Skew-symmetric identities of finitely generated alternative algebras

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Abstract

We prove that for every natural number $n$ there exists a natural number $N(n)$ such that every multilinear skew-symmetric polynomial on $N(n)$ or more variables which vanishes in the free associative algebra vanishes as well in any $n$-generated alternative algebra over a field of characteristic 0. Before this was proved only for a series of such polynomials constructed by the author in [7].

1 Introduction

Let $\mathcal{V}$ be a variety of algebras. Denote by $\mathcal{V}_n$ the subvariety of $\mathcal{V}$ generated by the $\mathcal{V}$-free algebra on $n$ free generators; then we have

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \mathcal{V} = \bigcup_n \mathcal{V}_n.$$ 

A minimal number $n$ for which $\mathcal{V}_n = \mathcal{V}$ (if such a number exists) is called the basic rank of the variety $\mathcal{V}$ and is denoted as $r_b(\mathcal{V})$ (see [4]). If $\mathcal{V} \neq \mathcal{V}_n$ for any $n$ then the basic rank of $\mathcal{V}$ is called to be infinite.

It is well known that for the varieties of associative and Lie algebras the basic rank is equal to 2. A.I.Shirshov in [1] posed a problem on the basic rank of the varieties of alternative $\text{Alt}$, Jordan $\text{Jord}$, Malcev $\text{Malc}$, and other algebras. In [7] the author constructed a series of multilinear skew-symmetric polynomials $g_m$ on $2m+1$ variables over a field of characteristic $\neq 2$ which vanish in any $n$-generated alternative or Malcev algebra for $m > \left\lceil \frac{n^3+5n}{12} \right\rceil$, but do not vanish in free alternative and free Malcev algebras of infinite rank. In particular, this implies that $r_b(\text{Alt})$ and $r_b(\text{Mal})$ are infinite.

Here we prove more general result for alternative algebras. Namely, we prove that for any natural number $n$ there exists a number $N = N(n)$, such that EVERY multilinear skew-symmetric polynomial on $N(n)$ variables which vanishes in the free associative algebra vanishes in any $n$-generated alternative algebra.

A similar result for Malcev algebras was proved in [11].

2 Skew-symmetric functions in flexible algebras

Let $F$ be a field of characteristic not 2. Recall that an algebra $A$ over $F$ is called flexible if it satisfies the identity

$$(x, y, x) = 0$$

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where \((x, y, z) = (xy)z - x(yz)\) denotes the **associator** of elements \(x, y, z\). Any flexible algebra satisfies the identity \([6] \text{ Lemma 1}\)

\[
[x^2, y] = x \circ [x, y],
\]

where \(x \circ y = xy + yx\) denotes the **Jordan product** of \(x, y\). We construct here a new series of skew-symmetric polynomials \(f_m\) on \(m\) variables in flexible algebras that have properties similar to those of polynomials \(g_m\) constructed in [7].

Set \(f_1(x) = x\), \(f_2(x_1, x_2) = [x_1, x_2]\), and then, by induction,

\[
f_{m+1}(x_1, x_2, \ldots, x_{m+1}) = \sum_{1 \leq i < j \leq m+1} (-1)^{i+j-1} f_m([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{m+1}),
\]

where \(\hat{x}_k\) means that the element \(x_k\) is omitted.

**Lemma 2.1.** Polynomials \(f_m(x_1, \ldots, x_m)\) for \(m > 1\) are skew-symmetric on its variables, that is, they vanish if any two of arguments \(x_i\) coincide.

**Proof.** The statement is true for \(m = 2\). Assume now that it is true for certain \(m \geq 2\) and prove it for \(m + 1\). Let \(x_i = x_j\), \(i < j\). Then we have, by induction,

\[
f_{m+1}(x_1, \ldots, x_{m+1}) = \sum_{k=1}^{i-1} (-1)^{k+i-1} f_m([x_k, x_i], x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{m+1})
\]

\[
+ \sum_{j \neq k = i+1}^{m+1} (-1)^{i+k-1} f_m([x_i, x_k], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_{m+1})
\]

\[
+ \sum_{j=1}^{i-1} (-1)^{k+j-1} f_m([x_k, x_j], x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_j, \ldots, x_{m+1})
\]

\[
+ \sum_{k=j+1}^{m+1} (-1)^{j+k-1} f_m([x_j, x_k], x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{m+1}).
\]

Now, if \(k < i\), we have

\[
f_m([x_k, x_i], x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_j, \ldots, x_{m+1}) = (-1)^{j-i-1} f_m([x_k, x_j], x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{m+1})
\]

The similar equality we have for \(k > j\). Finally, for \(i < k < j\) we have

\[
f_m([x_i, x_k], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_j, \ldots, x_{m+1}) = (-1)^{j-i-2} f_m([x_k, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, \hat{x}_j, \ldots, x_{m+1})
\]

\[
+ (-1)^{j-i-1} f_m([x_j, x_k], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, \hat{x}_j, \ldots, x_{m+1}).
\]

Therefore, the first two sums in the expression for \(f_{m+1}\) just differ by sign from the last two sums, and the whole sum is 0.

**Lemma 2.2.** Polynomials \(f_m\) are strongly skew-symmetric, that is, they satisfy the identity

\[
f_m(x^2, x, x_1, \ldots, x_{m-2}) = 0. \tag{2}
\]

**Proof.** The statement is true for \(m = 2\); assume that it is true for some \(m \geq 2\) and prove for \(m + 1\). Since the based filed \(F\) has at least 3 elements, identity (2) implies

\[
f_m(x^2, y, x_1, \ldots, x_{m-2}) + f_m(x \circ y, x, x_1, \ldots, x_{m-2}) = 0. \tag{3}
\]
By the induction assumption and skew-symmetry of $f_m$ we have

$$f_{m+1}(x^2, x, x_1, \ldots, x_{m-1}) = 
\sum_{i=1}^{m-1} (f([x^2, x_i], x, x_1, \ldots, x_i, \ldots, x_{m-1}) - f([x, x_i], x^2, x_1, \ldots, \bar{x}_i, \ldots, x_{m-1}))$$

Denote, for shortness, $f(a, b) = f_m(a, b, x_1, \ldots, \bar{x}_i, \ldots, x_{m-1})$. Then we have by (1) and (3)

$$f([x^2, x_i], x) = f(x \circ [x, x_i], x) = -f(x^2, [x, x_i]) = f([x, x_i], x^2),$$

which proves that $f_{m+1}(x^2, x, x_1, \ldots, x_{m-1}) = 0$. 

\[\square\]

The following theorem essentially generalizes the corresponding result from [7].

**Theorem 2.3.** Let $\mathcal{V}$ be a variety of flexible algebras over a field of characteristic \(\neq 2\) such that every finitely generated anticommutative algebra in $\mathcal{V}$ is nilpotent. Then for any $m$ there exists $N = N(m)$ such that any $m$-generated algebra in $\mathcal{V}$ satisfies the identity $f_N = 0$. If, moreover, for any $k$ polynomial $f_k$ is non-zero in the free $\mathcal{V}$-algebra of countable rank then $r_0(\mathcal{V})$ is infinite. The last condition holds if the characteristic of the based field is zero and $\mathcal{V}$ contains the variety of alternative algebras.

**Proof.** Let $A$ be the free $\mathcal{V}$-algebra on $m$ generators. Denote by $I_2(A)$ the subspace of $A$ generated by all the squares of elements of $A$. By [6], $I_2(A)$ is an ideal of $A$ and the quotient algebra $\bar{A} = A/I_2(A)$ is anticommutative. Since $A$ is $m$-generated, so is $\bar{A}$; therefore, $\bar{A}$ is nilpotent. Let $\bar{A}^n = 0$, then $A^n \subseteq I_2(A)$, that is, every monomial $u$ from $A$ with $\deg(u) \geq n$ may be written as a linear combination of Jordan products $u_i \circ v_i$, where $\deg u_i + \deg v_i = \deg u$. Denote by $M_k$ a subspace of $A$ generated by the set $\{f_{k+1}(u_1, \ldots, u_k, u)\}$, where $u_i, u$ are monomials on generators and $\deg u_i < n$. Let us show that $f_{k+1}(A, \ldots, A) \subseteq M_k$. Linearizing identity (3) we get in view of skew-symmetry of $f_{k+1}$

$$f_{k+1}(x \circ y, x_1, \ldots, x_{k-1}, z) = f_{k+1}(x, x_1, \ldots, x_{k-1}, z \circ y) + f_{k+1}(y, x_1, \ldots, x_{k-1}, z \circ x). \quad (4)$$

Consider $f_{k+1}(a_1, \ldots, a_k, a)$ where $a_i, a$ are monomials on generators. If $\deg a_1 \geq n$ then $a_1$ may be represented as a linear combination of Jordan products $u \circ v$ where $\deg u, \deg v < \deg a_1$. By (1), then $f_{k+1}(a_1, \ldots, a_k, a)$ is a linear combination of elements $f_{k+1}(u, a_2, \ldots, a_k, a')$ with $\deg u < \deg a_1$. Continuing in this way, we eventually obtain that $f_{k+1}(a_1, \ldots, a_k, a) \in M_k$, which proves that $f_{k+1}(A, \ldots, A, A) \subseteq M_k$.

Let now $N = \dim A/I_2(A)$, prove that $M_{N+1} = 0$. In fact, since $A^n \subseteq I_2(A)$, any $N + 1$ monomials $u_1, \ldots, u_{N+1}$ of degree less than $n$ in $A$ are linearly depending, hence in view of skew-symmetry, $f_{N+2}(u_1, \ldots, u_{N+1}, A) = 0$ and $M_{N+1} = 0$. By the previous considerations, this implies that $f_{N+2}(A, \ldots, A) = 0$.

Therefore, if the elements $f_k(x_1, \ldots, x_k) \neq 0$ in the free $\mathcal{V}$-algebra of infinite rank then $r_0(\mathcal{V})$ is infinite.

Finally, let $F$ be a field of characteristic zero. The similar proof as in [17] Chapter 13] shows that polynomials $f_k$ are nonzero in free metabelian alternative algebra on $k$ or more generators. Therefore, they are nonzero in the free $\mathcal{V}$-algebra on $k$ or more generators. 

\[\square\]

Recall that a flexible algebra is called *noncommutative Jordan* if it satisfies the identity

$$(x^2, y, x) = 0,$$
Corollary 2.4. Let \( \mathcal{V} \) be a variety of noncommutative Jordan algebras over a field of characteristic zero defined by the identity

\[
([x, y], z, z) = 0.
\]

Then the basic rank \( r_b(\mathcal{V}) \) is infinite.

Proof. In fact, it was proved in [6] that every finitely generated anticommutative algebra in this variety is nilpotent. It is also clear that this variety contains the variety of alternative algebras. \( \square \)

Corollary 2.5. Every \( m \)-generated alternative algebra over a field of characteristic \( \neq 2 \) satisfies the identity \( f_{n+1} = 0 \) for \( n > 1 + m + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} m \\ 3 \end{array} \right) \).

Proof. In fact, it is well known (see [17, p.122]) that an anticommutative alternative algebra \( A \) over a field of characteristic \( \neq 2 \) with a set of generators \( \{\alpha_i, i \in I\} \) is nilpotent of index 4 and is spanned by the elements of the type \( a_i a_j a_k \) where \( i < j < k \). Therefore, if \( A \) has \( m \) generators then \( \dim_F(A/I_2(A)) \leq m + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} m \\ 3 \end{array} \right) \). Hence we may take in this case \( N(m) = m + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} m \\ 3 \end{array} \right) \). \( \square \)

3 Skew-symmetric elements and one-generated superalgebras

In this section we will assume that the field \( F \) has characteristic 0.

Let us recall the definition of a superalgebra in a given variety of algebras \( \mathcal{V} \). In general, a superalgebra \( A \) is just a \( \mathbb{Z}_2 \)-graded algebra: \( A = A_0 \oplus A_1 \), where \( A_iA_j \subseteq A_{i+j}(\text{mod}2) \). Let \( G = \text{alg}(1, e_1, \ldots, e_n, \ldots | e_i e_j = -e_j e_i) \) be the Grassmann superalgebra, where \( G_0 \) is spanned by 1 and the even products \( e_{i_1} e_{i_2} \cdots e_{i_k} \), \( 1 < i_1 < i_2 < \cdots < i_k \), \( k \) even, and \( G_1 \) is spanned by the odd products \( e_{i_1} e_{i_2} \cdots e_{i_k} \), \( 1 < i_1 < i_2 < \cdots < i_k \), \( k \) odd. Then a superalgebra \( A = A_0 + A_1 \) is called a \( \mathcal{V} \)-superalgebra if its Grassmann envelope \( G(A) = G_0 \otimes A_0 + G_1 \otimes A_1 \), considered as an algebra, belongs to \( \mathcal{V} \). When working with superalgebras, we always assume that considered elements are homogeneous, that is, even (belonging to \( A_0 \)) or odd (belonging to \( A_1 \)). If the defining identities of the variety \( \mathcal{V} \) are known, one can easily write the defining super-identities for the variety of \( \mathcal{V} \)-superalgebras. Recall that an algebra \( A \) is called alternative (see ZSSS) if it satisfies the identities

\[
(x, y, y) = 0,
\]

\[
(x, x, y) = 0,
\]

where \( (x, y, z) = (xy)z - x(yz) \) denotes the associator of the elements \( x, y, z \). Now, a superalgebra \( A = A_0 \oplus A_1 \) is called an alternative superalgebra if it satisfies the superidentities

\[
(x, y, z) + (-1)^{|y||z|}(x, z, y) = 0,
\]

\[
(x, y, z) + (-1)^{|x||y|}(y, x, z) = 0,
\]
where $|z| = i$ if $z \in A_i$ denotes the parity of a homogeneous element $z$. Denote by $[x, y]_s = xy + (-1)^{|x||y|}yx$ the super-commutator of the homogeneous elements $x, y$, and by $x_s \circ y = xy + (-1)^{|x||y|}yx$ their super-Jordan product.

Recall some results on relation between the space of skew-symmetric elements in a free $\mathcal{V}$-algebra of countable rank and the free $\mathcal{V}$-superalgebra $F_{\mathcal{V}}[\emptyset; x]$ generated by one odd element $x$ (see [9, 10, 12]).

Let $u(x)$ be a homogenous of degree $n$ element from $F_{\mathcal{V}}[\emptyset; x]$. Write $u(x)$ in the form $u(x) = v(x, x, \ldots, x)$ where $v(x_1, \ldots, x_n)$ is a multilinear element. Set

$$(\text{Skew } u)(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in \Sigma_n} (-1)^{sgn \sigma} v(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

For example, if $u = xx$ then $(\text{Skew } u)(x_1, x_2) = [x_1, x_2] := x_1x_2 - x_2x_1$, the commutator of elements $x_1, x_2$; if $u = (xx)x$ then $(\text{Skew } u)(x_1, x_2, x_3) = \sum_{\sigma \in \Sigma_3} (-1)^{sgn \sigma} (x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}$.

**Proposition 3.1.** [9, 10, 12]. Let $F$ be a field of characteristic 0. The application $u \mapsto \text{Skew } u$ establishes an isomorphism between the linear space $F_{\mathcal{V}}[\emptyset; x]$ and the space of skew-symmetric elements in the free $\mathcal{V}$-algebra $F_{\mathcal{V}}[X]$, where $X = \{x_1, x_2, \ldots, x_0, \ldots, \}$. In particular, an element $u(x)$ is an identity in the superalgebra $F_{\mathcal{V}}[\emptyset; x]$ if and only if the element $(\text{Skew } u)$ is an identity in the free $\mathcal{V}$-algebra $F_{\mathcal{V}}[X]$.

Let $A = \text{Alt}[\emptyset; x]$ be the free alternative superalgebra generated by an odd generator $x$. Define by induction

$x^{[1]} = x, \ x^{[i+1]} = [x^{[i]}, x]_s, \ i > 0$,

and denote

$t = x^{[2]}, \ z^{[k]} = [x^{[k]}, t], \ u^{[k]} = x^{[k]} \circ_s x^{[3]}, \ k > 1$.

The following proposition summarizes some results from [13] on the structure of $A$.

**Proposition 3.2.** (i) For any $k > 0$, the element $z^{[k]}$ lies in the (super)center of $A$;

(ii) Elements

$t^m x^\sigma, \ m + \sigma \geq 1, \ t^m (x^{[k+2]} x^\sigma), \ t^m (u^{[4k+\varepsilon]} x^\sigma), \ t^m (z^{[4k+\varepsilon]} x^\sigma),$ (5)

where $k > 0, \ m \geq 0; \ \varepsilon, \sigma \in \{0, 1\}$, form a base of the superalgebra $A$.

**Lemma 3.3.** Let $f_m$ be the skew-symmetric polynomial constructed in Section 2. For any $m > 0$ there exist $\alpha_m, \beta_m \in F$ with $\alpha_m \neq 0$ such that

$f_m = \text{Skew } (\alpha_m x^{[m]} + \beta_m z^{[m-2]}).$ (6)

**Proof.** The element $f_m$ is a skew-symmetric polynomial on $m$ variables in the free alternative algebra $\text{Alt}[X]$. Moreover, it is a Malcev element, that is, it belongs to the free special Malcev superalgebra $SMalc[\emptyset; x]$. By [13], the superalgebra $SMalc[\emptyset; x]$ is isomorphic to the free Malcev superalgebra $Malc[\emptyset; x]$ generated by an odd element $x$, and by [10], the set $\{x^{[i]}, \ z^{[j]}, \ i, j > 0\}$ forms its base. By Proposition 3.1 there exists a homogenous element $u \in SMalc[\emptyset; x]$ of degree $m$ such that $f_m = \text{Skew } u$. Comparing the degrees, we have that $u = \alpha_m x^{[m]} + \beta_m z^{[m-2]}$ for some $\alpha_m, \beta_m \in F$. Since Skew $z^{[m]} = 0$ in any metabelian algebra, we have that $\alpha_m \neq 0$. \qed
Corollary 3.4. For any $m > 0$ there exist $\lambda_m, \nu_m \in F$ such that $\lambda_m \neq 0$ and

$$\text{Skew } x^{[m]} = \lambda_m f_m + \nu_m \left( \sum_{i < j} (-1)^{i+j} [f_{m-2}(x_i, \ldots, x_m), [x_i, x_j]] \right).$$  \hspace{1cm} (7)

**Proof.** From (1) we obtain since $z^{[k]} \in Z(A)$

$$\text{Skew } x^{[m]} = \lambda_m f_m + \nu_m \left( \sum_{i < j} (-1)^{i+j} [\text{Skew } x^{[m-2]}, [x_i, x_j]] \right)$$

$$= \lambda_m f_m + \nu_m \left( \sum_{i < j} (-1)^{i+j} ([\lambda_{m-2} f_{m-2} + \nu_{m-2} \text{Skew } z^{[m-4]}, [x_i, x_j]]) \right)$$

$$= \lambda_m f_m + \nu_m \left( \sum_{i < j} (-1)^{i+j} ([\lambda_{m-2} f_{m-2}, [x_i, x_j]]) \right),$$

which proves the Corollary. \hspace{1cm} $\Box$

Corollaries 2.5 and 3.4 imply

Corollary 3.5. Every $m$-generated alternative algebra over a field of characteristic 0 satisfies the identity $\text{Skew } x^{[n+3]} = 0$ for $n > 1 + m + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} m \\ 3 \end{array} \right)$.

Lemma 3.6. For every natural number $n$ there exists a number $M = M(m)$ such that every alternative algebra $A$ with $m$ generators which satisfies the identity

$$[[x, y]^2, x] = 0,$$  \hspace{1cm} (8)

satisfies also the identity

$$\text{Skew } t^{4M} = 0.$$

**Proof.** Let $A$ be the free $m$-generated alternative algebra in the variety defined by identity (4). Since $A$ is a finitely generated $PI$-algebra, by theorem 2.2 its Jacobson radical $J(A)$ is nilpotent, say of degree $M$. The quotient algebra $A/J(A)$ is a subdirect sum of primitive algebras satisfying identity (4), which by the Kaplansky [2] and Kleinfeld [17] theorems are isomorphic to matrix algebras of order no more than 2 or to Cayley-Dickson algebras over their centers. The first algebras satisfy the identity $\text{Skew } t^2 = 0$, and the second ones satisfy the identity $\text{Skew } t^4 = 0$. Therefore, the quotient algebra $A/J(A)$ satisfies the identity $\text{Skew } t^4 = 0$, and the algebra $A$ satisfies the identity $\text{Skew } t^{4M} = 0$.

Now we can prove our main result.

**Theorem 1.** For every natural number $n$ there exists a number $N(n)$ such that every skew-symmetric multilinear polynomial on more than $N(n)$ variables which is zero in the free associative algebra, vanishes in any $n$-generated alternative algebra $A$.

**Proof.** Let $f = f(x_1, \ldots, x_k)$ be a skew-symmetric polynomial from $\text{Alt}[X]$ which is zero in the free associative algebra $\text{Ass}[X]$. Then, as before, $f = \text{Skew } u$, where $u = u(x) \in A$ is a homogeneous element of degree $k$. Using base (5) and the fact that $f = 0$ in $\text{Ass}[X]$, we conclude that $u(x)$ is a linear combination of the following elements

1. $t^j(x^{i+2}x^\sigma)$, $2j + i + 2 + \sigma = k$, $i > 0$
2. $t^j(u^{4i+\varepsilon}x^\sigma)$, $2j + 4i + 3 + \varepsilon + \sigma = k$,
3. \( t^j (z^{[4i+\varepsilon]} x^\sigma) \), \( 2j + 4i + 2 + \varepsilon + \sigma = k \).

Let \( D = D(A) \) be the associator ideal of \( A \), that is, the ideal generated by all associators \((a, b, c) = (ab)c - a(bc)\), \( a, b, c \in A \), and let \( U = U(A) \) be the nucleus of \( A \), that is, the maximal ideal lying in the associative center of \( A \) (see [17]). Then \( U(A)D(A) = 0 \). Observe now that the second factors in products 1–3 lie in \( D(A) \). Moreover, by Corollary 3.5 if \( i > 3 + m + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} m \\ 3 \end{array} \right) \) then skew-symmetrizations of these factors are zero and Skew \( u(x) = 0 \).

Furthermore, by [16], the element \([x, y]^2, x] \in U(A)\). Therefore, if \( j \geq 4M(n) \), where \( M(n) \) is the number defined in Lemma 3.6 then Skew \( t^j \in U(A) \) and Skew \( u(x) \in U(A)D(A) = 0 \).

It is clear now that we can take \( N(n) = 2M(n) + 3 + m + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} m \\ 3 \end{array} \right) \).

\( \square \)

To conclude, we formulate the following open question:

Assume that an alternative algebra \( A \) over a field of characteristic zero satisfies all multilinear skew-symmetric identities of a fixed degree \( n \). Would \( A \) satisfy for some \( m \) all identities of the type

\[
\sum_{\sigma \in \Sigma_m} (-1)^{sgn(\sigma)} ((x_{\sigma(1)}W_1)(x_{\sigma(2)}W_2)\cdots (x_{\sigma(m)}W_m))_{q} = 0
\]

for arbitrary multiplication operators \( W_i \) and a fixed arrangement of brackets \( q \)?

This would be a generalization of the well known Kemer’s theorem claiming that in associative algebras over a field of characteristic zero the standard identity of degree \( n \) implies the Capelly identity of a certain degree \( m \) [3].

References

[1] Dnestrian Notebook (Non-solved problems in the theory of rings and modules) [in Russian], Novosibirsk, Institute of Mathematics, 1993; English transl.: in Non-associative algebra and its applications, Proceedings of the 5th international conference, Oaxtep, Mexico, July 27 – August 2, 2003. Lecture Notes in Pure and Applied Mathematics 246, (2006).

[2] I.N.Herstein: Noncommutative rings, Mathematical Association of America. doi:10.5948/UPO9781614440154.

[3] A.R.Kemer: Remark on the standard identity, Math. Notes, 23, no. 5 (1978), 753–757; English transl.: Math. Notes, 23, no. 5 (1978), 414–416.

[4] A.I. Malcev: Algebraic Systems, [in Russian], Nauka, 1970.

[5] A.I. Malcev: Selected Works [in Russian], v.1, Classical Algebra, Nauka, 1976.

[6] I.P. Shestakov: Certain classes of noncommutative Jordan rings, Algebra i Logika, (1971), 10, No. 4, 407–448; English transl.: Algebra and Logic, 10, No. 4 (1971), 252–280.

[7] I.P. Shestakov: On a problem by Shirshov, Algebra i Logika, (1977), 16, No.2, 227–246; English transl.: Algebra and Logic, v.16 (1977), no.2, 153–166.

[8] I. P. Shestakov: Finitely generated special Jordan and alternative PI-algebras, Math. USSR-Sb., (1985), 50, No. 1, 31–40.

[9] I.P.Shestakov: Alternative and Jordan superalgebras, Siberian Adv. Math., 9 No.2 (1999), 83–99.
[10] I.P. Shestakov: Free Malcev superalgebra on one odd generator, J. of Algebra and Its Applications, 2 No. 4 (2003), 451–461.

[11] I.P. Shestakov: Skew-symmetric identities of finitely generated Malcev algebras, Matematicheskii Zhurnal, 16 No. 2 (2016), 206–213.

[12] I. Shestakov, N. Zhukavets: Universal multiplicative envelope of free Malcev superalgebra on one odd generator, Communications in Algebra, 34 (2006), no.4, 1319–1344.

[13] I. Shestakov, N. Zhukavets: The free alternative superalgebra on one odd generator, Internat. J. Algebra Comp., 17 (5/6) (2007), 1215–1247.

[14] A.I. Shirshov: On special J-rings, Mat. Sb., 38 (1956), 149–166.

[15] A.I. Shirshov: On free Lie rings, Mat. Sb., 45 (1958), No.2, 113–122.

[16] S. R. Sverchkov: The composition structure of alternative and Malcev algebras, Commun. in Algebra, 44, no. 2 (2016), 457–478.

[17] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, Rings that are nearly associative [in Russian], Nauka, Moscow (1978), English transl.: Academic Press, New York - London (1982).