Estimation of dynamic mixed double factors model in high-dimensional panel data

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Abstract
This paper endeavors to develop some dimension reduction techniques in panel data analysis when the numbers of individuals and indicators are very large. We use principal component analysis method to represent a large number of indicators via minority common factors in the factor models. We propose the dynamic mixed double factor model (DMDFM for short) to reflect cross section and time series correlation with the interactive factor structure. DMDFM not only reduces the dimension of indicators but also deals with the time series and cross section mixed effect. Different from other models, mixed factor models have two styles of common factors. The regressors factors reflect common trend and the dimension reducing, while the error components factors reflect difference and weak correlation of individuals. The results of Monte Carlo simulation show that generalized method of moments estimators have good properties of unbiasedness and consistency. Simulation results also show that the DMDFM can improve the prediction power of the models effectively.

Keywords Panel data · Dynamic mixed double factor model · Identification · GMM estimation · Cross section and Time series correlation

1 Introduction
Processing of large-scale data sets of the macroeconomic has been one of the cumbersome problems in panel data analysis. Compared with micropanel data, macropanel data includes more indicators which usually are correlated with each other. Panel data is composed of cross section and time series data, so the correlation results from two aspects: periods and individuals dependency. If these dependencies exist, panel data model should be considered, regardless of the source, such as we compare economic development across different countries or regions. If every country or region is regarded as an individual and observed by continuous time, cross section and time series correlation occur since some of the items have the same economic structures and common trends. Similarly, we may encounter analogous issues in the micropanel data analysis, for example, the business industry and security market volatility are focused simultaneously when we study the assets allocation and portfolio management in stock market, which also can be seen as cross section and time series correlation. On the other hand, correlation between variables and individuals should be considered in high-dimensional panel data analysis, how to reduce the number of indicators is also very important, which is known as dimension reduction.

Factor models have been used to analyze large-scale macroeconomic data sets for a long time. These macro-data sets consist of hundreds of indicators and some common trends can be observed owing to co-movements of variables, reflecting the existence of correlation between cross sections. Chamberlain and Rothschild (1983) employed approximate factor structure to study risk-free arbitrage portfolio with weak correlation in large-scale assets analysis. They obtained the same conclusions as Ross (1976) did in the arbitrage pricing theory. Forni et al. (2000) proposed the method of identification and estimation in generalized dynamic factor
model (GDFM). GDFM is the factor model which includes the lag term of factors and cross-correlation of idiosyncratic components. In the factor model analysis, two parts of the inner structures, i.e., error components and regressors components could be considered, respectively. Some researchers focused only on the factor models with factor decomposition of error component, see, Ahn et al. (2001), Moon and Perron (2004), Fan et al. (2008), Bai (2009), among many others. They discussed unobservable interactive effects of individuals and periods in error components provided the heterogeneity structure between error and regressors, which extract error components through factor decomposition. Besides, factor decomposition with regressors is studied extensively, see Forni et al. (2000), Stock and Watson (2002), Bai (2003), Anderson and Deistler (2008), etc. In this case, the regressors are expressed by two unobservable orthogonal components. Common shocks are expressed by minority common factors which are used to reduce the dimension. Idiosyncratic components are expressed by factor loadings to reflect the differences of individuals. Furthermore, a complicated case was considered in the factor decomposition with both of error component and regressors, see, Andrews (2005), Pesaran (2006), among others. They discussed the multi-factor error structure and cross section dependency of individual due to the common shock effects.

The lag effects of general dynamic factor model come from the lag terms of common factors, i.e., AR or MA processes about common factors. These processes can reflect persistence effect on individual across periods. VAR processes of dynamic factor model also are built on lag terms of common factors (Stock and Watson 2005, etc.). Dynamics of common factors derive from regressors’ lag effects. Dependent variable in statistical model can be estimated by regressors. Current and past values of regressors will influence dependent variable if we introduce the lag terms of regressors into the model. In the real data analysis, lag terms of dependent variable also can influence current variable values. Stock and Watson (2002) used lag effects of dependent variable to forecast macroeconomy, but they didn’t consider time series correlation of regressors while transferring them to lag terms of common factors.

Not only the time series field does panel data model is used, but also the cross section correlation problems must be settled. In this paper, we propose a mixed double factor model (MDFM). The double factors refer to factor decomposition with regressors and error components, respectively. MDFM can capture the structural features of panel data with respect to time and individual. We introduce the common factors and factor loadings of regressors and error components to reflect cross section correlation, and lag terms of dependent variable can be seen as endogenous variables to reflect time series correlation. The mixed double factor model including lag effects of dependent variable is called dynamic mixed double factor model (DMDFM).

Different from time series and cross section data, panel data include three dimensions: individuals, periods, variables. Here, we consider short panel data case at first, where the number of individuals $N$ is larger than periods length $T$. Of course, we will relax this condition at the end of this article. Simultaneously, the number of observable variable $p$ can be larger than $N$ and $T$. Classic statistical modeling methods often face multi-collinearity problem. We decompose factors of regressors with principal component analysis (PCA) method. Large number of explanatory variables are represented by minority common factors (factor scores), and we reduce the number of indicators and parameters to be estimated. On the other hand, common factors reflect correlation among variables.

DMDFM includes lag terms of dependent variable in the right-hand side (RHS), and they are correlated with common factors of regressors and error component. So, we use generalized method of moments (GMM) to estimate the model. Arellano and Bover (1995) studied the linear moments conditions and choose the optimal weighting matrix in GMM estimation of dynamic panel data. DMDFM have more complicated structure than classic dynamic panel data model because they include double factors. In this case, the choice of optimal instrumental variables is very important. We divide the processes of DMDFM estimation into two steps. Firstly, we obtain idiosyncratic component correlated with regressors via GMM estimation, and then PCA method is used to decompose them, the result of which will be applied into original model. Secondly, we make transformations of the model and estimate the new model with error factors by GMM. By two-step iterative method, we acquire the uniform optimal estimators. The results of two-step estimation can be used to predict the future values of dependent variable.

The rest of this article is organized as follows. Section 2 gives some notations and the construction processes of DMDFM. Specification and assumptions of DMDFM are given in Sect. 3. Section 4 discusses two important problems in DMDFM, one of which is the choice of factors number and the other one is the choice of estimation method. Simulation results are given in Sect. 5, in which we will simulate the data generation processes of DMDFM. Some conclusion and remarks are provided in Sect. 6.

### 2 Panel data dynamic mixed double factor model

#### 2.1 Panel data factor model

In panel data model, let $X_{it}$ and $Y_{it}$ denote the observed value of regressors and respondon on the $t$th period across
the $i$th individual, $i = 1, \ldots, N; t = 1, \ldots, T$. $X_{it}$ is a $p$-dimensional column vector, $p$ is the number of regressors. Hsiao (2003) considered the following model, the slope term of which are constant and intercept term varies over individuals and time:

$$Y_{it} = \alpha_{it} + \sum_{k=1}^{p} \beta_k X_{kit} + u_{it}$$

If the intercept terms of above model are regarded as covariances, then the model can be rewritten as matrix form:

$$Y_{it} = X_{it}'B + u_{it}$$

where $B$ is $p \times 1$ vector to be estimated, $u_{it}$ is random error term.

Pesaran (2006) proposed an estimation method and gave the estimators’ statistical inference of linear heterogeneous panel data through multi-factors error structure model:

$$Y_{it} = A_i' D_i + X_{it}'B + u_{it}$$

where the error term has a multi-factors error structure:

$$u_{it} = G_t \Gamma_i' + \epsilon_{it}$$

where, $G_t$ is unobservable common effects, $\epsilon_{it}$ is the individual idiosyncratic error. If $G_t$ is correlated with $X_{it}, X_{it}$ can be expressed as linear combination of $G_t$, which is named as common correlated effect (CCE). Bai (2009) considered a special case when the numbers of individuals $N$ and periods $T$ are very large. Factor loadings and common factors are regarded as unobservable parameter of interactive fixed effects model:

$$Y_{it} = X_{it}'B + G_t \Gamma_i' + v_{it}$$

In the above model, identification, consistency, limiting distribution of the estimators were discussed.

In the case of high-dimensional panel data analysis, in order to reduce individual data dimension and reflect panel data dependent structure feature among individuals, Bai (2003) transferred the regressors of model (2.1) by common factors:

$$X_{it} = F_t \Lambda_i' + e_{it}$$

where $\Lambda_i$ represents factor loadings, $F_t$ is a common factors vector, and $e_{it}$ is idiosyncratic error. If the number of common factors is $r$, then $r$ common factors can be written as: $F_t \Lambda_i = \lambda_{1i} F_{1t} + \ldots + \lambda_{ri} F_{rt}$. Here, $\Lambda_i, F_t$ and $e_{it}$ are all unobservable. Model (2.1) can be rewritten as:

$$Y_{it} = F_t B^* + u_{it}^*$$

where $u_{it}^*$ is an unobservable idiosyncratic error, uncorrelated with $F_t$.

### 2.2 Panel data dynamic mixed double factor model

Since time series and cross section correlation may exist simultaneously among the indicators, we consider the situation that correlation exists both in regressors and lag terms of dependent variable when constructing panel data factor models. Stock and Watson (2002, 2005) discussed specification and estimation in multivariate time series dynamic factor model. They used it to extrapolate prediction in multivariate time series case, but do not extend it to panel data model. Meanwhile, idiosyncratic error component $u_{it}$ may exist unobservable interactive effects in panel data model. Considering these factors simultaneously, we propose AR(1) dynamic mixed double factor model with panel data as follows:

$$Y_{it} = Y_{iw} \beta_L + F_{it} \beta_F + G_t \Gamma_i' + \epsilon_{it}$$

where $Y_{it}$ is a dependent variable, representing observed value on $t$th period across $i$th individual; $Y_{iw}$ is a column vector composed of the lag terms of $Y_{it}, w = t-1, \ldots, t-h$; $\beta_L$ and $\beta_F$ are $h \times 1$ and $r \times 1$ parameter vectors to be estimated, $F_{it}$ is an unobservable $1 \times r$ common factors vector. Regressors $X_{it}$ can be decomposed as:

$$X_{it} = F_{it} \Lambda_i' + \epsilon_{it}$$

where $\Lambda$ is a $p \times r$ factor loadings matrix, and $r$ are common factors decomposed from $p$ regressors ($r < p$), while that in Eq. (2.3) is decomposed from $N$ individuals which are different from each other. Another group common factors $G_t$ and correspondent factor loadings $\Gamma_i$ are unobservable $1 \times s$ vector $s$, obtained from regression equation:

$$Y_{it} = Y_{iw} \beta_L + F_{it} \beta_F + u_{it}$$

Next, we decompose factors from idiosyncratic error $u_{it}$ as Eq. (2.2), i.e.,

$$u_{it} = Y_{it} - Y_{iw} \beta_L - F_{it} \beta_F = G_t \Gamma_i' + \epsilon_{it}$$

where $s$ common factors and corresponding factor loadings can be written as:

$$G_t \Gamma_i' = \gamma_{1t} G_{it} + \ldots + \gamma_{st} G_{st}. $$

Using matrix notation, we omit subscript of individuals and periods, rewrite Eq. (2.7) as a simplified style:

$$Y = Y_L \beta_L + F \beta_F + G \Gamma' + \epsilon$$

(2.8)
where $Y$ and $Y_L$ are $T \times N$ and $T \times N \times h$ matrix, respectively; $F$ is a $T \times N \times r$ matrix with $r$ indicator; $G$ and $\Gamma$ are $T \times s$ and $N \times s$ matrix, respectively; $\beta_L$ and $\beta_F$ are $h \times 1$ and $r \times 1$ coefficient vectors.

From model (2.8), we consider panel data models with interactive effect in time series and cross section dimension. In this model, lag terms $Y_L$ reflects time series correlation. Without loss of generality, we only consider AR(1) model below. In fact, high-order autoregressive model can be analyzed similarly AR(1). In this article, we propose a panel data factor modeling strategy when the number of indicators $p$ is very large. First group factor $F$ is used to reduce the dimension of regressors indicator and multi-collinearity among indicators. Second group factor $G$ reflect s interactive effects of the error component. After two times of factorization, idiosyncratic error component $\epsilon$ can satisfy model assumption.

Model (2.8) is a generalization of many previous approximate factor models. Bai (2009) proposed a interactive fixed effect model, considering interactive effect in the heterogeneity error term. If we regard the first factor decomposition as identical transformation of regressors without considering the lag effect, the DMDFM becomes the interactive fixed effect model. If we only decompose the factor to regressors, DMDFM becomes classic factor model.

Compared with Pesaran (2006)’s multi-error structure model, DMDFM can handle both individual effect of regressors and lag effect for dependent variable. In the processes of factor decomposition, if we decompose common factor $F$ and $G$ with the same method, DMDFM becomes the multi-error structure model.

Andrews (2005) proposed the common shocks of cross section regression which generalized classic common factor model, but using that model, the paper only discussed common shocks to cross section without giving specific form of common factors. If we regard factor decomposition of DMDFM as common shocks, the same conclusion as Andrews should be obtained.

The forecasting idea of DMDFM is slightly different from Stock and Watson (2002) because we introduce double style factors to reflect time and individual correlation. DMDFM generalized style of Stock and Watson from multivariate time series to panel data, and the more complex factors will be considered.

## 3 Identification and assumption of DMDFM

Generally, we assume that the number of individuals $N$ and periods length $T$ are very large when we investigate high-dimensional panel data. We pay more attention to large $N$ and $p$, where the dimensions of individuals and indicators are very large. The relative sizes of $N$ and $p$ aren’t restricted strictly.

### Assumption A: (Identification)

**A1.** $\Lambda' \Lambda / p \rightarrow I_r$.

**A2.** $E(FF') = \Sigma_{FF'}$, where $\Sigma_{FF'}$ is a order $r$ positive diagonal matrix; the subscript of $F_{it}$ is omitted for simplicity.

**A3.** $\Gamma' \Gamma / N \rightarrow I_s$.

**A4.** $E(G_t G_t') = \Sigma_{GG'}$, where $\Sigma_{GG'}$ is a order $s$ positive definite diagonal matrix.

We know that $F_{it} N' = F_{it} R R^{-1} \Lambda'$ and $G_t \Gamma_t' = G_t Q Q^{-1} \Gamma_t'$, where $R$ and $Q$ are arbitrary invertible matrix with order $r$ and $s$. If we do not add some constraint conditions to them, decomposition factor of regressors and error terms won’t be unique. Assumptions A1 and A2 can cause $r^2$ restrictions for first group common factors $F_{it}$ and factor loadings $\Lambda$. Assumptions A3 and A4 can lead to $s^2$ restrictions for second group common factors $G_t$ and factor loadings $\Gamma_t$. Stock and Watson (2002) argued that assumptions A2 and A4 can ensure covariance stationary if we introduced lag terms of common factors $F_{it}$ and $G_t$ into dynamic factor model (2.5). Bai (2009) proposed some invertible assumptions in coefficient matrix for identification and estimation of parameter $\beta_L$ and $\beta_F$.

### Assumption B: (Factors and factor loadings)

**b1.** $\|\lambda_i\| < \lambda_{\text{max}} \rightarrow \infty$.

**b2.** $E\|F\|^4 < \infty$, and $p^{-1} \sum_p F F' \rightarrow \Sigma_{FF'}$, the subscript of $F_{it}$ is omitted for simplicity.

**b3.** $\|\gamma_i\| < \gamma_{\text{max}} \rightarrow \infty$, $E\|G_i\|^4 < \infty$.

Frobenius norm of matrix $F$ is defined as $\|F\| = [tr(F'F)]^{1/2}$, where $tr(F)$ is the trace of matrix. Assumptions b1–b3 can assure common factors $F_{it}$ and $G_i$ with correspondence factor loadings are not infinity. Bai and Ng (2002) argued that the above factors and factor loadings can ensure factor model standardization and improve the efficiency of factor decomposition in primitive variable.

### Assumption C: (Errors component)

**c1.** $E(\epsilon_{it}) = 0$, $\text{Var}(\epsilon_{it}) = \sigma^2 \epsilon$, $E(Y_{it} Y_{it+h}) = \rho_l(h)$, $\lim_{N \to \infty} \sup_h \sum_N \|\rho_l(h)\| \leq M < \infty$.

**c2.** $E(Y_{it} Y_{jt}) = \tau_l(k)$.
c3. For every \((t, s)\), 
\[ E(N^{-1} \sum_{i} |e_{is} e_{it} - E(e_{is} e_{it})|^4) \leq \frac{M}{N} < \infty \]

\[ c4. \lim_{N \to \infty} \sup \sum_{i,j} \sum_{s,t,u,v} \|\text{cov}(e_{is} e_{it}, e_{ju} e_{jv})\| \leq M < \infty \]

Assumptions of error term and its moments come from three parts: mean, variance, and moments condition, which are also called weak correlation assumptions. Assumption c1 restricts weak correlation of time series and mean of error term ruled out by twice factor decomposition, where the weak correlation is ready to the follow discussion of dynamic factor model. Assumption c2 represents cross section correlation. Assumption c3 gives high-order moments condition with uniform bound. Assumption c4 is the covariance bound of TS/CS, which is more stricter than c1–c3.

The idiosyncratic error \(e_{it}\) from regressors \(X_{it}\) and error term \(u_{it}\) must satisfy the assumption of factor decomposition, i.e., idiosyncratic errors are mutually independent, mean 0, and diagonal covariance matrix with off-diagonal elements 0.

Assumption D: (Dependent variable, common factors and model parameters)

\[ d1. E(Y'_{it} G_t) = \xi, E(G'_it) = \psi. \]

\[ d2. E[G'_i e_{ii} G_i e_{ii}] = T^{-1} \sum (G'_i e_{ii} e_{ii}) \text{ (iff } t \to \infty), \]

\[ E[G'_i e_{ii} G_i e_{ii}] = \Sigma_{FG}, E[Y'_{iu} F_{it}] = \Sigma_{YF}, \text{ where } \Sigma_{FG} \text{ and } \Sigma_{YF} \text{ are block diagonal positive matrix.} \]

\[ d3. \|\beta_L\| < \infty, \|\beta_F\| < \infty. \]

Assumption D imposes on the relationship between regressors and error term, including the key conditions to be used in parametric estimation. Assumption d1 reflects the correlation of regressors in model (2.5), while assumption C gives some weak correlation in the other variables. Assumption d2 is very strong which ensure model (2.5) can be estimated. Assumption d3 restricts the bound of \(\beta_L\) and \(\beta_F\).

Assumption A–D describe inner structure of models (2.2)–(2.7), and guarantee that each model can be estimated. We will study how to estimate the model and discuss the asymptotic property of the estimator with large \(N\) and large \(p\).

4 Model estimation

4.1 Factor decomposition and choice of the number of factors

DMDFM does the factor decomposition twice, so that the method of factor decomposition and the choice of factor number are very important. Many studies have discussed the choice of lag orders and the number of factors, but the schemes they proposed are only adaptive to lag of factor, e.g., followed by Forni et al. (2000)’s generalized dynamic factor model (GDFM), Hallin and Liska (2007) proposed valid information to choose the number of common factors, whose method is based on spectral density matrix decomposition theory. Harding and Nair (2009) exploited random matrix theory and Stieltjes transformation in uniform estimation deriving processes to determine lag orders and the number of common factors for common shocks component. This is named as dynamic scree plot method, where the GDFM is conveyed as follows:

\[ R_t = \sum_{i=0}^{q} \Lambda_i F_{t-i} + \epsilon_t \]

where \(R_t\) is a \(N \times 1\) vector, and the dynamic refers to lag effect of factors, which is different from the dynamic model of dependent variable in this article.

We decompose factors twice in this paper. Firstly, we use Eq. (2.2) to handle weak correlation and reduce dimension of individual, where common factor is composed of common shocks by different individuals. Secondly, we use classic PCA method to decompose factor in Eqs. (2.2) and (2.6). We will apply two different methods to choose the number of factors. We can use nonparametric scree plot method to choose the factor number of regressors in model(2.6) because the common factors of model (2.6) extracted from large indicators as multivariate analysis, in which factors number determined by scree plot method through contribution rate of variance can reflect indicator information maximize.

Remark 1 We decompose factors on each period, and obtain different factor numbers varying with periods. It is very important to choose a unified number of factors, which can improve analysis efficiency. Here, we choose the maximum contribution rate of variance to determine the number of common factors.

Determining the factors’ number of idiosyncratic error \(u_{it}\) is more complicated because they are additional information after several times transformation. Bai and Ng (2002) proposed two choice strategies of number factors for panel data, and they are all derived from Mallows (1973) information criterion (\(C_p\)).

One of them is panel data \(C_p\) criteria (\(PC_p\)), including three styles, among which the basic one is:

\[ PC_{p1}(k) = V(k, \hat{F}_k) + k\hat{\sigma}^2 \left( \frac{N + T}{NT} \right) \ln \left( \frac{NT}{N + T} \right) \]
where $V(k, \hat{F}_k) = N^{-1} \sum_{t=1}^{N} \hat{\sigma}_t^2$, and $\hat{\sigma}_t^2 = \hat{\epsilon}_t^2 / T$. $PC_p$ is a minimizing criteria with square sum of errors plus a penalty function. $PC_{p2}$ and $PC_{p3}$ are similar with $PC_{p1}$.

The other one is panel information criteria ($IC_p$), corresponding to $PC_p$. They also have three styles, one of which is:

$$IC_{p1}(k) = V(k, \hat{F}_k) + k \left( \frac{N + T}{N T} \right) \ln \left( \frac{NT}{N + T} \right)$$

The advantage of this criteria is that it doesn’t depend on square error $\hat{\sigma}_t^2$, which may extend the application scope. Bai and Ng (2002) argued that both $IC_p$ criteria and $PC_p$ criteria can choose the number of factors in panel data analysis. $PC_p$ and $IC_p$ information criteria can both be used to factor number choice for panel data. DMDFM decompose factors for two times. Equation (2.6) is a multivariate PCA decomposition; however, Eq. (2.2) is a panel data factors decomposition condition of common factors and factor loadings. At the processes of idiosyncratic error $U_{it}$ decomposition, we use $PC_p$ and $IC_p$ minimization criteria in the choice of factors numbers. The regressors’ factors number will be chosen by variance contribution method or scree plot method.

### 4.2 Estimation processes of DMDFM

The estimation process of DMDFM (2.2)–(2.7) can be divided into the following four steps: firstly, decompose factors with regressors $X_{it}$; secondly, estimate model (2.7); thirdly, decompose factors with error term $u_{it}$; At last, estimate model (2.5). The two-step estimation and two-step factors decomposition are different from their realized processes, respectively.

At first, we reduce the dimension of multiple indicators of regressors $X_{it}$ from $p$ to $r$ ($r < p$), where the number of factors $r$ is determined by the rate of variance contribution. The results can be expressed as:

$$X_{it} = \tilde{F}_{it} \hat{\Lambda} + \epsilon_{it} \quad (4.1)$$

**Remark 2** Common factors $\tilde{F}_{it}$ and factor loadings $\hat{\Lambda}$ are unobservable, and the information of regressors $X_{it}$ is reflected by common factors $\hat{F}_{it}$. Here, we use factor scores in equation estimation rather than common factors. Factor scores can be obtained by weighting least square or other methods.

Next, we substitute $\tilde{F}_{it}$ and $Y_{it}$’s lag terms $Y_{iw}$ into model (7), and use Generalized Method of Moments (GMM) to obtain models’ initial parameter estimators $\hat{\beta}_L$ and $\hat{\beta}_F$. Furthermore, we calculate the error of model (7) from the results of GMM estimation:

$$\tilde{u}_{it} = Y_{it} - \tilde{Y}_{it} = Y_{it} - Y_{iw} \hat{\beta}_L - F_{it} \hat{\beta}_F$$

Then, we need to decompose factor with $u_{it}$, using $PC_p$ and $IC_p$ criteria to determine the number of common factors $s$. The results of decomposition can be expressed as:

$$\tilde{u}_{it} = \tilde{G}_{it} \tilde{\Gamma}_i + \epsilon_{it} \quad (4.2)$$

Finally, we substitute the results of twice factor decomposition into model (5), estimate model (5), and obtain the estimation parameters $\hat{\beta}_L$ and $\hat{\beta}_F$ as well as the prediction equation:

$$\tilde{Y}_{it} = Y_{iw} \hat{\beta}_L + \tilde{F}_{it} \hat{\beta}_F + \tilde{G}_{it} \hat{\Gamma}_i \quad (4.3)$$

When estimating model (2.5), we can get $\tilde{\Gamma} \hat{\Gamma} / N = I_r$ through assumptions A3 and A4, which provide the identification condition of common factors and factor loadings. At the same time, Eq. (4.2) provides the result of decomposition for common factors $G_i$ and factor loadings $\Gamma_i$, so $G_i \hat{\Gamma}_i$ in Eq. (4.3) can be observable. We consider the correlation between lag terms and regressors when we estimate model (2.5). Thus, we employ GMM to estimate the parameters of model (4.3).

The above four-step estimation methods include two-step factor decomposition and two-step model estimation. The first step factor decomposition makes the goal of indicators’ dimension reduction realized, identifying the typical factors and their scores to represent all covariates and their values. The second step factor decomposition mainly reflects idiosyncratic and interactive effects of individuals and periods. In the following, we consider the two-step estimation procedures provided in the model. The first step extracts idiosyncratic errors to decompose factor of interactive effects. The second step gives consistent estimator of model (2.5). The choice of correct estimation methods of given model is very important, otherwise we will get an incorrect estimation result. Here, we consider applying generalized moments method (GMM).

### 4.3 Realization of estimation processes

Model (2.5) includes lag term of common factors and dependent variable, therefore it is difficult to use maximum likelihood estimation method to get strong uniform convergence results. Arellano and Bond (1991) considered GMM estimation in individual random effect panel data autoregressive model with independent strict exogenous variables and predetermined variables. Arellano and Bover (1995) developed the method of instrumental variable selection through GMM estimation in panel data model which include predetermined variable, and they characterize the valid transformations for exogenous variables. GMM is more flexible for the panel data model estimates with lags and exogenous
variables, and it can also be regarded as a consistent parameters estimation method for DMDFM.

We need to determine moment conditions and choose optimal instrumental variable if we use GMM to estimate panel data DMDFM. Without loss of generality, we only discuss the AR(1) process of dependent variable below. Here, model (2.5) can be written as:

\[ Y_{it} = Y_{i,t-1} + F_{it} \beta_F + G_{it} \Gamma_i' + \epsilon_{it} \]  
\[ (4.4) \]

Because the common factor \( G_i \) and factor loading \( \Gamma_i \) are obtained from decomposition of Eq. (4.2), \( G_i \) and \( \Gamma_i \) are observable when estimating model (4.4), the estimators of which are denoted by \( \tilde{G}_i \) and \( \tilde{\Gamma}_i \), and model (4.4) becomes:

\[ Y_{it} = Y_{i,t-1} + F_{it} \beta_F + \tilde{G}_i \tilde{\Gamma}_i' + \epsilon_{it} \]  
\[ (4.5) \]

For simplicity, we still use notation \( \epsilon_{it} \) representing error components in model (4.5). Following the inspiration of Arellano and Bond (1991), Hsiao (2003), instrumental variables maybe choose lag terms of dependent variable (predetermined variable) and exogenous variables. For model (4.5), the choice of instrumental variables should be correlated with explanatory variable and orthogonal with the residual terms. So, implementing first-order difference transformation in model (4.5), we obtain

\[ Y_{it} - Y_{i,t-1} = (Y_{i,t-1} - Y_{i,t-2})\rho + (F_{it} - F_{it-1}) \beta_F + \tilde{G}_i \tilde{\Gamma}_i' + \epsilon_{it} - \epsilon_{i,t-1} \]

Here, \((\tilde{G}_i - \tilde{G}_{i,t-1}) \tilde{\Gamma}_i'\) is observable scalar variable, and it can be combined with constant term when we estimate model (4.4), or the model including a constant term in model (4.4).

Remark 3 We assume that the factor decomposition of error component can be substituted into constant terms, so they can be regarded as a constant factor among common factors \( F_{it} \). If so, we should replace \( F_{it} \) with new notations. For the sake of brevity, we still use the same notations as before, but the factorization results of error components are included in the error terms of model (4.5).

The first-order difference transformation of model (13) can be written as

\[ Y_{it} - Y_{i,t-1} = (Y_{i,t-1} - Y_{i,t-2})\rho + (F_{it} - F_{it-1}) \beta_F + \epsilon_{it} - \epsilon_{i,t-1} \]

rewritten as difference operator \( \Delta \)

\[ \Delta Y_{it} = \Delta Y_{i,t-1} \rho + \Delta F_{it} \beta_F + \Delta \epsilon_{it} \]  
\[ (4.6) \]

The lag terms of \( Y_{it}, Y_{i,t-2-j} (j = 0, 1, 2, \ldots, t - 2) \) is subject to \( E[Y_{i,t-2-j}(Y_{i,t-1} - Y_{i,t-2})] \neq 0 \) and \( E[Y_{i,t-2-j}\epsilon_{it} - \epsilon_{i,t-1}] = 0 \). For the \( i \)th individual which includes \( T(T - 1)/2 \) moment conditions, the difference of the error term, \( \epsilon_{it} - \epsilon_{i,t-1}, t = 2, \ldots, T \), is denoted as \( \Delta \epsilon_i \). Here \( r \) explanatory variables \( F_{it} \) have similar features with \( Y_{i,t-2-j} \).

\[ E[F_{it} \Delta \epsilon_i] = 0, t = 1, \ldots, T \]

Thus, we obtain \( r \times T \times (T - 1) \) moment conditions for \( i \)th individual, and predetermined variables and exogenous variables can determine \( T(T - 1)/2 + r \times T \times (T - 1) \) moment equations of residual term. Denote

\[ H_{it} = (Y_{i0}, \ldots, Y_{it-2}, F_{i1}', \ldots, F_{i,T}') \]

the \( T(T - 1)/2 + r \times T \times (T - 1) \) moment equations can be written as:

\[ E[H_{it} \Delta \epsilon_i] = 0, t = 2, \ldots, T \]

These moment equations provide some moment conditions to error terms. For simplicity, we omit the subscript \( t \) for all variables, and obtain matrix form of the model:

\[ \Delta Y_i = \Delta Y_{i-1} \rho + \Delta F_i \beta_F + \Delta \epsilon_i, i = 1, \ldots, N \]  
\[ (4.7) \]

Denote

\[ Z_i = \begin{bmatrix} H_{2} & 0 & \cdots & 0 \\ 0 & H_{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{T} \end{bmatrix} \]

for the \( i \)th individual, the previous moment equations can be written as:

\[ E[Z_i \Delta \epsilon_i] = 0, i = 1, \ldots, N \]  
\[ (4.8) \]

Because the number of moment equations in Eq. (4.8) is \( T(T - 1)/2 + r \times T \times (T - 1) \) which is much larger than the number of parameters to be estimated in model (4.7), \( r + 1 \). We impose some restriction conditions on it. The residual sum of squares of model (4.7) is defined as follows:

\[ V(\Delta Y, \Delta F; \rho, \beta) = \sum_{i=1}^{N} (\Delta Y_i - \Delta Y_{i-1} \rho - \Delta F_i \beta_F)^2 \]  
\[ (4.9) \]

We can obtain uniform optimal estimator of unknown parameters through minimizing objective function (4.9). Too many moment conditions cause the moment Eq. (4.8) insoluble. To acquire valid conditions of the parameter estimation, we
seek a positive definite matrix A, with which the transform objective function (4.9) is written as:

\[
\tilde{V}(\Delta Y, \Delta F; \rho, \beta) = \sum_{i=1}^{N} (\Delta Y_i - \Delta Y_{i-1}) \rho - \Delta F_i \rho F_i A (\Delta Y_i - \Delta Y_{i-1}) \rho - \Delta F_i \rho F_i \] (4.10)

Through minimizing objective function (4.10), we can obtain estimators \( \hat{\rho} \) and \( \hat{\beta}_F \) of parameter \( \rho \) and \( \beta_F \), by choosing an appropriate positive definite matrix. The covariance matrix of \( Z_i \Delta \epsilon_i \) is:

\[
V_N = N^{-1} \sum_{i=1}^{N} E(Z_i \Delta \epsilon_i \Delta \epsilon_i' Z_i')
\]

whose estimation results can be written as:

\[
\hat{V}_N = N^{-1} \sum_{i=1}^{N} E(Z_i \Delta \hat{\epsilon}_i \Delta \hat{\epsilon}_i' Z_i')
\]

From the results of Hansen (1982), we see that optimal alternative \( A_O \) of positive definite matrix A is \( \hat{V}_N^{-1} \). From previous assumption C, we know error term \( \epsilon_{it} \) is i.i.d., mean 0, variance \( \sigma^2_{\epsilon} \), so we have:

\[
A_O = \left(N^{-1} \sum_{i=1}^{N} Z_i U Z_i'\right)^{-1}
\]

According to the one-step estimation method of Arellano and Bond (1991), known transformation matrix can’t extract the information of error term thoroughly. We consider using two-step estimation method, and the residual \( \hat{\epsilon}_i^{(1)} \) of first step estimation to construct transformation matrix \( U_i = \sum_{i=1}^{N} \hat{\epsilon}_i^{(1)} \hat{\epsilon}_i^{(1)}' \). Then we minimize objective function (4.10) and obtain the estimators of \( \rho \) and \( \beta_F \) similar to Arellano and Bond (1991):

\[
(\hat{\rho}, \hat{\beta}_F) = \left((\Delta Y_{-1}, \Delta F)' Z' A_O Z (\Delta Y_{-1}, \Delta F)\right)^{-1}
\times (\Delta Y_{-1}, \Delta F)' Z' A_O Z \Delta Y
\] (4.11)

where \( \Delta Y_{-1} \) and \( \Delta F \) are \( N(T-1) \) vector and \( N(T-1) \times r \) matrix, respectively, which represent predetermined variables and exogenous variables. These two styles of variables can be estimated, respectively, or simultaneously as explanatory variables. The meaning of \( A_O \) and \( Z \), as mentioned before, represents optimal choice of transformation matrix and weighted matrix, respectively. \( Z \) is a block diagonal matrix composed by the instrumental variables.

### 4.4 Theory results

GMM estimation solve population moment equations through sample moment conditions, with regard to the case of over identification. We transform them for identification by weighted matrix or transformation matrix \( A \). If the optimal weighted matrix and the instrumental variable matrix have been correctly chosen, the GMM estimators have consistency and asymptotic normality. The sample estimators of parameters in Eq. (4.11) obtained by minimizing objective function (4.10) can be written as:

\[
(\hat{\rho}, \hat{\beta}_F) = \left[\sum_i (\Delta Y_{i-1}, \Delta F_i)' Z_i'\right]^{-1}
\times \left[\sum_i Z_i A_O Z_i\right]^{-1} Z_i Y_i,
\] (4.12)

The RHS of model (2.5) includes the high-order lag terms of dependent variable \( Y_{it} \), which are seen as IV in GMM estimation to obtain consistent efficiency estimators of regression parameter. After the previous assumption conditions are satisfied, we could draw more general conclusion as below.

**Theorem 4.1** (Consistency) Under assumption conditions A–D, GMM estimators \( \hat{\beta}_L \) and \( \hat{\beta}_F \) are the estimators of lag terms parameter \( \beta_L \) and common factor parameter \( \beta_F \), respectively. Suppose the number of explanatory variables \( p \) and period length \( T \) are given, when \( N \to \infty \), the following conclusions are found:

1. \( \hat{\beta}_L - \beta_L \to 0 \)
2. \( \hat{\beta}_F - \beta_F \to 0 \)

The proofs of Theorem 4.1 can be found in “Appendix A.”
\[
\text{var}(\hat{\rho}, \hat{\beta}_F) = \sigma^2 \left\{ \sum_i (\Delta Y_{i-1}, \Delta F_i)' Z_i \right\}^{-1} \left\{ \sum_i Z_i A \sigma Z_i \right\}^{-1}
\]

We rewrite objective function (4.10):

\[
O_N = N^{-1} \sum_{i=1}^{N} (\Delta Y_i - \delta Y_{i-1} \beta_L - \Delta F_i \beta_F)' A (\Delta Y_i - \delta Y_{i-1} \beta_L - \Delta F_i \beta_F)
\]

Calculating the first-order partial derivative to objective function \(O_N\) with respect to parameter \(\beta_L\) and \(\beta_F\), we have:

\[
R_L = \partial O_N / \partial \beta_L \quad \text{and} \quad R_F = \partial O_N / \partial \beta_F
\]

where \(R(\beta_L, \beta_F) = (\beta_L, \beta_F)'\) are first-order partial derivatives with respect to the parameters to be estimated, since we obtain the estimators form (4.12) via minimizing objective function (4.10), which converge to (4.11) consistently. Furthermore, notice that random matrix \(R\) converge to matrix \(R_1\) w.p.1., and denote

\[
\Sigma_1 = (R_1' A O R_1)^{-1} R_1' A O D_1 A O R_1 (R_1' A O R_1)^{-1}
\]

where \(D_1\) is the asymptotic variance of \(\sqrt{N} O_N\) when \(N \to \infty\),

\[
\sqrt{N} O_N \xrightarrow{d} N(0, D_1)
\]

Here we assume \(\sqrt{N} O_N\) converges in distribution to normal distribution with mean 0. The above analysis are all based on short panel data \((T < N)\). Furthermore, we consider long panel data whose periods length \(T\) and individual number \(N\) tend to infinity simultaneously, and \(R \xrightarrow{a.s.} R_2\). Let

\[
O_N = (NT)^{-1} \sum_{i=1}^{N} (\Delta Y_i - \delta Y_{i-1} \beta_L - \Delta F_i \beta_F)' A (\Delta Y_i - \delta Y_{i-1} \beta_L - \Delta F_i \beta_F)
\]

and the other notations remain unchanged. Denote

\[
\Sigma_2 = (R_2' A O R_2)^{-1} R_2' A O D_2 A O R_2 (R_2' A O R_2)^{-1}
\]

when \(N, T \to \infty\), we assume

\[
\sqrt{NT} O_N \xrightarrow{d} N(0, D_2)
\]

Under the given correlation assumptions, when periods length \(T \to \infty\), GMM estimators of dynamic double factors model have asymptotic normality. The conclusions can be seen in Theorem 2.

**Theorem 4.2** (CLT) Given some positive matrices \(\Sigma_1 - \Sigma_2\), under assumption conditions, the conclusions are as follows:

1. **Explanatory variables have serial correlation, and dependent variable have cross section correlation**, when \(N \to \infty\), \(T\) is fixed, and \(T/N \to 0\) (short panel data), then

\[
\sqrt{N} \left[ (\hat{\beta}_L, \hat{\beta}_F) - (\beta_L, \beta_F) \right] \xrightarrow{d} N(0, \Sigma_1);
\]

2. **Explanatory variables have serial correlation, and dependent variables do not have cross section correlation**, when \(N, T \to \infty\), and \(T/N \to C\), \(C\) is constant (long panel data), \(C \neq 0\), then

\[
\sqrt{NT} \left[ (\hat{\beta}_L, \hat{\beta}_F) - (\beta_L, \beta_F) \right] \xrightarrow{d} N(0, \Sigma_2).
\]

The proofs of Theorem 4.2 can be seen in “Appendix B.” From Theorem 4.2, we can get the conclusion that asymptotic normality of sample estimator for short panel \((T < N)\) and long panel \((T \text{ and } N \text{ is close})\) can be got. The values of \(\Sigma_1\) and \(\Sigma_2\) are correlated closely with asymptotic variance \(D_1\) and \(D_2\) of \(\sqrt{N} O_N\). Optimal weighted matrix \(A_O\) is generally substituted by a random given matrix to obtain \(D_1\) and \(D_2\), so \(D_1\) and \(D_2\) are mainly dependent on the variance of random error term. Furthermore, we assume

\[
E(\epsilon_i \epsilon_{i+h}) = 0
\]

and \(Var(\epsilon_i) = \sigma^2\), so variance of disturbance term influence asymptotic variance of estimator varied with the estimation method of given model. The choice of IV and weighted matrix also influence asymptotic variance, If

\[
E[\Delta \epsilon_i | Z_i] = 0, \quad i = 1, \ldots, N
\]

then the interactive effect of error term and IV aren’t considered, which is more stronger than \(E[\Delta \epsilon_i Z_i] = 0\).

Obviously, choosing different IV \(Z\) also influence asymptotic variance of \(\sqrt{N} O_N\), furthermore \(\Sigma_1\) and \(\Sigma_2\), so different number of IV will get different estimation results. For GMM estimation, appropriate IV comes from higher-order lag terms and exogenous variables, so it is important to choose the order of lag terms. Meanwhile, if every estimator of parameters to be estimated have asymptotic normality, by Slutsky’s lemma, the asymptotic properties of the sum of these estimators will be obtained.
5 Simulation study

DMDFM is concerned with time series correlation and cross section correlation simultaneously. To reflect these two styles of correlation, simulation processes permit that common factors of error term have lag effects. Common factors being decomposed by explanatory variables have individual correlation as well as series correlation. Factor loadings mainly reflect individual correlation. High-dimensional case includes a large number of explanatory variables, and we attempt to use minority common factors to extract information of explanatory variables to reduce dimension. So, in the simulation, we should consider not only correlation with explanatory variables, but also lag effects of explanatory variables in these common factors. Consider the following data generation process (DGP):

\[
y_{it} = \alpha_i + \beta_{it} y_{it-1} + \beta_{f1} f_{i1t} + \beta_{f2} f_{2it} \\
+ \gamma_{11} g_{11t} + \gamma_{12} g_{21t} + \epsilon_{it}
\]  

(5.1)

Compared with model (2.5), DGP add some restriction conditions to reflect existing issues in terms of five parts: Interception; first-order lag of dependent variable; common factors of covariates; common factors and factor loadings of error components; idiosyncratic errors. As mentioned above, we choose two common factors from each factor group.

Intercept terms are generated from normal distribution:

\[
\alpha_i \sim i.i.d. N(1, 2)
\]

To reflect series correlation, the error term of model (2.5) is generated from AR(1) processes:

\[
\begin{align*}
\epsilon_{it} &= \rho \epsilon_{i,t-1} + \eta_{it} \\
\rho &\sim i.i.d. U(0.05, 0.95) \\
\eta_{it} &\sim i.i.d. N(0, 1) \\
\epsilon_{i,0} &= 0
\end{align*}
\]

The errors in this part represent idiosyncratic error generated from factors decomposition. From the factor decomposition process of Eq. (2.2), we see that the other part of error components reflect in common factors and factor loadings of error term. Assume common factors of error component retain lag factors, and we express them as AR(1) processes from different idiosyncratic errors. The first-order correlation coefficients are generated from uniform distribution, two error components DGP can be written as:

\[
\begin{align*}
g_{jt} &= \rho_{ji} g_{j,t-1} + u_{jt} \quad (j = 1, 2) \\
\rho_{ji} &\sim i.i.d. U(0.05, 0.95), \quad g_{j,0} = 0 \\
u_{jt} &\sim i.i.d. N(0, 1)
\end{align*}
\]

where factor loadings of error component are always generated from uniform distribution or normal distribution, and here we use uniform distribution.

\[
\begin{align*}
\gamma_{k11} &\sim i.i.d. U(0.05, 0.95) \\
\gamma_{k21} &\sim i.i.d. U(0.05, 0.95)
\end{align*}
\]

Common factors extracted from explanatory variables should reflect correlation among individuals, periods and explanatory variables. Every common factor of different individuals retain main information of explanatory variables and idiosynchratic component of individuals. So the data generation process of each common factor consists of four parts: level term; error factors term; individual correlation component; error component, which can be generated from:

\[
f_{k1t} = a_{k1t} h_{11t} + \gamma_{k11} g_{11t} + \gamma_{k21} g_{21t} \\
+ \xi_{k1t} d_{1t} + \omega_{k1t} \quad (k = 1, 2)
\]

where level term is composed of an individual random coefficient multiplied by an AR(1) processes. First-order autocorrelation coefficients and initial value of AR(1) processes have been given, and the others are generated from AR(1) processes. Two common factors DGP of explanatory variables are:

\[
\begin{align*}
a_{k11} &\sim i.i.d. U(0.05, 0.95) \\
h_{11t} &= \rho_{h1} h_{11,t-1} + \tau_{h1} \\
\rho_{h1} &= 0.4, h_{11,0} = 0.2, \quad \rho_{h2} = 0.5, h_{21,0} = 0.3 \\
\tau_{h1} &\sim i.i.d. N(0, 1)
\end{align*}
\]

Random error of common factors terms are generated from normal distribution:

\[
\omega_{k1t} \sim i.i.d. N(0, 0.25)
\]

Individual correlation components are generated from spatial auto-regression SAR(1), which can be written as:

\[
\begin{align*}
q_{i1} &= \rho_q q_{i-1,1} + v_q \\
\rho_q &\sim i.i.d. U(0.05, 0.95), \quad q_{0,1} = 0.1 \\
v_q &\sim i.i.d. N(0, 1)
\end{align*}
\]

The coefficients of individual correlation components are generated from uniform distribution:

\[
\begin{align*}
\xi_{11t} &\sim i.i.d. U(0.05, 0.95) \\
\xi_{21t} &\sim i.i.d. U(0.05, 0.95)
\end{align*}
\]

The common factors of explanatory variables retain the common factors of error components to express extracted
Table 1  Bias and RMSE of simulation results

| (N,T)      | Bias   | RMSE   |
|------------|--------|--------|
| β₁        | β₁₂    | β₁₂_{f1} | β₁₂_{f2} | β₁₂ | β₁₂_{f1} | β₁₂_{f2} |
| (20,5)     | −0.0981| −0.0333 | −0.0319   | 0.01238 | 0.01313 | 0.01301  |
| (50,5)     | 0.00131| 0.02163 | 0.00161   | 0.00962 | 0.01013 | 0.01036  |
| (50,10)    | 0.03613| 0.00423 | 0.02274   | 0.00377 | 0.00655 | 0.00647  |
| (100,5)    | 0.01293| 0.02180 | 0.01876   | 0.00942 | 0.00909 | 0.00914  |
| (100,10)   | 0.04707| 0.00758 | 0.01898   | 0.00361 | 0.00607 | 0.00581  |
| (100,20)   | 0.07718| 0.00616 | 0.01829   | 0.00189 | 0.00459 | 0.00417  |
| (200,5)    | 0.03012| 0.02184 | 0.02596   | 0.00994 | 0.00808 | 0.00888  |
| (200,10)   | 0.05712| 0.00829 | 0.01733   | 0.00364 | 0.00591 | 0.00565  |
| (200,20)   | 0.08152| 0.00812 | 0.02530   | 0.00187 | 0.00432 | 0.00425  |
| (200,50)   | 0.08032| 0.01192 | 0.02292   | 0.00187 | 0.00448 | 0.00420  |

information, whose coefficients are generated from uniform distribution:

γ₁₁, γ₁₂ i.i.d. U(0.05, 0.95)

Based on the above thoughts, we should give an initial value of the explanatory variables y_{it}: y_{0} = 0, and β₁₁ = 0.6, β₁₂ = 0.8, β₁₂₂ = 1. To ensure the consistency of the data generation process, we discarded the first 15 simulation values. Every experiment was replicated 2000 times for the (N,T)=(20,5), (50,5), (50,10), (100,5), (100,10), (100,20), (200,5), (200,10), (200,20), (200,50), respectively. The estimation results of parameters β₁₁, β₁₂ and β₁₂₂ are derived from 2000 times replication, whose mean bias and root mean square error (RMSE) are calculated hereafter. The simulations results are summarized in Table 1.

As shown in Table 1, when the values of N and T are given, the first-order lag term of dependent variable in DMDFM has smaller bias and RMSE as well as coefficient estimation value of explanatory variables’ common factors. It indicates that GMM estimation can obtain consistent and efficient parameter estimator. Furthermore, considering the size of relative bias, we can see that the range of dependent variable and explanatory variables are in (−20, 20). The results of Table 1 is relatively smaller than initial values, so the estimators are consistent correspondence with population parameter. These satisfy the large sample properties of DMDFM and GMM estimation mentioned previously.

The results of simulation demonstrate that bias and RMSE of regression coefficients do not obviously vary with the number of individuals increment. The number of individuals increase from 20 to 200 and periods increase from 5 to 50, but bias do not increase with individual size, which indicates that the results of estimation have good properties in finite sample. For the short panel, periods are shorter than the number of individuals, bias and RMSE do not very obviously. When periods become longer, according to the results of Monte Carlo simulation, the bias of estimator becomes smaller. In the above simulation, the number of individuals is at least 4 times more than periods length, which reveals the high-dimensional feature. Bias of estimator becomes smaller with the increasing periods length. The results of estimation have higher uniform convergence speed.

Root Mean Square Error (RMSE) includes the information of sample bias and variance. The results of Table 1 demonstrate that bias and RMSE are smaller. It shows that the variance is also smaller due to the MSE is the sum of variance and square of bias. The smaller variance of estimation results indicate that this estimation method can not only obtain consistency estimator but also obtain efficient variance. This verified consistency and efficiency once again.

DMDFM can reduce the dimension of indicators and reflect the internal structure of panel data reasonably. Further-
more, the model estimation results can be used for predicting dependent variable. In order to test the prediction effect of DMDFM, we still use the DGP as before to generate a group training sets and testing sets. To enhance the observability of the graphics, we predict 20 periods values of dependent variable step by step. At first, we generate every periods value of explanatory variables and one period lag value of dependent variable, then predict dependent variable forward one period by two-step estimation method through models (2.5)–(2.7) to compare the predicted values and true values. Figure 1 shows the average of predicted values of 100 individuals compared with true values. Figure 2 shows 6 individuals which extracted randomly from 100 individual predicted values compared with true values.

As can be seen from Figs. 1 and 2, predicted values of all individual average and every individual have good prediction effect via GMM estimation. One-step predicted value have goodness fitting of trend as well as points. The constructed model and its estimation method reflect the data generation processes well, and prediction effect is better. Furthermore, if we consider mean absolute percentage error (MAPE), the similar conclusion should be obtained.

6 Conclusion

In this article, we propose a panel data double factors model which include both explanatory variables and error component factor decomposition. The mixed double factor model derives from the factor decomposition method, and the aim of twice decomposition analysis is different. Contrast to the general dynamic factor model, the dynamic of DMDFM refers to the lag terms of dependent variable. Theoretically, if panel data have first-order autocorrelation of time series and heterogeneity components of individual or periods (fixed effect or random effect), the lag term $Y_{it-1}$ of dependent variable $Y_{it}$ are determined by the expectation of two parts information: the lag information sets $I_{t-1}$ of explanatory variables $X_{it}$ and the remainder information given by $X_{it}$, i.e., $E(Y_{it}) = E(Y_{it}|I_{t-1}, X_{it}) = E(Y_{it}|Y_{it-1}, X_{it})$. The dynamic panel data models constructed by lag terms of explanatory variables and common factors are different; however, the results are excellent. Dynamic mixed double factor model is composed of four main parts: lag terms of response; common factor of regressors; factor error component and idiosyncratic error component.
RHS of dynamic panel data model includes the lag terms of dependent variable, so independent assumption of error term and dependent variable aren’t satisfied. We cannot get the consistent and efficient estimators using OLS or MLE of dynamic factor model, so generalized moment method (GMM) is a better alternative options. In this article, we propose an iteration GMM to estimate DMDFM. At first, we obtain the error component of the model through GMM estimation, furthermore decompose factor with the given error component. The factors decomposition results of estimated error component can be regarded as intercept term of new model which can be estimated by GMM to obtain parameter estimation value once again. The proof of the theorems and simulation results show that the two-step GMM estimation is able to get consistent estimators of the dynamic mixed double factor model. The estimation results of DMDFM have better explanatory power and prediction effects.

DMDFM reduces the dimension of large number of indicators. In which, a large number of explanatory variables are represented by few common factors, which extends the application scope of the model. However, every explanatory variable has its own implication in empirical analysis, and we should consider how to provide reasonable explanation of explanatory variables in the following step. The research scope of this article only aims at dimensional reduction, while variable selection for the explanatory ability of the indicators is not considered, which restrict the application effect of the model. Panel data usually has serial correlation and cross section correlation, and there perhaps exists other structural features. These structural features related to individuals are obvious in the spacial panel data, i.e., structural change, heteroscedasticity and variance magnitude, and so on, however DMDFM cannot solve these problem thoroughly. We will study how to improve DMDFM to reflect the structural features of panel data in the future.

The estimators of DMDFM mainly focus on expectation in this article, however variance of DMDFM also should be taken into account as well as multivariate time series heteroscedasticity model. Other issues of DMDFM include consistent asymptotic variance estimation, asymptotic efficiency of estimators, testing of estimators obtained by GMM estimation. In addition to theoretical analysis of model construction and estimation, empirical research also should be considered. Because high-dimensional panel data appears both in macroeconomic and microeconomic fields, empirical research combined with application background must be discussed in future.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Human and animal rights** This article does not contain any studies with human participants or animals performed by any of the authors.

**Appendix: Proof of theoretical results**

**A. Proof of Theorem 4.1.**

Denote \( b(z, \beta) = Z_i \Delta \epsilon_i, \) where \( \beta = (\beta_L, \beta_F) \). From Eq. (4.8), we have \( E[b(z, \beta)] = 0 \). We calculate partial derivative for each parameter to be estimated, \( \frac{\partial b(z, \beta)}{\partial \beta} \), then let

\[
Db(\beta_L, \beta_F) = \left( \frac{\partial b(z, \beta)}{\partial \beta_L}, \frac{\partial b(z, \beta)}{\partial \beta_F} \right)
\]

because the uniform consistency of random disturbance term, using Taylor series expansion around \( \hat{\beta}_L \) and \( \beta_F \):

\[
b(z, \hat{\beta}) = b(z, \beta) + Db(\beta_L, \beta_F)(b(z, \hat{\beta}) - b(z, \beta)) + o(b(z, \beta)) \quad (6.1)
\]

where \( \hat{\beta} = (\hat{\beta}_L, \hat{\beta}_F), \) \( \beta_L, \) and \( \beta_F, \) respectively, multiplied by weighting matrix \( A \) simultaneously:

\[
Ab(z, \hat{\beta}) = Ab(z, \beta) + ADb(\beta_L, \beta_F)(b(z, \hat{\beta}) - b(z, \beta)) + o(b(z, \beta)) \quad (6.2)
\]

Given the following three items:

(i) From assumptions as before, given optimal weighting matrix \( A_O \), we can obtain unique optimal estimator of \( \beta \). \( \beta \) is continuous vector defined on Euclid space \( R^n \), and space \( \Theta \) constituted by \( \beta \) is a subset of \( R^n \), and is closed and bounded.

(ii) For \( b(z, \beta) = Z_i \Delta \epsilon_i, \forall \epsilon > 0 \), from (6.1)

\[
E[b(z, \hat{\beta})] = b(z, \beta)
\]

so,

\[
\left| b(z, \hat{\beta}) - b(z, \beta) \right| \overset{D}{\rightarrow} 0 \quad (6.3)
\]

for given matrix \( A \), denote

\[
\hat{S}_N(\beta) = b(z, \hat{\beta})' Abv(z, \hat{\beta})
\]

and

\[
S_0(\beta) = b(z, \beta)' Ab(z, \beta)
\]
Using Cauchy–Schwartz inequalities

\[
\hat{S}_N(\beta) - S_0(\beta) = \left| b(z, \hat{\beta})' \hat{A}b(z, \hat{\beta}) - b(z, \beta)' Ab(z, \beta) \right|
\]

\[
= \left| \left( b(z, \hat{\beta}) - b(z, \beta) \right)' \hat{A} \left( b(z, \hat{\beta}) - b(z, \beta) \right) + b(z, \beta)' \hat{A} \left( b(z, \hat{\beta}) - b(z, \beta) \right) \\
+ b(z, \beta)' \hat{A} \left( b(z, \hat{\beta}) - b(z, \beta) \right) - b(z, \beta)' \hat{A}b(z, \beta) \right|
\]

Using triangle inequalities

\[
\leq \left| \left( b(z, \hat{\beta}) - b(z, \beta) \right)' \hat{A} \left( b(z, \hat{\beta}) - b(z, \beta) \right) \\
- b(z, \beta) \right| + \left| b(z, \beta)' \hat{A} + \hat{A}' \right| \left( b(z, \hat{\beta}) - b(z, \beta) \right)
\]

Using Cauchy–Schwartz inequalities

\[
\leq \| b(z, \hat{\beta}) - b(z, \beta) \|^2 \| \hat{A} \| + 2 \| b(z, \beta) \| b(z, \hat{\beta}) \\
- b(z, \beta) \| \| \hat{A} \| + \| b(z, \beta) \|^2 \| \hat{A} - A \|
\]

because

\[
b(z, \hat{\beta}) - b(z, \beta) \xrightarrow{p} 0 \\
\hat{A} - A \xrightarrow{p} 0
\]

we have

\[
\left| \hat{S}_N(\beta) - S_0(\beta) \right| \xrightarrow{p} 0
\]

By Newey and McFadden (1994), following uniform convergence theorem, the conclusion is obtained. □

## B. Proof of Theorem 4.2

(1) Because

\[
\partial R_1(\beta, \beta')/\partial \beta = \partial \left( b(z, \beta)' Ab(z, \beta) \right)/\partial \beta
\]

\[
= \partial \left( b(z, \beta)' / \partial \beta Ab(z, \beta) \right) + \partial \left( b(z, \beta)' / \partial \beta Ab(z, \beta) \right)
\]

\[
= 2 \partial \left( b(z, \beta)' / \partial \beta Ab(z, \beta) \right)
\]

where \( \beta = (\beta_L, \beta_F)' \) for notation simplicity. Following this notation, in order to estimate GMM, we solve first-order condition, so we obtain that

\[
R_1(\hat{\beta})' Ab(\hat{\beta}) = 0
\]

(6.4)

from (6.1), for optimal matrix \( A_O \), we have

\[
R_1(\beta)' A_O b(z, \hat{\beta}) = R_1(\beta)' A_O \sqrt{N} b(z, \hat{\beta})
\]

\[
+ o(\sqrt{N} b(z, \hat{\beta}))
\]

(6.5)

using Taylor series expansion around \( \beta \)

\[
R_1(\beta)' A_O b(z, \hat{\beta}) = R_1(\beta)' A_O \left( \sqrt{N} b(z, \hat{\beta}) \\
+ R_1(\beta) \sqrt{N} \left( \hat{\beta} - \beta \right) \right) + o(\sqrt{N} b(z, \hat{\beta}))
\]

from (6.4), we have

\[
R_1(\beta)' A_O R_1(\beta) \sqrt{N} \left( \hat{\beta} - \beta \right) = -R_1(\beta)' A_O \sqrt{N} b(z, \beta) + o(\sqrt{N} b(z, \hat{\beta}))
\]

so

\[
\sqrt{N} \left( \hat{\beta} - \beta \right) = -\left( R_1(\beta)' A_O R_1(\beta) \right)^{-1} \left( R_1(\beta)' A_O \sqrt{N} b(z, \beta) + o(\sqrt{N} b(z, \hat{\beta})) \right)
\]

by Eq. (4.15) as previous, we have

\[
\sqrt{N} b(z, \beta) \xrightarrow{d} N(0, D_1)
\]

and

\[
\left( R_1(\beta)' A_O R_1(\beta) \right)^{-1} R_1(\beta)' A_O
\]

is a determined matrix, so

\[
\sqrt{N} \left( \hat{\beta} - \beta \right) \xrightarrow{d} N(0, \Sigma_1)
\]
\[ \sqrt{N} \left( \left( \hat{\beta}_L, \hat{\beta}_F \right) - (\beta_L, \beta_F) \right) \xrightarrow{d} N(0, \Sigma_1) \]

(2) Similar to the proof of (1), omitted.

References

Ahn SG, Lee YH, Schmidt P (2001) GMM Estimation of linear panel data models with time-varying individual effects. J Econ 101:219–255

Andrews DWK (2005) Cross-section regression with common shocks. Econometrica 73:1551–1585

Anderson B, Deistler M (2008) Generalized linear dynamic factor models—a structure theory. In: 2008 IEEE conference on decision and control

Arellano M, Bond SR (1991) Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations. Rev Econ Stud 58:277–297

Arellano M, Bover O (1995) Another look at the instrumental variable estimation of error components models. J Econ 68:29–51

Bai J (2003) Inferential theory for factor models of large dimensions. Econometrica 71:135–173

Bai J (2009) Panel data models with interactive fixed effects. Econometrica 77:1229–1279

Bai J, Ng S (2002) Determining the number of factors in approximate factor models. Econometrica 70:191–221

Chamberlain G, Rothschild M (1983) Arbitrage, factor structure and mean-variance analysis in large asset markets. Econometrica 51:1281–1304

Fan J, Fan Y, Lv J (2008) High dimensional covariance matrix estimation using a factor model. J Econ 147:186–197

Forni M, Hallin M, Lippi M, Reichlin L (2000) The generalized dynamic factor model: identification and estimation. Rev Econ Stat 82:540–554

Hallin M, Liska R (2007) Determining the number of factors in the general dynamic factor model. J Am Stat Assoc 102:603–617

Hansen LP (1982) Large sample properties of generalized method of moments estimators. Econometrica 50:1029–1054

Harding M, Nair KK (2009) Estimating the number of factors and lags in high dimensional dynamic factor models. Mimeo

Hsiao C (2003) Analysis of panel data. Cambridge University Press, New York

Mallows CL (1973) Some comments on Cp. Technometrics 15:661–675

Moon HR, Perron B (2004) Testing for a unit root in panels with dynamic factors. J Econ 122:81–126

Newey W, McFadden D (1994) Large sample estimation and hypothesis testing. In: Engle RF, McFadden D (eds) Handbook of econometrics. North Holland, Amsterdam, pp 2111–2245

Pesaran MH (2006) Estimation and inference in large heterogeneous panels with a multifactor error structure. Econometrica 74:967–1012

Ross S (1976) The arbitrage theory of capital asset pricing. J Econ Theory 13:341–360

Stock JH, Watson MW (2002) Forecasting using principal components from a large number of predictors. J Am Stat Assoc 97:1167–1179

Stock JH, Watson MW (2005) Implications of dynamic factor models for VAR analysis. Princeton University, Princeton

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