On solutions of a class of matrix-valued convolution equations

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Abstract

We apply a relation between matrix-valued complete Bernstein functions and matrix-valued Stieltjes functions to prove that certain convolution equations for matrix-valued functions have unique solutions in a special class of functions. In particular the cases of the viscoelastic duality theorem and the Sonine equation are discussed, with applications in anisotropic linear viscoelasticity and a generalization of fractional calculus.

Keywords: viscoelasticity, completely monotone, Bernstein, complete Bernstein, Stieltjes, Sonine equation, fractional calculus

1 Introduction

A recent idea [6] of simplifying the proof of a viscoelastic duality relation between the tensor-valued relaxation modulus and the tensor-valued creep function [4], based on a relation between matrix-valued complete Bernstein functions and matrix-valued Stieltjes functions [5] has led me to consider a more general application of this method to convolutions of matrix-valued functions, including the Sonine equation [10] for matrix-valued functions.

Due to the similarities in the treatment of both equations I repeat the theorems and their proofs already published in [6]. The last reference is devoted to the aspects of the the duality relation relevant for viscoelasticity. In this paper I skip the details relevant for viscoelasticity and focus on proving the existence of solutions when one of the functions is either a locally integrable completely monotone (LICM) function or a Bernstein function.

In the case of the Sonine equation I assume that one of the functions is locally integrable and completely monotone (LICM). The Sonine equation was examined in much detail in [9], but the authors did not assume that one of the functions was LICM. They have constructed the inverse operator for the convolution equation $k(t) \ast x(t) = f(t)$. The inverse operator however involves
the solution \( l(t) \) of the Sonine equation. Existence of such a function is the subject of our investigation and we prove it for \( k \) in the LICM class. This problem is also studied in [7] for real-valued functions.

I reexamine Kochubei’s suggestion that the solutions of the Sonine equation could be used to construct a generalization of the concepts of a derivative and integration operators along the lines of fractional calculus. In our case it provides an ”anisotropic” derivative which might turn out to be useful in the context of non-linear relaxation equations. I also show that singularity of one of the LICM functions is essential for a correct construction of generalized ”fractional” calculus.

2 Convolutions equations for matrix-valued functions.

Consider the general convolutional equation

\[
A(t) \ast X(t) = R(t)
\]  

(1)

where \( A(t) \) and \( R(t) \) are two square matrix-valued functions defined for \( t \in [0, \infty) \) in a class to be specified, while \( X(t) \) is a square matrix-valued function defined be equation \([1]\). We shall determine the properties of the function \( X(t) \). The ranks of the matrices are equal and will be denoted by \( m \). Let \( S_m \) denote the space of real square matrices of rank \( m \).

The convolution is defined by the formula

\[
A(t) \ast B(t) := \int_0^t A(s) B(t - s) \, ds.
\]

**Definition 1** A matrix-valued function \( A : [0, \infty[ \to S_m \) is said to be completely monotone (CM) if it is infinitely differentiable and for every vector \( v \in \mathbb{R}^m \) the following inequalities are satisfied

\[
\forall t > 0 \, \forall n \in \mathbb{N} \, (-1)^n D^n v^T A(t) v \geq 0
\]

(2)

The above definition allows for a singularity at 0.

**Definition 2** A matrix-valued function \( A : [0, \infty[ \to S_m \) is said to be locally integrable completely monotone (LICM) if it is CM and integrable over \([0, 1]\).

**Definition 3** A matrix-valued function \( A : [0, \infty[ \to S_m \) is said to be a Bernstein function if it is differentiable and its derivative is CM.

If \( A \) is a matrix-valued Bernstein function, then for every \( v \in \mathbb{R}^m \) the function \( t \to v^T A(t) v, \, t > 0, \) is non-decreasing and continuous, hence it has a finite limit at \( t = 0 \). Its derivative is a LICM function.
The Laplace transform $\tilde{A}(p)$ of a matrix-valued function $A(t)$ is defined as usual by the formula

$$\tilde{A}(p) := \int_0^\infty e^{-pt} A(t) \, dt$$

for every $p \in \mathbb{C}$ such that the integral exists.

The Laplace transform exists for $p > 0$ and a LICM function $A$ because such a function is locally integrable and non-increasing. It also exists for $p > 0$ if $A$ is a matrix-valued Bernstein function because

$$\int_0^\infty e^{-pt} \int_0^t A(s) \, ds = \frac{1}{p} \int_0^\infty e^{-pt} B(t) \, dt,$$

where $B$ is the LICM derivative of $A$.

**Proposition 1** If $A$ is a symmetric matrix-valued LICM function and for each vector $v \in \mathbb{R}^m$ the function $v^T A(t) v$, $t > 0$ is not identically zero, then the matrix $\tilde{A}(p)$ is invertible for every $p > 0$.

**Proof.**

For each non-zero $v \in \mathbb{R}^m$ there is a real $t_1(v) > 0$ such that $v^T A(t_1(v)) v > 0$. The function $v^T A(t) v$ is continuous, hence it is positive on some interval $I \subset (0, \infty]$, while it is non-negative on $[0, \infty]$. Hence for every non-zero $v \in \mathbb{R}^m$, $p > 0$ we have $v^T \tilde{A}(p) v > 0$. The matrix $A(p)$ is symmetric and positive definite, hence it is invertible. \(\square\)

**Proposition 2** If $A$ is a symmetric matrix-valued Bernstein function and for each vector $v \in \mathbb{R}^m$ the function $v^T A(t) v$, $t > 0$ is not identically zero, then the matrix $\tilde{A}(p)$ is invertible for every $p > 0$.

The proof is analogous to the previous proposition.

It is easy to check the familiar identity

$$\int_0^\infty e^{-pt} A(t) \ast B(t) \, dt = \tilde{A}(p) \tilde{B}(p), \quad p > 0$$

provided both Laplace transforms on the right-hand side exist. Consequently equation \(\[\ref{eq:identity}\]\) implies the equation

$$\tilde{A}(p) \tilde{X}(p) = \tilde{R}(p)$$

(3)

If the matrix $\tilde{A}(p)$ is invertible for $p > 0$, then the unique solution of \((3)\) is

$$\tilde{X}(p) = \tilde{A}(p)^{-1} \tilde{R}(p), \quad p > 0$$

(4)

This is in particular true if $A$ is an $S_+\text{-valued}$ LICM function. For $R(t) = t I, t > 0$, where $I$ is the identity operator on $\mathbb{R}^m$, we have $R(p) = p^{-2} I$ and

$$p \tilde{X}(p) = \left[p \tilde{A}(p)\right]^{-1}$$

(5)
We shall show that under these assumptions the solution $X$ of equation (1) is an $S_+$-valued Bernstein function.

Similarly, if $A$ is an $S_+$-valued BF and $R(t) = tI, t > 0$, then $X$ is an $S_+$-valued LICM function.

By transposition the results obtained below also apply to equations of the form

$$X(t) * A(t) = R(t)$$

3 Main theorems and their proofs.

**Theorem 1** If $A$ is an $S_+$-valued LICM function and $(\ast)$ for each non-zero vector $v \in \mathbb{R}^n$ the function $v^T A(t) v, t > 0$, is not identically zero,

then equation (1) with $R(t) = tI$, has a unique solution $X$, which is a $S_+$-valued Bernstein function.

**Theorem 2** If $A$ is an $S_+$-valued Bernstein function satisfying Condition $(\ast)$,

then equation (1) with $R(t) = tI$, has a unique solution $X$,

$$X(t) = B \delta(t) + F(t) \quad (6)$$

where $B \in S_+$ and $F$ is a $S_+$-valued LICM function.

It follows from Theorems 1 and 2 that for every LICM function $\kappa(t)$ the equation $\kappa * \lambda = t$ for $t \geq 0$ has a unique solution $\lambda$. The solution function $\lambda$ is a BF. For every $\kappa$ which is a BF the same equation has a solution $\lambda$ which is a LICM function plus a Dirac delta term.

If $\kappa$ and $\lambda$ satisfy the same equation but $\kappa$ is neither a BF nor LICM then the same is true for $\lambda$. Do such pairs exist? Here is an example of such a pair of which none is CM nor BF: $\kappa(t) = t^{1/2} I_{\nu}(2^{1/2})$ and $\lambda(t) = t^{-1/2} J_{\nu}(t^{1/2})$ for $\nu > -1$. Indeed, according to (18) on p. 197 and (30) on p. 185 of [2] $\tilde{\kappa}(p) = p^{\nu-1} \exp(1/p)$ and $\tilde{\lambda}(p) = p^{\nu-1} \exp(-a/p)$. $\lambda$ changes sign hence it is neither LICM nor a BF.

**Proof of Theorem 1**

On account of (21)

$$p \tilde{A}(p) = p C + p \int_{[0,\infty]} (p + r)^{-1} G(r) \mu(dr),$$

where $C \geq 0$, hence $p \tilde{A}(p)$ is an $S_+$-valued CBF.

On account of $(\ast)$ and Proposition 1 the matrix $\tilde{A}(p)$ has an inverse for $p > 0$. Hence the function $p \tilde{A}(p)$ does not vanish identically and according to
Theorem 5 the inverse $p \tilde{X}(p)$ of $p \tilde{A}(p)$ is an $S_+$-valued Stieltjes function and has the form
\[ B + p^{-1} D + \int_{[0, \infty]} (p + r)^{-1} H(r) \mu(dr), \tag{7} \]
where $D, B \in S_+$, $\mu$ is a Borel measure on $]0, \infty[$ satisfying (18) and $H$ is a bounded measurable $S_+$-valued function defined $\mu$-almost everywhere on $]0, \infty[$.

On account of equation (4) $p \tilde{X}(p)$ has the form given by equation (7). It follows that $X(t) = B + tD + \int_0^t K(s) ds$, where
\[ K(t) := \int_{[0, \infty]} e^{-rt} H(r) \mu(dr) \]
is a LICM. It follows that $X$ is an $S_+$-valued Bernstein function.

Proof of Theorem 2.
Condition $(\ast)$ and Proposition ensures that the matrix $\tilde{A}(p)$ is invertible for $p \geq 0$.

We now note that $A(t) = \int_0^t L(s) ds$, where $L$ is an $S_+$-valued LICM function. By the Bochner Theorem there is a Borel measure $\mu$ satisfying (18) and an $S_+$-valued function $G(r)$, $r \geq 0$, bounded everywhere except perhaps on a set of $\mu$ measure zero, such that
\[ L(t) = \int_{[0, \infty]} e^{-rs} G(r) \mu(dr). \]

Consequently $p \tilde{A}(p) = \tilde{L}(p) = \int_{[0, \infty]} (p + s)^{-1} G(r) \mu(dr)$ is an $S_+$-valued Stieltjes function and it is not identically vanishing. By Theorem its inverse is an $S_+$-valued CBF and therefore it has the form
\[ pB + p \int_{[0, \infty]} (p + r)^{-1} G(r) \mu(dr), \]
for some $B \in S_+$, a Borel measure $\mu$ satisfying (18) and a measurable $S_+$-valued function $G$ bounded $\mu$-almost everywhere on $]0, \infty[$.

Equation (11) is satisfied if $X$ is given by equation (6) with
\[ F(t) := \int_{[0, \infty]} e^{-rt} G(r) \mu(dr). \]

Theorem 3 If $A$ is a non-zero $S_+$-valued LICM function and the limit $A_0 := \lim_{t \to 0} A(t)^{-1}$ exists, then equation (11) with $R(t) = I$ has a unique solution $X = A_0 \delta(t) + F(t)$, where $F$ is an $S_+$-valued LICM function.
The equation \( k \ast l = 1 \) in for locally integrable real-valued functions \( k \) and \( l \) is known as the Sonine equation \(^{10}\). If for a given \( k \in L^1_{\text{loc}}([0, \infty]) \) there is an \( l \in L^1_{\text{loc}}([0, \infty]) \) satisfying the above equation, then \( k \) is called a Sonine kernel, while \( k, l \) are known as a Sonine pair. Sonine pairs are studied in some detail in \(^9\). Theorem 3 asserts in particular that every LICM function or matrix-valued LICM function is a Sonine kernel and in this case the Sonine pair consists of two LICM functions. For real-valued functions this fact has apparently been discovered by Kochubei \(^7\).

However, not every Sonine pair consists of LICM functions. An counterexample is the Sonine pair \( k_\lambda(t) := t^{-\lambda/2} J_{-\lambda}(2t^{1/2}) \) with the Laplace transform \( \check{k}_\lambda(p) = \exp(-1/p) p^{\lambda-1} \) \(^2\) p. 185 (30)) and \( l_\lambda(t)t^{(\lambda-1)/2} I_{\lambda-1}(2t^{1/2}) \) with \( l_\lambda(p) = \exp(1/p) p^{-\lambda-2} \) \(^2\) p. 197 (18)) for \( \lambda > 0 \). The function \( k_\lambda \) changes sign and therefore is not CM, for example \( k_{1/2}(t) = \sqrt{2/\pi} \cos(2t^{1/2})/t^{3/4} \).

The following matrix-valued Sonine pairs are of particular interest:

1. \( k(t) \bm{K}_0 \) and \( l(t) \bm{K}_0^{-1} \), where \( k, l \) are a Sonine pair of LICM functions;

2. \( \text{diag}\{k_n(t), n = 1, \ldots, m\} \) and \( \text{diag}\{l_n(t), n = 1, \ldots, m\} \), where \( (k_n, l_n) \) are Sonine pairs of LICM functions for \( n = 1, \ldots, m \).

Many CM functions are known \(^8\), but it is often more difficult to find the other member of the Sonine pair. The simplest Sonine pair of CM functions is \( k(t) = t^{\alpha-1}/\Gamma(\alpha) \) and \( l(t) = t^{-\alpha}/\Gamma(1-\alpha), 0 < \alpha < 1 \). Using the Laplace transforms \( \check{k}(p) = (p+\lambda)^{-\alpha} \) and \( \check{L}[\Gamma(-\alpha, \alpha t)](p) = \Gamma(-\alpha) \lambda^{-\alpha} \left[ \chi^a - (\lambda + p)^a \right]/p \) one gets another pair \( k(t) = t^{\alpha-1} e^{-\lambda t}/\Gamma(\alpha), \lambda > 0 \), with \( l(t) = \lambda^{\alpha} \times [1 - \Gamma(-\alpha, \alpha t)/\Gamma(-\alpha)], \lambda \geq 0, 0 < \alpha < 1 \).

It is also interesting that for an arbitrary analytic function \( k(t) \) there is another analytic function \( l(t) \) such that \( k(t) t^{\alpha-1}/\Gamma(\alpha) \) and \( l(t) t^{-\alpha}/\Gamma(1-\alpha) \) are a Sonine pair and there is an algorithm for calculating the power series of \( l(t) \) given the power series for \( k(t) \) \(^9\) \(^11\).

**Proof of Theorem 3**

The Laplace transform \( \check{\bm{A}}(p) \) is a symmetric positive definite matrix for every \( p > 0 \). Equation \(^{11}\) is equivalent to \( \check{X}(p) = \left[ p \check{\bm{A}}(p) \right]^{-1} \). The right-hand side is the algebraic inverse of a matrix-valued CBF, hence it is a Stieltjes function of the form

\[
\bm{H}(p) = \bm{C} + \int_{[0, \infty]} (p + r)^{-1} \bm{G}(r) \mu(dr).
\]

where \( \bm{C} \geq 0, \bm{G} \) is a measurable symmetric matrix-valued function bounded \( \mu \)-almost everywhere and \( \mu \) is a Borel measure satisfying inequality \(^{13}\).

In view of inequality \(^{13}\) the Lebesgue Dominated Convergence Theorem implies that

\[
\bm{C} = \lim_{p \to \infty} \bm{H}(p) = \left[ \lim_{p \to \infty} (p \check{\bm{A}}(p)) \right]^{-1} = \bm{A}_0.
\]
The second term on the right-hand side of equation (8) is a Laplace transform \( \tilde{F}(p) \) of the LICM function

\[
F(t) := \int_{0,\infty} e^{-rt} G(r) \mu(dr)
\]

(9)

Inverting the Laplace transformation we conclude that

\[
X(t) = A_0 \delta(t) + F(t).
\]

□

The first application of this result is the solution of a convolution equation

\[
A(t) * v(t) = f(t)
\]

(10)

Since

\[
X(t) * A(t) = I,
\]

we have

\[
\int_0^t v(t) \, dt = X(t) * f(t)
\]

and therefore

\[
v(t) = D[X * f(t)]
\]

(11)

The LICM function \( A(t) \) can have a singularity at 0 such that for every \( v \in \mathbb{R}^m \) the limit \( \lim_{t \to 0} v^T A(t) v = \infty \). It then follows that \( A_0 = 0 \). In this case we define the generalized Caputo \( A \)-derivative by the formula

\[
D_A v(t) = D[A(t) * v(t)] - A(t) v(0) \equiv A * v'
\]

(12)

for every absolutely continuous function \( v : [0, \infty[ \to \mathbb{R}^m \). The term "derivative" is justified if the function \( A \) is singular at 0. If \( X = F \) is the solution of the convolution equation (1) with \( R(t) = I \), then the \( A \)-integral operator is defined by the formula

\[
J_A v(t) := F * 1 * v(t)
\]

(13)

where \( 1 * v(t) = \int_0^t v(s) \, ds \).

We then have

**Theorem 4**

Let \( A_0 = 0 \).

The following relations hold

\[
J_A D_A w(t) = w(t) - w(0) \quad \text{for } w \in AC([0, \infty])
\]

(14)

\[
D_A J_A v(t) = v(t) \quad \text{for } v \in L_{loc}^1([0, \infty])
\]

(15)

**Proof**

(1) The identity \( F * A = 1 \) implies that

\[
(J_A D_A w)(t) = \int_0^t F(s) \frac{d}{dt} \int_0^r A(\tau - r)w(r) \, dr \bigg|_{\tau = t-s} \, ds - w(0)
\]

The first term equals

\[
\int_0^t F(s) \frac{d}{dt} \int_0^{t-s} A(t-s-r)w(r) \, dr \, ds = \frac{d}{dt} (F * A * w)(t) = \frac{d}{dt} (1 * w)(t) = w(t)
\]
(2) Let \( w := F \ast v \). On account of the identity \( A \ast F = 1 \)
\[
D_A J_A v = \frac{d}{dt} (A \ast F \ast v)(t) - A(t) w(0) = v(t) - A(t) w(0)
\]

It remains to prove that \( w(0) = 0 \). \( F \) is a LICM function, hence it has the form (9) with \( \mu \) satisfying equation (18) and \(|H(r)| \leq 1\). Hence
\[
|w(t)| \leq \left[ \int_0^t \int_{[0,\infty]} e^{-rs} \mu(dr) \, ds \right] \int_0^t |v(t-s)| \, ds
\]

For \( t \leq 1 \) the second factor is bounded from above by a constant
\[
\int_0^1 |v(s)| \, ds < \infty
\]
because \( v \) is assumed locally integrable. The first factor equals
\[
\int_{[0,\infty]} \frac{1 - e^{-rt}}{r} \mu(dr) \tag{16}
\]
From the inequality \( e^x - 1 \leq xe^x \ (x \geq 0) \) follows the inequality \( 1 - e^{-x} \leq x \).
We shall apply this inequality for \( r \in [0,\infty[ \), noting that \( \mu([0,1]) < \infty \) because of (18) with the inequality \( 1 \leq 2/(1 + r) \) valid for \( r \leq 1 \). For \( r > 1 \) we shall note that \( 1/r \leq 2/(1 + r) \). Hence expression (16) is bounded by
\[
t \mu([0,1]) + 2 \int_{[1,\infty]} \left( 1 - e^{-rt} \right) \frac{1}{r} \mu(dr),
\]
which tends to 0 on account of (18) and the Lebesgue Dominated Convergence Theorem. Thus \( w(0) = 0 \) and the theorem has been proved. \( \square \)

The new derivative concept provides a new approach to modeling stress relaxation in anisotropic and non-linear viscoelastic media. A possible relaxation equation could have the form
\[
D_A \sigma = K(\sigma, \epsilon) \tag{17}
\]

Theorem applies only to (weakly) singular kernels \( A \). In particular it does not apply to Caputo-Fabrizio \( \mathbb{R} \) and Atangana-Baleanu \( \mathbb{R} \) fractional derivatives. In these cases \( A \) is non-singular but the corresponding \( X \) is no longer a superposition of Newtonian viscosity and a LICM relaxation function. There is no integral associated with these derivatives. For the Caputo-Fabrizio derivative we have \( A(t) = \exp(-\alpha t/(1-\alpha))/(1-\alpha) \) and \( F = (1-\alpha) \delta' + \delta \). In the Atangana-Baleanu case \( F \) involves \( \delta' \) and a strongly singular kernel.
4 Conclusions.

We have demonstrated a particular role of LICM and Bernstein kernels in two classes of convolution equations and the utility of the concepts of CBFs and Stieltjes derivatives in the study of existence problems for these equations.

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A Matrix-valued Stieltjes functions and Complete Bernstein functions.

Let $S_+$ denote the set of non-negative symmetric matrices.
An $S_+$-valued function $A(t)$ is CM if

$$(-1)^n D^n A(t) \geq 0 \quad \text{for } n = 0, 1, 2 \ldots$$

where $B \geq 0$ is equivalent to $v^T M v \geq 0$ for every $v \in \mathbb{R}^6$.

The function $A(t)$ is LICM if it is CM and locally integrable.

For every $S_+$-valued LICM function $A$ there is a Borel measure $\mu$ satisfying the inequality

$$\int_{[0, \infty]} (1 + s)^{-1} \mu(ds) < \infty$$

and an $S_+$-valued function $G$ on $[0, \infty]$ bounded on everywhere except for a set of measure zero such that

$$A(t) = \int_{[0, \infty]} e^{-rt} G(r) \mu(dr)$$

If $\mu(\{0\}) > 0$, then $G(0)$ is defined and (19) can be recast in the form

$$A(t) = B + \int_{[0, \infty]} e^{-rt} G(r) \mu(dr)$$

where $B := \mu(\{0\}) G(0)$ is a positive semi-definite symmetric matrix. If $\mu(\{0\}) = 0$ then $B = 0$.

The Laplace transform of the $S_+$-valued LICM $A(t)$ is given by the equation

$$\tilde{A}(p) = \int_{[0, \infty]} (p + r)^{-1} G(r) \mu(dr)$$

An $S_+$-valued Bernstein function is an indefinite integral of a $S_+$-valued LICM function.

We shall now recall some results from Appendix B of [5].

A matrix-valued Stieltjes function $Y(p)$ has the following integral representation:

$$Y(p) = B + \int_{[0, \infty]} (p + r)^{-1} G(r) \mu(dr)$$

where $B \in S_+$, $\mu$ is a Borel measure on $[0, \infty]$ satisfying (18) and $G(r)$ is an $S_+$-valued function defined $\mu$-almost everywhere on $[0, \infty]$. $D = \mu(\{0\}) G(0)$. Conversely, any matrix-valued function with the integral representation (22) is an $S_+$-valued Stieltjes function.

An $S_+$-valued CBF $Z(p)$ has the following integral representation:

$$Z(p) = p B + \int_{[0, \infty]} (p + r)^{-1} H(r) \nu(dr)$$

where $B \in S_+$, $\mu$ is a Borel measure on $[0, \infty]$ satisfying (18) and $G(r)$ is an $S_+$-valued function defined $\mu$-almost everywhere on $[0, \infty]$. $D = \mu(\{0\}) G(0)$. Conversely, any matrix-valued function with the integral representation (22) is an $S_+$-valued Stieltjes function.
where $B, D \in S_+$, $\nu$ is a Borel measure on $]0, \infty[$ satisfying (15) and $H(r)$ is an $S_+$-valued function defined $\nu$-almost everywhere on $]0, \infty[$, $D = H(0) \mu(\{0\})$.

Conversely, any matrix-valued function with the integral representation (24) is a $S_+$-valued CBF.

It follows immediately that the the function $p^{-1}Z(p)$, where $Z$ is an $S_+$-valued CBF function, is an $S_+$-valued Stieltjes function.

We quote Lemma 3 op. cit. in the form of the following theorem

**Theorem 5** If $Z(p)$ is an $S_+$-valued CBF and does not vanish identically, then $Z(p)^{-1}$ is an $S_+$-valued Stieltjes function.

Conversely, if $Y(p)$ is an $S_+$-valued function does not vanish identically then $Y(p)^{-1}$ is a CBF.