The Perfect Distinguishability of Quantum Operations

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We provide a feasible necessary and sufficient condition for when an unknown quantum operation (quantum device) secretly selected from a set of known quantum operations can be identified perfectly within a finite number of queries, and thus complete the characterization of the perfect distinguishability of quantum operations. We further design an optimal protocol which can achieve the perfect discrimination between two quantum operations by a minimal number of queries. Interestingly, employing the techniques from the theory of quantum Chernoff bound [1], it is clear that the perfect distinguishability of quantum states is uniquely in parallel. Due to these differences, it is quite difficult to identify the behavior of quantum operations when multiple queries are used. In particular, it is unclear when two quantum operations are perfectly distinguishable within a finite number of queries.

Several works have been devoted to the perfect distinguishability of special quantum operations including unitary operations [3, 4, 6, 7] and projective measurements [8]. Most notably, any two different unitary operations can be perfectly distinguishable by inputting an entangled state and applying the unknown unitary in parallel [4]. Such a perfect discrimination can also be achieved by applying the unitary operations on a single system sequentially, and entanglement or joint quantum operations are not necessary [6]. Interestingly, projective measurements also enjoy this kind of perfect distinguishability [5]. Very recently experimental results concerning the perfect discrimination of unitary operations and measurements have been reported [9]. All these progresses indicate that the notion of perfect distinguishability of general quantum operations would be much more complicated than that of quantum states. The minimum-error or unambiguous discrimination strategies for quantum states cannot be simply applied to quantum operations as they cannot fully reflect the fact that many quantum operations are essentially perfectly distinguishable in multi-use scenario although a perfect discrimination cannot be achieved by one single use.

The purpose of this Letter is to provide a complete characterization of the perfect distinguishability of quantum operations (See Theorem 1 below). We show that two simple properties are necessary and sufficient for the perfect discrimination between two quantum operations within a finite number of queries. The first property says that two quantum operations that are perfectly distinguishable should produce two quantum states

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with non-overlapping supports upon some common input state, which may entangled with an auxiliary system. The second property states that any such two quantum operations are capable of transforming some two non-orthogonal pure states, which are provided to the quantum operations as their respective inputs, into orthogonal states. These two properties reveal the key feature of the perfect distinguishability of quantum operations and thus provide new insight into this problem. It is also worth noting that both of these properties can be rephrased into analytical forms in terms of the Kraus operators of quantum operations to distinguish and can be verified quite efficiently. As a potential application, we show that the classical data hiding is possible by encoding the data into quantum devices instead of quantum states [12].

Furthermore, with the assistance of a mathematical notion of the maximal fidelity between quantum states, we can provide an optimal protocol which can distinguish two quantum operations with a minimal number of queries. This number can be efficiently determined using numerical iteration techniques. We further show that for distinguishing between two isometries (generalization of unitary operations), an optimal discrimination always can be achieved without auxiliary systems or entanglement by employing some results from the theory of q-numerical range. This generalizes our previous work on unitary operations [4].

2. Conditions for the perfect discrimination between quantum operations Consider a $d$-dimensional Hilbert space $\mathcal{H}_d$. The set of linear operators on $\mathcal{H}_d$ is denoted by $\mathcal{B}(\mathcal{H}_d)$. A general quantum state $\rho$ on $\mathcal{H}_d$ is given by a positive operator in $\mathcal{B}(\mathcal{H}_d)$ with trace one. A pure state $|\psi\rangle$ is a unit vector in $\mathcal{H}_d$. For simplicity, we will use $\psi$ to denote the density operator form $|\psi\rangle\langle\psi|$ of $|\psi\rangle$. Let $\rho$ be with the spectral decomposition $\rho = \sum_{k=1}^{d} p_k |\psi_k\rangle\langle\psi_k|$. The support of $\rho$ is given by $\text{supp}(\rho) = \text{span}\{|\psi_k\rangle : p_k > 0\}$. A quantum operation $\mathcal{E}$ from $\mathcal{B}(\mathcal{H}_d)$ to $\mathcal{B}(\mathcal{H}_d')$ is a trace-preserving completely positive map with the the form $\mathcal{E}(\rho) = \sum_{i=1}^{m} E_i \rho E_i^\dagger$, where $\{E_i\}_{i=1}^{m}$ are the Kraus operators of $\mathcal{E}$ satisfying $\sum_{i=1}^{m} E_i^\dagger E_i = I_d$. Quantum operations formalize all physically realizable operations allowed by quantum mechanics, including unitary operations, quantum measurements, and quantum channels. In particular, a quantum measurement $\mathcal{M}$ with measurement operators $\{E_1, \ldots, E_m\}$ is a special quantum operation with Kraus operations $\{E_k \otimes |k\rangle : k = 1 \cdots m\}$, where $\{|k\rangle\}$ is a classical system with $m$ distinguishable states. To emphasize the importance of the order among the measurement operators, a quantum measurement $\mathcal{M}$ can be represented as an $m$-tuple of matrices, say $(E_1, \ldots, E_m)$.

Two density operators $\rho_0$ and $\rho_1$ are said to be disjoint if $\text{supp}(\rho_0) \cap \text{supp}(\rho_1) = \{0\}$. Let us now introduce a notion to quantitatively describe the disjointness between two quantum states, which can be treated as a special inner product between two mixed states (actually two subspaces).

**Definition 1** The maximal fidelity between two quantum states $\rho_0$ and $\rho_1$ is defined as follows:

$$F(\rho_0, \rho_1) = \max\{||\psi_0\rangle\langle\psi_1|| : |\psi_k\rangle \in \text{supp}(\rho_k), k = 0, 1\}$$

It follows from the definition that $0 \leq F(\rho_0, \rho_1) \leq 1$. $F$ is vanishing iff $\rho_0$ and $\rho_1$ are orthogonal, and attains 1 iff $\rho_0$ and $\rho_1$ are not disjoint. The name “maximal fidelity” comes from the following simple connection to the ordinary fidelity:

$$\tilde{F}(\rho_0, \rho_1) = \max\{F(\rho_0', \rho_1') : \text{supp}(\rho_k') \subseteq \text{supp}(\rho_k)\},$$

where $F(\rho_0, \rho_1) = \text{tr}\sqrt{\rho_0^{1/2} \rho_1^{1/2}}$. Due to the above connection, the maximal fidelity $\tilde{F}$ enjoys some similar properties as $F$. For instance, for pure states both of them coincide with the ordinary inner product, and the maximal fidelity is also multiplicative according to tensor product. The most important property of the maximal fidelity is the following operational interpretation. Note that a similar operational interpretation of $F(\rho_0, \rho_1)$ has been found in [10]. The technical proof is put in the appendix.

**Lemma 1** For two pairs of quantum states $\{|\psi_0\rangle, |\psi_1\rangle\}$ and $\{|\tilde{\psi}_0\rangle, |\tilde{\psi}_1\rangle\}$, there is a quantum operation $\mathcal{T}$ such that $\mathcal{T}(\rho_k) = \tilde{\psi}_k$ for $k = 0, 1$ iff $\tilde{F}(\rho_0, \rho_1) = F(\tilde{\psi}_0, \tilde{\psi}_1)$. Thus we have

$$\tilde{F}(\rho_0, \rho_1) = \min\{||\psi_0\rangle\langle\tilde{\psi}_1|| : \exists \mathcal{T}, \mathcal{T}(\rho_k) = \tilde{\psi}_k\}. \quad (1)$$

It is straightforward to define two quantum operations are disjoint. Formally, we have the following

**Definition 2** $\mathcal{E}_0$ and $\mathcal{E}_1$ are said to be (unassisted) disjoint if there is an input state $|\psi\rangle \in \mathcal{H}_d$ such that $\mathcal{E}_0(|\psi\rangle)$ and $\mathcal{E}_1(|\psi\rangle)$ are disjoint. $\mathcal{E}_0$ and $\mathcal{E}_1$ are said to be entanglement-assisted disjoint if there is an input state $|\psi\rangle_R$ such that $(\mathcal{I}_R \otimes \mathcal{E}_0^Q)(|\psi\rangle_R)$ and $(\mathcal{I}_R \otimes \mathcal{E}_1^Q)(|\psi\rangle_R)$ are disjoint, where $R$ and $Q$ denote auxiliary and principal systems respectively, and $\mathcal{I}_R$ is the identity operation on $R$.

One can easily verify that the dimension of $R$ in the above definition can be assumed to be the same as $Q$ and larger dimension cannot make any difference. There is an efficient procedure to determine whether two quantum operations $\mathcal{E}_0$ and $\mathcal{E}_1$ are entanglement-assisted disjoint. Suppose that $\mathcal{S}_0 = \text{span}\{\mathcal{E}_0|\psi\rangle : k = 0, 1\}$ and $\text{dim}(\mathcal{S}_0)$ is $n_k$. If $\mathcal{S}_0 \cap \mathcal{S}_1 = \{0\}$ then $\mathcal{E}_0$ and $\mathcal{E}_1$ are entanglement-assisted disjoint and the input state can be chosen as $|\psi\rangle_R = 1/\sqrt{n_k} \sum_{k=1}^{n_k} |k\rangle_R |k\rangle_Q$. Otherwise, select an arbitrary basis $\{D_k\}_{k=1}^{n_k}$ for $\mathcal{S}_0 \cap \mathcal{S}_1$, and construct an operator $X = \sum_{k=1}^{n_k} D_k^\dagger D_k$. Let $P_1$ be the projector onto $\text{supp}(X)$, and consider two new channels.
$E'_0$ and $E'_1$, with respective Kraus operators $\{E_{0i}P_{i1}^+\}$ and $\{E_{1j}P_{j1}^+\}$, where $P_{i1}^+ = I_d - P_i$. The original problem is now reduced to decide whether $E'_0$ and $E'_1$ are entanglement-assisted disjoint, and a projector $P_2 \leq P_{i1}^+$ can be similarly constructed. Repeat this process $n \leq d$ times we can efficiently construct a sequence of mutual orthogonal projectors $P_1, \ldots, P_n$ such that $P_n = 0$ and $P_i \neq 0$ for any $i < n$. Let $P = I_d - \sum_{i=1}^{n-1} P_i$. Then $E_0$ and $E_1$ are entanglement-assisted disjoint iff $P \neq 0$. If satisfied, $|\psi\rangle = (I \otimes P)|\alpha\rangle$ is an eligible input state.

We are now ready to present a complete characterization of the perfect distinguishability of quantum operations.

**Theorem 1** Let $E_0$ and $E_1$ be two quantum operations from $B(H_d)$ to $B(H_{d'})$ with Kraus operators $\{E_{0i} : i = 1 \cdots n_0\}$ and $\{E_{1j} : j = 1 \cdots n_1\}$, respectively. Then $E_0$ and $E_1$ are perfectly distinguishable by a finite number of uses if i) $E_0$ and $E_1$ are entanglement-assisted disjoint, and ii) $I_d \not\in \text{span}(E_{0i}E_{1j})$.

**Proof.** Let us first show show that the conditions i) and ii) are necessary. Suppose $E_0$ and $E_1$ are perfectly distinguishable within $N$ uses, and assume $N$ is minimal. We claim that there is a minimal state $|\psi\rangle^{\text{RQ}}$ such that $(I^R \otimes E_0^Q)(|\psi\rangle)$ and $(I^R \otimes E_1^Q)(|\psi\rangle)$ are disjoint. By contradiction, assume that for any choice of $|\psi\rangle$, $(I^R \otimes E_0^Q)(|\psi\rangle)$ and $(I^R \otimes E_1^Q)(|\psi\rangle)$ are not disjoint. Then we can find a state $|\psi\rangle^{\text{RQ}}$ that lies in both supports. Thus in the next $N - 1$ uses we must be able to distinguish between $E_0$ and $E_1$ by inputing $|\psi\rangle^{\text{RQ}}$. That means $(N - 1)$ uses are sufficient to distinguish between $E_0$ and $E_1$ by inputing $|\psi\rangle^{\text{RQ}}$.

To show the necessity of ii), let’s consider the last use of the unknown quantum operation. Assume that the input states corresponding to $E_0$ and $E_1$ are $\rho_0$ and $\rho_1$, respectively. Both $\rho_0$ and $\rho_1$ are the output states of previous $(N - 1)$ uses and may be mixed states. As the last use must distinguish between $E_0$ and $E_1$ but all previously $(N - 1)$ uses cannot, we have $(I \otimes E_0)(\rho_0) \perp (I \otimes E_1)(\rho_1)$, $\rho_0 \not\perp \rho_1$. Thus there must be two states $|\psi_0\rangle = (I^R \otimes A_{0j}^Q)|\alpha\rangle^{\text{RQ}}$ and $|\psi_1\rangle = (I^R \otimes A_{1j}^Q)|\alpha\rangle^{\text{RQ}}$ from the supports of $\rho_0$ and $\rho_1$, respectively, such that

$$\text{tr}(I \otimes E_0)(|\psi_0\rangle)(I \otimes E_1)(|\psi_1\rangle) = 0, \text{ and } \langle\psi_0|\psi_1\rangle \neq 0.$$
not allowed to individually exactly recover $b$ while Charlie can reveal the bit at any time by supplying entanglement or asking them to move together. This kind of protocol has been shown to be possible if Charlie encodes the bit using two orthogonal bipartite mixed states $\rho_{0B}^1$ and $\rho_{1B}^1$ that are locally indistinguishable [12]. The new feature of hiding classical data using quantum devices instead of states is that the identified device can be reused in the future information processing tasks.

Applying Theorem [1], we can easily construct these kind of instances by imposing that $\{\mathcal{E}_0, \mathcal{E}_1\}$ satisfies only condition i) while $\{\mathcal{E}_0', \mathcal{E}_1'\}$ satisfies only condition ii).

An explicit example is as follows: $\mathcal{E}_0$ and $\mathcal{E}_1$ are quantum operations that prepare quantum states $|\psi_0\rangle = (|0\rangle + \sqrt{2}|1\rangle)/\sqrt{3}$ and $|\psi_1\rangle = (|0\rangle - \sqrt{2}|1\rangle)/\sqrt{3}$, respectively; while $\mathcal{E}_0' = (|0\rangle|0\rangle + 1/\sqrt{2}|1\rangle|1\rangle, 1/\sqrt{2}|1\rangle|0\rangle)$ and $\mathcal{E}_1' = (|0\rangle|0\rangle + 1/\sqrt{2}|1\rangle|1\rangle, 0, 1/\sqrt{2}|1\rangle|0\rangle)$ are two one-qubit measurements. One can easily verify that $\mathcal{E}_0'$ and $\mathcal{E}_1'$ are perfectly distinguishable upon the respective input states $|\psi_0\rangle$ and $|\psi_1\rangle$ as $tr(|\langle 0|0\rangle + 1/2|1\rangle|\langle 1)|\psi_0\rangle\langle \psi_1|) = 0$.

3. An optimal protocol for the perfect discrimination between two quantum operations The discrimination protocol we presented in Theorem [1] is not optimal in general. We shall now describe an optimal one. We need a notion of $q$-maximal fidelity, which is naturally induced from the maximal fidelity between quantum states, to quantitatively describe the disjointness between quantum operations.

Definition 3 For quantum operations $\mathcal{E}_0$ and $\mathcal{E}_1$, and $0 \leq q \leq 1$, the $q$-maximal fidelity is defined as follows:

$$F_q(\mathcal{E}_0, \mathcal{E}_1) = \min\{F(\mathcal{E}_0(|\psi_0\rangle, \mathcal{E}_1|\psi_1\rangle) : |\psi_0\rangle|\psi_1\rangle = q\}.$$  

The entanglement-assisted $q$-maximal fidelity is defined as follows

$$F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1) = \max\{F(\mathcal{E}_0(|\psi_0\rangle, \mathcal{E}_1|\psi_1\rangle) : |\psi_0\rangle|\psi_1\rangle = q\}.$$  

where $R$ is an auxiliary system with the same dimension as $Q$ (larger cannot make difference). When $q = 1$, $\tilde{F}_1(\mathcal{E}_0, \mathcal{E}_1)$ and $\tilde{F}_q^{ea}(\mathcal{E}_0, \mathcal{E}_1)$ are said to be the maximal fidelity and the entanglement-assisted maximal fidelity between $\mathcal{E}_0$ and $\mathcal{E}_1$, respectively.

Here we should point out that $\psi_0$ and $\psi_1$ in the above definition can be replaced by any $\rho_0$ and $\rho_1$ such that $\tilde{F}(\rho_0, \rho_1) = q$. However, in virtue of Lemma [1] we can verify that it is sufficient to consider pure states only.

The notion of $F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1)$ plays a crucial role in designing the optimal perfect discrimination protocol of quantum operations, which is mainly due to the following desirable property:

$$F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1) \leq \frac{q}{q'} F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1), 0 \leq q < q' \leq 1.$$  

(2)

This property can be understood as “more separable states will yield more separable output states.” It is true simply due to the fact that by appending an auxiliary qubit we can divide the input states for $f_q^{ea}$ into two parts: a pair of qubit states with inner product $q/q'$ and a pair of optimal input states for $f_q^{ea}$.

Let us start to describe an optimal perfect discrimination protocol between $\mathcal{E}_0$ and $\mathcal{E}_1$. Let $N_{\min}$ be the minimal number of uses of the unknown quantum operation required to perfectly distinguish between $\mathcal{E}_0$ and $\mathcal{E}_1$, and let $\{q_k\}$ be a sequence of $q$-maximal fidelities recursively defined as follows:

$$q_0 = 1, q_k = F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1), k \geq 1.$$  

Notice that $q_1 = \tilde{F}_1^{ea}(\mathcal{E}_0, \mathcal{E}_1)$ is just the entanglement-assisted maximal fidelity between $\mathcal{E}_0$ and $\mathcal{E}_1$. Let us further introduce $q_{\max}$ as follows:

$$q_{\max} = \{q : F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1) = 0\}.$$  

Then the following theorem shows that $N_{\min}$ is completely determined by the sequence of $\{q_k\}$ and $q_{\max}$ (indirectly).

**Theorem 2** Let $\mathcal{N}^{(k)}$ represent an arbitrary quantum discrimination network containing $k$ uses of the unknown quantum operation from $\{\mathcal{E}_0, \mathcal{E}_1\}$. Then

$$q_k \leq \tilde{F}_1^{ea}(\mathcal{N}^{(k)}(\mathcal{E}_0, \mathcal{N}^{(k)}(\mathcal{E}_1))).$$  

In other words, $q_k$ is the optimal maximal fidelity one can achieve by $k$ uses of the unknown quantum operation from $\{\mathcal{E}_0, \mathcal{E}_1\}$ and with the same input. Furthermore, $N_{\min} = \min\{k : q_k = 0, k \geq 1\} = \min\{k : q_{k-1} \leq q_{\max}\}$, and $q_k = 0$ for any $k > N_{\min}$.

**Proof.** By mathematical induction. By definition $q_1$ is the optimal maximal fidelity one can achieve by a single use. Assume that $q_k$ is optimal by $k$ uses of the unknown quantum operation. Consider any quantum discrimination network $\mathcal{N}^{(k+1)}$ containing $k+1$ uses of the unknown quantum operation. By induction assumption, we have $q_k = \tilde{F}(\rho_0(k), \rho_1(k)) \geq q_k$, where $\rho_0(k)$ and $\rho_1(k)$ are the output states of $\mathcal{N}^{(k+1)}$ except the last use of the unknown quantum operation. Clearly, $\rho_0(k)$ and $\rho_1(k)$ are the output states of a quantum discrimination network containing $k$ uses of the unknown quantum operation, and also the input states for the last use of the unknown quantum operation in $\mathcal{N}^{(k+1)}$. Let $\rho_0^{(k+1)}$ and $\rho_1^{(k+1)}$ be the final output states of $\mathcal{N}^{(k+1)}$. By Eq. (2), we have

$$\tilde{F}(\rho_0^{(k+1)}, \rho_1^{(k+1)}) \geq F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1) \geq q_k F_q^{ea}(\mathcal{E}_0, \mathcal{E}_1) \geq q_{k+1},$$  

where we have employed the assumption $q_k \geq q_k$ and the definition of $q_{k+1}$. The expression of $N_{\min}$ follows immediately.
Theorem 3  

More precisely, $\mathcal{E}_0$ and $\mathcal{E}_1$ are perfectly distinguishable iff $q_1 < 1$ and $q_{\max} > 0$, which is based on the following two simple observations: 1) $q_1 = 1$ implies $q_k = 1$ for any $k \geq 1$; or 2) $q_{\max} = 0$ implies $q_k > 0$ for any $k \geq 1$. One can also readily verify that $q_1 < 1$ and $q_{\max} > 0$ correspond to conditions i) and ii) in Theorem 1, respectively. As a consequence, we can obtain an upper bound of $N_{\min}$ in terms of $q_1$ and $q_{\max}$, say $N_{\min} \leq \frac{\ln q_{\max}}{\ln q_1}$. Note here $q_1 = 0$ implies $N_{\min} = 1$.

The sequence of $\{q_k\}$ and $q_{\max}$ can be calculated with arbitrary high precision using numerical iteration techniques as it is evident that $F_q(\mathcal{E}_0, \mathcal{E}_1)$ can be formulated into an optimization problem on a compact set. Hence we can estimate $N_{\min}$ for any two quantum operations $\mathcal{E}_0$ and $\mathcal{E}_1$ according to the above theorem. In many practical applications, a simple protocol like the one in Theorem 1 would be sufficient.

4. $q$-numerical range and the perfect distinguishability of isometries  

For general $\mathcal{E}_0$ and $\mathcal{E}_1$, it is normally very difficult to calculate the optimal fidelity sequence of $\{q_k\}$. Interestingly, if both $\mathcal{E}_0$ and $\mathcal{E}_1$ are isometries, the calculation becomes quite tractable. For isometries $U_0$ and $U_1$, we have

$$\bar{F}_q(U_0, U_1) = \bar{r}_q(A) = \min\{|z| : z \in W_q(A)\},$$

where $A = U_0^\dagger U_1$ and $W_q(A) = \{\\langle \psi_0 | A | \psi_1 \rangle : \langle \psi_0 | \psi_1 \rangle = q\}$. Similarly, $\bar{F}_q(U_0, U_1) = \bar{r}_q(I_d \otimes A)$. For $0 \leq q \leq 1$, $W_q(A)$ is said to be the $q$-numerical range of $A$ with $\bar{r}_q(A)$ the inner radius. When $q = 1$, $W(A) = W_1(A)$ is the classic numerical range of $A$. The theory of numerical range and its various generalizations including $q$-numerical range are an active and vast topic in linear algebra [14]. It has been recognized recently that these notions are quite useful in studying the local discrimination of unitary operations [13]. A somewhat surprising fact is that the optimal perfect discrimination of isometries can be achieved without auxiliary systems or entanglement.

Theorem 3  

For any isometries $U_0$ and $U_1$, and $0 \leq q \leq 1$, $F_q(U_0, U_1) = F_q(U_0, U_1)$.

Previously we have shown the same result for unitary operations [3]. We can derive the above result from an interesting result about the $q$-numerical range, say $W_q(I_d \otimes A) = W_q(A)$ for any linear operator $A$ and $0 \leq q \leq 1$. The equality for the case of $q = 1$ follows directly from the convexity of $W(A)$. For the general case we cannot find any existing reference to this important result and thus we provide a proof in the appendix.

There is no explicit expression for the $q$-inner radius $\bar{r}(A)$ of a general linear operator $A$. Hence it is generally impossible to obtain the analytical formula of $N_{\min}(U_0, U_1)$. Fortunately, it was known that $W_q(A)$ is a convex compact set for any linear operator $A$ and $0 \leq q \leq 1$, and efficient characterization of the boundary of $W_q(A)$ has been obtained [15]. As a consequence, it is quite feasible to compute $\bar{r}_q(A)$, and then determine the exact value of $N_{\min}$. It is also possible to obtain analytical results when $A$ belongs to normal or $2 \times 2$ matrices as efficient characterization of the $q$-numerical range has been found. In particular, the case that $A$ is unitary has been completely solved [3]. For the case that $A$ is positive definite, however, any parallel protocol cannot distinguish between $U_0$ and $U_1$, even assisted with arbitrary large amount of entanglement. In sharp contrast, we know from Theorem 1 that there is a sequential protocol that can achieve an optimal perfect discrimination. Furthermore, in this case $W_q(A)$ is an elliptical disk with eccentricity $q$, and foci $q\lambda_0$ and $q\lambda_1$, where $\lambda_0$ and $\lambda_1$ are the maximum and minimum of eigenvalues of $A$ [15]. Using this fact one can derive the following analytical formula:

$$N_{\min}(U_0, U_1) = \left[\frac{\ln 2 + \ln(1 - \lambda_1) - \ln(\lambda_1)}{\ln 2 - \ln(\lambda_0 + \lambda_1)}\right].$$

5. Discussions  

It would be highly desirable to identify the quantum Chernoff bound for quantum operations that are not perfectly distinguishable. Perhaps the first step to this problem is to identify the (asymptotically) optimal minimum-error discrimination strategy for quantum operations using the distance measure induced by diamond norm instead of the maximal fidelity. Many of our techniques can be generalized to multipartite setting, where distant parties share an unknown quantum operation and they are only allowed to perform arbitrary Local Operations and Communicate with each other Classically (LOCC). In a previous work we have shown that the perfect distinguishability of unitary operations is preserved under LOCC [15], benefiting from the local distinguishability of two orthogonal multipartite pure states [18]. With some additional efforts we can generalize Theorem 1 to a wider class ofmultipartite quantum operations, including all isometries and almost all quantum measurements. Unfortunately, the condition for the perfect distinguishability of general multipartite quantum operations remains unknown as it is still unknown when two general orthogonal mixed states can be locally distinguishable. We will continue to study these issues.

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It follows immediately from Ref. [11] that \( \tilde{a} \) is a set of orthonormal states from \( \text{supp}(P) \). Similarly \( \{\psi_1^{(k)}\} \) is a set of orthonormal states from \( \text{supp}(Q) \). Further more, we have \( \langle \psi_0^{(j)} | \psi^{(j)}_1 \rangle = \delta_{ij} \). So \( \{\psi_0^{(k)}; \psi_1^{(k)}\}_{k=1 \cdots r} \) are mutually orthogonal although \( \{\psi_0^{(k)}\} \) and \( \{\psi_1^{(k)}\} \) may not. Let \( P_k \) be the projector onto \( \text{span}\{\psi_0^{(k)}; \psi_1^{(k)}\} \) for each \( k = 1 \cdots r \). Let \( P_0 = P - \sum_{k=1}^{r} |\psi_0^{(k)}\rangle \langle \psi_0^{(k)}| \) and \( P_{r+1} = Q - \sum_{k=1}^{r} |\psi_1^{(k)}\rangle \langle \psi_1^{(k)}| \). Here both \( P_0 \) and \( P_{r+1} \) may be vanishing. One can readily verify that \( \{P_0, P_1, \ldots, P_r, P_{r+1}\} \) forms a complete projective measurement on \( \text{supp}(P + Q) \). Applying this measurement to \( \rho_0 \) and \( \rho_1 \). If the outcome is 0 or \( r+1 \) then the original state is in state \( \rho_0 \) or \( \rho_1 \), respectively. We can directly prepare a target state as \( \psi_0 \) or \( \psi_1 \). Otherwise the outcome is \( 1 \leq k \leq r \) and the post-measurement state should be \( |\psi_0^{(k)}\rangle \) or \( |\psi_1^{(k)}\rangle \), depending on the original state \( \rho_0 \) or \( \rho_1 \). Note that \( \langle \psi_0^{(k)} | \psi_1^{(k)} \rangle \leq \tilde{F}(\rho_0, \rho_1) \leq \langle \psi_0 | \psi_1 \rangle \). By Ref. [11] again we can further transform \( \{\psi_0^{(k)}; \psi_1^{(k)}\} \) into \( \{\psi_0; \psi_1\} \). With that we complete the proof of the lemma.

**Proof of\( W_q(I_d \otimes A) = W_q(A) \)**

We will introduce a more manageable representation of \( W_q(A) \) first. A relevant notion is the David-Wielandt shell of \( A \), which is the joint numerical range of \( A \) and \( A^*: \)

\[
DW(A) = \{ \langle \psi | A \psi \rangle, \langle \psi | A^* A \psi \rangle : \langle \psi | \psi \rangle = 1 \}. \quad (3)
\]

It was known that \( DW(A) \) is always convex for \( d \geq 3 \). For \( d = 2 \), \( DW(A) \) is convex if and only if \( A \) is normal. For the case that \( A \in B(H_2) \) is not normal, \( DW(A) \) is an ellipsoid without interior in \( \mathbb{R}^3 \), which is obviously not convex. We can introduce a scalar function on \( W(A) \) as follows:

\[
h_A(z) = \max \{ t : (z, t) \in DW(A) \}. \quad (15)
\]

By the Cauchy-Schwartz inequality, we have \( h_A(z) \geq |z|^2 \) for any \( z \in W(A) \). More importantly, \( h_A \) is a concave function (and thus continuous) on \( W(A) \) for any \( A \in B(H_2) \), as a consequence of the convexity of \( DW(A) \) for \( d > 3 \) or the geometric observation on the shape of \( DW(A) \) for \( d = 2 \). The significance of \( h_A \) is justified in the following representation of \( W_q(A) \) essentially due to Tsing [13]:

\[
W_q(A) = \{ qz + \tilde{q}w \sqrt{h_A(z)} - |z|^2 : z \in W(A), |w| \leq 1 \},
\]
where \( \bar{q} = \sqrt{1 - q^2} \). Hence the convexity of \( W_q(A) \) for \( 0 \leq q < 1 \) follows from the convexity of \( W(A) \) and the concaveness of \( h_A \).

From the above representation, it is clear that if \( DW(A) = DW(B) \), then \( W_q(A) = W_q(B) \) for any matrices \( A \) and \( B \), and \( 0 \leq q \leq 1 \). However, this requirement can be relaxed. To see this, introduce the upper boundary of \( DW(A) \), say \( \partial DW(A) \), as the set of \( (z, t) \in DW(A) \) such that \( (z, t') \notin DW(A) \) for any \( t' > t \). One can readily see this is exactly the set of \( \{(z, h_A(z)) : z \in W(A)\} \). Then it was shown in Ref. [16] that \( W_q(A) = W_q(B) \) for any \( 0 \leq q \leq 1 \), if and only if \( \partial DW(A) = \partial DW(B) \).

Now let us apply this result to the case of \( A \) and \( B = I_d \otimes A \). Our first observation is the following

\[
DW(A) = \text{Conv}\{ z : z \in DW(A) \},
\]

which can be verified directly from the definition of \( DW(A) \). Thus \( DW(I_d \otimes A) = DW(A) \) follows immediately if \( d \geq 3 \) or \( d = 2 \) and \( A \) is normal, as \( DW(A) \) is convex in this case. For \( d = 2 \) and \( A \) is not normal, we know that \( DW(A) \) is an ellipsoid without interior (by a direct calculation, or see Ref. [16]). Combining this observation with \( DW(I_2 \otimes A) = \text{Conv}(DW(A)) \), we know \( DW(I_2 \otimes A) \) is a solid ellipsoid with \( DW(A) \) as its surface. It is clear that \( \partial DW(I_2 \otimes A) = \partial DW(A) \) in this case.

\[\blacksquare\]