Prime chains are sequences \( p_1, \ldots, p_k \) of primes for which \( p_{j+1} \equiv 1 \pmod{p_j} \) for each \( j \). We introduce three new methods for counting long prime chains. The first is used to show that \( N(x; p) = O(x^{1+\varepsilon}) \), where \( N(x; p) \) is the number of chains with \( p_1 = p \) and \( p_k \leq px \). The second method is used to show that the number of prime chains ending at \( p \) is \( \approx \log p \) for most \( p \). The third method produces the first nontrivial upper bounds on \( H(p) \), the length of the longest chain with \( p_k = p \), valid for almost all \( p \). As a consequence, we also settle a conjecture of Erdős, Granville, Pomerance and Spiro from 1990. A probabilistic model of \( H(p) \), based on the theory of branching random walks, is introduced and analyzed. The model suggests that for most \( p \leq x \), \( H(p) \) stays very close to \( e \log \log x \).

1. Introduction

1.1. For positive integers \( a \) and \( b \), write \( a \prec b \) if \( b \equiv 1 \pmod{a} \). We are interested in properties of prime chains \( p_1 < p_2 < \cdots < p_k \), e.g. \( 3 < 7 < 29 < 59 \). Prime chains are multiplicative analogs of the well-studied additive prime \( k \)-tuples (sequences \( p_1 < \cdots < p_k \) of primes with \( p_k - p_1 \) small). Important quantities of study are \( N(x) \), the number of prime chains with \( p_k \leq x \) (\( k \) variable), \( N(x; p) \), the number of prime chains with \( p_1 = p \) and \( p_k/p_1 \leq x \), \( f(p) \), the number of prime chains with \( p_k = p \), and \( H(p) \), the length of the longest prime chain with \( p_k = p \). Estimates for these quantities have arisen in investigations of iterates of Euler’s totient function \( \phi(n) \) and Carmichael’s function \( \lambda(n) \) (e.g. [5], [6], [7], [19], [28], [29], [30]), the value distribution of \( \lambda(n) \) [21], common values of \( \phi(n) \) and the sum-of-divisors function \( \sigma(n) \) [22], and the complexity of primality certificates ([8], [33]).

In studying long chains, where the ratios \( \log p_{j+1}/\log p_j \) are small on average, we require information about the large prime factors of shifted primes \( p - 1 \). That is, we require good estimates for \( \pi(x; q, 1) \) when \( q \) is large, where \( \pi(x; q, a) = |\{p \leq x : p \equiv a \pmod{q}\}| \). Progress, however,
is hampered by our poor knowledge when \( q > \sqrt{x} \). Let \( \text{li}(x) = \int_2^x dt/\log t \). The Bombieri-Vinogradov theorem ([16], Ch. 28) implies that

\[
(1.1) \quad \sum_{m \leq Q, y \leq x} \max_{y \leq x} \left| \pi(y; m, 1) - \frac{\text{li}(y)}{\phi(m)} \right| \ll R,
\]

with \( Q = x^{1/2}(\log x)^{-B} \) and \( R = x(\log x)^{-A} \) (here \( A > 0 \) is arbitrary and \( B \) depends on \( A \)).

The corresponding statement with \( Q = x^\theta \) is not known for any fixed \( \theta > 1/2 \), however it is conjectured (the Elliott-Halberstam conjecture; abbreviated EH) that (1.1) holds with \( Q = x^\theta \) and \( R = x(\log x)^{-A} \) for any \( \theta < 1 \) and \( A > 0 \). The one-sided Brun-Titchmarsh inequality

\[
(1.2) \quad \pi(x; q, 1) \leq \frac{2x}{(q - 1) \log(x/q)},
\]

however, is useful in some situations.

If one asks just for the existence of many shifted primes \( p - 1 \) with a large prime factor, we can do a little bit better than Bombieri-Vinogradov. Let \( P^+(n) \) denote the largest prime factor of \( n \), and let \( \theta_0 \) be the supremum of real numbers \( \theta \) so that there are infinitely many primes \( p \leq x \) such that \( P^+(p - 1) \geq x^\theta \). EH implies \( \theta_0 = 1 \), and the best unconditional result is due to Baker and Harman [4], who showed that \( \theta_0 \geq 0.677 \).

In this paper, we prove new bounds for \( N(x; p), N(x), f(p) \) and \( H(p) \). At the core of our arguments is a kind of duality principle: in a chain \( p_1 \prec \cdots \prec p_k \), there are integers \( m_j \) with \( p_{j+1} = m_j p_j + 1 \), and there is an obvious bijection between the \( k \)-tuples \( (p_1, \ldots, p_k) \) and \( (p_1, m_1, \ldots, m_{k-1}) \). It is often more efficient to focus on properties of the latter vector rather than the former.

### 1.2. We begin with the problem of bounding \( N(x; p) \). By iterating (1.2), one arrives at a uniform bound (e.g. [19], Theorem 3.5)

\[
(1.3) \quad N_k(x; p) \ll \frac{x(\log x)^{k-1}}{\log x}
\]

for the number, \( N_k(x; p) \), of prime chains of length \( k \) with \( p_1 = p \) and \( p_k/p_1 \leq x \). Here \( \log_k x \) is the \( k \)-th iterate of the logarithm of \( x \), and \( c \) is some constant. Summing (1.3) over \( k \leq \frac{\log x}{\log 2} + 1 \), one obtains the weak estimate \( N(x; p) \ll x^{O(\log x)} \).

**Theorem 1.** For \( p \geq 2 \) and \( x \geq 20 \), we have the effective estimate

\[
N(x; p) \leq x \exp \left\{ \frac{\log x(\log x + O(1))}{\log_2 x} \right\} \quad (x \to \infty).
\]

In particular, for every \( \varepsilon > 0 \) there is an effective constant \( C(\varepsilon) \) so that \( N(x; p) \leq C(\varepsilon) x^{1+\varepsilon} \).

Theorem 1 has applications to problems which, at first glance, have nothing to do with prime chains. First, it is a crucial tool in the recent proof by Ford, Luca and Pomerance [22] that the equation \( \phi(a) = \sigma(b) \) has infinitely many solutions, settling a well-known 50-year old problem of Erdős. In [21], Theorem 1 is used to show that for some effective \( q_0 \), if \( \pi(p^{3a}; p^a, 1) - \pi(p^{a+1}; p^{a+1}, 1) \geq 113p^{3a-2a-3}/(\log(p^{a+1})) \) for all prime powers \( p^a \in (10^{10}, q_0) \), then for every positive integer \( n \), there is another positive integer \( m \) with \( \lambda(n) = \lambda(m) \). This nearly settles a conjecture from [5], the analog for \( \lambda \) of the famous Carmichael Conjecture for \( \phi \).
Theorem 1 is nearly best possible, since \( N(x; p) \gtrsim N_2(x; p) = \pi(px; p, 1) \), which is expected to be \( \gg x/(\log px) \) unless \( x \) is very small relative to \( p \).

**Conjecture 1.** We have \( N(x; p) \ll x \).

Conjecture 1 is easy to prove when \( p \) is bounded. Using \( f(2) = 1 \) and the recursive formula

\[
(1.4) \quad f(p) = 1 + \sum_{q \mid (p-1)} f(q),
\]

we have

\[
(1.5) \quad f(p) \leq \frac{2 \log p}{\log 2} - 1 \quad (\text{all } p).
\]

Summing on \( p \leq x \) using the prime number theorem gives \( N(x) \ll x \) and hence \( N(x; p) \ll px \).

Lower bounds on \( f(p) \) and \( N(x) \) are more difficult, since \( f(p) \) is sometimes very small, e.g. if \( p = 1 + 2^n 3^m \) then \( f(p) = 4 \) (it is conjectured that there are infinitely many such primes).

**Theorem 2.** (i) We have \( f(p) \geq 0.378 \log p \) for almost all primes \( p \). Hence, \( N(x) \gg x \).

(ii) For all \( x \geq 3 \) and any positive integer \( h \), \( |\{p \leq x : f(p) = h\}| \leq \left( \frac{6 \log x}{h} \right)^h \).

In particular, part (ii) implies that primes with \( f(p) = o(\log p) \) are exceptionally rare, the counting function being \( x^{o(1)} \) as \( x \to \infty \).

Also, by (i), we have \( N(x; 2) = \frac{1}{2} N(x) \gg x \).

It is likely that \( f(p) \) has normal order \( c \log p \) for some \( c \). A very similar problem was considered in Section 2 of [19], namely the behavior of \( I(n) = \min\{j : \phi_j(n) = 1\} \), where \( \phi_j \) is the \( j \)-th iterate of \( \phi \).

It turns out that \( F(n) = I(n) - \{1 \text{ if } n \text{ even} \} \) is completely additive, and \( F(p) = F(p-1) = \sum_{q^e \parallel p-1} a F(q) \) for odd primes \( p \). This is similar to (1.4), the only difference being the behavior at proper prime powers, which play an insignificant role in the arguments in [19]. Summing (1.4) over primes \( p \leq x \) gives

\[
N(x) = \pi(x) + \sum_{q \leq x/2} f(q) \pi(x; q, 1).
\]

Inserting this relation into the proof of Theorem 2.1 in [19] (combine Lemma 2.4, Corollary 2.5, (2.8) and Theorem 2.1 therein), we obtain the following.

**Theorem A.** If (1.1) holds with \( Q = x^{1-(\log_2 x)^{-1+\delta}} \) and \( R = x(\log x)^{-2} \) for some fixed \( \delta > 0 \), then \( N(x) \sim cx \) for some constant \( c > 0 \) and \( f(p) \) has normal order \( c \log p \).

Conjecture 1 implies that for all \( \varepsilon > 0 \) and prime \( q > (\log x)^{1+\varepsilon} \), for most \( p \leq x \) there is no prime chain \( q < \cdots < p \). This gives, conditionally, the first part of [19, Conjecture 1]. By contrast, the proof of Theorem 4.5 of [19] implies that if \( q \leq (\log x)^c \), for some small constant \( c > 0 \), then for almost all primes \( p \leq x \), there is a prime chain \( q < \cdots < p \).

\[\text{for every } \varepsilon > 0, |f(p) - c \log p| \ll \varepsilon \log p \text{ for all primes but } o(\pi(x)) \text{ exceptions up to } x.\]
1.3. The Pratt tree $T(p)$ for a prime $p$ is the tree with root node $p$, below $p$ are nodes labelled with the prime factors $q$ of $p-1$, below each $q$ are nodes labelled with the prime factors of $q-1$, and so on. In 1975, V. Pratt [33] used it in conjunction with Lucas’ primality test ([15], §4.1) to show that every prime has a short certificate (proof of primality). Pomerance [32] gave another method for producing primality certificates, but it is an open problem whether the Pratt certificate has longer complexity for most primes (see §1 of [32]). Two important statistics of the Pratt tree are the total number of nodes $f(p)$ and the height $H(p)$, the latter being the length of the longest prime chain ending at $p$. It is known (see [7], [18], [20]) that the number of primes at a fixed level $n$ in the Pratt tree for most $p$ is $\sim (\log p)^n/n!$. The idea is that for most primes $p$, $p-1$ has a multiplicative structure similar to that of a typical integer of its size; namely, $p-1$ has about $\log_2 p$ prime factors, uniformly distributed on a log log-scale (see [25], Ch. 1). This, however, does not give much information about $H(p)$.

Figure 1 shows histograms of $H(p)$ for all primes $p \leq 10^9$ and for 1000 randomly chosen primes near $10^{10}$. Very little is known about the distribution of $H(p)$, the extremal behavior being a case in point. First, $H(p) = 2$ if and only if $p$ is a Fermat prime, that is, $p = 2^{2^m} + 1$ for some $m$. It seems plausible that $H(p) = 3$ for infinitely many $p$, but this is hopeless to prove at this time. At the other extreme, we have the trivial upper bound $H(p) \leq \frac{\log p}{\log 2} + 1$. Large values of $H(p)$ may be obtained using the special chain $2 = q_1 \prec q_2 \prec \cdots$, where, for each $j$, $q_{j+1}$ is the smallest prime $\equiv 1 \pmod{q_j}$. By Linnik’s theorem, $q_{j+1} \leq q_j^L$ for some constant $L$, hence $H(q_j) \geq \frac{\log \log q_j}{\log L}$. It is conjectured that $q_{j+1} \leq q_j(\log q_j)^C$ for some fixed $C$ and this implies a far stronger bound $H(q_j) \gg \frac{\log q_j}{\log \log q_j}$. Even showing $H(p)/\log_2 p \to \infty$ for an infinite sequence of $p$ is extremely hard, as it implies $\theta_0 = 1$: a prime chain $p_1 \prec \cdots \prec p_k = p$ with $k = H(p)$ satisfies

$$\log p_k \over \log p_1 = \prod_{j=1}^{k-1} \frac{\log p_{j+1}}{\log p_j} \geq \left( \min_{1 \leq j \leq k-1} \frac{\log p_{j+1}}{\log p_j} \right)^{k-1},$$

and hence

$$\Lambda := \limsup_{p \to \infty} \frac{H(p)}{\log_2 p} \leq \frac{1}{-\log \theta_0}.$$
On the other hand, Kátai [27] proved that for some constant $c > 0$, $H(p) \geq c \log_2 p$ for almost all primes $p$ (for all primes $p \leq x$ with $o(x/\log x)$ exceptions). We prove a version with the constant made explicit in terms of the level of distribution of primes in progressions.

**Theorem 3.** (a) If (1.1) holds with $Q = x^\theta$ and $R = o(x/\log x)$, then for any $c < \frac{1}{e e^{-1} \log \theta}$, $H(p) > c \log_2 p$ for almost all primes $p$;

(b) If (1.1) holds with $Q = x^\theta$ and $R = x(\log x)^{-A}$ for every $A > 1$, then for every $c < \frac{1}{e \log \theta}$, there is a $K$ so that $H(p) > c \log_2 p$ for $x(\log x)^K$ primes $p \leq x$. Consequently, $\Lambda \geq \frac{1}{\log \theta}$.

**Corollary 1.** EH implies that for every $c < e$, $H(p) > c \log_2 p$ for almost all $p$.

**Remark 1.** By the Bombieri-Vinogradov theorem and (1.2), for every $\varepsilon > 0$, (1.1) holds with $Q = x^{1-\varepsilon}$ and $R = O_{\varepsilon}(x/\log x)$.

If $\Lambda \geq -\frac{1}{\log \theta}$ for some $\theta < 1$, then there are many chains $2 = p_1 < \cdots < p_k$ with $\frac{\log p_{j+1}}{\log p_j}$ at most $1/\theta$ on average. Thus, although (1.6) is likely an equality, to prove this would require strong information about the set of primes with $P^+(p-1)$ near $p^{\theta_0}$. The same difficulty arises when trying to prove (a) with $\theta = \theta_0$.

The bound on $H(p)$ in (a) is weaker than in (b), since it is unusual in a chain $p_1 < \cdots < p_k$ for most of the ratios $\frac{\log p_{j+1}}{\log p_j}$ to be close to $1/\theta$. The constant $e^{-1}$ appearing in (a) is likely best possible; see Conjecture 2 below.

Turning to upper bounds, before our work it was unknown if there is an infinite sequence of primes with $H(p) = o(\log p)$. A natural approach is to find $p$ such that $P^+(p-1)$ is small and use $H(p) = 1 + \max_{q_1(p-1)} H(q) \ll \max_{q_1(p-1)} \log q$. However, our knowledge of smooth shifted primes is very weak (the world record is $P^+(p-1) < p^{0.2961}$ infinitely often [4]). Using a new and very different approach, we give a much stronger upper bound.

**Theorem 4.** We have $H(p) \leq (\log p)^{0.9503}$ for almost all $p$.

The proof of Theorem 4 involves showing that for most primes $p$, all the primes at some bounded level of the tree are small. In particular, this settles [19] Conjecture 2.

**Theorem 5.** For every $\varepsilon > 0$ and $\delta > 0$, there is an integer $k$ so that for large $x$ and at least $(1 - \delta)x$ integers $n \leq x$, $P^+(\phi_k(n)) \leq x^{\varepsilon}$.

There is a folklore conjecture that $H(p) = O(\log_2 p)$ for most $p$.

**Conjecture 2.** $H(p)$ has normal order $e \log_2 p$.

**Remark 2.** The lower bound in Conjecture 2 follows from EH (Corollary 1). Conversely, by (1.6), the lower bound in Conjecture 2 implies that $\theta_0 \geq e^{-1/e} = 0.6922 \ldots$ (with a bit more work using (1.2), one can deduce $\theta_0 \geq 0.73$).

The upper bound in Conjecture 2 appears to be even more difficult. We cannot see a way to deduce it from standard conjectures in prime number theory, e.g. EH plus a uniform prime $k$-tuples conjecture, although Theorem 4 can be significantly improved under such hypotheses.

Understanding $H(p)$ requires detailed knowledge of the distribution of the large prime factors of shifted primes $p-1$. Making a reasonable assumption for this distribution (a consequence of EH), in Section 6 we model the Pratt tree by a branching random walk. The model provides a much more precise version of Conjecture 2.
Theorem 6. \( H(p) = e^{\log_2 p} - \frac{3}{2} \log_3 p + E(p) \), where for some fixed \( c, c' > 0 \) and any \( z \geq 0 \), the number of \( p \leq x \) for which \( E(p) \geq z \) is \( e^{-c'z} \pi(x) \) and \( \ll e^{-cz} \pi(x) \), and \( E(p) \leq -z \) for \( O(\exp\{-e^c\pi(x)\}) \) primes \( \leq x \).

Notable features of Conjecture 3 are (i) the tightness of \( E(p) \): the distribution of \( H(p) \) over \( p \leq x \) does not widen as \( x \to \infty \), and (ii) the pronounced asymmetry of the distribution of \( E(p) \). The analogs of these features for our probabilistic model are proved rigorously.

Unconditionally, we show that \( H(p) \) is tight to the left of its median.

Theorem 6. Suppose \( g \) and \( h \) are increasing, \( 0 \leq g(x) \leq h(x) \), \( h(x^2) - h(x) \leq K \) and \( g(x^2) - g(x) \leq K \) for \( x \geq 1 \). Suppose, for large \( x \), that \( H(p) \geq h(p) \) for at least \( c \pi(x) \) primes \( \leq x \). Then \( H(p) \geq h(p) - g(p) \) for all primes \( p \leq x \) with at most \( O(\pi(x) \exp\{-c \log^2 \frac{x}{K} g(x)\}) \) exceptions.

We conclude this section with a conjecture about prime chains, which follows from the prime \( k \)-tuples conjecture (with \( m = 2 \) below) but should be “easier”. It is a multiplicative analog of the statement that the primes contain arbitrarily long arithmetic progressions, recently proved by Green and Tao [23]. Even the case \( k = 3 \) is not known.

Conjecture 4. For each \( k \geq 3 \), there are infinitely many prime \( k \)-tuples \( (p_1, \ldots, p_k) \) where, for some \( m, p_{j+1} = mp_j + 1 \) for \( 1 \leq j \leq k - 1 \).

Notation. The letters \( p \) and \( q \), with or without subscripts, always denote primes. Constants implied by the \( O - \) and \( \ll - \) symbols do not depend on any parameter unless indicated. In Section 6 we use \( P \) for probability and \( E \) for probabilistic expectation.

2. Shifted chains: proof of Theorem 1

The underlying idea is a sieve; relax the condition that the numbers in the chain are prime, and only require that they do not have small prime factors. Let \( y \geq 2 \) and let \( r \) be the product of the primes \( \leq y \). For \( (a, r) = 1 \), let \( G_a(x; y) \) be the number of chains \( n_1 < \cdots < n_k \) with \( n_1 = a \), with \( n_k/n_1 \leq x \) and consisting of numbers coprime to \( r \). If \( p > y \), then \( N(x; p) \leq G_p(x; y) \). There are integers (“links”) \( m_1, \ldots, m_{k-1} \) with \( n_{j+1} = m_j n_j + 1 \) for \( 1 \leq j \leq k - 1 \). For positive integers \( a, b \) and real \( s > 1 \),
\[
S(a, b) = S(a, b; r, s) = \sum_{m > 1} m^{-s} \quad \text{for } am + 1 \equiv b \pmod{r}
\]
encodes the possible links \( m \) from a number \( n_i \equiv a \pmod{r} \) to a number \( n_{i+1} \equiv b \pmod{r} \).

Fix \( r, s \) and let \( U_r = (\mathbb{Z}/r\mathbb{Z})^* \). Let \( A_k(a_1, a_k) \) be the sum of \( (m_1 \cdots m_{k-1})^{-s} \) over all tuples \( (m_1, \ldots, m_{k-1}) \) which could serve as links in a chain starting from a number \( n_1 \equiv a_1 \pmod{r} \), ending with a number \( n_k \equiv a_k \pmod{r} \) and with all numbers in the chain coprime to \( r \). Then \( A_2(a_1, a_2) = S(a_1, a_2) \) and for \( k \geq 3 \),
\[
A_k(a_1, a_k) = \sum_{a_2, \ldots, a_{k-1} \in U_r} S(a_1, a_2) S(a_2, a_3) \cdots S(a_{k-1}, a_k).
\]
Let \( V_k(a_1) \) be the column vector \( (A_k(a_1, a_k) : a_k \in U_r) \). For consistency, let \( V_1(a_1) \) be a vector with all zero entries except for an entry of 1 in the \( a_1 \) position. Since
\[
A_{k+1}(a_1, a_{k+1}) = \sum_{a_k \in U_r} A_k(a_1, a_k) S(a_k, a_{k+1}),
\]
we obtain $V_{k+1}(a_1) = MV_k(a_1)$, where $M = M(r, s) = (S(a, b))_{b, a \in U_r}$. The rows of $M$ are indexed by $b$ and the columns are indexed by $a$. Finally, let $F_k(a_1) = \sum_{a_k} A_k(a_1, a_k)$, so that

$$F_k(a_1) = (1, \ldots, 1)V_k(a_1) = (1, \ldots, 1)M^{k-1}V_1(a_1);$$

i.e., $F_k(a_1)$ is the sum of the entries of column $a_1$ in $M^{k-1}$. Since $m_1 \cdots m_{k-1} \leq n_k/n_1 \leq x$,

$$G_a(x; y) \leq \inf_{s > 1} \left( x^s \sum_{1 \leq k \leq \log x \over \log y + 1} F_k(a) \right).$$

Observe that the sum on $k$ in (2.1), if extended to $k = \infty$, is convergent if and only if $M$ is a contracting matrix, i.e., all the eigenvalues of $M$ have modulus $< 1$. Since $M$ has positive real entries, the Perron-Frobenius Theorem implies that the eigenvalue with largest modulus is positive, real and simple. Call this eigenvalue $\lambda_s(y)$.

We show below that if $y$ is large and $s \geq 1 + \log_2 y$, then $\lambda_s(y) < 1$. Accurate estimation of $\lambda_s(y)$ is difficult for large $y$, but the largest row sum of $M$ serves as an upper bound. For a generic matrix $A$, let $R_b(A)$ be the sum of the entries in the row indexed by $b$, and let $R(A)$ be the maximum row sum of $A$. For row $b$ of $M$, write $d = (b - 1, r)$ and $b' = b - 1 \over d$. Then

$$R_b(M) = \sum_{a \in U_r, a = b \mod d} \sum_{a = b \mod d} m^{-s} = d^{-s} \sum_{k \cdot r/d = 1} k^{-s} \# \{ a \in U_r : ak \equiv b' \mod r/d \}. $$

The congruence $ak \equiv b' \mod r/d$ has a unique solution modulo $r/d$, and hence has $\phi(d)$ solutions $a \in U_r$. Thus,

$$R_b(M) = \frac{\phi(d)}{d^s} \sum_{k \cdot r/d = 1} k^{-s} = \frac{\phi(d)}{d^s} \prod_{p \mid r/d} (1 - p^{-s})^{-1} = \prod_{p \mid y} (1 - p^{-s})^{-1} \prod_{p \mid d} p - 1 \over p^s - 1. $$

Therefore, since $d$ is always even,

$$R(M) = \frac{1}{2^s - 1} \prod_{p \mid y} (1 - p^{-s})^{-1}. $$

Since $R(AB) \leq R(A)R(B)$, $R(M^{k-1}) \leq R(M)^{k-1}$. To bound $G_a(x; y)$, we need to bound the largest column sum of $M^{k-1}$. Lacking a better approach, we use the crude bound $\phi(r)R(M)^{k-1}$. Thus, $F_k(a) \leq \phi(q)R(M^{k-1}) \leq \phi(r)R(M)^{k-1}$. By (2.1),

$$G_a(x; y) \leq \phi(r) \inf_{s: R(M) < 1} \frac{x^s}{1 - R(M)}.$$

By standard prime number estimates, if $1 < s \leq 2$, then

$$- \sum_{p \mid y} \log (1 - p^{-s}) = O(1/y^{2s-1}) + \sum_{p \mid y} p^{-s} \ll \frac{e^{-(s-1)\log y}}{(s-1) \log y}. $$

Take $y = \log_2 x$ and $s = 1 + \log_2 y \over \log y$. Since $2^s - 1 = 1 + (2 \log 2)(s - 1) + O((s - 1)^2)$, (2.2) implies $1 - R(M) \sim (2 \log 2)(s - 1)$ as $x \to \infty$. Since $\phi(r) \leq r = o(1 + o(1))$ as $x \to \infty$, this proves Theorem I.
3. Proof of Theorem 2

Let \( Q(p) \) be the multiset of prime labels appearing in \( T(p) \), and let \( T'(p) \) be the subtree of \( T(p) \) consisting of nodes with odd prime labels. There is a natural bijection between \( T(p) \) and \( T'(p) \), obtained by adding to every node in \( T'(p) \) a child node with label 2. Let \( l(n) = \prod_{p \mid n} p^{d - 1} \). The quantity \( \frac{2}{q - 1} l(q - 1) \) measures the “loss of mass” when descending from a node labelled \( q \) to its child nodes: in fact, it is easy to see that

\[
\prod_{q \in Q(p)} \left( \frac{q}{q - 1} l(q - 1) \right) = p.
\]

If \( p > 2 \), exactly half of the nodes in \( T(p) \) are labelled with 2, thus

\[(3.1)\quad \prod_{q \in Q(p)} l(q - 1) \leq p2^{\frac{1}{2} f(p)}.\]

Consider the set of \( p \leq x \) with \( f(p) = h \), where \( h = 2n \) is positive and even. Let \( T \) be the set of rooted trees on \( n \) nodes. For each \( T' \in T \), we consider separately the primes \( p \) with \( T'(p) \) tree-isomorphic to \( T' \). Form the tree \( T \) on \( h \) nodes, by adding to each node of \( T' \) an additional child node. We count how many ways we can label with primes the nodes of \( T \), with the leafs having label 2 and the root having label \( p \leq x \). For a given prime \( p \), there may be more than one way to assign primes in the Pratt tree to the nodes of \( T' \); this occurs when some node has two or more child trees that are isomorphic (as rooted trees). Thus, for each \( T' \), we count ways to assign primes to the nodes, and divide by the number of ways in which we can permute the nodes; that is, the number \( I(T') \) of isomorphisms of \( T' \). Assign to each node an ordinal number 1, 2, \ldots, \( h \) so that the children of node \( j \) are assigned lower ordinals (e.g., node 1 will be a lowest leaf, and node \( h \) will be the root). To node number \( j \), we let \( q_j \) be its prime label and \( l_j = l(q_j - 1) \). Let \( B_j \) be the set of ordinal numbers of the children of node \( j \), and observe that \( B_1, \ldots, B_h \) depend only on \( T' \). With this notation,

\[(3.2)\quad q_j - 1 = l_j \prod_{k \in B_j} q_k.\]

With \( T \) fixed, (3.2) implies a natural bijection between \( (q_1, \ldots, q_h) \) and \( (l_1, \ldots, l_h) \) (recall that leafs have \( q_j = l_j = 2 \)). By (3.1), we have for any \( \beta > 0 \)

\[
|\{ p \leq x : f(p) = h \}| \leq \sum_{T' \in T} \frac{1}{I(T')} \sum_{l_1, \ldots, l_h} \left( \frac{x^{2-h/2}}{l_1 \cdots l_h} \right)^\beta.
\]

Suppose that \( j \geq 2 \) and that \( l_1, \ldots, l_{j-1} \) have been chosen. If node \( j \) is a leaf of \( T \), then \( l_j = 1 \). Otherwise, using (3.2), the primes \( q_k \) for \( k \in B_j \) are determined by \( l_1, \ldots, l_{j-1} \). Moreover, a prime \( r \mid l_j \) must equal one of these primes \( q_k \). Hence

\[
\sum l_j^\beta \leq \prod_{k \in B_j} \left( 1 - q_k^{-\beta} \right)^{-1} \leq (1 - 2^{-\beta})^{-1} (1 - 3^{-\beta})^{-|B_j| - 1}.\]
By Borchardt’s formula\footnote{commonly known as Cayley’s formula.} for counting labelled trees,
\[
\sum_{T' \in \mathcal{T}} \frac{1}{I(T')} = \frac{n^{n-2}}{(n-1)!} = \frac{n^{n-1}}{n!} \leq e^n.
\]
Since \(\sum_{|B_j| \geq 1} (|B_j| - 1) = h/2 - 1\), we conclude that
\[
(3.3) \quad |\{p \leq x : f(p) = h\}| \leq e^{h/2} (x^{2-h/2})^\beta (1 - 2^\beta)^{-h/2} (1 - 3^{-\beta})^{-h/2}
\]
Taking \(\beta = 0.37\), the right side of (3.3) is \(\leq x^{0.37} (27.8371)^{h/2} \leq x^{0.999}\) for \(h \leq 0.378 \log x\). For the second part of Theorem 2, assume \(h \leq (2/5) \log x\) (if \(h \geq 3 \log x\), there are no such \(p\) and for \((2/5) \log x < h \leq 3 \log x\), \((\frac{5 \log x}{h})^\beta \leq 1 \beta \log 2\) and \(1 - 3^{-\beta} \geq 0.889 \beta\). Hence, the right side of (3.3) is \(\leq x^\beta (4.412/\beta^2)^{h/2}\).

We remark that numerical improvements are possible by refining the above analysis; e.g. using the fact that leafs of \(T'\) must be labelled with a Fermat prime.

4. Lower Bounds for \(H(p)\): Proof of Theorem 3

We show part (b) first, as the proof is much easier.

**Proof of Theorem 3 (b).** Let \(c'\) and \(\theta'\) satisfy \(\theta' > 1/3\) and \(c < c' < \frac{1}{-\log \theta'} < \frac{1}{-\log \theta}\), and define \(K\) by \(8 \theta^K = \theta - \theta'\). Let \(x_0\) be large, depending on \(K, c, c', \theta, \theta'\) and put \(c'' = c' \log_2(x_0^3)\). Let \(\mathcal{P} = \{p : H(p) \geq c' \log_2 p - c''\}\). In particular, \(\mathcal{P}\) contains all primes \(\leq x_0^3\). We shall prove
\[
(4.1) \quad Q(x) := |\mathcal{P} \cap (x/2, x)| \geq \frac{x}{(\log x)^K}
\]
for \(x \geq x_0\), which implies the desired conclusion (since \(c' > c\)). By the prime number theorem and the fact that \(K > 1\), if \(x_0\) is large enough then (4.1) holds for \(x_0 \leq x \leq x_0^3\). Suppose \(y \geq x_0^3\) and (4.1) holds for \(x_0 \leq x \leq y\). Assume \(y < x \leq 2y\) and put \(I = \mathcal{P} \cap (x^{\theta'}, x^\theta]\). Suppose that \(x/2 < p \leq x\), and that \(q|p - 1\), where \(q \in I\). Then
\[
H(p) \geq 1 + H(q) \geq 1 + c' \log_2 q - c''
\]
\[
\geq 1 + c' \log_2 x + c' \log \theta' - c'' > c' \log_2 p - c''
\]
so that \(p \in \mathcal{P}\). For \(x/2 < p \leq x\), \(p - 1\) is divisible by at most two primes from \(I\), hence
\[
Q(x) \geq \frac{1}{2} \sum_{q \in I} \left( \pi(x; q, 1) - \pi(x/2; q, 1) \right) \geq \frac{x}{4 \log x} \sum_{q \in I} \frac{1}{q} + O\left( \frac{x}{(\log x)^{K+1}} \right).
\]
Since (4.1) holds for \(x^{\theta'} < y \leq x^\theta\), the sum on \(q \in I\) is
\[
\sum_{2^{\theta'} < x^{\theta'}} \frac{Q(2^{\theta'} x^{\theta'})}{2^{\theta'} x^{\theta'}} \geq \left( \frac{(\theta - \theta') \log x}{\log 2} - 1 \right) \frac{1}{(\log x)^K} \geq \frac{\theta - \theta'}{\theta^K (\log x)^{K-1}} = \frac{8}{(\log x)^{K-1}}.
\]
Therefore, (4.1) holds. By induction on dyadic intervals, (4.1) holds for all \(x \geq x_0\). \(\square\)
Remark. The same proof gives, assuming that (1.1) holds with \( Q = x^{1-\varepsilon(x)} \) and \( R(x) = x(\log x)^{-g(x)} \) where \( \varepsilon(x) \to 0 \) and \( g(x) \to \infty \) as \( x \to \infty \), that \( H(p) \geq h(p) \log_2 p \) for infinitely many \( p \), where \( h(p) \to \infty \) as \( p \to \infty \) (the function \( h \) depending on the functions \( \varepsilon, g \)).

Proof of Theorem 3 (a). We proceed by induction as in part (b), but instead we iterate by many levels in the chain at once rather than one level at a time. Suppose that \( c < h < c' < 1/(e^{-1} - \log(\theta)) \). For some constant \( c'' \), described below, let \( \mathcal{P} = \{ p : H(p) \geq c' \log_2 p - c'' \} \). We will show, for some \( \delta > 0 \), that
\[
\mathcal{P}(x) := |\{ p \leq x : p \in \mathcal{P} \}| \geq \delta \frac{x}{\log x}.
\]
Consequently, a positive proportion of primes \( p \) satisfy \( H(p) > h \log_2 p \), and Theorem 3 (a) follows from Theorem 6.

By Stirling’s formula, there is an integer \( k \geq 2 \) such that \( 1/c' > \frac{1}{k}(k!)^{1/k} - \log(\theta) \). Let \( \alpha, \beta \) satisfy \( e^{-k/c'} < \beta < \theta^k \exp\left(-(k!)^{1/k}\right) \) and \( \beta \exp\left((k!)^{1/k}\right) < \alpha < \theta^k \). Suppose that \( \delta \) is sufficiently small, depending only on the choice of \( c', \theta, k, \alpha, \beta \). Let \( x_0 \) be sufficiently large, depending on \( c', \theta, k, \alpha, \beta, \delta \), and put \( c'' = c' \log_2(x_0) \). Observe that (4.2) holds trivially for \( 2 \leq x \leq x_0 \), provided \( \delta \) is small enough. Throughout this proof, constants implied by the \( O - \) and \( \ll - \) symbols may depend on \( c', \theta, k, \alpha, \beta \), but not on \( \delta \).

Next, suppose that \( Y \geq x_0 \) and that inequality (4.2) holds for \( 2 \leq x \leq Y \). Let \( S \) be a subset of the primes in \( \mathcal{P} \cap [Y^\beta, Y^{\theta}] \). Let \( M(S) \) be the number of primes \( p_0 \in (Y, 2Y] \) so that there is a prime chain \( p_k < p_{k-1} < \cdots < p_0 \) with \( p_k \in S \). For such \( p_0 \), we have
\[
H(p_0) \geq k + H(p_k) \geq k + c' \log_2 p_k - c''
\]
\[
= c' \log_2(2Y) - c'' + c' \log \beta + k + O\left(\frac{1}{\log Y}\right) \geq c' \log_2 p_0 - c''
\]
if \( x_0 \) is large enough. We will show, for appropriate \( S \), that
\[
(4.3) \quad M(S) \geq \delta \frac{Y}{\log Y},
\]
which implies \( P(2Y) \geq P(Y) + M(S) \geq 2\delta Y/\log(2Y) \). By induction over dyadic intervals, (4.3) implies (4.2), and hence Theorem 3 (a).

To prove (4.3), we will consider chains satisfying not only \( p_k \in S \), but also
\[
(4.4) \quad p_{j+1} \leq p_j^\theta \quad (0 \leq j \leq k - 1), \quad p_1 \leq Y^\theta.
\]
With (4.4), we can use (1.1) to accurately count such chains. We have \( M(S) \geq M_1(S) - M_2(S) \), where \( M_1(S) \) is the number of chains satisfying \( p_k \in S \) and (4.4), and \( M_2(S) \) is the number of pairs of distinct chains satisfying these conditions with the same \( p_0 \). We begin with
\[
M_1(S) = \sum_{p_k \in S} \sum_{p_{k-1}} \cdots \sum_{p_1} \left( \pi(2Y, p_1, 1) - \pi(Y, p_1, 1) \right),
\]
where \( p_k < \cdots < p_0 \) and (4.4) in the summations. By (1.1) and induction on \( 1 \leq j \leq k \),
\[
(4.5) \quad M_1(S) = \frac{Y}{\log Y} \sum_{p_k \in S} \sum_{p_{k-1}} \cdots \sum_{p_j} \left( \log_2 Y^\theta - \log_2 p_j \right)^{j-1} \frac{1}{p_j(j-1)!} + o\left(\frac{Y}{\log Y}\right).
\]
Here, we used that for each $p_j$, there are $O(1)$ chains $p_k < \cdots < p_j$ with $p_k \geq Y^\beta$. By (4.5) with $j = k$,

$$M_1(\mathcal{P} \cap [Y^\beta, Y^\alpha]) \geq \frac{\delta Y}{\log Y} \int_{Y^\beta}^{Y^\alpha} \frac{\log \frac{\log_2 Y^\alpha - \log_2 t}{(k-1)! \log t}}{t \log t} \, dt + o\left(\frac{Y}{\log Y}\right) = \frac{\delta Y}{\log Y} \left(\frac{\log(\alpha/\beta)}{k!}\right) + o\left(\frac{Y}{\log Y}\right).$$

By hypothesis, $\log(\alpha/\beta) > (k!)^{1/k}$. The summands in (4.5) (with $j = k$) are $\asymp 1/p_k$ for $Y^\beta \leq p_k \leq Y^\alpha$. Hence, if $\delta$ is small enough, there is a set $S \subseteq \mathcal{P} \cap [Y^\beta, Y^\alpha]$ such that

$$\sum_{p_k \in S} \frac{1}{p_k} \ll \delta, \quad M_1(S) \geq (\delta + \delta^{3/2}) \frac{Y}{\log Y}. \tag{4.6}$$

We have $M_2 = M_{2,0} + \cdots + M_{2,k-1}$ where $M_{2,j}$ counts pairs of coupled chains

$$p_k < \cdots < p_{j+1} \Biggm| p_j < \cdots < p_0$$

with each of the two chains satisfying (4.4), $p_{j+1} \neq p_{j+1}'$ and $p_k, p_k' \in S$. We further write $M_{2,j} = M_{2,j}' + M_{2,j}''$, where $M_{2,j}'$ counts pairs of such chains with $p_j \leq p_{j+1}' Y^{\delta^2}$. As before, for each pair $(p_{j+1}, p_{j+1}')$, there are $O(1)$ choices for $p_k, p_k', \ldots, p_{j+2}, p_{j+2}'$. For $M_{2,0}'$, $p_1 p_1' \geq Y^{1-\delta^2}$, and for each $p_0$, there are $O(1)$ choices for $p_1, p_1'$. By sieve methods (e.g. Theorem 2.2 of [24]),

$$M_{2,0}' \ll \sum_{1 \leq k \leq Y^{\delta^2}} \left|\{n \leq Y : n \equiv 1 \pmod{k}, P^-(n(n-1)/k)) > Y^\beta\}\right| \ll \sum_{1 \leq k \leq Y^{\delta^2}} \frac{Y}{\phi(k) \log^2 Y} \ll \frac{\delta^2 Y}{\log Y}.$$  

Here, $P^-(m)$ is the smallest prime factor of $m$. For $j \geq 1$, an argument similar to that leading to (4.5), followed by the same sieve bound, gives

$$M_{2,j}' \ll \frac{Y}{\log Y} \sum_{p_{j+1}, p_{j+1}'} \sum_{p_j} \frac{1}{p_j} \ll \sum_{k \leq Y^{\delta^2}} \sum_{n \leq Y, n \equiv 1 \pmod{k, P^-(n(n-1)/k)) > Y^\beta}} \frac{1}{n} \ll \delta^2 \frac{Y}{\log Y}. \tag{4.7}$$

For chains counted by $M_{2,j}''$, the Brun-Titchmarsh inequality suffices for the estimations. When $j \geq 1$ and $p_j$ is given, as before we have

$$\sum_{p_{j-1}} \cdots \sum_{p_1} \pi(2Y, p_1, 1) \ll \frac{Y}{p_j \log Y}. \tag{1.2}$$

By partial summation and (1.2), given $p_{j+1}$ and $p_{j+1}'$,

$$\sum_{p_j} \frac{1}{p_j} \ll \frac{1}{p_{j+1} p_{j+1}' \delta^2 \log Y} + \frac{\log(1/\delta)}{p_{j+1} p_{j+1}'} \ll \frac{\log(1/\delta)}{p_{j+1} p_{j+1}'}.$$
For \( j + 1 \leq r \leq k - 1 \),
\[
\sum_{p_r} \frac{1}{p_r} \ll \frac{1}{p_{r+1}}, \quad \sum_{p_r'} \frac{1}{p_{r+1}'} \ll \frac{1}{p_{r+1}}.
\]
Finally, by (4.6), we arrive at
\[
M_{2j}'' \ll \delta^2 \log(1/\delta) \frac{Y}{\log Y}.
\]
In a similar way, when \( j = 0 \), we have by (1.2) and partial summation,
\[
\sum_{p_1 p_1'} \pi(2Y, p_1 p_1', 1) \ll \frac{\log(1/\delta)}{p_2 p_2'}.
\]
A second application of (4.7) then gives (4.8) in this case.

Finally, combining our estimates for \( M_{2j}' \) and \( M_{2j}'' \), we obtain
\[
M_2(S) \ll \delta^2 \log(1/\delta)Y/\log Y.
\]
Together with (4.6), if \( \delta \) is small enough then (4.3) holds, and this completes the proof.

5. Proof of Theorems 4 and 5

The proofs of Theorems 4 and 5 rely on the fact that the largest prime factor of \( p - 1 \) cannot be too large too often. At the core is a sieve upper bound for \( k \)-tuples of primes which is uniform in \( k \), and careful averages of the associated singular series. There is a potentially troublesome factor \( 2k k! \) in the sieve estimate, which is partly overcome by observing that if \( H(p) \) is large, then there must be a prime chain in the Pratt tree for \( p \) which is very condensed in a multiplicative sense.

**Lemma 5.1.** There is a positive constant \( \delta \) so that the following holds. Let \( a_1, \ldots, a_k \) be positive integers, let \( b_1, \ldots, b_k \) be integers with \((a_j, b_j) = 1\) for all \( j \), and let \( \xi(p) \) be the number of solutions of \( \prod_{i=1}^k (a_i n + b_i) \equiv 0 \pmod{p} \). If \( x \geq 10, 1 \leq k \leq \frac{\log x}{\log_2 x} \) and
\[
B := \sum_p \frac{k - \xi(p)}{p} \log p \leq \delta \log x,
\]
then the number of integers \( n \leq x \) for which \( a_1 n + b_1, \ldots, a_k n + b_k \) are all prime and \( > k \) is
\[
\ll \frac{2^k k! (\log x)^k \exp(O \left( \frac{k B + k^2 \log_2 x}{\log x} \right))}{\prod_p \left( 1 - \frac{\xi(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k}}.
\]

**Proof.** Since \( \xi(p) = k \) for large \( p \), \( \mathcal{S} > 0 \) if and only if \( \xi(p) < p \) for all \( p \). Also, \( \xi(p) \leq k \) for all \( p \). Hence, if \( \mathcal{S} = 0 \), the number of \( n \) is zero. If \( \mathcal{S} > 0 \), Montgomery’s large sieve estimate [12, Théorème 6] implies that the number of \( n \) in question is \( \ll x/G(\sqrt{x}) \), where
\[
G(z) = \sum_{n \leq z} g(n), \quad g(n) = \mu^2(n) \prod_{p | n} \frac{\xi(p)}{p - \xi(p)}.
\]
and \( \mu \) is the Möbius function. For fixed \( k \), the argument in \([24, \S 5.3]\) gives \( G(z) \sim (\log z)^k/(k! \mathcal{S}) \).

We sketch how to make explicit the dependence on \( k \). By the argument in \([24\text{ p. 147–148}]\),

\[
\sum_{d \leq z} g(d) \log d = \sum_{d \leq z} g(d) \sum_{p \leq z/d} \frac{\xi(p) \log p}{p} + \sum_{h \leq z} \sum_{p \mid h, p > z/h} \frac{\xi(p) \log p}{p} = k \sum_{d \leq z} g(d) \log \frac{z}{d} + O(G(z)(B + k \log_2 z)).
\]

Adding the sum on the right side to both sides yields

\[
G(z) \log z = (k + 1) \int_1^z \frac{G(t)}{t} dt + r(z)G(z) \log z,
\]

where \( r(z) \ll \frac{B + k \log z}{\log z} \). If \( \delta \) is small enough and \( z \geq \sqrt{x} \), then \( |r(z)| \leq \frac{1}{2} \). By the argument in \([24\text{ p. 150}]\), for some constant \( D \) and for \( z \geq \sqrt{x} \),

\[
(1 - r(z)) \frac{G(z)}{\log^k z} = D \exp \left\{ O \left( \frac{kB + k^2 \log_2 z}{\log z} \right) \right\}.
\]

By the argument on \([24\text{ p. 151–152}]\), \( D^{-1} = k! \mathcal{S} \). Taking \( z = \sqrt{x} \) completes the proof. \( \square \)

For given integers \( m_1, \ldots, m_{k-1} \geq 2 \), we will apply Lemma \([5.1]\) with the forms \( f_1(n) = n \), \( f_{j+1}(n) = m_jf_j(n) + 1 \ (1 \leq j \leq k - 1) \). We have \( f_j(n) = a_jn + b_j \), where

\[
(5.1) \quad a_j = m_1 \cdots m_{j-1} \ (j \geq 1), \quad b_1 = 0, \quad b_j = 1 + \sum_{i=2}^{j-1} m_i \cdots m_{j-1} \ (j \geq 2).
\]

Clearly, \( (a_j, b_j) = 1 \). Let \( \mathcal{S}(m) = \mathcal{S} \) be the associated singular series and let \( \xi(p, m) = \xi(p) \).

**Lemma 5.2.** There is a positive constant \( c_1 \) so that \( \mathcal{S}(m) \ll (c_1 \log_2(4m_1 \cdots m_{k-1}))^{k-1} \). Also,

\[
\sum_{p} \frac{k - \xi(p, m)}{p} \log p \ll k \log_2(4m_1 \cdots m_{k-1}) + O(1).
\]

**Proof.** We have \( \xi(p, m) = k \) if \( p \nmid N \), where \( N = m_1 \cdots m_{k-1} \prod_{i<j} |a_ib_j - a_jb_i| \). Also, \( \xi(p, m) \geq 1 \) for all \( p \). Let \( x = m_1 \cdots m_{k-1} \geq 2^{k-1} \). By \((5.1)\), \( a_j \leq x \) and \( b_j \leq 1 + \sum_{j=1}^{k-2} x/2^j \leq x \) for each \( j \). Thus, \( N \leq x^{k(k-1)+1} \leq \exp\{O((\log^3 x)\} \). Since \( 1 - k/p \leq (1 - 1/p)^k \) for \( p > k \), if \( \mathcal{S}(m) > 0 \) then \( \mathcal{S}(m) \leq \prod_{p \mid N} (1 - 1/p)^{1-k} \leq (c_1 \log_2 N)^{k-1} \) for a constant \( c_1 \). The second bound follows from \( \sum_{p \mid N} 1/p \leq \log_2 N + O(1) \). \( \square \)

**Lemma 5.3.** Let \( k \geq 1 \) and \( \mathcal{I} \subseteq \{1, \ldots, k-1\} \). If the variables \( m_i \) are fixed \( (1 \leq i \leq k-1, i \notin \mathcal{I}) \), then for any prime \( p \),

\[
\sum_{0 \leq m_i < p \ (i \in \mathcal{I})} \xi(p, (m_1, \ldots, m_{k-1})) \geq p^{\left|\mathcal{I}\right|+1} - (p - 1)^{\left|\mathcal{I}\right|+1}.
\]

**Proof.** Fix \( p \) and let \( N(k, \mathcal{I}) \) be the sum on the left side. We use induction on \( k \), the case \( k = 1 \) being trivial. Suppose \( k \geq 2 \) and the lemma holds with \( k \) replaced by \( k - 1 \). If \( k - 1 \notin \mathcal{I} \), then \( \xi(p, (m_1, \ldots, m_{k-1})) \geq \xi(p, (m_1, \ldots, m_{k-2})) \) implies \( N(k, \mathcal{I}) \geq N(k - 1, \mathcal{I}) \).
If \( k - 1 \in \mathcal{I} \), then \( N(k, \mathcal{I}) \) counts the number of \((|I| + 1)\)-tuples \((m_i \mid i \in \mathcal{I}), n) \) modulo \( p \) with \( p|f_1(n) \cdots f_{k-1}(n)(m_{k-1}f_{k-1}(n) + 1) \). The number of tuples with \( p|f_1(n) \cdots f_{k-1}(n) \) is \( pN(k-1, \mathcal{I} - \{k-1\}) \) and the number of remaining tuples is \( p^{\lceil |I| - N(k-1, \mathcal{I} - \{k-1\}) \rceil} \). By the inductive hypothesis, \( N(k, I) = p^{\lceil |I| - (p-1)|I|^+1 \rceil} \).

**Lemma 5.4.** Let \( k \geq 4 \), and suppose that \( M_i, N_i \) are integers satisfying \( M_i \geq 2 \) and \( 2 \leq N_i \leq 2kM_i \) for \( 1 \leq i \leq k - 1 \). For some positive constant \( c_2 \), we have

\[
\sum_{N_i < m_i < N_i + M_i \atop (1 \leq i \leq k-1)} \mathcal{G}(m) \ll M_1 \cdots M_{k-1} (c_2 \log k)^b \exp \left\{ O \left( \frac{k \log k}{\log k} \right) \right\},
\]

where \( b \) is the number of variables \( M_i \) which are \( \leq 2k^{2 \log^3 k} \).

**Proof.** Let \( L = \lceil \log k \rceil + 1 \) and \( r = k^2 L \). We will perform a precise averaging of the factors in \( \mathcal{G}(m) \) for primes \( p \leq r \), and use crude estimates for larger \( p \). If \( p \nmid m_1 \cdots m_{k-1} \), each congruence \( f_j(n) \equiv 0 \pmod{p} \) has exactly one solution. For \( h > j \), \( f_j(n) \equiv 0 \pmod{p} \) and \( f_h(n) \equiv 0 \pmod{p} \) have a common solution if and only if \( p|(a_j b_h - a_h b_j) \). Write

\[
a_j b_h - a_h b_j = m_1 \cdots m_{j-1} g_{j,h}(m), \quad g_{j,h}(m) := 1 + \sum_{i=j+1}^{h-1} m_i \cdots m_{h-1}.
\]

Define

\[
\psi_r(n) = \prod_{\substack{p \mid n \atop p > r}} \frac{p}{p-1}, \quad G_{j,h}(m) = \prod_{\substack{p \mid m_j \cdots m_{k-1} \atop p \nmid g_{j,h}(m)(j+1 \leq i \leq h-2)}} \frac{p}{p-1}.
\]

We then have

\[
\prod_{\substack{p \mid m_j \cdots m_{k-1} \atop p > r}} \left( 1 - \frac{\xi(p, m)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \leq \prod_{1 \leq j < h \leq k-1 \atop h \geq j+2} \psi_r(G_{j,h}(m)).
\]

Let

\[
\mathcal{J} = \{(j, h) : 1 \leq j < h \leq k - 1, h \geq j + 2, \max_{j+1 \leq i \leq h-1} M_i > 2k^2 \log^3 k \},
\]

and put \( J = |\mathcal{J}| \leq \frac{(k-3)(k-2)}{2} \). Also let \( \mathcal{I} = \{i : M_i > 2k^2 \log^3 k \} \). Write \( \mathcal{I} = \bigcup_{a=1}^{A} ([i_a, i_a') \cap \mathbb{N}) \), where \( i_{a+1} \geq i_a' + 2 \) for each \( a \). For each \( \alpha \) and \( 2 + i_a' \leq h \leq i_{a+1} \) (so \( h \notin \mathcal{I} \)),

\[
\prod_{i_a' \leq j < h-2} \psi_r(G_{j,h}(m)) = \psi_r(G_h), \quad G_h = \prod_{i_a' \leq j < h-2} G_{j,h}(m).
\]
Since \( G_h \leq (k2^k \log^2 k)^k \), \( G_h \) has \( O(k^3 \log^3 k) \) prime factors, and thus for some constant \( C > 1 \), \( \psi_r(G_h) \leq \exp\{\sum_{p \mid G_h, p > r} \frac{1}{p-1}\} \leq C \). There are \( b \) such numbers \( h \). Hence, by Hölder’s inequality,

\[
\sum_m \mathfrak{S}(m) \leq C^b \sum_m \prod_{p \leq r} \left( 1 - \frac{\xi(p, m)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \prod_{i=1}^{k-1} \psi_r(m_i)^{-1} \prod_{(j,h) \in \mathcal{J}} \psi_r(G_{j,h}(m))
\]

\[ (5.3) \]

\[ \leq C^b \left( \sum_m \left[ \prod_{p \leq r} \left( 1 - \frac{\xi(p, m)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \right] \right)^{1 \over r} \cdot \prod_{i=1}^{k-1} \left( \sum_m \psi_r(m_i)^{2L(k-1)^2} \right)^{1 \over r} \prod_{(j,h) \in \mathcal{J}} \left( \sum_m \psi_r(G_{j,h}(m))^{2L} \right)^{1 \over r}.
\]

If we write \( \psi_r = 1 * \beta_s \), then \( \beta_s \) is multiplicative and supported on square-free integers composed of primes \( > r \). Furthermore, if \( p > r \geq s + 1 \), then

\[
\beta_s(p) = \left( \frac{p}{p-1} \right)^{s} - 1 \leq e^{s/(p-1)} - 1 \leq \frac{4s}{p}.
\]

Thus, for each \( i \),

\[
\sum_{N_i < m_i \leq N_i + M_i} \psi_r(m_i)^{2L(k-1)^2} \leq \sum_{d \leq N_i + M_i} \beta_{2L(k-1)^2}(d) \frac{N_i + M_i}{d}
\]

\[ (5.5) \]

\[ \leq (2k + 1)M_i \prod_{p > r} \left( 1 + \frac{\beta_{2L(k-1)^2}(p)}{p} \right) \ll kM_i.
\]

For fixed \( (j,h) \in \mathcal{J} \), let \( M_l = \max(M_{j+1}, \ldots, M_{h-1}) > 2k^2 \log^3 k \) and write

\[
g_{j,h}(m) = m_1(m_{l+1} \cdots m_{h-1})g_{j,l}(m) + g_{j,h}(m).
\]

We’ll use

\[
G_{j,h}(m) | G'_{j,h}(m) := \prod_{p > r, p \mid g_{j,h}(m)} p,
\]

and note that \( G'_{j,h}(m) \leq g_{j,h}(m) \leq k(2k + 1)^k M_{j+1} \cdots M_{h-1} \leq (6kM_l)^k \) by (5.2). Fix all of \( m_{j+1}, \ldots, m_{h-1} \) except for \( m_l \). By (5.4) and (5.5),

\[
\sum_{m_l} \psi_r(G'_{j,h}(m))^{2L} = \sum_{\substack{d \leq (6kM_l)^k \ \text{and} \ d \mid G'_{j,h}(m) \prod_{m_i} m_i = 1 \ \text{for all} \ \substack{d \leq (6kM_l)^k \ \text{and} \ d \mid G'_{j,h}(m)}}} \beta_{2L}(d) \sum_{N_i < m_i \leq N_i + M_i} 1 \leq \sum_{d \leq (6kM_l)^k} \left( \frac{M_l}{d} + 1 \right) \beta_{2L}(d)
\]

\[ (5.7) \]

\[ \leq M_l \prod_{p > r} \left( 1 + \frac{8JL}{p^2} \right) + \prod_{r < p \leq (6kM_l)^k} \left( 1 + \frac{8JL}{p} \right) \ll M_l.
\]
Also,
\[
\prod_{p \leq r} \left( 1 - \frac{\xi(p, m)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \leq \prod_{p \leq r} \left( 1 - \frac{1}{p} \right)^{-k} = \exp \left\{ O \left( \frac{k \log_2 k}{\log k} \right) \right\}.
\]
Therefore, by (5.3), (5.5) and (5.7),
\[
\sum_m \mathcal{S}(m) \ll O \left( (M_1 \cdots M_{k-1})^{\frac{r}{r-1}} S^{1-\frac{r}{r-1}} \exp \left\{ O \left( \frac{k \log_2 k}{\log k} \right) \right\} \right),
\]
where
\[
S = \sum_m \prod_{p \leq r} \left( 1 - \frac{\xi(p, m)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k}.
\]
Let \( M' = \prod_{p \leq r} p \). Since \( M' \leq e^{2r} \), for each \( i \in \mathcal{I} \), \( M_i \gg kM' \). Hence, the number of \( m_i \in (N_i, N_i + M_i) \) lying in a given residue class modulo \( M' \) is \( \leq M_i/M' + 1 = (1 + (O(1/k))M_i/M' \). Thus, by Lemma 5.3 and the Chinese Remainder Theorem,
\[
S \leq \sum_{N_i < m_i \leq N_i + M_i} \prod_{(i \in \mathcal{I})} \frac{M_i}{M'} \left( 1 + O \left( \frac{1}{k} \right) \right) \sum_{m_i \mod M' \ p \leq r} \prod_{(i \in \mathcal{I})} \left( 1 - \frac{\xi(p, m)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k}
\]
\[
\leq \prod_{i \in \mathcal{I}} M_i \sum_{N_i < m_i \leq N_i + M_i} \prod_{p \leq r} \frac{1}{p^{\lceil |\mathcal{I}| \rceil}} \left( \frac{p}{p-1} \right)^k \left( \frac{p}{p-1} \right)^{-k} \sum_{0 \leq m_i < p} \xi(p, m)
\]
\[
\leq M_1 \cdots M_{k-1} \prod_{p \leq r} \left( \frac{p}{p-1} \right)^{k-1-|\mathcal{I}|}.
\]
Noting that \( b = k - 1 - |\mathcal{I}| \), the lemma follows from Mertens’ estimate.

**Theorem 7.** Suppose that \( \eta > 0 \), \( r \geq 1 \), and \( l \) and \( x \) are sufficiently large as a function of \( \eta \). There are
\[
\ll \frac{x}{\log x} \left( 2e^{1+\eta} \right)^{1/2} + \sum_{j=1}^{r} x(\log_2 x)^{O(jl)} \left( \frac{x^{(j)}(3+\eta)}{\log x} \right)^{\lfloor j/(\log_2 j) \rfloor^2}
\]
primes \( p \leq x \), such that there is a prime chain \( p_r l < p_{r-1} l < \cdots < p_0 = p \) with \( p_r l > x^{(r+1)\eta} \).

**Proof.** Suppose \( 2e^{1+\eta} < 1 \) and \( l \leq (\log x)^{1/(3+\eta)} \), else the theorem is trivial. Put \( k_j = j l \) and \( x_j = x^{(j+1)\eta} \) for \( 0 \leq j \leq r \). Suppose \( p \leq x \) and there are even integers \( h_1, \ldots, h_k \), so that
\[
(5.8) \quad p = p_0 = h_1 p_1 + 1, p_1 = h_2 p_2 + 1, \ldots, p_{k_r-1} = h_{k_r} p_{k_r} + 1,
\]
with \( p_0, \ldots, p_{k_r} \) prime and \( p_{k_r} \geq x_r \). The vector \( (h_1, \ldots, h_{k_r}) \) may not be unique, but we associate to each such \( p \) a single such vector. Each \( p \) lies in \( Q_1 \cup \cdots \cup Q_r \), where \( Q_j \) is the set of primes \( p \) so that \( p_{k_i} < x_i \) (\( i < j \)) and \( p_{k_j} \geq x_j \). By assumption, \( k_r \leq (\log x_r)^{1/3} \).

Fix \( j \) and even integers \( h_1, \ldots, h_{k_j} \) satisfying \( h_1 \cdots h_{k_j} \leq x/x_j \). By Lemma 5.2,
\[
\sum_p \frac{k_j - \xi(p; (h_1, \ldots, h_{k_j}))}{p} \log p \ll k_j \log_2 x \ll (\log x_r)^{1/3} \log_2 x.
\]
By Lemma 5.1 and Stirling’s formula, the number of \( p = p_0 \leq x \) satisfying (5.8) is
\[
(5.9) \quad \ll \frac{x}{h_1 \cdots h_{k_j}} \frac{(2k_j/e)^{k_j+3/2} \log(h_{k_j}, \ldots, h_1)}{(\log x_j)^{k_j+1}}.
\]

Let \( 1 \leq b_j \leq k_j \) be a parameter to be chosen later, and put \( A_j = 2^{2k_j^2 \log^3 k_j} \). Let \( Q_{j,1} \) be the set of \( p \in Q_j \) for which at least \( b_j \) of the variables \( h_1, \ldots, h_{k_j} \) are \( \leq A_j \), and \( Q_{j,2} = Q_j \backslash Q_{j,1} \).

To estimate \( |Q_{j,1}| \), fix a set \( B \subseteq \{1, \ldots, k_j\} \) of size \( b_j \) so that \( h_i \leq A_j \) for each \( i \in B \). Let \( I_j = \{1 \leq i \leq k_j : i \notin B\} \) and, for \( 0 \leq i \leq j-1 \), put \( a_i = |B \cap \{k_i + 1, \ldots, k_{i+1}\}| \), \( I_i = I_j \cap \{k_i + 1, \ldots, k_{i+1}\} \). By the definition of \( Q_j \),
\[
(5.10) \quad \prod_{g \in I_i \cup \cdots \cup I_{j-1}} h_g \leq h_{k_i+1} \cdots h_{k_j} \leq \frac{x_i}{x_j} \quad (0 \leq i \leq j-1).
\]

Since \( h_i \geq 2 \) for all \( i \),
\[
(5.11) \quad \prod_{g \in B} \sum_{2 \leq h_g \leq A_j} \frac{1}{h_g} \leq (2k_j^2 \log^3 k_j)^{b_j}.
\]

Let \( \alpha = l/\log x_j \). By (5.10) and the elementary estimate \( \sum_{2 \leq h \leq y} h^{-1-s} \leq 1/s \),
\[
\sum_{h_g (g \in I)} \prod_{g \in I} \frac{1}{h_g} \leq \sum_{h_g \geq 2 (g \in I)} \prod_{g \in I} \frac{1}{h_g} \prod_{i=0}^{j-1} \left( \frac{x_i}{x_j} \right)^{\alpha} \prod_{g \in I_i \cup \cdots \cup I_{j-1}} \frac{1}{h_g^{(i+1)/2}}
\]
\[
= \prod_{i=0}^{j-1} \left( \frac{x_i}{x_j} \right)^{\alpha} \sum_{h_g \geq 2 (g \in I_i)} \prod_{g \in I_i} \frac{1}{h_g^{1+1/2}}
\]
\[
\leq \prod_{i=0}^{j-1} \left( \frac{x_i}{x_j} \right)^{\alpha} \left( \frac{1}{(i+1)^{\alpha}} \right)^{k_{i+1} - k_i - a_i}
\]
\[
= \left( \frac{1}{\alpha} \right)^{k_j - b_j} \frac{1^{a_0} \cdot 2^{a_1} \cdots j^{a_{j-1}}}{(j!)^l} \exp \left\{ \sum_{i=0}^{j} \left[ \left( \frac{j+1}{i+1} \right)^{\eta} - 1 \right] \right\}.
\]

The last sum is \( \leq \frac{\eta}{1-\eta} (j+1) \leq 2j \). Also, \( 1^{a_0} \cdot j^{a_{j-1}} \leq j^{b_j} \) and \( j! \geq (j/e)^j \). Hence,
\[
(5.12) \quad \sum_{h_g (g \in I)} \prod_{g \in I} \frac{1}{h_g} \leq e^{3k_j} \left( \frac{\log x_j}{k_j} \right)^{k_j - b_j}.
\]
The number of choices for $B$ is $\binom{k_j}{b_j} \leq (ek_j/b_j)^{b_j}$. By (5.9), (5.11), (5.12), and Lemma 5.2,

$$|Q_{j,1}| \ll \frac{x \left( c_1 k_j \log_2 x \right)^{k_j+3/2} \left( \frac{2ek_j^3 \log^3 k_j}{b_j} \right)^{b_j} \left( \frac{\log x_j}{k_j} \right)^{k_j-b_j}}{(\log x_j)^{k_j+1}} \left( \frac{2ek_j^3 (k_j/b_j) \log^3 k_j (j+1)^n}{\log x} \right)^{b_j}$$

$$\ll x \exp \left\{ O(k_j \log_2 x) + b_j \left[ (3 + \eta) \log k_j + \log \left( \frac{k_j}{b_j} \right) - \log_2 x \right] \right\}.$$  

(5.13)

We next estimate $|Q_{j,2}|$. Place each variable $h_i$ into an interval $J_i$. If $h_i \leq A_j$, then take $J_i = (2^{l_i-1}, 2^{l_i}]$ for an integer $l_i \geq 1$, and if $h_i > A_j$, then take $J_i = \left[ A_j (1 + 1/k_j)^{h_i-1}, A_j (1 + 1/k_j)^{h_i} \right]$ for some integer $l_i \geq 1$. For brevity, write $J_i = (H_i, K_i]$ for each $i$. Since $K_i - H_i \geq H_i/(2k_j)$, there are at most $b_j$ values of $i$ with $K_i - H_i \leq A_j$. Lemma 5.4 then gives

$$\sum_{h_1 \in J_1} \ldots \sum_{h_{k_j} \in J_{k_j}} \Theta(h_{k_1}, \ldots, h_{k_j}) \leq 1 \frac{1}{H_1 \ldots H_{k_j}} \sum_{h_1 \in J_1} \ldots \sum_{h_{k_j} \in J_{k_j}} \Theta(h_{k_1}, \ldots, h_{k_j}) \ll (e_2 \log k_j)^{b_j} \exp \left\{ O \left( \frac{k_j \log_2 k_j}{\log k_j} \right) \right\} \prod_{i=1}^{k_j} \frac{K_i - H_i}{H_i}.$$ 

By our definition of the intervals $J_i$,

$$\prod_{i=1}^{k_j} \frac{K_i - H_i}{H_i} \leq \prod_{1 \leq i \leq k_j} \frac{1}{H_i} \sum_{K_i - H_i < A_j} \frac{1}{h_i} \prod_{1 \leq i \leq k_j} \left( 1 + O \left( \frac{1}{k_j} \right) \right) \sum_{H_i < h_i < K_i} \frac{1}{h_i} \ll 2^{b_j} \sum_{h_1 \in J_1} \ldots \sum_{h_{k_j} \in J_{k_j}} \frac{1}{h_1 \ldots h_{k_j}}.$$ 

Thus, after summing over all possibilities for $J_1, \ldots, J_{k_j}$, we obtain by (5.9)

$$|Q_{j,2}| \ll \frac{x (2k_j/e)^{k_j+3/2} \left( \frac{2\log x_j}{k_j+1} \right)}{(\log x_j)^{k_j+1}} \exp \left\{ O \left( \frac{k_j \log_2 k_j}{\log k_j} + b_j \log_2 k_j \right) \right\} \sum_{h_{k_1}', \ldots, h_{k_j}'} \frac{1}{h_{k_1}' \ldots h_{k_j}'},$$

where $h_{k_i+1}' \ldots h_{k_j}' \leq 2^{k_j-k_i} h_{k_i+1} \ldots h_{k_j} \leq 2^{k_j} x_i/x_j$ for $0 \leq i \leq j - 1$. For positive $\alpha_0, \ldots, \alpha_{j-1}$,

$$\sum_{h_{k_1}', \ldots, h_{k_j}'} \frac{1}{h_{k_1}' \ldots h_{k_j}'} \leq \prod_{i=0}^{j-1} \left( \frac{2k_j x_i}{x_j} \right)^{\alpha_i} \sum_{h_{k_i+1}' \ldots h_{k_{i+1}}'} \frac{1}{h_{k_i+1}' \ldots h_{k_{i+1}}'} \leq \prod_{i=0}^{j-1} \left( \frac{2k_j x_i}{x_j} \right)^{\alpha_i} \left( \frac{1}{\alpha_0 + \cdots + \alpha_i} \right)^t.$$
If we ignore the factors \(2^{k_j\alpha_i}\), the optimal choice of parameters is

\[
\alpha_i = \frac{l}{(j + 1)^n \log x_j} \left[ \frac{(i + 2)^n(i + 1)^n}{(i + 2)^n - (i + 1)^n} - \frac{(i + 1)^n i^n}{(i + 1)^n - i^n} \right], \quad i = 0, \ldots, j - 1.
\]

Since \((i + 2)^n - (i + 1)^n \geq \eta(i + 2)^{n-1}\),

\[
\alpha_0 + \cdots + \alpha_i \in \left[ \frac{l(i + 1)(i + 2)^n}{\eta(j + 1)^n \log x_j}, \frac{l(i + 1)(i + 1)^n}{\eta(j + 1)^n \log x_j} \right].
\]

Recalling \(k_j = j!l\), the sum on \(h'_1, \ldots, h'_j\) is at most

\[
2^{k_j(a_0 + \cdots + a_{j-1})} \exp \left\{ \sum_{i=0}^{j-1} \alpha_i \left[ \frac{(j + 1)^n}{i + 1} - 1 \right] \log x_j \right\} \left( \frac{\eta(j + 1)^n \log x_j}{l} \right)^{k_j} \frac{1}{(j!)(j + 1)^{\eta l}}.
\]

The exponential factor is \(e^{\epsilon j} = e^{k_j}\) and \((j + 1)! \geq e^{2(j + 1)^{\theta_j + 1}}\), so

\[
\sum_{h'_1, \ldots, h'_j} \frac{1}{h'_1 \cdots h'_j} \leq 2^{k_j^2/(\eta \log x_j)} \left( \frac{\eta e^{2 + \eta}}{k_j} \right)^{k_j}.
\]

Therefore,

\[
(5.14) \quad |Q_{j,2}| \ll \frac{x}{\log x} \left( 2^{\eta e^{1 + \epsilon}} \right)^{k_j} \exp \left\{ O \left( \frac{k_j \log_2 k_j}{\log k_j} + b_j \log_2 k_j \right) \right\}.
\]

Finally, put \(b_j = \lfloor k_j/(\log_2 k_j)^2 \rfloor\), and sum the inequalities (5.13) and (5.14) for \(1 \leq j \leq r\). \(\square\)

**Proof of Theorem 5** Let \(\eta = 0.15718\), \(l = \lfloor (\log x)^{\varepsilon} \rfloor\) and \(r = \lfloor (\log x)^{\beta} \rfloor\), where \(\varepsilon\) and \(\beta\) are fixed and satisfy \(0 < \varepsilon + \beta < \frac{1}{3 + \eta}\). Then \(\log x_r \approx (\log x)^{1-\eta\beta}\). For the primes \(p\) not counted in Theorem 7, the primes at level \(rl\) of the Pratt tree are all \(\leq x_r\), so \(H(p) \leq \log x_r \log_2 x_r + 1 + r l \ll (\log x)^{0.95022}\) if we take \(\beta\) sufficiently close to \(\frac{1}{3 + \eta}\). By Theorem 7, the number of exceptional primes \(p \leq x\) is \(O(x \exp \{-x(\log x)^{\delta}\})\) for some \(\delta > 0\). \(\square\)

**Proof of Theorem 4** Let \(x\) be large, \(x/\log x < n \leq x\) and suppose there is a prime \(p > x^{\varepsilon/2}\) such that \(p|\phi_k(n)\). Then either (i) there is a prime \(q > x^{\varepsilon/2}\) and \(0 \leq j < k\) such that \(q^2|\phi_j(n)\), or (ii) there is a prime chain \(p = p_k < p_{k-1} < \cdots < p_1 < p_0\) with \(p_0|n\). In case (i), let \(j\) be the smallest such index. Using the uniform estimate

\[
\sum_{p \equiv 1 \pmod{\phi(m)}} \frac{1}{p} \ll \frac{\log_2 x}{\phi(m)},
\]

coming from the Brun-Titchmarsh inequality, the number of integers in category (i) is

\[
\ll \sum_{q > x^{\varepsilon/2}} \frac{x}{q^2} + \sum_{j=1}^{k} \sum_{x^{\varepsilon/2} < q \leq x} \sum_{p_{j-1} \equiv 1 \pmod{q^2}} \sum_{p_{j-2} \equiv 1 \pmod{p_{j-1}}} \cdots \sum_{p_0 \equiv 1 \pmod{p_1}} \frac{x}{p_0} \ll x \frac{x^{1-\varepsilon/2}}{\log x} + \frac{k}{\log x} \frac{x^{1-\varepsilon/2}(\log_2 x)^j}{\log x} \ll \varepsilon. k \ x^{1-\varepsilon/2}.
\]
Consider $n$ in category (ii). Take $\eta = \frac{1}{4}$, let $r$ be the smallest integer with $(r + 1)^{-\eta} < \varepsilon/2$, let $l$ be sufficiently large, $l \leq \log_2 x$ and $k = rl$. By Theorem 7 for $x^{\varepsilon/2} < y \leq x$, the number of $p_y \leq y$ is $O(y/\log^2 y + y(2\eta e^{1+\eta})^{-1/2}/\log y)$. By partial summation, the number of $n$ is $\ll x(2\eta e^{1+\eta})^{-1/2}$.

Taking $l$ large enough, depending on $\varepsilon$ and $\delta$, completes the proof.

6. Stochastic model of Pratt trees

In this section, we develop a model of the Pratt trees which explains Conjectures 2 and 3. Factor $n$ as $n = \prod_{j=1}^{\Omega(n)} p_j (n)$, with $p_1 (n) \geq p_2 (n) \geq \cdots$. Put $p_j (n) = 1$ for $j > \Omega(n)$ and let

$$S(n) = \left( \frac{\log p_1 (n)}{\log n}, \frac{\log p_2 (n)}{\log n}, \cdots \right).$$

The distribution of the first component of $S(n)$ has been greatly studied, the results having wide application in the theory of numbers (see e.g. the comprehensive survey article [26]). We have\(^3\) $P(\log p_1 (n) \leq \frac{1}{u} \log n) = \rho(u)$, where $\rho$ is the \textit{Dickman function}, the unique continuous solution of the differential-delay equations $\rho(u) = 1$ ($0 \leq u \leq 1$), $u\rho(u) = -\rho(u-1)$ ($u > 1$). The complete distribution of $S(n)$, found by Billingsly in 1972 [11], corresponds to the Poisson-Dirichlet distribution with parameter 1, $PD(1)$ for short (more precisely, for each $j$, the first $j$ components of $S(n)$ are distributed as the first $j$ components of the $PD(1)$ distribution). The joint distribution of the components in the $PD(1)$ distribution can easily be expressed in terms of $\rho$. There is a simpler characterization of the distribution, found by Donnelly and Grimmett [17]. Let $U_1, U_2, \ldots$ be independent random variables with uniform distribution on $[0, 1]$. Let $x = (x_1, x_2, \ldots)$ be the infinite dimensional vector formed from the decreasing rearrangement of the numbers

$$y_1 = U_1, \quad y_2 = (1 - U_1) U_2, \quad y_3 = (1 - U_1)(1 - U_2) U_3, \ldots.$$

Then $x$ has the $PD(1)$ distribution. The paper [17] gives a simple, transparent proof that $(x_1, \ldots, x_k)$ and the first $k$ components of $S(n)$ have the same distribution.

Since $\sum x_i = 1$ with probability 1, we can interpret the $PD(1)$ distribution as a random partition of the unit interval $[0, 1]$ into an infinite number of parts achieved by cutting $[0, 1]$ at a random place (with uniform distribution), then cutting the right sub-interval at a random place, and so on.

**Conjecture 5.** As $p$ runs over the set of primes, $S(p - 1)$ has $PD(1)$ distribution.

Conjecture [5] is widely believed, and is a simple consequence of EH. Unconditionally, we know little about primes in progressions to very large moduli. Assume that $S(p - 1)$ has $PD(1)$ distribution, $S(q - 1)$ has $PD(1)$ distribution for each prime $q|(p - 1)$, the vectors $S(q - 1)$ for $q|(p - 1)$ are independent, and so forth. The primes in the first level of the tree, on a logarithmic scale, correspond to a random partition of $[0, 1]$. The primes in the second level correspond to randomly partitioning each of the parts of the original partition, etc. The entire procedure corresponds to what is known as a discrete-time \textit{random fragmentation process}. Random fragmentation processes have been used to model a variety of physical phenomena (e.g., genetic mutations, planet formation) and the growth of certain data structures in computer science. Discrete time fragmentation processes may be recast in the language of \textit{branching random walks}, which we now describe.

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\(^3\)If $B \subseteq A \subseteq \mathbb{N}$, we say $P(n \in B | n \in A) = \alpha$ if $\lim_{x \to \infty} \frac{\left| \{ n \in B : n \leq x \} \right|}{\left| \{ n \in A : n \leq x \} \right|} = \alpha$. 

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Let $M_n$ be the size of the largest object at time $n$. Then $M_n$ is a model of $Q_n := \frac{\log q_n}{\log p}$, where $q_n$ is the largest prime at level $n$ of the tree. The event $\{M_n < \frac{\log 2}{\log p}\}$ is a model of the statement “all the primes at level $n$ of the Pratt tree for $p$ are $< 2$”; that is, $H(p) < n$. Thus, $H(p)$ is modeled by the random variable $T\left(\frac{\log 2}{\log p}\right)$, where $T(\varepsilon) = \min\{n : M_n \leq \varepsilon\}$.

Assuming EH, Lamzouri [23] showed that $Q_n$ has the same distribution as $M_n$ for each fixed $n$ (he studies the distribution of $P^+(\phi_n(m))$ for all integers $m$; the same proofs give the distribution of $Q_n$). Further, on EH, Lamzouri shows that $\mathbb{P}\{Q_n \leq \frac{1}{u}\} = \mathbb{P}\{M_n \leq \frac{1}{u}\} = \rho_n(u)$, where, for each fixed $n$,

$$\rho_n(u) = \left(\frac{1 + o(1)}{\log_{n-1}(u) \log_n(u)}\right)^u \quad (u \to \infty),$$

with $\log_0(u) = u$. Our goal is to understand the distribution of $M_n$ as $n \to \infty$.

Create a tree structure from the random fragmentation process as follows: label the root node with zero, beneath the root node put an infinite number of child nodes, each corresponding to one of the fragments of the initial segment $[0, 1]$. Each of these nodes has an infinite number of child nodes, corresponding to the fragments in the second step of the process, and so on. Each node is labeled with the number $-\log x$, where $x$ is the fragment size. This randomly labeled tree corresponds to a branching random walk (BRW). More generally, an initial ancestor is at the origin, and who forms the zeroth generation. This parent then produces children, the first generation, which are randomly displaced from the parent according to some law. Each of these children behaves like an independent copy of the parent, their children randomly displaced from their parent according to the same law, and forming the second generation, and so on. In our case, each parent produces an infinite number of offspring, the displacements from their parent given by $V = \{-\log y : y \in Z\}$, where $Z$ is a point set with $PD(1)$ distribution. We’ll say that $V$ has $LPD$ (logarithmic Poisson-Dirichlet) distribution from now on.

Let $B_n$ be the minimum label of an individual at time $n$, so that $B_n = -\log M_n$. The first order behavior of the analog of $B_n$ (law of large numbers) for a general BRW was determined in the 1970s by Biggins, Hammersley and Kingman (see [9]). In our case, Biggins’ theorem [9] implies $B_n \sim \frac{n}{e}$ as $n \to \infty$ almost surely. Thus, $T\left(\frac{\log 2}{\log p}\right) \sim e \log 2 p$ as $p \to \infty$ almost surely, which justifies Conjecture [2].

Let $b_n = \text{median}(B_n)$. The study of $B_n$ naturally breaks into two parts: (i) global behavior: asymptotics for $b_n$, and (ii) local behavior: the distribution of $B_n - b_n$. A result of McDiarmid [31] can be used to prove $b_n = \frac{n}{e} + O(\log n)$, and this was sharpened by Addario-Berry and Ford [1] to

**Theorem 8.** We have $b_n = \frac{n}{e} + \frac{3}{2e} \log n + O(1)$.

**Corollary 2.** We have $\text{median}(T(\varepsilon)) = e \log(1/\varepsilon) - \frac{3}{2} \log_2(1/\varepsilon) + O(1)$.

This justifies part of Conjecture [3]. One ingredient in the proof is the following expectation identity. Let $Z_n(t)$ be the number of generation $n$ individuals with position $\leq t$, and let $Z^{(n)}$ be the set of positions of generation $n$ individuals. If $v = (v_1, v_2, \ldots)$ has $PD(1)$ distribution, (6.1) gives

$$\mathbb{E} \sum_{j=1}^{\infty} v_j^s = \mathbb{E} \sum_{k=1}^{\infty} \left(1 - U_1 \cdots (1 - U_{k-1}) U_k\right)^s = \sum_{k=1}^{\infty} \frac{1}{(1 + s)^k} = \frac{1}{s},$$
since \( E U_i^s = E (1 - U_i)^s = 1/(1 + s) \). By the branching property,

\[
E \sum_{z_n \in \mathbb{Z}^{(n)}} e^{-sz_n} = E \sum_{z_{n-1} \in \mathbb{Z}^{(n-1)}} e^{-sz_{n-1}} E \sum_{z_1 \in \mathbb{G}^{(1)}} e^{-sz_1} = \frac{1}{s} E \sum_{z_{n-1} \in \mathbb{Z}^{(n-1)}} e^{-sz_{n-1}}.
\]

By induction, the left side is \( 1/s^n \), so \( \int_0^\infty e^{-st}dE_{Z_n}(t) = 1/s^n \). Therefore, \( E_{Z_n}(t) = t^n/n! \). Because \( t^n/n! \approx 1 \) when \( t = \frac{n}{e} + \frac{1}{2e} \log n + O(1) \), a naive guess would be \( b_n = \frac{n}{e} + \frac{1}{2e} \log n + O(1) \). However, for reasons clearly explained in [2] and executed in [1], the leftmost point in the \( n \)-th generation of a branching random walk has an atypical ancestry with high probability. Denote the locations of points in the ancestral line of this leftmost point by \( B \). We next discuss the local behavior of \( B \). Under very general conditions on the BRW, it is known that \( B_n - b_n \) is a tight sequence.\(^4\) The basic idea is that a single individual, with high probability, produce many offspring a few generations later which are close by. In our situation, tightness on the left for \( H(p) \) is relatively easy to prove unconditionally:

**Proof of Theorem [6]** The conclusion is trivial if \( g(x^{1/2}) \leq 3K \), so we will assume that \( g(x^{1/2}) > 3K \). Let \( m = \lceil g(x^{1/2})/K \rceil \) so that \( m \geq 3 \). Put \( Q = x^{2-m} \) and let \( T \) be the set of primes \( x^{1/2} < p \leq x \) such that there is a prime \( q | (p - 1) \) with \( Q < q \leq x^{1/4} \) and \( H(q) \geq h(p) \). For \( p \in T \),

\[
H(p) \geq 1 + h(q) \geq h(Q) \geq h(x) - mK \geq h(p) - g(p),
\]

while by sieve methods (Theorem 4.2 of [24]), for large \( x \)

\[
|\{x^{1/2} < p \leq x : p \not\in T\}| \ll \frac{x}{\log x} \prod_{Q < q \leq x^{1/4}} \left(1 - \frac{1}{q}\right) \ll \frac{x}{\log x} 2^{-mc}. \quad \Box
\]

It is also known that under certain conditions on the displacement law of the BRW (e.g. [3]), the analog of \( B_n - b_n \) converges in probability to a random variable as \( n \to \infty \). This is not known in our case.

**Conjecture 6.** \( B_n - b_n \to X \) as \( n \to \infty \) for a random variable \( X \) with continuous distribution.

If \( X \) exists, and the medians satisfy \( b_{n+1} - b_n \to e^{-1} \) as \( n \to \infty \) (plausible in light of Theorem [8]), it is easy to see that \( X \overset{d}{=} -1/e + \min_i (z_i + X_i) \), where \( (z_1, z_2, \ldots) \) has LPD distribution, \( X_1, X_2, \ldots \) are independent copies of \( X \), and \( \overset{d}{=} \) means “has the same distribution as”. This follows by conditioning on the positions of the first generation individuals (the points \( z_i \)); that is, using \( B_n \overset{d}{=} \min_i (z_i + B_{n-1}^{(i)}) \), where \( B_{n-1}^{(i)} \) are independent copies of \( B_{n-1} \). The solutions \( X \) of this recursive distributional equation are not known, however.

Unconditionally (whether \( X \) exists or not), we prove that \( B_n - b_n \) has an exponentially decreasing left tail and doubly-exponentially decreasing right tail. Consequently, if Conjecture [5] holds, then all moments of \( X \) exist.

\(^4\) A sequence \( X_1, X_2, \ldots \) of random variables is tight if for every \( \varepsilon > 0 \) there is a number \( M \) so that for all \( j \), \( \mathbb{P}(|X_j| > M) \leq \varepsilon \). In other words, the distribution of \( X_j \) does not spread out as \( j \to \infty \).
Lemma 6.2. \( \theta \) and \( \eta > 0 \),

Proof. Suppose \( B_n - b_n \leq -x \) for any \( c_2 > 2e \log(2e) \) and \( \eta > 0 \),

\[
\mathbb{P}\{B_n - b_n \leq -x\} \geq c_2,\, \eta \, e^{-c_2 x} \quad (n \geq 1, 0 \leq x \leq (1/2e - \eta)n);
\]

(b) for any \( c_3 < 1 \) there is a constant \( c_4 \), depending on \( c_3 \), so that

\[
\mathbb{P}\{B_n \geq b_n + x\} \leq \exp\left(-e^{c_3(x-c_4)}\right) \quad (n \geq 1, x \geq 0).
\]

Lemma 6.3. For positive integers \( m, n \) and positive real numbers \( M, N \),

\[
\mathbb{P}\{B_{m+n} \geq M + N\} \leq \mathbb{E}[\{\mathbb{P}\{B_n \geq N\}\}^{Z_{m}(M)}].
\]

Proof. Suppose \( B_{m+n} \geq M + N \) and \( Z_{m}(M) = k \). For each of these \( k \) individuals, all of their
descendants in generation \( m + n \) are offset from their generation \( m \) ancestor by at least \( N \).

Lemma 6.4. For real \( t \geq 1 \) and integer \( k \), we have \( \mathbb{P}\{Z_1(t) \geq k\} \leq (et/k)^{k-1} \).

Proof. The conclusion is trivial if \( k \leq et \), so we suppose \( k > et \). Take \( s = \frac{k}{t} - 1 \). By (6.1),

\[
\mathbb{P}\{Z_1(t) \geq k\} \leq \mathbb{P}\{1 - U_1\cdots(1 - U_{k-1}) \geq e^{-t}\}
\]

\[
\leq e^{st} \int_0^1 \cdots \int_0^1 \left[(1 - u_1)\cdots(1 - u_{k-1})\right]^s du_1 \cdots du_{k-1} = \frac{e^{st}}{(1 + s)^{(k-1)}}.
\]

Remark 3. By (6.2), part (b) is nearly best possible; that is, the conclusion is false if \( c_3 > 1 \).

Theorem 9. (a) For any \( c_1 < e \), we have

\[
\mathbb{P}\{B_n - b_n \leq -x\} \ll c_1,\, e^{-c_1 x} \quad (n \geq 1, x \geq 0),
\]

and for any \( c_2 > 2e \log(2e) \) and \( \eta > 0 \),

\[
\mathbb{P}\{B_n - b_n \leq -x\} \geq c_2,\, \eta \, e^{-c_2 x} \quad (n \geq 1, 0 \leq x \leq (1/2e - \eta)n);
\]

(b) for any \( c_3 < 1 \) there is a constant \( c_4 \), depending on \( c_3 \), so that

\[
\mathbb{P}\{B_n \geq b_n + x\} \leq \exp\left(-e^{c_3(x-c_4)}\right) \quad (n \geq 1, x \geq 0).
\]
most $x$ from their parent. Let $I$ be the set of indices with $m_i \geq 100x$. Note that $m_1 + \ldots + m_j \geq \theta^x$. With $j, m_1, \ldots, m_j$ fixed, by Lemma 6.3, the probability that $Z_{r+1}(x) \geq \theta^x$ is at most

$$\prod_{i=1}^j P\{Z_i(x) \geq m_i\} \leq \prod_{i \in I} e^{-2m_i} \leq e^{-2(\theta^x - 100xj)} \leq \exp\{-\theta^x\}. $$

As $m_i \leq e^x$, the number of choices for $j, m_1, \ldots, m_j$ is at most $\exp\{(\theta - \varepsilon/4)^x\}$. For large $x$,

$$P\{Z_r(x) \geq \theta^x\} \leq \exp\{- (\theta - \varepsilon/2)^x\} + \exp\{(\theta - \varepsilon/4)^x - \theta^x\} \leq \exp\{- (\theta - \varepsilon)^x\}. $$

\[\square\]

**Proof of Theorem 9** Let $\alpha > 1/e$ and $0 < \eta < \alpha e/2$. By [10, Theorem 2], for large $r$ we have $P\{Z_r(\alpha r) \leq (\alpha e - \eta)^r\} \leq \frac{1}{5}$. Let $r$ be so large that, in addition, $b_{n+r} \geq b_n + (1/e - \eta)r$ for all $n$ ($r$ exists by Theorem 8). Apply Lemma 6.2 with $M = ar$, $m = r$, $\varepsilon = (ae - \eta)^{-r}$. For large integers $r$,

$$P\{B_n \leq b_n - (a - 1/e + \eta)r\} \leq P\{B_n \leq b_{n+r} - ar\} \leq (ae - \eta)^{-r}. $$

The first estimate follows with $c_1 = \frac{\log(\alpha e - \eta)}{(a - 1/e + \eta)}$. Fix $a$, let $\eta \to 0$, then let $a \to 1/e$, so that $c_1 \to e$.

For the second part of (a), take $\eta > 0$ and $r$ as before (but fixed here), and let $\delta = (1/e - \eta)r$, so that $b_{n+r} \geq b_n + \delta$ for all $n$. Since $\rho(u) = 1 - \log u$ for $1 \leq u \leq 2$, $P\{Z_1(\varepsilon) \geq 1\} = 1 - \rho(e^\varepsilon) = \varepsilon$ when $0 \leq \varepsilon \leq \log 2$. Considering the “leftmost child of the leftmost child of the . . . of the initial ancestor” in the branching random walk, we have $P\{B_{kr} \leq \delta k/2\} \geq P\{Z_1(\delta/2r) \geq 1\}^{kr} = (\delta/2r)^{kr}$ for every $k \geq 1$. Hence,

$$P\{B_n \leq b_{n-kr} + \delta k/2\} \geq P\{B_{n-kr} \leq b_{n-kr}\}P\{B_{kr} \leq \delta k/2\} \geq \frac{1}{2} (\frac{\delta}{2r})^{kr}. $$

By assumption, $b_{n-kr} + \delta k/2 \leq b_n - \delta k/2$. Hence, for $0 \leq k \leq n/r$ we have

$$P\{B_n \leq b_n - \delta k/2\} \geq \frac{1}{2} \left(\frac{\delta}{2r}\right)^{kr}. $$

This gives the desired bound when $0 \leq x \leq \frac{b_n}{2r}$, with $c_1' = \frac{2r}{\delta} \log \frac{2r}{\delta}$.

To show part (b), we use induction on $n$ to show that

$$P\{B_n \geq b_n + x\} \leq 2^{-\exp\{c_3(x-\delta_3)\}} $$

for $n \geq 1$ and $x > 0$, where $c_5$ is sufficiently large. Theorem 9 (c) then follows with $c_4 = c_5 - \frac{\log \log 2}{c_3}$. As (6.3) is trivial for $0 \leq x \leq c_5$, we may assume $x \geq c_5$. Let $r, \delta$ be such that $b_{n+r} - b_n \geq \delta > 0$ for all $n$ (the relationship between $r$ and $\delta$ is unimportant). Let $A$ be a large integer, so that if $R = Ar$ and $\Delta = A\delta$, then $2e^{2-\Delta} \leq 1 - c_3$. Also suppose $c_3 \geq \frac{1}{2}$. For $1 \leq n \leq R$, (6.2) implies $P\{B_n \geq b_n + x\} \leq P\{B_n \geq x\} = \rho_n(e^x) \leq \exp\{-e^x\}$ if $c_5$ is large enough. Suppose now that (6.3) has been proved for $1 \leq n \leq m - 1$, where $m - 1 \geq R$. Define $\lambda_j = \Delta + \frac{\log j - 1}{c_3}$ for $j \geq 1$. Let $j_0$ be the largest index $j$ with $\lambda_j \leq x + \Delta$. Let $z_1 \leq z_2 \leq \cdots$ be the points in $\mathbf{z}^{(R)}$. For $1 \leq j \leq j_0$, let $P_j$ be the event $\{z_i > \lambda_i \text{ for } i < j\}$, $z_j \leq \lambda_j$, and the $Q$ be the event $\{z_i > \lambda_i \text{ for } 1 \leq i \leq j_0\}$. If $P_j$, then the generation $m$ points descending from each of the $j$
points \( z_1, \ldots, z_j \) are offset from their generation \( R \) ancestor by at least \( b_m + x - \lambda_j \). So

\[
P\{B_m \geq b_m + x\} \leq \sum_{j=1}^{j_0} P[P_j] P\{B_{m-R} \geq b_m + x - \lambda_j\} + P[Q].
\]

Since \( b_m \geq b_{m-R} + \Delta \), the induction hypothesis implies that the sum on \( j \) is

\[
\leq \sum_{j=1}^{j_0} P[P_j] 2^{-j \exp\{c_3(x-c_5+\Delta-\lambda_j)\}} \leq \sum_{j=1}^{j_0} P[P_j] 2^{-\exp\{c_3(x-c_5+\Delta)\}} \leq 2^{-\exp\{c_3(x-c_5)\}}.
\]

Now suppose that \( Q \) holds. By the assumption on \( A \),

\[
\sum_{j \leq j_0} e^{-z_j} \leq \sum_{j=1}^{j_0} e^{-\lambda_j} \leq e^{-\Delta+1/c_3} \sum_{j=1}^{\infty} j^{-1/c_3} \leq \frac{1}{2}.
\]

As \( \lambda_j \geq x + \Delta - (\lambda_{j_0+1} - \lambda_j) \geq x \),

\[
\sum_{z \in \mathbb{Z}(R)} e^{-z} = 1 - \sum_{z \in \mathbb{Z}(R)} e^{-z} \geq \frac{1}{2}.
\]

Let \( \varepsilon = \frac{1}{3}(e - e^{-c_3}) \) and \( \theta = e - \varepsilon \), so that \( e^{-c_3} < \theta - \varepsilon < \theta < e \). For some integer \( k \geq x \), there are \( \geq \theta^k \) points of \( \mathbb{Z}(R) \) in \([k - 1, k)\), for otherwise

\[
\sum_{z \in \mathbb{Z}(R)} e^{-z} \leq \sum_{k \geq x} e\left(\frac{\theta}{e}\right)^k < \frac{1}{2}.
\]

By Lemma 6.4, \( P[Q] \leq \sum_k P\{Z_R(k) \geq \theta^k\} \leq 2e^{-(\theta-e)^x} \). This completes the proof of (b). \( \square \)

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