Study of the influence of spin on the angular distribution of synchrotron radiation for weakly excited particles

Vladislav Bagrov and Anastasia Burimova
Quantum Field Theory Department, Faculty of Physics, Tomsk State University, Lenina ave. 36, 634050, Tomsk, Russia
E-mail: bagrov@phys.tsu.ru, llefrith@yandex.ru

Abstract. In the framework of quantum theory, we obtain the precise analytical expressions for synchrotron radiation characteristics of first excited state charged particles. A detailed analysis of the total radiation power angular distribution and the angular distribution of polarization components is given for particles with spin 0 and 1/2. For both these types of particles one can calculate radiation characteristics and compare them demonstrating the dependance of the results on spin. It is demonstrated that the quantum effects (e.g. spin properties) accounting leads us to the results which seriously differ from classical theory predictions.

1. Introduction
The synchrotron radiation (SR) is essentially associated with a wide number of its practical applications. Nevertheless, during last 50 years the theory of SR stays a 'milker' as well, mostly for the physicists. The basic properties of the phenomenon gained a detailed analysis in terms of classical theory [1, 2]. However, the contemporary accelerators have their parameters being very close to the region where the quantum corrections become significant. Guided by the technique development, one obtains a motive to fill the gaps in quantum description created for SR a few decades ago [3, 4]. We focus on the theoretical research of SR characteristics, thus necessarily studying the influence of quantum corrections on the basic expressions received. Without any possibility for an adequate classical interpretation, spin is a property of a pure quantum nature. It is clear, that the investigation of spin properties can be carried out only with the use of quantum theory methods. To thoroughly study the influence of spin properties on the SR characteristics, we consider a spinless (scalar) particle (a boson) and a spinor particle, namely, a particle of 1/2 spin (an electron). We consider first excited state particles as a special case with unique frequency being radiated. It seriously differs from classical theory where one deals with a spectrum of radiated harmonics. Thus, this is a convenient way to observe quantum effects, especially spin effects.

In the framework of classical SR theory one interprets the phenomenon as a radiation of a particle moving around circular orbit in the plane straight perpendicular to the external magnetic field of strengths $H > 0$. Of course, the quantum picture differs completely, still we can consider the motion in a similar magnetic field. In this case the energy of a first excited state boson and
electron:
\[ E = m_0 c^2 \gamma, \text{ where } \gamma^2 = (1 - \beta^2)^{-1} = \begin{cases} 1 + 2B & \text{for an electron} \\ 1 + 3B & \text{for a boson} \end{cases}, \quad B = \frac{H}{H_0}, \quad H_0 = \frac{m_0^2 e^3}{\hbar} \]  
(1)

All the expressions received will be presented separately for a boson and an electron.

We use index ‘s’ to describe the polarization components as follows: if \( s = 2, 3 \) then it is \( \sigma \)- or \( \pi \)- component of linear polarized radiation respectively, \( s = -1, 1 \) for left and right circular polarization, \( s = 0 \) indicates the summed or so-called total radiation. Since, for an electron, we suppose the solutions of motion equation being the eigenfunctions of transverse spin polarization operator, thus two following cases are under consideration. The initial state of an electron could be characterized with the use of spin quantum number \( \zeta \) with its possible values \( \zeta = 1 \) and \( \zeta = -1 \). Evidently, the transverse spin polarization means that the spin of a particle is collinear to the external magnetic field direction. To be more precise, it could be co-directional to the field (we take \( \zeta = 1 \) in this case), or have an opposite direction ( \( \zeta = -1 \)).

2. Boson

Let us introduce a parameter \( \xi_0 \) and a variable \( \xi \)
\[ \xi_0 = \xi_0(\beta) = \frac{\sqrt{3 - \sqrt{3 - 2\beta^2}}}{\sqrt{3 + \sqrt{3 - 2\beta^2}}} \quad ; \quad \xi = \xi(\beta, \theta) = \frac{\sqrt{3 - \sqrt{3 - 2\beta^2 \sin^2 \theta}}}{\sqrt{3 + \sqrt{3 - 2\beta^2 \sin^2 \theta}}} \]  
(2)

Then for a boson we can write the angular distribution of radiated power as follows:
\[ \frac{dW^b_s(\beta; \theta)}{d\Omega} = \frac{Q_0 A(\beta)}{54}(1 + \xi)^3 e^{-\xi} \varphi^b_s(\beta; \theta); \quad Q_0 = \frac{e^2 m_0^2 c^3}{\hbar^2}, \quad A(\beta) = \frac{\beta^6}{1 - \beta^2} = \frac{(\gamma^2 - 1)^3}{\gamma^4}; \]
\[ \varphi^b_0(\beta; \theta) = \frac{1}{2} \varphi^b_0(\beta; \theta) + g(1 + \xi) \cos \theta, \quad \varphi^b_2(\beta; \theta) = 1 - \xi, \quad \varphi^b_3(\beta; \theta) = \frac{(1 + \xi)^2 \cos^2 \theta}{1 - \xi}, \]
\[ \varphi^b_0(\beta; \theta) = \varphi^b_0(\beta; \theta) + \varphi^b_3(\beta; \theta) = \varphi^b_{-1}(\beta; \theta) + \varphi^b_{1}(\beta; \theta); \quad d\Omega = \sin \theta d\theta, \quad g = \pm 1. \]  
(3)

From (3) we obtain the expression for the total power radiated by the boson.
\[ W^b_0(\beta) = \frac{4 Q_0 A(\beta)}{81} f^b(\beta), \quad f^b(\beta) = \frac{3(1 + \xi_0)^2}{8} f^b_0(\xi_0), \quad f^b_0(x) = f^b_{2}(x) + f^b_{3}(x). \]  
(4)

Here the following designations are used
\[ f^b_1(x) = \int_0^1 (1 - x^2 y^2) e^{-xy} dy = \frac{(1 + x)^2 e^{-x} - 1}{x}, \quad f^b_1(0) = 1, \]
\[ f^b_2(x) = \int_0^1 \frac{(1 + xy)(1 - xy)^2}{\sqrt{(1 - y)(1 - x^2 y)}} e^{-xy} dy, \quad f^b_2(0) = 2, \]
\[ f^b_3(x) = \int_0^1 (1 + xy) \sqrt{(1 - y)(1 - x^2 y)} e^{-xy} dy, \quad f^b_3(0) = \frac{2}{3}. \]  
(5)

One can easily rewrite (3)
\[ \frac{dW^b_s(\beta; \theta)}{d\Omega} = W^b_0(\beta) p^b_s(\beta; \theta), \quad p^b_0(\beta; \theta) = \frac{(1 + \xi)^3 e^{-\xi} \varphi^b_0(\beta; \theta)}{(1 + \xi_0)^2 f^b_0(\xi_0)}, \]
\[ p^b_s(\beta; \theta) = p^b_s(\beta; \pi - \theta) \quad (s = 0, 2, 3); \quad p^b_0(\beta; \theta) = p^b_0(\beta; \pi - \theta), \]  
in terms of \( p^b_s(\beta; \theta) \) defining the contribution of power radiated by \( s \)-polarization component in space angle \( d\Omega \) near the direction given by \( \theta \).
3. Electron

Let us introduce a parameter $x_0$ and a variable $x$

$$x_0 = x_0(\beta) = \frac{1 - \sqrt{1 - \beta^2}}{1 + \sqrt{1 - \beta^2}} = \frac{\gamma - 1}{\gamma + 1}; \quad x = x(\beta, \theta) = \frac{1 - \sqrt{1 - \beta^2 \sin^2 \theta}}{1 + \sqrt{1 - \beta^2 \sin^2 \theta}}; \quad (7)$$

$$0 \leq x_0(\beta) \leq 1; \quad 0 \leq x(\beta, \theta) \leq x_0(\beta).$$

As it was mentioned, our task includes the examination of the first excited state particles. To go on with an electron, one should keep in mind that in the ground state its spin is opposite to the external field. So, we can subdivide the transitions from the first excited state to the ground state into the spin-flip transitions ($\zeta = 1$) and the transitions without spin-flip ($\zeta = -1$).

For an electron we obtain:

$$\frac{dW_e^\epsilon(\zeta; \beta; \theta)}{d\Omega} = d(\zeta; \beta) \frac{Q_0 A(\beta)}{16(1 + x_0)} \frac{(1 + x)^3 e^{-x}}{1 - x} \varphi_e^\epsilon(\zeta; \beta; \theta);$$

$$\varphi_2^\epsilon(-1; \beta; \theta) = \varphi_3^\epsilon(1; \beta; \theta) = 1 - x_0 x,$$

$$\varphi_2^\epsilon(1; \beta; \theta) = \varphi_3^\epsilon(-1; \beta; \theta) = \frac{(1 + x)^2 \cos^2 \theta}{1 - x_0 x}, \quad (8)$$

$$\varphi_0^\epsilon(\zeta; \beta; \theta) = \varphi_0^\epsilon(\beta; \theta) = \varphi_2^\epsilon(-1; \beta; \theta) + \varphi_3^\epsilon(-1; \beta; \theta) = \varphi_2^\epsilon(1; \beta; \theta) + \varphi_3^\epsilon(1; \beta; \theta),$$

$$\varphi_0^\epsilon(\zeta; \beta; \theta) = \varphi_0^\epsilon(\beta; \theta) = \frac{\varphi_0^\epsilon(\beta; \theta)}{2} + g(1 + x) \cos \theta, \quad g = \pm 1.$$  

Here we use the function

$$d(\zeta; \beta) = \frac{1 - \zeta + x_0(1 + \zeta)}{2} = \begin{cases} x_0 & \text{at } \zeta = 1; \\ 1 & \text{at } \zeta = -1. \end{cases} \quad (9)$$

Integrating over $\theta$ from 0 to $\pi$ we obtain:

$$W_0^e(\zeta; \beta) = d(\zeta; \beta) \frac{Q_0 A(\beta)}{6} f^e(\beta), \quad f^e(\beta) = \frac{3(1 + x_0)}{8} f_0^e(x_0), \quad f_0^e(x) = f_2^e(x) + f_3^e(x). \quad (10)$$

Here

$$f_1^e(x) = \frac{2 - (2 + x) \exp(-x)}{x}; \quad 0 \leq x \leq 1;$$

$$f_2^e(x) = \int_0^1 (1 + xy) \exp(-xy) \sqrt{\frac{1 - x^2 y}{1 - y}} dy,$$

$$f_3^e(x) = \int_0^1 (1 + xy) \exp(-xy) \sqrt{\frac{1 - y}{1 - x^2 y}} dy.$$

We can create analogous functions $p_\epsilon^\epsilon(\zeta; \beta; \theta)$ for an electron

$$\frac{dW_e^\epsilon(\zeta; \beta; \theta)}{d\Omega} = W_0^e(\zeta; \beta)p_\epsilon^\epsilon(\zeta; \beta; \theta), \quad p_\epsilon^\epsilon(\zeta; \beta; \theta) = \frac{(1 + x)^3 e^{-x} \varphi_e^\epsilon(\zeta; \beta; \theta)}{(1 + x_0)^3 (1 - x) f_0^e(x_0)};$$

$$p_0^\epsilon(\zeta; \beta; \theta) = p_0^\epsilon(\beta; \theta), \quad p_2^\epsilon(\zeta; \beta; \theta) = p_2^\epsilon(\beta; \theta),$$

$$p_2^\epsilon(-1; \beta; \theta) = p_2^\epsilon(1; \beta; \theta), \quad p_2^\epsilon(1; \beta; \theta) = p_2^\epsilon(-1; \beta; \theta);$$

$$p_0^\epsilon(\zeta; \beta; \theta) = p_0^\epsilon(\zeta; \beta; \pi - \theta) \quad \text{for } s = 0, 2, 3; \quad p_2^\epsilon(\beta; \theta) = p_2^\epsilon(\beta; \pi - \theta). \quad (11)$$
4. The comparative analysis of basic SR characteristics

According to the formulae presented, one can perfectly see that the electron with $\zeta = 1$ radiates $x_0$ times less, than the electron with $\zeta = -1$. This is an expected result, because intuitively we understand that the electron seems to lose energy while changing its spin direction during the transition to the ground state. In ultrarelativistic case $x_0 \to 1$, thus providing the equiprobability of the transitions with and without spin-flip. Furthermore, we find an exciting fact: the angular distributions of SR linear polarization components for an electron with spin $\zeta = -1$ and an electron with spin $\zeta = 1$ are interchanging (subject to the scaling factor $x_0$). Evidently, it occurs so only for the first excited state electrons.

To compare the amount of radiation emitted by the boson and the electron we introduce the functions

$$k(\zeta; \beta) = \frac{W^{e}(\zeta; \beta)}{W^{b}(\beta)}; \quad k(-1; \beta) = \frac{27}{8} f^{e}(\beta), \quad k(1; \beta) = x_0 k(-1; \beta) \leq k(-1; \beta). \quad (12)$$

With the help of these functions it becomes convenient to compare the power radiated by the electron and the boson.

| $\beta$ | $k(-1; \beta)$ | $k(1; \beta)$ |
|---------|----------------|----------------|
| 0       | 3.375000000    | 0              |
| 0.1     | 3.377828182    | 0.008487059    |
| 0.2     | 3.386505200    | 0.034559774    |
| 0.3     | 3.401641477    | 0.080187921    |
| 0.4     | 3.424376702    | 0.149168461    |
| 0.5     | 3.456612178    | 0.248173589    |
| 0.6     | 3.501468648    | 0.389052072    |
| 0.7     | 3.564220897    | 0.594383401    |
| 0.8     | 3.654341288    | 0.913585322    |
| 0.9     | 3.789776916    | 1.488868656    |
| 1       | 3.716952519    | 3.716952519    |

Table 1. Here the values of $k$-functions at different $\beta$ are given to demonstrate the spin direction influence on the amount of radiated power.

The data of the above table shows that the electron with $\zeta = -1$ (left column) radiates more than boson at any $\beta$, however, the electron of $\zeta = 1$ (right column) starts radiating more than boson only within the relativistic region of parameters, i.e. when $\beta \to 1$.

5. The radiation polarization and spin

Now, let us consider the behavior of functions $p^{e,b}_s$ in details. The first important thing about the functions $p^e_s$ and $p^b_s$ is their being finite at any $\beta$ and $\theta$ (including $\beta = 1$, in contrast to the classical theory). The evolution of the functions $p^{b}_1$ and $p^{s}_2$ seems quite simple, both $p^{b}_2$ and $p^{s}_2$ are monotone increasing functions at $0 \leq \theta \leq \pi/2$ depending on $\theta$ (which qualitatively corresponds with classical theory). The functions $p^{b}_1$ are steadily decreasing within $[0; \pi/2]$ at any fixed value of $\beta$, whereas $p^{s}_2$ cease being monotone at $\beta > \sqrt{3}/2$. And here we also find a remarkable fact: the behavior of the electronic functions is more similar to the classical theory in comparison to the bosonic functions, though the spin itself has a pure quantum nature, and we deal exactly with a case where its influence is expected. We observe the same situation for the functions $p^{e,b}_0$ and $p^{e,b}_1$. The bosonic functions are monotone and decreasing on $[0; \pi/2]$ ([0; $\pi$] for $p^{b}_1$) at any $\beta$. 
Still, the electron radiation characteristics $p_{e1}$ and $p_{e0}$ are losing their monotony at $\beta > 1/\sqrt{2}$.
Finally, we see the function $p_{e1}(\beta)$ tending to 0 when $\beta \to 1$ within the interval $\pi/2 \leq \theta \leq \pi$, the fact which makes us sure about right polarization extinction in the lower half plane.

As an example, we present the figures demonstrating the behavior of functions $p_{e3}^b$ and $p_{e3}^c$.

![Figure 1. The graphs of functions $p_{e3}^b(\beta, \theta)$](image1)
![Figure 2. The graphs of functions $p_{e3}^b(\beta, \theta)$](image2)
![Figure 3. The graphs of functions $\theta_{max}(\beta)$](image3)

The above analysis provides a possibility to observe the absence of radiation power concentration near the orbit’s plane ($\theta = \pi/2$ neigbourhood), which contradicts classical theory predictions. For a more accurate analysis of polarization components contribution, we explore the character of the functions $\theta_{max}$ (Figure 3) - the maximum of $p_{e3}^c$ depending on $\beta$. It could be defined parametrically as follows

\[
\begin{align*}
\beta^2 &= \frac{2(2-a^3)}{(2-a)(2+a+a^2)}, \\
\cos^2 \theta_{max}^0(\beta) &= \frac{a^2}{(2-a^2)}, \\
\beta^2 &= \frac{4(2-a^3)^2}{(2-a)^2(2+a+a^2)(2+a-a^2)}, \\
\cos^2 \theta_{max}^1(\beta) &= \frac{a}{2-a^2}, \\
\beta^2 &= \frac{4(2-a^3)(2+a+a^2)}{(2-a)^2(2+a+a^2)(2+a-a^2)}, \\
\cos^2 \theta_{max}^3(\beta) &= \frac{a^2(1+a+a^2)}{(2-a^2)(2+a^2)}.
\end{align*}
\]

**Concluding comments**
The classical theory of SR provides a great number of adequate results. However, some properties of this phenomenon need to be described with the use of quantum theory methods. It turned out, that for the first excited state particles the presence of spin makes SR characteristics behave more similar to their classical analogues.

**References**
[1] Sokolov A A and Ternov I M 1968 Synchrotron Radiation Berlin
[2] Bagrov V G 2008 Russian Physics Journal, Special Features of the Angular Distribution of Synchrotron Radiation 51, 335-352
[3] Sokolov A A and Ternov I M 1986 Radiation from relativistic electrons New York
[4] Ternov I M and Mikhailin V V 1986 Synchrotron Radiation. Theory and Experiment. Moskow