HOMOMORPHISMS OF THE ALTERNATING GROUP $A_5$ INTO REDUCTIVE GROUPS

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Dedicated to Robert Steinberg on his 80-th birthday

INTRODUCTION

0.1. Let $A_5$ be the group defined by generators $x_2, x_3, x_5$ and relations $x_2^2 = x_3^3 = x_5^5 = x_2 x_3 x_5 = 1$. It is well known that $A_5$ is isomorphic to the alternating group in 5 letters. Let $k$ be an algebraically closed field of characteristic $p$ where $p = 0$ or a prime $\geq 7$. Let $G$ be a connected reductive algebraic group over $k$. We shall be interested in the classification of homomorphisms $A_5 \to G$ up to conjugation by $G$. When $G = GL_n$ this is the same as the classification of $n$-dimensional representations of $A_5$ over $k$ up to isomorphism; this can be reduced to the classification of irreducible representations of $A_5$ which is classical (there are five of them up to isomorphism, of degrees 1, 3, 3, 4, 5). In the general case, there is again something analogous to the notion of irreducible representation, which we call a regular homomorphism $A_5 \to G$. This is, by definition, a homomorphism $A_5 \to G$ whose image is not contained in a Levi subgroup of a proper parabolic subgroup of $G$. Again the classification of homomorphisms $A_5 \to G$ up to conjugacy can be more or less reduced to the analogous question for regular homomorphisms (for $G$ and also for smaller groups).

D.D.Frey [F1],[F2],[F3] has classified up to conjugacy the homomorphisms (resp. non-regular homomorphisms) $A_5 \to G$ for $G$ of type $E_6, E_7$ (resp. $E_8$) over $C$. For $G$ of type $E_8$ he showed that there is at least one regular homomorphism $A_5 \to G$, but the problem of classifying up to conjugacy the regular homomorphisms $A_5 \to G$ remained open.

Serre [S] has suggested that this problem could be attacked using the complex representation theory of $E_8(F_q), F_q$ a finite field. He showed that the number $d$ of conjugacy classes of regular homomorphisms $A_5 \to E_8(k)$ can be extracted from a certain sum over all irreducible characters of $E_8(F_q)$, if this sum can be computed.

The purpose of this paper is to show that this sum can be computed with enough precision so that a solution of the above problem is obtained. (See Theorem 4.5.) In fact, we show that $d = 1$. Thus, our results, in conjunction with the earlier

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work of Frey, complete the classification up to $G$-conjugacy of homomorphisms $A_5 \to G$ for $G$ adjoint simple.

One of the key ingredients in our proof is a collection of inequalities (see 2.2) involving representations of Weyl groups. While these inequalities are conjectural in general, enough of them can be verified (with the aid of a computer) so that the proof goes through.

I want to thank R. Griess for introducing me to this problem. I also want to thank J.-P. Serre for making [S] available to me and for some useful discussions.

1. Examples

1.1. If $G$ is a group we denote by $Z_G$ the centre of $G$ and by $G_{der}$ the derived group of $G$; we set $G_{ad} = G/Z_G$. If $g \in G$ let $Z_G(g)$ be the centralizer of $g$ in $G$ and let $C_G(g)$ be the conjugacy class of $g$ in $G$. If $H$ is a subgroup of $G$ let $Z_G(H)$ (resp. $N_G(H)$) be the centralizer (resp. normalizer) of $H$ in $G$. If $G,G'$ are groups, let $\text{Hom}(G,G')$ be the set of group homomorphisms $G \to G'$.

If $G$ is an affine algebraic group over $k$ (as in 0.1), let $G^0$ be the identity component of $G$ and let $\bar{G} = G/G^0$.

For a finite set $Y$ let $Y$ or $\#(Y)$ be the cardinal of $Y$.

1.2. Assume now that $G$ is connected, reductive algebraic group over $k$ of simply laced type. Let $\pi : \tilde{G} \to G$ be the simply connected covering of $G_{der}$. Let $T_G$ be the set of maximal tori in $G$. Let $r(G)$ be the rank of $G$ and let $\nu(G)$ be half the number of roots of $G$.

Since $A_5$ is perfect, any homomorphism $\psi : A_5 \to G$ has image contained in $G_{der}$. This gives us a bijection between the set of $G$-orbits on $\text{Hom}(A_5,G)$ and the set of $G_{der}$-orbits on $\text{Hom}(A_5,G_{der})$. Now, the obvious map from $G$-orbits on $\text{Hom}(A_5,G)$ to $G_{ad}$-orbits on $\text{Hom}(A_5,G_{ad})$ is injective.

(It is enough to show that, if $\psi,\psi' \in \text{Hom}(A_5,G)$ are such that $h\psi = h\psi'$ where $h : G \to G_{ad}$ is the obvious map, then $\psi = \psi'$. For $x \in A_5$ we have $\psi'(x) = \psi(x)z(x)$ where $z \in \text{Hom}(A_5,Z_G)$. Since $A_5$ is perfect and $Z_G$ is abelian, we have $z(x) = 1$ for all $x$, hence $\psi = \psi'$.)

Lemma 1.3 (Serre [S]). Let $\psi \in \text{Hom}(A_5,G)$. We have

$$\sum_{n=2,3,5} \dim Z_G(\psi(x_n)) = \dim(G) + 2 \dim Z_G(\psi(A_5)).$$

This is equivalent to its Lie algebra analogue

$$\sum_{n=2,3,5} \dim g^{\psi(x_n)} = \dim g + 2 \dim g^{A_5}$$

where $g = \text{Lie } G$, and the upper index denotes the fixed point set of the appropriate automorphism or group of automorphisms of $g$. More generally, this holds when $g$ is replaced by a finite dimensional $k$-vector space $V$ with an $A_5$-action; we may assume that $V$ is an irreducible representation of $A_5$ (we have complete reducibility since $\frac{1}{30} \in k$) and in that case the equality is checked by direct computation.
Lemma 1.4. If $\psi \in \text{Hom}(A_5, G)$, then $Z_G(\psi(A_5))$ is a reductive group.

Let $C(n) = C_G(\psi(x_n))$, $n = 2, 3, 5$. Then $C(n)$ is an affine variety since $\psi(x_n)$ is a semisimple element. Consider the affine variety

$$X = \{(g_2, g_3, g_5) \in C(2) \times C(3) \times C(5); g_2g_3g_5 = 1\}.$$ 

The group $G$ acts on $X$ by conjugation on all factors. By 1.3, the dimension of the $G$-orbit of $(g_2, g_3, g_5) \in X$ equals

$$\dim(G) - (\sum_{n=2,3,5} \dim Z_G(g_n) - \dim(G))/2 = \sum_{n=2,3,5} \dim C(n)/2;$$

in particular it is independent of the choice $(g_2, g_3, g_5)$. Since any $G$-orbit of minimum dimension must be closed in $X$, it follows that any $G$-orbit in $X$ is closed in $X$; hence it is affine. By a known criterion (Richardson) it follows that the isotropy group in $G$ of any point of $X$ is reductive. The lemma is proved.

Lemma 1.5. The following four conditions for $\psi \in \text{Hom}(A_5, G)$ are equivalent:

(i) $\dim Z_G(\psi(A_5)) = \dim Z_G$;
(ii) $Z_G(\psi(A_5))/Z_G$ is finite;
(iii) any subtorus of $Z_G(\psi(A_5))$ is contained in $Z_G^0$;
(iv) $\nu(G) - \sum_{n=2,3,5} \nu(Z_G^0(\psi(x_n))) = r(G_{ad})$.

This follows from Lemmas 1.3, 1.4.

1.6. We say that $\psi \in \text{Hom}(A_5, G)$ is regular if the conditions of Lemma 1.5 are satisfied. Let $\text{Hom}_{reg}(A_5, G) = \{\psi \in \text{Hom}(A_5, G); \psi \text{ regular}\}$. Note that $\psi \in \text{Hom}(A_5, G)$ is regular if and only if its image in $\text{Hom}(A_5, G_{ad})$ is regular.

We show that the classification of homomorphisms $\psi : A_5 \to G$ up to $G$-conjugacy can be essentially reduced to the analogous problem for regular homomorphisms. Let $\psi \in \text{Hom}(A_5, G)$. Let $S$ be a maximal torus of $Z_G(\psi(A_5))$ and let $L = Z_G(S)$. (A Levi subgroup of a parabolic subgroup of $G$.) Using 1.5(iii), we see that $\psi$ defines a regular homomorphism $A_5 \to L$. This gives us a bijection between the set of $G$-orbits on $\text{Hom}(A_5, G)$ and the disjoint union over all $G$-conjugacy classes of Levi subgroups $L$ of the sets of $N_G(L)$-orbits on $\text{Hom}_{reg}(A_5, L)$.

1.7. A 235-triple for $G$ is a triple $(C_2, C_3, C_5)$ of conjugacy classes in $G$ such that, if $g_n \in C_n$, then $g_n^n = 1$, $n = 2, 3, 5$. The type of a 235-triple is by definition the sequence $X_2, X_3, X_5$ where $X_n$ is the type of the root system of $Z_G^0(g_n)$. Clearly, $G$ has only finitely many 235-triples. As pointed out in [S], of particular interest are the 235-triples $(C_2, C_3, C_5)$ in $G$ such that

$$\nu(G) - \sum_{n=2,3,5} \nu(Z_G^0(g_n)) = r(G_{ad})$$
where \( g_n \in C_n \). Such 235-triples are called \textit{regular}. If \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) and \( C_n = C_G(\psi(x_n)) \), then \( (C_2, C_3, C_5) \) is a regular 235-triple.

Let \( \iota \) be an automorphism of order 2 of \( A_5 \) such that \( \iota(x_2) \) is conjugate to \( x_2 \), \( \iota(x_3) \) is conjugate to \( x_3 \), and \( \iota(x_5) \) is conjugate to \( x_5^2 \). For any \( \psi \in \text{Hom}(A_5, G) \) let \( \iota \psi = \psi \circ \iota \). If \( (C_2, C_3, C_5) \) is a regular 235-triple of \( G \), then \( \iota(C_2, C_3, C_5) = (C_2, C_3, C_5') \), where \( C_5' = \{ g^2; g \in C_5 \} \), is again a regular 235-triple of \( G \).

\[ 1.8. \text{ Assume that } G = G_{ad} \text{ is simple of type } A_{m-1}. \text{ If } m \in \{2, 3\} \text{ then } G \text{ has exactly two regular 235-triples (interchanged by } \iota \) and there are exactly two } \psi \in \text{Hom}_{\text{reg}}(A_5, G) \text{ up to conjugacy (interchanged by } \iota \). \text{ If } m = 4, \text{ then } G \text{ has exactly one regular 235-triple and there are exactly two } \psi \in \text{Hom}_{\text{reg}}(A_5, G) \text{ up to conjugacy. If } m \in \{5, 6\} \text{ then } G \text{ has exactly one regular 235-triple and there is exactly one } \psi \in \text{Hom}_{\text{reg}}(A_5, G) \text{ up to conjugacy. If } m \geq 7, \text{ then } G \text{ has no regular 235-triple and therefore } \text{Hom}_{\text{reg}}(A_5, G) = \emptyset. \]

The type of a regular 235-triple is \( (0,0,0) \) (if \( m = 2 \)), \( (A_1,0,0) \) (if \( m = 3 \)), \( (A_2^2, A_1,0) \) (if \( m = 4 \)), \( (A_2^1, A_1^2, A_1) \) (if \( m = 5 \)), \( (A_3^1, A_2^1, A_1) \) (if \( m = 6 \)).

For \( m = 3, 5 \) (resp. \( m = 2, 6 \)), \( \psi \) comes from an irreducible \( m \)-dimensional representation of \( A_5 \) (resp. of the double cover \( SL_2(F_5) \) of \( A_5 \), regarded as a projective representation of \( A_5 \)). For \( m = 4 \), one \( \psi \) comes from an irreducible 4-dimensional representation of \( A_5 \), the other \( \psi \) comes from an irreducible 4-dimensional representation of \( SL_2(F_5) \) which does not factor through \( A_5 \).

\[ 1.9. \text{ Assume that } G = G_{ad} \text{ is simple of type } D_m. \]

(a) If \( m = 4 \), then \( G \) has exactly four regular 235-triples: two of type \( (A_1^1, A_2^1, 0) \) (interchanged by \( \iota \)) and two of type \( (A_1^4, A_2, 0) \) (interchanged by \( \iota \)); correspondingly, it has exactly four \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) up to conjugacy: two of them (interchanged by \( \iota \)) come from 8-dimensional orthogonal representations of \( A_5 \) which decompose as \( 1 + 3 + 4 \) into irreducibles and two of them (interchanged by \( \iota \)) come from 8-dimensional orthogonal representations of \( A_5 \) which decompose as \( 3 + 5 \) into irreducibles.

(b) If \( m = 5 \), then \( G \) has exactly one regular 235-triple; it has type \( (A_3A_2^2, A_2A_1^2, A_1^2) \); also \( G \) has exactly two \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) up to conjugacy: one of them comes from a 10-dimensional orthogonal representation of \( A_5 \) which decomposes as \( 3 + 3 + 4 \) into irreducibles; the other comes from a 10-dimensional orthogonal representation of \( A_5 \) which decomposes as \( 1 + 4 + 5 \) into irreducibles.

(c) If \( m = 6 \), then \( G \) has exactly three regular 235-triples: one has type \( (A_3^2, A_3A_1^2, A_1^2) \) and the other two (interchanged by \( \iota \)) have type \( (A_3, A_4A_2, A_2A_1) \); correspondingly, it has exactly three \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) up to conjugacy: one of them comes from a 12-dimensional orthogonal representation of \( A_5 \) which decomposes as \( 1 + 3 + 3 + 5 \) into irreducibles; the other two (interchanged by \( \iota \)) come from 12-dimensional orthogonal representations of \( A_5 \) which decompose as \( 3 + 4 + 5 \) into irreducibles.

(d) If \( m = 8 \), then \( G \) has exactly one regular 235-triple; it has type \( (D_4A_3, A_4A_3, A_2^2A_1^2) \); it also has exactly one \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) up to conjugacy:
it comes from a 16-dimensional orthogonal representation of $A_5$ which decomposes as $1 + 3 + 3 + 4 + 5$ into irreducibles.

If $m = 7$ or $m > 9$ then $G$ has no regular $235$-triples; therefore, $\text{Hom}_{reg}(A_5, G) = \emptyset$.

We see that in these cases each regular $\psi$ comes from a homomorphism $A_5 \to SO_{2m}$.

1.10. Assume that $G = G_{ad}$ is simple of type $E_m$. From [F1], [F2],[F3] we see that:
if $m = 6$, then $G$ has exactly two regular $235$-triples (interchanged by $\iota$); they have type $(A_5A_1, A_3^2, A_2^2)$;
if $m = 7$, then $G$ has exactly two regular $235$-triples (interchanged by $\iota$); they have type $(A_7, A_5A_2, A_3A_2A_1)$;
if $m = 8$, then $G$ has exactly one regular $235$-triple; it has type $(D_8, A_8, A_4^2)$.

From [F1],[F2],[F3] we see also that in each of these cases, a regular $235$-triple is associated with at least one $\psi \in \text{Hom}_{reg}(A_5, G)$. (Another proof of this fact is given by Lemma 4.3.) This $\psi$ is in fact unique up to conjugacy. (See [F2], [F3] for $E_6, E_7$ and $p = 0$; see 4.5, 4.6 for the general case.)

2. Inequalities

2.1. Let $W$ be a Weyl group and let $V$ be the reflection representation of $W$ (over $C$). Let $V = \oplus_{j \geq 0} V_j$ be the algebra of polynomials $V \to C$ modulo the ideal generated by the $W$-invariant polynomials of degree $> 0$; here $V_j$ is the image of the space of polynomials of degree $j$. Now dim $V = \left[ W \right]$ and $W$ acts naturally on $V$ preserving the grading. Let $E(W)$ be the set of irreducible representations of $W$ over $C$ (up to isomorphism). For any $E \in E(W)$ let $b_W(E)$ (resp. $b'_W(E)$) be the smallest (resp. largest) integer $j$ such that $\text{Hom}_W(E, V_j) \neq 0$. Note that $b_W(E), b'_W(E)$ are well defined.

Assume now that $W$ is the Weyl group of $G$ as in 1.2. Let $\nu = \nu(G)$. Let $(C_2, C_3, C_5)$ be a $235$-triple for $G$. Let $(g_2, g_3, g_5) \in C_2 \times C_3 \times C_5$. We denote by $W_n$ (resp. by $\nu_n$) the Weyl group (resp. the number of positive roots) of $Z^0_G(g_n), n = 2, 3, 5$. We may regard $W_2, W_3, W_5$ as Weyl subgroups of $W$. Let $V^{(n)} = \oplus_j V^{(n)}$ be defined in terms of $W_n$ in the same way as $V = \oplus_j V_j$ is defined in terms of $W$. For any $E \in E(W)$ let $b_n(E)$ (resp. $b'_n(E)$) be the smallest (resp. largest) integer $j$ such that $\text{Hom}_{W_n}(E|W_n, \nu^{(n)}_j) \neq 0$. Note that $b_n(E), b'_n(E)$ are well defined for $n = 2, 3, 5$.

Let sgn be the sign representation of $W$. We have
(a) $b(E) = \nu - b'(E \otimes \text{sgn}), b_n(E) = \nu_n - b'_n(E \otimes \text{sgn}), \quad n = 2, 3, 5$.

Now $E(W)$ is a union of equivalence classes called families, see [L, 4.2]. For example the unit representation 1 is a family by itself. If $F$ is a family, then $\{ E \otimes \text{sgn}; E \in F \}$ is again a family. For any family $F$ of $W$ we set
$$a(F) = \min_{E \in F} b(E), \quad a'(F) = \max_{E \in F} b'(E),$$
\[ a_n(F) = \min_{E \in F} b_n(E), \quad a'_n(F) = \max_{E \in F} b'_n(E). \]

From (a) we deduce that

(b) \[ a(F) = \nu - a'(F \otimes \text{sgn}), \quad a_n(F) = \nu_n - a'_n(F \otimes \text{sgn}), \quad n = 2, 3, 5. \]

**Proposition 2.2.** In the setup of 2.1, assume that either

(1) \((C_2, C_3, C_5)\) is regular, or

(2) any simple component of \(G_{ad}\) has rank \(\leq 8\) and is not of type \(E_8\).

Then for any family \(F\) we have

(a) \[ a(F) - \sum_{n=2,3,5} a_n(F) \leq r(G_{ad}); \]

equivalently (see 2.1(b)),

(b) \[ \sum_{n=2,3,5} a'_n(F) - a'(F) \leq -\nu + \sum_n \nu_n + r(G_{ad}). \]

Moreover, if \((C_2, C_3, C_5)\) is not regular, then (a) and (b) are strict inequalities. If \((C_2, C_3, C_5)\) is regular, then (a) is strict for \(F \neq \{\text{sgn}\}\) and (b) is strict for \(F \neq \{1\}\). If \((C_2, C_3, C_5)\) is regular, then (a) is an equality for \(F = \{\text{sgn}\}\) and (b) is an equality for \(F = \{1\}\).

This can be checked with the aid of a computer. We may assume that \(G = G_{ad}\) is simple. In the case where \(G\) is of type \(E_8\) and \((C_2, C_3, C_5)\) is regular, I have used tables in [BL], [A]. If we are not in this case then \(G\) has rank \(\leq 8\) and is not of type \(E_8\). Then, instead of [BL], [A], one can use the CHEVIE package [C] to get tables for the quantities \(b_n(E)\) (or rather the analogous quantities where \(W_n\) is replaced by any reflection subgroup of \(W\)). From this information, one can check the required inequalities by hand. (I am grateful to M. Geck for his help with the CHEVIE package.)

**2.3.** We expect that 2.2 continues to hold even if we drop the assumptions (1),(2). One can check that 2.2(a),(b) hold for \(G\) of type \(A\) without the assumptions (1),(2). When \((C_2, C_3, C_5)\) is regular, then 2.2(a) for \(F = \{\text{sgn}\}\) is just the equality \(\nu - \sum_{n=2,3,5} \nu_n = r(G_{ad})\).

**2.4.** We illustrate 2.2(a) in the case when \(G = G_{ad}\) is of type \(E_8\) and \((C_2, C_3, C_5)\) is regular. The subgroups \(W_n\) of \(W\) are as in 1.10. In the following list, there is one line for each family \(F\) of \(W\), containing the information

\[ D; a(F) - a_2(F) - a_3(F) - a_5(F) = ? \]

where \(D\) is the degree of the "special" representation in \(F\). Note that \(F\) is determined by \(D\) and \(a(F)\).

| 1; 0 - 0 - 0 - 0 = 0 |
| 8; 1 - 1 - 1 - 1 = -2 |
| 35; 2 - 2 - 1 - 0 = -1 |
| 112; 3 - 0 - 0 - 0 = 3 |
| 210; 4 - 0 - 1 - 0 = 3 |
| 560; 5 - 1 - 1 - 0 = 3 |
| 567; 6 - 2 - 1 - 1 = 2 |
3. $\mathbb{F}_q$-structures

3.1. In this section we fix $G$ as in 1.2. We assume that $p$ is a prime $\geq 7$ and that $k$ is an algebraic closure of the field with $p$ elements. Let $q$ be a power of $p$ and
let \( F_q \) be the subfield of \( k \) with \( q \) elements. Let \( F : G \rightarrow G \) be the Frobenius map corresponding to an \( F_q \)-rational structure on \( G \).

For any integer \( m \geq 1 \) we write \( m = m_pm_{p'} \) where \( m_p \) is a power of \( p \) and \( m_{p'} \) is prime to \( p \). If \( \phi \) is a map of a set \( Y \) into itself, we set \( Y^\phi = \{ y \in Y; \phi(y) = y \} \).

**Lemma 3.3.** Let \( \rho \in \mathcal{E}_1(G^F) \). There exists a unique \( G^F \)-invariant function \( f^\rho : \mathcal{T}(G)^F \rightarrow \mathbb{C} \) and a unique \( \xi \in \mathcal{O}(G^F) \) such that

\[
\rho = \left[ G^F \right]^{-1} \sum_{T \in \mathcal{T}(G)^F} [T^F] f^\rho(T) R_{T,G}^1 + \xi.
\]

This follows immediately from the definition of \( \mathcal{E}_1(G^F) \) and from the orthogonality relations [DL, 6.8].

**Lemma 3.4.** Let \( g \in G^F \) be semisimple. Let \( T \in \mathcal{T}(G)^F \), \( \vartheta \in \text{Hom}(T^F, \mathbb{C}^*) \). We have

\[
R_T^\vartheta(g) = \epsilon_{Z_G(g)} \epsilon_T \left[ T^F \right]^{-1} \left[ Z_G(g) \right] g' \sum_{g' \in \mathcal{E}_{G,F}(g) \cap T} \vartheta(g').
\]

See [DL, 7.2].

**3.5.** For any \( G^F \)-invariant function \( f : \mathcal{T}(G)^F \rightarrow \mathbb{C} \) let

\[
R_{f,G} = \left[ G^F \right]^{-1} \sum_{T \in \mathcal{T}(G)^F} [T^F] f(T) R_{T,G}^1.
\]

If \( H \) is a connected reductive \( F \)-stable subgroup of \( G \) with \( r(H) = r(G) \) and \( f \) is as above, we shall write \( R_{f,H} \) instead of \( R_{f|_{\mathcal{T}(H)^F,H}} \).

**Lemma 3.6.** In the setup of 3.5 we have \( \sum_{T \in \mathcal{T}(G)^F} \epsilon_G \epsilon_T f(T) = \left[ G^F \right] R_{f,G}(1) \).

\( R_{f,G}(1) \) may be computed using the formula in [DL, 7.1] for \( R_{T,G}^1(1) \); the lemma follows.
Lemma 3.8. Assume that \( H, \theta \) where \((g)
\) and a bijection \( \rho \) (denoted by \( \rho \)) in \( \rho\). Let \( (H, \theta) \) be the set of all \( \theta \in H^F_\bullet \) such that \( \theta = \theta' \theta'' \) for some \( \theta' \in H^F_\bullet \). Let \( X_{G,F} \) be the set of all \( F \)-stable connected reductive subgroups \( H \) of \( G \) such that \( H^F_\bullet \neq \emptyset \).

Lemma 3.8. Assume that \( Z_G = Z^0_G \). Let \( H \in X'_{G,F} \). Then any orbit of the obvious \( N_{G}(H)^F \)-action on \( H^F_\bullet \) has cardinal \( \frac{N_{G}(H)^F}{H^F} \).

Let \( \theta \in H^F_\bullet \). Let \( S_\theta \) be the stabilizer of \( \theta \) in \( N_{G}(H)^F \). We show that \( S_\theta = H^F \).

The inclusion \( H^F \subset S_\theta \) is immediate. Now let \( g \in S_\theta \). We must show that \( g \in H^F \). Let \( T \in \mathcal{T}(H)^F \) be maximally split. Since \( gHg^{-1} = H \), we see that \( gTg^{-1} \in \mathcal{T}(H)^F \) is maximally split hence \( gTg^{-1} = hT'h^{-1} \) for some \( h \in H^F \). Replacing \( g \) by \( gh^{-1} \) we see that we may assume that \( gTg^{-1} = T \). Now conjugation by \( g \) keeps fixed \( \theta|_{TF} \). By \([DL, 5.13]\), \( g \) is a product of elements in \( N_H(T) \). In particular, \( g \in H \). Hence \( g \in H^F \). The lemma is proved.

3.9. Let \((H, \theta) \in X_{G,F} \). Let \( \mathcal{E}_{H,\theta}(G) \) be the set of all \( \rho \in \mathcal{E}(G^F) \) such that \( \rho : R_{T,G}^\theta_G \neq 0 \) for some \( T \in \mathcal{T}(H)^F \). (We write \( \theta_T \) instead of \( \theta|_{TF} \).) According to \([DL, 6.3, 5.20]\), if \( Z_G = Z^0_G \), we have a partition

(a) \( \mathcal{E}(G^F) = \sqcup_{(H, \theta)} \mathcal{E}_{H,\theta}(G^F) \)

where \((H, \theta) \) runs over a set of representatives for the \( G^F \)-orbits on \( X_{G,F} \).

According to \([L]\), if \( Z_G = Z^0_G \), for any \( (H, \theta) \in X_{G,F} \), there exists \( \kappa \in \{1, -1\} \) and a bijection

(b) \( \mathcal{E}_1(H^F) \leftrightarrow \mathcal{E}_{H,\theta}(G^F) \)

(denoted by \( \rho \leftrightarrow \rho^{H,\theta} \)) such that for any \( \rho \in \mathcal{E}_1(H^F) \) and any \( T \in \mathcal{T}(H)^F \) we have \( \langle \rho : R_{T,H}^\theta \rangle_T = \kappa(\rho^{H,\theta} : R_{T,G}^\theta) \).

Lemma 3.10. Assume that \( Z_G = Z^0_G \). Let \((H, \theta) \in X_{G,F} \). Let \( \kappa \) be as above. Let \( \rho \in \mathcal{E}_1(H^F) \). Let \( f^\rho : T(H)^F \to C \) be the \( H^F \)-invariant function such that

\[
\rho = H^F \sum_{T \in \mathcal{T}(H)^F} T^F f^\rho(T) R_{T,H}^\theta + \xi, \xi \in \mathcal{O}(H^F) \text{ (see 3.3).}
\]

(a) We have \( \rho^{H,\theta} = \kappa H^F \sum_{T \in \mathcal{T}(H)^F} T^F f^\rho(T) R_{T,G}^\theta + \xi' \) with \( \xi' \in \mathcal{O}(G^F) \).

(b) Let \( g \in G^F \) be semisimple. We have

\[
\rho^{H,\theta}(g) = \kappa Z_G(g)^F \sum_{g'} e_{Z^0_G(g)} e_{Z^0_H(g')} Z_H(g')^F \sum_{g''} R_{f^\rho, Z^0_H(g')(1) \theta(g')} \]

where \( g' \) runs through a set of representatives for the conjugacy classes in \( H^F \) that are contained in \( C_{GF}(g) \).

(c) We have \( \rho^{H,\theta}(1) = \kappa \left[ G^F \right]_{g'}^{H^F} \epsilon_{g'} \epsilon_{F}^{H} R_{F'}^{*} \epsilon_{H}(1) \).

To prove the identity in (a) it is enough to show that both sides have the same inner product with \( R_{T'}^{\theta} \) for any \( T' \in T(H)^F \). By [DL, 6.8], the inner product of the right hand side with \( R_{T'}^{\theta} \)

\[
\kappa \sum_{T \in T(H)^F} f^\rho(T) \epsilon(G^F) \theta(T) \]

For any \( g_1 \) in (d) we see that \( \text{Ad}(g_1) \) carries \( S(T,\theta) \) to \( S(T',\theta) \) hence it carries \( (H,\theta) \) to \( (H,\theta) \). By the argument in the proof of 3.8 we have \( g_1 \in H^F \). Thus, (d) is equal to

\[
\kappa \sum_{T \in T(H)^F} f^\rho(T) \epsilon(G^F) \theta(T) \]

Now

\[
(\rho^{H,\theta} : R_{T'}^{\theta})_{G} = \kappa(\rho : R_{T'}^{1})_{H}
\]

This proves (a). We prove (b). By [DL, 7.5] we have \( \xi'(g) = 0 \). Hence

\[
\rho^{H,\theta}(g) = \kappa \sum_{T \in T(H)^F} f^\rho(T) \epsilon_{H} \epsilon_{G}(g) R_{T}^{\theta}(g)
\]

The following result is well known.

[The rest of the document continues with more mathematical content, including equations and proofs.]
Lemma 3.11. Let $A, B, C$ be three conjugacy classes in $G^F$ and let $(a, b, c) \in A \times B \times C$. We have

$$(a) \quad \# \{(\bar{a}, \bar{b}, \bar{c}) \in A \times B \times C; \bar{a} \bar{b} \bar{c} = 1\} = \frac{\sum_{\rho \in \mathcal{E}(G^F)} \rho(a)\rho(b)\rho(c)}{\rho(1)}.$$ 

3.12. If $A, B, C$ are three semisimple conjugacy classes in $G^F$, we set

$$\mathcal{G}_{G,F;A,B,C} = \sum_{\rho \in \mathcal{E}(G^F)} q^{-\nu(Z^0_G(a)) - \nu(Z^0_G(b)) - \nu(Z^0_G(c)) + \nu(G) - \nu(G_{ad})} \times R_{f^\rho, Z^0_G(a)}(1) R_{f^\rho, Z^0_G(b)}(1) R_{f^\rho, Z^0_G(c)}(1) R_{f^\rho, G}(1)^{-1}$$

where $f^\rho : T(G)^F \to C$ are defined as in 3.3 and $(a, b, c) \in A \times B \times C$.

If, in addition, $H \in X_{G,F}'$ and $A', B', C'$ are three semisimple conjugacy classes in $H^F$ such that $A' \subset A, B' \subset B, C' \subset C$, we set

$$t_{G,F;H;A,B,C;A',B',C'} = \epsilon Z^0_G(a') \epsilon Z^0_H(a') \epsilon Z^0_G(b') \epsilon Z^0_H(b') \epsilon Z^0_G(c') \epsilon Z^0_H(c') \times \frac{H^F_{\rho'}}{H^F_{\rho'}} \times \frac{G^F_{\rho'}}{G^F_{\rho'}} \times q^{-\delta + \nu(Z^0_H(a')) + \nu(Z^0_H(b')) + \nu(Z^0_H(c')) - \nu(H) + \nu(H)}$$

where $(a', b', c') \in A' \times B' \times C'$ and

$$\delta = 3\nu(G) - \nu(Z^0_G(a)) - \nu(Z^0_G(b)) - \nu(Z^0_G(c)).$$

Note that $\epsilon Z^0_G(a) = \epsilon Z^0_G(a'), \epsilon Z^0_G(b) = \epsilon Z^0_G(b'), \epsilon Z^0_G(c) = \epsilon Z^0_G(c')$. We set

$$Y_{G,F;H;A,B,C;A',B',C'} = t_{G,F;H;A,B,C;A',B',C'} \mathcal{G}_{H,F;A',B',C'} q^{-\dim Z^0_H} \sum_{\theta \in H^F} \theta(a' b' c'),$$

$$X_{G,F;H;A,B,C} = \sum_{A', B', C'} Y_{G,F;H;A,B,C;A',B',C'},$$

where $A'$ (resp. $B', C'$) runs through the conjugacy classes in $H^F$ that are contained in $A$ (resp. $B, C$).

Let $X'_{G,F}$ be a set of representatives for the orbits of the natural $G^F$-action on $X_{G,F}'$. 
Lemma 3.13. Assume that $Z_G = Z_G^0$. We have

$$q^{-\delta} \sharp \{ (\tilde{a}, \tilde{b}, \tilde{c}) \in A \times B \times C; \tilde{a} \tilde{b} \tilde{c} = 1 \} = \sum_{H \in \mathcal{X}'_{G,F}} \frac{X_{G,F;H,A,B,C}}{N_G(H)^F / H^F}$$

We rewrite the right hand side of 3.11(a) using the partition 3.9(a), the bijection 3.9(b) and using Lemma 3.8; we obtain

$$\sharp \{ (\tilde{a}, \tilde{b}, \tilde{c}) \in A \times B \times C; \tilde{a} \tilde{b} \tilde{c} = 1 \} = \sum_{H \in \mathcal{X}'_{G,F}} N_G(H)^F / H^F X'_{G,F;H,A,B,C}$$

where for any $H \in \mathcal{X}'_{G,F}$ we set

$$X'_{G,F;H,A,B,C} = \frac{A | B | C}{G^F} \sum_{\theta \in H_F} \sum_{\rho \in E_1(H^F)} \frac{\rho H, \theta(a) \rho H, \theta(b) \rho H, \theta(c)}{\rho H, \theta(1)}.$$

We evaluate $\rho H, \theta(a), \rho H, \theta(b), \rho H, \theta(c)$ using 3.10(b), and $\rho H, \theta(1)$ using 3.10(c); we obtain $X'_{G,F;H,A,B,C} = q^d X_{G,F;H,A,B,C}$ (see 3.12). The lemma follows.

Lemma 3.14. (a) In the setup of 3.12 we have $|t_{G,F;H,A,B,C;A',B',C'}| \leq c$ where $c > 0$ is an integer depending only on the Coxeter graph on $G$ (and not on $q, F, H, A, B, C, A', B', C'$).

(b) More precisely,

$$|t_{G,F;H,A,B,C;A',B',C'} - \frac{\epsilon Z_G^0(a') \epsilon Z_G^0(b') \epsilon Z_G^0(c') \epsilon Z_G^0(\nu(a') \nu(b') \nu(c') \nu)}{Z(a') \epsilon Z_H(b') \epsilon Z_H(c')}| \leq c' q^{-1}$$

where $c' > 0$ is an integer depending only on the Coxeter graph on $G$ (and not on $q, F, H, A, B, C, A', B', C'$).

This is easily checked.

Lemma 3.15. Assume that $Z_G = Z_G^0$.

(a) There exists an integer $c > 0$ depending only on the Coxeter graph of $G$ (and not on $q, F$) such that $|\mathcal{X}'_{G,F}| \leq c$.

(b) There exists an integer $c' > 0$ depending only on the Coxeter graph of $G$ (and not on $q, F$) such that for any $H \in \mathcal{X}'_{G,F}$ we have $|H^F - H_F| \leq c' q^{\dim(Z_G^0)^{-1}}$.

Let $G^*$ be a connected reductive group of type dual to that of $G$. Let $F' : G^* \rightarrow G^*$ be the Frobenius map corresponding to an $F_q$-rational structure on $G$. Let $\mathcal{Y}_{G^*,F'}$ be the set of all pairs $(E, g)$ where $g \in G^* F'$ is semisimple and $E = Z_{G^*}(g)$. Let $\mathcal{Y}'_{G^*,F'}$ be the set of all subgroups $E$ of $G^*$ such that $(E, g) \in \mathcal{Y}_{G^*,F'}$ for some
g. For $E \in \mathcal{Y}'_{G^*,F'}$, let $Z_{E,\triangleright} = \{g \in Z_E; Z_{G^*}(g) = E\}$. Let $Z_{E,\triangleright}$ be the union of all connected components of $Z_E$ that contain some point in $Z_{E,\triangleright}'$. Let $\mathcal{Y}'_{G^*,F'}$ be a set of representatives for the orbits of the natural $G^*F'$-action on $\mathcal{Y}'_{G^*,F'}$.

Using [DL, 5.20, 5.24] we can reduce the statements of the lemma to statements about $G^*$:

(a') There exists an integer $c > 0$ depending only on the Coxeter graph of $G^*$ (and not on $q, F'$) such that $\overline{\mathcal{Y}'_{G^*,F'}} \leq c$.

(b') There exists an integer $c' > 0$ depending only on the Coxeter graph of $G^*$ (and not on $q, F'$) such that for any $(E, g) \in \mathcal{Y}'_{G^*,F'}$ we have $Z_{E,\triangleright}' - Z_{E,\triangleright}' \leq c'q^{\dim(Z_{E,\triangleright})-1}$.

The proof of (a') is standard. For (b') we note that any connected component of $Z_{E,\triangleright}$ is an irreducible variety isomorphic to $Z_E^0$ and its intersection with $Z_{E,\triangleright}$ is open and dense in it. The lemma is proved.

3.16. Let $H$ be an $F$-stable connected reductive subgroup of $G$ such that $r(H) = r(G)$. Let $W$ be the Weyl group of $G$ and let $W'$ be the Weyl group of $H$. Note that $F$ acts naturally on $W$ and on $W'$.

Let $T_0, B_0$ be a pair consisting of an $F$-stable maximal torus of $G$ and an $F$-stable Borel subgroup of $G$ containing $T_0$. Let $T'_0, B'_0$ be a pair consisting of an $F$-stable maximal torus of $H$ and an $F$-stable Borel subgroup of $H$ containing $T'_0$. We identify $W = N_G(T_0)/T_0$, $W' = N_H(T'_0)/T'_0$ in the standard way. We choose $\gamma \in G$ such that $\gamma T_0 \gamma^{-1} = T'_0$. Let $m = \gamma^{-1}F(\gamma)$. We have $m \in N_G(T_0)$. We have an imbedding $W' \subset W$ induced by $N_H(T'_0) \to N_G(T_0), h \mapsto \gamma^{-1}h \gamma$. We identify $W'$ with its image under this imbedding. The map $F : W' \to W'$ corresponds under this identification to the restriction of the map $F' : W \to W$ induced by $N_G(T_0) \to N_G(T_0), g \mapsto mF(g)m^{-1}$.

Let $E$ be an irreducible representation of $W$ with a given isomorphism $\lambda_E : E \to E$ of finite order such that $\lambda_E(we) = F(w)\lambda_E(e)$ for all $w \in W, e \in E$.

Define $f_E : T(G)^E \to C$ by $f_E(T) = \text{tr}(w_T\lambda_E : E \to E)$ where $w_T$ is the image in $W$ of $z^{-1}F(z) \in N_G(T_0)$ where $z \in G$ is such that $zT_0z^{-1} = T$. (Note that $f_E(T)$ is independent of the choice of $z$.)

Define $V_j'$ in terms of $W'$ in the same way as $V_j$ was defined in 2.1 in terms of $W$. Then $W'$ acts naturally on $V_j'$ and there is a natural isomorphism $\lambda_j : V_j' \to V_j'$ of finite order such that $\lambda_j(w'x) = F'(w')\lambda_j(x)$ for all $w' \in W', x \in V_j'$.

We define a linear map $\bar{\lambda}_j : \text{Hom}_{W'}(E, V_j') \to \text{Hom}_{W'}(E, V_j')$ by $(\bar{\lambda}_j(\xi))(e) = \lambda_j^{-1}(\xi(m\lambda_E(e)))$ for all $\xi \in \text{Hom}_{W'}(E, V_j'), e \in E$. (One checks easily that $\bar{\lambda}_j(\xi) \in \text{Hom}_{W'}(E, V_j').$) One also checks that $\bar{\lambda}_j$ is a map of finite order. With this notation, we have

\[
R_{f_E,H}(1) = \sum_{j \geq 0} \text{tr}(\bar{\lambda}_j, \text{Hom}_{W'}(E, V_j'))q^j.
\]
This follows in a standard way from the definitions using the formula [DL, 7.1] for $R^1_{T,G}(1)$.

**Lemma 3.17.** Let $H \in \mathcal{X}'_{G,F}$. Let $g \in H^F$ be a semisimple element. Assume that $Z^0_H(g)$ is split over $F_q$ and that $F$ acts trivially on $\tilde{Z}_H(g)$. Let $g' \in H^F$ be such that $g, g'$ are conjugate in $H$. Then:

(a) $\epsilon_{Z^0_H(g')} = \epsilon_{Z^0_H(g)}$;
(b) $F$ acts trivially on $\tilde{Z}_H(g')$.

We can find a maximal torus $T$ of $Z^0_H(g)$ such that $F(T) = T$ and $T$ is $F_q$-split. Let $T'$ be a maximal torus of $Z^0_H(g')$ such that $F(T') = T'$. We can find $x \in H$ such that $xgx^{-1} = g', xTx^{-1} = T'$. Let $m = x^{-1}F(x)$. Then $m \in Z_H(g)$ and $m \in N_H(T)$. Let $\mathcal{R}_G, \mathcal{R}_H$ (resp. $\mathcal{R}'_G, \mathcal{R}'_H$) be the set of roots of $Z^0_H(g), Z^0_H(g')$ (resp. $Z^0_H(g'), Z^0_H(g)$) with respect to $T$ (resp. $T'$). Then $F$ acts naturally on $\mathcal{R}_G, \mathcal{R}_H, \mathcal{R}'_G, \mathcal{R}'_H$. Let $\mathcal{R}'_G^+, \mathcal{R}'_H^+$ be a set of positive roots for $\mathcal{R}'_G, \mathcal{R}'_H$ respectively. It is well known that

$$\epsilon_T \epsilon_{Z^0_H(g')} = (-1)^{\sharp(\alpha \in \mathcal{R}'_G^+; F(\alpha) \in \mathcal{R}_G - \mathcal{R}'_G^+)},$$

$$\epsilon_T \epsilon_{Z^0_H(g')} = (-1)^{\sharp(\alpha \in \mathcal{R}'_H^+; F(\alpha) \in \mathcal{R}_H - \mathcal{R}'_H^+)}.$$

Now $\text{Ad}(x)$ establishes bijections $\mathcal{R}_G \cong \mathcal{R}'_G$ and $\mathcal{R}_H \cong \mathcal{R}'_H$ under which the action of $F$ on $\mathcal{R}'_G, \mathcal{R}'_H$ corresponds to the action given by $\text{Ad}(m)$ on $\mathcal{R}_G, \mathcal{R}_H$ (since $F$ acts trivially on $\mathcal{R}_G, \mathcal{R}_H$). Also under this bijection, $\mathcal{R}'_G^+$ (resp. $\mathcal{R}'_H^+$) corresponds to a set of positive roots $\mathcal{R}_G^+$ (resp. $\mathcal{R}_H^+$) for $\mathcal{R}_G$ (resp. $\mathcal{R}_H$). It follows that

$$\epsilon_T \epsilon_{Z^0_H(g')} = (-1)^{\sharp(\alpha \in \mathcal{R}_G^+; \text{Ad}(m)(\alpha) \in \mathcal{R}_G - \mathcal{R}_G^+)},$$

$$\epsilon_T \epsilon_{Z^0_H(g')} = (-1)^{\sharp(\alpha \in \mathcal{R}_H^+; \text{Ad}(m)(\alpha) \in \mathcal{R}_H - \mathcal{R}_H^+)}. $$

The right hand sides of the previous two equalities may be also interpreted as the determinant of the linear map induced by $\text{Ad}(m)$ on the group of characters of $T$. (We use that $m$ defines an element in the Weyl group of $Z^0_G(g)$ or $Z^0_H(g)$ with respect to $T$.) Hence those right hand sides coincide. It follows that (a) holds.

Now $\text{Ad}(x)$ also induces an isomorphism $Z_H(g) \cong Z_H(g')$ under which the action of $F$ on $Z_H(g')$ corresponds to the action of $\text{Ad}(m)$ on $Z_H(g)$ (since $F$ acts trivially on $Z_H(g)$). Hence to prove (b) it is enough to show that $\text{Ad}(m)$ acts trivially on $Z_H(g)$. Since $m \in Z_H(g)$, this follows from the well known fact that $Z_H(g)$ is commutative. The lemma is proved.

**3.18.** In this subsection we assume that $(H, F)$ is like $(G, F)$ in 3.1. We assume that $H = H_{der}$ and that $C$ is a semisimple conjugacy class in $H$. Let $a \in C$. We assume that $F(a) = a$ and that $F$ acts trivially on $\tilde{Z}_H(a)$. We assume also that $F$ acts trivially on $Z = Z_H\hat{}$.

(a) Define $\mu : Z \to H^F / \pi(\hat{H}^F)$ by $z \mapsto \pi(h)$ where $h \in \hat{H}$ satisfies $h^{-1}F(h) = z$. Then $\mu$ is a group isomorphism.

The proof is standard.

Let $Z'$ be the set of all $z \in Z$ such that $\tilde{a}, z\tilde{a}$ are conjugate in $\hat{H}$; here $\tilde{a} \in \pi^{-1}(a) \subset \hat{H}$. (This definition does not depend on the choice of $\tilde{a}$.) Note that $Z'$
is a subgroup of $Z$. Let $X$ be the set of $H^F$-conjugacy classes contained in $C^F$.

(b) Define $\mu' : Z' \to X$ by $z \mapsto C_{HF}(\pi(g)a\pi(g)^{-1})$ where $g \in \hat{H}$ is such that $g^{-1}F(g) = g_1g_1\tilde{g}_1^{-1} = za$. Then $\mu'$ is a bijection.

The proof is standard.

(c) The composition $Z' \to Z \xrightarrow{\mu'} H^F/\pi(\hat{H}^F)$ (the first map is the inclusion) coincides with the composition $Z' \xrightarrow{\mu'} X \to H^F/\pi(\hat{H}^F)$ (the second map is $A \mapsto \pi(g)$ where $g \in A$).

This follows from the definitions.

(d) Assume that $N \geq 1$ is an integer such that $a^N = 1$. Then $z^N = 1$ for all $z \in Z'$.

Indeed if $z \in Z'$ we have $z\tilde{a} = g\tilde{a}g^{-1}$ for some $g \in G$. Taking $N$-th powers gives $z^N\tilde{a}^N = g\tilde{a}^N g^{-1}$ hence (using $\tilde{a}^N \in Z$): $z^N\tilde{a}^N = gg^{-1}\tilde{a}^N = \tilde{a}^N$, hence $z^N = 1$.

Assume now that $(C''_2,C''_3,C''_5)$ is a 235-triple for $H$. Let $a_n \in C'_n$, $n = 2, 3, 5$.

We assume that $F(a_n) = a_n$ and $F$ acts trivially on $Z_H(a_n)$ for $n = 2, 3, 5$.

Let $Z'_n, X_n$ be defined like $Z', X$ in terms of $C'_n, a_n$ instead of $C, a$.

(e) The map $X_2 \times X_3 \times X_5 \to H^F/\pi(\hat{H}^F)$, given by $(A, B, C) \mapsto \pi(abc)$ where $(a, b, c) \in A \times B \times C$, is injective.

Using (a), (b), (c), we see that this is equivalent to the statement that the map $Z'_2 \times Z'_3 \times Z'_5 \to Z$ given by multiplication in $Z$ is injective. From (d) we see that any element $z \in Z'_n$ satisfies $z^n = 1$. Then the injectivity of the map above follows from the fact that an element of a finite abelian group can be written in at most one way as a product of elements of order dividing $2, 3, 5$.

In the remainder of this subsection we assume in addition that $H \in \mathcal{X}'_{G,F}$ where $(G, F)$ as in 3.1 is such that $G = G_{ad}$, $G$ simple of type $E_m$, $m = 6, 7, 8$. (Recall that $H = H_{der}$.)

(f) The image $I$ of the map in (e) is exactly $U = \{ x \in H^F/\pi(\hat{H}^F); x^{30} = 1 \}$.

By (d), we have $I \subset U$. Now $\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} Z'_2 & Z'_3 & Z'_5 \\ Z'_2 & Z'_3 & Z'_5 \end{bmatrix}$ and $\begin{bmatrix} U \\ I \end{bmatrix}$ can be easily determined by inspection of the various cases. We find that $\begin{bmatrix} I \\ U \end{bmatrix}$ hence $I = U$.

((f) ought to be true also in type $D$, but we have not checked it, as we don’t need it.) In particular, the map in (e) is a bijection except if $(G, H)$ is of type $(E_8, D_5 \times A_3)$ or $(E_7, A_3 \times A_3 \times A_1)$ in which case the image has index 2.

**Lemma 3.19.** Assume that $G = G_{ad}$. Let $F_0 : G \to G$ be the Frobenius map corresponding to an $\mathbf{F}_p$-rational structure on $G$. There exists an integer $s_0 \geq 1$ such that if $s \geq 1$ is an integer divisible by $s_0$, and $H \in \mathcal{X}'_{G,F_0}$ is semisimple, then $H$ is split over $\mathbf{F}_{p^s}$.

Let $G^*$ be as in the proof of 3.15. Let $F_0' : G^* \to G^*$ be the Frobenius map corresponding to an $\mathbf{F}_p$-rational structure on $G^*$. As in the proof of 3.15, we see that it is enough to verify the following statement:

(a) There exists an integer $s_0 \geq 1$ such that if $s \geq 1$ is an integer divisible by $s_0$, and $g \in G^{*F_0''}$ is a semisimple element such that $H = Z_{G^*}(g)$ is semisimple, then $H$ is split over $\mathbf{F}_{p^s}$. 
The set of all $g' \in G^*$ such that $g'$ is semisimple and $Z_{G^*}(g')$ is semisimple is a union of finitely many conjugacy classes $A_1, A_2, \ldots, A_m$ in $G^*$. Pick $g_r \in A_r$ for $r \in [1, m]$. Let $H_r = Z_{G^*}(g_r)$. We can find an integer $s_0 \geq 1$ such that $g_r \in G^{*F_0^{s_0}}$ and $H_r$ is split over $\mathbf{F}_{p^{s_0}}$, $r = 1, \ldots, m$. If $s \geq 1$ is an integer divisible by $s_0$ then $g_r \in G^{*F_0^{s}}$ and $H_r$ is split over $\mathbf{F}_{p^{s}}$, $r = 1, \ldots, m$. Now let $g \in G^{*F_0^{s}}$ be such that $H = Z_{G^*}(g)$ is semisimple. We have $g \in A_r$ for some $r$. Hence $g \in A_{r}^{F_0^{s}}$. Now $Z_{G^*}(g_r)$ is connected since $G^*$ is simply connected. It follows that $G^{*F_0^{s}}$ acts transitively (by conjugation) on $A_{r}^{F_0^{s}}$. In particular, $g, g_r$ are conjugate under $G^{*F_0^{s}}$. It follows that $H = Z_{G^*}(g)$ is split over $\mathbf{F}_{p^{s}}$. The lemma is proved.

4. Estimates

Lemma 4.1. Let $G, q, F$ be as in 3.1. Let $A, B, C$ be conjugacy classes in $G^F$ such that $A \subset C_2, B \subset C_3, C \subset C_5$ where $(C_2, C_3, C_5)$ is a 235-triple in $G$. Assume that the assumptions of 2.2 hold. Then $|\mathcal{G}_{G,F;A,B,C}| \leq c$ where $c$ is a constant depending only on $G$ (not on $F, q, A, B, C$). Moreover, if $(C_2, C_3, C_5)$ is not regular, then $|\mathcal{G}_{G,F;A,B,C}| \leq c'q^{-1}$ where $c'$ is a constant depending only on $G$ (not on $F, q, A, B, C$).

We use the notation of 2.1. We may assume that $(g_2, g_3, g_5) \in A \times B \times C$. Now $F$ acts naturally on $W$. Hence it acts naturally on the set of families of $W$. By [L, 4.23] we have a partition

$$\mathcal{E}_1(G^F) = \sqcup_{\mathcal{F}} \mathcal{E}_{1,\mathcal{F}}(G^F),$$

($\mathcal{F}$ runs over the $F$-stable families in $W$) such that the following holds: if $\rho \in \mathcal{E}_{1,\mathcal{F}}(G^F)$, then

$$f^\rho = \sum_{E} s_{\rho, E} f_E,$$

where $E$ runs through the representations in $\mathcal{F}$ that are "$F$-stable", $s_{\rho, E}$ are complex numbers that are bounded above (when $q$ varies) and $f_E : T(G)^F \to \mathbf{C}$ is as in 3.16. Using 3.16(a) we see that

$$R_{f_{\rho}, G}(1) = \sum_{E} s_{\rho, E} \sum_{j} \text{tr}(\sigma_j, \text{Hom}_W(E, V_j))q^j,$$

$$R_{f_{\rho}, Z_G(g_n)^0(1)} = \sum_{E} s_{\rho, E} \sum_{j} \text{tr}(\sigma_j^{(n)}, \text{Hom}_{W_n}(E|_{W_n}, V_j^{(n)}))q^j, \quad n = 2, 3, 5,$$

where $\sigma_j, \sigma_j^{(n)}$ are linear maps of finite order. Then

$$R_{f_{\rho}, Z_G(g_2)^0(1)}R_{f_{\rho}, Z_G(g_3)^0(1)}R_{f_{\rho}, Z_G(g_5)^0(1)}$$

is a linear combination with bounded coefficients of powers $q^j$ where

$$j \leq b'_2(E) + b'_3(E') + b'_5(E'')$$

with $E, E', E'' \in \mathcal{F}$ (hence $j \leq \sum_{n=2,3,5} a_n(\mathcal{F})$), while $R_{f_{\rho}, G}(1)$ is a linear combination with bounded coefficients of powers $q^j$ where $j \leq b'(E)$ with $E \in \mathcal{F}$ (hence $j \leq a'(\mathcal{F})$) and in
fact it is known that the coefficient of $q^{\alpha'(\mathcal{F})}$ is non-zero and its inverse is bounded above. Now

$$G_{G,F;A,B,C} = \sum_{\mathcal{F}} G_{G,F;A,B,C;\mathcal{F}}$$

where $\mathcal{F}$ runs over the $F$-stable families in $W$ and $G_{G,F;A,B,C;\mathcal{F}}$ is defined like $G_{G,F;A,B,C}$ but with $\rho$ restricted to $\mathcal{E}_1(\mathcal{F})$. The arguments above show, using 2.2(b), that each $G_{G,F;A,B,C;\mathcal{F}}$ is bounded above; and, in the case where $(C_2, C_3, C_5)$ is not regular, that each $\theta G_{G,F;A,B,C;\mathcal{F}}$ is bounded above. The lemma follows.

**4.2.** In the setup of 4.1, assume that $(C_2, C_3, C_5)$ is regular. By the method of 4.1, we see that

(a) $q(G_{G,F;A,B,C} - 1)$ is bounded above.

**Lemma 4.3.** Let $G$ be as in 3.1. Let $(C_2, C_3, C_5)$ be a regular 235-triple in $G_{der}$. Then there exists $\psi \in \text{Hom}(A_5, G)$ such that $\psi$ gives rise to $(C_2, C_3, C_5)$ as in 1.7.

Since this statement depends only on $G_{der}$ and we can find an imbedding of $G_{der}$ into a connected reductive group with connected centre and derived subgroup $G_{der}$, we see that we may assume that $G$ has connected centre. We may also assume that $\dim G > 0$ and that the lemma is true when $G$ is replaced by a group whose derived group has strictly smaller dimension than that of $G_{der}$. Assume that there is no $\psi$ as in the lemma (for $G$). Let $(g_2, g_3, g_5) \in C_2 \times C_3 \times C_5$. Since $g_n \in G_{der}$, we may choose $q, F$ as in 3.1 in such a way that $g_n \in \pi(\tilde{G}^F)$ for $n = 2, 3, 5$. We may also assume that $G$, $Z^0_G(g_2), Z^0_G(g_3), Z^0_G(g_5)$ are split over $F_q$ and that $q$ is large.

We write the identity in 3.13 for $(A, B, C) = (C_G(g_2), C_G(g_3), C_G(g_5))$. The left hand side of that identity is 0, by our assumption. We deduce that

(a) $\sum_{H \in \bar{X}_{G,F}} N_G(H)^F/H^F^{-1} X_{G,F;H;A,B,C} = 0$.

We show that

(b) $qX_{G,F;H;A,B,C}$ is bounded above (when $q, F$ vary) for any $H \in \bar{X}_{G,F}, H \neq G$.

With notation as in 3.12, we have $qX_{G,F;H;A,B,C} = \sum' + \sum''$ where $\sum', \sum''$ are given by

$$\sum_{(A', B', C')} t_{G,F;H;A,B,C;A', B', C'} G_{H,F;A', B', C'} q^{(- \dim Z^0_H \sum_{\theta \in H^F} \theta(a'b'c')]}$$

in $\sum'$, (resp. $\sum''$), $(A', B', C')$ runs over all triples of conjugacy classes in $H^F$ such that $A' \subset A, B' \subset B, C' \subset C$ and $A' \subset \tilde{A}', B' \subset \tilde{B}', C' \subset \tilde{C}'$ where $(\tilde{A}', \tilde{B}', \tilde{C}')$ is a 235-triple in $H$, and $(\tilde{A}', \tilde{B}', \tilde{C}')$ is regular (resp. non-regular) for $H$.

Now $\sum''$ is bounded above; indeed, in each term, $t_{G,F;H;A,B,C;A', B', C'}$ is bounded above (by 3.14), $qG_{H,F;A', B', C'}$ is bounded above (by 4.1) and $q^{- \dim Z^0_H \sum_{\theta \in H^F} \theta(a'b'c')}$ is bounded above since $q^{- \dim Z^0_H H^F}$ is bounded above.
We show that \( \sum' \) is bounded above. In each term, \( t_{G,F;H,A,B,C;A',B',C'} \) is bounded above (by 3.14), \( G_{H,F;A',B',C'} \) is bounded above (by 4.1) and it is enough to show that \( qq^{-\dim Z_H^0} \sum_{\theta \in H^F} \theta(a'b'c') \) is bounded above or that

\[
qq^{-\dim Z_H^0} \sum_{\theta \in H^F} \theta(a'b'c') - qq^{-\dim Z_H^0} \sum_{\theta \in H^F - H^F} \theta(a'b'c')
\]

is bounded above. Now \( qq^{-\dim Z_H^0} \sum_{\theta \in H^F - H^F} \theta(a'b'c') \) is bounded above since \( qq^{-\dim Z_H^0} \sum_{\theta \in H^F - H^F} \theta(a'b'c') \) is bounded above (by 3.15(b)). Hence it is enough to show that \( \sum_{\theta \in H^F} \theta(a'b'c') = 0. \) Since \( H^F \) is a union of \( H^F \)-cosets in \( H^F \), it is enough to show that \( a'b'c' \notin H_{der} \). Assume that \( a'b'c' \in H_{der} \) for some (hence any) \( (a',b',c') \in A' \times B' \times C' \). Since \( a',b',c' \) have orders dividing 2, 3, 5, it follows that each of \( a',b',c' \) is in \( H_{der} \). Hence \( A', B', C' \) are contained in \( H_{der} \). Since \( (A',B',C') \) is regular in \( H \), we may apply to it the induction hypothesis; we see that there exists \( \psi \in \text{Hom}(A_5, H) \) such that \( \psi(x_2) \in A', \psi(x_3) \in B', \psi(x_5) \in C' \). Since \( A' \subset C_2, B' \subset C_3, C' \subset C_5 \), we see that \( \psi(x_n) \in C_n \) for \( n = 2, 3, 5 \). This contradicts our assumption. Thus, we have \( a'b'c' \notin H_{der} \) and the boundedness of \( \sum' \) (hence (b)) is established.

Using now (a), we deduce that \( qX_{G,F;G;A,B,C} \) is bounded above. Setting \( t = t_{G,F;G;A,B,C;A,B,C} \), we have

\[
qX_{G,F;G;A,B,C} = qtGq^{-\dim Z_G^0} \sum_{\theta \in G^F} \theta(g_2g_3g_5) = qtGq^{-\dim Z_G^0} \sum_{\theta \in G^F} \theta(G^F)\theta_{G^F}
\]

since \( g_2g_3g_5 \in \pi(G^F) \). From 3.14(b) we see that \( t^{-1} \) is bounded above and it is clear that \( q^{\dim Z_G^0} \sum_{\theta \in G^F} \theta(G^F) \theta_{G^F} \) is bounded above. It follows that \( qG \) is bounded above. This contradicts 4.2. The lemma is proved.

4.4. Assume now that \( G \) (as in 1.2) is adjoint of type \( E_8 \) and that \( (C_2, C_3, C_5) \) is the unique regular 235 triple of \( G \). Let \( d \) be the number of \( G \)-orbits on \( \text{Hom}_{reg}(A_5, G) \). To determine \( d \) we may assume that \( k \) is as in 3.1. Let \( q, F \) be as in 3.1. We write the identity 3.11(a) for \( (A, B, C) = (C_2^F, C_3^F, C_5^F) \) (these are three conjugacy classes of \( G^F \)). Assume that we can evaluate the right hand side of 3.11(a) for infinitely many \( q \) and that it is of the form \( \Delta q + \) lower powers of \( q \) where \( \Delta \) is a constant. Then 3.11(a) implies that \( \{ (a, b, c) \in C_2 \times C_3 \times C_5; abc = 1 \} \) has exactly \( \Delta \) irreducible components of dimension \( D \) and it follows that \( d = \Delta \). (This strategy was suggested in [S].)

Theorem 4.5. Let \( G \) be as in 1.2 and let \( (C_2, C_3, C_5) \) be a regular 235-triple of \( G \). Assume that \( G = G_{ad} \). Let \( N = Z_G^0 \). Let \( g_n \in C_n \) and let \( N_n = Z_G(g_n) \).
n = 2, 3, 5. Then, up to G-conjugacy, there are exactly \( \frac{N}{N_2 N_3 N_5} \) elements \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) that give rise to \((C_2, C_3, C_5)\).

We may assume that \( G \) is simple. In the case where \( G \) is of type \( A \) or \( D \), the theorem can be obtained from the results in §1. In the rest of the proof we assume that \( G \) is of type \( E_m, m = 6, 7, 8 \) (although the same proof should work without this assumption).

We will carry out the strategy in 4.4, using the results in §2, §3. We may assume that \( k \) is as in 3.1. Let \( F_0 \) be a Frobenius map for an \( F_p \)-rational structure on \( G \). Let \( q = p^{s'} \), \( F = F_0^{s'} \) where \( s' \) is sufficiently large and divisible by a fixed integer \( s_0' \geq 1 \). By choosing \( s_0' \) appropriately we may assume that \( F \) acts trivially on \( Z_G \) that \( g_n \in \pi(\tilde{G}^F) \), that \( Z_G^0(g_n) \) is split over \( F_q \) and that \( F \) acts trivially on \( Z_G(g_n) \) for \( n = 2, 3, 5 \). (The choice of \( s_0' \) will be made more precise later in the proof.) Let \( \delta = 3\nu - \nu_2 - \nu_3 - \nu_5 \). (Notation of 2.1.) Let \( \Delta = \frac{N}{N_2 N_3 N_5} \). We will show that

\[
q(q^{-\delta} \#\{(g_2, g_3, g_5) \in C_2^F \times C_3^F \times C_5^F : g_2 g_3 g_5 = 1\} - \Delta)
\]

is bounded above when \( q \) varies or equivalently, that

\[
q(q^{-\delta} \sum_{(A,B,C) \in \mathcal{X}} \#\{(g_2, g_3, g_5) \in A \times B \times C : g_2 g_3 g_5 = 1\} - \Delta)
\]

is bounded above. Here \( \mathcal{X} \) is the set of triples \((A, B, C)\) of conjugacy classes in \( G^F \) such that \( A \subset C_2, B \subset C_3, C \subset C_5 \). By 3.13, it is enough to show that

\[
(b) \quad q(\sum_{(A,B,C) \in \mathcal{X}} X_{G,F;G;A,B,C} - \Delta) \text{ is bounded above},
\]

\[
(c) \quad q(\sum_{(A,B,C) \in \mathcal{X}} X_{G,F;H;A,B,C}) \text{ is bounded above for any } H \in \mathcal{X}^F, H \neq G.
\]

We have

\[
\sum_{(A,B,C) \in \mathcal{X}} X_{G,F;G;A,B,C} = \sum_{A,B,C} t_{G,F;G;A,B,C} G_{G,F;A,B,C} \sum_{\theta \in G_{\mathcal{X}}} \theta(abc)
\]

where \((a, b, c) \in A \times B \times C\). The sum over \( \theta \) is 0 unless \( abc \in \pi(\tilde{G}^F) \) (or equivalently, each of \( a, b, c \) is in \( \pi(\tilde{G}^F) \)), in which case it is \( N \). Thus, (b) is equivalent to the statement that

\[
q(t_{G,F;G;A,B,C} G_{G,F;A,B,C} N - \Delta) \text{ is bounded above};
\]

here \((A, B, C) \in \mathcal{X} \) is uniquely determined by the condition

\[
A \subset \pi(\tilde{G}^F), B \subset \pi(\tilde{G}^F), C \subset \pi(\tilde{G}^F).
\]

This follows from the fact that \( q(t_{G,F;G;A,B,C} - (N_2 N_3 N_5)^{-1}) \) is bounded above (see 3.14, 3.17) and \( q(G_{G,F;A,B,C} - 1) \) is bounded above (see 4.2). Thus, (b) holds.

Now, in (c), we have (as in the proof of 4.3)

\[
q(\sum_{(A,B,C) \in \mathcal{X}} X_{G,F;H;A,B,C} = \sum' + \sum'')
\]

where \( \sum', \sum'' \) are given by

\[
\sum_{(A',B',C')} t_{G,F;H;A,B,C;A',B',C'} G_{H,F;A',B',C'} qq^{-\dim Z^0_H} \sum_{\theta \in H_{\mathcal{X}}} \theta(a'b'c');
\]
in \(\sum'\), (resp. \(\sum''\)), \((A',B',C')\) runs over the set \(\mathcal{X}'\) (resp. \(\mathcal{X}''\)) consisting of all triples of conjugacy classes in \(H\), such that \(A' \subset \tilde{A}', B' \subset \tilde{B}', C' \subset \tilde{C}'\) where \((\tilde{A}',\tilde{B}',\tilde{C}')\) is a \(235\)-triple in \(H\), \(\tilde{A}' \subset C_2, \tilde{B}' \subset C_3, \tilde{C}' \subset C_5\) and \((\tilde{A}',\tilde{B}',\tilde{C}')\) is regular (resp. non-regular) for \(H\); moreover \((A,B,C)\) is the unique triple of conjugacy classes in \(G\) such that \(A' \subset A, B' \subset B, C' \subset C\).

Now \(\sum''\) is bounded above (exactly as in the proof of 4.3).

We now estimate \(\sum'.\)

Assume first that \(\dim Z_H > 0\). Again, \(\sum'\) is bounded above: in each term, \(t_{G,F;H,A,B,C;A',B',C'}\) is bounded above (by 3.14), \(\mathcal{G}_{H,F;A',B',C'}\) is bounded above (by 4.1) and it is enough to show that \(qq^{-\dim Z_H^0} \sum_{\theta \in H^\bullet} \theta(a'b'c')\) is bounded above or that

\[
qq^{-\dim Z_H^0} \sum_{\theta \in H^\xi} \theta(a'b'c') - qq^{-\dim Z_H^0} \sum_{\theta \in H^\xi-H^\bullet} \theta(a'b'c')
\]

is bounded above. Now \(qq^{-\dim Z_H^0} \sum_{\theta \in H^\xi-H^\bullet} \theta(a'b'c')\) is bounded above since

\[
qq^{-\dim Z_H^0} \boxed{H^\xi - H^\bullet}
\]

is bounded above (by 3.15(b)). Hence it is enough to show that

\[
\sum_{\theta \in H^\xi} \theta(a'b'c') = 0.
\]

Since \(H^\xi\) is a union of \(H^\xi\)-cosets in \(H^\xi\), it is enough to show that \(a'b'c' \notin H_{\text{der}}\). Assume that \(a'b'c' \in H_{\text{der}}\) for some (hence any) \((a',b',c') \in A' \times B' \times C'\). Since \(a',b',c'\) have orders dividing 2, 3, 5, it follows that each of \(a',b',c'\) is in \(H_{\text{der}}\). Hence \(\tilde{A}',\tilde{B}',\tilde{C}'\) are contained in \(H_{\text{der}}\). Since \((\tilde{A}',\tilde{B}',\tilde{C}')\) is regular in \(H\), we may apply to it Lemma 4.3; we see that there exists \(\psi \in \text{Hom}(A_5,H)\) such that \(\psi(x_2) \in \tilde{A}', \psi(x_3) \in \tilde{B}', \psi(x_5) \in \tilde{C}'\). Since \(\tilde{A}' \subset C_2, \tilde{B}' \subset C_3, \tilde{C}' \subset C_5\), we see that \(\psi(x_n) \in C_n\) for \(n = 2, 3, 5\). Since \((C_2,C_3,C_5)\) is regular for \(G\), we have \(\psi \in \text{Hom}_{\text{reg}}(A_5,G)\). This contradicts the fact that \(Z_G(\psi(A_5))\) contains the non-trivial torus \(Z_H^0\). Thus, we have \(a'b'c' \notin H_{\text{der}}\) and the boundedness of \(\sum'\) is established.

Next, assume that \(\dim Z_H = 0\). We show that \(\sum'\) is bounded above. Since for \((A',B',C') \in \mathcal{X}'\), \(q(\mathcal{G}_{H,F;A',B',C'} - 1)\) is bounded above (see 4.2) and \(t_{G,F;H,A,B,C;A',B',C'}\) is bounded above (see 3.14), it is enough to show that, for any regular \(235\)-triple \((C_2',C_3',C_5')\) of \(H\) such that \(C_n' \subset C_n, n = 2, 3, 5\), and for any \(\theta \in H^\bullet\),

\[
\sum_{(A',B',C') \in \mathcal{Y}} t_{G,F;H,A,B,C;A',B',C'} \theta(a'b'c')
\]

is bounded above. Here \(\mathcal{Y} = \{(A',B',C') \in \mathcal{X}' ; A' \subset C_2, B' \subset C_3, C' \subset C_5\}\) and \((a',b',c') \in A' \times B' \times C'\). Using 3.14(c) we see that it is enough to show that, for any \(\theta \in H^\bullet\),

\[
\sum_{(A',B',C') \in \mathcal{Y}} \tau_{A',B',C'}(a'b'c') = \frac{\varepsilon_{Z_H^0(a')}\varepsilon_{Z_H^0(b')}\varepsilon_{Z_H^0(c')}}{\varepsilon_{Z_H^0}(a')^r \varepsilon_{Z_H^0}(b')^r \varepsilon_{Z_H^0}(c')^r}.
\]

By choosing \(s'_0\) appropriately, we may assume that \(H\) is split over \(\mathbb{F}_q\) (see 3.19), that \(F\) acts trivially on \(\tilde{Z}_H\), that \(C_n'\) contains a \(\mathbb{F}_q\)-rational point \(h_n\) such that \(Z_H^0(h_n)\) is split over \(F_q\) and \(F\) acts trivially on \(Z_H(h_n)\), \(n = 2, 3, 5\). Using 3.17,
we see then that, in the last fraction, the numerator is 1 and the denominator is $\left[ \sum_{h_2} Z_H(h_2) \right] \left[ \sum_{h_3} Z_H(h_3) \right] \left[ \sum_{h_5} Z_H(h_5) \right]$; in particular, $\tau'_{A_5, B', C'}$ is independent of $A', B', C'$. Thus, we are reduced to showing that

$$\sum_{x \in \tilde{F}/\pi(\tilde{H}^F)} x^{30} \theta(a'b'c') = 0$$

Using the results at the end of 3.18, we see that the last equality is equivalent to the equality

$$\sum_{x \in \tilde{F}/\pi(\tilde{H}^F)} x^{30} \theta(x) = 0$$

where $\theta$ is regarded as a character of $H^F/\pi(\tilde{H}^F)$. More precisely, if $(G, \tilde{H})$ is not of type $(E_8, D_5 \times A_3)$ or $(E_7, A_4 \times A_3 \times A_1)$, we must show that $\sum_{x \in \tilde{F}/\pi(\tilde{H}^F)} x^{30} \theta(x) = 0$ and this follows from the fact that $\theta \neq 1$ (since $\theta \in \tilde{H}^\psi$). If $(G, \tilde{H})$ is of type $(E_8, D_5 \times A_3)$ or $(E_7, A_4 \times A_3 \times A_1)$, we must show that the restriction of $\theta$ to the subgroup of $H^F/\pi(\tilde{H}^F)$ ($\cong \mathbb{Z}/4\mathbb{Z}$ or $\cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$) consisting of all $x$ such that $x^2 = 1$ is $\neq 1$; this again follows from $\theta \in \tilde{H}^\psi$.

This proves (c). Thus, (a) is proved.

Let $X = \{(a, b, c) \in C_2 \times C_3 \times C_5; abc = 1\}$ (an algebraic variety defined over $F_q$). From (a) we see that there exists an integer $M > 0$ such that $q^s(q^{-\delta s}X^{F_\chi} - \Delta) \leq M$ for $s = 1, 2, \ldots$. As it is known, this implies that among the irreducible components of $X$ there are exactly $\Delta$ of maximum dimension, and that maximum dimension is $\delta$. By 1.3, any $G$-orbit on $X$ has dimension $\delta$. It follows that $X$ has exactly $\Delta$ $G$-orbits. Since $X$ is naturally in bijection with the set of all $\psi \in \text{Hom}(A_5, G)$ that give rise to $(C_2, C_3, C_5)$ we see that the theorem is proved.

4.6. For $G = G_{ad}$ of type $E_6, E_7, E_8$, the fraction $\frac{N}{N_{2N_3N_5}}$ is $\frac{3}{1 \cdot 3 \cdot 1} = 1$, $\frac{4}{2 \cdot 1 \cdot 1} = 1$, $\frac{6}{2 \cdot 3 \cdot 1} = 1$ respectively. We see that in type $E_8$ there is exactly one $\psi \in \text{Hom}_{reg}(A_5, G)$ up to $G$-conjugacy.

For $G = G_{ad}$ of type $A_1, A_2, A_3, A_4, A_5$, the fraction $\frac{N}{N_{2N_3N_5}}$ is $\frac{2}{2 \cdot 1 \cdot 1} = 1$, $\frac{3}{1 \cdot 3 \cdot 1} = 1$, $\frac{4}{2 \cdot 1 \cdot 1} = 2$, $\frac{5}{1 \cdot 1 \cdot 3} = 1$, $\frac{6}{2 \cdot 3 \cdot 1} = 1$ respectively.

For $G = G_{ad}$ of type $D_4, D_5, D_6, D_8$, the fraction $\frac{N}{N_{2N_3N_5}}$ is $\frac{4}{4 \cdot 1 \cdot 1} = 1$, $\frac{5}{2 \cdot 1 \cdot 1} = 2$, $\frac{4}{4 \cdot 1 \cdot 1} = 1$, $\frac{4}{4 \cdot 1 \cdot 1} = 1$ respectively.

5. Relation to homomorphisms $PGL_2(k) \to G$

5.1. In this section we assume that either $p = 0$ or $p$ is large enough. Let $G$ be as in 1.2. Let $\Phi : PGL_2(k) \to G$ be a homomorphism. Composing $\Phi$ with one of the two regular homomorphisms $A_5 \to PGL_2(k)$ (up to $PGL_2(k)$-conjugacy, see 1.8), we obtain two homomorphisms $A_5 \to G$ (which may or may not be $G$-conjugate).

In [S], it is pointed out that by applying this construction to a $\Phi$ whose image contains a regular unipotent element in $G$ (of type $E_m$) we obtain a regular $\psi$. Applying the same procedure for $G = G_{ad}$ of type $A_m$, we see that all regular $\psi$ in 1.8 can be thus obtained for $m = 2, 3, 5, 6$ but only one of the two regular $\psi$ is thus obtained for $m = 4$. This procedure can be applied for $G$ as in 1.9. In that case, for $m = 4$, the first two $\psi$ in 1.9(a) are obtained; for $m = 5$, the first $\psi$ in
1.9(b) is obtained; for \( m = 6 \), the first \( \psi \) in 1.9(c) is obtained; for \( m = 8 \), the first \( \psi \) in 1.9(d) is obtained.

Let us analyze what happens if we use some other \( \Phi \).

Let \( G \) be as in 1.9 and assume that \( \Phi \) is such that its image contains a subregular unipotent element of \( G \). For \( m = 4 \), we thus obtain the third and fourth \( \psi \) in 1.9(a); for \( m = 5 \), we thus obtain the second \( \psi \) in 1.9(b); for \( m = 6 \), we thus obtain the second and third \( \psi \) in 1.9(c).

Let \( G \) be as in 1.9(d). If \( \Phi \) is such that its image contains a unipotent element whose Jordan blocks in the standard representation have sizes 1, 3, 5, 7, then the associated \( \psi \) is regular.

If \( G \) (as in 1.10) is of type \( E_8 \) and \( \Phi \) is such that the centralizer of its image is the symmetric group in 5 letters, then the \( \psi \) obtained from \( \Phi \) by the procedure above is again regular (the corresponding 235-triple is the one described in 1.10). Using 4.5, the \( \psi \) thus obtained must be the same up to conjugacy as the one attached to a regular unipotent class.

These arguments together with 4.5 show that, if \( G = G_{ad} \) is simple, then any \( \psi \in \text{Hom}_{\text{reg}}(A_5, G) \) is obtained from some \( \Phi : PGL_2(k) \to G \), except for one case in type \( A_3 \).

References

[A] D. Alvis, Induce/restrict matrices for exceptional Weyl groups (1981), preprint, MIT.

[BL] W. M. Beynon and G. Lusztig, Some numerical results on the characters of exceptional Weyl groups, Mat. Proc. Camb. Phil. Soc. 84 (1978), 417-426.

[DL] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. Math. 103 (1976), 103-161.

[F1] D. D. Frey, Conjugacy of \( \text{Alt}_5 \) and \( SL(2,5) \)-subgroups of \( E_8(\mathbb{C}) \), Memoirs Amer. Math. Soc. 634 133 (1998).

[F2] D. D. Frey, Conjugacy of \( \text{Alt}_5 \) and \( SL(2,5) \)-subgroups of \( E_6(\mathbb{C}) \), \( F_4(\mathbb{C}) \) and a subgroup of \( E_8(\mathbb{C}) \) of type \( A_2E_6 \), J. Alg. 202 (1998), 414-454.

[F3] D. D. Frey, Conjugacy of \( \text{Alt}_5 \) and \( SL(2,5) \)-subgroups of \( E_7(\mathbb{C}) \), J. Group Th. 4 (2001), 277-323.

[C] M. Geck, G. Hiss, F. Lübeck, G. Malle, G. Pfeiffer, CHEVIE, a system for computing and processing generic character tables, AAECC 7 (1996), 175-210, available at http://www.math.rwth-aachen.de/~CHEVIE.

[L] G. Lusztig, Characters of reductive groups over a finite field, Ann. Math. Studies 107, Princeton Univ. Press, 1984.

[S] J.-P. Serre, Letter to D. D. Frey, July 23, 1998.