Fuzzy right (left) ideals in hypergroupoids and fuzzy bi-ideals in hypersemigroups

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Abstract. We introduce the concepts of fuzzy right and fuzzy left ideals of hypergroupoids and the concept of a fuzzy bi-ideal of an hypersemigroup and we show that a fuzzy subset $f$ of an hypergroupoid $H$ is a fuzzy right (resp. fuzzy left) ideal of $H$ if and only if $f \circ 1 \leq f$ (resp. $1 \circ f \leq f$) and for an hypersemigroup $H$, a fuzzy subset $f$ of $H$ is a bi-ideal of $H$ if and only if $f \circ 1 \circ f \leq f$. These characterizations are very useful for the investigation. The paper serves as an example to show the way we pass from fuzzy groupoids (semigroups) to fuzzy hypergroupoids (hypersemigroups).

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1 Introduction

In our paper in [2] we gave, among others, some equivalent definitions of fuzzy right (left) ideals and fuzzy bi-ideals in ordered semigroups which are very useful for applications. Besides, they show how similar is the theory of ordered semigroups based on fuzzy ideals with the theory of ordered semigroups based on ideals or on ideal elements. Using these definitions many results on fuzzy ordered semigroups or on fuzzy semigroups (without order) can be drastically simplified. The present paper is based on our paper in [2], and the aim is to show the way we pass from fuzzy groupoids (semigroups) to fuzzy hypergroupoids (hypersemigroups).

2 Main results

An hypergroupoid is a nonempty set $H$ with an hyperoperation

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$$
on $H$ and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of $H$) such that

$$A * B = \bigcup_{(a, b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ denotes the set of nonempty subsets of $H$).

The operation “*” is well defined. Indeed: If $(A, B) \in \mathcal{P}^*(H) \times \mathcal{P}^*(H)$, then $A * B = \bigcup_{(a, b) \in A \times B} (a \circ b)$. For every $(a, b) \in A \times B$, we have $(a, b) \in H \times H$, then $(a \circ b) \in \mathcal{P}^*(H)$, thus we get $A * B \in \mathcal{P}^*(H)$. If $(A, B), (C, D) \in \mathcal{P}^*(H) \times \mathcal{P}^*(H)$ such that $(A, B) = (C, D)$, then

$$A * B = \bigcup_{(a, b) \in A \times B} (a \circ b) = \bigcup_{(a, b) \in C \times D} (a \circ b) = C * D.$$

As the operation “*” depends on the hyperoperation “$\circ$”, an hypergroupoid can be also denoted by $(H, \circ)$ (instead of $(H, \circ, *)$).

If $H$ is an hypergroupoid then, for any $x, y \in H$, we have $x \circ y = \{x\} \ast \{y\}$. Indeed,

$$\{x\} \ast \{y\} = \bigcup_{u \in \{x\}, v \in \{y\}} u \circ v = x \circ y.$$

An hypergroupoid $H$ is called hypersemigroup if

$$(x \circ y) \ast \{z\} = \{x\} \ast (y \circ z)$$

for every $x, y, z \in H$. Since $x \circ y = \{x\} \ast \{y\}$ for any $x, y \in H$, an hypergroupoid $H$ is an hypersemigroup if and only if, for any $x, y, z \in H$, we have

$$\left(\{x\} \ast \{y\}\right) \ast \{z\} = \{x\} \ast \left(\{y\} \ast \{z\}\right).$$

Following Zadeh, if $(H, \circ)$ is an hypergroupoid, we say that $f$ is a fuzzy subset of $H$ (or a fuzzy set in $H$) if $f$ is a mapping of $H$ into the real closed interval $[0, 1]$ of real numbers, that is $f : H \to [0, 1]$. For an element $a$ of $H$, we denote by $A_a$ the subset of $H \times H$ defined as follows:

$$A_a := \{(y, z) \in H \times H \mid a \in y \circ z\}.$$
For two fuzzy subsets $f$ and $g$ of $H$, we denote by $f \circ g$ the fuzzy subset of $H$ defined as follows:

$$f \circ g : H \to [0, 1] \ a \to \begin{cases} \bigvee_{(y,z) \in A_a} \min \{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

Denote by $F(H)$ the set of all fuzzy subsets of $H$ and by "$\preceq$" the order relation on $F(H)$ defined by:

$$f \preceq g \iff f(x) \leq g(x) \text{ for every } x \in H.$$ 

We finally show by 1 the fuzzy subset of $H$ defined by:

$$1 : H \to [0, 1] \ x \to 1(x) := 1.$$ 

Clearly, the fuzzy subset 1 is the greatest element of the ordered set $(F(H), \preceq)$ (that is, $1 \geq f \ \forall f \in F(H)$).

For two fuzzy subsets $f$ and $g$ of an hypergroupoid $H$ we denote by $f \wedge g$ the fuzzy of $H$ defined as follows:

$$f \wedge g : H \to [0, 1] \ x \to (f \wedge g)(x) := \min \{f(x), g(x)\}.$$ 

One can easily prove that the fuzzy subset $f \wedge g$ is the infimum of the fuzzy subsets $f$ and $g$, so we write $f \wedge g = \inf \{f, g\}$.

We denote the hyperoperation on $H$ and the multiplication between the two fuzzy subsets of $H$ by the same symbol (no confusion is possible).

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

**Proposition 1.** Let $(H, \circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:

1. $x \in A \ast B \iff x \in a \circ b$ for some $a \in A$, $b \in B$.
2. If $a \in A$ and $b \in B$, then $a \circ b \subseteq A \ast B$.

**Lemma 2.** Let $H$ be an hypergroupoid. Then we have the following:

1. If $A \subseteq B$ and $C \subseteq D$, then $A \ast C \subseteq B \ast D$ and $C \ast A \subseteq D \ast B$.
2. $H \ast H \subseteq H$. 

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Proof. (1) Let $A \subseteq B$, $C \subseteq D$ and $x \in A \ast C$. By Proposition 1(1), there exist $a \in A$ and $c \in C$ such that $x \in a \circ c$. Since $a \in B$ and $c \in D$, by Proposition 1(2), we have $a \circ c \in B \ast D$, then $x \in B \ast D$. Similarly, $C \ast A \subseteq D \ast B$.

(2) $H \ast H \subseteq H$. Indeed: $H \ast H = \bigcup_{u \in H, v \in H} u \circ v$. On the other hand, for any $u, v \in H$, we have $u \circ v \subseteq H$. Thus we have $H \ast H \subseteq H$. □

Definition 3. Let $H$ be an hypergroupoid. A fuzzy subset $f$ of $H$ is called a fuzzy right ideal of $H$ if

$$f(x \circ y) \geq f(x)$$

for every $x, y \in H$, in the sense that: if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$.

Theorem 4. Let $H$ be an hypergroupoid and $f$ a fuzzy subset of $H$. Then $f$ is a fuzzy right ideal of $H$ if and only if

$$f \circ 1 \preceq f.$$

Proof. $\implies$. Let $a \in H$. Then $(f \circ 1)(a) \leq f(a)$. In fact:

(i) If $A_a = \emptyset$, then $(f \circ 1)(a) := 0$. Since $f$ is a fuzzy subset of $H$, we have $f(a) \geq 0$. Thus we have $(f \circ 1)(a) \leq f(a)$.

(ii) Let $A_a \neq \emptyset$. Then

$$(f \circ 1)(a) := \bigvee_{(y, z) \in A_a} \min\{f(y), 1(z)\} = \bigvee_{(y, z) \in A_a} f(y) \quad (\ast)$$

On the other hand, $f(y) \leq f(a)$ for every $(y, z) \in A_a$ \quad (\ast\ast)

Indeed, if $(y, z) \in A_a$, then $a \in y \circ z$ and, since $f$ is a fuzzy right ideal of $H$, we have $f(a) \geq f(y)$.

By (\ast\ast), we have $\bigvee_{(y, z) \in A_a} f(y) \leq f(a)$. Then, by (\ast), $(f \circ 1)(a) \leq f(a)$.

$\impliedby$. Let $x, y \in H$ and $u \in x \circ y$. Then $f(u) \geq f(x)$. Indeed:

Since $x, y \in H$ and $u \in x \circ y$, we have $(x, y) \in A_u$. Since $A_u \neq \emptyset$, we have

$$(f \circ 1)(u) := \bigvee_{(t, s) \in A_u} \min\{f(t), 1(s)\} = \bigvee_{(t, s) \in A_u} f(t) \geq f(t) \ \forall \ (t, s) \in A_u.$$ 

Since $(x, y) \in A_u$, we have $(f \circ 1)(u) \geq f(x)$. Since $u \in x \circ y \subseteq H$, we have $u \in H$. Since $u \in H$, by hypothesis, we have $(f \circ 1)(u) \leq f(u)$. Then we have $f(u) \geq f(x)$. □
Definition 5. Let $H$ be an hypergroupoid. A fuzzy subset $f$ of $H$ is called a fuzzy left ideal of $H$ if

$$f(x \circ y) \geq f(y)$$

for every $x, y \in H$, in the sense that: if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$.

In a similar way as in Theorem 4, we can prove the following theorem.

Theorem 6. A fuzzy subset $f$ of an hypergroupoid $H$ is a fuzzy left ideal of $H$ if and only if, for any fuzzy subset $f$ of $H$, we have

$$1 \circ f \preceq f.$$ 

Definition 7. Let $H$ be an hypergroupoid. A fuzzy subset $f$ of $H$ is called a fuzzy quasi-ideal of $H$ if

$$x \in b \circ s \text{ and } x \in t \circ c \implies f(x) \geq \min\{f(b), f(c)\} \forall x, b, s, t, c \in H.$$ 

Theorem 8. Let $H$ be an hypergroupoid. A fuzzy subset $f$ of $H$ is a fuzzy quasi-ideal of $H$ if and only if

$$(f \circ 1) \land (1 \circ f) \preceq f.$$ 

Proof. $\implies$. Let $x \in H$. Then $((f \circ 1) \land (1 \circ f))(x) \leq f(x)$, that is $\min\{(f \circ 1)(x), (1 \circ f)(x)\} \leq f(x)$. Indeed: For $A_x = \emptyset$ this is clear. Let $A_x \neq \emptyset$. Then

$$(f \circ 1)(x) := \bigvee_{(y,s) \in A_x} \min\{f(y), 1(s)\} = \bigvee_{(y,s) \in A_x} f(y)$$

and

$$(1 \circ f)(x) := \bigvee_{(t,z) \in A_x} \min\{1(t), f(z)\} = \bigvee_{(t,z) \in A_x} f(z).$$

If $f(x) \geq (f \circ 1)(x)$, then $f(x) \geq \min\{(f \circ 1)(x), (1 \circ f)(x)\}$.

Let $f(x) < (f \circ 1)(x)$. Then there exists $(y, s) \in A_x$ such that $f(y) > f(x)$ (otherwise $(f \circ 1)(x) \leq f(x)$ which is impossible).

We prove that $f(x) \geq f(z)$ for every $(t, z) \in A_x$. Then we have

$$f(x) \geq \bigvee_{(t,z) \in A_x} f(z) = (1 \circ f)(x) \geq \min\{(f \circ 1)(x), (1 \circ f)(x)\}.$$
and the proof is complete.

Let now \((t, z) \in A_x\). Then \(f(x) \geq f(z)\). Indeed: Since \((y, s) \in A_x\), we have \(y, s \in H\) and \(x \in y \circ s\). Since \((t, z) \in A_x\), we have \(t, z \in H\) and \(x \in t \circ z\).

Since \(x, y, s, t, z \in H\) such that \(x \in y \circ s\) and \(x \in t \circ z\), by hypothesis, we have \(f(x) \geq \min\{f(y), f(z)\}\). If \(\min\{f(y), f(z)\} = f(y)\), then \(f(x) \geq f(y)\) which is impossible by \((*)\). Thus we have \(\min\{f(y), f(z)\} = f(z)\), and \(f(x) \geq f(z)\).

\(\Leftarrow\). Let \(x, b, s, t, c \in H\) such that \(x \in b \circ s\) and \(x \in t \circ c\). Then \(f(x) \geq \min\{f(b), f(c)\}\). Indeed: By hypothesis, we have

\[
f(x) \geq \left((f \circ 1) \land (1 \circ f)\right)(x) := \min\{(f \circ 1)(x), (1 \circ f)(x)\}.
\]

Since \(x \in b \circ s\), we have \((b, s) \in A_x\), then

\[
(f \circ 1)(x) := \bigvee_{(u,v) \in A_x} \min\{f(u), 1(v)\} = \bigvee_{(u,v) \in A_x} f(u) \geq f(b).
\]

Since \(x \in t \circ c\), we have \((t, c) \in A_x\), then

\[
(1 \circ f)(x) := \bigvee_{(w,k) \in A_x} \min\{1(w), f(k)\} = \bigvee_{(w,k) \in A_x} f(k) \geq f(c).
\]

Thus we have

\[
f(x) \geq \min\{(f \circ 1)(x), (1 \circ f)(x)\} \geq \min\{f(b), f(c)\}.
\]

\(\square\)

The proof of the associativity of fuzzy sets on semigroups given in [1] can be naturally transferred to hypersemigroups in the following proposition.

**Proposition 9.** If \(H\) is an hypersemigroup, then the set of all fuzzy subsets of \(H\) is a semigroup.

**Proof.** Let \(f, g, h\) be fuzzy subsets of \(H\) and \(a \in H\). Then

\[
\left((f \circ g) \circ h\right)(a) = \left(f \circ (g \circ h)\right)(a).
\]

Indeed: If \(A_a = \emptyset\), then \(\left((f \circ g) \circ h\right)(a) = 0 = \left(f \circ (g \circ h)\right)(a)\).

Let \(A_a \neq \emptyset\). Then

\[
\left((f \circ g) \circ h\right)(a) = \bigvee_{(y,z) \in A_a} \min\{(f \circ g)(y), h(z)\}
\]
and
\[
(f \circ (g \circ h))(a) = \bigvee_{(u,v) \in A_a} \min\{f(u), (g \circ h)(v)\}.
\]

We put
\[
\begin{align*}
t &:= \bigvee_{(y,z) \in A_a} \min\{(f \circ g)(y), h(z)\} \\
s &:= \bigvee_{(u,v) \in A_a} \min\{f(u), (g \circ h)(v)\}.
\end{align*}
\]

We prove that \( t \geq \min\{(f(u), (g \circ h)(v)) \) for every \((u, v) \in A_a\). Then we have \( t \geq s \). In a similar way we prove that \( s \geq t \), and so \( s = t \).

Let \((u, v) \in A_a \) \( \Rightarrow t \geq \min\{f(u), (g \circ h)(v)\} \) ?

(A) Let \( t \geq f(u) \). Since \( f(u) \geq \min\{f(u), (g \circ h)(v)\} \), we have
\[
t \geq \min\{f(u), (g \circ h)(v)\}.
\]

(B) Let \( t < f(u) \). We consider the cases:

(a) Let \( A_v = \emptyset \). Then \((g \circ h)(v) := 0 \). Since \( f \) is a fuzzy set in \( H \), we have \( f(u) \geq 0 \), then \( \min\{f(u), (g \circ h)(v)\} = 0 \). Since \( t \in [0, 1] \), we have \( t \geq 0 \), so \( t \geq \min\{f(u), (g \circ h)(v)\} \).

(b) Let \( A_v \neq \emptyset \). Then \((g \circ h)(v) = \bigvee_{(c,d) \in A_v} \min\{g(c), h(d)\} \). We prove that
\[
t \geq \min\{g(c), h(d)\} \text{ for every } (c, d) \in A_v.
\]

Then we have \( t \geq (g \circ h)(v) \geq \min\{f(u), (g \circ h)(v)\} \).

Let now \((c, d) \in A_v \) \( \Rightarrow t \geq \min\{g(c), h(d)\} \) ?

(i) Let \( t \geq g(c) \). Since \( g(c) \geq \min\{g(c), h(d)\} \), we have \( t \geq \min\{g(c), h(d)\} \).

(ii) Let \( t < g(c) \). Since \((u, v) \in A_a \), we have \( a \in u \circ v \). Since \((c, d) \in A_v \), we have \( a \in u \circ v \) and \( v \in c \circ d \). Then we have
\[
a \in u \circ v = \{u\} \ast \{v\} \subseteq \{u\} \ast (c \circ d) \text{ (by Lemma 2)}
\]
\[
= (u \circ c) \ast \{d\} \text{ (the operation “\( \ast \)” is associative)}.
\]

By Proposition 1(1), \( a \in w \circ d \) for some \( w \in u \circ c \). Since \( a \in w \circ d \), we have \((w, d) \in A_a \) then, by \((*)\), we have
\[
t \geq \min\{(f \circ g)(w), h(d)\} \quad (**)
\]
Since \( w \in u \circ c \), we have \((u, c) \in A_w \), then
\[
(f \circ g)(w) = \bigvee_{(l, k) \in A_w} \min\{f(l), g(k)\} \geq \min\{f(u), g(c)\} \quad (***)
\]

Since \( t < f(u) \) and \( t < g(c) \), we have \( t < \min\{f(u), g(c)\} \). Then, by (***), \( t < (f \circ g)(w) \). Then, by (**), \( t \geq h(d) \). Since \( h(d) \geq \min\{g(c), h(d)\} \), we have \( t \geq \min\{g(c), h(d)\} \) and the proof of the proposition is complete. \( \square \)

According to Proposition 9, for any fuzzy subsets \( f, g, h \) of \( H \), we write
\[
(f \circ g) \circ h = f \circ (g \circ h) := f \circ g \circ h.
\]

**Definition 10.** Let \( H \) be an hypersemigroup. A fuzzy subset \( f \) of \( H \) is called a **fuzzy bi-ideal** of \( H \) if
\[
f\left((x \circ y) \ast \{z\}\right) \geq \min\{f(x), f(z)\} \ \forall \ x, y, z \in H,
\]
in the sense that if \( u \in (x \circ y) \ast \{z\} \), then \( f(u) \geq \min\{f(x), f(z)\} \).

**Theorem 11.** Let \( H \) be an hypersemigroup and \( f \) a fuzzy subset of \( S \). Then \( f \) is a fuzzy bi-ideal of \( H \) if and only if
\[
f \circ 1 \circ f \preceq f.
\]

**Proof.** \( \implies \). Let \( a \in H \). Then \((f \circ 1 \circ f)(a) \leq f(a)\). In fact: If \( A_a = \emptyset \), then
\[
(f \circ 1 \circ f)(a) = (f \circ 1 \circ f)(a) := 0 \leq f(a).
\]
Let \( A_a \neq \emptyset \). Then
\[
(f \circ 1 \circ f)(a) := \bigvee_{(y, z) \in A_a} \min\{(f \circ 1)(y), f(z)\}.
\]
It is enough to prove that
\[
\min\{(f \circ 1)(y), f(z)\} \leq f(a) \text{ for every } (y, z) \in A_a \quad (*)
\]

For this purpose, let \((y, z) \in A_a\). If \( A_y = \emptyset \), then \((f \circ 1)(y) := 0 \leq f(z)\), and \( \min\{(f \circ 1)(y), f(z)\} = 0 \leq f(a) \). Let now \( A_y \neq \emptyset \). Then
\[
(f \circ 1)(y) := \bigvee_{(x, w) \in A_y} \min\{f(x), 1(w)\} = \bigvee_{(x, w) \in A_y} f(x) \quad (1)
\]
We consider the following cases:

(i) Let $f(a) \geq (f \circ 1)(y)$. Then

$$f(a) \geq (f \circ 1)(y) \geq \min\{(f \circ 1)(y), f(z)\},$$

and condition (*) is satisfied.

(ii) Let $f(a) < (f \circ 1)(y)$. Then there exists $(x, w) \in A_y$ such that $f(a) < f(x)$ (2).

Indeed: If $f(a) \geq f(x)$ for every $(x, w) \in A_y$, then $f(a) \geq \bigvee_{(x, w) \in A_y} f(x)$. Then, by (1), $f(a) \geq (f \circ 1)(y)$ which is impossible.

Since $(y, z) \in A_a$, we have $y, z \in H$ and $a \in y \circ z$. Since $(x, w) \in A_y$, we have $x, w \in H$ and $y \in x \circ w$. Then we have

$$a \in y \circ z = \{y\} \ast \{z\} \subseteq (x \circ w) \ast \{z\}$$

and, since $f$ is a fuzzy bi-ideal of $H$, we have

$$f(a) \geq \min\{f(x), f(z)\}$$

If $f(x) \leq f(z)$ then $f(a) \geq f(x)$, which is impossible by (2). Thus we have

$$f(x) > f(z)$$

Then $\min\{f(x), f(z)\} = f(z)$ and, by (3), $f(a) \geq f(z)$.

Since $(x, w) \in A_y$, by (1), we have $f(x) \leq (f \circ 1)(y)$.

By (4) and (6), we have $f(z) < f(x) \leq (f \circ 1)(y)$, then

$$\min\{(f \circ 1)(y), f(z)\} = f(z) \leq f(a) \text{ (by (5))},$$

and condition (*) is satisfied.

$\iff$. Let $x, y, z \in H$ and $u \in (x \circ y) \ast \{z\}$. Then $f(u) \geq \min\{f(x), f(z)\}$.

Indeed: Since $u \in (x \circ y) \ast \{z\}$, by Proposition 1(1), there exist $a \in x \circ y$ such that $u \in a \circ z$. Since $a \in x \circ y$, we have $(x, y) \in A_a$. Since $u \in a \circ z$, we have $(a, z) \in A_u$. Since $(a, z) \in A_u$, $A_u$ is a nonempty set and we also have

$$(f \circ 1 \circ f)(u) = \bigvee_{(t, s) \in A_u} \min\{(f \circ 1)(t), f(s)\} \geq \min\{(f \circ 1)(a), f(z)\}.$$ 

Since $(x, y) \in A_u$, we have $A_u \neq \emptyset$ and we also have

$$(f \circ 1)(a) = \bigvee_{(z, w) \in A_u} \min\{f(z), 1(w)\} \geq \min\{f(x), 1(y)\} = f(x).$$

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Then we have

\[(f \circ 1 \circ f)(u) \geq \min\{f(x), f(z)\}.
\]

On the other hand, since \(u \in (x \circ y) \ast \{z\} \subseteq H \ast H \subseteq H\) and \(f \circ 1 \circ f \preceq f\), we have \((f \circ 1 \circ f)(u) \leq f(u)\). Thus we have

\[f(u) \geq (f \circ 1 \circ f)(u) \geq \min\{f(x), f(z)\},\]

and the proof of the theorem is complete. \(\square\)

**Note.** The characterization of fuzzy right (left) and fuzzy bi-ideals of an hyper-semigroup \(H\) using the greatest element 1 of the ordered set of fuzzy subsets of \(H\) is very useful for further investigation. Using these definitions many proofs can be drastically simplified. As an example, one can immediately see that every fuzzy right (or fuzzy left) ideal of \(H\) is a fuzzy bi-ideal of \(H\). Indeed, if \(f\) is a fuzzy right ideal of \(H\), then \(f \circ 1 \circ f = f \circ (1 \circ f) \preceq f \circ 1 \preceq f\).

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