Lectures on Factorization Homology, 
$\infty$-Categories, and 
Topological Field Theories

Notes by:
Araminta Amabel
Artem Kalmykov
Lukas Müller
Hiro Lee Tanaka
ABSTRACT. These are notes from an informal mini-course on factorization homology, infinity-categories, and topological field theories. The target audience was imagined to be graduate students who are not homotopy theorists.
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Introduction

The aim of these lectures is to give an expository and informal introduction to the topics in the title.

There will be two running themes in these lectures:

The first is the interaction between smooth topology and algebra. For example, we will see early on that the smooth topology of oriented, 1-dimensional disks (together with how they can embed into each other) captures exactly the algebraic structure of being a unital associative algebra. Factorization homology allows us to begin with such algebraic structures and construct invariants of both manifolds and algebras through formal, categorical constructions. As an example, the category of representations of a quantum group $U_q(g)$ is braided monoidal (in fact, ribbon), and hence defines invariants of 2-dimensional manifolds equipped with additional structures. These invariants come equipped with mapping class group actions that recover well known actions discovered in the representation theory of quantum groups $^{[19]}$.

Another place we will see this theme is in the cobordism hypothesis of Baez-Dolan $^{[10]}$ and Lurie $^{[62]}$. This articulates a precise way in which the smooth theory of framed cobordisms recovers the purely algebraically characterizable notion of objects having duals and morphisms having adjoints.

The second theme is the need to cohere various algebraic structures. A common question during the lectures was “so is this actually associative or not?” Coherence is a subtle beast, and we indicate some of the places where the language of $\infty$-categories is actually useful in articulating coherences, and in concluding the existence of continuous symmetries (thus allowing us to observe the existence of, say, mapping class group actions). The goal was to give an audience a feel for why this language is useful; we certainly did not give a user’s guide, but we hoped to make the ideas less foreign.

It must be said that there are many topics that could have been covered, but certainly were not. Though this list is intended as an apology, the reader may also take it to be a list of further reading, or at least a list of exciting storylines to follow.

- Most notably, there have been numerous works by Ayala and Francis; they have shown how factorization homology can be generalized in ways exhibiting the explicit coherence between Poincaré Duality for smooth manifolds and Koszul duality for $E_n$-algebras $^{[3,1]}$. They have also developed a program for proving the cobordism hypothesis that does not involve intricate Cerf
theory [4]. These are important developments that articulate how factorization homology and its generalizations may be central to smooth topology and higher algebra.

- To those interested in the interactions between representation theory and low-dimensional topology, let us mention another disservice. We do not touch on the strategy for producing quantum link invariants by computing invariants of chain maps

$$\int_{S^1 \times D^2} A \longrightarrow \int_{S^3} A$$

induced by framed embeddings of knots in $S^3$. (Here, $A$ is an $E_3$-algebra one can construct out of a semisimple Lie algebra.) As far as HLT understands, this is work in progress of Costello, Francis, and Gwilliam. We also do not touch upon the work of Ben-Zvi-Brochier-Jordan [19].

- Another perspective lacking in these notes is the work of Costello and Costello-Gwilliam [28] on factorization algebras, which makes more manifest the connections to perturbative quantum field theories and deformation theory. Aside from the physical considerations, these works also give deformation-theoretic sources of locally constant factorization algebras and exhibit fruitful connections to shifted Lie and Poisson algebras in characteristic 0. Indeed, we make no mention of Kontsevich formality theorems or of $P_n$-algebras in these notes.

- We do not touch the algebro-geometric avatars expressed through Ran spaces, as developed by Francis-Gaitsgory [34] and as utilized by Gaitsgory-Lurie [36] to prove Weil’s conjecture on Tamagawa numbers.

- Lurie’s topological account in [63] also uses the Ran space perspective; there he refers to (a slightly more general version of) factorization homology as topological chiral homology.

- We do not discuss applications to configuration spaces (though this is a manifestly natural topic to consider) and relations to Lie algebra homology. Representative works include those of Knudsen [55, 56]. The algebro-geometric techniques from the previous bullet point have also been used to prove homological stability for configuration spaces arising in positive characteristic [45]. Variants of factorization homology techniques to “partial algebra” settings have also been fruitful [57].

- There is no mention in these notes of the intersection with derived algebraic geometry. For example, when $A$ is a commutative cdga in non-positive degree, its Hochschild chain complex is at once (functions on) the derived loop space of $A$ (in the algebro-geometric sense) and the factorization homology of the circle with coefficients in $A$. Put another way, Hochschild chains also have an interpretation as the sheaf cohomology of the structure sheaf of the mapping stack $Map(S^1, Spec A)$ where $S^1$ is the constant stack.
On the other hand, when \( \text{Spec} \, A \) is replaced by a stack which is non-affine, \( \otimes \)-excision fails. Moreover, when one further considers stacks not on the site of (commutative) affine schemes, but on a site of \( \mathcal{E}_n \)-stacks (where “affine” objects are now \( \text{Spec} \) of \( \mathcal{E}_n \)-algebras), these mapping stacks are expected to be far more sensitive manifold invariants. Many are expected to arise as the result of deformation quantizations of shifted symplectic structures as discussed in the next bullet point. (For more, see [14] and [15] and [5].)

- Another open problem relating to derived algebraic geometry is the problem of quantizing shifted symplectic stacks; for example, the AKSZ formalism—as reinterpreted in [66]—suggests we should be able to quantize (the category of sheaves on) certain mapping stacks with shifted Poisson or shifted symplectic structures. This was carried out for some examples in [23], and for example, one obtains a braided monoidal quantization of the category of finite-dimensional representations of a Lie algebra \( \mathfrak{g} \). These quantizations are expected to lead to interesting examples of TFTs, in contrast to the non-quantized TFTs (for example, constructed out of the naive operation \( W \mapsto \text{Map}(W, BG) \)—this TFT is only sensitive to the homotopy type of manifolds \( W \)). The intuition that certain symplectic structures should give rise to quantizations is of course an old story with many modern narratives, but it should be mentioned that a hugely influential starting point of algebraically deformation-quantizing Poisson structures is Kontsevich’s formality theorem for Poisson manifolds [54].

For those seeking more algebraic consequences of framing data, we refer to the excellent notes of Teleman [72], who also takes a far more representation-theoretic basepoint for their exposition. Other resources include [1] and [5] and [37].

Finally, we note that the contents of these notes expand significantly on the contents of the delivered lectures. This was done in the hopes of having a somewhat more complete written account of the story. We warn future speakers that they should not attempt to fit a chapter of these notes into a single lecture.

\textbf{Convention .0.0.1.} Unless noted explicitly, every manifold in this work is smooth and paracompact. For simplicity the reader may assume that every manifold may be obtained by a finite sequence of open handle attachments—for example, every manifold arises as the interior of some compact manifold with boundary. One can treat larger manifolds (for example, countably infinite-genus surfaces), but we would have to say a few words about preserving filtered colimits. See Remark II.6.4.3.

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CHAPTER 1

The one-dimensional case

The goal of our first lecture is to see that the geometry of oriented embeddings of open, 1-dimensional disks gives rise to the algebra of associativity. That is, local building blocks of oriented 1-manifolds encode algebraic structure. We then explore how these local structures could be extended to define invariants of all oriented 1-dimensional manifolds. We will study an extension called factorization homology.

Of course, we know how to classify (oriented) 1-manifolds, so this invariant gains little for us as a manifold invariant. But we will see that an interesting algebraic invariant (the Hochschild chain complex) appears. Moreover, by exploiting the continuity of factorization homology, we will discover that the Hochschild complex naturally admits a circle action.

We will move onto higher-dimensional manifolds in the next chapter.

I.1. The algebra of disks.

DEFINITION I.1.0.1. Consider the category $\text{Disk}_{1,\text{or}}$ of oriented, 1-dimensional disks.

- Objects are finite disjoint unions of oriented 1-dimensional open disks. We can enumerate the objects as follows:
  \[
  \emptyset, \quad \mathbb{R}, \quad \mathbb{R} \coprod \mathbb{R}, \quad \ldots \quad \mathbb{R} \coprod \mathbb{R} \coprod \ldots,
  \]
  Note that we allow for the empty disjoint union.
- The collection of morphisms
  \[
  \text{hom}(\mathbb{R} \coprod \mathbb{R} \coprod \ldots, \mathbb{R} \coprod \mathbb{R} \coprod \ldots)
  \]
  consists of all smooth embeddings $j : \mathbb{R} \coprod \mathbb{R} \hookrightarrow \mathbb{R} \coprod \mathbb{R}$ respecting orientations.

This category comes with a natural operation of taking disjoint union: If $X$ is a union of $k$ oriented disks and $Y$ is a union of $l$ oriented disks, then $X \coprod Y$ is a union of $k + l$ oriented disks. There are natural isomorphisms

\[
X \coprod Y \cong Y \coprod X
\]

and $\coprod$ renders $\text{Disk}_{1,\text{or}}$ a symmetric monoidal category. The empty set $\emptyset$ is the monoidal unit of this category, as we can supply natural isomorphisms $X \coprod \emptyset \cong X \cong \emptyset \coprod X$. 

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**Notation I.1.0.2.** Fix a field \( k \). We denote by \( \text{Vect}_k^{\otimes} \) the category of \( k \)-vector spaces. The superscript \( \otimes_k \) indicates that we view \( \text{Vect}_k \) as a symmetric monoidal category equipped with the usual tensor product of vector spaces:

\[
\text{Vect}_k \times \text{Vect}_k \to \text{Vect}_k, \quad (V, W) \mapsto V \otimes_k W.
\]

That’s the set-up. For no good reason whatsoever, I want to now study functors \( F : \text{Disk}^\sqcup_{1, \text{or}} \to \text{Vect}_k^{\otimes} \)

where:

1. \( F \) is symmetric monoidal.
2. Isotopic embeddings are sent to the same linear map.

**Example I.1.0.3.** Thanks to the second requirement, such functors become quite tractable. For instance, consider the orientation-preserving smooth embeddings of \( \mathbb{R} \) into itself. This is the space of all smooth, strictly increasing functions \( \mathbb{R} \to \mathbb{R} \). These embeddings are all isotopic to a very simple one – the identity map \( \text{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R} \). You can check this fact as an exercise.

Therefore, if the functor \( F \) satisfies (2), \( F \) sends the whole collection of morphisms \( \text{hom}_{\text{Disk}^\sqcup_{1, \text{or}}} (\mathbb{R}, \mathbb{R}) \)

to the identity element

\[ id_{F(\mathbb{R})} \in \text{hom}_{\text{Vect}_k^{\otimes}} (F(\mathbb{R}), F(\mathbb{R})). \]

**Warning I.1.0.4.** If you are not used to monoidal categories, when you hear that \( F \) must be symmetric monoidal, you probably imagine that \( F \) “respects” the symmetric monoidal structures on both sides. But I put “respect” in quotes because being symmetric monoidal is not merely a property of \( F \); the data of \( F \) being symmetric monoidal means we also supply natural isomorphisms

\[
F(X \amalg Y) \cong F(X) \otimes_k F(Y).
\]

We also demand that the swap isomorphism \( X \amalg Y \cong Y \amalg X \) is sent to the swap isomorphism \( F(X) \otimes F(Y) \cong F(Y) \otimes F(X) \) in a way compatible with the above natural isomorphism.

Here is the main result I’d like us to understand. It is fundamental to everything that follows.

**Theorem I.1.0.5.** Suppose \( F : \text{Disk}^\sqcup_{1, \text{or}} \to \text{Vect}_k^{\otimes} \) is a symmetric monoidal functor such that if \( j \) and \( j' \) are isotopic embeddings, then \( F(j) = F(j') \). Then the data of \( F \) is equivalent to the data of a unital associative \( k \)-algebra.

**(Sketch of) Proof.** Let us set the notation

\[ A = F(\mathbb{R}). \]
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Since $F$ is symmetric monoidal, the empty set $\emptyset$ is sent to the monoidal unit of $\text{Vect}_k^{\otimes}$, which is the base field $k$. More generally, any object of $\text{Disk}_{1,\text{or}}$ is a disjoint union of $l$ disks, and $F$ sends $\mathbb{R}^l$ to the tensor power $A^\otimes l$.

We now study $F$'s effect on morphisms. Consider the morphism set $\text{hom}_{\text{Disk}_{1,\text{or}}} (\emptyset, \mathbb{R})$. This set consists of one element, given by the embedding $\eta : \emptyset \to \mathbb{R}$. $F$ determines a map

$$u = F(\eta) : k \to A$$

which we will call the unit.

Now consider $\text{hom}_{\text{Disk}_{1,\text{or}}} (\mathbb{R}^2, \mathbb{R})$. Up to isotopy, there are exactly two embeddings $\mathbb{R}^2 \to \mathbb{R}$. To see this, fix a smooth orientation-preserving embedding $j : \mathbb{R}^2 \to \mathbb{R}$. Let us order (non-canonically) the connected components of the domain $\mathbb{R}^2$, and consider the image of each component under $j$; the orientation of the codomain $\mathbb{R}$ respects this ordering, or does not. This distinguishes the two connected components to which $j$ can belong.

Let us denote an order-respecting embedding by $j_1$, and a non-respecting embedding by $j_2$.

Take

$$m = F(j_1) : A \otimes_k A \to A.$$

This is our multiplication. We claim that $m$ and $u$ determine a unital associative algebra structure on $A$.

- **Unit:** we need to verify that the diagram of vector spaces

\begin{align*}
A \xrightarrow{\cong} A \otimes_k k \xrightarrow{id_A \otimes_k u} A \otimes_k A \xrightarrow{m} A
\end{align*}

(I.1.0.6)

commutes. Consider the following diagram in $\text{Disk}_{1,\text{or}}$:

\begin{align*}
\mathbb{R} \xrightarrow{\cong} \mathbb{R} \coprod \emptyset \xrightarrow{id_\mathbb{R} \coprod \eta} \mathbb{R} \coprod \mathbb{R} \xrightarrow{j_i} \mathbb{R}.
\end{align*}

This diagram is not commutative—the embeddings of $\mathbb{R}$ to itself, given by $j_1 \circ (\eta \circ id_{\mathbb{R}})$ and $id_{\mathbb{R}}$, need not be equal—but there does exist an isotopy between the two embeddings. That is, the diagram commutes up to isotopy. Thus, after applying $F$, the induced diagram (I.1.0.6) is commutative.
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- **Associativity:** we need to check

\[
\begin{array}{c}
A \otimes_k A \otimes_k A \\
\downarrow \text{id}_A \otimes_k m \\
A \otimes_k A
\end{array}
\quad
\begin{array}{c}
m \circ \text{id}_A \\
\downarrow m \\
A
\end{array}
\]  

(I.1.0.7)

commutes. Consider the embeddings \(j_1 \circ (j_1 \square \text{id}_k)\) and \(j_1 \circ (\text{id}_k \square j_1)\), which we may draw as follows:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & (1) & \longrightarrow & (2) & \longrightarrow & (3) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(1) & \quad & (2) & \quad & (3) & \quad & (1) & \quad & (2) & \quad & (3)
\end{array}
\]

Depending on the specific choice of \(j_1\), the compositions \(j_1 \circ (j_1 \square \text{id}_k)\) and \(j_1 \circ (\text{id}_k \square j_1)\) may be non-equal; but there is an isotopy between the two embeddings. Because \(F\) collapses isotopic embeddings to the same linear map, the associativity axiom is satisfied. You can now check that the other embeddings produce no new restrictions on the data of \(m\) and of the unit.

For the reverse implication, let \(A\) be a unital associative algebra, i.e. we are given maps \(m : A \otimes_k A \rightarrow A\) and \(u : k \rightarrow A\) satisfying the unit and associativity axioms. We can reverse-engineer a functor \(F\) by the same rules as above. We leave it to the reader to check that all embeddings, up to isotopy, can be factored as a composition of (i) component-permutations \(\mathbb{R} \sqcup \mathbb{R} \rightarrow \mathbb{R} \sqcup \mathbb{R}\), and (ii) disjoint unions of \(\eta\) and of \(j_1\). Therefore, the property of \(A\) being associative and unital is enough to determine the whole functor \(F\).

\[\square\]

**Remark I.1.0.8.** What happened to \(j_2\), otherwise known as the “other” embedding \(\mathbb{R} \sqcup \mathbb{R} \rightarrow \mathbb{R}\)? We may express \(j_2\) as a composition \(j_1 \circ \sigma\) where \(\sigma\) is the swap map \(\mathbb{R} \sqcup \mathbb{R} \cong \mathbb{R} \sqcup \mathbb{R}\) switching the components. Because \(F\) was demanded to be symmetric monoidal, we have

\[
F(j_2) = F(j_1 \circ \sigma) = m \circ \sigma_{\text{Vect}^\otimes_k}.
\]

(Here, \(\sigma_{\text{Vect}^\otimes_k}\) is the swap map \(V \otimes W \cong W \otimes V\).) Hence \(F(j_2)\) is no new data; it encodes the canonical “opposite” multiplication given by any associative algebra.

Importantly, note that there is no reason to prefer \(j_2\) over \(j_1\), and in particular, we break symmetry when we decide whether a functor \(F\) ought to determine the algebra \(A\) with multiplication \(m = F(j_1)\), or with multiplication \(m = F(j_2)\). This symmetry is due to the automorphism of the category of associative algebras (which sends an
algebra to its opposite), and the automorphism of the category of symmetric monoidal functors $F$ (which precomposes $F$ with the “orientation-reversal” automorphism of the category $\text{Disk}_{1,\text{or}}$).

In a sense, a miracle happened: an algebraic operations turns out to be “encoded” in something geometric and rather simple.

**Dissatisfaction I.1.0.9.** However, some things leave us unsatisfied.

1. First, why restrict ourselves just to open disks? Could we extend $F$ to other 1-manifolds—for example, to a circle? Then $F(S^1)$ would be an invariant of a circle, and this may even be an interesting invariant of the algebra $A$.

2. Second, the isotopy business. It seems unnatural to require isotopy invariance from the functors. Let me be careful about this point.

Imagine that we have managed to assign an invariant to a circle, say $F(S^1)$. Of course, up to isotopy, any two orientation-preserving diffeomorphisms of the circle are equivalent. But there are *interesting* isotopies—for example, the identity map $\text{id}_{S^1} : S^1 \to S^1$ admits self-isotopies given by full rotations. Wouldn’t our invariant $F(S^1)$ be more interesting if it detected these self-isotopies?

We will come back to this last point in a moment. Let us first fantasize what an invariant of the circle might look like.

**I.2. The co-center: A first stab at a circle invariant**

**Definition I.2.0.1.** Let $\text{Mfld}_{1,\text{or}}$ be the category of oriented, 1-dimensional manifolds with finitely many connected components. Morphisms are smooth embeddings that respect orientation. This category is also symmetric monoidal with respect to the disjoint union $\coprod$.

**Remark I.2.0.2.** Note the (symmetric monoidal) inclusion of the category $\text{Disk}_{1,\text{or}}$ into the category $\text{Mfld}_{1,\text{or}}$.

Suppose we are given an associative algebra $A$, which we identify with a symmetric monoidal functor

$$F : \text{Disk}_{1,\text{or}} \to \text{Vect}_k$$

(using Theorem I.1.0.5). Can we extend $F$ to a dashed functor as below?

$$\begin{array}{ccc}
\text{Disk}_{1,\text{or}} & \xrightarrow{F} & \text{Vect}_k \\
\downarrow & & \downarrow \\
\text{Mfld}_{1,\text{or}} & \xrightarrow{-} & \text{Vect}_k
\end{array}$$

Let us imagine what such an extension would “want” to apply to $S^1$. We will denote by $F(S^1)$ the vector space assigned by this extension.
From embeddings \( \mathbb{R}^k \to S^1 \) we get maps \( \phi_k : A^\otimes k \to F(S^1) \) for any \( k \geq 0 \). In fact, any such embedding factors through a single connected interval of \( S^1 \), so these linear maps must factor through the multiplication of \( A \):

\[
\phi_k : A^\otimes k \to A \to F(S^1).
\]

Moreover, since we are on a circle, one can isotope a configuration of \( k \) intervals cyclically. For instance, for \( k = 2 \), fixing an embedding \( h : \mathbb{R} \coprod \mathbb{R} \to S^1 \), we witness an isotopy

\[
h \circ \sigma \sim h.
\]

See Image I.2.0.3

**Image I.2.0.3.** An image of an isotopy realizing a cyclic permutation of embedded intervals in a circle. Note that the vertical direction is the time direction.

In particular, the linear map \( \phi_2 \) satisfies the property that \( \phi_2 \circ \sigma = \phi_2 \) where \( \sigma \) is the swap map \( A \otimes A \cong A \otimes A \). For example, for elements \( x_1, x_2 \in A \) to \( \phi_2 \), we have

\[
\phi_1(x_2 x_1) = \phi_2(x_2 \otimes x_1) = \phi_2(x_1 \otimes x_2) = \phi_1(x_1 x_2).
\]

Thus, whatever the map \( \phi_1 : A \to F(S^1) \) is, it renders multiplication commutative; this means \( \phi_1 \) must factor through the quotient

\[
A/[A, A].
\]

There is at this point a naive

**Guess I.2.0.4.** The invariant \( F(S^1) \) of the circle is isomorphic to \( A/[A, A] \).
Remark I.2.0.5. Sometimes, $A/[A, A]$ is called the *cocenter* of $A$. If $A$ is commutative, the projection map from $A$ to the cocenter is an isomorphism.

The vector space $A/[A, A]$ has another presentation:

**Exercise I.2.0.6.** Check that $A/[A, A]$ is isomorphic to $A \otimes_{A \otimes_k A^{op}} A$.

(To make sense of the tensor product, note that the algebra $A$ is naturally an $(A, A)$-bimodule by left and right multiplication. This means $A$ can be considered both a right- and left-module over the ring $A \otimes_k A^{op}$, where $A^{op}$ is the algebra with the opposite multiplication.)

Remark I.2.0.7. If you have never done this exercise, I highly encourage you to do it.

We will come back to this expression by the end of this lecture.

### I.3. $\infty$-categories

Let us return to (2) of Dissatisfaction I.1.0.9. What to do with isotopies? The frustration of (2) might inspire us to contemplate a category whose collection of morphisms has a topology. If our invariants are functors that respect these topologies, we have a hope of seeing the topology of the space of embeddings from $S^1$ to $S^1$, and hence these non-trivial rotational isotopies.

We will talk more about this next lecture, but let’s use this motivation to talk a bit about $\infty$-categories. On a first pass, you can think of an $\infty$-category as having a collection of objects and, for each pair of objects $\langle X_0, X_1 \rangle$, a *topological space* of morphisms $\text{hom}(X_0, X_1)$ (instead of a mere set of morphisms). We demand that the composition maps

$$\text{hom}(X_1, X_2) \times \text{hom}(X_0, X_1) \longrightarrow \text{hom}(X_0, X_2), \quad (f_{12}, f_{01}) \longmapsto f_{12} \circ f_{01}$$

be continuous.

For example, one can endow the collection of smooth, orientation-preserving embeddings

$$\{ j : X \longrightarrow Y \}$$

with a topology—e.g., the Whitney $C^\infty$ topology. Composition of embeddings is a continuous operation. In this way, we can define

**Definition I.3.0.1.** We let

$$\text{Disk}_{1, or}$$

denote the $\infty$-category whose objects are (disjoint unions of) oriented 1-dimensional open disks, and whose morphism spaces are given by the space of smooth, orientation-preserving embeddings.

**Warning I.3.0.2.** There is a font difference between $\text{Disk}$ (the category from before) and $\text{Disk}$ (the $\infty$-category, which sees the topology of embedding spaces).
**Definition I.3.0.3** (Homotopy). Given an $\infty$-category $C$ and two objects $X, Y$, fix two morphisms $f_0, f_1 \in \text{hom}_C(X, Y)$. A continuous path from $f_0$ to $f_1$ in the space $\text{hom}_C(X, Y)$ is called a homotopy from $f_0$ to $f_1$.

**Warning I.3.0.4.** Consider the example of $\text{Disk}_{1,\text{or}}$. Confusingly, a homotopy is not the same thing as a smooth isotopy, as one might choose a continuous path in $\text{hom}_{\text{Disk}_{1,\text{or}}}$ which does not give rise to a smooth isotopy; regardless, by smooth approximation, two morphisms are homotopic if and only if they are smoothly isotopic as embeddings.

For now, you can think of a functor between $\infty$-categories as a functor in the usual sense, and for which the maps between morphism spaces are continuous.

What are other examples of $\infty$-categories?

**Example I.3.0.5.** Every ordinary category is an $\infty$-category—we simply treat the set of morphisms as a discrete topological space.

**Example I.3.0.6.** As a sub-example, the category of vector spaces $\text{Vect}_k$ is an $\infty$-category with a discrete morphism spaces. That is, two linear maps $f : V \to W$ are homotopic if and only if they are equal.

**Example I.3.0.7 (Chain complexes).** We are now going to sketch the idea of an $\infty$-category $\text{Chain}_k$

of cochain complexes. Its objects are cochain complexes, and we will content ourselves with only sketching the space of morphisms. To this end, fix two cochain complexes $V$ and $W$. One can construct a space $\text{hom}(V, W)$ which is combinatorially defined—i.e., built of simplices:

- Vertices of $\text{hom}(V, W)$ are usual morphisms of chain complexes—that is, maps $f : V \to W$ such that $df = fd$.
- An edge is the data of a triplet $(f_0, f_1, H)$ where $f_0, f_1$ are vertices and $H$ is a chain homotopy, i.e. a degree -1 map $H : V \to W$ satisfying $dH + Hd = f_1 - f_0$.
- Simplices of dimension $k$ are degree $-k$ maps exhibiting homotopies between homotopies. For example, a triangle is the data

$$(f_0, f_1, f_2, H_{01}, H_{02}, H_{12}, G)$$

where the $H_{ij}$ are homotopies from $f_i$ to $f_j$, and $G$ is a degree -2 map $G : V \to W$ exhibiting a homotopy between $H_{02}$ and $H_{12} + H_{01}$.

This space is called the Dold-Kan space of the usual hom cochain complex $\text{Hom}^\bullet(V, W)$. We won’t talk much about it, though we will talk a little more in the next lecture about the general philosophy of what algebraically motivated $\infty$-categories look like.

For more on Dold-Kan, the interested reader may consult III.2 of [39]. Original sources are [29] and [50].
Remark I.3.0.8. For now, I’d like to say that this combinatorially defined space has well-understood homotopy groups:
\[ \pi_i(\text{hom}(V,W)) \cong H^{-i}(\text{Hom}^\bullet(V,W)), \quad i \geq 0. \]
That is, there is a natural isomorphism between the non-positive cohomology groups of the cochain complex \( \text{Hom}^\bullet \) and the homotopy groups of the space \( \text{hom} \).

The upshot being—in a way I’ll elaborate on next lecture—one should think of a functor of \( \infty \)-categories
\[ \text{Disk}_{1,\text{or}} \to \text{Chain}_k \]
as assigning a chain complex to each object of \( \text{Disk}_{1,\text{or}} \), a chain map to every embedding, and a chain homotopy to every isotopy of embeddings, and so forth.

One can also speak of symmetric monoidal \( \infty \)-categories; we don’t define these here, but you should stick with the intuition that these are symmetric monoidal categories whose symmetric monoidal structures are continuous with respect to the morphisms.

Example I.3.0.9. \( \text{Disk}^{\Pi}_{1,\text{or}} \) is a symmetric monoidal \( \infty \)-category with symmetric monoidal structure given by disjoint union.

I.4. Algebra of disks, revisited

Definition I.4.0.1. Fix a symmetric monoidal \( \infty \)-category \( C^\otimes \). A \( \text{Disk}_{1,\text{or}} \)-algebra in \( C^\otimes \) is a symmetric monoidal functor
\[ F : \text{Disk}^{\Pi}_{1,\text{or}} \to C^\otimes. \]

Example I.4.0.2. Take \( C^\otimes = \text{Vect}_k^\otimes \) as the target category. Because this is a discrete category, the conclusion of Theorem I.1.0.5 holds verbatim: The data of a symmetric monoidal functor \( F \) is equivalent to the data of a unital associative \( k \)-algebra. Indeed, the condition that “isotopic embeddings must be sent to the same linear maps” from before is re-expressed as our functor \( F \) being a functor of \( \infty \)-categories.

Let us next consider the \( \infty \)-category \( \text{Chain}_k \) with symmetric monoidal structure given by the usual tensor product \( \otimes_k \). As before, let us set \( A = F(\mathbb{R}) \).

Then for every embedding \( j : \mathbb{R}^{\Pi^2} \to \mathbb{R} \) we have a multiplication \( F(j) =: m_j : A \otimes A \to A \). The choice of \( j \) is by no means canonical, and specifying all these \( m_j \) is indeed an enormous amount of data. But the space of order-preserving \( j \) is contractible. So \( m_j \) is determined up to contractible choice. We will come back to this in the next lecture; for now, you should imagine that this data is manageable—in

\[ ^1 \text{It is important here that we are working over a field } k; \text{ otherwise we would take the derived tensor product, but this would lead us afield, pun intended.} \]

\[ ^2 \text{In the sense we used in the proof of Theorem I.1.0.5} \]
fact, applying a contraction to a point, you should imagine that specifying all the \( \{m_j\} \) continuously is tantamount to just producing a single \( m \).

But what to make of associativity of \( m_j \)? Tracing through the same proof as before, we find that the square (I.1.0.7) is commutative only up to chain homotopy. That is, the data of the functor \( F \) is not supplying a multiplication which is associative on the nose, but only associative up to homotopy.

You can now imagine that the data of the spaces of embeddings

\[
\mathbb{R} \coprod \ldots \coprod \mathbb{R} \to \mathbb{R}
\]

supplies even further complicated data. But in fact, the space of these embeddings is also contractible up to swapping components of the domain. All told, it turns out that the immense amount of data specifying an algebra “associative up to specified homotopies” has a name already:

**Theorem I.4.0.3.** The data of a functor between \( \infty \)-categories \( F : \text{Disk}^{1,\text{or}} \to \text{Chain}^k \) is equivalent to the data of a unital \( A_\infty \)-algebra.

More precisely, there is an equivalence of \( \infty \)-categories between the \( \infty \)-category of symmetric monoidal functors \( F \), and the \( \infty \)-category of unital \( A_\infty \)-algebras.

**Remark I.4.0.4.** The word “\( A_\infty \)” is not important at all for what we will speak of next. I just wanted to emphasize that a \( \text{Disk}^{1,\text{or}} \)-algebra is simply an algebra whose multiplication is associative up to certain specified homotopies; and if this is too much, you will lose little intuition imagining these algebras to be actually associative.

**Example I.4.0.5.** It is a classical fact that any unital \( A_\infty \)-algebra in \( \text{Chain}^k \) is equivalent to an associative, unital dg-algebra. In particular, unital dg-algebras are examples of \( \text{Disk}^{1,\text{or}} \)-algebras. Examples of such algebras include the de Rham cochains of a manifold (which in fact form a commutative dg algebra) and the endomorphism hom-complex of a cochain complex.

**Example I.4.0.6 (Maps of \( A_\infty \)-algebras.)**. However, you will gain something if you become accustomed to the fact that maps between these algebras need not respect multiplication and associativity on the nose. (This is also visible if one contemplates what a natural transformation between two symmetric monoidal functors \( \text{Disk}^{1,\text{or}} \to \text{Chain}^k \) looks like.) For example, an \( A_\infty \)-algebra map between two dg-algebras is not the same thing as a map of dg algebras.

Just to give you a feel for what an \( A_\infty \)-algebra map \( f : A \to B \) between two dg-algebras might look like, let us say that an \( A_\infty \)-algebra map is not simply a map

\[
f_1 : A \to B
\]

This follows, for example, by embedding an \( A_\infty \)-algebra into its category of modules through the Yoneda embedding. See for example Section (2g) of [70].
satisfying the equation

\[ f_1(da) = df_1(a) \quad \text{for all } a \in A. \]

An \(A_\infty\)-algebra map also contains the data of a map

\[ f_2 : A \otimes A \to B[-1] \]

satisfying

(I.4.0.7) \( f_1(m(a_2, a_1)) - m(f_1 a_2, f_1 a_1) = \pm f_2(da_2, a_1) + \pm f_2(a_2, da_1) + \pm df_2(a_2, a_1) \).

Note that (I.4.0.7) states that \( f_1 \) does not necessarily respect multiplication on the nose, but the failure to do so is controlled by an exact element in the hom cochain complex \( \text{Hom}^\bullet(A \otimes A, B) \) (as expressed on the righthand side). By definition, an \(A_\infty\)-algebra map also comes equipped with maps \( f_k : A^\otimes k \to B[k - 1] \) satisfying higher analogues of (I.4.0.7) that cohere associativity properties up to homotopy.

For more on \(A_\infty\)-algebras, we refer the interested reader to sources such as [52].

**Remark I.4.0.8.** The above is also a useful principle to keep in mind for \(\infty\)-categories. While a single \(\infty\)-category may be thought of as a category enriched in topological spaces, it is of course the maps and equivalences between them that make \(\infty\)-categories interesting. In particular, while any functor between \(\infty\)-categories can be modeled appropriately by an actual functor between topologically enriched categories, it is at times helpful to think instead of functors as lacking a “strict” respect for compositions, but equipped with higher coherences that “make up” for the lack of strict respect.

### I.5. Factorization homology of the circle

Now we are ready to define an invariant of \(S^1\).

**Definition I.5.0.1.** Denote by \(\mathcal{Mfld}_{1,or}^{\Pi} \) the \(\infty\)-category of oriented, one-dimensional manifolds with finitely many connected components. The morphism space

\[ \text{hom}_{\mathcal{Mfld}_{1,or}}(X, Y) \]

is the space of smooth, orientation-preserving embeddings from \(X\) to \(Y\).

**Remark I.5.0.2.** There is an obvious symmetric monoidal inclusion functor \(\mathcal{Disk}_{1,or}^{\Pi} \to \mathcal{Mfld}_{1,or}^{\Pi} \).

**Definition I.5.0.3.** Let \(\mathcal{C}^\otimes\) be given by \(\text{Vect}_k^\otimes\) or by \(\text{Chain}_k^\otimes\). Let \(A\) be a \(\mathcal{Disk}_{1,or}^{\Pi}\)-algebra in \(\mathcal{C}^\otimes\). The **factorization homology** of \(A\) is the left Kan extension

\[
\begin{array}{ccc}
\mathcal{Disk}_{1,or}^{\Pi} & \xrightarrow{A} & \mathcal{C}^\otimes \\
\downarrow & & \\
\mathcal{Mfld}_{1,or}^{\Pi} & \xrightarrow{f A} & \mathcal{C}^\otimes
\end{array}
\]
and we denote this functor by \( \int A \) as in the diagram. Given a smooth, oriented 1-dimensional manifold \( X \in \text{Mfld}_{1,\text{or}} \), we denote the value of the left Kan extension by

\[
\int_X A
\]

and we call this the factorization homology of \( X \) with coefficients in \( A \).

Let me not tell you what left Kan extension exactly is, for the time being. (See next lecture.) But let me tell you one theorem we can prove about this left Kan extension:

**Theorem I.5.0.4** (\( \otimes \)-excision for \( S^1 \)). Fix a \( \text{Disk}_{1,\text{or}} \)-algebra \( A \) in \( C^\otimes \) and fix an orientation on \( S^1 \). Then factorization homology of the circle admits an equivalence

\[
\int_{S^1} A \simeq A \otimes_{A \otimes A^{op}} A.
\]

**Remark I.5.0.6.** Let \( A \) be an associative \( k \)-algebra. As we saw in Exercise I.2.0.6, \( A \) is a bimodule over itself, and in particular, an \( A \otimes A^{op} \)-module (on the right, or on the left). This explains the righthand side of (I.5.0.5) when \( C^\otimes = \text{Vect}_k^{\otimes k} \).

To make sense of (I.5.0.5) when \( C^\otimes = \text{Chain}^{\otimes k} \), let us simply state that the notion of (bi)modules makes sense for \( A_\infty \)-algebras as well, and the notion of tensoring modules over an algebra also makes sense, for instance by articulating a model for the bar construction.

**Example I.5.0.7.** If \( A \) is a unital associative algebra in \( C^\otimes = \text{Vect}_k^{\otimes k} \), we already guessed that \( \int_{S^1} A \simeq A/[A, A] \) in Section I.2.

**Example I.5.0.8.** Let \( C^\otimes = \text{Chain}^{\otimes k} \). Then the bar construction models the derived tensor product:

\[
\int_{S^1} A \simeq A \otimes_{A \otimes A^{op}} A.
\]

This tensor product already has a name: It is the *Hochschild chain complex* of \( A \). (See [43] and [25].)

**Remark I.5.0.9.** Let \( A \) be an ordinary unital associative \( k \)-algebra, concentrated in degree 0. Then one may consider \( A \) to be a \( \text{Disk}_{1,\text{or}} \)-algebra in \( C = \text{Vect}^{\otimes k} \) and in \( C = \text{Chain}^{\otimes k} \). Then the bar construction \( A \otimes_{A \otimes A^{op}} A \) constructed in \( C = \text{Vect}_k \) yields a vector space given by the 0th cohomology of the Hochschild complex, otherwise known as \( A/[A, A] \). On the other hand, the bar construction constructed in \( C = \text{Chain}_k \) encodes more homotopically rich information, giving rise to the entire Hochschild chain complex of \( A \).
We mentioned at some point that the circle invariant naturally possesses an $S^1$-action (more correctly, an action of the orientation-preserving diffeomorphism group $Diff^+(S^1)$; and this group is homotopy equivalent to $S^1$). Assume that $A$ is a smooth algebra over a perfect field $k$. Then there is a Hochschild-Kostant-Rosenberg isomorphism \[ H^{-i}(\text{Hochschild complex}) \cong \Omega^i(A), \quad i \geq 0 \]
where $\Omega^i(A)$ is the space of algebraic de Rham $i$-forms. The latter can be equipped with the de Rham differential, and this is precisely the circle action in this case. We will elaborate on this in the start of next lecture.

### I.6. Further exercises

**Exercise I.6.0.1.** Let $C^\otimes = \mathbf{Cat}^\times$ be the $\infty$-category of categories. Its objects are categories, and given two objects $D, E$ we define the hom-space combinatorially as follows. We let $\text{hom}_{\mathbf{Cat}}(D, E)$ have vertices given by functors $D \to E$, edges natural isomorphisms, triangles commutative triangles of natural isomorphisms, and $k$-simplices commutative diagrams of natural isomorphisms in the shape of $k$-simplices.

Show that a $\text{Disk}_{1,or}$-algebra in $\mathbf{Cat}^\times$ is equivalent to a (unital) monoidal category.

**Exercise I.6.0.2.** Let $\mathcal{D}\text{isk}_1$ be the category of 1-disks without any orientation condition on morphisms. Then the space of embeddings $\text{Emb}(\mathbb{R}, \mathbb{R}) \simeq O(1) \simeq S^0$ is no longer contractible. Given a symmetric monoidal functor $F : \mathcal{D}\text{isk}_1 \to \mathbf{Vect}_{k}^\otimes$
let $A = F(\mathbb{R})$. We get a map $\text{Id} : A \to A$ and a map $\tau : A \to A$ from orientation reversal.

How does $\tau$ interact with the multiplication map?

Note that now there are two distinct isotopy classes of maps $S^1 \to S^1$; one of them contains an orientation-reversing diffeomorphism. Can you describe the induced map on cocenters?
CHAPTER 2

Factorization homology in higher dimensions

The goal today is to introduce higher-dimensional versions of associative algebras. The simplest of these are the $E_n$-algebras. To many of you, these will contain new kinds of algebraic structure. Informally, they have more commutativity than associative algebras, but they do not quite have all the commutativity one could wish for. This is a feature, not a bug; the lack of higher commutativity is in some sense what makes these algebras appropriate building blocks of manifolds of fixed dimension.

We will also define factorization homology. This is a local-to-global invariant satisfying a generalization of the $\otimes$-excision we saw last time for the circle.

The last section, Section II.6, contains various commentary on the notions we did not touch on in-depth during the spoken lecture.

II.1. Review of last talk

- We defined $\mathcal{D}isk_{1,or}$ as a symmetric monoidal $\infty$-category under disjoint union. We defined also $\mathcal{M}fld_{1,or}$.
- We considered symmetric monoidal functors
  \[ F : \mathcal{D}isk_{1,or} \to \text{Vect}_k^\otimes \]
  and saw that the data of such a functor was the same as the data of an associative algebra over $k$.
- We defined factorization homology with coefficients in an algebra $A := F(\mathbb{R})$ as the left Kan extension along the inclusion $\mathcal{D}isk_{1,or} \hookrightarrow \mathcal{M}fld_{1,or}$. This was left opaque and unexplained.
- We stated that factorization homology for the circle satisfies $\otimes$-excision.

II.2. More on $\infty$-Categories

II.2.1. A first pass through. Recall that last time, we told you to think of an $\infty$-category $\mathcal{C}$ as just a category such that for every pair of objects $x, y \in \mathcal{C}$, the collection of maps $\text{hom}_\mathcal{C}(x, y)$ is a topological space and composition is continuous. Such data is usually called a topologically enriched category, and this is one way you can think about what an $\infty$-category is.

In this model, a functor $F : \mathcal{C} \to \mathcal{D}$ of $\infty$-categories is a functor in the usual sense with the property that the induced maps
\[ \text{hom}_\mathcal{C}(x, y) \to \text{hom}_\mathcal{D}(Fx, Fy) \]
are continuous.

In an \(\infty\)-category \(\mathcal{C}\), one can talk about equivalences of objects.

**Definition II.2.1.1.** An equivalence \(x \to y\) between objects \(x, y\) in an \(\infty\)-category \(\mathcal{C}\) is the data of a map \(f : x \to y\) such that there exists a map \(g : y \to x\) and homotopies \(f \circ g \simeq \text{Id}\) and \(g \circ f \simeq \text{Id}\).

For reasons that are not always obvious at your first rodeo, it turns out that the question “what are \(\infty\)-categories?” is just as important as the question “when are two \(\infty\)-categories equivalent?”

**Definition II.2.1.2.** A functor \(F : \mathcal{C} \to \mathcal{D}\) of \(\infty\)-categories is an equivalence if \(F\) is essentially surjective, and the induced maps \(\text{hom}_\mathcal{C}(x, y) \to \text{hom}_\mathcal{D}(Fx, Fy)\) are weak homotopy equivalences.

**Remark II.2.1.3.** Recall that a map is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups. That is, a continuous map of topological spaces \(g : X \to Y\) induces functions

\[
\pi_0(X) \to \pi_0(Y), \quad \pi_1(X) \to \pi_1(Y), \quad \pi_2(X) \to \pi_2(Y), \quad \ldots
\]

and we say \(g\) is a weak homotopy equivalence if these maps are bijections \(\pi_i(X) \to \pi_i(Y)\) for all \(i \geq 0\). Of course, for \(i \geq 1\), we demand that these are bijections for any choice of connected component of \(X\).

As an example, if \(Y\) is a contractible space, then any choice of map \(* \to Y\) is a weak homotopy equivalence, and the unique map \(Y \to *\) is also a weak homotopy equivalence.

**Remark II.2.1.4.** One should think of the notion of a weak homotopy equivalence of spaces as analogous to the definition of quasi-isomorphisms for chain complexes. It is not always true that weak homotopy equivalences can be inverted, even up to homotopy (nor can quasi-isomorphisms). Regardless, both weak homotopy equivalences and quasi-isomorphisms induce isomorphisms on the most tractable algebraic invariants we have: homotopy groups (for spaces) and cohomology groups (for chain complexes).

And just as for chain complexes, one must have some technology to really consider equivalent objects to behave as though they are equivalent—for example, if \(f : \mathcal{C} \to \mathcal{D}\) is an equivalence of \(\infty\)-categories, we had better have a functor \(g : \mathcal{D} \to \mathcal{C}\) exhibiting some notion of invertibility of \(f\). In homological algebra, this was classically dealt with via derived categories. For \(\infty\)-categories (and in more recent approaches to homological algebra), this can be dealt with through the language of model categories and various localization techniques. For example, Lurie [61] and Joyal [48] construct model categories of \(\infty\)-categories.
II.2. MORE ON $\infty$-CATEGORIES

**Remark II.2.1.5.** We end with a final remark on contractibility. When you hear “There is a contractible space of BLAH,” you should think “There is a unique BLAH.” This is because given $BLAH$, there is a canonical map $\{BLAH\} \to \ast$ to the point, and contractibility means that this map is an homotopy equivalence.

**Example II.2.1.6.** The space of oriented embeddings from $\mathbb{R}$ to itself is contractible, and contains the identity morphism.

For this reason, when we are given a functor of $\infty$-categories $\mathcal{D}_{\text{isk1,or}} \to \mathcal{C}$, the induced continuous map

$$\text{hom}_{\mathcal{D}_{\text{isk1,or}}} (\mathbb{R}, \mathbb{R}) \to \text{hom}_{\mathcal{C}} (F(\mathbb{R}), F(\mathbb{R}))$$

can fruitfully be thought of as sending the identity of $\mathbb{R}$ to the identity of $F(\mathbb{R})$, and as no more information.

**II.2.2. A second pass.** The comments so far are meant to make $\infty$-categories seem less foreign and familiar; if you know what categories, spaces, and weak homotopy equivalences are, you can more or less follow a conversation or discussion.

But that’s just our first pass at infinity-categories.

Let’s now take a second look; I’ll try to address some more of the utility of $\infty$-categories throughout these lectures.

**Remark II.2.2.1 ($\infty$-categorical practice in algebraic settings).** It was “obvious” that the discrete category $\text{Disk}_{1,\text{or}}$ admitted a topology on its morphism set, so we were naturally led to consider the $\infty$-category $\mathcal{D}_{\text{isk1,or}}$. But how do we construct $\infty$-categories in algebra?

In most algebraic examples, the morphism space $\text{hom}_{\mathcal{C}}(x, y)$ is almost always combinatorially defined. We saw this yesterday in the Dold-Kan space: We already had a set called the chain maps, but rather than try to topologize the collection of chain maps, we added on edges (for every homotopy) and higher simplices (for homotopies between homotopies).

And in such cases, we do not construct functors by constructing truly flimsy continuous maps; instead, we usually construct these continuous maps combinatorially, by first mapping vertices to vertices, then edges to edges, and so on. Of course, when spaces are combinatorially defined, any combinatorially well-behaved function automatically induces a continuous map.

**Example II.2.2.2.** Pavel Safronov is giving lectures on Poisson structures in derived algebraic geometry. When he speaks of the $\infty$-category of cdgas, one can likewise construct a combinatorial space of maps. Given $A, B$ two cdgas, $\text{hom}(A, B)$ has vertices given by honest cdga maps, edges given by homotopies between these (which is not just the data of a homotopy of the underlying chain maps), and so forth.

**Remark II.2.2.3.** Pavel has also spoken of $\infty$-groupoids; these are $\infty$-categories in which every morphism is an equivalence.
A general philosophy going back to Grothendieck is that any ∞-groupoid is equivalent to a space. That is, given a space, one can obtain an ∞-category whose objects are points of the space, and whose morphisms are paths in the space; and any ∞-groupoid is equivalent as an ∞-category to such a thing. This is called the Homotopy Hypothesis; it is provable in any model of ∞-categories.

Thus you will often hear of the space of objects of an ∞-category (obtained by throwing out all non-equivalences), or of people treating a space as an ∞-category.

Remark II.2.2.4. One of the apparently hard things about constructing $\text{Disk}_{1,\text{or}}$-algebras is that $\text{Disk}_{1,\text{or}}$ and $\text{Chain}_{k}$ are ∞-categories in different ways. One is algebraic and one is geometric—for example, the domain is topologized using continuous techniques while the target has combinatorially defined spaces.

Normally, we overcome this by making the more topological thing behave more combinatorially. For example, out of any space $X$, one can construct a combinatorial gadget whose vertices are points of $X$, whose edges are paths in $X$, and whose higher simplices are continuous maps from higher simplices into $X$. (The combinatorial output is called the singular complex of $X$; see for example [39].) As in the previous remark, constructing a functor of ∞-categories then usually boils down to combinatorially assigning simplices to simplices.

Example II.2.2.5 (A combinatorial model for spaces of smooth embeddings). Fix two smooth manifolds $X$ and $Y$. The reader will soon realize that a continuous path in the topological space of smooth embeddings $\text{Emb}(X, Y)$ need not represent a smooth isotopy. So the previous remark’s example of the singular complex construction doesn’t fit the idea that an edge (in a combinatorial model for the space of smooth embeddings) should represent a smooth isotopy.

So here is yet another technique that can be used to model $\text{Disk}_{1,\text{or}}$ as an ∞-category (see for example [4, 5], though this idea goes back much further). One declares a vertex to be a smooth embedding $j : X \to Y$. One declares an edge to be the data of a smooth embedding $X \times \Delta^1 \to Y \times \Delta^1$ which respects the projection to $\Delta^1$. More generally, a $k$-simplex is a smooth embedding $X \times \Delta^k \to Y \times \Delta^k$ respecting the projections to $\Delta^k$.

Remark II.2.2.6 (Simplicial sets). We have avoided the term simplicial set to lower the bar for entry into these discussions. But in actuality, every “combinatorially defined” gadget I’ve spoken of is more specifically an example of a simplicial set—i.e., a functor $\Delta^{\text{op}} \to \text{Set}$. While $\Delta$ is simple enough to define, I wanted to keep the discussion focused, and there are plenty of other resources on simplicial sets out there. A simple online search may do.

Remark II.2.2.7 (∞-categories as weak Kan complexes). It is healthy to think of any category (in the classical sense) as a bunch of vertices (for objects) and edges (for morphisms) and triangles (for commutative triangles). There is likewise a way to think of an ∞-category as given by such combinatorial data, rather than specifying
morphism spaces for every pair of objects. This is the model often referred to as the model of quasi-categories, or weak Kan complexes; these are also examples of simplicial sets. This model for $\infty$-categories goes back to Boardman-Vogt \[21\] and is the model developed by Joyal \[48\] and Lurie \[61\].

**Remark II.2.2.8.** All models of $\infty$-categories are equivalent in a precise sense. (See for example the work of Julie Bergner \[16, 17\].) As I have said before: If you do not work with this stuff, then for this lecture series, it is probably healthiest (and least technical) to think of an $\infty$-category as a topologically enriched category.

Regardless, let us mention that working with $\infty$-categories is often quite combinatorial in nature. For algebraic necessities, one can often invoke category-theoretic intuitions from classical category theory (for example, the theory of adjunctions); and for coherence results, one often needs to simply check lifting properties rather than construct difficult compatibilities.

**Example II.2.2.9.** There are models for $\infty$-categories—such as the weak Kan complex model—for which we do *not* specify a composition law for an $\infty$-category. Regardless, in such models, if one fixes two composable morphisms $f$ and $g$, one may study the space of all coherent triangles

$$
\begin{array}{ccc}
& h & \\
f & \downarrow & g \\
& h & \downarrow \\
\end{array}
$$

Informally, the space of such triangles may be thought of as the space of all $h$ equipped with a homotopy between $h$ and a putative composition $g \circ f$. For a weak Kan complex (which is a model of $\infty$-category, see Remark \[112.2.7\]), this space is always contractible. That is, by Remark \[112.2.1.5\] there is a homotopically *unique* way to compose $g$ with $f$.

This illustrates one of the most useful operating principles of $\infty$-categories: One can often avoid defining a specific operation (such as multiplication in an algebra, or composition in a category). Instead, it is often easier to construct a gigantic gadget containing a large space of possible choices for an operation, and to prove the contractibility of such choice-spaces. The insight here is that it is often difficult to finagle coherences of operations defined in particular ways; it is easier to describe properties about the spaces of possible operations.

**Warning II.2.2.10.** Now that we have come this far, let us point out some intuitions that can be *misleading* if one only thinks of $\infty$-categories as topologically enriched categories. (This is why it’s worth taking a second look.)

1. A notion of composition need not always be defined for an $\infty$-category; instead one may provide a contractible space of ways in which composition can be interpreted. (Example \[111.2.2.9\]) This is not some impossible amount
of data; that one can construct such a contractible space is often a consequence of the model one is using, such as the weak Kan complex model, and constructing an \( \infty \)-category is often either a combinatorial or formal task.

(2) Likewise, a functor need not “respect” composition in the classical sense—especially when composition may not even be strictly defined! (Remark I.4.0.8.)

(3) Given an \( \infty \)-category \( C \), there need not be an “underlying category” that we have topologized to obtain \( C \). (In this sense, both \( \text{Chain} \) and \( \text{Mfld}_n \) are somewhat misleading examples, as both had natural starting points that we sought to topologize.) The reason that there is no “underlying category” to be topologized is because “underlying set” is not a notion preserved under weak homotopy equivalences; hence “underlying category” is not a notion invariant under equivalences of \( \infty \)-categories. (Definition II.2.1.2.)

II.3. The example of Hochschild chains

Back to factorization homology. Let us elaborate on our last example from last lecture.

Let \( F: \text{Disk}_{1, \text{or}} \rightarrow \text{Chain}^\otimes_k \) be a symmetric monoidal functor and set \( A := F(\mathbb{R}) \). I claimed that factorization homology

\[
\int (\_): \text{Mfld}_{1, \text{or}} \rightarrow \text{Chain}_k
\]

allows us to see an action of the group of orientation-preserving diffeomorphisms \( \text{Diff}^+(S^1) \) on \( \int S^1 A \). I now give details and a warning (Warning II.3.0.3).

Since factorization homology is a functor, we get a map

\[
\text{hom}_{\text{Mfld}_{1, \text{or}}}(S^1, S^1) \rightarrow \text{hom}_{\text{Chain}_k}(\int_{S^1} A, \int_{S^1} A)
\]

and this map is continuous. Now \( S^1 \) acts on itself by diffeomorphisms, and we thus have the inclusion

\[
S^1 \hookrightarrow \text{Diff}^+(S^1) = \text{hom}_{\text{Mfld}_{1, \text{or}}}(S^1, S^1).
\]

(This inclusion is a weak homotopy equivalence, though this will not matter for us.) So we get a continuous map

\[
S^1 \rightarrow \text{hom}_{\text{Chain}_k}(\int_{S^1} A, \int_{S^1} A).
\]

Because this map is continuous, we may study its effect on homotopy groups. The effect on \( \pi_0 \)

\[
\pi_0 S^1 \rightarrow \pi_0 \left( \text{hom}_{\text{Chain}_k}(\int_{S^1} A, \int_{S^1} A) \right) \simeq H^0 \text{hom}_{\text{Chain}_k}(\int_{S^1} A, \int_{S^1} A)
\]
II.3. THE EXAMPLE OF HOCHSCHILD CHAINS

is not interesting; by definition, a functor must send the identity component \([\text{id}] \in \pi_0 S^1\) to the homology class of the identity chain map of \(\int_{S^1} A\). However, what can we say about the map on fundamental groups?

**Theorem II.3.0.1.** Assume that \(A\) is a smooth commutative \(\Bbbk\)-algebra and \(\Bbbk\) is perfect. The map

\[
\pi_1 S^1 \to \pi_1 \left( \text{hom}_{\text{chains}} \left( \int_{S^1} A, \int_{S^1} A \right) \right) \simeq H^{-1} \text{hom}_{\text{chains}} \left( \int_{S^1} A, \int_{S^1} A \right)
\]

induced by factorization homology sends a generator \(1 \in \pi_1 S^1 = \BbbZ\) to the de Rham differential \([d_{dR}]\).

We will not prove this theorem here. (The reader may consult sources such as [46], [58], Section 1.4 of [15], Example 5.5.3.14 of [63], and Proposition 2.2 of [59].) Regardless, let us explain its content.

Recall that by excision

\[
\int_{S^1} A \simeq A \bigotimes_{A \otimes \hat{A}} A
\]

where the righthand side is a well-known chain complex, called the Hochschild chain complex of \(A\) (Example I.5.0.8). By the Hochschild-Kostant-Rosenberg theorem [44], when \(A\) is a smooth commutative algebra and \(\Bbbk\) is a perfect field, there is an isomorphism

\[
H^{-l} (\text{Hochschild chains on } A) \cong \{\text{degree } l \text{ algebraic de Rham forms}\} = \Omega^l_{A/\Bbbk}.
\]

Yes, you read that correctly. The cohomology of something recovers forms. Moreover, things seem to be in wonky degrees: For example, the 1-forms are concentrated in cohomological degree minus 1. And of course, when we write down de Rham forms, we can also usually write down a de Rham differential. Where is the de Rham differential here? The above claim states that the degree -1 element picked out by the generator of \(\pi_1 S^1\) is precisely the de Rham differential.

**Remark II.3.0.2.** Historically, people combinatorially exhibited an \(S^1\) action on the Hochschild chains. This story goes at least as far back as Connes [26]. Factorization homology exhibits this action more geometrically.

In higher dimensions, you might appreciate that combinatorially modeling actions of (complicated) diffeomorphism groups is not an easy task. It may even be an ad hoc task one must do one manifold at a time. Factorization homology, for free, exhibits actions of diffeomorphism groups on our invariants.

---

1Here we are using the fact from last time that the homotopy groups of the Dold-Kan space recover the cohomology groups of the \(Hom\) cochain complex. See Example I.3.0.7
Warning II.3.0.3 (The “action” of diffeomorphism groups). An astute reader may have heeded the repeated warnings that functors of ∞-categories need not respect composition on the nose (Remark I.4.0.8 and Warning II.2.2.10). Now, because of the equivalence of the different models of ∞-categories, if one has a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ of ∞-categories, it is true that one can write an honest continuous, group homomorphism

$$\text{Aut}_\mathcal{C}(X) \rightarrow \text{Aut}_\mathcal{D}(fX)$$

for any object $X \in \mathcal{C}$. However, one can only do this after possibly passing to very particular representatives of the homotopy equivalence classes of the automorphism spaces in both the domain and target. So even if $\mathcal{C} = \mathcal{Mfld}_{n, fr}$ or $\mathcal{Mfld}_{n, or}$ (which are ∞-categories of manifolds that we will introduce below), it is not always possible, nor advisable, to write a strict and continuous group action of $\text{Diff}_{or}(X)$ or $\text{Diff}_{fr}(X)$ on its factorization homology. Instead, the natural output of factorization homology is a homotopy coherent (not strict) action of $\text{Diff}$ on the target.

Indeed, at this point, it is the opinion of this lecturer that one ought to get used to the idea of a functor—and of an action—that need not respect composition on the nose, and embrace models like the weak Kan complex model that gets one used to thinking about maps and functors that simply supply extra homotopies.

II.4. The algebra of disks in higher dimensions

We will now discuss the algebra of disks in higher dimensions.

II.4.1. Framings and orientations, a first glance. Before we get on with it, let us recall the exercises at the end of the last chapter. They showed that by considering disks with different kinds of tangential structure (an orientation, or no orientation at all)—and altering our embedding spaces accordingly—we discover different kinds of algebraic structures. The same is true in higher dimensions.

While there is a different kind of $n$-dimensional disk algebra for any choice of group $G$ equipped with a continuous homomorphism $G \rightarrow GL_n(\mathbb{R})$, we will mainly consider the cases of $G = SO_n(\mathbb{R})$ (the oriented case), and of $G = \{e\}$ (the case of framed manifolds). As we will see, the framed case will have the simplest algebraic description.

Definition II.4.1.1. Let $\mathcal{Disk}_{n, or}$ be the ∞-category whose objects are finite disjoint unions of oriented, $n$-dimensional open disks. Morphisms are orientation-preserving smooth embeddings.

Remark II.4.1.2. As before, any object of $\mathcal{Disk}_{n, or}$ is equivalent to $(\mathbb{R}^n)^{\sqcup k}$ for some $k \geq 0$. Disjoint union renders $\mathcal{Disk}_{n, or}$ a symmetric monoidal ∞-category.

Definition II.4.1.3. Fix $\mathcal{C}^\otimes$ a symmetric monoidal ∞-category. A $\mathcal{Disk}_{n, or}$-algebra in $\mathcal{C}^\otimes$ is a symmetric monoidal functor

$$F : \mathcal{Disk}_{n, or}^{\sqcup} \rightarrow \mathcal{C}^\otimes.$$
II.4. THE ALGEBRA OF DISKS IN HIGHER DIMENSIONS

Here is the framed variant:

**Definition II.4.1.4.** Fix $X$ a smooth $n$-manifold. A *framing* on $X$ is a choice of trivialization $\phi : TX \sim - \to X \times \mathbb{R}^n$ of vector bundles.

**Definition II.4.1.5 (Informal.).** Let $\mathcal{D}isk_{n,fr}$ be the $\infty$-category with objects finite disjoint unions of framed $n$-disks and morphisms smooth embeddings *equipped* with a compatibility of framings. A $\mathcal{D}isk_{n,fr}$-algebra in $\mathcal{C}^\otimes$ is a symmetric monoidal functor

$$F : \mathcal{D}isk_{n,fr}^\Pi \to \mathcal{C}^\otimes.$$ 

**Remark II.4.1.6.** This “compatibility” of framings doesn’t have an expression that you learn in a typical differential geometry class. Let me just say for now that a morphism isn’t simply a smooth embedding $j : X \to Y$ satisfying a property, but additional data on $j$. See Section II.6.1 and Definition II.6.1.4 at the end of this chapter for details. We will see there that the most natural way to think of framings (and maps of framed manifolds) is by constructing pullbacks of certain natural $\infty$-categories.

Let us also say that—just as before—one can enumerate the objects of $\mathcal{D}isk_{n,fr}$ up to equivalence as follows:

$$\emptyset, \quad \mathbb{R}^n, \quad \mathbb{R}^n \bigsqcup \mathbb{R}^n, \quad \ldots.$$ 

To see the difference between $\mathcal{D}isk_{n,or}$- and $\mathcal{D}isk_{n,fr}$-algebras, let us begin to unpack the definitions. Fix a symmetric monoidal functor $F$ out of $\mathcal{D}isk_{n,or}$. The reader may benefit from setting $n = 2$ for ease of drawing, though we will work with arbitrary $n$ for now.

As in the one-dimensional case, we denote by $A_\cdot = F(\mathbb{R}^n)$ the value of $F$ on a single disk. On objects, $F$ sends the empty manifold to $\mathbb{k}$, a single disk $\mathbb{R}^n$ to $A$, and a disjoint union $(\mathbb{R}^n)\bigsqcup \mathbb{k}$ to $A^\otimes \mathbb{k}$. $F$ also induces a map

$$\text{hom}_{\mathcal{D}isk_{n,or}}(\mathbb{R}^n, \mathbb{R}^n) =: \text{Emb}^\text{or}(\mathbb{R}^n, \mathbb{R}^n) \to \text{hom}\mathcal{C}(A, A)$$

So let us first understand $\text{Emb}^\text{or}(\mathbb{R}^n, \mathbb{R}^n)$, the space of oriented embeddings of $\mathbb{R}^n$ to itself.

**Lemma II.4.1.7.** For any $n$, the inclusion of orthogonal transformations

$$SO_n(\mathbb{R}) \to \text{Emb}^\text{or}(\mathbb{R}^n, \mathbb{R}^n)$$

is a homotopy equivalence.

**Proof.** Let $\text{Emb}_0^\text{or}(\mathbb{R}^n, \mathbb{R}^n)$ denote the subspace consisting of those embeddings that send the origin to the origin. By translating, the inclusion

$$\text{Emb}_0^\text{or}(\mathbb{R}^n, \mathbb{R}^n) \to \text{Emb}^\text{or}(\mathbb{R}^n, \mathbb{R}^n)$$

is a homotopy equivalence. (Any embedding $j$ can be translated to the sum $j - tj(0)$; running this from time $t = 0$ to $t = 1$ gives the retraction.)
Moreover, let us send an origin-preserving embedding $f$ to its difference quotient at the origin:

$$\frac{f(x_0 + t\vec{v}) - f(x_0)}{t} = \frac{f(t\vec{v})}{t} \quad (at \ x_0 = 0).$$

Running this different quotient from time $t = 1$ to $t = 0$ exhibits a deformation retraction of $\text{Emb}_0^c(\mathbb{R}^n, \mathbb{R}^n)$ to $GL_n^+(\mathbb{R})$ (invertible matrices with positive determinant). Finally, the Gram-Schmidt process—whose formulas are all continuous in its parameters—defines a deformation retraction of $GL_n^+(\mathbb{R})$ onto $SO_n(\mathbb{R})$. □

**Remark II.4.1.8.** So an oriented $n$-disk-algebra specifies an object $A = F(\mathbb{R}^n)$ with some action of $SO_n(\mathbb{R})$, as articulated by the data of $F$ on morphism spaces:

$$SO_n(\mathbb{R}) \simeq \text{hom}_{\text{Disk}_{n,or}}(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \text{hom}_{\text{Chain}}(A, A).$$

This is already a lot of data; actions by (special) orthogonal groups do not grow on trees.

Demanding that our embeddings $j$ be equipped with framing compatibilities simplifies the situation considerably. In fact, one can run the deformation retraction in the proof above so that the derivative of $j$ at the origin is equipped with a homotopy to the $n \times n$ identity matrix; so in fact, the inclusion of a point,

$$* = \{I_{n \times n}\} \longrightarrow \text{hom}_{\text{Disk}_{n,fr}}(\mathbb{R}^n, \mathbb{R}^n),$$

is a homotopy equivalence. That is, the framed embedding space is contractible. (See also Remark II.6.1.6.)

In summary so far, a $\text{Disk}_{n,or}$-algebra specifies to $\mathbb{R}^n$ an object $A$ with an $SO_n(\mathbb{R})$ action. The framed version—a $\text{Disk}_{n,fr}$-algebra—simply sends $\mathbb{R}^n$ to a chain complex (with no specified $SO_n(\mathbb{R})$ action).

**II.4.2. $E_n$-algebras.** What of the multiplication maps? As before, setting $A := F(\mathbb{R}^n)$, we would like to interpret morphisms

$$j : \mathbb{R}^n \coprod \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

as inducing multiplication maps $F(j) : A \otimes A \longrightarrow A$. When $n \geq 2$, up to isotopy, there is one oriented embedding $j : \mathbb{R}^n \coprod \mathbb{R}^n \longrightarrow \mathbb{R}^n$. So up to chain homotopy, we have a chain map $m : A \otimes A \longrightarrow A$ specified by $F$. How does this interact with the swap map?

Let’s begin with the two-dimensional case and let $\sigma : \mathbb{R}^2 \sqcup \mathbb{R}^2 \to \mathbb{R}^2 \sqcup \mathbb{R}^2$ be the swap map. Note that, unlike in the 1-dimensional case, one can exhibit an isotopy

$$j \simeq j \circ \sigma.$$

Which is to say, the multiplication admits a homotopy $m \circ \sigma \sim m$; it is commutative up to homotopy.
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II.4.2.1. A homotopy swapping the placement of two embedded disks.

\begin{center}
\includegraphics[width=0.3\textwidth]{image1.png}
\end{center}

\textbf{Remark II.4.2.2.} You have most likely seen this picture before; it is the same picture one draws when proving that the homotopy groups $\pi_n$ for $n \geq 2$ are all commutative. This argument is an example of the so-called Eckmann-Hilton argument.

However, the above isotopy of disks is far from unique—one could wind a pair of disks around each other one more time.

\textbf{Image II.4.2.3.} An isotopy from a multiplication to itself.

\begin{center}
\includegraphics[width=0.3\textwidth]{image2.png}
\end{center}

So the space of isotopies is not contractible (in fact, we see a winding number obstruction).

In any case, $F$ takes $j$ to an operation homotopic to $j \circ \sigma$. So multiplication is commutative up to homotopy; but the \textit{space} of ways in which this commutativity is exhibited is non-trivial, as there are at least $\mathbb{Z}$ many ways to do so (given by the winding number of disks moving past each other).

\textbf{Remark II.4.2.4.} In higher dimensions, this winding number obstruction can be trivialized.
A disk exhibiting a null homotopy of the loop from Figure II.4.2.3.

In the image above, we have trivialized the winding by choosing a disk whose boundary is the winding isotopy. But we note that another mathematician could have chosen a disk presenting the lower hemisphere of a sphere, rather than the upper. That is, the space of exhibiting commutativity now has a non-trivial two-sphere appearing.

In general, in dimension $n$, a $\text{Disk}_{n, \text{fr}}$-algebra yields an algebra with a multiplication which is commutative up to homotopy, and the canonicity of the commutativity is obstructed by an $(n - 1)$-dimensional sphere. In a precise sense, the limit as $n \to \infty$ gives rise to the notion of a “truly” commutative algebra for homotopy theorists, but we won’t get into the details here. The interested reader may look up the word $E_\infty$-algebra.

**Definition II.4.2.6.** Fix a symmetric monoidal $\infty$-category $C^\otimes$. For any $n \geq 1$, an $E_n$-algebra in $C^\otimes$ is a symmetric monoidal functor $\text{Disk}_{n, \text{fr}} \to C^\otimes$.

**Remark II.4.2.7.** The notion of an $E_n$-algebra was defined (differently) far before being expressed the way we have expressed it. Historically, $E_n$-algebras were defined by considering not the space of framed smooth embeddings, but by considering spaces of rectilinear embeddings of bounded cubes—that is, embeddings that component-wise can be written as a composition of scalings and translations. This definition goes back at least to the work of Boardman and Vogt [20], and was famously studied by May in [64].

We will proceed using our definition. For now the reader may simply have the intuition that an $E_n$-algebra for $n \geq 2$ is an algebra equipped with some commutativity data, but this data is not quite canonical.

**Remark II.4.2.8.** We have already seen that a $\text{Disk}_{n, \text{or}}$-algebra (i.e., the oriented setting) specifies at the very least an object $A$ with an action of $SO_n(\mathbb{R})$. In a way we do not articulate here, one can informally think of a $\text{Disk}_{n, \text{or}}$-algebra as an $E_n$-algebra equipped with an $SO_n(\mathbb{R})$-action which is compatible with the multiplication and commutativity. (It is this compatibility that we do not articulate.)

II.4.3. Examples of $E_n$-algebras.
Example II.4.3.1. Fix $R$ a base ring. We let $\text{Chain}_R^{\otimes}$ be the symmetric monoidal $\infty$-category whose objects are chain complexes of $R$-modules, and whose symmetric monoidal structure is given by derived tensor product over $R$. (If one takes $R = \mathbb{k}$ to be a field, one need not derive the tensor product as all chain complexes are flat.)

Then a commutative $R$-algebra is an $E_n$-algebra in $\text{Chain}_R^{\otimes}$ for any $n$. More generally, a cdga (commutative dg algebra) is an $E_n$-algebra for any $n$.

Exercise II.4.3.2. When $n = 1$, show that a framing on a 1-manifold is the same thing as an orientation; moreover, show that the space of framed embeddings (i.e., embeddings equipped with framing compatibility) is equivalent to the space of orientation-preserving embeddings. You will need to consult Section II.6.1 for this.

Remark II.4.3.3. Thus, an $E_1$-algebra is the same thing as a $\text{Disk}_{1,\text{or}}$-algebra, which we grew to love in the last lecture. In fact, $E_1$ and $A_\infty$ are synonyms to a homotopy-theorist.

Exercise II.4.3.4. Let $C^{\otimes}$ be $\text{Vect}_k^{\otimes}$. Show that an $E_n$-algebra in $\text{Vect}_k^{\otimes}$ is a commutative algebra over $\mathbb{k}$ for $n \geq 2$. Likewise, show that an $E_n$-algebra in the category of sets $\text{Set}^\otimes$ is a commutative monoid.

Exercise II.4.3.5. Take $C^{\otimes} = \text{Cat}^{\otimes}$. Show that an $E_2$-algebra in $\text{Cat}$ is a braided monoidal category \cite{49}. (You can see where the word “braided” comes from in this picture: What does a movie of moving 2-dimensional disks around each other look like?)

Show that a $\text{Disk}_{2,\text{or}}$-algebra is a balanced monoidal category. (In particular, ribbon categories are examples of $\text{Disk}_{2,\text{or}}$-algebras.)

What about $E_n$-algebras in $\text{Cat}$ for $n \geq 3$?

Example II.4.3.6 (Dunn additivity). Let $C^{\otimes}$ be a symmetric monoidal $\infty$-category. One can then show that the $\infty$-category $E_1\text{Alg}(C^{\otimes})$ of $E_1$-algebras in $C^{\otimes}$ is again a symmetric monoidal $\infty$-category under $\otimes$. Hence one can iterate: What are the $E_1$-algebras in $E_1$-algebras?

The Dunn additivity theorem states that there exists an equivalence

$$E_n\text{Alg}(C^{\otimes}) \simeq E_1\text{Alg}(E_1\text{Alg}(\ldots(E_1\text{Alg}(C^{\otimes}))))$$

between the $\infty$-category of $E_n$-algebras in $C^{\otimes}$, and the $\infty$-category of $E_1$-algebras in $\ldots$ in $E_1$-algebras in $C^{\otimes}$. For example, an $E_2$-algebra is the same data as an $E_1$-algebra structure on an $E_1$-algebra. By induction, an $E_n$-algebra is the same data as an $E_1$-algebra structure on an $E_{n-1}$-algebra.

Informally, one may thus think of an $E_n$-algebra as an object of $C^{\otimes}$ equipped with $n$ mutually compatible multiplications, each of which is associative up to coherent homotopy.

See \cite{32} and a modern account in Section 5.1.2 of \cite{63}. 
Exercise II.4.3.7. Show than an associative monoid in the category of associative monoids is simply a commutative monoid. (We are testing Dunn additivity for \( n = 2 \) when \( \mathcal{C}^\otimes = \text{Set}_x \) the category of sets under direct product.)

Likewise, show that an associative algebra in the category of associative \( k \)-algebras is a commutative \( k \)-algebra. Compare with Exercise II.4.3.4.

Example II.4.3.8. The Hochschild cochain complex (not the chain complex) of any associative algebra (or dg-algebra, or \( A_\infty \)-algebra) is an example of an \( \mathbb{E}_2 \)-algebra in \( \mathcal{C}^\otimes = \text{Chain}_{k}^\otimes \). For definitions of the complex, see for example [43] or Chapter 9 of [73].

It was historically observed that the cohomology of the Hochschild cochain complex (i.e., the Hochschild cohomology) of an associative algebra had an action by the homology groups of the framed embedding spaces of 2-dimensional disks. Deligne conjectured that this action lifts to the chain level—i.e., that the Hochschild cochain complex admitted an \( \mathbb{E}_2 \)-algebra structure—in 1993. Since then many proofs have been given of this conjecture, the first being Tamarkin’s [71] as far as we know.

Remark II.4.3.9. A reader may be familiar with the fact that the Hochschild cochain complex of an \( A_\infty \)-category \( \mathcal{A} \) may be computed as the natural transformations of the identity functor. (When \( \mathcal{A} \) has a single object, one recovers the Hochschild cochain complex of \( \mathcal{A} \) considered as an \( A_\infty \)-algebra.) That the Hochschild cochain complex has an \( \mathbb{E}_2 \)-algebra structure is compatible with Dunn additivity (Example II.4.3.6) in the following sense: The collection of self-natural transformations of the identity functor has two natural composition maps—one by compositing natural transformations, and the other by composing the identity functor with itself (which induces an a priori different—but homotopic—multiplication operation from composition on the collection of natural transformations).

Example II.4.3.10 (\( n \)-fold loop spaces). Let \( \text{Top}^\times \) denote the \( \infty \)-category of topological spaces under direct product. Let \( X \) be a topological space and choose a basepoint \( x_0 \in X \). Let \( \Omega X \) denote the space of continuous maps \( \gamma : [0,1] \to X \) such that \( \gamma(0) = \gamma(1) = x_0 \). (This is what one normally calls the based loop space of \( X \).) Then \( \Omega X \) is an \( \mathbb{E}_1 \)-algebra in \( \text{Top}^\times \).

This is a “homotopical lift” of the well-known statement that the fundamental group \( \pi_1(X,x_0) \) of \( X \) at \( x_0 \) is associative under path concatenation.

More generally, let \( \Omega^n X = \Omega(\Omega(\ldots(\Omega X))) \) denote the \( n \)-fold based loop space. (An element of \( \Omega^n X \) is given by a continuous map \( \gamma : [0,1]^n \to X \) for which the boundary of \( [0,1]^n \) is sent to \( x_0 \).) Then \( \Omega^n X \) is an \( \mathbb{E}_n \)-algebra in \( \text{Top}^\times \).

Example II.4.3.11 (Configuration spaces as free algebras). The free \( \mathbb{E}_n \)-algebra on one generator (in \( \text{Top}^\times \)) is the topological space

\[
\prod_{l \geq 0} \text{Conf}_l(\mathbb{R}^n)
\]
whose \(l\)th component is the space of unordered configurations of \(l\) disjoint points in \(\mathbb{R}^n\).

**Exercise II.4.3.12.** Spell out \(E_n\)-algebra structures for Examples [II.4.3.10] and [II.4.3.11].

**Example II.4.3.13.** Given any \(E_n\)-algebra \(X\) in \(\mathcal{T}_{\text{op}}\), the singular chain complex \(C_*(X; k)\) is an \(E_n\)-algebra in \(\text{Chain}_k\). (This is true even when \(k\) is not a field.)

In particular, chains on based loop spaces are examples of \(A_\infty\)-algebras in chain complexes.

**Remark II.4.3.14.** \(E_n\)-algebras are often constructed as deformations of commutative or cocommutative objects. For example, the braided monoidal category of representations of \(U_q(\mathfrak{g})\) is constructed from deforming a cocommutative Hopf algebra (the universal enveloping algebra of \(\mathfrak{g}\)) [31, 47].

Another source using combinatorially articulable deformation problems is found in [51].

### II.5. Factorization homology

A \(\text{Disk}_{n,fr}\)-algebra is the local input data of a manifold invariant. Factorization homology is the (global) invariant.

**Remark II.5.0.1.** In the previous talk, we took \(\mathcal{C}\) to be \(\text{Vect}_k\) or \(\text{Chain}_k\). We could have taken \(\mathcal{C}\) to be any symmetric monoidal \(\infty\)-category such that \(\otimes\) preserves sifted colimits in each variable. See Section II.6.3 below.

**Definition II.5.0.2 ([1, 8]).** Let \(\mathcal{C}^{\otimes}\) be a symmetric monoidal \(\infty\)-category admitting all sifted colimits. Fix an \(E_n\)-algebra in \(\mathcal{C}^{\otimes}\), which we will denote by \(A\) by abusing notation. Factorization homology with coefficients in \(A\) is the left Kan extension,

\[
\begin{array}{c}
\text{Disk}_{n,fr} \\
\downarrow \\
\mathcal{M}_{\text{fld}} \end{array} \xrightarrow{A} \begin{array}{c}
\mathcal{C} \\
\downarrow \\
\int_A \\
\mathcal{M}_{\text{fld}}_{n,fr}
\end{array}
\]

Given a framed manifold \(X \in \mathcal{M}_{\text{fld}}_{n,fr}\), we let

\[
\int_X A
\]

denote the value of factorization homology, and we call it *factorization homology of \(X\) with coefficients in \(A\).*

**Remark II.5.0.3.** In the oriented version, we still have \(\text{Disk}_{n,or} \to \mathcal{M}_{\text{fld}}_{n,or}\), and factorization homology is defined to be the left Kan extension as before. The \(\otimes\)-excision theorem below still holds in the oriented setting.
Remark II.5.0.4. Not every smooth manifold $X$ admits a framing. For example, if $X$ is a compact, orientable, boundary-less 2-manifold, $X$ admits a framing only when $X$ is a torus. Note, however, that a given manifold may have many inequivalent framings. Thus factorization homology in the framed setting is an invariant not just of $X$, but of $X$ equipped with its framing.

Remark II.5.0.5. Assume $\otimes$ commutes with sifted colimits in each variable. Then factorization homology can be made symmetric monoidal \cite{1, 8}. That is, one has natural equivalences

$$\int_{X \coprod Y} A \simeq \int_X A \otimes \int_Y A.$$ 

So to understand this invariant, it suffices to compute it on connected manifolds.

The interested reader may consult Section II.6.2 below for details on what a left Kan extension is. This will not concern us at present; let me just tell you how one in principle computes factorization homology.

Theorem II.5.0.6 ($\otimes$-Excision, \cite{1, 8}). Fix a framed $n$-manifold $X$ and a decomposition

\begin{equation}
X = X_0 \cup_{W \times \mathbb{R}} X_1
\end{equation}

where each of $X_0, X_1$ is an open subset of $X$, and we have given their intersection $X_0 \cap X_1 \cong W \times \mathbb{R}$ a direct product decomposition as a smooth manifold.

Let $C^\otimes$ be a symmetric monoidal $\infty$-category admitting all sifted colimits and such that $\otimes$ preserves sifted colimits in each variable. Then there is an equivalence

\begin{equation}
\int_X A \simeq \int_{X_0} A \otimes \int_{W \times \mathbb{R}} A \int_{X_1} A.
\end{equation}

Both the decomposition (II.5.0.7) and the tensor product on the righthand side of (II.5.0.8) warrant an explanation. First, the decomposition:

Remark II.5.0.9. Note that $X_0$ and $X_1$ form an open cover of $X$, and the crucial part of the decomposition is the choice of direct product decomposition $X_0 \cap X_1 \cong W \times \mathbb{R}$, where $W$ is an $(n-1)$-dimensional manifold. Of course, being open subsets of $X$, each of $X_0, X_1, W \times \mathbb{R}$ inherits a framing from $X$. Moreover, since $\mathbb{R}$ is contractible, the framing map from $W \times \mathbb{R}$ factors through the projection $W \times \mathbb{R} \to W$. (See Section II.6.1 for details on what we mean by a framing.) The direct product decomposition thus allows us to interpret the inherited framing on $W \times \mathbb{R}$ as a framing on the direct sum bundle $TW \oplus \mathbb{R}$ over $W$. Note in particular that $W$ itself need admit a framing. See also Definition III.8.2.4.

This discussion also holds if one replaces the notion of framing by any other $G$-structure (Definition II.6.1.2).

Remark II.5.0.10. The decomposition (II.5.0.7) is an example of a collar-gluing in the sense of \cite{7, 8}. 

Now, let us explain the tensor product in (II.5.0.7):

**Proposition II.5.0.11.** \( \int_{W \times \mathbb{R}} A \) is an \( E_1 \)-algebra. Moreover, each \( \int_{X_i} A \) for \( i = 0, 1 \) is a module over \( \int_{W \times \mathbb{R}} A \).

**Sketch.** One has an induced symmetric monoidal functor

\[
\text{Disk}^I_{1,or} \rightarrow \mathcal{Mfld}^I_{n,fr} \quad \mathbb{R}^I \hookrightarrow W \times \mathbb{R}^I.
\]

So we see that \( W \times \mathbb{R} \) is an \( E_1 \)-algebra (in \( \mathcal{Mfld}^I_{n,fr} \)). Because factorization homology can be made symmetric monoidal by Remark II.5.0.5 this exhibits the \( E_1 \)-algebra structure on \( \int_{W \times \mathbb{R}} A \).

(If you don’t like the above paragraph for whatever reason, just consider those embeddings

\[
j : (W \times \mathbb{R}) \coprod \ldots \coprod (W \times \mathbb{R}) \rightarrow W \times \mathbb{R}
\]

which are the identity on the \( W \) direction. That is, they are given as a direct product \( \text{id}_W \times h \) where \( h : \mathbb{R} \coprod \ldots \coprod \mathbb{R} \rightarrow \mathbb{R} \) is an orientation-respecting embedding. The images of these embeddings under \( \int A \) exhibit the “associative” algebra structure on \( \int_{W \times \mathbb{R}} A \). We leave as an exercise to the concerned reader how to cohere this with the framing data.)

On the other hand, because \( W \times \mathbb{R} \) is a collar for \( X_1 \), we have embeddings

\[
\rho : X_1 \coprod (W \times \mathbb{R}) \rightarrow X_1
\]

given by “squeezing” \( X_1 \) into itself along the collar, then inserting a copy of \( W \times \mathbb{R} \) in the available collar-space.

**Image II.5.0.12.** The left module structure on \( X_1 \). On the right, all of \( X_1 \) has been “squeezed” into \( X_1 \), but we are showing in purple only what happens to the indicated collaring region of \( X_1 \). The grey cylinder is the manifold \( W \times \mathbb{R} \), and on the right we have indicated its image under \( \rho \).

Then the functor \( \int A \) exhibits maps

\[
\int_{W \times \mathbb{R}} A \otimes \int_{X_1} A \simeq \int_{(W \times \mathbb{R}) \coprod X_1} A \rightarrow \int_{X_1} A
\]

which gives \( \int_{X_1} A \) a module structure. The same argument shows the module structure on \( \int_{X_0} A \). \( \square \)
Remark II.5.0.13. The word “tensor” strictly speaking is only defined for usual linear objects over some base ring $R$. However, there are notions of tensor product for modules in arbitrary categories and arbitrary $\infty$-categories; given a left module and a right module over an algebra, one writes down a simplicial object called the bar construction. When sifted colimits (and in particular colimits of simplicial objects) exist, the colimit of the bar construction is what one usually calls the tensor product of two modules over an algebra. Most readers will not lose any intuition by imagining that the tensor product in Theorem II.5.0.6 is simply a derived tensor product.

Remark II.5.0.14. Factorization homology is always pointed, meaning that for any framed manifold $X$, the factorization homology $\int_X A$ is always equipped with a map from the monoidal unit of $C^\otimes$. This is because the empty set admits an embedding into any manifold, and uniquely so.

As an example, if $C^\otimes = \text{Cat}^\times$ is the $\infty$-category of categories, the monoidal unit is the trivial category with one object, and the pointing $\ast \to \int_X A$ picks out an object of the category $\int_X A$.

If $C^\otimes = \text{Chain}_k^{\otimes^2}$, then the pointing picks out a degree 0 cohomology class of $\int_X A$, specified by a map $k \to \int_X A$.

II.5.1. Examples. Factorization homology exhibits mapping group actions:

Example II.5.1.1. Fix a $\text{Disk}_{2,\sigma}$-algebra $\mathcal{R}$ in $C^\otimes$. For any oriented genus $g$ surface $\Sigma_g$, let $\text{Diff}^+(\Sigma_g)$ denote the space of orientation-preserving diffeomorphisms of $\Sigma_g$. Then factorization homology induces a map

$$\text{hom}_{\text{Mfld}_{2,\sigma}}(\Sigma_g, \Sigma_g) = \text{Diff}^+(\Sigma_g) \to \text{hom}_C(\int_{\Sigma_g} \mathcal{R}, \int_{\Sigma_g} \mathcal{R}).$$

Taking connected components, we get an action of the mapping class group on $\pi_0$ of the right hand side.

When $C^\otimes$ is the $\infty$-category of $k$-linear categories, \cite{19} recovers well-known mapping class group actions on certain invariants of quantum groups.

It can also be computed iteratively for product manifolds:

Example II.5.1.2. Fix the product framing of the torus $S^1 \times S^1 = T^2$. For an $E_2$-algebra $A$, let $B = \int_{S^1 \times \mathbb{R}} A$ be Hochschild chains, which now has an $E_1$-algebra structure by virtue of the $\mathbb{R}$ factor. We have

$$\int_{T^2} A \simeq \int_{S^1 \times \mathbb{R}} A \bigotimes_{\int_{(S^1 \times \mathbb{R})^{\otimes 12}} A} \int_{S^1 \times \mathbb{R}} A \simeq B \bigotimes_{B \otimes B^{op}} B$$

is Hochschild chains of Hochschild chains.

This example can also be exhibited using the following:
II.5. FACTORIZATION HOMOLOGY

**Theorem II.5.1.3 (Fubini Theorem).** There is an equivalence
\[ \int_{X \times Y} A \simeq \int_X \int_{Y \times \mathbb{R}^{\dim X}} A \]

See Proposition 3.23 of [1] and Corollary 2.29 of [8].

**Remark II.5.1.4.** To make sense of the Fubini theorem, note that factorization homology for \( Y \times \mathbb{R}^{\dim X} \) has the structure of an \( \mathbb{E}_d \)-algebra for \( d = \dim X \).

Factorization homology also computes mapping spaces: (Of course, it is an old problem of topology to be able to compute invariants of mapping spaces.)

**Theorem II.5.1.5 (Non-abelian Poincaré Duality).** Let \( C^\otimes = \mathcal{T} \text{op}^\times \). Fix a topological space \( X \) which has trivial homotopy groups in dimensions \( \leq n - 1 \), choose a basepoint \( x_0 \in X \), and consider the \( \mathbb{E}_n \)-algebra \( \Omega^n X \) (Example II.4.3.10). Then for any framed manifold \( M \), we have that
\[ \int_M \Omega^n X \simeq \text{Map}_c(M, X) \]

That is, factorization homology is homotopy equivalent to the spaces of compactly supported maps from \( M \) to \( X \). (Here, compact support of \( f : M \to X \) means that outside some compact subset of \( M \), \( f \) is constant with value given by the basepoint of \( X \).)

**Remark II.5.1.6.** The above equivalence is called non-abelian Poincaré duality and is a theorem due in several guises to Salvatore [68], Lurie [63], Ayala-Francis [1], and Ayala-Francis-T [8].

In the case \( n = 1 \) with \( M = S^1 \), it states that Hochschild homology of a based loop space \( \Omega X \) is homotopy equivalent to the free loop space of \( X \) (for \( X \) connected). This recovers a theorem of Burghelea-Fiedorowicz [22] and Goodwillie [40] after applying the singular chains construction.

Factorization homology is intimately tied to configuration spaces, which are also recurring characters in topology:

**Theorem II.5.1.7.** Let \( C^\otimes = \mathcal{T} \text{op}^\times \) once more, and let \( A \) be the free \( \mathbb{E}_n \)-algebra on one generator (Example II.4.3.11). Then
\[ \int_M A \simeq \prod_{l \geq 0} \text{Conf}_l(M) \]

That is, factorization homology of \( M \) with coefficients in the free \( \mathbb{E}_n \)-algebra on one generator is homotopy equivalent to the disjoint union over \( l \geq 0 \) of the configuration spaces of \( l \) disjoint, unordered points in \( M \).

**Remark II.5.1.8.** Theorem II.5.1.7 is a simple case of Proposition 5.5 of [1] and Proposition 4.12 of [8].
Example II.5.1.9. Another example illustrating some of the ingredients of factorization homology is Bandklayder’s alternative proof \([11]\) of the Dold-Thom theorem \([30]\) for manifolds. Recall that the Dold-Thom theorem states the following: Fix a connected, reasonable topological space \(X\) along with a basepoint \(x_0\) and an abelian group \(A\). Then there exist natural isomorphisms
\[
\pi_k(\text{Sym}(X; A)) \cong \widetilde{H}_k(X; A), \quad k \geq 1,
\]
between the homotopy groups of the \textit{infinite symmetric product of \(X\) with labels in \(A\)} and the reduced homology groups of \(X\) with coefficients in \(A\). Informally, \(\text{Sym}(X; A)\) is a configuration space of disjoint, unordered points in \(X\) labeled by elements of \(A\)—topologized so that any element labeled by the identity \(e \in A\) disappear; so that when points collide, their labels add; and so that any point that collides with \(x_0\) loses its label, or “disappears.” (One may reasonably think of the basepoint \(x_0\) as a point at infinity where labeled points go to be forgotten.)

Finally, the more “commutative” the coefficient algebra, the less sensitive factorization homology is to the smooth isomorphism type of the manifold (and becomes more an invariant of its homotopy type).

Example II.5.1.10 (Usual homology.). See Exercise [I.7.0.3] for how one recovers usual homology by using factorization homology with coefficients in an abelian group (which is a commutative algebra in \(C^\otimes = \text{Chain}^\otimes\)).

Example II.5.1.11 (Pirashvili’s higher order Hochschild invariants). Fix a base ring \(R\). Recall that any commutative \(R\)-algebra \(A\) is an \(E_n\) algebra in \(\text{Chain}_{R}^{\otimes/}\) for all \(n\) (Example [II.4.3.1]). Hence one may compute factorization homology on a framed manifold of any dimension.

Such invariants were studied under the name of \textit{Higher order Hochschild homology} by Pirashvili [65]—there, rather than take a framed manifold, Pirashvili constructed invariants associated to any simplicial set \(Y\) (Pirashvili also studied the case where one may take a bimodule \(M \neq A\) as an additional datum—see Section II.6.7). The way this intersects with our story is as follows: Given any manifold \(X\), equip it with a homotopy equivalence to a simplicial set \(Y\) (for example by taking the singular complex of \(X\), see Remark [II.2.2.4]). Then can construct an explicit chain complex out of \(Y\) and \(A\) that computes factorization homology of \(X\) with coefficients in \(A\).

These methods can be generalized more generally to cdgas (commutative differential graded algebras) over \(R\). Indeed, let \(\text{Cdga}_{R}\) be the \(\infty\)-category of cdgas as sketched in Example [II.2.2.2]. Fixing a cdga \(A\), and thinking of \(A\) as an \(E_n\)-algebra, for any framed \(n\)-manifold we have the following equivalence:
\[
\int_X A \simeq X \otimes_{\text{Cdga}_{R}} A \in \text{Chain}_{R}.
\]

\(^2\)E.g., what we have called a “combinatorially defined” space before, see Remark [II.2.2.6].

\(^3\)Note that ordinary commutative algebras are examples of cdgas concentrated in degree 0.
Here we have used the tensoring of the ∞-category of cdgas over spaces—i.e., $X \otimes_{\text{cdga}_R} A$ is the colimit of $A$ in $\text{cdga}_R$ indexed by the constant functor in the shape of $X$. For example, we have equivalences

$$\text{hom}_{\text{cdga}}(X \otimes_{\text{cdga}} A, B) \simeq \text{hom}_{\text{Top}}(X, \text{hom}_{\text{cdga}}(A, B)).$$

See for instance Section 3.2 of [38] and Proposition 5.7 of [5].

Remark II.5.1.12. The above example hints at how, when $A$ is a commutative algebra, one may in general construct invariants of reasonable topological spaces—say, those that admit a proper embedding into $\mathbb{R}^N$ for large $N$ (for example, suitably finite CW complexes). One may then take a small open neighborhood of the embedded space, which is a framed $N$-manifold, and compute its factorization homology with coefficients in $A$ considered as an $E_N$-algebra. All such invariants will be invariant under homotopy equivalences.

II.6. Leftovers and elaborations

II.6.1. Framings and other tangential structures. Let’s address this framing business.

We are motivated to make the category $\text{Disk}_{n,fr}$ have an easy-to-enumerate list of objects. This motivation is a little disingenuous, because the data of an algebra over a complicated ∞-category may still be interesting and deserve study. But let’s just say that we really want to recreate the notion of $E_n$-algebra from framed disks, for whatever reason.

Suppose you naively define a framed embedding to simply be one that respects framings, in the sense that

$$\begin{align*}
TX & \xrightarrow{\phi_X} X \times \mathbb{R}^n \\
TY & \xrightarrow{\phi_Y} Y \times \mathbb{R}^n
\end{align*}$$

commutes. Here, $j : X \longrightarrow Y$ is the smooth embedding, $Dj$ is its derivative, and the $\phi$ are the framings on $X$ and $Y$. This is a property of $j$, and merits no extra data on $j$.

Already when $X = Y = \mathbb{R}^n$, it becomes nearly impossible to find a $j$ making the above diagram commute for arbitrary choices of $\phi_X$ and $\phi_Y$. For example, endow $X$ with the canonical framing one constructs from the $\mathbb{R}$-vector space structure on $X$. Then any smooth function $\alpha : \mathbb{R}^n \longrightarrow GL_n(\mathbb{R})$ defines a framing $\phi_Y$ on $Y = \mathbb{R}^n$. It is a highly difficult differential geometry problem to find a map $j : X \longrightarrow Y$ such that the derivative $Dj$ recovers exactly the matrices $\alpha$. (Indeed, this integrability problem almost always lacks a solution.) The upshot is that the category of framed disks would have many non-equivalent objects whose underlying manifold is $\mathbb{R}^n$. 
On the other hand, to have an “algebra” structure, we need to be able to embed copies of \( \mathbb{R}^n \coprod \mathbb{R}^n \) into \( \mathbb{R}^n \). To do this in a way satisfying (II.6.1.1) is impossible if each component is given the vector space framing.

So we see that the naive notion of “embeddings that respect framings” will at least lead us down a complicated path; we can’t enumerate objects easily, and articulating the multiplications \( \mathbb{R}^n \coprod \mathbb{R}^n \to \mathbb{R}^n \) seems to require us to grapple with this difficult enumeration.

So let us instead articulate some technology to simplify things. Any manifold \( X \) comes equipped with a canonical continuous map classifying the tangent bundle:

\[ \tau_X : X \to BGL_n(\mathbb{R}) \]

Here, \( n = \dim X \) and \( BGL_n(\mathbb{R}) \) is the classifying space of principle \( GL_n(\mathbb{R}) \)-bundles.

**Definition II.6.1.2.** Fix a topological group \( G \) with a map \( G \to GL_n(\mathbb{R}) \). Note this induces a continuous map of classifying spaces \( p : BG \to BGL_n(\mathbb{R}) \). A *tangential \( G \)-structure* on a smooth manifold \( X \) is the data of a map \( X \to BG \) and a homotopy from the composite \( X \to BG \to BGL_n(\mathbb{R}) \) to the map \( X \to BGL_n(\mathbb{R}) \) classifying the tangent bundle.

We draw this as a triangle

\[
\begin{array}{ccc}
BG & \to & \tau_X : X \to BGL_n(\mathbb{R})
\end{array}
\]

where it is understood that the triangle need not commute on the nose, but is supplied with a homotopy \( p \circ \phi \sim \tau_X \).

**Example II.6.1.3.** The data of a homotopy \( p \circ \phi \sim \tau_X \) reduces the structure group of \( TX \) from \( GL_n(\mathbb{R}) \) to \( G \). When \( G = SO_n(\mathbb{R}) \), the triangle above is a choice of orientation on \( X \). When \( G = * \) is trivial, the above is the data of a framing on \( X \).

In fact, manifolds with a framing arise as objects of a very natural fiber product of \( \infty \)-categories:

**Definition II.6.1.4 (\( \mathcal{Mfld}_{n,fr} \)).** See Definition 2.7 of [1] and Definition 5.0.2 of [7]. Consider the \( \infty \)-category \( \mathcal{Mfld}_n \) whose objects are smooth \( n \)-dimensional manifolds and whose morphisms consist of all smooth embeddings; we note that the tangent bundle construction defines a functor

\[ \mathcal{Mfld}_n \to Top_{/BGL_n(\mathbb{R})}, \quad X \mapsto (\tau_X : X \to BGL_n(\mathbb{R})) \]

to the \( \infty \)-category of all topological spaces equipped with a map to \( BGL_n(\mathbb{R}) \). On the other hand, if one fixes a map \( G \to GL_n(\mathbb{R}) \) (and hence a map \( BG \to BGL_n(\mathbb{R}) \)) we have an induced functor \( Top_{/BG} \to Top_{/BGL_n(\mathbb{R})} \). When \( G = * \) is trivial, we
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define $\mathcal{M}frd_{n,fr}$ to be the pullback (i.e., fiber product)

\[
\begin{array}{ccc}
\mathcal{M}frd_{n,fr} & \longrightarrow & \mathcal{T}op \simeq \mathcal{T}op_{/s} \\
\downarrow & & \downarrow \tau \\
\mathcal{M}frd_n & \longrightarrow & \mathcal{T}op_{/BGL_n(\mathbb{R})}.
\end{array}
\]

of $\infty$-categories. Likewise, $\mathcal{D}isk_{n,fr}$ is the full subcategory of $\mathcal{M}frd_{n,fr}$ obtained by pulling back as below:

\[
\begin{array}{ccc}
\mathcal{D}isk_{n,fr} & \longrightarrow & \mathcal{T}op \simeq \mathcal{T}op_{/s} \\
\downarrow & & \downarrow \tau \\
\mathcal{D}isk_n & \longrightarrow & \mathcal{T}op_{/BGL_n(\mathbb{R})}.
\end{array}
\]

**Warning II.6.1.5.** We have not defined what slice categories are in the $\infty$-categorical setting. Let us informally state the following: Given two objects $(X \rightarrow B)$ and $(Y \rightarrow B)$ in $\mathcal{C}/B$, a morphism is given by a homotopy-commutative triangle

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
B & \longrightarrow & B
\end{array}
\]

where both the map $X \rightarrow Y$ and the homotopy rendering the triangle homotopy-commutative is part of the data of the morphism. See Section 1.2.9 of [61].

**Remark II.6.1.6.** Morphism spaces of pullback $\infty$-categories can be computed as pullbacks of the original morphism spaces. Fix $\mathbb{R}^n \in \mathcal{M}frd_{n,fr}$ and let us compute its endomorphism space. Because $\mathbb{R}^n$ is contractible (and the $\infty$-category of spaces identifies weakly homotopy equivalent spaces), we have that

(II.6.1.7) \[ \hom_{\mathcal{T}op_{/BGL_n(\mathbb{R})}}(\mathbb{R}^n, \mathbb{R}^n) \simeq \hom_{\mathcal{T}op_{/BGL_n(\mathbb{R})}}(\ast, \ast) \simeq \Omega BGL_n(\mathbb{R}) \simeq GL_n(\mathbb{R}) \]

where the $\Omega$ denotes the based loop space. We have also already seen in the proof of Lemma [II.4.1.7]—using the trick of translation and derivative-taking—that the inclusion

(II.6.1.8) \[ GL_n(\mathbb{R}) \rightarrow \hom_{\mathcal{M}frd_n}(\mathbb{R}^n, \mathbb{R}^n) \]

is a homotopy equivalence. Consider the pullback diagram of mapping spaces

\[
\begin{array}{ccc}
\hom_{\mathcal{M}frd_{n,fr}}(\mathbb{R}^n, \mathbb{R}^n) & \longrightarrow & \ast = \hom_{\mathcal{T}op}(\ast, \ast) \\
\downarrow & & \downarrow \sim \\
\hom_{\mathcal{M}frd_n}(\mathbb{R}^n, \mathbb{R}^n) & \longrightarrow & \hom_{\mathcal{T}op_{/BGL_n(\mathbb{R})}}(\mathbb{R}^n, \mathbb{R}^n).
\end{array}
\]
The bottom arrow is an equivalence by tracing through (II.6.1.7) and (II.6.1.8). Since pullbacks of equivalences are equivalences, we conclude that \( \text{hom}_{\mathcal{M}fd_n}([\mathbb{R}^n, \mathbb{R}^n]) \to \ast \) is an equivalence. That is, as promised before, the space of endomorphisms of framed \( \mathbb{R}^n \) is contractible.

**Remark II.6.1.9.** More generally, after fixing a continuous group homomorphism \( G \to GL_n(\mathbb{R}) \), one can define the \( \infty \)-category of \( G \)-structured manifolds by the pullback of \( \mathcal{M}fd_n \) along \( \mathcal{T}op/BG \to \mathcal{T}op/BGL_n(\mathbb{R}) \). The same computation as above shows that the endomorphism space of \( \mathbb{R}^n \) in this category is weakly homotopy equivalent to the topological group \( G \).

**Remark II.6.1.10.** Thus for any continuous group homomorphism \( G \to GL_n(\mathbb{R}) \), one can define the \( \infty \)-category of \( \mathcal{D}isk_n,G \) as a fiber product \( \mathcal{D}isk_n \times \mathcal{T}op/BGL_n(\mathbb{R}) \to \mathcal{T}op/BG \). Informally, an object is a disjoint union of \( n \)-dimensional Euclidean spaces, each equipped with a tangential \( G \)-structure. This still has a symmetric monoidal structure given by disjoint union, and one can study \( \mathcal{D}isk_n,G \)-algebras, which are symmetric monoidal functors

\[
\mathcal{D}isk_{n,G} \to C^\otimes.
\]

(For example, when \( G = SO(n) \), we have the oriented disk-algebras.) One can define factorization homology as a left Kan extension as before (thereby obtaining invariants of \( G \)-structured \( n \)-dimensional manifolds) and one still has the \( \otimes \)-excision theorem. This is proven for example in [1, 8].

**Remark II.6.1.11.** Now we can state what we mean by equipping embeddings with compatibilities of framings: Fix framings on manifolds \( X \) and \( Y \), and fix a smooth embedding \( j : X \to Y \). Then a compatibility of \( j \) with the framings is the data of a homotopy rendering the following tetrahedral diagram homotopy-commutative:

```
BG
\phi_x
\phi_y
X \longrightarrow_{\tau_x} BGL_n(\mathbb{R}) \quad Y
\tau_y
```

This involves the data of the faces containing both \( X \) and \( Y \), and the three-cell defining the interior of the tetrahedron.

**Remark II.6.1.12.** Let \( X \) be a framed \( n \)-dimensional manifold, and let \( A \) be an \( \mathbb{E}_{n+k} \)-algebra. Note that the thickening \( X \times \mathbb{R}^k \) may admit many framings that do not decompose as the “direct product” of a framing on \( X \) with a framing on \( \mathbb{R}^k \). For different framings, factorization homology recovers different invariants of \( A \).
II.6.2. **Left Kan extensions.** Let me at least say something about left Kan extensions.

Fix two functors $F : \mathcal{C} \to \mathcal{E}$ and $a : \mathcal{C} \to \mathcal{D}$. One can ask if there is a “canonical” way to extend $F$ to a functor emanating from $\mathcal{D}$:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{E} \\
\downarrow & \rotatebox{90}{$\bowtie$} & \downarrow \\
\mathcal{D} & \xrightarrow{a} & \mathcal{E}
\end{array}
\]

Suppose you are given the data of a pair $(G, \eta)$ where $G$ is a functor $\mathcal{D} \to \mathcal{E}$, and $\eta : F \to G \circ a$ is a natural transformation. Such a pair is called a *left Kan extension* if it is initial with respect to all possible $(G, \eta)$; that is, given any other $(G', \eta')$, one can find a unique map from $(G, \eta)$ to $(G', \eta')$.

The word “extension” is slightly misleading, as $G \circ a$ may not agree with $F$. However, if $a$ is a fully faithful inclusion (as in the case $\text{Disk} \to \text{Mfld}$ of interest to us), $G \circ a$ can be made naturally equivalent to $F$.

A left Kan extension exists if $\mathcal{E}$ admits certain colimits. In fact, one can prove the following formula for left Kan extensions:

\[G(d) = \text{colim}_{x \in \mathcal{C}/d} F(x).\]

The colimit is diagrammed by the slice category of objects in $\mathcal{C}$ over $d \in \mathcal{D}$. In the case of factorization homology, we have

(II.6.2.1) \[\int_X A = \text{colim}_{(\text{Disk}_n)/X} F.\]

Concretely, the colimit is indexed by collections of (disjoint unions of) disks equipped with embeddings into the manifold $X$, and we evaluate the symmetric monoidal functor $F$ on these disks. In this way, factorization homology may be interpreted as “the most efficient invariant glued out of ways to embed disks into $X$.” (We have omitted mention of framings and orientations from the notation.)

**Remark II.6.2.2.** While the idea of a left Kan extension has been cast aside here as an afterthought, we would like to remark that our ability to use left Kan extensions for $\infty$-categories depends on a robust enough theory of $\infty$-categories and colimits within an $\infty$-category, just as the classical notion depends on a good development of the language of categories and colimits. Though it seems we can guarantee that a left Kan extension induces a continuous map of morphism spaces “for free;” this lunch indeed is not free, and it was made possible by machinery developed by Joyal [48] and Lurie [61].

**Remark II.6.2.3.** We have not defined the notion of colimits in an arbitrary $\infty$-category, but colimits satisfy the universal property analogous to the classical notion: Colimits are initial objects receiving a map from a diagram. See 1.2.13 of [61].
Remark II.6.2.4 ($\infty$-categorical colimits are homotopy colimits). Let us also make a comment for readers familiar with the notion of homotopy colimits. The notion of a colimit in an $\infty$-category agrees with the notion of a homotopy colimit whenever the $\infty$-category arises from a setting in which homotopy colimits make sense. (For example, as the nerve of a model category.) See 4.2.4 of [61] for details.

II.6.3. What’s up with sifted colimits? Recall that colimits are a way to “glue together” objects in a category. Likewise, colimits can be articulated in $\infty$-categories, though a given $\infty$-category may not always have the colimits you want (just as in ordinary category theory).

We often classify colimits by the shape of the diagram encoding the gluing. A particular class of diagrams is given by the sifted diagrams, and a sifted colimit is a colimit glued out of a sifted diagram.\footnote{A diagram (i.e., an $\infty$-category) $D$ is called sifted if it is non-empty and if the diagonal inclusion $D \to D \times D$ is left final (i.e., cofinal in some works). Typical examples include filtered diagrams and $\Delta^{op}$.}

It turns out that for any $X \in \mathcal{Mfld}$—framed or not—the colimit in (II.6.2.1) is a sifted colimit, so the left Kan extension is guaranteed to exist if $C$ has sifted colimits.

Moreover, one can also extend $\int A$ to be a symmetric monoidal functor if $\otimes$ commutes with sifted colimits in each variable.

Finally, let us say that the tensor product

$$M \otimes_R N$$

of two modules over an algebra can be presented as a bar construction, and the bar construction is indexed by a simplicial diagram, which in particular is a sifted diagram. That is to say, the bar construction (the tensor product) always exists if $C$ admits sifted colimits.

This explains why we assume that $C$ admits sifted colimits and why we assume that its symmetric monoidal structure preserves sifted colimits in each variable.

Example II.6.3.1. Let $C^\otimes = \text{Ch}_{k}$. Note that the symmetric monoidal structure here is the direct sum of cochain complexes; i.e., the coproduct. $\oplus$ does preserve sifted colimits in each variable, but it does not preserve all colimits in each variable. (For example, it doesn’t preserve itself in each variable!)

II.6.4. How many excisive theories are there? Let $C^\otimes$ be a symmetric monoidal $\infty$-category admitting sifted colimits, and for which $\otimes$ preserves sifted colimits in each variable. Fix an $\mathbb{E}_n$-algebra $A$.

We have seen that factorization homology results in a functor $\int A : \mathcal{Mfld}_{n,fr} \to C^\otimes$ which is symmetric monoidal and $\otimes$-excisive (Remark II.5.0.5 and Theorem II.5.0.6). One could ask the following question: Are there other functors other than factorization homology that could satisfy these properties?

The answer is no. More precisely, we have the following:
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Theorem II.6.4.1. Restriction to the full subcategory $\mathcal{D}isk_{n,fr} \subset \mathcal{M}fld_{n,fr}$ of disks defines an equivalence of $\infty$-categories

$$\{ \otimes\text{-excisive, symmetric monoidal } F : \mathcal{M}fld_{n,fr}^{\Pi} \to \mathcal{C}^\otimes \} \to \mathcal{D}isk_{n,fr}\text{-algebras in } \mathcal{C}^\otimes$$

with an inverse functor implemented by factorization homology.

See Theorem 3.24 of [1] and Theorem 2.45 of [8].

This means that any symmetric monoidal, $\otimes$-excisive invariant arises as factorization homology of some $E_n$-algebra. The proof itself is not too difficult once the machinery is set up: One uses the assumption of $\otimes$-excision to show by induction that the values of an excisive theory on handles are completely determined by their values on disks, then one uses handle decompositions to show the same for arbitrary smooth manifolds. The proofs of the theorems in [1] and [8] cited above are slightly more complicated because [1] addresses the case of topological manifolds (where handlebody decompositions are not guaranteed in dimension 4) and because [8] addresses a stratified generalization for which one must rely on (analogous, but involved to set up and verify) decomposition results for stratified spaces as developed in [7].

Remark II.6.4.2. Theorem II.6.4.1 holds true also in the generality of arbitrary $G$-structures (Remark II.6.1.10): Any symmetric monoidal, $\otimes$-excisive functor for $G$-structured manifolds arises as factorization homology of a $\mathcal{D}isk_{n,G}$-algebra. See again Theorem 3.24 of [1] and Theorem 2.45 of [8].

Remark II.6.4.3. If one wants to consider manifolds that are larger than our Convention 0.0.1, the above classification theorem must be modified slightly. Every symmetric monoidal, $\otimes$-excisive functor that preserves sequential colimits of countable open exhaustions arises as factorization homology of some algebra. (See for example Definition 2.37 of [8].)

II.6.5. Locally constant factorization algebras. The origins of factorization homology are rooted in the work on chiral algebras of Beilinson and Drinfeld [13]. Factorization algebras are another perspective on how to take local algebraic structures and form global invariants; see the work of Costello-Gwilliam [28].

I am often asked about the equivalence between locally constant factorization algebras on $\mathbb{R}^n$, and $E_n$-algebras. I refer the reader to Section 2.4 of [8] for one formulation, where we exhibit the $\infty$-category of disk embeddings as a localization of a discrete version.

Another equivalence proven using the formulation of cosheaves on the Ran space of $\mathbb{R}^n$ can be found in [63]. This approach works for not-necessarily-unital $E_n$-algebras.

II.6.6. How good a manifold invariant is factorization homology? This is also a natural question. Roughly speaking, it seems to be “about as good as the homotopy type of configuration spaces.” It is unknown how good a manifold invariant the homotopy type of configuration spaces are; for some time they were conjectured...
to be only sensitive to the homotopy type of a manifold, but now they are known
to distinguish homotopy-equivalent (but non-diffeomorphic) manifolds of the same dimension [60].

To see why configuration spaces enter the picture, recall that the free $E_n$-algebra
(in topological spaces) generated by a single element is the configuration space of
unordered points in $\mathbb{R}^n$ (Example II.4.3.11). By the free-forget adjunction, any $E_n$
-algebra may be resolved by algebra maps from this configuration space, and thus
factorization homology with coefficients in an arbitrary $E_n$-algebra receives maps
from (a diagram made up of) the factorization homology with coefficients in the free
algebra. On the other hand, factorization homology of a manifold with coefficients
in the free algebra is homotopy equivalent to the configuration space of the manifold
(Example II.5.1.7).

II.6.7. Are we stuck with algebras only? There is a variant of factorization
homology where one can obtain invariants of not only algebras, but algebras equipped
with the data of bimodules, and also of higher categories. This is developed in [7, 8]
(in the case of algebras and bimodules) and in [6] (for higher categories). For example,
by articulating what we mean by a category of 1-manifolds equipped with marked
points, factorization homology over a circle with marked points recovers Hochschild
homology of an algebra (assigned to open 1-dimensional disks) with coefficients in
the desired bimodules (assigned to marked points).

II.7. Exercises

Exercise II.7.0.1. Let $C^\otimes = \text{Vect}_k^\otimes$ be the ordinary category of vector spaces
and fix a unital associative $k$-algebra $A$. Using the general excision theorem, Theorem II.5.0.6 of this chapter, recover the excision theorem for the circle from the previous chapter (Theorem I.5.0.4).

Be very careful about orientations of circles; the point is to understand why the
tensor product contains an $A^{\text{op}}$ factor.

Exercise II.7.0.2. Let $C^\otimes = \text{Chain}^\otimes$ be the $\infty$-category of cochain complexes,
but with direct sum as the symmetric monoidal structure. Show that any cochain
complex $V$ admits an $E_n$-algebra structure by defining $V \oplus V \to V$ to be the addition
map. Show further that this structure is unique.

Note that I have left out the subscript $k$; this exercise is valid for cochain complexes
over an arbitrary base ring, including $\mathbb{Z}$.

Exercise II.7.0.3. Let $C^\otimes = \text{Chain}^\otimes$ and fix $A$ any abelian group (considered as
a cochain complex in degree 0). By the previous exercise, addition endows $A$ with
an $E_n$-algebra structure for any $n$. 
Show that for any framed manifold $X$, there is a quasi-isomorphism of cochain complexes

$$\int_X A \simeq C_*(X; A)$$

where $C_*(X; A)$ is the cochain complex of singular chains. (Hint: Show that $\otimes$-excision for $\oplus$ gives rise to the Mayer-Vietoris sequence; this is a consequence of the usual excision theorem for singular homology.)

This exercise shows that factorization homology generalizes ordinary homology, hence the word “homology.” Your proof also shows why the $\otimes$-excision theorem has the word “excision” in it.

Can you recover the homology of any compact manifold using factorization homology? How about for any finite CW complex?
CHAPTER 3

Topological field theories and the cobordism hypothesis

III.1. Review of Lecture 2

DEFINITION III.1.0.1. Let $\mathcal{C}^\otimes$ be a symmetric monoidal $\infty$-category, e.g. $\text{Vect}^\otimes$, $\text{Chain}^\otimes$ or $\text{Cat}^\otimes$. An $E_n$-algebra in $\mathcal{C}$ is a symmetric monoidal functor $\text{Disk}_{n,fr} \rightarrow \mathcal{C}^\otimes$.

We give some examples from the previous lectures in the following table:

| $E_1$ | $\text{Vect}^\otimes$ | Monoidal categories | $A_\infty$-algebras |
|-------|------------------------|---------------------|---------------------|
| $E_2$ | Unital commutative $K$-algebras | Braided monoidal categories | $E_2$-algebras |
| $E_3$ | " | Symmetric monoidal categories | $E_3$-algebras |
| ...   | " | " | ... |

What we see is that the $\text{Vect}_{k}$ could not see the difference between $E_n$-algebras for $n \geq 2$. This is because the morphism spaces of $\text{Vect}_{k}$ are discrete, hence there is “no room” for the interesting homotopies to show up. Likewise, $\text{Cat}^\otimes$ sees the difference between $E_1$-algebras and $E_2$-algebras, but because its only “higher” morphisms are given by natural isomorphisms, there is no room to see the higher-dimensional homotopies that we need to detect $E_3$-structures. That is, any $E_n$-algebra in categories for $n \geq 3$ is a symmetric monoidal category.

Finally, it turns out that we do not have other names for an $E_n$-algebra in cochain complexes other than “$E_n$-algebra.”

We also stated:

DEFINITION III.1.0.2. Let $\mathcal{C}^\otimes$ be a symmetric monoidal $\infty$-category which has all sifted colimits and let $A$ be an $E_n$-algebra in $\mathcal{C}^\otimes$. Factorization homology is the left Kan extension

$$
\text{Disk}_{n,fr} \rightarrow \mathcal{C}^\otimes
$$

We write $\int_X A$ for factorization homology evaluated on a framed manifold $X$.

REMARK III.1.0.3. Let us collect some remarks.
• Factorization homology can also be defined for manifolds whose tangent bundles are equipped with a $G$-reduction; the necessary algebraic input there is—informally—an $\mathbb{E}_n$-algebra with $G$-action.

• Factorization homology is also functorial in the $A$ variable. That is, given a map of $\mathbb{E}_n$-algebras $A \to B$, we also have induced maps $\int_X A \to \int_X B$ for any framed manifold $X$. In this way, you can think of factorization homology $\int_X A$ as not only an invariant of the framed manifold $X$, but also an invariant of the $\mathbb{E}_n$-algebra $A$. For example, if we take $C^\otimes = \text{Ch}_{k}^\otimes$, and if we know that $\int_X A$ is not quasi-isomorphic to $\int_X B$, then we can conclude that $A$ and $B$ are not equivalent $\mathbb{E}_n$-algebras.

The tool which makes factorization homology computable is tensor excision:

**Theorem III.1.0.4 (\otimes excision).** Assume that $C^\otimes$ is a symmetric monoidal $\infty$-category with all sifted colimits, and assume that $\otimes$ preserves sifted colimits in each variable. Fix a framed manifold $X$ and a decomposition $X = X_0 \cup_{W \times \mathbb{R}} X_1$.

Then $\int_X A$ can be computed as the relative tensor product

$$\int_X A = \int_{X_0} A \otimes \int_{W \times \mathbb{R}} A \int_{X_1} A.$$ 

**III.2. Cobordisms and higher categories**

We switch topics a bit to work toward the notion of a topological field theory.

$\otimes$-excision shows that factorization homology is a ‘local-to-global’ invariant. Thus the invariant can be computed from a decomposition of $X$, and importantly, the invariant is insensitive to the way in which we decompose $X$. Our next goal is to capture this property categorically.

**Warning III.2.0.1.** A lot what we say in this section regarding the (higher) category of cobordisms is informal. For a slightly more rigorous treatment, see Section [III.8.1] below.

**Definition III.2.0.2 (Informal).** Let $\text{Cob}_{n,n-1}$ be the category whose objects are compact $n - 1$ dimensional, oriented manifolds. Given $W_0$ and $W_1$ objects of $\text{Cob}_{n,n-1}$, an element of the set $\text{hom}(W_0, W_1)$ is a compact $n$-dimensional manifold $X$ together with an oriented identification

$$\partial X \cong W_0 \coprod W_1^{\text{op}}$$

where $W_1^{\text{op}}$ denotes $W^1$ equipped with the opposite orientation. For technical reasons, we will demand that this identification extends to define a collar of $\partial X$,

$$\partial X \times [0, 1] \cong (W_0 \coprod W_1^{\text{op}}) \times [0, 1]$$
where we also identify $\partial X \times [0, 1]$ with a small neighborhood of $\partial X \subset X$.

Composition is given by gluing together manifolds along their boundary:

$$\circ = W_2 \circ W_1 \circ W_0 \circ W_1 \circ W_2 \circ W_0$$

Finally, we note that the identity morphism of any $W$ is given by the cylinder $W \times [0, 1]$.

**Remark III.2.0.3.** Note that in the figure, we have read the cobordism “from right to left.” This is to make the usual notation for composition more easily compatible with our pictures.

**Remark III.2.0.4.** Usually categories are named after their objects. The category $\text{Cob}_{n,n-1}$ is one of the few exceptions where a category is named after its morphisms.

**Definition III.2.0.5.** The pair

$$(X, \partial X \cong W_0 \coprod W_1^{op})$$

is called a *cobordism* from $W_0$ to $W_1$.

The definition of $\text{Cob}_{n,n-1}$ achieves our goal of articulating local-to-global invariants in the following way: Every functor

$$(\text{III.2.0.6}) \quad Z : \text{Cob}_{n,n-1} \longrightarrow \mathcal{C}$$

satisfies a decomposition property.

To see this, fix an element

$$X \in \text{hom}_{\text{Cob}_{n,n-1}}(\emptyset, \emptyset),$$

i.e., a smooth oriented compact manifold $X$ without boundary. The evaluation of $Z$ on $X$ is an element

$$Z(X) \in \text{hom}_\mathcal{C}(Z(\emptyset), Z(\emptyset)).$$

Let $X = X_0 \coprod_W X_1$ be a decomposition of $X$ into two cobordisms $X_0 : \emptyset \rightarrow W$ and $X_1 : W \rightarrow \emptyset$. The functoriality of $Z$ implies $Z(X) = Z(X_1) \circ Z(X_0)$, i.e., we can compute $Z(X)$ from the decomposition of $X$. Moreover, every decomposition of $X$ gives the same result, namely $Z(X)$.

**Definition III.2.0.7.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. An $n$-dimensional (oriented) topological field theory is a symmetric monoidal functor

$$Z : \text{Cob}_{n,n-1} \longrightarrow \mathcal{C}^\otimes.$$

**Remark III.2.0.8.** Classically, for instance in ideas of Atiyah and Segal, one would take $\mathcal{C}^\otimes$ to be the category of vector spaces with symmetric monoidal structure given by $\otimes_k$. 
This is tantalizing. So we must naturally be led to ask:

**Question III.2.0.9.** Can we classify such functors?

The answer for high values of \( n \) is a resounding no. For example, to classify such functors, it may help to have a good handle on the objects of \( \text{Cob}_{n,n-1} \); but classifying all closed \((n - 1)\)-dimensional manifolds is near impossible. We cannot do it when \((n - 1) = 4\).

**Remark III.2.0.10.** Of course, one need not classify all objects to define a functor. And the difficulty of classification of manifolds is one reason one would seek a functor as above. The real reason I am moaning on about this is to get to the fully extended cobordism category below.

Let me emphasize that the difficulty of classifying manifolds is not a convincing reason for one to abandon the search for functors out of \( \text{Cob}_{n,n} \) for values such as \( n = 2, 3, \) or 4. For toy examples in representation theory, the reader can look up Dijkgraaf-Witten theory, Seiberg-Witten invariants or Chern-Simons theory come also “close” to defining functors out of \( \text{Cob}_{n,n-1} \) in our sense, though I won’t go into it here. At the very least, it has certainly been difficult to construct meaningful and powerful examples. See [53] and [35].

But what if we can further decompose \((n - 1)\)-dimensional manifolds? For example, what if instead of classifying \((n - 1)\)-dimensional manifolds, we allow our “category” to decompose these further into \((n - 2)\)-dimensional manifolds, and so forth and so forth? Then to understand our objects, we would need only understand 0-dimensional manifolds. This we can do.

I put “category” in quotes above because it turns out the natural algebraic structure to look for is that of an \( n \)-category, of which a category is a special case when \( n = 1 \). Let me ease us into this notion by discussing the example of \( n = 2 \).

**Definition III.2.0.11 (Informal).** We denote by \( \text{Cob}_{0,1,2} \), or \( \text{Cob}_2 \)

for short, the “category” whose data are given by

- **Objects:** Oriented 0-dimensional compact manifolds. (Which is to say, an object is a possibly empty collection of points, each point equipped with a plus or a minus.)
- **Given two objects** \( W_0, W_1 \), we declare

\[
\text{hom}(W_0, W_1) = \{ \text{cobordisms } X : W_0 \to W_1 \}
\]

to be the collection of cobordisms from \( W_0 \) to \( W_1 \). Then,

- **Given any two cobordisms** \( X \) and \( Y \) having the same source and target, we declare an element of

\[
\text{hom}(X, Y)
\]
to be a compact 2-dimensional oriented manifold $Q$ with corners, equipped with an identification

$$\partial Q \cong (X \coprod Y^{\text{op}}) \cup (W_0 \times [0, 1] \coprod W_1^{\text{op}} \times [0, 1]).$$

Informally, $Q$ is a cobordism between the cobordisms $X$ and $Y$. The $\cup$ above is a gluing along the subspace $(W_0 \times \{0, 1\}) \coprod (W_1 \times \{0, 1\})$.

**Example III.2.0.12.** There’s lots to unpack here. Let’s first begin with an example of a $Q$, the saddle:

![Diagram](image.png)

In this example, I have taken $W_0 = W_1 = \ast^+ \coprod \ast^-$ to equal two disjoint points; one with positive orientation and one with negative orientation. $X$ is a morphism given by a horseshoe and a co-horseshoe—it is a disconnected, oriented 1-manifold with boundary given by $W_0 \coprod W_1^{\text{op}}$. Finally, I have taken $Y$ to be the product $W_0 \times [0, 1]$, the identity morphism of $W_0$.

**Remark III.2.0.13 (Cobordisms, classically).** Cobordisms have long been a way to prescribe ways to change the differential topology of a smooth manifold. You may have heard smooth topologists talking about attaching handles to change a manifold, and the above cobordism is an example of attaching a 1-handle to change a 2-manifold. Indeed, the (co)-horseshoes are examples of attaching 0- and 1-handles to change 1-manifolds.

**Remark III.2.0.14 (Composing the 2-manifolds).** We knew from before that we could compose the 1-manifolds, which are cobordisms in the previous sense—we glue $X$’s along a common $W$. We now note that given two $Q$’s (i.e., two 2-dimensional manifolds with corners) we can glue them to each other in possibly two distinct senses:

1. If the target of $Q$ is a cobordism $Y$, and if the domain of $Q'$ is that same cobordism, we can glue $Q$ and $Q'$ along $Y$. (We can compose along 1-manifolds.)
(2) If \(X\) and \(Y\) are morphisms from \(W_0\) to \(W_1\), and if \(X', Y'\) are morphisms from \(W_1\) to \(W_2\), we can glue \(Q\) and \(Q'\) to each other along \(W_1 \times [0,1]\). (We can compose along “constant” 0-manifolds.)

All this data, satisfying conditions that are natural but cumbersome to articulate, is some form of a 2-category. The \(W\)’s are called objects, the \(X\)’s and \(Y\)’s are morphisms, or 1-morphisms, and each \(Q\) is a 2-morphism.

**Remark III.2.0.15** (The “\(n\)” in \(n\)-morphism.). We referred to \(Q\) as a 2-morphism. This “2” can stand for many things in your mind: The dimension of the manifold, the “height” above the notion of being an object (an object is a 0-morphism and a usual morphism is a 1-morphism, for example), or the number of distinct senses in which you can compose \(Q\)’s (as indicated in Remark [III.2.0.14](#)).

Now that we have somewhat given intuition for \(n = 2\), let us state:

**Definition III.2.0.16** (Informal). We let \(\text{Cob}_n\) denote the \(n\)-category whose objects are oriented 0-manifolds, whose morphisms are oriented 1-manifolds with boundary (expressed as cobordisms between 0-dimensional manifolds), and whose \(k\)-morphisms are \(k\)-manifolds with corners (expressed as cobordisms between \((k-1)\)-dimensional manifolds).

If we could formalize the notion of \(n\)-category (I have only sketched an idea here), then it’s clear that the following algebraic gadget can encode “local-to-global” invariants: Functors

\[ \text{Cob}_n \rightarrow \mathcal{C} \]

of \(n\)-categories. Whatever we mean by functor, all compositions should be respected; this is the sense in which a functor from \(\text{Cob}_n\) defines local-to-global invariants.

**Example III.2.0.17.** Let us be concrete in the case \(n = 2\). We have defined a 2-category \(\text{Cob}_2\) whose 2-categorical structure encodes the notion of decomposing 2-dimensional manifolds: Fixing a 2-morphism \(Q\) between the empty cobordism and itself (i.e., a compact 2-dimensional manifold), a functor \(Z\) assigns a 2-morphism of \(\mathcal{C}\), and this is an invariant of the manifold. We can decompose this 2-dimensional manifold by expressing it as various compositions of other 2-morphisms (i.e., of other 2-dimensional cobordisms), which in turn may be be glued along 1-morphisms that themselves are expressed as compositions of various 1-morphisms. And regardless of how we decompose a 2-manifold—regardless of how we factor the 2-morphism—the original invariant \(Z(Q)\) can be recovered by composing along the corresponding factorizations in \(\mathcal{C}\).

And, in the spirit that once we have understood connected manifolds, we have understood the disconnected manifolds, we may as well seek functors that are symmetric monoidal. Finally, in the mean spirit of your homotopy theorist lecturer, we
may as well treat everything in sight as an \((\infty, n)\)-category. Indeed, \(\mathcal{C}_{ob_n}\) may be constructed as an \((\infty, n)\)-category, whatever that is.\footnote{This is not just for the sake of using the symbol \(\infty\). See Section III.8.1}

**Definition III.2.0.18.** Let \(\mathcal{C}^\otimes\) be a symmetric monoidal \((\infty, n)\)-category. A **fully extended, oriented** \(n\)-dimensional topological field theory with values in \(\mathcal{C}\), or \(n\)-dimensional oriented TFT for short, is a symmetric monoidal functor

\[
Z : \mathcal{C}_{ob_n} \longrightarrow \mathcal{C}^\otimes.
\]

One can define a version of \(\mathcal{C}_{ob_n}\), which every manifold in sight is appropriately framed. (See Definition III.8.2.6) We denote it by \(\mathcal{C}_{ob_n}^{fr}\). A symmetric monoidal functor

\[
Z : (\mathcal{C}_{ob_n}^{fr})^\otimes \longrightarrow \mathcal{C}^\otimes
\]

is called a *framed* \(n\)-dimensional TFT (with values in \(\mathcal{C}\)).

**Remark III.2.0.19.** The “fully extended” refers to the fact that we have “extended” a functor in the sense of (III.2.0.6) to capture manifolds of lowest possible dimension (zero).

Now, let us ask two natural questions:

**Question III.2.0.20 (Question One).** Can we make the definition of \(\mathcal{C}_{ob_n}\), and of \((\infty, n)\)-categories, precise?

**Question III.2.0.21 (Question Two).** Can we classify fully extended topological field theories, i.e. symmetric monoidal functors \(Z : \mathcal{C}_{ob_n} \longrightarrow \mathcal{C}^\otimes\) where \(\mathcal{C}\) is a symmetric monoidal \((\infty, n)\)-category?

Both have “yes” as an answer.

**Remark III.2.0.22.** The first question, Question One (III.2.0.20), we will touch upon more in Section III.8.1 below. But let us say for now that an \((\infty, n)\)-category consists of a collection of objects, and for every pair of objects \(X, Y\), the collection of morphisms \(\text{hom}(X, Y)\) can be considered an \((\infty, n-1)\)-category. This gives some feel of the notion by induction, beginning with the case \(n = 0\), where an \((\infty, 0)\)-category can be thought of as the same thing as a topological space.

Put another way, an \(n\)-category (or \((\infty, n)\)-category) is roughly meant to be a category enriched in \((n-1)\)-categories (or in \((\infty, n-1)\)-categories). In particular, given a pair of \(k\)-morphisms \(f, g\), the collection \(\text{hom}(f, g)\) forms an \((\infty, n-k)\)-category.

We for now focus on Question Two (III.2.0.21).
III.3. The cobordism hypothesis

The answer to Question Two (III.2.021) is as follows:

HYPOTHESIS III.3.0.1 (Cobordism Hypothesis). Fix a symmetric monoidal \((\infty, n)\)-category \(C^\otimes\). Then there exists an equivalence between

\[
\text{Fun}^\otimes((\text{Cob}_n^{fr})^!, C^\otimes)
\]

(the \(\infty\)-category of symmetric monoidal functors from \((\text{Cob}_n^{fr})^!\) to \(C^\otimes\)) and

\(C_{f.d.}\)

the space of fully dualizable objects of \(C^\otimes\). This equivalence is given by the evaluation map

\[
\text{Fun}^\otimes((\text{Cob}_n^{fr})^!, C^\otimes) \longrightarrow C_{f.d.} \quad Z \longmapsto Z(\ast^+)
\]

at the positively framed point.

This was an idea first proposed by Baez and Dolan [10], and for this reason is also referred to as the Baez-Dolan cobordism hypothesis.

Remark III.3.0.2. The reader will note that the above is stated as a hypothesis, and not a theorem; this is because the statement is widely expected to be true. Lurie has outlined a proof in [62] and there are certainly individuals working toward a proof. (One of the announced proof methods does not follow Lurie’s outline and instead uses techniques inspired by factorization homology [4].)

Remark III.3.0.3. We have stated that there is an equivalence of spaces. Indeed, buried in the statement is the following claim: If we have any symmetric monoidal natural transformation \(Z \longrightarrow Z'\), then this must actually be a natural equivalence.

(Following the philosophy of \(\infty\)-categories, any higher category in which all morphisms are invertible is equivalent to the \(\infty\)-category obtained from a space. See Remark [II.2.2.3].)

So now we must understand the notion of full dualizability. We begin in dimension one.

III.4. The cobordism hypothesis in dimension 1, for vector spaces

Before I go on, let me just say

Definition III.4.0.1. An \((\infty, 1)\)-category is an \(\infty\)-category.

Remark III.4.0.2. That is, the word “\(\infty\)-category” is just shorthand for the notion of \((\infty, 1)\)-category. This is simply how the culture has come to talk about these things.

Regardless, at least in dimension 1, there is no mystery about what the target category of a TFT is.
Under disjoint union the objects of $(\text{Cob}_1^\text{fr})^\Pi$ are generated by the empty set $\emptyset$, a positive point $*^+$ and a negative point $*^-$. We give names to the values of a topological field theory on these 0-dimensional manifolds

$$Z: (\text{Cob}_1^\text{fr})^\Pi \rightarrow \text{Vect}^{\otimes k}$$

$$*^+ \mapsto V$$

$$*^- \mapsto W$$

$$\emptyset \mapsto k.$$ 

Now, the identity morphism $*^+ \times [0, 1]$ can be factored as indicated on the lefthand column of the following figure:

**Remark III.4.0.4.** Let us explain Image [III.4.0.3]. We have drawn a “horseshoe” cobordism

$$\left(*^+ \coprod *^-\right) \leftarrow \emptyset : u$$

and a cohorseshoe cobordism

$$\emptyset \leftarrow \left(*^- \coprod *^+\right) : \epsilon,$$

where as before, are we reading our cobordisms as propagating from right to left. Thus, the bottom-left composition of the image is read as

$$(\text{id}_{*^+} \coprod \epsilon) \circ (u \coprod \text{id}_{*^+}).$$
Notation III.4.0.5. Out of sloth, and to save notational clutter, we will denote $Z(u)$ by the $u$ as well. Likewise for $\epsilon$. We hope that from context, it will be clear in which category ($\text{Cob}_n$ or $\mathcal{C}_\otimes$) these morphisms live.

From the decomposition we can directly deduce the following lemma.

**Lemma III.4.0.6 (Zorro’s Lemma).** Let $Z: \text{Cob}_1 \rightarrow \mathcal{C}$ be a topological field theory and let $V = Z(\ast^+)$. Then

(III.4.0.7) $\text{id}_V = (\text{id}_V \otimes \epsilon) \circ (u \otimes \text{id}_V)$.

**Proof.** Consider the “snake-like” factorization in the bottom-left of Image III.4.0.3. This composite is equivalent to the identity cobordism as a cobordism, hence applying $Z$ to the relation

$$\text{id}_{\ast^+} = (\text{id}_{\ast^+} \prod \epsilon) \circ (u \prod \text{id}_{\ast^+})$$

in $\text{Cob}_1$, the result follows. \hfill $\Box$

**Remark III.4.0.8.** Zorro is not a mathematician, but a masked avenger.

This actually puts a strong restriction on the values that $Z(\ast^+)$ can take:

**Lemma III.4.0.9.** $V = Z(\ast^+)$ must be finite dimensional.

**Proof.** As before let $W = Z(\ast^-)$. We have linear maps

$$\epsilon: V \otimes W \rightarrow k, \quad v \otimes w \mapsto \langle v, w \rangle$$

and

$$\epsilon: k \rightarrow W \otimes V, \quad 1 \mapsto \sum_{i,j} a_{i,j} w_i \otimes v_j,$$

where we note that the summation is *finite* (by the definition of the tensor product of vector spaces). The composition of linear maps on the right side of (III.4.0.7) is

$$v \mapsto v \otimes \left( \sum_{i,j} a_{i,j} w_i \otimes v_j \right) \mapsto \sum_{i,j} a_{i,j} \langle v, w_i \rangle v_j.$$

We get from (III.4.0.7) that $V$ must be spanned by finitely many vectors $v_j$. \hfill $\Box$

**Remark III.4.0.10.** This puts us well on the way to verifying the cobordism hypothesis in dimension 1, where there is again no difference between the framing and orientation conditions. Also, it will follow from the definition that a fully dualizable object of $\text{Vect}_k^{\otimes}$ will be a finite-dimensional $k$-vector space.

What remains to prove is that $Z(\ast^+)$ determines all of $Z$, and that any natural transformation of symmetric monoidal functors $Z \rightarrow Z'$ actually induces an *isomorphism* of vector spaces $Z(\ast^+) \rightarrow Z'(\ast^+)$. (This is what we mean by “space” of objects; an $\infty$-category where all morphisms are invertible).

We will leave this as an exercise to the reader (Exercise III.9.0.1).

This concludes our example for $n = 1$ and $\mathcal{C}_\otimes = \text{Vect}_k^{\otimes}$.
III.5. Full dualizability

From the previous example we can extract the following slogan: full dualizability is a generalization of “being finite dimensional” to higher categories.

**Definition** III.5.0.1. Fix a monoidal ∞-category $C$. An object $V \in C$ is dualizable, or 1-dualizable if

- $V$ admits a left dual. That is, there exists an object $V_L$ and morphisms $u: 1_C \to V \otimes V_L$ and $\epsilon: V_L \otimes V \to 1_C$ satisfying the formula
  \[(III.5.0.2) \quad id_V = (id_V \otimes \epsilon) \circ (u \otimes id_V).\]

(The reader should compare this to (III.4.0.7).)
- $V$ admits a right dual. That is, there exists an object $V_R$ and morphisms $u_R: 1_C \to V_R \otimes V$ and $\epsilon_R: V \otimes V_R \to 1_C$ satisfying an appropriate analogue of (III.5.0.2).

**Remark** III.5.0.3. The monoidal unit $1_C$ is always dualizable. When $C$ is symmetric monoidal then $V_L \cong V_R$ and the existence of a left/right dual implies that $V$ is dualizable.

Thus, from hereon (because $C^\otimes$ will always be symmetric monoidal for us) we will make no distinction between having right and left duals.

**Example** III.5.0.4. Being dualizable is a delicate balance between the symmetric monoidal structure chosen, and properties of the objects and morphisms in the category. Here is a list of dualizable objects in various categories:

| Category | Dualizable Objects |
|----------|-------------------|
| $C^\otimes$ | dualizable objects |
| Set$^\times$ | pt |
| Cat$^\times$ | pt |
| Vect$^\otimes$ | 0 |
| ComAlg$^\otimes_k$ | $k$ |
| AlgBimod | all algebras |

Here AlgBimod is the $\infty$-category with objects algebras, morphisms bimodules and 2-morphisms bimodule equivalences. The composition is given by the relative tensor product.

Now we turn our attention to the definition of higher notions of dualizability; this relies on defining when a morphism (not an object) is dualizable.

**Example** III.5.0.5. As a motivation we consider the classical notion of adjunctions. This illustrates when 2 functors are dualizable.

Let $\mathcal{D}$ and $\mathcal{E}$ be categories together with a pair of functors

$L: \mathcal{D} \rightleftarrows \mathcal{E} : R$.
Recall that \( L \) is left adjoint to \( R \) (and \( R \) is right adjoint to \( L \)) if and only if there exists a pair of natural transformations
\[
u: \text{id}_D \to R \circ L \quad \epsilon: L \circ R \to \text{id}_E
\]
such that the compositions
\[
L \xrightarrow{id_L \circ u} LRL \xrightarrow{\epsilon \circ L} L
\]
and
\[
R \xrightarrow{u \circ \text{id}_R} RLR \xrightarrow{id_R \circ \epsilon} R
\]
agree with the identity natural transformations (from \( L \) to itself, and from \( R \) to itself, respectively).

These are the classic examples of dualizable morphisms in an \((\infty, 2)\)-category:

**Definition III.5.0.6.** Let \( C \) be an \((\infty, 2)\)-category and fix two objects \( D \) and \( E \). Then a morphism \( f: D \to E \) is right dualizable if there exists another morphism \( R: E \to D \), along with two 2-morphisms
\[
u: \text{id}_D \to Rf, \quad \epsilon: fR \to \text{id}_E
\]
satisfying the adjointness conditions:
\[
(\epsilon \circ \text{id}_f) \circ (id_f \circ u) \simeq \text{id}_f, \quad (id_R \circ \epsilon) \circ (u \circ \text{id}_R) \simeq \text{id}_R.
\]
(That is, there exists an equivalence between the indicated compositions and the identity morphisms.) We then call \( f \) a left dual to \( R \), and we call \( R \) a right dual to \( f \). Likewise, we say that \( f \) is left dualizable if it admits a left dual.

We say \( f \) is dualizable, or admits adjoints, or is 2-dualizable, if it admits both left and right duals.

One may begin to be bothered by the notation, and also bothered by the similarities apparent in the dualizability condition for objects (Definition III.5.0.1), for morphisms (Definition III.5.0.6), and the example of adjunctions in Example III.5.0.5.

Let us clarify the sense in which “every formula that has appeared is actually the same formula.”

**Example III.5.0.7.** The notion of an object being dualizable can be phrased as a case of when a morphism is 2-dualizable.

That is, let \( C \) be a monoidal category. Denote by \( BC \) the 2-category with one object \( * \) and \( \text{End}(*) = C \). The composition in \( BC \) is given by the tensor product \( \otimes: C \times C \to C \). The 1-morphisms in this 2-category correspond to the objects of \( C \), and the 2-morphisms in \( BC \) are the morphisms of \( C \).

Then an object of \( C^\otimes \) is dualizable if and only if it is dualizable as a morphism in \( BC \).
EXAMPLE III.5.0.8. Let $\text{Cat}$ be the 2-category of categories: Objects are categories, morphisms are functors, and 2-morphisms are natural transformations (not just natural isomorphisms, as in the case of $\text{Cat}$).

Then a morphism in $\text{Cat}$ is dualizable if and only if it admits both a right and a left adjoints.

We would recommend that the reader only memorize the adjointness relations for functors; this is enough to recover all the relations.

Finally, the notion of 2-dualizability for morphisms is enough to define full dualizability:

**Definition** III.5.0.9. A $k$-morphism $X \rightarrow Y$ in an $(\infty, n)$-category is called dualizable, or $(k+1)$-dualizable, if it is 2-dualizable as a 1-morphism in $\text{hom}(X, Y)^{(2)}$.

Here, note that $\text{hom}(X, Y)$ is naturally an $(\infty, n-k)$-category. The notation $\text{hom}(X, Y)^{(2)}$ indicates the $(\infty, 2)$-category obtained by discarding non-invertible morphisms above degree 3.

**Definition** III.5.0.10 ([62]). An object $X$ inside an symmetric monoidal $(\infty, n)$-category $C^\otimes$ is fully dualizable if and only if

- $X$ is dualizable (this requires the existence of certain 1-morphisms).
- These 1-morphisms, and their duals, and all their composites, are 2-dualizable (this requires the existence of certain 2-morphisms).
- These 2-morphisms, and their duals, and all their composites, are in turn 3-dualizable,
- And this pattern continues until we verify that the required $(n-1)$-morphisms are $n$-dualizable.

**Remark** III.5.0.11. Inductive definitions can seem opaque at times, but let me emphasize that the hardest part of understanding this definition is getting used to what an $n$-category, or $(\infty, n)$-category, is.

Once one then remembers the adjointness relations (which are fairly quick to internalize, as it turns out), it is very much possible to verify whether or not an object is fully dualizable.

**Sadness** III.5.0.12. We are already two hours overtime so we won’t be able to give great examples in-depth; they’ll remain as exercises.

**III.6. The point is fully dualizable**

Now that we have seen the definition of full dualizability (Definition [III.5.0.10]), let us put the cobordism hypothesis to the test in the simplest possible example: Since the identity functor $(\text{Cob}_n^\text{fr})^! \rightarrow (\text{Cob}_n^\text{fr})^!$ is surely a TFT with values in $C^\otimes = (\text{Cob}_n^\text{fr})^!$, can we verify that $\ast^+$ is a fully dualizable object in $(\text{Cob}_n^\text{fr})^!$?

**Warning** III.6.0.1. In what follows, we only illustrate that the point in $\text{Cob}_n$ (the oriented case) is dualizable. We must be more careful in the framed case. See
Section III.8.2. We are illustrating only in the oriented case mainly to give a feel for what kinds of pictures one must draw in this business.

**Example III.6.0.2.** Let us first see that \( *^+ \in \mathcal{C}ob_1 \) is a dualizable, or 1-dualizable, object. Indeed, the proof of Zorro’s Lemma relies on this fact. (See Image III.4.0.3.)

What we discovered there can be summarized as follows using our new vocabulary: The point with opposite orientation \( *^- \) is both a left and right dual object to \( *^+ \), and the unit and counit maps exhibiting the adjunctions are the (co)horseshoes, 

\[
\begin{align*}
*^+ & \quad = \ u \\
*^- & \quad = \ \epsilon
\end{align*}
\]

We recall that our conventions are to read cobordisms from right to left, so \( u \) is a map from \( \emptyset \) to \( *^+ \coprod *^- \) while \( \epsilon \) is a map from \( *^- \coprod *^+ \) to \( \emptyset \).

**Example III.6.0.3.** Now let us see that \( *^+ \) is fully dualizable in \( \mathcal{C}ob_2 \). For this, we must verify that the 1-morphisms that arose in verifying that \( *^+ \) is 1-dualizable are 2-dualizable (by Definition III.5.0.10), and repeat this process for any dual 1-morphisms that arise, along with all possible composites.

Let us check only that \( u \) is right dualizable. (To finish the proof, one must also verify that \( u \) is left dualizable, then verify that \( \epsilon \) is 2-dualizable; then that the “opposite orientation” cobordisms \( u^\vee \) and \( \epsilon^\vee \) are also 2-dualizable. In the framed case, “new” cobordisms with distinct framings will appear—these all have underlying manifolds given by cohorseshoes and their composites, but with various framings. One must check for all these as well.)

Let \( u^\vee \) be \( u \) with the opposite orientation; we read this as a horseshoe-shaped cobordism from \( *^+ \coprod *^- \) to \( \emptyset \). We claim this is the right adjoint to \( u \). For example, that the composite \( u^\vee \circ u \) is a circle; and hence a unit for this adjoint pair must be some cobordism from the (identity cobordism of the) empty manifold to the circle. Here is an obvious candidate:

![Diagram](image)

which one might call the “cap.” The other composite \( u \circ u^\vee \) is a disjoint union of a horseshoe and a cohorseshoe, giving a morphism from \( *^+ \coprod *^- \) to itself. A counit
for this adjoint pair must be a cobordism from this disjoint union to the identity cobordism of $*^+ \coprod *^-$. We have seen such a cobordism before, given by the saddle:

![Diagram of a cobordism]

The picture below proves that the cap and the saddle indeed exhibit $u^\vee$ as the right adjoint to $u$. (Composition is read from top to bottom.)

**Image III.6.0.4.** The right dualizability of $u$:

![Diagram of right dualizability]

**Example III.6.0.5.** Let us now sketch the dualizability of an arbitrary $k$-morphism in $\text{Cob}_n$. To that end, fix two $(k-1)$-morphisms $V$ and $W$, and let $W \leftarrow V : X$ be a cobordism from $V$ to $W$. We draw $X$ below as a jagged line, to indicate some orientation on $X$. When the jagged curve is flipped upside down, we see $X$ with the opposite orientation.

We claim the following picture illustrates the fact that $X^\vee$ ($X$ with the opposite orientation, read as a cobordism from $W$ to $V$) is the right dual to $X$.

![Diagram of dualizability example]
Let us read the image starting in the middle, with the leftmost copy of $X$. We have “rotated” $X$ in a semicircle. The equator of this semicircle is hence labeled by the composition $X \circ X^\vee$. Then we have extended the rotation of $X$ by a constant copy of $W$ in the bottom-left region, so that we obtain a cobordism from $X \circ X^\vee$ to $\text{id}_W$. This explains the lower-left region of the image. (There may of course appear to be singularities in the figure; but we can guarantee that the apparent singularities of this process takes place in a collared region of the cobordism $X$, hence this cobordism has no singularities.)

The bottom-right region is simply a copy of $X$ that is extended constantly in the vertical direction of the page. Thus, the bottom region as a whole depicts a cobordism from $X \circ X^\vee \circ X$ to $\text{id}_W \coprod X$.

Likewise, the top-right region is obtained by rotating $X^\vee$ and extended by a constant copy of $V$, and so forth.

We then note that this colon-shaped picture may be isotoped to obtain a single, straight copy of the cobordism $X$; this is the proof of one adjunction relation.

Remark III.6.0.7. Informally, the image above is obtained by drawing the “direct product” of Zorro’s Lemma with $X$, while filling in the empty regions of Zorro’s Lemma with the obvious constant copies of $V$ and $W$.

Remark III.6.0.8. The reader may benefit from projecting the adjunction picture from Image III.6.0.4 to $\mathbb{R}^2$ to visualize the colon-shaped figure in Image III.6.0.6. For example, the saddle is indeed obtained by extending a rotation of $u \circ u^\vee$.  

III.7. Factorization homology as a topological field theory

Now let us explain how factorization homology provides an interesting example of TFTs. For this, we must describe the target \((\infty, n)\)-category. We will not give a rigorous construction of it. (The interested reader may consult [42].)

**Definition III.7.0.1 (Informal.)** Let \(D\) denote a symmetric monoidal \(\infty\)-category. Then the Morita category of \(E_n\)-algebras in \(D\) is an \((\infty, n)\)-category

\[ \text{Morita}_{E_n}(D) \]

with the following description:

- An object is an \(E_n\)-algebra in \(D\).
- Given two objects \(A\) and \(A'\), a morphism between them is an \((A, A')\) bimodule \(M\), equipped with a compatible \(E_{n-1}\)-algebra structure.
- More generally, given two \(k\)-morphisms \(M\) and \(M'\), which are in particular \(E_{n-k}\)-algebras, a \((k + 1)\)-morphism between them is an \((M, M')\) bimodule equipped with a compatible \(E_{n-k-1}\)-algebra structure.
- Finally, given two \((n-1)\)-morphisms \(M\) and \(M'\) (which are, in particular, \(E_1\)-algebras), the space of \(n\)-morphisms between them is the space of pointed \((M, M')\) bimodules. Which is to say, vertices are given by bimodules, edges are given by equivalences of bimodules, and so forth. We demand that these bimodules are all pointed, meaning that they receive maps from the monoidal unit of \(D\).

**Remark III.7.0.2.** Let us leave a cryptic remark. Our definition of \(E_n\)-algebras allowed us to imagine algebras as living in an open disk, say \(\mathbb{R}^n\). In the above, one should think of a 1-morphism \(M\) as living on \(\mathbb{R}^n\) equipped with the data of an embedded hyperplane. \(M\) itself "lives on" the hyperplane, while the two half-spaces on either side of the hyperplane admit embeddings from disks labeled by \(A\) (for disks embedded in one half-space), or by \(A'\) (for disks embedded in the other half-space).

Likewise, a \(k\)-morphism \(M\) is an object that one imagines living on a flag of linear subspaces of \(\mathbb{R}^n\), where the flag consists of planes of codimension 1 through \(k\) inside \(\mathbb{R}^n\). The \(k\)-morphism itself lives on the codimension \(k\) plane, and this plane cuts the codimension \((k-1)\) plane into two half-spaces. Each of these half-spaces is labeled by the \((k-1)\)-morphisms that act in a way rendering \(M\) a bimodule over the algebras living on these half-spaces.

For ways to make this intuition precise, we refer the reader to [41] and [7, 8].

**Remark III.7.0.3.** There is another gadget one might call the Morita category of \(E_n\)-algebras, and this gadget is an \((\infty, n + 1)\)-category. The \(n\)-morphisms are given by bimodules of \(E_1\)-algebras, and given two such bimodules \(M\) and \(M'\) over the same pair of algebras, we define \(\text{hom}(M, M')\) to be the space of intertwiners (i.e., of bimodule maps). In particular, there are \((n+1)\)-morphisms which are not invertible,
as not all bimodule maps are equivalences. (Put another way, \( \text{Morita}_\mathbb{E}_n \) is obtained by throwing out all non-invertible \((n+1)\)-morphisms.)

It is a far more subtle business to determine which objects are fully dualizable in this \((\infty, n+1)\)-category. But as we will see below, the cobordism hypothesis will tell us that every \( \mathbb{E}_n \)-algebra is fully dualizable in the \((\infty, n)\)-category \( \text{Morita}_\mathbb{E}_n \) from Definition III.7.0.1.

Fix a positive integer \( n \). We begin to see how factorization homology might define a TFT valued in \( \text{Morita}_\mathbb{E}_n \) as follows.

- When given a point \( * \), one imagines a fattened neighborhood \( * \times \mathbb{R}^n \), and we equip this with a framing. Assign to it an \( \mathbb{E}_n \)-algebra \( A \).
- Given a 1-morphism \( X \) in \( \text{Cob}^f_\mathbb{E}_n \), we think of it as a framed fattened neighborhood, \( X \times \mathbb{R}^{n-1} \). Because \( X \) is a cobordism, it may be collared; but by the usual arguments we saw when discussing factorization homology, this collaring guarantees that the factorization homology of \( X \) is a bimodule over its collaring boundary manifolds. So this defines a 1-morphism in \( \text{Morita}_\mathbb{E}_n \).
- More generally, given a \( k \)-morphism \( X \), it is collared by two \((k-1)\)-dimensional manifolds. Again thinking of everything as thickened, we view \( X \times \mathbb{R}^{n-k} \) as a framed \( n \)-manifold, and take its factorization homology. The collaring endows this with a bimodule structure receiving actions from the factorization homology of the two \((k-1)\)-dimensional boundary manifolds.
- Finally, given a framed \( n \)-manifold \( X \), we obtain a bimodule which is pointed by virtue of receiving a map from the empty manifold.

The above intuition can be made precise:

**Theorem III.7.0.4** (Scheimbauer [69]). Fix an \( \mathbb{E}_n \)-algebra \( A \) in a symmetric monoidal \((\infty, 1)\)-category \( D^{\otimes} \). Factorization homology defines an \( n \)-dimensional framed topological field theory

\[
(Cob^f_\mathbb{E}_n)^{\Pi} \to \text{Morita}_\mathbb{E}_n(D^{\otimes})
\]

\[
X^k \mapsto \int_{X^k \times \mathbb{R}^{n-k}} A.
\]

**Remark III.7.0.5.** One can also prove that every \( \mathbb{E}_n \)-algebra is fully dualizable in \( \text{Morita}_\mathbb{E}_n \) without utilizing the cobordism hypothesis, see [41].

**Remark III.7.0.6.** Indeed, one can define notions of TFT for cobordism categories “with colors,” or with defects, in more physical language. An example may be obtained by marking cobordisms with marked points, or by coloring different markings and different regions of a cobordisms with various labels (called colors). While there is no written account of this defining a topological field theory, the framework of factorization homology for such manifolds was constructed in [8].
III.8. Leftovers and elaborations

III.8.1. The cobordism categories. To at least give some indication of the answer to Question One (III.2.0.20), we give a slightly more rigorous account of how we think of \( \text{Cob}_n \).

**Remark III.8.1.1.** First, let us say that there does exist a good theory of \((\infty, n)\)-categories, thanks to works of Barwick [12], Rezk [67], and many others. The most common model of \((\infty, n)\)-categories is called an \(n\)-fold Segal space.

**Remark III.8.1.2.** It is not trivial to create a cobordism category fitting into the above framework. Let us reference that the idea of how to define such a thing was given in [62], and made rigorous in [24].

Without again getting into the technical details, we give some indication of how one can start to construct an \((\infty, n)\)-category of cobordisms. We first ignore framings and orientations.

The general idea is that we think of the collection of all 0-morphisms (i.e., the collection of all 0-dimensional manifolds) as the collection of all subsets \( W \subset \mathbb{R}^\infty \) for which \( W \) happens to be a compact, smooth 0-manifold. (I.e., happens to be a finite subset.)

Then, the collection of all 1-morphisms is the collection of all subsets \( X \subset \mathbb{R}^\infty \times \mathbb{R} \) which happen to be smooth 1-manifolds. Further, we demand that

1. Given the projection map \( X \to \mathbb{R} \), \( X \) is “constant” outside a compact interval \( I = [t_0, t_1] \subset \mathbb{R} \). This means \( X \) is equal to a product \( W_0 \times (-\infty, t_0] \) and \( W_1 \times [t_1, \infty) \) as a subset of \( \mathbb{R}^\infty \times \mathbb{R} \).
2. Moreover, the map \( X \to \mathbb{R} \) is a submersion at \( t_0 \) and \( t_1 \), which guarantees that \( W_0 \) and \( W_1 \) are smooth submanifolds of \( \mathbb{R}^\infty \). (This also guarantees the existence of collars.)
3. We finally demand that the projection \( X \to \mathbb{R} \) is proper, so that in particular the preimage of \( [t_0, t_1] \) is a compact subset of \( \mathbb{R}^\infty \times [t_0, t_1] \).

Likewise, we think of the collection of \( k \)-morphisms as the collection of all subsets \( Q \subset \mathbb{R}^\infty \times \mathbb{R}^k \) which happens to be a smooth submanifold, and which satisfy analogous properties to the above. Informally, one demands the existence of some \( k \)-dimensional compact cube in \( \mathbb{R}^k \) above which \( Q \) is collared in a standard way that allows us to compose \( Q \) along the obvious faces of the cube. Note also that we demand that, if \( t_i \) labels the \( i \)th coordinate of \( \mathbb{R}^k \), the cube’s \( t_i = \text{constant} \) face is collared by a manifold which is “constant” in all directions \( t_j \) with \( j > i \). (The asymmetry induced by this \( j > i \) condition is what allows us to say that \( Q \) is a \( k \)-morphism.)

By doing this for all \( 0 \leq k \leq n \), we have more or less specified the necessary data. For example, given a \( k \)-morphism \( Q \subset \mathbb{R}^\infty \times \mathbb{R}^k \), the restriction of \( Q \) to the various subsets, not extra structure.
faces of the $k$-dimensional cube in $\mathbb{R}^k$ tells us what the source morphisms and target morphisms of $Q$ are.

**Remark III.8.1.3.** We note that the collection of subsets of $\mathbb{R}^\infty \times \mathbb{R}^k$ has a topology—it makes sense to say when two compact subsets are nearby (or, when two “constant outside a compact subset” are nearby). And the operation of restricting a cobordism to a face of a cube is a continuous operation. This topology is part of the reason we have the fancy symbol “$\infty$” in describing $\mathcal{C}ob_n$ as an $(\infty, n)$-category.

**Remark III.8.1.4.** Let us at least motivate why one might want to consider such a thing. Consider the collection $\mathcal{E}nd_n(\emptyset)$ of $n$-morphisms between the empty cobordisms; i.e., the collection of subsets of $\mathbb{R}^\infty \times \mathbb{R}^n$ which happen to be compact smooth manifolds of dimension $n$. Choosing a particular $n$-dimensional submanifold $X \subset \mathbb{R}^\infty \times \mathbb{R}^n$, we pick out a connected component of $\mathcal{E}nd_n(\emptyset)$.

We claim that this connected component is interesting: It is homotopy equivalent to $B\text{Diff}X$, the classifying space for the space of diffeomorphisms of $X$.

To see this, consider the space of all smooth embeddings $j : X \to \mathbb{R}^\infty \times \mathbb{R}^n$. By standard arguments in smooth topology, this space is contractible. On the other hand, clearly the space of diffeomorphisms of $X$ acts on the space of all $j$ by precomposition. This action is free because each $j$ is an injection. On the other hand, if I quotient the collection of embeddings by the collection of reparametrizations, I obtain the collection of all possible images—i.e., the collection of all subsets of $\mathbb{R}^\infty \times \mathbb{R}^n$ that happen to be diffeomorphic to $X$. This is exactly the connected component of $\mathcal{E}nd_n(\emptyset)$ we picked out.

And, of course, the quotient of a contractible space by a free continuous action of a group $G$ is precisely a model for the space $BG$.

In particular, this illustrates how an $n$-dimensional TFT $Z : \mathcal{C}ob_n \to \mathcal{C}$ assigns to $X$ an object with the action of $\text{Diff}(X)$ (or, of the orientation-preserving diffeomorphisms, for example, if we demand all our manifolds be oriented).

**III.8.2. Framings.** Let us also elaborate on tangential structures. Fix a group homomorphism $G \to \text{GL}_n(\mathbb{R})$.

**Remark III.8.2.1.** The “$n$” here is the same $n$ as in $\mathcal{C}ob_n$.

For a given $k$-dimensional manifold $X$, one has a map $\tau_X : X \to B\text{GL}_k(\mathbb{R})$ classifying the tangent bundle of $X$. (See Section [II.6.1].) If $X$ is a $k$-manifold in $\mathbb{R}^\infty \times \mathbb{R}^k$, we define a $G$-structure on $X$ to be a homotopy-commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\tau_X} & B\text{GL}_k(\mathbb{R}) \\
& & \downarrow \\
& & B\text{GL}_n(\mathbb{R}).
\end{array}
\]

Here, $B\text{GL}_k(\mathbb{R}) \to B\text{GL}_n(\mathbb{R})$ is the map induced by the usual inclusion $\text{GL}_k(\mathbb{R}) \to \text{GL}_n(\mathbb{R})$. 

III.8. LEFTOVERS AND ELABORATIONS

Put a more concrete way, consider the rank $n$ bundle $TX \oplus \mathbb{R}^{n-k}$, where $\mathbb{R}^{n-k}$ is a trivialized bundle on $X$ of rank $n - k$. Then a $G$-structure is a reduction of structure group of this bundle to $G$.

**Warning III.8.2.3.** The $n$ lurking in the background is important. For example, suppose $n > k$. When $G = \ast$, a $G$-structure on a $k$-manifold $X$ is not the same thing as a framing on $X$ in the sense of Section II.6.1. There, we trivialized $TX$ itself. Here, we are trivializing a higher-rank bundle, $TX \oplus \mathbb{R}^{n-k}$. Indeed, many $k$-manifolds that do not admit a framing may admit a $G$-structure for $n$ large enough.

To heed this warning, we define the following:

**Definition III.8.2.4.** Fix $n > k$ and fix a $k$-dimensional manifold $X$. An $n$-dimensional framing of $X$ is a trivialization of the bundle $TX \oplus \mathbb{R}^{n-k}$.

More generally, fix a continuous group homomorphism $G \rightarrow GL_n(\mathbb{R})$. Then data as in (III.8.2.2), is called an $n$-dimensional $G$-structure.

**Remark III.8.2.5.** Fix $G = \ast$ to be the trivial group—i.e., the case of a framing. It is often said that an $n$-dimensional framing could be thought of as a framing on a small trivial neighborhood $X \times \mathbb{R}^{n-k}$ of $X$. If you prefer to think this way, let me caution that it is most natural to think of an $n$-dimensional framing as a framing on $X \times \mathbb{R}^{n-k}$ such that the trivialization factors through the collapse map $X \times \mathbb{R}^{n-k} \rightarrow X$. Informally, such a framing is “constant” along the $\mathbb{R}^{n-k}$ directions, and that is the kind of framing that an $n$-dimensional framing captures.

So let us retroactively set:

**Definition III.8.2.6.** Fix $n > k$. A $k$-morphism of $\text{Cob}^{\text{fr}}_{n}$ is a $k$-dimensional submanifold $X \subset \mathbb{R}^{\infty} \times \mathbb{R}^{k}$ such that $X$ is equipped with an $n$-dimensional framing.

Note that the boundary $\partial X \subset X$ comes equipped with a canonical isomorphism $T(\partial X) \oplus \mathbb{R} \rightarrow TX|_{\partial X}$, and hence we have an isomorphism

$$T(\partial X) \oplus \mathbb{R}^{n-k+1} \cong TX|_{\partial X} \oplus \mathbb{R}^{n-k}$$

and this allows us to articulate what the sources and targets—as $n$-dimensionally framed manifolds—of a $k$-morphism are.

**Remark III.8.2.7.** This definition has real consequences. For example, we should still prove that the point is fully dualizable in $\text{Cob}^{\text{fr}}_{2}$. Then what is the right adjoint to $u$ as described in Example II.6.0.3?

First, we must specify “which” framed horseshoe $u$ we mean, as $Tu \oplus \mathbb{R}$ admits several inequivalent framings once we have fixed the 2-dimensional framing data on the boundary points.

Further, whatever the right adjoint $r$ is, the composition $r \circ u$ should be exhibited as the boundary of a disk (the cap), but this must be done in a framed way. Because not every 2-dimensional framing of a circle extends to a framing of the tangent bundle of the cap, this puts a restriction on the possible 2-dimensional framing(s) that $r$ can be equipped with.
III.8.3. Structure groups and homotopy fixed points. Finally, we note that we can always alter an \( n \)-dimensional framing on a \( k \)-manifold \( X \) by acting on \( \mathbb{R}^n \) by \( GL_n(\mathbb{R}) \). From this observation, one can in fact exhibit a continuous action of \( GL_n(\mathbb{R}) \) on the category \( \text{Cob}^\text{fr}_n \) itself.

In particular, given a group homomorphism \( G \to GL_n \), one can ask for \( G \)-fixed points of the induced action on \( \text{Cob}^\text{fr}_n \).

**Warning III.8.3.1.** We warn the reader that here, by a \( G \)-fixed point we mean a functor from \( EG \) to \( \text{Cob}^\text{fr}_n \) extending a functor \( G \to \text{Cob}^\text{fr}_n \) (given by picking out an object). That is, exhibiting a \( G \)-fixed point in the homotopical world is not finding an object of \( \text{Cob}^\text{fr}_n \), but exhibiting a way in which we can trivialize the \( G \)-action along a particular orbit of the \( G \)-action on \( \text{Cob}^\text{fr}_n \).

To emphasize this distinction, what we are calling a fixed point is sometimes called a homotopy fixed point.

**Example III.8.3.2.** In what follows, the reader may assume \( G \) is a discrete group, or should otherwise topologize the collection of simplices by the topology of \( G \).

\( EG \) can be described combinatorially; it has \( G \) many vertices, \( G \times G \) many edges, \( G^l \) many \( l \)-simplices, and so forth. Informally, one obtains \( EG \) by constructing the “complete Cayley complex” on the generating set \( G \) itself; unlike a Cayley graph which inserts edges for every \( G \)-translation, we insert triangles and higher simplices each time a sequence of \( G \)-translations commutes. It follows straightforwardly that \( EG \) is contractible and enjoys a free \( G \) action.

If \( G \) acts on a category \( \mathcal{C} \), any object \( X \in \mathcal{C} \) determines a functor \( G \to \mathcal{C} \) by acting on \( X \). An extension of \( G \to \mathcal{C} \) to \( EG \) (i.e., a factorization through \( EG \)) in particular determines equivalences \( X \cong gX \) for any \( g \in G \), and these equivalences are compatible with multiplication in \( G \) by definition of the higher simplices of \( EG \).

Thus, while \( X \) may not equal \( gX \) on the nose, they may abstractly be isomorphic, and the functor from \( EG \) specifies these isomorphisms in a way coherent with the \( G \)-action.

As above, fix a continuous homomorphism \( G \to GL_n(\mathbb{R}) \). Since \( G \) then acts on \( \text{Cob}^\text{fr}_n \), we obtain a \( G \)-action on the space of TFTs; by the Cobordism hypothesis, this induces a \( G \)-action on the space of fully dualizable objects of the target \( \mathcal{C}^\otimes \).

**Example III.8.3.3.** When \( n = 1 \) and \( \mathcal{C}^\otimes = \text{Vect}^\otimes_k \), the \( GL_1(\mathbb{R}) \simeq O(1) \)-action sends a finite dimensional vector space \( V \) to its dual. Note the necessity of restricting to fully dualizable objects (and isomorphisms between them; not all morphisms). Sending a vector space to its dual is usually a contravariant operation.

This action allow us to identity TFTs with any \( G \)-structure:
Theorem III.8.3.4 (Cobordism hypothesis for $G$-structures.) Fix a continuous group homomorphism $G \rightarrow GL_n(\mathbb{R})$ and let $\text{Cob}_n^G$ be the $(\infty, n)$-category of cobordism with $n$-dimensional $G$-structures. Then there is an equivalence of $\infty$-categories

$$\text{Fun}^\otimes(\text{Cob}_n^G \amalg, \mathcal{C} \otimes) \simeq (\mathcal{C}f.d.)^G$$

between the $\infty$-category of fully extended $n$-dimensional TFTs for $G$-structures, and the space of $G$ fixed points of the space $\mathcal{C}f.d.$.

Example III.8.3.5. Let us consider the $(\infty, 2)$-category $\mathcal{C}$ of $\infty$-categories with $\otimes = \times$, and suppose we have a fully dualizable object $D$. (In the $\mathbb{C}$-linear setting, an example of such a thing is the dg-category $D^b\text{Coh}$ of a smooth and proper variety.)

Fixing $D$, we have a map $O(2) \rightarrow \mathcal{C}f.d.$, and in particular a map from $SO(2)$ to $\mathcal{C}f.d.$ A loop based at the identity in $SO(2) \simeq S^1$ is thus the data of an automorphism of $D$; a generator of $\pi_1(S^1) \cong \mathbb{Z}$ gives a distinguished automorphism.

When $\mathcal{C}$ is the $(\infty, 2)$-category of $\mathbb{C}$-linear $\infty$-categories with $\otimes = \otimes_{\mathbb{C}}$, let us fix $D = D^b\text{Coh}(\mathbb{Y})$ for some smooth and proper complex variety $\mathbb{Y}$. Then this automorphism may be identified with the Serre automorphism.

III.9. Exercises

Exercise III.9.0.1. Show that an object of $\text{Vect}_k^\otimes$, thought of as a symmetric monoidal $\infty$-category as usual, is fully dualizable if and only if it is a finite-dimensional $k$-vector space.

Verify the cobordism hypothesis in this case; be careful in proving that any natural transformation of symmetric monoidal functors $Z \rightarrow Z'$ must actually be a natural isomorphism.

Exercise III.9.0.2. In the case $G = O(1)$, consider $\text{Cob}_1^G$. (The cobordism category of unoriented 0- and 1-dimensional manifolds.) Verify the $G$-structured version of the cobordism hypothesis.

In this case, show that a $G$-structured TFT is the same thing as a vector space $V$ equipped with a symmetric non-degenerate pairing. (Hint: Identify $V$ with $W$ using the $O(1)$-fixed point structure.)

Exercise III.9.0.3. Verify Remark [III.5.0.3]

Exercise III.9.0.4. Let $\mathcal{C}^\otimes$ be a symmetric monoidal $(\infty, n)$-category.

Verify that any equivalence $k$-morphism (i.e., any equivalence between two $(k-1)$-morphisms) is $(k+1)$-dualizable.

Verify that the unit of $\otimes$ is fully dualizable.

Exercise III.9.0.5. Let $*^+$ be a point with some fixed framing, considered as an object of $\text{Cob}^\text{fr}_n$. Prove that the space of self-equivalences of $*^+$ in $\text{Cob}^\text{fr}_n$ is homotopy equivalent to $\Omega GL_n(\mathbb{R})$, the based loop space of $GL_n(\mathbb{R})$ at the identity matrix. (Note that this is in turn homotopy equivalent to $\Omega SO_n(\mathbb{R})$.)

You will want to read Section [III.8.2] to think carefully about this exercise.
EXERCISE III.9.0.6. Fix \( n = 2 \) and fix a point \( *^+ \in \text{Cob}^2 \). Draw pictures of the automorphisms of \( *^+ \); note that the space of automorphisms is given by \( \Omega SO(2) \simeq \Omega S^1 \simeq \mathbb{Z} \). Can you quantify/describe the sense in which there is an integers’ worth of invertible cobordisms from a point to itself?

EXERCISE III.9.0.7. In stating the cobordism hypothesis, we have claimed that the collection of TFTs forms a space, which informally means that every symmetric monoidal natural transformation \( Z \to Z' \) is actually a natural equivalence. (See Remark [III.3.0.3].)

Prove this fact.

EXERCISE III.9.0.8. Fix \( n \geq 1 \) and let \( \text{Morita}_n \) denote the Morita \((\infty, n)\)-category (Definition [III.7.0.1]). Let \( \mathbb{Z}/2\mathbb{Z} \cong \pi_0 O(n) \) denote the group of orientations. Convince yourself that the action of \( \mathbb{Z}/2\mathbb{Z} \) from the cobordism hypothesis sends an \( \mathbb{E}_n \)-algebra \( A \) to its opposite algebra, \( A^{\text{op}} \).

EXERCISE III.9.0.9 (Morita invariance for compact, top-dimensional manifolds). Fix a symmetric monoidal \( \infty \)-category \( \mathcal{D} \) admitting sifted colimits, for which \( \otimes \) preserves sifted colimits in each variable.

Recall Scheimbauer’s result (Theorem [III.7.0.4]), that factorization homology defines a framed, fully extended \( n \)-dimensional TFT with values in the Morita \((\infty, n)\)-category \( \text{Morita}_{\mathbb{E}_n}(\mathcal{D}^{\otimes}) \) of \( \mathbb{E}_n \)-algebras (Definition [III.7.0.1]).

Now assume that two \( \mathbb{E}_n \)-algebras \( A \) and \( B \) are Morita equivalent. That is, there exist two \( \mathbb{E}_{n-1} \)-algebras \( M \) and \( N \) that are \((A, B)\) and \((B, A)\)-bimodules, respectively, equipped with equivalences

\[
M \otimes_B N \simeq A, \quad N \otimes_A M \simeq B
\]

in the Morita category, meaning we are supplied with a sequence of further bimodules (for example, between \( M \otimes_B N \) and \( A \)) exhibiting the Morita equivalences above, all the way until we reach an actual equivalence of objects in \( \mathcal{D} \) as bimodules between \( \mathbb{E}_1 \)-algebras.

(For instance, when \( n = 1 \), the equivalences in (III.9.0.10) are actual equivalences of objects in \( \mathcal{D} \), while if \( n = 2 \), the equivalences in (III.9.0.10) are equivalences exhibited by invertible bimodules; in particular, these need not imply \( M \otimes_B N \) and \( A \) are equivalent as objects in \( \mathcal{D} \).)

Using Scheimbauer’s result, show that if two \( \mathbb{E}_n \)-algebras \( A \) and \( B \) are Morita equivalent, then for any compact framed \( n \)-manifold \( X \), we have an equivalence

\[
\int_X A \simeq \int_X B
\]
as objects in \( \mathcal{D} \).
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