Linear vs. nonlinear speed selection of the front propagation into unstable states

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Abstract

In this paper, we mainly consider the speed selection problem for the Lotka-Volterra competition-diffusion system. For the first time, we propose a sufficient and necessary condition for this long-standing problem from a new point of view, which differs from the classical result on single equations established in [Lucia-Muratov-Novaga, CPAM 2004]. Moreover, our results can also reveal the essence of the linearly selected problem for the monostable dynamical system from the

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observation on the decay rate of the minimal traveling wave solution.

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1 Introduction

The phenomenon of the front propagation into unstable states is a classical issue and has been discussed by many physical researchers in early works, e.g., [4, 5, 36, 37, 38]. A typical model for this phenomenon is the well-known Fisher-KPP equation ([3, 10, 23]), depicting the spatial propagation of organisms such as dominant genes and invasive species in a homogeneous environment. In the long survey paper [38], Saarloos highlighted the practical significance of this problem, pointing out that it is not only esoteric from purely academic interest, but also plays an important role in reality, as there are numerous important experimental examples for which the fronts propagate rapidly into an unstable state. Among other things, he also emphasized the importance of the connection between pulled fronts and pushed fronts, which is crucial in studying the speed selection problem of front propagation.

A typical model to describe the transition from a stable state to an unstable one in reaction-diffusion equations is

\[
\begin{cases}
w_t = w_{xx} + f(w), & t > 0, \ x \in \mathbb{R}, \\
w(0, x) = w_0(x), & x \in \mathbb{R},
\end{cases}
\]  

(1.1)

where \( f \) is of the monostable type that satisfies

\[
f(0) = f(1) = 0, \quad f'(0) > 0 > f'(1), \quad \text{and} \quad f(w) > 0 \quad \text{for all} \ w \in (0, 1).
\]

It is well known that the global dynamics of (1.1) is highly related to the properties of traveling wave solutions, which are particular solutions in the form \( w(t, x) = W(x - ct) \) satisfying

\[
\begin{cases}
W'' + cW' + f(W) = 0, & \xi \in \mathbb{R}, \\
W(-\infty) = 1, \quad W(+\infty) = 0, \quad W'(\cdot) < 0.
\end{cases}
\]

(1.2)

It has been proved that (see [3, 10, 23, 39]) there exists \( c^* \geq 2 \sqrt{f'(0)} > 0 \) such that (1.2) admits a solution if and only if \( c \geq c^* \). Thus, \( c^* \) is called the minimal traveling wave speed. In general, the minimal speed \( c^* \) depends on the shape of \( f \) and cannot be characterized explicitly except for some specific \( f \); for example, it was shown in [23] that \( c^* = 2 \sqrt{f'(0)} \) if \( f \) satisfies the KPP condition:

\[
f'(0)w \geq f(w) \quad \text{for all} \ w \in [0, 1].
\]

(1.3)

In the literature, the minimal traveling wave is classified into two types: pulled front and pushed front [34, 35, 38]. The minimal traveling wave \( W \) with the speed \( c^* \) is called a pulled front if \( c^* = 2 \sqrt{f'(0)} \). In this case, the front is pulled by the leading edge with speed determined by the linearized problem at the unstable state \( w = 0 \). Therefore, the minimal speed \( c^* \) is said to be linearly selected. On the other hand, if \( c^* > 2 \sqrt{f'(0)} \), the minimal traveling wave \( W \) with the speed \( c^* \) is called a pushed front since the propagation speed is determined by the whole wave, not only by the behavior of the leading edge, and thus the minimal speed \( c^* \) is said to be nonlinearly selected.
In the remarkable work [27], Lucia, Muratov, and Novaga proposed a variational approach to rigorously establish a mechanism to determine speed linear and nonlinear selection on the scalar monostable reaction-diffusion equations. Roughly speaking, the following two conditions are equivalent:

(i) the minimal traveling wave speed of \( u_t = u_{xx} + f(u) \) is nonlinearly selected;

(ii) \( \Phi_c[u] \leq 0 \) holds for some \( c > 2\sqrt{f'(0)} \) and \( u(\neq 0) \in C^\infty_0(\mathbb{R}) \), where

\[
\Phi_c[u] := \int_\mathbb{R} e^{cx} \left[ \frac{1}{2} u_x^2 - \int_0^u f(s)ds \right] dx.
\]

As an application, some explicit and easy-to-check results can be obtained to determine linear and nonlinear selection (see Section 5 in [27]). A related issue can be found in [28] using the theory of abstract monotone semiflow.

In this paper, we first revisit the speed selection problem for the minimal traveling wave speed of scalar reaction-diffusion equations with monostable nonlinearity. We will establish a new sufficient and necessary condition for determining the linear or nonlinear selection mechanism by considering a family of continuously varying nonlinearities. By varying the parameter within the nonlinearity, we obtain a comprehensive understanding of how the decay rate of the minimal traveling wave at infinity affects the minimal traveling wave speed, through a novel construction of super-subsolutions. This approach enables us to gain insight into the essence underlying the transition from linear selection to nonlinear selection. The propagation phenomenon and inside dynamics of the front for more general scalar equations have been widely discussed in the literature. We may refer to, e.g., [6, 9, 11, 26, 32, 34, 35] and references cited therein.

Our methodology for studying the speed selection problem in scalar equations can extend to monostable monotone systems, in which the variational approach may not be applicable due to the lack of a variational structure. Specifically, we will focus on the following two-species Lotka-Volterra competition-diffusion model, which is of significant biological relevance [30]:

\[
\begin{align*}
  &u_t = u_{xx} + u(1 - u - au), \quad t > 0, \quad x \in \mathbb{R}, \\
  &v_t = dv_{xx} + rv(1 - v - bu), \quad t > 0, \quad x \in \mathbb{R}.
\end{align*}
\]

(1.4)

In this model, \( u = u(t,x) \) and \( v = v(t,x) \) represent the population densities of two competing species at the time \( t \) and position \( x \); \( d \) and \( r \) stand for the diffusion rate and intrinsic growth rate of \( v \), respectively; \( a \) and \( b \) represent the competition coefficient of \( v \) and \( u \), respectively. Here, all parameters are assumed to be positive and satisfy the monostable structure, i.e. \( a, b > 0 \) and \( 0 < x < 1 \).

Similar to the scalar equation, linear and nonlinear selection of the minimal traveling wave speed \( c_{LV}^* \) can be defined. It is linearly selected if \( c_{LV}^* = 2\sqrt{1-a} \) since the linearization of (1.4) at the unstable state \( (u,v) = (0,1) \) results in the linear speed \( 2\sqrt{1-a} \). This situation is also called pulled front case since the propagation speed is determined only by the leading edge of the distribution of the population. In the case \( c_{LV}^* > 2\sqrt{1-a} \), we say that the minimal traveling wave speed \( c_{LV}^* \) is nonlinearly selected. This situation is also called pushed front case since the propagation speed is not only determined by the behavior of the leading edge of the population distribution, but by the whole wave. We also refer to the work of Roques et al. [33] that introduced another definition of pulled and pushed fronts for (1.4).
Sufficient conditions for linear or nonlinear selection mechanism for (1.4) with \(0 < a < 1 < b\) have been investigated widely. Okubo et al. [31] used a heuristic argument to conjecture that the minimal speed \(c^*_{LV}\) is linearly selected, and applied it to study the competition between gray squirrels and red squirrels. Hosono [18] suggested that \(c^*_{LV}\) can be nonlinear selected in some parameter regimes. It has been proved by Lewis, Li and Weinberger [24] that linear selection holds when

\[
0 < d < 2 \quad \text{and} \quad r(ab - 1) \leq (2 - d)(1 - a).
\]

(1.5)

An improvement for the sufficient condition for linear selection was found by Huang [20]:

\[
\frac{(2 - d)(1 - a) + r}{rb} \geq \max \left\{ a, \frac{d - 2}{2|d - 1|} \right\}.
\]

(1.6)

Note that (1.5) and (1.6) are equivalent when \(d \leq 2\). Although Huang [20] strongly believed that the condition (1.6) is optimal for linear determinacy, Roques et al. [33] numerically reported that the region of the parameter for linear determinacy can still be improved. For the minimal speed \(c^*_{LV}\) being nonlinearly selected, Huang and Han [21] constructed examples in which linear determinacy fails to hold. Holzer and Scheel [16] showed that, for fixed \(a, b,\) and \(r,\) the minimal speed \(c^*_{LV}\) becomes nonlinear selection as \(d \to \infty.\) For related discussions, we also refer to, e.g., [1, 2, 13, 17, 19] and the references cited therein.

To the best of our knowledge, a comprehensive understanding on the sufficient and necessary condition of linear or nonlinear selection mechanism for (1.4) under assumption (H) has not been completely understood in the literature. In particular, previous works on speed selection problem for (1.4) primarily focused on the strong-weak competition case \((0 < a < 1 < b).\) However, as we will demonstrate in Remark 1.8 below, there are some cases that the speed \(c^*_{LV}\) is nonlinearly selected for all \(b > 1.\) This indicates that the speed selection problem for (1.4) cannot be fully explained by dealing with only the strong-weak competition case. Therefore, in this paper we will fix \(a, r,\) and \(d,\) and set the competition rate \(b \in \mathbb{R}^+\) as a continuously varying parameter. By analyzing the asymptotic behavior of the minimal traveling wave at \(\pm \infty\) and constructing novel super-subsolutions, we can extend the idea of working on the scalar equation to establish the threshold behavior between linear selection and nonlinear selection with respect to \(b.\) Our result reveals the fundamental mechanism underlying the transition of speed linear selection to nonlinear selection for the system (1.4).

1.1 Main results

In this subsection, we provide the main results of this paper. We divide this subsection into two parts; the first part is for the scalar monostable equation, the other for the two-species competition-diffusion system with monostable nonlinearity.

1.1.1 Speed selection for the scalar reaction-diffusion equation

We consider the following scalar reaction-diffusion equation

\[
w_t = w_{xx} + f(w; s),
\]

where \(\{f(\cdot; s)\} \subset C^2\) is a one-parameter family of nonlinear functions satisfying monostable condition, and varying continuously and monotonously on the parameter \(s \in [0, \infty).\) The assumptions on \(f\) are as follows:
(A1) (monostable condition) \( f(\cdot; s) \in C^2([0, 1]), f(0; s) = f(1; s) = 0, f'(0; s) := \beta^2 > 0 > f'(1; s), \) and \( f(w; s) > 0 \) for all \( s \in \mathbb{R}^+ \) and \( w \in (0, 1), \) where \( \beta \) is a given positive constant independent of \( s. \)

(A2) (Lipschitz continuity) \( f(\cdot; s), f'(\cdot; s), \) and \( f''(\cdot; s) \) are Lipschitz continuous on \( s \in \mathbb{R}^+ \) uniformly in \( w. \) In other words, there exists \( L_0 > 0 \) such that
\[
|f^{(n)}(w; s_1) - f^{(n)}(w; s_2)| \leq L_0|s_1 - s_2| \quad \text{for all} \quad w \in [0, 1] \quad \text{and} \quad n = 0, 1, 2, \text{red}
\]
where \( f^{(n)} \) mean the \( n \)th derivative of \( f \) with respect to \( w \) for \( n \in \mathbb{N}, \) i.e. \( f^{(0)} = f, f^{(1)} = f', \) and \( f^{(2)} = f'' \).

(A3) (monotonicity condition) \( f(w; \hat{s}) > f(w; s) \) for all \( w \in (0, 1) \) if \( \hat{s} > s, \) and \( f''(0; \hat{s}) > f''(0; s) \) if \( \hat{s} > s. \)

**Remark 1.1** Note that, in the present paper, we always assume \( \{f(\cdot; s)\} \subset C^2 \) as that in (A1) for the simplicity of the proof. As a matter of fact, our approach still works for weaker regularity of \( f, \) say \( \{f(\cdot; s)\} \subset C^{1,\alpha} \) for some \( \alpha \in (0, 1). \) If we consider a higher degree of regularity for \( f, \) such as \( f(\cdot; s) \subset C^k \) for some \( k > 2, \) then the condition in (A3) for \( f''(0; \cdot) \) will be replaced by \( f^{(i)}(0; \cdot) \) for some \( 1 < i \leq k. \)

Thanks to (A1), there exists the minimal traveling wave speed for all \( s \in [0, \infty), \) denoted by \( c^*(s), \) such that the system
\[
\begin{aligned}
W'' + cW' + f(W; s) &= 0, \quad \xi \in \mathbb{R}, \\
W(-\infty) &= 1, \quad W(+\infty) = 0, \\
W' > 0, \quad \xi \in \mathbb{R},
\end{aligned}
\]  
(1.7)
admits a unique (up to translations) solution \((c, W)\) if and only if \( c \geq c^*(s). \)

We further assume that, linear (resp., nonlinear) selection mechanism can occur at some \( s. \) More precisely, \( f(\cdot; s) \) satisfies

(A4) there exists \( s_1 > 0 \) such that \( f(w; s_1) \) satisfies KPP condition (1.3), and thus \( c^*(s_1) = 2\beta. \)

(A5) there exists \( s_2 > s_1 \) such that \( c^*(s_2) > 2\beta. \)

**Remark 1.2** In view of (A3), a simple comparison yields that \( c^*(\hat{s}) \geq c^*(s) \) if \( \hat{s} \geq s. \) Together with (A4), (A5) and the fact \( c^*(s) \geq 2\beta \) for all \( s \geq 0, \) we see that \( c^*(s) = 2\beta \) for all \( 0 \leq s \leq s_1 \) and \( c^*(s) > 2\beta \) for all \( s \geq s_2. \)

A typical example for understanding the link between pulled fronts and pushed fronts is the following equation [15]:
\[
\partial_t w - w_{xx} = w(1 - w)(1 + sw),
\]
(1.8)
where \( s \geq 0 \) is the continuously varying parameter. It is easy to check that (1.8) satisfies assumptions (A1)-(A5). Moreover, the KPP condition (1.3) is satisfied if and only if \( 0 \leq s \leq 1. \) If \( s > 1, \) (1.3) is not satisfied, and such \( f(w; s) \) is called weak Allee effect. The minimal traveling wave speed \( c^*(s) \) is characterized in [15] as:
\[
c^*(s) = \begin{cases} 
2 & \text{if } 0 \leq s \leq 2, \\
\sqrt{\frac{2}{s}} + \sqrt{\frac{s}{2}} & \text{if } s > 2.
\end{cases}
\]
Then it is easy to see that the minimal speed $c^*(s)$ is linearly selected for $0 < s \leq 2$, while it is nonlinearly selected for $s > 2$. Note particularly that, for $s \in (1, 2]$, the minimal speed $c^*(s)$ is still linearly selected even though the KPP condition (1.3) is not satisfied. For the critical case $s = 2$, we call the minimal traveling wave as the pulled to pushed transition front from linear selection to nonlinear selection. These can be also realized from [27, Theorem 5.2].

Our first main result describes how a pulled front evolves to a pulled to pushed transition front in terms of the varying parameter $s$, which gives a new necessary and sufficient condition to determine linear and nonlinear selection. The key point is to completely characterize the evolution of the decay rate of the minimal traveling wave $W_s(\xi)$ with respect to $s$. It is well known ([3]) that if $c^*(s) = 2\beta$, then

$$W_s(\xi) = A \xi e^{-\beta \xi} + B e^{-\beta \xi} + o(e^{-\beta \xi}) \quad \text{as} \quad \xi \to \infty,$$

(1.9)

where $A \geq 0$ and $B \in \mathbb{R}$, and $B > 0$ if $A = 0$. On the other hand, if $c^*(s) > 2\beta$, then

$$W_s(\xi) = A e^{-\lambda_s \xi} + o(e^{-\lambda_s \xi}) \quad \text{as} \quad \xi \to \infty,$$

(1.10)

where $A > 0$ and

$$\lambda_s := \frac{c^*(s) + \sqrt{(c^*(s))^2 - 4\beta^2}}{2} > \beta.$$

(1.11)

As we will see, the key point to understand the speed selection problem is to determine the leading order of the decay rate of $W_s(\xi)$, i.e. whether $A > 0$ or $A = 0$ in (1.9).

**Theorem 1.3** Assume that (A1)-(A5) hold. Then there exists the threshold value $s^* \in [s_1, s_2]$ such that the minimal traveling wave speed of (1.7) satisfies

$$c^*(s) = 2\beta \quad \text{for all } s \in [0, s^*]; \quad c^*(s) > 2\beta \quad \text{for all } s \in (s^*, \infty),$$

(1.12)

Moreover, the minimal traveling wave $W_s(\xi)$ satisfies

$$W_s(\xi) = B e^{-\beta \xi} + o(e^{-\beta \xi}) \quad \text{as} \quad \xi \to +\infty \quad \text{for some} \quad B > 0,$$

(1.13)

if and only if $s = s^*$. 
Remark 1.4  

(1) Note that (1.13) in Theorem 1.3 indicates that, as $\xi \to +\infty$, the leading order of the decay rate of $W_s(\xi)$ switches from $\xi e^{-\beta \xi} \to e^{-\beta \xi}$ as $s \to s^*$ from below.

(2) In our proof of (1.12) and the sufficient condition for (1.13), the condition in (A3) that $f''(0; \hat{s}) > f''(0; s)$ for $\hat{s} > s$ is not required.

Combining (1.9), (1.10) and Theorem 1.3, we can fully understand how the decay rates of the minimal traveling wave depend on $s$, which is formulated as follows:

Corollary 1.5  
Assume that (A1)-(A5) hold. Then asymptotic behaviors of the pulled front, pushed front, and pulled to pushed transition front are shown as follows.

1. Pulled front: if $s \in [0, s^*)$, then $W_s(\xi) = A\xi e^{-\beta \xi} + Be^{-\beta \xi} + o(e^{-\beta \xi})$ as $\xi \to +\infty$ with $A > 0$ and $B \in \mathbb{R}$;

2. Pulled to pushed transition front: if $s = s^*$, then $W_s(\xi) = Be^{-\beta \xi} + o(e^{-\beta \xi})$ as $\xi \to +\infty$ with $B > 0$;

3. Pushed front: if $s \in (s^*, \infty)$, then $W_s(\xi) = Ae^{-\lambda_s \xi} + o(e^{-\lambda_s \xi})$ as $\xi \to +\infty$ with $A > 0$.

Here $\beta > 0$ is defined in (A1), and $\lambda_s$ is defined as (1.11).

1.1.2 Speed selection for the Lotka-Volterra competition-diffusion system

Next, we state our main result regarding the speed selection mechanism for the competition-diffusion system (1.4) under (H).

Depending on the different dynamics of the related ODE systems, the assumption (H) can be classified as three cases: (I) $0 < a < 1 < b$ (the strong-weak competition case), (II) $0 < a < 1$ and $0 < b < 1$ (the weak competition case), (III) $0 < a < 1$ and $b = 1$ (the critical case). Regarding the traveling waves of (1.4) for the case (I), Kan-on [22] showed that there exists the minimal traveling wave speed $c_{LV}^* \in [2\sqrt{1-a}, 2]$ such that (1.4) admits a positive solution $(u, v)(x, t) = (U, V)(x-ct)$ satisfying

$$
\begin{align*}
U'' + cU' + U(1 - U - aV) &= 0, \\
\frac{dV''}{(1-V-bU)} &= 0, \\
(U, V)(-\infty) &= (1, 0), \quad (U, V)(\infty) = (0, 1), \\
U' < 0, \quad V' > 0,
\end{align*}
$$

if and only if $c \geq c_{LV}^*$. For the case (II), it has been shown in [25, Example 4.2] that there exists the minimal traveling wave speed $c_{LV}^* > 0$ such that (1.4) admits a positive solution $(u, v)(x, t) = (U, V)(x-ct)$ satisfying

$$
\begin{align*}
U'' + cU' + U(1 - U - aV) &= 0, \\
\frac{dV''}{(1-V-bU)} &= 0, \\
(U, V)(-\infty) &= \left(\frac{1}{1-ab}, \frac{1-b}{1-ab}\right), \quad (U, V)(\infty) = (0, 1), \\
U' < 0, \quad V' > 0,
\end{align*}
$$

if and only if $c \geq c_{LV}^*$. Additionally, the existence of the minimal wave speed for Case (III) can be established by a certain approximation argument. Further details are given in the Appendix.
As seen in the literature, the minimal traveling wave speed depends on system parameters $d$, $r$, $a$, and $b$, but whether linear selection holds is not completely understood until now. According to our methodology used for studying the scalar equation, we shall choose the competition rate $b \in \mathbb{R}^+$ as a continuously varying parameter. Then we can establish a threshold behavior between linear selection and nonlinear selection in terms of $b$. To do so, we always assume (H) and fix $d$, $r > 0$ and $a \in (0, 1)$. We will show that there exists $b^* \in (0, +\infty)$ such that $c^*_{LV}(b)$ is linearly selected for $0 < b \leq b^*$ and is nonlinearly selected for $b > b^*$.

A key role for characterizing the transition from linear selection to nonlinear selection is the asymptotic behavior of the pulled to pushed transition front $U_{b^*}$ at $+\infty$. In this case, $c^*_{LV}(b^*) = 2\sqrt{1-a}$. It well know (see [12] or [29]) that,

$$U_{b^*}(\xi) = A\xi e^{-\lambda_u \xi} + Be^{-\lambda_u \xi} + o(e^{-\lambda_u \xi}) \quad \text{as} \quad \xi \to +\infty,$$

where $\lambda_u := \sqrt{1-a} > 0$, $A \geq 0$, $B \in \mathbb{R}$, and if $A = 0$, then $B > 0$. We gain a full understanding of how the decay rate of $U$-fronts at infinity impact the mechanism of speed by showing that $A = 0$ occurs if and only if $b = b^*$. Namely, the leading order term of the decay rate of $U_{b^*}(\xi)$ at $\xi = +\infty$ is $e^{-\lambda_u \xi}$.

We state our main results as follows.

**Theorem 1.6** For any $d > 0$, $r > 0$ and $a \in (0, 1)$, there exists $b^* \in (0, +\infty)$ such that

$$c^*(b) = 2\sqrt{1-a} \quad \text{for} \quad b \in (0, b^*]; \quad c^*(b) > 2\sqrt{1-a} \quad \text{for} \quad b \in (b^*, +\infty),$$

Furthermore, for the minimal traveling wave $(c^*(b), U_b, V_b)$ defined as in (1.14), the following three conditions are equivalent:

(i) $b = b^*$;

(ii) $U_b(\xi) = Be^{-\lambda_u \xi} + o(e^{-\lambda_u \xi})$ as $\xi \to +\infty$ for some $B > 0$;

(iii) $\int_{-\infty}^{\infty} e^{\lambda_u \xi} U_b(\xi) [a(1-V_b) - U_b(\xi)] d\xi = 0$,

where $\lambda_u := \sqrt{1-a}$.

Theorem 1.6 indicates that $(U_{b^*}, V_{b^*})$ is the pulled to pushed transition front.

Note that the sub-solution for $U$-component constructed in [20] has the asymptotic behavior $\xi e^{-\lambda_u \xi}$ as $\xi \to \infty$, which cannot capture the transition front $U_{b^*}$ with the asymptotic behavior $e^{-\lambda_u \xi}$ as $\xi \to \infty$ reported in Theorem 1.6. This observation gives a natural reason for why the condition (1.6) for linear selection can still be improved (see, e.g., [1, 33]). We formulate this as a corollary as follows.

**Corollary 1.7** The condition (1.6) for linear selection is not optimal.

**Remark 1.8** We would like to give some remarks for Theorem 1.6 and emphasize the main features of the present paper.

1. It is the first time to provide a sufficient and necessary condition for the speed selection problem of the Lotka-Volterra competition system under (H). We have improved the understanding for this problem by considering a wide range of competition coefficients $0 < a < 1$ and $0 < b < +\infty$, not just the previously studied case of $0 < a < 1 < b$. In addition, we expect that in some
cases, $c^*(b) > 2\sqrt{1-a}$ for all $b > 1$, indicating that the threshold $b^*$ may not be well-defined by only considering $b > 1$.

For instance, numerical simulations suggest that for any fixed $0 < a < 1$ and $r > 0$, there exists $d_0 > 0$ sufficiently large such that $c^*(b) > 2\sqrt{1-a}$ for all $b > 1$ if $d > d_0$. In Figure 1, we consider (1.4) with $a = b = 1/2$, $r = 1$, $v_0(x) \equiv 2/3$ and $u_0(x)$ satisfying

$$u_0(x) = 1 \text{ for } x \leq 10, \quad u_0(x) = 0 \text{ for } x > 10.$$  

Set

$$x(t) := \sup_{x \geq 0} \{x > 0 | u(t, x) = 1/2\}.$$

A numerical simulation suggests that $\lim_{t \to \infty} [x(t)/t] > 2\sqrt{1-a} = \sqrt{2}$ when $d = 50$. Together with the comparison principle, it indicates that the wave speed should be nonlinearly selected for all $b > 1/2$ when $a = 1/2$, $r = 1$, and $d = 50$.

On the other hand, numerical simulations suggest that for any fixed $0 < a < 1$ and $d > 0$, there exists $r_0 > 0$ sufficiently small such that $c^*(b) > 2\sqrt{1-a}$ for all $b > 1$ if $r < r_0$. In Figure 2, we consider (1.4) with $a = b = 1/2$, $d = 1$, and the initial data $(u_0, v_0)$ is taken as the same as the one in Figure 1. Together with the comparison principle, it suggests that the wave speed should be nonlinearly selected for all $b > 1/2$ when $a = 1/2$, $d = 1$ and $r = 0.00001$.

(2) Why should we not expect an explicit formula for the speed selection problem of the system (1.4)? The condition (iv) in Theorem 1.6 shows that the linear selection is influenced by the whole traveling wave $(U, V)$, not only the leading edge of the wave. Thus, it should be impossible to have an explicit expression.

(3) We show the process of how pulled front changes to pushed front through the pulled to pushed transition front for scalar monostable equations and the two-species Lotka-Volterra competition system. We expect that such a phenomenon can be found in more complicated models.

(4) Our approach can be applied to more general monotone systems, which leaves as our future work. As stated in [38], many natural elements, like advection and periodicity, need to be
considered in the propagation problem. Since the limitation of variational approach, [27] can only treat homogeneous scalar reaction-diffusion equations. In contrast, our method could extend to the equations and systems with advection or periodicity as long as the comparison principle holds.

The rest of this paper is organized as follows. In Section 2, we study the speed selection mechanism for the scalar monostable equations and prove Theorem 1.3. Section 3 is devoted to the speed selection mechanism for a two-species competition-diffusion system, where Theorem 1.6 is established. In the end of this paper, the Appendix section presents the results for the existence of the minimal speed of (1.4) and the asymptotic behavior of traveling wave at ±∞ under condition (H). It also includes proofs for results that have not been previously established in the literature.

## 2 Threshold of scalar monostable equation

In this section, we aim to prove Theorem 1.3. First, it is well known that for each \( s \geq 0 \), under (A1), the minimal traveling wave is unique (up to a translation). Together with (A2), one can use standard compactness argument to conclude that \( c^*(s) \) is continuous for all \( s \geq 0 \). It follows from (A3)-(A5) and Remark 1.2 that \( c^*(s) \) is nondecreasing in \( s \). Thus, we immediately obtain the following result.

**Lemma 2.1** Assume that (A1)-(A5) hold. Then there exists a threshold \( s^* \in [s_1, s_2) \) such that (1.12) holds.

Thanks to Lemma 2.1, to prove Theorem 1.3, it suffices to show that (1.13) holds if and only if \( s = s^* \). Let \( W_{s^*} \) be the minimal traveling wave of (1.2) with \( s = s^* \) and speed \( c^*(s^*) = 2\beta \). For simplicity, we denote \( W_s := W_{s^*} \). The first and the most involved step is to show that if \( s = s^* \), then (1.13) holds. To do this, we shall use a contradiction argument. Assume that (1.13) is not true. Then, it holds that (cf. [3])

\[
\lim_{\xi \to +\infty} \frac{W_s(\xi)}{\xi e^{-\beta \xi}} = A_0 \quad \text{for some } A_0 > 0.
\]

Under the condition (2.1), we shall prove the following proposition.

**Proposition 2.2** Assume that (A1)-(A5) hold. In addition, if (2.1) holds, then there exists an auxiliary continuous function \( R_w(\xi) \) defined in \( \mathbb{R} \) satisfying

\[
R_w(\xi) = O(\xi e^{-\beta \xi}) \quad \text{as } \xi \to \infty,
\]

Figure 2: the blue curve represents the evolution of \( x(t)/t \) on different \( r \).
such that \( \bar{W}(\xi) := \min\{W_*(\xi) - R_w(\xi), 1\} \geq (\neq) 0 \) is a super-solution satisfying
\[
N_0[\bar{W}] := \bar{W}'' + 2\beta \bar{W}' + f(\bar{W}; s^*) + \delta_0 \leq 0, \quad \text{a.e. in } \mathbb{R}, \tag{2.3}
\]
for all sufficiently small \( \delta_0 > 0 \), where \( \bar{W}'(\xi^+) \) exists and \( \bar{W}'(\xi^-) \leq \bar{W}'(\xi^+) \) if \( \bar{W}' \) is not continuous at \( \xi_0 \).

Next, we shall go through a lengthy process to prove Proposition 2.2. Hereafter, (A1)-(A5) are always assumed.

From (A1), by shifting the coordinates, we can immediately obtain the following lemma.

**Lemma 2.3** Let \( \beta \) be given in (A1) and \( \nu_1 > 0 \) be an arbitrary constant. Then there exist
\[
\xi_1, \xi_2 \in \mathbb{R} \quad \text{with} \quad -\infty < \xi_2 < \xi_1 < +\infty
\]
such that the following hold:

1. \( f(W_*(\xi); s^*) = \beta^2 W_*(\xi) + \frac{f''(0; s^*)}{2} W_*(\xi) + o(W_*(\xi)) \) for all \( \xi \in [\xi_1, \infty) \):
2. \( f'(W_*(\xi); s^*) < 0 \) for all \( \xi \in (-\infty, \xi_2] \).

### 2.1 Construction of the super-solution

Let us define \( R_w(\xi) \) as (see Figure 2.1)
\[
R_w(\xi) = \begin{cases} 
\varepsilon_1 \sigma(\xi)e^{-\beta \xi}, & \text{for } \xi \geq \xi_1 + \delta_1, \\
\varepsilon_2 e^{\lambda_1 \xi}, & \text{for } \xi_2 + \delta_2 \leq \xi \leq \xi_1 + \delta_1, \\
\varepsilon_3 \sin(\delta_4 (\xi - \xi_2)), & \text{for } \xi_2 - \delta_3 \leq \xi \leq \xi_2 + \delta_2, \\
-\varepsilon_4 e^{\lambda_3 \xi}, & \text{for } \xi \leq \xi_2 - \delta_3,
\end{cases} \tag{2.4}
\]
where \( \delta_i, \lambda_i > 0 \), and \( \sigma(\xi) > 0 \) will be determined such that \( \bar{W}(\xi) \) satisfies (2.3). Moreover, we should choose \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \lambda_1, \lambda_3 > 0 \) and \( \lambda_2 > 0 \) such that
\[
|f''(W_*(\xi); s^*)| < K_1, \quad |f'(W_*(\xi); s^*)| < K_2 \quad \text{for all } \xi \in \mathbb{R}. \tag{2.5}
\]
We set \( \lambda_1 > K_2 \) large enough such that
\[
-2\beta \lambda_1 - \lambda_1^2 + K_2 < 0. \tag{2.6}
\]
Furthermore, there exists \( K_3 > 0 \) such that
\[
f'(W_*(\xi); s^*) \leq -K_3 < 0 \quad \text{for all } \xi \leq \xi_2. \tag{2.7}
\]
We set 0 \( < \lambda_2 < \lambda_w := \sqrt{\beta^2 - f'(1; s^*)} - \beta \) sufficiently small such that
\[
\lambda_2^2 + 2\beta \lambda_2 - K_3 < 0. \tag{2.8}
\]
We now divide the proof into several steps.
By assumption (A1) and the statement (1) of Lemma 2.3, since $W_{2}$, in this case, we have

Then, from (2.12), up to enlarging $\xi$ for some small $\varepsilon_{1} \ll A_{0}$.

Note that $W_{s}$ satisfies (1.2) with $c = 2\beta$. By some straightforward computations, we have

By assumption (A2) and the statement (1) of Lemma 2.3, there exists $C_{1} > 0$ such that

By assumption (A1) and the statement (1) of Lemma 2.3, since $W_{s} \ll 1$ and $R_{w} \ll W_{s}$ for $\xi \in [\xi_{1} + \delta_{1}, \infty)$, we have

By assumption (A2) and the statement (1) of Lemma 2.3, there exists $C_{1} > 0$ such that

From (2.5), (2.9), (2.10), (2.11), and Lemma 2.3, we have

Now, we define

which satisfies $\sigma(\xi_{1}) = 0$, $\sigma'(\xi) = \frac{1}{\beta} - \frac{2}{\beta} e^{-\frac{2}{\beta}(\xi - \xi_{1})}$, $\sigma''(\xi) = e^{-\frac{2}{\beta}(\xi - \xi_{1})}$. Moreover, $\sigma(\xi) = O(\xi)$ as $\xi \to \infty$ implies that $R_{w}$ satisfies (2.2).

Due to (2.1) and the equation of $W_{s}$, we may also assume

Then, from (2.12), up to enlarging $\xi_{1}$ if necessary, we always have

Figure 2.1: the construction of $R_{w}(\xi)$.

**Step 1:** We consider $\xi \in [\xi_{1} + \delta_{1}, \infty)$ where $\delta_{1} > 0$ is small enough and will be determined in Step 2. In this case, we have

for some small $\varepsilon_{1} \ll A_{0}$.

$W_{s}$ satisfies (2.2). Moreover, $\sigma(\xi) = O(\xi)$ as $\xi \to \infty$ implies that $R_{w}$ satisfies (2.2).

Due to (2.1) and the equation of $W_{s}$, we may also assume

Then, from (2.12), up to enlarging $\xi_{1}$ if necessary, we always have

$$N_{0}[\tilde{W}] \leq -\varepsilon_{1} e^{-\beta(\xi - \xi_{1})} e^{-\beta\xi} + K_{1} \left(\frac{R_{w}^{2}}{2} + W_{s} R_{w}\right) + C_{1} \delta_{0} W_{s}^{2} + o((W_{s})^{2}) \leq 0$$
for all sufficiently small \( \delta_0 > 0 \) since \( R_w^2(\xi), W_s R_w(\xi), \) and \( W_s^2(\xi) \) are \( o(e^{-3\beta \delta}) \) for \( \xi \geq \xi_1 \) from (2.13) and the definition of \( R_w \). Therefore, \( N_0[\tilde{W}] \leq 0 \) for \( \xi \geq \xi_1 \).

Step 2: We consider \( \xi \in [\xi_2 + \delta_2, \xi_1 + \delta_1] \) for some small \( \delta_2 > 0 \), and \( \delta_1 > 0 \) satisfying

\[
\frac{1}{\beta} + \frac{3}{\beta(1 - e^{-\frac{\alpha_1}{2}})} - 2\delta_1 > 0.
\] (2.14)

From the definition of \( R_w \) in Step 1, it is easy to check that \( R_w'((\xi_1 + \delta_1)^+) > 0 \) under the condition (2.14). In this case, we have \( R_w(\xi) = \varepsilon e^{\lambda_1 \xi} \) for some large \( \lambda_1 > 0 \) satisfying (2.6).

We first choose

\[
\varepsilon_1 = \frac{4}{\beta^2} e^{-\frac{\alpha_1}{2}} - \frac{4}{\beta^2} + \frac{4}{\beta} \delta_1 \left(e^{-(\beta + \lambda_1)\xi} - \lambda_1 R_w(\xi + \delta_1) \right)
\] (2.15)

such that \( R_w(\xi) \) is continuous at \( \xi = \xi_1 + \delta_1 \). Then, from (2.15), we have

\[
R_w'((\xi_1 + \delta_1)^+) = \varepsilon_1 \sigma'((\xi_1 + \delta_1)^+) e^{-\beta(\xi_1 + \delta_1)} - \beta R_w(\xi_1 + \delta_1) > R_w'((\xi_1 + \delta_1)^- - \lambda_1 R_w(\xi_1 + \delta_1)
\]

is equivalent to

\[
\beta + (3\beta + 2\lambda_1)(1 - e^{-\frac{\alpha_1}{2}}) > 2(\beta + \lambda_1)\beta \delta_1
\]

which holds by taking \( \delta_1 \) sufficiently small. This implies that \( \angle \alpha_1 < 180^\circ \).

By some straightforward computations, we have

\[
N_0[\tilde{W}] = -(2\beta + \lambda_1^2) R_w - f(W_s; s^*) + f(W_s - R_w; s^* + \delta_0)
\]
\[
= -(2\beta + \lambda_1^2) R_w - f(W_s; s^*) + f(W_s - R_w; s^*)
\]
\[
- f(W_s - R_w; s^*) + f(W_s - R_w; s^* + \delta_0).
\]

Thanks to (2.5), we have

\[
-f(W_s; s^*) + f(W_s - R_w; s^*) < K_2 R_w.
\]

Moreover, by assumption (A2),

\[
-f(W_s - R_w; s^*) + f(W_s - R_w; s^* + \delta_0) \leq L_0 \delta_0.
\]

Then, since \( \lambda_1 \) satisfies (2.6), we have

\[
L_0 \delta_0 < \varepsilon_2 e^{\lambda_1 \xi} + 2\beta \lambda_1 - K_2 e^{\lambda_1 (\xi_2 + \delta_2)}
\]

for all sufficiently small \( \delta_0 > 0 \), which implies that \( N_0[\tilde{W}] \leq 0 \) for all \( \xi \in [\xi_2 + \delta_2, \xi_1 + \delta_1] \).

Step 3: We consider \( \xi \in [\xi_2 - \delta_3, \xi_2 + \delta_2] \) for some small \( \delta_2, \delta_3 > 0 \). We first verify the following Claim.

Claim 2.4 For any \( \delta_2 \) with \( \delta_2 > \frac{1}{\lambda_1} \), there exist \( \varepsilon_3 > 0 \) and small \( \delta_4 > 0 \) such that \( R_w((\xi_2 + \delta_2)^+) = R_w((\xi_2 + \delta_2)^-) \) and \( \angle \alpha_2 < 180^\circ \).

Proof. Note that \( R_w((\xi_2 + \delta_2)^+) = \varepsilon e^{\lambda_1 (\xi_2 + \delta_2)} \), \( R_w((\xi_2 + \delta_2)^-) = \varepsilon_3 \sin(\delta_4 \delta_2) \). Therefore, we may take

\[
\varepsilon_3 = \frac{\varepsilon e^{\lambda_1 (\xi_2 + \delta_2)}}{\sin(\delta_4 \delta_2)} > 0
\] (2.16)
such that \( R_w((\xi + \delta_2)^+) = R_w((\xi + \delta_2)^-) \).

By some straightforward computations, we have \( R'_w((\xi + \delta_2)^+) = \lambda_1 \varepsilon_2 e^{\lambda_1 (\xi + \delta_2)} \) and
\[
R'_w((\xi + \delta_2)^-) = \varepsilon_3 \delta_4 \cos(\delta_4 \xi) = \frac{\varepsilon_2 e^{\lambda_1 (\xi + \delta_2)}}{\sin(\delta_4 \xi)} \delta_4 \cos(\delta_4 \xi),
\]
which yields that \( R'_w((\xi + \delta_2)^-) \to \varepsilon_2 e^{\lambda_1 (\xi + \delta_2)} / \delta_2 \) as \( \delta_4 \to 0 \). In other words, as \( \delta_4 \to 0 \), \( R'_w((\xi + \delta_2)^+) > R'_w((\xi + \delta_2)^-) \) is equivalent to \( \delta_2 > \frac{1}{\lambda_1} \). Therefore, we can choose \( \delta_4 > 0 \) sufficiently small so that \( \angle \alpha_3 < 180^\circ \). This completes the proof of Claim 2.4.

Next, we verify the differential inequality of \( N_0[\tilde{W}] \) for \( \xi \in [\xi_2 - \delta_3, \xi_2 + \delta_2] \). By some straightforward computations, we have
\[
N_0[\tilde{W}] = \delta_2^2 R_w - 2\beta \varepsilon_3 \delta_4 \cos(\delta_4 (\xi - \xi_2)) + f(W_\ast; s^\ast) - f(W_\ast - R_w; s^\ast) - f(W_\ast - R_w; s^\ast + \delta_0).
\]
The same argument as in Step 2 implies that
\[
-f(W_\ast; s^\ast) + f(W_\ast - R_w; s^\ast) \leq K_2 R_w \quad \text{and} \quad -f(W_\ast - R_w; s^\ast) + f(W_\ast - R_w; s^\ast + \delta_0) \leq L_0 \delta_0,
\]
which yields that
\[
N_0[\tilde{W}] \leq \delta_2^2 R_w - 2\beta \varepsilon_3 \delta_4 \cos(\delta_4 (\xi - \xi_2)) + K_2 R_w + L_0 \delta_0.
\]
We first focus on \( \xi \in [\xi_2 - \delta_3, \xi_2 + \delta_2] \). By (2.16) and the definition of \( \lambda_1 \) (see (2.6)), we can choose \( \delta_2 \in (1/\lambda_1, 1/K_2) \) such that
\[
\min_{\xi \in [\xi_2 - \delta_3, \xi_2 + \delta_2]} \delta_2^2 \left( \delta_4 \cos(\delta_4 (\xi - \xi_2)) \right) \frac{\varepsilon_2 e^{\lambda_1 (\xi + \delta_2)}}{\delta_2} > R_w((\xi + \delta_2)^+) \quad \text{as} \quad \delta_4 \to 0.
\]
Thus, we have
\[
\min_{\xi \in [\xi_2 - \delta_3, \xi_2 + \delta_2]} \delta_4 \varepsilon_3 \cos(\delta_4 (\xi - \xi_2)) > (K_2 + \delta_2^2) R_w((\xi + \delta_2)^+) \quad \text{for all sufficiently small} \quad \delta_4 > 0.
\]
Then, for all sufficiently small \( \delta_0 > 0 \), we see that \( N_0[\tilde{W}] \leq 0 \) for \( \xi \in [\xi_2, \xi_2 + \delta_2] \).

For \( \xi \in [\xi_2 - \delta_3, \xi_2] \), we fix \( \delta_3 > 0 \) small enough such that \( N_0[\tilde{W}] \leq 0 \) can be verified easily by the same argument since \( R_w < 0 \). This completes the Step 3.

**Step 4:** We consider \( \xi \in (-\infty, \xi_2 - \delta_3] \). In this case, we have \( R_w(\xi) = -\varepsilon_4 e^{\lambda_2 \xi} \). Recall that we choose \( 0 < \lambda_2 < \lambda_1 \) and \( 1 - W_\ast(\xi) \approx C_2 e^{-\lambda_2 \xi} \) as \( \xi \to -\infty \). Then, there exists \( M > 0 \) such that \( \tilde{W} := \min\{W_\ast - R_w, 1\} \equiv 1 \) for all \( \xi \leq -M \), and thus \( N_0[\tilde{W}] \leq 0 \) for all \( \xi \leq -M \). Therefore, we only need to show \( N_0[\tilde{W}] \leq 0 \) for all \( -M \leq \xi \leq -\xi_2 - \delta_3 \).

We first choose \( \varepsilon_4 = \varepsilon_3 \sin(\delta_4 \delta_3) e^{\lambda_2 (\xi - \delta_3)} \) such that \( R_w \) is continuous at \( \xi_2 - \delta_3 \). It is easy to check that \( R'_w((\xi_2 - \delta_3)^+) > 0 > R'_w((\xi_2 - \delta_3)^-) \) and so \( \angle \alpha_3 < 180^\circ \).

By some straightforward computations, we have
\[
N_0[\tilde{W}] = - (\lambda_2^2 + 2\beta \lambda_2) R_w - f(W_\ast; s^\ast) - f(W_\ast - R_w; s^\ast + \delta_0)
\]
\[
= - (\lambda_2^2 + 2\beta \lambda_2) R_w - f(W_\ast; s^\ast) + f(W_\ast; s^\ast + \delta_0)
\]
\[
- f(W_\ast - R_w; s^\ast) + f(W_\ast - R_w; s^\ast + \delta_0).
\]
From (2.7), we have \( -f(W_\ast; s^\ast) + f(W_\ast - R_w; s^\ast) < K_3 R_w < 0 \). Together with (A2), we have
\[
N_0[\tilde{W}] \leq - (\lambda_2^2 + 2\beta \lambda_2 - K_3) R_w + L_0 \delta_0 \quad \text{for all} \quad \xi \in [-M, \xi_2 - \delta_3].
\]
In view of (2.8), we can assert that \( N_0[\tilde{W}] \leq 0 \) for all \( \xi \in [-M, \xi_2 - \delta_3] \), provided that \( \delta_0 \) is sufficiently small. This completes the Step 4.
2.2 Proof of Theorem 1.3

We first complete the proof of Proposition 2.2.

Proof of Proposition 2.2. From the discussion from the Step 1 to Step 4 in §2.1, we are now equipped with a suitable function $R_w(\xi)$ defined as in (2.4) such that $\bar{W}(\xi) = \min\{W_s(\xi) - R_w(\xi), 1\}$, which is independent of the choice of all sufficiently small $\delta_0 > 0$, forms a super-solution satisfying (2.3). Therefore, we complete the proof of Proposition 2.2.

Now, we are ready to prove Theorem 1.3 as follows.

Proof of Theorem 1.3. In view of Lemma 2.1, we have obtained (1.12). It suffices to show that (1.13) holds if and only if $s = s^*$. First, we show that

$$s = s^* \implies (1.13) \text{ holds.} \quad (2.17)$$

Suppose that (1.13) does not hold. Then $W_s$ satisfies (2.1). In view of Proposition 2.2, we can choose $\delta_0 > 0$ sufficiently small such that $\bar{W}(\xi) = \min\{W_s(\xi) - R_w(\xi), 1\} \geq (\neq)0$ satisfies (2.3). Next, we consider the following Cauchy problem with compactly supported initial datum $w_0 \geq (\neq)0$:

$$\begin{aligned}
\begin{cases}
\partial_t w = w_{xx} + f(w; s^* + \delta_0), & t \geq 0, x \in \mathbb{R}, \\
w(0, x) = w_0(x), & x \in \mathbb{R},
\end{cases}
\end{aligned} \quad (2.18)$$

where $w_0(x) \leq \bar{W}(x)$ for $x \in \mathbb{R}$. Then, in view of (1.12), we see that $c^*(s^* + \delta_0) > 2\beta$ (the minimal speed is nonlinearly selected). Therefore, we can apply Theorem 2 of [34] to conclude that the spreading speed of the Cauchy problem (2.18) is strictly greater than $2\beta$.

On the other hand, we define $\bar{w}(t, x) := \bar{W}(x - 2\beta t)$, and hence $\bar{w}(0, x) = \bar{W}(x) \geq w_0(x)$ for $x \in \mathbb{R}$. Since $\bar{W}$ satisfies (2.3), $\bar{w}$ forms a super-solution of (2.18). This immediately implies that the spreading speed of the solution, namely $w(t, x)$, of (2.18) is slower than or equal to $2\beta$, due to the comparison principle. This immediately contradicts to the spreading speed of the Cauchy problem (2.18), which is strictly greater than $2\beta$. Thus, we obtain (2.17).

Finally, we prove that

$$(1.13) \text{ holds } \implies s = s^*. \quad (2.19)$$

Note that for $s > s^*$, from (1.12) we see that $c^*(s) > 2\beta$; so the asymptotic behavior of $W_s$ at $+\infty$ satisfies (1.10). This implies that (1.13) does not hold for any $s > s^*$; Therefore, we only need to show that if $s < s^*$, then (1.13) does not hold. We assume by contradiction that there exists $s_0 \in (0, s^*)$ such that the corresponding minimal traveling wave satisfies

$$W_{s_0}(\xi) = B_0 e^{-\beta\xi} + o(e^{-\beta\xi}) \quad \text{as } \xi \to +\infty \quad (2.20)$$

for some $B_0 > 0$. For $\xi \approx -\infty$, we have

$$1 - W_{s_0}(\xi) = C_0 e^{\lambda_0 \xi} + o(e^{-\beta\xi}) \quad \text{as } \xi \to -\infty \quad (2.21)$$

for some $C_0 > 0$, where $\lambda_0 := \sqrt{\beta^2 - f'(1; s_0)} - \beta$. Recall that the asymptotic behavior of $W_{s^*}$ at $\pm\infty$ satisfies

$$W_{s^*}(\xi) = B e^{-\beta\xi} + o(e^{-\beta\xi}) \quad \text{as } \xi \to +\infty; \quad 1 - W_{s^*}(\xi) = C e^{\lambda_0 \xi} + o(e^{-\beta\xi}) \quad \text{as } \xi \to -\infty \quad (2.22)$$
for some $B, C > 0$, where $\lambda_w := \sqrt{\beta^2 - f''(0; s^*)} - \beta$. In view of assumption (A3), we have $\lambda_w > \lambda_0$. In particular, we have

$$\lim_{\xi \to -\infty} \frac{1 - W_s(\xi) - L^*}{1 - W_s(\xi)} = 0.$$  

(2.23)

Combining (2.20), (2.21) and (2.22), there exists $L > 0$ sufficiently large such that $W_s(\xi - L^*) > W_s(\xi)$ for all $\xi \in \mathbb{R}$. Now, we define

$$L^* := \inf\{ L \in \mathbb{R} \mid W_s(\xi - L) \geq W_s(\xi) \text{ for all } \xi \in \mathbb{R} \}.$$  

By the continuity, we have $W_s(\xi - L^*) \geq W_s(\xi)$ for all $\xi \in \mathbb{R}$. If there exists $\xi^* \in \mathbb{R}$ such that $W_s(\xi^* - L^*) = W_s(\xi^*)$, by the strong maximum principle, we have $W_s(\xi - L^*) = W_s(\xi)$ for $\xi \in \mathbb{R}$, which is impossible since $W_s(\cdot - L^*)$ and $W_s(\cdot)$ satisfy different equations. Consequently, $W_s(\xi - L^*) > W_s(\xi)$ for $\xi \in \mathbb{R}$. In particular, we have

$$\lim_{\xi \to \infty} \frac{W_s(\xi - L^*)}{W_s(\xi)} \geq 1.$$  

Furthermore, we can claim that

$$\lim_{\xi \to \infty} \frac{W_s(\xi - L^*)}{W_s(\xi)} = 1. \quad (2.24)$$

Otherwise, if the limit in (2.24) is strictly bigger than 1, together with (2.23), we can easily find $\varepsilon > 0$ sufficiently small such that $W_s(\xi - (L^* + \varepsilon)) > W_s(\xi)$ for $\xi \in \mathbb{R}$, which contradicts the definition of $L^*$. As a result, from (2.20), (2.22) and (2.24), we obtain $B_0 = Be^{\beta L^*}$.

On the other hand, we set $\hat{W}(\xi) = W_s(\xi - L^*) - W_s(\xi)$. Then $\hat{W}(\xi)$ satisfies

$$\hat{W}'' + 2\beta \hat{W}' + \beta^2 \hat{W} + J(\xi) = 0, \quad \xi \in \mathbb{R},$$  

(2.25)

where $J(\xi) = f(W_s; s^*) - \beta^2 W_s - f(W_s; s) + \beta^2 W_s$. By (A1) and Taylor’s Theorem, there exist $\eta_1 \in (0, W_s)$ and $\eta_2 \in (0, W_s)$ such that

$$J(\xi) = f''(\eta_1; s^*) W_s^2 - f''(\eta_2; s) W_s^2$$

$$= f''(\eta_1; s^*)(W_s^2 - W_s^2) + [f''(\eta_1; s^*) - f''(\eta_2; s)] W_s^2$$

$$= f''(\eta_1; s^*) (W_s + W_s) \hat{W} + [f''(\eta_1; s^*) - f''(\eta_2; s)] W_s^2.$$  

Since $f''(0; s^*) > f''(0; s)$ (from (A3)), we can find small $\delta > 0$ such that $\min_{\eta \in [0, \delta]} f''(\eta; s^*) > \max_{\eta \in [0, \delta]} f''(\eta; s)$ and thus there exist $\kappa > 0$ such that

$$[f''(\eta_1; s^*) - f''(\eta_2; s)] W_s^2(\xi) \geq \kappa e^{-2\beta \xi} \quad \text{for all large } \xi.$$  

(2.26)

Using $B_0 = Be^{\beta L^*}$ and (2.26), we obtain

$$\hat{W}'' + 2\beta \hat{W}' + \beta^2 \hat{W} + J(\xi) > 0 \quad \text{for all large } \xi,$$

which contradicts to (2.25). Therefore, (2.19) holds, and the proof is complete. \qed
\section{Threshold of the Lotka-Volterra competition-diffusion system}

This section is devoted to the proof of Theorem 1.6. Let us fix the parameters \( a \in (0, 1), d > 0, \) and \( r > 0. \) It is well known (cf. [22, Lemma 5.6]) that the minimal traveling wave speed \( c_{LV}^*(b) \) is a continuous function on \((0, +\infty)\). Moreover, by Theorem 1.4 of [41] and a simple comparison argument, we see that \( c_{LV}^*(b) \) is nondecreasing on \( b. \) We first introduce a crucial proposition which corresponds to assumptions (A4) and (A5) for the scalar equation.

\textbf{Proposition 3.1} \textit{For any fixed} \( a \in (0, 1), d > 0, \) \textit{and} \( r > 0, \) \textit{there exists} \( \varepsilon > 0 \) \textit{very small such that}

\[ c_{LV}^*(\varepsilon) = 2\sqrt{1-a}. \]

\textit{On the other hand, there exists} \( b_0 > 0 \) \textit{sufficiently large such that} \( c_{LV}^*(b) > 2\sqrt{1-a} \) \textit{for all} \( b > b_0. \)

Together with Proposition 3.1, we immediately obtain Lemma 3.2. The proof of Proposition 3.1 is given in Subsection 3.4.

\textbf{Lemma 3.2} \textit{For any} \( d > 0, r > 0 \) \textit{and} \( a \in (0, 1), \) \textit{there exists} \( 0 < b^* < \infty \) \textit{such that}

\[ c_{LV}^*(b) = 2\sqrt{1-a} \text{ for } b \in (1, b^*] \text{ and } c_{LV}^*(b) > 2\sqrt{1-a} \text{ for } b \in (b^*, +\infty). \]

Let \( (c_{LV}^*, U_s, U_v) \) be the minimal traveling wave of system (1.14) with \( b = b^* > 0 \) and \( e_{LV}^* = c_{LV}^*(b^*) = 2\sqrt{1-a}. \) Hereafter, for the simplicity we denote

\[ \lambda_u := -\lambda_u^-(c_{LV}^*(b^*)) > 0, \quad \lambda_v := -\lambda_v^-(c_{LV}^*(b^*)) > 0, \]

where \( \lambda_u^- \) and \( \lambda_v^- \) are defined in the subsection A.2.

We shall establish the following result parallel to Proposition 2.2 for the scalar equation.

\textbf{Proposition 3.3} \textit{Assume that (H) holds. In addition, if}

\[ \lim_{\xi \to +\infty} \frac{U_s(\xi)}{\xi e^{-\lambda_u \xi}} = A_0 \text{ for some } A_0 > 0, \quad (3.1) \]

\textit{then there exist two continuous functions} \( R_u(\xi), R_v(\xi) \) \textit{defined in} \( \mathbb{R} \) \textit{with}

\[ R_u(\xi) = O(\xi e^{-\lambda_u \xi}), \quad R_v(\xi) = O(e^{-\lambda_v \xi}) \text{ as } \xi \to \infty, \quad (3.2) \]

\textit{such that} \((W_u, W_v)(\xi) := \left( \min\{(U_s - R_u)(\xi), 1\}, \max\{(U_v + R_v)(\xi), 0\} \right)\) \textit{is a super-solution satisfying}

\[ \begin{aligned}
N_3[W_u, W_v] := W_u'' + 2\sqrt{1-a}W_u' + W_u(1 - W_u - aW_v) &\leq 0, \text{ a.e. in } \mathbb{R}, \\
N_4[W_u, W_v] := dW_v'' + 2\sqrt{1-a}W_v' + rW_v(1 - W_v - (b^* + \delta_0)W_u) &\geq 0, \text{ a.e. in } \mathbb{R}, 
\end{aligned} \quad (3.3) \]

\textit{for all small} \( \delta_0 > 0, \) \textit{where} \( W_u'((\xi_0^\pm), \text{ resp. } W_v'((\xi_0^\pm)) \) \textit{exists and}

\[ W_u'(\xi_0^+) \leq W_u'(\xi_0^-) \quad (\text{resp. } W_v'(\xi_0^+) \geq W_v'(\xi_0^-)) \]

\textit{if} \( W_u' \) \textit{(resp., } \( W_v' \) \textit{is not continuous at} \( \xi_0. \)

In the following discussion, we divide the construction of \( R_u \) and \( R_v \) into two subsections: \( b^* \geq 1 \) (the strong-weak competition case and the critical case); \( 0 < b^* < 1 \) (the weak competition case).
3.1 Construction of the super-solution for $b^* \geq 1$

In this subsection, we always assume $b^* \geq 1$.

First, since $(U_*, V_*)(-\infty) = (1, 0)$ and $(U_*, V_*)(+\infty) = (0, 1)$, for any given small $\rho > 0$, we can take $M_0 > 0$ sufficiently large such that

$$
\begin{aligned}
0 < U_*(\xi) &< \rho, \quad 1 - \rho < V_*(\xi) < 1 & \text{for all } \xi \geq M_0, \\
0 < V_*(\xi) &< \rho, \quad 1 - \rho < U_*(\xi) < 1 & \text{for all } \xi \leq -M_0.
\end{aligned}
$$

(3.4)

For $\xi$ being close to $\infty$, we have the following for later use. First, due to (3.1) and Lemma A.3(ii), up to enlarging $M_0$ if necessary, we may assume that for some positive constant $A_0$,

$$
U_*(\xi) \leq 2A_0\xi e^{-\lambda_0\xi} \quad \text{for all } \xi \geq M_0.
$$

(3.5)

Moreover, we may also assume there exists $C_0 > 0$ such that

$$
V_*(\xi) \geq 1 - C_0\xi^2 e^{-\min(\lambda_0, \lambda_1)\xi} \quad \text{for all } \xi \geq M_0.
$$

(3.6)

We also define

$$
C_1 := \max_{\xi \in [M_0, \infty)} \left| \left(\frac{d-2}{r} - 1 + 2V_*(\xi) - (b^* + \delta_0)U_*(\xi)\right) \right| > 0.
$$

(3.7)

We now define $(R_u, R_v)(\xi)$ as (see Figure 3.1)

$$
(R_u, R_v)(\xi) := \begin{cases} 
(\epsilon_1 \sigma(\xi)e^{-\lambda_0\xi}, \eta_1(\xi - \xi_1)e^{-\lambda_0\xi}), & \text{for } \xi_1 + \delta_1 \leq \xi, \\
(\epsilon_2 \sin(\delta_2(\xi - \xi_1 + \delta_3)), \eta_2 e^{\lambda_1\xi}), & \text{for } \xi_1 - \delta_4 \leq \xi \leq \xi_1 + \delta_1, \\
(\epsilon_3, \eta_3 e^{\lambda_1\xi}), & \text{for } \xi_1 - \delta_4 \leq \xi \leq \xi_1 - \delta_4, \\
(-\epsilon_4(-\xi)^\theta[1 - U_*(\xi)], \eta_3 \sin(\delta_6(\xi - \xi_2))), & \text{for } \xi_2 - \delta_7 \leq \xi \leq \xi_2 + \delta_5, \\
(-\epsilon_4(-\xi)^\theta[1 - U_*(\xi)], -\eta_4(-\xi)^\theta V_*(\xi)), & \text{for } \xi \leq \xi_2 - \delta_7,
\end{cases}
$$

where $0 < \theta < 1$, and $\lambda_1 > 0$ is fixed such that

$$
d\lambda_1^2 + 2\sqrt{1 - \delta_1} \lambda_1 - r(2 + b^*) > 0 \quad \text{and} \quad \lambda_1 > \frac{2r(b^* + 1)}{\sqrt{1 - \delta_1}}.
$$

(3.8)

Here $\xi_1 > M_0$, $\xi_2 < -M_0$, $\epsilon_1, \ldots, 4 > 0$ and $\eta_1, \ldots, 4 > 0$, $\delta_1, \ldots, 7 > 0$, and $\sigma(\xi)$ will be determined later.

Figure 3.1: $(R_u, R_v)$ for the case $b^* \geq 1$. 

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Next, we divide the proof into several steps.

**Step 1:** We consider \( \xi \in [\xi_1 + \delta_1, \infty) \) with \( \xi_1 > M_0 \) (\( M_0 \) is defined in (3.4)) and some small \( \delta_1 \) satisfying

\[
0 < \delta_1 < \frac{1}{\lambda_u + \lambda_1} \quad \text{and} \quad \frac{1}{\lambda_u} + \frac{3}{\lambda_u}(1 - e^{-\frac{\lambda_u\delta_1}{2}}) - 2\delta_1 > 0. \tag{3.9}
\]

In Step 1, we aim to verify that (3.10), we have

\[
2(1 - d) = \frac{1}{\lambda_u} + \frac{3}{\lambda_u}(1 - e^{-\frac{\lambda_u\delta_1}{2}}) - 2\delta_1 > 0.
\]

Then, up to enlarging \( \xi \) relatively small to \( A_0, \eta_1 \) relatively smaller than \( \varepsilon_1 \), and

\[
0 < \delta_0 < \frac{\varepsilon_1 r b^* - 2r C_1 \lambda_u \eta_1 - 2[1 - d] \lambda_u^2 \eta_1}{r A_0 \lambda_u}, \tag{3.10}
\]

where \( C_1 \) is defined in (3.7).

Similar as the construction of \( R_w(\xi) \) for the scalar equation problem, we define

\[
\sigma(\xi) := \frac{4}{\lambda_u} e^{-\frac{\lambda_u}{2}(\xi - \xi_1)} - \frac{4}{\lambda_u} \xi - \frac{4}{\lambda_u} \xi_1
\]

which satisfies \( \sigma(\xi_1) = 0, \sigma'(\xi) = \frac{4}{\lambda_u} - \frac{2}{\lambda_u} e^{-\frac{\lambda_u}{2}(\xi - \xi_1)}, \sigma''(\xi) = e^{-\frac{\lambda_u}{2}(\xi - \xi_1)}, \) and \( \sigma(\xi) = O(\xi) \) as \( \xi \to \infty \). Therefore, \( R_u \) satisfies (3.2). Moreover, by some straightforward computations, we obtain \( R_u((\xi_1 + \delta_1)^+) > 0 \), while \( R_u((\xi_1 + \delta_1)^+) > 0 \) follows from (3.9).

Recall that, \( (U_*, V_*) \) is the minimal traveling wave satisfying (1.14) with \( c = 2\sqrt{1 - a} \). By some straightforward computations, we have

\[
N_3[W_u, W_v] = -\varepsilon_1 \sigma''(\xi)e^{-\lambda_u \xi} - R_u(a - 2U_*) + R_u - aV_* - aR_v - aR_vU_*,
\]

and

\[
N_4[W_u, W_v] = r R_v \left[ \frac{(d - 2)(1 - a)}{r} + 1 - 2V_* - R_v - (b^* + \delta_0)U_* + (b^* + \delta_0)R_u \right]
+ 2(1 - d) \lambda_u \eta_1 e^{-\lambda_u \xi} + r V_* [(b^* + \delta_0) R_u - \delta_0 U_*].
\]

Then, from (3.5) and (3.6), by setting \( \varepsilon_1 > 0 \) and \( \eta_1 > 0 \) relatively small to \( A_0 \), for all \( \xi \in [\xi_1 + \delta_1, \infty) \), it holds

\[
a - 2U_* + R_u - aV_* - aR_v = o(e^{-\frac{\lambda_u}{2} \xi}).
\]

Then, up to enlarging \( \xi_1 \) if necessary, since \( R_v > 0 \), we obtain that \( N_3[W_u, W_v] \leq 0 \) for all \( \xi \in [\xi_1 + \delta_1, \infty) \).

Next, we deal with the inequality of \( N_4[W_u, W_v] \). For \( \xi \in [\xi_1 + \delta_1, \infty) \), from (3.4) and (3.7), we have

\[
N_4[W_u, W_v] \geq -r R_v(C_1 + R_v) + 2(1 - d) \lambda_u \eta_1 e^{-\lambda_u \xi} + r(1 - \rho) (b^* + \delta_0) R_u - r \delta_0 U_*.
\]

From the definition of \( \sigma(\xi) \), we can find a \( M_1 > \xi_1 \) such that \( \sigma(\xi) \approx \frac{4}{\lambda_u} \xi \). In view of (3.5) and

\[
2(1 - d) \lambda_u \eta_1 e^{-\lambda_u \xi} = o(R_u) \quad \text{as} \quad \xi \to \infty,
\]

by further choosing \( \eta_1/\varepsilon_1 \) sufficiently small and \( \delta_0 \) satisfying (3.10), we have \( N_4[W_u, W_v] > 0 \) for \( \xi \geq M_1 \). For \( \xi \in [\xi_1 + \delta_1, M_1] \), by choosing \( \eta_1/\varepsilon_1 \) sufficiently small, we have

\[
-r R_v(C_1 + R_v) + 2(1 - d) \lambda_u \eta_1 e^{-\lambda_u \xi} + r(1 - \rho) (b^* + \delta_0) R_u > 0.
\]
We first verify the following claim:

**Claim 3.4** There exist $\varepsilon_2 > 0$ and $\eta_2 > 0$ sufficiently small such that

$$R_u((\xi_1 + \delta_1)^+) = R_u((\xi_1 + \delta_1)^-), \quad R_v((\xi_1 + \delta_1)^+) = R_v((\xi_1 + \delta_1)^-),$$

$$\angle \alpha_1 < 180^\circ, \quad \text{and} \quad \angle \alpha_2 < 180^\circ,$$

provided that $\delta_1, \delta_3$ satisfy (3.9) and (3.12), and $\delta_2$ is sufficiently small.

**Proof.** By some straightforward computations, we have

$$R_u((\xi_1 + \delta_1)^+) = \varepsilon_1 \sigma(\xi_1 + \delta_1) e^{-\lambda_u(\xi_1 + \delta_1)}, \quad R_u((\xi_1 + \delta_1)^-) = \varepsilon_2 \sin(\delta_2 + \delta_3),$$

$$R_v((\xi_1 + \delta_1)^+) = \varepsilon_1 \sigma(\xi_1 + \delta_1) e^{-\lambda_u(\xi_1 + \delta_1)} - \lambda_u R_u(\xi_1 + \delta_1),$$

$$R'_v((\xi_1 + \delta_1)^-) = \varepsilon_2 \delta_2 \cos(\delta_2 + \delta_3)).$$

We first choose $\varepsilon_2 = \varepsilon_1 \sigma(\xi_1 + \delta_1) e^{-\lambda_u(\xi_1 + \delta_1)} / \sin(\delta_2 + \delta_3)$ such that

$$R_u((\xi_1 + \delta_1)^+) = R_u((\xi_1 + \delta_1)^-).$$

Then, by applying (3.12) and the fact $\frac{x \cos x}{\sin x} \to 1$ as $x \to 0$, we have $R'_v((\xi_1 + \delta_1)^+) - R'_v((\xi_1 + \delta_1)^-) > 0$ is equivalent to

$$\frac{2}{\lambda_u} + \frac{2}{\lambda_u} (1 - e^{-\lambda_u \delta_1}) > \left( \frac{1}{\delta_1 + \delta_3} + \lambda_u \right) \sigma(\xi_1 + \delta_1),$$

which holds since $\sigma(\xi_1 + \delta_1) \to 0$ as $\delta_1 \to 0$ and (3.12). It follows that $\angle \alpha_1 < 180^\circ$.

On the other hand, by some straightforward computations, we have

$$R_v((\xi_1 + \delta_1)^-) = \eta_2 e^{\lambda_1(\xi_1 + \delta_1)}, \quad R_v((\xi_1 + \delta_1)^+) = \eta_1 \delta_1 e^{-\lambda_u(\xi_1 + \delta_1)},$$

$$R'_v((\xi_1 + \delta_1)^-) = \lambda_1 \eta_2 e^{\lambda_1(\xi_1 + \delta_1)}, \quad R'_v((\xi_1 + \delta_1)^+) = \eta_1 (1 - \delta_1 \lambda_u) e^{-\lambda_u(\xi_1 + \delta_1)},$$

where $\lambda_1$ satisfies (3.8). We take

$$\eta_2 = \eta_1 \delta_1 e^{-(\lambda_u + \lambda_1)(\xi_1 + \delta_1)} > 0,$$

which implies $R_v((\xi_1 + \delta_1)^-) = R_v((\xi_1 + \delta_1)^+).$ Then, from (3.9), we have

$$R'_v((\xi_1 + \delta_1)^+) - R'_v((\xi_1 + \delta_1)^-) = \eta_1 e^{-\lambda_u(\xi_1 + \delta_1)} (1 - \delta_1 \lambda_u - \delta_1 \lambda_1) > 0,$$

which yields that $\angle \alpha_2 < 180^\circ$. The proof of Claim 3.4. \qed
To finish Step 2, it suffices to take a small $\delta_2 > 0$ and suitable $0 < \delta_3 < \delta_4$ such that
\[ N_3[W_u, W_v] \leq 0 \quad \text{and} \quad N_4[W_u, W_v] \geq 0 \quad \text{for} \quad \xi \in [\xi_1 - \delta_4, \xi_1 + \delta_1]. \tag{3.13} \]

By some straightforward computations, for $\xi \in [\xi_2 - \delta_3, \xi_1 + \delta_1]$ we have
\[
N_3[W_u, W_v] = -2\sqrt{1-a}\alpha^2 \cos(\delta_2(\xi - \xi_1 + \delta_3)) - a(U_* - R_u)R_v \\
- R_u(1 - \delta_2^2 - 2U_* + R_u - aV_*),
\]
\[
N_4[W_u, W_v] = R_v \left[ d\lambda_2^2 + 2\sqrt{1-a}\lambda_1 + r[1 - 2V_* - R_v - (b^* + \delta_0)(U_* - R_u)] \right] \\
+ rV_*[(b^* + \delta_0)R_u - \delta_0U_*].
\]

To estimate $N_3[W_u, W_v]$, we consider $\xi \in [\xi_1 - \delta_3, \xi_1 + \delta_1]$ and $\xi \in [\xi_1 - \delta_4, \xi_1 - \delta_3]$ separately as follows:

- For $\xi \in [\xi_1 - \delta_3, \xi_1 + \delta_1]$, we have $0 \leq R_u(\xi) \leq \epsilon_2 \sin(\delta_2(\delta_1 + \delta_3))$ and $R_v(\xi) \geq 0$. Then, from (3.12), we have
  \[
  N_3[W_u, W_v] \leq -R_u(\xi) \left( 2\sqrt{1-a}\alpha^2 \cos(\delta_2(\delta_1 + \delta_3)) \right) \\
  + 1 - \delta_2^2 - 2U_* - aR_v - aV_* \leq 0,
  \]
  provided that $\delta_2$ is sufficiently small.

- For $\xi \in [\xi_1 - \delta_4, \xi_1 - \delta_3]$, we have $R_u(\xi) \leq 0$, $R_v(\xi) \geq 0$, and $R_v(\xi) \geq 0$. Note that, $|R_u(\xi)| \ll R_v(\xi)$ for $\xi \in [\xi_1 - \delta_4, \xi_1 - \delta_3]$ and
  \[
  \max_{\xi \in [\xi_1 - \delta_4, \xi_1 - \delta_3]} |R_u(\xi)| \to 0 \quad \text{as} \quad |\delta_3 - \delta_4| \to 0. \tag{3.14}
  \]
  Then, since $a(U_* - R_u)R_v > 0$, we have
  \[
  N_3[W_u, W_v] \leq -a(U_* - R_u)R_v - R_u(1 - \delta_2^2 - 2U_* + R_u - aV_*) \leq 0,
  \]
  provided $|\delta_3 - \delta_4| > 0$ is chosen sufficiently small.

From the above discussion, we assert that $N_3[W_u, W_v] \leq 0$ for $\xi \in [\xi_1 - \delta_4, \xi_1 + \delta_1]$, provided that $\delta_1, \delta_3, \delta_4$ satisfy (3.11) and (3.12), and $\delta_2$ is small enough.

On the other hand, thanks to (3.8), we have
\[
N_4[W_u, W_v] \geq R_v \left[ r(2 + b^*) + r[1 - 2V_* - R_v - (b^* + \delta_0)(U_* - R_u)] \right] \\
+ rV_*[(b^* + \delta_0)R_u - \delta_0U_*] \geq \frac{r}{2} R_v + rV_*[(b^* + \delta_0)R_u - \delta_0U_*].
\]

Note that $R_u(\xi) \geq 0$ for $\xi \in [\xi_1 - \delta_3, \xi_1 + \delta_1]$; $R_u(\xi) < 0$ but satisfies (3.14) for $\xi \in [\xi_1 - \delta_4, \xi_1 - \delta_3]$. Consequently, we assert that $N_4[W_u, W_v] \geq 0$ for $\xi \in [\xi_2 - \delta_3, \xi_1 + \delta_1]$ up to decreasing $|\delta_3 - \delta_4|$ and $\delta_0$ if necessary. This completes the proof of (3.13), and Step 2 is finished.

**Step 3**: We consider $\xi \in [\xi_2 + \delta_5, \xi_1 - \delta_4]$ for some $\delta_5 > 0$ such that $\xi_2 + \delta_5 < -M_0$ and satisfies
\[
\delta_5 > \frac{1}{\lambda_1} \quad \text{and} \quad \left| \delta_5 - \frac{1}{\lambda_1} \right| \text{is sufficiently small.} \tag{3.16}
\]
In this case, we have

$$(R_u, R_v)(\xi) = (-\varepsilon_3, \eta_2 e^{\lambda_1 \xi}).$$

First, we choose $\varepsilon_3 = R_u(\xi_1 - \delta_4)$ such that $R_u(\xi)$ is continuous at $\xi = \xi_1 - \delta_4$. Clearly, by setting $|\delta_3 - \delta_4|$ very small as in Step 2, we have $R'_u((\xi_1 - \delta_4)^+) > 0 = R'_u((\xi_1 - \delta_4)^-)$, i.e., $\angle \alpha_3 < 180^\circ$.

By some straightforward computations, we have

$$N_3[W_u, W_v] = -R_u(1 - 2U_s + R_u - a(V_s + R_v)) - aU_s R_v,$$

and $N_4[W_u, W_v]$ satisfies (3.15). Note that, $|\delta_3 - \delta_4| \to 0$ implies that $\varepsilon_3 \to 0$, and $|R_v(\xi)|$ does not depend on $|\delta_3 - \delta_4|$. It follows that $|R_u(\xi)| \ll |R_v(\xi)|$ for all $\xi \in [\xi_2 + \delta_5, \xi_1 - \delta_4]$ up to decreasing $|\delta_3 - \delta_4|$ if necessary. Also, we have $\min_{\xi \in (-\infty, \xi_1 - \delta_4)} U_s(\xi) > 0$. Therefore, we see that $N_3[W_u, W_v] \leq 0$ for $\xi \in [\xi_2 + \delta_5, \xi_1 - \delta_4]$ by taking $|\delta_3 - \delta_4|$ sufficiently small. On the other hand, by using the same process in Step 2, we see that $N_4[W_u, W_v] \geq 0$ for $\xi \in [\xi_2 + \delta_5, \xi_1 - \delta_4]$ up to decreasing $|\delta_3 - \delta_4|$ and $\delta_0$ is necessary. Thus, Step 3 is finished.

**Step 4:** We consider $\xi \in [\xi_2 - \delta_7, \xi_2 + \delta_5]$. In this case, we have

$$(R_u, R_v)(\xi) = \left(-\varepsilon_4(-\xi)^\theta[1 - U_s(\xi)], \eta_3 \sin(\delta_6(\xi - \xi_2))\right),$$

where $\theta \in (0, 1)$ is fixed, while $\varepsilon_4 > 0$, $\eta_3 > 0$, $\delta_6 > 0$, and $\delta_7 > 0$ are determined below.

We first choose

$$\varepsilon_4 = \frac{\varepsilon_3}{(-\xi_2 - \delta_5)^\theta[1 - U_s(\xi_2 + \delta_5)]}$$

such that $R_u(\xi)$ is continuous at $\xi = \xi_2 + \delta_5$, where $\varepsilon_3$ and $\delta_5$ are fixed in Step 3. Then, from the behavior of $1 - U_s$ for both cases $b^* > 1$ and $b^* = 1$ as $\xi \to -\infty$ in the Appendix, we may assume that $R'_u(\xi) < 0$ for all $\xi \leq \xi_2 + \delta_5$. In particular, we have

$$R'_u((\xi_2 + \delta_5)^+) = 0 > R'_u((\xi_2 + \delta_5)^-),$$

and thus $\angle \alpha_5 < 180^\circ$. Next, we verify the following claim to obtain the continuity of $R_v$ at $\xi_2 + \delta_5$ and the right angle of $\alpha_6$:

**Claim 3.5** For any $\delta_5$ with $\delta_5 > \frac{1}{\lambda_1}$, there exist $\eta_3 > 0$ and small $\delta_6 > 0$ such that $R_v(\xi)$ is continuous at $\xi = \xi_2 + \delta_5$ and $\angle \alpha_6 < 180^\circ$.

**Proof.** First, we take

$$\eta_3 = \frac{\eta_2 e^{\lambda_1(\xi_2 + \delta_5)}}{\sin(\delta_5 \delta_6)} > 0$$

such that $R_v((\xi_2 + \delta_5)^+) = R_v((\xi_2 + \delta_5)^-)$. By some straightforward computations, we have $R'_v((\xi_2 + \delta_5)^+) = \lambda_1 \eta_2 e^{\lambda_1(\xi_2 + \delta_5)}$, and from (3.18),

$$R'_v((\xi_2 + \delta_5)^-) = \eta_3 \delta_6 \cos(\delta_5 \delta_6) = \eta_2 e^{\lambda_1(\xi_2 + \delta_5)} \frac{\delta_6 \cos(\delta_5 \delta_6)}{\sin(\delta_5 \delta_6)},$$

which yields that

$$R'_v((\xi_2 + \delta_5)^-) \to \eta_2 e^{\lambda_1(\xi_2 + \delta_5)} / \delta_6 \text{ as } \delta_6 \to 0.$$Thus, $R'_v((\xi_2 + \delta_5)^+) > R'_v((\xi_2 + \delta_5)^-)$ is equivalent to $\delta_5 > \frac{1}{\lambda_1}$ by setting $\delta_6$ sufficiently small. This completes the proof of Claim 3.5. \qed

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Next, we prove the following claim to determine $\delta_\tau$. Note that the choice of $\delta_\tau$ is rather technical and crucial when we prove the differential inequalities later.

**Claim 3.6** There exists $0 < \delta_\tau \leq \delta_5$ such that $R_v(\xi_2 - \delta_\tau) = -\varepsilon_4(-\xi_2 + \delta_\tau)^\theta V_*(\xi_2 - \delta_\tau)$ and

$$-\varepsilon_4(-\xi)^\theta V_*(\xi) < R_v(\xi) < 0 \text{ for all } \xi \in (\xi_2 - \delta_\tau, \xi_2). \quad (3.19)$$

**Proof.** Recall from Step 3 and (3.17) that $R_v(\xi_2 + \delta_5) \gg \varepsilon_3 = \varepsilon_4(-\xi_2 - \delta_5)^\theta[1 - U_*(\xi_2 + \delta_5)]$. Together with Lemma A.6 (if $b^* > 1$) and Lemma A.8 (if $b^* = 1$), we may assume that

$$R_v(\xi_2 + \delta_5) > \varepsilon_4(-\xi_2 - \delta_5)^\theta V_*(\xi_2 + \delta_5), \quad (3.20)$$

if necessary, we may choose $\varepsilon_3$ smaller by decreasing $|\delta_3 - \delta_4|$. Furthermore, by the asymptotic behavior of $V_*(\xi)$ as $\xi \to -\infty$, $-\varepsilon_4(-\xi)^\theta V_*(\xi)$ is strictly decreasing for all $\xi < \xi_2 + \delta_5$. Together with (3.20), we obtain that

$$-\eta_3 \sin(\delta_5 \delta_6) = -R_v(\xi_2 + \delta_5) < -\varepsilon_4(-\xi_2 - \delta_5)^\theta V_*(\xi_2 + \delta_5) < -\varepsilon_4(-\xi_2 - \delta_5)^\theta V_*(\xi_2 - \delta_5).$$

Define $F(\xi) := \eta_3 \sin(\delta_5(\xi_2 - \xi)) + \varepsilon_4(-\xi)^\theta V_*(\xi)$. Clearly, $F$ is continuous and strictly increasing for $\xi \in [\xi_2 - \delta_5, \xi_2]$. Also, we have $F(\xi_2) > 0$ and $F(\xi_2 - \delta_5) < 0$. Then, by the intermediate value theorem, there exists a unique $\delta_\tau \in (0, \delta_5)$ such that Claim 3.6 holds. \hfill $\Box$

We now prove the differential inequalities. Note that, it suffices to assume $V_* + R_v \geq 0$. By some straightforward computations, $N_3[W_u, W_v]$ satisfies

$$N_3[W_u, W_v] = \varepsilon_4(-\xi)^\theta \left(-U''_v - c^*U'_v - \theta(1 - \theta)(-\xi)^{-2}(1 - U_v) + 2\theta(-\xi)^{-1}U'_v \right.$$

$$- c^*\theta(-\xi)^{-2}(1 - U_v) - R_u(1 - 2U_v + R_u - a(V_* + R_v)) - aU_*R_v \right.$$ 

$$\leq \varepsilon_4(-\xi)^\theta \left(U_v(1 - U_v - aV_* + c^*\theta(-\xi)^{-1}(1 - U_v) \right.$$ 

$$- R_u(1 - 2U_v + R_u - a(V_* + R_v)) - aU_*R_v. \quad (3.21)$$

The last inequality holds due to $\theta \in (0, 1)$ and $U'_v < 0$.

We next divide our discussion into two parts: $\xi \in [\xi_2, \xi_2 + \delta_5]$ and $\xi \in [\xi_2 - \delta_\tau, \xi_2]$. Notice that, $R_v(\xi) < 0 < R_v(\xi)$ and $(V_* + R_v)(\xi) \geq 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$ since $\xi_2 < -M_0$. Then, for $\xi \in [\xi_2, \xi_2 + \delta_5], (3.21)$ reduces to

$$N_3[W_u, W_v] \leq \varepsilon_4(-\xi)^\theta \left(U_v(1 - U_v - aV_* + (1 - 2U_v - c^*\theta(-\xi)^{-1})(1 - U_v) \right).$$

- For $b^* = 1$, we see from Lemma A.8 and Corollary A.9 that $U_v(1 - U_v - aV_*) = o(\xi^{-1})$ and $1 - U_v \approx (-\xi)^{-1}$ as $\xi \to -\infty$, we conclude that $N_3[W_u, W_v] \leq 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$ as long as $M_0$ in (3.4) is chosen sufficiently large.

- For $b^* > 1$, we have

$$N_3[W_u, W_v] \leq \varepsilon_4(-\xi)^\theta[(1 - U_v) + c^*\theta(-\xi)^{-1}](1 - U_v).$$

By Lemma A.6, since $1 - U_v$ decays exponentially as $\xi \to -\infty$, we obtain $N_3[W_u, W_v] \leq 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$ as long as $M_0$ is chosen sufficiently large.
On the other hand, for $\xi \in [\xi_2 - \delta_7, \xi_2]$, by using (3.19) and $\varepsilon_4(\xi) U_*(1 - U_* U_* - aV_*) = -R_u U_*$, from (3.21) we have

$$N_4[W_u, W_v] \leq -R_u U_* - a\varepsilon_4(\xi) U_*(1 - U_* - aV_*) - R_u (1 - 2U_* - aV_*) - R^2_u + aR_u R_v + a\varepsilon_4(\xi) U_*(1 - U_* - aV_*) - R^2_u + aR_u R_v.$$ 

Denote that

$$I_1 := c^\theta(\xi)^{-1} - R_u, \quad I_2 := -R_u (1 - U_* - aV_*), \quad I_3 := -R^2_u + aR_u R_v.$$

- For the case $b^* = 1$, by the equation satisfied by $U$, $1 - U_* - aV_*$ > 0 for all $\xi \leq -M_0$ (if necessary, we may choose $M_0$ larger). Therefore,

$$I_3 = -R^2_u + aR_u R_v \leq R_u \varepsilon_4(\xi) (1 - U_* - aV_*) < 0 \quad \text{for} \quad \xi \in [\xi_2 - \delta_7, \xi_2].$$

Moreover, in view of Corollary A.9, we have $I_2 = o(I_1)$ as $\xi \to -\infty$.

- For the case $b^* > 1$, since $1 - U_* - aV_* \to 0$ exponentially (See Lemma A.6), we have $I_2, I_3 \approx o((\xi)^{-1} R_u)$. Then, up to enlarging $M_0$ if necessary, we have $N_3[W_u, W_v] \leq 0$ for $\xi \in [\xi_2 - \delta_7, \xi_2]$.

We next deal with the inequality of $N_4[W_u, W_v]$. By some straightforward computations, we have

$$N_4[W_u, W_v] = rR_u (1 - 2V_* - R_v - (b^* + \delta_0)(U_* - R_u) - \frac{d\delta_0^2}{r}) + 2\sqrt{1 - a\delta_6\eta_3 \cos(\delta_6(\xi - \xi_2)) + rV_e((b^* + \delta_0) U_* - \delta_0 U_*).$$

(3.22)

For $\xi \in [\xi_2, \xi_2 + \delta_5]$, (3.18) and the fact $\frac{x\cos x}{\sin x} \to 0$ as $x \to 0$ yield that

$$\min_{\xi \in [\xi_2, \xi_2 + \delta_5]} \delta_6 \eta_3 \cos(\delta_6(\xi - \xi_2)) \to \eta_2 e^{1/(\xi_2 + \delta_5)} = \frac{R_v(\xi_2 + \delta_5)}{\delta_5} \quad \text{as} \quad \delta_6 \to 0.$$ 

In view of (3.8) and (3.16), we have

$$\min_{\xi \in [\xi_2, \xi_2 + \delta_5]} \delta_6 \eta_3 \cos(\delta_6(\xi - \xi_2)) > \lambda_1 R_v(\xi_2 + \delta_5) > R_v(\xi_2 + \delta_5) \frac{2r(b^* + 1)}{\sqrt{1 - a}}.$$ 

(3.23)

By applying (3.4), (3.22), and (3.23), we have

$$N_4[W_u, W_v] \geq -rR_v(\xi_2 + \delta_5) \left(1 + R_v + (b^* + \delta_0) + \frac{d\delta_0^2}{r}\right) + 2r(b^* + 1) R_v(\xi_2 + \delta_5) + r\rho(b^* + \delta_0) R_u(\xi_2 + \delta_5) - r\rho\delta_0.$$ 

Recall that, $|R_v(\xi)| \ll R_v(\xi_2 + \delta_5)$ for all $\xi \in [\xi_2, \xi_2 + \delta_5]$ up to decreasing $|\delta_3 - \delta_4|$. Therefore, we assert that $N_4[W_u, W_v] \geq 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$, and for all small $\delta_0 > 0$.

For $\xi \in [\xi_2 - \delta_7, \xi_2]$, since $R_v < 0$ and $\xi_2 < -M_0$, by applying (3.4), (3.22), and a similar discussion as for $[\xi_2, \xi_2 + \delta_5]$, we have

$$N_4[W_u, W_v] \geq rR_v(1 - R_u - \frac{d\delta_0^2}{r}) + 2\sqrt{1 - a\delta_6\eta_3 \cos(\delta_6(\xi - \xi_2)) + rV_e((b^* + \delta_0) U_* - \delta_0 U_*).$$

$$\geq rR_v(\xi_2 - \delta_7) \left(1 - R_v(\xi_2 - \delta_7) - \frac{d\delta_0^2}{r}\right) + 2r(b^* + 1) R_v(\xi_2 + \delta_5) + r\rho(b^* + \delta_0) R_u(\xi_2 + \delta_5) - r\rho\delta_0.$$ 

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Since $0 < \delta_7 \leq \delta_5$ and $|R_v(\xi_2 - \delta_7)| \leq R_v(\xi_2 + \delta_5)$. $N_4[W_u, W_v] \geq 0$ holds in $[\xi_2 - \delta_7, \xi_2]$ for all small $\delta_0 > 0$. From the above discussion, the construction for Step 5 is finished.

**Step 5:** We consider $\xi \in (-\infty, \xi_2 - \delta_7]$. In this case, we have

$$(R_u, R_v)(\xi) = \left( - \varepsilon_4(-\xi)^\theta[1 - U_*(\xi)], - \eta_4(-\xi)^\theta V_*(\xi) \right).$$

Let us take

$$\eta_4 = \frac{\eta_3 \sin(\delta_0 \delta_7)}{(\delta_7 - \xi_2)^\theta V_*(\xi_2 - \delta_7)}$$

such that $R_v(\xi)$ is continuous at $\xi = \xi_2 - \delta_7$. Also, since $0 < \delta_7 \leq \delta_5$, we have $R'_v((\xi_2 - \delta_7)^+) > 0 > R'_v((\xi_2 - \delta_7)^-)$, and hence $\angle \alpha_6 < 180^\circ$.

Finally, we verify the differentiable inequalities. Since $\theta > 0$, there exists $M_1 > M_0$ sufficiently large such that $W_u = 1$ and $W_v = 0$ for all $\xi \in (-\infty, -M_1]$. More precisely, Claim 3.6 implies $\eta_4 = \varepsilon_4$. From the definition of $(R_u, R_v)$, we may define $M_1$ satisfying $1 - \eta_4(M_1)^\theta = 0$, and thus $W_u(\xi) = 1$, $W_v(\xi) = 0$ for all $\xi \in (-\infty, -M_1]$, which implies that $N_3[W_u, W_v] \leq 0$ and $N_4[W_u, W_v] \geq 0$ for $\xi \in (-\infty, -M_1]$.

It suffices to consider $\xi \in [-M_1, \xi_2 - \delta_7]$. Without loss of generality, we may assume $\xi_2 - \delta_7 < \xi_0$, where $\xi_0$ is defined in Corollary A.9. Additionally, by the definition of $M_1$ and $\eta_4 = \varepsilon_4$, we have

$$1 - \varepsilon_4(-\xi)^\theta = 1 - \eta_4(-\xi)^\theta > 0 \quad \text{for all } \xi \in (-M_1, \xi_2 - \delta_7],$$

which yields $W_u < 1$ and $W_v > 0$ on $[-M_1, \xi_2 - \delta_7]$. Note that, $R_u, R_v < 0$ in $[-M_1, \xi_2 - \delta_7]$, and $N_3[W_u, W_v]$ satisfies (3.21). By applying the same argument as that in Step 4 for $\xi \in [\xi_2 - \delta_7, \xi_2]$, we obtain $N_3[W_u, W_v] \leq 0$ for all $\xi \in [-M_1, \xi_2 - \delta_7]$.

On the other hand, by some straight computations, we have

$$N_4[W_u, W_v] = d \left( V_v'' + \theta(1 - \theta)\eta_4(-\xi)^\theta V_v - 2\theta \eta_4(-\xi)^\theta V_v + \eta_4(-\xi)^\theta V_v'' \right)$$

$$+ c^* \left( V_v' + \theta \eta_4(-\xi)^\theta V_v - \eta_4(-\xi)^\theta V_v' \right)$$

$$+ r(\xi + R_v) \left( 1 - V_v - R_v - (b^* + \delta_0)(U_*, R_v) \right).$$

Notice that, in Claim 3.19, we choose a suitable $\delta_7$ such that $\varepsilon_4 = \eta_4$. Then, from $V_v' > 0$ and $\theta \in (0, 1)$, we further have

$$N_4[W_u, W_v] \geq r \eta_4(-\xi)^\theta V_v \left( V_v - b^*(1 - U_*) + \frac{c^* \theta}{r} (-\xi)^{-1} + R_v - b^* R_v \right)$$

$$- r(U_* - R_u)(V_v + R_v) \delta_0.$$  \hspace{1cm} (3.25)

- For the case $b^* > 1$, both $1 - U_* \to 0$ and $V_v \to 0$ exponentially as $\xi \to -\infty$. From (3.4), up to enlarging $M_0$ if necessary, we obtain $1 - U_* = o((-\xi)^{-1})$ and $R_v = o((-\xi)^{-1})$ for $\xi \in [-M_1, \xi_2 - \delta_7]$.

- For the case $b^* = 1$, (3.25) reduces to

$$N_4[W_u, W_v] \geq r \eta_4(-\xi)^\theta V_v \left( (\eta_4(-\xi)^\theta - 1)(1 - U_*) \right)$$

$$- r(U_* - R_u)(V_v + R_v) \delta_0.$$  \hspace{1cm} (3.25)

By Corollary A.9 and (3.24), up to enlarging $M_0$, we have $\eta_4(-\xi)^\theta - 1(1 - U_* - V_v) > 0$ for $\xi \in [-M_1, \xi_2 - \delta_7]$.

It follows that $N_4[W_u, W_v] \geq 0$ for $\xi \in [-M_1, \xi_2 - \delta_7]$ for all small $\delta_0 > 0$. Therefore, the construction for Step 5 is finished.
3.2 Construction of the super-solution for $b^* < 1$

In this subsection, we always assume $0 < b^* < 1$. Let $(c_{LV}^*, U_*, V_*)$ be the minimal traveling wave of (1.15) with $b = b^*$ and $c_{LV}^* = 2\sqrt{1-a}$. Different from the strong-weak competition case and the critical case, since $(U_*, V_*)(+\infty) = (0, 1)$ and

$$(U_*, V_*)(-\infty) = \left( \frac{1-a}{1-ab^*}, \frac{1-b^*}{1-ab^*} \right) := (\hat{u}, \hat{v}),$$

and $U_0' < 0 < V_0'$, for any given small $\rho > 0$, up to enlarging $M_0 > 0$ if necessary, we have

$$\begin{cases} 0 < U_*(\xi) < \rho, & 1 - \rho < V_*(\xi) < 1 \quad \text{for all } \xi \geq M_0, \\ \hat{u} - \rho < U_*(\xi) < \hat{u}, & \hat{v} < V_*(\xi) < \hat{v} + \rho \quad \text{for all } \xi \leq -M_0. \end{cases} \quad (3.26)$$

We consider $(R_u, R_v)(\xi)$ defined as (see Figure 3.2)

$$(R_u, R_v)(\xi) := \begin{cases} (\varepsilon_1 \sigma(\xi) e^{-\lambda u \xi}, \eta_1(\xi - \xi_1) e^{-\lambda u \xi}), & \text{for } \xi \geq \xi_1 + \delta_1, \\ (\varepsilon_2 \sin(\delta_2(\xi - \xi_1 - \delta_3)), \eta_2 e^{\lambda v \xi}), & \text{for } \xi_1 - \delta_4 \leq \xi \leq \xi_1 + \delta_1, \\ (-\delta_u, \eta_2 e^{\lambda v \xi}), & \text{for } \xi_2 + \delta_5 \leq \xi \leq \xi_1 - \delta_4, \\ (-\delta_u, \eta_3 \sin(\delta_6(\xi - \xi_2))), & \text{for } \xi_2 - \delta_7 \leq \xi \leq \xi_2 + \delta_5, \\ (-\delta_u, \delta_6), & \text{for } \xi \leq \xi_2 - \delta_7, \end{cases}$$

where $\xi_1 > M_0$ and $\xi_2 < -M_0$ are fixed points, and $\lambda_1$ satisfies

$$d\lambda_1^2 + 2\sqrt{1-a} \lambda_1 - r(2 + b^*) > 0 \quad \text{and} \quad \lambda_1 > \frac{r(\hat{u} + 1)}{2\sqrt{1-a}}. \quad (3.27)$$

Here $\varepsilon_{1,2} > 0$, $\eta_{k=1,2,3} > 0$, and $\delta_{i=1,\ldots,7} > 0$ are chosen as same as that in Subsection 3.1. Therefore, from (3.26) and $|R_u|, |R_v| \ll 1$, up to enlarging $M_0$, there exist $C_2 > 0$ and $C_3 > 0$ such that, for all $\xi \in (-\infty, \xi_2 + \delta_5]$, it holds

$$1 - 2U_* + R_u - aV_* - aR_v < -C_2 < 0, \quad (3.28)$$

and

$$-(\hat{v} + \hat{u} \delta_1 + C_3 \rho) < 1 - 2V_* - R_v - (b^* + \delta_0)U_* < C_3 \rho. \quad (3.29)$$
Moreover, we set
\[
\delta_u := \varepsilon_2 \sin(\delta_2 (\delta_4 - \delta_3)) \quad \text{and} \quad \delta_v := \eta_3 \sin(\delta_6 \delta_7),
\]
(3.30)
which yield that \((R_u, R_v)(\xi)\) is continuous on \(\mathbb{R}\).

Note that, for the construction in Subsection 3.1, we only set \(|\delta_3 - \delta_4|\) sufficiently small to obtain \(|R_u(\xi)| \ll |R_v(\xi + \delta_5)|\) in \([\xi_2 + \delta_5, \xi_1 - \delta_3]\). However, for the weak competition case, we will subtly set \(\delta_6\) and \(\delta_v\) to satisfy
\[
\delta_v = \frac{b^*}{\sqrt{a}} \delta_u,
\]
(3.31)
which can be done by modifying \(|\delta_3 - \delta_4|\) and \(|\delta_7|\).

Now, we define
\[
(W_u, W_v)(\xi) := \left( \min \{ (U_s - R_u)(\xi), 1 \}, \max \{ (V_s + R_v)(\xi), 0 \} \right),
\]
and show that \((W_u, W_v)\) satisfies (3.3). In fact, thanks to (3.26) and (3.27), for \(\xi \in [\xi_1 - \delta_4, \infty)\), \(N_3[W_u, W_v] \leq 0\) and \(N_4[W_u, W_v] \geq 0\) follow from the same argument as that in Subsection 3.1. Therefore, it suffices to deal with \(\xi \in (-\infty, \xi_1 - \delta_4]\). Next, we divide the discussion into three steps as follows.

**Step 1:** We consider \(\xi \in [\xi_2 + \delta_5, \xi_1 - \delta_4]\) with \(\xi_2 < -M_0\). In this case, we have
\[
(R_u, R_v)(\xi) = (-\delta_u, \eta_2 e^{\lambda_1 \xi}),
\]
where \(\lambda_1\) satisfies (3.27), and \(\delta_u\) is fixed as that in (3.30). Note that, \(\delta_u \to 0\) as \(|\delta_3 - \delta_4| \to 0\), and thus \(R_u^{'}((\xi_1 - \delta_4) +) > 0 = R_u^{'}((\xi_1 - \delta_4) -)\), i.e., \(\angle \alpha_1 < 180^\circ\).

By some straightforward computations, since \(R_v \geq 0\),
\[
N_3[W_u, W_v] = \delta_u(1 - 2U_s - \delta_v - a(V_s + R_v)) - aU_s R_v,
\]
(3.32)
and \(N_4[W_u, W_v]\) satisfies (3.15).

Note that, \(\lambda_1\) and \(\eta_2\) have already been determined by the construction on \(\xi \in [\xi - \delta_4, \infty)\). Since \(U_s \leq \hat{u}\) and \(\hat{v} \leq V_s \leq 1\), by setting \(|\delta_3 - \delta_4|\) small enough such that
\[
\delta_u < \min \left\{ \frac{a \rho}{2 \hat{u}}, \frac{1}{4 b^*} \right\} \eta_2 e^{\lambda_1 (\xi_2 + \delta_5)},
\]
(3.33)
we have \(N_3[W_u, W_v] \leq \delta_u(1 - aV_s) - aU_s R_v \leq 0\) for \(\xi \in [\xi_2 + \delta_5, \xi_1 - \delta_4]\). Up to decreasing \(\delta_0\) is necessary, \(N_4[W_u, W_v] \geq 0\) follows immediately from (3.15) and (3.33). Thus, Step 1 is finished.

**Step 2:** We consider \(\xi \in [\xi_2 - \delta_7, \xi_2 + \delta_5]\) with \(\delta_7 > 0\) very small satisfying
\[
\frac{2\sqrt{1-a}}{r \delta_7} - C_3 \rho > 2.
\]
(3.34)

In this case, we have
\[
(R_u, R_v)(\xi) = \left( -\delta_u, \eta_3 \sin(\delta_6 (\xi - \xi_2)) \right),
\]
with \(\delta_6, \delta_7 \geq 0\) very small, and \(\delta_5\) satisfies (3.16). It follows from the same argument as Claim 3.5 that, there exist \(\eta_3 > 0\) and small \(\delta_6 > 0\) such that \(R_v(\xi)\) is continuous at \(\xi = \xi_2 + \delta_5\) and \(\angle \alpha_3 < 180^\circ\).
Note that $N_3[W_u, W_v]$ still satisfies (3.32). Then, from (3.28) and $R_v(\xi) > 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$, we have $N_3[W_u, W_v] \leq 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$. For $\xi \in [\xi_2 - \delta_7, \xi_2]$, since $-\delta_0 \leq R_v \leq 0$, from (3.26), (3.31), and (3.32), we obtain that $N_3[W_u, W_v] \leq 0$ up to increasing $M_0$ if necessary.

We next deal with the inequality of $N_4[W_u, W_v]$. For $\xi \in [\xi_2, \xi_2 + \delta_5]$, by applying the same argument as Step 5 in Subsection 3.1, from (3.22), (3.23), (3.27), and (3.29), we have

$$N_4[W_u, W_v] \geq r R_v \left[1 - 2V_s - R_v - (b^* + \delta_0)U_s - \frac{d}{r} \delta_6^2 \right] + 2 \sqrt{1 - \alpha} \delta_6 \eta_3 \cos(\delta_6(\xi - \xi_2))$$

$$+ r V_s((b^* + \delta_0)R_u - \delta_0 U_s)$$

$$\geq r \left( \frac{2 \sqrt{1 - \alpha}}{r} \lambda_1 - \hat{\nu} \delta_0 - C_3 \rho - \frac{d}{r} \delta_6^2 \right) R_v(\xi_2 + \delta_5)$$

$$- r(\hat{\nu} + \rho)(b^* + \delta_0)\delta_u - r \delta_0$$

$$\geq r(1 - \hat{\nu} \delta_0 - C_3 \rho - \frac{d}{r} \delta_6^2) R_v(\xi_2 + \delta_5) - r(\hat{\nu} + \rho)(b^* + \delta_0)\delta_u - r \delta_0$$

Recall that, $\delta_u \ll R_v(\xi_2 + \delta_5)$ up to decreasing $|\delta_3 - \delta_4|$, and $\delta_0$ can be chosen sufficiently small such that $\frac{d}{r} \delta_6^2 < \frac{1}{4}$. Then we have $N_4[W_u, W_v] \geq 0$ for $\xi \in [\xi_2, \xi_2 + \delta_5]$, up to enlarging $M_0$ and decreasing $\delta_0$ if necessary.

For $\xi \in [\xi_2 - \delta_7, \xi_2]$, since $R_v < 0$ and $\xi_2 < -M_0$, from (3.22), (3.29), and (3.34), we have

$$N_4[W_u, W_v] \geq r \left( \frac{2 \sqrt{1 - \alpha}}{r} \lambda_1 - \hat{\nu} \delta_0 - C_3 \rho \right) \delta_v - r(\hat{\nu} + \rho)(b^* + \delta_0)\delta_u - r \delta_0 \hat{\nu}$$

$$\geq 2r \delta_v - r(\hat{\nu} + \rho)(b^* + \delta_0)\delta_u - r \delta_0 \hat{\nu}.$$

Then, from (3.31), we assert that $N_4[W_u, W_v] \geq 0$ up to decreasing $\delta_0$ if necessary. Hence, the construction for Step 2 is finished.

**Step 3:** We consider $\xi \in (-\infty, \xi_2 - \delta_7]$. In this case, we have

$$(R_u, R_v)(\xi) = (-\delta_u, -\delta_v).$$

Since $\delta_7 \ll \frac{\pi}{2\delta_6}$, we see $R_v(\xi_2 - \delta_7) = 0 < R_v(\xi_2 - \delta_7)$, i.e., $\angle \alpha_2 < 180^\circ$.

By applying the same argument as Step 2 above, $N_3[W_u, W_v] \leq 0$ for $\xi \in (-\infty, \xi_2 - \delta_7]$. Therefore, it suffices to verify the inequality of $N_4[W_u, W_v]$. By some straightforward computations, from (3.26), we have

$$N_4[W_u, W_v] = - r V_s(b^* + \delta_0)\delta_u - r \delta_v \left[1 - 2V_s + \delta_v - (b^* + \delta_0)(U_s + \delta_u) \right]$$

$$\geq - r(\hat{\nu} + \rho)(b^* + \delta_0)\delta_u - r \delta_v \left[1 - 2\hat{\nu} + \delta_v - (b^* + \delta_0)(\hat{\nu} + \rho + \delta_u) \right].$$

Then, from (3.31) and $0 < b^* < 1$, we have $N_4[W_u, W_v] \geq 0$ up to enlarging $M_0$ and decreasing $\delta_0$ if necessary. The construction for Step 3 is complete.

### 3.3 Proof of Theorem 1.6

We first prove Proposition 3.3.
Proof of Proposition 3.3. Combining the construction of \((R_u, R_v)\) in Subsection 3.1 and Subsection 3.2, we are now equipped with a super-solution

\[(W_u, W_v) = (\min\{U_* - R_u, 1\}, \max\{V_* + R_u, 0\})\]

satisfying (3.3). Moreover, at the points of discontinuity of \(W'_u\) and \(W'_v\), the corresponding one-sided derivatives have the right sign. Therefore, we complete the proof of Proposition 3.3.

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. In view of Lemma 3.2, it suffices to show that three conditions (i), (ii), and (iii) are equivalent. We now deal with \((i) \iff (ii)\). To prove \((i) \implies (ii)\), we use the contradiction argument and assume that (ii) is not true, by Lemma A.3 (ii), we see that \(U_*\) satisfies (3.1) and thus Proposition 3.3 is available.

To reach a contradiction, we consider the Cauchy problem

\[
\begin{aligned}
\partial_t u &= u_{xx} + u(1 - u - av), & t > 0, \ x \in \mathbb{R}, \\
\partial_t v &= dv_{xx} + rv(1 - v - (b^* + \delta_0)u), & t > 0, \ x \in \mathbb{R}, \\
\ u(0, x) &= u_0(x), \ v(0, x) = 1, & x \in \mathbb{R},
\end{aligned}
\]

(3.35)

where \(u_0(x)\) is compactly supported continuous function. Additionally we assume \(\max_{x \in \mathbb{R}} |u_0(x)| < \frac{1}{1 - ab}\) and \(\delta_0 > 0\) is sufficiently small such that \(b^* + \delta_0 \neq 1\) if \(b^* < 1\). By the definition of \(b^*\), we see that the minimal wave speed \(c_{LV}^*(b^* + \delta_0)\) corresponding to the system (1.14) with \(b = b^* + \delta_0\) satisfies \(c_{LV}^*(b^* + \delta_0) > 2\sqrt{1 - a}\). Then, according to results from [24, 25], the spreading speed of (3.35) is exactly \(c_{LV}^*(b^* + \delta_0)\), strictly greater than \(2\sqrt{1 - a}\).

Let \((W_u, W_v)\) be constructed in Proposition 3.3. Then, thanks to Proposition 3.3, it is easy to see that \((\bar{u}, \bar{v})(t, x) := (W_u, W_v)(x - (2\sqrt{1 - a})t + \eta)\), forms a super-solution for (3.35) for all \(t \geq 0\) and \(x \in \mathbb{R}\), where \(\eta \in \mathbb{R}\) is chosen to have \(\bar{u}(0, x) \geq u_0(x)\) and \(\bar{v}(0, x) \leq v_0(x)\) for \(x \in \mathbb{R}\). By applying the comparison principle, we assert that the spreading speed of (3.35) is smaller than or equal to \(2\sqrt{1 - a}\), which reaches a contradiction. The proof of \((i) \implies (ii)\) is finished.

Next, we show \((ii) \implies (i)\). Note that for \(b > b^*\), the speed is nonlinearly selected, which together with Lemma A.3 implies that \((ii)\) cannot hold. Therefore, it suffices to show that \((ii)\) cannot happen with \(b < b^*\). We assume by contradiction that that there exists \(b^1 \in (0, b^*)\) such that

\[U_{b^1}(\xi) = B^1 e^{-\lambda_0 \xi} + o(e^{-\lambda_0 \xi}) \quad \text{as} \ \xi \to \infty\]

for some \(B^1 > 0\). In view of the asymptotic behavior of fronts at \(\pm \infty\) given in the Appendix, we can define

\[L^* := \inf\{L \in \mathbb{R} | U_*(\xi - L) \geq U_{b^1}(\xi), \ V_*(\xi - L) \leq V_{b^1}(\xi), \ \forall \xi \in \mathbb{R}\}.
\]

Note that, we should divide the discussion into several cases: \(b^* > 1\) and \(b^1 > 1, = 1, \text{or} < 1\); \(b^* = 1\) and \(b^1 = 1, \text{or} < 1\); \(b^* < 1\) and \(b^1 < 1\). But to define \(L^*\) we only need \(0 < b^1 < b^*\). Therefore, we can apply the sliding method as we used in the proof of Theorem 1.3 to reach a contradiction.

By the continuity, we have \(U_{b^1}(\xi - L^*) \geq U_{b^1}(\xi)\) and \(1 - V_{b^1}(\xi - L^*) \geq 1 - V_{b^1}(\xi)\) for all \(\xi \in \mathbb{R}\). If there exists \(\xi^* \in \mathbb{R}\) such that \(U_{b^1}(\xi^* - L^*) = U_{b^1}(\xi^*)\) or \(1 - V_{b^1}(\xi^* - L^*) = 1 - V_{b^1}(\xi^*)\), by the strong maximum principle, we have \((U_{b^1}, V_{b^1})(\xi^* - L^*) = (U_{b^1}, V_{b^1})(\xi^*)\) for all \(\xi \in \mathbb{R}\), which is impossible since they satisfy different equations. Consequently,

\[
U_{b^1}(\xi - L^*) > U_{b^1}(\xi), \quad V_{b^1}(\xi - L^*) < V_{b^1}(\xi) \quad \text{for all} \ \xi \in \mathbb{R}.
\]

Furthermore, we claim that the touch point cannot happen at \(-\infty\).
Claim 3.7 It holds

(I) \( \lim_{\xi \to -\infty} \frac{1 - U_b^*(\xi - L^*)}{1 - U_b^*(\xi)} < 1 \) and (II) \( \lim_{\xi \to -\infty} \frac{V_b^*(\xi - L^*)}{V_b^*(\xi)} < 1. \)

**Proof.** Without loss of generality, we only deal with the case \( 1 < b^l < b^u \). The others can be proved by the same argument. Recall that \( \mu_u^+ (c^*) \) and \( \mu_v^+ (c^*) \) defined as that in Lemma A.6. Let us denote for simplicity that

\[
\mu_u = \mu_u^+(c^*), \quad \mu_v,1 = \mu_v^+(c^*, b^u), \quad \mu_v,2 = \mu_v^+(c^*, b^l).
\]

Note that \( \mu_u \) is independent of \( b \).

Clearly, it follows from the definition of \( \mu_v, i, i = 1, 2 \), that \( \mu_{v,1} > \mu_{v,2} \). Then (II) immediately follows from Lemma A.6.

Next, we deal with (I). First, we consider the case \( \mu_u > \mu_{v,i} \). Since \( \mu_{v,1} > \mu_{v,2} \), (I) follows immediately from Lemma A.6 since \( 1 - U_b^*(\xi) \) decays faster than \( 1 - U_b^*(\xi) \) as \( \xi \to -\infty \).

For the case \( \mu_{v,2} > \mu_u \), we assume by the contradiction that

\[
\lim_{\xi \to -\infty} \frac{1 - U_b^*(\xi - L^*)}{1 - U_b^*(\xi)} = 1.
\]

Then from Lemma A.6, there exist \( C_1, C_2 > 0 \) satisfying \( C_1 = C_2e^{\mu_u L^*} \) such that \( 1 - U_b^*(\xi - L^*) \approx C_1e^{\mu_u \xi} \) and \( 1 - U_b^*(\xi) \approx C_2e^{\mu_u \xi} \) as \( \xi \to -\infty \).

To reach a contradiction, we set

\[
U_1(\xi) = (1 - U_{b^l}(\xi)) - (1 - U_b^*(\xi - L^*)), \quad V_1(\xi) := V_{b^l}(\xi) - V_b^*(\xi - L^*).
\]

Then, by (3.36), \( U_1(\xi) > 0 \) and \( V_1(\xi) > 0 \) for all \( \xi \in \mathbb{R} \). Moreover, \( U_1 \) satisfies

\[
U_1'' + c^*U_1' - U_1 + aV_1 + g_1(\xi) + g_2(\xi) = 0, \quad \xi \in \mathbb{R}. \tag{3.37}
\]

where

\[
g_1(\xi) = [2 - U_{b^l}(\xi) - U_b^*(\xi - L^*) - aV_b^*(\xi - L^*)]U_1(\xi), \quad g_2(\xi) = -a(1 - U_{b^l}(\xi))V_1(\xi).
\]

However, using \( C_1 = C_2e^{\mu_u L^*}, 1 - U_{b^l}(\xi) \approx C_2e^{\mu_u \xi} \) and \( V_1(\xi) \approx C_3e^{\mu_u \xi} \) (for some \( C_3 > 0 \) as \( \xi \to -\infty \)), we obtain

\[
U_1'' + c^*U_1' - U_1 + aV_1 + g_1(\xi) + g_2(\xi) > 0 \quad \text{for all large negative } \xi,
\]

which contradicts (3.37) and thus (II) holds. \( \square \)

Now, we are ready to prove that the touch point always happens on \( U \)-equation at \( +\infty \).

**Claim 3.8** It holds

\[
\lim_{\xi \to +\infty} \frac{U_b^*(\xi - L^*)}{U_b^*(\xi)} = 1.
\]
Proof. Let $\lambda^-_v(c^*) < 0$ be defined as in Lemma A.3. For the case $\lambda^-_v(c^*) \leq -\sqrt{1-a}$, we are going to prove

$$\lim_{\xi \to +\infty} \frac{U_b^*(\xi - L^*)}{U_b^*(\xi)} > 1 \implies \lim_{\xi \to +\infty} \frac{1 - V_b^*(\xi - L^*)}{1 - V_b^*(\xi)} > 1. \quad (3.38)$$

We divide our discussion into three cases:

1. If $\lambda^-_v(c^*) < -\sqrt{1-a}$, then by Lemma A.3, we see that $U_b^*(\xi)$ and $1 - V_b^*(\xi)$ have the same decay rate at $+\infty$ and there exists a positive constant $A_1$ such that

$$\lim_{\xi \to +\infty} \frac{U_b^*(\xi)}{1 - V_b^*(\xi)} = A_1.$$ 

Therefore, one has

$$\lim_{\xi \to +\infty} \frac{1 - V_b^*(\xi - L^*)}{1 - V_b^*(\xi)} = \lim_{\xi \to +\infty} \left[ \frac{1 - V_b^*(\xi - L^*) U_b^*(\xi - L^*)}{U_b^*(\xi - L^*) U_b^*(\xi)} \right] \frac{U_b^*(\xi)}{1 - V_b^*(\xi)} = \frac{1}{A_1} \left( \lim_{\xi \to +\infty} \frac{U_b^*(\xi - L^*)}{U_b^*(\xi)} \right) A_1 > 1.$$ 

Hence (3.38) holds.

2. If $\lambda^-_v(c^*) = -\sqrt{1-a}$, then by Lemma A.3, there exists a positive constant $A_2$ such that

$$\lim_{\xi \to +\infty} \frac{\xi U_b^*(\xi)}{1 - V_b^*(\xi)} = A_2.$$ 

Therefore, one has

$$\lim_{\xi \to +\infty} \frac{1 - V_b^*(\xi - L^*)}{1 - V_b^*(\xi)} = \lim_{\xi \to +\infty} \left[ \frac{1 - V_b^*(\xi - L^*)}{\xi U_b^*(\xi - L^*)} \right] \frac{U_b^*(\xi - L^*)}{U_b^*(\xi)} \frac{\xi U_b^*(\xi)}{1 - V_b^*(\xi)} = \frac{1}{A_2} \left( \lim_{\xi \to +\infty} \frac{U_b^*(\xi - L^*)}{U_b^*(\xi)} \right) A_2 > 1,$$

which yields (3.38).

3. If $\lambda^-_v(c^*) > -\sqrt{1-a}$, we assume by contradiction that

$$\lim_{\xi \to +\infty} \frac{1 - V_b^*(\xi - L^*)}{1 - V_b^*(\xi)} = 1. \quad (3.39)$$

Then from Lemma A.3 and (3.39), there exist $C_1, C_2 > 0$ satisfying $C_1 = C_2 e^{\lambda^-_v(c^*)L^*}$ such that $1 - V_b^*(\xi - L^*) \approx C_1 e^{\lambda^-_v(c^*)\xi}$ and $1 - V_b^*(\xi) \approx C_2 e^{\lambda^-_v(c^*)\xi}$. To reach a contradiction, similar to the proof of Claim 3.7, we set

$$U_1(\xi) := U_b^*(\xi - L^*) - U_b^*(\xi), \quad V_1(\xi) := (1 - V_b^*(\xi)) - (1 - V_b^*(\xi - L^*)).$$ 

Considering the equation satisfied by the positive function $V_1$:

$$cV_1'' + dV_1'' = -rV_1 + rbU_1 + h_1(\xi) + h_2(\xi) = 0, \quad \xi \in \mathbb{R},$$

where

$$h_1(\xi) = r[2 - V_b^*(\xi) - V_b^*(\xi - L^*)]V_1(\xi),$$
$$h_2(\xi) = -rb(1 - V_b^*(\xi - L^*))U_1(\xi).$$

Using a similar argument as in Claim 3.7, we can reach a contradiction, and thus (3.38) holds.
As a result, if Claim 3.8 is not true, from Claim 3.7 and (3.38), it is easy to see that there exists \( \varepsilon > 0 \) sufficiently small such that \( W_s^\ast(\xi - (L^\ast + \varepsilon)) > W_{s_0}(\xi) \) for \( \xi \in \mathbb{R} \), which contradicts the definition of \( L^\ast \). Therefore, the proof of Claim 3.8 is finished.

Now, we are ready to finish the proof of (ii) \( \Rightarrow \) (i) by the help of Claim 3.7 and Claim 3.8. For this, we set

\[
U_2(\xi) := U_b^\ast(\xi - L^\ast) - U_b^\ast(\xi), \quad V_2(\xi) := (1 - V_b^\ast(\xi)) - (1 - V_b^\ast(\xi - L^\ast)).
\]

Then we focus on the equation satisfied by \( U_2 \) and use a similar argument as in Claim 3.7, we can again reach a contradiction. Consequently, we obtain (ii) \( \Rightarrow \) (i).

Finally, note that (ii) \( \Leftrightarrow \) (iii) follows from Lemma A.3 and Proposition A.5. This completes the proof of Theorem 1.6.

### 3.4 Proof of Proposition 3.1

At the end of Section 3, we complete the proof of Proposition 3.1.

**Proof of Proposition 3.1.** We first show \( b^\ast < \infty \). For contradiction, we assume that \( b^\ast = \infty \). Due to the monotonicity of \( c_{LV}^\ast(b) \), we have \( c_{LV}^\ast(b) = 2\sqrt{1 - a} \) for all \( b > 0 \). To reach a contradiction, we take a sequence \( b_n \uparrow \infty \) and write \( (U_n, V_n) \) as the solution of (1.14) with \( c = c_{LV}^\ast(b_n) = 2\sqrt{1 - a} \) and \( b = b_n \). By a translation, we may assume that \( U_n(0) = 1/2 \) for all \( n \). Since \( 0 \leq U_n, V_n \leq 1 \) in \( \mathbb{R} \), by standard elliptic estimates, we have \( |U_n|_{C^{2+a}(\mathbb{R})} \leq C \) for some \( C > 0 \) independent of \( n \).

We now fix any \( R > 0 \). Then there exists \( \varepsilon > 0 \) such that

\[
U_n(\xi) \geq \varepsilon \quad \text{for all } \xi \in [-R, R] \text{ and } n \in \mathbb{N}.
\]  

(3.40)

Next, we define an auxiliary function

\[
\bar{V}_n(\xi) = \frac{e^{-\lambda_n(\xi+2R)} + e^{\lambda_n(\xi-2R)}}{1 + e^{-4\lambda_nR}}, \quad \xi \in [-2R, 2R],
\]

where

\[
\lambda_n := \frac{-c + \sqrt{c^2 + 4dR(\varepsilon b_n - 1)}}{2d} \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{and } c = 2\sqrt{1 - a}.
\]

Clearly, \( \bar{V}_n(\pm 2R) = 1, 0 \leq \bar{V}_n(\xi) \leq 1 \) for all \( \xi \in [-2R, 2R] \) and \( n \in \mathbb{N} \), and \( \bar{V}_n \rightarrow 0 \) uniformly in \([-R, R] \) as \( n \rightarrow \infty \). Furthermore, by direct computation, for all large \( n \) we have

\[
c\bar{V}_n'' + d\bar{V}_n'' + r\bar{V}_n(1 - \bar{V}_n) - rb_n\varepsilon \bar{V}_n \leq 0, \quad \xi \in [-2R, 2R].
\]

Together with (3.40), one can apply the comparison principle to conclude that \( V_n \leq \bar{V}_n \) in \([-2R, 2R] \) for all large \( n \). In particular, we have

\[
\sup_{\xi \in [-R, R]} |V_n(\xi)| \to 0 \quad \text{as } n \to \infty.
\]  

(3.41)
Thanks to (3.41) and the $C^{2+\alpha}$ bound of $U_n$, up to subtract a subsequence, we may assume that $U_n \to U_R$ uniformly in $[-R, R]$ as $n \to \infty$, where $U_R$ is defined in $[-R, R]$ and satisfies $U_R(0) = 1/2, U_R' \leq 0$ in $[-R, R]$ and

$$cU'_R + U''_R + U_R(1 - U_R) = 0, \quad \xi \in [-R, R].$$

Next, by standard elliptic estimates and taking $R \to \infty$, up to subtract a subsequence, we may assume that $U_R \to U_\infty$ locally uniformly in $\mathbb{R}$ as $n \to \infty$, where $U_\infty$ satisfies

$$cU'_\infty + U''_\infty + U_\infty(1 - U_\infty) = 0, \quad \xi \in \mathbb{R}, \quad U_\infty(0) = 1/2, \quad U'_\infty \leq 0.$$  

It is not hard to see that $U_\infty(-\infty) = 1$ and $U_\infty(+\infty) = 0$. Therefore, $U_\infty$ forms a traveling front with speed $c = 2\sqrt{1 - a}$, which is impossible since $c \geq 2$ (see [23]). This contradiction shows that $b^* < \infty$.

Next, we prove $b^* > 0$. To do this, we assume by contradiction that $b^* = 0$ and let $W_*(\xi)$ be the minimal traveling wave satisfying

$$\begin{cases} W''_\ast + 2\sqrt{1 - a}W'_\ast + W_\ast(1 - a - W_\ast) = 0, \quad \xi \in \mathbb{R} \\ W_\ast(-\infty) = 1, \quad W_\ast(+\infty) = 0. \end{cases}$$

We look for continuous function $(R_u(\xi), R_v(\xi))$ defined in $\mathbb{R}$, such that

$$(W_u, W_v)(\xi) := \left( \min\{W_\ast - R_u, 1\}, 1 + R_v \right)(\xi)$$

forms a super-solution satisfying

$$\begin{cases} N_3[W_u, W_v] := W''_u + 2\sqrt{1 - a}W'_u + W_u(1 - W_u - aW_v) \leq 0, \text{ a.e. in } \mathbb{R}, \\ N_4[W_u, W_v] := dW''_v + 2\sqrt{1 - a}W'_v + rW_v(1 - W_v - \delta_0 W_u) \geq 0, \text{ a.e. in } \mathbb{R}, \end{cases}$$

for some sufficiently small $\delta_0 > 0$. By some straightforward computations, we have

$$N_3[W_u, W_v] = -R''_u - c^* R'_u - R_u(1 - a - 2W_\ast + R_u - aR_v) - aW_*R_v,$$  

and

$$N_4[W_u, W_v] = dR''_v + c^* R'_v - rR_v(1 + R_u) - \delta_0 r(1 + R_v)(W_\ast - R_u).$$

We consider $(R_u, R_v)(\xi)$ defined as (see Figure 3.4)

$$(R_u, R_v)(\xi) := \begin{cases} (\varepsilon_1 \sigma(\xi)e^{-\lambda_0 \xi}, -\eta_1 e^{-\lambda_1 \xi}), & \text{for } \xi \geq \xi_1 + \delta_1, \\ (\varepsilon_2 e^{\lambda_2 \xi}, -\delta_2), & \text{for } \xi_2 + \delta_2 \leq \xi \leq \xi_1 + \delta_1, \\ (\varepsilon_3 \sin(\delta_3 (\xi - \xi_2)), -\delta_v), & \text{for } \xi_2 - \delta_4 \leq \xi \leq \xi_2 + \delta_2, \\ (-\delta_u, -\delta_v), & \text{for } \xi \leq \xi_2 - \delta_4, \end{cases}$$

where $\lambda_u := \sqrt{1 - a}, \xi_1 > M_0$ and $\xi_2 < -M_0$ are fixed points. Since $|R_u|, |R_v| \ll 1$, up to enlarging $M_0$, for all $\xi \in (-\infty, \xi_2]$, it holds

$$1 - 2W_\ast - a < -1 + a + \rho,$$  

(3.45)
with arbitrarily small \( \rho > 0 \). We also set \( 0 < \lambda_1 < \lambda_u \) satisfies
\[
d\lambda_1^2 - 2\sqrt{1 - a}\lambda_1 - r =: -C_0 < 0, \tag{3.46}
\]
and \( \lambda_2 \) satisfies
\[
\lambda_2^2 + 2\sqrt{1 - a}\lambda_2 - 3 + a =: C_1 > 0. \tag{3.47}
\]
Here \( \varepsilon_{1.2.3} > 0 \) and \( \eta_1 > 0 \) make \( (R_u, R_v) \) continuous on \( \mathbb{R} \), while \( \delta_1, ..., 4 > 0 \) will be determined later such that \( (W_u, W_v) \) satisfies (3.42). Moreover, we set
\[
\delta_u = \varepsilon_3 \sin(\delta_3\delta_4) \quad \text{and} \quad \delta_v = \eta_1 e^{-\lambda_1(\xi_1 + \delta_1)}, \tag{3.48}
\]
which yield \( (R_u, R_v)(\xi) \) are continuous on \( \mathbb{R} \).

Next, we will divide the construction into several steps.

**Step 1:** We consider \( \xi \in [\xi_1 + \delta_1, \infty) \) with \( \xi_1 > M_0 \) and some small \( \delta_1 \) satisfying
\[
0 < \delta_1 < \frac{1}{2(\lambda_2 + \lambda_u)}. \tag{3.49}
\]
In Step 1, we aim to verify that \( (W_u, W_v)(\xi) = (U_\ast - R_u, 2 + R_v)(\xi) \), with
\[
(R_u, R_v)(\xi) = (\varepsilon_1 \sigma(\xi)e^{-\lambda_1\xi}, -\eta_1 e^{-\lambda_1\xi}),
\]
satisfies (3.42) by setting \( \delta_0 \) sufficiently small.

Similar as the construction of \( R_u(\xi) \) for single equation problem, we define
\[
\sigma(\xi) := \frac{4}{\lambda_1^2} e^{-\lambda_1^2(\xi - \xi_1)} - \frac{4}{\lambda_1^2} \lambda_1 \xi - \frac{4}{\lambda_1^2} \xi_1
\]
which satisfies \( \sigma(\xi_1) = 0 \), \( \sigma'(\xi) = \frac{4}{\lambda_1} - \frac{2}{\lambda_1^2} e^{-\lambda_1^2(\xi - \xi_1)} \), \( \sigma''(\xi) = e^{-\lambda_1^2(\xi - \xi_1)} \), and \( \sigma(\xi) = O(\xi) \) as \( \xi \to \infty \). From (3.43), we have
\[
N_3[W_u, W_v] \leq -e^{-\lambda_1^2(\xi - \xi_1)}R_u + R_u(2W_\ast - R_u + aR_v) - aW_uR_v.
\]
Since \( W_\ast(\xi) = O(\xi e^{-\lambda_u\xi}) \) as \( \xi \to \infty \) and \( 0 < \lambda_1 < \lambda_u \), we obtain \( N_3[W_u, W_v] \leq 0 \) for all \( \xi \in [\xi_1 + \delta_1, \infty) \) up to enlarging \( M_0 \) if necessary.
Next, we deal with the inequality of $N_4[W_u, W_v]$. From (3.44) and (3.46), we have

\[ N_4[W_u, W_v] = -C_0 R_v - r R_v - \delta_0 r(1 + R_v) W_u. \]

Since $0 < \lambda_1 < \lambda_u$ and $R_v < 0$, by setting $\delta_0$ sufficiently small, then we have $N_4[W_u, W_v] \geq 0$ for all $\xi \in [\xi_1 + \delta_1, \infty)$ up to enlarging $M_0$ if necessary.

**Step 2:** We consider $\xi \in [\xi_2 + \delta_2, \xi_1 + \delta_1]$ with $\delta_1$ satisfying (3.49). In this case, we have $(R_u, R_v)(\xi) = (\varepsilon_2 e^{\lambda_2 \xi}, -\delta_v)$ with $\lambda_2$ satisfying (3.47) and $\delta_v$ defined as (3.48). It is easy to see that $R_v(\xi)$ is continuous on $\xi = \xi_1 + \delta_1$, and $\angle \alpha_v < 180^\circ$ since $R_v((\xi_1 + \delta_1)^+) > 0 = R_v((\xi_1 + \delta_1)^-)$.

On the other hand, we set

\[ \varepsilon_2 = \frac{\varepsilon_1 \sigma(\xi_1 + \delta_1) e^{-\lambda_u (\xi_1 + \delta_1)}}{e^{\lambda_2 (\xi_1 + \delta_1)}} \]

such that $R_u(\xi)$ is continuous on $\xi = \xi_1 + \delta_1$. Then, by some straightforward computations, we have

\[ R_u'(\xi_1 + \delta_1) = \varepsilon_1 \sigma'(\xi_1 + \delta_1) e^{-\lambda_u (\xi_1 + \delta_1)} - \varepsilon_1 \lambda_u \sigma(\xi_1 + \delta_1) e^{-\lambda_u (\xi_1 + \delta_1)}, \]

\[ R_u'((\xi_1 + \delta_1)^-) = \lambda_2 R_u(\xi_1 + \delta_1). \]

Thus, $R_u'((\xi_1 + \delta_1)^+) > R_u'((\xi_1 + \delta_1)^-)$ is equivalent to

\[ (\lambda_2 + \lambda_u) \sigma(\xi_1 + \delta_1) < \sigma'(\xi_1 + \delta_1), \]

which holds since (3.49) and $\delta_1$ sufficiently small.

From (3.43), (3.47), and $R_v < 0$, we have

\[ N_3[W_u, W_v] \leq -C_1 R_u + aW_u \delta_v. \]

Notice that, we can set $\eta_1 \ll \varepsilon_1$ such that $\delta_v \ll |R_u|$ for all $\xi \in [\xi_2 + \delta_2, \xi_1 + \delta_1]$. Therefore, we have $N_3[W_u, W_v] \leq 0$ for all $\xi \in [\xi_2 + \delta_2, \xi_1 + \delta_1]$. On the other hand, from (3.44) and $R_v < 0$, we have

\[ N_4[W_u, W_v] = r \delta_u (1 - \delta_v) - \delta_0 r(1 - \delta_v)(W_u - R_u). \]

Therefore, up to decreasing $\delta_0$ if necessary, we have $N_4[W_u, W_v] \geq 0$ for all $\xi \in [\xi_2 + \delta_2, \xi_1 + \delta_1]$. Moreover, it is easy to see that $N_4[W_u, W_v] \geq 0$ for all $\xi \in (-\infty, \xi_1 + \delta_1]$ as long as $\delta_0 > 0$ is sufficiently small since $W_u - R_u \leq 1$ in $\mathbb{R}$. Therefore, hereafter it suffices to prove the inequality of $N_3[W_u, W_v]$.

**Step 3:** We consider $\xi \in [\xi_2 - \delta_4, \xi_2 + \delta_2]$ with

\[ \delta_2 > \frac{1}{\lambda_2}. \]  

(3.50)

In this case, we have $(R_u, R_v) = (\varepsilon_3 \sin(\delta_3 (\xi - \xi_2)), -\delta_v)$. We first set

\[ \varepsilon_3 = \frac{\varepsilon_2 e^{\lambda_2 (\xi_2 + \delta_2)}}{\sin(\delta_2 \delta_3)} \]

such that $R_u(\xi)$ is continuous on $\xi = \xi_2 + \delta_2$. Then, by some straightforward computations, we have

\[ R_u'((\xi_2 + \delta_2)^+) = \lambda_2 R_u(\xi_2 + \delta_2) \quad \text{and} \quad R_u'((\xi_2 + \delta_2)^-) = \varepsilon_3 \delta_3 \cos(\delta_2 \delta_3). \]
Thus, from \( \frac{x \cos x}{\sin x} \to 1 \) as \( x \to 0 \), \( R''_u((\xi_2 + \delta_2)^+) > R'_u((\xi_2 + \delta_2)^-) \) (namely, \( \angle \alpha_3 < 180^\circ \)) follows by taking \( \delta_3 \) sufficiently small and \( \delta_2 \) satisfying (3.50).

It suffices to only verify the inequality of \( N_3[W_u,W_v] \). From (3.43), we have

\[
N_3[W_u,W_v] = \delta_2^2 R_u - c^*\delta_3\varepsilon_3 \cos(\delta_3(\xi - \xi_2)) - R_u(1 - a - 2W_s + R_u - aR_v) - aW_sR_v.
\]

For \( \xi \in [\xi_2,\xi_2 + \delta_2] \), we have

\[
N_3[W_u,W_v] \leq (\delta_2^2 + 1 + a)\varepsilon_3 \sin(\delta_3\alpha_3) - c^*\delta_3\varepsilon_3 \cos(\delta_3\alpha_3) - aW_sR_v.
\]

Note that, from \( \frac{x \cos x}{\sin x} \to 1 \) as \( x \to 0 \),

\[
(\delta_2^2 + 1 + a)\varepsilon_3 \sin(\delta_3\alpha_3) - c^*\delta_3\varepsilon_3 \cos(\delta_3\alpha_3) \leq 0
\]

is equivalent to \( \delta_2 < \frac{\delta_3c^*}{\delta_2 + 1 + a} \) which holds since \( \lambda_2 \) in (3.50) can be chosen arbitrarily large. For \( \xi \in [\xi_2 - \alpha_4,\xi_2] \), from \( R_u \leq 0 \) and (3.45), up to enlarging \( M_0 \), we have

\[
N_3[W_u,W_v] \leq -c^*\delta_3\varepsilon_3 \cos(\delta_2\alpha_3) - aW_sR_v.
\]

Then, by setting

\[
0 < \delta_4 < \delta_2 < \frac{\delta_3c^*}{\delta_2^2 + 1 + a}
\]

we have \( N_3[W_u,W_v] \leq 0 \) for all \( \xi \in [\xi_2 - \delta_4,\xi_2 + \delta_2] \) up to decreasing \( \delta_0 \) if necessary.

**Step 4:** We consider \( \xi \in (-\infty,\xi_2 - \delta_4] \). In this case, we have \( (R_u,R_v) = (-\delta_u,-\delta_v) \). From (3.48), \( R_u(\xi) \) is continuous on \( \xi = \xi_2 - \delta_4 \). It is easy to see that \( R'_u((\xi_2 - \delta_4)^+) > 0 = R'_u((\xi_2 - \delta_4)^-) \) and \( \angle \alpha_4 < 180^\circ \). Moreover, from (3.51) and \( \delta_v \ll R_u(\xi_2 + \delta_2) \), we assert that \( \delta_v \ll \delta_u \).

From (3.43) and (3.45), we have

\[
N_3[W_u,W_v] = \delta_u(1 - a - 2W_s - \delta_u + a\delta_v) + aW_s\delta_v \leq 0
\]

since \( \delta_v \ll \delta_u \). The construction of \( (R_u,R_v)(\xi) \) is complete.

**Step 5:** We are ready to complete the proof of Proposition 3.1. From Step 1 to Step 4, we are equipped with a super-solution \( (W_u,W_v)(\xi) \). Next, we consider the Cauchy problem (3.35) with \( b^* = 0 \). By applying the same argument in the proof of Theorem 1.6, we can reach a contradiction and thus \( b^* = 0 \) is impossible. This completes the proof of Proposition 3.1.

\[\Box\]

**A Appendix**

**A.1 Existence of traveling waves for (1.4) under (H)**

**Proposition A.1** Assume that (H) holds. There exists the minimal speed \( c^*_{LV} \in [2\sqrt{1 - a},2] \) such that (1.4) admits a positive solution \( (u,v)(x,t) = (U,V)(x - ct) \) satisfying

\[
\begin{cases}
U'' + cU' + U(1 - U - aV) = 0, \\
dV'' + cV' + rV(1 - V - bU) = 0, \\
(U,V)(-\infty) = \omega, (U,V)(\infty) = (0,1), \\
U' < 0, V' > 0,
\end{cases}
\]

(A.1)
if and only if $c \geq c_{LV}^*$, where

$$
\omega = \begin{cases} 
(1, 0) & \text{if } b \geq 1, \\
(u^*, v^*) := \left( \frac{1 - a}{1 - ab}, \frac{1 - b}{1 - ab} \right) & \text{if } 0 < b < 1.
\end{cases}
$$

Moreover, the minimal traveling wave speed $c_{LV}^*(b)$ is continuous and monotone increasing on $b \in (0, \infty)$.

**Proof.** For the existence of the minimal speed $c_{LV}^*$, it suffices to deal with the case $b = 1$ since the case $b > 1$ and $0 < b < 1$ have been proved in [22] and [25, Example 4.2], respectively.

**Claim A.2** Suppose that, for each $n \in \mathbb{N}$, $(\hat{c}, U_n, V_n)$ is a solution of (A.1) with $b = b_n$ and $b_n \searrow 1$ as $n \to \infty$. Then (A.1) has a monotone solution with $b = 1$ and $c = \hat{c}$.

**Proof of Claim A.2.** First, by translation, we may assume that $V_n(0) = 1/2$ for all $n$. Also, by transferring the equation into integral equations (using variation of constants formula), it is not hard to see that $U_n'$ and $V_n'$ are uniformly bounded. Together with the fact that $0 \leq U_n(\xi), V_n(\xi) \leq 1$ for all $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$, Arzelà-Ascoli Theorem allows us to take a subsequence that converges to a pair of limit functions $(U, V) \in [C(\mathbb{R})]^2$ with $0 \leq U, V \leq 1$, locally uniformly in $\mathbb{R}$. Moreover, using Lebesgue’s dominated convergence theorem to integral equations, we can conclude that $(\hat{c}, U, V)$ satisfies (A.1) with $b = 1$ (since $b_n \searrow 1$). Moreover, we can see from the equations satisfied by $U$ and $V$ that $(U, V) \in [C^2(\mathbb{R})]^2$ and $U' \leq 0$ and $V' \geq 0$ (since $U_n' \leq 0$ and $V_n' \geq 0$ for all $n$), which implies that $(U, V)(\pm \infty)$ exists.

It remains to show that

$$(U, V)(-\infty) = (1, 0), \quad (U, V)(+\infty) = (0, 1). \quad (A.2)$$

Note that we must have

$$U(\pm \infty)[1 - U(\pm \infty) - aV(\pm \infty)] = 0, \quad V(\pm \infty)[1 - V(\pm \infty) - U(\pm \infty)] = 0. \quad (A.3)$$

Hence, $U(\pm \infty), V(\pm \infty) \in \{0, 1\}$. Since $V_n(0) = 1/2$ for all $n$, we have $V(0) = 1/2$ and thus

$$V(-\infty) = 0, \quad V(+\infty) = 1. \quad (A.4)$$

Also, note that from (A.3) we see that $V(+\infty) = 1$ implies that

$$(U(+\infty) = 0. \quad (A.5)$$

If $U(-\infty) = 0$, then $U \equiv 0$ due to $U' \leq 0$. However, by integrating the equation of $V$ over $(\infty, +\infty)$, it follows that

$$\hat{c} + r \int_{-\infty}^{\infty} V(\xi)(1 - V(\xi))d\xi = 0,$$

which implies that $\hat{c} < 0$. This contradicts with $\hat{c} > 0$ (more precisely, from [22] we see that $2\sqrt{T - a} \leq \hat{c} \leq 2$). As a result, we have $U(-\infty) = 1$, which together with (A.4) and (A.5) implies (A.2). We, therefore, obtain a monotone solution with $b = 1$ and $c = \hat{c}$. \qed
Let us define
\[ c_{LV}^i := \min\{\hat{c} > 0 \mid \text{(A.1) has a solution with } c = \hat{c}\}. \]

We write \( c_{LV}^i = c_{LV}^i(b) \) to emphasize the dependence of \( c_{LV}^i \) on \( b \). It follows from [22] and [25, Example 4.2] that \( c_{LV}^i(b) \) is well defined for all \( b > 0 \) except \( b = 1 \). We next prove the existence of \( c_{LV}^i(1) \).

Next, we show \( c_{LV}^i(1) \) is well defined and \( c_{LV}^i(1) = \lim_{b \to 1^+} c_{LV}^i(b) \). Note that \( c_{LV}^i(b) \) is continuous for \( b \in (1, \infty) \) (see [22]). Also, by simple comparison argument, we see that \( c_{LV}^i(b) \) is monotone increasing. Therefore, for any fixed \( \hat{b} > 1 \), we conclude that for each \( c \geq c_{LV}^i(\hat{b}) \), \( (A.1) \) has a monotone solution for each \( b \in (1, \hat{b}) \). Together with the conclusion of Claim A.2, we see that for any \( c \geq c_{LV}^i(\hat{b}) \), \( (A.1) \) has a monotone solution with \( b = 1 \). Therefore, we conclude that for any \( c > \lim_{b \to 1^+} c_{LV}^i(\hat{b}) =: \hat{c} \), \( (A.1) \) has a monotone solution with \( b = 1 \). This also shows that \( c_{LV}^i(1) \) is well defined. In fact, \( (A.1) \) still admits a monotone solution with \( b = 1 \) for \( c = \hat{c} \). For this, we can consider \( \{c_i, U_i, V_i\} \) as a sequence of monotone solutions of \( (A.1) \) such that \( c_i \searrow \hat{c} \) as \( i \to \infty \).

Using a similar argument as in Claim 1, we can show that there is a subsequence that converges to a monotone solution of \( (A.1) \) with \( b = 1 \) and \( c = \hat{c} \).

Next, we show that \( c_{LV}^i(1) = \hat{c} \). If \( \hat{c} = 2\sqrt{1-a} \), then we immediately obtain \( c_{LV}^i(1) = \hat{c} = 2\sqrt{1-a} \). If \( \hat{c} > 2\sqrt{1-a} \), we use a contradiction argument and assume that \( c_{LV}^i(1) < \hat{c} \). Then there exists \( \hat{c} \in (2\sqrt{1-a}, \hat{c}) \) such that \( (A.1) \) admits a monotone solution with \( b = 1 \) for \( c = \hat{c} \).

Applying the continuation method (see [22]), we can conclude that \( (A.1) \) has a monotone solution with \( |b - 1| < \delta \) and \( |c - \hat{c}| < \delta \), for some sufficiently small \( \delta > 0 \) such that \( \hat{c} + \delta < c_{LV}^i(1 + \delta) \). However, this contradicts the definition of \( c_{LV}^i(1 + \delta) \). Thus, we must have \( c_{LV}^i(1) = \hat{c} \) and the proof of Proposition A.1 is complete.

## A.2 Asymptotic behavior of \((U_c, V_c)\) near \(\pm \infty\) for \(0 < a < 1\) and \(b > 0\)

In this subsection, we provide the asymptotic behavior of \((U_c, V_c)\) near \(\pm \infty\) for \(0 < a < 1\) and \(b > 0\), where \((U_c, V_c)\) satisfies either (1.14) or (1.15) with speed \(c\). Some results are reported in [29]. We further establish an implicit condition that determines the asymptotic behavior of the pushed transition front \(U_{c_{LV}}\) at \(+\infty\).

Hereafter, for simplicity, we write \((U_c, V_c)\) as \((U, V)\). To describe the asymptotic behavior of \((U, V)\) near \(+\infty\), we define
\[
\lambda_\pm^-(c) := \frac{-c \mp \sqrt{c^2 - 4(1-a)}}{2} < 0,
\]
\[
\lambda_\pm^+(c) := \frac{-c \pm \sqrt{c^2 + 4rd}}{2d} \quad < 0 \quad \text{for } \lambda_\pm^+(c) := \frac{-c + \sqrt{c^2 + 4rd}}{2d}.
\]

The asymptotic behavior of \((U, V)\) near \(+\infty\) for \(0 < a < 1\) and \(b > 1\) can be found in [29]. Note that the conclusions presented in [29] are still applicable for \(b > 0\) since \(b\) is not present in the linearization at the unstable equilibrium \((0, 1)\). Therefore, we have the following result.

**Lemma A.3** ([29]) Assume that \(0 < a < 1\) and \(b > 0\). Let \((c, U, V)\) be a solution of the system (1.14). Then there exist positive constants \(\ell_i = 1, \ldots, s\) such that the following hold:
(i) For $c > 2\sqrt{1-a}$,

\[
\lim_{\xi \to +\infty} \frac{U(\xi)}{e^{\Lambda(c)\xi}} = l_1,
\]

\[
\lim_{\xi \to +\infty} \frac{1 - V(\xi)}{e^{\Lambda(c)\xi}} = l_2 \quad \text{if } \lambda_v^+(c) < \Lambda(c),
\]

\[
\lim_{\xi \to +\infty} \frac{1 - V(\xi)}{\xi e^{\lambda_v^+(c)\xi}} = l_3 \quad \text{if } \lambda_v^+(c) = \Lambda(c),
\]

\[
\lim_{\xi \to +\infty} \frac{1 - V(\xi)}{e^{\lambda_v^+(c)\xi}} = l_4 \quad \text{if } \lambda_v^+(c) > \Lambda(c),
\]

where $\Lambda(c) \in \{\lambda_v^+(c), \lambda_u^+(c)\}$.

(ii) For $c = 2\sqrt{1-a}$,

\[
\lim_{\xi \to +\infty} \frac{U(\xi)}{\xi p e^{\Lambda(c)\xi}} = l_5,
\]

\[
\lim_{\xi \to +\infty} \frac{1 - V(\xi)}{\xi p e^{\Lambda(c)\xi}} = l_6 \quad \text{if } \lambda_v^-(c) < \Lambda(c),
\]

\[
\lim_{\xi \to +\infty} \frac{1 - V(\xi)}{\xi p e^{\Lambda(c)\xi}} = l_7 \quad \text{if } \lambda_v^-(c) = \Lambda(c),
\]

\[
\lim_{\xi \to +\infty} \frac{1 - V(\xi)}{\xi e^{\lambda_v^-(c)\xi}} = l_8 \quad \text{if } \lambda_v^-(c) > \Lambda(c),
\]

where $\Lambda(c) = \lambda_v^+(c) = -\sqrt{1-a}$ and $p \in \{0, 1\}$.

**Remark A.4** In Lemma A.3, if $c = c_{LV}^* > 2\sqrt{1-a}$ (pushed front case), then from [8, Lemma 2.3] we have $\Lambda(c) = \lambda_u^-(c)$.

When $c = 2\sqrt{1-a}$, it is not clear whether $p = 0$ or $p = 1$. By applying a similar argument used in [14] that considered the discrete version of (1.4), we can derive an implicit criterion for determining whether $p = 0$ or $p = 1$, which is given in the following proposition.

**Proposition A.5** Assume that $0 < a < 1$ and $b > 0$. Let $(c, U, V)$ be a solution of (1.14) with $c = 2\sqrt{1-a}$ and $p$ be given in (ii) of Lemma A.3. Then

\[
p = \begin{cases} 
1 & \text{if and only if } \int_{-\infty}^{\infty} e^{-\Lambda(c)\xi} U(\xi)[a(1 - V(\xi)) - U(\xi)]d\xi \neq 0, \\
0 & \text{if and only if } \int_{-\infty}^{\infty} e^{-\Lambda(c)\xi} U(\xi)[a(1 - V(\xi)) - U(\xi)]d\xi = 0,
\end{cases}
\]

(A.6)

where $\Lambda(c) = \lambda_u^+(c) = -\sqrt{1-a}$.

**Proof.** In fact, modifying the process used in [14], we can prove Lemma A.3 and (A.6) independently, however the proof is quite long. Instead of giving the detailed proof, we here simply assume that Lemma A.3 hold and derive (A.6). For this, let us recall modified Ikehara’s Theorem (see [7]).

For a positive non-increasing function $U$, we define

\[
F(\lambda) := \int_{0}^{+\infty} e^{-\lambda \xi} U(\xi) d\xi, \quad \lambda \in \mathbb{C} \text{ with } \text{Re}\lambda < 0.
\]
If $F$ can be written as $F(\lambda) = H(\lambda)/(\lambda + \gamma)^{p+1}$ for some constants $p > -1, \gamma > 0$, and some analytic function $H$ in the strip $-\gamma \leq \text{Re}\lambda < 0$, then

$$\lim_{\xi \to +\infty} \frac{U(\xi)}{\xi^p e^{-\gamma \xi}} = \frac{H(-\gamma)}{\Gamma(\gamma + 1)}.$$ 

Let us define the bilateral Laplace transform of $U$ as

$$\mathcal{L}(\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda \xi} U(\xi) d\xi;$$

which is well-defined for $\Lambda(c) < \text{Re}\lambda < 0$ (since we have assumed that Lemma A.3 holds). Using the equation of $U$ and integration by parts several times, we have

$$\Phi(\lambda) \mathcal{L}(\lambda) + I(\lambda) = 0, \quad \Lambda(c) < \text{Re}\lambda < 0,$$ 

(A.7)

where

$$\Phi(\lambda) := c\lambda + \lambda^2 + 1 - a, \quad I(\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda \xi} U[a(1 - V) - U](\xi) d\xi.$$ 

To apply Ikehara’s Theorem, we rewrite (A.7) as

$$F(\lambda) := \int_{0}^{+\infty} e^{-\lambda \xi} U(\xi) d\xi = -\frac{I(\lambda)}{\Phi(\lambda)} - \int_{-\infty}^{0} e^{-\lambda \xi} U(\xi) d\xi;$$

as long as $\Phi(\lambda)$ does not vanish. Also, we define

$$H(\lambda) := Q(\lambda) - (\lambda - \Lambda(c))^{p+1} \int_{-\infty}^{0} e^{-\lambda \xi} U(\xi) d\xi,$$ 

(A.8)

where $\Lambda(c) = -\sqrt{1 - a}, p \in \mathbb{N} \cup \{0\}$, and

$$Q(\lambda) := -\frac{I(\lambda)}{\Phi(\lambda)/[\lambda - \Lambda(c)]^{p+1}}.$$ 

(A.9)

We now prove that $H$ is analytic in the strip $S := \{\Lambda(c) \leq \text{Re}\lambda < 0\}$. Since the second term on the right-hand side of (A.8) is always analytic for $\text{Re}\lambda < 0$, it suffices to show that $Q$ is analytic in the strip $S$. Since $\mathcal{L}$ is well-defined for $\Lambda(c) < \text{Re}\lambda < 0$, we see that $Q$ is analytic for $\Lambda(c) < \text{Re}\lambda < 0$. Therefore, it suffices to prove the analyticity of $Q$ on $\{\text{Re}\lambda = \Lambda(c)\}$. For this, we claim that the only root of $\Phi(\lambda) = 0$ is the real root $\lambda = \Lambda(c)$. To see this, let $\lambda = \alpha + \beta i$ for $\alpha, \beta \in \mathbb{R}$ and $i := \sqrt{-1}$. If $\Phi(\alpha + \beta i) = 0$, then by simple calculations we see that $\beta = 0$ and $\alpha = \Lambda(c)$. Therefore, from (A.9) we see that $Q$ is analytic on $\{\text{Re}\lambda = \Lambda(c)\}$ and is also analytic in $S$. Then, Ikehara’s Theorem can be applied to assert that

$$\lim_{\xi \to +\infty} \frac{U(\xi)}{\xi^p e^{-\Lambda(c)\xi}} = \frac{H(\Lambda(c))}{\Gamma(-\Lambda(c) + 1)} = \frac{Q(\Lambda(c))}{\Gamma(-\Lambda(c) + 1)}.$$ 

Finally, we need to prove $Q(\Lambda(c)) \neq 0$ by taking suitable $p$. To do so, note that (A.9) and the fact that $\Phi(\lambda) = 0$ imply that $\lambda = \Lambda(c)$. We see that, if $I(\Lambda(c)) \neq 0$, then $Q(\Lambda(c)) \neq 0$ if and only if $p = 1$. On the other hand, when $I(\Lambda(c)) = 0$, then $\lambda = \Lambda(c)$ must be simple root of $I(\lambda) = 0$. Otherwise, we have $Q(\Lambda(c)) = 0$ for any $p \in \mathbb{N} \cup \{0\}$, which contradicts the conclusion (ii) of Lemma A.3. Therefore, when $I(\Lambda(c)) = 0$, we have $Q(\Lambda(c)) \neq 0$ if and only if $p = 0$, so (A.6) holds. This completes the proof. \hfill \square
To describe the asymptotic behavior of \((U, V)\) near \(-\infty\), we define
\[
\mu_u^-(c) := \frac{-c - \sqrt{c^2 + 4}}{2}, \quad \mu_u^+(c) := \frac{-c + \sqrt{c^2 + 4}}{2},
\]
\[
\mu_v^-(c) := \frac{-c - \sqrt{c^2 + 4rd(b - 1)}}{2d}, \quad \mu_v^+(c) := \frac{-c + \sqrt{c^2 + 4rd(b - 1)}}{2d}.
\]

**Lemma A.6** ([29]) Assume that \(0 < a < 1 \) and \(b > 1\). Let \((c, U, V)\) be a solution of the system (1.14). Then there exist two positive constants \(l_{i=9,\ldots,12}\) such that
\[
\lim_{\xi \to -\infty} \frac{V(\xi)}{e^{\mu_u^+(c)\xi}} = l_9,
\]
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{e^{\mu_v^+(c)\xi}} = l_{10} \quad \text{if } \mu_u^+(c) > \mu_v^+(c),
\]
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{\xi e^{\mu_u^+(c)\xi}} = l_{11} \quad \text{if } \mu_u^+(c) = \mu_v^+(c),
\]
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{e^{\mu_u^+(c)\xi}} = l_{12} \quad \text{if } \mu_u^+(c) < \mu_v^+(c).
\]

**Lemma A.7** Assume that \(0 < a, b < 1\). Let \((c, U, V)\) be a solution of the system (1.14). Then there exist two positive constants \(l_{13}\) and \(l_{14}\) such that
\[
\lim_{\xi \to -\infty} \frac{u^* - U(\xi)}{e^{\nu\xi}} = l_{13}, \quad \lim_{\xi \to -\infty} \frac{V(\xi) - v^*}{e^{\nu\xi}} = l_{14}
\]
where \(\nu\) is the smallest positive zero of
\[
\rho(\lambda) := (\lambda^2 + c\lambda - u^*)(d\lambda^2 + c\lambda - rv^*) - rabu^*v^*.
\] (A.10)

**Proof.** Set \(g_u(\lambda) := \lambda^2 + c\lambda - u^*\) and \(g_v(\lambda) := d\lambda^2 + c\lambda - rv^*\). Then \(g_u\) (resp., \(g_v\)) has two zeros \(\mu_u^+\) (resp., \(\mu_v^+\)) with \(\mu_u^+ < 0 < \mu_v^+\) (resp., \(\mu_u^- < 0 < \mu_v^-\)). More precisely, we have
\[
\mu_u^+ = \frac{-c + \sqrt{c^2 + 4u^*}}{2}, \quad \mu_v^+ = \frac{-c - \sqrt{c^2 + 4rdv^*}}{2d}.
\]
Note that \(\rho(\lambda) = g_u(\lambda)g_v(\lambda) - rabu^*v^*\). Since \(\rho(\pm\infty) = +\infty\), \(\rho(\mu_u^+) < 0\), \(\rho(\mu_v^+ < 0, \rho(0) = ru^*v^*(1 -hk) > 0\), we see that \(\rho\) has exactly four distinct real zeros \(\lambda = \nu_i (i = 1, 2, 3, 4)\), two negative and two positive zeros, such that
\[
\nu_4 < \min\{\mu_u^-, \mu_v^-, \mu_u^+, \mu_v^+\} \leq \max\{\mu_u^-, \mu_v^-, \mu_u^+, \mu_v^+\} < \nu_3 < 0 < \nu_2 < \min\{\mu_u^+, \mu_v^+\} \leq \max\{\mu_u^-, \mu_v^-, \mu_u^+, \mu_v^+\} < \nu_1.
\]
Set \(P = U'\) and \(Q = V'\). Then from (1.4), we have
\[
U' = P, \quad P' = -cP - U(1 - U - aV), \quad V' = Q, \quad Q' = -\frac{c}{d}Q - \frac{r}{d}V(1 - V - bU).
\] (A.11)
Linearizing (A.11) at \((U, P, V, Q) = (u^*, 0, v^*, 0)\) yields that \(Y' = JY\), where \(Y = (Y_1, Y_2, Y_3, Y_4)^T\) and
\[
J := \begin{pmatrix}
0 & 1 & 0 & 0 \\
-u^* & -c & au^* & 0 \\
0 & 0 & 0 & 1 \\
-\frac{rb}{d}v^* & 0 & \frac{r}{d}v^* & -\frac{c}{d}
\end{pmatrix}.
\]
Using cofactor expansions, one has \( \det(J - \lambda I) = \rho(\lambda) \), where \( \rho(\lambda) \) is defined in (A.10). Hence, \( J \) has four distinct real eigenvalues \( \nu_4 < \nu_3 < 0 < \nu_2 < \nu_1 \). By straightforward calculations, for each eigenvalue \( \nu_i \), the corresponding eigenvector \( w_i \) is given by

\[
 w_i := \left( 1, \nu_i, \frac{g_u(\nu_i)}{au^*}, \nu_i \frac{g_u(\nu_i)}{au^*} \right)^T, \quad i = 1, 2, 3, 4.
\]

Therefore, the general solution of \( \mathbf{Y}' = J \mathbf{Y} \) with \( \mathbf{Y}(-\infty) = 0 \) is given by \( \mathbf{Y}(\xi) = \sum_{i=1}^{2} K_i e^{\nu_i \xi} w_i \) for some constants \( K_i \in \mathbb{R}, i = 1, 2 \). By standard ODE theory, as \( \xi \to -\infty \),

\[
 \begin{pmatrix}
 U(\xi) \\
 U'(\xi) \\
 V(\xi) \\
 V'(\xi)
\end{pmatrix} =
\begin{pmatrix}
 u^* + K_1 e^{\nu_1 \xi} + K_2 e^{\nu_2 \xi} \\
 K_1 \nu_1 e^{\nu_1 \xi} + K_2 \nu_2 e^{\nu_2 \xi} \\
 v^* + K_1 \frac{g_u(\nu_1)}{au^*} e^{\nu_1 \xi} + K_2 \frac{g_u(\nu_2)}{au^*} e^{\nu_2 \xi} \\
 K_1 \nu_1 \frac{g_u(\nu_1)}{au^*} e^{\nu_1 \xi} + K_2 \nu_2 \frac{g_u(\nu_2)}{au^*} e^{\nu_2 \xi}
\end{pmatrix} + h.o.t. \quad \text{(A.12)}
\]

Clearly, \( K_1^2 + K_2^2 \neq 0 \). If \( K_2 = 0 \), then \( K_1 \neq 0 \) and it follows from (A.12) that \( U'(\xi) \sim K_1 \nu_1 e^{\nu_1 \xi} \) and \( V'(\xi) \sim K_1 \nu_1 \frac{g_u(\nu_1)}{au^*} e^{\nu_1 \xi} \). Since \( \nu_1 > \max \{ \mu_1^+, \mu_2^+ \} \), we see that \( g_1(\nu_1) > 0 \). This implies that \( U' \) and \( V' \) have the same sign as \( \xi \to -\infty \), which is impossible since \( U' < 0 \) and \( V' > 0 \) in \( \mathbb{R} \). Therefore, we obtain \( K_2 \neq 0 \). Moreover, we have \( K_2 < 0 \) due to the monotonicity of \( U \) and \( V \). The proof is thus complete by taking \( \nu = \nu_2, l_{13} = -K_2 \) and \( l_{14} = K_2 g_u(\nu_2)/au^* \).

For the strong-weak competition case \((b > 1)\) (resp., the weak competition case \((b < 1)\)), Lemma A.6 and Lemma A.7 show that \((U, V)(\xi)\) converges to \((1, 0)\) (resp., \((u^*, v^*)\)) exponentially as \( \xi \to -\infty \). However, in the critical case \((b = 1)\), the convergence rates may be of polynomial orders due to the degeneracy of the principal eigenvalue.

We now apply the center manifold theory to establish the decay rate of \( U \) and \( V \) at \( \xi = -\infty \) when \( b = 1 \). Let \( W(\xi) = 1 - U(\xi) \). Then by simple calculations, \((W, V)\) satisfies

\[
\begin{align*}
 W'' + cW' - (1 - W)(W - aV) &= 0, & \xi \in \mathbb{R}, \\
 dV'' + cV' + rV(W - V) &= 0, & \xi \in \mathbb{R}, \\
 (W, V)(-\infty) &= (0, 0), & (W, V)(+\infty) &= (1, 1) 
\end{align*}
\]

To reduce (A.13) to first-order ODEs, we introduce

\[
 X_1(\xi) = V(\xi), \quad X_2(\xi) = V'(\xi), \quad X_3(\xi) = W(\xi), \quad X_4(\xi) = W'(\xi).
\]

Then \( X := (X_1, X_2, X_3, X_4)(\xi) \) satisfies \( X' = G(X) \), which is described as

\[
\begin{align*}
 X_1' &= X_2, & \xi \in \mathbb{R}, \\
 X_2' &= -\frac{c}{d} X_2 - \frac{r}{d} X_1 (X_3 - X_1), & \xi \in \mathbb{R}, \\
 X_3' &= X_4, & \xi \in \mathbb{R}, \\
 X_4' &= -c X_4 + (1 - X_3)(X_3 - a X_1), & \xi \in \mathbb{R},
\end{align*}
\]

By linearizing (A.14) at \((0, 0, 0, 0)\), we obtain \( \mathbf{Y}' = J \mathbf{Y} \), where \( \mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)^T \) and

\[
 J := \begin{pmatrix}
 0 & 1 & 0 & 0 \\
 0 & -\frac{c}{d} & 0 & 0 \\
 0 & 0 & 1 & 1 \\
 -a & 0 & 1 & -c
\end{pmatrix}.
\]

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It is easy to calculate that $J$ has four eigenvalues

$$
\mu_1 = 0, \quad \mu_2 = -\frac{c}{d}, \quad \mu_3 := \frac{-c - \sqrt{c^2 + 4}}{2} < 0, \quad \mu_4 := \frac{-c + \sqrt{c^2 + 4}}{2} > 0,
$$

and the corresponding eigenvector $v_i$ with respect to $\mu_i$ is given by

$$
v_1 = (1, 0, a, 0)^T, \quad v_2 = \left(\omega, -\frac{c}{d}\omega, -ad, ac\right)^T, \quad v_3 = (0, 0, 1, \mu_3)^T, \quad v_4 = (0, 0, 1, \mu_4)^T,
$$

where

$$
\omega := -d - c^2 + \frac{c^2}{d}.
$$

To reduce (A.14) into the normal form, we set $Z := Q^{-1}X$, where $Z := (Z_1, Z_2, Z_3, Z_4)^T$ and $Q := (v_1 \ v_2 \ v_3 \ v_4)^T \in \mathbb{R}^{4 \times 4}$. By some tedious computations, we have

$$
\begin{align*}
X_1 &= Z_1 + \omega Z_2, & X_2 &= -\frac{c}{d}\omega Z_2, \\
X_3 &= aZ_1 - adZ_2 + Z_3 + Z_4, & X_4 &= acZ_2 + \mu_3 Z_3 + \mu_4 Z_4,
\end{align*}
$$

and

$$
Q^{-1} := \begin{pmatrix}
1 & \frac{\mu_4}{\mu_3 - \mu_4} & \frac{\mu_3}{\mu_4} & 0 \\
0 & -\frac{\mu_3}{\mu_4} & -\frac{\mu_4}{\mu_3} & 0 \\
\frac{\mu_4}{\mu_3 - \mu_4} & \frac{\mu_3}{\mu_4} & \frac{1}{\mu_3 - \mu_4} & 0 \\
\frac{\mu_3}{\mu_3 - \mu_4} & \frac{1}{\mu_4} & \frac{\mu_3}{\mu_4} & \frac{1}{\mu_3 - \mu_4}
\end{pmatrix},
$$

where $\omega$ is defined in (A.15). By (A.14), (A.16) and (A.17), some tedious computations yield that

$$
\begin{align*}
Z'_1 &= g_1(Z), \\
Z'_2 &= -\frac{c}{d}Z_2 + g_2(Z), \\
Z'_3 &= \mu_3 Z_3 + g_3(Z), \\
Z'_4 &= \mu_4 Z_4 + g_4(Z),
\end{align*}
$$

where

\begin{align*}
g_1(Z) &:= -r \frac{Z_1 + \omega Z_2}{\omega c} h_1(Z), & g_2(Z) &:= \frac{r}{\omega c} (Z_1 + \omega Z_2) h_1(Z), \\
g_3(Z) &:= -g_2 r \frac{Z_1 + \omega Z_2}{d} h_1(Z) + q_3 h_2(z) h_3(z), \\
g_4(Z) &:= -g_2 r \frac{Z_1 + \omega Z_2}{d} h_1(Z) + q_4 h_2(z) h_3(z), \\
h_1(Z) &:= (a - 1) Z_1 - (\omega + ad) Z_2 + Z_3 + Z_4, \\
h_2(Z) &:= a Z_1 - ad Z_2 + Z_3 + Z_4, \\
h_3(Z) &:= a (\omega + d) Z_2 - Z_3 - Z_4.
\end{align*}

Here $g_{ij}$ is defined as the $i, j$ entry of the matrix $Q^{-1}$. Note from the definition of $g_i$ and $h_i$, we see that $g_i$ does no linear term of $Z_i$ for $i = 1, 2, 3, 4$, and thus

$$
g_i(0) = 0, \quad Dg_i(0) = 0, \quad i = 1, 2, 3, 4.
$$
Therefore, we can apply the center manifold theory (see [40, Chapter 18]) to conclude that there exists a one-dimensional center manifold for (A.18), and \(Z_i, i = 2, 3, 4\) can be represented by a smooth function \(Z_i = H_i(Z_1), i = 2, 3, 4\), for small \(Z_1\). We assume that
\[
H_i(Z_1) = C_i Z_i^2 + o(|Z_1|^2), \quad i = 2, 3, 4,
\]
for some \(C_i \in \mathbb{R}\). Indeed, \(C_i\) is determined such that
\[
H'_i(Z_1)g_i(Z) - \left[ - \frac{c}{d} Z_2 + g_2(Z) \right] = o(|Z_1|^2), \quad (A.19)
\]
\[
H'_3(Z_1)g_1(Z) - (\mu_3 Z_3 + g_3(Z)) = o(|Z_1|^2), \quad (A.20)
\]
\[
H'_4(Z_1)g_1(Z) - (\mu_4 Z_4 + g_4(Z)) = o(|Z_1|^2). \quad (A.21)
\]
By comparing the coefficients in front of \(Z_i^2\) on the both sides of (A.19), we need \(C_2 = - \frac{r_2}{\omega c} (1 - a)\). Also, from (A.20) and (A.21), with some tedious computations, we see that \(C_3 = C_4 = 0\). Moreover, the flow on the center manifold is defined by
\[
Z'_1 = g_1(Z_1, H_2(Z_1), H_3(Z_1), H_4(Z_1)) = \frac{r}{c} (1 - a) Z_1^2 + o(|Z_1|^2),
\]
for sufficiently small \(Z_1(\xi)\), which implies that
\[
Z_1(\xi) = - \frac{c}{r(1 - a)} \xi^{-1} + o(|\xi|^{-1}) \quad \text{as} \quad \xi \to -\infty.
\]
Therefore, the center manifold theory yields that if \(0 < Z_1(\xi) \ll 1\), we have
\[
Z_1(\xi) \sim - \frac{c}{r(1 - a)} \xi^{-1}, \quad Z_2(\xi) \sim - \frac{d}{r \omega (1 - a)} \xi^{-2} \quad \text{as} \quad \xi \to -\infty.
\]
Therefore, in view of (A.16) and the definition of \(X_i\), together with the fact that \(0 < U, V < 1\) in \(\mathbb{R}\), we see that there exists \(l_{15} > 0\) such that
\[
\lim_{\xi \to -\infty} \frac{V(\xi)}{\xi^{-1}} = l_{15}, \quad \lim_{\xi \to -\infty} \frac{1 - U(\xi)}{\xi^{-1}} = a l_{15}, \quad (A.22)
\]
Furthermore, it holds that
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{V(\xi)} = a < 1. \quad (A.23)
\]
Combining (A.22) and (A.23), we have the following result.

**Lemma A.8** Assume that \(0 < a < 1\) and \(b = 1\). Let \((c, U, V)\) be a solution of the system (1.14). Then there exist two positive constants \(l_{15}\) such that
\[
\lim_{\xi \to -\infty} \frac{V(\xi)}{\xi^{-1}} = l_{15}, \quad \lim_{\xi \to -\infty} \frac{1 - U(\xi)}{\xi^{-1}} = a l_{15}, \quad \lim_{\xi \to -\infty} \frac{1 - U(\xi)}{V(\xi)} = a < 1.
\]
Hence, we immediately obtain a Lemma as follows:

Thanks to Lemma A.6, Lemma A.7 and Lemma A.8, we immediately obtain
Corollary A.9 Assume that $0 < a < 1$ and $b > 0$. Let $(U, V)$ be a solution of the system (1.14) with speed $c$. Then it holds that

$$1 - U(\xi) - aV(\xi) = o(|\xi|^{-1}).$$

In particular, for the case $b = 1$, there exists $\xi_0$ near $-\infty$ such that $(1 - U - V)(\xi) < 0$ for all $\xi \in (-\infty, \xi_0]$.

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