Perfect Mannheim, Lipschitz and Hurwitz weight codes

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Abstract

In this paper, upper bounds on codes over Gaussian integers, Lipschitz integers and Hurwitz integers with respect to Mannheim metric, Lipschitz and Hurwitz metric are given.

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1 Introduction

If a code attains an upper bound (the sphere-packing bound) in a given metric, then it is called a perfect code. Perfect codes have always drawn the attention of coding theorists and mathematicians since they play an important role in coding theory for theoretical and practical reasons. All perfect codes with respect to Hamming metric over finite fields are known [1]-[4]. For non-field alphabets only trivial codes are known and by similar methods it was proved in [5].

Perfect codes have been investigated not only with respect to Hamming metric but also other metrics, for example Lee metric. Lee metric was introduced in [6]. Some perfect codes with respect to Lee metric were discovered in [7].

Later, Mannheim metric was introduced by Huber in [8]. It is well known that the Euclidean metric is the relevant metric for maximum-likelihood decoding. Although Mannheim metric is a reasonable approximation to it, it is not a priori, a natural choice. However, the codes being proposed are very useful in coded modulation schemes based on quadrature amplitude modulation (QAM)-type constellations for which neither Hamming nor Lee metric is appropriate.

Two classes of codes over Gaussian integers were considered in [8], namely, the one Mannheim error-correcting codes (OMEC), and codes having minimum Mannheim distance greater than three. The OMEC codes are perfect with respect to Mannheim metric. Thus, some perfect codes were discovered. But, dimension $k$ of OMEC codes with parameters $[n, k, d]$ are only $n - 1$. In the present study, we obtain some perfect codes with respect to Mannheim metric. The dimension of these perfect codes are not only $n - 1$ but also $n - k$, ($k < n$).

On the other hand, Lipschitz metric was presented and some perfect codes over Lipschitz integers with respect to Lipschitz metric were introduced in [9].

In this paper, we consider the existence and nonexistence of perfect codes with respect to Mannheim metric and Lipschitz metric over Gaussian integers,
Lipschitz integers and Hurwitz integers. Also, we introduce Hurwitz metric and we give upper bounds on these codes over Hurwitz integers.

In what follows, we consider the following:

**Definition 1** [4] An \((n, k)\) linear code is said to be perfect if for a given positive integer \(t\), the code corrects all errors of weight \(t\) or less and no error of weight greater than \(t\). For a perfect code correcting errors of weight \(t\) or less, number of vectors of weight \(t\) or less including the vector of all zeros is equal to the number of available cosets.

**Definition 2** [8, 10] Let \(G\) denotes the set of all Gaussian integers and \(G_\pi\), the residue class of \(G\) modulo \(\pi\), where \(\pi \pi^* = p \equiv 1 \pmod 4\) and \(\pi^*\) is conjugate of \(\pi\). For \(\beta, \gamma \in G_\pi\), consider \(a + bi\) in the class of \(\beta - \gamma\) with \(|a| + |b|\) minimum. Mannheim distance \(d_M\) between \(\beta\) and \(\gamma\) is

\[d_M(\beta, \gamma) = |a| + |b|.

Note that Mannheim distance is not a true metric. The metric given by Def. (2) is a true metric [10]. We will use this metric as Mannheim metric in the present paper.

More information which are related with Mannheim metric and Mannheim weight can be found in [8, 9, 10].

**Definition 3** [11] The Hamilton Quaternion Algebra over the set of the real numbers \((\mathbb{R})\), denoted by \(H(\mathbb{R})\), is the associative unital algebra given by the following representation:

- \(\text{i) } H(\mathbb{R})\) is the free \(\mathbb{R}\) module over the symbols \(1, e_1, e_2, e_3\), that is, \(H(\mathbb{R}) = \{a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}\);
- \(\text{ii) } 1\) is the multiplicative unit;
- \(\text{iii) } e_1^2 = e_2^2 = e_3^2 = -1\);
- \(\text{iv) } e_1 e_2 = -e_2 e_1 = e_3, e_3 e_1 = -e_1 e_3 = e_2, e_2 e_3 = -e_3 e_2 = e_1\).

The set of Lipschitz integers \(H(\mathbb{Z})\), which is defined by \(H(\mathbb{Z}) = \{a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 : a_0, a_1, a_2, a_3 \in \mathbb{Z}\}\), is a subset of \(H(\mathbb{R})\), where \(\mathbb{Z}\) is the set of all integers. If \(q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3\) is a quaternion integer, its conjugate quaternion is \(q^* = a_0 - (a_1 e_1 + a_2 e_2 + a_3 e_3)\). The norm of \(q\) is \(N(q) = qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2\). The units of \(H(\mathbb{Z})\) are \(\pm 1, \pm e_1, \pm e_2, \pm e_3\).

**Definition 4** [11] Let \(\pi\) be an odd. If there exist \(\delta \in H(\mathbb{Z})\) such that \(q_1 - q_2 = \delta \pi\) then \(q_1, q_2 \in H(\mathbb{Z})\) are right congruent modulo \(\pi\) and it is denoted as \(q_1 \equiv_R q_2\).

This equivalence relation is well-defined. We can consider the ring of the quaternion integers modulo this equivalence relation, which we denote as

\[H(\mathbb{Z})_\pi = \{ q \ (\text{mod} \pi) \mid q \in H(\mathbb{Z})\}\ [10].\]

Except as noted otherwise, we will use right congruent modulo \(\pi\) in the present paper. Analogous result hold for left congruent modulo \(\pi\).
Theorem 1 [9] Let $\alpha \in H(\mathbb{Z})$. Then $H(\mathbb{Z})_\alpha$ has $(N(\alpha))^2$ elements.

Definition 5 [10] Let $\pi$ be a quaternion integer. Given $\alpha, \beta \in H(\mathbb{Z})_\pi$, then Lipschitz distance between $\alpha$ and $\beta$ is computed as $|a_0| + |a_1| + |a_2| + |a_3|$ and denoted by $d_L(\alpha, \beta)$, where

$$\alpha - \beta \equiv r_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \pmod{\pi}$$

with $|a_0| + |a_1| + |a_2| + |a_3|$ minimum.

Lipschitz weight of the element $\gamma$ is defined as $|a_0| + |a_1| + |a_2| + |a_3|$ and is denoted by $w_L(\gamma)$, where $\gamma = \alpha - \beta$ with $|a_0| + |a_1| + |a_2| + |a_3|$ minimum.

More information which are related with the arithmetic properties of $H(\mathbb{Z})$ can be found in [10] [11].

Definition 6 [12] The set of all Hurwitz integers is

$$\mathcal{H} = \left\{ a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in H(\mathbb{R}) : a_0, a_1, a_2, a_3 \in \mathbb{Z} \text{ or } a_0, a_1, a_2, a_3 \in \mathbb{Z} + \frac{1}{2} \right\}.$$ 

It can be checked that $\mathcal{H}$ is closed under quaternion multiplication and addition, so that it forms a subring of the ring of all quaternions. The units of $\mathcal{H}$ are $\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm \frac{1}{2} e_1 \pm \frac{1}{2} e_2 \pm \frac{1}{2} e_3$.

Definition 7 Let $\pi$ be a prime in $H(\mathbb{Z})$. If there exists $\delta \in H(\mathbb{Z})$ such that $q_1 - q_2 = \delta \pi$ then $q_1, q_2 \in \mathcal{H}$ are right congruent modulo $\pi$ and it is denoted as $q_1 \equiv_r q_2$.

This equivalence relation is well-defined. We can consider the ring of the Hurwitz integers modulo this equivalence relation, which we denote as

$$\mathcal{H}_\pi = \left\{ q \pmod{\pi} \mid q \in \mathcal{H} \right\}.$$ 

Theorem 2 Let $\alpha$ be a prime integer quaternion. Then $\mathcal{H}_\alpha$ has $2N(\alpha)^2 - 1$ elements.

Proof. Let $\pi 0$ be a prime integer quaternion. According to Theorem 1, the cardinal number of $H(\mathbb{Z})_\pi$ is equal to $N(\pi)^2$. Also, the cardinal number of $H(\mathbb{Z} + \frac{1}{2})_\pi$ is equal to $N(\pi)^2$. $(H(\mathbb{Z})_\pi - \{0\}) \cap (H(\mathbb{Z} + \frac{1}{2})_\pi - \{0\}) = \emptyset$ since the elements of the set $H(\mathbb{Z} + \frac{1}{2})_\pi - \{0\}$ are defined in the form $q - \delta \pi = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 w$, where $q \in H(\mathbb{Z} + \frac{1}{2})$, $\delta, \pi \in H(\mathbb{Z})$, $a_0, a_1, a_2, a_3 \in \mathbb{Z}$ and $a_4$ is an odd integer. But the additive identity is an element of both sets $H(\mathbb{Z})_\pi$ and $H(\mathbb{Z} + \frac{1}{2})_\pi$. Hence the proof is completed.

Note that if $\delta$ is chosen from $\mathcal{H}$ instead of $H(\mathbb{Z})$ then, Theorem 2 does not hold.

In the following definition, we introduce Hurwitz metric.

Definition 8 Let $\pi$ be a prime quaternion integer. Given $\alpha = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 w, \beta = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 w \in H_\pi$, then the distance between $\alpha$ and $\beta$ is computed as $|c_0| + |c_1| + |c_2| + |c_3| + |c_4|$ and denoted by $d_H(\alpha, \beta)$, where

$$\gamma = \alpha - \beta \equiv_r c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 w \pmod{\pi}$$

with $|c_0| + |c_1| + |c_2| + |c_3| + |c_4|$ minimum.
Also, we define Hurwitz weight of \( \gamma = \alpha - \beta \) as

\[ w_H(\gamma) = d_H(\alpha, \beta). \]

It is possible to show that \( d_H(\alpha, \beta) \) is a metric. We only show that the triangle inequality holds since the other conditions are straightforward. For this, let \( \alpha, \beta, \) and \( \gamma \) be any three elements of \( \mathcal{H}_\pi \). We have

i) \( d_H(\alpha, \beta) = w_H(\delta_1) = |a_0| + |a_1| + |a_2| + |a_3| + |a_4|, \) where \( \delta_1 \equiv \alpha - \beta = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4w \pmod{\pi} \) is an element of \( \mathcal{H}_\pi \), and \(|a_0| + |a_1| + |a_2| + |a_3| + |a_4| \) is minimum.

ii) \( d_H(\alpha, \gamma) = w_H(\delta_2) = |b_0| + |b_1| + |b_2| + |b_3| + |b_4|, \) where \( \delta_2 \equiv \alpha - \gamma = b_0 + b_1e_1 + b_2e_2 + b_3e_3 + b_4w \pmod{\pi} \) is an element of \( \mathcal{H}_\pi \), and \(|b_0| + |b_1| + |b_2| + |b_3| + |b_4| \) is minimum.

iii) \( d_H(\gamma, \beta) = w_H(\delta_3) = |c_0| + |c_1| + |c_2| + |c_3| + |c_4|, \) where \( \delta_3 \equiv \gamma - \beta = c_0 + c_1e_1 + c_2e_2 + c_3e_3 + c_4w \pmod{\pi} \) is an element of \( \mathcal{H}_\pi \), and \(|c_0| + |c_1| + |c_2| + |c_3| + |c_4| \) is minimum.

Thus, \( \alpha - \beta = \delta_2 + \delta_3 \pmod{\pi} \). However, \( w_H(\delta_2 + \delta_3) \geq w_H(\delta_1) \) since \( w_H(\delta_1) = |a_0| + |a_1| + |a_2| + |a_3| + |a_4| \) is minimum. Therefore,

\[ d_H(\alpha, \beta) \leq d_H(\alpha, \gamma) + d_H(\gamma, \beta). \]

Note that Hurwitz metric is not Lipschitz metric. To see this, Lipschitz weight of the element \( w = \frac{1}{2} + \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \) is \( w_L(w) = 2 \) and Hurwitz weight of the same element is \( w_H(w) = 1 \).

The rest of this paper is organized as follows. In Section 2, an upper bound on the number of parity check digits for linear Mannheim weight codes correcting errors of Mannheim weight 1 and Mannheim weight 2 or less over \( G_\pi(\pi\pi^* = p \geq 5, \) a prime) is obtained. Also, the bound with equality for the existence of perfect codes is examined and an example of a perfect code correcting errors of Mannheim weight 1 over \( G_{2+i} \) is given. In the third section of the present paper, a similar study for linear Lipschitz weight codes correcting errors of Lipschitz weight 1 and Lipschitz weight 2 or less over \( H(\mathbb{Z})_\pi \) is given. In the fourth section, a similar study for linear Lipschitz weight codes correcting errors of Lipschitz weight 1 and Lipschitz weight 2 or less over \( \mathcal{H}_\pi \) is presented. In fifth section, upper bounds on linear Hurwitz weight codes are defined.

2 Perfect codes over Gaussian integers

2.1 Perfect codes correcting errors of Mannheim weight 1

First, an upper bound on the number of parity check digits for one Mannheim error correcting codes over \( G_\pi (p \equiv 1 \pmod{4}) \) is obtained. Note that a Mannheim error of weight 1 takes on one of the four values \( \pm 1, \pm i, \) where \( i^2 = -1 \).

**Theorem 3** An \((n, k)\) linear code over \( G_\pi \) corrects all errors of Mannheim weight 1 provided that the bound \( p^{n-k} \geq 4n + 1, \) where \( p \equiv 1 \pmod{4}. \) Here and thereafter, \( p \) will denote an odd prime number.
Proof. Error vectors of Mannheim weight 1 have just one nonzero component. The nonzero component of the above stated error vectors can take on one of the four values \( \pm 1, \pm i \). The number of errors of Mannheim weight 1 including the vector of all zeros over \( G_\pi \) is

\[
4 \binom{n}{1} + 1 = 4n + 1.
\]

We have

\[
p^{n-k} \geq 4n + 1, \tag{1}
\]

because all these vectors must be elements of distinct cosets of the standard array and we have \( p^{n-k} \) cosets.

To investigate the parameters of perfect codes, we must consider the inequality (1) as

\[
p^{n-k} = 4n + 1. \tag{2}
\]

We now examine the values of \( n \) and \( k \) satisfying Eq. (2). Some values of \( n \) and \( k \) satisfying Eq. (2) are

\[
(n, k) = \{(3, 2), (4, 3), (6, 4), (7, 6), (9, 8), \ldots, (31, 28), (42, 40), \ldots, (549, 546), \ldots\}.
\]

These values show that possible perfect codes over \( G_\pi \) correcting all error patterns of Mannheim weight 1 are \((3, 2), (4, 3), (6, 4), (7, 6), (9, 8)\ldots\). Note that one Mannheim error correcting codes (OMEC) introduced by Huber in \([8]\) are perfect. An OMEC code have parameters \((n, n-1)\), where \( n = (p-1)/4 \).

We suppose that \( p \) equals 5 in Eq. (2). Then, we have

\[
5^{n-k} = 4n + 1. \tag{3}
\]

An integral solution of Eq. (3) is \((6, 4)\). In the following, we give an example of a \((6, 4)\) perfect code correcting errors of Mannheim weight 1. The \((6, 4)\) code is not an OMEC code.

Example 1 Consider the following parity check matrix for \((6, 4)\) perfect code over \( G_{2+i} \):

\[
H = \begin{bmatrix}
1 & 0 & 1 & 1 & i & 1 \\
0 & 1 & 1 & -1 & 1 & i
\end{bmatrix}.
\]

The code which is the null space of \( H \) can correct all errors of Mannheim weight 1 over \( G_{2+i} \). In Table I, we give all the error vectors of Mannheim weight 1 and their corresponding syndromes over \( G_{2+i} \) which can be seen to be distinct altogether and detailed.
Table I: Error patterns of Mannheim weight 1 and their corresponding syndromes.

| Error pattern | Syndrome |
|---------------|----------|
| (100000)      | (1, 0)   |
| (-100000)     | (-1, 0)  |
| (i00000)      | (i, 0)   |
| (-i00000)     | (-i, 0)  |
| (010000)      | (0, 1)   |
| (0 - 10000)   | (0, -1)  |
| (0i0000)      | (0, i)   |
| (0 - i0000)   | (0, -i)  |
| (001000)      | (1, 1)   |
| (00 - 1000)   | (-1, -1) |
| (00i000)      | (i, i)   |
| (00 - i000)   | (-i, -i) |
| (000100)      | (1, -1)  |
| (000 - 100)   | (-1, 1)  |
| (000i00)      | (i, -i)  |
| (000 - i00)   | (-i, i)  |
| (000010)      | (i, 1)   |
| (0000 - 10)   | (-i, -1) |
| (0000i0)      | (-1, i)  |
| (0000 - i0)   | (1, -i)  |
| (000001)      | (1, i)   |
| (00000 - 1)   | (-1, -i) |
| (00000i)      | (i, -1)  |
| (00000 - i)   | (-i, 1)  |

Therefore, [6, 4, 3] code is a perfect code over $G_{2+i}$ correcting errors of Mannheim weight 1.

2.2 Perfect codes correcting errors of Mannheim weight 2 or less

In this section, we get a bound for an $(n, k)$ linear code which corrects all error patterns of Mannheim weight 2 or less over $G_{2+i}$ and $G_{\pi}$ ($\pi^* = p \geq 13$, a prime). Hence, we obtain possible perfect codes. In this sequence, the first theorem is as follows.

**Theorem 4** An $(n, k)$ linear code over $G_{2+i}$ corrects all errors of Mannheim weight 2 or less provided that the bound

$$5^{n-k} \geq 8n^2 - 4n + 1.$$  \hspace{1cm} (4)

**Proof.** We first enumerate error vectors of Mannheim weight 2 or less. The number of error vectors of Mannheim weight 1 including the vector of all zeros over $G_{2+i}$ is $4n + 1$. There are only one type error vectors of Mannheim weight 2 over $G_{2+i}$.

Those vectors that have two nonzero components and the nonzero components could be one of the four values $\pm 1, \pm i$. 
The number of such vectors is $16 \left( \binom{n}{2} \right) = 8n^2 - 8n$.

Thus, total number of error vectors of Mannheim weight 2 or less over $G_{2+i}$ is equal to $8n^2 - 4n + 1$. Also, the number of available cosets is equal to $5^{n-k}$. Therefore, in order to correct all errors of Mannheim weight 2 or less, the code must satisfy $5^{n-k} \geq 8n^2 - 4n + 1$. Hence, the proof is completed. ■

To obtain the parameters of perfect codes, we must consider the inequality (4) as

$$5^{n-k} = 8n^2 - 4n + 1.$$  \hspace{1cm} (5)

The integral solutions of Eq. (5) for $n$ and $k$ are $(1, 0), (2, 0)$. The solution $(1, 0), (2, 0)$ are not feasible since $k$ must greater than or equal to 1. So, we conclude that there does not exist a perfect code over $G_{2+i}$ correcting all errors of Mannheim weight 2 or less.

Theorem 5 An $(n, k)$ linear code over $G_\pi$ corrects all errors of Mannheim weight 2 or less provided that the bound

$$p^{n-k} \geq 8n^2 + 1,$$  \hspace{1cm} (6)

where $p \equiv 1 \pmod{4}, p \geq 13$.

Proof. We first enumerate error vectors of Mannheim weight 1.

The number of error vectors of Mannheim weight 1 including the vector of all zeros over $G_\pi$ is $4n + 1$.

There are two types error vectors of Mannheim weight 2 over $G_\pi$.

(1) Those vectors that have two nonzero components and the nonzero components could be one of the four values $\pm 1, \pm i$.

The number of such vectors is $16 \left( \binom{n}{2} \right) = 8n^2 - 8n$.

(2) Those error vectors that have only one nonzero component and the nonzero component could be one of the four values $\pm 2, \pm 2i$.

The number of such vectors is $4n$.

Thus, total number of error vectors of Mannheim weight 2 or less over $G_\pi$ is equal to

$$8n^2 + 1.$$  

Also, the number of available cosets is $p^{n-k}$.

Therefore, in order to correct all error vectors of Mannheim weight 2 or less, the code must satisfy the bound $p^{n-k} \geq 8n^2 + 1$. Hence, the proof is completed. ■

To obtain the parameters of perfect codes, we must consider the inequality (6) as

$$p^{n-k} = 8n^2 + 1.$$  \hspace{1cm} (7)

One can obtain that some integral solutions of Eq. (7) for $n, k$ are $(3, 2), (6, 4), (12, 11), (15, 14), (18, 17), (21, 20), (33, 32), ..., (204, 202), ...$
These values show that possible perfect codes over $G_\pi$ correcting all errors of Mannheim weight 2 or less are $(3,2)$, $(6,4)$, $(12,11)$, $(15,14)$, $(18,17)$, $(21,20)$, $(33,32)$, ..., $(204,202)$, ...

Using a computer programme, for $n - k = 1$, we show that there does not exist any perfect code correcting errors of Mannheim weight 2 or less. However, the existence/nonexistence of perfect codes correcting errors of Mannheim weight 2 or less over $G_\pi$ ($n - k \geq 2$) is still unknown (except some special works [8, 10]).

3 Perfect codes over Lipschitz integers with respect to Lipschitz metric

3.1 Perfect codes correcting errors of Lipschitz weight 1

We first obtain an upper bound on the number of parity check digits for one Lipschitz error correcting codes over $H(\mathbb{Z})_\pi$. Note that a Lipschitz error of weight 1 takes on one of the eight values $\pm 1$, $\pm e_1$, $\pm e_2$, $\pm e_3$, at position $l$ ($0 \leq l \leq n - 1$).

Theorem 6 An $(n, k)$ linear code over $H(\mathbb{Z})_\pi$ corrects all errors of Lipschitz weight 1 provided that $(p^2)^{n-k} \geq 8n+1$, where $p = \pi \pi^*$ and $p$ is a prime integer.

Proof. We know that the cardinal number of $H(\mathbb{Z})_\pi$ is $p^2$ (see Thm. 1). Error vectors of Lipschitz weight one are those vectors which have only one nonzero component and the nonzero component could be one of the eight elements $\pm 1, \pm e_1, \pm e_2, \pm e_3$.

The number of such vectors is equal to $8n$. Therefore, the number of error vectors of Lipschitz weight 1 including the vector of all zeros is equal to $8 \binom{n}{1} + 1 = 8n + 1$.

Since all these vectors must elements of distinct cosets of the standard array and we have $(p^2)^{n-k}$ cosets in all, therefore, we obtain

\[(p^2)^{n-k} \geq 8n + 1.\]  

(8)

Hence, the proof is completed.

To obtain the parameters of perfect codes, we must consider the inequality (8) as

\[(p^2)^{n-k} = 8n + 1.\]  

(9)

A set of some integral solutions of Eq. (9) is

\[\{ (3,2), (6,5), (10,8), \ldots, (3570,3568), \ldots \}.\]

These values show that the parameters of possible perfect codes correcting all error patterns of Lipschitz weight 1 and no others over $H(\mathbb{Z})_\pi$ are $(3,2)$, $(6,5)$, $(10,8)$, $(15,14)$, $(21,20)$, ... We suppose that $p$ is equal to 5 in Eq. (9). Then, we get

\[(5^2)^{n-k} = (1 + 8n).\]  

(10)
The integral solutions of Eq. (10) for \( n \) and \( k \) are
\[(3, 2), (553, 550), \ldots\]
These values show the possibility of the existence of (3, 2), (553, 550), \ldots perfect codes correcting errors of Lipschitz weight 1 over \( H(\mathbb{Z})_{2+e_1} \).

In the following, we give an example of a (3, 2) perfect code correcting errors of Lipschitz weight 1 over \( H(\mathbb{Z})_{2+e_1} \).

**Example 2** Let \( C \) be a code defined by the parity check matrix
\[
H = \begin{bmatrix} 1 & 1 + e_3 & 1 + e_2 \end{bmatrix}.
\]
The code \( C \) have the parameters (3, 2). It can correct all errors of Lipschitz weight 1. In Table II, we give all the error vectors of Lipschitz weight 1 and their corresponding syndromes over \( H(\mathbb{Z})_{2+e_1} \).

| Error pattern | Syndrome |
|---------------|----------|
| (1, 0, 0)     | 1        |
| (e_1, 0, 0)   | e_1      |
| (e_2, 0, 0)   | e_2      |
| (e_3, 0, 0)   | e_3      |
| (-1, 0, 0)    | -1       |
| (-e_1, 0, 0)  | -e_1     |
| (-e_2, 0, 0)  | -e_2     |
| (-e_3, 0, 0)  | -e_3     |
| (0, 1, 0)     | 1 + e_3  |
| (0, e_1, 0)   | e_1 - e_2|
| (0, e_2, 0)   | e_1 + e_2|
| (0, e_3, 0)   | 1 + e_3  |
| (0, -1, 0)    | 1 - e_3  |
| (0, -e_1, 0)  | -e_1 + e_2|
| (0, -e_2, 0)  | -e_1 - e_2|
| (0, -e_3, 0)  | 1 - e_3  |
| (0, 0, 1)     | 1 + e_2  |
| (0, 0, e_1)   | e_1 + e_3|
| (0, 0, e_2)   | -1 + e_2 |
| (0, 0, e_3)   | -e_1 + e_3|
| (0, 0, -1)    | -1 - e_2 |
| (0, 0, -e_1)  | -e_1 - e_3|
| (0, 0, -e_2)  | 1 - e_2  |
| (0, 0, -e_3)  | e_1 - e_3|

The code with parameters [3, 2, 3] over \( H(\mathbb{Z})_{2+e_1} \) is a perfect code correcting all errors of Lipschitz weight 1.

**Remark 7** We have investigated solutions of Eq. (9) for \( p = 5 \). We have been able to obtain a perfect code for one of the solutions. One can similarly solve the existence of perfect codes correcting errors of Lipschitz weight 1 over \( H(\mathbb{Z})_{2+e_1+e_2+e_3}, H(\mathbb{Z})_{1+e_1+e_2}, H(\mathbb{Z})_{3+e_1+e_2}, \ldots \) by taking \( \pi = 2 + e_1 + e_2 + e_3, 1 + e_1 + e_2, 3 + e_1 + e_2, \ldots \), respectively, in Eq. (9) and finding the solutions for \( n \) and \( k \).
Note that for \( n - k = 1 \), there always exists a perfect code corresponding to the parameters obtained by Eq. (9).

### 3.2 Perfect codes correcting errors of Lipschitz weight 2 or less

In this section, we obtain bound on the number of parity check digits for an \((n, k)\) linear code correcting all error patterns of Lipschitz weight 2 or less over \( H(Z)_{1+e_1+e_2}, H(Z)_{2+e_1}, H(Z)_{2+e_1+e_2+e_3}, H(Z)_{3+e_1+e_2} \) and over \( H(Z)_\pi \) (\( \pi \pi^* = p \geq 13 \), a prime) and then we investigate the existence of corresponding perfect codes. In this sequence, the first theorem is as follows.

**Theorem 8** An \((n, k)\) linear code over \( H(Z)_{1+e_1+e_2} \) corrects all errors of Lipschitz weight 2 or less provided that the bound

\[
(p^2)^{n-k} \geq 32n^2 - 24n + 1
\]

**Proof.** We first enumerate error vectors of Lipschitz weight 2 or less over \( H(Z)_{1+e_1+e_2} \).

The number of error vectors of Lipschitz weight 1 including the vector of all zeros over \( H(Z)_{1+e_1+e_2} \) is \( 8n + 1 \).

There is only one type error vectors of Lipschitz weight 2 over \( H(Z)_{1+e_1+e_2} \).

Those vectors which have two nonzero components and the nonzero components could be one of the eight values \( \pm 1, \pm e_1, \pm e_2, \pm e_3 \).

The number of such vectors is equal to \( 64 \left( \begin{array}{c} n \\ 2 \end{array} \right) = 32n^2 - 32n \).

Thus, total number of error vectors of Lipschitz weight 2 or less over \( H(Z)_{1+e_1+e_2} \) is equal to \( 32n^2 - 24n + 1 \). Also, the number of available cosets is \( (3^2)^{n-k} \). In order to correct all error patterns of Lipschitz weight 2 or less over \( H(Z)_{1+e_1+e_2} \), the code must satisfy

\[
(3^2)^{n-k} \geq 32n^2 - 24n + 1. \tag{11}
\]

Hence, the proof is completed. ■

To obtain the parameters of perfect codes, we must consider the inequality (11) as

\[
(3^2)^{n-k} = 32n^2 - 24n + 1. \tag{12}
\]

The integral solutions of Eq. (12) are \( n = 1, k = 0 \) and \( n = 2, k = 0 \). the solution \( n = 1, k = 0 \) and \( n = 2, k = 0 \) are not feasible as \( n \geq 2 \) and \( k > 0 \).

So, we conclude that there does not exists a perfect code over \( H(Z)_{1+e_1+e_2} \) correcting all error patterns of Lipschitz weight 2 or less.

Now, we obtain the bound on the number of parity check digits for an \((n, k)\) linear code over \( H(Z)_{2+e_1}, H(Z)_{2+e_1+e_2+e_3} \) and \( H(Z)_{3+e_1+e_2} \) correcting errors of Lipschitz weight 2 or less.

**Theorem 9** An \((n, k)\) linear code over \( H(Z)_{2+e_1} \) corrects all errors of Lipschitz weight 2 or less provided that the bound

\[
(p^2)^{n-k} \geq 32n^2 - 8n + 1
\]

**Proof.** We first enumerate error vectors of Lipschitz weight 2 or less over \( H(Z)_{2+e_1} \).
The number of error vectors of Lipschitz weight 1 including the vector of all zeros over $H(Z)_{2+e_1}$ is $8n + 1$.

There are two types error vectors of Lipschitz weight 2 over $H(Z)_{2+e_1}$.

(1) Those vectors which are also error vectors of Lipschitz weight 2 over $H(Z)_{1+e_1+e_2}$.

The number of such vectors is equal to $64 \left( \frac{n}{2} \right) = 32n^2 - 32n$.

(2) Those error vectors which have only one nonzero component and the nonzero component could be one of the sixteen values $\pm (1+e_2)$, $\pm (1+e_3)$, $\pm (e_1+e_2)$, $\pm (e_1+e_3)$, $\pm (1-e_2)$, $\pm (1-e_3)$, $\pm (e_1-e_2)$, $\pm (e_1-e_3)$.

The number of such vectors is equal to $16n$.

Thus, total number of error vectors of Lipschitz 2 or less over $H(Z)_{2+e_1}$ is equal to $32n^2 - 8n + 1$. Also, the number of available cosets is $(5^2)^{n-k}$. In order to correct all error patterns of Lipschitz weight 2 or less over $H(Z)_{2+e_1}$, the code must satisfy

$$\begin{align*}
(5^2)^{n-k} & \geq 32n^2 - 8n + 1. \\
\end{align*}$$

Hence, the proof is completed. ■

To obtain the parameters of perfect codes, we must consider the inequality (13) as

$$\begin{align*}
(5^2)^{n-k} & = 32n^2 - 8n + 1. \\
\end{align*}$$

The only integral solution of Eq. (14) is $n = 1, k = 0$. The solution $n = 1, k = 0$ is not feasible. So, we conclude that there does not exists a perfect code over $H(Z)_{2+e_1}$ correcting all error patterns of Lipschitz weight 2 or less.

**Theorem 10** An $(n, k)$ linear code over $H(Z)_{2+e_1+e_2+e_3}$ and $H(Z)_{3+e_1+e_2}$ corrects all errors of Lipschitz weight 2 or less provided that the bound $(p^2)^{n-k} \geq 32n^2 + 1$.

**Proof.** We first enumerate error vectors of Lipschitz weight 2 or less over $H(Z)_{2+e_1+e_2+e_3}$ and $H(Z)_{3+e_1+e_2}$.

The number of error vectors of Lipschitz weight 1 including the vector of all zeros over $H(Z)_{2+e_1+e_2+e_3}$ and $H(Z)_{3+e_1+e_2}$ is equal to $8n + 1$.

There are two types error vectors of Lipschitz weight 2 over $H(Z)_{2+e_1+e_2+e_3}$ and $H(Z)_{3+e_1+e_2}$.

(1) Those vectors which are also error vectors of Lipschitz weight 2 over $H(Z)_{1+e_1+e_2}$.

The number of such vectors is $64 \left( \frac{n}{2} \right) = 32n^2 - 32n$.

(2) Those error vectors which have only one nonzero component and the nonzero component could be one of the twenty four values $\pm (1+e_2)$, $\pm (1+e_3)$, $\pm (e_1+e_2)$, $\pm (e_1+e_3)$, $\pm (e_2+e_3)$, $\pm (1-e_1)$, $\pm (1-e_2)$, $\pm (1-e_3)$, $\pm (e_1-e_2)$, $\pm (e_1-e_3)$, $\pm (e_2-e_3)$.

The number of such vectors is $24n$. 

11
Thus, total number of error vectors of Lipschitz 2 or less over \( H(\mathcal{Z})_{2+e_1+e_2+e_3} \) and \( H(\mathcal{Z})_{3+e_1+e_2} \) is equal to \( 32n^2 + 1 \). Also, the number of available cosets is equal to \((7^2)^{n-k}\) and \((11^2)^{n-k}\), respectively. In order to correct all error patterns of the Lipschitz weight 2 or less over \( H(\mathcal{Z})_{2+e_1+e_2+e_3} \) and \( H(\mathcal{Z})_{3+e_1+e_2} \), the code must satisfy
\[
(7^2)^{n-k} \geq 32n^2 + 1, \quad (11^2)^{n-k} \geq 32n^2 + 1, \quad \text{(15)}
\]
respectively. Hence, the proof is completed.

**Theorem 11.** An \((n,k)\) linear code over \( H(\mathcal{Z})_n \) \((\pi \pi^* = p \geq 13 \text{ a prime})\) corrects all errors of Lipschitz weight 2 or less provided that the bound \((p^2)^{n-k}\) \(\geq 32n^2 + 8n + 1\).

**Proof.** The number of error vectors of the Lipschitz weight 1 including the vector of all zeros over \( H(\mathcal{Z})_n \) \((\pi \pi^* = p \geq 13)\) is \(8n + 1\).

There are two types error vectors of Lipschitz weight two over \( H(\mathcal{Z})_n \) \((\pi \pi^* = p \geq 13)\).

1. Those vectors which are also error vectors of Lipschitz weight 2 over \( H(\mathcal{Z})_{2+e_1}, H(\mathcal{Z})_{2+e_2+e_3} \) and \( H(\mathcal{Z})_{3+e_1+e_2} \). The number of such vectors is \(32n^2 - 32n\).

2. Those error vectors which have only one nonzero component and the nonzero component could be one of the Thirty-two values \(\pm 2, \pm 2e_1, \pm 2e_2, \pm 2e_3,\)
\(\pm(1 + e_1), \pm(1 + e_2), \pm(1 + e_3), \pm(e_1 + e_2), \pm(e_1 + e_3), \pm(e_2 + e_3),\)
\(\pm(1 - e_1), \pm(1 - e_2), \pm(1 - e_3), \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3).\)

The number of such vectors is \(32n\).

Thus, total number of error vectors of Lipschitz 2 or less over \( H(\mathcal{Z})_n \) \((\pi \pi^* = p \geq 13)\) is equal to \(32n^2 + 8n + 1\). Also, the number of available cosets is equal to \((p^2)^{n-k}\). In order to correct all error patterns of Lipschitz 2 or less over \( H(\mathcal{Z})_n \), the code must satisfy
\[
(p^2)^{n-k} \geq 32n^2 + 8n + 1. \quad \text{(17)}
\]
Hence, the proof is completed.

To obtain the parameters of perfect codes, we must consider the inequality (17) as
\[
(p^2)^{n-k} = 32n^2 + 8n + 1. \quad \text{(18)}
\]
Take \(p = 29\) in Eq. (18), we get
\[
(29^2)^{n-k} = 32n^2 + 8n + 1. \quad \text{(19)}
\]
The only integral solution of Eq. (19) for \( n \) and \( k \) is \( n = 5, k = 4 \).

Take \( p = 33461 \) in Eq. (18), we get

\[
(33461^2)^{n-k} = 32n^2 + 8n + 1.
\]  (20)

The only integral solution of Eq. (20) for \( n \) and \( k \) is \( n = 5915, k = 5914 \).

There is no other integral solution of Eq. (18) other than the above mentioned solutions.

These values show the possibility of the existence of \((5,4),(5915,5914)\) \((n \leq 10000)\) perfect codes over \( H(\mathbb{Z})_\pi \) \( (\pi^\pi = p \geq 13) \) correcting all error patterns of Lipschitz weight 2 or less and no others.

4 Perfect codes over Hurwitz integers with respect to Lipschitz metric

4.1 Perfect codes correcting errors of Lipschitz weight 1 over Hurwitz integers

We first obtain an upper bound on the number of parity check digits for one Lipschitz error correcting codes over \( H_\pi \). Note that a Lipschitz error of weight 1 takes on one of the eight values \( \pm 1, \pm e_1, \pm e_2, \pm e_3 \), at position \( l(0 \leq l \leq n-1) \).

**Theorem 12** An \((n,k)\) linear code over \( H_\pi \) corrects all errors of Lipschitz weight 1 provided that \((2p^2 - 1)^{n-k} \geq 8n + 1\), where \( p = \pi^\pi \) and \( p \) is a prime integer.

**Proof.** We know that the cardinal number of \( H_\pi \) is \( 2p^2 - 1 \) (see Thm. 2). Error vectors of Lipschitz weight one are those vectors which have only one nonzero component and the nonzero component could be one of the eight elements \( \pm 1, \pm e_1, \pm e_2, \pm e_3 \).

The number of such vectors is equal to \( 8n \). Therefore, the number of error vectors of Lipschitz weight 1 including the vector of all zeros is equal to \( 8 \binom{n}{1} + 1 = 8n + 1 \).

Since all these vectors must elements of distinct cosets of the standard array and we have \((2p^2 - 1)^{n-k}\) cosets in all, therefore, we obtain

\[
(2p^2 - 1)^{n-k} \geq 8n + 1.
\]  (21)

Hence, the proof is completed.

To obtain the parameters of perfect codes, we must consider the inequality (21) as

\[
(2p^2 - 1)^{n-k} = 8n + 1.
\]  (22)
We suppose that \( p \) is equal to 3 in Eq. (22). Then, we get
\[
(17)^{n-k} = (1 + 8n). \tag{23}
\]
The integral solutions of Eq. (23) for \( n \) and \( k \) are
\[
(2, 1), (36, 34), (614, 611), (10440, 10436), (177482, 177477), ... 
\]
We suppose that \( p \) is equal to 5 in Eq. (22). Then, we get
\[
(49)^{n-k} = (1 + 8n). \tag{24}
\]
The integral solutions of Eq. (24) for \( n \) and \( k \) are
\[
(6, 5), (300, 298), (14706, 14703), (720600, 720596), ... 
\]
We suppose that \( p \) is equal to 7 in Eq. (22). Then, we get
\[
(97)^{n-k} = (1 + 8n). \tag{25}
\]
The integral solutions of Eq. (25) for \( n \) and \( k \) are
\[
(12, 11), (1176, 1174), (114084, 114081), ... 
\]
There always exist a perfect code which its parameters corresponding to above parameters. These perfect codes are not known before.

In the following, we give an example of a \((2, 1)\) perfect code over \( H_{1+e_1+e_2} \).

**Example 3** Consider the following parity check matrix \( H \) for \((2, 1)\) perfect code over \( H_{1+e_1+e_2} \):

\[
H = \begin{bmatrix} 1, & \frac{1}{2} + \frac{e_1}{2} + \frac{e_2}{2} + \frac{e_3}{2} \end{bmatrix}.
\]

The code which is the null space of \( H \) can correct all errors of Lipschitz weight 1 over \( H_{1+e_1+e_2} \) and no others. In Table III, we list all the error vectors of Lipschitz weight 1 and their corresponding syndromes over \( H_{1+e_1+e_2} \) which can be seen to be distinct altogether and exhaustive.

| Error pattern | Syndrome |
|---------------|----------|
| (1, 0)        | 1        |
| \((e_1, 0)\)  | \(e_1\)  |
| \((e_2, 0)\)  | \(e_2\)  |
| \((e_3, 0)\)  | \(e_3\)  |
| \((-1, 0)\)   | \(-1\)   |
| \((-e_1, 0)\) | \(-e_1\) |
| \((-e_2, 0)\) | \(-e_2\) |
| \((-e_3, 0)\) | \(-e_3\) |
| \((0, 1)\)    | \(\frac{1}{2} + \frac{e_1}{2} + \frac{e_2}{2} + \frac{e_3}{2}\) |
| \((0, e_1)\)  | \(-\frac{1}{2} + \frac{e_2}{2} + \frac{e_3}{2} - \frac{e_1}{2}\) |
| \((0, e_2)\)  | \(-\frac{1}{2} + \frac{e_1}{2} + \frac{e_3}{2} - \frac{e_2}{2}\) |
| \((0, e_3)\)  | \(-\frac{1}{2} + \frac{e_2}{2} + \frac{e_1}{2} - \frac{e_3}{2}\) |
| \((0, -1)\)   | \(-\frac{1}{2} - \frac{e_1}{2} - \frac{e_2}{2} - \frac{e_3}{2}\) |
| \((0, -e_1)\) | \(-\frac{1}{2} - \frac{e_2}{2} - \frac{e_3}{2} + \frac{e_1}{2}\) |
| \((0, -e_2)\) | \(-\frac{1}{2} - \frac{e_1}{2} - \frac{e_3}{2} + \frac{e_2}{2}\) |
| \((0, -e_3)\) | \(-\frac{1}{2} - \frac{e_2}{2} - \frac{e_1}{2} + \frac{e_3}{2}\) |
Therefore, (2, 1) code is a perfect code correcting errors of Lipschitz weight 1 over \( \mathcal{H}_{1+e_1+e_2} \).

To the best of our knowledge, above perfect code is not known before.

4.2 Perfect codes correcting errors of Lipschitz weight 2 or less over Hurwitz integers

In this section, we obtain bound on the number of parity check digits for an \((n, k)\) linear code correcting all error patterns of Lipschitz weight 2 or less over \( \mathcal{H}_{1+e_1+e_2}, \mathcal{H}_{2+e_1}, \mathcal{H}_{2+e_1+e_2+e_3}, \mathcal{H}_{3+e_1+e_2} \) and \( \mathcal{H}_\pi \) (\( \pi \pi^* = \pi \geq 7, \) a prime). In this sequence, the first theorem is as follows.

**Theorem 13** An \((n, k)\) linear code over \( \mathcal{H}_{1+e_1+e_2} \) corrects all errors of Lipschitz weight 2 or less provided that the bound
\[
17n - k \geq 32n^2 - 16n + 1.
\]

**Proof.** We first enumerate error vectors of Lipschitz weight 2 or less over \( \mathcal{H}_{1+e_1+e_2} \).

The number of error vectors of Lipschitz weight 1 including the vectors of all zeros over \( \mathcal{H}_{1+e_1+e_2} \) is equal to \( 8n + 1 \).

There are two types error vectors of Lipschitz weight 2 over \( \mathcal{H}_{1+e_1+e_2} \).

1. Those vectors which have two nonzero components and the nonzero components could be one of the eight values \( \pm 1, \pm e_1, \pm e_2, \pm e_3 \).

The number of such vectors is \( 64 \left( \frac{n}{2} \right) = 32n^2 - 32n \).

2. Those error vectors which have only one nonzero component and the nonzero component could be one of the eight values
\[
\frac{1}{2} + \frac{e_1}{2} + \frac{e_2}{2}, -\frac{1}{2} + \frac{e_1}{2} + \frac{e_2}{2}, \frac{1}{2} + \frac{e_1}{2} + \frac{e_3}{2}, -\frac{1}{2} + \frac{e_1}{2} + \frac{e_3}{2},
\]
\[
\frac{1}{2} - \frac{e_1}{2} + \frac{e_3}{2}, -\frac{1}{2} - \frac{e_1}{2} + \frac{e_3}{2}, \frac{1}{2} - \frac{e_1}{2} - \frac{e_3}{2}, -\frac{1}{2} - \frac{e_1}{2} - \frac{e_3}{2}.
\]

The number of such vectors is \( 8n \).

Thus, total number of error vectors of Lipschitz weight 2 or less over \( \mathcal{H}_{1+e_1+e_2} \) is \( 32n^2 - 16n + 1 \). Also, the number of available cosets is equal to \( 17^{n-k} \). In order to correct all error patterns of Lipschitz weight 2 or less over \( \mathcal{H}_{1+e_1+e_2} \), the code must satisfy
\[
17^{n-k} \geq 32n^2 - 16n + 1.
\]

Hence, the proof is completed. \( \blacksquare \)

To obtain the parameters of perfect codes, we must consider the inequality (26) as
\[
17^{n-k} = 32n^2 - 16n + 1.
\]

The only integral solution of Eq. (27) is \( n = 1; k = 0 \). The solution \( n = 1; k = 0 \) is not feasible. There is no other integral solution of Eq. (27) other than the above mentioned solutions. So, we conclude that there is not exist a perfect code over \( \mathcal{H}_{1+e_1+e_2} \).

**Theorem 14** An \((n, k)\) linear code over \( \mathcal{H}_{2+e_1} \) corrects all errors of Lipschitz weight 2 or less provided that the bound
\[
49^{n-k} \geq 32n^2 + 8n + 1.
\]
**Proof.** We first enumerate error vectors of Lipschitz weight 2 or less over $\mathcal{H}_{2+e_1}$.

The number of error vectors of Lipschitz weight 1 including the vectors of all zeros over $\mathcal{H}_{2+e_1}$ is equal to $8n + 1$.

There are two types error vectors of Lipschitz weight 2 over $\mathcal{H}_{2+e_1}$.

(1) Those vectors which have two nonzero components and the nonzero components could be one of the eight values $\pm 1, \pm e_1, \pm e_2, \pm e_3$.

The number of such vectors is $64 \left( \begin{array}{c} n \\ 2 \end{array} \right) = 32n^2 - 32n$.

(2) Those error vectors which have only one nonzero component and the nonzero component could be one of the thirty two values $\pm \frac{1}{2}, \pm \frac{e_1}{2}, \pm \frac{e_2}{2}, \pm \frac{e_3}{2}, \pm (1 + e_2), \pm (1 - e_2), \pm (1 + e_3), \pm (1 - e_3), \pm (e_1 + e_2), \pm (e_1 - e_2), \pm (e_1 - e_3)$.

The number of such vectors is $32n$.

Thus, total number of error vectors of Lipschitz weight 2 or less over $\mathcal{H}_{2+e_1}$ is $32n^2 + 8n + 1$. Also, the number of available cosets is equal to $49^{n-k}$. In order to correct all error patterns of Lipschitz weight 2 or less over $\mathcal{H}_{2+e_1}$, the code must satisfy

$$49^{n-k} \geq 32n^2 + 8n + 1. \tag{28}$$

Hence, the proof is completed. ■

To obtain the parameters of perfect codes, we must consider the inequality (28) as

$$49^{n-k} = 32n^2 + 8n + 1. \tag{29}$$

There is no integral solution of Eq. (29). So, there does not exists a perfect code over $\mathcal{H}_{2+e_1}$ correcting all error patterns of Lipschitz weight 2 or less.

**Theorem 15** An $(n, k)$ linear code over $\mathcal{H}_{2+e_1+e_2+e_3}$ and $\mathcal{H}_{3+e_1+e_2}$ corrects all errors of Lipschitz weight 2 or less provided that the bound $97^{n-k} \geq 32n^2 + 16n + 1$ and $241^{n-k} \geq 32n^2 + 16n + 1$, respectively.

**Proof.** We first enumerate error vectors of Lipschitz weight 2 or less over $\mathcal{H}_{2+e_1+e_2+e_3}$ and $\mathcal{H}_{3+e_1+e_2}$.

The number of error vectors of Lipschitz weight 1 including the vectors of all zeros over $\mathcal{H}_{2+e_1+e_2+e_3}$ and $\mathcal{H}_{3+e_1+e_2}$ is equal to $8n + 1$.

There are two types error vectors of Lipschitz weight 2 over $\mathcal{H}_{2+e_1+e_2+e_3}$ and $\mathcal{H}_{3+e_1+e_2}$.

(1) Those vectors which have two nonzero components and the nonzero components could be one of the eight values $\pm 1, \pm e_1, \pm e_2, \pm e_3$.

The number of such vectors is $64 \left( \begin{array}{c} n \\ 2 \end{array} \right) = 32n^2 - 32n$.

(2) Those error vectors which have only one nonzero component and the nonzero component could be one of the forty values $\pm \frac{1}{2}, \pm \frac{e_1}{2}, \pm \frac{e_2}{2}, \pm \frac{e_3}{2}, \pm (1 + e_2), \pm (1 - e_2), \pm (1 + e_3), \pm (1 - e_3), \pm (e_1 + e_2), \pm (e_1 - e_2), \pm (e_1 - e_3)$.
$\pm (e_1 + e_3), \pm (e_1 - e_3), \pm (e_2 + e_3), \pm (e_2 - e_3).$

The number of such vectors is $40n$.

Thus, total number of error vectors of Lipschitz weight 2 or less over $H_{2+e_1+e_2+e_3}$ and $H_{3+e_1+e_2}$ is $32n^2 + 8n + 1$. Also, the number of available cosets are equal to $97^{n-k}$ and $241^{n-k}$, respectively. In order to correct all error patterns of Lipschitz weight 2 or less over $H_{2+e_1+e_2+e_3}$ and $H_{3+e_1+e_2}$, the code must satisfy

$$97^{n-k} \geq 32n^2 + 16n + 1$$  \hspace{1cm} (30)

and

$$241^{n-k} \geq 32n^2 + 16n + 1,$$  \hspace{1cm} (31)

respectively. Hence, the proof is completed. 

To obtain the parameters of perfect codes, we must consider the inequality (30) and (31) as

$$97^{n-k} = 32n^2 + 16n + 1,$$  \hspace{1cm} (32)

and

$$241^{n-k} = 32n^2 + 16n + 1.$$  \hspace{1cm} (33)

There is no integral solution of Eq. (32) and (33). So, there does not exists a perfect code over $H_{2+e_1+e_2+e_3}$ and $H_{3+e_1+e_2}$ correcting all error patterns of Lipschitz weight 2 or less.

Now, we obtain the bound on the number of parity check digits for an $(n,k)$ linear code over $H_p$ $(\pi \pi^* = p \geq 13$ a prime) correcting errors of Lipschitz weight 2 or less.

**Theorem 16** An $(n,k)$ linear code over $H_p$ $(\pi \pi^* = p \geq 13$ a prime) corrects all errors of the Lipschitz weight 2 or less provided that the bound $(2p^2 - 1)^{n-k} \geq 32n^2 + 24n + 1$.

**Proof.** The number of error vectors of Lipschitz weight one including the vector of all zeros over $H_p$ $(\pi \pi^* = p \geq 13$ is equal to $8n + 1$.

There are two types error vectors of Lipschitz weight 2 over $H_p$ $(\pi \pi^* = p \geq 13$.

1. Those vectors which have two nonzero components and the nonzero components could be one of the eight values $\pm 1, \pm e_1, \pm e_2, \pm e_3$.

The number of such vectors is $32n^2 - 32n$.

2. Those error vectors which have only one nonzero component and the nonzero component could be one of the forty eight values $\pm 2, \pm 2e_1, \pm 2e_2, \pm 2e_3, \pm (e_1 \pm e_2) \pm (e_1 \pm e_3), \pm (e_1 \pm e_2), \pm (e_1 + e_2), \pm (e_1 - e_3)$.

The number of such vectors is $48n$.

Thus, total number of error vectors of Lipschitz weight 2 or less over $H_p$ $(\pi \pi^* = p \geq 13$ is $(32n^2 + 24n + 1)$. Also, the number of available cosets is equal to $(2p^2 - 1)^{n-k}$. In order to correct all error patterns of Lipschitz weight 2 or less over $H_p$, the code must satisfy

$$(2p^2 - 1)^{n-k} \geq 32n^2 + 24n + 1.$$  \hspace{1cm} (34)
Hence, the proof is completed.

To obtain the parameters of perfect codes, we must consider the inequality (34) as

\[(2p^2 - 1)^{n-k} = 32n^2 + 24n + 1.\]  \hspace{1cm} (35)

There is no integral solution of Eq. (35) for \(n \leq 1000000\).

5 Perfect codes over Hurwitz integers with respect to Hurwitz metric

5.1 Perfect codes correcting errors of Hurwitz weight 1 over Hurwitz integers

We first obtain an upper bound on the number of parity check digits for one Hurwitz error correcting codes over \(H_p\). Note that a Hurwitz error of weight 1 takes on one of the ten values \(\pm 1, \pm \epsilon_1, \pm \epsilon_2, \pm \epsilon_3, \pm w = \pm (\frac{1}{2} + \frac{1}{2} \epsilon_1 + \frac{1}{2} \epsilon_2 + \frac{1}{2} \epsilon_3)\) at position \(l(0 \leq l \leq n - 1)\).

Theorem 17 An \((n, k)\) linear code over \(H_p\) corrects all errors of Hurwitz weight 1 provided that \((2p^2 - 1)^{n-k} \geq 8n + 1\), where \(p = \pi \pi^*\) and \(p\) is a prime integer.

Proof. Error vectors of Hurwitz weight one are those vectors which have only one nonzero component and the nonzero component could be one of the ten elements \(\pm 1, \pm \epsilon_1, \pm \epsilon_2, \pm \epsilon_3, \pm w = \pm (\frac{1}{2} + \frac{1}{2} \epsilon_1 + \frac{1}{2} \epsilon_2 + \frac{1}{2} \epsilon_3)\).

The number of such vectors is equal to \(10n\). Therefore, the number of error vectors of Hurwitz weight 1 including the vector of all zeros is equal to

\[10 \binom{n}{1} + 1 = 10n + 1.\]

Since all these vectors must elements of distinct cosets of the standard array and we have \((2p^2 - 1)^{n-k}\) cosets in all, therefore, we obtain

\[(2p^2 - 1)^{n-k} \geq 8n + 1.\] \hspace{1cm} (36)

Hence, the proof is completed.

To obtain the parameters of perfect codes, we must consider the inequality (36) as

\[(2p^2 - 1)^{n-k} = 10n + 1.\] \hspace{1cm} (37)

We suppose that \(p\) is equal to 3 in Eq. (37). Then, we get

\[17^{n-k} = 10n + 1.\] \hspace{1cm} (38)

The integral solutions of Eq. (38) for \(n\) and \(k\) are

\((83520, 83516), (6975757440, 6975757432), ...

These values show the possibility of the existence of \((83520, 83516), (6975757440, 6975757432), ...
perfect codes correcting errors of Hurwitz weight 1 over \(H_{1+\epsilon_1+\epsilon_2}\).
We suppose that $p$ is equal to 5 in Eq. (37). Then, we get

$$49^n - k = 10n + 1.$$  \hspace{1cm} (39)

The integral solutions of Eq. (39) for $n$ and $k$ are

$$(2400, 2398), (5764800, 5764796), ...$$

There are always integral solutions of Eq. (37) for primes $p \geq 7$.
These values show the possibility of the existence of $$(2400, 2398), (5764800, 5764796), ...$$
perfect codes correcting errors of Hurwitz weight 1 over $\mathcal{H}_{2+e_1}$.

5.2 Perfect codes correcting errors of Hurwitz weight 2 or less over Hurwitz integers

In this section, we obtain bound on the number of parity check digits for an $(n, k)$ linear code correcting all error patterns of Hurwitz weight 2 or less over $\mathcal{H}_{1+e_1+e_2}$ and $\mathcal{H}_π (\pi \pi^* = p \geq 17, \text{a prime})$. In this sequence, the first theorem is as follows.

**Theorem 18** An $(n, k)$ linear code over $\mathcal{H}_{1+e_1+e_2}$ corrects all errors of Hurwitz weight 2 or less provided that the bound $17n - k \geq 50n^2 - 34n + 1$.

**Proof.** We first enumerate error vectors of Hurwitz weight 2 or less over $\mathcal{H}_{1+e_1+e_2}$.

- **The number of error vectors of Hurwitz weight 1 including the vectors of all zeros over $\mathcal{H}_{1+e_1+e_2}$ is equal to $10n + 1$.**

  - **There are two types error vectors of Hurwitz weight 2 over $\mathcal{H}_{1+e_1+e_2}$.**
    - (1) Those vectors which have two nonzero components and the nonzero components could be one of the eight values $\pm 1, \pm e_1, \pm e_2, \pm e_3$.
      The number of such vectors is $100 \binom{n}{2} = 50n^2 - 50n$.
    - (2) Those error vectors which have only one nonzero component and the nonzero component could be one of the six values $1 - w, -1 + w, i - w, -i + w, j - w, -j + w$.
      The number of such vectors is $6n$.

  Thus, total number of error vectors of Hurwitz weight 2 or less over $\mathcal{H}_{1+e_1+e_2}$ is $50n^2 - 34n + 1$. Also, the number of available cosets is equal to $17n - k$. In order to correct all error patterns of Hurwitz weight 2 or less over $\mathcal{H}_{1+e_1+e_2}$, the code must satisfy

$$17n - k \geq 50n^2 - 34n + 1.$$  \hspace{1cm} (40)

Hence, the proof is completed. ■

To obtain the parameters of perfect codes, we must consider the inequality (40) as

$$17n - k = 50n^2 - 34n + 1.$$  \hspace{1cm} (41)

The only integral solution of Eq. (41) is $n = 1; k = 0$. The solution $n = 1; k = 0$ is not feasible. There is no other integral solution of Eq. (41) other than the
above mentioned solutions. So, we conclude that there is not exist a perfect code over $H_{1+e_1+e_2}$.

Now, we obtain the bound on the number of parity check digits for an $(n, k)$ linear code over $H$ $(p = p \geq 17$ a prime) correcting errors of Hurwitz weight 2 or less.

**Theorem 19** An $(n, k)$ linear code over $H$ $(p = p \geq 13$ a prime) corrects all errors of the Hurwitz weight 2 or less provided that the bound $(2p^2 - 1)^{n-k} \geq 50n^2 + 10n + 1$.

**Proof.** The number of error vectors of Hurwitz weight one including the vector of all zeros over $H$ $(p = p \geq 17$ is equal to $10n + 1$.

There are two types error vectors of Hurwitz weight 2 over $H$ $(p = p \geq 17$.

1. Those vectors which have two nonzero components and the nonzero components could be one of the ten values $\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm w$.

The number of such vectors is $50n^2 - 50n$.

2. Those error vectors which have only one nonzero component and the nonzero component could be one of the fifty values $\pm 2, \pm 2e_1, \pm 2e_2, \pm 2e_3, \pm 1 \pm w, \pm i \pm w, \pm j \pm w, \pm k \pm w, \pm 1 \pm i, \pm 1 \pm j, \pm 1 \pm k, \pm i \pm j, \pm i \pm k, \pm j \pm k, \pm 2w$.

The number of such vectors is $50n$.

Thus, total number of error vectors of Hurwitz weight 2 or less over $H$ $(p = p \geq 17$ is $(50n^2 + 10n + 1)$. Also, the number of available cosets is equal to $(2p^2 - 1)^{n-k}$. In order to correct all error patterns of Hurwitz weight 2 or less over $H$, the code must satisfy

$$(2p^2 - 1)^{n-k} \geq 50n^2 + 10n + 1. \quad (42)$$

Hence, the proof is completed.

To obtain the parameters of perfect codes, we must consider the inequality

$$(2p^2 - 1)^{n-k} = 50n^2 + 10n + 1. \quad (43)$$

There is no integral solution of Eq. (35) for $n \leq 1000000, n - k \leq 23$.

If we restrict $H$ to $R = \{a + bw : a, b \in Z, w = \frac{1}{2}(1 + e_1 + e_2 + e_3)\}$ then, we obtain perfect codes corresponding to codes given in [13].

**6 Conclusion**

In this paper, we have investigated the existence/nonexistence of perfect codes correcting errors of Mannheim, Lipschitz, and Hurwitz weight 1, errors of Mannheim weight 2 or less, Lipschitz 2 or less, and Hurwitz weight 2 or less over $G_\pi$, $H(Z)_\pi$ and $H$. We have been able to obtain perfect codes correcting errors of Mannheim, Lipschitz, and Hurwitz weight 1. To the best of our knowledge, some of these codes are not known before.
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