Photons in the Quantum World

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Abstract

Einstein’s photo-electric effect allows us to regard electromagnetic waves as massless particles. Then, how is the photon helicity translated into the electric and magnetic fields perpendicular to the direction of propagation? This is an issue of the internal space-time symmetries defined by Wigner’s little group for massless particles. It is noted that there are three generators for the rotation group defining the spin of a particle at rest. The closed set of commutation relations is a direct consequence of Heisenberg’s uncertainty relations. The rotation group can be generated by three two-by-two Pauli matrices for spin-half particles. This group of two-by-two matrices is called $SU(2)$, with two-component spinors. The direct product of two spinors leads to four states leading to one spin-0 state and one spin-1 state with three sub-states. The $SU(2)$ group can be expanded to another group of two-by-two matrices called $SL(2,c)$, which serves as the covering group for the group of Lorentz transformations. In this Lorentz-covariant regime, it is possible to Lorentz-boost the particle at rest to its infinite-momentum or massless state. Also in this $SL(2,c)$ regime, there are four spin states for each particle, as in the case of the Dirac equation. The direct product of two $SL(2,c)$ spinors thus leads sixteen states. Among them, four of them can be used for the electromagnetic four-potential, and six for the Maxwell tensor. The gauge degree of freedom is shown to be a Lorentz-boosted rotation. The polarization of massless neutrinos is interpreted as a consequence of the gauge invariance.

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1 Introduction

The algebra of quantum mechanics starts from Heisenberg’s commutation relations

\[ [x_i, p_j] = i \delta_{ij}, \]  

(1)

with

\[ p_i = -i \frac{\partial}{\partial x_i}. \]  

(2)

These expressions are well known.

Let us next consider the operators

\[ J_i = -i \epsilon_{ijk} x_j \frac{\partial}{\partial x_k} = -i \left( x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right), \]  

(3)

which satisfy the commutation relations

\[ [J_i, J_j] = i \epsilon_{ijk} J_k. \]  

(4)

This closed set of commutation relations is the Lie algebra of the three-dimensional rotation group \( O(3) \). This Lie algebra is a direct consequence of Heisenberg’s uncertainty relations given in Eq.(1).

The simplest matrices representing this Lie algebra are

\[ S_i = \frac{1}{2} \sigma_i, \]  

(5)

where \( \sigma_i \) are the two-by-two Pauli spin matrices. We use the spinors

\[ u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]  

(6)

for the spin-up and spin-down states, respectively.

With these spinors, we can construct the spin-0 and spin-1 states in the following manner. For the spin-0 state, we make the anti-symmetric combination

\[ \frac{1}{\sqrt{2}} (uv - vu). \]  

(7)

There are three spin-1 states. They are

\[ uu, \quad \frac{1}{\sqrt{2}} (uv + vu), \quad vv, \]  

(8)

for the \( z \)-component spin 1, 0, and -1 respectively.

Next, we all know photons are massless particles with spin 1 parallel or anti-parallel to its momentum. They are called helicities. We are also familiar with the expressions

\[ A_\mu = \begin{pmatrix} A_0 \\ A_z \\ A_x \\ A_y \end{pmatrix}, \quad \text{and} \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E_z & -E_x & -E_y \\ E_z & 0 & -B_y & B_x \\ E_x & B_y & 0 & -B_z \\ E_y & -B_x & B_z & 0 \end{pmatrix}, \]  

(9)
These are Maxwell’s four-vector and second-rank four-tensor for electromagnetic fields. These expressions belong to Einstein’s Lorentz-covariant world. For convenience, we use the Minkowskian four-vector convention of \((t, z, x, y)\) throughout the paper.

Here is the question. Is it possible to derive these Maxwell vector and tensor from Heisenberg’s relations given in Eq. (1)? The answer is No, but is it possible to construct a bridge between them? The answer is Yes, but this question has a stormy history. The purpose of this paper is to provide a simple answer to this question. The bridge consists of the set of three two-by-two “imaginary” Pauli matrices \(i \sigma_i\):

\[
K_i = \frac{i}{2} \sigma_i. 
\]

They correspond to

\[
K_i = -i \left( t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t} \right),
\]

applicable to the four-dimensional Minkowskian space.

The six matrices consisting of \(S_i\) and \(K_i\) become the generators of the group \(SL(2,c)\) isomorphic to the group of Lorentz transformations. This group thus allows us to Lorentz-transform spinors which will eventually lead us to the electromagnetic four-vector and four-tensor.

It was of course Einstein who unified the energy-momentum relation for both massive and massless particles. We are now faced with the problem of unifying internal space-time symmetries. Einstein’s photon has spin-one parallel or anti-parallel to its momentum. For a particle at rest, we all know how to construct spin-1 states from two spinors as was the case in Eq. (8). The issue is how to Lorentz-boost those spin-1 states to reach the Maxwell tensor and four-vector for electromagnetic field.

It was Eugene Wigner who pioneered this research line. In 1939 [1], he constructed the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. He called these subgroups “little groups.” Thus, Wigner’s little groups dictate the internal space-time symmetries of particles in the Einstein’s Lorentz-covariant world which includes both massive and massless particles, as shown in Table 1. This table was first published in 1986 [2]. Indeed, the photon polarization and the gauge degree of freedom are the issues of the internal space-time symmetries of massless particles.

In Sec. 2 we present two-by-two representation of the Lorentz group, and Wigner’s little groups in Sec. 3. In Sec. 4 we discuss massless particles as large-momentum or small-mass limit of massive particles. It is shown that there are four spin states in the Lorentz-covariant world. Thus there are sixteen different ways to combine two spinors. In Sec. 5 we construct explicitly those sixteen states. Among them are the electromagnetic four-vector and the Maxwell tensor. It is pointed out that the polarization of massless neutrinos is a consequence of gauge invariance.
Table 1: Extension of the concepts contained in Einstein’s $E = mc^2$ to photons helicity and gauge. Under the Lorentz boost along the $z$ direction, the $z$ component of the spin remains as the helicity, but the transverse components collapse into one gauge degree of freedom.

|       | Slow | Relativistic | Fast |
|-------|------|--------------|------|
| Energy-momentum | $p^2/2m$ | $\sqrt{p^2 + m^2}$ | $E = p$ |
| Helicity | $S_3$ | Wigner’s | Helicity |
| Spin & Gauge | $S_1, S_2$ | Little Groups | Gauge Trans. |

2 Lorentz Group and Its Representations

In addition to the rotation generators of Eq.(3), we can consider another set of three operators, namely

$$K_i = -i \left( x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i} \right).$$

These operators are known to generate Lorentz boosts in the Minkowskian space of one time direction and three spatial dimensions, and they satisfy the commutation relations

$$[K_i, K_j] = -i \epsilon_{ijk} J_k.$$  

These three boost generators do not lead to a closed set of commutation relations. However, with the $J_i$ generators, they satisfy the commutation relations

$$[J_i, K_j] = i \epsilon_{ijk} K_k.$$  

Let us write the commutation relations of Eqs. (13) and (14) as one closed set of commutation relations

$$[J_i, J_j] = i \epsilon_{ijk} J_k, \quad [J_i, K_j] = i \epsilon_{ijk} K_k, \quad [K_i, K_j] = -i \epsilon_{ijk} K_k.$$  

This set is called the Lie algebra of the Lorentz group.

In terms of four-by-four matrices these generators take the form:

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for rotations around and boosts along the $z$ direction, respectively. Here, again the ordering of the coordinates in Minkowskian space-time are $(t, z, x, y)$, where the transformation matrices corresponding to the generators above are applicable.
Table 2: Spinors in the relativistic world. The spinors \( u \) and \( v \) are for spin-up and spin-down states respectively. Under the Lorentz boost, the dotted spinors are boosted in the opposite direction.

|            | undotted | dotted |
|------------|----------|--------|
| Spin up    | \( u \)  | \( \dot{u} \) |
| Spin down  | \( v \)  | \( \dot{v} \) |

Similar expressions can be written for the \( x \) and \( y \) directions. We see that the rotation generators \( J_i \) are Hermitian, but the boost generators \( K_i \) are anti-Hermitian.

Four-by-four matrices that leave the quantity \( (t^2 - z^2 - x^2 - y^2) \) invariant, in the four-dimensional Minkowski space forms the basis of the group of Lorentz transformations. Since there are three rotation and three boost generators, the Lorentz group is a six-parameter group.

The Lorentz group can also be represented by two-by-two matrices. If we choose

\[
J_i = \frac{1}{2} \sigma_i, \quad K_i = \frac{i}{2} \sigma_i. \tag{17}
\]

They satisfy the set of commutation relations given in Eq. (15). Thus, to each two-by-two transformation matrix, there is a corresponding four-by-four matrix applicable to the Minkowskian space.

The algebra of Eq. (17) is invariant under the sign change of the \( K_i \) matrices. Let us introduce the notation

\[
\dot{K}_i = -K_i. \tag{18}
\]

Then

\[
J_i = \frac{1}{2} \sigma_i, \quad \dot{K}_i = -\frac{i}{2} \sigma_i. \tag{19}
\]

Corresponding to these two-by-two matrices, we can construct one set of two-component spinor (spin-up and spin-down) for the undotted representation, and another set for the dotted representation. There are thus four spin states in the Lorentz-covariant world as shown in Table 2. This is the reason why the Dirac spinor has four components.

As far as rotations are concerned, the representation constructed from the Lie algebra of Eq. (19) is transformed in the same way as that of Eq. (17). However, the Lorentz boosts are performed in opposite directions.

If two spinors are coupled, there are 16 (\( = 4 \times 4 \)) states, which can be partitioned into to the spin-0 and spin-1 states. We shall come back to this problem in Sec. 5.
3 Wigner’s Little Groups

In 1939 [1], Wigner considered the subgroups of the Lorentz group the transformations of which leave the four-momentum of a given particle invariant. It is well-known that the momentum of a massive particle at rest is invariant under rotations. On the other hand massless particles do not have rest frames. Then how shall we study the space-time symmetries of massless particles? In this paper we shall address this problem, primarily focusing on the case of the photons.

The generators of Eq.(15) lead to the group of two-by-two unimodular matrices of the form

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

(20)

whose elements are complex numbers, with det($G$) = 1. Six independent real parameters are required to generate a group, where its six generators are given in Eq.(17). This type of matrices form a group and are called $SL(2, c)$.

The generators $K_i$ are not Hermitian, therefore the matrix $G$ is not always unitary. Moreover, its Hermitian conjugate is not necessarily its inverse. This two-by-two representation has extensively been studied in the literature [4, 5, 6, 7, 8, 9].

In this two-by-two representation, the space-time four-vector can be written as

$$\begin{pmatrix} t + z \\ x - iy \\ x + iy \\ t - z \end{pmatrix},$$

(21)

whose determinant is $t^2 - z^2 - x^2 - y^2$, and remains invariant under the Hermitian transformation:

$$X' = G X G^\dagger.$$

(22)

Therefore this is a Lorentz transformation, which can explicitly be written as:

$$\begin{pmatrix} t' + z' \\ x' - iy' \\ x' + iy' \\ t' - z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t + z \\ x - iy \\ x + iy \\ t - z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ -\beta^* & \delta^* \end{pmatrix}.$$

(23)

The transformation matrices of the Lorentz group applicable to four dimensional Minkowskian space-time can be constructed with these six independent real parameters of $SL(2, c)$ [3, 8]. In the sequel we shall only be using the two-by-two representations of the Lorentz group. We give them in Table 3.

In a similar manner to that of Eq.(21), the four-momentum can be expressed as a two-by-two matrix in the form

$$P = \begin{pmatrix} p_0 + p_z & px - ip_y \\ px + ip_y & p_0 - p_z \end{pmatrix},$$

(24)

where $p_0$ is defined through Einstein’s equation $p_0^2 - (p_z^2 + px^2 + py^2) = m^2$. The transformation property of Eq.(23) is also applicable to this energy-momentum matrix. In 1939 [1], Wigner considered the following two-by-two matrices:

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(25)
Table 3: Two-by-two representations of the Lorentz group. Rotations take the same form for both dotted and undotted representations, but boosts are performed in opposite directions.

| Generators | Transformation Matrices for Undotted Representation | Transformation Matrices for Dotted Representation |
|------------|-----------------------------------------------------|--------------------------------------------------|
| $J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} \exp(-i\phi/2) & 0 \\ 0 & \exp(i\phi/2) \end{pmatrix}$ same | $\begin{pmatrix} \exp(-i\phi/2) & 0 \\ 0 & \exp(i\phi/2) \end{pmatrix}$ |
| $K_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ | $\begin{pmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{pmatrix}$ inverse | $\begin{pmatrix} \exp(-\eta/2) & 0 \\ 0 & \exp(\eta/2) \end{pmatrix}$ |
| $J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ same | $\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ |
| $K_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ | $\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ inverse | $\begin{pmatrix} \cosh(\lambda/2) & -\sinh(\lambda/2) \\ -\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ |
| $J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ same | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ |
| $K_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\begin{pmatrix} \cosh(\lambda/2) & -i\sinh(\lambda/2) \\ i\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ inverse | $\begin{pmatrix} \cosh(\lambda/2) & i\sinh(\lambda/2) \\ -i\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$ |
whose determinants are 1, 0, and −1, respectively, corresponding to the four-momenta of massive, massless, and imaginary-mass particles, as shown in Table I. Wigner’s little groups are the subgroups of the Lorentz group whose transformations leave $P_i$ invariant:

$$W P_i W^\dagger = P_i,$$

where $i = +, 0, −$. Since the momentum of the particle is fixed, these little groups define the internal space-time symmetries of the particle.

The rotation matrix around this axis is expressed as:

$$Z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \tag{26}$$

Then we have

$$Z(\phi) P_i Z^\dagger(\phi) = P_i,$$

which means that for all the three cases the four-momentum remains invariant under rotations around the $z$ axis.

Let us now explicitly give the transformation matrices of the little groups for i) massive, ii) massless and iii) imaginary mass particles.

**Case i)** For a massive particle at rest whose momentum is $P_+$, the little group is to satisfy:

$$W P_+ W^\dagger = P_+. \tag{27}$$

This four-momentum remains invariant under rotations around the $y$ axis, whose transformation matrix is

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{28}$$

This matrix together with $Z(\phi)$ leads to the rotation also around the $x$ axis. Thus, Wigner’s little group for the massive particle is the three-dimensional rotation subgroup of the Lorentz group generated by $S_i$ given in Eq.(5).

**Case ii)** For a massless particle whose momentum is $P_0$, the triangular matrix of the form:

$$T(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \hat{T}(\gamma) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \tag{31}$$

satisfies the Wigner condition of Eq.(26). For rotations around the $z$ axis, these triangular matrices become

$$T(\gamma e^{-i\phi}) = \begin{pmatrix} 1 & -\gamma \exp(-i\phi) \\ 0 & 1 \end{pmatrix}, \quad \hat{T}(\gamma e^{-i\phi}) = \begin{pmatrix} 1 & 0 \\ \gamma \exp(i\phi) & 1 \end{pmatrix}. \tag{32}$$

The $T$ matrix is generated by:

$$N_1 = J_2 - K_1 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad N_2 = J_1 + K_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{33}$$

Its dotted matrix is generated by

$$\bar{N}_1 = J_2 + K_1 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \bar{N}_2 = J_1 - K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{34}$$
Thus, the little group is generated by $J_3$, $N_1$, and $N_2$. Together with $J_3$ they satisfy the following sets of commutation relations:

$$\left[ N_1, N_2 \right] = 0, \quad \left[ J_3, N_1 \right] = iN_2, \quad \left[ J_3, N_2 \right] = -iN_1,$$

(35)

and

$$\left[ \bar{N}_1, \bar{N}_2 \right] = 0, \quad \left[ J_3, \bar{N}_1 \right] = i\bar{N}_2, \quad \left[ J_3, \bar{N}_2 \right] = -i\bar{N}_1.$$

(36)

Wigner in 1939 [1] observed that the first set given in Eq. (35) is the same as that of the generators for the two-dimensional Euclidean group with one rotation and two translations. For massless particles rotation corresponds to the helicity of the particle and its physical meaning is well understood. On the other hand, the physical interpretation of $N_1$ and $N_2$ has not been clarified, until it was completely resolved in 1990 [10]. We now know that they generate gauge transformations [11, 12, 13].

**Case iii)** For an imaginary mass particle whose momentum is $P_-$, the little group matrix is of the form:

$$S(\lambda) = \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix},$$

(37)

and satisfies the Wigner condition of Eq. (26). This corresponds to the Lorentz boost along the $x$ direction generated by $K_1$ as shown in Table 3. Because of the rotational symmetry around the $z$ axis, condition in Eq. (26) is also satisfied by the boost along the $y$ axis. Therefore, the little group is generated by $J_3, K_1,$ and $K_2$, and they satisfy the commutation relations:

$$\left[ J_3, K_1 \right] = iK_2, \quad \left[ J_3, K_2 \right] = -iK_1, \quad \left[ K_1, K_2 \right] = -iJ_3.$$

(38)

This is the $O(2, 1)$ subgroup of the Lorentz group applicable to two space-like and one time-like dimensions. The same commutation relations are satisfied for the dotted matrices.

To summarize the setting, in Table 4 we give Wigner momentum matrices along with their corresponding transformation matrices.

### 4 Massive and Massless Particles

Indeed, the massive particle at rest remains invariant under rotations. Let us Lorentz-boost this particle along the $z$ direction. The boost matrix is given in Table 3 and it takes the form

$$B(\eta) = \begin{pmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{pmatrix}.$$

(39)

Then its momentum becomes

$$p_z = m \sinh(\eta), \quad \text{or} \quad e^\eta = \frac{p_z + \sqrt{p_z^2 + m^2}}{m}.$$

(40)

This momentum remains invariant under rotations around the $z$ axis. The rotation matrix $Z(\phi)$ given in Eq. (27) commutes with the boost matrix $B(\eta)$ of Eq. (39).
Table 4: Wigner four-momentum matrices. Their two-by-two matrix forms are given, together with their corresponding transformation matrices in the dotted and undotted representations. These momentum matrices have determinants that are positive, zero, and negative for the massive, massless, and imaginary-mass particles respectively.

| Mass      | Four-momentum | Transformation Matrices for Undotted Representation | Transformation Matrices for Dotted Representation |
|-----------|---------------|--------------------------------------------------|--------------------------------------------------|
| Massive   | (1 0)         | \((\cos(\theta/2)\quad -\sin(\theta/2))\) \((\sin(\theta/2)\quad \cos(\theta/2))\) | \((\cos(\theta/2)\quad -\sin(\theta/2))\) \((\sin(\theta/2)\quad \cos(\theta/2))\) |
| Massless  | (1 0)         | \((1\quad -\gamma)\) \((0\quad 1)\) | \((1\quad 0)\) \((\gamma\quad 1)\) |
| Imaginary mass | (1 0) | \((\cosh(\lambda/2)\quad \sinh(\lambda/2))\) \((\sinh(\lambda/2)\quad \cosh(\lambda/2))\) | \((\cosh(\lambda/2)\quad -\sinh(\lambda/2))\) \(( -\sinh(\lambda/2)\quad \cosh(\lambda/2))\) |
Table 5: $T(\gamma)$ and $\dot{T}(\gamma)$ transformations on the spinors. Due to the parity invariance of the Lie algebra of the Lorentz group, we should consider the triangular matrices and their dots applicable to both $u$ and $v$, and and also to $\dot{u}$ and $\dot{v}$.

|                  | $T(\gamma)$ with $+\eta$ | $\dot{T}(\gamma)$ with $-\eta$ |
|------------------|---------------------------|---------------------------------|
| **Spinors**      | $T(\gamma)u = u$          | $\dot{T}(\gamma)u = u + \gamma v$ |
| $T(\gamma)v = v - \gamma u$ | $\dot{T}(\gamma)v = v$ |
| **Dotted spinors** | $T(\gamma)\dot{u} = \dot{u}$ | $\dot{T}(\gamma)\dot{u} = \dot{u} + \gamma \dot{v}$ |
| $T(\gamma)\dot{v} = \dot{v} - \gamma \dot{u}$ | $\dot{T}(\gamma)\dot{v} = \dot{v}$ |

The story is different for rotations around an axis perpendicular to the $z$ axis. Let us pick the rotation around the $y$ axis given in Eq. (30). This matrix can be boosted as $B(\eta)R(\theta)B^\dagger(\eta)$, to become

$$
\begin{pmatrix}
\cos(\theta/2) & -e^{\eta} \sin(\theta/2) \\
e^{-\eta} \sin(\theta/2) & \cos(\theta/2)
\end{pmatrix},
$$

(41)

where the boost matrix $B(\eta)$ is that of Eq. (39). According to Eq. (40), $\eta$ becomes infinite as the mass becomes smaller. If we decide to keep all the quantities in Eq. (41) finite, the upper-right element $e^{\eta} \sin(\theta/2)$ must be finite. Let that be $\gamma$. The lower-left element then becomes $e^{-2\eta}\gamma$ which vanishes as $\eta$ becomes infinite. The angle $\theta$ becomes zero. Thus, the boosted rotation matrix becomes the triangular matrix

$$
T(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \dot{T}(\gamma) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix},
$$

(42)

which are the triangular Wigner matrices given in Eq. (31). When they are applied to the spinors given in Table 2, $u$ and $v$ remain invariant, but $\dot{u}$ and $\dot{v}$ become changed as shown in Table 5.

Here again, there is the rotational degree of freedom around the $z$ axis. The matrix of Eq. (41) is generalized into

$$
\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -e^{\eta} \sin(\theta/2) \\ e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix},
$$

(43)

which becomes

$$
\begin{pmatrix} \cos(\theta/2) & -e^{-i\phi} e^{\eta} \sin(\theta/2) \\ e^{i\phi} e^{-\eta} \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.
$$

(44)

In the large-$\eta$ limit, this expression leads to the triangular matrices of Eq. (32).
Table 6: Sixteen combinations of the $SL(2,c)$ spinors. In the non-relativistic sector, there are two spinors leading to four bilinear forms. In its relativistic extension, there are two undotted and two dotted spinors. These four-spinors add up to sixteen independent bilinear combinations [18]

|       | Spin 1                                                                 | Spin 0                                                                 |
|-------|------------------------------------------------------------------------|------------------------------------------------------------------------|
| uu    | $\frac{1}{\sqrt{2}}(uv + vu)$, $vv$, $\frac{1}{\sqrt{2}}(uv - vu)$    |                                                                        |
| $\dot{u}\dot{u}$ | $\frac{1}{\sqrt{2}}(\dot{u}\dot{v} + \dot{v}\dot{u})$, $\dot{v}\dot{v}$, $\frac{1}{\sqrt{2}}(\dot{u}\dot{v} - \dot{v}\dot{u})$ |                                                                        |
| $u\dot{u}$    | $\frac{1}{\sqrt{2}}(u\dot{v} + v\dot{u})$, $\dot{v}\dot{v}$, $\frac{1}{\sqrt{2}}(u\dot{v} - v\dot{u})$ |                                                                        |
| $\dot{u}u$    | $\frac{1}{\sqrt{2}}(\dot{u}v + u\dot{v})$, $\dot{v}\dot{v}$, $\frac{1}{\sqrt{2}}(\dot{u}v - u\dot{v})$ |                                                                        |

5 Representations of Scalars, Vectors, and Tensors for Photon Helicity, Gauge-dependent Four-Potentials and Gauge-independent Field Tensor

In the non-relativistic regime the process of constructing three spin-1 states and one spin-0 state from two spinors is guided by well known spin addition mechanisms of quantum mechanics [17]. On the other hand in the relativistic regime, due the dotted representation of $SL(2,c)$ there are two more two-component spinors for each spin-1/2 particle [3, 8, 14, 15, 16]. When all types of the two-component spinors are combined in this regime, totally there are 16 states. The construction mechanism of those 16 states are similar to those that are constructed in the $SU(2)$ regime, and its details were given in our earlier work [9]. For the purpose of this work it will suffice to give the results in Table 5.

The spinors in Table 6 can be partitioned into the following states:

1. scalar with one state,
2. pseudo-scalar with one state,
3. four-vector with four states,
4. axial vector with four states,
5. second-rank tensor with six states.
Figure 1: Wigner excursion. We are interested in transforming a massive particle at rest into a massless particle with the same energy as illustrated in fig.(a), but this transformation is not allowed within the framework of the Lorentz group where the mass is an invariant quantity. Thus we boost the system to the infinite-momentum state where the mass hyperbola coincides with the light-cone. Then we can come back to a finite-momentum state along the light cone as illustrated in fig.(b) [18].

Furthermore, it can be observed from Table 5 that the combinations
\[ S = \frac{1}{\sqrt{2}} (uv - vu), \quad \text{and} \quad \dot{S} = \frac{1}{\sqrt{2}} (\dot{u}v - \dot{v}u) \] (45)
are invariant both under rotations and boosts. Therefore, they are scalars in the relativistic world. Let us next consider the following combinations.
\[ S_+ = \frac{1}{\sqrt{2}} \left( S + \dot{S} \right), \quad \text{and} \quad S_- = \frac{1}{\sqrt{2}} \left( S - \dot{S} \right). \] (46)
Under the dot conjugation, \( S_+ \) remains invariant, but \( S_- \) changes sign. As was noted in Sec. 2, the dot conjugation corresponds to space inversion. Therefore, \( S_+ \) is a scalar, while \( S_- \) is called a pseudo-scalar.

5.1 Four-vectors, Four-potential and the Gauge Transformation

Let us rewrite the expression for the space-time four-vector given in Eq. (21) as
\[ \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}. \] (47)
It becomes
\[ \begin{pmatrix} t - z & -x + iy \\ -x - iy & t + z \end{pmatrix}, \] (48)
under the parity operation. The off-diagonal elements undergo sign changes, and the diagonal elements become interchanged. We can now construct a four-vector and its “dot” conjugation.
in terms of spinors, in the following form:

\[
V \simeq \begin{pmatrix}
\dot{v}u - u\dot{v} \\
u\dot{u} - \dot{u}u
\end{pmatrix}
\quad \text{and} \quad
\dot{V} \simeq \begin{pmatrix}
\dot{v}\dot{u} - u\dot{v} \\
\dot{u}\dot{u} - \dot{u}u
\end{pmatrix},
\]  
(49)

whose transformation properties are like those of Eq.(47) and Eq.(48), respectively.

Accordingly, we write the electromagnetic four-potential as

\[
A = \begin{pmatrix}
A_0 + A_z \\
A_x + iA_y
\end{pmatrix}
\begin{pmatrix}
A_x - iA_y \\
A_0 - A_z
\end{pmatrix},
\]  
(50)

If boosted along the \(z\) direction as \(B(\eta)AB(\eta)\), the matrix \(A\) becomes

\[
A_\eta = \begin{pmatrix}
(A_0 + A_z) e^\eta \\
A_x + iA_y
\end{pmatrix}
\begin{pmatrix}
A_x - iA_y \\
(A_0 - A_z) e^{-\eta}
\end{pmatrix}.
\]  
(51)

We can then make the Wigner excursion as illustrated in Fig. 1, which transforms this matrix for a massive particle at rest to that of a massless particle with the same energy. The net result is

\[
A = \begin{pmatrix}
A_0 + A_z \\
A_x + iA_y
\end{pmatrix}
\begin{pmatrix}
A_x - iA_y \\
0
\end{pmatrix},
\]  
(52)

resulting in \(A_0 = A_z\) which is widely known as the Lorentz condition. To the language of spinors this condition is translated as \(v\dot{u} = \dot{u}v\).

If we perform the \(T(\gamma)\) on \(u\) and \(v\), while \(\dot{T}(\gamma)\) on \(\dot{u}\) and \(\dot{v}\) in \(V\) of Eq.(49), since \(A\) and \(V\) have the same transformation properties, now the matrix \(A\) becomes:

\[
A + 2\gamma \begin{pmatrix}
0 \\
A_0
\end{pmatrix}.
\]  
(53)

This results in the addition of \(2\gamma A_0\) to \(A_x\). It is a translation in the plane of \(A_x\) and \(A_y\).

On the other hand, if we perform the \(\dot{T}(\gamma)\) on \(u\) and \(v\), while \(T(\gamma)\) on \(\dot{u}\) and \(\dot{v}\), \(A\) becomes

\[
A - 2\gamma \begin{pmatrix}
A_x \\
0
\end{pmatrix}.
\]  
(54)

The triangular matrices \(T(\gamma)\) and \(\dot{T}(\gamma)\) are given in Table 5.

The question next is which one we choose between Eq.(53) and Eq.(54) for our transformation. In order to decide, let us go to the transformation rule given in Eq.(23) for the four-vector, and apply the triangular matrix \(T(\gamma)\) as the Lorentz transformation matrix, resulting in the matrix multiplication

\[
\begin{pmatrix}
1 & -\gamma \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
A_0 + A_z \\
A_x + iA_y
\end{pmatrix}
\begin{pmatrix}
1 \\
-\gamma
\end{pmatrix},
\]  
(55)

which becomes

\[
A - 2\gamma \begin{pmatrix}
A_x \\
0
\end{pmatrix}.
\]  
(56)

This form is the same as the form given in Eq.(54). Thus, we choose the transformation of Eq.(54) as the \(T(\gamma)\) and \(\dot{T}(\gamma)\) transformations applicable to the spinors.
Both in Eq. (54) and Eq. (56), $A_x$ and $A_y$ remain invariant while there is an additional quantity to $A_0 + A_z$. The $T(\gamma)$ matrices therefore lead to a gauge transformation.

What we have done so far can be rotated around $z$ axis. Then, $\gamma$ is replaced by $\gamma e^{-i\phi}$. The transformed $A$ of Eq.(54) becomes

$$A - 2\gamma \begin{pmatrix} A_x \cos \phi + A_y \sin \phi & 0 \\ 0 & 0 \end{pmatrix}. \quad (57)$$

It is possible to reach the same conclusion using the four-by-four formulation of the Lorentz group. This larger representation contains geometries leading to Eq.(53) and Eq.(57) \[10, 11, 12, 13\]. We now know from Eq.(57) that $T(\gamma e^{-i\phi})$ performs a gauge transformation.

Let us go back to the limiting process discussed in Sec. 4. According to Sec. 4, the transverse rotational degrees of freedom collapse into one gauge degree of freedom in the infinite-momentum or zero-mass limit, as illustrated in Table 1. This aspect was observed first by Han et al. in 1983 \[19\], and its geometry was given by Kim and Wigner in 1990 \[10\]. The most recent version of this geometry was given by the present authors in 2017 \[9\].

The matrices given in Eq.(49) were given in our earlier paper on this subject \[9\]. However, as we go deeper into the problem by starting from the transformation property of each spinor, it was inevitable to make a number of minus-sign adjustments. These changes do not alter the conclusions given there and those given here.

5.2 Second-Rank Tensor for the Electromagnetic Field and Gauge Independence

As was noted in our earlier publication \[9\], it is possible to construct the 2nd-rank tensor from bilinear combinations of spinors, and the tensor can take the form

$$\begin{pmatrix} 0 & -f_z & -f_x & -f_y \\ f_z & 0 & -g_y & g_x \\ f_x & g_y & 0 & -g_z \\ f_y & -g_x & g_z & 0 \end{pmatrix}, \quad (58)$$

which can be used for the gauge-invariant electromagnetic field tensor.

Let us first write its $z$ components as

$$f_z \simeq \frac{1}{2} [(uv + vu) - (\dot{u}\dot{v} + \dot{v}\dot{u})], \quad g_z \simeq \frac{1}{2i} [(uv + vu) + (\dot{u}\dot{v} + \dot{v}\dot{u})]. \quad (59)$$

These quantities are invariant under the boost along the $z$ direction. They are also invariant under rotations around this axis, but they are not invariant under boosts along or rotations around the $x$ or $y$ axis. The parity operation on spinors correspond to dot conjugation, and thus $f_z$ and $g_z$ are respectively anti-symmetric and symmetric under the parity operation, as in the case of the electric an magnetic fields.

As to the $x$ and $y$ components, they can be constructed as:

$$f_x \simeq \frac{1}{2} [(uu + vv) - (\dot{u}\dot{v} + \dot{v}\dot{u})],$$

$$f_y \simeq \frac{1}{2i} [(uu - vv) - (\dot{u}\dot{v} - \dot{v}\dot{u})], \quad (60)$$
and:

\[ g_x \simeq \frac{1}{2t} [(uu + vv) + (\dot{u}\dot{u} + \dot{v}\dot{v})], \]

\[ g_y \simeq -\frac{1}{2} [(uu - vv) + (\dot{u}\dot{u} - \dot{v}\dot{v})]. \]  

(61)

At this point, we note that \( f_x \) and \( f_y \) are anti-symmetric under dot conjugation, while \( g_x \) and \( g_y \) are symmetric. The \( f_i \) of Eqs. (59) and (60) transform like a three-dimensional vector, as in the case of the electric field. As for \( g_i \) of Eqs. (59) and (61), they remain invariant under the dot conjugation. They form a pseudo-vector like the magnetic field.

We can now investigate the symmetry of photons by taking the Wigner excursion as illustrated in Fig. 1. If, in Eq.(60) and Eq.(61), we keep only the terms that become larger for larger values of \( \eta \), they can be identified as the transverse components of the electric and magnetic fields with:

\[ E_x \simeq \frac{1}{2} (uu - \dot{v}\dot{v}), \quad E_y \simeq \frac{1}{2t} (uu + \dot{v}\dot{v}), \]

\[ B_x \simeq \frac{1}{2t} (uu + \dot{v}\dot{v}), \quad B_y \simeq -\frac{1}{2} (uu - \dot{v}\dot{v}). \]  

(62)

The \( T(\gamma) \) transformations applicable to \( u \) and \( \dot{v} \) are the two-by-two matrices

\[
\begin{pmatrix}
1 & -\gamma \\
0 & 1
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
\gamma & 1
\end{pmatrix},
\]  

(63)

respectively, as given in Eq.(31). Both \( u \) and \( \dot{v} \) are invariant under these transformations. Therefore, these electric and magnetic fields are invariant under the gauge transformation.

The electric and magnetic field components are perpendicular to each other. Furthermore,

\[ B_x = E_y, \quad B_y = -E_x. \]  

(64)

In order to examine how the photon helicity is translated into the electric and magnetic fields perpendicular to the direction of propagation, let us first construct

\[ E_+ \simeq E_x + iE_y, \quad B_+ \simeq B_x + iB_y, \]  

(65)

\[ E_- \simeq E_x - iE_y, \quad B_- \simeq B_x - iB_y. \]  

(66)

Thus,

\[ B_+ \simeq E_+ \simeq uu, \quad B_- \simeq E_- \simeq \dot{v}\dot{v}, \]  

(67)

which means the photon spin is either in the direction of momentum or in the opposite direction.

Under the parity operation, the direction of momentum is reversed, while \( uu \) and \( \dot{v}\dot{v} \) become \( \dot{u}\dot{u} \) and \( vv \) respectively. This means that the spinors are replaced by the surviving terms in during the Wigner excursion in the opposite direction. The direction of the spin remains unchanged. This is what we expect from the parity rule for momentum and angular momentum.
Figure 2: Unified picture of massive and massless particles. Wigner’s little group for massless spin-1 particles generate gauge transformations, whereas the helicity of the photon is left intact under $T$ and $\dot{T}$. The handedness of spin-1/2 massless particles is a consequence of the gauge invariance condition.

Some of the formulas presented in this Subsection are from our previous publication [9]. In the present paper, we have given a more precise definition of the zero-mass limit in terms of the Wigner excursion as illustrated in Fig. 1 as well as a more detailed application of the $T(\gamma)$ transformation to each spinor. In this way, we were able to study the effect of parity operation on the electromagnetic tensor in terms of the $SL(2, c)$ spinors.

5.3 Higher Spins

Since Wigner’s original book of 1931 [20, 21], the rotation group, has been extensively discussed in the literature [15, 22, 23]. On the other hand we see that the Lorentz group has not been fully exploited. Although there has been some efforts as to the construction of the most general spin states from the two-component spinors in the Lorentz-covariant world it has not been examined in all its details [14].

All possible spin states that can be constructed from two $SL(2, c)$ spin-1/2 states are presented in Table 6.

In the non-relativistic $SU(2)$ regime, the common practice for higher spin constructions is to resort to the well-known angular momentum addition mechanisms. For instance, with three spinors it is possible to construct four spin-3/2 states and two spin-1/2 states, which adds up to six states. Compared to two spin combinations, this partition process is much more complicated [24, 25]. In the Lorentz-covariant world, there should be 64 states for three spinors and 256 states for four spinors.

Since we now know how to Lorentz-boost spinors and take their infinite-$\eta$ limit, we have a better understanding of the differences between the massive and massless particle representations and their symmetry properties. We also observe that the transverse rotations become gauge transformations in the limit of zero-mass which is basically the infinite-$\eta$ limit. Thus, we are able to combine them all into the table given in Figure 2.
Figure 3: Polarization of massless neutrinos. This polarization is a consequence of gauge invariance.

Photons and gravitons are both relativistic integer spin massless particles that we are focusing on in this work. As was mentioned in Subsections 5.1 and 5.2, the observable components have terms that they become largest for large values of \( \eta \). They are the terms that are invariant under gauge transformations.

Photons have two helicity states. They can be parallel or anti-parallel. It can be seen from Section 5.2 that, terms consisting of \( uu \) correspond to parallel states while terms with \( \dot{v}\dot{v} \) are for the anti-parallel states.

We have seen in Section 5.2 that \( uu \) and \( \dot{v}\dot{v} \) represent photon states, whose spins are parallel and anti-parallel to the momentum, respectively.

In 1964, Weinberg constructed massless particle states [26], specifically for photons and gravitons [27] by introducing the conditions on the states as:

\[
N_1|\text{state}\rangle = 0, \quad \text{and} \quad N_2|\text{state}\rangle = 0,
\]

where \( N_1 \) and \( N_2 \) are defined in Eq.(33). Now, we know that \( N_1 \) and \( N_2 \) are the generators of gauge transformations, therefore the states in Eq.(68) are gauge invariant. Thus, \( uu \) and \( \dot{v}\dot{v} \) are Weinberg's states for photons.

Moreover, we can construct \( uuuu \) and \( \dot{v}\dot{v}\dot{v} \dot{v} \) to represent spin-2 gravitons. Since they obey the conditions in Eq.(68), they correspond to Weinberg’s graviton states.

### 5.4 Polarization of Massless Neutrinos

We have established that the triangular matrices \( T \) and \( \dot{T} \) generate gauge transformations when applied to four-vectors. Let us go back to Table 5. They also perform gauge transformations on massless spin-half particles [2, 28].

Let us go back to Table 3. If we insist on gauge invariance of the world, massless spin-half particles are polarized. The dotted particle becomes left-handed, while the undotted spinor becomes right-handed [2, 28, 29]. Indeed, this is what we observe in the real world. Massless neutrinos and anti-neutrinos are left- and right-handed respectively.

Yes, neutrinos have non-zero masses [30, 31], but they are so small compared with their momenta that they can be regarded as small corrections to their massless states. In other words, their massless states will play important roles in physics.
Concluding Remarks

From the mathematical point of view, this paper is about the expansion of the group $SU(2)$ to $SL(2, c)$ within the world of two-by-two matrices. From the physical point of view, we studied here an issue of building a bridge between Heisenberg’s uncertainty relations and Maxwell’s Lorentz-covariant electromagnetic fields.

It was Einstein who defined the photon as a massless particle in the quantum world from his photo-electric effect. However, he did not consider the “wings” of the electromagnetic wave. In the classical picture, there are electric and magnetic fields perpendicular to the direction of propagation. This aspect is translated into the polarization of photons. The question is how?

This question belongs in the subject area pioneered by Wigner in 1939 [1]. His 1939 paper deals with the internal space-time symmetries, as specified in Table I. However, the issue of the electromagnetic four-potential with its gauge degree of freedom has a stormy history and was settled in later papers [10, 11, 13]. As for the Maxwell tensor, the present authors dealt with the problem in their recent publications [8, 9]. In this paper, we have presented further details of this problem starting from the transformation properties of the four spinors defined for the Lorentz-covariant world.

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