Massive IIA String Theory and Matrix Theory Compactification

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Abstract

We propose a Matrix Theory approach to Romans’ massive Type IIA supergravity. It is obtained by applying the procedure of Matrix Theory compactifications to Hull’s proposal of the Massive Type IIA String Theory as M-Theory on a twisted torus. The resulting Matrix Theory is a super-Yang Mills theory on large $N$ three-branes with a space dependent non-commutativity parameter, which is also independently derived by a T-duality approach. We give evidence showing that the energies of a class of physical excitations of the super-Yang Mills theory show the correct symmetry expected from Massive Type IIA string theory in a lightcone quantization.
1 Introduction

Sometime ago Romans [1] found a massive deformation of ten-dimensional Type IIA supergravity. This ten-dimensional theory has remained something of a mystery from the string theory viewpoint. Polchinski [2] argued this supergravity theory should lift to a massive Type IIA string theory, corresponding to ordinary Type IIA string theory in the background of a constant 10-form Ramond-Ramond field strength.

The problem of lifting this theory into M-theory has been considered by a number of authors, including Hull [3]. His proposal is similar in spirit to the idea of [4, 5] of obtaining ten-dimensional Type IIB by compactifying M-theory on a 2-torus of vanishing size but fixed complex structure. Instead one considers M-theory compactified on a 2-torus bundle over $S^1$, $B(A, R)$, and takes the limit of zero size. We review this construction in detail in section 2.

We propose a nonperturbative formulation of M-theory in this background using Matrix theory techniques. In section 3 we generalize the construction of Seiberg and Sen [6, 7], which provides us with a formulation of the discretized light-cone quantization (DLCQ) of M-theory on the twisted torus. The end result is a decoupled system of D3-branes in a background $B_{\mu \nu}$ field, which we represent as a noncommutative Yang-Mills theory with 8 linearly realized supercharges. An important novel feature of our construction is the space dependent noncommutativity parameter $\theta$.

In section 4, we construct the noncommutative Yang-Mills degrees of freedom directly by compactifying an infinite system of D0-branes on the twisted torus, following the general Matrix theory [8, 9] procedure of [8, 10, 11, 12, 13]. Since we do not have the full zero-brane action in the original curved background we cannot proceed to derive the super-Yang-Mills theory as in the commutative case or the case with constant $\theta$. Nevertheless we obtain some useful information about the nature of fields in the theory. Concretely we give a construction of the covariant derivatives acting on an appropriate space of fields, and obeying the compactification constraints of the twisted torus.

In section 5, we take advantage of known results about the star products in the presence of space-dependent non-commutativity and the result of section 3 concerning the emergence of space dependent non-commutative Yang-Mills in a generalized Sen-Seiberg limit in order to elaborate on the form of the action.

The spectrum of states for a D8-brane background of massive Type IIA is examined in section 6 and we provide evidence for an $SO(7)$ invariant spectrum of states, as expected for DLCQ string theory in this background. This provides further evidence supporting the Matrix formulation of the DLCQ string theory. In section 7 we consider a holographic dual spacetime to the noncommutative gauge theory, generalizing [14, 15], and we end with conclusions and discussion in section 8. We comment on the extension of these Matrix compactification methods to other massive reductions of M Theory which admit de Sitter space solutions.
2 Review of Hull’s duality

It is unknown how to lift the massive Type IIA string theory \[3\] and its D8 background solution directly into M theory. While M theory does not seem to admit a cosmological constant, a direct lifting of Romans’ massive ten-dimensional supergravity \[1\] would yield an eleven-dimensional cosmological constant. One possible way around this is to obtain the ten-dimensional mass via a generalized Scherk-Schwarz reduction on a circle. The standard implementation of such a reduction requires a global symmetry in eleven dimensions. The action of the eleven dimensional supergravity does not have such a symmetry but the equations of motion do have a scaling symmetry, which was exploited in \[16\] to reduce to a massive ten-dimensional supergravity. However, one obtains not Romans’ massive supergravity but a different supergravity in ten dimensions. That massive supergravity can also be obtained as a usual reduction of a modified M theory \[17\].

Hull \[3\] was able to embed the massive supergravity \[1\] and the D8 background in M theory by introducing two extra T dualities, one of which is a “massive T duality” as defined in \[18\]. Let us describe this in detail. Scherk-Schwarz reduction is a mechanism for generating masses by compactification in the presence of a global symmetry

\[
\phi(x^\mu, y) = g_y(\phi(x^\mu)) \tag{1}
\]

For the simplest case, of a \(U(1)\) invariance, we can write \(\phi(x, y) = e^{2\pi i q m y} \phi(x)\), and obtain a mass \(q m\) for \(\phi(x)\).

In \[18\] it was shown that the Scherk-Schwarz reduction of 10d IIB supergravity, using a \(U(1)\) subgroup of the \(SL(2, \mathbb{R})\) global invariance is T dual to massive IIA supergravity, using a modified set of “massive T duality” rules. The reduction is given by

\[
g_y = \begin{pmatrix} 1 & \frac{m y}{R} \\ 0 & 1 \end{pmatrix} \tag{2}
\]

which implies

\[
\tau(x, y) \equiv a + i e^{-\phi} = \tau(x) + \frac{m y}{R} \tag{3}
\]

The monodromy (obtained for \(y = R\)) must be a symmetry of the full quantum theory, that is it must be an element of \(SL(2, \mathbb{Z})\), which implies that \(m\) must be an integer. Then this compactification is mapped by massive T duality into the usual compactification of the massive IIA supergravity in ten dimensions.

On the other hand, a Type IIB compactification on \(S^1\) with nontrivial \(\tau(x, y)\) is equivalent to M theory compactified on a space \(B\) which is a \(T^2\) bundle over \(S^1\), where the \(T^2\) has modulus \(\tau(x, y)\) fixed and area \(A \rightarrow 0\). Equivalently, it is an F theory compactification on \(B\) where \(A\) is fixed.

We consider ten-dimensional massive Type IIA string theory, so the T dual (Type IIA) radius must go to infinity, hence the IIB radius \(R\) goes to zero. If we also impose that \(\tau(x) = \tau_0 = i R_2/R_1\), then massive IIA supergravity is equivalent to M theory on the space \(B(A, R)\), in the limit \(A \rightarrow 0, R \rightarrow 0\). The metric is (renaming \(y\) as \(x_3\) and \(R\) as \(R_3\))

\[
ds^2_B = R_3^2(dx_3)^2 + \frac{A}{\text{Im}(\tau)}|dx_1 + \tau(x_3) dx_2|^2 = R_3^2(dx_3)^2 + R_2^2(dx_2)^2 + R_1^2(dx_1 + mx_3 dx_2)^2 \tag{4}
\]
with all the radii going to zero, and the \( x_i \) with periodicity 1, \( x_i \sim x_i + 1 \). In the limit, we should keep the massive IIA quantities fixed, so

\[
g_A = \frac{l_s}{\text{Im}(\tau_0)} R_3 = \text{fixed}, \quad l_s = \frac{j^{3/2}}{R_1^{1/2}} = \text{fixed}, \quad m = \text{fixed}.
\]

A comment is in order regarding the quantization of the 10d IIA mass \( m \) and the massive T duality. The relevant terms in the string frame supergravity actions are, for IIA

\[
S_{IIA} = \frac{1}{k^2} \int \sqrt{g} (e^{-2\phi} R + \tilde{M}^2) + ... = \frac{1}{k^2} \int \sqrt{g} (e^{-2(\phi - \phi_0)} R + (g_A^s)^2 \tilde{M}^2) + ...
\]

whereas on the IIB side we have, similarly

\[
S_{IIB} = \frac{1}{k^2} \int \sqrt{g} (e^{-2(\phi - \phi_0)} R + (g_B^s)^2 (\partial \mu a)^2) + ...
\]

\( \tilde{M} \) the supergravity mass parameter, is quantized in units of \( 1/l_s \), and it remains so when we reduce to 9d, whereas on the IIB side, \( a = a_0 + \frac{m x_3}{R_3} \), so the string frame masses are indeed equal

\[
m_A^s(9) \equiv M = g_A^s \tilde{M} = \frac{g_A^s m}{l_s} = \frac{m g_B^s}{R_3} = m_B^s(9)
\]

When talking about a duality, we have to specify the background as well. The question is nontrivial, as the massive supergravity does not admit a Minkowski background, not even a maximally supersymmetric one. It does admit a half supersymmetric background, namely the D8 brane solution.

The D8 has the string metric and dilaton (\( d\sigma^2_{8,1} \) is the 8 + 1-dimensional Minkowski metric)

\[
ds^2 = H^{-1/2}(d\sigma^2_{8,1}) + H^{1/2} dx^2
\]
\[e^\phi = H^{-5/4}
\]
\[H = c + |\tilde{M}| |x| = c + \frac{m}{l_s} |x|
\]

where \( c \) is an arbitrary constant of integration or (by the usual rescaling for p-branes)

\[
ds^2 = \tilde{H}^{-1/2}(d\sigma^2_{8,1}) + \tilde{H}^{1/2} d\bar{x}^2
\]
\[e^\phi = e^{\phi_0} \tilde{H}^{-5/4} = g_s \tilde{H}^{-5/4}
\]
\[\tilde{H} = H/c = 1 + g_s |\tilde{M}| |\bar{x}| = 1 + \frac{g_s m}{l_s} |\bar{x}|
\]

where \( g_s \) is defined as the coupling constant at the position of the D8 brane. The solution is obtained by promoting \( \tilde{M} \) to a field \( \tilde{M}(x) \) and dualizing it to a 10-form field strength \( F_{(10)} = \tilde{M} \epsilon_{i_1...i_{10}} dx^{i_1} \wedge ... \wedge dx^{i_{10}} \). Then \( \tilde{M}(x) = \pm \tilde{H}^i \), so the mass is piecewise constant, and jumps at the positions of the D8 branes. The \( \pm \) in the mass corresponds to D8 branes vs. anti-D8 branes (since \( F_{10} = *\tilde{M} \) is the field strength for D8’s), so for a D8 the supergravity
mass jumps by a positive amount, whereas for an anti-D8 by a negative amount. Note though that the tension of both is positive (the metric is the same for both).

On a compact space we should think of the D8’s as being part of a D8-O8 system, with 16 D8’s canceling the charge of the orientifold 8-plane O8. If the transverse space is noncompact we can assume that the O8 and the rest of the D8’s are far away, and concentrate on the local physics of single, or coincident D8’s.

We now have to find the M theory dual of the D8 solution. Dimensionally reducing to eight dimensions, one finds a 6-brane solution (domain wall), which can be oxidized on the space $B(A, R)$ to the Ricci-flat M theory background

$$ds^2 = H^{1/2}(H^{-1/2}d\sigma_{6,1}^2 + H^{1/2}dx^2) + ds_B^2 = d\sigma_{6,1}^2 + Hdx^2 + ds_B^2$$

(11)

where the moduli parameters of $ds_B^2$ are

$$R_3 = H^{1/2}, \tau = mx_3 + iH.$$  

(12)

Equivalently, introducing the constants $r_i$, we define

$$R_1 = r_1/\sqrt{H}, \quad R_2 = r_2\sqrt{H}, \quad R_3 = r_3\sqrt{H},$$

(13)

and the limit becomes $r_i \to 0$ with

$$\frac{r_1 l_s}{r_2 r_3} = \text{fixed, and } l_s \text{ fixed.}$$

(14)

Counting parameters, we find 5 parameters in the M-theory compactification, $l_P$, $R_i$ and $m$. This limit sends 2 parameters to zero (e.g. $A = R_1 R_2 \to 0$ then $R_3 \to 0$), so that we are left with the 3 parameters of massive IIA, $l_s$, $g_s$ and $m$.

3 Matrix theory description in D8 background and T duality approach

Hull’s prescription tells us how to relate massive IIA string theory to M theory. In this section we construct a Matrix description of the M-theory compactified on $B(A, R)$.

The problem is nontrivial for two reasons. The first is that the space is curved, and moreover, if we write

$$ds^2 = (dz_3)^2 + (dz_1 + \alpha z_3 dz_2)^2 + (dz_2)^2$$

(15)

we find the Ricci tensor components $^1$

$$R_{11} = \frac{\alpha^2}{2}, \quad R_{12} = \frac{\alpha^3 z_3}{2},$$

$$R_{22} = -\frac{\alpha^2}{2}(1 - \alpha^2(z_3)^2), \quad R_{33} = -\frac{\alpha^2}{2}.$$  

(18)

$^1$We have $g^{11} = 1 + \alpha^2(z_3)^2$, $g^{22} = 1$ and $g^{12} = -\alpha z_3$, and

$$\Gamma^3_{12} = -\frac{\alpha}{2}, \quad \Gamma^3_{13} = -\frac{\alpha^2 z_3}{2}, \quad \Gamma^3_{23} = \frac{\alpha}{2}(1 - \alpha^2(z_3)^2),$$

$$\Gamma^3_{22} = -\alpha^2 z_3, \quad \Gamma^3_{13} = -\frac{\alpha}{2}, \quad \Gamma^3_{23} = \frac{\alpha^2 z_3}{2}.$$  

(16)
The curvature scalar is
\[ R = -\frac{\alpha^2}{2}. \]

The metric \([15]\) is invariant under the following isometries
\[ T_1 : \quad z_1 \to z_1 + a_1, \quad z_2 \to z_2, \quad z_3 \to z_3 \]
\[ T_2 : \quad z_2 \to z_2 + a_2, \quad z_1 \to z_1, \quad z_3 \to z_3 \]
\[ T_3 : \quad z_3 \to z_3 + a_3, \quad z_1 \to z_1 - \alpha z_2 a_3, \quad z_2 \to z_2 \]

with Killing vectors \( V_1 = \partial_1, \quad V_2 = \partial_2 \) and \( V_3 = \partial_3 - \alpha z_2 \partial_1 \).

We also note that \([T_2, T_3] \neq 0\). By identifying under the isometries with \( a_i = R_i \) we obtain the space \( B(A, R) \), and then we have (since \( z_i = R_i x_i \))
\[ \alpha = \frac{m R_1}{R_2 R_3} = M. \]  

We note therefore that we can trust supergravity as long as
\[ \alpha l_P = \frac{m R_1 l_P}{R_2 R_3} \ll 1 \]  
which is true in our limit \((\frac{R_i}{R_2 R_3} \text{ fixed, } l_P \to 0)\).

In the following we choose to work with the D8 background, corresponding to the M theory metric \([11]\) with radii \([13]\). We propose a Matrix description is obtained by considering the action of \( N \) D0-branes in the D8 background \([11]\). Since the radii of \( B \) go to zero, we have to make T dualities in the 3 directions of \( B \), and so the Matrix model describing massive IIA will be the action of \( N \) D3 branes in the T dual background. It is understood that the general procedure used to obtain the dual Matrix model will be the same for any massive IIA background with a light-like symmetry.

First, however, we must define correctly the limit taken on the M theory, and see what kind of limit we obtain for the D3 brane. This is described in detail in the appendix, but we will give here only the relevant facts. Sen \([7]\) and Seiberg \([6]\) give a prescription for the discretized light-cone quantization (DLCQ) of M theory (with light-like radius \( R \) and finite \( l_P \)) on a torus of finite radii \( R_i \). One goes to an equivalent \( \bar{M} \) theory with \( \bar{l}_P \to 0 \) and spacelike 11th direction of radius \( R_s \to 0 \) and compactification radii \( \bar{R}_i \to 0 \) such that
\[ \frac{R_s}{\bar{l}_P^2} = \frac{R}{l_P^2}, \quad \frac{\bar{R}_i}{l_P} = \frac{R_i}{\bar{l}_P} \]  
are held fixed in the \( \bar{l}_P \to 0 \) limit. Then one makes T dualities in the compact directions and all the rest are zero. Then we use
\[ R_{ab} = \partial_c \Gamma^c_{ab} - \frac{1}{2} \partial_a \partial_b l n g + \frac{1}{2} \Gamma^c_{ad} \partial_d l n g - \Gamma^c_{ad} \Gamma^d_{cb} \]  
and the fact that \( g = 1 \).
and gets a decoupled theory of Dp-branes (D3 in our case) with finite $g_{YM}$ and dual radii

$$\tilde{R}_i = \frac{\tilde{r}_i^2}{R_i} = \frac{\tilde{l}_i^2}{R_i R},$$

$$\tilde{g}^2_{YM} = \tilde{g}_s = \frac{R^3}{\tilde{l}_P} \prod_i \tilde{R}_i. \quad (24)$$

As we can see, the parameters of the dual D3 matrix model do not depend on the parameters of the $\bar{M}$ theory, which was introduced just to prove the duality. Therefore we can apply another limit to this construction (independent of the Sen-Seiberg $\bar{l}_P \to 0$ limit), namely $\bar{l}_P \to 0$ and $R_i \to 0$, with $\tilde{l}_P^3/R_1 = (l_A^s)^2 = \tilde{l}_s^2$ and $R_1/(R_2 R_3)$ kept fixed. We also need to make a “9-11 flip”, namely to reinterpret the lightcone coordinate $R$ as the 11th direction (since in the M theory construction of massive IIA $R_1$ takes the role of 11th coordinate).

The parameters of the super-Yang-Mills are

$$\tilde{R}_1 = \frac{\tilde{l}_s^2}{R}, \quad \tilde{R}_2 = \frac{R_1}{R_2} \tilde{R}_1, \quad \tilde{R}_3 = \frac{R_1}{R_3} \tilde{R}_1, \quad \tilde{g}_s = \tilde{g}^2_{YM} = \frac{l_s g_A^s}{R_1} \quad (25)$$

and the inverse relations are, if $R = N l_s$,

$$R = N l_s, \quad R_1 = \frac{N}{g_{YM}} \sqrt{\tilde{R}_2 \tilde{R}_3}, \quad g_A^s = \sqrt{\tilde{g}^2_{YM} \frac{\tilde{R}_2 \tilde{R}_3}{\tilde{R}_1^2}}, \quad l_s = N \tilde{R}_1. \quad (26)$$

In order to still have decoupling of the string theory from the D3 brane theory, we need to have $\bar{l}_s \to 0$ and the S-dual string length $\tilde{g}_s \bar{l}_s^2 \to 0$, which is satisfied in the $\bar{M}$ theory, since

$$\tilde{l}_s^2 = \frac{\tilde{l}_P^3 \bar{l}_P}{R \bar{l}_P} \Rightarrow \tilde{g}_s \bar{l}_s^2 = \frac{l_s^3 g_A^s}{R \bar{l}_P} \to 0. \quad (27)$$

Let us now follow this procedure in order to find the Matrix model description of the background \[11\]: 9-11 flip, going to $\bar{M}$ theory, dimensional reduction to string theory, followed by 3 T dualities. The string theory background in the $\bar{M}$ theory is

$$ds^2 = d\sigma_{5,1}^2 + H dx^2 + ds_B^2 \quad (28)$$

with the radii given in \[13\] and constant dilaton $\phi_0$, and now we need to perform 3 T dualities. We will concentrate on the space $B$, with metric

$$ds_B^2 = H(\bar{r}_3^2 dx_3^2 + \bar{r}_2^2 dx_2^2) + \frac{\bar{r}_1^2}{H} (dx_1 + m x_3 dx_2)^2 \quad (29)$$

and work with string metrics, on which the T dualities act in a simple way. We will work in units of $l_s$. If we want to restore the $l_s$ dependence we can formally put $\bar{r}_i \to \bar{r}_i/l_s, x_i \to x_i l_s, m \to m/l_s$. 

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The Buscher T duality rules \[19, 20, 21\] are

\[
\begin{align*}
\hat{g}_{00} &= \frac{1}{g_{00}} \\
\hat{g}_{0i} &= \frac{B_{0i}}{g_{00}} \\
\hat{g}_{ij} &= g_{ij} - g_{0i}g_{0j} - \frac{B_{0i}B_{0j}}{g_{00}} \\
\hat{B}_{0i} &= \frac{g_{0i}}{g_{00}} \\
\hat{B}_{ij} &= B_{ij} + \frac{g_{0i}B_{0j} - B_{0i}g_{0j}}{g_{00}} \\
\hat{\phi} &= \phi - \frac{1}{2} \log(g_{00})
\end{align*}
\] (30)

Here the coordinate 0 of the T duality is defined such that \(\partial_0\) is the Killing vector of an isometry. It is worth noting here that one might be worried that we have to use the massive T duality rules at some point, however the \(m\)-dependent terms are only in the transformation rules of the RR fields (see [18]).

As we saw, we have 3 isometries, \(T_1, T_2, T_3\). \(T_2\) and \(T_3\) do not commute, so the order of T dualities matters. We will choose to do \(T_1\), then \(T_2\), then \(T_3\). We begin by considering the simpler case of T dualities on the twisted torus with the radii and \(\bar{L}_s, H\) set to 1 corresponding to the core of the D8 background \(x = 0\), and later we will generalize this to the complete background. After \(T_1\) we have:

\[
\begin{align*}
\frac{ds^2}{\bar{g}^{00}} &= (dx_3^2 + dx_2^2 + dx_1^2) \\
B_{12} &= mx_3 \Rightarrow H_{123} = m \\
e^\phi &= e^{\phi_0}.
\end{align*}
\] (31)

After \(T_2\) we have

\[
\begin{align*}
\frac{ds^2}{\bar{g}^{00}} &= (dx_3^2 + dx_1^2) + (dx_2 + mx_3dx_1)^2 \\
e^\phi &= e^{\phi_0}.
\end{align*}
\] (32)

\(T_3\) is generated by the vector \(V_3 = \partial_3 - mx_1\partial_2\). Transforming to coordinates

\[
x'_3 = x_3, x'_2 = x_2 + mx_1x_3
\] (33)

implies \(V_3 = \partial'_3\) so that we can apply the usual T duality rules.

The metric in the new coordinates (after dropping primes on coordinates)

\[
\begin{align*}
\frac{ds^2}{\bar{g}^{00}} &= (dx_3^2 + dx_1^2) + (dx_2 - mx_1dx_3)^2 \\
e^\phi &= e^{\phi_0}.
\end{align*}
\] (34)
After the third $T$-duality, we have
\[ ds^2 = dx_1^2 + \frac{(dx_2^2 + dx_3^2)}{1 + m^2x_1^2} \]
\[ B_{23}dx^2 \wedge dx^3 = -\frac{mx_1}{1 + m^2x_1^2} dx_2 \wedge dx_3 \]
\[ e^{\phi} = \frac{e^{\phi_0}}{(1 + m^2x_1^2)^{1/2}}. \] (35)

The open string metric and $\theta$-field are
\[ ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \]
\[ \theta^{23} = mx_1. \] (36)

The closed string metric in (35) is no longer periodic in $x_1$. The metric in (32) has the property that the (23) torus at $x_1 + 1$ is related to that at $x_1$ by an $SL(2, \mathbb{Z})$ transformation
\[ A = \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix}. \] (37)

This $2 \times 2$ matrix is embedded in the full $O(2, 2; \mathbb{Z})$ T-duality group as
\[ S = \begin{pmatrix} A & 0 \\ 0 & A^{T-1} \end{pmatrix}. \] (38)

as explained for example in the review [22].

The closed string metric and B-field in (35) obey the property that the background matrix $E = G + B$ of the (23) torus and the dilaton are related
\[ E(x_1 + 1) = \frac{aE(x_1) + b}{cE(x_1) + d} \]
\[ e^{\phi(x_1+1)} = e^{\phi(x_1)} \left( \frac{\det g(x_1+1)}{\det g(x_1)} \right)^{1/4} \] (39)

where $a, b, c, d$ are $2 \times 2$ matrices entering a $4 \times 4 O(2, 2)$ matrix
\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \] (40)

One easily checks (39) when $x_1 = 0$ and with a little more work for general $x_1$. The $a, b, c, d$ are calculated by observing that the shift by $x_1$ in (35) can be accomplished by first T-dualizing to (34), doing the shift and T-dualizing back. The $O(2, 2; \mathbb{Z})$ matrix $T_3$ for the T-duality along $x_3$ in (34) is
\[ T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \] (41)
and \( M = TST^{-1} \). This gives

\[
\begin{align*}
a &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
b &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
c &= \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} \\
d &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

(42)

It is also interesting to observe that the open string background in (36) characterized by the matrices \( G \) and \( \Theta \) transforms under a shift of \( x_1 \) by the same \( O(2,2;\mathbb{Z}) \) matrix \( M \) when we use the action of \( O(2,2;\mathbb{Z}) \) given by Seiberg-Witten:

\[
\begin{align*}
G(x_1 + 1) &= G(x_1) = (a + b\Theta(x_1)) G(x_1) (a + b\Theta(x_1))^T \\
\Theta(x_1 + 1) &= (c + d\Theta(x_1))(a + b\Theta(x_1))^{-1} \\
g_{YM}(x_1 + 1) &= g_{YM}(x_1) (\det(a + b\Theta(x_1)))^{\frac{1}{4}}
\end{align*}
\]

(43)

The final background is to be viewed as a \( T^2 \) bundle over \( S^1 \) where the \( T^2 \) is twisted by an element of the full \( T \)-duality group of the torus upon transport along the \( S^1 \). This structure of the closed string background obtained after \( T \)-dualizing the twisted torus has been observed recently in [23] and related work appears in [24, 25].

We now describe the \( T \)-dualities on the full background. The above remarks on the \( O(2,2) \) carry over. After \( T \) duality on \( T_1 \) we have (the full 10d metric)

\[
\begin{align*}
&d\sigma^2_{5,1} + H(dx_2^2 + \bar{r}_2^2dx_3^2 + \bar{r}_2^2dx_2^2 + 1/\bar{r}_1^2dx_1^2) \\
&B_{12} = mx_3 \Rightarrow H_{123} = m \\
e^\phi &= \frac{e^{\phi_0}}{\bar{r}_1}\sqrt{H}
\end{align*}
\]

(44)

which we recognize as nothing but the NS5 brane metric smeared over the transverse directions 1,2,3.

After \( T \) duality on \( T_2 \) we have

\[
\begin{align*}
&H(\bar{r}_3^2dx_3^2 + 1/\bar{r}_1^2dx_1^2) + \frac{1}{\bar{r}_2^2H}(dx_2 + mx_3dx_1)^2 \\
e^\phi &= \frac{e^{\phi_0}}{\bar{r}_1\bar{r}_2}
\end{align*}
\]

(45)

which is the same metric as we started from, with inverted radii \( \bar{r}_1, \bar{r}_2 \) and with 1 and 2 interchanged.
Then in these dual coordinates, $T_3$ has Killing vector $V_3 = \partial_3 - mx_1 \partial_2$. Applying the coordinate transformation (33) and $T_3$ T duality we get (restoring also the $l_s$ dependence for later use)

$$ds^2 = \tilde{l}_s^4 \left( H(dx_1^2/\tilde{r}_1^2) + \frac{H^{-1}(dx_2^2/\tilde{r}_2^2 + dx_3^2/\tilde{r}_3^2)}{1 + \left(\frac{\tilde{r}_2 \tilde{r}_3}{H \tilde{r}_1^4}\right)^2} \right)$$

$$B_{23} dx^2 \wedge dx^3 = -\tilde{l}_s^4 \frac{m \tilde{r}_1}{\tilde{r}_2 \tilde{r}_3} H^2 \frac{x_1/\tilde{r}_1}{1 + \left(\frac{\tilde{r}_2 \tilde{r}_3}{H \tilde{r}_1^4}\right)^2} dx_2/\tilde{r}_2 \wedge dx_3/\tilde{r}_3$$

$$e^{\phi} = \frac{\tilde{l}_s^3 e^{\phi_0}}{\tilde{r}_1 \tilde{r}_2 \tilde{r}_3 \sqrt{H}} H^{-1/2} \left( 1 + \left(\frac{m \tilde{r}_1 x_1}{H \tilde{r}_2 \tilde{r}_3}\right)^2 \right)^{-1/2}.$$  (46)

Let us now define $\tilde{y}_i = \tilde{l}_s^2 x_i/\tilde{r}_i$ and calculate open string variables, to find the metric $G$ and noncommutativity parameter $\theta$ the D3-brane sees [26]. Using

$$\left( G + \frac{\theta}{\tilde{l}_s^2} \right)^{ij} = \left( \frac{1}{g + \tilde{l}_s^2 B} \right)^{ij}$$  (47)

we get for the full 10d metric

$$ds^2 = d\sigma^2_{5,1} + H(dx^2 + d\tilde{y}_1^2) + \frac{d\tilde{y}_2^2 + d\tilde{y}_3^2}{H}$$

$$e^{\tilde{\phi}} = \frac{\tilde{l}_s^3 e^{\phi_0}}{\tilde{r}_1 \tilde{r}_2 \tilde{r}_3 \sqrt{H}}$$

$$[\tilde{y}_2, \tilde{y}_3] = -i(m \tilde{r}_1 \tilde{l}_s^2 \tilde{y}_1) = -i\tilde{\alpha} \tilde{y}_1$$

$$H = 1 + \frac{m \tilde{r}_1}{\tilde{r}_2 \tilde{r}_3} |x| = 1 + \frac{m \tilde{r}_1 l_s^2}{\tilde{r}_2 \tilde{r}_3} |X| = 1 + \tilde{\alpha} |X|,$$  (48)

where we have defined $X = x/\tilde{l}_s^2$.

Recalling that $\tilde{l}_s$ goes to zero in the infinite boost limit, and then, with $\tilde{y}_i$ and $\tilde{\alpha}$ fixed, we have

$$\tilde{l}_s^2 \sim \epsilon^{1/2}, \quad g_{ij} \propto \tilde{l}_s^4 \sim \epsilon, \quad G_{ij} \sim \frac{g_{ij}}{\tilde{l}_s^4} = \text{fixed}$$  (49)

which is nothing other than the Seiberg-Witten limit for noncommutative geometry, which means that the theory on the D3 branes is nothing other than noncommutative super Yang-Mills theory with variables (metric, dilaton and noncommutativity) given in (48). The noncommutativity parameter is

$$\tilde{\alpha} = \frac{m \tilde{r}_1 l_s^2}{\tilde{r}_2 \tilde{r}_3} = \frac{m g_{s} l_s^4}{R_1} = \frac{m g_{s} l_s A l_s^4}{R} = \frac{m \tilde{R}_2 \tilde{R}_3}{R_1}.$$  (50)
4 Space-dependent non-commutativity and solution of Matrix Theory constraints

Let us now see that the super-Yang-Mills defined on (48) can also be obtained from an algebraic approach in Matrix theory. Our goal will be to reproduce the noncommutative structure of the Matrix degrees of freedom $X^i(t)$. We follow the general procedure for compactification [8,10,11,12,13]. We need to find T-dual variables $y_i$ such that the matrices $X^i$ can be represented as covariant derivatives. For the simple case of circle compactification, one represents the algebra

$$\Omega X^i \Omega^{-1} = X^i + R$$

by $X = iD_y \equiv i\partial_y + A, \Omega = e^{iRy}$, since $[i\partial_y + A, e^{iRy}] = -Re^{iRy}$.

Let us review this in a bit more detail. We start with a finite number $k$ of zero branes on a circle. This is equivalent to having an infinite number of copies of $k$ zero branes along a line, with zero branes separated by a constant periodic shift $R$ related by a gauge transformation $\Omega$ in $U(\infty)$ as in (51). One writes $X^{ab}$ matrices as $X^{a'b'}_{mn}$ where $a',b'$ run over the $k$ zero branes and $m,n$ are integers. These matrices are now operators on states labeled by an integer $m$ corresponding to $e^{imRy} = |m\rangle$, tensored by finite $k \times k$ matrices. The operators on the states $|m\rangle$ can be viewed as the algebra of functions on the T-dual space. The form

$$A(y) = \sum_p A_p e^{ipRy},$$

allows us to read off the T-dual radii. The matrix elements of $X$ and $\Omega$ are given by $X_{n,m} = -nR\delta_{n,m} + A_p\delta_{n,m+p}$ and $\Omega_{nn} = \delta_{n,n+1}$. So we have the T dual Matrix model description in terms of D1-branes (and by generalization, Dp-branes). Fluctuations in the compact $X$ are mapped to fluctuations in the gauge field $A$, and part of the original Matrix degrees of freedom were used to generate functions of the worldvolume direction $y$.

We will now try and apply this procedure to our case. We will treat first the case when the harmonic function $H = 1$, and will see later what complications $H$ introduces. We can think of it as working in the near core region $x \simeq 0$. We will also put for simplicity $R_i = 1$ for the moment, and return to the general case later on.

In our case we have 3 isometries, $T_1, T_2, T_3$, which means that we need to impose constraints on the D0 Matrix model analog to (51) and try to solve them in terms of a T dual space. The constraints are

$$T_1: \quad \Omega_1 X_1 \Omega_1^{-1} = X_1 + 1$$
$$\quad \Omega_1 X_2 \Omega_1^{-1} = X_2$$
$$\quad \Omega_1 X_3 \Omega_1^{-1} = X_3$$

where $\Omega_1$ is a transformation acting on the X’s which corresponds to the isometry $T_1$, and
similarly

\[ T_2 : \quad \Omega_2 X_1 \Omega_2^{-1} = X_1 \]
\[ \Omega_2 X_2 \Omega_2^{-1} = X_2 + 1 \]
\[ \Omega_2 X_3 \Omega_2^{-1} = X_3 \]

\[ T_3 : \quad \Omega_3 X_1 \Omega_3^{-1} = X_1 - m X_2 \]
\[ \Omega_3 X_2 \Omega_3^{-1} = X_2 \]
\[ \Omega_3 X_3 \Omega_3^{-1} = X_3 + 1 . \]

(53) (54)

We noted that \( T_2 \) and \( T_3 \) don’t commute, and we can see that therefore \( \Omega_2 \) and \( \Omega_3 \) don’t commute. Namely, if we put \( M = \Omega_2^{-1} \Omega_3^{-1} \Omega_2 \Omega_3 \), then we have

\[ MX_1 M^{-1} = X_1 - m \]
\[ MX_2 M^{-1} = X_2 \]
\[ MX_3 M^{-1} = X_3 . \]

(55)

From the relations defining \( T_1 \) and \( T_2 \) we can see that we can put \( X_1 = iD_1, X_2 = iD_2 \) and \( \Omega_1 = e^{i y_1}, \Omega_2 = e^{i y_2} \) just as in flat space. The commutation of relation of \( \Omega_3 \) with \( X_3 \) can also be solved by \( X_3 = iD_3 \) and \( \Omega_3 = e^{i \alpha y_3} \) (here \( \alpha = m \)). The relations (55) allow us to solve for \( M = e^{-i m y_1} \). Hence we deduce that \( y_2 \) and \( y_3 \) don’t commute, and we get exactly the noncommutativity relations

\[ [y_2, y_3] = i \theta_{23} = i \alpha y_1, \quad [y_1, y_2] = [y_1, y_3] = 0 \]

(56)

as we obtained from the T duality approach of the last section.

The relations (55) also imply nontrivial commutations for derivatives and coordinates. Indeed, by taking the commutator with various derivatives of the relations (55), we get a set of equations for \([\partial_1, y_i]\). We will not list them here but just mention a solution, namely \([\partial_1, y_i] = 1\) as usual, but also \([\partial_1, y_3] = -i \alpha \partial_2\). The relevant equation is obtained from \([\partial_1, [y_2, y_3]]\)

\[ [[\partial_1, y_2], y_3] + [y_2, [\partial_1, y_3]] = -i \alpha \]

(57)

and we can see that it is indeed solved by \([\partial_1, y_3] = -i \alpha \partial_2\).

It is useful to observe that a change of variables maps the non-commutativity parameter to a constant. Indeed, in the open string metric

\[ ds^2 = dy_1^2 + dy_2^2 + dy_3^2 + ds_{tr}^2 \]

(58)

with the noncommutativity \( \theta_{23} = \alpha y_1 \) we can make the change of variables \( y_1 = y'_1, y_2 = y'_2 y_1, y_3 = y'_3 \), after which the theory has metric

\[ ds^2 = dy'_1^2 + dy'_2^2 + d(y'_1 y'_2)^2 \]

(59)
and constant noncommutativity $\theta'_{23} = \alpha$. The closed string metric and B field in (60) become, under the transformation:

$$
\begin{align*}
\text{ds}^2 &= dy_1'^2 (1 + \frac{y_2'^2}{1 + \alpha^2 y_1'^2}) + \frac{dy_3'^2 + y_1'^2 dy_2'^2 + 2 y_2'y_1'dy_2'dy_1'}{1 + \alpha^2 y_1'^2} \\
\text{l}_s^4 B &= \frac{\alpha y_1'(y_1'dy_2' + y_2'dy_1') \wedge dy_3'}{1 + \alpha^2 y_1'^2}.
\end{align*}
$$

To put back the $R_i$ and $l_s$ dependence, we only need to substitute $\alpha = \tilde{\alpha}/l_s^2$ in the above.

Since the noncommutativity is constant in the new coordinates $y'_i$, we can realize the commutation relations

$$
\begin{align*}
[y'_i, y'_j] &= i\theta'_{ij} \\
[\partial'_i, y'_j] &= \delta'_j
\end{align*}
$$

as we explain further below. The commutation relations (61) imply the following relations for the unprimed coordinates

$$
\begin{align*}
[y_i, y_j] &= i\theta_{ij}(y_1) = i\theta'_{ij}(y_1) \\
[\partial_i, y_j] &= \delta_i + \Psi^i_j(\partial)
\end{align*}
$$

(62)

where $\theta_{ij}(y)$ has nontrivial components $\theta_{23} = -\theta_{32} = \alpha y_1$, $\Psi^i_j$ has nontrivial components $\Psi^1_3 = i\alpha \frac{\partial}{\partial w_2}$, as we wanted (see (56) and (57)). These guarantee that if we set $X'_i = i \frac{\partial}{\partial y_i}$ and $e^{iy_i}$ we correctly obey the constraints in (52) (53) (54). This provides the foundation for the general solution including gauge fields but we first need to review the construction of the covariant derivatives including gauge fields in the case of constant non-commutativity.

We recall some facts about the construction of a non-commutative gauge theory from covariant derivatives in the context of an ordinary constant noncommutativity $\theta'_{ij}$. To have notation which agrees with our set-up we will let $i, j$ run over 2, 3 and we will let the noncommutative torus algebra be generated by $y'_2, y'_3$. Consider compactification constraints generated by $\Omega'_2 = e^{i\tilde{y}'_2}$ and $\Omega'_3 = e^{i\tilde{y}'_3}$ where $\Omega'_i\Omega'_j(\Omega'_i)^{-1}(\Omega'_j)^{-1} = e^{i\theta'_{ij}}$, with the only nontrivial components being $\theta'_{23} = -\theta'_{32} = \theta$. The $\Omega'_i$ and $X'_i$ are represented in a Hilbert space where there is non-trivial commutant generated by $\tilde{\Omega}'_2, \tilde{\Omega}'_3$ (i.e. $[\Omega'_i, \tilde{\Omega}'_j] = 0$) which have a non-commutativity parameter $\tilde{\theta} = -\theta$. Writing $\tilde{\Omega}'_2 = e^{i\tilde{y}'_2}$ and $\tilde{\Omega}'_3 = e^{i\tilde{y}'_3}$, we have $[\tilde{y}'_2, \tilde{y}'_3] = -i\theta$. Explicit construction of the $y'_i$ and $\tilde{y}'_i$’s or equivalently the $\Omega'_i$ and $\tilde{\Omega}'_i$ in terms of coordinates $w_2, w_3$ which commute with each other and satisfy standard commutation relations with their derivatives $\frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_3}$ and together describe a four-dimensional phase space.
are given in \((26)\)

\[
\begin{align*}
y'_2 &= w_2 + \frac{i\theta}{2} \frac{\partial}{\partial w_3} \\
y'_3 &= w_3 - \frac{i\theta}{2} \frac{\partial}{\partial w_2} \\
\tilde{y}'_2 &= w_2 - \frac{i\theta}{2} \frac{\partial}{\partial w_3} \\
\tilde{y}'_3 &= w_3 + \frac{i\theta}{2} \frac{\partial}{\partial w_2}.
\end{align*}
\]

These formulas can be used to check that the correct mutual non-commutativity of \(y'_i\) and of \(\tilde{y}'_i\) are reproduced, as well as the vanishing commutators of any \(y'_i\) with \(\tilde{y}'_i\).

We can define derivatives with respect to \(y'_i\) and \(\tilde{y}'_i\)

\[
[\frac{\partial}{\partial y'_i}, y'_j] = \delta^i_j, \quad [\frac{\partial}{\partial y'_i}, \tilde{y}'_j] = 0
\]

and

\[
[\frac{\partial}{\partial \tilde{y}'_i}, y'_j] = \delta^i_j, \quad [\frac{\partial}{\partial \tilde{y}'_i}, \tilde{y}'_j] = 0.
\]

These constraints can be solved by defining the derivatives in terms of appropriate commutator actions with elements in the algebra of \(y, \tilde{y}\)

\[
\frac{\partial}{\partial y'_i} = -i(\theta^{-1})^{ij} y'_j, \\
\frac{\partial}{\partial \tilde{y}'_i} = +i(\theta^{-1})^{ij} \tilde{y}'_j
\]

and the non-trivial commutation relations of derivatives follow

\[
[\frac{\partial}{\partial y'_i}, \frac{\partial}{\partial y'_j}] = -i(\theta^{-1})^{ij} \\
[\frac{\partial}{\partial \tilde{y}'_i}, \frac{\partial}{\partial \tilde{y}'_j}] = +i(\theta^{-1})^{ij}.
\]

The fact that the derivatives can be expressed in terms of commutator action with elements in the algebra plays an important role in \(27\) in the context of a discussion of solutions of Matrix Theory describing extended objects in \(R^2\) and having a non-commutative worldvolume theory derived from Matrix Theory.

The presence of the commutant generated by the \(\tilde{y}'\) is important in getting solutions to the constraints with non-trivial gauge fields. The simplest gauge theories are in fact obtained when we take the covariant derivatives to be \(\partial y'_i + \partial \tilde{y}'_i - iA_i(e^{i\tilde{y}_i})\). Such a choice of derivative was implicit for example in \(13\). It is useful to note that

\[
[\frac{\partial}{\partial y'_i} + \frac{\partial}{\partial \tilde{y}'_i}, \frac{\partial}{\partial y'_j} + \frac{\partial}{\partial \tilde{y}'_j}] = 0
\]
which means that there is no background magnetic field. As far as solving the compactification constraints we could work with a more general set of partial derivatives \((\partial_{y'} + \phi_{ij} \partial_{y'_j})\).

We can now write a solution for the \(X\) operators acting on a Hilbert space of functions. Since the periodicities are simple in the \(y\)-coordinates, we are led to consider functions of \(e^{iy_1}, e^{iy_2}\) and \(e^{iy_3}\). Recalling the discussion above, where we saw that the constraints are expressed in terms of \(y'\) variables whereas the fields are functions of variables \(\tilde{y}'\), we are led to work with the Hilbert space of functions of the form

\[
\psi = \sum_{n_1, n_2, n_3} \psi_{n_1, n_2, n_3} \psi_{n_1, n_2, n_3} e^{in_1 y'_1} e^{in_2 y'_2} e^{in_3 y'_3}
\]

where \(n_i\) are arbitrary integers. The \(y_i\) and \(\tilde{y}'_i\), for \(i = 2, 3\) are constructed in terms of a four dimensional phase space as in \((63)\).

On this Hilbert space we can write operators

\[
X_1 = i \frac{\partial}{\partial y'_1} - i y'_2 \left( \frac{1}{y'_1} \frac{\partial}{\partial y'_2} - i A_2(e^{iy'_1}, e^{iy'_2}, e^{iy'_3}) \right) - i y'_2 \frac{\partial}{\partial y'_2} + A_1(e^{iy'_1}, e^{iy'_2}, e^{iy'_3})
\]
\[
X_2 = \frac{i}{y'_1} \frac{\partial}{\partial y'_2} + A_2(e^{iy'_1}, e^{iy'_2}, e^{iy'_3})
\]
\[
X_3 = i \frac{\partial}{\partial y'_3} + A_3(e^{iy'_1}, e^{iy'_2}, e^{iy'_3})
\]
\[
\Phi^a = \Phi^a(e^{iy'_1}, e^{iy'_2}, e^{iy'_3})
\]

The constraints are generated by

\[
\Omega_1 = e^{iy'_1}
\]
\[
\Omega_2 = e^{iy'_2}
\]
\[
\Omega_3 = e^{iy'_3}
\]

With these expressions we can check that the constraints in \((52)\), \((53)\), \((54)\) are satisfied and that the \(X\)’s correctly act in the Hilbert space. We elaborate on some aspects of these properties. The combination \(i \frac{\partial}{\partial y'_1} - i y'_2 \frac{\partial}{\partial y'_2}\) is necessary in \(X_1\) because it allows \(X_1\) to be well defined on the Hilbert space. Consider for example \(i \frac{\partial}{\partial y'_1}\) acting on \(e^{iy'_1} \tilde{y}'_2\). It gives \(-\tilde{y}'_2 e^{iy'_1} \tilde{y}'_2\) which does not belong to the Hilbert space we defined. The combination \(i \frac{\partial}{\partial y'_1} - i y'_2 \frac{\partial}{\partial y'_2}\) does map elements in the Hilbert space back to themselves. For similar reasons, the appearance of \(i \frac{\partial}{\partial y'_1} - i y'_2 \frac{\partial}{\partial y'_2}\) guarantees that the constraint \(\Omega_2 X_1 \Omega_2^{-1} = X_1\) is satisfied. The appearance of \(A_2\) in \(X_1\) may seem surprising but is necessary to make sure that the conjugation of \(X_1\) by \(\Omega_3\) does correctly reproduce the shift \(-m X_2\) in \((53)\). It is also worth noting that the change of variables to \(y'\) coordinates is a useful guide in constructing the solution but the periodicity conditions are not simple in these coordinates. A consequence is that what we might call the gauge fields in the primed coordinates, deduced from those in the unprimed coordinates.
are not good operators that act in the Hilbert space. For example $A'_2 = \frac{1}{y_i} A_2$ acting on the Hilbert space gives functions of the form $\frac{1}{y_i} \psi$ which do not belong to the Hilbert space. Finally while the above is a relatively simple solution including gauge fields, it is not the most general. Just as in the commutative case, we can also consider $k$-vectors $\psi_k$ acted on by the above operators. By analogy to the commutative case [11] or the case of constant $\theta$ [12, 13], where the appropriate Hilbert spaces could be generalized to include magnetic fluxes we expect similar generalizations here.

Let us see what happens now in the presence of the harmonic function $H$ [18], and let us restore also the $R_i$ factors. The isometries of the metric continue to be the same, $H$ does not affect the identifications, so we can write down the same constraints as before, where now $X^i$ correspond to the dimensionless coordinates $x^i$.

\[
\begin{align*}
\Omega_1 X_1 \Omega_1^{-1} &= X_1 + 1 \\
\Omega_2 X_2 \Omega_2^{-1} &= X_2 + 1 \\
\Omega_3 X_3 \Omega_3^{-1} &= X_3 + 1 \\
\Omega_3 X_1 \Omega_3^{-1} &= X_1 - m X_2
\end{align*}
\] (72)

We have only written the nontrivial relations above. These have solutions described above. If we rescale $\tilde{X}_i = r_i X_i$ and correspondingly $\tilde{y}_i = l^2_s y_i / r_i$ for the dual variables, then $\tilde{X}_i = i l^2_s \tilde{D}_i \Omega_i = e^{i \frac{\tilde{\alpha}}{2 l^2_s}}$, and one obtains

\[
\begin{align*}
[\tilde{y}_2, \tilde{y}_3] &= i m \frac{l^2_s \tilde{r}_1}{r_2 r_3}, \\
\tilde{y}_1 &= i \tilde{\alpha} \tilde{y}_1, \\
[\tilde{y}_1, \tilde{y}_3] &= [\tilde{y}_1, \tilde{y}_2] = 0
\end{align*}
\] (73)

We have thus obtained the same noncommutativity as from the T duality approach [50]. Unfortunately, now there is no independent way to verify the open string metric and dilaton, which are nontrivial in the presence of the harmonic function $H$.

Still, this fact gives us some useful information about other backgrounds. We notice that the constraints were not modified by the presence of the D8-brane background (i.e. by the nontrivial $H$). We can guess that for a general massive IIA background, the M theory lift will be again a “dressing” of the same space $B(A, R)$ with the same identifications for the dimensionless $x_i$’s, therefore the constraints (72) are unmodified. So by the above procedure we will get a super-Yang-Mills on a space with the same noncommutativity. Again we will have a nontrivial metric and dilaton as well as possible other RR fluxes, which will have to be derived from the T duality approach. The identifications in M theory and correspondingly the noncommutativity of the D3 brane space are of a topological nature, and so insensitive to local modifications.

5 NCSYM action and stability

In this section we describe the D3 brane action we are getting for the Matrix model. First let us check that we can put D0 branes at $x = 0$ in the background [28] (and they are stable).
We will also check whether they can stay at nonzero $x$ (in the D3 Matrix model, whether we can have a nonzero vev for $X$).

A probe D0 brane in the background \((28)\) will have the action

$$S_{1D0} = \int dt e^{-\phi} \sqrt{1 + g_{ij}(\dot{X}^i \dot{X}^j)} \quad (74)$$

The equations of motion of this action in the background \((28)\) are

\[
\begin{align*}
2r_3^2 \frac{d}{dt} [H \dot{X}_3] &= 2r_1^2 m \dot{X}_2 H^{-1}(\dot{X}_1 + mX_3 \dot{X}_2) \\
2 \frac{d}{dt} [H \dot{X}] &= H'(\dot{X}^2 + r_3^2 \dot{X}_3^2 + r_2^2 \dot{X}_2^2) - r_1^2 H^{-1}(\dot{X}_1 + mX_3 \dot{X}_2)^2 \\
\frac{d}{dt} [H^{-1}(\dot{X}_1 + mX_3 \dot{X}_2)] &= 0 \\
\frac{d}{dt} \left( \frac{2mX_3}{H}(\dot{X}_1 + mX_3 \dot{X}_2) + Hr_2^2 \dot{X}_2 \right) &= 0.
\end{align*}
\] (75)

It follows that if $\dot{X}_1 = \dot{X}_2 = \dot{X}_3 = 0$ and $X$ small (so that $H \simeq 1$, then the only remaining equation is

$$2\ddot{X} = -H'(\dot{X})^2 \quad (76)$$

hence the static potential vanishes.

Let us compare this with what happens for the D0-D8 system. There we have a 1-loop Chern-Simons term \(k \int dt (X + A_0), k \in \mathbb{Z}\), which gives a potential for the D0's. One can calculate it in string theory \((28)\) or directly from arguments about the supersymmetric quantum mechanics \((29,30,31)\). But one can understand it from charge conservation. When a D0 passes through a D8 charge conservation requires the creation of an elementary string (Hanany-Witten effect), which will generate a linear potential. In its absence, the D0-D8 is not supersymmetric and has a linear repulsive potential \(V(R) = -T_0 R\). Such a Chern-Simons term (and consequently the linear potential) are absent in the geometric background we consider \((28)\). One can also see this from the D0 worldline perspective. The CS term of the D0-D8 system appeared by integrating out the massive \((0,8)\) fermion, which is absent from our case.

Now that we have established that we can have D0 branes at fixed $x$ (and correspondingly D3 branes in the T dual picture), we would like to describe the action of these D branes in more detail. The prescription of Myers \((32)\) for the (bosonic part of the) Dirac-Born-Infeld (DBI) action in a general background is

$$S_{DBI} = -T_p \int d\sigma^{p+1} \text{STr} \left( e^{-\phi} \sqrt{-\det(P[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb}] + \lambda F_{ab}) \det Q_{ij}^3} \right)$$

\[
\begin{align*}
Q_{ij}^3 &= \delta_{ij} + i\lambda [X^i, X^j] E_{kj} \\
E_{\mu
u} &= (g + B)_{ij}
\end{align*}
\] (77)

and a corresponding Chern-Simons piece. Here the fields are in closed string variables and if the fields depend on the transverse scalars, the prescription is to write the fields as functions
of the adjoint-valued scalars and then take a completely symmetric trace over all adjoint indices.

So the DBI action in our background \([46]\) will be

\[
S = -T_3 \int dt \left( \prod_{i=1}^{3} dx_i \right) \text{STr} \left( e^{-\phi(\bar{X}^{mn}, x_1)} \sqrt{-\det(g_{ab}(\bar{X}^{mn}, x_1) + B_{ab}(\bar{X}^{mn}, x_1) + \lambda F_{ab})} \right),
\]

(78)

here \(\bar{X}^{mn}\) is the Matrix scalar corresponding to the coordinate transverse to the D8-brane, and \(x_1\) is as defined before. If we assume that the Seiberg-Witten map continues to hold in the presence of the nontrivial \(\bar{X}^{mn}\) (which is not entirely obvious, but should probably be true in a Taylor expansion for small values of \(\bar{X}^{mn}\)), then we get

\[
S = -T_3 \int dt \left( \prod_{i=1}^{3} dx_i \right) \text{STr} \left( e^{-\phi(\bar{X}^{mn})} \sqrt{-\det(G_{ab}(\bar{X}^{mn}) + \lambda \tilde{F}_{ab})} \right),
\]

(79)

where \(\tilde{F}\) is the noncommutative field strength. Moreover, we saw that the Sen-Seiberg procedure implied that we take the Seiberg-Witten limit for noncommutative geometry on the D3 action, so we are left with noncommutative super-Yang-Mills with \(X\)-dependent metric and dilaton.

The D8 brane had 16 supersymmetries, and correspondingly the M theory background had also 16 supersymmetries, which means that the D3 brane action (noncommutative super-Yang-Mills) will have 8 linearly realized supersymmetries. The fermionic field content is the same as for the flat D3 brane, but half the supersymmetries are broken by the nonzero \(X^{mn}\) and the nontrivial noncommutativity.

In the near horizon region (at \(x = 0\)), the D8 background is flat, so it has 32 supersymmetries. Correspondingly, the DBI action has 16 supersymmetries if we put \(\phi\) and \(G_{ab}\) constant (and keep only the noncommutativity), as in the \(\theta\) constant case.

Let us now examine the star product, since it is nontrivial (space-dependent). For constant noncommutativity, the noncommutativity of the space can be traded for a modified product, the star product,

\[
f \star g = e^{i\theta^{ij}\partial_i |_{x=0}} (f(x)g(x'))\]

(80)

but when \(\theta\) is space-time dependent, we have to be more careful.

The first observation we can make is that our \(\theta^{ij}\) satisfies the associativity condition

\[
\theta^{il} \partial_i \theta^{jk} + \theta^{jl} \partial_j \theta^{ki} + \theta^{kl} \partial_k \theta^{ij} = 0
\]

(81)

and so we can define an associative star product. As an aside, we have a nonzero \(H_{123}\), yet the product is still associative. This is possible because \(\theta_{ij}\) is not invertible in the whole space \((1, 2, 3)\), but just in \((2, 3)\) (if it would be, then associativity and zero \(H\) field would be the same, see \([34]\)). Associative star products in the case of space dependent \(\theta\) can be defined with the prescription given by Kontsevich \([33]\).
The abstract formula is

\[ f \star g = \sum_n h^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma, \theta}(f, g) \]

\[ w_\Gamma = \frac{1}{n! (2\pi)^n} \int_{H_n} \Lambda_n \left( d\phi e^1 \wedge d\phi e^2 \right) \]

and where explicitly, derivatives which can act either on \( f \) and \( g \), or on \( \theta \), are contracted with other \( \theta \)'s. For example, up to second order in \( \theta \) we have

\[ f \star g = fg + \hbar \theta^{ij} \partial_i f \partial_j g + \frac{\hbar^2}{2} \theta^{ij} \theta^{kl} \partial_i \partial_k f \partial_j \partial_l g \]

\[ + \frac{\hbar^2}{3} (\theta^{ij} \partial_j \theta^{kl} (\partial_i \partial_k f \partial_l g - \partial_k \partial_i f \partial_l g) + O(\hbar^3). \] (83)

But in our case we have not only the associativity condition (81) but also the more restrictive condition

\[ \theta^{ij} \partial_j \theta^{kl} = 0 \] (84)

which implies that there will be no corrections (since derivatives on \( \theta \) will always appear in the above combination, as the only object with contravariant indices is \( \theta \)). Then the Kontsevich product will be the same as the usual star product, a fact which is obvious in the expanded form. The exponential form will also be the same, and we therefore have

\[ f \star g = e^{i\theta^{ij}(x) \partial_j f (x')} g(x') \bigg|_{x=x'} = e^{i\theta^{ij}(x') \partial_i f (x)} g(x) \bigg|_{x=x'} . \] (85)

As we saw, we can change coordinates by \( x_2 = x_2' x_1', x_1 = x_1' \), and then \( \theta^{23} = -\tilde{\alpha} \), but then the metric is not flat anymore. We can however obtain a third form for the star product. Since in these new coordinates the product is

\[ f \star g = e^{i\theta^{ij}(x') \partial_i f (x')} g(x') \bigg|_{x=x'} \]

by going back to unprimed coordinates we get

\[ f \star g = e^{i\theta^{ij} \partial_i \partial_j f (x')} g(y') \bigg|_{x'=y'} . \] (86)

As this point it is interesting to observe the similarity of our noncommutative theory to the one described by Hashimoto and Sethi in [35]. Moreover, if we take a Penrose-like limit (infinite boost, while taking a relevant mass parameter to zero) we obtain “half” their solution. Indeed, take an infinite boost in the \( x_1 \) direction, and also take \( \tilde{\alpha} \) to zero as

\[ x_1 \simeq \frac{e^\epsilon}{2} (x_1' + t') = \frac{e^\epsilon}{2} x'^+, \quad \tilde{\alpha} = \frac{e^{-\epsilon}}{2} \tilde{\alpha}' \] (88)

and drop the primes. Then the open string variables (88) become the flat metric (and constant string coupling), with \( \theta^{23} = -\alpha x^+ \) (notice that \( H = 1 \) in this limit, since \( \tilde{\alpha} \rightarrow \))
Their solution has also $\theta^{-3} = -\alpha x^2$. In these coordinates (with spacetime dependent noncommutativity), their closed string variables (metric, B field and dilaton) are

\[
\begin{align*}
    ds^2 &= \left[-2dx^+dx^- + \frac{R^2(dx_2^2 + dx_3^2)}{R^2 + (x^+)^2}\right] - \frac{x_2^2(dx^+)^2}{(x^+)^2} + \frac{2x_2x^+dx_2dx^+}{R^2 + (x^+)^2}, \\
    B &= \frac{Rx^+dx_2 \wedge dx_3}{R^2 + (x^+)^2} - \frac{Rx_2dx^+ \wedge dx_3}{R^2 + (x^+)^2}, \\
    e^\phi &= g_s \sqrt{\frac{R^2}{R^2 + (x^+)^2}}
\end{align*}
\]

where the terms in brackets correspond to the “Penrose” limit of our solution. In these coordinates, their open string metric is flat and the open string dilaton constant, just as in our case. So we are obtaining “half” the solution in [35], which seems to suggest that both are part of a 1-parameter set of solutions.

Another observation is that in [35] there is also a transformation of coordinates which makes $\theta^{ij}$ constant, namely

\[
\begin{align*}
    x^+ &= x'^+, \\
    x_2 &= x'_2x'^+, \\
    x^- &= x'^- + \frac{1}{2}x'^+x_2^2
\end{align*}
\]

whereas for the “Penrose” limit of our solution it is just

\[
x^+ = x'^+, \quad x_2 = x'_2x'^+. \tag{91}
\]

However, in their case (84) is not satisfied, while (81) is still satisfied, so in their case the Kontsevich product is different from the usual star product, even though there is a coordinate transformation which makes $\theta$ constant.

Finally we note that an example of spacetime dependent noncommutativity has been analyzed in [36], and the Seiberg-Witten analysis still holds (even though $\theta$ is varying).

### 6 Spectrum of states

Now we take a step toward deriving the duality between the noncommutative super-Yang-Mills on (48) and massive IIA in the D8 background, by studying the spectrum of BPS states.

Type IIB string theory in ten dimensions can be obtained by compactifying M-theory on a 2-torus of vanishing area, but fixed complex structure. In this case the Sethi-Susskind [5] and Banks-Seiberg [4] constructions gave evidence for the duality. We will follow the Sethi-Susskind construction, which is defined in 3+1 dimensions, setting it up so that we can go smoothly to our case. The IIB Matrix model is 2+1 super-Yang-Mills which has naturally an $SO(7)$ invariance, but the claim is that at strong coupling it develops an $SO(8)$ invariance (which is consistent with the supersymmetry algebra and is the maximal $R$ symmetry). The
easiest way to see it is to go to 3+1d super-Yang-Mills and use electric-magnetic duality. There we have only an $SO(6)$ manifest invariance (6 scalars), which will be enhanced to $SO(8)$. In our case we naturally have 3+1d super-Yang-Mills, so it should be our starting point. In the massless case ($m = 0$), we still have an $SO(6)$ (6 scalars) enhanced to $SO(8)$, but in the massive case we have an $SO(5)$ (the scalar $X$ vev corresponding to the direction transverse to the D8 brane in IIA - is special), which should be enhanced to $SO(7)$.

Let us then set up 3+1d super-Yang-Mills for our use. The super-Yang-Mills lives on a dual torus of lengths $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3$. Sethi and Susskind have $R_2, R_3 \to 0, R_1 \to \infty$. The mass of a membrane on the shrinking torus $R_2, R_3$ is identified via the M theory-IIB duality with the momentum mode on another direction $Y$ in IIB,

$$\frac{R_2 R_3}{l_p^3} = \frac{1}{R_Y}$$  \hspace{1cm} (92)

with the limit $R_Y$ to infinity, and we set $R_Y = R_1$. By the above formula, we see that $R_Y = l_p^2/R_2$ (if $R_3$ is the M theory direction), and so as we said $R_Y$ is the extra transverse direction in the lightcone IIB theory which appears when the M theory torus shrinks to zero size. To obtain $SO(8)$ invariance, we indeed need to choose $R_Y = R_1 = R_\perp$, so that all the IIB lightcone coordinates, $X_1, X_Y, X_4, ..., X_9$ have length $R_\perp$. Then the 3+1d super-Yang-Mills coupling,

$$\tilde{g}^2_{YM} = \frac{l_p^2}{R_1 R_2 R_3} = 1$$ \hspace{1cm} (93)

so we are at the self-dual point, and we have electric-magnetic duality. As usual by the Sen-Seiberg procedure, $\hat{M}$ theory was introduced in order to show massive string degrees of freedom decouple from the super-Yang-Mills, but the T dual super-Yang-Mills variables depend only on M theory quantities.

The electric flux along $\tilde{R}_1$ corresponds to the momentum conjugate to $X_1$ under T duality $\partial A_1 \to \partial X_1$, and so it goes together with the other momenta to increase $SO(6)$ to $SO(7)$ invariance. Because of electric-magnetic duality however, $SO(7)$ becomes $SO(8)$.

In our case, when the 3d space $B$ in M theory shrinks to zero size, we have 2 extra transverse lightcone coordinates appearing in type IIB,

$$\frac{R_1 R_2}{l_p^3} = \frac{1}{R_{Y_2}} \text{ and } \frac{R_1 R_3}{l_p^3} = \frac{1}{R_{Y_3}}$$  \hspace{1cm} (94)

and so $SO(8)$ invariance (in the massless case) should be recovered when $X_{Y_2}, X_{Y_3}, X_4, ..., X_9$ have the same length,

$$R_{Y_2} = R_{Y_3} = R_\perp$$ \hspace{1cm} (95)

For Sethi and Susskind, the magnetic flux $F_{23}$ in the D2 theory (and correspondingly in the D3 theory as well) was T dual to wrapping number of membranes, which by the M-IIB duality (and 9-11 flip) was identified with momentum on $R_Y$ (see [92]). For our case, the invariance we seek is with the momentum in $R_{Y_2}$ and $R_{Y_3}$, which corresponds in the D0 theory by (94) to wrapping number on 12 and 13 respectively. By T duality, in the D3 theory, this is magnetic flux $F_{12}$ and $F_{13}$.
So let us see the $SO(8)$ invariance in the 2 cases from the YM energy. The energy of magnetic fluxes and electric fluxes is deduced from

$$\int dx^2 dx^3 tr F_{23} = n_{m}^{23}$$
$$\int dx^2 dx^3 tr F_{01} = n_{e}^{23} g_{YM}$$

(for magnetic flux on 23 and electric flux on 1) so that

$$tr F_{23} = \frac{n_{m}^{23}}{R_2 R_3}$$
$$tr F_{01} = \frac{n_{e}^{23} g_{YM}^{2}}{R_2 R_3}.$$  \hfill (96)

If we add momentum modes on $x_1, x_2, x_3$ ($p^i = n^i/\tilde{R}_i$), the energy

$$E = \frac{1}{2 g_{YM}^2} \int dx^4 dx^3 [tr F_{0i}^2 + \sum_{j<k} tr F_{jk}^2 + (\partial_{\mu} X^i)^2]$$ \hfill (98)

becomes (using that $tr_{U(N)} F_{\mu\nu}^2 = tr_{SU(N)} F_{\mu\nu}^2 + 1/N(tr F_{\mu\nu})^2$ and concentrating on the U(1) piece)

$$NE = \frac{\tilde{R}_3}{2R_1 R_2} [\frac{n_{m12}^2}{g_{YM}^2} + \frac{n_{e12}^2 g_{YM}^2}{g_{YM}^2}] + \frac{\tilde{R}_1}{2R_2 R_3} [\frac{n_{m23}^2}{g_{YM}^2} + \frac{n_{e23}^2 g_{YM}^2}{g_{YM}^2}] + \frac{\tilde{R}_2}{2R_1 R_3} [\frac{n_{m13}^2}{g_{YM}^2} + \frac{n_{e13}^2 g_{YM}^2}{g_{YM}^2}] + \sqrt{(\frac{n_{1}}{R_1})^2 + (\frac{n_{2}}{R_2})^2 + (\frac{n_{3}}{R_3})^2}.$$ \hfill (99)

This formula is also in accord with \cite{13, 37}. The elementary excitations of the theory are the momentum modes $n_i$, but the Matrix theory prescription tells us to look at the energy of excitations on the moduli space, in other words for excitations with energy much smaller than that of momentum modes.

In the Sethi-Susskind case ($\tilde{g}_{YM} = 1$), the smallest elementary excitation (momentum mode) is of order $1/\tilde{R}_{2,3}$ ($\tilde{R}_1 \ll \tilde{R}_{2,3}$), and the 12 and 13 fluxes have energy much bigger than that, whereas the 23 fluxes have much smaller energy, so they should be thought of as moduli.

In our case, ($\tilde{g}_{YM} \to \infty, \tilde{R}_1 \gg \tilde{R}_{2,3}$), the smallest elementary excitation is of order $1/\tilde{R}_1$, and the 23 fluxes have energy much bigger than that. For the 12 and 13 fluxes, choosing $\tilde{R}_2 = \tilde{R}_3$ (as we have seen we need in \cite{94}), the prefactor (energy scale) of the fluxes is also $1/\tilde{R}_1$, as for the momentum modes. However, since $\tilde{g}_{YM}$ is infinite, the electric fluxes have energy much larger than the momentum modes, whereas the magnetic fluxes have energy much smaller than the momentum modes, so are real moduli.

Now let us see what should we compare the energy of those moduli against. The energy on the moduli space of the 6 scalar fields is

$$NE = \frac{\tilde{p}_1^2}{2M_0} = \frac{\tilde{n}_1^2}{2\tilde{R}_1^2 M_0} = \frac{\tilde{n}_2^2}{2M_0 \tilde{R}_2^2 \tilde{p}_2} = \frac{\tilde{n}_3^2}{2M_0 \tilde{R}_3^2 \tilde{p}_3}$$ \hfill (100)
where we have put the transverse space in a box of size \(R_\perp\), to be equated with \(R_{Y_2} = R_{Y_3}\) as above, and \(M_0\) is the mass of those moduli, but we have taken into account that we are calculating energies in the \(\bar{M}\) theory. \(M_0\) comes from the fact that we are really expanding the DBI action of the D3 in order to get (100). But then

\[
M_0 = V_0 T_p = \frac{\tilde{R}_1 \tilde{R}_2 \tilde{R}_3}{\bar{g}_{s} l_s} = \frac{1}{\bar{g}_{s} l_s} NE = \frac{\bar{g}_{s} l_s^2 \tilde{p}_\perp^2}{2M_0} = \frac{\bar{g}_{s} l_s^2}{2R_1^2} \tilde{p}_\perp^2 = R \frac{\bar{g}_{s} l_s^2}{2R_1^2} \tag{101}
\]

Here \(\tilde{l}_s\) is the string length for the \(\bar{M}\) theory. Then in the Sethi-Susskind case we have the energy of the moduli (using (93))

\[
E = l_s^2 \bar{g}_{s} l_s = \frac{\bar{g}_{s} l_s^2}{2N} \left[ n_{m12}^2 + n_{e12}^2 \right] + \tilde{\bar{g}}^2 \left[ \bar{R}_1^2 \right]
\]

which is \(SO(8)\) invariant if we put \(R_y = R_1 = R_\perp\), and in our case (with \(m = 0\)) we get (using that \(\bar{g}_{Y_M}^2 = l_s^2/(R_2 R_3)\), \(\tilde{R}_1 = l_s^2/R\) and (94))

\[
E = R \frac{\bar{g}_{s} l_s^2}{2N} \left[ n_{m_{13}}^2 + n_{m_{12}}^2 + \tilde{\bar{g}}^2 \left[ \bar{R}_1^2 \right] \right] \tag{102}
\]

which is \(SO(8)\) invariant if we put \(R_{Y_2} = R_{Y_3} = R_\perp\). We notice that the formulas (102) and (103) are exactly what we expect from supergravity and from the BFSS model [8] for the free supergravitons. Of course it would be more interesting to derive the interaction piece.

Finally, what happens in the massive case \(m \neq 0\) (in the \(X = 0\) sector, which has still 16 supersymmetries)? As we mentioned, one of the scalars \((X)\) corresponds to the direction transverse to the D8 brane, so we have manifest \(SO(5)\) invariance of the scalars which should be lifted to a \(SO(7)\) invariance, of the lightcone string theory in the D8 background. So the above formulas should apply to 5 of the scalars, but not to the \(X\) direction.

The noncommutative super-Yang-Mills is obtained by replacing the usual product with the star product. To first order in \(\theta\), the action (see [26], equation 4.27) is

\[
S = \int [F_{ij} F_{mn} G^{im} G^{jn} (1 - 1/2 F_{ij} \theta^{ij}) - 2 \theta^{kl} F_{ki} F_{lj} F_{mn} G^{im} G^{jn}] - \int [F_{ij}^2 (1 + 3 F_{23} \theta^{23})] \tag{104}
\]

and we see that if \(F_{23} = 0\), the action is unmodified, and so the energy formula is unmodified as well, as expected. The higher order terms will just contain terms with derivatives of \(F\), and so a constant \(F_{23}\) and \(F_{13}\) will still be a solution, and the energy formula of the magnetic fluxes will again be unmodified. As for the momentum modes \(\tilde{p}_\perp\), they are modes on the moduli space, not in worldvolume, so there is no reason for their energy to be modified. We can easily verify that there are no \(\theta\) corrections to the energy in our case by applying the general formula in [13] to the moduli space of our theory.

As for the action of the elementary strings, that is easy to understand. In the Type IIB case, [4] considered the limit of small coupling, when \(R_3 \ll R_2 \rightarrow \tilde{R}_2 \ll \tilde{R}_3\), so Yang-Mills
moduli space excitations occur only in the 3 direction. Then the string action is just the sigma model action on the moduli space, with worldvolume given by time and the 3 direction. In our case, there is no need to take small coupling, since already $R_1 \ll R_{2,3} \to \bar{R}_1 \gg R_{2,3}$, and so Yang-Mills moduli space excitations already occur just in the worldvolume 1 direction, and the string action is again the sigma model.

What about the supergravity mass? Massive IIA supergravity \[1\] has a massless graviton, a dilaton and an antisymmetric tensor $B_{\mu\nu}$ which acquire a mass proportional to $m$, a massless 3-form $A_{(3)}$, and massive fermions (gravitino and spin 1/2). The 1-form $A_{(1)}$ is gauged away, since it appears in the combination $F_{(2)} + MB_{(2)}$. In the D8 brane background, one has to study the wave equation for each field. The massless fields (graviton and $A_{(3)}$) can still have a constant wavefunction in the $x$ direction (transverse to the D8), and then

$$ p^2 = -2p^+ p^- + \vec{p}_\perp^2 = 0 \Rightarrow E = p^- = \frac{\vec{p}_\perp^2}{2p^+} $$

(105)

(where $p$ is the momentum along the D8) which reproduces (103). A nontrivial wavefunction in the $x$ direction implies that $p^2 \neq 0$, and correspondingly an extra term in the energy.

For one of the massive fields, we have to study the wave equation in the D8 background. The Einstein metric is

$$ ds^2_E = H^{1/8}dy_i^2 + H^{9/8}dx^2 $$

(106)

which means that the wave equation for a scalar of mass $aM$ in this background (the dilaton is such a scalar), with $a$ a constant, is

$$ (\Box - a^2M^2)\phi = 0 \Rightarrow (\partial^2_i + H^{-1}\partial^2_x - a^2M^2)\phi = 0 $$

(107)

and then for a separated solution,

$$ \phi = e^{jny_i}\phi(x) $$

(108)

we get

$$ (\partial^2_x - p^2(1 + M|x|) - a^2M^2(1 + M|x|)^{9/8})\phi(x) = 0 . $$

(109)

Notice that this equation does not admit a constant wavefunction, since $\phi(x) = c, -p^2 - a^2M^2 = 0$ is not a solution. At large $x$, it has the asymptotic solutions

$$ \phi(x) = e^{\pm \frac{16}{25}a(Mx)^{25/16}} $$

(110)

so we can keep only the decaying solution and at small $x$ it becomes a combination of the oscillatory solutions

$$ \phi(x) = e^{ix\sqrt{-p^2 - a^2M^2}} $$

(111)

(for $p^2 < -a^2M^2$). So from matching the wavefunction and its derivative over $x = 0$ we get a condition on $p^2$, which will be that the coefficient of the $\sin$ should be zero, which will imply a quantization condition, of the type

$$ p^2 = -M^2_n(a, M) . $$

(112)
The same type of analysis should hold for every massive field in supergravity, so in general we will get a formula for the lightcone energy of the type

$$E = \frac{\vec{p}_+^2 + M_0^2(a, M)}{2p^+}. \tag{113}$$

Correspondingly, we expect to find in super-Yang-Mills that various moduli have such an additional mass term, which will depend on the detailed structure of the interactions permitted by the 8 linearly realized supersymmetries. However, these massive moduli will appear only when we look at nontrivial wavefunctions for the super-Yang-Mills scalar $X$ (other than $X =$ constant and small), so it is hard to analyze. The moduli with trivial wavefunctions in the $X$ direction will correspond to the massless supergravity modes, and as we saw, they have the right lightcone energy. Moreover, even if we would find the massive super-Yang-Mills moduli, on the supergravity side it is also hard to get any results (although one could of course use numerical methods to find the mass terms).

But we can make one observation. On the supergravity side, all the fermions are massive, so we expect that also fermionic super-Yang-Mills moduli will have a mass term in the energy. One hint that this might happen is that we expect the D3 brane fermions to have a worldvolume mass term of the type

$$\bar{\psi}D\psi,$$

where $D$ is the spacetime Killing spinor operator pulled back on the worldvolume \[38\]. The kinetic term then contains a term of the type

$$\alpha' H_{\mu\nu\rho} \bar{\psi} \Gamma^{\mu\nu\rho} \psi \tag{114}$$

which would imply a mass proportional to $l_s^2 H_{abc}$ (flat indices). But for our closed string background in the Seiberg-Witten limit \[40\], $B_{23} = 1/(\hat{\alpha} x_1)$ and $g_{22} = g_{33} = (l_s^2/(\hat{\alpha} x_1))^2$, and so the fermion mass will be proportional to

$$M = \frac{\hat{\alpha}}{l_s^2} = \frac{Mr_1}{r_2 r_3 l_P} = M l_P \tag{115}$$

with $M$ being the supergravity mass. Then the moduli mass will be $M_0(a, M) = M_n(a, M) l_P / \hat{\alpha}$. If we would get an energy $M_n^2(a, M)/2NM_0 = M_n^2(a, M)/2p^+$, it would be as desired. It would be of course very interesting to see whether one can recover all the supergravity mass terms for the lightcone energy, but as we saw, the analysis looks quite difficult.

## 7 Holographic dual

Let us try to write down the holographic dual of our noncommutative super-Yang-Mills defined on \[45\] in the spirit of the AdS/CFT correspondence. We have to write down a solution for D3 branes in the closed string background \[46\] corresponding to \[45\] and then take a decoupling limit. It turns out however to be easier to start with D1 branes in the background \[14\] and then make two T dualities. Indeed, as we saw, the background \[14\]
corresponds to NS5 branes smeared over the directions 1,2,3. We have to put D1 branes at $x = 0$ along time and $x_1$. But this background is one of D1 ending on NS5’s, smeared over the D1 direction, as well as 2 others, transverse to both NS5 and D1. The S dual of this (type IIB) configuration is F1 ending on D5, which we know that exists. Then the original IIA metric is D0’s parallel to KK monopoles, and after $T_1$ and $T_2$ we have D2 ending on KK monopoles, and finally after $T_3$ we have D3 ending on an unusually T dualized KK monopole (an 8 dimensional worldvolume).

The solution for D1 ending on NS5’s, depending only on the coordinate $x$ can be found pretty easily, namely
\[
\begin{align*}
    ds^2 &= -dt^2 H_1^{-1/2} + H_1^{1/2} d\phi_5^2 + H H_1^{-1/2} dx_1^2 / r_1^2 + H H_1^{1/2} (dx^2 + r_2^2 dx_2^2 + r_3^2 dx_3^2) \\
    B_{12} &= mx_3 \\
    e^\phi &= e^{\phi_0} H_1^{1/2} H_1^{1/2}.
\end{align*}
\]

T dualizing on $T_2$ we get
\[
\begin{align*}
    ds^2 &= -dt^2 H_1^{-1/2} + H_1^{1/2} d\phi_5^2 + H H_1^{-1/2} dx_1^2 / r_1^2 + H H_1^{1/2} (dx^2 + r_2^2 dx_2^2 + r_3^2 dx_3^2) \\
    &\quad + \frac{H^{-1} H_1^{-1/2}}{r_2} (dx_2 + mx_3 dx_1)^2 \\
    e^\phi &= \frac{e^{\phi_0}}{r_1 r_2} H_1^{1/4}.
\end{align*}
\]

And finally, after the coordinate transformation $\frac{r_1}{r_2 r_3}$ and $T_3$ T duality, we get (putting back the $l_s$ dependence)
\[
\begin{align*}
    ds^2 &= l_s^4 [-dt^2 l_s^{-4} H_1^{-1/2} + H_1^{1/2} l_s^{-4} d\phi_5^2 + H H_1^{-1/2} dx_1^2 / r_1^2 + H H_1^{1/2} l_s^{-4} dx^2 \\
    &\quad + \frac{H^{-1} H_1^{-1/2}}{1 + l_s^4 (\frac{r_1}{r_2 r_3})^2 m^2 x_1^2 / r_1^2} (dx_2^2 / r_2^2 + dx_3^2 / r_3^2)] \\
    B_{23} &= -l_s^4 \frac{mx_1}{r_2^2 r_3^2 H^2 H_1} \frac{1}{1 + l_s^4 (\frac{r_1}{r_2 r_3})^2 m^2 x_1^2 / r_1^2} \\
    e^\phi &= \frac{l_s^3 e^{\phi_0}}{r_1 r_2 r_3 H^{1/2}} (1 + l_s^4 (\frac{r_1}{r_2 r_3})^2 m^2 x_1^2 / H^2 H_1) -1/2.
\end{align*}
\]

However we need to generalize this to the fully localized solution, where the D3-branes are not smeared over the transverse directions.

The first thing we can do is to look in the near core region. In the near core region, $H$ is constant ($\simeq c$), and then there is no obstruction for making the harmonic function $H_1$ depend on all its transverse coordinates. Indeed, since $H$ is constant, we can ask to find a D1 brane solution in the corresponding flat background, and that is just the usual D1 brane with a nontrivial $B$ field, i.e. (116), with $H = c$ and $H_1(\phi_5, x_2, x_3, x)$ the usual harmonic function. Then after the two T dualities one gets the solution (118), where $H = c$ and
\[
    H_1 \simeq 1 + \frac{4\pi g_s N \alpha'^2}{(\phi^2 + c x^2)^2} \simeq 1 + \frac{4\pi g_s N \alpha'^2}{\phi^4}.\]
\[
(119)
\]

27
where in the last line we have used the fact that we work near \( x \simeq 0 \).

Let us now derive the equation for the full solution (outside the core). Partially localized intersections, where brane 1 with harmonic function \( H_1 \) lives on \( t, \vec{w}, \vec{x} \), and brane 2 with harmonic function \( H_2 \) lives on \( t, \vec{w}, \vec{y} \), with overall transverse space \( \vec{z} \), are written in terms of harmonic functions \( H_1 \) and \( H_2 \) in the usual way, except that now \( H_1 \) and \( H_2 \) satisfy the equations (e.g. \[39\], \[40\])

\[
\partial_x^2 H_1(z, x) + H_2(z) \partial_y^2 H_1(z, y) = 0, \quad \partial_y^2 H_2(z, y) = 0 \quad \text{or} \quad \partial_x^2 H_2(z, x) + H_1(z) \partial_x^2 H_2(z, x) = 0, \quad \partial_x^2 H_1 = 0. \tag{120}
\]

In other words, we delocalize one brane (say brane 2) over the worldvolume coordinates of the other brane (1), and then \( H_1 \) is harmonic (obeys the Laplace equation) in the background of brane 2. This is true for any kind of branes, but in particular \[40\] has derived explicitly these equations for the 11d intersection of M2 and M5 (over a string). This intersection is related to our D1-NS5 solution as follows. Dimensionally reduce to type IIA on the common string, to an F1-D4(0) solution, T dualize to IIB on a transverse direction to a F1-D5(0), and then S dualize to D1-NS5(0).

In our case, the harmonic function \( H \) is delocalized over the \( \vec{D} \) and \( \vec{W} \), as well as over \( x_2, x_3 \), over which we need to \( \mathbb{T} \) dualize, and \( H_1 \) is delocalized over \( x_2, x_3 \). So the full solution is given by \[16\], where \( H_1 \) satisfies the equation

\[
[\partial_x^2 + H(x) \partial_\vec{D}^2] H_1(\vec{D}, x) = Q \delta(\vec{D}) \delta(x) \tag{121}
\]

where we have put explicitly the source term \( Q = 16\pi^4 g_s N (\alpha')^2 \). Then also \[17\] and \[18\] are the corresponding \( \mathbb{T} \) dual solutions. We notice that near the core \( x = 0 \), \( H \simeq c \), so the solution is indeed \[19\].

In order to solve \[121\], we separate variables, by writing

\[
H_1(\vec{D}, x) = 1 + \int \frac{d^5p}{(2\pi)^5} e^{i\vec{p} \cdot \vec{D}} H_{1,p}(x) = 1 + \frac{1}{8\pi^3 r^2} \int_0^\infty dp p^2 \left( \frac{\sin(pr)}{pr} - \cos(pr) \right) H_{1,p}(x) \tag{122}
\]

and get the equation

\[
H_{1,p}''(x) - p^2 H(x) H_{1,p}(x) = Q \delta(x). \tag{123}
\]

By putting \( H = c + m|x| \) (we will keep this form for now and replace it later with \( c = 1 \) and \( m = \tilde{a}/l_s^2 \)) and

\[
\vec{x} = \left( \frac{p}{m} \right)^{1/3}(c + m|x|) \tag{124}
\]

we get the Airy equation

\[
\frac{d^2 H_p}{d\vec{x}^2} - \vec{x} H_p(\vec{x}) = Q m^{-1/3} p^{-2/3} \delta(\vec{x} - c(p/m)^{2/3}) \tag{125}
\]

which has solutions in terms of the Bessel functions \( I_{1/3} \) and \( K_{1/3} \). We choose \( K_{1/3} \) which decays exponentially at infinity, and get

\[
H_p(\vec{x}) = c_p \vec{x}^{1/2} K_{1/3} \left( \frac{2}{3} \vec{x}^{3/2} \right). \tag{126}
\]
The coefficient $c_p$ can be fixed by matching with the normalization of the $\delta$ function source. We get
\[ c_p = \frac{Q\sqrt{c}}{2p^{1/3}m^{2/3}[K_{1/3}(\frac{2p}{3}mc^{3/2}) - \frac{2p}{3}mc^{3/2}K_{4/3}(\frac{2p}{3}mc^{3/2})]} \]
Therefore the final formula for the harmonic function is
\[ H_1(r, x) = 1 + \frac{Q\sqrt{c}}{8\pi^3m^{2/3}} \frac{\beta^{1/3}}{r^2} \int dp^2 \frac{(\sin(pr) - \cos(pr))K_{1/3}(\frac{2\beta p}{3})}{K_{1/3}(\frac{2p}{3}mc^{3/2}) - \frac{2p}{3}mc^{3/2}K_{4/3}(\frac{2p}{3}mc^{3/2})} \]
with
\[ \beta = \frac{(c + m|x|)^{2/3}}{m} \]
So we have found the full solution for the D3 branes in the background.

We can now write down the decoupling limit for the holographic dual in the near core ($x = 0$), namely
\[ \alpha' \to 0, U = \frac{|\vec{\sigma}|}{\alpha'} = \text{fixed}, \quad X = \frac{x}{\alpha'} = \text{fixed}, \quad g_sN = \lambda = \text{fixed} \]
but we have to supplement it with
\[ r_i \to 0, \bar{y}_i = \frac{t^s_{ix_i}}{r_i} = \text{fixed}, \quad \bar{\alpha} = \frac{mr_{1s}}{r_2r_3} = \text{fixed} \]
and then we have the holographic dual
\[ ds^2 = \alpha'\left[ \frac{U^2}{\sqrt{\lambda}}(-dt^2 + d\bar{y}_1^2 + \frac{d\bar{y}_2^2 + d\bar{y}_3^2}{1 + \Delta^4U^4}) + \frac{\sqrt{\lambda}}{U^2}(dX^2 + dU^2 + U^2d\Omega_4^2) \right] \]
This metric is then dual to super-Yang-Mills with
\[ [\bar{y}_2, \bar{y}_3] = i\bar{\alpha}\bar{y}_1, \quad ds^2 = -dt^2 + d\bar{y}_2^2 + d\bar{y}_1d\bar{y}_3 + d\bar{y}_3^2 \]
We note that the holographic dual in the near core region is just what we would have expected from the usual noncommutative case, with holographic dual
\[ ds^2 = \alpha'\left[ \frac{U^2}{\sqrt{\lambda}}(-dt^2 + d\bar{y}_1^2 + \frac{d\bar{y}_2^2 + d\bar{y}_3^2}{1 + \Delta^4U^4}) + \frac{\sqrt{\lambda}}{U^2}(dU^2 + U^2d\Omega_3^2) \right] \]
and $\Delta^2 = \theta^{23}$.
To get the full holographic dual, since (remembering just for the purpose of next formula that what we call $l_s$ is really $\bar{l}_s$, whereas $t_s^A$ still appears in $H$ and also $m$ denoting the integer D8 number)
\[ H = 1 + \frac{mg_s^A}{l_s}X + 1 + \frac{mg_s^A}{l_s}\bar{X} = 1 + \tilde{\alpha}|X| \]
we replace $c = 1$, $m = \tilde{\alpha}/l_s^2$, $Q = 16\pi^4 g_s N l_s^4$, together with the rest of the limit into (128), and rescaling the integration variable as $p = P/l_s^2$ we get

$$H_1(r, x) \simeq \frac{h_1(U, X)}{l_s^4}$$

where

$$h_1(U, X) = \frac{2\pi g_s N}{\tilde{\alpha}^{2/3}} \frac{\tilde{\beta}^{1/3}}{U^2} \int_0^\infty dPP^2 \frac{(\sin(PU) - \cos(PU))K_{1/3}(\frac{2\tilde{\beta}P}{3\tilde{\alpha}}) - \frac{P}{\tilde{\alpha}}K_{4/3}(\frac{2\tilde{\beta}P}{3\tilde{\alpha}})}{K_{1/3}(\frac{2\tilde{\beta}P}{3\tilde{\alpha}}) - \frac{P}{\tilde{\alpha}}K_{4/3}(\frac{2\tilde{\beta}P}{3\tilde{\alpha}})}$$

and

$$\tilde{\beta} = \frac{(1 + \tilde{\alpha}|X|)^{2/3}}{\tilde{\alpha}}.$$ 

Then the full holographic dual is

$$ds^2 = \alpha'[h_1^{-1/2}(U, X)(-dt^2 + H d\vec{y}_1^2 + H^{-1} \frac{d\vec{y}_2^2 + d\vec{y}_3^2}{\tilde{\alpha}^2 y_1^2 U^4}) + h_1^{1/2}(HdX^2 + dU^2 + U^2 d\Omega_4^2)]$$

We note that the Seiberg-Witten limit is a subset of the holographic limit, as it should be.

8 Conclusions and Discussion

We have proposed a new nonperturbative formulation of massive Type IIA string theory in terms of a noncommutative Yang-Mills theory with space dependent noncommutativity parameter. There remains much to study. In particular, it would be very interesting to construct in more detail the interaction terms in the action, the energies of physical excitations and to study the S-duality properties of this noncommutative gauge theory. A more direct derivation of the non-commutative Yang Mills in section 5 starting from the solution of the Matrix Theory constraints of section 4, using information about the action of zero branes in the curved space of the twisted torus will be useful. In fact it may be easier to try and guess the form of the zero brane action which would lead to the actions in section 5, using the $X$ matrices constructed in section 4. Progress in these directions is likely to also be useful in flux compactifications since T-duality of the twisted torus gives a background with $H$-flux as discussed in section 3. These compactifications offer promising avenues toward the problem of fixing moduli in string phenomenology [23].

We note that we have described massive IIA theory in terms of a matrix model of D3 branes with noncommutativity, a theory which has a holographic dual. As a limit, massless IIA theory is described by a matrix model of D3 branes, which is dual to string theory in $AdS_5 \times S_5$. But there are two things we should observe:

1) The D3 branes are on a torus, which translates in making identifications in $AdS_5$ (in Poincare coordinates, $ds^2 = g^2(-dt^2 + d\vec{x}^2) + dy^2/y^2$, and the $\vec{x}$ coordinates are identified on a torus).

2) There are different observables in the D3 brane theory which describe flat space IIA string theory and $AdS_5 \times S_5$ string theory. For $AdS_5 \times S_5$, we look at gauge invariant observables in the D3 brane theory, whereas for the IIA matrix model we look at wavefunctions.
on the moduli space, thereby spontaneously breaking gauge invariance. Holographic duals in the context of 8-brane solutions have also been discussed recently in [41].

We comment on the relation of this construction to Type IIA string theory [42], where D8-branes and O8-planes coexist. The massive Type IIA physics is recovered by focusing on the local physics between a pair of separated D8-branes, or equivalently, by sending the D8-branes and O8-planes off to infinity. There exists a Matrix proposal for the complete nonperturbative Type IIA system [43, 44, 45, 46] which is related by S-duality to the $E_8 \times E_8$ heterotic string. It would be interesting to recover the noncommutative theory described in this paper by integrating out degrees of freedom in these heterotic Matrix models.

As we mentioned in section 2, a generalized Scherk-Schwarz reduction based on a scaling symmetry of the equations of motion gives a ten dimensional supergravity which has de Sitter solution [47]. It was observed in [16] that these can be viewed in terms a Euclidean radial reduction from M Theory. This suggests that a Matrix Model could be found by generalizing the dimensional reduction methods of Matrix Theory that we have used to radial reductions. This is of course a non-trivial generalization since the spacetime of M-Theory, and hence a Euclidean radial direction, appears very indirectly in Matrix Theory. Rather than imposing the constraints directly on a few $X$ fields corresponding to the compactified directions, one has to scale all the $X$ matrices as well as the worldline time coordinate. This approach appears non-trivial and very different from proposals made for a Matrix Model for de Sitter made so far [48], [49], and is an interesting avenue for the future.

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Appendix A. Limits

In this appendix we review the various Matrix theory limits, and derive the correct limit in our case. For completeness, let us recall the formulas relating the M theory parameters on a spatial circle to the IIA string theory parameters. They are obtained from

\[
\frac{1}{g_s l_s} = \frac{1}{R_{11}}, \quad \frac{1}{l_s^2} = \frac{R_{11}}{l_P^3}.
\]  

(A.1)

Sen \[7\] and Seiberg \[6\] used a construction for M theory compactified on $T^p$ in a limit of vanishing radii. We refer to this as $\bar{M}$ theory, taking

\[
\bar{g}_s, \bar{l}_s \to 0 \Rightarrow \bar{g}_s, \bar{l}_s \to 0 \quad \text{(A.2)}
\]

such that

\[
a_i = \frac{\bar{R}_i}{l_P}, \quad M = \frac{\bar{R}_{11}}{l_P^2}
\]

(A.3)

are held fixed. After dimensionally reducing on $\bar{R}_{11}$ to string theory and making T dualities on all the $\bar{R}_i$, the T dual variables are

\[
\bar{l}_s = \bar{l}_s = M^{-1} \bar{g}_s^{1/3}
\]

\[
\bar{R}_i = \frac{l_s^2}{R_i} = \frac{1}{M a_i}
\]

\[
\tilde{g}_s = \frac{\bar{g}_s}{\prod_{i=1}^p (\bar{R}_i/l_s)} = \bar{g}_s^{1-p/3} \prod_{i=1}^p a_i^{-1}
\]

and moreover

\[
\frac{1}{g_{YM}^2} = \frac{\bar{l}_s^3}{\bar{g}_s}, \quad \frac{1}{\tilde{g}_{YM}^2} = \frac{\bar{l}_s^{3-p}}{\bar{g}_s}
\]

(A.4)

such that

\[
g_{YM}^2 = M^3, \quad \tilde{g}_{YM}^2 = M^{3-p} \prod_{i=1}^p a_i^{-1}
\]

(A.5)

So the limit was chosen to decouple string theory both in the original and in the dual theory ($\bar{g}_s, \bar{l}_s, \bar{g}_s, \bar{l}_s \to 0$), while keeping the Yang-Mills couplings ($g_{YM}$ of the D0-branes and $\tilde{g}_{YM}$ of the Dn-branes) and the dual radii finite.

Let us now review the BFSS point of view, which is also advocated for the Matrix string of type IIA, and the Matrix theory of type IIB, and then apply it for our case. After that, we will look at the relation between Sen-Seiberg and BFSS and apply it to the Matrix models, finally deriving our limit.

BFSS \[8\] chose the limit

\[
R_{11} \sim N \to \infty, \quad l_P = \text{fixed}.
\]  

(A.7)

So that

\[
g_s = (R_{11}/l_P)^{3/2} \to \infty, \quad l_s = \frac{l_P^{3/2}}{R_{11}} \to 0
\]

(A.8)
and thus one obtained an $U(\infty)$ D0 brane theory. The argument being that in the limit, string theory does decouple (even though $g_s$ is infinite), because there are no string states other than D0 branes which have momentum on the 11th direction (that is the D0 charge), and so if we look at fixed momentum $N/R_{11}$, strings decouple.

The analogous statement happened for the IIA Matrix string. One added to the above construction a compactification on a finite $R_9$, and then made a 9-11 flip, meaning one reinterprets 9 as the 11-th direction. Since $l_P$ was finite, after the flip

$$g_s = \left( R_9 / l_P \right)^{3/2} = \text{finite}, \quad l_s = \frac{l_P^{3/2}}{R_9^{3/2}} = \text{finite}. \quad (A.9)$$

The IIB Matrix theory was similar. Add to the BFSS two extra radii $R_1, R_2 \to 0$, with $R_1/R_2$ finite. We know that M theory on this space gives IIB with finite coupling. Then take the BFSS construction and consider $l_P \to 0$, but independent of $N$ (which is consistent with the BFSS limit), such that one holds $R_1/l_P^2$ and $R_2/l_P^2$ fixed (the $(p,q)$ type IIB string tensions fixed), then flip 9-11.

Then

$$g_s^B = \frac{R_1}{R_2}, \quad l_s^2 = \frac{l_P^3}{R_1} \quad (A.10)$$

are fixed in this limit.

Similarly for our case, for the new Matrix theory of type IIA obtained by compactifying on a radius of zero size, we have, first for the massless case: compactify on an extra $R_3 \to 0$ and make a T duality so that

$$g_s^A = g_s^B l_s = \frac{R_1 l_s}{R_2 R_3} = \text{fixed}, \quad l_s^2 = \frac{l_P^3}{R_9} = \text{fixed} \quad (A.11)$$

where we have as before flipped 9-11 (so this corresponds to IIA with $R_{11} = N$), and we note that now $g_s^B$ goes to zero, and it is $g_s^A$ which is finite. For the massive IIA case, everything is similar, with the addition of the new parameter $m$.

The equivalence of the BFSS limit and the Seiberg-Sen limit was derived as follows. The light-like circle compactification for finite $N$ (DLCQ, see [52]) with $p^+ = N/R$ finite (BFSS corresponds to $N$, $R$ infinite, keeping $p^+$ finite, with other possible compactified directions of fixed radii $R_i$),

$$\left( \begin{array}{c} x \\ t \end{array} \right) \sim \left( \begin{array}{c} x \\ t \end{array} \right) + \left( \begin{array}{c} R/\sqrt{2} \\ -R/\sqrt{2} \end{array} \right) \quad (A.12)$$

is understood as the $R_s \ll R$ limit of

$$\left( \begin{array}{c} x \\ t \end{array} \right) \sim \left( \begin{array}{c} x \\ t \end{array} \right) + \left( \begin{array}{c} \sqrt{R^2/2 + R_s^2} \\ -R/\sqrt{2} \end{array} \right) \quad (A.13)$$

which is the infinite boost limit ($\beta = R/\sqrt{R^2 + 2R_s^2}$) of

$$\left( \begin{array}{c} x \\ t \end{array} \right) \sim \left( \begin{array}{c} x \\ t \end{array} \right) + \left( \begin{array}{c} R_s \\ 0 \end{array} \right) \quad . \quad (A.14)$$
So the light-like compactification of M theory on $R$ is related to the $R_s \to 0$ limit of the spatial compactification of another M theory. If we subsequently rescale Planck’s constant such that $p^-$ and $p^i$ are held fixed we obtain the $\bar{M}$ theory described above. So

$$g_s = (\bar{R}_{11}\bar{M}_P)^{3/2}, \quad \bar{M}_s^2 = \bar{R}_{11}\bar{M}_P^3$$

$$p^+ = N/R, \quad p^- \sim R\bar{M}_P^2, \quad p^i \sim R_i\bar{M}_P$$

$$\bar{p}_{i1} = N/\bar{R}_{11}, \quad \bar{p}^- \sim \bar{R}_{11}\bar{M}_P^2, \quad \bar{p}^i \sim \bar{R}_i\bar{M}_P$$

(A.15)

and then

$$\bar{R}_{11}\bar{M}_P^2 = R\bar{M}_P^2, \quad \bar{R}_i\bar{M}_P = R_i\bar{M}_P$$

(A.16)

are held fixed in the $\bar{M}_P \to \infty$ limit. Then

$$g_s^2 = (\bar{R}_{11}\bar{M}_P)^{3/2} = \bar{R}_{11}^{3/4}(R\bar{M}_P^2)^{3/4} \to 0, \quad \bar{M}_s^2 = \bar{R}_{11}\bar{M}_P^3 = \bar{R}_{11}^{-1/2}(R\bar{M}_P^2)^{3/2} \to \infty$$

(A.17)

so string theory decouples and the D0 coupling is fixed

$$g_{YM}^2 = g_s^2\bar{M}_s^3 = (\bar{R}_{11}\bar{M}_P^2)^3 = (R\bar{M}_P^2)^3$$

(A.18)

If the (BFSS) M theory is compactified on a torus of fixed radii $R_i$ then one T dualizes the string theory coming from the $\bar{M}$ theory and gets

$$\bar{R}_i = \frac{1}{R_i\bar{M}_s^2} = \frac{1}{\bar{R}_iR_i\bar{M}_P^3}, \quad \bar{g}_s = g_s\bar{M}_s^p\prod_i \tilde{R}_i = \bar{M}_s^{p-3}\bar{R}_i^{p-3}\bar{M}_P^{p-3}\prod_i \tilde{R}_i \to 0 \text{ if } p < 3$$

$$g_{YM,Dp}^2 = \frac{\bar{g}_s^2}{\bar{M}_s^{p-3}} = R^3\bar{M}_P^p\prod_i \tilde{R}_i$$

(A.19)

So again string theory decouples and one gets a Dp brane theory of fixed Yang-Mills coupling and dual radii.

Applying this to the Matrix string theory, one relates again M theory on light-like $R \sim N$, with $l_P$ fixed and $R_9$ fixed to the $\bar{M}$ theory with $R_s \to 0, \bar{M}_P \to \infty, \bar{R}_9 \to 0$, that is a D0 brane theory on a vanishing circle. After T duality, it becomes a D1 brane theory with fixed $\bar{R}_9$.

On the M theory side, one flips 9-11, reinterpreting it as string theory with a light-like coordinate $R$, so

$$g_s = (R_9\bar{M}_P)^{3/2}, \quad l_s = l_P^{3/2}R_9^{-1/2}, R \sim Nl_s$$

(A.20)

Then on the $\bar{M}$ theory side, string theory decouples:

$$\bar{g}_s = R_{s}^{1/2}(R\bar{M}_P^2)^{3/2}\bar{R}_9 \to 0$$

$$\bar{M}_s^2 = R_{s}^{-1/2}(R\bar{M}_P^2)^{3/2} \to \infty$$

(A.21)

and the Yang-Mills parameters are

$$\bar{R}_9 = \frac{l_s^2}{R}, \quad \bar{g}_{YM}^2 = \frac{1}{g_s^2\bar{R}_9^2}$$

(A.22)
In this way a D1 theory with fixed parameters is related to a string theory with fixed parameters.

For our IIA Matrix model, M theory with $R_i \to 0, l_P \to 0, R \sim N l_s$ with

$$g_s^A = \frac{R_1 l_s}{R_2 R_3}, \quad l_s = \frac{l_P^{3/2}}{R_1^{1/2}} \quad (A.23)$$

fixed. Passing to the $\bar{M}$-theory we see we are left with a decoupled D3-brane theory on the T-dual space with parameters

$$\tilde{R}_1 = \frac{l_s^2}{R}, \quad \tilde{R}_2 = g_s^A l_s R_3, \quad \tilde{R}_3 = g_s^A l_s R_2, \quad \tilde{g}_s = g_{YM}^2 = \frac{l_s g_s^A}{R_1}. \quad (A.24)$$

We notice though that the Yang-Mills coupling and 2 of the radii are actually not finite, so there is probably a better description, but one has to find it. In particular, since the Yang-Mills coupling is going to infinity, one should S dualize, but the problem is in the presence of the noncommutativity it is not very obvious what that means, so we will stick with this description. At $\theta = 0$ the S-dual is a good description, and

$$\tilde{g}_{s,D} = \tilde{g}_{YM,D}^2 = \frac{R_1}{l_s g_s^A} \to 0. \quad (A.25)$$

But under S duality, the “dimensionless Newton constant” $\bar{k}^2/R^8 \sim \tilde{g}_s^2 (\bar{l}_s/R)^8$ (with R a fixed length scale in the metric) is invariant (since $g_s \to 1/g_s, R \to R/\sqrt{g_s}$ and we would like to have $\bar{l}_s/R \to 0$ as well as $\bar{l}_s/R_D \to 0$. (then, the S dual theory is decoupled, and therefore so is the original theory) The condition can be written as $g_s (\bar{l}_s/R)^2 \to 0$, that is, the coefficient of the first loop correction to the action should be negligible, and this condition is satisfied if we have an $\bar{M}$ theory.

We also notice that $\tilde{R}_1$ fixed, but $\tilde{R}_{2,3} \to 0$, but all $\tilde{R}_i/\bar{l}_s \to \infty$ (so we are talking about a 3+1d super-Yang-Mills!) and there are 2 fixed quantities, a dimensionless one, $\tilde{g}_{YM} g_s^2$, and a dimensionful one, $\tilde{R}_1$, which will be related to $g_s^A$ and $l_s^A$, respectively.

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