Schreier Numbers and Nontrivial Small Divisors Satisfying Linear Recurrence of Order at Most Two

Karthik Nataraj
Institute for Computational Mathematics and Engineering
Stanford University
Stanford, CA, U.S.A.
kartnat@stanford.edu

Abstract: Schreier sets have been an object of study since first introduced in 1930 by Jozef Schreier to construct a counterexample to a conjecture of Banach. In 1974 George Andrews found interesting connections between these sets and Fibonacci numbers, and since then more results of a combinatorial flavor were proven by Chu, Beanland, and Finch-Smith. In parallel Iannucci introduced the concept of a small divisor and characterized all natural numbers whose small divisors are in arithmetic progression, results which were generalized by Chentouf and Chu. Then combining these two ideas, Chu introduced the notion of a Schreier number, one whose nontrivial small divisor set (small divisors excluding 1) is Schreier. Our main results are twofold: we first prove the asymptotic density of these numbers is 0 and that there are infinitely many non-prime Schreier pairs with difference 2 or 4. Then motivated by Chu’s generalization of Iannucci’s result we characterize all natural numbers whose nontrivial small divisors satisfy a linear recurrence with order no larger than 2.

1 Introduction

A nonempty set $A$ is said to be a Schreier set if $\min A \geq |A|$. The empty set by convention is considered Schreier. Schreier sets were first considered
by Jozef Schreier in the first half of the twentieth century to illustrate a counterexample to a problem in the field of Banach Space Theory. Later in 2012 A. Bird [2] proved that the number of Schreier sets whose greatest element is \( n \) is exactly the \( n \)th Fibonacci number \( F_n \). This was followed by a flurry of interesting observations relating these sets to higher order Fibonacci sequences, Turán graphs, and Schreier-Zeckendorf sets [1,7–10].

Another thread of work was introduced by Iannucci [13] where he studied the set \( S_n \) consisting of the so-called small divisors of a number \( n \), namely

\[
S_n := \{ d \mid n \mid, 1 \leq d \leq \sqrt{n} \}.
\]

In particular he characterized all numbers \( n \) for which the elements of \( S_n \) are in arithmetic progression. Chentouf [4] and Chu [5,6] extended this work—just as Chu [6] generalized Iannucci’s work [13] to the nontrivial small divisors, defined as

\[
S'_n := \{ d \mid n \mid, 1 < d \leq \sqrt{n} \},
\]

here we generalize Chentouf’s work [4] and characterize those numbers for which the elements of \( S'_n \) satisfy a linear recurrence of order at most two (following his language we will call such numbers recurrent). Furthermore we expand upon Chu’s notion of a Schreier number, one for which \( S'_n \) is itself Schreier. Our four main results are as follows:

**Theorem 1.** The asymptotic density of the Schreier numbers exists and equals 0. Namely,

\[
\lim_{N \to \infty} \frac{|\{n \leq N \mid n \in S\}|}{N} = 0.
\]

**Theorem 2.** For any natural number \( g \), there exists a pair of non-Schreier numbers \( n_1 < n_2 \) such that \( n_2 - n_1 = g \).

**Theorem 3.** Call a Schreier pair an ordered pair of Schreier numbers, neither of which is prime. Then at least one of the following statements is true: (i) There are infinitely many Schreier pairs of the form \((s, s + 2)\), and/or (ii) There are infinitely many Schreier pairs of the form \((s, s + 4)\).

**Theorem 4.** Let \( n \) be a recurrent number, \( p < q < r \) distinct primes, and \( k, l, m \) natural numbers. Then \( n \) falls into one of the following categories:

1. \( n = p^k \) or \( p^k q \) with \( q > p^k \). Here \( S'_{p^k} = \{p, p^2, \ldots, p^{\left\lfloor \frac{k}{2} \right\rfloor}\} \) and \( S'_{p^k,q} = \{p, p^2, \ldots, p^k\} \).
(2) \( n = p^k q^m, m < 4 \) or \( p^k q^m r, m < 3 \) with \( p^k q^m < r \). Here \( S'_{p^k q^m} = \{ p, q, p^2, pq, p^3, p^2 q, p^4, p^3 q, \ldots \} \) ending either in \( (p^l q, p^{l+2}), (p^l, p^{l+1} q) \), or \( (p^l q, p^{l+1} q) \), \( S'_{p^k q^m r} = \{ p, q, \ldots, p^{k+3} \} \) if \( p^3 \leq q^2 \), and otherwise \( S'_{p^k q^2 r} = \{ p, q, \ldots, p^{k+1} q \} \). Finally \( S'_{p^k q^m} = \{ p, \ldots, p^k q \} \).

(3) \( n = p^2 q^m \), where \( S'_n = \{ p, q, pq, q^2, \ldots, pq^{\lfloor \frac{m}{2} \rfloor} \} \) if \( m \) is even, or odd with \( q > p^2 \), and otherwise \( S'_n = \{ p, q, pq, q^2, \ldots, q^{\frac{m+1}{2}} \} \).

(4) \( n = pq^m \), where \( S'_n = \{ p, q, pq, \ldots, pq^{\frac{m}{2}} \} \) for \( m \) odd and \( \{ p, q, pq, \ldots, q^{\frac{m}{2}} \} \) for \( m \) even.

(5) \( n = pq^m r \) with \( S'_n = \{ p, q, pq, q^2, \ldots, q^m, pq^m \} \).

(6) \( n = 60 \) or \( n = pqr \) with \( S'_n = \{ p, q, r \} \).

Comparing Theorem 4 with that of Chentouf’s [4, Theorem 15], we may confirm that the class of recurrent numbers is strictly wider when we don’t constrain the first element of the linear recurrence to be one. The remainder of the paper will be split up into five sections. In Sections 2 and 3 we will prove Theorems 1, 2, and 3. Section 4 will be devoted to extending the analysis [4] to prove Theorem 4. Denoting \( s(n) \) the number of nontrivial small divisors, we discuss the cases of \( s(n) \leq 5 \) in Section 5. Finally Section 6 concludes with options for future work.

2 Proof of Theorem 1

We first note the simple observation relating \( s(n) \) and the number of divisors function \( \tau(n) \), defined as

\[
\tau(n) = \sum_{d \mid n} 1,
\]

which follows from the fact that the divisors of \( n \) come in pairs:

\[
\tau(n) = \begin{cases} 
2s(n) + 1, & \text{if } n \text{ is a perfect square;} \\
2s(n) + 2 & \text{otherwise.}
\end{cases}
\]

The following lemma uses this relationship to bound the number of distinct primes that occur in a Schreier number \( n \):

**Lemma 1.** Let \( n = p^m k, m \geq 1 \) be Schreier. Suppose the smallest prime in the factorization of \( k \) is larger than \( p \). Then the maximum number of distinct primes in the prime factorization of \( k \) is \( 1 + \log_2(\frac{m+1}{2}) \).
Proof. Note that $\tau(n) \leq 2p + 2$, as otherwise $\tau(n) > 2p + 2$ implies (from above) that $s(n) > p$, which contradicts that $p \geq s(n)$ in the Schreier $n$. But $\tau(n) \geq (m + 1)2^i$, $i$ the number of distinct primes in $k$’s prime factorization. Solving for $i$ and noting that $m \geq 1$ yields the claim. 

We then easily obtain the following corollary:

**Corollary 1.** The density of Schreier numbers with smallest prime $p$ is 0.

**Proof.** From Lemma 1 the Schreier numbers whose prime factorization has smallest prime equal to $p$ are of the form $p^m k$, where $m$ is bounded above (by for example, $2p + 2$) and the number of distinct primes in $k$ is bounded by $u_p := \lfloor 1 + \log_2(p + 1/2) \rfloor$. Note in fact this bound is achieved, at least for $m = 1$. Hence

$$\sum_{n \leq x, n = p^m k, n \in S} 1 \sim \sum_{k \leq x, k = p_1 \cdots p_{u_p}, p_1 < \cdots < p_{u_p}} 1 \leq \frac{1}{(u_p - 1)!} x (\log \log x)^{u_p - 1} \frac{1}{\log x}$$

by a result of Landau \[14\]. (Indeed, from the same result of Landau, if $k$ has fewer than $u_p$ distinct primes it’s density will be $o()$ of the above). Now divide this by $x$ and see that it has limit of 0.

Finally we can prove Theorem 1.

**Proof.** Considering the density of Schreier numbers divisible by at least 1 of the first $N$ primes $p_1, \ldots, p_N$, amongst all such numbers, we have

$$\frac{|\{n \leq x \mid n \in S, p_i | n, 1 \leq i \leq N\}|}{|\{n \leq x \mid p_i | n, 1 \leq i \leq N\}|} = \sum_{k=1}^{N} \frac{|\{ n \leq x \mid n \in S, p_k = \min_{1 \leq j \leq N, p_j | n} p_j \}|}{|\{n \leq x \mid p_k | n\}|} \leq \sum_{k=1}^{N} \frac{|\{ n \leq x \mid n \in S, p_k = \min_{1 \leq j \leq N, p_j | n} p_j \}|}{|\{n \leq x \mid p_k | n\}|} = \sum_{k=1}^{N} \frac{|\{n \leq x \mid n \in S, p_k = \min_{1 \leq j \leq N, p_j | n} p_j\}|/x}{|\{n \leq x \mid p_k | n\}|/x} \rightarrow 0$$

as $x \rightarrow \infty$, from Corollary 1 and the fact that the denominator tends to $1/p_k$. Hence almost all of the natural numbers that are multiples of at least one of
the first $N$ prime numbers are non-Schreier. That is to say, the density of non-Schreiers is

$$\geq 1 - \prod_{i \leq N} \left(1 - \frac{1}{p_i}\right) \to 1$$

as $N \to \infty$.

3 Proofs of Theorems 2 and 3

Various results have been obtained regarding gaps in primes [16], and the arithmetic progressions that occur in the primes [15]. Especially since the Schreier numbers include primes and products of primes, we may ask similar questions for the Schreier/non-Schreier numbers as well. The Green-Tao theorem guarantees arbitrarily long arithmetic progressions of these numbers. It’s not hard to construct such examples explicitly amongst the non-Schreier numbers as well. Furthermore, amongst the non-Schreier numbers we can in fact find such pairs with arbitrary spacing, which is our Theorem 2:

Proof. For $g > 1$, $(g^k, g^k+g)$ should work for sufficiently large $k$. The number of divisors of $g^k \to \infty$ as $k \to \infty$ while the smallest prime in $g$’s factorization remains constant, hence such a $k$ that makes $g^k$ non-Schreier definitely exists.

When $g$ is odd $g^k + g = g(g^{k-1} + 1)$ is divisible by 2 and at least two other distinct primes (since $\gcd(g, g^{k-1} + 1) = 1$), making it non-Schreier. When $g$ is even but not a power of 2 the same argument holds. Now suppose that $g = 2^u$ for some $u \geq 1$. For $u = 1$ note the above pair with $k = 6$ works, and for $u = 2, k = 4$ works. For $u \geq 3$ the small divisor set of $g^k + g$ will consist of at least 2, $2^2$ and $2^3$ for $k$ sufficiently large, automatically making it non-Schreier.

Last, for $g = 1$ note that $\{(7^{20}m - 1 + 6 \cdot 7^{20}, 7^{20}m + 6 \cdot 7^{20}), m \equiv 1 \pmod{6}\}$ are all non-Schreier.

We now conclude with a partial answer to an interesting question regarding twin pairs of Schreier numbers (Theorem 3 from the introduction): are there infinitely many ordered pairs of Schreier numbers of the form $(s, s+2)$, wherein neither $s$ nor $s+2$ are prime (we will refer to such ordered pairs “Schreier pairs”)? We note that the same question without the restriction on the non-primality is true as a corollary of Chen’s Theorem [3] that there are infinitely many primes $p$ wherein $\Omega(p+2) \leq 2$. In our case, the work of
Goldston et al. [12] on small gaps between almost primes provides a partial answer, and the proof of Theorem 3 is as follows:

Proof. Apply [12, Basic Theorem] to the linear forms

\[ L_1 = m + 1, L_2 = 3m + 1, L_3 = 5m + 1, \]

and notice that

\[ 5L_1 = L_3 + 4, 3L_1 = L_2 + 2, 5L_2 = 3L_3 + 2. \]

Thus there is a pair of forms \((L'_i, L'_j) \subset \{(L_3, 5L_1), (L_2, 3L_1), (3L_3, 5L_2)\}\) with prime factorizations \((u p_1 q_1, v p_2 q_2)\), \(p_i \neq q_i\) arbitrarily large and \(u, v \subset \{1, 3, 5\}\) for infinitely many values of \(m\). This is a Schreier pair since each form has 1 or 3 nontrivial small divisors and the smallest prime in each is \(\geq 3\).

This is better than what one gets as an immediate corollary of their work [11], that there are infinitely many pairs of \(E_2\) numbers (numbers that are products of exactly 2 distinct primes) with distance \(\leq 6\). The difficulty in directly applying the Basic Theorem to obtain stronger results (like the existence of infinitely many Schreier pairs of the form \((s, s + 1)\)) is the requirement that simultaneously the \(u_{ij}\) in \(L_j = u_{ij} L_i + v_{ij}\) be either 1 or an odd prime, while \(v_{ij}\) is also small. Future work will investigate modifications of the basic theorem to better accommodate this restriction.

4 Study of numbers whose proper small divisors satisfy order 2 linear recurrence

As mentioned in the introduction, Chentouf extended Iannucci’s work [13] wherein 1 was considered a divisor. In that setting the initial conditions of the order 2 linear recurrence were fixed to be 1 and \(p, p\) being some prime. As in Chu’s generalization to arithmetic progressions in \(S'_n\), we extend Chentouf’s work [4] to \(S'_n\), as opposed to \(S_n\). Hence in our case we can expect arguments of a similar “elementary” nature, yet with complications because initial conditions are now one of either \((p, q)\) or \((p, p^2)\), instead of only \((1, p)\).

We however find that the general techniques do carry over, and therefore in complete analogy to [4, Theorem 3] we have the following:
Proposition 1. Let $U(u,v,a,b)$ denote an order 2 linear recurrence in the sequence $(d_i)_{i=1}^\infty$ given by

$$d_i = \begin{cases} u, & \text{if } i = 1 \\ v, & \text{if } i = 2 \\ ad_{i-1} + bd_{i-2} & \text{otherwise.} \end{cases}$$

Then given a recurrent number $n$ associated with either $U(p,q,a,b)$ or $U(p,p^2,a,b)$ (there are two cases now), at least one of the following statements is true:

1. $\gcd(a,b) = 1$.
2. All elements of $S'_n$ are powers of $p$.
3. $S'_n = \{ p, q, p^2, pq, p^3, p^2q, p^4, p^3q, \ldots \}$ ending either in a $(p^kq, p^{k+2})$, a $(p^k, p^{k-1}q)$, or a $(p^kq, p^{k+1}q)$
4. $S'_n = \{ p, q, p^2, pq, p^3, p^2q, p^4, p^3q, \ldots \}$
5. $S'_n = \{ p, q, r \}$

Proof. Suppose $p \nmid \gcd(a,b)$. If $(d_1, d_2) = (p, p^2)$ then we have situation (2). Otherwise $(d_1, d_2) = (p, q)$; if $d_3 = pq$ then $pq | d_4 \implies d_4 = p^2q$, impossible as then $p^2 \not\in S'_n$. This is contained in situation (4). Otherwise $d_3 = p^2$ and the rest of the small divisors are divisible by $p$. So $d_4 = pq$ and $d_5 \in \{ p^2q, p^3 \}$ because $p^2 | d_5$. If $p^2q$ then the small divisor list ends there, since the only candidates for $d_6$ are $p^3q$ and $p^2q^2$, both of which can’t happen since $p^3, q^2$, resp., haven’t appeared yet. Hence $d_5 = p^3$ and $p^2 | d_6$ implies $d_6 = p^2q$. $d_7 \in \{ p^3q, p^4 \}$ and for the same reason as before we must have $d_7 = p^4$. Continue this process to see that the small divisor list is of the form $\{ p, q, p^2, pq, p^3, p^2q, p^4, p^3q, \ldots \}$, ending either in a $(p^kq, p^{k+2})$, a $(p^k, p^{k-1}q)$, or a $(p^kq, p^{k+1}q)$, yielding situation (3).

Now suppose $p \nmid \gcd(a,b) \neq 1$, so either $d_3 = aq + bp$ or $d_3 = ap^2 + bp$. In the latter case $p | d_4$ and iterating the recurrence reveals we are in situation (2) again. In the former case there are 3 sub-cases, corresponding to (i) $d_3 = p^2$, (ii) $d_3 = pq$, or (iii) $d_3 = r$, $r$ some other prime. If (i) then $p^2 = aq + bp \implies \gcd(a,b) | p^2 \implies \gcd(a,b) = p$, contradiction to $p \nmid \gcd(a,b)$. In (ii) $\gcd(a,b) | d_3$, so $\gcd(a,b) \in \{ p, q \}$. It cannot equal $p$ as $p \nmid \gcd(a,b)$. If it equals $q$ then note $q^2 | d_4 \implies d_4 = q^2$, and iterating leads to situation (4) again.

Finally (iii) would mean $r = aq + pb \implies \gcd(a,b) | r \implies \gcd(a,b) = r$. Thus $d_4$ is a positive multiple of $r$, and the smallest of these that is a divisor
of $n$ would be $rp > pq$, contradiction as then $pq \not\in S'_n$. Hence there can be no $d_4$ in this case and we have situation (5).

The following then classifies the numbers identified in conditions (2) through (5) above:

**Proposition 2.** Recurrent numbers $n$ with property (2) are of the form $p^k$ or $p^kq$ with $q > p^k$; with property (3) of the form $p^kq^m, m < 4$ or $p^kq^mr, m < 3$ and $p^kq^m < r$; with property (4) of the form $p^kq^m, k < 3$ or $pq^mr$ with $pq^m < r$; with property (5) of the form $pqr$.

**Proof.** The first part follows from [4, Proposition 4]. For the second note that a composite large divisor $r$ has at most one prime factor distinct from $p$ and $q$, which is itself a large divisor. Hence $n$ must take the form given above, with the restrictions on $m$ from the fact that $q^2 \not\in S'_n$. The third part follows from the same logic, and the fourth because $pqr$ is the only form for a number with three distinct prime divisors having $s(n) = 3$.

Now all that remains is the gcd$(a, b) = 1$ case. Henceforth we assume the small divisor set of the recurrent number $n$ contains two distinct primes $p < q$. We have the following lemma:

**Lemma 2.** Suppose $s(n) \geq 4$. Then gcd$(b, d_i) = 1$ for all $i$ unless $d_3 = r$, $r > q > p$ prime, in which case gcd$(b, d_i) = 1$ for $i > 1$.

**Proof.** Suppose for the sake of contradiction that some prime divides gcd$(b, d_i)$. There are 3 cases corresponding to $d_3 \in \{p^2, pq, r\}$. Suppose $d_3 = p^2 \implies d_2 = q$. It then cannot be that $i = 1$ since then $p|b$, and from $d_3 = p^2 = aq + bp$ we have $p|a$. So $p|\gcd(a, b)$, contradiction to $\gcd(a, b) = 1$. If it is true for $i = 2$ then $q|b$ and we’d have $p^2 = aq+kpq$ for $k > 0$, impossible as $q \not| p^2$. Last if true for $i > 2$, namely some prime $r|\gcd(b, d_i)$, then $d_i = ad_{i-1} + bd_{i-2} \implies r|ad_{i-1} \implies r|d_{i-1}$, and inductively $r|d_3 \implies r|p^2 \implies r = p$, and then $p^2 = aq + bp \implies r|a \implies \gcd(a, b) > 1$, contradiction.

Now for $d_3 = pq \implies d_2 = q$, almost an identical argument implies that $i \neq 1$. For $i = 2$, $pq = aq+kpq$ again; $k = 0 \implies b = 0, a = p$, contradiction to $\gcd(a, b) = 1$. $k > 0 \implies a = (1-k)p \implies d_4 = (1-k)p^2q + kq^2$. Of course $q|d_4 \implies d_4 \in \{q^2, p^2q\}$, and quick calculations reveal no primes $p, q$ satisfy that equality on $d_4$. A very similar argument to the one just given covers the case of $i > 2$.

Lastly when $d_3 = r$, again a very similar argument implies $i$ cannot be
greater than 2. For \( i = 2 \) we have \( q | b \) and \( r = aq + bp \), a problem as \( q \nmid r \). Hence the only break can occur when \( i = 1 \), wherein \( p | b \). \hfill \Box

Fortunately the above proviso when \( d_3 = r \) doesn’t change the corollary \cite{c} Corollary 7 that \( \gcd(d_i, d_{i+1}) = 1 \), which in our case is surely true when \( s(n) \geq 4 \) and \( (d_1, d_2) \neq (p, p^2) \) (which has already been covered in part (2) of Proposition \cite{c}). Now our version of \cite{c} Lemma 8 is the following:

**Proposition 3.** Suppose \( s(n) \geq 4 \) and \( (d_1, d_2) = (p, q) \). The allowable configurations for \( (d_1, d_2, d_3, d_4) \) are \((p, q, p^2, r), (p, q, r, p^2), (p, q, r, s), \) and \((p, q, r, pq)\). If \((p, q, r, pq)\) then \( S'_n = \{p, q, r, pq\} \).

**Proof.** We’ll go case by case depending on the value of \( d_3 \). If \( d_3 = p^2 \), the possibilities for \( d_4 \) are \( pq \) and \( r \). By \cite{c} Corollary 7 \( (\gcd(d_i, d_{i+1}) = 1) \) the only allowable configuration is \((p, q, p^2, r)\). And by the same corollary \( d_5 = pq \) is not possible.

Now if \( d_3 = r \) the possibilities for \( d_4 \) are \( p^2, pq \), and \( s, s \) prime and \( > r \). Take \( d_4 = pq \). Then we note, in analogy to \cite{c} Corollary 11, that \( p | d_i \) iff \( i \) is 1 more than a multiple of 3, and \( q | d_i \) iff \( i \) is even. Since \( 5 \) is neither even nor 1 more than a multiple of 3, \( d_5 = s \) a new prime. Likewise \( d_6 = q^2, d_7 = rp, d_8 \in \{sq, q^3\} \), contradiction as then \( rq \) is a small divisor that does not appear in the list. So the only configuration starting with \((p, q, r, pq)\) is possibly \((p, q, r, pq, s, q^2, rp) = S'_n \). This is impossible as \( sq^2rp|n \Rightarrow ps \in S'_n \), which it of course isn’t. That leaves us with \((p, q, r, pq, s, q^2) = S'_n \), which is again not possible as \( rsq^2|n \Rightarrow rq \in S'_n \), which it isn’t. And finally \((p, q, r, pq, s)\) isn’t possible as then \( pqr|n \Rightarrow s(n) \geq 7 \). Hence the small divisor set must exactly be \((p, q, r, pq)\). \hfill \Box

In fact, further analysis restricts the options for a recurrent number with \( \{p, q, p^2, r\} \subseteq S'_n \):

**Proposition 4.** \((d_1, d_2, d_3, d_4) = (p, q, p^2, r)\) occurs only in a number of the form \( n = p^2qr, p < q < r \). Further, there is no recurrent number with \( S'_n = \{p, q, r, pq\} \).

**Proof.** In this case \( d_5 \in \{p^3, pq\} \), since \( p | d_{2i+1} \). \( d_5 = p^3 \Rightarrow p^2 | a, \) and \( p^3 = aq + bp \Rightarrow p | b \), contradiction to \( \gcd(a, b) = 1 \). So \( d_5 = pq \), and we note that \( q | d_i \) iff \( q \) is 2 more than a multiple of 3. With these, \( d_5 = pq \Rightarrow d_6 = s \Rightarrow d_7 \in \{p^3, rp\} \), and in either case the only option for \( d_8 \) would be \( q^2 \).
Here we employ the same line of reasoning as in [4, Theorem 12] to show that if $s(n) \geq 8$, then in fact $s(n)$ isn’t finite. This is equivalent to proving that $s(n) \geq 8 \implies s(n) \geq 3i + 2$ for all $i \geq 2$. The base case is true trivially. Now if this is true for some particular $i$, then indeed $q^2d_{3i}d_{3i+1} | n \implies qd_{3i} \in S'_{n}$. If $qd_{3i} > d_{3i+2}$ then $qd_{3i} \geq d_{3i+5} \implies s(n) \geq 3i + 5$, as required. Otherwise $qd_{3i} = d_{3i+2}$ and $d_{3i+2} = ad_{3i+1} + bd_{3i}$, combined with $\gcd(d_i, d_{i+1}) = 1$, implies $d_{3i} | a$. Looking at the residues of $d_i \pmod{a}$ reveals that $d_{3i}$ must divide a number of the form $b^k p$ or $b^k q$, and since $\gcd(d_i, b) = 1$ it must be that $d_{3i} \in \{p, q\}$, which is impossible as those divisors have already occurred. Hence at most $s(n) = 8$ with $S'_n = \{p, q, p^2, r, pq, s, p^3/rp, q^2\}$. But this would imply $p^2qrs | n \implies rq \in S'_n$, which it isn’t, hence we can shorten to $S'_n = \{p, q, p^2, r, pq\}$. Thus indeed $n$ is of the form $p^2q r$.

And if $S'_n = \{p, q, r, pq\}$, there is no $n$ with at least 3 distinct prime divisors having $\tau(n) \in \{9, 10\}$, thus this isn’t possible.

So there remain two cases to cover, $(p, q, r, p^2)$ and $(p, q, r, s)$. Fortunately Chentouf’s method to outlaw the $(p, q, r, p^2)$ case transfers here as well:

**Proposition 5.** The $(p, q, r, p^2)$ configuration does not occur in a recurrent number.

**Proof.** Again $p | d_i$ iff $i$ is 1 more than a multiple of 3. Just as in the proof of Proposition 4 we’ll use induction to show $s(n) \geq 3i + 1$ for all $i \geq 1$. It’s clearly true for $i = 1$. Then $\gcd(d_{3i-1}, d_{3i}) = 1 \implies p^2d_{3i-1}d_{3i} | n \implies pd_{3i-1}$ is a small divisor. If $pd_{3i-1} > d_{3i+1}$ then it’s $\geq d_{3i+4}$. Otherwise $d_{3i+1} = pd_{3i-1} = ad_{3i} + bd_{3i-1} \implies d_{3i-1} | a$, hence looking at the residue classes of $d_i \pmod{a}$ we see $d_{3i-1} \in \{p, q, bp, bq, b^2p, b^2q, \ldots\}$. The argument finishes the same way as above, namely $d_{3i-1} \in \{p, q\}$, which is impossible for any value of $i > 1$. 

Similarly the set $(p, q, r, s)$ cannot occur as well. The argument from [4, Lemma 13] transfers exactly, except here $t$ can be either $p$ or $q$ considering the sequence $d_i \pmod{t}$, along with the facts $t | a$ and $\gcd(b, t) = 1$. (Note to use [4, Lemma 13] we require $i \geq 2$ which is fine as trivially $\gcd(d_1, d_3) = 1$.) However if $t = p$ then $p | a \implies p | r$, contradiction, and if $t = q$ then $q | a \implies q | s$, contradiction. Then Theorem 14 transfers exactly.
5 Case of small $s(n)$

We can ask for the recurrent numbers $n$ with $s(n) \in \{1, 2, 3, 4\}$. Now note a necessary condition for $s(n) = k$ is the number of divisors function $\tau(n) = 2k + 2$ or $2k + 1$ (perfect square). For $s(n) \in \{1, 2\}$ the answer is parts (ii) and (iii) of Section 4 in [4]. For $s(n) = 3$ the candidates which aren’t powers of $p$ (which are easily shown to be recurrent) are $\{p^3q, pq^3, pqr\}$. All $p^3q$ are recurrent except when $p^3 > q > p^2$, in which case $S'_n = \{p, p^2, q\} \implies p|q$, impossible. All $pq^3$ are recurrent, with $S'_n = \{p, q, pq\}$. Also all $pqr$ are recurrent: if $r < pq$ then $\gcd(p, q) = 1$ implies the existence of $a, b$ for which $r = aq + bp$. And if $pq < r$ then $(a, b) = (p, 0)$ gives rise to the small divisor list $(p, q, pq)$. For $s(n) = 4, n \in \{p^2, p^2q^2, pq^4, p^4q\}$. The first two are clearly recurrent, $p^2q^2$ is if $q < p^2$, in which case the list is $(p, q, p^2, pq)$. All $pq^4$ are recurrent with list $(p, q, pq, q^2)$, and $p^4q$ is recurrent if either $q > p^4$ $(S'_n = \{p, p^2, p^3, p^4\})$ or $q < p^2$ $(S'_n = \{p, q, p^2, pq\})$.

For $s(n) = 5, we can easily deduce similar relationships between $p, q$, and $r$ in order to achieve a converse to Theorem 4 in the cases when $n$ is of the form given in parts $(1) - (5)$. More interesting and difficult is understanding the relationships between $p, q, r$ and $p$ and $q$ to obtain recurrent $n = p^2qr$ with $S'_n = \{p, q, p^2, r, pq\}$; as part (6) of Theorem 4 indicates, we have the following:

**Proposition 6.** The only recurrent $n$ with $(d_1, d_2, d_3, d_4) = (p, q, r, p^2)$ is 60.

**Proof.** The proof in [4] Theorem 10 does not directly translate as the relationships between $p, q$, and $r$ differ here without $d_1 = 1$, although similar algebraic manipulations bear fruit. First note that $p^2 = ap + bp \implies p|a$, so let’s set $a = kp$ for $k > 0$. Now the divisor list implies $r = ap^2 + bp, q = ar + bp^2$, and after substituting $a$ for $kp$ in all 3 equations we obtain the relations $p = kq + b, r = kp^3 + bq, q = kr + bp$. We can substitute for $r, b$ in the latter equation: $q = kr + bp = k(kp^3 + bq) + bp = k^2p^3 + (p+kq) = k^2p^3 + (p-kq)(p+kq)$, and solving for $k$ yields $k = \sqrt{\frac{q-p^2}{p-q}} := u$. Solving for $q$ in this equation yields $q = \frac{-1 + \sqrt{1 + 4u^2p^2(1+p)}}{2u^2} \implies \sqrt{1 + 4u^2p^2(1+p)} \equiv 1 \pmod{q} \implies 1 + 4u^2p^2(1+p) \equiv 1 \pmod{q} \implies q|p^2(1+p) \cdot q|p+1 \implies (p, q) = (2, 3)$ and we obtain the small divisor set of 60. If this is not the case then $q|u$ as $q$ cannot divide $p^2$. Letting $u = cq$ we obtain $q - p^2 = c^2q^2(p^3 - q^2) \implies q|p$, impossible. Thus the only option meeting these constraints is $n = 60$. \qed
6 Future work

There is much more we can ask relating to the questions in Section 3—for example, we only proved a difference condition on pairs of non-Schreier numbers \((n_1, n_2)\). It would be interesting to prove a corresponding gap condition. Just as there are arbitrarily large gaps in the primes, are there such gaps between Schreier numbers? How large can the gaps be between non-Schreier numbers? Also, is it possible to prove that there are arbitrarily large arithmetic progressions amongst the Schreier non-primes (such as, amongst almost primes)? And of course, it’d be nice to decisively improve our Theorem 3 to there being infinitely many Schreier pairs \((s, s + 2)\), or even better, of the form \((s, s + 1)\). One could either try to approach this in an elementary way or through a strengthening of [11, Theorem 3], to there being infinitely many \(n\) for which \(n, n - 2 \in E_2\), as opposed to just \(n, n - 2 \in \mathcal{P}_2\).

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