Research Article

Representation Theorem for Generators of BSDEs Driven by $G$-Brownian Motion and Its Applications

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We obtain a representation theorem for the generators of BSDEs driven by $G$-Brownian motions and then we use the representation theorem to get a converse comparison theorem for $G$-BSDEs and some equivalent results for nonlinear expectations generated by $G$-BSDEs.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and, for fixed $T \in [0, +\infty)$, let $(B_t)_{t \leq T}$ be a standard Brownian motion and let $\mathcal{F}_t$ be the augmentation of $\sigma(B_s, 0 \leq s \leq t)$. Then Pardoux and Peng [1] introduced the backward stochastic differential equations (BSDEs) and proved the existence and uniqueness result of the BSDEs. In 1997, Peng [2] promoted $g$-expectations based on BSDEs. One of the important properties of $g$-expectations is comparison theorem or monotonicity. Chen [3] first considers a converse result of BSDEs under equal case. After that, Briand et al. [4] obtained a converse comparison theorem for BSDEs under general case. They also derived a representation theorem for the generator $g$. Following this paper, Jiang [5] discussed a more general representation theorem then, in his another paper [6], showed a more general converse comparison theorem. Here the representation theorem is an important method in solving the converse comparison problem and other problems (see Jiang [7]).

Peng [8–13] defined the $G$-expectations and $G$-Brownian motions ($G$-BMs) and proved the representation theorem of $G$-expectation by a set of singular probabilities, which differs from nonlinear $g$-expectations because $g$-expectations are equivalent with a group of absolutely continuous probabilities with respect to the probability measure $P$. Soner et al. [14] obtained an existence and uniqueness result of 2 BSDEs. Recently, Hu et al. [15] proved another existence and uniqueness result on BSDEs driven by $G$-Brownian motions ($G$-BSDEs).

An important advantage of $G$-BSDEs is the easiness to define the nonlinear expectations. Hu et al. in [16] gave a comparison theorem for $G$-BSDEs and talked about the properties of corresponding nonlinear expectations. In this paper, we consider the representation theorem for generators of $G$-BSDEs and then consider the converse comparison theorem of $G$-BSDEs and some equivalent results for nonlinear expectations generated by $G$-BSDEs. In the following, in Section 2, we review some basic concepts and results about $G$-expectations. We give the representation theorem of $G$-BSDEs in Section 3. In Section 4, we consider the applications of representation theorem of $G$-BSDEs, which contain the converse comparison theorem and some equivalent results for nonlinear expectations generated by $G$-BSDEs.

2. Preliminaries

We review some basic notions and results of $G$-expectation, the related spaces of random variables, and the backward stochastic differential equations driven by a $G$-Brownian motion. The readers may refer to [10, 13, 15, 17–19] for more details.

Definition 1. Let $\Omega$ be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on $\Omega$, namely, $c \in \mathcal{H}$ for each
constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. A superlinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, one has

(a) monotonicity: if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;

(b) constant preservation: $\mathbb{E}[c] = c$;

(c) subadditivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;

(d) positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for each $\lambda \geq 0$.

In this paper, we only consider nondegenerate $G$-normal distribution; that is, there exists some $c^2 > 0$ such that $G(A) - G(B) \geq c^2 \text{tr}[A - B]$ for any $A \geq B$.

**Definition 5.** (i) Let $\Omega = C^0_{\alpha}(\mathbb{R}^d)$ denote the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$ and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$L_{ip}(\Omega) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0, \infty), \varphi \in C_{b, Lip}(\mathbb{R}^{d^m}) \}.$$  

(ii) For each fixed $t \in [0, \infty)$, the conditional $G$-expectation $\mathbb{E}_t$ for $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \in L_{ip}(\Omega)$, where without loss of generality we suppose $t_i = t$, is defined by

$$\mathbb{E}_t [\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] = \psi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}),$$

where

$$\psi(x_1, \ldots, x_t) = \mathbb{E} [\varphi(x_1, \ldots, x_t, B_{t_{m-1}} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] .$$

For each fixed $T > 0$, we set

$$L_{ip}(\Omega_T) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b, Lip}(\mathbb{R}^{d^m}) \} .$$

For each $p \geq 1$, we denote by $L^p_G(\Omega)$ (resp., $L^p_{ip}(\Omega_T)$) the completion of $L^p_{ip}(\Omega)$ (resp., $L^p_{ip}(\Omega_T)$) under the norm $\|\|_{L^p_G} = (\mathbb{E}[|\varphi|^p])^{1/p}$. It is easy to check that $L^q_{ip}(\Omega) \subset L^p_{ip}(\Omega)$ for $1 \leq p \leq q$ and $\mathbb{E}[|\varphi|]$ can be extended continuously to $L^1_G(\Omega)$.

For each fixed $a \in \mathbb{R}^d$, $B^a_t = \langle a, B_t \rangle$ is a 1-dimensional $G_a$-Brownian motion, where $G_a(\alpha) = (1/2)(\sigma_{aa^T} \alpha - \sigma_{aa^T} \alpha^T)$, $\sigma_{aa^T} = 2G(a a^T)$ and $\sigma_{aa^T}^2 = -2G(-a a^T)$. Let $n_i^N = \{ t_0^N, \ldots, t_i^N \}, N = 1, 2, \ldots$, be a sequence of partitions of $[0, t]$ such that $\mu(n_i^N) = \|n_i^N - n_j^N\|_F < 1$ when $i = 0, \ldots, N - 1 \to 0$; the quadratic variation process of $B^a$ is defined by

$$\langle B^a \rangle_t = \lim_{\mu(n_i^N) \to 0} \sum_{j=0}^{N-1} (B^a_{t_j^N} - B^a_{t_{j+1}^N})^2 .$$

For each fixed \(a, \bar{a} \in \mathbb{R}^{d}\), the mutual variation of \(B^a\) and \(B^{\bar{a}}\) is defined by

\[
\langle B^a, B^{\bar{a}} \rangle_t = \frac{1}{4} \left[ \langle B^{a+\bar{a}} \rangle_t - \langle B^{a-\bar{a}} \rangle_t \right].
\] (11)

**Definition 6.** For fixed \(T > 0\), let \(M^d_G(0, T)\) be the collection of processes in the following form: for a given partition \(\{t_0, \ldots, t_N\} = \pi_T \) of \([0, T]\),

\[
\eta_t^d(\omega) = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(\omega),
\] (12)

where \(\xi_j \in L^p_{\pi_T}(\Omega_t), j = 0, 1, 2, \ldots, N - 1\). For \(p \geq 1\), one denotes by \(H^p_G(0, T), M^p_G(0, T)\) the completion of \(M^p_G(0, T)\) under the norms \(\|\eta\|_{H^p_G} = \{\mathbb{E}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}, \|\eta\|_{M^p_G} = \{\mathbb{E}[\int_0^T |\eta_s|^p ds]\}^{1/p}\), respectively.

For each \(\eta \in M^1_G(0, T)\), we can define the integrals \(\int_0^T \eta_s dt\) and \(\int_0^T \eta_s d\langle B^a, B^{\bar{a}} \rangle_s\) for each \(a, \bar{a} \in \mathbb{R}^{d}\). For each \(\eta \in H^p_G(0, T; \mathbb{R}^d)\) with \(p \geq 1\), we can define Itô’s integral \(\int_0^T \eta_s dB_s\).

Let \(S^d_G(0, T) = \{h(t, B_0, Z_t, \cdots, B_n) : t_1, \ldots, t_n \in [0, T], h \in C_{D,\pi_T}(\mathbb{R}^{n+1})\}\). For \(p \geq 1\) and \(\eta \in S^p_G(0, T)\), set \(\|\eta\|_{S^p_G} = \{\mathbb{E}[(\sup_{t \in [0, T]} |\eta_t|^p)]\}^{1/p}\). Denote by \(S^d_G(0, T)\) the completion of \(S^d_G(0, T)\) under the norm \(\|\cdot\|_{S^p_G}\).

We consider the following type of G-BSDEs (in this paper, we always use Einstein convention):

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t),
\]

where

\[
f(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R},
\]

satisfy the following properties.

(H1) There exists some \(\beta > 1\) such that for any \(y, z, f(\cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M^p_G(0, T)\).

(H2) There exists some \(L > 0\) such that

\[
|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L(|y - y'| + |z - z'|).
\] (15)

**Definition 7.** Let \(\xi \in L^\beta_G(\Omega_T)\) and \(f, g_{ij}\) satisfy (H1) and (H2) for some \(\beta > 1\). A triplet of processes \((Y, Z, K)\) is called a solution of (13) if for some \(1 < \alpha \leq \beta\) the following properties hold:

(a) \((Y, Z, K) \in \mathcal{G}^\alpha_G(0, T)\);

(b) \(Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)\).

**Theorem 8** (see [15]). Assume that \(\xi \in L^\beta_G(\Omega_T)\) and \(f, g_{ij}\) satisfy (H1) and (H2) for some \(\beta > 1\). Then, (13) has a unique solution \((Y, Z, K)\). Moreover, for any \(1 < \alpha < \beta\), one has \(Y \in \mathcal{G}^\alpha_G(0, T)\), \(Z \in H^\alpha_G(0, T; \mathbb{R}^d)\), and \(K \in L^\alpha_G(\Omega_T)\).

We have the following estimates.

**Proposition 9** (see [15]). Let \(\xi \in L^\beta_G(\Omega_T)\) and \(f, g_{ij}\) satisfy (H1) and (H2) for some \(\beta > 1\). Assume that \((Y, Z, K) \in \mathcal{G}^\alpha_G(0, T)\) for some \(1 < \alpha < \beta\) is a solution of (13). Then, there exists a constant \(C_\alpha > 0\) depending on \(\alpha, T, G, L\) such that

\[
|Y_t|^\alpha \leq C_\alpha \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^\alpha \right] + \mathbb{E} \left[ \left( \int_0^T |h_s|^\alpha ds \right)^{\alpha/\alpha'} \right]^{1/2},
\]

\[
\mathbb{E} \left[ \left( \int_0^T |Z_s|^\beta ds \right)^{\alpha/2} \right] \leq C_\alpha \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^\alpha \right] + \mathbb{E} \left[ \left( \int_0^T |h_s|^\alpha ds \right)^{\alpha/\alpha'} \right]^{1/2} \right\},
\]

\[
\mathbb{E} \left[ |K_t|^\alpha \right] \leq C_\alpha \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^\alpha \right] + \mathbb{E} \left[ \left( \int_0^T |h_s|^\alpha ds \right)^{\alpha/\alpha'} \right] \right\},
\]

where \(h_s^\alpha = |f(s, 0, 0)| + \sum_{i,j=1}^d |g_{ij}(s, 0, 0)|\).

**Proposition 10** (see [15, 20]). Let \(\alpha \geq 1\) and \(\delta > 0\) be fixed. Then, there exists a constant \(C\) depending on \(\alpha, T, G, L\) such that

\[
\mathbb{E} \left[ \left( \int_0^T |\xi|^\alpha ds \right)^{\alpha/(\alpha + \delta)} \right] \leq C \left\{ \mathbb{E} \left[ \left( \int_0^T |\xi|^\alpha \right)^{\alpha/(\alpha + \delta)} ds \right] \mathbb{E} \left[ |\xi|^\alpha \right] \right\},
\]

\[
\forall \xi \in L^{\alpha + \delta}_G(\Omega_T).
\]
Theorem 11 (see [16]). Let \((Y_l, Z_l, K_l), l = 1, 2\), be the solutions of the following G-BSDEs:

\[ Y^1_t = \xi + \int_t^T f(s, Y^1_s, Z^1_s) \, ds + \int_t^T g_{ij}(s, Y^1_s, Z^1_s) \, dB^i_s + \int_t^T Z^1_s \, dB^i_s - \left(K^1_t - K^1_t\right), \]

where \(\xi \in L^\beta_G(\Omega_T)\), \(f \) and \(g_{ij}\) satisfy (H1) and (H2) for some \(\beta > 1\) and \((Y_l)_{l \in T}\) are RCLL processes in \(M^\beta_G(0, T)\) such that \(E[\sup_{t \in [0,T]} |Y^1_t|^\beta] < \infty\). If \(V^1_l - V^2_l\) is an increasing process, then \(V^1_l \geq V^2_l\) for \(t \in [0, T]\).

In this paper, we also need the following assumptions for G-BSDE (13).

(H3) For each fixed \((\omega, y, z) \in \Omega_T \times \mathbb{R} \times \mathbb{R}^d\), \(t \to f(t, \omega, y, z)\) and \(t \to g_{ij}(t, \omega, y, z)\) are continuous.

(H4) For each fixed \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d\), \(g_{ij}(t, y, z) \in L^\beta_G(\Omega_T)\), and

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \left[ \int_0^\varepsilon \left| f(u, y, z) - f(t, y, z) \right|^\beta du \right. \]

\[ \left. + \sum_{i,j=1}^d |g_{ij}(u, y, z) - g_{ij}(t, y, z)|^\beta du \right] = 0. \]

(H5) For each \((t, \omega, y) \in [0, T] \times \Omega_T \times \mathbb{R}\), \(f(t, \omega, y, 0) = g_{ij}(t, \omega, y, 0) = 0\).

Assume that \(\xi \in L^\beta_G(\Omega_T)\); \(f \) and \(g_{ij}\) satisfy (H1), (H2), and (H5) for some \(\beta > 1\). Let \((Y^{T_\varepsilon}, Z^{T_\varepsilon}, K^{T_\varepsilon})\) be the solution of G-BSDE (13) corresponding to \(\xi, f\), and \(g_{ij}\) on \([0, T]\). It is easy to check that \(Y^{T_\varepsilon} = Y^{T_\varepsilon}\) on \([0, T]\) for \(T' > T\). Following [16], we can define consistent nonlinear expectation

\[ \mathbb{E}_t[Y] = Y_t^{T_\varepsilon} \text{ for } t \in [0, T] \]

and set \(\mathbb{E}[\xi] = \mathbb{E}_0[\xi] = Y_0^{T_\varepsilon}\).

3. Representation Theorem of Generators of G-BSDEs

We consider the following type of G-FBSDEs:

\[ X^{t,x}_u = x + \int_t^u b\left(X^{t,x}_s\right) \, du \]

\[ + \int_t^u h_{ij}\left(X^{t,x}_s\right) d\langle B^i, B^j \rangle_s + \int_t^u \sigma\left(X^{t,x}_s\right) dB_s, \]

where \(h_{ij} = h_{ji}, \sigma_{ij} = \sigma_{ji}, 1 \leq i, j \leq d\).

We now give the main result in this section.

Theorem 12. Let \(b : \mathbb{R}^n \to \mathbb{R}^n\), \(h_{ij} : \mathbb{R}^n \to \mathbb{R}^n\), and \(\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}\) be Lipschitz functions and let \(f\) and \(g_{ij}\) satisfy (H1), (H2), (H3), and (H4) for some \(\beta > 1\). Then, for each \((t, x, y, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) and \(\alpha \in (1, \beta)\), one has

\[ \mathbb{E}_t[Y^\alpha_t] = \frac{1}{\mathbb{E}_t[Y^\alpha_t]} \mathbb{E}_t \left[ \int_t^\varepsilon f(u, Y^\alpha_u, p, X^{t,x}_u) \, du \right. \]

\[ + \int_t^\varepsilon g_{ij}(u, Y^\alpha_u, p, X^{t,x}_u) \, dB^i_s + \int_t^\varepsilon \sigma(u, Y^\alpha_u, p, X^{t,x}_u) dB_s \]

\[ \left. - \int_t^\varepsilon \mathbb{E}_t[Z^{T_\varepsilon}_u] \, dB_u - \left(K^{T_\varepsilon}_t - K^{T_\varepsilon}_t\right) \right\} \text{ for simplicity. We have } \mathbb{E}[|Y_t^{T_\varepsilon}|] < \infty \text{ for each } \eta \geq 1 \text{ (see [16, 19]). Thus, by Theorem 8, G-BSDE (22) has a unique solution } (Y^{T_\varepsilon}, Z^{T_\varepsilon}, K^{T_\varepsilon}) \text{ and } Y^{T_\varepsilon} \in L^\beta_G(\Omega_T). \text{ We set, for } s \in [t, t + \varepsilon],

\[ \tilde{Y}^{T_\varepsilon}_s = Y^{T_\varepsilon}_s - \left( y + \langle p, X^{t,x}_s - x \rangle \right), \]

\[ \tilde{Z}^{T_\varepsilon}_s = Z^{T_\varepsilon}_s - \sigma^T(x^{t,x}_s) p, \quad K^{T_\varepsilon}_s = K^{T_\varepsilon}_s. \]

Applying Itô’s formula to \(\tilde{Y}^{T_\varepsilon}_s\) on \([t, t + \varepsilon]\), it is easy to verify that \((\tilde{Y}^{T_\varepsilon}, \tilde{Z}^{T_\varepsilon}, K^{T_\varepsilon})\) solves the following G-BSDE:

\[ \tilde{Y}^{T_\varepsilon}_s = \int_t^s \left[ \int_t^\eta f\left(u, \tilde{Y}^{T_\varepsilon}_u + \langle p, X^{t,x}_u - x \rangle \right) \, du \right. \]

\[ + \int_t^\eta \langle p, b\left(X^{t,x}_u\right) \rangle \, du \]

\[ + \int_t^\eta g_{ij}\left(u, \tilde{Y}^{T_\varepsilon}_u + \langle p, X^{t,x}_u - x \rangle \right) \, dB^i_s \]

\[ + \int_t^\eta \sigma\left(u, \tilde{Y}^{T_\varepsilon}_u + \langle p, X^{t,x}_u - x \rangle \right) \, dB_s \]

\[ \left. - \int_t^\eta \mathbb{E}_t[Z^{T_\varepsilon}_u] \, dB_u - \left(K^{T_\varepsilon}_t - K^{T_\varepsilon}_t\right) \right\]
From Proposition 9,

\[
|Y_t^\epsilon|^\alpha \leq C_n \mathbb{E} \left[ \left( \int_t^{t+\epsilon} \left( f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p \right) + \langle p, b(X_u^{t,x}) \rangle \right) + \sum_{i,j=1}^d |g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p) \right) \right] \leq C_2 \alpha/\beta \]

\[
\mathbb{E} \left[ \left( \int_t^{t+\epsilon} \left| \int_{\mathbb{Z}} \right|^\alpha \right) \right] \leq C_n \mathbb{E} \left[ \left( \int_t^{t+\epsilon} \left( f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p) \right) \right) \right] \leq C_1 \alpha/\beta \]

Together with assumption (H4), we get

\[
\mathbb{E} \left[ \sup_{\mathbb{Z}} |Y_t^\epsilon|^\alpha + \left( \int_t^{t+\epsilon} \left| \int_{\mathbb{Z}} \right|^\alpha \right) \right] \leq C_3 \alpha, \quad (29)
\]

where \(C_3\) depends on \(x, y, p, \alpha, \beta, T, G, \) and \(L\). Now, we prove (23). Let us consider

\[
1/\epsilon \left| Y_t^\epsilon - y \right| = 1/\epsilon \mathbb{E} \left[ \mathbb{E} \left[ \left( \int_t^{t+\epsilon} f(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p) \right) \right] \right]
\]

\[
+ \int_t^{t+\epsilon} \langle p, b(X_u^{t,x}) \rangle du
\]

\[
+ \int_t^{t+\epsilon} g_{ij}(u, y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p) \right) \times d\langle B^i, B^j \rangle
\]

\[
+ \int_t^{t+\epsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle \times d\langle B^i, B^j \rangle + L_{\epsilon}, \quad (30)
\]

where

\[
L_{\epsilon} = 1/\epsilon \left( \epsilon \left( \int_t^{t+\epsilon} \right) \right)
\]

\[
\mathbb{E} \left[ \left( \int_t^{t+\epsilon} f(u, Y_u^{t,x} + y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p) \right) \right]
\]

\[
+ \int_t^{t+\epsilon} \langle p, b(X_u^{t,x}) \rangle du
\]

\[
+ \int_t^{t+\epsilon} g_{ij}(u, Y_u^{t,x} + y + \langle p, X_u^{t,x} - x \rangle, \sigma^T(X_u^{t,x}) p) \right) 
\]

\[
	imes d\langle B^i, B^j \rangle
\]

\[
+ \int_t^{t+\epsilon} \langle p, h_{ij}(X_u^{t,x}) \rangle \times d\langle B^i, B^j \rangle + L_{\epsilon}.
\]

(31)
It is easy to check that $|L_\varepsilon| \leq (C_4/\varepsilon)\tilde{E} \int_t^{t+\varepsilon} (|\tilde{Y}_{\varepsilon}| + |\tilde{Z}_{\varepsilon}|) du$, where $C_4$ depends on $G$, $L$, and $T$. Thus, by (29), we get

\[
\mathbb{E} \left[ |L_{\varepsilon}|^\alpha \right] \\
\leq \frac{C_4}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} (|\tilde{Y}_{\varepsilon}| + |\tilde{Z}_{\varepsilon}|) du \right]^\alpha \\
\leq \frac{2^{\alpha-1} C_4}{\varepsilon} \mathbb{E} \left[ \left( \int_t^{t+\varepsilon} |\tilde{Y}_{\varepsilon}| du \right)^\alpha + \left( \int_t^{t+\varepsilon} |\tilde{Z}_{\varepsilon}| du \right)^\alpha \right] \\
\leq \frac{2^{\alpha-1} C_4}{\varepsilon} \mathbb{E} \left[ \sup_{s \in [t, t+\varepsilon]} |\tilde{Y}_{\varepsilon}|^\alpha + \varepsilon^{\alpha/2} \mathbb{E} \left[ \left( \int_t^{t+\varepsilon} |\tilde{Z}_{\varepsilon}|^2 du \right)^{\alpha/2} \right] \right] \\
\leq \frac{2^{\alpha-1} C_4 C_7 (\varepsilon^\alpha + \varepsilon^{\alpha/2})},
\]
which implies $L_G^\alpha - \lim_{\varepsilon \to 0^+} M_{\varepsilon} = 0$. We set

\[
M_{\varepsilon} = \frac{1}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} f \left( u, y + \langle p, X^{t,x}_\varepsilon \rangle, \sigma^T \left( X^{t,x}_\varepsilon \right) p \right) du \right. \\
+ \int_t^{t+\varepsilon} \langle p, b \left( X^{t,x}_\varepsilon \right) \rangle du \\
+ \int_t^{t+\varepsilon} g_y \left( u, y + \langle p, X^{t,x}_\varepsilon \rangle, \sigma^T \left( X^{t,x}_\varepsilon \right) p \right) \\
x d \langle B^i, B^i \rangle_u \\
+ \int_t^{t+\varepsilon} \langle p, h_{ij} \left( X^{t,x}_\varepsilon \right) \rangle d \langle B^i, B^j \rangle_u \\
- \mathbb{E} \left[ \int_t^{t+\varepsilon} f \left( u, y, \sigma^T \left( x \right) p \right) du + \langle p, b \left( x \right) \rangle \varepsilon \right. \\
+ \int_t^{t+\varepsilon} g_y \left( u, y, \sigma^T \left( x \right) p \right) d \langle B^i, B^i \rangle_u \\
+ \int_t^{t+\varepsilon} \langle p, h_{ij} \left( x \right) \rangle d \langle B^i, B^j \rangle_u \left. \right].
\]
\[
\leq \frac{C_7}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} \left( |f \left( u, y, \sigma^T \left( x \right) p \right) - f \left( t, y, \sigma^T \left( x \right) p \right) \right) \right. \\
+ \sum_{i,j=1}^d \left| g_y \left( u, y, \sigma^T \left( x \right) p \right) \right|^\alpha du \\
\leq \frac{C_7}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} \left( |f \left( u, y, \sigma^T \left( x \right) p \right) \right. \\
\left. \left. - f \left( t, y, \sigma^T \left( x \right) p \right) \right) \right]\left. \right]^\alpha du \\
\leq \frac{C_7}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} \left( \left| f \left( u, y, \sigma^T \left( x \right) p \right) \right. \\
\left. \left. - f \left( t, y, \sigma^T \left( x \right) p \right) \right) \right]\left. \right]^\alpha du \\
\leq C_7 \mathbb{E} \left( \left| f \left( u, y, \sigma^T \left( x \right) p \right) \right. \\
\left. \left. - f \left( t, y, \sigma^T \left( x \right) p \right) \right) \right]\left. \right]^\beta du \right).\]

By the Lipschitz condition, we can get $|M_{\varepsilon}| \leq (C_5/\varepsilon)\tilde{E} \int_t^{t+\varepsilon} |X^{t,x}_\varepsilon - x| du$, where $C_5$ depends on $p$, $G$, $L$, and $T$. Noting that $\tilde{E} \sup_{s \in [t, t+\varepsilon]} |X^{t,x}_s - x| \leq C_6 (1 + |x|)^{\alpha/2}$ (see [16, 19]), where $C_6$ depends on $L$, $G$, and $\alpha$, we obtain

\[
\mathbb{E} \left[ |M_{\varepsilon}|^\alpha \right] \leq C_5 \mathbb{E} \left( \sup_{s \in [t, t+\varepsilon]} |X^{t,x}_s - x| \right)^\alpha \\
\leq C_5 C_6 (1 + |x|)^{\alpha/2},
\]
which implies $L_G^\alpha - \lim_{\varepsilon \to 0^+} N_{\varepsilon} = 0$. Now, we set

\[
N_{\varepsilon} = \frac{1}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} f \left( u, y, \sigma^T \left( x \right) p \right) du + \langle p, b \left( x \right) \rangle \varepsilon \right. \\
+ \int_t^{t+\varepsilon} g_y \left( u, y, \sigma^T \left( x \right) p \right) d \langle B^i, B^i \rangle_u \\
+ \int_t^{t+\varepsilon} \langle p, h_{ij} \left( x \right) \rangle d \langle B^i, B^j \rangle_u \left. \right].
\]

It is easy to deduce that $|N_{\varepsilon}| \leq (C_7/\varepsilon)\tilde{E} \int_t^{t+\varepsilon} |f(u, y, \sigma^T(x)p - f(t, y, \sigma^T(x)p) + \sum_{i,j=1}^d \left| g_y \left( u, y, \sigma^T(x)p \right) \right. \\
\left. g_y \left( t, y, \sigma^T(x)p \right) \right| d \langle B^i, B^i \rangle_u \left. \right] du$, where $C_7$ depends on $G$. Then,

\[
\mathbb{E} \left[ |N_{\varepsilon}|^\alpha \right] \\
\leq \frac{C_7}{\varepsilon} \mathbb{E} \left[ \int_t^{t+\varepsilon} \left| f \left( u, y, \sigma^T \left( x \right) p \right) - f \left( t, y, \sigma^T \left( x \right) p \right) \right| \right. \\
+ \sum_{i,j=1}^d \left| g_y \left( u, y, \sigma^T \left( x \right) p \right) \right. \\
\left. - g_y \left( t, y, \sigma^T \left( x \right) p \right) \right| d \langle B^i, B^i \rangle_u \left. \right]\left. \right]\left. \right]^\alpha du \\
\leq C_7 \mathbb{E} \left( \left| f \left( u, y, \sigma^T \left( x \right) p \right) \right. \\
\left. - f \left( t, y, \sigma^T \left( x \right) p \right) \right| \right. \\
+ \sum_{i,j=1}^d \left| g_y \left( u, y, \sigma^T \left( x \right) p \right) \right. \\
\left. - g_y \left( t, y, \sigma^T \left( x \right) p \right) \right| d \langle B^i, B^i \rangle_u \left. \right]^\beta du \right).\]

Take limit on both sides of the above inequality and use assumption (H4); then, we have

\[
L_G^\alpha - \lim_{\varepsilon \to 0^+} L_{\varepsilon} = 0.
\]
On the other hand,

\[
\mathbb{E} \left[ \int_{t}^{t+\varepsilon} f(t, y, \sigma^T(x)p) \, du + \langle p, b(x) \rangle \varepsilon \right. \\
+ \int_{t}^{t+\varepsilon} g_{ij}(t, y, \sigma^T(x)p) \, d\langle B^i, B^j \rangle_u \\
+ \int_{t}^{t+\varepsilon} \langle p, h_{ij}(x) \rangle \, d\langle B^i, B^j \rangle_u \\
= f(t, y, \sigma^T(x)p) \varepsilon + \langle p, b(x) \rangle \varepsilon \\
+ \mathbb{E} \left[ \left( g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle \right) \right. \\
\times \left( \langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t \right) \\
= \left( f(t, y, \sigma^T(x)p) + \langle p, b(x) \rangle \\
+ 2G \left( g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle \right)_{i,j=1} \right) \varepsilon.
\]

Then, we have

\[
L_G^\alpha - \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \{ Y_t^\varepsilon - y \} = f(t, y, \sigma^T(x)p) + \langle p, b(x) \rangle \\
+ 2G \left( g_{ij}(t, y, \sigma^T(x)p) + \langle p, h_{ij}(x) \rangle \right)_{i,j=1}.
\]

The proof is complete. \( \square \)

4. Some Applications

4.1. Converse Comparison Theorem for G-BSDEs. We consider the following G-BSDEs:

\[
Y_t^{2,\xi} = \xi + \int_{t}^{T} f^2(s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, ds \\
+ \int_{t}^{T} g_{ij}^2(s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, d\langle B^i, B^j \rangle_s \\
- \int_{t}^{T} Z_s^{2,\xi} \, dB_s - \left( K_T^{2,\xi} - K_t^{2,\xi} \right), \quad l = 1, 2,
\]

where \( g_{ij}^l = g_{ji}^l \).

We first generalized the comparison theorem in [16].

Proposition 13. Let \( f^l \) and \( g_{ij}^l \) satisfy (H1) and (H2) for some \( \beta > 1, l = 1, 2 \). If \( f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}) \leq 0 \), then, for each \( \xi \in L_G^\beta(\Omega_T) \), one has \( Y_t^{1,\xi} \geq Y_t^{2,\xi} \) for \( t \in [0, T] \).

Proof. From the above G-BSDEs, we have

\[
Y_t^{2,\xi} = \xi + \int_{t}^{T} f^2(s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, ds \\
+ \int_{t}^{T} g_{ij}^2(s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, d\langle B^i, B^j \rangle_s \\
- \int_{t}^{T} Z_s^{2,\xi} \, dB_s - \left( K_T^{2,\xi} - K_t^{2,\xi} \right),
\]

where

\[
V_t = \int_{0}^{t} \left( f^2 - f^1 \right) (s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, ds \\
+ \int_{0}^{t} \left( g_{ij}^2 - g_{ij}^1 \right) (s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, d\langle B^i, B^j \rangle_s \\
= \int_{0}^{t} \left( f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}) \right) (s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, ds \\
+ \int_{0}^{t} \left( g_{ij}^2 - g_{ij}^1 \right) (s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, d\langle B^i, B^j \rangle_s \\
- \int_{0}^{t} 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}) (s, Y_s^{2,\xi}, Z_s^{2,\xi}) \, ds.
\]
where $h_{ij} = h_{ji} \in \mathbb{R}^d$. By Theorem 12, we have, for each $\alpha \in (1, \beta)$,

$$L^\alpha_G - \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \left( \mathbb{E}^\xi [\eta] - y \right) = f^i (t, y, z) + 2G \left( \left( g^i_j (t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right).$$

(44)

Since $\mathbb{E}^\xi [\eta] \geq \mathbb{E}^\xi [\eta]$, 

$$f^i (t, y, z) + 2G \left( \left( g^i_j (t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right) \geq f^i (t, y, z) + 2G \left( \left( g^i_j (t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right) \text{ q.s.}$$

(45)

Take a $h_{ij}$ such that $(z, h_{ij}) = -g^i_j (t, y, z)$. Therefore, $(f^2 - f^1 + 2G((g^2_j - g^1_j)_{i,j=1}^d)) (t, y, z) \leq 0$ q.s. By the assumptions (H2) and (H3), it is easy to deduce that $f^2 - f^1 + 2G((g^2_j - g^1_j)_{i,j=1}^d) \leq 0$ q.s.

In the following, we use the notation $\mathbb{E}^\xi [\xi] = Y_t^\xi, l = 1, 2$.

**Corollary 16.** Let $f^i$ and $g^i_j$ be deterministic functions and satisfy (H1), (H2), (H3), and (H5) for some $\beta > 1, l = 1, 2$. If $\mathbb{E}^\xi [\xi] \geq \mathbb{E}^\eta [\xi]$ for each $\xi \in L^\beta_G (\Omega_T)$, then $f^2 - f^1 + 2G((g^2_j - g^1_j)_{i,j=1}^d) \leq 0$ q.s.

**Proof.** Taking $\eta$ as in Theorem 15, since $f^i$ and $g^i_j$ are deterministic, we could get $\mathbb{E}^\xi [\eta] = \mathbb{E}^\eta [\eta]$, for $l = 1, 2$. And the proof in Theorem 15 still holds true.

4.2. Some Equivalent Relations. We consider the following G-BSDE:

$$Y_t^\xi = \xi + \int_t^T f (s, Y_s, Z_s, d) ds + \int_t^T g^i_j (s, Y_s, Z_s, d) \left( B^i_s, B^j_s \right)_s$$

$$- \int_t^T Z_s dB_s - (K^i_T - K^i_t),$$

(46)

where $g^i_j = g^j_i$. We use the notation $\mathbb{E}^\xi [\xi] = Y_t^\xi$.

**Proposition 17.** Let $f$ and $g^i_j$ satisfy (H1), (H2), (H3), (H4), and (H5) for some $\beta > 1$ and fix $\alpha \in (1, \beta)$. Then, one has

(1) $\mathbb{E}^\xi [\xi + \eta] = \mathbb{E}^\xi [\xi] + \eta$ for $t \in [0, T], \xi, \eta \in L^\alpha_G (\Omega_T)$, and $\eta \in L^\alpha_C (\Omega_T)$ if and only if for each $t \in [0, T], y, y' \in \mathbb{R}^d, z \in \mathbb{R}^d$,

$$f (t, y, z) - f (t, y', z') + 2G \left( \left( g^i_j (t, y, z) - g^j_i (t, y', z') \right)_{i,j=1}^d \right) = 0;$$

(47)

(2) $\mathbb{E}^\xi [\xi + \eta] \leq \mathbb{E}^\xi [\xi] + \mathbb{E}^\xi [\eta]$ for $t \in [0, T], \xi \in L^\alpha_C (\Omega_T)$, and $\eta \in L^\alpha_C (\Omega_T)$ if and only if for each $t \in [0, T], y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^d$,

$$0 \geq f (t, y + y', z + z') - f (t, y, z) - f (t, y', z') + 2G \left( \left( g^i_j (t, y + y', z + z') - g^j_i (t, y, z) \right)_{i,j=1}^d \right) - g^i_j (t, y, z);$$

(48)

(3) $\mathbb{E}^\xi [\xi + (1 - \lambda) \eta] \leq \lambda \mathbb{E}^\xi [\xi] + (1 - \lambda) \mathbb{E}^\xi [\eta]$ for $t \in [0, T], \lambda \in [0, 1], \xi \in L^\alpha_C (\Omega_T)$, and $\eta \in L^\alpha_C (\Omega_T)$ if and only if for each $t \in [0, T], y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^d, \lambda \in [0, 1]$,

$$0 \geq f (t, \lambda y + (1 - \lambda) y', \lambda z + (1 - \lambda) z') - \lambda f (t, y, z) - (1 - \lambda) f (t, y', z') + 2G \left( \left( g^i_j (t, \lambda y + (1 - \lambda) y', \lambda z + (1 - \lambda) z') \right)_{i,j=1}^d \right) - \lambda g^i_j (t, y, z);$$

(49)

(4) $\mathbb{E}^\xi [\xi] = \mathbb{E}^\eta [\xi]$ for $t \in [0, T], \xi \geq 0$, and $\xi \in L^\alpha_C (\Omega_T)$ if and only if for each $t \in [0, T], y, z \in \mathbb{R}^d, \lambda \geq 0$,

$$f (t, \lambda y, z) - \lambda f (t, y, z) = 2G \left( \left( \lambda g^i_j (t, y, z) - g^i_j (t, \lambda y, \lambda z) \right)_{i,j=1}^d \right)$$

$$= -2G \left( \left( g^i_j (t, \lambda y, \lambda z) - \lambda g^i_j (t, y, z) \right)_{i,j=1}^d \right).$$

(50)

Proof. (1) "⇒" part. For each fixed $t \in [0, T], y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^d$, we take

$$\xi_t = y + \left( \left( B^i_s, B^j_s \right)_{t \to s} - \left( B^i_s, B^j_s \right)_t \right)_t$$

$$+ \left( z, B_{t \to s} - B_t \right)_t, \quad \eta = y',$$

where $h_{ij} = h_{ji} \in \mathbb{R}^d$. Then, by Theorem 12 and $\mathbb{E}^\xi [\xi + \eta] = \mathbb{E}^\xi [\xi] + \eta$, we can obtain

$$f (t, y', z) + 2G \left( \left( g^i_j (t, y', z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right) = f (t, y, z) + 2G \left( \left( g^i_j (t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right).$$

(52)

We choose $h_{ij}$ such that $g^i_j (t, y', z) + \langle z, h_{ij} \rangle = 0$, which implies (47).

"⇐" part. Let $(Y, Z, K)$ be the solution of G-BSDE (46) corresponding to terminal condition $\xi$. We claim that $(Y_t + \eta, Z_t, K_t)_{t \in [0, T]}$ is the solution of G-BSDE (46) corresponding
to terminal condition $\xi + \eta$ on $[t, T]$. For this, we only need to check that, for $s \in [t, T]$,
\begin{align*}
\int_s^T f(u, Y_u, Z_u) \, du + \int_s^T g_{ij}(u, Y_u, Z_u) \, d\langle B^i, B^j \rangle_u \\
&= \int_s^T f(u, Y_u + \eta, Z_u) \, du \\
&+ \int_s^T g_{ij}(u, Y_u + \eta, Z_u) \, d\langle B^i, B^j \rangle_u.
\end{align*}
(53)
By (47) we can get
\begin{align*}
\int_s^T (g_{ij}(u, Y_u + \eta, Z_u) - g_{ij}(u, Y_u, Z_u)) \, d\langle B^i, B^j \rangle_u \\
&- 2 \int_s^T G \left( \left( g_{ij}(u, Y_u + \eta, Z_u) - g_{ij}(u, Y_u, Z_u) \right)_{i,j=1}^d \right) \, du \\
&= \int_s^T (g_{ij}(u, Y_u, Z_u) - g_{ij}(u, Y_u + \eta, Z_u)) \, d\langle B^i, B^j \rangle_u \\
&+ \int_s^T (f(u, Y_u + \eta, Z_u) - f(u, Y_u, Z_u)) \, du \leq 0,
\end{align*}
which implies (53). The proof of (1) is complete.

Finally, we could prove (3) as in (2) and (4) as in (1). \qed

Proposition 18. One has the following.

1. If $G(A) + G(-A) > 0$ for any $A \in \mathbb{S}_d$ and $A \neq 0$, then (47) holds if and only if $f$ and $g_{ij}$ are independent of $y$.

We choose $h_{ij}, h'_{ij}$ such that $g_{ij}(t, y, z) + \langle z, h_{ij} \rangle = 0$ and $g_{ij}(t, y', z') + \langle z', h'_{ij} \rangle = 0$, which implies (48).

"⇐" part. Let $(Y, Z, K)$ and $(Y', Z', K')$ be the solutions of G-BSDE (46) corresponding to terminal condition $\xi$ and $\eta$, respectively. Then, $(Y + Y', Z + Z', K)$ solves the following G-BSDE:
\begin{align*}
Y_t + Y'_t &= \xi + \eta + \int_t^T f(s, Y_s + Y'_s, Z_s + Z'_s) \, ds \\
&+ \int_t^T g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) \, d\langle B^i, B^j \rangle_s \\
&+ V_T - V_t - \int_t^T (Z_s + Z'_s) \, dB_s - (K_T - K_t),
\end{align*}
where
\begin{align*}
V_t &= -K'_t - \int_0^t \left( f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \right) \, ds \\
&\quad - \int_0^t g_{ij}(s, Y_s, Z_s, Z'_s) \, d\langle B^i, B^j \rangle_s \\
&\quad - g_{ij}(s, Y'_s, Z'_s) \, d\langle B^i, B^j \rangle_s \\
&\quad + \left\{ \int_0^t \left( g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) \right. \\
&\quad - g_{ij}(s, Y'_s, Z'_s) \left. \right) \, d\langle B^i, B^j \rangle_s \right\} \\
&\quad - \int_0^t \left( f(s, Y_s, Z_s, Z'_s) \right. \\
&\quad - f(s, Y'_s, Z'_s) \left. \right) \, ds \\
&\quad + 2G \left( g_{ij}(s, Y_s + Y'_s, Z_s + Z'_s) - g_{ij}(s, Y_s, Z_s) \right. \\
&\quad - g_{ij}(s, Y'_s, Z'_s) \left. \right) \, ds.
\end{align*}
By (48), it is easy to check that $V_t$ is an increasing process. Then, by Theorem 11, we can get $E_t[\xi + \eta] \leq \bar{E}_t[\xi] + \bar{E}_t[\eta]$. The proof of (2) is complete.

Finally, we could prove (3) as in (2) and (4) as in (1). \qed
(2) If there exists an $A \in S_d$ with $A \neq 0$ such that $G(A) + G(-A) = 0$ and $G(A) \neq 0$, then, for any fixed $g(t, y, z)$ satisfying $(H1)-(H5)$, one has $f(t, y, z) = -2G(A)g(t, y, z)$ and $(g_{ij}(t, y, z))_{i,j=1}^d = g(t, y, z)A$ satisfying (47).

Proof. It is easy to verify (2), and we only need to prove (1). If (47) holds, it is easy to check that $G((g_{ij}(t, y, z) - g_{ij}(0, 0, z))_{i,j=1}^d) + G((g_{ij}(t, 0, z) - g_{ij}(t, y, z))_{i,j=1}^d) = 0$ holds. Then, from the assumption, we get $g_{ij}(t, y, z) = g_{ij}(t, 0, 0)$. Therefore, by (47), we have $f(t, y, z) = f(t, 0, z)$, which implies that $f$ and $g_{ij}$ are independent of $y$. The converse part is obvious.

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