ON THE ASYMPTOTIC BEHAVIOR
OF THE MAXIMUM ABSOLUTE VALUE
OF GENERALIZED GEGENBAUER POLYNOMIALS

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Abstract. Using well-known facts on Jacobi polynomials, we derive some asymptotic estimates for the maximum absolute value of generalized Gegenbauer polynomials.

Key words and phrases: orthogonal polynomials, Jacobi polynomial, Gegenbauer polynomial, generalized Gegenbauer polynomial, asymptotic behavior

MSC 2010: 33C45

Introduction and main result

In this section, we introduce some classes of orthogonal polynomials on $[-1,1]$, including the so-called generalized Gegenbauer polynomials. For a background and more details on the orthogonal polynomials, the reader is referred to [1–3, 5]. Here, we also formulate the main result of the publication.

Let $\alpha, \beta > -1$. The Jacobi polynomials, denoted by $P_{n}^{(\alpha,\beta)}(\cdot)$, where $n = 0, 1, \ldots$, are orthogonal with respect to the Jacobi weight function $w_{\alpha,\beta}(t) = (1 - t)^{\alpha}(1 + t)^{\beta}$ on $[-1,1]$, namely,

$$
\int_{-1}^{1} P_{n}^{(\alpha,\beta)}(t) P_{m}^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) dt = \begin{cases} 
2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1) \\
(2n + \alpha + \beta + 1) \Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)
\end{cases}, \quad n = m,
$$

$$
0, \quad n \neq m.
$$

Here, as usual, $\Gamma$ is the gamma function.

For $\lambda > -\frac{1}{2}$, $\mu \geq 0$, and $n = 0, 1, \ldots$, the generalized Gegenbauer polynomials $C_{n}^{(\lambda,\mu)}(\cdot)$ are defined by

$$
C_{2n}^{(\lambda,\mu)}(t) = a_{2n}^{(\lambda,\mu)} P_{n}^{(-1/2,\mu-1/2)}(2t^2 - 1), \quad a_{2n}^{(\lambda,\mu)} = \frac{(\lambda + \mu)n}{(\mu + \frac{1}{2})n},
$$

$$
C_{2n+1}^{(\lambda,\mu)}(t) = a_{2n+1}^{(\lambda,\mu)} t P_{n}^{(-1/2,\mu+1/2)}(2t^2 - 1), \quad a_{2n+1}^{(\lambda,\mu)} = \frac{(\lambda + \mu)n+1}{(\mu + \frac{1}{2})n+1},
$$

where $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_0 = 1, \quad (\lambda)_n = \lambda(\lambda + 1) \cdots (\lambda + n - 1) \quad \text{for} \quad n = 1, 2, \ldots .$$

They are orthogonal with respect to the weight function

$$
v_{\lambda,\mu}(t) = |t|^{2\mu}(1 - t^2)^{-1/2}, \quad t \in [-1,1].
$$

For $\mu = 0$, these polynomials, denoted by $C_{n}^{(\lambda)}(\cdot)$, are called the Gegenbauer polynomials:

$$
C_{n}^{(\lambda)}(t) = C_{n}^{(\lambda,0)}(t) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})n} P_{n}^{(-1/2,\lambda-1/2)}(t).
$$
For $\lambda > -\frac{1}{2}$, $\mu > 0$, and $n = 0, 1, \ldots$, we have the following connection:

$$C_n^{(\lambda, \mu)}(t) = c_\mu \int_{-1}^{1} C_n^{\lambda+\mu}(tx)(1+x)(1-x^{2})^{\mu-1} \, dx, \quad c_\mu^{-1} = 2 \int_{0}^{1} (1-x^{2})^{\mu-1} \, dx.$$  

Denote by $\{\tilde{C}_n^{(\lambda, \mu)}(\cdot)\}_{n=0}^{\infty}$ the sequence of orthonormal generalized Gegenbauer polynomials. It is easily verified that these polynomials are given by the following formulae:

$$\tilde{C}_{2n}^{(\lambda, \mu)}(t) = \tilde{a}_{2n}^{(\lambda, \mu)} P_n^{(\lambda-1/2, \mu-1/2)}(2t^2 - 1),$$

$$\tilde{a}_{2n}^{(\lambda, \mu)} = \left( \frac{(2n + \lambda + \mu)\Gamma(n + \lambda)\Gamma(n + \lambda + \mu)}{\Gamma(n + \lambda + 1/2)\Gamma(n + \mu + 1/2)} \right)^{1/2},$$

$$\tilde{C}_{2n+1}^{(\lambda, \mu)}(t) = \tilde{a}_{2n+1}^{(\lambda, \mu)} t P_n^{(\lambda-1/2, \mu+1/2)}(2t^2 - 1),$$

$$\tilde{a}_{2n+1}^{(\lambda, \mu)} = \left( \frac{(2n + \lambda + \mu + 1)\Gamma(n + \lambda + 1)\Gamma(n + \lambda + \mu + 1)}{\Gamma(n + \lambda + 1/2)\Gamma(n + \mu + 3/2)} \right)^{1/2}.  \tag{1}$$

Throughout the paper we use the following asymptotic notation: $f(n) \lesssim g(n)$, $n \to \infty$, or equivalently $g(n) \gtrsim f(n)$, $n \to \infty$, means that there exist a positive constant $C$ and a positive integer $n_0$ such that $0 \leq f(n) \leq C g(n)$ for all $n \geq n_0$ (asymptotic upper bound); if there exist positive constants $C_1$, $C_2$, and a positive integer $n_0$ such that $0 \leq C_1 g(n) \leq f(n) \leq C_2 g(n)$ for all $n \geq n_0$, then we write $f(n) \asymp g(n)$, $n \to \infty$ (asymptotic tight bound).

To simplify the writing, we will omit “$n \to \infty$” in the asymptotic notation.

It follows directly from Stirling’s asymptotic formula that

$$\frac{\Gamma(n + \alpha)}{\Gamma(n + \beta)} \asymp n^{\alpha-\beta}$$

for arbitrary real numbers $\alpha$ and $\beta$. For appropriate values of $q$, $n$ the formula $\frac{\Gamma(q+n)}{\Gamma(q)} = (q)_n$ holds. Thus, for $q > 0$,

$$\frac{(q)_n}{n!} = \frac{(q)_n}{\Gamma(n+1)} \asymp n^{q-1}.$$  

Hence, by (1),

$$\tilde{a}_{2n}^{(\lambda, \mu)} \asymp n^{1/2}, \quad \tilde{a}_{2n+1}^{(\lambda, \mu)} \asymp n^{1/2}. \tag{2}$$

Define the uniform norm of a continuous function $f$ on $[-1, 1]$ by

$$\|f\|_\infty = \max_{-1 \leq t \leq 1} |f(t)|.$$  

The maximum of two real numbers $x$ and $y$ is denoted by $\max(x, y)$.

Now we can formulate the main result.

**Theorem 1.** Let $\lambda > -\frac{1}{2}$, $\mu > 0$. Then

$$\|\tilde{C}_n^{(\lambda, \mu)}\|_\infty \asymp n^{\max(\lambda, \mu)}.$$  

The proof of the above theorem is contained in Section 2. In the next section, we give some important properties of Jacobi polynomials needed for our purpose.
1. Some facts on Jacobi polynomials

For $\alpha, \beta > -1$, the following formulae are valid \[4, \S 2.3, \text{Corollary 3.2}\]

$$P_n^{(\alpha,\beta)}(-t) = (-1)^n P_n^{(\beta,\alpha)}(t),$$

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha + 1)n}{n!} \asymp n^\alpha, \quad |P_n^{(\alpha,\beta)}(-1)| = \frac{(\beta + 1)n}{n!} \asymp n^\beta.$$  

(3)

We have [5, Theorem 7.32.1]

$$\|P_n^{(\alpha,\beta)}\|_\infty = \left\{ \begin{array}{cl}
P_n^{(\alpha,\beta)}(1), & \alpha \geq \beta, \quad \alpha \geq -\frac{1}{2}, \\
|P_n^{(\alpha,\beta)}(-1)|, & \alpha \leq \beta, \quad \beta \geq -\frac{1}{2}
\end{array} \right.$$  

(4)

and

$$\max_{0 \leq t \leq 1} |P_n^{(\alpha,\beta)}(t)| \lesssim n^{\max(\alpha,-1/2)}.$$  

(5)

It follows from Theorem 8.1.1 in [5] that, for $\alpha > -\frac{1}{2}$,

$$|P_n^{(\alpha,\beta)}(\cos n^{-1})| \asymp n^\alpha.$$  

(6)

It is known (see, for example, [5, Theorem 7.32.2]) that

$$\max_{\frac{1}{n} \leq \theta \leq \frac{\pi}{2}} \left\{ \theta^{\alpha + \frac{1}{2}} |P_n^{(\alpha,\beta)}(\cos \theta)| \right\} \lesssim n^{-\frac{1}{2}},$$

$$\max_{0 \leq \theta \leq \frac{\pi}{2}} |P_n^{(\alpha,\beta)}(\cos \theta)| \lesssim n^\alpha.$$  

(7)

Lemma 1. Let $\alpha > \frac{1}{2}$. Then

$$\max_{\frac{1}{n} \leq \theta \leq \frac{\pi}{2}} \left\{ \sin \frac{\theta}{2} |P_n^{(\alpha,\beta)}(\cos \theta)| \right\} \lesssim n^{\alpha-1},$$

$$\max_{\frac{1}{n} \leq \theta \leq \frac{\pi}{2}} \left\{ \sin \frac{\theta}{2} |P_n^{(\alpha,\beta)}(\cos \theta)| \right\} \lesssim n^{\max(\beta,-1/2)}.$$  

(8)

Proof. Using the assumption $\alpha > \frac{1}{2}$ and the asymptotic equality $\sin \frac{n^{-1}}{2} \asymp n^{-1}$, it can be easily seen that, for sufficiently large $n$,

$$\max_{\frac{1}{n} \leq \theta \leq \frac{\pi}{2}} \left\{ \theta^{\alpha - \frac{1}{2}} \sin \frac{\theta}{2} \right\} = n^{\alpha + \frac{1}{2}} \sin \frac{n^{-1}}{2} \asymp n^{\alpha-1/2}.$$  

Hence, by (8), we obtain

$$\max_{0 \leq \theta \leq \frac{\pi}{2}} \left\{ \sin \frac{\theta}{2} |P_n^{(\alpha,\beta)}(\cos \theta)| \right\} \lesssim n^{\alpha-1},$$  

(9)
\[
\max_{\frac{1}{4} \leq \theta \leq \frac{3}{4}} \left\{ \sin \frac{\theta}{2} | P_n^{(\alpha, \beta)}(\cos \theta)| \right\} \leq \\
\leq \max_{\frac{1}{4} \leq \theta \leq \frac{3}{4}} \left\{ \theta^{-\alpha-\frac{1}{2}} \sin \frac{\theta}{2} \right\} \cdot \max_{\frac{1}{4} \leq \theta \leq \frac{3}{4}} \left\{ \theta^{\alpha+\frac{1}{2}} | P_n^{(\alpha, \beta)}(\cos \theta)| \right\} \lesssim n^{\alpha-1}.
\]

So, we get the desired estimate on \([1, 3]\), and Lemma 1 with \(\alpha \)

Thus, Theorem 1 is proved for even orthonormal generalized Gegenbauer polynomials.

The asymptotic upper bound on \(\lambda < \mu\) follows from (3), (6). Indeed,

\[
\max_{\frac{1}{2} \leq \theta \leq \pi} \left\{ \sin \frac{\theta}{2} | P_n^{(\alpha, \beta)}(\cos \theta)| \right\} \approx \max_{\frac{1}{2} \leq \theta \leq \pi} | P_n^{(\alpha, \beta)}(\cos(\pi - \theta))| = \\
= \max_{0 \leq \theta \leq \frac{\pi}{2}} | P_n^{(\alpha, \beta)}(-\cos \theta)| = \max_{0 \leq \theta \leq \frac{\pi}{2}} | P_n^{(\beta, \alpha)}(\cos \theta)| \lesssim n^{\max(\beta, 1/2)}.
\]

\[\square\]

2. Proof of Theorem 1

According to (1), (5), (4), and (2), we get

\[
\| \tilde{C}_{2n}^{(\lambda, \mu)} \|_{\infty} = \tilde{a}_{2n}^{(\lambda, \mu)} \max_{-1 \leq t \leq 1} | P_n^{(\lambda-1/2, \mu-1/2)}(2t^2 - 1)| = \\
= \tilde{a}_{2n}^{(\lambda, \mu)} \max_{-1 \leq t \leq 1} | P_n^{(\lambda-1/2, \mu-1/2)}(t)| \approx n^{\max(\lambda, \mu)}.
\]

Thus, Theorem 1 is proved for even orthonormal generalized Gegenbauer polynomials.

Consider the case that \(\lambda \geq \mu + 1\).

In particular, we have \(\lambda > \mu\). By (5), (4), the assumption \(\lambda \geq \mu + 1\) implies that

\[
\| P_n^{(\lambda-1/2, \mu+1/2)} \|_{\infty} = P_n^{(\lambda-1/2, \mu+1/2)}(1) \approx n^{\lambda-1/2}.
\]

Hence, it follows from the above asymptotic equality, (1), and (2), that

\[
\| \tilde{C}_{2n+1}^{(\lambda, \mu)} \|_{\infty} \lesssim n^{\lambda}.
\]

Consider the case that \(\lambda < \mu + 1\).

Note that \(2 \sin^2 \frac{\theta}{2} - 1 = -\cos \theta\). Making the change of variable \(x = \sin \frac{\theta}{2}\) and applying (1) - (3), and Lemma 1 with \(\alpha = \mu + \frac{1}{2}, \beta = \lambda - \frac{1}{2}\), we get

\[
\| \tilde{C}_{2n+1}^{(\lambda, \mu)} \|_{\infty} = \tilde{a}_{2n+1}^{(\lambda, \mu)} \max_{-1 \leq t \leq 1} | x P_n^{(\lambda-1/2, \mu+1/2)}(2x^2 - 1)| = \\
= \tilde{a}_{2n+1}^{(\lambda, \mu)} \max_{0 \leq x \leq 1} | x P_n^{(\lambda-1/2, \mu+1/2)}(2x^2 - 1)| = \\
= \tilde{a}_{2n+1}^{(\lambda, \mu)} \max_{0 \leq \theta \leq \pi} \left\{ \sin \frac{\theta}{2} | P_n^{(\lambda-1/2, \mu+1/2)}(-\cos \theta)| \right\} = \\
= \tilde{a}_{2n+1}^{(\lambda, \mu)} \max_{0 \leq \theta \leq \pi} \left\{ \sin \frac{\theta}{2} | P_n^{(\mu+1/2, \lambda-1/2)}(\cos \theta)| \right\} \lesssim \begin{cases} 
\mu^{\mu}, & \mu > \lambda, \\
n^{\lambda}, & \mu \leq \lambda.
\end{cases}
\]
Using (1), (4), and (2), we obtain
\[
\| \tilde{C}^{(\lambda,\mu)}_{2n+1} \|_{\infty} \geq \tilde{C}^{(\lambda,\mu)}_{2n+1}(1) = \tilde{a}^{(\lambda,\mu)}_{2n+1} n^{-1/2} (\cos \theta) \bigg| P_{n}^{(\lambda-1/2,\mu+1/2)}(\cos \theta) \bigg| \geq n^\lambda.
\] (10)

Because of (9), (2), and (7), we have
\[
\| \tilde{C}^{(\lambda,\mu)}_{2n+1} \|_{\infty} = \tilde{a}^{(\lambda,\mu)}_{2n+1} \max_{0 \leq \theta \leq \pi} \bigg\{ \sin \frac{\theta}{2} \bigg| P_{n}^{(\mu+1/2,\lambda-1/2)}(\cos \theta) \bigg| \bigg\} \geq \tilde{a}^{(\lambda,\mu)}_{2n+1} \sin \frac{n-1}{2} \bigg| P_{n}^{(\mu+1/2,\lambda-1/2)}(\cos n^{-1}) \bigg| \geq n^\mu.
\] (11)

Combining (9), (10), and (11), we observe the desired asymptotic behavior in the considered situation.

Theorem 1 is completely proved.

3. Conclusion

The generalized Gegenbauer polynomials play an important role in Dunkl harmonic analysis (see, for example, [2, 3]). So, the study of these polynomials for different purposes is very natural.

As an application of Theorem 1, we are going to establish the Hausdorff – Young and the Hardy – Littlewood-type inequalities for orthonormal generalized Gegenbauer polynomials in future publications.

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