THE GEOMETRY OF CONVERGENCE IN NUMERICAL ANALYSIS

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Abstract. The domains of mesh functions are strict subsets of the underlying space of continuous independent variables. Spaces of partial maps between topological spaces admit topologies which do not depend on any metric. Such topologies geometrically generalize the usual numerical analysis definitions of convergence.

1. Introduction. The numerical analysis of ordinary differential equations is usually within the context of open subsets of $\mathbb{R}^n$. In the machine everything is anyway a tuple of floating point numbers. However, dynamical systems are most naturally expressed in the category of differentiable manifolds, and there is a significant geometric integration literature specifically targeting manifolds ([15, 20, 21], and their references and citations, and also [9, 12, 27]).

By definition [2, 14, 30, 31], a discrete approximation $y_{h,k}$ to a solution $y(t)$ of an ordinary differential equation ($h$ is the timestep, $k$ is an integer, $0 \leq k \leq nh$, and the sequence $y_{h,k}$ approximates $y(t)$ at time $t = t_{h,k}$) converges if

$$
\lim_{h \to 0^+} \max_{0 \leq k \leq nh} \|y_{h,k} - y(t_{h,k})\| = 0.
$$

(1)

The purpose of this article is to derive a purely topological backward-compatible replacement of the $\mathbb{R}^n$-based and metric-dependent (1), for future work in coordinate-independent numerical analysis, and to establish the metric independence of existing work. The conclusions (Section 4) are generally accessible: essentially, for a sequence of approximations $\langle y_i(t) \rangle$ and a solution $y(t)$, an elaboration of

$$
y_i \to y \iff \lim_{i \to \infty} y_i(t_i) = y(t) \text{ whenever } t_i \in \text{domain}(y_i) \text{ and } t_i \to t
$$

(2)

may be used, in the context of second countable locally compact Hausdorff spaces, given that the approximating sequences to bounded solutions are bounded. The underlying technical theory is based on the general topology of spaces of partial maps [3–6, 8, 11, 13, 16–19, 22, 23, 25], which is reviewed and developed in Sections 2 and 3. An Appendix reviews the generalities of defining topologies from a priori convergence definitions such as (2).

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2. Geometric hypertopologies. Although the simplicity of (2) is compelling, the topological view diminishes sole reliance on just that. For example, (2) is inconsistent with the usual properties of topological convergence (Theorem A.4): the sequence \( (g_i) \) of partial maps
\[
g_i: (-1)^{i+1}[0,1] \subseteq \mathbb{R} \to \mathbb{R}, \quad g_i(t) = (-1)^{i+1}t,
\]
converges to any function \( g \) such that \( g(0) = 0 \), because the logical predicate
\[
t_i \in \text{domain}(g_i) \text{ and } t_i \to t \iff t_i \in (-1)^{i+1}[0,1] \text{ and } t_i \to t
\]
is vacuous unless \( t = 0 \), but the constant even and odd subsequences converge to their constant values.

Let \( X \) be a topological space. A hypertopology is a topology on a subset of the power set of \( X \); a hyperspace is such a topological space. The geometric hypertopologies are those defined using only the topology of \( X \). Sequences are the objective in this article because the target application is numerical analysis, but sequence convergence does not generally suffice for topology. Net convergence [29,32], where the indices are from general directed ordered sets, does suffice in the general.

2.1. Kuratowski-Painlevé convergence.

**Definition 2.1.** Let \( \langle A_\lambda \rangle_{\lambda \in \Lambda} \) be a net of subsets of a topological space \( X \).

(a) \( x \in X \) is a limit point of \( \langle A_\lambda \rangle \) if, for all open \( U \ni x \), \( A_\lambda \cap U \neq \emptyset \) finally i.e., for all open \( U \ni x \) there is a \( \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \) for all \( \lambda \geq \lambda^* \). The lower closed limit or Kuratowski limit inferior of \( \langle A_\lambda \rangle \), denoted \( \text{Li}_A \lambda \), is the set of its limit points.

(b) \( x \in X \) is a cluster point of \( \langle A_\lambda \rangle \) if, for all open \( U \ni x \), \( A_\lambda \cap U \neq \emptyset \) cofinally i.e., for all open \( U \ni x \) and all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \). The upper closed limit or Kuratowski limit superior of \( \langle A_\lambda \rangle \), denoted \( \text{Ls}_A \lambda \), is the set of its cluster points.

(c) \( \langle A_\lambda \rangle \) Kuratowski-Painlevé converges to \( A \), denoted \( A = \text{K-lim} A_\lambda \), if LiA \lambda = A, and LsA \lambda = A.

The notations conform to [5]. In concept, \( x \) is a limit point of a net of subsets if the subsets are eventually close to \( x \), and a cluster point if the subsets are perpetually close. From LiA \lambda \subseteq LsA \lambda follows that \( A = \text{K-lim} A_\lambda \) if and only if \( A \subseteq \text{Li}_A \lambda \) and LsA \lambda \subseteq A ([5], Lemma 5.2.4). LiA \lambda and LsA \lambda are both closed: If \( x \in \text{cl}((\text{Li}_A \lambda)) \) and \( U \ni x \) is open then \( U \cap \text{Li}_A \lambda \neq \emptyset \). Choose \( y \in U \cap \text{Li}_A \lambda \). Then \( y \in U \), so there is a \( \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \) for \( \lambda \geq \lambda^* \), which suffices to show that \( x \in \text{Li}_A \lambda \). The proof that LsA \lambda is closed is similar. That LiA \lambda and LsA \lambda are necessarily closed means that KP-convergence can occur only to closed sets, which thereby have an elevated significance.

Limits and cluster points are defined in terms of neighbourhoods of the underlying topology, but they may also be expressed in terms of convergence:

**Proposition 2.2.** If \( \langle A_\lambda \rangle_{\lambda \in \Lambda} \) is a net of subsets of \( X \) then

(a) \( x \in X \) is a cluster point of \( \langle A_\lambda \rangle \) if and only if there is a subnet \( \langle A_{\mu_\lambda} \rangle_{\mu \in M} \) and a net \( \langle x_{\mu} \rangle_{\mu \in M} \) such that \( x_{\mu} \in A_{\mu_\lambda} \) and \( \langle x_{\mu} \rangle \) converges to \( x \);

(b) \( x \in X \) is limit point of \( \langle A_\lambda \rangle \) if and only if it is a cluster point of every subnet of \( \langle A_\lambda \rangle \).

If \( X \) is first countable and \( \langle A_\lambda \rangle \) is a sequence of subsets of \( X \) then
Corollary 2.3. If and only if every convergent sequence the elements of by Prop. 2.2(c), and hence is a limit point by KP-convergence, so by Prop. 2.2(d) U point. Then there is an open and every limit point is a cluster point. Conversely, suppose that \( x \in A_i \) such that \( x \in A_i \) and \( x \in A_i \). Then \( (x, U) \in A_i \) and all convergent sequences \( x \in A_i \), admits a convergent extension to all of \( A_i \), i.e. a convergent sequence \( (x, U) \) such that \( x \in A_i \) and \( x \in A_i \), for all \( j \). For example, this immediately clarifies that the limit of \( (-1)^i [0, 1] \) does not exist. Indeed, given KP-convergence of \( A_i \), the limit \( x \) of \( (x, U) \) is a cluster point by Prop. 2.2(c), and hence is a limit point by KP-convergence, so by Prop. 2.2(d) there is an \( x \in A_i \) such that \( x \in A_i \), and replacing \( x \) with in the sequence \( \langle x_i \rangle \) provides the required extension. Conversely, if \( x \) is a cluster point of \( A_i \), then there are \( x_i \in A_i \) such that \( x_i \to x \), then the extension provides that \( x \) is a limit point, and KP-convergence holds.

Proof. (a) Suppose \( x \) is a cluster point of \( A_i \). \{\( (\lambda, U) \}\}, where \( \lambda \in \Lambda \) and \( U \) is a neighbourhood of \( x \), is directed with the ordering \( (\lambda_1, U_1) \geq (\lambda_2, U_2) \) if \( \lambda_i \geq \lambda_2 \) and \( U_1 \subseteq U_2 \). The subset \( M = \{ (\lambda, U) \mid A_i \cap U \neq \emptyset \} \) is directed by Lemma A.1. Define \( A_{(\lambda, U)} = A_i \setminus \lambda \setminus U ) \in M \), and note that \( A_{(\lambda, U)} \) is a subnet of \( A_i \).

Picking \( x(\lambda, U) \in A_i \setminus U \), \( \lambda \setminus U \) there is a subnet \( x(\lambda, U) \in A_i \setminus U \). Given open \( V \ni x \), choose \( \lambda^* \) such that \( A_i \cap V \neq \emptyset \) and let \( U^* = V \). Then \( (\lambda^*, U^*) \in M \), and \( (\lambda, U) \geq (\lambda^*, U^*) \) implies \( x(\lambda, U) \in U \subseteq U^* = V \), from which \( x_{A_i} \to x \). Conversely, suppose \( A_{(\lambda, U)} \) is a subnet of \( A_i \), \( x \in A_{(\lambda, U)} \), and \( x \to x \), and \( \lambda^* \in A_i \). Given an open \( U \ni x \) there is a \( \mu_i \in U \) if \( \mu \geq \mu_i \), and given \( \lambda^* \in A_i \) there is a \( \mu_i \) such that \( \lambda \mu \geq \lambda^* \), for \( \mu \geq \mu_i \). Choose \( \mu \) such that \( \mu \geq \mu_i \) and \( \mu \geq \mu_i \). For that \( \mu \), \( x \in A_{(\mu, U) \cap U} \), providing \( \lambda \mu \geq \lambda^* \) and \( A_{(\mu, U) \cap U} \neq \emptyset \).

(b) If \( x \) is a limit point of \( A_i \cap U \) then it is a limit point of every subnet of \( A_i \), and every limit point is a cluster point. Conversely, suppose that \( x \) is not a limit point. Then there is an open \( U \ni x \) such that for all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( A_i \cap U = \emptyset \), and the set \( \{ \lambda \mid A_i \cap U = \emptyset \} \) provides a subnet of \( A_i \) for which \( x \) is not a cluster point.

(c,d) Suppose \( x \) is a cluster point of \( A_i \) and \( (U_i) \) be a countable neighbourhood base at \( x \). Inductively choose \( i \), strictly increasing such that \( A_i \cap U_i ) \neq \emptyset \) and choose \( x \) in that set. The converse is immediate from (a) because any subsequence of \( A_i \) is a subnet of that. For (d), if \( x \) is a limit point of \( A_i \) then, for all \( j \), there is an \( N_j \) such that \( i \geq N_j \) implies \( A_i \cap U_j ) \neq \emptyset \). Without loss of generality assume \( N_j \) is increasing. Choosing a sequence \( x_i \in A_i \) such that \( x_i \in U_1 \) for \( N_1 \leq i < N_2 \), \( x_i \in U_2 \) for \( N_2 \leq i < N_3 \), and so on, provides \( x_i \in A_i \) such that \( x_i \to x \). Conversely, any such sequence provides final nonempty intersection of \( A_i \) with any neighbourhood of \( x \).

Corollary 2.3. If \( X \) is first countable then \( A_i = K \lim A_i \) if and only if \( A_i \) is exactly the elements of \( X \) that can be approximated in sequence from within the \( A_i \), i.e.

(a) if \( x \in A_i \) is a sequence \( (x_i) \) such that \( x \in A_i \) and \( x_i \to x \); and
(b) \( \lim x_j \in A_i \) for all subsequences \( A_i \) of \( A_i \) and all convergent sequences \( (x_j) \), \( x_j \in A_i \).

Also in the case of first countable \( X \), a sequence of subsets \( A_i \) KP-converges if and only if every convergent sequence \( (x_j) \), \( x_j \in A_i \), admits a convergent extension to all of \( A_i \), i.e. a convergent sequence \( (x_i) \) such that \( x_i \in A_i \) and \( x_i \to x \), for all \( j \). For example, this immediately clarifies that the limit of \( (-1)^i [0, 1] \) does not exist. Indeed, given KP-convergence of \( A_i \), the limit \( x \) of \( (x_j) \) is a cluster point by Prop. 2.2(c), and hence is a limit point by KP-convergence, so by Prop. 2.2(d) there is an \( x_i \in A_i \) such that \( x_i \to x \), and replacing \( x_i \) with \( x_i \) in the sequence \( (x_i) \) provides the required extension. Conversely, if \( x \) is a cluster point of \( A_i \), then there are \( x_j \in A_i \) such that \( x_j \to x \), then the extension provides that \( x \) is a limit point, and KP-convergence holds.
2.2. The Vietoris and co-compact hypertopologies.

**Definition 2.4.** Let $X$ be a topological space.

(a) The lower Vietoris topology [lower co-compact topology] on $2^X$ is generated by the collection $\{ \{ A \subseteq X \mid A \cap U \neq \emptyset \} \}$ over $U \subseteq X$ open [U \subseteq X co-compact].

(b) The upper Vietoris topology [upper co-compact topology] on $2^X$ is generated by the collection $\{ \{ A \subseteq X \mid A \subseteq U \} \}$ over $U \subset X$ open [U \subset X co-compact].

There is a variety of hypertopologies and the literature is extensive, e.g. [5, 13, 17, 18, 23, 25], and [19] presents a systematization and a convenient summary table. The notations are not quite standard: the upper co-compact topology is usually the co-compact topology, and the lower co-compact topology does not appear. [22] calls the lower Vietoris topology the Thurston topology. The Vietoris topology or exponential topology [17] is the join of the upper and lower Vietoris topologies. Often the closed subsets are used instead of the full power set; here $2^X$ will be either unless otherwise noted. These are hit-and-miss topologies, since they are defined in terms of subsets that set theoretically meet an open or co-compact set, or do not intersect the complement of a closed or compact set, i.e. for the upper Vietoris [upper co-compact] topology there is the equivalent $\{ \{ A \mid A \cap K = \emptyset \} \}$ over closed [compact] $K$, corresponding to a “miss”. The use of co-compact sets is posteriori justified for example by Lemma 2.8, where the upper co-compact topology is related to the alignment of the topologies with KP convergence. If $X$ is Hausdorff then every compact set is closed, every co-compact set is open and the Vietoris and co-compact topologies are the same.

**Proposition 2.5.** The Vietoris and co-compact topologies on $2^X$ are natural with respect to closed subspaces, i.e. if $Y \subseteq X$ is closed then the topology on $2^Y$ as a subspace of $2^X$ is the corresponding topology of $2^Y$ ($2^X$ and $2^Y$ are both all subsets or both the closed subsets).

**Proof.** The left and right sides of the collections

$$2^Y \cap \{ \{ A \in 2^X \mid A \cap U \neq \emptyset \} \} = \{ A \in 2^Y \mid A \cap (U \cap Y) \neq \emptyset \},$$
over open $U \subseteq X$, generate lower Vietoris subspace topology on $2^Y$ and the lower Vietoris topology of $2^X$, respectively. Similarly,

$$2^Y \cap \{ \{ A \in 2^X \mid A \subseteq U \} \} = \{ \{ A \in 2^Y \mid A \subseteq Y \cap U \} \},$$

shows the two upper Vietoris are the same. For the upper co-compact topology, the collections

$$2^Y \cap \{ \{ A \in 2^X \mid A \cap K \neq \emptyset \} \}, \quad \{ \{ A \in 2^Y \mid A \cap L \neq \emptyset \} \},$$

over compact $K \subseteq X$ and $L \subseteq Y$ generate the subspace topology on $2^Y$ and the topology of $2^Y$. These correspond for choices of $K$ and $L$: if $Y$ is closed, and $K$ is compact in $X$, then $L = K \cap Y$ is compact in $Y$, while if $L$ is compact in $Y$ then it is compact in $X$. For the upper co-compact topology, the collections

$$2^Y \cap \{ \{ A \in 2^X \mid A \cap (X \setminus K) \neq \emptyset \} \}, \quad \{ \{ A \in 2^Y \mid A \cap (Y \setminus L) \neq \emptyset \} \},$$

correspond over compact $K \subseteq X$ and compact $L \subseteq Y$: if $Y$ is closed, and $K$ is compact in $X$, then $L = K \cap Y$ is compact in $Y$ and $A \cap (X \setminus K) = A \cap (X \setminus Y \setminus K) = A \cap (Y \setminus (K \cap Y)) = A \cap (Y \setminus L)$, while if $L$ is compact in $Y$ then $K = L$ is compact in $X$ and $A \subseteq Y$ implies $A \cap (X \setminus K) = A \cap (X \setminus Y) = A \cap (Y \setminus L)$. \hfill \Box

Continuity of intersection is related to localization, e.g. determining that $A_i \cap U \rightarrow A \cap U$ if $A_i \rightarrow A$, $A_i$ and $A$ are subsets of a manifold, and $U$ is the domain of a chart. For the upper Vietoris and co-compact topologies, where the neighbourhoods are defined by containment within an open set, containment of $A \cap B$ is stable under small changes in $A$, and continuity follows. However, for the lower Vietoris topology, where the neighbourhoods are defined by nonempty intersection, and where $A$ and $B$ meet at their boundaries, $A \cap B$ may turn empty under small arbitrarily small changes to $A$. This is a variety of such results for the exponential topology in [17].

**Proposition 2.6.**

(a) If $B$ is closed then $A \mapsto A \cap B$ is continuous in the upper Vietoris and upper co-compact topologies on $2^X$.

(b) If $A_0 \cap B \subseteq \text{cl}(A_0 \cap \text{int } B)$ then $A \mapsto A \cap B$ is continuous at $A_0$ in the lower-Vietoris topology.

**Proof.** (a) If $A_0 \subseteq X$ and $A_0 \cap B \subseteq V$, where $V$ is open, then define $U = (X \setminus B) \cup V$, which is open since $B$ is closed. Then $A_0 \subseteq U$, and if $A \subseteq U$ then $A \cap B \subseteq V$, so $A \mapsto A \cap B$ maps the upper Vietoris subbasic open neighbourhood $\{ A \in 2^X \mid A \subseteq U \}$ of $A_0$ into the (arbitrary) subbasic open neighbourhood $\{ A \in 2^X \mid A \subseteq V \}$ of $A_0 \cap B$. If $V$ is assumed to be co-compact then $U$ as defined is co-compact because $X \setminus U = B \cap (X \setminus V)$ and the same argument holds for the upper co-compact topology.

(b) Suppose $V \subseteq X$ is open and $(A_0 \cap B) \cap V \neq \emptyset$. Choose $x \in A_0 \cap B \cap V$, so $x \in \text{cl}(A_0 \cap \text{int } B)$, and there is an $\hat{x} \in A_0 \cap \text{int } B$ such that $\hat{x} \in V$. Choose open $U \ni \hat{x}$ such that $U \subseteq B$. Then $A_0 \cap U \neq \emptyset$ because $\hat{x} \in A_0 \cap \text{int } B \cap U$, and if $A \cap U \neq \emptyset$ then $(A \cap B) \cap U = A \cap U \neq \emptyset$. \hfill \Box

Sequences (as opposed to nets) are the objective here because numerical analysis is the target application, so the countability of these hypotopologies is a focus. The most natural route to sequential convergence, first countability, is a difficult problem [4]. However, the (general) topologies of dynamical state spaces in the applications are simple: the usual assumption is at least a second countable Hausdorff manifold, and therefore a metrizable space. Second countability is stronger than first and is easily passed:
Proposition 2.7. If $X$ is second countable [and locally compact Hausdorff] then the lower Vietoris [upper co-compact, where $2^X$ is the collection of closed sets] topology is second countable.

Proof. Suppose $X$ has a countable basis $\mathcal{B}_0$. For the lower Vietoris topology, define $\mathcal{N}(U) = \{A \subseteq 2^X \mid A \cap U \neq \emptyset\}$. It suffices to show that $\mathcal{B} = \{\mathcal{N}(U) \mid U \text{ is open}\}$ and $\mathcal{B}' = \{\mathcal{N}(U) \mid U \in \mathcal{B}_0\}$ are equivalent subbases, the first generating the lower Vietoris topology by definition and the second being countable. Indeed, $\mathcal{B}' \subseteq \mathcal{B}$, while if $\mathcal{N}(U) \in \mathcal{B}$ and $A \in \mathcal{N}(U)$, then let $x \in A \cap U$ and choose $V \in \mathcal{B}_0$ such that $x \in V \subseteq U$, and the result follows because $A \in \mathcal{N}(V) \subseteq \mathcal{N}(U)$. For the upper co-compact topology, define $\mathcal{N}(K) = \{A \subseteq X \mid A \cap K = \emptyset\}$. Because $X$ is locally compact and Hausdorff, without loss of generality $\mathcal{B}_0$ consists of relatively compact subsets. It suffices to show $\mathcal{B} = \{\mathcal{N}(K) \mid K \text{ is compact}\}$ and $\mathcal{B}' = \{\mathcal{N}(\text{cl}U) \mid U \in \mathcal{B}_0\}$ are equivalent subbases. Indeed, $\mathcal{B}' \subseteq \mathcal{B}$, while if $K$ is compact and $A \in \mathcal{N}(K)$, then using regularity of $X$ and $A$ closed, choose finitely many $U_1, \ldots, U_n \in \mathcal{B}_0$ that cover $K$ with closures contained in $X \setminus A$, from which $A \in \mathcal{N}(\text{cl}(U_1)) \cap \ldots \cap \mathcal{N}(\text{cl}(U_n)) \subseteq \mathcal{N}(K)$.

The convergence of the Vietoris and co-compact topologies can be expressed in terms of the Kuratowski limit inferior and superior of Section 2.1, thus establishing a relationship to the convergence expressed by Proposition 2.2 and Corollary 2.3.

Lemma 2.8. Let $X$ be a topological space, and let $(A_\lambda)_{\lambda \in \Lambda}$ be a net of subsets of $X$.

(a) If $X$ is regular and $A_\lambda \to A$ (upper Vietoris) then $\text{cl}A \supseteq \text{Ls}A_\lambda$.
(b) If $X$ is locally compact Hausdorff and $A_\lambda \to A$ (upper co-compact) then $\text{cl}A \supseteq \text{Ls}A_\lambda$.
(c) If $X$ is compact and $A \supseteq \text{Ls}A_\lambda$ then $A_\lambda \to A$ (upper Vietoris).
(d) If $A \supseteq \text{Ls}A_\lambda$ then $A_\lambda \to A$ (upper co-compact).
(e) $A_\lambda \to A$ (lower Vietoris) if and only if $A \subseteq \text{Li}A_\lambda$.
(f) If $X$ is Hausdorff and $A \subseteq \text{Li}A_\lambda$ then $A_\lambda \to A$ (lower co-compact).
(g) If $X$ is compact and $A_\lambda \to A$ (lower co-compact) then $A \subseteq \text{Li}A_\lambda$.

Proof. (a) Suppose $A_\lambda \to A$ (upper Vietoris) and that $X$ is regular. If $x \notin \text{cl}A$ then there are open $U \supseteq \text{cl}A$ and $V \ni x$ such that $U \cap V = \emptyset$. So there is a $\lambda^*$ such that $A_{\lambda^*} \subseteq U$, and hence $A_{\lambda^*} \cap V = \emptyset$ for all $\lambda \geq \lambda^*$, i.e. $x$ is not a cluster point of $A_{\lambda^*}$.

(b) Suppose $A_\lambda \to A$ (upper co-compact) and that $X$ is locally compact Hausdorff. If $x \notin \text{cl}A$ then there is a compact neighbourhood $U$ of $x$ such that $A \cap U = \emptyset$. So there is a $\lambda^*$ such that $A_{\lambda^*} \cap U = \emptyset$ for all $\lambda \geq \lambda^*$, i.e. $x$ is not a cluster point of $A_{\lambda^*}$.

(c) Suppose $A_\lambda \nrightarrow A$ (upper Vietoris) i.e. there is an open $U$ such that $A \subseteq U$, and, for all $\lambda^*$ there is $\lambda$ such that $\lambda \geq \lambda^*$ and $A_{\lambda^*} \not\subseteq U$. The set $\Lambda' = \{\lambda \in \Lambda \mid A_{\lambda} \not\subseteq U\}$ is directed. For each $\lambda \in \Lambda'$ choose $x_{\lambda^*} \in A_{\lambda^*}$ such that $x_{\lambda^*} \notin U$. Since $X$ is compact, a subnet $(x_{\lambda^*_\nu})$ converges, say to $x$. Then $x \in \text{Ls}A_{\lambda^*}$ by Prop. 2.2(a), but $x \notin U$ since $X \setminus U$ is closed and $x_{\lambda^*_\nu} \notin U$, from which $x \notin A$ since $A \subseteq U$.

(d) Suppose $A_\lambda \nrightarrow A$ (upper co-compact) i.e. there is a compact $K$ such that $A \cap K = \emptyset$ and such that, for all $\lambda^*$, there is a $\lambda$ such that $\lambda \geq \lambda^*$ and $A_{\lambda^*} \cap K \neq \emptyset$. The set $\Lambda' = \{\lambda \in \Lambda \mid A_{\lambda} \cap K \neq \emptyset\}$ is directed. For each $\lambda \in \Lambda'$ choose $x_{\lambda} \in A_{\lambda}$ such that $x_{\lambda} \notin K$. Since $K$ is compact, there is a subnet $(x_{\lambda^*_\nu})$ that converges, say to $x \in K$, so $x$ is a cluster point of $A_{\lambda}$ and $x \notin A$. 
(e) If \( A_\lambda \to A \) (lower Vietoris) and \( x \in A \) and \( U \ni x \) is open then \( U \cap A \neq \emptyset \) and there is a \( \lambda^* \) such that \( U \cap A_\lambda \neq \emptyset \) for all \( \lambda \geq \lambda^* \), so \( x \) is a limit point. The converse is similar.

(f,g) By (e), \( A_\lambda \to A \) in the lower Vietoris topology. In a Hausdorff space, every co-compact set is open, so the lower Vietoris topology is finer than the co-compact topology and therefore \( A_\lambda \to A \) in the co-compact topology, while if \( X \) is compact then every open set is co-compact.

The following technical result is used in Proposition 2.11. The purpose is to identify a property of certain maximal subsets, in preparation for an application of Zorn’s lemma.

**Lemma 2.9.** Let \( \langle A_\lambda \rangle_{\mu \in M} \) be a subnet of the net \( \langle A_\lambda \rangle_{\lambda \in \Lambda} \) of subsets of \( X \). Then:

(a) \( \operatorname{Li} A_\lambda \geq \operatorname{Li} A_\mu \);

(b) if \( A \subseteq X \) is maximal in the collection of all lower limit sets of subsets of \( \langle A_\lambda \rangle \), and if \( A = \operatorname{Li} A_\lambda \), then \( A_{\operatorname{Li} A} \to A \) in the upper co-compact topology.

**Proof.** (a) If \( x \in \operatorname{Li} A_\lambda \), and \( U \ni x \) is open, then choose \( \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \) for all \( \lambda \geq \lambda^* \). Choose \( \mu^* \) so that \( \lambda_{\mu^*} \geq \lambda^* \). Then \( \mu \geq \mu^* \) implies \( \lambda_{\mu^*} \geq \lambda^* \) and \( A_{\lambda_{\mu^*}} \cap U \neq \emptyset \).

(b) If \( A = \operatorname{Li} A_\lambda \) and \( A_{\lambda_{\mu^*}} \neq A \) (upper co-compact) then there is a compact set \( K \) such that \( K \cap A = \emptyset \) and such that, for all \( \mu^* \) there is a \( \mu > \mu^* \) such that \( A_{\lambda_{\mu}} \cap K \neq \emptyset \). The set \( \{ \mu \mid A_{\lambda_{\mu}} \cap K \neq \emptyset \} \) is directed; choose an \( x_\mu \in A_{\lambda_{\mu}} \cap K \) for each such \( \mu \). Since \( K \) is compact, a subnet of \( \langle x_\mu \rangle \) converges, say to \( x \in K \). But then the limit inferior of the corresponding subnet of \( \langle A_{\lambda_{\mu}} \rangle \) contains both \( A \) and \( x \), and \( x \not\in A \) because \( A \cap K = \emptyset \), so \( A \) is not maximal.

2.3. The geometric topology. By Corollary 2.3, KP-convergence, i.e. \( A \subseteq \operatorname{Li} A_\lambda \) and \( A \supseteq \operatorname{Ls} A_\lambda \), corresponds to approximation from within a sequence of subsets, in the first countable case. By Lemma 2.8(e), the criterion \( A \subseteq \operatorname{Li} A_\lambda \) unrestrictedly corresponds to the lower Vietoris topology. For \( A \supseteq \operatorname{Ls} A_\lambda \), the relevant results are Lemma 2.8(a–d), from which correspondence with the upper-Vietoris topology only occurs within the overly restrictive assumption of compactness, while correspondence with the upper co-compact topology occurs within the less restrictive (and applicable) context of closed subsets of a locally compact Hausdorff space. Proposition A.7(c) provides that the combination of \( A \subseteq \operatorname{Li} A_\lambda \) and \( A \supseteq \operatorname{Ls} A_\lambda \) is the same as the convergence of the join of the lower Vietoris and upper co-compact topologies, which topology thereby gains significance.

**Definition 2.10.** The geometric topology on \( 2^X \) is the join of the lower Vietoris and upper co-compact topologies.

So-called in [7], this topology first occurs in [8] and [11]; see also [4]. It is often just the Fell topology. The term Chabauty-Fell topology is suggested in [10], while [22] (and also [7]) call it the Chabauty topology.

**Proposition 2.11.**

(a) The collection of closed subsets of \( X \) is compact in the geometric topology (for any topological space \( X \)).

(b) If \( X \) is locally compact and Hausdorff then a net of subsets \( \langle A_\lambda \rangle \) converges to \( A \) in the geometric topology if and only if it KP-converges to \( \operatorname{cl} A \).

(c) If \( X \) is locally compact and Hausdorff then the collection of closed subsets of \( X \) is Hausdorff in the geometric topology.
(d) If $X$ is locally compact, Hausdorff, and second countable then the geometric topology is second countable.

Proof. (a) Let $(A_{\lambda})_{\lambda \in \Lambda}$ be a net of closed subsets of $X$. Let $M_{\alpha}, \alpha \in \mathcal{A}$, be a collection of directed sets, and suppose $f_\alpha: M_\alpha \to \Lambda, \alpha \in \mathcal{A}$, are monotone cofinal maps. Then we have subnets $\mu \mapsto A_{f_\alpha(\mu)}, \alpha \in \mathcal{A}$, each with a lower limit set, say $A_{\alpha}$. The co-product $M = \bigvee M_\alpha = \{ (\alpha, \mu) \mid \mu \in M_\alpha, \alpha \in \mathcal{A} \}$ may be ordered by $(\alpha_1, \mu_1) \geq (\alpha_2, \mu_2)$ if $\alpha_1 = \alpha_2$ and $\mu_1 \geq \mu_2$, $(\alpha_1, \mu_1)$ and $(\alpha_2, \mu_2)$ are incomparable if $\alpha_1 \neq \alpha_2$. With this ordering, $(\alpha, \mu) \mapsto A_{f_\alpha(\mu)}$ is a subnet of $A_\lambda$ and $\operatorname{Li} f \supseteq \bigcup_\alpha A_{\alpha}$. Particularly, every chain of limit sets of subnets of $(A_\lambda)$ has an upper bound, Zorn’s lemma provides a maximal lower limit set, and there is a convergent subnet of $A_\lambda$ by Lem. 2.9(b). A simpler proof, but based on ultranets, is provided in 11.

(b) $A_\lambda \to A$ in the geometric topology if and only if $A_\lambda \to A$ in both the upper co-compact and the lower Vietoris topology. So, assuming that, Lem. 2.8 implies $A \subseteq \operatorname{Li} A_\lambda$ and $\operatorname{Ls} A_\lambda \subset \operatorname{cl} A$. Since $\operatorname{Li} A_\lambda$ is closed this implies $\operatorname{cl} A \subset \operatorname{Li} A_\lambda \subset \operatorname{Ls} A_\lambda \subset \operatorname{cl} A$ so $\operatorname{Li} A_\lambda = \operatorname{Ls} A_\lambda = \operatorname{cl} A$ i.e. $(A_\lambda)$ KP-converges to $\operatorname{cl} A$. Conversely, $\operatorname{Li} A_\lambda = \operatorname{Ls} A_\lambda = \operatorname{cl} A$ implies both $A \subset \operatorname{cl} A = \operatorname{Li} A_\lambda$ and $\operatorname{cl} A = \operatorname{Ls} A_\lambda \subseteq \operatorname{Ls} A_\lambda$.

(c) If $A_\lambda \to A$ and $A_\lambda \to A'$ in the geometric topology, then, from (b), $A_\lambda$ KP-converges both to $\operatorname{cl} A$ and to $\operatorname{cl} A'$. Both of those are equal to both $\operatorname{Li} A_\lambda$, so $\operatorname{Ls} A_\lambda$ and $\operatorname{cl} A = \operatorname{cl} A'$. On the closed sets this implies $A = A'$ and that limits are unique.

(d) If $X$ is a second countable locally compact Hausdorff space then Prop. 2.7 provides countable bases for both the lower Vietoris and upper co-compact topologies, and together those are countable basis of the join. \hfill \Box

To give some indication of the meaning of Proposition 2.11(a), note that the sequence of subsets $[i, i+1]$ converges to the null set in the geometric topology. The geometric topology does not exert control at infinity. Generally, when defining a set as a hypertopological limit, it may be essential to ensure that the limit is not empty.
Proposition 2.12.

(a) The collection
\[
\left\{ \{ A \in 2^X \mid A \cap K \subseteq U \text{ and } A \cap V_i \neq \emptyset \text{ for all } 1 \leq i \leq n \} \right\}
\] 
over all open \(U,V_1,\ldots,V_n \subseteq X\) and compact \(K \subseteq X\), is a base for the geometric topology (Figure 2 left).

(b) Suppose \(X\) is locally compact and Hausdorff, and let \(B \subseteq X\) be compact subset. Then the collection
\[
\left\{ \{ A \in 2^X \mid A \subseteq U \text{ and } A \cap V_i \neq \emptyset \text{ for all } 1 \leq i \leq n \} \right\}
\]
over all relatively compact open \(U\) with \(B \subseteq U\), and all open subsets \(V_1,\ldots,V_n\) such that \(B \cap V_i \neq \emptyset\), \(1 \leq i \leq n\), is a neighbourhood base of the geometric topology at \(B\) (Figure 2 right).

Proof. (a) The collection (3) is the same as
\[
\left\{ \{ A \in 2^X \mid A \cap K = \emptyset \text{ and } A \cap V_i \neq \emptyset \text{ for all } 1 \leq i \leq n \} \right\}
\]
over all closed \(L\), open \(V_i\), and compact \(K\), while the collection
\[
\left\{ \{ A \in 2^X \mid A \cap K = \emptyset \text{ and } A \cap V_i \neq \emptyset \text{ for all } i \} \right\}
\]
over all open \(V_i\) and compact \(K\) is a base, by definition of the join of the upper co-compact and lower Vietoris topologies. (5) and (6) are the same: Given \(K, L,\) and \(V_i\) corresponding to an member of (5), that same member is in the collection (6) by using \(K \cap L\) for the \(K\) in (6). Conversely, the member of (6) corresponding to \(K\) and \(V_i\) is in the collection (5), by using the same \(K\) and \(V_i\), and by setting \(L = X\).

(b) If \(U\) is relatively compact then set \(K = clU\) and note that \(A \cap K \subseteq U\) if and only if \(A \subseteq U\), so the collection (4) is an extraction from the base (3). It remains to show that every basic neighbourhood (3) of \(B\) contains a neighbourhood from (4) which also contains \(B\). Suppose \(U, V_i\), and \(K\) are such that \(B \cap K \subset U\) and \(B \cap V_i \neq \emptyset\). Then \(B\) and \(K \setminus U\) are disjoint compact sets, so there are disjoint open relatively compact \(U' \supseteq B\) and \(U'' \supseteq K \setminus U\). If \(A \subseteq U'\) then \(A \cap K \subseteq U' \cap K \subseteq K \cap \left( X \setminus U'' \right) \subseteq K \cap \left( X \setminus \left( K \setminus U \right) \right) = K \cap U \subseteq U\), and setting \(U = U'\) in (4), and using the same \(V_i\), provides the required neighbourhood.

Proposition 2.12 remains valid if the \(V_i\) of (3) or (4) are restricted to given bases of \(X\). In the case of (4), \(K\) may also be restricted to a covering collection of compact sets, i.e. any compact subset of \(X\) is contained in some compact set of the collection.

3. Partial maps. Given topological spaces \(X\) and \(Y\), we seek a topology on the space of partial maps \(\mathcal{P}(X,Y) = \{ f: A \to Y \mid A \subseteq X \}\) with convergence along the lines of (2), in some applicable class of topological spaces.

Definition 3.1. Let \(X\) and \(Y\) be topological spaces.

(a) The topologies on \(\mathcal{P}(X,Y)\), corresponding to the hypertopologies on the power set of \(X \times Y\) in Definitions 2.4 and 2.10, are the subspace topologies obtained from the identification of maps with graphs.

(b) The compact-open topology on \(\mathcal{P}(X,Y)\) is the topology generated by the collection \(\{ \{ f: A \subseteq X \to Y \mid f(K) \subseteq V \} \}\), where \(K \subseteq X\) is compact and \(V \subseteq Y\) is open.

(c) The generalized compact-open topology on \(\mathcal{P}(X,Y)\) is the join of the compact-open topology, and the weak topology obtained from the domain map \((f: A \subseteq X \to Y) \mapsto A\) and the lower Vietoris topology on \(2^X\).
Spaces of partial maps are first considered in [16], where a convergence criterion occurs even at the beginning: a sequence \( \langle f_i \rangle \) with domains \( A_i \) converges to \( f \) with domain \( A \) if and only if \( A_i \) converges to \( A \) in the Hausdorff distance, and \( x_i \rightarrow x \) and \( x \in A \) implies \( f_i(x_i) \rightarrow f(x) \). A recent reference is [6]. The spaces are often restricted to the continuous partial maps, especially in the case of the compact-open topologies.

The compact-open topology reverts to textbook [32] on the space of total functions, but is weak on \( P(X,Y) \), e.g. the sequence \( f_i: [i,i+1] \rightarrow \mathbb{R} \) with constant value 1 converges to any function at all because \( f_i(K) = \emptyset \) for any compact \( K \) and large enough \( i \) (and in particular the compact-open topology is not Hausdorff). The generalized compact-open topology is from [3], and it may be referred to just as the compact-open topology or the Back topology. It is generated by the collection

\[
\left\{ \{ f: A \subseteq X \rightarrow Y \mid A \cap U \neq \emptyset \text{ and } f(K) \subseteq V \} \right\},
\]

over open \( U \subseteq X \), compact \( K \subseteq X \) and open \( V \subseteq Y \). Including \( A \cap L \) over compact \( L \subseteq X \) corresponds to using the geometric instead of the lower Vietoris topology in Definition 3.1(c), but

\[
\{ f: A \subseteq X \rightarrow Y \mid f(L) \subseteq \emptyset \} = \{ f: A \subseteq X \rightarrow Y \mid A \cap L = \emptyset \}
\]

so the two definitions, and even the two subbases, would be the same. The generalized compact-open topology is finer than the weak topology induced by the domain map, which is therefore continuous with the geometric topology (or the coarser lower Vietoris or co-compact topologies) on \( 2^X \).

**Definition 3.2.** \( K_c(X,Y) \subseteq P(X,Y) \) is the collection of continuous partial maps with closed domains, as a subspace of \( P(X,Y) \) with the generalized compact-open topology.

The domains are required to be closed to align KP-convergence on \( 2^X \) with the geometric topology (Proposition 2.11). The partial maps are required to be continuous with the defining \( f(K) \subseteq U \) of the compact-open topology in the case that \( X \) is locally compact. The result below is the first in this article that directly connects to (2) and is, essentially, Lemma 1 of [3].

**Proposition 3.3.** If \( X \) is locally compact and \( X \) and \( Y \) are Hausdorff then a net \( \langle f_\lambda: A_\lambda \subseteq X \rightarrow Y \rangle \) in \( K_c(X,Y) \) converges to \( f: A \subseteq X \rightarrow Y \) if and only if

(a) \( A_\lambda \rightarrow A \) (KP); and (b) \( \lim f_{\lambda_\mu}(x_\mu) = f(\lim x_\mu) \), for all monotone cofinal \( \lambda_\mu \) and all convergent nets \( \langle x_\mu \rangle, x_\mu \in A_{\lambda_\mu} \) such that \( \lim x_\mu \in A \).

**Proof.** If \( f_\lambda \rightarrow f \) in \( K_c(X,Y) \) then \( A_\lambda \rightarrow A \) in the geometric topology (a) follows from Prop. 2.11(b). For (b), suppose \( x_\mu \in A_{\lambda_\mu}, x_\mu \rightarrow x, x \in A, \) and suppose \( V \supseteq f(x) \) is open. By continuity of \( f \), there is a compact neighbourhood \( K \) \( \ni x \) such that \( f(K) \subseteq V \). Choose \( \lambda^* \) such that \( f_\lambda(K) \subseteq V \) for \( \lambda \geq \lambda^* \). Choose \( \mu^* \) such that \( \mu \geq \mu^* \) implies both \( \lambda_\mu \geq \lambda^* \) and \( x_\mu \in K \). Then \( \mu \geq \mu^* \) implies \( x_\mu \in K \) and \( f_{\lambda_\mu}(K) \subseteq V \), from which \( f_{\lambda_\mu}(x_\mu) \in V \), so \( f_{\lambda_\mu}(x_\mu) \rightarrow x \), since \( V \) was arbitrary.

Conversely, let \( \gamma_a \) and \( \gamma_b \) denote the convergence criteria (a) and (b), and let \( \gamma \) be topological convergence in \( K_c(X,Y) \). These generate the topologies \( \gamma_a \) and \( \gamma_b \)†, as discussed in the Appendix. As was just proved, \( \gamma \Rightarrow \gamma_a \land \gamma_b \) so \( \gamma \uparrow \geq (\gamma_a \land \gamma_b) \uparrow \) by Prop. A.7(a). The generalized compact-open topology \( \gamma \uparrow \) is the coarsest topology such that the subsets in its defining subbase are open, so it suffices to show those are open in \( (\gamma_a \land \gamma_b) \uparrow \). For open \( U \subseteq X \), \( \{ A \subseteq X \mid A \cap U \neq \emptyset \} \) is open in the geometric topology on \( 2^X \), while the domain map is continuous in \( \gamma_a \) to \( 2^X \) with the geometric
topology by Prop. A.6(c). Thus \( \{ f : A \to Y \mid A \cap U \neq \emptyset \} \) is open in \( \gamma^\dagger_a \) and hence also in \( (\gamma_a \land \gamma_b)^\dagger \). It remains to show that each \( \mathcal{N}(K,V) = \{ f : A \subseteq X \to Y \mid f(K) \subseteq V \} \) is open in \( (\gamma_a \land \gamma_b)^\dagger \), for which one may use Prop. A.6(a) and the criterion \( \gamma_a \land \gamma_b \) to show the complement of \( \mathcal{N}(K,V) \) is closed. Suppose \( \langle f_\lambda \rangle_{\lambda \in \Lambda} \) is a net such that \( f_\lambda \notin \mathcal{N}(K,V) \) and \( f_\lambda \to f \) in \( \gamma_a \land \gamma_b \). Then for all \( \lambda \in \Lambda \) there is an \( x_\lambda \in K \) such that \( x_\lambda \in A_\lambda \) and \( f_\lambda(x_\lambda) \in Y \setminus V \). A subnet of \( x_\lambda \), converges, say to \( x \in K \), which is a cluster point of \( \langle A_\lambda \rangle \) by Prop. 2.2(a). Hence \( f \notin \mathcal{N}(K,V) \), because \( A = \text{K-lim} \ A_\lambda \) by \( \gamma_a \), so \( x \in L \ A_\lambda = A, f_{\lambda_n}(x_{\lambda_n}) \to f(x) \) by \( \gamma_b \), and \( f(x) \notin V \) because \( Y \setminus V \) is closed.

**Corollary 3.4.** If \( X \) is locally compact and Hausdorff and \( Y \) is Hausdorff then \( \mathcal{K}_c(X,Y) \) is Hausdorff.

**Proof.** A net \( \langle f_\lambda : A_\lambda \subseteq X \to Y \rangle \) converges in \( \mathcal{K}_c(X,Y) \) to \( f : A \subseteq X \to Y \) and also to \( f' : A' \subseteq X \to Y \) then \( A = \text{K-lim} \ A_\lambda = A' \). If \( x \in A \) then \( x \) is a cluster point of \( \langle A_\lambda \rangle \) and so there is a subnet \( \langle A_{\lambda_n} \rangle \) such that \( x_{\mu} \in A_{\lambda_n} \) and \( x_{\mu} \to x \). Every subnet of \( \langle f_\lambda \rangle \) also converges and Prop. 3.3 implies \( f(x) = \lim_{\mu} f_{\lambda_n}(x_{\lambda_n}) = f'(x) \) because limits in \( \mathcal{K}_c(X,Y) \) are unique.

**Proposition 3.5.** If \( X \) is locally compact Hausdorff and \( X \) and \( Y \) are second countable then \( \mathcal{K}_c(X,Y) \) is second countable.

**Proof.** Let \( \mathcal{B}_X \) and \( \mathcal{B}_Y \) be countable bases of \( X \) and \( Y \) respectively. Without loss of generality \( \mathcal{B}_X \) consists of compact sets. Defining

\[
\mathcal{N}(K,V) = \{ \{ f : A \subseteq X \to Y \mid A \cap U \neq \emptyset \text{ and } f(K) \subseteq V \} \},
\]

the collections \( \{ \mathcal{N}(K,V) \} \) over compact \( K \subseteq X \) and open \( V \subseteq Y \), and \( \{ \mathcal{N}(L,W) \} \) over \( L \in \mathcal{B}_X \) and \( W \in \mathcal{B}_Y \), are equivalent subbases of the compact-open topology, as follows: Let \( K \subseteq X \) be compact and \( V \subseteq Y \) be open, and suppose \( \{ f : A \subseteq X \to Y \} \in \mathcal{N}(K,V) \), where \( A \) is closed and \( f \) is continuous. The subsets \( L \in \mathcal{B}_X \) such that \( f(L) \subseteq W, W \in \mathcal{B}_Y \), and \( W \subseteq V \), cover the compact subset \( K \) (if \( x \in A \cap K \) then this follows from the continuity of \( f \); while if \( x \in K \) and \( x \notin A \) then there is a \( L \in \mathcal{B}_X \) such that \( L \cap A = \emptyset \), implying \( f(L) = \emptyset \), and there are finitely many \( L_i, W_i \) such that \( K \subseteq \bigcup L_i \) and \( f(L_i) \subseteq W_i \). So \( f \in \mathcal{N}(L_i, W_i) \), and if \( g \) is also in that intersection and \( x \in K \) then \( x \in L_i \) for one such \( i \), \( g(x) \in W_i \) for that \( i \), and \( g(x) \in V \), so \( g \in \mathcal{N}(K,V) \) as required. Thus \( \mathcal{K}_c(X,Y) \) is second countable, because the lower Vietoris topology is second countable by Prop. 2.7.

**Theorem 3.6.** If \( X \) and \( Y \) are second countable locally compact Hausdorff spaces then \( \mathcal{K}_c(X,Y) \) is second countable and Hausdorff, and a sequence \( \langle f_i : A_i \to Y \rangle \) converges to \( f : A \to Y \) if and only if

(a) for all \( x \in A \) there is a sequence \( \langle x_i \rangle \) such that \( x_i \in A_i \) and \( x_i \to x \); and

(b) \( \lim x_j \in A \) and \( \lim f_{i_j}(x_j) = f(\lim x_j) \), for all strictly increasing \( i_j \) and convergent \( \langle x_j \rangle \), \( x_j \in A_{i_j} \).

**Proof.** Since \( X, Y \), and \( \mathcal{K}_c(X,Y) \) are all second countable, Prop. 3.3 with sequences replacing nets holds with the analogous proof, and Cor. 2.3 provides the equivalence of (a) and the \( \lim x_j \in A \) part of (b) with the KP-convergence as required by Prop. 3.3(a).

Every sequence is a subsequence, and every sequence \( \langle x_i \rangle, x_i \in A_i \) such that \( x_i \to x \) gives \( x_j = x_{i_j} \), and then \( x_j \in A_{i_j} \) and \( x_j \to x \), so (a) may be replaced by the existence of \( x_j \in A_{i_j} \) such that \( x_j \to x \) for all strictly increasing \( i_j \). If this is
done then both (a) and (b) can be neatly re-written to refer to subsequences \( \langle f_{i_j} \rangle \) of \( \langle f_i \rangle \). This also applies to Theorem 3.8 below, and it is the final form of these convergence criteria in the conclusions.

The main goal has been achieved: the generalized compact-open topology in the context of second countable locally compact Hausdorff spaces and partial functions with closed domains, is second countable, with sequential convergence an elaboration of the notion (2) introduced at the beginning. This excludes for example maximal integral curves of smooth vector fields, the domains of which are never proper closed intervals. The restriction to closed domains may be relaxed with another geometric topology (and a different criterion).

**Definition 3.7.** \( \mathcal{G}_c(X,Y) \subseteq \mathcal{P}(X,Y) \) is the collection of partial maps with closed graphs, as a subspace of \( \mathcal{P}(X,Y) \) with the geometric topology.

Similarly to Definition 3.2, the graphs are required to be closed to align KP-convergence with the geometric topology. The graphs of \( \tanh(nx) \) converge in the geometric topology \( \{ (x,-1) \mid x \leq 0 \} \cup \{ (0) \times [-1,1] \} \cup \{ (x,1) \mid x \geq 0 \} \), which is not the graph of a function, so the set of graphs is not necessarily closed in the geometric topology on the closed subsets of \( X \times Y \). \( \mathcal{G}_c(X,Y) \) is not necessarily compact, even though the geometric topology on the power set of \( X \times Y \) is compact.

**Theorem 3.8.** Suppose \( X \) and \( Y \) are second countable locally compact Hausdorff spaces. Then \( \mathcal{G}_c(X,Y) \) is second countable and Hausdorff, and a sequence \( \langle f_i : A_i \rightarrow Y \rangle \) converges to \( f : A \rightarrow Y \) if and only if

(a) for all \( x \in A \) there is a sequence \( \langle x_i \rangle \), \( x_i \in A_i \), such that \( x_i \rightarrow x \) and \( \langle f_i(x_i) \rangle \) converges; and

(b) \( \lim x_j \in A \) and \( f(\lim x_j) = \lim f_i(x_j) \), for all strictly increasing \( i_j \) and \( x_j \), \( x_j \in A_j \), such that both \( \langle x_j \rangle \) and \( \langle f_i(x_j) \rangle \) converge.

**Proof.** If \( X \) and \( Y \) are second countable locally compact Hausdorff spaces then \( X \times Y \) is also that and the geometric topology of the power set of \( X \times Y \) is second countable and Hausdorff by Prop. 2.11. Subspaces of second countable Hausdorff spaces are second countable Hausdorff ([32], Theorems 13.8 and 16.2), so \( \mathcal{G}_c(X,Y) \) is second countable and Hausdorff. By Prop. 2.11, convergence in \( \mathcal{G}_c(X,Y) \) is the same as KP-convergence on graphs, which is equivalent to the convergence criteria of Corollary 2.3 (on \( X \times Y \)).

Suppose \( f_i \rightarrow f \) in \( \mathcal{G}_c(X,Y) \). If \( x \in A \) then \( (x,f(x)) \in \text{graph } f \) and by Corollary 2.3(a) there is a sequence \( \langle (x_i,y_i) \rangle \), \( (x_i,y_i) \in \text{graph } f_i \), such that \( (x_i,y_i) \rightarrow (x,y) \), implying that \( \langle x_i \rangle \) converges to \( x \) and that \( \langle f_i(x_i) \rangle \) converges. For (b), if \( i_j \) is strictly increasing, and \( x_j \in A_j \), \( x_j \rightarrow x \) and \( f_i(x_j) \rightarrow y \), then \( (x_j,f_i(x_j)) \in \text{graph } f_i \) and \( (x_j,f_i(x_j)) \rightarrow (x,y) \), and then Corollary 2.3(b) implies \( (x,y) \in \text{graph } f \), from which \( x \in A \) and \( \lim f_i(x_j) = y = f(x) = f(\lim x_j) \). Conversely, suppose (a) and (b). If \( (x,y) \in \text{graph } f \) then \( x \in A \) and by (a) there is an \( x_i \in A_i \) such that \( x_i \rightarrow x \) and \( \langle f_i(x_i) \rangle \) converges. By (b), \( y = f(x) = f(\lim x_i) = \lim f_i(x_i) \) so \( (x_i,f_i(x_i)) \in \text{graph } f_i \) and \( (x_i,f_i(x_i)) \rightarrow (x,y) \). Also, if \( i_j \) is strictly increasing and \( (x_j,y_j) \in \text{graph } f_i \) and \( (x_j,y_j) \rightarrow (x,y) \) then \( x_j \rightarrow x \) and \( y_j = f_i(x_j) \), and then by (b) \( x = \lim x_j \in A \) and \( y = \lim y_j = \lim f_i(x_j) = f(\lim x_j) = f(x) \), i.e. \( (x,y) \in \text{graph } f \). \( \square \)

Now there are two topologies, the generalized compact-open and the geometric, two corresponding topological spaces identified by Definitions 3.2 and 3.7, and the
two corresponding convergence criteria of Theorems 3.6 and 3.8, respectively. The generalized compact-open convergence is finer (and so compact-open convergence implies geometric), and they are the same if the domains are closed and co-domain $Y$ is compact:

**Proposition 3.9.** Let $X$ be a locally compact Hausdorff space, $Y$ be a Hausdorff space, $(f_\lambda)$ be a net in $K_c(X,Y)$, and $f \in K_c(X,Y)$.

(a) If $f_\lambda \to f$ in $K_c(X,Y)$ then graph $f_\lambda \to$ graph $f$ (KP).

(b) If $Y$ is compact Hausdorff and graph $f_\lambda \to$ graph $f$ (KP) then $f_\lambda \to f$ in $K_c(X,Y)$.

**Proof.** (a) If $(x,y)$ is a cluster point of graph $f_\lambda$ then there is a monotone increasing $\lambda_\mu$ sequence such that $(x_\mu,y_\mu) \to (x,y)$. $x_\mu \to x$, so $x$ is a cluster point of $(A_\lambda)$, and $x \in A$ by Prop. 3.3(a) and $y = \lim y_\mu = \lim f_\lambda(x_\mu) = f(\lim x_\mu) = f(x)$ by Prop. 3.3(b), so $Ls(graph f_\lambda) \subseteq graph f$. If $(x,y) \in graph f$, i.e. if $x \in A$ and $y = f(x)$, then $x$ is a limit point of $(A_\lambda)$ by Prop. 3.3(a), $x$ is a cluster point of any subnet of $(A_\lambda)$ by Prop. 2.2(b), and again Prop. 3.3(b) implies $(x,y)$ is a cluster point of the corresponding subnet of graph $f_\lambda$, from which graph $f \subseteq Ls(graph f_\lambda)$ by Prop. 2.2(b).

(b) It suffices to show Prop. 3.3(a,b). If $x \in A$ and $(A_\lambda)$ is a subnet of $(A_\lambda)$, then $(x, f(x))$ is a cluster point of $(graph f_\lambda)$, because $(x, f(x)) \in graph f \subseteq Ls(graph f_\lambda)$. Thus $x$ is a cluster point of $A_\lambda$, $x$ is a limit point of $A_\lambda$ because $\lambda_\mu$ was arbitrary, and $A \subseteq Ls A_\lambda$ because $x$ was arbitrary. $x \in Ls A_\lambda$ means that there is $a$ is a monotone cofinal $\lambda_\mu$ and a net $x_\mu$ such that $x_\mu \in A_\lambda$, and $x_\mu \to x$. A subnet of $f_\lambda(x_\mu)$ converges and the corresponding subnet of $(x_\mu, f_\lambda(x_\mu))$ converges, say to $(x,y) \in Ls(graph f) = graph f$, so $x \in A$ and $Ls A_\lambda \subseteq A$. For Prop. 3.3(b), if $\lambda_\mu$ is monotone cofinal, and $(x_\mu), x_\mu \in A_\lambda$ is a net such that $x_\mu \to x \in A$, then any subnet of $f_\lambda(x_\mu)$ has a subnet that converges. The corresponding subnet of $(x_\mu, f_\lambda(x_\mu)) \in graph f_\lambda$ converges to an element of $Ls(graph f_\lambda) = graph f$, and the corresponding subnet of $f_\lambda(x_\mu)$ converges to $f(x)$, and $f_\lambda(x_\mu)$ converges to $f(x)$ by Thm. A.4(d).

**Corollary 3.10.** Let $X$ be a locally compact Hausdorff space, let $Y$ be a compact Hausdorff space, let $(f_\lambda: A_\lambda \to Y)$ be a net in $K_c(X,Y)$, and let $(f: A \subseteq X \to Y) \in K_c(X,Y)$. Then the following are equivalent:

(a) $f_\lambda \to f$ in $K_c(X,Y)$;

(b) $A_\lambda \to A$ (KP); and $\lim f_\lambda(x_\mu) = f(\lim x_\mu)$, for all monotone cofinal $\lambda_\mu$ and all convergent nets $(x_\mu), x_\mu \in A_\lambda$ such that $\lim x_\mu \in A$;

(c) graph $f_\lambda$ converges to graph $f$ (KP).

The remaining results in this section are anticipated to have significance in the applications, and have to do with the restrictions of partial maps, both in the codomain and domain, and with how $K_c(X,Y)$ and $G_c(X,Y)$ transform under continuous transformations of $X$ and $Y$.

**Theorem 3.11.** The generalized compact-open topology (geometric topology) is natural with respect to subspaces, i.e. if $\bar{X} \subseteq X$ is closed and $\bar{Y} \subseteq Y$ is closed then the subspace topology of $P(\bar{X},\bar{Y}) \subseteq P(X,Y)$ is the generalized compact-open geometric topology of $P(\bar{X},\bar{Y})$.

**Proof.** In the case of the geometric topology, $\{graph \tilde{f} \mid f \in P(\bar{X},\bar{Y})\} = \{graph f \mid f \in P(X,Y)\} \cap (\bar{X} \times \bar{Y})$ and the result follows from Prop. 2.5. For the generalized
compact-open topology, the two collections
\[ \mathcal{P}(\tilde{X}, \tilde{Y}) \cap \{ \{ f: A \subseteq X \rightarrow Y \mid A \cap U \neq \emptyset \text{ and } f(K) \subseteq V \} \}, \]
\[ \{ \{ \tilde{f}: \tilde{A} \subseteq \tilde{X} \rightarrow \tilde{Y} \mid \tilde{A} \cap \tilde{U} \neq \emptyset \text{ and } \tilde{f}(	ilde{K}) \subseteq \tilde{V} \} \}, \]
correspond over compact \( K \subseteq X \) and \( \tilde{K} \subseteq \tilde{X} \), and open \( U \subseteq X, V \subseteq Y, \tilde{U} \subseteq \tilde{X} \), and \( \tilde{V} \subseteq \tilde{Y} \); given \( U, K, V \) take \( \tilde{U} = U \cap X, \tilde{K} = K \cap X, \text{ and } V = V \cap Y \), while given \( \tilde{U}, \tilde{K}, \tilde{V} \) choose \( U, V \) such that \( \tilde{U} = U \cap X, \tilde{V} = V \cap \tilde{X} \), and set \( K = \tilde{K} \).

**Theorem 3.12.** Let \( f \in \mathcal{P}(X, Y) \), \( f: A \subseteq X \rightarrow Y \), let \( B \subseteq X \) be closed, and suppose \( A \cap B \subseteq \text{cl}(A \cap \text{int } B) \). Then the restriction map \( f' \mapsto f'|B \) from \( \mathcal{K}_c(X, Y) \) to itself [from \( \mathcal{G}_c(X, Y) \) to itself] is continuous at \( f \) [if \( f \) is continuous].

**Proof.** Restriction to \( B \) is continuous with respect to the compact-open topology since \( (f|B)(K) \subseteq V \) is equivalent to \( f(K \cap B) \subseteq V \) and \( K \cap B \) is compact if \( K \) is compact and \( B \) is closed. Continuity at \( f \) of restriction to \( B \) in the generalized compact-open topology follows by definition and Prop. 2.6(b). In the case of the geometric topology, it suffices by Prop. 2.6(a,b) to show that graph \( f \cap (B \times Y) \subseteq \text{cl}(\text{graph } f \cap (\text{int } B \times Y)) \). If \( x \in A, y = f(x) \), and \( x \in B \), and if \( U \times V \ni (x, y) \) is open, then by continuity of \( f \) there is an open \( U' \ni x \) such that \( f(U') \subseteq V \). By hypothesis \( x \in \text{cl}(A \cap \text{int } B) \) so there is an \( \tilde{x} \) such that \( \tilde{x} \in (U \cap U') \cap (A \cap \text{int } B) \neq \emptyset \), so \( (\tilde{x}, f(\tilde{x})) \in \text{graph } f \cap (\text{int } B \times Y) \).

If \( X, \tilde{X}, Y \) and \( \tilde{Y} \) are topological spaces then we have the spaces \( \mathcal{P}(X, Y) \) and \( \mathcal{P}(\tilde{X}, \tilde{Y}) \) of partial maps. By covariance is meant the relationship between \( \mathcal{P}(X, Y) \) and \( \mathcal{P}(\tilde{X}, \tilde{Y}) \) as a result of relationships among the topological spaces \( X, \tilde{X}, Y \) and \( \tilde{Y} \). Such covariance bears on invariance of convergence under change of coordinates or manifold chart.

**Theorem 3.13.** If \( \varphi: \tilde{X} \rightarrow X \) and \( \psi: Y \rightarrow \tilde{Y} \) are continuous and \( \varphi \) is open, then \( f \mapsto \psi \circ f \circ \varphi \) is a continuous map from \( \mathcal{P}(X, Y) \) to \( \mathcal{P}(\tilde{X}, \tilde{Y}) \) with the generalized compact-open topologies.

**Proof.** Suppose \( f: A \subseteq X \rightarrow Y \), and let \( \tilde{f}: \tilde{A} \subseteq \tilde{X} \rightarrow \tilde{Y} \) be such that \( \tilde{A} = \varphi^{-1}(A) \). Suppose \( \tilde{K} \subseteq \tilde{X} \) is compact, \( \tilde{U} \subseteq \tilde{X} \) and \( \tilde{V} \subseteq \tilde{Y} \) are open, denote
\[ \mathcal{N}(\tilde{U}, \tilde{K}, \tilde{V}) = \{ g: A \subseteq X \rightarrow Y \mid A \cap \tilde{U} \neq \emptyset \text{ and } g(\tilde{K}) \subseteq \tilde{V} \}, \]
and suppose \( \tilde{f} \in \mathcal{N}(\tilde{U}, \tilde{K}, \tilde{V}) \) i.e. \( A \cap \tilde{U} \neq \emptyset \) and \( \tilde{f}(\tilde{K}) \subseteq \tilde{V} \). Define the open set \( U = \varphi(\tilde{U}) \), the compact set \( K = \varphi(\tilde{K}) \), and the open set \( V = \psi^{-1}(\tilde{V}) \). Then \( f \in \mathcal{N}(U, K, V) \), because \( \varphi(A \cap \tilde{U}) = \varphi(\varphi^{-1}(A) \cap \tilde{U}) \subseteq \varphi(\varphi^{-1}(A)) \cap \varphi(\tilde{U}) = A \cap U \) implies \( A \cap U \neq \emptyset \), and \( \psi f(K) = \psi(\varphi(K)) = \tilde{f}(\tilde{K}) = \tilde{V} \). Suppose \( g: B \subseteq X \rightarrow Y \) and \( g \in \mathcal{N}(U, K, V) \), i.e. \( B \cap U \neq \emptyset \) and \( g(K) \subseteq V \) Choose \( x \in B \cap U \). Then there is an \( \tilde{x} \in \tilde{U} \) such that \( \varphi(\tilde{x}) = x \), so \( \tilde{x} \in \varphi^{-1}(B) = B \) and \( \tilde{B} \cap \tilde{U} \neq \emptyset \). Also \( \tilde{g}(\tilde{K}) = \psi g(K) \subseteq \psi V \subseteq \tilde{V} \).

Covariance of the geometric topology is more intricate. Recall that a map \( f: X \rightarrow Y \) is proper [26] if \( f^{-1}(K) \) is compact for all compact \( K \subseteq Y \). If \( f \) is continuous and proper and \( y \in Y \) and \( U \) is an open set containing \( f^{-1}(\{ y \}) \), then there is an open \( V \ni y \) such that \( f^{-1}(V) \subseteq U \), or else there would be a net \( \langle x_\lambda \rangle \) such that \( x_\lambda \in X \setminus U \) and \( f(x_\lambda) \rightarrow y \), and a subnet of \( x_\lambda \) would converge, say to \( x \in X \setminus U \), and \( f(x) = y \) by continuity of \( f \), contradicting \( f^{-1}(\{ y \}) \subseteq U \).
Theorem 3.14. If $X$ is Hausdorff, $\varphi: \tilde{X} \to X$ is continuous, surjective, and open, and $\psi: \tilde{Y} \to \tilde{Y}$ is continuous, proper, and closed, then $f \mapsto \psi f \varphi$ is a continuous map from $\mathcal{G}_c(X, Y)$ to $\mathcal{G}_c(\tilde{X}, \tilde{Y})$.

Proof. Denote $\tilde{f} = \phi f \varphi$; by definition, if $f: A \subseteq X \to Y$ and $\tilde{f}: \tilde{A} \to \tilde{Y}$ then $\tilde{A} = \varphi^{-1}(A)$. The following diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & \tilde{X} \\
\downarrow{f} & \swarrow & \searrow{\tilde{f}} \\
Y & \xrightarrow{\psi} & \tilde{Y}
\end{array}
$$

are helpful, and also there is the heterogeneous relation $(x, y) \sim (\tilde{x}, \tilde{y})$ if $\varphi(\tilde{x}) = x$ and $\psi(y) = \tilde{y}$. If $A \subseteq X \times Y$ and $\tilde{A} \subseteq \tilde{X} \times \tilde{Y}$ then define

$$
A' = \{ \tilde{p} \in \tilde{X} \times \tilde{Y} \mid p \sim \tilde{p} \text{ for some } p \in A \} = (1_{\tilde{X}} \times \psi)(\varphi \times 1_Y)^{-1}A,
$$

$$
\tilde{A}' = \{ p \in X \times Y \mid p \sim \tilde{p} \text{ for some } \tilde{p} \in \tilde{A} \} = (\varphi \times 1_Y)(1_{\tilde{X}} \times \psi)^{-1}\tilde{A},
$$

and there are the elementary set theoretic verifications $(A \cap B)' \subseteq A' \cap B', A' \cap \tilde{B} \subseteq (A \cap \tilde{B})'$, and $(\text{graph } f)' = \text{graph } \tilde{f}$ (there are a variety of such properties). $\tilde{f}$ has closed graph because graph $\tilde{f} = (1_{\tilde{X}} \times \psi)(\varphi \times 1_Y)^{-1}(\text{graph } f)$, $\varphi$ is continuous and $\psi$ is closed.

Suppose open $\tilde{U} \subseteq \tilde{X} \times \tilde{Y}$ and finitely many open $V_i \subseteq \tilde{X} \times \tilde{Y}$, and compact $K \subseteq X$ and $L \subseteq Y$, and $\tilde{f} \in \mathcal{N}(\tilde{U}, \tilde{K} \times \tilde{L}, (\tilde{V}_i))$ i.e. graph $\tilde{f} \cap (\tilde{K} \times \tilde{L}) \subseteq \tilde{U}$ and, for all $i$, graph $\tilde{f} \cap \tilde{V}_i \neq \emptyset$, as in (3). Required are open $U \subseteq X \times Y$, finitely many open $V_i \subseteq X \times Y$, and compact $K \subseteq X$ and $L \subseteq Y$, such that $\tilde{f} \in \mathcal{N}(U, K \times L, (V_i))$, and such that $\tilde{g} \in \mathcal{N}(\tilde{U}, \tilde{K} \times \tilde{L}, (\tilde{V}_i))$ whenever $g \in \mathcal{N}(U, K \times L, (V_i))$. Define $K = \varphi(\tilde{K})$, $L = \psi^{-1}(\tilde{L})$ and $V_i = \tilde{V}_i'$ for all $i$. The $V_i$ are open because $V_i = (\varphi \times 1_Y)(1_{\tilde{X}} \times \psi)^{-1}\tilde{V}_i$ and $\varphi$ is open and $\psi$ is continuous, and $K$ and $L$ are compact because $\varphi$ is continuous and $\psi$ is proper, respectively. Note that

$$
(\tilde{K} \times \tilde{L})' = (\varphi \times 1_Y)(1_{\tilde{X}} \times \psi)^{-1}(\tilde{K} \times \tilde{L}) = \varphi(\tilde{K}) \times \psi^{-1}(\tilde{L}) = K \times L,
$$

so $K \times L$ has been formed from $\tilde{K} \times \tilde{L}$ in the same way as $V_i$ has been formed from $\tilde{V}_i$. Defining $U = U'$, however, would not suffice, but rather it is shown below that there is an open $U \subseteq X \times Y$ such that graph $\tilde{f} \cap (K \times L) \subseteq U$ and such that $(U \cap (K \times L))' \cap (\tilde{K} \times \tilde{L}) \subseteq \tilde{U}$. Assuming that, if graph $g \cap (K \times L) \subseteq U$ then

$$
\text{graph } \tilde{g} \cap (\tilde{K} \times \tilde{L}) = (\text{graph } g)' \cap (\tilde{K} \times \tilde{L})
$$

$$
\subseteq (\text{graph } g \cap (K \times L))' \cap (\tilde{K} \times \tilde{L})
$$

$$
= (\text{graph } g \cap (K \times L))' \cap (\tilde{K} \times \tilde{L})
$$

$$
\subseteq (U \cap (K \times L))' \cap (\tilde{K} \times \tilde{L})
$$

$$
\subseteq \tilde{U}.
$$

Also, if $(x, y) \in \text{graph } g \cap V_i$ then $(x, y) \in V_i = \tilde{V}_i'$, so there is an $(\tilde{x}, \tilde{y}) \in \tilde{V}_i$ such that $\varphi(\tilde{x}) = x$ and $\psi(y) = \tilde{y}$. Then $\tilde{g}(\tilde{x}) = \psi g \varphi(\tilde{x}) = \tilde{y}$ and graph $\tilde{g} \cap \tilde{V}_i \neq \emptyset$, as required.

The purpose of the aforementioned $U \subseteq X \times Y$ is to control graphs $\tilde{g}$ to within $\tilde{U}$ by controlling graphs $g$ to within $U$. This is problematic since $\varphi$ is not necessarily injective and since $U$ may vary over different $\tilde{x} \in \tilde{X}$ that have been collapsed.
Lemma [24]. For each $x \mapsto f$.

Corollary 3.15. If $\psi$ is required.

by $\varphi$ to a single $x \in X$. The proof is similar to that of the well-known Tube Lemma [24]. For each $x \in A \cap K$ such that $f(x) \in L$, set $y = f(x)$ and $\tilde{y} = \psi(y)$, so $\tilde{y} \in \tilde{L}$. Then the subset on the left side of the inclusion $((x) \cap \tilde{K}) \times \{\tilde{y}\} \subseteq \text{graph } \tilde{f} \cap (\tilde{K} \times \tilde{L}) \subseteq \tilde{U}$ is covered by finitely many open sets $\tilde{W}_i \times \tilde{Z}_i$, $\tilde{W}_i \subseteq \tilde{X}$, $\tilde{Z}_i \subseteq \tilde{Y}$, such that $\tilde{W}_i \times \tilde{Z}_i \subseteq \tilde{U}$. The $\tilde{W}_i$ cover $\tilde{K}$, and $\varphi(\tilde{K}) \cap K$ is proper (X is Hausdorff, K and $\tilde{K}$ are both compact, and $\varphi$ is continuous), so there is an open neighbourhood $W_x \ni x$ such that $\varphi^{-1}(W_x \cap K) \ni K \subseteq \bigcup \tilde{W}_i W_i$. Define $Z_x = \bigcap_i \psi^{-1}(\tilde{Z}_i)$ and note that $y \in Z_x$ without loss of generality. Then $(x, y) \in W_x \times Z_x$ and $(W_x \times Z_x) \cap (K \times \tilde{L}) \subseteq \tilde{U}$: Indeed, if $(\tilde{x}', \tilde{y}') \in (W_x \times Z_x) \cap (K \times \tilde{L})$, then there is a $(x', y') \in W_x \times Z_x$ such that $\varphi(\tilde{x}') = x'$ and $\psi(y') = \tilde{y}'$. So $x' \in W_x$ and $x' \in \varphi(K) = K$, from which $\tilde{x}' \in \varphi^{-1}(W_x \cap K) \cap \tilde{K}$, and there is an $i$ such that $\tilde{x}' \in W_i \cap \tilde{K}$. $y' \in \psi^{-1}(\tilde{Z}_i)$ for that $i$ since $Z_x$ is defined as the intersection of all such, so $(\tilde{x}', \tilde{y}') \in \tilde{W}_i \times \tilde{Z}_i \subseteq \tilde{U}$. Define $U = \bigcup_{x \in A \cap K} W_x \times Z_x$. Then $(x, y) \in \text{graph } f \cap (K \times L)$ implies $x \in A \cap K$ and $(x, y) \in W_x \times Z_x \subseteq U$, and

$$\begin{align*}
(U \cap (K \times L))' \cap (K \times \tilde{L}) &= \bigcup_{x \in A \cap K} ((W_x \times Z_x) \cap (K \times L))' \cap (K \times \tilde{L}) \\
&\subseteq \bigcup_{x \in A \cap K} (W_x \times Z_x)' \cap (K \times L)' \cap (K \times L) \\
&= \bigcup_{x \in A \cap K} (W_x \times Z_x)' \cap (K \times \tilde{L}) \\
&\subseteq \tilde{U},
\end{align*}$$

as required. \hfill \Box

Corollary 3.15. If $\varphi: X \to \bar{X}$ and $\psi: Y \to \bar{Y}$ are homeomorphisms, then the map $f \mapsto \psi \varphi^{-1}$ restricts to homeomorphisms $K_c(X, Y) \to K_c(\bar{X}, \bar{Y})$ and $G_c(X, Y) \to G_c(\bar{X}, \bar{Y})$.

4. Conclusions. The numerical analysis of ordinary differential equations involves approximations in the form of partial functions of a real independent variable, with values in $\mathbb{R}^n$, or with values in a differentiable manifold. For partial differential equations, the independent variables may range within a manifold. Every manifold is a topological space, and the following definitions are applicable:

Definition 4.1. Let $X$ and $Y$ be topological spaces. A sequence of partial functions $f_i: A_i \subseteq X \to Y$ converges pointwise to $f: A \subseteq X \to Y$ if, for all subsequences $f_{i_j}$ of $f_i$, we have $f_{i_j}(x) \to f(x)$ for all $x \in A$.
Y

\[ y \]

\[ y_1 \]

\[ y_2 \]

\[ y_3 \]

\[ \vdots \]

Y

\[ \text{Figure 3. Left: the discrete approximations } y_i, i = 1, 2, 3 \ldots \text{ (circles) are limiting to a continuous } y. \text{ Shown (squares on red curves) are three selections of subsequences from the graphs of } y_i. \text{ Every such subsequence converges to the graph of } y, \text{ and that graph is the limit of such subsequences. Right: an open neighbourhood of the red graph is defined by a compact set } K, \text{ an open set } U, \text{ and open sets } V_i. \text{ These sets may be restricted to a product of compact sets, may be thought of as a frame within which proximity to the graph is controlled by } U. \text{ The other black curves are in the neighbourhood because they also contact the } V_i. \text{ Larger } K, \text{ smaller } U, \text{ and more and smaller } V_i, \text{ correspond to smaller neighbourhoods.} \]

\[
(a) \text{ for all } x \in A \text{ there is a sequence } (x_j), x_j \in A_{ij}, \text{ such that } x_j \to x \text{ and } (f_{ij}(x_j)) \text{ converges; and } \\
(b) \lim x_j \in A \text{ and } f(\lim x_j) = \lim f_{ij}(x_j), \text{ for all convergent } (x_j), x_j \in A_{ij} \text{ such that } (f_{ij}(x_j)) \text{ converges.}
\]

**Definition 4.2.** Let \( X \) and \( Y \) be topological spaces. A sequence of partial functions \( f_i : A_i \subseteq X \to Y \) converges \textit{uniformly} to \( f : A \subseteq X \to Y \) if, for all subsequences \( f_{ij} \) of \( f_i \),

(a) for all \( x \in A \) there is a sequence \( (x_j), x_j \in A_{ij} \) and \( x_j \to x \); and 

(b) \( \lim x_j \in A \) and \( f(\lim x_j) = \lim f_{ij}(x_j), \text{ for all convergent } (x_j), x_j \in A_{ij}. \)

Imagine a \textit{route to } \((x, y)\) \textit{through the partial functions } \( f_i \) (Figure 3, left) to be a strictly increasing \( i_j \) and a sequence \( (x_j), x_j \in A_{ij} \) such that \( x_j \to x \) and \( f_{ij}(x_j) \to y \), then \( f_i \to f \) pointwise if and only if the graph of \( f \) is exactly the destinations of the routes. Divergent routes may co-exist with pointwise convergence; uniform convergence imposes the convergence of all routes obtained by convergent evaluations. Uniform convergence implies pointwise convergence. The converse is true if the sequence \( f_i \) is bounded, i.e. if there is a compact \( K \subseteq Y \) such that \( f_i(A_i) \subseteq K \) for all \( i \): if \( x_j \in A_j \) and \( x_j \to x \) then a subsequence of \( f_{ij}(x_j) \) converges, and to \( f(x) \) by Definition 4.1(b), and then \( f_{ij}(x_j) \to f(x) \) after an application of Theorem A.4(d).

The manifolds occurring in applications are almost always second countable, locally compact, and Hausdorff, after which Theorems 3.8 and 3.6 provide that pointwise and uniform convergence correspond to convergence in the second countable Hausdorff geometric topology and generalized compact-open topologies, on the space of partial functions with closed graphs, and the space of continuous partial functions with closed domains, respectively. The closed sets of these spaces are readily accessible via the convergence criteria in Definitions 4.1 and 4.2, the open sets by the neighbourhood bases provided by Proposition 2.12 (Figure 3, right) and Definition 3.1(c). Within these spaces, pointwise and uniform convergence satisfy
the topological convergence properties listed in Theorem A.4. Localization is provided by Theorems 3.11 and 3.12. Covariance, or independence of coordinates, is provided by Theorems 3.14 and 3.13.

While there are any number of plausible convergence criteria, Definitions 4.1 and 4.2 correspond to natural topologies on spaces of partial maps. Alterations may result in unexpected behaviour. Were the definitions to refer to sequences, as opposed to all subsequences, and were 4.1(a) to be replaced by 4.2(a), then \( f_i: [0,1] \to \mathbb{R} \) by \( f_i(x) = (-1)^i \) would converge pointwise to any partial function whatsoever, because 4.1(b) would be satisfied vacuously. Replacing 4.1(b) by \( f(\lim x_j) = \lim f_i(x_j) \), when both sides are defined, would allow the domain of the limit \( f \) to be a strict superset of where the approximations \( f_i \) converge, corresponding to a false prediction that a solution exists.

Definition 4.2 may be deployed as a drop-in replacement for the usual numerical analysis definitions of convergence. This is backward-compatible, as follows: A mesh \( A \) of an interval \( [a,b] \subset \mathbb{R} \) is a tuple \( t_k \), \( 0 \leq k \leq n \), such that \( a = t_0 < t_1 < \cdots < t_n = b \); the norm \( \|A\| \) of the mesh is the number \( \max_{1 \leq k \leq n} (t_k - t_{k-1}) \). A mesh function is a partial function \( f: A \subset [a,b] \to \mathbb{R}^n \), where \( A \) is a mesh. Mesh functions are continuous because their domains are discrete.

**Theorem 4.3.** Let \( [a,b] \subset \mathbb{R} \) and let \( \| \cdot \| \) be any \( \mathbb{R}^n \) norm. Suppose \( f_i: A_i \to \mathbb{R}^n \), \( A_i = \{ t_{ik} \mid 1 \leq k \leq n_i \} \), is a sequence of mesh functions such that \( \|A_i\| \to 0 \). Then \( \langle f_i \rangle \) converges uniformly to a continuous function \( f: [a,b] \to \mathbb{R}^n \) if and only if

\[
\lim_{i \to \infty} \max_{k=0,1,\ldots,n_i} \|f_i(t_{ik}) - f(t_{ik})\| = 0. \tag{7}
\]

**Proof.** Suppose \( f_i \to f \) uniformly, and let the maxima in (7) occur at \( \hat{t}_i \). Since \( [a,b] \) is compact, the sequence \( \hat{t}_i \) has a convergent subsequence \( \hat{t}_{i_j} \). By Definition 4.2(b), \( f_i(\hat{t}_{i_j}) \to f(\hat{t}) \), without loss of generality, and \( f(\hat{t}_{i_j}) \to f(\hat{t}) \) by continuity of \( f \), so \( \|f_i(\hat{t}_{i_j}) - f(\hat{t}_{i_j})\| \to 0 \). The foregoing may be applied to any subsequence of \( f_i \) to conclude that every subsequence of \( \|f_i(\hat{t}_{i_j}) - f(\hat{t}_{i_j})\| \) has a subsequence that converges to 0, and (7) follows from Theorem A.4(d). Conversely, if \( t \in [a,b] \), then \( \|A_i\| \to 0 \) provides a sequence \( t_i \in A_i \) such that \( t_i \to t \), while if \( f_{i_j} \) is a subsequence of \( f_i \), and if \( t_j \in A_{i_j} \) and \( t_j \to t \), then

\[
\|f_{i_j}(t_j) - f(t)\| \leq \|f_{i_j}(t_j) - f(t_j)\| + \|f(t_j) - f(t)\|
\]

implies \( f_{i_j}(t_j) \to f(t) \) by (7) and continuity of \( f \). \( \Box \)

If the objective is convergence to solutions, say of an ordinary differential equation, that are defined on closed intervals and are therefore not maximal solutions, then uniform convergence is serviceable. Pointwise convergence is applicable for open domains, for example where convergence to an unbounded maximal solution is of interest, as in Theorem 2.1.26 of [1], where the (fixed final time) convergence domain of (constant step size) one-step methods is shown to be exactly the domain of the exact continuous flow. However, the sequence \( \langle f_i \rangle \), \( f_i: [0,1] \to \mathbb{R} \) defined by \( f_i(t) = 1/i \) if \( t \in [1/2i, 1/i] \), and zero otherwise, pointwise converges to the zero function on \( [0,1] \) because, if \( t_j \in A_{i_j} \) then either \( t_j \) is perpetually in \( [1/2i_j, 1/i_j] \), in which case 4.1(a) is vacuously satisfied, or it is eventually not in \([1/2i_j, 1/i_j]\), in which case \( f_j(t_{i_j}) \to 0 \). Without a priori boundedness of the approximations, particular evaluations of the approximations may diverge. If pointwise convergence is used generally then it should be prerequisite to establish that bounded solutions
on closed domains imply bounded approximations, so that the convergence is uniform to those. Given that prerequisite, pointwise convergence and the geometric topology can be used by default as a geometric generalization of convergence.

It is routine to validate a numerical simulation by observing machine convergence from some fixed initial condition to some fixed final time. However, this natural coordinate-invariant definition is not localizable, meaning that restrictions of the independent variable, e.g. to proper subintervals in the case of ordinary differential equations, inherently lead to varying initial conditions and final times. Convergence in the geometric topology is localizable, and exploring its systematic use as a theoretical foundation for geometric integration on manifolds may be the subject of further work.

Appendix A. Appendix: Topologies from convergence criteria. A directed set is a set \( \Lambda \) with a directed preorder: a relation \( \geq \) which satisfies: (1) reflexive: \( \lambda \geq \lambda \); (2) transitive: \( \lambda_1 \geq \lambda_2 \) and \( \lambda_2 \geq \lambda_3 \) implies \( \lambda_1 \geq \lambda_3 \); and (3) upper bounds: for all \( \lambda_1 \) and \( \lambda_2 \) there is a \( \lambda^* \) such that \( \lambda^* \geq \lambda_1 \) and \( \lambda^* \geq \lambda_2 \). A property \( P(\lambda) \) is final if there is a \( \lambda^* \in \Lambda \) such that \( P(\lambda) \) is true for all \( \lambda \geq \lambda^* \). A property \( P(\lambda) \) is cofinal if for all \( \lambda^* \in \Lambda \) there is a \( \lambda \in \Lambda \) such that \( \lambda \geq \lambda^* \) and \( P(\lambda) \) is true. A net \( \langle x_\lambda \rangle_{\lambda \in \Lambda} \) is a function \( \lambda \mapsto x_\lambda \) on a directed set \( \Lambda \) with values in a topological space. \( x \in X \) is a limit point \( [\text{cluster point}] \) of \( x_\lambda \) if \( x_\lambda \in U \) finally [cofinally]. \( \langle x_\lambda \rangle \) converges to \( x \), written \( x_\lambda \to x \), or \( x = \lim_\lambda x_\lambda \), if \( x \) is a limit point, i.e. if for all neighbourhoods \( U \ni x \) and all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( x_\lambda \in U \). A subnet of \( \langle x_\lambda \rangle \) is a net \( \langle x_{\lambda_u} \rangle_{\mu \in M} \) where \( \mu \mapsto \lambda_u \) is map from a directed set \( M \) to \( \Lambda \) that satisfies (1) monotone: \( \mu_1 \geq \mu_2 \) implies \( \lambda_{\mu_1} \geq \lambda_{\mu_2} \); and (2) cofinal: for all \( \lambda \in \Lambda \) there is a \( \mu \in M \) such that \( \lambda_{\mu} \geq \lambda \).

Every sequence in a topological space is a net, but the concepts differ essentially because the index set of a subsequence is a subset of the original, whereas the index set of a subnet may be more complicated than the original. The following three results are illustrative.

**Lemma A.1.** Suppose \( \Lambda \) is directed and \( \Lambda' \subseteq \Lambda \) is cofinal, i.e. for all \( \lambda \in \Lambda \) there is a \( \lambda' \geq \lambda \) such that \( \lambda' \in \Lambda' \). Then \( \Lambda' \) is directed.

**Proof.** If \( \lambda_1', \lambda_2' \in \Lambda' \) then choose \( \lambda_3 \in \Lambda \) such that \( \lambda_3 \geq \lambda_1' \) and \( \lambda_3 \geq \lambda_2' \). Choose \( \lambda_3' \in \Lambda' \) such that \( \lambda_3' \geq \lambda_3 \). This suffices because, by transitivity, \( \lambda_3' \geq \lambda_1' \) and \( \lambda_3' \geq \lambda_2' \).

**Lemma A.2.** (Theorem 11.5 of \([32]\)). A net \( \langle x_\lambda \rangle \) has a cluster point \( x \) if and only if it has a subnet that converges to \( x \).

**Proof.** If \( x \) is a cluster point of \( \langle x_\lambda \rangle \), then \( \{ (\lambda, U) \} \) such that \( U \) is a neighbourhood of \( x \) and \( x_\lambda \in U \), is directed by \( (\lambda_1, U_1) \leq (\lambda_2, U_2) \iff \lambda_1 \leq \lambda_2 \) and \( U_1 \supseteq U_2 \), and \( (\lambda, U) \to x \) and defines a subnet which converges to \( x \). Conversely, given a subnet \( \langle x_{\lambda_u} \rangle \), a neighbourhood \( U \ni x \), and a \( \lambda^* \), choose \( \mu \) such that \( \lambda_{\mu} > \lambda^* \) and \( x_{\lambda_{\mu}} \in U \).

**Lemma A.3.** Let \( C \) be a subbase for a topology \( \tau \). Then \( x_\lambda \to x \) if and only if, for all \( V \in C \), there is a \( \lambda^* \) such that \( x_\lambda \in V \) for all \( \lambda \geq \lambda^* \).

**Proof.** If \( V \in C \) then \( V \) is open so \( x_\lambda \) is finally in \( V \) by definition of net convergence. For the converse, every open \( V \) contains the finite intersection of sets \( V_i \in C \). For each \( i \) choose \( \lambda_i^* \) such that \( x_\lambda \in V_i \) for all \( \lambda \geq \lambda_i^* \). The upper bound property of
directed sets, and transitivity, provide an upper bound \(\lambda^*\) for all \(\lambda_i^*\). If \(\lambda \geq \lambda^*\) then \(x_{\lambda} \in V_i\) for all \(i\) and hence \(x_{\lambda} \in V\). \(\square\)

Here is the standard result regarding convergence and topologies, which in [32] is relegated to an exercise.

**Theorem A.4.** Net convergence on a topological space has the following properties:

(a) the constant net \(\langle x_\lambda \rangle\), where \(x_\lambda = x\), converges to \(x\);

(b) if \(\langle x_\lambda \rangle\) converges to \(x\) then every subnet of \(\langle x_\lambda \rangle\) also converges to \(x\);

(c) if \(\langle x_\lambda \rangle\) converges to \(x\) and \(\langle x_{\lambda \mu} \rangle_{\mu \in M_\lambda}\) converges to \(x_\lambda\) for each fixed \(\lambda\) (lexicographic ordering on the two character words \(\lambda \mu\)) then \(\langle x_{\lambda \mu} \rangle\) has a subnet that converges to \(x\);

(d) if every subnet of \(\langle x_\lambda \rangle\) has a subnet that converges to \(x\), then \(\langle x_\lambda \rangle\) converges to \(x\).

Conversely, given a logical statement predicated on nets \(\langle x_\lambda \rangle\) and limits \(x\), and satisfying (a)–(c), \(\text{cl} E = \{\lim x_\lambda \mid x_\lambda \in E\}\) is a Kuratowski closure defining a topology (the convergence topology) within which a net that satisfies the statement also converges in the topology. If the statement also satisfies (d) then every net which converges in that topology also satisfies the statement.

**Proof.** Net convergence on topological spaces does satisfy (a)–(d):

(a) The constant net with \(x_\lambda = x\) converges because if \(U \ni x\) is open then any \(\lambda^*\) provides \(x_{\lambda} \in U\) for all \(\lambda \geq \lambda^*\).

(b) Suppose \(\langle x_{\lambda_\mu} \rangle\) is a subnet of \(\langle x_\lambda \rangle\) and \(x_\lambda \rightarrow x\). If \(U \ni x\) is open then choose \(\lambda^*\) so that \(\lambda \geq \lambda^*\) implies \(x_{\lambda} \in U\). By the definition of a subnet, there is a \(\lambda^*\) such that \(\mu \geq \lambda^*\) implies \(\lambda_{\mu} \geq \lambda^*\), so \(\mu \geq \lambda^*\) implies \(x_{\lambda_{\mu}} \in U\).

(c) By Lem. A.2, it suffices to show that \(x\) is a cluster point of \(\langle x_{\lambda \mu} \rangle\). Suppose \(U \ni x\) is open, \(\lambda^* \in \Lambda\) and \(\mu^* \in M_{\lambda^*}\). Since \(x_\lambda \rightarrow x\), there is a \(\lambda > \lambda^*\) such that \(x_{\lambda} \in U\). Since \(x_{\lambda \mu} \rightarrow x_{\lambda}\) in \(\mu \in M_{\lambda}\) and since \(x_{\lambda} \in U\), there is a \(\mu \in M_{\lambda}\) such that \(x_{\lambda \mu} \in U\) (\(\mu^*\) is irrelevant because \(\lambda > \lambda^*\) and the ordering is lexicographic).

(d) If \(\langle x_\lambda \rangle\) does not converge to \(x\) then there is an open \(U \ni x\) such that, for all \(\lambda^*\) there is a \(\lambda \geq \lambda^*\) with \(x_{\lambda} \notin U\). Then \(M = \{\mu \in \Lambda \mid x_{\lambda} \notin U\}\) is directed: if \(\mu_1, \mu_2 \in M\) then choose \(\lambda^* > \mu_1\) and \(\lambda^* > \mu_2\), and then there is a \(\mu \geq \lambda^*\) such that \(x_{\mu} \notin U\), so \(\mu \in M\) and \(\mu \geq \lambda^* \geq \mu_1\) and \(\mu \geq \lambda^* \geq \mu_2\). The restriction of \(\langle x_\lambda \rangle\) to \(M\) is a subnet which has no subnet that converges to \(x\).

Assuming (a–c), \(E \rightarrow \text{cl} E\) is a closure operation, and so provides a topology with closed sets exactly those \(E\) such that \(\text{cl} E = E\), as follows:

- \(\text{cl} \emptyset = \emptyset\): otherwise, after choosing \(x \in \text{cl} \emptyset\) there is an \(x_{\lambda} \rightarrow x\) with \(x_{\lambda} \in \emptyset\), which is impossible.
- \(E \subseteq \text{cl} E\): if \(x \in E\) then any constant net with all values equal to \(x\) is a net in \(E\) converging to \(x\), so \(x \in \text{cl} E\).
- \(\text{cl} \text{cl} E = \text{cl} E\): By the above, \(E \subseteq \text{cl} E \subseteq \text{cl} \text{cl} E\). On the other hand, suppose \(x_{\lambda} \rightarrow x\) with \(x_{\lambda} \in \text{cl} E\) and choose \(x_{\lambda \mu}\) such that \(\lim x_{\lambda \mu} = x_{\lambda}\) with \(x_{\lambda \mu} \in E\). Then \(x \in \text{cl} E\) because there is a subnet of \(\langle x_{\lambda \mu} \rangle\) in the lexicographic ordering such that \(x_{\lambda \mu} \rightarrow x\).
- If \(x \in \text{cl} A\) then there is an \(x_{\lambda} \rightarrow x\) with \(x_{\lambda} \in A\). Since \(x_{\lambda} \in A \cup B\) also, this implies \(x \in \text{cl}(A \cup B)\), and similarly \(\text{cl} B \subseteq \text{cl}(A \cup B)\), so \(\text{cl} A \cup \text{cl} B \subseteq \text{cl}(A \cup B)\). Conversely, if \(x \in \text{cl}(A \cup B)\) then there is a net \(\langle x_{\lambda} \rangle\) converging to \(x\) with
A convergence criterion \( \gamma \) generating be referred to as criteria: the given one and net convergence in the topology itself. The first will sets will be denoted \( \gamma \) two closed sets is closed is as in the proof of Theorem A.4. The collection of open

\[
E \subset X \text{ is closed if and only if } \text{cl} E \subseteq E, \text{ i.e. if and only if, for all nets } (x_\lambda), x_\lambda \in E \text{ for all } \lambda \text{ and } x_\lambda \to x \text{ (statement) implies } x \in E. \]

It remains to show that the nets defined by the logical statement do converge in the topology defined by that, and, given (d), the converse. Assume (a)–(c), and suppose that \( x_\lambda \to x \) (statement). Let \( U \ni x \) be open, so that \( K = X \setminus U \) is closed, and suppose that for all \( \lambda' \) there is a \( \lambda \geq \lambda' \) such that \( x_\lambda \in K \). Then \( \Lambda' = \{ \lambda \in \Lambda \mid x_\lambda \in K \} \) is directed and \( (x_\lambda) \) restricted to \( \Lambda' \) converges to \( x \) (statement), from which \( x \in K \) because \( K \) is closed. Conversely, assume (a)–(d), and let \( (x_\lambda)_{\lambda \in \Lambda} \) converge to \( x \) (topology). Then \( x \in \text{cl} \{ x_{\lambda'} \mid \lambda' \geq \lambda \} \) for any \( \lambda \in \Lambda \), so, by the definition the closure of the topology, for all \( \lambda \in \Lambda \) there is a net \( (x_\mu)_{\mu \in M_\lambda} \) of elements of that set such that \( x_\mu \to x \) (statement). Since the constant nets converge (statement), there is a directed set \( N \) and a monotone cofinal map \( \nu \mapsto \lambda_\nu \) to the two letter words \( \{ \lambda_\nu \mid \mu \in M_\lambda \} \) (lexicographic ordering) such that \( (x_\mu) \) converges. This provides a subnet of \( (x_\lambda) \) converging to \( x \) (statement), by selection from within the \( \nu \) having the same \( \lambda_\nu \), of one \( \lambda \) such that \( x_\lambda = x_\mu \). Thus every convergent net (topology) has a convergent subnet (statement). Since every subnet of \( (x_\lambda) \) also converges (topology), every subnet also has a convergent subnet (statement), and (d) implies \( (x_\lambda)_{\lambda \in \Lambda} \) converges (statement).

There is a variety of well-known function-space topologies where a central aim seems to be the capture of some particular notion of convergence, for which the primary definition of the topology does not have an immediately obvious relationship. The topology for distributional test function spaces, and even the familiar compact-open topology are examples. Theorem A.4 suggests that, to define a topology from a logical statement, one should verify all of Theorem A.4(a–d). In the case of success the result is topological: not only is the topology obtained, but also a complete characterization of all convergent nets. If the aim is just to define a topology, then (b) alone suffices.

**Definition A.5.** A convergence criterion on a set \( X \) is a logical statement \( \gamma \) that depends on functions \( (x_\lambda)_{\lambda \in \Lambda} \) and elements \( x \in X \), where \( \Lambda \) is a directed set and \( x_\lambda \in x \), such that, for all directed sets \( M \) and monotone cofinal \( \lambda_\mu, \mu \in M, \gamma \) is true for \( (x_\lambda)_{\mu \in M} \) and \( x \) if \( \gamma \) is true for \( (x_\lambda)_{\lambda \in \Lambda} \) and \( x \).

If \( \gamma \) is true for \( (x_\lambda) \) and \( x \) then we write \( \lim_\Lambda x_\lambda = x \) or \( x_\lambda \to x \). By definition the closed sets of the topology generated by a convergence criterion are those \( E \subseteq X \) that contain all \( x \) such that \( x_\lambda \to x \) and \( x_\lambda \in E \) for all \( \lambda \). That the union of two closed sets is closed is as in the proof of Theorem A.4. The collection of open sets will be denoted \( \gamma \). In such a topology, there are (at least) two convergence criteria: the given one and net convergence in the topology itself. The first will be referred to as generating, while the second will be referred to as topological. If \( x_\lambda \to x \) (generating) then \( x_\lambda \to x \) (topological), as in the proof of Theorem A.4. A convergence criterion \( \gamma \) is called topological if every convergent net in \( \gamma \) also
satisfies $\gamma$. The topology $\gamma^\dagger$ may be thought of as the most exact carrier of the convergence criterion $\gamma$: it is the finest such that the nets satisfying $\gamma$ converge in the topology. Indeed, if $\tau$ is such a topology and $A$ is closed in $\tau$, and $x_\lambda \to x$ (generating) with $x_\lambda \in A$, then $x_\lambda \to x$ (in $\tau$) so $x \in A$, and hence $A$ is closed in the topology generated by $\gamma$.

This set-up is operationally effective, because the principal topological notions can be expressed in the usual way by the generating convergence:

**Proposition A.6.** Let $x_\lambda \to x$ be a convergence criterion.

(a) $A \subset X$ is closed if and only if $x_\lambda \in A$ and $x_\lambda \to x$ (generating) implies $x \in A$.

(b) $U \subset X$ is open if and only if, for all $x_\lambda \to x$ (generating) such that $x \in U$, there is an $x^* \subset U$ whenever $\lambda \geq \lambda^*$.  

(c) $f: X \to Y$ is continuous if and only if $x_\lambda \to x$ (generating) implies $f(x_\lambda) \to f(x)$.

**Proof.** The first statement is by definition of the closed sets in the convergence topology. For the next two statements, it suffices to show the converse, i.e. it suffices to show that generating convergence suffices. For the second statement, if $x \in U$ implies that every net $x_\lambda \to x$ (generating) is finally in $U$, then $x_\lambda \in X \setminus U$ and $x_\lambda \to x$ (generating) implies $x \in X \setminus U$ and hence $X \setminus U$ is closed, or else $x_\lambda$ is both in $X$ and $X \setminus U$ for large enough $\lambda$. For the third statement, suppose that $\lim f(x_\lambda) = f(x)$ for all nets $x_\lambda \to x$ (generating). If $B \subseteq Y$ is closed and $x_\lambda \in f^{-1}(B)$ with $x_\lambda \to x$ (generating), then $f(x) = f(\lim x_\lambda) = \lim f(x_\lambda) \in B$ so $x \in f^{-1}(B)$. This shows that $f$ if continuous because $f^{-1}(B)$ is closed whenever $B$ is.

There may be nets that converge in the generated topology which do not satisfy the criterion itself. For example, an element of $X$ may be approximable from $A \subset X$ via nets in the convergence topology, but inaccessible from $A$ via the generating convergence. It is an error to assert the existence of a net $x_\lambda \to x$ (generating) only from the knowledge $x \in \text{cl} A$. In fact, there is the following: let $X = \mathbb{R}$ and use the convergence criterion $x_\lambda \to x$ if $|x - x_\lambda| \leq 1$ for all $\lambda$. If $A = [0, 1]$ then $\text{pcl} A = [-1, 2]$ while $\text{cl} A = \mathbb{R}$, where $\text{pcl} A$, or the pre-closure, denotes the limits of all convergent nets in $A$. The pre-closure is not necessarily a closed set because it only contains those limit of generating nets but not necessarily limits of those limits. Incidentally, Proposition A.6 explains why sequences suffice for continuity of linear maps on the test function spaces of distribution theory ([28], Theorem 6.6). The restriction to linear maps arises because the topology is not the relevant generating topology but rather the convexification of that.

If $X$ and $Y$ are topological spaces generated by convergence criteria, then the topology generated by the product criterion $(x_\lambda, y_\lambda) \to (x, y)$ if $x_\lambda \to x$ (generating) and $y_\lambda \to y$ (generating) may be strictly finer than the product topology, which is after all the coarsest topology with continuous projections to the factors. Indeed, the criterion $x_\gamma \to x$ if $|x_\gamma - x| \to 0$ through powers of $1/2$, generates a topology on $\mathbb{R}$ strictly finer than the usual (for example, the set $\{1/3^n\}$ is closed because it does not contain any nonconstant net). In the product topology of two copies of that, an open line segment with irrational slope does not contain any nonconstant net in the product criterion and hence is closed in the topology generated by that. However, open intervals are open in the factor topologies, and their products are open neighbourhoods in the product topology. Therefore such irrational sloped open segments are not closed in the product topology because they
do not contain the endpoints which are in their closure. This has an important operational consequence: to show that a bivariate function \( f(x, y) \) is continuous in the product topology, it is (generally) insufficient to show that \( f(x, y) \to f(x, y) \) whenever \( x \lambda \to y \) (generating) and \( y \lambda \to y \) (generating).

**Proposition A.7.** Suppose \( \gamma_1 \) and \( \gamma_2 \) are convergence criteria.

(a) If \( \gamma_1 \Rightarrow \gamma_2 \) then \( \gamma_1^\dagger \supseteq \gamma_2^\dagger \) (relaxed convergence criteria generate finer topologies). The convergence criterion that all nets converge to any value [exactly the constant nets converge to their constant values] generates the discrete [indiscrete] topology.

(b) \( (\gamma_1 \lor \gamma_2)^\dagger = \gamma_1^\dagger \land \gamma_2^\dagger \), (the logical “or” of two criteria generates the intersection of the topologies generated by the criteria separately).

(c) \( (\gamma_1 \land \gamma_2)^\dagger \supseteq \gamma_1^\dagger \lor \gamma_2^\dagger \), (the logical “and” of two criteria generates a topology finer than the join of the topologies generated by the criteria separately). If \( \gamma_1 \) and \( \gamma_2 \) are both topological then \( (\gamma_1 \land \gamma_2)^\dagger = \gamma_1^\dagger \lor \gamma_2^\dagger \).

**Proof.** (a) Suppose \( E \) is closed in \( \gamma_2^\dagger \) and \( x \lambda \to x \) (\( \gamma_1 \) generating) with \( x \lambda \in E \). Then \( x \lambda \to x \) (\( \gamma_2 \) generating) and \( x \in E \), so \( E \) is closed in \( \gamma_1^\dagger \). The last two statements follow from this, or directly: If every net converges then any constant net of any point in any set converges to any point not in that set, from which the only closed sets are the empty set and the whole space, and the topology generated is indiscrete. If no net converges then the condition that a set be closed is vacuously true for any set, and the topology generated is indiscrete.

(b) If \( E \) is closed in \( \gamma_1^\dagger \land \gamma_2^\dagger \) then \( E \) is closed in both \( \gamma_1^\dagger \) and \( \gamma_2^\dagger \) because \( \gamma_1^\dagger \land \gamma_2^\dagger \subseteq \gamma_1^\dagger \) and \( \gamma_1^\dagger \land \gamma_2^\dagger \subseteq \gamma_2^\dagger \). So, if \( x \lambda \to x \) (\( \gamma_1 \lor \gamma_2 \) generating) then at least one of \( x \lambda \to x \) (\( \gamma_1 \) generating) or \( x \lambda \to x \) (\( \gamma_2 \) generating), and \( x \in E \) in either case, which implies \( E \) is closed in \( (\gamma_1 \lor \gamma_2)^\dagger \) and hence that \( \gamma_1^\dagger \land \gamma_2^\dagger \subseteq (\gamma_1 \lor \gamma_2)^\dagger \). Conversely, \( \gamma_1 \Rightarrow \gamma_1 \lor \gamma_2 \) and \( \gamma_2 \Rightarrow \gamma_1 \lor \gamma_2 \), from which \( \gamma_1^\dagger \supseteq (\gamma_1 \lor \gamma_2)^\dagger \), \( \gamma_2^\dagger \supseteq (\gamma_1 \lor \gamma_2)^\dagger \), and \( (\gamma_1 \lor \gamma_2)^\dagger \subseteq \gamma_1^\dagger \land \gamma_2^\dagger \).

(c) If \( \gamma_1 \land \gamma_2 \Rightarrow \gamma_1 \) and \( \gamma_1 \land \gamma_2 \Rightarrow \gamma_2 \) then \( (\gamma_1 \land \gamma_2)^\dagger \supseteq \gamma_1^\dagger \) and \( (\gamma_1 \land \gamma_2)^\dagger \supseteq \gamma_2^\dagger \), and \( \gamma_1^\dagger \lor \gamma_2^\dagger \) is the coarsest topology containing both \( \gamma_1^\dagger \) and \( \gamma_2^\dagger \), so \( \gamma_1^\dagger \lor \gamma_2^\dagger \subseteq (\gamma_1 \land \gamma_2)^\dagger \).

For the second part, let \( \gamma_1, \gamma_2 \) be the convergence criteria defined by the topologies \( \gamma_1^\dagger \lor \gamma_2^\dagger \), \( \gamma_1^\dagger \), and \( \gamma_2^\dagger \), respectively. The join topology \( \gamma_1^\dagger \lor \gamma_2^\dagger \) contains both \( \gamma_1^\dagger \) and \( \gamma_2^\dagger \), so \( \gamma_1 \Rightarrow \gamma_1 \) and \( \gamma_2 \Rightarrow \gamma_2 \). Since \( \gamma_1 = \gamma_1 \) and \( \gamma_2 = \gamma_2 \), it follows that \( \gamma \Rightarrow \gamma_1 \land \gamma_2 \), so \( \gamma_1 \lor \gamma_2 = \gamma \lor (\gamma_1 \land \gamma_2) \). \( \square \)

With respect to Proposition A.7(c), \( (\gamma_1 \land \gamma_2)^\dagger \supseteq \gamma_1^\dagger \lor \gamma_2^\dagger \) is possible: Let \( \gamma_1 \) be the convergence criterion defined as every constant net converges to its value, the constant net 1 also converges to 2, and the constant net 2 also converges to 3. As is easily verified, the closed sets of \( \gamma_1^\dagger \) are \( \emptyset, 3, 23, 123 \) (23 abbreviates \{2, 3\} etc.), so \( \gamma_1^\dagger = \{123, 12, 1, \emptyset\} \), which is one of the known 29 topologies on three point sets. In \( \gamma_1^\dagger \), the only open set containing 3 is 123, so every net converges to 3, and in particular, the constant net 1 converges to 3, so \( \gamma_1 \) is not topological. Similarly, the convergence criterion \( \gamma_2 \) defined as every constant net converging to its value, the constant net 1 converges to 3, and the constant net 3 converges to 2, generates the topology \( \gamma_2^\dagger = \{123, 13, 1, \emptyset\} \). The join topology is \( \gamma_1^\dagger \lor \gamma_2^\dagger = \{1, 2, 12, 13, 123\} \), but \( \gamma_1 \land \gamma_2 \) generates the discrete topology.
Definition A.8. Let $X$ be a set suppose $f_\alpha : X \to Y_\alpha$ are maps, where $Y_\alpha$ are topological spaces. The weak topology defined by $f_\alpha$ is the topology with subbase $\bigcup_\alpha \{ f_\alpha^{-1}(V) \mid V \subseteq Y_\alpha \text{ is open} \}$. 

Proposition A.9. A net $x_\lambda$ converges to $x$ in the weak topology defined by $f_\alpha : X \to Y_\alpha$ if and only if for all $V \subseteq Y_\alpha$ there is a $\lambda_0$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in f_\alpha^{-1}(V)$, and the latter is equivalent to $f_\alpha(x_\lambda) \in V$. 

Proof. By Lem. A.3, $x_\lambda \to x$ if and only if for all $V \subseteq Y_\alpha$ there is a $\lambda_0$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in f_\alpha^{-1}(V)$, and the latter is equivalent to $f_\alpha(x_\lambda) \in V$. 

Suppose that $I$ is a directed set, $X_i \subseteq X$, $X = \bigcup_{i \in I} X_i$, and $X_i$ are topological spaces, such that $X_i \subseteq X_j$ with continuous inclusion whenever $i \leq j$. The inductive limit topology is the finest topology on $X$ such that every inclusion $X_i \to X$ is continuous; $U \subseteq X$ is open in the inductive limit topology if and only if $U \cap X_i$ is open for all $i$. By eventual membership convergence is meant $x_\lambda \to x$ if there are $\lambda_0$ and $i$ such that $x_\lambda \in X_i$ whenever $\lambda \geq \lambda_0$, and $\langle x_\lambda \rangle_{\lambda \geq \lambda_0}$ converges to $x$ in the topology of $X_i$. 

Proposition A.10. Eventual membership convergence generates the inductive limit topology. 

Proof. There is a general approach for such results: to show a convergence topology is the same as a given topology, show first that convergence in the criterion converges in the topology, so that Prop. A.7(a) implies the convergence topology is finer. In the case that the given topology is the finest satisfying some condition, then establishing that the convergence topology satisfies the same condition completes the proof. In the case of the inductive limit topology, the condition is that the inclusions $X_i \to X$ are continuous. Supposing that $x_\lambda \to x$ (eventual membership), choose $\lambda_0$ and $i$ as in the definition, and note that $x_\lambda$ converges in the inductive limit topology because the inclusions are continuous in that. And, the inclusions in the topology on $X$ generated by membership convergence are continuous by Prop. A.6, since if $x_\lambda \in X_i$ and $x_\lambda \to x$ in the topology of $X_i$, then inclusion does not change any $x_\lambda$ and $\langle x_\lambda \rangle$ satisfies eventual membership convergence. 

Suppose $X$ is a topological space, $\pi : X \to Y$ is onto, and define the convergence criterion $\gamma$ by $y_\lambda \to y$ if there is a net $\langle x_\lambda \in X \rangle$ such that $\pi(x_\lambda) = y_\lambda$, $x_\lambda \to x$, and $\pi(x) = y$. Let $\tau$ be the quotient topology on $Y$. Then $\pi$ is continuous (quotient topology), so $\gamma$-convergence implies $\tau$-convergence and $\gamma^\dagger \supseteq \tau$. $\tau$ is the finest topology such that $\gamma$ is continuous, and, if $x_\lambda \to x$ in $X$ then $\pi(x_\lambda) \to \pi(x)$ (criterion $\gamma$), so $\pi$ is continuous with respect to $\gamma^\dagger$. Hence $\gamma$ generates the quotient topology, as in the proof of Proposition A.10. Let $X = \{ (x, 1/x) \mid x > 0 \} \cup \{ 0 \} \times \mathbb{R}$, i.e. the union of the graph of $y = 1/x$ and the $y$-axis, with the subspace topology from $\mathbb{R}^2$. Define $Y = [0, \infty)$, with the usual topology, and $\pi : X \to Y$ by $\pi(x, y) = x$. Then $\pi$ is a quotient map and $1/i \to 0$ in $Y$ but there is no convergent $(x_i, y_i) \in X$ and $y$ such that $(x_i, y_i) \to (0, y)$. Thus there are convergent nets in the quotient $Y$ that do not satisfy the criterion $\gamma$, while still, by Proposition A.6, $f : Y \to Z$ is continuous if and only if $f(\pi(x_i)) \to f(\pi(x))$ for all $x_i \to x$.

The topological view is advantageous over sole reliance on a convergence criterion. In the theory of distributions, for example, it is common to use sequential uniform convergence of derivatives on compact sets for the test function spaces, which is the same as convergence in an inductive limit topology (Proposition A.10). However the inductive limit is not a locally convex topological vector space. Of interest is the
dual space of the test functions, but the Hahn-Banach theorem fails without local convexity ([28], esp. Example 1.47). Addition of test functions is not continuous in the inductive limit, which in the theory is subsequently convexified. The resulting topological convergence is not the same as the a priori convergence, which can then be used only restrictedly, for example to establish continuity of a linear operator.

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