Partition functions of integrable lattice models and combinatorics of symmetric polynomials

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Abstract

We review and present new studies on the relation between the partition functions of integrable lattice models and symmetric polynomials, and its combinatorial representation theory based on the correspondence, including our work. In particular, we examine the correspondence between the wavefunctions of the XXZ type, Felderhof type and the boson type integrable models and symmetric polynomials such as the Schur, Grothendieck and symplectic Schur functions. We also give a brief report of our work on generalizing the correspondence between the Felderhof models and factorial Schur and symplectic Schur functions.

1 Introduction

Integrable lattice models $^1$ $^2$ $^3$ $^4$ are special classes of statistical mechanics, which interesting connections with many subjects of mathematics have been found in the past, and will be found in the future. In particular, it plays a very important role in representation theory and combinatorics, and many notions and objects in modern representation theory have their origin in integrable models, the quantum group $^5$ $^6$ for example. Many objects were introduced by investigating the mathematical structures of microscopic or quasimacroscopic quantities such as a single $R$-matrices and monodromy matrices. From the point of view of statistical mechanics, the most fundamental quantity in statistical mechanics is the partition function, which is the most macroscopic bulk quantity constructed from the monodromy matrices and characterizes the whole system. For non-integrable models, partition functions

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are just numbers, which do not usually have interesting connections with mathematics. The situation changes for integrable models: one can introduce the spectral parameter. There are many advantages of introducing spectral parameter. One thing from the point of physics is that if one introduces the spectral parameter, we can construct a generating function of conserved quantities for integrable models. One advantage from the viewpoint of mathematics is that we find it corresponds to the symmetric variables of symmetric polynomials when one investigates a class of partition functions called the wave functions. Another advantage is that it plays a role of the variable for the generating function of the enumeration of alternating sign matrices \[7, 8, 9, 10\]. In fact, these facts are deeply related with each other.

In this article, we review and present new studies on the relation between partition functions of integrable lattice models and combinatorics of symmetric polynomials. We mainly deal with two integrable vertex models: (i) the XXZ-type models and (ii) the Felderhof models. These models are related with the quantum group of Drinfeld-Jimbo type and the colored representation, respectively. In the next two sections, We explain how symmetric polynomials such as the Schur, Grothendieck polynomials and their generalizations arise from a particular type of partition functions called the wave functions. In section 4, we review how Cauchy and dual Cauchy identities can be derived by dealing with scalar products and domain wall boundary partition functions by taking the XXZ-type and the Felderhof models as an example, respectively. In section 5, we make comments on other six-vertex models and give a brief report of our study on investigating symmetric polynomials from the generalized Felderhof models. Section 6 is devoted to conclusion.

2 XXZ-type models

We first investigate the Yang-Baxter integrability associated with the \(U_q(sl_2)\) quantum group, following the Appendix of [11].

The most fundamental object in quantum integrable models is the \(R\)-matrix satisfying the Yang-Baxter relation

\[
R_{ab}(u_1/u_2)R_{aj}(u_1)R_{bj}(u_2) = R_{bj}(u_2)R_{aj}(u_1)R_{ab}(u_1/u_2),
\]

holding in \(\text{End}(W_a \otimes W_b \otimes V_j)\) for arbitrary \(u_1, u_2 \in \mathbb{C}\). In this section, we take \(W\) and \(V\) as a complex two-dimensional vector spaces \(W = V = \mathbb{C}^2\) spanned by the “empty state” \(|0\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\) and the “particle occupied state” \(|1\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\), and take the \(R\)-matrix acting on the tensor product \(W \otimes V\) or \(V \otimes V\) as the following one

\[
R(u) = \begin{pmatrix}
  u - qu^{-1} & 0 & 0 & 0 \\
  0 & q(u - u^{-1}) & 1 - q & 0 \\
  0 & 1 - q & u - u^{-1} & 0 \\
  0 & 0 & 0 & u - qu^{-1}
\end{pmatrix},
\]

which is the \(R\)-matrix associated with the quantum group \(U_q(sl_2)\) \[5, 6\]. The quantum integrable model constructed from the \(R\)-matrix (2.2) is called the XXZ chain.

In the original Yang-Baxter relation (2.1), every \(R\)-matrix is the same. However, one can generalize this relation to the following \(RLL\) relation still keeping the integrability

\[
R_{ab}(u_1/u_2)L_{aj}(u_1)L_{bj}(u_2) = L_{bj}(u_2)L_{aj}(u_1)R_{ab}(u_1/u_2),
\]

(2.3)
of $B$ with acting on the quantum space $V$. The four elements of the monodromy matrix $A$ which is obtained by setting $\beta$ parameter $\alpha$ are regarded as a particularly important $L$-object. Among the above generalized $L$-operator in more detail since we find that the $\beta$-operator in (2.4), the following one $L(u) = \begin{pmatrix} (1-q)\alpha_1\alpha_2 + \alpha_3\alpha_6 - \alpha_4\alpha_5 = 0, \\
(q^2 - q)\alpha_1\alpha_2 + q^2\alpha_3\alpha_6 - \alpha_4\alpha_5 = 0. \\
\end{pmatrix}$

Among the above generalized $L$-operator (2.4), the following one $L(u) = \begin{pmatrix} u + q\beta u^{-1} & 0 & 0 & 0 \\
0 & q(u + \beta u^{-1}) & 1 - q & 0 \\
0 & 1 - q & -\beta^{-1}u - u^{-1} & 0 \\
0 & 0 & 0 & -\beta^{-1}u - qu^{-1} \\
\end{pmatrix},$ (2.7)

which is obtained by setting $\alpha_1 = \alpha_2 = \alpha_3 = 1, \alpha_4 = q\beta, \alpha_5 = -\beta^{-1}, \alpha_6 = -1$ can be regarded as a particularly important $L$-operator. For example, (2.7) is a $\beta$-deformation of the $U_q(sl_2)$ $R$-matrix (2.2). Another observation is that the wavefunction constructed from the $L$-operator at $q = 0$ gives the $\beta$-Grothendieck polynomials of the Grassmannian variety.

Now we introduce and examine a class of partition function which is usually called as the wavefunction.

First, let us consider the monodromy matrix:

$$T_a(u) = \prod_{j=1}^{M} L_{aj}(u) = \begin{pmatrix} A(u) & B(u) \\
C(u) & D(u) \end{pmatrix}.$$ (2.8)

The four elements of the monodromy matrix $A(u), B(u), C(u)$ and $D(u)$ are the operators acting on the quantum space $V_1 \otimes \cdots \otimes V_M$.

The arbitrary $N$-particle state $|\psi\{\{u\}_N\}\rangle$ (resp. its dual $\langle\psi\{\{u\}_N\}|$) (not normalized) with $N$ spectral parameters $\{u\}_N = \{u_1, u_2, \ldots, u_N\}$ is constructed by a multiple action of $B$ (resp. $C$) operator on the vacuum state $|\Omega\rangle := |0^M\rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$ (resp. $\langle\Omega|$ := $\langle0^M|$ := $\langle0|_1 \otimes \cdots \otimes |0\rangle_M$):

$$|\psi\{\{u\}_N\}\rangle = \prod_{j=1}^{N} B(u_j)|\Omega\rangle, \quad \langle\psi\{\{u\}_N\}| = \prod_{j=1}^{N} \langle\Omega| C(u_j).$$ (2.9)
Next, we introduce the wavefunction \( \langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle \) and its dual \( \langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle \) as the overlap between an arbitrary off-shell \( N \)-particle state \( \psi(\{u\}_N) \) and the (normalized) state with an arbitrary particle configuration \( |x_1 \cdots x_N\rangle \) \( (1 \leq x_1 < \cdots < x_N \leq M) \), where \( x_j \) denotes the positions of the particles. The particle configurations are explicitly defined as

\[
|x_1 \cdots x_N\rangle = \prod_{j=1}^N \sigma_{x_j}^+ | x_1 \cdots x_N\rangle = \prod_{j=1}^N \sigma_{x_j}^- | \Omega \rangle.
\] (2.10)

We find the following form on the wavefunction.

**Theorem 2.1.** The wavefunction \( \langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle \) constructed from the generalized L-operator (2.7) has the following form

\[
\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \prod_{j=1}^N \frac{(1-q)(u_j + q \beta u_j^{-1})^M}{(-\beta^{-1} u_j - u_j^{-1})} \prod_{1 \leq j < k \leq N} \frac{q - u_j^{-2} u_k^2}{1 - u_j^{-2} u_k^2} \times \sum_{\sigma \in S_N} \prod_{1 \leq j < k \leq N; \sigma(j) > \sigma(k)} \frac{1}{q - u_j^2 u_k^2} \prod_{j=1}^N \left( \frac{-\beta^{-1} u_{\sigma(j)} - u_{\sigma(j)}^{-1}}{u_{\sigma(j)} + q \beta u_{\sigma(j)}^{-1}} \right)^{x_j}.
\] (2.11)

We can regard this wavefunction as a \( q \)-deformed \( \beta \)-Grothendieck polynomials since this generalizes the expression for the wavefunction constructed from the L-operator at the point \( q = 0 \) [12], which gives the \( \beta \)-Grothendieck polynomials. The sum of the expression is exactly proved from the method of matrix product representation [13, 14] and the domain wall boundary partition function [15, 16]. A similar proof is given for the wavefunction of the Felderhof model in the next section. Details for the XXZ-type models will appear elsewhere [17].

We check below that the wavefunction (2.11) reduces essentially to the \( \beta \)-Grothendieck polynomials.

\[
\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \prod_{j=1}^N \frac{u_j^M}{-\beta^{-1} u_j - u_j^{-1}} \prod_{1 \leq j < k \leq N} \frac{-u_j^{-2} u_k^2}{1 - u_j^{-2} u_k^2} \times \sum_{\sigma \in S_N} \prod_{1 \leq j < k \leq N; \sigma(j) > \sigma(k)} \frac{1}{-u_j^2 u_k^2} \prod_{j=1}^N \left( \frac{-\beta^{-1} u_{\sigma(j)} - u_{\sigma(j)}^{-1}}{u_{\sigma(j)} + q \beta u_{\sigma(j)}^{-1}} \right)^{x_j}.
\]

\[
= \prod_{j=1}^N \frac{u_j^{M+1}}{-\beta^{-1} u_j^2 - 1} \prod_{1 \leq j < k \leq N} \frac{u_k^2}{u_k^2 - u_j^2} \times \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{j=1}^N \prod_{1 \leq j < k \leq N; \sigma(j) > \sigma(k)} \frac{-u_{\sigma(j)}^{-2} u_{\sigma(k)}^2}{u_{\sigma(j)}^2 - u_{\sigma(k)}^2} \prod_{j=1}^N \left( -\beta^{-1} u_{\sigma(j)}^2 - 1 \right)^{x_j}.
\]
where \( z \) with weakly decreasing nonnegative integers \( \lambda \) denotes a Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \) with weakly decreasing nonnegative integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \). The correspondence between the Young diagram and the configuration of particles is given by \( \lambda_j = x_{N-j+1} - N + j - 1 \). We also relate the symmetric variables \( z \) and the spectral parameters by \( z_j = -\beta^{-1} - u_j^{-2} \).

We remark that the overall factor of the right hand side of the correspondence between the wavefunction and the \( \beta \)-Grothendieck polynomials (2.12) can easily corrected to be one by a simple gauge transformation of the \( L \)-operator, which has combinatorial description in terms of pipedream [19], excited Young diagrams [21], set-valued tableaux [23] and so on in the world of Schubert calculus.

### 3 Felderhof models

We review the studies on the Felderhof model [24, 25, 26, 27, 28]. We start from the following \( L \)-operator

\[
L(u, p, q) = \begin{pmatrix}
1 - pqu & 0 & 0 & 0 \\
0 & -p^2(1 - p^{-1}qu) & 1 - q^2 & 0 \\
0 & (1 - p^2)u & u - p^{-1}q & 0 \\
0 & 0 & 0 & u - pq
\end{pmatrix},
\]

(3.1)

The generalized \( R \)-matrix (3.1) can be shown to satisfy the Yang-Baxter relation

\[
R_{ab}(z_1/z_2, p_1, p_1)R_{aj}(z_1, p_1, p_2)R_{bj}(z_2, p_1, p_2) = R_{bj}(z_2, p_1, p_2)R_{aj}(z_1, p_1, p_2)R_{ab}(z_1/z_2, p_1, p_1),
\]

(3.2)

holding in \( \text{End}(W_a \otimes W_b \otimes V_j) \).

The generalized \( R \)-matrix (3.1) can be constructed from a class of an exotic quantum group called the colored representation or the nilpotent representation [24, 25]. The colored
representation becomes a finite-dimensional highest weight representation when the parameter of the quantum group is fixed at roots of unity. Each colored representation space is allowed to have a free parameter, and since the $R$-matrix is understood as an intertwiner acting on the tensor product of two representation spaces, one can include two free parameters $p$ and $q$, which can be regarded to be associated with the auxiliary and quantum spaces respectively.

Let us state the explicit form of the wavefunction constructed from the generalized $R$-matrix \((3.1)\).

**Theorem 3.1.** The wavefunction of the generalized $R$-matrix \((3.1)\) has the following form

\[
\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \prod_{j=1}^N (1 - q^2) (1 - pq u_j)^{M-1} \prod_{1 \leq j < k \leq N} \frac{u_k - p^2 u_j}{u_j - u_k} \frac{\text{det}_N \left( \left( \frac{u_j - p^{-1} q}{1 - pq u_j} \right)^{x_k - 1} \right)}{\lambda_k + N - k}, \tag{3.3}
\]

or, in terms of Young diagrams

\[
\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \prod_{j=1}^N (1 - q^2) (1 - pq u_j)^{M-1} \prod_{1 \leq j < k \leq N} \frac{u_k - p^2 u_j}{u_j - u_k} \frac{\text{det}_N \left( \left( \frac{u_j - p^{-1} q}{1 - pq u_j} \right)^{\lambda_k + N - k} \right)}{\lambda_k + N - k}, \tag{3.4}
\]

where $\lambda_j = x_{N-j+1} + N - j - 1$, which reduces to the ordinary Schur polynomials for the case $q = 0$ \([27]\), and is also a special case of the factorial Schur functions \([28]\) by an appropriate transformation of variables.

To explain how to derive a dual Cauchy identity from the domain wall boundary partition function in the next section, we also state the following theorem on the overlap between the wavefunction $\langle 1 \cdots M | B(u_1) \cdots B(u_N) \rangle$ constructed by acting $B$-operators on the state $\langle 1 \cdots M := \langle 1^M := 1 \rangle_1 \otimes \cdots \otimes M \rangle_1$, and the (normalized) state with an arbitrary hole configuration $|x_1 \cdots x_N\rangle$ ($1 \leq x_1 < \cdots < x_N \leq M$), where $x_j$ denotes the positions of the holes. Explicitly,

\[
|x_1 \cdots x_N\rangle = \prod_{j=1}^N \sigma_{x_j} (|1\rangle_1 \otimes \cdots \otimes |1\rangle_M), \tag{3.5}
\]

**Theorem 3.2.** The wavefunction of the generalized $R$-matrix \((3.1)\) has the following form

\[
\langle 1 \cdots M | B(u_1) \cdots B(u_N) | x_1 \cdots x_N \rangle = \prod_{j=1}^N (1 - q^2) (-pq (1 - p^{-1} qu_j))^{M-1} \prod_{1 \leq j < k \leq N} \frac{p^2 u_k - u_j}{p^2 (u_k - u_j)} \frac{\text{det}_N \left( \left( \frac{u_j - pq}{-p^2 (1 - p^{-1} qu_j)} \right)^{x_k - 1} \right)}{\lambda_k + N - k}, \tag{3.6}
\]

or, in terms of Young diagrams

\[
\langle 1 \cdots M | B(u_1) \cdots B(u_N) | x_1 \cdots x_N \rangle = \prod_{j=1}^N (1 - q^2) (-pq (1 - p^{-1} qu_j))^{M-1} \prod_{1 \leq j < k \leq N} \frac{p^2 u_k - u_j}{p^2 (u_j - u_k)} \frac{\text{det}_N \left( \left( \frac{u_j - pq}{-p^2 (1 - p^{-1} qu_j)} \right)^{\lambda_k + N - k} \right)}{\lambda_k + N - k}, \tag{3.7}
\]
where $\lambda_j = p_{N+j+1} - N + j - 1$.

The above theorems can be proved by combining the matrix product method and the domain wall boundary partition function, as in [12]. Let us prove Theorem 3.1. The strategy is as follows. We first rewrite the wavefunctions into a matrix product form, following [13]. The matrix product form can be expressed as a determinant with some overall factor which remains to be calculated. The information of the particle configuration \{x_1, x_2, \ldots, x_N\} is encoded in the determinant. On the other hand, the overall factor is independent of the particle positions, and therefore we can determine this factor by considering the specific configuration: we explicitly evaluate the overlap of the consecutive configuration (i.e. $x_j = j$) which is essentially the same with the domain wall boundary partition function.

Let us begin to compute the wavefunction $\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle$. We first rewrite it into the matrix product representation. With the help of graphical description, one finds that the wavefunction can be written as

$$\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \text{Tr}_{W^\otimes N} \left[ Q \langle x_1 \cdots x_N | \prod_{a=1}^{N} T_a(u_a) | \Omega \rangle \right],$$

where $Q = |1^N \rangle \langle 0^N|$ is an operator acting on the tensor product of auxiliary spaces $W_1 \otimes \cdots \otimes W_N$. The trace here is also over the auxiliary spaces.

Changing the viewpoint of the products of the monodromy matrices, we have

$$\prod_{a=1}^{N} T_a(u_a) = \prod_{j=1}^{M} T_j(\{u\}_N),$$

where $T_j(\{u\}_N) := \prod_{a=1}^{N} L_{a_j}(u_a) \in \text{End}(W^\otimes N \otimes V_j)$ can be regarded as a monodromy matrix consisting of $L$-operators acting on the same quantum space $V_j$ (but acting on different auxiliary spaces). The monodromy matrix is decomposed as

$$T_j(\{u\}_N) := \left( \begin{array}{cc} A_N(\{u\}_N) & B_N(\{u\}_N) \\ C_N(\{u\}_N) & D_N(\{u\}_N) \end{array} \right)_j,$$

where the elements ($A_N$, etc.) act on $W_1 \otimes \cdots \otimes W_N$. The wavefunction $\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle$ can then be rewritten by $T_j(\{u\}_N)$ as

$$\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \text{Tr}_{W^\otimes N} \left[ Q \langle x_1 \cdots x_N | \prod_{j=1}^{M} T_j(\{u\}_N) | \Omega \rangle \right] = \text{Tr}_{W^\otimes N} \left[ Q \mathcal{D}_{n+1}^{M-x_N} \mathcal{C}_{n} \mathcal{D}_{n+1-x_{N-1}}^{x_N} \cdots \mathcal{C}_{n} \mathcal{D}_{n+1-x_{1}}^{x_1} \mathcal{C}_{n} \mathcal{D}_{n+1-1} \right].$$

For these operators, one finds the following recursive relations:

$$\mathcal{D}_{n+1}(\{u\}_{n+1}) = \left( \begin{array}{cc} 1 - pqu_{n+1} & 0 \\ 0 & u_{n+1} - p^{-1}q \end{array} \right) \otimes \mathcal{D}_n(\{u\}_n) + \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \otimes C_n(\{u\}_n),$$

$$\mathcal{C}_{n+1}(\{u\}_{n+1}) = \left( \begin{array}{cc} 1 - q^2 & 0 \\ 0 & 0 \end{array} \right) \otimes \mathcal{D}_n(\{u\}_n) + \left( \begin{array}{cc} -p^2(1 - p^{-1}q)u_{n+1} & 0 \\ 0 & u_{n+1} - pq \end{array} \right) \otimes C_n(\{u\}_n),$$

where

$$\lambda_j = p_{N+j+1} - N + j - 1.$$
with the initial condition
\[ D_1 = \begin{pmatrix} 1 - pqu_1 & 0 \\ 0 & u_1 - p^{-1} q \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 - q^2 \\ 0 & 0 \end{pmatrix}. \] (3.14)

By using the recursive relations (3.12) and (3.13), one sees that these operators satisfy the following simple algebra.

**Lemma 3.3.** There exists a decomposition of \( C_n : C_n = \sum_{j=1}^n C_n^{(j)} \) such that the following algebraic relations hold for \( D_n \) and \( C_n^{(j)} \):

\[ C_n^{(j)} D_n = \frac{u_j - p^{-1} q}{1 - pqu_j} D_n C_n^{(j)}, \] (3.15)
\[ (C_n^{(j)})^2 = 0, \] (3.16)
\[ C_n^{(j)} C_n^{(k)} = \frac{(pu_j - q)(1 - pqu_k)}{(pu_k - q)(1 - pqu_j)} C_n^{(k)} C_n^{(j)}, \quad (j \neq k). \] (3.17)

**Proof.** This can be shown by induction on \( n \). For \( n = 1 \), from (3.14) \( D_1 \) is diagonal and one directly sees that the relations are valid. For \( n \), we assume that \( D_n \) is diagonalizable and write the corresponding diagonal matrix as \( D_n = G_n^{-1} D_n G_n \). Also writing \( C_n = G_n^{-1} C_n G_n \) and \( C_n = \sum_{j=1}^n \epsilon_n^{(j)} \), and noting that the algebraic relations above do not depend on the choice of basis, we suppose by the induction hypothesis that the same relations are satisfied by \( D_n \) and \( C_n^{(j)} \).

Now we shall show that they also hold for \( n + 1 \). To this end, first we construct \( G_{n+1} \). Noting from (3.12) that \( D_{n+1} \) is an upper triangular block matrix whose block diagonal elements are written in terms of \( D_n \), we assume that \( G_{n+1} \) is written as
\[ G_{n+1} = \begin{pmatrix} G_n & 0 \\ G_n H_n & G_n \end{pmatrix}, \] (3.18)
where \( 2n \times 2n \) matrix \( H_n \) remains to be determined. Using the induction hypothesis for \( n \), one obtains
\[ G_{n+1}^{-1} D_{n+1} G_{n+1} = \begin{pmatrix} (1 - pqu_{n+1}) D_n \\ (u_{n+1} - p^{-1} q) D_n H_n + (1 - p^2) u_{n+1} \epsilon_n - (1 - pqu_{n+1}) H_n D_n \end{pmatrix}. \]
(3.19)

The above matrix is guaranteed to be diagonal when
\[ (u_{n+1} - p^{-1} q) D_n H_n + (1 - p^2) u_{n+1} \epsilon_n - (1 - pqu_{n+1}) H_n D_n = 0 \] (3.20)

Utilizing the above relation and recalling \( D_n \) and \( \epsilon_n^{(j)} \) satisfy the relation same as that in (3.13), one finds
\[ H_n = D_n^{-1} \sum_{j=1}^n \frac{(1 - p^2) u_{n+1} (1 - pqu_j)}{(1 - q^2)(u_j - u_{n+1})} \epsilon_n^{(j)}, \] (3.21)
One thus obtains the diagonal matrix $\mathcal{D}_{n+1}$:

$$\mathcal{D}_{n+1} = \begin{pmatrix} (1 - pqu_{n+1}) \mathcal{D}_n & 0 \\ 0 & (u_{n+1} - p^{-1}q) \mathcal{D}_n \end{pmatrix}. \tag{3.22}$$

The remaining task is to derive $\mathcal{C}^{(j)}_{n+1}$ and to prove the relations (3.15)--(3.17) hold for $n + 1$. Combining (3.13), (3.18) and (3.21), and also inserting the relations (3.16) and (3.17), one arrives at

$$\mathcal{C}_{n+1} = \sum_{j=1}^{n+1} \mathcal{C}^{(j)}_{n+1}$$

where

$$\mathcal{C}^{(j)}_{n+1} = \begin{cases} \frac{1}{u_j - u_{n+1}} \left( (u_{n+1} - p^2 u_j) (1 - pqu_{n+1}) \mathcal{C}^{(j)}_{n} \right) \\ \frac{1}{u_{n+1} - p^{-1}q} (p^2 u_j - u_{n+1}) \mathcal{C}^{(j)}_{n} \end{cases}$$

for $1 \leq j \leq n$

$$\begin{cases} 0 \ (1 - q^2) \mathcal{D}_n \\ 0 \ 0 \end{cases}$$

for $j = n + 1$. \tag{3.23}

Finally recalling that $\mathcal{D}_n$ and $\mathcal{C}^{(j)}_n$ are supposed to satisfy the relations (3.15)--(3.17) and using the explicit form of $\mathcal{D}_{n+1}$ (3.22) and $\mathcal{C}^{(j)}_{n+1}$ (3.23), one sees they satisfy the same algebraic relations as those in (3.15)--(3.17) for $n + 1$. \hfill \Box

Due to the algebraic relations (3.15), (3.16) and (3.17) in Lemma 3.3, the matrix product form for the wavefunction (3.11) can be rewritten as

$$\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = \prod_{j=1}^N \left( \frac{1 - pq u_j}{u_j - p^{-1}q} \right)^j \text{Tr}_{W \otimes N} \left[ Q D_{N}^{M-N} C_{N}^{(N)} \cdots C_{N}^{(1)} \right]$$

$$\times \sum_{\sigma \in \mathfrak{S}_N} (-1)^{\sigma} \prod_{j=1}^N \left( \frac{u_{\sigma(j)} - p^{-1}q}{1 - pq u_{\sigma(j)}} \right)^{x_{\sigma(j)}} \tag{3.24}$$

where $\mathfrak{S}_N$ is the symmetric group of order $N$. One easily notes that (3.24) can be further rewritten in the following determinant form:

$$\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = K \prod_{j=1}^N \left( \frac{1 - pq u_j}{u_j - p^{-1}q} \right)^j \det_N \left( \left( \frac{u_j - p^{-1}q}{1 - pq u_j} \right)^{x_{\sigma(j)}} \right), \tag{3.25}$$

where the prefactor $K$ given below remains to be determined:

$$K = \text{Tr}_{W \otimes N} \left[ Q D_{N}^{M-N} C_{N}^{(N)} \cdots C_{N}^{(1)} \right]. \tag{3.26}$$

In (3.26), we notice that the information of the particle configuration $\{x_1, x_2, \ldots, x_N\}$ is encoded in the determinant, while the overall factor $K$ is independent of the configuration. This fact allows us to determine the factor $K$ by evaluating the overlap for a particular particle configuration. In fact, we find the following explicit expression for the case $x_j = j$ ($1 \leq j \leq N$):
Proposition 3.4. The wavefunction \( \langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle \) for the case \( x_j = j \) \((1 \leq j \leq N)\) has the following form:

\[
\langle 12 \cdots N | \psi(\{u\}_N) \rangle = (1 - q^2)^{N(N+1)/2} \prod_{j=1}^{N} (u_j - p^{-1}q)^{M-N} \prod_{1 \leq j < k \leq N} (u_k - p^2u_j).
\] (3.27)

**Proof.** We can easily show by its graphical description that \( \langle 12 \cdots N | \psi(\{u\}_N) \rangle \) can be factorized as

\[
\langle 12 \cdots N | \psi(\{u\}_N) \rangle = (1 - q^2)^{N(N+1)/2} \prod_{j=1}^{N} (u_j - p^{-1}q)^{M-N} Z_N(\{u\}_N),
\] (3.28)

where \( Z_N(\{u\}_N) \) is the domain wall boundary partition function. The domain wall boundary partition function on an \( M \times M \) lattice is defined as

\[
Z_M(\{u\}_M) = (1 \cdots M | B(u_1) \cdots B(u_M) | \Omega),
\] (3.29)

where \( M \) \( B \)-operators are inserted between the vacuum vector \( | \Omega \rangle \) and the state of particles \( \langle 1 \cdots M | = 1 \langle 1 | \otimes \cdots \otimes M \langle 1 | \).

One can show a more general result for the domain wall boundary partition function with inhomogeneities

\[
Z_M(\{u\}_M|\{v\}_M,\{q\}_M) = (1 \cdots M | B(u_1|\{v\}_M,\{q\}_M) \cdots B(u_M|\{v\}_M,\{q\}_M) | \Omega),
\] (3.30)

where

\[
B(u|\{v\}_M,\{q\}_M) = a(0) L_{aN}(u/v_M,q_M) \cdots L_{a1}(u/v_1,q_1)|1\rangle a.
\] (3.31)

**Lemma 3.5.** cf. [26] The domain wall boundary partition function with inhomogeneities has the following form.

\[
Z_M(\{u\}_M|\{v\}_M,\{q\}_M) = \prod_{j=1}^{M} \frac{1 - q_j^2}{q_j^2 - 1} \prod_{1 \leq j < k \leq M} (v_k - q_j v_j)(u_k - p^2u_j).
\] (3.32)

**Lemma 3.5** can be proved by using the Izergin-Korepin technique, i.e., show that both hand sides of (3.32) satisfy the same recursive relation, initial condition and the degree counting of polynomials.

Taking the homogeneous limit \( q_j \to 1, v_j \to 1 \) \((j = 1, \cdots , M)\) of (3.32) and inserting into (3.28) gives (3.27).

At last, Theorem 3.1 can be proved by checking that it has the determinant form (3.25) and satisfies the particular case (3.27).

4 Combinatorial identities

In this section, we derive combinatorial identities by investigating partition functions in more detail. We show that Cauchy identities are derived from scalar products, while dual Cauchy identities are obtained from domain wall boundary partition functions, which we explain by XXZ-type models and Felderhof models respectively.
4.1 Cauchy identities

The scalar product \([4]\) between the arbitrary off-shell state vectors is defined as

\[
\langle \psi(\{u\}_N)|\psi(\{v\}_N) \rangle = \langle \Omega | \prod_{j=1}^N C(u_j) \prod_{k=1}^N B(v_k) |\Omega \rangle
\]

(4.1)

with \(u_j, v_k \in \mathbb{C}\). Here we illustrate a way to derive a Cauchy identity for the \(\beta\)-Grothendieck polynomials from the scalar products of the \(q = 0\) limit of the \(L\)-operator \([4, 2]\).

\[
L(u) = \begin{pmatrix}
    u & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & -\beta^{-1}u - u^{-1} & 0 & 0 \\
    0 & 0 & -\beta^{-1}u
\end{pmatrix}.
\]

(4.2)

First, let us recall the following correspondence between the wavefunction constructed from

\[
\text{five-vertex model (4.2)}.
\]

Next, we recall that one can show the following determinant form \([29, 4, 11]\). Note that the Young diagram \(\lambda^\vee\) is the complementary part of the Young diagram \(\lambda\) in the \(N \times (M - N)\) rectangular Young diagram.

Theorem 4.1. The (off-shell) wavefunction and its dual wave-function of the integrable five-vertex model \([1, 2]\) are, respectively, given by the Grothendieck polynomials as

\[
\langle x_1 \cdots x_N | \psi(\{u\}_N) \rangle = (-\beta^{-1})^{N(N-1)/2} \prod_{j=1}^N u_j^{M-1} G_{\lambda}(z; \beta),
\]

(4.3)

\[
\langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle = (-\beta^{-1})^{N(N-1)/2} \prod_{j=1}^N u_j^{M-1} G_{\lambda^\vee}(z; \beta),
\]

(4.4)

where \(z_j = -\beta^{-1} - u_j^{-2}\), and \(\lambda = (\lambda_1, \ldots, \lambda_N)\) \((M - N \geq \lambda_1 \geq \cdots \geq \lambda_N \geq 0)\) and \(\lambda^\vee = (\lambda_1^\vee, \ldots, \lambda_N^\vee)\) \((M - N \geq \lambda_1^\vee \geq \cdots \geq \lambda_N^\vee \geq 0)\) are the Young diagrams related to the particle configuration \(x = (x_1, \ldots, x_N)\) as \(\lambda_j = x_{N-j+1} - N + j - 1\) and \(\lambda_j^\vee = M - N + j - x_j\), respectively.

Note that the Young diagram \(\lambda^\vee\) is the complementary part of the Young diagram \(\lambda\) in the \(N \times (M - N)\) rectangular Young diagram.

Next, we recall that one can show the following determinant form \([29, 4, 11]\).

Theorem 4.2. The scalar product is given by a determinant form:

\[
\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \prod_{1 \leq j < k \leq N} \frac{1}{(u_j^2 - u_k^2)(v_j^2 - v_k^2)} \det_N Q(\{u\}_N|\{v\}_N),
\]

(4.5)

where \(\{u\}_N\) and \(\{v\}_N\) are arbitrary sets of complex values (i.e. off-shell conditions), and \(Q\) is an \(N \times N\) matrix with matrix elements

\[
Q(\{u\}_N|\{v\}_N)_{jk} = \frac{u_j^M(-\beta^{-1}v_k - v_k^{-1}) v_k^{2(N-1)} - v_k^M(-\beta^{-1}u_j - u_j^{-1}) u_j^{2(N-1)}}{v_k/u_j - u_j/v_k}.
\]

(4.6)
The Cauchy formula for the $\beta$-Grothendieck polynomials can be derived by combining Theorem 4.1 and 4.2. The key is to substitute the completeness relation,

$$\sum_{\{x\}} |x_1 \cdots x_N \rangle \langle x_1 \cdots x_N| = \text{Id},$$

and decompose the scalar product as

$$\langle \psi(\{u\}_N) | \psi(\{v\}_N) \rangle = \sum_{1 \leq x_1 < \ldots < x_N \leq M} \langle \psi(\{u\}_N) | x_1 \cdots x_N \rangle \langle x_1 \cdots x_N | \psi(\{v\}_N) \rangle.$$  \hspace{1cm} (4.8)

Using Theorem 4.2 and 4.1 in the left and right hand side of the equality (4.8) respectively, one has the following Cauchy identity.

**Theorem 4.3.** \[11\] The following Cauchy identity for the Grothendieck polynomials holds.

$$\sum_{\lambda \subseteq L^N} G_\lambda(z; \beta) G_{\lambda^\vee}(w; \beta) = \prod_{1 \leq j < k \leq N} \frac{1}{(z_j - z_k)(w_k - w_j)} \det_N \left[ \frac{z^{L+N}(1 + \beta w_k)^{N-1} - w_k^{L+N}(1 + \beta z_j)^{N-1}}{z_j - w_k} \right],$$

where the Young diagram $\lambda^\vee = (\lambda_1^\vee, \ldots, \lambda_N^\vee)$ is given by the Young diagram $\lambda = (\lambda_1, \ldots, \lambda_N)$ as $\lambda_j^\vee = L - \lambda_{N+1-j}$.

Here we have set $L = M - N$, but the above formula holds for any $L \geq 0$.

### 4.2 Dual Cauchy identities

The dual Cauchy identities can be derived by dealing with domain wall boundary partition functions \[15, 16\]

$$\langle 1 \cdots M | B(u_1) \cdots B(u_M) | \Omega \rangle,$$

where $M$ $B$-operators are inserted between the vacuum vector $|\Omega\rangle$ and the state of particles $\langle 1 \cdots M | = 1(1) \otimes \cdots \otimes_M 1$. This class of partition function has found applications to the enumeration of alternating sign matrices in the 1990s, and it was only noticed in recent years to have applications to the dual Cauchy identities \[27, 28\]. We illustrate this by the Felderhof models.

First, we rewrite the wavefunctions in the following forms

$$\langle x_1 \cdots x_N | B(u_1) \cdots B(u_N) | \Omega \rangle = \prod_{j=1}^N \frac{(1 - q^2)^M}{(1 + pqw_j)^{M-1}}$$

$$\times \prod_{1 \leq j < k \leq N} \frac{p^2q(p^2 - 1)w_jw_k + p(p^2 - q^2)w_k + p(p^2q^2 - 1)w_j + q(p^2 - 1)}{p(q^2 - 1)} s_\lambda(w),$$ \hspace{1cm} (4.11)
where we make transformation of variables from \( u_j \) to \( w_j = \frac{u_j - p^{-1}q}{1 - pq w_j} \), and

\[
|1 \cdots M \rangle B(u_1) \cdots B(u_M) |x_1 \cdots x_N\rangle = \prod_{j=1}^{N} \frac{(-p^2)_{M-1}(1- q^2)_{M}}{(1 + p^{-1}q w_j)_{M-1}}
\]

\[
\times \prod_{1 \leq j < k \leq M} \frac{q(1-p^2)z_j z_k + p(q^2 - p^2)z_k + p(1-p^2q^2)z_j + p^2q(1-p^2)}{p(q^2 - 1)} s_{\lambda}(\frac{z}{-p^2}), \tag{4.12}
\]

where we also transform from \( u_j \) to \( z_j = \frac{u_j - pq}{1 - p^{-1}q w_j} \).

The dual Cauchy identities is derived by evaluating the domain wall boundary partition function in two ways.

First, we evaluate the domain wall boundary partition function by viewing it as a particular limit of the wavefunction (4.11). One has

\[
|1 \cdots M \rangle B(u_1) \cdots B(u_M) |\Omega\rangle = \prod_{j=1}^{M} \frac{(1-q^2)^{M}}{(1 + pq w_j)_{M-1}}
\]

\[
\times \prod_{1 \leq j < k \leq M} \frac{p^2q(p^2-1)w_j w_k + p(p^2-q^2)w_k + p(p^2q^2-1)w_j + q(p^2-1)}{p(q^2 - 1)} \tag{4.13}
\]

Another way is to insert the completeness relation

\[
\sum_{\{x\}} |x_1 \cdots x_N\rangle \langle x_1 \cdots x_N| = \text{Id}, \tag{4.14}
\]

between the \( B \)-operators

\[
|1 \cdots M \rangle B(u_1) \cdots B(u_M) |\Omega\rangle
= \sum_{\{x\}} |1 \cdots M \rangle B(u_1) \cdots B(u_{M-N}) |x_1 \cdots x_N\rangle \langle x_1 \cdots x_N| B(u_{M-N+1}) \cdots B(u_M) |\Omega\rangle
\]

\[
= \sum_{\{x\}} |1 \cdots M \rangle B(u_1) \cdots B(u_{M-N}) |x_1 \cdots x_{M-N}\rangle \langle x_1 \cdots x_{M-N}| B(u_{M-N+1}) \cdots B(u_M) |\Omega\rangle, \tag{4.15}
\]

and insert (4.11) and (4.12) into the right hand side of (4.15). Comparing (4.13) and (4.15), we find

\[
\sum_{\chi \subseteq (M-N)^N} s_{\lambda}(\frac{z}{-p^2}) s_{\lambda}(\overline{w}) = (-p^2)^{(N-M)N} \prod_{j=1}^{M-N} \frac{1 + p^{-1} q z_j}{1 + pq w_j} \left( \frac{A}{BC} \right)^{M-1}, \tag{4.16}
\]

where

\[
A = \prod_{1 \leq j < k \leq M} \frac{p^2q(p^2-1)w_j w_k + p(p^2-q^2)w_k + p(p^2q^2-1)w_j + q(p^2-1)}{p(q^2 - 1)}, \tag{4.17}
\]

\[
B = \prod_{M-N+1 \leq j < k \leq M} \frac{p^2q(p^2-1)w_j w_k + p(p^2-q^2)w_k + p(p^2q^2-1)w_j + q(p^2-1)}{p(q^2 - 1)}, \tag{4.18}
\]

\[
C = \prod_{1 \leq j < k \leq M-N} \frac{q(1-p^2)z_j z_k + p(q^2 - p^2)z_k + p(1-p^2q^2)z_j + p^2q(1-p^2)}{-p(q^2 - 1)}, \tag{4.19}
\]

13
and \( W = \{ w_{M-N+1}, \ldots, w_M \} \), \( z = \{ z_1, \ldots, z_{M-N} \} \). Note also that the sum over all particle configurations \( \{ x \} \) is translated to the sum over all Young diagrams \( \lambda \) satisfying \( \lambda \subseteq (M-N)^N \).

\((4.16)\) is nothing but the dual Cauchy formula for the Schur functions. In fact, if we set \( q = 0 \) and \( t = -p^2 \), \((4.16)\) becomes

\[
\sum_{\lambda \subseteq (M-N)^N} s_\lambda (\mathbf{u}) s_\lambda (\mathbf{w}) = \prod_{j=1}^{M-N} \prod_{k=M-N+1}^{M} \left( \frac{u_j}{t} + u_k \right),
\]

which becomes the celebrated dual Cauchy identity by scaling \( u \) to \( tu \). See [27] for example for more results on this direction of research of deriving other combinatorial identities, by changing boundary conditions for example.

5 Other models, formulae and generalizations

We give several remarks on other integrable models.

5.1 Combinatorial formula for the Schur polynomials

First, we remark that there are other interesting six-vertex models. For example, the following \( L \)-operator

\[
L_{aj}(v) = \begin{pmatrix}
1 - \beta v & 0 & 0 & 0 \\
0 & 1 + \beta v & 2v & 0 \\
0 & 1 & v & 0 \\
0 & 0 & 0 & v
\end{pmatrix}_{aj}.
\]

\((5.1)\)
can be shown to be integrable. In fact this \( L \)-operator is another special limit of the generalized XXZ-type six-vertex models in section 2. By investigating the wavefunction itself in detail, one can find the following combinatorial formula for the Schur polynomials.

**Theorem 5.1.** [32] We have the following combinatorial formula for the Schur polynomials

\[
s_\lambda(z) = \prod_{1 \leq j < k \leq N} (z_j + z_k + 2\beta z_j z_k) \sum_{x^{(N)} \succ x^{(N-1)} \succ \cdots \succ x^{(0)} = \phi} N \prod_{k=1}^{N} \left\{ \sum_{j=1}^{k} x_{j}^{(k)} - \sum_{j=1}^{k-1} x_{j}^{(k-1)} - 1 \right\} \times \left( \frac{2(1 + \beta z_k)}{1 + 2\beta z_k} \right)^{\#(x^{(k)})\#(x^{(k-1)}) - 1} \prod_{j=1}^{k-1} \left( 1 + 2\beta z_k (1 - \delta_{x_{j}^{(k-1)} x_{j+1}^{(k-1)}}) \right),
\]

\((5.2)\)

where \( \beta \) is an arbitrary parameter. \( x_{j}^{(k)} = (x_{1}^{(k)}, \ldots, x_{k}^{(k)}) \), \( k = 0, 1, \ldots, N \) are strict partitions satisfying the interlacing relations \( x^{(N)} \succ x^{(N-1)} \succ \cdots \succ x^{(0)} = \phi \), and \( x^{(N)} \) is fixed by the Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_N) \) as \( \lambda_j = x_{j}^{(N)} - N + j - 1 \), and \( \#(y|x) \) denotes the number of parts in \( y \) which are not in \( x \).

This type of formula of expressing Schur polynomials using an additional parameter resembles, but is different from the Tokuyama formula [34, 55]. The modern understanding of
the Tokuyama formula comes from the fact that Schur $Q$-functions factorizes into an overall factor and Schur functions when the length of the Young diagram which labels the Schur $Q$-functions is the same with the number of symmetric variables [36]. The proof of Theorem 5.1 also relies essentially on this fact.

5.2 Boson model

In this section, we remark the relation between the wavefunction of an integrable boson model (nonhermitian phase model) [37] and the Grothendieck polynomials. The nonhermitian phase model is a boson system characterized by the generators $\phi$, $\phi^\dagger$, $N$ and $\pi$ acting on a bosonic Fock space $\mathcal{F}$ spanned by orthonormal basis $|n\rangle$ ($n = 0, 1, \ldots, \infty$). Here the number $n$ indicates the occupation number of bosons. The generators $\phi$, $\phi^\dagger$, $N$ and $\pi$ are, respectively, the annihilation, creation, number and vacuum projection operators, whose actions on $\mathcal{F}$ are, respectively, defined as

$$\phi(0) = 0, \quad \phi|n\rangle = |n-1\rangle, \quad \phi^\dagger|n\rangle = |n+1\rangle, \quad N|n\rangle = n|n\rangle, \quad \pi|n\rangle = \delta_{n,0}|n\rangle. \quad (5.3)$$

Thus the operator forms are explicitly given by

$$\phi = \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \quad \phi^\dagger = \sum_{n=0}^{\infty} |n+1\rangle\langle n|, \quad N = \sum_{n=0}^{\infty} n|n\rangle\langle n|, \quad \pi = |0\rangle\langle 0|. \quad (5.4)$$

These operators generate an algebra referred to as the phase algebra:

$$[\phi, \phi^\dagger] = \pi, \quad [N, \phi] = -\phi, \quad [N, \phi^\dagger] = \phi^\dagger. \quad (5.5)$$

We consider the tensor product of Fock spaces $\otimes_{j=0}^{M-1} \mathcal{F}_j$, whose basis is given by $|\{n\}_M\rangle := \otimes_{j=0}^{M-1} |n_j\rangle_j$, $n_j = 0, 1, \ldots, \infty$. We denote a dual state of $|\{n\}_M\rangle$ as $\langle \{n\}_M| := \otimes_{j=0}^{M-1} \langle n_j|$. The operators $\phi_j$, $\phi_j^\dagger$, $N_j$ and $\pi_j$ act on the Fock space $\mathcal{F}_j$ as $\phi$, $\phi^\dagger$, $N$ and $\pi$, and the other Fock spaces $\mathcal{F}_k, k \neq j$ as an identity.

The $L$-operator for the nonhermitian phase model [37] is given by

$$L_{aj}(v) = \begin{pmatrix} v^{-1} - \beta v \pi_j & \phi_j^\dagger \\ \phi_j & v \end{pmatrix}, \quad (5.6)$$

acting on the tensor product $W_a \otimes \mathcal{F}_j$ of the complex two-dimensional space $W_a$ and the Fock space at the $j$th site $\mathcal{F}_j$. The $L$-operator satisfies the intertwining relation (RLL-relation)

$$R_{ab}(u/v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u/v), \quad (5.7)$$

which acts on $W_a \otimes W_b \otimes \mathcal{F}_j$. The $R$-matrix $R(u)$ is the same as the one for the integrable five-vertex model which is a $q = 0$ limit of the $U_q(sl_2)$ $R$-matrix [22]. The auxiliary space $W_a$ is the complex two-dimensional space, which is the same as that for the integrable five-vertex model, while the quantum space $\mathcal{F}_j$ is the infinite-dimensional bosonic Fock space.

From the $L$-operator, we construct the monodromy matrix

$$T_a(v) = L_{aM-1}(v) \cdots L_{a0}(v) = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix}_a, \quad (5.8)$$
which acts on $W_0 \otimes (\mathcal{F}_0 \otimes \cdots \otimes \mathcal{F}_{M-1})$.

The arbitrary $N$-particle state $|\Psi\{\{v\}_N\}\rangle$ (resp. its dual $\langle \Psi\{\{v\}_N\} |$ (not normalized) with $N$ spectral parameters $\{v\}_N = \{v_1, \ldots, v_N\}$ is constructed by a multiple action of $\mathcal{B}$ (resp. $\mathcal{C}$) operator on the vacuum state $|\Omega\rangle := |0^M\rangle := |0\rangle_0 \otimes \cdots \otimes |0\rangle_{M-1}$ (resp. $\langle \Omega | := \langle 0^M | := 0\langle 0 | \otimes \cdots \otimes |0\rangle_{M-1}$):

$$|\Psi\{\{v\}_N\}\rangle = \prod_{j=1}^{N} B(v_j) |\Omega\rangle, \quad \langle \Psi\{\{v\}_N\} | = \prod_{j=1}^{N} C(v_j). \quad (5.9)$$

The orthonormal basis of the $N$-particle state $|\Psi\{\{v\}_N\}\rangle$ and its dual $\langle \Psi\{\{v\}_N\} |$ is given by $|\{n\}_M\rangle := |n_0\rangle_0 \otimes \cdots \otimes |n_{M-1}\rangle_{M-1}$ and $\langle \{n\}_M | := \langle n_0 | \otimes \cdots \otimes \langle n_{M-1} |_{M-1}$, where $n_0 + n_1 + \cdots + n_{M-1} = N$. The wavefunctions can be expanded in this basis as

$$|\Psi\{\{v\}_N\}\rangle = \sum_{0 \leq n_0, \ldots, n_{M-1} \leq N} \langle \{n\}_M | \psi\{\{v\}_N\} | \{n\}_M \rangle, \quad (5.10)$$

$$\langle \Psi\{\{v\}_N\} | = \sum_{0 \leq n_0, \ldots, n_{M-1} \leq N} \langle \{n\}_M | \psi\{\{v\}_N\} | \{n\}_M \rangle. \quad (5.11)$$

There is a one-to-one correspondence between the set $\{n\}_M \in \{n_0, n_1, \ldots, n_{M-1}\}$ ($n_0 + n_1 + \cdots + n_{M-1} = N$) and the Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ ($M - 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$). Namely, each Young diagram $\lambda$ under the constraint $\ell(\lambda) \leq N, \lambda_1 \leq M - 1$ can be labeled by a set of integers $\{n\}_M$ as $\lambda = ((M - 1)^{n_{M-1}}, \ldots, 1^{n_1}, 0^{n_0})$. We have the following correspondence between the wavefunctions and the $\beta$-Grothendieck polynomials.

**Theorem 5.2.** [11] The wavefunctions can be expressed by the Grothendieck polynomials as

$$\langle \{n\}_M | \Psi\{\{v\}_N\} \rangle = \prod_{j=1}^{N} (v_j^{-1} - \beta v_j)^{M-1} G_\lambda(z_1, \ldots, z_N; \beta), \quad (5.12)$$

$$\langle \Psi\{\{v\}_N\} | \{n\}_M \rangle = \prod_{j=1}^{N} (v_j^{-1} - \beta v_j)^{M-1} G_{\lambda^\vee}(z_1, \ldots, z_N; \beta), \quad (5.13)$$

where $z_j^{-1} = v_j^{-2} - \beta$ and $\lambda^\vee = (\lambda_1^\vee, \lambda_2^\vee, \ldots, \lambda_N^\vee)$ ($M - 1 \geq \lambda_1^\vee \geq \cdots \geq \lambda_N^\vee \geq 0$) is given by the Young diagram $\lambda$ as $\lambda_j^\vee = M - 1 - \lambda_{N+1-j}$.

We remark that there exists a $q$-deformation of the nonhermitian phase model introduced in [38]. The six-vertex model [5,11] in the former section can also be regarded as a certain degeneration point $q = i$ of the integrable $q$-boson $L$-operator. We also remark that besides the quantum inverse scattering approach, there is another approach to this model based on the affine Hecke algebra [13, 44]. See [11, 37, 38, 39, 40, 41, 42, 43, 44] for more about the integrable boson models, the correspondence between the wavefunctions and the Grothendieck, Hall-Littlewood polynomials and their generalizations, and its relation with the Verlinde algebra etc, for example.
5.3 Generalized Felderhof models and Generalized factorial Schur functions

Second, we remark that the correspondence between the Felderhof models and Schur functions [27] was generalized to factorial Schur functions [28]. We furthermore generalized this correspondence and find the following: First we introduce the following generalized $L$-operator for the Felderhof model

\[
L_{aj}(z,t,\alpha_j,\gamma_j) = \begin{pmatrix}
1 - \gamma_jz & 0 & 0 & 0 \\
0 & t + \gamma_jz & 1 & 0 \\
0 & (t+1)z & \alpha_j + (1 - \alpha_j\gamma_j)z & 0 \\
0 & 0 & 0 & -t\alpha_j + (1 - \alpha_j\gamma_j)z
\end{pmatrix}, \quad (5.14)
\]

and introduce the $N$-particle state

\[
\Phi(\{z\}_{N},t,\{\alpha\},\{\gamma\}) = B(z_1,t,\{\alpha\},\{\gamma\}) \cdots B(z_N,t,\{\alpha\},\{\gamma\})|\Omega\rangle,
\]

where

\[
B(z,t,\{\alpha\},\{\gamma\}) = a(0|L_{aN}(z,t,\alpha_N,\gamma_N) \cdots L_{a1}(z,t,\alpha_1,\gamma_1)|1\rangle_a. \quad (5.16)
\]

We found the correspondence between the wavefunction $\langle x_1 \ldots x_N|\Phi(\{z\}_{N},t,\{\alpha\},\{\gamma\})\rangle$ and the generalized factorial Schur functions defined below.

**Definition 5.3.** We define the generalized factorial Schur functions to be the following determinant:

\[
s_\lambda(\{z\}|\{\alpha\}|\{\gamma\}) = \frac{F_{\lambda+\delta}(\{z\}|\{\alpha\}|\{\gamma\})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)}, \quad (5.17)
\]

where $\{z\} = \{z_1, \ldots, z_N\}$ is a set of variables and $\lambda$ denotes a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with weakly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$, and $\delta = (N-1, N-2, \ldots, 0)$. $F_{\mu}(\{z\}|\{\alpha\}|\{\gamma\})$ is an $N \times N$ determinant

\[
F_{\mu}(\{z\}|\{\alpha\}|\{\gamma\}) = \det_N(f_{\mu_j}(z_k|\{\alpha\}|\{\gamma\})), \quad (5.18)
\]

where

\[
f_{\mu}(z|\{\alpha\}|\{\gamma\}) = \prod_{j=1}^{\mu}(\alpha_j + (1 - \alpha_j\gamma_j)z) \prod_{j=\mu+2}^{M}(1 - \gamma_jz). \quad (5.19)
\]

**Theorem 5.4.** The wavefunction $\langle x_1 \ldots x_N|\Phi(\{z\}_{N},t,\{\alpha\},\{\gamma\})\rangle$ is expressed by the generalized factorial Schur functions $s_\lambda(\{z\}|\{\alpha\}|\{\gamma\})$ as

\[
\langle x_1 \ldots x_N|\Phi(\{z\}_{N},\{\alpha\},\{\gamma\})\rangle = \prod_{1 \leq j < k \leq N}(z_j + tz_k)s_\lambda(\{z\}|\{\alpha\}|\{\gamma\})), \quad (5.20)
\]

under the relation $\lambda_j = x_{N-j+1} - N + j - 1$, $j = 1, \ldots, N$.

Based on this correspondence, one can show the following dual Cauchy formula for the generalized factorial Schur functions.
Theorem 5.5. The following dual Cauchy formula holds for the generalized factorial Schur functions with sets variables \( x = \{ x_1, \ldots, x_N \}, y = \{ y_1, \ldots, y_M \}, \alpha = \{ \alpha_1, \ldots, \alpha_{N+M} \}, \gamma = \{ \gamma_1, \ldots, \gamma_{N+M} \}, \)
\[
\sum_{\lambda \subseteq M^N} s_\lambda(x|\{\alpha\}|\{\gamma\})s_\lambda(y|\{-\alpha\}|\{-\gamma\}) = \prod_{j=1}^N \prod_{k=1}^M (x_j + y_k) \prod_{1 \leq j < k \leq N+M} (1 + \alpha_j(\gamma_k - \gamma_j)),
\]
where \( \{-\alpha\} = \{-\alpha_1, \ldots, -\alpha_{N+M} \}, \{-\gamma\} = \{-\gamma_1, \ldots, -\gamma_{N+M} \}, \)
and \( \hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_M) \) is the partition of the Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_N) \) given by
\[
\hat{\lambda}_i = |\{ j \mid \lambda_j \leq M - i \}|.
\]

It is also known that changing the boundary condition of the partition functions to the Tsuchiya boundary condition [45], which is a class of partition function with reflecting end, what appears is the symplectic Schur functions [46 47]. We have also generalized this to a more general setting, and find the following dual Cauchy formula for the generalized factorial symplectic Schur functions.

Definition 5.6. We define the generalized symplectic Schur functions to be the following determinant:
\[
sp_{\lambda}(z|\{\tilde{\alpha}\}|\{\tilde{\gamma}\}) = \frac{G_{\lambda+\delta}(z|\{\tilde{\alpha}\}|\{\tilde{\gamma}\})}{\det_N(z_k^{N-j+1} - z_k^{N-j-1})}.
\]
Here, \( G_{\mu}(z|\{\tilde{\alpha}\}|\{\tilde{\gamma}\}) \) is an \( N \times N \) determinant
\[
G_{\mu}(z|\{\tilde{\alpha}\}|\{\tilde{\gamma}\}) = \det_N(g_{\mu_j}(z_k|\{\tilde{\alpha}\}|\{\tilde{\gamma}\}) - g_{\mu_j}(z_k^{-1}|\{\tilde{\alpha}\}|\{\tilde{\gamma}\})),
\]
where
\[
g_{\mu}(z|\{\tilde{\alpha}\}|\{\tilde{\gamma}\}) = \prod_{j=0}^\mu (\alpha_j + (1 - \alpha_j\gamma_j)z) \prod_{j=\mu+2}^M (1 - \gamma_j z) \prod_{j=1}^M (1 - \gamma_j z^{-1}).
\]

Theorem 5.7. The following dual Cauchy formula holds for the generalized factorial symplectic Schur functions with sets variables \( x = \{ x_1, \ldots, x_N \}, y = \{ y_1, \ldots, y_M \}, \tilde{\alpha} = \{ \alpha_0, \ldots, \alpha_{N+M} \}, \tilde{\gamma} = \{ \gamma_0, \ldots, \gamma_{N+M} \}, \)
\[
\sum_{\lambda \subseteq M^N} sp_{\lambda}(x|\{\tilde{\alpha}\}|\{\tilde{\gamma}\})sp_{\lambda}(y|\{-\tilde{\alpha}\}|\{-\tilde{\gamma}\})
\]
\[
= \prod_{j=1}^M y_j^{-N} \prod_{j=1}^N \prod_{k=1}^M (1 + x_jy_k)(1 + x_j^{-1}y_k) \prod_{0 \leq j < k \leq N+M} (1 + \alpha_j(\gamma_k - \gamma_j)) \prod_{1 \leq j < k \leq N+M} (1 - \gamma_j \gamma_k),
\]
where \( \{-\tilde{\alpha}\} = \{-\alpha_0, \ldots, -\alpha_{N+M} \}, \{-\tilde{\gamma}\} = \{-\gamma_0, \ldots, -\gamma_{N+M} \} \) and \( \hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_M) \) is the partition of the Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_N) \) given by
\[
\hat{\lambda}_i = |\{ j \mid \lambda_j \leq M - i \}|.
\]

Details of the theorems on this subsection will be given elsewhere [33].
6 Conclusion

In this paper, we reviewed and presented new results on the relation between integrable lattice models and combinatorial representation theory of symmetric polynomials.

The philosophy of statistical mechanics is to investigate macroscopic quantities from microscopic description of the system. For the case of integrable lattice models, this means to study symmetric polynomials as partition functions constructed from the local $L$-operators and global boundary conditions. The correspondence between symmetric polynomials and partition functions allows us to investigate and find various new combinatorial formulae which seems to be extremely hard to discover or to prove without integrable models.

Some of the correspondence is essentially equivalent to the notions in the field of algebraic combinatorics, especially Schubert calculus. To make further progresses on the representation theory of symmetric polynomials, we think the point of view of quantum integrability is indispensable. For example, the wavefunction (2.11) is “the” (not just “a”) $q$-deformation of the $\beta$-Grothendieck polynomials, which the parameter $q$ is nothing but the parameter for the quantum group $U_q(sl_2)$, and we believe quantum integrability is essential to find this deformation. We expect further advances will be made from the interplay between quantum integrable models and representation theory in the future.

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