SOME INEQUALITIES FOR THE MATRIX HERON MEAN

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Abstract. Let $A, B$ be positive definite matrices, $p = 1, 2$ and $r \geq 0$. It is shown that

$$||A + B + r(A^t_1 B + A_{1-t}^t B)||_p \leq ||A + B + r(A^{1/2} B^{1/2} + A^{1/2} B^{1/2})||_p.$$  

We also prove that for positive definite matrices $A$ and $B$

$$\det(P_t(A, B)) \leq \det(Q_t(A, B))$$

where $Q_t(A, B) = (A^{1/2} B^{1/2})^{1/t}$ and $P_t(A, B)$ is the $t$-power mean of $A$ and $B$. As a consequence, we obtain the determinant inequality for the matrix Heron mean: for any positive definite matrices $A$ and $B$,

$$\det(A + B + 2(A\sharp B)) \leq \det(A + B + A^{1/2} B^{1/2} + A^{1/2} B^{1/2}).$$

These results complement those obtained by Bhatia, Lim and Yamazaki (LAA, 501 (2016) 112-122).

1. Introduction

Let $M_n$ be the space of $n \times n$ complex matrices and $M_n^+$ the positive part of $M_n$. Denote by $I$ the identity element of $M_n$. For self-adjoint matrices $A, B \in M_n$ the notation $A \preceq B$ means that $B - A \in M_n^+$. For a real-valued function $f$ of a real variable and a self-adjoint matrix $A \in M_n$, the value $f(A)$ is understood by means of the functional calculus.

For $0 \leq t \leq 1$ the $t$-geometric mean of $A$ and $B$ is defined as

$$A_t \sharp B = A^{1/2} (A^{-1/2} B A^{-1/2})^{t} A^{1/2}.$$  

The geometric mean $A^{\sharp 2} := A^{\sharp 1/2} B$ is the midpoint of the unique geodesic $A_t \sharp B$ connecting two points $A$ and $B$ in the Riemannian manifold of positive matrices.

Recently, Bhatia et al. [1] proved that for any positive definite matrices $A$ and $B$ and for $p = 1, 2$

$$||A + B + 2r A\sharp B||_p \leq ||A + B + r(A^{1/2} B^{1/2} + B^{1/2} A^{1/2})||_p. \quad (1)$$

When $r = 1$, inequality holds for $p = \infty$.

For the case $p = 2$ the proof of (1) is based on the following fact: for any positive definite $A$ and $B,$

$$\lambda(A^{1/2} (A^{\sharp 2} B) A^{1/2}) \prec_{\log} \lambda(A^{3/4} B^{1/2} A^{3/4}), \quad (2)$$

2000 Mathematics Subject Classification. 46L50, 15A45, 15B57.

Key words and phrases. operator $(r, s)$-convex functions, operator Jensen type inequality, operator Hansen-Pedersen type inequality, operator Popoviciu inequality.
where the notation $\lambda$ is used for the $n$-tuple of eigenvalues of a matrix $A$ in decent order and $\lambda(A) \prec_{\log} \lambda(B)$ means that
\[
\prod_{j=1}^{k} \lambda_i(A) \leq \prod_{j=1}^{k} \lambda_i(B), \quad 1 \leq k \leq n
\]
and inequality holds when $k = n$.

For $p = \infty$ inequality (11) was proved by using a result of Lim and Yamazaki [3, Theorem 4.1]
\[
||P_t(A_1, A_2, \cdots, A_n)||_{\infty} \leq ||Q_t(A_1, A_2, \cdots, A_n)||_{\infty},
\]
where the power mean $P_t(A_1, A_2, \cdots, A_n)$ of $A_1, A_2, \cdots, A_n$ [2] and is the unique solution of the matrix equation
\[
X = \frac{1}{m} \sum_{i=1}^{m} X_{i}^{\ast} A_i
\]
and
\[
Q_t(A_1, A_2, \cdots, A_n) = \left( \frac{1}{m} \sum_{i=1}^{m} A_i^{\ast} \right)^{1/t}.
\]
For $m = 2$ Lim and Pálfia [2, Remark 3.10] show that
\[
P_t(A, B) = A_{1/2}^{\ast} \left( \frac{1}{2} (A + A_{1/2}^{\ast} B) \right) = A^{1/2} \left( I + \frac{(A^{-1/2} B A^{-1/2})^t}{2} \right)^{1/2} A^{1/2}.
\]
Hopefully, for $t = 1/2,$
\[
P_{1/2}(A, B) = \frac{1}{4} (A + B + A_{1/2}^{\ast} B), \quad Q_{1/2}(A, B) = \frac{1}{4} (A + B + A^{1/2} B^{1/2} + B^{1/2} A^{1/2}).
\]
And so, the inequality (11) for $p = \infty$ is obtained from (13) choosing $m = 2$ and $t = 1/2.$

Recall that the family of Heron mean [6] for nonnegative number $a, b$ is defined as
\[
H_t(a, b) = (1 - t) \left( \frac{a + b}{2} \right) + t \sqrt{ab}, \quad 0 \leq t \leq 1.
\]
The Kubo-Ando extension of this to matrices is
\[
(1 - t) \frac{A + B}{2} + t (A_{1/2}^{\ast} B)
\]
that connects the arithmetic mean and the geometric mean, and a naive extension is
\[
(1 - t) \frac{A + B}{2} + t A^{1/2} B^{1/2} + B^{1/2} A^{1/2}
\]
that connects the arithmetic mean and the midpoint of the Heinz mean $\frac{A^{1/2} B^{1/2} + B^{1/2} A^{1/2}}{2}.$

So, inequality (11) is a special case of the following
\[
|| (1 - t) \frac{A + B}{2} + t (A_{1/2}^{\ast} B) ||_p \leq || (1 - t) \frac{A + B}{2} + t A^{1/2} B^{1/2} + B^{1/2} A^{1/2} ||_p
\]
with $t = 1/2.$

Notice that another naive extension of the Heron mean for positive definite matrices $A$ and $B$ is defined as
\[
(1 - t) \frac{A + B}{2} + t A^{1/2} B^{1/2} + A^{1/2} B^{1/2} - A^{1/2} B^{1/2}.
\]
In this paper, we extend the inequality [2] to $t$-geometric means. More precisely, we prove that for any positive definite matrices $A, B$ and for any $t \in [0, 1]$

$$\lambda(A^{1/2}(A_t^*B)A^{1/2}) \prec_{\log} \lambda(A^{1-t}B^tA^{1-t}).$$

Using this extension, we prove the following result:

**Theorem 1.1.** Let $A, B$ be positive definite matrices and $p = 1, 2$ and $r \geq 0$. Then

$$||A + B + r(A_t^*B + A_{1-t}^*B)||_p \leq ||A + B + r(A^tB^{1-t} + A^{1-t}B^t)||_p.$$  \hspace{1cm} (5)

Also, using the approach in [4] we show that for positive definite matrices $A, B$ and for any $z$ in the strips $S_{1/4} = \{z \in \mathbb{C} : \text{Re}(z) \in [1/4, 3/4]\}$,

$$|\text{Tr}(A^{1/2}BzA^{1/2}B^{1-z})| \leq \text{Tr}(AB).$$

2. Inequalities

**Proposition 2.1.** Let $A, B$ be positive definite matrices. Then for any $t \in [0, 1]$

$$\lambda(A^{1/2}(A_t^*B)A^{1/2}) \prec_{\log} \lambda(A^{1-t/2}B^tA^{1-t/2}).$$ \hspace{1cm} (6)

**Proof.** Firstly, let’s prove

$$\lambda_1(A^{1/2}(A_t^*B)A^{1/2}) \leq \lambda_1(A^{1-t/2}B^tA^{1-t/2}).$$ \hspace{1cm} (7)

This inequality is equivalent to the statement

$$A^{1-t/2}B^tA^{1-t/2} \leq I \implies A^{1/2}(A_t^*B)A^{1/2} \leq I,$$

which in turn is equivalent to

$$B^t \leq A^{t-2} \implies (A^{-1/2}BA^{-1/2})^t \leq A^{-2}.$$ \hspace{1cm} (8)

That can be proved by using the Furuta inequality which states that if $0 \leq Y \leq X$, then for all $p \geq 1$ and $r \geq 0$ we have

$$(X^rY^pX^r)^{1/2} \leq (X^{p+2r})^{1/p}.$$ \hspace{1cm} (9)

Let apply (9) to $X = A^{t-2}$, $Y = B^t$, $p = \frac{1}{7}$ and $r = -\frac{1}{2(t-2)}$, we get (8), and hence (7).

Denote by $C_k(X)$ the $k$-th compound of $X \in M_n$, $k = 1, \ldots, n$. Note that for any positive definite matrices $X, Y$,

$$C_k(A^{1/2}(A_t^*B)A^{1/2}) = C_k(A(A^{-1/2}BA^{-1/2})^tA)$$

$$= C_k(A)C_k((A^{-1/2}BA^{-1/2})^t)C_k(A)$$

$$= C_k^{1/2}(A)(C_k^{1/2}(A)(C_k^{-1/2}(A)C_k(B)C_k^{-1/2}(A))^tC_k^{1/2}(A))C_k^{1/2}(A)$$

$$= C_k^{1/2}(A)(C_k(A)^{t}C_k(B))C_k^{1/2}(A).$$ \hspace{1cm} (10)
In the other hand,
\[ \lambda_1(C_k(A^{1/2}(A^{1/2}_t B)A^{1/2})) = \prod_{i=1}^k \lambda_i(A^{1/2}(A^{1/2}_t B)A^{1/2}), \quad k = 1, \ldots, n - 1, \quad (11) \]

On account of (7) and (11) for \( 1 \leq k \leq n \) we have
\[
\prod_{i=1}^k \lambda_i(A^{1/2}(A^{1/2}_t B)A^{1/2}) = \lambda_1(C_k(A^{1/2}(A^{1/2}_t B)A^{1/2})) \\
\leq \lambda_1(C_k(A)^{1-t}C_k(B)^tC_k(A)^{1-t}) \\
= \prod_{i=1}^k \lambda_i(A^{1-t}B^tA^{1-t}).
\]

The equality holds for \( k = n \), since \( \text{det}(A^{1/2}(A^{1/2}_t B)A^{1/2}) = \text{det}(A^{1-t}B^tA^{1-t}) \).

Thus, we have proved (8). \( \square \)

The following special case of Proposition will be used in the proof of the main result.

**Corollary 2.2.** For any positive definite matrices \( A \) and \( B \),
\[ \text{Tr}(A(A^{1/2}_t B)) \leq \text{Tr}(A^{2-t}B^t), \quad t \in [0, 1]. \]

In order to prove the next result, let’s recall the generalized Hölder inequality for trace [5, Theorem 2.8]: let \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \) for \( p, q, r \geq 1 \) and \( X, Y, Z \) be matrices in \( M_n \), then
\[ \text{Tr}(XYZ) \leq ||XYZ||_1 \leq ||X||_p||Y||_q||Z||_r. \]

We also need the famous Lieb-Thirring inequality: \( \text{Tr}((AB)^m) \leq \text{Tr}(A^m B^m) \).

**Theorem 2.3.** Let \( X, Y \) be positive definite matrices and \( z \in S_{1/4} = \{ z \in \mathbb{C} : \text{Re}(z) \in \left[\frac{1}{4}, \frac{3}{4}\right]\} \). Then
\[ |\text{Tr}(X^{1/2}Y^{1/2}Z^{1/2}Y^{-1/2}Y^{-1})| \leq \text{Tr}(XY). \quad (12) \]

**Proof.** Let \( z = \frac{1}{2} + iy, y \in \mathbb{R} \) denote any point in the vertical line of the complex plane passing \( x = 1/2 \). Then we have
\[
|\text{Tr}(X^{1/2}Y^{1/2}Z^{1/2}X^{1/2}Y^{1/2}Y^{-1-z})| = |\text{Tr}(X^{1/2}Y^{1/2}Y^{iy}X^{1/2}Y^{1/2}Y^{iy-1})| \\
\leq \text{Tr}(|X^{1/2}Y^{1/2}Y^{iy}X^{1/2}Y^{1/2}Y^{iy-1}|) \\
\leq ||X^{1/2}Y^{1/2}Y^{iy}||_2||X^{1/2}Y^{1/2}Y^{iy-1}||_2 \\
= ||X^{1/2}Y^{1/2}||_2^2 \\
= \text{Tr}(XY).
\]

The first inequality is obvious, the second one follows from the Cauchy-Schwarz inequality for trace, and the second equality is from the fact that \( Y^{iy} \) and \( Y^{-iy} \) are unitary operators.
Now let consider $z = \frac{1}{4} + iy, y \in \mathbb{R}$, a generic point in the vertical line over $x = 1/4$, then by using the Hölder inequality with $\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$ and the Araki-Lieb-Thirring inequality we have

\[
|\text{Tr}(X^{1/2}Y^2X^{1/2}Y^{1-z})| = |\text{Tr}(X^{1/2}Y^{1/4}Y^{iy}X^{1/2}Y^{-1/2}Y^{iy}Y^{-1/4})| \\
= |\text{Tr}(Y^{1/4}X^{1/4}Y^{iy}X^{1/2}Y^{-1/2}Y^{iy}Y^{-1/2})| \\
\leq ||Y^{1/4}X^{1/4}||_1^{2/4}||X^{1/2}Y^{-1/2}||_2 \\
\leq ||Y^{1/2}X^{1/2}||_2 ||X^{1/2}Y^{-1/2}||_2 \\
= \text{Tr}(XY).
\]

Mention that the map $x \mapsto A^x = e^{x\ln A} = \sum_k z^k \frac{\ln A^k}{k!}$ is analytic for $A > 0$, the product of matrices is also analytic and the trace is complex linear, the function $f(z) = \text{Tr}(X^{1/2}Y^2X^{1/2}Y^{1-z})$ is entire. Moreover, by the similar above argument for $z = x + iy$ it is easy to see that if $0 \leq x \leq 1$ then the function is bounded. By the Hadamard three-lines theorem the supremum $M(x) = \sup\{|f(x + iy)| : y \in \mathbb{R}\}$ of the function $\text{Tr}(X^{1/2}Y^2X^{1/2}Y^{1-z})$ is log-convex, that means, for any $\lambda \in [0, 1]$,

\[
M(\lambda x_1 + (1 - \lambda)x_2) \leq M(x_1)^\lambda M(x_2)^{1-\lambda} \leq \text{Tr}(XY)^\lambda \text{Tr}(XY)^{1-\lambda} = \text{Tr}(XY).
\]

Therefore, the bound $\text{Tr}(XY)$ is valid in the vertical strip $1/4 \leq \text{Re}(z) \leq 1/2$. Invoking the symmetry $z \mapsto 1 - z$ and exchanging the roles of $A$ and $B$ give the desired bound on the full strip $S_{1/4} = \{1/4 \leq \text{Re}(z) \leq 3/4\}$.

As a consequence, we have the following inequality (see [11, Inequality (39)]):

**Corollary 2.4.** For any positive definite matrices and for $t \in [0, 1]$,

\[
\text{Tr}((A^{2t}_x B)(A^{2t}_{1-t} B)) \leq \text{Tr}(AB).
\]

Now we are ready to prove the main result in this paper.

**Theorem 2.5.** Let $A, B$ be positive definite matrices, $p = 1, 2$ and $r \geq 0$. Then

\[
||A + B + r(A^{2t}_x B + A^{2t}_{1-t} B)||_p \leq ||A + B + r(A^t B^{1-t} + A^{1-t} B^t)||_p. \tag{13}
\]

**Proof.** Since $A + B + r(A^{2t}_x B + A^{2t}_{1-t} B) \geq 0$, the left hand side of (13) is $\text{Tr}(A + B + r(A^{2t}_x B + A^{2t}_{1-t} B))$. It is well-known that $\text{Tr}(A^{2t}_x B) \leq \text{Tr}(A^{1-t} B^t)$ and $\text{Tr}(A^{2t}_{1-t} B) \leq \text{Tr}(A^t B^{1-t})$. We have

\[
\text{Tr}(A + B + r(A^{2t}_x B + A^{2t}_{1-t} B)) \leq \text{Tr}(A + B + r(A^t B^{1-t} + A^{1-t} B^t)) \\
\leq \text{Tr}(|A + B + r(A^t B^{1-t} + A^{1-t} B^t)|).
\]

So for $p = 1$ the inequality (13) follows.
Next consider the case $p = 2$. Notice again that $\text{Tr}((A^2 tB)^2) \leq \text{Tr}(B^2 t A^2(1-t))$ (see [1], pape 121). Similarly, we also have $\text{Tr}((A^2 t_{1-t}B)^2) \leq \text{Tr}(A^2 B^2(1-t))$. Then

$$\text{Tr}((A^2 tB)^2) + (A^2 t_{1-t}B)^2) \leq \text{Tr}(A^2 B^2(1-t) + B^2 A^2(1-t)). \quad (14)$$

By Proposition 2.1 we have

$$\text{Tr}((A + B)(A^2 t B + A^2 t_{1-t}B)) \leq \text{Tr}(A^{t_1} B^{1-t} + A^{2-t} B^t + A^t B^{2-t} + A^{1-t} B^{1+t}). \quad (15)$$

Now, squaring both sides of (13), we need to show

$$\text{Tr}((A + B)^2 + r^2(A^2 tB)^2 + s^2(A^2 t_{1-t}B)^2 + 2r(A + B)(A^2 t B + A^2 t_{1-t}B) + 2r^2(A^2 t B)(A^2 t_{1-t}B)) \leq \text{Tr}((A + B)^2 + 2r(A^{t_1} B^{1-t} + A^{2-t} B^t + A^t B^{2-t} + A^{1-t} B^{1+t}) + r^2 A^2 B^2(1-t) + r^2 B^2 A^2(1-t) + 2r^2 \text{Tr}(AB)).$$

The last inequality follows from (13), (15) and Corollary 2.4.

Remark 2.6. From Theorem 2.5 for $s \in [0, 1]$ we

$$|| (1-s)(A + B) + s(A^2 t B + A^2 t_{1-t}B)||_p \leq ||(1-s)(A + B) + s(A^t B^{1-t} + A^{1-t} B^t)||_p.$$ 

When $t = 1/2$ we obtain one kind of inequality for the matrix Heron mean

$$|| \frac{1-s}{2}(A + B) + s(A^2 B)||_p \leq || \frac{1-s}{2}(A + B) + s A^{1/2} B^{1/2}||_p.$$ 

Remark 2.7. By the same arguments, one can show that

$$||A + B + A^2 t B + A^2 t_{1-t}B||_p \leq ||A + B + A^t B^{1-t} + A^{1-t} B^t||_p.$$ 

But is we use another version of the Heinz mean $(A^t B^{1-t} + B^t A^{1-t})/2$ and realize the same proof in Theorem 2.5 the inequality in Corollary 2.4 could be as follows

$$\text{Tr}((A^2 t B)(B^2 t A)) \leq \text{Re} \text{Tr}(A^t B^t A^{1-t} B^{1-t}). \quad (16)$$

Notice that both sides are bounded by $\text{Tr}(AB)$ but it is not clear that (16) is true or not.

From the proof of the main theorem, it is natural to ask the following question: Is it true that for $0 \leq X, Y \leq Z$ such that $X \prec_{\log} Y$

$$Z^{1/2} X Z^{1/2} \prec_{\log} Z^{1/2} Y Z^{1/2}? \quad (17)$$

Unfortunately, the answer is negative. Indeed, let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Now $s(X) = s(Y) = (1, 1)$ but $s(Z^{1/2} X Z^{1/2}) = (1, 2) \not\prec_w (\sqrt{2}, \sqrt{2}) = s(Z^{1/2} Y Z^{1/2})$. So it is not even true for diagonal positive definite matrices.
3. Determinant Inequality for the Heron mean

Let’s recall a recent result of Audeanert [9]: for any positive semidefinite matrices $A$ and $B$

$$\det(I + A^* B) \leq \det(I + A^{1/2} B^{1/2}).$$  \hspace{1cm} (18)

The author used the well-known fact that $\lambda(A^* B) \prec_{\log} \lambda(A^{1/2} B^{1/2})$ and the function $\Phi(X) = \sum_{i=1}^n \log(1 + e^{x_i}) (X = (x_1, x_2, \cdots, x_n))$ is isotone (i.e. the function preserving weak majorization: $x \prec y \Rightarrow \Phi(x) \prec_w \Phi(y)$.)

In fact, for matrices $A$ and $B$ such that $\lambda(A) \prec_{\log} \lambda(B)$ we have

$$\det(I + A) \leq \det(I + B).$$ \hspace{1cm} (19)

A useful characterization of isotone functions in the case $m = 1$ is as follows:

**Lemma 3.1.** A differentiable function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is isotope if and only if it satisfy

1. $\Phi$ is permutation invariant;
2. for all $X \in \mathbb{R}^n$ and for all $i, j$:

$$ (x_i - x_j) \left( \frac{\partial \Phi}{\partial x_i}(x) - \frac{\partial \Phi}{\partial x_j}(x) \right) \geq 0. $$

Do the similar argument as in [9] one can prove the following

$$\det(I + A^* t B) \leq \det(I + A^{1-t} B^t).$$ \hspace{1cm} (20)

Now we can use this fact to obtain some inequality for the Heron mean.

**Theorem 3.2.** For any positive definite matrices $A$ and $B$

$$\det(P_t(A, B)) \leq \det(Q_t(A, B)).$$ \hspace{1cm} (21)

**Proof.** The inequality (21) is equivalent to the following

$$\det^{1/t}(A^t + B^t) = \det(A) \det^{1/t}(I + A^{-t/2} B^t A^{-t/2})$$

$$\geq \det(A^{1/t}(A + A^* t B))$$

$$= \det(A) \cdot \det^{1/t}(I + (A^{-1/2} B A^{-1/2})^t)$$

or

$$\det(I + A^{-t/2} B^t A^{-t/2}) \geq \det(I + (A^{-1/2} B A^{-1/2})^t).$$ \hspace{1cm} (22)

By the Araki-Lieb-Thirring inequality we have

$$\lambda(I + (A^{-1/2} B A^{-1/2})^t) \prec_{\log} \lambda(I + A^{-t/2} B^t A^{-t/2}).$$

Therefore, the inequality (22) follows from the last inequality and (19). $\Box$

As a consequence, we obtain a determinant inequality for the Heron mean.
Corollary 3.3. For any positive definite matrices $A$ and $B$,

$$\det(A + B + 2(A^*B)) \leq \det(A + B + A^{1/2}B^{1/2} + A^{1/2}B^{1/2}).$$

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