COHERENT PRESENTATIONS OF STRUCTURE MONOIDS
AND THE HIGMAN-THOMPSON GROUPS

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Abstract. Structure monoids and groups are algebraic invariants of equational varieties. We show how to construct presentations of these objects from coherent categorifications of equational varieties, generalising several results of Dehornoy. We subsequently realise the higher Thompson groups $F_{n,1}$ and the Higman-Thompson groups $G_{n,1}$ as structure groups. We go on to obtain presentations of these groups via coherent categorifications of the varieties of higher-order associativity and of higher-order associativity and commutativity, respectively. These categorifications generalise Mac Lane's pentagon and hexagon conditions for coherently associative and commutative bifunctors.

1. Introduction

Thompson’s group $F$ is a finitely presented infinite simple group that appears in a number of guises. For us, the most useful description is that of Brown [Bro87], which casts the elements of $F$ as pairs of finite binary trees having the same number of leaves, subject to a certain equivalence relation on the pairs. This description suggests that $F$ may in fact have something to do with associativity, with the elements representing pairs of equivalent terms in some free semigroup. This observation turns out to be fruitful and Dehornoy [Deh05] has exploited it in order to realise $F$ as an algebraic invariant of the variety of semigroups and subsequently to construct a “geometric” presentation of $F$. In a similar manner, Dehornoy realises Thompson’s group $V$ as an algebraic invariant of the variety of commutative semigroups and constructs a geometric presentation of $V$.

The relations in Dehornoy’s presentations consist of two parts. First, there are the so-called geometric relations, which arise purely from the fact that a semigroup is, in the first instance, a magma. The second class of relations arise from the particular equational structure of the variety at hand. In the case of $F$, one additional class of relations are added corresponding to the Stasheff-Mac Lane pentagon [ML63] and in the case of $V$, the presentation further contains a class of relations corresponding to the Mac Lane hexagon, which encodes the essential interaction between associativity and commutativity.

The first goal of this paper is to place Dehornoy’s constructions in a more general context. More precisely, instead of a set with operations and equations, we consider a category with functors and natural isomorphisms. Within this setting, Dehornoy’s geometric relations correspond to functoriality and naturality of the associated categorical structure. The second class of relations correspond to so-called “coherence axioms”, which are a collection of equations making the free categorical structure equivalent to a preorder.

Dehornoy’s relation of $F$ and $V$ to particular equational varieties is a special case of a more general construction [Deh93] associating an inverse monoid to any balanced equational variety. This monoid is termed the “structure monoid” of the variety. In Section 2 we begin by recalling the construction from [Deh93]. We
then go on to describe “categorifications” of equational varieties and show that a coherent categorification of an equational variety gives rise to a presentation of the associated structure monoid. In certain favourable situations, the structure monoid turns out to be a group and we show that the construction of a presentation from a coherent categorification carries over to this setting.

Higman [Hig74] has shown that Thompson’s group $V$ is in fact part of an infinite family of finitely presented groups $G_{n,r}$, which are either simple or have a simple subgroup of index 2. Brown [Bro87] subsequently showed that Thompson’s group $F$ fits into a similar infinite family $F_{n,r}$. We recall the definitions of $F_{n,1}$ and $G_{n,1}$ in Section 6. In Section 3 we show that the groups $F_{n,1}$ arise as the structure groups of $n$-catalan algebras, which encode a notion of associativity for an $n$-ary function symbol. Similarly, we show that the groups $G_{n,1}$ arise as the structure groups of symmetric $n$-catalan algebras, which contain an action of the symmetric group $S_n$ on the variables of an $n$-ary function symbol.

In Section 5 we construct a coherent categorification of $n$-catalan algebras, which we call $n$-catalan categories. The coherence axioms for $n$-catalan categories directly generalise the Stasheff-Mac Lane pentagon axiom, with a new class of axioms appearing when $n \geq 3$. Following from the results of sections 2 and 4, we obtain new presentations for $F_{n,1}$. In Section 5 we construct symmetric $n$-catalan categories and show that these form a coherent categorification of symmetric $n$-catalan algebras, thus obtaining new presentations for $G_{n,1}$. As in the case of $n$-catalan categories, additional classes of coherence axioms are required when $n \geq 3$.

Throughout this paper, we read $f \cdot g$ as “$f$ followed by $g$”.

2. Free categories and structure monoids

We begin this section by recalling Dehornoy’s construction of an inverse monoid associated to a balanced equational theory [Deh93]. Following this, we describe a process for obtaining a categorical version of an equational theory and a method of constructing a monoid from such a categorification. Finally, we link the two constructions together by showing that coherent categorifications give rise to presentations of structure monoids. We base our analysis at the level of theories, rather than of equational varieties. While this is seemingly at odds with Dehornoy’s result [Deh93] that structure monoids are independent of the particular equational presentation of a variety, differing presentations of the same variety lead to distinct categorifications and thence to distinct presentations of the structure monoid.

2.1. Structure monoids associated to equational theories. For a graded set of function symbols $F$ and a set $X$, we denote by $F_F(X)$ the absolutely free term algebra generated by $F$ on $X$. An equational theory is a tuple $(V, F, E)$, where $V$ is a set of variables, $F$ is a graded set of function symbols and $E$ is an equational theory on $F_F(V)$. A map $\varphi : V \to F_F(V)$ is called a substitution and it extends inductively to an endomorphism $F_F(V) \to F_F(V)$. By abuse of notation, we label this latter map by $\varphi$ as well. We use $[V, F_F(V)]$ to denote the set of all substitutions. For a term $s \in F_F(V)$ and a substitution $\varphi \in [V, F_F(V)]$, we use $s^\varphi$ to denote the image of $s$ under $\varphi$. The support of a term $s$ is the set of variables appearing in it. A pair of terms $(s, t)$ is balanced if they have the same support.

Definition 2.1. Given a balanced pair of terms $(s, t)$ in $F_F(V)$, we use $\rho_{s,t}$ to denote the partial function $F_F(V) \to F_F(V)$ with graph

$$\{(s^\varphi, t^\varphi) \mid \varphi \in [V, F_F(V)]\}.$$  

For a balanced pair of terms $(s, t)$, the partial function $\rho_{s,t}$ is functional since the support of $t$ is a subset of the support of $s$. The stronger restriction that the
Definition 2.3 (Orthogonal). Given a graded set \( \mathcal{F} \) and addresses \( \alpha, \beta \in A^*_{\mathcal{F}} \), we say that \( \alpha \) and \( \beta \) are orthogonal and write \( \alpha \perp \beta \) if neither \( \alpha \) nor \( \beta \) is a prefix of the other. Given a term \( t \), and addresses \( \alpha \) and \( \beta \), the subterms \( \text{sub}(t, \alpha) \) and \( \text{sub}(t, \beta) \) are orthogonal if \( \alpha \perp \beta \).

Our current addressing system is sufficient to describe translated copies of the basic operators.

Definition 2.4. Given a graded set of function symbols \( \mathcal{F} \), a variable set \( \mathcal{V} \), a balanced pair of terms \((s, t)\) \( \in \mathbb{F}_\mathcal{F}(\mathcal{V}) \) and an address \( \alpha \in A^*_{\mathcal{F}} \), the \( \alpha \)-translated copy of \( \rho_{s,t} \) is denoted \( \rho^\alpha_{s,t} \) and is the partial map \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \to \mathbb{F}_\mathcal{F}(\mathcal{V}) \) defined as follows:

- A term \( u \in \mathbb{F}_\mathcal{F}(\mathcal{V}) \) is in the domain of \( \rho^\alpha_{s,t} \) if \( \text{sub}(u, \alpha) \) is defined and is in the domain of \( \rho_{s,t} \).

For a graded set \( \mathcal{F} := \coprod_n \mathcal{F}_n \), we set

\[
A^*_\mathcal{F} := \bigcup_n \bigcup_{F \in \mathcal{F}_n} \{ (F, 1), \ldots, (F, n) \}.
\]

The set of addresses associated to \( \mathcal{F} \) is denoted by \( A^*_\mathcal{F} \) and is the free monoid generated by \( A \mathcal{F} \) under concatenation, with the unit being the empty string \( \lambda \). For a term \( t \in \mathbb{F}_\mathcal{F}(\mathcal{V}) \) and an address \( \alpha \in A^*_\mathcal{F} \), we use \( \text{sub}(t, \alpha) \) to denote the subterm of \( t \) at the address \( \alpha \). Note that \( \text{sub}(t, \alpha) \) only exists if the term tree of \( t \) contains the path \( \alpha \) and that \( \text{sub}(t, \lambda) = t \).

Example 2.2. Suppose that \( \mathcal{F} := \{ F, G \} \), where \( F \) is a binary function symbol and \( G \) is a ternary function symbol. Suppose that \( \mathcal{V} \) is a set of variables. Then, the term \( t := F(w, G(x, y, z)) \) is in \( \mathbb{F}_\mathcal{F}(\mathcal{V}) \). The term tree of \( t \) is:

```
     F
    / \  \
   w  G
     / \
    x  y
      / \
     z
```

The term \( t \) has the following subterms:

- \( \text{sub}(t, (F, 1)) = w \)
- \( \text{sub}(t, (F, 2)) = G(x, y, z) \)
- \( \text{sub}(t, (F, 1)(G, 1)) = x \)
- \( \text{sub}(t, (F, 1)(G, 2)) = y \)
- \( \text{sub}(t, (F, 1)(G, 3)) = z \)
For \( u \in \text{dom}(\rho^\alpha_{s,t}) \), the image \( \rho^\alpha_{s,t}(u) \) is defined by
\[
\text{sub}(\rho^\alpha_{s,t}(u), \alpha) = \rho_{s,t}(\text{sub}(u, \alpha))
\]
and \( \text{sub}(\rho^\alpha_{s,t}(u), \beta) = \text{sub}(u, \beta) \) for every address \( \beta \) orthogonal to \( \alpha \).

Note that \( \rho^\alpha_{s,t} = \rho_{s,t} \).

We are finally in a position to introduce the structure monoid generated by an equational theory.

**Definition 2.5** (Structure Monoid). Given an equational theory \( T := (V,F,E) \), the structure monoid of \( T \), denoted \( \text{Struct}(T) \), is the monoid of partial endomorphisms of \( F_X(V) \) generated by the following maps under composition:
\[
\{ \rho^\alpha_{s,t} \mid (s,t) \text{ or } (t,s) \in E \text{ and } \alpha \in A^*_F \}
\]

The structure monoid of an equational theory is readily seen to completely capture the equational theory.

**Lemma 2.6** (Dehornoy [Deh93]). Let \( T := (V,F,E) \) be a balanced equational theory and let \( t, t' \in F_X(V) \). Then \( t =_T t' \) if and only if there is some \( \rho \in \text{Struct}(T) \) such that \( \rho(t) = t' \).

Given an equational theory \( T = (V,F,E) \) and maps \( \rho_{s_1,t_1}, \rho_{s_2,t_2} \in \text{Struct}(T) \), the composition \( \rho_{s_1,t_1} \cdot \rho_{s_2,t_2} \) may be empty. It is nonempty precisely when there exist substitutions \( \varphi, \psi \in [V,F_P(V)] \) such that \( t^*_1 = s^*_2 \). In this case, we say that the pair \( (s_1, s_2) \) is *unifiable* and that \( (\varphi, \psi) \) is a *unifier* of the pair. In the case where \( (s_1, s_2) \) is not unifiable, the composition \( \rho_{s_1,t_1} \cdot \rho_{s_2,t_2} \) results in the empty operator, which we denote by \( \varepsilon \). Note that, for any operator \( \rho \in \text{Struct}(T) \), we have \( \rho \cdot \varepsilon = \varepsilon \cdot \rho = \varepsilon \). The existence of the empty operator makes freely computing with inverses in \( \text{Struct}(T) \) impossible.

**Definition 2.7** (Composable). An equational theory \( (V,F,E) \) is *composable* if any pair of terms in \( \bigcup_{(s,t) \in E} \{s,t\} \) are unifiable.

\( \text{Struct}(T) \) always forms an inverse monoid [Deh06] and contains the empty operator precisely when \( T \) is not composable. One way in which to transform \( \text{Struct}(G) \) into a group is by passing to the universal group of \( \text{Struct}(T) \), which we denote by \( \text{Struct}_G(T) \), by collapsing all idempotents to \( 1 \). In the case where \( T \) is composable, the idempotent elements of \( \text{Struct}(T) \) are precisely those operators that act as the identity on their domain. A particular class of composable theories is provided by a certain class of linear theories. Recall that an equation \( s = t \) is linear if it is balanced and each variable appears precisely once in both \( s \) and \( t \). An equational theory is linear if each of its defining equations is linear.

**Lemma 2.8** (Dehornoy [Deh06]). A linear equational theory containing precisely one function symbol is composable.

It follows from the above lemma that each linear equational theory containing precisely one function symbol gives rise to a structure group.

**Example 2.9.** The equational theories for semigroups, \( S \), and for commutative semigroups, \( C \), are both linear. Since these theories involve a single binary operator, Lemma 2.8 implies that they are composable. In this case we have that \( \text{Struct}_G(S) \) is Thompson’s group \( F \) and \( \text{Struct}_G(C) \) is Thompson’s group \( V \) [Deh05].
2.2. Categorification of equational theories. An equational theory \( T = (V, F, \mathcal{E}) \) defines an algebraic structure on a set. In passing to the structure monoid \( \text{Struct}(T) \) we abstract away from the underlying set and focus instead on the partial operations generated by the congruence \([\mathcal{E}]\). This suggests passing to a structure where the operations generated by \( \mathcal{E} \) are given first-class status. In order to achieve this goal, we pass from a set with algebraic structure to a category with algebraic structure.

**Definition 2.10 (Precategorification).** Given a balanced equational theory \( T = (V, F, \mathcal{E}) \), the precategorification of \( T \) is the structure \( \hat{T} := (\hat{V}, \hat{F}, \hat{\mathcal{E}}) \), which consists of:

1. The discrete category \( \hat{V} \) generated by \( V \).
2. For every function symbol \( F \in F_n \), a functor \( \hat{F} : \hat{V}^n \to \hat{V} \) in \( \hat{F} \). The functor \( \hat{f} \) for a term \( t \in \mathcal{F}(\hat{V}) \) is defined inductively.
3. For every equation \( (s, t) \in \mathcal{E} \), a natural isomorphism \( \hat{\rho}_{s,t} : \hat{s} \to \hat{t} \). We use the notation \( \hat{\rho}_{s,t} := (\hat{\rho}_{s,t})^{-1} \).

A precategorification of an equational theory should be thought of as being akin to a graded set of function symbols, rather than to an equational theory. The reason for this is that, although it contains all of the information of an equational theory, it does not contain enough information to ensure that it faithfully represents the equational structure. A precategorification generates a category whose objects are the absolutely free term algebra and whose morphisms are “iterated substitutions” of the basic maps. Before making this statement precise, we introduce some notation. Given a term \( \hat{s} \in \mathcal{F}(\hat{V}) \), its support, \( \text{supp}(\hat{s}) \), is the set of objects appearing in it. For a morphism \( \hat{\rho} : \hat{s} \to \hat{t} \), if \( \text{supp}(\hat{s}) = \text{supp}(\hat{t}) = \{x_1, \ldots, x_n\} \), then we often write \( \hat{\rho}(x_1, \ldots, x_n) : \hat{s}(x_1, \ldots, x_n) \to \hat{t}(x_1, \ldots, x_n) \) to specifically refer to the objects in the support. Note that a particular \( x_i \) may appear more than once in \( \hat{s} \) or \( \hat{t} \).

**Definition 2.11.** Given a precategorification \( \hat{T} := (\hat{V}, \hat{F}, \hat{\mathcal{E}}) \) of a balanced equational theory \( (V, F, \mathcal{E}) \), we denote by \( \mathcal{F}(\hat{V}) \) the category whose objects are \( \mathcal{F}(\hat{V}) = (\text{Ob}(\hat{V})) \) and whose morphisms are constructed inductively as follows:

1. \( \mathcal{F}(\hat{V}) \) contains the identity \( 1 : \hat{V} \to \hat{V} \).
2. \( \mathcal{F}(\hat{V}) \) contains \( \hat{\mathcal{E}} \).
3. If \( \{ \hat{\rho}_i : \hat{s}_i \to \hat{t}_i \}_{i=1}^n \in \mathcal{F}(\hat{V}) \) and \( \hat{F} \in \hat{F}_n \), then
   \[ \hat{F}(\hat{\rho}_1, \ldots, \hat{\rho}_n) : \hat{F}(\hat{s}_1, \ldots, \hat{s}_n) \to \hat{F}(\hat{t}_1, \ldots, \hat{t}_n) \]
   is in \( \mathcal{F}(\hat{V}) \).
4. If \( \hat{\rho} : \hat{s} \to \hat{t} \) is in \( \mathcal{F}(\hat{V}) \) and \( \hat{\phi} \in \hat{V} \), then \( \hat{\phi} \cdot \hat{\rho} : \hat{s} \to \hat{t} \) is a substitution, then
   \[ \hat{\rho}(x_1^\hat{\phi}, \ldots, x_n^\hat{\phi}) : \hat{s}(x_1, \ldots, x_n) \to \hat{t}(x_1^\hat{\phi}, \ldots, x_n^\hat{\phi}) \]
   is in \( \mathcal{F}(\hat{V}) \).
5. If \( \hat{\rho}_1 : \hat{s} \to \hat{u} \) and \( \hat{\rho}_2 : \hat{u} \to \hat{t} \) are in \( \mathcal{F}(\hat{V}) \), then \( \hat{\rho}_1 \cdot \hat{\rho}_2 : \hat{s} \to \hat{t} \) is in \( \mathcal{F}(\hat{V}) \).

It is straightforward to show that for \( \hat{T} := (\hat{V}, \hat{F}, \hat{\mathcal{E}}) \), the category \( \mathcal{F}(\hat{V}) \) is the free category on \( \hat{V} \) containing all of the functors in \( \hat{F} \) and all of the natural transformations in \( \hat{\mathcal{E}} \), so that \( \mathcal{F}(\hat{V}) \) forms our analogue of the absolutely free term algebra.

Categorical structures very rarely arise as precategorifications of equational theories. Far more common is to require in the definition of a structure that certain
diagrams commute. In particular, given a collection of diagrams \( \mathcal{D} \), each of which consists of a parallel pair of morphisms \( \hat{\rho}_1, \hat{\rho}_2 : \hat{s} \to \hat{t} \) in \( F_\mathcal{T}(\hat{\mathcal{V}}) \), we may build a congruence \([\mathcal{D}]\) on the set of morphisms of \( F_\mathcal{T}(\hat{\mathcal{V}}) \). Factoring out by this congruence yields the free \( \mathcal{T} \)-structure on \( \hat{\mathcal{V}} \) satisfying the property that all of the diagrams in \( \mathcal{D} \) commute. For particular categorical structures, it is possible to obtain a set of such diagrams, whose commutativity implies the commutativity of any diagram in \( F_{(\mathcal{T},\mathcal{D})}(\hat{\mathcal{V}}) := F_\mathcal{T}(\hat{\mathcal{V}})/[\mathcal{D}] \). This phenomenon is termed “coherence” and it is equivalent to requiring that \( F_{(\mathcal{T},\mathcal{D})}(\hat{\mathcal{V}}) \) is a preorder. Coherence was first investigated by Mac Lane [ML63] in relation to monoidal and symmetric monoidal categories and has subsequently formed a major part of categorical universal algebra, with an abstract categorical treatment having been provided by Kelly [Kel72].

**Definition 2.12 (Categorification).** A categorification of a balanced equational theory \( \mathcal{T} := (\mathcal{V}, \mathcal{F}, \mathcal{E}) \) is a pair \( (\hat{\mathcal{T}}, \mathcal{D}) \), where \( \hat{\mathcal{T}} \) is a precategorification of \( \mathcal{T} \) and \( \mathcal{D} \) is a collection of parallel pairs of morphisms in \( F_\mathcal{T}(\hat{\mathcal{V}}) \). We say that \( (\hat{\mathcal{T}}, \mathcal{D}) \) is coherent if \( F_{(\hat{\mathcal{T}},\mathcal{D})}(\hat{\mathcal{V}}) \) is a preorder.

**Example 2.13.** The precategorification of the theory of semigroups consists of a binary functor \( \otimes \), together with a natural isomorphism:

\[
\alpha(x,y,z) : x \otimes (y \otimes z) \to (x \otimes y) \otimes z.
\]

Mac Lane [ML63] showed that, in order to obtain a coherent categorification of the theory of semigroups, we need only the “pentagon axiom”, which states that the following diagram commutes:

The precategorification of the theory of commutative semigroups has an additional natural isomorphism \( \tau \) with the following components:

\[
\tau(x,y) : x \otimes y \to y \otimes x.
\]

Mac Lane [ML63] went on to show that a coherent categorification of the theory of commutative semigroups is obtained via the pentagon axiom, together with the axiom that \( \tau \cdot \tau = 1 \) and the “hexagon axiom”, which states that the following diagram commutes:

**2.3. Monoid presentations from coherent categorifications.** Dehornoy’s utilisation of the pentagon and hexagon coherence axioms in order to obtain presentations of Thompson’s groups [Deh05] is indicative of a more general relationship between structure monoids and coherent categorifications of equational theories.
The first step on the road to formalising this relationship is to construct a monoid presentation out of a categorification of an equational theory. The initial difficulty is to construct a set of generators for the monoid corresponding to the generators of the structure monoid.

**Definition 2.14** (Singular morphisms). Given a balanced equational theory $\mathcal{T} = (\mathcal{V}, \mathcal{F}, \mathcal{E})$ and a precategorification $\hat{\mathcal{T}} = (\hat{\mathcal{V}}, \hat{\mathcal{F}}, \hat{\mathcal{E}})$, the set of singular morphisms of $\hat{\mathcal{F}}(\hat{\mathcal{V}})$ is denoted $\text{Sing}(\hat{\mathcal{T}})$ and is generated as follows:

1. Every natural isomorphism in $\hat{\mathcal{T}}$ is singular.
2. If $\hat{\rho}$ is singular and $\hat{\phi}$ is a substitution, then $\hat{\rho}\hat{\phi}$ is singular.
3. If $\hat{\rho}$ is singular, $\hat{F} \in \hat{\mathcal{F}}_n$ and $1 \leq i \leq n$, then

$$\hat{F}(1, \ldots, 1, \hat{\rho}, 1, \ldots, 1),$$

is singular.

In essence, the set of singular morphisms are those that contain precisely one instance of a generating natural isomorphism. Their suitability to act as generators is highlighted by the following lemma.

**Lemma 2.15.** Let $\hat{\mathcal{T}} := (\hat{\mathcal{V}}, \hat{\mathcal{F}}, \hat{\mathcal{E}})$ be a precategorification of a balanced equational theory $(\mathcal{V}, \mathcal{F}, \mathcal{E})$. Every morphism in $\hat{\mathcal{F}}(\hat{\mathcal{V}})$ is a composite of finitely many singular morphisms.

**Proof.** The only potential problem is a morphism of the form $\hat{F}(\hat{\rho}_1, \ldots, \hat{\rho}_n)$. However, by functoriality of $\hat{F}$, we have

$$\hat{F}(\hat{\rho}_1, \ldots, \hat{\rho}_n) = \hat{F}(\hat{\rho}_1, 1, \ldots, 1) \cdot \hat{F}(1, \hat{\rho}_2, 1, \ldots, 1) \cdot \ldots \cdot \hat{F}(1, \ldots, 1, \hat{\rho}_n).$$

$\square$

In order to make the relationship between the monoid we construct from a categorification and the structure monoid more perspicuous, we introduce an addressing system for singular morphisms.

**Definition 2.16** (Type/Address). The type, $\text{Type}(\hat{\rho})$ of a singular morphism $\hat{\rho}$ of a precategorification $(\hat{\mathcal{V}}, \hat{\mathcal{F}}, \hat{\mathcal{E}})$ is the generating natural isomorphism that appears as in instance in it. The address, $\text{Address}(\hat{\rho})$ is the word of $A^*_{\hat{\mathcal{E}}}$ constructed as follows:

$$A(\hat{\rho}) = \begin{cases} (\hat{F}, i)A(\hat{\sigma}) & \text{if } \hat{\rho} = \hat{F}(1, \ldots, 1, \hat{\sigma}, 1, \ldots, 1) \\ \lambda & \text{otherwise} \end{cases}$$

Given a categorification $(\hat{\mathcal{T}}, \hat{\mathcal{D}})$, we can now construct a monoid whose generators are the singular morphisms of $\hat{\mathcal{T}}$ and whose relations are generated by functoriality, naturality and the diagrams in $\hat{\mathcal{D}}$.

**Definition 2.17.** Let $\mathcal{T} := (\mathcal{V}, \mathcal{F}, \mathcal{E})$ be a balanced equational theory and let $(\hat{\mathcal{T}}, \hat{\mathcal{D}})$ be a categorification of $\mathcal{T}$. The monoid $\mathbb{P}(\hat{\mathcal{T}}, \hat{\mathcal{D}})$ is the monoid generated by

$$\{\text{Type}(\hat{\rho}) A(\hat{\rho}) \mid \hat{\rho} \in \text{Sing}(\hat{\mathcal{T}})\} \cup \{\hat{\rho}_{s,t}^\alpha \mid (s, t) \text{ or } (t, s) \text{ in } \hat{\mathcal{E}} \text{ and } \hat{\alpha} \in A^*_{\hat{\mathcal{E}}}\}$$

if $\mathcal{T}$ is composable and by

$$\{\text{Type}(\hat{\rho}) A(\hat{\rho}) \mid \hat{\rho} \in \text{Sing}(\hat{\mathcal{T}})\} \cup \{\hat{\rho}_{s,t}^\alpha \mid (s, t) \text{ or } (t, s) \text{ in } \hat{\mathcal{E}} \text{ and } \hat{\alpha} \in A^*_{\hat{\mathcal{E}}}\} \cup \{\hat{\varepsilon}\}$$

otherwise, subject to the following relations.
• **Inverse:**
\[ \hat{\rho}_{s,t} \cdot \hat{\rho}_{s,s} = \hat{\rho}_{s,t} \]
\[ \hat{\rho}_{s,s} \cdot \hat{\rho}_{s,t} = \hat{\rho}_{s,t} \]
\[ \hat{\rho}_{s,t} \cdot \hat{\rho}_{t,s} = \hat{\rho}_{s,t} \]

• **Composition:** If \( t_1 \) and \( s_2 \) are not unifiable then
\[ \hat{\rho}_{s_1,t_1} \cdot \hat{\rho}_{s_2,t_2} = \hat{\epsilon} \]

• **Empty operator:**
\[ \hat{\rho}_{s,t} \cdot \hat{\epsilon} = \hat{\epsilon} \]
\[ \hat{\epsilon} \cdot \hat{\rho}_{s,t} = \hat{\epsilon} \]

• **Functoriality:** For \( \alpha \vdash \beta \):
\[ \hat{\rho}_{s,t} \cdot \hat{\rho}_{u,v} = \hat{\rho}_{u,v} \cdot \hat{\rho}_{s,t} \]

• **Naturality:** Suppose that \( \hat{\rho}_{s,t} \) is a generator and that some variable \( x \) appears at addresses \( \hat{\beta}_1, \ldots, \hat{\beta}_p \) \( s \) and at addresses \( \hat{\gamma}_1, \ldots, \hat{\gamma}_q \) \( t \). Then, for all addresses \( \hat{\alpha}, \hat{\delta} \) and each generator \( \hat{\rho}_{u,v} \):
\[ \hat{\rho}_{s,t} \cdot \hat{\rho}_{u,v} \cdot \hat{\beta}_1 \cdots \hat{\beta}_p = \hat{\rho}_{u,v} \cdot \hat{\alpha} \cdot \hat{\beta}_1 \cdots \hat{\beta}_p \cdot \hat{\rho}_{s,t} \]

• **Coherence:** For \( (\sigma_1, \ldots, \sigma_p, \tau_1, \ldots, \tau_q) \in D \), where each \( \sigma_i \) and \( \tau_j \) is singular, set:
\[ T(\sigma_1)^{A(\sigma_1)} \cdots T(\sigma_p)^{A(\sigma_p)} = T(\tau_1)^{A(\tau_1)} \cdots \cdot T(\tau_q)^{A(\tau_q)} \]

The relations for functoriality and naturality in \( P(\hat{T}, D) \) are adapted from [Deh06]. The functoriality relation is precisely the requirement that each operator \( \hat{T} \in \hat{\mathcal{F}} \) is a functor. The naturality condition is, in turn, precisely the requirement that each \( \hat{T} \in \hat{\mathcal{F}} \) is a natural transformation. The rather involved addressing system in the naturality condition is due to the fact that the same variable may appear multiple times in different positions on either side of an equation. For naturality, one needs to apply a map to each of these instances of the variable simultaneously. We now set about relating \( P(\hat{T}, D) \) to \( \text{Struct}(T) \).

**Lemma 2.18.** Let \( T \) be a balanced equational theory and let \( (\hat{T}, D) \) be a categorification of \( T \). Then \( P(\hat{T}, D) \) is an inverse monoid.

**Proof.** For nonempty \( \hat{\rho} := \hat{\rho}_{i_1, j_1} \cdots \hat{\rho}_{k, j_k} \), set \( \hat{\rho}^{-1} := \hat{\rho}_{k, j_k} \cdots \hat{\rho}_{i_1, j_1} \). Then it follows from the Inverse relations that
\[ \hat{\rho} \cdot \hat{\rho}^{-1} \cdot \hat{\rho} = \hat{\rho} \]
\[ \hat{\rho}^{-1} \cdot \hat{\rho} \cdot \hat{\rho}^{-1} = \hat{\rho}^{-1} \]
Since we also have that \( \hat{\epsilon} \cdot \hat{\epsilon} \cdot \hat{\epsilon} = \hat{\epsilon} \), it follows that \( P(\hat{T}, D) \) forms an inverse monoid. \( \Box \)

**Theorem 2.19.** Let \( T \) be a balanced equational theory and let \( (\hat{T}, D) \) be a categorification of \( T \). The following map is an epimorphism of inverse monoids and it is an isomorphism if and only if \( (\hat{T}, D) \) is coherent:
\[ P(\hat{T}, D) \overset{\Theta}{\longrightarrow} \text{Struct}(T) \]
\[ \hat{\rho}_{i_1, j_1} \cdots \hat{\rho}_{k, j_k} \mapsto \rho_{i_1, j_1} \cdots \rho_{k, j_k} \]
COHERENT PRESENTATIONS OF STRUCTURE MONOIDS AND THE HIGMAN-THOMPSON GROUPS

Proof. By construction, \( \Theta \) is a homomorphism of inverse monoids. For surjectivity, we need only show that every generator \( \rho_{s,t}^{\alpha} \in \text{Struct}(T) \) corresponds to some singular morphism \( S(\rho_{s,t}^{\alpha}) \in \hat{T} \). This singular morphism can be constructed recursively as follows:

\[
S(\rho_{s,t}^{\alpha}) = \begin{cases} 
\hat{F}(1, \ldots, 1, S(\rho_{s,t}^{(i)}), 1, \ldots, 1) & \text{if } \alpha = (\hat{F}, i) \beta \\
\rho_{s,t} & \text{if } \alpha = \lambda
\end{cases}
\]

It remains to show that \( \Theta \) is faithful if and only if \( (\hat{T}, D) \) is coherent.

Suppose that \( \Theta \) is faithful and let \( \hat{\rho}_1, \hat{\rho}_2 \) be a parallel pair of morphisms in \( \hat{T} \). Then \( \Theta(\hat{\rho}_1) = \Theta(\hat{\rho}_2) \), since \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) have the same source and target. Since \( \Theta \) is faithful, it follows that \( \hat{\rho}_1 = \hat{\rho}_2 \).

Conversely, suppose that \( (\hat{T}, D) \) is coherent and that \( \Theta(\hat{\rho}_1) = \Theta(\hat{\rho}_2) \). Then, \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) have the same source and target. Since \( (\hat{T}, D) \) is coherent, it follows that \( \hat{\rho}_1 = \hat{\rho}_2 \).

As in the case of structure monoids, when the theory \( T \) is balanced and composable, we may construct a group \( P_G(\hat{T}, D) \) from a categorification \( (\hat{T}, D) \).

**Definition 2.20.** Let \( T \) be a balanced composable equational theory and let \( (\hat{T}, D) \) be a categorification of \( T \). The group \( P_G(\hat{T}, D) \) is generated by

\[
\{ T(\hat{\rho})^{A(\hat{\rho})} \mid \hat{\rho} \in \text{Sing}(T) \},
\]

subject to the functoriality, naturality and coherence relations from Definition 2.17 together with the following relation:

\[
(\hat{\rho}_{s,t}^{\alpha})^{-1} = \hat{\rho}_{t,s}^{\alpha}.
\]

Following the same line of reasoning as in the proof of Theorem 2.19, we obtain the following relationship between \( P_G(\hat{T}, D) \) and \( \text{Struct}_G(T) \).

**Theorem 2.21.** Let \( T \) be a balanced, composable equational theory and let \( (\hat{T}, D) \) be a categorification of \( T \). The following map is an epimorphism of groups and it is an isomorphism if and only if \( (\hat{T}, D) \) is coherent:

\[
\begin{array}{ccc}
P_G(\hat{T}, D) & \xrightarrow{\Theta} & \text{Struct}_G(T) \\
\hat{\rho}_{s_1,t_1}^{\alpha_1} \cdots \hat{\rho}_{s_k,t_k}^{\alpha_k} & \mapsto & \rho_{s_1,t_1}^{\alpha_1} \cdots \rho_{s_k,t_k}^{\alpha_k}
\end{array}
\]

\( \square \)

**Example 2.22.** In Example 2.18, we obtained a coherent categorification of the theory of semigroups, \( S \), consisting of Mac Lane’s pentagon axiom. It follows from Theorem 2.21 that we can construct a presentation for \( \text{Struct}_G(S) \). We saw in Example 2.14 that \( \text{Struct}_G(S) \) is isomorphic to Thompson’s group \( F \) and we thereby obtain a presentation for \( F \). Similarly, we obtain a presentation of Thompson’s group \( V \) using the pentagon and hexagon coherence axioms from Example 2.13. The resulting presentations are the same as those constructed by Dehornoy [Deh05].

In the following section, we describe generalisations of Thompson’s groups \( F \) and \( V \) due to Higman [Hig74] and Brown [Bro87].

3. THE GROUPS \( F_{n,1} \) AND \( G_{n,1} \)

In the previous section, we have seen that Thompson’s groups \( F \) and \( V \) arise as structure groups of certain balanced equational theories and we have subsequently obtained presentations for these groups via coherent presentations of their associated categorical theories. In this section, we introduce generalisations of these
groups due to Brown [Bro87] and Higman [Hig74], which we call $F_{n,1}$ and $G_{n,1}$, respectively. In the following sections, we shall see how the aforementioned process of constructing presentations for $F$ and $V$ generalises to this broader class of groups.

There are several paths to defining the groups $F_{n,1}$ and $G_{n,1}$, all of which relate to the fact that each of these groups arises as a subgroup of the automorphism group of a Cantor set. Of the myriad of definitions available, we choose to follow the description of Brown [Bro87], which utilises certain equivalence classes of pairs of finite rooted trees.

**Definition 3.1 (Tree).** The set of $n$-ary trees is defined inductively as follows:

- The graph consisting solely of a single vertex is an $n$-ary tree.
- If $T_1, \ldots, T_n$ are $n$-ary trees then the following is also an $n$-ary tree:

  \[
  \begin{array}{c}
  \bullet \\
  T_1 \quad T_2 \quad \cdots \quad T_n
  \end{array}
  \]

  The root of an $n$-ary tree is the unique vertex of valence 0 or $n - 1$. The leaves of a rooted tree $T$ are the vertices of valence 0 or 1 and we denote this set by $\ell(T)$.

**Definition 3.2 (Expansion).** A simple expansion of an $n$-ary tree $T$ is the tree obtained by replacing a leaf $v$ of $T$ with the following:

  \[
  \begin{array}{c}
  v \\
  \alpha_1(v) \quad \alpha_2(v) \quad \cdots \quad \alpha_n(v)
  \end{array}
  \]

  An expansion of an $n$-ary tree is a tree obtained by making finitely many sucessive simple expansions.

Given two trees $T_1$ and $T_2$ having a common expansion $S$, we say that $S$ is a minimal common expansion if any other expansion $S'$ of $T_1$ and $T_2$ is an expansion of $S$.

**Lemma 3.3 (Higman [Hig74]).** Any two finite $n$-ary trees have a minimal common expansion. $\square$

The underlying sets of the groups $F_{n,1}$ and $G_{n,1}$ consist of certain formal expressions called tree diagrams.

**Definition 3.4 (Tree diagram).** An $n$-ary tree diagram is a triple $(T_1, T_2, \sigma)$, where $T_1$ and $T_2$ are $n$-ary trees having the same number of leaves and $\sigma$ is a bijection $\ell(T_1) \to \ell(T_2)$.

As in the case of trees, we may talk about expansions of tree diagrams.

**Definition 3.5.** A simple expansion of an $n$-ary tree diagram $(T_1, T_2, \sigma)$ is an $n$-ary tree diagram $(T'_1, T'_2, \sigma')$ obtained by the following procedure:

- $T'_1$ is a simple expansion of $T_1$ along the leaf $l$.
- $T'_2$ is the simple expansion of $T_2$ along the leaf $\sigma(l)$.
- $\sigma'$ is the bijection $\ell(T'_1) \to \ell(T'_2)$ defined by setting $\sigma'(k) = \sigma(k)$ for $k \in \ell(T_1) \setminus \{l\}$ and $\sigma'(\alpha_i(l)) = \alpha_i(\sigma(l))$.

An expansion of an $n$-ary tree diagram $(T_1, T_2, \sigma)$ is any $n$-ary tree diagram obtained by making finitely many sucessive simple expansions of $(T_1, T_2, \sigma)$.
Let $\sim$ be the equivalence relation on the set of $n$-ary tree diagrams obtained by setting $(T_1, T_2, \sigma) \sim (T'_1, T'_2, \sigma')$ whenever $(T_1, T_2, \sigma)$ and $(T'_1, T'_2, \sigma')$ possess a common expansion. Let $[(T_1, T_2, \sigma)]$ denote the equivalence class of $(T_1, T_2, \sigma)$ modulo $\sim$. We call $[(T_1, T_2, \sigma)]$ an $n$-ary tree symbol.

**Definition 3.7.** For $n \geq 2$, the order-preserving tree lexicographically, which is equivalent to ordering the leaves left-to-right when drawn on a page. We say that an $n$-ary tree symbol is order-preserving if $\sigma$ preserves this ordering.

It follows from the definitions that any $n$-ary tree is an expansion of the tree consisting solely of a single vertex. Thus, the leaves of an $n$-ary tree may be seen as a subset of the free monoid on $\{1, \ldots, n\}$. Therefore, we may order the leaves of the tree lexicographically, which is equivalent to ordering the leaves left-to-right when drawn on a page. We say that an $n$-ary tree symbol is order-preserving if $\sigma$ is an isomorphism of ordered sets; that is, if $\sigma$ preserves this ordering.

**Definition 3.7.** For $n \geq 2$, we set $F_{n,1}$ to be the subgroup of $G_{n,1}$ consisting of the order-preserving $n$-ary tree symbols.

The groups $F_{n,1}$ and $G_{n,1}$ generalise Thompson’s original groups $F$ and $V$, since we have $F_{2,1} \cong F$ and $G_{2,1} \cong V$. They also share several of the interesting properties of $F$ and $V$ as surveyed in [Sco92]. In the following section, we shall realise $F_{n,1}$ as the structure group of higher-order associativity and $G_{n,1}$ as the structure group of higher order associativity and commutativity.

4. **$F_{n,1}$ and $G_{n,1}$ as Structure Groups**

Our goal in this section is to realise $F_{n,1}$ and $G_{n,1}$ as structure groups. Since both of these groups are built using maps between $n$-ary trees, we take our set of function symbols to be $\mathcal{F} := \{\otimes\}$, where $\otimes$ is an $n$-ary function symbol. For a set of variables $\mathcal{V}$, there is an obvious bijection between $\mathbb{F}_\mathcal{V}^{\otimes}(\mathcal{V})$ and the set of $n$-ary trees whose leaves are labelled by members of $\mathcal{V}$. We denote the absolutely free term algebra generated by $\{\otimes\}$ on the set $\mathcal{V}$ by $\mathbb{F}_\otimes(\mathcal{V})$ and we denote the free monoid generated by $\mathcal{V}$ by $\mathcal{V}^\star$.

Our basic strategy is to first realise $F_{n,1}$ as a structure group by constructing an equational theory $\mathcal{E}$ such that $[\mathcal{E}]$ equates any two terms $t_1, t_2 \in \mathbb{F}_\mathcal{V}(\mathcal{V})$ that contain precisely the same variables in the same order and such that no variable appears more than once in either $t_1$ or $t_2$. In the binary case, there is an obvious candidate for $\mathcal{E}$: associativity. So, $\mathcal{E}$ needs to be an analogue of associativity for $n > 2$. Once we have this realisation of $F_{n,1}$ we need only add the ability to arbitrarily permute variables in order to obtain a realisation of $G_{n,1}$ as a structure group.

4.1. **Catalan Algebras and $F_{n,1}$**. Associativity of a binary function symbol is sufficient to establish that any two bracketings of the same string are equal. The way in which one establishes this fact is to show that any bracketing of a string is equal to the left most bracketing. So, for an $n$-ary function symbol to be associative, we need equations which imply that any bracketing of a term is equivalent to the left most one. In order to simplify notation, for integers $i \leq j$, we use the symbol $x^i_j$ to denote the list $x_i, x_{i+1}, \ldots, x_j$. If $i > j$, then $x^i_j$ is the empty list.
Definition 4.1 (n-Catalan algebras). For $n \geq 2$, the theory of $n$-Catalan algebras consists of an $n$-ary function symbol $\odot$ together with the following equations, where $0 < i < n$:
\[
\odot(x_1^i, \odot(x_{i+1}^{i+n}, x_{i+n+1}^{2n-1})) = \odot(x_i^{i-1}, \odot(x_{i+n-1}^{i+n-1}, x_{i+n}^{2n-1})
\]
We denote the theory of $n$-Catalan algebras by $C_n$.

The reason for the name of $n$-catalan algebras is that the set of all terms having $k$ occurrences of the symbol $\odot$ and containing precisely one variable is in bijective correspondence with the set of $n$-ary trees having $k$ internal nodes, which has cardinality equal to the generalised Catalan number \[
\frac{1}{(n-1)k+1} \binom{n}{k}
\]
A rather opaque equational theory of $n$-Catalan algebras is rendered somewhat more understandable by viewing the induced equations on the term trees which, for $n = 3$, yields the following:

\[
\begin{align*}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & x_5 & \cdot \\
x_3 & x_4 & x_5 \\
x_1 & x_2 & x_3
\end{array}
\end{align*}
\]
\[
\begin{align*}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & x_5 & \cdot \\
x_2 & x_3 & x_4 \\
x_1 & x_2 & x_3
\end{array}
\end{align*}
\]
\[
\begin{align*}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & x_5 & \cdot \\
x_1 & x_2 & x_3 \\
x_4 & x_5 & \cdot
\end{array}
\end{align*}
\]

Definition 4.2 (Underlying list). Let $t \in F_{\odot}(V)$. The underlying list of $t$ is the word of $V^*$ defined inductively by
\[
U(t) = \begin{cases} 
U(t_1) \cdots U(t_n) & \text{if } t = \odot(t_1, \ldots, t_n) \\
t & \text{otherwise}
\end{cases}
\]

Definition 4.3 (Left-most bracketing). Let $t \in F_{\odot}(V)$. If $U(t) = t_1 \cdots t_{n+k(n-1)}$, then the left-most bracketing of $t$ is defined recursively by
\[
\lmb(t_1^{n+k(n-1)}) = \lmb(\odot(t_1^n), t_{n+1}^{n+k(n-1)}).
\]

We wish to establish that any term $F_{\odot}(V)$ is equal, in $F_{C_n}(V)$, to its left most bracketing. To this end, we define the rank and the length of a term, which will be useful again for a related problem in Section 5.

Definition 4.4. Let $t \in F_{\odot}(V)$. Define the length of $t$ to be:
\[
L(t) = \begin{cases} 
\sum_{i=1}^{n} L(t_i) & \text{if } t = \odot(t_1^n) \\
1 & \text{otherwise}.
\end{cases}
\]

Define the rank, $R(t)$, of $t$ inductively by setting $R(t) = 0$ if $t \in V$ and
\[
R(\odot(t_1^n)) = \sum_{i=1}^{n} R(t_i) + \sum_{i=2}^{n} (i - 1)L(t_i) - \frac{n(n - 1)}{2}.
\]

Note that $R(t) = 0$ precisely when $t = \lmb(t)$.

We may now proceed to show that any term is equivalent to its left-most bracketing.

Lemma 4.5. For any $t \in F_{\odot}(V)$, we have $t =_{C_n} \lmb(t)$

Proof. Let $t \in F_{\odot}(V)$. Let $R(t)$ and $L(t)$ be as in Definition 4.4. We proceed by double induction on $R(t)$ and $L(t)$ to show that $t =_{C_n} \lmb(t)$. If $L(t) = 0$ then the statement is trivial. We also have that $R(t) = 0$ if and only if $t = \lmb(t)$.
Suppose that $L(t) > 1$ and $R(t) > 0$, so that $t = \otimes(t^n_i)$. Let $i$ be the greatest integer with the property that $t_i \notin \mathcal{V}$. If $i = 1$, then $t = \text{Imb}(t)$ by induction on $L(t)$. If $i > 1$, then $t_i = \otimes(u^n_i)$ and set

$$t' = \otimes((t_1^{i-2}, \otimes(t_{i-1}^{n-1}, u^n_i, t_{i+1}^{n})).$$

A single application of one of the equations in $C_n$ establishes that $t = \text{C}_n t'$. Since $R(t) - R(t') = \sum_{i=1}^{n-1} L(u_i)$, we have $R(t') < R(t)$ and the statement follows by induction on $R(t)$.

In order to manipulate elements of $\text{Struct}(C_n)$ effectively, we introduce the notion of a seed.

**Definition 4.6 (Seed).** Let $\mathcal{F}$ be a graded set of function symbols on some set $\mathcal{V}$ and let $\rho$ be a partial function $\mathcal{F}_\mathcal{F}(\mathcal{V}) \rightarrow \mathcal{F}_\mathcal{F}(\mathcal{V})$. A seed for $\rho$ is a pair of terms $s, t \in \mathcal{F}_\mathcal{F}(\mathcal{V})$ such that the graph of $\rho$ is equal to $\{(s^\rho, t^\rho) \mid \varphi \in [\mathcal{V}, \mathcal{F}_\mathcal{F}(\mathcal{V})]\}$.

In particularly nice cases, we can construct seeds for any operator in a structure monoid.

**Lemma 4.7 (Dehornoy [Deh00]).** Let $\mathcal{T}$ be a balanced equational theory that contains precisely one function symbol. Then, each operator $\rho \in \text{Struct}(\mathcal{T})$ admits a seed.

It follows from Lemma 4.6 that $C_n$ is composable and we may, therefore, form the group $\text{Struct}_G(C_n)$. In order to facilitate the passage from members of $\text{Struct}_G(C_n)$, to members of $F_{n,1}$, we introduce the tree generated by a term.

**Definition 4.8.** For a term $t \in \mathcal{F}_\otimes(\mathcal{V})$, let $T(t)$ denote the $n$-ary tree obtained via the following construction:

- If $t = \otimes(t_1, \ldots, t_n)$, then $T(t)$ is equal to:
  $$\begin{array}{ccc}
  & & \\
  & T(t_1) & T(t_2) \\
  T(t) & & T(t_n)
  \end{array}$$

- Otherwise, $T(t) = \cdot$

**Theorem 4.9.** $\text{Struct}_G(C_n) \cong F_{n,1}$.

**Proof.** We denote the seed of $\rho \in \text{Struct}_G(C_n)$, which exists by Lemma 4.7, by $(s_\rho, t_\rho)$. We claim that the following map is an isomorphism:

$$\begin{array}{ccc}
  \text{Struct}_G(C_n) & \rightarrow & F_{n,1} \\
  \rho & \mapsto & [(T(s_\rho), T(t_\rho), \text{id})]
  \end{array}$$

It is routine to see that $\Theta$ is a homomorphism. Suppose that $\rho, \rho' \in \text{Struct}_G(C_n)$ and that $\Theta(\rho) = \Theta(\rho')$. It follows that $\rho$ and $\rho'$ have the same seed, so $\rho = \rho'$ and $\Theta$ is faithful.

By Lemma 2.6 in order to establish that $\Theta$ is surjective, we need only show that $t_1 = \text{C}_n t_2$ whenever $t_1, t_2 \in \mathcal{F}_\otimes(\mathcal{V})$ and $U(t_1) = U(t_2)$. By Lemma 4.6 we have $t_1 = \text{Imb}(t_1) = \text{C}_n \text{Imb}(t_2) = \text{C}_n t_2$, so $\Theta$ is surjective and, hence, an isomorphism.
4.2. Symmetric Catalan Algebras and $G_{n,1}$. We saw in Section 3 that the leaves of a tree may be ordered by the lexicographic ordering on their addresses. An $n$-ary tree symbol $[(T_1, T_2, \sigma)]$ may thereby be viewed as a pair of tree diagrams, together with a permutation of the leaves of $T_1$. Thus, in order to obtain an equational theory whose structure group is $G_{n,1}$ we need to add the ability to arbitrary permute variables to Catalan algebras. Recalling that the symmetric group is generated by transpositions of adjacent elements, we are led to the following definition.

**Definition 4.10 (Symmetric $n$-Catalan Algebras).** The theory of symmetric $n$-catalan algebras extends that of $n$-catalan algebras with the following equations, where $1 \leq i < n$:

$$\otimes(x_1^{i-1}, x_i, x_{i+1}, x_{i+2}) = \otimes(x_1^{i-1}, x_{i+1}, x_i, x_{i+2}).$$

We denote the theory of symmetric $n$-catalan algebras by $SC_n$.

Symmetric $n$-catalan algebras essentially add an action of the symmetric group on the indices of $\otimes$. In general, this is sufficient to induce an action of a symmetric group on the variables of any term in $\mathbb{F}_{\otimes}(\mathcal{V})$. In the binary case, we recover the definition of commutative semigroups.

**Theorem 4.11.** $\text{Struct}_{G}(SC_n) \cong G_{n,1}$.

**Proof.** For $\rho \in \text{Struct}_{G}(SC_n)$, let $(s_\rho, t_\rho)$ represent its seed, which exists by Lemma 4.7. Since $SC_n$ is linear, $s_\rho$ and $t_\rho$ are linear and $\text{supp}(s_\rho) = \text{supp}(t_\rho)$. Let $\pi(\rho)$ be the permutation of $\text{supp}(s_\rho)$ induced by the permutation $U(s_\rho) \rightarrow U(t_\rho)$. Consider the following map:

$$\text{Struct}_{G}(C_n) \xrightarrow{\Theta} G_{n,1} \quad \rho \mapsto [(T(s_\rho), T(t_\rho), \pi(\rho))]$$

A similar argument to the proof of Theorem 4.9 establishes that $\Theta$ is an isomorphism.

We now know that $F_{n,1}$ and $G_{n,1}$ are the structure groups of catalan algebras and of symmetric catalan algebras, respectively. We also know that if we can construct coherent categorifications of these algebras, then we can apply Theorem 2.21 to obtain presentations of these groups. In the following section, we set about the task of constructing a coherent categorification of catalan algebras.

5. Catalán categories and $F_{n,1}$

In order to obtain a presentation for $\text{Struct}_{G}(C_n)$ and, hence, for $F_{n,1}$ along the lines of that provided by Dehornoy for $F$ [Deh05], we need to obtain a coherent categorification of $C_n$. The immediate problem is discerning a set of diagrams whose commutativity imply the commutativity of all diagrams in $\mathbb{F}_{\hat{\mathcal{C}}_n}(\hat{\mathcal{V}})$. As we shall see in this section, the following definition suffices for this purpose. While the coherence axioms that we have chosen may seem slightly cryptic, the reason for their choice will become apparent in the proof that the resulting categorification is coherent. We shall make frequent use of the following useful shorthand: For $1 \leq i \leq n$ and a morphism $\rho : t_i \rightarrow t'_i$, we set

$$\otimes^i(\rho) = \otimes(1_{t_i}, \ldots, 1_{t_{i-1}}, \rho, 1_{t_{i+1}}, \ldots, 1_{t_n}).$$

**Definition 5.1.** A discrete $n$-catalan category is the categorification of $C_n$ that consists of:

- A discrete category $\hat{\mathcal{V}}$.
- A functor $\otimes : \hat{\mathcal{V}}^n \rightarrow \hat{\mathcal{V}}$. 

For 1 ≤ i < n, a natural isomorphism α_i with the following components:

\[ \alpha_i(x_1^{2i-1}) : \otimes(x_1^i, \otimes(x_{i+1}^{2i-1}), x_{i+n+1}^{2n-1}) \to \otimes(x_1^{i-1}, \otimes(x_i^{2i-1}), x_{i+n+1}^{2n-1}) \]

**Pentagon axiom:** For 1 ≤ i ≤ n - 1, the following diagram commutes, where \( X = x_1^{i-1} \) and \( Z = z_1^{n-i-1} \):

\[ \begin{array}{c}
\otimes(X, y_1, \otimes(y_2^n, \otimes(y_2^{2n-1})), Z) \\
\alpha_i \\
\otimes(X, \otimes(y_1^n, (y_2^{2n-1}), Z) \\
\alpha_i \\
\otimes(X, \otimes(\otimes(y_1^n, y_2^{2n-1}), y_{n+1}), Z) \\
\otimes(X, \otimes(\otimes(y_2^n, y_{n+1}), y_{n+1}), Z) \\
\otimes'(\alpha_{i-1} \cdots \alpha_1)
\end{array} \]

**Adjacent associativity axiom:** For 1 ≤ i ≤ n - 2, the following diagram commutes, where \( X = x_1^{i-1} \) and \( Z = z_1^{n-i-2} \):

\[ \begin{array}{c}
\otimes(X, y_1, \otimes(y_2^{i+1}, \otimes(y_2^{2i+1})), Z) \\
\alpha_i \\
\otimes(X, \otimes(y_1^n, y_{i+1}, \otimes(y_2^{2n+1})), Z) \\
\alpha_{i+1} \\
\otimes(X, \otimes(\otimes(y_1^n, y_{i+1}), y_{2n+1}), Z) \\
\otimes(X, \otimes(\otimes(y_2^n, y_{i+1}), y_{2n+1}), Z) \\
\otimes'(\alpha_i)
\end{array} \]

We denote the theory of discrete n-catalan categories by \( \mathbb{C}_n \) and the free \( \mathbb{C}_n \) category on \( \hat{\mathcal{V}} \) by \( \mathbb{F}_{\mathbb{C}_n}(\hat{\mathcal{V}}) \).

In the case where \( n = 2 \), the pentagon axiom reduces to Mac Lane’s pentagon axiom for monoidal categories from Example 2.13 and the adjacent associativity axiom is empty, so we recover the usual definition of a coherently associative bifunctor.

**Definition 5.2 (Positive/Negative).** A singular morphism of \( \mathbb{F}_{\mathbb{C}_n}(\hat{\mathcal{V}}) \) that contains an instance of \( \alpha_i \) is called positive and one that contains an instance of \( \alpha_i^{-1} \) is called negative. A morphism in \( \mathbb{F}_{\mathbb{C}_n}(\hat{\mathcal{V}}) \) is called positive if it is an identity or a composite of positive morphisms and negative if it is a composite of negative morphisms.
It follows from the proof of Lemma 4.5 that there is always a positive morphism $t \rightarrow \text{lm}(t)$ in $\mathbb{F}_{C_n}(\hat{V})$. In order to show that $C_n$ is a coherent categorification of $C_n$, we need to show that any diagram built out of the singular and identity morphisms of $C_n$ commutes. As our first step towards this goal, we show that there is a unique positive morphism $t \rightarrow \text{lm}(t)$.

**Lemma 5.3.** Let $t \in \text{Ob}(\mathbb{F}_{C_n}(\hat{V}))$. There is a unique positive morphism $t \rightarrow \text{lm}(t)$ in $\mathbb{F}_{C_n}(\hat{V})$.

**Proof.** Let $t \in \text{Ob}(\mathbb{F}_{C_n}(\hat{V}))$. It follows from the proof of Lemma 4.5 that there is a positive morphism $t \rightarrow \text{lm}(t)$. Suppose that $\varphi, \psi : t \rightarrow \text{lm}(t)$, that $\varphi = \varphi_1 \cdot \varphi_2$ and that $\psi = \psi_1 \cdot \psi_2$. By Lemma 2.15, we may assume that $\varphi_1$ and $\psi_1$ are singular. Let $R(t)$ and $L(t)$ be defined as in Definition 4.4. We proceed by double induction on $R(t)$ and $L(t)$ to show that there exists an object $w$ in $\mathbb{F}_{C_n}(\hat{V})$ making the following diagram commute:

![Diagram](image)

By induction on $R(t)$, we have that the subdiagrams labelled (2) and (3) commute. So, we need only establish the existence and commutativity of the subdiagram labelled (1). If $R(t) = 0$ or $L(t) = 1$, then the statement is trivial, so suppose that $R(t) > 0$ and $L(t) > 1$. If $\varphi_1 = \psi_1$, then take $w = u = v$ and subdiagram (1) commutes trivially.

Suppose that $\varphi_1 \neq \psi_1$. Then, either $\varphi_1 = \otimes^i(\varphi'_1)$ or $\varphi_1 = \alpha_i(t^n_1)$ and there are similar possibilities for $\psi_1$. We proceed by case analysis on the form of $\varphi_1$ and $\psi_1$.

Suppose that $\varphi_1 = \otimes^i(\varphi'_1)$ and $\psi_1 = \otimes^j(\psi'_1)$. If $i = j$, then the whole diagram commutes by induction on $L(t)$. Suppose that $i \neq j$. Then, without loss of generality, $i < j$ and we may take (1) to be the following diagram, which commutes by the functoriality of $\otimes$:

![Diagram](image)

Suppose that $\varphi_1 = \otimes^i(\varphi'_1)$ and $\psi_1 = \alpha_j(t^n_1)$. If $i \neq j$, then without loss of generality $i < j$ and $t_j = \otimes(u^j_1)$. If $i \neq j - 1$, then we may take (1) to be the
following square, which commutes by the naturality of $\alpha_i$:

\[
\begin{array}{ccc}
\otimes(t_1^{i-1}, \otimes(u_1^n, t_{j+1}^n)) & \otimes(t_1^{i-1}, t_i, t_{i+1}, \otimes(u_1^n, t_{j+1}^n)) & \otimes(t_1^{i-2}, \otimes(t_{j-1}, u_1^{n-1}, u_n, t_{j+1}^n)) \\
\varphi_1 & \psi_1 & \varphi_i' \\
\psi_i & \varphi_i & \otimes(t_1^{i-1}, t_i, t_{i+1}, \otimes(t_{j-1}, u_1^{n-1}, u_n, t_{j+1}^n))
\end{array}
\]

If $i = j - 1$, then we may take (1) to be a similar naturality square. Suppose that $i = j$. If $\varphi'_1 = \otimes^k(\varphi''_1)$, then we may take (1) to be a naturality square. If $\varphi'_1 = \alpha_k(u_1^n)$ then we have two cases. If $k \neq (n-1)$, then we may take (1) to be a naturality square. If $k = n - 1$, then we may take (1) to be an instance of the pentagon axiom, which commutes by assumption.

Finally, we are left with the case where $\varphi_1 = \alpha_i(t_1^n)$ and $\psi_1 = \alpha_j(t_1^n)$. If $|i-j| > 1$, then we may take (1) to be a naturality diagram. If $|i-j| = 1$, then we may take (1) to be an instance of the adjacent associativity axiom.

We now know that every object in $\mathbb{F}_{C_n}(\hat{V})$ has a unique positive morphism to its left-most bracketing. With a little work, we can bootstrap this result in order to show that there is a unique morphism - positive, negative or otherwise - between any two arbitrary objects in $\mathbb{F}_{C_n}(\hat{V})$.

**Theorem 5.4.** $C_n$ is a coherent categorification of $C_n$.

**Proof.** Suppose that $\varphi : s \to t$ is a reduction in $\mathbb{F}_{C_n}(\hat{V})$. By Lemma 2.15

\[
\varphi = s \varphi_1 s_1 \varphi_2 s_2 \cdots \varphi_{n-1} s_n \varphi_n t,
\]

where each $\varphi_i$ is singular. By Lemma 5.3 each term $t \in \mathbb{F}_{C_n}(\hat{V})$ has a unique map $N_t : t \to \text{imb}(t)$. We claim that each rectangle in the following diagram commutes:

\[
\begin{array}{cccccc}
s & s_1 & s_2 & s_{n-1} & s_n & t \\
\varphi_1 & \varphi_2 & \varphi_3 & \cdots & \varphi_{n-1} & \varphi_n \\
N_{s_1} & N_{s_2} & N_{s_{n-1}} & N_{s_n} & N_t \\
\text{imb}(s) & \text{imb}(s_1) & \text{imb}(s_2) & \cdots & \text{imb}(s_{n-1}) & \text{imb}(t)
\end{array}
\]

If $\varphi_i$ is positive, then it follows immediately from Lemma 5.3 that $\varphi_i \cdot N_{s_i} = N_{s_{i-1}}$. If $\varphi_i$ is negative, then Lemma 5.3 implies that $\varphi_i^{-1} \cdot N_{s_{i-1}} = N_{s_i}$, which implies that $\varphi_i \cdot N_{s_i} = N_{s_{i+1}}$. Since each rectangle commutes, we have $\varphi \cdot N_t = N_s$, which implies that $\varphi = N_s \cdot N_t^{-1}$. Since $N_s$ and $N_t$ are unique and we did not rely on a particular choice of $\varphi$, we conclude that $C_n$ is coherent. \[\square\]

With Theorem 5.4 in hand, we can obtain a presentation for $F_{n,1}$, which generalises the presentation for $F$ given in [Deh05].

**Corollary 5.5.** $\mathbb{F}_G(C_n) \cong F_{n,1}$

**Proof.** By Theorem 5.4 and Theorem 2.21 we have $\mathbb{F} - G(C_n) \cong \text{Struct}_G(C_n)$. It follows then from Theorem 4.9 that $\mathbb{F}_G(C_n) \cong F_{n,1}$. \[\square\]

In the following section, we shall obtain a coherent categorification of $SC_n$ and, thereby, a presentation of $G_{n,1}$. 

6. Symmetric catalan categories and $G_{n,1}$

Our goal in this section is to construct a coherent categorification of symmetric catalan algebras. The coherence theorem for catalan categories, Theorem 5.4, reduces this problem to ensuring that any two sequences of transpositions of the objects appearing in a term realise the same permutation. In other words, our categorification needs to somehow encode a presentation of the symmetric group whose generators correspond to transpositions of adjacent variables. Such a presentation is well known, having been constructed by Moore [Moo96]. This presentation has generators $T_1, \ldots, T_{n-1}$ and the following relations:

\[
T_i^2 = 1 \quad \text{for } 1 \leq i \leq n - 1 \\
(T_i T_{i+1})^3 = 1 \quad \text{for } 1 \leq i \leq n - 2 \\
(T_i T_k)^2 = 1 \quad \text{for } 1 \leq i \leq k - 2
\]

With this presentation in mind, we may now construct a reasonable categorification of $SC_n$. Recall our shorthand that for $1 \leq i \leq n$ and a morphism $\rho : t_i \to t_i'$, we have

\[
\otimes^i(\rho) = \otimes (1_{t_1}, \ldots, 1_{t_{i-1}}, \rho, 1_{t_{i+1}}, \ldots, 1_{t_n}).
\]

**Definition 6.1.** For $n \geq 2$, a discrete symmetric $n$-catalan category, is a discrete $n$-catalan category on the category $\hat{\mathcal{V}}$, together with, for $1 \leq i \leq n - 1$, a natural isomorphism $\tau_i$ with components

\[
\tau_i(t_1^n) : \otimes (t_1^{i-1}, t_i, t_{i+1}, t_{i}^{n-2}) \to \otimes (t_1^{i-1}, t_{i+1}, t_i, t_{i+2}^{n}),
\]

satisfying the following axioms:

**Involution axiom:** For $1 \leq i \leq n - 1$, the following diagram commutes:

\[
\begin{array}{c}
\otimes (t_1^n) \\
1 \\
\otimes (t_1^n) \\
\end{array}
\quad \tau_i \\
\tau_i \\
\tau_i \\
\otimes (t_1^{i-1}, t_{i+1}, t_i, t_{i+2}^{n})
\]

**Compatibility axiom:** For $2 \leq i \leq n$ and $1 \leq j \leq n - 2$, the following diagram commutes, where $W = w_1^i$ and $Z = z_1^{n-i}$:

\[
\begin{array}{c}
\otimes (W, \otimes (y_1^n), Z) \\
\alpha_{i-1} \\
\otimes (W, \otimes (x, y_1^{n-1}), y_n, Z) \\
\otimes (W, \otimes (x, y_1^{n-1}), y_n, Z) \\
\otimes (W, \otimes (x, y_1^{n-1}), y_n, Z)
\end{array}
\quad \alpha_{i-1} \\
\otimes^i(\tau_j) \\
\otimes (W, \otimes (y_1^{j-1}, y_j, y_j^{n+2}), Z) \\
\otimes (W, \otimes (y_1^{j-1}, y_j, y_j^{n+2}), Z)
\]
**3-cycle axiom:** For $1 \leq i \leq n - 2$, the following diagram commutes:

![Diagram 3-cycle axiom]

Hexagon axiom: For $1 \leq i \leq n - 1$, the following diagram commutes, where $W = u_1^{i-1}$ and $Z = x_2^{-i-1}$.

![Diagram Hexagon axiom]

We denote the theory of discrete symmetric $n$-catalan categories by $\mathcal{S}C_n$ and the free $\mathcal{S}C_n$-category on $\hat{\mathcal{V}}$ by $\mathbb{F}_{\mathcal{S}C_n}(\hat{\mathcal{V}})$.

The hexagon axiom ensures that we may replace a transposition of the form $\tau_i(t_1^{i-1}, \otimes(u_1^n), t_1^{n-1})$ with a sequence of transpositions involving only the terms $t_1^{n-1}$ and $u_1^n$. One might posit the commutativity of a diagram that serves the same purpose for a morphism of the form $\tau_i(t_1^{i-1}, \otimes(u_1^n), t_1^{n-1})$. Doing so leads to the dual hexagon diagram, which has the following form, where $2 \leq i \leq n$ and
Lemma 6.2. The dual hexagon diagram commutes in $\mathbb{F}_{\mathcal{SC}_n}(\hat{V})$.

Proof. By the involution axiom, we have
\[ \tau_i(t_1^n, \otimes(u^n_i), t_{i+1}^{n-1}) = [\tau_i(t_1^{i-1}, \otimes(u_1^n), t_i^{n-1})]^{-1}. \]
Using the hexagon axiom and ignoring component labels, we have:
\[ \tau_i = \tau_i^{-1} \]
\[ = (\alpha_i^{-1} \cdot \otimes^{i+1}(\tau_{n-1} \cdot \ldots \cdot \tau_2 \cdot \alpha_i \cdot \otimes^i(\tau_1) \cdot \alpha_i^{-1})^{-1} \]
So, we have:
\[ \alpha_i^{-1} \cdot \otimes^{i+1}(\tau_{n-1}) \cdot \alpha_i = \alpha_i \cdot \otimes^i(\tau_1) \cdot \alpha_i^{-1} \cdot \otimes^{i+1}(\tau_{n-1}) \]  
(1)
From the compatibility axiom, we have $\otimes^{i+1}(\tau_j) = \alpha_i \cdot \otimes^i(\tau_{j+1}) \cdot \alpha_i^{-1}$. This implies:
\[ \boxed{\cdot \otimes^{i+1}(\tau_{n-1}) \cdot \alpha_i^{-1} = \alpha_i \cdot \otimes^i(\tau_1 \cdot \ldots \cdot \tau_{n-1}) \cdot \alpha_i^{-1} \}
Therefore, the dual hexagon diagram commutes in $\mathbb{F}_{\mathcal{SC}_n}(\hat{V})$. \(\square\)

In the $n = 2$ case, the axiomatisation of $\mathcal{SC}_2$ reduces to the theory of a coherently associative and commutative bifunctor given in Example 2.13. The main result of this section establishes that $\mathcal{SC}_n$ is a suitable generalisation of this case.

Theorem 6.3. $\mathcal{SC}_n$ is a coherent categorification of $\mathcal{SC}_n$.

Proof. By Theorem 5.4 we may assume that all of the associativity maps are strict equalities. Thus, an object of $\mathbb{F}_{\mathcal{SC}_n}(\hat{V})$ may be represented as $\otimes(t_1^n)$, where each $t_i$ is an object in $\hat{V}$ and $m = n + k(n - 1)$, for some $k \geq 0$. Lemma 6.2 and the hexagon axiom imply that it suffices to consider transpositions of adjacent variables that are objects of $\hat{V}$. So, for a given object $t := \otimes(t_1^n)$, we need only consider the $m - 1$ induced transposition natural isomorphisms
\[ T_{i}(t_1^n) : \otimes(t_1^{i-1}, t_i, t_{i+1}, t_{i+2}^m) \rightarrow \otimes(t_1^{i-1}, t_{i-1}, t_i, t_{i+2}^m). \]
In order to establish coherence, we have to show that every permutation of $t_1^n$ is unique. That is, we have to show that the induced transposition maps satisfy the defining relations for the symmetric group of order $m$. 

\[ W = w_1^{n-2} \text{ and } Z = z_1^{n-1}. \]
The compatibility axiom implies that each \( T_i \) is unique. By the naturality of the maps \( T_i \), we have \( T_i \cdot T_k = T_k \cdot T_i \) for all \( 1 \leq i \leq k - 2 \). The involution axiom implies that \( T_i^2 = 1 \). Thus, it only remains to establish that \( (T_i \cdot T_{i+1})^3 = 1 \). For \( n = 2 \), we may use the proof from Mac Lane [ML63]. Suppose that \( n \geq 3 \). Since the associativity maps are taken to be strict equalities, we may assume that \( t \) has the form \( \otimes (R \otimes (S, t_i, t_{i+1}, t_{i+2}, U), V) \), where \( R, S, U \) and \( V \) are sequences of objects of \( \mathcal{V} \). The result then follows from the 3-cycle axiom. □

With the coherence theorem in hand, we can construct a presentation of \( \text{Struct}_G(\mathcal{S}C_n) \) and, therefore, of \( G_{n,1} \), which generalises the presentation for \( V \) given in [Deh05].

**Corollary 6.4.** \( \mathcal{P}_G(\mathcal{S}C_n) \cong G_{n,1} \)

**Proof.** By Theorem 6.3 and Theorem 2.21 we have

\[
\mathcal{P}_G(\mathcal{S}C_n) \cong \text{Struct}_G(\mathcal{S}C_n).
\]

It follows then from Theorem 1.11 that \( \mathcal{P}_G(\mathcal{C}_n) \cong G_{n,1} \). □

7. Conclusions and Further Work

We have demonstrated, by way of Theorem 2.19 and Theorem 2.21, that there is a close relationship between structure monoids and coherent categorical theories. This relationship is quite powerful, as illustrated by the fact that we were able to exploit it in order to obtain new presentations of \( F_{n,1} \) and \( G_{n,1} \).

While we only dealt with invertible categorical structures, it is straightforward to extend the constructions of Section 2 to structures involving a mix of invertible and non-invertible natural transformations. Within this setting, it is possible to develop an abstract coherence theorem that applies to a large array of structures and, inter alia, yields presentations of a wide variety of structure monoids. A general coherence theorem along the lines of the proof of Theorem 5.4 is developed in [CJ]. A more powerful, though more difficult to apply, general coherence theorem applying mainly to badly behaved non-invertible structure is developed in [Coh].

The presentations that arise from coherence theorems are all infinite, regardless of whether or not the associated algebraic structure is finitely presentable. In particular, this is the case for the presentations of \( F_{n,1} \) and \( G_{n,1} \) that arise from the coherence theorems for catalan categories and symmetric catalan categories. However, these presentations arise from finitely presented categorical structures and so retain some amount of finiteness. Conditions on a finitely presented categorical structure ensuring that it is coherent via finitely many coherence conditions are obtained in [Coh]. The relation between the finite presentability of a coherent categorical structure and the finite presentability of the monoid or group that arises from it remains unclear and much remains to be done in this direction.

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