Curvature grafted by instantons

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Received: 13 November 2020 / Accepted: 04 March 2021 / Published online: 5 June 2021

Abstract: We show that an instanton with high charge can provoke the creation of extra curvature on the space that holds it. Geometrically, this corresponds to a new surgery operation, which we name grafting. Curvature around a sphere increases by grafting when the charge of an instanton decays.

Keywords: Particle decay; Characteristic classes; Grafting; Holomorphic surgery

1. Particle decay and local characteristic classes

The concept of local characteristic classes has its motivation in the description of particle decay phenomena. Recent news on particle decay have brought to light the need to describe a general theory of local invariants. The new observation of decay was presented at the 40th International Conference on High Energy Physics. Scientists at CERN have reported on their first significant evidence for a process predicted by theory, paving the way for searches for evidence of new physics in particle processes that could explain dark matter and other mysteries of the universe. CERN NA62 presented the first significant experimental evidence for the ultra-rare decay of the charged kaon into a charged pion and two neutrinos.

Here, we will discuss the concepts of local characteristic classes, in particular Chern class and local holomorphic Euler characteristic, and how these occur within the mathematical descriptions of particle decay. We will focus in four dimensions, recalling results obtained in joint work with Pushan Majumdar about instanton decay, and we will propose generalizations to higher dimensions. We are especially interested in the loss of charge that can be provoked by the contraction of a 2-sphere. The local Chern classes measure local manifestations of curvature concentrated around points or around subvarieties.

2. Instantons

We first review the basic mathematical setup of instantons. The original definition of instanton says it is a connection

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In memory of Pushan Majumdar.

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minimizing the Yang–Mills functional. Let us make this concept mathematically precise.

We start up by recalling the concept of connection on a principal bundle. Let $P$ be a principal $G$-bundle over a smooth manifold $M$. A connection $\omega$ on $P$ is a differential 1-form $\omega$ on $P$ with values in the Lie algebra satisfying:

- $\text{Ad}_g R^*_g \omega = \omega$ ($G$-equivariance),
- $\omega(X_i) = \xi$ (Fundamental vector fields),

where $R_g$ denotes the right translation by $g$, upper $*$ the pull back of forms, and $X_i$ denotes the fundamental vector field associated to $\xi$.

Let $F_A$ denote the curvature of the connection $A$, and assume $M$ has real dimension 4. An instanton on $M$ is a connection minimizing the Yang–Mills Functional:

$$\text{YM}(A) = \int |F_A|^2 \text{dvol}.$$ 

Note that the functional applies to 4 dimensions, since the integrand is a 4-form. The Yang–Mills equations are the Euler–Lagrange equations corresponding to the Yang–Mills Functional. Hence, an instanton is a solution of the YM equations. These are nonlinear partial differential equations, which are well known to be very difficult to solve.

One of the few cases where an explicit solution is written in terms of connections is the basic $SU(2)$ instanton on $S^4$:

$$A = \frac{1}{1 + |x|^2} (\theta_1 \mathbf{i} + \theta_2 \mathbf{j} + \theta_3 \mathbf{k}),$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is a basis for $\mathfrak{sl}(2)$ and

$$\theta_1 = x_1 dx_2 - x_2 dx_1 - x_3 dx_4 + x_4 dx_3,$$
$$\theta_2 = x_1 dx_3 - x_3 dx_1 - x_4 dx_2 + x_2 dx_3,$$
$$\theta_3 = x_1 dx_4 - x_4 dx_1 - x_2 dx_3 + x_3 dx_2.$$ 

However, for more general manifolds explicit connections solving the YM equations are not known. A simpler way to look for instantons is to consider the linearized version of the YM equations, namely:

$$\star F = - F,$$

where $\star$ is the Hodge operator. Connections whose curvatures satisfy this equation are called anti-self-dual.

These ASD connections are solutions of the YM equations. Using ASD connections, alternative approaches to find instantons appeared. A celebrated construction of instantons in Euclidean space was obtained by Atiyah–Drinfeld–Hitchin–Manin in [1] in terms of the following linear data:

- a $k \times N$ complex matrix $I$,
- a $N \times k$ complex matrix $J$,
- a real moment map $\mu_r = [B_1, B_1^i] + [B_2, B_2^i] + II^t - J^t J$,
- a complex moment map $\mu_c = [B_1, B_2] + JJ$.

The ADHM construction was explored and generalized in a variety of ways by several authors. Subsequently, this approach using complex vector spaces was significantly improved as to include a far wider variety of instantons by using holomorphic vector bundles, giving rise to a very general principle called the Kobayashi–Hitchin correspondence, see [12], which provides an identification:

**ASDSU(r) connections $\leftrightarrow$ rank $r$ holomorphic bundles.**

The correspondence was proved by Donaldson [6] and Uhlenbeck–Yau [14]. We will concern ourselves with the case of $SU(2)$ instantons:

$$\{ \text{irreducible SU(2)-instantons} \} \leftrightarrow \{ \text{stable SL(2, C)-bundles} \}$$

If we represent the connection in terms of a covariant derivative on a vector bundle, then the association is:

$$\nabla = \partial + \partial \leftrightarrow \partial.$$ 

The method of using holomorphic bundles to study instantons is very fruitful, and has been used widely in the study of instanton moduli spaces, as it allows us to profit from all the technology of complex algebraic geometry to explore properties of instantons. In what follows we will identify instantons with holomorphic bundles via the Kobayashi–Hitchin correspondence, using the particular version of this conjecture proved in [9] for instantons on the surfaces $\mathbb{Z}_r$. We can then talk about instanton charges in terms of characteristic classes.

### 3. Chern classes

Let us recall the basic properties of Chern classes of vector bundles. Let $E$ be a vector bundle on a complex manifold $M$. The Chern polynomial of $E$ is written as

$$c(E) = c_0(E) + c_1(E) t + c_2(E) t^2 + \cdots + c_n(E) t^n,$$

where $c_i(E) \in H^{2i}(M, \mathbb{Z})$ is the $i$-th Chern class of $E$. These Chern classes satisfy the following four axioms:

1. $c_0(E) = 1$
2. $c_i(f^*(E)) = f^*(c_i(E))$
3. $0 \to F_1 \to E \to F_2 \to 0 \Rightarrow c(E) = c(F_1)c(F_2)$
4. $c_1(\gamma) = -1$, 

- complex vector spaces $V$ and $W$ of dimension $k$ and $N$,
- $k \times k$ complex matrices $B_1, B_2$,
where $\gamma$ is the universal bundle on $\mathbb{P}^1$ whose fiber at a given point is the line in $\mathbb{C}^2$ it represents, $n = \dim \mathbb{R} M$ and $f : N \to M$ is a map between complex manifolds.

Observe that the single axiom setting the first Chern class of $\gamma$ to $-1$ is the only one among the 4 axioms that guarantees nontriviality of the theory, else, without this axiom, the other properties would all have been satisfied by $c = 1$ for all bundles. The line bundle $\gamma$ is usually denoted in algebraic geometry by $O_\mathbb{P}^1(-1)$ and its total space is isomorphic to the blow-up of $\mathbb{C}^2$ at the origin:

$$\tilde{\mathbb{C}}^2 := \{(z, l) \in \mathbb{C}^2 \times \mathbb{P}^1 : z \in l\}.$$  

We will return to $\tilde{\mathbb{C}}^2$ in the section about local surfaces, where it will be denoted by $Z_1$ being the first in the collection of surfaces $Z_k := \text{Tot} O_{\mathbb{P}^1}(-k)$ we will use, where $O_{\mathbb{P}^1}(-k) = O_{\mathbb{P}^1}(-1) \otimes \cdots \otimes O_{\mathbb{P}^1}(-1)$ tensored $k$ times has first Chern class $c_1(O_{\mathbb{P}^1}(-k)) = -k$ by axiom 3. Given its importance in the context of characteristic classes and also for our calculations of local Chern classes, it is worth depicting the geometry of $Z_1$ in terms of the real analogue. The blow up of the real plane is

$$\tilde{\mathbb{R}}^2 := \{(z, l) \in \mathbb{R}^2 \times \mathbb{R} \mathbb{P}^1 : z \in l\}$$

which can be depicted as

[Diagram of $\tilde{\mathbb{R}}^2$]

and is isomorphic to an infinite Möbius band (figure by C. Varea). Regarded as a vector bundle over $\mathbb{R} \mathbb{P}^1$, the space $\tilde{\mathbb{R}}^2$ is used within the axioms of characteristic classes of real vector bundles for the nontriviality axiom of Stiefel–Whitney classes.

Presented axiomatically the theory of Chern classes looks rather abstract. A more geometric approach is to write Chern classes in terms of polynomials on the curvature of a connection, as follows. Choosing any connection $A$ on a complex vector bundle $E$, the first Chern class is actually the cohomology class of the curvature of the connection, that is:

$$c_1(E) = \left[ \frac{\sqrt{-1}}{2\pi} F_A \right] \in H^2(M)$$



and the higher Chern class $c_i$ is given by the class of the elementary invariant polynomial $P^i$ of degree $i$

$$c_i(E) = \left[ P^i \left( \frac{\sqrt{-1}}{2\pi} F_A \right) \right] \in H^{2i}(M),$$

see [10, p. 407].

The Gauss–Bonnet theorem then tells us that Chern classes measure obstruction to constructing nowhere vanishing sections. For example, the top Chern class gives the number of zeros of a generic section. In further generality, the $i$-th Chern class of a rank $r$ vector bundle is Poincaré dual to the degeneracy cycle of $r - i + 1$ sections [10, p. 413].

4. Local surfaces and local characteristic classes

We will consider the local surfaces $Z_k := \text{Tot} O_{\mathbb{P}^1}(-k)$. We begin by setting some notation:

We fix once and for all coordinate charts on $Z_k$, which we refer to as canonical coordinates, given by

$$U = \mathbb{C}^2_{z,u} = \{(z, u)\} \quad \text{and} \quad V = \mathbb{C}^2_{\xi,v} = \{(\xi, v)\},$$

such that on $U \cap V = \mathbb{C}^* \times \mathbb{C}$ we identify

$$((\xi, v), (z^{-1}, z\xi u)) = (z^{-1}, z\xi u)$$

We denote by $\ell$ the subvariety of $Z_k$ corresponding to the zero section of $O(-k)$, thus $\ell \simeq \mathbb{P}^1$. Let $X_k$ denote the surface obtained from $Z_k$ by contracting $\ell$ to a point $x$, with

$$\pi : Z_k \to X_k$$

the contraction map. Then, $X_1 \simeq \mathbb{C}^2$ and $X_k$ is singular at $x$ for $k > 1$. Note that $X_k \simeq \mathbb{C}^2 / \Gamma$, where $\Gamma \subset GL(2, \mathbb{C})$ is a cyclic group of order $k$ with a generator acting on $\mathbb{C}^2$ via multiplication by $\gamma = \left( \begin{array}{cc} \omega & 0 \\ 0 & \omega \end{array} \right)$ for $\omega$ a primitive $k$th root of unity. Thus, $X_2$ is the $A_1$ surface singularity, and instantons on $X_2$ occur in a variety of papers on instantons.
on ALE spaces, see for instance [5] and references therein. However, for \( k \geq 3 \) we have \( \det \gamma = \omega^2 \neq 1 \), so that \( \Gamma \not\subset SU(2) \) and consequently \( Z \geq 3 \) is not an ALE space.

If \( E \) is a vector bundle on \( Z_k \) then the direct image \( \pi_*E \) happens to be a vector bundle on \( X_k \) only in the case when \( E|_\ell \) is trivial, otherwise, it is a non-locally free sheaf with the stalk at \( x \) strictly larger than at the smooth points of \( X_k \).

We will consider rank 2 vector bundles and we wish to define the local second Chern class of a bundle \( E \) around \( \ell \) using information from the sheaf \( \pi_*E \). This local second Chern class will correspond to the local charge of an instanton on \( Z_k \).

Considering \( \pi_*E \) then takes us to the realm of Chern classes of sheaves on singular varieties. Quoting T. Suwa we observe that for a singular complex algebraic or analytic variety \( X \), there are at least three (in general different) kinds of Chern classes in the cohomology \( H^*(X) \).

- the Chern–Schwarz–MacPherson class \( c_*(X) \)
- the Chern–Mather class \( c^M(X) \)
- the canonical class or Fulton–Johnson’s Chern class \( c^{FJ}(X) \).

These three classes all reduce to the standard Chern classes \( c(X) \) when the variety has no singularities. Since there is an a priori no reason to prefer one of these definitions over the others, we chose instead to consider the concept of local holomorphic Euler characteristic \( \chi(\ell,E) \) whose definition is uncontroversial.

If \( \pi: (\bar{X}, \ell) \to (X,x) \) is a resolution of an isolated singularity, and \( \mathcal{F} \) is a sheaf of rank \( n \) on \( \bar{X} \), then the local holomorphic Euler characteristic of \( \mathcal{F} \) is

\[
\chi(x, \mathcal{F}) = \chi(\ell, \mathcal{F}) = h^0(x, \mathcal{Q}) + \sum_{i=1}^{n} (-1)^i h^0(x, R^i \pi_* \mathcal{F}),
\]

where \( h^0 \) is the dimension of the 0-th Čech cohomology, \( R^i \pi_* \mathcal{F} \) is the \( i \)-th higher derived image of \( \mathcal{F} \) (see [11, p. 250]) and \( \mathcal{Q} \) is the skyscraper sheaf supported at \( x \) defined by the exact sequence

\[
0 \to \pi_* \mathcal{F} \to (\pi_* \mathcal{F})^\vee \to \mathcal{Q} \to 0.
\]

If \( X \) is a compact orbifold and \( \mathcal{F} \) a sheaf over \( X \), there exists the global holomorphic orbifold Euler characteristic of \( \mathcal{F} \) (see [4])

\[
\chi_{orb}(X, \mathcal{F}) = \int_X \text{ch} (\mathcal{F}) \text{td} (X).
\]

It satisfies:

\[
\chi(X, \mathcal{F}) = \chi_{orb}(X, \mathcal{F}) + \sum_{x \in \text{Sing}(X)} \mu(x, \mathcal{F}).
\]

Here \( \mu(x, \mathcal{F}) \) is a rational number that depends both on the class of the sheaf \( \mathcal{F} \) in the Grothendieck group of the variety and on the order of singularity, with the denominator at an orbifold singularity given by the order of the corresponding group that gives the local quotient.

In the examples, we wish to consider when \( E \) is a rank 2 vector bundle on \( Z_k \) then we refer to the local holomorphic Euler characteristic of \( E \) around \( \ell \) as the charge of \( E \) for short, since it corresponds to the contribution to the charge in the language of instantons. It is given by

\[
\chi(\ell, E) = h^0(\mathcal{Q}) + h^0(R^1 \pi_*E),
\]

where

\[
0 \to \pi_*E \to (\pi_*E)^\vee \to \mathcal{Q} \to 0.
\]

The numbers \( h^0(\mathcal{Q}) \) and \( h^0(R^1 \pi_*E) \) are independent analytic invariants, called the width and the height of \( E \), respectively. These 2 invariants can be calculated using the theorem on formal functions (see [11, p. 277])

\[
R^f(E)_c = \lim_{\nu \to \infty} H^0(\nu^*E_c)
\]

by calculating Čech cohomology in canonical coordinates as given in (1). It is the local holomorphic characteristic \( \chi(\ell, E) \) that gets identified with the local instanton charge under the Kobayashi–Hitchin correspondence, and for this reason we call it the charge of the bundle, it gives the local contribution to the second Chern class \( c_2^{orb}(X, E) \), which we will denote simply by \( c_2^{orb}(X) \).

To provide examples of these invariants, we first recall some properties of vector bundles on local surfaces. To start with, holomorphic vector bundles on \( Z_2 \) are filtrable and algebraic [7, Lem. 3.1, Thm. 3.2]. Filtrability for a rank 2 bundle \( E \) means that it is an extension of line bundles. While filtrability always happens for bundles on complex curves (Riemann surfaces) this is a very unusual property for complex surfaces. For example, on the projective space \( \mathbb{P}^2 \) only split bundles are filtrable, so that the entire moduli spaces of stable vector bundles on \( \mathbb{P}^2 \) are made of nonfiltrable bundles.

If \( E \) is a rank 2 bundle on \( Z_k \) with \( c_1(E) = 0 \) then for some integer \( j \geq 0 \) there is a short exact sequence

\[
0 \to \mathcal{O}(-j) \to E \to \mathcal{O}(j) \to 0,
\]

where \( E|_{\ell} = \mathcal{O}(-j) \oplus \mathcal{O}(j) \) splits by Grothendieck’s splitting principle. The integer \( j \) is called the splitting type of \( E \). Note that here we use the symbol \( \mathcal{O}(j) \) for the line bundle with first Chern class \( j \) both on \( \ell \simeq \mathbb{P}^1 \) and on \( Z_2 \), but there should be no confusion given that one is obtained from the other by pullback.
Giving a short exact sequence $0 \rightarrow \mathcal{O}(-j) \rightarrow E \rightarrow \mathcal{O}(j) \rightarrow 0$ corresponds to giving an extension class $p \in \text{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))$. This extension class can be written in canonical coordinates as a polynomial such that the transition matrix of $E$ in canonical coordinates is 
\[
\begin{pmatrix}
  z^j & p \\
  0 & z^{-j}
\end{pmatrix}.
\]
Therefore, the bundle $E$ is completely determined by the data $(j, p)$ (Table 1).

The following table contains examples for the values of width, height and charge for vector bundles on $\mathbb{Z}_1$ with splitting type $j = 3$ defined by the monomials in the first column:

| Monomial | Width | Height | Charge |
|----------|-------|--------|--------|
| $z^{-1}u$ | 3     | 2      | 5      |
| $u$      | 1     | 2      | 3      |
| $zu$     | 1     | 2      | 3      |
| $z^2u$   | 3     | 2      | 5      |
| $u^2$    | 3     | 3      | 6      |
| $zu^2$   | 2     | 3      | 5      |
| $z^2u^2$ | 3     | 3      | 6      |
| $zu^3$   | 4     | 3      | 7      |
| $z^2u^3$ | 4     | 3      | 7      |
| $z^2u^4$ | 5     | 3      | 8      |
| zero     | 6     | 3      | 9      |

Once this is done, then [3, Thm. 4.11] obtained: The moduli space of holomorphic bundles of rank 2 on $\mathbb{Z}_k$ with $c_1 = 0$ and splitting type $j \geq k \geq 2$ is a quasi-projective variety of dimension $2j - k - 2$.

Gasparim, Köppe, and Majumdar [9, Cor. 5.5] showed that a rank 2 bundle on $\mathbb{Z}_k$ corresponds to an SU(2) instanton if and only if its splitting type is a multiple of $k$. The correspondence also takes gauge transformation of instantons to isomorphism of holomorphic bundles. Furthermore, their choice of ad-hoc stability concept is such that stable instantons with splitting type $j$ have topological charge $j$. It then follows that the moduli space of SU(2) instantons of charge $j = nk \geq 2$ on $\mathbb{Z}_k$ is a quasi-projective variety of dimension $2nk - k - 2$. There is a single instanton of charge 1 on $\mathbb{Z}_1$ and, very surprisingly, there are no charge 1 instantons on $\mathbb{Z}_k$ for $k > 2$, as we shall see next.

5. Moduli spaces

To define moduli of vector bundles, we first fix topological invariants, and then mod out by holomorphic isomorphisms. Next, as it is well known in algebraic geometry, we need to select a good subset of bundles to obtain moduli spaces. This means that we have to select a proper subset of the set of all vector bundles with the chosen topology to obtain a well behaved quotient, else the quotient may be a stack, or something worse, but will not be an algebraic variety, at times not even a complex space. In geometric invariant theory, a well behaved moduli space is obtained using the concept of stable bundle. However, the concept of stability is defined for bundles on compact varieties and works well in such case, but it does not apply to bundles on noncompact varieties. For this reason, a somewhat ad-hoc choice of good moduli space was obtained for the surfaces $\mathbb{Z}_k$ by choosing those bundles that do not split on the first formal neighborhood of $\ell$. In practice this means that the extension class has a nontrivial linear term on $u$, where $u = 0$ is the equation that cuts out $\ell$ inside the surface.

6. Instanton decay

Sharp bounds for local charges of SU(2) instantons on $\mathbb{Z}_1$ with splitting type $j$ are as follows:

\[
j \leq c^\text{loc}_j(Z_1, E) \leq j^2.
\]

The minimum local charge of a splitting type $j$ instanton on $\mathbb{Z}_1$ equals $j$ and is attained by those bundles that do not split on the first formal neighborhood, such as those with local data $(j, p = au + buu)$ for any constants $a, b, u$ not both 0. The maximum local charge is always attained by the split bundle, thus the case of the instanton with data $(j, p = 0)$ which has local charge $j^2$. Furthermore, every pair of admissible numerical values of (width, height) occurs for some instanton on $\mathbb{Z}_1$ [2, Thm. 0.2]. Therefore, there exist instantons on $\mathbb{Z}_1$ for any values of local charge, width and height.

In contrast, existence of instantons on $\mathbb{Z}_2$ with prescribed value of local charge is still an open problem. Indeed, even though sharp bounds are known for each fixed splitting type, it is not known whether all intermediate values of width and height actually occur. This issue is already featured in algebraic geometry, since the corresponding existence of vector bundles is unknown. The current state of affairs is as follows. Assume $k \geq 2$. Let $E$ be a holomorphic rank 2 bundle on $\mathbb{Z}_k$ with $c_1 = 0$ and splitting type $j = nk + r$ with $0 \leq r < k$, then the following bounds are sharp [3, Cor. 2.18]:

\[
j - 1 \leq c^\text{loc}_j(Z_2, E) \leq \begin{cases} n^2k + r(2n + 1) - 1 & \text{if } r \neq 0 \\ n^2k & \text{if } r = 0. \end{cases}
\]

Even though the question of existence of instantons on $\mathbb{Z}_k$ has not been completely understood when $k \geq 2$, the current
partial knowledge generated strong conclusions regarding the possibilities for instanton decay. The main result obtained in [3, Thm. 6.8] is that if \( k \geq 2 \), then the minimal local charge of a nontrivial instanton on \( Z_k \) is \( k - 1 \). As a corollary we conclude that instanton decay is obstructed by the self-intersection \( \ell^2 = -k \) of the line in \( Z_k \), given that there are no instantons of charge 1 on \( Z_k \) when \( k > 2 \).

**Open question:** For what admissible values of local charge \( c \) fitting into bounds (3) do there exist instantons on \( Z_k \) of charge \( c \)?

**Open question:** Assume \( A \) is an SU(2) instanton on \( Z_k \) of minimal charge, hence the local charge satisfies \( c(A) = k - 1 > 1 \) if \( k > 2 \). Then, such an instanton \( A \) ought to decay, because maintaining such local charge costs too much energy. However, \( A \) can not decay to other instantons on \( Z_k \), because there are no nontrivial instantons of lower charge in this space. What is the result of the decay of such an instanton? We will give a partial solution to this question in the following section.

7. Holomorphic surgery and instanton decay

We define a new type of holomorphic surgery which we call grafting.

We will discuss two types of grafting: open and closed. Closed grafting is a specific new type of surgery on the base space such that, when the space holds an instanton (or a vector bundle), then the four dimensional manifold changes under grafting, but the instanton (or holomorphic bundle) on it remains essentially the same, although losing some of its charge. We will carry out the closed grafting by replacing a \( Z_k \) with another \( Z_{k'} \). Holomorphic patching of a \( Z_1 \) is possible at any smooth point, and corresponds to the operation of blowing up a point which is well known in algebraic geometry. However, the new operation considered here is not just that of changing the surface by adding a \( Z_1 \), but also carrying along with it an instanton (or holomorphic vector bundle).

We may regard the blow up \( \pi : \tilde{X} \to X \) of a point \( x \in X \) as a holomorphic patching \( \tilde{X} = (X - \{x\}) \cup_{Z_1} Z_1 \).

Given holomorphic bundles \( E \) on \( X \) and \( F \) on \( Z_1 \) with the same rank, we may construct an open grafting of \( E \) and \( F \), written \( E \bowtie F \), read “\( E \) graft \( F \)”, by making \( E \bowtie F \) isomorphic to \( E \) on \( \tilde{X} - \ell \) and isomorphic to \( F \) on \( Z_1 \); this requires specifying a choice of gluing. So, the holomorphic type of the resulting bundle depends on the gluing, but the resulting second Chern class of \( E \bowtie F \) depends only on \( E \) and \( F \), see [8]. It is worth noticing that such grafting is always possible, because of the fact that every holomorphic bundle on \( Z_1 \) is trivial on \( Z_1^0 := Z_1 - \{\ell\} \), see [7]. Similarly, we can graft a bundle locally by exchanging it on a \( Z_4 \) type of neighborhood inside a complex surface. We have

\[
c_2(E \bowtie F) = c_2(E) + c_{2\text{loc}}(F).
\]

The second type of grafting, which is a somewhat more surprising operation is to essentially keep the vector bundle while changing the base manifold. Suppose that we have a complex surface \( X \) that contains a \(-k\) line. Hence, \( X \) contains a line \( \ell \) with a tubular neighborhood of the same holomorphic type as a neighborhood of \( \ell \) in \( Z_k \). We now replace \( k \) by \( k' \) by cutting out the \(-k\) line and gluing in a \(-k'\) line. If \( E \) is a bundle on \( X \), we keep all of its transition matrices, and they now define a new bundle on \( (X - \ell) \cup_{Z_{k'}} Z_{k'} \) which we denote by \( E \bowtie_{Z_{k'}} \) read “\( E \) graft \( k' \)”. In general, we will need to also modify other charts of \( X \) because of the cocycle condition in triple intersections. But, let us consider the simplest case when \( X \) is one of the \( Z_k \) themselves, so that we just exchange \( Z_k \) by \( Z_{k'} \) and keep the transition matrix for the bundle. This might seem a trivial thing to do, nevertheless, the effect on the local charge is very strong, as the following examples illustrate.

First observe that over \( Z_k \) the bundle \( E = \mathcal{O}(j) \oplus \mathcal{O}(-j) \), that is, the split bundle with type \( j = nk \) and \( p = 0 \) over \( Z_k \), has local charge \( c_{2\text{loc}}(Z_k,E) = n^2 k \), in fact, it realizes the upper bound in 3. Now, let us see the effect of grafting in a couple of cases, by considering the split bundles with type \( j = 6 \) over \( Z_1 \) and \( Z_2 \). So that, in each case, we are considering a bundle defined by transition matrix \( T = \begin{pmatrix} z^6 & 0 \\ 0 & z^{-6} \end{pmatrix} \).

**Case 1.** When we consider the split bundle \( E \to Z_1 \) given by transition matrix \( T \) we are in the case \( j = 6 \times 1 \), so

\[
c_{2\text{loc}}(E) = 6^2 \times 1 = 36
\]

by grafting to \( Z_3 \) we get the bundle \( E \bowtie Z_3 \) also given by transition matrix \( T \) but now we are in the case \( j = 2 \times 3 \), so
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\[ c_2^{\text{loc}}(E \otimes_3) = 2^2 \times 3 = 12. \]

So that in this grafting the instanton bundles lost 24 units of charge.

**Case 2.** When we consider the split bundle \( E \to Z_2 \) given by transition matrix \( T \) we are in the case \( j = 3 \times 2 \), so

\[ c_2^{\text{loc}}(E) = 3^2 \times 2 = 18 \]

by grafting to \( Z_6 \) we get the bundle \( E \otimes_6 \) also given by transition matrix \( T \) but now we are in the case \( j = 1 \times 6 \), so

\[ c_2^{\text{loc}}(E \otimes_6) = 1^2 \times 6 = 6. \]

So that in this grafting the instanton bundles lost 12 units of charge.

In both cases, we can interpret the loss of charge as the energy used to provoke the grafting in the base.

Now let \( j \geq 2 \) and \( E \to Z_1 \) the bundle defined by the transition matrix \( T' = \begin{pmatrix} z^j & zu \\ 0 & z^{-j} \end{pmatrix} \). As \( E \) does not split, we have that

\[ c_2^{\text{loc}}(E) = j. \]

By grafting to \( Z_3 \) we have another bundle defined by the same matrix \( T' \), and as shown in [3], the extension class \( zu \) always gives the lower bound stated in Formula (3). Therefore,

\[ c_2^{\text{loc}}(E \otimes_3) = j - 1, \]

so in this grafting the instanton bundles lost only 1 unit of charge.

Here is a picture of closed grafting, as it exchanges the local surfaces \( Z_k \) and \( Z_{k'} \) inside a complex surface, while essentially maintaining the instanton bundle globally. The grafted bundle is still given by the same transition matrices, however, loses global charge, that is, undergoes a loss of second Chern class.

Note that the first Chern number \( c_1(\mathcal{O}_{\mathbb{P}^1}(-k)) = -k \) can be seen as a measure of curvature of the total space \( Z_k \) itself given that the first Chern class of this line bundle is by definition the class of the curvature of any connection on it and higher \( k \) implies stronger negative curvature.

In conclusion, we propose that an instanton with high charge provokes a grafting on the 4 dimensional base manifold which holds it. The grafting happens around a 2-sphere where the instanton charge is highly concentrated. The instanton loses charge in the process, while the curvature of the base manifold around the 2-sphere increases. In other words, we have explained that the decay of an instanton can happen when the instanton inflicts larger concentrations of curvature around 2-spheres, thereby losing charge. We then say that the addition to the curvature is grafted by the instanton.

**Acknowledgements** I first met Pushan Majumdar in 1998 and we immediately engaged into scientific discussions. Shortly after that, I found him reading about moduli of instantons, and inquired why he was reading about such a theme, which was not part of his PhD project. He said that he wanted to be my friend, and if he did not learn about the themes that interested me, then soon we would not have anything to talk about. Decades later, this remains the most loyal offer of friendship I have ever received. We thank Koushik Ray for inviting us to contribute to the memorial volume of Pushan and for suggesting several improvements to our article. B.S. acknowledges support of ANID/FAPESP 2019/13204-0, E.G. acknowledges support of VRIDT-UCN, Chile.

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