Determinant versus Permanent: salvation via generalization?

The algebraic complexity of the Fermionant and the Immanant

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July 3, 2013
1. Algebraic complexity

2. Immanant
Definition

The size of an arithmetic circuit is the number of operational gates.

\[ f(x, y) = (x + y)^2(z + 3) + 2(x + y)^2 + (z + 3)^2 \]
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A family $F = (f_n)$ of polynomials is in VP if there exists a family of circuits $C_n$ of polynomial size such that for any $n$

$$f_n \text{ is computed by } C_n$$

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A family $F = (f_n)$ is in VNP if there is a family $G = (g_n)$ in VP such that

$$f_n (\bar{x}) = \sum_{\bar{\epsilon} \in \{0,1\}^n} g_n (\bar{\epsilon}, \bar{x})$$
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Definition

Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The determinant is

$$\det_n(x) = (-1)^n \sum_{\pi \in S_n} (-1)^{c(\pi)} \prod_{i=1}^{n} x_{i\pi(i)}$$

Theorem (Valiant 79)

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Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The permanent is

$$\text{per}_n(x) = \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\pi(i)}$$

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The family $\text{per} = (\text{per}_n)_{n \in \mathbb{N}}$ is VNP-complete.

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**Conjecture (Valiant hypothesis)**

\[ \text{VP} \neq \text{VNP} \]

**Theorem (Bürgisser 2000)**

Under Generalized Riemann Hypothesis,

\[ \text{VP} = \text{VNP} \Rightarrow P/\text{poly} = NP/\text{poly} \]
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The main approaches of Valiant Hypothesis:
- Geometric Complexity Theory (GCT)
- Lower bounds
- The study of complexity classes (Characterization, complete polynomials)
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*Under Generalized Riemann Hypothesis,*

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The main approaches of Valiant Hypothesis:

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- Lower bounds
- The study of complexity classes (Characterization, complete polynomials)
Definition (informal)

A generalization of the determinant and the permanent is a series of family $F^k = (f_n)^k$ indexed by some $k$ such that

- For certain $k$, $F^k$ are in $VP$.
- For others $k$, $F^k$ are $VNP$-complete.

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Determinant versus Permanent: salvation via generalization?
Definition

A young diagram is a collection of boxes in left adjusted row with decreasing row length.

Young diagrams

\[
\begin{bmatrix} 4, 4 \\ 4, 2 \end{bmatrix}
\]

Definition

Let $\chi_Y$ be an irreducible character of $S_n$. Then

\[
\text{im}_\chi(\bar{x}) = \sum_{\pi \in S_n} \chi_Y(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}
\]
If $Y$ is a single row, then $\text{im}_Y(\bar{x}) = \text{per}(\bar{x})$.

If $Z$ is a single column, then $\text{im}_Z(\bar{x}) = \text{det}(\bar{x})$. 
Theorem (Bürgisser 2000)

If \((Y_n)\) is a family of Young diagrams with only a constant number of boxes at the right of the first column, then

\((\text{im} Y_n)\) is in \(\text{VP}\)
Theorem (Brylinski 2003)

Let $Y_n$ be Young diagrams such that the maximal difference between the size of two consecutive rows is $\Omega(n)$, then

$$(\text{im}_{Y_n}) \text{ is VNP-complete}$$
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Theorem (2)

Let $[n, n]$ be the Young diagram with two columns, each with $n$ boxes. Then $(\text{im}_{[n,n]})$ is VNP-complete.
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Theorem (3)

If \((Y_n)\) has a polynomial number of boxes at the right of the first column and a constant number of columns, then

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(\text{im}_{Y_n}) \text{ is VNP-complete}
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Theorem (2)

Let \([n, n]\) be the Young diagram with two columns, each with \(n\) boxes. Then

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If \((Y_n)\) has a polynomial number of boxes at the right of the first column and a constant number of columns, then

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Theorem (Conclusion)

Let $(Y_n)$ be a family of Young diagrams with a constant number of columns such that $|Y_n| = \Omega(n)$. Then

- If the number of boxes at the right of the first column is constant $c$, then $(\text{im}_{Y_n})$ is in $\text{VP}$.
- If the number of boxes at the right of the first column is logarithmic, then $(\text{im}_{Y_n})$ is not $\text{VNP}$-complete.
- If the number of boxes at the right of the first column is polynomial, $(\text{im}_{Y_n})$ is $\text{VNP}$-complete.

Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a $\text{VP}$-complete family!
Theorem (Conclusion)

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Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a \(\text{VP}\)-complete family!
Thank you!
Definition

If $\pi$ is a permutation, $c(\pi)$ is its number of cycles.

$$Ferm_n^k A = (-1)^n \sum_{\pi \in S_n} (-k)^{c(\pi)} \prod_{i=1}^n A_{i,\pi(i)}$$

Let $Ferm^k$ the family of $(Ferm_n^k)$

- If $k = 1$, then $Ferm_1^1(x) = \det(x)$
- If $k = -1$, then $Ferm_{-1}(\bar{x}) = \per(\bar{x})$
Definition

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Let \text{Ferm}_n^k the family of $(\text{Ferm}_n^k)$

- If $k = 1$, then $\text{Ferm}_n^1(\bar{x}) = \det(\bar{x})$
- If $k = -1$, then $\text{Ferm}_n^{-1}(\bar{x}) = \text{per}(\bar{x})$
Theorem (1)

- $Ferm^0 = 0$.
- $Ferm^1$ is in $VP$
- for $k \in \mathbb{Q}$ different from 0, 1 $Ferm^k$ is VNP-complete.