Deformation quantization of submanifolds and reductions via Duflo-Kirillov-Kontsevich map.

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Abstract

We propose the following receipt to obtain the quantization of the Poisson submanifold $N$ defined by the equations $f_i = 0$ (where $f_i$ are Casimirs) from the known quantization of the manifold $M$: one should consider factor algebra of the quantized functions on $M$ by the images of $D(f_i)$, where $D : \text{Fun}(M) \to \text{Fun}(M) \otimes \mathbb{C}[\hbar]$ is Duflo-Kirillov-Kontsevich map. We conjecture that this algebra is isomorphic to quantization of $\text{Fun}(N)$ with Poisson structure inherited from $M$. Analogous conjecture concerning the Hamiltonian reduction saying that ”deformation quantization commutes with reduction” is presented. The conjectures are checked in the case of $S^2$ which can be quantized as a submanifold, as a reduction and using recently found explicit star product. It’s shown that all the constructions coincide.
1 Introduction

1.1 Quantization

Consider a manifold $M$ with the Poisson bracket on it. In [11] it was proposed to find the new associative multiplication (usually called star product and denoted by $f \ast g$) on $\text{Fun}(M)[[\hbar]]$ such that:

$$f \ast g = fg + \hbar(\text{quantum corrections}) \quad \text{and} \quad f \ast g - g \ast f = i\{f, g\} \mod \hbar^2 \quad (1)$$

The algebra with the new multiplication pretends to be the algebra of quantum observables associated with the given classical algebra of observables which is $\text{Fun}(M)$. The problem was ingeniously solved by Kontsevich in [2], (for the symplectic manifolds it was done before in [5]) who also obtained the classification of the star products, which includes the following desired result of the uniqueness: there is bijection between the star products up to equivalence and Poisson brackets up to equivalence (see theorem in section 1.3

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4 Comparison with the explicit star product

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in [2]). The main property that equivalent star products define isomorphic algebras on $\text{Fun}(M)[[\hbar]]$.

Hence despite that for the given Poisson bracket one can construct different star products satisfying \[\Box\] but the algebra corresponding via Kontsevich’s bijection to the given Poisson bracket is unique up to isomorphism.

**Notation** we will denote such algebra by $\widehat{\text{Fun}}(M)$.

Let us mention that from the physical perspective the construction of the algebra of quantum observables is not the full solution of the problem of quantization. One also needs to define the Hilbert space where such algebra acts unitary and irreducibly. For the symplectic manifold it is believed to be the only one such representation (for the algebra $\widehat{\text{Fun}}(M)$ with $\hbar = 1$). This problem is not yet solved in full generality, but in some cases this can be done (see [6, 7, 8, 9]). In our paper one also finds the simplest example confirming this belief.

**Notation** we will denote such Hilbert space associated to the symplectic manifold $M$ as $H(M)$.

### 1.2 Main conjectures

The main aim of this paper is to propose and present some evidences for the conjectures below. Consider some Poisson manifold $M$ and some Casimir function $f$ on this manifold (i.e. $f$ Poisson commute with any other function). Then submanifold $N : f = \text{Const}$ inherits the Poisson structure from $M$ (see section 2.1 for explanations). It is clear that:

$$\text{Fun}(N) = \text{Fun}(M)/(f - c) \quad (2)$$

as Poisson algebras, so it is natural to try to quantize this isomorphism. Let us denote by $D$ a Duflo-Kirillov-Kontsevich map from $\text{Fun}(M) \to \widehat{\text{Fun}}(M)$ (see section 1.3 for explanations).

**Conjecture 1:** There is isomorphism of algebras:

$$\widehat{\text{Fun}}(M) = \widehat{\text{Fun}}(N)/D(f - c) \quad (3)$$

One should possibly add some regularity property for $f$ like $f = c$ is a smooth manifold, (for several $f_i$ one should request transversal intersection).

Let us mention that due to results of Duflo, Kirillov, Kontsevich $D(f - c)$ is Casimir in $\widehat{\text{Fun}}(M)$ i.e. $D(f - c)$ star product commutes with everything, so ideal generated by it is both sided.

The conjecture 1 above can be generalized to the following more general situation (called Hamiltonian reduction): consider functions (called constraints) $f_i \in \text{Fun}(M)$ such that they generate Poisson closed ideal (such constraints are called the first class constraints following Dirac). Let us denote by $I = I(f_i)$ the ideal generated by $f_i$. Let $N = N(f_i)$ be Poisson normalizer of ideal $I$. Let us consider Poisson factor algebra $N/I$, it is known (at least for general $f_i$) that it is algebra of functions on the quotient of the manifold $f_i = 0$ by the vector fields generated by the Hamiltonian vector fields corresponding to $f_i$. This manifold is called the Hamiltonian reduced manifold by the
constraints \( f_i \) and denoted by \( M//f_i \), so there is the following isomorphism of Poisson algebras:

\[
\text{Fun}(M//f_i) = N/I.
\] (4)

The quantization of the Hamiltonian reduction works as follows: denote by \( \hat{I} \) left ideal in \( \text{Fun}(M) \) generated by \( D(f_i) \), by \( \hat{N} \) the right normalizer of it in \( \text{Fun}(M) \), so \( \hat{I} \) is both sided ideal in \( \hat{N} \).

**Conjecture 2:** There is isomorphism of algebras:

\[
\text{Fun}(M//f_i) = \hat{N}/\hat{I}.
\] (5)

One should most probably add some regularity properties for \( f_i \) like transversal intersection and existence of smooth quotient. Obviously conjecture 1 is particular case of conjecture 2, because if \( f_i \) are Casimirs, then Hamiltonian vector fields corresponding to them are zero and so \( M//f_i \) is just the submanifold \( f_i = 0 \).

(The general scheme of the Hamiltonian reduction is due to Dirac [10], our contribution is the remark that one should use the map \( D \) to obtain the answer which is the quantization of the classically reduced space).

This conjecture means that "deformation quantization commutes with reduction" on the level of algebras of observables. The same should be true for the Hilbert spaces associated to the both manifolds i.e.

**Conjecture 3:**

\[
H(M//f_i) = \{ v \in H(M) : D(f_i)v = 0 \}
\] (6)

In the case when \( f_i \) are generators of some compact Lie group this conjecture is due to Guillemin and Sternberg [11]. (In this case there is no need to use the map \( D \)). In their work \( H(M) \) was described in holomorphic polarization. Their conjecture has been proved recently (see [12, 13] for surveys).

Let us remark that our conjectures depends on some auxiliary choices like the choice of concrete generators in \( f_i \) defining the ideal and the choice of map \( D \), which also not canonical, we believe that the conjectures are true for arbitrary choices mentioned above.

### 1.3 Duflo-Kirillov-Kontsevich map

In [2] M. Kontsevich proposed a universal method for deformation quantization. Namely, for any Poisson bracket \( \{ \cdot, \cdot \} \) on a manifold \( M \) one can construct a star product on \( \text{Fun}(M) \) such that \( a \star b - b \star a = i\hbar \{a, b\} \mod \hbar^2 \) for all \( a, b \in \text{Fun}(M) \). The gauge-isomorphism class of this star-product is defined canonically by the gauge-isomorphism class of the Poisson bracket. This star-product satisfies another very nice property: there is a natural mapping

\[
D : \text{Fun}(M) \to \hat{\text{Fun}(M)}
\] (7)

whose restriction to the Poisson center of \( \text{Fun}(\mathbb{R}^n) \) gives an algebra isomorphism onto the center of \( \hat{\text{Fun}(M)} \). We will call this map Duflo-Kirillov-Kontsevich map.
The construction of star-product in $\mathbb{R}^n$ for arbitrary Poisson bracket is given by explicit, but very complicated formula. The same can be said about the construction of the map $D$. We postpone this definition to the appendix. To our luck it was proved by Kontsevich that in the case when Poisson manifold is a vector space with linear Poisson bracket (i.e. it is dual space to some Lie algebra with Kirillov’s bracket) it is true the following: quantization of such manifold is isomorphic the universal enveloping algebra and the map $D$ coincides with the rather explicit map called Duflo-Kirillov map. In this paper we will consider only such Poisson manifolds. So here we will recall the definition of the map $D$ in this situation.

In the case when Poisson manifold is $\mathbb{R}^{2n}$ with the standard Poisson bracket $\{p_k, q_j\} = \delta_{kj}$ then the map $D$ is just the symmetrization.

\[
D(ab) = \frac{1}{2!} (a * b + b * a), \quad D(a_1 a_2 \ldots a_n) = \frac{1}{n!} \left( \sum_{\sigma \in S_n} a_{\sigma(1)} * a_{\sigma(2)} * \ldots * a_{\sigma(n)} \right) \tag{8}
\]

where $a, b, a_i$ are any linear combinations of $p_k, q_j$. It is easy to check that the map $D$ gives an $sp_{2n}$-module isomorphism between $Fun(\mathbb{R}^{2n})$ and $\hat{Fun}(\mathbb{R}^{2n})$.

The more general case is the following: Poisson manifold is a $g^*$ with the Kirillov’s bracket, where $g^*$ is the dual space of a finite-dimensional Lie algebra. The algebra $Fun(g^*)$ is the symmetric algebra $S(g)$ and $\hat{Fun}(g^*)$ is isomorphic to the universal enveloping algebra $U(g)$. To define the Duflo-Kirillov-Kontsevich map in this case we introduce some notations.

1. Let $\text{Tr}_{2k}$ be invariant polynomials on $g$, $x \mapsto \text{Tr}_g(adx)^{2k}$, considered as differential operators on $g^*$ with constant coefficients.

2. Let $a_{2k}$ bee the sequence of real numbers, such that

\[
\sum_{k \geq 0} a_{2k} t^{2k} = \frac{1}{2} \text{Log} \frac{e^t - e^{-t}}{t}. \tag{9}
\]

3. Let $\sigma : S(g) \to U(g)$ be the symmetrization map.

The Duflo-Kirillov-Kontsevich map $D : S(g) \to U(g)$ is given by the formula (see [2, section 8.3])

\[
s \mapsto \sigma \left( \sum_{k \geq 0} \frac{(i\hbar)^{2k} a_{2k} \text{Tr}_{2k}}{k} s \right). \tag{10}
\]

Note that in the case of nilpotent $g$ we have $\text{Tr}_{2k} = 0$ for all $k$, and hence $D = \sigma$. The different proofs that this formula gives the isomorphisms of centers of $S(g)$ and $U(g)$ can be found in [3] and [4].

### 1.4 Plan of the paper

The main text of this paper is devoted to the successful check of our conjectures in the first nontrivial example of the sphere $S^2$ with standard $SO(3)$ invariant symplectic form.
Sphere $S^2$ with this symplectic form can be obtained in two different ways first as coadjoint orbit for $SO(3)$ i.e. as a submanifold in $so(3)$ second as the Hamiltonian reduction of $\mathbb{R}^4$. Both constructions can be quantized according to our receipts and we see that the results coincide. This is done in section 2. Conjecture 3 is also true in this example see section 3.3.

The third thing - we compare the quantizations above with the explicit star product construction recently found in [19]. And also we find the complete agreement. This is done in section 4.

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2 Quantization of $S^2$ as a submanifold

In this section we consider $S^2$ as a submanifold in $\mathbb{R}^3 = so(3)$ and we quantize it by the receipt of the conjecture 1 (see claim 1 in section 2.2). We also show that our considerations completely confirm the belief that for the symplectic manifold there is only one irreducible and unitary representation of the algebra $\hat{\text{Fun}}(M)$ (see section 2.3).

2.1 Inheriting the symplectic structure on $S^2$ from $\mathbb{R}^3 = so(3)$.

In this subsection we will explain how to inherit the symplectic structure from Poisson bracket in $\mathbb{R}^3$ and calculate the volume. (It does not coincide with Euclidean volume $4\pi R^2$, but is given by $2\pi R$).

Consider the sphere in $\mathbb{R}^3$ given by the equation $x_1^2 + x_2^2 + x_3^2 = R^2$.

The space $\mathbb{R}^3$ can be identified with the Lie algebra $so(3)$ (more precisely with its dual space $so(3)^*$ but in the case of semisimple Lie algebras like $so(3)$ one can identify $so(3)$ and $so(3)^*$ with the help of Killing form), so it can be endowed with the Poisson bracket $[x_i, x_j] = 2\epsilon_{ijk}x_k$, where $\epsilon_{ijk}$ is totally antisymmetric tensor.

The element $C = x_1^2 + x_2^2 + x_3^2$ is Casimir element for this bracket, so the sphere $x_1^2 + x_2^2 + x_3^2 = R^2$ can be endowed with the Poisson bracket. This is the general trivial fact that: if any element $C$ is Casimir for any Poisson bracket on any manifold $M$, then the submanifold $N$: $C = \text{Const}$, inherits the Poisson bracket from $M$. Which goes as follows: the functions on $N$ are factor by the ideal generated by $C$ of functions on $M$, but the ideal $I$ generated by the Casimir $C$ is ideal with respect to the Poisson bracket, i.e. for $f \in I, g \in \text{Fun}(M)$ holds $\{f, g\} \in I$. Hence the Poisson bracket can pushed down to the factor algebra $\text{Fun}(M)/I = \text{Fun}(N)$.

Let us mention that the only property needed for restricting the Poisson bracket to the submanifold is the property that ideal $I$ is Poisson ideal. Geometrically this can be reformulated as bivector $\pi$ is tangent to the submanifold $N$. (Polyvector is tangent to
some submanifold iff it can be presented as sum of products of tangent to this submanifold vectors).

Obviously Poisson bracket on $S^2$ is nondegenerate, so we obtain symplectic form on $S^2$. It is easy to see that $S^2$ is coadjoint orbit for $so(3)$, and the symplectic structure above is Kirillov’s symplectic structure on the coadjoint orbit. It is obviously $so(3)$ invariant.

**Lemma 1** volume of $S^2$ with respect to this symplectic form is $2\pi R$.

**Proof** at the upper half sphere one can consider $x_1, x_2$ as local coordinates and the Poisson bracket is given by $\{x_1, x_2\} = 2x_3 = 2\sqrt{R^2 - x_1^2 - x_2^2}$, so the symplectic form is given by $\omega = \frac{1}{2\sqrt{R^2 - x_1^2 - x_2^2}}dx_1 \wedge dx_2$. So the volume of the semisphere can be calculated as

$$\int_{x_1^2 + x_2^2 < R^2} \frac{1}{2\sqrt{R^2 - x_1^2 - x_2^2}}dx_1 \wedge dx_2 = \int_{0 < r < R, 0 < \phi < 2\pi} \frac{1}{2\sqrt{R^2 - r^2}} rd\phi = \int_{0 < r < R} \frac{R}{4\sqrt{1 - r^2}} R^2 dr = 2\pi \int_{0 < u < 1} \frac{R}{4\sqrt{1 - u}} du = -2\pi R \frac{1}{2} \sqrt{1 - u_0} = \pi R$$

(11)

Now recalling that it is volume of semisphere we multiply it by two and obtain the volume of sphere is $2\pi R$. □

### 2.2 Quantization and Duflo-Kirillov map.

The algebra of functions on $\mathbb{R}^3$ with respect to star product corresponding to Poisson bracket $\{x_i, x_j\} = 2\epsilon_{ijk} x_k$ is isomorphic to $U(so(3))$. (This is true for any Lie algebra see for example [2] section 8.3.1). In this section we will never use the star-product, but we will work with $U(so(3))$. Let us denote the multiplication in $U(so(3))$ by $\circ$. So $[x_i, x_j]_\circ = 2i\hbar \epsilon_{ijk} x_k$.

Later on we put $\hbar = 1$ for simplicity.

**Proposition 1** The image of Casimir element $C = \sum_i x_i^2$ under Duflo-Kirillov isomorphism is $D(C) = \sum_i x_i \circ x_i + 1 \in U(so(3))$.

**Proof**: Explicit computation shows that $a_2 = \frac{1}{48}$ and $Tr_2 = -8 \sum_i \partial_i^2$. Hence we have

$$D(C) = \sigma(C + 1) = \sum_i x_i \circ x_i + 1.$$  

(12)

□

As a corollary of this proposition we obtain that modula the conjecture 1 the following theorem is obtained. (We call ”claim” because it is proved here only modula conjecture 1, later we will prove it by explicit star product construction, so it will be really the theorem):
Claim 1  Quantization of $S^2$ with standard $SO(3)$ invariant symplectic form of the volume $R$ (i.e. the algebra $\text{Fun}(S^2)$ with $\hbar = 1$) is isomorphic to the algebra

$$U(so(3))/\left(\sum_i x_i \circ x_i + 1 = R^2\right),$$

(13)

where $x_i$ are generators of $so(3)$ obeying the relations: $[x_i, x_j]_\circ = 2i\epsilon_{ijk}x_k$.

So we have described the quantization of $S^2$ as a submanifold in $\mathbb{R}^3$.

2.3 Hilbert space from representation theoretic point of view.

Now let us describe the (unique) finite-dimensional representation of the algebra $U(so(3))/\left(D(C) = R^2\right)$.

Recall that the isomorphism of $sl(2)$ and $so(3)$ is given by the formulas

$$h = x_1, \ e = \frac{1}{2}(x_2 + ix_3), \ f = \frac{1}{2}(x_2 - ix_3).$$

(14)

The commutator relations os $sl(2)$ are standard $[e, f]_\circ = h, \ [h, e]_\circ = 2e, \ [h, f]_\circ = -2f$.

The Casimir element $D(C) \in U(so(3)) = U(sl(2))$ can be rewritten as

$$D(C) = \sum_i x_i \circ x_i + 1 = 4e \circ f + h \circ h - 2h + 1 = 4f \circ e + e \circ h + 2h + 1.$$  (15)

Let $V_\lambda$ is the irreducible representation of the Lie algebra $so(3) = sl(2)$ with the highest weight vector $|0 >$ and weight $\lambda = R - 1$, i.e. $h|0 > = \lambda |0 >, \ e|0 > = 0$. The Casimir operator $D(C) = \sum_i x_i \circ x_i + 1 = 4e \circ f + h \circ h - 2h + 1 = 4f \circ e + e \circ h + 2h + 1$ acts on it as scalar operator on $V_\lambda$ and the scalar can be easily computed

$$D(C)|0 > = (4f \circ e + e \circ h + 2h + 1)|0 > = (\lambda^2 + 2\lambda + 1)|0 > = R^2|0 >.$$  (16)

So we come to the following lemma:

Lemma 2  We see from representation theoretic point of view that the belief that the algebra of quantized functions has the only representation in the Hilbert space finds complete confirmation. The only representation is $V_{R-1}$. Its dimension is equal to $R$. Other representation of $sl(2)$ should be dropped out because either Casimir will act by the irrelevant constant or because they cannot be made unitary (like Verma modules).

2.4 Hilbert space from geometric quantization

According to general optimistic belief the deformation quantization of the algebra of functions (with $\hbar = 1$) on the symplectic manifold $M$ has unique irreducible unitary (i.e. real-valued functions acts as self-adjoint operators) representation in the Hilbert space. (For the Poisson manifold the representations are related to the symplectic leaves). Moreover the dimension of the such representation is expected to be given by the formula $\int_M \exp(\omega) \hat{A}(M)$, where $\hat{A}(M)$ is $A$-genus of the manifold $M$. This is predicted by the geometric quantization with half-forms and by Fedosov’s index theorem [14] (one usually
requests $\omega$ to be integer 2-form on $M$, but possibly for non integer 2-forms all the same can be done making from $\text{Fun}(M)$ von Neumann algebra and using von Neumann’s fractional dimension). (If $K$ is trivial and in some other cases this coincides with the naive prediction of physicists that the dimension of the Hilbert space is $1/n!(\text{symplectic volume})$, usually this is said in textbooks as one quantum state takes $\prod dp \prod dq (2\pi \hbar)^n$ of the phase space (see for example section 48 in [15])). If $K$ is trivial and in some other cases this coincides with the naive prediction of physicists that the dimension of the Hilbert space is $1/n!(\text{symplectic volume})$, usually this is said in textbooks as one quantum state takes $\prod dp \prod dq (2\pi \hbar)^n$ of the phase space (see for example section 48 in [15])). If the form $\omega$ is Kahler and sufficiently positive form then Hilbert space can be realized as the space of holomorphic sections $H^0(L \otimes \sqrt{K})$, where $L$ is such line bundle that: $c_1(L) = \omega$, $K$ canonical line bundle (the line bundle of holomorphic exterior forms of highest degree). The line bundle $\sqrt{K}$ is such bundle that $\sqrt{K} \otimes \sqrt{K} = K$, it exists if $w_2(M) = 0$ and unique if $M$ is simply connected. The sections of such bundle are called half-forms. Note that by the Riemann-Roch and vanishing theorems $\dim H^0(L \otimes \sqrt{K}) = \int_M \exp(\omega - \frac{1}{2}c_1(M))T\text{odd}(M) = \int_M \exp(\omega)\hat{A}(M)$, due to the equality $\exp(-\frac{1}{2}c_1(M))T\text{odd}(M) = \hat{A}(M)$.

One receipt which is due to Kostant and Souriau how to construct the representation of the algebra of functions with deformed product in the space of sections of some line bundle is called geometric quantization (see [6] for survey). (Let us mention that it was developed before deformation quantization, and there is some misunderstanding that sometimes people insist on the exact equality $i\{f, h\} = [f, g]$ in the geometric quantization approach. This is not really true. This is true only for consideration of representation on the non-polarized sections, but when one needs to find the representation in the Hilbert space i.e. in the space of polarized sections - this commutation relation does not hold). Though it was never realized in full generality, it is known to work in the case of semisimple orbits of semisimple Lie algebras. (Another receipt is the so-called Berezin-Toeplitz quantization which succeeds in the case of compact Kahler manifolds [7, 8, 9].)

Turning from the generalities to our concrete example of $S^2 = \mathbb{C}P^1$ we see that geometric quantization predicts that the algebra of functions with the star product should have the irreducible unitary representation realized in the sections of line bundle $L \otimes \sqrt{K}$, where $L = O(R)$, on $\mathbb{C}P^1$ it is well-known that $K = O(-1)$ and so $L \otimes \sqrt{K} = O(R - 1)$. Hence the dimension of the Hilbert space is $\dim H^0(L \otimes \sqrt{K})$ and it is equal to:

$$\dim H(S^2) = \int_{S^2} \exp\left(\frac{1}{2\pi} \omega\right) \exp\left(-\frac{1}{2} c_1(S^2)\right) \hat{A}(S^2) = \int_{S^2} \left(\frac{1}{2\pi} \omega\right) = R. \quad (17)$$

**Remark 1** We see that dimension of the Hilbert space in this example coincides with the symplectic volume up to $2\pi$.

**Corollary 1** We see the complete agreement for the dimension of Hilbert space prescribed from the representation theoretic of view (see Lemma 2) and from the point of view of geometric quantization with half-forms.

### 3 Quantization of $S^2$ by Hamiltonian reduction.

In this section we recall the Hamiltonian reduction procedure and we show how to obtain $S^2$ as a reduction of $\mathbb{R}^4$, and proceed with quantization of reduction by the receipt of the conjecture 2 (see claim 2 in section 3.2). As an evidence for our conjectures we show...
that the result is the same as in the previous section (see corollary \[1\] in section \[3.2\]). We also confirm the conjecture 3 describing the Hilbert space from the point of view of the reduction.

The procedure of Hamiltonian reduction is due to Dirac \[10\] (see \[16\] for recent short exposition and very nice remark that non reduced constraints like \(x^n = 0\) leads to appearance of matrix degrees of freedom, which was possibly motivated by string theorists belief that coincident D-branes leads to appearance of \(U(n)\) gauge group as "brane volume" theory). The geometric sense of the Hamiltonian reduction in the case of arbitrary symplectic manifolds was explained to mathematicians in \[17\].

### 3.1 Classical Hamiltonian reduction of \(\mathbb{R}^4\) by \(\frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)\).

The procedure of hamiltonian reduction has been briefly described in the introduction, we will follow the described scheme.

The symplectic structure on \(S^2\) considered above can be obtained as a Hamiltonian reduction of the constant symplectic structure on \(\mathbb{R}^4 = \mathbb{C}^2\). Namely, let \(p_1, p_2, q_1, q_2\) be coordinates on \(\mathbb{R}^4\) with the standard Poission bracket (i.e. \(\{p_i, q_j\} = \delta_{ij}\)), and let \(z_1 = \frac{1}{\sqrt{2}}(q_1 + ip_1), z_2 = \frac{1}{\sqrt{2}}(q_2 + ip_2)\), then \(\{z_1, \bar{z}_1\} = i, \{z_2, \bar{z}_2\} = i\). Let us consider the constraint, which is Hamiltonian for the harmonic oscillator:

\[
E = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) = (\bar{z}_1 z_1 + \bar{z}_2 z_2).
\]

**Lemma 3** Let \(N\) be the commutant in \(\text{Fun}(\mathbb{R}^4)\) of the element \(E\) with respect to the Poisson bracket. The algebra \(N\) is generated by:

\[
E, \ x_1 = \frac{1}{2}(p_1^2 + q_1^2 - p_2^2 - q_2^2) = (z_1 \bar{z}_1 - z_2 \bar{z}_2),
\]

\[
x_2 = (q_1 q_2 + p_1 p_2) = 2 \text{Re}(z_1 \bar{z}_2), \ x_3 = (p_1 q_2 - q_1 p_2) = 2 \text{Im}(z_1 \bar{z}_2)
\]

**The elements** \(x_i\) **satisfy the relations**: \(\{x_i, x_k\} = 2 \epsilon_{ijk} x_k\), **which are so(3) relations**.

**Proof.** Clear.

Let us recall that we have defined Casimir element in so(3) as \(C = \sum_i x_i^2\)

**Lemma 4** \(C = E^2\).

**Proof.** \(C = x_1^2 + (x_2 + ix_3)(x_2 - ix_3) = (|z_1|^2 - |z_2|^2)^2 + 4|z_1|^2|z_2|^2 = (|z_1|^2 + |z_2|^2)^2\).

The element \(E\) is central in \(N\), hence the Poisson bracket on the algebra \(S = N/(E = R)\), where \(R\) is a constant, is well-defined.

**Corollary 2** The classical Hamiltonian reduction of \(\mathbb{R}^4\) by the constraint \(E - R = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) - R\) is sphere \(S^2\) of symplectic volume \(2\pi R\). On the level of functions this mean that: \(N/(E = R)\) is isomorphic to \(\text{Fun}(S^2)\) as Poisson algebra and isomorphism is given by formulas \(12\).

**Proof:** the calculation of volume follows from proposition \(11\) the other things are clear.
3.2 Quantum Hamiltonian reduction of $\mathbb{R}^4$ by $\frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)$.

As we have already mentioned in the introduction the quantization of $\mathbb{R}^{2n}$ can be explicitly described by the Moyal formula [18]:

$$r * s = (e^{i\hbar \sum_{i=1,2} \partial_p^i \partial_{q_i}} r(p, q)s(\tilde{p}, \tilde{q}))_{p_i = \tilde{p}_i, q_i = \tilde{q}_i}$$

(20)

Let us put $\hbar = 1$.

All commutators in this section are with respect to the Moyal’s product.

It’s obviously true that $[p_i, q_j] = i\delta_{ij}$ (hence $z_i$ satisfy the relations: $[z_i, \bar{z}_i] = 1$).

Recall that the Duflo-Kirillov-Kontsevich map in this case is given just by the symmetrization:

$$D(a_1...a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} * ... * a_{\sigma(n)},$$

(21)

where $a_k$ is any linear combination of $p_i, q_j$.

Thus, on the quantum level we have

$$D(E) = \hat{E} = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) = \frac{1}{2}(p_1 * p_1 + q_1 * q_1 + p_2 * p_2 + q_2 * q_2) =$$

$$= \frac{1}{2} \sum_{i=1,2} z_i * \bar{z}_i + \bar{z}_i * z_i. \quad (22)$$

**Remark 2** so let us mention that in this case if one works in generators $p_i, q_j$ then there is no need to use the symmetrization due to $E = D(E)$, but working in generators $z_i, \bar{z}_i$ really shows that symmetrization is really essential due to $D(E) = \frac{1}{2} \sum_{i=1,2} z_i * \bar{z}_i + \bar{z}_i * z_i \neq \sum_{i=1,2} z_i * \bar{z}_i$.

**Lemma 5** For any $s \in Fun(\mathbb{R}^4)$ we have $[D(E), D(s)] = D(\{E, s\})$.

**Proof.** Indeed, let $s$ be homogeneous of degree $m_i$ with respect to $z_i$ and of degree $n_i$ with respect to $\bar{z}_i$. Then $D(E), D(s) = (n_1 + n_2 - m_1 - m_2) D(s) = D((n_1 + n_2 - m_1 - m_2)s) = D(\{E, s\})$. \qed

**Corollary 3** Denote by $\hat{N}$ the commutant of $D(E)$. This algebra is generated by

$$D(E), \ x_1 = \frac{1}{2}(p_1^2 + q_1^2 - p_2^2 - q_2^2) = \frac{1}{2}(z_1 * \bar{z}_1 - z_2 * \bar{z}_2 + \bar{z}_1 * z_1 - \bar{z}_2 * z_2),$$

$$x_2 = (q_1 * q_2 + p_1 * p_2) = (q_1 q_2 + p_1 p_2) = 2Re(z_1 * \bar{z}_2),$$

$$x_3 = (p_1 * q_2 - q_1 * p_2) = (p_1 q_2 - q_1 p_2) = 2Im(z_1 * \bar{z}_2) = 2Im(z_1 \bar{z}_2) \quad (23)$$

**Lemma 6** Elements $x_i$ satisfy the relations $[x_i, x_j] = 2i\epsilon_{ijk} x_k$, and hence elements $h = x_1, e = \frac{1}{2}(x_2 + ix_3) = z_1 \bar{z}_2, f = \frac{1}{2}(x_2 - ix_3) = \bar{z}_1 z_2$ satisfy the sl(2) relations: $[e, f] = h, [h, e] = 2e, [h, f] = -2f$.

Let us recall that according to conjecture 2 about the Hamiltonian reduction the quantizations of functions on $S^2$ should be $\hat{N}/(D(E) = R)$. So we see that:
Claim 2 Quantum of $S^2$ with standard $SO(3)$ invariant symplectic form of the volume $2\pi R$ (i.e. the algebra $\text{Fun}(S^2)$ with $h = 1$) is isomorphic to the algebra $\hat{N}/D(E) = R$, where $\hat{N}$ is described in corollary and lemma and $D(E)$ is given by the formula.

Let us prove that this quantization is the same as in the previous subsection. The subalgebra generated by $x_i$ with respect to the star product is isomorphic to $U(so(3))$. Recall that Casimir element $D(C)$ in $U(so(3))$ was defined: $D(C) := \sum_{i=1,2,3} x_i^2 x_i + 1 = 4e * f + h * h - 2 h + 1 = 4f * e + h * h + 2 h + 1 = 2(e * f + f * e) + h * h + 1$.

Lemma 7 $D(C) = D(E)^2$.

Proof. The algebra $\text{Fun}(\mathbb{R}^4)$ acts by differential operators on the polynomial algebra $\mathbb{C}[z_1, z_2]$ (where $\bar{z}_i$ acts as $\partial_{z_i}$ and $z_i$ acts as multiplication by $z_i$). The kernel of this action is zero. Therefore, it suffices to check that the elements $D(C)$ and $D(E)^2$ act by the same operator. The space of homogeneous polynomials of degree $n$ is isomorphic to $V_n$ as $sl(2)$-module, and $D(C)$ acts as the Casimir operator on this space. Thus, according to lemma for any homogeneous polynomial $P$ of degree $n$ we have

$$D(C)P = (n^2 + 2n + 1)P = (n + 1)^2 P.$$  (24)

On the other hand

$$D(E)^2 P = \left(\frac{1}{2} \sum_{i=1,2} \partial_{z_i} z_i + z_i \partial_{z_i}\right)^2 P = (1 + \sum_{i=1,2} z_i \partial_{z_i})^2 P = (n + 1)^2 P.$$  (25)

□

So we come to the main corollary of this section:

Corollary 4 There is a natural isomorphism

$$\hat{N}/(D(E) = R) \simeq U(so(3))/(D(C) = R^2).$$  (26)

This proves the desired result that both quantizations of $S^2$ are the same.

3.3 Hilbert space from Hamiltonian reduction.

Let us recall that in the method of Hamiltonian reduction for the constraints $f_i = 0$ the Hilbert space of the reduced system is defined (according to the conjecture 3) as the subspace of the nonreduced system such that constraints $D(f_i)$ acts as zero: $H^{\text{red}} = \{ v \in H : D(f_i)v = 0 \}$, it is clear that the reduced algebra of functions acts on this space.

In our case we have the constraint $E - R = 0$, where $E = D(E) = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)$. The Hilbert space for the quantization of $\mathbb{R}^4 = \mathbb{C}^2$ with the standard symplectic structure $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ is known from any textbook to be $L^2(q_1, q_2)$ with the standard measure $dq_1 dq_2$, another realization for the same space is the so-called holomorphic realization $\mathbb{C}[z_1, z_2]$, (more precisely we should consider holomorphic functions of $z_1, z_2$ which are square integrable with the measure $exp(-|z_1|^2 - |z_2|^2)$, but it does not matter for our questions). In this representation $\bar{z}_i$ acts as $\partial_{z_i}$. So we come to:
Proposition 2 The reduced space for the constraint $E = R$, i.e. subspace in $\mathbb{C}[z_1, z_2]$, where $D(E) - R$ acts as zero is the space of homogeneous polynomials of degree $R - 1$, it has the dimension $R$, the quantization of functions on $S^2$ acts irreducible and unitary on this space.

Proof is clear from the formula and description of representations of $sl(2)$.

Corollary 5 We obtain that the unique unitary irreducible representation of $(\text{Fun}(S^2))$ can be obtain by the method of Hamiltonian reduction so completely confirming the conjecture 3. Also we obtain the complete agreement of the description of the Hilbert space from the point of view of Hamiltonian reduction method and all other points of view: representation theoretic, geometric quantization with half-forms and Fedosov’s index theorem (see section 2.4).

4 Comparison with the explicit star product

In this section we recall the explicit $SO(3)$-invariant star product on $S^2$ following and show that it gives the same quantization as predicted by our conjectures, despite that from the first sight we see some contradiction.

4.1 Explicit $SO(3)$-invariant star product on $\mathbb{R}^3$.

Explicit $SO(3)$-invariant star product on $\mathbb{R}^3$ was found in [19] using earlier work [20], later in [21] there was proposed invariant star product on arbitrary coadjoint orbits of semisimple group. In [22] it was shown that the last star product coincides with the one from [19] in the case of $S^2$.

Let us recall (by cut and paste from [22]) the invariant star product on $\mathbb{R}^3$ from [19]. Let $x_i, i = 1, 2, 3$ be the coordinates in $\mathbb{R}^3$, $r^2 = \sum x_i^2$, $\epsilon_{abc}$ - totally antisymmetric tensor.

$$f \star g = fg + \sum_{n=1}^{\infty} C_n(\frac{\hbar}{r}) J^{a_1 b_1} \ldots J^{a_n b_n} \partial_{a_1} \ldots \partial_{a_n} f \partial_{b_1} \ldots \partial_{b_n} g,$$

(27)

where

$$C_n(\frac{\hbar}{r}) = \frac{\left(\frac{\hbar}{r}\right)^n}{n!(1 - \frac{\hbar}{r})(1 - 2\frac{\hbar}{r}) \cdots (1 - (n - 1)\frac{\hbar}{r})},$$

(28)

and

$$J^{ab} = r^2 \delta_{ab} - x_a x_b + i r \epsilon_{abc} x_c.$$ 

(29)

The star product is defined on $\mathbb{R}^3 \setminus \{0\}$, but can be restricted to two-spheres centered at the origin since because of the property (see [19]) $f(r^2) \star g(x) = g(x) \star f(r^2) = f(r^2)g(x)$ so this property guarantees, that the ideal generated by $r^2$ is two-sided and the algebra of functions on $S^2$ with respect to this star product is factor by this ideal of the algebra of functions on $\mathbb{R}^3$ with respect to the star product above, and it is rotation invariant since $J^{ab}$ is a covariant 2-tensor.
4.2 Apparent contradiction of our proposal and explicit star product calculation

Lemma 8

\[ [x_a, x_b]_s = 2i\hbar\epsilon_{abc}x_c \]
\[ \sum_{i=1,2,3} x_i \ast x_i = r^2 + 2\hbar r \]  

Proof: \( C_1 = \frac{\hbar}{r}, \quad J^1 = r^2 - x_1^2, \quad J^{12} = -x_1x_2 + irx_3, \) so
\[ [x_1, x_2]_s = x_1 \ast x_2 - x_2 \ast x_1 = (x_1x_2 + \frac{\hbar}{r}(-x_1x_2 + irx_3)) - (x_2x_1 + \frac{\hbar}{r}(-x_2x_1 - irx_3)) = 2i\hbar x_3 \]
\[ x_1 \ast x_1 = x_1^2 + C_1J^{11} = x_1^2 + \frac{\hbar}{r}(r^2 - x_1^2), \] hence
\[ \sum_{i=1,2,3} x_i \ast x_i = \sum_{i=1,2,3} x_i^2 + \frac{\hbar}{r}(r^2 - x_i^2) = r^2 + 2\hbar r \]

Corollary 6 So we obtain the apparent contradiction: as it follows from the lemma above the quantization of the \( S^2 \) given by the \( \sum_i (x_i^{\text{classical}})^2 = R^2 \) and the Poisson structure is \( \{x_a^{\text{classical}}, x_b^{\text{classical}}\} = 2\epsilon_{abc}x_c^{\text{classical}} \) is the algebra with generators \( x_1, x_2, x_3 \) with the relations: \( [x_a, x_b]_s = 2i\hbar\epsilon_{abc}x_c, \) \( \sum_{i=1,2,3} x_i \ast x_i = R^2 + 2\hbar R, \) but our proposal from the previous sections predicts that as a quantization of \( S^2 \) we should obtain the algebra with the other answer for the second relation: \( \sum_{i=1,2,3} x_i \ast x_i = R^2 - 1 \)

4.3 Solution to the contradiction

As one can see from the previous corollary the difference between the two answers is not very big: if we put \( \hbar = 1 \), then our methods of quantization gives \( \sum_{i=1,2,3} x_i \ast x_i = R^2 - 1 \) and star product gives \( \sum_{i=1,2,3} x_i \ast x_i = R^2 + 2R = (R + 1)^2 - 1 - \) the same as our method, but with the change \( R \rightarrow R + 1 \).

So in order to solve the puzzle we need to explain that star product quantizes the sphere of radius \( R + 1 \) not of the radius \( R \) as is seems.

To our luck this essentially has already been done in \[22\] where the characteristic class of the invariant quantization was found. Let us mention that it is rather nontrivial calculation which used the results of Karabegov \[24\] and Fedosov-Nest-Tsygan index theorem \[14, 25\].

Proposition 3 \[22\] the characteristic class of the invariant star product \( \theta = \frac{\omega}{2\pi\hbar} + \frac{1}{2}\hbar c_1(S^2) \).

Putting \( \hbar = 1 \) we get that \( \int_{S^2} \theta = R + 1 \). From this we conclude:

Claim: the invariant star product \[27\] quantizes the symplectic structure on \( S^2 \) which corresponds to sphere with radius \( R + 1 \), not \( R \).

This is more or less by definition of the characteristic class of deformation quantization. Which measures the difference between the given star product and the isomorphism class of star products which canonically corresponds to the given symplectic structure.

It follows from the claim above, that:

We come to complete agreement of our method of quantization and the explicit star product computation, due to putting \( R - 1 \) instead of \( R \) in the construction with the invariant star product quantization of \( S^2 \) we obtain that it has the characteristic class precisely \( \omega \) and it gives the same quantum algebra as our methods of quantization.
4.4 Characteristic classes of deformation quantization

In this section we briefly recall the material related to the classification of the star products, in order to clarify our point of view. (It seem that even among experts there are some ”dark places” in these matters).

The naive requirement \( f \ast g - g \ast f = i\hbar \{f, g\} \mod \hbar^2 \) is not enough to uniquely define the correspondence between Poisson bracket \( \{,\} \) and the star products. This can be seen from the trivial example: one can consider zero Poisson bracket \( \forall f, g \{f, g\} = 0 \) and arbitrary nonzero Poisson bracket which is multiple of \( \hbar \) i.e. \( \{f, g\} = \hbar(\text{something}) \). The star products for the both of this brackets will satisfy \( f \ast g - g \ast f = 0 \mod \hbar^2 \).

So the question what is the star product corresponding to the given Poisson bracket arises. And more generally what is the correspondence between the brackets and star products. To this question answers the fundamental theorem of Kontsevich (see section 1.3 in [2]). Which says roughly speaking the following to the given Poisson bracket one can construct class of star products, but the algebra of functions with respect to all these star products are isomorphic. So to the given Poisson bracket one can construct one algebra up to isomorphism. Moreover Kontsevich theorem works in the back direction it states that for a given star product one can describe the class of Poisson brackets (depending on \( \hbar \) in general) deformation quantization of which leads to star product equivalent to the initial one.

In the case of symplectic manifolds (i.e. when the Poisson bracket is nondegenerate) the other classification exists: star products corresponding to given symplectic form \( \omega \) are classified by \( H^2(M)[[\hbar]] \). More precisely, to the given star product with the property \( f \ast g - g \ast f = \hbar \{f, g\}\omega \mod \hbar^2 \), one can canonically associate the element of the affine space \( -\frac{\omega}{\hbar} + H^2(M)[[\hbar]] \) (see for example [23]) and the star-products with the same element from \( H^2(M)[[\hbar]] \) defines the isomorphic algebras. (There is no contradiction with example described above and this theorem because in our example we started with the zero Poisson bracket which is degenerate so it is not symplectic). The characteristic class is naturally constructed as a cocycle in the Cech complex of \( M \), so it is very hard to write it down as a de Rham cocycle.

The following conjecture (which states that both classification are agreed) should be true, but we do not know the reference:

**Conjecture 4:** If one takes the star product on the symplectic manifold with the characteristic class \( \theta \) then under the bijections between Poisson brackets and star products defined by Kontsevich this star product corresponds to \( \theta^{-1} \)

More precisely one should take nondegenerate representative in the cohomology class \( \theta \), hopefully it can be done.

So this means: if the characteristic class of the quantization is \( \theta \) then this quantization really quantizes symplectic manifold with the symplectic form \( \theta \), but not \( \omega \).

This clarifies the claim made in previous section.

5 Discussion

The ”weak version” of Conjecture 1 can be formulated as follows. Let \( Z(M) \) be the center of the algebra \( \widehat{\text{Fun}}(M) \) (by Kotsevich theorem it is the same as the Poisson center
of $Fun(M)$).

"Weak Conjecture" 1 $Fun(M)$ and $Fun(M)$ are isomorphic as modules over $Z(M)$

It is probably a weaker form of our conjecture 1, it can be explained rather informally, in the following way: our conjecture says that $Fun(M)/D(C - \alpha)$ is quantization of $Fun(M)/(C - \alpha)$, where $\alpha$ is arbitrary constant, so they are isomorphic as modules of constants, Casimir $C$ here acts as a constant $\alpha$, due to it is true for all $\alpha$ it should be true before the factorization.

In [26] B. Shoikhet proved that for any Lie algebra $\mathfrak{g}$ there is a natural isomorphism of $ZU(\mathfrak{g})$-modules

$$S(\mathfrak{g})/\{S(\mathfrak{g}), S(\mathfrak{g})\} \simeq U(\mathfrak{g})/[U(\mathfrak{g}), U(\mathfrak{g})].$$ \hspace{1cm} (32)

The question is if this isomorphism can be extended to a $ZU(\mathfrak{g})$-module isomorphism between $S(\mathfrak{g})$ and $U(\mathfrak{g})$.

Let us mention that our conjectures requires the explicit choice of generators $f_i$ defining the submanifold. It is of course not satisfactory, because the same manifolds can be defined by different choices of generators. At the moment the situation with this question is not clear for us. It is quite obvious that linear change of generators leads to the same quantization. About the general case it is not quite clear: hopefully for arbitrary choice of generators $f_i \in I D(f_i)$ generate the same ideal in $Fun(M)$, but possibly the ideals generated by them are different, and there is only isomorphism of $\hat{N}_{f_i}/\hat{I}_{f_i}$ for different of $f_i$.

Moreover the map $D$ is defined not uniquely, but depends on the auxiliary structures like the choice of coordinate system in the approach of [2], nevertheless we hope that our conjectures are true for arbitrary choice of the map $D$.

Let us mention that the next step to check our conjectures can be the attempt to check it for the other coadjoint orbits of semisimple Lie groups, for example $gl(n)$. By the definition they are submanifolds in $gl(n)$ on the other hand there is the explicit star product found in [21], and they also can be represented by the hamiltonian reduction (quiver like description of coadjoint orbits). One can hope that all the three approaches coincide.

It would be also very interesting to find the generalization of the conjectures above to the case of infinite-dimensional Lie algebras, because many interesting spaces like moduli spaces of flat connections, instantons, etc. can be obtained by the Hamiltonian reduction by the action of infinite-dimensional groups. But in this case seems nothing to be known about Duflo-Kirillov map, it is believed that it should be basically the same but there should be some corrections.

6 Appendix. General definition of the Duflo-Kirillov-Kontsevich map

Let us recall the definition of the map $D$. We will follow the letter from V. Dolgushev to whom we deeply indebted for the clarifications. One is reffered to [2, 27, 28] for further details.
To any manifold $M$ one can associate two differential graded Lie algebras (DGLA). The first is the algebra

$$P = \bigoplus_{k \geq -1} \Gamma(\wedge^{k+1} TM), \quad \Gamma(\wedge^0 TM) = \text{Fun}(M)$$

of smooth polyvector fields. The structure of Lie algebra is given by the Schouten-Nijenhuis bracket $[,]_{SN}$, and the differential is zero.

The second DGLA is the algebra $H$ of polydifferential operators with the Gerstenhaber bracket $[,]_G$ and the differential given by

$$\partial = [m, \bullet]_G,$$

where $m$ is the commutative product $\text{Fun}(M) \otimes \text{Fun}(M) \to \text{Fun}(M)$.

The solutions of the Maurer-Cartan equation in the first algebra are Poisson brackets, and the solutions of the Maurer-Cartan equation in the second algebra are star-products.

The formality quasi-isomorphism of Kontsevich is a (nonlinear) $L_\infty$-morphism $F$ from the DGLA of polyvector fields to the DGLA of polydifferential operators. The structure maps $F_n$ of this quasi-isomorphism are described in terms of integrals over configuration spaces related with the Lobachevsky plane (see [2] for more details). For any solution $\alpha$ of the Maurer-Cartan equation in the DGLA of polyvector fields (i.e. for any Poisson bracket) the bidifferential operator

$$F(h\alpha) = \sum_{k=0}^\infty \frac{1}{k!} F_{k+1}(h\alpha, \ldots, h\alpha)$$

satisfies Maurer-Cartan equation as well.

Furthermore, the formality quasi-isomorphism of Kontsevich gives a quasi-isomorphism $I$ from the complex of polyvector fields with the differential

$$d_\alpha = [h\alpha, \bullet]_{SN},$$

to the complex of polydifferential operators with the differential

$$\partial_\alpha = [m + F(h\alpha), \bullet]_G.$$

This quasi-isomorphism of complexes is given by the formula:

$$I(\gamma) = \sum_{k=0}^\infty \frac{1}{k!} F_{k+1}(h\alpha, \ldots, h\alpha, \gamma),$$

where $\gamma$ is a cochain in $P$, and $F_n$ are the structure maps of Kontsevich’s quasi-isomorphism. This quasi-isomorphism is compatible with the cup-product on cohomology of these complexes.

The fact that a function $C \in \text{Fun}(M)$ is central is equivalent to the fact that $C$ defines a cocycle in $P^{-1}$ with respect to $d_\alpha$, and the cup-product of such functions is their ordinary product. Analogously, the fact that a function $C \in \hat{\text{Fun}}(M)$ is central is equivalent to
the fact that $C$ defines a cocycle in $H^{-1}$ with respect to $\partial_\alpha$, and the cup-product of such functions is their star-product.

Thus, the formula

$$D(C) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{k+1}(h\alpha, \ldots, h\alpha, C)$$

(39)

defines the desired mapping $D$.

It was proved in [2] that it coincides with the map Duflo-Kirillov map in the case of Lie algebras.

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