COMPLETENESS AND ABSOLUTE $H$-CLOSEDNESS OF TOPOLOGICAL SEMILATTICES

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Abstract. We find (completeness type) conditions on topological semilattices $X, Y$ guaranteeing that each continuous homomorphism $h : X \to Y$ has closed image $h(X)$ in $Y$.

1. INTRODUCTION

It is well-known that a topological group $X$ is complete in its two-sided uniformity if and only if for any isomorphic topological embedding $h : X \to Y$ into a Hausdorff topological group $Y$ the image $h(X)$ is closed in $Y$.

In this paper we prove a similar result for topological semilattices. A topological semilattice is a topological space $X$ endowed with a continuous binary operation $X \times X \to X$, $(x, y) \mapsto xy$, which is associative, commutative and idempotent in the sense that $xx = x$ for all $x \in X$.

We define a Hausdorff topological semigroup $X$ to be

• $H$-closed if for any isomorphic topological embedding $h : X \to Y$ to a Hausdorff topological semigroup $Y$ the image $h(X)$ is closed in $Y$;
• absolutely $H$-closed if for any continuous homomorphism $h : X \to Y$ to a Hausdorff topological semigroup $Y$ the image $h(X)$ is closed in $Y$.

(Absolutely) $H$-closed topological semigroups were introduced by Stepp in [20] (resp. [21]) who called them (absolutely) maximal topological semigroups. More information on (absolutely) $H$-closed topological semigroups can be found in [1, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23].

In this paper we are concentrated at the problem of detecting (absolutely) $H$-closed topological semilattices. For discrete topological semilattices this problem has been resolved by combined efforts of Stepp [21] and Banakh, Bardyla [2] who proved that a discrete topological semilattice $X$ is absolutely $H$-closed if and only if $X$ is $H$-closed if and only if all chains in $X$ are finite.

A subset $C$ of a semilattice $X$ is called a chain if $xy \in \{x, y\}$ for all $x, y \in C$. This is equivalent to saying that any two elements of $C$ are comparable in the natural partial order $\leq$ on $X$, defined by $x \leq y$ iff $xy = x$. Endowed with this partial order, each semilattice becomes a poset, i.e., a set endowed with a partial order. In a Hausdorff topological semilattice $X$ the partial order $\{(x, y) \in X \times X : x \leq y\}$ is a closed subset of $X \times X$, which means that $X$ is a pospace, i.e., a topological space endowed with a closed partial order. A semilattice $X$ is linear if the natural partial order on $X$ is linear (i.e., $X$ is a chain in $X$).

The following characterization of (absolutely) $H$-closed discrete topological semilattices is a combined result of Stepp [21] and Banakh, Bardyla [2] (who proved the equivalences (2) $\iff$ (3) and (1) $\iff$ (3), respectively).

Theorem 1.1 (Stepp, Banakh, Bardyla). For a discrete topological semilattice $X$ the following conditions are equivalent:

1. $X$ is $H$-closed;
2. $X$ is absolutely $H$-closed;
3. all chains in $X$ are finite.

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The implication (3) \(\Rightarrow\) (2) in this theorem can be also derived from the following characterization proved by Banakh and Bardyla in [2].

**Theorem 1.2** (Banakh, Bardyla). For a Hausdorff topological semilattice \(X\) the following conditions are equivalent:

1. all closed chains in \(X\) are compact;
2. all maximal chains in \(X\) are compact;
3. each non-empty chain \(C \subset X\) has \(\inf C \in \overline{C}\) and \(\sup C \in \overline{C}\);
4. each closed subsemilattice of \(X\) is absolutely \(H\)-closed;
5. each closed chain in \(X\) is \(H\)-closed.

Here by \(\overline{C}\) we denote the closure of a set \(C\) in a topological space \(X\). A subset \(A\) of a poset \((X, \leq)\) is called lower (resp. upper) if \(A = \downarrow A\) (resp. \(A = \uparrow A\)) where

\[
\downarrow A = \{x \in X : \exists a \in A (x \leq a)\} \quad \text{and} \quad \uparrow A = \{x \in X : \exists a \in A (x \geq a)\}.
\]

In 2008 Gutik and Repovš proved the following characterization of (absolutely) \(H\)-closed linear semilattices.

**Theorem 1.3** (Gutik-Repovš). For a linear Hausdorff topological semilattice \(X\) the following conditions are equivalent:

1. \(X\) is \(H\)-closed;
2. \(X\) is absolutely \(H\)-closed;
3. any non-empty chain \(C \subset X\) has \(\inf C \in \overline{C}\) and \(\sup C \in \overline{C}\) in \(X\).

In [23] Yokoyama extended Gutik-Repovš Theorem 1.3 to topological pospaces with finite antichains.

Trying to extend Gutik-Repovš characterization to all (not necessarily linear) topological semilattices we have discovered that the last condition of Theorem 1.3 admits at least three non-equivalent versions, defined as follows.

**Definition 1.4.** A topological semilattice \(X\) is defined to be

- \(k\)-complete if each non-empty chain \(C \subset X\) has \(\inf C \in \overline{C}\) and \(\sup C \in \overline{C}\);
- \(s\)-complete if each non-empty subsemilattice \(S \subset X\) has \(\inf S \in \overline{S}\) and each non-empty chain \(C \subset X\) has \(\sup C \in \overline{C}\);
- \(c\)-complete if for each closed upper set \(F \subset X\), each non-empty chain \(C \subset F\) has \(\inf C \in F\) and \(\sup C \in \overline{C}\) in \(X\).

In Proposition 9.1 we shall prove that for any topological semilattice the following implications hold:

\[ k\text{-complete} \Rightarrow s\text{-complete} \Rightarrow c\text{-complete}.\]

By Theorem 1.2 a Hausdorff topological semilattice is \(k\)-complete if and only if all maximal chains in \(X\) are compact. Theorems 1.1 and 1.3 imply that a discrete or linear Hausdorff topological semilattice \(X\) is \(s\)-complete if and only if \(X\) is \(c\)-complete if and only if \(X\) is (absolutely) \(H\)-closed.

These completeness properties of topological semilattices will be paired with the following notions.

**Definition 1.5.** We say that a pospace \((X, \leq)\) is

- well-separated if for any points \(x < y\) there exists a neighborhood \(V \subset X\) of \(y\) such that \(x \notin \overline{V}\);
- down-open if for every open set \(U \subset X\) the lower set \(\downarrow U\) is open in \(X\);
- a topological lattice if any two points \(x, y \in X\) have \(\inf \{x, y\}\) and \(\sup \{x, y\}\) and the binary operations \(\wedge : X \times X \to X\), \(\wedge : (x, y) \mapsto \inf \{x, y\}\), and \(\vee : X \times X \to X\), \(\vee : (x, y) \mapsto \sup \{x, y\}\) are continuous.

In Lemma 6.1 we shall prove that for a Hausdorff topological semilattice the following implications hold:

\[ \text{linear} \Rightarrow \text{topological lattice} \Rightarrow \text{down-open} \Rightarrow \text{well-separated}.\]
The main result of this paper is the following theorem, which will be proved in Section 8, after some preparatory work made in Sections 2–7.

**Main Theorem 1.6.** Let \( h : X \to Y \) be a continuous homomorphism from a topological semilattice \( X \) to a Hausdorff topological semigroup \( Y \). The image \( h(X) \) is closed in \( Y \) if one of the following conditions is satisfied:

1. \( X \) is \( k \)-complete;
2. \( X \) is \( s \)-complete and \( Y \) is well-separated;
3. \( X \) is \( c \)-complete and \( X \) or \( Y \) is down-open.

Some corollaries of Theorem 1.6 can be formulated using the notion of a \( \vec{C} \)-closed topological semilattice for a category \( \vec{C} \) whose objects are topological semigroups and morphisms are continuous homomorphisms between topological semigroups.

Given a class \( C \) of topological semigroups by \( h: C \) and \( e: C \) we denote the category whose objects are topological semigroups in the class \( C \) and morphisms are continuous homomorphisms and isomorphic topological embeddings of topological semigroups in the class \( C \), respectively.

**Definition 1.7.** Let \( \vec{C} \) be a category whose objects are topological semigroups and morphisms are continuous homomorphisms between topological semigroups. An object \( X \) of the category \( \vec{C} \) is called \( \vec{C} \)-closed if for any morphism \( h : X \to Y \) of the category \( \vec{C} \) the image \( h(X) \) is closed in \( Y \).

In particular, for a class \( C \) of topological semigroup, a topological semigroup \( X \in C \) is called

- \( h:C \)-closed if for any continuous homomorphism \( f : X \to Y \in C \) the image \( f(X) \) is closed in \( Y \);
- \( e:C \)-closed if for each isomorphic topological embedding \( f : X \to Y \in C \) the image \( f(X) \) is closed in \( Y \).

Therefore a topological semigroup \( X \) is (absolutely) \( H \)-closed if and only if it is \( e:TS \)-closed (resp. \( h:TS \)-closed) for the class \( TS \) of Hausdorff topological semigroups.

We shall be interested in \( h:C \)-closedness for the following classes \( C \) of topological semilattices:

- \( TsL \) of all Hausdorff topological semilattices;
- \( TsL_w \) of all well-separated Hausdorff topological semilattices;
- \( TsL_o \) of all down-open Hausdorff topological semilattices;
- \( TL \) of Hausdorff topological lattices.

The results of this paper allow us to draw the following diagram, containing implications between various completeness and closedness properties of a Hausdorff topological semilattice. The implications from this diagram will be proved in Section 9.
2. Upper and Lower Sets in Topological Semilattices

In this section we prove some auxiliary results related to upper and lower sets in topological semilattices.

Simple examples show that in general, the closure of an (upper) lower set in a pospace is not necessarily an (upper) lower set in the pospace. However, in semilattices we have the following positive result.

**Lemma 2.1.** Let $X$ be a topological semilattice.

1. For any open set $U \subset X$ the upper set $\uparrow U$ is open in $X$.
2. The interior of any upper set $U \subset X$ is an upper set in $X$.
3. The closure $\bar{L}$ of any lower set $L \subset X$ is a lower set in $X$.

**Proof.** Given an open set $U \subset X$, for every point $x \in \uparrow U$ choose a point $u \in U$ with $u \leq x$ and observe that $O_x = \{z \in X : uz \in U\}$ is an open neighborhood of $x$, contained in $\uparrow U$.

Given any upper set $P \subset X$ consider its interior $P^o$ in $X$ and observe that for any point $x \in P^o$ there exists an open set $U \subset P$ containing the point $x$. Taking into account that the upper set $\uparrow U$ is open, we conclude that $\uparrow x \subset \uparrow U \subset P^o$, which means that $P^o$ is an upper set in $X$.

For any lower set $L \subset X$, the complement $X \setminus L$ is an upper set, whose interior $(X \setminus L)^o$ is an upper set in $X$. Then the complement $X \setminus (X \setminus L)^o = \bar{L}$ is a lower set in $X$.  

Applying Lemma 2.1(1) to the continuous operation $(x, y) \mapsto \sup\{x, y\}$ in a topological lattice, we obtain the following simple (but important) fact.

**Corollary 2.2.** Each topological lattice $X$ is down-open.

The following proposition is a straightforward corollary of [22, Lemma 1].

**Proposition 2.3.** Each linear pospace is a topological lattice.

**Lemma 2.4.** Any points $x \nleq y$ of a pospace $X$ have open neighborhoods $V_x, V_y \subset X$ such that $\uparrow V_x \cap \downarrow V_y = \emptyset$.

**Proof.** Since the partial order $\leq$ is closed, the points $x, y$ have open neighborhoods $V_x, V_y$ such that $V_x \times V_y$ is disjoint with the partial order $\leq$ in $X \times X$. Consequently, $\tilde{x} \nleq \tilde{y}$ for any $\tilde{x} \in V_x$ and $\tilde{y} \in V_y$. We claim that $\uparrow V_x \cap \downarrow V_y = \emptyset$. Assuming that this intersection contains some point $v$, we could find points $\tilde{x} \in V_x$ and $\tilde{y} \in V_y$ such that $\tilde{x} \leq v \leq \tilde{y}$, which contradicts the choice of $V_x$ and $V_y$.

**Corollary 2.5.** Any points $x \nleq y$ in Hausdorff topological semilattice $X$ have open neighborhoods $V_x, V_y \subset X$ such that $\uparrow V_x \cap \downarrow V_y = \emptyset$.

3. Chain-Closed Sets in Semilattices

A subset $A$ of a poset $X$ is called

- **lower chain-closed in $X$** if $\inf C \in A$ for any chain $C \subset A$ possessing $\inf C \in X$;
- **upper chain-closed in $X$** if $\sup C \in A$ for any chain $C \subset A$ possessing $\sup C \in X$;
- **chain-closed in $X$** if $A$ is lower chain-closed and upper chain-closed in $X$.

A poset $(X, \leq)$ is defined to be **chain-complete** if each chain $C \subset X$ has $\inf C$ and $\sup C$ in $X$. Definition 1.3 implies that each $\epsilon$-complete topological semilattice is chain-complete.

The following lemma can be easily derived from the definitions.

**Lemma 3.1.** Any closed upper or lower set in a $\epsilon$-complete topological semilattice is chain-closed.

For two subsets $A, B$ of a semilattice $X$ let $AB = \{ab : a \in A, b \in B\}$ be their product in $X$. The proof of the following simple fact can be found in Lemma III-1.2 of [10].

**Lemma 3.2.** Let $X$ be a semilattice and $A, B \subset X$ be two sets that have $\inf A$ and $\inf B$ in $X$. Then $(\inf A) \cdot (\inf B) = \inf(A \cdot B)$. 

A semilattice $X$ is called *chain-continuous* if for any chain $C \subset X$ possessing $\sup C \in X$ we get $x \cdot \sup C = \sup xC$.

**Lemma 3.3.** A Hausdorff topological semilattice $X$ is chain-continuous if $\sup C \in \overline{C}$ for any chain $C \subset X$ possessing $\sup C \in X$. Consequently, each $c$-complete Hausdorff topological semilattice is chain-continuous.

**Proof.** Let $C \subset X$ be a chain possessing $\sup C \in X$. It is easy to see that for any $a \in X$ the product $a \cdot \sup C$ is an upper bound for the set $aC$. To show that $a \cdot \sup C = \sup aC$, we need to check that $a \cdot \sup C \leq b$ for any upper bound $b \in X$ of the set $aC$ in $X$. Taking into account that the set $\downarrow b$ is a closed lower set in $X$, we conclude that $\sup aC \in \downarrow(aC) \subset \downarrow b$ and hence $a \cdot \sup C \leq b$ and $a \cdot \sup C = \sup aC$. \hfill $\square$

For a subset $A$ of a poset $X$ by the *chain-closure* $\overrightarrow{A}$ of $A$ we understand the smallest chain-closed subset of $X$ that contain the set $A$. It is equal to the intersection of all chain-closed subsets of $X$ containing $A$.

**Lemma 3.4.** Let $X$ be a chain-continuous semilattice. For any point $a \in X$ and set $B \subset X$ we get $a \cdot \overrightarrow{B} \subset \overrightarrow{aB}$.

**Proof.** Consider the shift $s_a : X \to X$, $s_a : x \mapsto ax$. We claim that the set $P = s_a^{-1}(\overrightarrow{B})$ is chain-closed in $X$. Indeed, for any chain $C \subset P$ possessing $\inf C$, Lemma 3.2 implies that $a \cdot \inf C = \inf aC$. Since the set $\overrightarrow{aB} \supset s_a(P) \supset s_a(C) = aC$ is chain-closed in $X$, $s_a(\inf C) = a \cdot \inf C = \inf aC \in \overrightarrow{aB}$ and hence $\inf C \in P$.

By analogy we can prove that for any chain $C \subset P$ possessing $\sup C$, we get $\sup C \in P$. This means that the set $P$ is chain-closed, contains $B$ and hence $\overrightarrow{B} \subset P$, which implies $a \overrightarrow{B} = s_a(\overrightarrow{B}) \subset s_a(P) \subset \overrightarrow{aB}$. \hfill $\square$

**Corollary 3.5.** For any subsets $A, B \subset X$ of a chain-continuous semilattice $X$, we get $\overrightarrow{A \cdot B} \subset \overrightarrow{AB}$.

**Proof.** For any point $a \in A$, by Lemma 3.4, we get $a \overrightarrow{B} \subset \overrightarrow{aB} \subset \overrightarrow{AB}$ and hence $A \overrightarrow{B} \subset \overrightarrow{AB}$. Applying Lemma 3.4 once more, we conclude that for every $b \in \overrightarrow{B}$, we get

\[
\overrightarrow{A} \cdot \overrightarrow{b} \subset \overrightarrow{Ab} \subset \overrightarrow{A} \overrightarrow{B} \subset \overrightarrow{AB} = \overrightarrow{A} \overrightarrow{B}
\]

and hence $\overrightarrow{A \cdot B} \subset \overrightarrow{AB}$. \hfill $\square$

4. Countable chain-closures of sets in semilattices

For a subset $A$ of a chain-complete poset $X$ let

\[
\downarrow A = \{ \inf C : C \text{ is a countable chain in } A \},
\]

\[
\uparrow A = \{ \sup C : C \text{ is a countable chain in } A \},
\]

\[
\downarrow \uparrow A = \uparrow \downarrow A.
\]

**Lemma 4.1.** For any subsets $A, B \subset G$ of a semilattice $X$ we get $\downarrow A \cdot \downarrow B \subset \downarrow (AB)$. If the semilattice $X$ is chain-continuous, then $\uparrow A \cdot \uparrow B \subset \uparrow (AB)$.

**Proof.** Given two points $a \in \downarrow A$ and $b \in \downarrow B$, find countable chains $\overrightarrow{A} \subset A$ and $\overrightarrow{B} \subset B$ such that $a = \inf \overrightarrow{A}$ and $b = \inf \overrightarrow{B}$. Since $\overrightarrow{A}$ and $\overrightarrow{B}$ are countable, we can find decreasing sequences $\{a_n\}_{n \in \omega} \subset \overrightarrow{A}$ and $\{b_n\}_{n \in \omega} \subset \overrightarrow{B}$ such that $a = \inf \{a_n\}_{n \in \omega}$ and $b = \inf \{b_n\}_{n \in \omega}$. It follows that $\{a_nb_n\}_{n \in \omega}$ is a decreasing sequence in $AB$. Moreover, for any $m, k \in \omega$ there exists $n \in \omega$ such that $a_nb_n \leq a_mb_k$, which implies that $a
b \leq \inf \{a_nb_n\}_{n \in \omega} = \inf \{a_nb_n\}_{n \in \omega} = \inf (AB)$.

By Lemma 3.2 for every $n \in \omega$,

\[
a_nb_n = a_n \cdot \inf \{b_k\}_{k \in \omega} \geq \inf \{a_nb_n\}_{n \in \omega} \geq \inf \{a_nb_k\}_{m \in \omega} = \inf (AB).
\]
Applying the Lemma 3.2 once more, we get \( ab = \inf_{n \in \omega} a_n b \geq \inf(A \bar{B}) \) and hence

\[
ab = \inf(\bar{A} \bar{B}) = \inf\{a_n b_n\}_{n \in \omega} \in \Downarrow AB.
\]

By analogy we can prove that the chain-continuity of \( X \) implies \( \uparrow A \cdot \uparrow B \subset \uparrow (AB) \).

**Lemma 4.2.** Let \( \{V_n\}_{n \in \omega} \) be a decreasing sequence of non-empty sets in a chain-continuous chain-complete semilattice \( X \) such that \( V_{n+1} V_{n+1} \subset V_n \) for all \( n \in \omega \). Then the set \( L = \bigcap_{n \in \omega} \uparrow V_n \) is a non-empty subsemilattice of \( X \).

**Proof.** To see that the set \( L \) is a subsemilattice of \( X \), observe that for every positive integer \( n \), Lemma 4.1 guarantees that

\[
LL \subset (\uparrow V_n) \cdot (\uparrow V_n) \subset \uparrow (V_n \cdot V_n) \subset \uparrow (V_n V_n) \subset \uparrow V_{n-1}
\]

and hence, \( LL \subset \bigcap_{n \in \omega} \uparrow V_n = L \), which means that \( L \) is a subsemilattice of \( X \).

To see that \( L \) is not empty, for every \( n \in \omega \), choose a point \( v_n \in V_n \). For any numbers \( n < m \), consider the product \( v_n \cdots v_m \in X \). Using the inclusions \( V_k V_k \subset V_{k-1} \) for \( k \in \{m, m-1, \ldots, n+1\} \), we can show that \( v_n \cdots v_m \in V_n \). Observe that for every \( n \in \omega \), the sequence \( (v_n \cdots v_m)_{n > m} \) is decreasing and by the chain-completeness of \( X \), this chain has the greatest lower bound \( \bar{v}_n := \inf m > n v_n \cdots v_m \) in \( X \), which belongs to \( \downarrow V_n \).

Taking into account that \( v_n \cdots v_m \preceq v_k \cdots v_m \) for any \( n \leq k < m \), we conclude that \( \bar{v}_n \preceq \bar{v}_k \) for any \( n \leq k \). By the chain-completeness of \( X \) the chain \( \{\bar{v}_n\}_{n \in \omega} \) has the least upper bound \( \bar{v} = \sup \{\bar{v}_n\}_{n \in \omega} \) in \( X \). Since for every \( k \in \omega \), the chain \( \{\bar{v}_n\}_{n > k} \) is contained in \( V_k \), the least upper bound \( \bar{v} = \sup \{\bar{v}_n\}_{n \in \omega} = \sup \{\bar{v}_n\}_{n > k} \) belongs to \( \uparrow V_k \) for all \( k \in \omega \). Then \( \bar{v} \in \bigcap_{n \in \omega} \uparrow V_n = L \), so \( L \neq \emptyset \).

\[ \Box \]

### 5. The Key Lemma

In this section we shall prove a Key Lemma for the proof of Theorem 1.6.

**Lemma 5.1.** Let \( h : X \to Y \) be a homomorphism of topological semilattices. The subsemilattice \( S = \{h(x) \in Y \} \) is closed in \( Y \) if the following conditions are satisfied:

1. \( X \) is chain-complete and chain-continuous;
2. for any \( y \in \bar{S} \setminus S \subset Y \) and \( x \in X \) there exists a sequence \( (U_n)_{n \in \omega} \) of neighborhoods of \( y \) in \( Y \) such that such that the point \( x \) does not belong to the chain-closure \( \bigcup_{\alpha \in \kappa} U_{\alpha,n} \) of the set \( U = \bigcap_{n \in \omega} \uparrow h^{-1}(U_n) \) in \( X \).

**Proof.** Given any point \( y \in \bar{S} \), we should prove that \( y \notin S \). To derive a contradiction, assume that \( y \in S \). By transfinite induction, for every non-zero cardinal \( \kappa \) we shall prove the following statement:

\[ (\ast_\kappa) \quad \text{for every family } \{U_{\alpha,n}\}_{(\alpha,n) \in \kappa \times \omega} \text{ of neighborhoods of } y \text{ in } Y, \text{ the set } \bigcap_{\alpha \in \kappa} \bigcap_{n \in \omega} \uparrow h^{-1}(U_{\alpha,n}) \text{ is not empty.} \]

To prove this statement for the smallest infinite cardinal \( \kappa = \omega \), fix an arbitrary double sequence \( (U_{\alpha,n})_{(\alpha,n) \in \omega \times \omega} \) of neighborhoods of \( y \). Using the continuity of the semilattice operation at \( y \), construct a decreasing sequence \( (W_n)_{n \in \omega} \) of neighborhoods of \( y \) such that \( W_n \subset \bigcap_{\alpha+k \leq n} U_{\alpha,k} \) and \( W_{n+1} \subset W_n \) for all \( n \in \omega \). Then the preimages \( V_n = h^{-1}(W_n) \) form a decreasing sequence \( (V_n)_{n \in \omega} \) of non-empty sets in \( X \) such that \( V_n V_n \subset V_{n-1} \) for all \( n > 0 \). By Lemma 1.2, the set \( \bigcap_{n \in \omega} \downarrow V_n \) is not empty. Then the set

\[
\bigcap_{\alpha \in \omega} \bigcap_{n \in \omega} \uparrow h^{-1}(U_{\alpha,n}) \supset \bigcap_{\alpha \in \omega} \bigcap_{n \in \omega} \uparrow h^{-1}(U_{\alpha,n}) \supset \bigcap_{n \in \omega} \uparrow h^{-1}(W_n) = \bigcap_{n \in \omega} \downarrow V_n
\]

is not empty as well.

Now assume that for some infinite cardinal \( \kappa \) and all cardinals \( \lambda < \kappa \) the statement \((\ast_\lambda)\) has been proved. To prove the statement \((\ast_\kappa)\), fix any family \( (U_{\alpha,n})_{(\alpha,n) \in \kappa \times \omega} \) of neighborhoods of \( y \) in the topological semilattice \( Y \).
Using the continuity of the semilattice operation on $Y$, for every $\alpha \in \kappa$ choose a decreasing sequence $(W_{\alpha,n})_{n \in \omega}$ of neighborhoods of $y$ such that $W_{\alpha,n} \subset U_{\alpha,n}$ and $W_{\alpha,n+1} \subset W_{\alpha,n}$ for all $n \in \omega$. By Lemma 2.2, the intersection $L_\alpha = \bigcap_{n \in \omega} \downarrow h^{-1}(W_{\alpha,n}) \subset \bigcap_{n \in \omega} \downarrow h^{-1}(U_{\alpha,n})$ is a non-empty subsemilattice in $X$. Lemma 3.5 implies that $\prod L_\alpha \cdot \prod L_\alpha \subset \prod h^{-1}(U_{\alpha,n}) = \prod_\alpha$, which means that $\prod L_\alpha$ is a chain-closed subsemilattice in $X$. Then for every $\beta \in \kappa$ the intersection

$$L_{<\beta} = \bigcap_{\alpha < \beta} \prod L_\alpha = \bigcap_{\alpha < \beta} \bigcap_{n \in \omega} h^{-1}(W_{\alpha,n})$$

is a chain-closed subsemilattice of $X$. By the inductive assumption ($\ast_{[\beta \times \omega]}$), the semilattice $L_{<\beta}$ is not empty.

Choose any maximal chain $M$ in $L_{<\beta}$. The chain-completeness of $X$ guarantees that $M$ has inf $M \in X$. Taking into account that $L_{<\alpha}$ is chain-closed in $X$, we conclude that inf $M \in L_{<\alpha}$. We claim that $x_\alpha := \inf M$ is the smallest element of the semilattice $L_{<\alpha}$. In the opposite case, we could find an element $z \in L_{<\alpha}$ such that $x_\alpha \not< z$ and hence $x_\alpha z < x_\alpha$. Then $\{x_\alpha z\} \cup M$ is a chain in $L_{<\alpha}$ that properly contains the maximal chain $M$, which is not possible. This contradiction shows that $x_\alpha = \inf M$ is the smallest element of the semilattice $L_{<\alpha}$.

Observe that for any ordinals $\alpha < \beta < \kappa$ the inclusion $L_{<\beta} \subset L_{<\alpha}$ implies $x_\alpha \leq x_\beta$. So, $\{x_\beta\}_{\beta \in \kappa}$ is a chain in $Y$ and by the chain-completeness of $X$, it has sup $\{x_\beta\}_{\beta \in \kappa} \in \bigcap_{\beta < \kappa} L_{<\beta} \subset \bigcap_{\alpha \in \kappa} \bigcap_{n \in \omega} \downarrow h^{-1}(U_{\alpha,n})$. So, the latter set is not empty and the statement ($\ast_\kappa$) is proved.

By condition (2), for every point $x \in X$ there exists a sequence $(U_{x,n})_{n \in \omega}$ of open neighborhoods of $y$ in $Y$ such that $x \not\in \bigcap_{n \in \omega} \downarrow h^{-1}(U_{x,n})$. Then the set $\bigcap_{x \in X} \bigcap_{n \in \omega} \downarrow h^{-1}(U_{x,n})$ is empty, which contradicts the property ($\ast_\kappa$) for $\kappa = |X|$.

6. Well-separated semilattices

We recall that a topological semilattice $X$ is well-separated if for any points $x < y$ in $X$ there exists a neighborhood $U \subset X$ of $y$ such that $x \not\in \uparrow \uparrow \uparrow U$.

**Lemma 6.1.** A Hausdorff topological semilattice $X$ is well-separated if one of the following conditions is satisfied:

1. $X$ admits a continuous injective homomorphism into a well-separated topological semilattice;
2. the pospace $X$ is down-open;
3. $X$ is a topological lattice.

**Proof.** 1. Assume that $h : X \to Y$ is a continuous injective homomorphism of $X$ into a well-separated topological semilattice $Y$. Given any points $x, y \in X$ with $x < y$, consider their images $\bar{x} = h(x)$ and $\bar{y} = h(y)$ and observe that $\bar{x} < \bar{y}$ (by the injectivity of $h$). Since $Y$ is well-separated, the point $\bar{y}$ has a neighborhood $V \subset Y$ such that $\bar{x} \not\in \uparrow \uparrow \uparrow V$. By the continuity and monotonicity of $h$, the neighborhood $U := h^{-1}(V)$ of $y$ witnesses that $X$ is well-separated as $x \not\in \uparrow \uparrow \uparrow U$.

2. Assume that the lower set of any open set in $X$ is open. To show that $X$ is well-separated, take any points $x, y \in X$ with $x < y$. By Lemma 2.4, the points $x, y$ have open neighborhoods $V_x, V_y$ in $X$ such that $\downarrow V_x \cap \uparrow V_y = \emptyset$. By our assumption, the lower set $\downarrow V_x$ is open in $X$ and its complement $F = X \setminus \downarrow V_x$ is a closed upper set. Then for the neighborhood $V_y \subset F$ of $y$ we get

$$\uparrow \uparrow \uparrow V_y \subset \uparrow \uparrow \uparrow F = F \subset X \setminus \{x\}.$$ 

3. If $X$ is a topological lattice, then by Corollary 2.2 $X$ is down-open and by the preceding item, the topological semilattice $X$ is well-separated.

**Problem 6.2.** Find an example of a Hausdorff topological semilattice $X$ which is not well-separated.

**Problem 6.3.** Is each regular topological semilattice well-separated?
7. A Main Technical Result

Theorem 7.1. For a continuous homomorphism \( h : X \to Y \) from a c-complete topological semilattice \( X \) to a Hausdorff topological semigroup \( Y \), the image \( h(X) \) is closed in \( Y \) if one of the following conditions is satisfied:

1. \( X \) or \( Y \) is a down-open topological semilattice;
2. \( \uparrow U \subset \uparrow U \) for any open set \( U \subset X \);
3. \( Y \) is well-separated topological semilattice and \( \bar{S} \subset \uparrow S \) for any subsemilattice \( S \subset X \).

Proof. Observing that the closure \( \bar{S} \) of the semilattice \( S \) in the Hausdorff topological semigroup \( Y \) is a semilattice, we can assume that \( Y = \bar{S} \) is a topological semilattice.

Being c-complete, the semilattice \( X \) is chain-complete. By Lemma 3.3, the topological semilattice \( X \) is chain-continuous. The closedness of the set \( S := h(X) \) in \( Y \) will follow from Lemma 5.1 as soon as we check the condition (2) of this lemma.

Given any points \( y \in \bar{S} \setminus S \subset Y \) and \( x \in X \) we need to find a sequence \( (U_n)_{n \in \omega} \) of neighborhoods of \( y \) in \( Y \) such that \( x \notin \bigcap_{n \in \omega} \downarrow h^{-1}(U_n) \). Depending on the relation between the points \( y \) and \( s := h(x) \), we consider two cases.

If \( s \not\leq y \), then by Lemma 2.4, the points \( s \) and \( y \) have neighborhoods \( V_s \) and \( V_y \) in \( Y \) such that \( \uparrow V_s \cap \downarrow V_y = \emptyset \). By Lemma 2.1(1), the upper set \( \uparrow V_s \) is open in \( Y \). Then \( F := h^{-1}(Y \setminus \uparrow V_s) \) is a closed lower set in \( X \) that contains \( h^{-1}(V_y) \) but does not contain the point \( x \in h^{-1}(V_s) \). By Lemma 3.1, the closed lower set \( F \) is chain-closed in \( X \). Then for the neighborhoods \( U_n := V_y, n \in \omega \) we get

\[
\bigcap_{n \in \omega} \downarrow h^{-1}(U_n) = \downarrow h^{-1}(V_y) = h^{-1}(V_y) \subset h^{-1}(\bar{S}) = F = F \subset X \setminus \{x\}.
\]

The case \( s \leq y \) is more complicated. In this case \( y \not\leq s \) and we can apply Lemma 2.3 to find two open neighborhoods \( V_s, V_y \subset Y \) of \( s, y \) such that \( \downarrow V_s \cap \uparrow V_y = \emptyset \). By Lemma 2.1(1), the upper set \( \uparrow V_y \) is open in \( Y \). The continuity of the homomorphism \( h \) implies that \( h^{-1}(\uparrow V_y) \) is an open upper set in \( X \).

If \( X \) is down-open, then the lower set \( \downarrow h^{-1}(V_s) \) is open in \( X \) and \( X \setminus \downarrow h^{-1}(V_s) \) is an upper closed set that contains \( h^{-1}(V_y) \) and is disjoint with \( h^{-1}(V_s) \).

If \( Y \) is down-open, then the lower set \( \downarrow V_s \) is open in \( Y \) and \( h^{-1}(\downarrow V_s) \) is an open lower set in \( X \). Then its complement \( X \setminus h^{-1}(\downarrow V_s) \) is an upper closed set that contains \( h^{-1}(V_y) \) and is disjoint with \( h^{-1}(V_s) \).

In both cases we have found an upper closed set \( F \subset X \), containing the set \( h^{-1}(V_y) \) and disjoint with the set \( h^{-1}(V_s) \). By the c-completeness of \( X \), the upper closed set \( F \) is chain-closed. For every \( n \in \omega \) put \( U_n := V_y \) and observe that

\[
\bigcap_{n \in \omega} \uparrow h^{-1}(U_n) \subset \uparrow h^{-1}(V_y) = h^{-1}(V_y) \subset \bar{S} = F \subset X \setminus \{x\}.
\]

If the condition (2) of the theorem holds, then we put \( U_n := \uparrow V_y \) for all \( n \in \omega \) and observe that

\[
\bigcap_{n \in \omega} \uparrow h^{-1}(U_n) = \uparrow h^{-1}(\uparrow V_y) = h^{-1}(\uparrow V_y) \subset h^{-1}(\bar{S}) \subset X \setminus h^{-1}(V_s) \subset X \setminus \{x\}.
\]

Finally, assume that the condition (3) holds. In this case we can replace \( V_y \) by a smaller neighborhood and additionally assume that \( s \not\leq \uparrow \uparrow \uparrow V_y \). By the continuity of the semilattice operation at \( y \), there exists a decreasing sequence \( (U_n)_{n \in \omega} \) of open neighborhoods of \( y \) such that \( U_n \subset U_{n+1} \subset U_{n+2} \cap \uparrow V_y \) for \( n \in \omega \). Replacing each neighborhood \( U_n \) by \( \uparrow U_n \), we can assume that \( U_n = \uparrow U_n \) is an upper set. Then \( W_n := h^{-1}(U_n) \) is an open upper set in \( X \) such that \( W_n W_n \subset W_{n+1} \cap h^{-1}(\uparrow V_y) \) for all \( n \in \omega \). Lemma 4.1 guarantees that \( \downarrow W_n \cdot \downarrow W_n \subset \downarrow W_{n+1} \subset \downarrow W_{n+1} \) and hence \( \downarrow W_n \cdot \downarrow W_n \subset \uparrow \uparrow W_{n-1} \), which implies that \( F = \bigcap_{n \in \omega} \downarrow W_n \) is a subsemilattice of \( X \) with
By Lemma 4.2, the semilattice $\bigcap_{n \in \omega} \uparrow W_n \subset \bigcap_{n \in \omega} \uparrow W_n = F$ is not empty. By our assumption, $\overrightarrow{F} \subset \uparrow \uparrow F = \uparrow F$. Then

$$\bigcap_{n \in \omega} \uparrow h^{-1}(U_n) \subset \bigcap_{n \in \omega} \uparrow W_n \subset \overrightarrow{F} \subset \uparrow F.$$  

It remains to show that $x \notin \uparrow F$. Observe that $F \subset \uparrow \uparrow h^{-1}(V_y)$. We claim that $\downarrow h^{-1}(V_y) \subset \uparrow \uparrow h^{-1}(V_y)$. Indeed, for any $a \in \downarrow h^{-1}(V_y)$, we can find a countable chain $C \subset h^{-1}(V_y)$ with $a = \inf C$ and observe that $C$ is a semilattice such that $\uparrow C \subset \uparrow h^{-1}(V_y) = h^{-1}(V_y)$. By our assumption, $\overrightarrow{C} \subset \uparrow C$ and hence

$$a = \inf C \subset \overrightarrow{C} \subset \uparrow \uparrow h^{-1}(V_y).$$

Therefore $\downarrow h^{-1}(V_y) \subset \uparrow \uparrow h^{-1}(V_y)$ and $F \subset \uparrow \uparrow h^{-1}(V_y) \subset \uparrow \uparrow h^{-1}(V_y) = \uparrow h^{-1}(V_y)$. By the continuity and monotonicity of $h$,

$$h(F) \subset h(\uparrow \uparrow h^{-1}(V_y)) \subset \uparrow h(h^{-1}(V_y)) \subset \uparrow \uparrow h^{-1}(V_y) \subset \uparrow \uparrow V_y.$$  

Then $h(F) \subset \overrightarrow{h(F)} \subset \uparrow \uparrow V_y$ and $h(F) \subset \uparrow \uparrow V_y \subset Y \setminus \{s\}$, which implies the desired non-inclusion $x \notin \uparrow F$.

Now it is legal to apply Lemma 5.1 to conclude that the set $S$ is closed in $Y$.

8. Proof of Theorem 1.6

In this section we shall prove a more general version of Theorem 1.6.

Let $h : X \to Y$ be a continuous homomorphism from a $c$-complete topological semilattice $X$ to a Hausdorff topological semigroup $Y$. We shall prove that the image $S := h(X)$ is closed in $Y$ if one of the following conditions is satisfied:

1. $X$ is $k$-complete;
2. $X$ is $s$-complete and $Y$ is well-separated;
3. $X$ or $Y$ is down-open;
4. $X$ or $Y$ is a topological lattice.

1. Assume that $X$ is $k$-complete. Then for each closed subset $F \subset X$ and each non-empty chain $C \subset X$ we get $\{\inf C, \sup C\} \subset C \subset \overrightarrow{F} = F$, which means that $F$ is chain-closed. In particular, for any open subset $U \subset X$ the closed set $\overleftarrow{U}$ is chain-closed and hence contains the chain-closure $\overrightarrow{U}$ of the set $U$. Now we see that the condition (2) of Theorem 7.1 is satisfied and hence $h(X)$ is closed in $Y$.

2. Assume that $X$ is $s$-complete and $Y$ is well-separated. Then for any non-empty subsemilattice $S \subset X$ we get $\inf S \in \overrightarrow{S}$. By the $c$-completeness of $X$, the closed upper set $\uparrow \inf S$ is chain-closed and hence $\overrightarrow{S} \subset \uparrow \inf S = \uparrow S$. Now we can apply Theorem 7.1(3) and conclude that $h(X)$ is closed in $Y$.

3. If $X$ or $Y$ is down-open, then $h(X)$ is closed in $Y$ by Theorem 7.1(1).

4. If $X$ or $Y$ is a topological lattice, then $X$ or $Y$ is down-open according to Corollary 2.2. By the preceding item $h(X)$ is closed in $Y$.

9. $h$-$C$-Closed topological semilattices

In this section we prove the implications drawn in the diagram at the end of the introduction. These implications can be derived from Corollary 2.2 (saying that each topological lattice is down-open), Lemma 6.1 (establishing the embeddings of the classes $\mathcal{T}_{\mathcal{L}_0} \subset \mathcal{T}_{\mathcal{L}_w} \subset \mathcal{T}_{\mathcal{L}}$), Theorem 1.2 (implying that a Hausdorff topological semilattice is $k$-complete if and only if it has compact maximal chains) and the following two propositions.

**Proposition 9.1.** For any Hausdorff topological semilattice $X$ the following statements hold.

1. If $X$ is $k$-complete, then $X$ is $s$-complete.
(2) If $X$ is $s$-complete, then $X$ is $c$-complete.
(3) If $X$ has $c$-complete maximal chains, then $X$ is $c$-complete.

Proof. 1. Suppose $X$ is $k$-complete. To show that $X$ is $s$-complete, fix any non-empty subsemilattice $S \subset X$ and observe that $\uparrow \! S$ is a closed subsemilattice in $X$. Using Zorn’s Lemma, choose any maximal chain $M$ in $\uparrow \! S$. By the $k$-completeness of $X$ the chain $M$ has $\inf M \in M \subset \uparrow \! S$. We claim that $a := \inf M$ is a lower bound for $S$. Given any element $s \in S$, observe that $as \leq a$ and $as \in S$. By the maximality of the chain $M$, $as \in M$ and hence $a \leq as$, which means that $a = as \leq s$ and hence $a$ is a lower bound for the semilattice $S$. To see that $a$ is the largest lower bound for $S$, take any lower bound $b$ for $S$ and observe that $S \subset \uparrow b$ implies $a \in \uparrow \! S \subset \uparrow b$ and hence $b \leq a$. So, $\inf S = a \in \uparrow \! S$.

On the other hand, the $k$-completeness of $X$ guarantees that each non-empty chain $C \subset X$ has $\sup C \in \overline{C} \subset \uparrow \! \overline{C}$. Hence $X$ is a $c$-complete semilattice.

2. Suppose that $X$ is $s$-complete semilattice. Let $F \subset X$ is an arbitrary closed upper set and $C$ be an arbitrary chain in $F$. Observe that the upper set $\uparrow C \subset F$ and its closure $\overline{\uparrow C} \subset F = F$ are subsemilattices in $X$. Since $X$ is $s$-complete, the semilattice $\uparrow \! C$ has $\inf \uparrow C \in \overline{\uparrow \! C} \subset F$. It is clear that $\inf C = \inf \uparrow C \in F$.

Since $X$ is $s$-complete semilattice each non-empty chain $C \subset X$ has $\sup C \in \overline{C}$. Hence $X$ is $c$-complete semilattice.

3. Let $X$ be a semilattice with $c$-complete maximal chains. Let $F \subset X$ be an arbitrary closed upper set and $C$ be an arbitrary chain in $F$. Using Zorn’s Lemma, extend the chain $C$ to a maximal chain $M \subset X$. Since the closure of a chain in a semilattice is a chain, the maximal chain $M$ is closed in $X$.

Consider the upper set $U := M \cap \uparrow C \subset M \cap F$ in $M$ and its closure $\overline{U} \subset M \cap F \subset M \cap \overline{F} = M \cap F$. By Proposition 2.3, the linear pospace $M$ is a topological lattice. Applying Lemma 2.1(3) (to the continuous semilattice operation $M \times M \to M, (x, y) \mapsto \sup \{x, y\}$), we conclude that $\overline{U}$ is an upper set in $M$.

The $c$-completeness of the maximal chain $M$ ensures that the chain $C$ has the greatest lower bound $\inf_M C \in \overline{U} \subset F$ in $M$. We claim that $\inf_M C$ is the greatest lower bound of $C$ in $X$. Given any lower bound $b \in X$ for $C$, we conclude that $U = M \cap \uparrow C \subset \uparrow b$ and hence $\inf_M C \in \overline{U} \subset \uparrow b$. Then $b \leq \inf_M C$, which means that $\inf C = \inf_M C \subset F$.

On the other hand, the $c$-completeness of $M$ ensures that the chain $C \subset M$ has the least upper bound $\sup_M C \in \overline{M} \cap \uparrow C$ in $M$. We claim that $\sup_M C$ is the least upper bound for $C$ in $X$. Given any upper bound $b \in X$ for $C$, observe that $C \subset \downarrow b$ and hence $\sup_M C \in \overline{M} \cap \downarrow C \subset \downarrow b$, which means that $\sup_M C \leq b$ and $\sup_M C$ is the least upper bound for $C$ in $X$. Then $\sup C = \sup_M C \in \overline{M} \cap \uparrow C \subset \overline{C}$, which completes the proof of the $c$-completeness of the topological semilattice $X$. \hfill $\Box$

Corollary 9.2. For any Hausdorff topological semilattice $X$ the following statements hold.

1. If $X$ is $k$-complete, then $X$ is $hTSL$-closed.
2. If $X$ is $s$-complete, then $X$ is $hTSL_w$-closed.
3. If $X$ is $c$-complete, then $X$ is $hTSL_o$-closed.
4. If $X$ is $c$-complete and down-open, then $X$ is $hTSL$-closed.

10. SOME COMMENTS AND OPEN PROBLEMS

In this section we shall discuss some results and open problems related to Main Theorem 1.6. Theorems 1.1, 1.2, 1.3, 1.6 suggest the following intriguing open problems.

Problem 10.1. Is any (regular) absolutely $H$-closed topological semilattice $X$ $c$-complete?

Problem 10.2. Is each $s$-complete (regular) Hausdorff topological semilattice $X$ absolutely $H$-closed?

Problem 10.3. Is a Hausdorff topological semilattice $X$ (absolutely) $H$-closed if all maximal chains in $X$ are $c$-complete?
Problem 10.4. Is a Hausdorff topological semilattice $X$ s-complete if all maximal chain in $X$ are $s$-complete?

Problem 10.5. Is each c-complete Hausdorff topological semilattice s-complete?

Problem 10.6. Is each absolutely $H$-closed topological semilattice chain-complete?

The following example shows that $H$-closed topological semilattices need not be c-complete.

**Example 10.7.** There exists a topological semilattice $X$ such that

1. $X$ is metrizable, countable, and locally compact;
2. $X$ is $H$-closed;
3. $X$ is not chain-complete;
4. $X$ contains a closed upper set $U$, which is not chain-closed in $X$, so $X$ is not c-complete;
5. there exists an injective continuous homomorphism $h : X \to Z$ to a compact Hausdorff topological semilattice $Z$ whose image $h(X)$ is not closed in $Z$.

**Proof.** Let $\mathbb{Z} = \{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ be the set of integer numbers with attached elements $-\infty, +\infty$ such that $-\infty < z < +\infty$ for all $z \in \mathbb{Z}$. We endow $\mathbb{Z}$ with the semilattice operation of minimum. Let $\{0, 1\}$ be the discrete two-element semilattice with the semilattice operation of minimum. In the product semilattice $\mathbb{Z} \times \{0, 1\}$, consider the subsemilattice

$$X := (\mathbb{Z} \times \{0\}) \cup (\mathbb{Z} \times \{1\}).$$

Endow $X$ with the topology

$$\tau = \{U \subset X : (-\infty, 0) \in U \Rightarrow (\exists n \in \mathbb{Z} \forall m \leq n \ (-2m, 0) \in U)\} \cup$$

$$\cup \{U \subset X : (+\infty, 0) \in U \Rightarrow (\exists n \in \mathbb{Z} \forall m \geq n \ (2m, 0) \in U)\}.$$  

By [4] Theorem 4, the topological semilattice $X$ is $H$-closed. On the other hand, the upper closed set $\mathbb{Z} \times \{1\}$ is not chain-closed in $X$ as $(-\infty, 0) = \inf(\mathbb{Z} \times \{1\}) \notin \mathbb{Z} \times \{1\}$. So, $X$ is not c-complete. Also the set $\mathbb{Z} \times \{1\}$ has no upper bound in $X$, so the semilattice $X$ is not chain-complete.

Next, endow $\mathbb{Z}$ with the compact metrizable topology

$$\tau_k = \{U \subset X : (-\infty, 0) \in U \Rightarrow (\exists n \in \mathbb{Z} \forall m \leq n \ (-m, 0) \in U)\} \cap$$

$$\cap \{U \subset X : (+\infty, 0) \in U \Rightarrow (\exists n \in \mathbb{Z} \forall m \geq n \ (m, 0) \in U)\}.$$  

Then the identity map $\text{id} : X \to \mathbb{Z} \times \{0, 1\}$ is an injective continuous homomorphism whose image $\text{id}(X)$ is not closed in $\mathbb{Z} \times \{0, 1\}$. \hfill \Box

In [23] Yokoyama asked the question: Is each $H$-closed pospace chain-complete? The following example gives a negative answer to this question.

**Example 10.8.** Let $\mathbb{I} := [0, 1]$ be the unit interval endowed with the usual topology $\tau$ and let $L = \{1/n : n \in \mathbb{N}\}$. Let $\tau_1$ be the topology on $\mathbb{I}$, generated by the base $\{V \setminus L : V \in \tau\}$. On the space $X$ consider the partial order $\leq$ in which $x \leq y$ iff either $x = y$ or $x, y \in L$ and $x \leq y$. By [9] 3.12.5, $X = ([0, 1], \tau_1)$ is an $H$-closed topological space, which implies that $(X, \leq)$ is an absolutely $H$-closed pospace. On the other hand, $L$ is a maximal chain in $(X, \leq)$ without lower bound in $X$. 
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