A TOPOLOGICAL CHARACTERIZATION OF LF-SPACES

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Abstract. We present a topological characterizations of LF-spaces and some other spaces of the form \( \Omega \times \mathbb{R}^\infty \). Those characterizations are applied to recognizing the topology of small box-product and uniform direct limits of Polish ANR-groups.

1. Introduction

In this paper we shall present a simple criterion for recognizing topological spaces that are homeomorphic to LF-spaces. In [2] this criterion will be applied to recognizing the topology of some homeomorphism groups.

We recall that an LF-space is the direct limit \( \text{lc-lim} X_n \) of an increasing sequence \( X_0 \subset X_1 \subset X_2 \subset \ldots \) of Fréchet (= locally convex complete linear metric) spaces in the category of locally convex spaces. The simplest example of a non-metrizable LF-space is the inductive limit \( \mathbb{R}^\infty = \text{lc-lim} \mathbb{R}^n \) of the sequence \( \mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \ldots \) of Euclidean spaces, where each space \( \mathbb{R}^n \) is identified with the hyperplane \( \mathbb{R}^n \times \{0\} \) in \( \mathbb{R}^{n+1} \).

The space \( \mathbb{R}^\infty \) can be identified with the direct sum \( \oplus_{n \in \omega} \mathbb{R} \) of one-dimensional Fréchet spaces in the category of locally convex spaces.

P. Mankiewicz in [16] obtained a topological classification of LF-spaces and proved that each LF-space is homeomorphic to the direct sum \( \oplus_{n \in \omega} l_2(\kappa_i) \) of Hilbert spaces for some sequence \((\kappa_i)_{i \in \omega}\) of cardinals. Here \( l_2(\kappa) \) stands for the Hilbert space with orthonormal base of cardinality \( \kappa \). A more precise version of the Mankiewicz’s classification says that the spaces

- \( l_2(\kappa) \) for some cardinal \( \kappa \geq 0 \);
- \( \mathbb{R}^\infty \);
- \( l_2(\kappa) \times \mathbb{R}^\infty \) for some \( \kappa \geq \omega \), and
- \( \oplus_{n \in \omega} l_2(\kappa_i) \) for a strictly increasing sequence of infinite cardinals \((\kappa_i)_{i \in \omega}\)

are pairwise non-homeomorphic and represent all possible topological types of LF-spaces. In particular, each infinite-dimensional separable LF-space is homeomorphic to one of the following spaces: \( l_2 \), \( \mathbb{R}^\infty \) or \( l_2 \times \mathbb{R}^\infty \).

The topological characterizations of the spaces \( l_2 \) and \( \mathbb{R}^\infty \) were given by H. Toruńczyk [22] and K. Sakai [19], respectively. Those characterizations belong to the best achievements of the

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classical Infinite-Dimensional Topology. In this paper we shall present a topological characterization of other LF-spaces, in particular, \( l_2 \times \mathbb{R}^\infty \).

First we recall the topological characterization of the spaces \( l_2(\kappa) \) and \( \mathbb{R}^\infty \), and manifolds modeled on such spaces.

By a manifold modeled on a space \( E \) (briefly, an \( E \)-manifold) we understand any paracompact space \( M \) that can be covered by open subsets homeomorphic to open subsets of the model space \( E \). Manifolds modeled on Hilbert spaces are called Hilbert manifolds (this definition includes also the case of manifolds modeled on finite-dimensional Hilbert spaces).

**Theorem 1.1** (Toruńczyk). A topological space \( X \) is homeomorphic to (a manifold modeled on) an infinite-dimensional Hilbert space \( l_2(\kappa) \) if and only if \( X \) is a completely-metrizable absolute (neighborhood) retract of local density \( \leq \kappa \) such that each map \( f : C \to X \) from a completely-metrizable space \( C \) of density \( \leq \kappa \) can be uniformly approximated by closed topological embeddings.

Now we explain some concepts appearing in this theorem. We say that a topological space has local density \( \leq \kappa \) if each point \( x \in X \) has a neighborhood \( O(x) \) whose density is \( \leq \kappa \). A topological space is completely-metrizable if it is homeomorphic to a complete metric space. A Polish space is a separable completely-metrizable space.

An absolute (neighborhood) retract is a metrizable space \( X \) that is a (neighborhood) retract in each metrizable space which contains \( X \) as a closed subspace.

Finally, we shall say that a map \( f : C \to X \) can be uniformly approximated by maps with certain property \( \mathcal{P} \) if for each open cover \( U \) of \( Y \) there is a map \( g : C \to X \) with the property \( \mathcal{P} \) that is \( U \)-near to \( f \) in the sense that for each point \( c \in C \) the set \( \{ f(c), g(c) \} \) lies in some set \( U \in U \).

Next, we recall the characterization of the LF-space \( \mathbb{R}^\infty \) due to K.Sakai [19]. This characterization is based on the observation that the LF-space \( \mathbb{R}^\infty = \text{l}c\text{-lim} \mathbb{R}^n \) carries the topology of the topological direct limit of the tower \( (\mathbb{R}^n)_{n \in \omega} \) of finite-dimensional Euclidean spaces.

By the topological direct limit \( \text{t-lim} X_n \) of a tower
\[
X_0 \subset X_1 \subset X_2 \subset \cdots
\]
of topological spaces we understand the union \( X = \bigcup_{n \in \omega} X_n \) endowed with the largest topology turning the identity inclusions \( X_n \to X, \ n \in \omega \), into continuous maps.

**Theorem 1.2** (Sakai). A topological space \( X \) is homeomorphic to the space \( \mathbb{R}^\infty \) if and only if

1. \( X \) is homeomorphic to the topological direct limit \( \text{t-lim} X_n \) of a tower \( (X_n)_{n \in \omega} \) of finite-dimensional metrizable compacta;

2. each embedding \( f : B \to X \) of a closed subset \( B \subset C \) of a finite-dimensional metrizable compact space \( C \) extends to an embedding \( \bar{f} : C \to X \).

Replacing the class of finite-dimensional compact metrizable spaces in this theorem by the class of Polish spaces, E.Pentsak [18] obtained a topological characterization of the topological direct limit \( \text{t-lim} (l_2)^n \) of the tower of Hilbert spaces
\[
l_2 \subset l_2 \times l_2 \subset \cdots \subset l_2^n \subset \cdots
\]
where each space \( l_2^n \) is identified with the subspace \( l_2^n \times \{ 0 \} \) of the Hilbert space \( l_2^{n+1} \). However, the topology of the topological direct limit \( \text{t-lim} l_2^n \) is strictly stronger that the topology of the

1By a map we mean a continuous function. In contrast, a function need not be continuous.
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direct limit $\lim_{\to}^X_{n}$ of that tower in the category of locally convex spaces. Moreover, $\lim_{\to}^X_{n}$ is not even homeomorphic to a topological group, see [1]. In fact, an LF-space $X$ is homeomorphic to the topological direct limit of a tower of metrizable spaces if and only if $X$ is either metrizable or is topologically isomorphic to $\mathbb{R}^\infty$, see [7].

This means that topological direct limits cannot be used for describing the topology of non-metrizable LF-spaces which are different from $\mathbb{R}^\infty$. On the other hand, it was discovered in [4] that for any tower $(X_n)_{n \in \omega}$ of Fréchet spaces the topology of the LF-space $X = \lim_{\to}^X_{n}$ coincides with the topology of the direct limit $\lim_{\to}^X_{n}$ of this tower in the category of uniform spaces!

By the uniform direct limit $\lim_{\to}^X_{n}$ of a tower $X_0 \subset X_1 \subset X_2 \subset \ldots$

of uniform spaces we understand the union $X = \bigcup_{n \in \omega} X_n$ endowed with the largest uniformity turning the identity inclusions $X_n \to X$ into uniformly continuous maps. Each topological group $G$ (in particular, each locally convex space) is considered as a uniform space endowed with the left uniformity generated by the entourages $\{ (x, y) \in G^2 : x \in y \} \subseteq U$ where $U = U^{-1}$ runs over all symmetric neighborhoods of the neutral element in $G$.

For any tower $(X_n)_{n \in \omega}$ of Fréchet spaces the identity map $\lim_{\to}^X_{n} \to \lim_{\to}^X_{n}$ is continuous (because each continuous group homomorphism is uniformly continuous). A less trivial fact established in [4] is the continuity of the inverse map $\lim_{\to}^X_{n} \to \lim_{\to}^X_{n}$. This means that we can identify LF-spaces with uniform direct limits of Fréchet spaces and reduce the problem of topological characterization of LF-spaces to the problem of recognizing uniform direct limits that are homeomorphic to LF-spaces. The answer to this problem will be given in Theorem 1.3 which involves the following definitions.

The uniformity of a uniform space $X$ will be denoted by $U_X$. A uniform space is metrizable if its uniformity is generated by a metric. For a point $a \in X$, a subset $A \subseteq X$, and an entourage $U \in U_X$, let $B(a, U) = \{ x \in X : (x, a) \in U \}$ and $B(A, U) = \bigcup_{a \in A} B(a, U)$ be the $U$-balls around $a$ and $A$, respectively.

**Definition 1.** A uniform space $X$ is defined to be uniformly locally equiconnected if there is an entourage $U_0 \in U_X$ and a continuous map $\lambda : [0, 1] \to X$ such that

- $\lambda(x, y, 0) = x$ and $\lambda(x, y, 1) = y$ for all $x, y \in U_0$, and
- for every entourage $V \in U_X$ there is an entourage $V' \in U_X$ such that for each pair $(x, y) \in V \cap U_0$ and any $t, \tau \in [0, 1]$ we get $(\lambda(x, y, \tau), \lambda(x, y, t)) \in U$;

If $\lambda$ is defined on the entire product $X^2 \times [0, 1]$, then the uniform space $X$ is called uniformly equiconnected.

**Definition 2.** A subset $A$ of a uniform space $X$ is defined to be a uniform neighborhood retract in $X$ if there is an entourage $V_0 \in U_X$ and a retraction $r : B(A, V_0) \to A$ such that for every entourage $U \in U_X$ there is an entourage $V \in U_X$ such that $(r(x), x) \in U$ for any point $x \in B(A, V \cap V_0)$.

Such a retraction $r$ is called regular. If $r$ can be defined on the whole space $X$, then $A$ is called a uniform retract of $X$.

**Definition 3.** We say that a subset $A$ of a uniform space $X$ has a uniform collar in $X$ if there is a continuous map $e : [0, 1] \to X$ such that

- for any entourage $U \in U_X$ there is a number $\delta > 0$ such that $e(a, t) \in B(a, U)$ for any $(a, t) \in A \times [0, \delta]$, and
• for every $\delta \in (0, 1]$ there is an entourage $U \in U_X$ such that $e(a, t) \notin B(A, U)$ for any $(a, t) \in A \times [\delta, 1]$.

Now we are able to formulate our topological characterization of LF-spaces.

**Theorem 1.3.** A topological space $X$ is homeomorphic to a non-metrizable LF-space if and only if $X$ is homeomorphic to the uniform direct limit $\lim_{\to} X_n$ of a tower $(X_n)_{n \in \omega}$ of metrizable uniform spaces such that each space $X_n$

1. is uniformly locally equiconnected,
2. is a uniform neighborhood retract in $X_{n+1}$,
3. has a uniform collar in $X_{n+1}$,
4. is contractible in $X_{n+1}$, and
5. is a Hilbert manifold.

A special case of this theorem is the following topological characterization of the LF-space $l_2 \times \mathbb{R}^\infty$.

**Corollary 1.4.** A topological space $X$ is homeomorphic to the LF-space $l_2 \times \mathbb{R}^\infty$ if and only if $X$ is homeomorphic to the uniform direct limit $\lim_{\to} X_n$ of a tower $(X_n)_{n \in \omega}$ of metrizable uniform spaces such that each space $X_n$

1. is uniformly locally equiconnected,
2. is a uniform neighborhood retract in $X_{n+1}$,
3. has a uniform collar in $X_{n+1}$,
4. is contractible in $X_{n+1}$, and
5. is an $l_2$-manifold.

In fact, Corollary 1.4 is a special case of the topological characterization of spaces of the form $\Omega \times \mathbb{R}^\infty$, where $\Omega$ is a model space, defined as follows.

**Definition 1.5.** Let $\mathcal{C}$ be a class of topological spaces and $X_0$ be a closed subset of a topological space $X$. A pair $(X, X_0)$ is called strongly $\mathcal{C}$-universal if for each open cover $\mathcal{U}$ of $X \setminus X_0$ and each continuous map $f : C \to X$ from a space $C \in \mathcal{C}$, the map $f$ restricted to the subset $V = f^{-1}(X \setminus X_0) \subset C$ can be approximated by a closed topological embedding $g : V \to X \setminus X_0$ that is $\mathcal{U}$-near to $f|_V$.

Observe that a pair $(X, X_0)$ is strongly $\mathcal{C}$-universal if each map $f : U \to X \setminus X_0$ defined on an open subspace $U \subset C$ of a space $C \in \mathcal{C}$, can be uniformly approximated by closed embeddings. This observation implies that for each closed subset $X_0$ of an $l_2(\kappa)$-manifold $X$ the pair $(X, X_0)$ is strongly $\mathcal{C}_{\leq \kappa}$-universal for the class $\mathcal{C}_{\leq \kappa}$ of completely-metrizable spaces of density $\leq \kappa$.

For a topological space $X$ let $\mathcal{F}_0(X)$ denote the class of topological spaces, homeomorphic to closed subsets of $X$. Also let $\mathbb{I} = [0, 1]$ be the closed unit interval. Each $n$-cube $\mathbb{I}^n$ will be identified with the face $\mathbb{I}^n \times \{0\}$ of the cube $\mathbb{I}^{n+1}$.

**Definition 1.6.** A metrizable topological space $\Omega$ is called a model space if $\Omega$ is an absolute retract, $\Omega \times \mathbb{I} \in \mathcal{F}_0(\Omega)$, and for every $n \in \omega$ the pair $(\Omega \times \mathbb{I}^{n+1}, \Omega \times \mathbb{I}^n)$ is strongly $\mathcal{F}_0(\Omega)$-universal.

Toruńczyk’s characterization Theorem 1.1 implies that each infinite-dimensional Hilbert space is a model space. Many other examples of (incomplete) model spaces can be found in [9] and [3].

For a model space $\Omega$, the topology of the product $\Omega \times \mathbb{R}^\infty$ can be characterized as follows:
Theorem 1.7. A topological space $X$ is homeomorphic to the space $\Omega \times \mathbb{R}^\infty$ for a model space $\Omega$ if and only if $X$ is homeomorphic to the uniform direct limit $\varinjlim_{\leftarrow n} X_n$ of a tower $(X_n)_{n \in \omega}$ of metrizable uniform spaces such that for every $n \in \omega$

1. $X_n$ is uniformly locally equiconnected,
2. $X_n$ is a uniform neighborhood retract in $X_{n+1}$,
3. $X_n$ has a uniform collar in $X_{n+1}$,
4. $X_n$ is contractible in $X_{n+1}$,
5. $X_n$ is an ANR, and
6. the pair $(X_{n+1}, X_n)$ is strongly $\mathcal{F}_0(\Omega)$-universal.

This theorem immediately implies:

Corollary 1.8. If $\Omega$ and $\Omega'$ are two model spaces with $\mathcal{F}_0(\Omega) = \mathcal{F}_0(\Omega')$, then the products $\Omega \times \mathbb{R}^\infty$ and $\Omega' \times \mathbb{R}^\infty$ are homeomorphic.

By a bit more elaborated technique we shall show in [6] that after deleting the condition (4), Theorems 1.3 and 1.7 turn into characterizations of topological spaces homeomorphic to open subspaces in LF-spaces or spaces $\Omega \times \mathbb{R}^\infty$.

In Sections 10 and 8 these characterization results will be applied to detecting topological groups and small box products homeomorphic to LF-spaces or spaces of the form $\Omega \times \mathbb{R}^\infty$ for a model space $\Omega$.

2. Describing the topology of uniform direct limits

We start by recalling some notions related to uniform spaces. For more detail information, see Chapter 8 of the Engelking’s book [13].

As we have already said, the uniformity of a uniform space $X$ is denoted by $U_X$. Elements of $U_X$ are called entourages (of the diagonal $\Delta_X = \{(x, x) : x \in X\} \subset X \times X$). For a point $x \in X$, a subset $A \subset X$, and a subset $U \subset X^2$ let

$$B(a, U) = \{x \in X : (x, a) \in U\}$$

and

$$B(A, U) = \bigcup_{a \in A} B(a, U).$$

Any subset $O(A) \subset X$ that contains a set of the form $B(A, U)$ for some $U \in U$ will be called a uniform neighborhood of $A$ in the uniform space $X$.

Given two subsets $U, V \subset X^2$ (thought of as relation), consider their composition

$$U + V = \{(x, z) \in X^2 : \exists y \in X\text{ with } (x, y) \in U\text{ and } (y, z) \in V\} \subset X^2.$$  

This operation can be extended to sequences of subsets $(U_i)_{i \in \omega}$ of $X^2$ by the formulas:

$$\sum_{i \leq n} U_i = U_0 + \cdots + U_n$$

and

$$\sum_{i \in \omega} U_i = \bigcup_{n \in \omega} \sum_{i \leq n} U_i.$$

An increasing sequence of spaces

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

is called a tower. Such a tower is closed if each space $X_n$ is a closed subspace of $X_{n+1}$.

For a point $x \in X = \bigcup_{n \in \omega} X_n$ let

$$|x| = \min\{n \in \omega : x \in X_n\}$$

denote the height of $x$ in $X$. 

By the uniform direct limit \( u\lim X_n \) of a tower of uniform space \((X_n)_{n\in\omega}\) we understand their union \(X = \bigcup_{n\in\omega} X_n\) endowed with the largest uniformity turning the identity inclusions \(X_n \to X\) into uniformly continuous maps.

The topology of the space \( u\lim X_n \) has been described in [4] as follows.

**Theorem 2.1.** For a tower of uniform spaces \((X_n)_{n\in\omega}\) the family

\[
\mathcal{B} = \left\{ B(x, \sum_{i \geq |x|} U_i) : (U_i)_{i \geq |x|} \in \prod_{i \geq |x|} \mathcal{U}_X, \right\}
\]

is a base of the topology of the uniform direct limit \( u\lim X_n \).

This description implies the following two results established in [4].

**Proposition 2.2.** Let \((X_n)_{n\in\omega}\) be a tower of uniform spaces. If each space \(X_n\) is locally compact, then the identity map \( t\lim X_n \to u\lim X_n \) is a homeomorphism.

**Proposition 2.3.** For two towers of uniform spaces \((X_n)\) and \((Y_n)\), the identity map \( u\lim (X_n \times Y_n) \to u\lim X_n \times u\lim Y_m \) is a homeomorphism. In particular, for any uniform space \(\Omega\), the spaces \(\Omega \times u\lim X_n\) and \( u\lim (\Omega \times X_n)\) are naturally homeomorphic.

This multiplicativity property distinguishes uniform direct limits from the topological direct limits: as we already know the product \(l_2 \times (t\lim \mathbb{R}^n)\) is not homeomorphic to \(t\lim (l_2 \times \mathbb{R}^n)\).

Following [4], we define a map \(f : X \to Y\) between two uniform spaces to be regular at a subset \(A \subset X\) if for any entourages \(U \in \mathcal{U}_Y\) and \(V \in \mathcal{U}_X\) there is an entourage \(W \in \mathcal{U}_X\) such that for each point \(x \in B(A, W)\) there is a point \(a \in A\) with \((x, a) \in V\) and \((f(x), f(a)) \in U\).

The description of the topology of the uniform direct limits given in Theorem 2.1 implies the following Continuity Theorem proved in [4].

**Theorem 2.4.** A function \(f : u\lim X_n \to Y\) from the uniform direct limit of a tower \((X_n)_{n\in\omega}\) of uniform spaces and with values in a uniform space \(Y\) is continuous if for every \(n \in \mathbb{N}\) the restriction \(f|X_n\) is continuous and regular at the subset \(X_{n-1}\).

### 3. The Category of Retral Spaces

Theorems 2.2 and 2.3 will be proved by the standard back-and-forth argument in the category of retral spaces.

A **retral space** is a pair \(\langle X, r \rangle\) consisting of a uniform space \((X, \mathcal{U}_X)\) and a (not necessarily continuous) function \(r : X \to X\) such that for every entourage \(U \in \mathcal{U}_X\) there is an entourage \(V \in \mathcal{U}_X\) such that for each \(x \in B(rX, V)\) we get \((r(x), x) \in U\). The function \(r\) is called the **retral function**; its image \(r(X) \subset X\) will be denoted by \(rX\).

The Hausdorff property of the uniform space \(X\) implies that \(r(x) = x\) for all \(x \in rX\). The definition implies the continuity of \(r\) at each point of the set \(rX\); consequently, \(rX\) is a closed subset of \(X\).

Each uniform space \(X\) will be identified with the retral space \(\langle X, \text{id} \rangle\) where \(\text{id} : X \to X\) is the identity map of \(X\).

By a **retral subspace** of a retral space \(\langle X, r \rangle\) we understand any pair \(\langle A, r|A \rangle\) where \(A\) is a subspace of the uniform space \(X\) such that \(r(A) \subset A\).

A map \(f : X \to Y\) from a retral space \(\langle X, r \rangle\) to a uniform space \(Y\) is called \(r\)-**regular** if for each entourage \(U \in \mathcal{U}_Y\) there is an entourage \(V \in \mathcal{U}_X\) such that \((f(x), f \circ r(x)) \in U\) for each \(x \in B(rX, V)\).
By a retral map between retral spaces \((X, r)\) and \((Y, \rho)\) we understand any continuous \(r\)-regular map \(f : X \to Y\). A retral embedding of \((X, r)\) into \((Y, \rho)\) is a topological embedding \(f : X \to Y\) such that \(\rho(f(X)) \subset f(X)\) and both maps \(f : (X, r) \to (Y, \rho)\) and \(f^{-1} : (f(X), \rho|f(X)) \to (X, r)\) are retral. If, in addition, \(f(X) = Y\), then \(f\) is called a retral homeomorphism.

By a retral tower we understand a sequence \(\mathcal{X} = (\langle X_n, r_n \rangle)_{n \in \omega}\) of retral spaces such that for every \(n \in \mathbb{N}\) the uniform space \(X_{n-1}\) coincides with the subspace \(r_n X_n\) of the uniform space \(X_n\). For any numbers \(k < n\) let \(r^n_k : X_n \to X_k\) be the composition of the retral functions \(r_{k+1} \circ \cdots \circ r_n\). The functions \(r^n_k\) are called structure retral functions of the retral tower \(\mathcal{X}\).

### 4. Extendably \(\mathcal{C}\)-universal towers and Uniqueness Theorem

In this section we introduce the notion of an extendably \(\mathcal{C}\)-universal tower and prove the Uniqueness Theorem \[\text{4.2}\] for uniform direct limits of such towers.

Let \(\mathcal{C}\) be a class of retral spaces. A retral tower \((\langle X_n, r_n \rangle)_{n \in \omega}\) is called a \(\mathcal{C}\)-tower if \(\langle X_n, r_n \rangle \in \mathcal{C}\) for all \(n \in \omega\).

**Definition 4.1.** A retral tower \((\langle X_n, r_n \rangle)_{n \in \omega}\) is defined to be extendably \(\mathcal{C}\)-universal if for any retral space \((C, r) \in \mathcal{C}\) any retral homeomorphism \(f : (A, r|A) \to (X_n, \text{id})\) defined on a closed subset \(A \subset C\) with \(rC \subset A\) extends to a closed retral embedding \(\tilde{f} : (C, r) \to (X_n, r^n_k)\) for some \(n > k\).

**Theorem 4.2.** The uniform direct limits \(u\-\lim X_n\) and \(u\-\lim Y_n\) of any two extendably \(\mathcal{C}\)-universal \(\mathcal{C}\)-towers \((\langle X_n, r_n \rangle)_{n \in \omega}\) and \((\langle Y_n, \rho_n \rangle)_{n \in \omega}\) are homeomorphic.

**Proof.** A homeomorphism \(f : u\-\lim X_n \to u\-\lim Y_n\) will be constructed by a standard back-and-forth argument.

We shall construct by induction increasing number sequences \((n_k)_{k \in \omega}\) and \((m_k)_{k \in \omega}\) and sequences of closed retral embeddings \(f_k : \langle X_{n_k}, r^n_{n_k+1} \rangle \to \langle Y_{m_k}, r^m_{m_k+1} \rangle\) and \(g_k : \langle Y_{m_k}, r^m_{m_k+1} \rangle \to \langle X_{n_k}, r^n_{n_k+1} \rangle\), \(k \in \omega\), such that

- \(f_{k+1}|_{X_{n_k}} = f_k\) and \(g_{k+1}|_{Y_{m_k}} = g_k\),
- \(g_k|f_k(X_{n_k}) = f_k^{-1}\) and \(f_{k+1}|g_k(Y_{m_k}) = g_k^{-1}\),

for every \(k \in \omega\).

Let \(X_{-1} = Y_{-1} = \emptyset\), \(n_{-1} = m_{-1} = -1\), and \(f_{-1} : X_{-1} \to Y_{-1}\) be the identity map on the empty space \(X_{-1} = Y_{-1} = \emptyset\).

Assume that for some \(k \in \omega\), the numbers \(n_i, m_i\) and closed retral embeddings

\[
f_i : \langle X_{n_i}, r^m_{m_{i-1}} \rangle \to \langle Y_{m_i}, r^m_{m_{i-1}} \rangle \quad \text{and} \quad g_{i-1} : \langle Y_{m_{i-1}}, r^m_{m_{i-1}} \rangle \to \langle X_{n_i}, r^n_{n_{i-1}} \rangle
\]

have been constructed for all \(i \leq k\).

Consider the retral space \(\langle C, r \rangle = \langle Y_{m_k}, r^m_{m_k+1} \rangle\) and its closed subspace \(\langle A, r|A \rangle\) where

\[
A = f_k(X_{n_k}) \supset f_k \circ g_{k-1}(Y_{m_{k-1}}) = Y_{m_{k-1}} = \rho^m_{m_{k-1}}(Y_{m_k})
\]

Since \(f_k : \langle X_{n_{k}}, r^n_{n_{k+1}} \rangle \to \langle A, r|A \rangle\) is a retral homeomorphism, the inverse map \(f_k^{-1} : \langle A, r|A \rangle \to \langle X_{n_{k}}, \text{id} \rangle\) is a retral homeomorphism, too. The extendable \(\mathcal{C}\)-universality of the tower \((\langle X_n, r_n \rangle)_{n \in \omega}\) guarantees that the retral homeomorphism \(f_k^{-1}\) extends to a closed retral embedding \(g_k : \langle C, r \rangle = \langle Y_{m_k}, r^m_{m_k+1} \rangle \to \langle X_{n_{k+1}}, r^n_{n_{k+1}} \rangle\) for some number \(n_{k+1} > n_k\).

By the same argument, the extendable \(\mathcal{C}\)-universality of the tower \((\langle Y_n, \rho_n \rangle)_{n \in \omega}\) implies that the retral homeomorphism \(g_k^{-1} : \langle g_k(Y_{m_k}), r^n_{n_{k+1}} | g_k(Y_{m_k}) \rangle \to \langle Y_{m_k}, \text{id} \rangle\) extends to a closed retral
embedding \( f_{k+1} : (X_{n+1}, r_{n+1}^k) \to (Y_{m+1}, \rho_{m+1}^k) \) for some \( m_{k+1} > m_k \). This completes the inductive step.

After completing the inductive construction, consider the maps \( f : u\lim X_n \to u\lim Y_m \) and \( g : u\lim Y_m \to u\lim X_n \) defined by \( f|X_n = f_k \) and \( g|Y_m = g_k \) for all \( k \in \omega \). The Continuity Theorem 2.4 guarantees that the maps \( f \) and \( g \) are continuous. Since both compositions \( f \circ g \) and \( g \circ f \) are identity maps, we conclude that the uniform direct limits \( u\lim X_n = u\lim Y_m \) and \( u\lim Y_m = u\lim Y_m \) are homeomorphic.

\[ \square \]

5. Recognizing extendably \( \mathcal{C} \)-universal \( \mathcal{C} \)-towers

In this section we give a criterion for recognizing extendably \( \mathcal{C} \)-universal \( \mathcal{C} \)-towers for some special classes \( \mathcal{C} \) of retral spaces.

Given a class \( \mathcal{C} \) of topological spaces, consider the class \( \downarrow \mathcal{C} \) of all retral spaces \( (C, r) \) such that

- \( C \) is homeomorphic to a space from the class \( \mathcal{C} \),
- the uniformity of \( C \) is generated by a metric, and
- for some entourage \( U \in \mathcal{U}_C \) the restriction \( r|B(rC, U) \) is continuous.

Retral spaces from the class \( \downarrow \mathcal{C} \) are called \( \mathcal{C} \)-retral spaces.

**Lemma 5.1.** Let \( \mathcal{C} \) be a class of topological spaces. A retral tower \( (\langle X_n, r_n \rangle)_{n \in \omega} \) is extendably \( \downarrow \mathcal{C} \)-universal provided that

1. each uniform space \( X_n, n \in \omega \), is metrizable, uniformly locally equiconnected, and has a uniform collar in \( X_{n+1} \),
2. each space \( X_n \) is an ANR and is contractible in \( X_{n+1} \), and
3. for any space \( C \in \mathcal{C} \) and a number \( k \in \omega \) there is a number \( n \geq k + 2 \) such that the pair \( (X_n, X_k) \) is strongly \( \{C\} \)-universal.

**Proof.** In order to check the extendable \( \downarrow \mathcal{C} \)-universalty of the tower \( (\langle X_n, r_n \rangle)_{n \in \omega} \), fix a retral space \( (C, r) \in \downarrow \mathcal{C} \), a closed subspace \( A \subset C \) with \( rC \subset A \), and a retral homeomorphism \( f : ([A, r|A]) \to (X_k, \text{id}) \) for some \( k \in \omega \).

It follows from our assumption that the uniformities of the spaces \( C \) and \( X_{k+2} \) are generated by metrics \( d_C \) and \( d_X \), respectively. Replacing \( d_C \) by \( \min \{1, d_C \} \), if necessary, we can assume that \( d_C \leq 1 \).

The \( r \)-regularity of the retral map \( f \) implies the existence of a continuous function \( \varphi : [0, \infty) \to [0, \infty) \) such that \( \varphi^{-1}(0) = 0 \) and \( d_X(f(a), f \circ r(a)) \leq \varphi(d_C(a, rC)) \) for all \( a \in A \).

The uniform space \( X_k \) is uniformly locally equiconnected and hence admits a (not necessarily continuous) function \( \lambda : X_k \times X_k \times [0, 1] \to X_k \) such that

- \( \lambda(x, y, 0) = x, \lambda(x, y, 1) = y \) for all \( x, y \in X_k \),
- for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( d_X(x, \lambda(x, y, t)) < \varepsilon \) for all \( t \in [0, 1] \) and \( x, y \in X_k \) with \( d_X(x, y) < \delta \), and
- \( \lambda \) is continuous on the set \( \{(x, y, t) \in X_k \times X_k \times [0, 1] : d_X(x, y) < \varepsilon_0 \} \) for some positive \( \varepsilon_0 > 0 \).

Since \( (C, r) \in \downarrow \mathcal{C} \), the function \( r \) is continuous on the closure \( \overline{U} \) of some uniform neighborhood \( U \subset C \) of \( rC \). We can assume that \( U \) is so small that \( \varphi(d_C(x, rC)) \leq \varepsilon_0/3 \) for all \( x \in \overline{U} \).

Since \( X_k \) is an ANR, the map \( f : A \to X_k \) admits a continuous extension \( f_1 : O(A) \to X_k \) to an open neighborhood \( O(A) \) of \( A \) in \( C \). The continuity of the retraction \( r|\overline{U} \) implies that the
set
\[ V = \{ x \in \mathcal{U} \cap O(A) : d_X(f_1(x), f \circ r(x)) < 2 \varphi(d_C(x, rC)) \} \]
is an open neighborhood of the set \( A \cap \mathcal{U} \setminus rC \) in \( \mathcal{U} \setminus rC \).

Since \( \mathcal{U} \setminus (V \cup rC) \) and \( A \setminus rC \) are disjoint closed subsets of the metrizable space \( \mathcal{U} \cup A \setminus rC \), there is a continuous function \( \xi : \mathcal{U} \cup A \setminus rC \to [0, 1] \) such that \( \xi(A \setminus rC) \subset \{ 0 \} \) and \( \xi(\mathcal{U} \setminus (V \cup rC)) \subset \{ 1 \} \).

Extend \( \xi \) to a (not necessarily continuous) function \( \tilde{\xi} : \mathcal{U} \cup A \to [0, 1] \) and define a function \( \tilde{f} : \mathcal{U} \cup A \to X_k \) by the formula
\[
\tilde{f}(x) = \lambda(f_1(x), f \circ r(x), \tilde{\xi}(x)).
\]
One can readily check that this map is continuous and \( r \)-regular.

Since the ANR-space \( X_k \) is contractible in \( X_{k+1} \), the function \( \tilde{f} \) admits a continuous extension \( \bar{f} : C \to X_{k+1} \).

Let \( \gamma : X_{k+1} \times [0, 1] \to X_{k+2} \) be a uniform collar of \( X_{k+1} \) in the uniform space \( X_{k+2} \). By definition, the map \( \gamma \) has the following two properties:

- for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \gamma(x, t) \in B(x, \varepsilon) \) for all \( (x, t) \in X_{k+1} \times [0, \delta] \);
- for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \gamma(x, t) \notin B(X_{k+1}, \delta) \) for any \( (x, t) \in X_k \times [\varepsilon, 1] \).

Consider the continuous function
\[
g : C \to X_{k+2}, \quad g : c \mapsto \gamma(\tilde{f}(c), d_C(c, A)).
\]

Find \( n \geq k+2 \) such that the pair \((X_n, X_k)\) is strongly \( \{C\} \)-universal. Fix any metric \( d_n \) generating the uniformity of \( X_n \) and approximate the map \( g : C \setminus A \to X_n \setminus X_k \) by a closed embedding \( \bar{g} : C \setminus A \to X_n \setminus X_k \) such that
\[
\text{dist}(\bar{g}(c), g(c)) < \text{dist}(g(c), X_k)/2 \text{ for all } c \in C \setminus A.
\]

The interested reader can check that the map \( \bar{g} : C \to X_n \) defined by \( \bar{g}|A = f \) and \( \bar{g}|C \setminus A = \tilde{g} \) is a closed embedding determining a closed retral embedding \( \bar{g} : \langle C, r \rangle \to \langle X_n, r^n_k \rangle \).

\[ \square \]

6. A TOPOLOGICAL CHARACTERIZATION OF LF-SPACES

In this section we shall prove a topological characterization of LF-spaces that implies Theorem \( \text{[13]} \) announced in the Introduction.

Let \( \mathcal{C}_{<\omega} \) denote the class of finite-dimensional metrizable compact spaces and for an uncountable cardinal \( \kappa \) let \( \mathcal{C}_{<\kappa} \) be the class of completely-metrizable spaces of density \( < \kappa \).

**Theorem 6.1.** For a topological space \( X \) the following conditions are equivalent:

1. \( X \) is homeomorphic to a non-metrizable LF-space.
2. \( X \) is homeomorphic to the uniform direct limit \( \mathop{\text{u-lim}}_{\downarrow} X_n \) of an extendably \( \downarrow \mathcal{C}_{<\kappa} \)-universal \( \downarrow \mathcal{C}_{<\kappa} \)-tower \( \langle (X_n, r_n) \rangle_{n \in \omega} \) for some infinite cardinal \( \kappa \).
3. \( X \) is homeomorphic to the uniform direct limit \( \mathop{\text{u-lim}}_{\uparrow} X_n \) of a tower \( (X_n)_{n \in \omega} \) of metrizable uniform spaces such that each space \( X_n \):
   a. is uniformly locally equiconnected;
   b. is a uniform neighborhood retract in \( X_{n+1} \);
   c. has a uniform collar in \( X_{n+1} \);
   d. is contractible in \( X_{n+1} \);
   e. is a Hilbert manifold.
Proof. (1) ⇒ (3). Assume that \( X \) is homeomorphic to a non-metrizable LF-space. Then \( X \) can be identified with the direct limit \( \text{lc-lim} X_n \) of a tower of Fréchet spaces \( (X_n)_{n \in \omega} \). Since \( X \) is non-metrizable, \( X_n \neq X_{n+1} \) for infinitely many numbers. Passing to a suitable subsequence, we can assume that \( X_n \neq X_{n+1} \) for all \( n \).

Next, we shall show that each space \( X_n \) satisfies the conditions (a)–(e) from the assertion (3). Fix any \( n \in \omega \).

(a) The natural convex structure of the Fréchet space \( X_n \) turns it into a uniformly equiconnected space.

(b) In was remarked in [17] that (the proof of) the Dugundji Extension Theorem [12] implies that each convex subset \( C \) of a locally convex metric space is a uniform retract in each metric space that contains \( C \) as a closed subspace. In particular, there is a regular retraction \( r_{n+1} : X_{n+1} \to X_n \).

(c) To show that \( X_n \) has a uniform collar in \( X_{n+1} \), fix any point \( a \in X_{n+1} \setminus X_n \) and observe that the embedding
\[
e : X_n \times [0, 1] \to X_{n+1}, \quad e : (x, t) \mapsto x + ta
\]
determined a uniform collar of \( X_n \) in \( X_{n+1} \).

(d) The contractibility of each \( X_n \) (in \( X_{n+1} \)) is obvious.

(e) By the Anderson-Kadec-Toruńczyk Theorem [22], the Fréchet space \( X_n \) is homeomorphic to a Hilbert space.

(3) ⇒ (2). Assume that \( X \) is homeomorphic to the uniform direct limit \( \text{u-lim} X_n \) of a tower of uniform spaces \( (X_n)_{n \in \omega} \) satisfying the conditions (a)–(e). For every \( n \in \omega \) fix a function \( r_{n+1} : X_{n+1} \to X_n \) whose restriction to some uniform neighborhood of \( X_n \) witnesses that \( X_n \) is a uniform neighborhood retract in \( X_{n+1} \). Also let \( r_0 : X_0 \to X_0 \) be the identity map.

By (e), each space \( X_n \) is an \( l_2(\kappa_i) \)-manifold for some cardinal \( \kappa_i \). Let \( \kappa = \sup_{i \in \omega} \kappa_i^+ \). We claim that the cardinal \( \kappa \) is infinite. In the opposite case, we would obtain for some \( i \in \omega \) that \( \kappa_i = \kappa_{i+1} \) is finite and then \( X_i \subset X_{i+1} \) are two manifolds of the same finite dimension. By the Open Domain Principle [13, 1.8.11], \( X_i \) is an open subset in \( X_{i+1} \), which is not possible as \( X_i \) has a uniform collar in \( X_{i+1} \).

Now consider the class \( C_{\angle \kappa} \). For uncountable \( \kappa \) this class consists of completely-metrizable spaces of weight \( \kappa \) and for \( \kappa = \omega \) the class \( \mathcal{C}_{\omega} \) consists of finite-dimensional metrizable compacta.

First consider the case of uncountable cardinal \( \kappa = \sup_{i \in \omega} \kappa_i \). We claim that \( \mathcal{X} = (\langle X_n, r_n \rangle)_{n \in \omega} \) is an extendably \( \downarrow C_{\angle \kappa} \)-universal retral \( \downarrow C_{\angle \kappa} \)-tower.

For every \( n \in \omega \) the space \( X_n \) lies in a connected subspace of the \( l_2(\kappa_{n+1}) \)-manifold \( X_{n+1} \) and hence \( X_n \) has density \( \text{dens}(X_n) \leq \kappa_{n+1} < \kappa \). Being a Hilbert manifold, the space \( X_n \) is completely-metrizable. Thus all retral spaces \( \langle X_n, r_n \rangle \) belong to the class \( \downarrow C_{\angle \kappa} \) and hence the retral tower \( \mathcal{X} \) is a \( \downarrow C_{\angle \kappa} \)-tower.

Now we are going to show that for every space \( C \in C_{\angle \kappa} \) and every \( n \in \omega \) there is \( m \geq n + 2 \) such that the pair \( (X_m, X_n) \) is strongly \( \{C\} \)-universal. Since \( \text{dens}(C) < \kappa = \sup_{i \in \omega} \kappa_i \), there is \( m \geq n + 2 \) such that \( \kappa_m \geq \text{dens}(C) \). We claim that the pair \( (X_m, X_n) \) is strongly \( \{C\} \)-universal.

Given a map \( f : C \to X_m \), and an open cover \( \mathcal{V} \) of the set \( X_n \setminus X_k \), we need to approximate the restriction \( f|C \) unto the open subset \( C \setminus f^{-1}(X_m \setminus X_n) \) by a closed embedding \( g : C \setminus X_n \setminus X_k \) that is \( \mathcal{V} \)-near to \( f|C \). The existence of such a closed embedding immediately follows from the Toruńczyk’s Characterization Theorem [1,1] because \( X_n \setminus X_k \) is an \( l_2(\kappa_n) \)-manifold and \( C \in C_{\angle \kappa_n} \).

Applying Lemma [5,1] we see that the retral \( \downarrow C_{\langle \kappa \rangle} \)-tower \( \mathcal{X} \) is extendably \( \downarrow C_{\langle \kappa \rangle} \)-universal.
Next, assume $\kappa = \omega$. In this case, each space $X_n$ is a finite-dimensional manifold. Being contractible in $X_{n+1}$, the space $X_n$ is separable and thus is a topological direct limit of a tower of finite-dimensional compacta. Since each space $X_n$ is locally compact, the identity map $t\lim X_n \to u\lim X_n$ is a homeomorphism by Proposition 2.2. Now we see that $X = t\lim X_n = u\lim X_n$ is homeomorphic to the topological direct limit of a tower of finite-dimensional metrizable compacta.

In order to apply Sakai’s characterization Theorem [1.2], we need to show that each embedding $f : B \to X$ of a closed subset $B$ of a finite-dimensional metrizable compact space $C$ extends to an embedding $\bar{f} : C \to X$. Being a compact subset of the topological direct limit $t\lim X_n$, the subset $f(B)$ lies in some space $X_n$. Since $\sup_{i \in \omega} \kappa_i^+ = \kappa = \omega$, we can take $n$ so large that $\kappa_n \geq 2 \dim(C) + 1$.

Since the space $X_n$ is contractible in the ANR-space $X_{n+1}$, the map $f : B \to X_n \subset X_{n+1}$ is null-homotopic and hence admits a continuous extension $f_1 : C \to X_{n+1}$.

Let $\xi : C \to [0, 1]$ be a continuous map such that $\xi^{-1}(0) = B$. Since $X_{n+1}$ has a uniform collar in $X_{n+2}$, there is a continuous map $\gamma : X_{n+1} \times [0, 1] \to X_{n+2}$ such that $\gamma(x, 0) = x$ and $\gamma(x, t) \notin X_{n+1}$ for all $x \in X_{n+1}$ and $t \in (0, 1]$. Observe that the map

$$f_2 : C \to X_{n+2}, \ f_2 : c \mapsto \gamma(f_1(c), \xi(c)),$$

extends $f|B$ and has the property $f(B) \cap f(C \setminus B) \subset X_n \cap (X \setminus X_{n+1}) = \emptyset$.

By our assumption, the uniformity of the space $X_{n+2}$ is generated by some metric $d$. Since the dimension of the manifold $X_{n+2}$ equals $\kappa_{n+2} \geq 2 \dim(C) + 1$, the map $f_2 : C \setminus B \to X_{n+2}$ can be approximated by an injective map $\bar{f}_2 : C \setminus B \to X_{n+2}$ such that $d(\bar{f}_2(c), f_2(c)) < \operatorname{dist}(f_2(c), f(B))$ for all $c \in C \setminus B$ (this follows from a local version of the classical Menger-Pontryagin-Nöbeling-Tolstova Embedding Theorem [13, 1.11.4]). Then the map $\bar{f} : C \to X_{n+2}$ defined by $\bar{f}|B = f|B$ and $\bar{f}|C \setminus B = \bar{f}_2$ is continuous and injective. Since $C$ is compact, $\bar{f} : C \to X_{n+2} \subset X$ is the desired embedding that extends the embedding $f$. Now it is legal to apply Theorem [1.2]' and conclude that the space $X = t\lim X_n$ is homeomorphic to $\mathbb{R}^\infty$.

Theorem [1.2] also implies that $\mathbb{R}^\infty$ is homeomorphic to the topological direct limit $t\lim \mathbb{I}^n$ of the tower $\{\mathbb{I}^n\}_{n \in \omega}$. By Proposition 2.2, $t\lim \mathbb{I}^n$ is homeomorphic to the uniform direct limit of that tower. Endowing each cube $\mathbb{I}^n$, $n > 1$, with the uniform retraction

$$r_n : (t_1, \ldots, t_m) \to (t_1, \ldots, t_{m-1}),$$

we turn the tower $\{\mathbb{I}^n\}_{n \in \omega}$ into an extendably $\mathcal{C}_{\leq \kappa}$-universal retraction $\mathcal{C}_{\leq \kappa}$-tower $\{\langle \mathbb{I}^n, r_n \rangle\}_{n \in \omega}$ (here we put $r_0 : \mathbb{I}^0 \to \mathbb{I}^0$ be the identity map). The space $X = u\lim X_n$ is homeomorphic to the uniform direct limit $u\lim \mathbb{I}^n$ of the extendably $\mathcal{C}_{\leq \kappa}$-universal $\mathcal{C}_{\leq \kappa}$-tower $\{\langle \mathbb{I}^n, r_n \rangle\}_{n \in \omega}$. This completes the proof of (2).

The implication (2) $\Rightarrow$ (1) follows from the Uniqueness Theorem [4.2] and the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2). \hfill $\square$

7. A topological characterization of spaces $\Omega \times \mathbb{R}^\infty$

In this section we present a topological characterization of spaces of the form $\Omega \times \mathbb{R}^\infty$ where $\Omega$ is a model space. This characterization implies Theorem [1.7] announced in the Introduction.

**Theorem 7.1.** Let $\Omega$ be a model space and $\mathcal{C} = \mathcal{F}_0(\Omega)$. For a topological space $X$ the following conditions are equivalent:

1. $X$ is homeomorphic to $\Omega \times \mathbb{R}^\infty$;
(2) $X$ is homeomorphic to the uniform direct $\Ulim \Omega_n$ of an extendably $\mathcal{C}$-universal $\mathcal{C}$-tower $(\langle X_n, r_n \rangle)_{n \in \omega}$.

(3) $X$ is homeomorphic to the uniform direct limit $\Ulim X_n$ of a tower $(X_n)_{n \in \omega}$ of metrizable uniform spaces such that for each space $X_n$:

(a) is uniformly locally equiconnected,
(b) is a uniform neighborhood retract in $X_{n+1}$,
(c) has a uniform collar in $X_{n+1}$,
(d) is an ANR, contractible in $X_{n+1}$,
(e) belongs to the class $\mathcal{C}$, and
(f) together with $X_{n+1}$ forms a strongly $\mathcal{C}$-universal pair $(X_{n+1}, X_n)$.

Proof. (1) $\Rightarrow$ (3, 2). Assume that $X$ is homeomorphic to $\Omega \times \mathbb{R}^\infty$. By Theorem 2.3 of [21], the AR-space $\Omega$, is homeomorphic to a uniform retract of a normed space, and hence has an admissible metric turning it into a uniformly equiconnected uniform space.

It follows from Theorem 1.2 that $\mathbb{R}^\infty$ is homeomorphic to the topological direct limit $\Tlim \mathbb{I}^n$ of the tower

$$[0, 1] = \mathbb{I} \subset \mathbb{I}^2 \subset \mathbb{I}^3 \subset \ldots$$

where each cube $\mathbb{I}^n$ is identified with the face $\mathbb{I}^n \times \{0\}$ of the cube $\mathbb{I}^{n+1} \subset \mathbb{R}^{n+1}$. Since each cube $\mathbb{I}^n$ is compact, the topological direct limit $\Tlim \mathbb{I}^n$ is naturally homeomorphic to the uniform direct limit $\Ulim \mathbb{I}^n$, see Proposition 2.2. Consequently, the space $\Omega \times \mathbb{R}^\infty$ is homeomorphic to the space $\Omega \times \Ulim \mathbb{I}^n$. By Proposition 2.3, the latter product is naturally homeomorphic to the uniform direct limit $\Ulim (\Omega \times \mathbb{I}^n)$ of the tower $(\Omega \times \mathbb{I}^n)_{n \in \omega}$.

It is clear that for every $n$ the projection

$$r_{n+1} : \Omega \times \mathbb{I}^{n+1} \rightarrow \Omega \times \mathbb{I}^n, \quad r_n : (z, t_1, \ldots, t_n, t_{n+1}) \mapsto (z, t_1, \ldots, t_n),$$

is a uniformly continuous retraction witnessing that $\Omega \times \mathbb{I}^n$ is a uniform retract in $\Omega \times \mathbb{I}^{n+1}$. For $n = 0$ let $r_0 : \Omega \times \mathbb{I}^0 \rightarrow \Omega \times \mathbb{I}^0$ be the identity map. In such a way, we obtain a retral $\mathcal{C}$-tower $(\langle \Omega \times \mathbb{I}^n, r_n \rangle)_{n \in \omega}$.

It is clear that each uniform space $\Omega \times \mathbb{I}^n$:

(a) is uniformly equiconnected;
(b) is a uniform retract in $\Omega \times \mathbb{I}^{n+1}$;
(c) has a uniform collar in $\Omega \times \mathbb{I}^{n+1}$;
(d) is an AR;
(e) belongs to the class $\mathcal{C} = \mathcal{F}_0(\Omega)$ (because $\Omega \times \mathbb{I} \in \mathcal{F}_0(\Omega)$);
(f) includes in the strongly $\mathcal{C}$-universal pair $(\Omega \times \mathbb{I}^{n+1}, \Omega \times \mathbb{I}^n)$.

The latter two conditions follows from the definition of a model space $\Omega$.

Therefore, the space $\Omega \times \mathbb{R}^\infty$ satisfies the condition (3). Moreover, Lemma 5.1 ensures that the retral $\mathcal{C}$-tower $(\langle \Omega \times \mathbb{I}^n, r_n \rangle)_{n \in \omega}$ is extendably $\mathcal{C}$-universal and hence $X$ satisfies also the condition (2).

(3) $\Rightarrow$ (2) Assume that $X = \Ulim X_n$ is the uniform direct limit of a tower $(X_n)_{n \in \omega}$ of uniform spaces that satisfy the conditions (a)–(f). By (b) for every $n > 0$ there is a (not necessarily continuous) function $r_n : X_n \rightarrow X_{n-1}$ whose restriction on some uniform neighborhood of $X_{n-1}$ is continuous and witnesses that $X_{n-1}$ is a uniform neighborhood retract in $X_n$. Also let $r_0 : X_0 \rightarrow X_0$ be the identity map. By Lemma 5.1 the retral $\mathcal{C}$-tower $(\langle X_n, r_n \rangle)_{n \in \omega}$ is extendably $\mathcal{C}$-universal.
The implication (2) ⇒ (1) follows from the Uniqueness Theorem 4.2 and the fact that \( \Omega \times \mathbb{R}^\infty \) is homeomorphic to the uniform direct limit \( \xrightarrow{\text{u-lim}} (\Omega \times \prod) \) of the extendably \( \downarrow \mathcal{C} \)-universal \( \downarrow \mathcal{C} \)-tower \( (\langle \Omega \times \prod, r_n \rangle)_{n \in \omega} \).

\[ \text{□} \]

8. Small box products homeomorphic to LF-spaces

In this section we shall detect small box-products, homeomorphic to spaces of the form \( \Omega \times \mathbb{R}^\infty \).

By the small box-product of pointed topological spaces \( (X_n, *_n), n \in \omega \), we understand the subspace

\[ \square_{i \in \omega} X_i = \{ (x_i)_{i \in \omega} \in \square_{i \in \omega} X_i : \exists n \geq n \ (x_i = *_i) \} \]

of the box-product \( \square_{i \in \omega} X_i \). The latter space in the Cartesian product \( \prod_{i \in \omega} X_i \) endowed with the box-topology generated by the boxes \( \prod_{i \in \omega} U_i \) where \( U_i \subset X_i, i \in \omega \), are open sets.

The small box-product \( \square_{i \in \omega} X_i \) can be written as the countable union of the tower

\[ X_0 \subset X_0 \times X_1 \subset \ldots \subset \prod_{i \leq n} X_i \subset \ldots \]

where each finite product \( \prod_{i \leq n} X_i \) is identified with the subspace

\[ \{ (x_i) \in \square_{i \in \omega} X_i : \forall i > n \ (x_i = *_i) \} \subset \square_{i \in \omega} X_i. \]

**Theorem 8.1.** The small box-product \( \square_{i \in \omega} X_i \) of pointed spaces is homeomorphic to \( \Omega \times \mathbb{R}^\infty \) for some model space \( \Omega \), provided that

1. each space \( X_i \) is an absolute retract that contains more than one point,
2. each finite product \( \prod_{i \leq n} X_i \) belongs to the class \( \mathcal{F}_0(\Omega) \), and
3. each pair \( (\prod_{i \leq n} X_i, \prod_{i < n} X_i), n \in \mathbb{N} \), is strongly \( \mathcal{F}_0(\Omega) \)-universal.

**Proof.** By Theorem 2.3 of [21], each AR-space \( X_i \) carries a metric that turns it into a uniformly equiconnected uniform space. Those metrics turn the tower \( (\prod_{i \leq n} X_i)_{n \in \omega} \) into a tower of uniform spaces.

The union of this tower is equal to the small box-product \( \square_{i \in \omega} X_i \). By Proposition 5.5 from [4], the identity map \( \text{id} : \text{u-lim} \prod_{i \leq n} X_i \rightarrow \square_{i \in \omega} X_i \) is a homeomorphism.

So the proof of the theorem will be complete as soon as we prove that the uniform direct limit of the tower \( (\prod_{i \leq n} X_i)_{n \in \omega} \) is homeomorphic to \( \Omega \times \mathbb{R}^\infty \). For this it suffices to check the condition (a)–(f) of Theorem 7.1 for each space \( \prod_{i \leq n} X_i \) of the tower.

The condition (a) follows from the fact that the uniform spaces \( X_i, i \leq n \), are uniformly equiconnected, and so is their product.

The condition (b) follows from the fact that the coordinate projection

\[ r_n : \prod_{i \leq n+1} X_i \rightarrow \prod_{i \leq n} X_i \]

is a uniform retraction.

To see (c), find an injective map \( \gamma : [0, 1] \rightarrow X_{n+1} \) such that \( \gamma(0) = *_{n+1} \). Such a map exists because \( X_{n+1} \) is an AR containing more than one point. Then the map \( e : \prod_{i \leq n} X_i \times [0, 1] \rightarrow \prod_{i \leq n+1} X_i \) defined by

\[ e : (x_0, \ldots, x_n, t) \mapsto (x_0, \ldots, x_n, \gamma(t)) \]

witnesses that \( \prod_{i \leq n} X_i \) has a uniform collar in \( \prod_{i \leq n+1} X_i \).

The condition (d) trivially follows from the contractibility of the absolute retracts \( X_i, i \leq n \).
The final conditions (e), (f) are included into the hypothesis (3). □

Applying this theorem to infinite-dimensional Hilbert spaces, we get

**Corollary 8.2.** Let $\kappa$ be an infinite cardinal. The small box-product $\boxtimes_{i \in \omega} X_i$ of pointed spaces is homeomorphic to the LF-space $l_2(\kappa) \times \mathbb{R}^\infty$ if

1. some space $X_i$ is homeomorphic to $l_2(\kappa)$,
2. infinitely many spaces $X_i$ contain more than one point, and
3. each space $X_i$ is a completely-metrizable absolute retract of density $\leq \kappa$.

**Proof.** Consider the model space $\Omega = l_2(\kappa)$ and observe that the class $F_0(\Omega)$ coincides with the class $C_{< \kappa}$ of all completely-metrizable spaces of density $\leq \kappa$.

Assume that a sequence of pointed spaces $X_i, i \in \omega$, satisfies the conditions (1)–(3). Then some space $X_n$, say $X_0$, is homeomorphic to $l_2(\kappa)$ and all other spaces $X_i$ are absolute retracts from the class $C_{< \kappa}$. Since $X_0$ is homeomorphic to $l_2(\kappa)$, we can apply the ANR-Theorem for Hilbert manifolds \cite{21} and conclude that each product $\prod_{i \leq n} X_i$ is homeomorphic to $l_2(\kappa)$. Now Theorem \cite{11} ensures that for every $n > 0$ the pair $(\prod_{i \leq n} X_i, \prod_{i < n} X_i)$ is strongly $C_{< \kappa}$-universal.

Applying Theorem \cite{31} we conclude that the small box-product $\boxtimes_{i \in \omega} X_i$ is homeomorphic to $l_2(\kappa) \times \mathbb{R}^\infty$. □

**9. Direct limits of topological groups**

In this section we discuss the interplay between various direct limit topologies on the union $G = \bigcup_{n \in \omega} G_n$ of a closed tower

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

of topological groups. The group $G = \bigcup_{n \in \omega} G_n$ carries at least 6 natural direct limit topologies. The strongest one is the topology of the direct limit $\rightarrow t\lim G_n$ of the tower $(G_n)_{n \in \omega}$ in the category of topological spaces. By definition, $\rightarrow t\lim G_n$ is the group $G = \bigcup_{n \in \omega} G_n$ endowed with the strongest topology making the identity inclusions $G_n \to G$ continuous.

On the other extreme there is the topology of the direct limit $\rightarrow g\lim G_n$ of the tower $(G_n)_{n \in \omega}$ in the category of topological groups. This topology is the strongest topology turning $G$ into a topological group and making the identity inclusions $G_n \to G$ continuous.

Between the direct limits $\rightarrow t\lim G_n$ and $\rightarrow g\lim G_n$ there are four uniform direct limits of the topological groups $G_n$ endowed with one of four canonical uniformities: $\mathcal{U}^L$, $\mathcal{U}^R$, $\mathcal{U}^{LR}$, or $\mathcal{U}^{RL}$.

Here for a topological group $H$

- $\mathcal{U}^L$ is the *left uniformity* generated by the entourages $U^L = \{(x, y) \in H^2 : x \in yU\},$
- $\mathcal{U}^R$ is the *right uniformity* generated by the entourages $U^R = \{(x, y) \in H^2 : x \in Uy\},$
- $\mathcal{U}^{LR}$ is the *two-sided uniformity* generated by the entourages $U^{LR} = \{(x, y) \in H^2 : x \in yU \cap Uy\},$ and
- $\mathcal{U}^{RL}$ is the *Roelcke uniformity* generated by the entourages $U^{RL} = \{(x, y) \in H^2 : x \in UyU\}$

where $U = U^{-1}$ runs over open symmetric neighborhoods of the neutral element 1 of $H$.

The group $H$ endowed with one of the uniformities $\mathcal{U}^L, \mathcal{U}^R, \mathcal{U}^{LR}, \mathcal{U}^{RL}$ is denoted by $H^L, H^R, H^{LR}, H^{RL}$, respectively. These four uniformities on $H$ coincide if and only if $H$ is a SIN-group, which means that $H$ has a neighborhood base at $e$ consisting of open sets $U \subset H$ that are *invariant* in the sense that $U^H = U$ where $U^H = \{huh^{-1} : h \in H, u \in U\}$. Observe that a topological group $G$ is a SIN-group if and only if for each neighborhood $U \subset G$ of the neutral element $e$ its $G$-root $\sqrt{U} = \{x \in G : x^G \subset U\}$ is a neighborhood of $e$ in $G$. 
For a tower \( G_0 \subset G_1 \subset G_2 \subset \cdots \) of topological groups let \( u \lim G_n^L, u \lim G_n^R, u \lim G_n^{LR}, u \lim G_n^{RL} \) be the uniform direct limits of the towers of uniform spaces \( (G_n^L)_{n \in \omega}, (G_n^R)_{n \in \omega}, (G_n^{LR})_{n \in \omega}, (G_n^{RL})_{n \in \omega} \), respectively.

For these uniform direct limits we get the following diagram in which an arrow indicates that the corresponding identity map is continuous:

\[
\begin{array}{ccc}
\text{u-lim } G_n^L & \longrightarrow & \text{u-lim } G_n^{LR} \\
\text{t-lim } G_n & \longrightarrow & \text{u-lim } G_n^R \\
& & \text{g-lim } G_n \\
& & \text{u-lim } G_n^{RL}
\end{array}
\]

The following theorem proved in [11] and [7] shows that the identity map \( t \lim G_n \to g \lim G_n \) rarely is a homeomorphism.

**Theorem 9.1.** Let \( (G_n)_{n \in \omega} \) be a closed tower of topological groups.

1. If each group \( G_n, n \in \omega \), is locally compact, then the identity map \( t \lim G_n \to g \lim G_n \) is a homeomorphism;
2. If the groups \( G_n, n \in \omega \), are metrizable, then the identity map \( t \lim G_n \to g \lim G_n \) is a homeomorphism if and only if each group \( G_n \) is locally compact or there is \( n \in \omega \) such that for every \( m \geq n \) the group \( G_m \) is open in \( G_{m+1} \).

The following result proved in [5] indicates the cases when the identity map \( u \lim G_n^{LR} \to g \lim G_n \) is a homeomorphism.

**Theorem 9.2.** For a closed tower \( (G_n)_{n \in \omega} \) of topological groups the following conditions are equivalent:

1. the identity map \( u \lim G_n^{LR} \to g \lim G_n \) is a homeomorphism;
2. the identity map \( u \lim G_n^L \to g \lim G_n \) is a homeomorphism;
3. the identity map \( u \lim G_n^R \to g \lim G_n \) is a homeomorphism.

These equivalent conditions hold if the tower \( (G_n) \) satisfies one of the following conditions:

**SIN:** Every topological group \( G_n \) is a SIN-group.

**PTA:** Every group \( G_n \) has a neighborhood base \( B_n \) at the identity \( e \), consisting of open symmetric neighborhoods \( U \subset G_n \) such that for every \( m \geq n \) and every neighborhood \( V \subset G_m \) of \( e \) there is a neighborhood \( W \subset G_m \) of \( e \) such that \( WU \subset UV \).

**3-SIN:** For every \( n \in \omega \) and any neighborhoods \( V \subset G_{n+1} \) and \( U \subset G_{n+2} \) of the neutral element \( e \) the product \( V \cdot x^{G_n} \) is a neighborhood of \( e \) in the group \( G_{n+2} \). Here \( x^{G_n} \) is the product \( V \cdot x^{G_n} = \{ x \in G : x^{G_n} \subset U \} \) and \( x^{G_n} = \{ g \in G_n : g^{G_n} = \} \).

10. Topological groups homeomorphic to spaces \( \Omega \times \mathbb{R}^\infty \)

In this section we shall detect direct limits of topological groups, which are homeomorphic to LF-spaces or spaces of the form \( \Omega \times \mathbb{R}^\infty \). We shall do that for uniform direct limits of the form \( u \lim G_n \) where \( (G_n)_{n \in \omega} \) is a tower of topological groups. Theorem 9.2 gives conditions under which the uniform direct limit \( u \lim G_n^L \) is homeomorphic to \( u \lim G_n^R, u \lim G_n^{LR} \) or \( g \lim G_n \).
Theorem 10.1. For a tower \( (G_n)_{n \in \omega} \) of topological groups the uniform direct limit \( \varprojlim G_n \) is homeomorphic to an LF-space if for every \( n \in \omega \)

1. the uniform space \( G_n \) a uniform neighborhood retract in \( G_{n+1} \);
2. the topological space \( G_n \) is contractible in \( G_{n+1} \);
3. the topological space \( G_n \) a Hilbert manifold.

Proof. By (3), each group \( G_i \) is an \( l_2(\kappa_i) \)-manifold for some cardinal \( \kappa_i \). Let \( \kappa = \sup_{i \in \omega} \kappa_i \) and consider three cases.

Case 1. The cardinal \( \kappa \) is finite. Then there is \( m \in \omega \) such that \( \kappa_i = \kappa \) for all \( i \geq m \). For every \( i \geq m \), the groups \( G_i \subset G_{i+1} \) are Hilbert manifolds of the same finite dimension. Consequently, \( G_i \) is a closed-and-open subgroup of \( G_{i+1} \). Being closed-and-open and contractible in \( G_{n+1} \), the space \( G_n \) is contractible. The same is true for the space \( G_{i+1} \). Then the group \( G_i \), being a closed-and-open subset in the connected space \( G_{i+1} \), coincides with \( G_{i+1} \). It follows from the structure theorem of Iwasawa \([14]\) that the group \( G_i \), being a contractible \( \mathbb{R}^{\kappa} \)-manifold, is homeomorphic to \( \mathbb{R}^{\kappa} \). Since \( G_i = G_{i+1} \) for all \( i \geq m \), we see that \( \varprojlim G_n = G_m \) is homeomorphic to the LF-space \( \mathbb{R}^{\kappa} \).

Case 2. The cardinal \( \kappa \) is infinite but there is \( m \in \omega \) such that for every \( n \in \omega \) the group \( G_n \) is open in \( G_{n+1} \). Repeating the argument from the preceding case, we can show that \( \varprojlim G_n = G_m \), being a contractible \( l_2(\kappa) \)-manifold, is homeomorphic to the Hilbert space \( l_2(\kappa) \) according to the Classification Theorem for Hilbert manifolds, see \([8 \text{ IX.7.3}]\).

Case 3. For infinitely many numbers \( n \) the group \( G_n \) is not open in \( G_{n+1} \). Passing to a suitable subsequence \( (G_{n_k})_{k \in \omega} \), we may assume that each group \( G_n \) is not open in \( G_{n+1} \). Then also each group \( G_n \) is nowhere dense in \( G_{n+1} \). In this case we can apply Theorem 6.1 and show that the uniform direct limit \( \varprojlim G_n \) is homeomorphic to a non-metrizable LF-space. The conditions (2), (4), and (5) of that theorem hold according to our hypothesis. The remaining conditions (1) and (3) are established in the following two lemmas.

Lemma 10.2. If \( H \) is a closed nowhere dense subgroup of a locally path-connected topological group \( G \), then \( H^L \) has a uniform collar in the uniform space \( G^L \).

Proof. Since \( H \) is nowhere dense in the locally path-connected group \( G \) there is a continuous map \( \gamma : [0, 1] \to G \) such that \( \gamma(0) = e \) and \( \gamma(1) \notin H \). We may additionally assume that \( \gamma^{-1}(H) = \{0\} \). In the opposite case take the real number \( b = \max \gamma^{-1}(H) \in [0, 1) \) and consider the map

\[ \gamma' : [0, 1] \to G, \quad \gamma' : t \mapsto \gamma(b)^{-1} \cdot \gamma(b + (1 - b)t) \text{ for } t \in [0, 1]. \]

The map \( \gamma' \) will have the required property: \( \gamma'(0) = \gamma(b)^{-1} \cdot \gamma(b) \) and \( \gamma'(t) \notin H \) for \( t > 0 \).

It is easy to check that the map

\[ \alpha : H^L \times [0, 1] \to G^L, \quad \alpha : (h, t) \mapsto h \cdot \gamma(t), \]

determines a uniform collar of \( H^L \) in \( G^L \).

Lemma 10.3. If a topological group \( G \) is locally contractible, then the uniform space \( G^L \) is uniformly locally equiconnected.

Proof. Since \( G \) is locally contractible, there is a neighborhood \( V = V^{-1} \) of the neutral element \( e \in G \) and a continuous map \( \gamma : V \times [0, 1] \to G \) such that \( \gamma(x, 0) = x \) and \( \gamma(x, 1) = e \) for all \( x \in V \). Replacing the map \( \gamma \) by the map

\[ \gamma' : V \times [0, 1] \to G, \quad \gamma' : (x, t) \mapsto \gamma(e, t)^{-1} \cdot \gamma(x, t), \]
we may additionally assume that \( \gamma(e, t) = e \) for all \( t \in [0, 1] \).

The neighborhood \( V \) determines an entourage \( U = \{(x, y) \in G^2 : x \in yV\} \) that belongs to the left uniformity on \( G \). Then the function
\[
\lambda : U \times [0, 1] \to G, \quad \lambda : (x, y, t) \mapsto y \cdot \gamma(y^{-1}x, t),
\]
witnesses that the group \( G \) endowed with the left uniformity is uniformly locally equiconnected.

According to [3], each Polish ANR-group is a Hilbert manifold. This fact combined with Theorem 10.1 implies:

**Corollary 10.4.** For a tower \((G_n)_{n \in \omega}\) of Polish ANR-groups the uniform direct limit \(u-lim G_n^L\) is homeomorphic to a separable LF-space if each uniform space \(G_n^L\) is a uniform neighborhood retract in \(G_{n+1}^L\) and \(G_n\) is contractible in \(G_{n+1}\).

Combining Theorem 7.1 with Lemmas 10.2 and 10.3 we obtain

**Corollary 10.5.** For a closed tower \((G_n)\) of topological groups the uniform direct limit \(u-lim G_n^{LR}\) is homeomorphic to the product \(\Omega \times \mathbb{R}^\infty\) for some model space \(\Omega\) if for every \(n \in \omega\)

(1) the uniform space \(G_n^{L_n}\) is a uniform neighborhood retract in \(G_{n+1}^L\),

(2) the space \(G_n\) is nowhere dense, and contractible in \(G_{n+1}\),

(3) the space \(G_n\) is an ANR, and

(4) the pair \((G_{n+1}, G_n)\) is an \(F_0(\Omega)\)-universal pair.

If the class \(F_0(\Omega)\) is sufficiently rich, then we can prove a much stronger result.

**Definition 10.6.** A class \(\tilde{\mathcal{C}}\) of pairs of topological spaces is called:

- **compact** if for each pair \((K, C) \in \tilde{\mathcal{C}}\), the space \(K\) is compact and metrizable, and \(C \subset K\);
- **\(2^\omega\)-stable** if for any pair \((K, C) \in \tilde{\mathcal{C}}\) the pair \((K \times 2^\omega, C \times 2^\omega)\) belongs to \(\tilde{\mathcal{C}}\);
- **quotient-stable** if for any pair \((K, C) \in \tilde{\mathcal{C}}\) and a closed subset \(B \subset K\) the pair \((K/B, C \setminus B)\) belongs to \(\tilde{\mathcal{C}}\);
- **\(F_\sigma\)-additive** if for every pair \((K, C) \in \tilde{\mathcal{C}}\) and every \(F_\sigma\)-set \(F \subset K\) the pair \((K, C \cup F)\) belongs to \(\tilde{\mathcal{C}}\);
- **\(G_\delta\)-multiplicative** if for every pair \((K, C) \in \tilde{\mathcal{C}}\) and every \(G_\delta\)-subset \(G \subset K\) the pair \((K, C \cap G)\) belongs to \(\tilde{\mathcal{C}}\).

Classes of pairs having these five properties will be called **rich**.

A class \(\mathcal{C}\) of spaces is defined to be **rich** if there is a rich class \(\tilde{\mathcal{C}}\) of pairs such that \(\mathcal{C} = \{C : (K, C) \in \tilde{\mathcal{C}}\}\).

It should be noted that many classes considered in the Descriptive Set Theory are rich. In particular, all the projective classes \(\Pi_{n_1}^1\), \(\Sigma_{n_1}^1\), \(n \in \mathbb{N}\), and all the Borel classes \(\Pi_{\alpha}^0\), \(\Sigma_{\alpha}^0\), \(\alpha \geq 2\), are rich.

We recall that a topological space \(X\) is **\(\mathcal{C}\)-universal** for a class \(\mathcal{C}\) of topological spaces if each space \(X \in \mathcal{C}\) admits a closed topological embedding into \(X\).

The following non-trivial fact was implicitly proved in Theorem 4.2.3 of [3].

**Theorem 10.7.** Let \(\mathcal{C}\) be a rich class of spaces. An infinite-dimensional separable ANR-group \(G\) is \(\mathcal{C}\)-universal if and only if for each closed subset \(F\) the pair \((G, F)\) is strongly \(\mathcal{C}\)-universal.

Combining this theorem with Theorem 10.5 we obtain
Corollary 10.8. Let $\Omega$ be a model space such that the class $\mathcal{F}_0(\Omega)$ is rich. For a closed tower $(G_n)_{n \in \omega}$ of separable ANR-groups the uniform direct limit $\varprojlim G_n^L$ is homeomorphic to $\Omega \times \mathbb{R}^\infty$ if the group $G_0$ contains a closed topological copy of the space $\bar{\Omega}$, each group $G_n$ contractible in $G_{n+1}$ and each uniform space $G_n^L$ is a uniform neighborhood retract in $G_{n+1}$.

Finally, let us discuss the problem of detecting subgroups which are uniform neighborhood retracts in ambient groups (endowed with their left uniformity).

For a closed subgroup $H$ of a topological group let $G/H = \{ Hx : x \in G \}$ denote the quotient space endowed with the quotient topology. The subgroup $H$ is called (locally) topologically complemented in $G$ if the quotient map $q : G \to G/H$ is a (locally) trivial bundle. This happens if and only if the quotient map $q : G \to G/H$ has a (local) section.

Proposition 10.9. If $H$ is a locally topologically complemented subgroup of a topological group $G$, then the uniform space $H^L$ is a uniform neighborhood retract in $G^L$.

Proof. By our hypothesis, the quotient map $q : G \to G/H$ is a locally trivial bundle and as such, has a local continuous section $s : U \to G$ defined on an open neighborhood $U \subset G/H$ of the distinguished element $\bar{e} = He \in G/H$ (here $e$ stands for the neutral element of $H$).

Replacing the section $s$ by the section $s' : y \mapsto s(\bar{e})^{-1} \cdot s(y)$, we can additionally assume that $s(\bar{e}) = e$. Now it is easy to check that the formula
\[ r(x) = x \cdot (s \circ q(x))^{-1}, \quad x \in q^{-1}(U), \]
determines a uniform retraction of the uniform neighborhood $q^{-1}(U) \subset G$ of $H$ onto $H$, witnessing that $H^L$ is a uniform neighborhood retract in $G^L$. □

The other obvious condition, which implies that a closed subgroup $H \subset G$ is a uniform neighborhood retract in $G^L$ is that $H^L$ is a uniform absolute neighborhood retract.

Following [17], we define a metric space $X$ to be a uniform absolute (neighborhood) retract if $X$ is a uniform (neighborhood) retract in each metric space $M$ that contains $X$ as a closed isometrically embedded subspace. Modifying the proof of the Dugundji Extension Theorem [12], E. Michael [17] proved that each convex subset of a locally convex linear metric space is a uniform absolute retract. By [21] and [17] each absolute (neighborhood) retract is homeomorphic to a uniform absolute (neighborhood) retract.

By Birkhoff-Kakutani Metrization Theorem [20], the left uniformity of any first countable topological group $G$ is generated by some left-invariant metric. So we can think of topological groups as metric spaces endowing them with a left-invariant metric.

Problem 10.10. Detect topological groups that are uniform absolute neighborhood retracts.

This problem is not trivial because of the following

Example 10.11. There is a linear metric space $L$ which is an AR but not a uniform AR.

Proof. According to a famous result of R. Cauty [10], there exists a $\sigma$-compact linear metric space $E$, which is not an absolute retract. Let $D \subset E$ be a countable dense subset and $L$ be its linear hull in $E$. The space $L$ is an AR, being a countable union of finite-dimensional compacta, see [15].

By [17, 1.4] a metric space is a uniform AR if it contains a dense subspace that is a uniform AR. Since $E$ fails to be a (uniform) AR, its dense AR-subspace $L$ fails to be a uniform AR. □
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