ON THE DELIGNE-LUSZTIG INVOLUTION FOR CHARACTER SHEAVES

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ABSTRACT. For a reductive group $G$, we study the Drinfeld-Gaitsgory functor of the category of conjugation-equivariant $D$-modules on $G$. We show that this functor is an equivalence of categories, and that it has a filtration with layers expressed via parabolic induction of parabolic restriction. We use this to provide a conceptual definition of the Deligne-Lusztig involution on the set of isomorphism classes of irreducible character $D$-modules, which was defined previously in [Lu1, §15].

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1. Introduction

1.1. Background and motivation.

1.1.1. The Deligne-Lusztig involution. Let $G$ be a connected reductive algebraic group over a finite field $F$. On the set of isomorphism classes of irreducible representations of the finite group $G(F)$ (over $\mathbb{Q}_\ell$) one has an involution $DL$, the Deligne-Lusztig\footnote{Other names relevant for this involution are Alvis, Curtis and Kawanaka; See [Lu2, 47] for the history of this involution.} involution. Namely, in the $K$-group of representations, given an irreducible representation $V$, the irreducible representation $DL(V)$ is given, up to a sign, by

$$\sum_{I \subseteq \Sigma} (-1)^{|I|} \text{pind}^G_{P_I^{-}} \text{pres}^G_{P_I} V.$$

Here $\Sigma$ is the set of simple roots, $P_I, P_I^{-}$ are opposite standard parabolics associated to the subset $I \subseteq \Sigma$, and $\text{pres}, \text{pind}$ denote parabolic restriction and induction.

In [Lu1, §15], an involution given by an analogous formula is defined on the set of isomorphism classes of irreducible character sheaves on $G$ (it is denoted “$d$" there, we will denote it by $DL$).

In the present paper, we give a conceptual definition of this involution on irreducible character sheaves. Technically, we work with character $D$-modules rather than character sheaves, but one can transport everything to $\ell$-adic sheaves as well and, anyhow, in this introduction we will be vague about such details.

What we actually do is to study the Drinfeld-Gaitsgory functor of the $DG$-category of conjugation-equivariant sheaves on a connected reductive algebraic group $G$. The desired involution on irreducible character sheaves is then simply induced by this functor. The main technical result is a filtration of this functor whose layers are expressed as parabolic inductions of parabolic restrictions (with some cohomological shifts), in particular showing that our definition of the involution coincides with that of [Lu1, §15].

A point which seems nice for us to stress is that our definition of the Deligne-Lusztig involution for irreducible character sheaves is “abstract" - its input is the category of conjugation-equivariant sheaves as a category, so it is not “informed" about the more specific structure of Levi subgroups, parabolic induction functors and so on, as in the formula above (that is what we meant by the adjective “conceptual" above).

Let us next try to motivate the two main objects of this paper, character sheaves and the Drinfeld-Gaitsgory functor, independently of the utility of defining the Deligne-Lusztig involution.
1.1.2. Character sheaves. In the representation theory of finite groups, a prominent object is the space of class functions. In somewhat fancy terms, one might say that the space of class functions is the cocenter of the category of (finite-dimensional) representations, and so one can assign to every representation an element of the space of class functions - its character.

Thus, after some familiarity with categorification, one might suspect that for an algebraic group $G$, the category of conjugation-equivariant sheaves on $G$ is of some similar basic importance. Restricting ourselves to the case when $G$ is a connected reductive group, it turns out that indeed a certain subcategory of this category (discovered by Lusztig), the subcategory of character sheaves, is a central object of study. It should roughly be seen as a cocenter of some 2-category of categorical representations of $G$. A fundamental prior role of character sheaves, studied in great depth by Lusztig, is their tight match with actual characters of irreducible representations of finite groups of Lie type under the sheaf-to-function dictionary.

The whole category of conjugation-equivariant sheaves can be seen as some “direct integral” of the subcategories of character sheaves with various “central characters”.

1.1.3. The Drinfeld-Gaitsgory functor. For any compactly-generated presentable $DG$-category $\mathcal{C}$, one constructs an endo-functor

$$DG_\mathcal{C} : \mathcal{C} \to \mathcal{C},$$

the Drinfeld-Gaitsgory functor. It seems to be a basic homological construction, describing some “duality” phenomena. To get some feeling of that, consider the following examples:

- In [GaYo], it is shown that under some conditions on the category $\mathcal{C}$, the Drinfeld-Gaitsgory functor is an equivalence of categories, whose inverse is the Serre functor (which traditionally embodies some sort of “duality”). However, these conditions (which can be thought of as “smallness” conditions) oftentimes do not hold, and it is our feeling that in such cases the Drinfeld-Gaitsgory functor is more “correct” than the Serre functor.
- The Drinfeld-Gaitsgory functor has a fondness to intertwine left and right adjoints - see claim 3.1 for a general statement of this sort.
- For a ring $A$, the Drinfeld-Gaitsgory functor of the $DG$-category of $A$-modules is given by tensoring with the bimodule

$$\text{Hom}_{A \otimes A^{op}}(A, A \otimes A^{op})$$

(where $A$ is considered as a bimodule over itself in the usual way, and the Hom is in the derived sense). Thus, the relation to Hochschild cohomology, Grothendieck-style duality, etc. is seen.

See the introduction to [GaYo] (as well as that paper itself) for some appearances of the Drinfeld-Gaitsgory functor (there called the Pseudo-Identity functor) in representation theory.

1.2. The results of this paper in short. Let us summarize the main results of this paper in a more technical way. Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$ of characteristic zero. Consider the category $D(G^{\vee}G)$ of $D$-modules on the quotient-stack of $G$ by the conjugation action of $G$, and consider the corresponding Drinfeld-Gaitsgory (a.k.a. Pseudo-Identity) functor

$$DG_{D(G^{\vee}G)} : D(G^{\vee}G) \to D(G^{\vee}G).$$

We prove:
• (proposition 5.1) Given opposite parabolic subgroups $P, P^- \subset G$ with Levi $L$, there
is a commutation relation $DG_{D(G,G)} \circ \text{pind}^G_P \cong \text{pind}^G_{P^-} \circ DG_{D(L,L)}$, where $\text{pind}$ is the
parabolic induction. In view of the second adjunction, this is shown quite formally,
since $DG$ “likes” to intertwine between left and right adjoints.

• (theorem 5.2) The functor $DG_{D(G,G)}$ is “glued” in a specific way from cohomological
shifts of functors of the form $\text{pind}^G_P \circ \text{pres}^G_P$, where $P, P^- \subset G$ are opposite parabolic
subgroups and $\text{pind}, \text{pres}$ are functors of parabolic induction and restriction. This
is done by “resolving” the kernel governing $DG_{D(G,G)}$, by means of the wonderful
compactification (technically, by means of the Vinberg semigroup).

• (proposition 5.3) The functor $DG_{D(G,G)}$ is invertible (i.e. an equivalence of categories).
This follows quite formally from $DG_{D(G,G)}$ being proper (i.e. sending compact objects
to compact objects), which in turn follows from the previous result.

• (proposition 5.7) Fixing an integer $d$, the functor $DG_{D(G,G)}[d]$ is $t$-exact when restricted
to the subcategory of character $D$-modules which, roughly, are obtained via parabolic
induction of cuspidal character $D$-modules from Levi’s whose center has dimension $d$.

• (claim 5.9) The previous item provides an auto-bijection of the set of isomorphism
classes of irreducible character $D$-modules; This auto-bijection is an involution. This is
the Deligne-Lusztig involution, appearing in [Lu1, §15] (where it is denoted by “$d$”).

• (proposition 5.11) Here, perhaps as an exercise, we reprove a partial case of [Lu1,
Corollary 15.8.(c)]. Namely, we calculate that when applied to the irreducible unipotent
character $D$-modules of the “principal series”, i.e. constituents of the Springer
$D$-module, which are parametrized by irreducible representations $\alpha$ of the Weyl group $W$,
the Deligne-Lusztig involution swaps $\alpha$ with $\text{sgn} \otimes \alpha$, where $\text{sgn}$ is the one-dimensional
sign representation of $W$. This is proven simultaneously with the curious formula

$$\sum_{I \in \Sigma} (-1)^{|I|} \cdot \text{ind}^W_I \text{res}^W_I V = \text{sgn} \otimes V$$

in the $K$-group of finite-dimensional representations of the Weyl group $W$ (here $\Sigma$ is
the set of simple reflections and $W_I \subset W$ are the various “parabolic” subgroups).

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2. Notations and conventions

2.1. Categories. We fix an algebraically closed field $k$ of characteristic zero. We will work with
the $(\infty, 1)$-category $\text{Lin}_k$ of $k$-linear, stable and presentable $(\infty, 1)$-categories, where morphisms
are continuous $k$-linear functors (i.e. $k$-linear functors preserving colimits). By a category
we will mean an object in $\text{Lin}_k$ unless remarked otherwise, and by a functor we will mean a
morphism in $\text{Lin}_k$, unless remarked otherwise.

By a subcategory of a category we will mean a full subcategory closed under colimits which
is presentable (and thus itself an object of $\text{Lin}_k$).

For a functor $F$, we denote by $F^R$ and $F^L$ the right and left adjoints of $F$. We say that
a continuous functor between compactly generated categories is proper, if it sends compact
objects to compact objects. This is equivalent to the functor admitting a continuous right adjoint.
Given a $t$-structure on $\mathcal{C} \in \text{Lin}_k$, we will say that an object $\mathcal{F} \in \mathcal{C}$ is irreducible if $\mathcal{F}$ lies in $\mathcal{C}^\circ$ and is irreducible as an object of this abelian category. We will say that an object $\mathcal{F} \in \mathcal{C}$ is bounded if $H^n(\mathcal{F}) \neq 0$ only for finitely many $n \in \mathbb{Z}$. We will say that an object $\mathcal{F} \in \mathcal{C}$ has finite length if it is bounded and each of its cohomologies has finite length in the abelian category $\mathcal{C}^\circ$ (denote by $\mathcal{C}^\text{fl} \subset \mathcal{C}$ the non-cocomplete subcategory of objects of finite length). We will say that an object $\mathcal{F} \in \mathcal{C}$ is semisimple if it is isomorphic to a finite direct sum of cohomological shifts of semisimple objects of finite length in $\mathcal{C}^\circ$ (denote by $\mathcal{C}^{ss} \subset \mathcal{C}$ the non-cocomplete subcategory of semisimple objects).

**Definition 2.1.** Let $\mathcal{C} \in \text{Lin}_k$. We say that an object $x \in \mathcal{C}$ is glued from a list of objects $x_0, \ldots, x_n \in \mathcal{C}$ if there exist fiber sequences

$$y_{n-1} \rightarrow x \rightarrow x_n,$$

$$y_{n-2} \rightarrow y_{n-1} \rightarrow x_{n-1},$$

$$\ldots$$

$$x_0 \rightarrow y_1 \rightarrow x_1.$$

2.2. Spaces. By a scheme we will mean a scheme of finite type over $k$. For convenience, by a space we will mean a QCA stack of finite type over $k$ (see [DrGa1] Definition 1.1.8 for the notion of a QCA stack).

For a space $X$, we will denote by $D(X)$ the category of $D$-modules on $X$. Recall (see [DrGa1]) that $D(X)$ is compactly generated and self-dual (by Verdier duality). We denote by $\omega \in D(X)$ the “dualizing sheaf” ($\omega = \pi^!k$ where $\pi : X \rightarrow \mathbf{•}$), by $C \in D(X)$ the “constant sheaf” ($C = \mathbb{D}^Vc(\omega) = \pi^*k$) and, if $X$ is smooth, by $\mathcal{O}$ the constant $D$-module ($\mathcal{O} = \pi^e k = \pi^!k[n] = \pi^*k[-n]$ where $n$ is the dimension of $X$).

2.3. The kernel formalism. Given spaces $X$ and $Y$, we identify

$$D(X \times Y) \cong \text{Hom}_{\text{Lin}_k}(D(X), D(Y))$$

by matching a “kernel” $X \in D(X \times Y)$ with the functor $T_X(\mathcal{F}) = (\pi_2)_!(X \otimes \pi_1^!\mathcal{F})$, where $(\pi_2)_!$ is the renormalized direct image (see [DrGa1] Definition 9.3.2)).

2.4. The group. We fix a connected reductive group $G$.

We denote by $T$ the universal Cartan of $G$. We denote by $\Sigma \subset X^*(T)$ the set of simple roots, and by $W \subset \text{Aut}(T)$ the Weyl group.

We fix a Torel $T_{\text{sub}} \subset B \subset G$, i.e. a Cartan subgroup $T_{\text{sub}}$ contained in a Borel subgroup $B$. We then have an identification $\phi : T \rightarrow T_{\text{sub}}$. For every $I \subset \Sigma$, we denote by $P_I \subset G$ the corresponding standard parabolic subgroup containing $B$ (with the convention $P_{\emptyset} = B$ and $P_{\Sigma} = G$), and by $P_I^{-}$ we denote the corresponding opposite parabolic containing $T_{\text{sub}}$. We denote $G_I := P_I \cap P_{I}^{-}$, and mostly think of it in the usual way as a quotient of $P_I$ and a quotient of $P_{I}^{-}$.

For $0 \leq i \leq |\Sigma|$ we denote $d_i := \dim T - i = \dim Z(G) + |\Sigma| - i$ and for $I \subset \Sigma$ we denote $d_I := \dim Z(G_I) = d_{|I|}$.

3. The Drinfeld-Gaitsgory functor

In this section we recall the Drinfeld-Gaitsgory functor and prove some properties of it (which mostly can be extracted from [Ga1], but given here in greater generality).
3.1. Recollection. Let us recall the definition of the Drinfeld-Gaitsgory functor $\text{DG}_C : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C} \in \text{Lin}_k$ is compactly generated (see [Ga1] and also [GaYo, Section 1.4.2], where it is denoted $\text{Ps-Id}_C$).

For a compactly generated $\mathcal{C} \in \text{Lin}_k$, we have a colimit-preserving functor $\mathcal{C} \to \mathcal{C}$ characterized by

$$(\cdot)^\vee : \mathcal{C} \to (\mathcal{C}^\vee)^{\text{op}}$$

for $m \in \mathcal{C}, c \in \mathcal{C}^c$ (and $c^{\text{op}} \in (\mathcal{C}^c)^{\text{op}}$ denotes the corresponding object in the opposite category). An object $m \in \mathcal{C}$ is called reflexive, if the natural map $m \to (m^\vee)^\vee$ is an isomorphism. Compact objects are reflexive, and [Ga1, Corollary 6.1.8] shows that coherent objects in $D(X)$, for a space $X$, are reflexive as well.

It will be convenient in what follows to keep in mind the identification

$$\text{Hom}(\mathcal{C}, \mathcal{D}) \cong \mathcal{C}^\vee \otimes \mathcal{D}.$$ 

Given two compactly generated categories $\mathcal{C}, \mathcal{D} \in \text{Lin}_k$ we denote

$$\text{DG}_{\mathcal{C}, \mathcal{D}} : \text{Hom}(\mathcal{C}, \mathcal{D}) \cong \mathcal{C}^\vee \otimes \mathcal{D} \xrightarrow{(\cdot)^\vee} (\mathcal{D}^\vee \otimes \mathcal{C})^{\text{op}} \cong \text{Hom}(\mathcal{D}, \mathcal{C})^{\text{op}}.$$ 

Then the Drinfeld-Gaitsgory functor $\text{DG}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ is given as

$$\text{DG}_\mathcal{C} := \text{DG}_{\mathcal{C}, \mathcal{C}}(\text{Id}_\mathcal{C}).$$

Let us recall (see [GaYo, Lemma 2.1.3]) that for a space $X$, the functor $\text{DG}_{D(X)}$ is given by the kernel $\Delta_!$ where $\Delta : X \to X \times X$ is the diagonal.

3.2. Intertwining left and right adjoints.

Claim 3.1. Let $\mathcal{C}, \mathcal{D} \in \text{Lin}_k$ be compactly generated categories, and let $F : \mathcal{C} \to \mathcal{D}$ admit a continuous right adjoint as well as a left adjoint. Then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\text{DG}_\mathcal{D}} & \mathcal{D} \\
\mathcal{C} & \xleftarrow{\text{DG}_\mathcal{C}} & \mathcal{C}
\end{array}$$

$F^R$ \quad $F^L$

Proof. Recall the for a continuous functor admitting a left adjoint, the conjugate of the the left adjoint is the same as the dual (see [Ga1] for all these terms). We thus see that we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^\vee \otimes \mathcal{D} & \xrightarrow{\text{DG}_\mathcal{D}^\gamma \phi} & (\mathcal{D}^\vee \otimes \mathcal{D})^{\text{op}} \\
\mathcal{C}^\vee \otimes \mathcal{D} & \xrightarrow{\text{DG}_\mathcal{C}^\gamma \phi} & (\mathcal{D}^\vee \otimes \mathcal{C})^{\text{op}} \\
\mathcal{C}^\vee \otimes \mathcal{C} & \xleftarrow{\text{DG}_\mathcal{C}^\gamma \phi} & (\mathcal{C}^\vee \otimes \mathcal{C})^{\text{op}}
\end{array}$$

(3.1)

\[2\text{Here, the target is not an object of Lin}_k - \text{i.e. we step briefly outside of our “world”.}\]
Then, \( \text{Id}_C \) in the bottom left and \( \text{Id}_D \) in the top left are mapped to the same thing in the middle right as is evident by going vertically and then horizontally. Going now horizontally and then vertically, we get the desired identity.

### 3.3. Invertibility.

**Claim 3.2.** Let \( \mathcal{C} \in \text{Lin}_k \) be a compactly generated category. Suppose that \( \text{Id}_C \in \text{Hom}(\mathcal{C}, \mathcal{C}) \) is reflexive. If \( DG_\mathcal{C} \) is proper, then it is right-invertible. In particular, if \( DG_\mathcal{C} \) and \( DG_{\mathcal{C}^*} \) are proper, then \( DG_\mathcal{C} \) is invertible.

**Proof.** Notice that the condition that \( \text{Id}_C \in \text{Hom}(\mathcal{C}, \mathcal{C}) \) is reflexive means

\[
\bar{DG}_\mathcal{C} \circ (DG_\mathcal{C}) \cong \text{Id}_C.
\]

Considering again the lower square of diagram 3.1 with \( D := \mathcal{C} \) and \( F := DG_\mathcal{C} \), evaluating on the object \( \text{Id}_C \) in the left bottom, we obtain

\[
\text{Id}_C \cong DG_\mathcal{C} \circ (DG_\mathcal{C})^R.
\]

Thus, \( DG_\mathcal{C} \) is right-invertible.

The second assertion follows by recalling that \( DG_{\mathcal{C}^*} \cong (DG_\mathcal{C})^\vee \), so that if \( DG_{\mathcal{C}^*} \) is proper then, by what we just proved, \( (DG_\mathcal{C})^\vee \) is right-invertible, and thus \( DG_\mathcal{C} \) is left-invertible. □

**Corollary 3.3 ([Gal Corollary 6.7.2]).** Let \( X \) be a space. If \( DG_{D(X)} \) is proper, then \( DG_{D(X)} \) is invertible.

**Proof.** First, notice that \( \text{Id}_{D(X)} \in \text{Hom}(D(X), D(X)) \) is reflexive, since it is given by a coherent kernel (as is easily seen from preservation of holonomicity by functors!), and as recalled above, coherent objects are reflexive. Second, recall that \( D(X)^\vee \cong D(X) \) via Verdier duality. Thus the corollary follows from the claim. □

### 4. Adjoint-equivariant \( D \)-modules

In this section we gather information regarding conjugation-equivariant \( D \)-modules, their parabolic restriction and induction, and character \( D \)-modules.

#### 4.1. Parabolic induction and restriction.

Let \( P \subset G \) be a parabolic, with Levi quotient \( M \). The functor of parabolic restriction

\[
pres_P^G : D(G \setminus G) \to D(M \setminus M)
\]

is defined as \( q_* p^! \) where

\[
\begin{array}{ccc}
P \setminus P & \xrightarrow{p} & P \setminus P \\ \downarrow & & \downarrow q \\ G \setminus G & \xleftarrow{G \setminus G} & M \setminus M \\
\end{array}
\]

is the natural correspondence.

The map \( p \) is projective and the map \( q \) is smooth of relative dimension 0 (notice that \( q \) is also safe!). Hence \( \text{pres}_P^G \) admits a left adjoint, the functor of parabolic induction

\[
\text{pind}_P^G : D(M \setminus M) \to D(G \setminus G)
\]

given by \( p q^* \cong p_* q! \).
For $I \subset \Sigma$, we abbreviate:

\[ \text{pres}_I := \text{pres}^G_{P_I}, \quad \text{pres}^-_I := \text{pres}^G_{P^-_I}, \quad \text{pind}_I := \text{pind}^G_{P_I}, \quad \text{pind}^-_I := \text{pind}^G_{P^-_I}. \]

**Remark 4.1.** For $J \subset I \subset \Sigma$, let us denote by $\text{pind}^J_I : D(G \backslash J / G) \to D(G \backslash I / G)$ the analogous functor for $G_I$. One has $\text{pind}^J_I \circ \text{pind}^J_J \cong \text{pind}^J_J$. Also, “Mackey theory” shows that for $I, J \subset \Sigma$, the functor $\text{pres}_J \circ \text{pind}_I$ can be glued from functors of the form $\text{pind}^J_{K_2} \circ ? \circ \text{pres}^J_{K_1}$ where $K_1 \subset I$ and $K_2 \subset J$ with $|K_1| = |K_2|$ and $?$ is an equivalence given by conjugation by a suitable element in $G$. For example, in the special case $I = \emptyset$, the functor $\text{pres}_\emptyset \circ \text{pind}_\emptyset$ can be glued from functors of the form $\text{pind}^\emptyset_{\emptyset} \circ \hat{w}$ where $\hat{w} : T \setminus T \to T \setminus T$ is the conjugation by $\hat{w} \in N_G(T)$. See [DrGa3] for explicit details in the case $G \backslash g$ (everything transfers to the case $G \backslash G$ word-by-word).

### 4.2. Second adjointness and exactness.

The proof of the following theorem using Braden’s hyperbolic localization theorem is sketched in [DrGa3, Section 0.2.1]:

**Theorem 4.2** (Second adjointness). The functor $\text{pres}_I$ is left adjoint to $\text{pind}^I_I$.

**Corollary 4.3.** The functors $\text{pres}_I$ and $\text{pind}_I$ admit all iterated left and right adjoints (in particular, $\text{pres}_I$ is proper).

The following theorem is proved in [BeYo]:

**Theorem 4.4** ([BeYo Theorem 5.4]). The functors $\text{pres}_I$ and $\text{pind}_I$ are t-exact.

#### 4.3. Character $D$-modules.

The functor

\[ ch : D(G \backslash (G/U \times G/U) / T) \to D(G \backslash G) \]

is defined (perhaps up to a cohomological shift, which is irrelevant to us) as $p_* q^!$ where

\[ G \backslash (G \times G/B) \]

\[ G \backslash G \]

\[ \downarrow p \]

\[ \overset{q}{\rightarrow} \]

\[ G \backslash (G/U \times G/U) / T \]

(here $B \subset G$ is a Borel subgroup with unipotent radical $U$ and Levi quotient $T$, and the maps are $p(g, xB) = g, q(g, xB) = (xU, gxU)$). Notice that $p$ is proper and $q$ is smooth, hence $ch$ is proper.

**Remark 4.5.** Denoting by $ch^I$ the analogous functor for $G_I$, similarly to remark one shows that $\text{pind}^I_I \circ ch^I \cong ch^I$ for some functor $?$, and also that $\text{pres}_I \circ ch$ can be glued from functors of the type $ch^I \circ ?$ where $?$ is some functor.

**Definition 4.6.** The subcategory $\text{CH}(G) \subset D(G \backslash G)$ of character $D$-modules is the subcategory generated under colimits by the image of $ch$ on $T$-monodromic objects.

The following are standard properties:

**Lemma 4.7.**

1. The irreducible subquotients of cohomologies of any character $D$-module are again character $D$-modules.
2. Every irreducible character $D$-module is of geometric origin (in particular, holonomic with regular singularities).
3. Every compact character $D$-module has finite length.
4. $\text{CH}(G) \subset D(G \backslash G)$ is closed under truncation.
Lemma 4.8. The functors $\text{pind}_I$ and $\text{pres}_I$ preserve $\text{CH}(\cdot)$.

Proof. This follows from remark 4.3. □

Lemma 4.9. The functors $\text{pind}_I$ and $\text{pres}_I$ preserve $\text{CH}(\cdot)^{ss}$ (and hence also $\text{CH}(\cdot)^{fl}$).

Proof. In view of lemma 4.4, every irreducible character $D$-module is of geometric origin. Hence by the decomposition theorem, $\text{pind}_I$ sends irreducible character $D$-modules to semisimple ones.

To show that $\text{pres}_I$ preserves semisimplicity of character $D$-modules, one uses its preservation of purity - see [BeYo] §5.3. □

Lemma 4.10. The functors $\text{pind}_I$ and $\text{pind}_J$ induce the same map

$$K_0(\text{CH}(G_I)^{fl}) \to K_0(\text{CH}(G)^{fl}).$$

Proof. Let $\mathcal{G} \in \text{CH}(G_I)^{\vee}$ be irreducible; We want to show that $\text{pind}_I(\mathcal{G})$ and $\text{pind}_J(\mathcal{G})$ are equal in the $K_0$-group. Since these objects are in the heart (by theorem 4.4) and semisimple (by lemma 4.9), it is enough to show that for every irreducible $\mathcal{F} \in \text{CH}(G)^{\vee}$ we have

$$[\text{pind}_I(\mathcal{G}) : \mathcal{F}] = [\text{pind}_J(\mathcal{G}) : \mathcal{F}]$$

(where $[- : \mathcal{F}]$ denotes the amount of times $\mathcal{F}$ enters the semisimple $-$. And indeed:

$$[\text{pind}_I(\mathcal{G}) : \mathcal{F}] = \dim H^0\text{Hom}(\text{pind}_I(\mathcal{G}), \mathcal{F}) = \dim H^0\text{Hom}(\mathcal{G}, \text{pres}_I(\mathcal{F})) = [\text{pres}_I(\mathcal{F}) : \mathcal{G}] =$$

$$= \dim H^0\text{Hom}(\text{pres}_I(\mathcal{F}), \mathcal{G}) = \dim H^0\text{Hom}(\mathcal{F}, \text{pind}_J(\mathcal{G})) = [\text{pind}_J(\mathcal{G}) : \mathcal{F}]$$

(where we have also used $\text{pres}_I(\mathcal{F})$ being in the heart (by theorem 4.4) and semisimple (by lemma 4.9)). □

4.4. Decomposition w.r.t. cuspidal rank. For $0 \leq i \leq |\Sigma|$, we denote by $\text{CH}(G)^{(\leq i)} \subset \text{CH}(G)$ the subcategory generated under colimits by the images of the functors $\text{pind}_I$, where $|I| \leq i$. We also denote by $\text{CH}(G)^{(i)}$ the right-orthogonal of $\text{CH}(G)^{(\leq i-1)}$ in $\text{CH}(G)^{(\leq i)}$ (since the $\text{pind}_I$’s are proper, these again are subcategories in the sense of subsection 2.1). In particular, we set $\text{CH}(G)^{\text{cusp}} := \text{CH}(G)^{(\leq |\Sigma|)}$ (the subcategory of cuspidal objects).

Lemma 4.11. Let $\mathcal{F} \in \text{CH}(G)^{\vee}$ be irreducible. Then there exists $I \subset \Sigma$ and a cuspidal irreducible $\mathcal{G} \in \text{CH}(G_I)^{\vee}$ such that $\mathcal{F}$ is isomorphic to a direct summand of $\text{pind}_I \mathcal{G}$. One has then $\mathcal{F} \in \text{CH}(G)^{(I)}$.

Proof. Consider a minimal $I$ for which there exists irreducible $\mathcal{G} \in \text{CH}(G_I)^{\vee}$ such that $\mathcal{F}$ is a direct summand of $\text{pind}_I \mathcal{G}$ (such $I$ exists because $\Sigma$ always suits). We want to show that $\mathcal{G}$ is cuspidal. Otherwise, we would have $J \subseteq I$ such that $\text{pres}_J \mathcal{G} \neq 0$. Taking an irreducible quotient $\text{pres}_I \mathcal{G} \to \mathcal{K}$, by adjunction we get a non-zero map $\mathcal{K} \to \text{pind}_I \mathcal{H}$. By semisimplicity, $\mathcal{G}$ is a direct summand of $\text{pind}_I \mathcal{H}$ and hence $\text{pind}_I \mathcal{G}$ is a direct summand of $\text{pind}_I \text{pind}_I \mathcal{K} \cong \text{pind}_I \mathcal{H}$, so $\mathcal{F}$ is a direct summand of $\text{pind}_I \mathcal{H}$, contradicting the minimality of $I$.

Let $\mathcal{F}$ be as above. Clearly $\mathcal{F} \in \text{CH}(G)^{(\leq |I|)}$. Moreover, from remark 4.1, we see that $\mathcal{F}$ is in the right-orthogonal to $\text{CH}(G)^{(\leq |I|-1)}$. □

Definition 4.12. Let $\mathcal{F} \in \text{CH}(G)^{\vee}$ be irreducible. We define the cuspidal rank of $\mathcal{F}$ as the integer $0 \leq i \leq |\Sigma|$ for which $\mathcal{F} \in \text{CH}(G)^{(i)}$. 

Lemma 4.13. Let $\mathcal{F}_1, \mathcal{F}_2 \in \text{CH}(G)^{\infty}$ be irreducibles of cuspidal ranks $i_1, i_2$. If $i_1 \neq i_2$, then $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$.

Proof. If $i_1 < i_2$, the claim is immediate. Suppose that $i_1 > i_2$. By Lemma 4.11 we can find $|I| = i_2$ and irreducible cuspidal $\mathcal{G} \in \text{CH}(G_I)^{\infty}$ such that $\mathcal{F}_2$ is a direct summand of $\text{pind}_I(\mathcal{G})$. Thus it is enough to show that $\text{Hom}(\mathcal{F}_1, \text{pind}_I(\mathcal{G})) = 0$. By second adjunction, this $\text{Hom}$ is the same as $\text{Hom}(\text{pres}_I^{-1}\mathcal{F}_1, \mathcal{G})$, and hence it is zero since $\text{pres}_I^{-1}\mathcal{F}_1 = 0$.

Let us denote by $\text{inc}^{(\leq i-1)} : \text{CH}(G)^{(\leq i-1)} \to \text{CH}(G)^{(\leq i)}$, $\text{inc}^{(i)} : \text{CH}(G)^{(i)} \to \text{CH}(G)^{(\leq i)}$ the inclusion functors. Let us denote by $P^{(\leq i-1)} = \text{inc}^{(\leq i-1)} \circ (\text{inc}^{(\leq i-1)})^R$, $P^{(i)} = \text{inc}^{(i)} \circ (\text{inc}^{(i)})^L$ the corresponding “projection” endo-functors of $\text{CH}(G)^{(\leq i)}$. One has a fiber sequence $P^{(\leq i-1)} \to \text{Id} \to P^{(i)}$.

Lemma 4.14. Let $\mathcal{F} \in \text{CH}(G)^{(\leq i)}$ be of finite length. Then the fiber sequence

$P^{(\leq i-1)}(\mathcal{F}) \to \mathcal{F} \to P^{(i)}(\mathcal{F})$

splits.

Proof. Step 1: Let us show first that $P^{(i)}(\mathcal{F})$ (resp. $P^{(\leq i-1)}(\mathcal{F})$) is of finite length, with all irreducible constituents being cuspidal of rank $i$ (resp. $\leq i-1$). We reduce to $\mathcal{F}$ being irreducible. Then if $\mathcal{F}$ has cuspidal rank $i$, $\mathcal{F} \to P^{(i)}(\mathcal{F})$ is an isomorphism. If $\mathcal{F}$ has cuspidal rank $\leq i-1$, $P^{(\leq i-1)}(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Step 2: The fiber sequence splits, since by the first step and by Lemma 4.13 we have

$\text{Hom}(P^{(i)}(\mathcal{F}), P^{(\leq i-1)}(\mathcal{F})) = 0$.

□

Corollary 4.15. The inclusion $\text{CH}(G)^{(i)} \to \text{CH}(G)^{(\leq i)}$ is proper.

Proof. The objects $(\text{inc}^{(i)})^L(\mathcal{F})$, where $\mathcal{F} \in \text{CH}(G)^{(\leq i)}$ are compact, are compact generators of $\text{CH}(G)^{(i)}$. Hence it is enough to show that, for compact $\mathcal{F} \in \text{CH}(G)^{(\leq i)}$, the object

$\text{inc}^{(i)}((\text{inc}^{(i)})^L(\mathcal{F})) = P^{(i)}(\mathcal{F}) \in \text{CH}(G)^{(\leq i)}$

is compact. This follows from Lemma 4.13 (recall that compact objects in $\text{CH}(G)$ have finite length).

□

Proposition 4.16. $\text{CH}(G)^{(i)}$ is the left-orthogonal of $\text{CH}(G)^{(\leq i-1)}$ in $\text{CH}(G)^{(\leq i)}$.

Proof. Let $\mathcal{F} \in \text{CH}(G)^{(i)}$ and $\mathcal{G} \in \text{CH}(G)^{(\leq i-1)}$. We want to show that $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$. One can assume that $\mathcal{F}$ is compact in $\text{CH}(G)^{(i)}$, and hence in $\text{CH}(G)^{(\leq i)}$ by corollary 4.15. This allows to assume that $\mathcal{G}$ is compact. One then reduces to $\mathcal{G}$ being of the form $\text{pind}_I(\mathcal{H})$ where $|I| \leq i-1$ and $\mathcal{H}$ is compact. By second adjointness one has $\text{Hom}(\mathcal{F}, \text{pind}_I(\mathcal{H})) \cong \text{Hom}(\text{pres}_I^{-1}(\mathcal{F}), \mathcal{H})$.

But $\text{pres}_I^{-1}(\mathcal{F}) = 0$, and we are done. □
Theorem 4.17. One has a direct sum decomposition
\[ \text{CH}(G) = \bigoplus_{0 \leq i \leq |\Sigma|} \text{CH}(G)^{(i)} \]
compatible with the t-structure.

Proof. One splits the filtration
\[ \text{CH}(G)^{(\leq 0)} \subset \ldots \subset \text{CH}(G)^{(\leq |\Sigma|)} \]
using proposition 4.10. The compatibility with the t-structure follows from the t-exactness of the \text{pres}_I's. \qed

Remark 4.18. See [Gu2] for a statement, in the case of \( G_z \), which both generalizes to all \( D \)-modules (rather than character \( D \)-modules) and also takes into account the more refined “cuspidal support” (versus only “cuspidal rank”).

5. The main results
In this section we state the main results of this paper.

5.1. Commutation with parabolic induction.

Proposition 5.1. The following diagram is commutative:

\[
\begin{array}{ccc}
D(G \backslash G) & \xrightarrow{\text{DG}_{D(G;G)}} & D(G \backslash G) \\
pind_I & & pind_I \\
\downarrow & & \downarrow \\
D(G_I \backslash G_I) & \xrightarrow{\text{DG}_{D(G;G)}_{G_I}} & D(G_I \backslash G_I)
\end{array}
\]

Proof. Apply claim 3.1 to \( F = \text{pres}_I \) (taking into account theorem 4.2). \qed

5.2. Filtration.

Theorem 5.2. For \( 0 \leq i \leq |\Sigma| \), denote
\[ M_i := \bigoplus_{|I|=i} \text{pind}_I \circ \text{pres}_I[ -d_I ] . \]
Then the functor \( \text{DG}_{D(G;G)} \) is glued\(^3\) from the list of functors \( M_0, \ldots, M_{|\Sigma|} \).

Proof. The proof is postponed to section 7. \qed

Corollary 5.3. The functor \( \text{DG}_{D(G;G)} \) is isomorphic to \( \text{Id}_{D(G;G)}[-d_{\Sigma}] \) when restricted to \( D(G \backslash G)^{\text{cusp}} \) (the subcategory consisting of objects \( \mathcal{F} \) for which \( \text{pres}_I(\mathcal{F}) = 0 \) for all \( I \neq \Sigma \)).

Corollary 5.4. The functor \( \text{DG}_{D(G;G)} \) preserves the subcategory \( \text{CH}(G) \subset D(G \backslash G) \).

Proof. Clear by theorem 5.2 and lemma 4.9. \qed

\(^3\)see definition 2.3
5.3. Invertibility.

**Proposition 5.5.** The functor $DG_{D(G;G)}$ is invertible.

*Proof.* By corollary 3.3, it is enough to show that $DG_{D(G;G)}$ is proper. This follows from theorem 5.2, since $pind_I$ and $pres_I$ are proper. \[\square\]

**Proposition 5.6.** For $0 \leq i \leq |\Sigma|$, denote

$$N_i := \bigoplus_{|I|=i} \text{pind}_I \circ \text{pres}_I[d_I].$$

Then the functor $DG^{-1}_{D(G;G)}$ is glued from the list of functors $N_{|\Sigma|}, \ldots, N_0$.

*Proof.* Apply left adjoints to the gluing of theorem 5.2. \[\square\]

5.4. Deligne-Lusztig involution.

**Proposition 5.7.** Let $0 \leq i \leq |\Sigma|$, denote

1. The functor $DG_{D(G;G)}$ preserves $CH_{D(G;G)}^{i}$.
2. The functor $DG_{D(G;G)}[d_I]$ is t-exact when restricted to $CH_{D(G;G)}^{i}$.

*Proof.*

1. This follows from proposition 5.1.
2. It is enough to show that if $F \in (CH_{D(G;G)}^{i})^{\geq 0}$ then $DG_{D(G;G)}(F) \in CH_{D(G;G)}^{\geq-d_I}$ and $DG_{D(G;G)}^{-1}(F) \in CH_{D(G;G)}^{\geq-d_I}$. Since the t-structure is compatible with filtered colimits, we can reduce to $F$ being of the form $\tau_{\geq 0} F'$ where $F' \in CH_{D(G;G)}^{i}$ is compact, and hence in particular to $F$ being of finite length. Thus we may reduce to $F$ being irreducible. By lemma 4.11, we may reduce to $F$ being of the form $pind_I S$ for some $|I| = i$ and cuspidal irreducible $S$. But then:

$$DG_{D(G;G)}(pind_I S) \cong pind_I(DG_{D(G;G)}(S)) \cong pind_I(S[-d_I]) \in CH_{D(G;G)}^{\geq d_I}$$

and similarly

$$DG_{D(G;G)}^{-1}(pind_I S) \cong pind_I(DG_{D(G;G)}^{-1}(S)) \cong pind_I(S[d_I]) \in CH_{D(G;G)}^{\geq-d_I}.$$

\[\square\]

**Corollary 5.8.** We obtain an auto-bijection (the Deligne-Lusztig involution)

$$DL \subset \text{Irr} \left( CH(G)^{\varnothing} \right),$$

by sending an irreducible object $E \in (CH(G)^{i})^{\varnothing}$ to $DG_{D(G;G)}(E)[d_I]$.

Let us show that DL is indeed involutive:

**Claim 5.9.** One has $DL \circ DL = \text{Id}$.

*Proof.* In view of theorem 5.2 and proposition 5.6 coupled with lemma 4.10, it is clear that $DG_{D(G;G)}$ and $DG_{D(G;G)}^{-1}$ induce the same map

$$K_0(CH(G)^{i}) \rightarrow K_0(CH(G)^{i}).$$

\[\square\]

**Remark 5.10.** From theorem 5.2, it is clear that the involution DL coincides with that defined in [Lu1] §15 (where it is denoted “d”).
5.5. Computation of the Deligne-Lusztig involution on unipotent principal series.
Let us denote $\text{Spr} := \text{pind}_G(\mathcal{O})$ (this is the Springer $D$-module). The irreducible character $D$-modules which are isomorphic to a direct summand of $\text{Spr}$ will be called unipotent principal series. The set of isomorphism classes of unipotent principal series character $D$-modules is in bijection with the set $\text{Irr}(W)$ of isomorphism classes of irreducible representations of $W$ (see section 6.1). Denote by $\text{Spr}_\alpha$ the isomorphism class of unipotent principal series character $D$-modules corresponding to $\alpha \in \text{Irr}(W)$.

Proposition 5.11. For every $\alpha \in \text{Irr}(W)$, one has

$$DL(\text{Spr}_\alpha) = \text{Spr}_{\text{sgn} \otimes \alpha}$$

(where $\text{sgn}$ is the one-dimensional sign representation of $W$).

Proof. The proof is postponed to the end of section 6.2. □

Remark 5.12. The above proposition is just a special case of [Lu1, Corollary 15.8.(c)]. We include it here for completeness.

6. Proof of Proposition 5.11

In this section we provide a proof for proposition 5.11. We will work with abelian categories; all $\text{Hom}$’s will be understood to be $H^0\text{Hom}$’s, etc.

6.1. The abelian category $\text{PS}(G)$ and its relation to $\text{Rep}_k^f(W)$. Let us denote by $\text{PS}(G) \subset \text{CH}(G)^\otimes$ the abelian subcategory consisting of objects which are isomorphic to finite direct sums of direct summands of the Springer $D$-module $\text{Spr} := \text{pind}_G(\mathcal{O})$. Notice that $\text{PS}(G) \subset (\text{CH}(G)^{(0)})^\otimes$.

Lemma 6.1. The functors $\text{pind}_I$, $\text{pres}_I$ and $DL_{D(G;G)}[d_G]$ preserve $\text{PS}(\cdot)$.

Proof. For $\text{pind}_I$ and $\text{pres}_I$ the claim is clear by remark 6.1.1 and lemma 4.9. For $DL_{D(G;G)}[d_G]$ the claim is clear because $DL_{D(G;G)}[d_G](\text{Spr}) = \text{Spr}$ in view of proposition 5.11. □

Recall that when restricted to $W \backslash T^{reg} \cong G^{rs} \subset G \backslash G$, the $D$-module $\text{Spr}$ becomes identified with the pushforward of $\mathcal{O}$ via $T^{reg} \to W \backslash T^{reg}$. By permuting the fibers of this Galois cover, one obtains an action of $W$ on $\text{Spr}$, which induces an isomorphism $\text{End}(\text{Spr}) \cong k[W]$. One therefore obtains an equivalence of abelian categories

$$F : \text{PS}(G) \cong \text{Rep}_k^f(W),$$

given by $\text{Hom}(\text{Spr}, -)$.

Claim 6.2. The following diagrams are commutative:

$$\begin{array}{ccc}
\text{PS}(G) & \xrightarrow{F} & \text{Rep}_k^f(W) \\
\text{pres}_I \downarrow & & \text{pres}_W \downarrow \\
\text{PS}(G_I) & \xrightarrow{F_I} & \text{Rep}_k^f(W_I)
\end{array}$$
Proof. One has $\text{Spr} \cong \text{pind}_I(\text{Spr}_I)$. This identification can be chosen so that the two actions of $W_I$ on $\text{Spr}$, one obtained by restriction from the action of $W$, and the other obtained by functoriality of $\text{pind}_I$ applied to the $W_I$ action on $\text{Spr}_I$, coincide.

Then $\text{pres}_W^W(F(-)) = \text{pres}_W^W \text{Hom}(\text{Spr}, -) \cong \text{Hom}(\text{pind}_I(\text{Spr}_I), -) \cong \text{Hom}(\text{Spr}_I, \text{pres}_I(-)) = F_I(\text{pres}_I(-))$.

The commutativity of the second diagram follows by taking left adjoints in the first diagram. □

6.2. The Deligne-Lusztig operator for the Weyl group.

Definition 6.3. Define the Deligne-Lusztig operator

$$DL_W : K_0(\text{Rep}_k^d(W)) \to K_0(\text{Rep}_k^d(W))$$

as follows:

$$DL_W(V) := \sum_{I \in \Sigma} (-1)^{|I|} \cdot \text{ind}_W^W I \text{res}_W^W V.$$ 

Lemma 6.4. One has, on the level of $K_0$-groups:

$$DL_W \circ F = F \circ DG_{D(G \setminus G)}[d_G].$$

Proof. This is clear by theorem 5.2 and claim 6.2. □

Lemma 6.5. The operator $DL_W$ sends irreducibles to irreducibles.

Proof. This follows from lemma 6.4 and the fact that $DG_{D(G \setminus G)}[d_G]$ sends irreducibles to irreducibles when restricted to $(D(G \setminus G)^{G'})^\circ$, by proposition 5.11. □

Lemma 6.6. Let $I \subset \Sigma$ and denote by

$$DL_{W_I} : K_0(\text{Rep}_k^d(W_I)) \to K_0(\text{Rep}_k^d(W_I))$$

the corresponding operator. Then one has

$$DL_W \circ \text{ind}_W^W I = \text{ind}_{W_I}^W \circ DL_{W_I}.$$ 

Proof. This follows, using lemma 6.4 from proposition 5.1 together with lemma 4.10. □

Lemma 6.7. One has

$$DL_W(\text{triv}) = \text{sgn}.$$ 

Proof. Since $DL_W$ map irreducibles to irreducibles, it is enough to check that

$$\langle \text{sgn}, DL_W(\text{triv}) \rangle = 1.$$ 

Indeed:

$$\langle \text{sgn}, DL_W(\text{triv}) \rangle = \langle \text{sgn}, \sum_{I \in \Sigma} (-1)^{|I|} \cdot \text{ind}_W^W I \text{res}_W^W \text{triv} \rangle =$$
= \sum_{I \subseteq \Sigma} (-1)^{|I|} \langle \text{res}_{W_I}^W \text{sgn}, \text{res}_{W_I}^W \text{triv} \rangle.

Now notice that the inner product inside the sum is 0 if $I \neq \emptyset$, and 1 if $I = \emptyset$. \hfill \square

Lemma 6.8. The elements $\text{pin}^W_{W_I}(\text{triv})$, for $I \subseteq \Sigma$, span $K_0(\text{Rep}^{fd}_k(W))$.

Proof. When $|\Sigma| \leq 1$, the claim is clear. Let us assume thus that $|\Sigma| > 1$ and let us assume by induction that the claim is true for $W_{I'}$ whenever $I' \neq \Sigma$. Then given $V \in K_0(\text{Rep}^{fd}_k(W))$ and $I \neq \Sigma$ we get, for all $J \subseteq I$:

$$\langle \text{res}_{W_I}^W V, \text{ind}_{W_J}^W(\text{triv}) \rangle = \langle V, \text{ind}_{W_J}^W(\text{triv}) \rangle = 0$$

and by the induction hypothesis we get $\text{res}_{W_I}^W V = 0$. Since the subgroups $W_I$, for $I \neq \Sigma$, generate $W$ (because $|\Sigma| > 1$), we obtain $V = 0$. \hfill \square

Proposition 6.9. One has

$$\text{DL}_W = \text{sgn} \otimes \cdot.$$

Proof. Let us assume by induction that the claim is true for $W_{I'}$ whenever $I' \neq \Sigma$. Then for $I \neq \Sigma$ we obtain

$$\text{DL}_W(\text{ind}_{W_{I'}}^W(\text{triv})) = \text{ind}_{W_I}^W(\text{DL}_W(\text{triv})) = \text{ind}_{W_I}^W(\text{sgn} \otimes \cdot \text{ind}_{W_{I'}}^W(\text{triv}))$$

(the third equality is by the projection formula). By lemma 6.8 we are done. \hfill \square

Proof (of proposition 5.11). One has

$$F(\text{DG}_{\Delta}(\mathcal{G}; G)[d_\emptyset](\text{Spr}_\alpha)) = \text{DL}_W(\alpha) = \text{sgn} \otimes \alpha = F(\text{Spr}_{\text{sgn} \otimes \alpha}).$$

so

$$\text{DG}_{\Delta}(\mathcal{G}; G)[d_\emptyset](\text{Spr}_\alpha) = \text{Spr}_{\text{sgn} \otimes \alpha}.$$

\hfill \square

7. Proof of theorem 5.2

In this section we provide a proof for theorem 5.2.

7.1. The Vinberg monoid. Our reference is [DrGa2, Appendix D].

Denote $T_{\text{adj}} := T/Z(G)$. We have an isomorphism $T_{\text{adj}} \rightarrow \mathbb{G}_m^\Sigma$ given by the simple roots. We denote by $\hat{\mathbb{G}}_m$ the affine line with its multiplicative monoid structure, and $\hat{T}_{\text{adj}} := \mathbb{G}_m^\Sigma$.

Let us denote by $V$ the Vinberg monoid of $G$. The group of invertible elements $V^\times \subset V$ is $G \times T$. One has a homomorphic map $\text{deg} : V \rightarrow \hat{T}_{\text{adj}}$ such that $\text{deg}^{-1}(T_{\text{adj}}) = V^\times$. Let us denote by $\hat{V} \subset V$ the non-degenerate locus (it contains $V^\times$). The restriction of $\text{deg}$ to $\hat{V} \subset V$ is smooth.

We consider $V$ as a $(G \times G)$-space on the left, and a $T$-space on the right, by $(g_1, g_2) \ast v \ast t = g_1vg_2^{-1}t$. The right action of $T$ on $\hat{V}$ is free, and the quotient by this action, as a $(G \times G)$-space, is the wonderful compactification of $G_{\text{adj}} := G/Z(G)$.

Given $I \subseteq \Sigma$, let us denote by $(T_{\text{adj}})_I \subset T_{\text{adj}}$ the subspace of elements whose $(\Sigma - I)$-coordinates are 0. Let us denote by $e_I \in (T_{\text{adj}})_I$ the element whose $I$-coordinates are 1. Denote by $T^I \subset T$ the subgroup consisting of elements whose $I$-coordinates are 1.

Recall our fixed choice of a Torel $T_{\text{sub}} \subset B \subset G$, giving rise to an identification $\phi : T \xrightarrow{\sim} T_{\text{sub}}$. Using this choice, we obtain a homomorphic section $s : \hat{T}_{\text{adj}} \rightarrow \hat{V}$ of $\text{deg}$, which sends $t \in T_{\text{adj}}$.
to \((\phi(t)^{-1}, t) \in V^\times\). Given \(I \subset \Sigma\), the action of \(G \times G\) on the fiber \(\tilde{V}_{e_1}\) is transitive, and the stabilizer of \(s(e_1)\) in \(G \times G \times T^{op}\) consists of triples \((p_1, p_2, t)\) for which \(p_1 \in P, p_2 \in P^-, t \in T^I\) and \([p_1] \cdot \phi(t) = [p_2]\). In particular, the stabilizer of \(s(e_1)\) in \(G \times G\) is \(P \times P^-\).

### 7.2. Filtration of the kernel.

Let us denote by \(S \subset G \times G \times \tilde{V}\) the subgroup scheme of the constant group scheme over \(\tilde{V}\), consisting of \((g_1, g_2, m)\) for which \(g_1 mg_2^{-1} = m\) (see [DrGa2, subsection D.4.6], where \(S\) is denoted \(\text{Stab}_{G \times G}\)). Notice that \(G \times G\) acts on \(S\) on the left and \(T\) acts on \(S\) on the right (compatibly with these actions on \(\tilde{V}\)).

We consider the following diagram, with Cartesian squares:

\[
\begin{array}{ccccccccc}
G \times G \setminus G \times G & \xrightarrow{pr_{1,2}} & (G \times G) \setminus S_{e_1} & \xrightarrow{\tau''} & (G \times G) \setminus S_{e_2} & \xrightarrow{\tau} & (G \times G) \setminus T / \hat{G} & \xrightarrow{\pi} & (G \times G) \setminus S_{e_1} & \xrightarrow{\sigma} & (e_1) \\
(G \times G) \setminus S_{e_2} & \xrightarrow{\hat{G}} & (G \times G) \setminus S_{e_1} & \xrightarrow{\hat{G}} & (G \times G) \setminus S_{e_1} & \xrightarrow{\hat{G}} & (G \times G) \setminus S_{e_1} & \xrightarrow{\hat{G}} & (G \times G) \setminus S_{e_1} & \xrightarrow{\hat{G}} & (G \times G) \setminus S_{e_1} \\
\end{array}
\]

Here, \(\pi\) is the projection induced by \(S \xrightarrow{pr_3} \tilde{V} \xrightarrow{deg} \tilde{T}_{\text{adj}}\). Notice that \(\pi\) is smooth because \(pr_3\) is (see [DrGa2, Section D.4.6]) and \(deg\) is (see [DrGa2, Section D.4.5]).

The map \(pr_{1,2}\) is projective - this follows from \(\tilde{V} / T\) being projective.

Let us describe explicitly the \((G \times G)\)-space \(S_{e_1}\). It can be identified with the subspace of \(G \times G \times \left((G \times G)/(P_1 \times G_1)\right)\) consisting of \((g_1, g_2; x_1, x_2)\) for which \(x_1 g_1 x_1^{-1} g_2 x_2 \in P_1 \times G_1\). The identification is obtained by sending \((g_1, g_2; x_1, x_2)\) to \((g_1, g_2, x_1 s(e_1) x_2^{-1})\).

In particular, one sees that \((G \times G) \setminus S_{e_1}\) can be identified with \(G \setminus G\), in such a way that \(pr_{1,2} \circ j \circ \tilde{\tau}\) becomes identified with the diagonal embedding for \(G \setminus G\).

The Cousin complex now allows us to glue the kernel representing \(D_G(D_G(G))\), which is \((pr_{1,2})^* \tilde{\tau}_{\tilde{G}} C_{\tilde{G}}\), from kernels of the form \((pr_{1,2} \circ \tilde{i}_*)^* \tilde{\tau}_{\tilde{G}} C_{\tilde{G}}\). We will thus prove theorem [5.2] if we show:

**Claim 7.1.** The kernel \((pr_{1,2} \circ \tilde{i}_*)^* \tilde{\tau}_{\tilde{G}} C_{\tilde{G}} \in D(G \setminus G \times G)\) corresponds to the functor \(\text{pind}^* \circ \text{pres}_{e_1} [-d_I]\).

### 7.3. Calculation of the filtrants.

**Lemma 7.2.** One has \(\text{pind}^* \tilde{\tau}_{\tilde{G}} C_{\tilde{G}} \cong \text{res}_{e_1} \omega[-d_I]\).

**Proof.** By the contraction principle (see [DrGa3, Proposition 3.2.2]), denoting by \(r : \tilde{T}_{\text{adj}} / T \rightarrow (\tilde{T}_{\text{adj}})_{t/T}\) the map equating all \((\Sigma - I)\)-coordinates to 0, one has

\[
\text{pind}^* \tilde{\tau}_{\tilde{G}} C_{\tilde{G}} \cong r_! \tilde{\tau}_{\tilde{G}} C = (r \circ j \circ \tau)_! C = \sigma_! C = \sigma_! \omega.
\]
Notice that one has $\sigma_1 = \sigma_* [-d_I]$ (for example, this follows from claim 6.1 and the calculation of Pseudo-identity in [Ga1, Section 6.7.3]) and thus the claim follows.

**Lemma 7.3.** One has $\tilde{\tau}_{i|j} \tilde{\eta}_C \cong \tilde{\sigma}_\omega [-d_I]$.

**Proof.** We have
$$\tilde{\tau}_{i|j} \tilde{\eta}_C = \pi_j^* \pi_i^* \tilde{\eta}_C \cong \pi_j^* \sigma_* [-d_I] = \tilde{\sigma}_\omega (-d_I) = \tilde{\sigma}_\omega [-d_I].$$

**Lemma 7.4.** One has $(pr_{1,2} \circ \tilde{\tau}_{i|j}) \tilde{\eta}_C \cong (pr_{1,2} \circ \tilde{\tau} \circ \tilde{\sigma}) \omega [-d_I]$.

**Proof.** Follows from 7.3.

7.4. **Description by correspondence.** From lemma 7.4 we see that the kernel $(pr_{1,2} \circ \tilde{\tau}_{i|j}) \tilde{\eta}_C \subset D(G \setminus G \times G \setminus G)$ corresponds to the shift by $-d_I$ of $(pr_1)_* pr_2^*$ where
$$\begin{align*}
(G \times G) \setminus S_{e_I} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad