Orbifolds of $AdS_3$ and fixpoints of the CFT

Klaus Behrndt

Humboldt-University Berlin, Invalidenstrasse 110, 10115 Berlin, Germany

Abstract

The 3-d BTZ black hole represents an orbifold of $AdS_3$ gravity. The UV as well as the IR region of the CFT is governed by a gauged $SL(2, \mathbb{R})$ WZW model. In the UV it corresponds to a light-cone gauging (Liouville model) whereas in the IR it is a space-like gauging (2-d black hole).

1 Introduction and summary

In the AdS/CFT correspondence [1], [2], [3] the radial coordinate of the AdS space sets the energy scale of the world volume field theory. The brane does not reside at infinity, instead the complete AdS space is expected to be dual (in proper limits) to the world volume field theory [4]. In the maximal supersymmetric case the world volume field theory is scale invariant and thus the field theory is conformal for any energy scale. This means in the AdS picture that we could have set a cutoff at any other radius, it would reproduce the same boundary conformal field theory (CFT).

The situation changes however if we break supersymmetry, which can be done by orbifolding either in the spherical part or in the AdS part. Consider e.g. the case of $AdS_3 \times S_3$. As we will discuss below AdS orbifolding introduces a black hole, which obviously break the scale invariance. To be concrete, the orbifold $AdS_3/Z_n$ describes a BTZ black hole [5] (see also [6]) and it corresponds to the near-horizon geometry of a black string with momentum modes, where $n$ is the momentum number. On the other hand an $S_3$ orbifold corresponds to the embedding of a Taub-NUT space, which changes also the AdS part. E.g. the $AdS_3$ part of the $D1 \times D5$ intersection reads $ds^2 = r^2 du dv + (dr/r)^2$ and it becomes $ds^2 = r du dv + (dr/r)^2$ for $D1 \times D5 + $Taub-NUT, i.e. the power of $r$ has

---

1Email: behrndt@physik.hu-berlin.de ; Talks presented at the “32nd International Symposium Ahrens-choop on the Theory of Elementary Particles”, Buckow 1998 and at ”Quantum aspects of gauge theories, supergravity and unification”, Corfu, September 1998. Work is supported by the DFG.
changed. This is obvious due to the different harmonic functions. So, this orbifolding breaks the scale invariance $r \to e^{\lambda D} r$ and $x_i \to e^{-\lambda D} x_i$, where $x_i$ are the world volume coordinates. Of course, in both cases we can restore the scale invariance locally by a proper coordinate transformation, but nevertheless they are globally different. Notice, coordinate transformations in the AdS space corresponds to an operator reparameterization of the boundary field theory.

In this letter we show, that the boundary CFT in the UV as well as in IR can be described by different gauged WZW models. In AdS gravity the asymptotic region (large radius) translates into the UV region of the CFT and the IR corresponds to the near-horizon region of the BTZ black hole. The $AdS_3$ orbifolding fixes the subgroup, that has to be gauged: in the UV (asymptotic AdS region) the CFT is given by an $SL(2,\mathbb{R})/SO(1,1)$ WZW model and in the IR (near horizon region) one gets an $SL(2,\mathbb{R})/U(1)$ WZW model. Since the central charges of both CFT’s differ, there is no exactly marginal deformation connecting both regions, i.e. there is a non-trivial renormalization group flow. At many points we will be very brief and refer to [7] and [8] for more details and more references.

## 2 BTZ black hole as $AdS_3$ orbifolding

A three-dimensional anti-de Sitter space-time is defined as a hyperboloid in a 4-d space with the signature $(-++-)$, i.e.

$$-l^2 = -(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 = -z_1^+ z_1^- + z_2^+ z_2^-$$

where we introduced the coordinates

$$z_1^\pm = l e^{\pm \frac{1}{2}(\theta_R + \theta_L)} \cosh \frac{\lambda}{2}, \quad z_2^\pm = l e^{\pm \frac{1}{2}(\theta_R - \theta_L)} \sinh \frac{\lambda}{2}.$$  

The metric reads

$$ds^2 = -dz_1^+ dz_1^- + dz_2^+ dz_2^- = \frac{l^2}{4} \left( d\lambda^2 + d\theta_R^2 + d\theta_L^2 + 2 \cosh \lambda d\theta_R d\theta_L \right).$$

Defining $\theta_R = (y + t)/l$, $\theta_L = (y - t)/l$ and $\lambda = 2\rho$ the metric can also be written as

$$ds^2 = l^2 d\rho^2 + \cosh^2 \rho dy^2 - \sinh^2 \rho dt^2.$$

Furthermore, the $AdS_3$ space is the $SL(2,\mathbb{R})$ group space and a group element $g$ can be written as

$$g = e^{\theta_L T_1} e^{\lambda T_2} e^{\theta_R T_1} = \left( \begin{array}{cc} \cosh \frac{\theta_R}{2} & \sinh \frac{\theta_R}{2} \\ \sinh \frac{\theta_R}{2} & \cosh \frac{\theta_R}{2} \end{array} \right) \left( \begin{array}{cc} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{array} \right) \left( \begin{array}{cc} \cosh \frac{\theta_L}{2} & \sinh \frac{\theta_L}{2} \\ \sinh \frac{\theta_L}{2} & \cosh \frac{\theta_L}{2} \end{array} \right)$$

with the $SL(2,\mathbb{R})$ algebra

$$[T_a, T_b] = \epsilon_{ab}^c T_c, \quad \text{Tr}(T_a T_b) = \frac{1}{2} \eta_{ab}$$
\( \eta = \text{diag}(-1,1,1), \, e^{012} = 1 \). A representation is
\[
T_0 = \frac{1}{2} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad T_1 = \frac{1}{2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

In terms of the \( SL(2) \) current
\[
J = J^a T_a = \frac{l}{2} g^{-1} d g = \frac{l}{2} [- \sinh \theta_L \, d \lambda + \sinh \lambda \, \cosh \theta_L \, d \theta_R] T_0 + \frac{l}{2} [\cosh \lambda \, d \theta_R + d \theta_L] T_1 + \frac{l}{2} [\cosh \theta_L \, d \lambda - \sinh \lambda \, \sinh \theta_L \, d \theta_R] T_2
\]
the \( AdS_3 \) metric (3) becomes
\[
ds^2 = \text{tr}(J^2) = -(J^0)^2 + (J^1)^2 + (J^3)^2.
\]

Let us now turn to the BTZ black hole [5] which reads
\[
ds^2 = -e^{-2V(r)} \, dt^2 + e^{2V(r)} \, dr^2 + \left( \frac{T}{r} \right)^2 \left( dy - \frac{r-r_+}{r^2} \, dt \right)^2
\]
with
\[
e^{-2V(r)} = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}.
\]

This black hole can also be written in terms of two chiral currents
\[
\frac{l}{2} g^{-1} d g \equiv \frac{l}{2} A = \frac{l}{2} \left( e^{-V} T_0 + \frac{r}{T} (1 - \frac{r-r_+}{r^2}) T_1 \right) dv + \frac{l}{2} e^V (1 + \frac{r-r_+}{r^2}) T_2 \, dr,
\]
\[
\frac{l}{2} \bar{g}^{-1} d \bar{g} \equiv \frac{l}{2} \bar{A} = \frac{l}{2} \left( e^{-V} T_0 - \frac{r}{T} (1 + \frac{r-r_+}{r^2}) T_1 \right) du - \frac{l}{2} e^V (1 - \frac{r-r_+}{r^2}) T_2 dr.
\]

with \( g, \bar{g} \in SL(2, \mathbb{R}) \) and
\[
v/u = l \, \theta_{R/L} = y \pm t.
\]

Combining the chiral currents to a non-chiral, the metric of the BTZ black hole is
\[
ds^2 = \frac{l}{2} \text{tr}(A - \bar{A})^2.
\]

For the BTZ black hole the \( y \) coordinate is periodic, which can be expressed by \( y \simeq y + \frac{4\pi n}{l} \).

This means
\[
\theta_{R/L} \simeq \theta_{R/L} + \frac{4\pi}{n} \quad \text{or} \quad z^\pm_1 \simeq e^{\pm \frac{4\pi}{n}} z^\pm_1
\]
which corresponds to an orbifolding for the \( AdS_3 \) space. Notice, this orbifold breaks all supersymmetries, which coincides with the non-extremality of the BTZ black hole metric (10). In the next section we will see, that in the supersymmetric case \( r_+ = r_- \) one chiral current drops out and we get an orbifolding only in \( \theta_L \)
\[
\theta_L \simeq \theta_L + \frac{4\pi}{n} \quad \text{or} \quad z^\pm_1 \simeq e^{\pm \frac{2\pi}{n}} z^\pm_1, \quad z^\pm_2 \simeq e^{\pm \frac{2\pi}{n}} z^\pm_2.
\]

This orbifold can be seen as lens space for \( AdS_3 \), by appropriate Wick rotation we can obtain the \( S_3 \) lens space [9].
3 Conformal fixpoints

In the previous section we have argued that “adding” a BTZ black hole to the AdS$_3$ space corresponds to an orbifolding. One may regard this black hole as a (small) perturbation of the asymptotic CFT and the corresponding states, which are invariant under this orbifold, can be related to microstates of the BTZ black hole and therewith to 4-d black holes, see e.g. [10], [11]. On the other hand, since the radial coordinate corresponds to the energy scale, this asymptotic CFT describes the UV fixpoint. One goal of this letter is, to investigate the IR fixpoint and the CFT that describes this fixpoint. Obviously, a flow towards the IR corresponds to the radial movement and if there is a further fixpoint we may expect them near the horizon.

To make this more explicit we can employ the fact, that 3-d AdS gravity can be described by a Chern-Simons theory and is exactly solvable [12], [13]. Decomposing the AdS$_3$ diffeomorphism group $SO(2,2) \cong SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$, the 3-dimensional action can be written as

$$S = S_{CS}[A] - S_{CS}[ar{A}]$$

with

$$S_{CS}[A] = \frac{k}{4\pi} \int_{M_3} d^3x \text{Tr} \left( A dA + \frac{2}{3} A^3 \right)$$

where we will adopt the notation where the level $k = l$ (in other notations $k = 2\pi l/\kappa_3$).

The gauge field one-forms are

$$A = (\omega^a + \frac{1}{l} e^a) T_a \in SL(2,\mathbb{R})_R, \quad \bar{A} = (\omega^a - \frac{1}{l} e^a) \bar{T}_a \in SL(2,\mathbb{R})_L.$$  

where $\omega^a \equiv \frac{1}{2} \epsilon^{abc} \omega_{bc}$ are given by the spin-connections $\omega_{bc}$ and $e^a$ are the dreibeine. Due to boundaries the Chern-Simons theory is not invariant under gauge transformations and as a consequence gauge degrees of freedom do not decouple and become dynamical on the boundaries. These are the degrees of freedom of the conformal field theories living at the boundaries.

In the following we will discuss this procedure for the BTZ black hole. The geometry of the manifold is $M_3 = \mathbb{R} \times \Sigma$, where $\mathbb{R}$ corresponds to the time of the covering space of AdS$_3$ and $\Sigma$ represents an “annulus” $r_+ \leq r < \infty$.

Calculating the gauge connections $A = A^a T_a$ and $\bar{A} = \bar{A}^a \bar{T}_a$ for the BTZ solution (10) one finds exactly the fields (12), for details see [8]. In order to extract the CFTs at the boundaries we have to perform two steps: (i) we have to add boundary terms that impose the correct boundary conditions and (ii) we have to mod out the isometry group related to the orbifold, that corresponds to the Killing direction that has been periodically identified in constructing the BTZ black hole.

(i) As dictated by the Chern-Simons solution (12) we will impose as boundary conditions

$$A_u = \bar{A}_v = 0$$

4
and therefore we add as boundary term to the action

$$\delta S = \frac{k}{8\pi} \int_{\partial M} Tr(A_v A_u + \bar{A}_v \bar{A}_u) .$$  \hspace{1cm} (21)$$

Since we have flat gauge connection we insert \( A = g^{-1}dg \) and \( \bar{A} = \bar{g}^{-1}d\bar{g} \) into the action and obtain as result an \( SL(2, \mathbb{R}) \) WZW model [14].

(ii) In a second step we translate the orbifolding (15) into a gauging of an \( SL(2, \mathbb{R}) \) group direction. From the Chern-Simons fields (12) we find

\[
\begin{align*}
A & = \frac{r}{\ell}(T_1 + T_0) \frac{dv}{r} + T_2 \frac{dr}{r} = \frac{r}{\ell} T_+ \frac{dv}{r} + T_2 \frac{dr}{r} \\
\bar{A} & = -\frac{r}{\ell}(T_1 - T_0) \frac{du}{r} - T_2 \frac{dr}{r} = -\frac{r}{\ell} T_- \frac{du}{r} - T_2 \frac{dr}{r} .
\end{align*}
\]

with \( T_\pm = T_1 \pm T_0 \). Therefore, the orbifolding in \( \theta_{L/R} \sim u/v \) corresponds to a gauging of the lightcone group directions \( T_\pm \). On the other hand, in the IR (\( \lambda \to 0 \)) we reach the horizon boundary \( r \to r_+ \) and the gauge fields become

\[
\begin{align*}
A & = \frac{r}{\ell}(r_+ - r_-) T_1 \frac{dv}{r} + T_2 d\lambda \\
\bar{A} & = -\frac{r}{\ell}(r_+ + r_-) T_1 \frac{du}{r} - T_2 d\lambda
\end{align*}
\]

where \( \lambda \) is defined by the radial coordinate for which \( A_r \) and \( \bar{A}_r \) are constant. So we see, that in this case the orbifolding translates into a gauging of the \( T_1 \) group direction.

Both gauged WZW models have been discussed some time ago. The gauging of a lightcone group direction, i.e. an \( \frac{SL(2, \mathbb{R})}{SO(1,1)} \) WZW model corresponds to a Liouville model [15], [16], where the Liouville field corresponds to the radial coordinate [17]; see also [3]. The gauging of a spatial direction yields an \( \frac{SL(2, \mathbb{R})}{U(1)} \) WZW model, which describes a 2-d black hole solution [18], [16]. Both solutions are known to be exact CFTs, but with different central charges. For the Liouville model, describing the CFT in the UV region, the central charge is [16]

$$c_{UV} = 1 + 6(k - 2)Q^2 = \frac{3k}{k - 2} - 2 + 6k$$  \hspace{1cm} (24)$$

where \( Q = \frac{k-1}{k+2} \) is the background charge of the Liouville field (\( k = \frac{r}{4\alpha'} \)). On the other hand the CFT in the IR has the central charge

$$c_{IR} = \frac{3k}{k - 2} - 1 .$$  \hspace{1cm} (25)$$

Due to the difference in the central charges there cannot be an exactly marginal deformation between both CFT’s. So, there is a non-trivial renormalization group flow (see also [19]), which corresponds to the bulk physics of the BTZ black hole. Due to the holographic nature, the complete bulk physics will be fixed by these boundary CFTs. However, let us
stress that at any finite point in space time one can promote the background to an exact CFT. On one hand the gauged WZW model can be made exact by changing the renormalization group scheme (field redefinitions) [20] and on the other hand the BTZ black hole is locally at any point $AdS_3$. The different central charges indicate the non-trivial global structure of the model and it is better thought of as an interpolating solution between two (different) CFTs.

Finally, we will comment on the supersymmetric case where the BTZ black hole becomes extremal, i.e. $r_− = r_+$. In this case the Chern-Simons field $A$ decouples and the CFT is covered by one chiral current ($\bar{A}$) only. In fact, if we first discuss the IR limit ($r → r_+$) we see from (23) that $A_0 = 0$ and due to the boundary condition (20) also $A_u = 0$ and thus $A$ drops completely ($A_v$ was fixed due to the choice of the coordinates). On the other hand in the UV we get the same asymptotic behavior as for the non-extremal case. But by performing the orbifolding in $θ_L$ (gauging of the $T_-$ direction) one truncates the theory already to Liouville\(^2\), i.e. one has to gauge only one light-cone direction. Hence, in the extremal case the Chern-Simons field $A$ decouples from the CFT in the IR as well as in the UV. These two points are boundaries for the Chern-Simons theory and since gauge degrees of freedom can become dynamical only on these boundaries, we can disregard the $A$ field completely. Therefore, as expected the supersymmetric case is governed by only one chiral Chern-Simons field and corresponds only to the orbifold of $θ_L$ in (16).

References

[1] J. Maldacena, “The large n limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* 2 (1998) 231, hep-th/9711200.

[2] E. Witten, “Anti-de sitter space and holography,” *Adv. Theor. Math. Phys.* 2 (1998) 253, hep-th/9802150.

[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett.* B428 (1998) 105, hep-th/9802109.

[4] J. Maldacena, “Talk at Strings98,” www.itp.ucsb.edu/online/strings98/maldacena.

[5] M. Banados, C. Teitelboim, and J. Zanelli, “The black hole in three-dimensional space-time,” *Phys. Rev. Lett.* 69 (1992) 1849–1851, hep-th/9204099; M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, “Geometry of the (2+1) black hole” *Phys. Rev.* D48 (1993) 1506, gr-qc/9302012.

\(^2\)By gauging a light-cone direction one gets rid of two degrees of freedom [16].
[6] S. Aminneborg, I. Bengtsson, D. Brill, S. Holst, and P. Peldan, “Black holes and wormholes in (2+1)-dimensions,” *Class. Quant. Grav.* **15** (1998) 627, gr-qc/9707036.
M. Banados, A. Gomberoff, and C. Martínez, “Anti-de sitter space and black holes,” *Class. Quant. Grav.* **15** (1998) 3575, hep-th/9805087.

[7] K. Behrndt, “AdS gravity and field theories at fixpoints,” hep-th/9809015.

[8] K. Behrndt, I. Brunner, and I. Gaida, “AdS$_3$ gravity and conformal field theories,” hep-th/9806195, “Entropy and conformal field theories of AdS$_3$ models”, *Phys. Lett.* **B432** (1998) 310, hep-th/9804159.

[9] M. J. Duff, H. Lu, and C. N. Pope, “AdS$_3 \times S^3$ (un)twisted and squashed, and an O(2,2,Z) multiplet of dyonic strings,” hep-th/9807173.

[10] J. Maldacena and A. Strominger, “AdS$_3$ black holes and a stringy exclusion principle,” hep-th/9804085.

[11] M. Cvetic and F. Larsen, “Microstates of four-dimensional rotating black holes from near horizon geometry,” hep-th/9805146.

[12] A. Achucarro and P. K. Townsend, “A Chern-Simons action for three-dimensional anti-de sitter supergravity theories,” *Phys. Lett.* **B180** (1986) 89.

[13] E. Witten, “(2+1)-dimensional gravity as an exactly soluble system,” *Nucl. Phys.* **B311** (1988) 46.

[14] S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, “Remarks on the canonical quantization of the Chern-Simons-Witten theory,” *Nucl. Phys.* **B326** (1989) 108.

[15] A. Alekseev and S. Shatashvili, “Path integral quantization of the coadjoint orbits of the Virasoro group and 2-d gravity,” *Nucl. Phys.* **B323** (1989) 719.

[16] R. Dijkgraaf, H. Verlinde, and E. Verlinde, “String propagation in a black hole geometry,” *Nucl. Phys.* **B371** (1992) 269–314.

[17] S. Carlip, *Inducing Liouville theory from topologically massive gravity*, *Nucl. Phys.* **B362** (1991) 111.
O. Coussaert, M. Henneaux and P. van Driel, *The asymptotic dynamics of three-dimensional Einstein gravity with negative cosmological constant*, *Class. Quant. Grav.* **12** (1995 ) 2961, gr-qc/9506019.

[18] E. Witten, “On string theory and black holes,” *Phys. Rev.* **D44** (1991) 314–324.

[19] C. Schmidhuber and A. A. Tseytlin, “On string cosmology and the RG flow in 2-d field theory,” *Nucl. Phys.* **B426** (1994) 187–202, hep-th/9404180.
[20] K. Sfetsos and A. A. Tseytlin, “Antisymmetric tensor coupling and conformal invariance in sigma models corresponding to gauged WZNW theories,” *Phys. Rev. D* **49** (1994) 2933–2956, hep-th/9310159.