilde{E} - F$ is nef on $\tilde{X}$.

Now, we show that $D(a, b)$ is nef for $a \geq 1$ and $b \geq 0$. Since $F|_V \in \text{Pic}(\tilde{V})$ corresponds to $S' \in \text{Pic}(V')$ via the isomorphism $\beta|_V : \tilde{V} \to V'$, the divisor $D(a, b)|_V$ is identified with the following:

$$(2H' + L' - E)|_{V'} - S' \quad (a = 1, b = 0)
(2H' + bL' - E)|_{V'} - S' \quad (a = 1, b \geq 1)
(aH' + L' - E)|_{V'} - S' \quad (a \geq 2, b = 0)
(aH' + bL' - E)|_{V'} - S' \quad (a \geq 2, b \geq 1).$$

Note that $\pi|_{V'} : V' \to V$ is the blow-up at the point $y_0 = S \cap C$ and the exceptional divisor is $E \cap V'$. Hence, we have

$$S' \sim (\pi|_{V'})^*S - E|_{V'} \sim (\pi|_{V'})^*(U|_V) - E|_{V'} \sim (aH' + bL')|_{V'} - E|_{V'}$$

$$= \begin{cases} (H' - E)|_{V'} & (a = 1, b = 0) \\
(H' + bL' - E)|_{V'} & (a = 1, b \geq 1) \\
(aH' - E)|_{V'} & (a \geq 2, b = 0) \\
(aH' + bL' - E)|_{V'} & (a \geq 2, b \geq 1). \end{cases}$$

Therefore, $D(a, b)|_V$ corresponds to:

$$(H' + L')|_{V'} \quad (a = 1, b = 0)
H'||_{V'} \quad (a = 1, b \geq 1)
L'|_{V'} \quad (a \geq 2, b = 0)
0 \quad (a \geq 2, b \geq 1),$$

which is nef in any case.

On the other hand, since $\tilde{V} \sim \tilde{H} + \tilde{L} - F$, we have

$$D(a, b) - \tilde{V} \sim \begin{cases} \tilde{H} - \tilde{E} & (a = 1, b = 0) \\
\tilde{H} + (b - 1)\tilde{L} - \tilde{E} & (a = 1, b \geq 1) \\
(a - 2)\tilde{H} + (\tilde{H} - \tilde{E}) & (a \geq 2, b = 0) \\
(a - 2)\tilde{H} + (b - 1)\tilde{L} + (\tilde{H} - \tilde{E}) & (a \geq 2, b \geq 1). \end{cases}$$
Recall that $\tilde{H}, \tilde{L}$ and $\tilde{H} - \tilde{E}$ are nef. Hence, so is $D(a, b) - \tilde{V}$. By Lemma 3, we conclude that $D(a, b)$ is nef.

We have the following table of intersection numbers.

|     | $\tilde{H}$ | $\tilde{L}$ | $\tilde{E}$ | $F$ |
|-----|-------------|-------------|-------------|-----|
| $l$ | 1           | 0           | 0           | 0   |
| $\tilde{h}$ | 0 | 1           | 0           | 0   |
| $\tilde{e}_0$ | 0 | 0           | 1           | 0   |
| $f$ | 0           | 0           | 0           | 1   |

By definition of $l(a), h(b)$ and $D(a, b)$, for $(a, b) = (0, 1)$ and for any $(a, b)$ such that $a \geq 1$ and $b \geq 0$, we have:

|     | $\tilde{H}$ | $\tilde{L}$ | $\tilde{H} - \tilde{E}$ | $D(a, b)$ |
|-----|-------------|-------------|----------------|-----------|
| $l(a)$ | 1 | 0           | 0           | 0         |
| $h(b)$ | 0 | 1           | 0           | 0         |
| $\tilde{e}_0$ | 0 | 0           | 1           | 0         |
| $f$ | 0           | 0           | 0           | 1         |

Now, the proposition follows from Lemma 2 because we have $\rho(\tilde{X}) = 4$.

Remark. In the proof, we have also shown that $\overline{\text{NE}(\tilde{X})} = \mathbb{R}^+[l(a)] + \mathbb{R}^+[h(b)] + \mathbb{R}^+[\tilde{e}_0] + \mathbb{R}^+[f]$.

4 Self intersection numbers of anti-canonical divisors

The purpose of this section is to prove the following:

**Proposition 2.** If $a = 1$, we have

$$(-K\tilde{X})^n = \frac{(7 - b)n}{2}(n - 1)^{n-1} - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n.$$ 

If $a \neq 1$, we have

$$(-K\tilde{X})^n = (n - a)^{n-1} \left( -3a + 2 + ab \right) n + a^2 - ab \left( \frac{a^2 - b}{a - 1} \right) + (n - 1)^{n-1} \left( \frac{a^2 - b}{a - 1} \right) - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n.$$ 

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We prepare some lemmas.

**Lemma 4.** Let $D$ be a divisor on a smooth projective variety $Y$ of dimension $n \geq 3$ and let $S$ be a smooth subvariety in $Y$ of codimension $r \geq 2$. Let $\mu : Z \to Y$ be the blow-up along $S$. Let $F$ be the exceptional divisor of $\mu$. Then, for $k = 1, 2, \cdots, n$, we have

$$(\mu^*D)^{n-k}F^k = (-1)^{r-1}(D|_S)^{n-k}s_{k-r}(N^*_S/Y)$$

where $s_{k-r}(N^*_S/Y)$ denotes the Segre classes of the conormal bundle $N^*_S/Y$.

**Proof.** We follow the notation in [4] Chapter 3 and Appendix B, i.e. for a vector space $V$, the projectivization $\mathbb{P}(V)$ denotes the set of lines in $V$. Consider the $\mathbb{P}^{r-1}$-bundle $\mu|_F : F = \mathbb{P}(N_S/Y) \to S$. Let $\mathcal{O}(1)$ be the dual bundle of the tautological line bundle $\mathcal{O}(-1)$ associated to $N_S/Y$. By a definition of Segre classes, we have

$$(\mu|_F)^*(D|_S)^{n-k}O(1)^{k-1} = ((\mu|_F)^*(D|_S))^{n-k}O(1)^{(r-1)+(k-r)} = (D|_S)^{n-k}s_{k-r}(N_S/Y).$$

This yields

$$(\mu^*D)^{n-k}F^k = (\mu^*D|_F)^{n-k}(F|_F)^{k-1}
= ((\mu|_F)^*(D|_S))^{n-k}O(1)^{k-1}
= (-1)^{k-1}(D|_S)^{n-k}s_{k-r}(N_S/Y)
= (-1)^{k-1}(D|_S)^{n-k}(-1)^{k-r}s_{k-r}(N^*_S/Y)
= (-1)^{r-1}(D|_S)^{n-k}s_{k-r}(N^*_S/Y).$$

Now, we return to our situation (Notations (*) in Section 2). However, in what follows, we put $h := H|_S$ and $l := L|_S$.

**Lemma 5.** We have

$$h^{n-2} = a + b, \ h^{n-3}l = a, \ l^2 \equiv 0.$$
Proof. Note that $S = UV \equiv (aH+bL)(H+L)$. Since $H^n = 0$, $H^{n-1}L = 1$ and $L^2 \equiv 0$, we obtain

\[
\begin{align*}
h^{n-2} &= H^{n-2}S = aH^n + (a+b)H^{n-1}L = a + b, \\
h^{n-3}l &= H^{n-3}LS = aH^{n-1}L + (a+b)H^{n-2}L^2 = a, \\
l^2 &= L^2S \equiv 0.
\end{align*}
\]

\[\square\]

For $a \geq 1$, we put $P(m) := \sum_{i=0}^{m} a^i$ and $Q(m) := \sum_{i=0}^{m} (ia^{i-1}b+(m-i)a^i)$.

**Lemma 6.** For $m = 1, 2, \cdots, n-2$, the $m$-th Segre classe is given by

\[s_m(N^*_{S/Y}) = P(m)h^m + Q(m)h^{m-1}l.\]

**Proof.** Put $u := U|_S$ and $v := V|_S$. Since $S = U \cap V$ is a complete intersection, we have

\[N_{S/Y} = N_{U/Y}|_S \oplus N_{V/Y}|_S = U|_S \oplus Y|_S = u \oplus v.\]

Hence, $N^*_{S/Y} = (-u) \oplus (-v)$. By Whitney formula, we obtain

\[c(N^*_{S/Y}) = (1-u)(1-v).\]

By the equality $c \cdot s = 1$ between the total Chern classe and the total Segre classe, we get

\[s(N^*_{S/Y}) = \frac{1}{1-u} \cdot \frac{1}{1-v} = (1+u+u^2+\cdots) \cdot (1+v+v^2+\cdots),\]

whose homogeneous part of degree $m$ equals $\sum_{i+j=m} u^i v^j$. Since $l^2 \equiv 0$, we have

\[u^i v^j = (ah+bl)^i(h+l)^j = a^i h^{i+j} + (ia^{i-1}b + ja^i)h^{i+j-1}l.\]

Therefore,

\[s_m(N^*_{S/Y}) = \sum_{i+j=m} u^i v^j = \sum_{i+j=m} (a^i h^{i+j} + (ia^{i-1}b + ja^i)h^{i+j-1}l) = \left( \sum_{i+j=m} a^i \right)h^m + \left( \sum_{i+j=m} (ia^{i-1}b + ja^i) \right)h^{m-1}l.\]
Put

\[ I_n := \sum_{k=2}^{n} \binom{n}{k} (-1)^k P(k-2)n^{n-k}, \]

\[ I'_n := \sum_{k=2}^{n} \binom{n}{k} (-1)^k kP(k-2)n^{n-k}, \]

\[ J_n := \sum_{k=2}^{n} \binom{n}{k} (-1)^k Q(k-2)n^{n-k}. \]

Lemma 7. If \( a = 1 \), we have

\[ I_n = n^n - (2n-1)(n-1)^{n-1}, \]

\[ I'_n = n(n-1)^{n-1}, \]

\[ J_n = \frac{b+1}{2}((5n-2)(n-1)^{n-1} - 2n^n). \]

If \( a \geq 2 \), we have

\[ I_n = \frac{(n-a)^n + (a-1)n^n - a(n-1)^n - a(n-1)^n}{a(a-1)}, \]

\[ I'_n = \frac{n}{a-1}((n-1)^{n-1} - (n-a)^{n-1}), \]

\[ J_n = \frac{(a+b-2ab)(n-a) - ab(a-1)n}{a^2(a-1)^2}(n-a)^{n-1} \]

\[ + \frac{(a-1)n + (a+b-2)(n-1)}{(a-1)^2}(n-1)^{n-1} - \frac{a+b}{a^2}n^n. \]

Proof. For \( a = 1 \), we have

\[ P(k-2) = k-1, \quad Q(k-2) = \frac{b+1}{2}(k^2 - 3k + 2). \]

If \( a \geq 2 \), we put \( \theta := 1/(a^2 - a) \). Then, we have

\[ P(k-2) = \theta(a^k - a), \]

\[ Q(k-2) = \theta^2((a+b-2ab)a^k + b(a-1)ka^k - a^2(a-1)k + a^2(a+b-2)). \]
The statement is verified by direct computations using the following equalities for \( x = a \) and \( x = 1 \):

\[
\sum_{k=2}^{n} \binom{n}{k} (-x)^k n^{n-k} = (n-x)^n + (x-1)n^n,
\]

\[
\sum_{k=2}^{n} \binom{n}{k} k(-x)^k n^{n-k} = xn^n - xn(n-x)^{n-1},
\]

\[
\sum_{k=2}^{n} \binom{n}{k} k^2(-x)^k n^{n-k} = x(x-1)n^2(n-x)^{n-2} + xn^n.
\]

\[\square\]

**Proof of Proposition 2** First, we consider the case \((a, b) = (0, 1)\). Put \(L_0 := q^{-1}(q(y_0))\) and \(H_0 := p^{-1}(p(S))\). Note that \(S\) is a hyperplane in \(L_0 \cong \mathbb{P}^{n-1}\). Let \(L'_0\) and \(H'_0\) be the strict transforms by \(\pi\) of \(L_0\) and \(H_0\). Then \(S'\) is the complete intersection of \(L'_0\) and \(H'_0\). Since \(H'_0 \sim H' - E, L'_0 \sim L'\) and \(L'|_{S'} \sim 0\), we have \(N^*_{S'/X} \cong O_{S'}(H' - E) \oplus O_{S'}\). As in the proof of Lemma 6 this yields

\[
s_m(N^*_{S'/X}) = (H'|_{S'} - E|_{S'})^m \quad \text{for} \quad m = 1, 2, \cdots, n - 2.
\]

On the other hand, we have

\[-K_X|_{S'} \sim (nH' + 2L' - (n - 2)E)|_{S'} \sim nH'|_{S'} - (n - 2)E|_{S'}.
\]

We observe that \(\pi|_{S'} : S' \to S \cong \mathbb{P}^{n-2}\) is the blow-up at \(y_0\). We have \(H'|_{S'} \sim (\pi|_{S'})^* O_{\mathbb{P}^{n-2}}(1)\) and \(\text{Exc}(\pi|_{S'}) = E|_{S'}\). Note that

\[
(H'|_{S'})(E|_{S'}) \equiv 0, \quad (H'|_{S'})^{n-2} = 1 \quad \text{and} \quad (E|_{S'})^{n-2} = (-1)^{n-3}.
\]

By Lemma 4 for \(r = 2\), we obtain \(\beta^*(-K_X)^{n-1}F = 0\) and for \(k = 2, \cdots, n\),

\[
\beta^*(-K_X)^{n-k}F^k = -(nH'|_{S'} - (n - 2)E|_{S'})^{n-k}(H'|_{S'} - E|_{S'})^{k-2} = (n - 2)^{n-k} - n^{n-k}.
\]

Since \(X\) is isomorphic to \(\mathbb{P}^1 \times Bl_z(\mathbb{P}^{n-1})\) where \(z\) is a point in \(\mathbb{P}^{n-1}\), we have

\[
(-K_X)^n = 2n(n^{n-1} - (n - 2)^{n-1}).
\]
It follows that
\[
(-K_X)^n = (\beta^*(-K_X) - F)^n
\]
\[
= (-K_X)^n + \sum_{k=2}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k \beta^*(-K_X)^{n-k} F^k
\]
\[
= 2n(n^{n-1} - (n-2)^{n-1}) + \sum_{k=2}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k ((n-2)^{n-k} - n^{n-k})
\]
\[
= 2n^n - (n-1)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n.
\]

In case \((a, b) \neq (0, 1)\), since we cannot necessarily describe \(S' \subset X\) as a complete intersection (remark that \(U' \cap V' = S' \cup E_0\) where \(U'\) and \(V'\) are the strict transforms by \(\pi\) of \(U\) and \(V\)), it seems hard to compute \((-K_X)^n\) directly from \((-K_X)^n\). We avoid this difficulty by considering a flip of \(\tilde{X}\):

**Step 1.** Let \(\mu : Z \to Y = \mathbb{P}^{n-1} \times \mathbb{P}^1\) be the blow-up along \(S = U \cap V\). Let \(F_Z\) be the exceptional divisor of \(\mu\). We have
\[
\mu^*(-K_Y)^n = (-K_Y)^n = (nH + 2L)^n = 2n^n.
\]
By Lemma 4 for \(r = 2\), we have \(\mu^*(-K_Y)^{n-1}F_Z = 0\) and
\[
\mu^*(-K_Y)^{n-k}F_Z^k = -(-K_Y|_S)^{n-k}s_{k-2}(N_{S/Y}^*) \text{ for } k = 2, \ldots, n.
\]
Therefore,
\[
(-K_Z)^n = (\mu^*(-K_Y) - F_Z)^n
\]
\[
= \mu^*(-K_Y)^n + \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \mu^*(-K_Y)^{n-k}(-F_Z)^k
\]
\[
= 2n^n - \sum_{k=2}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^k (-K_Y|_S)^{n-k}s_{k-2}(N_{S/Y}^*).
\]

Here, we have \(-K_Y|_S \sim (nH + 2L)|_S = nh + 2l\). By Lemma 5 and 6
\((-K_Y|_S)^{n-k}s_{k-2}(N_{S/Y}^*)\) is equal to
\[
(3a + b)P(k - 2)n^{n-k} - 2aP(k - 2)kn^{n-k-1} + aQ(k - 2)n^{n-k}.
\]
Thus,
\[
(-K_Z)^n = 2n^n - (3a + b)I_n + \frac{2a}{n}I'_n - aJ_n.
\]
Therefore, we conclude that if \( a = 1 \), we have
\[
(-K_Z)^n = \frac{(7 - b) n}{2} (n - 1)^{n-1},
\]
if \( a \geq 2 \), we have
\[
(-K_Z)^n = (n - a)^{n-1} \frac{(-3a + 2 + ab)n + a^2 - ab}{(a - 1)^2} + (n - 1)^{n-1} \frac{(a^2 - b)n - a + b}{(a - 1)^2}.
\]

Step 2. Let \( \alpha : \widetilde{Z} \to Z \) be the blow-up along the curve \( C' := \mu_*^{-1}C \) with the exceptional divisor \( G \). Note that \( N_{C'/Z} = \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-1) \). Hence, we have \( -K_Z \cdot C' = 1 \) and \( \deg(N_{C'/Z}) = 1 \). Using Lemma 4 for \( r = n - 1 \), we have the following:
\[
\begin{align*}
\alpha^*(-K_Z)^{n-k}G^k & = 0 \quad \text{for } k = 1, 2, \cdots, n - 2, \\
\alpha^*(-K_Z)G^{n-1} & = (-1)^n(-K_Z \cdot C') = (-1)^n, \\
G^n & = (-1)^n s_1(N_{C'/Z}) = (-1)^{n+1}.
\end{align*}
\]
Therefore,
\[
(-K_{\widetilde{Z}})^n = (\alpha^*(-K_Z) - (n - 2)G)^n \\
= (\alpha^*(-K_Z))^n + (-1)^{n-1} n(n - 2)^{n-1} \alpha^*(-K_Z)G^{n-1} + (-1)^n(n - 2)^n G^n \\
= (-K_Z)^n - n(n - 2)^{n-1} - (n - 2)^n \\
= (-K_Z)^n - 2(n - 1)(n - 2)^{n-1}.
\]

Step 3. We observe that \( \widetilde{X} \) and \( \widetilde{Z} \) are connected by a flip. We have
\[
(-K_{\widetilde{X}})^n = (-K_{\widetilde{Z}})^n + (n - 3)^n.
\]
To see this, put \( \Gamma_0 := \mu^{-1}(y_0) \) and \( \widetilde{\Gamma}_0 := \alpha_*^{-1}\Gamma_0 \). Let \( \gamma : W \to \widetilde{Z} \) be the blow-up along the curve \( \widetilde{\Gamma}_0 \) and let \( M \) be the exceptional divisor of \( \gamma \). Note that \( M \simeq \mathbb{P}^{n-2} \times \mathbb{P}^1 \) and \( N_{M/W} \simeq \mathcal{O}_{\mathbb{P}^{n-2} \times \mathbb{P}^1}(-1, -1) \). The contraction map sending \( M \) to \( \mathbb{P}^{n-2} \) is nothing but the blow-up \( \delta : W \to \widetilde{X} \) along \( \widetilde{E}_0 \simeq \mathbb{P}^{n-2} \).

We have \( K_W \sim \delta^*K_{\widetilde{X}} + M \) and \( K_W \sim \gamma^*K_{\widetilde{Z}} + (n - 2)M \). Hence,
\[
\delta^*(-K_{\widetilde{X}}) \sim \gamma^*(-K_{\widetilde{Z}}) - (n - 3)M.
\]
Note that \( N_{\widetilde{E}_0/\widetilde{Z}} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\oplus(n-1)} \). Hence, we have \( -K_{\widetilde{Z}} \cdot \widetilde{\Gamma}_0 = 3 - n \) and \( \deg(N_{\widetilde{E}_0/\widetilde{Z}}) = n - 1 \). As in Step 2, we obtain \( \gamma^*(-K_{\widetilde{Z}})M^{n-1} = (-1)^{n+1}(n-3) \), \( M^n = (-1)^{n+1}(n - 1) \) and \( \gamma^*(-K_{\widetilde{Z}})^{n-k}M^k = 0 \) for \( k = 1, \cdots, n - 2 \).
Thus,
\[(−K_\tilde{X})^n = (δ^*(-K_\tilde{X}))^n \]
\[= (γ^*(-K_\tilde{Z}) - (n - 3)M)^n \]
\[= (γ^*(-K_\tilde{Z}))^n + (-1)^{n-1}n(n-3)^{n-1}γ^*(-K_\tilde{Z})M^{n-1} + (-1)^n(n-3)^nM^n \]
\[= (-K_\tilde{Z})^n + (n-3)^n. \]

By Step 2 and Step 3, we obtain
\[(−K_\tilde{X}) = (−K_\tilde{Z})^n - 2(n-1)(n-2)^{n-1} + (n-3)^n. \]

Substituting the result of Step 1, we complete the proof of Proposition 2.

**Proposition 3.** For \((a, b) = (0, 1)\) and for \((a, b)\) such that \(1 ≤ a ≤ 3\) and \(0 ≤ b ≤ 3\), we have \((−K_\tilde{X})^n > 0\) for any \(n ≥ 3\).

**Proof.** For each case, we compute \((−K_\tilde{X})^n\) using Proposition 2. Note that we have \((n - 1)^n - 2(n-1)(n-2)^{n-1} > 0\) for any \(n ≥ 3\).

If \((a, b) = (0, 1)\), then
\[(−K_\tilde{X})^n = 2n^n - (n - 1)^n - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \]
\[≥ 2(n - 1)^n - (n - 1)^n - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \]
\[≥ (n - 1)^n - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \]
\[> 0. \]

Assume \(b ≤ 3\). If \(a = 1\),
\[(−K_\tilde{X})^n = 7 - \frac{b}{2}n(n - 1)^{n-1} - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \]
\[≥ 2n(n - 1)^{n-1} - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \]
\[> (n - 1)^n - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \]
\[> 0. \]
If \( a = 2 \),

\[
(-K_{\tilde{X}})^n = (4n - 2)(n - 1)^{n-1} - 6(n - 1)(n - 2)^{n-1} - b(n - 1)^n - 2(n - 1)(n - 2)^{n-1} + (n - 3)^n \\
\geq (4n - 2)(n - 1)^{n-1} - 6(n - 1)(n - 2)^{n-1} - 3((n - 1)^n - 2(n - 1)(n - 2)^{n-1}) + (n - 3)^n \\
= (n - 1)^n + (n - 3)^n \\
> 0.
\]

If \( a = 3 \),

\[
4(-K_{\tilde{X}})^n = -3(n - 1)(n - 3)^{n-1} - 8(n - 1)(n - 2)^{n-1} + (9n - 3)(n - 1)^{n-1} \\
- b((n - 1)^n - 3(n - 1)(n - 3)^{n-1}) \\
\geq -3(n + 1)(n - 3)^{n-1} - 8(n - 1)(n - 2)^{n-1} + (9n - 3)(n - 1)^{n-1} \\
- 3((n - 1)^n - 3(n - 1)(n - 3)^{n-1}) \\
= (6n - 12)(n - 3)^{n-1} + 6n(n - 1)^{n-1} - 8(n - 1)(n - 2)^{n-1} \\
> 6n(n - 1)^{n-1} - 8(n - 1)(n - 2)^{n-1} \\
> 4((n - 1)^n - 2(n - 1)(n - 2)^{n-1}) \\
> 0.
\]

\( \Box \)

Remark. More precise estimations show that we have \((-K_{\tilde{X}})^n > 0\) for any \( n \geq 3 \) in the cases: \( a = 1 \) and \( b \leq 5 \); \( a = 2 \) and \( b \leq 6 \); \( a = 3 \) and \( b \leq 8 \). In case \( a \geq 4 \), the positivity of \((-K_{\tilde{X}})^n\) is independent of the value \( b \). For example, if \( a = 15 \) we have \((-K_{\tilde{X}})^4 = -306b - 285 < 0\) and \((-K_{\tilde{X}})^5 = 3056b + 1344 > 0\) for any \( b \in \mathbb{N} \).

5 Proof of Theorem

By the canonical bundle formula for the blow-ups \( \pi \) and \( \beta \), we have

\[
K_{\tilde{X}} \sim \beta^*K_X + F \sim \beta^*(\pi^*K_Y) + (n - 2)\tilde{E} + F.
\]

Combining with \(-K_Y \sim nH + 2L\), we get

\[
-K_{\tilde{X}} \sim n\tilde{H} + 2\tilde{L} - (n - 2)\tilde{E} - F.
\]
First, we consider the case $a \geq 2$ and $b \geq 1$. We rewrite $-K_{\tilde{X}}$ by means of the generators of $Nef(\tilde{X})$:

$$
-K_{\tilde{X}} = (3 - a)\tilde{H} + (2 - b)\tilde{L} + (n - 3)(\tilde{H} - \tilde{E}) + (a\tilde{H} + b\tilde{L} - \tilde{E} - F).
$$

By Proposition 1, we see that $-K_{\tilde{X}}$ is nef if and only if $3 - a \geq 0$, $2 - b \geq 0$ and $n - 3 \geq 0$. Since the numerical equivalence classes of ample divisors are interior points of the nef cone ([7] Theorem 1.4.23), it follows that $-K_{\tilde{X}}$ is ample if and only if $3 - a > 0$, $2 - b > 0$ and $n - 3 > 0$.

In the other cases, we argue similarly in the following forms:

- For $a = 0$ and $b = 1$,
  $$
  -K_{\tilde{X}} \sim 2\tilde{H} + \tilde{L} + (n - 3)(\tilde{H} - \tilde{E}) + (\tilde{H} + \tilde{L} - \tilde{E} - F).
  $$

- For $a = 1$ and $b = 0$,
  $$
  -K_{\tilde{X}} \sim \tilde{H} + \tilde{L} + (n - 3)(\tilde{H} - \tilde{E}) + (2\tilde{H} + \tilde{L} - \tilde{E} - F).
  $$

- For $a = 1$ and $b \geq 1$,
  $$
  -K_{\tilde{X}} \sim \tilde{H} + (2 - b)\tilde{L} + (n - 3)(\tilde{H} - \tilde{E}) + (2\tilde{H} + b\tilde{L} - \tilde{E} - F).
  $$

- For $a \geq 2$ and $b = 0$,
  $$
  -K_{\tilde{X}} \sim (3 - a)\tilde{H} + \tilde{L} + (n - 3)(\tilde{H} - \tilde{E}) + (a\tilde{H} + \tilde{L} - \tilde{E} - F).
  $$

Finally, we conclude that $-K_{\tilde{X}}$ is nef if and only if $n \geq 3$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1) \text{ or } (3, 2).$$

Moreover, $-K_{\tilde{X}}$ is ample if and only if $n \geq 4$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (2, 0) \text{ or } (2, 1).$$

In general, a nef divisor $D$ is big if and only if $D^n > 0$ ([7] Theorem 2.2.16). Thus, the proof of Theorem is completed by Proposition 3. $\square$


6 Other examples

Details in this section can be verified by the methods in Sections 3 and 4. We keep all the notations (*) in Section 2 except that the divisors $H$ and $L$ are replaced by appropriate ones.

In the following examples 1, 2 and 3, we put $Y = \mathbb{P}^n$ with $n \geq 4$.

**Example 1.** Let $C$ be a line and $S$ an $(n - 2)$-plane. Assume $C \cap S \neq \emptyset$. We consider $H := \mathcal{O}_{\mathbb{P}^n}(1)$ and $\bar{H} := (\pi \circ \beta)^*H$. Then, we have

$$
\text{Nef}(\bar{X}) = \mathbb{R}^+[\bar{H}] + \mathbb{R}^+[\bar{H} - \bar{E}] + \mathbb{R}^+[2\bar{H} - \bar{E} - F],
$$

$$-K_{\bar{X}} \sim (n + 1)\bar{H} - (n - 2)\bar{E} - F = 2\bar{H} + (n - 3)(\bar{H} - \bar{E}) + (2\bar{H} - \bar{E} - F).$$

Hence, $\bar{X}$ is a Fano manifold for any $n \geq 4$.

**Example 2.** Let $C$ be a line. Let $S$ be the complete intersection of a hyperplane and a hyperquadric. Assume that $C$ and $S$ intersect transversally at one point. Then $\bar{X}$ is a Fano manifold for any $n \geq 4$. Indeed, the structure of nef cone and the description of the anti-canonical divisor for $\bar{X}$ are completely same as in Example 1. Even if the intersection $C \cap S$ consists of two points, $\bar{X}$ remains Fano, while the exceptional locus of the small contraction has two irreducible components.

**Example 3.** Let $P \subset Y = \mathbb{P}^n$ be a 2-plane and $C$ a smooth conic on $P \cong \mathbb{P}^2$. Let $S$ be an $(n - 2)$-plane such that $\sharp(C \cap S) = 2$. Then, we have

$$
\text{Nef}(\bar{X}) = \mathbb{R}^+[\bar{H}] + \mathbb{R}^+[2\bar{H} - \bar{E}] + \mathbb{R}^+[3\bar{H} - \bar{E} - F],
$$

$$-K_{\bar{X}} \sim (n + 1)\bar{H} - (n - 2)\bar{E} - F = (4 - n)\bar{H} + (n - 3)(2\bar{H} - \bar{E}) + (3\bar{H} - \bar{E} - F).$$

We see that $-K_{\bar{X}}$ is nef only for $n = 4$. Moreover, we have $(-K_{\bar{X}})^4 = 353 > 0$. Hence $\bar{X}$ is a weak Fano manifold for $n = 4$.

**Example 4.** Let $Y := \mathbb{P}^2 \times \mathbb{P}^2$. Let $C$ be a line in a fiber of a projection $Y \to \mathbb{P}^2$ and $S$ a fiber of the other projection such that $C \cap S \neq \emptyset$. Then $\bar{X}$ is a Fano 4-fold. Indeed, we are able to show:

**Proposition 4.** Let $Y := \mathbb{P}^{n-2} \times \mathbb{P}^2$ with $n \geq 3$. Let $C$ be a smooth plane curve of degree $d$ in a fiber of the projection $p : Y \to \mathbb{P}^{n-2}$. Let $S$ be a fiber of the projection $q : Y \to \mathbb{P}^2$ such that $C \cap S \neq \emptyset$. Then $\bar{X}$ is a weak Fano manifold if and only if

$$(n, d) = (3, 1), (3, 2), (3, 3), (4, 1) \text{ or } (5, 1).$$

Moreover, $\bar{X}$ is a Fano manifold if and only if $(n, d) = (4, 1)$. 

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Proof. Put $H := p^*\mathcal{O}_{\mathbb{P}^m}(1)$ and $L := q^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then we have

$$\text{Nef}(\tilde{X}) = R^+[\tilde{H}] + R^+[[\tilde{L} + \tilde{E} + \tilde{E} - F]].$$

Since $K_{\tilde{X}} \sim (\pi \circ \beta)^*K_Y + (n-2)\tilde{E} + F$ and $-K_Y \sim (n-1)H + 3L$, we have

$$-K_{\tilde{X}} \sim (n-1)\tilde{H} + 3\tilde{L} - (n-2)\tilde{E} - F$$
$$= \tilde{H} + (3 - d(n-2))\tilde{L} + (n-3)(\tilde{H} + d\tilde{L} - \tilde{E}) + (\tilde{H} + d\tilde{L} - \tilde{E} - F).$$

Hence, $-K_{\tilde{X}}$ is nef (resp. ample) if and only if $3 - d(n-2)$ and $n-3$ are positive (resp. strictly positive). On the other hand, we obtain

$$(-K_{\tilde{X}})^n = 4n(n-1)^{n-1} + (n-2)^{n-1}(d(d-3)n-2d^2+2) + (n-3)^n,$$

which is strictly positive for $(n, d) = (3, 1), (3, 2), (3, 3), (4, 1)$ and $(5, 1)$. □

Acknowledgements. The author would like to thank Kento Fujita and Kazunori Yasutake for helpful comments.

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