Some combinatorial sequences associated with context-free grammars

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Abstract

The purpose of this paper is to show that some combinatorial sequences, such as second-order Eulerian numbers and Eulerian numbers of type $B$, can be generated by context-free grammars.

Keywords: Context-free grammars; Combinatorial sequences; Permutations; Partitions

1 Introduction

The grammatical method was introduced by Chen [2] in the study of exponential structures in combinatorics. Let $A$ be an alphabet whose letters are regarded as independent commutative indeterminates. A context-free grammar $G$ over $A$ is defined as a set of substitution rules replacing a letter in $A$ by a formal function over $A$. Following Chen [2], the formal derivative $D$ is a linear operator defined with respect to a context-free grammar $G$. For any formal functions $u$ and $v$, we have

$$D(u + v) = D(u) + D(v), \quad D(uv) = D(u)v + uD(v) \quad \text{and} \quad D(f(u)) = \frac{\partial f(u)}{\partial u} D(u),$$

where $f(x)$ is a analytic function. By definition, we have $D^{n+1}(u) = D(D^n(u))$ for all $u$. For example, if $G = \{x \to xy, y \to y\}$, then

$$D(x) = xy, \quad D(y) = y, \quad D^2(x) = x(y + y^2), \quad D^3(x) = x(y + 3y^2 + y^3).$$

In [5], Dumont considered chains of general substitution rules on words. It is a hot topic to explore the connection between combinatorics and context-free grammars. The reader is referred to [3, 4, 6, 11] for recent progress on this subject.

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We now recall some definitions, and fix some notation, that will be used throughout the rest of this paper. Let $[n] = \{1, 2, \ldots, n\}$. Let $S_n$ denote the symmetric group of all permutations of $[n]$. The Eulerian number $\langle \binom{n}{k} \rangle$ enumerates the number of permutations in $S_n$ with $k$ descents (i.e., $i < n, \pi(i) > \pi(i+1)$) as well as the number of permutations in $S_n$ which have $k$ excedances (i.e., $i < n, \pi(i) > i$) (see [12, A008292]). The numbers $\langle \binom{n}{k} \rangle$ satisfy the recurrence relation

$$\langle \binom{n}{k} \rangle = (k + 1) \langle \binom{n-1}{k} \rangle + (n - k) \langle \binom{n-1}{k-1} \rangle,$$

the initial condition $\langle \binom{0}{0} \rangle = 1$ and boundary conditions $\langle \binom{0}{k} \rangle = 0$ for $k \geq 1$. Let

$$A_n(t) = \sum_{k=0}^{n-1} \langle \binom{n}{k} \rangle t^k$$

be the Eulerian polynomial. The exponential generating function for $A_n(t)$ is

$$A(t, z) = 1 + \sum_{n \geq 1} t A_n(t) \frac{z^n}{n!} = \frac{1 - t}{1 - te^{(1-t)}}.$$ (1)

We now consider a restricted version of Eulerian numbers. Let $r$ be a nonnegative integer. Denote by $P(n, n-r)$ the set of permutations of $n$ numbers taken $n-r$ at a time. Let $\sigma \in P(n, n-r)$. If $\sigma(i) > i$, then we say that $\sigma$ has an excedance at position $i$, where $1 \leq i \leq n-r$. The $r$-restricted Eulerian number, denoted by $\langle \langle \binom{n}{k} \rangle \rangle$, is defined as the number of permutations in $P(n, n-r)$ having $k$ excedances (see [12, A144696, A144697, A144698, A144699] for details).

A Stirling permutation of order $n$ is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that for each $i$, $1 \leq i \leq n$, the elements lying between the two occurrences of $i$ are greater than $i$. The second-order Eulerian number $\langle \langle \binom{n}{k} \rangle \rangle$ is the number of Stirling permutation of order $n$ with $k$ ascents (see [12, A008517]). The combinatorial interpretations for the second-order Eulerian numbers $\langle \langle \binom{n}{k} \rangle \rangle$ have been extensively investigated (see [11][13][19]). It is well known that the numbers $\langle \langle \binom{n}{k} \rangle \rangle$ satisfy the recurrence relation

$$\langle \langle \binom{n+1}{k} \rangle \rangle = (2n - k + 1) \langle \langle \binom{n}{k-1} \rangle \rangle + (k+1) \langle \langle \binom{n}{k} \rangle \rangle,$$ (2)

with initial condition $\langle \langle \binom{1}{0} \rangle \rangle = 1$ and boundary conditions $\langle \langle \binom{n}{k} \rangle \rangle = 0$ for $n \leq k$ or $k < 0$ (see [12, A008517]).

Let $B_n$ denote the set of signed permutations of $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all $i$, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let

$$B_n(x) = \sum_{k=0}^{n} B(n,k) x^k = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)},$$

where

$$\text{des}_B = |\{i \in [n] : \pi(i-1) > \pi(i)\}|$$
with $\pi(0) = 0$. The polynomial $B_n(x)$ is called an Eulerian polynomial of type B, while $B(n,k)$ is called an Eulerian number of type B (see [12, A060187]). The first few of these polynomials are listed below:

\[
B_0(x) = 1, B_1(x) = 1 + x, B_2(x) = 1 + 6x + x^2, B_3(x) = 1 + 23x + 23x^2 + x^3.
\]

The numbers $B(n,k)$ satisfy the recurrence relation

\[
B(n + 1, k) = (2n - 2k + 3)B(n, k - 1) + (2k + 1)B(n, k),
\]

with initial condition $B(0,0) = 1$ and boundary conditions $B(0,k) = 0$ for $k \geq 1$. An explicit formula for $B(n,k)$ is given as follows:

\[
B(n, k) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (2k - 2i + 1)^n
\]

for $0 \leq k \leq n$ (see [7] for details).

The unsigned Stirling number of the first kind $\left[ \begin{array}{c} n \\ k \end{array} \right]$ is the number of permutations in $S_n$ with exactly $k$ cycles (see [12, A132393]). The Stirling number of the second kind $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the number of ways to partition $[n]$ into $k$ blocks (see [12, A008277]). Let $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle$ denote the number of ways to partition $[n]$ into $k$ nonempty linearly ordered subsets. The numbers $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle$ are called the unsigned Lah numbers (see [12, A105278]).

We recall some known results on context-free grammars.

**Proposition 1** ([2, Eq. 4.8]). If $G = \{ x \to xy, y \to y \}$, then

\[
D^n(x) = x \sum_{k=1}^{n} \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle y^k.
\]

**Proposition 2** ([5, Section 2.1]). If $G = \{ x \to xy, y \to xy \}$, then

\[
D^n(x) = x \sum_{k=0}^{n-1} \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle x^k y^{n-k}.
\]

**Proposition 3** ([4]). If $G = \{ x \to x^2y, y \to x^2y \}$, then

\[
D^n(x) = \sum_{k=0}^{n-1} \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle x^{2n-k} y^{k+1}.
\]

**Proposition 4** ([11]). If $G = \{ x \to y^2, y \to xy \}$, then

\[
D^n(x) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} W_{n,k} x^{n-2k-1} y^{2k+2}, \quad D^n(y) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} W_{n,k}^l x^{n-2k} y^{2k+1},
\]

where $W_{n,k}$ is the number of permutations in $S_n$ with $k$ interior peaks and $W_{n,k}^l$ is the number of permutations in $S_n$ with $k$ left peaks.

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2 Results

For \( n \geq 0 \), we always assume that

\[
(xD)^{n+1}(x) = (xD)(xD)^n(x) = xd((xD)^n(x)).
\]

The following theorem is in a sense “dual” to Proposition 3.

**Theorem 5.** If \( G = \{ x \rightarrow xy, y \rightarrow xy \} \), then

\[
(xD)^n(x) = \sum_{k=0}^{n-1} \binom{n}{k} x^{2n-k} y^{k+1} \quad \text{for } n \geq 1.
\]

**Proof.** For \( n \geq 1 \), we define

\[
(xD)^n(x) = \sum_{k=0}^{n-1} E(n,k) x^{2n-k} y^{k+1}.
\]

Note that

\[
(xD)(x) = x^2 y, (xD)(x^2 y) = x^4 y + 2x^3 y^2.
\]

Then \( E(1,0) = \langle \langle 1 \rangle \rangle = 1 \), \( E(2,0) = \langle \langle 2 \rangle \rangle = 1 \) and \( E(2,1) = \langle \langle 2 \rangle \rangle = 2 \). Using (4), we obtain

\[
(xD)(xD)^n(x) = \sum_{k=0}^{n-1} (2n-k)E(n,k)x^{2n-k+1} y^{k+2} + \sum_{k=0}^{n-1} (k+1)E(n,k)x^{2n-k+2} y^{k+1}.
\]

Therefore,

\[
E(n+1,k) = (2n-k+1)E(n,k-1) + (k+1)E(n,k).
\]

Comparing with (2), we see that the coefficients \( E(n,k) \) satisfy the same recurrence and initial conditions as \( \langle \langle n \rangle \rangle \), so they agree. \( \square \)

Now we present the main result of this paper.

**Theorem 6.** For \( n \geq 1 \), we have the following results:

\( (c_1) \) If \( G = \{ x \rightarrow xy^2, y \rightarrow x^2 y \} \), then

\[
D^n(xy) = xy \sum_{k=0}^{n} B(n,k)x^{2n-2k} y^{2k}.
\]

\( (c_2) \) If \( G = \{ x \rightarrow xy^2, y \rightarrow x^2 y \} \), then

\[
D^n(x^2 y^2) = 2^n x^2 y^2 \sum_{k=0}^{n} \binom{n+1}{k} x^{2n-2k} y^{2k}.
\]

\( (c_3) \) If \( G = \{ x \rightarrow xy^2, y \rightarrow x^2 y \} \), then

\[
D^n(x) = x \sum_{k=1}^{n} N(n,k)x^{2n-2k} y^{2k},
\]

where the number \( N(n,k) \) enumerates perfect matchings of \([2n]\) with the restriction that only \( k \) matching pairs have odd smaller entries (see [12, A185411]).
(c4) If \( G = \{x \to xy, y \to xy\} \), then
\[
D^n(xy^r) = x \sum_{k=0}^n \binom{n+r}{k} y^{n+r-k}.
\]

(c5) If \( G = \{x \to xy^2, y \to xy\} \), then
\[
D^n(x) = x \sum_{k=0}^{n-1} 2^k \binom{n}{k} x^k y^{2n-2k}.
\]

(c6) Consider the numbers \( T(n, k) \) with generating function
\[
\sqrt{A(2t, z)} = 1 + \sum_{n \geq 1} \sum_{k=1}^n T(n,k) \frac{t^k z^n}{n!},
\]
where \( A(t, z) \) is given by [11] (see [12, A156920]). If \( G = \{x \to xy^2, y \to xy\} \), then
\[
D^n(y) = \sum_{k=1}^n T(n,k) x^k y^{2n-2k+1}.
\]

(c7) If \( G = \{x \to x^2y, y \to y\} \), then
\[
D^n(x) = x \sum_{k=1}^n k! \binom{n}{k} x^k y^k.
\]

(c8) If \( G = \{x \to x^2y, y \to y^2\} \), then
\[
D^n(x) = x \sum_{k=1}^n k! \binom{n}{k} x^k y^n.
\]

(c9) If \( G = \{x \to xy^2, y \to y^2\} \), then
\[
D^n(x) = x \sum_{k=1}^n \left[ \binom{n}{k} y^{n+k} \right].
\]

(c10) If \( G = \{x \to xy^3, y \to y^3\} \), then
\[
D^n(x) = x \sum_{k=1}^n b(n,k) y^{2n+k},
\]
where \( b(n,k) \) is the number of forests with \( k \) rooted ordered trees with \( n \) non-root vertices labeled in an organic way (see [12, A035342]).

(c11) For a fixed positive integer \( r \geq 4 \), if \( G = \{x \to xy^r, y \to y^r\} \), then
\[
D^n(x) = x \sum_{k=1}^n a(n,k;r) y^{(r-1)n+k},
\]
where \( a(n,k;r) \) enumerates unordered \( n \)-vertex \( k \)-forests composed of \( k \) plane increasing \( r \)-ary trees (see [12, A035469, A049029, A049385, A092082]).
(c12) If \( G = \{ x \rightarrow x^2 y, y \rightarrow xy \} \), then
\[
D^n(y) = x^n \sum_{k=1}^{n} d(n, k)y^k,
\]
where \( d(n, k) \) is the number of increasing mobiles (circular rooted trees) with \( n \) nodes and \( k \) leaves (see [12, A055356]).

Proof. We only prove \((c_1)\) and the others can be proved in a similar way. Note that \( D(x) = xy^2 \) and \( D(y) = x^2 y \). Then
\[
D(xy) = xy(x^2 + y^2), D^2(xy) = D(D(xy)) = xy(x^4 + 6x^2y^2 + y^4).
\]

For \( n \geq 1 \), we define
\[
D^n(xy) = xy \sum_{k=0}^{n} G(n, k)x^{2n-2k}y^{2k}.
\]
Hence \( G(1, 0) = B(1, 0) \) and \( G(1, 1) = B(1, 1) \). Since
\[
D^{n+1}(xy) = D(D^n(xy)) = \sum_{k=0}^{n} (2n-2k+1)G(n, k)x^{2n-2k+1}y^{2k+3} + \sum_{k=0}^{n} (2k+1)G(n, k)x^{2n-2k+3}y^{2k+1},
\]
there follows
\[
G(n+1, k) = (2n - 2k + 3)G(n, k-1) + (2k + 1)G(n, k).
\]
It follows from \([3]\) that \( G(n, k) \) satisfies the same recurrence and initial conditions as \( B(n, k) \), so they agree.

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