A general solution for particle motion in a plane wave with arbitrary initial conditions

This section summarises the analytical solution for relativistic particle motion in a plane electromagnetic wave with an arbitrary initial phase presented in Ref.\textsuperscript{1}. Particle energy lost due to the classical radiation reaction is included in the model.

The wave amplitude is taken to be constant, and there is no feedback on the wave itself. According to Ref\textsuperscript{3}, relativistic 4-velocity within a linearly polarized wave can be expressed as:

$$u^\mu(\phi) = \frac{1}{h(\phi)} \left\{ u^\mu_0 + \frac{1}{\rho_0} \left[ h^2(\phi) - 1 \right] n^\mu - \frac{1}{\rho_0} I_1(\phi) \frac{e f^{\mu\nu}_1}{m} u_{0,\nu} + \frac{1}{2\rho_0} \xi_1^2 I_1(\phi) n^\mu \right\}, \quad (1)$$

where $\xi_1^2 = \xi_1^2 = (eE_0/m\omega_0)^2$, $f^{\mu\nu}_1 = n^\mu a_1^\nu - n^\nu a_1^\mu$, $a_1^\mu = (0, a_0, 0, 0)^T$ is the normalized vector potential and $n^\mu = (1, \vec{n})$. Here, $\vec{n}$ is a unit vector parallel to the laser direction of motion $\vec{n} \parallel u_1$ and $\rho_0 \equiv (nu_0)$. Einstein notation was used (a repeated Greek index is equivalent to a summation over all dimensions and $(ab) \equiv a^\mu b_\mu$). Functions $I_1(\phi)$ and $h(\phi)$ carry the imprint of the radiation reaction. They are expressed as:

$$h(\phi) = 1 + \frac{1}{3} \alpha \eta_0 \xi_1^2 \omega_0 |\alpha_0(0) - \sin(\omega_0\phi) \cos(\omega_0\phi) + \sin(\omega_0\phi_0)\cos(\omega_0\phi_0)|$$

$$I_1(\phi) = \cos(\omega_0\phi) h(\phi) - \cos(\omega_0\phi_0) \frac{2}{3} \alpha \eta_0 (\sin(\omega_0\phi) - \sin(\omega_0\phi_0)) \times$$

$$\left[ 1 + \frac{\xi_1^2}{3} (\sin^2(\omega_0\phi) + \sin(\omega_0\phi) \sin(\omega_0\phi_0) + \sin^2(\omega_0\phi_0)) \right], \quad (2)$$

where $\phi_0$ denotes the initial phase, $\phi$ current phase of the particle within the field of the electromagnetic wave, $\alpha$ is the fine structure constant, and $\eta_0 = (k_0 u_0)/m$ is the wave frequency measured in the electron rest frame in units of the electron mass $m$.

Other quantities denote elementary charge $e$ and electromagnetic field amplitude $E_0$. Without radiation reaction, Eqs. (2) reduce to:

$$h(\phi) = 1, \quad I_1(\phi) = \cos(\omega_0\phi) - \cos(\omega_0\phi_0) \quad (3)$$

The equations (1)-(3) are presented in natural units, as in Ref.\textsuperscript{1}. In those units $h = c = 1$, and the fine-structure constant is defined by $\alpha = e^2/\hbar c \simeq 1/137$. We now switch to the dimensionless units more common in laser-plasma interactions. Here, all the quantities are normalized to the laser frequency $\omega_0$, such that $t \rightarrow t \omega_0$, $p \rightarrow p/mc$, $E \rightarrow E/mc^2$, $E \rightarrow eE/mc\omega_0$ and $B \rightarrow eB/mc\omega_0$. In these units, $\omega_0 = 1$, $\lambda_0 = 2\pi$, $c = 1$, mass is normalized to the electron mass $m$ and charge is normalized to the elementary charge $e$. The particle equations of motion by component for the case without radiation reaction starting at arbitrary laser phase with an arbitrary initial momentum can be written as:

$$\gamma(\phi) = \gamma_0 + a_0 \frac{p_{0,1}}{\gamma_0 - p_{0,3}} I_1(\phi) + \frac{1}{2(\gamma_0 - p_{0,3})} \alpha_0^2 I_1(\phi)^2$$

$$p_1(\phi) = p_{0,1} + a_0 I_1(\phi) \quad \quad \quad \quad \quad (4)$$

$$p_2(\phi) = p_{0,2}$$

$$p_3(\phi) = p_{0,3} + a_0 \frac{p_{0,1}}{\gamma_0 - p_{0,3}} I_1(\phi) + \frac{1}{2(\gamma_0 - p_{0,3})} \alpha_0^2 I_1(\phi)^2$$
We can compute the spatial displacement of the particle as a function of the phase $\phi$. We have $x_3 = \int_0^t p_3(t) / \gamma(t) dt = \int_0^\phi p_3(\phi) / (\gamma(\phi) - p_3(\phi)) d\phi$, where we used a relation between the phase and time $d\phi = dt / (\gamma - p_3) / \gamma$ to perform the change of variables. The denominator is an integral of motion, such that $\gamma(\phi) - p_3(\phi) = \gamma_0 - p_{0,3}$. The spatial displacement in the laser direction of motion is then given by:

$$x_3 = \frac{p_{0,3}}{\gamma_0 - p_{0,3}} (\phi - \phi_0) + \frac{a_0 p_{0,1}}{(\gamma_0 - p_{0,3})^2} \left[ \sin \phi - \sin \phi_0 - (\phi - \phi_0) \cos \phi_0 \right]$$

$$+ \frac{a_0^2}{2(\gamma_0 - p_{0,3})^2} \left[ (\phi - \phi_0) \left( \cos^2 \phi_0 + \frac{1}{2} \right) - 2 \cos \phi_0 \sin \phi + \frac{1}{4} \sin(2\phi) + \frac{3}{4} \sin(2\phi_0) \right]$$

(5)

The standard result for a particle starting at rest can be retrieved by taking $\gamma_0 = 1$ and $p_{0,1} = p_{0,2} = p_{0,3} = 0$. However, our particles are not created at rest. Electron-positron pairs are created by scattering an LWFA electron bunch with an intense laser at normal incidence. We can assume as a first approximation that $p_{0,1} = p_{0,3} = 0$, but $p_{0,2} \neq 0$, i.e. the particle initial velocity is perpendicular to both the laser polarization direction $x_1$ and to the laser propagation direction $x_3$.

**Maximum energy and plane wave acceleration length for particles initially at 90 degrees**

For a particle born at 90 degrees of incidence to the laser axis with an initial Lorentz factor $\gamma_0$, the maximum attainable energy according to Eqs. (4) is given by $\epsilon_{\text{max}} = 2a_0^2 / \gamma_0$. This is achieved when $I_1(\phi)^2$ is at the maximum. For simplicity, let us consider a particle with $\phi_0 = 0$, as an example that allows to obtain that maximum. Here, the expression for longitudinal displacement given by Eq. (5) can be simplified:

$$x_3 = \frac{a_0^2}{2\gamma_0^2} \left( \frac{2}{2} \phi - 2 \sin \phi + \frac{1}{4} \sin(2\phi) \right).$$

(6)

The maximum energy $\epsilon_{\text{max}}$ is reached for $\phi = \pi$, after the particle has traveled a distance of $l_{\text{pwa}} = 3a_0^2 \pi / (4\gamma_0^2)$ or $l_{\text{pwa}} = \lambda_0 \times 3a_0^2 / (8\gamma_0^2)$. We define this distance as the plane wave acceleration length. If our particle would start at a different $\phi_0$ in otherwise identical conditions, its maximum energy can be smaller or equal to $\epsilon_{\text{max}}$. For particles born at normal incidence, the particles with the lowest initial $\gamma_0$ are the ones that can attain the highest energy in the interaction with the plane wave (because $\epsilon_{\text{max}} \propto \gamma_0^{-1}$). However, one should note here that the higher energy gain is possible due to the slower dephasing, and the plane wave acceleration length is reversely proportional to the square of the initial energy $l_{\text{pwa}} \propto \gamma_0^{-2}$. Such particles, therefore, propagate a longer distance before reaching the $\epsilon_{\text{max}}$.

**Correction of the maximum energy due to the pulse focusing**

The maximum energy and the acceleration length in the previous section are valid for a plane wave interaction. However, the lasers are not plane waves, and the laser de-focusing affects the acceleration. Particles are expected to attain an energy cutoff similar to $\epsilon_{\text{max}}$ in a focused laser only if $R_L \gg l_{\text{pwa}}$, where $R_L = \pi W_0^2 / \lambda_0$ is the Rayleigh range, $W_0$ and $\lambda_0$ are the laser waist and the wavelength respectively. If $R_L \simeq l_{\text{pwa}}$, we require an estimate on how the laser de-focusing affects the acceleration. To this end, we perform a series of test particle simulations, where the particles are initialized with the most favourable phase for the highest energy gain within the shortest propagation distance. The test particles are born at the laser central axis, in focus, at $\phi_0 = 0$ of the wave. The 3D analytical expression for the fields of the focused laser pulse is taken from Ref.2. We varied the laser intensity, and the initial particle energy is chosen according to $\gamma_0 = 5 \times 10^4 / a_0$ to be consistent with the lowest available energy expected in the system.

The results displayed in Fig. 1 correspond to a range of laser intensities between $a_0 = 100$ and $a_0 = 4000$ and three different waist sizes: $W_0 = 3.2 \, \mu m$, $W_0 = 5 \, \mu m$ and $W_0 = 10 \, \mu m$. For every example, the maximum energy attained by the particle born in the laser focus with $\phi_0 = 0$ is divided by the maximum energy such a particle could attain in a plane wave. An interesting point to note here is that the distances between the case with and without radiation reaction are not strongly pronounced. This is consistent with the fact that a quickly acquired parallel component of the momentum reduces the $\chi_2$ and the radiation reaction stops being relevant for the particle dynamics. Figure 1 indicates that the ratio $\epsilon / \epsilon_{\text{max}}$ is a function of $R_L / l_{\text{pwa}}$ and for $R_L / l_{\text{pwa}} < 1$ this can be approximated as:

$$\frac{\epsilon}{\epsilon_{\text{max}}} = 0.5 \sqrt{\frac{R_L}{l_{\text{pwa}}}}$$

(7)
Figure 1. Ratio of the maximum attainable energy for a focused laser $\mathcal{E}$ and the maximum attainable energy for the same laser intensity in a plane wave $\mathcal{E}_{\text{max}}$ as a function of the ratio of the Rayleigh range $R_L$ to the acceleration length $l_{\text{pwa}}$. For $0.05 < R_L/l_{\text{pwa}} < 1$, one can approximate the correction of maximum energy due to the pulse focusing using the Eq. (7).

For $R_L/l_{\text{pwa}} > 10$, we consider the condition $R_L \gg l_{\text{pwa}}$ satisfied and $\mathcal{E} \simeq \mathcal{E}_{\text{max}}$. For $l_{\text{pwa}} < R_L < 10 l_{\text{pwa}}$, Eq. (7) is not a good estimate, due to its value being on the same order of $\mathcal{E}_{\text{max}}$. For $l_{\text{pwa}} < R_L < 10 l_{\text{pwa}}$, the maximum energy takes values between $\mathcal{E}_{\text{max}}/2$ and $\mathcal{E}_{\text{max}}$.

Equation (7) shows the defocusing correction for the maximum energy of a particle born in the very centre of the laser field. We can assume that for an arbitrary particle born anywhere in the laser field (with the same $\gamma_0$), the effective interaction length cannot be more than twice the value for the particle born in the centre. As the effective interaction length is proportional to $R_L$, we should not expect to obtain particles with energies higher than

$$\frac{\mathcal{E}}{\mathcal{E}_{\text{max}}} = 0.5 \sqrt{\frac{2R_L}{l_{\text{pwa}}}}.$$  \hspace{1cm} (8)

Full-scale 3D simulations in the main manuscript are in very good agreement with the predictions for the cutoff of the particle energy spectrum given by Eq. (8).

References

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