SMOOTHNESS OF ISOMETRIC FLOWS ON ORBIT SPACES
AND MOLINO’S CONJECTURE

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Abstract. We prove here that given a proper isometric action $K \times M \to M$ on a complete Riemannian manifold $M$ then each continuous isometric flow on the orbit space $M/K$ is smooth, i.e., it is the projection of an $K$-equivariant smooth flow on the manifold $M$. The first application of our result concerns with the dynamical behaviour of singular Riemannian foliations and Molino’s conjecture, which states that the partition of a Riemannian manifold into the closures of the leaves of a singular Riemannian foliation is still a singular Riemannian foliation. We prove Molino’s conjecture for the main class of foliations considered in his book, namely orbit-like foliations. As another direct application of our result we remark that, if $G$ is a connected closed group of isometries of the leaf space $M/F$, then $G$ acts smoothly on the leaf space, as long as $F$ is a closed orbit-like foliation on a compact manifold $M$.

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1. Introduction

Given a Riemannian manifold $M$ on which a compact Lie group $K$ acts by isometries, the quotient $M/K$ is in general not a manifold. Nevertheless, the canonical projection $\pi : M \to M/K$ gives $M/K$ the structure of a Hausdorff metric space. Moreover, following [14], one can define a “smooth structure” on $M/K$ to be the $\mathbb{R}$-algebra $C^{\infty}(M/K)$ consisting of functions $f : M/K \to \mathbb{R}$ whose pullback $\pi^* f$ is a smooth, $K$-invariant function on $M$. If $M/K$ is a manifold, the smooth structure defined here corresponds to the more familiar notion of smooth structure. A

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map $F : M/K \to M'/K'$ is called smooth if the pull-back of a smooth function $f \in C^\infty(M'/K')$ is a smooth function $F^*f$ on $M/K$.

These concepts can actually be formulated in the wider context of singular Riemannian foliations. A singular foliation $\mathcal{F}$ is called Riemannian if every geodesic perpendicular to one leaf is perpendicular to every leaf it meets. The decomposition of a Riemannian manifold into the orbits of some isometric action is a special example of a singular Riemannian foliation, that is called Riemannian homogeneous foliation. Given a singular Riemannian foliation $(M, \mathcal{F})$ with compact leaves, one can define a quotient $M/\mathcal{F}$ and again endow it with a metric structure and a smooth structure, exactly as for group actions.

In [14, Corollary 2.4] Schwarz proved that given a proper action $K \times M \to M$ each smooth flow on the orbit space $M/K$ is a projection of an $K$-equivariant smooth flow on the manifold $M$, and hence solved the Isotopy Lift Conjecture due to Bredon; see details in [14].

Our main result concerns with smoothness of continuous flow of isometries (i.e., continuous 1-parameter groups of isometries) on orbit spaces.

**Theorem 1.1.** Let $M$ be a complete Riemannian manifold and $\mathcal{F}$ be a singular Riemannian foliation. Assume that the leaves of $\mathcal{F}$ are orbits of a proper action $K \times M \to M$. Let

$$\varphi : M/K \times I \to M/K$$

be a continuous flow of isometries on the orbit space. Then $\varphi$ is smooth, and hence it is the projection of an $K$-equivariant flow on $M$.

Flows of isometries on the leaf spaces of foliations appear naturally in the study of the dynamical behavior of non closed singular Riemannian foliations. Recall that a (locally closed) singular Riemannian foliation $(M, \mathcal{F})$ is locally described by submetries $\pi_\alpha : U_\alpha \to U_\alpha/\mathcal{F}_\alpha$, where $\{U_\alpha\}$ is an open cover of $M$ and $\mathcal{F}_\alpha$ denotes the restriction of $\mathcal{F}$ to $U_\alpha$. If a leaf $L$ is not closed, one might be interested to understand how it intersects a given neighborhood $U_\alpha$, and in particular how the closure $\overline{L}$ of $L$ intersects $U_\alpha$. It turns out that the projection $\pi_\alpha(\overline{L} \cap U_\alpha)$ (that is contained in the local quotient of a stratum) is a submanifold, which is spanned by continuous flows of isometries $\varphi_\alpha$ on $U_\alpha/\mathcal{F}_\alpha$, cf. [10, Thm 5.2]. Therefore, in order to better understand the closure of $\overline{L}$, it would be relevant to understand if these flows admit smooth lifts.

The above discussion already suggests that Theorem 1.1 should be a useful tool in the study of dynamical behavior of singular Riemannian foliations and should help to solve Molino’s conjecture in important cases.

**Conjecture 1.2** (Molino). Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Then the partition $\mathcal{F}$ given by the closures of the leaves of $\mathcal{F}$ is again a singular Riemannian foliation.

Molino himself proved the conjecture for regular Riemannian foliations, i.e., foliations where all the leaves have the same dimension; see [10]. In [1], the first author proved the conjecture for polar foliations, i.e., foliations admitting a totally geodesic submanifold transverse to the regular leaves and which meets every leaf perpendicularly. Recently, the first author and Lytchak remarked in [4] that Molino’s Conjecture holds for infinitesimally polar foliations, i.e., foliations for which each point admits a neighborhood on which the restricted foliation is diffeomorphic to a polar foliation (this is equivalent to saying that $U_\alpha/\mathcal{F}_\alpha$ is an orbifold); see also [3].
In this paper we prove Molino’s conjecture for the class of singular foliations considered in his book, namely orbit-like foliations; see [10] p. 210 for Molino’s description about the state of the art of the known foliations at that time. Recall that a singular Riemannian foliation is called orbit-like foliation, if its restriction to each slice is diffeomorphic to a homogeneous foliation; see Section 3 for definitions, examples and remarks and Appendix A for some properties.

**Theorem 1.3.** Let $\mathcal{F}$ be an orbit-like foliation on a complete Riemannian manifold $M$. Then the closure of the leaves of $\mathcal{F}$ is a singular Riemannian foliation.

Together with the known results, Molino’s Conjecture now holds true for regular foliations, polar and infinitesimally polar foliations, and homogeneous and orbit like foliations. New foliations can be obtained from these ones by taking products, quotients, or by surgery. This leads us to the following interesting questions:

1. Can Molino’s Conjecture be proved for all the foliations generated by the foliations listed above? In this case, one should be more precise about the exact meaning of “generate”.
2. Is there a singular Riemannian foliation on the flat Euclidean space that is not a product of foliations listed above?

Roughly speaking, one of the ideas behind the proof of Theorem 1.1 is to reduce the problem to the study of a weak solution of certain parabolic equation; see Section 2. In order to prove regularity of this solution, we have used desingularizations of foliations (blow-ups). These methods motivate the following questions:

3. Let $\rho : (\hat{M}, \hat{\mathcal{F}}) \to (M, \mathcal{F})$ be the desingularization of the singular Riemannian foliation $\mathcal{F}$ (see the definition in Section 4.1) and let $f$ be a basic function of $(M, \mathcal{F})$ that is smooth on the principal part of $M$. Under what conditions does the smoothness of the pull back function $\rho^*f$ imply the differentiability of $f$ on $M$?
4. Let $\mathcal{E}$ be a transverse operator defined on the principal part of $M$ and let $f$ be a smooth basic function on $M$. Under what conditions is $\mathcal{E}(f)$ a well defined smooth basic function on $M$?

As we will see in Remark 4.7, each closed group $G$ of isometries of $M/\mathcal{F}$ can be lifted to a closed Lie pseudogroup of isometries of an orbifold if $M$ is compact and $\mathcal{F}$ is closed and hence, according to Salem [10] Appendix D, $G$ admits a Lie pseudogroup structure. Therefore Theorem 1.1 and [13] p.64 imply the next corollary.

**Corollary 1.4.** Let $\mathcal{F}$ be a closed orbit-like foliation on a compact Riemannian manifold $M$ and $G$ be a connected closed group of isometries of $M/\mathcal{F}$. Then $G$ acts smoothly on $M/\mathcal{F}$.

The above corollary leads us to our last question.

5. Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold $M$ and $G$ be a closed pseudogroup of isometries of $M/\mathcal{F}$. Under which conditions is the action of $G$ on $M/\mathcal{F}$ smooth?

This paper is organized as follows. In Section 2 we briefly sketch the proofs of Theorems 1.1 and 1.3. In Section 3 we review, based on [5], some basic concepts such as singular Riemannian foliations, infinitesimal foliations, orbit-like foliations, flows of isometries on leaf spaces. In Section 4 we introduce the main new tools.
used in the proof of the theorems. In the first part, after reviewing the concept of desingularization of a foliation, we introduce blow-up functions and we prove some regularity results. In the second part, we introduce the concept of reduction of a foliation. In Section 5 we prove Theorem 1.1. Finally in Section 6 we prove Theorem 1.3 using Theorem 1.1 and the results presented in Section 4.

In Appendix A we review the concept of linearized vector fields and discuss some of theirs properties. In particular, we show that a locally closed foliation is linearizable if and only if it is orbit-like.

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2. Sketch of proof of the main results

For the sake of motivation, in this section we provide some ideas of the proofs of the main results. Technical aspects of the proofs will be discussed in later sections of the paper.

2.1. Sketch of proof of Theorem 1.1 in a particular case. In what follows we briefly give an idea of proof of Theorem 1.1 in a particular but important case.

On the Euclidean space \( \mathbb{R}^n = X \times Y \) where \( X = \mathbb{R}^{n-1} \) and \( Y = \mathbb{R} \), let \( F \) be the foliation whose leaves are the orbits of a subgroup \( K \) of \( \text{Iso}(X) \) (the group of isometries of \( X \)) acting linearly on \( X \), and trivially on \( Y \).

Assume that there exists a \( C^1 \)-flow \( \varphi \) on \( \mathbb{R}^n / F \) and that \( Y^* := (\{0\} \times Y) / F \equiv Y \) is an integral curve of this flow. We want to prove that the flow \( \varphi \) is smooth.

We will divide the proof of the smoothness of \( \varphi \) into two steps. In the first step, we fix the fiber \( X_0 := X \times \{0\} \) and prove the smoothness on \( X_0 \times I \). In other words, for each basic function \( h \), i.e., a \( K \)-invariant function, we prove that the map \( (x, 0, t) \mapsto \varphi^* h(x, 0, t) \) is smooth. In the second step of the proof, we extend the smoothness of \( \varphi \) to the hole \( \mathbb{R}^n / K \times I \). In other words, we prove that the map \( (x, y, t) \mapsto \varphi^* h(x, y, t) \) is smooth for each basic function \( h \).

We start the proof fixing \( q = (0, 0) \). Then there exists a curve \( \gamma(t) \subset \{0\} \times Y \) that projects to \( Y^* \). For each fixed \( t \), the flow induces an an isometry

\[
\phi(t) : X_0 / F := (X \times \{0\}) / F \to X_t / F := (X \times \{\gamma(t)\}) / F
\]

defined as \( \phi(t)(x^*) := \varphi(x^*, t) \).

Let \( h : X \times Y \to \mathbb{R} \) be a smooth basic function, i.e., a \( K \)-invariant function on \( \mathbb{R}^n \).

Firstly, recall that the mean curvature vector field \( \vec{H} \) of the principal \( K \)-orbits is tangent to every \( X_t \), it projects to a well defined vector field on \( \mathbb{R}^n / K \) and its projection is preserved by every isometry \( \phi(t) \); see [5].

Secondly, note that isometry \( \phi(t) \) also preserves the Laplacian operator on the orbit space of the principal strata.

Finally, recall that the Laplacian \( \Delta \) of the principal stratum \( (X_0)_{\text{princ.}} \) can be obtained from the Laplacian \( \underline{\Delta} = \Delta((X_0)_{\text{princ.}}) \) on its orbit space \( (X_0 / K)_{\text{princ.}} \) by

\[
\Delta h = \underline{\Delta} h - \langle \nabla h, \vec{H} \rangle.
\]

Since \( \phi(t) \) preserves both summands in the right-hand side of the above equation, we infer that \( \Delta \phi(t)^* h = \phi(t)^* \Delta h \) holds in the principal stratum.
Therefore, as remarked in [5], the following equation holds in a weak sense
\[ \triangle \phi(t)^* h = \phi(t)^* \Delta h, \]
where \( \Delta h \) denotes the Laplacian operator on \( X \).

Set \( u_h(t) := \phi(t)^* h \) and \( f(t) := \phi(t)^* \Delta h + \frac{d}{dt} u_h(t) \). From the equation above the following equation holds in a weak sense
\[ \frac{d}{dt} u_h + \Delta u_h = f. \]

This equation motivates us to consider the results of regularity of parabolic equations to prove that \( u_h : X_0 \times I \rightarrow \mathbb{R} \) is smooth. The regularity theory of parabolic equations requires some compatibility conditions that can be checked using the desingularization of \( F \) and some properties of a class of functions that we have called “blow-up functions”. Once these conditions have been checked, we can apply the regularity theory and a bootstrap type argument to conclude that \( u_h \) is smooth on \( X_0 \times I \), i.e., the map \( (x,0,t) \mapsto \phi^* h(x,0,t) \) is smooth.

Using the inverse function theorem on orbit spaces [14] and the fact that \( \varphi \) is a flow, we then prove that the map \( (x,y,t) \mapsto \phi^* h(x,y,t) \) is smooth. In other words, we prove that \( \varphi \) is a smooth flow and this concludes the proof of the theorem in this particular case.

Roughly speaking the proof of the theorem in the general case will consist of finding a way to reduce the problem to the particular case discussed above. We will also have to prove the compatibility conditions, other assumptions and claims that were made here without any explanation.

2.2. Sketch of proof of Theorem 1.3 We first need some definitions and notations. Let

- \( \Sigma \) be a minimal stratum,
- \( D \subset \Sigma \) be a slice at \( q \in \Sigma \) to the foliation \( F|_\Sigma \),
- \( N := \exp(\nu \Sigma|_D) \cap B_\epsilon(D) \) where \( \nu \Sigma|_D \) denotes the restriction of the normal bundle of \( \Sigma \) to the slice \( D \),
- \( F_0^N := F \cap N \) be the foliation induced on \( N \).

The difficult part of the proof of Molino’s conjecture is to prove that the closure \( \bar{F} \) of \( F \) is a singular foliation; see Definition 3.1 below. More precisely, our goal is to prove that for each given vector \( v \in T_q D \) tangent to \( L_q \subset \Sigma \), there exists a smooth vector field \( \vec{Y} \) tangent to the leaves of \( \bar{F} \) so that \( \vec{Y}(q) = v \).

Using singular holonomy and desingularization we prove the following fact: there exists a continuous flow of isometries \( \varphi \) on \((N, \tilde{g})/F_0^N \) such that \( \pi_{F_0^N}^* v \) is tangent to the integral curve of \( \varphi \). Here \( \pi_{F_0^N} \) is the canonical projection and \( \tilde{g} \) is a metric on \( N \) so that \((N, F_0^N)\) is a singular Riemannian foliation with the same transverse metric of \( F \), i.e., the distance between leaves of \( F_0^N \) is the same as the distance between the plaques of \( F \) that contain such leaves.

Since \( F_0^N \) is homogeneous, we can apply Theorem 11 and conclude that \( \varphi \) is smooth. The existence of \( \vec{Y} \) follows from Schwarz’s Theorem [14] that allows us to lift smooth flows on the orbit space to smooth flows on the manifold.
3. Preliminaries

3.1. Singular Riemannian foliations. Let us recall the definition of a singular Riemannian foliation.

Definition 3.1 (SRF). A partition $\mathcal{F}$ of a complete Riemannian manifold $M$ by connected immersed submanifolds (the leaves) is called a singular Riemannian foliation (SRF for short) if it satisfies condition (a) and (b):

(a) $\mathcal{F}$ is a singular foliation, i.e., for each leaf $L$ and each $v \in TL$ with footpoint $p$, there is a smooth vector field $\vec{Y}$ with $\vec{Y}(p) = v$ that belongs to $\mathfrak{X}(\mathcal{F})$, i.e., that is tangent at each point to the corresponding leaf.

(b) $\mathcal{F}$ is a transnormal system, i.e., every geodesic perpendicular to one leaf is perpendicular to every leaf it meets.

A leaf $L$ of $\mathcal{F}$ (and each point in $L$) is called regular if the dimension of $L$ is maximal, otherwise $L$ is called singular. In addition a regular leaf is called principal if it has trivial holonomy; for a definition of holonomy, see e.g. [10, page 22].

A typical example of a singular Riemannian foliation is the partition of a Riemannian manifold into the connected components of the orbits of an isometric action. Such singular Riemannian foliations are called Riemannian homogeneous. In this case the principal leaves coincide with the principal orbits. We will sometimes denote a Riemannian homogeneous foliation, given by the action of a Lie group $K$, by $(M,K)$, provided the $K$-action is understood.

If a singular Riemannian foliation $(M,\mathcal{F})$ is spanned by a smooth action of a Lie group, which does not necessarily act by isometries, then we call such a foliation homogeneous.

3.2. The infinitesimal foliation at a point. Let $(M,\mathcal{F})$ be a singular Riemannian foliation. Given a point $p \in M$ and some small $\epsilon > 0$, let $S_p = \exp_p(\nu_p L_p) \cap B_\epsilon(p)$ be a slice at $p$, where $B_\epsilon(p)$ is the distance ball of radius $\epsilon$ around $p$. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}|_{S_p}$ on $S_p$ by letting the leaves of $\mathcal{F}|_{S_p}$ be the connected components of the intersection between $S_p$ and the leaves of $\mathcal{F}$. In general the foliation $(S_p,\mathcal{F}|_{S_p})$ is not a singular Riemannian foliation with respect to the induced metric on $S_p$. Nevertheless, the pull-back foliation $\exp^*_p(\mathcal{F})$ is a singular Riemannian foliation on $\nu_p L_p \cap B_\epsilon(0)$ equipped with the Euclidean metric (cf. [10 Proposition 6.5]), and it is invariant under homotheties fixing the origin (cf. [10 Lemma 6.2]). In particular, it is possible to extend $\exp^*_p(\mathcal{F})$ to all of $\nu_p L_p$, giving rise to a singular Riemannian foliation $(\nu_p L_p, \mathcal{F}_p)$ called the infinitesimal foliation of $\mathcal{F}$ at $p$.

If $(M,K)$ is Riemannian homogeneous, the infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$ is again Riemannian homogeneous, given by the action of (the identity component of) the isotropy group $K^0_p$ on $\nu_p L_p$ (the slice representation).

The converse however is not true: namely, there are examples of non-Riemannian homogeneous foliation all of whose infinitesimal foliations are.

Definition 3.2. A SRF $\mathcal{F}$ on $M$ is called an orbit-like foliation if for each point $q$ there exists a compact group $K_q$ of isometries of $\nu_q L_q$ such that the infinitesimal foliation $\mathcal{F}_q$ is the partition of $\nu_q L_q$ into the orbits of the action of $K_q$. 

Examples of orbit-like foliations are given by the closures of (regular) Riemannian foliations. Other examples can be obtained via a procedure called suspension of homomorphism; for more details see e.g. [10, sec. 3.7].

**Example 3.3.** Consider a nonhomogeneous manifold $Q$ and let $(V = V_1 \times V_2, \tilde{F})$ be a homogeneous Riemannian foliation given by the orbits of an isometric action of a closed subgroup $K \subseteq \text{Iso}(V_1)$. Take a homomorphism $\rho : \pi_1(Q, q_0) \rightarrow H \subset \text{Iso}(V_2)$. By construction, $\rho$ induces an action $\rho' = \rho \times \text{id} : \pi_1(Q, q_0) \rightarrow \text{Iso}(V_1 \times V_2)$ that preserves the foliation $\tilde{F}$. Then $\pi_1(Q, q_0)$ acts diagonally on the product $\tilde{M} = \tilde{Q} \times V$ where $\tilde{Q}$ is the universal cover of $Q$ and the action on the first factor is the action of deck transformations. Since the action is free, the quotient $M := \tilde{M}/\pi_1(Q, q_0)$ is a manifold, and there is a map

$$P : M \rightarrow Q$$

$$[\tilde{q}, v] \mapsto [\tilde{q}]$$

that makes $M$ the total space of a fiber bundle with base $Q$. In addition, the fiber is $V$ and the structural group is given by the image of $\rho'$. Finally, the SRF $(\tilde{M}, \tilde{Q} \times \tilde{F})$ induces via $\tilde{M} \rightarrow M$ a SRF $(M, F)$, which can be proved to be an orbit-like foliation.

### 3.3. Smooth structure of a leaf space

Let $(M, F)$ be a SRF with closed leaves. The quotient $M/F$ is equipped with the natural quotient metric and a natural quotient “$C^k$ structure”. The $C^k$ structure on $M/F$ is given by the sheaf $C^k_b(M, F)$ of $C^k$ basic functions on $M$, i.e., those functions that are constant along the leaves of $F$. A function $f \in C^k_b(M, F)$ is called of class $C^k$, while $f \in C^\infty(M, F)$ is called smooth. One says that a map $\varphi : M_1/F_1 \rightarrow M_2/F_2$ between two leaf spaces of SRF is of class $C^k$ if the pull-back of a smooth function $f \in C^\infty(M_2/F_2)$ is a function $\varphi^* f \in C^k(M_1/F_1)$. When $\varphi$ is smooth, this definition coincides with the definition of Schwarz [14].

### 3.4. Flow in a leaf space

Consider a SRF $F$ on a complete Riemannian manifold $M$ with closed leaves.

**Definition 3.4.** A continuous map $\varphi : M/F \times I \rightarrow M/F$ is called a flow on $M/F$ if the following conditions hold:

(a) $\varphi$ is a one-parameter local group;
(b) for each $p \in M/F$ each integral curve $t \rightarrow \varphi(p, t)$ is contained in the quotient of a stratum;
(c) there exists a locally bounded derivative $\tilde{Y}$ on $M/F$ associated to the flow, such that for each smooth basic function $h$ we have $\tilde{Y} \cdot h(p) = \frac{d}{dt}h(\varphi(p, t))|_{t=0}$.

**Remark 3.5.** These conditions are independent and do not imply each other. However, in the particular case where $\varphi$ is a one-parameter local group of isometries of $M/F$, one can prove that (b) holds (see e.g., [5, section 5.1]) and that (a) and (b) imply (c); see e.g. Remark [15].
3.5. SRF’s with disconnected leaves. Sometimes one has to consider Riemannian foliations with non-connected leaves. This kind of foliations come up naturally: consider for example a Riemannian homogeneous foliation \((N, K)\). Even when \(K\) itself is connected, some isotropy subgroup \(K_p\) might not be, and its orbits under the slice representation might also be disconnected. Therefore the Riemannian homogeneous foliation \((\nu_p(K\cdot p), K_p)\) would be an example of a disconnected singular Riemannian foliation. In general, a singular Riemannian foliation with disconnected leaves \((N, F_N)\) is a triple \((N, \tilde{F}_N, H)\) where \(H\) is a group of isometries of \(N/\tilde{F}_N\), and the non-connected leaves of \(F_N\) are just the orbits \(H\cdot L_p\), for \(L_p \in \tilde{F}_N\).

A slight generalization of such a triple, which we still call singular Riemannian foliation with disconnected leaves and still denote \((N, F_N)\), is a triple \((N, \tilde{F}_N, H)\) where \(H\) is a pseudogroup of local isometries of \(N/\tilde{F}_N\). Again, the non-connected leaves of \(F_N\) are \(H\)-orbits of leaves of \(\tilde{F}_N\). Such a foliation appears naturally when dealing with the singular holonomy around a non-closed leaf (cf. Definition 4.16).

4. New tools

In the following sections we define the main new technical tools used in this paper. These are of independent interest and we hope they might be used in other contexts, from which the decision of devoting a section to them is made.

The first tool is the blow-up of a singular Riemannian foliation. This object has already been studied in \([2]\), but here we will further the analysis and in particular we will define blow-up functions. The second tool is the concept of local reduction.

4.1. Blow-up. Let \(M\) be a complete manifold, and \((M, F)\) a SRF with closed leaves. As in the classical theory of isometric actions, it is possible to construct, via compositions of projective blow-up’s, a surjective map \(\rho : M_\epsilon \to M\) with the following properties:

- \(M_\epsilon\) is a smooth complete Riemannian manifold foliated by a regular Riemannian foliation \(F_\epsilon\).
- The map \(\rho_\epsilon\) sends leaves of \(F_\epsilon\) to leaves of \(F\).

This map is called a desingularization map. If \(M/F\) is compact, then for each small \(\epsilon > 0\) one can choose \(M_\epsilon\) and \(\rho_\epsilon\) so that \(d_{GH}(M_\epsilon/F_\epsilon, M/F) < \epsilon\) where \(d_{GH}\) is the Gromov-Hausdorff distance; see \([2]\).

In this section we briefly recall the construction of the first blow-up along the minimal stratum \(\Sigma\) (see \([11, 2, 3]\)) and present Proposition 4.4, Proposition 4.9 and Proposition 4.10 that will be used in the proof of the main theorems.

Following a procedure analogous to the blow-up of isometric actions one has the next lemma.

Lemma 4.1. Let \(B := \text{Tub}_r(\Sigma)\) be a small neighborhood of the minimal stratum \(\Sigma\). Then

(a) \(\hat{B} := \{(x, [\xi]) \in B \times \mathbb{P}(\nu\Sigma) | x = \exp^\perp(t\xi)\text{ for } |t| < r\}\) is a smooth manifold (called blow-up of \(B\) along \(\Sigma\)) and the projection or blow-up map \(\hat{\rho} : \hat{B} \to B\), defined as \(\hat{\rho}(x,[\xi]) = x\) is also smooth.

(b) \(\hat{\Sigma} := \hat{\rho}^{-1}(\Sigma) = \{([\pi([\xi])], [\xi]) \in \hat{B} = \mathbb{P}(\nu\Sigma), \text{ where } \pi : \mathbb{P}(\nu\Sigma) \to \Sigma \text{ is the canonical projection.}\)
(c) There exists a singular foliation \( \tilde{F} \) on \( \tilde{B} \) so that \( \tilde{\rho} : (\tilde{B} - \tilde{\Sigma}, \tilde{F}) \to (B - \Sigma, F) \) is a foliated diffeomorphism. In addition if \( F \) is homogeneous then the leaves of \( \tilde{F} \) are also homogeneous.

Getting the right metric on \( \tilde{B} \) is a bit more complicated.

**Lemma 4.2** (\([2]\)). There exists a metric \( \check{g} \) on \( \tilde{B} \) such that \( \tilde{F} \) is a SRF.

Let us briefly recall the construction of this metric, that will be important in the proof of the results of this section.

Consider the smooth distribution \( S \) on \( B \) defined as \( S_{\exp(q)} := T_{\exp(q)}S_q \) where \( \xi \in \nu_q \Sigma \) and \( S_q \) is a slice of \( L_q \) at \( q \) with respect to the original metric \( g \).

First we find a metric \( \check{g} \) with the following properties:

(a) The distance between the leaves of \( F \) on \( B \) with respect to \( \check{g} \) and to respect to \( g \) are the same.

(b) The normal space of each plaque of \( F|_B \) (with respect to \( \check{g} \)) is contained in \( S \). In fact those spaces are the orthogonal projection (with respect to \( g \)) of the normal spaces (with respect to \( g \)) of \( F|_B \).

(c) If a curve \( \gamma \) is a unit speed geodesic segment orthogonal to \( \Sigma \) with respect to the original metric \( g \), then \( \gamma \) is a unit speed geodesic segment orthogonal to \( \Sigma \) with respect to the new metric \( \check{g} \).

We now come to the second step of our construction, in which we change the metric \( \check{g} \) in some directions, getting a new metric \( \hat{g}^B \) on \( B - \Sigma \).

First note that, for small \( \xi \in \nu_q \Sigma \), we can decompose \( T_{\exp(q)}M \) as a direct sum of orthogonal subspaces (with respect to the metric \( \check{g} \))

\[
T_{\exp(q)}M = S^1_{\exp(q)} \oplus S^2_{\exp(q)} \oplus S^3_{\exp(q)},
\]

where \( S^i_{\exp(q)} \) is orthogonal to \( S^i_{\exp(q)} \) and \( S^i_{\exp(q)} \subset S^i_{\exp(q)} \), \( i=1,2,3 \), are defined below:

- \( S^1_{\exp(q)} \) is the orthogonal complement of \( S^1_{\exp(q)} \) and \( S^i_{\exp(q)} \subset S^i_{\exp(q)} \), \( i=1,2,3 \), as defined below:

Now we define a new metric \( \hat{g}^B \) on \( B - \Sigma \) as follows:

\[
\hat{g}^B_{\exp(q)}(Z,W) := \check{g}(Z_{\perp},W_{\perp}) + \frac{r^2}{||\xi||^2} \check{g}(Z_1,W_1) + \check{g}(Z_2,W_2) + \check{g}(Z_3,W_3),
\]

where \( Z_i, W_i \in S^i_{\exp(q)} \) and \( Z_{\perp}, W_{\perp} \in S^i_{\exp(q)} \).

Finally we define the pullback metric \( \check{g} := \hat{\rho}^* \hat{g}^B \).

We have recalled the construction of the blow-up an \( F \)-invariant neighborhood \( B \) along \( \Sigma \). We have explained the case where \( B = \text{Tub}_r(\Sigma) \) because we will only be concerned with this kind of neighborhood \( B \) and with this first blow-up \( \hat{\rho} \).

**Remark 4.3.** For the sake of completeness let us explain the rest of the construction, e.g. when \( M \) is compact. We simply glue \( \tilde{B} \) with a copy of \( M - B \) and construct the space \( \tilde{M}_r(\Sigma) \) and the projection \( \rho_r : \tilde{M}_r(\Sigma) \to M \). A natural singular foliation \( \tilde{F}_r \) is induced on \( \tilde{M}_r(\Sigma) \) in analogy to the blow-up of isometric actions. To define the appropriate metric \( \check{g}_r \) on \( \tilde{M}_r(\Sigma) \) consider a partition of unity of \( \tilde{M}_r(\Sigma) \) by two functions \( \tilde{f} \) and \( \tilde{h} \) such that
Proposition 4.4. Each flow of isometries $\varphi : B/F \times I \to B/F$ can be lifted to a flow of isometries $\hat{\varphi} : \hat{B}/\hat{F} \times I \to \hat{B}/\hat{F}$.

Proof. Since $\varphi_t$ maps geodesics orthogonal to the minimal stratum to geodesics orthogonal to the minimal stratum, the lift $\hat{\varphi}_t$ is well defined and continuous.

Let $x$ be a principal point and $H$ be the transverse space of the leaf $L_x$. Then $H$ is decomposed into a direct sum of subspaces $H_1 \oplus H_2 \oplus H_3$, where $H_1 = S_1 \cap H$; for the definition of $S_1$ recall equation (4.1). Let $\hat{H}$ be the transversal space of $\hat{L}_{\hat{x}}$ where $\hat{\rho}(\hat{x}) = x$. Then $\hat{H}$ also decomposes into a direct sum $\hat{H}_j$ and $d\hat{\varphi} : (\hat{H}_j, \hat{g}_J) \to (H_j, g_j)$ is an isometry where $\hat{g}_J$ is the transverse metric of $\hat{F}$ and $g_j$ is the restriction of transverse metric $g_T$ of $F$ to $H_j$, if $j \neq 1$ and $g_1 = \frac{x^2}{\|x\|^2} g_T$.

Note that $\varphi_t$ (respectively $\hat{\varphi}_t$) preserves the decomposition $H_i$ (respectively $\hat{H}_i$). Since the function $\frac{x^2}{\|x\|^2}$ is invariant under the action of $\varphi_t$ we infer that $\hat{\varphi}_t$ is a local isometry on $(\hat{\rho})^{-1}(B_0) / \hat{F}$, where $B_0$ is the union of principal leaves of $B$. Using the density of principal points in the quotient space $\hat{B}/\hat{F}$ and the fact that a minimal geodesic segment joining principal points does not contain singular points, we conclude that the each $\hat{\varphi}_t$ is a global isometry on $\hat{B}/\hat{F}$.

Remark 4.5. Using blow-ups, one can also check that the derivative $\hat{Y}$ of the flow $\varphi$ is locally bounded (cf. Definition 4.4). In fact, by successive blow-ups one can lift a continuous one parameter local group $\varphi$ on $B/F$ to an isometric flow on an orbifold (see Remark 4.7 below) where they are locally bounded by more classical results, see e.g. [10, Salem appendix D]. Since the blow-up’s are distance non-increasing maps between the leaf spaces (see [2, Remark 3.8]) the result follows.

Although along the paper we will consider foliations $\mathcal{F}$ whose leaves are homogeneous but not necessarily Riemannian homogeneous, we present the next result for the sake of completeness.

Proposition 4.6. Let $(B, G)$ be a Riemannian homogeneous SRF. Then $G$ acts on $\hat{B}$, and there exists a new metric $\hat{g}^G$ such that $G$ acts by isometries and $(\hat{B}, \hat{F})$ is the Riemannian homogeneous foliation induced by $G$. The transverse metric of $\hat{g}^G$ coincides with the transverse metric of $\hat{g}$.

According to this proposition, in particular, a flow of isometries $\varphi$ on the orbit space $B/F$ can be lifted to a flow of isometries $\hat{\varphi}$ on the orbit space $\hat{B}/\hat{F}$ with respect to the new metric $\hat{g}^G$.

Proof. We first claim that the action of $G$ on each stratum preserves the normal bundle (with respect to $\hat{g}$) of each orbit in this stratum. In addition $G$ acts isometrically on the fibers of this bundle.

The above claim is a direct consequence of the following facts:

(1) The distribution $S$ is invariant under the action of $G$. 

(2) $\hat{f}$ and $\hat{h}$ are constant on the cylinders $\partial\text{Tub}_\delta(\hat{S})$ for $\delta < 2r$. 

Set $\hat{g}_r := \hat{f}\hat{g} + \hat{h}g$ The desingularization $\rho_r$ mentioned in the beginning of this section is then the composition of the blow-up’s along the strata.
The normal bundle of the orbits (with respect to the original metric $g$) is invariant under the action of $G$.

(3) The orthogonal projection (with respect to the original metric $g$) is also invariant under the action of $G$.

Now, since the action preserves the decomposition $H_1$, $H_2$ and $H_3$ the claim is also valid for the metric $\hat{g}^B$ and hence to the blow-up metric $\hat{g} = \hat{\rho}^* \hat{g}^B$. Finally one can define the new metric as

$$\hat{g}^G(X,Y) := \int_G \hat{g}(d\hat{g}X, d\hat{g}Y) \omega$$

where $\omega$ is a right-invariant volume form of the compact group $G$. The rest of the proof follows from Proposition 4.4.

$\square$

Remark 4.7. When $M$ is compact, we can take into account Remark 4.3 and generalize the above results as follows:

(a) Each flow of isometries $\varphi : M/F \times I \to M/F$ can be lifted to a (smooth) flow of isometries $\varphi_\epsilon : M_e/F_e \times I \to M_e/F_e$.

(b) If $F$ is the partition of $M$ into orbits of a compact group $G$ of isometries of $M$, then there exists a metric $g_e^G$ on $M_e$ so that $F_e$ turns out to be the partition of $M$ into orbits of an isometric action of $G$ on $M_e$. In addition the transverse metric associated to this new metric $g_e^G$ coincides with the transverse metric of the original metric $g^G$ of $M$.

4.2. Blow-up functions. We now introduce a class of basic functions on $B$ that will be used in Lemma 5.4 to check some regularity conditions necessary to prove the smoothness of solutions of a (weak) parabolic equation. Consider the blow-up $\hat{\rho} : (\hat{B}, \hat{F}) \to (B, F)$ of $B$ along its minimal stratum $\Sigma$.

Definition 4.8. We say that a continuous $F$-basic function $h$ on $B$ belongs to $\mathcal{B}$ or is a blow-up function, if

(a) $h \circ \hat{\rho}$ is a smooth $\hat{F}$-basic function on $\hat{B}$.

(b) The restriction of $h$ to $\Sigma$ and to $B - \Sigma$ is smooth.

(c) $X \cdot h = 0$ for each $X \in \nu \Sigma$.

In what follows we prove two important properties of these functions.

Proposition 4.9. If $h \in \mathcal{B}$ then $h$ is a $C^1$ function.

Proof. Let $q \in \Sigma$. We claim that it suffices to show that

$$v_n \cdot h(p_n) \to 0$$

for every sequence $(p_n, v_n) \to (q, v_0)$ such that $(p_n, v_n) \in TB$, and the vectors $v_n$ are tangent to the distance spheres in the normal bundle of $\Sigma$.

Consider first a simple example, namely when $B = \mathbb{R}^2$ is foliated by concentric circles around the origin and $\Sigma = \{q\} = \{(0, 0)\}$. Note that $\frac{\partial h}{\partial \alpha}|_{p_n} = a_1 \frac{\partial h}{\partial \alpha}|_{p_n} + a_2 v_n$ and $\frac{\partial h}{\partial \beta}|_{p_n} = b_1 \frac{\partial h}{\partial \beta}|_{p_n} + b_2 v_n$ where $|a_i|$ and $|b_i|$ are bounded. Equation (4.3) and item (c) of Definition 4.8 imply that $\nabla h(p_n) \to 0$, which is what one needs to ensure that $h$ is $C^1$.

In the general case, consider a converging sequence $(p_n, v_n) \to (q, v)$. The tangent spaces $T_{p_n}B$ split as $\ker(p_e) \oplus \ker(p_e)^\perp$, where $p : B \to \Sigma$ is the metric
projection. Every $v_n$ then splits as $v_n = v_n^{\ker p^*} + v_n^{\ker p^*_0}$, and in order to prove the proposition we need to prove

\[(4.4) \quad v_n^{\ker p^*} \cdot h(p_n) \xrightarrow{n \to \infty} v^{\ker p^*} \cdot h(q)\]

\[(4.5) \quad v_n^{\ker p^*_0} \cdot h(p_n) \xrightarrow{n \to \infty} v^{\ker p^*_0} \cdot h(q)\]

By \cite{2, proof of Lemma 3.5}, \eqref{4.5} is satisfied. Moreover, since $h$ is a blow-up function, by condition (c) we have $v^{\ker p^*} \cdot h(q) = 0$, and it is enough to check that $v^{\ker p^*_0} \cdot h(p_n) \to 0$. Since moreover the derivatives in the radial direction go to zero, this reduces to check \eqref{4.3}, as claimed.

Set $g := h \circ \hat{\rho}$ and let $\{\hat{p}_n\} \subset \hat{B}$ be the lift of $\{p_n\}$. We can assume without loss of generality that $\{\hat{p}_n\}$ is contained in a relatively compact neighborhood in $\hat{B}$ that admits cylindrical coordinates $(r, \theta, z_i)$.

For each $p_n$, define a frame $\{\vec{e}_j\}$ of $(\ker p^*_0)_{p_n} \cap \partial \hat{B}$. Also, let $\tilde{\theta} = (\theta_1, \ldots, \theta_{\text{codim}(\Sigma, B) - 1})$ and $\tilde{z} = (z_1, \ldots, z_{\text{dim} \Sigma})$.

By definition of $\vec{e}_j$, in order to prove equation \eqref{4.3} it suffices to show

\[(4.6) \quad \lim_{n \to \infty} \vec{e}_j(p_n) \cdot h(p_n) = \lim_{n \to \infty} \frac{1}{r(n)} \frac{\partial g}{\partial \theta_i}(r(n), \theta(n), z(n)) = 0\]

for the sequence $\hat{p}_n = (r(n), \tilde{\theta}(n), \tilde{z}(n))$ that converges to the fiber $(0, \tilde{\theta}, \tilde{z})$.

From the definition of $g$ we have

\[(4.7) \quad \frac{\partial g}{\partial \theta_i}(0, \tilde{\theta}, \tilde{z}) = 0\]

Condition (c) implies

\[(4.8) \quad \frac{\partial g}{\partial r}(0, \tilde{\theta}, \tilde{z}) = 0\]

and hence, since $g$ is smooth (see condition (a)), we conclude that

\[(4.9) \quad \frac{\partial^2 g}{\partial r \partial \theta_i}(0, \tilde{\theta}, \tilde{z}) = \frac{\partial^2 g}{\partial \theta_i \partial r}(0, \tilde{\theta}, \tilde{z}) = 0\]

From mean value theorem we also have

\[(4.10) \quad \left| \frac{1}{r(n)} \frac{\partial g}{\partial \theta_i}(r(n), \tilde{\theta}(n), \tilde{z}(n)) - \frac{1}{r(n)} \frac{\partial g}{\partial \theta_i}(0, \tilde{\theta}(n), \tilde{z}(n)) \right| \leq \left| \frac{\partial^2 g}{\partial r \partial \theta_i}(\tilde{r}(n), \tilde{\theta}(n), \tilde{z}(n)) \right|\]

Now Eq. \eqref{4.6} follows direct from equations \eqref{4.7}, \eqref{4.9} and \eqref{4.10}.

Let $\varphi$ be a flow of isometries on $B/F$ and consider the flow of isometries $\hat{\varphi}$ on $\hat{B}/\hat{F}$ defined in Proposition \ref{4.4}. Let $\hat{Y}$ be the associated derivative in the quotient $B/F$; recall Definition \ref{4.4}.

**Proposition 4.10.** Assume that $\hat{\varphi}$ is smooth. Then for each $h \in B$ we have $\varphi^*_a(\hat{Y} \cdot h) \in B$. 
Proof. We must check that the function $(\bar{Y} \cdot h) \circ \varphi_s$ satisfies the conditions of the Definition 4.8.

Condition (b) of Definition 4.8 follows from hypothesis.

Now we want to check condition (a) of Definition 4.8. Note that

$$\hat{\rho}(\varphi_s + z) = \hat{\rho}(\hat{\varphi}_{s+z}).$$

Finally we have to check condition (c) of Definition 4.8. Let $\gamma$ be a geodesic orthogonal to the minimal stratum $\Sigma$ and $\hat{\gamma} \subset \hat{B}$ a lift of $\gamma$. Consider the smooth function $g(z,t) := h(\hat{\rho}(\hat{\varphi}_{s+z}(\hat{\gamma}(t))).$ Note that $\hat{\varphi}_{s+z} \circ \hat{\gamma}$ is a horizontal geodesic orthogonal to the lift of $\Sigma$ and hence that $\hat{\rho} \circ \hat{\varphi}_{s+z} \circ \hat{\gamma}$ is orthogonal to $\Sigma$. This fact and the fact that $h \in B$ (in particular satisfies condition (c) of Definition 4.8) imply that $\frac{\partial}{\partial z} g(z,0) = 0$. We conclude that

$$\frac{d}{dt} (\bar{Y} \cdot h) \circ \varphi_s \circ \gamma(t))_{t=0} = \frac{\partial^2}{\partial t \partial z} h(\varphi_s + z(\gamma(t)))_{z,t=0}$$

$$= \frac{\partial^2}{\partial t \partial z} h(\varphi(s+z(\hat{\gamma}(t))))_{z,t=0}$$

$$= \frac{\partial^2}{\partial z \partial t} g(z,t)_{z,t=0}$$

$$= \frac{\partial^2}{\partial z \partial t} g(z,t)_{z,t=0} = 0.$$

□

Remark 4.11. As we prove Theorem 1.1 it will be clear that the hypothesis in Proposition 4.10, i.e., the smoothness of $\hat{\varphi}$, is always satisfied when $F$ is homogeneous.

Remark 4.12. The above results are also valid for foliation with disconnected leaves.

4.3. The local reduction. Let $(M,F)$ be a closed SRF, and let $\Sigma \subseteq M$ be a stratum of $F$. Let $Y$ be a submanifold contained in a slice (a transverse submanifold) of the regular foliation $(\Sigma,F|_{\Sigma})$. Consider $Y_{\pi} := \pi^{-1}_{\pi}(\pi_{\pi}(Y))$ the saturation of $Y$. We also assume that $Y$ coincides with the intersection of $Y_{\pi}$ with the slice, i.e., that $Y$ is invariant under the action of the holonomy pseudogroup of the (regular) foliation $(\Sigma,F|_{\Sigma})$.

Remark 4.13. Notice that $\dim Y \leq \dim(\Sigma/F)$. Throughout the paper we will be especially interested in the cases $\dim Y = 1$ and $\dim Y = \dim(\Sigma/F)$. More precisely, in the proof of Theorem 1.3 we will take $Y$ to be a curve that projects to an integral curve of the flow $\varphi$, while for Theorem 1.3 we will consider $Y$ to be a slice of $(\Sigma,F|_{\Sigma})$. 
Suppose that the normal exponential \( \exp : \nu(Y_F) \rightarrow M \) is well defined on a tubular neighborhood of radius \( \varepsilon \) around \( Y_F \), and call \( B_\varepsilon Y_F \) the image of such tube. \( B_\varepsilon Y_F \) exists if for example \( Y_F \) is relatively compact. Define \( N := \exp (\nu(Y_F)|_Y) \cap B_\varepsilon Y_F \), together with the metric projection \( p_Y : N \rightarrow Y \). The fiber of \( p_Y \) at a point \( p \in Y \) will be denoted by \( N_p \).

The foliation \( \mathcal{F} \) intersects \( N \) in a foliation \( \mathcal{F}_N^0 := \mathcal{F} \cap N \). Notice that the leaves of \( \mathcal{F}_N^0 \) are contained in the fibers of \( p_Y \).

**Proposition 4.14.** There exists a metric \( \tilde{g} \) on \( N \) that preserves the transverse metric of \( \mathcal{F} \), i.e., the distance between leaves of \( \mathcal{F}_N^0 \) is the same as the distance between the plaques of \( \mathcal{F} \) that contain such leaves. In particular \( \mathcal{F}_N^0 \) is a SRF on \((N, \tilde{g})\).

**Proof.** This metric can be constructed as follows. Consider the regular distribution \( S \) defined as \( S_z := T_z S_p \) where \( z \in N_p \) and \( S_p \) is the slice through \( p \). According to [2, Proposition 3.1] there exists a metric \( \tilde{g} \) on a neighborhood of \( N \) so that the normal space of \( \mathcal{F} \) (with respect to \( \tilde{g} \)) is contained in \( S \) and the SRF \( \mathcal{F} \) (with respect to \( \tilde{g} \)) has the same transverse metric of \( \mathcal{F} \) (with respect to the original metric). Let \( \Pi : TM|_N \rightarrow TN \) be the orthogonal projection (with respect to original metric) and define a metric on \( TN \) as \((\Pi|_N^*)^* \tilde{g} \). Let us denote this new metric on \( N \) also as \( \tilde{g} \). Following [2, Proposition 2.17] we conclude that \( \mathcal{F}_N^0 \) is a SRF on \((N, \tilde{g})\).

\( \square \)

**Definition 4.15.** We will call the Riemannian manifold \((N, \tilde{g})\) described above the **local reduction of** \((M, \mathcal{F})\) along \( Y \).

Suppose \( N \) is the local reduction of \((M, \mathcal{F})\) along \( Y \), and let \( q \in Y \). If \( q' \) is another point in \( L_q \), we can similarly find \( Y' \) through \( q' \) and a local reduction \((N', \mathcal{F}_N')\) along \( Y' \). Moreover we can do it so that there is a flow of a vector field \( X \) tangent to the leaves that sends \((N, \mathcal{F}_N^0)\) foliated diffeomorphically to \((N', \mathcal{F}_N')\). By the properties of the metrics \( \tilde{g} \), \( g' \) on \( N \), \( N' \) proved in Proposition 4.14 this diffeomorphism induces a local isometry

\[ \tau : (N, \tilde{g})/\mathcal{F}_N^0 \longrightarrow (N', \tilde{g}')/\mathcal{F}_N'. \]

This isometry does not depend on the choice of \( X \), but only on the homotopy class of the integral curve \( \alpha \) of \( X \) joining \( q \) to \( q' \), so we refer to \( \tau \) as \( \tau_{[\alpha]} \).

Notice that \( Y \) can meet \( L_q \) in several points \( q_i \). For every such \( q_i \), and every curve \( \alpha \) from \( q \) to \( q_i \) contained in a leaf, there is an associated local isometry

\[ \tau_{[\alpha]} : N/\mathcal{F}_N^0 \longrightarrow N/\mathcal{F}_N'. \]

Let \( \mathcal{H} \) be the pseudogroup of \( N/\mathcal{F}_N^0 \) generated by all isometries \( \tau_{[\alpha]} \), for all curves \( \alpha \subseteq L_q \) with initial and final point in \( N \).

**Definition 4.16.** The pseudogroup \( \mathcal{H} \) defined above will be called **singular holonomy pseudogroup**. The triple \((N, \mathcal{F}_N^0, \mathcal{H})\) is an example of singular Riemannian foliation with disconnected leaves (cf. Section [4.5]) which we denote by \( \mathcal{F}_N \). The leaves of \( \mathcal{F}_N \) are precisely the (possibly disconnected) intersections of \( N \) with the leaves of \( \mathcal{F} \).

**Remark 4.17.** By Proposition 4.14 the inclusion \( N \rightarrow B_\varepsilon Y_F \) induces an isometry \( N/\mathcal{F}_N \rightarrow B_\varepsilon Y_F/\mathcal{F} \) that preserves the codimension of the leaves. By the main result
in [5], this map is smooth and in particular every smooth basic function in \((N, \mathcal{F}_N)\) extends to a smooth basic function in \((B, Y_F, \mathcal{F})\).

Let \(\pi_{\mathcal{F}_N}^0 : N \rightarrow N/\mathcal{F}_N^0\) be the quotient map, \(Y^* := \pi_{\mathcal{F}_N}^0(Y)\) and \(p_{Y^*} : N/\mathcal{F}_N^0 \rightarrow Y^*\) be the submetry with fibers \(N_p/\mathcal{F}_N^0\). Note that \(Y^*\) can be identified with \(Y\). It is easy to see that

\[
(4.11) \quad p_{Y^*} = p_Y \circ \pi_{\mathcal{F}_N}^0
\]

or, equivalently, that the following diagram commutes

\[
\begin{array}{ccc}
N & \xrightarrow{\pi_{\mathcal{F}_N}^0} & N/\mathcal{F}_N^0 \\
\downarrow{p_{Y^*}} & & \downarrow{p_Y} \\
Y & = & Y^*
\end{array}
\]

A local reduction satisfies the following nice property that relates the foliated structure of \((N, \mathcal{F}_N)\) with the submersion \(p_Y : N \rightarrow Y^*\).

**Proposition 4.18.** Any horizontal basic vector field \(\tilde{\xi}\) for \(p_Y\) is a horizontal foliated vector field of \(\mathcal{F}_N^0\) and for each fixed \(q\) the geodesic \(t \mapsto \exp_q(t\tilde{\xi}(q))\) is always contained in the same stratum.

**Proof.** Let \(\tilde{\xi}\) be a horizontal basic vector field of \(p_Y\). We first claim that \(\tilde{\xi}\) restricted to the regular stratum of \(\mathcal{F}_N^0\) is a foliated vector field, i.e., a basic vector field with respect to \(\pi_{\mathcal{F}_N}^0\). For \(q \in N_p\) a regular point, let \(\xi\) be the horizontal foliated vector field along \(L_q\) such that \(\tilde{\xi}(q) = \xi(q)\). Since \(p_Y\) is foliated and the \(\mathcal{F}_N^0\) leaves in \(Y\) are just points, the fibers of \(p_Y\) are saturated by the leaves of \(\mathcal{F}_N^0\) and therefore \(\tilde{\xi}\) is everywhere normal to \(N_p\).

Let \(q' \in L_q\). By equation (4.11) we note that the \(p_Y\)-horizontal geodesics \(\alpha_q(t) = \exp_q(t\tilde{\xi}(q))\), \(\alpha_{q'}(t) = \exp_{q'}(t\tilde{\xi}(q'))\) project to the same geodesic in \(Y\). This implies that \(\tilde{\xi}(q') = \tilde{\xi}(q)\) for every \(q' \in L_q\) and concludes the proof of the claim.

By the equifocality property of singular Riemannian foliations (cf. [6]), \(\alpha_q\) cannot contain a singular point \(\alpha_q(t_0)\). In fact if such a point existed then there would be two lifts of a geodesic in \(Y\) intersecting at \(\alpha_q(t_0)\), contradiction. Therefore \(\alpha_q\) is always contained in the regular stratum.

By induction on the stratification, one can now prove that \(\tilde{\xi}\) restricted to each stratum is foliated, and for every \(q \in N\) the horizontal geodesic \(\alpha_q\) is always contained in the same stratum.

**□**

**Proposition 4.19.** There exists a metric \(g_N\) on \(N\) with following properties:

- (a) The submersion \(p_Y : N \rightarrow Y\) is Riemannian.
- (b) Each fiber \(N_p\) is flat.
- (c) The foliation \(\mathcal{F}_N^0\) is a SRF on \((N, g_N)\).

**Proof.** Consider the regular distribution \(S\) defined as \(S_z := T_zS_p\) where \(z \in N_p\) and the metric \(g_0\) on \(S\) so that \(d(exp_p)_v : T_vT_pS_p \rightarrow S_{exp_p(v)}\) is an isometry, for each \(v\) normal to \(Y\). As before, let \(\Pi : TM|_N \rightarrow TN\) be the orthogonal projection (with
respect to original metric) and define a metric on $TN$ as $(\Pi|_p^{-1})^*g_0$. Let us denote this new metric on $N$ also as $g_0$. Following [2] Proposition 2.17 we conclude that $\mathcal{F}_N^0$ is a SRF on $(N, g_0)$. Moreover, since $T_s N_p \subset S_s$ and $d(exp_p) : T_{s} T_p S_p \to S_{exp_p(s)}$ is an isometry, every fiber $N_p$ is flat with respect to $g_0$. Consider $H$ the distribution orthogonal to the fibers of $N$. We will change the metric of $H$ in order to get the appropriate metric $g_N$ satisfying (a) and (c). Let 

$$g_N := g_0|_{H^\perp} + p_Y^* g_Y$$

Notice that the metric on the fibers of $p_Y$ is still $g_0$, thus condition (b) remains satisfied. Moreover, the submersion is now Riemannian by construction.

In order to prove that $\mathcal{F}_N^0$ is a singular Riemannian foliation, it is enough to compute the Lie derivative $\mathcal{L}_{\tilde{X}} g_N$ and check that $\mathcal{L}_{\tilde{X}} g_N |_H = 0$ for each vector field $\tilde{X}$ tangent to the leaves. To this scope, let $\xi_1, \xi_2$ be $p_Y$-basic vector fields. By Proposition 4.20 these vectors are foliated, and in particular for every vector field $\tilde{X}$ tangent to the leaves, $[\tilde{X}, \xi_i]$ is tangent to the leaves as well, $i = 1, 2$. Since clearly $\xi_1, \xi_2 \in H$, we can compute

$$\mathcal{L}_{\tilde{X}} g_N (\xi_1, \xi_2) = \tilde{X} \cdot g_N (\xi_1, \xi_2) - g_N \left( [\tilde{X}, \xi_1], \xi_2 \right) - g_N \left( [\xi_1, \xi_2], \tilde{X} \right)$$

$$= \tilde{X} \cdot g (p_Y \xi_1, p_Y \xi_2) - 0 - 0 = 0.$$

Since $\mathcal{F}_N^0$ is a SRF with respect to some metric (e.g., the metric $\tilde{g}$ constructed before), by [2] Proposition 2.14 $\mathcal{F}_N^0$ is a SRF with respect to $g_N$.

Remark 4.20. The metric $g_N$ and the metric $\tilde{g}$ on $N$ will be used in Section 5 and Section 4 respectively.

Notice that the metric $g_N$ does not preserve the transverse metric of $\tilde{g}$. In particular, an isometry $\phi : (N, \tilde{g}) / \mathcal{F}_N^0 \to (N, \tilde{g}) / \mathcal{F}_N^0$ will not be an isometry of $(N, g_N) / \mathcal{F}_N^0$. Nevertheless, we still have the following result.

**Proposition 4.21.** Let $\phi : (N, \tilde{g}) / \mathcal{F}_N^0 \to (N, \tilde{g}) / \mathcal{F}_N^0$ be an isometry preserving $Y^\ast$. Then $\phi$ preserves the fibers of $p_Y$, and

$$\phi |_{N_{p_1}, \mathcal{F}_N^0} : (N_{p_1}, g_N) / \mathcal{F}_N^0 \to (N_{p_2}, g_N) / \mathcal{F}_N^0$$

is still an isometry.

**Proof.** The metric projection $p_Y$ sends a point $q^\ast$ to the point $p^\ast \in Y^\ast$ which is closest to $q^\ast$. This is a metric condition, and since $\phi$ preserves the metric, in particular it preserves the fibers of $p_Y$.

Given $\lambda \in (0, 1)$ the homothetic transformation $h_\lambda : N_p \to N_p$, $exp_p v \mapsto exp_p \lambda v$ is a foliated map (cf. [10]) and one can define $\tilde{g}_\lambda := \frac{1}{\lambda} h_\lambda^* \tilde{g}$ such that $(N_p, \tilde{g}_\lambda, \mathcal{F}_N^0)$ is still a singular Riemannian foliation. Moreover, since

$$\phi |_{N_{p_1}, \mathcal{F}_N^0} : (N_{p_1}, \tilde{g}) / \mathcal{F}_N^0 \to (N_{p_2}, \tilde{g}) / \mathcal{F}_N^0$$

is an isometry, it will still be an isometry with respect to $\tilde{g}_\lambda$. Since the restrictions of $\tilde{g}_\lambda$ to the fibers of $p_Y$ converge smoothly to the metric $g_N$, the proposition is proved. See a similar argument in [2] Theorem 1.2.
Remark 4.22. Suppose that the leaves in \(Y_F\) meet \(Y\) only once, for example in the proof of Theorem 1.1. In this case the isometric action of the singular holonomy pseudogroup (cf. Definition 4.16) \(\mathcal{H}\) on \(N/F_N\) (as in Section 4.2) preserves the fibers \(N_p/F_N\). Moreover given an isometry \(\phi : (N, \tilde{g})/F_N \rightarrow (N, \tilde{g})/F_N\), Proposition 4.21 can be reproved after replacing \(F_N\) by \(F_N\). In particular, \(\phi\) induces \(g_N\)-isometries
\[
\phi|_{N_{p_1}/F_N} : (N_{p_1}, g_N) / F_N \rightarrow (N_{p_2}, g_N) / F_N.
\]
whenever \(\phi(p_1) = p_2\).

5. ISOMETRIC FLOWS ON ORBIT SPACES: PROOF OF THEOREM 1.1

The goal of this section is to prove Theorem 1.1. In particular, we are assuming that the leaves of \((M, F)\) are the orbits of a smooth proper action of a Lie group \(K\) on \(M\).

In order to avoid cumbersome notations, we will denote each basic function \(h : M \rightarrow \mathbb{R}\) and the induced function on \(M/F\) by the same letter \(h\).

We know that Theorem 1.1 is true when all orbits have the same dimension, see e.g. [10, Salem appendix D] and Swartz [15]. We assume by induction that Theorem 1.1 is true for a number of strata lower or equal to \(n - 1\).

Let \(q\) be a point of the minimal stratum, \(q^* \in M/F\) its projection in the quotient, and \(Y^*\) a neighborhood of \(q^*\) in the orbit of \(\varphi\) through \(q^*\). The preimage \(Y_F = \pi_x^{-1}(Y^*)\) is a regularly saturated submanifold of \(M\) (cf. Section 4.3) such that \(F|_{Y_F}\) has codimension 1. Let \(Y\) be the intersection of \(Y_F\) with a slice at \(q\) for the action of \(K\) on \(M\), and \(N\) the local reduction of \((M, F)\) along \(Y\); cf. Definition 4.15. Unless explicitly stated otherwise, we will always consider the Riemannian metric \(g_N\) on \(N\) defined in Proposition 4.19.

Before we go through the details, let us briefly explain the main idea of the proof. By Remark 4.14, it is enough to show that \(\varphi\) is smooth on \(N/F_N \times I\). In other words, for a given smooth \(F_N\)-basic function \(h\) on \(N\) we will prove that \(\varphi^*h\) is a smooth basic function on \(N \times I\) with respect to the foliation \(F_N \times \{\ast\}\).

We will divide the proof of the smoothness of \(\varphi\) into two steps.

Step 1) We restrict our attention to a fiber \(X := N_{q_0}\) of the metric projection \(p_Y : N \rightarrow Y\) (cf. Section 4.3), and we prove in Proposition 5.8 that the restriction of \(\varphi^*h\) to \(X \times I\) is smooth. The main idea here is to use some some arguments of [5] to check that \(u_h(x, t) := \varphi^*h(x, t)\) is a weak solution of a parabolic equation; see equation (5.4). We apply regularity theory of solutions of linear parabolic equations to prove that \(u_h\) is smooth. This theory requires some initial regularity conditions that will be checked using Propositions 4.9 and 4.10; see details in Lemma 5.4.

Step 2) We extend the smoothness of \(\varphi\) to the whole \(N/F_N \times I\) using the inverse function theorem for orbit spaces; see [14, page 45]. This is proved in Proposition 5.7.

Since we will be working on \((N, F_N)\) instead of \((M, F)\), we should better make sure that \((N, F_N)\) is still homogeneous.

Lemma 5.1 (The group \(G\)). The points on the curve \(Y\) have the same isotropy group \(G := K_g\). Also note that the restriction of \(F_N\) to \(N\) is the partition of \(N\) into the orbits of the action of \(G\).
Proof. Let us denote $\tilde{\varphi}_t$ a flow of isometries isometry on $Y$ which is a lift of $\varphi_t$; see e.g. [15]. We want to prove that

$$K_{\tilde{\varphi}_t(q)} = K_q.$$  \hfill (5.1)

Consider the action $\mu : K_q \times Y \to Y$ and the induced homomorphism $\mu : K_q \to \text{Iso}(Y)$. Since we are dealing with isotropy groups, in order to prove equation (5.1) it suffices to prove that

$$\mu(K_{\tilde{\varphi}_t(q)}) = \mu(K_q).$$  \hfill (5.2)

We first claim that $\tilde{\varphi}_t \mu(K_q) \tilde{\varphi}_t^{-1} = \mu(K_{\tilde{\varphi}_t(q)})$. Let $p$ be a principal point in $Y$ and $k \in \mu(K_q)$. Set $k_1 := \tilde{\varphi}_t(k) \tilde{\varphi}_t^{-1}$. Note that $k_1 \tilde{\varphi}_t(p) = \tilde{\varphi}_t(kp)$. On the other hand, since $\varphi_t$ is an isometry in the quotient, and in particular sends loops into loops, there exists a unique $k_2 \in \mu(K_{\tilde{\varphi}_t(q)})$ such that $k_2 \tilde{\varphi}_t(p) = \tilde{\varphi}_t(kp)$. Therefore, since the same argument applies to other principal points near $p$ (recall that the set of principal points is an open set) we infer that $k_1 = k_2$ and hence $\tilde{\varphi}_t \mu(K_q) \tilde{\varphi}_t^{-1} \subset \mu(K_{\tilde{\varphi}_t(q)})$. The proof of the other inclusion is identical and hence the claim has been proved.

Now, since $Y$ is contained in a slice at $q$, we have that $K_{\tilde{\varphi}_t(q)}, K_q$ are compact Lie subgroups and $K_{\tilde{\varphi}_t(q)} \subset K_q$. These facts and the above claim imply equation (5.2).

\[\Box\]

Remark 5.2. In the particular case where $F$ is Riemannian homogeneous, the fact that $G$ acts isometrically follows from Proposition 4.19 and Remark 4.22.

5.1. The first step. Let us fix a fiber $X := N_{q_0}$ of $p_Y : N \to Y$, and denote $N_t := N_{\tilde{\varphi}_t(q_0)}$. Since the flow $\varphi : (N, \tilde{g})/F_N \times I \to (N, \tilde{g})/F_N$ acts by isometries and $Y^*$ is an orbit, then by Proposition 4.24 each $\varphi_t$ is an isometry between $(X, g_N)/F_N \to (N_t, g_{N_t})/F_{N_t}$, where the metrics on the fibers are now flat.

Since $X$ and $N_t$ are flat, it follows from [5] Proposition 2.1] that the mean curvature vector fields of the leaves in $X$ and $N_t$ project to well defined vector fields in the regular strata of $X/F_N$, $N_t/F_N$ respectively, and moreover $\phi(t) := \varphi_t$ sends one vector field to the other. On the other hand since $\phi(t)$ is an isometry, it preserves the Laplacian operator in the principal part of $X/F_N$.

Set $u_h(\cdot, t) := \phi(t)^* h$. From what said above, we obtain that the following equation holds a in weak sense (cf. [5] Lemma 2.5)):

$$\triangle u_h = \triangle \phi(t)^* h = \phi(t)^* \triangle h = u_{\Delta h}$$  \hfill (5.3)

where $\triangle h$ denotes the Laplacian operator of $N_t$ applied to the restriction $h|_{N_t}$.

As in [5] Proposition 2.3], one can prove the next lemma, taking in consideration item (c) of Definition 3.4.

Lemma 5.3. The restriction of the flow $\varphi$ to $N/F_N \times (-\epsilon, \epsilon)$ is a $C^1$ map.

From the lemma above, $\frac{d}{dt} u_h$ is continuous and we can define

$$f(\cdot, t) := \phi(t)^* (\triangle h) + \frac{d}{dt} (\phi(t)^* h) = u_{\Delta h} + \frac{d}{dt} u_h(\cdot, t)$$
and thus we have
\[(5.4) \quad \frac{d}{dt}u_h + \Delta u_h = f \quad \text{in } X \times I\]
in a weak sense.

The goal is to use regularity properties of parabolic equations to prove that \(u_h\) is smooth in \(X \times I\). We first need to prove some initial regularity for \(u_h\).

**Lemma 5.4.** For \(n \geq 0\) we have
\[(a) \quad \frac{d^n}{dt^n} u_h(\cdot, t) \in C^\infty(X) \text{ for each } t.\]
\[(b) \quad \frac{d^n}{dt^n} u_h \in L^2(0, T, H^1(X)).\]

**Proof.** Let us prove the case where \(n = 1\); the other cases are identical. Consider the first blow-up \(\hat{\rho} : \hat{N} \to N\) of \((N, F_N)\) along its minimal stratum, cf. Section 4.1. Since \(\varphi\) is a flow of isometries on \((N, \tilde{g})/F_N\), by Proposition 4.4 there is an induced flow of isometries \(\hat{\varphi}\) on \(\hat{N}/\hat{F}_N\). Since the number of strata in \((\hat{N}, \hat{F}_N)\) is strictly smaller than the number of strata in \((N, F_N)\), then by induction Theorem 1.1 holds, and \(\hat{\varphi}\) is smooth. Therefore the conditions of Proposition 4.10 are met, and \(\varphi^*(\bar{Y} \cdot h)\) is a blow-up function as well. Since
\[\frac{d}{dt} u_h = \frac{d}{dt} \varphi^* h = \varphi^*(\bar{Y} \cdot h)\]
then by Proposition 4.9 \(\frac{d}{dt} u_h(\cdot, t) \in C^1(X)\) for each \(t\). The above equation also implies that \(\frac{d}{dt} u_h\) is continuous.

Note that in the regular stratum
\[\Delta \left( \frac{d}{dt} u_h \right) = \frac{d}{dt} (\Delta u_h) = \frac{d}{dt} u_{\Delta h}\]
Since \(\frac{d}{dt} u_h(\cdot, t)\) and \(\frac{d}{dt} u_{\Delta h}(\cdot, t) \in C^1(X)\) we can apply the same argument as in [4] to infer that the following equation holds weakly:
\[(5.5) \quad \Delta \left( \frac{d}{dt} u_h \right) = \frac{d}{dt} u_{\Delta h} \quad \text{in } X\]
From regularity theory of solutions of elliptic partial differential equations [7] we conclude that \(\frac{d}{dt} u_h(\cdot, t)\) lies in the Sobolev space \(H^3(X)\). Applying the argument successively, we obtain \(\frac{d}{dt} u_h(\cdot, t) \in C^\infty(X)\) and this conclude the proof of part (a).

Now since \(\bar{Y} \cdot h \in \mathcal{B}\) and \(\varphi_t\) preserves the geodesics orthogonal to the minimal stratum, we have that the directional derivatives of \(\frac{d}{dt} u_h\) exist. Moreover
\[
\left\| \nabla \left( \frac{d}{dt} u_h(\cdot, t) \right) \right\| = \left\| \nabla \phi(t)^*(\bar{Y} \cdot h) \right\|.
\]
The term in the right-hand side of this equation is a continuous function due to Lemma 5.3.

Therefore, since \(\frac{d}{dt} u_h\) and \(\left\| \nabla \left( \frac{d}{dt} u_h \right) \right\|\) are both continuous the part (b) follows.

□

Let us recall the next result that can be found in Evans [7, Theorem 6, page 365]; for definitions and notations about Sobolev spaces see [4].
Theorem 5.5 (Regularity of parabolic equations). Assume that \( g \in H^{2m+1}(X) \), \( \frac{d^k f}{dt^k} \in L^2(0, T, H^{2m-2k}(X)) \) (\( k = 0, \ldots, m \)). Suppose also the following \( m \)-th order compatibility condition holds:

\[
\begin{align*}
g_0 &:= g \in H^1_0(X) \\
g_1 &:= f(0) - \Delta g_0 \in H^1_0(X) \\
&
\vdots \\
g_m &:= \frac{d^{m-1}}{dt^{m-1}} f(0) - \Delta g_{m-1} \in H^1_0(X)
\end{align*}
\]

Let \( u \in L^2(0, T, H^1_0(X)) \) with \( \frac{du}{dt} \in L^2(0, T, H^{-1}(X)) \) be a weak solution of

\[
\begin{align*}
\frac{du}{dt} + \Delta u &= f & \text{in} & X \times [0, T] \\
u &= 0 & \text{on} & \partial X \times [0, T] \\
u &= g & \text{on} & X \times \{t = 0\}
\end{align*}
\]

Then \( \frac{d^k u}{dt^k} \in L^2(0, T, H^{2m+2-2k}(X)) \).

We can now finally prove the proposition below.

Proposition 5.6. \( u_h \) is smooth on \( X \times (-\epsilon, \epsilon) \).

Proof. Here let us make the following convenient definitions:

- \( u_h^{(n)} := \frac{d^n}{dt^n} u_h \).
- \( f^{(n)} := u_{\Delta h}^{(n)} + u_h^{(n+1)} \).
- \( g^{(n)} := u_h^{(n)}(\cdot, 0) \).

By differentiating equation (5.4), one gets the following family of parabolic equations, parameterized by \( n \):

\[
\begin{align*}
\frac{d}{dt} u_h^{(n)} + \Delta u_h^{(n)} &= f^{(n)} & \text{in} & X \times [0, T] \\
u_h^{(n)} &= g^{(n)} & \text{on} & X \times \{t = 0\}
\end{align*}
\]

By Lemma 5.4, \( f^{(n)}(\cdot, 0) \) and \( g^{(n)} \) are \( C^\infty(X) \) for any \( n > 0 \), and in particular all the \( m \)-th order compatibility conditions hold. Therefore the only condition that has to be checked is

\[
f^{(n)} \in L^2(0, \epsilon, H^{2m}(X))
\]

If one first applies Theorem 5.5 with \( m = 0 \), then the condition (5.6) holds (recall Lemma 5.4), and from the Theorem 5.5 one gets \( u_h^{(n)} \in L^2(0, \epsilon, H^2(X)) \).

Now suppose by induction (on \( r \)) that \( u_h^{(n)} \in L^2(0, \epsilon, H^{2r}(X)) \) for every \( n \). Then \( f^{(n)} \in L^2(0, \epsilon, H^{2r}(X)) \) as well, and one can apply Theorem 5.5 with \( m = r \). Again the only condition to be checked is (5.6), which holds, and by the Regularity Theorem we obtain, in particular, that \( u_h^{(n)} \in L^2(0, \epsilon, H^{2r+2}(X)) \).

By induction, we obtain that

\[
u_h, \frac{d^n}{dt^n} u_h \in L^2(0, \epsilon, H^m(X)) \quad \forall m \in \mathbb{N}
\]

and from this it follows that \( u_h \in C^\infty(X \times [0, \epsilon]) \).
5.2. The second step.

**Proposition 5.7.** \( \varphi : (\mathcal{N}/\mathcal{F}_N) \times I \rightarrow N/\mathcal{F}_N \) is smooth.

**Proof.** If \( Y \) is a point, then the result was already proved in the previous proposition. Let us assume that \( Y \) is not a point.

We know from Proposition 5.6 that the restriction \( \psi := \varphi \mid_{X/\mathcal{F}_N \times I} : X/\mathcal{F}_N \times I \rightarrow N/\mathcal{F}_N \) is smooth, and therefore we can apply the inverse function theorem on orbit space (see [14, page 45]) to conclude that \( \psi^{-1} \) is smooth. Note that, since for each fixed \( s \) the function \( p_{N/\mathcal{F}_N} \circ \psi(\cdot, s) \) is a constant \( k(s) \), it is not necessary to reduce \( N_q \) in order to define \( \psi^{-1} \). We claim that the diagram below commutes, and hence \( \varphi \) is a composition of smooth maps and therefore is a smooth map.

\[
\begin{array}{ccc}
N/\mathcal{F}_N \times I & \xrightarrow{\varphi} & N/\mathcal{F}_N \\
\downarrow{\psi^{-1} \times \text{Id}} & & \downarrow{\psi^{-1}} \\
(X/\mathcal{F}_N \times I) \times I & \xrightarrow{(p_1, p_2 + p_3)} & (X/\mathcal{F}_N \times I)
\end{array}
\]

In fact, set \( z = \psi(\tilde{x}, s) \). Then we have

\[
\varphi(z, t) = \varphi_t(z) = \varphi_t(\psi(\tilde{x}, s)) = \varphi(\tilde{x}, s + t) = \psi(p_1 \circ \psi^{-1}(z), p_2 \circ \psi^{-1}(z) + t).
\]

This proves the commutativity of the diagram. The smoothness of the arrows of the diagram can be proved using the smoothness of \( \psi^{-1} \) and Schwarz’s Lemma [13]; see also comments in the beginning of proof of the main Theorem [13, page 65].

\[\square\]

6. Molino’s conjecture: proof of Theorem 1.3

Let \((M, \mathcal{F})\) be a singular Riemannian foliation, and let \( \mathcal{F} \) be the partition of \( M \) by the closures \( \overline{L} \) of leaves \( L \in \mathcal{F} \). In Molino [10] Theorem 6.2, page 214] (cf. [10] Appendix D) when \( M \) non compact) it is proved that each closure \( \overline{L} \) is a closed submanifold, and that the partition \( \mathcal{F} = \{\overline{L} \}_{L \in \mathcal{F}} \) is a transnormal system, i.e., the leaves of \( \mathcal{F} \) are locally equidistant (cf. Definition 3.1). In fact, the equifocality of \( \mathcal{F} \) (cf. [6]) implies that plaques of \( \mathcal{F} \) are equidistant to any fixed plaque of \( \overline{T_q} \) and so are the plaques of \( \mathcal{F} \); see a similar argument in [2, Proposition 2.13].

In what follows we will prove that this partition is a smooth singular foliation.

**Proposition 6.1.** \((M, \mathcal{F})\) is a singular Riemannian foliation.

**Proof.** Let us fix a point \( q \in M \) and consider \( \Sigma \) the stratum containing the point \( q \). Consider a slice \( D \) of the (regular) foliation \((\Sigma, \mathcal{F}_\Sigma)\). Finally let \( v \in \nu_q L_q \cap T_q \overline{L_q} \).
We want to prove that there exists a vector field $\vec{Y}$ around $q$, tangent to the leaves of $\mathcal{F}$ so that $\vec{Y}(q) = v$.

Let $(N, \mathcal{F}_N^0)$ be the local reduction of $(M, \mathcal{F})$ along the slice $D$; recall Definition 4.11. In what follows we will only make use of the metric $\tilde{g}$ on $N$ introduced in Proposition 4.11 so we will give it as understood. Since $\mathcal{F}$ is orbit-like, $(N_q, \mathcal{F}_N^0)$ is homogeneous given by the orbits of some group $G$. By flowing along the foliated basic vector fields defined in Proposition 4.18 we can make $G$ act smoothly on the whole $N$, even though not by isometries.

Recall that there is a pseudogroup $\mathcal{H}$ of local isometries of $N/\mathcal{F}_N^0$ that describes how the leaves around $\Sigma$ intersect $N$ (cf. Definition 4.16).

By applying Molino’s theorem to the regular foliation $\mathcal{F}|_\Sigma$, $\pi_{\mathcal{F}_N^0} v$ is tangent to the orbit of the closure of $\mathcal{H}$; see [10, Theorem 5.1, page 156] and [10, Section 3.4, page 287].

Let $(\tilde{N}, \mathcal{F}_N^0)$ be the desingularization of $(N, \mathcal{F}_N^0)$ (cf. Section 4.1), with projections $\tilde{\rho} : \tilde{N} \to N$ and $\tilde{\rho} : \tilde{N}/\mathcal{F}_N^0 \to N/\mathcal{F}_N^0$.

For the sake of simplicity, here we are using the notation $\tilde{N}$ to denote the desingularized space and not just the first blow-up along the minimal stratum. As in Proposition 4.3 we can lift any isometry $\tau|_N$ to an isometry $\tilde{\tau}|_{\tilde{N}}$ of the orbifold $(\tilde{N}, \tilde{g})/\mathcal{F}_{\tilde{N}}$; see also Remark 4.7 applied to small relatively compact neighborhoods. Let $\hat{\mathcal{H}}$ be pseudogroup generated by all isometries $\tilde{\tau}|_{\tilde{N}}$ constructed in this way, and let $\overline{\mathcal{H}}$ its closure. By Salem [10, Appendix D] the closure $\overline{\mathcal{H}}$ is a Lie pseudogroup.

We claim that we can find a Killing vector field $\vec{Y}$ in the orbifold $\tilde{N}/\mathcal{F}_N^0$ with flow $\phi_t \in \hat{\mathcal{H}}$, that projects to a flow $\varphi_t$ on $N/\mathcal{F}_N^0$ and such that $\frac{d}{dt} \varphi_t(q) = \pi_{\mathcal{F}_N^0} v$.

Indeed, there is an étale morphism $\hat{\mathcal{H}} \to \mathcal{H}$.

Set $\hat{D} := \tilde{\rho}^{-1}(D)$. For an appropriate choice of a point $\hat{q}$ with $\tilde{\rho}(\hat{q}) = q$, we have that the restriction of $(\tilde{\rho})|_{\hat{D}}$ to a neighborhood of the orbit $\overline{\mathcal{H}}(\hat{q})$ is a submersion. Note that the restriction of $\tilde{\rho}$ to the orbit $\overline{\mathcal{H}}(\hat{q})$ is a surjective smooth map

$\tilde{\rho}_{\hat{q}} : \overline{\mathcal{H}}(\hat{q}) \to \overline{\mathcal{H}}(q)$.

Moreover, if the differential has rank $d$ at some $\hat{p} \in \overline{\mathcal{H}}(\hat{q})$, it will be $d$ on the whole orbit $\overline{\mathcal{H}}(\hat{p}) \subseteq \overline{\mathcal{H}}(\hat{q})$, and this implies that the differential of $\tilde{\rho}_{\hat{q}}$ is everywhere constant, and hence $\tilde{\rho}_{\hat{q}}$ is a submersion. In particular, given $\pi_{\mathcal{F}_N^0} v$ tangent to $\overline{\mathcal{H}}(q)$, there exists a vector $\hat{v}$ tangent to $\overline{\mathcal{H}}(\hat{q})$ that projects to $\pi_{\mathcal{F}_N^0} v$. We can now take a flow $\phi \in \hat{\mathcal{H}}$ with vector field $\vec{Y}$ such that $\vec{Y}(\hat{q}) = \hat{v}$. Due to the construction of $\hat{\mathcal{H}}$, it makes sense to project $\phi$ to a flow $\varphi$ in $N/\mathcal{F}_N^0$, and this proves the claim.

Since the foliation $(N, \mathcal{F}_N^0)$ is homogeneous, it follows from Theorem 1.31 that $\varphi$ is smooth. From Schwarz [14, Corollary 2.4] we can lift $\varphi$ and produce the smooth vector field $\vec{Y}$ on $N$ tangent to $v$. Finally, using the flow of vector fields tangent to the leaves, one can extend the vector field $\vec{Y}$ on $N$ to a vector field on an
neighborhood of $q$ in $M$ that is tangent to the leaves of $\mathcal{F}$ and this concludes the proof.

**Appendix A. Linearized vector fields and orbit-like foliations**

In this appendix we review the concept of linearized vector field introduced in \cite{10} and prove, among other results, the following:

1. The linearization of the infinitesimal foliation $\mathcal{F}_p$ gives the Riemannian homogenous part of $\mathcal{F}_p$. Therefore a locally closed foliation $\mathcal{F}$ is orbit-like if and only if the infinitesimal foliation is linearizable.

2. There are vector fields in $\mathfrak{X}(\mathcal{F})$ that are invariant by homothetic transformation; here, as in Definition 3.1, we denote $\mathfrak{X}(\mathcal{F})$ as the module of vector fields that are always tangent to the leaves of $\mathcal{F}$.

3. There exists a linearized flow that provides an isometry between tangent spaces of slices at points in the same leaf.

Let $(M, \mathcal{F})$ be a singular Riemannian foliation on a manifold $M$, and let $L$ be a singular leaf. Given $p \in L$, let $P_p$ denote a plaque (i.e., a neighborhood of $p$ in $L$). In \cite{6} new metrics $\tilde{g}$, $\hat{g}_\lambda$ and $g_0$ were created on $Tub(P_p) \simeq \nu(P_p)$. Let’s point out that $\tilde{g}$ and $g_0$ depend on the choice of a family $\mathcal{X}$ of vertical vector fields $\{\tilde{X}_1, \ldots, \tilde{X}_k\}$ on $Tub(P_p)$ that are everywhere linearly independent and provide a basis for the points of $P_p$. Nevertheless, the metric of $g_0|_{\nu_pP_p}$ is intrinsically defined.

A family $\mathcal{X}$ defined as above gives rise to a sub-foliation $\mathcal{F}^2$, and a $\tilde{g}$-orthogonal splitting $T(Tub(P_p)) := A \oplus B$, where $A = T\mathcal{F}^2$ and $B$ is the tangent space to the slices. From now on, we will look at $\nu_p(P_p)$ instead of $Tub(P_p)$. In particular, if $\tilde{X}$ is a vector field with components $\tilde{X} = \tilde{X}^A + \tilde{X}^B$, then $\tilde{X}^B$ is always tangent to the fibers, and the restriction to one fiber $\tilde{X}^B|_{\nu_pP_p}$ can be seen as a vector field on a vector space.

For $P_p$ small enough, $\nu P_p \simeq P_p \times \mathbb{R}^m$, and we can write a point in $\nu P_p$ as a couple $(q, v)$. Again for every vector field $\tilde{X}$, the projection $\overline{X}^B$ is a vector field tangent to the fibers, so $\overline{X}^B|_{\nu_qP_p}$ is a vector field on a vector space.

**Definition A.1.** Consider $\overline{Y} \in \mathfrak{X}(\mathcal{F})$. For any $\lambda \in (0, 1)$, define $\overline{Y}_\lambda$ as

$$ (\overline{Y}_\lambda)_p = (h_\lambda)_*^{-1}\overline{Y}_{h_\lambda(p)}, $$

and let $\overline{Y}^L = \lim_{\lambda \to 0} \overline{Y}_\lambda$. Such vector field $\overline{Y}^L$ is called linearization of $\overline{Y}$. If $\overline{Y}^L = \overline{Y}$, then $\overline{Y}$ is called linearized.

**Example A.2.** Consider a SRF foliation $\mathcal{F}$ on $\mathbb{R}^n$, where $\{0\}$ is a leaf. Given a smooth vector field $\tilde{X} \in \mathfrak{X}(\mathcal{F})$ defined around 0, the corresponding linearized vector field is given by

$$ \tilde{X}^L = \lim_{\lambda \to 0} (h_\lambda)_*^{-1}\tilde{X}_{h_\lambda(0)} = \lim_{\lambda \to 0} \frac{1}{\lambda} \tilde{X}_\lambda = \left(\nabla_v \tilde{X}\right)_0. $$

In other words, $\tilde{X}^L = \left(\nabla_v \tilde{X}\right)_0$ and in particular it is a linear vector field. Moreover, since $\tilde{X}^L \in \mathfrak{X}(\mathcal{F})$, it is tangent to the distance spheres around 0, and therefore

$$ 0 = \langle \tilde{X}^L, v \rangle = \langle \left(\nabla_v \tilde{X}\right)_0, v \rangle. $$
In other words, the linear vector field \( \vec{X}_L^{(i)} = (\nabla_{(i)} \vec{X})_0 \) is determined by a matrix which is skew-symmetric and hence \( \vec{X}_L \) is a Killing vector field.

With this example in mind, we can proceed to describe the generic situation:

**Proposition A.3.** Let \( P_p, \text{Tub}(P_p) \) defined as before, and let \( \pi : \text{Tub}(P_p) \to P_p \) be the closest-point projection map. Let \( \vec{Y} \in \mathfrak{X}(\mathcal{F}) \). Then:

(a) \( \vec{Y}_L \) is well defined, and is smooth.
(b) \( \vec{Y}_L \) belongs to \( \mathfrak{X}(\mathcal{F}) \).
(c) \( \vec{Y}_L \) is invariant under the homothetic transformations \( h_{\lambda s} \). In particular \( (\vec{Y}_L)_L = \vec{Y}_L \), i.e., the linearization of a vector field is a linearized vector field, and the terminology makes sense.
(d) \( \vec{Y}_L \) is basic with respect to \( \pi \).
(e) \( \vec{Y}_L|_{P_p} = \vec{Y}|_{P_p} \).

Proof of (a) Notice that this statement is local, and only depends on the structure of vector space on \( \pi : \text{Tub}(P_p) \to P_p \). Everything can thus be reduced to the case of \( \mathbb{R}^{n+k} \to \mathbb{R}^n \), and if \( \vec{X} \) splits as \( \vec{X}_A + \vec{X}_B, \vec{X}_A \in \mathbb{R}^n, \vec{X}_B \in \mathbb{R}^k \), then one can easily see that, as in example A.2, the following holds:

\[
\vec{X}_L^{(v,w)} = (\vec{X}_A(v,0)) + (\nabla_w \vec{X}_B(v,0))
\]

where \( \nabla \) is the Euclidean connection. This convergence is pointwise, and the resulting limit is smooth. This implies that the convergence is uniform on compact sets.

Proof of (b) This part follows since every \( \vec{Y}_\lambda \in \mathfrak{X}(\mathcal{F}) \), and therefore \( \vec{Y}_L = \lim_{\lambda \to 0} (\vec{Y}_\lambda) \) belongs to \( \mathfrak{X}(\mathcal{F}) \) as well.

Proof of (c) We compute:

\[
h_{s\ast}s\ast \vec{X}^{(v,w)}_L = h_{s\ast}\lim_{\lambda \to 0} (h_{\lambda s})\ast \vec{X}_{h_{\lambda}(q)}
\]

\[
= \lim_{\lambda \to 0} h_{s\ast}(h_{\lambda s})\ast \vec{X}_{h_{\lambda}(q)}
\]

\[
= \lim_{\lambda \to 0} (h_{\lambda s})\ast \vec{X}_{h_{\lambda}(q)}
\]

\[
= \lim_{\lambda/s \to 0} (h_{\lambda/s})\ast \vec{X}_{h_{\lambda/s}(h_{\lambda}(q))}
\]

\[
= \vec{X}^{(v,w)}_{h_{\lambda s}(q)}.
\]

Proof of (d) Notice first of all that for any \( q \in \text{Tub}(P_p) \), \( \pi(q) = \lim_{\lambda \to 0} h_{\lambda}(q) \) and \( \pi_s = \lim_{\lambda \to 0} h_{\lambda s} \). Let’s now compute:

\[
\pi_s \vec{X}^{(v,w)}_L = \lim_{\lambda \to 0} h_{\lambda s}(\vec{X}^{(v,w)}_L)
\]

\[
= \lim_{\lambda \to 0} \vec{X}^{(v,w)}_{h_{\lambda}(q)}
\]

\[
= \vec{X}^{(v,w)}_{\pi(q)}.
\]

In the second equation we used the invariance of linearized vector fields under homoteties, as proved in the last item. In the third and fourth equation we used the continuity of \( \vec{X}_L \) and the observation that \( \pi = \lim_{\lambda \to 0} h_{\lambda} \).
Proof of (e) This simply follows from the fact that $\tilde{Y}_\lambda|_{P_p} = \tilde{Y}_1|_{P_p}$.

□

From the last point in the proposition above it follows that given a family $\mathcal{X} = \{\tilde{X}_1, \ldots, \tilde{X}_k\}$ of vertical vector fields as above, the “linearized family” $\mathcal{X}^L = \{\tilde{X}_1^L, \ldots, \tilde{X}_k^L\}$ is also linearly independent.

**Proposition A.4.** Let $\tilde{Y}$ be a linearized vector field on $\text{Tub}(P_p)$. Moreover, let $\mathcal{X}$ be a family of vector fields as above. Then

(a) $(\tilde{Y}^L)^A, (\tilde{Y}^L)^B$ belong to $\mathcal{X}(\mathcal{F})$.

(b) If $\mathcal{X}$ is a linearized family, and $z = z^A + z^B$ is a vertical vector at $(q, v)$,

\[ g_0(\tilde{Y}^L, z)_{(q,v)} = g(\pi_* \tilde{Y}^A, \pi_* (z^A))_{(q,0)} + g(\nabla_\lambda \tilde{Y}^B, z^B)_{(q,0)}, \]

where the connection is the Euclidean connection on the fiber $\nu_q P_p$. In particular $(\tilde{Y}^L)^A$ is the horizontal lift of $\tilde{Y}^A_{(q,0)}$, and $(\tilde{Y}^L)^B$ can be thought of as the linear vector field $(\nabla \tilde{Y}^B)_{(q,0)}$.

(c) $(\tilde{Y}^L)^B = (\tilde{Y}^B)^L$ is a Killing vector field. In addition, if $G_p$ is the connected Lie subgroup of isometries whose Lie algebra is the span of the subspace $\mathcal{X}(\mathcal{F}_p)^L$ then the orbits of $G_p$ are the maximal homogeneous subfoliation of the infinitesimal foliation $\mathcal{F}_p$. In particular, for locally closed foliations, $\mathcal{F}_p$ is Riemannian homogeneous if and only if the orbits of the closure of $G_p$ coincides with the leaves of $\mathcal{F}_p$.

(d) $(\phi_{\tilde{Y}^L})_{(q,v)} = (\phi_{\tilde{Y}^L})_{(q)}$ and therefore the flow of $\tilde{Y}^L$ is a linear map between fibers (and therefore providing a reason for the terminology).

(e) The flow of $Y^L$ is an isometry between fibers.

(f) $(\nu_{P_p}, g_0)$ splits metrically as $(P_p, g|_{P_p})\times (\nu_{P_p}, g_p)$.

**Proof of (a)**

We know that both $\tilde{Y}^L$ and $(\tilde{Y}^L)^A$ belong to $\mathcal{X}(\mathcal{F})$, and so must be $(\tilde{Y}^L)^B$.

**Proof of (b)** Let’s compute:

\[
\tilde{g}_\lambda(\tilde{Y}^L, z)_{(q,v)} = \tilde{g}^1(\frac{1}{\lambda}(h_\lambda)_*(z^A)_B, \frac{1}{\lambda}(h_\lambda)_* z^B)_{(q,\lambda v)} + g(\pi_* \tilde{Y}^A, \pi_* (z^A))_{(q,0)}
\]

Since $\mathcal{X}$ is linearized, it is preserved under the homothetic transformations, and for every vector $x$ we have $(h_\lambda)_* x^B = ((h_\lambda)_* x)^B$. The equation above then becomes

\[
\tilde{g}_\lambda(\tilde{Y}^L, z)_{(q,v)} = \tilde{g}^1(\frac{1}{\lambda} \tilde{Y}^B, \frac{1}{\lambda}(h_\lambda)_* z^B)_{(q,\lambda v)} + g(\pi_* \tilde{Y}^A, \pi_* (z^A))_{(q,0)}
\]

Since $\tilde{Y} \in \mathcal{X}(\mathcal{F})$, it is parallel to $\mathcal{F}^2$ at the zero section, and so $\tilde{Y}^B_{(q,0)} = 0$. In particular $\lim_{\lambda \to 0} \frac{1}{\lambda} \tilde{Y}^B_{(q,\lambda v)} = \nabla_\lambda \tilde{Y}^B_{(q,v)}$, where the connection is the Euclidean connection on the fiber $\nu_q P_p$. Moreover $\lim_{\lambda \to 0} \frac{1}{\lambda}(h_\lambda)_* z^B = z^B$. Therefore taking the limit of the equation above, we get

\[
g_0(\tilde{Y}^L, z)_{(q,v)} = \lim_{\lambda \to 0} \tilde{g}^1(\frac{1}{\lambda} \tilde{Y}^B, \frac{1}{\lambda}(h_\lambda)_* z^B)_{(q,\lambda v)} + g(\pi_* \tilde{Y}^A, \pi_* (z^A))_{(q,0)}
\]

\[= \tilde{g}^1(\nabla_\lambda \tilde{Y}^B, z^B)_{(q,0)} + g(\pi_* \tilde{Y}^A, \pi_* (z^A))_{(q,0)}
\]
that \( \gamma \) flow \( \phi \) \( p \) call \( \nu \) and \( (t, i) \text{ stands the relation between the smooth structure of local and global quotient.} \)

\[ \text{infinitesimal foliation} \ F \ (\text{subgroup of Iso}(A)) \]  

\[ \text{send each leaf to itself (but not necessarily fixing each point of the fixed leaf).} \]

\[ \text{differential of the flow of } \vec{Y} \ \text{leaves and therefore } \| \phi_{\nu} \| = \| v \| \text{ for every } v \in \nu(p). \]

\[ \text{Moreover, since the flow preserves the foliation, it preserves the distance between} \]

\[ \text{fibers is given by} \ F_{\lambda}(v) = \frac{1}{\lambda} F(\lambda v). \]

\[ \text{Taking the limit as } \lambda \to 0, \text{ we obtain} \]

\[ \lim_{\lambda \to 0} F_{\lambda} = \{F_q\}_0. \]

\[ \text{In other words, the flow of } \gamma^L = \gamma^L \text{ is given by the} \]

\[ \text{differential of the flow of } \gamma^L \text{ at } 0. \]

\[ \text{Here we have used the fact that } \gamma^L \text{ converges to } \gamma^L \text{ uniformly on compact sets, see proof of item (a) Proposition \[ \ref{prop:linearization}. \]i.} \]

\[ \text{Proof of (c) We said in the paragraph above that the flow of } \vec{Y} \text{ is a linear map.} \]

\[ \text{Moreover, since the flow preserves the foliation, it preserves the distance between} \]

\[ \text{leaves and therefore } \| \phi_{\nu} (v) \| = \| v \| \text{ for every } v \in \nu(p). \]

\[ \text{Therefore it is an isometry.} \]

\[ \text{Proof of (f) From the formula at point (b) it follows that } \pi : (\text{Tub}(P_p), g_0) \to (P_p, g|_{p}) \text{ is a Riemannian submersion.} \]

\[ \text{Moreover, the holonomy maps (in the riemannian submersion sense) are flows of} \]

\[ \text{linearized vector fields always tangent to } A, \text{ and therefore they are isometries between fibers.} \]

\[ \text{This is equivalent to the} \]

\[ \text{fibers being totally geodesic and the } S-\text{tensor of the submersion } \nu(P_p) \to P_p \text{ is zero.} \]

\[ \text{Since moreover the horizontal distribution is integrable (the leaves of } \mathcal{F}^2 \text{ are the} \]

\[ \text{integral manifolds) then the O'Neill tensor is zero as well. By \[ \ref{thm:oneill} \text{ Theorem 1.4.1,} \]

\[ \text{having } S = 0, A = 0 \text{ implies that the distribution locally splits, and since } \nu(P_p) \text{ can be taken simply connected, then the metric splitting becomes global.} \]

\[ \square \]

We conclude this section discussing how the above result can be used to understand the relation between the smooth structure of local and global quotient.

Let \( \text{Iso}(\nu_p L_p, \mathcal{F}_p) \) denote the isometries of \( \nu_p L_p \) whose elements preserve the infinitesimal foliation \( \mathcal{F}_p \), i.e., that send leaves to leaves. Let us also denote \( G_p \) the subgroup of \( \text{Iso}(\nu_p L_p, \mathcal{F}_p) \) whose elements fix the leaves of the foliation \( \mathcal{F}_p \), i.e., that send each leaf to itself (but not necessarily fixing each point of the fixed leaf).

The foliation induced by \( G_p \) on \( \nu^1_p L \) will be called the homogeneous kernel of \( \mathcal{F}_p \).

Note that a SRF \((M, \mathcal{F})\) is orbit-like at a point \( p \in M \) if the homogeneous kernel of \( \nu^1_p L \) is the whole infinitesimal foliation \( \mathcal{F}_p \).

Let \( L_p \) be a closed leaf through a point \( p \), and let \( \gamma : [0, 1] \to L_p \) be a closed path in \( L_p \) with base point \( p \). We want to define a singular holonomy map \( \tau : \pi_1(L_p, p) \to \text{Iso}(\nu_p L_p, \mathcal{F}_p)/G_p \) in the following way: take a partition \( 0 = t_0 < t_1 < \ldots < t_k = 1 \in [0, 1] \), such that each \( \gamma(t_i, t_{i+1}) \) is contained in a plaque \( P_i \), and call \( p_i = \gamma(t_i) \). For each plaque, choose a linearized vector field \( \vec{X}_i \) on \( \text{Tub}(P_i) \) such that \( \gamma(t_i, t_{i+1}) \) is an integral curve for \( \vec{X}_i \). According to proposition \[ \ref{prop:linearization} \] above, the flow \( \phi_i \) of \( \vec{X}_i \) defines an isometry between the infinitesimal foliations \((\nu^1_p L_p, \nu^1_i \mathcal{F})\) and \((\nu^1_{p_{i+1}} L_p, \nu^1_{p_{i+1}} \mathcal{F})\). Finally, define

\[ \tau(\gamma, X_0, \ldots, X_{k-1}) = \phi_{k-1} \circ \ldots \circ \phi_1 \circ \phi_0 \in \text{Iso}(\nu^1_p L_p, \mathcal{F}_p). \]
Of course, this map depends on the choice of the linearized vector fields $\vec{X}_i$. Still, if $(\vec{Y}_1, \ldots, \vec{Y}_k)$ is a different choice of linearized vector fields, $\tau(\gamma, \vec{Y}_1, \ldots, \vec{Y}_k)$ differs from $\tau(\gamma, \vec{X}_1, \ldots, \vec{X}_k)$ by an element of $G_p$. In particular, it makes sense to define $\tau_\gamma$ as the projection of some $\tau(\gamma, \vec{X}_1, \ldots, \vec{X}_k)$ on $\text{Iso}(\nu_p L_p, \mathcal{F}_p)/G_p$.

Notice moreover that if $\gamma_1$ and $\gamma_2$ are homotopic, then $\tau_{\gamma_1} = \tau_{\gamma_2}$, and it is well defined a map

$$\tau : \pi_1(L_p, p) \to \text{Iso}(\nu_p L_p, \mathcal{F}_p)/G_p.$$ 

Since $G_p$ is normal in $\text{Iso}(\nu_p L_p, \mathcal{F}_p)$, we have that $\text{Iso}(\nu_p L_p, \mathcal{F}_p)/G_p$ is a group, and $\tau$ is a group homomorphism. Call $H$ the image $\tau(\pi_1(L_p, p))$. $H$ acts in a natural way on $C^\infty_b(\nu_p L_p, \mathcal{F}_p)$: in fact, given $[h] \in H$, choose a representative $h_0 \in \text{Iso}_p$ of $[h]$, and define $[h] \cdot f = h_0^* f$. Moreover we have the following “foliated slice theorem for functions”:

**Proposition A.5.** Let $(M, \mathcal{F})$ be a SRF, $L$ a closed leaf, and $U_\varepsilon$ a tubular neighborhood of radius $\varepsilon$ around $L$. Let $S_\varepsilon$ be a slice at a point $p \in L$, Let $(\nu_p L, \mathcal{F}_p)$ be the infinitesimal foliation at $p$, and let $\rho = \exp : \nu^L \to U_\varepsilon$. Then

$$\rho^* C^\infty_b(U_\varepsilon, \mathcal{F}) = C^\infty_b(\nu_p^L, \mathcal{F}_p)^H.$$

**Proof.** the inclusion $\subseteq$ is obvious. As for the inclusion $\supseteq$: take a function $f \in C^\infty_b(\nu_p^L, \mathcal{F}_p)^H$, that we can think of as a function on the slice $S_p$. We are going to construct a basic function $\tilde{f}$, defined on the whole $U_\varepsilon$ whose restriction to $S_p$ is $f$. If $q$ is another point of $L_p$, we can define $\tilde{f}$ on $S_q$ by joining $q$ and $p$ with a simple curve $\gamma$, take a vector field $X$ around $\gamma$ such that $X(\gamma(t)) = \gamma'(t)$, extend $X$ locally, to a vertical vector field $\vec{X}$, and finally take the linearized vector field $\vec{X}^L$. By the results proven above, the flow $\phi_{\vec{X}}$ takes $(S_q, g_0)$ isometrically to $(S_p, g_0)$, and one can define $\tilde{f}|_{S_q} = (\phi_{\vec{X}})_* f|_{S_p}$. Notice that, since $f$ is constant on the leaves and it is invariant under $H$, the result does not depend on the choices made and $\tilde{f}$ is well defined. Moreover, by local considerations $\tilde{f}$ is also smooth, and the proposition is proved.

\[\square\]

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