Ornstein–Uhlenbeck semigroups in infinite dimension

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This is a survey paper about Ornstein–Uhlenbeck semigroups in infinite dimension and their generators. We start from the classical Ornstein–Uhlenbeck semigroup on Wiener spaces and then discuss the general case in Hilbert spaces. Finally, we present some results for Ornstein–Uhlenbeck semigroups on Banach spaces.

This article is part of the theme issue ‘Semigroup applications everywhere’.

1. Introduction

In this article we present the basic results on Ornstein–Uhlenbeck (O-U) semigroups on infinite dimensional spaces. We refer to the survey The Ornstein–Uhlenbeck semigroup in finite dimension by A. Lunardi, G. Metafune and D. Pallara in this volume for a general introduction to the finite dimensional case.

Infinite dimensional O-U operators, semigroups and processes find their motivations in statistical mechanics, quantum theory, analysis of PDEs, control theory, random processes and stochastic PDEs. In the quantum field theory, the classical O-U operator is the ‘number operator’, whose eigenvalues represent the number of bosons in a quantum field, and, indeed, classical results like hypercontractivity and logarithmic Sobolev inequalities have their origins in the quantum theory community. In analysis, the O-U operator appears as the generator of Chebyshev–Hermite orthogonal polynomials, which eventually led to the Wiener chaos decomposition mentioned in §3. The classical O-U semigroup plays an essential role in Malliavin calculus.
This theory began to provide a probabilistic proof of Hörmander hypoellipticity theorem and found important applications in the regularity theory of probability distributions of functionals of underlying Gaussian processes and of solutions of stochastic differential equations, as well as multiple stochastic integrals.

The principal motivation to study non-symmetric O-U semigroups comes from stochastic evolution equations. The connection is explained in §4a, see (4.6) and (4.7).

This article is organized as follows. After an introductory section with preliminaries and notation, the classical O-U semigroup on separable Banach spaces is discussed in §3; we refer to the survey paper [1] for many details and historical notes.

The main body of this article is §4, where we describe the theory of O-U semigroups on separable Hilbert spaces. Readers can refer to ref. [2], where one can find the basic ideas, many examples and applications and connections with stochastic analysis in Hilbert spaces.

In the last section we consider O-U semigroups on separable Banach spaces. There are many more technicalities and far fewer examples than in the Hilbert setting, and in this short survey, we have not room to give details, so we only briefly list some extensions of the results of §4 to the Banach case.

2. Preliminaries

Throughout this article, $X$ is a separable real Banach space, with norm $\| \cdot \|$. $B_b(X)$, $C_b(X)$ and $BUC(X)$ denote the spaces of Borel measurable, continuous, uniformly continuous and bounded functions from $X$ to $\mathbb{R}$, respectively, endowed with the sup norm $\| \cdot \|_{\infty}$. Occasionally, we will be concerned also with the mixed topology in $C_b(X)$, which is the finest locally convex topology that agrees on every bounded set in $C_b(X)$ with the topology of uniform convergence on compact sets.

As we are concerned with Gaussian measures on $X$ and the relative Cameron–Martin Hilbert space $H \subset X$ is separable, we state the standing assumption that $X$ itself is separable: in fact, Gaussian measures are always concentrated on the closure of $H$ in $X$. See §2d.

If $Y$ is any Banach space, $\mathcal{L}(X, Y)$ is the space of linear bounded operators from $X$ to $Y$; as usual, if $Y = X$, it is denoted by $\mathcal{L}(X)$, and if $X = \mathbb{R}$, it is denoted by $X^*$. For $2 \leq h \in \mathbb{N}$, $\mathcal{L}^h(X)$ is the space of continuous $h$-linear operators from $X^h$ to $\mathbb{R}$.

The Borel $\sigma$-algebra $\mathcal{B}(X)$ coincides with the $\sigma$-algebra $\mathcal{E}(X)$ generated by the cylindrical sets, i.e., the sets of the form $C = \{ x \in X : (f_1(x), \ldots, f_n(x)) \in C_0 \}$, where $f_1, \ldots, f_n \in X^2$ and $C_0 \in \mathcal{B}(\mathbb{R}^n)$, see e.g. [3, Ch. 1]. Accordingly, a function $f : X \to \mathbb{R}$ is called cylindrical if there are $f_1, \ldots, f_n \in X^*$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = \varphi(f_1(x), \ldots, f_n(x))$.

If $X$ is a Hilbert space, we denote by $\langle \cdot, \cdot \rangle$ its inner product. $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$ denote the subspaces of $\mathcal{L}(X)$ of nuclear self-adjoint and Hilbert–Schmidt operators, respectively.

(a) Symmetric and positive operators

An operator $Q \in \mathcal{L}(X^*, X)$ is called symmetric if $g(Qf) = f(Qg)$ for every $f, g \in X^*$ and positive if $f(Qf) \geq 0$ for every $f \in X^*$ (in fact, the right word should be ‘non-negative’ but we adopt the common terminology). As usual, if $X$ is a Hilbert space, we identify $X$ and $X^*$, and the above notions correspond respectively to a self-adjoint and non-negative $Q \in \mathcal{L}(X)$.

If $Q$ is symmetric and positive, there exists a unique Hilbert space $H_Q$ continuously embedded in $X$, such that $Q(X^*)$ is dense in $H_Q$ and $(Qf, Qg)_{H_Q} = g(Qf)$, for every $f, g \in X^*$, see [3, Prop. III.1.6]. Denoting by $i : H_Q \to X$ the embedding we have $\| i \|_{\mathcal{L}(H_Q, X)} = |Q|^{1/2}_{\mathcal{L}(X^*, X)}$ and $i \circ i^* = Q$; see [3, Chapter III]. $H_Q$ may be equivalently constructed by completing $Q(X^*)$ with respect to the norm associated with the inner product $(Qf, Qg) \mapsto g(Qf)$. It is easily seen that every Cauchy sequence $(Qf_n)$ in such norm converges in $X$, and two equivalent Cauchy sequences converge in $X$ to the same limit. Identifying (the equivalence class of) any Cauchy sequence $(Qf_n)$ with its $X$-limit $h$, the completion is identified with $H_Q$.
If $X$ is a Hilbert space and $Q \in \mathcal{L}(X)$ is self-adjoint and non-negative, $H_Q$ is just $Q^{1/2}(X)$ with the inner product $(Q^{1/2}x, Q^{1/2}y)_{H_Q} = (x, y)$ for every $x, y \in X$, or equivalently $(h, k)_{H_Q} = (Q^{-1/2}h, Q^{-1/2}k)$. Here, if $Q^{1/2}$ is not one to one, $Q^{-1/2}$ denotes its pseudo-inverse.\(^1\)

The space $H_Q$ is sometimes called reproducing kernel Hilbert space associated with $Q$, but since the expression ‘reproducing kernel Hilbert space’ has several different meanings in the literature, we will not use it.

(b) Regular functions

Let $Y$ be any Banach space, $\alpha \in (0, 1), k \in \mathbb{N}$.

$C^k_\alpha(X; Y)$ is the space of bounded and $\alpha$-Hölder continuous functions from $X$ to $Y$, endowed with its standard norm $\|f\|_{C^k_\alpha(X; Y)} := \|f\|_\infty + \|f\|_{C^k(X; Y)}$, where $\|f\|_{C^k(X; Y)} = \sup_{x, y \in X, x \neq y} \|f(x) - f(y)\| / \|x - y\|^\alpha$. If $Y = \mathbb{R}$, we set $C^k_\alpha(X; \mathbb{R}) := C^k_\alpha(X)$.

$C^\infty_\alpha(X)$ is the space of $k$ times Fréchet differentiable functions $F : X \to \mathbb{R}$, with continuous and bounded derivatives $D^jF : X \to \mathcal{L}(X)$ for every $j = 1, \ldots, k$. The first-order Fréchet derivative $D^1$ is denoted by $D$.

If $X$ is a Hilbert space and $f : X \to \mathbb{R}$ is Fréchet differentiable at $x$, by the Riesz isometry, there is a unique $z \in X$, such that $Df(x)(h) = (z, h)$ for every $h \in X$. Such $z$ is denoted by $\nabla f(x)$.

If $X$ is a Hilbert space and $f : X \to \mathbb{R}$ is $H$-differentiable at $x$, by $H$-Hilbert continuous if there is $\alpha \in (0, 1)$, such that $|\langle \Delta f(x)(h), h \rangle| < \infty$. $C^\infty_H(X, Y)$ is the space of the functions in $C^\infty(X, Y)$ that are $H$-Hilbert continuous.

A function $\varphi : X \to Y$ is $H$-differentiable at $x \in X$ if there exists $G \in \mathcal{L}(H, Y)$ such that $\langle x + h - \varphi(x) - G(h) \rangle = o(h)$ as $h \to 0$ in $H$. In this case, the operator $G$ is unique and denoted by $D_H\varphi(x)$. Again, if $Y = \mathbb{R}$, there is a unique $y \in H$ such that $G(h) = (y, h)_H$ for each $h \in H$. Such $y$ is denoted by $\nabla_H\varphi(x)$.$^1$

(c) Semigroups of bounded operators on $C_b(X)$

Let $T(t)$ be a semigroup of bounded operators on $C_b(X)$, such that $\|T(t)\|_{\mathcal{L}(C_b(X))} \leq Me^{\omega t}$ for some $M > 0$, $\omega \in \mathbb{R}$ and for every $t \geq 0$. Assume in addition that the function $(t, x) \mapsto T(t)f(x)$ is continuous in $[0, +\infty) \times X$.

Since we are going to deal with resolvent and spectrum, it is convenient to extend $T(t)$ to the space $C_b(X; \mathbb{C})$, setting $T(t)(f + ig) = T(t)f + iT(t)g$ for $f, g \in C_b(X)$.

This allows to define a generator through its resolvent,

$$R_\lambda f(x) := \int_0^\infty e^{-\lambda t}T(t)f(x) \, dt, \quad Re\lambda > \omega, f \in C_b(X; \mathbb{C}), x \in X. \tag{2.1}$$

Indeed, in the space $C_b(X, \mathbb{C})$, the family $\{R_\lambda : Re\lambda > \omega\}$ satisfies the resolvent identity $R_\lambda R_\mu = (R_\mu - R_\lambda)(\lambda - \mu)$ in the half-plane $\Pi := \{\lambda \in \mathbb{C} : Re\lambda > \omega\}$ since $T(t)$ is a semigroup. Moreover, such identity implies that if $R_\mu f = 0$ for some $\mu \in \Pi$, then $R_\lambda f = 0$ for every $\lambda \in \Pi$. In particular,

$^1$If $T \in \mathcal{L}(X)$ is self-adjoint and non-negative, for every $h \in T(X) T^{-1}h$ is the element of minimal norm in the set $T^{-1}(\{h\})$. We have $T^{-1}h = P_y$ for every $y \in T(\{h\})$, where $P$ is the orthogonal projection on $\overline{T(X)} = (Ker T)$.\(^1\)}
for every \( x \in X \), the Laplace transform \( G \) of the function \( g(t) := e^{-\alpha t}T(t)f(x) \) vanishes for \( \Re \alpha > 0 \); since \( g \in C_b([0, +\infty)) \), then \( g(0) = \lim_{\lambda \to -\infty} \lambda G(\lambda) = 0 \), so that \( f(x) = g(0) = 0 \). Therefore, \( R_\mu \) is one to one for every \( \mu \in \Pi \), and, by e.g. \([4, \S VIII.4]\), there exists a unique closed operator whose resolvent operator is \( R_\mu \) for every \( \mu \) with \( \Re \mu > \omega \). The part \( L \) of such operator in \( C_b(X) \) preserves \( C_b(X) \), and it is called \textit{generator} of \( T(t) \) in \( C_b(X) \) although it is not an infinitesimal generator in the classical sense.

From the definition, it follows \( T(t)L = LT(t) \) on \( D(L) \). For every \( x \in X \), the continuity of \( T(t)f(x) \) in \([0, +\infty) \) yields easily its differentiability provided \( f \in D(L) \), see e.g. \([5, \text{Prop. 4.2}] \).

We recall that a Borel probability measure \( \mu \) in \( X \) is called \textit{invariant} for \( T(t) \) if

\[
\int_X T(t)f \, d\mu = \int_X f \, d\mu, \quad t > 0, \quad f \in C_b(X).
\]

(2.2)

\((d)\) \textbf{Gaussian measures}

We list here notation and results that will be used in this article, referring to \([6]\) for their proofs and for the general theory.

A probability measure \( \gamma \) on \( (X, B(X)) \) is \textit{Gaussian} if \( \gamma \circ f^{-1} \) (defined as \( \gamma \circ f^{-1}(B) = \gamma(f^{-1}(B)) \) for every \( B \in B(\mathbb{R}) \)) is a Gaussian measure on \( \mathbb{R} \) for every \( f \in X^* \). The measure \( \gamma \) is called \textit{centred} if all the measures \( \gamma \circ f^{-1} \) have zero mean, and it is called \textit{non-degenerate} if for any \( f \neq 0 \) the measure \( \gamma \circ f^{-1} \) is absolutely continuous with respect to the Lebesgue measure.

We fix a centred Gaussian measure \( \gamma \). By the Fernique theorem, see \([6, \text{Thm. 2.8.5}] \), \( \gamma \) has finite moments of any order. For every \( g \in X^* \), the mapping \( R : X^* \to \mathbb{R} \), \( Rf := \int_X f(x)g(x) \, d\gamma(dx) \) belongs to \( X^{**} \), and even if \( X \) is not reflexive, there exists a unique \( y \in X \) such that \( Rf = f(y) \), for every \( f \in X^* \). We set \( y = Qg \). The operator \( Q \in \mathcal{L}(X^*, X) \) is called \textit{covariance operator}, it is symmetric and positive, and it is represented by the Bochner integral

\[
Qf = \int_X f(x) \, x \, d\gamma(dx), \quad f \in X^*.
\]

(Such a formula may be used as an equivalent definition of \( Q \)). If \( X \) is a Hilbert space, we identify as usual \( X \) and \( X^* \), and, therefore, \( Q \in \mathcal{L}(X) \) is defined by

\[
\langle Qx_0, y_0 \rangle = \int_X \langle x_0, x \rangle \, \langle y_0, x \rangle \, d\gamma(dx), \quad x_0, y_0 \in X.
\]

Moreover, \( Q \) belongs to \( \mathcal{L}_1(X) \). Conversely, if a linear self-adjoint non-negative operator \( Q \) is nuclear, then it is the covariance of a centred Gaussian measure called \( N_0, Q \).

Let us go back to general Banach spaces. The closure of \( X^* \) in \( L^2(X, \gamma) \) is denoted by \( X^*_\gamma \). For every \( g \in X^*_\gamma \), the mapping \( R \) defined above still has the representation \( Rg = g(y) \) for a suitable (unique) \( y \in X \), and we set \( y = R\gamma g \). So, \( R\gamma \) is the natural extension of \( Q \) to the whole \( X^*_\gamma \).

The Cameron–Martin space \( H \) consists of the elements \( h \in X \) such that the measure \( \gamma_h(B) := \gamma(B - h) \), \( B \in B(X) \), is absolutely continuous with respect to \( \gamma \). An important characterization of \( H \), which yields a Hilbert space structure in it, is the following: we have \( H = R\gamma(X^*_\gamma) \), namely \( h \in X \) belongs to \( H \) if and only if there is \( \widehat{h} \in X^*_\gamma \) such that \( \int_X \widehat{h}(x)g(x) \, d\gamma(dx) = g(h) \) for every \( g \in X^* \). In this case, \( \|h\|_H = \|\widehat{h}\|_{L^2(X, \gamma)} \). Therefore, \( R\gamma : X^*_\gamma \to H \) is an isometry, and \( H \) is a Hilbert space with the inner product \( \langle h, k \rangle := \int_X \langle \widehat{h}, \widehat{k} \rangle \, d\gamma(dx) \) whenever \( h = R\gamma \widehat{h}, k = R\gamma \widehat{k} \).

\textbf{Remark 2.1.} The triplet \((X, \gamma, H)\) is usually referred to as \textit{abstract Wiener space}. In our discussion, we have followed the presentation in ref. \([6]\). As we have seen, \( \gamma \) (or equivalently, the covariance operator \( Q \)) determines \( H \), but it is possible to go the other way around as follows. If a separable Hilbert space \( H \) is given together with a continuous inclusion mapping \( i : H \to X \), setting \( Q = i \circ i^* \), it turns out that \( Q : X^* \to X \) is a positive symmetric operator. If \( Q \) is the covariance operator of a Gaussian measure \( \gamma \), then

\[
\|i^*f\|_H^2 = \|Qf\| = \int_X (f(x))^2 \, d\gamma(dx) = \|f\|_{L^2(X, \gamma)}^2, \quad f \in X^*.
\]
Since the range of $i^* \gamma$ is dense in $H$, this shows that the mapping $i^* f \mapsto f$ has a unique extension to an isometric embedding of $H$ into $L^2(X, \gamma)$. The image of every $h \in H$ under this embedding is just $\hat{h}$, so that the range of this embedding is the space $X^\gamma_\gamma$ defined above.

For every $h \in H$, the density of $\gamma_h$ with respect to $\gamma$ is given by $e^{-\|h\|_H^2/2 + \hat{h}}$. It yields the integration by parts formula:

$$
\int_X \frac{\partial \varphi}{\partial h} \psi \gamma(\text{d}x) = - \int_X \varphi \frac{\partial \psi}{\partial h} \gamma(\text{d}x) + \int_X \varphi \hat{h} \psi \gamma(\text{d}x), \quad \varphi, \psi \in C^\infty_b(\mathbb{R})(X). \tag{2.3}
$$

Moreover, for every $h \in H$, the function $\hat{h}$ is a real Gaussian random variable with law $\mathcal{N}_{0,\|h\|_H^2}$. In particular, $\hat{h} \in L^q(X,\gamma)$ for every $q \in [1,\infty)$ and $\|\hat{h}\|_{L^q(X,\gamma)} = (\int_X \|\xi\|_q \mathcal{N}_{0,1}(\text{d}\xi))^{1/q} \|h\|_H =: c_q \|h\|_H$. Recalling that for $f \in X^*$ we have $\int_X f(x)g(x) \gamma(\text{d}x) = \gamma(Qf)$ for every $g \in X^*$, we see that $H = HQ$ (the space introduced in subsection (i)), with the same inner product. More precisely, referring to the construction of $HQ$ in [3, Chapter III] and the operators $A \in \mathcal{A}$ involved there, we can take $A : X^* \rightarrow X^*_\gamma, \Lambda f = f$, so that $A^* = R_\gamma$. If $X$ is a Hilbert space, the Cameron–Martin space is the range of $\gamma_h$ independent of $k$, and isometric embedding of $k$ the graph norm, independent of $k$, we set $\text{div} = \frac{\partial \varphi}{\partial h} \gamma(\text{d}x)$, so that $\gamma = \gamma_h$. We denote by $\mathcal{C}_b^k(X)$ the space of the cylindrical functions $f : X \rightarrow \mathbb{R}$ such that $f(x) = \varphi(f_1(x), \ldots, f_n(x))$ with $f_1, \ldots, f_n \in X^*$ and $\varphi \in C^\infty_b(\mathbb{R})$. Any such function is $k$ times Fréchet differentiable, and we have $Df(x) = \sum_{j=1}^n D_j \varphi(f_1(x), \ldots, f_n(x)) f_j$, $\nabla f \gamma(x) = Q Df(x)$. Using (2.3), one proves that for every $p \in [1,\infty)$ and $k \in \mathbb{N}$, the operator $\nabla : \mathcal{C}_b^k(X) \subset L^p(X,\gamma) \rightarrow L^p(X,\gamma;\mathbb{R})$ is closable, and the domain of its closure (still denoted by $\nabla$) is a Banach space endowed with the graph norm, independent of $k$, called $W^{1,p}(X,\gamma)$. Moreover, for $k \geq 2$, the operator $(\nabla, D_{\nabla}^2) : \mathcal{C}_b^k(X) \subset L^p(X,\gamma) \rightarrow L^p(X,\gamma;\mathbb{R}) \times L^p(X,\gamma;\mathbb{R}^2)$ is closable too, and the domain of its closure, endowed with the graph norm, is independent of $k$ and $\mathbb{R}^2$. The Gaussian divergence is defined as minus the formal adjoint of $\nabla$ and is denoted by $\nabla H$. More precisely, a vector field $F \in L^1(X,\gamma;H)$ has Gaussian divergence if there exists a (unique) $\beta \in L^1(X,\gamma)$ such that $\int_X \langle \nabla H \varphi, F \rangle \gamma(\text{d}x) = \int_X \varphi(x) \beta(x) \gamma(\text{d}x)$, for every $\varphi \in \mathcal{C}_b^k(X)$. In this case, we set $\text{div} H f := -\beta$.

3. The classical Ornstein–Uhlenbeck semigroup

Here $X$ is a separable Banach space endowed with a centred Gaussian measure $\gamma$. The proofs of the statements of this section may be found in the book [6], unless otherwise specified.

The O-U semigroup is defined through the Mehler formula by

$$
T(0)f = f, \quad T(t)f(x) := \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy), \quad t > 0, \quad f \in C_b(X). \tag{3.1}
$$

It is a contraction semigroup on $C_b(X)$, and $\gamma$ is its unique invariant measure. It is not strongly continuous, not even on $BUC(X)$. In fact, it is easily seen that for every $f \in BUC(X)$ we have $\lim_{t \rightarrow 0^+} \|T(t)f - f\|_\infty = 0$ if and only if $\lim_{t \rightarrow 0^+} \|f(e^{-t}) - f\|_\infty = 0$. However, for every $f \in C_b(X)$, the function $(t,x) \mapsto T(t)f(x)$ is continuous on $[0,\infty) \times X$ by the dominated convergence theorem, and this allows to define the generator $L$ as in §2c. Moreover, $T(t)$ is strongly continuous in the mixed topology, see refs [8,9].

Coming back to the norm topology, $T(t)$ is not analytic and even not continuous in norm on $(0,\infty)$, since $\|T(t) - T(s)\|_{L^2(X)} \geq 2$ for $t \neq s$, as a consequence of [10, Prop. 2.4]. The semigroup
$T(t)$ is smoothing along the Cameron–Martin space $H$. More precisely, for every $f \in C_b(X)$ and $t > 0$, $T(t)f$ is $H$-differentiable at every $x \in X$, and we have

$$
\langle \nabla_H T(t)f(x), h \rangle_H = \frac{e^{t}}{\sqrt{1 - e^{-2t}}} \int_X f(x + \sqrt{1 - e^{-2t}}h(y))\gamma(dy), \quad h \in H.
$$

(3.2)

Therefore, using the Hölder inequality and recalling that $\|\hat{h}\|_{L^1(X,\gamma)} \leq \|h\|_H$, $\|\hat{h}\|_{L^\gamma(X)} = c_1\|h\|_H$, for every $f \in C_b(X)$ and $x \in X$, we get

$$
(i) \|\nabla_H T(t)f(x)\|_H \leq e^{-t}(\sqrt{1 - e^{-2t}})^{-1/2}\|f\|_\infty,
(ii) \|\nabla_H T(t)f(x)\|_H \leq c_p e^{-t}(\sqrt{1 - e^{-2t}})^{-1/2}[\langle T(t)(f^p)(x)\rangle]^{1/p}, \quad p \in (1, \infty),
$$

(3.3)

and, moreover, $\nabla_H T(t)f : X \to H$ is continuous. If in addition $f \in C^1_b(X)$, then $T(t)f \in C^1_b(X)$ for any $t \geq 0$, and

$$
\frac{\partial T(t)f}{\partial h}(x) = DT(t)f(h) = e^{-t}T(t)(Df(\cdot)(h)), \quad x, h \in H,
$$

(3.4)

so that $\sup_{x \in X} \|DT(t)f(x)\|_\infty \leq e^{-t} \sup_{x \in X} \|Df(x)\|_\infty$. Iterating, we get $T(t)C^k_b(X) \subset C^k_b(X)$ for any $t \geq 0$, $k \in \mathbb{N}$, and $\sup_{x \in X} \|DT(t)f(x)\|_{L^q(X)} \leq e^{-kt} \sup_{x \in X} \|D^k f(x)\|_{L^q(X)}$.

Notice that (3.2) and (3.3) describe a smoothing property of $T(t)$, while the subsequent statements assert that $T(t)$ preserves the spaces $C^k_b(X)$ and it is contractive there. However, $T(t)$ regularizes only along $H$ and it does not map $C_b(X)$ into $C^1(X)$.

The continuity of $\nabla_H T(t)f$ for $f \in C_b(X)$ and estimate (3.3)(i) yield the embedding $D(L) \subset C^1_b(X)$ through the representation formula (2.1) for $R(\lambda, L)$. Here, $L$ is the generator of $T(t)$ defined in §2c. Moreover, for every $f \in D(L)$, $D_H f \in C^0_b(X, H)$ for every $\theta \in (0, 1)$, and it also satisfies a Zygmund condition along $H$, see ref. [11]. A Schauder type theorem was proved in ref. [11] for $H$-Hölder continuous functions, and precisely: for every $\alpha \in (0, 1)$, $\lambda > 0$ and $f \in C^\alpha_b(X)$, $R(\lambda, L)f \in C^\alpha_H(X)$ and $D^\alpha_H R(\lambda, L)f \in C^\alpha_H(X, L^2(H))$.

The semigroup $T(t)$ is readily extended to $L^p(X,\gamma)$, for every $p \in [1, \infty)$. Indeed, we have

$$
\int_X |T(t)f(x)|^p \gamma(dx) \leq \int_X |T(t)|^p \gamma(dx) = \int_X |f|^p \gamma(dx), \quad t > 0, f \in C_b(X),
$$

(3.5)

by the Hölder inequality and the invariance of $\gamma$. Hence, $(T(t) : t \geq 0)$ is uniquely extendable to a contraction semigroup $\{T_p(t) : t \geq 0\}$ on $L^p(X,\gamma)$. Moreover,

(i) $(T_p(t) : t \geq 0)$ is strongly continuous on $L^p(X,\gamma)$, for every $p \in [1, \infty);
(ii) T_2(t)$ is self-adjoint and non-negative on $L^2(X,\gamma)$ for every $t > 0;
(iii) \int_X T_p(t)f \gamma(dx) = \int_X f \gamma(dx)$, for every $f \in L^p(X,\gamma);
(iv) (hypercontractivity) for any $p, q > 1$ and $t > 0$ such that $q \leq 1 + (p - 1)e^{2t}$, $T(t)$ maps $L^p(X,\gamma)$ into $L^q(X,\gamma)$ and $\|T(t)f\|_{L^q(X,\gamma)} \leq \|f\|_{L^p(X,\gamma)}$ for every $f \in L^p(X,\gamma)$. For $q > 1 + (p - 1)e^{2t}$, $T(t)(L^p(X,\gamma))$ is not contained in $L^q(X,\gamma)$.

For $p \in (1, \infty)$, $L^p$ estimates for $\|\nabla_H T_p(t)f\|_H$ are obtained similarly to (3.5). For every $f \in C_b(X)$, (3.3)(ii) yields

$$
\int_X \|\nabla_H T_p(t)f(x)\|_H^p \gamma(dx) \leq \frac{c_p e^{-t}}{\sqrt{1 - e^{-2t}}} \int_X |T_p(t)|^p \gamma(dx) = \frac{c_p e^{-t}}{\sqrt{1 - e^{-2t}}} \int_X |f|^p \gamma(dx).
$$

This argument fails for $p = 1$, since (3.3)(ii) holds only for $p > 1$. Indeed, $T(t)$ does not map $L^1(X,\gamma)$ into $W^{1,1}(X,\gamma)$ for $t > 0$, even in the simplest case $X = \mathbb{R}$ where $\gamma$ is the standard Gaussian measure (see for instance [12, Corollary 5.1]). For $1 \leq p < \infty$, using formulae (3.4) in $C^1_b(X)$, one obtains that $T_p(t)$ preserves $W^{1,p}(X,\gamma)$ for every $t > 0$, and $\|T_p(t)f\|_{W^{1,p}(X,\gamma)} \leq \|f\|_{W^{1,p}(X,\gamma)}$ for every $f \in W^{1,p}(X,\gamma)$.Keep calm and write on!
Let us denote by $L_p$ the infinitesimal generator of $T_p(t)$ in $L^p(X, \gamma)$. It is not hard to see that every $f \in \mathcal{F}C_b^2(X)$ belongs to $D(L_p)$, and using (2.3), we get
\[
L_pf(x) = \text{div}_\gamma \nabla_H f(x) = \sum_{j=1}^{\infty} \left( \partial_{h_j}^2 f(x) - \hat{h}_j(x) \partial_{h_j} f(x) \right), \quad \gamma - \text{a.e. } x \in X,
\]
where $(h_j : j \in \mathbb{N})$ is any orthonormal basis of $H$. Moreover, $\mathcal{F}C_b^2(X)$ is a core of $L_p$ for every $p \in [1, \infty)$. In other words, $D(L_p)$ consists of all $f \in L^p(X, \gamma)$ such that there exists a sequence $(f_n)$ in $\mathcal{F}C_b^2(X)$ which converges to $f$ in $L^p(X, \gamma)$ and such that $L_p f_n = \text{div}_H \nabla_H f_n$ converges in $L^p(X, \gamma)$. The Meyer inequalities, see [13], yield
\[
D(L_p) = W^{2,p}(X, \gamma), \quad 1 < p < \infty,
\]
with equivalence of the respective norms (an independent analytic proof is in [6, Section 5.5]). Moreover, (3.6) holds for every $f \in W^{2,p}(X, \gamma)$.

For $p = 2$, $L_2$ is the operator associated with the Dirichlet form
\[
D(f, g) = \int_X \langle \nabla_H f, \nabla_H g \rangle_H \text{d}\gamma, \quad f, g \in W^{1,2}(X, \gamma),
\]
namely we have
\[
D(L_2) = \{ u \in W^1(X, \gamma) : \exists f \in L^2(X, \gamma) \text{ s.t. } D(u, g) = -\langle f, g \rangle_{L^2(X, \gamma)} \forall g \in W^{1,2}(X, \gamma) \},
\]
\[
L_2 u = f.
\]
In particular, $(L_2 u, u)_{L^2(X, \gamma)} = -\| \nabla_H u \|_{L^2(X, \gamma; H)}^2 \leq 0$ for every $u \in D(L_2)$.

Having a self-adjoint and dissipative generator, $T_2(t)$ is an analytic semigroup with angle of analyticity $\pi/2$; classical results about symmetric Markov semigroups (e.g. [14, Thm. 1.4.2]) yield that $T_p(t)$ is an analytic semigroup on $L^p(X, \gamma)$ with the angle of analyticity $\geq \pi(1 - |2/p - 1|)/2$, for every $p \in (1, +\infty)$. The optimal analyticity angle in finite dimension, $\theta_p = \pi/2 - \arctan(|\pi - 2|/2\sqrt{p - 1})$, was shown to be optimal also in infinite dimension in the previous study [15]. Functional calculus for $L_p$ in the sector $\{ z \in \mathbb{C} : \arg z < \theta_p \}$ was considered in refs [16,17].

A complete description of the spectral properties of $L_2$ is available. Even more, there is an explicit orthonormal basis of $L^2(X, \gamma)$ made by eigenfunctions of $L_2$, which are the Hermite polynomials, defined for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, by
\[
H_\alpha(x) = \prod_{j=1}^{\infty} H_{\alpha_j}(\hat{h}_j(x)), \quad x \in X,
\]
where for $k \in \mathbb{N} \cup \{0\}$, $H_k : \mathbb{R} \to \mathbb{R}$ is the polynomial
\[
H_k(\xi) = \frac{(-1)^k}{\sqrt{k!}} \exp\{\xi^2/2\} (d^k/d\xi^k) \exp\{-\xi^2/2\},
\]
for every $\xi \in \mathbb{R}$.

All the polynomials $H_\alpha$ belong to $L^p(X, \gamma)$ for every $p \in [1, \infty)$, and the set $\{H_\alpha : \alpha \in A\}$ is an orthonormal basis of $L^2(X, \gamma)$. Moreover, denoting by $X_k$ the closure of span $\{H_\alpha : \alpha \in A, |\alpha| = k\}$ in $L^2(X, \gamma)$, we have the Wiener chaos decomposition,
\[
L^2(X, \gamma) = \bigoplus_{k \in \mathbb{N} \cup \{0\}} X_k.
\]
The spectrum of $L_2$ is equal to $-\mathbb{N} \cup \{0\}$. For every $k \in \mathbb{N} \cup \{0\}$, $X_k$ is the eigenspace of $L_2$ with eigenvalue $-k$. $X_0$ is the kernel of $L_2$, consisting of constant functions, and $X_1 = X_1^*$.

4. Ornstein–Uhlenbeck semigroups on Hilbert spaces

Here, $X$ is a separable real Hilbert space, $Q \in \mathcal{L}(X)$ is a self-adjoint non-negative operator and $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$ on $X$. 
We consider the O-U operator formally defined by
\[
\mathcal{L}u(x) = \frac{1}{2} \text{Tr}[Q(D^2 u(x)) + (Ax, \nabla u(x))].
\] (4.1)

The standing assumption of this section is that the linear operators \(Q_t\) defined by
\[
Q_t x = \int_0^t e^{sA} Q e^{sA^*} x \, ds, \quad t > 0, \ x \in X,
\] (4.2)

are nuclear (\(Q\) itself does not need to be nuclear). Under such assumption, in [2, Ch. 6], it was proved that for very good initial data, namely, \(f \in BUC^2(X)\) such that \(QD^2 f \in BUC(X; \mathcal{L}_1(X))\),

the initial value problem
\[
u_t(t, x) = \mathcal{L}u(t, x), \quad t \geq 0, \ x \in D(A); \quad u(0, x) = f(x), \ x \in X,
\] (4.3)

has a unique strict solution, which is a continuous function \(u : [0, +\infty) \times X \to \mathbb{R}\) such that \(u(t, \cdot) \in BUC^2(X)\) for every \(t \geq 0\), \(QD^2 u(t, x) \in \mathcal{L}_1(X)\) for every \(t \geq 0\) and \(x \in X\), \(u(\cdot, x)\) is continuously differentiable in \([0, +\infty)\) for every \(x \in D(A)\), and satisfies (4.3). Moreover, the solution is given by
\[
u(t, x) = \int_X f(e^{tA} x + y) \mu_t(dy), \quad t \geq 0, \ x \in X,
\] (4.4)

where \(\mu_t\) is the centred Gaussian measure \(N_{0, Q_t}\) with covariance \(Q_t\) for \(t > 0\), and \(\mu_0 = \delta_0\).

(a) Ornstein–Uhlenbeck semigroups on spaces of continuous functions

The right-hand side of (4.4) is meaningful for every \(f \in B_b(X)\). Setting
\[
T(t)f(x) := \int_X f(e^{tA} x + y) \mu_t(dy), \quad t \geq 0, \ f \in B_b(X), \ x \in X,
\] (4.5)

\(T(t)\) is a contraction semigroup on \(B_b(X)\). The fact that \(T(t)\) maps \(B_b(X)\) into itself and it is a contraction is obvious. The fact that \(T(t)\) is a semigroup is less obvious. It can be proved rewriting \(T(t + s)\), for \(t, s > 0\), as
\[
T(t + s)f(x) = \int_X f(e^{(t+s)A} x + w)(\mu_t \circ (e^{sA})^{-1} \ast \mu_s)(dw), \quad f \in B_b(X), \ x \in X,
\]

and checking that \(\mu_t \circ (e^{sA})^{-1} \ast \mu_s = \mu_{t+s}\), or else recalling that \(T(t)\) is the transition semigroup of the stochastic differential equation
\[
\d X_t = A X_t \, dt + \sqrt{Q} \, dW_t, \ t > 0, \ X(0) = x,
\] (4.6)

where \(W_t\) is any cylindrical Wiener process on \(X\). Indeed, for every \(x \in X\), the unique mild solution of (4.6) is \(X_t = e^{tA} x + \int_0^t e^{(t-s)A}Q^{1/2} \, dW_s\), and the law of the stochastic convolution \(\int_0^t e^{(t-s)A}Q^{1/2} \, dW_s\) is precisely \(N_{0, Q_t}\), see [7, Ch. 5]. Therefore,
\[
T(t)f(x) = \mathbb{E}(f(X_t)), \quad t \geq 0, \ f \in B_b(X), \ x \in X.
\] (4.7)

If \(A = -I\) and \(Q\) is nuclear, setting \(\nu := N_{0, Q_t}\), \(T(t)\) coincides with the classical Ornstein–Uhlenbeck semigroup considered in §3. If \(A = 0\), \(T(t)\) may be called heat semigroup. In this case, \(Q_t = tQ\) so that setting \(\nu = \sqrt{t}z\) in the right-hand side of (4.5), we get a simpler representation formula for \(T(t)\),
\[
T(t)f(x) := \int_X f(x + \sqrt{t}z) \mu(dz), \quad t \geq 0, \ f \in B_b(X), \ x \in X,
\]

where \(\mu := N_{0, Q_t}\).

Going back to the general case, the representation formula (4.5) yields that \(T(t)\) is a Feller semigroup, namely, it maps \(C_b(X)\) into itself and, in fact, it maps the subspaces \(BUC(X), C^0_b(X),\)
\[ (\nabla T(t)f(x), h) = \int_X (e^{tA^*} \nabla f(e^{tA}x + y), h) \mu_t(dy), \quad x, h \in X. \quad (4.8) \]

\[ T(t) \text{ is strong-Feller (namely, it maps } B_b(X) \text{ into } C_b(X) \text{) iff (see [7, Remark 9.20])} \]

\[ e^{tA}(X) \subset Q_t^{1/2}(X), \quad t > 0. \]

In this case, \( T(t) \) maps \( B_b(X) \) into \( C_b^k(X) \) for every \( k \in \mathbb{N} \), and the operators

\[ \Lambda_t = Q_t^{-1/2} e^{tA}, \quad t > 0, \]

play an important role in the rest of the theory. First, \( \Lambda_t \in L(X) \) for every \( t > 0 \). Moreover, for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that

\[ \| D^k T(t)f(x) \|_{L(X)} \leq C_k \| \Lambda_t \|_{L(X)}^{k} \| f \|_{\infty}, \quad t > 0, f \in B_b(X), \ x \in X. \quad (4.11) \]

A proof for \( k = 1, 2 \) is in [2, Ch. 6]. For general \( k \), (4.11) follows e.g. from [18, Sect. 5.1, Prop. 3.3(ii)].

Condition (4.9) is called controllability condition since it is equivalent to null controllability in any time \( t \) of an associated linear evolution equation in \( X \), see e.g. [7, Appendix B] and [2, Chapter III]. It is not satisfied if \( A = -I \), and, more generally, if \( A \) generates a strongly continuous group. Instead, it is satisfied if \( Q = I \) and \( A \) generates an analytic semigroup, and, in this case, \( \| \Lambda_t \|_{L(X)} \leq M e^{\omega t} t^{-1/2} \) for some \( M > 0, \omega \in \mathbb{R} \), and for every \( t > 0 \). See refs [2, Appendix B] and [9, Thm. 3.5(3)].

Anyway, smoothing properties along \( H := Q^{1/2}(X) \) are available also in the case where \( H \) is properly contained in \( X \), provided that \( e^{tA} \) maps \( H \) into itself, and that \( S_H(t) := e^{tA} \|_H : H \to H \) is a strongly continuous semigroup on \( H \). In this case, \( e^{tA} \) maps \( H \) into \( Q_t^{1/2}(X) \) for every \( t > 0 \), and \( \sup_{0 < t < 1} \| e^{tA} \|_{L(H, Q_t^{1/2}(X))} < \infty \) by [9, Thm. 3.5]. This allows to prove that \( T(t) \) is smoothing along \( H \), by arguments similar to the ones that led to (3.3)(ii). See refs [19, Sect. 2] and [18] for representation formulae and estimates for any order \( H \)-derivatives of \( T(t)f \) when \( f \in C_b(X) \).

Let us discuss strong continuity. Even in the case \( X = \mathbb{R} \), \( T(t) \) is not strongly continuous on \( BUC(X) \) unless \( A = 0 \) (let alone on \( C_b(X) \)). However, it is not hard to show that \( \mu_t \) converges weakly to \( \delta_0 \) as \( t \to 0 \) (namely, \( \lim_{t \to 0} \int_X f(y) \mu_t(dy) = f(0) \) for every \( f \in C_b(X) \)), and this implies

\[ \lim_{t \to 0} \| T(t)f - f(e^{tA}) \|_{\infty} = 0, \quad f \in BUC(X). \]

So, the subspace \( BUC_S(X) \) of strong continuity of \( T(t) \) on \( BUC(X) \) is \( \{ f \in BUC(X) : \| T(t)f - f(e^{tA}) \|_{\infty} \to 0 \text{ as } t \to 0 \} \). If (4.11) holds, \( T(t)(C_b(X)) \subset BUC(X) \) for every \( t > 0 \), and therefore, \( BUC_S(X) \) coincides with the subspace of strong continuity of \( T(t) \) in \( C_b(X) \). In the general case, the subspace of strong continuity of \( T(t) \) in \( C_b(X) \) is not known. However, \( T(t) \) is strongly continuous on \( C_b(X) \) with respect to the mixed topology, see refs [5,8]. In particular, the function \( (t, x) \mapsto T(t)f(x) \) is continuous on \([0, +\infty) \times X \) for every \( f \in C_b(X) \), and this allows to define a generator \( L \) as in \( \S 2c \). Moreover, setting \( \Delta_t f = (T(h)f - f)/h \) for \( h > 0 \), we have

\[ D(L) = \{ f \in C_b(X) : \lim_{h \to 0} \sup_t \| \Delta_t f \|_{\infty} < +\infty, \exists g \in C_b(X) \text{ s.t.} \lim_{h \to 0} \| \Delta_t f \|_{\infty} = g(x) \text{ uniformly on compact sets}, \quad Lf = g. \quad (4.12) \]

See refs [8,9]. An analogous characterization with the space \( C_b(X) \) replaced by \( BUC(X) \) is in demonstrated in ref. [20]. Still in ref. [8], it was proved that (similarly to the case of strongly continuous semigroups on Banach spaces) any subspace \( D \subset D(L) \) that is dense in \( C_b(X) \) in the mixed topology and such that \( T(t)(D) \subset D \) is a core for \( L \), namely for every \( f \in D(L) \), there exists a net \( (f_\alpha) \subset D \) such that \( f_\alpha \to f \) and \( Lf_\alpha \to Lf \) in the mixed topology. In ref. [9, Thm. 6.6], see also
been proved that the spectrum of the part of (ii) implies that the domain of the part of
studied.

\[ Lf(x) = \frac{1}{2} \text{Tr}[QD^2 f(x)] + \langle x, A^\alpha \nabla f(x) \rangle, \quad f \in \mathcal{F}_0, \ x \in X, \]  

where the right-hand side is equal to \( Lf(x) \) for every \( x \in D(A) \). Related results with \( BU\mathcal{C}(X) \) replacing \( C_b(X) \) are presented in refs [5,20,21]. In some papers, e.g. [8], also the realization of \( T(t) \) in the weighted spaces \( C_n(X) = \{ f \in C(X, \mathbb{R}) : \| f \|_{C_n(X)} = \sup_{x \in X} |f(x)|/(1 + \| x \|^n) < \infty \} \) has been studied.

In finite dimension, \( T(t) \) is analytic iff \( A = 0 \). Instead, if \( X \) is infinite dimensional, we have
\[
\| T(t) - T(s) \|_{\mathcal{L}(BU\mathcal{C}(X))} = 2 \quad \text{and therefore} \quad \| T(t) - T(s) \|_{\mathcal{L}(C(X))} \geq 2 \text{ whenever } \mu_t \text{ and } \mu_s \text{ are singular (which is the case for every } t, s > 0 \text{ if } A = 0 \).
\]
The same equality holds if \( e^{tA} \neq e^{sA} \), see refs [10,22]. Therefore, \( T(t) \) is not norm continuous, and hence not analytic, both in the case \( A = 0 \) and in the case \( A \neq 0 \). See refs [9,10,22].

An alternative proof of norm discontinuity in the case \( A = 0 \) comes from ref. [23], where it has been proved that the spectrum of the part of \( L \) in \( BU\mathcal{C}(X; \mathbb{C}) \) is the halfplane \( \lambda \in \mathbb{C} : Re \lambda \leq 0 \), and for every \( t > 0 \), the spectrum of \( T(t) \) in \( BU\mathcal{C}(X; \mathbb{C}) \) is the whole closed unit disk.

Schauder type results in the usual Hölder spaces are available if (4.9) holds, under the further assumption
\[
\exists M, \theta > 0, \omega \in \mathbb{R} : \quad \| \Lambda_t \|_{\mathcal{L}(X)} \leq \frac{M e^{\omega t}}{t^\theta}, \quad t > 0.
\]  

Easy examples such that (4.2) and (4.15) hold (with any \( \theta \geq 1/2 \)) are given in ref. [2, Ex. 6.2.11]. The following theorem is taken from ref. [18, Sect. 5.1].

**Theorem 4.1.** Let (4.2) and (4.15) hold. For every \( f \in C_b(X) \) and \( \lambda > 0 \), let \( u = R(\lambda, L)f \). Then,

(i) If \( 1/ \theta \notin \mathbb{N} \), then \( u \in C_b^{1/\theta}(X) \), and there is \( C > 0 \), independent of \( f \), such that \( \| u \|_{C_b^{1/\theta}(X)} \leq C \| f \|_{\infty} \).

(ii) If in addition \( f \in C_b^\alpha(X) \) with \( \alpha \in (0,1) \) and \( \alpha + 1/ \theta \notin \mathbb{N} \), then \( u \in C_b^{\alpha + 1/\theta}(X) \) and there is \( C > 0 \), independent of \( f \), such that \( \| u \|_{C_b^{\alpha + 1/\theta}(X)} \leq C \| f \|_{C_b^\alpha(X)} \).

Statement (ii) was already proved in ref. [24] in the case that \( L \) is the realization of a second-order elliptic system with general boundary conditions in \( X = L^2(\Omega) \), \( \Omega \) being a bounded open set in \( \mathbb{R}^n \), and suitable assumptions on \( Q \) that yield \( \theta = 1/2 \). See also ref. [25] for an earlier result.

Statement (i) implies that \( D(L) \subset C_b^{1/\theta}(X) \) if \( 1/ \theta \notin \mathbb{N} \), with continuous embedding. Statement (ii) implies that the domain of the part of \( L \) in \( C_b^\alpha(X) \) is continuously embedded in \( C_b^{\alpha + 1/\theta}(X) \) if \( \alpha + 1/ \theta \notin \mathbb{N} \). In both cases, we gain ‘\( 1/ \theta \) degrees’ of regularity.

Both for \( \alpha = 0 \) and for \( \alpha > 0 \), in the critical cases, \( \alpha + 1/ \theta = k \in \mathbb{N} \), we cannot expect that \( u \in C_b^k(X) \); in ref. [18], it is proved that \( u \) belongs to a suitable Zygmund space, which is continuously embedded in all spaces \( C_b^{k-\varepsilon}(X) \) with \( \varepsilon \in (0,1) \). This difficulty arises even in finite dimension, for instance if \( X = \mathbb{R}^n \), \( A = 0 \), \( Q = 2I \) and \( \| f \|_{C_b^\alpha(X)} \) with \( \theta = 1/2 \), but if \( \lambda u - \Delta u = f \in C_b(\mathbb{R}^n) \) with \( n \geq 2 \), \( u \) is not necessarily a \( C^2 \) function.

Even in the case that (4.9) and (4.15) do not hold, if \( e^{tA} \) maps \( H = Q^{1/2}(X) \) into itself, and \( S_H(t) = e^{tA} : H \to H \) is a strongly continuous semigroup on \( H \), Schauder theorems similar to the ones stated in §3 were proved in ref. [18]: for every \( \alpha \in (0,1) \), \( \lambda > 0 \) and \( f \in C_H^\alpha(X) \), \( R(\lambda, L)f \in C_b^2(X) \) and \( D_H^2 R(\lambda, L)f \in C_b^\alpha(X, L^2(H)) \).

Schauder type regularity results are available also for evolution equations with bounded and continuous data, see ref. [18].
The asymptotic behaviour of \( T(t) \) is well understood if

\[
\sup_{t>0} \text{Tr} \left( Q_t \right) = \int_0^\infty \text{Tr}(e^{sA} Q e^{sA^*}) ds < +\infty. \tag{4.16}
\]

Next statements are taken from refs [2, Sect. 10.1] and [7, Sect. 11.3]. If (4.16) holds, there exists a nuclear self-adjoint operator \( Q_\infty \), given by

\[
Q_\infty x = \int_0^\infty e^{sA} Q e^{sA^*} x \, ds, \quad x \in X,
\]

which maps \( D(A^*) \) into \( D(A) \) and satisfies the identity (called Lyapunov equation)

\[
Q_\infty A^* x + A Q_\infty x = -Q x, \quad x \in D(A^*). \tag{4.18}
\]

Such identity is easily obtained recalling that \( \langle Q_\infty e^{tA^*} x, e^{tA^*} y \rangle = \langle Q_\infty x, y \rangle - \langle Q_\infty x, y \rangle \) for every \( x, \ y \in D(A^*) \), differentiating in time and taking \( t = 0 \), we get \( \langle Q_\infty A^* x, y \rangle + \langle Q_\infty A^* x, y \rangle = \langle Q x, y \rangle \) and (4.18) follows by the density of \( D(A^*) \).

Moreover, the Gaussian measure \( \mu_\infty := N_0 Q_\infty \) is invariant for \( T(t) \), namely

\[
\int_X T(t)f(x) \mu_\infty(dx) = \int_X f(x) \mu_\infty(dx), \quad t > 0, \; f \in C_b(X).
\]

In fact, it is possible to show that (4.16) holds iff there exists a probability invariant measure for \( T(t) \) iff there exists a self-adjoint non-negative nuclear operator \( P \) mapping \( D(A^*) \) into \( D(A) \) and such that \( PA^* x + AP x = -Q x \) for every \( x \in D(A^*) \) (which is equivalent to \( 2(PA^* x) + (Q x, x) = 0 \) for every \( x \in D(A^*) \)). Moreover, any invariant measure is given by \( \nu * \mu_\infty \), \( \nu \) being a probability invariant measure for the semigroup \( R(t) \) defined by \( R(t)f(x) = f(e^{tA} x) \) (e.g. [26], [7, Thm. 11.17]).

So, if \( R(t) \) has no invariant measure except \( \delta_0, \mu_\infty \) is the unique invariant measure for \( T(t) \). In particular, this happens if \( \lim_{t \to \infty} e^{tA} x = 0 \) for every \( x \).

If \( \|e^{tA}\|_{\mathcal{L}(X)} \) vanishes as \( t \to \infty \), namely if there are \( M, \omega > 0 \) such that

\[
\|e^{tA}\|_{\mathcal{L}(X)} \leq M e^{-\omega t}, \quad t > 0,
\]

it is not hard to see that (4.16) holds (e.g. [7, Thm. 11.20]), and therefore, \( \mu_\infty \) is well defined, and it is the unique invariant measure for \( T(t) \). Moreover, if (4.19) holds, then \( A \) is invertible.

Notice that if \( Q \) commutes with \( e^{tA} \) for every \( t \) and in addition \( A \) is self-adjoint then \( Q_\infty = -QA^{-1}/2 = -A^{-1}Q/2 \). The equality \( Q_\infty = -A^{-1}Q/2 \) holds even in a more general situation, see the remarks after theorem 4.2.

It is interesting to compare kernels and ranges of \( Q^{1/2}, Q_t^{1/2} \) and \( Q_\infty^{1/2} \) for \( t > 0 \), which play an important role in the theory. We set

\[
H := Q^{1/2}(X), \quad H_t := Q_t^{1/2}(X), \quad H_\infty := Q_\infty^{1/2}(X),
\]

endowing them with their natural inner products, described in §2a. Using the Lyapunov equation (4.18), one gets easily (e.g. [27, Lemma 2.1])

\[
e^{tA} H_\infty \subset H_\infty, \quad \|Q^{-1/2} e^{tA} Q_\infty^{1/2}\|_{\mathcal{L}(X)} \leq 1, \quad t > 0.
\]

Therefore, \( e^{tA}_H : H_\infty \to H_\infty \) is a contraction semigroup, called \( S_\infty(t) \). Its infinitesimal generator is the part \( A_\infty \) of \( A \) in \( H_\infty \). Since \( \langle Q_t(x, x) \rangle \leq \langle Q_\infty(x, x) \rangle \) for every \( t > 0 \) and \( x \in X \), then \( \ker Q_\infty = \ker Q^{1/2} \subset \ker Q_t^{1/2} \subset \ker Q^{1/2} = \ker Q \), and \( H_\infty \subset H_\infty \) (we recall that, given self-adjoint operators \( T_1, T_2 \in \mathcal{L}(X) \), we have \( T_1(X) \subset T_2(X) \) iff there exists \( C > 0 \) such that \( \|T_1 x\| \leq C \|T_2 x\| \) for every \( x \in X \)). Instead, the converse inclusion \( H_\infty \subset H_t \) is not necessarily satisfied, and by ref. [27, Prop. 4.1] or [28, Lemma 4], it is equivalent to

\[
\|Q^{-1/2} e^{tA} Q_\infty^{1/2}\|_{\mathcal{L}(X)} < 1, \tag{4.20}
\]

namely, to \( \|S_\infty(t)\|_{\mathcal{L}(H_\infty)} < 1 \).

In the proof of theorem 11.22 of [7], it was shown that if (4.9) holds, then \( H_\infty \subset H_t \) (so that (4.20) holds), and moreover, the operators \( Q_t^{-1/2} Q_\infty Q_t^{-1/2} - I \) are Hilbert–Schmidt on \( H_\infty \) for every
that (4.16) holds with denoted by $T_\rho$ is contained in covariance of Ornstein–Uhlenbeck semigroup has obtained in ref. [9]. In both papers, the key tool was the representation of the invariant measure $\gamma$ itself as unique invariant measure (we recall that the covariance of $\gamma$ is 2Q).

(b) Ornstein–Uhlenbeck semigroups on $L^p$ spaces with respect to invariant measures

Throughout this section, we assume that (4.16) holds, and we consider $L^p$ spaces with respect to the invariant measure $\mu_\infty$, $1 \leq p < \infty$.

For every $f \in C_0(X)$ and $t > 0$, the Hölder inequality and the invariance of $\mu_\infty$ yield

$$\int_X |T(t)f(x)|^p \mu_\infty(dx) \leq \int_X f(x)^p \mu_\infty(dx),$$

and therefore, since $C_0(X)$ is dense in $L^p(X, \mu_\infty)$, $T(t)$ has a bounded extension to $L^p(X, \mu_\infty)$, denoted by $T_p(t)$. The aforementioned inequality implies that $T_p(t)$ is a contraction semigroup on $L^p(X, \mu_\infty)$. By the dominated convergence theorem, $\lim_{t \to 0} |T(t)f - f|_{L^p(X, \mu_\infty)} = 0$ for every $f \in C_0(X)$, and this yields $\lim_{t \to 0} \|T_p(t)f - f\|_{L^p(X, \mu_\infty)} = 0$ for every $f \in L^p(X, \mu_\infty)$.

The generator of $T_p(t)$ is denoted by $L_p$. Since $T_p(t)f = T_q(t)f$ for $p \leq q$ and $f \in L^q(X, \mu_\infty)$, then $L_q$ is the part of $L_p$ in $L^q(X, \mu_\infty)$, and the subindex $p$ will be written only if needed.

Notice that, for every $f \in D(L_p)$, letting $t \to 0$ in the equality $\int_X (T(t)f - f)/t \, d\mu_\infty = 0$, we obtain $\int_X L_p f \, d\mu_\infty = 0$.

Concerning asymptotic behaviour, for every $f \in L^p(X, \mu_\infty)$, we have

$$\lim_{t \to \infty} \left\| T_p(t)f - \int_X f(y) \mu_\infty(dy) \right\|_{L^p(X, \mu_\infty)} = 0. \tag{4.22}$$

If $f \in C_0(X)$, (4.22) is a consequence of (4.21) through the dominated convergence theorem; if $f \in L^p(X, \mu_\infty)$, it follows approximating $f$ by a sequence of continuous and bounded functions.

Using the dominated convergence theorem, it is easy to see that the space $F_0$ defined in (4.13) is contained in $D(L_p)$ for every $p \in [1, \infty)$, and it is a core for $L_p$ since it is invariant under $T(t)$ and dense in $L^p(X, \mu_\infty)$. Another convenient core, used in ref. [2], is the subspace of $F_0$ defined by

$$E_A(X) := \text{span} \{ \cos(\langle \cdot, h \rangle), \sin(\langle \cdot, k \rangle); h, k \in D(A^*)\}.$$

Necessary and sufficient conditions for $T_2(t)$ to be self-adjoint for every $t > 0$ (or, equivalently, for $L_2$ be self-adjoint) were given in ref. [29] under the assumption that $Q_0$ is one to one, that was later removed in ref. [9]. In both papers, the key tool was the representation of $T_2(t)$ as the second quantization operator of the operator $S_\infty(t)^*$, that goes back to ref. [28].

**Theorem 4.2.** The following conditions are equivalent.

(i) $T_2(t) = T_2(t)^*$ for every $t > 0$;
(ii) $Q(D(A^*)) \subset D(A)$, and $AQx = QA^*x$ for every $x \in D(A^*)$;
(iii) $e^{tA}Q = Qe^{tA^*}$, for every $t > 0$;
(iv) $e^{tA}Q_\infty = Q_\infty e^{tA^*}$, for every $t > 0$;
(v) $e^{tA}(H) \subset H$, and $S_\infty(t) := e^{tA}_{|H}: H \to H$ is a self-adjoint strongly continuous semigroup on $H$.

We refer to the conditions of theorem 4.2 as ‘the symmetric case’. In such a case, by the general theory of semigroups, the infinitesimal generator $L_2$ of $T_2(t)$ is self-adjoint too. Moreover $T_2(t)$

$t > 0$, and therefore, $\mu_t$ and $\mu_\infty$ are equivalent measures, for every $t > 0$, by the Feldman–Hájek Theorem (see e.g. [7, Thm. 2.25]). If (4.16) holds, we have (see [7, Thm. 11.20])

$$\lim_{t \to \infty} T(t)f(x) = \int_X f(y) \mu_\infty(dy), \quad f \in C_0(X), \ x \in X. \quad \tag{4.21}$$

We notice that if $A = 0$, then $\text{Tr} \ Q_t = t \text{Tr} \ Q$, so that (4.16) does not hold, and the heat semigroup has no invariant measure. Instead, if $A = -\omega I$ with $\omega > 0$, then $\text{Tr} \ Q_t = (1 - e^{-2\omega t}) \text{Tr} \ Q/(2\omega)$, so that (4.16) holds with $Q_\infty = Q/(2\omega)$. In particular, as we already mentioned in §3, the classical Ornstein–Uhlenbeck semigroup has $\gamma$ itself as unique invariant measure (we recall that the covariance of $\gamma$ is 2Q).
is a symmetric Markov semigroup on $L^2(X, \mu_\infty)$, according to the terminology of ref. [14], and therefore, $T_p(t)$ is an analytic semigroup on $L^p(X, \mu_\infty)$ for every $p \in (1, \infty)$ with angle of analyticity $\geq \pi(1 - [2/p - 1])/2$, by [14, Thm. 1.4.2]. In addition, $(iv)$ yields that $Q_\infty$ maps $DA^*$ into $D(A)$, and on $D(A^*)$ we have $AQ_\infty = Q_\infty A^* = -Q/2$ by the Lyapunov equation. In particular, if $0$ belongs to the resolvent set $\rho(A)$, we get an explicit formula for $Q_\infty = -\frac{1}{2}A^{-1}Q = -\frac{1}{2}Q(A^*)^{-1}$.

About condition $(v)$, we remark that $S_H(t)$ is self-adjoint and strongly continuous on $H$ iff $Q^{-1/2}e^{tA}Q^{1/2}$ is self-adjoint and strongly continuous on $X$. Moreover, in the symmetric case, not only $S_H(t)$ is strongly continuous but also there are $M_1$, $\beta > 0$ such that

$$\|S_H(t)\|_{L^2(H)} \leq M_1 e^{-\beta t}, \quad t > 0.$$ (4.23)

See [9, Thm. 4.5]. Such estimate plays an important role in the asymptotic behaviour of $T_p(t)$.

In the non-symmetric case, $T_p(t)$ is not in general analytic, even in finite dimension: see the counterexamples in ref. [30]. Necessary and sufficient conditions for analyticity were studied in the previous studies [9,15,22,30–32]. In particular, ref. [9] contains extensions and improvements of the previous ones, which are summarized in the next theorem.

**Theorem 4.3.** The following conditions are equivalent:

(i) $T_2(t)$ is an analytic semigroup on $L^2(X, \mu_\infty)$;

(ii) there exists $M > 0$ such that $|(Q_\infty A^*x, y)| \leq M|Qx, x|^{1/2}(Qy, y)^{1/2}$, for every $x, y \in D(A^*)$;

(iii) $S_\infty(t)$ is an analytic contraction semigroup$^2$ in $H_\infty$.

If in addition $Q$ has a bounded inverse, the aforementioned conditions are also equivalent to

(iv) The operator $AQ_\infty$ has an extension belonging to $L^2(X)$;

(v) the operator $Q_\infty A^*$ has an extension belonging to $L^2(X)$.

We refer to the conditions of theorem 4.3 as ‘the analytic case’. As in the symmetric case, if $T_2(t)$ is analytic on $L^2(X, \mu_\infty)$, then $T_p(t)$ is analytic on $L^p(X, \mu_\infty)$ for every $p \in (1, \infty)$, by a simple application of the Stein interpolation theorem (e.g. [33, Sect. 6.2]). Moreover, $T_p(t)$ is an analytic contraction semigroup and the optimal angle of analyticity $\theta_p$ has been determined in ref. [15]; in ref. [17] it has been proved that such angle coincides with the optimal angle for the bounded $F^\infty$ calculus of $-L_p$. In addition, in the analytic case, the semigroup $e^{tA}$ maps $H$ into itself, and the semigroup $S_H(t)$ is a strongly continuous, bounded analytic semigroup on $H$, see ref. [19, Thm. 3.3]. For $p = 1$, $T_1(t)$ is not analytic, even in finite dimension. Characterizations of the domains $D(L_p)$ as suitable Sobolev spaces are known only in the analytic case.

The definition of the proper Sobolev spaces relies on the closability of the operator $\nabla_H: F_0 \subset L^p(X, \mu_\infty) \rightarrow L^p(X, \mu_\infty; H)$, with $p \in [1, \infty)$. If $f \in F_0$, $f(x) = \varphi((x, x_1), \ldots, (x, x_n))$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $x_k \in D(A^*)$, we have $\nabla_H f(x) = \sum_{k=1}^n D_k \varphi((x, x_1), \ldots, (x, x_n)) Qx_k$. Recalling (4.14) and (2.3) and using the Lyapunov equation, it is easy to see that for $f, g \in F_0$, we have

$$\int_X (Lf + f Lg) \mu_\infty(dx) = -\int_X \langle Q^\infty f - \nabla g, \nabla g \rangle \mu_\infty(dx) = -\int_X \langle \nabla_H f, \nabla_H g \rangle_{H} \mu_\infty(dx).$$ (4.24)

According to ref. [34, Sect. 6], a sufficient condition for $\nabla_H$ be closable is that $Q$ is one to one and the operator $W: H_\infty \rightarrow X, W(x) = Q^{1/2}Q^{1/2}x$, is closable in $X$. Another sufficient condition, see ref. [34, Cor. 5.6], is that $e^{tA}$ maps $H$ into itself and $S_H(t)$ is strongly continuous on $H$. So, in the analytic case (and, in particular, in the symmetric case), $\nabla_H$ is closable in $L^p(X, \mu_\infty)$ for every $p \in [1, \infty)$. See also refs [9, Prop. 8.3] and [34] for counterexamples to the closability of the gradient.

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$^2$An analytic semigroup $T(t)$ on a real Banach space $X$ is called ‘analytic contraction semigroup’ if there exists a sector $\Sigma := \{z \neq 0 : |\arg z| < \theta\}$ with $\theta > 0$ such that the analytic extension $T(z)$ satisfies $\|T(z)\|_{L(X^c)} \leq 1$ for every $z \in \Sigma$. $X^c$ is the complexification of $X$. 
Whenever $\nabla H$ is closable in $L^p(X, \mu_\infty)$, the Sobolev space $W^{1,p}_H(X, \mu_\infty)$ is defined as the domain of its closure (still called $\nabla H$), and it is a Banach space endowed with the graph norm

$$\|f\|_{W^{1,p}_H(X, \mu_\infty)} = \|f\|_{L^p(X, \mu_\infty)} + \int_X |\nabla_H f(x)|^p \mu_\infty(dx).$$

In particular, for $p = 2$, it is a Hilbert space with inner product $\langle f, g \rangle_{W^{1,2}_H(X, \mu_\infty)} = \langle f, g \rangle_{L^2(X, \mu_\infty)} + \langle \nabla_H f, \nabla_H g \rangle_{L^2(X, \mu_\infty; H)}$. In its turn, the operator $D^2_H : \mathcal{F}_0 \subset L^p(X, \mu_\infty) \to L^p(X, \mu_\infty; L^2(H))$ is closable, and $W^{2,p}_H(X, \mu_\infty)$ is defined as the domain of the closure (still called $D^2_H$), endowed with the graph norm

$$\|f\|_{W^{2,p}_H(X, \mu_\infty)} = \|f\|_{W^{1,p}_H(X, \mu_\infty)} + \int_X \|D^2_H f(x)\|_{L^2(H)} \mu_\infty(dx).$$

Another involved Sobolev-type space is the domain of the closure of $\nabla^*_\infty \nabla H : \mathcal{F}_0 \subset L^p(X, \mu_\infty) \to L^p(X, \mu_\infty; H_\infty)$ in $L^p(X, \mu_\infty)$, called $W^{1,p}_{AQ}(X, \mu_\infty)$ (we recall that $A_\infty$ is the part of $A$ in $H_\infty$).

Using the notation in (4.13), for $f \in \mathcal{F}_0$, we have $\|\nabla_H f(x)\|_H = \|Q^{1/2} \nabla f(x)\|_H$, $\|D^2_H f(x)\|_{L^2(H)} = \text{Tr} (QD^2 f(x))^2$, and $\|A^*_\infty \nabla H_\infty f(x)\|_{H_\infty} = (A^* \nabla \psi(x), Q_\infty A^* \nabla \psi(x))$. In the symmetric case, using the Lyapunov equation, we get $\|A^*_\infty \nabla H_\infty f(x)\|_{H_\infty}^2 = (\nabla \psi(x), -AQ \nabla \psi(x))/2$. In the case of the classical Ornstein–Uhlenbeck operator, we have $A = -I$, $Q_\infty = 2Q$, and the spaces $W^{1,p}_H(X, \mu_\infty) = W^{1,p}_{AQ}(X, \mu_\infty)$, $W^{2,p}_H(X, \mu_\infty)$ considered here coincide respectively with the spaces $W^{1,p}(X, \gamma)$, $W^{2,p}(X, \gamma)$ described in §2d, with $\gamma = N_0, 2Q$.

Before going on, we observe that the quadratic form

$$Q(\psi, \psi) := \frac{1}{2} \int_X (\nabla H_\infty \psi(x), \nabla H_\infty \psi(x))_H \mu_\infty(dx), \quad \psi \in W^{1,2}_H(X, \mu_\infty),$$

is closed, and in the symmetric case, $-L_2$ is the operator associated with the form $Q$ in $L^2(X, \mu_\infty)$, namely

$$D(L_2) = \{f \in W^{1,2}_H(X, \mu_\infty); \exists g \in L^2(X, \mu_\infty) \text{ s.t. } Q(f, \psi) = \langle f, g \rangle_{L^2(X, \mu_\infty)} \}, \quad L_2 f = -g,$$

and therefore, $D(-L_2)^{1/2} = W^{1,2}_H(X, \mu_\infty)$. Even in the non-symmetric case, recalling that $\mathcal{F}_0$ is a core for $L_2$, formula (4.24) yields $D(L_2) \subset W^{1,2}_H(X, \mu_\infty)$ and (4.24) holds for any $f, g \in D(L_2)$. In particular, taking $f = g$, we get

$$\int_X L_2 f(x) \mu_\infty(dx) = -\frac{1}{2} \int_X \|\nabla H_\infty \|^2_H \mu_\infty(dx), \quad f \in D(L). \quad (4.25)$$

In the analytic case (see condition (ii) of theorem 4.3), there is a sort of bounded extension of $Q_\infty A^*$ to $H$; more precisely, see ref. [15], there exists an operator $B \in \mathcal{L}(H)$ such that $BQ_\infty x = Q_\infty A^* x$ for $x \in D(A^*)$ and that satisfies $B + B^* = -I$ in $H$ by the Lyapunov equation. Moreover, $L_2 f = \nabla^*_H B \nabla H_\infty f$, for every $f$ in the core $\mathcal{F}_0$. In the symmetric case, we have $B = -I/2$, and this statement coincides with (3.6) for the classical Ornstein–Uhlenbeck operator.

The next theorem follows from [19,29,35,36] and generalizes an earlier result of [37].

**Theorem 4.4.** In the symmetric case for every $p \in (1, \infty)$, we have $D(L_p) = W^{2,p}_H(X, \mu_\infty) \cap W^{1,p}_{AQ}(X, \mu_\infty)$, $D((-L_p)^{1/2}) = W^{1,p}_H(X, \mu_\infty)$, with equivalence of the respective norms.

The next theorem follows from refs [19,38]. We recall that in the analytic case $e^{tA}$ maps $H$ into itself, and $S_H(t) = e^{tA}_{[H]} : H \to H$ is a strongly continuous semigroup. We denote by $A_H$ its infinitesimal generator.

**Theorem 4.5.** Let $1 < p < \infty$. In the analytic case, the following conditions are equivalent.

(i) $D((-L_p)^{1/2}) = W^{1,p}_H(X, \mu_\infty)$, with equivalence of the respective norms;

(ii) the operator $-A_H$ admits bounded $H^\infty$ functional calculus in $H$. 
If such equivalent conditions are satisfied, we have $D(L_p) = W^{2,p}_H(X, \mu_\infty) \cap W^{1,p}_A(X, \mu_\infty)$, with equivalence of the respective norms.

Theorem 4.5 is a generalization of 4.4, since in the symmetric case (i) and (ii) are satisfied.

In ref. [36], sufficient conditions were given in order that $D(L_p) \subset W^{2,p}_H(X, \mu_\infty)$ for $p \in (1, 2]$, even in the non-analytic case.

Concerning summability improving, the following hypercontractivity result holds.

**Theorem 4.6.** Fix $t > 0$ and let $1 \leq p < q$ be such that

$$q - 1 \leq (p - 1)\|Q^{-1/2} e^{tA} Q^{1/2} \|_{\mathcal{L}(X)}.$$  \hspace{1cm} (4.26)

Then $T_p(t)(L^p(X, \mu)) \subset L^q(X, \mu)$, and $\|T_p(t)\|_{L^p(X, \mu)} \leq \|f\|_{L^p(X, \mu)}$ for every $f \in L^p(X, \mu)$.

The proof is presented in ref. [27] and (in the case that $Q^{-1/2}$ is one to one) in ref. [28]. Of course, the statement is meaningful only if (4.20) is satisfied. As we mentioned earlier, if (4.9) holds, then (4.20) holds for every $T > 0$. Another simple example is the case that $Q$ commutes with $e^{tA}$ and $e^{tA}$ holds; then, $Q^{-1/2} e^{tA} Q^{1/2} = e^{tA}$ and (4.20) is satisfied for large $t$ if $M > 1$, for every $t > 0$ if $M = 1$, independently of the validity of (4.9). In particular, if $A = -\omega I$ with $\omega > 0$, (4.9) is not satisfied but (4.20) holds for every $t > 0$.

For the classical Ornstein–Uhlenbeck semigroup of §3, condition (4.26) coincides with the hypercontractivity property stated there.

It is well known, see refs [39,40], that under appropriate assumptions the hypercontractivity of a semigroup is equivalent to the occurrence of a suitable logarithmic Sobolev inequality. But, for general Ornstein–Uhlenbeck semigroups the assumptions of [40] are not necessarily satisfied, as shown in ref. [27]. In the symmetric case, namely, under the conditions of theorem 4.2, they are satisfied, and by ref. [29, Thm. 4.2] for every $\beta > 0$, the following conditions are equivalent.

(i) $\|Q^{-1/2} e^{tA} Q^{1/2} \|_{\mathcal{L}(X)} \leq e^{-\beta t}$, for every $t > 0$;
(ii) $\|Q^{-1/2} e^{tA} Q^{1/2} \|_{\mathcal{L}(X)} \leq e^{-\beta t}$, for every $t > 0$;
(iii) for every $f \in D(L_2)$ we have

$$\int_X |f(x)|^2 \log(\|f(x)\|_{\mu_\infty}(dx)) \leq \frac{2}{\beta} (-L_2 f, f)_{L^2(X, \mu_\infty)} + \|f\|^2_{L^2(X, \mu_\infty)} \log(\|f\|_{L^2(X, \mu_\infty)}),$$

(iv) $T(t)$ is a contraction from $L^p(X, \mu_\infty)$ to $L^q(X, \mu_\infty)$ for every $t > 0, 1 \leq p \leq q$ such that $q - 1 \leq (p - 1) e^{2\beta t}$.

In ref. [27], it was remarked that if (4.20) holds for some $t > 0$, then there exist $K, \nu > 0$ such that

$$\left\| T_2(t) f - \int_X f(x) \mu_\infty(dx) \right\|_{L^2(X, \mu_\infty)} \leq K e^{-\nu t} \|f\|_{L^2(X, \mu_\infty)}, \hspace{1cm} t > 0, f \in L^2(X, \mu_\infty).$$

Notice that the operator $\Pi : L^2(X, \mu_\infty) \rightarrow L^2(X, \mu_\infty), (\Pi f)(x) = \int_X f(x) \mu_\infty(dx)$ for a.e. $x \in X$, is just the orthogonal projection on the subspace of constant functions.

In general, exponential convergence of $T_2(t)$ to $\Pi f$ is related to the behaviour of the semigroup $S_H(t)$. Indeed, if $e^{tA}$ maps $H$ into itself, for every $f \in C^1_b(X), t > 0$ and $h \in H$ formula (4.8) yields

$$\frac{\partial T(t)}{\partial h} (x) = \int_X \langle \nabla f(e^{tA}x + y), e^{tA}h \rangle_X \mu_\nu(dx) = \int_X \langle \nabla Hf(e^{tA}x + y), e^{tA}h \rangle_X \mu_\nu(dx),$$

and therefore, if $\|S_H(t)\|_{\mathcal{L}(H)} \leq M_1 e^{-\beta t}$ for some $M_1, \beta > 0$, we argue as in §3 and we obtain

$$\|\nabla H T(t)(f(x), h)\|_H = \left| \frac{\partial T(t)}{\partial h} (x) \right| \leq M_1 e^{-\beta t} \|h\|_H \int_X \|\nabla Hf(e^{tA}x + y)\|_H \mu_\nu(dx) \leq M_1 e^{-\beta t} \|h\|_H \left( \int_X \|\nabla Hf(e^{tA}x + y)\|_H^2 \mu_\nu(dy) \right)^{1/2} = M_1 e^{-\beta t} \|h\|_H(T(t)(\|\nabla Hf\|^2_H(x)))^{1/2}$$
namely,
\[ \| \nabla_H T(t)f(x) \|_H \leq M_1 e^{-\beta t} \left( T(t)(\| \nabla_H f \|^2(x)) \right)^{1/2}, \quad t > 0, \ x \in X. \] (4.27)

Squaring and integrating with respect to \( \mu_\infty \), we get, for every \( t > 0 \),
\[ \int_X \| \nabla_H T(t) \|^2_2 \ d\mu_\infty \leq M_1^2 e^{-2\beta t} \int_X T(t)(\| \nabla_H f \|^2) \ d\mu_\infty = M_1^2 e^{-2\beta t} \int_X \| \nabla_H f \|^2_2 \ d\mu_\infty. \]

In the analytic case, this estimate and (4.25) allow to obtain a Poincaré inequality,
\[ \int_X |f - \Pi f|^2 \ d\mu_\infty \leq \frac{M_1^2}{2\beta} \int_X \| \nabla_H f \|^2_2 \ d\mu_\infty, \quad f \in W^1_2(X, \mu_\infty) \]
(4.28)

by a classical method that seems to go back to ref. [41] (the proof given in [2, Prop. 10.5.2] for a particular case works as well in general, using as main ingredients (4.25) and (4.27)).

By the invariance of \( \mu_\infty \), \( T_2(t) \) maps \( L^2_0(X, \mu_\infty) := (I - \Pi)(L^2(X, \mu_\infty)) \) into itself. Moreover, (4.28) and (4.25) yield \( (L^2_0 f, f)_{L^2(X, \mu_\infty)} \leq -\| \beta/M_1^2 \|^2 \| f \|^2_{L^2(X, \mu_\infty)} \) for every \( f \in D(L_2) \cap L^2_0(X, \mu_\infty) \). By the general theory of semigroups (e.g. [4, Section IX.8]), \( \| T_2(t) \|_{L^2_0(X, \mu_\infty)} \leq e^{-\beta t/M_1^2} \) for \( t > 0 \), and therefore,
\[ \| T_2(t) f - \Pi f \|^2_{L^2_0(X, \mu_\infty)} \leq e^{-\beta t/M_1^2} \| f \|^2_{L^2_0(X, \mu_\infty)}, \quad t > 0, \ f \in L^2_0(X, \mu_\infty). \] (4.29)

If in addition (4.20) holds for some \( t > 0 \), the rate of convergence of \( T_{p}(t)f \) to \( \Pi f \) is the same in all spaces \( L^p(X, \mu_\infty) \), \( 1 \leq p < \infty \). Indeed, if \( p > 2 \), we fix \( \tau > 0 \) such that \( T(\tau) \) is a contraction from \( L^2(X, \mu_\infty) \) to \( L^p(X, \mu_\infty) \) (such a \( \tau \) exists, since \( Q_{1/2} \ e^{tA}Q_{1/2} \) is a semigroup, and therefore if (4.20) holds for some \( t > 0 \), then \( \lim_{\tau \to \infty} \| Q_{1/2} e^{tA}Q_{1/2} \|_\infty = 0 \)). For every \( t \geq \tau \) and \( f \in L^p(X, \mu_\infty) \), we have
\[ \| T(t)f - \Pi f \|_{L^p(X, \mu_\infty)} = \| T(\tau)(T(t - \tau)f - \Pi f) \|_{L^p(X, \mu_\infty)} \leq \| T(t - \tau)f - \Pi f \|_{L^2(X, \mu_\infty)} \]
by theorem 4.6, and using (4.29), we get
\[ \| T(t)f - \Pi f \|_{L^p(X, \mu_\infty)} \leq e^{-\beta (t - \tau)/M_1^2} \| f \|_{L^2(X, \mu_\infty)} \leq e^{\beta t/M_1^2} e^{-\beta \tau/M_1^2} \| f \|_{L^p(X, \mu_\infty)}, \quad t \geq \tau. \]

Similarly, if \( p < 2 \), we fix \( \tau > 0 \) such that \( T(\tau) \) is a contraction from \( L^p(X, \mu_\infty) \) to \( L^2(X, \mu_\infty) \). For every \( t \geq \tau \) and \( f \in L^p(X, \mu_\infty) \), we have
\[ \| T(t)f - \Pi f \|_{L^p(X, \mu_\infty)} \leq \| T(t)f - T(\tau)f - \Pi (T(\tau)f) \|_{L^2(X, \mu_\infty)} = \| T(t - \tau)(T(\tau)f - \Pi (T(\tau)f)) \|_{L^2(X, \mu_\infty)} \]
so that using (4.29) and then theorem 4.6, we get
\[ \| T(t)f - \Pi f \|_{L^p(X, \mu_\infty)} \leq e^{-\beta (t - \tau)/M_1^2} \| f \|_{L^2(X, \mu_\infty)} \leq e^{\beta t/M_1^2} e^{-\beta \tau/M_1^2} \| f \|_{L^p(X, \mu_\infty)}, \quad t \geq \tau. \]

5. Ornstein–Uhlenbeck semigroups on Banach spaces

Many of the results of §3 have been extended to the case where \( X \) is a separable Banach space. In fact, refs [9,10,15,17,19,20,32,34,36,38] deal with the Banach space case. A survey of the state of the art up to 2003 is referred to in ref. [9].

As in §4, \( Q \in \mathcal{L}(X^*, X) \) is a symmetric positive operator, and \( A : D(A) \subset X \to X \) is the infinitesimal generator of a strongly continuous semigroup \( e^{tA} \) on \( X \). As in the Hilbert case, the basic assumption of this section is that for every \( t > 0 \) the operator \( Q_t \), defined by (4.2) is the covariance of a Gaussian measure \( \mu_t \), and in this case, the Ornstein–Uhlenbeck semigroup \( T(t) \) is defined by (4.5).

If \( Q \) itself is a covariance and \( A = -I \), \( T(t) \) is the classical Ornstein–Uhlenbeck semigroup of §3, provided \( \gamma \) is the centred Gaussian measure on \( X \) with covariance \( 2Q \).

As in the Hilbert case, it is the transition semigroup of a stochastic differential equation in \( X \), with a proper notion of mild solution, see Refs. [42, 43], and it is a contraction semigroup on \( B_b(X) \) that leaves invariant the spaces \( C_b(X), BUC(X), C^\alpha_b(X), C^\alpha_k(X), C^{\alpha+k}_b(X) \) for \( \alpha \in (0, 1), k \in \mathbb{N} \).
The strong-Feller property of $T(t)$ is not easily recognizable as in the Hilbert case. Characterizations and sufficient conditions for $T(t)$ to be strong-Feller are in presented in ref. [9, Sect. 6.1].

Concerning the behaviour of $T(t)$ on $C_b(X)$, it is strongly continuous in the mixed topology, and the space $F_0$ defined now by

\[ F_0 := \{f \in C_b(X) : f = \varphi (\cdot , a_1), \ldots , (\cdot , a_n) ; \varphi \in C^2_b(\mathbb{R}^n), \quad n \in \mathbb{N}, a_i \in D(A^*), A^*f (\cdot ) \in C_b(X) \} \tag{5.1} \]

is a core of the generator $L$ of $T(t)$ in the mixed topology, by ref. [9, Thm.6.6]. The domain of $L$ is still given by (4.12), see ref. [9, Section 6.1].

The spaces $H := H_Q$ and $H_t := H_{Q_t}$ introduced in §2a play the role of the spaces $Q^{1/2}(X)$, $Q_t^{1/2}(X)$ in the Hilbert case. We recall that $H_t$ is the Cameron–Martin space of the measure $\mu_t$.

As mentioned in §4 in the Hilbert space case, an important hypothesis to get smoothing properties of $T(t)$ along $H$ is that $e^{tA}$ maps $H$ into itself, and $S_{H(t)} := e^{tA}_H : H \to H$ is a strongly continuous semigroup on $H$. Indeed, in this case $e^{tA}$ maps $H$ into $H_t$ for every $t > 0$, and $\sup_{0 < t < \xi} \| e^{tA} \|_{\mathcal{L}(H,H_t)} < \infty$, by ref. [9, Thm. 3.5]. As a consequence, $T(t)$ is smoothing along $H$. See refs. [19, Sect. 2] and [18] for representation formulae and estimates for any order $H$-derivatives of $T(t)$ when $f \in C_b(X)$. Again, as in the Hilbert case, Schauder type theorems were proved in ref. [18], which generalize the one stated in §3, and precisely for every $\alpha \in (0,1)$, $\lambda > 0$ and $f \in C^{1,\lambda}_H(X,R(\lambda)L)f \in C^{1,\lambda}_H(X)$ and $D^2_H R(\lambda,L)f \in C^{1,\lambda}_H(X,L^2(H))$. Notice that $H$ is invariant under $e^{tA}$ in the analytic case, see ref. [19, Th. 3.3].

Concerning asymptotic behaviour and existence of invariant measures, assumption (4.16) is generalized as follows.

\[
\begin{align*}
(i) & \quad \forall f \in X^* \exists \text{ weak} - \lim_{t \to \infty} Q_tf := Q_{\infty}f, \\
(ii) & \quad Q_{\infty} \text{ is the covariance of a centred Gaussian measure } \mu_{\infty}.
\end{align*}
\tag{5.2}
\]

Condition (i) is satisfied if (4.19) holds, in which case a representation formula similar to (4.17) holds, namely $Q_{\infty}f = \int_0^\infty e^{\lambda t}Q e^{\lambda t}f \, ds$ for every $f \in X^*$, where now the integral converges as a Pettis integral, see ref. [9, Sect. 2]. As in the Hilbert case, if (i) holds the operator $Q_{\infty}$ maps $D(A^*)$ into $D(A)$ and satisfies the Lyapunov equation (4.18); moreover (i) holds iff there exists a symmetric and positive operator $P \in \mathcal{L}(X^*,X)$ mapping $D(A^*)$ into $D(A)$ such that $PA^*f + APf = -Qf$ for every $f \in D(A^*)$, see ref. [9, Sect. 4].

However, establishing whether a given symmetric positive operator is the covariance of a Gaussian measure is not as simple as in the Hilbert case. Necessary and sufficient conditions are presented in ref. [43]. If (5.2) holds, denoting by $H_{\infty} := H_{Q_{\infty}}$ the Cameron–Martin space of $\mu_{\infty}$ (as in the Hilbert case), several statements of the previous section are extendable to the Banach setting. In particular,

(a) $e^{tA}$ maps $H_{\infty}$ into itself, and $e^{tA}_H : H_{\infty} \to H_{\infty}$ is a strongly continuous contraction semigroup, still denoted by $S_{\infty}(t)$. Moreover, for any $t > 0$, we have $H_{\infty} = H_t$ iff $\| S_{\infty}(t) \|_{\mathcal{L}(H,H_{\infty})} < 1$.

(b) $\mu_{\infty}$ is an invariant measure of $T(t)$, and the arguments used in §§3 and 4 yield that $T(t)$ extends to a contraction $C_0$-semigroup $T_p(t)$ on $L^p(X,\mu_{\infty})$, for every $p \in [1, +\infty)$.

(c) Conditions (i) and (iii) of theorem 4.2 are still equivalent, see [9, Thm. 7.4]; if they hold $T_p(t) \mu_{\infty}$ is an analytic contraction semigroup on $L^p(X,\mu_{\infty})$ for every $p \in (1, +\infty)$.

(d) Conditions (i), (ii), and (iii) of theorem 4.3 are still equivalent, see Ref. [9, Sect. 8]; if they hold $T_p(t)$ is an analytic contraction semigroup on $L^p(X,\mu_{\infty})$ for every $p \in (1, +\infty)$. The optimal angle of analyticity and the optimal angle for the bounded $H^\infty$ calculus of $-L_p$ were determined in [15,17], respectively, in the present Banach setting.

(e) Theorems 4.2 and 4.3 still hold, where the involved Sobolev spaces $W^{1,p}_H(X,\mu_{\infty})$, $W^{2,p}_H(X,\mu_{\infty})$, $W^{1,p}_Q(X,\mu_{\infty})$ are defined in a similar way to the Hilbert case. See Refs. [19,36,38].
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