Quantum description of classical apparatus: Zeno effect and decoherence

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Abstract

We study the measurement process by treating classical detectors entirely quantum mechanically. As a generic model we use a point-contact detector coupled to an electron in a quantum dot and tunneling into the continuum. Transition to the classical description and the mechanism of decoherence are investigated. We concentrate on the influence of the measurement on the electron decay rate to the continuum. We demonstrate that the Zeno (or the anti-Zeno) effect requires a nonuniform density of states in the continuum. In this case we show that the anti-Zeno effect relates only to the average decay rate, whereas for sufficiently small time the Zeno effect always takes place. We discuss the experimental consequences of our results and the role of the projection postulate in a measurement process.
I. INTRODUCTION

The description of a measurement process has been a topic debated from the early development of quantum mechanics [1, 2]. Nevertheless, the understanding of quantum-mechanical measurements has not been achieved yet. The main problem is still the nature of the projection postulate [1], according to which the wave-function of the observed system is projected onto an eigenstate of the observable under consideration. During recent years the measurement problem received a great deal of attention due to exiting opportunities offered by developments in experimental techniques of optics and mesoscopic structures. The problem also has close connections to the rapidly growing fields of quantum cryptography and quantum computing.

One of the most striking problems, directly related to the projection postulate, is the so-called “Zeno paradox” (or “Zeno effect”), which suggests that frequent or continuous observations can inhibit (or slow down) the decay an unstable quantum system [3]. During the last two decades the Zeno effect has become a topic of great interest. It has been discussed in the areas of radioactive decay [4], polarized light [5], physics of atoms [6, 7], neutron physics [8], quantum optics [9], mesoscopic physics [10, 11] and even in cognitive science [12]. Recently, it was proposed that under some conditions repeated observations could accelerate the average transition rate of a quantum system, so called the anti-Zeno effect [10, 13, 14]. This effect has been further analyzed in Refs. [15, 16, 17, 18].

The Zeno paradox was originally introduced as an effect of continuous observation of an unstable state. Consider for instance a particle localized initially in a potential well, which decays to the continuum via tunneling through the barrier, Fig. 1. It is well-known that the probability of finding the particle inside the well (the probability of survival) drops down exponentially, \( P_0(t) = e^{-\Gamma t} \). For small \( t \), however, \( P_0(t) = 1 - at^2 \) [3, 19]. Indeed, the probability of survival is

\[
P_0(t) = |\langle \Phi_0 | e^{-iHt/\hbar} |\Phi_0 \rangle|^2 = 1 - (\Delta H)^2 t^2/\hbar^2 + \cdots \tag{1.1}
\]

where \( (\Delta H)^2 = \langle \Phi_0 | H^2 |\Phi_0 \rangle - (\langle \Phi_0 | H |\Phi_0 \rangle)^2 \).
Let us assume that a particle inside the well is continuously monitored (by the “eye”, shown in Fig. 1). This process can be viewed as $n$ discrete measurements, where each one takes some small measurement time $\Delta t$. Then after the first measurement one finds the particle inside the well with a probability $P_0(\Delta t) = 1 - a(\Delta t)^2$. According to the projection postulate [1], the measurement projects the system into the state which is actually observed. As a result the system continues its evolution with the new initial conditions. Hence, after $n$ consecutive measurements the probability of finding the system undecayed at time $t = n\Delta t$ is

$$P_0(n\Delta t) = [1 - a(\Delta t)^2]^n,$$  \hspace{1cm} (1.2)

Taking the limit of continuous monitoring, $\Delta t \rightarrow 0$ and $n \rightarrow \infty$, where $t=\text{const}$, one finds

$$P_0(t) = [1 - a(\Delta t)^2]^{t/\Delta t} \simeq 1 - a(\Delta t)t \rightarrow 1 \quad \text{for} \quad \Delta t \rightarrow 0.$$ \hspace{1cm} (1.3)

Therefore the continuously observed system cannot decay.

The Zeno paradox in quantum mechanics is still not so famous as the EPR or the Schrödinger cat paradoxes. Yet, the Zeno paradox represents a real dynamical effect of the projection postulate and not only an interpretation problem of Quantum Mechanics with no experimental consequences [20]. For a proper understanding of the Zeno paradox and therefore a role of the projection postulate in quantum mechanics, it is absolutely necessary to include the measurement device in the Schrödinger equation for the entire system. In this case quantum-mechanical description of the measurement device would allow us to study thoroughly the measurement process without explicit use of the projection postulate. The main difficulty with such an approach, however, is that the measurement device is a macroscopic system and therefore its quantum mechanical analysis is very complicated. For this reason one would expect that mesoscopic systems, which are between the microscopic and macroscopic scales, would be very useful for this investigation [21].
In the following we concentrate on measurements of quantum dots in a two-dimensional electron gas. As a generic example of the measurement device (detector) we consider a point contact (tunneling junction) created electrostatically by two electrodes. This junction separates two reservoirs, Fig. 2a, which are filled up to their Fermi levels $\mu_L$ and $\mu_R$, respectively, with $\mu_L > \mu_R$. As a result, a macroscopic current $I$ flows through the point contact, as shown schematically in Fig. 2b, where the point contact is represented by a potential barrier.

![Diagram of point contact](image)

**Fig. 2.** (a) Point-contact in a two-dimensional electron gas and (b) its schematic representation. $\Omega_{lr}$ is the coupling between the levels $E_l$ and $E_r$ in the left and the right reservoirs, and $n$ denotes the number of electrons arriving the right reservoir by time $t$.

The electron current $I$ flowing though the point-contact is very sensitive to nearby electrostatic fields that modulate the size of the opening. Thus, one can use the point contact as a detector to monitor the charging of a quantum dot. For instance, it can be used for monitoring a single electron in a coupled-dot (electrostatic q-bit), Fig. 3. Indeed, if the electron occupies the lower dot, located far away from the point contact (Fig. 3a), its electric field does not affect the conductivity of the point contact. However, if the electron occupies the
upper dot, close to the point-contact, its electrostatic field diminishes the conductivity of the point contact. As a result the point-contact current decreases, \( I_2 < I_1 \) (Fig. 3b). Thus, by observing a variation of the detector current one can monitor the electron’s transitions between the dots.

In a similar way, by using the point-contact detector one can investigate the influence of measurement on the decay of an unstable system. An appropriate setup is shown in Fig. 4. Here the point-contact detector is placed near the quantum dot, opening into the continuum. Again, the detector current increases when the electron leaves the quantum dot.

![Fig. 3. Point-contact as an electrostatic detector of an electron in a coupled quantum dot. The detector current decreases upon transition of the electron from the lower to the upper dot.](image)

It should be pointed out that although a quantum point contact has successfully been used as a “Which Path Detector” in different types of experiments (for instance in measurements of quantum interference [22, 23]) measurements of a “single electron” using the point-contact detector, as in Figs. 3 and 4, have not been achieved. Nevertheless, the rapid progress of nano-technology should make such measurements feasible in the near future.

II. QUANTUM DESCRIPTION OF THE DETECTOR

Let us consider the point-contact detector and the measured electron (Figs. 3,4) as one quantum system described by the Schrödinger equation. The point-contact detector, how-
ever, represents a macroscopic system and therefore it should exhibit classical behavior. This is an essential condition for a “measurement” device. Now we demonstrate how such a classicality of the point-contact detector emerges from the Schrödinger equation.

\[ H_{PC} = \sum_l E_l a_l^\dagger a_l + \sum_r E_r a_r^\dagger a_r + \sum_{l,r} \Omega_{lr} (a_l^\dagger a_r^\dagger + a_r^\dagger a_l^\dagger), \]  

where \( a_l^\dagger (a_l) \) and \( a_r^\dagger (a_r) \) are the creation (annihilation) operators in the left and the right reservoirs, respectively, and \( \Omega_{lr} \) is the hopping amplitude between the states \( E_l \) and \( E_r \) in the right and the left reservoirs. (We choose the gauge where \( \Omega_{lr} \) is real).

We assume that all the levels in the emitter (left) and the collector (right) are initially filled up to their Fermi energies \( \mu_L \) and \( \mu_R \) respectively. We shall call this the “vacuum” state, \( |0\rangle \). The Hamiltonian Eq. (2.1) requires the vacuum state \( |0\rangle \) to decay exponentially to a continuum state having the form: \( a_l^\dagger a_l |0\rangle \) with an electron in the collector reservoir and a hole in the emitter reservoir; \( a_r^\dagger a_r^\dagger a_l a_l |0\rangle \) with two electrons in the collector reservoir and two holes in the emitter reservoir, and so on. In order to treat such a system one usually uses the Keldysh non-equilibrium Green’s function technique [24]. Here we use a different, simpler and more transparent technique developed by us in Ref. [25]. It consists of reduction
of the Schrödinger equation to Bloch-type rate equations for the density matrix obtained by integrating over the reservoir states. Such a procedure is described below, and can be carried out in the strong nonequilibrium limit without any stochastic assumptions.

Let us consider the many-body wave function describing the point-contact detector. It can be written in the occupation number representation as

\[
|\Psi(t)\rangle = b_0(t) + \sum_{l,r} b_{lr}(t) a_l^\dagger a_r + \sum_{l<l',r<r'} b_{l'r'}(t) a_l^\dagger a_{r'}^\dagger a_{l'} a_r + \cdots |0\rangle,
\]

(2.2)

where \(b(t)\) are the time-dependent probability amplitudes for finding the system in the corresponding states given the initial condition \(b_0(0) = 1\), with all the other \(b(0)\)'s being zeros. In order to find these amplitudes we substitute Eq. (2.2) into the Schrödinger equation

\[i |\dot{\Psi}(t)\rangle = H_{PC} |\Psi(t)\rangle.\]

As a result we obtain an infinite system of linear differential equations for the amplitudes \(b(t)\), which completely determines the quantum behavior of the point contact. It is useful to use the Laplace transform, \(\tilde{b}(E) = \int_0^\infty e^{iEt} b(t) dt\), so that these differential equations become the following algebraic coupled equations for the amplitudes \(\tilde{b}(E)\)

\[
\begin{align*}
E\tilde{b}_0(E) - \sum_{l,r} \Omega_{lr} \tilde{b}_{lr}(E) &= i \\
( E + E_l - E_r )\tilde{b}_{lr}(E) - \Omega_{lr} \tilde{b}_0(E) - \sum_{l'l',r'} \Omega_{l'r'} \tilde{b}_{l'r'}(E) &= 0 \\
( E + E_l + E_{l'} - E_r - E_{r'} )\tilde{b}_{l'r'}(E) - \Omega_{l'r'} \tilde{b}_{lr}(E) + \Omega_{lr} \tilde{b}_{l'r'}(E) \\
&\quad - \sum_{l''r''} \Omega_{l'r'} \tilde{b}_{l'l'r'r''}(E) = 0
\end{align*}
\]

(2.3)

Eqs. (2.3) can be substantially simplified by replacing the amplitude \(\tilde{b}\) in the term \(\sum \Omega \tilde{b}\) of each of the equations by its expression obtained from the subsequent equation \(\Omega_{lr} \equiv \Omega(E_l, E_r) = \Omega\). Since the states in the reservoirs are very dense (essential a continuum), one can replace the sums over \(l\) and \(r\) by integrals, for instance \(\sum_{l,r} \rightarrow \int \rho_L(E_l)\rho_R(E_r) dE_l dE_r\), where \(\rho_{L,R}\) are the densities of states in the emitter and collector.
Then the first sum in Eq. \((2.4)\) becomes an integral which can be split into the sum of its singular and principal value parts. The singular part yields \(i\pi\Omega^2\rho_L\rho_R V_d\), and the principal part is merely absorbed into a redefinition of the energy levels. The second sum in Eq. \((2.4)\) can be neglected. Indeed, by replacing \(\tilde{b}_{ll'}(E) \equiv \tilde{b}(E, E_l, E_r, E_{l'})\) and the sums by the integrals we find that the integrand has the poles on the same sides of the integration contours. It means that the corresponding integral vanishes for infinite integration limits. This corresponds to strongly non-equilibrium limit, \(V_d/\Omega^2\rho \to \infty\).

Applying analogous considerations to the other equations of the system \((2.3)\), we finally arrive at the following set of equations:

\[
(E + iD/2)\tilde{b}_0(E) = i \tag{2.5a}
\]

\[
(E + E_l - E_r + iD/2)\tilde{b}_{lr}(E) - \Omega \tilde{b}_0(E) = 0 \tag{2.5b}
\]

\[
(E + E_l + E_{l'} - E_r - E_{r'} + iD/2)\tilde{b}_{ll'}(E) - \Omega \tilde{b}_{lr}(E) + \Omega \tilde{b}_{lr'}(E) = 0, \tag{2.5c}
\]

where \(D = 2\pi\Omega^2\rho_L\rho_R V_d\).

### A. Rate equations

Eqs. \((2.5)\) can be transformed to differential equations for the reduced density matrix \(\sigma^{(nn')}(t)\) of electrons in the right reservoir (collector). This density matrix is directly related to the amplitudes \(\tilde{b}(t)\). For instance the diagonal density-matrix elements

\[
\sigma^{(00)}(t) = |\tilde{b}_0(t)|^2
\]

\[
\sigma^{(11)}(t) = \sum_{l,r} |\tilde{b}_{lr}(t)|^2
\]

are the probabilities of finding 0, 1, etc. electrons in the collector. The corresponding off-diagonal matrix elements (coherences)

\[
\sigma^{(01)}(t) = \sum_{l,r} \tilde{b}_0(t)\tilde{b}_{lr}^*(t), \quad \sigma^{(12)}(t) = \sum_{l < l', r < r'} \tilde{b}_{lr}(t)\tilde{b}_{lr'}^*(t), \ldots \tag{2.6}
\]
have no classical equivalent and describe electrons in the linear superposition of the states in different reservoirs.

Let us rewrite \( \sigma^{(n,n')}(t) \) in terms of the amplitudes \( \tilde{b}(E) \) via the inverse Laplace transform

\[
\sigma^{(n,n')}(t) = \sum_{l,\ldots,r} \int dE dE' \tilde{b}_{n'} \ldots \tilde{r}_{n'}(E) \tilde{b}^*_{l'} \ldots \tilde{r}^*_{l'}(E') e^{i(E'-E)t} \tag{2.7}
\]

Using Eq. \(2.7\) one can transform Eqs. \(2.5\) directly into equations for \( \sigma^{(n,n')}(t) \) (c.f. [10, 25]). For instance, consider Eq. \(2.5b\) for the amplitude \( \tilde{b}_{l'}(E) \). Multiplying this equation by \( \tilde{b}^*_{l'}(E') \) and subtracting the same equation for \( \tilde{b}^*_{l'}(E') \) multiplied by \( \tilde{b}_{l'}(E) \) we obtain

\[
(E' - E - iD) \tilde{b}_{l'}(E) \tilde{b}^*_{l'}(E') - \Omega \left[ \tilde{b}_{l'}(E) \tilde{b}^*_{0}(E') - \tilde{b}^*_{l'}(E') \tilde{b}_{0}(E) \right] = 0. \tag{2.8}
\]

Substituting Eq. \(2.8\) into Eq. \(2.7\) we find

\[
\dot{\sigma}^{(1,1)}(t) = -D\sigma^{(1,1)}(t) - i\Omega \left[ \sigma^{0,1}(t) - \sigma^{1,0}(t) \right] \tag{2.9}
\]

A similar procedure can be performed for all other equations \(2.5\). As a result we arrive at the following system of equations for the density matrix \( \sigma^{(n,n')} \)

\[
\dot{\sigma}^{(n,n)}(t) = -D\sigma^{(n,n)}(t) - i\Omega \left[ \sigma^{(n-1,n)}(t) - \sigma^{(n,n-1)}(t) \right] \tag{2.10a}
\]

\[
\sigma^{(n-1,n)}(t) = i(D/\Omega)\sigma^{(n-1,n-1)}(t) \tag{2.10b}
\]

Substituting Eq. \(2.10b\) into Eq. \(2.10a\) we find a linear differential equation for the probabilities only

\[
\frac{d}{dt}\sigma^{(n,n)}(t) = -D\sigma^{(n,n)}(t) + D\sigma^{(n-1,n-1)}(t), \tag{2.11}
\]

where \( D = 2\pi\Omega^2 \rho_L \rho_R (\mu_L - \mu_R) \) is the rate of electrons arriving in the right reservoir.

Equation \(2.11\) is a classical rate equation. On the other hand it was obtained from the Schrödinger equation entirely in the framework of Quantum Mechanics. No Markov approximations have been used in its derivation. It is important to note that this classical limit of the Schrödinger equation does not require the vanishing of the nondiagonal density-matrix elements (coherences), as follows from Eq. \(2.10b\). The transition from quantum to classical description is provided by a decoupling of coherences from probabilities in the equation of motion, Eq. \(2.11\).

Equation \(2.11\) can easily be solved for the initial condition \( \sigma^{(n,n)}(0) = \delta_{n,0} \). One finds the Poisson distribution

\[
\sigma^{(n,n)}(t) = \frac{(D\ t)^n}{n!} e^{-D\ t} \simeq \frac{1}{\sqrt{2\pi Dt}} \exp \left[ -\frac{(Dt - n)^2}{2Dt} \right] \tag{2.12}
\]
Thus the average electric current flowing into the right reservoir is $I = e < n > / t = eD$.

III. MEASUREMENT OF THE DECAY TIME

Consider now the measurement of the Zeno effect for decay into the continuum using a point-contact detector, as in Fig. 4. The tunneling Hamiltonian describing the entire system consists of three parts [16, 26]:

$$\mathcal{H} = \mathcal{H}_{PC} + \mathcal{H}_{QD} + \mathcal{H}_{int}. \quad (3.1)$$

The first term describes tunneling through the point-contact detector, Eq. (2.1). The second term describes the quantum dot coupled to the continuum.

$$\mathcal{H}_{QD} = E_0 c_0^\dagger c_0 + \sum_{\alpha} E_\alpha c_\alpha^\dagger c_\alpha + \sum_{\alpha} \Omega_{\alpha} (c_\alpha^\dagger c_0 + c_0^\dagger c_\alpha) \quad (3.2)$$

Here the operators $c_0^\dagger (c_0)$ and $c_\alpha^\dagger (c_\alpha)$ create (annihilate) an electron inside the dot or in the continuum, respectively, and $\Omega_{\alpha}$ is the corresponding coupling between these states.

The last term describes the interaction of the detector with the electron in the quantum dot,

$$\mathcal{H}_{int} = -\sum_{l,r} \delta \Omega_{lr} c_0^\dagger c_0 (a_l^\dagger a_r + a_r^\dagger a_l). \quad (3.3)$$

This term modulates the detector current. That is whenever the electron occupies the dot, i.e. $c_0^\dagger c_0 \rightarrow 1$, the coupling between the states $E_{lr}$ of the detector decreases $\Omega'_{lr} = \Omega_{lr} - \delta \Omega_{lr} < \Omega_{lr}$. In this case the detector current becomes $I' = eD'$, where $D' = 2\pi(\Omega - \delta \Omega_{lr})^2 \rho_L \rho_R (\mu_L - \mu_R)$.

A. Dynamics of an unstable system

Consider first the decay of an electron into the continuum with no interaction with the point-contact detector, $\delta \Omega_{lr} = 0$. In this case the electron evolution is determined only by $\mathcal{H}_{QD}$, Eq. (3.2). This Hamiltonian is essentially equivalent to that of the Lee model. The latter reproduces an exponential decay for homogeneous reservoirs and a constant coupling $\Omega_{\alpha}$, without any Markovian approximations [27]. Let us demonstrate this result by solving the time-dependent Schrödinger equation $i \partial_t |\Psi(t)\rangle = \mathcal{H}_{QD} |\Psi(t)\rangle$. The electron wave
function can be written in the most general way as

$$|\Psi(t)\rangle = \left[b_0(t)c_0^\dagger + \sum_\alpha b_\alpha(t)c_\alpha^\dagger\right]|0\rangle,$$

(3.4)

where $b_{0,\alpha}(t)$ are the time-dependent probability amplitudes for finding the electron inside the dot or in the reservoir in the state $E_\alpha$. The initial condition is $b_0(0) = 1$ and $b_\alpha(0) = 0$. Performing the Laplace transform, $b(t) \rightarrow \tilde{b}(E)$, we easily find that the Schrödinger equation can be written as

$$(E - E_0)\tilde{b}_0(E) - \sum_\alpha \Omega_\alpha \tilde{b}_\alpha = i,$$

(3.5a)

$$(E - E_\alpha)\tilde{b}_\alpha(E) - \Omega_\alpha \tilde{b}_0(E) = 0.$$

(3.5b)

In order to solve these equations we replace the amplitude $\tilde{b}_\alpha$ in Eq. (3.5a) by its expression obtained from Eq. (3.5b). One then obtains

$$\left[E - E_0 - \sum_\alpha \frac{\Omega_\alpha^2}{E - E_\alpha}\right] \tilde{b}_0(E) = i.$$

(3.6)

Since the states in the reservoir are very dense, one can replace the sum over $\alpha$ by an integral over $E_\alpha$.

$$\sum_\alpha \frac{\Omega_\alpha^2}{E - E_\alpha} = \int \frac{\Omega_\alpha^2(E_\alpha)\rho(E_\alpha)}{E - E_\alpha} dE_\alpha,$$

(3.7)

where $\rho(E_\alpha)$ is the density of states in the reservoir. To evaluate this integral, we can split the integral into its principal and singular parts, $-i\delta(E - E_\alpha)$. As a result the original Schrödinger equation (3.5a) is reduced to

$$\left[E - E_0 - \Delta(E) + i\frac{\Gamma(E)}{2}\right] \tilde{b}_0(E) = i,$$

(3.8a)

$$(E - E_\alpha)\tilde{b}_\alpha(E) - \Omega(E_\alpha)\tilde{b}_0(E) = 0,$$

(3.8b)

where $\Gamma(E) = 2\pi \rho(E)\Omega_\alpha^2(E)$ and $\Delta(E)$ is the energy-shift due to the principal part of the integral.

Let us assume that $\Omega_\alpha^2(E_\alpha)\rho(E_\alpha)$ is weakly dependent on the energy $E_\alpha$. As a result the width becomes a constant $\Gamma(E) = \Gamma_0$ and the energy shift $\Delta(E)$ tends to zero. Using Eqs. (3.8) and the inverse Laplace transform one obtains the occupation probabilities of the levels $E_0$ and $E_\alpha$. Yet, Eqs. (3.8) are not convenient if we wish to include the effects of a measurement (or of an environment) on the electron behavior. These effects can
be determined in a natural way only in terms of the density matrix. For this reason we transform Eqs. (3.8) into equations for the density matrix $\sigma_{ij}(t) \equiv b_i(t)b_j^*(t)$. The latter is directly related to the amplitudes $\tilde{b}(E)$ via the inverse Laplace transform, Eq. (2.7). One finds

$$\dot{\sigma}_{00}(t) = -\Gamma_0 \sigma_{00}(t),$$

$$\dot{\sigma}_{\alpha\alpha}(t) = i\Omega_{\alpha}(\sigma_{\alpha0}(t) - \sigma_{0\alpha}(t))$$

$$\dot{\sigma}_{\alpha0}(t) = i\epsilon_{\alpha0}\sigma_{\alpha0}(t) - i\Omega_{\alpha}\sigma_{00}(t) - \frac{\Gamma_0}{2}\sigma_{\alpha0}(t),$$

with $\epsilon_{0\alpha} = E_0 - E_\alpha$ and $\sigma_{0\alpha} = \sigma_{\alpha0}^*$. Here $\sigma_{00}(t)$ and $\sigma_{\alpha\alpha}(t)$ are the probabilities of finding the electron in the dot or in the continuum at the level $E_\alpha$, respectively. The off-diagonal density-matrix elements $\sigma_{\alpha0}(t)$ (coherences) describe the electron in a linear superposition. These matrix elements decrease exponentially due to the last term in Eq. (3.9c), generated by decay into the continuum.

Eqs. (3.9) represent a generalization of the optical Bloch equations describing quantum transitions between two isolated levels [25, 29] to transitions between one isolated level and the continuum [16, 26, 30]. In this case the coherence term $\sigma_{\alpha0}$ is coupled to the corresponding probability term $\sigma_{00}$, but not with that of the continuum spectrum $\sigma_{\alpha\alpha}$, as one would expect for usual optical Bloch equations.

Solving Eqs. (3.9) we find the following expressions for the occupation probabilities, $\sigma_{00}$ and $\sigma_{\alpha\alpha}$, of the levels $E_0$ and $E_\alpha$, respectively [28]:

$$\sigma_{00}(t) = e^{-\Gamma_0 t},$$

$$\sigma_{\alpha\alpha}(t) = \frac{\Omega^2_{\alpha}}{(E_\alpha - E_0)^2 + (\Gamma_0/2)^2} \left[1 - 2\cos[(E_\alpha - E_0)t] e^{-\Gamma_0 t/2} + e^{-\Gamma_0 t}\right]$$

Notice that the line shape, $P(E_\alpha) \equiv \sigma_{\alpha\alpha}(t \to \infty)\rho$, given by Eq. (3.10b) is the standard Lorentzian distribution,

$$P(E_\alpha) = \frac{\Gamma_0/(2\pi)}{(E_\alpha - E_0)^2 + (\Gamma_0/2)^2},$$

with the width $\Gamma_0$ corresponding to the inverse life-time of the quasi-stationary state, Eq. (3.10a).
B. Continuous measurement of an unstable system.

Now we introduce the coupling with the point-contact detector, \( \delta \Omega_{lr} \neq 0 \). The many-body wave function describing the entire system can be written in the same way as Eq. (2.2)

\[
|\Psi(t)\rangle = \left[ b_0(t)c_0^\dagger + \sum_{l,r} b_{lr}(t)a_l^\dagger a_l c_0^\dagger + \sum_\alpha b_\alpha(t)c_\alpha^\dagger c_0 + \sum_{\alpha,l,r} b_{\alpha lr}(t)a_l^\dagger c_\alpha^\dagger a_l c_0 + \cdots \right] |0\rangle \tag{3.12}
\]

where \( b(t) \) are the time-dependent probability amplitudes for finding the system in the corresponding states, given the initial condition \( b_0(0) = 1 \) with all other \( b(0) \)'s equal to zero. These amplitudes are obtained from the Schrödinger equation

\[
i |\dot{\Psi}(t)\rangle = \mathcal{H}|\Psi(t)\rangle,
\]

where \( \mathcal{H} \) is given by Eq. (3.1). Using the same technique as in the previous case, Eqs. (2.11) and (3.9), the Schrödinger equation for the amplitudes \( b(t) \) is reduced to quantum rate equations for the reduced density matrix \( \sigma_{nn}^{(n)}(t) \equiv \sigma_{ij}^{n}(t) \) by integration over the reservoir states of the detector. One finds \[16, 26\]

\[
\dot{\sigma}^{(n)}_{00} = -\left( \Gamma + D' \right)\sigma^{(n)}_{00} + D'\sigma^{(n-1)}_{00},
\]

\[
\dot{\sigma}^{(n)}_{\alpha\alpha} = -D\sigma^{(n)}_{\alpha\alpha} + D\sigma^{(n-1)}_{\alpha\alpha} + i\Omega_{\alpha}\left( \sigma^{(n)}_{\alpha\alpha} - \sigma^{(n)}_{\alpha\alpha} \right),
\]

\[
\dot{\sigma}^{(n)}_{\alpha0} = i(E_0 - E_\alpha)\sigma^{(n)}_{\alpha0} - i\Omega_{\alpha}\sigma^{(n)}_{00} - \frac{\Gamma_0 + D + D'}{2}\sigma^{(n)}_{\alpha0} + \sqrt{DD'}\sigma^{(n-1)}_{\alpha0},
\tag{3.13a}
\]

where \( \Gamma_0 = 2\pi\rho(E_0)\Omega_\alpha^2(E_0) \) corresponds to the inverse lifetime of the quasi-stationary state, Eq. (3.10a). The index \( n \) denotes the number of electrons arriving the left reservoir by time \( t \) and the indices \( i, j = 0, \alpha \) denote the state of the observed electron. One finds that only the density-matrix elements diagonal in \( n \) enter into the rate equations, similar to Eq. (2.11). However, we are not integrating over the final states of the escaped electron. As a result the off-diagonal terms, describing the superposition of the electron in the dot and in the continuum, enter the rate equations (c.f. Eqs. (3.9)).

The density matrix \( \sigma_{ij}^{(n)}(t) \) given by Eqs. (3.13) describes both the detector and the escaped electron. Indeed the probability of finding \( n \) electrons in the collector, \( \sigma^{(n)}(t) \), is obtained by tracing over the escaped electron variables

\[
\sigma^{(n)}(t) = \sigma_{00}^{(n)}(t) + \sum_\alpha \sigma_{\alpha\alpha}^{(n)}(t). \tag{3.14}
\]

The average detector current is therefore

\[
<I(t)> = e \sum_n n\dot{\sigma}^{(n)}(t) = eD'\sigma_{00}(t) + eD[1 - \sigma_{00}(t)], \tag{3.15}
\]

\[13\]
where $\sigma_{00}(t) = \sum_n \sigma_{00}^{(n)}(t)$ is the probability of finding the electron inside the dot. The latter is obtained by tracing over the detector variables in the total density matrix: $\sigma_{ij}(t) = \sum_n \sigma_{ij}^{(n)}(t)$. One easily finds from Eqs. (3.13) that

$$
\dot{\sigma}_{00} = -\Gamma_0 \sigma_{00} \quad (3.16a)
$$
$$
\dot{\sigma}_{\alpha\alpha} = i\Omega_\alpha (\sigma_{\alpha0} - \sigma_{0\alpha}) \quad (3.16b)
$$
$$
\dot{\sigma}_{\alpha0} = i(E_0 - E_\alpha)\sigma_{\alpha0} - i\Omega_\alpha \sigma_{00} - \frac{\Gamma_0 + \Gamma_d}{2} \sigma_{\alpha0} \quad , \quad (3.16c)
$$

where $\Gamma_d = (\sqrt{D} - \sqrt{D'})^2$ is the decoherence rate generated by the detector, in addition to the “intrinsic” decoherence rate $\Gamma_0$ generated by tunneling. Here we would like to point out the important distinction between these two different origins of decoherence. Tunneling into an infinite continuum is the only intrinsically irreversible process encountered in ordinary quantum mechanics. On the other hand $\Gamma_d$ is related to the “effective” irreversibility that occurs when a simple quantum system is coupled to a macroscopic measurement apparatus, averaged over unobservable degrees of freedom.

Equation (3.15) displays a direct connection between the averaged detector current and the probability of finding the electron inside the dot. Its escape to the continuum results in an increase of the detector current at $t = 1/\Gamma_0$, as shown in Fig. 5. Therefore the continuous measurement process is completely described by the rate equations (3.13).

Comparing Eqs. (3.16) with Eqs. (3.9) we find that the decay rate is not modified by the detector. Indeed, the probability of finding the electron inside the dot drops down with the same exponential, $\sigma_{00}(t) = \exp(-\Gamma_0 t)$, as in the noninteracting case ($\delta\Omega_{lr} = 0$). Therefore no Zeno effect can be observed in the exponential decay of an unstable system.
Fig. 5. The average detector current as a function of time.

Nevertheless, the influence of the measurement can be seen in the last term of Eq. (3.16c), which constitutes an additional decoherence rate $\Gamma_d$ generated by the detector. This affects the energy spectrum of the tunneling electron, given by $P(E_{\alpha}) = \sigma_{\alpha\alpha}(t \to \infty)$. Indeed, on solving Eqs. (3.16b), (3.16c) in the limit $t \to \infty$ we obtain

$$P(E_{\alpha}) = \frac{\Gamma_0 + \Gamma_d}{(E_{\alpha} - E_0)^2 + \frac{(\Gamma_0 + \Gamma_d)^2}{4}}$$

Comparing with Eq. (3.11) one finds that the measurement results in a broadening of the line width, which becomes $\Gamma_0 + \Gamma_d$.

To understand these results, one might think of the following argument. Due to the measurement, the energy level $E_0$ suffers an additional broadening of the order of $\Gamma_d$. However, this broadening does not affect the decay rate of the electron $\Gamma_0$, since the exact value of $E_0$ relative to $E_{\alpha}$ is irrelevant to the decay process. In contrast, the probability distribution $P(E_{\alpha})$ is affected because it does depend on the position of $E_0$ relative to $E_{\alpha}$, as can be seen in Eq. (3.17).

Although our result has been proved for a specific detector, we expect it to be valid for the general case, provided that the density of states $\rho$ and the transition amplitude $\Omega_{\alpha}$ for the observed electron vary slowly with energy. This condition is sufficient to ensure a pure exponential decay of the state $E_0$. On the other hand, if the product $\Omega_{\alpha}^2 \rho(E_{\alpha})$ depends sharply on energy $E_{\alpha}$, it yields strong $E$-dependence of $\Gamma$ and $\Delta$ in Eqs. (3.8). This would result in a deviation from a pure exponential decay and consequently to the Zeno effect in the case of continuous measurement.
IV. NONUNIFORM DENSITY OF STATE AND ZENO EFFECT.

Consider the electron escape to the reservoir, Fig. 4, where the density of the reservoir states $\rho(E_\alpha)$ does depend on the energy. For the definiteness we take a Lorentzian form of the density of states

\[
\rho(E_\alpha) = \frac{\Gamma_1/2\pi}{(E_\alpha - E_1)^2 + \Gamma_1^2/4} \quad (4.1)
\]

One can demonstrate [16] that such a system can be mapped onto that shown in Fig. 6, where the Lorentzian states are represented by a resonance cavity coupled to the quantum dot and the reservoir.

\[\text{Fig. 6. A point-contact detector near a quantum dot coupled with a resonance cavity.}\]

Using the same treatment as in the previous section and summing over the states $n$ of the detector one arrives at the following rate equations for the electron density matrix
\[
\sigma_{ij}(t) = \sum_n \sigma_{ij}^{(n)}(t),
\]

\[
\dot{\sigma}_{00} = i\Omega_\alpha (\sigma_{01} - \sigma_{10})
\]

\[
\dot{\sigma}_{11} = -\Gamma_1 \sigma_{11} + i\Omega_\alpha (\sigma_{10} - \sigma_{01})
\]

\[
\dot{\sigma}_{01} = i\epsilon_{10} \sigma_{01} + i\Omega_\alpha (\sigma_{00} - \sigma_{11}) - \frac{\Gamma_1 + \Gamma_d}{2} \sigma_{01}
\]

\[
\dot{\sigma}_{\alpha\alpha} = i\Omega_\alpha (\sigma_{\alpha0} - \sigma_{0\alpha})
\]

\[
\dot{\sigma}_{0\alpha} = i\epsilon_{\alpha0} \sigma_{0\alpha} + i\Omega_\alpha (\sigma_{00} - \sigma_{1\alpha}) - \frac{\Gamma_d}{2} \sigma_{0\alpha}
\]

\[
\dot{\sigma}_{1\alpha} = i\epsilon_{\alpha1} \sigma_{1\alpha} + i\Omega_\alpha (\sigma_{10} - \sigma_{0\alpha}) - \frac{\Gamma_1}{2} \sigma_{1\alpha}
\]

where the index “1” relates to the cavity state \((E_1)\) and \(\Gamma_d = (\sqrt{D} - \sqrt{D'})^2\) is the decoherence rate generated by the detector.

Consider first the case of no measurement, \(\Gamma_d = 0\). Solving Eqs. (4.2) we find that the decay is not a pure exponential one \([16]\). In particular, the probability of finding the electron in the initial state for small \(t\) is \(\sigma_{00}(t) = 1 - \Omega_\alpha^2 t^2\), in contrast with Eq. (3.10a). This second-order dependence of \(\sigma_{00}\) on \(t\) is due to the fact that decoupling it from the off-diagonal \(\sigma_{01}\) results in a second-order differential equation. Note that the absence of a term linear in \(t\) in \(\sigma_{00}(t)\) is a prerequisite for the Zeno effect. However, at large values of \(t\) the decay becomes an exponential one, i.e.

\[
\sigma_{00}(t) \simeq \exp \left( -\frac{4\Omega_\alpha^2}{\Gamma_1} t \right) \quad \text{for} \quad t \gg 1/\Omega_\alpha.
\]  

Consider now the case of measurement, i.e. \(\Gamma_d \neq 0\). Solving Eqs. (4.2) for \(t \gg \Omega_\alpha^{-1}\) we discover that the probability of finding the electron inside the dot, \(\sigma_{11}(t)\), drops down exponentially as

\[
\sigma_{00}(t) = \exp \left( -\frac{4(\Gamma_1 + \Gamma_d)\Omega_\alpha^2}{4(E_1 - E_0)^2 + (\Gamma_1 + \Gamma_d)^2} t \right) \quad \text{for} \quad t \gg 1/\Omega_\alpha.
\]  

Let us compare Eq. (4.4) with Eq. (4.3). We see that the decay rate decreases with \(\Gamma_d\) (Zeno effect) in the presence of the detector, but only for aligned levels, \(|E_0 - E_1| \ll \Gamma_1 + \Gamma_d\). If, however, the levels \(E_0\) and \(E_1\) are not aligned, \(|E_0 - E_1| \gg \Gamma_1 + \Gamma_d\), we find that the decay rate increases (the anti-Zeno effect \([10, 13, 14, 15, 16, 17, 18]\)). Such an increase of the decay rate due to the measurement is shown in Fig. 7a for \(E_1 - E_0 = 10\Omega_\alpha\). However, for very short times we always observe the decrease of the decay rate i.e. the Zeno effect even for misaligned levels, as shown in Fig. 7b \([16]\). Thus the anti-Zeno effect discussed recently...
in the literature represents an increase in the “average” transition rate. For small enough $t$, however, no anti-Zeno effect can be found.

![Graph](image)

**Fig. 7.** (a) The probability of finding the electron inside the quantum dot at the level $E_0$ where $E_1 - E_0 = 10\Omega_\alpha$. The solid line corresponds to $\Gamma_d = 0$ (no measurement) while the dashed line, which displays the anti-Zeno effect, corresponds to $\Gamma_d = 10\Omega_\alpha$. (b) The same for small $t$, where the dashed line displays the Zeno effect.

The Zeno and anti-Zeno effects described above can be interpreted in terms of broadening of the level $E_0$ induced by the detector. One expects that this broadening would always lead to spreading of the energy distribution. On the other hand, its influence on the decay rate depends whether the levels $E_0$ and $E_1$ are in resonance or not. If $E_0 = E_1$ the broadening of the level $E_0$ destroys the resonant-tunneling condition, so that the decay into continuum slows down. If on the other hand $E_0 \neq E_1$, the same broadening would effectively diminish the levels displacement (c.f. with [31]). As a result, the decay rate should increase. Yet, such qualitative arguments do not work at very short times, since the decay rate slows down even for $E_0 \neq E_1$, as in Fig. 7b., except in the case of a flat density of states, for which the decay rate is not affected by measurement.

V. ORIGIN OF THE WAVE-FUNCTION COLLAPSE.

We demonstrated in this paper that the inclusion of a measurement device in the Schrödinger equation made it possible to describe the measurement process without ex-
licit use of the projection postulate (the wave-function collapse). Even the Zeno effect
is described in terms of the decoherence generated by a macroscopic detector. Thus one
might assume that wave-function collapse is a redundant assumption and can be avoided by
including the detector in the Schrödinger equation for an entire system. Yet this is not the
case: wave-function collapse is the indispensable component of Quantum Mechanics.

Let us explain this point again taking the point-contact detector in Fig. 2 as an ex-
ample. We demonstrated above how tracing over the quantum dot subsystem reduces the
Schrödinger equation describing such a detector to the classical rate equations Eq. (2.11),
\[
\dot{P}_n(t) = -D P_n(t) + D P_{n-1}(t),
\]
where \( P_n(t) \equiv \sigma^{(n,n)}(t) \) is the probability of finding \( n \) electrons in the right reservoir by
time \( t \). Solving these equations for the initial conditions \( P_n(0) = \delta_{n0} \), we obtain a Poisson
distribution for \( P_n(t) \), Eq. (2.12). Consider for simplicity the case of \( t \gg 1/D \). Then \( P_n(t) \)
can be written as
\[
P_n(t) \approx \frac{1}{\sqrt{2\pi D t}} \exp \left[ -\frac{(Dt - n)^2}{2Dt} \right].
\]

Let us assume that the detector displays \( N_1 \) electrons at \( t = t_1 \), and the corresponding
information is directly available to the observer. One can ask whether such an information
affects the distribution function \( P_n(t) \), Eq. (5.2). Simple arguments show that it does.
Indeed, Eq. (5.1) represents classical rate equation and therefore it obeys Bayes principle
[32], as any probabilistic description. This implies that one has to solve Eq. (5.1) with the
new initial condition, determined by the information obtained by the observer. One obtains
[33]
\[
P_n(t) \approx \frac{1}{\sqrt{2\pi D(t-t_1)}} \exp \left[ -\frac{(Dt - n + \Delta N)^2}{2D(t-t_1)} \right]
\]
where \( \Delta N = N_1 - Dt_1 \). Obviously, this distribution is different from that given by Eq. (5.2):
it has a narrower width, but the same group velocity. This result is not surprising, since
the probabilistic description of classical systems is not a complete one. The measurement
improves our knowledge of the system, so the statistical uncertainty diminishes.

The above arguments must also be applicable to Eq. (5.1) considered as a pure quantum
mechanical equation. Indeed, Eq. (5.1) has been obtained directly from the Schrödinger
equation in the limit of high bias voltage and without any use of the Markov-type anzatz.
Thus the Schrödinger evolution must be subject to Bayes principle too. As a matter of

fact, Bayes principle, extended to the off-diagonal density-matrix elements, is essentially equivalent to wave function collapse [34].

Despite the importance of the Bayes principle in any probabilistic description, it does not appear in standard calculations of Quantum Mechanics, since the latter does not predict individual events but only ensemble averages of observables and their correlations. This allowed us to avoid explicit use of the projection postulate in the above evaluations of the detector average current and the measured electron distributions.

Now it would be interesting to compare our result for the Zeno effect with alternative predictions involving the projection postulate. On first sight some of our results contradict such predictions. For instance, we predict that continuous measurement does not affect the decay rate for a flat density of final states, contrary to the the projection postulate arguments leading to the Zeno effect, Eq.(1.3). However, for a flat density of states and infinite reservoirs, the expansion (1.1) is not applicable due to a discontinuity in the derivative $\dot{P}_0(t)$ at $t = 0$. In this case the probability of survival drops linearly with $t$ for small $t$, and no Zeno effect is expected from the projection postulate argument. For a nonuniform distribution, however, we obtain a decrease of the decay rate for small $t$, as shown in Fig. 7b.

In any case, such a comparison of standard quantum mechanical calculations with those involving the projection postulate is far from being completed. For a proper understanding of the measurement process we need to extend our quantum description of the detector to a chain of measurement devices representing the von Neumann hierarchy [1] (a system “measured” by another system etc). Only then one can properly investigate a possible dynamical role of the projection postulate in Quantum Mechanics. We believe that our quantum rate equations, described in this paper, represent the proper tool for the realization of this program.

VI. SUMMARY

In this paper we have proposed Bloch-type rate equations as a very useful approach to the quantum mechanical treatment of measurement devices. These quantum rate equations were derived from the microscopic Schrödinger equation without any stochastic assumptions.

First we applied this approach to a quantum mechanical treatment of the point-contact detector. The latter represented a generic example of a measurement device for the contin-
uous monitoring of an unstable system. We found that the transition to the classical regime of the detector takes place due to decoupling of the nondiagonal density-matrix elements from the equations of motion for the diagonal. The latter does not require the vanishing of these terms.

Then we used the same approach for a description of a larger system consisting of an observed electron which escapes into continuum together with the point-contact detector. The decoherence mechanism is clearly displayed in the resulting rate equations, where the corresponding decoherence rate is determined by the averaged detector outcome.

With respect to Zeno effect, we found that the measurement would not affect the decay rate of an unstable system, providing that the density of final states is a flat one and the reservoir is infinite. If this is not the case, we predict either Zeno or anti-Zeno effects, except for the short-time behavior where only the Zeno effect is found.

All our results were obtained without explicit use of the projection postulate. Nevertheless, the latter cannot be discarded in Quantum Mechanics, as we demonstrated using the example of point-contact detector evolution. We have also shown that our quantum mechanical predictions for the decay rate measurements are not in contradiction with the projection postulate argument, although a more detailed analysis is needed.

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