A CHARACTERIZATION OF HARDY SPACES ASSOCIATED WITH CERTAIN SCHRÖDINGER OPERATORS

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Abstract. Let \( \{K_t\}_{t>0} \) be the semigroup of linear operators generated by a Schrödinger operator \( -L = \Delta - V(x) \) on \( \mathbb{R}^d, \ d \geq 3, \) where \( V(x) \geq 0 \) satisfies \( \Delta^{-1} V \in L^\infty. \) We say that an \( L^1 \)-function \( f \) belongs to the Hardy space \( H^1_L \) if the maximal function \( M_L f(x) = \sup_{t>0} |K_t f(x)| \) belongs to \( L^1(\mathbb{R}^d). \) We prove that the operator \( (-\Delta)^{1/2} L^{-1/2} \) is an isomorphism of the space \( H^1_L \) with the classical Hardy space \( H^1(\mathbb{R}^d) \) whose inverse is \( L^1(\mathbb{R}^d)/(-\Delta)^{-1/2}. \) As a corollary we obtain that the space \( H^1_L \) is characterized by the Riesz transforms \( R_j = \frac{\partial}{\partial x_j} L^{-1/2}. \)

1. Introduction and statement of the result

Let \( K_t(x,y) \) be the integral kernels of the semigroup \( \{K_t\}_{t>0} \) of linear operators on \( \mathbb{R}^d, \ d \geq 3, \) generated by a Schrödinger operator \( -L = \Delta - V(x), \) where \( V(x) \) is a non-negative locally integrable function which satisfies

\[
\Delta^{-1} V(x) = -c_d \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} V(y) dy \in L^\infty(\mathbb{R}^d).
\]

Since \( V(x) \) is non-negative, the Föhrman-Kac formula implies that

\[
0 \leq K_t(x,y) \leq (4\pi t)^{-d/2} e^{-|x-y|^2/4t} =: P_t(x-y).
\]

It is known, see [14], that for \( V(x) \geq 0 \) the condition (1.1) is equivalent to the lower Gaussian bounds for \( K_t(x,y), \) that is, there are \( c,C > 0 \) such that

\[
c t^{-d/2} e^{-C|x-y|^2/t} \leq K_t(x,y).
\]

We say that an \( L^1 \)-function \( f \) belongs to the Hardy space \( H^1_L \) if the maximal function \( M_L f(x) = \sup_{t>0} |K_t f(x)| \) belongs to \( L^1(\mathbb{R}^d). \) Then we set

\[
\|f\|_{H^1_L} = \|M_L f\|_{L^1(\mathbb{R}^d)}.
\]

The Hardy spaces \( H^1_L \) associated with Schrödinger operators with nonnegative potentials satisfying (1.1) were studied in [10]. It was proved that the map \( f(x) \mapsto w(x) f(x) \) is an isomorphism of \( H^1_L \) onto the classical Hardy space \( H^1(\mathbb{R}^d), \) where

\[
w(x) = \lim_{t \to \infty} \int K_t(x,y) dy,
\]

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which in particular means that
\[(1.5) \quad \|fw\|_{H^1(\mathbb{R}^d)} \sim \|f\|_{H^1_L(\mathbb{R}^d)},\]
see [10, Theorem 1.1]. The function \(w(x)\) is \(L\)-harmonic, that is, \(K_tw = w\), and satisfies \(0 < \delta \leq w(x) \leq 1\).

Let us remark that the classical real Hardy space \(H^1(\mathbb{R}^d)\) can be thought as the space \(H^1_L\) associated with the classical heat semigroup \(e^{t\Delta}\), that is, \(L = -\Delta + V\) with \(V \equiv 0\) in this case. Obviously, the constant functions are the only bounded harmonic functions for \(\Delta\).

The present paper is a continuation of [10]. Our goal is to study the mappings
\[
L^{1/2}(-\Delta)^{-1/2} \quad \text{and} \quad (-\Delta)^{1/2}L^{-1/2}
\]
which turn out to be bounded on \(L^1(\mathbb{R}^d)\) (see Lemma 2.6). Our main result is the following theorem, which states another characterization of \(H^1_L\).

**Theorem 1.6.** Assume that \(L = -\Delta + V(x)\) is a Schrödinger operator on \(\mathbb{R}^d, d \geq 3\), with a locally integrable non-negative potential \(V(x)\) satisfying (1.1). Then the mapping \(f \mapsto (-\Delta)^{1/2}L^{-1/2}f\) is an isomorphism of \(H^1_L\) onto the classical Hardy space \(H^1(\mathbb{R}^d)\), that is, there is a constant \(C > 0\) such that
\[(1.7) \quad \|(-\Delta)^{1/2}L^{-1/2}f\|_{H^1(\mathbb{R}^d)} \leq C\|f\|_{H^1_L(\mathbb{R}^d)},\]
\[(1.8) \quad \|L^{1/2}(-\Delta)^{-1/2}f\|_{H^1_L(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)}.
\]

As a corollary we immediately obtain the following Riesz transform characterization of \(H^1_L\).

**Corollary 1.9.** Under the assumptions of Theorem 1.6 an \(L^1\)-function \(f\) belongs to the space \(H^1_L\) if and only if \(R_jf = \frac{\partial}{\partial x_j}L^{-1/2}f\) belong to \(L^1(\mathbb{R}^d)\) for \(j = 1, 2, \ldots, d\). Moreover, there is a constant \(C > 0\) such that
\[(1.10) \quad C^{-1}\|f\|_{H^1_L(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d}\|R_jf\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{H^1_L(\mathbb{R}^d)}.
\]

**Example 1.** It is not hard to see that if for a function \(V(x) \geq 0\) defined on \(\mathbb{R}^d, d \geq 3\), there is \(\varepsilon > 0\) such that \(V \in L^{d/2-\varepsilon}(\mathbb{R}^d) \cap L^{d/2+\varepsilon}(\mathbb{R}^d)\), then \(V\) satisfies (1.1).

**Example 2.** Assume that (1.1) holds for a function \(V : \mathbb{R}^d \to [0, \infty), d \geq 3\). Then \(V(x_1, x_2) := V(x_1)\) defined on \(\mathbb{R}^d \times \mathbb{R}^n, n \geq 1\), fulfils (1.1).

The reader interested in other results concerning Hardy spaces associated with semigroups of linear operators, and in particular semigroups generated by Schrödinger operators, is referred to [1], [2], [4], [5], [6], [7], [8], [12].
2. Boundedness on $L^1$

We define the operators:

$(-\Delta)^{-1} f(x) = \int_0^\infty P_t f(x) \, dt = c_d \int \frac{f(y)}{|x-y|^{d-2}} \, dy =: \int \Gamma_0(x-y) f(y) \, dy,$

$L^{-1} f(x) = \int_0^\infty K_t f(x) \, dt =: \int \Gamma(x,y) f(y) \, dy,$

$(-\Delta)^{-1/2} f = c_1 \int_0^\infty P_t f \frac{dt}{\sqrt{t}} = c'_d \int \frac{1}{|x-y|^{d-1}} f(y) \, dy =: \int \tilde{\Gamma}_0(x-y) f(y) \, dy,$

$L^{-1/2} f = c_1 \int_0^\infty K_t f \frac{dt}{\sqrt{t}} =: \int \tilde{\Gamma}(x,y) f(y) \, dy,$

where $c_1 = \Gamma(1/2)^{-1}$. Clearly,

$$0 \leq \tilde{\Gamma}(x,y) \leq c'_d |x-y|^{-d+1}, \quad 0 < \Gamma(x,y) \leq c_d |x-y|^{-d+2}. \quad (2.1)$$

The perturbation formula asserts that

$$P_t(x-y) = K_t(x,y) + \int_0^t \int P_{t-s}(x-z)V(z)K_s(z,y) \, dz \, ds$$

$$= K_t(x,y) + \int_0^t \int K_{t-s}(x,z)V(z)P_s(z-y) \, dz \, ds. \quad (2.2)$$

Multiplying the second inequality in (2.2) by $w(x)$ and integrating with respect to $dx$ we get

$$\int P_t(x-y)w(x) \, dx = w(y) + \int_{\mathbb{R}^d} \int_0^t w(z)V(z)P_s(z,y) \, ds \, dx,$$

since $w$ is $L$-harmonic. The left-hand side of (2.3) tends to a harmonic function, which is bounded from below by $\delta$ and above by 1, as $t$ tends to infinity. Thus there is a constant $0 < c_w \leq 1$ such that

$$c_w = w(y) + \int_{\mathbb{R}^d} w(z)V(z)\Gamma_0(z-y) \, dz. \quad (2.4)$$

Similarly, integrating the first equation in (2.2) with respect to $x$ and taking limit as $t$ tends to infinity, we get

$$1 = w(y) + \int_{\mathbb{R}^d} V(z)\Gamma(z,y) \, dz. \quad (2.5)$$

For a reasonable function $f$ the following operators are well defined in the sense of distributions:

$$(-\Delta)^{1/2} f = c_2 \int_0^\infty (P_t f - f) \frac{dt}{t^{3/2}}, \quad c_2 = \Gamma(-1/2)^{-1},$$

$$L^{1/2} = c_2 \int_0^\infty (K_t f - f) \frac{dt}{t^{3/2}}.$$
Lemma 2.6. There is a constant $C > 0$ such that

\begin{align}
\|(−Δ)^{1/2}L^{-1/2}f\|_{L^1} & \leq C\|f\|_{L^1}, \\
\|L^{1/2}(-Δ)^{-1/2}f\|_{L^1} & \leq C\|f\|_{L^1}.
\end{align}

Proof. From the perturbation formula (2.2) we get

\begin{align}
(−Δ)^{1/2}L^{-1/2}f(x) & = c_2 \int_0^\infty (P_t - I)L^{-1/2}f(x) \frac{dt}{t^{3/2}} \\
& = c_2 \int_0^\infty (P_t - K_t)L^{-1/2}f(x) \frac{dt}{t^{3/2}} + c_2 \int_0^\infty (K_t - I)L^{-1/2}f(x) \frac{dt}{t^{3/2}} \\
& = c_2 \int_0^\infty \int_0^t \int\int P_{t-s}(x-z)V(z)K_s(z,y)L^{-1/2}f(y) dy dz ds \frac{dt}{t^{3/2}} + f(x).
\end{align}

Consider the integral kernel $W(x,u)$ of the operator

\[ f \mapsto \int_0^\infty \int_0^t \int\int P_{t-s}(x-z)V(z)K_s(z,y)L^{-1/2}f(y) dy dz ds \frac{dt}{t^{3/2}}, \]

that is,

\[ W(x,u) = \int_0^\infty \int_0^t \int\int P_{t-s}(x-z)V(z)K_s(z,y)\bar{Γ}(y,u) dy dz ds \frac{dt}{t^{3/2}}. \]

Clearly $0 \leq W(x,u)$. Integration of $W(x,u)$ with respect to $dx$ leads to

\begin{align}
\int W(x,u) \, dx & = \int_0^\infty \int_0^t \int\int V(z)K_s(z,y)\bar{Γ}(y,u) dy dz ds \frac{dt}{t^{3/2}} \\
& = 2 \int_0^\infty \int\int V(z)K_s(z,y)\bar{Γ}(y,u) dy dz \frac{ds}{\sqrt{s}} \\
& \leq 2c_1^{-1} \int\int V(z)\bar{Γ}(z,y)\bar{Γ}(y,u) dy dz \\
& = 2c_1^{-1} \int V(z)\bar{Γ}(z,u) dz.
\end{align}

Using (2.1) we see that $\int W(x,u) \, dx \leq 2c_1^{-1}\|Δ^{-1}V\|_{L^\infty}$, which completes the proof of (2.7). The proof of (2.8) goes in the same way. We skip the details.

We finish this section by proving the following two lemmas, which will be used in the sequel.

Lemma 2.11. Assume that $f \in L^1(\mathbb{R}^d)$. Then

\begin{align}
\int (−Δ)^{1/2}L^{-1/2}f(x) \, dx & = f(x)w(x) \, dx.
\end{align}
Proof. From (2.9) and (2.10) we conclude that

\[
\int (-\Delta)^{1/2} L^{-1/2} f(x) \, dx = c_2 \int \int W(x, u) f(u) \, dudx + \int f(x) \, dx
\]

\[
= 2c_2c_1^{-1} \int V(z) \Gamma(z, u) f(u) \, dz \, du + \int f(x) \, dx
\]

\[
= \int (w(u) - 1) f(u) \, du + \int f(x) \, dx,
\]

where in the last equality we have used (2.5). □

Lemma 2.13. Assume that \( f \in L^1(\mathbb{R}^d) \). Then

\[
(2.14) \quad \int (L^{1/2}(-\Delta)^{-1/2} f)(x) w(x) \, dx = c_w \int f(x) \, dx.
\]

Proof. The proof is similar to that of Lemma 2.11. Indeed, by the perturbation formula (2.2) we have

\[
\int (L^{1/2}(-\Delta)^{-1/2} f)(x) w(x) \, dx
\]

\[
= c_2 \int \int_0^\infty (K_t - P_t)((-\Delta)^{-1/2} f)(x) \frac{dt}{t^{3/2}} w(x) \, dx
\]

\[
+ c_2 \int \int_0^\infty (P_t - I)((-\Delta)^{-1/2} f)(x) \frac{dt}{t^{3/2}} w(x) \, dx
\]

\[
= -c_2 \int \int_0^\infty \int_0^t \int w(x)K_{t-s}(x, z)V(z)
\]

\[
\times P_s(z-y)((-\Delta)^{-1/2} f)(y) \, dydz \, ds \frac{dt}{t^{3/2}} \, dx
\]

\[
+ \int w(x) f(x) \, dx
\]

\[
= -c_2 \int \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w(z)V(z)P_s(z-y)((-\Delta)^{-1/2} f)(y) \, dydz \, ds \frac{dt}{t^{3/2}} \, dx
\]

\[
+ \int w(x) f(x) \, dx,
\]
where in the last equality we have used that \( w \) is \( L \)-harmonic. Integrating with respect to \( dt \) and then with respect to \( ds \) yields

\[
\int (L^{1/2}(-\Delta)^{-1/2} f)(x) w(x) \, dx = -\frac{2c_2}{c_1} \int \int w(z) V(z) \tilde{\Gamma}_0(z - y)((-\Delta)^{-1/2} f)(y) \, dy \, dz + \int f(x) w(x) \, dx
\]

\[
= \int w(z) V(z) \Gamma_0(z - u) f(u) \, du \, dz + \int f(x) w(x) \, dx
\]

\[
= \int c_w f(x) \, dx - \int w(y) f(y) \, dy + \int f(x) w(x) \, dx,
\]

where in the last equality we have used (2.4). \( \square \)

3. Atoms and molecules

Fix \( 1 < q \leq \infty \). We say that a function \( a \) is an \((1, q, w)\)-atom if there is a ball \( B \subset \mathbb{R}^d \) such that \( \text{supp} \, a \subset B \), \( \|a\|_{L^q(\mathbb{R}^d)} \leq |B|^\frac{1}{q} - 1 \), \( \int a(x) w(x) \, dx = 0 \). The atomic norm \( \|f\|_{H^1_{at,q,w}} \) is defined by

\[
(3.1) \quad \|f\|_{H^1_{at,q,w}} = \inf \left\{ \infty \sum_{j=1}^{\infty} |\lambda_j| \right\},
\]

where the infimum is taken over all representations \( f = \sum_{j=1}^{\infty} \lambda_j a_j \), where \( \lambda_j \in \mathbb{C} \), \( a_j \) are \((1, q, w)\)-atoms.

Clearly, if \( w_0(x) \equiv 1 \), then the \((1, q, w_0)\)-atoms coincide with the classical \((1, q)\)-atoms for the Hardy space \( H^1(\mathbb{R}^d) \), which can be thought as \( H^1_{-\Delta} \).

As a direct consequence of Theorem 1.1 of [10] (see (1.5)) and the results about atomic decompositions of the classical real Hardy spaces (see, e.g., [3], [13], [15]), we obtain that the space \( H^1_L \) admits atomic decomposition into \((1, q, w)\)-atoms, that is, there is a constant \( C_q > 0 \) such that

\[
(3.2) \quad C_q^{-1} \|f\|_{H^1_{at,q,w}} \leq \|f\|_{H^1_L} \leq C_q \|f\|_{H^1_{at,q,w}}.
\]

Let \( \varepsilon > 0 \), \( 1 < q < \infty \). We say that a function \( b \) is a \((1, q, \varepsilon, w)\)-molecule associated with a ball \( B = B(x_0, r) \) if

\[
(3.3) \quad \left( \int_B |b(x)|^q \, dx \right)\frac{1}{q} \leq |B|^{\frac{1}{q} - 1}, \quad \left( \int_{2kB \setminus 2k-1B} |b(x)|^q \, dx \right)\frac{1}{q} \leq |2kB|^{\frac{1}{q} - 1} 2^{-\varepsilon k}
\]

and

\[
(3.4) \quad \int b(x) w(x) \, dx = 0.
\]

Obviously every \((1, q, w)\)-atom is a \((1, q, \varepsilon, w)\)-molecule. It is also not hard to see that for fixed \( q > 1 \) and \( \varepsilon > 0 \) there is a constant \( C > 0 \) such that every \((1, q, \varepsilon, w)\) molecule
constant $C > 0$ where for every $r > H$ that the space \( H \). Let (3.6) 
\[ \| b \| \leq \int_{R^d} \left( \int_{R^d} \left| b(x) \right| \right)^{1/q} \left( 1 + \frac{|x - y_0|}{r} \right)^{\delta/q} \left( \frac{d}{q} \right) \] 
then \( b \) is a \((1, q, \varepsilon, w)\)-molecule associated with \( B(y_0, r) \).

In order to prove Theorem 1.6 we shall use general results about Hardy spaces as-
mentioned above. The proof follows from functional calculi (see, e.g., [11]). Note that
Proof. Then we say that \( a \) is a ball \( B \) is a ball. We say that a function \( f \) is define by means of the maximal function, that is,
\[ H^1 = \{ f \in L^1(R^d) : \| f \|_{H^1} = \sup_{t>0} |T_t f(x)| \| L^1(R^d) \| < \infty \}. \]
We say that a function \( a \) is a generalized \((1, \infty, L)\)-atom for the Hardy space \( H^1 \) if there is a ball \( B = B(y_0, r) \) and a function \( b \) such that
\[ \sup \| b \|_{L^\infty} \leq |B|^{-1}, \quad a = (I - T_r x)b. \]
Then we say that \( a \) is associated with the ball \( B(y_0, r) \). It was proved in Section 6 of [9] that the space \( H^1 \) admits atomic decompositon with the generalized \((1, \infty, L)\)-atoms, that is, \( \| f \|_{H^1} \sim \| f \|_{H^*_{\infty, \infty, \infty}} \), where the norm \( \| f \|_{H^*_{\infty, \infty, \infty}} \) is defined as in (3.1) with \( a_j(x) \) replaced by the general \((1, \infty, L)\)-atoms \( a_j(x) \).

Lemma 3.7. There is a constant \( C > 0 \) such that for every \( a \) being a generalized \((1, \infty, L)\) atom associated with \( B(y_0, r) \) one has
\[ |L^{-1/2}a(y)| \leq C|r|^{-d}(1 + \frac{|y - y_0|}{r})^{-d}. \]
Proof. The proof follows from functional calculi (see, e.g., [11]). Note that \( L^{-1/2}a = m_{(r)}(L)b \) with \( m_{(r)}(\lambda) = r(r^2 \lambda)^{-1/2}(e^{-r^2 \lambda} - 1) \) and \( b \) such that \( \sup \| b \|_{L^\infty} \leq |B(y_0, r)|^{-1} \). From [11] we conclude that there is a constant \( C > 0 \) such that for every \( r > 0 \) one has
\[ m_{(r)}(L)f(x) = \int_{R^d} m_{(r)}(x, y)f(y) dy, \]
with \( m_{(r)}(x, y) \) satisfying
\[ |m_{(r)}(x, y)| \leq C|y|^{-d}(1 + \frac{|x - y|}{r})^{-d}. \]
Now the lemma can be easily deduced from (3.8) and the size and support property of \( b \).

4. Proof of Theorem 1.6

For real numbers \( n > 2, \beta > 0 \) let

\[
g(x) = (1 + |x|)^{-n-\beta}, \quad g_s(x) = s^{-n/2}g\left(\frac{x}{\sqrt{s}}\right).
\]

One can easily check that

\[
\int_0^t g_s(x) \, ds \leq C |x|^{-n}(1 + |x|^{-\beta}),
\]

\[
\int_\mathbb{R}^d g_s(x) \, ds \leq C r^{-n}(1 + |x|)^{-n+2} \quad \text{for } r > 0.
\]

Moreover, it is easily to verify that for \( 1 < q < \infty, d(1 - \frac{1}{q}) < \alpha \leq d, \beta > 0 \) one has

\[
\left\| |x|^{\alpha-d} \left(1 + \frac{|x|}{\sqrt{t}}\right)^{-\beta}\right\|_{L^q(\mathbb{R}^d, dx)} = C_{\alpha, \beta} t^{(\alpha-d+d/q)/2}
\]

and

\[
\int |z - y|^{2-d} \left(1 + \frac{|z - y|}{r}\right)^{-\beta} \left(1 + \frac{|y|}{r}\right)^{-d+\gamma} \, dy \leq C r^2 \left(1 + \frac{|z|}{r}\right)^{-d+\gamma+2-\beta}
\]

for \( 0 < \gamma < \beta < 2 \).

Lemma 4.5. Assume that \( V(x) \) satisfies the assumptions of Theorem 1.6. Then for \( 0 < \gamma \leq 2 \) and \( r > 0 \) one has

\[
\int_{\mathbb{R}^d} V(z) \left(1 + \frac{|z - y|}{r}\right)^{-d+\gamma} \, dz \leq c_d^{-1} r^{d-2} \| \Delta^{-1} V \|_{L^\infty}.
\]

Proof. The left-hand side of (4.6) is bounded by

\[
\int_{|z-y| \leq r} V(z) \left(\frac{r}{|z-y|}\right)^{d-2} \, dz + \int_{|z-y| > r} V(z) \left(\frac{z-y}{r}\right)^{-d+2} \, dz \leq c_d^{-1} r^{d-2} \| \Delta^{-1} V \|_{L^\infty}.
\]

Proof of Theorem 1.6. We already have known that the operators \((-\Delta)^{1/2} L^{-1/2}\) and \(L^{1/2} (-\Delta)^{-1/2}\) are bounded on \( L^1(\mathbb{R}^d) \). It suffices to prove (1.7) and (1.8). Set \( \gamma = \frac{1}{10} \) and fix \( q > 1 \) and \( \varepsilon > 0 \) such that \( \gamma > d(1 - \frac{1}{q}) + \varepsilon \). Set \( w_0(x) \equiv 1 \). According to the atomic and molecular decompositions (see Section 3) the proof of (1.7) will be done if we verify that \((-\Delta)^{1/2} L^{-1/2} a\) is a multiple of a \((1, q, \varepsilon, w_0)\)-molecule for every generalized \((1, \infty, L)\)-atom \( a \) with a multiple constant independent of \( a \). Identical arguments can be then applied to show that \(L^{1/2} (-\Delta)^{-1/2} a\) is a \((1, q, \varepsilon, w)\)-molecule for \( a \) being a generalized atom for the classical Hardy space \( H^1(\mathbb{R}^d) = H^1_{-\Delta} \) with a multiple constant independent of \( a \).
Let $a = (I - K_x) b$ be a generalized $(1, \infty, L)$-atom for $H^1_L$ associated with $B(y_0, r)$.

By Lemma 2.11, since $\int w(x)a(x)\,dx = 0$, we have that

$$\int (-\Delta)^{1/2} L^{-1/2} a(x)\,dx = 0.$$ 

Set

$$J(x) = \int_0^x \int_0^t \int P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}a)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}}$$

(4.7)

$$= \int_0^{x^2} \int_0^t \int P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}a)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}} + \int_0^{x^2} \int_0^t \int \int P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}a)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}} + ...$$

$$= J_1(x) + J_2(x) + J_3(x).$$

Thanks to (2.9) and Lemma 3.5 it suffices to show that there is a constant $C_q > 0$, independent of $a(x)$ such that

$$\left\| \left(1 + \frac{|x-y_0|}{r}\right)^\gamma J_1(x) \right\|_{L^q(\mathbb{R}^d)} \leq C_q r^{-d+d/q}. \tag{4.8}$$

Applying Lemma 3.7 and (4.1) with $n = d + 1$, we obtain

$$|J_1(x)| = \left| \int_0^{x^2} \int_0^t \int P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}a)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}} \right|$$

(4.9)

$$\leq C \int_0^{x^2} \int_0^t \int P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}a)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}}$$

$$\leq C \int_0^{x^2} \int P_{s}(x-z)V(z)K_s(z,y)(L^{-1/2}a)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}}$$

$$\leq C_N \int |x-z|^{-d}(1 + \frac{|x-z|}{r})^{-N} V(z) r^{-d}(1 + \frac{|z-y_0|}{r})^{-d} dz.$$ 

Consequently,

$$|J_1(x)| \left(1 + \frac{|x-y_0|}{r}\right)^\gamma$$

(4.10)

$$\leq C_N r^{-d} \int |x-z|^{-d+1}(1 + \frac{|x-z|}{r})^{-N+\gamma} V(z) \left(1 + \frac{|z-y_0|}{r}\right)^{-d+\gamma} dz.$$

Therefore, using the Minkowski integral inequality together with (4.3) and (4.6), we get

$$\left\| J_1(x) \left(1 + \frac{|x-y_0|}{r}\right)^\gamma \right\|_{L^q(dx)} \leq C r^{-d+d/q}. \tag{4.11}$$

In order to estimate $J_2(x)$ we use Lemma 3.7 and (4.1) with $n = d$ to obtain
\[ |J_2(x)| \left(1 + \frac{|x - y_0|}{r} \right)^\gamma \]
\[ \leq C \int_{r^2} \left(1 + \frac{|x - y_0|}{r} \right)^{\gamma} \int_0^{t/2} \int t^{-d/2} e^{-c|x-z|^2/t} V(z) \]
\[ \times K_s(z, y) r^{1-d} \left(1 + \frac{|y - y_0|}{r} \right)^{-d} dy dz ds \frac{dt}{t^{3/2}} \]
(4.12)
\[ \leq C \int_{r^2} \int t^{(2\gamma - d)/2} e^{-c|x-z|^2/t} V(z) \]
\[ \times |z - y|^{2-d} \left(1 + \frac{|z - y|}{\sqrt{t}} \right)^{-N+\gamma} r^{1-d-2\gamma} \left(1 + \frac{|y - y_0|}{r} \right)^{-d+\gamma} dy dz dt. \]

Setting \( N = \beta + \gamma \) with \( 0 < \gamma < \beta < 2 \) and applying the Minkowski integral inequality together with (4.4) and (4.6) we conclude that
\[ \left\| J_2(x) \left(1 + \frac{|x - y_0|}{r} \right)^\gamma \right\|_{L^q(dx)} \]
\[ \leq C \int_{r^2} \int t^{-(d+3-2\gamma-d/q)/2} V(z) \]
\[ \times |z - y|^{2-d} \left(1 + \frac{|z - y|}{\sqrt{t}} \right)^{-\beta} r^{1-d-2\gamma} \left(1 + \frac{|y - y_0|}{r} \right)^{-d+\gamma} dy dz dt \]
(4.13)
\[ \leq C \int_{r^2} \int t^{-(d+3-2\gamma-d/q)/2} V(z) \]
\[ \times |z - y|^{2-d} \left(1 + \frac{|z - y|}{r} \right)^{-\beta} \left(\frac{\sqrt{t}}{r}\right)^\beta r^{1-d-2\gamma} \left(1 + \frac{|y - y_0|}{r} \right)^{-d+\gamma} dy dz dt \]
\[ \leq C \int r^{-2d+2+d/q} V(z) \left(1 + \frac{|z - y_0|}{r} \right)^{-d+2+\gamma-\beta} dz \]
\[ \leq C r^{-d+d/q}. \]
By Lemma 3.7 and (4.1) with \( n = d \), we have
\[ |J_3(x)| \]
\[ \leq C \int_{r^2} \int \int_{\frac{t}{4}} t P_{t-s}(x-z) V(z) t^{-d} e^{-d|x-y|^2/r} \]
\[ \times \left(1 + \frac{|y - y_0|}{r} \right)^{-d} r^{1-d} dy dz ds \frac{dt}{t^{3/2}} \]
\[ \leq C_N \int_{r^2} \int |x - z|^{2-d} \left(1 + \frac{|x - z|}{\sqrt{t}} \right)^{-N} V(z) \]
\[ \times t^{-d} e^{-c|x-y|^2/t} \left(1 + \frac{|y - y_0|}{r} \right)^{-d} r^{1-d} dy dz ds \frac{dt}{t^{3/2}}. \]
Proof of Corollary 1.9. Assume that $g \in H^1_l$. Then, thanks to Theorem 1.6, there is $g \in H^1(\mathbb{R}^d)$ such that $f = L^{1/2}(-\Delta)^{-1/2}g$. By the characterization of the classical Hardy space $H^1(\mathbb{R}^d)$ by the Riesz transforms we have

\begin{equation}
\frac{\partial}{\partial x_j}L^{-1/2}f = \frac{\partial}{\partial x_j}L^{-1/2}L^{1/2}(-\Delta)^{-1/2}g = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}g \in L^1(\mathbb{R}^d).
\end{equation}

Conversely, assume that for $f \in L^1(\mathbb{R}^d)$ we have $\frac{\partial}{\partial x_j}L^{-1/2}f \in L^1(\mathbb{R}^d)$ for $j = 1, 2, \ldots, d$. Set $g = (-\Delta)^{1/2}L^{-1/2}f$. Then by Lemma 2.6, $g \in L^1(\mathbb{R}^d)$ and

\begin{equation}
\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}g = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}(-\Delta)^{1/2}L^{-1/2}f = \frac{\partial}{\partial x_j}L^{-1/2}f \in L^1(\mathbb{R}^d),
\end{equation}

with

\[\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}(-\Delta)^{1/2}L^{-1/2}f = \frac{\partial}{\partial x_j}L^{-1/2}f \in L^1(\mathbb{R}^d)\]
which implies that \( g \in H^1(\mathbb{R}^d) \). Consequently, by Theorem 1.6, \( f \in H^1_L \). Finally (1.10) can be deduced from (5.1), (5.2), and Theorem 1.6.

\[ \square \]

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