SIMPLE BUT EFFICIENT TESTS
OF QCD FROM TAU DECAYS

S. Narison
Theoretical Physics Division, CERN
CH - 1211 Geneva 23
and
Laboratoire de Physique Mathématique
Université de Montpellier II
Place Eugène Bataillon
34095 - Montpellier Cedex 05

Abstract

We review the determinations of the QCD coupling $\alpha_s$ from the inclusive and exclusive modes of $\tau$-decays. The most recent $\tau$-data provide the average value $\alpha_s(M^2_\tau) \simeq 0.347 \pm 0.030$ corresponding to $\alpha_s(M^2_\tau) \simeq 0.121 \pm 0.003 \pm 0.001$. The values of the QCD vacuum condensates extracted from the inclusive weighted-moment distributions are consistent with the ones from QCD spectral sum rules. Accurate estimates and further precision tests should be reached in the next $\tau$C/B-factory machines. The $M_\tau$-stability test from the $e^+e^-\rightarrow I = 1$ hadron data is also discussed.

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1 INTRODUCTION

The $\tau$ semileptonic decay modes have been shown \cite{1}-\cite{4} to be a “good laboratory” for testing the perturbative and non-perturbative aspects of QCD. In particular, due to their inclusive nature, which is even greater than in the case of $e^+e^-$ into hadrons, these decay modes can provide a measurement of the QCD coupling $\alpha_s(M_\tau)$ with “unprecedented” accuracy.

Before going to the content of the talk, let me quote the following statements, which will give the flavour of the subject that I shall discuss:

Indeed, it is for this process that more work has been done concerning higher power corrections than for any other processes. The non-perturbative condensates apparently give a small contribution thus making this relatively low-energy process a prime place to determine the $\Lambda$-parameter of QCD... (A. Mueller \cite{7}).

Another entry in the table of Figure 3 (see Figs.3 and 4) is particularly interesting and provocative, and suggests some later developments, so I want to go into a little more detail regarding it. It is the determination of $\alpha_s$ from QCD with corrections to the tau lepton lifetime. Tau lepton decay of course is a very low energy process by the standards of LEP or most other QCD tests. So we can expect, in line with the previous discussion, that the prediction will perhaps be delicate but on the other hand it will have a favorable lever arm for determining $\alpha_s$... (F. Wilczek \cite{8}).

Tau-decay is a lucky process... (G. Veneziano).

In this talk (desolé du peu!), I shall review the determinations of $\alpha_s$ from the inclusive and from the sum of the exclusive $\tau$-decay modes. I shall also discuss the weighted-moment distributions for simultaneously measuring $\alpha_s$ and the non-perturbative QCD condensates. The $e^+e^- \rightarrow I = 1$ hadron data will be used for testing the stability of the result for arbitrary values of the $\tau$-mass.

2 $\alpha_s(M_\tau)$ FROM THE INCLUSIVE MODE

This section is mainly based on the paper in \cite{2}.

2.1 The naïve quark-parton model

From the well-known naïve quark-parton model, one predicts :

$$R_\tau \equiv \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau \bar{\nu}_e)} = N_c,$$  \hspace{1cm} (1)

very analogously to
as the two processes can be related to each other through an $SU(2)_I$ rotation. The present data average from the $\tau$-lifetime:

\[
R^\Gamma_\tau \equiv \frac{\Gamma_\tau - \sum_{e,\mu} \Gamma_{\tau \rightarrow \ell}}{\Gamma_{\tau \rightarrow \ell}} = 3.55 \pm 0.06
\]  

(3)

and from the $\tau$-leptonic branching ratios:

\[
R^B_\tau \equiv \frac{1 - B_\mu - B_\mu}{B_\mu} = 3.64 \pm 0.03
\]  

(4)

gives:

\[
R_\tau = 3.62 \pm 0.03
\]  

(5)

This experimental value is indeed a good evidence for the existence of colour but it is 20% higher than the quark-parton model estimate.

### 2.2 QCD formulation of the tau decay

Here, we propose to study the different QCD corrections on $R_\tau$ and show the ability of QCD to resolve this 20% discrepancy. In so doing, we shall be concerned with the decay rate:

\[
R_\tau = 12\pi \int_0^{M_\tau^2} ds \frac{d^2}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left\{\left(1 + \frac{2s}{M_\tau^2}\right) \text{Im} \Pi^{(1)}_{(s)} + \text{Im} \Pi^{(0)}_{(s)}\right\},
\]

(6)

where $\text{Im} \Pi^{(J)}$ is the hadronic spectral function of a hadron of spin $J$. In QCD, these two-point correlators can be expressed as:

\[
\Pi^{(J)} = \sum_{q=d,s} |V_{uq}|^2 \left(\Pi^{(J)}_{uq,V} + \Pi^{(J)}_{uq,A}\right);
\]

(7)

$V_{uq}$ is the CKM mixing matrix, and the correlators

\[
\Pi^{\mu\nu}_{ij,V} \equiv i \int d^4 x \, e^{iqx} \left\langle \frac{0}{T} \, V^{\mu}_{ij}(x) \left(V^{\nu}(0)\right)^\dagger \right\rangle,
\]  

\[
\Pi^{\mu\nu}_{ij,A} \equiv i \int d^4 x \, e^{iqx} \left\langle \frac{0}{T} \, A^{\mu}_{ij}(x) \left(A^{\nu}(0)\right)^\dagger \right\rangle,
\]  

\[
\Pi^{\mu\nu}_{ij \, V/A} = - (g^{\mu\nu}q^2 - q^\mu q^\nu) \, \Pi_{(1)}^{(1)}_{ij \, V/A} + q^\mu q^\nu \Pi_{(0)}^{(0)}_{ij \, V/A},
\]

(8)
are associated to the quark vector and axial-vector local currents

\[ V_{ij}^\mu \equiv \bar{\psi}_i \gamma_\mu \psi_j : \quad \text{and} \quad A_{ij}^\mu \equiv : \bar{\psi}_i \gamma_\mu \gamma_5 \psi_j : . \]  

(9)

It is clear that \( R_\tau \) in (6) cannot be calculated directly from QCD for \( s \leq \Lambda^2 \). However, exploiting the analyticity of the correlators \( \Pi^{(J)}(s) \) and the Cauchy theorem, one can express \( R_\tau \) as a contour integral in the complex \( s \)-plane running counter-clockwise around the circle of radius \( |s| = M_\tau^2 \) (see Fig. 1):

\[ R_\tau = 6i\pi \oint_{|s|=M_\tau^2} \frac{ds}{s} \left( 1 - \frac{s}{M_\tau^2} \right)^2 \left\{ \left[ 1 + \frac{2s}{M_\tau^2} \right] \Pi^{(1)}(s) + \Pi^{(0)}(s) \right\}. \]  

(10)

One should notice the existence of the double zero at \( s = M_\tau^2 \), which suppresses the uncertainties near the time-like axis. As \( |s| = M_\tau^2 \gg \Lambda^2 \), one can use the standard operator product expansion (OPE) à la SVZ [9] for the estimate of the correlators:

\[ \Pi^{(J)} = \sum_{D=0,2,4,...} \left( -s \right)^{D/2} \sum_{\text{dim } O=D} \frac{1}{\text{dim } O} C^{(J)}(s, \mu) \langle O(\mu) \rangle, \]  

(11)

where \( \mu \) is an arbitrary scale that separates the long- and short-distance dynamics; \( C^{(J)} \) are the Wilson coefficients calculable in perturbative QCD, while \( \langle O \rangle \) are the non-perturbative operators.

### 2.3 The (non-)perturbative operators \( \langle O \rangle \)

In this part, we shall limit ourselves to the discussions of the local gauge-invariant operators \( \langle O \rangle \) of dimension \( D \) which appear in the standard SVZ-expansion. Discussions of some eventual effects not included in this expansion will be done later on.

- \( \langle O_2 \rangle \equiv \bar{m}_i^2, \bar{m}_i \bar{m}_j \) and \( \bar{m}_j^2 \) are products of the running quark masses of the QCD Lagrangian which are the only possible dimension-2 local gauge-invariant operators that can be built from the quark and/or gluon fields. The values of the quark masses determined from the sum rules are [10], [11]:

\[ \hat{m}_u = (8.7 \pm 1.5) \text{MeV} \quad \hat{m}_d = (15.4 \pm 1.5) \text{MeV} \quad \hat{m}_s = (270 \pm 30) \text{MeV}, \]  

(12)

where the invariant mass \( \hat{m} \) is related to the running quark-mass as

\[ \hat{m}_i = (\log M_\tau/\Lambda)^{\gamma_1/\beta_1} \bar{m}_i \left( M_\tau^2 \right) \left\{ 1 + O \left( \frac{\alpha_s}{\pi} \right) \right\}, \]  

(13)

where \( \gamma_1 = 2 \) and for 3 flavours \( -\beta_1 = 9/2 \) are respectively the first coefficients of the quark anomalous dimension and of the \( \beta \)-function.
the quark and gluon condensates. The former is known from PCAC and from the sum rules analysis of the SU(3)F breaking. The second is determined from a combined analysis of the charmonium and e+e− into I = 1 hadron data. Their values are \[\hat{\mu}_u = \hat{\mu}_d = (189 \pm 7)\text{MeV}, \quad \hat{\mu}_s = (160 \pm 10)\text{MeV}\] and \[\langle \alpha_s G^2 \rangle = (0.06 \pm 0.03)\text{GeV}^4,\] where \(\hat{\mu}_i\) are the invariant condensates defined as:

\[\langle \bar{\psi}_i \psi_i \rangle (M_\tau) = -\hat{\mu}_i^3 (\log M_\tau / \Lambda)^{2/3} \{1 + O(\alpha_s)\}.\] (16)

However, due to the operator mixings in the massive quarks case, the building of the RGI quark and gluon condensates needs the inclusion of a tiny \(m^4\) perturbative terms due to the light quark masses as given in the Appendix B of [2].

− \(\langle O_6 \rangle \equiv \alpha_s \langle \bar{\psi}_i \Gamma_1 \psi_j \bar{\psi}_j \psi_i \rangle\) are the dimension-6 operators in the chiral limit \(m_i = 0\); \(\Gamma_{1,2}\) are generic notations for any Dirac and colour matrices. One can express it as:

\[\langle O_6 \rangle = \frac{1}{16N_c^2} \{T_i \Gamma_1 T_j \Gamma_2 - T_r (\Gamma_1 \Gamma_2)\} \rho \alpha_s \langle \bar{\psi} \psi \rangle^2,\] (17)

where \(\rho\) is the parameter controlling the deviation from the vacuum saturation assumption. The sum-rules analysis of the vector and axial-vector channels gives [11]:

\[\rho \alpha_s \langle \bar{\psi} \psi \rangle^2 \simeq (3.8 \pm 2.0) 10^{-4}\text{GeV}^6,\] (18)

which signals a large deviation from the naïve vacuum saturation assumption.

− \(\langle O_8 \rangle\) are the dimension-8 operators whose strengths are poorly known as they involve large numbers of operators. Some of their effects have been calculated in Ref. [12] and have been estimated in Ref. [2] to be about \(10^{-5}\).

We are now in a position to write down the theoretical QCD expression of \(R_\tau\) in the form

\[R_\tau \equiv 3 \left( |V_{ud}|^2 + |V_{us}|^2 \right) S_{EW} \left\{ 1 + \delta_{EW} + \delta^{(0)} + \sum_{D=2,4} \delta^{(D)} \right\},\] (19)

where \(|V_{ud}| \simeq 0.9753 \pm 0.0006\), and \(|V_{us}| \simeq 0.221 \pm 0.003\) are the CKM mixing angles.
2.4 The size of the different corrections

The different corrections are:

- **Electroweak**

$S_{EW} = 1.0194$ from summing, via the RGE, the leading “log contribution” [13].

$\delta_{EW} = \frac{5}{12} \frac{\alpha}{\pi} = 0.0010$ is the next-to-leading electroweak contribution [14].

- **Perturbative**

We use the available calculations from the $e^+e^- \rightarrow$ hadron process [15], from which we deduce [2, 3]:

$$
\delta^{(0)} = \frac{\alpha_s}{\pi} + 5.2023 \left( \frac{\alpha_s}{\pi} \right)^2 + 26.366 \left( \frac{\alpha_s}{\pi} \right)^3 + (78 \pm 50) \left( \frac{\alpha_s}{\pi} \right)^4, \quad (20)
$$

where, for a typical value of $\alpha_s(M_\tau) = 0.35$, they are respectively 11.1, 6.5, 3.6 and $(1.1 \pm 0.7)$ %, of the leading-order term. The error has been estimated using an algebraic growth of the coefficient $(50 = 2K_3(K_3/K_2)$, where in the $\overline{MS}$-scheme $K_2 = 1.6398$ and $K_3 = 6.3711$ are the coefficients of the $\alpha_s^2$ and $\alpha_s^3$-terms of the D-function obtained in [15]. The error has been multiplied by a factor 2 in order to be more conservative. The convergence of the perturbative series for $\alpha_s \geq 0.35$ has been improved in Ref. [3] after a resummation of the series and by using an expansion other than the $\alpha_s$ one used in (20). Such a modified expansion appears to be less dependent on the subtraction point $\mu$ and on the choice of the renormalization scheme. For a typical value of $\delta^{(0)} \simeq 0.22$, these different uncertainties in the perturbative series induce an error of about .0017 for the value of $\alpha_s(M_Z^2)$.

- **Quark mass**

Using the quark mass values quoted in (12), one has

$$
\delta^{(2)} \simeq -(0.7 \pm 0.2)\%, \quad (21)
$$

which comes mainly from $\hat{m}_s$.

- **Non-perturbative**

$\delta^{(4)} = -(8 \pm 1)10^{-3}$ using (15), where one should notice that due to the $s$-structure of $R_\tau$ and the Cauchy theorem, the leading-order contribution of the $\langle O_4 \rangle$ effect is zero, explaining the small value of $\delta^{(4)}$, which is only induced by the radiative corrections responsible for the $s$-dependence of $\langle O_4 \rangle$.

$\delta^{(6)} \simeq -(7 \pm 4)10^{-3}$ using (18). The relative smallness of this contribution is due not only to the $1/M_\tau^6$ suppression, but also to some compensation between the vector and axial-vector contributions in $R_\tau$. This nice compensation also happens to order $\alpha_s$ [16], where the remaining radiative corrections are much smaller than the errors in (18).
\( \delta^{(8)} \simeq 10^{-5} \) using the vacuum-saturation estimate of the calculated contributions.

Adding these different non-perturbative contributions, one obtains:

\[
\delta_{SVZ} \equiv \sum_{D=2}^{8} \delta_D \simeq -(15 \pm 5)10^{-3}
\]  

(22)

One can remark that the sum of the non-perturbative effects is tiny as it is of the order of the estimated perturbative error. This is mainly due to the vanishing of the \( \langle O_4 \rangle \) effects to leading order and to the fact that operators of dimension \( D \geq 6 \) effects are highly suppressed in powers of \( 1/M_r \).

### 2.5 Exotic contributions beyond the SVZ-expansion

From the theoretical point of view, one has also made some progress (which should be pursued) in the understanding of some eventual “exotic effects” not contained in the SVZ-expansion. Instanton-like effects, though not under good control, have been shown to be negligible, as they induce a correction in the range \( 10^{-6} - 10^{-3} \) [17], which is much smaller than the non-perturbative effects within the SVZ-expansion retained previously. The most dangerous effect might be due to the eventual existence of dimension-two operators, which might appear in the massless limit and which are not contained in the original SVZ-expansion as argued by Altarelli [18]:

* I think it is a fair statement that there is no theorem that guarantees the absence of \( 1/M_r^2 \) terms in \( R_r \) in the massless limit by proving that terms of order \( (\Lambda/M_r)^2 \) cannot arise...

Phenomenological constraints on this term (assuming its existence) from QCD spectral sum rules analysis of the \( e^+e^- \) data [19] provide an estimate of about \(-0.5 \pm 3.1\)% correction and indicates that this term (if there) contributes as an imaginary mass. However, this constraint is not strong enough for excluding radically this possibility. Some constraints of this type should be derived from other sources. Moreover, the absence of these terms have been shown in a formal way by [7, 20] using arguments based on U.V renormalon. The proof is well summarized by F. Wilczek [8] as:

* Mueller has given an important, although not entirely rigorous, argument that no \( \Lambda^2/Q^2 \) term can appear. The argument is a little technical, so I won’t be able to do it full justice here but I will attempt to convey the main idea. The argument is based on the idea that at each successive power of \( 1/Q^2 \) one can make the perturbation series in QCD, which is a badly divergent series in general, at least almost convergent, that is Borel summable, by removing a finite number of obstructions. Furthermore the obstructions are captured and parametrized by the low dimension operators mentioned before. Once these obstructions are removed, the remaining (processed) perturbation expression converges on the correct result for the full theory. Neither in the obstructions nor in the residual perturbative expression do the potentially dangerous terms occur—which means that they don’t occur at all.

But, Altarelli [18] continues with another statement:

* I also stress that the advantage from the absence of \( 1/M_r^2 \) terms in \( R_r \) could be an illusion...
when comparing $\alpha_s(M^2_\tau)$ measured from $R_\tau$ with $\alpha_s(Q)$ derived from some other process, because one needs control of $\alpha_s(Q)$ down to terms of order $\Lambda^2/M^2_\tau$, while only the asymptotic form of $\alpha_s(Q)$ is known. For example, any freezing mechanism at $Q \simeq \Lambda$ introduces typical corrections of order $\alpha_s^2(M^2_\tau)\Lambda^2/M^2_\tau$.

I have worked out explicitly a check of this statement, by using the following expression

$$1/\log((Q^2 + C^2)/\Lambda^2),$$

of $\alpha_s$ at low-energy, instead of the usual asymptotic $1/\log(Q^2/\Lambda^2)$-behaviour at high-energy. Using the generous range $\Lambda \leq C \leq M_\rho$, where $M_\rho$ is a typical hadronic scale, this effect, which is an $\alpha_s^2$ effect, induces a correction less than $5 \times 10^{-3}$ in the tau-decay rate, which is about the same as the error from the non-perturbative effects in (21).

Then, one can conclude, without any doubts, that the different non-perturbative effects within or beyond the SVZ-expansion are tiny and make the $\tau$-decay a prime place for determining $\alpha_s$.

## 2.6 The value of $\alpha_s$

By confronting the QCD predictions in Table 1 with the data in (5), one deduces

$$\alpha_s(M^2_\tau) = 0.36 \pm 0.03. \quad (23)$$

We run this value at $M_\tau$ up to $M_Z$, by using the matching conditions at the heavy quark thresholds à la [21], i.e at the value of the running c and b quark masses, which we deduce

| $\alpha_s(M^2_\tau)$ | $R_{\tau,V}$ | $R_{\tau,A}$ | $R_{\tau,S}$ | $R_\tau$ |
|----------------------|--------------|--------------|--------------|----------|
| 0.16                 | 1.59 ± 0.02  | 1.49 ± 0.03  | 0.145 ± 0.004 | 3.23 ± 0.01 |
| 0.18                 | 1.61 ± 0.02  | 1.51 ± 0.03  | 0.145 ± 0.004 | 3.26 ± 0.01 |
| 0.20                 | 1.62 ± 0.02  | 1.53 ± 0.03  | 0.145 ± 0.005 | 3.29 ± 0.01 |
| 0.22                 | 1.64 ± 0.02  | 1.54 ± 0.03  | 0.145 ± 0.005 | 3.33 ± 0.02 |
| 0.24                 | 1.66 ± 0.02  | 1.56 ± 0.03  | 0.145 ± 0.005 | 3.37 ± 0.02 |
| 0.26                 | 1.68 ± 0.02  | 1.58 ± 0.03  | 0.145 ± 0.005 | 3.41 ± 0.02 |
| 0.28                 | 1.70 ± 0.02  | 1.61 ± 0.03  | 0.145 ± 0.005 | 3.45 ± 0.02 |
| 0.30                 | 1.72 ± 0.02  | 1.63 ± 0.03  | 0.145 ± 0.006 | 3.50 ± 0.02 |
| 0.32                 | 1.75 ± 0.02  | 1.65 ± 0.03  | 0.145 ± 0.006 | 3.54 ± 0.03 |
| 0.34                 | 1.77 ± 0.02  | 1.67 ± 0.03  | 0.145 ± 0.006 | 3.58 ± 0.03 |
| 0.36                 | 1.79 ± 0.02  | 1.69 ± 0.03  | 0.144 ± 0.006 | 3.63 ± 0.03 |
| 0.38                 | 1.81 ± 0.03  | 1.71 ± 0.03  | 0.144 ± 0.007 | 3.67 ± 0.04 |
| 0.40                 | 1.83 ± 0.03  | 1.73 ± 0.03  | 0.143 ± 0.007 | 3.71 ± 0.04 |
| 0.42                 | 1.85 ± 0.03  | 1.75 ± 0.04  | 0.143 ± 0.007 | 3.75 ± 0.04 |
| 0.44                 | 1.87 ± 0.03  | 1.77 ± 0.04  | 0.142 ± 0.008 | 3.79 ± 0.04 |

Table 1: QCD predictions [2,3,5] for the different components of the $\tau$ hadronic width.
from the QCD spectral sum rule estimates of the \textit{perturbative} pole masses \cite{22}:

\begin{align}
M_c(p^2 = M_c^2) &\simeq 1.45 \pm 0.05 \text{ GeV} \quad M_b(p^2 = M_b^2) \simeq 4.58 \pm 0.05 \text{ GeV.} \quad (24)
\end{align}

One should notice that the values of the pole masses given above come from the standard relativistic sum rule estimate through the perturbative euclidian mass, such that they are not affected by renormalon-type contributions which induce a non-perturbative effect of the order $\Lambda$ in the pole mass used in non-relativistic sum rule and heavy quark effective theory. Such an effect makes these non-relativistic masses slightly larger as in the potential models but their definition is still ambiguous at present.

The previous matching procedure already includes in it \cite{21} the tiny $(\alpha_s/\pi)^2$ effects of about $\delta(H) \simeq 5 \times 10^{-4}$ from virtual heavy quark loops obtained in \cite{23}. At the end of the day, one obtains:

\begin{align}
\alpha_s \left( M_Z^2 \right) &\simeq 0.122 \pm 0.003 \pm 0.001, \quad (25)
\end{align}

where the matching procedure has induced the last error which is a conservative error. This result is in nice agreement with, and slightly more precise than, the present LEP average \cite{24} $\alpha_s (M_Z^2) \simeq 0.125 \pm 0.005$, done without including the $\tau$-decay source. This precision indicates that a modest accuracy at $M_\tau$ leads to a high-precision measurement at $M_Z$ as the error bars run like $\alpha_s^2$. The agreement between the independent determinations at $M_\tau$ and $M_Z$ also shows that $\alpha_s$ runs as expected in a QCD asymptotically-free theory.

### 2.7 ALEPH test from the weighted-moment distributions

The ALEPH collaboration at LEP \cite{4} has tested the previous result by working with the weighted moments distributions \cite{3}:

\begin{align}
R_{\tau}^{k\ell} &\equiv \int_0^{M_\tau^2} ds \left( 1 - \frac{s}{s_0} \right)^k \left( \frac{s}{M_\tau^2} \right)^\ell \frac{dR_\tau}{ds} \\
D_{\tau}^{k\ell} &\equiv \frac{R_{\tau}^{k\ell}}{R_{\tau}^{00}}, \quad (26)
\end{align}

which are sensitive to $\alpha_s$ and to the non-perturbative condensates. These moments have the advantage of being directly measurable, thanks to the hadronic invariant-mass squared distribution $dR_\tau/ds$. The factor $(1 - s/M_\tau^2)^k$ supplements $(1 - s/M_\tau^2)^2$ and squeezes the integrand near the positive real axis. This improves the reliability of the OPE and of the analysis. Using the present LEP data, which are still statistically limited, one obtains from a 4-parameter fit of 5 observables ($R_\tau$, $D_{\tau}^{1l}$ $l=0\text{--}3$):

\begin{align}
\alpha_s (M_\tau) &= (0.34 \pm 0.04) \\
\langle \alpha_s G^2 \rangle &= (0.06 \pm 0.05) \text{ GeV}^4 \\
\rho \alpha_s \langle \bar{\psi}\psi \rangle^2 &= (4 \pm 4) \times 10^{-4} \text{ GeV}^6 \\
\langle O_8 \rangle &= (3 \pm 2) \times 10^{-3} \text{ GeV}^8, \quad (27)
\end{align}
where the results are strongly correlated. The value of $\alpha_s$ is compatible with the previous ones from the decay modes. The values of the condensates are still inaccurate, but they are compatible with the ones from QSSR analysis of charmonium and $e^+e^- \rightarrow I = 1$ hadron data used previously. However, due to these strong correlations, it will be also useful and it is possible to extract with a much better accuracy the ratios of different condensates in order to also test the sum rule predictions of these quantities. Adding these different correlated non-perturbative contributions, one also obtains:

$$\delta_{SVZ} \simeq (3 \pm 5) \times 10^{-3},$$

(28)

which is consistent with (22) and confirms the smallness of the non-perturbative contributions in $R_\tau$. The experiments which can produce millions of $\tau$ such as the $\tau$C/B-factory are the best place for improving these interesting results on $\alpha_s$ and on the size of the QCD condensates.

### 3 $\alpha_s$ FROM THE SUM OF EXCLUSIVE MODES

This section is based on the work in [3].

#### 3.1 The vector channel

It is known that using an $SU(2)_I$ rotation, one can relate the vector component of the $\tau$-decay with the $e^+e^- \rightarrow I = 1$ hadrons data:

$$R_{\tau,V} = \frac{\Gamma (\tau \rightarrow \nu_\tau V)}{\Gamma (\tau \rightarrow \nu_\tau e\nu e)} = \frac{3 \cos^2 \theta_c}{2 \pi \alpha^2} S_{EW} \int_0^{M^2_{\tau}} ds \frac{M^2_{\tau}}{s \left(1 - \frac{s}{M^2_{\tau}}\right)^2 \left(1 + \frac{2s}{M^2_{\tau}}\right)^2} s \sigma_{e^+e^- \rightarrow V_0(s)}\delta^{(D)}_{ud,V}.$$  

(29)

The values of $R_{\tau,V}$ from the $\tau$-data and estimated from (19) are given in Table 2, where one should notice that the estimates of $K^-K^0$ and $\pi^-K^+K^-$ have been done using $SU(3)$ rotations with the appropriate phase space factor. By combining the data and the estimated results, one can obtain the best value:

$$R_{\tau,V}^{exp} = 1.78 \pm 0.03.$$  

(30)

Comparing this result with the QCD expression:

$$R_{\tau,V} = \frac{3}{2} |V_{ud}|^2 \left(1 + \delta^{(0)} + \sum_{D=2,4,...} \delta^{(D)}_{ud,V}\right),$$

(31)
| $V^-$ | $\tau$-data | $e^+e^-$ (Ref. [26]) | $e^+e^-$ (our estimate) |
|-------|-------------|-----------------|-----------------|
| $\pi^-\pi^0$ | 1.355 ± 0.021 | 1.349 ± 0.046 | 1.346 ± 0.040 |
| $2\pi^-\pi^0\pi^0$ | 0.307 ± 0.013 | 0.248 ± 0.015 | 0.268 ± 0.040 |
| $\pi^-3\pi^0$ | 0.063 ± 0.009 | 0.061 ± 0.003 | 0.057 ± 0.010 |
| $\pi^-\omega$ | 0.090 ± 0.028 | 0.128 ± 0.018 | 0.129 ± 0.023 |
| $3\pi^-2\pi^+\pi^0$ | 0.003 ± 0.001 | - | - |
| $(6\pi)^-$ | - | 0.011 ± 0.002 | 0.008 ± 0.003 |
| $\pi^-\pi^0\eta$ | 0.010 ± 0.002 | 0.007 ± 0.001 | 0.008 ± 0.003 |
| $K^-K^0$ | $\leq0.015$ | 0.006 ± 0.002 | 0.009 ± 0.001 |
| $\pi^-K^-K^0$ | 0.011 ± 0.005 | - | 0.009 ± 0.003 |
| $R_{\tau,V}$ | 1.768 ± 0.032 | 1.693 ± 0.049 | 1.725 ± 0.069 |

Table 2: Contributions of different exclusive $\tau$-decay modes $\tau^- \to \nu, V^-$ to $R_{\tau,V}$.

where

$$\delta_{ud,V}^{(2)} \simeq -(0.6 \pm 0.2)10^{-3},$$

$$\delta_{ud,V}^{(4)} \simeq (0.8 \pm 0.3)10^{-3},$$

$$\delta_{ud,V}^{(6)} \simeq (2.4 \pm 1.3)10^{-2},$$

(32)

one should notice that the strength of the $\langle O_6 \rangle$ effect is larger here than in the inclusive mode $R_{\tau}$. In addition to the inaccuracy of the exclusive data, this fact limits the accuracy on the determination of $\alpha_s$ from the vector channel. We deduce from Table 2:

$$\alpha_s \left( M_{\tau}^2 \right) \simeq 0.35 \pm 0.05.$$  

(33)

3.2. The axial-vector channel

We give in Table 3 the exclusive decays in the axial-vector channel and their sum, using the more recent data quoted in [23]. The QCD expression of $R_{\tau,A}$ is

$$R_{\tau,A} = \frac{3}{2} |V_{ud}|^2 \left( 1 + \delta^{(0)} + \sum_{D=2,4,...} \delta_{ud,A}^{(D)} \right).$$

(34)

The non-perturbative corrections are :

$$\delta_{A}^{(2)} \simeq -(1 \pm 0.2)10^{-3}$$

$$\delta_{A}^{(4)} \simeq -(4.6 \pm 0.7)10^{-3}$$

$$\delta_{A}^{(6)} \simeq -(3.8 \pm 2.0)10^{-2}.$$  

(35)
A comparison of the data and of $R_{\tau,A}$ in Table 1 leads to :

$$\alpha_s \left( M_\tau^2 \right) \simeq 0.34 \pm 0.05, \quad (36)$$

where the central value is slightly lower than the one from the inclusive and vector channels, although consistent. This slightly lower value of $\alpha_s$ still signals a remaining though small deficit in this exclusive channel, but compared with the previous data used in [5], the new data [25] have provided an improvement of the value of $\alpha_s$ from this axial channel. A more precise value of $\alpha_s$ from this channel needs a better measurement of the $3\pi$ and of some other multipion channels.

### 3.3. The Cabibbo-suppressed channel

We show the data in Table 4, where we have used the most recent data for $K^-$ [25], with an improved error by about a factor 3. The QCD expression is :

$$R_{\tau,S} = 3 |V_{us}|^2 \left( 1 + \delta^{(0)} + \sum_{D=2,4} \delta^{(D)}_{us} \right), \quad (37)$$

which predicts from Table 1:
\[ R^{QCD}_{\tau,S} \simeq 0.145 \pm 0.006. \] (38)

One should notice that the estimate is “almost” insensitive to the value of \( \alpha_s \). This is mainly due to the “almost” cancellation of the \( \alpha_s \) and \( m_s^2 \) contributions, while the higher dimension condensates “almost” cancel with the \( \alpha_s^2 \) and \( \alpha_s^3 \) effects. These different cancellations make \( R^{QCD}_{\tau,S} \) to be “almost” equal to the prediction of the naïve quark-parton model:

\[ R^{\text{naive}}_{\tau,S} = N_c |V_{us}|^2 \simeq 0.147. \] (39)

At present, one cannot extract useful informations on the structure of QCD from the data. But a high-accuracy measurement of this channel can provide a measurement of the strange quark mass or a constraint on some exotic \( D = 2 \) “operator” not contained in the SVZ-expansion.

### 3.2 The sum of the exclusive modes

\( R_{\tau}^{\text{exclusive}} \) can be obtained by adding (30) to Tables 3 and 4. In this way one obtains:

\[ R_{\tau}^{\text{exclusive}} = 3.59 \pm 0.05, \] (40)

which leads to:

\[ \alpha_s \left( M^2_{\tau} \right) \simeq 0.34 \pm 0.04, \] (41)

in good agreement with the one from the inclusive mode in (23), though the central value in (41) is slightly lower.

### 4 STABILITY TEST FROM e^+ e^- DATA

We test the stability of the previous result by varying the \( \tau \)-mass. In so doing, we use (29) for arbitrary values of \( M_{\tau} \equiv M \) and we use the \( e^+ e^- \rightarrow I = 1 \) hadron data. Our result is shown in Fig. 2. The bars come from the \( e^+ e^- \) data. The continuous line is the fit for \( \alpha_s \left( M_{\tau} \right) = 0.33, \delta^{(6)}_V = 0.024 \) and \( \delta^{(8)}_V = -0.010 \). The shaded region shows the effect of the errors in \( \delta^{(6)}_V \), which are \( \pm 0.013 \). The hatched regions show the effects of the errors in \( \alpha_s \left( M^2_{\tau} \right) \), taken to be \( \pm 0.03 \) at fixed values of the condensates. It is clear that there is a good agreement between the theory and the data, except at low \( M \) where the role of the higher dimension \( D \geq 8 \) condensates is important as it changes completely the predicted behaviour below 1.2 GeV. One can impose an agreement of the theory with the data until 1 GeV by fitting the value of the \( D = 8 \) operators. One should however notice that the ratio of the obtained value of \( \delta^{(8)}_V \) over the \( D = 6 \) corrections is 1.25, signaling presumably the breaking of the OPE at such a low scale of 1 GeV.

Here, I should also mention that Ref. [27] has also used the \( e^+ e^- \) data in order to test the accuracy of the estimate of \( \alpha_s \) from \( \tau \)-decay. However, after a careful reading and check of the method used there, one can realize that the analysis emphasizes the region above 1.8
Observables & $\alpha_s(M^2_\tau)$
\hline
$R_\tau$ & $0.36 \pm 0.03$ \\
$D^{kl}_\tau$ & $0.330 \pm 0.046$ \\
$R_{\tau,V}$ & $0.35 \pm 0.05$ \\
$R_{\tau,A}$ & $0.34 \pm 0.05$ \\
$R^{\text{exclusive}}_\tau$ & $0.34 \pm 0.04$ \\
\hline
average & $0.347 \pm 0.030$ \\
\hline

Table 5: Values of $\alpha_s$ from different observables in $\tau$-decays.

GeV where the data are in contradiction when available. Indeed, the author works with a difference of two finite energy sum rules of radius $M^2_\tau$ and $s_0$. Moreover, one can also notice that in the uses of the usual FESR and dispersion relation for the D-function, the results depend strongly on the way the parametrization of the data in the energy region above 1.8 GeV is done. That is due to the well-known sensitivity of these methods on the medium-energy behaviour of the spectral function because of the usual $s^n$ weight factors entering in the sum rules. Fortunately, this is not the case of $R_\tau$ thanks to the $(1-s/M^2_\tau)^2$ threshold effect weight factor which suppresses this source of uncertainty. One can fairly conclude that the analysis done in [27] has nothing to do with the estimate of $\alpha_s$, but instead, only shows the already known evidence of the unstability of the results from the FESR and usual dispersion relation approaches due to the medium-energy behaviour of the spectral function. One could instead consider this analysis as a test of the validity of different parametrizations used in this medium-energy regime and how fast they reach the asymptotic regime of QCD. However, a more definite conclusion needs a careful inclusion of the error bars from the different data parametrizations.

5 CONCLUSION

We have reviewed the different determinations of the QCD coupling $\alpha_s$ from the inclusive [1]-[5] and exclusive [6] $\tau$-decay data, which we summarize in Table 5, from which one can deduce:

$$\alpha_s(M^2_\tau) \simeq 0.347 \pm 0.030 \implies \alpha_s(M^2_Z) \simeq 0.121 \pm 0.003 \pm 0.001,$$

where the last error quoted here at $M_Z$ is a conservative error induced by the one of the heavy quark masses and by the procedure of the matching conditions at the quark thresholds [28]. We compare this result with the ones from the other determinations (see Figs. 3 and 4 which are updated from [8, 24] as we have included the new average of $R_\tau$ and the global fit from the elecroweak data from LEP). As one can see in these figures, it turns out that $\tau$-decays are a good and presumably one of the best place for accurately measuring $\alpha_s$. The agreement with the other LEP results at $M_Z$ is a strong indication of the 1/log-running of $\alpha_s$ as expected from the asymptotically free theory of QCD. The ALEPH [4] measurement of the condensates also constitutes a test, in a method-independent way, of the underlying non-perturbative aspects of QCD within and beyond the SVZ expansion. Indeed, one can also use this result the other way around:
the accurate agreement between the values of $\alpha_s$ obtained from $\tau$-decay and from other high-energy processes such as LEP, is evidence against the possible existence of huge exotic non-perturbative effects beyond the SVZ-expansion. At present, the errors in the determinations of these different QCD fundamental or more properly universal parameters are dominated by the statistical errors, which can be notably reduced in the future $\tau$C/B-factory experiments.

As I started my talk with some quotations, I shall conclude it in the same way:

*I went into some detail into the analysis of tau decay because I think it’s not only important in itself but quite fundamental, and it connects with many other issues. In particular, this kind of argument could potentially provide a rigorous foundation for the QCD or ITEP sum rules which are the basis of a very successful phenomenology... *(F. Wilczek [8]).

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**Figure captions**

Fig. 1 : Integration contour in the complex s-plane, used to obtain (10).

Fig. 2 : $R_{\tau,V}$ in (29) as function of $M \equiv M_\tau$.

Fig. 3 : Different sources for evaluating $\alpha_s$ at different energies.

Fig. 4 : Different values of $\alpha_s$ from Fig. 3, but evaluated at $M_Z$. 
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