An energy-based discontinuous Galerkin method for dynamic Euler-Bernoulli beam equations

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Abstract

In this paper, an energy-based discontinuous Galerkin method for dynamic Euler-Bernoulli beam equations is developed. The resulting method is energy-dissipating or energy-conserving depending on the simple, mesh-independent choice of numerical fluxes. By introducing a velocity field, the original problem is transformed into a first-order in time system. In our formulation, the discontinuous Galerkin approximations for the original displacement field and the auxiliary velocity field are not restricted to be in the same space. In particular, a given accuracy can be achieved with the fewest degrees of freedom when the degree for the approximation space of the velocity field is two orders lower than the degree of approximation space for the displacement field. In addition, we establish the error estimates in an energy norm and demonstrate the corresponding optimal convergence in numerical experiments.

Keywords: Discontinuous Galerkin, Euler-Bernoulli beam equations, Upwind method
AMS subject: 65M06, 65M12

1 Introduction

The discontinuous Galerkin (DG) method is a class of finite element methods using a piecewise polynomial basis for both numerical solutions and test functions in the spatial variables. They have been proved to be very efficient when solving the initial-boundary value hyperbolic partial differential equations (PDE) in first-order Friedrichs form [15] since proposed in 1973 by Reed and Hill [19]. Because of their attractive properties, such as arbitrary high-order accuracy, local time evolution, element-wise conservation, geometrical flexibility, hp-adaptivity, etc., they have been widely used to solve the problems in many fields of science, engineering, and industry. For the details of the applications, we refer to [8, 9, 10] and the references therein.

However, the wave equations arising in physical theories are not only in first-order Friedrichs form. For the problems that involve high-order spatial derivatives, it is unclear that they can always be rewritten in Friedrichs form. Thus the methods which are able to deal with the high-order spatial derivative wave equations are needed. In the past few decades, the interior penalty discontinuous Galerkin (IPDG) methods, the symmetric interior penalty discontinuous Galerkin (SIPDG) methods [5, 21], the local discontinuous Galerkin (LDG) methods [24] have been used to solve the equations in high-order form. However, the stability of the IPDG and SIPDG methods depend on the penalty term, typically proportional to the jumps of the solution, which are mesh-dependent and order-dependent; the LDG methods introduce the first-order spatial derivatives as auxiliary variables, which already doubling/tripling/quadrupling the number of fields needed to be solved for a wave equation with the second/third/fourth-order spatial derivatives even in one dimension.

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In 2015, a new type of DG method – energy-based DG method was proposed by Appelö and Hagstrom [1] to solve a general form of second-order wave equations. The idea is to introduce the velocity as an auxiliary variable to reduce the second-order in time problem to the first-order in time system. Since the velocity is naturally connected to the kinetic energy and displacement, their formulation mimics the dynamics of the energy of the problem directly and the stability of the scheme only depends on the simple choices of the mesh-independent numerical fluxes which are based on the energy flux at element boundaries. In addition, since the auxiliary variable is the velocity – first-order derivative in time, only two fields needed to be solved. In the same year, they applied the method to solve the linear elastic wave equations [2]. Later, the authors in [12] establish a superconvergence rigorously for the scheme developed in [1] by using postprocessing techniques. The methods then are extended to the linear advective wave equation, where the energy is not restricted to be the sum of kinetic and potential energy [26]. In [3], a generalization of the energy-based DG methods to the semi-linear wave equations was demonstrated. For the recent work, a staggered grid together with energy-based DG methods [4] and the combination of Galerkin difference basis with energy-based DG method [25] are investigated to enlarge the stability region of the time integrator.

Fourth-order wave equations are widely used to describe the physical problems in science and engineering such as surface modeling, flexible body dynamics, propagation of shallow-water waves, surface diffusion of thin solid films, and vibrations of building and etc. [11], they have drawn a lot of attention from researchers. To simulate wave propagation problems, it is essential that the numerical methods can keep the accuracy over a long-time simulation, that is, they will not generate non-physical growth of the solution in time. In [17], a class of summation-by-parts methods is generated to solve the high-order wave equations (higher than second-order in the spatial variable). In [20, 23, 5, 24], a class of finite elements methods (FEM), such as spline-based FEM, IPDG, SIPDG, LDG, are constructed to solve the high-order wave equations. Recently, [16] proposed Galerkin difference methods for the simulation of high-order wave equations, [22] proposed a new discontinuous Galerkin formulation for the fourth-order wave equation. There they introduce second order spacial derivative of \( u \) as a new variable and rewrite the fourth-order in space problem into a second-order in space system.

In this work, we extend the energy-based DG methods to problems with fourth order spatial derivatives. In particular, we consider the dynamic Euler-Bernoulli beam equations (1) which admits the following properties:

- The stability of the variational formulation only depends on simply defined, mesh-independent numerical fluxes at the boundaries of elements,
- Only one auxiliary field, \( v = \frac{\partial u}{\partial t} \), is introduced,
- The DG approximations of \( u \) and \( v \) are not restricted to be in the same space.

The rest of the paper is organized as follows. We present the governing equations and the energy-based DG formulation for the problem in Section 2. We also establish the numerical fluxes at both inter-element boundaries and physical boundaries guaranteeing the stability of the proposed methods in Section 2.2. In Section 3, we derive error estimates in an energy norm for different numerical fluxes. Optimal convergence with alternating fluxes is presented in Section 3.1. Numerical experiments are presented in Section 4 to verify the theoretical convergence order in an energy norm. Last, we draw a conclusion and future work in Section 5.

## 2 Problem formulation

We consider the scalar dynamic Euler-Bernoulli beam equations describing the free vibrations of thin objects subject to a small deformation,

\[
\mu(x)u_{tt} = -(D(x)u_{xx})_{xx} + f(x,t), \quad x \in I = (a,b), \quad t \geq 0,
\]  

(1)
where \( u(x,t) \) is the displacement in the normal direction, \( \mu(x) = \rho(x)A(x) \) and \( \rho(x) > 0 \) is the mass per unit volume, \( A(x) > 0 \) is the cross-sectional area of the beam, \( D(x) > 0 \) is the flexural rigidity of the model, and \( f(x,t) \) is an external force loaded on the objects. The initial conditions are given by

\[
 u(x, 0) = g_1(x), \quad u_t(x, 0) = g_2(x), \quad x \in I = (a, b).
\]

Particularly, when the objects are homogeneous, \( \rho, A \) and \( D \) are constants, then (1) leads to the uniform Euler-Bernoulli beam model

\[
 \mu u_{tt} = -EIu_{xxxx} + f(x,t),
\]

where \( \mu = \rho A \) and \( D = EI \). Here, \( E \) is Young's modulus and \( I \) is the area moment of inertia;

To derive an energy-based DG formulation for the problem (1), we introduce an auxiliary scalar variable \( v = u_t \) to produce a first-order in time system

\[
 \begin{cases}
     u_t = v \\
     \mu v_t = -(Du_{xx})_{xx} + f(x,t).
\end{cases}
\]  

Here, for simplicity, we have dropped the dependence \( x \) in \( \mu \) and \( D \). The energy for (2) takes the form

\[
 E(t) = \frac{1}{2} \int_a^b \mu v^2 + D(u_{xx})^2 \, dx.
\]  

Denote \( \lbrack g(x) \rbrack_c \coloneqq g(d) - g(c) \) and \( \lbrack g(x) \rbrack_c \coloneqq g(c) \).

Then the change of energy is given by both the external force and boundary flux contributions

\[
 \frac{dE}{dt} = \int_a^b f v \, dx + \lbrack -v(Du_{xx})_x + Du_{xx}v_x \rbrack_a^b.
\]

2.1 Semi-discrete DG formulation

We develop an energy-based DG scheme for problem (1) through the reformulation (2). Let the domain \( I \) be discretized by non-overlapping elements \( I_j \) with

\[
 I = \bigcup_j I_j = \bigcup_j (x_{j-1}, x_j), \quad j = 1, \cdots, n,
\]

and

\[
 x_0 = a, \quad x_n = b, \quad h_j = x_j - x_{j-1}, \quad h = \max_j h_j
\]

with mesh regularity requirement \( \frac{h}{\min h_j} < \sigma, \sigma \) is fixed during mesh refinement. Then on each element \( I_j \), we approximate \((u,v)\) by \((u^h,v^h)\), where \( u^h \) and \( v^h \) belong to the piecewise polynomial spaces of degree \( q \) and \( s \),

\[
 U^h_q = \{ u^h(x,t), u^h(x,t) \in P^q(I_j), x \in I_j, t \geq 0 \},
\]

\[
 V^h_s = \{ v^h(x,t), v^h(x,t) \in P^s(I_j), x \in I_j, t \geq 0 \},
\]

respectively. We now seek an approximation to the first-order in time system (2). To this end, on each element \( I_j \), we first consider a discrete energy which is analogous to (3),

\[
 E^h_j(t) = \frac{1}{2} \int_{I_j} \mu(v^h)^2 + D \left( u^h_{xx} \right)^2 \, dx,
\]
and its corresponding time derivative

\[
\frac{dE^h_j(t)}{dt} = \int_{I_j} \mu v^h_t v^h + D u^h_{xx} u^h_{xt} \, dx. \tag{5}
\]

Then to obtain a weak form which is compatible with the discrete energy (4) and (5), we choose \(\phi \in U^h_{n}, \psi \in V^h_{n}\) and test the first equation in (2) with \((D\phi_{xx})_{xx}\), the second equation in (2) with \(\psi\), and add boundary integrals which vanish for the continuous problem. We then obtain the following equations,

\[
\int_{I_j} (D\phi_{xx})_{xx} \left(u^h_t - v^h\right) \, dx = \left[(D\phi_{xx})_x \left(u^h_t - v^h\right) + D\phi_{xx} \left((v_x)^* - u^h_{tx}\right)\right]_{x_{j-1}}^{x_j},
\]

and

\[
\int_{I_j} \mu \psi v^h_t + \psi(Du^h_{xx})_{xx} \, dx = \int_{I_j} \psi f \, dx + \left[\psi \left((Du^h_{xx})_x - ((Du_{xx})_x)^*\right) + \psi_x \left((Du_{xx})^* - Du^h_{xx}\right)\right]_{x_{j-1}}^{x_j},
\]

where \(v^*, (v_x)^*, (Du_{xx})^*\) and \(((Du_{xx})_x)^*\) are numerical fluxes on both interelement and physical boundaries. We then apply integration by parts to get the following alternative formulations,

\[
\int_{I_j} D\phi_{xx} (u^h_t - v^h)_{xx} \, dx = \left[(D\phi_{xx})_x (v^h - v^*) + D\phi_{xx} ((v_x)^* - v^h)\right]_{x_{j-1}}^{x_j}, \tag{6}
\]

and

\[
\int_{I_j} \mu \psi^h_t + D\psi_{xx} u^h_{xx} \, dx = \int_{I_j} \psi f \, dx + \left[-\psi((Du_{xx})_x)^* + \psi_x(Du_{xx})^*\right]_{x_{j-1}}^{x_j}. \tag{7}
\]

Last, we must supplement (6) with two extra equations to uniquely solve the problem: one determines the mean value of \(u^h_t\); the other determines the mean value of \(u^h_{tx}\). Precisely, for an arbitrary constant \(\tilde{\phi} \in P^0(I_j)\) we enforce

\[
\int_{I_j} \tilde{\phi} (u^h_t - v^h) \, dx = 0, \tag{8}
\]

and for any \(\tilde{\phi} \in P^1(I_j)\) we require

\[
\int_{I_j} \tilde{\phi}_x (u^h_t - v^h) \, dx = 0. \tag{9}
\]

Note that the equations (8) and (9) do not change the numerical energy \(E^h_j\).

Denoting \(\Phi = (\phi, \psi, \tilde{\phi}, \tilde{\phi})\) and \(U^h = (u^h, v^h)\), we arrive at the final form

\[
\mathcal{B}(\Phi, U^h) = \sum_{j=1}^n \int_{I_j} \left(D\phi_{xx} \frac{\partial^2}{\partial x^2} + \tilde{\phi} + \tilde{\phi}_x \frac{\partial}{\partial x}\right) \left(u^h_t - v^h\right) + \mu \psi v^h_t
\]

\[
+ D\psi_{xx} u^h_{xx} \, dx - \sum_{j=1}^n \left[\left((D\phi_{xx})_x (v^h - v^*) + D\phi_{xx} ((v_x)^* - v^h)\right)\right]_{x_{j-1}}^{x_j}
\]

\[
- \sum_{j=1}^n \left[-\psi((Du_{xx})_x)^* + \psi_x(Du_{xx})^*\right]_{x_{j-1}}^{x_j}.
\]

Denote by \(\mathcal{N}\) the space of functions belonging to \(P^1(I_j)\), we then have the semi-discrete problem
Problem 1. Find $U = (u^h, v^h) \in \mathcal{P}^q(I_j) \times \mathcal{P}^s(I_j)$ such that for all $\Phi \in \mathcal{P}^q(I_j) \times \mathcal{P}^s(I_j) \times \mathcal{N}$,

$$B(\Phi, U) = \sum_{j=1}^n \int_{I_j} \psi f \, dx.$$ 

We have the following result.

**Theorem 1.** Let $U = (u^h, v^h)$ and the fluxes $v^\ast, (v_x)^\ast, ((Du_{xx})_x)^\ast$ and $(Du_{xx})^\ast$ be given. Then $\frac{dU}{dt}$ is uniquely determined, and the energy identity

$$E_t^h = \sum_{j=1}^n \frac{dE_j}{dt} = \sum_{j=1}^n \int_{I_j} v^h f \, dx + \sum_{j=1}^n \left[ (Du_{xx})_x (v^h - v^\ast) - v^h ((Du_{xx})_x)^\ast + Du_{xx} (v_x)^\ast - v_x^h + v_x^h (Du_{xx})^\ast \right]_{x_{j-1}}^{x_j}$$

holds.

**Proof.** The system on the element $I_j$ is linear in the time derivatives, and the mass matrix of $v_t^h$ is nonsingular. The number of linear equations for $u_t^h$, which equals the number of independent equations in (6) plus the equations in (8) and (9), matches the dimensionality of $\mathcal{P}^q(I_j)$. If the data $v^h, v^h_x, v^\ast, (v_x)^\ast$ vanish in (6), we must have $u_t^h = 0$, and so the linear system is invertible. By setting $\Phi = (U, \phi, \tilde{\phi})$, $\phi \in \mathcal{P}^0(I_j), \tilde{\phi} \in \mathcal{P}^1(I_j)$ in Problem 1, we obtain (10) directly. \hfill \Box

### 2.2 Fluxes

To complete the energy-based DG formulations proposed in Section 2.1, we also need to specify the numerical fluxes $v^\ast, ((Du_{xx})_x)^\ast, (v_x)^\ast, (Du_{xx})^\ast$ both at interelement and physical boundaries. Denote

$$\eta^\pm(x) := \lim_{\varepsilon \to 0^\pm} \eta(x + \varepsilon),$$

then introduce the common notations for averages and jumps,

$$\{\{v^h\}\} = \frac{v^h_+ + v^h_-}{2}, \quad [[v^h]] = v^h_- - v^h_+,$$

$$\{\{v^h_x\}\} = \frac{v^h_+ + v^h_-}{2}, \quad [[v^h_x]] = v^h_x^+ - v^h_x^-,$$

$$\{\{Du_{xx}^h\}\} = \frac{D^- u_{xx}^h - D^+ u_{xx}^h}{2}, \quad [[[Du_{xx}^h]]] = D^- u_{xx}^h - D^+ u_{xx}^h,$$

$$\{\{(Du_{xx}^h)_x\}\} = \frac{(D^- u_{xx}^h)_x + (D^+ u_{xx}^h)_x}{2}, \quad [[[Du_{xx}^h]_x]] = (D^- u_{xx}^h)_x - (D^+ u_{xx}^h)_x.$$

#### 2.2.1 Interelement boundaries

We first consider the net contribution to the discrete energy derivative $E_t^h$ from the interelement numerical fluxes

$$\sum_{j=1}^{n-1} [J^h]_{x_j},$$

where

$$J^h = \left( (D^- u_{xx}^h)_x (v^h_- - v^\ast) - v^h_- ((Du_{xx})_x)^\ast \right.$$

$$+ D^- v_{xx}^h (v_x)^\ast - v_x^h + v_x^h (Du_{xx})^\ast - \left( (D^+ u_{xx}^h)_x (v^h_+ - v^\ast) - v^h_+ ((Du_{xx})_x)^\ast + D^+ v_{xx}^h (v_x)^\ast - v_x^h + v_x^h (Du_{xx})^\ast \right).$$

(11)
To develop an energy stable scheme, we must choose numerical fluxes such that \( J^h \leq 0 \). In particular, \( J^h < 0 \) leads to a dissipating scheme and \( J^h = 0 \) yields a conserving scheme. Precisely, we introduce the following numerical fluxes:

\[
\begin{align*}
(v^*)^h &= \alpha_1 v^h - (1 - \alpha_1) v^h + \beta_1 (D^- u^h_x)_x - (D^+ u^h_x)_x, \\
((Du)_{xx})^h &= (1 - \alpha_1)(D^- u^h_x)_x + \alpha_1 (D^+ u^h_x)_x + \tau_1 (v^h - v^h), \\
(v_x)^h &= \alpha_2 v^h + (1 - \alpha_2) (D^- u^h_x)_x - \beta_2 (D^- u^h_x)_x - (D^+ u^h_x)_x, \\
(Du_{xx})^h &= (1 - \alpha_2)D^- u^h_x + \alpha_2 D^+ u^h_x - \tau_2 (v^h - v^h),
\end{align*}
\]

where \( 0 \leq \alpha_1, \alpha_2 \leq 1, \beta_1, \beta_2, \tau_1, \tau_2 \geq 0 \). Note that, to match the dimensionality, we also require:

(i) both \( \alpha_1 \) and \( \alpha_2 \) are dimensionless; (ii) \( \beta_1 \) has the same unit as \( L^3/(TD) \), where \( L \) represents length, \( T \) represents time; (iii) \( \tau_1 \) has the same unit as \( 1/T \); (iv) \( \beta_2 \) has the same unit as \( L/(TD) \); (v) \( \tau_2 \) has the same unit as \( 1/\beta_2 \). Plugging (12) into (11), we have

\[ J^h = -\tau_1[(v^h)^2 - \beta_1([(Du_{xx})^h_x]^2) - \beta_2[[Du_{xx}^h]]^2 - \tau_2[[v^h_x]^2]. \]

In particular, when

\[ \alpha_1 = \alpha_2 = 0.5, \quad \beta_1 = \beta_2 = \tau_1 = \tau_2 = 0, \]

we have so-called central flux,

\[ v^* = \{(v^h), ((Du_{xx})^h_x)\} = \{(Du_{xx})^h_x\}, (v_x)^h = \{v^h_x\}, (Du_{xx})^h = \{Du_{xx}^h\} \]

which gives an energy conserving scheme with \( J^h = 0 \); when \( \beta_1 = \beta_2 = \tau_1 = \tau_2 = 0 \) and \( \alpha_1, \alpha_2 \) belongs to the following cases:

(a) \( \alpha_1 = 0, \alpha_2 = 0 \); (b) \( \alpha_1 = 0, \alpha_2 = 1 \); (c) \( \alpha_1 = 1, \alpha_2 = 0 \); (d) \( \alpha_1 = 1, \alpha_2 = 1 \),

we have alternating flux which also gives an energy conserving scheme with \( J^h = 0 \). Finally, when \( 0 \leq \alpha_1, \alpha_2 \leq 1 \) and at least one of \( \{\beta_1, \beta_2, \tau_1, \tau_2\} \) larger than 0, we have an upwind flux which leads to \( J^h < 0 \), which is an energy dissipating scheme.

### 2.2.2 Physical boundaries

In this subsection, we focus on the approximation of physical boundary conditions. Particularly, we consider the following four boundary conditions:

(i). Simply supported boundary conditions: \( Du_{xx} = 0, u = 0 \);

(ii). Free (natural) boundary conditions: \( Du_{xx} = 0, (Du_{xx})_x = 0 \);

(iii). Clamped boundary conditions: \( u = 0, u_x = 0 \);

(iv). Sliding boundary conditions: \( u_x = 0, (Du_{xx})_x = 0 \).

With these physical boundary conditions, a complete energy identity can be obtained. In particular, the energy flux through the physical boundary is zero since

\[
\frac{dE}{dt}\bigg|_{\partial I} = \left[-v(Du_{xx})_x + Du_{xx}v_x\right]_{x_0}^{x_n} = 0.
\]

For the simplicity of analysis, we introduce a general notation for the above four different conditions,

\[
\begin{align*}
-a_1 v_x + b_1 Du_{xx} &= 0, \quad x = x_0, \\
a_2 v + b_2 (Du_{xx})_x &= 0, \quad x = x_0, \\
a_1 v_x + b_1 Du_{xx} &= 0, \quad x = x_n, \\
a_2 v - b_2 (Du_{xx})_x &= 0, \quad x = x_n,
\end{align*}
\]
where \( a_1 = b_2 = 0, b_1 = a_2 = 1 \) corresponds to the condition (i); \( a_1 = a_2 = 0, b_1 = b_2 = 1 \) corresponds to the condition (ii); \( b_1 = b_2 = 0, a_1 = a_2 = 1 \) corresponds to the condition (iii); \( b_1 = a_2 = 0, a_1 = b_2 = 1 \) corresponds to the condition (iv).

To approximate the physical boundary conditions we choose \( v^*, ((Du_{xx})_x)^*, (v_x)^*, (Du_{xx})^* \) to be consistent with (13) as

\[
\begin{align*}
-a_1(v_x)^* + b_1(Du_{xx})^* &= 0, \quad x = x_0, \\
a_2v^* + b_2((Du_{xx})_x)^* &= 0, \quad x = x_0, \\
a_1(v_x)^* + b_1(Du_{xx})^* &= 0, \quad x = x_n, \\
a_2v^* - b_2((Du_{xx})_x)^* &= 0, \quad x = x_n.
\end{align*}
\] (14)

We first denote

\[
\zeta_1 := \begin{cases} -a_1v_x^h + b_1Du_{xx}^h, & x = x_0, \\
a_1v_x^h + b_1Du_{xx}^h, & x = x_n, \end{cases} \text{ and } \zeta_2 := \begin{cases} a_2v_x^h + b_2(Du_{xx}^h)_x, & x = x_0, \\
a_2v_x^h - b_2(Du_{xx}^h)_x, & x = x_n. \end{cases}
\] (15)

Then solving (14), we find a one parameter family of consistent choices:

\[
\begin{align*}
(v_x)^* &= v_x^h + (a_1 - \eta_1 b_1)\zeta_1 \\
(Du_{xx})^* &= Du_{xx}^h - (b_1 + \eta_1 a_1)\zeta_1 \\
v^* &= v_x^h - (a_2 - \eta_2 b_2)\zeta_2 \\
((Du_{xx})_x)^* &= (Du_{xx}^h)_x - (b_2 + \eta_2 a_2)\zeta_2
\end{align*}
\] (16)

at \( x = x_0 \) and

\[
\begin{align*}
(v_x)^* &= v_x^h - (a_1 - \eta_1 b_1)\zeta_1 \\
(Du_{xx})^* &= Du_{xx}^h - (b_1 + \eta_1 a_1)\zeta_1 \\
v^* &= v_x^h - (a_2 - \eta_2 b_2)\zeta_2 \\
((Du_{xx})_x)^* &= (Du_{xx}^h)_x + (b_2 + \eta_2 a_2)\zeta_2
\end{align*}
\] (17)

at \( x = x_n \). Here \( \eta_1, \eta_2 \) in (16) and (17) are arbitrary constants. Then the contribution to the discrete energy from the physical boundaries is given by

\[
\begin{align*}
E^h_{\partial I} &= \left[ -(v^* - v_x^h)(Du_{xx}^h)_x - v_x^h((Du_{xx}^h)_x)^* + Du_{xx}^h ((v_x)^* - v_x^h) \right]_{x_0}^{x_n} \\
&\quad + v_x^h(Du_{xx}^h)_x \Big|_{x_0}^{x_n} + \frac{\partial}{\partial x}\left((v_x)^* - v_x^h\right)_{x_0}^{x_n} \\
&\quad + \left[ -Du_{xx}^h(a_1 - \eta_1 b_1)\zeta_1 + ((v_x)^* + (a_1 - \eta_1 b_1)\zeta_1)(Du_{xx})^* \right]_{x_0}^{x_n} \\
&\quad - \left[ Du_{xx}^h(a_1 - \eta_1 b_1)\zeta_1 + ((v_x)^* - (a_1 - \eta_1 b_1)\zeta_1)(Du_{xx})^* \right]_{x_0}^{x_n} \\
&\quad \quad =: \left[ \sigma_1(x) \right]_{x_0}^{x_n} + \left[ \sigma_2(x) \right]_{x_0}^{x_n},
\end{align*}
\] (18)

where

\[
\begin{align*}
\sigma_1(x) &= (a_2 - \eta_2 b_2)\zeta_2(Du_{xx}^h)_x - (a_2 - \eta_2 b_2)\zeta_2((Du_{xx})_x)^*, \\
\sigma_2(x) &= -Du_{xx}^h(a_1 - \eta_1 b_1)\zeta_1 + (a_1 - \eta_1 b_1)\zeta_1(Du_{xx})^*.
\end{align*}
\]

Then simplify (18) by continuing to use (16) and (17). We obtain

\[
\begin{align*}
E^h_{\partial I} &= -\left[ \zeta_2^2\eta_2(a_2^2 - b_2^2) + \zeta_1^2\eta_1(a_1^2 - b_1^2) \right]_{x_0}^{x_n} - \left[ \zeta_2^2\eta_2(a_2^2 - b_2^2) + \zeta_1^2\eta_1(a_1^2 - b_1^2) \right]_{x_0}^{x_n},
\end{align*}
\] (19)

where we have used the fact \( a_1b_1 = 0 \) and \( a_2b_2 = 0 \). Then (19) yields a nonincreasing contribution to the energy if

\[
\eta_1(a_1^2 - b_1^2) \geq 0 \quad \text{and} \quad \eta_2(a_2^2 - b_2^2) \geq 0.
\]

Now, we are ready to establish the stability of the proposed energy-based DG scheme.
Theorem 2. The discrete energy $E^h(t) = \sum_j E^h_j(t)$ with $E^h_j(t)$ defined in (4) satisfies

$$E^h_j = \sum_{j=1}^n \int_{I_j} v^h f \, dx - \sum_{j=1}^{n-1} \left[ \tau_1 [(u^h)]^2 + \beta_1 [[(D u^h)_{xx}]_x]^2 + \beta_2 [[D u^h_{xx}]_x]^2 + \tau_2 [[v^h]]^2 \right]_{x_j}$$

$$- \left[ \zeta_2 \eta_2 (a^2 - b^2_2) + \zeta_1 \eta_1 (a^2_1 - b^2_1) \right]_{x_n} - \left[ \zeta_2 \eta_2 (a^2 - b^2_2) + \zeta_1 \eta_1 (a^2_1 - b^2_1) \right]_{x_0},$$

where $\zeta_1, \zeta_2$ are defined in (15) and

$$\begin{align*}
\eta_1 &\leq 0, \eta_2 \geq 0, \text{ simply supported BC: } a_1 = b_2 = 0, b_1 = a_2 = 1, \\
\eta_1 &\leq 0, \eta_2 \leq 0, \text{ free BC: } a_1 = a_2 = 0, b_1 = b_2 = 1, \\
\eta_1 &\geq 0, \eta_2 \geq 0, \text{ clamped BC: } b_1 = b_2 = 0, a_1 = a_2 = 1, \\
\eta_1 &\geq 0, \eta_2 \leq 0, \text{ sliding BC: } b_1 = a_2 = 0, a_1 = b_2 = 1.
\end{align*}$$

If the flux parameters $\tau_1, \beta_1, \beta_2$ are nonnegative and the external forcing $f = 0$, then

$$E^h_t \leq 0.$$

3 Error estimate

To analyze the numerical error of the scheme, we define the errors by

$$e^u = u - u^h, \quad e^v = v - v^h,$$

and compare $(u^h, v^h)$ with arbitrary functions $(\tilde{u}^h, \tilde{v}^h) \in U_h^q \times V_h^s$, $q - 4 \leq s \leq q$. To proceed, we denote the differences

$$\tilde{e}^u = \tilde{u}^h - u^h, \quad \tilde{e}^v = \tilde{v}^h - v^h, \quad \delta^u = \tilde{u}^h - u, \quad \delta^v = \tilde{v}^h - v,$$

and define the numerical error energy

$$\mathcal{E}^h = \frac{1}{2} \sum_{j=1}^n \int_{I_j} \mu (\tilde{e}^v)^2 + D (\tilde{e}^u_{xx})^2 \, dx.$$  (22)

Since both continuous solution $(u, v)$ and DG numerical solution $(u^h, v^h)$ satisfy semidiscrete schemes (6)–(7), the error $(e^u, e^v)$ satisfies the following equations

$$\int_{I_j} D \phi_{xx} (e^u - e^v)_{xx} \, dx = \left[ (D \phi_{xx})_x (e^v - e^{v, \ast}) + D \phi_{xx} (e^{v, \ast} - e^v) \right]_{x_j}^{x_{j-1}},$$

and

$$\int_{I_j} \mu \psi (e^v)_t + D \psi_{xx} e^u_{xx} \, dx = \left[ - \psi (D e^u_{xx})_x \ast + \psi_x (D e^u_{xx}) \ast \right]_{x_j}^{x_{j-1}}.$$  (24)

Then choosing $(\phi, \psi) = (\tilde{e}^u, \tilde{e}^v)$, using the relations $e^v = \tilde{e}^u - \delta^u$, $e^v = \tilde{e}^v - \delta^v$, and summing (23) and (24), we have

$$\sum_{j=1}^n \int_{I_j} D \tilde{e}^u_{xx} \tilde{e}^u_{xx} + \mu \tilde{e}^v \tilde{e}^v \, dx = \sum_{j=1}^n \int_{I_j} D \tilde{e}^u_{xx} (\delta^u - \delta^v)_{xx} + \mu \tilde{e}^v \delta^v + D \tilde{e}^x_{xx} \delta^x_{xx} \, dx$$

$$+ \sum_{j=1}^n \left[ (D \tilde{e}^u_{xx})_x (e^v - e^{v, \ast}) + D \tilde{e}^u_{xx} (e^v - e^v) - \tilde{e}^v ((D e^u_{xx})_x \ast + \tilde{e}^v (D e^u_{xx})_x \ast \right]_{x_j}^{x_{j-1}}.$$  (25)
Now, following the same steps as in the derivation of (20) in Theorem 2, we obtain
\[
\frac{d\bar{\epsilon}^h}{dt} = A^h - \sum_{j=1}^{n-1} \left[ \tau_1 [\bar{\epsilon}^v]^2 + \beta_1 [(D\bar{\epsilon}^u_{x,x})_x]^2 + \tau_2 [\bar{\epsilon}^v]^2 + \beta_2 [(D\bar{\epsilon}^u_{x,x})]^2 \right]_{x_j} \\
- \left[ \eta_2 (a_2^2 - b_2^2)(a_2 \bar{\epsilon}^v - b_2 (D\bar{\epsilon}^u_{x,x})_x)^2 + \eta_1 (a_1^2 - b_1^2)(a_1 \bar{\epsilon}^v + b_1 D\bar{\epsilon}^u_{x,x}) \right]_{x_n} \\
- \left[ \eta_2 (a_2^2 - b_2^2)(a_2 \bar{\epsilon}^v + b_2 (D\bar{\epsilon}^u_{x,x})_x)^2 + \eta_1 (a_1^2 - b_1^2)(-a_1 \bar{\epsilon}^v + b_1 D\bar{\epsilon}^u_{x,x}) \right]_{x_0} \tag{25}
\]
where
\[
A^h = \sum_{j=1}^{n} \int_{I_j} D\bar{\epsilon}^u_{x,x} (\delta^v - \bar{\delta}^v)_{x,x} + \mu \bar{\epsilon}^v \delta^v_x + D\bar{\epsilon}^u_{x,x} \delta^v_x \, dx \\
- \sum_{j=1}^{n} \left[ (D\bar{\epsilon}^u_{x,x})_x (\delta^v - \bar{\delta}^v) + D\bar{\epsilon}^u_{x,x} (\bar{\delta}^v_x - \delta^v_x) - \bar{\epsilon}^v ((D\delta^u_{x,x})_x)^* + \bar{\epsilon}^v (D\delta^u_{x,x})_x^* \right]_{x_{j-1}} \tag{26}
\]
Here, the fluxes \(\bar{\delta}^v, \bar{\delta}^v_x, ((D\delta^u_{x,x})_x)^*, (D\delta^u_{x,x})_x^*\) are built from \(\delta^v, (\delta^v_x), (D\delta^u_{x,x})_x, D\delta^u_{x,x}\) with the same specification as in Section 2.2. To achieve an acceptable error estimate, we must choose suitable \((\bar{u}^h, \bar{v}^h)\). Note that in what follows we will assume for simplicity that \((u^h, v^h) = (\bar{u}^h, \bar{v}^h)\) at \(t = 0\), though we do not satisfy this condition in the numerical experiments. Particularly, on each \(I_j\), we impose for all times \(t\) and all test functions \((\phi, \psi, \phi, \psi)\) with \((\phi, \psi) \in \mathcal{P}^q(I_j) \times \mathcal{P}^q(I_j), \phi \in \mathcal{P}^0(I_j)\) and \(\tilde{\phi} \in \mathcal{P}^1(I_j)\),
\[
\int_{I_j} D\phi_x \delta^u_{x,x} \, dx = \int_{I_j} D\psi \delta^v \, dx = \int_{I_j} \tilde{\phi} \delta^v \, dx = \int_{I_j} \tilde{\phi}_x \delta^v_x \, dx = 0, \tag{27}
\]
where we use the \(L^2\) projection of \(v\) and a projection of \(u\) in the \(H^2\) seminorm. The solvability of both the \(H^2\) projection equation for \(\bar{u}^h\) and the \(L^2\) projection equation for \(\bar{v}^h\) follows from counting and uniqueness arguments which is the same as the arguments in Theorem 1.

To use (27), we rewrite \(A^h\) in (26) through integration by parts to get
\[
A^h = \sum_{j=1}^{n} \int_{I_j} D\bar{\epsilon}^u_{x,x} \delta^v_{x,x} - D_{x,x} \bar{\epsilon}^u_{x,x} \delta^v_{x,x} - D_{x,x,x} \bar{\epsilon}^u_{x,x} \delta^v_{x,x} - 2D_{x} \bar{\epsilon}^u_{x,x} \delta^v_{x} + \mu \bar{\epsilon}^v \delta^v_x \\
+ D\bar{\epsilon}^u_{x,x} \delta^v_x \, dx - \sum_{j=1}^{n} \left[ - (D\bar{\epsilon}^u_{x,x})_x \delta^v_x + D\bar{\epsilon}^u_{x,x} \delta^v_x - \bar{\epsilon}^v ((D\delta^u_{x,x})_x)^* + \bar{\epsilon}^v (D\delta^u_{x,x})^*_x \right]_{x_{j-1}} \tag{28}
\]
Since \(q - 4 \leq s \leq q\), the volume integral terms containing \(\delta^u_{x,x}, \delta^u_{x,x,x}\) and \(\delta^u_{x,x}\) in \(A^h\) vanish. Then combining the contributions from the neighboring elements we obtain
\[
A^h = \sum_{j=1}^{n} \int_{I_j} -D_{x,x} \bar{\epsilon}^u_{x,x} \delta^v_{x} - 2D_{x} \bar{\epsilon}^u_{x,x} \delta^v_{x} + \mu \bar{\epsilon}^v \delta^v_x \, dx \\
- \sum_{j=1}^{n-1} \left[ - [(D\bar{\epsilon}^u_{x,x})_x] \delta^v_x + [(D\bar{\epsilon}^u_{x,x})] \delta^v_x - [(\bar{\epsilon}^v)]((D\delta^u_{x,x})_x)^* + [(\bar{\epsilon}^v)](D\delta^u_{x,x})^*_x \right]_{x_j} \\
- \left[ - (D\bar{\epsilon}^u_{x,x})_x \delta^v_x + D\bar{\epsilon}^u_{x,x} \delta^v_x - \bar{\epsilon}^v ((D\delta^u_{x,x})_x)^* + \bar{\epsilon}^v (D\delta^u_{x,x})^*_x \right]_{x_0}. \tag{29}
\]
In what follows, \(C\) will be a constant independent of the element diameter \(h\). It may differ from line to line. We denote Sobolev norms by \(\| \cdot \|\) and the associated seminorms by \(| \cdot |\). Here we will assume the solution is sufficiently smooth up to some time, \(T\). We then have the following error estimate.
Theorem 3. Suppose $\mu(x), D(x), D_x(x), D_{xx}(x)$ are bounded. Let $\bar{q} = \min(q - 3, s - 1), q - 4 \leq s \leq q$. Then there exist numbers $C_0, C_1$ depending only on $s, q$, but independent of $h$, such that for smooth solutions $u, v$ and time $0 \leq t \leq T$

$$\|e^u\|^2_{L^2(I)} + \|D_e^u\|^2_{L^2(I)} \leq CT^2 \max_{t \leq T} \left[ h^\gamma \left( |u(\cdot, t)|^2_{H^{q-1}(I)} + |v(\cdot, t)|^2_{H^{q-1}(I)} \right) + h^{2(s+1)} |v(\cdot, t)|^2_{H^{q-1}(I)} \right], \quad (29)$$

where

$$\gamma = \begin{cases} \bar{q}, & \beta_1, \tau_1, \beta_2, \tau_2 \geq 0, \ \eta_1, \eta_2 \text{ satisfy (21),} \\ \bar{q} + \frac{1}{2}, & \beta_1, \tau_1, \beta_2, \tau_2 > 0, \ \eta_1, \eta_2 \text{ satisfy (21), } \eta_1 \eta_2 \neq 0. \end{cases}$$

Proof. From the basic Bramble-Hilbert lemma [7], we have

$$\begin{cases} \|D_e^u\|^2_{L^2(\partial I)} \leq C h^{-1} \|D_e^u\|^2_{L^2(I)}, \quad \|D_e^{\delta^u}\|^2_{L^2(\partial I)} \leq C h^{2q-3} |u(\cdot, t)|^2_{H^{q-1}(I)}, \\ \|D_e^{\delta^v}\|^2_{L^2(\partial I)} \leq C h^{2q-1} |v(\cdot, t)|^2_{H^{q-1}(I)}, \quad \|D_e^{\delta^v}\|^2_{L^2(\partial I)} \leq C h^{2q-3} |D_e^u|^2_{L^2(I)}, \quad (30) \\
\|D_e^{\delta^v}\|^2_{L^2(\partial I)} \leq C h^{2q-1} \|v(\cdot, t)|^2_{H^{q-1}(I)}, 
\|D_e^{\delta^v}\|^2_{L^2(\partial I)} \leq C h^{2q-1} \|D_e^u|^2_{L^2(I)}, 
\|D_e^{\delta^v}\|^2_{L^2(\partial I)} \leq C h^{2q-1} |u(\cdot, t)|^2_{H^{q-1}(I)}, \quad \|D_e^{\delta^v}\|^2_{L^2(\partial I)} \leq C h^{2q-3} |v(\cdot, t)|^2_{H^{q-1}(I)}, \quad (31) \end{cases}$$

Let’s first consider the case where $\beta_1, \tau_1, \beta_2, \tau_2 \geq 0$ and $\eta_1, \eta_2$ satisfy (21): applying the same process that leads to (20) in Theorem 2, we have the time derivative of the error energy (22) satisfies

$$\frac{dE^h}{dt} \leq A^h.$$ 

To bound $A^h$, we use the Cauchy-Schwartz inequality to get

$$A^h \leq C \sum_j \|\tilde{e}_x^u\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)} + \|\tilde{e}_x^u\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)} + \|\tilde{e}_x^v\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)} + \|\tilde{e}_x^v\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)}$$

$$+ \|\tilde{e}_x^v\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)} + \|\tilde{e}_x^v\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)} + \|\tilde{e}_x^v\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)} + \|\tilde{e}_x^v\|_{L^2(I_j)} \|\delta^x\|_{L^2(I_j)},$$

then combining with (30)–(31), we obtain

$$A^h \leq C \sqrt{E} \left( h^{s-1} |v(\cdot, t)|_{H^{q-1}(I)} + h^{s-1} |v(\cdot, t)|_{H^{q-1}(I)} + h^{q-3} |u(\cdot, t)|_{H^{q-1}(I)} \right).$$

Then a direct integration in time combined with the assumption that $(\tilde{e}^u, \tilde{e}^v) = (0, 0)$ at $t = 0$ yields

$$E^h(T) \leq CT^2 \max_{t \leq T} \left[ h^\gamma \left( |v(\cdot, t)|^2_{H^{q-1}(I)} + |u(\cdot, t)|^2_{H^{q-1}(I)} \right) + h^{2(s+1)} |v(\cdot, t)|^2_{H^{q-1}(I)} \right].$$

For dissipative fluxes, $\beta_1, \beta_2, \tau_1, \tau_2 > 0$ and $\tau_1, \tau_2$ satisfy (21) with $\eta_1 \eta_2 \neq 0$, we can improve the estimate above. Precisely, the contribution for $\frac{dE^h}{dt}$ in (25) from the interelement boundaries is

$$- \sum_{j=1}^{n-1} \left[ - \|\tilde{e}_x^u\| \|\delta^x\| - \|\tilde{e}_x^v\| \|\delta^x\| + \|\tilde{e}_x^u\| \|\delta^x\| + \|\tilde{e}_x^v\| \|\delta^x\| \right] \right.$$ 

$$+ \|\tilde{e}_x^u\| \|\delta^x\| + \|\tilde{e}_x^v\| \|\delta^x\| \right] \leq C \left( h^{2q-3} |u(\cdot, t)|^2_{H^{q-1}(I)} + h^{2s-1} |v(\cdot, t)|^2_{H^{q-1}(I)} \right), \quad (32)$$
On the other hand, we have the contribution for $\frac{d\mathcal{E}^h}{dt}$ in (25) from the physical boundaries $\Theta_1 + \Theta_2$ with

$$\Theta_1 := \left[ (D\tilde{c}_x^u(x)\delta v_x) + \tilde{e}^v((D\delta u^v(x))^* - \eta_2(a_2^2 - b_2^2)(a_2\tilde{e}^v - b_2(D\tilde{c}_x^u)) \right]_{x_n}$$

$$- \left[ (D\tilde{c}_x^u(x)\delta v_x) + \tilde{e}^v((D\delta u^v(x))^* + \eta_2(a_2^2 - b_2^2)(a_2\tilde{e}^v + b_2(D\tilde{c}_x^u)) \right]_{x_0},$$

(33)

and

$$\Theta_2 := - \left[ D\tilde{c}_x^u(x)\delta v_x + \tilde{e}^v(D\delta u^v(x))^* + \eta_1(a_1^2 - b_1^2)\left(a_1\tilde{e}^v + b_1 D\tilde{c}_x^u\right) \right]_{x_n}$$

$$+ \left[ D\tilde{c}_x^u(x)\delta v_x + \tilde{e}^v(D\delta u^v(x))^* - \eta_1(a_1^2 - b_1^2)\left(-a_1\tilde{e}^v + b_1 D\tilde{c}_x^u\right) \right]_{x_0}. \quad (34)$$

Now, let’s consider four different physical boundary conditions.

(i). Simply supported boundary condition with $a_1 = b_2 = 0, b_1 = a_2 = 1, \eta_1 < 0$ and $\eta_2 > 0$, we have $(D\delta u^v)^* = 0, \delta v^* = 0$, then

$$\Theta_1 + \Theta_2 \leq C \left( \|\delta v^*\|_{L^2(\partial I)} + \|\delta u^v\|_{L^2(\partial I)} \right)$$

$$\leq C \left( h^{2q-5} |u(\cdot,t)|_{H^{q+1}(I)}^2 + h^{2s-1} |v(\cdot,t)|_{H^{s+1}(I)}^2 \right). \quad (35)$$

(ii). Free boundary condition with $a_1 = a_2 = 0, b_1 = b_2 = 1, \eta_1 < 0$ and $\eta_2 < 0$, we have $(D\delta u^v)^* = 0$ and $(D\delta v^*) = 0$, then

$$\Theta_1 + \Theta_2 \leq C \left( \|\delta v^*\|_{L^2(\partial I)}^2 + \|\delta u^v\|_{L^2(\partial I)}^2 \right)$$

$$\leq C \left( h^{2s+1} |v(\cdot,t)|_{H^{s+1}(I)}^2 + h^{2q-1} |v(\cdot,t)|_{H^{q+1}(I)}^2 \right). \quad (36)$$

(iii). Clamped boundary condition with $b_1 = b_2 = 0, a_1 = a_2 = 1, \eta_1 > 0$ and $\eta_2 > 0$, we have $\delta v^* = 0$ and $(\delta u^v)^* = 0$, then

$$\Theta_1 + \Theta_2 \leq C \left( \|\delta v^*\|_{L^2(\partial I)}^2 + \|D\delta u^v\|_{L^2(\partial I)}^2 \right)$$

$$\leq C \left( h^{2q-5} |u(\cdot,t)|_{H^{q+1}(I)}^2 + h^{2q-3} |u(\cdot,t)|_{H^{q+1}(I)}^2 \right). \quad (37)$$

(iv). Sliding boundary condition with $b_1 = a_2 = 0, a_1 = b_2 = 1, \eta_1 > 0$ and $\eta_2 < 0$, we have $(\delta u^v)^* = 0$ and $(\delta v^*) = 0$, then

$$\Theta_1 + \Theta_2 \leq C \left( \|\delta v^*\|_{L^2(\partial I)}^2 + \|D\delta u^v\|_{L^2(\partial I)}^2 \right)$$

$$\leq C \left( h^{2s+1} |v(\cdot,t)|_{H^{s+1}(I)}^2 + h^{2q-3} |u(\cdot,t)|_{H^{q+1}(I)}^2 \right). \quad (38)$$

Combine (25) and (32)–(38), we get

$$\frac{d\mathcal{E}^h}{dt} \leq C\sqrt{\mathcal{E}^h} \left( h^s |v(\cdot,t)|_{H^{s+1}(I)} + h^{s+1} |v(\cdot,t)|_{H^{s+1}(I)} \right)$$

$$+ C \left( h^{2q-5} |u(\cdot,t)|_{H^{q+1}(I)}^2 + h^{2q-3} |u(\cdot,t)|_{H^{q+1}(I)}^2 \right).$$

Integrating in time from 0 to $T$ gives

$$\mathcal{E}^h(T) \leq \mathcal{E}^h(0) + CT(T + 1) \max_{t \leq T} \left( h^{2s} |v(\cdot,t)|_{H^{s+1}(I)}^2 + h^{2(s+1)} |v(\cdot,t)|_{H^{s+1}(I)}^2 \right)$$

$$+ CT \max_{t \leq T} \left( h^{2q-5} |u(\cdot,t)|_{H^{q+1}(I)}^2 + h^{2q-3} |v(\cdot,t)|_{H^{q+1}(I)}^2 \right) + \frac{1}{2} \int_0^T \frac{1}{t+1} \mathcal{E}^h(t) \, dt.$$
From the assumption that \( \tilde{e}^u, \tilde{e}^v = (0, 0) \) at \( t = 0 \), \( \mathcal{E}^h(0) = 0 \). Now, by using Grönwall inequality [6], we get
\[
\mathcal{E}^h(T) \leq CT(T + 1)^2 \max_{t \leq T} \left( h^{2\kappa} |u(\cdot, t)|_H^2 + h^{2(\kappa+1)} |v(\cdot, t)|^2_H + h^{2\mu} |v(\cdot, t)|^2_H + h^{2\mu} |v(\cdot, t)|^2_H \right) + CT(T + 1) \max_{t \leq T} \left( h^{2\kappa-5} |u(\cdot, t)|_H^2 + h^{2\mu-1} |v(\cdot, t)|^2_H \right).
\]
Thus we obtain
\[
\mathcal{E}^h(T) \leq CT^2 \max_{t \leq T} \left( h^{2\kappa-5} |u(\cdot, t)|_H^2 + h^{2\kappa-1} |v(\cdot, t)|^2_H + h^{2(\kappa+1)} |v(\cdot, t)|^2_H \right).
\]
Finally, invoking triangle inequality with the relations \( e^v = \tilde{e}^v - \delta^v, e^u = \tilde{e}^u - \delta^u \) concludes the result (29).
\[
\square
\]

The requirements for proving optimal error estimates are more restrictive. In the following, we demonstrate optimal error estimates when \( s = q - 2 \) with alternating fluxes.

### 3.1 Optimal convergence for special fluxes

In this section, we consider the problems with \( \mu, D \) positive constants
\[
\frac{\partial^2 u}{\partial t^2} = -D \frac{\partial^4 u}{\partial x^4}, \quad x \in (0, L), \quad t \in (0, T),
\]
where \( D = \mu / D \). We further assume that the degree of approximation spaces satisfies \( s = q - 2 \).

To avoid the units of the fluxes parameters in (12), we introduce the dimensionless variables
\[
X = \frac{x}{L}, \quad \Gamma = \frac{t}{T}, \quad \Lambda = \frac{D T^2}{L^4}.
\]

Then we can use the proposed DG scheme in Section 2 to solve the following dimensionless problem
\[
\frac{\partial^2 u}{\partial \Gamma^2} = -\Lambda \frac{\partial^4 u}{\partial X^4}, \quad X \in (0, 1), \quad \Gamma \in (0, 1).
\]

For the rest of the analysis in this section, we will focus on the dimensionless problem (40), the results can be easily generated to the original problem (39). As discussed in [18, 1], we seek \( \tilde{u}^h, \tilde{v}^h \) to make \( \delta^v, (\frac{\partial \delta^v}{\partial X})^*, (\frac{\partial^2 \delta^v}{\partial X^2})^*, (\frac{\partial^3 \delta^v}{\partial X^3})^* \) vanish at both interelement and physical boundaries, that is,
\[
\delta^v = \frac{\partial \delta^v}{\partial X} = \frac{\partial^2 \delta^v}{\partial X^2} = \frac{\partial^3 \delta^v}{\partial X^3} = 0.
\]

To this end, by analyzing the general form of the numerical fluxes in (12), we find that (41) can be achieved by the following conditions
\[
\alpha_1 (1 - \alpha_1) = \beta_1 \tau_1, \quad \alpha_2 (1 - \alpha_2) = \beta_2 \tau_2.
\]

Note that (42) is satisfied by the alternating flux and the upwind flux, but not the central flux. We now substitute (42) into (12) and use (41), then rewrite the resulting equations into independent equations only containing data from one single element, \( I_j := [X_{j-1}, X_j] \),

\[
\begin{align*}
(1 - \alpha_1 + \tau_1) \delta^v - (\alpha_1 + \beta_1) \frac{\partial^3 \delta^v}{\partial X^3} &= 0, \quad X = X_{j-1}, \\
(1 - \alpha_2 + \tau_2) \frac{\partial \delta^v}{\partial X} + (\alpha_2 + \beta_2) \frac{\partial^2 \delta^v}{\partial X^2} &= 0, \quad X = X_{j-1}, \\
(\alpha_1 + \tau_1) \delta^v + (1 - \alpha_1 + \beta_1) \frac{\partial^2 \delta^v}{\partial X^2} &= 0, \quad X = X_j, \\
(\alpha_2 + \tau_2) \frac{\partial \delta^v}{\partial X} - (1 - \alpha_2 + \beta_2) \frac{\partial^2 \delta^v}{\partial X^2} &= 0, \quad X = X_j.
\end{align*}
\]
Here, we have assumed that the choices of \( \alpha_1 \) and \( \alpha_2 \) are consistent, that is, the values for \( \alpha_1 \) and \( \alpha_2 \) are the same for all elements. (43) gives us four independent equations on \( I_j \) to determine \( \tilde{u}^h \) and \( \tilde{v}^h \). We are now ready to construct \( \delta^u \) and \( \delta^v \) by requiring

Case I : \( q \geq 4 \)

- For all \( \phi \in \mathcal{P}^{q-1}(I_j) \)
  \[
  \int_{X_{j-1}}^{X_j} \phi \frac{\partial^2 \delta u}{\partial X^2} dX = \int_{X_{j-1}}^{X_j} \phi \delta v \ dX = 0, \tag{44}
  \]

- Zero average error of \( \frac{\partial u}{\partial X} \)
  \[
  \int_{X_{j-1}}^{X_j} \frac{\partial \delta u}{\partial X} dX = 0, \tag{45}
  \]

- Zero average error of \( u \)
  \[
  \int_{X_{j-1}}^{X_j} \delta u \ dX = 0, \tag{46}
  \]

- Equations (43) holds.

Case II : \( q = 3 \)

- Equations (43), (45) and (46) hold.

To show the uniqueness of \( \tilde{u}^h \) and \( \tilde{v}^h \) defined above, we further assume that

\[
\alpha_1(1 - \alpha_1) = \alpha_2(1 - \alpha_2) = 0, \quad \beta_1 = \beta_2 = \tau_1 = \tau_2 = 0, \tag{47}
\]

that is we have alternating fluxes. Now we are ready to state the following lemma.

**Lemma 1.** The function \((\tilde{u}^h, \tilde{v}^h)\) is uniquely defined by (44)–(46) combining with (43) and \((u, v) \in H^{q+1}(0,1) \times H^{q-1}(0,1) \) when \( q \geq 4 \); uniquely defined by (45)–(46) combining with (43) and \((u, v) \in H^{2}(0,1) \times H^{2}(0,1) \) when \( q = 3 \). In addition, there exists a constant \( C \) such that for \( h = \max |X_j - X_{j-1}| \)

\[
\begin{align*}
& \left\| \frac{\partial \delta^u}{\partial \Gamma} \right\|_{H^2(0,1)} + \left\| \frac{\partial \delta^v}{\partial \Gamma} \right\|_{L^2(0,1)} \leq CH^{q-1} \left( \left\| \frac{\partial u}{\partial \Gamma} \right\|_{H^{q+1}(0,1)} + \left\| \frac{\partial v}{\partial \Gamma} \right\|_{H^{q-1}(0,1)} \right). \tag{48}
\end{align*}
\]

**Proof.** Case I : \( q \geq 4 \),

The dimensionality of the local polynomial space \( \mathcal{P}^q(I_j) \times \mathcal{P}^{q-2}(I_j) \) is 2\( q \), which matches the number of linear equations. Suppose \( u = v = 0 \), then on \( I_j \) condition (44) and the boundary condition (43) together with the condition (47) imply that \( \frac{\partial^2 \tilde{v}^h}{\partial X^2} = \tilde{u}^h = 0 \). A subsequent use of condition (45) implies \( \frac{\partial \tilde{v}^h}{\partial X} = 0 \). Further combine with the condition (46), we have \( \tilde{u}^h = 0 \). Thus \((\tilde{u}^h, \tilde{v}^h)\) is uniquely defined and preserves the space \( \mathcal{P}^q(I_j) \times \mathcal{P}^{q-2}(I_j) \).

Case II : \( q = 3 \),

The dimensionality of the local polynomial space \( \mathcal{P}^3(I_j) \times \mathcal{P}^1(I_j) \) is 6, which matches the number of linear equations. Suppose \( u = v = 0 \), then \( \frac{\partial^3 \tilde{v}^h}{\partial X^3} \) and \( \frac{\partial \tilde{v}^h}{\partial X} \) are constants on \( I_j \). The condition (43) together with the condition (47) imply that \( \frac{\partial^3 \tilde{v}^h}{\partial X^3} = \frac{\partial \tilde{v}^h}{\partial X} = 0 \) on \( I_j \). Thus we have \( \frac{\partial^2 \tilde{v}^h}{\partial X^2} \) and \( \tilde{v}^h \) are constants on \( I_j \). Using the boundary condition (43) and the condition (47) one more time, we obtain \( \frac{\partial^2 \tilde{v}^h}{\partial X^2} = \tilde{v}^h = 0 \) on \( I_j \). Thus \( \frac{\partial \tilde{v}^h}{\partial X} \) is a constant on \( I_j \). A subsequent use of condition (45) implies \( \frac{\partial \tilde{v}^h}{\partial X} = 0 \). Further combine with the condition (46), we have \( \tilde{u}^h = 0 \). Thus \((\tilde{u}^h, \tilde{v}^h)\) is uniquely defined and preserves the space \( \mathcal{P}^3(I_j) \times \mathcal{P}^1(I_j) \).

To derive the error estimates we note that the polynomial approximation system commutes with time differentiation. Using the Bramble–Hilbert lemma, the result follows.
We now state an optimal error estimate in the energy norm.

**Theorem 4.** Suppose that \( s = q - 2, q \geq 3, \mu, D \) are positive constants, and the solution satisfies \((u, v) \in H^{q+1}(0, 1) \times H^{q-1}(0, 1)\). Suppose further that the initial condition for the discontinuous Galerkin solution satisfies error estimates commensurate with Lemma 1. Then there exists \( C \) independent of \( u, v \) and \( h \) such that

\[
\|e^v\|_{L^2(0,1)}^2 + \|e^{u_\tau}\|_{L^2(0,1)}^2 \leq C h^{2(q-1)\max} \left( \left\| \frac{\partial u}{\partial t} \right\|_{H^{q+1}(0,1)}^2 + \left\| \frac{\partial v}{\partial t} \right\|_{H^{q-1}(0,1)}^2 \right).
\]

**Proof.** Repeating the proof of Theorem 3 using \((\tilde{u}^h, \tilde{v}^h)\) defined by \((43), (44)-(46)\), we get

\[
A^h = \sum_{j=1}^{n} \int_{X_{j-1}}^{X_j} \Lambda \frac{\partial^2 \tilde{e}^u}{\partial X^2} \frac{\partial^2 (\partial \delta^u)}{\partial \sigma^2} - \Lambda \frac{\partial^2 \tilde{e}^u}{\partial X^4} \frac{\partial \delta^v}{\partial \sigma^2} + \mu \frac{\partial \tilde{e}^v}{\partial X^2} \frac{\partial^2 \delta^u}{\partial \sigma^2} \frac{\partial^2 \delta^u}{\partial \sigma^2} \frac{\partial^2 \sigma^2}{\partial \sigma^2} \frac{\partial \delta^u}{\partial \sigma^2} \frac{\partial \delta^u}{\partial \sigma^2} \frac{\partial \delta^u}{\partial \sigma^2} \frac{\partial \delta^u}{\partial \sigma^2} \frac{\partial \delta^u}{\partial \sigma^2} dX
\]

\[
= \sum_{j=1}^{n} \int_{X_{j-1}}^{X_j} \Lambda \frac{\partial^2 \tilde{e}^u}{\partial X^2} \frac{\partial^2 (\partial \delta^u)}{\partial \delta^u \partial \delta^u} + \mu \frac{\partial \tilde{e}^v}{\partial X^2} \frac{\partial \delta^u}{\partial \delta^u} dX. \tag{49}
\]

Combining \((25), (49)\), the estimate \((48)\) in Lemma 1 and the nonpositive boundary contribution, we have

\[
\frac{d\tilde{e}^h}{d\tau} \leq C h^{q-1} \sqrt{\tilde{e}^h} \left( \left\| \frac{\partial u}{\partial t} \right\|_{H^{q+1}(0,1)} + \left\| \frac{\partial v}{\partial t} \right\|_{H^{q-1}(0,1)} \right).
\]

Integrating in time, then using the triangle inequality with the relations \( e^v = \tilde{e}^v - \delta^v, e^u = \tilde{e}^u - \delta^u \) completes the proof. \( \square \)

**Remark 1.** Note that we are only able to prove an optimal convergence in the energy norm with alternating fluxes in the current work, but we observe optimal convergence for both the upwind flux and the central flux in the numerical experiments.

### 4 Numerical Simulation

In this section, we present numerical experiments to investigate the convergence of our method in the energy norm. The reason to use the energy norm is that it is the starting point of the energy-based DG method in establishing both stability and the rate of convergence as discussed in [26]. We use a standard modal basis and in all cases we choose \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \tau_1 = \tau_2 = 0.5 \) for an upwind flux and choose \( \alpha_1 = \alpha_2 = 0 \) for an alternating flux. In addition, we denote the alternating flux by A.-flux, the central flux by C.-flux and the upwind flux by U.-flux.

#### 4.1 The spectral deferred correction (SDC) time-stepping algorithm

The energy-based DG method for \((1)\) is derived based on the first order in time system \((2)\). However, for PDEs containing higher-order spatial derivatives, classical stability analysis of explicit schemes requires an extremely small time step. This can be computationally inefficient. To alleviate this problem, we consider an SDC method. As discussed in [13, 14, 16], the SDC methods are based on low order time-stepping methods, and then iterative corrections on a defect equation to obtain a desired order of accuracy. Consider the following ODE system

\[
\begin{cases}
y_t = Ay + F(t), & t \in [0, T], \\
y(0) = y_0.
\end{cases}
\]


where \( y_0, y(t) \in \mathbb{C}^l, A \in M_{1 \times l}(\mathbb{C}) \) and \( F : \mathbb{R} \rightarrow \mathbb{C}^l \). Suppose the time interval \([0, T]\) is partitioned into \( N_t\) subintervals as \( 0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T \). Denote \( \Delta t_n = t_{n+1} - t_n \) and \( y_n := y(t_n) \), we then present the SDC time stepping algorithm used in the current work as follows:

**Algorithm 1 SDC time stepping algorithm**

1. **Input**: \( y_{n,0} = y_0, \quad t_n, \quad t_{n+1}, \quad m, \quad J \)
2. **Compute** \( m \) Gauss-Radau points \( \tau_i \in (t_n, t_{n+1}] \) and set \( \tau_0 = t_n, \quad k_i = \tau_i - \tau_{i-1} \)
3. **for** \( i = 1, \ldots, m \) **do**
   4. \( \text{Solve} \quad y_{n,i} = y_{n,i-1} + k_i \left( Ay_{n,i} + F(\tau_i) \right) \quad \triangleright \text{Compute the initial approximation by backward Euler} \)
5. **end for**
6. **for** \( j = 1, \ldots, J \) **do**
7. Initialize \( \epsilon_{n,0} = 0, \delta_{n,0} = 0 \)
8. **for** \( i = 1, \ldots, m \) **do**
9. Compute \( \epsilon_{n,i} = y_n - y_{n,i-1} + I_{n,0,i}^m (Ay^j(\tau) + F(\tau)) \)
10. Solve \( \delta_{n,i} = \delta_{n,i-1} + k_i A\delta_{n,i} + (\epsilon_{n,i} - \epsilon_{n,i-1}) \)
11. Update \( y_{n+1} = y_{n,i} + \delta_{n,i} \)
12. **end for**
13. **end for**
14. **return** \( y_{n,m} \)

Here, \( I_{n,0,i}^m (Ay^j(\tau) + F(\tau)) \) is the integral of the \((m-1)\)-th degree interpolating polynomial on the \( m \) nodes \( (\tau_i, Ay_{n,i} + F(\tau_i))_{i=1}^m \) over the subinterval \([t_n, \tau_i]\), which is the numerical quadrature approximation of

\[
\int_{t_n}^{\tau_i} Ay^j(\tau) + F(\tau) d\tau.
\]

Here, we have omitted the details of standard derivations of the above SDC algorithm, one can refer to [16] and references therein for details. For the rest of numerical experiments, we set \( m = 5, \quad J = 15 \) to ensure the convergence order in time \((2m - 1)\) is larger than the convergence order in space \((6th \ order \ in \ space)\).

**4.2 Numerical verification**

In this section, we verify the theoretical convergence rates for the energy-based DG method derived in Section 2 for time-dependent Euler-Bernoulli beam equations.

**4.2.1 Time dependent uniform Euler-Bernoulli beam**

Consider the following problem

\[
\begin{align*}
    u_{tt} &= -u_{xxxx}, \quad x \in (0, 10), \quad t > 0, \\
    u(0, t) &= u(10, t) = 0, \quad u_{xx}(0, t) = u_{xx}(10, t) = 0
\end{align*}
\]

with \( u(0, t) = u(10, t) = 0, u_{xx}(0, t) = u_{xx}(10, t) = 0 \) and with initial data so that the exact solution is given by

\[
    u(x, t) = \sin(0.6\pi x) \sin((0.6\pi)^2 t).
\]

The discretization is performed on a uniform mesh with DG element vertices \( x_i = ih, i = 0, \ldots, N, h = 10/N \). The problem is evolved until the final time \( T = 1 \) with time step size \( \mathcal{O}(h) \). We present results for the degree of the approximation space of \( u^h \) being \( q = (4, 5, 6) \) and \( v^h \) being \( s \) with \( s = q \) or \( q - 1 \) or \( q - 2 \) or \( q - 3 \) or \( q - 4 \).

The energy norm errors \( \left( \|v - v^h\|^2_{L^2} + \|u - u^h\|^2_{H^2} \right)^{1/2} \) are presented from Table 1 to Table 3 with the alternating flux, the central flux and the upwind flux, respectively. We observe:

(a). a suboptimal convergence rate \( \bar{q} \) when \( s \neq q - 2 \) for both the central flux and the alternating flux,
Table 6 when $q_L$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$.

Table 1: Energy norm errors ($\|v - v^h\|_{L^2}^2 + |u - u^h|_{H^1}^2$) for problem 1 (50) when the A.-flux is used. $q$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$.

Table 2: Energy norm errors ($\|v - v^h\|_{L^2}^2 + |u - u^h|_{H^1}^2$) for problem 1 (50) when the C.-flux is used. $q$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$.

(b). a suboptimal convergence rate $\bar{q} + \frac{1}{2}$ for most of cases when $s \neq q - 2$ with the upwind flux. When $(q, s) = (5, 2), (5, 5), (6, 3)$, we note a suboptimal convergence $\bar{q} + 1$,

(c). an optimal convergence rate $q - 1$ when $s = q - 2$ for all three different fluxes.

As discussed in previous works [3, 25], the accuracy analysis of our scheme is based on an energy norm. For the numerical results of problem (50), we observe that the performance of the $L^2$ error ($\|u - u^h\|_{L^2}$) is irregular. In most of the cases with $s \neq q - 2$, we observe the same convergence rate for the $L^2$ errors and the energy norm errors, but the $L^2$ errors are much smaller than the energy norm errors. Here, we also present the $L^2$ errors for $u$, $|u - u^h|_{L^2}$, in Table 4, $L^2$ errors for $v$, $\|v - v^h\|_{L^2}$, in Table 5 and $H^2$ semi-norm errors for $u$, $|u - u^h|_{H^2}$ in Table 6 when $q = s - 2$ with three different numerical fluxes and $q = (2, 3, 4, 5, 6)$. We observe

(a). $q = 2$ leads to zero convergence for all three different fluxes and all types of errors, which agrees with the theoretical findings,

(b). optimal convergence is observed for the $L^2$ errors of $v$ and the $H^2$ semi-norm errors of $u$ when $q \geq 3$ and the alternating flux or the central flux is used,
Table 3: Energy norm errors $\| u - v_h \|^2_s + \| u - u_h \|^2_{L^2}$ for problem 1 (50) when the U.-flux is used. 
$q$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$.

Table 4: $L^2$ errors $\| u - u_h \|_s$ for problem 1 (50) when $s = q - 2$. $q$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$.

Table 5: $L^2$ errors $\| v - v_h \|^2_s$ for problem 1 (50) when $s = q - 2$. $q$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$. 
| $L^2$ error | flux  | N=10            | N=20            | N=40            | N=80            | N=160           |
|-------------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $q = 2$     | A.-flux | 3.18e-00(–)     | 3.18e-00(0.00)  | 3.18e-00(0.00)  | 3.18e-00(0.00)  | 3.18e-00(0.00)  |
|             | C.-flux | 3.18e-00(–)     | 3.18e-00(0.00)  | 3.18e-00(0.00)  | 3.18e-00(0.00)  | 3.18e-00(0.00)  |
|             | U.-flux | 3.18e-00(–)     | 3.18e-00(0.00)  | 3.18e-00(0.00)  | 3.18e-00(0.00)  | 3.18e-00(0.00)  |
| $q = 3$     | A.-flux | 9.62e-00(–)     | 3.63e-00(1.41)  | 9.49e-01(1.93)  | 2.40e-01(1.98)  | 6.02e-02(2.00)  |
|             | C.-flux | 2.55e-00(–)     | 2.35e-00(0.12)  | 5.12e-01(2.20)  | 1.22e-01(2.07)  | 3.01e-02(2.02)  |
|             | U.-flux | 3.77e-00(–)     | 4.12e-00(-.13)  | 3.42e-00(0.27)  | 2.22e-00(0.62)  | 1.22e-00(0.86)  |
| $q = 4$     | A.-flux | 1.05e+00(–)     | 4.48e-02(4.55)  | 4.86e-01(3.20)  | 5.34e-04(3.19)  | 6.42e-06(3.07)  |
|             | C.-flux | 8.38e-01(–)     | 1.10e-01(2.93)  | 1.25e-02(3.14)  | 1.33e-03(3.23)  | 1.55e-04(3.10)  |
|             | U.-flux | 1.40e-00(–)     | 1.62e-01(3.11)  | 1.49e-02(3.44)  | 1.48e-03(3.34)  | 1.64e-04(3.17)  |
| $q = 5$     | A.-flux | 5.56e-02(–)     | 1.63e-03(5.09)  | 1.04e-04(5.97)  | 6.51e-06(4.00)  | 4.07e-07(4.00)  |
|             | C.-flux | 4.20e-02(–)     | 1.01e-03(5.39)  | 5.45e-05(4.21)  | 3.29e-06(4.05)  | 2.04e-07(4.01)  |
|             | U.-flux | 5.86e-02(–)     | 2.57e-03(4.51)  | 1.51e-04(4.09)  | 9.18e-06(4.04)  | 5.65e-07(4.02)  |
| $q = 6$     | A.-flux | 4.54e-03(–)     | 7.33e-05(5.95)  | 2.29e-06(5.00)  | 7.15e-08(5.00)  | 2.24e-09(5.00)  |
|             | C.-flux | 1.61e-03(–)     | 6.68e-05(4.59)  | 2.33e-06(4.84)  | 7.49e-08(4.96)  | 2.36e-09(4.99)  |
|             | U.-flux | 2.82e-03(–)     | 8.03e-05(5.13)  | 2.48e-06(5.02)  | 7.60e-08(5.01)  | 2.38e-09(5.01)  |

(c). optimal convergence is observed for the $L^2$ errors of $v$ and the $H^2$ semi-norm errors of $u$ when $q \geq 4$ and the upwind flux is used. When $q = 3$, we only obtain first order convergence for the $L^2$ errors of $v$ and the $H^2$ semi-norm errors of $u$,

(d). the same orders of convergence are observed for the $L^2$ errors of $v$ compared with the $L^2$ errors of $v$ and the $H^2$ semi-norm errors of $u$ in most of cases. But the $L^2$ errors of $u$ are much smaller than the other two types of error. Specially, when $q = 5$ and the upwind flux is used, we observe optimal convergence in the $L^2$ norm errors of $u$.

![Figure 1: The plots of discrete energy for DG solutions of (50) with three different numerical fluxes and approximation degrees $q = 5$, $s = 3$. For the left graph, the discrete energy for the alternating flux, the central flux, and the upwind flux are presented. For the right graph, the discrete energy with the alternating flux and the central flux are displayed.](image-url)

Finally, for the uniform beam problem (50), we present the numerical energy history of the proposed energy-based DG scheme in Figure 1 with the central flux, the alternating flux, and the upwind flux. In particular, we show the results for the case with $q = 5$ and $s = 3$ until the final time $T = 100$ with $N = 40$. We note that the magnitude of energy error for the conservative schemes, the alternating flux and the central flux, is smaller than $10^{-7}$. Since the SDC time stepping scheme is energy dissipating, we can conclude that both the alternating flux and the central flux conserve the discrete energy. Even for the dissipating scheme, the upwind flux, the discrete energy is still conserved up to 2 digits.
Table 7: Energy norm errors ($\|v - v^h\|_{L^2} + \|u - u^h\|_{H^1}^2$) for problem 1 (52) when $s = q - 2$. $q$ is the degree of $u^h$, $s$ is the degree of $v^h$, and $N$ is the number of the cells with uniform mesh size $h = 10/N$.

### 4.2.2 Time dependent nonuniform Euler-Bernoulli beam

In this section, we consider the following model

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2}{\partial x^2} \left( D(x) \frac{\partial^2 u}{\partial x^2} \right) + f(x, t), \quad x \in (0, 10), \quad t > 0, \quad (52)$$

where $D(x) = 1 + 0.1 \cos(\pi x)$. For the simplicity of description, we use the same exact solution (51) as in Section 4.2.1 to determine the external forcing $f(x, t)$. Note that $\frac{\partial D}{\partial x} = 0$ at $x = 0, 10$, so the simply supported boundary condition is also satisfied. We use the same discretization as those in Section 4.2.1.

We display the energy norm errors ($\|v - v^h\|_{L^2} + \|u - u^h\|_{H^1}^2$) in Table 7. We observe the same results as in the constant coefficients case: optimal convergence for the central flux and the alternating flux when $q \geq 3$; optimal convergence for the upwind flux when $q \geq 4$ and a suboptimal convergence $q - 2$ when $q = 3$; when $q = 2$, again, we don’t observe any convergence, since when $s = q - 2$ the scheme is not defined with $q = 2$.

The numerical solution $u^h$ and the errors $u - u^h$ at time $t = 100$ with three different numerical fluxes are shown in Figure 2. From this figure we observe our numerical results profiles match well. In addition, the errors show no severe error localized at the element boundaries and the central flux has the best performance (relatively smaller errors) for this problem.

### 5 Conclusions and future work

In conclusion, we have developed an energy-based DG method for dynamic Euler-Bernoulli beam equations. In our formulation, only one extra velocity field is introduced, velocity field $v = u_t$. The approximation spaces of discontinuous Galerkin solutions for the displacement field and the velocity field can be different. The stability of the scheme only depends on simple, mesh-independent numerical fluxes. We also prove error estimates in the energy norm and obtain sub-optimal convergence for general cases. When $q \geq 3$, optimal convergence in the energy norm is also proven when problems are in one spatial dimension with constant coefficients and particular numerical fluxes. Numerical experiments demonstrate the theoretical findings.

As stated in the Remark 1, one future work is to investigate the mechanism of the optimal convergence behind for central flux and the upwind flux. Another extension is to establish schemes for the problem with nonlinearities which will enable applications to a wider variety of problems of physical interest.
Figure 2: Left panel: numerical solution and exact solution at $t = 100$ for problem (52) when the approximation order $q = 5$ and $s = 3$ on a uniform mesh of $N = 40$. Right panel: error $u - u^h$ as a function of $x$. From the top to the bottom are the alternating flux, the central flux and the upwind flux.
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