Mathematical Modelling and Identification of a Quadrotor

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Abstract. Motivated by the important growth of VTOL vehicles research such as quadrotors and to a small extent autonomous flight, a quadrotor dynamical model is presented in this work. The purpose of this study is to get a better understanding of its flight dynamics. It is an underactuated system. So, a simplified and clear model is needed to implement controllers on these kind of unmanned aerial systems. In addition, a computational tool is used for validation purposes. For future works embedded or intelligent control systems can be developed to control them. Gyroscopic and some aerodynamics effects are neglected.

Keywords: Quadrotor · VTOL · Flight dynamics · UAV

1 Introduction

Humanity has been having a significant development increase that centuries ago it would not have been possible to believe that the first manned airplane had been built and flown by the Wright brothers in the early 1900s. Aircrafts are not the only ones with flight capacities. There are other flying vehicles like drones or unmanned aerial vehicles (UAVs) which can do so. Hence, there have been many researchers and engineers from different areas interested in developing aerial vehicles without the influence of humans. Several engineering areas such as aerodynamics, control, embedded electronics are associated to this type of systems. These type of vehicles can have small designs that favor their abilities for carrying or payload. The term drones has been used because of the autonomy constraints they could have. This is the reason why embedded and guidance control systems are applied to drones which permit autonomous flight tasks. One of the categories of UAVs is the multirotor which has the possibility of vertical takeoff and landing (VTOL). Nowadays, unmanned aerial vehicles
play a very meaningful role in the current aerospace industry. They can provide
different autonomous flight applications such as environmental research, rescue,
traffic monitoring, agricultural inspections, image and video, scientific research,
inspections of places with very difficult access and even more recently, home
delivery of products. Therefore, it is important to remark that their uses are
not just limited to dangerous roles. Due to the great progress of technology,
UAVs have been evolving up to autonomous systems. They are capable of run-
ning by themselves without any kind of interventions from humans and with a
pre-determined flight mission. Therefore, this fact has drawn the attention to
research more in-depth about autonomous aircrafts [3,10,16,17].

2 Reference Frames

It is necessary to have at least one reference frame to describe any position
or motion. The use of additional reference frames will make the derivation of
the equations of motions easier. When using multiple reference frames there is
one important issue which is related to the transformation of vector coordinates
from one frame to another. The rotation matrices used are based on the Euler
angles. A reference fixed frame is applied to determine distance and direction.
A coordinate system is used to represent measurements in a frame. In flight
dynamics there are two reference frames clearly defined, the Earth Fixed Frame
and Body Fixed Frame [5,12,14,17]. The E-frame is chosen as the inertial frame.
\((0_E X_E Y_E Z_E)\). The origin is at \(O_E\). In this reference frame the linear position
\((\xi^E)\) and angular position \((\Theta^E)\) of the quadrotor are defined. The other reference
frame required is the body frame \((0_B X_B Y_B Z_B)\) which is attached to the
quadrotor body. The origin is at the vehicle’s reference point \(O_B\). In this B-
frame the linear velocity, the angular velocity and the forces and torques are
defined.

3 Quadrotor Assumptions for Modeling

Considering that computation of any model is just an approximation to real
conditions found in the real world, some assumptions need to be made for sim-
ulation purposes:

- The quadrotor is treated as a rigid structure.
- The propellers are rigid. Aerodynamics effects such as blade flapping are
  neglected.
- Gyroscopic effects and aerodynamic torques can be ignored.
- The quadrotor structure is treated as symmetrical.
- Wind disturbances are ignored.
4 Quadrotor System Variables

The quadrotor motion has six degrees of freedom (6DOF) which are defined as follows: $\xi = (x, y, z)$ that represents the linear position of the quadrotor and $\Theta = (\theta, \phi, \psi)$ is the attitude or orientation. These are also known as the Euler angles pitch, roll and yaw. Therefore, if $\Theta = (\theta, \phi, \psi)$ and $\xi = (x, y, z)$, so the general position vector $\Phi$ is (Fig. 1):

$$\Phi = \begin{bmatrix} \xi \\ \Theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \\ \psi \end{bmatrix}$$  \hspace{1cm} (1)

Let $v$ and $\omega$ be the quadrotor linear and angular velocities $v = (u, v, w)$ $\omega = (p, q, r)$. Therefore, the linear and angular velocities are:

$$v = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$  \hspace{1cm} (2)

4.1 Quadrotor Kinematics

To be able to transform the vectors from the E-frame to the B-frame a direction cosine matrix is required. If the rotations are done first around $x$
axis then around $y$ and the final one around $z$ axis, the rotation matrix is:

$$R_T = R(\phi, \theta, \psi) = R_x(\phi)R_y(\theta)R_z(\psi) \quad [14, 15, 17, 18].$$

Where:

Roll rate: $p = \dot{\phi} - \dot{\psi} \sin \theta$

Pitch rate: $q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta$

Yaw rate: $r = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi$

$$R_x(\phi) = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix} \quad (3)$$

$$R_y(\theta) = 
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix} \quad (4)$$

$$R_z(\psi) = 
\begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (5)$$

Then, the complete rotation matrix is the product of the three rotation matrices:

$$R_{TE} = 
\begin{bmatrix}
\cos \theta \cos \phi \cos \psi \sin \theta \sin \phi - \sin \psi \cos \phi \sin \phi \cos \theta + \cos \phi \cos \psi \sin \theta \\
\sin \phi \cos \theta \cos \psi \cos \psi \sin \phi \sin \theta + \cos \phi \cos \psi \sin \phi \\
-\sin \psi \cos \theta \cos \psi \cos \psi \sin \phi \sin \theta \cos \phi \cos \phi \cos \theta - \cos \phi \cos \psi \sin \theta
\end{bmatrix} \quad (6)$$

The relationship between the angular velocity $\dot{\omega}$ in (E-frame) and the angular velocity in (B-frame) is given by: $\dot{\omega} = T_\chi \omega$. The transformation matrix $T_\chi$ is found by computing the attitude rates ($\dot{\phi}, \dot{\theta}, \dot{\psi}$) and body rates ($p, q, r$) as follows:

$$v^E = x^E = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = R_T v^B \quad (7)$$

$$\Rightarrow$$

$$v^E = R_T = 
\begin{bmatrix}
\cos \theta \cos \phi \cos \psi \sin \theta \sin \phi - \sin \psi \cos \phi \sin \phi \cos \theta + \cos \phi \cos \psi \sin \theta \\
\sin \phi \cos \theta \cos \psi \cos \psi \sin \phi \sin \theta + \cos \phi \cos \psi \sin \phi \\
-\sin \psi \cos \theta \cos \psi \cos \psi \sin \phi \sin \theta \cos \phi \cos \phi \cos \theta - \cos \phi \cos \psi \sin \theta
\end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (8)$$

The angular body rates $p, q, r$ are shown in Fig. 2. First, roll rotates around the angle $\phi$ and angular velocity $\dot{\phi}$. Then, pitch which rotates around the $\theta$ angle with $\dot{\theta}$ angular velocity. Likewise, yaw rotates through angle $\psi$ with angular velocity $\dot{\psi}$. Hence, the relationship between the aerial vehicle body rates and attitude rates is given as:
Fig. 2. Angular velocities transformation

Roll rate:
\[ p = \dot{\phi} - \dot{\psi} \sin \theta \]  
(9)

Pitch rate:
\[ q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta \]  
(10)

Yaw rate:
\[ r = \dot{\psi} \cos \phi + \cos \theta - \dot{\theta} \sin \phi \]  
(11)

When the quadrotor Euler angles \((\phi, \theta, \psi)\) are considered small \((\phi, \theta, \psi \approx 0)\) the body rates equations can be taken as: [4,17,18].

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]  
(12)

Since \([x, y, z, \phi, \theta, \psi]^T\) is the vector that contains the linear and angular position of the quadrotor in the E-frame and \([u, v, w, p, q, r]^T\) is the vector which contains the linear and angular velocities in the B-frame, then these two reference frames are related as follows:

\[ v = R_T \cdot v_B \]  
(13)

\[ \omega = T_\chi \cdot \omega_B \]  
(14)

Finally, the kinematic model of a quadrotor is:

\[
\begin{align*}
\dot{x} &= u[\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta)] - v[\cos(\psi)\sin(\psi) - \cos(\phi)\sin(\theta)] + w[\cos(\psi)\cos(\theta)] \\
\dot{y} &= v[\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta)] - w[\cos(\psi)\sin(\psi) - \cos(\phi)\sin(\theta)] + u[\cos(\psi)\cos(\theta)] \\
\dot{z} &= w[\cos(\phi)\cos(\theta)] - u[\sin(\theta)] + v[\cos(\theta)\sin(\phi)] \\
\dot{\phi} &= p + r[\cos(\phi)\tan(\theta)] + q[\sin(\phi)\tan(\theta)] \\
\dot{\theta} &= q\cos(\phi) - r\sin(\phi) \\
\dot{\psi} &= q\sin(\phi) + r\cos(\phi)\cos(\theta)
\end{align*}
\]  
(15)
4.2 Quadrotor Dynamics

The second Newton’s law states that a forced applied in the E reference frame is equal to the product of the vehicle’s mass and its acceleration. i.e, 

\[ \text{Force} = \text{mass} \times \text{acceleration} \]

Therefore, it can be inferred that:

\[ F = m \left( \frac{dv}{dt} \right) = m \left( \frac{d^2\xi}{dt^2} \right) \]  

(16)

\[ \text{Force} = \text{mass} \times \text{acceleration} \]  

Hence,

\[ F = m \left( \frac{dv}{dt} \right) = m \left( \frac{d^2\xi}{dt^2} \right) \]  

(17)

From Eq.1:

\[ \left( \frac{d^2\xi^E}{dt^2} \right) = \ddot{\xi}^E = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \]  

(18)

The total force exerted on the quadrotor is given by the following matrix relation:

\[ m(\omega^B \times v^B + \dot{v}^B) = F^B \]  

(19)

The total force exerted on the quadrotor is given by the following matrix relation:

\[ m(\omega^B \times v^B + \dot{v}^B) = F^B \]  

(20)

Where \( \times \) is the cross product, \( m \) is the quadcopter mass and \( F^B = [F_x \ F_y \ F_z]^T \) which is the total force. So, a skew symmetric matrix is got and expressed as: [17,18].

\[ \omega^B \times v^B = \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} \]  

(21)

And if the differential of the linear velocity is:

\[ v = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} \]  

(22)

Therefore, the following equation is obtained:

\[ F^B = m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix} \]  

(23)
If the column vector $F^B = [F_x \ F_y \ F_z]^T$ is substituted and dividing by $m$, we get:

$$
\frac{1}{m} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix}
$$

(24)

Rewriting, the following is acquired:

$$
\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} - \begin{bmatrix} qw - rv \\ ru - pw \\ pv - qu \end{bmatrix}
$$

(25)

Then, the total torque applied to the quadrotor is given by:

$$
I \ddot{\omega}^B + \omega^B \times (I \omega^B) + M_G = M^B
$$

(26)

Where $\omega^B \times v^B$ is known as the Coriolis term, $[M_x \ M_y \ M_z] = M^B$ is the total torque, $M_G$ gyroscopic effects due to rotor’s inertia and $I$ known as the inertia matrix $[7,17,18]$.

A quadcopter is assumed as symmetric:

$$
I_{xy} = I_{xz} = I_{yx} = I_{yz} = I_{zx} = I_{zy} = 0
$$

(27)

$$
I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}
$$

(28)

The angular acceleration is given by:

$$
\dot{\omega}^B = I^{-1}[T_\chi \tau^B - (\omega^B \times I \omega^B)]
$$

(29)

Where $\tau^B$ is given as:

$$
\tau^B = [\tau_r \ \tau_p \ \tau_y]^T
$$

(30)

Therefore $T_\chi \tau^B$ is:

$$
T_\chi \tau^B = \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} = I \dot{\omega}^B + [\omega^B \times (I \omega^B)]
$$

(31)

The inverse of matrix $28$ is:

$$
I^{-1} = \begin{bmatrix} \frac{1}{I_{xx}} & 0 & 0 \\ 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix}
$$

(32)
If the angular velocities in Eq. 2 are derived we obtain:

\[
\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}
\]

(33)

Now substituting in Eq. 29 results:

\[
\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = I^{-1}\left[ T_\chi \tau^B - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right]
\]

(34)

Simplifying:

\[
\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = I^{-1}\left[ \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} - \begin{bmatrix} qr I_{zz} - rq I_{yy} \\ rp I_{xx} - pr I_{zz} \\ pq I_{yy} - qp I_{xx} \end{bmatrix} \right]
\]

(35)

Substituting \( I^{-1} \) in Eq. 35 we get:

\[
\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} I_{zz} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} I_{yy} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} I_{xx}
\]

(36)

Simplifying:

\[
\dot{\omega}^B = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} - \begin{bmatrix} qr I_{zz} - rq I_{yy} \\ rp I_{xx} - pr I_{zz} \\ pq I_{yy} - qp I_{xx} \end{bmatrix}
\]

(37)

### 4.3 Thrust and Moment

It is assumed that the torque and thrust caused by each rotor acts particularly in the \( z \) axis of the B-frame. Accordingly, the net propulsive force in the \( z^B \) direction is given by:

\[
F_t = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}
\]

(38)

Moment results from the thrust action of each rotor around the center of mass that induces a pitch and roll motion. Furthermore, there is a reactive torque of the rotors on the vehicle which produces a yaw reply. The moment vector is therefore: [9,17].

\[
M_t = \begin{bmatrix} \tau_r \\ \tau_p \\ \tau_y \end{bmatrix} = \begin{bmatrix} lb(F_2 - F_4) \\ lb(F_3 - F_1) \\ (F_1 - F_2 + F_3 - F_4)d \end{bmatrix}
\]

(39)
The external forces used are based on the model found in [4,14]:

\[ F_x = -W \sin \theta + x = m(\dot{u} + qw - rv) \]
\[ F_y = W \cos \theta \cos \psi + x = m(\dot{v} + ru - pw) \]
\[ F_z = W \cos \theta \cos \psi + z = m(\dot{w} + pv - qu) \]

The Matrix 42 shows the relationship between the net torque that is performed on the vehicle and the angular acceleration. The total torque is the summation of the aerodynamic torque \( \tau_a \) and the torque generated by the rotors [13].

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\begin{bmatrix}
\text{Aero} \\
\text{Quad}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\begin{bmatrix}
\text{Aero} \\
\text{Quad}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{I}_{xx} \dot{p} + (I_{zz} - I_{yy}) qr \\
\dot{I}_{yy} \dot{q} + (I_{xx} - I_{zz}) rp \\
\dot{I}_{zz} \dot{r} + (I_{yy} - I_{xx}) pq
\end{bmatrix}
\]

Therefore, the quadrotor dynamic model in the B-frame is:

\[
\begin{cases}
F_x = m(\dot{u} + qw - rv) \\
F_y = m(\dot{v} - pw + ru) \\
F_z = m(\dot{w} + pv - qu) \\
M_x = I_{xx} \dot{p} + (I_{zz} - I_{yy}) qr \\
M_y = I_{yy} \dot{q} + (I_{xx} - I_{zz}) rp \\
M_z = I_{zz} \dot{r} + (I_{yy} - I_{xx}) pq
\end{cases}
\]

The rotation of the propellers combined with the rotation of the body results in a gyroscopic torque which is given by:

\[
M_G = J_m(\omega_B \times z^B)(\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4)
\]

Where \( \Omega \) (i = 1, 2, 3, 4) is propeller angular velocity, \( J_m \) is the inertia of each motor. Taking into an account that \( \tau_B = [\tau_p \ \tau_r \ \tau_y] \) are the torques generated by the actuators action and \( \tau_a = [\tau_{ax} \ \tau_{ay} \ \tau_{az}] \) are the aerodynamic torques, finally, the complete quadrotor dynamic model is:

\[
\begin{cases}
-mg[\sin(\theta)] + F_{ax} = m(\dot{u} + qw - rv) \\
mg[\cos(\theta)\sin(\phi)] + F_{ay} = m(\dot{v} - pw + ru) \\
mg[\cos(\theta)\cos(\phi)] + F_{az} - ft = m(\dot{w} + pv - qu) \\
\tau_p + \tau_{ax} = \dot{p}I_{xx} - qrI_{yy} + qrI_{zz} \\
\tau_r + \tau_{ay} = \dot{q}I_{yy} + prI_{xx} - prI_{zz} \\
\tau_y + \tau_{az} = \dot{r}I_{zz} - pqI_{xx} + pqI_{yy}
\end{cases}
\]
\[ F^B = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \text{[N]}, \quad R_T \text{ is the rotation matrix and } m \text{ [kg] is the mass of the quadrotor. So, it is possible to combine them as follows:}^{[17,18]} \]

\[ F^B = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = mR_T \frac{d}{dt} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (46) \]

They can be arranged as:

\[ \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{m} R^{-1}_T \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \quad (47) \]

Making substitutions the equation is simplified as:

\[ \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \frac{1}{m} R^{-1}_T \begin{bmatrix} -mgsin\theta \\ mgsin\phi cos\theta \\ mgcos\phi cos\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix} \quad (48) \]

Therefore:

\[ \begin{aligned}
\ddot{x} &= \frac{F_{\text{total}}}{m} [sin(\phi)sin(\psi) + cos(\phi)cos(\psi)sin(\theta)] \\
\ddot{y} &= \frac{F_{\text{total}}}{m} [cos(\phi)sin(\psi)sin(\theta) - cos(\psi)sin(\phi)] \\
\ddot{z} &= -g \frac{F_{\text{total}}}{m} [cos(\phi)cos(\theta)]
\end{aligned} \quad (49) \]

As stated earlier, for small angles of movement \([\dot{\phi} \; \dot{\theta} \; \dot{\psi}]^T = [p \; q \; r]^T\), hence, the dynamic model of quadrotor in the E-frame is:

\[ \begin{aligned}
\ddot{x} &= \frac{F_{\text{total}}}{m} [sin(\phi)sin(\psi) + cos(\phi)cos(\psi)sin(\theta)] \\
\ddot{y} &= \frac{F_{\text{total}}}{m} [cos(\phi)sin(\psi)sin(\theta) - cos(\psi)sin(\phi)] \\
\ddot{z} &= -g \frac{F_{\text{total}}}{m} [cos(\phi)cos(\theta)] \\
\dot{\phi} &= \frac{I_{yy} - I_{zz}}{I_{xx}} \dot{\psi} + \frac{\tau_x}{I_{xx}} \\
\dot{\theta} &= \frac{I_{zz} - I_{xx}}{I_{yy}} \dot{\phi} + \frac{\tau_y}{I_{yy}} \\
\dot{\psi} &= \frac{I_{xx} - I_{yy}}{I_{zz}} \dot{\theta} + \frac{\tau_z}{I_{zz}} 
\end{aligned} \quad (50) \]

### 4.4 Motor and Propeller Dynamics

The following is the differential equation for a DC motor:

\[ J_m \dot{\Omega}_m = -\frac{K_E}{R} K_q \Omega_m - \tau_l + \frac{K_q}{R} V \quad (51) \]

Where \(J_m\) is the motor inertia \([N m s^2]\), \(\dot{\Omega}_m\) is the motor angular speed \([rad/s^2]\), \(\tau_l\) is the load torque \([N m]\). There is an acceleration or deceleration of the angular speed of the motor \(\Omega_m\) when the torques \(\tau_m\) (motor torque) and \(\tau_l\) are not the same. The motor torque \(\tau_m\) is proportional to the current \(i\) and through the
torque constant $K_q$ [Amps/Nm], $K_E$ is known as the voltage motor constant [rad s/V].

The power required to hover is given as:

$$P = TV_i = TV_{rh} = T \sqrt{\frac{T}{2\rho A_b}} = \frac{T^{3/2}}{\sqrt{2\rho A_b}}$$

(52)

Where $A_b$ is the area cleaned out by the rotor blades and $\rho$ is the air density. The above expression can be related to the induced velocity $v_{rh}$ in forward flight as:

$$v_i = \frac{v_{rh}^2}{\sqrt{(v_\infty \cos \alpha)^2 + (v_i - v_\infty \sin \alpha)^2}}$$

(53)

Where $\alpha$ is rotor angle of attack and $v_\infty$ is velocity stream flow [8,11]. If Eq. 52 is substituted for $v_{rh}$, it results;

$$fom \eta_m \frac{\tau_m}{K_q} V = T \sqrt{\frac{T}{2\rho A_b}}$$

(54)

The rotor power in forward flight is given by:

$$P = T(v_\infty \sin \alpha + v_i)$$

(55)

So, the ideal thrust $T$ when a power input $P$ can be calculated as follows:

$$T = \frac{P}{v_\infty \sin \alpha + v_i}$$

(56)

Where the denominator is the airspeed across the rotors.

The motor torque is proportional to the thrust constant $K_t$:

$$\tau_m = K_t T$$

(57)

If Eq. 52 is substituted for $\tau_m$, it gives:

$$fom \eta_m \frac{K_t T}{K_q} V = T \sqrt{\frac{T}{2\rho A_b}}$$

(58)

Torque $\tau_m$ is provided by each motor that is balanced by the drag torque. Accordingly, the torque acting on the propeller is:

$$\tau_m = J_m \dot{\Omega} + \tau_{drag}$$

(59)

Finally, the relationship between voltage and thrust is given by:

$$T = 2\rho A_b \left[fom \frac{\eta_m K_t}{K_q}\right]^2 V^2$$

(60)
A quadrotor is controlled by providing four torques \( \tau_1, \tau_2, \tau_3, \tau_4 \) to the rotors which produce four thrust forces \( F_1, F_2, F_3, F_4 \) in the z axis. The net torques acting on the body can be calculated by using the inputs \( U = (U_1, U_2, U_3, U_4) \) that can be applied to control the quadrotor. As an aerodynamic consideration forces and torques are proportional to the squared propeller’s speed [1].

Therefore the relationship between motions and propellers’ squared speed is as follows:

\[
F_T = b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 - \Omega_4^2)
\]
\[
\tau_x(\theta) = bl(\Omega_1^2 - \Omega_3^2)
\]
\[
\tau_y(\phi) = bl(\Omega_2^2 - \Omega_4^2)
\]
\[
\tau_z(\psi) = K_{drag}(\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2)
\]

Where \( l \) [m] is the distance between any rotor and the center of the drone, \( b \) is the thrust factor \([N \cdot m^2]\), \( K_{drag} \) is a drag constant \([N \cdot m \cdot s^2]\) and \( \Omega_i \) [rad s\(^{-1}\)] is the propeller angular acceleration [7,17,18]. Then the dynamic model of the quadrotor is:

\[
\begin{bmatrix}
-mg[sin(\theta)] + F_{ax} = m(\dot{u} + qw - rv) \\
mg[cos(\theta)sin(\phi)] + F_{ay} = m(\dot{v} - pw + ru) \\
mg[cos(\theta)cos(\phi)] + F_{az} - b(\Omega_1^2 + \Omega_2^2 - \Omega_3^2 - \Omega_4^2) = m(\dot{w} + pv - qu) \\
bl(\Omega_3^2 - \Omega_1^2) + \tau_{ax} = \dot{p}I_x - qrI_y + qrI_z \\
bl(\Omega_4^2 - \Omega_2^2) + \tau_{ay} = \dot{q}I_y - prI_x + prI_z \\
K_{drag}(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) + \tau_{az} = \dot{r}I_z - pqI_x + pqI_y
\end{bmatrix}
\]

The overall propeller’s velocities \( \Omega \) [rad s\(^{-1}\)] and propeller’s velocities vector \( \Omega \) are defined as: [1].

\[
\Omega = -\Omega_1 + \Omega_2 - \Omega_3 + \Omega_4
\]

\[
\Omega = \begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\Omega_3 \\
\Omega_4
\end{bmatrix}
\]

The following equations system shows the relationship between the control inputs and the propellers’ squared speed:

\[
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4
\end{bmatrix}
= \begin{bmatrix}
b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\
bl(\Omega_2^2 - \Omega_4^2) \\
bl(\Omega_3^2 - \Omega_1^2) \\
d(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2)
\end{bmatrix}
\]

Where \( l \) [m] is the distance from the center of the quadrotor and the propeller and \( [U_1 U_2 U_3 U_4] \) are the propeller’s speed control inputs. \( U_1 \) is the combination of thrust forces responsible for the quadrotor altitude \( z \). \( U_2 \) is the differential
thrust between rotors 2 and 4 which generate the roll moment. $U_3$ is the thrust differential between rotors 1 and 3 that create the pitch moment. Finally, $U_4$ is the combination of the individual torques between the clockwise and counterclockwise rotors in charge of generating yaw rotation. Consequently, $U_1$ generates the desired quadrotor altitude, $U_2$ and $U_3$ generate the respective roll and pitch angles whereas $U_4$ creates the yaw angle $[1, 2, 6, 7, 18]$. Simplifying and including $U$ variables we get:

$$f(x,u) = \begin{pmatrix}
\dot{\phi} = \dot{\theta} \psi \lambda_1 + \dot{\theta} \lambda_2 \Omega_r + b_1 U_2 \\
\dot{\theta} = \dot{\phi} \psi \lambda_3 + \dot{\phi} \lambda_4 \Omega_r + b_2 U_3 \\
\dot{\psi} = \dot{\theta} \dot{\phi} \lambda_5 + b_3 U_4 \\
\ddot{x} = \mu_x \frac{1}{m} U_1 \\
\ddot{y} = \mu_y \frac{1}{m} U_1 \\
\ddot{z} = g - (\cos \phi \cos \theta) \frac{1}{m} U_1
\end{pmatrix} \quad (66)$$

$J_m \ [N \ m/ \ s^2]$ is the motor inertia.

5 Simulations and Results

Table 1 presents the selected variables for simulation purposes which are based on the AR drone. Since a complete math model is provided, other variables such as thrust and drag coefficients as well as motor inertia are involved in the flight control analysis. Therefore, after performing several control simulations and some gain adjustments, it can be seen that PD control has a good response getting a better and quick stabilization of the vehicle. Also, when the moments of inertia are of very small values, attitude stabilization is more difficult. Some overshoot or peaks are also noticed in attitude angles. For altitude, peaks are not really evident. The PID gains are computed as well. Additional corrections are required as some signal deviations from the desired references are observed. The mean of dynamics values are helpful for more appropriate results (Fig. 3).

| Dynamic variables | Value       |
|-------------------|-------------|
| $m$               | 0.1 kg      |
| $I_x$             | 0.45 kg · m$^2$ |
| $I_y$             | 0.51 kg · m$^2$ |
| $I_z$             | 0.95 kg · m$^2$ |
| $l$               | 0.5 m       |
6 Conclusions

Vertical takeoff and landing aircrafts have a complex and highly non-linear dynamics. A complete model is given using the Newton-Euler approach. It could be used to predict changes in position and orientation of a quadcopter at a given point in time. It can be achieved by varying the rotors speed. Equations are inverted to find inputs needed for a certain position. Additionally, there are six degrees of freedom and just four inputs. Therefore, it is not appropriate and a simplified dynamics model would be required to be then implemented for control purposes. Also, there are some sensor measurements that are based on the E-frame and not on sensors frames. This model approach can be adapted to consider voltage inputs and not just \( U \) since there is an important association between thrust and voltage. It is necessary to adjust control gains. So, the smaller the moments of inertia and mass the more difficult for the platform to stabilize. State estimation would have a significant role because of an estimation of sensor measurements is necessary for autopilots design.

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