**ZN symmetric chiral Rabi model: a new N-level system**

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We present a new tractable quantum Rabi model for N-level atoms by extending the $Z_N$ symmetry of the two-state Rabi model. The Hamiltonian is $Z_N$ symmetric and allows the parameters in the level separation terms to be complex while remaining hermitian. This latter property means that the new model is chiral, which makes it differ from any existing N-state Rabi models in the literature. The $Z_N$ symmetry provides partial diagonalization of the general Hamiltonian. The exact isolated (i.e. exceptional) energies of the model have the Rabi-like form but are $N$-fold degenerate. For the three-state case ($N = 3$), we obtain three transcendental functions whose zeros give the regular (i.e. non-exceptional) energies of the model.

PACS numbers: 03.65.Ge, 02.30.Ik, 42.50.Pq

**INTRODUCTION**

Matter-light interactions are ubiquitous in nature. In modern physics, they are modeled by systems of atoms interacting with boson modes (i.e. spins coupled to harmonic oscillators). One of the best-known spin-boson systems is the phenomenological quantum Rabi model [1–3], which continues to be a subject of significant interest [4–12]. This model describes the interaction of a two-level atom with a cavity mode of quantized electromagnetic field, i.e. a single spin-1/2 particle coupled to a harmonic oscillator. Due to the simplicity of its Hamiltonian, the Rabi model has served as the theoretical basis for understanding the interactions between matter and light, and has found a variety of applications ranging from quantum optics, solid state semiconductor systems, molecular physics to quantum information. With experimental techniques now available to access ultra-strong atom-cavity coupling regimes [17], there is also much ongoing interest in experimental realizations of the Rabi interactions in both circuit and cavity quantum electrodynamics (QED) [18–21].

The quantum Rabi model is the simplest spin-boson system without the rotating wave approximation (RWA). To understand more sophisticated spin-boson interactions, e.g. multi-level atom-cavity interaction [25–27], there is a need for extensions of the two-level Rabi model. In this regard, let us mention three research directions which have attracted significant attention. One is the study of the Dicke model which couples $N$ two-level systems to a single radiation mode and is relevant to experimental realization and applications in quantum computing. There is a vast literature on the Dicke model. Analytic solutions for the $N = 2, 3$ Dicke models have recently been studied in [28–31]. Another one concerns models of a two-level atom interacting with multi harmonic modes or with a higher-order harmonic generation. Examples include the two-mode [31] and 2-photon [32] Rabi models recently solved analytically in [11, 33]. These models can be experimentally realized in circuit QED systems [19] and have established applications in e.g. Rubidium atoms [34] and quantum dots [35, 36]. The third direction is to consider multi-state atom-cavity interactions, i.e. multi-level atoms coupled to harmonic boson modes. Most previous multi-level extensions [37, 39–41] are neither tractable nor applicable to atom-cavity systems. Recently the author in [42] proposed a tractable quantum Rabi model for $N$-state atoms.

In this Letter we introduce a different, tractable quantum Rabi model for $N$-level atoms. The Hamiltonian of the new model is $Z_N$ symmetric, extending the $Z_2$ symmetry of the two-state model. One of the unique features to our model is that it allows the parameters in the level-splitting terms ($\alpha_m$ in (2) below) to be complex while keeping the Hamiltonian hermitian. It is therefore a chiral system [43] and is referred to as $Z_N$ symmetric $N$-state chiral Rabi model. The $Z_N$ symmetry provides a partial diagonalization of the Hamiltonian. It is found that the exact isolated (i.e. exceptional) energies of the model have the Rabi-like form but are $N$-fold degenerate. They correspond to polynomial solutions of the Schrödinger equation and appear when the model parameters satisfy certain constraints. For the three-state case ($N = 3$), we analytically determine three transcendental functions whose zeros give the regular energies of the system. Our results pave the way for applications to multi-level atom-cavity experiments.

**ZN SYMMETRIC RABI HAMILTONIAN**

The two-state Rabi Hamiltonian is $Z_2$ symmetric. So the most natural $N$-state generalization of the Rabi model would be given by a Hamiltonian with $Z_N$ symmetry. We can proceed in the following intuitive and mathematically rigorous way. Similar to the Rabi case where a two-level atom is modeled by spins with two states (Pauli matrices $\sigma_z$ and $\sigma_x$), we model an $N$-level atom by “spins” with $N$ states. Then the Hilbert space of the $N$-state atom is the $N$-dimensional vector space $\mathbb{C}^N$. Let $Z$ and $X$ be the basic operators which generalize the
Paul matrices $\sigma_z$ and $\sigma_x$ to $\mathbb{C}^N$, respectively. Instead of anti-commutation relations, these operators satisfy
\[
Z^N = X^N = 1, \quad Z^\dagger = Z^{N-1}, \quad X^\dagger = X^{N-1},
\]
\[
ZX = \omega XZ, \quad \omega = e^{2\pi i/N}.
\] (1)

It is useful to keep in mind some explicit representations of these operators. Diagonalizing the the operator $Z$ gives $Z = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{N-1})$ and $X_{l,m} = \delta_{l,m+1} \, (\text{mod } N)$. On the other hand, from (1) we have $X^\dagger Z = \omega ZX^\dagger$. Thus the representation in which $X$ is diagonal is given by $X^\dagger = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{N-1})$ and $Z_{l,m} = \delta_{l,m+1} \, (\text{mod } N)$. This latter representation is useful in what follows.

Then the most natural and mathematically manageable $N$-state generalization of the two-state Rabi model can be obtained by replacing the Pauli matrices in the latter model with the “spins” with $N$ states. It is convenient to label the $N$-states by $1, \omega, \omega^2, \ldots, \omega^{N-1}$. We thus arrive at the following Hamiltonian of a $N$-level atom interacting with a boson mode,

\[
H_N = \Omega b^\dagger b + \Delta \sum_{m=1}^{N-1} \alpha_m Z^m + \lambda (X^\dagger b + Xb),
\] (2)

where $\Omega, \Delta, \lambda$ are real parameters, and the couplings $\alpha_m$ are complex and obey $\alpha_m^* = \alpha_{N-m}$ in order for the Hamiltonian to be hermitian. The above $H_N$ is the simplest possible, hermitian and $N$-state generalization of the two-state Rabi Hamiltonian. For $N = 2$, the Hamiltonian (2) simplifies to the well-studied two-state Rabi model,

\[
H_2 = \Omega b^\dagger b + \Delta \sigma_z + \lambda \sigma_x (b^\dagger + b). \quad (3)
\]

Let us state clearly that our model differs from the one proposed in [42] in several important aspects. Firstly, the atom-cavity interaction term of the model in (2) (see eq (6) of that paper) is, in our notation, $\lambda(X^\dagger b + Xb)$, while it is $\lambda(X^\dagger b + Xb)$ in our model. Although both of them reduce to the same Rabi model interaction term, for $N \geq 3$ they represent quite different atom-cavity interactions since $X$ is non-hermitian. It is interesting that the Rabi model interaction has two different $N$-state generations. Secondly, perhaps more importantly, the parameters $\alpha_m$ in our model (2) are in general complex (while the level-splitting parameters in [42] are real). Allowing these parameters to be complex may result in interesting behaviour not possible in the Rabi model. To understand this, consider the three-state system $N = 3$ and let $\alpha_1 = \alpha_2 = e^{i\phi}$, where $\phi$ is a real parameter. Then we obtain

\[
H_3 = \Omega \lambda b^\dagger b + \Delta (e^{i\phi} Z + e^{-i\phi} Z^\dagger) + \lambda (X^\dagger b^\dagger + Xb).
\]

(4)

There are thus three physical important parameters in $H_3$: $\Omega/\lambda, \Delta/\lambda$ and $\phi$. When the phase $\phi = 0$ so that $\alpha_1$ and $\alpha_2$ are real, the Hamiltonian is invariant if $Z$ and $Z^\dagger$ are interchanged. For $\phi \neq 0$, the Hamiltonian is no longer invariant if $Z$ is interchanged with $Z^\dagger$. This means that spatial parity symmetry in any direction is broken. For this reason, our model with non-zero $\phi$ is chiral [43].

Working in the representation in which $X$ is diagonal,

\[
X^\dagger = \text{diag}(1, \omega, \omega^2), \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \omega = e^{2\pi i/3},
\]

we can express (4) in the explicit matrix form

\[
H_3 = \begin{pmatrix} \Omega b^\dagger b + \lambda (b^\dagger + b) & \Delta e^{-i\phi} & \Delta e^{i\phi} \\ \Delta e^{i\phi} & \Omega b^\dagger b + \lambda (\omega b^\dagger + \omega^2 b) & \Delta e^{-i\phi} \\ \Delta e^{-i\phi} & \Delta e^{i\phi} & \Omega b^\dagger b + \lambda (\omega^2 b^\dagger + \omega b) \end{pmatrix}. \quad (5)
\]

The Hamiltonian (2) is invariant under the cyclic group $Z_N$ and the corresponding symmetry generator is

\[
\Pi = Z e^{i\frac{2\pi b^\dagger b}{N}}, \quad (6)
\]

which is an operator acting in the Hilbert space $\mathbb{C}^N \otimes \mathcal{H}_b$.

Throughout $\mathcal{H}_b$ denotes the Hilbert space of the boson degree of freedoms. Indeed it can be shown that $\Pi$ satisfies $\Pi^N = 1$ and commutes with the Hamiltonian, $[H_N, \Pi] = 0$. Thus $\mathbb{C}^N \otimes \mathcal{H}_b$ splits into $N$ invariant subspaces $|\omega^{k-1}\rangle \otimes \mathcal{H}_b$, $k = 1, 2, \ldots, N$ labeled by the eigenvalues $1, \omega, \omega^2, \ldots, \omega^{N-1}$ of $Z$. This invariance can be used to partially diagonalize $H_N$ via the generalized Fulton-Gouterman transformation $U_N$ [43],

\[
U_N = \frac{1}{\sqrt{N}} \sum_{\gamma=1}^{N} \sum_{r=1}^{N} \omega^{(r-1)(\gamma-1)} \langle r | \langle \gamma | R_r, \quad (7)
\]

where $R_r = R^{r-1}$ with $R = e^{i\frac{2\pi b^\dagger b}{N}}$. For example, when $N = 3$, we have from (7)

\[
U_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ R & \omega R & \omega^2 R \\ R^2 & \omega^2 R^2 & \omega R^2 \end{pmatrix}, \quad R = e^{i\frac{2\pi b^\dagger b}{3}}. \quad (8)
\]
Then it can be checked that
\[
U_3^1 H_3 U_3 = \begin{pmatrix} H^{(0)} & 0 & 0 \\ 0 & H^{(1)} & 0 \\ 0 & 0 & H^{(2)} \end{pmatrix},
\]
(9)
where
\[
H^{(s)} = \Omega b^\dagger b + \lambda (b^\dagger + b) + \Delta e^{-i\phi} \omega^s R + \Delta e^{i\phi} \omega^{2s} R^2
\]
(10)
act in three mutually orthogonal subspaces \(\mathcal{H}_b \otimes |\omega^s\rangle\) with fixed eigenvalue \(\omega^s\) of \(Z\). Here and throughout \(s = 0, 1, 2\).

**EXACT ISOLATED ENERGIES**

We represent the continuous boson degree of freedom \(b^\dagger, b\) as differential operators in the Bargmann-Hilbert space \(B\) of entire functions which is isomorphic to \(\mathcal{H}_b \otimes \mathbb{C}^N\). The scalar product of any two elements \(f(z), g(z)\) in the Bargmann-Hilbert space is given by
\[
(f, g) = \int \overline{f(z)} g(z) d\mu(z), \quad d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dx dy.
\]

An orthonormal basis of the Bargmann-Hilbert space is provided by the monomials \(\{z^n/\sqrt{n!}\}\). So if \(f(z) = \sum_{n=0}^\infty c_n z^n\), then \(||f||^2 = \sum_{n=0}^\infty |c_n|^2 n!\) and \(f(z)\) is entire iff this sum converges. With respect to the orthonormal basis, we have the Bargmann realization \(b^\dagger = z\) and \(b = \frac{d}{dz}\).

In the Bargmann representation the eigenvectors of \[\complement\] can be written as \(N\)-component, \(z\)-dependent vectors \(\psi(z) = (\psi_1(z), \ldots, \psi_N(z))^t\), satisfying the time-independent matrix Schrödinger equation
\[
\left[ (\Omega z + \lambda X) \frac{d}{dz} + \lambda z X^\dagger + \Delta \sum_{m=1}^{N-1} \alpha_m Z^m \right] \psi(z) = E \psi(z).
\]
(11)

Working in the representation in which \(X\) is diagonal, we write this equation in terms of the components \(\psi_k(z)\),
\[
(\Omega z + \lambda(\omega^s)^{k-1}) \frac{d\psi_k}{dz} = (E - \lambda(\omega^s)^{k-1}) \psi_k
\]
\[
-\Delta \sum_{m=1}^{N-1} \alpha_m \sum_{l=1}^N (Z^m)_{kl} \psi_l,
\]
(12)
where \(k = 1, 2, \ldots, N\). This is a system of \(N\) coupled differential equations of Fuchsian type. In general it has \(N\) independent solutions. The required solution must be analytic in the whole complex plane, i.e. entire, in order for \(E\) to belong to the spectrum of the system. The singular points of the system are at \(z = -\frac{\lambda}{\Omega}(\omega^s)^{k-1}, \ k = 1, 2, \ldots, N\). Assuming that \(\psi(z)\) behaves like \((z + \frac{\lambda}{\Omega}(\omega^s)^{k-1})^\rho\) in the vicinity of each \(z = -\frac{\lambda}{\Omega}(\omega^s)^{k-1}\), we are lead to the following indicial equation:
\[
\rho - \frac{E}{\Omega} + \frac{\lambda^2}{\Omega^2} = 0
\]
(13)
for all \(k = 1, 2, \ldots, N\), where we have used the fact that \((Z^m)_{kk} = 0\) for \(m = 1, 2, \ldots, N - 1\). So for all \(N\) independent solutions \(\psi_k(z)\) to be analytic at the singular points \(z = -\frac{\lambda}{\Omega}(\omega^s)^{k-1}\), we must have
\[
E = \Omega \left( N - \frac{\lambda^2}{\Omega^2} \right), \quad N = 0, 1, 2, \ldots.
\]
(14)
This gives the exact isolated energies of the \(N\)-state Rabi model. These energies have the Rabi-like form but are \(N\)-fold degenerate. They correspond to polynomial solutions (wavefunctions) of the Schrödinger equation and appear when the model parameters fulfill certain constraints.

**REGULAR ENERGIES**

Here for the purpose of illustration, we will consider the three-state (\(N = 3\)) case. Generalization to the \(N\)-state model is straightforward. Throughout this section \(\omega = e^{i\pi/3}\).

In the Bargmann representation, \(R\) in [3] can be realized as \(R = e^{i\frac{2\pi}{3}} \frac{d}{dz}\), which acts on elements \(f(z)\) of \(B\) as \((R f)(z) = f(\omega z)\). Thus we can express the Hamiltonians \(H^{(s)}, s = 0, 1, 2\), as the differential operators in \(B\)
\[
H^{(s)} = (\Omega z + \lambda) \frac{d}{dz} + \lambda z + \Delta e^{-i\phi} \omega^{s+1} \frac{d}{dz} + \Delta e^{i\phi} \omega^{2s+2} \frac{d}{dz}.
\]
(15)

The corresponding time-independent Schrödinger equations are
\[
\left[ (\Omega z + \lambda) \frac{d}{dz} + \Delta e^{-i\phi} \omega^{s+1} \frac{d}{dz} + \Delta e^{i\phi} \omega^{2s+2} \frac{d}{dz} + \lambda z - E^{(s)} \right] \psi^{(s)}(z) = 0
\]
(16)
for \(s = 0, 1, 2\). Here we have written \(E^{(s)}\) since in general the spectra of \(H^{(s)}\) are not the same. Solutions to these differential equations must be analytic in the whole complex plane if \(E^{(0)}, E^{(1)}, E^{(2)}\) belong to the spectra of \(H^{(0)}, H^{(1)}, H^{(2)}\), respectively. In other words, we are seeking solutions of the form
\[
\psi^{(s)}(z) = \sum_{n=0}^\infty K_n^{(s)}(E^{(s)}) z^n,
\]
(17)
which converge in the whole complex plane, i.e. are entire.

Substituting (17) into (16), we obtain the 3-term recurrence relations
\[
K^{(s)}_1 + A^{(s)}_0 K^{(s)}_0 = 0,
\]
\[
K^{(s)}_{n+1} + A^{(s)}_n K^{(s)}_n + B^{(s)}_n K^{(s)}_{n-1} = 0, \quad n \geq 1,
\]
(18)
where

\[ A_n^{(s)} = \frac{n\Omega + \Delta e^{-i\phi} \omega^{n+s} + \Delta e^{i\phi} \omega^{2(n+s)} - E^{(s)}}{\lambda(n+1)}, \]
\[ B_n^{(s)} = \frac{1}{n+1}. \]  \hspace{1cm} (19)

The coefficients \( A_n^{(s)} \), \( B_n^{(s)} \) have the asymptotic behavior when \( n \to \infty \),

\[ A_n^{(s)} \sim \frac{\Omega}{\lambda}, \quad B_n \sim n^{-1}. \]  \hspace{1cm} (20)

Applying the Perron-Kreuser theorem (i.e. Theorem 2.3 of [43]), it follows that for each \( s = 0,1,2 \), the truly 3-term part (i.e. the \( n \geq 1 \) part) of the recurrence relations [18] has two linearly independent solutions \( K_n^{(s)} \), \( K_{n,1}^{(s)} \) for which, when \( n \to \infty \)

\[ \frac{K_{n+1}^{(s)}}{K_n^{(s)}} \sim \frac{-\Omega}{\lambda}, \quad \frac{K_{n+2}^{(s)}}{K_n^{(s)}} \sim -\frac{\lambda}{\Omega} n^{-1} \]  \hspace{1cm} (21)

So \( K_n^{(s)\min} \equiv K_n^{(s)} \) for each \( s \) value is a minimal solution of the truly 3-term part of [18]. The corresponding infinite power series solutions, generated by substituting \( K_n^{(s)\min} \) for the \( K_n^{(s)} \)'s in [17], converge in the whole complex plane, i.e. they are entire.

By the Pincherle theorem, i.e. Theorem 1.1 of [43], the ratios of successive elements of the minimal solution sequences \( K_n^{(s)\min} \), \( s = 0,1,2 \), are expressible in terms of infinite continued fractions. Proceeding in the direction of increasing \( n \), we find

\[ S_n^{(s)} = \frac{K_{n+1}^{(s)\min}}{K_n^{(s)\min}} = -\frac{B_{n+1}^{(s)}}{A_n^{(s)} - A_{n+2}^{(s)} - A_{n+3}^{(s)} - \cdots}, \]  \hspace{1cm} (22)

which for \( n = 0 \) gives

\[ S_0^{(s)} = \frac{K_{1}^{(s)\min}}{K_0^{(s)\min}} = -\frac{B_1^{(s)}}{A_1^{(s)} - A_2^{(s)} - A_3^{(s)} - \cdots}. \]  \hspace{1cm} (23)

Note that the ratio \( S_0^{(s)} = \frac{K_{1}^{(s)\min}}{K_0^{(s)\min}} \) involve \( K_0^{(s)\min} \), although the above continued fraction expressions are obtained from the truly 3-term part of [18], i.e. the recurrence [18] for \( n \geq 1 \). However, for single-ended sequences such as those appearing in the infinite power series expansions [17], the ratios \( S_0^{(s)} = \frac{K_{1}^{(s)\min}}{K_0^{(s)\min}} \) of the first two terms of minimal solutions are unambiguously fixed by the \( n = 0 \) part (i.e. the first equation) of the recurrence [18], namely,

\[ S_0^{(s)} = -A_0^{(s)} = -\frac{1}{\lambda} \left[ \Delta \left( \omega^s e^{-i\phi} + \omega^{2s} e^{i\phi} \right) - E^{(s)} \right]. \]  \hspace{1cm} (24)

In general, the \( S_0^{(s)} \) computed from [23] are not the same as those from [24] (i.e. [23] and [24] are not both satisfied) for arbitrary values of recurrence coefficients \( A_n^{(s)} \) and \( B_n^{(s)} \). As a result, general solutions to the recurrence [18] are dominant and are usually generated by simple forward recursion from a given value of \( K_0^{(s)} \). Physical meaningful solutions are those that are entire in the Bargmann-Hilbert spaces. They can be obtained if \( E^{(s)} \) can be adjusted so that equations [23] and [24] are both satisfied for each \( s \). Then the resulting solution sequences \( K_n^{(s)}(E^{(s)}) \) will be purely minimal and the power series expansions [17] will converge in the whole complex plane.

Therefore, if we define the functions \( F^{(s)}(E^{(s)}) = S_0^{(s)} + A_0^{(s)} \) with \( S_0^{(s)} \) given by the continued fraction in [23], then the zeros of \( F^{(s)}(E^{(s)}) \) correspond to the points in the parameter space where the condition [23] is satisfied. In other words, \( F^{(s)}(E^{(s)}) = 0 \) yield the eigenvalue equations, which may be solved for \( E^{(s)} \) by standard nonlinear root-search techniques [46]. Only for the denumerable infinite values of \( E^{(s)} \) which are the roots of \( F^{(s)}(E^{(s)}) = 0 \), do we get entire wavefunction solutions of the Schrödinger differential equations.

**CONCLUSIONS AND DISCUSSIONS**

This work introduces a new \( \mathbb{Z}_N \) symmetric \( N \)-state extension of the two-state Rabi model. A unique feature to the model is that it allows parameters \( \alpha_m \) to be complex without violating the hermiticity of the Hamiltonian. This is not possible in the Rabi model since \( \sigma^2 = \sigma_z \). Our model is a \( N \)-state Rabi model with complex level separation terms and is thus a chiral system [15]. This is one of the main differences between our model and the existing \( N \)-state models in the literature. The \( \mathbb{Z}_N \) symmetry can be used to partially diagonalize the Hamiltonian of the model. Analytic solutions of the model has been investigated in the Bargmann-Hilbert space. It is found that the exact isolated energies have the Rabi-like form but are \( N \)-fold degenerate. They correspond to polynomial wavefunctions and special model parameters. The regular energies are given by zeros of suitable transcendental functions, similar to the Rabi case. This is shown for the three-state \( N = 3 \) system.

From the 3-term recurrence relations [18], it is not difficult to show that the wavefunction expansion coefficients in [17] are related to orthogonal polynomials [15]. Thus it is expected that the regular energies of the \( N \)-state model can be determined as the polynomial zeros by a procedure similar to that in [10, 12].

The new \( N \)-state Rabi Hamiltonian presented in this work is the simplest possible, hermitian and \( \mathbb{Z}_N \) symmetric generalization of the two-state Rabi model, using a minimal number of system parameters. More sophisticated extensions are possible. For example, the Hamil-
tonian of another $N$-state generalization which preserves the $\mathbb{Z}_N$ symmetry has the form

$$\hat{H}_N = \Omega b^\dagger b + \Delta \sum_{m=1}^{N-1} \alpha_m Z^m + \lambda \sum_{m=1}^{N-1} \beta_m [X^m (b^\dagger)^{N-m} + (X^\dagger)^m b^{N-m}]$$

(25)

where $\Omega, \Delta, \lambda$ are real and the couplings $\alpha_m, \beta_m$ are complex and satisfy $\alpha_m^\dagger = \alpha_{N-m}$ and $\beta_m^\dagger = \beta_{N-m}$ in order for the Hamiltonian to be hermitian. It can be checked that this Hamiltonian commutes with the $\mathbb{Z}_N$ operator $\Pi$. For $N = 2$, (25) also simplifies to the two-state Rabi model Hamiltonian. However, for $N \geq 3$ the Hamiltonian contains non-linear terms of spin and boson operators. A detailed analysis of this extension is interesting but beyond the scope of this paper.

We would like to thank Victor Albert for critical comments and email conversations, and Daniel Braak for comments and useful suggestions. We also thank Jacques H.H. Perk for pointing out a misprint and some references on the chiral Potts model. This work was partially supported by the Australian Research Council through Discovery-Projects grants DP110103434 and DP140101492.

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