LENGTH FORMULAS FOR THE HOMOLOGY OF GENERALIZED KOSZUL COMPLEXES

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Abstract

Let $M$ be a finite module over a noetherian ring $R$ with a free resolution of length 1. We consider the generalized Koszul complexes $C_{\lambda}(t)$ associated with a map $\lambda : M \to \mathcal{H}$ into a finite free $R$-module $\mathcal{H}$ (see [IV], section 3), and investigate the homology of $C_{\lambda}(t)$ in the special setup when grade $I_M = \text{rank } M = \dim R$. ($I_M$ is the first non-vanishing Fitting ideal of $M$.) In this case the (interesting) homology of $C_{\lambda}(t)$ has finite length, and we deduce some length formulas. As an application we give a short algebraic proof of an old theorem due to Greuel (see [G], Proposition 2.5). We refer to [HM] where one can find another proof by similar methods.

Introduction

Let $R$ be a noetherian ring and $M$ an $R$-module with a presentation

$$0 \longrightarrow \mathcal{F} \xrightarrow{\chi} \mathcal{G} \longrightarrow M \longrightarrow 0$$

where $\mathcal{F}$, $\mathcal{G}$ are free modules of finite rank. We consider an $R$-homomorphism $\lambda : M \to \mathcal{H}$ into a finite free $R$-module $\mathcal{H}$. With $\lambda$ we associate the generalized
Koszul complexes $C_\lambda(t)$. The main object of our interest is the homology of these complexes in the special setup when grade $I_\chi = \text{rank } M = \text{dim } R$. Here $I_\chi$ denotes the ideal generated by the maximal minors of a matrix representing $\chi$.

In section 1 we give a short survey of generalized Koszul complexes and Koszul bicomplexes. Section 2 contains some results concerning the grade sensitivity of these and some related complexes.

The main part of the paper is concentrated in section 3. In case grade $I_\chi$ has the greatest possible value $\text{rank } M + 1$, Theorem 6.10 in [IV] provides a satisfactory description of the homology of $C_\lambda(t)$ depending on the grade $h$ of $I_\chi$, $\lambda : G \to H$ being the corresponding lifted map. In a sense the situation described above is another extremal case. Here the homology modules $\tilde{H}_i$ of $C_\lambda(t)$ have finite length for $i \leq \min(h - 1, 2 \text{rank } M)$, and they are connected by some length formulas which mainly involve the symmetric powers of the cokernel of $\chi^\ast$ (Theorem 3.1 and Corollary 3.3).

In section 4 we treat the very special situation of a quasi-homogeneous isolated complete intersection singularity. We can extend the length formulas of section 3, in order to give a purely algebraic proof of an old theorem due to Greuel (see [G], 2.5). The formulas and – at least implicitly – an algebraic proof of Greuel’s Theorem are also contained in [HM].

We use [E] for a general reference in commutative algebra. As there we denote by $\bigwedge M$ ($\bigwedge^p M$) the exterior power algebra ($p$th exterior power) of an $R$-module $M$, by $S(M)$ ($S_p(M)$) the symmetric power algebra ($p$th symmetric power) of $M$, and by $D(M)$ ($D_p(M)$) the divided power algebra ($p$th divided power) of $M$. The reader should always have in mind that for a finite free $R$-module $F$ there are canonical isomorphisms $D(F^\ast) \cong S(F)^\ast$ and $D(F) \cong S(F^\ast)^\ast$ which we shall use implicitly several times in the following. Here $S(F)^\ast = \bigoplus S_p(F)^\ast$ is the so called graded dual of $S(F)$.

1 Koszul Complexes and Koszul Bicomplexes

Generalizations of the classical Koszul complex have been known since a long time, e. g. the complexes due to Eagon and Northcott or those introduced by Buchsbaum and Rim. We refer to [E], Chapter A2.6, for a comprehensive treatment. Some related material may also be found in [BV1] and [BV2].

So the following constructions, generalizing the classical Koszul complex and its dual, are not really new. They should be viewed as appropriate components of the Koszul bicomplex, we define at the end of this section and which is the main tool of our investigations.

Let $R$ be a commutative ring and $\psi : G \to F$ a homomorphism of an $R$-module $G$ into a finite free $R$-module $F$. Let $f_1, \ldots, f_m$ be a basis of $F$ and $f_1^\ast, \ldots, f_m^\ast$ the dual basis of $F^\ast$. Then we may consider $\psi$ as an element of $\bigwedge^* G^\ast \otimes S(F)$ with the
presentation \( \psi = \sum_{j=1}^{m} \psi^*(f_j^*) \otimes f_j \). We set

\[
\partial_\psi(y_1 \wedge \ldots \wedge y_n \otimes z) = \sum_j (y_1 \wedge \ldots \wedge y_n \leftarrow \psi^*(f_j^*)) \otimes z \cdot f_j.
\]

for all \( y_i \in G \) and \( z \in S(F)^* \). The right multiplication \( \leftarrow \) of \( \wedge G^* \) on \( \wedge G \) is given by

\[
y_1 \wedge \ldots \wedge y_n \leftarrow y_1^* \wedge \ldots \wedge y_p^* = \sum_\sigma \varepsilon(\sigma) \det_{1 \leq i, j \leq p} (y_j^*(y_{\sigma(i)})) y_{\sigma(p+1)} \wedge \ldots \wedge y_{\sigma(n)}
\]

for \( y_1, \ldots, y_n \in G \) and \( y_1^*, \ldots, y_p^* \in G^* \), where \( \sigma \) runs through the set \( \mathfrak{S}_{n,p} \) of permutations of \( n \) elements which are increasing on the intervals \([1, p]\) and \([p + 1, n]\). The generalized Koszul complexes we have in mind, are the complexes

\[
C_\psi(t) : \ldots \to \wedge^{t+m+p} G \otimes S_p(F)^* \xrightarrow{d_0} \ldots \xrightarrow{d_2} \wedge^{t+m} G \otimes S_0(F)^* \xrightarrow{\nu^\psi} \wedge^t G \otimes S_0(F) \xrightarrow{d_0} \ldots \xrightarrow{d_{n+2}} \wedge^0 G \otimes S_t(F) \to 0;
\]

\( \nu^\psi \) is the right multiplication by \( \psi^*(f_1^*) \wedge \ldots \wedge \psi^*(f_m^*) \in \wedge G^* \).

Similarly we can associate complexes with a map \( \varphi : H \to G \) from a finite free \( R \)-module \( H \) into \( G \), generalizing the dual version of the classical Koszul complex. Let \( h_1, \ldots, h_t \) be a basis of \( H \) and \( h_1^*, \ldots, h_t^* \) the dual basis of \( H^* \). Then \( \varphi \), as an element of \( S(H^*) \otimes \wedge G \), has the presentation \( \varphi = \sum_{j=1}^{t} h_j^* \otimes \varphi(h_j) \). Define \( d_\varphi \) to be the left multiplication by \( \varphi \) on \( D(H) \otimes \wedge G \), i.e.

\[
d_\varphi(x_1^{(k_1)} \ldots x_p^{(k_p)} \otimes y) = \sum_j x_1^{(k_1)} \ldots x_j^{(k_j-1)} \ldots x_p^{(k_p)} \otimes \varphi(x_j) \wedge y
\]

for \( x_i \in H \) and \( y \in \wedge G \), and on \( S(H^*) \otimes \wedge G \) (in an obvious way). We obtain the family of complexes

\[
D_\varphi(t) : 0 \to D_t(H) \otimes \wedge^0 G \xrightarrow{d_0^\varphi} \ldots \xrightarrow{d_2^\varphi} D_0(H) \otimes \wedge^t G \xrightarrow{\nu^\varphi} S_0(H^*) \otimes \wedge^{t+1} G \xrightarrow{d_0^\varphi} \ldots \xrightarrow{d_2^\varphi} S_p(H^*) \otimes \wedge^{t+l} G \to \ldots
\]

where \( \nu^\varphi \) is the left multiplication by \( \varphi(h_1) \wedge \ldots \wedge \varphi(h_m) \in \wedge G \).

Now let \( \psi \circ \varphi = 0 \). Then there is a simple associativity formula (see Proposition 2.1 in [IV]) concerning the right and left multiplications from above, which allows
us to assemble the complexes $C_\psi(t)$ and $D_\varphi(t)$ to the Koszul bicomplexes $K_n(t)$

\[
\begin{array}{cccccccc}
\cdots H \otimes \Lambda^{t+m} G \otimes F^* & \longrightarrow & \Lambda^{t+m+1} G \otimes F^* & \longrightarrow & \Lambda^{t+m+2} G \otimes F^* & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots H \otimes \Lambda^{t+m-1} G & \longrightarrow & \Lambda^{t+m} G & \longrightarrow & \Lambda^{t+m+1} G & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots H \otimes \Lambda^{t-2} G \otimes F & \longrightarrow & \Lambda^{t-1} G \otimes F & \longrightarrow & \Lambda^{t} G \otimes F & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots & & \cdots & & \cdots & & \\
\end{array}
\]

The rows in the upper half arise from $D_\varphi(t+m+j)$ tensored with $S_j(F)^*$, $j = 0, 1, \ldots$, while the rows below are built from $D_\varphi(t-j)$ tensored with $S_j(F)$, $j = 0, 1, \ldots$; we abbreviate $d_\varphi \otimes 1_{S(F)^*}$ and $d_\varphi \otimes 1_{S(F)}$ to $d_\varphi$, and correspondingly $\nu_\varphi \otimes 1_{S(F)^*}$ and $\nu_\varphi \otimes 1_{S(F)}$ to $\nu_\varphi$. The columns are obtained analogously: in western direction we have to tensorize $D_i(H)$ with $C_\psi(t-i)$, $i = 0, 1, \ldots$, while going east we must tensorize $S_i(H^*)$ with $C_\psi(t+l+i)$, $i = 0, 1, \ldots$; as before we shorten the complex maps to $\partial_\psi$ and $\nu_\psi$. The signs of $\nu_\varphi$ and $\nu_\psi$ are determined by the associativity formula.

A detailed and more general treatment of the generalized Koszul complexes from above and of the Koszul bicomplexes just defined may be found in [I].

## 2 Grade Sensitivity

The following contains some preparing material for the main results in the next section. In a sense it is a matter of generalizing part of the considerations of section 5 in [IV].

Our general assumption throughout the rest of this section will be that $R$ is noetherian, that $H$, $G$ and $F$ are free $R$-modules of finite ranks $l$, $n$ and $m$, and that

\[
H \xrightarrow{\varphi} G \xrightarrow{\psi} F
\]

is a complex. Although much of what we will do, holds formally for any $l$, $n$ and $m$, the applications will refer to the case in which $n \geq m$ and $n \geq l$. So we suppose that $r = n - m \geq 0$, $s = n - l \geq 0$. By $I_\psi$ ($I_\varphi$) we denote the ideals in $R$ generated by the maximal minors of a matrix representing $\psi$ ($\varphi$). We set $g = \text{grade } I_\psi$ and $h = \text{grade } I_\varphi$.  

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Since $G$ is finitely generated, the generalized Koszul complexes $C_\psi(t)$ and $D_\varphi(t)$ have only a finite number of non-vanishing components. To identify the homology, we fix their graduations as follows: position 0 is held by the leftmost non-zero module. As $C_\psi(t)$ and $D_\varphi(t)$ are isomorphic to well-known generalizations of the classical Koszul complex (see [IV], section 3), their homology behaves grade sensitively in the sense of the following theorem (see Theorem 5.2 in [IV]).

**THEOREM 2.1.** Set $C = \text{Coker } \psi$ and $D = \text{Coker } \varphi^*$. Furthermore let $S_0(D) = R/I_\varphi$, $S_{-1}(D) = \bigwedge^{s+1} \text{Coker } \varphi$, $S_0(C) = R/I_\psi$ and $S_{-1}(C) = \bigwedge^{r+1} \text{Coker } \psi^*$. Then the following hold.

(a) $H^i(D_\varphi(t)) = 0$ for $i < h$. Moreover, if $t \leq s + 1$ and grade $I_k(\varphi) \geq n - k + 1$ for all $k$ with $l \geq k \geq 1$, then $D_\varphi(t)$ is a free resolution of $S_{s-l}(D)$. (If $-1 \leq t \leq s + 1$, then it suffices to require that grade $I_\varphi \geq s + 1$.)

(b) $H^i(C_\psi(t)) = 0$ for $i < g$. Moreover, if $t \geq -1$ and grade $I_k(\psi) \geq n - k + 1$ for all $k$ with $m \geq k \geq 1$, then $C_\psi(t)$ is a free resolution of $S_t(C)$. (If $-1 \leq t \leq r + 1$, then it suffices to require that grade $I_\psi \geq r + 1$.)

Finally, if $I_\varphi = R$ ($I_\psi = R$), then all sequences $D_\varphi(t)$ ($C_\psi(t)$) are split exact.

By $C_{\psi}(t)$ we shall denote the bicomplex which is the lower part of the Koszul bicomplex $K_{\psi}(t)$ (the rows below the second row in the last diagram of the previous section). In other words,

$$C^{0,0} = C_{0,0}(t) = \begin{cases} D_{1}(H) \otimes \bigwedge^{0} G \otimes S_{0}(F) & \text{if } 0 \leq t, \\ S_{0}(H^*) \otimes \bigwedge^{t+1} G \otimes S_{0}(F) & \text{if } -l \leq t < 0, \\ S_{-t-l}(H^*) \otimes \bigwedge^{0} G \otimes S_{0}(F) & \text{if } t < -l. \end{cases}$$

The row homology at $C^{p,q} = C_{p,q}(t)$ is denoted by $H^{p,q}_\psi$, the column homology by $H^{p,q}_\varphi$. Thus $H^{p,0}_\varphi$ is the $p$-th homology module of $D_\varphi(t)$.

Set $N^p = \text{Ker } (C^{p,0} \xrightarrow{d_\psi} C^{p,1})$. The canonical injections $N^p \rightarrow C^{p,0}$ yield a complex morphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^0 & \longrightarrow & N^1 & \longrightarrow & \cdots & N^p & \xrightarrow{d_\psi} & N^{p+1} & \cdots \\
\| & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^{0,0} & \longrightarrow & C^{1,0} & \longrightarrow & \cdots & C^{p,0} & \xrightarrow{d_\psi} & C^{p+1,0} & \cdots \\
\end{array}$$

where the maps $d_\varphi$ are induced by $d_\psi$. The homology of the first row $N(t)$ at $N^p$ is denoted by $H^p$. We shall now investigate this homology under the assumption $r \leq g$.

For this purpose we extend $C_{\psi}(t)$ by $N(t)$ to the complex $\tilde{C}_{\psi}(t)$. So $C^{p,-1} = \tilde{C}_{p-1}(t) = N^p$. We record some facts about the homology of $\tilde{C}_{\psi}(t)$. To avoid
new symbols, the column homology at $C^{p,q}$ is again denoted by $H_{\psi}^{p,q}$ (actually it differs from that of $C_\cdot(t)$ only at $C^{p,0}$). By construction $H_{\psi}^{p,q} = 0$ for $q = -1, 0$. Furthermore we draw from Theorem 2.1 (a) that
\[ H_{\psi}^{p,q} = 0 \quad \text{for } p < h \text{ and } q \neq -1. \]  
Let $r \leq g$. Then in case $0 < p \leq t$, we get from Theorem 2.1 (b) that
\[ H_{\psi}^{p,q} = \begin{cases} 0 & \text{for } q < \min(p - 1, r), \\ D_{t-p}(H) \otimes S_p(C) & \text{for } q = p, \end{cases} \]  
and if $0 \leq t < p$, we obtain
\[ H_{\psi}^{p,q} = \begin{cases} 0 & \text{for } q < \min(p + l - 2, r), \\ S_{p-t-1}(H^*) \otimes S_{p+l-1}(C) & \text{for } q = p + l - 1, \end{cases} \]  
where $C = \text{Coker } \psi$ as in Theorem 2.1.

For $q \geq -1$ we consider the $q$th row $\tilde{C}_{\cdot,q}$ of $\tilde{C}_{\cdot}$ and its image complex $\partial_{\psi}(\tilde{C}_{\cdot,q})$ in $\tilde{C}_{\cdot,q+1}$. We set $E_{\psi}^{q,p} = H^p(\partial_{\psi}(\tilde{C}_{\cdot,q-1}))$ for $q \geq 0$. There are exact sequences
\[ H_{\psi}^{i-(j+1),j} \rightarrow E_{\psi}^{i-(j+1),j+1} \rightarrow E_{\psi}^{i-j,j} \rightarrow H_{\psi}^{i-j,j} \]  
if $H_{\psi}^{i-(j+1),j} = H_{\psi}^{i-j,j} = 0$, and because of (2) and (3) this holds if
\[ 0 < i - (j + 1) \leq t \text{ and } j < \min(i - (j + 1) - 1, r) \text{ or} \]
\[ 0 \leq t < i - (j + 1) \text{ and } j < \min(i - (j + 1) + l - 2, r). \]

THEOREM 2.2. Let $t \geq 0$ be an integer. Assume that $1 \leq r \leq g$. Then, with the notation introduced above, $H^i = 0$ for $i = 0, \ldots, \min(2, h-1)$. Set $C = \text{Coker } \psi$.

(a) For $i$ odd, $3 \leq i < \min(h-1, 2r, 2t + 2)$, one has a natural exact sequence
\[ 0 \rightarrow H^i \rightarrow D_{t-i-1}(H) \otimes S_{i-1}(C) \rightarrow H_{\psi}^{i+1,i-1} \rightarrow H^{i+1} \rightarrow 0. \]

(b) Suppose that $l > 1$.

(i) If $3 \leq 2t + 1 < h$, then $H^{2t+1} = D_0(H) \otimes S_t(C)$.

(ii) $H^i = 0$ for $2t + 2 \leq i < \min(h, 2t + l + 1)$.

(iii) If $2t + l + 1 < h$, then $H^{2t+l+1} = H_{\psi}^{i+1,i+l-1}$.

(c) For $i - l$ even, $2t + l + 2 \leq i < \min(h-1, 2r - l + 2)$, one has a natural exact sequence
\[ 0 \rightarrow H^i \rightarrow S_{i-1}(H^*) \otimes S_{i-1}(C) \rightarrow H_{\psi}^{i+1,i-l-1} \rightarrow H^{i+1} \rightarrow 0. \]
Proof. For the proof of the first statement we refer to [IV], Theorem 5.3. Let \( i \) be an odd integer, \( 3 \leq i < \min(h-1, 2r, 2t+2) \). Using (4), we obtain the “southwest” isomorphisms

\[
\tilde{H}^i = E^{i,0} \cong E^{i-1,1} \cong \cdots \cong E^{i+1,1} \cong E^{i+1,0} \cong E^{i,1} \cong \cdots \cong E^{i+3,1} \cong E^{i,2}.
\]

We abbreviate \( \partial_{\psi}^{p,q} = (C^{p,q} \xrightarrow{\partial} C^{p,q+1}) \) and \( \partial_{\psi}^{p,q} = (C^{p,q} \xrightarrow{d_{\psi}} C^{p+1,q}) \). The diagram

\[
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & \\
\to \text{Im} \partial_{\psi}^{i-1,1} & \to \text{Im} \partial_{\psi}^{i+1,1} & \to \text{Im} \partial_{\psi}^{i+3,1} & \to \text{Im} \partial_{\psi}^{i+5,1} & 0 \\
\downarrow & & & & \\
\to \text{Im} \partial_{\psi}^{i,1} & \to \text{Ker} \partial_{\psi}^{i,1} & \to \text{Ker} \partial_{\psi}^{i+3,1} & \to \text{Ker} \partial_{\psi}^{i+5,1} & 0 \\
\downarrow & & & & \\
0 & \to H_{\psi}^{i+1,1} & \to 0 & \to 0 & 0 \\
\end{array}
\]

is induced by \( C_{\psi}(t) \) and has exact columns. Its row homology at \( \text{Im} \partial_{\psi}^{i+1,1} \) is \( E^{i+1,1} = \tilde{H}^i \), and at \( \text{Ker} \partial_{\psi}^{i+1,1} \) it coincides with \( D_{t-\frac{i+1}{2}}(H) \otimes S_{\frac{i+1}{2}}(C) \) as the following diagram with exact columns and exact middle row shows (we write \( D_{t-\frac{i-1}{2}} \) for \( D_{t-\frac{i+1}{2}}(H) \) and \( S_{\frac{i-1}{2}} \) for \( S_{\frac{i+1}{2}}(C) \)):

\[
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & \\
\text{Im} \partial_{\psi}^{i+1,1} & \to \text{Ker} \partial_{\psi}^{i+1,1} & \to \text{Ker} \partial_{\psi}^{i+3,1} & \to \text{Ker} \partial_{\psi}^{i+5,1} & \text{Im} \partial_{\psi}^{i+1,1} \\
\downarrow & & & & \\
0 & \to C^{i+1,1} & \to C^{i+3,1} & \to C^{i+5,1} & 0 \\
\downarrow & & & & \\
0 & \to D_{t-\frac{i-1}{2}} \otimes S_{\frac{i-1}{2}} & \to \text{Im} \partial_{\psi}^{i+1,1} & \xrightarrow{d_{\psi}} \text{Im} \partial_{\psi}^{i+3,1} & \text{Im} \partial_{\psi}^{i+1,1} \\
\downarrow & & & & \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

In this diagram the row homology at \( \text{Im} \partial_{\psi}^{i+1,1} \) vanishes since \( d_{\psi}^{i+1,1} \) is injective. So in the preceding diagram the row homology at \( \text{Ker} \partial_{\psi}^{i+1,1} \) is zero.
Altogether we obtain an exact sequence

$$0 \to \bar{H}^i \to D_{t - \frac{i}{2}}(H) \otimes S_{\frac{i-t}{2}}(C) \to H_{\psi}^{\frac{i-t+1}{2}, \frac{i-t+1}{2}} \to E_{\frac{i-t+1}{2}, \frac{i-t+1}{2}} \to 0.$$  

(*)

Since

$$\bar{H}^{i+1} \cong E_{\frac{i+1}{2}, \frac{i+1}{2}},$$

(a) has been proved.

For (b) we note that \( l > 1 \) implies \( g \geq r > r - l + 1 \) so we may use [IV], Theorem 5.3.

The proof of (c) is similar to the proof of (a). We just mention that, for all \( i \) under consideration, we have the "southwest" isomorphisms

$$\bar{H}^i = E^{i,0} \cong E^{i-1,1} \cong \ldots \cong E^{\frac{i-t}{2}+1, \frac{i-t}{2}-1}, \quad \bar{H}^{i+1} = E^{i+1,0} \cong E^{i,1} \cong \ldots \cong E^{\frac{i-t}{2}+2, \frac{i-t}{2}-1}.$$

If \( t \) is a negative integer, we obtain a similar result. We indicate it without the proof which is completely analogous with the proof of the previous theorem.

**Theorem 2.3.** Let \( t < 0 \) be an integer. Assume that \( 1 \leq r \leq g \), and use the notation from above.

(a) Suppose that \( t + l > 0 \) (this implies \( l > 1 \)). Then

(i) \( \bar{H}^i = 0 \) for \( 0 \leq i < \min(h, \max(2, t + l)) \);

(ii) if \( 2 \leq t + l < h \), then \( \bar{H}^{t+l} = H^{0,t-l-1}_\psi \);

(iii) if \( i - t - l \) is odd, and \( t + l + 1 \leq i < h - 1 \), then one has a natural exact sequence

$$0 \to \bar{H}^i \to S_{\frac{i-t-l}{2}}(H^*) \otimes S_{\frac{i-t-l}{2}}(C) \to H^{\frac{i-t-l+1}{2}, \frac{i-t-l+1}{2}}_\psi \to \bar{H}^{i+1} \to 0.$$

(b) Suppose that \( t + l \leq 0 \). Then \( \bar{H}^i = 0 \) for \( i = 0, \ldots, \min(2, h - 1) \). For \( i \) odd, \( 3 \leq i < \min(h - 1, 2r) \), one has a natural exact sequence

$$0 \to \bar{H}^i \to S_{\frac{i-t-l}{2}}(H^*) \otimes S_{\frac{i-t-l}{2}}(C) \to H^{\frac{i-t-l+1}{2}, \frac{i-t-l+1}{2}}_\psi \to \bar{H}^{i+1} \to 0.$$

We supply Theorem 2.2 by some simple results concerning the homology of \( \mathcal{N}(t) \) at \( N^h \).

**Proposition 2.4.** As in Theorem 2.2 assume that \( 1 \leq r \leq g \) and set \( C = \text{Cok } \psi \). Let \( \mu = \min(h, 2r + 1) \) and \( t \geq \frac{\mu}{2} - 1 \).
(a) There is an exact sequence
\[ 0 \to E_{\mu-1}^{\mu-1,1} \to \bar{H}_{\mu} \to H_{\mu}^{\mu,0}, \]
\[ \text{in particular, if } \mu < 3, \text{ then there is an exact sequence} \]
\[ 0 \to \bar{H}_{\mu} \to H_{\mu}^{\mu,0}. \]

(b) For \( \mu \geq 3 \) odd, there is an exact sequence
\[ 0 \to E_{\mu-1}^{\mu-1,1} \to D_{t - \frac{\mu-1}{2}}(H) \otimes S_{\frac{\mu-1}{2}}(C) \to H_{\psi}^{\mu-1, \frac{\mu-1}{2}} \to \bar{H}_{\mu} \to H_{\mu}^{\mu,0}. \]

and

(c) for \( \mu \geq 3 \) even, there is an exact sequence
\[ 0 \to \bar{H}_{\mu-1} \to D_{t - \frac{\mu-2}{2}}(H) \otimes S_{\frac{\mu-2}{2}}(C) \to H_{\psi}^{\mu-1, \frac{\mu-2}{2}} \to \bar{H}_{\mu} \to H_{\mu}^{\mu,0}. \]

REMARK 2.5. One may easily deduce similar sequences in case \( t < \frac{\mu}{2} - 1 \).

Proof. (a) This follows immediately from (4) and the fact that \( H_{\psi}^{p,0} = H_{\psi}^{p,1} = 0 \) for all \( p \).

(b) In order to cover the case \( \mu = 2r + 1 \), we modify the first diagram in the proof of Theorem 2.2, the diagram
\[
\begin{array}{cccccc}
0 & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\to & \to & \to & \to & \to & 0 \\
\end{array}
\]

has exact columns. Then, as in the proof of Theorem 2.2, we obtain an exact sequence
\[ 0 \to E_{\mu}^{\mu-1,1} \to D_{t - \frac{\mu-1}{2}}(H) \otimes S_{\frac{\mu-1}{2}}(C) \to \text{Ker}(H_{\psi}^{\mu,0}) \to \bar{H}_{\mu} \to H_{\mu}^{\mu,0}. \]
Since $E^{\mu+1, \mu-1} \cong E^{\mu-1,1}$ in this case, we are done.

(c) Set $i = \mu - 1$ and use the sequence $(\ast)$ in the proof of Theorem 2.2 to get the exact sequence

$$0 \to \bar{H}^{\mu-1} \to D_{t, \mu-2}(H) \otimes S_{\mu-2}(C) \to \bar{H}^{\mu-1}_{\psi} \to E^{\mu-2, \mu-2}_{\mu-2} \to 0.$$  

Since $E^{\mu+2, \mu-2} \cong E^{\mu-1,1}$, we can glue this sequence and the sequence obtained under (a) to get the result.  

\[\square\]

### 3 Length Formulas

In this section $R$ is a noetherian ring, and $M$ an $R$-module which has a presentation

$$0 \longrightarrow F \xrightarrow{\chi} G \longrightarrow M \longrightarrow 0$$

where $F$, $G$ are free modules of ranks $m$ and $n$. Then in particular $r = n - m \geq 0$.

Let $\lambda : M \to \mathcal{H}$ be an $R$-homomorphism into a finite free $R$-module $\mathcal{H}$ of rank $l \leq n$. By $\lambda : \mathcal{G} \to \mathcal{H}$ we denote the corresponding lifted map. In case grade $I_\chi$ has the greatest possible value $r + 1$, Theorem 6.10 in [IV] provides a comparably satisfactory description of the homology of the generalized Koszul complex $C_\lambda(t)$

$$0 \to \wedge^n M \otimes S_p(\mathcal{H})^{*} \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_3} \wedge^{t+1} M \xrightarrow{\nu} \wedge^t M \xrightarrow{\partial_3} \cdots \xrightarrow{\partial_3} S_l(\mathcal{H}) \to 0,$$

depending on the grade of $I_\lambda$.

In the following we deal with another extremal case. We assume that grade $I_\chi = \dim R$. Then $R/I_\chi$ has finite length $\ell(R/I_\chi)$. (Generally the length of an $R$-module $N$ is denoted by $\ell(N)$).

First we dualize the sequence $\mathcal{F} \xrightarrow{\chi} \mathcal{G} \xrightarrow{\lambda} \mathcal{H}$. Then we set $F = F^*$, $G = G^*$, $H = H^*$ and $\psi = \chi^*$, $\varphi = \lambda^*$, to attain the setup which we studied in the previous sections.

We recall to some notation and facts from section 6 in [IV]. As there we consider the upper part of the Koszul bicomplex $\mathcal{K}_{\chi}(t)$ which we rewrite as

\[\begin{array}{cccccccc}
\vdots & \vdots & \vdots \\
\cdots \longrightarrow B_{t-1}^{0,-1} \longrightarrow \cdots \longrightarrow B_{t-1}^{t-1} \longrightarrow & B_{t}^{t-1} \longrightarrow & B_{t}^{t+1,-1} \longrightarrow & \cdots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots \longrightarrow B_{t}^{0,0} \longrightarrow \cdots \longrightarrow B_{t}^{t,0} \longrightarrow & B_{t}^{t+1,0} \longrightarrow & \cdots \\
\end{array}\]
where
\[ B_{t_i}^{0,0} = \begin{cases} 
D_t(H) \otimes \wedge^m G \otimes S_0(F)^* & \text{if } 0 \leq t, \\
S_0(H^*) \otimes \wedge^{l+i+m} G \otimes S_0(F)^* & \text{if } -l \leq t < 0, \\
S_{-t-l}(H^*) \otimes \wedge^m G \otimes S_0(F)^* & \text{if } t < -l.
\]

Set \( M^p = \text{Coker}(B_{t_i}^{p-1} \stackrel{\partial^p}{\to} B_{t_i}^{p,0}) \). The canonical surjection \( B_{t_i}^{p,0} \to M^p \) yields a complex morphism
\[
\cdots \longrightarrow B_{t_i}^{-1,0} \longrightarrow B_{t_i}^{0,0} \longrightarrow \cdots \xrightarrow{d^p} B_{t_i}^{p,0} \longrightarrow B_{t_i}^{p+1,0} \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow \cdots \xrightarrow{d^p} M^p \longrightarrow M^{p+1} \longrightarrow \cdots
\]
where the maps \( d^p \) are induced by \( d^p \). The lower row is denoted by \( \mathcal{M}(t) \). By [IV], Proposition 6.8, there is an isomorphism \( \mathcal{M}(\rho - t) \to \mathcal{C}_\lambda(t) \) whose inverse composed with the induced complex map \( \mathcal{M}(\rho - t) \to \mathcal{N}(\rho - t) \) yields a complex map \( \mu : \mathcal{C}_\lambda(t) \to \mathcal{N}(\rho - t) \).

**Theorem 3.1.** Let \( M, \overline{\lambda}, \text{ and } \lambda \) be as above. Set \( \rho = r - l, \ g = \text{grade } I_\chi, \ h = \text{grade } I_\lambda. \) Suppose that \( g = \dim R = r. \) Equip \( \mathcal{C}_\lambda(t) \) with the graduation induced by the complex morphism \( \mu : \mathcal{C}_\lambda(t) \to \mathcal{N}(\rho - t) \). Then the homology modules \( \tilde{H}^i \) of \( \mathcal{C}_\lambda(t) \) have finite length for \( i \leq \min(h - 1, 2r). \)

Set \( C = \text{Coker } \chi^*, \ S_0(C) = R/I_\chi, \) and assume that \( h > 0. \)

(a) Let \( l = 1. \) Then for all \( t \in \mathbb{Z} \) and \( i \) odd, \( 0 < i < \min(h - 1, 2r), \)
\[ \ell(\tilde{H}^i) - \ell(\tilde{H}^{i+1}) = \ell(S_{i+1}(C)) - \ell(S_{i+1}(C)). \]

(b) Let \( l > 1. \) We distinguish four cases.

(i) For all \( t \leq \frac{\rho}{2} \) and \( i \) odd, \( 0 < i < h - 1, \)
\[ \ell(\tilde{H}^i) - \ell(\tilde{H}^{i+1}) = \left( \frac{r - t - \frac{i+1}{2}}{l - 1} \right) \ell(S_{i+1}(C)) - \left( \frac{r - t - \frac{i+3}{2}}{l - 1} \right) \ell(S_{i+1}(C)). \]

(ii) Suppose that \( \frac{\rho}{2} < t \leq \rho. \) If \( i \) is odd, \( 0 < i < \min(h - 1, 2(\rho - t)), \) then
\[ \ell(\tilde{H}^i) - \ell(\tilde{H}^{i+1}) = \left( \frac{r - t - \frac{i+1}{2}}{l - 1} \right) \ell(S_{i+1}(C)) - \left( \frac{r - t - \frac{i+3}{2}}{l - 1} \right) \ell(S_{i+1}(C)). \]

If \( i - l \) is even, \( 2(\rho - t) + l + 2 \leq i < h - 1, \) then
\[ \ell(\tilde{H}^i) - \ell(\tilde{H}^{i+1}) = \left( \frac{i+l}{2} - \rho + t - 2 \right) \ell(S_{\frac{i+l+1}{2}-1}(C)) - \left( \frac{i+l}{2} - \rho + t - 1 \right) \ell(S_{\frac{i+l+1}{2}}(C)). \]
If $2(\rho - t) + l + 1 < h$, then $\ell(\widetilde{H}^{2(\rho - t) + l + 1}) = \ell(S_{r-t}(C))$. Moreover

$$\widetilde{H}^i = \begin{cases} S_{\rho-t}(C) & \text{if } i = 2(\rho - t) + 1 < h, \\ 0 & \text{if } 2(\rho - t) + 2 \leq i < \min(h, 2(\rho - t) + l + 1). \end{cases}$$

(iii) Suppose that $\rho < t < r$. If $i + r - t$ is odd, $r - t + 1 \leq i < h - 1$, then

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \left(\frac{i-r+t-3}{2} + l\right)\ell(S_{\frac{i+r-t-1}{2}}(C)) - \left(\frac{i-r+t-1}{2} + l\right)\ell(S_{\frac{i+r-t+1}{2}}(C)).$$

If $r - t < h$, then $\ell(\widetilde{H}^{r-t}) = \ell(S_r(t-C))$. Moreover $\widetilde{H}^i = 0$ if $0 \leq i < \min(h, r-t)$.

(iv) Suppose that $r \leq t$ and $i$ odd, $0 < i < h - 1$. Then

$$\ell(\widetilde{H}^i) - \ell(\widetilde{H}^{i+1}) = \left(t - \rho + \frac{i-3}{2}\right)\ell(S_{\frac{i}{2}}(C)) - \left(t - \rho + \frac{i-1}{2}\right)\ell(S_{\frac{i+1}{2}}(C)).$$

REMARK 3.2. Observe that in the above formulas we use the fact that $h$ can reach its maximal value only if it is even. More precisely, if $l \geq 2$ and $g = h = r$, then $l = 2$ and $r, h, g$ are even (see Corollary 6.2 in [IV]). In this case $i$ odd, $i < h - 1$, means $i \leq r - 3$.

We further notice that for $h < \infty$ the formulas under (b) cover the case in which $l = 1$. We specified them for the readers convenience since we shall apply the $l = 1$ case in section 4.

Proof. If $r = 0$, then $M = 0$ since $\chi$ is injective. So we may assume that $r \geq 1$.

The graduation induced by $\mathcal{N}(\rho - t)$ on $C_\chi(t)$ is completely determined by

$$C_\chi^q(t) = \begin{cases} \Lambda^t M \otimes S_{\rho-t}(\mathcal{H})^* & \text{if } t \leq \rho, \\ \Lambda^t M \otimes S_0(\mathcal{H}) & \text{if } \rho < t < r, \\ \Lambda^t M \otimes S_{t-r}(\mathcal{H}) & \text{if } r \leq t. \end{cases}$$

For $q > r$ the support of $\Lambda^q M$ is contained in the variety of $I_\chi$. Consequently $C_\chi^q(t)$ has finite length if $i < 0$, which in turn implies that $\widetilde{H}^i$ has finite length. In particular, there remains nothing to prove if $h = 0$. Let $h > 0$. Then $\rho \geq 0$ by Proposition 5.1. in [IV].

By [IV], Proposition 6.9, $\mu : C_\chi(t) \to \mathcal{N}(\rho - t)$ induces the following commutative
For arbitrary $t$ and arbitrary $i$ the maps $\mu_i$ are isomorphisms at all prime ideals which do not contain $I_\chi$. Consequently Ker $\mu_i$ and Coker $\mu_i$ have finite length. In particular Ker $\mu_0$ equals the torsion submodule of $C^0_\lambda$ since $N^0$ is free. On the other hand, $C^1_\lambda$ is a torsion free module. So the torsion submodule of $C^0_\lambda$ is contained in Ker $\partial_0$. If we denote by $\bar{\mu}_0$ and $\bar{\partial}_0$ the maps induced by $\mu_0$ and $\partial_0$ on $C^0_\lambda / \text{Ker} \mu_0$, we get the following commutative diagram

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
C^0_\lambda / \text{Ker} \mu_0 & C^1_\lambda & C^2_\lambda & C^3_\lambda \\
\bar{\mu}_0 & \bar{\partial}_0 & \mu_1 & \mu_2 & \mu_3 \\
0 & N^0 & N^1 & N^2 & N^3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Coker} \mu_0 & \text{Coker} \mu_1 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

with exact columns.

Since $h > 0$, $\bar{\partial}_0$ is injective. Then $\bar{\partial}_0$ must be injective, which in turn implies that Ker $\mu_0 = \text{Ker} \bar{\partial}_0$. As $\bar{H}^0$ is a factor of Ker $\bar{\partial}_0$, we deduce that $\bar{H}^0$ has finite length.
In case $\rho < t < r$, $\mu_0$ is injective by Proposition 6.9,(3) in [IV]. So $\partial_0^\rho$ is injective and $\tilde{H}^0 = 0$. In particular, we proved the only statement for $h = 1$.

Assume that $h \geq 2$. If $h = \infty$, then $l = 1$ (by Theorem 6.1 in [IV]), and we refer to the next paragraph of the proof. So let $\infty > h \geq 2$. Then, in particular, $r \geq 2$. The row homologies at $N^0$ and $N^1$ vanish (see Theorems 2.2 and 2.3). Of course, $\tilde{H}^1 = \ker \alpha$ has finite length. We focus on the last statements under (ii) and (iii) about $\tilde{H}^1$. Since alternatively $t = \rho$, $t = r - 1$, $\rho < t \leq r - 2$, another application of Proposition 6.9,(3) in [IV] shows that $\coker \mu_1 = 0$ in all these cases. So

$$
\tilde{H}^1 = \coker \mu_0 = \begin{cases} 
S_0(C) & \text{if } t = \rho, \\
H^r(C_\psi(1)) & \text{if } t = r - 1, \\
0 & \text{if } \rho < t \leq r - 2.
\end{cases}
$$

By Proposition 2.3 in [BV1] we have $\ell(H^r(C_\psi(1))) = \ell(S_1(C))$. With that we proved all claims for $h = 2$.

Now suppose that $h \geq 3$. If $l = 1$, then the row homologies at $N^0$, $N^1$ and $N^2$ vanish (see Theorems 2.2 and 2.3), and we obtain

$$
\ell(\tilde{H}^1) - \ell(\tilde{H}^2) = \ell(\ker \alpha) - \ell(\coker \alpha) = \ell(\coker \mu_0) - \ell(\coker \mu_1).
$$

But $\coker \mu_0 = S_0(C)$ and $\coker \mu_1 = H^r(C_\psi(1))$, so

$$
\ell(\tilde{H}^1) - \ell(\tilde{H}^2) = \ell(S_0(C)) - \ell(H^r(C_\psi(1))).
$$

If $i$ is odd, $3 \leq i < h - 1$, then we deduce directly from Theorem 2.2(a) and (c) and from Theorem 2.3(b) that

$$
\ell(\tilde{H}^i) - \ell(\tilde{H}^{i+1}) = \ell(S_{i+1}(C)) - \ell(H^r(C_\psi(\frac{i+1}{2}))).
$$

Proposition 2.3 in [BV1] implies that $\ell(H^r(C_\psi(k))) = \ell(S_k(C))$ whenever $0 \leq k \leq r$. It remains to prove that $\tilde{H}^{h-1}$ has finite length if $h$ is even. But Proposition 2.4(c) provides an injection of $\tilde{H}^{h-1}$ into a module of finite length. So we settled the case in which $l = 1$.

Let $(h \geq 3$ and $l > 1$. Only the case in which $\rho \leq t < r$, deserves special attention. The other cases are similar to the case $l = 1$. We computed already $\tilde{H}^0$ and $\tilde{H}^1$. If $\rho \leq t < r - 2$ or $\rho = t = r - 2$, then the row homologies at $N^0$, $N^1$, $N^2$ vanish as $\coker \mu_1$ does. So $\tilde{H}^2 = \coker \mu_1 = 0$ in these cases. It remains to show that $\ell(\tilde{H}^2) = \ell(S_2(C))$ if $\rho < t = r - 2$. Of course, the row homologies at $N^0$ and $N^1$ vanish. Since $\coker \mu_1 = 0$, we get that $\tilde{H}^2$ equals the row homology at $N^2$ which is $H^0_\psi(-l + 1)$ (see Theorem 2.3(a) (ii)). By Proposition 2.3 in [BV1] we obtain the desired length equality. The remaining claims follow easily if one uses as pattern the proof for the $l = 1$ case. 

\[ \square \]
Let $\lambda$ be as in the theorem, and denote by $\tilde{C}_\lambda(t)$ the complex obtained from $C_\lambda(t)$ by replacing $C_i(t)$ with 0 whenever $i < 0$. As for $C_\lambda(t)$ let $\tilde{H}^k$ denote the homology of $\tilde{C}_\lambda(t)$ at $\tilde{C}_k(t)$. Actually it differs from the corresponding homology in $C_\lambda(t)$ only for $k \leq 0$.

**Corollary 3.3.** We adopt the assumptions and the notation of the first paragraph in Theorem 3.1. As there we set $C = \text{Coker } \chi$ and $S_0(C) = R/I_\chi$.

(a) If $h = \infty$, then
\[ \ell(S_0(C)) = \ell(S_1(C)) = \ldots = \ell(S_r(C)). \]

(b) If $h$ is odd and $t \leq \frac{\rho}{2}$, then
\[ \sum_{k=0}^{h-1} (-1)^k \ell(\tilde{H}^k) = \left( r - t - \frac{h+1}{2} \right) \ell(S_{\frac{h-1}{2}}(C)). \]

(c) If $h$ is even and $t \leq \frac{\rho}{2}$, then
\[ \sum_{k=0}^{h-2} (-1)^k \ell(\tilde{H}^k) = \left( r - t - \frac{h}{2} \right) \ell(S_{\frac{h-2}{2}}(C)). \]

In case $t > \frac{\rho}{2}$ one can easily deduce formulas similar to (b) and (c).

**Proof.** (a) If $h = \infty$, then $l$ must be 1 (see Theorem 6.1 in [IV]). We notice that this result is also an easy consequence of Proposition 2.8 in [BV1].

(b) and (c) We may obviously suppose that $h > 0$. So $\rho \geq 0$. We have to prove that
\[ \ell(\tilde{H}^0(t)) = \ell(D_{\rho-t}(H) \otimes S_0(C)) \]
if $t \leq \frac{\rho}{2}$. From the proof of Theorem 3.1 we deduce that
\[ \tilde{H}^0(t) \cong D_{\rho-t}(H) \otimes H^r(C_\psi(0)), \]
and from [BV1], Proposition 2.3 we draw that
\[ \ell(H^r(C_\psi(0))) = \ell(S_0(C)). \]
4 An application to quasi-homogeneous icis

In this final section we extend the formulas of the previous section in a very special case. Following the line of argumentation in [BV1] we shall give a purely algebraic proof of an old theorem due to Greuel (see [G], Proposition 2.5). Though the length formulas and - at least implicitly - the proof of Greuel’s Theorem are contained in [HM], we present our considerations as a byproduct of a more general approach.

We specialize to the case in which $R$ is a quasi-homogeneous complete intersection with isolated singularity. More precisely, we let $S = k[[X_1, \ldots, X_n]]$ where $k$ is a field of characteristic zero, assign positive degrees $a_i$ to the variables $X_i$, and set $R = S/(p_1, \ldots, p_m) = k[[x_1, \ldots, x_n]]$ where the $p_i \in (X_1, \ldots, X_n)^2$ form a regular sequence of homogeneous polynomials of degrees $b_i$. By the Euler formula

$$b_j p_j = \sum_{i=1}^{n} a_i \frac{\partial p_j}{\partial X_i} X_i.$$  

Since $\sum_{j=1}^{m} S p_j = \sum_{j=1}^{m} S (b_j p_j)$, the $b_j p_j$ may be viewed as defining elements for $R$. If we set $p_j' = b_j p_j$ and $X_i' = a_i X_i$, we get

$$p_j' = \sum_{i=1}^{n} \frac{\partial p_j}{\partial X_i} X_i'.$$

We suppose $m < n$ and $R_p$ to be regular for all prime ideals $p$ different from the maximal ideal. As usual we denote by $\Omega_{R/k}$ the module of Kähler-differentials of $R$ over $k$. There is a presentation

$$0 \rightarrow F \xrightarrow{\chi} G \rightarrow \Omega_{R/k} \rightarrow 0$$

where $F$, $G$ are free $R$-modules of ranks $m$, $n$ and grade $I_\chi = r$. Moreover the Euler derivation $\bar{\lambda}$ gives rise to an exact sequence

$$\bigwedge^r \Omega_{R/k} \rightarrow \bigwedge^{r-1} \Omega_{R/k} \rightarrow \cdots \rightarrow \Omega_{R/k} \xrightarrow{\bar{\lambda}} R \rightarrow k \rightarrow 0$$

which is in fact the non-negative grade part of $C_\chi$. Let $\lambda : G \rightarrow R$ be the corresponding lifted map. Set $\varphi = \lambda^*, \psi = \chi^*$ as above. As in the proof of Theorem 3.1 we can complement (**) to an exact sequence

$$0 \rightarrow \tau(\bigwedge^r \Omega_{R/k}) \rightarrow \bigwedge^r \Omega_{R/k} \rightarrow \bigwedge^{r-1} \Omega_{R/k} \rightarrow \cdots \rightarrow \Omega_{R/k} \rightarrow R \rightarrow k \rightarrow 0$$

where $\tau$ denotes the torsion submodule.

THEOREM 4.1. Set $C = \text{Coker } \psi$ and $S_0(C) = R/I_\chi$. If $0 \leq i \leq r - 1$, then

$$H^i(C_\psi(i + 1)) \cong S_i(C).$$
Proof. We choose bases $g_1, \ldots, g_n$ for $G$ and $f_1, \ldots, f_m$ for $F$ such that $\psi$ is represented by the matrix $(\frac{\partial p_i}{\partial x_j})_{i,j}$, while $\varphi$ is represented by $(x'_1, \ldots, x'_n)$ (we denote by $\frac{\partial p_i}{\partial x_j}$ the image in $R$ of $\frac{\partial p_i}{\partial x_j}$ and by $x'_i$ the image of $X'_i$). Then we associate the bicomplex $K_{\ast}(0)$ with the sequence $R \xrightarrow{\varphi} G \xrightarrow{\psi} F$ (see section 1). The commutative diagram

\[
\begin{array}{ccccccccc}
\vdots & \longrightarrow & \wedge^{n-r} G & \longrightarrow & \cdots & \longrightarrow & d_\varphi & \wedge^n G & \cong R & \longrightarrow & 0 \\
\pm \nu_0 \downarrow & & \pm \nu_r \downarrow & & & & \pm \nu_r \downarrow & & \pm \nu_r \downarrow & \\
0 & \longrightarrow & \wedge^0 G & \cong R & \longrightarrow & \cdots & \longrightarrow & d_\varphi & \wedge^r G & \longrightarrow & \cdots
\end{array}
\]

is the ‘middle’ part of this bicomplex. First we prove the following

Claim (1): $\nu_r(g_1 \wedge \ldots \wedge g_n)$ generates the homology in the second row at $\wedge^r G$.

(For the original proof see the second part of the proof of Theorem 3.1 in [HM]). We have

\[
\nu_r(g_1 \wedge \ldots \wedge g_n) = g_1 \wedge \ldots \wedge g_n \leftarrow \psi^*(f^*_1) \wedge \ldots \wedge \psi^*(f^*_m)
\]

where $\psi$ runs through the set of permutations of $n$ elements which are increasing on the intervals $[1,m]$ and $[m+1,n]$ (see [IV], section 2).

On the other hand there is a (non-canonical) complex isomorphism

\[
\begin{array}{cccccc}
0 & \longrightarrow & \wedge^0 G & \longrightarrow & \cdots & \longrightarrow & d_\varphi & \wedge^n G & \longrightarrow & 0 \\
\pm \Omega_0 \downarrow & & \pm \Omega_n \downarrow & & \pm \Omega_n \downarrow & & \pm \Omega_n \downarrow & \\
0 & \longrightarrow & \wedge^n G & \longrightarrow & \cdots & \longrightarrow & \partial_\lambda & \wedge^0 G & \longrightarrow & 0
\end{array}
\]

induced by the isomorphism $\Omega_n : \wedge^n G \to R$, $\Omega(g_1 \wedge \ldots \wedge g_n) = 1$. The lower row is the Koszul complex associated with $\lambda$. If we denote by $H_i(R)$ the row homology at $\wedge^i G$, then $H_m(R) \cong \wedge^m H_1(R)$ by a theorem of Tate and Assmus (see Theorem 2.3.11 in [BH]). The relations $(\ast)$ imply that $H_1(R)$ is generated by the homology classes of the cycles $\psi^*(f^*_j)$, $j = 1, \ldots, m$ (see Chapter 2.3 in [BH]), so $H_m(R)$ is generated by $\psi^*(f^*_1) \wedge \ldots \wedge \psi^*(f^*_m)$. An easy computation shows that

\[
\Omega_r(\nu_r(g_1 \wedge \ldots \wedge g_n)) = \pm \sum_{1 \leq i, j \leq m} \det_{\sigma} (\psi^*(f^*_j)(g_{\sigma(1)})) g_{\sigma(i)} \wedge \ldots \wedge g_{\sigma(m)}
\]

where $\sigma$ runs as above. Consequently $\nu_r(g_1 \wedge \ldots \wedge g_n)$ generates the homology at $\wedge^r G$ as we claimed.
The complex map $\nu$ in the first diagram induces the commutative diagram (D)

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \wedge^r \Omega_{R/k} / \tau(\wedge^r \Omega_{R/k}) & \rightarrow & \wedge^{r-1} \Omega_{R/k} & \rightarrow & \wedge^{r-2} \Omega_{R/k} & \rightarrow & \cdots \\
\mu_0 & & \mu_1 & & \mu_2 & & \\
0 & \rightarrow & N^0 = R & \rightarrow & N^1 & \rightarrow & N^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S_0(C) & \rightarrow & H^r(C_\psi(1)) & \rightarrow & 0 & \rightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & \\
\end{array}
\]

with exact columns where the $N^i$ are defined as in section 2. It is the specialized version of the second diagram in the proof of Theorem 3.1. The homology of the lowest row is denoted by $h^r$. The claim implies that $\nu_r(\wedge^n G) \not\subseteq \text{Im } d_\psi$. So $\mu_r(R) \not\subseteq \text{Im } d_\psi$.

Let $r = 1$. $R$ being a complete intersection, the homology of $D_\psi$ at $\wedge^r G$ is $k$ (for arbitrary $r \geq 1$). From the diagram

\[
\begin{array}{ccccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
0 & \rightarrow & N^0 & \rightarrow & N^1 & \rightarrow & 0 & \rightarrow & \\
\| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R & \rightarrow & C^{1,0} & \rightarrow & C^{2,0} & \rightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Im } \partial_\psi & \rightarrow & \text{Im } \partial_\psi & \rightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & \\
\end{array}
\]

with exact columns, we deduce that the homology of $N$ at $N^1$ is also $k$, since the first non-trivial maps in the second and in the third row are injective. So we obtain an exact sequence

\[
0 \rightarrow h^0 \rightarrow k \xrightarrow{\beta} k \rightarrow h^1 \rightarrow 0.
\]

Since $\mu_r(R) \not\subseteq \text{Im } d_\psi$, $\beta$ must be an isomorphism, and consequently $\alpha$ is an isomorphism.
Let $r = 2$. First we show that $H^2(C_\psi(1)) \cong S_0(C)$. The row homologies at $N^0$, $\Omega_{R/k}$, and $N^1$ vanish. Therefore $h^0 = 0$, and we get an exact sequence

$$0 \longrightarrow h^1 \longrightarrow k \xrightarrow{\gamma} H(N^2)$$

where $H(N^2)$ denotes the row homology at $N^2$. Because $\gamma$ is induced by $\mu_2$ and $\mu_r(R) \not\subseteq \text{Im } \bar{d}_\varphi$, $\gamma$ must be injective, so $h^1 = 0$, and $\alpha$ is an isomorphism. Next we show that $H^2(C_\psi(2)) \cong S_1(C)$. Set $m = (x_1, \ldots, x_n)$. By Proposition 2.3 in [BV1] and the local duality theorem (see 3.5.8 in [BH]) we have

$$H^2(C_\psi(2)) \cong \text{Ext}^1(\bigwedge^2 \Omega_{R/k}, R) \cong (H^1_m(\bigwedge^2 \Omega_{R/k}))^\vee \cong (S_0(C))^\vee \cong (H^2(C_\psi(1)))^\vee \cong (H^1_m(\Omega_{R/k}))^\vee \cong \text{Ext}^1(\Omega_{R/k}, R) \cong S_1(C).$$

Finally let $r \geq 3$. Adopting the notation of section 2, we shall prove

Claim (2): If $1 \leq j \leq \frac{r+1}{2}$, then $E^i,j = 0$ for $i = j, \ldots, r - j + 1$, and $H^r(C_\psi(j)) = S_{j-1}(C)$.

We argue by induction on $j$. Let $j = 1$. In the diagram (D) the row homologies at $N^0$, $N^1$, $N^2$ vanish, so $\alpha$ must be an isomorphism which proves that $H^r(C_\psi(1)) \cong S_0(C)$. Furthermore $\mathcal{N}$ has homology only at $N^r$, namely $k$. Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & N^{r-2} & \longrightarrow & N^{r-1} & \xrightarrow{d_\varphi} & N^r & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & C^{r-2,0} & \longrightarrow & C^{r-1,0} & \xrightarrow{d_\varphi} & C^{r,0} & \longrightarrow & C^{r+1,0} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \text{Im } \partial^{r-2,0}_\psi & \longrightarrow & \text{Im } \partial^{r-1,0}_\psi & \longrightarrow & \text{Im } \partial^{r,0}_\psi & \longrightarrow & \text{Im } \partial^{r+1,0}_\psi & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

with exact columns. The second row is the Koszul complex associated with $\varphi$. It follows immediately that $E^{i,j} = 0$ for $i = j, \ldots, r - 2$. Furthermore we get an exact sequence

$$0 \longrightarrow E^{r-1,1} \longrightarrow k \xrightarrow{\beta} k \longrightarrow E^{r,1} \longrightarrow 0.$$

Since $\iota(N^r) = \nu_r(\bigwedge^n G)$ ($\mu_r$ is an isomorphism), $\beta$ must be an isomorphism. We deduce $E^{r-1,1} = E^{r,1} = 0$. 

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Now let $1 \leq j + 1 \leq \frac{r+1}{2}$. The commutative diagram
\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Im} \tilde{\partial}_{i,j}^{j+1} & \text{Im} \tilde{\partial}_{i,j}^{j+1,j-1} & \text{Im} \tilde{\partial}_{i,j}^{j+2,j-1} & \cdots & \text{Im} \tilde{\partial}_{i,j}^{r-j+1,j-1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & C_j^{j,j} & C_{j+1,j} & C_{j+2,j} & \cdots & C_{r-j+1,j} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & S_j(C) & \text{Coker} \tilde{\partial}_{i,j}^{j+1,j-1} & \tilde{\delta} & \text{Im} \tilde{\partial}_{i,j}^{j+2,j} & \cdots & \text{Im} \tilde{\partial}_{i,j}^{r-j+1,j-1} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 
\end{array}
\]
has exact columns and its second row is the Koszul complex of $\varphi$ tensored with $S_j(F)$, and therefore its homology vanishes at least up to $C_{r-j}$. By our induction hypothesis the homology of the first row vanishes up to $\text{Im} \tilde{\partial}_{i,j}^{r-j+1,j-1}$. In particular the row homologies at $S_j(C)$ and $\text{Coker} \tilde{\partial}_{i,j}^{j+1,j-1}$ vanish. Since $\text{Im} \tilde{\partial}_{i,j}^{j+1,j} \xrightarrow{\delta} \text{Im} \tilde{\partial}_{i,j}^{j+2,j}$ is injective and is induced by $\delta$, we obtain
\[H^r(C_{\psi}(j + 1)) \cong \ker \delta = \text{Im} \gamma \cong S_j(C) \quad \text{and} \quad \text{Im} \delta = \text{Im} \tilde{\partial}_{\psi}^{j+1,j}.
\]
Together with the last equation we deduce $E_{i,j}^{i+1} = 0$ for $i = j + 1, \ldots, r - j$. This proves our claim.

The claim implies that $H^r(C_{\psi}(i + 1)) = S_i(C)$ for $0 \leq i < \frac{r}{2}$. For $\frac{r}{2} \leq i < r$, we use Proposition 2.3 in [BV1] and the local duality theorem as in the $r = 2$ case to get the desired isomorphism.

**COROLLARY 4.2.** With notation and the assumptions from above we obtain
\[\ell(S_0(C)) = \ell(S_1(C)) = \ldots = \ell(S_r(C)).\]

**Proof.** Use the isomorphisms of Theorem 4.1 and Corollary 2.2 in [BV1].

**REMARK 4.3.** In [BV1], section 3, the length formulas of Corollary 4.2 have been proved for $r$ odd. The reader may also find a proof in [HM], Proposition 4.9. Our approach follows the line of [BV1]. The isomorphisms of Theorem 4.1 were previously obtained only for $0 \leq i \leq r - 2$, and consequently only the formula $\ell(S_0(C)) = \ldots = \ell(S_{r-1}(C))$.

**PROPOSITION 4.4.** Set $M_{\varphi} = \text{Coker} \varphi$, and let $\tilde{\psi} : M_{\varphi} \to F$ be the map induced by $\psi$. Then for the homology of $C_{\tilde{\psi}}(r)$, the following holds:
\[H^0(C_{\tilde{\psi}}(r)) = H^1(C_{\tilde{\psi}}(r)) = 0, \quad H^{i+1}(C_{\tilde{\psi}}(r)) = S_i(C).
\]
If $i + r$ is odd, $0 \leq i \leq r - 1$, then
\[\ell(H^i(C_{\tilde{\psi}}(r))) = \ell(H^{i+1}(C_{\tilde{\psi}}(r))).
\]
If $r$ is even, then $H^2(C_{\tilde{\psi}}(r)) = 0$. (To avoid misunderstandings: the graduation of $C_{\tilde{\psi}}(r)$ is fixed in such a way that position 0 is held by $\wedge^n M_{\varphi}$.)
Proof. The bicomplex

\[
\begin{array}{cccc}
0 & \longrightarrow & 0 & \longrightarrow \\
\downarrow & & \downarrow & \\
N^{r-1} & \longrightarrow & N^r & \longrightarrow \\
\downarrow & & \downarrow & \\
\longrightarrow & \longrightarrow & \longrightarrow & k \\
\downarrow & & \downarrow & \\
\Lambda^{r-1} G \otimes S_0(F) & \longrightarrow & \Lambda^r G \otimes S_0(F) & \longrightarrow \\
\downarrow & & \downarrow & \\
\longrightarrow & \longrightarrow & \longrightarrow & 0 \\
\downarrow & & \downarrow & \\
\Lambda^{r-2} G \otimes S_1(F) & \longrightarrow & \Lambda^{r-1} G \otimes S_1(F) & \longrightarrow \\
\downarrow & & \downarrow & \\
\longrightarrow & \longrightarrow & \longrightarrow & 0 \\
\downarrow & & \downarrow & \\
\Lambda^{r-3} G \otimes S_2(F) & \longrightarrow & \Lambda^{r-2} G \otimes S_2(F) & \longrightarrow \\
\downarrow & & \downarrow & \\
\longrightarrow & \longrightarrow & \longrightarrow & 0 \\
\vdots & & \vdots & \\
\end{array}
\]

arises from \( \tilde{C}_{\psi}(r) \) by truncation at the \((r + 1)\)th column and taking cokernels. Its last column is just \( C_{\psi}(r) \). (Observe that \( \Lambda^n M_{\varphi} \cong k \).) From Claim (1) in the proof of Theorem 4.1, we draw that \( H^0(C_{\psi}(r)) = 0 \).

To prove \( H^1(C_{\psi}(r)) = 0 \), let \( Q \) be the total ring of fractions of \( R \). Consider the commutative diagram

\[
\begin{array}{ccc}
\Lambda^r M_{\varphi} & \xrightarrow{\partial_{\psi}} & \Lambda^{r-1} M_{\varphi} \otimes F \\
\downarrow & & \downarrow \\
\Lambda^r M_{\varphi} \otimes Q & \xrightarrow{\partial_{\psi} \otimes Q} & \Lambda^{r-1} M_{\varphi} \otimes F \otimes Q
\end{array}
\]

with canonical vertical arrows. The kernel of the left vertical arrow is the torsion of \( \Lambda^r M_{\varphi} \). The right vertical arrow and \( \partial_{\psi} \otimes Q \) are injective since \( \Lambda^{r-1} M_{\varphi} \) is torsionfree and \( C_{\psi}(r) \otimes Q \) is split exact. So \( \ker(\Lambda^r M_{\varphi} \xrightarrow{\partial_{\psi}} \Lambda^{r-1} M_{\varphi} \otimes F) \) is the torsion submodule of \( \Lambda^r M_{\varphi} \). Next we note that this torsion equals \( H^r(D_{\varphi}) \) where \( D_{\varphi} = D_{\varphi}(t) \) is the dual version of the Koszul complex associated with \( \varphi \). Of course \( H^r(D_{\varphi}) = H_m(R) \). Furthermore \( H_1(R) = k^n \). Since \( R \) is a complete intersection, the Koszul algebra \( H(R) \) is isomorphic with the exterior algebra of \( H_1(R) \). Therefore \( H_m(R) = k \).

Since \( \Lambda^n M_{\varphi} = k \) and \( \nu_{\psi} : \Lambda^n M_{\varphi} \rightarrow \Lambda^r M_{\varphi} \) is injective, we get \( H^1(C_{\psi}(r)) = 0 \).

The isomorphism \( H^{r+1}(C_{\psi}(r)) \cong S_r(C) \) is obvious.

By the same method we used to prove Theorem 2.2, we obtain exact sequences

\[
0 \rightarrow H^1(C_{\psi}(r)) \rightarrow S_{i+r-1}(C) \rightarrow H^r(C_{\psi}(\frac{i + r + 1}{2})) \rightarrow H^{i+1}(C_{\psi}(r)) \rightarrow 0.
\]
for \( i + r \) odd, \( 0 \leq i \leq r - 1 \). Since
\[
S_{\frac{i+r-1}{2}}(C) \cong H^r(C_{\psi}(\frac{i+r+1}{2}))
\]
by Theorem 4.1, the length formula follows. Moreover \( H^2(C_{\psi}(r)) = 0 \) if \( r \) is even.

\[\square\]

REMARK 4.5. We conjecture that actually
\[
H^i(C_{\psi}(r)) = \begin{cases} S_r(C) & \text{if } i = r + 1, \\ 0 & \text{otherwise.} \end{cases}
\]

The conjecture is true if \( m = \text{rank } F = 1 \). \textbf{Proof.} In the case under consideration we have an the exact sequence of complex morphisms
\[
C_{\psi}(r)[1] \xrightarrow{\iota} C_{\psi}(r) \xrightarrow{\pi} C_{\psi}(r) \rightarrow 0
\]
where \([\ ]\) means shift and
\[
\pi_p = \begin{cases} \wedge^n(G \rightarrow M_\varphi) & \text{for } p = 0 \\ \wedge^{r-p+1}(G \rightarrow M_\varphi) & \text{for } p = 1, \ldots, r + 1. \end{cases}
\]

Obviously \( \iota_p \) is injective for \( p > 0 \). Since \( H^p(C_{\psi}(r)) = 0 \) for \( p = 0, \ldots, r - 1 \) and \( H^0(C_{\psi}(r)) = H^1(C_{\psi}(r)) = 0 \), we obtain \( H^p(C_{\psi}(r)) = 0 \) for \( p = 0, \ldots, r \) as desired.

\textbf{References}

[BH] W.Bruns and J. Herzog. \textit{Cohen-Macaulay Rings}. Cambridge Uni. Press 1996.

[BV1] W. Bruns and U. Vetter. \textit{Length formulas for the Local Cohomology of Exterior Powers}. Math Z. \textbf{191} (1986), 145–158.

[BV2] W. Bruns and U. Vetter. \textit{Determinantal rings}. Lect. Notes Math. \textbf{1327}, Springer 1988.

[E] D. Eisenbud. \textit{Commutative Algebra with a View Toward Algebraic Geometry}. Grad. Text in Math. \textbf{150}, Springer 1995.

[G] G.M. Greuel. \textit{Dualität in der lokalen Kohomologie isolierter Singularitäten}. Math. Ann. \textbf{250}, (1980), 157–173.

[HM] J. Herzog and A. Martsinkovsky. \textit{Glueing Cohen-Macaulay modules with applications to quasihomogeneous complete intersections with isolated singularities}. Comment. Math. Helv. \textbf{68} (1993), 365–384.
[I]  B. Ichim. *Generalized Koszul Complexes*. Thesis, Universität Oldenburg (Germany), 2004.

[IV] B. Ichim and U. Vetter. *Koszul Bicomplexes and generalized Koszul complexes in projective dimension one*. Submitted to Comm. in Algebra.