Quantum subroutine problem and the robustness of quantum complexity classes

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Abstract

This paper positively solves the quantum subroutine problem for fully quantum oracles. The quantum subroutine problem asks whether a quantum computer with an efficiently computable oracle can be efficiently simulated by a non-oracle quantum computer. We extend the earlier results obtained by Bennett, Bernstein, Brassard, and Vazirani, and by Aharonov, Kitaev, and Nisan to the case where the oracle evaluates a unitary operator and the computer is allowed to be in the superposition of a query state and a non-query state during computation. We also prove the robustness of $\text{EQP}$, $\text{BQP}$, and $\text{ZQP}$ under the above general formulation, extending the earlier results on the robustness of $\text{BQP}$ shown by Bennett et al.

Keywords: quantum computation, quantum Turing machines, complexity theory, oracles, quantum complexity classes

1 Introduction

In computational complexity theory, an oracle is described as a device for computing some Boolean function $f$ at unit cost per evaluation. This allows us to formulate questions such as, “If we added the power of computing $f$ to a Turing machine, which functions could be efficiently computed by that Turing machine?” Many researchers have investigated the computational power of a quantum Turing machine (QTM) with an oracle which computes a Boolean function. Berthiaume and Brassard \cite{Berthiaume-1995} constructed an oracle relative to which...
the QTM is exponentially more efficient than any deterministic Turing machine, recasting the promise problem of Deutsch and Jozsa [2]. Bernstein and Vazirani [3] subsequently constructed an oracle which produces a superpolynomial gap between the quantum and probabilistic Turing machines. This result was improved by Simon [4], who constructed an oracle which produces an exponential gap between the quantum and probabilistic Turing machines. Extending Simon’s idea and using some new techniques, Shor [5] gave quantum polynomial time algorithms for factoring problems and discrete logarithms. On the other hand, Bennett, Bernstein, Brassard, and Vazirani (BBBV) [6] showed that relative to an oracle chosen uniformly at random, with probability 1, NP-complete problems cannot be solved by a QTM in polynomial time.

The notion of oracles for quantum computers can be naturally extended to a device for carrying out a unitary operator $U$ at unit cost per evaluation. This allows us to formulate questions such as, “If we added the power of carrying out $U$ to a QTM, which functions could be efficiently computed by that QTM?” If $U$ is efficiently carried out, an oracle QTM with $U$ seems no more powerful than a non-oracle QTM. In fact, in the classical case, if a language $L$ is efficiently computable, a non-oracle Turing machine can efficiently simulate an oracle Turing machine with $L$ by substituting a machine computing $L$ for a query to $L$. However, in the case of quantum computing, we need to consider a superposition of a query state and a non-query state. Moreover, quantum states with query strings of different lengths may superpose, even if each element of the superposition is a query state. In these cases, if we merely substitute a QTM computing $U$ for a query to $U$, quantum coherence will collapse. Thus, in this paper, we discuss the following problem. If a unitary transformation $U$ is efficiently computable by a QTM, is there a QTM efficiently simulating an oracle QTM with $U$? This problem is called the quantum subroutine problem. BBBV [6] solved the quantum subroutine problem in the case where an oracle evaluates a deterministic function and the machine enters a query state deterministically. Aharonov, Kitaev, and Nisan [7] solved the problem for quantum circuits instead of QTMs in the case where an oracle evaluates a probabilistic function. We will positively solve the quantum subroutine problem for fully quantum oracles, extending their results to the case where the oracle evaluates a unitary operator and the computer is allowed to be in the superposition of a query state and a non-query state during computation. We can solve
the quantum subroutine problem using the simulation of QTM's by quantum circuits \cite{8} (See \cite{9} more formally), generalized quantum controls \cite{10}, and the simulation of quantum circuits by QTM's. However, we use a quantum analog of a time constructible function. Because, this method is simple and it can reduce the polynomial slowdown caused by inserting subroutines as much as possible, comparing with the method of using quantum circuits.

For a complexity class $\mathcal{C}$, we say that $\mathcal{C}$ is \textit{robust} if it holds the relation $\mathcal{C}^c = \mathcal{C}$. In computational complexity theory, it is known that the complexity classes $\mathsf{P}$, $\mathsf{BPP}$, and $\mathsf{ZPP}$ are robust, while it remains still open whether several classes such as $\mathsf{NP}$ and $\mathsf{RP}$ are robust or not. In this paper, we investigate the robustness of the quantum complexity classes $\mathsf{EQP}$, $\mathsf{BQP}$, and $\mathsf{ZQP}$, the quantum analogs of $\mathsf{P}$, $\mathsf{BPP}$, and $\mathsf{ZPP}$, extending the earlier result due to BBBV \cite{6}, who showed the robustness of $\mathsf{BQP}$ in the case where the machine enters a query state deterministically. Using a solution for the subroutine problem and the method of the proof of BBBV, we can show that $\mathsf{EQP}$ and $\mathsf{BQP}$ are robust in the general case where a query state and a nonquery state may superpose. By the method of BBBV, a query step of an oracle QTM can be replaced by a Monte Carlo non-oracle QTM, but their method does not work for Las Vegas algorithms. In order to prove the robustness of $\mathsf{ZQP}$, we improve their method by keeping a witness to distinguish the case where a QTM queries an oracle correctly from other cases.

This paper is organized as follows. In Section 2 we give definitions and basic theorems on QTM's. In Section 3 we introduce a stationary time constructible function, and solve the quantum subroutine problem by using this function. This section also contains the rigorous formulation of oracle QTM's. In Section 4 we show that $\mathsf{EQP}$, $\mathsf{BQP}$, and $\mathsf{ZQP}$ are robust in general form, improving the method of BBBV and using a solution of the quantum subroutine problem.

2 Preliminaries

A quantum Turing machine (QTM) is a quantum system consisting of a processor, a bilateral infinite tape, and a head to read and write a symbol on the tape. The formal definition of a QTM as a mathematical structure is given as follows. A \textit{processor configu-
ration set is a finite set with two specific elements \( q_0 \) and \( q_f \), where \( q_0 \) represents the initial processor configuration and \( q_f \) represents the final processor configuration. A symbol set is a finite set of the cardinality at least 2 with a specific element denoted by \( B \) and called the blank. A Turing frame is a pair \((Q, \Sigma)\) of a processor configuration set \( Q \) and a symbol set \( \Sigma \). In what follows, let \((Q, \Sigma)\) be a Turing frame. A tape configuration from a symbol set \( \Sigma \) is a function \( T \) from the set \( \mathbb{Z} \) of integers to \( \Sigma \) such that \( T(m) = B \) except for finitely many \( m \in \mathbb{Z} \). The set of all the possible tape configurations is denoted by \( \Sigma^\# \). The configuration space of \((Q, \Sigma)\) is the product set \( \mathcal{C}(Q, \Sigma) = Q \times \Sigma^\# \times \mathbb{Z} \). A configuration of \((Q, \Sigma)\) is an element \( C = (q, T, \xi) \) of \( \mathcal{C}(Q, \Sigma) \). Specifically, if \( q = q_0 \) and \( \xi = 0 \) then \( C \) is called an initial configuration of \((Q, \Sigma)\), and if \( q = q_f \) then \( C \) is called a final configuration of \((Q, \Sigma)\). The quantum state space of \((Q, \Sigma)\) is the Hilbert space \( \mathcal{H}(Q, \Sigma) \) spanned by \( \mathcal{C}(Q, \Sigma) \) with the canonical basis \( \{|C\rangle| C \in \mathcal{C}(Q, \Sigma)\} \) called the computational basis. A quantum transition function for \((Q, \Sigma)\) is a function from \( Q \times \Sigma \times Q \times \Sigma \times \{-1, 0, 1\} \) into the complex number field \( \mathbb{C} \). A (single tape) prequantum Turing machine is defined to be a triple \( M = (Q, \Sigma, \delta) \) consisting of a Turing frame \((Q, \Sigma)\) and a quantum transition function \( \delta \) for \((Q, \Sigma)\).

Let \( M = (Q, \Sigma, \delta) \) be a prequantum Turing machine. An element of \( Q \) is called a processor configuration of \( M \), the set \( \Sigma \) is called the alphabet of \( M \), the function \( \delta \) is called the quantum transition function of \( M \), and an (initial or final) configuration of \( (Q, \Sigma) \) is called the (initial or final) configuration of \( M \). A unit vector in \( \mathcal{H}(Q, \Sigma) \) is called a state of \( M \). The evolution operator of \( M \) is a linear operator \( M_\delta \) on \( \mathcal{H}(Q, \Sigma) \) such that

\[
M_\delta|q, T, \xi\rangle = \sum_{p \in Q, \tau \in \Sigma, d \in \{-1, 0, 1\}} \delta(q, T(\xi), p, \tau, d)|p, T_\tau^\tau, \xi + d\rangle
\]

for all \((q, T, \xi) \in \mathcal{C}(Q, \Sigma)\), where \( T_\tau^\tau \) is a tape configuration defined by

\[
T_\tau^\tau(m) = \begin{cases} \tau & \text{if } m = \xi, \\ T(m) & \text{if } m \neq \xi. \end{cases}
\]

Eq. (1) uniquely defines the bounded operator \( M_\delta \) on the space \( \mathcal{H}(Q, \Sigma) \) \([1]\). A (single tape) prequantum Turing machine is said to be a (single tape) quantum Turing machine (QTM) if the evolution operator is unitary.

The following theorem proved in \([1]\) characterizes the quantum transition functions that give rise to QTMs. If it is assumed that the head must move either to the right or
to the left at each step, condition (c) of Theorem 2.1 is automatically satisfied. In this case, Theorem 2.1 is reduced to the result due to Bernstein and Vazirani [3].

**Theorem 2.1** A prequantum Turing machine $M = (Q, \Sigma, \delta)$ is a QTM if and only if $\delta$ satisfies the following condition.

(a) For any $(q, \sigma) \in Q \times \Sigma$,

$$\sum_{p \in Q, \tau \in \Sigma, d \in \{-1, 0, 1\}} |\delta(q, \sigma, p, \tau, d)|^2 = 1.$$  

(b) For any $(q, \sigma), (q', \sigma') \in Q \times \Sigma$ with $(q, \sigma) \neq (q', \sigma')$,

$$\sum_{p \in Q, \tau \in \Sigma, d \in \{-1, 0, 1\}} \delta(q', \sigma', p, \tau, d)^* \delta(q, \sigma, p, \tau, d) = 0.$$  

(c) For any $(q, \sigma, \tau), (q', \sigma', \tau') \in Q \times \Sigma^2$,

$$\sum_{p \in Q, d = 0, 1} \delta(q', \sigma', p, \tau', d - 1)^* \delta(q, \sigma, p, \tau, d) = 0.$$  

(d) For any $(q, \sigma, \tau), (q', \sigma', \tau') \in Q \times \Sigma^2$,

$$\sum_{p \in Q} \delta(q', \sigma', p, \tau', -1)^* \delta(q, \sigma, p, \tau, 1) = 0.$$  

Let $S \subseteq Q \times \Sigma$. A complex-valued function on $S \times Q \times \Sigma \times \{-1, 0, 1\}$ is unidirectional, if we have $d = d'$ whenever $\delta(p, \sigma, \tau, q, d)$ and $\delta(p', \sigma', \tau', q, d')$ are both non-zero, where $q \in Q$, $(p, \sigma), (p', \sigma') \in S$, $\tau, \tau' \in \Sigma$, and $d, d' \in \{-1, 0, 1\}$. A prequantum Turing machine (or QTM) is said to be unidirectional if the quantum transition function is unidirectional. It is easy to see that a unidirectional prequantum Turing machine is a unidirectional QTM if the quantum transition function is unidirectional. It is easy to see that a unidirectional prequantum Turing machine is a unidirectional QTM if it satisfies conditions (a) and (b) of Theorem 2.1. We can show the following lemma for a unidirectional QTM by a way similar to [3]. This lemma allows us to extend a partially defined unidirectional quantum transition function so that it can characterize a QTM.

**Lemma 2.2 (completion lemma)** Let $\delta'$ be a unidirectional function on $S \times Q \times \Sigma \times \{-1, 0, 1\}$, where $S \subseteq Q \times \Sigma$. Assume that $\delta'$ satisfies the following conditions (a) and (b).
(a) For any \((q, \sigma) \in S\),
\[
\sum_{p \in Q, \tau \in \Sigma, d \in \{-1,0,1\}} |\delta'(q, \sigma, p, \tau, d)|^2 = 1.
\]
(b) For any \((q, \sigma), (q', \sigma') \in S\) with \((q, \sigma) \neq (q', \sigma')\),
\[
\sum_{p \in Q, \tau \in \Sigma, d \in \{-1,0,1\}} \delta'(q', \sigma', p, \tau, d)^* \delta'(q, \sigma, p, \tau, d) = 0.
\]

Then there is a unidirectional QTM \(M = (Q, \Sigma, \delta)\) such that \(\delta(p, \sigma, q, \tau, d) = \delta'(p, \sigma, q, \tau, d)\) whenever \(\delta'(p, \sigma, q, \tau, d)\) is defined.

We shall give a formal definition of simulation. Let \(M = (Q, \Sigma, \delta)\) and \(M' = (Q', \Sigma', \delta')\) be QTMs. Let \(t\) be a positive integer and \(\epsilon > 0\). Let \(e : \mathcal{C}(Q, \Sigma) \to \mathcal{C}(Q', \Sigma')\) be an injection computable in polynomial time, \(d : \mathcal{C}(Q', \Sigma') \to \mathcal{C}(Q, \Sigma)\) be a function computable in polynomial time satisfying \(d \cdot e = \text{id}\), and \(f\) a function from \(\mathbb{N}^2\) to \(\mathbb{N}\). We say that \(M'\) simulates \(M\) for \(t\) steps with accuracy \(\epsilon\) and slowdown \(f\) (under the encoding \(e\) and the decoding \(d\)), if for any \(C_0 \in \mathcal{C}(Q, \Sigma)\), we have
\[
\sum_{C' \in \mathcal{C}(Q, \Sigma)} \left| \langle C' | M_{\delta}^{f(t,1)} | C_0 \rangle \right|^2 - \sum_{C \in e^{-1}(C')} \left| \langle C | M_{\delta'}^d | e(C_0) \rangle \right|^2 \leq \epsilon.
\]
If \(f\) depends only on \(t\) and the above relation is satisfied for \(\epsilon = 0\), we merely say that \(M'\) simulates \(M\) for \(t\) steps with slowdown \(f\).

Let \(M = (Q, \Sigma, \delta)\) be an \(m\)-track QTM. Then \(\Sigma\) can be factorized as \(\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_m\) and \(T \in \Sigma^\#\) can be written in the form \((T^1, T^2, \ldots, T^m)\), where \(T^i \in \Sigma^\#_i\) for \(i = 1, \ldots, m\). The function \(T^i\) is called an \(i\)-th track configuration. For a string \(x = x_0 x_1 \cdots x_{k-1}\) of length \(k\), we denote by \(T[x]\) a tape (or track) configuration such that \(T[x](i) = x_i\) \((0 \leq i \leq k - 1)\), \(B\) (otherwise). For any tape configuration \(T\), we will write \(T = (T^1, \ldots, T^j)\) if \(T = (T^1, \ldots, T^j, T[\varepsilon], \ldots, T[\varepsilon])\), where \(\varepsilon\) denotes the empty string. Henceforth, \((q, (T^1, \ldots, T^k), 0)\) abbreviate a configuration \((q, (T^1, \ldots, T^k, T[\varepsilon], \ldots, T[\varepsilon]), (0, \ldots, 0))\). Let \(E(\hat{\xi} = j), E(\hat{q} = p)\), \(E(\hat{T} = T_0)\) and \(E(\hat{T^i} = T_0)\) be respectively projections on \(\text{span}\{[q, T, j] | q \in Q, T \in \Sigma^\#\}\), \(\text{span}\{[p, T, \xi] | T \in \Sigma^\#, \xi \in \mathbb{Z}\}\), \(\text{span}\{[q, T_0, \xi] | q \in Q, \xi \in \mathbb{Z}\}\) and \(\text{span}\{[q, T, \xi] | q \in Q, T = (T^1, \ldots, T_0, \ldots, T^m) \in \Sigma^\#, \xi \in \mathbb{Z}\}\). A QTM \(M = (Q, \Sigma, \delta)\) is said to be stationary, if given an initial configuration \(C\), there exists some \(t \in \mathbb{N}\) satisfying \(||E(\hat{\xi} = 0) E(\hat{q} =
\[ q_f \rangle M_s \rangle |C\rangle |^2 = 1 \] and for all \( s < t \) we have \[ \| E(q = q_f) M_s \rangle |C\rangle |^2 = 0. \] The positive integer \( t \) is called the computation time of \( M \) for input state \( |C\rangle \), and \( M_s \rangle |C\rangle \) is called the final state of \( M \) for \( |C\rangle \). Specifically, if \( |C\rangle = |q_0, T[x], 0\rangle \), the integer \( t \) is called the computation time on input \( x \). A polynomial time QTM is a stationary QTM such that the computation time on every input is a polynomial in the length of the input. It is easy to see that a polynomial time bounded QTM (i.e., a QTM whose computation time on every input is bounded by a polynomial in the length of the input) can be simulated by a polynomial time QTM with at most a polynomial slowdown. Moreover, we say that \( M = (Q, \Sigma, \delta) \) is in normal form if \( \delta(q_f, \sigma, q_0, \sigma, 1) = 1 \) for any \( \sigma \in \Sigma \). Henceforth, we shall consider only unidirectional stationary normal form QTMs, since such restricted QTMs are computationally equivalent to general QTMs independent of constraints on the error probability of algorithms [4].

We have discussed solely single tape QTMs, but our arguments can be easily adapted to multi-tape QTMs. See [11] for the formulation of the multi-tape QTMs.

### 3 Solution of the quantum subroutine problem

A stationary time constructible (ST-constructible) QTM of a function \( f : \mathbb{N} \to \mathbb{N} \) is defined to be a QTM such that if the initial state is \( |q_0, T[x], 0\rangle \), then the final state is \( |q_f, T[x], 0\rangle \) and that the computation time is \( f(|x|) \), where \( |x| \) denotes the length of \( x \). A function \( f : \mathbb{N} \to \mathbb{N} \) is said to be stationary time constructible (ST-constructible) if there exists a stationary time-constructible QTM of \( f \).

**Lemma 3.1** For any \( k \geq 2 \), there is an ST-constructible monic polynomial of degree \( k \).

**Proof.** We show this theorem by induction on \( k \). First, when \( k = 2 \), we consider a two-track QTM \( M_2 = (Q, \Sigma \times \{B, 1\}, \delta) \) satisfying the following transition rules, where \( Q = \{q_0, q_1, \ldots, q_5, q_f\} \) and \( \Sigma \) is an arbitrary symbol set. Henceforth, let \( \sigma \in \Sigma \setminus \{B\} \) and let \( s_i \) be an arbitrary symbol in the alphabet of the \( i \)-th track.
\[
\delta(q_0, (\sigma, B), q_1, (\sigma, B), -1) = 1, \quad \delta(q_4, (\sigma, 1), q_4, (\sigma, 1), 1) = 1,
\]
\[
\delta(q_1, (B, B), q_2, (B, B), 1) = 1, \quad \delta(q_4, (s_1, B), q_5, (s_1, B), -1) = 1,
\]
\[
\delta(q_2, (\sigma, B), q_2, (\sigma, 1), 1) = 1, \quad \delta(q_5, (\sigma, 1), q_3, (\sigma, B), -1) = 1,
\]
\[
\delta(q_2, (B, B), q_3, (B, B), -1) = 1, \quad \delta(q_5, (B, B), q_f, (B, B), 1) = 1
\]
\[
\delta(q_3, (\sigma, 1), q_3, (\sigma, 1), -1) = 1,
\]
\[
\delta(q_3, (B, B), q_4, (B, B), 1) = 1
\]

The above partially defined function \(\delta\) can be extended to be total by the completion lemma. Assuming that the input is written on the first track, \(M_2\) implements the following steps.

Step 1. The head of \(M_2\) changes each scanned symbol \(B\) to the symbol 1 on the second track with moving one cell to the right until it scans \(B\) on the first track. If the head scans \(B\) on the first track, it goes to the left until it scans \(B\) on the first track again and then moves one cell to the right.

Step 2. We iterate the following operation until the second track comes to be empty, where we say that the \(i\)-th track \(T_i\) is empty if \(T_i = T[\varepsilon]\). The head goes to the right until it scans \(B\) on the second track and then moves one cell to the left. Afterward, the head changes the scanned 1 to \(B\) on the second track and moves one cell to the left, goes to the left until it scans \(B\) again on the second track, and moves one cell to the right if it scans \(B\).

The computation time of \(M_2\) is \((2n + 4) + \sum_{i=0}^n(2i + 2) = n^2 + 5n + 6\).

Next, we assume that there exists a QTM \(M'\) such that the initial state and the final state are equal except for the processor configuration and that the computation time is a monic polynomial \(p(n)\) of degree \(k\). Then, we consider a QTM \(M\) which implements the following steps.

Step 1. The head of \(M\) changes each scanned symbol \(B\) to the symbol 1 on an auxiliary track with moving one cell to the right until it scans \(B\) on the first track. If the head scans \(B\) on the first track, it goes to the left until it scans \(B\) on the first track again and then moves one cell to the right.

Step 2. We iterate the following operation until the auxiliary track comes to be empty. Firstly, the head goes to the right until it scans \(B\) on the auxiliary track and then moves
one cell to the left. Secondly, the head changes the scanned 1 to B on the auxiliary track and moves one cell to the left, goes to the left until it scans B again on the auxiliary track, and moves one cell to the right if it scans B. Thirdly, the machine runs $M'$. Lastly, the head goes to the right until it scans B on the auxiliary track and afterward it goes to the left until it scans B again on the auxiliary track.

We can construct a partially defined unidirectional quantum transition function implementing the above steps similar to the case $k = 2$. Thus, we obtain the quantum transition function of $M$ by the completion lemma. The computation time of $M$ is $(2n + 4) + \sum_{i=1}^{n} (c_1 i + c_2 + p(n)) = np(n) + O(n^2)$, where $c_1$ and $c_2$ are constant positive integers. By induction hypothesis on $k$, the computation time of $M$ is a monic polynomial of degree $k + 1$. Therefore, the proof is completed. QED

It can be verified that the following lemma follows from Lemma 3.1.

**Lemma 3.2** For any polynomial $p$ of degree $k$, there is an ST-constructible function $f$ such that $p + f$ is an ST-constructible (and monotone increasing) polynomial of degree $k$.

**Proof.** For $k \geq 2$, there exists an ST-constructible monic polynomial of degree $k$ by Lemma 3.1. Moreover, it can be verified that $2n + 4$, $3n + 4$, and a constant function are ST-constructible. For example, we can provide an ST-constructible QTM of $3n + 4$ whose quantum transition function $\delta$ satisfies the following condition, where $\sigma$ is an arbitrary non-blank element in the alphabet of that QTM.

$$\delta(q_0, \sigma, q_1, \sigma, -1) = 1, \quad \delta(q_2, B, q_4, B, -1) = 1,$$

$$\delta(q_1, B, q_2, B, 1) = 1, \quad \delta(q_4, \sigma, q_4, \sigma, -1) = 1,$$

$$\delta(q_2, \sigma, q_3, \sigma, 0) = 1, \quad \delta(q_4, B, q_f, B, 1) = 1,$$

$$\delta(q_3, \sigma, q_2, \sigma, 1) = 1,$$

Thus, any polynomial $p(n) = \sum_{j=0}^{k} a_j n^j$ of degree $k$ is written in the form

$$p(n) = b_k f_k + b_{k-1} f_{k-1} + \ldots + b_2 f_2 + b_1 (2n + 4) + b_0 (3n + 4) + b_{-1}, \quad (2)$$

where $b_k, \ldots, b_0, b_{-1} \in \mathbb{Z}$, and $f_k, \ldots, f_2$ are ST-constructible monic polynomials of degree $k, \ldots, 2$, respectively. Now let $f_1 = 2n + 4$, $f_0 = 3n + 4$, and $f_{-1} = b_{-1}$. Let $\{g_1, \ldots, g_l\} = \{b_j f_j \mid b_j < 0\}$ and $\{h_1, \ldots, h_m\} = \{b_j f_j \mid b_j \geq 0\}$. Then, from Eq. (2) we have

$$p(n) - g_1 - \ldots - g_l = d_1 + \ldots + d_m. \quad (3)$$
We can see that the left hand side of Eq. (3) is an ST-constructible polynomial in the form \( p + f \), where \( f \) is ST-constructible. Moreover, it can be easily verified that Eq. (3) can be modified to an equation such that its left hand side is monotone increasing. \( \text{QED} \)

BBBV [6] defined an oracle quantum Turing machine as the following special QTM. An oracle quantum Turing machine has a special tape called an oracle tape. Its processor configuration set contains special elements \( q_q \) and \( q_a \), which are respectively called the prequery processor configuration and the postquery processor configuration. All cells of the oracle tape are blank except for a single block of non-blank cells. Given a language \( L \) called an oracle language, this machine evolves as follows.

(1) If the processor configuration is \( q_q \) and the string \((x, b)\) is written on the oracle tape, where \((x, b) \in \{0, 1\}^* \times \{0, 1\}\), the processor enters \( q_a \) while the contents of the oracle tape change to \((x, b \oplus L(x))\) deterministically in a single step, where \( \oplus \) denotes the exclusive-or.

(2) If the processor configuration is not \( q_q \), then the machine evolves according to the quantum transition function.

Moreover, BBBV mentioned the notion of more general oracle quantum Turing machines, which has an oracle unitary transformation instead of an oracle language. Now we formulate a quantum Turing machine with an oracle unitary transformation, and give its elementary properties. We assume without loss of generality that the processor enters \( q_q \) only when the head position of the oracle tape is zero and that an oracle unitary transformation are length-preserving, i.e., a state representing a string of length \( n \) is transformed into a superposition of states representing strings of length \( n \).

Let \( Q \) be a processor configuration set with \( q_q \) and \( q_a \), let \( \Sigma \) be a symbol set, let \( \delta \) be a function from \((Q \setminus \{q_a\}) \times \Sigma \times (Q \setminus \{q_a\}) \times \{−1, 0, 1\} \) to \( \mathbb{C} \), and let \( U \) be a unitary transformation such that \( U|x\rangle \in \text{span}\{|z\rangle | z \in \{0, 1\}^n\} \) for any \( x \in \{0, 1\}^n \). Then \( M = (Q, \Sigma, \delta, U) \) is said to be an oracle prequantum Turing machine (with \( U \)). The evolution operator of \( M \) is defined to be a linear operator \( U_M \) on \( \mathcal{H}(Q, \Sigma) \) such that

\[
U_M|q, T, \xi\rangle = \begin{cases} 
\sum_{p \in Q \setminus \{q_a\}, \tau \in \Sigma, \delta \in \{-1, 0, 1\}} \delta(q, T(\xi), p, \tau, d)|p, T^\tau_\xi, \xi + d\rangle & (q \neq q_q) \\
\sum_{y \in \{0, 1\}^n} \langle y|U|x\rangle|q_a, T[y], 0\rangle & (q = q_q, T = T[x], \xi = 0) \\
|q_a, T, \xi\rangle & \text{(otherwise)}.
\end{cases}
\]
If $U_M$ is unitary, $M$ is said to be an oracle quantum Turing machine (oracle QTM). Then we can obtain the following necessary and sufficient conditions by a way similar to the proof of Theorem 2.1 [11].

**Theorem 3.3** An oracle prequantum Turing machine $M = (Q, \Sigma, \delta, U)$ is an oracle QTM if and only if the following quantum transition function $\delta'$ for $((Q \cup \{r\}) \setminus \{q_q, q_a\}, \Sigma)$ satisfies conditions (a)–(d) of Theorem 2.1. Here, $r$ is an element which is not in $Q$.

$$\delta'(q, \sigma, p, \tau, d) = \begin{cases} 
\delta(q_a, \sigma, q, \tau, d) & (q = p = r) \\
\delta(q_a, \sigma, q, \tau, d) & (q = r, \ p \neq r) \\
\delta(q, \sigma, q, \tau, d) & (p = r, \ q \neq r) \\
\delta(q, \sigma, p, \tau, d) & (q \neq r, \ p \neq r).
\end{cases}$$

Similarly we can define a multi-tape oracle QTM. For example, if $M$ is a $k$-tape oracle QTM and the state $|\psi\rangle$ of $M$ is $|q_q, (T^1, \ldots, T^{k-1}, T[x]), (d_1, \ldots, d_{k-1}, 0)\rangle$, the state $U_M|\psi\rangle$ is defined to be

$$U_M|\psi\rangle = \sum_{y \in \{0,1\}^{|x|}} \langle y|U|x\rangle |q_q, (T^1, \ldots, T^{k-1}, T[y]), (d_1, \ldots, d_{k-1}, 0)\rangle.$$  

Then, the $k$-th tape is called an oracle tape. We can consider an oracle QTM with a language $L$, defined by BBBV, to be a multi-tape oracle QTM with the unitary transformation $U_L$ such that $U_L|x, b\rangle = |x, b \oplus L(x)\rangle$ for all $(x, b) \in \{0,1\}^* \times \{0,1\}$. In what follows, we denote by $M'^U$ (or $M'^L$) an arbitrary oracle QTM with a unitary transformation $U$ (or a language $L$).

We introduce a notion necessary for a solution of the quantum subroutine problem. We denote by $\mathcal{D}(M, x)$ the set

$$\mathcal{D}(M, x) = \{C \in \mathcal{C}(Q, \Sigma) | \exists s \leq t \ [\langle C|M_{\delta}^s|q_0, T[x], 0\rangle \neq 0]\},$$

where $t$ is the computation time of $M$ on input $x$. Let $M = (Q, \Sigma, \delta)$ be a QTM and $M' = (Q', \Sigma_1 \times \Sigma_2, \delta')$ be a QTM such that $Q \times \Sigma \subseteq Q' \times \Sigma_1$. We say that $M'$ carries out $M$ with slowdown $f$, if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for any input $x$ of $M$ and $C \in \mathcal{D}(M, x)$ there exists some $T' \in \Sigma_2^f$ (depending on $x$), and that

$$M_{\delta}^{f(|x|)}|C\rangle|T'\rangle = \sum_{C' \in \mathcal{C}(Q, \Sigma)} \langle C'|M_{\delta}|C\rangle|C'|T'\rangle,$$
where $|C⟩|T⟩$ denotes $|q, (T, T'), ξ⟩$ for $C = (q, T, ξ)$. It is easy to verify that if a QTM $M'$ can prepare a track configuration $T'$ satisfying the above condition in polynomial time and if $M'$ carries out a QTM $M$, then $M'$ simulates $M$ for any arbitrary steps.

We can define analogous notions for multi-tape QTMs and oracle QTMs. For any QTM $M = (Q, Σ, δ)$ and any $r \in Q$, we obtain the oracle QTM $M^U = (Q', Σ, δ', U)$ with $Q' = (\{(Q \setminus \{r\}) \times \{0, 1\}\} \cup \{q_0, q_a\}$ satisfying the following conditions.

$$
\begin{align*}
\delta'(q_0, 0, σ, (p, 1), τ, d) &= \delta(q, σ, p, τ, d), \\
\delta'(q_1, 0, σ, (q, 0), σ, 0) &= 1, \\
\delta'(q_0, 0, σ, q_q, τ, d) &= \delta(q, σ, r, τ, d), \\
\delta'(q_a, σ, (p, 1), τ, d) &= \delta(r, σ, p, τ, d), \\
\delta'(q_a, σ, q_q, τ, d) &= \delta(r, σ, r, τ, d).
\end{align*}
$$

In particular, if $U$ is the identity operator, then $M^U$ carries out $M$. Thus, we can consider a QTM to be a special case of an oracle QTM.

We say that a unitary transformation $U$ is polynomial time computable by a QTM $M$, if the final state of $M$ for the initial state $|q_0, T[x], 0⟩$ with $|x| = n$ is

$$
\sum_{y \in \{0, 1\}^n} ⟨y|U|x⟩|q_f, T[y], 0⟩
$$

and the computation time of $M$ is a polynomial in $n$.

The following theorem gives us the positive answer for the quantum subroutine problem.

**Theorem 3.4** If a unitary transformation $U$ is polynomial time computable by a QTM $M$, there are a polynomial $p$ and a polynomial time QTM $M'$ such that $M'$ carries out a polynomial time oracle QTM $M^U$ with slowdown $p$.

**Proof.** Let the quantum transition functions of $M$, $M'$, and $M^U$ be $δ$, $δ'$, and $δ^u$ respectively. Let the computation times of $M^U$ and $M$ be $f(n)$ and $g(n)$ respectively. By Lemma 3.2 we can assume that $g$ is monotone increasing ST-constructible. Let $h = g \circ f + f$. Now we consider a QTM $M'$ which implements the following steps on input $x$, where $M'$ has three tapes and the third tape consists of two tracks.

Step 1. $M'$ writes $1^{h(|x|)}$ on the second tape.
Step 2. $M'$ carries out a single step of $M^U$ by the following steps 2.1–2.3.

Step 2.1. If the processor configuration is $q_q$ and $y$ is written on the third tape of $M'$, which corresponds to the oracle tape of $M^U$, then the head of the second tape goes to the right until it scans $B$. At the same time, $M'$ runs a QTM carrying out $M$ for $g(n)$ steps on the third tape. Afterward, the head of the third tape goes to the right while writing a special symbol $*$ on each cell of the second track. Here, let $q$ and $p$ be processor configurations of $M$, let $q_0$ and $q_f$ be respectively the initial and final processor configurations of $M$, and let $\sigma$ and $\tau$ be arbitrary elements in the alphabet of $M$. Moreover, throughout this proof, let $s_1$ be an arbitrary first tape symbol of $M'$, and let $s_3$ be an arbitrary first track symbol of the third tape of $M'$.

$$\delta'(q_1, (s_1, 1, \sigma), (q, 1), (s_1, 1, \tau), (0, 1, d)) = \delta(q_0, \sigma, q, \tau, d),$$
$$\delta'(q, 1), (s_1, 1, \sigma), (p, 1), (s_1, 1, \tau), (0, 1, d)) = \delta(q, \sigma, p, \tau, d), \quad (q \neq q_0, q_f)$$
$$\delta'(q_f, 1), (s_1, 1, \sigma), q_2, (s_1, 1, \sigma), (0, 1, -1)) = 1,$$
$$\delta'(q_2, (s_1, 1, B), q_3, (s_1, 1, B), (0, 1, 1)) = 1,$$
$$\delta'(q_3, (s_1, 1, s_3), q_3, (s_1, 1, (s_3, *)) (0, 1, 1)) = 1.$$

If the head of the second tape scans $B$, then the heads of the second and third tape move to the left. Specifically, the head of the third tape changes each scanned special symbol $*$ to $B$ while going to the left.

$$\delta'(q_3, (s_1, B, s_3), q_4, (s_1, B, s_3), (0, -1, -1)) = 1,$$
$$\delta'(q_4, (s_1, 1, (s_3, *)) q_4, (s_1, 1, s_3), (0, -1, -1)) = 1.$$

If the head of the third tape scans $B$, then $M'$ carries out an ST-constructible QTM $M_{g+1}$ of the function $g + 1$ on input $y$ after moving one cell to the right. Here, $q$ and $p$ are processor configurations of $M_{g+1}$, the symbols $\sigma$ and $\tau$ are arbitrary elements in the alphabet of $M_{g+1}$, and $\delta_{g+1}$ is the quantum transition function of $M_{g+1}$.

$$\delta'(q_4, (s_1, 1, B), q_5, (s_1, 1, B), (0, -1, 1)) = 1,$$
$$\delta'(q_5, (s_1, 1, \sigma), (q, 6), (s_1, 1, \tau), (0, -1, d)) = \delta_{g+1}(q_0, \sigma, q, \tau, d),$$
$$\delta'((q, 6), (s_1, 1, \sigma), (p, 6), (s_1, 1, \tau), (0, -1, d)) = \delta_{g+1}(q, \sigma, p, \tau, d) \quad (q \neq q_0, q_f).$$

If the head of the second tape scans $B$, then it moves one cell to the right and after three
steps the processor enters $q_a$.

\[
\delta'((q_f, 6), (s_1, B, s_3), q_7, (s_1, B, s_3), (0, 1, 0)) = 1,
\]
\[
\delta'((q_7, (s_1, 1, s_3), q_8, (s_1, 1, s_3), (0, 0, 0)) = 1,
\]
\[
\delta'((q_8, (s_1, 1, s_3), q_9, (s_1, 1, s_3), (0, 0, 0)) = 1,
\]
\[
\delta'((q_9, (s_1, 1, s_3), q_a, (s_1, 1, s_3), (0, 0, 0)) = 1.
\]

Step 2.2. If the processor configuration is not $q_q$, then the head of the second tape goes to the right. Here, $q$ and $p$ are processor configurations of $M^U$.

\[
\delta'(q, (s_1, 1, s_3), q, 2), (s_1, 1, s_3), (0, −1, 0)) = 1,
\]
\[
\delta'((q, 2), (s_1, B, s_3), (q, 3), (s_1, B, s_3), (0, 1, 0)) = 1,
\]
\[
\delta'((q, 3), (s_1, 1, s_3), (q, 3), (s_1, 1, s_3), (0, 1, 0)) = 1.
\]

If the head of the second tape scans $B$, then it goes to the left.

\[
\delta'((q, 3), (s_1, B, s_3), (q, 4), (s_1, B, s_3), (0, −1, 0)) = 1,
\]
\[
\delta'((q, 4), (s_1, 1, s_3), (q, 4), (s_1, 1, s_3), (0, −1, 0)) = 1.
\]

If the head of the second tape scans $B$ again, then it moves one cell to the right. Afterward, $M'$ carries out a single step of $M^U$ on the first and the third tapes while the head of the second tape stays during one step.

\[
\delta'(q, 4), (s_1, B, s_3), (q, 5), (s_1, B, s_3), (0, 1, 0)) = 1,
\]
\[
\delta'((q, 5), (s_1, 1, s_3), (q, 5), (s_1, 1, s_3), (d_1, 0, d_3))) = \delta^u(a, (s_1, 1, s_3), (d_1, 0, d_3))).
\]

Step 2.3. If the processor configuration is $q_q$ and the query string is not correctly written on the third tape, then the head of the second tape goes to the right, goes to the left after it scans $B$, moves one cell to the right after it scans $B$ again, and $M'$ enters the postquery configuration without changing the tape configuration.

We can implement step 1 since the function $h$ is polynomial time computable by a QTM. In step 2.1, $M'$ must carry out $M$, since if $M'$ only simulates $M$, then $M'$ may leave extra information and computational paths with different extra information do not interfere. The partially defined function $\delta'$ satisfies the unitary conditions of quantum transition functions of multi-tape QTMs \[\text{II}\], so that there exists a QTM implementing step 2 by the completion lemma. It is easy to see that the QTM implementing step 2 carries out $M'^U$ with slowdown $2h(|x|) + 5$ and that the computation time of $M'$ is a polynomial in $n$. QED
Using ST-constructible functions, we have reduced polynomial slowdown caused by the insertion of subroutines as much as possible. The degree of this polynomial slowdown is same as the case of deterministic or probablistic Turing machines.

**Corollary 3.5** If a unitary transformation $U$ is computable by a QTM $M$ in linear time, there is a QTM $M'$ which carries out a linear time oracle QTM $M^U$ in quadratic time.

Next, we consider the bounded error version of the quantum subroutine problem (If a unitary transformation $U$ is efficiently computable by a QTM with any accuracy, is there an efficient QTM simulating an oracle QTM $M^U$ with any accuracy?).

A function $f : \mathbb{N}^2 \to \mathbb{N}$ is said to be **stationary time constructible (ST-constructible)** if there is a QTM satisfying the following condition: the final state of the QTM for the initial state $|q_0, (T[x], T[y]), 0\rangle$ is $|q_f, (T[x], T[y]), 0\rangle$ and the computation time is $f(|x|, |y|)$.

Then the following lemma holds by a proof similar to Lemmas 3.1 and 3.2.

**Lemma 3.6** For any polynomial $p(n, m) = \sum_{c=1}^{k} \sum_{d=1}^{l} a_{c,d} n^{c} m^{d}$ over $\mathbb{Z}$, where $a_{k,l} \neq 0$, there is an ST-constructible function $f(n, m) = \sum_{c=1}^{k} \sum_{d=1}^{l} b_{c,d} n^{c} m^{d}$ with $b_{k,l} \neq 0$ such that $p + f$ is ST-constructible.

We say that a unitary transformation $U$ is **approximately polynomial time computable** by a QTM $M$ if the following conditions hold.

1. $U|x\rangle \in \text{span}\{|z\rangle | z \in \{0, 1\}^n\}$ for any $x \in \{0, 1\}^n$.
2. There is a family of unitary transformations $\{U'_i\}$ such that the final state of $M$ for the initial state $|q_0, T[x, 1^l], 0\rangle$ with $|x| = n$ is
   $$\sum_{y \in \{0, 1\}^*} \langle y|U'_i|x\rangle |q_f, T[y, 1^l], 0\rangle$$
   and $||U|x\rangle - U'_i|x\rangle|| \leq 1/2^l$ for any $x \in \{0, 1\}^*$.
3. The computation time of $M$ is a polynomial in $n$ and $l$.

Let $M = (Q, \Sigma, \delta)$ be a QTM and $M' = (Q', \Sigma_1 \times \Sigma_2, \delta')$ be a QTM such that $Q \times \Sigma \subseteq Q' \times \Sigma_1$. We say that $M'$ carries out $M$ with bounded error and slowdown $f$, if there exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any input $y = (x, 1^l)$ of $M$, where $l \in \mathbb{N}$, and any $C \in \mathcal{D}(M, y)$ there exists some $T' \in \Sigma_2^{\pm}$ (depending on $y$), and that $||M'_f(x|y)|C[T'] - (M_5|C) \otimes |T'|| | \leq 1/2^l$. Now using Lemma 3.6 the following theorem
holds by a way similar to the proof of Theorem 3.4, and gives the positive answer for the bounded error version of the quantum subroutine problem of QTM.

**Theorem 3.7** If a unitary transformation $U$ is approximately polynomial time computable by a QTM $M$, for any $l \in \mathbb{N}$ there are a polynomial $p(n, l)$ and a polynomial time QTM $M'$ such that $M'$ carries out a polynomial time oracle QTM $M^U$ with bounded error and slowdown $p$.

## 4 Robustness of quantum complexity classes

In this section, we identify a language $L$ with its characteristic function $c_L$, and we denote $c_L(x)$ by $L(x)$. We shall now define complexity classes for oracle QTMs. These definitions naturally extend the notion of complexity classes for QTMs [3, 9]. In what follows, we assume that the ranges of quantum transition functions are the polynomial time computable numbers.

We say that an oracle QTM $M$ accepts (or rejects) $x \in \{0, 1\}^*$ with probability $p$ if the final state $|\psi\rangle$ of $M$ for the initial state $|q_0, T[x], 0\rangle$ satisfies

$$||E(\hat{T}^1 = T[x])E(\hat{T}^2 = T[1])|\psi\rangle||^2 = p \quad \text{(or } ||E(\hat{T}^1 = T[x])E(\hat{T}^2 = T[0])|\psi\rangle||^2 = p).$$

We say that $M$ recognizes a language $L$ with probability $p$ if $M$ accepts $x$ with probability at least $p$ for any $x \in L$ and rejects $x$ with probability at least $p$ for any $x \notin L$. Moreover, we say that $M$ recognizes $L$ with probability uniformly larger than $p$, if there is a constant $0 < \eta \leq 1 - p$ such that $M$ recognizes $L$ with probability $p + \eta$. A language $L'$ is in $\text{BQP}^L$ (or $\text{EQP}^L$) if there is a polynomial time oracle QTM $M^L = (Q, \Sigma, \delta, U_L)$ that recognizes $L'$ with probability uniformly larger than $\frac{1}{2}$ (with probability 1). Then, $M^L$ is called a $\text{BQP}$-machine (or $\text{EQP}$-machine). A language $L'$ is in $\text{ZQP}^L$ if there is a polynomial time QTM $M^L = (Q, \Sigma, \delta, U_L)$ satisfying the following conditions: (1) $M^L$ recognizes $L$ with probability uniformly larger than $\frac{1}{2}$; (2) If $M$ accepts (rejects) $x$ with a positive probability, $M$ rejects (accepts) $x$ with probability 0. Such a QTM $M^L$ is called a $\text{ZQP}$-machine. For classes $C$ and $D$ of languages, let $C^D = \bigcup_{L \in D} C^L$. If $C^C = C$, the class $C$ is said to be robust.

We can apply Theorems 3.4 and 3.7 to the robustness of the quantum complexity classes $\text{EQP}$ and $\text{BQP}$. If $L$ is in $\text{EQP}$, then we can construct a polynomial time oracle
QTM such that only the input $x$ and the answer $L(x)$ are written on the tape of the final state with probability 1 by the method of Bennett in reversible computation \[12\]. In other words we can assume that an EQP-machine has only one accepting configuration. His method is implemented in the following steps. We compute $L(x)$, copy $L(x)$ into an extra track, and carry out the reverse of the process of computing $L(x)$ in order to get rid of the scratch work. In the case of QTM$s$, reverse computation can be implemented by using the reversal lemma due to Bernstein and Vazirani \[3\]. By the method of Bennett, we can see that for any $L \in \text{EQP}$, a unitary transformation $U_L$ such that $U_L|x, b\rangle = |x, b \oplus L(x)\rangle$ is polynomial time computable. Thus, EQP is robust by Theorem 3.4.

**Theorem 4.1** $\text{EQP}^{\text{EQP}} = \text{EQP}$.

BBBV \[6\] showed the following theorem, which ensures the use of a Monte Carlo quantum algorithm as a subroutine of another quantum algorithm.

**Theorem 4.2** If a language $L$ is in BQP, for any $l \in \mathbb{N}$ there is a QTM $M$ which recognizes $L$ with probability $1 - 1/2^l$ and has the following property (A): The computation time of $M$ for $|q_0, T[x], 0\rangle$ is a polynomial in $|x|$ and $l$, and the final state is $\alpha|q_f, T, 0\rangle + |\psi\rangle$, where $|\alpha|^2 \geq 1 - 1/2^l$ and $T = (T[x], T[L(x)])$.

**Remark.** The QTM $M$ obtained in the proof in \[6\] is not always stationary. However, we can construct a stationary QTM with property (A) by using the construction of a universal QTM \[3, 8, 9\].

Theorem 4.2 guarantees that without loss of generality a BQP-machine recognizing $L$ has a clean tape with only the input $x$ and the answer $L(x)$ with arbitrary large probability after computation. In other words we can assume that a BQP-machine has only one accepting configuration. BBBV \[6\] claimed that BQP is robust as the corollary of Theorem 4.2, since this theorem allows us to use a QTM recognizing an oracle language instead of the oracle itself. However, they considered the case where the machine enters a query state deterministically, i.e., they did not discuss the possibility that the coherence of different computation paths collapses by the insertion of a QTM recognizing an oracle language. We have already solved this problem by Theorem 3.4, so that we can show that BQP is robust in the general setting where a query state and a nonquery state may superpose.
Theorem 4.3 \( \text{BQP}^{\text{BQP}} = \text{BQP} \).

Now we consider the robustness of \( \text{ZQP} \). If we apply Theorem 4.2 to a language in \( \text{ZQP} \), the obtained algorithm will not be Las Vegas. Thus, we need the following theorem, which means that we can also assume that a \( \text{ZQP} \)-machine has only one accepting configuration.

**Theorem 4.4** If a language \( L \) is in \( \text{ZQP} \), for any \( l \in \mathbb{N} \) there is a QTM \( \text{M} \) which recognizes \( L \) with probability \( 1 - 1/2^l \) and has the following property (B): The computation time of \( \text{M} \) for \( |q_0, T[x], 0 \rangle \) is a polynomial in \( |x| \) and \( l \), and the final state is \( \alpha|q_f, T, 0 \rangle + |\psi \rangle \), where \( \alpha^2 \geq 1 - 1/2^l \), \( T = (T[x], T[L(x)]) \), and \( E(\hat{q} = q_f)E(\hat{T}^2 = T[\star])|\psi \rangle = |\psi \rangle \). Here, we denote by \( \star \) a special symbol of the second track of \( M \).

**Proof.** For simplicity, we represent a state of a QTM by its track configurations. We denote by \( |T^1\rangle|T^2\rangle \cdots |T^n\rangle \) a computational basis vector of a QTM such that for each \( i = 1, \ldots, n \), the \( i \)-th track configuration is \( T^i \). Let \( L \in \text{ZQP} \). Then, we can assume that there is a \( \text{ZQP} \)-machine \( M' \) which recognizes \( L \) with probability \( 1 - 1/2^{l+1} \) in time polynomial in the length of input and \( l \). Let the final state of \( M' \) for the initial state \( |x\rangle \) be

\[
\sum_w \alpha(w)|x\rangle|L(x)\rangle|w\rangle + \sum_{y,z} \sum_v \beta(y,z,v)|y\rangle|z\rangle|v\rangle,
\]

where \( x \) and \( L(x) \) denote \( T[x] \) and \( T[L(x)] \) respectively, the summations \( \sum_v \) and \( \sum_w \) are respectively taken over all the third track strings, and \( \sum_{y,z} \) is taken over all the pairs \( (y,z) \) of the first and second track strings such that \( (y,z) \neq ((T[x], T[0]), (T[x], T[1])) \). Then, we have \( \sum_w |\alpha(w)|^2 \geq 1 - 1/2^{l+1} \). Now we consider a \( \text{ZQP} \)-machine \( M \) with seven tracks which implements the following steps.

Step 1. \( M \) on input \( x \) writes \( 1^{p_l(|x|)} \) between cell 1 and cell \( p_l(|x|) \), and \( 0^{p_l(|x|)} \) between cell \(-1\) and cell \(-p_l(|x|)\) of the seventh track. Here, \( p_l(|x|) \) is the computation time of \( M' \) on input \( x \).

Step 2. \( M \) runs \( M' \).

Step 3. \( M \) respectively copies the first and second track strings to the fourth and fifth tracks.

Step 4. \( M \) runs the reverse of \( M' \). This step is implementable by the reversal lemma.
Step 5. If $x$ is respectively written on the first and fourth tracks, the symbol 0 or 1 is written on the fifth track, and other tracks are empty, then $M$ writes no symbol. Otherwise, $M$ writes a special symbol $\star$ in the cell 0 of the sixth track.

Step 6. If the first track string and the fourth track string are equal, then $M$ erases the fourth track string.

Step 7. If $\star$ is written in the cell 0 of the sixth track, then $M$ exchanges the contents of the second track for those of the sixth track. Otherwise, $M$ exchanges the contents of the second track for those of the fifth track.

Now we shall verify that the desired state is obtained after steps 1–7. The state of the system after step 2 is represented by Eq. (4). After step 3 the system will evolve into the state

$$
\sum_w \alpha(w)|x\rangle|L(x)\rangle|w\rangle|x\rangle|L(x)\rangle + \sum_{y,z,v} \beta(y,z,v)|y\rangle|z\rangle|v\rangle|y\rangle|z\rangle
$$

$$
= \left( \sum_w \alpha(w)|x\rangle|L(x)\rangle|w\rangle + \sum_{y,z,v} \beta(y,z,v)|y\rangle|v\rangle \right) |x\rangle|L(x)\rangle
$$

$$
+ \sum_{y,z,v} \beta(y,z,v)|y\rangle|z\rangle|v\rangle(|y\rangle|z\rangle - |x\rangle|L(x)\rangle).
$$

Since the unitary transformation $U$ implementing step 4 is identical on the fourth and fifth tracks, there is a unitary transformation $U'$ such that

$$
U(|T^1\rangle|T^2\rangle|T^3\rangle|T^4\rangle|T^5\rangle) = (U'|T^1\rangle|T^2\rangle|T^3\rangle) \otimes (|T^4\rangle|T^5\rangle),
$$

where $T^i$ is an arbitrary $i$-th track configuration for $i = 1, \ldots, 5$. Then, the state of the system is

$$
|x\rangle|B\rangle|B\rangle|x\rangle|L(x)\rangle + U\left( \sum_{y,z,v} \beta(y,z,v)|y\rangle|z\rangle|v\rangle(|y\rangle|z\rangle - |x\rangle|L(x)\rangle) \right)
$$

$$
= |x\rangle|B\rangle|B\rangle|x\rangle|L(x)\rangle + \sum_{y,z} U'\left( \sum_{v} \beta(y,z,v)|y\rangle|z\rangle|v\rangle \right) \otimes (|y\rangle|z\rangle - |x\rangle|L(x)\rangle).
$$

Since we have $(y,z) \neq ((T[x], T[0]), (T[x], T[1]))$, if we write

$$
U \left( \sum_{y,z,v} \beta(y,z,v)|y\rangle|z\rangle|v\rangle(|y\rangle|z\rangle - |x\rangle|L(x)\rangle) \right) = \gamma|x\rangle|B\rangle|B\rangle|x\rangle|L(x)\rangle + |\psi\rangle,
$$

then $\langle\psi|x, B, B, x, T[1 - L(x)]\rangle = 0$, where $|x, B, B, x, T[1 - L(x)]\rangle$ denotes $|x\rangle|B\rangle|B\rangle|x\rangle|T[1 - L(x)]\rangle$. Moreover, we can see that

$$
|||\psi|||^2 \leq \left|\left| U \left( \sum_{y,z,v} \beta(y,z,v)|y\rangle|z\rangle|v\rangle(|y\rangle|z\rangle - |x\rangle|L(x)\rangle) \right) \right|^2 \leq \sum_{y,z,v} 2|\beta(y,z,v)|^2 = 1/2^4.
$$

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Steps 5–7 are implementable by using the symbol strings written on the seventh track and the branching lemma \[3\]. From the above, \(M\) satisfies the statement of this theorem. QED

Let \(M = (Q, \Sigma, \delta)\) be a QTM and \(M' = (Q', \Sigma_1 \times \Sigma_2 \times \Sigma_3, \delta')\) be a QTM such that \(Q \times \Sigma \subseteq Q' \times (\Sigma_1 \times \Sigma_2)\). We say that \(M' \) carries out \(M\) with zero error and slowdown \(f\), if there exists a function \(f : \mathbb{N}^2 \rightarrow \mathbb{N}\) such that for any input \(y = (x, 1^l)\) of \(M\), where \(l \in \mathbb{N}\), and any \(C \in D(M, y)\) there exists some \(T' \in \Sigma_3^\#\) (depending on \(y\)) satisfying the following condition: If \(M_\delta^{L(x,l)}(C) |T'\rangle = |\phi\rangle + |\psi_1\rangle\) and \(E(T^2 = T[\star])(|\phi\rangle + |\psi_1\rangle) = |\psi_1\rangle\), then \((M_\delta(C)) \otimes |T'\rangle = |\phi\rangle + |\psi_2\rangle\), \(|||\psi_1\rangle - |\psi_2\rangle|| \leq 1/2^l\), and \(E(T^2 = T[\star])|\psi_2\rangle = 0\). Using Theorem 4.4 we can show the following lemma by a way similar to the proof of Theorem 3.4.

**Lemma 4.5** If \(L\) is in \(ZQP\), there are a polynomial \(p(n,l)\) and a polynomial time QTM \(M\) such that \(M\) carries out a polynomial time oracle QTM \(M^L\) with zero error and slowdown \(p\).

We shall show that \(ZQP\) is robust by using Lemma 4.5. To this end, we need to construct our algorithm so that we cannot erase the symbol \(*\) written as a witness of an error in the subsequent steps.

**Theorem 4.6** \(ZQP^{ZQP} = ZQP\).

*Proof.* Let \(L \in ZQP^{ZQP}\). Then there is a language \(L' \in ZQP\) such that \(L \in ZQP^{L'}\).

We can assume that an oracle QTM \(M_1^{L'}\) recognizes \(L\) with probability \(1 - 1/2^l\). Let the computation times of \(M_1^{L'}\) be \(p(n,l) = p(n, l)\). By Lemma 4.5 there are a polynomial \(f(n,l)\) and a polynomial time QTM \(M'\) that carries out a polynomial time oracle QTM \(M_1^{L'}\) with zero error and slowdown \(f\). Now we consider a \(ZQP\)-machine \(M_t\) which implements the following algorithm. We assume that the length of the input \(x\) of \(M_t\) is \(n\).

1. \(M_t\) writes \(1^l\) and \(1^{p_l(n)}\) on the second and third tapes.
2. \(M_t\) repeats the following operation \(p_l(n)\) times: If the special symbol \(*\) is written in the cell 0 of the first tape, then \(M_t\) changes the string \(0^m 1^{p_l(n) - m}\) on the third tape to \(0^{m+1} 1^{p_l(n) - m - 1}\) in \(g(n, l)\) steps. Otherwise, \(M_t\) carries out \(f(n,l)\) steps of \(M'\) on \((x, 1^l)\) in \(g(n, l)\) steps. Here, \(g\) is an ST-constructible function.
Since the probability that our algorithm incorrectly carries out a single step of \( M_1^{L'} \) and the error probability of \( M_1^{L'} \) are both at most \( 1/2^l \), the probability that \( M_l \) produces a correct answer is at least \( (1 - 1/2^l)^{p_L(n)+1} \). Thus, if \( 1/2^l \leq 1/c'(p_L(n) + 1) \) with some constant \( c' \), then \( M_l \) recognizes \( L \) uniformly larger than \( 1/2 \). By Lemma 3.6, the branching lemma and the looping lemma, step 2 is implementable by a stationary QTM. If our algorithm incorrectly carries out a single step of \( M_1^{L'} \) then the special symbol \( \star \), a witness of an error, is written on the second track of the first tape by Theorem 4.4. We can see that the construction of our algorithm ensures that the symbol \( \star \) is not erased in the subsequent steps. Therefore, our algorithm is Las Vegas type. Now we can choose \( l \) such that \( 1/2^l \leq 1/c'(p_L(n) + 1) \) and that \( l \) is a polynomial in \( n \), and then the computation time of \( M_l \) is a polynomial in \( n \). QED

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