An embedding theorem for weighted Sobolev classes on a John domain: case of weights that are functions of a distance to a certain $h$-set

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (an open connected set), and let $g, v : \Omega \to \mathbb{R}_+$ be measurable functions. For each measurable vector-valued function $\varphi : \Omega \to \mathbb{R}^m$, $\varphi = (\varphi_k)_{1 \leq k \leq m}$, and for each $p \in [1, \infty]$ we put

$$\|\varphi\|_{L^p(\Omega)} = \left\| \max_{1 \leq k \leq m} |\varphi_k| \right\|_p.$$ 

Let $\vec{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d$, $|\vec{\beta}| = \beta_1 + \ldots + \beta_d$. For any distribution $f$ defined on $\Omega$ we write $\nabla^f f = \left( \frac{\partial f}{\partial x^\beta} \right)_{|\beta|=r}$ (here partial derivatives are taken in the sense of distributions), and denote by $m_r$ the number of components of the vector-valued distribution $\nabla^f$. Set

$$W_{p,g}^r(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \exists \varphi : \Omega \to \mathbb{R}^{m_r} : \|\varphi\|_{L^p(\Omega)} \leq 1, \nabla^f f = g \cdot \varphi \}$$

(we denote the corresponding function $\varphi$ by $\nabla^f f_g$),

$$\|f\|_{L^q,v(\Omega)} = \|f\|_{L^q,v} = \|fv\|_{L^q(\Omega)}, \quad L_{q,v}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \|f\|_{q,v} < \infty \}.$$

We call the set $W_{p,g}^r(\Omega)$ a weighted Sobolev class.

For properties of weighted Sobolev spaces and their generalizations, see the books [15, 30, 55, 57, 58, 60] and the survey paper [33]. Sufficient conditions for boundedness and compactness of embeddings of weighted Sobolev spaces into weighted $L_q$-spaces were obtained by Kudryavtsev [32], Nečas [47], Kufner [34, 36], Yakovlev [63], Triebel [55], Lizorkin and Otelbaev [43], Gurka and Opic [22, 24], Besov [5, 8], Antoci [4], Gol’dshtein and Ukhlov [21], and other authors. Notice that in these papers weighted

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Sobolev classes were defined as $W_{p,g}^r(\Omega) \cap L_{p,w}(\Omega)$ for some weight $w$, or as $\bigcap_{i=0}^r W_{p,g_i}^i(\Omega)$ for some different weight functions $g_i$.

For a Lipschitz domain $\Omega$, a $k$-dimensional manifold $\Gamma \subset \Omega$, and for weights depending only on distance from $x$ to $\Gamma$, the following results were obtained. The case $r = 1$, $p = q$ was considered in papers of Nečas [17] (the case of power weights and $\Gamma = \partial \Omega$), Kufner [34] (weights are powers of distance from a fixed point), Yakovlev [63] (weights depend on distance to $k$-dimensional manifold), Kadlec and Kufner [30, 31] (here weights are powers with a logarithmic factor, $\Gamma = \partial \Omega$), Kufner [35] (here weights are arbitrary functions of distance from $\partial \Omega$). For $p = q$, $r \in \mathbb{N}$, $\Gamma = \partial \Omega$ and for power type weights, the embedding theorem was obtained by El Kolli [16]. By using Banach space interpolation, Triebel [54] extended this result to the case $p \leq q$. For $p = q$, $r = 1$, a $k$-dimensional manifold $\Gamma$ and general weights Kufner and Opic [37] obtained some sufficient conditions for compactness of embeddings. For $p > q$, $r \in \mathbb{N}$, for an arbitrary $k$-dimensional manifold $\Gamma$ and power type weights the criterion of the embedding was obtained in [27–29]. In addition, in [29] for $r = 1$ the criterion was obtained for arbitrary functions depending on distance from the manifold $\Gamma$.

Notice that for $p \geq q$ in the proof of embedding theorems two-weighted Hardy-type inequalities were applied.

In [22] sufficient conditions for the embedding were obtained for $r = 1$ and general weights. The norm in the weighted Sobolev space was defined by $\|f\|_{g,w} = \left\| \sum f \right\|_{L_p(\Omega)} + \|wf\|_{L_p(\Omega)}$. The idea of the proof was the following. First the Besikovitch covering of $\Omega$ was constructed, then for each ball of this covering the Sobolev embedding theorem was applied. After that the obtained estimates were summarized. Here it was essential to use the second weight $w$, which satisfied rather tight restrictions. If the boundary $\partial \Omega$ is Lipschitz and weight functions are powers of distance from $\partial \Omega$, then it is possible to take more weak restrictions on $w$. To this end, the other method of proof is used (employing the Hardy inequality). In [23] embedding theorems were obtained for a Hölder domain $\Omega$ and power type weights depending on distance from $\partial \Omega$.

It is also worth noting the paper [38], where the result on embedding of $W_{p,g}^1(\Omega)$ into $L_{p,v}(\Omega)$ was obtained for $r = 1$, $p = q$ and weights that are powers of the distance from the irregular boundary of $\partial \Omega$.

In the present paper, we consider a John domain $\Omega$, an $h$-set $\Gamma \subset \partial \Omega$ and weight functions depending on distance from $\Gamma$ (their form will be written below).

Let $X$, $Y$ be sets, $f_1$, $f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \lesssim f_2(x, y)$ (or $f_2(x, y) \gtrsim f_1(x, y)$) if, for any $y \in Y$, there exists $c(y) > 0$ such that $f_1(x, y) \leq c(y)f_2(x, y)$ for each $x \in X$; $f_1(x, y) \asymp f_2(x, y)$ if $f_1(x, y) \lesssim f_2(x, y)$ and $f_2(x, y) \lesssim f_1(x, y)$.

For $x \in \mathbb{R}^d$ and $\alpha > 0$ we shall denote by $B_a(x)$ the closed Euclidean ball of radius $a$ in $\mathbb{R}^d$ centered at the point $x$. 

2
Let \(| \cdot |\) be an arbitrary norm on \(\mathbb{R}^d\), and let \(E, E' \subset \mathbb{R}^d, x \in \mathbb{R}^d\). We set
\[
\text{diam}_{|\cdot|} E = \sup \{|y - z| : y, z \in E\}, \quad \text{dist}_{|\cdot|} (x, E) = \inf \{|x - y| : y \in E\},
\]
\[
\text{dist}_{|\cdot|} (E', E) = \inf \{|x - y| : x \in E, y \in E'\}.
\]

**Definition 1.** Let \(\Omega \subset \mathbb{R}^d\) be a bounded domain, and let \(a > 0\). We say that \(\Omega \in \text{FC}(a)\) if there exists a point \(x_* \in \Omega\) such that, for any \(x \in \Omega\), there exists a curve \(\gamma_x : [0, T(x)] \to \Omega\) with the following properties:
1. \(\gamma_x \in AC[0, T(x)], |\dot{\gamma}_x| = 1\) a.e.,
2. \(\gamma_x(0) = x, \gamma_x(T(x)) = x_*\),
3. \(B_{\text{at}}(\gamma_x(t)) \subset \Omega\) for any \(t \in [0, T(x)]\).

**Definition 2.** We say that \(\Omega\) satisfies the John condition (and call \(\Omega\) a John domain) if \(\Omega \in \text{FC}(a)\) for some \(a > 0\).

For a bounded domain, the John condition is equivalent to the flexible cone condition (see the definition in [9]). Reshetnyak in the papers [50, 51] constructed the integral representation for functions defined on a John domain \(\Omega\) in terms of their derivatives of order \(r\). This integral representation yields that in the case \(r = \left(\frac{1}{p} - \frac{1}{d}\right)_+ > 0\) the class \(W^r_* (\Omega)\) is compactly embedded in the space \(L_q(\Omega)\) (i.e., the conditions of the compact embedding are the same as for \(\Omega = [0, 1]^d\)).

**Remark 1.** If \(\Omega \in \text{FC}(a)\) and a point \(x_*\) is such as in Definition 1, then
\[
\text{diam}_{|\cdot|} \Omega \lesssim \text{dist}_{|\cdot|} (x_*, \partial \Omega).
\]  \hspace{1cm} (1)

Denote by \(\mathbb{H}\) the set of all non-decreasing positive functions defined on \((0, 1]\). We introduce the concept of \(h\)-set according to [10].

**Definition 3.** Let \(\Gamma \subset \mathbb{R}^d\) be a compact set, and let \(h \in \mathbb{H}\). We say that \(\Gamma\) is an \(h\)-set if there exists a finite measure \(\mu\) on \(\mathbb{R}^d\) such that \(\text{supp} \mu = \Gamma\) and \(\mu(B_t(x)) \asymp h(t)\) for each \(x \in \Gamma, t \in (0, 1]\).

Notice that the measure \(\mu\) is non-negative.

The concept of \(h\)-sets for functions \(h\) of a special type appeared earlier (see papers of Edmunds, Triebel and Moura [13, 14, 45, 50]). In these and some other papers (see, for example, [11, 12, 48, 49, 59]) properties of the operator \(\text{tr}|_{\Gamma}\) in Besov and Triebel–Lizorkin spaces and its composition with the operator \((\Delta)^{-1}\) were studied. Here \(\text{tr}|_{\Gamma}\) is the operator of restriction on the \(h\)-set \(\Gamma\). In [25] Besov spaces with Muckenhoupt weights were studied; weight functions depending on the distance from a certain \(h\)-set were considered as examples.

3
In the sequel we suppose that
\[ h(t) = t^\theta \Lambda(t), \quad 0 \leq \theta < d, \] (2)
where \( \Lambda : (0, +\infty) \to (0, +\infty) \) is an absolutely continuous function such that
\[ \frac{t\Lambda'(t)}{\Lambda(t)} \to 0. \] (3)

Let \( \Omega \in \mathbf{FC}(a) \) be a bounded domain, and let \( \Gamma \subset \partial \Omega \) be an \( h \)-set. In the sequel for convenience we suppose that \( \Omega \subset [-\frac{1}{2}, \frac{1}{2}]^d \) (the general case can be reduced to this case). Let \( 1 < p \leq \infty, 1 \leq q < \infty, r \in \mathbb{N}, \delta := r + \frac{d}{q} - dp > 0, \)
\[ g(x) = \varphi_g(\text{dist}_i(x, \Gamma)), \quad v(x) = \varphi_v(\text{dist}_i(x, \Gamma)), \] (4)
with absolutely continuous functions \( \Psi_g, \Psi_v \) such that
\[ \frac{t\Psi'_g(t)}{\Psi_g(t)} \to 0, \quad \frac{t\Psi'_v(t)}{\Psi_v(t)} \to 0; \] (5)
in addition, we suppose that
\[ -\beta_v q + d - \theta > 0. \] (6)

Also we assume that
\[ \text{a) } \beta_g + \beta_v < \delta - \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+, \quad \text{or b) } \beta_g + \beta_v = \delta - \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+. \] (7)

In the case b) we suppose that
\[ \Lambda(t) = |\log t|^\gamma \tau(|\log t|), \quad \Psi_g(t) = |\log t|^{-\alpha_g} \rho_g(|\log t|), \quad \Psi_v(t) = |\log t|^{-\alpha_v} \rho_v(|\log t|), \] (8)
functions \( \rho_g, \rho_v, \tau \) are absolutely continuous,
\[ \lim_{y \to +\infty} \frac{y\tau'(y)}{\tau(y)} = \lim_{y \to +\infty} \frac{y\rho'_g(y)}{\rho_g(y)} = \lim_{y \to +\infty} \frac{y\rho'_v(y)}{\rho_v(y)} = 0, \] (9)
\[ \gamma < 0 \text{ and } \alpha := \alpha_g + \alpha_v > (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right)_+. \] (10)

It is easy to show that the functions \( \Lambda, \Psi_g \) and \( \Psi_v \) satisfy (3) and (5).

**Remark 2.** If functions \( \Psi_g \) and \( \Psi_v \) (respectively \( \rho_g \) and \( \rho_v \)) satisfy (3) (respectively (4)), then their product and each degree of these functions satisfies the similar condition.
Denote
\[ \beta = \beta_g + \beta_v, \quad \rho(y) = \rho_g(y)\rho_v(y), \quad \Psi(y) = \Psi_g(y)\Psi_v(y), \]
\[ Z = (r, d, p, q, \beta_g, \beta_v, \theta, \Lambda, \Psi_g, \Psi_v, a). \]

Let \( P_{r-1}(\mathbb{R}^d) \) be the space of polynomials on \( \mathbb{R}^d \) of degree not exceeding \( r - 1 \). For a measurable set \( E \subset \mathbb{R}^d \), we put \( P_{r-1}(E) = \{ f \in P_{r-1}(\mathbb{R}^d) \} \).

**Theorem 1.** For any function \( f \in \text{span} \, W^r_{p,g}(\Omega) \) there exists a polynomial \( Pf \in P_{r-1}(\Omega) \) such that
\[ \| f - Pf \|_{L_{q,v}(\Omega)} \lesssim \| \nabla^r f \|_{L_p(\Omega)}. \]

Here the mapping \( f \mapsto Pf \) can be extended to a linear continuous operator \( P : L_{q,v}(\Omega) \to P_{r-1}(\Omega) \).

Later we shall give a more general formulation of this theorem. It can be used in problems on estimating of approximation of the class \( W^r_{p,g}(\Omega) \) by piecewise polynomial functions in the space \( L_{q,v}(\Omega) \) and in problems on estimating of \( n \)-widths.

We may assume that the norm on \( \mathbb{R}^d \) is given by
\[ |(x_1, \ldots, x_d)| = \max_{1 \leq i \leq d} |x_i|. \]

The paper is organized as follows. In Sections 2 and 3, we give necessary notations and formulate the results which will be required in the sequel. In Section 4, we describe the domain \( \Omega \) in terms of a tree \( \mathcal{T} \) (see [61]) and construct a special partition of this tree. In Section 5, the discrete weighted Hardy-type inequality on a combinatorial tree is obtained for \( p = q \). If the tree is regular, i.e., the number of vertices that follow the given vertex depends only on the distance between this vertex and the root of the tree, then we employ some convexity arguments and reduce the problem to the proof of a Hardy-type inequality for sequences. The tree which was constructed in Section 4 is not regular in general; however, it satisfies some more weak condition of regularity. For such trees it is possible to reduce the problem to the case of regular trees. To this end, a discrete analogue for theorem of Evans – Harris – Pick [20] is proved. At this step, some quantity \( B_D \) emerges; it is defined for subtrees \( D \) and can be calculated recursively. Under some conditions on weights, we prove that \( B_D \) can be estimated by some more simple quantity \( S_D \). Then for any subtree \( D \) we construct a subtree \( \hat{D} \) in some regular tree \( \hat{A} \), such that \( S_D \) can be estimated from above by \( S_{\hat{D}} \). In Section 6, the discrete Hardy-type inequality on a tree is proved for \( p \neq q \). To this end, the problem is reduced to consider the cases \( p = q \) and \( p = \infty \); here the Hölder inequality is applied. In Section 7, the embedding theorem is proved. The problem is reduced to considering the case \( r = d \) and employing the discrete Hardy-type inequality on a tree.

Embedding theorems and related results for function classes on metric and combinatorial trees were studied by different authors. Naimark and Solomyak [46].
obtained Hardy-type inequalities on regular metric trees. For a weighted summation operator (i.e., a Hardy-type operator) on a combinatorial tree acting from $l_2$ into $l_\infty$, Lifshits and Linde \cite{40,42} obtained estimates of entropy numbers. In \cite{18,19,53} Evans, Harris, Lang and Solomyak obtained estimates of approximation numbers for weighted Hardy-type operators on metric trees. Also it is worth noting results of Evans and Harris \cite{17} on embeddings of Sobolev classes on ridged domains into Lebesgue spaces; here the definition of a ridged domain was given in terms of metric trees.

\section{Notation}

In what follows $\overline{A}$ (int $A$, mes $A$, card $A$, respectively) be, respectively, the closure (interior, Lebesgue measure, cardinality) of $A$. If a set $A$ is contained in some subspace $L \subset \mathbb{R}^d$ of dimension $(d - 1)$, then we denote by int$_{d-1}A$ the interior of $A$ with respect to the induced topology on the space $L$. We say that sets $A$, $B \subset \mathbb{R}^d$ do not overlap if $A \cap B$ is a Lebesgue nullset. For a convex set $A$ we denote by $\dim A$ the dimension of the affine span of the set $A$.

A set $A \subset \mathbb{R}^d$ is said to be a parallelepiped if there are $s_j \leq t_j$, $1 \leq j \leq d$, such that

$$\prod_{j=1}^d [s_j, t_j] \subset A \subset \prod_{j=1}^d (s_j, t_j).$$

If $t_j - s_j = t_1 - s_1$ for any $j = 1, \ldots, d$, then a parallelepiped is referred to as a cube.

Let $\mathcal{K}$ be a family of closed cubes in $\mathbb{R}^d$ with axes parallel to coordinate axes. For a cube $K \in \mathcal{K}$ and $s \in \mathbb{Z}_+$ we denote by $\Xi_s(K)$ the set of $2^sd$ closed non-overlapping cubes of the same size that form a partition of $K$, and write $\Xi(K) := \bigcup_{s \in \mathbb{Z}_+} \Xi_s(K)$. We generally consider that these cubes are close (except the proof of Lemma \cite{P}).

We recall some definitions from graph theory. Throughout, we assume that the graphs have neither multiple edges nor loops.

Let $\Gamma$ be a graph containing at most countable number of vertices. We shall denote by $V(\Gamma)$ and by $E(\Gamma)$ the set of vertices and the set of edges of $\Gamma$, respectively. Two vertices are called adjacent if there is an edge between them. We shall identify pairs of adjacent vertices with edges that connect them. Let $\omega_i \in V(\Gamma)$, $1 \leq i \leq n$. The sequence $(\omega_1, \ldots, \omega_n)$ is called a path, if the vertices $\omega_i$ and $\omega_{i+1}$ are adjacent for any $i = 1, \ldots, n - 1$. We say that a graph is connected if any two vertices are connected by a finite path. A connected graph is a tree if it has no cycles.

Let $(\mathcal{T}, \omega_0)$ be a tree with a distinguished vertex (or a root) $\omega_0$. We introduce a partial order on $V(\mathcal{T})$ as follows: we say that $\omega' \succ \omega$ if there exists a path $(\omega_0, \omega_1, \ldots, \omega_n, \omega')$ such that $\omega = \omega_k$ for some $k \in 0, n$. In this case, we set $\rho_{\mathcal{T}}(\omega, \omega') = n + 1 - k$ and call this quantity the distance between $\omega$ and $\omega'$. In addition, we set $\rho_{\mathcal{T}}(\omega, \omega) = 0$. If $\omega' \succ \omega$ or $\omega' = \omega$, then we write $\omega' \geq \omega$ and denote
Given \( \omega \) sets of maximal and minimal vertices in \( W \) if adjacent in \( G \) partial order on \( \Delta \) vertices, and let \( \lambda \) (metric tree. A point on the edge \( \Delta(t, \omega, \omega') \) if \( \omega \in V(T) \), we denote by \( T_\omega = (T_\omega, \omega) \) a subtree of \( T \) with the set of vertices

\[
\omega' \in V(T) : \omega' \geq \omega.
\]  

(11)

Let \( G \) be a subgraph in \( T \). Denote by \( V_{\max}(G) \) and by \( V_{\min}(G) \), respectively, the sets of maximal and minimal vertices in \( G \).

Let \( W \subset V(T) \). We say that \( G \subset T \) is a maximal subgraph on the set of vertices \( W \) if \( V(G) = W \) and if any two vertices \( \xi', \xi'' \in W \) that are adjacent in \( T \) are also adjacent in \( G \).

We need the concept of a metric tree. Let \( (T, \omega, \omega') \) be a tree with a finite set of vertices, and let \( \Delta : E(T) \to 2^\mathbb{R} \) be a mapping such that for any \( \lambda \in E(T) \) the set \( \Delta(\lambda) = [a_\lambda, b_\lambda) \) is a non-trivial segment. Then the pair \( T = (T, \Delta) \) is called a metric tree. A point on the edge \( \lambda \) of the metric tree \( T \) is a pair \((t, \lambda) \), \( t \in [a_\lambda, b_\lambda] \), \( \lambda \in E(T) \) (if \( \omega' \in V_1(\omega), \omega'' \in V_1(\omega'), \lambda = (\omega, \omega'), \lambda' = (\omega', \omega'') \), then we set \( (b_\lambda, \lambda) = (a_\lambda, \lambda') \)). The distance between two points of \( T \) is defined as follows: if \( (\omega_0, \omega_1, \ldots, \omega_n) \) is a path in the tree \( T \), \( n \geq 2 \), \( \lambda_i = (\omega_{i-1}, \omega_i) \), \( x = (t_1, \lambda_1) \), \( y = (t_n, \lambda_n) \), the we set

\[
|y - x|_T = |b_{\lambda_1} - t_1| + \sum_{i=2}^{n-1} |b_{\lambda_i} - a_{\lambda_i}| + |t_n - a_{\lambda_n}|;
\]

if \( x = (t', \lambda), y = (t'', \lambda) \), then \( |y - x|_T = |t' - t''| \).

We say that \((t', \lambda') \leq (t'', \lambda'') \) if \( \lambda' \leq \lambda'' \) and \( t' \leq t'' \) in the case \( \lambda' = \lambda'' \). If \((t', \lambda') \leq (t'', \lambda'') \) and \((t', \lambda') \neq (t'', \lambda'') \), then we write \((t', \lambda') < (t'', \lambda'') \). If \( a, x \in T \), \( a \leq x \), then we set \([a, x] = \{y \in T : a \leq y \leq x\} \).

A subset \( A = \{(t, \lambda) : \lambda \in E(T), t \in A_\lambda\} \) is said to be measurable, if \( A_\lambda \) is measurable for any \( \lambda \in E(T) \). The Lebesgue measure of \( A \) is defined by

\[
|A| = \sum_{\lambda \in E(T)} |A_\lambda|.
\]

A function \( f : A \to \mathbb{R} \) is said to be integrable if \( f_\lambda := f|_{\{(t, \lambda) : t \in A_\lambda\}} \) is integrable for any \( \lambda \in E(T) \) and the sum \( \sum_{\lambda \in E(T)} \int_{A_\lambda} |f_\lambda(t)| \, dt \) is finite. In this case, we set

\[
\int_{A} f(x) \, dx = \sum_{\lambda \in E(T)} \int_{A_\lambda} f_\lambda(t) \, dt.
\]

Let \( D \subset T \) be a connected subset. Denote by \( T_D \) the maximal subtree in \( T \) such that for any \( \lambda \in E(T_D) \) the set \( \Delta(\lambda) \cap D \) is a non-trivial segment. Set \( \Delta_D(\lambda) = \)
Δ(λ) ∩ D, λ ∈ E(T). Then (TD, ∆D) is a metric tree, which will be identified with the set D and which will be called a metric subtree in T.

Let D be a metric subtree in T. A point t ∈ D is said to be maximal if x ∈ T \ D for any x > t.

3 Preliminary results

Let Δ be a cube with a side of length 2\(^{-m}\), m ∈ Z. Set m(Δ) = m. In particular, if Δ ∈ Ξ([-\(\frac{1}{2}\), \(\frac{1}{2}\)\(^d\)], Δ ∈ Ξ\(m(Δ)\)([-\(\frac{1}{2}\), \(\frac{1}{2}\)\(^d\)]).

We shall need Whitney’s covering theorem (see, e.g., [39, p. 562]).

**Theorem A.** Let Ω ⊂ [-\(\frac{1}{2}\), \(\frac{1}{2}\)\(^d\)] be an open set. Then there exists a family of closed pairwise non-overlapping cubes Θ(Ω) = {Δ\(_j\)}\(_{j∈N}\) ⊂ Ξ([-\(\frac{1}{2}\), \(\frac{1}{2}\)\(^d\)]) with the following properties:

1. Ω = ∪\(_{j∈N}\) Δ\(_j\);
2. dist (Δ\(_j\), ∂Ω) \(\asymp\) 2\(^{-m(Δ\(_j\))}\);
3. for any j ∈ N
   \(\text{card}\{i ∈ N : \dim(Δ\(_i\) ∩ Δ\(_j\)) = d - 1\} \leq 12^d\); (12)
4. if dim(Δ\(_i\) ∩ Δ\(_j\)) = d - 1, then
   \(m(Δ\(_j\)) - 2 \leq m(Δ\(_i\)) \leq m(Δ\(_j\)) + 2\). (13)

Andersen and Heinig in [3, 26] proved discrete analogues of the two-weighted Hardy-type inequality. We formulate a particular case of their result, which will be used in the sequel.

**Theorem B.** Let 1 ≤ p ≤ q < ∞, and let \(\{u_n\}_{n∈Z}\), \(\{w_n\}_{n∈Z}\) be nonnegative sequences such that

\[ C := \sup_{m ∈ Z} \left( \sum_{n=-\infty}^{∞} u_n^p \right)^{1/p} \left( \sum_{n=-\infty}^{∞} w_n^q \right)^{1/q} < ∞. \]

Then, for any sequence \(\{a_n\}_{n∈Z}\),

\[ \left( \sum_{n=-\infty}^{∞} |w_n| \sum_{k=-\infty}^{n} u_k a_k \right)^{1/q} \leq C \left( \sum_{n∈Z} |a_n|^p \right)^{1/p}. \] (14)
Evans, Harris and Pick in [20] proved a criterion for boundedness of a two-weighted Hardy-type operator on a metric tree.

Let $T = (\mathcal{T}, \Delta)$ be a metric tree, $x_0 \in T$, and let $u, w : T \to \mathbb{R}_+$ be measurable functions. We set $T_{x_0} = \{x \in T : x \geq x_0\}$,

$$I_{u,w,x_0}f(x) = w(x) \int_{x_0}^{x} u(t)f(t)\,dt.$$ 

Denote by $J_{x_0} = J_{x_0}(T)$ a family of metric subtrees $\mathcal{D} \subset T$ with the following properties:

1. $x_0$ is a minimal vertex in $\mathcal{D}$;
2. if $x \in \partial \mathcal{D}\backslash\{x_0\}$, then $x$ is a maximal point in $\mathcal{D}$.

For $\mathcal{D} \in J_{x_0}$, we set

$$\alpha_{\mathcal{D}} = \inf \left\{ \|f\|_{L_p(T)} : \int_{x_0}^{t} |f(x)|u(x)\,dx = 1 \text{ for any } t \in \partial \mathcal{D} \right\}.$$

**Theorem C.** Let $1 \leq p \leq q \leq \infty$. Then the operator $I_{u,w,x_0} : L_p(T_{x_0}) \to L_q(T_{x_0})$ is bounded if and only if

$$C_{u,w} := \sup_{\mathcal{D} \in J_{x_0}} \frac{\|w\chi_{T_{x_0}\backslash \mathcal{D}}\|_{L_q(T)}}{\alpha_{\mathcal{D}}} < \infty.$$ 

Moreover, $C_{u,w} \leq \|I_{u,w}\|_{L_p(T_{x_0}) \to L_q(T_{x_0})} \leq 4C_{u,w}$.

The quantity $\alpha_{\mathcal{D}}$ is calculated recursively. The following theorem is also proved in [20].

**Theorem D.** Let $\mathcal{D} \in J_{x_0}$, $\mathcal{D} = \bigcup_{j=0}^{m} \mathcal{D}_j$, $\mathcal{D}_0 = [x_0, y_0]$, $x_0 < y_0$, $\mathcal{D}_j \in J_{y_0}$, $1 \leq j \leq m$, $\mathcal{D}_i \cap \mathcal{D}_j = \{y_0\}$, $i \neq j$. Then

$$\frac{1}{\alpha_{\mathcal{D}}} = \left\| (\alpha_{\mathcal{D}_0}^{-1}, (\alpha_{\mathcal{D}_i})_{i=1}^{m})_{p}^{-1}\right\|_{p'}.$$ 

Notice that if $x_0 = (t', \lambda)$, $\lambda \in E(T)$, $t' \in \Delta(\lambda)$, and $y_0$ is such as in Theorem D then $y_0$ is a right end of $\Delta(\lambda)$.

The following theorem is proved in [1, 2, 52]; see also [44, p. 51] and [39, p. 566].

**Theorem E.** Let $1 < p < q < \infty$, $d \in \mathbb{N}$, $r > 0$, $\frac{1}{p} + \frac{1}{q} - \frac{1}{d} = 0$. Then the operator

$$Tf(x) = \int_{\mathbb{R}^d} f(y)|x - y|^{-d}\,dy$$

is bounded from $L_p(\mathbb{R}^d)$ in $L_q(\mathbb{R}^d)$. 

\[9\]
Reshetnyak [50, 51] constructed the integral representation for smooth functions defined on a John domain $\Omega$ in terms of their derivatives of order $r$. We shall use the following form of his result (see also [61]).

**Theorem F.** Let $\Omega \in \text{FC}(a)$, let the point $x_*$, the curves $\gamma_x$ and the numbers $T(x)$ be such as in Definition 4 and let $R_0 = \text{dist}_{||\cdot||_g}(x_*, \partial \Omega)$, $r \in \mathbb{N}$. Then there exist measurable functions $H_{\beta}: \Omega \times \Omega \rightarrow \mathbb{R}$, $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d$, $|\beta| = r$, such that the inclusion $\text{supp} H_{\beta}(x, \cdot) \subset \bigcup_{t \in [0, T(x)]} B_{R_0}(\gamma_x(t))$ and the inequality $|H_{\beta}(x, y)| \lesssim a, d, r, |x - y|^{d - r}$ hold for any $x \in \Omega$. Moreover, the following representation holds:

$$f(x) = \sum_{|\beta| = r} \int H_{\beta}(x, y) \frac{\partial^r f(y)}{\partial^r y} dy, \quad f \in C^\infty(\Omega), \quad f|_{B_{R_0/2}(x_*)} = 0.$$ 

The proof of the following lemma is straightforward and will be omitted.

**Lemma 1.** Let $\Phi : (0, +\infty) \rightarrow (0, +\infty)$, $\rho : (0, +\infty) \rightarrow (0, +\infty)$ be absolutely continuous functions and let $\lim_{t \rightarrow +0} \frac{t\Phi'(t)}{\Phi(t)} = 0$, $\lim_{y \rightarrow +\infty} \frac{\rho'(y)}{\rho(y)} = 0$. Then for any $\varepsilon > 0$

$$t^\varepsilon \lesssim \Phi(t) \lesssim t^{-\varepsilon}, \quad \text{if} \quad t \in (0, 1], \quad t^{-\varepsilon} \lesssim \rho(t) \lesssim t^\varepsilon, \quad \text{if} \quad t \in [1, \infty).$$

Let $\sigma \in \mathbb{R}$, $\mu < -1$. Then for any sequence $\{k_j\}_{j=0}^l \subset \mathbb{Z}_+$ such that $k_0 < k_1 < \cdots < k_l$, the following estimates hold:

$$\sum_{j=0}^l 2^{\sigma k_j} \Phi(2^{-k_j}) \lesssim 2^{\sigma k_0} \Phi(2^{-k_0}), \quad \text{if} \quad \sigma < 0,$$

$$\sum_{j=0}^l 2^{\sigma k_j} \Phi(2^{-k_j}) \lesssim 2^{\sigma k_l} \Phi(2^{-k_l}), \quad \text{if} \quad \sigma > 0,$$

$$\sum_{j=0}^l k_j^\mu \rho(k_j) \lesssim k_0^{1+\mu} \rho(k_0).$$

**4 Construction of the partition of the tree**

Let $\Theta \subset \Xi \left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ be a set of non-overlapping cubes.

**Definition 4.** Let $\mathcal{G}$ be a graph, and let $F : \text{V}(\mathcal{G}) \rightarrow \Theta$ be a one-to-one mapping. We say that $F$ is consistent with the structure of the graph $\mathcal{G}$ if the following condition holds: for any adjacent vertices $\xi', \xi'' \in \text{V}(\mathcal{G})$ the set $\Gamma_{\xi', \xi''} := F(\xi') \cap F(\xi'')$ has dimension $d - 1$. 


Let \((T, \xi_*)\) be a tree, and let \(F : V(T) \to \Theta\) be a one-to-one mapping consistent with the structure of the tree \(T\). For any adjacent vertices \(\xi', \xi''\), we set \(\Gamma^{\xi',\xi''} = \text{int}_{d-1}\Gamma_{\xi',\xi''}\), and for each subtree \(T'\) of \(T\), we put

\[
\Omega_{T',F} = \left(\bigcup_{\xi \in V(T')} \text{int} F(\xi)\right) \cup \left(\bigcup_{(\xi',\xi'') \in E(T')} \Gamma^{\xi',\xi''}\right). \tag{15}
\]

For \(\xi \in V(T)\), \(\Delta = F(\xi)\), denote \(m_\xi = m(\Delta)\), \(\Omega_{\xi,\Delta} = \Omega_{[\xi,\xi],F}\).

Let \(\Theta(\Omega)\) be a Whitney covering of \(\Omega\) (see Theorem \(\text{A}\)). The following lemma is proved in [61].

**Lemma 2.** Let \(\Omega \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^d\), \(\Omega \in \text{FC}(a)\). Then there exist a tree \(T\) and a one-to-one mapping \(F : V(T) \to \Theta(\Omega)\) consistent with the structure of \(T\) and which satisfies the following properties:

1. for any subtree \(T'\) of \(T\),
\[
\Omega_{T',F} \in \text{FC}(b_\Delta), \quad \text{where} \quad b_\Delta = b_\Delta(a, d) > 0; \tag{16}
\]

2. if \(x \in F(\xi)\), then a curve \(\gamma_x\) from Definition \(\text{I}\) can be chosen so that \(B_{b_\Delta}(\gamma_x(t)) \subset \Omega_{\xi,F(\xi)}\) for any \(t \in [0, T(x)]\); if \(\xi_{T'}\) is a minimal vertex of \(T'\), then the center of the cube \(F(w_{T'})\) can be taken as a point \(x_*\) from Definition \(\text{I}\) with \(\Omega_{T',F} \in \text{FC}(b_\Delta)\).

For \(\xi \in V(T)\), we set \(\Omega_\xi = \Omega_{\xi,F}\); the number \(k_\xi \in \mathbb{Z}_+\) is chosen so that

\[
2^{-k_\xi} \leq \text{dist}_{T}(F(\xi), \Gamma) < 2^{-k_\xi + 1}. \tag{17}
\]

By Theorem \(\text{A}\)

\[
2^{-m_\xi} \leq \text{dist}_{T}(F(\xi), \partial \Omega) \leq \text{dist}_{T}(F(\xi), \Gamma) \leq 2^{-k_\xi}; \tag{18}
\]

hence, there exists \(\delta(d) \in \mathbb{Z}_+\) such that

\[
k_\xi \leq m_\xi + \delta(d). \tag{19}
\]

Let \(z_\xi \in F(\xi)\) be such that \(\text{dist}_{T}(z_\xi, \Gamma) = \text{dist}_{T}(F(\xi), \Gamma)\), and let \(\tilde{z}_\xi\) be a center of the cube \(F(\xi)\). Then the first relation in (18) together with

\[
|z_\xi - \tilde{z}_\xi| \leq 2^{-m_\xi} \tag{20}
\]

imply that for any \(x \in \Omega_\xi\)

\[
|x - z_\xi| \leq \text{diam}_\xi \Omega_\xi \leq \text{dist}_{T}(\tilde{z}_\xi, \partial \Omega) \leq \text{dist}_{T}(F(\xi), \partial \Omega) + |z_\xi - \tilde{z}_\xi| \leq \frac{18}{d} 2^{-m_\xi}. \tag{18, 20}
\]

11
Hence, there exists $c(a, d) > 0$ such that
\[|x - z_\xi| \leq c(a, d) \cdot 2^{-m_\xi}, \ x \in \Omega_\xi. \tag{21}\]
Prove that
\[
dist_{|\cdot|}(x, \Gamma) \sim_d 2^{-k_\xi} \tag{22}\]
for any $x \in F(\xi)$. Indeed,
\[
2^{-k_\xi} \leq \text{dist}_{|\cdot|}(F(\xi), \Gamma) \leq \text{dist}_{|\cdot|}(x, \Gamma) \leq \text{dist}_{|\cdot|}(F(\xi), \Gamma) + |x - z_\xi| \lesssim_d \tag{17}\]
\[\lesssim 2^{-k_\xi + 1} + 2^{-m_\xi} \lesssim_d 2^{-k_\xi}. \tag{19}\]
Denote
\[
\hat{W} = \{\xi \in V(\mathcal{T}) : m_\xi \leq k_\xi + 1 + \log c(a, d)\}. \tag{23}\]
From (19) it follows that for any $\xi \in \hat{W}$
\[
2^{-m_\xi} \gtrsim_d 2^{-k_\xi}. \tag{24}\]
Let $\xi \notin \hat{W}$. We show that for any $x \in \Omega_\xi$
\[
dist_{|\cdot|}(x, \Gamma) \gtrsim_d 2^{-k_\xi}. \tag{25}\]
Indeed,
\[
\text{dist}_{|\cdot|}(x, \Gamma) \leq \text{dist}_{|\cdot|}(z_\xi, \Gamma) + \text{dist}_{|\cdot|}(z_\xi, \Gamma) + |x - z_\xi| \\lesssim_d \tag{17}, \tag{21}\]
\[2^{-k_\xi} + 2^{-m_\xi} \lesssim_d 2^{-k_\xi}, \tag{19}\]
\[\text{dist}_{|\cdot|}(x, \Gamma) \geq \text{dist}_{|\cdot|}(z_\xi, \Gamma) - |x - z_\xi| \\gtrsim \tag{17}, \tag{21}\]
\[2^{-k_\xi} - c(a, d) \cdot 2^{-m_\xi} \gtrsim 2^{-k_\xi - 1}. \tag{23}\]
Denote
\[
\hat{W}_\nu = \{\xi \in \hat{W} : k_\xi = \nu\}. \tag{26}\]
Then (21) and (21) imply that for any $\xi \in \hat{W}_\nu$ and for any tree $\mathcal{T}' \subset \mathcal{T}_\xi$ rooted at $\xi$
\[
\text{diam}_{|\cdot|} \Omega_{\mathcal{T}', F} \gtrsim_d 2^{-\nu}. \tag{27}\]

**Lemma 3.** There exist a partition of the tree $\mathcal{T}$ into subtrees $\mathcal{T}_{k,i}$ with minimal vertices $\xi_{k,i}$, $k \in \mathbb{Z}_+$, $i \in I_k$, $I_k \neq \emptyset$, and numbers $\nu_k \in \mathbb{N}$, satisfying the following conditions:
1. \( \nu_0 < \nu_1 < \cdots < \nu_k < \cdots \); 

2. \( \xi_{k,i} \in \hat{W}_{\nu_k} \); 

3. \( \text{dist}_{|\cdot|}(x, \Gamma) \gtrsim 2^{-\nu_x} \) for any \( x \in \Omega_{T_k,F} \); 

4. if \( \xi_{k',i} < \xi_{k,i} \), then \( k' < k \).

**Proof.** Let \( \nu \in \mathbb{Z}_+ \), \( \hat{\xi} \in \hat{W}_\nu \). Denote by \( \mathfrak{T}(\hat{\xi}) \) a set of subtrees \( T' \subset T_{\hat{\xi}} \) with the minimal vertex \( \hat{\xi} \) such that 

\[
\mathcal{V}(T') \cap \left( \bigcup_{l \geq \nu+1} \hat{W}_l \right) = \emptyset
\]  

(this set is nonempty, since \( \{ \hat{\xi} \} \in \mathfrak{T}(\hat{\xi}) \)). Denote by \( \mathcal{S}(T_{\hat{\xi}}) \) a subtree in \( T_{\hat{\xi}} \) such that \( \mathcal{V}(\mathcal{S}(T_{\hat{\xi}})) = \bigcup_{S \in \mathfrak{T}(\hat{\xi})} \mathcal{V}(S) \). Then \( \mathcal{S}(T_{\hat{\xi}}) \in \mathfrak{T}(\hat{\xi}) \).

Prove that there exists \( \hat{\nu} = \hat{\nu}(a, d) \in \mathbb{N} \) such that for any \( x \in \Omega_{\mathcal{S}(T_{\hat{\xi}}),F} \)

\[
2^{-\nu-\hat{\nu}} \leq \text{dist}_{|\cdot|}(x, \Gamma) \leq 2^{-\nu+\hat{\nu}}.
\]

Indeed,

\[
\text{dist}_{|\cdot|}(x, \Gamma) \leq |x - z_{\hat{\xi}}| + \text{dist}_{|\cdot|}(z_{\hat{\xi}}, \Gamma) \lesssim_{a,d} 2^{-m_{\hat{\xi}}} + 2^{-\nu} \lesssim_{d} 2^{-\nu}.
\]

Prove the estimate from below. Let \( x \in F(\eta), \eta \in \mathcal{V}(\mathcal{S}(T_{\hat{\xi}})) \). Set 

\[
\hat{\eta} = \max\{\mathcal{W} \cap [\hat{\xi}, \eta]\}.
\]

Then \( \hat{\eta} \in \hat{W}_j \) for some \( j \in \mathbb{Z}_+ \); since \( \mathcal{S}(T_{\hat{\xi}}) \in \mathfrak{T}(\hat{\xi}) \), we have \( j \leq \nu \). If \( \eta = \hat{\eta} \), then 

\[
\text{dist}_{|\cdot|}(F(\eta), \Gamma) \gtrsim 2^{-j} \geq 2^{-\nu}.
\]

Let \( \eta > \hat{\eta}, \hat{\xi} \in [\hat{\eta}, \eta] \cap \mathcal{V}_1(\hat{\eta}) \). Then \( \hat{\xi} \notin \hat{W} \), \( \dim(F(\hat{\eta}) \cap \mathcal{F}(\hat{\xi})) = d - 1 \), and for any \( x \in F(\eta) \)

\[
\text{dist}_{|\cdot|}(x, \Gamma) \gtrsim_{a,d} 2^{-k_{\hat{\xi}}} \gtrsim_{d} 2^{-m_{\hat{\xi}}} \gtrsim_{a,d} 2^{-m_{\eta}-2} \lesssim_{a,d} 2^{-k_{\eta}} \gtrsim_{d} 2^{-\nu}.
\]

Let \( \xi_0 \) be a minimal vertex of the tree \( T \). Prove that \( \xi_0 \in \hat{W} \). Indeed, otherwise \( \text{(25)} \) imply that \( \text{dist}_{|\cdot|}(x, \Gamma) \gtrsim_{a,d} 2^{-k_{\xi_0}} \) for any \( x \in \Omega_{T,F} \). Hence, \( \Gamma \not\subset \partial\Omega \), which leads to a contradiction.

The further arguments are the same as the arguments in Lemma 2 from [62]. \( \square \)
Proposition 1. Let \( \xi_{k,i} < \xi_{k',i'} \), \( \{ \xi : \xi_{k,i} \leq \xi < \xi_{k',i'} \} \subset V(T_{k,i}) \). Then \( \nu_{k'} \leq \nu_k + \overline{\sigma} \), with \( \overline{\sigma} = \overline{\sigma}(a, d) \).

Proof. Let \( \xi \in [\xi_{k,i}, \xi_{k',i'}] \) be the direct predecessor of \( \xi_{k',i'} \). Then \( \xi \in V(T_{k,i}) \).

By Assertion 3 of Lemma 3, \( \text{dist}_{a,d}(x, \Gamma) \approx a,d^2 - \nu_k \) for any \( x \in F(\xi) \), as well as \( \text{dist}_{a,d}(x, \Gamma) \approx a,d^2 - \nu_{k'} \) for any \( x \in F(\xi_{k',i'}) \). Since the mapping \( F \) is consistent with the structure of the tree \( T \), then \( F(\xi) \cap F(\xi_{k',i'}) \neq \emptyset \). Hence, \( 2^{-\nu_k} \approx 2^{-\nu'} \). This completes the proof.

Lemma 4. Let \( \hat{\xi} \in \hat{W}_{\nu_0} \). In addition, suppose that there exists \( c_0 \geq 1 \) such that
\[
\forall j \in \mathbb{N}, \ t, s \in [2^{-j-1}, 2^{-j+1}] \quad c_0^{-1} \leq \frac{h(t)}{h(s)} \leq c_0.
\]

Given \( \nu \geq \nu_0 \), we denote \( \hat{W}_{\nu}(\hat{\xi}) = \hat{W}_{\nu} \cap V(T_{\hat{\xi}}) \).

Then
\[
\text{card} \hat{W}_{\nu}(\hat{\xi}) \lesssim \frac{h(2^{-\nu_0})}{h(2^{-\nu})}.
\]

If, in addition, \( \hat{\xi} \) is a root of the tree \( T \), then there is \( \hat{k} = \hat{k}(a, d) \in \mathbb{N} \) such that
\[
\sum_{j=\nu}^{\nu+k} \text{card} \hat{W}_j(\hat{\xi}) \gtrsim \frac{h(2^{-\nu_0})}{h(2^{-\nu})}.
\]

Proof. Throughout the proof of this lemma we suppose that a cube is a product of semi-intervals \( \prod_{j=1}^d [a_j, b_j] \). Then any two non-overlapping cubes do not intersect.

If \( \xi \in \hat{W}_{\nu} \), then it follows from (24) that there is \( k_*(a, d) \) such that \( \nu - k_*(a, d) \leq m_\xi \leq \nu + k_*(a, d) \). Further,
\[
\text{diam} \Omega_\xi \gtrsim \frac{2^{-\nu_0}}{a,d}.
\]

Therefore, if \( \xi \in \hat{W}_{\nu}(\hat{\xi}) \), then \( 2^{-\nu} \gtrsim 2^{-m_\xi} \leq \text{diam} \Omega_\xi \approx 2^{-\nu_0} \). Hence,
\[
2^\nu \gtrsim 2^{\nu_0}, \quad \text{if} \quad \hat{W}_{\nu}(\hat{\xi}) \neq \emptyset.
\]

Let \( k \in \mathbb{Z}, -k_*(a, d) \leq k \leq k_*(a, d) \). Denote by \( \hat{W}_{\nu,k}(\hat{\xi}) \) the set of \( \xi \in \hat{W}_{\nu}(\hat{\xi}) \) such that \( m_\xi = \nu + k \).

It follows from (17) and (26) that
\[
\text{dist}_{a,d}(F(\xi), \Gamma) \leq 2^{-\nu+1}, \quad \xi \in \hat{W}_{\nu}(\hat{\xi}).
\]
This together with \((33)\) implies that there exists a cube \(\Delta_0\) and a number \(k_0 = k_0(a, d) \in \mathbb{Z}_+\) such that

\[
F(\xi) \in \Xi(\Delta_0), \quad \nu_0 - k_0(a, d) \leq m(\Delta_0) \leq \nu_0 + k_0(a, d),
\]

\[
\Omega_\xi \subset \Delta_0, \quad \exists \tau \in \Gamma \cap \Delta_0 : \text{dist}_{a,d}(x, \partial \Delta_0) \gtrsim 2^{-\nu_0},
\]

\[
\forall \xi \in \hat{W}_\nu(\xi) \exists x \in \Gamma \cap \Delta_0 : \text{dist}_{a,d}(x, F(\xi)) \leq 2^{-\nu+1}.
\]

Let \(j \in \mathbb{N}, j \geq \nu_0 + k_0(a, d)\). In this case, if \(\Delta \in \Xi_j\left([\frac{1}{2}, \frac{1}{2}]^d\right)\), then either \(\Delta \subset \Delta_0\) or \(\Delta\) does not overlap with \(\Delta_0\). It follows from the conditions \(F(\xi) \in \Xi(\Delta_0)\) and \(j \geq \nu_0 + k_0(a, d) \geq m(\Delta_0)\). Denote by \(\Delta_{0,j}\) a cube that is obtained from \(\Delta_0\) by a dilatation in respect to its center, with a side length \(m(\Delta_0) + 2 \cdot 2^{-j}\). Set

\[
\Theta_j(\Delta_0) = \left\{ \Delta \in \Xi_j\left([\frac{1}{2}, \frac{1}{2}]^d\right) : \Delta \subset \Delta_0, \Delta \cap \Gamma \neq \emptyset \right\},
\]

\[
\tilde{\Theta}_j(\Delta_0) = \left\{ \Delta \in \Xi_j\left([\frac{1}{2}, \frac{1}{2}]^d\right) : \Delta \subset \Delta_{0,j}, \Delta \cap \Gamma \neq \emptyset \right\}.
\]

Prove that

\[
\text{card } \Theta_j(\Delta_0) \asymp \frac{h(2^{-\nu_0})}{h(2^{-j})}.
\]

Let \(\Delta \in \Theta_j(\Delta_0)\). Since \(\Delta \cap \Gamma \neq \emptyset\), there is a cube \(K_{\Delta}\) centered at \(z_\Delta\), such that

\[
\Delta \in \Xi_1(K_{\Delta}), \quad \text{dist}_{a,d}(z_\Delta, \Gamma) \leq 2^{-m(\Delta)-1}.
\]

Then there are

\[
\tilde{z}_\Delta \in \Gamma, \quad t_\Delta \gtrsim 2^{-j}, \quad \tilde{t}_\Delta \lesssim 2^{-j} \quad \text{such that} \quad B_{t_\Delta}(\tilde{z}_\Delta) \subset K_{\Delta} \subset B_{\tilde{t}_\Delta}(\tilde{z}_\Delta).
\]

Let \(\mu\) be a measure from the definition \((3)\) (in particular, \(\text{supp } \mu \subset \Gamma\)). Then \((30)\) and \((32)\) imply that \(\mu(K_{\Delta}) \asymp h(2^{-j})\). On the other hand, by \((30)\), \((34)\) and \((36)\),

\[
h(2^{-\nu_0}) \asymp \mu(\Delta_0) = \sum_{\Delta \in \Theta_j(\Delta_0)} \mu(\Delta),
\]

\[
h(2^{-\nu_0}) \asymp \mu(\Delta_{0,j}) = \sum_{\Delta \in \tilde{\Theta}_j(\Delta_0)} \mu(\Delta).
\]
Therefore, in order to prove \((38)\) it is sufficient to check that
\[
\sum_{\Delta \in \Theta_j(\Delta_0)} \mu(K_\Delta) \lesssim \sum_d \mu(\Delta).
\]
The first inequality holds since the measure \(\mu\) is nonnegative. Prove the second inequality. Since \(\Delta \in \Xi_1(K_\Delta)\), we have \(K_\Delta \subset \Delta_{0,j}\).

Denote
\[
\Theta_{j,\Delta} = \{ \Delta' \in \tilde{\Theta}_j(\Delta_0) : \Delta' \subset K_\Delta \} \quad \text{for} \quad \Delta \in \Theta_j(\Delta_0),
\]
\[
\Theta'_{j,\Delta'} = \{ \Delta \in \Theta_j(\Delta_0) : \Delta' \subset K_\Delta \} \quad \text{for} \quad \Delta' \in \tilde{\Theta}_j(\Delta_0).
\]

Since \(\text{card } \Theta'_{j,\Delta'} \leq 1\) for any \(\Delta' \in \tilde{\Theta}_j(\Delta_0)\), we have
\[
\sum_{\Delta \in \Theta_j(\Delta_0)} \mu(K_\Delta) = \sum_{\Delta \in \Theta_j(\Delta_0)} \sum_{\Delta' \in \Theta_{j,\Delta}} \mu(\Delta') \leq \sum_{\Delta' \in \Theta_j(\Delta_0)} \sum_{\Delta \in \Theta'_{j,\Delta'}} \mu(\Delta') \lesssim \sum_d \mu(\Delta').
\]

This proves \((38)\).

Show that if \(-k_*(a, d) \leq k \leq k_*(a, d)\), \(\nu \geq \nu_0 + 2k_*(a, d)\), then
\[
\text{card } \hat{W}_{\nu,k}(\hat{\xi}) \lesssim \text{card } \Theta_{\nu+k}(\Delta_0)
\]
(recall that \(\Theta_j(\Delta_0)\) was defined for \(j \geq \nu_0 + k_*(a, d)\)).

Set
\[
A := \left\{ \Delta' \in \Xi_{\nu+k} \left( \left[ -\frac{1}{2}, \frac{1}{2} \right]^d \right) : \exists \Delta \in \Theta_{\nu+k}(\Delta_0) : \text{dist}_{\nu}(\Delta', \Delta) \leq 2^{-\nu+1} \right\}.
\]

From \((37)\) it follows that \(\{ F(\xi) : \xi \in \hat{W}_{\nu,k}(\hat{\xi}) \} \subset A\) and \(\text{card } \hat{W}_{\nu,k}(\hat{\xi}) \leq \text{card } A \leq \text{card } \Theta_{\nu+k}(\Delta_0)\).

If \(\nu \geq \nu_0 + 2k_*(a, d)\), then \((30)\), \((38)\) and \((41)\) imply \((31)\). If \(\hat{W}_{\nu}(\hat{\xi}) \neq \emptyset\) and \(\nu < \nu_0 + 2k_*(a, d)\), then by \((34)\) we get \(2^{-\nu} \approx 2^{-\nu_0}\); hence, \(\frac{h(2^{-\nu_0})}{h(2^{-\nu})} \lesssim 1\). This together with \((24)\), \((26)\), \((35)\) and \((36)\) yield \((31)\).

Let us prove \((32)\). By \((38)\), it is sufficient to check
\[
\sum_{j=\nu}^{\nu+k} \text{card } \hat{W}_{\nu,k}(\hat{\xi}) \geq \text{card } \Theta_{\nu}(\Delta_0).
\]

Let \(\Delta \in \Theta_l(\Delta_0)\), and let \(K_\Delta\) be the cube defined above (see \((39)\)),
\[
x_\Delta \in K_\Delta \cap \Gamma, \quad |x_\Delta - z_\Delta| \leq 2^{-m(\Delta)-1}.
\]

Since \(\Gamma \subset \partial \Omega\), by Theorem \[A\] and Lemma \[2\] for any \(m \in \mathbb{N}\) there is a vertex \(\xi_\Delta \in V(\mathcal{T})\) such that
\[
\text{dist} (x_\Delta, F(\xi_\Delta)) < 2^{-m}, \quad 2^{-m} |z_\Delta| \leq 2^{m}.\]
If $m$ is sufficiently large, then it follows from (43) and (44) that $F(\xi_\Delta) \subset K_\Delta$. Denote $\eta_\Delta = \min\{\eta \in [\xi, \xi_\Delta] : F(\eta) \subset K_\Delta\}$. Show that

$$2^{-m_\Delta} \asymp 2^{-l} \quad \text{and} \quad \eta_\Delta \in \hat{W} \quad \text{for sufficiently large } m. \quad (45)$$

Indeed, since $\Delta \in \Theta_l(\Delta_0)$, we have $2^{-m(\Delta)} = 2^{-l}$. The inclusion $F(\eta_\Delta) \subset K_\Delta$ and the first relation in (39) imply that $2^{-m_\Delta} \lesssim 2^{-m(\Delta)}$. Check that $2^{-m_\Delta} \gtrsim 2^{-m(\Delta)}$.

It follows from the definition of $\eta_\Delta$ that $\partial F(\eta_\Delta) \cap \partial K_\Delta \neq \emptyset$. Let $\hat{x} \in \partial F(\eta_\Delta) \cap \partial K_\Delta$. Then $|\hat{x} - z_\Delta| = 2^{-m(\Delta)}$. By (44), there is a point $\hat{y} \in F(\xi_\Delta)$ such that $|x_\Delta - \hat{y}| \leq 2^{-m}$. Since $F(\xi_\Delta) \subset \Omega_{\eta_\Delta}$, we have

$$2^{-m_{\eta_\Delta}} \gtrsim_{a,d} \text{diam}_{\eta_\Delta} \gtrsim |\hat{x} - \hat{y}| \gtrsim |\hat{x} - z_\Delta| - |z_\Delta - x_\Delta| - |x_\Delta - \hat{y}| \gtrsim 2^{-m(\Delta) - 1} - 2^{-m} \gtrsim 2^{-m(\Delta) - 2}$$

for large $m$. The first relation in (45) is proved. Check the second relation. Let $\eta_\Delta \notin \hat{W}$. Then, by (25), for any $x \in \Omega_{\eta_\Delta}$ we have $\text{dist}_{\eta_\Delta}(x, \Gamma) \asymp 2^{-k_{\eta_\Delta}}$. Taking $x \in F(\xi_\Delta) \subset \Omega_{\eta_\Delta}$, we get

$$2^{-m_{\eta_\Delta}} \lesssim_{a,d} 2^{-k_{\eta_\Delta}} \asymp \text{dist}_{\eta_\Delta}(x, \Gamma) \lesssim |x - x_\Delta| \lesssim_{d} 2^{-m}. \quad (44)$$

It follows from the proved first relation in (45) that $2^{-l} \lesssim_{a,d} 2^{-m}$. It is impossible for large $m$.

It follows from (21), (26) and (45) that $\eta_\Delta \in \hat{W}_j$ for some $j \in \mathbb{Z}_+$ such that $2^{-j} \asymp 2^{-l}$. Therefore,

$$l - l_* \leq j \leq l + l_*, \quad \text{with} \quad l_* = l_*(a, d) \in \mathbb{N}. \quad (46)$$

Set $\hat{k} = 2l_*$. In order to prove (32), we take $j = \nu + l_*$ and apply (46). We get

$$\text{card} \{\eta_\Delta : \Delta \in \Theta_{\nu + l_*}(\Delta_0)\} \leq \sum_{l = \nu}^{\nu + \hat{k}} \text{card} \hat{W}_l.$$

Hence, in order to prove (42) it is sufficient to check

$$\text{card} \Theta_j(\Delta_0) \lesssim \sum_{d} \text{card} \{\eta_\Delta : \Delta \in \Theta_j(\Delta_0)\} \quad (47)$$

and to apply (58) with (30). Let $\Delta, \Delta' \in \Theta_j(\Delta_0)$, $\eta_\Delta = \eta_{\Delta'}$. Then $K_\Delta \cap K_{\Delta'} \supset \Delta''$, $\Delta'' \in \Xi_1(K_\Delta)$ and $\Delta'' \in \Xi_1(K_{\Delta'})$. Therefore, $\text{card} \{\eta_{\Delta'} : \Delta' \in \Theta_j(\Delta_0), \eta_{\Delta'} = \eta_\Delta\} \lesssim 1$, which implies (47). This completes the proof. \qed
Let \( m \in \mathbb{N} \). For \( 0 < t_0 < t_1 \leq \infty \) denote by \( G^{t_0, t_1} \) the maximal subgraph in \( T \) on the set of vertices
\[
V(G^{t_0, t_1}) := \bigcup_{t_0 \leq t < t_1} \bigcup_{i \in I_k} T_{k, i}
\]
(the index set \( I_k \) was defined in Lemma 3); by \( \{ D_{j, i} \} \subseteq \overline{I}_j \) we denote the set of all connected components of the graph \( G^{1 + mj, 1 + m(j + 1)} \); by \( \hat{\xi}_{j, i} = \hat{\xi}^m_{j, i} \) denote the minimal vertex of the tree \( D_{j, i} \); \( j \in \mathbb{Z}_+ \). Then

1. \( \hat{\xi}_{j, i} \in \hat{\mathcal{W}}_{\nu_k} \) for some \( \nu_k \in [1 + mj, 1 + m(j + 1)] \); in particular,
\[
\text{diam} \Omega_{D_{j, i}, F} \gtrsim a, d, m 2^{-mj};
\]
2. for any \( x \in \Omega_{D_{j, i}, F} \)
\[
\text{dist}_{|\cdot|}(x, \Gamma) \sim a, d, m 2^{-mj}
\]
(it follows from Assertion 3 of Lemma 3);
3. if \( \hat{\xi}_{j, i} < \hat{\xi}_{j', i'} \), then \( j < j' \) (indeed, \( \hat{\xi}_{j, i} = \xi_{k, t} \) and \( \hat{\xi}_{j', i'} = \xi_{k', t'} \) for some \( k, t, k', t' \); by Assertions 4 and 1 of Lemma 3 \( \nu_k < \nu_{k'} \); it implies that \( j \leq j' \); the equality \( j = j' \) is impossible; indeed, in this case the vertices \( \hat{\xi}_{j, i} \) and \( \hat{\xi}_{j', i'} \) are incomparable).

Let \( \hat{\xi}_{j, i} < \hat{\xi}_{j', i'} \),
\[
\{ \xi : \hat{\xi}_{j, i} \leq \xi < \hat{\xi}_{j', i'} \} \subset D_{j, i}.
\]
Then we say that the tree \( D_{j', i'} \) follows the tree \( D_{j, i} \).

**Remark 3.** Let \( \mathfrak{s} \) be such as in Proposition 1, let \( m \geq \mathfrak{s} \), and let \( D_{j', i'} \) follow the tree \( D_{j, i} \). Then \( j' = j + 1 \).

Indeed, let \( \xi_{t, s} \in D_{j, i} \) \( \{ \xi : \xi_{t, s} < \xi < \hat{\xi}_{j', i'} \} \subset V(T_{t, s}) \); \( \hat{\xi}_{j', i'} = \xi_{t', s'} \). By Proposition 1 \( 1 + mj' \leq \nu_{t'} \leq \nu_t + \mathfrak{s} < 1 + m(j + 1) + \mathfrak{s} \). Hence, \( m(j' - j - 1) < \mathfrak{s} \). Since \( m \geq \mathfrak{s} \), the last inequality is possible only for \( j' = j + 1 \).

Given \( j \in \mathbb{Z}_+, l \in \mathbb{N}, t \in I_j \), we denote
\[
\tilde{I}_{j, t}^l = \tilde{I}_{j, t}^{m, l} = \{ i \in \tilde{I}_{j + l, t} : \hat{\xi}_{j + l, i}^m > \hat{\xi}_{j, t}^m \}.
\]

**Lemma 5.** Let \( m \in \mathbb{N} \) be divisible by \( \mathfrak{s} \). Suppose that (30) holds for some \( c_0 \geq 1 \). Then
\[
\text{card} \tilde{I}_{j, t}^l \lesssim a, d, c_0 \frac{h(2^{-mj})}{h(2^{-m(j + l)})}.
\]
Proof. First consider the case \( m = s \).

By the property 1 of the trees \( D_{j,t} \) and \( D_{j+l,i} \),

\[
\hat{\xi}_{j,t} \in \bigcup_{\nu' = 1+\bar{\pi}_j}^{\bar{\pi}_{j(l+1)}} \hat{W}_{\nu'}, \quad \hat{\xi}_{j+l,i} \in \bigcup_{\nu = 1+\bar{\pi}_{j+l}}^{\bar{\pi}_{j(l+1)}} \hat{W}_{\nu}(\hat{\xi}_{j,t})
\]

(recall that \( \hat{W}_{\nu}(\hat{\xi}_{j,t}) = \hat{W}_{\nu} \cap V(\bar{T}_{\hat{\xi}_{j,t}}) \)). Therefore, from Lemma 3 and (30) it follows that

\[
\text{card } \tilde{I}_{j,t} \leq \sum_{\nu = 1+\bar{\pi}_{j+l}}^{\bar{\pi}_{j+l+1}} \text{card } \hat{W}_{\nu}(\hat{\xi}_{j,t}) \lesssim \frac{h(2-s_j)}{h(2-\pi_{j+l})}.
\]

Consider the case \( m = m' \). Then \( \hat{\xi}_{j,t} = \hat{\xi}_{j',t'} \) for some \( j' \geq m' j \), \( \hat{\xi}_{j+l,i} = \hat{\xi}_{m'(j+l),i'} \) (by Remark 3). Hence,

\[
\text{card } \tilde{I}_{j,t} \leq \text{card } \tilde{I}_{m'j,t'} \lesssim \frac{h(2-m'j)}{h(2-\pi_{m'(j+l)})} = \frac{h(2-mj)}{h(2-\pi_{m(j+l)})}.
\]

This completes the proof. \( \Box \)

5 The discrete Hardy-type inequality on a tree: case \( p = q \)

5.1 The analogue of Evans – Harris – Pick theorem

Let \((A, \xi_0)\) be a tree with a finite vertex set, let \( 1 \leq p \leq \infty \), and let \( u, w : V(A) \to \mathbb{R}_+ \) be weight functions. Denote by \( \mathcal{S}_{A,u,w} \) the minimal constant \( C \) in the inequality

\[
\left( \sum_{\xi \in V(A)} w^p(\xi) \left( \sum_{\xi' \preceq \xi} u(\xi') f(\xi') \right)^p \right)^{1/p} \leq C \left( \sum_{\xi \in V(A)} f^p(\xi) \right)^{1/p}, \quad f : V(A) \to \mathbb{R}_+.
\]

(52)

Remark 4. If \( D \subset A \) is a subtree, then \( \mathcal{S}_{D,u,w} \leq \mathcal{S}_{A,u,w} \).

Let us obtain two-sided estimates for \( \mathcal{S}_{A,u,w} \). We reduce this problem to estimating the constant in the Hardy-type inequality on a metric tree and use the result from the article [20].

Let \( \xi \in V(A) \), \( D \subset A_\xi \). We say that \( D \in T_\xi \) if the following conditions hold:

1. \( \hat{\xi} \) is the minimal vertex in \( D \),
2. if \( \xi \in V(D) \) is not a maximal vertex \( D \), then \( V(D) \subset \{ T_\xi \} \).
Denote by $D$ the subtree in $D$ such that $V(D) = V(D) \setminus V_{\max}(D)$. For any subgraph $G \subset A$ and for any function $f : V(G) \to \mathbb{R}$, we denote

$$\|f\|_{l_p(G)} = \left( \sum_{\omega \in V(G)} |f(\omega)|^p \right)^{1/p}.$$  

(53)

By $l_p(G)$ we denote the space of functions $f : V(G) \to \mathbb{R}$ equipped with the norm $\|f\|_{l_p(G)}$.

For $D \in J'_\xi$ we set

$$\beta_D = \inf \left\{ \|f\|_{l_p(A)} : \sum_{\xi \leq \xi' \leq \xi} f(\xi') u(\xi') = 1, \quad \forall \xi \in V_{\max}(D) \right\}.$$  

(54)

Notice that if $D = \{\xi\}$, then

$$\beta_{\{\xi\}} = \inf \{ |f(\xi)| : f(\xi) u(\xi) = 1 \} = u^{-1}(\xi).$$  

(55)

**Lemma 6.** Suppose that there exists $\hat{C} \geq 1$ such that for any $\xi \in V(A)$

$$\text{card } V_1(\xi) \leq \hat{C},$$  

(56)

and let for any adjacent vertices $\xi, \xi' \in V(A)$

$$\hat{C}^{-1} \leq \frac{u(\xi)}{u(\xi')} \leq \hat{C}, \quad \hat{C}^{-1} \leq \frac{w(\xi)}{w(\xi')} \leq \hat{C}.$$  

(57)

Then

$$S_{A, \xi, u, w} \lesssim \sup_{p, \hat{C}'} D \in J'_\xi \frac{\|w \chi_{A \setminus \hat{D}}\|_{l_p(A)}}{\beta_D}.$$  

Proof. If $V(A) = \{\xi_0\}$, then the assertion is trivial.

Let $V(A) \neq \{\xi_0\}$. Add to the set $V(A)$ a vertex $\xi_*$ and join it with $\xi_0$ by an edge. Thus we obtain the tree $(\hat{A}, \xi_*)$. Define the mapping $\Delta$ by $\Delta(\lambda) = [0, 1], \lambda \in E(\hat{A})$. Thus we get the metric tree $A = (\hat{A}, \Delta)$. For any function $\psi : V(\hat{A}) \to \mathbb{R}$ we define $\psi^\#: A \to \mathbb{R}$ as follows. Let $e = (\xi', \xi) \in E(A), \xi > \xi'$. Then we set $\psi^\#|_{\Delta(e)} = \psi(\xi)$.

Let $\lambda_{\xi} \in E(A)$ be an edge with the end $\xi, x_0 = (0, \lambda_{\xi}) \in A$. By H"older inequality,

$$\|I_{u^\#, w^\#, x_0}\|_{L_p(\lambda_{x_0}) \to L_p(\lambda_{x_0})} \lesssim S_{A, \xi, u, w}.$$  

It follows from Theorem C that

$$S_{A_{\xi}, u, w} \lesssim \sup_{D \in J_{\xi_0}} \frac{\|u^\# \chi_{\lambda_{x_0} \setminus D}\|_{l_p(\lambda_{x_0})}}{\alpha_D},$$  

(58)
prove that

\[
\alpha_D = \inf \left\{ \| \phi \|_{L_p(A_{x_0})} : \int_{x_0}^t | \phi(x) | u^#(x) \, dx = 1 \quad \forall t \in \partial \mathbb{D} \right\}.
\]

Applying the Hölder inequality once again (see also [20]), we obtain that

\[
\alpha_D = \inf \left\{ \| \phi \|_{L_p(A_{x_0})} : \phi \in L^\text{discr}_p(A_{x_0}), \int_{x_0}^t | \phi(x) | u^#(x) \, dx = 1 \quad \forall t \in \partial \mathbb{D} \right\};
\]

here \( L^\text{discr}_p(A) \) is the set of functions \( \phi : A \to \mathbb{R} \) that are constants on each edge of the metric tree \( A \).

Let \( \mathbb{D} = (D, \Delta) \in J_{x_0} \). Set \( \mathbb{D}^+ = (\hat{D}, \Delta), \mathbb{D}^- = (\hat{D}, \Delta) \). Prove that \( D \in J'_\xi \).

Indeed, let \( \xi \in \mathbb{V}(D) \), and suppose that there exist vertices \( \xi' \in \mathbb{V}_1(\xi) \setminus \mathbb{V}(D) \) and \( \xi'' \in \mathbb{V}_1(\xi) \cap \mathbb{V}(D) \). Let \( \eta \) be a vertex in \( A \) that is the direct predecessor of \( \xi \). Then the point \((1, (\eta, \xi)) = (0, (\xi, \xi')) = (0, (\xi, \xi'')) \) belongs to the boundary of \( \mathbb{D} \), as well as it is not maximal.

We have

\[
\| u^# \chi_{A_{x_0} \setminus D} \|_{L_p(A)} \leq \| u^# \chi_{A_{x_0} \setminus D} \|_{L_p(A)} = \| u \chi_{A_{x_0} \setminus D} \|_{L_p(A)};
\]

\[
\alpha_D \geq \inf \left\{ \| \phi \|_{L_p(A_{x_0})} : \phi \in L^\text{discr}_p(A_{x_0}), \int_{x_0}^t | \phi(x) | u^#(x) \, dx = 1 \quad \forall t \in \partial D^+ \right\} = \beta_D.
\]

This implies the upper estimate for \( G_{A_{x_0}, u, w} \). Prove the lower estimate. Notice that if \( \mathbb{D} = \mathbb{D}^+ \), then \( \alpha_D = \beta_D \). If in addition \( \mathbb{V}_{\text{max}}(D) \cap \mathbb{V}_{\text{max}}(A) = \emptyset \), then

\[
\| u^# \chi_{A_{x_0} \setminus D} \|_{L_p(A_{x_0})} = \| u \chi_{A_{x_0} \setminus D} \|_{L_p(A_{x_0})} \geq \frac{56, 57}{p, c} \| u \chi_{A_{x_0} \setminus D} \|_{L_p(A_{x_0})}.
\]

Hence,

\[
G_{A_{x_0}, u, w} \geq \sup_{p, c} \left\{ \frac{\| u \chi_{A_{x_0} \setminus D} \|_{L_p(A_{x_0})}}{\beta_D} : D \in J'_\xi, \mathbb{V}_{\text{max}}(D) \cap \mathbb{V}_{\text{max}}(A) = \emptyset \right\} =: \Sigma.
\]

Prove that

\[
\Sigma \lesssim \sup_{p, c} \left\{ \frac{\| u \chi_{A_{x_0} \setminus D} \|_{L_p(A_{x_0})}}{\beta_D} : D \in J'_\xi \right\}.
\]

To this end, it is sufficient to show that if \( \mathbb{V}(D) \neq \{ \hat{\xi} \} \), then

\[
\frac{\| u \chi_{A_{x_0} \setminus D} \|_{L_p(A_{x_0})}}{\beta_D} \lesssim \Sigma.
\]
Indeed, set \( D_1 = \mathcal{D} \). Then from (54), (55) and (57) it follows that \( \|w_{\mathcal{A}_j \setminus \mathcal{D}}\|_{\psi(\mathcal{A}_j)} \prec \|w_{\mathcal{A}_j \setminus \mathcal{D}}\|_{\psi(\mathcal{A}_j)} \) and \( \beta_\mathcal{D} \prec \beta_\mathcal{D}_1 \). It remains to observe that \( D_1 \in \mathcal{J}_\xi \) and \( V_{\max}(D_1) \cap V_{\max}(\mathcal{A}) = \emptyset \).

**Proposition 2.** Let \( \xi_* \in V(\mathcal{A}), V_1(\xi_*) = \{\xi_1, \ldots, \xi_m\}, \mathcal{D}_j \in \mathcal{J}_\xi, 1 \leq j \leq m, \mathcal{D} = \{\xi_*\} \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_m \). Then

\[
\beta_\mathcal{D}^{-1} = \left\| \left( \beta_{\mathcal{D}_j}^{-1} \right)_j \right\|_{\psi(\mathcal{D})}. \tag{58}
\]

This assertion follows from Theorem \( \square \)

### 5.2 The reduction lemma

Let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an increasing function, \( \psi(0) = 0 \), let \( (\mathcal{A}, \xi_0) \) be a tree with a finite vertex set. In addition, suppose that there exists \( C_* \geq 1 \) such that for any \( j_0, j \in \mathbb{Z}_+, j \geq j_0, \xi \in V_{\xi_j}(\xi_0) \)

\[
\text{card} \ V_{\xi_{j-j_0}}(\xi) \leq C_* 2^{\psi(j)-\psi(j_0)} \tag{59}
\]

Let \( u : V(\mathcal{A}) \to (0, +\infty), u(\xi) = u_j \) for \( \xi \in V_{\xi_j}(\xi_0) \). Suppose that there is \( \sigma \in (0, 1) \) such that for any \( j \in \mathbb{N} \)

\[
\frac{u_j^{-\frac{\psi(j)}{\rho}}}{u_{j-1}^{-\frac{\psi(j-1)}{\rho}}} \geq \sigma^{-\frac{1}{\rho}} \tag{60}
\]

For each \( \xi_* \in V(\mathcal{A}) \) and for any subtree \( \mathcal{D} \in \mathcal{J}_\xi \), we define the quantity \( \beta_{\mathcal{D}} \) by (54). Then \( \beta_{\mathcal{D}_j} = u_j^{-1} \) for \( \xi \in V_{\xi_j}(\xi_0) \), and if \( \mathcal{D} \neq \{\xi_*\} \), then (58) holds.

We set

\[
B_{\mathcal{D}} = \frac{1}{\beta_{\mathcal{D}}}, \quad S_{\mathcal{D}} = \left( \sum_{\xi \in V_{\max}(\mathcal{D})} u^{-p}(\xi) \right)^{-\frac{1}{p}}.
\]

Let \( \mathcal{D} \in \mathcal{J}_\xi, \xi \in V(\mathcal{D}), V_1(\xi) = \{\xi_1, \ldots, \xi_m\} \). Then

\[
\mathcal{D}_\xi = \{\xi\} \cup \mathcal{D}_{\xi_1} \cup \cdots \cup \mathcal{D}_{\xi_m}, \mathcal{D}_{\xi_i} \in \mathcal{J}_{\xi_i}, 1 \leq j \leq m_1.
\]

Let \( \varepsilon > 0, 1 \leq i \leq m_1 \). A vertex \( \xi_i \) is said to be \((\varepsilon, \mathcal{D})\)-regular if

\[
B_{\mathcal{D}_{\xi_i}}^{-p} \geq \frac{B_{\mathcal{D}_{\xi_1}}^{-p} + \cdots + B_{\mathcal{D}_{\xi_{m_1}}}^{-p}}{m_1} \tag{61}
\]

Notice that if \( \varepsilon < 1 \), then at least one of the vertices \( \xi_i \) is \((\varepsilon, \mathcal{D})\)-regular. A path \((\eta_0, \ldots, \eta_l)\) in \( \mathcal{D} \) is said to be \((\varepsilon, \mathcal{D})\)-regular if \( \eta_0 < \eta_1 < \cdots < \eta_l \) and for any \( 1 \leq j \leq l \) the vertex \( \eta_j \) is \((\varepsilon, \mathcal{D})\)-regular.
Lemma 7. There exists \( \hat{\sigma} = \hat{\sigma}(p, C_\ast) > 0 \) such that if (60) holds with \( \sigma \in (0, \hat{\sigma}) \), then for any \( \xi_* \in V(A) \) and for any subtree \( D \in J_{\gamma}' \),

\[
S_D \leq B_D \leq 2S_D. \tag{62}
\]

Proof. Let

\[
\nu_D = \max\{j \in \mathbb{Z}_+: V_j(\xi_*) \neq \emptyset\}.
\]

If \( \nu_D = 0 \), then it follows from the definition that \( S_D = B_D \). Let us prove the assertion for \( \nu_D > 0 \). In this case,

\[
D = \{\xi_*\} \cup D_1 \cup \cdots \cup D_{m_1}, \quad D_j \in J_{\gamma}'_{\xi_j}, \quad \xi_j \in V_1(\xi_*) \tag{63}
\]

Notice that

\[
S_D^{-p} = \sum_{i=1}^{m_1} S_{D_i}^{-p}. \tag{64}
\]

Prove the first inequality, i.e.,

\[
S_D \leq B_D. \tag{65}
\]

Let \( \nu \in \mathbb{Z}_+ \), and let the assertion be proved for any \( D \) such that \( \nu_D \leq \nu \). Prove the assertion for \( \nu_D = \nu + 1 \). From (58) and the induction assumption it follows that

\[
B_D' = B_{\{\xi_*\}}' + \left( \sum_{j=1}^{m_1} B_{D_j}^{-p} \right)^{-\frac{\nu'}{p}} \geq \left( \sum_{j=1}^{m_1} S_{D_j}^{-p} \right)^{-\frac{\nu'}{p}} = S_D'. \tag{66}
\]

Prove the second inequality. It is sufficient to check that

\[
B_D \leq \left( \prod_{j=1}^{\infty} (1 + \sigma_j^{j/2})^{\frac{j}{p'}} \right) S_D \tag{67}
\]

holds for \( \sigma \in (0, \hat{\sigma}(p, C_\ast)) \).

Let \( \varepsilon \in (0, 1) \) (it will be chosen later). Then the end of any \((\varepsilon, D)\)-regular path that has a maximal length and starts from \( \xi_* \) is a maximal vertex in \( D \) (otherwise one of its direct successors is \((\varepsilon, D)\)-regular). Denote by \( l_D \) the maximal length of \((\varepsilon, D)\)-regular paths that start in \( \xi_* \). We show by induction on \( \nu_D \) that for \( \sigma \in (0, \hat{\sigma}(p, C_\ast)) \)

\[
B_D \leq \left( \prod_{j=1}^{l_D} (1 + \sigma_j^{j/2})^{\frac{j}{p'}} \right) S_D. \tag{68}
\]

This implies (67).

If \( \nu_D = 0 \), then \( D \) is a single vertex. Therefore, \( B_D = S_D \) and (67) is true.
Let $\nu_D > 0$. Then (63) holds, and by (59)
\[
m_1 \leq C_*2^{\psi(j_0 + 1) - \psi(j_0)}.
\]

Let $l_i = l_{D_i}$. Denote by $I_1$ the set of $i \in \{1, \ldots, m_1\}$ such that $\xi_i$ is $(\varepsilon, D)$-regular, $I_2 = \{1, \ldots, m_1\} \setminus I_1$. Set
\[
\hat{l} = \max\{l_i : i \in I_1\} + 1.
\]

Then $\hat{l} = l_D$.

Prove that there exists $\sigma_* = \sigma_*(\varepsilon, C_*, p) > 0$ such that for any $\sigma \in (0, \sigma_*)$
\[
\beta_{\{\xi_i\}}^{-p'} \leq \sigma^{\frac{\hat{l}}{p}} \left( \sum_{i=1}^{m_1} B_{D_i}^{-p} \right)^{-\frac{p'}{p}}.
\]
(68)

Suppose the converse, i.e.,
\[
\beta_{\{\xi_i\}}^{-p'} > \sigma^{\frac{\hat{l}}{p}} \left( \sum_{i=1}^{m_1} B_{D_i}^{-p} \right)^{-\frac{p'}{p}}.
\]
(69)

Let $\xi* \in V_{j_0}(\xi_0)$. Then
\[
\beta_{\{\xi_i\}}^{-1} = u_{j_0} = 2^{\psi(j_0)} \cdot \left( \prod_{j=1}^{\hat{l}} \frac{u_{j_0+j-1}^{\psi(j_0+j-1)}}{u_{j_0+j}^{\psi(j_0+j)}} \right) \cdot u_{j_0+1}^{2^{\psi(j_0+1)}} \leq
\]
\[
\leq u_{j_0+1}^{2^{\psi(\hat{l})}} \cdot \sigma^{\frac{\hat{l}}{p}}.
\]

This together with (69) yields
\[
\left( \sum_{i=1}^{m_1} B_{D_i}^{-p} \right)^{-\frac{p'}{p}} < \sigma^2 u_{j_0+1}^{2^{-p'}} \cdot \sigma^\frac{\hat{l}}{p},
\]
i.e.,
\[
\sum_{i=1}^{m_1} B_{D_i}^{-p} > \sigma^{\frac{\hat{l}}{p'}} 2^{\psi(j_0+\hat{l}) - \psi(j_0)} u_{j_0+1}^{-p}.
\]
(70)

Let $(\xi_*, \eta_1, \ldots, \eta_{\hat{l}})$ be an $(\varepsilon, D)$-regular path in $D$. Then
\[
B_{D_{n_1}}^{-p} \geq \frac{\varepsilon}{m_1} \sum_{i=1}^{m_1} B_{D_i}^{-p} > \frac{\varepsilon}{m_1} \sigma^{\frac{\hat{l}}{p'}} 2^{\psi(j_0+\hat{l}) - \psi(j_0)} u_{j_0+1}^{-p}.
\]
(71)
Let \(2 \leq j \leq i\), \(\mathcal{D}_{\eta_j} = \{\eta_{j-1}\} \cup \mathcal{D}_{j,1} \cup \cdots \cup \mathcal{D}_{j,m_j}\). Then

\[
B_{\mathcal{D}_{\eta_j}} = B_{\{\eta_{j-1}\}} + \left( \sum_{i=1}^{m_j} B_{\mathcal{D}_{j,i}}^{-p} \right)^{-\frac{2}{p}} \geq \left( \sum_{i=1}^{m_j} B_{\mathcal{D}_{j,i}}^{-p} \right)^{-\frac{2}{p}},
\]

i.e., \(B_{\mathcal{D}_{\eta_j}}^{-p} \leq \sum_{i=1}^{m_j} B_{\mathcal{D}_{j,i}}^{-p}\). Since the vertex \(\eta_j\) is \((\varepsilon, \mathcal{D})\)-regular, we have

\[
B_{\mathcal{D}_{\eta_j}}^{-p} \geq \frac{\varepsilon}{m_j} \sum_{i=1}^{m_j} B_{\mathcal{D}_{j,i}}^{-p} > \frac{\varepsilon}{m_j} B_{\mathcal{D}_{\eta_{j-1}}}. \tag{71}
\]

Therefore,

\[
B_{\mathcal{D}_{\eta_j}}^{-p} \geq \frac{\varepsilon}{m_j} \sum_{i=1}^{m_j} B_{\mathcal{D}_{j,i}}^{-p} \geq \frac{\varepsilon^2}{m_{j-1} m_j} B_{\mathcal{D}_{\eta_{j-1}}}^{-p} \geq \cdots \geq \frac{\varepsilon^{i-1}}{m_2 \cdots m_i} B_{\mathcal{D}_{\mathcal{D}_{\eta_{i-1}}}} \geq \frac{\varepsilon}{m_1 m_2 \cdots m_i} 2^{\psi(j_0+i)-\psi(j_0)} u_{j_0+i}^{-p} \sigma^{-\frac{i}{p}}. \tag{71}
\]

The vertex \(\eta_l\) is maximal in \(\mathcal{D}\). Hence, \(B_{\mathcal{D}_{\mathcal{D}_{\eta_l}}} = B_{\{\eta_l\}} = u_{j_0+l}\). In addition,

\[
m_1 m_2 \cdots m_i \leq C_s \prod_{i=0}^{l-1} 2^{\psi(j_0+i)-\psi(j_0)} = C_s 2^{\psi(j_0+l)-\psi(j_0)}. \tag{59}
\]

Thus,

\[
u_{j_0+l}^{-p} \geq \frac{\varepsilon^l}{C_s 2^{\psi(j_0+l)-\psi(j_0)}} 2^{\psi(j_0+i)-\psi(j_0)} u_{j_0+i}^{-p} \sigma^{-\frac{i}{p}}, \]

i.e., \(1 \geq \varepsilon^l C_s^{-1} \sigma^{-\frac{l}{p}}, \) or \(\sigma^{\frac{p}{l}} \geq \varepsilon C_s^{-1}\). For \(0 < \sigma \leq \frac{(C_s^{-1})^{\frac{2p}{l}}}{2}\) we get the contradiction. This proves \((68)\).

Now let us prove \((67)\). We have

\[
B_{\mathcal{D}}^{-p} = \beta_{\{\xi_s\}}^{-p} + \left( \sum_{i=1}^{m_1} B_{\mathcal{D}_i}^{-p} \right)^{-\frac{2}{p}} \leq \left( \sum_{i=1}^{m_1} B_{\mathcal{D}_i}^{-p} \right)^{-\frac{2}{p}}. \tag{72}
\]

Show that there exists \(\varepsilon_* = \varepsilon_* (p) \in (0, 1)\) such that for any \(\varepsilon \in (0, \varepsilon_*)\), \(0 < \sigma < \min \left( \frac{1}{2}, \sigma_*(\varepsilon, C_s, p) \right)\)

\[
\left( B_{\mathcal{D}_1}^{-p} + \cdots + B_{\mathcal{D}_{m_1}}^{-p} \right)^{-\frac{1}{p}} \leq \left( S_{\mathcal{D}_1}^{-p} + \cdots + S_{\mathcal{D}_{m_1}}^{-p} \right)^{-\frac{1}{p}} \left( \prod_{j=1}^{l-1} (1 + \sigma^{j/2} \hat{\psi}) \right) \cdot (1 + \sigma^{l/2} \hat{\psi}). \tag{73}
\]
Then (72), (64) and (73) yield (67).

The relation (73) is equivalent to

\[
\sum_{i=1}^{m_1} B_{D_i}^p - \frac{\sum_{i=1}^{m_1} S_{D_i}^{-p}}{(1 + \sigma^{j/2})^{\frac{p}{2}} \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}}} \geq 0. \tag{74}
\]

Consider separately sums in \( i \in I_1 \) and in \( i \in I_2 \). Let \( l = \max_{1 \leq i \leq m_1} l_i + 1 \). By the induction hypotheses,

\[
B_{D_i} \leq \left( \prod_{j=1}^{l_i} (1 + \sigma^{j/2})^{\frac{p}{2}} \right) S_{D_i} \leq \left( \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}} \right) S_{D_i}, \quad i \in I_1, \tag{75}
\]

\[
B_{D_i} \leq \left( \prod_{j=1}^{l_i} (1 + \sigma^{j/2})^{\frac{p}{2}} \right) S_{D_i} \leq \left( \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}} \right) S_{D_i}, \quad i \in I_2. \tag{76}
\]

Hence,

\[
\sum_{i \in I_1} B_{D_i}^p - \frac{\sum_{i \in I_1} S_{D_i}^{-p}}{(1 + \sigma^{j/2})^{\frac{p}{2}} \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}}} \geq \left( \sum_{i \in I_1} S_{D_i}^{-p} \right) (1 + \sigma^{j/2})^{\frac{p}{2}} \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}} \geq \left( \sum_{i \in I_1} B_{D_i}^{-p} \right) \sigma^{j/2}
\]

(the penultimate relation holds for \( 0 < \sigma < \frac{1}{2} \)). Therefore, there exists \( C_1(p) > 0 \) such that

\[
\sum_{i \in I_1} B_{D_i}^p - \frac{\sum_{i \in I_1} S_{D_i}^{-p}}{(1 + \sigma^{j/2})^{\frac{p}{2}} \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}}} \geq C_1(p) \left( \sum_{i \in I_1} B_{D_i}^{-p} \right) \sigma^{j/2}. \tag{77}
\]

If \( l = \hat{l} \), then the sum in \( i \in I_2 \) is estimated similarly. In this case, (74) is proved. Let \( l \geq \hat{l} + 1 \). Then we have for \( 0 < \sigma < \min \left( \frac{1}{4}, \sigma_*(\varepsilon, C_*, p) \right) \)

\[
\sum_{i \in I_2} B_{D_i}^p - \frac{\sum_{i \in I_2} S_{D_i}^{-p}}{(1 + \sigma^{j/2})^{\frac{p}{2}} \prod_{j=1}^{l-1} (1 + \sigma^{j/2})^{\frac{p}{2}}} \geq \left( \sum_{i \in I_2} B_{D_i}^{-p} \right) \sigma^{j/2}. \tag{78}
\]

26
where $C_2(p) > 0$, $C_3(p) > 0$. Thus,

$$
\sum_{i \in I_2} B_{D_i}^p - \frac{\sum_{i \in I_1} S_{D_i}^{-p}}{(1 + \sigma^{i/2})^{\frac{2p}{p}}} \prod_{j=1}^{l-1}(1 + \sigma^{j/2})^{\frac{2p}{p}} \geq -C_3(p) \left( \sum_{i \in I_2} B_{D_i}^p \right) \sigma^{i/2}.
$$

(78)

From definitions of $I_1$ and $I_2$ we get

$$
\sum_{i \in I_2} B_{D_i}^p \leq \sum_{i \in I_2} \frac{\varepsilon}{m_1} \sum_{j=1}^{m_1} B_{D_j}^p \leq \varepsilon \sum_{j=1}^{m_1} B_{D_j}^p,
$$

$$
\sum_{i \in I_1} B_{D_i}^p = \sum_{i=1}^{m_1} B_{D_i}^p - \sum_{i \in I_2} B_{D_i}^p \geq (1 - \varepsilon) \sum_{i=1}^{m_1} B_{D_i}^p.
$$

This together with (77) and (78) implies that

$$
\sum_{i=1}^{m_1} B_{D_i}^p - \frac{\sum_{i=1}^{m_1} S_{D_i}^{-p}}{(1 + \sigma^{i/2})^{\frac{2p}{p}}} \prod_{j=1}^{l-1}(1 + \sigma^{j/2})^{\frac{2p}{p}} \geq C_1(p) \left( \sum_{i \in I_1} B_{D_i}^p \right) \sigma^{i/2} - C_3(p) \left( \sum_{i \in I_2} B_{D_i}^p \right) \sigma^{i/2} \geq
$$

$$
\sigma^{i/2} ((1 - \varepsilon)C_1(p) - \varepsilon C_3(p)) \sum_{i=1}^{m_1} B_{D_i}^p > 0
$$

for sufficiently small $\varepsilon$. This completes the proof of (74). \(\square\)

Let $w : V(A) \to (0, \infty)$, $w(\xi) = w_j$ for $\xi \in V^j(\xi_0)$. Suppose that there exists $\sigma \in (0, \frac{1}{2})$ such that for any $j \in \mathbb{N}$

$$
\frac{w_j \cdot 2^{\sigma(j)}}{w_{j-1} \cdot 2^{\sigma(j-1)}} \leq \sigma^{\frac{1}{p}}.
$$

(79)
Given $D \in \mathcal{J}_t$, we denote
\[
R_D = \left( \sum_{\xi \in \mathcal{V}_{\text{max}}(D)} \sum_{\xi' \geq \xi} w^p(\xi') \right)^{1/p}, \quad Q_D = \left( \sum_{\xi \in \mathcal{V}_{\text{max}}(D)} w^p(\xi) \right)^{1/p}.
\] (80)

From (79) and (59) it follows that there exists $\sigma_\ast = \sigma_\ast(p, C_\ast) > 0$ such that for any $0 < \sigma < \sigma_\ast$
\[
Q_D \leq R_D \leq 2Q_D.
\] (81)

Construct the function $\psi_\ast$ by induction as follows:
\[
\psi_\ast(0) = 0, \quad 2^{\psi_\ast(j) - \psi_\ast(j-1)} = [2^{\psi(j) - \psi_\ast(j-1)}], \quad j \in \mathbb{N}.
\] (82)

Then $2^{\psi_\ast(j) - \psi_\ast(j-1)} \in \mathbb{N}$ and
\[
2^{\psi_\ast(j)} \leq 2^{\psi(j)} \leq 2^{\psi_\ast(j)+1}.
\] (83)

Let $\xi_\ast \in \mathcal{V}_{j_0}(\xi_0)$, and let $(\hat{A}, \hat{\xi})$ be a tree such that
\[
\text{card } \mathcal{V}_\hat{A}(\hat{\xi}) = 2^{\psi(\hat{\xi}+1) - \psi_\ast(\hat{\xi})}, \quad \xi \in \mathcal{V}_\hat{A}(\hat{\xi}), \quad j \geq j_0.
\] (84)

**Lemma 8.** Let $\mathcal{D} \subset \mathcal{A}$ be a tree rooted at $\xi_\ast \in \mathcal{V}_{j_0}(\xi_0)$. Then there exists $\sigma_0 = \sigma_0(p, C_\ast) > 0$ satisfying the following property: if (60) and (79) hold for some $\sigma \in (0, \sigma_0)$, then there exists a tree $\mathcal{D} \subset \mathcal{A}$ rooted at $\xi$ such that $S_\mathcal{D} \leq S_D$ and $Q_\mathcal{D} \leq Q_D$.

**Proof.** Set
\[
\{j_1, \ldots, j_s\} = \{j \in \mathbb{N} : \mathcal{V}_{\text{max}}(D) \cap \mathcal{V}_{j-j_0}(\xi_\ast) \neq \emptyset\}, \quad j_1 < \cdots < j_s.
\]

For each $1 \leq l \leq s$, we denote
\[
\mathcal{V}_{l,D} = \mathcal{V}_{\text{max}}(D) \cap \mathcal{V}_{j-l}(\xi_\ast),
\]
\[
\mathcal{U}_s = \mathcal{V}_{j-1}(\xi_\ast) \cap \mathcal{V}(D) \setminus \mathcal{V}_{\text{max}}(D).
\]

Then
\[
\mathcal{V}_{s,D} \subset \bigcup_{\xi \in \mathcal{U}_s} \mathcal{V}_{j_s-j_s-1}(\xi).
\] (85)

By (59) and (79), there exists $\sigma_1 = \sigma_1(p, C_\ast)$ such that for any $\sigma \in (0, \sigma_1)$, $1 \leq \nu \leq s$
\[
w_{j_\nu}^p \cdot \text{card } \mathcal{V}_{j_\nu-j_\nu-1}(\xi) \leq w_{j_\nu}^p.
\] (86)

Show that for any $\sigma \in (0, \sigma_1)$
\[
\sum_{l=\nu}^s w_{j_\nu}^p \cdot \text{card } \mathcal{V}_{l,D} \leq w_{j_\nu}^p \cdot \text{card } \mathcal{V}_{j_\nu-j_\nu}(\xi_\ast).
\] (87)
We use induction on \( s - \nu \). If \( s - \nu = 0 \), then the inequality is trivial. Let \( s - \nu \geq 1 \). Denote by \( \mathcal{D} \) the subtree in \( \mathcal{A}_\xi \) with the set of maximal vertices \( (\cup_{t=1}^{s-1} V_{t,\mathcal{D}}) \cup U_s \) and the root \( \xi_\ast \). Then

\[
\sum_{t=\nu}^{s} w_{t,\mathcal{D}}^p \text{card } V_{t,\mathcal{D}} \leq \sum_{t=\nu}^{s-1} w_{t,\mathcal{D}}^p \text{card } V_{t,\mathcal{D}} + w_{j_{s-1},\mathcal{D}}^p \text{card } U_s = \sum_{t=\nu}^{s-1} w_{t,\mathcal{D}}^p \text{card } V_{t,\mathcal{D}} \leq w_{j_{\nu-1},\mathcal{D}}^p \text{card } V_{j_{\nu-1},\mathcal{D}}^\ast (\xi_\ast)
\]

(the last inequality holds by the induction assumption). This completes the proof of (84).

Applying induction on \( l \), construct the set \( V_{l,\hat{\mathcal{D}}} \subset V(\hat{\mathcal{A}}_\xi) \) with the following properties:

1. if \( 1 \leq t < \nu \leq l \), then \( V_{t,\mathcal{D}} \cap \left( \cup_{\xi \in V_{t,\mathcal{D}}} V_{j_{t-1},\mathcal{D}}^\ast (\xi) \right) = \emptyset \); (88)

2. if \( \cup_{t=1}^{l} \cup_{\xi \in V_{t,\mathcal{D}}} V_{j_{t-1},\mathcal{D}}^\ast (\xi) = V_{j_{l-1},\mathcal{D}}^\ast (\hat{\xi}) \), (89)

then the tree \( \hat{\mathcal{D}} \) with the set of vertices

\[
V(\hat{\mathcal{D}}) = \cup_{t=1}^{l} \cup_{\xi \in V_{t,\mathcal{D}}} [\hat{\xi}, \xi]
\]

satisfies \( V_{\max}(\hat{\mathcal{D}}) = \cup_{1 \leq t \leq l} V_{t,\hat{\mathcal{D}}} \), \( S_{\hat{\mathcal{D}}} \lesssim S_{\mathcal{D}} \) and \( Q_{\hat{\mathcal{D}}} \lesssim Q_{\mathcal{D}} \);

3. if \( \cup_{t=1}^{l} \cup_{\xi \in V_{t,\mathcal{D}}} V_{j_{t-1},\mathcal{D}}^\ast (\xi) \neq V_{j_{l-1},\mathcal{D}}^\ast (\hat{\xi}) \), (90)

then \( \text{card } V_{t,\hat{\mathcal{D}}} = \text{card } V_{t,\mathcal{D}} \) for any \( 1 \leq t \leq l \).

If (89) holds for some \( l \), then the construction is interrupted. In this case, \( \hat{\mathcal{D}} \) is the desired tree. If (90) holds for any \( l \leq s \), then we take as \( \hat{\mathcal{D}} \) the tree with the vertex set \( \cup_{1 \leq t \leq s} \cup_{\xi \in V_{t,\mathcal{D}}} [\hat{\xi}, \xi] \). In this case, \( S_{\hat{\mathcal{D}}} = S_{\mathcal{D}} \) and \( Q_{\hat{\mathcal{D}}} = Q_{\mathcal{D}} \).

**The base of induction.** Let \( l = 1 \). If \( \text{card } V_{1,\mathcal{D}} < \frac{1}{2} 2^{\psi_{j_{1}} - \psi_{j_{0}}} \), then we take as \( V_{1,\hat{\mathcal{D}}} \) an arbitrary subset \( E_1 \subset V_{j_{1-1},\mathcal{D}}^\ast (\hat{\xi}) \) such that \( \text{card } E_1 = \text{card } V_{1,\mathcal{D}} \). By (84), we have (90).

Let \( \text{card } V_{1,\mathcal{D}} \geq \frac{1}{2} 2^{\psi_{j_{1}} - \psi_{j_{0}}} \). Then we set \( V_{1,\hat{\mathcal{D}}} = V_{j_{1-1},\mathcal{D}}^\ast (\hat{\xi}) \) (in this case, (89) holds). Hence, \( V(\hat{\mathcal{D}}) = \cup_{j=0}^{j_{l-1}} V_{j,\mathcal{D}}^\ast (\hat{\xi}) \), \( V_{\max}(\hat{\mathcal{D}}) = V_{1,\mathcal{D}} \) and

\[
S_{\hat{\mathcal{D}}}^{-p} = 2^{\psi_{j_{1}} - \psi_{j_{0}}} w_{j_{1}}^{-p} , \quad Q_{\hat{\mathcal{D}}}^{p} = 2^{\psi_{j_{1}} - \psi_{j_{0}}} w_{j_{1}}^{p}.
\]
Further,
\[ S_D^{-p} \geq \text{card } V_{1,D} \cdot u_{j_1}^{-p} \geq \frac{1}{2} 2^{\psi_+(j_1)-\psi_+ (j_0)} u_{j_1}^{-p}, \]
which implies \( S_D \leq S_D^{-p} \). Prove that \( Q_D \leq Q_D^{-p} \). Indeed,

\[ Q_D = \sum_{t=1}^{s} \text{card } V_{t,D} \cdot w_j^p \leq w_j^p \cdot \text{card } V_{j_1-j_0} (\xi_*) \leq \frac{1}{2} 2^{\psi_+(j_1)-\psi_+ (j_0)} = Q_D^p. \]

**The induction step.** Let \( 1 \leq l < s \),

\[ \sum_{t=1}^{l} \text{card } V_{t,D} \cdot 2^{\psi_+(j_l)-\psi_+ (j_l)} < \frac{1}{2} 2^{\psi_+(j_l)-\psi_+ (j_0)}. \]  

(91)

Suppose that there are the sets \( V_{t,D} \subset V_{j_l-j_0} (\hat{\xi}) \), \( 1 \leq t \leq l \), satisfying (88) and

\[ \text{card } V_{t,D} = \text{card } V_{t,D}, \quad 1 \leq t \leq l. \]  

(92)

Then

\[ \sum_{t=1}^{l} \sum_{\xi \in V_{t,D}} \text{card } V_{j_{l-1}-j_l} (\xi) \leq \sum_{t=1}^{l} \text{card } V_{t,D} \cdot 2^{\psi_+(j_l)-\psi_+ (j_l)} \leq \frac{1}{2} 2^{\psi_+(j_l)-\psi_+ (j_0)} \leq \text{card } V_{j_l-j_0} (\hat{\xi}). \]

Therefore, properties 1–3 of the sets \( V_{t,D} \) hold (property 2 is trivial, since (90) holds instead of (89); property 3 follows from (92), property 1 holds since we supposed that the sets satisfy (88)).

Construct the set \( V_{l+1,D} \subset V_{j_l-j_0} (\hat{\xi}) \setminus \bigcup_{t=1}^{l} \bigcup_{\xi \in V_{t,D}} V_{j_{l+1}-j_l} (\xi) \).

Let

\[ \text{card } V_{l+1,D} + \sum_{t=1}^{l} \text{card } V_{t,D} \cdot 2^{\psi_+(j_{l+1})-\psi_+ (j_l)} < \frac{1}{2} 2^{\psi_+(j_{l+1})-\psi_+ (j_0)}. \]  

(93)

In this case, we take an arbitrary subset

\[ V_{l+1,D} \subset V_{j_{l+1}-j_0} (\hat{\xi}) \setminus \bigcup_{t=1}^{l} \bigcup_{\xi \in V_{t,D}} V_{j_{l+1}-j_l} (\xi), \quad \text{card } V_{l+1,D} = \text{card } V_{l+1,D}. \]

This set exists, since

\[ \text{card } V_{l+1,D} + \sum_{t=1}^{l} \sum_{\xi \in V_{t,D}} \text{card } V_{j_{l+1}-j_l} (\xi) \leq \text{card } V_{l+1,D} + \sum_{t=1}^{l} \text{card } V_{t,D} \cdot 2^{\psi_+(j_{l+1})-\psi_+ (j_l)} \]

\[ < \frac{1}{2} 2^{\psi_+(j_{l+1})-\psi_+ (j_0)} \leq \text{card } V_{j_{l+1}-j_0} (\hat{\xi}). \]
Then we have (88), (91) and (92) with \( l + 1 \) instead of \( l \). Hence, properties 1–3 for the sets \( \{ V_{t,D} \}_{t=1}^{l+1} \) hold.

Let

\[
\text{card } V_{l+1,D} + \sum_{t=1}^{l} \text{card } V_{t,D} \cdot 2^{\psi^*(j_{t+1}) - \psi^*(j_t)} \geq \frac{1}{2} 2^{\psi^*(j_{l+1}) - \psi^*(j_0)}. \tag{94}
\]

Then we set

\[
V_{l+1,D} = V_{j_{l+1}-j_0}(\hat{\xi}) \setminus \bigcup_{t=1}^{l} \bigcup_{\xi \in V_{t,D}} V_{j_{t+1}-j_t}(\xi). \tag{95}
\]

By construction, we have property 1 of the sets \( \{ V_{t,D} \}_{t=1}^{l+1} \) and (89) (with \( l + 1 \) instead of \( l \)); i.e.,

\[
V_{\nu,D} \cap \left( \bigcup_{\xi \in V_{t,D}} V_{j_{t+1}-j_t}(\xi) \right) = \emptyset, \quad 1 \leq t < \nu \leq l + 1,
\tag{96}
\]

\[
\bigcup_{t=1}^{l+1} \bigcup_{\xi \in V_{t,D}} V_{j_{t+1}-j_t}(\xi) = V_{j_{l+1}-j_0}(\hat{\xi}).
\]

Therefore, it is sufficient to check property 2. Define the tree \( \hat{D} \) by

\[
V(\hat{D}) = \bigcup_{t=1}^{l+1} \bigcup_{\xi \in V_{t,D}} [\hat{\xi}, \xi].
\]

From (96) it follows that

\[
V_{\text{max}}(\hat{D}) = \bigcup_{t=1}^{l+1} V_{t,D}. \tag{97}
\]

We claim that \( S_D \lesssim_p S_{\hat{D}} \) and \( Q_D \lesssim_p Q_{\hat{D}} \). Indeed,

\[
S_D^{-p} \overset{(92), (97)}{=} \sum_{t=1}^{l} u_{jt}^{-p} \text{card } V_{t,D} + u_{j_{t+1}}^{-p} \text{card } V_{l+1,D}, \tag{98}
\]

\[
S_D^{-p} \geq \sum_{t=1}^{l} u_{jt}^{-p} \text{card } V_{t,D} + u_{j_{t+1}}^{-p} \text{card } V_{l+1,D}, \tag{99}
\]

\[
Q_D^{p} \overset{(92), (97)}{=} \sum_{t=1}^{l} w_{jt}^{p} \text{card } V_{t,D} + w_{j_{t+1}}^{p} \text{card } V_{l+1,D}, \tag{100}
\]

\[
Q_D^{p} \overset{(92), (97)}{=} \sum_{t=1}^{l} w_{jt}^{p} \text{card } V_{t,D} + \sum_{t=1}^{s} w_{jt}^{p} \text{card } V_{t,D} \lesssim \sum_{t=1}^{l} w_{jt}^{p} \text{card } V_{t,D} + w_{j_{t+1}}^{p} \text{card } V_{j_{t+1}-j_0}(\xi), \tag{87}
\]

\[
\lesssim \sum_{t=1}^{l} w_{jt}^{p} \text{card } V_{t,D} + w_{j_{t+1}}^{p} \cdot 2^{\psi^*(j_{t+1}) - \psi^*(j_0)}.
\tag{101}
\]
In addition,
\[
\text{card } V_{l+1,D} \leq \text{card } V_{j_{l+1}-j_0}^{\hat{A}}(\hat{\xi}) \overset{(84)}{=} 2^{\psi_*(j_{l+1})-\psi_*(j_0)}.
\] (102)

**Case 1.** Let
\[
\sum_{t=1}^l \text{card } V_{t,D} \cdot 2^{\psi_*(j_{t+1})-\psi_*(j_t)} < \frac{1}{4} 2^{\psi_*(j_{l+1})-\psi_*(j_0)}.
\] (103)

Then
\[
\text{card } V_{l+1,D} \geq 2^{\psi_*(j_{l+1})-\psi_*(j_0)-2}.
\] (104)

Indeed,
\[
\text{card } V_{l+1,D} \overset{(84), (103)}{\geq} \frac{1}{2} 2^{\psi_*(j_{l+1})-\psi_*(j_0)} - \frac{1}{4} 2^{\psi_*(j_{l+1})-\psi_*(j_0)} = \frac{1}{4} 2^{\psi_*(j_{l+1})-\psi_*(j_0)}.
\]

From (98), (99), (102) and (104) it follows that \( S_D \lesssim S_\ell_D \).

Prove that \( Q_D \overset{p, C_*}{\lesssim} Q_\ell_D \). By (100) and (101), it suffices to check that \( \text{card } V_{l+1,D} \geq 2^{\psi_*(j_{l+1})-\psi_*(j_0)-1} \). We have
\[
\text{card } V_{l+1,D} \overset{(84), (92), (95)}{\geq} \text{card } V_{j_{l+1}-j_0}(\hat{\xi}) - \sum_{t=1}^l \text{card } V_{t,D} \cdot 2^{\psi_*(j_{t+1})-\psi_*(j_t)} \overset{(103)}{\geq}
\]\[
= 2^{\psi_*(j_{l+1})-\psi_*(j_0)} - \frac{1}{4} 2^{\psi_*(j_{l+1})-\psi_*(j_0)} \geq 2^{\psi_*(j_{l+1})-\psi_*(j_0)-1}.
\]

**Case 2.** Let
\[
\sum_{t=1}^l \text{card } V_{t,D} \cdot 2^{\psi_*(j_{t+1})-\psi_*(j_t)} \geq \frac{1}{4} 2^{\psi_*(j_{l+1})-\psi_*(j_0)}.
\] (105)

Then by (60), (79) and (83), there exists \( \sigma_1' = \sigma_1'(p, C_*) \) such that for any \( \sigma \in (0, \sigma_1') \)
\[
\sum_{t=1}^l u_{j_t}^{-p} \text{card } V_{t,D} \geq \sum_{t=1}^l u_{j_{t+1}}^{-p} 2^{\psi_*(j_{t+1})-\psi_*(j_t)} \text{card } V_{t,D} \overset{(105)}{\geq} \frac{u_{j_t}^{-1}}{4} 2^{\psi_*(j_{t+1})-\psi_*(j_0)},
\]
\[
\sum_{t=1}^l w_{j_t}^{-p} \text{card } V_{t,D} \geq \sum_{t=1}^l w_{j_{t+1}}^{-p} 2^{\psi_*(j_{t+1})-\psi_*(j_t)} \text{card } V_{t,D} \overset{(105)}{\geq} \frac{u_{j_{t+1}}^{-1}}{4} 2^{\psi_*(j_{t+1})-\psi_*(j_0)}.
\]

This together with (98), (99), (100), (101) and (102) implies that
\[
S_D^{-p} \geq \sum_{t=1}^l u_{j_t}^{-p} \text{card } V_{t,D} \lesssim S_\ell_D^{-p},
\]
\[
Q_D^{-p} \lesssim \sum_{t=1}^l w_{j_t}^{-p} \text{card } V_{t,D} \lesssim Q_\ell_D^{-p}.
\]

This completes the proof. \( \Box \)
Let (56) and (57) hold, let $\hat{\sigma}$ be such as in Lemma 7, and let $\sigma_0$ be such as in Lemma 8. Take $\sigma \in (0, \min\{\hat{\sigma}, \sigma_0\})$. By Lemma 6

$$\mathcal{G}_{A_{\xi, u, w}} \leq \sup_{p, C} \sup_{D \in J_{\xi}^t} \|w\chi_{A_{\xi, u, w}}\|_{L^p(A_{\xi, u, w})} B_D \leq \sup_{D \in J_{\xi}^t} R_D B_D \leq \sup_{D \in J_{\xi}^t} Q_D S_D.$$  \hspace{1cm} (106)

Lemma 9. Let $\xi_* \in V_{j_0}^A(\xi_0), \hat{u}(\xi) = u_j, \hat{\omega}(\xi) = w_j$ for any $\xi \in V_{j_0}^A(\hat{\xi})$. Then there exists $\sigma_2 = \sigma_2(p, C_*) > 0$ such that $\mathcal{G}_{A_{\xi, u, w}} \leq \mathcal{G}_{A_{\hat{\xi}, \hat{u}, \hat{\omega}}}$ for any $\sigma \in (0, \sigma_2)$.

Proof. Suppose that the supremum of the right-hand side in (106) is attained at the tree $D \in J_{\xi}^t$. Apply Lemma 8 and construct the tree $\hat{D} \subset A$ rooted at $\xi$ such that $S_D \lesssim S_{\hat{D}}$ and $Q_D \lesssim Q_{\hat{D}}$. Apply (106) to the trees $D$ and $\hat{D}$ and notice that

$$\mathcal{G}_{A_{\xi, u, w}} \leq \mathcal{G}_{A_{\hat{\xi}, \hat{u}, \hat{\omega}}}$$

(see Remark 4).

5.3 Estimates for the special class of weights

Let $r = d$, $p = q$ and let the conditions (2), (3), (4), (5), (6), (7), (8), (9), (10) hold. From (7) it follows that $\beta \leq d$.

Let $T, F$ be the tree and the mapping such as in Lemma 2 and let $\mathfrak{F} = \mathfrak{F}(a, d) \in \mathbb{N}$ be such as in Proposition 1. Let $m \in \mathbb{N}$ be divisible in $\mathfrak{F}$. Consider the partition $\{D_{j,i}\}_{j \in \mathbb{Z}_+, i \in I_j}$ of the tree $T$ defined at the page 12. Fix $N \in \mathbb{N}$. Let $A = A(m)$ be the tree with the set of vertices $\{\eta_{j,i}\}_{0 \leq j < N, i \in I_j}$ and with the set of edges defined by

$$V_1^A(\eta_{j,i}) = \{\eta_{j+1,s}\}_{s \in I_{j,i}}.$$  \hspace{1cm} (107)

Here $I_{j,i}$ is defined in (50). By Remark 3, if $D_{j', i'}$ follows the tree $D_{j,i}$, then $j' = j + 1$ and $i' \in I_{j,i}$. Hence, card $V_1^A(\eta_{j,i}) = \text{card} I_{j,i}$ for any $l \in \mathbb{Z}_+$.

By Lemma 5 for any $j_0, j \in \{0, \ldots, N\}, j \geq j_0$, and for any $\xi \in V_{j_0}^A(\eta_{b, 1})$ we have

$$\text{card} V_{j-j_0}^A(\xi) \lesssim_{a, d, co} h(2^{-mj_0}) \leq 2^{\psi(j)-\psi(j_0)}$$ \hspace{1cm} (107)

with

$$\psi(j) = m\theta j - \log_2 \Lambda(2^{-mj}).$$  \hspace{1cm} (108)
Denote $\xi_0 = \eta_{0.1}$. Set

$$
u_j := u(\xi) = \varphi_u(2^{-mj}) \cdot 2^{-\frac{mj}{p}} \frac{1}{2} 2^{mj} (\beta_d^\frac{d}{p} - \frac{d}{p}) \Psi_g(2^{-mj}),$$

$$w_j := w(\xi) = \varphi_v(2^{-mj}) \cdot 2^{-\frac{mj}{p}} \frac{1}{2} 2^{mj} (\beta_d^\frac{d}{p} - \frac{d}{p}) \Psi_v(2^{-mj}), \quad \xi \in V^A_\xi(\xi_0).\tag{109}$$

Lemma 10. There exists $m_\ast = m_\ast(3) \in \mathbb{N}$ such that for any $m \geq m_\ast$, $\xi_\ast \in V^A_{0.1}(\xi_0)$ we have $\mathcal{S}_{A_{\xi}, u, w} \lesssim 2^{mj_\ast(\beta_d - d)} \Psi(2^{-mj_\ast})$ in the case a) of (11); in the case b) for $\alpha > 0$ we have $\mathcal{S}_{A_{\xi}, u, w} \lesssim 2^{-\alpha j_\ast(\beta_0 - \rho_0)}$. If $\alpha = 0$ and $\rho \equiv 1$, then $\mathcal{S}_{A_{\xi}, u, w} \lesssim \frac{1}{3}$.

**Proof.** First suppose that

$$\beta_d - \frac{d}{p} = \frac{\theta}{p} > 0.\tag{110}$$

We have

$$\nu_j \cdot 2^{-\frac{v(j)}{p}} = 2^{mj} (\beta_d^\frac{d}{p} - \frac{d}{p}) \cdot 2^{mj} (\beta_d^\frac{d}{p} - \frac{d}{p}) \Psi_g(2^{-mj}),$$

$$w_j \cdot 2^{-\frac{v(j)}{p}} = 2^{mj} (\beta_d^\frac{d}{p} - \frac{d}{p}) \cdot 2^{mj} (\beta_d^\frac{d}{p} - \frac{d}{p}) \Psi_v(2^{-mj}).$$

From (6) and (110) it follows that (60) and (79) hold with $\psi = \psi_\ast$ defined in (82), and let $\hat{\psi}(\xi) = \nu_j$, $\hat{\psi}(\xi) = w_j$ for $\xi \in V_{j_\ast}(\xi_0)$. By Lemma 9, $\mathcal{S}_{A_{\xi}, u, w} \lesssim \mathcal{S}_{A_{\xi}, u, w}$.

Denote by $\mathcal{S}_{A_{\xi}, \hat{\nu}, \hat{\psi}}$ equals to the minimal constant $C$ in

$$\sup_{\xi \in V_{A_{\xi}}} \hat{\nu}^p(\xi) \left( \sum_{\xi \leq \xi' \leq \xi} \hat{\nu}(\xi') f^{1/p}(\xi') \right)^p \leq C^p \sum_{\xi \in V_{A_{\xi}}} f(\xi), \quad f : V(A_{\xi}) \to \mathbb{R}_+.\tag{111}$$

We claim that the function $\mathcal{F}$ is concave. Indeed, let $\lambda \in [0, 1]$, $f_1, f_2 : V(A_{\xi}) \to \mathbb{R}_+$. Applying the inverse Minkowski inequality and the homogeneity property, we get

$$\sup_{\xi \in V(A_{\xi})} \hat{\nu}^p(\xi) \left( \sum_{\xi \leq \xi' \leq \xi} \hat{\nu}(\xi') f_1^{1/p}(\xi') \right)^p \geq$$

$$\geq (1 - \lambda) \sum_{\xi \in V(A_{\xi})} \hat{\nu}^p(\xi) \left( \sum_{\xi \leq \xi' \leq \xi} \hat{\nu}(\xi') f_1^{1/p}(\xi') \right)^p +$$

$$+ \lambda \sum_{\xi \in V(A_{\xi})} \hat{\nu}^p(\xi) \left( \sum_{\xi \leq \xi' \leq \xi} \hat{\nu}(\xi') f_2^{1/p}(\xi') \right)^p.$$
Set $n_j = \text{card } V_j^A(\xi)$, $\xi \in V_j^A(\xi)$, $j \in \mathbb{Z}_+$. It follows from (84) that this quantity does not depend on $\xi$. Prove that

$$
\sup \{ F(f) : \| f \|_{l_1(A)} \leq 1 \} = \sup \{ F(f) : \| f \|_{l_1(A)} \leq 1, \forall j \in \mathbb{Z}_+, \forall \xi', \xi'' \in V_j(\xi) f(\xi') = f(\xi'') \}
$$

(112)

(see the notation (53)).

Construct $f_{k;i_1,...,i_k,i_{k+1}}$ by induction on $k \in \{0, 1, \ldots, N - j_0\}$. Set $f_0 = f(\hat{x})$.

Let $0 \leq k \leq N - j_0 - 1$, $f_{k;i_1,...,i_k} = f(\xi)$ for some $\xi \in V_k^A(\hat{x})$. Then we define $f_{k+1;i_1,...,i_k,i_{k+1}}$ for $1 \leq i_{k+1} \leq n_k$ so that

$$
\{ f_{k+1;i_1,...,i_k,i_{k+1}} \}_{i_{k+1}=1}^{n_k} = \{ f(\xi') : \xi' \in V_k^A(\xi) \}.
$$

Denote by $S_j$ the set of permutations of $j$ elements.

For $0 \leq t \leq N - j_0 - 1$, $\sigma \in S_n$, we get

$$(f^t,\sigma)_{k;i_1,...,i_k} = \begin{cases} f_{k;i_1,...,i_k} & \text{for } k \leq t, \\ f_{k;i_1,...,\sigma(i_{k+1}),...,i_k} & \text{for } k > t, \end{cases}$$

$$
\phi(t)(f) = \frac{1}{\text{card } S_n} \sum_{\sigma \in S_n} f^t,\sigma.
$$

Since the function $F$ is concave and $F(f^t,\sigma) = F(f)$, we get

$$
F(f) = F(\phi(0)(f)) \leq F(\phi(1)\phi(0)(f)) \leq \ldots \leq F(\phi(N - j_0 - 1) \ldots \phi(0)(f)).
$$

It remains to observe that $(\phi(N - j_0 - 1) \ldots \phi(0)(f))(\xi') = (\phi(N - j_0 - 1) \ldots \phi(0)(f))(\xi'')$ for any $\xi', \xi'' \in V_k(\hat{x})$, $0 \leq k \leq N - j_0$.

Thus, (112) holds. Hence, it suffices to find the minimal constant $C$ in (111) for the family of functions $f$ such that $f_{\hat{x}}(\xi) = f_k$, $0 \leq k \leq N - j_0$. Set $m_k = n_0 \ldots n_k$.

From (84) it follows that $n_k = 2^{\psi_{(j_0+k+1)-\psi_{(j_0+k)}}}$ and

$$
m_k = 2^{\psi_{(j_0+k+1)-\psi_{(j_0+k)}}} \lesssim 2^{b_{mk} \Lambda(2^{-m_{j_0}})} \Lambda(2^{m_j} \Lambda(2^{-m_{j_0}})).
$$

(113)

Let $x_k = (m_k - 1|f_k|)^{1/p}$, $m_1 = 1$. Then it follows from the definition of $\hat{u}$ and $\hat{w}$ that (111) can be written as

$$
\left( \sum_{k=0}^{N-j_0} m_k w_{k,j_0} \sum_{l=0}^{k} u_{l+j_0} m_{l-1}^{-\frac{1}{p}} x_l \right)^{1/p} \leq C \left( \sum_{k=0}^{N-j_0} x_k^{p} \right)^{1/p}.
$$

(114)

Applying Theorem 3 we get

$$
C \lesssim \sup_{0 \leq k \leq N - j_0} \left( \sum_{l=k}^{N-j_0} m_{l-1} w_{l+j_0}^{p} \right)^{1/p} \left( \sum_{l=0}^{k} u_{l+j_0} m_{l-1}^{-\frac{1}{p}} \right)^{1/p}.
$$

35
Apply Lemma 11, taking into account Remark 2. From (6), (109) and (113) it follows that \( \sum_{l=k}^{N-j_0} m_{l-j_0} w_{l+j_0}^p \lesssim m_{k-1} w_{k+j_0}^p \). The condition \( \beta_g > \frac{d}{p} + \frac{\theta}{p} \) yields the inequality
\[
\sum_{l=0}^{k} w_{l+j_0} m_{l-1} \lesssim m_{k-1} w_{k+j_0}.
\]
Therefore,
\[
C \lesssim \sup_{0 \leq k \leq N-j_0} u_{k+j_0} w_{k+j_0} \quad \text{sup}_{j_0 \leq l \leq N} 2^{m(l(\beta - \beta_d))} \Psi(2^{-m}) =: M.
\]
In the case (7), a), we have \( M \gtrsim 2^{m_{j_0}(\beta - \beta_d)} \Psi(2^{-m_{j_0}}) \). In the case (7), b) for \( \alpha > 0 \) we get \( M \gtrsim j_0^{-\alpha} \rho(j_0) \). If \( \alpha = 0 \) and \( \rho \equiv 1 \), then \( M = 1 \).

Let, now, \( \beta_g - \frac{d}{p} - \frac{\theta}{p} \leq 0 \). Since \( \beta_e < \frac{d}{p} \), there exists \( \beta_g' > \frac{d}{p} + \frac{\theta}{p} \) such that \( \beta_g + \beta_e < d \). Set \( \tilde{u}(\xi) = u(\xi) \cdot 2^{(\beta_g - \beta_g') m_{j_0}} \), \( \xi \in \mathcal{V}_j(\xi_0) \). Then
\[
\mathcal{S}_\mathcal{A}_{t_w} \lesssim \mathcal{S}_\mathcal{A}_{t_w} \tilde{u}(\xi) \cdot 2^{m_{j_0}(\beta_g - \beta_g')} \lesssim \frac{1}{5} 2^{m_{j_0}(\beta - \beta_d)} \Psi(2^{-m_{j_0}}).
\]
This completes the proof.

6 The discrete Hardy-type inequality on the tree: case \( p \neq q \)

Let the tree \( \mathcal{A} \) be such as in the previous section, and let
\[
u(\xi) = \varphi_\gamma(2^{-m_{j_0}}) \cdot 2^{-\frac{m_{j_0}}{p}} = 2^{m_{j_0}(\beta_g - \beta_e)} \Psi_\gamma(2^{-m_{j_0}}),\]
\[
u(\xi) = \varphi_e(2^{-m_{j_0}}) \cdot 2^{-\frac{m_{j_0}}{q}} = 2^{m_{j_0}(\beta_e - \beta_f)} \Psi_e(2^{-m_{j_0}}), \quad \xi \in \mathcal{V}_j(\xi_0).
\]

Let \( \xi \in \mathcal{V}_j(\xi_0) \). Denote by \( \mathcal{S}_{\mathcal{A}_{t_w}}^{p,q} \) the minimal constant \( C \) in the inequality
\[
\left( \sum_{\xi \in \mathcal{V}(\mathcal{A}_{t_w})} w(\xi) \left( \sum_{\xi' \in \xi} u(\xi') f(\xi') \right)^q \right)^{\frac{1}{q}} \leq C \| f \|_{L_p(\mathcal{A}_{t_w})}.
\]

**Lemma 11.** Let \( p > q \). Then there exists \( m_* = m_*(3) \in \mathbb{N} \) such that for any \( m \geq m_* \)
\[
\mathcal{S}_{\mathcal{A}_{t_w}}^{p,q} \lesssim \frac{1}{5} 2^{m_{j_0}(\beta_e - \beta_f)} \Psi(2^{-m_{j_0}})
\]
for the case a) in (7); in the case b), for \( \alpha > \left( \frac{1}{q} - \frac{1}{p} \right) (1 - \gamma) \)
\[
\mathcal{S}_{\mathcal{A}_{t_w}}^{p,q} \lesssim \frac{1}{5} 2^{-(\beta_e - \beta_f) m_{j_0}} \rho(j_0).
\]

36
Proof. First consider the case \( p = \infty \). Let \( \beta_g > d \). Then
\[
\left( \sum_{\xi \in \mathcal{V}(A_{\xi_2})} w^q(\xi) \left( \sum_{\xi_* \leq \xi' \leq \xi} u(\xi') f(\xi') \right)^q \right)^{1/q} \leq \frac{107}{108} \sum_{a,d,e,m} \| f \|_{l_\infty(A_{\xi_1})}.
\]
This together with the condition \( \beta_g > d \), Lemma 1 and Remark 2 yield
\[
\mathcal{S}^{\infty,q}_{A_{\xi_2},u,w} \leq \left( \sum_{j=j_0}^{N} 2^{m_j(\beta_g-d-q)} \psi_q(2^{-m_j}) \cdot 2^{m_j(\beta_g-d-q)} \psi_q(2^{-m_j}) \right)^{1/q}.
\]
In the case (7), a), the right-hand side can be estimated from above up to a multiplicative constant by \( 2^{m_j(\beta_g-d-q)} \psi_q(2^{-m_j}) \); in the case (7), b) it is estimated by
\[
\left( \sum_{j=j_0}^{N} (m_j)^{-\alpha q} \rho_q(m_j) \cdot 2^{-\theta m_j \rho_j(2^{-m_j})} \right)^{1/q} \leq 2^{-\theta m_j \rho_j(2^{-m_j})}.
\]
(here we use Lemma 1 and Remark 2 again). If \( \beta_g \leq d \), then we choose \( \beta_g > d \) so that \( \beta_g + \beta_v < d + \frac{d-q}{q} \) (it is possible by (10)), and we set \( \tilde{u}(\xi) = u(\xi) \cdot 2^{m_j(\beta_g-\beta_g)} \), \( \xi \in \mathcal{V}_j(\xi_0) \). Then
\[
\mathcal{S}^{\infty,q}_{A_{\xi_2},u,w} \leq \mathcal{S}^{\infty,q}_{A_{\xi_2},\tilde{u},w} \cdot 2^{m_j(\beta_g-\beta_g)} \leq \frac{107}{108} \sum_{a,d,e,m} \| f \|_{l_\infty(A_{\xi_1})}.
\]
Let, now, \( q < p < \infty \). Let \( \beta_{g,1} + \beta_{g,2} = \beta_g \), \( \beta_{v,1} + \beta_{v,2} = \beta_v \),
\[
u_1(\xi) = 2^{m_j(\beta_g-d-q)} \psi_q(2^{-m_j}), \quad \nu_2(\xi) = 2^{m_j(\beta_g-d-q)} \psi_q(2^{-m_j}),
\]
\[
u_1(\xi) = 2^{m_j(\beta_v-d-q)} \psi_q(2^{-m_j}), \quad \nu_2(\xi) = 2^{m_j(\beta_v-d-q)} \psi_q(2^{-m_j}), \quad \xi \in \mathcal{V}_j(\xi_0).
\]
Then \( \nu_1(\xi) \nu_2(\xi) = u(\xi), \nu_1(\xi) \nu_2(\xi) = u(\xi) \). Applying the Hölder inequality, we get
\[
\left( \sum_{\xi \in \mathcal{V}(A_{\xi_2})} w_1^q(\xi) w_2^q(\xi) \left( \sum_{\xi_* \leq \xi' \leq \xi} \nu_1(\xi') \nu_2(\xi') f(\xi') \right)^q \right)^{1/q} \leq 37.
\]
\[
\left( \sum_{\xi \in \mathcal{V}(A_{\xi}, u)} w_1^p(\xi) w_2^q(\xi) \left( \sum_{\xi_s \leq \xi' \leq \xi} u_1^{p/2}(\xi') \right)^{\frac{q}{2}} \left( \sum_{\xi_s \leq \xi' \leq \xi} u_2^{p/2}(\xi') \right)^{\frac{q}{2}} \right)^{1/q} \leq \\
\left( \sum_{\xi \in \mathcal{V}(A_{\xi}, u)} w_1^p(\xi) \left( \sum_{\xi_s \leq \xi' \leq \xi} u_1^{p/2}(\xi') \right)^{\frac{q}{2}} \left( \sum_{\xi_s \leq \xi' \leq \xi} u_2^{p/2}(\xi') \right)^{\frac{q}{2}} \right)^{1/q} = \\
\left( \sum_{\xi \in \mathcal{V}(A_{\xi}, u)} \tilde{w}_1^p(\xi) \left( \sum_{\xi_s \leq \xi' \leq \xi} \tilde{w}_1(\xi') \right)^{\frac{q}{2}(1-\frac{q}{p})} \left( \sum_{\xi_s \leq \xi' \leq \xi} \tilde{w}_2(\xi') \right)^{\frac{q}{2}} \right)^{1/q}, \\
\text{with } \tilde{w}_1(\xi) = u_1^{\frac{p}{2q}}(\xi), \quad \tilde{w}_2(\xi) = u_1^{\frac{p}{2q}}(\xi)
\]

by Remark 2, since the functions \(\Psi_g\) and \(\Psi_v\) satisfy (5) (\(\rho_g\) and \(\rho_v\) satisfy (9), respectively), we observe that their powers satisfy the similar conditions. First choose \(\beta_{v,1}\) and \(\beta_{v,2}\) so that

\[
\beta_{v,1} < (d - \theta) \left( \frac{1}{q} - \frac{1}{p} \right), \quad \beta_{v,2} < \frac{d - \theta}{p}, \quad \beta_{v,1} + \beta_{v,2} = \beta_v
\]

hold (it is possible, since \(\beta_v < \frac{d - \theta}{q}\)). Then we choose \(\beta_{g,1}\), \(\beta_{g,2}\). In the case (7), a) require

\[
\beta_{g,1} + \beta_{v,1} < \left(1 - \frac{q}{p}\right) \left( d + \frac{d}{q} - \frac{\theta}{q} \right), \quad \beta_{g,2} + \beta_{v,2} < \frac{qd}{p}, \quad \beta_{g,1} + \beta_{g,2} = \beta_g.
\]

It is possible, since \(\beta_g + \beta_v < d + \frac{d}{q} - \frac{\theta}{q} \left( \frac{1}{q} - \frac{1}{p} \right) = \left(1 - \frac{q}{p}\right) \left( d + \frac{d}{q} - \frac{\theta}{q} \right) + \frac{qd}{p}\). In the case (7), b) we require

\[
\beta_{g,1} + \beta_{v,1} = \left(1 - \frac{q}{p}\right) \left( d + \frac{d}{q} - \frac{\theta}{q} \right), \quad \beta_{g,2} + \beta_{v,2} = \frac{qd}{p}.
\]

The condition \(\alpha_{p,q} \frac{p}{p-q} > \frac{1-\frac{q}{p}}{q}\) holds by (10).

Also observe that \(\|f_2\|_{l_q(A_{\xi})} = \|f\|_{l_p(A_{\xi})}\).

Thus, in the case (7), a)

\[
\mathcal{G}_{A_{\xi}, u, w}^{p,q} \lesssim \left[ 2^{mj_0} (\beta_{g,1} + \beta_{v,1}) \frac{d}{p} - d - \frac{d}{q} \right] \frac{p}{p-q} \left(2^{mj_0}\right)^{1-\frac{q}{p}} \times
\]
Proof. Set in the case a) of (7); in the case b), if as well as in the case (7), b) \( \lambda \leq \frac{\mu}{\rho} \sum_{\xi} \left( \sum_{\xi' \in \xi} \lambda \right)^{1-p} = 2^{\frac{m_{jo}}{\rho}} \left( \frac{2^{m_{jo}}}{\rho} \right)^{1-p} \rho(j_0). \)

This completes the proof. \( \square \)

Lemma 12. Let \( p < q. \) Then there exists \( m_{*} = m_{*}(3) \in \mathbb{N} \) such that for any \( m \geq m_{*} \)

\[ \mathcal{S}_{A_{\epsilon},-u,w}^{p,q} \leq \left( \frac{2^{m_{jo}}}{\rho} \right)^{1-p} \mathcal{S}_{A_{\epsilon},-u,w}^{p,q} \]

in the case a) of (3); in the case b), if \( \alpha > 0, \) then

\[ \mathcal{S}_{A_{\epsilon},-u,w}^{p,q} \leq \left( \frac{2^{m_{jo}}}{\rho} \right)^{1-p} \rho(j_0). \]

Proof. Set \( \lambda = \frac{1}{p} - \frac{1}{q}, \) and define the quantity \( p_1 \) by \( \frac{1}{p} = \frac{1}{p_1} + \lambda. \) Then \( \frac{1}{q} = \frac{1}{p_1}. \)

Applying the Hölder inequality, we get

\[
S := \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} u^q(\xi) \left( \sum_{\xi' \in \xi} u(\xi') f(\xi') \right)^{q} \right)^{1/q} = \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} w^q(\xi) \left( \sum_{\xi' \in \xi} u(\xi') f^{\frac{q}{p}}(\xi') f^{1-\frac{q}{p}}(\xi') \right)^{q} \right)^{1/q} \leq \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} \left( \sum_{\xi' \in \xi} u^{\frac{1}{1-\lambda}}(\xi') f^{\frac{p}{1-\lambda}}(\xi') \right)^{(1-\lambda)q} \left( \sum_{\xi \in \xi' \in \xi} \left( \sum_{\xi' \in \xi} f^{\frac{1-\lambda}{p}}(\xi') \right)^{\lambda q} \right)^{\frac{1}{\lambda}} \right) \leq \left( \max_{\xi \in \mathcal{V}(A_{\epsilon},)} \sum_{\xi' \in \xi} f^p(\xi') \right)^{\lambda} \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} w^{\frac{p}{1-\lambda}}(\xi) \left( \sum_{\xi' \in \xi} u^{\frac{1}{1-\lambda}}(\xi') f^{\frac{p}{1-\lambda}}(\xi') \right)^{p_1} \right)^{\frac{1-\lambda}{p_1}} \leq \| f \|_{lp(A_{\epsilon},)}^{\lambda p} \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} \tilde{w}^{p_1}(\xi) \left( \sum_{\xi' \in \xi} \tilde{u}(\xi') \tilde{f}(\xi') \right)^{p_1} \right) \right) \right),
\]

with \( \tilde{w}(\xi) = w^{\frac{1}{1-\lambda}}(\xi), \) \( \tilde{u}(\xi) = u^{\frac{1}{1-\lambda}}(\xi), \) \( \tilde{f}(\xi) = f^{\frac{1}{1-\lambda}}(\xi). \) We have

\[
\| \tilde{f} \|_{lp_1(A_{\epsilon},)}^{1-\lambda} = \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} f^{\frac{p_1}{p}}(\xi) \right)^{\frac{1-\lambda}{p_1}} = \left( \sum_{\xi \in \mathcal{V}(A_{\epsilon},)} f^p(\xi) \right)^{\frac{1}{q}} = \| f \|_{lp(A_{\epsilon},)}^{\frac{p_1}{p}}.
\]

39
Hence,
\[ S \leq \|f\|_{L_p(A_{\zeta})}^{1-\frac{d}{p}} \cdot (\mathbb{S}^{p_1,p_1}_{A_{\zeta},\bar{u},\bar{w}})^{1-\lambda} \|f\|_{L_p(A_{\zeta})}^{\frac{d}{p}} = (\mathbb{S}^{p_1,p_1}_{A_{\zeta},\bar{u},\bar{w}})^{1-\lambda} \|f\|_{L_p(A_{\zeta})}. \]

If \( \xi \in V_j(\zeta_0) \), then
\[ \tilde{w}(\xi) = 2^{mj} \left( \frac{\beta}{1-\lambda} - \frac{d}{p} \right) \Psi_v^{\frac{1}{1-\lambda}} (2^{-mj}) = 2^{mj} \left( \frac{\beta}{1-\lambda} - \frac{d}{p} \right) \Psi_v^{\frac{1}{1-\lambda}} (2^{-mj}), \]
\[ \tilde{u}(\xi) = 2^{mj} \left( \frac{\beta}{1-\lambda} - \frac{d}{p} \right) \Psi_v^{\frac{1}{1-\lambda}} (2^{-mj}), \]
with \( \Psi_v = \Psi_v^{\frac{1}{1-\lambda}}, \Psi_v = \Psi_v^{\frac{1}{1-\lambda}}, \tilde{\beta}_v = \frac{\beta}{1-\lambda} - \frac{d}{p} \) and \( \tilde{\beta}_v = \frac{\beta_v}{1-\lambda} \). Then \(-\tilde{\beta}_v p_1 + d - \theta > 0 \).

In the case (7), a), we have \( \beta < d + \frac{d}{q} - \frac{d}{p} \). Check that \( \tilde{\beta}_g + \tilde{\beta}_v < d \). It is equivalent to \( \beta_g + \beta_v - \frac{d}{p} + \frac{(1-\lambda)}{p_1} < d(1-\lambda) \), i.e., \( \beta - d + \frac{d}{q} - \frac{(1-\lambda)}{p_1} < 0 \). It remains to observe that \( \frac{1-\lambda}{p_1} = \frac{1}{q} \). Hence, by Lemma 10 \((\mathbb{S}^{p_1,p_1}_{A_{\zeta},\bar{u},\bar{w}})^{1-\lambda} \leq 2^{mjjo} \left( \frac{d-\frac{d}{q} + \frac{d}{p}}{1-\lambda} \right) \Psi_v(2^{-mj}). \)

Consider the case (7), b). Then \( \beta = d + \frac{d}{q} - \frac{d}{p} \). Hence, \( \tilde{\beta}_g + \tilde{\beta}_v = d \), and by Lemma 10 we get \((\mathbb{S}^{p_1,p_1}_{A_{\zeta},\bar{u},\bar{w}})^{1-\lambda} \leq 2^{mjjo} \rho(j_0). \)

\[ \square \]

7 The proof of the embedding theorem

In this section we prove the main result of this article. In particular, we obtain Theorem 11.

Let \( m = m(3) \) (see Lemmas 10, 11, and 12), and let \( \{D_{j,i}, \xi_{j,i}\} \) be the partition of \( T \) for \( m = m(3) \) (see the definition on page 13).

**Theorem 2.** Let \( D \subset T_{\xi_{j_0},i_0} \) be a subtree with the minimal vertex \( \xi_{j_0,i_0} \). Then for any function \( f \in \text{span} W_{p,g}^r(\Omega) \) there exists a polynomial \( P \) of degree not exceeding \( r - 1 \) such that
\[ \|f - Pf\|_{L_q,v(\Omega_{D,r})} \lesssim 2^{m_{j_0} \beta \Psi(2^{-m_{j_0}})} \left\| \nabla^r f \right\|_{L_p(\Omega_{D,r})} \]
(117)
in the case (7), a),
\[ \|f - Pf\|_{L_q,v(\Omega_{D,r})} \lesssim 2^{-m_{j_0} \beta (\frac{d}{q} - \frac{1}{p})} \left\| \nabla^r f \right\|_{L_p(\Omega_{D,r})} \]
(118)
in the case (7), b). Here the mapping \( f \mapsto Pf \) can be extended to a linear continuous operator \( P : L_q,v(\Omega) \to P_{r-1}(\Omega) \).
Proof. We shall denote \( \tilde{\Omega} = \Omega_{D,F} \).

Step 1. The set \( C^\infty(\tilde{\Omega}) \cap W_2^{r,q}(\tilde{\Omega}) \) is dense \( W_2^{r,q}(\tilde{\Omega}) \) (it can be proved in the same way as for a non-weighted case, see [14], p. 16]). Therefore, it is sufficient to check (117) and (118) for \( f \in C^\infty(\tilde{\Omega}) \).

By Lemma 2, \( \tilde{\Omega} \in FC(b_\ast) \), \( b_\ast = b_\ast(a, d) \). Let \( x_\ast \in \tilde{\Omega} \), \( \gamma_\ast(\cdot), T(x) \) be such as in Definition 1 and let \( R_0 = \text{dist}_\parallel \parallel \ast \parallel_2(x_\ast, \partial \tilde{\Omega}) \). From assertion 2 of Lemma 2 it follows that we can take the center of the cube \( F(\hat{\xi}_{j_0,i_0}) \) as the point \( x_\ast \).

It is sufficient to show that if \( f \in C^\infty(\Omega^m) \), \( f|_{B_{R_0/2}(x_\ast)} = 0 \), then (117), (118) hold with \( Pf = 0 \) (the general case can be proved in the same way as in [62]; here we can take as \( f \mapsto Pf \) the Sobolev’s projection operators).

Let \( \varphi(x) = \frac{|Vf(x)|}{g(x)} \). By Theorem 1 for any \( x \in \tilde{\Omega} \) there exists a set \( G_x \subset \cup_{t \in [0, T(x)]} B_{b_\ast}(\gamma_\ast(t)) \) such that

\[
\{(x, y) \in \tilde{\Omega} \times \tilde{\Omega} : x \in \tilde{\Omega}, y \in G_x\} \quad \text{is measurable,}
\]

\[
|f(x)| \lesssim \int_{G_x} |x - y|^{-d} g(y) \varphi(y) \, dy.
\]

By Assertion 2 of Lemma 2

\[
\text{if } x \in \Delta, \Delta \in \Theta(\Omega), \text{ then } G_x \subset \Omega_{\leq \Delta}. \tag{119}
\]

Thus, it is sufficient to prove that

\[
\left( \int_{\tilde{\Omega}} v^q(x) \left( \int_{G_x} g(y) \varphi(y)|x - y|^{-d} \, dy \right)^q \, dx \right)^{1/q} \lesssim C(j_0) \| \varphi \|_{L^p(\tilde{\Omega})}, \tag{120}
\]

with \( C(j_0) = 2^{m_\ast j_0 (\beta - \delta)} \Psi(2^{-m_\ast j_0}) \) in the case (7), a), and

\[
C(j_0) = 2^{-m_\ast \theta(\frac{1}{2} - \frac{1}{p}) j_0} \rho(j_0)^{\frac{1}{4} + \frac{\alpha - \frac{1}{4} - \frac{1}{2}}{\delta}} \tag{121}
\]

in the case (7), b).

Extending the function \( \varphi \) by zero to \( \Omega_{T_{\hat{\xi}_{j_0,i_0}}^{\ast}} \setminus \Omega_{D,F} \) and applying the B. Levi’s theorem, we may assume that \( \mathbf{V}(D) = \{ \xi \in \mathbf{V}(T_{\hat{\xi}_{j_0,i_0}}) : \rho_T(\hat{\xi}_{j_0,i_0}, \xi) \leq N \} \) for some \( N \in \mathbb{N} \).

Step 2. Consider the case \( r = d \). Let \( (A, \xi_0) = (A(m_\ast), \xi_0) \) be the tree defined on the page 383. If \( \xi = \eta_{j,i} \in \mathbf{V}(A) \), then we set \( D[\xi] = D_{j,i}, \Omega[\xi] = \Omega_{D[\xi]}, F \)

\[
g_\xi = 2^{\delta m_\ast j} \Psi_g(2^{-m_\ast j}), \quad v_\xi = 2^{\beta_\ast m_\ast j} \Psi_v(2^{-m_\ast j}). \tag{121}
\]

\(^1\text{Here } C^\infty(\Omega) \text{ is the space of functions that are smooth on the open set } \Omega, \text{ but not necessarily extendable to smooth functions on the whole space } \mathbb{R}^d.\)
By (120), the property 2 of the partition \( \{ D_j, i \} \) \( j \in \mathbb{Z}_+, i \in I_j \) and (4), we have
\[
\text{diam} \Omega[\xi] \lesssim \frac{2^{-m_j}}{a_d}, \quad g(x) \gtrsim g_\xi, \quad v(x) \gtrsim v_\xi, \quad x \in \Omega_{D_j, i}. \tag{122}
\]

Set \( \xi = \eta_{j_0, i_0} \). Then
\[
\left( \int_{\Omega} v^q(x) \left( \int_{G_x} g(y) \varphi(y) \, dy \right) \right) \left( \int_{\Omega} v^q(x) \varphi(y) \, dy \right) \right) \frac{1}{q} \leq \tag{119}
\]
\[
\leq \left( \sum_{\xi \in V(A_{\xi})} \int_{\Omega[\xi]} v^q(x) \left( \sum_{\xi \leq \xi' < \xi} \int_{\Omega[\xi']} g(x) \varphi(y) \, dy \right) \right) \frac{1}{q} \leq \tag{122}
\]
\[
\leq \left( \sum_{\xi \in V(A_{\xi})} \int_{\Omega[\xi]} v^q(x) \left( \sum_{\xi \leq \xi' < \xi} \int_{\Omega[\xi']} \varphi(y) \, dy \right) \right) \frac{1}{q} \leq \tag{115, 121, 122}
\]
\[
\leq C(j_0) \left( \sum_{\xi \in V(A_{\xi})} \| \varphi \|_{L^p(\Omega[\xi])} \right) \frac{1}{p} = C(j_0) \| \varphi \|_{L^p(\tilde{\Omega})}
\]

(the penultimate relation follows from Lemmas 10, 11 and 12).

**Step 3.** Let \( r \neq d \). Set
\[
G^1_x = \{ y \in G_x : |x - y| \geq 2 \text{dist}_{1, \Gamma}(x), \Gamma \}, \quad G^2_x = \{ y \in G_x : |x - y| < 2 \text{dist}_{1, \Gamma}(x), \Gamma \}.
\]

Then in order to prove (120) it suffices to check the inequalities
\[
\left( \int_{\tilde{G}^1_x} v^q(x) \left( \int_{\tilde{G}^1_x} g(y) \varphi(y) |x - y|^r \, dy \right) \right) \frac{1}{q} \leq C(j_0) \| \varphi \|_{L^p(\tilde{\Omega})}, \tag{123}
\]
\[
\left( \int_{\tilde{G}^2_x} v^q(x) \left( \int_{\tilde{G}^2_x} g(y) \varphi(y) |x - y|^r \, dy \right) \right) \frac{1}{q} \leq C(j_0) \| \varphi \|_{L^p(\tilde{\Omega})}. \tag{124}
\]

Prove (123). At first we check that for \( y \in G^1_x \)
\[
|x - y| \gtrsim \text{dist}_{1, \Gamma}(y, \Gamma). \tag{125}
\]
Indeed, let \( z_x \in \Gamma, |x - z_x| = \text{dist}_{|\Gamma|}(x, \Gamma) \). Then
\[
\text{dist}_{|\Gamma|}(y, \Gamma) \leq |y - z_x| \leq |y - x| + |x - z_x| = |x - y| + \text{dist}_{|\Gamma|}(x, \Gamma) \leq |x - y| + \frac{|x - y|}{2} = \frac{3|x - y|}{2}.
\]
Prove the inverse inequality. Let \( y \in F(\omega), \omega \in V(T) \). From (119) it follows that \( x \in \Omega_{\mathcal{T}_\omega,F} \). Since \( \Omega_{\mathcal{T}_\omega,F} \in \mathbf{FC}(b_\omega) \), we have \( \text{diam}(\Omega_{\mathcal{T}_\omega,F}) \lesssim 2^{-m_\omega} \). From assertion 2 of Theorem A it follows that \( \text{dist}_{|\Gamma|}(y, \partial \Omega) \gtrsim \frac{2^{-m_\omega}}{a_d} \). Hence,
\[
\text{dist}_{|\Gamma|}(y, \Gamma) \geq \text{dist}_{|\Gamma|}(y, \partial \Omega) \gtrsim \frac{2^{-m_\omega}}{a_d} \gtrsim \frac{\text{diam} \Omega_{\mathcal{T}_\omega,F}}{a_d} \gtrsim |x - y|.
\]
Thus, (125) is proved, and
\[
\left( \int_{\Omega} v^q(x) \left( \int_{G_x^2} g(y) \varphi(y) |x - y|^{-r-d} dy \right)^{q} \right)^{1/q} \gtrsim \left( \int_{\Omega} v^q(x) \left( \int_{G_x^2} \tilde{g}(y) \varphi(y) dy \right)^{q} \right)^{1/q},
\]
with \( \tilde{g}(y) = \varphi_{\tilde{g}}(\text{dist}_{|\Gamma|}(y, \Gamma)) \),
\[
\varphi_{\tilde{g}}(t) = \varphi_g(t) \cdot t^{-d} = t^{-\beta \tilde{g}} \Psi_g(t), \quad \beta \tilde{g} = \beta_g + d - r.
\]
Since \( \beta - \delta = \beta_g + \beta_v - r - \frac{4}{q} + \frac{4}{p} = \beta_g + \beta_v - d - \frac{4}{q} + \frac{4}{p} \), it remains to apply the estimate which was obtained at the previous step.

Prove (124). If \( y \in G_x^2 \), then
\[
\text{dist}_{|\Gamma|}(y, \Gamma) \leq \text{dist}_{|\Gamma|}(x, \Gamma) + |x - y| \leq 3 \text{dist}_{|\Gamma|}(x, \Gamma).
\]
Let \( x \in \Omega[\eta_{j,i}], y \in \Omega[\eta_{j',i'}] \). From (119) and property 3 of the partition \( \{\mathcal{D}_{j,i}\}_{j \in Z_{+}, i \in I_j} \) it follows that \( j' \leq j \). By (100), \( \text{dist}_{|\Gamma|}(x, \Gamma) \asymp 2^{-m_j}, \text{dist}_{|\Gamma|}(y, \Gamma) \asymp 2^{-m_{j'}} \). This together with (126) yield that there exists \( j_* = j_*(a, d, m_*) \) such that \( j - j_* \leq j' \leq j \). Notice that
\[
|x - y| \lesssim \frac{2^{-m_j}}{a_{d,m_*}}.
\]
Denote by \( \mathcal{I}_{\eta_{j,i,j_*}} \) the maximal subgraph on the vertex set
\[
V(\mathcal{I}_{\eta_{j,i,j_*}}) = \bigcup_{j' \geq j - j_*} \bigcup_{\eta' \in \eta_{j,i}} V(\mathcal{D}_{\eta',i'})
\]
and set $\hat{\Omega}[\eta_{j,i}] = \Omega_{I[\eta_{j,i}] \cdot F}$. Then for any $x \in \Omega[\eta_{j,i}]$, the inclusion $G^2_x \subset \hat{\Omega}[\eta_{j,i}]$ holds. In addition, from (121) and (122) it follows that

$$g(y) \lesssim g_{\eta_{j,i}}, \quad y \in \hat{\Omega}[\eta_{j,i}].$$  \hspace{1cm} (128)

By (107), for any $\xi' \in V(A_{\xi})$

$$\text{card} \{ \xi \in V(A_{\xi}) : \xi' \in \Omega[\xi] \} \lesssim 1.$$  \hspace{1cm} (128)

Therefore,

$$\left( \sum_{\xi \in V(A_{\xi})} \| \varphi \|_{L^p(\hat{\Omega}[\xi])}^p \right)^{1/p} \lesssim \| \varphi \|_{L^p(\tilde{\Omega}).} \hspace{1cm} (129)$$

We have

$$\left( \int_{\tilde{\Omega}} v^q(x) \left( \int_{\Omega[\xi]} g(y) \varphi(y) |x - y|^{-d} dy \right)^q dx \right)^{1/q} \lesssim \left( \sum_{\xi \in V(A_{\xi})} \int v^q(x) \left( \int_{\Omega[\xi]} g(y) \varphi(y) |x - y|^{-d} dy \right)^q dx \right)^{1/q} \leq \frac{1}{3} \times \left( \sum_{\xi \in V(A_{\xi})} \int v^q(x) \left( \int_{\tilde{\Omega}[\xi]} \varphi(y) |x - y|^{-d} dy \right)^q dx \right)^{1/q} =: S.$$  \hspace{1cm} (130)

Let $\xi = \eta_{j,i}$. By (122) and (127), $\Omega[\xi]$ and $\tilde{\Omega}[\xi]$ are contained in a ball of radius

$$R_{\xi} \lesssim 2^{-m \cdot j}. \hspace{1cm} (130)$$

Applying Theorem E and the Hölder inequality, we get

$$S \lesssim \left( \sum_{\xi \in V(A_{\xi})} g^q_{\xi} v^q_{\xi} R^d_{\xi} \| \varphi \|_{L^p(\tilde{\Omega}[\xi])}^q \right)^{1/q} \lesssim S_1.$$  \hspace{1cm} (130)

If $p \leq q$, then

$$S_1 \leq \max_{\xi \in V(A_{\xi})} g^q_{\xi} v^q_{\xi} R^d_{\xi} \left( \sum_{\xi \in V(A_{\xi})} \| \varphi \|_{L^p(\tilde{\Omega}[\xi])}^p \right)^{\frac{1}{p}} \lesssim \frac{1}{3} \left( \sum_{\xi \in V(A_{\xi})} \| \varphi \|_{L^p(\tilde{\Omega}[\xi])}^p \right)^{\frac{1}{p}} \lesssim \frac{1}{3} \left( \sum_{\xi \in V(A_{\xi})} \| \varphi \|_{L^p(\tilde{\Omega}[\xi])}^p \right)^{\frac{1}{p}}.$$  \hspace{1cm} (130)
\[ \lesssim 2^{m_j\nu(\beta-\delta)}\Psi(2^{-m_j})\|\varphi\|_{L_p(\Omega)} = C(j_0)\|\varphi\|_{L_p(\Omega)}. \]

If \( p > q \), then by the Hölder inequality
\[
S_1 \leq \left( \sum_{\xi \in \mathcal{V}(A_{\xi})} (g_{\xi}v_{\xi}R_{\xi}^\delta)^{\frac{p}{pq}} \right)^{\frac{1}{p}} \left( \sum_{\xi \in \mathcal{V}(A_{\xi})} \|\varphi\|_{L_p(\Omega(\xi))}^p \right)^{\frac{1}{p}} \lesssim_{\frac{121}{129}} \frac{1}{\delta} \lesssim_{\frac{129}{130}} \frac{1}{\delta}
\]
(see Lemma 1).

Notice that if the condition (7), a) is replaced by
\[ \beta_g + \beta_v > \delta - \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+, \]
or if (7), b) holds and \( \alpha < (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right)_+ \), then the set \( W_{p,q}(\Omega) \cap C_0^\infty(\Omega) \) is unbounded in \( L_q(\Omega) \). Indeed, let \( \varphi \in C_0^\infty([0, 1]^d), \varphi \geq 0, \int_{[0, 1]^d} |\nabla^\ast \varphi(x)|^p \, dx = 1 \), let \( \hat{k} = \hat{k}(a, d) \in \mathbb{N} \) be such as in Lemma 4, and let \( \hat{\xi} \) be the minimal vertex of the tree \( \mathcal{T}, \hat{\xi} \in \mathcal{W}_{v_0} \). Then for sufficiently large \( \nu \in \mathbb{N} \)
\[
\sum_{l=\nu}^{\nu+k} \text{card} \mathcal{W}_l \lesssim_{\frac{32}{3}} \frac{h(2-v_0)}{2^{v_0}} \gtrsim_{\frac{32}{3}} \frac{2^{v_0}}{\Lambda(2^{-v})}.
\]
Set
\[
\{\Delta_j\}_{j \in J_\nu} = \left\{ F(\xi) : \xi \in \bigcup_{l=\nu}^{\nu+k} \mathcal{W}_l \right\}.
\]
Then \( \Delta_j = z_j + t_j[0, 1]^d, t_j \lesssim_{\frac{24}{3}} 2^{-\nu}. \) In addition, from (4) and (25) it follows that
\[
g(x) \lesssim 2^{\nu_3} \Psi_g(2^{-\nu}), \quad v(x) \lesssim 2^{\nu_3} \Psi_v(2^{-\nu}), \quad x \in \Delta_j, \quad j \in J_\nu.
\]

Let \( p \leq q \). Take \( j \in J_\nu \) and set \( \varphi_\nu(x) = c_{\nu} \varphi \left( \frac{x - z_j}{t_j} \right) \), with \( c_{\nu} > 0 \) such that
\[
\left\| \nabla^\ast \varphi_\nu \right\|_{L_p(\Omega)} = 1. \text{ Then } c_{\nu} \lesssim_{\frac{132}{3}} 2^{\nu_3} \Psi_g(2^{-\nu})^{2^{\nu}}. \text{ Hence,}
\]
\[
\left\| \varphi_\nu \right\|_{L_{q,v}(\Omega)} \lesssim_{\frac{132}{3}} \frac{c_{\nu}}{2^{\nu_3}} 2^{\nu_3} \Psi_v(2^{-\nu})^{2^{\nu}} \lesssim_{\frac{132}{3}} 2^{\nu_3(\beta-\delta)} \Psi(2^{-\nu}).
\]
If \( \beta - \delta > 0 \), then \( \|\varphi_\nu\|_{L_{q,v}(\Omega)} \to \infty \). If \( \beta = \delta \) and \( \alpha < 0 \), then \( \|\varphi_\nu\|_{L_{q,v}(\Omega)} \lesssim_{\frac{45}{3}} \nu^{-\alpha} \rho(\nu) \to \infty \).
Let \( p > q \). First consider the case \( \beta > \delta - \theta \left( \frac{1}{q} - \frac{1}{p} \right) \). Set \( \varphi_{\nu}(x) = c_{\nu} \sum_{j \in J_{\nu}} \varphi \left( \frac{x - x_{j}}{t_{j}} \right) \), where \( c_{\nu} > 0 \) is such that \( \| \nabla \varphi_{\nu} \|_{L_{p}(\Omega)} = 1 \). Then

\[
(\text{card } J_{\nu})^{-\frac{1}{p}} \approx c_{\nu} \cdot 2^{\nu \beta} \Psi_{g}(2^{-\nu \beta}) \cdot 2^{-\frac{\nu \beta}{p}} (\text{card } J_{\nu})^{\frac{1}{p}} \ approximate \frac{2^{\nu (\beta - \delta + (\frac{1}{q} - \frac{1}{p}))} \Psi(2^{-\nu \beta}) \left( A(2^{-\nu \beta}) \right)^{\frac{1}{p} - \frac{1}{q}}. \]

Therefore, \( \| \varphi_{\nu} \|_{L_{q,p}(\Omega)} \rightarrow \infty \).

Let \( \beta = \delta - \theta \left( \frac{1}{q} - \frac{1}{p} \right) \), \( \alpha < (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right) \). For \( s \in \mathbb{N} \) denote

\[
N_{s} = \{ s + l(k + 1) : l \in \mathbb{Z}_{+}, \ l(k + 1) \leq s \}. \]

Then \( \text{card } N_{s} \approx s \). Set \( \psi_{s}(x) = \sum_{\nu \in N_{s}} \sum_{j \in J_{\nu}} c_{\nu} \varphi \left( \frac{x - x_{j}}{t_{j}} \right) \), where \( c_{\nu} > 0 \) is such that

\[
\| \nabla \psi_{s} \|_{L_{p}(\Omega)} = (\text{card } J_{\nu})^{-\frac{1}{p}} (\text{card } N_{s})^{-\frac{1}{p}}, \ j \in J_{\nu}, \ \nu \in N_{s}. \]

Then \( \| \nabla \psi_{s} \|_{L_{p}(\Omega)} = 1 \) and

\[
c_{\nu} \approx \frac{2^{\nu \beta} \Psi_{g}(2^{-\nu \beta}) \cdot 2^{\nu (\beta - \delta)} (\text{card } J_{\nu})^{-\frac{1}{p}} s^{-\frac{1}{p}}. \] (133)

Hence,

\[
\| \psi_{s} \|_{L_{q,p}(\Omega)} \approx \left( \sum_{\nu \in N_{s}} 2^{\nu \beta} q_{\nu}^{\beta} 2^{-\nu \beta} (\text{card } J_{\nu}) \right)^{\frac{1}{p}} \] (132), (131), (133)

\[
\approx \left( \sum_{\nu \in N_{s}} 2^{\nu \beta} q_{\nu}^{\beta} 2^{-\nu \beta} (\text{card } J_{\nu})^{-\frac{1}{p}} s^{-\frac{1}{p}} \right)^{\frac{1}{p}} \] (132), (131), (133)

\[
\times s^{-\alpha} \rho(s) \cdot s^{\left( \frac{1}{p} - \frac{1}{q} \right) (1 - \gamma) [\tau(s)]^{-\frac{1}{p}} \frac{1}{p}} \rightarrow \infty. \]

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