A Novel and Efficient Method for Computing the Resistance Distance

MUHAMMAD SHOAIB SARDAR\textsuperscript{1}, JIA-BAO LIU\textsuperscript{2}, IMRAN SIDDIQUE\textsuperscript{3}, AND MOHAMMED M. M. JARADAT\textsuperscript{4}

\textsuperscript{1}School of Mathematical Sciences, Anhui University, Hefei, Anhui 230601, China
\textsuperscript{2}School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China
\textsuperscript{3}Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan
\textsuperscript{4}Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar

Corresponding author: Mohammed M. M. Jaradat (mmjst4@qu.edu.qa)

This work was supported by the Qatar National Library.

\textbf{ABSTRACT} The resistance distance is an intrinsic metric on graphs that have been extensively studied by many physicists and mathematicians. The resistance distance between two vertices of a simple connected graph \(G\) is equal to the resistance between two equivalent points on an electrical network, constructed to correspond to \(G\), with each edge being replaced by a unit resistor. Hypercube \(Q_n\) is one of the most efficient and versatile topological structures of the interconnection networks, which received much attention over the past few years. The folded \(n\)-cube graph is obtained from hypercube \(Q_n\) by merging vertices of the hypercube \(Q_n\) that are antipodal, i.e., lie at a distance \(n\). Folded \(n\)-cube graphs have been studied in parallel computing as a potential network topology. The folded \(n\)-cube has the same number of vertices but half the diameter as compared to hypercubes which play an important role in analyzing the efficiency of interconnection networks. We intend to minimize the diameter. In this study, we will compute the resistance distance between any two vertices of the folded \(n\)-cube by using the symmetry method and classic Kirchhoff’s equations. This method is beneficial for distance-transitive graphs. As an application, we will also give an example and compute the resistance distance in the Biggs-Smith graph, which shows the competency of the proposed method.

\textbf{INDEX TERMS} Resistance distance, resistance diameter, networks, folded \(n\)-cube.

\section{I. INTRODUCTION}

For undetermined symbols and terminology, please refer to the book by Bollobás [1].

The computation of two-vertex resistances in electrical networks is a very old problem considered by many researchers over many years [2]. The computation of resistance is pertinent to a wide selection of problems extending from random walks [4], opinion formation [12], classical transport in disordered media [3], robustness of coupled oscillators network [5]–[7], first-passage processes [8], identifying the influential spreader node in a network [11], lattice Greens functions [9], [10], resistance distance [13]–[15], to graph theory [10], [16]. There are numbers of techniques and formulae have been developed for calculating the resistance distance, i.e., algebraic formulae [18], [20]–[25], series and parallel rules, combinatorial formula [4], delta-wye transformation [17], sum rules [18], [19], star-triangle transformation [17], probabilistic formulae [4], [26], star-mesh transformation, the principle of elimination, recursion formula [27], the principle of substitution and so forth. By employing the above methods and formulas, resistance distance in many networks and graphs has been discussed before, i.e., Potting network [41], circulant graphs [28], Sailboat fractal networks [40], Cayley graphs [29], complete \(n\)-partite graphs [30], wheels and fans [31], Double graphs; graph with an involution [32], regular graphs [33], [34], pseudo-distance-regular graphs [35], distance-regular graphs [51], some fullerene graphs [36], Sierpinski gasket network [37], ring-type network [38], maximum and minimum resistance distance in \(n\)-dimensional hypercubes [39], and others [42]–[48]. But, it is not straightforward to get the resistance distance in complex networks.

In this paper, we study simple connected graphs, i.e., graphs without loops and multiple edges. The vertices and edges of a graph \(G\) are symbolized by \(V(G)\) and \(E(G)\), respectively. The distance \(d(u, v)\) is the shortest-path distance...
between two vertices \( u \) and \( v \) in a graph \( G \). The length of the longest shortest-path in a graph \( G \) is called its diameter and it is denoted by \( D \). The resistance diameter \( D_r(G) \) of a graph \( G \) is defined by the maximum resistance distance between all pairs of vertices in \( G \) [39]. We need to minimize the diameter of a graph to improve the efficiency of interconnection networks. In this study, we employ symmetry method to compute the resistance distance. This idea was previously discussed by van Steenwijk [49], when he calculated the resistance of regular polyhedral resistive structures.

A graph can be viewed as an electrical network in which each edge is corresponding to a resistor of 1-ohm resistance \( r \). If there is a potential difference \( p \) between any edge of vertices \( i \) and \( j \) then an electric current \( w \) will flow in the edge according to the ohm’s law:

\[
w = \frac{p}{r}.
\]

In many practical problems, the electric current is made to compelled the network at a single point and leave it to others. The famous laws of Kirchhoff govern these currents. Kirchhoff’s potential law states that the sum of potential differences round any cycle \( a_1, a_2, \ldots, a_k \) equal to zero:

\[
p_{a_1a_2} + p_{a_2a_3} + \ldots + p_{a_{k-1}a_k} + p_{a_ka_1} = 0.
\]

Kirchhoff’s current law states that for any vertex the total current entering the vertex is exactly equal to the total current leaving the same vertex. In this study, we utilize the symmetry structure and determine which vertices have the same potential. This problem is to model the network in such a way that when a current \( w \) is entered into one vertex while it is allowed to leave the vertex at the remaining \( n - 1 \) nodes in equal portions \( w/(n-1) \) (see Figure 1). For more details, see the paper by van Steenwijk [49]. Now we have solved the current scheme on all sides and superimposed it to a similar network, where all currents are ignored and rotated, so that the current \( w \) now leaves the node of interest.

In the superposed system, the current \( nw/(n-1) \) reaches one vertex and leaves another vertex, and zero current reaches or leaves each other’s vertex. We will draw a layered graph by using breadth-first search technique for a network \( N \) and we choose a vertex \( s \) as a starting vertex through which the external current \( w \) is passed. We number these layers as their distance away from a starting vertex \( s \). We then define a layer matrix \( I_{i,j} \) (number of vertices in layer \( i \) connected to any vertex in layer \( j \)). Then we can obtain the potential difference by using ohm’s from any starting vertex \( s \) to any desired vertex \( t \) as follows:

\[
(w_{s,v_1} + w_{v_1,v_2} + \ldots + w_{v_{d-1},t})r.
\]

\( s = v_0, v_1, \ldots, v_d = t \) is a walk from \( s \) to \( t \).

In the resolving system, the potential difference among similar vertices is

\[
-(w_{t,v_{d-1}} + w_{v_{d-1},v_{d-2}} + \ldots + w_{v_1,t})r.
\]

It is easy to verify that

\[
w_{v_{i+d}} = -w_{d-i,d-1} \cdot (i = 0, \ldots, d-1).
\]

So by equations 1, 2 and 3, the potential difference between \( s \) and \( t \) in the superimposed system is

\[
2(w_{s,v_1} + w_{v_1,v_2} + \ldots + w_{v_{d-1},t})r.
\]

The equivalent resistor \( r \) between vertices \( s \) and \( t \) is found by superposition of the situation described above as follows:

\[
r_{s,t} = 2(w_{s,v_1} + w_{v_1,v_2} + \ldots + w_{v_{d-1},t})\frac{(n-1)}{wn}.
\]

### II. RESISTANCE DISTANCE IN THE FOLDED N-CUBES

The graph of the \( n \)-hypercube is given by the graph Cartesian product [50] of complete graphs \( K_2 \square K_2 \square \cdots \square K_2 \). A hypercube of order \( n \) is \( n \)-regular, bipartite, with diameter \( n \), \( 2^n \) vertices and \( n2^{n-1} \) edges. The folded \( n \)-cube graph is a graph obtained by merging vertices of the \( n \)-hypercube graph \( Q_n \) that are antipodal, i.e., lie at a distance \( n \) (the graph diameter of \( Q_n \)). The folded \( n \)-cube graph has a diameter \( D = \lceil \frac{n}{2} \rceil \), \( n + 1 \) regular, \( 2^n \) vertices and \( (n + 1)2^{n-1} \) edges (see Figure 2). We use the symbol \( Q_n \) for \( n \)-hypercube and \( F(Q_n) \) for the folded \( n \)-cube graph.

**Theorem 1:** The resistance distance between two vertices of the folded \( n \)-cube \( F(Q_n) \) equals

\[
r_{n,k}(F(Q_n)) = \frac{2^n - 1}{2^{n-1}} \sum_{i=1}^{k} w_i,
\]

where \( k \) is the distance between two vertices and \( 1 \leq k \leq \lceil \frac{n}{2} \rceil \), and \( w_k \), as shown at the bottom of the next page.

**Proof:** Suppose that the resistance of each edge of folded \( n \)-cube \( F(Q_n) \) is 1-ohm. We will use the symmetry method to compute the resistance distance between any two vertices.

![Figure 1. Superposition of symmetric current distributions.](image1)

![Figure 2. Constructions of \( F(Q_2) \) and \( F(Q_3) \) by merging vertices of \( Q_2 \) and \( Q_3 \).](image2)
of a folded $n$-cube $F(Q_n)$. For that, we solve the sets of equations. These sets of equations are obtained by entering a current $w$ through any vertex and taking a current $\frac{w}{2^{n-1}}$ out through the all other vertices in the network $F(Q_n)$. Since the $F(Q_n)$ is a distance transitive so we can choose any vertex $s$ as a starting vertex through which the external current $w$ is passed. This cleaves the network $F(Q_n)$ into different layers of equipotential vertices according to their distances away from $s$, i.e., vertices in the $k^{th}$ layer are at a distance $k$ away from $s$, where $1 \leq k \leq \lceil \frac{n}{2} \rceil$ (see Figure 3).

We select a vertex for each layer and make a Kirchhoff current equations to express the current reaching and exiting that vertex. Each vertex in the $k^{th}$ layer is adjacent to $k$ vertices in layer $k - 1$ when $1 \leq k \leq \lceil \frac{n}{2} \rceil$ and $(n + 1 - k)$ vertices in the layer $k + 1$ when $0 \leq k \leq \lceil \frac{n}{2} \rceil - 1$. For odd $n$, each vertex at $k = D$ layer is adjacent to $n + 1$ vertices in the layer $k - 1$. The number of vertices in the $k^{th}$ layer is $\frac{(n+1)^k}{k!(n+1-k)!}$, where $0 \leq k \leq \lceil \frac{n}{2} \rceil$. For odd $n$, the number of vertices at $k = D$ layer is $\frac{2\times4\times8\times\cdots\times(n+1)!}{(n+1-k)!}$.

Now we set up a layer matrix $l_{ij}$. In Figure 3, the layer graph shows that starting vertex $s$ (Layer 0) connected to $n + 1$ vertices only in layer 1 and it is not connected to any other vertex in any other layer. So we can write the first row of matrix as follows:

$$
\begin{pmatrix}
0 & n + 1 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
$$

Now we select any vertex in layer 1 which is connected to $n$ vertices in layer 2 and one vertex in layer 0 gives us the 2nd row, i.e.,

$$
\begin{pmatrix}
1 & 0 & n & 0 & \cdots & 0 \\
\end{pmatrix}
$$

Similarly, we create the layer matrix for all other vertices. For even $n$, we have

$$
\begin{pmatrix}
0 & n + 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & n & 0 & \cdots & 0 \\
0 & 2 & 0 & n - 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & D - 1 & 0 & n + 2 - D \\
0 & \cdots & 0 & D & 0 & \phantom{w} \\
\end{pmatrix}^{(\lceil \frac{n}{2} \rceil + 1) \times (\lceil \frac{n}{2} \rceil + 1)}
$$

and for odd $n$, we have

$$
\begin{pmatrix}
0 & n + 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & n & 0 & \cdots & 0 \\
0 & 2 & 0 & n - 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & D - 1 & 0 & n + 2 - D \\
0 & \cdots & 0 & D & 0 & \phantom{w} \\
\end{pmatrix}^{(\lceil \frac{n}{2} \rceil + 1) \times (\lceil \frac{n}{2} \rceil + 1)}
$$

From matrices 6 and 7, we have the following two sets of equations:

$$
\begin{align*}
\frac{w}{2^n - 1} & = 0 \\
\frac{w}{2^n - 1} & = 0 \\
\frac{w}{2^n - 1} & = 0 \\
\frac{w}{2^n - 1} & = 0
\end{align*}
$$

where $w_i$ is the current of an edge between the layer $i - 1$ and the layer $i$.\[107106 VOLUME 9, 2021\]
Here there are \( \lceil \frac{n}{2} \rceil + 1 \) equations and \( \lceil \frac{n}{2} \rceil + 1 \) unknown variables in both sets of equations. Because there is a no connection between vertices that are at a distance greater than one in different layers. The current among layers can be procured by the simple recursion relation:

\[
\begin{align*}
w_1 & = \frac{w}{n + 1}, \\
w_k & = \frac{w}{n + 2 - k}((k - 1)w_{k-1} - \frac{1}{2^n - 1}) \quad (2 \leq k \leq \lceil \frac{n}{2} \rceil).
\end{align*}
\]

Hence, let \( w = 1 \), we can get (8), as shown at the bottom of the page.

So by using equations 8 and 5, we can find the resistance distance between any two vertices in the folded \( \text{n-cube} \) \( F(Q_n) \),

\[ r_{n,k}(F(Q_n)) = \frac{2^n - 1}{2^{n-1}} \sum_{i=1}^{k} w_i, \]

where \( 1 \leq k \leq \lceil \frac{n}{2} \rceil \).

**Corollary 1:** The resistance distance between two vertices of the folded \( \text{n-cube} \) \( F(Q_n) \) is maximum at \( k = D \).

**Proof:** From Theorem 1, we have

\[ r_{n,k-1}(F(Q_n)) = \frac{2^n - 1}{2^{n-1}} \sum_{i=1}^{k-1} w_i. \]

So,

\[ r_{n,k}(F(Q_n)) = \frac{2^n - 1}{2^{n-1}} \sum_{i=1}^{k} w_i = r_{n,k-1}(F(Q_n)) + \frac{2^n - 1}{2^{n-1}} w_k, \]

where \( \frac{2^n - 1}{2^{n-1}} w_k > 0 \). So the resistance distance between two vertices of the folded \( \text{n-cube} \) \( F(Q_n) \) is maximum at \( k = D \).

**Remark 1:** Since the \( F(Q_n) \) is a distance transitive graph, the \( F(Q_n) \) is a distance regular graph. Biggs [51] (or see also [52], [53]) presented a set of potentials, described in terms of the intersection arrays of distance-regular graphs, which allow one to compute the resistance between any two vertices. The resistance distance among any two vertices of \( F(Q_n) \) can be computed by the method of [51]. In Theorem 1, the current distribution satisfying the recursive relation is considered and then the resistance distance between any two points is calculated according to Ohm’s law.

**Remark 2:** It appears in Fig 5 that resistance diameter \( D_r(F(Q_n)) \) of the folded \( \text{n-cube} \) strictly decreases as \( n \) increases while the ordinary diameter of the folded \( \text{n-cube} \) in Fig 4 is strictly increased as \( n \) increases. In many communication aspects, the folded \( \text{n-cubes} \) has proven to be superior to the hypercubes. The diameter is halved, the average distance is better, the communication link delay is shorter, and lower cost make this new structure very promising. The resistance diameter of folded \( \text{n-cubes} \) is also less than resistance diameter of hypercubes [39]. The reason is that we have more paths between pairs of vertices in folded \( \text{n-cubes} \) as compared to the hypercubes. Due to the reduction in the resistance diameter, it improves the efficiency of the folded \( \text{n-cube} \) in message transmission and parallel computing.

**III. APPLICATIONS**

In this section, as an application, we will compute the resistance distance of folded 4-cube and Biggs-Smith graph to show the efficacy of the suggested method.

**Example 1:** The graph for the folded 4-cube is shown in Figure 6 (a). We take the vertex 1 as a starting vertex and draw a layered graph for the folded 4-cube as depicted.
M. S. Sardar et al.: Novel and Efficient Method for Computing Resistance Distance

FIGURE 6. a. The folded 4-cube. b. The layered graph of the folded 4-cube.

FIGURE 7. The Biggs-Smith graph.

in Figure 6 (b). The starting vertex 1 is adjacent to 5 vertices in layer 1, each vertex in layer 1 is adjacent to 4 vertices in layer 2 and 1 vertex in layer 0 and each vertex in layer 2 is adjacent to 2 vertices in layer 1. The layer matrix and vertex equations for the currents, as shown below:

\[
l_4 = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 2 & 0 \end{pmatrix}.
\]

\[
w - 5w_1 = 0 \quad (9)
\]

\[
w_1 - 4w_2 - \frac{w}{15} = 0 \quad (10)
\]

\[
2w_2 - \frac{w}{15} = 0 \quad (11)
\]

From the above equations, we have \( w_1 = \frac{1}{5}w \) and \( w_2 = \frac{1}{30}w \). Now by putting these values in equation 5, we obtain the resistance distance between any two vertices of the folded 4-cube, i.e.,

\[
r_{4,1} = (w_1) \frac{30}{16w} = \frac{3}{8},
\]

\[
r_{4,2} = (w_1 + w_2) \frac{30}{16w} = \frac{7}{16},
\]

where \( r_{4,1} \) and \( r_{4,2} \) are the resistance distances in the folded 4-cube at a distance 1 and 2, respectively.

FIGURE 8. The layered graph of the Biggs-Smith graph.

Example 2: The Biggs-Smith graph is a 3-regular graph on 102 vertices and 153 edges (see Figure 7). Since Biggs-Smith graph is a distance-transitive, so it does not matter which vertex we choose to draw a layer graph. We draw a layered graph by using a breadth-first search technique by choosing 102 as a starting vertex (see Figure 8). The vertex 102 is adjacent to 3 vertices in layer 1, each vertex in layer 1 is adjacent to 2 vertices in layer 2 and 1 vertex in layer 0 and so on. So we can write the layer matrix and vertex equations for the currents as follows:

\[
\begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}
\]

\[
w - 3w_1 = 0 \quad (12)
\]

\[
w_1 - 2w_2 - \frac{w}{101} = 0 \quad (13)
\]

\[
w_2 - 2w_3 - \frac{w}{101} = 0 \quad (14)
\]

\[
w_3 - 2w_4 - \frac{w}{101} = 0 \quad (15)
\]

\[
w_4 - w_5 - \frac{w}{101} = 0 \quad (16)
\]

\[
w_5 - w_6 - \frac{w}{101} = 0 \quad (17)
\]

\[
w_6 - w_7 - \frac{w}{101} = 0 \quad (18)
\]

\[
3w_7 - \frac{w}{101} = 0 \quad (19)
\]

After solving above equations, we get

\[
w_1 = \frac{1}{3}w, \quad w_2 = \frac{49}{303}w, \quad w_3 = \frac{23}{303}w, \quad w_4 = \frac{10}{303}w, \quad (20)
\]

\[
w_5 = \frac{7}{303}w, \quad w_6 = \frac{4}{303}w, \quad w_7 = \frac{1}{303}w. \quad (21)
\]
We obtain the resistance distance in Biggs-Smith graph by using equation 5 and the values obtained in 20.

\[
\begin{align*}
    r_{3,1} &= \frac{202}{102} w = \frac{101}{153}, \\
    r_{3,2} &= \left( w_1 + w_2 \right) \frac{202}{102} w = \frac{50}{51}, \\
    r_{3,3} &= \left( w_1 + w_2 + w_3 \right) \frac{202}{102} w = \frac{173}{153}, \\
    r_{3,4} &= \left( w_1 + w_2 + w_3 + w_4 \right) \frac{202}{102} w = \frac{61}{51}, \\
    r_{3,5} &= \left( w_1 + w_2 + w_3 + w_4 + w_5 \right) \frac{202}{102} w = \frac{190}{153}, \\
    r_{3,6} &= \left( w_1 + w_2 + w_3 + w_4 + w_5 + w_6 \right) \frac{202}{102} w = \frac{194}{153}, \\
    r_{3,7} &= \left( w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 \right) \frac{202}{102} w = \frac{65}{51}.
\end{align*}
\]

IV. CONCLUSION

Over the last few years, the formula for computing resistance distance is usually obtained by using (pseudo)-inversion or eigenvalues and eigenfunctions of the Laplacian matrix. We cannot apply these formulas to further study because they contain Chebyshev polynomials or trigonometric functions. So in this study, we developed a novel and efficient method for computing the resistance distance. The resistance distance between any two vertices in a folded n-cubes is obtained by using the symmetry method and classic Kirchhoff’s equations. The method is more suitable to graphs that are distance-transitive. As an application, we also compute the resistance distance for Biggs-Smith graph by using the suggested method. It is also shown that the resistance diameter of folded n-cubes is also less than that of hypercubes which could play an important role in analyzing the efficiency of interconnection networks.

ACKNOWLEDGMENT

The authors would like to express their sincere gratitude to the anonymous referees for valuable suggestions, which led to great deal of improvement of the original manuscript. The publication of this article was supported by the Qatar National Library.

REFERENCES

[1] B. Bollobás, Modern Graph Theory. New York, NY, USA: Springer-Verlag, 1998.
[2] G. Kirchhoff, “Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird,” Annalen der Physik, vol. 148, pp. 497–508, 1847.
[3] S. Kirkpatrick, “Percolation and conduction,” Rev. Mod. Phys., vol. 45, no. 5, pp. 574–588, 1973.
[4] P. G. Doyle and J. L. Snell, Random Walks and Electric Networks. Washington, DC, USA: The Mathematical Association of America, 1984.
[5] T. W. Grunberg and D. F. Gayme, “Performance measures for linear oscillator networks over arbitrary graphs,” IEEE Trans. Control Netw. Syst., vol. 5, no. 1, pp. 456–468, Mar. 2018.
[6] M. Tyloo, T. Coletta, and P. Jacqoud, “Robustness of synchrony in complex networks and generalized Kirchhoff indices,” Phys. Rev. Lett., vol. 120, no. 8, Feb. 2018, Art. no. 084101.
[7] M. Tyloo, L. Pagnier, and P. Jacqoud, “The key player problem in complex oscillator networks and electric power grids: Resistance centralities identify local vulnerabilities,” Sci. Adv., vol. 5, no. 11, Nov. 2019, Art. no. eaaw8359.
[8] S. Redner, A Guide to First-Passage Processes. Cambridge, U.K.: Cambridge Univ. Press, 2001.
[9] S. Katsura and S. Inawashiro, “Lattice Green’s functions for the rectangular and the square lattices at arbitrary points,” J. Math. Phys., vol. 12, no. 8, pp. 1622–1630, Aug. 1971.
[10] W. Kook, “Combinatorial Green’s function of a graph and applications to networks,” Adv. Appl. Math., vol. 46, nos. 1–4, pp. 417–423, Jan. 2011.
[11] P. Van Mieghem, K. Devriendt, and H. Cetinay, “Pseudoinverse of the Laplacian and best spreader node in a network,” Phys. Rev. E. Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top., vol. 96, no. 3, Sep. 2017, Art. no. 032311.
[12] F. Baumann, I. M. Sokolov, and M. Tyloo, “A Laplacian approach to stubborn agents and their role in opinion formation on influence networks,” Phys. A, Stat. Mech. Appl., vol. 557, Nov. 2020, Art. no. 124869.
[13] D. J. Klein and M. Randić, “Resistance distance,” J. Math. Chem., vol. 12, pp. 81–95, Dec. 1993.
[14] M. Li, S. Zhou, J. Liu, G. Chen, and Q. Zhou, “Phase transition in spectral clustering based on resistance matrix,” Phys. A, Stat. Mech. Appl., vol. 566, Mar. 2021, Art. no. 125598.
[15] W. Xiao and I. Gutman, “Resistance distance and Laplacian spectrum,” Theor. Chem. Accounts, vol. 110, no. 4, pp. 284–289, Nov. 2003.
[16] L. Novak and A. Gibbons, Hybrid Graph Theory and Network Analysis. Cambridge, U.K.: Cambridge Univ. Press, 2009.
[17] A. E. Kennelly, “The equivalence of triangles and three-pointed stars in conducting networks,” Electr. World Eng., vol. 34, no. 12, pp. 413–414, 1899.
[18] H. Chen and F. Zhang, “Resistance distance local rules,” J. Math. Chem., vol. 44, no. 2, pp. 405–415, Aug. 2008.
[19] Y. Yang and H. Zhang, “Some rules on resistance distance with applications,” J. Phys. A, Math. Theor., vol. 41, no. 44, Nov. 2008, Art. no. 445203.
[20] D. J. Klein, “Graph geometry, graph metrics and Wiener,” MATCH Commun. Math. Comput. Chem., vol. 35, pp. 7–27, Mar. 1997.
[21] C. R. Rao and S. K. Mitra, Generalized Inverse of Matrices and Its Applications. New York, NY, USA: Wiley, 1971.
[22] G. E. Sharpe and G. P. H. Styan, “A note-on equicofactor matrices,” Proc. IEEE, vol. 55, no. 7, pp. 1226–1227, Jul. 1967.
[23] S. Seshu and M. B. Reed, Linear Graphs and Electrical Networks. Reading, MA, USA: Addison-Wesley, 1961.
[24] G. Sharpe and B. Spain, “On the solution of networks by means of the equicofactor matrix,” IRE Trans. Circuit Theory, vol. 7, no. 3, pp. 296, 1960.
[25] G. Sharpe and G. Styan, “Circuit duality and the general network inverse,” IEEE Trans. Circuit Theory, vol. CT-12, no. 1, pp. 22–27, Mar. 1965.
[26] C. S. J. A. Nash-Williams, “Random walk and electric currents in networks,” Proc. Cambridge Philos. Soc., vol. 55, no. 2, pp. 181–194, 1959.
[27] Y. Yang and D. J. Klein, “A recursion formula for resistance distances and its applications,” Discrete Appl. Math., vol. 161, nos. 16–17, pp. 2702–2715, Nov. 2013.
[28] H. Zhang and Y. Yang, “Resistance distance and Kirchhoff index in circulant graphs,” Int. J. Quantum Chem., vol. 107, no. 2, pp. 330–339, 2007.
[29] X. Gao, Y. Luo, and W. Liu, “Resistance distances and the Kirchhoff index in Cayley graphs,” Discrete Appl. Math., vol. 159, no. 17, pp. 2050–2057, Oct. 2011.
[30] S. V. Gervacio, “Resistance distance in complete n-partite graphs,” Discrete Appl. Math., vol. 203, pp. 53–61, Apr. 2016.
[31] R. B. Bapat and S. Gupta, “Resistance distance in wheels and fans,” Indian J. Pure Appl. Math., vol. 41, no. 1, pp. 1–13, Feb. 2010.
[32] L. Shi and H. Chen, “Resistance distances and Kirchhoff index of graphs with an involution,” Discrete Appl. Math., vol. 215, pp. 185–196, Jan. 2016.
[33] I. Lukovits, S. Nikolić, and N. Trinajstić, “Resistance distance in regular and the square lattices at arbitrary points,” J. Math. Phys., vol. 12, no. 8, pp. 2702–2715, Nov. 2013.
[34] H. Zhang and Y. Yang, “Resistance distance and Kirchhoff index in circulant graphs,” Int. J. Quantum Chem., vol. 71, no. 3, pp. 217–225, 1999.
[35] D. J. Klein, I. Lukovits, and I. Gutman, “On the definition of the hyper-Wiener index for cycle-containing structures,” J. Chem. Inf. Comput. Sci., vol. 35, no. 1, pp. 50–52, Jan. 1995.
[36] S. Jafarizadeh, R. Sufiani, and M. A. Jafarizadeh, “Evaluation of effective resistances in pseudo-distance-regular resistor networks,” J. Stat. Phys., vol. 139, no. 1, pp. 177–199, Apr. 2010.
[36] P. W. Fowler, “Resistance distances in fullerene graphs,” *Croatica Chem. Acta*, vol. 75, no. 2, pp. 401–408, 2002.
[37] Z. Jiang and W. Yan, “Some two-point resistances of the Sierpinski gasket network,” *J. Stat. Phys.*, vol. 172, no. 3, pp. 824–832, Aug. 2018.
[38] Z. Jiang and W. Yan, “Resistance between two nodes of a ring network,” *Phys. A, Stat. Mech. Appl.*, vol. 484, pp. 21–26, Oct. 2017.
[39] M. S. Sardar, H. Hua, X.-F. Pan, and H. Raza, “On the resistance diameter of hypercubes,” *Phys. A, Stat. Mech. Appl.*, vol. 540, Feb. 2019, Art. no. 123076, doi: 10.1016/j.physa.2019.123076.
[40] J. Zha, L. Tian, J. Fan, and L. Xi, “Two-point resistances in sailboat fractal networks,” *Fractals*, vol. 28, no. 2, Mar. 2020, Art. no. 2050027, doi: 10.1142/S0218348X20500279.
[41] J. Fan, J. Zha, L. Tian, and Q. Wang, “Resistance distance in potting networks,” *Phys. A, Stat. Mech. Appl.*, vol. 540, Feb. 2020, Art. no. 123053.
[42] M. S. Sardar, M. Alaeiyan, M. R. Farahani, M. Cancan, and S. Ediz, “Resistance distance in some classes of rooted product graphs obtained by Laplacian generalized inverse method,” *J. Inf. Optim. Sci.*, vol. 42, no. 4, pp. 1–21, Apr. 2021.
[43] J. Liu, X.-F. Pan, Y. Wang, and J. Cao, “The Kirchhoff index of folded hypercubes and some variant networks,” *Math. Problems Eng.*, vol. 2014, pp. 1–9, Jan. 2014, doi: 10.1155/2014/380874.
[44] J. Liu, J. Cao, X.-F. Pan, and A. Elaiw, “The Kirchhoff index of hypercubes and related complex networks,” *Discrete Dyn. Nature Soc.*, vol. 2013, pp. 1–7, Jan. 2013.
[45] M. S. Sardar, X.-F. Pan, and S.-A. Xu, “Computation of resistance distance and Kirchhoff index of the two classes of silicate networks,” *Appl. Math. Comput.*, vol. 381, Sep. 2020, Art. no. 125283.
[46] M. S. Sardar, X.-F. Pan, and Y.-X. Li, “Some two-vertex resistances of the three-towers Hanoi graph formed by a fractal graph,” *J. Stat. Phys.*, vol. 181, no. 1, pp. 116–131, Oct. 2020, doi: 10.1007/s10955-020-02569-1.
[47] M. S. Sardar, X.-F. Pan, and S.-A. Xu, “Some two-vertex resistances of nested triangle network,” *Circuits, Syst., Signal Process.*, vol. 40, no. 3, pp. 1511–1524, Mar. 2021, doi: 10.1007/s00034-020-01541-4.
[48] M. S. Sardar, M. Cancan, S. Ediz, and W. Sajjad, “Some resistance distance and distance-based graph invariants and number of spanning trees in the tensor product of $P_2$ and $K_2$,” *Proyecciones, J. Math.*, vol. 39, no. 4, pp. 919–931, Aug. 2020, doi: 10.22199/issn.0717-6279-2020-04-0057.
[49] F. J. van Steenwijk, “Equivalent resistors of polyhedral resistive structures,” *Amer. J. Phys.*, vol. 66, no. 1, pp. 90–91, Jan. 1998.
[50] V. G. Vizing, “The Cartesian product of graphs,” *Vycisl. Sistemy*, vol. 34, no. 4, pp. 770–786, May 2013.

**JIA-BAO LIU** received the B.S. degree in mathematics and applied mathematics from Wanxi University, China, in 2005, and the M.S. and Ph.D. degrees in mathematics and applied mathematics from Anhui University, China, in 2009 and 2016, respectively. From September 2013 to July 2014, he was a Visiting Researcher with the School of Mathematics, Southeast University, China, where he was a Postdoctoral Fellow with the School of Mathematics, in March 2017. He is currently working as a Professor with the School of Mathematics and Physics, Anhui Jianzhu University, Hefei, China. He is the author or coauthor of more than 100 journals articles and two edited books. His current research interests include graph theory and its applications, fractional calculus theory, neural networks, and complex dynamical networks. He is also a Reviewer of *Mathematical Reviews* and *Zentralblatt Math*.

**IMRAN SIDDIQUE** received the Ph.D. degree in mathematics from ASSMS GC, University of Lahore, Pakistan. He is currently working as a Full Professor with UMT, Lahore, Pakistan. He has published more than 100 research articles in well reputed international journals of *Mathematical and Engineering Sciences*. His research interests include Newtonian and non-Newtonian fluid mechanics, ordinary and partial differential equations, fractional calculus, integral transforms, soliton theory, lubrication theory, graph theory, fuzzy algebra and decision making, and numerical analysis. He is also a referee and an editor of several international mathematical journals.

**MOHAMMED M. M. JARADAT** received the bachelor’s and master’s degrees from Yarmouk University, in 1993 and 1995, respectively, and the Ph.D. degree from Pittsburgh University, USA, in 2001. Then directly he joined Yarmouk University, as an Assistant Professor, where he promoted to an Associate Professor and a Full Professor, in 2006 and 2013, respectively. In 2017, he promoted to a Full Professor once again in Qatar University. He has been a Faculty Member with the Math, Stat and Physics Department, Qatar University, since 2006. He is the author or coauthor for more than 90 articles published in international journals. His research interests include graph theory, fixed point theory, and game theory.

---

**MUHAMMAD SHOAIB SARDAR** received the M.S. degree in mathematics from the University of Management and Technology, Lahore, Pakistan, in 2016, and the Ph.D. degree in mathematics and pure mathematics from Anhui University, Hefei, China, in 2020. He is currently working as a Visiting Assistant Professor with the Department of Mathematics, Riphah International University at Faisalabad, Pakistan. He has published more than 28 research articles in well-known journals. His research interests include graph theory and combinatorics, algebraic graph theory, spectral graph theory, and chemical graph theory.