ON EQUIVALENCE OF THIRD ORDER LINEAR DIFFERENTIAL OPERATORS ON TWO-DIMENSIONAL MANIFOLDS

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Abstract. We study differential invariants of the third order linear differential operators and use them to find conditions for equivalence of differential operators acting in line bundles on two dimensional manifolds with respect to groups of authomorphisms.

1. Introduction

This paper is a continuation of ([5]) and here we analyze equivalence of the third order linear differential operators, acting in sections of line bundles over an oriented 2-dimensional manifold with respect to the group of automorphisms of the line bundle. The method we applied is very similar to the method used in ([5]) and the novel idea we used is to get global equivalence by using natural coordinates delivering by differential invariants. Also we use Chern ([3],[2],[9]) and Wagner ([1]) connections instead of Levi-Civita connection in ([5]) and show that any regular third order linear differential operators acting in sections of line bundles defines natural affine connections in the bundles. This allows us, by using the quantization, to substitute the operator by its total symbol and then find its field of rational differential invariants.

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2. Differential operators

2.1. Notations. The following notations we use in this paper. We denote by $\tau : TM \to M$ and $\tau^* : T^*M \to M$ the tangent and respectively cotangent bundles for a manifold $M$. By $1 : \mathbb{R} \times M \to M$ we denote the trivial line bundle.

The symmetric and exterior powers of a vector bundle $\pi : E(\pi) \to M$ we’ll denote by $S^k(\pi)$ and $\Lambda^k(\pi)$. The module of smooth sections of bundle $\pi$ we denote by $C^\infty(\pi)$, and for the cases tangent, cotangent and the trivial bundles we’ll use the following notations:

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\[ \Sigma_k(M) = C^\infty(S^k(\tau)) \] is the module of symmetric \( k \)-vectors (or symbols), \( \Sigma^k(M) = C^\infty(S^k(\tau^*)) \) is the module of symmetric \( k \)-forms, \( \Omega_k(M) = C^\infty(\Lambda^k(\tau)) \) is the module of skew-symmetric \( k \)-vectors, \( \Omega^k(M) = C^\infty(\Lambda^k(\tau^*)) \) is the module of exterior \( k \)-forms, and \( C^\infty(1) = C^\infty(M) \).

The bundles of \( k \)-jets of sections of bundle \( \pi \) we denote by \( \pi_k : J^k(\pi) \rightarrow M \) and by \( \pi_{k,l} : J^k(\pi) \rightarrow J^l(\pi) \) we denote the reduction of \( k \)-jets to \( l \)-jets, \( k \geq l \).

There is the following exact sequence of vector bundles

\[ 0 \rightarrow S^k(\tau^*) \otimes \pi \rightarrow J^k(\pi) \xrightarrow{\pi_{k,k-1}} J^{k-1}(\pi) \rightarrow 0, \]

connecting \((k - 1)\) and \( k \)-jets.

Differential operators of order \( k \) acting from a vector bundle \( \alpha \) to another bundle, say \( \beta \), \( \Delta : C^\infty(\alpha) \rightarrow C^\infty(\beta) \), could be lifted to operators \( \hat{\Delta} : C^\infty(\hat{\alpha}) \rightarrow C^\infty(\hat{\beta}) \), where \( \hat{\alpha} \) and \( \hat{\beta} \) are vector bundles over \( J^\infty(\pi) \), induced by the projection \( \pi_{\infty} : J^\infty(\pi) \rightarrow M \), and operator \( \hat{\Delta} \) is defined by the universal property

\[ j_{k+1}(h)^* \circ \hat{\Delta} = \Delta \circ j_l(h)^*, \]

for all sections \( h \in C^\infty(\pi) \) and \( l = 0, 1, \ldots \).

We call operators \( \hat{\Delta} \) as total lift of \( \Delta \). Here we’ll especially use the following two cases. The de Rham operator \( \Delta = d : \Omega^0(M) \rightarrow \Omega^{i+1}(M) \).

Then sections \( C^\infty(\Lambda^i(\tau^*)) = \Omega^i_h(J^\infty(\pi)) \) are horizontal differential \( i \)-forms on \( J^\infty(\pi) \) and \( \hat{d} \) the total differential.

In the other case, when \( \Delta = d_{\nabla} : C^\infty(\alpha) \rightarrow C^\infty(\alpha) \otimes \Omega^1(M) \) is a covariant differential of a connection \( \nabla \) in the bundle \( \alpha \), operator \( \hat{d}_{\nabla} : C^\infty(\hat{\alpha}) \rightarrow C^\infty(\hat{\alpha}) \otimes \Omega^1_h(J^\infty(\pi)) \) is the total covariant differential.

Linear combinations of total lifts of vector fields are called total derivations and linear combinations of compositions of total derivations are total differential operators.

2.2. Symbols. Let \( M \) be an oriented 2-dimensional oriented manifold and let \( \xi : E(\xi) \rightarrow M \) be a line bundle. We denote by \( \text{Diff}_k(\xi) \) the \( C^\infty(M) \)-module of differential operators of order \( \leq k \), acting in the bundle: \( A : C^\infty(\xi) \rightarrow C^\infty(\xi) \), if \( A \in \text{Diff}_k(\xi) \).

By leading or principal symbol \( \sigma_k(A) \) of operator \( A \in \text{Diff}_k(\xi) \) we mean the equivalence class

\[ \sigma_{k,A} = A \bmod \text{Diff}_{k-1}(\xi). \]

It is known that the symbol could be also viewed as a fibre-wise homogeneous polynomial of degree \( k \) on the cotangent bundle \( \tau^* : T^*M \rightarrow M \) and the following sequence

\[ 0 \rightarrow \text{Diff}_{k-1}(\xi) \rightarrow \text{Diff}_k(\xi) \xrightarrow{\sigma} \Sigma_k \rightarrow 0 \]
is exact.

If \((x, y)\) are local coordinates on \(M\) and \(e\) is a local basic section of \(\xi\), then differential operator \(A \in \text{Diff}_3(\xi)\) can be represented in the form

\[
A = a_1 \partial_x^3 + 3a_2 \partial_x^2 \partial_y + 3a_3 \partial_x \partial_y^2 + a_4 \partial_y^3 + b_1 \partial_x^2 + 2b_2 \partial_x \partial_y + b_3 \partial_y^2 + c_1 \partial_x + c_2 \partial_y + a_9,
\]

where all coefficients are smooth functions and the action of operator \(A\) on a section \(s = he\), where \(h\) is a smooth function, equals \(A(h)e\).

The symbol of operator \(A\) is a symmetric 3-vector

\[
\sigma_{3,A} = a_1 \partial_x^3 + 3a_2 \partial_x^2 \cdot \partial_y + 3a_3 \partial_x \cdot \partial_y^2 + a_4 \partial_y^3 \in \Sigma_3(M)
\]

where dots (and degrees) stand for symmetric product of vector fields.

In canonical coordinates \((x, y, p_x, p_y)\) on \(T^*M\) this tensor is a cubic polynomial (=Hamiltonian)

\[
\sigma_{3,A} = a_1 p_x^3 + 3a_2 p_x^2 p_y + 3a_3 p_x p_y^2 + a_4 p_y^3
\]
on \(T^*M\).

We say that the symbol \(\sigma_{3,A}\) is regular (at a point or domain) if \(\sigma_{3,A}\), as a polynomial on \(T^*M\), has distinct roots. Denote by

\[
\Delta(\sigma_{3,A}) = 6a_1 a_2 a_4 - 4(a_1 a_3^3 + a_4 a_2^3) + 3a_2^2 a_3^2 - a_1^2 a_2^2
\]
the polynomial proportional to the discriminant of \(\sigma_{3,A}\).

Then polynomial \(\sigma_{3,A}\) has three distinct real roots if \(\Delta(\sigma_{3,A}) > 0\) and one real and two complex roots if \(\Delta(\sigma_{3,A}) < 0\).

We’ll say that the differential operator \(A\) is hyperbolic if \(\Delta(\sigma_{3,A}) > 0\) and ultrahyperbolic, if \(\Delta(\sigma_{3,A}) < 0\).

Locally, symbol of a hyperbolic operator could be presented as a symmetric product of pair wise linear independent vector fields: \(\sigma = X_1 \cdot X_2 \cdot X_3\), and therefore there are local coordinates \((x, y)\) such that

\[
\sigma_{3,A} = (a \partial_x + b \partial_y) \cdot \partial_x \cdot \partial_y,
\]
where \(a\) and \(b\) are smooth functions and \(ab \neq 0\).

For the case of ultrahyperbolic symbols we have \(\sigma_{3,A} = X \cdot q\), where \(X\) is a non zero vector and \(q\) is a positive symmetric 2-vector. Therefore, there are local coordinates \((x, y)\) such that

\[
\sigma = (a \partial_x + b \partial_y) \cdot (\partial_x^2 + \partial_y^2),
\]
where \(a\) and \(b\) are smooth functions and \(a^2 + b^2 > 0\).

2.3. Groups actions. In this paper we study orbits of the 3rd order differential operators with respect to the following groups:

(1) For the scalar differential operators \(A \in \text{Diff}_3(1)\), the group is the group \(\mathcal{G}(M)\) of diffeomorphisms of \(M\) with the natural action

\[
\phi_* : A \mapsto \phi_* \circ A \circ \phi_*^{-1},
\]
where $\phi = \phi^* : C^\infty (M) \to C^\infty (M)$ is the induced algebra morphism and $\phi^* (f) = f \circ \phi$, $f \in C^\infty (M)$.

(2) For general linear bundles $\xi$ and operators $A \in \text{Diff}_3 (\xi)$ we’ll take the group of automorphisms $\text{Aut} (\xi)$ with the following actions.

For sections $s \in C^\infty (\xi)$ and elements $\tilde{\phi} \in \text{Aut} (\xi)$, we define action as follows

$\tilde{\phi}_s : s \mapsto \tilde{\phi} \circ s \circ \phi^{-1},$

and

$\tilde{\phi}_A : A \mapsto \tilde{\phi} \circ A \circ \tilde{\phi}^{-1}.$

These groups are connected by the sequence

$1 \to F (M) \to \text{Aut} (\xi) \to G (M) \to 1,$

where $F (M)$ is the multiplicative group of smooth nowhere vanishing functions on $M$.

Later on we’ll get conditions under which a diffeomorphism $\phi \in G (M)$ has a lift in $\text{Aut} (\xi)$.

Assume now that $e$ is a local base for $\xi$, i.e. nowhere vanishing section in a domain. Then for any operator $A \in \text{Diff}_3 (\xi)$ we define a scalar operator $A_e \in \text{Diff}_3 (1)$ in such a way that

$A (fe) = A_e (f) e,$

for $f \in C^\infty (M)$.

It is easy to see that change of base leads us to the following transformation of scalar models:

$A_e \mapsto \tilde{A}_e = h \circ A \circ h^{-1},$

if $\tilde{e} = he$, and $h \in F (M)$.

Therefore, (locally) action (5) could be considered as composition of actions (4) and (6).

3. Connections and Symbols

In this section $\sigma \in \Sigma_3 (M)$ is a regular symmetric 3-vector.

3.1. Wagner connection. The following result due to Wagner, ([1]).

**Theorem 1.** For any regular symbol $\sigma \in \Sigma_3 (M)$ there exists and unique connection $\nabla^\sigma$ in the tangent bundle such that

$d_{\nabla^\sigma} (\sigma) = 0,$

where $d_{\nabla^\sigma} : \Sigma_3 (M) \to \Omega^1 (M) \otimes \Sigma_3 (M)$ is the covariant differential.

**Proof.** Let’s choose a local coordinates $(x, y)$ on $M$, and let $\Gamma^k_{ij}$ be the Christoffel symbols of connection $\nabla$ that satisfies the theorem.

Then

$\partial_i a_{ijk} + \Gamma^i_{lm} a_{mjk} + \Gamma^j_{lm} a_{imk} + \Gamma^k_{im} a_{ijm} = 0,$
where \( a_{ijk} \) are components of \( \sigma \) and in the previous notations \( a_{111} = a_1, a_{112} = a_{211} = a_2, a_{122} = a_{221} = a_3 \) and \( a_{222} = a_4 \).

Consider this as a system of linear equations with respect to Christoffel symbols \( \Gamma^k_{ij} \). This is an 8 \( \times \) 8 system (we did not assume that \( \nabla \) is a torsion-free connection). The determinant of this system equals \( 81 \Delta (\sigma)^2 \neq 0 \), therefore (7) determines the unique connection. □

We’ll call connection defining by (7), Wagner connection.

In the case of hyperbolic symbols we’ll choose local coordinates in such a way that \( \sigma \) has formula (2) holds. Then the non-zero Christoffel coefficients of the Wagner connection could be found from the above system of linear equations:

\[
\begin{align*}
\Gamma^1_{11} &= \frac{1}{3} \left( \ln \frac{b}{a^2} \right)_x, \\
\Gamma^2_{22} &= \frac{1}{3} \left( \ln \frac{a}{b^2} \right)_y, \\
\Gamma^1_{12} &= \frac{1}{3} \left( \ln \frac{b}{a^2} \right)_y, \\
\Gamma^2_{21} &= \frac{1}{3} \left( \ln \frac{a}{b^2} \right)_x.
\end{align*}
\]

In the similar way, for the ultrahyperbolic symbols (3), we get

\[
\begin{align*}
\Gamma^1_{11} &= \Gamma^2_{21} = -\frac{1}{6} \left\{ \ln \left( a^2 + b^2 \right) \right\}_x, \\
\Gamma^1_{12} &= \Gamma^2_{22} = -\frac{1}{6} \left\{ \ln \left( a^2 + b^2 \right) \right\}_y, \\
\Gamma^1_{21} &= -\Gamma^2_{11} = \frac{ab - ab_x}{a^2 + b^2}, \\
\Gamma^1_{22} &= -\Gamma^2_{12} = \frac{ab_x - ab_y}{a^2 + b^2}.
\end{align*}
\]

**Corollary 2.**

1. The curvature of the Wagner connection equals zero.
2. The torsion form of the Wagner connection equals

\[
\theta^b_u = \frac{1}{3} \left( \ln \frac{b^2}{a} \right)_x dx + \frac{1}{3} \left( \ln \frac{a^2}{b} \right)_y dy,
\]

for the hyperbolic case, and

\[
\theta^u_u = \frac{ab_y - ab_x}{a^2 + b^2} dx + \frac{ab_x - ab_y}{a^2 + b^2} dy - \frac{1}{6} d\left[ \ln \left( a^2 + b^2 \right) \right],
\]

for the ultrahyperbolic case.

3.2. Wagner metric. Denote by \( \Sigma^h_0 (M) \subset \Sigma^h (M) \) and \( \Sigma_{k,0} (M) \subset \Sigma_k (M) \) subsets of regular symmetric differential \( k \)-forms and regular symbols, respectively, i.e. symmetric tensors with non zero discriminant.

The following theorem holds for two-dimensional manifolds and it is also due to Wagner ([1]).

**Theorem 3.** There is a natural mapping

\[
W : \Sigma_{3,0} (M) \to \Sigma^3_0 (M),
\]

i.e. a mapping commuting with the action of the diffeomorphism group.
Proof. Let’s take a point \( a \in M \) and let \( V = T^*_a \) be the cotangent space and \( \sigma \) be a regular symbol, which we’ll consider as homogeneous cubic polynomial on \( V \). Denote by \( \text{Hess} (f) \) the Hessian of a function \( f \) on \( V \), computing in a fixed coordinates on the vsector space.

Then, \( \text{Hess} (\lambda f) = \lambda^2 \text{Hess} (f), \) for \( \lambda \in \mathbb{R} \), and

\[
\text{Hess} (A^* (f)) = (\det A)^2 A^* (\text{Hess} (f)),
\]

for any linear map \( A : V \to V \).

Remark, that \( \text{Hess} (f) \) is a quadratic function on \( V \), when \( f \) is a cubic. Therefore,

\[
\text{Hess}^2 (\sigma) = \text{Hess} (\text{Hess} (\sigma)) \in \mathbb{R},
\]

is a scalar.

The straightforward computations show that \( \text{Hess}^2 (\sigma) \) proportional to the discriminant of \( \sigma \) and, therefore \( \text{Hess}^2 (\sigma) \neq 0 \), if \( \sigma \) regular tensor.

On the other hand, we have

\[
\text{Hess}^2 (A^* (\sigma)) = \text{Hess} (\text{Hess} (A^* (\sigma))) = \text{Hess} (\det (A)^2 A^* (\text{Hess} (\sigma))) = \\
\det (A)^4 \text{Hess} (A^* (\text{Hess} (\sigma))) = \det (A)^6 \text{Hess}^2 (\sigma).
\]

Put

\[
g_k (\sigma) = \text{Hess}^2 (\sigma)^k \text{Hess} (\sigma), \tag{8}
\]

for \( k \in \mathbb{R} \).

Then,

\[
g_k (A^* (\sigma)) = \text{Hess}^2 (A^* (\sigma))^k \text{Hess} (A^* (\sigma)) = \det (A)^{6k+2} A^* (g_k (\sigma)).
\]

Therefore, quadratic form \( g_{-1/3} \) behave in the natural way with respect to linear transformations:

\[
A^* (g_{-1/3} (\sigma)) = g_{-1/3} (A^* (\sigma)),
\]

and the mapping

\[
W (\sigma) = \sqrt[3]{\text{Hess}^2 (\sigma) \text{Hess} (\sigma)^{-1}},
\]

where we denoted by \( \text{Hess} (\sigma)^{-1} \in S^2 V^* \) the metric inverse to \( \text{Hess} (\sigma) \in S^2 V^* \), satisfies the conditions of the theorem. \( \square \)

**Remark 4.** If tensor \( \sigma \in \Sigma_{0,3} (M) \) in local coordinates has form \( (1) \), then:

\[
\text{Hess} (\sigma) = (a_1 a_3 - a_2^2) \partial_x^2 + (a_1 a_4 - a_3 a_2) \partial_x \partial_y + (a_2 a_4 - a_3^2) \partial_y^2,
\]

\[
\text{Hess}^2 (\sigma) = \Delta (\sigma) = 6a_1 a_2 a_3 a_4 - 4 (a_1 a_3^2 + a_2^2 a_4) + 3 a_2^2 a_3^2 - a_1^2 a_4^2,
\]

and

\[
W (\sigma) = \frac{4}{\Delta (\sigma)^{2/3}} \left( (a_2 a_4 - a_3^2) dx^2 + (a_2 a_3 - a_1 a_4) dx \cdot dy + (a_1 a_3 - a_2^2) dy^2 \right).
\]
Remark that the type of this metric depends on sign of $\Delta (\sigma)$. Namely, $W (\sigma)$ is the definite metric in the hyperbolic case and indefinite for the ultrahyperbolic one.

**Remark 5.** In the similar way one might show that there is also natural map $\Sigma^3_0 (M) \rightarrow \Sigma^2_0 (M)$.

### 3.3. Group-type symbols

As the first application of the Wagner connection we’ll analyze the case when a symbol $\sigma \in \Sigma^3_{3,0} (M)$ has a group nature, i.e. there is a hidden group Lie that acts in a transitive way on the manifold, and the symbol is a group invariant.

It is known that if on a manifold there is a connection $\nabla$ with zero curvature then the torsion tensor $T_{\nabla}$ defines a skew symmetric bracket on vector fields. If we consider only covariant constant vector fields and assume that torsion tensor also covariant constant, the bracket given by $T_{\nabla}$ convert the vector space of covariant constant vector fields into a Lie algebra. Moreover, this construction shows that parallel transports with respect to $\nabla$ are exactly left multiplications in the corresponding (local) Lie group.

Applying these remarks to the Wagner connections $\nabla^\sigma$, corresponding to the regular symbols $\sigma$ we get the following result.

**Theorem 6.** Let $\sigma \in \Sigma^3_{3,0} (M)$ be a regular symbol and let $\nabla^\sigma$ be the corresponding Wagner connection. Then:

1. Symbol $\sigma$ is locally equivalent to the symbol with constant coefficients
   \[
   \sigma = c_1 p_1^3 + 3c_2 p_1^2 p_2 + 3c_3 p_1 p_2^2 + c_4 p_2^3, \quad c_i \in \mathbb{R},
   \]
   if and only if $T_{\nabla^\sigma} = 0$.

2. Symbol $\sigma$ is locally equivalent to the symbol of the form
   \[
   \sigma = c_1 \exp (3y) p_1^3 + 3c_2 \exp (2y) p_1^2 p_2 + 3c_3 \exp (y) p_1 p_2^2 + c_4 p_2^3, \quad c_i \in \mathbb{R},
   \]
   if and only if $T_{\nabla^\sigma} \neq 0$, but
   \[
d_{\nabla^\sigma} (T_{\nabla^\sigma}) = 0.
   \]

**Remark 7.** In the second case the discussed above Lie algebra $\mathfrak{g}$ is the solvable 2-dimensional Lie algebra generated by vector fields
\[
\mathfrak{g} = \langle \partial_x, \partial_y + x \partial_x \rangle.
\]

### 3.4. Chern connection

Together with the Wagner connection we’ll consider also another connection, which we call *Chern connection*. This connection is similar to the connection used by Chern in geometry of plane 3-webs (see, for example,[3], [2],[9]).

Namely, by Chern connection, associated with symbol $\sigma$, we’ll understand a linear connection $\nabla^\omega$ on manifold $M$, that preserves conformal classes of $\sigma$:

\[
d_{\nabla^\omega} (\sigma) = \omega \otimes \sigma,
\]
for some differential form $\omega \in \Omega^1(M)$.

**Theorem 8.** For any regular symmetric 3-vector $\sigma \in \Sigma_3$ there exist and unique torsion free Chern connection, and

$$d\nabla_\sigma - d\nabla_c = \alpha \otimes \text{Id},$$

(10)

where $\alpha = \theta_\sigma$ is the torsion form of the Wagner connection, and $\omega = -3\theta_\sigma$.

Curvature of the Chern connection equals $d\omega$.

**Proof.** Consider (9) as a linear system of equations with respect to components of symmetric connection $\nabla^c$ and differential form $\omega$. It gives us a $8 \times 8$ system of linear equations with determinant $-9\Delta(\sigma)^2 \neq 0$.

On the other hand we have relation (10) because of stationary Lie algebra of a regular symmetric 3-vector is trivial, and therefore $\theta_\sigma = \alpha$. □

**Example 9.** The Chern curvature of hyperbolic symbol (2) equals to

$$\ln \left( \frac{a}{b} \right)_{xy} \ dx \wedge dy.$$  

In the case of ultrahyperbolic symbol we can choose a representative in the conformal class of symbol (3) with $a = \sin(\phi(x,y))$ and $b = \cos(\phi(x,y))$. Then the Chern curvature equals

$$(\phi_{xx} + \phi_{yy}) \ dx \wedge dy.$$  

**Theorem 10.** Conformal classes of regular symbols are locally equivalent to conformal classes of symbols with constant coefficients if and only if the curvature of their Chern connection vanishes.

**Proof.** The statement easily to check for the hyperbolic case. In the ultrahyperbolic case it can be seen in complex coordinates $z = x + \sqrt{-1}y, \bar{z} = x - \sqrt{-1}y$. □

4. **Classification of regular symmetric 3-vectors**

Let $\pi : S^3\tau(M) \to M$ be the vector bundle of symmetric 3-vectors, $C^\infty(\pi) = \Sigma_3(M)$, and let $\pi_k : J^k(\pi) \to M$ be the bundles of $k$-jets of symmetric 3-vectors.

The group of diffeomorphisms $\mathcal{G}(M)$ acts in the natural way in the bundle $\pi$ as well as (by prolongations) in jet bundles $\pi_k$.

In this section we study orbits of this actions. First of all, because the action of $\mathcal{G}(M)$ on $M$ transitive, we could fix a point $a \in M$ and restrict ourselves by actions of subgroup $\mathcal{G}_a(M) = \{ \phi \in \mathcal{G}(M) | \phi(a) = a \}$ on fibres $J^k_a = \pi^{-1}_k(a)$. Secondly, for the case $k = 0$, we have two open orbits $\mathcal{O}_h$ and $\mathcal{O}_u$ that correspond to the hyperbolic and ultrahyperbolic symbols. The complement to them consists of symbols having multiple roots.
4.1. **Invariant coframe.** Consider now the orbits in 1-jets space. As we have seen, the torsion forms $\theta_\sigma$, for regular symbols $\sigma \in \Sigma_3(M)$, depend on 1-jet of $\sigma$ and therefore defines a horizontal differential 1-form $\Theta$ on $\pi_{1,0}(O_h \cup O_u) \subset J^1(\pi)$, such that

$$j_1(\sigma)^*(\Theta) = \theta_\sigma,$$

in the domain where $\sigma$ is regular.

Here we denoted by $j_1(\sigma) : M \to J^1(\pi)$ the section that corresponds to 1-jet of $\sigma$.

In the similar way the Wagner metric $W(\sigma) \in \Sigma^2(M)$ defines a quadratic horizontal quadratic form $W$ on $\pi_{1,0}(O_h \cup O_u) \subset J^1(\pi)$ in such a way, that

$$j_1(\sigma)^*(W) = W(\sigma),$$

in the domain where $\sigma$ is regular.

Both tensors $\Theta$ and $W$ are $\mathcal{G}(M)$-invariants. Using orientation on $M$ we construct an oriented $\mathcal{G}(M)$ invariant coframe $\langle \Theta, \Theta' \rangle$ on $\pi_{1,0}(O_h \cup O_u)$, where $\Theta'$ is a horizontal form such that

$$W(\Theta, \Theta') = 0, \quad W(\Theta, \Theta) = W(\Theta', \Theta').$$

In order to get form $\Theta'$ we need that $W(\Theta, \Theta) \neq 0$. (11)

Therefore, the above coframe exist in domain $\pi_{1,0}(O_h \cup O_u)$, where (11) holds. We denote the last domain by $O^{(1)} \subset J^1(\pi)$ and call symbols $\sigma$ 1-regular if its 1-jet belong to $O^{(1)}$.

4.2. **Universal symbol.** Let’s denote by $\hat{X}$ the total derivation, $\hat{X} : C^\infty(J^k(\pi)) \to C^\infty(J^{k+1}(\pi))$, that corresponds to vector field $X$ on manifold $M$, (see, [4]), and let $\mathcal{D}_H(\pi)$ be the module over $C^\infty(J^\infty(\pi))$, generated by the total derivatives. Elements of this module we call horizontal fields.

Elements of the symmetric cube of this module, i.e. $S^3(\mathcal{D}_H(\pi))$, we’ll call horizontal symbols.

The basic property of horizontal fields consist of the fact that they preserve and tangent to the Cartan distribution and therefore they could be restricted on sections of $k$-jet bundles $\pi_k$ of the form $j_k(\sigma)$.

The proof of the following theorem is standard for universal constructions ([4],[5]).

**Theorem 11.** There exists and unique an universal horizontal symbol $\Xi_3$ such that the restriction of this symbol on $j_0(\sigma)$ coincides with $\sigma$. The symbol is an invariant of the diffeomorphism group.
4.2.1. Coordinates. Denote by \((x, y)\) local coordinates on \(M\) and lets \((x, y, u^1, u^2, u^3, u^4)\) be the corresponding standard local coordinates in \(S^3(T)(M)\). Then the section \(j_0(\sigma)\), corresponding to symbol,
\[
\sigma = a_1(x, y) \partial_x^3 + 3a_2(x, y) \partial_x^2 \cdot \partial_y + 3a_3(x, y) \partial_x \cdot \partial_y^2 + a_4(x, y) \partial_y^3,
\]
has the form
\[
u^1 = a_1(x, y), u^2 = a_2(x, y), u^3 = a_3(x, y), u^4 = a_4(x, y).
\]
It is easy to check, that the universal symbol has the form:
\[
\Xi_3 = u^1 \left(\frac{d}{dx}\right)^3 + 3u^2 \left(\frac{d}{dx}\right)^2 \cdot \left(\frac{d}{dy}\right) + 3u^3 \left(\frac{d}{dx}\right) \cdot \left(\frac{d}{dy}\right)^2 + u^4 \left(\frac{d}{dy}\right)^3,
\]
in these coordinates.

4.3. Differential invariants of symbols.

4.3.1. Symbols. First of all, as we have seen, there are no none constant functions on \(J^0(\pi)\), which are invariant with respect to the diffeomorphism pseudo group. Indeed, in this case we have to open or bits \(O_h\) and \(O_u\) such that the closure of their union coincide with \(J^0(\pi)\).

The action of the diffeomorphism pseudo group into 1-jet space \(J^1(\pi)\) has invariant coframe \(\langle \Theta, \Theta' \rangle\) and invariant universal symbol.

Let horizontal vector fields \(\delta_1\) and \(\delta_2\) constitute a frame dual to coframe \(\langle \Theta, \Theta' \rangle\) and let
\[
\Xi_3 = I_1 \delta_1^3 + 3I_2 \delta_1^2 \cdot \delta_2 + 3I_3 \delta_1 \cdot \delta_2^2 + I_4 \delta_2^3,
\]
be the decomposition of the universal symbol in this frame.

Then functions \(I_i, i = 1, ..., 4\), are defined on the domain \(O^{(1)} \subset J^1(\pi)\) and they are invariants with respect to the diffeomorphism group action.

Their structure we can described in the following way. Let's denote by \(F_1\) the field of rational functions on the fibre \(J^1_b(\pi)\), for some fixed point \(a \in M\), and let \(F_1 \left(\sqrt[3]{\Delta}\right)\) be the field extension of \(F_1\) by \(\sqrt[3]{\Delta}\), where \(\Delta(j_1(\sigma)) = \Delta(\sigma)\). Then the restrictions of functions \(I_i\) on fibres \(J^1_b(\pi)\) do no depend on \(b \in M\) and therefore its enough to consider their restriction on the fibre \(J^1_a(\pi)\). On the other hand, their construction shows, that \(I_i \in F_1 \left(\sqrt[3]{\Delta}\right)\).

Moreover, the horizontal vector fields \(\delta_1, \delta_2\) are also invariants and therefore their action on invariants give us new invariants.

Let's call by natural differential invariants of symbols functions \(I\) defined on \(J^k(\pi)\) which are invariants with respect to the diffeomorphism group action and which belong to fields \(F_k \left(\sqrt[3]{\Delta}\right)\), where \(F_k\) are the fields of rational functions on fibres \(J^k_b(\pi)\), for \(b \in M\). The number \(k\) is the order of the invariant.

Summing up this we arrive at the following.
Theorem 12.

1. The field of natural differential invariants of symbols is generated by basic invariants $I_1, I_2, I_3, I_4$ and the Tresse derivations $\delta_1, \delta_2$.

2. This field separate regular orbits.

4.3.2. Conformal classes of symbols. Let $[\pi] : \mathbb{P} (S^3 \tau) (M) \to M$ be the projectivization of the symbol bundle $\pi : S^3 \tau (M) \to M$.

Sections of this bundle we’ll consider as conformal classes of symbols and denote by $[\sigma] = \{ f \sigma, f \in \mathcal{F} (M) \}$ the conformal class of a symbol $\sigma$.

Let $[\pi]_k : J^k ([\pi]) \to M$ be the bundle of $k$-jets of conformal classes. The actions of group $\mathcal{G} (M)$ of diffeomorphisms in the bundles $\pi_k$ induce the actions in bundles $[\pi]_k$ and in this section we study orbits and invariants of these actions.

First of all, it is clear that this action in $J^0 ([\pi])$ has two open orbits $O_h$ and $O_u$, where $\Delta \neq 0$, and singular orbits that correspond to symbols with multiple roots.

The Chern connection $\nabla^{[\sigma]}$ depends on conformal class of the regular symbol $\sigma$, and

$$d_{\nabla^{[\sigma]}} (\sigma) = \omega_\sigma \otimes \sigma,$$

where differential 1-form $\omega_\sigma$ depends on representative $\sigma$ in the conformal class $[\sigma]$ in the following way:

$$\omega_f = \omega_\sigma + d \ln |f|,$$

where $f \in \mathcal{F} (M)$.

The curvature of the Chern connection $\nabla^{[\sigma]}$, as a tensor in $\Omega^2 (M) \otimes \text{End} (\tau)$ is the scalar operator $d\omega_\sigma \otimes \text{Id}$.

As we have seen, the Christoffel symbols of the Chern connection $\nabla^{[\sigma]}$ depend on the first order jets of $\sigma$ and therefore the curvature 2-form $d\omega_\sigma$ depends on the second jets.

As above, let’s denote by $\Omega$ an universal horizontal 2-form on the manifold of second jets $\mathcal{O}^{(2)} = [\pi]_{2,0}^{-1} (\mathcal{O})$, where $\mathcal{O} = \mathcal{O}_h \cup \mathcal{O}_u$, such that restrictions of $\Omega$ on 2-jet sections $j^2 ([\sigma]) : M \to J^2 ([\pi])$ coincide with curvature form $d\omega_\sigma$:

$$j^2 ([\sigma])^* (\Omega) = d\omega_\sigma,$$

for regular symbols.

Applying the total covariant differential

$$\widehat{d_{\nabla}} : \Omega^2_h (J^2 ([\pi])) \to \Omega^2_h (J^3 ([\pi])) \otimes \Omega^1_h (J^3 ([\pi])),$$

$\nabla = \nabla^{[\sigma]}$, to the universal 2-form we get

$$\widehat{d_{\nabla}} (\Omega) = \Omega \otimes \theta,$$

where $\theta \in \Omega^1_h (J^3 ([\pi]))$ is a horizontal 1-form on the space of 3-jets.
Repeating this procedure and applying the total covariant differential
\[ \hat{\nabla} : \Omega^1_h (J^3 ([\pi])) \rightarrow \Omega^1_h (J^4 ([\pi])) \otimes \Omega^2_h (J^4 ([\pi])) \]
to the horizontal 1-form \( \theta \) we get tensor \( \hat{\nabla} (\theta) \) on the space of 4-jets. We take the symmetric part of this tensor (it is easy to check that the skew symmetric part is proportional to \( \Omega \)) and get horizontal quadratic differential form \( G \) on the space of 4-jets.

Denote by \( O^{(4)} \subset [\pi]_{4,0} (\mathcal{O}) \) the domain of regular 4-jets, i.e. such 4-jets where \( G \) is non degenerated and \( G (\theta, \theta) \neq 0, \Omega \neq 0 \). Then, similar to the above, we construct the \( G \)-orthogonal coframe \( \langle \theta, \theta' \rangle \) in the domain \( O^{(4)} \). This coframe is invariant of the diffeomorphism group and decomposing the projective class of the universal symbol \( \Xi \) we get \( \mathcal{G} (M) \)-invariant mapping:

\[ I : O^{(4)} \rightarrow [I_1 : I_2 : I_3 : I_4 ] \in \mathbb{P}^3, \]

where functions \( I_i/I_1 \) are natural invariants of order 4.

**Theorem 13.**
The field of natural differential invariants of conformal classes symbols is generated by basic invariants \( I_1/I_1 \) and the Tresse derivations \( \delta_1, \delta_2 \), where frame \( \langle \delta_1, \delta_2 \rangle \) is dual to coframe \( \langle \theta, \theta' \rangle \). This field separates regular orbits.

4.3.3. Coordinates. In hyperbolic case we’ll take such coordinates \((x, y)\) that the symbol has the form:

\[ \sigma = (\partial_x + \exp (h) \partial_y) \cdot \partial_x \cdot \partial_y, \]

where \( h = h (x, y) \) is a smooth function.

Then, the non zero Christoffel symbols are

\[ \Gamma^1_{1,1} = h_x, \quad \Gamma^2_{2,2} = -h_y, \tag{12} \]

and the curvature form

\[ \Omega = -3h_{xy} dx \wedge dy, \]

The corresponding 1-form equals

\[ \theta = d \ln (h_{xy}) - h_x dx + h_y dy. \]

The covariant differential of an 1-form

\[ \alpha = Pdx + Qdy \]

equals

\[ d\nabla (\alpha) = (P_x - Ph_x) dx \otimes dx + P_y dx \otimes dy + Q_x dy \otimes dx + (Q_y + Qh_y) dy \otimes dy, \]

and therefore

\[ G = (P_x - Ph_x) dx \cdot dx + (P_y + Q_x) dx \cdot dy + (Q_y + Qh_y) dy \cdot dy, \]
where

\[ P = \frac{h_{xyy} - h_x}{h_{xy}}, \quad Q = \frac{h_{xyy} + h_y}{h_{xy}}. \]

In the ultrahyperbolic case we take such coordinates \((x, y)\) that the symbol has the form

\[ \sigma = (\sin (h) \partial_x + \cos (h) \partial_y) \cdot (\partial_x^2 + \partial_y^2), \]

where, as above, \(h = h(x, y)\) is a smooth function.

Then, the non-zero Christoffel symbols are

\[ \Gamma^1_{1,1} = h_y, \quad \Gamma^1_{1,2} = -h_x, \quad \Gamma^2_{1,2} = h_y, \quad \Gamma^2_{2,2} = -h_x, \quad (13) \]

and the curvature form

\[ \Omega = -3 \left( h_{xx} + h_{yy} \right) dx \wedge dy. \]

The corresponding 1-form equals

\[ \theta = d \ln (h_{xx} + h_{yy}) - 2h_y dx + 2h_x dy. \]

The covariant differential of an 1-form \(\alpha\) equals

\[ d^\nabla (\alpha) = (P_x - Ph_y - Qh_x) dx \otimes dx + (P_y + Ph_x - Qh_y) dx \otimes dy + (Q_x + Ph_x - Qh_y) dy \otimes dx + (Q_y + Ph_x + Qh_y) dy \otimes dy, \]

and therefore

\[ G = (P_x - Ph_y - Qh_x) dx \cdot dx + (P_y + Q_x + 2(Ph_x - Qh_y)) dx \cdot dy + (Q_y + Ph_x + Qh_y) dy \cdot dy, \]

where

\[ P = \frac{h_{xxx} + h_{xyy}}{h_{xx} + h_{yy}} - 2h_y, \quad Q = \frac{h_{xyy} + h_{yy}}{h_{xx} + h_{yy}} + 2h_x. \]

5. Quantization and splitting of scalar differential operators

5.1. Quantization. In this section we apply the quantization procedure outlined in ([5] and [6]).

Let \(\Sigma = \bigoplus_{k \geq 0} \Sigma^k (M)\) be the graded algebra of symmetric differential forms on the surface \(M\). Then any affine connection \(\nabla\) on \(M\) defines a derivation

\[ d^\nabla : \Sigma \to \Sigma^{+1} \]

of degree one in this algebra.

Because any such derivation is defined by its action on generators of the algebra we define \(d^\nabla\) as follows:

\[ d^\nabla = d : C^\infty (M) \to \Omega^1 (M) = \Sigma^1 (M), \]

\[ d^\nabla : \Omega^1 (M) = \Sigma^1 (M) \xrightarrow{d\Sigma} \Omega^1 (M) \otimes \Omega^1 (M) \xrightarrow{\text{Sym}} \Sigma^2 (M). \]
Let choose local coordinate \((x_1, x_2)\) on \(M\), and let \((x_1, x_2, w_1, w_2)\) be the induced local coordinates in the tangent bundle. Then elements of the symbol algebra \(\Sigma\) we shall write as polynomials of the form
\[
\Sigma_\alpha a_\alpha (x_1, x_2) w^\alpha,
\]
where \(\alpha = (\alpha_1, \alpha_2)\) are multi indices and \(a_\alpha (x_1, x_2)\) are smooth functions.

Then the derivation \(d_\nabla\) takes the following form
\[
d_\nabla = w_1 \partial_{x_1} + w_2 \partial_{x_2} - \Sigma_{j,k} \Gamma^1_{jk} w_j w_k \partial_{w_1} - \Sigma_{j,k} \Gamma^2_{jk} w_j w_k \partial_{w_2}.
\]
Remark that the symbol of the operator (as of any derivation) at a covector equals to symmetric multiplication in the algebra by the covector. Therefore, the value of the symbol of \(k\)-th power \((d_\nabla)^k : \Sigma \to \Sigma^{+k}\) at a covector equals to symmetric multiplication by the \(k\)-th degree of the covector.

Let’s \(\sigma \in \Sigma_k\) be a symbol. We define a differential operator \(\hat{\sigma} \in \text{Diff}_k (1)\) as follows:
\[
\hat{\sigma} (f) \overset{\text{def}}{=} \frac{1}{k!} \left\langle \sigma, (d_\nabla)^k (f) \right\rangle.
\]
Then, due to the above remark, the symbol of operator \(\hat{\sigma}\) equals to \(\sigma\).

The correspondence
\[
\mathcal{Q} : \Sigma \to \text{Diff}_k (1) = \cup_{k \geq 0} \text{Diff}_k (1),
\]
\[
\mathcal{Q} : (\sigma_0, \ldots, \sigma_k) \mapsto \Sigma_{i \geq 0} \hat{\sigma}_i,
\]
we call quantization associated with connection.

It is worth to note that \(\mathcal{Q}\) is the module isomorphism only but not the algebra morphism.

Thus, for any operator \(A \in \text{Diff}_k (1)\), there is a tensor \(\sigma_A = (\sigma_0, \ldots, \sigma_k)\), where \(\sigma_i \in \Sigma_i\), such that
\[
\mathcal{Q} (\sigma_A) = A.
\]
We call \(\sigma_A\) the total symbol of operator \(A\). The term \(\sigma_k\) is the principal or leading symbol of the operator.

The leading symbol \(\sigma_k\) does not depend on a connection but the total symbol does.

5.1.1. Wagner quantization. We consider here the Wagner quantization, i.e. the quantization associated with the Wagner connection. We’ll assume that local coordinates \((x, y)\) are chosen in such a way that
\[
\sigma_h = 3 \exp(a + b) \left( \exp(b) \partial_x + \exp(a) \partial_y \right) \cdot \partial_x \cdot \partial_y
\]
in the hyperbolic case, and
\[
\sigma_u = \exp(3r) \left( \sin(h) \partial_x + \cos(h) \partial_y \right) \cdot (\partial^2_x + \partial^2_y)
\]
in the ultrahyperbolic case.

Here \(a, b, r, h\) are some smooth functions.
Then in the hyperbolic case the Wagner connection has the following non zero Christoffel coefficients:

\[ \Gamma^1_{11} = -b_x, \quad \Gamma^1_{12} = -b_y, \quad \Gamma^2_{21} = -a_x, \quad \Gamma^2_{22} = -a_y, \]

and the derivation \( d_V^e \) has the following form

\[ d_V^e = w_1 \partial_x + w_2 \partial_y + (b_x w_1^2 + b_y w_1 w_2) \partial_{w_1} + (a_x w_1 w_2 + a_y w_2^2) \partial_{w_2}. \]

Then the quantization of the third order symbol

\[ \sigma_3 = a_{111} \partial^3_1 + 3a_{112} \partial^2_1 \cdot \partial_2 + 3a_{122} \partial_1 \cdot \partial^2_2 + a_{222} \partial^3_2 \]

is the following operator

\[
\hat{\sigma}_3 = a_{111} \partial^3_1 + 3a_{112} \partial^2_1 \cdot \partial_2 + 3a_{122} \partial_1 \cdot \partial^2_2 + a_{222} \partial^3_2 + \\
+ a_{111} (b_{xx}^2 + 2b_x^2 \partial_x + 3b_x \partial_x^2) + \\
a_{112} (3b_y \partial_x^2 + 3(a_x + b_x) \partial_x \partial_y + (2b_{xy} + 3b_x b_y + a_x b_y) \partial_x + \\
(a_{xx}^2 + a_x^2 + a_x b_y) \partial_y) + \\
a_{122} (3(a_y + b_y) \partial_x \partial_y + 3a_x \partial_y^2 + (b_{yy} + b_y^2 + a_y b_y) \partial_x + \\
(2a_{xy} + a_x b_y + 3a_x a_y) \partial_y) + \\
a_{222} (3a_y \partial_y^2 + (a_{yy} + a_y^2) \partial_y). 
\]

In particular, for the symbol \( \sigma_h \) we get

\[
\hat{\sigma}_h = 3 \exp(a + b) \left( \exp(b) \partial_x + \exp(a) \partial_y \right) \partial_x \partial_y + \\
3 \exp(a + 2b) \left( b_y \partial_x^2 + (a_x + b_y) \partial_x \partial_y \right) + \\
3 \exp(2a + b) \left( (a_y + b_y) \partial_x \partial_y + a_x \partial_y^2 \right) + \\
\exp(a + 2b) \left( (2b_{xy} + 3b_x b_y + a_x b_y) \partial_x + (a_{xx} + a_x^2 + a_x b_y) \partial_y \right) + \\
\exp(2a + b) \left( (b_{yy} + b_y^2 + a_y b_y) \partial_x + (2a_{xy} + a_x b_y + 3a_x a_y) \partial_y \right). 
\]

The Wagner quantization of the general second order symbol

\[ \sigma_2 = a_{11} \partial^2_1 + 2a_{12} \partial_1 \cdot \partial_2 + a_{22} \partial^2_2 \]

equals

\[
\hat{\sigma}_2 = a_{11} \partial^2_1 + 2a_{12} \partial_1 \partial_2 + a_{22} \partial^2_2 + \\
a_{11} (b_x \partial_x) + a_{12} (b_y \partial_x + a_x \partial_y) + a_{22} (a_y \partial_y). 
\]

5.1.2. Chern quantizations. Let’s apply the quantization procedure using hyperbolic Chern connections \((12)\).

Then the derivation \( d_V^e \) has the following form

\[ d_V^e = w_1 \partial_x + w_2 \partial_y - h_x w_1^2 \partial_{w_1} + h_y w_2^2 \partial_{w_2}, \]

and the quantization of the third order symbol

\[ \sigma_3 = a_{111} \partial^3_1 + 3a_{112} \partial^2_1 \cdot \partial_2 + 3a_{122} \partial_1 \cdot \partial^2_2 + a_{222} \partial^3_2. \]
will be the following operator
\[
\hat{\sigma}_3 = a_{111}\partial_1^3 + 3a_{112}\partial_1^2\partial_2 + 3a_{122}\partial_1\partial_2^2 + a_{222}\partial_2^3 + a_{111} (2h_x^2\partial_x - h_{xx}\partial_x - 3h_x\partial_y^2) - a_{112} (h_{xy}\partial_x + 3h_x\partial_y) + a_{112} (h_{xy}\partial_y + 3h_y\partial_x\partial_y) + a_{222} (2h_y^2\partial_y + h_{yy}\partial_y + 3h_y\partial_y^2).
\]
In particular, for the symbol
\[
\sigma_3 = \partial_1^2 \cdot \partial_2 + \exp(h)\partial_1 \cdot \partial_2^2,
\]
we get
\[
\hat{\sigma}_3 = (\partial_1 + \exp(h)\partial_2 + h_x (e^h - 1)) \partial_1 \partial_2 - \frac{h_{xy}}{3} \partial_x + \frac{h_{xx}}{3} \partial_y.
\]
Quantization of the second order symbols
\[
\sigma_2 = a_{111}\partial_1^2 + 2a_{112}\partial_1 \cdot \partial_2 + a_{222}\partial_2^2
\]
are operators of the following form
\[
\hat{\sigma}_2 = a_{111}\partial_1^2 + 2a_{112}\partial_1 \cdot \partial_2 + a_{222}\partial_2^2 - a_{111}h_x\partial_x + a_{222}h_y\partial_y.
\]
For the case of ultrahyperbolic Chern connection we have the similar formulae.
The derivation \(d^\nu_{cu}\) has the form
\[
d^\nu_{cu} = w_1\partial_x + w_2\partial_y \left(-h_y w_1^2 + 2h_x w_1 w_2 + h_y w_2^2\right) \partial_{w_1} + \left(-h_x w_1^2 - 2h_y w_1 w_2 + h_x w_2^2\right) \partial_{w_2},
\]
and the quantization of the third order symbol is the following
\[
\hat{\sigma}_3 = a_{111}\partial_1^3 + 3a_{112}\partial_1^2\partial_2 + 3a_{122}\partial_1\partial_2^2 + a_{222}\partial_2^3 + a_{111} (6h_x\partial_x^2 - 9h_y\partial_x\partial_y - 3h_x\partial_y^2 + (2h_{xx} - 12h_x h_y - h_{yy})\partial_x
\]
\[- (3h_{xy} - 6h_x^2 + 6h_y^2)\partial_y) + a_{112} (6h_y\partial_x^2 + 9h_x\partial_x\partial_y - 3h_x\partial_y^2 + (6h_x^2 + 3h_{xy} - 6h_y^2)\partial_x
\]
\[+ (h_{xx} - 12h_x h_y - 2h_{yy})\partial_y) + a_{122} (6h_y\partial_x^2 + 9h_x\partial_x\partial_y - 6h_y\partial_y^2 + (6h_x^2 + 3h_{xy} - 6h_y^2)\partial_x
\]
\[+ (2h_x^2 + h_{xy} - 2h_y^2)\partial_y) + a_{222} (3h_x\partial_x\partial_y + 3h_x\partial_y^2 + (4h_x h_y + h_{yy})\partial_x
\]
\[+ (2h_x^2 + h_{xy} - 2h_y^2)\partial_y).
\]
The quantization of the second order symbols has the following form
\[
\hat{\sigma}_2 = a_{111}\partial_1^2 + 2a_{112}\partial_1 \cdot \partial_2 + a_{222}\partial_2^2 - a_{111}h_x\partial_x + a_{222}h_y\partial_y + (a_{22} - a_{11}) (h_y\partial_x + h_x\partial_y) + 2a_{12} (h_x\partial_x - h_y\partial_y).
\]
6. Differential invariants of differential operators

6.1. Jets of differential operators. Denote by \( \pi : \text{Diff}_3(1) \to M \) the bundle of linear scalar differential operators of the third order, and by \( \tau_i = S^i \tau : S^i(TM) \to M \) the bundles of symmetric contravariant tensors. Sections of these bundles are linear scalar differential operators of the third order and symmetric contravariant tensors (symbols).

If \((x,y)\) are local coordinates on \(M\), then the induced local coordinates in these bundles we will denote by \((x,y,u^\alpha)\), where \(\alpha = (\alpha_1,\alpha_2)\) are multi indexes of length \(0 \leq |\alpha| \leq 3\) - for the case of bundle \(\pi\) and \(|\alpha| = i\) - for bundles \(\tau_i\).

Thus, for example, if an operator \(A\) has the form
\[
A = \sum_{\alpha,0 \leq |\alpha| \leq 3} \frac{a_\alpha(x,y)}{\alpha!} \partial^\alpha,
\]
in local coordinates \((x,y)\), then the corresponding section \(S_A : M \to \text{Diff}_3(1)\), has the form \(u^\alpha = a_\alpha(x,y)\), in the canonical coordinates \((x,y,u^\alpha)\).

The similar conversion valids also for contravariant tensors.

Let \(\pi_k: J^k(\pi) \to M\), \(k \geq 4\), be the bundles of \(k\)-jets of sections of bundle \(\pi\), i.e. \(k\)-jets of linear differential operators of the third order. The induced canonical coordinates in the bundle \(\pi_k\) we’ll denote by \(u^\beta\), where multi indexes \(\beta\) have length \(0 \leq |\beta| \leq k\). If we denote by \([A]^k_p \in J^k(\pi)\) the \(k\)-jet of the operator \(A\) at a point \(p \in M\), then values of \(u^\beta\) at \([A]^k_p\) equal \(\frac{\partial^{|eta|} a_\alpha}{\partial x^\beta}(p)\).

6.2. Splitting mapping and invariants. Denote by \(O \subset J^2(\pi)\), and \(O^{(k)} \subset J^k(\pi)\), \(k \geq 4\), the domains of regular \(k\)-jets, i.e. such \(k\)-jets \([A]^k_p\) that jets of the symbols \(\sigma_3, A\) belong to domains of regular jets \([\sigma_3, A]^2_p \in O \subset J^2(\tau_3)\) and 4-jets of the symbols belong to the domain of regular jets of symbols \(O^{(4)} \subset J^4(\tau_3)\).

Consider differential operator
\[
\nu : O \subset J^2(\pi) \to \tau_{\leq 3} = \tau_3 \oplus \tau_2 \oplus \tau_1 \oplus \tau_0,
\]
which sends 2-jet \([A]^2_p\) to the total symbol \(\sigma_A = (\sigma_{3, A}, \sigma_{2, A}, \sigma_{1, A}, a_0)\) with respect to the Chern connection \(\nabla_{\sigma_{3, A}}\).

It follows from the construction of the Chern connection that this operator is natural, i.e. commutes with the action of the diffeomorphism group.

Regularity conditions allow to construct invariant coframe \((\theta, \theta')\) on the space of regular 4-jets \(J^4(\tau_3)\). Let, as in section 4.2., denote by \(\Xi_i\) the universal symbols in \(\tau_i, i = 0, 1, 2, 3\), and let \(I_{i, \alpha}\) be their components in the coframe \((\theta, \theta')\).
By a natural differential invariant of total symbols we mean a function on $J^k(\tau_3 \oplus \tau_2 \oplus \tau_1 \oplus \tau_0)$ which is rational along fibres of the projection

$$J^k(\tau_3 \oplus \tau_2 \oplus \tau_1 \oplus \tau_0) \to M$$

and which is invariant with respect to the action of the diffeomorphism group.

Then, similar to section 4.3, we get the following result.

**Theorem 14.** The field of natural differential invariants of total symbols is generated by the basic invariants $I_{i,\alpha}$ and the Tresse derivatives $\delta_1, \delta_2$.

Let’s $\nu_k : J^{k+2}(\pi) \to J^k(\tau_3 \oplus \tau_2 \oplus \tau_1 \oplus \tau_0)$ be the $k$-th prolongations of the differential operator $\nu$. Then these mappings are also natural and the induced morphisms $\nu_k^*$ maps invariants of total symbols to invariants of differential operators.

**Theorem 15.** The field of natural differential invariants of linear scalar differential operators of the third order is generated by the basic invariants $\nu_2^*(I_{i,\alpha})$ and the Tresse derivatives $\delta_1, \delta_2$.

### 6.3. Universal differential operator of the third order

We apply here the construction of universal differential operator of the second order (see [5]) to the operators of the third order.

We define a total operator

$$\Box_3 : C^\infty (J^k\pi) \to C^\infty (J^{k+3}\pi),$$

by the same universal property

$$j_{k+3}(A)^* (\Box_3(f)) = A(j_k(A)^*(f)),$$

for all functions $f \in C^\infty (J^k\pi)$ and $k = 0, 1, ..., k$, and for all operators of the third order $A \in \text{Diff}_3(1)$.

The standard arguments show that this operator exists and unique and in local coordinates has the form

$$\Box_3 = 6 \sum_{\alpha, 0 \leq |\alpha| \leq 3} \frac{u^\alpha}{\alpha!} \left(\frac{d}{dx}\right)^\alpha. \quad (15)$$

**Theorem 16.** (1) The universal total differential operator of the third order $\Box_3$ exists and unique and has representation (15) in the standard local coordinates.

(2) The universal total differential operator of the third order $\Box_3$ commutes with action of the diffeomorphism group $\mathcal{G}(M)$ on the jet bundles $\pi_k$.

**Corollary 17.** If $f$ is a natural differential invariant of order $k$ for differential operators of the third order, then $\Box_3(f)$ is a natural invariant of the $(k + 3)$ order.
Here, as above, by *natural differential invariant* of order \( k \) we mean a function on \( J^{k}\pi \) which is \( \mathcal{G} (M) \)-invariant and which is rational along fibres of the projection \( \pi_k \).

We’ll also say that two natural differential invariants \( I_1, I_2 \) are in *general position* if
\[
\hat{d}I_1 \wedge \hat{d}I_2 \neq 0.
\]

**Theorem 18** (The principle of \( n \)-invariants,([7])). Let natural differential invariants \( I_1, I_2 \) be in general position and let
\[
J^\alpha = \Box_3 (I^\alpha),
\]
where \( \alpha = (\alpha_1, \alpha_2) \) and \( 0 \leq |\alpha| \leq 3 \).
Then the field of natural differential invariants is generated by invariants \( (I_1, I_2, J^\alpha) \) and all their Tresse derivatives
\[
\frac{d^l J^\alpha}{dI_1^{l_1} dI_2^{l_2}},
\]
where \( l = l_1 + l_2 \).

**Proof.** First of all remark that condition (16) defines open and dense domains in the \( k \)-jet bundles. Take an operator \( A \in \text{Diff}_3 (1) \) such that locally the section \( j_k (A) \) belongs to the domain. Let \( f \) be a natural invariant of order \( k \) and let
\[
f (A) = j_k (A)^* (f)
\]
be its value on the operator \( A \).

Then functions \( I_1 (A) \) and \( I_2 (A) \) are local coordinates on \( M \) and functions \( J^\alpha (A) \) give us a coefficients of \( A \) in these coordinates. By definition, the value \( f (A) \) depends on these coefficients and their derivatives only, and its true for almost all operators \( A \). Therefore, function \( f \) itself is a rational function of \( J^\alpha \) and its Tresse derivatives. \( \square \)

7. Differential operators in linear bundles

Here we extend results of the previous chapter on differential operators acting in line bundles \( \xi : E (\xi) \to M. \)

7.1. **Quantization.** Let \( \xi : E (\xi) \to M \) be a line bundle. Denote by \( \Sigma (\xi) = C^\infty (\xi) \otimes \Sigma (M) \) the graded module of symmetric \( \xi \)-valued differential forms on the surface.

Assume now that both surface \( M \) and bundle \( \xi \) are equipped with connections \( \nabla^M \) and \( \nabla^\xi \) respectively and let \( d_{\nabla^M} : \Omega^1 (M) \to \Omega^1 (M) \otimes \Omega^1 (M) \) and \( d_{\nabla^\xi} : C^\infty (\xi) \to C^\infty (\xi) \otimes \Omega^1 (M) \) be their covariant differentials.

Similar to ([5]) we define a derivation \( d_{\nabla^\xi}^\alpha : \Sigma (\xi) \to \Sigma^{+1} (\xi) \) of degree one over the derivation \( d_{\nabla^M}^\alpha : \Sigma (M) \to \Sigma^{+1} (M) \) by requirement that
\[
d_{\nabla^\xi}^\alpha (s \otimes \theta) = s \otimes d_{\nabla^M}^\alpha \theta + d_{\nabla^\xi}^\alpha (s) \cdot \theta, \text{ where } s \in C^\infty (\xi), \theta \in \Sigma (M).
\]
It is important to note that both these derivations are differential operators of the first order and the values of their symbols on a covector $v$ coincide with the symmetric product on the covector.

Let’s now $\sigma \in \Sigma_k (M)$ be a symbol of order $k$. We define its quantization $\hat{\sigma}$ to be a linear differential operator of order $k$ in the line bundle $\xi$, having symbol $\sigma$ and acting in the following way
\[
\hat{\sigma} (s) = \frac{1}{k!} \left< \sigma, (d_{\nabla^M})^k (s) \right>.
\] (17)

Let $(x_1, x_2)$ be local coordinates on $M$ and let $e$ be a basic section of $\xi$ over the coordinate neighborhood.

Then, as above, we will write down elements of $\Sigma_k (\xi)$ in the form
\[
e \otimes \sum_{|\alpha| = k} a_{\alpha} (x) w^\alpha,
\]
where coefficients $a_{\alpha} (x)$ are smooth functions on the neighborhood.

Remark that formula (17) allows us to define prolongations of operator $A \in \text{Diff}^3_3 (\xi)$ to operators $A^{[k]} \in \text{Diff}^3_3 (\xi \otimes k)$ acting in the tensor powers $\xi \otimes k$ of the bundle, for all $k \in \mathbb{Z}$.

**7.2. Connection defined by the 3rd order differential operator.**

Let $A \in \text{Diff}^3_3 (\xi)$ be the third order linear differential operator, acting in the bundle $\xi$. Denote by $\sigma_{3,A} = \text{smbl}_3 (A) \in \Sigma_3 (M)$ the principal symbol of this operator and assume that $\sigma_{3,A}$ is regular. Let $\nabla = \nabla_{\sigma_{3,A}}$ be the Chern connection associated with this symbol.

Assume that a connection $\nabla^{\xi}$ is given, then we define subsymbol $\sigma_{2,A} (\xi) \in \Sigma_2 (M)$ as follows
\[
\sigma_{2,A} (\xi) = \text{smbl}_2 (A - \sigma_{3,A}) .
\]

**Theorem 19.** There is and unique a connection $\nabla^{\xi}$ in the linear bundle such that
\[
\sigma_{2,A} (\xi) = \lambda \ g_{-1/3} (\sigma_{3,A}) ,
\] (19)
where $g_{-1/3} (\sigma_{3,A})$ is the Wagner metric (8), associated with symbol $\sigma_{3,A}$, and $\lambda \in C^\infty (M)$ is a multiplier.

**Proof.** Assume that operator $A$ has the form
\[
A = a_1 \partial_1^3 + 3a_2 \partial_1^2 \partial_2 + 3a_3 \partial_1 \partial_2^2 + a_4 \partial_2^2 +
a_11 \partial_1^2 + 2a_12 \partial_1 \partial_2 + a_22 \partial_2^2 + b_1 \partial_1 + b_2 \partial_2 + b_0
\]
in a local coordinates $(x_1, x_2)$. 

Then equation (19) gives us a linear system

\[
3a_1 \theta_1 + 3a_2 \theta_2 + a_{11} = \left( a_1 a_3 - a_2^2 \right) \Delta (\sigma_{3,A})^{-1/3} \lambda,
\]

\[
3a_3 \theta_1 + 3a_4 \theta_2 + a_{22} = \left( a_2 a_4 - a_3^2 \right) \Delta (\sigma_{3,A})^{-1/3} \lambda,
\]

\[
6a_2 \theta_1 + 6a_3 \theta_2 + 2a_{12} = \left( a_1 a_4 - a_2 a_3 \right) \Delta (\sigma_{3,A})^{-1/3} \lambda,
\]

on coefficients \((\theta_1, \theta_2)\) of the connection form and multiplier \(\lambda\).

The determinant of this system equals \(\Delta (\sigma_{3,A})\) and therefore the system has unique solution \((\theta_1, \theta_2, \lambda)\).

**Remark 20.** The similar result also valids for the Wagner connection and therefore there are two natural prolongations of the operators \(A \in \text{Diff}_3 (\xi)\) to the operators \(A^{(k)} \in \text{Diff}_3 (\xi^{\otimes k})\).

**7.3. Invariants of automorphism group.** By \(\pi^\xi : \text{Diff}_3 (\xi) \rightarrow M\) denote the bundle of linear differential operators, acting in the bundle \(\xi\) and having order 3.

Let

\[
\pi_\xi : J^k (\pi^\xi) \rightarrow M
\]

be the bundles of k-jets of sections of the bundle \(\pi^\xi\).

As above, denote by \(\mathcal{O} \subset J^2 (\pi^\xi)\) and \(\mathcal{O}^{(k)} \subset J^k (\pi^\xi), k \geq 4\), the domains where the jets of the leading symbol \(\sigma_{3,A} \in \Sigma_3, A \in \text{Diff}_3 (\xi)\), satisfy the same regularity conditions that in the scalar case, \(\xi = 1\).

As we have seen, differential operators with such regular symbols define:

1. the Chern or the Wagner connections on \(M\) and
2. the affine connection in the line bundle \(\xi\).

Therefore, the quantization, defining by these connections, allows us to split operator \(A\) into the sum

\[
A = \widehat{\sigma}_{3,A} + \widehat{\sigma}_{2,A} + \widehat{\sigma}_{1,A} + \widehat{\sigma}_{0,A},
\]

where \(\sigma_{i,A} \in \Sigma_i\).

Therefore, the operator \(A\) is defined by the total symbol \(\sigma_A = \oplus_{0 \leq i \leq 3} \sigma_{i,A}\) and the connection \(\nabla^A\) in the line bundle.

Remark, that all this construction and the splitting is defined by the second jet of the operator.

In other words, similar to the scalar case, we get a mapping

\[
\nu^\xi : J^2 (\pi^\xi) \supset \mathcal{O} \rightarrow \oplus_{0 \leq i \leq 3} \tau_i \oplus \Lambda^2 \tau^*,
\]

where the component in \(\Lambda^2 \tau^*\) is the curvature form \(\omega_A\) of the connection \(\nabla^A\).

Remark that mapping \(\nu^\xi\) is invariant of the action of the automorphism group \(\text{Aut}(\xi)\) on the left hand side and the induced action of the diffeomorphism group on the right hand side.

Therefore, we have \(\text{Aut}(\xi)\)-invariants of operators in terms of invariants of the total symbols and the curvature form.
If we consider now operators with $j_4 (\sigma_{3, A}) \in \mathcal{O}^{(4)}$ and take coordinates of the total symbol and the curvature form in the invariant coframe $\langle \theta, \theta' \rangle$ we get the basic $\text{Aut}(\xi)$-invariants $J_{\alpha,i}^\xi$ for the total symbol and $K$ for the curvature form. Let $\nu_{4}^{\xi} \colon J_{\alpha}^{4} (\pi^{\xi}) \to J_{\alpha}^{2} (\oplus_{0 \leq i \leq 3} \tau_i \oplus \Lambda^2 \tau^*)$, be the second prolongation of $\nu^\xi$.

Then the following theorem valids.

\textbf{Theorem 21.} The field of rational $\text{Aut}(\xi)$-invariants is generated by the basic invariants $(\nu_{4}^{\xi})^* (J_{\alpha,i}^\xi), (\nu_{4}^{\xi})^* (K)$ and the Tresse derivations $\delta_1, \delta_2$.

8. \textbf{Equivalence}

8.1. \textbf{Scalar operators.} In this section we consider the equivalence problem for scalar linear differential operators of the third order, $A \in \text{Diff}_3 (1)$.

We say that this operators are \textit{in general position} if for any point $a \in M$ there are two natural invariants, say $I_1, I_2$, such that their values $I_1 (A), I_2 (A)$ are independent in a neighborhood $U$ of the point, i.e. $dI_1 (A) \wedge dI_2 (A) \neq 0$.

We call functions $I_1 (A), I_2 (A)$ \textit{natural coordinates} in $U$.

We also call an atlas $\{ U_\alpha, \phi_\alpha : U_\alpha \to D_\alpha \subset \mathbb{R}^2 \}$ \textit{natural} if coordinates $\phi_\alpha = (I_1^\alpha (A), I_2^\alpha (A))$ are given by distinct natural invariants: $(I_1^\alpha (A), I_2^\alpha (A)) \neq (I_1^\beta (A), I_2^\beta (A))$, when $\alpha \neq \beta$.

We denote by $D_{\alpha \beta} = \phi_\alpha (U_\alpha \cap U_\beta)$ and will assume that domains $D_\alpha$ and $D_{\alpha \beta}$ are connected and simply connected.

Let $A_\alpha = \phi_\alpha^* (A|_{U_\alpha}), A_{\alpha \beta} = \phi_\alpha^* (A|_{U_\alpha \cap U_\beta})$ be the images of the operator $A$ in these coordinates. Then $\phi_{\alpha \beta}^* (A_{\alpha \beta}) = A_{\beta \alpha}$, where $\phi_{\alpha \beta} : D_{\alpha \beta} \to D_{\beta \alpha}$ are the transition mappings.

We call collection $(D_\alpha, D_{\alpha \beta}, \phi_{\alpha \beta}, A_\alpha, A_{\alpha \beta}) = \mathcal{D} (A)$ \textit{natural model or natural atlas for differential operator} $A$.

\textbf{Theorem 22.} Let $A, A' \in \text{Diff}_3 (1)$ be operators in general position. Then these operators are equivalent with respect to group of diffeomorphisms $G (M)$ if and only if the following conditions hold:

Open sets $U_\alpha' = (\phi_{\alpha}^{'-1}) (D_\alpha')$, where $\phi_{\alpha}' = (I_1^\alpha (A'), I_2^\alpha (A'))) : M \to \mathbb{R}^2$ constitute a natural atlas for
operator $A', \phi_{\alpha\beta}' = \phi_{\alpha\beta} : D_{\alpha\beta} \to D'_{\beta\alpha}$, and

$$A_{\alpha} = \phi'_{\alpha*} \left( A'_{\alpha|U'_\alpha} \right), A_{\alpha\beta} = \phi'_{\alpha*} \left( A'_{\alpha|U'_\alpha \cap U'_\beta} \right).$$

**Proof.** Indeed, any diffeomorphism $\psi : M \to M$ such that $\psi_* (A) = A'$ transform natural atlas to the natural one and because of $\psi^{-1} (I (A)) = I (\psi_* (A))$, for any natural invariant $I$, this diffeomorphism has the form of the identity map in the natural coordinates.

**Remark 23.** In natural coordinates operators $A_{\alpha}$ are defined by 10 functions.

### 8.2. Operators acting in line bundles

First of all we discuss the problem of lifting diffeomorphisms $\psi : M \to M$ to automorphisms $\psi : E (\xi) \to E (\xi)$ of line bundles.

**Lemma 24.** A diffeomorphism $\psi : M \to M$ admits a lifting to an automorphism $\tilde{\psi} : E (\xi) \to E (\xi)$ if and only if $\psi^* (w_1 (\xi)) = w_1 (\xi)$, where $w_1 (\xi) \in H^1 (M, \mathbb{Z}_2)$ is the first Stiefel-Whitney class of the bundle.

**Proof.** Remark that a real linear bundle $\xi$ is trivial if and only if $w_1 (\xi) \in H^1 (M, \mathbb{Z}_2)$ vanishes ([8]).

Therefore, in the case when $w_1 (\xi) = 0$ the statement of the lemma trivial.

Let now $w_1 (\xi) \neq 0$ and let $\psi^* (\xi)$ be the line bundle induced by a diffeomorphism $\psi$. Then, we have

$$w_1 (\xi^* \otimes \psi^* (\xi)) = w_1 (\xi^*) + w_1 (\psi^* (\xi)) = w_1 (\xi) + \psi^* (w_1 (\xi)) = 0,$$

if $\psi^* (w_1 (\xi)) = w_1 (\xi)$.

Therefore, any nowhere vanishing section of the bundle $\xi^* \otimes \psi^* (\xi)$ give us an isomorphism between bundle $\xi$ and $\psi^* (\xi)$ covering the identity map and then the lift of diffeomorphism $\psi$.

Similar to the scalar case we say that operator $A \in \text{Diff}_3 (\xi)$ is in general position if for any point $a \in M$ there are two $\text{Aut} (\xi)$-invariants, say $I_1^\xi, I_2^\xi$, such that $dI_1^\xi (A) \wedge dI_2^\xi (A) \neq 0$ in some neighborhood $U^\xi$ of the point.

As above we call functions $I_1^\xi (A), I_2^\xi (A)$ natural coordinates in $U$, and we also call an atlas $\{ U_0^\xi, \phi_0^\xi : U_0^\xi \to D_0^\xi \subset \mathbb{R}^2 \}$ natural if coordinates $\phi_0^\xi = \left( I_{a_1}^\xi (A), I_{a_2}^\xi (A) \right)$ are given by distinct natural invariants:

$$\left( I_{a_1}^\xi (A), I_{a_2}^\xi (A) \right) \neq \left( I_{b_1}^\xi (A), I_{b_2}^\xi (A) \right), \text{ when } \alpha \neq \beta.$$

We denote by $D_{\alpha \beta}^\xi = \phi_0^\xi (U_0^\xi \cap U_0^\xi)$ and assume that domains $D_{\alpha}^\xi$ and $D_{\alpha \beta}^\xi$ are connected and simply connected.
Let $\sigma_{\alpha a} = \phi_{a*}^\xi \left( \sigma_A|_{U_\alpha^\xi} \right)$, $\sigma_{\alpha a,\beta} = \phi_{a*}^\xi \left( \sigma_A|_{U_\alpha^\xi \cap U_\beta^\xi} \right)$, $\omega_{\alpha a} = \phi_{a*}^\xi \left( \omega_A \right)$ be the images of the total symbol $\sigma_A$ and the curvature form $\omega_A$ in these coordinates.

Then $\phi_{a\alpha*} \left( \sigma_{\alpha a,\beta} \right) = \sigma_{\beta a,\alpha}$ and $\phi_{a\alpha*} \left( \omega_{\alpha a,\beta} \right) = \omega_{\alpha a,\beta}$, where $\phi_{a\alpha} : D_{a,\alpha} \to D_{a,\beta}$ are the transition maps.

We call collection $(D_{a,\alpha}, D_{a,\beta}, \phi_{a,\alpha}, \sigma_{\alpha a}, \sigma_{\alpha a,\beta}, \omega_{\alpha a,\beta}) = D(\xi)$ the natural model or natural atlas for differential operator $A \in \text{Diff}_3(\xi)$.

Let $A' \in \text{Diff}_3(\xi)$ be another operator such that natural model $D(\xi)$ satisfies the conditions of the above theorem and therefore defines the natural model for operator $A'$.

Denote by $\psi_D : M \to M$ the diffeomorphism which equalizes total symbols and curvature forms: $(\psi_D)_* \left( \sigma_A \right) = \sigma_{A'}, (\psi_D)_* \left( \omega_A \right) = \omega_{A'}$.

Due to the above lemma there is a lifting $\psi_D \in \text{Aut}(\xi)$ of this diffeomorphism, if $\psi_D(w_1(\xi)) = w_1(\xi)$.

Let $A'' = \tilde{\psi}_D(A')$ be the image of the operator $A'$.

Then $\sigma_{A''} = \sigma_A$, $\omega_{A''} = \omega_A$, and therefore $d\psi_{A''} - d\psi_A = \theta \otimes \text{Id}$, where $\theta \in \Omega^1(M)$.

This form is closed $d\theta = 0$, because $\omega_{A''} = \omega_A$, and its cohomology class we denote by $\chi(A, A') \in H^1(M, \mathbb{R})$.

If this class is trivial, then $\theta = df$ for some function $f \in C^\infty(M)$, and therefore multiplication by function $\exp(f) \in \mathcal{F}(M)$ establishes the equivalence between operators $A$ and $A''$.

Summarizing, we get the following result.

**Theorem 25.** Let $A, A' \in \text{Diff}_3(\xi)$ be operators in general position. Then these operators are equivalent with respect to group of automorphisms $\text{Aut}(\xi)$ if and only if the following conditions hold:

1. **Open sets**
   $$U_{\alpha^\xi} = \left( \phi_{a*}^\xi \right)^{-1} \left( D_{a,\alpha}^\xi \right),$$
   where $\phi_{a*}^\xi = (I_{a,1}^\xi(A'), I_{a,2}^\xi(A')) : M \to \mathbb{R}^2$, constitute a natural atlas for operator $A'$, $\phi_{a*}^\xi = \phi_{a*}^\xi : D_{a,\alpha}^\xi \to D_{a,\beta}^\xi$, and
   $$\sigma_{\alpha a} = \phi_{a*}^\xi \left( \sigma_{A'}|_{U_{\alpha^\xi}} \right),$$
   $$\sigma_{\alpha a,\beta} = \phi_{a*}^\xi \left( \sigma_{A'}|_{U_{\alpha^\xi} \cap U_{\beta^\xi}} \right),$$
   $$\omega_{\alpha a} = \phi_{a*}^\xi \left( \omega_{A'}|_{U_{\alpha^\xi}} \right),$$
   $$\omega_{\alpha a,\beta} = \phi_{a*}^\xi \left( \omega_{A'}|_{U_{\alpha^\xi} \cap U_{\beta^\xi}} \right).$$

2. **The obstruction** $\chi(A, A') \in H^1(M, \mathbb{R})$ is trivial and $\psi_D(w_1(\xi)) = w_1(\xi)$. 
8.3. **Equivalence of differential equations of the 3rd order.** Let $A \in \text{Diff}_3(\xi)$ be an operator with the regular symbol. Then the operator

$$A_0 = \lambda(A)^{-\frac{2}{3}}A,$$

where $\lambda(A) = W_{\sigma_3,A}(\theta_A, \theta_A)$, $W_{\sigma_3,A}$ is the Wagner metric and $\theta_A$ is the first covector in the invariant coframe, we call a normalization of operator $A$.

It is easy to see that $W_{f\sigma} = f^{-\frac{2}{3}}W_{\sigma}$ and therefore the normalization of operator $B = fA$, $f \in \mathcal{F}(M)$, and the normalization of operator $A$ related as follows

$$B_0 = \text{sign}(f)A_0.$$ 

A linear differential equation of 3rd order $\mathcal{E}_A \subset J^3(\xi)$ is defined by a conformal class $\{fA\}$, where $f \in \mathcal{F}(M)$. Using the normalizations we get the following result.

**Theorem 26.** Linear differential equations of the 3rd order $\mathcal{E}_A \subset J^3(\xi)$ and $\mathcal{E}_B \subset J^3(\xi)$ are equivalent with respect to group of automorphisms $\text{Aut}(\xi)$ if and only if the normalization $A_0$ is equivalent to normalization $B_0$ or $-B_0$.

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