Nonlinear Dynamic Behaviors of the Arc Tooth Connected Rotor under Different Loads

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Abstract. Due to the wear of wheels during operation, the unbalance is induced. To study the nonlinear vibration of the arc tooth connected rotor under different loads, the motion equation was built based on Lagrange equation, considering contact property of the arc tooth and the oil-film force. From the GW rough surface model, the interface contact stiffness for the arc end-tooth was calculated. The eventual contact force was simulated by a nonlinear spring. Rich nonlinear dynamic phenomena, such as periodic doubling bifurcation phenomenon, multiperiodic motion, quasi periodic motion, oil whirl, etc., were observed. The results show distinct phenomena caused by the load variations. To reduce the nonsynchronous vibration, the relative phase is suggested to be kept at 120°.

Keyword: rod fastening rotor; arc end tooth; nonlinear dynamics; unbalance; contact stiffness; bifurcation; vibration characteristics; curvic coupling

1. Introduction

Mass unbalance is the main vibration source of rotating machinery, is also a variety of vibration trigger factors. It may cause the rotor deflection, make the machine produce noise, and accelerate the wear of bearing. If not effectively controlled, the working efficiency is gradually decreased. Different wear position between disks will make the unbalance load different.

Most of gas turbines and aero-engines adopt the disc-rod-fastening rotor structure for rotor design [1-2]. The disks are compressed by a set of pre-tightening tie rod bolts and then connected into one body. Structural discontinuous feature makes the vibration characteristics more complex [3-4]. Hu et al. [5] studied the nonlinear vibration of the circumferential rod fastened rotor with rub-impact, where the contact effect between the disk and the rod was equivalent as a spring. Hei et al. [6] researched the nonlinear bifurcation of the tilting pad bearing-rod fastened rotor. Wang et al. [7-8] introduced the nonlinear stiffness into the dynamic equation due to the disk contact. On this basis, the stability was investigated through multiple methods. Meng et al. [9] proposed a method that considers the tooth contact between the disks. The equivalent stiffness correction coefficient was obtained through ANSYS, and the central rod rotor was established to analyze the natural frequency. Jam et al. [10] equivalent the stiffness and mass of the rod to the outer wheel disk. Cheng Li et al. [11] found that an engine has a sudden vibration phenomenon.

Most scholars study the dynamics of the combined rotor for the eccentricity size. The influence of various loads on the rotor response has not performed sufficient studies. The previous references all show that eccentricity can change the rotor response behavior. It is important to know the load effect for this special rotor with arc end-tooth.
2. Rod Fastening Rotor Model

The discs are connected by the arc end-tooth through a pre-tightening central rod and a nut. The rotor schematic diagram is depicted in figure 1. The mathematical model is simplified with some assumptions.

![Diagram of a rotor](image)

**Figure 1.** Schematic diagram of an arc end-tooth connected rotor.

(a) The rotor is discrete of 4 nodes, and the shaft is considered massless.
(b) Two bearings are identical in structure and lubrication.
(c) The nonlinearity of arc end-tooth contact is equivalent to the nonlinear flexible spring.

2.1. Stiffness Model of Contact

Based on the GW rough surface model, stiffness formulas for an arc end-tooth in the normal and tangential directions [12] was derived as follows:

\[
\begin{align*}
k_n &= \frac{F}{\sigma} \frac{\sin \theta}{\mu \cos \theta + \sin \theta}, \\
k_t &= \frac{F}{\sigma} \frac{\cos \theta}{\mu \cos \theta + \sin \theta},
\end{align*}
\]

where, \( F \) is the pre-tightening force of rod, \( \theta \) is the pressure angle of arc end-tooth, \( \sigma \) is the roughness, and \( \mu \) is the friction coefficient. After obtaining the surface contact stiffness and the body stiffness, the combined stiffness of the arc end tooth section considering the contact effect can be obtained.

\[
\frac{1}{K_{\text{combined}}} = \frac{1}{K_{\text{contact}}} + \frac{1}{K_{\text{body}}}
\]

In this paper, to consider the effect of the arc end-tooth, the equivalent beam element is used to replace the linking stiffness, only considering the direct lateral stiffness. The elastic modulus of the equivalent element can be calculated through \( E=KL/A \).

Then, the equivalent flexural stiffness of the section for arc end-tooth is \( K_e=12EJ/L^3 \). The nonlinear nature of the contact is simulated by the third power function. Therefore, the combined stiffness is

\[
\begin{pmatrix}
F_{x1} \\
F_{x2} \\
F_{y1} \\
F_{y2}
\end{pmatrix} =
\begin{pmatrix}
k_x(x_2-x_3) + k^3(x_2-x_3)^3 \\
k_x(y_2-y_3) + k^3(y_2-y_3)^3
\end{pmatrix}
\]

2.2. System Governing Equations

Based on the Lagrange equation [13], the governing equations for the arc end-tooth connected rod fastening rotor system are derived as follows:
\[
\begin{align*}
    m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 (x_1 - x_2) - F_x(x_1, y_1, \dot{x}_1, \dot{y}_1) &= 0 \\
    m_2 \ddot{y}_1 + c_1 \dot{y}_1 + k_1 (y_1 - y_2) - F_y(x_1, y_1, \dot{x}_1, \dot{y}_1) + m_1 g &= 0 \\
    m_2 \ddot{x}_2 + c_2 \dot{x}_2 + c_2 (x_2 - \dot{x}_2) + k_2 (x_2 - x_1) + F_{\alpha 1} - m_2 e_2 \omega^2 \cos \omega t = 0 \\
    m_2 \ddot{y}_2 + c_2 \dot{y}_2 + c_2 (y_2 - \dot{y}_2) + k_2 (y_2 - y_1) + F_{\alpha 1} - m_2 e_2 \omega^2 \sin \omega t + m_2 g &= 0 \\
    m_3 \ddot{x}_3 + c_3 \dot{x}_3 + c_3 (x_3 - \dot{x}_3) + k_3 (x_3 - x_2) + F_{\alpha 1} - m_3 e_3 \omega^2 \cos (\omega t + \phi) = 0 \\
    m_3 \ddot{y}_3 + c_3 \dot{y}_3 + c_3 (y_3 - \dot{y}_3) + k_3 (y_3 - y_2) + F_{\alpha 1} - m_3 e_3 \omega^2 \sin (\omega t + \phi) + m_3 g &= 0 \\
    m_4 \ddot{x}_4 + c_4 \dot{x}_4 + k_4 (x_4 - x_3) - F_y(x_1, y_1, \dot{x}_1, \dot{y}_1) = 0 \\
    m_4 \ddot{y}_4 + c_4 \dot{y}_4 + k_4 (y_4 - y_3) - F_y(x_1, y_1, \dot{x}_1, \dot{y}_1) + m_4 g &= 0 
\end{align*}
\]

where \( m \) denotes the lumped mass. \( k_s \) is the stiffness of shaft section. \( c \) represents the damping. \( F_x \) and \( F_y \) are the oil-film forces from Capone [14]. \( e \) is the eccentric offset, and \( e_2 = e_3 = 0.02 \) mm. \( \phi \) is the relative phase.

Equation 5 can be rewritten as the dimensionless motion equation through dimensionless replacement, and the matrix form is shown as follows:

\[
X^* + \frac{C}{M \omega^2} X^* + \frac{K}{M \omega^2} X = \frac{F}{M \omega^2 \epsilon} + \frac{g}{\omega^2 \epsilon},
\]

### Table 1. System Parameters.

| Property       | Value | Property       | Value |
|----------------|-------|----------------|-------|
| Mass \( m_1, m_2/\text{kg} \) | 4.0   | Stiffness \( k_s, k_f/\text{N/m} \) | 2.5e+7 |
| Mass \( m_2, m_3/\text{kg} \) | 32.1  | Stiffness \( k_r/\text{N/m^3} \) | 2.5e+7 |
| Damping \( c_1/(\text{N-s/m}) \) | 1050  | Length \( L/\text{mm} \) | 12 |
| Damping \( c_2, c_3/(\text{N-s/m}) \) | 2100  | Clearance \( c/\text{mm} \) | 0.11 |
| Radius \( R/\text{mm} \) | 25    | Viscosity \( \eta/(\text{Pa-s}) \) | 0.018 |

3. **Vibration Characteristics under Different Loads**

The Runge-Kutta method [15-16] is employed to solve equation (5). The rotor parameters are listed in Table 1. The load is taken as control parameter. The nonlinear dynamic behavior for different loads at the operation speed of 1800 rad/s is investigated, in which the considering speed is above the 2nd critical speed and within the whip instability region [17], to know the complex nonlinear phenomena of the rotor at high speed.

![Figure 2](attachment:figure2.png)

(a) Bifurcation diagram  
(b) Vibration response

**Figure 2.** Bifurcation diagram and vibration response at 1# bearing.
Figure 3. Vibration response at $\phi = 100^\circ$.

Figure 4. Waterfall diagram of $X_1$ at $\phi \in [0^\circ, 180^\circ]$.

Figure 5. Rotor orbits at different loads.
Figure 6. Poincaré map at different loads.

Figure 2a and figure 2b are the bifurcation diagram and vibration response of dimensionless displacement $X_1$ under varying loads, from 0 rad to $2\pi$ rad. The system response keeps quasi-periodic motion at $\phi < 120^\circ$. As shown in figure 2a, the system behaves quasi-periodic motion, where a closed cycle in Poincaré map and some discrete frequencies in spectrum for 100°, and the oil whip of frequency $fn1$ dominates the motion. When $\phi$ increases to 120°, the system will take place tangential bifurcation and enter into periodic-3 motion at $\phi \in [120^\circ, 128^\circ]$. And then, the response returns to the quasi-periodic motion when $\phi > 128^\circ$.

All of vibration RMS value, single peak value and maximus value continuously follow a downtrend with the increase in $\phi$, but the drop is small, as shown in figure 3. Surprisingly, the dimensionless vibration RMS level of 120° is decreased by 0.14 compared to that of 128°, which reduces by almost 20.14 %. Hence, a suitable $\phi$ for the existed unbalance can reduce the rotor vibration to some extent, even at instability zone.

Figure 4 shows a waterfall from 0° to 180°, it shows that the complicated frequency components always exist in every $\phi$, which include whirl frequency of $fn1$ and $fn2$, rotating frequency of $fr$, and the combined frequencies, such as $fr$-$2fn1$, $2fn1$, $fr$-$fn1$, $2fr$-$4fn1$, $fr$+$fn1$, $2fr$-$2fn1$, $2fr$-$2fn1$+$fn2$, $2fr$-$fn1$, etc. Moreover, the frequencies will occur a special bifurcation phenomenon at 120°, or at 128°, which confirm the quasi-periodic motion is mainly caused by the frequencies of $fr$-$2fn1$ and $fr$+$fn2$. At the speed in the whip instability region, the response and the onset speed of instability are not sensitive to $\phi$ due to the motion controlled by subsynchronous.

The orbit and the Poincaré map are shown in figure 5 and figure 6, respectively. The orbits show that the journal motion may be from the aperiodic to the periodic at some certain loads, such as 120°. The change from a circular ring to three isolated points in the Poincaré map are also occurs at a certain load.

Above all, it is not recommended to design a rod fastening rotor within the instability region, such as 1800 rad/s for this investigated rotor, due to the irregular rotor orbits and the unstable motion pattern with large amplitude.

4. Conclusions

Based upon the GW model, the Capone oil film model and Lagrange Equation, the mathematical model of the arc end-tooth connected rod fastening rotor system is built, to investigate the nonlinear vibration performance under different loads. In comparison with the response under different loads, the result of 120° shows that the system can perform period-3 motion, while perform quasi-periodic motion at the other loads; In addition, the vibration is small at 120°, which reduces by almost 20.14 % compared to 128°. A suitable load from existed unbalance can reduce the vibration to some extent,
even in the instability region. The research helps to further understand the vibration behavior of an arc end-tooth connected rod fastening rotor.

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