Collision Avoidance for Dynamic Obstacles with Uncertain Predictions using Model Predictive Control

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Abstract—We propose a Model Predictive Control (MPC) for collision avoidance between an autonomous agent and dynamic obstacles with uncertain predictions. The collision avoidance constraints are imposed by enforcing positive distance between convex sets representing the agent and the obstacles, and tractably reformulating them using Lagrange duality. This approach allows for smooth collision avoidance constraints even for polytopes, which otherwise require mixed-integer or non-smooth constraints. We consider three widely used descriptions of the uncertain obstacle position: 1) Arbitrary distribution with polytopic support, 2) Gaussian distributions and 3) Arbitrary distribution with first two moments known, and obtain deterministic reformulations of the collision avoidance constraints. The proposed MPC formulation optimizes over feedback policies to reduce conservatism in satisfying the collision avoidance constraints. The proposed approach is validated using simulations of traffic intersections in CARLA.

I. INTRODUCTION

Autonomous vehicle technologies have seen a surge in popularity over the last decade, with the potential to improve flow of traffic, safety and fuel efficiency [1]. While existing technology is being gradually introduced into structured scenarios such as highway driving and low-speed parking, autonomous driving in urban settings is challenging in general, due to the uncertainty in surrounding agents’ behaviours. Significant research has been devoted to building predictive models for these behaviours, to provide nominal trajectories for these agents and also characterize the uncertainty in deviating from the nominal predictions [2].

In this work, we use these uncertain predictions of the surrounding agents (denoted as obstacles) to design a planning framework for the controlled agent for collision avoidance. Our main focus is to design a planner that can solve the planning problem 1) efficiently (measured as time to compute the solution, critical for real-time deployment), and 2) reliably (measured as low rate of infeasibility, critical for fewer interventions of backup planners). We investigate this problem in the context of constrained optimal control and use Model Predictive Control (MPC), the state-of-the-art technique for real-time optimal control [3], [4].

MPC is a popular technique for real-time collision avoidance for autonomous driving [5], [6] and robotics [7]. A typical MPC algorithm computes control inputs by solving a finite horizon, constrained optimal control problem in a receding horizon fashion. The collision avoidance problem involves checking if there is a non-empty intersection between two general sets (corresponding to the geometries of the agent and obstacle). This is a non-convex problem and NP-hard in general [8], and application-specific simplifications are commonly used for the improving the tractability of the resulting optimal control problem. The most common simplifications involve convexifying one or both of the sets as a 1) point, 2) affine space, 3) sphere, 4) ellipsoid or 5) polytope. Combinations of 1)-4) are convenient for their simplicity and computational tractability, but tend to be conservative since the shape of an actual car is not well represented by such sets. Polytopes offer compact representations for obstacles in tight environments and are popular in autonomous driving applications, but using them in the collision avoidance problem results in non-smooth constraints, which require specialized solvers or mixed-integer reformulations. The authors of [9] consider the dual perspective of collision avoidance for static obstacles; checking for existence of a separating hyperplane between two sets. This perspective provides smooth collision checking conditions for convex sets (including polytopes) by introducing additional dual variables.

We use this dual formulation in the MPC for avoiding collisions with dynamic obstacles, in the presence of prediction uncertainty in the pose of the agent and obstacles. The MPC finds an optimal sequence of parameterized agent-obstacle state feedback policies. Compared to optimizing over open-loop sequences, optimizing over feedback policies enhances feasibility of the MPC optimization problem due to the ability to react to different trajectory realizations of the agent and obstacles along the prediction horizon. For computational efficiency, we convexify the problem for finding optimal solutions or detecting infeasibility quickly. The work [10] is the closest to our approach, which proposes a nonlinear Robust MPC scheme that optimizes over policies and uses the dual collision avoidance formulation with safety guarantees (under assumptions of a safety region and a backup policy). However, their policies do not account for feedback from the obstacles, and their formulation is specific to uncertainty distributions with polytopic support. The contributions of this work are summarised as follows:

- MPC formulations for collision avoidance with dynamic obstacles, for three prediction uncertainty descriptions: 1) Arbitrary distributions with polytopic support, 2) Gaussian distributions, and 3) Arbitrary distributions with first two moments known.
- Deterministic reformulations of collision avoidance between the agent and uncertain, dynamic obstacles, with convex geometries in closed-loop with a parameterized feedback policy over the agent’s and obstacles’ states.
II. PROBLEM FORMULATION

Notation: The index set $\{k_1, k_1 + 1, \ldots, k_2\}$ is denoted by $I_{k_2}^{k_1}$. For a proper cone $K$ and $x, y \in K$, we have $x \succeq_K y \Leftrightarrow x - y \in K$. The dual cone of $K$ is given by the convex set $K^* = \{y | y^T x \geq 0, \forall x \in K\}$, $\parallel \cdot \parallel_p$ denotes the $p$-norm. $\otimes$ denotes the Kronecker product.

A. Dynamics and Geometry of Agent and Obstacles

Consider a controlled agent described by a linear time-varying discrete-time model

$$x_{t+1} = A_t x_t + B_t u_t + E_t w_t, \quad p_t = C x_t + e_t \tag{1}$$

where $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$, $w_t \in \mathbb{R}^{n_w}$ are the state, input and process noise respectively and $A_t, B_t, E_t, C, e_t$ are the system matrices at time $t$. The vector $p_t \in \mathbb{R}^{n_p}$ describes the position of the autonomous agent in a global, Cartesian coordinate system. Given the rotation matrix $R_t$ describing the orientation of the autonomous agent (with respect to the global coordinate system) at time $t$, define the space occupied by the autonomous agent as the set

$$S_t(x_t) = \{z \in \mathbb{R}^n | \exists y \in \mathbb{R}^n : G y \succeq_K g, z = R_t y + p_t\} \tag{2}$$

where $p_t = C x_t + e_t, G \in \mathbb{R}^{l \times n}, g \in \mathbb{R}^l$ and $K \subset \mathbb{R}^l$ is a closed convex cone with non-empty interior. The set $\{y | G y \succeq_K g\}$ is non-empty, convex and compact, and describes the space occupied by the un-oriented agent at the origin, and can denote various shapes for an appropriate choice of $K$ (e.g., a polytope when $K$ is the positive orthant or an ellipsoid when $K$ is the second-order cone).

Now suppose that there are $M$ obstacles, each described by the affine time-varying discrete-time dynamics

$$o_{i+1}^{t} = T_{i} o_{i}^{t} + q_{i}^{t} + F_{i}^{t} n_{i}^{t}, \quad p_{i}^{t} = C o_{i}^{t}, \quad \forall i \in I_{M}^{t} \tag{3}$$

where $o_{i}^{t} \in \mathbb{R}^{n_{o}}, p_{i}^{t} \in \mathbb{R}^{n_{o}}, n_{i}^{t} \in \mathbb{R}^{n_{w}}$ are the state, position and process noise respectively and $T_{i}^{t}, q_{i}^{t}, F_{i}^{t}, C^{t}$ are the system matrices of the $i^{th}$ obstacle at time $t$. Now suppose that the orientation $R_{i}$ of the $i^{th}$ obstacle at time $t$ is given, and define the space occupied by the obstacle as the set

$$S_{i}^{t}(o_{i}^{t}) = \{z \in \mathbb{R}^n | \exists y \in \mathbb{R}^n : G y \succeq_K g, z = R_{i} y + p_{i}^{t}\} \tag{4}$$

where $p_{i}^{t} = C_{i} o_{i}^{t}$ is the non-empty, convex and compact set $\{y | G y \succeq_K g\}$ describes the un-oriented shape of the obstacle at the origin. We also introduce the notation $o_{t} = [o_{1}^{T}, \ldots, o_{M}^{T}]^{T}, n_{t} = [n_{1}^{T}, \ldots, n_{M}^{T}]^{T}$ to denote the stacked obstacle state and process noise vectors at time $t$, and $T_{t} = \text{blkdiag}(T_{1}^{t}, \ldots, T_{M}^{t}), F_{t} = \text{blkdiag}(F_{1}^{t}, \ldots, F_{M}^{t}), q_{t} = [q_{1}^{T}, \ldots, q_{M}^{T}]^{T}$ to define the combined obstacle dynamics as $o_{t+1} = T_{t} o_{t} + q_{t} + F_{t} n_{t}.$

B. Uncertainty Description

The presence of process noises $w_t$ and $n_t$ in the dynamics of the controlled agent (1) and the obstacles (3) adds uncertainty in the prediction of their state trajectories. In this paper, we consider three different descriptions of the process noise distributions as follows:

- **D1**: The joint process noise $[w_{t}^{T}, n_{t}^{T}]^{T}$ are i.i.d. $\forall t \geq 0$ and are given by an unknown distribution with compact support, $[w_{t}^{T}, n_{t}^{T}]^{T} \in \mathcal{D} = \{d \in \mathbb{R}^{(1+M)n_{x}} | \|\Gamma d\|_{\infty} \leq \gamma\}$ for $\gamma > 0$ and non-singular $\Gamma$.

- **D2**: The joint process noise $[w_{t}^{T}, n_{t}^{T}]^{T}$ are i.i.d. $\forall t \geq 0$ and Gaussian $[w_{t}^{T}, n_{t}^{T}]^{T} \sim \mathcal{N}(0, \Sigma).

- **D3**: The joint process noise $[w_{t}^{T}, n_{t}^{T}]^{T}$ are i.i.d. $\forall t \geq 0$ and are given by an unknown distribution with known mean, $E([w_{t}^{T}, n_{t}^{T}]^{T}) = 0$ and covariance, $E(([[w_{t}^{T}, n_{t}^{T}]^{T}(([w_{t}^{T}, n_{t}^{T}]^{T})]) = \Sigma.$

C. Model Predictive Control Formulation

We aim to design a state-feedback control $u_t = \pi(x_t, o_t)$ for the controlled agent such that it avoids collisions with the obstacles while respecting polytopic state-input constraints given by $\mathcal{X}_t = \{x, u | F_{j}^{x} x + F_{j}^{u} u \leq f_{j}, \forall j \in I_{n}^{t}\},$ where $F_{j}^{x} \in \mathbb{R}^{l \times n_{x}}, F_{j}^{u} \in \mathbb{R}^{l \times n_{u}}, f_{j} \in \mathbb{R}^{l} \forall j \in I_{n}^{t}.$ We propose to compute the feedback control using MPC, by solving the following finite-horizon constrained optimal control problem.

$$\text{OPT}_t(\mathcal{D} \in \{D_1, D_2, D_3\}) : \min_{\theta_t} J_t(x_t, u_t) \tag{5a}$$

s.t. $x_{k+1} = A_k x_{k} + B_k u_{k} + E_k w_{k}, \quad o_{k+1} = T_k o_{k} + q_{k} + F_k n_{k}, \tag{5b}$

$u_t = \Pi_{t}(x_t, o_t), \tag{5c}$

$x_{t+1} = x_t, \quad o_{t+1} = o_t, \tag{5d}$

$\forall k \in I_{n+1}^{t} \tag{5e}$

where $x_t = [x_{1}^{T}, \ldots, x_{M}^{T}]^{T}$ (similar notation for $o_t$) and $u_t = [u_{1}^{T}, \ldots, u_{M+1}^{T}]^{T}$ (similar notation for $w_t, u_t$). The feedback control is given by the optimal solution of (5) as

$$u_t = \pi_{\text{MPC}}(x_t, o_t) = u_{\text{opt}} \tag{6}$$

where the feedback over the agent’s and obstacles’ states starts as (5g). The objective (5a) penalizes deviations of the agent’s trajectory from a desired reference. The collision avoidance constraints, and state-input constraints along the prediction horizon are summarised as $C(\mathcal{D})$ in (5e), and depend on the uncertainty description assumed in (5d). In (5f), the control inputs $u_t$ along the prediction horizon are given by a parameterized policy class that depends on predictions of the agent’s and obstacles’ trajectories. We solve (5) for the MPC (6) in batch form by explicitly substituting for the equality constraints (5b), (5c) and optimize over the policy parameters $\theta_t$.

III. COLLISION AVOIDANCE FOR DYNAMIC OBSTACLES WITH UNCERTAINTY PREDICTIONS

In this section, we detail MPC formulations for avoiding collisions with uncertain and dynamically moving obstacles. Section III-A describes our policy parameterization for (5f). In III-B, we derive a continuous reformulation of the collision avoidance problem $S_k(x_{k|t}) \cap S_k(o_{k|t}) = \emptyset, \quad \forall t$, keeping $x_{k|t}, o_{k|t}$ fixed. Then we introduce the prediction uncertainties in $x_{k|t}, o_{k|t}$, and derive deterministic reformulations of the collision avoidance constraints and state-input constraints for each uncertainty description in III-C. Section III-D describes the MPC cost function, and the MPC design is consolidated in III-E. Proofs and additional notation are deferred to the Appendix of [11] in the interest of space.
A. Policy Parameterization

In (5f), we use parameterised feedback policies \( \Pi_{\theta_t}(x_t, o_t) \) for the control actions \( u_t \) (as in (5f) along the prediction horizon. Consider the following input policy for time \( k \),

\[
    u_{k|t} = h_{k|t} + \sum_{i=t}^{k-1} M_i x_{k|t} \|w_i\| + K_i \|o_{k|t} - o_k\| \tag{7}
\]

which uses state feedback for the obstacles’ states but affine disturbance feedback for feedback over the agent’s states (cf. [12] for equivalence of state and disturbance feedback). The nominal states \( o_k \) are obtained as \( o_{k+1} = T_k o_k + q_k \) \( \forall k \in \mathbb{T}_t^{i+N-1} \), with \( o_0 = o_t \).

In Appendix A of [11], we define the matrices \( A_t, B_t, E_t \) to express the agent’s trajectory as a function of \( (x_t, u_t, w_t) \) as \( x_t = A_t x_t + B_t u_t + E_t w_t \). Similarly, the matrices \( T_t, q_t, F_t \) give the obstacles’ trajectory as a function of \( (o_t, u_t) \) as \( o_t = T_t o_t + q_t + R_t u_t \). The matrices \( h_t, M_t, K_t \) define the control policies along the prediction horizon as \( u_t = \Pi_{\theta_t}(x_t, o_t) = h_t + M_t w_t + K_t F_t u_t \), parameterized by \( \theta_t = (h_t, M_t, K_t) \).

Note that although \( o_t \) doesn’t necessarily depend on \( x_t \) (\( n_t \) may be independent from \( w_t \)), the policies \( \Pi_{\theta_t}(x_t, o_t) \) modify the distribution of \( x_t \) in response to \( o_t \). Solving (5) over open-loop sequences (i.e., \( \Pi_{\theta_t}(x_t, o_t) = h_t \)) can be conservative because the agent-obstacle trajectories \( (x_t, o_t) \) from a single control sequence \( u_t = h_t \) must satisfy all the constraints regardless of the realizations of \( w_t, n_t \).

B. Collision Avoidance Reformulation by Dualization

Given the states \( x_{k|t} \), \( o_{k|t} \) of the agent and \( j \)th obstacle respectively, the collision avoidance constraint is given by \( S_k(x_{k|t}) \cap \mathbb{S}_k(o_{k|t}) = \emptyset \). This can be equivalently expressed as \( \text{dist}(S_k(x_{k|t}), \mathbb{S}_k(o_{k|t})) > 0 \) where \( \text{dist}(S_k(x_{k|t}), \mathbb{S}_k(o_{k|t})) \) is defined as the solution of the convex optimization problem

\[
    \min \{ |x_1 - z_2| \geq \delta |z_1 \in S_k(x_{k|t}), z_2 \in S_k(o_{k|t}) \} = \min \{ \|z_1 - z_2\| \geq G_k R_k^T (z_1 - p_{k|t}) \leq \kappa g, G_k R_k (z_1 - p_{k|t}) \leq \kappa g \} \tag{8}
\]

In the following proposition, we use the above formulation (8) and Lagrange duality to express the set intersection problem \( S_k(x_{k|t}) \cap \mathbb{S}_k(o_{k|t}) = \emptyset \) as a convex feasibility problem of finding a separating hyperplane.

Proposition 1: Given the state and orientation of the agent \( x_{k|t}, R_k \), and state and orientation of the \( j \)th obstacle \( o_{k|t}, R_k \) at the \( k \)th prediction time step, we have

\[
    \text{dist}(S_k(x_{k|t}), \mathbb{S}_k(o_{k|t})) > 0 \nonumber \tag{9}
\]

and substituting for \( \mu = \lambda_{k|t}^T G_k R_k^T \), we thus get \( \mu^T z_1 - \mu^T z_2 \geq \lambda_{k|t}^T (G_k R_k (p_{k|t} - p_{k|t} + g)) - \lambda_{k|t}^T g > 0 \).

Next, we reformulate (9) to address the non-determinism arising from the uncertainty in positions \( p_{k|t} = C x_{k|t} + c_k \), \( p_{k|t} = C o_{k|t} \) along the prediction horizon due to \( w_t, n_t \).

C. Deterministic Constraint Reformulation

Deterministic reformulations for the collision avoidance constraints (9) together with the state-input constraints \( \mathcal{XU} = \{ (x, u) | F x + F^T u \leq f_j \} \) for the state predictions \( x_t, o_t \) in closed-loop with (7), is presented next for each uncertainty description: D1, D2 and D3.

We use the constant matrices \( S_x, S_o, \mathbb{S}_x, \mathbb{S}_o \) such that \( S_x x_t = x_{k|t}, S_o u_t = u_{k|t}, S_o w_t = w_{k|t}, S_o n_t = o_{k|t} \). Let \( \mathbb{P}_v \) be a permutation matrix such that \( [w_t^T n_t^T] = \mathbb{P}_v v_t \) where \( v_t = [w_{t+1}^T n_{t+1}^T ... w_{t+N-1}^T n_{t+N-1}^T] \). Also, define \( \lambda_{k|t} = [\lambda_{k|t}^T ... \lambda_{k|t}^T] \), \( \lambda_0 = [\lambda_0^T ... \lambda_0^T] \) (similarly for \( v_{k|t}, \mu_{v} \)). Given a sequence of noise realisations \( (w_t, n_t) \), define the set of feasible agent-obstacles joint realizations \( (x_t, u_t, o_t) \) in the lifted-space \( (x_t, u_t, o_t) \) as

\[
    S_l(w_t, n_t) = \begin{bmatrix}
    \{ \text{Collision avoidance constraints} \} \\
    \{ \text{State-input constraints} \} \\
    \{ \text{Agent & obstacles’ predictions} \}
\end{bmatrix}
\]

We now express the reformulations for the considered uncertainty descriptions using this set.

1) Robust Formulation for Uncertainty Description D1:

We seek to tighten the obstacle avoidance constraints, and state-input constraints to find \( u_t \) such that the tuple \( (x_t, u_t, o_t) \) satisfies the aforementioned constraints for all realisations of \( [w_t^T n_t^T] \in D, \forall k \in \mathbb{T}_t^{i+N-1} \). We can write this formally as

\[
    C(D1) = \bigcap_{w_t, n_t \in D} S_l(w_t, n_t) \tag{11}
\]

where \( D^N = \{ w_t \} \|w_t\| \leq \gamma, \Gamma = \mathbb{I}_N \otimes \Gamma \).

2) Chance Constraint Formulation for Uncertainty Description D2:

For uncertainty description \( w_t^T n_t^T \sim \mathcal{N}(0, \Sigma) \), i.i.d. \( \forall t \geq 0 \). Since the uncertainties have unbounded support, we adopt a chance constrained formulation, where for some \( 0 < \varepsilon << 1 \), we find \( u_t \) such that the tuple \( (x_t, u_t, o_t, \lambda_t, \mu_t) \) satisfies the obstacle avoidance constraints (9) and state-input constraints with probability greater than \( 1 - \varepsilon \), given that \( [w_t^T n_t^T] \sim \mathcal{N}(\mu, \Sigma) \), \( \forall k \in \mathbb{T}_t^{i+N-1} \). Formally, we write this set as

\[
    C(D2) = \bigcap_{w_t, n_t \in D} S_l(w_t, n_t) \geq 1 - \varepsilon \tag{12}
\]
where the probability measure $\mathbb{P}(\cdot)$ is over $v_t = P^\top [w^\top n^\top]^\top$, and constructed as the product measure of $N$ i.i.d. Gaussian distributions $N(0, \Sigma)$.

3) Distributionally Robust Formulation for Uncertainty Description D3: For uncertainty description D3, we have that $[w^t, n^t]^\top$ are i.i.d. $\forall t \geq 0$ and have known mean and covariance, $E[[w^t, n^t]^\top] = 0$, $E[[w^t, n^t]^\top [w^t, n^t]^\top] = \Sigma$. Denote the mean and covariance of the stacked random variables $v_t$ as $0 = [0 \ldots, 0]^\top$, $\Sigma = I_N \otimes \Sigma$. Now define the ambiguity set $[13]$ as $\mathcal{P} = \{\text{probability distributions with } E[v_t] = 0, E[v_t v_t^\top] = \Sigma \}$. We adopt a distributionally robust, chance constrained formulation, where for some $0 < \epsilon << 1$, we find $u_t$ such that the tuple $(x_t, u_t, o_t, \lambda_t, \nu_t)$ satisfies the obstacle avoidance constraints (9) and state-inout constraints with probability greater than $1 - \epsilon$, for all probability distributions in $\mathcal{P}$. Formally, we write this set as

$$C(D3) = \left\{ \begin{array}{l} x_t \\ o_t \\ \lambda_t \\ u_t \\ \nu_t \end{array} \right| \inf_{P \in \mathcal{P}}, \forall v_t \sim P E_t(w_t, n_t) \geq 1 - \epsilon \}.$$ (13)

The next theorem provides deterministic reformulations of the constraint sets presented above, and establishes the feasible set of (5) in terms of the policy parameters $\theta_t = (h_t, M_t, K_t)$ in (7) and Lagrange multipliers $\lambda_t, \nu_t$ in (9).

**Theorem 1:** For the agent (1) in closed-loop with policy (7) and obstacles modelled by (3), define the following functions $\forall k \in \mathcal{T}^t_{i+1}, \forall i \in \mathcal{I}^t_i, \forall j \in \mathcal{I}^t_j$ in the dual variables $L_t, \nu_t$ and policy parameters $\theta_t = (h_t, M_t, K_t)$:

$$Y^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j}) = -\lambda^T_{k|j} g - \nu^T_{k|j} g - \lambda^T_{k|j} G R_{k|j} c_t - L^T_{k|j} G R_{k|j} C(S^T_{k|j}(A_i x_t + B h_t) - S^T_{k|j} (T_i o_t + q_t)), \tag{14}$$

$$Z^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j}) = \lambda^T_{k|j} G R_{k|j} C \left( \left( S^T_{k|j} (B_i M_t + E_t) \right) - \left( F^T_{k|j} S_{k|j} B_i K_t - S_{k|j} M_i F_t \right) \right) \tag{15}$$

$$J_t(x_t, u_t) = ||Q(x_t^{ref} - x_t)||^2 + ||R(u_t^{ref} - u_t)||^2. \tag{20}$$

**E. Convexified MPC Formulation**

We linearize $Y^i_{k|j}(\cdot), Z^i_{k|j}(\cdot) \forall k \in \mathcal{T}^t_{i+1}, i \in \mathcal{I}^t_i$ from (14), (15) for time $t$, about the previous solution $\theta^*_{t-1}, L^*_{t-1}$ to get affine functions $LY^i_{k|j}(\cdot), LZ^i_{k|j}(\cdot)$ given by

$$LY^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j}) = Y^i_{k|j}(\theta^*_{t-1}, \lambda^*_{k|j-1}, \nu^*_{k|j}), \tag{16}$$

$$LZ^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j}) = Z^i_{k|j}(\theta^*_{t-1}, \lambda^*_{k|j-1}, \nu^*_{k|j}). \tag{17}$$

When $\gamma > 0, \epsilon < \min \{NM, NJ\}$, the constraints

$$LY^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j}) > \gamma \|LY^i_{k|j}(\theta^*_{t-1}, \lambda^*_{k|j-1}, \nu^*_{k|j})\|_1,$$

$$LZ^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j}) > \gamma \|LZ^i_{k|j}(\theta^*_{t-1}, \lambda^*_{k|j-1}, \nu^*_{k|j})\|_1$$

are second-order cone (SOC) representable (LP representable in the first case) because the composition of a SOC constraint with an affine map is still an SOC constraint. Substituting these affine functions in the definitions of $F_t(D_1), F_t(D_2), F_t(D_3)$, the resulting constraints $F_t(D_1), F_t(D_2), F_t(D_3)$ are convex because of 1) convexity of the dual cone $K^*$, 2) convexity of constraints $\|\lambda^T_{k|j} G R_{k|j}\|_2 \leq 1, \lambda^T_{k|j} G R_{k|j} = -\nu^T_{k|j} g - \lambda^T_{k|j} G R_{k|j} c_t$, and 3) convexity of constraints $\gamma \|Z^i_{k|j}(\theta_t, \lambda^i_{k|j}, \nu^i_{k|j})\|_1, \gamma \|Y^i_{k|j}(\theta_t)\|_1$ are affine. The resulting MPC optimization problem for either description D1, D2 or
D3 is given by the convex optimization problem:
\[
\min_{h_t, K_t, M_t, \lambda_t, \nu_t} J_t(\bar{x}_t, \bar{u}_t)
\]
\[
s.t \quad \bar{x}_t = A_t x_t + B_t h_t, \quad \bar{u}_t = h_t, \quad \{h_t, K_t, M_t, \lambda_t, \nu_t\} \in F_t(D).
\]

When the cone $K$ is given by the positive orthant (for polytopic shapes) or the second-order cone (for ellipsoidal shapes), the optimization problem (21) is given by a second-order cone program which can be efficiently solved. The optimal solution to (21) is used to obtain the control action $u^*_t$ given by (7).

Remark 1: The feasible set of (21) is not a convex inner-approximation of the original problem with $F_t(D)$ from Theorem 1. However, at the cost of introducing several new variables, a convex inner-approximation can be obtained by enforcing the collision avoidance constraints for all points in the convex relaxation of the bilinear equalities (14), (15) given by McCormick envelopes. An investigation along these lines is left for future research.

IV. SIMULATIONS

In this section, we demonstrate our MPC formulation via two numerical examples of a traffic intersection: 1) A longitudinal control example comparing the MPC formulations for each uncertainty description $D \in \{D_1, D_2, D_3\}$, and 2) An unprotected left turn in CARLA, comparing the proposed approach against [14] to highlight the benefit of the proposed collision avoidance formulation.

A. Longitudinal Control Example

1) Models and Geometry: We simulate an autonomous vehicle as the controlled agent and $M = 2$ surrounding vehicles as obstacles at a traffic intersection as in Figure 1. The vehicles are modelled as 4.8m x 2.8m rectangles, and their dynamics are given by Euler-discretized double integrator dynamics with $\Delta t = 0.1s$, states: $s$ (longitudinal position), $v$ (speed), control input: $a$ (acceleration) and keep the lateral coordinate constant. For obstacle predictions (3), we use forecasts of the acceleration inputs for each obstacle.

2) Process Noise Distribution: We model $[w^T_t, n^T_t]$ as a product of 6 (2 for agent, 2x2 for obstacles) independent uni-variate truncated normal random variables (cf. [6], [7]), $\text{truncNorm}(\mu, \sigma, a, b)$, with $\mu = 0, a = -2, b = 2$ and $\sigma = 0.01$ for $w_t$, $\sigma = 0.1$ for $n_t$. The resulting distribution for $[w^T_t, n^T_t]$ has mean 0, variance $\Sigma = 7.7 \cdot \text{blkdiag}(10^{-5}I_{2 \times 2}, 10^{-3}I_{4 \times 4})$ and support $D = \{\gamma \cdot\} = \text{blkdiag}(10^{-5}I_{2 \times 2}, 10^{-3}I_{4 \times 4})$, which is used for defining the uncertainty descriptions $D_1$, $D_2$, $D_3$.

3) Constraints and Cost: We set horizon $N = 12$ and choose cost matrices $Q = 10I_{2N}$, $R = 20I_N$ to penalise deviations from set-point $[2s_{\text{final}}, 0]$, along with constraints on speed $v \in [0, 12]ms^{-1}$ and acceleration $a \in [-6, 5]ms^{-2}$. Every chance constraint is imposed with the same risk level, $\epsilon = 0.0228$ to yield $\gamma_{ca} = \gamma_{ca} = 6.55$ for $D_3$.

4) Simulation Setup: We compare the following control policies for the agent corresponding to the different uncertainty descriptions: 1) Robust MPC (RMPC) for $D_1$, 2) Stochastic MPC (SMPC) for $D_2$, and 3) Distributionally Robust MPC (DRMPC) for $D_3$. We run 10 simulations for each policy, starting from $x_0 = [3m, 11.8ms^{-1}]$ until the agent reaches $s_{\text{final}} = 50m$. If (21) is infeasible, the brake $a = -6ms^{-2}$ is applied. In Figure 1, the first obstacle has a PD controller to go south across the intersection at high speed, while the second obstacle has a PD controller to stop at the intersection. The first obstacle is re-spawned after crossing the intersection by 20m. Casadi is used for modeling the problem (21) with Gurobi as the solver.

5) Results: The performance metrics for all the runs are recorded in Table I. We record % of time steps where constraint violations (in particular, the collision avoidance and speed constraints) and MPC infeasibility were detected, along with average times for solving (21) and reaching $s_{\text{final}}$ and finally, the average values of the closest distance from the obstacles (computed using (9)). In Figure 1, we compare various formulations for a particular run.

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Fig. 1: Snapshots of agents with RMPC, SMPC and DRMPC for a particular run. Darker colors correspond to later time steps. On the left, all agents slow down and cross behind the first obstacle. On the right, the agents speed up to cross the intersection, after slowing down for the uncertain second obstacle at the stop. Video: https://youtu.be/wgqO36a1SU8

| Performance metric | RMPC | SMPC | DRMPC |
|--------------------|------|------|------|
| Constraint violations (%) | 1.07 | 3.88 | 3.69 |
| Feasibility (%) | 95.24 | 97.46 | 96.58 |
| Avg. solve time (ms) | 32.62 | 54.10 | 54.33 |
| Avg. task completion time (s) | 9.15 | 8.59 | 8.73 |
| Avg. min. distance from obstacles (m) | 0.52 | 0.19 | 0.36 |

6) Discussion: In Figure 1 and Table I, we see that in terms of conservativeness (feasibility, time to reach $s_{\text{final}}$, constraint violations) the policies are ordered as: SMPC > DRMPC > RMPC, with SMPC being least conservative. Using the equivalence of norms and $\Sigma = 0.88\Gamma^{-1}$ for our example, it can be seen that $F_t(D_1) < F_t(D_2)$,
The relation between $\tilde{F}_t(D_1), \tilde{F}_t(D_3)$ can’t be established this way and needs further study. The increased conservatism for RMPC, however, results in fewer constraint violations and its LP formulation of collision avoidance constraints yields faster solve times.

**B. Unprotected Left Turn in CARLA**

We use our setup from [14] for the next couple of experiments, with an autonomous vehicle and $M = 1$ other vehicle as the obstacle at a traffic intersection in CARLA. The agent is tasked to turn left while avoiding collision with the oncoming obstacle.

1) **Models and Geometry:** The agent’s dynamics (1) are modelled by the kinematic bicycle model linearized about the reference, and the obstacle’s predictions (3) are given by our implementation of Multipath [2] for $N = 10$ steps. We use uni-modal predictions for Experiment 1 and multi-modal predictions with 3 modes for Experiment 2. The vehicles’ geometries are given by $4.9m \times 2.8m$ rectangles.

2) **Process Noise Distribution:** $[w^T_n \quad n^T_\gamma]^T$ is given by the Gaussian distribution used in [14].

3) **Constraints and Cost:** We set $N = 10$ and choose $Q = \text{blkdiag}(Q_1, \ldots, Q_N, 10, 1)$, where $Q_i = 5R_i \otimes \text{diag}(10^{-2})R_i$, and $R = I_N \otimes \text{diag}(10, 10^2)$ for the cost to penalize deviations from the reference. We impose constraints on speed $v \in [0, 12] \text{ms}^{-1}$, acceleration $a \in [-3, 2] \text{ms}^{-2}$ and steering $\delta \in [-0.5, 0.5]$. Every individual chance constraint is imposed with $\epsilon = 0.05$ to yield $\gamma_{ca} = \gamma_{ca}^0 = 1.64$ for $D_2$, and $\gamma_{ca} = \gamma_{ca}^0 = 4.36$ for $D_3$. For the multi-modal predictions in Experiment 2, the constraints (9) are imposed for each mode of the obstacle, and the SMPC finds a single policy sequence (7) that satisfies the tightened constraints for all modes.

**Experiment 1: SMPC vs DRMPC:** Since the prediction uncertainties are unbounded, we only compare SMPC for $D_2$ and DRMPC for $D_3$. We run 10 simulations for each policy with different initial conditions. If (21) is infeasible, the brake $a = -6 \text{ms}^{-2}$ is applied. The results are tabulated in Table II, where we see that compared to SMPC, the agent with DRMPC stays further away from the obstacle and deviates more from the reference. However, this enables maintaining a higher speed and completing the task faster.

**Experiment 2: SMPC vs SMPC of [14]:** We compare the SMPC approach in this paper with that of [14], where collision is modelled as the intersection of an ellipse (for the obstacle) and a circle (for the agent), and the free space is inner-approximated using an affine constraint. While the latter is robust to deviations of the agent’s orientation along the predictions, this approach induces conservative and undesirable maneuvers for collision avoidance. We summarise our findings in Figure 2, and observe that the new approach allows for a tighter left-turn in Figure 2b.