A PROBABILISTIC APPROACH TO INTERIOR REGULARITY OF FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS IN SMOOTH DOMAINS

WEI ZHOU

Abstract. We consider probabilistic solutions to Dirichlet problems for degenerate Bellman equations, given by value functions of controlled diffusion processes. We investigate the regularity and estimate the derivatives up to second order of the value function, under the assumption of non-degeneracy along the normal to the boundary and an interior condition weaker than the non-degeneracy. The value function is the unique solution to the associated Dirichlet problem in our setting. Our approach is probabilistic.

1. Introduction

We consider the Dirichlet problem for the Bellman equation

(1.1) \[ \sup_{\alpha \in \mathcal{A}} \left[ L^\alpha v(x) - c(\alpha, x)v(x) + f(\alpha, x) \right] = 0 \quad \text{in } D \\
\quad v = g \quad \text{on } \partial D, \]

where \( L^\alpha v(x) := a^{ij}(\alpha, x)v_{x^i x^j}(x) + b^i(\alpha, x)v_{x^i}(x), \) and summation convention is understood. On the one hand, it is known that under appropriate conditions the Dirichlet problem for the fully nonlinear convex elliptic equation

(1.2) \[ F(v_{x^i x^j}(x), v_{x^i}(x), v(x), x) = 0 \quad \text{in } D \\
\quad v = g \quad \text{on } \partial D \]

can be rewritten as a Bellman equation in the form of (1.1). On the other hand, under suitable regularity assumptions on \( a, b, c, f, g \) and \( D, \) the Bellman equation (1.1) is satisfied by the value function

(1.3) \[ v(x) = \sup_{\alpha \in \mathcal{A}} v^\alpha(x), \]

where

(1.4) \[ v^\alpha(x) = E \left[ g(x^\alpha_{x^\alpha}, x) e^{-\phi^\alpha_{x^\alpha}} + \int_0^{x^\alpha} f^\alpha s(x^\alpha_s) e^{-\phi^\alpha_s} ds \right], \]

with \( \phi^\alpha_t = \int_0^t e^{\phi^\alpha_s(x^\alpha_s)} ds, \)
in a control problem associated with the family of Itô equations

\[(1.5) \quad x_t^{\alpha,x} = x + \int_0^t \sigma^{\alpha_s}(x_s^{\alpha,x}) dw_s + \int_0^t b^{\alpha_s}(x_s^{\alpha,x}) ds,\]

where \(\tau^{\alpha,x}\) is the first exit time of \(x_t^{\alpha,x}\) from \(D\).

However, in general, \(v\) defined by (1.3) is not sufficiently smooth, or even continuous, so \(v\) in (1.3) is known as a probabilistic solution, or viscosity solution, to (1.1). We are interested in understanding under what conditions, \(v\) given by (1.3) is twice differentiable and is the unique solution of (1.1) in an appropriate sense. The main difficulties in dealing with this problem are the fully nonlinearity, the degeneracy of the operator and the inhomogeneous boundary condition.

The results stated and proved here are closely related to those obtained by M. V. Safonov [10] (1977), [11] (1978); P.-L. Lions [9] (1983) and N. V. Krylov [6] (1989). In [10] and [11], the domain \(D\) is two-dimensional, and the arguments are based on the fact that the controlled processes are in a plane region. In [9], the regularity results are proved by a combination of probabilistic and PDE arguments, which heavily rely on the assumption that the discount coefficient \(c^{\alpha}(x)\) is sufficiently large to bound first derivatives of \(\sigma^{\alpha}(x)\) and \(b^{\alpha}(x)\). In [6], the boundary data \(g\) is assumed to be of class \(C^4\), and under certain assumptions, it is proved that \(v\) has second derivatives bounded up to the boundary. The results are obtained in a purely probabilistic approach by introducing and using quasiderivatives and a reduction of controlled processes in a domain to controlled processes on a surface without boundary in the space having four more dimensions.

In this article, under a more general setting, we prove that the first and second derivatives of \(v\) given by (1.3) exist almost everywhere in \(D\), which implies the existence and uniqueness for the Dirichlet problem (1.1). Moreover, since, as discussed in [12], the derivatives of \(v\) may not be bounded up to the boundary of the domain under our setting, we also estimate first and second derivatives. The main result is stated in Section 2, and the proof is given in Section 3. Our approach is also probabilistic by using quasiderivatives. However, to deal with the boundary, instead of adding four more dimensions, we construct two families of supermartingales to bound the moments of quasiderivatives near the boundary and in the interior of the domain, respectively. For the background and motivations of quasiderivative method, we refer to [8, 12] and the references therein.

To conclude this section, we introduce the notation: For \(k = 1, 2\), let \(C^k(D)\) be the space of \(k\)-times continuously differentiable functions in \(D\) with finite norm given by

\[|g|_{1,D} = |g|_{0,D} + |g_x|_{0,D}, \quad |g|_{2,D} = |g|_{1,D} + |g_{xx}|_{0,D},\]

respectively, where

\[|g|_{0,D} = \sup_{x \in D} |g(x)|,\]
$g_x$ is the gradient vector of $g$, and $g_{xx}$ is the Hessian matrix of $g$. For $\beta \in (0, 1]$, the Hölder spaces $C^{k, \beta}(\bar{D})$ are defined as the subspaces of $C^k(\bar{D})$ consisting of functions with finite norm

$$|g|_{k, \beta, D} = |g|_{k, D} + [g]_{\beta, D}, \quad \text{with} \quad [g]_{\beta, D} = \sup_{x,y \in D} \frac{|g(x) - g(y)|}{|x - y|^{\beta}}.$$

$\mathbb{R}^d$ is the $d$-dimensional Euclidean space with $x = (x^1, x^2, ..., x^d)$ representing a typical point in $\mathbb{R}^d$, and $(x, y) = \sum_{i=1}^d x^i y^i$ is the inner product for $x, y \in \mathbb{R}^d$. For $x, y, z \in \mathbb{R}^d$, set

$$u(y) = \sum_{i=1}^d u_{x^i} y^i, \quad u(y)(z) = \sum_{i,j=1}^d u_{x^i,x^j} y^i z^j,$$

$$u_{(y)}^2 = (u(y))^2.$$

For any matrix $\sigma = (\sigma^{ij})$,

$$\|\sigma\|^2 := \text{tr} \sigma \sigma^* = \sum_{i,j} (\sigma^{ij})^2.$$  

We also use the notation

$$s \wedge t = \min(s, t), \quad s \vee t = \max(s, t).$$

Constants $K, M$ and $N$ appearing in inequalities are usually not indexed. They may differ even in the same chain of inequalities.

### 2. Main results

Assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space and $\{\mathcal{F}_t; t \leq 0\}$ an increasing filtration of $\sigma$-algebras $\mathcal{F}_t \subset \mathcal{F}$ which are complete with respect to $\mathcal{F}, P$. Let $\{w_t, \mathcal{F}_t; t \geq 0\}$ be a $d_1$-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$.

Let $A$ be a separable metric space. Suppose that the following have been defined for each $\alpha \in A$ and $x \in \mathbb{R}^d$: a $d \times d_1$ matrix $\sigma^\alpha(x)$, a $d$-dimensional vector $b^\alpha(x)$ and real scalars $c^\alpha(x) \geq 0$ and $f^\alpha(x)$. We assume that $\sigma, b, c$ and $f$ are Borel measurable on $A \times \mathbb{R}^d$, and $g(x)$ is a Borel measurable function on $\mathbb{R}^d$. We also assume that $\sigma^\alpha$, $b^\alpha$, $c^\alpha$ and their first and second derivatives are all continuous in $x$ uniformly with respect to $\alpha$.

Let $D \in C^4$ be a bounded domain in $\mathbb{R}^d$, then there exists a function $\psi \in C^4$ satisfying

$$\psi > 0 \text{ in } D, \quad \psi = 0 \text{ and } |\psi_x| \geq 1 \text{ on } \partial D.$$  

Additionally, we assume that

$$\sup_{\alpha \in A} L^\alpha \psi \leq -1 \text{ in } D,$$

with

$$L^\alpha := (a^\alpha)^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + (b^\alpha)^i(x) \frac{\partial}{\partial x^i},$$

where $a^\alpha$, $b^\alpha$ and $c^\alpha$ are Borel measurable on $A \times \mathbb{R}^d$.
where \( a = 1/2(\sigma \sigma^*) \). We also assume that
\[
|\langle \sigma^\alpha \rangle^{ij} \rangle_{2D} + |\langle b^\alpha \rangle^i \rangle_{2D} + |c^\alpha \rangle_{2D} + |\psi|_{4D} \leq K_0,
\]
with \( K_0 \in [1, \infty) \), not depending on \( \alpha \).

By \( \mathfrak{A} \), we denote the set of all functions \( \alpha_r(\omega) \) on \( \Omega \times [0, \infty) \) which are \( \mathcal{F}_r \)-adapted and measurable in \((\omega, r)\) with values in \( A \).

For \( \alpha \in \mathfrak{A} \) and \( x \in D \), we consider the Itô equation
\[
\begin{align*}
x_t^{\alpha, x} &= x + \int_0^t \sigma_{\alpha}(x_s^{\alpha, x})dw_s + \int_0^t b_{\alpha}(x_s^{\alpha, x})ds.
\end{align*}
\]
The solution of this equation is known to exist and to be unique by our assumptions on \( \sigma^\alpha \) and \( b^\alpha \).

Let \( \tau_{\alpha, x} \) be the first exit time of \( x_t^{\alpha, x} \) from \( D \):
\[
\tau_{\alpha, x} = \inf\{t \leq 0 : x_t^{\alpha, x} \notin D\}.
\]

For any \( t \geq 0 \), we define
\[
\phi_t^{\alpha, x} = \int_0^t c_{\alpha}(x_s^{\alpha, x})ds.
\]

Set
\[
(2.3) \quad v(x) = \sup_{\alpha \in \mathfrak{A}} v^\alpha(x),
\]
with
\[
(2.4) \quad v^\alpha(x) = E^\alpha_x \left[ g(x_\tau) e^{-\phi_\tau} + \int_0^\tau f^\alpha(x_s) e^{-\phi_s}ds \right],
\]
where we use common abbreviated notation, according to which we put the indices \( \alpha \) and \( x \) beside the expectation sign instead of explicitly exhibiting them inside the expectation sign for every object that can carry all or part of them. Namely,
\[
E^\alpha_x \left[ g(x_\tau) e^{-\phi_\tau} + \int_0^\tau f^\alpha(x_s) e^{-\phi_s}ds \right] = E \left[ g(x_{\tau_{\alpha, x}}^{\alpha, x}) e^{-\phi_{\tau_{\alpha, x}}^{\alpha, x}} + \int_0^{\tau_{\alpha, x}} f^\alpha(x_s^{\alpha, x}) e^{-\phi_{s}^{\alpha, x}}ds \right].
\]

The value function \( v(x) \) given by (2.3) and (2.4) is the probabilistic solution of the Dirichlet problem for the Bellman equation:
\[
\begin{align*}
&\quad \sup_{\alpha \in \mathcal{A}} [L^\alpha v - c^\alpha v + f^\alpha] = 0 \quad \text{in } D \\
&\quad v = g \quad \text{on } \partial D.
\end{align*}
\]

Define
\[
(2.5) \quad \mu(x, \xi) := \inf_{\zeta} \sup_{\alpha \in \mathcal{A}} a^{ij}(\alpha, x) \zeta^i \zeta^j,
\]
\begin{align}
\mu(x) &:= \inf_{|\zeta|=1} \sup_{\alpha \in A} a^{ij}(\alpha, x) \zeta^i \zeta^j. 
\end{align}

The condition $\mu(x, \xi) > 0$ means that $v(\xi(\xi))(x)$ is actually "present" in the Bellman equation in (2.5). More precisely, for any fixed $x \in D$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, $\mu(x, \xi) > 0$ if and only if there exists a control $\alpha \in A$ such that the corresponding diffusion matrix $a^\alpha(x)$ is non-degenerate in the direction $\xi$. For example, consider the linear equation
\begin{align}
(2.8) \quad u_{x_1 x_1} + 2 u_{x_1 x_2} + u_{x_2 x_2} = 0.
\end{align}

By (2.6), here
\begin{align}
\mu(x, \xi) &= \inf_{(\xi, \zeta)=1} (\zeta^1 + \zeta^2)^2.
\end{align}

$\mu(x, \xi) > 0$ if and only if $\xi \parallel \xi_0 = (1, 1)$. So only $u(\xi_0)(\xi_0)$ is "present" in (2.8).

Also, it is not hard to see that
\begin{align}
\mu(x) = \inf_{|\xi|=1} \mu(x, \xi).
\end{align}

Note that we have $\mu(x) > 0$ at a point $x$ if and only if for any $\xi \neq 0$, there exists a control $\alpha \in A$, such that the corresponding diffusion term $a^\alpha(x)$ is non-degenerate in the direction $\xi$.

Let $\mathcal{B}$ be the set of all skew-symmetric $d_1 \times d_1$ matrices. For any positive constant $\lambda$, define
\begin{align}
D_\lambda = \{x \in D : \psi(x) > \lambda\}.
\end{align}

Our main result is the following:

**Theorem 2.1.** Suppose that
\begin{enumerate}
\item [(1)] (uniform non-degeneracy along the normal to the boundary) There exists a positive constant $\delta_0$, such that
\begin{align}
(a^\alpha n, n) \geq \delta_0 \text{ on } \partial D, \forall \alpha \in A,
\end{align}
where $n$ is the unit normal vector.
\item [(2)] (interior condition to control the moments of quasiderivatives, weaker than the non-degeneracy) There exist a function $\rho^\alpha(x) : A \times D \to \mathbb{R}^d$, bounded on every set in the form of $A \times D_\lambda$ for all $\lambda > 0$, a function $Q^\alpha(x, y) : A \times D \times \mathbb{R}^d \to \mathcal{B}$, bounded with respect to $(\alpha, x)$ on every set in the form of $A \times D_\lambda$ for all $\lambda > 0, y \in \mathbb{R}^d$ and linear in $y$, and a function $M^\alpha(x) : A \times D \to \mathbb{R}$, bounded on every set in the form of $A \times D_\lambda$ for all $\lambda > 0$, such that for any $\alpha \in A, x \in D$ and $|y| = 1$,
\begin{align}
||\sigma^\alpha(\psi(x)) + \rho^\alpha(x, y) \sigma^\alpha(x) + \sigma^\alpha(x) Q^\alpha(x, y)||^2 +
2(\sigma^\alpha(\psi(x)) + \rho^\alpha(x, y) \sigma^\alpha(x)) \leq c^\alpha(x) + M^\alpha(x) (a^\alpha(x)y, y).
\end{align}
\end{enumerate}
Then we have
(1) If for any \( \alpha \in A, f^\alpha, g \in C^{0,1}(\bar{D}), \) satisfying
\[
\sup_{\alpha \in A} |f^\alpha|_{0,1,D} + |g|_{0,1,D} \leq K_0,
\]
then \( v \in C^{0,1}(D) \), and for any \( \xi \in \mathbb{R}^d \),
\[
(2.11) \quad |v(\xi)(x)| \leq N\left( |\xi| + \frac{\psi(\xi)}{\psi^\xi} \right), \text{a.e. in } D,
\]
where the constant \( N \) depends only on \( d, d_1 \) and \( K_0 \).

(2) If for any \( \alpha \in A, f^\alpha \in C^{0,1}(\bar{D}), g \in C^{1,1}(\bar{D}), \) satisfying
\[
\sup_{\alpha \in A} |f^\alpha|_{0,1,D} + |g|_{1,1,D} \leq K_0,
\]
and \( f^\alpha_{xx} \) exists almost everywhere in \( D \), satisfying
\[
f^\alpha_{xx} + K_0 I \geq 0, \text{ a.e. in } D,
\]
then for any \( \xi \in \mathbb{R}^d \),
\[
(2.12) \quad v(\xi)(x) \geq -N\left( |\xi|^2 + \frac{\psi^2(\xi)}{\psi} \right), \text{a.e. in } D,
\]
\[
(2.13) \quad v(\xi)(x) \leq \mu(x,\xi/|\xi|)^{-1}N\left( \frac{|\xi|^2}{\psi} \right), \text{a.e. in } D(\xi),
\]
where \( D(\xi) := \{ x \in D : \mu(x,\xi) > 0 \} \), and the constant \( N \) depends only on \( d, d_1 \) and \( K_0 \).

If \( \mu(x) > 0 \) in \( D \), then \( v \in C^{1,1}_{loc}(D) \). In addition, \( v \) given by (2.4) is the unique solution in \( C^{1,1}_{loc}(D) \cap C^{0,1}(\bar{D}) \) of
\[
(2.14) \quad \begin{cases} 
\sup_{\alpha \in A} \left[ L^\alpha v(x) - c(\alpha,x)v(x) + f(\alpha,x) \right] = 0 & \text{a.e. in } D \\
v = g & \text{on } \partial D.
\end{cases}
\]

**Remark 2.1.** The author doesn't know whether the estimates (2.11), (2.12), and (2.13) are sharp.

### 3. Auxiliary Convergence Results

Let \( U \) be a connected open subset in \( \mathbb{R}^d \). Assume that, for any \( \alpha \in \mathcal{A}, \omega \in \Omega, t \geq 0, \) and \( x \in U \), we are given a \( d \times d_1 \) matrix \( \kappa_x^\alpha(x) \) and a \( d \)-dimensional vector \( \nu_x^\alpha(x) \). We assume that \( \kappa_x^\alpha \) and \( \nu_x^\alpha \) are continuous in \( x \) for any \( \alpha, \omega, t, \) measurable in \( (\omega,t) \) for any \( \alpha, x, \) and \( \mathcal{F}_t \)-measurable in \( \omega \) for any \( \alpha, t, x \). Assume that for any \( \alpha \in \mathcal{A} \), the Itô equation
\[
(3.1) \quad d\zeta_t^{\alpha,\xi} = \kappa_t^\alpha(\zeta_t^{\alpha,\xi})dw_t + \nu_t^\alpha(\zeta_t^{\alpha,\xi})dt
\]
has a unique solution.

We suppose that for an \( \epsilon_0 \in (0,1] \) and for each \( \epsilon \in [0,\epsilon_0] \), we are given
\[
\kappa_t^\alpha(\epsilon) = \kappa_t^\alpha(x, \epsilon), \quad \nu_t^\alpha(\epsilon) = \nu_t^\alpha(x, \epsilon)
\]
having the same meaning and satisfying the same assumptions as those of \( \kappa_t^\alpha \) and \( \nu_t^\alpha \). Assume that for any \( \alpha \in \mathfrak{A} \), the Itô equation \( (3.1) \) corresponding to \( \kappa_t^\alpha (\epsilon) \) and \( \nu_t^\alpha (\epsilon) \) with initial condition \( \zeta(\epsilon) \in U \)

(3.2) \[ d\zeta_t^\alpha (\epsilon) = \kappa_t^\alpha (\zeta_t^\alpha (\epsilon), \epsilon)dw_t + \nu_t^\alpha (\zeta_t^\alpha (\epsilon), \epsilon)dt \]

has a unique solution denoted by \( \zeta_t^\alpha (\epsilon) \).

**Lemma 3.1.** Let \( q \in [2, \infty) \), \( \theta \in (0, 1) \), \( M \in (0, \infty) \) be constants and \( M_t^\alpha \) be a \( \mathcal{F}_t \)-adapted nonnegative process for any \( \alpha \in \mathfrak{A} \).

1. If for any \( \alpha \in \mathfrak{A}, t \geq 0, x \in U \),

(3.3) \[ ||\kappa_t^\alpha (x)|| + ||\nu_t^\alpha (x)|| \leq M|x| + M_t^\alpha, \]

then for any bounded stopping times \( \gamma^\alpha \leq \tau_U^{\alpha, \zeta} \), \( \forall \alpha \)

(3.4) \[ \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} e^{-Nt}|\zeta_t|^q \]

\[ \leq |\zeta|^q + (2q-1) \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^\gamma M_t^q e^{-Nt}dt, \]

(3.5) \[ \sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma} e^{-Nt}|\zeta_t|^q \theta \]

\[ \leq \frac{2 - \theta}{1 - \theta} \left( |\zeta|^q + (2q-1)^\theta \sup_{\alpha \in \mathfrak{A}} E^\alpha \left( \int_0^\gamma M_t^q e^{-Nt}dt \right)^\theta \right), \]

where \( N = N(q, M) \).

2. If for any \( \alpha \in \mathfrak{A}, t \geq 0, x \in U \), and some \( \epsilon \in [0, \epsilon_0] \),

(3.6) \[ ||\kappa_t^\alpha (x) - \kappa_t^\alpha (y, \epsilon)|| + ||\nu_t^\alpha (x) - \nu_t^\alpha (y, \epsilon)|| \leq M|x - y| + \epsilon M_t^\alpha, \]

then for any bounded stopping times \( \gamma^\alpha \leq \tau_U^{\alpha, \zeta} \land \tau_U^{\alpha, \zeta}(\epsilon), \forall \alpha \)

(3.7) \[ \sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha} e^{-Nt}|\zeta_t^\alpha (\epsilon)|^q \theta \]

\[ \leq |\zeta(\epsilon) - \zeta|^q + \epsilon^q(2q-1) \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^\gamma M_t^q e^{-Nt}dt, \]

(3.8) \[ \sup_{\alpha \in \mathfrak{A}} E \sup_{t \leq \gamma^\alpha} e^{-Nt}|\zeta_t^\alpha (\epsilon)|^q \theta \]

\[ \leq \frac{2 - \theta}{1 - \theta} \left( |\zeta(\epsilon) - \zeta|^q + \epsilon^q(2q-1)^\theta \sup_{\alpha \in \mathfrak{A}} E^\alpha \left( \int_0^\gamma M_t^q e^{-Nt}dt \right)^\theta \right), \]

where \( N = N(q, M) \).

**Remark 3.1.** Observe that \( q \theta \) covers \((0, \infty)\).

**Proof.** It suffices to prove the uncontrolled version of \((3.1), (3.3), (3.7)\) and \((3.8)\), so we drop the index \( \alpha \) in what follows for simplicity of notation. We also denote \( \zeta_t = \zeta_t^\alpha \), \( \zeta_t(\epsilon) = \zeta_t^\alpha (\epsilon) \).

Also, choosing a localizing sequence of stopping times \( \gamma_n \uparrow \infty \) such that \( \int_0^{t \land \gamma_n} M_s^q e^{-Ns}ds \) are bounded for every \( n \), we see, in view of the Monotone
Due to Lemma 7.3(ii) in [7], we conclude that mimicking the argument for proving (3.4) and (3.5), which can play the same role as (3.3). So (3.7) and (3.8) can be proved by Convergence Theorem, that it will suffice to consider the case in which \( \int_0^t M_s^q e^{-Nt} ds \) are bounded with respect to \((\omega, t)\).

By Itô’s formula, we have
\[
de^{-Nt}|\zeta_t|^q = e^{-Nt} \left[ q|\zeta_t|^{q-2} (\zeta_t, \nu_t(\zeta_t)) + \frac{q}{2} |\zeta_t|^{q-2} \|\kappa_t(\zeta_t)\|^2 \right.
\]
\[
+ \frac{q(q-2)}{2} |\zeta_t|^{q-4} \|\kappa_t^*(\zeta_t)\zeta_t\|^2 - N \right] dt + dm_t,
\]
where \(m_t\) is a local martingale starting at zero. By \((3.3)\) and Young’s inequality
\[
q|\zeta_t|^{q-2} (\zeta_t, \nu_t(\zeta_t)) \leq (qM + q - 1)|\zeta_t|^q + M_t^q
\]
\[
\frac{q}{2} |\zeta_t|^{q-2} \|\kappa_t(\zeta_t)\|^2 \leq \left( \frac{qM}{2} + (q - 1)M \frac{q^2}{2} + \frac{q - 2}{2} \right)|\zeta_t|^q + 2M_t^q
\]
\[
\frac{q(q-2)}{2} |\zeta_t|^{q-4} \|\kappa_t^*(\zeta_t)\zeta_t\|^2 \leq (q - 2) \left[ \left( \frac{qM}{2} + (q - 1)M \frac{q^2}{2} + \frac{q - 2}{2} \right)|\zeta_t|^q + 2M_t^q \right]
\]
So for sufficiently large constant \(N = N(q, M)\), we have
\[
e^{-Nt}|\zeta_t|^q \leq |\zeta|^q + (2q - 1) \int_0^t M_s^q e^{-Nt} dt.
\]
Applying Lemma 7.3(i) in [7], we get
\[
E \sup_{t \leq \gamma} e^{-Nt}|\zeta_t|^q \leq |\zeta|^q + (2q - 1)E \int_0^\gamma M_s^q e^{-Nt} dt.
\]
Due to Lemma 7.3(ii) in [7], we conclude that
\[
E \sup_{t \leq \gamma} e^{-Nt} \|\zeta_t\|^{q\theta} \leq \frac{2 - \theta}{1 - \theta} E \left( |\zeta|^q + (2q - 1) \int_0^\gamma M_s^q e^{-Nt} dt \right)^\theta
\]
\[
\leq \frac{2 - \theta}{1 - \theta} \left( |\zeta|^{q\theta} + (2q - 1)^\theta E \left( \int_0^\gamma M_s^q e^{-Nt} dt \right)^\theta \right).
\]
Similarly, by Itô’s formula,
\[
de^{-Nt}|\zeta_t(\epsilon) - \zeta_t|^q
\]
\[
e^{-Nt} \left[ q|\zeta_t|^{q-2} (\zeta_t, \nu_t(\zeta_t, \epsilon) - \nu_t(\zeta_t)) + \frac{q}{2} |\zeta_t|^{q-2} \|\kappa_t(\zeta_t, \epsilon) - \kappa_t(\zeta_t)\|^2 \right.
\]
\[
+ \frac{q(q-2)}{2} |\zeta_t|^{q-4} \|\kappa_t^*(\zeta_t, \epsilon) - \kappa_t^*(\zeta_t)\zeta_t\|^2 - N \right] dt + dm_t,
\]
where \(m_t\) is a local martingale starting at zero. By \((3.6)\), we have
\[
\|\kappa_t(\zeta_t, \epsilon) - \kappa_t(\zeta_t)\| + |\nu_t(\zeta_t, \epsilon) - \nu_t(\zeta_t)| \leq M|\zeta_t(\epsilon) - \zeta| + \epsilon M_t,
\]
which can play the same role as \((3.3)\). So \((3.7)\) and \((3.8)\) can be proved by mimicking the argument for proving \((3.4)\) and \((3.5)\). \(\square\)
Next, we introduce the quasiderivatives to be used in the proof of the main theorem and apply Lemmas 3.1 to estimate moments of these quasiderivatives.

For any $\alpha \in \mathfrak{A}$, let $r_t^{\alpha}, \hat{r}_t^{\alpha}, \pi_t^{\alpha}, \hat{\pi}_t^{\alpha}, P_t^{\alpha}, \hat{P}_t^{\alpha}$ be jointly measurable adapted processes with values in $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_1}$, Skew($d_1, \mathbb{R}$), Skew($d_1, \mathbb{R}$), respectively, where Skew($d_1, \mathbb{R}$) denotes the set of all $d_1 \times d_1$ skew-symmetric real matrices. Let $\epsilon$ be a small positive constant. For each $\alpha \in \mathfrak{A}$, $x, y, z \in D$, $\xi, \eta \in \mathbb{R}^{d_1}$, we consider the Itô equation (2.2) and the following four other Itô equations:

\begin{align}
(3.9) \quad dy_t^{\alpha,y}(\epsilon) &= \sqrt{1 + 2er_t^{\alpha}} \sigma^{\alpha}(y_t^{\alpha,y}(\epsilon))e^{\epsilon P_t^{\alpha}} dw_t \\
&+ \left[(1 + 2er_t^{\alpha})b^{\alpha}(y_t^{\alpha,y}(\epsilon)) - \sqrt{1 + 2er_t^{\alpha}} \sigma^{\alpha}(y_t^{\alpha,y}(\epsilon))e^{\epsilon P_t^{\alpha}} \epsilon \pi_t^{\alpha}\right] dt,
\end{align}

\begin{align}
(3.10) \quad dx_t^{\alpha,z}(\epsilon) &= \sqrt{1 + 2er_t^{\alpha}} + \epsilon^2 r_t^{\alpha} \sigma^{\alpha}(z_t^{\alpha,z}(\epsilon))e^{\epsilon P_t^{\alpha}} e^{\frac{\epsilon^2}{2} \hat{P}_t^{\alpha}} dw_t \\
&+ \left[(1 + 2er_t^{\alpha}) + \epsilon^2 r_t^{\alpha} b^{\alpha}(z_t^{\alpha,z}(\epsilon)) - \sqrt{1 + 2er_t^{\alpha}} \sigma^{\alpha}(z_t^{\alpha,z}(\epsilon))e^{\epsilon P_t^{\alpha}} e^{\frac{\epsilon^2}{2} \hat{P}_t^{\alpha}} (\epsilon \pi_t^{\alpha} + \frac{\epsilon^2}{2} \hat{\pi}_t^{\alpha})\right] dt,
\end{align}

\begin{align}
(3.11) \quad d\xi_t^{\alpha,\xi} &= \left[\sigma^{\alpha}(\xi_t^{\alpha,\xi}) + r_t^{\alpha} \sigma^{\alpha} + \sigma^{\alpha} P_t^{\alpha}\right] dt \\
&+ \left[b^{\alpha}(\xi_t^{\alpha,\xi}) + 2r_t^{\alpha} b^{\alpha} - \sigma^{\alpha} \pi_t^{\alpha}\right] dt,
\end{align}

\begin{align}
(3.12) \quad d\eta_t^{\alpha,\eta} &= \left[\sigma^{\alpha}(\eta_t^{\alpha,\eta}) + \hat{r}_t^{\alpha} \sigma^{\alpha} + \sigma^{\alpha} \hat{P}_t^{\alpha} + \sigma^{\alpha}(\eta_t^{\alpha,\xi})(\xi_t^{\alpha,\xi}) + 2r_t^{\alpha} \sigma^{\alpha}(\xi_t^{\alpha,\xi})\right] dt \\
&+ 2\sigma^{\alpha}(\xi_t^{\alpha,\xi}) P_t^{\alpha} + 2r_t^{\alpha} \sigma^{\alpha} P_t^{\alpha} - \left(r_t^{\alpha} \right)^2 \sigma^{\alpha} + \sigma^{\alpha}(P_t^{\alpha})^2 dt \\
&+ \left[b^{\alpha}(\xi_t^{\alpha,\eta}) + 2r_t^{\alpha} b^{\alpha} - \sigma^{\alpha} \pi_t^{\alpha} + b^{\alpha}(\xi_t^{\alpha,\xi})(\xi_t^{\alpha,\xi}) + 4r_t^{\alpha} b^{\alpha}(\xi_t^{\alpha,\xi})\right] dt \\
&- 2\sigma^{\alpha}(\xi_t^{\alpha,\xi}) \pi_t^{\alpha} - 2r_t^{\alpha} \sigma^{\alpha} \pi_t^{\alpha} - 2\sigma^{\alpha} \hat{P}_t^{\alpha} \pi_t^{\alpha}\right] dt,
\end{align}

where $\sigma^{\alpha}$ and $b^{\alpha}$ satisfy (2.1) and we drop the arguments $x_t^{\alpha,x}$ in $\sigma^{\alpha}$ and $b^{\alpha}$ and their derivatives in (3.11) and (3.12).

Let $\tau_D^{\alpha,y}(\epsilon)$ be the first exit time of $y_t^{\alpha,y}(\epsilon)$ from $D$, and $\tau_D^{\alpha,z}(\epsilon)$ be the first exit time of $z_t^{\alpha,z}(\epsilon)$ from $D$. It is known that if

\begin{align}
(3.13) \quad \int_0^T (|r_t^{\alpha}|^2 + |\pi_t^{\alpha}|^2 + |P_t^{\alpha}|^2) dt < \infty, \\
\forall T \in [0, \infty), \forall \alpha \in \mathfrak{A},
\end{align}
then (3.9) and (3.11) have unique solutions on \([0, \tau_{D}^\alpha(\epsilon))\) and \([0, \tau_{D}^\alpha(\epsilon))\), respectively. If

\[
\int_0^T \left( |\dot{r}_t^\alpha|^2 + |\dot{\tau}_t^\alpha|^2 + |\dot{P}_t^\alpha|^2 + |\dot{\tau}_t^{\alpha}|^4 + |P_t^\alpha|^4 \right) dt < \infty,
\]

\(\forall T \in [0, \infty), \forall \alpha \in \mathfrak{A},\)

then (3.10) and (3.12) have unique solutions on \([0, \tau_{D}^\alpha(\epsilon))\) and \([0, \tau_{D}^\alpha(\epsilon))\), respectively. It is shown in Theorem 2.1 in [12] that for each \(\alpha \in \mathfrak{A}\), \(\xi_t^\alpha,\xi_t^\alpha\) is a first quasiderivative of \(x_t^{\alpha,x}\) in \(D\) in the direction of \(\xi\) at \(x\) and

\[
\xi_t^0,\alpha = \int_0^t \pi_s^\alpha dw_s
\]

is its first adjoint process, and \(\eta_t^{0,\eta}\) is a second quasiderivative of \(x_t^{\alpha,x}\) associated with \(\xi_t^{\alpha,\xi}\) in \(D\) in the direction of \(\eta\) at \(x\) and

\[
\eta_t^{0,\alpha} = (\xi_t^{0,\alpha})^2 - (\xi_t^{0,\alpha}) + \int_0^t \pi_s^\alpha dw_s
\]

is its second adjoint process.

**Theorem 3.1.** Given constants \(p \in (0, \infty), p' \in [0, p), T \in [1, \infty), x \in D, \xi \in \mathbb{R}^d\). Suppose (3.13) is satisfied. Assume that there exists a constant \(K \in [1, \infty)\) and for any \(\alpha \in \mathfrak{A}\), an adapted nonnegative process \(K_t^\alpha\), such that

\[
|\dot{r}_t^\alpha| + |\dot{\tau}_t^\alpha| + |\dot{P}_t^\alpha| \leq K|\xi_t^{\alpha,\xi}| + K_t^\alpha, \forall \alpha.
\]

(1) Given stopping times \(\gamma^\alpha \leq \tau_{D}^\alpha, \alpha \in \mathfrak{A},\) if

\[
sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma \wedge T} K_t^{2v^p} dt < \infty,
\]

then we have

\[
\sup_{\alpha \in \mathfrak{A}} E^\alpha \sup_{t \leq \gamma \wedge T} |\xi_t|^p < \infty.
\]

(2) Let the constant \(\epsilon_0\) be sufficiently small so that \(B(x, \epsilon_0|\xi|) \subset D\). For any \(\epsilon \in [0, \epsilon_0]\), given stopping times \(\gamma^\alpha(\epsilon) \leq \tau_{D}^\alpha x_{D}^{\alpha,x} + \epsilon(\epsilon), \alpha \in \mathfrak{A},\) if

\[
\sup_{\epsilon \in [0, \epsilon_0]} \sup_{\alpha \in \mathfrak{A}} E^\alpha \int_0^{\gamma(\epsilon) \wedge T} K_t^{2(2v^p)} dt < \infty,
\]

then we have

\[
\limsup_{\epsilon \downarrow 0} E^\alpha \sup_{\gamma(\epsilon) \wedge T} \frac{|y_t^{\alpha,x+\xi}(\epsilon) - x_t^{\alpha,x}|^p}{\epsilon^{p'}} = 0,
\]

\[
\lim_{\epsilon \downarrow 0} E^\alpha \sup_{\gamma(\epsilon) \wedge T} \frac{|y_t^{\alpha,x+\xi}(\epsilon) - x_t^{\alpha,x}|}{\epsilon} - \xi_t^{\alpha,\xi}|^p = 0.
\]
Proof. In the proof, we drop the superscripts \( \alpha, \alpha_t \), etc., when this will not cause confusion.

To prove (1) we consider the Itô equation (3.1) in which \( \zeta_t^{x, \xi} = \xi_t^{x, \xi} \). By conditions (2.1) and (3.17), we have

\[
\| \sigma(\xi_t) + r_t \sigma + \sigma P_t \| + \| b(\xi_t^{x, \xi}) + 2r_t b - \sigma \pi_t \| \leq M |\xi_t| + M_t, \forall \alpha,
\]

where \( M = N(K, K_0), M_t^\alpha = N(K_0)K_t^\alpha \). Applying Lemma 3.1(1), we have

\[
\sup_{\alpha \in \mathbb{A}} E_\xi^{\alpha} \sup_{t \leq \gamma \wedge T} |\xi_t|^p \leq \begin{cases} 
\epsilon^{NT}(|\xi|^p + (2p - 1) \sup_{\alpha \in \mathbb{A}} E^{\alpha} \int_0^{\gamma \wedge T} M_t^p \, dt) \text{ if } p \geq 2 \\
\epsilon^{NT} \frac{4 - p}{2 - p} (|\xi|^p + 3\epsilon^2 (\sup_{\alpha \in \mathbb{A}} E^{\alpha} \int_0^{\gamma \wedge T} M_t^2 \, dt)^\frac{p}{2}) \text{ if } p < 2.
\end{cases}
\]

To prove (2) we first consider the Itô equations (3.1) and (3.2) in which

\[
\zeta_t^{\alpha, \zeta} = x_t^{\alpha, x}, \quad \zeta_t^{\alpha, \zeta}(\epsilon) = y_t^{\alpha, x + \epsilon(\epsilon)},
\]

Notice that

\[
\| \kappa_t(y, \epsilon) - \kappa_t(x) \| = \| \sqrt{1 + 2 \epsilon r_t} \sigma(y) e^{\epsilon P_t} - \sigma(x) \| \\
\leq \sqrt{1 + 2 \epsilon r_t} - 1 \| \sigma(y) e^{\epsilon P_t} \| + \| \sigma(y) \| \| e^{\epsilon P_t} - I_{d_1 \times d_1} \| \\
+ \| \sigma(y) - \sigma(x) \| \\
\leq 2 \epsilon r_t |K_0 + K_0 e e^{\epsilon P_t} + K_0| |y - x| \\
\leq M |y - x| + \epsilon M_t,
\]

where \( M = K_0, M_t^\alpha = N(K, K_0)(|\zeta_t^{x, \xi}| + (K_0)^2 \wedge 1) \). Applying Lemma 3.1(2), we have

\[
\sup_{\alpha \in \mathbb{A}} E^{\alpha} \sup_{t \leq \gamma \wedge T} |y_t^{\alpha, x + \epsilon(\epsilon)}(\epsilon) - x_t^{\alpha, x}|^p \leq \begin{cases} 
\epsilon^{p} e^{NT}(|\xi|^p + (2p - 1) \sup_{\alpha \in \mathbb{A}} E^{\alpha} \int_0^{\gamma \wedge T} M_t^p \, dt) \text{ if } p \geq 2 \\
\epsilon^{p} e^{NT} \frac{4 - p}{2 - p} (|\xi|^p + 3\epsilon^2 (\sup_{\alpha \in \mathbb{A}} E^{\alpha} \int_0^{\gamma \wedge T} M_t^2 \, dt)^\frac{p}{2}) \text{ if } p < 2.
\end{cases}
\]

Due to (3.20) and (3.19), we have

\[
\sup_{|0, \epsilon_0|} \sup_{\alpha \in \mathbb{A}} E^{\alpha} \int_0^{\gamma(\epsilon) \wedge T} M_t^{2p} \, dt < \infty,
\]

which completes the proof of (3.21).

Next, we first consider the Itô equations (3.1) and (3.2) in which

\[
\zeta_t^{x, \xi} = \xi_t^{x, \xi}, \quad \zeta_t^{x, \xi}(\epsilon) = \xi_t^{x, \xi}(\epsilon) := \frac{y_t^{\alpha, x + \epsilon(\epsilon)}(\epsilon) - x_t^{\alpha, x}}{\epsilon}.
\]

Observe that

\[
\frac{\| \sigma(y_t(\epsilon)) - \sigma(x_t) \|}{\epsilon} = \| \sigma(\xi_t(\epsilon)) (y_t(\epsilon)) - \sigma(\xi_t) (x_t) \|.
\]
\[
= \|\sigma(\xi_t(\epsilon)) (y_t(\epsilon)) - \sigma(\xi_t(\epsilon)) (x_t)\| + \|\sigma(\xi_t(\epsilon)) (x_t) - \sigma(\xi_t) (x_t)\|
\leq K_0 \frac{|y_t(\epsilon) - x_t|^2}{\epsilon} + K_0 |\xi_t(\epsilon) - \xi_t|,
\]
\[
\frac{|b(y_t(\epsilon)) - b(x_t)|}{\epsilon} - b(\xi_t) (x_t) \leq K_0 \frac{|y_t(\epsilon) - x_t|^2}{\epsilon} + K_0 |\xi_t(\epsilon) - \xi_t|,
\]
\[
|\sqrt{1 + 2er_t} - 1| = r_t (\frac{2}{\sqrt{1 + 2er_t} + 1} - 1) = |r_t (\frac{-2er_t}{1 + \sqrt{1 + 2er_t}^2})| \leq 2|r_t|^2,
\]
\[
\|e^{\epsilon P_t} - 1 - P_t\| = \frac{\epsilon}{2} \|P_t^2 e^{\epsilon P_t}\| \leq \frac{\epsilon}{2} \|P_t\|^2.
\]

The equation (3.22) can be proved by mimicking the proof of (3.21). \( \square \)

**Theorem 3.2.** Given constants \( p \in (0, \infty), p' \in [0, p), T \in [1, \infty), x \in D, \xi \in \mathbb{R}^d, \eta \in \mathbb{R}^d. \) Suppose (3.14) is satisfied. Assume that there exists a constant \( K \in [1, \infty) \) and for any \( \alpha \in \mathfrak{A}, \) an adapted nonnegative process \( K_\alpha^t, \) such that

(3.23) \[ |\tilde{\eta}\tilde{\alpha} + \tilde{\pi}\tilde{\alpha} + \tilde{P}\tilde{\alpha}| + |\tilde{\tau}\tilde{\alpha} + \tilde{\eta}\tilde{\alpha}^2 + |P\tilde{\alpha}^2| \leq K(|\eta^\alpha| + |\xi^\alpha|^2) + K_\alpha^t, \forall \alpha. \]

1. Given stopping times \( \gamma^\alpha \leq \tau^\alpha, x, \alpha \in \mathfrak{A}, \) if (3.18) holds, then we have (3.19) and

(3.24) \[ \sup_{\alpha \in \mathfrak{A}} E^\alpha_{\eta^\alpha} \sup_{t \leq \gamma^\wedge T} |\eta_t|^p < \infty. \]

2. Let the constant \( \epsilon_0 \) be sufficiently small so that \( B(x, \epsilon_0|\xi) \subset D. \) For any \( \epsilon \in [0, \epsilon_0], \) let

\[ x(\epsilon) = x + \epsilon \xi + \frac{\epsilon^2}{2} \eta. \]

If (3.20) holds for given stopping times \( \gamma^\alpha_2 (\epsilon) \) satisfying

\[ \gamma^\alpha_2 (\epsilon) \leq \tau^\alpha_2, x \wedge \tau^\alpha_2, x (\epsilon), \alpha \in \mathfrak{A}, \]

then we have

(3.25) \[ \lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha_{\eta^\alpha} \sup_{t \leq \gamma^\wedge T} \left| z_t^\alpha, x(\epsilon) - x_t^\alpha, x \right|^p e^{\epsilon p'} = 0, \]

(3.26) \[ \lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha_{\eta^\alpha} \sup_{t \leq \gamma^\wedge (-\epsilon) T} \left| z_t^\alpha, x(-\epsilon) - x_t^\alpha, x \right|^p e^{\epsilon p'} = 0, \]

(3.27) \[ \lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{A}} E^\alpha_{\eta^\alpha} \sup_{t \leq \gamma^\wedge T} \left| z_t^\alpha, x(\epsilon) - x_t^\alpha, x \right|^p \epsilon = 0. \]

If (3.22) holds for given stopping times \( \gamma^\alpha_3 (\epsilon) \) satisfying

\[ \gamma^\alpha_3 (\epsilon) \leq \tau^\alpha_2, x \wedge \tau^\alpha_2, x (\epsilon) \wedge \tau^\alpha_2, x (-\epsilon) \leq \tau^\alpha_2, x (-\epsilon), \alpha \in \mathfrak{A}, \]
then we have

\[ \limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathbb{N}} \sup_{t \leq \gamma T} \left| \frac{z_t^{\alpha}(\epsilon)}{\epsilon} - 2x_t^{\alpha,x} + z_t^{\alpha}(\epsilon) - \eta_t^{\alpha,\eta} \right|^p = 0. \]

**Proof.** Again, we drop superscripts \( \alpha, \alpha_t, \) etc., when this will cause no confusion.

The inequality (3.24) can be proved by observing that (3.23) and (3.18) imply that

\[ \sup_{\alpha \in \mathbb{N}} E \sup_{t \leq \gamma T} |\xi_t|^{2p} < \infty \]

and then mimicking the proof of (3.19).

The equations (3.25) and (3.27) are obtained by repeating the proof of (3.24) and (3.22). The equation (3.26) is obvious once we get (3.25).

To proof (3.28), we observe that, for example,

\[ \sigma(z_t(\epsilon)) - 2\sigma(x_t) + \sigma(z_t(-\epsilon)) \]

\[ \frac{1}{\epsilon^2} \left[ \sigma(z_t(\epsilon) - x_t) - \sigma(z_t(-\epsilon) - x_t) \right] \left( \bar{z}_t(\epsilon) \right) \]

\[ + \sigma(z_t(\epsilon) - x_t) - \sigma(z_t(-\epsilon) - x_t) \left( \bar{z}_t(-\epsilon) \right) \]

\[ = \sigma(\eta_t(\epsilon)) + \frac{1}{2} \left[ \sigma(\xi_t(\epsilon)) \left( \bar{z}_t(\epsilon) \right) + \sigma(\xi_t(-\epsilon)) \left( \bar{z}_t(-\epsilon) \right) \right], \]

where

\[ \eta_t(\epsilon) = \frac{z_t(\epsilon) - 2x_t + z_t(-\epsilon)}{\epsilon}, \quad \xi_t(\epsilon) = \frac{z_t(\epsilon) - x_t}{\epsilon}, \]

\( \bar{z}_t(\epsilon) \) is a point on the straight line segment with endpoints \( x_t \) and \( z_t(\epsilon) \), and \( \bar{z}_t(-\epsilon) \) is a point on the straight line segment with endpoints \( x_t \) and \( z_t(-\epsilon) \).

It follows that

\[ \left| \frac{\sigma(z_t(\epsilon)) - 2\sigma(x_t) + \sigma(z_t(-\epsilon))}{\epsilon} - \sigma(\eta_t)(x_t) - \sigma(\xi_t)(x_t) \right| \]

\[ \leq K_0 \left| z_t(\epsilon) - 2x_t + z_t(-\epsilon) \right| \left( \frac{1}{\epsilon^2} \right) - \eta_t | + K_0 \left| z_t(\epsilon) - x_t \right|^3 \left( \frac{1}{\epsilon^2} \right) + + K_0 \left| z_t(-\epsilon) - x_t \right|^3 \left( \frac{1}{\epsilon^2} \right) + K_0 \left| \frac{\bar{z}_t(\epsilon) - x_t}{\epsilon} \right|^2 + K_0 \left| \frac{\bar{z}_t(-\epsilon) - x_t}{\epsilon} \right| - \xi_t |^2. \]

It remains to mimic the proof of (3.22). \( \square \)

We end up this section by showing a convergence result about the stopping times which will be applied in the next section.

**Theorem 3.3.** Let \( \delta \) be a positive constant such that \( D_\delta = \{ x \in D : \psi > \delta \} \) is nonempty, and \( \delta_1, \delta_2 \) be positive constants satisfying \( \delta_1 < \delta_2 \). Let \( D_{\delta_1} = \{ x \in D : \delta_1 < \psi < \delta_2 \} \). Then for any \( x \in D \), if (3.27) holds with

\[ \gamma^\alpha(\epsilon) = t_D^\alpha \wedge t_D^\alpha + \epsilon \xi(\epsilon), \]

where \( t_D^\alpha \) is the first time when the process hits \( D_\alpha \), and \( \xi(\epsilon) \) is a standard normal random variable.
for $p = 1, p' = 0$ and $\forall T \in [1, \infty)$, then we have

$$\lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{S}} E(\tau_D^{\alpha,x} - \tau_D^{\alpha,x} \wedge \tau_D^{\alpha,x+\epsilon}(\epsilon)) = 0.$$  

(3.29)

For any $x \in D$, if (3.25) and (3.26) hold with

$$\gamma^\alpha(\epsilon) = \tau_D^{\alpha,x} \wedge \tau_D^{\alpha,x}(\epsilon)$$

and

$$\gamma^\alpha(-\epsilon) = \tau_D^{\alpha,x} \wedge \tau_D^{\alpha,x}(-\epsilon),$$

respectively, for $p = 1, p' = 0$ and $\forall T \in [1, \infty)$, then we have

$$\lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{S}} E(\tau_D^{\alpha,x} - \tau_D^{\alpha,x} \wedge \tau_D^{\alpha,x}(\epsilon) \wedge \tau_D^{\alpha,x}(-\epsilon)(\epsilon)) = 0.$$  

(3.30)

The statement still holds when replacing $D$ by $D_\delta$ or $D_\delta^2$, provided that $\delta_2$ is sufficiently small.

**Proof.** We drop the subscript $D$ and the argument $\epsilon$ for simplicity of notation. Notice that, for any $\alpha \in \mathfrak{S}$,

$$E(\tau^{\alpha,x} - \tau^\alpha) = E \int_{\tau^\alpha}^{\tau^{\alpha,x}} 1 dt$$

$$\leq - E \int_{\tau^\alpha}^{\tau^{\alpha,x}} L^\alpha \psi(x_t^{\alpha,x}) dt$$

$$= - E \left( \psi(x_{\gamma^{\alpha,x}}) - \psi(x_{\gamma^\alpha}) \right) I_{\gamma^{\alpha,x} < \tau^{\alpha,x}}$$

$$= E \psi(x_{\tau^{\alpha,x+\epsilon}}) I_{\tau^{\alpha,x+\epsilon} < \tau^{\alpha,x}}$$

$$\leq E \left( \psi(x_{\tau^{\alpha,x+\epsilon}}) - \psi(y_{\tau^{\alpha,x+\epsilon}}) \right) I_{\tau^{\alpha,x+\epsilon} < \tau^{\alpha,x} \leq T} + 2K_0 P_x(\tau > T).$$

Due to (3.21), we have

$$\lim_{\epsilon \downarrow 0} \left( \sup_{\alpha} E(\psi(x_{\tau^{\alpha,x+\epsilon}}) - \psi(y_{\tau^{\alpha,x+\epsilon}})) I_{\tau^{\alpha,x+\epsilon} < \tau^{\alpha,x} \leq T}) \right)$$

$$\leq \sup_{D} |\psi| \lim_{\epsilon \downarrow 0} \left( \sup_{\alpha} E \left| x_{\tau^{\alpha,x+\epsilon}} - y_{\tau^{\alpha,x+\epsilon}} \right| I_{\tau^{\alpha,x+\epsilon} < \tau^{\alpha,x} \leq T} \right)$$

$$= 0.$$

Also, notice that for any $\alpha \in \mathfrak{S}, T \in [1, \infty)$,

$$P_x(\tau > T) \leq \frac{1}{T} E_x^\alpha \tau \leq \frac{1}{T} E_x^\alpha \int_{0}^{\tau} (-L^\alpha \psi(x_t)) dt$$

$$= \frac{1}{T} \left( \psi(x) - \psi(x_{\tau^{\alpha,x}}) \right) \leq \frac{K_0}{T}.$$  

It turns out that

$$\lim_{\epsilon \downarrow 0} \sup_{\alpha \in \mathfrak{S}} E(\tau_D^{\alpha,x} - \tau_D^{\alpha,x} \wedge \tau_D^{\alpha,x+\epsilon}(\epsilon)) \leq \frac{2K_0^2}{T} \to 0, \text{ as } T \uparrow \infty.$$  

To prove (3.20), we just need to notice that for any stopping times $\tau, \gamma_1, \gamma_2$

$$\tau - \tau \wedge \gamma_1 \wedge \gamma_2 = (\tau - \tau \wedge \gamma_1) I_{\gamma_1 < \gamma_2} + (\tau - \tau \wedge \gamma_2) I_{\gamma_1 \geq \gamma_2}.$$
By noticing that
\[ \psi - \delta = 0 \text{ on } D_\delta, \quad \psi - \delta > 0, \text{ sup}_{\alpha \in \mathbb{A}} L^\alpha(\psi - \delta) = \text{sup}_{\alpha \in \mathbb{A}} L^\alpha \psi \leq -1 \text{ in } D_\delta, \]
we see that the statement is true in the subdomain \( D_\delta \).

Similarly, notice that
\[ (\psi - \delta_1)(\delta_2 - \psi) = 0 \text{ on } D_{\delta_1}^{\delta_2}, \quad (\psi - \delta_1)(\delta_2 - \psi) > 0 \text{ in } D_{\delta_1}^{\delta_2}, \]
\[ L^\alpha((\psi - \delta_1)(\delta_2 - \psi)) = (\delta_1 + \delta_2 - 2\psi)L^\alpha \psi - 2(a^\alpha \psi_x, \psi_x) \]
\[ \leq (\delta_1 + \delta_2)[L^\alpha \psi] - 2|\psi_x^* \sigma^\alpha|^2 \text{ in } D_{\delta_1}^{\delta_2}, \forall \alpha \in \mathbb{A}. \]

On \( \partial D \) it holds that \( \psi_x = |\psi_x|n \), where \( n(x) \) is the unit inward normal vector at \( x \in \partial D \). So due to (2.9) and the compactness of \( \partial D \),
\[ |\psi_x^* \sigma^\alpha|^2 = 2|\psi_x|^2(a^\alpha n, n) \geq 2|\psi_x|^2 \delta_0 \geq 2\delta_0' \text{ on } \partial D, \]
where \( \delta_0' \) is a positive constant. By continuity
\[ |\psi_x^* \sigma^\alpha|^2 \geq \delta_0' \text{ in } D_{\delta_1}^{\delta_2}, \]
if \( \delta_1 \) and \( \delta_2 \) are sufficiently small. It turns out that
\[ \text{sup}_{\alpha \in \mathbb{A}} L^\alpha \frac{(\psi - \delta_1)(\delta_2 - \psi)}{\delta_0} \leq -1, \]
when \( \delta_1 \) and \( \delta_2 \) are sufficiently small. So the statement is still true in the subdomain \( D_{\delta_1}^{\delta_2} \) when \( \delta_1, \delta_2 \) are sufficiently small.

□

4. Proof of Theorem 2.1

Before proving the main theorem, we state two remarks and one lemma. Remarks 4.1 and 4.2 are about two reductions of the problem, and Lemma 4.3 will be used when estimating the second derivatives. They are nonlinear counterparts of Remarks 3.2 and 3.3 and Lemma 3.2 in [12], and there is no essential change when extending them from linear case to nonlinear case.

**Remark 4.1.** Without loss of generality, we may assume that \( c^\alpha \geq 1, \forall \alpha \in \mathbb{A} \), and replace condition (2.10) by
\[ (\sigma^\alpha(y)) \leq c^\alpha(x) - 1 + M^\alpha(x)(a^\alpha(x)y, y). \]

**Remark 4.2.** Without loss of generality, we may assume that \( v \in C^1(D) \) and \( f^\alpha, g \in C^1(D) \) when investigating first derivatives of \( v \), and \( v \in C^2(D) \) and \( f^\alpha, g \in C^2(D) \) when investigating second derivatives of \( v \).

**Lemma 4.1.** If \( f^\alpha, g \in C^2(\bar{D}) \), and \( v \in C^1(\bar{D}) \), then for any \( y \in \partial D \) we have
\[ |v(y)| \leq K(|g|_{2,D} + \sup_{\alpha \in \mathbb{A}} |f^\alpha|_{0,D}), \]
where $n$ is the unit inward normal on $\partial D$ and the constant $K$ depends only on $K_0$.

Let $\delta$ and $\lambda$ be constants satisfying $0 < \delta < \lambda^2 < \lambda < 1$ and that the three sets defined below are nonempty:

$$D_\delta := \{ x \in D : \delta < \psi(x) \}$$
$$D_\lambda^\alpha := \{ x \in D : \delta < \psi(x) < \lambda \}$$
$$D_{\lambda^2} := \{ x \in D : \lambda^2 < \psi(x) \}$$

For each $\alpha \in \mathfrak{A}$, we use the same quasiderivatives and barrier functions constructed in [12]. Their properties are collected in the following two lemmas.

Lemma 4.2. In $D_\lambda^\alpha$, introduce

$$\varphi(x) = \lambda^2 + \psi(1 - \frac{1}{4\lambda}\psi), \quad B_1(x, \xi) = [\lambda + \sqrt{\psi(1 + \sqrt{\psi})}]|\xi|^2 + K_1\varphi^\alpha\frac{\psi^2(\xi)}{\psi},$$

where $K_1 \in [1, \infty)$ is a constant only depending on $K_0$.

For each $\alpha$, we define the first and second quasiderivatives by (3.11) and (3.12), in which

$$r(x, \xi) := r(x, \xi) = \frac{\psi(\xi)}{\psi}, \quad r_t := r(x_t, \xi_t),$$

with $\rho(x, \xi) := -\frac{1}{\Upsilon} \sum_{k=1}^{d_1} \psi(\sigma_k)(\psi(\sigma_k))(\xi), \quad \Upsilon := \sum_{k=1}^{d_1} \psi^2(\sigma_k);$

$$\hat{r}(x, \xi) := \frac{\psi^2(\xi)}{\psi^2}, \quad \hat{r}_t := \hat{r}(x_t, \xi_t);$$

$$\pi^k(x, \xi) := \frac{2\psi(\sigma_k)(\psi(\xi))}{\varphi(\psi)}, \quad k = 1, ..., d_1, \quad \pi_t := \pi(x_t, \xi_t);$$

$$P^{ik}(x, \xi) := \frac{1}{\Upsilon} \left[ (\psi(\sigma_k)(\psi(\sigma_i))(\xi) - \psi(\sigma_i)(\psi(\sigma_k))(\xi)) \right], \quad i, k = 1, ..., d_1, \quad P_t := P(x_t, \xi_t);$$

$$\hat{\pi}_t^k = \hat{P}_t^{ik} = 0, \quad \forall i, k = 1, ..., d_1, \forall t \in [0, \infty).$$

where we drop the superscript $\alpha$ or $\alpha_i$ without confusion. Then (3.19), (3.21), (3.22), (3.23), (3.24), (3.25), (3.26), (3.27) and (3.28) all hold for any constants $p \in (0, \infty), p' \in [0, p), T \in [1, \infty), x \in D_\delta^\alpha, \xi, \eta \in \mathbb{R}^d$ and stopping times

$$\gamma^\alpha \leq \tau^{\alpha, x}_{D_\delta^\alpha}, \gamma^\alpha(\epsilon) \leq \tau^{\alpha, x}_{D_\delta^\alpha} \wedge \tau^{\alpha, x+\epsilon}(\epsilon), \gamma^\alpha_2(\epsilon) \leq \tau^{\alpha, x}_{D_\delta^\alpha} \wedge \tau^{\alpha, x}(\epsilon),$$

$$\gamma^\alpha_3(\epsilon) \leq \tau^{\alpha, x}_{D_\delta^\alpha} \wedge \tau^{\alpha, x}(\epsilon) \wedge \tau^{\alpha, x-\epsilon}(-\epsilon),$$

where $x(\epsilon) = x + \epsilon \xi + \frac{\epsilon^2}{2\Upsilon} \eta$.

When $\lambda$ is sufficiently small, for $x \in D_\alpha^\lambda, \xi \in \mathbb{R}^d$ and $\eta = 0$, we have
(1) For each $\alpha \in \mathfrak{A}$, $B_1(x_t^{\alpha,x}, \xi_t^{\alpha,\xi})$ and $\sqrt{B_1(x_t^{\alpha,x}, \xi_t^{\alpha,\xi})}$ are local supermartingales on $[0, \tau_1^\delta]$, where $\tau_1^\delta = \tau_{D,\delta}^\alpha$;

(2) $\sup_{\alpha \in \mathfrak{A}} E_{x,\xi}^\alpha \int_0^{\tau_1^\delta} |\xi_t|^2 + \frac{\psi_0^2(t)}{\psi_2^2} dt \leq NB_1(x, \xi)$;

(3) $\sup_{\alpha \in \mathfrak{A}} \sup_{t \leq \tau_1^\delta} |\xi_t|^2 \leq NB_1(x, \xi)$;

(4) $\sup_{\alpha \in \mathfrak{A}} E_{0}^\alpha |\eta_{t_1}| \leq \sup_{\alpha \in \mathfrak{A}} E_{0}^\alpha \sup_{t \leq \tau_1^\delta} |\eta_t| \leq NB_1(x, \xi)$;

(5) $\sup_{\alpha \in \mathfrak{A}} E_{0}^\alpha \left( \int_0^{\tau_1^\delta} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq NB_1(x, \xi)$;

where $N$ is a constant depending on $K_0$ and $\epsilon$.

**Proof.** Notice that $\sup_{\alpha \in \mathfrak{A}} |Y_0|_{[0, D_1]}$ is bounded from below by a positive constant due to (2.9), so conditions (3.17) and (3.23) hold with $K_1^\alpha = 0$.

The properties (1)-(5) are nothing but Lemma 3.5 in [12] because the constant $N$ there doesn’t depend on $\alpha$. \qed

**Lemma 4.3.** In $D_{\lambda,2}$, introduce

$$B_2(x, \xi) = \lambda^2 |\xi|^2.$$

For each $\alpha \in \mathfrak{A}$, we define the first and second quasiderivatives by (3.11) and (3.12), in which

$$r(x, y) := \rho(x, y), \quad r_t := r(x_t, \xi_t), \quad \hat{r}_t := r(x_t, \eta_t),$$

$$\pi(x, y) := \frac{M(x)}{2} \sigma^*(x) y, \quad \pi_t := \pi(x_t, \xi_t), \quad \hat{\pi}_t := \pi(x_t, \eta_t),$$

$$P(x, y) := Q(x, y), \quad P_t := P(x_t, \xi_t), \quad \hat{P}_t := P(x_t, \eta_t),$$

where $\rho(x), M(x)$ and $Q(x, y)$ are defined in the statement of the main theorem and satisfy (2.10), and again, we drop the superscript $\alpha$ or $\alpha_t$ without confusion. Then (3.19), (3.21), (3.22), (3.24), (3.25), (3.26), (3.27) and (3.28) hold for any constants $p \in (0, \infty), p' \in [0, p), T \in [1, \infty), x \in D_\delta, \xi, \zeta \in \mathbb{R}^d$ and stopping times

$$\gamma_1^\alpha \leq t_{D,2}^\alpha, \quad \gamma_2^\alpha (\epsilon) \leq t_{D,2}^\alpha \wedge \frac{\alpha x + \epsilon}{p} \gamma_2^\alpha (\epsilon) \leq t_{D,2}^\alpha \wedge \frac{\alpha x - \epsilon + \frac{2}{p} \eta \epsilon}{p} \gamma_2^\alpha (\epsilon),$$

$$\gamma_3^\alpha (\epsilon) \leq t_{D,2}^\alpha \wedge \frac{\alpha x + \epsilon}{p} \gamma_3^\alpha (\epsilon) \wedge t_{D,2}^\alpha \wedge \frac{\alpha x - \epsilon + \frac{2}{p} \eta \epsilon}{p} \gamma_3^\alpha (\epsilon).$$

where $x(\epsilon) = x + \epsilon \xi + \frac{\epsilon^2}{2} \eta$.

Furthermore, for $x \in D_{\lambda,2}, \xi \in \mathbb{R}^d$ and $\eta = 0$, we have

(1) $e^{-\phi_{t_1}^\alpha} B_2(x_t^{\alpha,x}, \xi_t^{\alpha,\xi})$ and $\sqrt{e^{-\phi_{t_1}^\alpha} B_2(x_t^{\alpha,x}, \xi_t^{\alpha,\xi})}$ are local supermartingales on $[0, \tau_2)$, where $\tau_2 = \tau_{D,2}^\alpha$;

(2) $\sup_{\alpha \in \mathfrak{A}} E_{x,\xi}^\alpha \int_0^{\tau_2} e^{-\phi_t^\alpha} |\xi_t|^2 dt \leq NB_2(x, \xi)$.
(3) \(\sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,\xi} \sup_{t \leq \tau_2} e^{-\phi_t} |\xi_t|^2 \leq NB_2(x, \xi)\)

(4) \(\sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,0} e^{-\phi_{\tau_2}} |\eta_{\tau_2}| \leq \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,0} \sup_{t \leq \tau_2} e^{-\phi_t} |\eta_t| \leq NB_2(x, \xi)\)

(5) \(\sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,0} \left( \int_0^{\tau_2} e^{-2\phi_t} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq NB_2(x, \xi)\)

(6) The above inequalities are still all true if we replace \(\phi^\alpha_{t,x}\) by \(\phi^\alpha_{t,x} - \frac{1}{2} t\). More precisely, we have

\[
\sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,\xi} \int_0^{\tau_2} e^{-\phi_t + \frac{1}{2} t} |\xi_t|^2 dt \leq NB_2(x, \xi), \sup_{\alpha \in \mathfrak{A}} E^\alpha_{\xi,0} \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2} t} |\xi_t|^2 \leq NB_2(x, \xi)
\]

\[
\sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,0} \left( \int_0^{\tau_2} e^{-2\phi_t + t} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq NB_2(x, \xi), \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x,0} \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2} t} |\eta_t| \leq NB_2(x, \xi)
\]

where \(N\) is constant depending on \(K_0\) and \(\lambda\).

Proof. The same as Lemma 4.2.

We split the proof of Theorem 2.1 into three parts. Note that in the proof, for simplicity of notation, we may drop the superscripts such as \(\alpha\) when it will cause no confusion.

Proof of (2.11). First, we fix an \(x \in D^\lambda_3\) and a \(\xi \in \mathbb{R}^d \setminus \{0\}\). Choose \(\epsilon_0 > 0\) sufficiently small, so that \(B(x, \epsilon_0 |\xi|) := \{ y : |y - x| \leq \epsilon_0 |\xi| \} \subset D^\lambda_3\). For any \(\epsilon \in (0, \epsilon_0)\), by Bellman’s principle (Theorem 1.1 in 11, in which \(Q\) is defined by \(D \times [-1, T + 1]\), where \(T\) is an arbitrary positive constant), we have, with the stopping time \(\gamma^\alpha = \tau^\alpha_{D^\lambda_3} \wedge \tau^\alpha_{\partial D^\lambda_3} \wedge T\),

\[
\frac{v(x + \epsilon \xi) - v(x)}{\epsilon} = \frac{1}{\epsilon} \left\{ \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x+\epsilon \xi} \left[ v(x_{\gamma}) e^{-\phi_{x}} + \int_0^{\gamma} f^\alpha_s(x_s) e^{-\phi_s} ds \right] \right.
\]

\[
- \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x} \left[ v(x_{\gamma}) e^{-\phi_x} + \int_0^{\gamma} f^\alpha_s(x_s) e^{-\phi_s} ds \right].
\]

By Theorem 2.1 in 3 and Lemmas 2.1 and 2.2 in 4,

\[
\sup_{\alpha \in \mathfrak{A}} E^\alpha_{x+\epsilon \xi} \left[ v(x_{\gamma}) e^{-\phi_{x}} + \int_0^{\gamma} f^\alpha_s(x_s) e^{-\phi_s} ds \right]
\]

\[
= \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x+\epsilon \xi} \left[ v(y_{\gamma}(\epsilon)) p^\alpha_{\gamma}(\epsilon) e^{-\phi_{x}} + \int_0^{\gamma} (1 + 2\epsilon r_s) f^\alpha_s(y_s(\epsilon)) p_s(\epsilon) e^{-\phi_s(\epsilon)} ds \right],
\]

in which \(y^\alpha_{t,Y}(\epsilon)\) is the solution to the Itô equation 3.9,

\[
\dot{\phi}^\alpha_{t,Y}(\epsilon) := \int_0^t (1 + 2\epsilon r_s^\alpha) e^\alpha_s(y^\alpha_{s,Y}(\epsilon)) ds,
\]

and

\[
(4.3) \quad p^\alpha_t(\epsilon) := \exp \left( \int_0^t \epsilon \pi^\alpha_s dw_s - \frac{1}{2} \int_0^t |\epsilon \pi^\alpha_s|^2 ds \right).
\]

with \(\alpha \in \mathfrak{A}, r^\alpha_s, \pi^\alpha_s, P^\alpha_s\) defined in Lemma 4.2.
Let
\[ q_t^\alpha(\epsilon) = \int_0^t (1 + 2\epsilon t^\alpha) f^\alpha(y_s(\epsilon)) p_s(\epsilon) e^{-\phi_s(\epsilon)} ds, \]
\[ \bar{y}_t^\alpha y = (y_t^\alpha y(\epsilon), -\phi_t^\alpha(\epsilon), p_t^\alpha(\epsilon), q_t^\alpha(\epsilon)), \]
\[ \bar{x}_t^\alpha x = (x_t^\alpha x, -\phi_t^\alpha(0), p_t^\alpha(0), q_t^\alpha(0)). \]

For any \( \bar{x} = (x, x^{d+1}, x^{d+2}, x^{d+3}) \in D \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \), introduce
\[ (4.4) \quad V(\bar{x}) = v(x) \exp(x^{d+1})x^{d+2} + x^{d+3}. \]

Then we have
\[ \frac{v(x + \epsilon \xi) - v(x)}{\epsilon} = \frac{1}{\epsilon} \left( \sup_{\alpha \in A} E^\alpha_{x+\epsilon \xi} V(\bar{y}_\gamma(\epsilon)) - \sup_{\alpha \in A} E^\alpha_{x} V(\bar{x}_\gamma) \right). \]

Since the difference of two supremums is less than the supremum of the differences, and the supremum of a sum is less than the sum of the supremums, we have
\[ \frac{v(x + \epsilon \xi) - v(x)}{\epsilon} \leq \sup_{\alpha \in A} \frac{E^\alpha_{x+\epsilon \xi} V(\bar{y}_\gamma(\epsilon)) - V(\bar{x}_\gamma)}{\epsilon} \]
\[ \leq \sup_{\alpha \in A} \frac{E^\alpha_{x+\epsilon \xi} V(\bar{y}_\gamma(\epsilon)) - V(\bar{x}_\gamma)}{\epsilon} - V(\bar{y}_\gamma) + \sup_{\alpha \in A} E^\alpha(\bar{x}_\gamma) \]
\[ := I_1(\epsilon, T) + I_2(\epsilon, T), \]

where
\[ (4.5) \quad \bar{\xi}_t^{\alpha, \xi} = (\xi_t^{\alpha, \xi}, \xi_t^{\alpha, d+1, \alpha}, \xi_t^{\alpha, d+2, \alpha}, \xi_t^{\alpha, d+3, \alpha}), \]

with
\[ \xi_t^{d+1, \alpha} := -\int_0^t \left[ C_{\xi_s^\alpha} (x_s^{\alpha, x}) + 2r_s^{\alpha} C_{\xi_s^\alpha} (x_s^{\alpha, x}) \right] ds, \]
\[ \xi_t^{d+2, \alpha} := \xi_t^{0, \alpha} = \int_0^t \pi_s^{\alpha} dw_s, \]
\[ \xi_t^{d+3, \alpha} := \int_0^t e^{-\phi_s^{\alpha, x}} \left[ f_{\xi_s^\alpha}^\alpha (x_s^{\alpha, x}) + (2r_s^{\alpha} + \xi_s^{d+1, \alpha} + \xi_s^{d+2, \alpha}) f_{\xi_s^\alpha}^\alpha (x_s^{\alpha, x}) \right] ds. \]

We claim that
\[ (4.6) \quad \lim_{\epsilon \downarrow 0} I_1(\epsilon, T) = 0. \]

To show it, bearing in mind that for any \( h^\alpha(x) \in C^1(\bar{D}_\delta) \), we have, for any \( x, y \in D_\delta \) and \( \xi \in \mathbb{R}^d, r \in \mathbb{R}^d \) and \( n \in \mathbb{N} \),
\[ (4.7) \quad \frac{|(1 + 2\epsilon r)h^\alpha(y) - h^\alpha(x)|}{\epsilon} = \frac{|h^\alpha(y - x) - h^\alpha(x)|}{\epsilon} \]
\[ = |h^\alpha_{(y - x)}(y) - h^\alpha_{(y)}(x)| + 2r(h^\alpha(y) - h^\alpha(x))| \]
\[ \leq |(h_x^\alpha(y^*) - h_x^\alpha(x), \frac{y - x}{\epsilon})| + |(h_x^\alpha(x), \frac{y - x}{\epsilon} - \xi)| + 2K_0(\epsilon r^2 + \frac{|y - x|^2}{\epsilon}) \]

where \( y^* \) is a point on the line segment with ending points \( x \) and \( y \).

First, by Lemma 4.2, for any contants \( p \) and \( p' \) satisfying \( 0 \leq p' < p < \infty \), we have

\[
\sup_{\alpha \in \mathbb{R}} E^\alpha \sup_{t \leq \gamma} |\xi_t|^p < \infty, \tag{4.8}
\]

\[
\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathbb{R}} \sup_{t \leq \gamma^\alpha} \frac{|y_t^{\alpha,x+\epsilon}(\tau) - x_t^{\alpha,x}|}{\epsilon^{p'}} = 0, \tag{4.9}
\]

\[
\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathbb{R}} \sup_{t \leq \gamma^\alpha} \frac{|y_t^{\alpha,x+\epsilon}(\tau) - x_t^{\alpha,x}|}{\epsilon} - \xi_t^{\alpha,\tau} = 0. \tag{4.10}
\]

Second, apply (4.7) to \( c^\alpha(x) \) we get

\[
\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathbb{R}} \sup_{t \leq \gamma} \left| \phi_t(0) - \phi_t(\epsilon) \right| - \xi_t^{d+1} = 0. \tag{4.11}
\]

Third, we notice that

\[
\frac{p_t(\epsilon) - p_t(0)}{\epsilon} = \frac{p_t(\epsilon) - 1}{\epsilon} = \int_0^\epsilon p_\alpha(\tau) \pi_\tau d\tau.
\]

It follows that

\[
E^\alpha \sup_{t \leq \gamma} \left| \frac{p_t(\epsilon) - 1}{\epsilon} - \xi_t^{d+2} \right|^p \leq N(p) E^\alpha \left( \int_0^\gamma \left( \frac{p_t(\epsilon) - 1}{\epsilon} \right)^2 |\pi_t|^2 dt \right)^{p/2} \leq \epsilon^p N(p) E^\alpha \left( \sup_{t \leq \gamma} \left| \frac{p_t(\epsilon) - 1}{\epsilon} \right|^2 + \int_0^\gamma |\pi_t|^2 dt \right) \leq \epsilon^p N(p) E^\alpha \left( \int_0^\gamma p_{t\epsilon}^2(\epsilon) |\pi_t|^2 dt + \int_0^\gamma |\pi_t|^{2p} dt \right). \]

Hence

\[
\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathbb{R}} \sup_{t \leq \gamma} \left| \frac{p_t(\epsilon) - p_t(0)}{\epsilon} - \xi_t^{d+2} \right|^p = 0. \tag{4.12}
\]

Fourth, bearing in mind that

\[
\left| \frac{f(\epsilon)g(\epsilon) - fg}{\epsilon} - f'g - fg' \right| \leq \left| \frac{f(\epsilon) - f}{\epsilon} \right| - f'||g(\epsilon)|| + \left| \frac{g(\epsilon) - g}{\epsilon} \right| - g'||f(\epsilon)| - g| \leq \left| \frac{f(\epsilon) - f}{\epsilon} \right| - f'||g(\epsilon)|| + \left| \frac{g(\epsilon) - g}{\epsilon} \right| - g'||f(\epsilon)| + \epsilon(|f'|^2 + \frac{|g(\epsilon) - g|^2}{\epsilon^2}).
\]

Therefore, to prove

\[
\limsup_{\epsilon \downarrow 0} E \sup_{\alpha \in \mathbb{R}} \sup_{t \leq \gamma} \left| \frac{q_t(\epsilon) - q_t(0)}{\epsilon} - \xi_t^{d+3} \right|^p = 0. \tag{4.13}
\]
it suffices to show that
\[
\limsup_{\epsilon \downarrow 0} E^\alpha \sup_{t \leq \gamma} \left( \frac{1 + 2\epsilon r_I}{\epsilon} f^{\alpha_t}(y_I(\epsilon)) \right) - f^{\alpha_t}(x_I) - f^{\alpha_t}(x_I) - 2r_I f^{\alpha_t}(x_I) \right)^p = 0,
\]
\[
\limsup_{\epsilon \downarrow 0} E^\alpha \sup_{t \leq \gamma} \left( \frac{e^{-\phi_t(\epsilon)} - e^{-\phi_t(0)}}{\epsilon} + \xi^{d+1} e^{-\phi_t(0)} \right)^p = 0.
\]
The first equation is true due to (4.7) with \( h^\alpha = f^\alpha \). The second one is true by a similar argument.

Finally, observe that for any \( \bar{x} = (x, x^{d+1}, x^{d+3}) \), \( \bar{y} = (y, y^{d+1}, y^{d+2}, y^{d+3}) \), \( \bar{\xi} = (\xi, \xi^{d+1}, \xi^{d+2}, \xi^{d+3}) \in D \times \mathbb{R}^+ \times \mathbb{R} \), we have
\[
V(\bar{y}) - V(\bar{x}) = v(y)e^{d+1}y^{d+2} - v(x)e^{d+1} + y^{d+3} - x^{d+3} - e^{d+1}v(\xi(x) + v(x)(\xi^{d+1} + \xi^{d+2}) - \xi^{d+3}.
\]
It is not hard to see (4.6) is true with (4.10), (4.11), (4.12) and (4.13) in hand.

To estimate \( I_2(\epsilon, T) \), we notice that \( V_{(\xi^{0,\alpha})}(x^{0,\alpha}) \) is exactly \( X^{\alpha}_r \) defined by (2.8) in [12], in which \( u \) is replaced by \( v \). More precisely,
\[
V_{(\xi^{0,\alpha})}(x^{0,\alpha}) = X^{\alpha}_r := e^{-\phi_t^{\alpha,x}} \left[ v(x^{0,\alpha}) + \xi^{d+1} + \xi^{d+1} \right]
\[
+ \int_0^t e^{-\phi_s^{\alpha,x}} \left[ f^{\alpha_s}(x^{0,\alpha}) + (2\gamma_s + \xi^{0,\alpha}) f^{\alpha_s}(x^{0,\alpha}) \right] ds,
\]
where
\[
\xi^{d+1} = \xi^{0,\alpha} + \xi^{d+1,\alpha}.
\]
It follows that
\[
I_2(\epsilon, T) = \sup_{\alpha \in \mathbb{A}} E^\alpha X_\gamma \leq \sup_{\alpha \in \mathbb{A}} E e^{-\phi_t^{\alpha,x}} v(x^{0,\alpha}) + \sup_{\alpha \in \mathbb{A}} E \left( X^{\alpha}_r - e^{-\phi_t^{\alpha,x}} v(x^{0,\alpha}) \right).
\]

It is proved in [12] in the proof of (3.5), that for each \( \alpha \),
\[
E \sup_{t \leq \gamma \leq \gamma \alpha \beta} \left( X^{\alpha}_r - e^{-\phi_t^{\alpha,x}} v(x^{0,\alpha}) \right) \leq N \sqrt{B_1(x, \xi)},
\]
where \( N \) is independent of \( \alpha \). So
\[
I_2(\epsilon, T) \leq \sup_{\alpha \in \mathbb{A}} E e^{-\phi_t^{\alpha,x}} v(x^{0,\alpha}) + N \sqrt{B_1(x, \xi)}.
\]

We next notice that
\[
\sup_{\alpha \in \mathbb{A}} E v(x^{0,\alpha}) = \sup_{\alpha \in \mathbb{A}} E \left( v(x^{0,\alpha}) \right) / B_1(x^{0,\alpha}, \xi^{0,\alpha}) \cdot \sqrt{B_1(x^{0,\alpha}, \xi^{0,\alpha})}
\]
\[
\leq \sup_{\alpha \in \mathbb{A}} E \left( \frac{v(x^{0,\alpha})}{B_1(x^{0,\alpha}, \xi^{0,\alpha})} - \frac{\frac{v(x^{0,\alpha})}{\xi^{0,\alpha}}}{\sqrt{B_1(x^{0,\alpha}, \xi^{0,\alpha})}} \right) \cdot \sqrt{B_1(x^{0,\alpha}, \xi^{0,\alpha})}.
\]
\[\begin{align*}
+ \sup_{\alpha \in \mathbb{A}} & \frac{v_\xi(x) \alpha_x (x_{\alpha,x}^\gamma, \xi_\gamma)}{\sqrt{B_1(x, \xi)} \sqrt{B_1(x_{\alpha,x}^\gamma, \xi_\gamma)}} \\
& \cdot \sqrt{B_1(x_{\alpha,x}^\gamma, \xi_\gamma)} \\
& := J_1(h, T) + J_2(h, T).
\end{align*}\]

Notice that

\[\frac{v_\xi(x)}{\sqrt{B_1(x, \xi)}} = \frac{v_\xi(x/|\xi|)}{\sqrt{B_1(x, \xi/|\xi|)}}\]

is a continuous function from \(D_\delta^\lambda \times S_1\) to \(\mathbb{R}\), where \(S_1\) is the unit sphere in \(\mathbb{R}^d\). By Weierstrass Approximation Theorem, there exists a polynomial \(W(x, \xi) : D_\delta^\lambda \times S_1 \to \mathbb{R}\), such that

\[\sup_{x \in D_\delta^\lambda, \xi \in S_1} \left| \frac{v_\xi(x)}{\sqrt{B_1(x, \xi)}} - W(x, \xi) \right| \leq 1.\]

It follows that

\[J_1(h, T) \leq \sup_{\alpha \in \mathbb{A}} E[W(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha) - W(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha)| \sqrt{B_1(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha)}\]

\[+ 2 \sup_{\alpha \in \mathbb{A}} E \sqrt{B_1(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha)}\]

\[\leq N \sup_{\alpha \in \mathbb{A}} E| x_{\alpha,x}^\gamma - x_{\alpha,x}^\gamma | \sqrt{B_1(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha)} + 2 \sqrt{B_1(x, \xi)}\]

\[\leq N \sqrt{B_1(x, \xi)} \sup_{\alpha \in \mathbb{A}} E| x_{\alpha,x}^\gamma - x_{\alpha,x}^\gamma |^2 + \sup_{\alpha \in \mathbb{A}} E B_1(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha)\]

\[+ 2 \sqrt{B_1(x, \xi)}\]

\[\leq N \sqrt{B_1(x, \xi)} \left( E|x_{\alpha,x}^\gamma - x_{\alpha,x}^\gamma |^2 + E|x_{\alpha,x}^\gamma - x_{\alpha,x}^\gamma |^2 \right)\]

\[+ 3 \sqrt{B_1(x, \xi)}\]

\[\leq N \sqrt{B_1(x, \xi)} \left( E|\gamma^\alpha - \tau_{D_\delta^\lambda} \wedge T| + E|\tau_{D_\delta^\lambda} \wedge T - \tau_{D_\delta^\lambda} | \right) + 3 \sqrt{B_1(x, \xi)}\]

Thus

\[\lim_{T \to \infty} \lim_{\epsilon \downarrow 0} J_1(\epsilon, T) \leq 3 \sqrt{B_1(x, \xi)}\]

Also, notice that

\[J_2(\epsilon, T) \leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \{0\}} \frac{v_\zeta(y)}{\sqrt{B_1(y, \zeta)}} \sup_{\alpha \in \mathbb{A}} E \sqrt{B_1(x_{\alpha,x}^\gamma, \xi_\gamma^\alpha)}\]

\[\leq \sup_{y \in \partial D_\delta^\lambda, \zeta \in \mathbb{R}^d \{0\}} \frac{v_\zeta(y)}{\sqrt{B_1(y, \zeta)}} \cdot \sqrt{B_1(x, \xi)}.\]
Hence,
\[
\lim_{T \uparrow \infty} \lim_{\epsilon \to 0} I_2(\epsilon, T) \leq \sup_{y \in \partial D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}} \frac{v(\xi)(y)}{\sqrt{B_1(y, \xi)}} \cdot \sqrt{B_1(x, \xi)} + N \sqrt{B_1(x, \xi)}.
\]

We conclude that
\[
\frac{v(\xi)(x)}{\sqrt{B_1(x, \xi)}} \leq \sup_{y \in \partial D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}} \frac{v(\xi)(y)}{\sqrt{B_1(y, \xi)}} + N, \ \forall x \in D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}.
\]

Notice that \(B_1(x, \xi) = B_1(x, -\xi)\). Replacing \(\xi\) by \(-\xi\), we have
\[
\frac{-v(\xi)(x)}{\sqrt{B_1(x, \xi)}} \leq \sup_{y \in \partial D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}} \frac{v(\xi)(y)}{\sqrt{B_1(y, \xi)}} + N, \ \forall x \in D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\},
\]
which implies that
\[
\frac{v(\xi)(x)}{\sqrt{B_1(x, \xi)}} \leq \sup_{y \in \partial D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}} \frac{v(\xi)(y)}{\sqrt{B_1(y, \xi)}} + N, \ \forall x \in D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}.
\]

Repeating the argument above in \(D^{\lambda_2}\), we have
\[
\frac{v(\xi)(x)}{\sqrt{B_2(x, \xi)}} \leq \sup_{y \in \partial D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}} \frac{v(\xi)(y)}{\sqrt{B_2(y, \xi)}} + N, \ \forall x \in D^{\lambda_2}, \xi \in \mathbb{R}^d \setminus \{0\}.
\]

The inequalities (4.14) and (4.15) are the same as (3.16) and (3.18) in [12]. So by repeating the argument after (3.18) in [12], we get
\[
v(\xi)(x) \leq N \left( |\xi| + \frac{|v(\xi)(x)|}{\psi_2(x)} \right), \quad \text{a.e. in } D.
\]

(2.11) is proved.

Proof of (2.12). The idea is the same as the first order case. Fix \(x \in D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}\) and sufficiently small positive \(\epsilon_0\), so that \(B(x, \epsilon_0|\xi|) \subset D^\lambda_0\). For each \(\alpha \in \mathfrak{A}\), let \(\gamma^\alpha := \tau^\alpha_{D^\lambda_0}(x + \epsilon \xi) \cap \tau^\alpha_{D^\lambda_0}(x) \cap \tau^\alpha_{D^\lambda_0}(x - \epsilon \xi) \cap T\), where \(T \in [1, \infty)\). We have
\[
\frac{v(x + \epsilon \xi) - 2v(x) + v(x - \epsilon \xi)}{\epsilon^2} = \frac{1}{\epsilon^2} \left\{ - \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x + \epsilon \xi} \left[ v(x_\gamma) e^{-\phi_\gamma} + \int_0^{\gamma} f^\alpha(x,s)e^{-\phi_\gamma} ds \right] \\
+ 2 \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x} \left[ v(x_\gamma) e^{-\phi_\gamma} + \int_0^{\gamma} f^\alpha(x,s)e^{-\phi_\gamma} ds \right] \\
- \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x - \epsilon \xi} \left[ v(x_\gamma) e^{-\phi_\gamma} + \int_0^{\gamma} f^\alpha(x,s)e^{-\phi_\gamma} ds \right] \right\} \\
= \frac{1}{\epsilon^2} \left\{ - \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x + \epsilon \xi} \left[ v(z_\gamma(\epsilon)) q_\gamma(\epsilon) e^{-\phi_\gamma(\epsilon)} + \int_0^{\gamma} f^\alpha(z,s)q_\gamma(\epsilon)e^{-\phi_\gamma(\epsilon)} ds \right] \\
+ 2 \sup_{\alpha \in \mathfrak{A}} E^\alpha_{x} \left[ v(x_\gamma) q_\gamma e^{-\phi_\gamma} + \int_0^{\gamma} f^\alpha(x,s)q_\gamma e^{-\phi_\gamma} ds \right] \right\}
\]
we get
\[ z_t^{\alpha,\xi}(\epsilon) = \int_0^t (1 + 2\epsilon r_s^\alpha + \epsilon^2 r_s^\alpha) e^{\alpha s}(z_s^{\alpha,\xi}(\epsilon)) ds, \]
and
\[ \tilde{p}_t^\alpha(\epsilon) := \exp \left( \int_0^t (\epsilon \pi_s^\alpha + \epsilon^2 \hat{\pi}_s^\alpha) dw_s - \frac{1}{2} \int_0^t \| \epsilon \pi_s^\alpha + \frac{\epsilon}{2} \hat{\pi}_s^\alpha \|^2 ds \right). \]

with \( \alpha \in \mathfrak{A}, r_s^\alpha, \pi_s^\alpha, P_s^\alpha, \hat{\pi}_s^\alpha, \hat{\pi}_s^\alpha, \tilde{p}_s^\alpha \) defined in lemma (4.2).

By introducing
\[ \hat{q}_t^\alpha(\epsilon) = \int_0^t (1 + 2\epsilon r_s^\alpha + \epsilon^2 r_s^\alpha) f^{\alpha s}(z_s^\alpha(\epsilon)) \hat{p}_s(\epsilon) e^{-\phi_s(\epsilon)} ds, \]
\[ \bar{z}_t^{\alpha,\xi}(\epsilon) = (\bar{z}_t^{\alpha,\xi}(\epsilon), -\hat{q}_t^\alpha(\epsilon), \tilde{p}_t^\alpha(\epsilon), \hat{q}_t^\alpha(\epsilon)), \]
\[ \bar{x}_t^{\alpha,\xi} = (x_t^{\alpha,\xi}, -\hat{q}_t^\alpha(0), \tilde{p}_t^\alpha(0), \hat{q}_t^\alpha(0)), \]
we get
\[ -v(x + \epsilon \xi) = 2v(x) + v(x - \epsilon \xi) \]
\[ = \frac{1}{\epsilon^2} \left( -\sup_{\alpha \in \mathfrak{A}} E_x^{\alpha} \mathbb{E}(\bar{z}_\gamma(\epsilon) + 2 \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha} \mathbb{E}(\bar{x}_\gamma) - \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha} \mathbb{E}(\bar{z}_\gamma(-\epsilon)) \right) \]
\[ \leq \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha} \mathbb{E}(\bar{z}_\gamma^{\alpha,x+\epsilon \xi}(\epsilon) + 2V(x^{\alpha,x}) - V(z^{\alpha,x-\epsilon \xi}(-\epsilon)) \]
\[ = \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha} \mathbb{E}(\bar{z}_\gamma^{\alpha,x+\epsilon \xi}(\epsilon) + 2V(x^{\alpha,x}) - V(z^{\alpha,x-\epsilon \xi}(\epsilon)) \]
\[ \leq \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha} \mathbb{E}(\bar{z}_\gamma^{\alpha,x+\epsilon \xi}(\epsilon) + 2V(x^{\alpha,x}) - V(z^{\alpha,x-\epsilon \xi}(\epsilon)) \]
\[ + \sup_{\alpha \in \mathfrak{A}} \left[ V(\bar{x}_\gamma^{\alpha,x})(\xi^{\alpha,x}) + V(\bar{x}_\gamma^{\alpha,x})(\tilde{x}_\gamma^{\alpha,x}) \right] \]
\[ := G_1(\epsilon, T) + G_2(\epsilon, T), \]
where \( V \) and \( \xi^{\alpha,x} \) are defined by (4.4) and (4.5), respectively, and
\[ \eta^{0,\eta} := (\eta_t^{0,\eta}, \eta_t^{d+1,\alpha}, \eta_t^{d+2,\alpha}, \eta_t^{d+3,\alpha}), \]
with
\[ \eta_t^{d+1,\alpha} := - \int_0^t c_{(x^{\alpha,x})(\xi^{\alpha,x})}(x^{\alpha,x}) + c_{(\eta_0^{\alpha,\eta})}(x^{\alpha,x}) + 4r_s^\alpha c_{(x^{\alpha,x})}(x^{\alpha,x}) + 2\rho_s^\alpha c(x^{\alpha,x}) ds, \]
\[ \eta_t^{d+2,\alpha} := \eta_t^{0,\alpha} = \left( \int_0^t \pi_s^\alpha dw_s \right)^2 - \int_0^t |\pi_s^\alpha|^2 ds + \int_0^t \hat{\pi}_s ds, \]
\[ \eta_t^{d+3,\alpha} := \int_0^t e^{-\phi_t^{d,\alpha}} \left[ f_{(x^{\alpha,b})}^{(x^{\alpha,b})}(x^{\alpha,b}) + f_{(\eta^{\alpha,b})}^{(\eta^{\alpha,b})}(x^{\alpha,b}) + (2\xi^{d+1,\alpha} + 4r^{d+1,\alpha})f_{(\xi^{d+1,\alpha})}^{(\xi^{d+1,\alpha})}(x^{\alpha,b}) \right] ds. \]

We first claim that
\[ \lim_{\epsilon \to 0} G_1(\epsilon, T) = 0. \]

The proof is similar as that of (4.6) with the help of the following two second-order counterparts.

First, if \( h^\alpha(x) \in C^2(D_\delta), \) then for any \( x, z, z' \in D_\delta, \xi, \eta \in \mathbb{R}^d, r, \hat{r} \in \mathbb{R} \) and \( n \in \mathbb{N}, \) we have
\[
\begin{align*}
&h^\alpha(z) - 2h^\alpha(x) + h^\alpha(z') \\
&= \frac{1}{\epsilon^2} \left[ \frac{1}{2} h^\alpha(z-x)(z-x) + h^\alpha(z)(z-x) \right] + \frac{1}{2} h^\alpha(z-x)(z-x) \left( z' \right) \\
&= \frac{1}{\epsilon^2} \left[ \frac{1}{2} h^\alpha(z-x)(z-x) + h^\alpha(z)(z-x) \right] + \frac{1}{2} h^\alpha(z-x)(z-x) \left( z' \right),
\end{align*}
\]

where \( z^* \) and \( z'^* \) are on the line segments \( \overline{xx} \) and \( \overline{xx'}, \) respectively. Hence,
\[
\begin{align*}
&\left| (1 + 2\epsilon \eta + \epsilon^2 \hat{r}) h^\alpha(z) - 2h^\alpha(x) + (1 - 2\epsilon \eta + \epsilon^2 \hat{r}) h^\alpha(z') \right| \\
&\leq \left| \frac{h^\alpha(z) - 2h^\alpha(x) + h^\alpha(z')}{\epsilon^2} \right| \\
&\quad + \frac{2|\eta||h^\alpha(z) - h^\alpha(z')|}{\epsilon} - 2h^\alpha(x) | + |\hat{r}| |h^\alpha(z) + h^\alpha(z') - 2h^\alpha(x)| \\
&\leq \left| \frac{h^\alpha(z) - 2h^\alpha(x) + h^\alpha(z')}{\epsilon^2} \right| \\
&\quad + \frac{2|\eta||h^\alpha(z) - h^\alpha(z')|}{\epsilon} - 2h^\alpha(x) | + |\hat{r}| |h^\alpha(z) + h^\alpha(z') - 2h^\alpha(x)|
\end{align*}
\]

Second, by noticing that
\[
\begin{align*}
\frac{\hat{p}_t(\epsilon) - 2\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{\epsilon^2} &= \int_0^t \left( \hat{p}_t(\epsilon) - \hat{p}_t(-\epsilon) \right) \pi_s + \frac{\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{2} \hat{p}_t(\epsilon) ds, \\
\eta_t^{d+2} &= \eta_t^0 = 2 \int_0^t \xi_s^0 \pi_s dw_s,
\end{align*}
\]

we have
\[
E_\alpha \sup_{t \leq T} \left( \frac{\hat{p}_t(\epsilon) - 2\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{\epsilon^2} - \eta_t^{d+2} \right)^p \\
\leq N(p) E_\alpha \left( \int_0^T \left( \frac{\hat{p}_t(\epsilon) - \hat{p}_t(-\epsilon)}{\epsilon} - 2\xi_t^0 \right)^2 \pi_t^2 + \left( \frac{\hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon)}{2} - 1 \right)^2 |\hat{p}_t|^2 dt \right)^{p/2}
\]
Therefore, is exactly $Y_t^\alpha$ defined by (2.9) in [12], in which $u$ is replaced by $v$, that is

$$V_{(\xi_t^0,0)}(x_t^{\alpha,x}) + V_{(\xi_t,\zeta_t)}(\xi_t^0,\zeta_t) = Y_t^\alpha$$

$$= e^{-\phi_{t,x}^{\alpha,x}} \left[ v_{(\xi_t^0,0)}(x_t^{\alpha,x}) + 2\xi_t^0 v_{(\xi_t^0,0)}(x_t^{\alpha,x}) + \eta_t^0 v(x_t^{\alpha,x}) \right] + \int_0^t e^{-\phi_{s,x}^{\alpha,x}} \left[ f_{(\xi_s^0,\zeta_s)}^{\alpha,s}(x_s^{\alpha,x}) + f_{(\xi_s^0,0)}^{\alpha,s}(x_s^{\alpha,x}) + (4r_s^\alpha + 2\xi_s^0) f_{(\xi_s^0,\zeta_s)}^{\alpha,s}(x_s^{\alpha,x}) \right] ds,$$

where

$$\eta_t^0 = \eta_t^d + 2\xi_t^d + 2\xi_t^{d+1} + (\xi_t^{d+1})^2 + \eta_t^{d+1}.$$

In [12], it is proved that for each $\alpha$,

$$E \sup_{t \leq x_t^\alpha} \left( Y_t^\alpha - e^{-\phi_{t,x}^{\alpha,x}} v_{(\xi_t^0,0)}(x_t^{\alpha,x}) \right) \leq NB_1(x, \xi),$$

where $N$ is independent of $\alpha$. So

$$G_2(\epsilon, T) \leq E e^{-\phi_{t,x}^{\alpha,x}} v_{(\xi_t^0,\zeta_t)}(\xi_t^0,\zeta_t)(x_t^{\alpha,x}) + NB_1(x, \xi).$$

By mimicking the argument in the proof of (2.11), we have

$$\lim_{T \to \infty} \limsup_{\epsilon \to 0} E e^{-\phi_{t,x}^{\alpha,x}} v_{(\xi_t^0,\zeta_t)}(\xi_t^0,\zeta_t)(x_t^{\alpha,x}) \leq \sup_{y \in \partial D_\alpha^s \cup \mathbb{R}^d \setminus \{0\}} \frac{v_{(\xi,\zeta)}(y)}{B_1(x, \zeta)} + 3B_1(x, \xi).$$

So we conclude that

$$\lim_{T \to \infty} \limsup_{\epsilon \to 0} G_2(\epsilon, T) \leq \sup_{y \in \partial D_\alpha^s \cup \mathbb{R}^d \setminus \{0\}} \frac{v_{(\xi,\zeta)}(y)}{B_1(y, \zeta)} \cdot B_1(x, \xi) + NB_1(x, \xi),$$

where $\xi = |\pi_t|^2 + 2\xi_t^2 + \xi_t^{2d+2} + (\xi_t^{2d+1})^2 + \eta_t^{2d+1}$. 

Therefore,

$$\limsup_{\epsilon \to 0} E \sup_{t \leq \gamma} \left| \hat{p}_t(\epsilon) - \hat{p}_t(-\epsilon) \right|^2 - 2\xi_t^0 \left( E \sup_{t \leq \gamma} \left| \hat{p}_t(\epsilon) + \hat{p}_t(-\epsilon) \right|^2 - 1 \right)^{2p} + e^p \int_0^\gamma \left| \pi_t \right|^{2p} dt + e^p \int_0^\gamma \left| \hat{p}_t \right|^{2p} dt \right).$$
which implies that

\[
(4.16) \quad \frac{v(\xi(\xi))(x)}{B_1(x, \xi)} \leq \sup_{y \in \partial D_\lambda, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta(\zeta))(y)}{B_1(y, \zeta)} + N, \quad \forall x \in D_\lambda, \xi \in \mathbb{R}^d \setminus \{0\}.
\]

Repeating the argument above for \(D_\lambda^2\), we have

\[
(4.17) \quad \frac{v(\xi(\xi))(x)}{B_1(x, \xi)} \leq \sup_{y \in \partial D_{\lambda^2}, \zeta \in \mathbb{R}^d \setminus \{0\}} \frac{v(\zeta(\zeta))(y)}{B_1(y, \zeta)} + N, \quad \forall x \in D_{\lambda^2}, \xi \in \mathbb{R}^d \setminus \{0\}.
\]

Since (4.16) and (4.17) are similar as (3.24) and (3.26) in \([12]\), by repeating the argument after (3.26) in \([12]\), we get

\[-v(\xi(\xi))(x) \leq N \left( |\xi|^2 + \frac{\psi^2(\xi)}{\psi(x)} \right), \quad \text{a.e. in } D.\]

The inequality (2.12) is proved. \(\square\)

**Proof of (2.13).** Fix an \(x \in D\). For simplicity of the notations we will drop the argument \(x\) through the proof below.

From (2.12) we have

\[
v(\xi(\xi)) + N \left( |\xi|^2 + \frac{\psi^2(\xi)}{\psi(x)} \right) \geq 0, \forall \xi \in \mathbb{R}^d.
\]

It follows that

\[
v(\xi(\xi)) + \frac{N}{\psi} |\xi|^2 \geq 0, \forall \xi \in \mathbb{R}^d.
\]

Let

\[V = v_{xx} + \frac{N}{\psi} I + I,
\]

where \(I\) is the identity matrix of size \(d \times d\).

Then we have

\[(V \xi, \xi) \geq |\xi|^2 > 0, \forall \xi \in \mathbb{R}^d \setminus \{0\}.
\]

Fix a \(\xi \in \mathbb{R}^d\) such that \(\mu(\xi) > 0\). Introduce

\[
\kappa = \sqrt{V} \xi, \quad \theta = |\kappa|^{-2} \kappa, \quad \zeta = \sqrt{V} \theta.
\]

Then

\[
\text{tr}(a^\alpha V) = \text{tr}(\sqrt{V} a^\alpha \sqrt{V}) \geq |\theta|^{-2}(\sqrt{V} a^\alpha \sqrt{V} \theta, \theta) = |\kappa|^2(a^\alpha \zeta, \zeta) = (V \xi, \xi)(a^\alpha \zeta, \zeta).
\]

Taking the supremum and noticing that \((\xi, \zeta) = (\kappa, \theta) = 1\), we get

\[
\sup_{\alpha \in A} \text{tr}(a^\alpha V) \geq (V \xi, \xi) \sup_{\alpha \in A} (a^\alpha \zeta, \zeta) \geq (V \xi, \xi) \mu(\xi).
\]

It follows that

\[
v(\xi(\xi)) \leq (V \xi, \xi) \leq \mu^{-1}(\xi) \sup_{\alpha \in A} \text{tr}(a^\alpha V) \leq \mu^{-1}(\xi) \left[ \sup_{\alpha \in A} \text{tr}(a^\alpha v_{xx}) + \frac{N}{\psi} \sup_{\alpha \in A} \text{tr}(a^\alpha) \right].
\]
Notice that
\[ \mu(\xi) = \left| \xi \right|^{-2} \mu(\xi/|\xi|), \]
so it remains to estimate \( \sup_{\alpha \in A} \text{tr}(a^\alpha v_{x\bar{x}}) \) from above. The equation
\[
\sup_{\alpha \in A} \left[ L^\alpha v - c^\alpha v + f^\alpha \right] = 0
\]
implies that
\[ L^\alpha v - c^\alpha v + f^\alpha \leq 0, \forall \alpha \in A. \]

Thus
\[
\text{tr}(a^\alpha v_{x\bar{x}}) = (a^\alpha)^{ij} v_{x^i x^j} \leq \left| (b^\alpha)^{i} \right|_{0,D} |v_{x^i}|_{0,D} + |e^\alpha|_{0,D} |v|_{0,D} + |f^\alpha|_{0,D} \leq K.
\]

\( \square \)

Proof of the existence and uniqueness of (2.14). The fact that \( u \) given by (2.4) satisfies (2.14) follows from Theorem 1.3 in [5].

To prove the uniqueness, assume that \( v_1, v_2 \in C^{1,1}_{\text{loc}}(D) \cap C^0(\bar{D}) \) are solutions of (2.14). Let \( \Lambda = |v_1|_{0,D} \vee |v_2|_{0,D} \). For constants \( \delta \) and \( \varepsilon \) satisfying \( 0 < \delta < \varepsilon < 1 \), define
\[
\Psi(x,t) = \varepsilon(1 + \psi(x)) \Lambda e^{-\delta t}, \quad V(x,t) = v(x)e^{-\varepsilon t} \quad \text{in} \quad \bar{D} \times (0, \infty),
\]
\[
F[V] = \sup_{\alpha \in A} \left( V_t + L^\alpha V - c^\alpha V + f^\alpha \right) \quad \text{in} \quad D \times (0, \infty).
\]

Notice that a.e. in \( D \), we have
\[
F[V_1 - \Psi] \geq -\varepsilon e^{-\varepsilon t} v_1 + \delta \Psi - \varepsilon \Lambda e^{-\delta t} \sup_{\alpha} L^\alpha \psi + \inf_{\alpha} c^\alpha \Psi \geq \varepsilon \Lambda(e^{-\delta t} - e^{-\varepsilon t}) \geq 0,
\]
\[
F[V_2 + \Psi] \leq \varepsilon e^{-\varepsilon t} v_2 - \delta \Psi + \varepsilon \Lambda e^{-\delta t} \sup_{\alpha} L^\alpha \psi - \inf_{\alpha} c^\alpha \Psi \leq \varepsilon \Lambda(e^{-\varepsilon t} - e^{-\delta t}) \leq 0.
\]

On \( \partial D \times (0, \infty) \), we have
\[
V_1 - V_2 - 2\Psi = -2\Psi \leq 0.
\]
On \( \bar{D} \times T \), where \( T = T(\varepsilon, \delta) \) is a sufficiently large constant, we have
\[
V_1 - V_2 - 2\Psi = (v_1 - v_2)e^{-\varepsilon T} - 2\varepsilon(1 + \psi)\Lambda e^{-\delta T} \leq 2\Lambda(e^{-\varepsilon T} - e^{-\delta T}) \leq 0.
\]

Applying Theorem 1.1 in [2], we get
\[
V_1 - V_2 - 2\Psi \leq 0 \quad \text{a.e. in} \quad \bar{D} \times (0, T).
\]

It follows that
\[
v_1 - v_2 \leq 2\varepsilon(1 + \psi)\Lambda e \to 0, \quad \text{as} \quad \varepsilon \to 0, \quad \text{a.e. in} \quad D.
\]
Similarly, \( v_2 - v_1 \leq 0 \) a.e. in \( D \). The uniqueness is proved.

\( \square \)

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127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455