Identification of physical networks through structured polynomial models

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Abstract—Physical dynamic networks most commonly consist of interconnections of physical components that can be described by diffusive couplings. These diffusive couplings imply that the cause-effect relationships in the interconnections are symmetric and therefore physical dynamic networks can be represented by undirected graphs. This paper shows how (prediction error) identification methods developed for polynomial linear time-invariant systems can be configured to consistently identify the parameters and the interconnection structure of (undirected) physical networks. Further, a multi-step least squares (convex) optimization algorithm is developed to solve the nonconvex optimization problem that results from the identification method.

I. INTRODUCTION

Physical networks can describe many physical processes from different domains, such as mechanical, magnetic, electrical, hydraulic, acoustic, thermal, and chemical processes. Physical networks are typically considered as undirected dynamic interconnections between node signals, where he interconnections represent diffusive couplings [1], [2], [3]. The model is typically described by a vector differential equation of maximum second order. The most well-known example is a mechanical mass-spring-damper system, with positions of masses as node (state) signals and the dynamics being described by a second order vector differential equation.

Identification of these physical models can be done by conversion of the model into a state space form, after which matrix transformations [4], [5] or eigenvalue decompositions [6], [7] are being applied to estimate the model parameters. However, during these operations the network structure in the model is generally lost. Physical models can also be converted into transfer function models or polynomial models, of which the parameters are being identified through prediction error methods [8]. Though, in general the network structure in the model is lost again. Physical networks can also be considered to be directed dynamic networks with specific structural properties [9]. Dynamic networks are directed interconnections of transfer function modules [10], [11] for which an identification framework has been developed in [11]. However, the structural properties of physical networks cannot easily be taken into account in the identification algorithms for dynamic networks.

The overall objective of this research is to develop a comprehensive theory for the identification of individual interconnections in physical (undirected) networks, where the order of the dynamics is not restricted and possibly correlated disturbances can be present. The objective includes questions like which nodes to measure (sense) and which nodes to excite (actuate) in order to identify a particular (local) dynamics in the network or to identify the full dynamics and topology of the network. In addition, consistency and minimum variance properties of estimates have to be specified.

This paper addresses the problem of identifying the full dynamics and topology of physical (undirected) networks. The physical networks that will be considered in this paper are defined in Section II. Next, the set-up for identification of the full network dynamics is described in Section III. In order to be able to consistently identify the network dynamics, data informativity and network identifiability conditions need to be satisfied. These conditions are formulated in Section IV as well as the results for consistent identification of physical networks. Section V contains a multi-step algorithm for consistently identifying the network dynamics and Section VI consists of a simulation example that supports these results. Section VII contains some extensions after which Section VIII concludes the paper.

Consider the following notation throughout the paper. A polynomial matrix $A(q^{-1})$ consists of matrices $A_{t}$ and $(j,k)$-th polynomial elements $a_{jk}(q^{-1})$ such that $A(q^{-1}) = \sum_{t=0}^{n} A_{t} q^{-t}$ and $a_{jk}(q^{-1}) = \sum_{t=0}^{n} a_{jk,t} q^{-t}$. Hence, the $(j,k)$-th element of the matrix $A_{t}$ is denoted by $a_{jk,t}$. Physical components are indicated in sans serif font: $\mathbb{A}$ or a.

II. PHYSICAL NETWORK

Physical networks can describe many physical processes from different domains, such as mechanical (translational, rotational), magnetic, electrical, hydraulic, acoustic, thermal, and chemical processes. Physical networks, such as mass-spring-damper systems and resistor-inductor-capacitor (RLC) circuits, are often described by second order differential equations. They can be considered to consist of $L$ interconnected node signals $w_{j}(t)$, $j = 1, \ldots, L$, of which the behavior is described according to

$$M_{j0} \dot{w}_{j}(t) + D_{j0} \ddot{w}_{j}(t) + \sum_{k \in N_{j}} D_{jk}[\dot{w}_{j}(t) - \dot{w}_{k}(t)] + K_{j0}w_{j}(t) + \sum_{k \in N_{j}} K_{jk}[w_{j}(t) - w_{k}(t)] = u_{j}(t) ,$$

with real-valued coefficients $M_{j0} \geq 0$, $D_{jk} \geq 0$, $K_{jk} \geq 0$, $D_{jj} = 0$, $K_{jj} = 0$, $N_{j}$ is the set of indices of node signals...
$w_k(t)$ for $k \neq j$ with connections to node signals $w_j(t)$, $u_j(t)$ are the external signals and $\dot{w}_j(t)$ and $\ddot{w}_j(t)$ are the first and second order derivative of the node signals $w_j(t)$, respectively. In the physical networks that will be considered in this paper, all connections are symmetric, meaning that the strength of the connection from node $w_i$ to node $w_k$ is equal to the strength of the connection (in opposite direction) from node $w_k$ to node $w_i$. This means that the interconnections of the nodes are diffusive couplings, which emerge in from the symmetric connections: $D_{jk} = D_{kj}$ and $K_{jk} = K_{kj}$ for all $k \neq j$. Moreover, all systems that can be described by diffusive (symmetric) couplings are included in this paper.

An example of a physical networks with diffusive couplings is the mass-spring-damper system shown in Figure 1 in which masses $M_{j0}$ are interconnected through dampers $D_{jk}$ and springs $K_{jk}$ with $k \neq 0$ and are connected to the ground with dampers $D_{j0}$ and springs $K_{j0}$. The positions $x_j(t)$ of the masses $M_{j0}$ are the signals of interest and therefore chosen to be the node signals: $w_j(t) := x_j(t)$. The couplings between the masses are diffusive, because springs and dampers are symmetric components.

Further, a system as the one shown in Figure 1 would require at least a two-dimensional position vector $x_j(t)$, but for notational convenience, without loss of generality, we will restrict our attention to scalar-valued node signals $w_j(t)$. Physical processes from different fields can be combined in a single physical networks using power conversion components. The corresponding linear transformations between node signals can be included in , resulting in a physical networks that contains node signals and external signals of multiple physical quantities.

**A. Higher order network**

A physical network as described above is typically of second order, where all node signals are collected in $w(t)$. Network models that explain only a selection of the node signals can be constructed by removing nodes from the network through a Gaussian elimination procedure that is referred to as Kron reduction [2] or immersion [12], which will generally lead to higher order dynamics between the remaining node signals. In order to accommodate this, we will include higher order terms in our model.

**Definition 1 (Physical network):** A physical network is a network consisting of $L$ node signals $w_1(t), \ldots, w_L(t)$ interconnected through diffusive couplings and with possible connections of nodes to a ground node. The behavior of the node signals $w_j(t)$, $j = 1, \ldots, L$, is described by

$$
\sum_{\ell=0}^{n} x_{jj,\ell} w_j^{(\ell)}(t) + \sum_{k \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} y_{jk,\ell}[w_j^{(\ell)}(t) - w_k^{(\ell)}(t)] = u_j(t),
$$

with real-valued coefficients $x_{jj,\ell} \geq 0$, $y_{jk,\ell} \geq 0$, $y_{jk,\ell} = y_{kj,\ell}$, where $w_j^{(\ell)}(t)$ is the $\ell$-th derivative of $w_j(t)$ and where $u_j(t)$ is the external signal entering the $j$-th node.

The graphical interpretation of the coefficients is as follows: $x_{jj,n}$ represent the buffers, that is the components intrinsically related to the nodes $w_j$; $x_{jj,\ell}$ with $\ell \neq n$ represent the components connecting the node $w_j$ to the ground node; and $y_{jk,\ell}$ represent the components in the diffusive couplings between the nodes $w_j$ and $w_k$. The ground node is characterized by $w_{\text{ground}}(t) = 0$ and therefore can be seen as a node with an infinite buffer, see also [3].

A graphical representation of a physical network is shown in Figure 2. The network dynamics are represented by the blue boxes containing the polynomials $x_{jj}(q^{-1}) = \sum_{\ell=0}^{n} x_{jj,\ell}$ and $y_{jk} = \sum_{\ell=0}^{n-1} y_{jk,\ell}$ and the node signals are represented by the blue circles, which sum the diffusive couplings and the external signals. For example $w_5(t) = x_{55}(w_5(t) - 0) + y_{55}(w_5(t) - w_4(t)) + u_5(t)$. Furthermore, every matrix $X_\ell$ composed of elements $x_{jj,\ell} := x_{jj,\ell}$ is diagonal and every matrix $Y_\ell$ composed of elements $y_{jj,\ell} := \sum_{k \in \mathcal{N}_j} y_{jk,\ell}$ and $y_{jk,\ell} := -y_{kj,\ell}$ for $k \neq j$ is Laplacian representing an undirected graph of a specific physical component (i.e. the diffusive couplings of a specific order).

**B. Discretization**

Typically, identification of linear time-invariant systems takes place in discrete time and the identification tools for dynamic networks that are available in literature are developed for discrete-time models as well. Therefore, it is desirable to convert the continuous-time network into an equivalent discrete-time network.

There exist several discretization methods that are applicable to interconnected systems [14], which means that they possess the property that first discretizing the systems and

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1A Laplacian matrix is a symmetric matrix with nonpositive off-diagonal elements and with nonnegative diagonal elements that are equal to the negative sum of all other elements in the same row (or column) [13].
then interconnecting them in series, parallel and/or feedback results in the same discrete-time network as first interconnecting the systems in series, parallel and/or feedback and then discretizing the network. These discretization methods include the step invariant method, (slightly adapted) impulse invariant method, matched Z-transform method, and direct mappings (forward difference method, backward difference method, and Tustin’s method). All these discretization methods possess the property that asymptotic stability and instability are preserved for sufficiently high sampling frequencies [13].

In this paper, the backward difference method is chosen as it is relatively simple, results in a causal network representation and describes a (unique) bijective mapping between the continuous-time model and the discrete-time model. This discretization method assumes a linear (first order) relation between consecutive samples.

**Proposition 1 (Discrete-time physical network):** By using the approximation
\[
\frac{dw(t)}{dt} \bigg|_{t=t_d} = \frac{w(t_d) - w(t_{d-1})}{T_s},
\]
with discrete time sequence \( t_d = dT_s, d = 0, 1, \ldots \) and time interval \( T_s \), the continuous-time physical network [2] can be described in discrete time by
\[
\sum_{\ell=0}^{n} \bar{x}_{jj,\ell} q^{-\ell} w_j(t_d) + \sum_{k \in N_j, \ell=0}^{n-1} \bar{y}_{jk,\ell} q^{-\ell} [w_j(t_d) - w_j(t_d-1)] = w_j(t_d),
\]
with \( q^{-1} \) the shift operator meaning \( q^{-1} w_j(t_d) = w_j(t_{d-1}) \) and with
\[
\bar{x}_{jj,\ell} = (-1)^{\ell} \sum_{i=0}^{n} \binom{i}{\ell} T^{-i}_{ss} \bar{x}_{ji,i},
\]
\[
\bar{y}_{jk,\ell} = (-1)^{\ell} \sum_{i=0}^{n-1} \binom{i}{\ell} T^{-i}_{ss} \bar{y}_{kj,i},
\]
where \( \binom{i}{\ell} \) is a binomial coefficient.

**Proof:** Equation [2] is discretized by a similar approach as described in [15] by using a backward shift [3].

In the sequel, \( t - i \) is used for \( t_{d-i} = t_d - i T_s \). The expressions for the node signals [4] can be combined in a matrix equation describing the network as
\[
\begin{align*}
\bar{X}(q^{-1}) w(t) + \bar{Y}(q^{-1}) w(t) &= u(t),
\end{align*}
\]
with \( \bar{X}(q^{-1}) \) and \( \bar{Y}(q^{-1}) \) polynomial matrices in the shift operator \( q^{-1} \) and composed of elements
\[
\bar{X}_{jk}(q^{-1}) = \begin{cases}
\sum_{\ell=0}^{n} \bar{x}_{jj,\ell} q^{-\ell}, & \text{if } k = j \\
0, & \text{otherwise}
\end{cases}
\]
\[
\bar{Y}_{jk}(q^{-1}) = \begin{cases}
\sum_{m \in N_j} \sum_{\ell=0}^{n-1} \bar{y}_{jm,\ell} q^{-\ell}, & \text{if } k = j \\
- \sum_{\ell=0}^{n-1} \bar{y}_{jk,\ell} q^{-\ell}, & \text{if } k \in N_j \\
0, & \text{otherwise}
\end{cases}
\]

Observe that \( \bar{X}(q^{-1}) \) is diagonal and \( \bar{Y}(q^{-1}) \) is Laplacian, implying that the structural properties of [2] are maintained in [7].

**III. Identification set-up**

As mentioned before, the objective of this paper is to identify the full dynamics and the topology of physical networks. In this section, the identification setting is described, which includes the physical network model, the network predictor, the model set, and the identification criterion.

The node signals in the network might be affected by a user-applied excitation signal and subject to a disturbance signal. This needs to be included in the network description, which is achieved by splitting the external signal as
\[
u(t) := B(q^{-1}) r(t) + F(q) e(t),
\]
where, the known excitation signals \( r(t) \) enter the network through dynamics described by polynomial matrix \( B(q^{-1}) \) and where the unknown disturbance signals acting on the network are modeled as a filtered stationary white noise process \( F(q) e(t) \) with \( F(q) \) a rational matrix.

For system identification purposes, we will use a slightly different, but equivalent, physical network description than the one in [7], where the partitioning [10] is included.

**Definition 2 (Physical network model):** The physical network that will be considered during identification consists of \( L \) node signals \( w(t) \) and \( K \) excitation signals \( r(t) \) and is defined as
\[
A(q^{-1}) w(t) = B(q^{-1}) r(t) + F(q) e(t),
\]
with
\[
A(q^{-1}) \in \mathcal{A} := \{ A \in \mathbb{R}^{L \times L}[q^{-1}] \mid a_{kj}(q^{-1}) = a_{jk}(q^{-1}), \ \forall k,j \}
\]
\[
B(q^{-1}) \in \mathcal{B} := \{ B \in \mathbb{R}^{L \times K}[q^{-1}] \mid B \text{ proper} \}
\]
\[
F(q) \in \mathcal{F} := \{ F \in \mathbb{R}^{L \times L}(q) \mid F \text{ monic, stable and stably invertible} \}
\]
\[
\Lambda \succ 0 \text{ the covariance matrix of the noise } e(t).
\]

Further, the network is assumed to be well-posed and stable, implying that \( A^{-1}(q^{-1}) \) exists and is proper and stable. It is also assumed that the network is connective, which means that there is a path between every pair of nodes [2].

Observe that \( A(q^{-1}) = \bar{X}(q^{-1}) + \bar{Y}(q^{-1}) \). Often, \( B(q^{-1}) \) is chosen to be binary, diagonal and known, which represents the assumption that each external excitation signal directly enters the network at a distinct node (without dynamics in between).

As a result, physical networks lead to polynomial models with the particular properties that \( A(q^{-1}) \) is symmetric and nonmonic. Moreover, if \( F(q) \) is polynomial or even stronger if \( F(q) = I \), the physical network [11] leads to an ARMAX-like or ARX-like model structure with these particular properties, respectively.

Now the physical network representation and its properties have been defined, the next step is to formulate the identification setting.

2The network is connective if its Laplacian matrix (i.e. the degree matrix minus the adjacency matrix) has a positive second smallest eigenvalue [2].
3Polynomial models are linear time-invariant dynamic models of the form \( A(q^{-1}) y(t) = E^{-1}(q^{-1}) B(q^{-1}) w(t) + D^{-1}(q^{-1}) C(q^{-1}) e(t) \), where \( A(q^{-1}), B(q^{-1}), C(q^{-1}), D(q^{-1}), \) and \( E(q^{-1}) \) are polynomials in \( q^{-1} \) that are all monic except for \( B(q^{-1}) \) [8, 13].
4The structure is formally only an ARMAX (autoregressive-moving average with exogenous variables) or ARX (autoregressive with exogenous variables) structure if the \( A(q^{-1}) \) polynomial is monic [16].
A. Network predictor

The objective is to identify the dynamics of the complete physical network. This estimation is performed using a prediction error method, which is the most common system identification method and it is applicable to networks \(^{(11)}\). In order to identify the complete network dynamics, all node signals \(w(t)\) are predicted based on the measured signals that are available in the network. This leads to the following predictor.

**Definition 3 (Network predictor):** In line with \(^{(17)}\), the network predictor is defined as the conditional expectation

\[
\hat{w}(t|t-1) = \mathbb{E}\{w(t) | w^{t-1}, r^t\}. \tag{12}
\]

where \(w^{t-1}\) represents the past of \(w(t)\), that is \(w(t-1), \ldots, w(t-2)\), and \(r^t\) represents \(r(t), r(t-1), \ldots\). The current node signals, \(w(t)\), are not used for prediction as the network contains direct feedthrough terms.

**Proposition 2 (Network predictor):** For a network model \(^{(11)}\), the one-step-ahead network predictor \(^{(13)}\) is given by (omitting arguments \(q, t\))

\[
\hat{w}(t|t-1) := [I - A_0^{-1}F^{-1}A] w + A_0^{-1}F^{-1}Br, \tag{13}
\]

where \(A_0 := \lim_{t \to \infty} A(z)\).

**Proof:** The physical network \(^{(11)}\) can be described by

\[ Aw = Br + Fe = Br + FA_0^{-1}e. \]

Premultiplying with \(A_0^{-1}F^{-1}\) gives

\[ A_0^{-1}F^{-1}Aw = A_0^{-1}F^{-1}Br + A_0^{-1}e. \tag{14} \]

Adding \(w\) to both sides of the equality and rewriting gives

\[
w = [I - A_0^{-1}F^{-1}A] w + A_0^{-1}F^{-1}Br + A_0^{-1}e. \tag{15} \]

where the factor \(A_0^{-1}\) makes the filter \([I - A_0^{-1}F^{-1}A]\) strictly proper and where \(A_0^{-1}F^{-1}B\) is proper. The one-step-ahead network predictor \(^{(13)}\) follows directly by applying its definition \(^{(13)}\) to \(^{(14)}\).

**Proposition 3 (Innovation):** The innovation corresponding to the network predictor \(^{(13)}\) is

\[
\tilde{e}(t) := w(t) - \hat{w}(t|t-1) = A_0^{-1}e(t), \tag{16} \]

which has covariance matrix \(\bar{\Lambda} = A_0^{-1}A_0^{-1}\).

**Proof:** This follows directly from subsequently substituting \(\hat{w}(t|t-1) \) \(^{(13)}\) and \(w(t) \) \(^{(11)}\) into \(^{(15)}\). The innovation is a scaled version of the driving noise process. As \(A_0\) is not necessarily diagonal, the scaling possibly causes correlations among the noise channels, but the innovation signal \(\tilde{e}(t)\) remains a white noise process.

B. Model set and prediction error

The physical network models that will be considered during identification are gathered in the network model set.

**Definition 4 (Physical network model set):** The model set is defined as a set of parametrized functions as

\[ M := \{ M(\theta), \theta \in \Theta \}, \tag{16} \]

with all particular models

\[ M(\theta) := (A(q^{-1}, \theta), B(q^{-1}, \theta), F(q, \theta), \Lambda(\theta)) \] \tag{17} \]

satisfying the properties in Definition \(^{(2)}\).

The experimental data that are available for identification are generated by the true system.

**Definition 5 (Data generating system):** The data generating system \(S\) is denoted by the model

\[ M^0 := (A^0, B^0, F^0, \Lambda^0). \tag{18} \]

The true system is in the model set, i.e. \((S \in M)\), if \(\exists \theta^0 \in \Theta\) such that \(M(\theta^0) = M^0\), where \(\theta^0\) indicates the true parameters. Using the parametrized physical network model set, the parametrized one-step-ahead network predictor is defined.

**Definition 6 (Parametrized predictor):** The parametrized network predictor is defined in accordance with \(^{(13)}\) as

\[
\hat{w}(t|t-1; \theta) = [I - W_w(q, \theta)] w(t) + W_r(q, \theta)r(t), \tag{19} \]

with filters

\[ W_w(q, \theta) = A_0^{-1}(\theta)F^{-1}(q, \theta)A(q^{-1}, \theta), \tag{20} \]

\[ W_r(q, \theta) = A_0^{-1}(\theta)F^{-1}(q, \theta)B(q^{-1}, \theta). \tag{21} \]

The parametrized predictor leads to the prediction error.

**Proposition 4 (Prediction error):** The prediction error corresponding to the parametrized predictor \(^{(19)}\) is defined as

\[ \varepsilon(t, \theta) := w(t) - \hat{w}(t|t-1; \theta), \tag{22} \]

which is obtained as (omitting argument \(q\))

\[ \varepsilon(t, \theta) = A_0^{-1}(\theta)F^{-1}(\theta) \left[ A(\theta)w(t) - B(\theta)r(t) \right], \tag{23} \]

\[ = W_w(q, \theta)w(t) - W_r(q, \theta)r(t). \tag{24} \]

which equals the innovation \(\tilde{e}(t) \) \(^{(15)}\) for \(\theta = \theta^0\).

**Proof:** The expression for the parametrized prediction error \(^{(23)}\) directly follows from its definition \(^{(22)}\) and the network predictor \(^{(19)}\). Expressing the parametrized prediction error \(^{(23)}\) in terms of \(r(t)\) and \(e(t)\) yields

\[ \varepsilon(t, \theta) = W_{\tilde{e}}(q, \theta)r(t) + W_{\tilde{e}}(q, \theta)e(t) + (A_0^{-1})^{-1}e(t), \tag{25} \]

with (omitting argument \(q\))

\[ W_{\tilde{e}}(q, \theta) = A_0^{-1}(\theta)F^{-1}(\theta) \left[ A(\theta)(A^0)^{-1}B^0 - B(\theta) \right], \tag{26} \]

\[ W_{\tilde{e}}(q, \theta) = A_0^{-1}(\theta)F^{-1}(\theta)A(\theta)(A^0)^{-1}F^0 - (A_0^0)^{-1}. \tag{27} \]

with \(W_{\tilde{e}}(q; \theta)\) strictly proper. The latter two terms in \(^{(25)}\) are uncorrelated since \(e(t)\) is white noise. If the true system is in the model set, the prediction error for the true system is equal to the innovation \(^{(15)}\):

\[ \varepsilon(t, \theta^0) = (A_0^0)^{-1}e(t) = \tilde{e}(t). \tag{26} \]
C. Identification criterion

In order to estimate the parameters, a weighted least squares identification criterion is applied:

\[
\hat{\theta}_N = \arg \min_{\theta} V_N(\theta),
\]

\[
V_N(\theta) := \frac{1}{N} \sum_{t=1}^{N} \varepsilon^T(t, \theta) S \varepsilon(t, \theta),
\]

\[
\hat{\Lambda}(\hat{\theta}_N) := \frac{1}{N} \sum_{t=1}^{N} \varepsilon(t, \hat{\theta}_N)^T \varepsilon(t, \hat{\theta}_N),
\]

with weight \( S > 0 \) that has to be chosen by the user. For analysis of the asymptotic properties of the parameter estimate, we consider the asymptotic criterion

\[
\theta^* := \arg \min_{\theta} \hat{V}(\theta),
\]

\[
\hat{V}(\theta) := \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \mathbb{E} \{ \varepsilon^T(t, \theta) S \varepsilon(t, \theta) \}. \tag{31}
\]

It can be shown that under some mild conditions, see [8], the solution of the weighted least squares criterion (27) converges with probability 1 to the solution of the asymptotic criterion (30).

IV. Consistent identification

In order to be able to consistently identify the network, the experimental data need to satisfy certain conditions. These conditions are referred to as data informativity conditions. In addition, the network itself needs to satisfy certain conditions, such that it can be uniquely recovered. These conditions are referred to as network identifiability conditions. This section describes these conditions, after which the results for consistent network identification can be formulated.

The physical network (11) can be represented as

\[
w(t) = T_{wr}(q) r(t) + \bar{v}(t), \quad \bar{v}(t) = T_{we}(q) \bar{e}(t), \tag{32}
\]

where \( \bar{e}(t) \) is the innovation [15] and

\[
T_{wr}(q) = A^{-1}(q^{-1}) B(q^{-1}),
\]

\[
T_{we}(q) = A^{-1}(q^{-1}) F(q) A_0.
\]

For estimating a network model, prediction error identification methods typically use the second order statistical properties of the measured data, which are represented by the spectral densities of \( w(t) \) and \( r(t) \). As \( r(t) \) is measured, but \( e(t) \) is not, the second order properties of \( w(t) \) are generated by transfer function \( T_{wr}(q) \) and spectral density

\[
\Phi_{\bar{e}}(\omega) := \mathcal{F} \{ \mathbb{E}[\bar{v}(t) \bar{v}^*(t - \tau)] \},
\]

\[
= T_{we}(e^{j\omega}) \hat{\Lambda} T_{we}(e^{j\omega}), \tag{36}
\]

with \( \mathcal{F} \) the discrete-time Fourier transform and \( (\cdot)^* \) the complex conjugate transpose. Observe that the spectral factorization in (36) is unique, as \( T_{we}(q) \in \mathcal{F} \) and \( \hat{\Lambda} \succ 0 \) [18].

A. Data informativity

The data are called informative if it contains sufficient information such that \( T_{wr}(q, \theta) \) and \( \Phi_{\bar{e}}(\omega) \) can uniquely be obtained from the spectral density of the measured data: \( \Phi_{\bar{e}}(\omega) \). We define this for quasi-stationary data sequences in line with [17].

Definition 7 (Data informativity): A quasi-stationary data sequence \( \{w(t), r(t)\} \) is called informative with respect to the model set \( \mathcal{M} \) if for any two \( \theta_1, \theta_2 \in \Theta \)

\[
\mathbb{E} \left\{ (\varepsilon(t, \theta_1) - \varepsilon(t, \theta_2))^T S (\varepsilon(t, \theta_1) - \varepsilon(t, \theta_2)) \right\} = 0
\]

\[
\Rightarrow \begin{cases} W_w(e^{j\omega}, \theta_1) = W_w(e^{j\omega}, \theta_2), \\ W_r(e^{j\omega}, \theta_1) = W_r(e^{j\omega}, \theta_2) \end{cases} \tag{37}
\]

for almost all \( \omega \).

Applying this definition to physical networks, leads to the following conditions for data informativity.

Proposition 5 (Data informativity): The quasi-stationary data sequence \( \{\varepsilon(t), r(t)\} \) is informative with respect to the model set \( \mathcal{M} \) if, in the situation \( K \geq 1, \Phi_r(\omega) \succ 0 \) for a sufficiently high number of frequencies.

Proof: The premise of implication (37) is satisfied if and only if \( \Delta_{\varepsilon} := \varepsilon(t, \theta_1) - \varepsilon(t, \theta_2) = 0 \), i.e. (omitting argument \( q \))

\[
\Delta_{\varepsilon} = \Delta_{\varepsilon}(\theta) z(t) = 0,
\]

with

\[
\Delta_{\varepsilon}(\theta) = \begin{bmatrix} W_w(\theta_1) - W_w(\theta_2) \\ W_r(\theta_1) - W_r(\theta_2) \end{bmatrix},
\]

\[
z(t) = \begin{bmatrix} w(t) \\ r(t) \end{bmatrix}.
\]

Applying Parseval’s theorem gives

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta_{\varepsilon}(e^{j\omega}, \theta) \Phi_{\varepsilon}(\omega) \Delta_{\varepsilon}^T(e^{-j\omega}, \theta) d\omega = 0.
\]

This implies \( \Delta_{\varepsilon}(\theta) = 0 \) only if \( \Phi_{\varepsilon}(\omega) \succ 0 \) for a sufficiently high number of frequencies. As \( w(t) \) depends on \( r(t) \), substituting the open-loop response (11) for \( w(t) \) gives

\[
z(t) = J(q) \kappa(t),
\]

with

\[
J(q) = \begin{bmatrix} A^{-1} F & A^{-1} B \\ 0 & I \end{bmatrix}, \quad \kappa(t) = \begin{bmatrix} e(t) \\ r(t) \end{bmatrix}.
\]

As \( J(q) \) has always full rank, \( \Phi_{\varepsilon}(\omega) \succ 0 \) if and only if \( \Phi_r(\omega) \succ 0 \). As \( e(t) \) and \( r(t) \) are assumed to be uncorrelated and \( E\{e(t)\} = 0 \), we have that \( \Phi_{re} = \Phi_{er} = 0 \) and

\[
\Phi_{r} = \begin{bmatrix} \Phi_{rr} & \Phi_{re} \\ \Phi_{er} & \Phi_{ee} \end{bmatrix} = \begin{bmatrix} \Phi_{r} & 0 \\ 0 & \Lambda \end{bmatrix}.
\]

Then \( \Phi_{e}(\omega) \succ 0 \) if and only if \( \Lambda \succ 0 \) (which is assumed) and \( \Phi_r(\omega) \succ 0 \). The condition \( \Phi_{r}(\omega) \succ 0 \) reduces to \( \Phi_r(\omega) \succ 0 \).

The condition that \( \Phi_{r}(\omega) \succ 0 \) for a sufficiently high number of frequencies seems to be a general condition. However, observe that the dimensions of \( \Phi_{\varepsilon}(\omega) \) depend on the number of excitation signals \( r(t) \), denoted by \( K \), which is specified in the model set. Thus all excitation signals \( r(t) \) that are
models that describe the same data \cite{19}. where also the relation between identifiability and equivalent
B. Network identifiability
identifiability \cite{19}. As a result, we consider the following definition for network
Definition 9 (Network identifiability): The network model set \mathcal{M} \cite{16} is globally network identifiable from measured data \{w(t), r(t)\} if the following conditions are satisfied:

1) The polynomials \(A(q^{-1})\) and \(B(q^{-1})\) are left coprime.
2) At least one matrix \(A_k\) or \(B_k\) is diagonal.

Proof: Condition 3 implies that \(T_{wr}(q, \theta)\) is nonzero. According to Lemma \[1\] condition 1 and 2 imply that \(A(q^{-1}, \theta)\) and \(B(q^{-1}, \theta)\) are found up to a scalar factor \(\alpha\). \(T_{wr}(q, \theta)\) and \(\Lambda(\theta)\) are uniquely recovered from \(\Phi_v(\omega, \theta)\) as \(T_{w}(q, \theta) \in \mathcal{F}\) and \(\Lambda \succ 0\) \cite{18}. Together with the fact that \(A(q^{-1}, \theta)\) is found up to a scalar factor \(\alpha\), \(T_{wr}(q, \theta)\) gives a unique \(F(q, \theta)\), and \(\Lambda(\theta)\) gives \(\Lambda(\theta)\) up to a scalar factor \(\alpha^2\). Finally, condition 4 implies that the parameters cannot be scaled anymore and therefore \(\alpha\) is fixed.

The coprimeness of \(A(q^{-1})\) and \(B(q^{-1})\) ensures that \(A(q^{-1})\) and \(B(q^{-1})\) have no common factors. This condition is also necessary for global identifiability of typical polynomial model structures, see Theorem 4.1 of \cite{18}. The parameter \(\alpha\) is a scaling factor that is introduced by the nonmonicity of \(A(q^{-1})\).

In physical model structures the scaling factor needs to be fixed by additional constraints induced by condition 2 and condition 4 in Proposition 6. The parameter constraint in condition 4 of Proposition 6 can for example be

- One nonzero element in \(B(q^{-1}, \theta)\) is known, i.e. one excitation signal enters a node through known dynamics.
- One nonzero parameter is known.
- The fraction of two nonzero parameters is known.
- The sum of some nonzero parameters is known.

Remark 1: In general dynamic networks conditions for global network identifiability typically include algebraic conditions verifying the rank of particular transfer functions from external signals to internal node signals \cite{19}. For the generic version of network identifiability this entails a related graph-based check on vertex disjoint paths in the network model \cite{22}.

In contrast to these conditions, the current conditions in Proposition 6 are very simple and require only a single excitation signal \(r(t)\) to be present in the network. This is induced by the structural properties of the diffusive couplings between the nodes, reflected in the fact that the polynomial matrix \(A(q^{-1})\) is restricted to be diagonal.

C. Consistency
Now, we can formulate the consistency result as follows.

Theorem 1 (Consistency): Consider a data generating system \(\mathcal{S}\) as defined in Definition 5 Then, under mild conditions (see Section II-C), \(M(\bar{\theta}_N)\) is a consistent estimate of \(M^0\) if the following conditions hold:

1) The polynomials \(A(q^{-1})\) and \(B(q^{-1})\) are left coprime.
2) At least one matrix \(A_k\) or \(B_k\) is diagonal.

Proof: According to \cite{20}, the LMFD of any two polynomial and left coprime matrices is unique up to a premultiplication with a unimodular matrix. To preserve diagonality of \(A_k\) or \(B_k\), the unimodular matrix is restricted to be diagonal. To preserve symmetry of \(A(q^{-1})\), this diagonal matrix is further restricted to have equal elements.

Hence, both condition 2 in Lemma \[1\] and the scaling factor freedom are a result of the fact that \(A(q^{-1})\) is not necessarily monic.

Now the conditions for global network identifiability in physical networks can be formulated.

Proposition 6 (Physical network identifiability): A physical network model set \(\mathcal{M}\) \cite{16} is globally network identifiable from measured data \{\(w(t), r(t)\)\} if the following conditions are satisfied:

1) The polynomials \(A(q^{-1})\) and \(B(q^{-1})\) are left coprime.
2) At least one matrix \(A_k\) or \(B_k\) is diagonal.
3) At least one excitation signal \(r_j(t)\) is present: \(K \geq 1\).
4) There is at least one constraint on the parameters of \(A(q^{-1}\theta_a)\) and \(B(q^{-1}\theta_a)\) of the form \(\Gamma\theta = \gamma \neq 0\), with \(\theta := [\theta_a, \theta_b]^\top\).

Proof: Condition 3 implies that \(T_{wr}(q, \theta)\) is nonzero. According to Lemma \[1\] condition 1 and 2 imply that \(A(q^{-1}, \theta)\) and \(B(q^{-1}, \theta)\) are found up to a scalar factor \(\alpha\). \(T_{wr}(q, \theta)\) and \(\Lambda(\theta)\) are uniquely recovered from \(\Phi_v(\omega, \theta)\) as \(T_{w}(q, \theta) \in \mathcal{F}\) and \(\Lambda \succ 0\) \cite{18}. Together with the fact that \(A(q^{-1}, \theta)\) is found up to a scalar factor \(\alpha\), \(T_{wr}(q, \theta)\) gives a unique \(F(q, \theta)\), and \(\Lambda(\theta)\) gives \(\Lambda(\theta)\) up to a scalar factor \(\alpha^2\). Finally, condition 4 implies that the parameters cannot be scaled anymore and therefore \(\alpha\) is fixed.

The coprimeness of \(A(q^{-1})\) and \(B(q^{-1})\) ensures that \(A(q^{-1})\) and \(B(q^{-1})\) have no common factors. This condition is also necessary for global identifiability of typical polynomial model structures, see Theorem 4.1 of \cite{18}. The parameter \(\alpha\) is a scaling factor that is introduced by the nonmonicity of \(A(q^{-1})\).

In physical model structures the scaling factor needs to be fixed by additional constraints induced by condition 2 and condition 4 in Proposition 6. The parameter constraint in condition 4 of Proposition 6 can for example be

- One nonzero element in \(B(q^{-1}, \theta)\) is known, i.e. one excitation signal enters a node through known dynamics.
- One nonzero parameter is known.
- The fraction of two nonzero parameters is known.
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In contrast to these conditions, the current conditions in Proposition 6 are very simple and require only a single excitation signal \(r(t)\) to be present in the network. This is induced by the structural properties of the diffusive couplings between the nodes, reflected in the fact that the polynomial matrix \(A(q^{-1})\) is restricted to be diagonal.

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3) At least one excitation signal \(r_j(t)\) is present: \(K \geq 1\).
4) There is at least one constraint on the parameters of \(A(q^{-1}\theta_a)\) and \(B(q^{-1}\theta_a)\) of the form \(\Gamma\theta = \gamma \neq 0\), with \(\theta := [\theta_a, \theta_b]^\top\).

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In contrast to these conditions, the current conditions in Proposition 6 are very simple and require only a single excitation signal \(r(t)\) to be present in the network. This is induced by the structural properties of the diffusive couplings between the nodes, reflected in the fact that the polynomial matrix \(A(q^{-1})\) is restricted to be diagonal.
1) The true system is in the model set ($S \in M$).
2) The data are informative with respect to the model set.
3) The model set is globally network identifiable.

Proof: The proof consists of three steps. First, convergence of $V_N(\theta)$ to $\hat{V}(\theta)$ for $N \to \infty$ follows directly from applying Theorem 2B.1 of [8] and the fact that $S \supseteq 0$ as the conditions for convergence are satisfied by the network model set. Second, by condition 1, $\theta^0$ is a minimum of $V(\theta)$, which can be seen as follows. As $r(t)$ and $e(t)$ are uncorrelated and $W_{w}(q, \theta)$ is strictly proper, the power of any cross term between the three terms in the prediction error (25) is zero, so the power of each term can be minimized individually. As a result, $W_{w}(q, \theta^0) = 0$ and $W_{v}(q, \theta^0) = 0$ and thus the cost function reaches its minimum value when the prediction error is equal to the innovation as in (26). Third, following the cost function reaches its minimum value when the prediction error is equal to the innovation as in (26). Third, following the cost function reaches its minimum value when the prediction error is equal to the innovation as in (26). Third, following the cost function reaches its minimum value when the prediction error is equal to the innovation as in (26).

Observe that any weight $S \supseteq 0$ leads to consistent estimates, but that minimum variance is only achieved for $S = \Lambda^{-1}$.

Now it has been proved that physical networks can be identified consistently, the next step is to formulate algorithms for obtaining these estimates.

V. A MULTI-STEP ALGORITHM

The parametrized prediction error (25) is not affine in the parameters $\theta$. Only in the very special situation where $F(q, \theta) = I$ and $A_0(q, \theta) = I$, the structure of (25) is affine. This situation causes the optimization problem (27) to be nonconvex. Especially for networks with many nodes, this results high computational complexity and occurrences of local optima. One approach to reduce the problem is to solve multiple multi-input single-output (MISO) problems instead of one large multi-input multi-output (MIMO) problem [11], [12], [24]. However, since the dynamics are coupled (that is, $A(q^{-1})$ is symmetric and therefore its elements are not independently parametrized), a decomposition into MISO problems cannot be made without loss of accuracy.

In this section, as an alternative, a multi-step algorithm is developed, where in each step a quadratic problem is solved using a linear regression scheme. With that, the developed method contains steps that are similar to sequential least squares (SLS) [25], weighted null-space fitting (WNSF) [26], and the multi-step least squares method in [27], but particularly tuned to the physical network model structure of the current paper. As only quadratic problems are solved, the optimizations are convex and have unique solutions. In this way, the formulated algorithm achieves a consistent parameter estimation with minimum variance and limited computation complexity. This makes the algorithm also applicable to networks with many nodes.

A. Physical network with an ARMAX-like model structure

Consider a data generating system $S = (A^0, B^0, F^0, \Lambda^0)$ with $F^0(q) := C^0(q^{-1})$ being a monic polynomial, representing the physical network

$$A^0(q^{-1})w(t) = B^0(q^{-1})r(t) + C^0(q^{-1})e(t),$$

(41)

which would have an ARMAX structure if $A^0(q^{-1})$ would be monic. Multiplying both sides of (41) with $[C^0(q^{-1})A_0^0]^{-1}$ leads to

$$\hat{A}^0(q^{-1})w(t) = \hat{B}^0(q^{-1})r(t) + \hat{e}(t),$$

(42)

where $\hat{A}^0(q^{-1})$ is monic, $\hat{e}(t)$ is the innovation (15), and

$$\hat{A}^0(q^{-1}) = [\hat{C}^0(q^{-1})]^{-1}A^0(q^{-1}),$$

(43)

$$\hat{B}^0(q^{-1}) = [\hat{C}^0(q^{-1})]^{-1}B^0(q^{-1}),$$

(44)

$$\hat{C}^0(q^{-1}) = C^0(q^{-1})A_0^0.$$ (45)

Now consider the model structure $A(q^{-1}, \theta_0), B(q^{-1}, \theta_0)$, and $C(q^{-1}, \eta_0)$, as models of $A^0(q^{-1})$, $B^0(q^{-1})$, and $C^0(q^{-1})$, respectively, with $C(q^{-1}, \eta_0) = C(q^{-1}, \theta_0)A_0(\theta_0)$, and with $\theta := [\theta_0^\top \eta_0^\top]^\top$. The exact parametrization is given in Appendix A.

Step 1: Estimating the nonparametric ARX model

As a first step, we are going to estimate a nonparametric ARX model for $\hat{e}(t)$, by parametrizing the infinite series expansions $\hat{A}^0(q^{-1})$ and $\hat{B}^0(q^{-1})$ by high order polynomial (finite) expansions $\hat{A}(q^{-1}, \zeta^n)$ and $\hat{B}(q^{-1}, \zeta^n)$, according to

$$\hat{e}(t) = \hat{A}(q^{-1}, \zeta^n)w(t) - \hat{B}(q^{-1}, \zeta^n)r(t),$$

(46)

$$\hat{e}(t) = \hat{A}(q^{-1}, \zeta^n)w(t) - \hat{B}(q^{-1}, \zeta^n)r(t),$$

(47)

with $n$ the finite order of the polynomials, which is typically chosen to be high. The parameter vector $\zeta^n$ is given in Appendix A and the matrix $[\phi^n(t)]^\top$ is given in Appendix B. The nonparametric ARX model (42) is then estimated through estimating its parameters $\zeta^n$. This step is equivalent to the first step of SLS [25] and WNSF [26], [27]. As this step serves to make an initial estimate of the network, the network structure is not taken into account. Further, consistency of this step is only achieved if the order $n$ tends to infinity as function of the data length $N$ at an appropriate rate, according to [28]. However, the bias will be negligibly small if the order $n$ is chosen sufficiently large. The least-squares estimate of $\zeta^n$ is found by

$$\hat{\zeta}_N = \left[ \frac{1}{N} \sum_{t=n+1}^{N} \phi^n(t)[\phi^n(t)]^\top \right]^{-1} \left[ \frac{1}{N} \sum_{t=n+1}^{N} \phi^n(t)w(t) \right],$$

(48)

Under conditions of consistent estimation, and so if $n$ and $N$ approach infinity, $\hat{e}(t, \hat{\zeta}_N^N)$ will be an accurate estimate of the innovation $\hat{e}(t)$. The covariance of the innovation is estimated as the covariance of the residual as

$$\hat{\Lambda}(\hat{\zeta}_N^N) = \frac{1}{N} \sum_{t=n+1}^{N} \hat{e}(t, \hat{\zeta}_N^N)\hat{e}(t, \hat{\zeta}_N^N)^\top,$$

(49)
with residual evaluated at . The covariance of the estimation error , is estimated by

\[
P(\hat{\zeta}_N) = \left[ \frac{1}{N} \sum_{t=n+1}^{N} \varphi^T(t)\bar{\Lambda}^{-1}(\hat{\zeta}_N)\varphi^T(t) \right]^{-1}. \tag{50}
\]

Remark 2: As each row in (47) is independently parametrized, the parameters can be estimated for each row independently, resulting in L MISO problems instead of one MIMO problem. This is attractive for networks with many nodes.

Step 2: Reducing to the physical network model

The high order ARX model is used to identify the physical network model. This step is similar to the second step of WNSF [26], [27], where the difference lies in the parametrization structure. In this step, the structural properties of are incorporated and the parameter constraint is taken into account to fix the scaling parameter and obtain a unique solution.

The relations (51) and (52) are equivalently written as

\[
\begin{align*}
A^0(q^{-1}) - \bar{C}^0(q^{-1})\hat{A}(q^{-1}) &= 0, \\
B^0(q^{-1}) - \bar{C}^0(q^{-1})\hat{B}(q^{-1}) &= 0.
\end{align*}
\tag{51, 52}
\]

Then from (51) and (52) we can extract:

\[
-\bar{Q}(\zeta^0)\bar{\vartheta}^0 = 0, \tag{53}
\]

where and are incorporated in , where represents the actual underlying system described by and , and where the polynomial terms from (51) and (52) are considered up to time lag and , where the row dimension of is equal to . The matrix is given in Appendix C.

Together with the linear parameter constraint described by , where with full rank, the initial least-squares estimate of is obtained by the linear optimization problem

\[
\hat{\vartheta}^0_N = \min_{\hat{\vartheta}} \hat{\vartheta}^TQ(\zeta^0)\hat{\vartheta} \tag{54}
\]

subject to \(\Gamma \hat{\vartheta} = \gamma\),

which can be solved using the Lagrangian and the Karush–Kuhn–Tucker conditions, giving

\[
\begin{bmatrix}
\hat{\vartheta}^0_N \\
\hat{\lambda}^0_N
\end{bmatrix} = \begin{bmatrix}
Q^T(\zeta^0)Q(\zeta^0) & \Gamma^T \\
\Gamma & 0
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
\gamma
\end{bmatrix}, \tag{56}
\]

where are the estimated Lagrange multipliers. The covariance of the residuals is updated according to (for )

\[
\bar{\Lambda}(\hat{\vartheta}^{(k)}_N) = \frac{1}{N} \sum_{t=n+1}^{N} \bar{\varepsilon}(t, \hat{\vartheta}^{(k)}_N)\bar{\varepsilon}^T(t, \hat{\vartheta}^{(k)}_N), \tag{57}
\]

with residual

\[
\bar{\varepsilon}(t, \hat{\vartheta}^{(k)}_N) = \bar{C}^{-1}(q^{-1}, \hat{\vartheta}^{(k)}_N) \left[ A(q^{-1}, \hat{\vartheta}^{(k)}_N)w(t) - B(q^{-1}, \hat{\vartheta}^{(k)}_N)r(t) \right]. \tag{58}
\]

Weighted least-squares can be used as well (see Step 3) with weighting matrix \(W(\zeta^0_N) = P^{-1}(\zeta^0_N)\) [29].

Step 3: Improving the physical network model

This step aims to correct for the residuals in that are not accounted for in [56], due to the fact that only a high order approximation of the nonparametric ARX model is used. This step is similar to the third step of WNSF [26], [27], where again the difference lies in the parametrization structure. Substituting and for , respectively, into (51) and (52) gives

\[
\begin{align*}
A^0(q^{-1}) - \bar{C}^0(q^{-1})\bar{A}(q^{-1}, \zeta^o) &= \bar{C}^0(q^{-1})[\bar{A}(q^{-1}, \zeta^o) - \bar{A}(q^{-1})], \\
B^0(q^{-1}) - \bar{C}^0(q^{-1})\bar{B}(q^{-1}, \zeta^o) &= \bar{C}^0(q^{-1})[\bar{B}(q^{-1}, \zeta^o) - \bar{B}(q^{-1})],
\end{align*}
\tag{59, 60}
\]

which are equivalently written as (by using (53))

\[
-\bar{Q}(\zeta^o)\bar{\vartheta} = T(\vartheta)(\zeta^o - \zeta^{no}), \tag{61}
\]

where the matrix is given in Appendix C. The estimate of with minimum variance is obtained recursively by the weighted linear constraint optimization problem

\[
\hat{\vartheta}^{(k)}_N = \min_{\hat{\vartheta}} \hat{\vartheta}^TQ(\zeta^0_N)W(\hat{\vartheta}^{(k-1)}_N)Q(\zeta^0_N)\hat{\vartheta} \tag{62}
\]

subject to \(\Gamma \hat{\vartheta} = \gamma\),

where the weighting matrix is iteratively updated for \(k = 1, 2, \cdots\) according to

\[
W(\hat{\vartheta}^{(k-1)}_N) = T^{-1}(\hat{\vartheta}^{(k-1)}_N)P^{-1}(\hat{\vartheta}^{(k-1)}_N)T^{-1}(\hat{\vartheta}^{(k-1)}_N), \tag{64}
\]

where is updated according to

\[
P^{-1}(\hat{\vartheta}^{(k-1)}_N) = \frac{1}{N} \sum_{t=n+1}^{N} \varphi(t)\bar{\varepsilon}^{-1}(\hat{\vartheta}^{(k-1)}_N)[\varphi(t)]^T. \tag{65}
\]

Similar to Step 2, this optimization problem can be solved through

\[
\begin{bmatrix}
\hat{\vartheta}^{(k)}_N \\
\hat{\lambda}^{(k)}_N
\end{bmatrix} = \begin{bmatrix}
Q^T(\zeta^0_N)W(\hat{\vartheta}^{(k-1)}_N)Q(\zeta^0_N) & \Gamma^T \\
\Gamma & 0
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
\gamma
\end{bmatrix}, \tag{66}
\]

where are the estimated Lagrange multipliers. Finally, the covariance of the residuals is updated according to (57).

Remark 3: Although this step is asymptotically efficient, iterating may improve the estimate for finite data length \(N\). The cost

\[
V_N(\vartheta) = \frac{1}{N} \det \sum_{t=1}^{N} \bar{\varepsilon}^T(t, \vartheta)\bar{\varepsilon}(t, \vartheta). \tag{67}
\]

is evaluated at each iteration to decide whether the parameter estimation has improved. However, as is not affine in the parameters, an improved cost may still result in deteriorated parameter estimates. The cost is used as it is independent of \(\Lambda(\vartheta)\) and under Gaussian assumptions, results in minimum variance of the estimates if \(\Lambda(\vartheta)\) is independently parametrized from \(A(q^{-1}, \vartheta), B(q^{-1}, \vartheta),\) and \(F(q, \vartheta)\), because then the asymptotic (minimum) variance resulting from (31) is equal to the asymptotic variance of the maximum likelihood estimator [8].
Step 4: Obtaining the noise model

With $A(q^{-1}, \hat{\theta}_N)$ and $B(q^{-1}, \hat{\theta}_N)$, the dynamics of the physical network have been estimated. In this step, $C(q^{-1}, \theta)$ and $\Lambda(\theta)$ are estimated from $A(q^{-1}, \theta)$, $C(q^{-1}, \eta)$, and $\Lambda(\theta)$. As $C(q^{-1}) = A_0(q^{-1})A_0^T$, $C(q^{-1}, \theta)$ is estimated by

$$C(q^{-1}, \hat{\theta}_N^k) = C(q^{-1}, \eta_N^k)A_0^{-1}(\hat{\theta}_N^k).$$

Further, as $A_0 = A_0^T\bar{A}_0^T$, $\Lambda(\theta)$ is estimated by

$$\Lambda(\hat{\theta}_N^k) = A_0(\hat{\theta}_N^k)\bar{A}(\hat{\theta}_N^k)A_0(\hat{\theta}_N^k).$$  

Step 5: Estimating the continuous-time parameters

With $A(q^{-1}, \hat{\theta}_N)$ and $B(q^{-1}, \hat{\theta}_N)$ from Step 3, the dynamics of the discrete-time physical network have been estimated. The dynamics of the continuous-time physical network, represented by $A(\frac{d}{dt}, \hat{\theta}_N^k)$ and $B(\frac{d}{dt}, \hat{\theta}_N^k)$, are obtained through the inverse mapping of $\hat{X}_N$, given by

$$\tilde{A}_{jk,i} = (-T_s)^i \sum_{q=0}^{n_q} (q^i) A_{jk,i},$$  

$$\tilde{B}_{jj,i} = (-T_s)^i \sum_{q=0}^{n_q} (q^i) B_{jj,i}. $$

The complete algorithm

The above steps describe the procedure for identifying the parameters of a physical network with an ARMAX-like model structure. This procedure leads to the following algorithm.

**Algorithm 1 (ARMAX-like model structure):** Consider a data generating system $S$ with $F^0(q) := C^0(q^{-1})$ a monic polynomial and a physical network model set $M$ with $F(q, \theta) := C(q^{-1}, \theta)$ a monic polynomial. Then $\bar{M}(\theta_N)$, a consistent estimate of $M^0$, is obtained through the following steps:

1. Estimate the nonparametric ARX model [42] by least squares [48] to obtain $\hat{\zeta}_N^0$.
2. Reduce the nonparametric ARX model to a parametric model [11] by weighted least-squares [56] to obtain $\hat{\theta}_N^0$.
3. Improve the parametric model [11] by weighted least-squares [56] to obtain $\hat{\theta}_N^k$ for $k = 1, 2, \ldots$.
4. Obtain the noise model by calculating [68] and [69] to obtain $C(q^{-1}, \hat{\theta}_N^k)$ and $\Lambda(\hat{\theta}_N^k)$.
5. Estimate the continuous-time parametric model from the discrete-time parametric model through [70] and [71] to obtain $\tilde{A}(\frac{d}{dt}, \hat{\theta}_N^k)$ and $\tilde{B}(\frac{d}{dt}, \hat{\theta}_N^k)$.

Consistency and minimum variance of the estimates obtained with Algorithm 1 follows from the similarity with WNSF and its proof [26]. The main difference is that $A(q^{-1}, \theta)$ is nonmonic and symmetrically parametrized, resulting in a different structure in [53]. For consistency, $Q(\zeta_N^0)$ needs to have full column rank, which can be shown to be satisfied if the identifiability conditions in Proposition 6 are satisfied. Consistency of Step 4 follows naturally.

**Remark 4:** In order to perform Algorithm 1, the measured data $\{w(t), r(t)\}$ is needed; the order $n$ of the ARX model needs to be chosen; and the true orders $n_x$, $n_y$, and $n_c$ of $A(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$, respectively, need to be known.

**Remark 5 (Simplification to an ARX-like model structure):** If the noise is not filtered, that is $F(q) := C(q^{-1}) = I$, the physical network has an ARX-like model structure and the ARX model [42] can exactly describe the physical network, where $A(q^{-1})$ and $B(q^{-1})$ are of the same order as $A(q^{-1})$ and $B(q^{-1})$, respectively. Algorithm 1 improves in the sense that Step 1 is consistent for sufficiently large data length $N$ and therefore, no additional estimation error is made in Step 2, which makes Step 3 superfluous.

**Remark 6 (Simplification to an ARX model structure):** If $A_0 = I$ in addition to unfiltered noise ($F(q) := C(q^{-1}) = I$), the physical network has an ARX model structure. In this case, the physical network can consistently be identified in a single step, by incorporating the symmetric structure in Step 1 of Algorithm 1 and by choosing the order of $\bar{A}(q^{-1})$ and $\bar{B}(q^{-1})$ equivalent to the order of $A(q^{-1})$ and $B(q^{-1})$, respectively. The resulting identification procedure has been described in [9].

VI. SIMULATION EXAMPLE

This section contains a simulation example that serves to illustrate the theory and to show that indeed the topology and the parameters of a physical network can be identified using a single excitation signal only. The identification is performed with the algorithm presented above.

A. Experimental setup

Consider the continuous-time physical network [2] consisting of four one-dimensional nodes, with external signal $u(t) = B_0 r(t) + v(t)$, described by

$$\tilde{A}_w(t) + \tilde{A}_1 \frac{d}{dt} w(t) + \tilde{A}_2 \frac{d^2}{dt^2} w(t) = B_0 r(t) + v(t),$$

where $\tilde{A}_i = X_i + Y_i$, $r(t)$ is one-dimensional and known, and $B_0$ has dimension $4 \times 1$ and has only the first element nonzero. Figure 3 shows the structure of this physical network, where it can be seen that the excitation signal $r(t) = r_1(t)$ enters the network only at node $w_1$. One can think of this network as a mechanical mass-spring-damper network as explained in Section II where $X_0$ and $Y_0$ contain the spring constants, $X_1$ and $Y_1$ contain the damper coefficients, $X_2$ contains the masses, the node signals $w(t)$ represent the positions of the masses, and the excitation signal $r(t)$ is a force. One can also
think of this network as an electrical circuit with nodes that are interconnected through capacitors, resistors, and inductors (in parallel). The matrices $X_0$ and $Y_0$ contain the capacitances, $X_1$ and $Y_1$ contain the conductance values of the resistors, $X_2$ contain the inverses of the inductances, the node signals $w(t)$ represent the electric potentials of the interconnection points, and the excitation signal $r(t)$ is the derivative of a current flow (in this case, the external signal would typically be of the form $u(t) = B_0 \frac{d}{dt} i(t) + v(t)$ with $i(t)$ the current flow).

The discrete-time representation is obtained by applying Proposition 1 with sampling frequency $f_s = 100$ Hz. In addition, the disturbance $v(t)$ acting on the network is modeled in discrete time as a white noise filtered by a first order filter. This results in the discrete-time physical network model (11) $[A_0 + A_1 q^{-1} + A_2 q^{-2}]w(t) = B_0 r(t) + [I + C_1 q^{-1}]e(t)$. (73)

The network topology is assumed to be unknown reflected by the situation that in the model there are parametrized second order connections between all pairs of nodes. The location where $r(t)$ enters and the first nonzero parameter of $B_0$ are assumed to be known, which induces that $B_0$ is fixed and not parametrized. This guarantees that the identifiability conditions 2 and 4 in Proposition 6 are satisfied.

The symmetric structure of $A_0(q^{-1})$ and $A_1(q^{-1})$ is taken into account in the parametrization of the continuous-time model and discrete-time model, respectively. The continuous-time model matrices (72) and the discrete-time model matrices (73) are, respectively, parametrized as

$$A_0 = \begin{bmatrix} \theta_1^c & \theta_1^f & \theta_1^g & \theta_1^h & \theta_1^i \\ \theta_2^c & \theta_2^f & \theta_2^g & \theta_2^h & \theta_2^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_n^c & \theta_n^f & \theta_n^g & \theta_n^h & \theta_n^i \end{bmatrix}, \quad A_1 = \begin{bmatrix} \theta_1^c & \theta_1^f & \theta_1^g & \theta_1^h & \theta_1^i \\ \theta_2^c & \theta_2^f & \theta_2^g & \theta_2^h & \theta_2^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_n^c & \theta_n^f & \theta_n^g & \theta_n^h & \theta_n^i \end{bmatrix}.$$

TABLE I

| Parameter | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ | $\theta_7$ | $\theta_8$ |
|-----------|------------|------------|------------|------------|------------|------------|------------|------------|
| True value | 10.57 x 10^{-3} | -2.68 x 10^{-3} | 0.001 | 0 | 0.001 | 0 | 0.001 | 0 |
| Mean | 10.57 x 10^{-3} | -2.68 x 10^{-3} | 0.001 | 0 | 0.001 | 0 | 0.001 | 0 |
| Variance | 8.35 x 10^{-4} | 3.94 x 10^{-4} | 1.87 x 10^{-6} | 4.48 x 10^{-14} | 7.09 x 10^{-3} | 4.49 x 10^{-9} | 1.24 x 10^{-8} | 7.33 x 10^{-6} |

TABLE II

| Parameter | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ | $\theta_7$ | $\theta_8$ |
|-----------|------------|------------|------------|------------|------------|------------|------------|------------|
| True value | 13.3 | -23.0 | 10.0 | -3.3 | 0 | 0 | 0 | 0 |
| Mean | 13.3 | -23.0 | 10.0 | -3.3 | 0 | 0 | 0 | 0 |
| Variance | 5.19 x 10^{-4} | 4.31 x 10^{-5} | 1.78 x 10^{-3} | 0.04234 | 0.02818 | 2.232 x 10^{-5} | 0.1116 | 0.08815 |

The network topology is assumed to be unknown reflected by the situation that in the model there are parametrized second order connections between all pairs of nodes. The location where $r(t)$ enters and the first nonzero parameter of $B_0$ are assumed to be known, which induces that $B_0$ is fixed and not parametrized. This guarantees that the identifiability conditions 2 and 4 in Proposition 6 are satisfied.
The true exact parameter values, represented by $\theta^{0}$ and $\theta^{0}$ for the continuous-time and discrete-time model parameters, respectively, are given in Table I and Table II.

The external excitation signal $r_{1}(t)$ is an independent white noise process with mean 0 and variance $\sigma_{r}^{2} = 1$. All nodes are subject to disturbances $e_{i}(t)$, which are independent white noise processes (uncorrelated with $r_{1}(t)$) with mean 0 and variance $\sigma_{e}^{2} = 10^{-4}$. The order of the ARX model in Step 1 of the algorithm is $n = 4$. In Step 2 of the algorithm, the possibility to apply the weighting $W(\zeta(N)) = P^{-1}(\zeta(N))$ is exploited. In Step 3 of the algorithm, at most 50 iterations are allowed to improve the result of Step 2. However, only one or none of these iterations are executed due to numerical issues that are caused by $P(\hat{\theta}_{N})$ being ill-conditioned.

The left figure in Figure 5 shows a Boxplot of the relative parameter estimation errors for the discrete-time model parameters (left) and continuous-time model parameters (right). Missing relative parameter errors indicate that the true parameter value is 0, which is related to an absent component or interconnection in the network. Table III contains the mean and variance of the estimated continuous-time and discrete-time model parameters, respectively. From these tables it can be seen that parameters with a true value equal to 0 are estimated with mean values close to 0 and small variance. Table III shows that the parameters of a higher order polynomial matrix have smaller variance. That is, the parameters of $A_{2}$ have the smallest variance and the parameters of $A_{0}$ have the largest variance, even if the true values are in the same order of magnitude.

The simulation results for the second experiment are shown in Figure 4 and in Table I and Table II. Figure 4 shows the relative parameter estimation errors for the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right). The simulation results for the second experiment are shown in Figure 4. It shows a Boxplot of the relative mean squared error (RMSE) of the discrete-time model parameters (left) and continuous-time model parameters (right).
VII. DISCUSSION

In this section, three extensions of the presented theory are discussed. First, the connection with dynamic networks is made. Second, networks with unmeasured nodes are considered. Third, parameter constraints are discussed.

A. Dynamic networks

A commonly used description of dynamic networks is the module representation \( \{11\} \), in which a network is considered to be the interconnection of directed transfer functions (modules) through measured node signals as

\[
 w(t) = G(q)w(t) + R(q)r(t) + H(q)\hat{e}(t),
\]

with white noise process \( \hat{e}(t) \) and with proper rational transfer function matrices \( G(q), R(q), \) and \( H(q) \in \mathcal{F} \), where the matrix entries \( G_{jk}(q), R_{jk}(q), \) and \( H_{jk}(q) \) describe the dynamics in the paths from \( w_k(t), r_k(t), \) and \( \hat{e}_k(t) \) to \( w_j(t) \), respectively. A physical network \( \{11\} \) can be described as a module representation with the following particular symmetrical properties: \( \{9\} \)

- The transfer functions \( G_{jk}(q) \) and \( G_{kj}(q) \) have the same numerator for all \( j, k \).
- The transfer functions \( G_{jk}(q) \) and \( R_{jm}(q) \) have the same denominator for all \( k, m \).
- The transfer functions \( G_{jk}(q) \) and \( H_{jm}(q) \) have the same denominator for all \( k, m \) if \( F(q) \) is polynomial.

Moreover, conditions for a unique mapping between a module representation and a physical network are formulated in \( \{9\} \). The structure of \( G(q) \) and \( R(q) \) corresponding to a physical network with three nodes is illustrated by Figure 6. It shows that the modules \( G_{jk}(q) = -a_{jk}^{-1}(q^{-1})a_{jk}(q^{-1}) \) and \( G_{kj}(q) = -a_{kj}^{-1}(q^{-1})a_{jk}(q^{-1}) \) have the same numerator.
Fig. 6. A module representation of a physical network with three nodes.

\((a_{jk}(q^{-1}))\) and all transfer functions in the paths towards a specific node \(w_j\) have the same denominator \((a_{jj}(q^{-1}))\). Since \(G_{jk}(q)\) and \(G_{kj}(q)\) have the same numerator, they will either be both present or both absent, which is in accordance with the fact that they represent a single physical interconnection. In addition, the connections to the ground node are only present in the denominators, because they are only present in \(a_{jj}(q^{-1})\). This means that they do not have an effect on the topology in the module representation, although they are part of the topology in the physical network.

**B. Partial measurements**

Throughout this paper we assumed that all node signals are measured, although this is not always possible. As mentioned in Section II-A, unmeasured node signals can be removed from the representation by Gaussian elimination, which is equivalent to Kron reduction [2] and immersion [12]. The resulting network representation does not necessarily directly have a polynomial form with a symmetric \(A(q^{-1})\). However, with a unique additional step, it can always be transformed into a representation that satisfies these physical network properties, see Definition [9].

Thus, the experimental setting considered in this paper is that all node signals in the physical network are measured. As a result, at least one excitation signal is necessary for identifiability, see Proposition [6]. Observe that this excitation always excites a measured node. The dual problem considers the experimental setting where all nodes are excited. For this problem the identifiability conditions change in the sense that at least one node signal needs to be measured instead of that at least one excitation signal needs to be present (condition 3 of Proposition [6]). In practice, this dual situation is rare as it is uncommon that all nodes can be excited in a physical network. Therefore, the focus is on the situation where all nodes are measured. In the practical situation where only a subset of node signals can be excited and only a subset of nodes can be measured, it can be shown that identifiability can be achieved if all nodes are excited or measured and one node is both excited and measured. This is in accordance with the literature [3], [5].

**C. Parameter constraints**

Physical networks that consist of interconnected physical components, such as mass-spring-damper systems and RLC circuits, are known to have positive real-valued coefficient values in the continuous-time representation [2]. This is because the coefficients in the continuous-time network model \(\hat{A}(q^{-1}, \zeta^n)\) represent the values of the physical component in the network. This also leads to coefficients with known signs in the corresponding discrete-time representation. In both the theoretical consistency proof and the practical algorithm, these sign constraints are not taken into account, which means that the theory presented in this paper also holds for networks without these sign constraint.

The sign constraints can be taken into account in the algorithm by adding inequality constraints of the form \(\Gamma_u \vartheta < 0\) to the optimization problems \((54)\) and \((62)\). As explained before, known parameter values can easily be taken into account by the equality constraint \(\Gamma \vartheta = \gamma\) in the optimization problems \((54)\) and \((62)\). Known (continuous-time) component values can be taken into account as well, by splitting \(\vartheta\) as \(\vartheta = \vartheta_u + \vartheta_k\), where \(\vartheta_u\) and \(\vartheta_k\) represent the unknown and known part of \(\vartheta\), respectively. Then the linear form \(-Q(\zeta^n)\vartheta = 0\) leads to \(-Q(\zeta^n)\vartheta_u + \gamma_k(\zeta^n, \vartheta_k) = 0\), where \(\gamma_k(\zeta^n, \vartheta_k) := -Q(\zeta^n)\vartheta_k\) is known.

**VIII. Conclusion**

The undirected network description of physical networks has been extended by allowing for higher order diffusive couplings. Undirected network descriptions of physical systems with diffusive couplings can be represented as polynomial systems with particular structural properties. This allows for effective identification of the global dynamics and topology of the physical network, for which only a single excitation signal is needed. The identification is performed through a multi-step algorithm that relies on convex optimizations and the results of identifying the topology and parameters are illustrated in a Monte Carlo simulation example. It is shown how undirected physical networks form a particular subclass of directed (module) dynamic networks with structural constraints. The considered situation of all network nodes being measured can be extended to the situation of a selected set of measured and excited node signals.

**APPENDIX**

**A. Parametrization**

The model structure \(\hat{A}(q^{-1}, \zeta^n)\) and \(\hat{B}(q^{-1}, \zeta^n)\) of the non-parametric ARX model \((42)\) is parametrized in terms of the parameters \(\zeta^n\). The parameter vector \(\zeta^n := \begin{bmatrix} \zeta^n_a \top & \zeta^n_b \top \end{bmatrix} \top\) is given by

\[
\zeta^n_a = \begin{bmatrix} \zeta^n_{a_1} \\ \vdots \\ \zeta^n_{a_L} \end{bmatrix}, \quad \zeta^n_b = \begin{bmatrix} \zeta^n_{b_1} \\ \vdots \\ \zeta^n_{b_L} \end{bmatrix}, \quad \zeta^n_{a_{ij}} = \begin{bmatrix} \hat{a}_{1,ij} \\ \hat{a}_{2,ij} \\ \vdots \\ \hat{a}_{n,ij} \end{bmatrix},
\]

(80)

\[
\zeta^n_b = \begin{bmatrix} \hat{b}_{0,ij} \\ \hat{b}_{1,ij} \\ \vdots \\ \hat{b}_{n,ij} \end{bmatrix},
\]

(81)

The model structure \(A(q^{-1}, \theta_a), B(q^{-1}, \theta_b),\) and \(C(q^{-1}, \theta_c)\) of the physical network model \((41)\) is parametrized in
terms of the parameters $\theta$ and $\eta$, where $C'(q^{-1}, \eta_c) = C(q^{-1}, \eta_c) A_0(\theta_a)$. The parameter vectors $\theta_a$, $\theta_b$, $\theta_c$, and $\eta_c$ are given by

$$
\begin{align*}
\theta_a &= \begin{bmatrix} \theta_{a_1} \\ \vdots \\ \theta_{a_{n_A}} \end{bmatrix}, & \theta_b &= \begin{bmatrix} \theta_{b_1} \\ \vdots \\ \theta_{b_{n_B}} \end{bmatrix}, & \theta_c &= \begin{bmatrix} \theta_{c_1} \\ \vdots \\ \theta_{c_{n_C}} \end{bmatrix}, & \eta_c &= \begin{bmatrix} \eta_{c_1} \\ \vdots \\ \eta_{c_{n_C}} \end{bmatrix}, \\
\theta_{a_{(i+1)j}} &= \begin{bmatrix} \theta_{a_{(i+1)j}} \\ \vdots \\ \theta_{a_{(n_A+1)j}} \end{bmatrix}, & \theta_{a_{ij}} &= \begin{bmatrix} a_{0,ij} \\ a_{1,ij} \\ \vdots \\ a_{n_A,ij} \end{bmatrix}, & a_{0,ij} &= \begin{bmatrix} a_{0,ij} \\ a_{1,ij} \\ \vdots \\ a_{n_A,ij} \end{bmatrix}
\end{align*}
$$

and observe that $\Pi^a_\circ$ has dimensions $L_n \times (n_A + 1)$ and that $\Pi^b_\circ$ has dimensions $(n+1) \times (n_A + 1)$. For $x \in \{a, b\}$, define the block matrix

$$
\Pi^x_\circ := \begin{bmatrix} Z^x_{0,L} & Z^x_{1,L-1} & \cdots & Z^x_{L-1,1} \\
R(\Pi^x_{1,L}) & R(\Pi^x_{2,L}) & \cdots & R(\Pi^x_{L-2,1}) \\
\vdots & \vdots & \ddots & \vdots \\
S_{L-1}(\Pi^x_{1,L}) & S_{L-2}(\Pi^x_{2,L}) & \cdots & S_{L-1}(\Pi^x_{L-2,1}) 
\end{bmatrix},
$$

with $Z^a_i$ an $i \times j$ block matrix with blocks $0_{L,n_A(n_A+1)}$ and with $Z^b_i$ an $i \times j$ block matrix with blocks $0_{K(n+1),n_A(n_A+1)}$ (that is $Z^a_i = 0_{L,n_A(n_A+1)}$ and $Z^b_i = 0_{K(n+1),n_A(n_A+1)}$), with

$$
R(\Pi^x_{i,j}) := \begin{bmatrix} \Pi^x_{i-1,j} & \cdots & \Pi^x_{i,j+1} \end{bmatrix}^T, \quad \text{for } i \leq j,
$$

and with

$$
S_i(\Pi^x_{i,j}) := \begin{bmatrix} Z^x_{1,i} & D_i(\Pi^x_{i,j}) \end{bmatrix},
$$

with $D_i(\Pi^x_{i,j})$ a block diagonal matrix consisting of $i$ blocks of $\Pi^x_{i,j}$.

Observe that $A(q^{-1})$ is parametrized symmetrically and that $\bar{c}_{0,i,j} = a_{0,i,j}$ and therefore is parametrized as such. Remember that $\theta = \begin{bmatrix} \theta^a_T & \theta^b_T & \eta_c^T \end{bmatrix}^T$.

B. Matrix $[\varphi^n(t)]^T$

The regressor $[\varphi^n(t)]^T$ in (47) is given by $[\varphi^n(t)]^T = \begin{bmatrix} [\varphi^n_0(t)]^T \\ [\varphi^n_1(t)]^T \end{bmatrix}$ with

$$
|\varphi^n_0(t)]^T = \begin{bmatrix} \varphi^n_0(y_1) \\ \varphi^n_0(y_2) \\ \vdots \\ \varphi^n_0(y_L) \end{bmatrix}^T,
$$

$|\varphi^n_1(t)]^T = \begin{bmatrix} \varphi^n_1(y_1) \\ \varphi^n_1(y_2) \\ \vdots \\ \varphi^n_1(y_L) \end{bmatrix}^T$,

$|\varphi^n_2(t)]^T = \begin{bmatrix} \varphi^n_2(y_1) \\ \varphi^n_2(y_2) \\ \vdots \\ \varphi^n_2(y_L) \end{bmatrix}^T$,

$|\varphi^n_{n_C}(t)]^T = \begin{bmatrix} \varphi^n_{n_C}(y_1) \\ \varphi^n_{n_C}(y_2) \\ \vdots \\ \varphi^n_{n_C}(y_L) \end{bmatrix}^T$.

C. Matrices $Q(\zeta^n)$ and $T(\theta)$

In order to construct $Q(\zeta^n)$ and $T(\theta)$, we first define some other matrices.

1) Zero and identity: Let $0_{i,j}$ denote a matrix of dimension $i \times j$ with all its elements equal to 0. Let $I_{i,j}$ denote an identity matrix of dimension $i \times j$, where $I_{i,j} = \begin{bmatrix} I_{i,i} & 0_{i,j-1} \end{bmatrix}$ for $i \leq j$ and $I_{i,j} = \begin{bmatrix} 0_{j-i,j} & I_{j,j} \end{bmatrix}^T$ for $i > j$. Let $L_{k(i,j)}$ denote a block diagonal matrix of $k$ blocks of $I_{i,j}$ and let $L_{k(i,j)}$ denote a block diagonal matrix of $\ell$ blocks of $I_{k(i,j)}$.

2) $\Pi$: Define the matrices

$$
\begin{align*}
\Pi^a_i &= \begin{bmatrix} \zeta^n_{n_A} & 0_{n_A,n_A} \\
\cdots \\ -\zeta^n_{n_{n_A}} \\ \zeta^n_{n_{n_A+1}} & 0_{n_{n_A+1},n_{n_A+1}} \\
\cdots & \cdots \\ \zeta^n_{n_{n_A+1}} & 0_{n_{n_A+1},n_{n_A+1}} \\
\end{bmatrix}, & \Pi^b_i &= \begin{bmatrix} \zeta^n_{0,n_A} & 0_{0,n_A} \\
\cdots \\ \zeta^n_{n_{n_A}-1} & 0_{n_{n_A}-1,n_A} \\
\cdots & \cdots \\ \zeta^n_{n_{n_A}} & 0_{n_{n_A},n_A} \\
\end{bmatrix}
\end{align*}
$$

and $\Pi^a_\circ$ has dimensions $L_n \times (n_A + 1)$ and that $\Pi^b_\circ$ has dimensions $(n+1) \times (n_A + 1)$. For $x \in \{a, b\}$, let $\bar{T}_{m(k,e)}(\bar{X})$ denote a block diagonal matrix consisting of $m$ blocks of $T_{k,e}(\bar{X})$. Observe that $\bar{T}_{n,n_c}(A)$ has dimensions $L_n \times L_{n_c}$ and that $\bar{T}_{n+1,n_c}(B)$ has dimension $K(n+1) \times L_{n_c}$.

Let $\bar{T}_{m(k,e)}(\bar{c}_{ij})$ denote a block diagonal matrix consisting of $m$ blocks of $\bar{T}_{k,e}(\bar{c}_{ij})$. Observe that $\bar{T}_{L,(n_{n_c})}(\bar{c}_{ij})$ is an $L \times L_n$ block diagonal matrix consisting of $L$ blocks of $\bar{T}_{n,n_c}(\bar{c}_{ij})$ and

$$
\begin{align*}
\bar{T}_{k,e}(A) &= \begin{bmatrix} T_{k,e}(A_{11}) & \cdots & T_{k,e}(A_{1L}) \\
\vdots & \ddots & \vdots \\
T_{k,e}(A_{L1}) & \cdots & T_{k,e}(A_{LL}) 
\end{bmatrix},
\bar{T}_{k,e}(B) &= \begin{bmatrix} T_{k,e}(B_{11}) & \cdots & T_{k,e}(B_{1L}) \\
\vdots & \ddots & \vdots \\
T_{k,e}(B_{L1}) & \cdots & T_{k,e}(B_{LL}) 
\end{bmatrix}
\end{align*}
$$

where $\bar{T}_{k,e}(\bar{A})$ has dimensions $Lk \times L\ell$ and $\bar{T}_{k,e}(\bar{B})$ has dimensions $Kk \times L\ell$.

For $x \in \{a, b\}$, let $\bar{T}_{m(k,e)}(\bar{X})$ denote a block diagonal matrix consisting of $m$ blocks of $T_{k,e}(\bar{X})$. Observe that $\bar{T}_{n,n_c}(A)$ has dimensions $L_n \times L_{n_c}$ and that $\bar{T}_{n+1,n_c}(B)$ has dimension $K(n+1) \times L_{n_c}$.
that } \bar{T}_{K(n+1,n+1)}(\bar{c}_{ij}) \text{ is an } K(n+1) \times K(n+1) \text{ block diagonal matrix consisting of } K \text{ blocks of } \bar{T}_{n+1,n+1}(\bar{c}_{ij}). \text{ Finally, define }

\begin{align}
T_{m(k,t)}(C) := \begin{bmatrix}
T_{m(k,t)}(\bar{c}_{11}) & \cdots & T_{m(k,t)}(\bar{c}_{1L}) \\
\vdots & & \vdots \\
T_{m(k,t)}(\bar{c}_{n1}) & \cdots & T_{m(k,t)}(\bar{c}_{nL})
\end{bmatrix}.
\end{align}

(100)

4) Matrix } Q(\zeta^n) \text{: With the matrices defined above, we can now describe the matrix } Q(\zeta^n) \text{ in (53) by }

Q(\zeta^n) = \begin{bmatrix}
\Pi_A^n & 0 \\
\Pi_L^n & -I_{L(K(n+1,m))}
\end{bmatrix} \begin{bmatrix}
\bar{T}_{L(n,n)}(\bar{A}) \\
\bar{T}_{L(n+1,n+1)}(\bar{B})
\end{bmatrix},

(101)

which has row dimension } Ln^2 + LK(n+1) \text{ and column dimension } n_1L(n_1+1) + LK(n_B+1) + L^2n_C.

5) Matrix } \bar{T}(\vartheta) \text{: With the matrices defined above, we can now describe the matrix } T(\vartheta) \text{ in (61) by }

T(\vartheta) = \begin{bmatrix}
-\bar{T}_{L(n,n)}(\bar{C}) & 0 \\
0 & -\bar{T}_{K(n+1,n+1)}(\bar{C})
\end{bmatrix},

(102)

which has dimensions } [Ln^2 + LK(n+1)] \times [Ln^2 + LK(n+1)].

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