Verification of the Thin-walled Beam Theory with Application of FEM and Shell Modelling

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Abstract. This article presents the results of a systematic comparison of predictions of the thin-walled beam theory and shell theory in the context of bending of elastic thin-walled beams with an open or closed cross-section. The results for shell theory were obtained using FEM and the ABAQUS/Standard program. The bending task of the thin-walled beam was decomposed into the task with a symmetrical and anti-symmetrical loading, so that the bending and torsion of the thin-walled element would be better visible. In order to objectify the comparison, two coefficients characterizing the mutual similarity of functions were introduced: root mean square deviation coefficient determined for the selected cross-section (in relation to the entire contour) and a coefficient determining the maximum relative difference of a given quantity. The comparison was made for both the stress and the displacement state.

1. Introduction

Thin-walled structures are commonly used in modern civil engineering. When designing these structures, the classic theory of thin-walled beams [1, 2, 3, 4, 5] is still used. This theory, along with the results obtained from experimental research, is the basis for regulations and standard guidelines regarding their design [6, 7, 8, 9]. For the majority of metal element structures, the solutions obtained using Bernoulli's beam theory are the basis. When the thickness of the element walls is distinctly smaller than the other dimensions, e.g. in PE-type or cold-formed PE-elements, it is reasonable to use the theory of thin-walled beams. This theory should be treated as a kind of generalization of Bernoulli's theory of beams, and the correctness of its predictions can be evaluated in the light of the theory of shells.

This paper presents a systematic comparison of solution results of bended thin-walled bars obtained using the thin-walled beam’s theory with the results obtained from the shell theory. Attention is paid to the influence of modeling of boundary conditions and wall thickness of a thin-walled element. Solutions within the shell theory [10, 11] were obtained using the finite element method (FEM) [12] and the ABAQUS Standard program [13], assuming the linear constitutive relationships and geometrical relationships (solutions were obtained within the small displacement theory). Calculations for a thin-walled bar with open and closed trapezoidal section were made, therefore, on the basis of the presented results, it is also possible to assess the impact of the closure of the cross-section on the work of the element during bending. A general load case is analyzed, which can be decomposed over the so-called symmetrical and anti-symmetrical loading cases. Thanks to this operation, it is possible to
emphasize the effects of dominant bending in the case of symmetrical load and dominant torsion in the case of antisymmetrical load.

The comparison of the results was based on two norms characterizing the accuracy of the TWB (thin-walled beam) solution representation in relation to the FEM solution. For this purpose, a coefficient of variation was defined understood as the relative mean square error and the relative maximum difference. Both the mean square error and the maximum difference were referred to the maximum results obtained from FEM solutions. The first parameter allows for evaluation of the average fit of the function while the second parameter indicates the so-called extreme difference between solutions. The introduction of this type of parameters allows for a more objective comparison of results in situations where the graphic evaluation leaves doubts. In connection with the large number of calculation results obtained, the results in the graphic form were presented only for selected characteristic solutions.

This paper is a continuation of the considerations regarding the stability of thin-walled bars carried out, inter alia, by the authors [14, 15] in relation to thin-walled perforated bars [16] also with regard to the influence of geometrical imperfections [17] on overall buckling resistance of thin-walled elements.

2. Verification example and its discussion

2.1. Formulation of the task and results presentation

The TWB verification was carried out on the example of a thin-walled bar, which geometry, support scheme and load along with the adopted material parameters are shown in figure 1. The trapezoidal cross-section of the rod has been adopted in two variants. The first as a thin-walled open cross-section with a small discontinuity at the point $D$, and the second one as closed in point $D$. The material of the rod was assumed as isotropic and linearly elastic by adopting: $E = 21000 \text{kN/cm}^2$, $\nu = 0.3$. The load values were assumed depending on the thickness successively as: 1, 2, 5 and 10 kN/m. Loads were selected so that the results obtained were in the same numerical range and could be easily compared. In figure 1 the loading was given as $2q$ due to the analysis method adopted hereinafter, in which the loading is divided into a symmetric and asymmetric part.

![Figure 1. Static scheme, loading pattern, the cross sectional shape and values of material parameters used in the calculation.](image)

The aim of the formulated above task was to compare the results of calculations obtained from analytical solutions, i.e. from the classical theory of thin-walled beams (TWB) [1, 2, 3] with the results of calculations made with the finite element method [12] using the ABAQUS program [13]. Graphs of the variation of normal stresses and displacement components ($u$, $v$, $w$) along the length of the centerline in selected cross-sections were compared graphically. In addition, mutual mean square error and maximum deviations of results obtained by TWB and FEM were determined.

In order to facilitate discussions of the results obtained, the task formulated in figure 1 was analyzed by its decomposition into symmetric and asymmetric tasks as shown in figure 2. This allows the analytical solution to be considered in symmetrical case as bending of the thin-walled beam and in asymmetric case as its constrained torsion. According to what has been said above, two cases of cross-section are analyzed: an open (figure 2 a, b, c) and closed (figure 2 d, e, f) cross-section.
All presented results will be given for a symmetrical, asymmetrical and summary loading tasks. It is possible to apply the superposition principle, because at all given loads, in none of the points the stress value does not exceed the limit of applicability of the linear elastic theory. All values appearing in the calculations were assumed to be expressed in [kN] and in [cm] as well as in their products or quotients.

2.2. Analytical solution

The task was solved by splitting into a symmetrical (S) and asymmetrical (A) case according to figure 2, that is, bending relative to the axis \( y \) uniformly distributed load \( 2q \) and torsion around the main pole line \( B \) with a distributed moment \( m = q \cdot b \), (see figure 1). The solution of bending as well as constrained torsion are well known. The deflection function of the geometric center of the cross-section \( O \) was assumed in the following form:

\[
w(x) = \frac{q x^2}{24 E J_y} \left( 2x^2 - 5lx + 3l^2 \right),
\]

(1)

In the case of torsion, both the left and right displacement type boundary conditions were modelled as “limited plane torsion”. It means that on the right side a rigid diaphragm wall for the cross-section has been used without limiting its freedom to rotation about the \( y \)-axis. However, the rotation around the \( x \)-axis is limited. The left side boundary conditions were assumed as typical for a fixed end (all rotations and displacements blocked). Under these boundary conditions, the solution of the differential equation takes the form of [18]:

\[
\theta(x) = \frac{B(0)}{GJ_s} \left( 1 - \cosh \alpha x \right) + \frac{M_s(0)}{GJ_s} \left( x - \frac{\sinh \alpha x}{\alpha} \right) - \frac{m}{GJ_s \alpha^2} \left[ 1 + \frac{(\alpha x)^2}{2} - \cosh \alpha x \right],
\]

(2)

where \( \alpha \) coefficient, parameters \( B(0) \) and \( M_s(0) \) (bimoment and total torsional moment) determined with the initial parameters method for the open cross-section are assumed in the form:
\[ \alpha = \frac{G J_s}{E_i J_{\omega}}, \quad B(0) = \frac{m l}{\alpha} \frac{1 + \cosh \alpha l}{2 \alpha \sinh \alpha l} \quad \text{and} \quad M_x(0) = \frac{m l}{2}, \quad (3) \]

Cross-sectional forces accompanying torsion can be determined using the following differential relations:

\[ M_s = G J_s \theta'(x), \quad B = -E_i J_{\omega} \theta'(x), \quad M_{\omega} = -E_i J_{\omega} \theta'(x), \quad M_x = M_{\omega} + M_s. \quad (4) \]

Similarly, the following calculations may be done in the case of a thin-walled rod with a closed cross-section. However, due to the high value of the coefficient \( \alpha \), the high torsional stiffness of the analyzed closed cross-section, the solution in the form (2) does not lead to reasonable numerical results, because the hyperbolic sine and cosine for small \( x \) values very quickly reach very high values. In that case practically the torsional angle and the cross-sectional forces associated with it with high torsional stiffness are close to zero.

The normal component of the stress tensor at any point on the center line of the thin-walled bar was determined from a known formula, separating into a symmetrical and asymmetrical part:

\[ \sigma_x = \sigma_y(x,s) = \sigma_x^M + \sigma_x^B = \frac{M_y(x)z(s)}{J_y} + \frac{B(x)\theta(s)}{J_{\omega}} = \sigma_x^S + \sigma_x^A. \quad (5) \]

Displacement components \( u, v, w \) (interpreted as displacements in direction of \( x, y, z \)) at any point \( K \) of the center line were determined on the basis of the determined deflection functions from (1) and the torsional angle from (2) as follows:

\[ u_K = u_K(x,s) = u_K^M + u_K^B = -w'(x)z(s) - \theta'(x)\theta(s) = u_K^S + u_K^A, \]
\[ v_K = v_K(x,s) = v_K^M + v_K^B = 0 - \theta(x)(z_B + z(s)) = v_K^S + v_K^A, \]
\[ w_K = w_K(x,s) = w_K^M + w_K^B = w(x) + \theta(x)y(s) = w_K^S + w_K^A. \quad (6) \]

2.3. Formulation of the FEM task for bending a thin-walled bar with the use of shell elements

Bent, thin-walled elements with an open and closed cross-section as in subsection 2.1 were modelled using shell elements of the S4R type (four-node elements with linear shape functions and reduced integration) with a division into 60000 finite elements, schematically given in figure 3.
Figure 3. Geometry of the FEM model, characteristic points, FEM mesh

The ABAQUS input files were generated without the use of the CAE graphical interface, so as to allow full control over the numbering of nodes and elements, and consequently the easier operation of the analysis results used for comparison with those obtained with the use of thin-wall beams theory.

Displacement boundary conditions were implemented in the so-called reference points RP1 and RP2. Previously, using the MPC option, bar member contour nodes at \( x = 0 \) cm with node RP1 and contour nodes at \( x = 600 \) cm with node RP2 were identified. It means that rigid diaphragm walls were created in these sections. In the RP1 node, full restraint was realized: \( u = v = w = 0 \) and \( \varphi_x = \varphi_y = \varphi_z = 0 \). At the other end, i.e. at RP2 node, it was assumed \( v = w = 0 \). The geometry is the same for a bar with an open and closed cross-section, i.e. on the P3-P4 line, the nodes are duplicated and in the case of a closed section bar they are additionally connected to each other using the MPC option. The loads of the analyzed thin-wall bars were adopted in accordance with the diagram shown in figure 2. All tasks were solved as linear problems in one increment of the Newton-Raphson algorithm.

3. Results and discussions

In accordance with the stated goal of the work, selected mutual comparisons of the results of the solutions formulated in section 2.1 will be presented in this section. The graphs of the cross-section displacements that illustrate its deformation are shown in figure 4. Displacement graphs are shown in the section with the coordinate \( x = 360 \) cm, in which they are close to extreme. The diagrams are presented separately for the case of symmetrical, asymmetrical and summary loading. These deformations were obtained by adding 10 times scaled displacement values of \( v \) and \( w \) to the coordinates of the cross-section points. These diagrams are presented for two extreme cases of analyzed wall thicknesses of the bar i.e: \( \delta = 3 \) and \( 20 \) mm, which corresponds to the ratio of section height to its thickness \( h / \delta = 53 \) and \( 8 \). In the first case, with a smaller thickness, we are rather dealing with shell than the rod, in the second the proportions correspond to those commonly used in manufactured rods.
Figure 4. Deformation of the contour view of the cross-section with a coordinate x = 360 cm with two thicknesses of the cross-sectional walls $\delta = 3$ mm ($h/\delta = 53$) and $\delta = 20$ mm ($h/\delta = 8$) (description in the text, scale factor=10).

The individual deformed contours differ in colour. The black colour corresponds to the initial configuration, while the blue colour to the results obtained from the analytical solution consistent with the theory of thin-walled bars (TWB). The red colour is used for the results of calculations made using the finite element method of an open section bar (FEM-OS), while the green colour corresponds to the results of calculations made with the finite element method of a closed section element (FEM-CS).

Figure 5 presents the variation of normal stress at individual cross-section on the centreline corresponding to the symmetrical and asymmetrical loading and their sum.
Figure 5. The variation of normal stresses on the center-line of the cross-section with coordinate x = 360 cm with two thicknesses of the cross-sectional walls $\delta = 3$ mm ($h/\delta = 53$) (left) and $\delta = 20$ mm ($h/\delta = 8$) (right) (h/δ = 8).

Comparison of the graphs shown in figure 5 leads to the conclusion that in the case of a small wall thickness $\delta$ of the rod, that is, when the ratio $h/\delta$ is high, in the case of a symmetrical load, the differences in the stress values determined for the open sections from FEM and TWB are significant. In the asymmetrical case, however, these differences are insignificant.

In order to numerically discuss this problem and for the purpose of comparative analysis, the notion of the coefficient of variation in stresses and displacements was introduced in selected sections of the bar in the following form:

$$c_v = \left(\frac{1}{N} \sum_{n=1}^{N} (w_{FEM} - w_{TWB})^2\right)^{1/2} / w_{FEM \ max} \times 100\%,$$

where: $w_{FEM}$ – result of the calculated quantity $w$ at a given cross-sectional point obtained using FEM, $w_{TWB}$ – result of the calculated quantity $w$ at a given cross-sectional point obtained using TWB, $w_{FEM \ max}$ – the maximum from the absolute values of the extremes of the calculated quantity $w$ at a given cross-sectional point obtained using FEM, $N$ – the number of nodes/elements in chosen cross-section. In order to supplement the information on the compliance of the results of the TWB predictions with the FEM results, the following coefficient (coefficient determining the maximum relative difference of a given quantity) was also defined:

$$c_{max} = \max_N \left(\left|w_{FEM} - w_{TWB}\right|/w_{FEM \ max}\right) \times 100\%,$$
where the parameters are defined as in the formula (7).

The \( c_v \) coefficient was defined in a percentage to express the mutual deviation of results obtained from FEM and TWB in the case of an open section bar. At zero value of this coefficient, we have full results. In other cases, it was assumed that if the value of this coefficient does not exceed 5%, compliance can be considered good. When it is in the range of \( 5\% \text{to} 10\% \), it can be considered satisfactory, while above 10% it is unsatisfactory. The nature of the compatibility of the results of TWB calculations with FEM can be better observed using bar charts.

![Graphs showing mapping error](image)

**Figure 6.** Mapping error \( c_{\text{max}} \) for: a) normal stress and for b) displacement components in the case of a thin-walled bar with an open cross-section for summary loading

For example, figure 6 shows the values of the coefficient \( c_{\text{max}} \) in case of summary loading. Next in figure 7 values of the coefficient \( c_v \) for normal stresses in particular cross-sections (figure 7a), and the displacements in a cross-section located with coordinate \( x = 360 \text{ cm} \) (figure 7b) are presented. As before, the graphs show separately the effect of symmetric, asymmetric and summary loads depending on the \( h/\delta \) ratio value. When assessing the results, one should assume that the lower the bar, the greater the compatibility of solutions.
Analysing the above results, it can be stated that in cross-sections with a coordinate \( x=0 \) and \( x=600 \) cm we have unsatisfactory compliance of the FEM and TWB results. To a large extent it depends on the modelling of boundary conditions in FEM, or if we assume that those in FEM have a simple physical interpretation, it rather indicates the weakness of the TWB. In other sections only for a very high value of the ratio \( h/\delta \) the consistency of the results is unsatisfactory especially for symmetric load, i.e. when bending. In the case of an asymmetrical load, the compliance of results is satisfactory regardless of the ratio of \( h/\delta \). The conformity of displacements, which in figure 7 are only shown for cross-section located with coordinate \( x = 360 \) cm, is much smaller, and displacement values have significant absolute values. Interestingly in that case compatibility of stresses is very good.

4. Conclusions

The paper presents a comparison of results obtained using the theory of thin-walled bars with the results obtained from the shell theory implemented in FEM program ABAQUS. The comparisons were made on the example of a thin-walled bar with a trapezoidal cross-section. The cross-section and static scheme of the beam are shown in figure 1. Both the closed and open cross-section were considered. The bar loading pattern was chosen so that it could be decomposed into a symmetrical and asymmetrical one. In the case of a symmetrical loading pattern, bending of the rod dominated, and in the case of asymmetrical loading pattern the torsion dominated. Multivariate calculations were carried out using both the theory of thin-walled bars and the theory of shells. Different wall thicknesses of the bar were considered while maintaining its cross-section dimensions and length. Also different boundary conditions on the right side and the effect of the diaphragm walls on the determined values.
were considered. The results of the calculations are compiled in the form of bar charts, of which the selected ones are presented in this paper.

The subject of comparisons were all the components of the displacement vector and the normal component of stresses in selected contour points and cross-sections. The trajectory of displacements and stresses along the length of the rod and along the center line of selected sections of the thin-walled bar were compared. Comparisons were made by introducing two coefficients. The first of them is the so-called relative root-mean-square deviation and the second is used for determining the maximum relative difference of a given quantity. Both coefficients were calculated for each analyzed quantity (displacement or stress) in selected cross-sections and were related to their maximum absolute value determined from the shell solution and expressed in %. As a result of comparisons, it was found that in the case of open cross-sections:

- TWB maps well the FEM shell solutions in sections sufficiently distant from the displacement type of boundary conditions and for moderate thin-walledness, i.e. at \( h/\delta \) close to 10.
- While maintaining the above conditions, TWB significantly better reflects the state of stress than displacement.
- TWB significantly better reflects torsion than bending, regardless of the thickness of the cross-sectional walls both in relation to displacements and stresses.
- The application of diaphragm walls and rigid supports introduces high stress concentrations causing even their 50% increase in very thin walls (\( h/\delta \approx 50 \)) and about 20% in moderate thickness (\( h/\delta \approx 10 \)).

In case of the closed cross-sections:

- TWB does not lead to correct solution of the torsion problem, which, however, is not so important as results from the FEM analysis and the shell solution. Torsion does not cause significant displacements and normal stresses.
- Torsional resistance is very high and bending is dominant regardless of the load and thickness. With small thicknesses, local deformation of the cross-section is possible.
- In the case of simultaneous bending and torsion, it is possible to use the classic Bernoulli beams theory.

Summarizing the above considerations, it can be concluded that the use of TWB is justified, however, with certain restrictions mentioned above.

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