Second and first order phase transition in three dimension Gross-Neveu model*

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Symmetry restoring phase transitions in three dimension Gross-Neveu model are shown to be second order at finite temperature $T$ and first order at $T = 0$ and finite chemical potential $\mu$ by critical analysis of the dynamical fermion mass based on the gap equation. The latter is further verified by effective potential analysis. The resulting tricritical point is $(T, \mu) = (0, m(0))$, where $m(0)$ is the dynamical fermion mass at $T = \mu = 0$. Physical difference between the above second and first order phase transition is illustrated by means of variations of thermodynamical particle density.

Keywords: thermal gap equation and effective potential, symmetry breaking and restoration, second and first order phase transition, tricritical point, particle density

The Gross-Neveu (GN) model is a good laboratory to research symmetry restoring phase transitions at high temperature and high density. One believes that some phase structure of Quantum Chromodynamics (QCD), e.g. a tricritical point in temperature-$\mu$ phase diagram, i.e. the point where second order and first order phase transition meets, could appear in such type of models. Existence of a tricritical point is of both theoretical and experimental significance. Since the three dimension (3D) GN model is easy to deal with analytically, and in addition, in this model there is still a question to be clarified further of that whether first order phase transition exists or not in it, it is interesting to make a careful analysis of the phase transition of this model at finite temperature, especially at high density. In this paper, we will report our results and indicate that a tricritical point exists indeed in 3D GN model.

Our method is to analyze critical behaviors of the dynamical fermion mass as the order parameter of symmetry breaking based on the gap equation obeyed by the dynamical fermion mass and on the corresponding effective potential at finite temperature and chemical potential. The discussions will be made in the fermion bubble diagram approximation which is equivalent to the leading order of $1/N$ expansion.

We first briefly review the used model. The Lagrangian will be written by

$$\mathcal{L}(x) = \sum_{k=1}^{N} \bar{\psi}^k(x) i\gamma^\mu \partial_\mu \psi_k(x) + \frac{g}{2} \sum_{k=1}^{N} \bar{\psi}^k(x) \psi^k(x),$$

(1)

where $\psi_k(x)$ is a spinor with $N$ "color" components. In 3 dimensions, the coupling constant $g$ has mass dimension -1 thus the model is perturbatively non-renormalizable however it is renormalizable in $1/N$ expansion. We take the $\gamma^\mu$ $(\mu = 0, 1, 2)$ to be $2 \times 2$ matrices and relate them to the Pauli matrices $\sigma^i (i = 1, 2, 3)$ by $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$ and $\gamma^2 = i\sigma^2$. Noting that in three dimensions there is no "$\gamma_5$" matrix which anticommutes with all the three $\gamma^\mu$ matrices. Consequently $\mathcal{L}(x)$ has no chiral symmetry. However, the action of the system is invariant under the special $\mathcal{P}_1$ and $\mathcal{P}_2$ parity transformation (omitting the index $k$ of $\psi$)

$$\psi(t, x^1, x^2) \overset{\mathcal{P}_1}{\rightarrow} \gamma^1 \psi(t, -x^1, x^2),$$

$$\psi(t, x^1, x^2) \overset{\mathcal{P}_2}{\rightarrow} \gamma^2 \psi(t, x^1, -x^2)$$

and the time reversal

$$\psi(t, x^1, x^2) \overset{\mathcal{T}}{\rightarrow} \gamma^0 \psi(-t, x^1, x^2).$$

If the four-fermion interactions in Eq.(1) will lead to the dynamical mass term $-m(0)\psi$, then the discrete symmetries $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{T}$ will be spontaneously broken. These broken symmetries, however, could be restored at high temperature and/or high density. This is just the problem we will discussed.

At zero-temperature $T = 0$ and zero chemical potential $\mu = 0$, the four-fermion interactions $g \sum_{k=1}^{N} (\bar{\psi}^k \psi_k)^2$ can lead to the fermion condensates $\left< \sum_{k=1}^{N} (\bar{\psi}^k \psi_k) \right> \neq 0$, then we will obtain the gap equation obeyed by the dynamical

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fermion mass \( m(0) \)

\[
1 = 2gN \int \frac{id^3l}{(2\pi)^3} \frac{1}{l^2 - m^2(0) + i\varepsilon} \tag{2}
\]

After the Wick rotation, angular integration and introduction of 3D Euclidean momentum cut-off \( \Lambda \), Eq.(2) becomes

\[
1 = \frac{gN\Lambda}{\pi^2} \left[ 1 - \frac{m(0)}{\Lambda} \arctan \frac{\Lambda}{m(0)} \right]. \tag{3}
\]

For sufficiently strong coupling \( gN \) that \( gN\Lambda/\pi^2 > 1 \), Eq.(3) can be satisfied. When \( T \neq 0 \), we must replace the vacuum expectation value \( \sum_{k=1}^{N}(\bar{\psi}^k\psi_k) \) by the thermal expectation value \( \sum_{k=1}^{N}(\bar{\psi}^k\psi_k)_T \). In the real-time formalism of thermal field theory, this means the substitution of the fermion propagator \( \bar{\psi}^k\psi_k \)

\[
\frac{i}{l - m(0) + i\varepsilon} \rightarrow \frac{i}{l - m + i\varepsilon} - 2\pi\delta(l^2 - m^2)
\times (l + m) \sin^2 \theta(l^0, \mu) \tag{4}
\]

with

\[
\sin^2 \theta(l^0, \mu) = \frac{\theta(l^0)}{\exp[\beta(l^0 - \mu)] + 1} + \left( l^0 \rightarrow -l^0, \mu \rightarrow -\mu \right) \tag{5}
\]

where \( \beta = 1/T \) and \( m \equiv m(T, \mu) \) is the dynamical fermion mass at finite temperature \( T \) and finite chemical potential \( \mu \). As a result, the gap equation at \( T \neq 0 \) becomes

\[
1 = \frac{gN\Lambda}{\pi^2} \left[ 1 - \frac{m}{\Lambda} \arctan \frac{\Lambda}{m} - \frac{\pi}{2\Lambda} F_2(T, \mu, m) \right], \tag{6}
\]

Substituting Eq.(9) with \( \mu_c \) replaced by \( \mu \) into Eq.(10) we will obtain the critical behavior of \( m^2 \) near \( T_c \) when \( \mu \) is fixed

\[
m^2 = 2T \left( 1 + \sinh \frac{\mu}{T} \right) \left\{ \ln \left( 2 \left( 1 + \cosh \frac{\mu}{T_c} \right) \right) - \frac{\mu}{T_c} \tanh \frac{\mu}{2T_c} \right\} (T_c - T) \quad \text{when} \quad T \lesssim T_c \tag{11}
\]

Substituting Eq.(9) with \( T_c \) replaced by \( T \) into Eq.(10) we will obtain the critical behavior of \( m^2 \) near \( \mu_c \) when \( T \) is fixed

\[
m^2 = 2T \sinh(\mu/T)(\mu_c - \mu), \quad \text{when} \quad \mu \lesssim \mu_c \quad \text{and} \quad T \neq 0. \tag{12}
\]

where

\[
F_2(T, \mu, m) = T \ln \left[ 1 + e^{-\mu/T} \right] + (-\mu \rightarrow \mu) . \tag{7}
\]

Considering the gap equation (3) at \( T = 0 \) and the fact that the momentum cut-off \( \Lambda \) may be high enough so that \( \arctan \frac{\Lambda}{m} = \frac{\pi}{2} \), we will reduce Eq.(6) to

\[
m(0) = m + F_2(T, \mu, m). \tag{8}
\]

It is easy to verify that \( \partial F_2(T, \mu, m)/\partial T > 0 \) and \( \partial F_2(T, \mu, m)/\partial \mu > 0 \), i.e. \( F_2(T, \mu, m) \) is respectively a monotone increasing function of temperature \( T \) and chemical potential \( \mu \). As a result, it can be seen from Eq.(8) that, as \( T \) and/or \( \mu \) increase, \( m \) must decrease and finally go to zero at a critical point \((T_c, \mu_c)\). The critical equation to determine the point \((T_c, \mu_c)\) will be

\[
m(0) = F_2(T_c, \mu_c, m = 0)
= T_c \left[ \ln(1 + e^{\mu_c/T_c}) + (\mu_c \rightarrow -\mu_c) \right] \tag{9}
\]

In the above discussions, we have actually used an implicit assumption, i.e. \( m \) could go to zero continuously as \( T \) and/or \( \mu \) increase. This just represents the characteristic of a second order phase transition. Hence the \( T - \mu \) phase diagram given by Eq.(9) will be the one of second order phase transition. The critical behavior of \( m \) near a critical point can be derived from Eqs.(8) and (9). Since near a critical point \((T_c, \mu_c)\), \( m \approx 0 \), thus \( m/T \ll 1 \) for a finite temperature \( T \). Then Eq.(8) with Eq. (7) will lead to

\[
m(0) = T \ln \left( 1 + e^\mu/T \right) + T \ln \left( 1 + e^{-\mu/T} \right) + \frac{m^2}{2T(1 + \cosh(\mu/T))} + O \left( \frac{m^3}{T^3} \right). \tag{10}
\]

Substituting Eq.(9) with \( \mu_c \) replaced by \( \mu \) into Eq.(10) we will obtain the critical behavior of \( m^2 \) near \( T_c \) when \( \mu \) is fixed

\[
m^2 = 2T \left( 1 + \cosh \frac{\mu}{T} \right) \left\{ \ln \left( 2 \left( 1 + \cosh \frac{\mu}{T_c} \right) \right) - \frac{\mu}{T_c} \tanh \frac{\mu}{2T_c} \right\} (T_c - T) \quad \text{when} \quad T \lesssim T_c \tag{11}
\]
For $T = 0$, we must come back to the gap equation (8). Considering that

$$
\lim_{T \to 0} F_2(T, \mu, m) = \begin{cases} 
0 & \text{when } \mu \leq m \\
\mu - m & \text{when } \mu > m
\end{cases},
$$

(13)

we can obtain from the $T \to 0$ limit of Eq.(8) that

$$
m(0) = m, \text{ when } \mu \leq m \quad (14)$$

$$
m(0) = \mu, \text{ when } \mu > m \quad (15)
$$

Eq.(14) can be changed into that $m = m(0)$, when $\mu \leq m = m(0)$. Then Eq.(15) will become $m(0) = \mu$, when $\mu > m(0)$, two contradictory formulas each other. This implies that the gap equation can not be satisfied when $\mu > m(0)$ thus there is no symmetry breaking and the order parameter $m$ must be zero. The above result can be summarized as

$$
m = \begin{cases} 
m(0) & \text{when } \mu \leq m(0) \\
0 & \text{when } \mu > m(0)
\end{cases}.
$$

(16)

Eq.(16) indicates that when $T \to 0$, the order parameter $m$ will jump from $m(0)$ down to 0 as $\mu$ crosses over the critical chemical potential $\mu_c = m(0)$, hence the phase transition is first order. This conclusion can also be verified by an effective potential analysis. It can be proven that the gap equation (8) may also come from the extreme value condition of the effective potential $V_{eff}^{(3)}(T, \mu, m)$, i.e. \[ \partial V_{eff}^{(3)}(T, \mu, m) / \partial m = 0, \]

where

$$
\frac{\partial V_{eff}^{(3)}(T, \mu, m)}{\partial m} = \frac{m}{2\pi} \left[ m - m(0) + F_2(T, \mu, m) \right],
$$

(17)

Thus we can obtain from Eq.(17) by integration

$$
V_{eff}^{(3)}(T, \mu, m) = \frac{1}{2\pi} \int_0^m \left[ m' - m(0) + F_2(T, \mu, m') \right] dm',
$$

(18)

which has been normalized to $V_{eff}^{(3)}(T, \mu, m = 0) = 0$. By means of Eq.(13) we may obtain the $T \to 0$ limit of $V_{eff}^{(3)}(T, \mu, m)$

$$
V_{eff}^{(3)}(T = 0, \mu, m) = \frac{1}{12\pi} \left\{ 3m^2[\mu \theta(\mu - m)
-m(0)] + \theta(m - \mu)(2m^3 + \mu^3) \right\}.
$$

(19)

The shape of $V_{eff}^{(3)}(T = 0, \mu, m)$ will vary as $\mu$ increases.

1) $\mu = 0$. $V_{eff}^{(3)}(T = 0, \mu, m)$ will have the maximum point $m = 0$ and the minimum point $m = m(0)$ with $V_{eff}^{(3)}(T = 0, \mu, m(0)) = -\frac{1}{12\pi} m^3(0)$. This simply represents the spontaneous symmetry breaking at $T = 0$.

2) $\mu < m(0)$. $V_{eff}^{(3)}(T = 0, \mu, m)$ has the same maximum point $m = 0$ and the same minimum point $m = m(0)$, but the minimum $V_{eff}^{(3)}(T = 0, \mu, m = m(0)) = \frac{1}{12\pi} [\mu^3 - m^3(0)]$ will rise as $\mu$ increases.

3) $\mu = m(0)$. The effective potential now becomes

$$
V_{eff}^{(3)}(T = 0, \mu = m(0), m) = \frac{1}{12\pi} \theta[m - m(0)] [m^3(0) - 3m^2 m(0) + 2m^3]
$$

which indicates that the total real axis segment $0 \leq m \leq m(0)$ including the original $m = m(0)$ when $\mu < m(0)$ will be the minimum points of $V_{eff}^{(3)}(T = 0, \mu = m(0), m)$.

4) $\mu > m(0)$. $V_{eff}^{(3)}(T = 0, \mu > m(0), m)$ will only have a minimum point $m = 0$ and this implies that the broken symmetries will be restored. Obviously, the critical chemical potential $\mu_c$ should be determined by the condition

$$
\frac{\partial^2 V_{eff}^{(3)}(T = 0, \mu, m)}{\partial m^2} \bigg|_{m=0} = \mu - m(0) = 0,
$$

(20)

i.e. at $\mu_c$, $m = 0$ must change from a maximum point into a minimum point. Eq.(20) gives that $\mu_c = m(0)$. As $\mu$ increases and crosses over $m(0)$, the minimum point $m$ of $V_{eff}^{(3)}(T = 0, \mu, m)$ will jump from $m(0)$ down to 0. This will reproduce Eq.(16) and verify that $(T, \mu) = (0, m(0))$ is a first order phase transition point. On the other hand, it is noted that $(T, \mu) = (0, m(0))$ is also a solution of the critical equation (9) corresponding to second order phase transition when we take $\mu_c = \mu$ and the limit $T_c = T \to 0$. Hence we can reasonably conclude that $(T, \mu) = (0, m(0))$ will be a tricritical point in the $T - \mu$ phase diagram of 3D GN model.

The physical difference between the above second order and first order phase transition can be clearly displayed by variations of thermodynamical particle density near critical points. We will respectively consider the phase transitions at $T \neq 0$ and $T = 0$. The general total density of fermions with mass $m$ in 3 dimensions may be expressed by
\[ n(T, \mu, m) = \int \frac{d^3 p}{(2\pi)^2} \left[ \frac{1}{e^{\beta(\sqrt{p^2 + m^2} - \mu)} + 1} - (-\mu \to \mu) \right] \]
\[ = \frac{m^2}{2\pi} \int_1^\infty dz \frac{1}{e^{y z - r} + 1} - (-r \to r), \quad y = \beta m, \ r = \beta \mu, \]
\[ = \frac{T m}{2\pi} \ln \left( 1 + e^{-y z + r} \right) + \frac{T^2}{\pi} \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2} \sinh(kr) e^{-ky}. \quad (21) \]

For the model of dynamical spontaneous symmetry breaking, \( m \) should be identified with the dynamical fermion mass. When \( T \neq 0 \), the phase transition is second order thus near a critical point \( (T_c, \mu_c) \), \( y < 1 \), we can make the expansion of \( n(T, \mu, m) \) in the powers of \( y \) up to \( O(y^2) \) order and obtain

\[ n(T, \mu, m) \simeq n(T, \mu, m = 0) - \frac{m^2}{4\pi} \frac{e^r - 1}{e^r + 1}, \] when \( \frac{m}{T} \ll 1. \] \quad (22)

Now if we consider the phase transition at finite chemical potential and \( T \neq 0 \), then can get by using Eq.(12)

\[ n(T, \mu, m) = \begin{cases} n(T, \mu, m = 0) - \frac{T e^{(\mu - 1)}}{2\pi(e^r + 1)} (\sinh \frac{\mu}{T}) (\mu_c - \mu), & \text{when } \mu \lesssim \mu_c, \\ n(T, \mu, m = 0), & \text{when } \mu > \mu_c. \end{cases} \quad (23) \]

However,

\[ \frac{\partial n(T, \mu, m)}{\partial \mu} \simeq \begin{cases} \frac{\partial n(T, \mu, m = 0)}{\partial \mu} + \frac{T e^{(\mu - 1)}}{2\pi(e^r + 1)} \sinh \frac{\mu}{T}, & \text{when } \mu \lesssim \mu_c, \\ \frac{\partial n(T, \mu, m = 0)}{\partial \mu}, & \text{when } \mu > \mu_c. \end{cases} \quad (24) \]

Eqs.(23) and (24) indicate that when \( \mu \to \mu_c \), \( n(T, \mu, m) \) will increase continuously up to \( n(T, \mu, m = 0) \) but \( \partial n(T, \mu, m)/\partial \mu \) will have a discontinuous jump across over \( \mu_c \). These just show second order feature of the phase transition. For discussion of the phase transition at \( T = 0 \), we first find out the \( T \to 0 \) limit of \( n(T, \mu, m) \) from Eq.(21)

\[ n(T = 0, \mu, m) = \theta(\mu - m)\frac{\mu^2 - m^2}{4\pi}. \quad (25) \]

Since the phase transition at \( T = 0 \) and \( \mu = \mu_c = m(0) \) is first order, we can substitute Eq.(16) into Eq.(25) and obtain

\[ n(T = 0, \mu, m) = \begin{cases} 0, & \text{when } \mu \leq m(0), \\ \frac{m^2}{4\pi}, & \text{when } \mu > m(0). \end{cases} \quad (26) \]

Eq.(26) shows that when \( T = 0 \), as \( \mu \) crosses over the critical point \( \mu_c = m(0) \), the particle density will jump from 0 to \( \mu^2/4\pi \) (the density of massless fermions) and this just represents a characteristic of first order phase transition.

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