Distance-two coloring of sparse graphs

Zdeněk Dvořák*, Louis Esperet†

* Computer Science Institute, Charles University, Prague, Czech Republic
† CNRS – Laboratoire G-SCOP, Grenoble, France

Abstract
Consider a graph $G = (V, E)$ and, for each vertex $v \in V$, a subset $\Sigma(v)$ of neighbors of $v$. A $\Sigma$-coloring is a coloring of the elements of $V$ so that vertices appearing together in some $\Sigma(v)$ receive pairwise distinct colors. An obvious lower bound for the minimum number of colors in such a coloring is the maximum size of a set $\Sigma(v)$, denoted by $\rho(\Sigma)$. In this paper we study graph classes $\mathcal{F}$ for which there is a function $f$, such that for any graph $G \in \mathcal{F}$ and any $\Sigma$, there is a $\Sigma$-coloring using at most $f(\rho(\Sigma))$ colors. It is proved that if such a function exists for a class $\mathcal{F}$, then $f$ can be taken to be a linear function. It is also shown that such classes are precisely the classes having bounded star chromatic number. We also investigate the list version and the clique version of this problem, and relate the existence of functions bounding those parameters to the recently introduced concepts of classes of bounded expansion and nowhere-dense classes.

Keywords: Distance-two coloring; star coloring; sparse graphs.

1 Introduction
Consider an undirected simple graph $G = (V, E)$ and suppose that for each vertex $v \in V$, we are given a subset $\Sigma(v) \subseteq N(v)$, where $N(v)$ denotes the set of vertices adjacent to $v$ in $G$. We call such a collection a neighborhood system for $G$.

A $\Sigma$-coloring is an assignment of colors to the elements of $V$ so that any pair of vertices appearing together in some $\Sigma(v)$ must receive different colors. It can also be seen as a strong coloring of the natural hypergraph defined by $\Sigma$ on the vertex set $V$. Note that this coloring may not be a proper coloring of $G$ (a condition that was required in [3]). It turns out that adding this condition would not affect the results, so we prefer to use the simpler definition here. We let $\chi(\Sigma)$ denote the minimum number of colors required for a $\Sigma$-coloring to exist.

When additionally each vertex $v$ has its own list $L(v)$ of colors from which its color must be chosen, we talk about a $\Sigma$-L-coloring. We define $ch(\Sigma)$ as the minimum integer $k$ such that for each assignment $L$ of lists of at least $k$ colors to vertices $v \in V$, there a $\Sigma$-L-coloring.

This work was partially supported by ANR Project HEREDIA, under grant ANR-10-JCJC-0204-01, and by the project LL1201 (Complex Structures: Regularities in Combinatorics and Discrete Mathematics) of the Ministry of Education of Czech Republic.
Given a graph $G = (V, E)$ and a neighborhood system $\Sigma$ for $G$, consider the graph $G_\Sigma$ with vertex set $V$ and with the edge set defined as follows: two vertices $u$ and $v$ are adjacent in $G_\Sigma$ if and only if $u$ and $v$ have a common neighbor $w$ such that $\{u, v\} \subseteq \Sigma(w)$. Observe that $\chi(\Sigma)$ is precisely $\chi(G_\Sigma)$, the chromatic number of $G_\Sigma$, and $ch(\Sigma)$ is precisely $ch(G_\Sigma)$, the choice number of $G_\Sigma$.

We denote by $\rho(\Sigma)$ the quantity $\max_{v \in V} |\Sigma(v)|$. Since $G_\Sigma$ contains a clique of size $\rho(\Sigma)$, we clearly have $\rho(\Sigma) \leq \chi(\Sigma) \leq ch(\Sigma)$. In [3], it was proved that if $\mathcal{F}_S$ is the class of all graphs embeddable on a given surface $S$, then for any $G \in \mathcal{F}_S$ and any neighborhood system $\Sigma$ for $G$, $ch(\Sigma) \leq \frac{3}{4} \rho(\Sigma) + o(\rho(\Sigma))$. As direct consequences of this result, asymptotical versions of a conjecture of Wegner [22] and a conjecture of Borodin [4] (see also [12]) were obtained.

A natural problem is to give a precise characterization of classes $\mathcal{F}$ for which there exists a function $f_\mathcal{F}$ such that every $G \in \mathcal{F}$ and every neighborhood system $\Sigma$ for $G$ satisfies $\chi(\Sigma) \leq f_\mathcal{F}(\rho(\Sigma))$. If such a function $f_\mathcal{F}$ exists, we say that $\mathcal{F}$ is $\sigma$-bounded. Consider $C_\mathcal{F} = \sup_{G \in \mathcal{F}, \Sigma} \chi(\Sigma)/\rho(\Sigma)$, where the supremum is taken over all graphs $G \in \mathcal{F}$ and non-empty neighborhood systems $\Sigma$ for $G$. If $C_\mathcal{F} < \infty$, we say that $\mathcal{F}$ is linearly $\sigma$-bounded.

As mentioned above, the class of graphs embeddable on a fixed surface is linearly $\sigma$-bounded. It is not difficult to derive from this result and the Graph Minor Theorem [21] that any proper minor-closed class $\mathcal{F}$ is also linearly $\sigma$-bounded [3], but the constant $C_\mathcal{F}$ obtained from the second result is huge.

For a graph $G$, the $1$-subdivision $G^*$ of $G$ is the graph obtained from $G$ by subdividing every edge exactly once, i.e. by replacing every edge by a path with 2 edges. For some integer $n \geq 4$, consider $K_n^*$, the 1-subdivision of $K_n$, and for every vertex of degree two in $K_n^*$, set $\Sigma(v) = N(v)$, and otherwise set $\Sigma(v) = \emptyset$. A $\Sigma$-coloring is then exactly a proper coloring of $K_n$, so $\chi(\Sigma) \geq n$, while $\rho(\Sigma) = 2$. Hence, the class containing the 1-subdivisions of all complete graphs is not $\sigma$-bounded.

It follows that there exist classes that have bounded maximum average degree (or degeneracy, or arboricity), but that are not $\sigma$-bounded.

A star coloring of a graph $G$ is a proper coloring of the vertices of $G$ so that every pair of color classes induces a forest of stars. The star chromatic number of $G$, denoted $\chi_s(G)$, is the least number of colors in a star coloring of $G$. We say that a class $\mathcal{F}$ of graphs has bounded star chromatic number if the supremum of $\chi_s(G)$ for all $G \in \mathcal{F}$ is finite.

In Section 2, we will prove that a class $\mathcal{F}$ is $\sigma$-bounded if and only if it is linearly $\sigma$-bounded. In order to do so, we will prove that both statements are equivalent to the fact that $\mathcal{F}$ has bounded star chromatic number. One of the consequences of our main result is that there is a constant $c$ such that every $K_t$-minor free graph $G$ and every neighborhood system $\Sigma$ for $G$ satisfies $\chi(\Sigma) \leq ct^4(\log t)^2\rho(\Sigma)$.

In Section 4, we will extend this result to the list version of the problem using results about arrangeability proved in Section 3. In Section 5 we will consider the clique version of the problem, which has interesting connections with the 2VC-dimension of hypergraphs.

Finally, in Section 6 we will analyze the connections between the extension of the problem to balls of higher radius and the recently introduced notions of classes of bounded expansion and nowhere-dense classes.
2 Star Coloring

In this section, we will prove the main result of this paper.

Theorem 1 Let $\mathcal{F}$ be a class of graphs. The following four propositions are equivalent:

(i) $\mathcal{F}$ is linearly $\sigma$-bounded.

(ii) $\mathcal{F}$ is $\sigma$-bounded.

(iii) The supremum of $\chi(\Sigma)$, over all $G \in \mathcal{F}$ and all neighborhood systems $\Sigma$ for $G$ with $\rho(\Sigma) = 2$, is finite.

(iv) $\mathcal{F}$ has bounded star chromatic number.

The chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial. The implication (iii) $\Rightarrow$ (iv) can be derived from Lemmas 2 and 3 below, which follow from the results of Dvořák [8].

Lemma 2 [8, Lemma 7] There exists a function $g_2$ such that for every $k > 0$, every graph with minimum degree at least $g_2(k)$ contains as a subgraph the 1-subdivision of a graph with chromatic number $k$.

The acyclic chromatic number of a graph $G$, denoted by $\chi_a(G)$, is the least number of colors in a proper coloring of the vertices of $G$ such that every pair of color classes induces a forest. Since a star coloring is also an acyclic coloring, we have $\chi_a(G) \leq \chi_s(G)$. On the other hand, Albertson et al. [1] proved that $\chi_s(G) \leq \chi_a(G)(2\chi_a(G) - 1)$ for any graph $G$.

The following claim is a simple combination of Lemma 2 and the main result of [8].

Lemma 3 There exists a function $g$ such that for every $k > 0$, every graph with star chromatic number at least $g(k)$ contains as a subgraph the 1-subdivision of a graph with chromatic number $k$.

Proof. By Theorem 3 of [8], there exists a function $g_1$ such that for every $k > 0$ and every graph $G$ with acyclic chromatic number at least $g_1(k)$, there exists a graph $H$ with chromatic number $k$ such that either $H$ or the 1-subdivision of $H$ is a subgraph of $G$. Let $g_2$ be the function of Lemma 2. Let $g(k) = 2(g_1(\max(k, g_2(k) + 1)))^2$.

Suppose that $G$ is a graph with star chromatic number at least $g(k)$. By Albertson et al. [1], $G$ has acyclic chromatic number at least $\sqrt{g(k)/2} = g_1(\max(k, g_2(k) + 1))$. Hence, there exists a graph $H$ with chromatic number at least $g_2(k)$ containing a subgraph with minimum degree at least $g_2(k)$. Consequently, there exists a graph $H'$ with chromatic number $k$ such that the 1-subdivision of $H'$ is a subgraph of $H \subseteq G$. In both cases, the 1-subdivision of a graph with chromatic number $k$ is a subgraph of $G$. \qed

Proof of (iii) $\Rightarrow$ (iv) of Theorem 1 Suppose that there exists a constant $k$ such that for every $G \in \mathcal{F}$ and every neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$, we have $\chi(\Sigma) \leq k$. Let $g$ be the function of Lemma 3.
Suppose that a graph $G \in \mathcal{F}$ has star chromatic number at least $g(k+1)$. Then, there exists a graph $H$ with chromatic number $k+1$ such that $G$ contains the 1-subdivision $H^*$ of $H$ as a subgraph. For every vertex $v \in V(H^*)$ that has degree two in $H^*$, let $\Sigma(v) = N_{H^*}(v)$. For all other vertices of $G$, set $\Sigma(v) = \emptyset$. Note that a $\Sigma$-coloring of $G$ by $k$ colors induces a proper coloring of $H$, which is a contradiction since $H$ has chromatic number $k+1$. Therefore, the star chromatic number of every graph in $\mathcal{F}$ is less than $g(k+1)$.

The idea in the proof above will be used repeatedly in this paper: for all graphs $G$ and $H$ such that $G$ contains the 1-subdivision $H^*$ of $H$ as a subgraph, there is a neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$ such that $G_\Sigma$ is isomorphic to $H$ together with a set of isolated vertices.

Before we prove the implication $(iv) \Rightarrow (i)$, we need to introduce a couple of definitions. For a graph $G$, a colored in-orientation of $G$ is a pair $(c, \overrightarrow{G})$, where $\overrightarrow{G}$ is an orientation of $G$ and $c$ is a proper coloring of $G$ such that for every 2-colored path on 3 vertices in $G$, the edges of the path are oriented toward the middle vertex. Albertson et al. [14], and independently Nešetřil and Ossona de Mendez [14], observed the following:

**Observation 4.** A proper coloring $c$ of a graph $G$ is a star coloring if and only if there exists an orientation $\overrightarrow{G}$ of $G$ such that $(c, \overrightarrow{G})$ is a colored in-orientation of $G$. Consequently, $G$ has star chromatic number at most $k$ if and only if $G$ has a colored in-orientation using $k$ colors.

We now prove the implication $(iv) \Rightarrow (i)$ in Theorem [1] that is, if $\mathcal{F}$ has bounded star chromatic number, then $\mathcal{F}$ is linearly $\sigma$-bounded. This is a direct consequence of Lemma 5 below.

**Lemma 5.** Every graph $G$ and every neighborhood system $\Sigma$ for $G$ satisfies $\chi(\Sigma) \leq \chi_s(G)^2 \rho(\Sigma)$.

**Proof.** Take a graph $G$ and a neighborhood system $\Sigma$ for $G$. Consider a colored in-orientation $(c_1, \overrightarrow{G})$ of $G$ using at most $k = \chi_s(G)$ colors. Note that by definition of a colored in-orientation, every vertex has out-degree at most $k$ in $\overrightarrow{G}$. Let $G_2$ be the graph with vertex set $V(G_2) = V(G)$ and edge set $E(G_2)$ defined as follows: $uv \in E(G_2)$ if and only if there exists $w$ such that $\overrightarrow{uw}, \overrightarrow{vw} \in E(\overrightarrow{G})$ and $u, v \in \Sigma(w)$. For every $v \in V(G_2)$, the degree of $v$ in $G_2$ is at most the sum of $|\Sigma(w)| - 1$ over all vertices $w \in V(G)$ satisfying $\overrightarrow{vw} \in E(\overrightarrow{G})$. Since every vertex has out-degree at most $k$ in $\overrightarrow{G}$, we conclude that the maximum degree of $G_2$ is at most $k(\rho(\Sigma) - 1)$. Hence, there is a proper coloring $c_2$ of $G_2$ using at most $k(\rho(\Sigma) - 1) + 1 \leq k\rho(\Sigma)$ colors.

For any vertex $v \in V(G)$, define $c(v) = (c_1(v), c_2(v))$. This coloring uses at most $k^2 \rho(\Sigma)$ colors. We now prove that $c$ is a $\Sigma$-coloring of $G$. Take a vertex $w$ of $G$ and two neighbors $u, v$ of $w$ with $u, v \in \Sigma(w)$. If $c_1(u) = c_1(v)$, we know by the definition of an colored in-orientation that $\overrightarrow{uw}, \overrightarrow{vw} \in E(\overrightarrow{G})$. In this case, by the definition of $G_2$ and $c_2$, we have $c_2(u) \neq c_2(v)$. This shows that $\chi(\Sigma) \leq k^2 \rho(\Sigma)$ and concludes the proof of Lemma 5.

It follows from [14] that graphs with no $K_t$-minor have star chromatic number $O(t^2 \log t)$. Hence, we obtain the following corollary of Lemma 5.

**Corollary 6.** There is a constant $c$ such that every $K_t$-minor free graph $G$ and every neighborhood system $\Sigma$ for $G$ satisfies $\chi(\Sigma) \leq ct^4(\log t)^2 \rho(\Sigma)$.
3 Arrangability

Recall that the graph $G_\Sigma$ with vertex set $V$ was defined as follows: two vertices $u$ and $v$ are adjacent in $G_\Sigma$ if and only if $u$ and $v$ have a common neighbor $w$ such that $\{u, v\} \subseteq \Sigma(w)$. We define $\text{mad}(\Sigma)$ as the maximum average degree over all subgraphs of $G_\Sigma$. Note that we have $\chi(\Sigma) \leq \text{ch}(\Sigma) \leq \lfloor \text{mad}(\Sigma) \rfloor + 1$.

The arrangeability of a graph $G$ is the least $k$ such that there exists a linear ordering $\prec$ of the vertices of $G$ satisfying the following: for each vertex $v$, there are at most $k$ vertices smaller than $v$ that have a common neighbor $u$ with $v$ such that $v \prec u$. It can be observed that any $k$-arrangeable graph is $(k+1)$-degenerate and has acyclic chromatic number at most $2k+2$, see [8].

A class $\mathcal{F}$ has bounded arrangeability if there exists a constant $k$ such that every graph in $\mathcal{F}$ has arrangeability at most $k$. We now prove the following result relating the maximum average degree of neighborhood system $\Sigma$ for graphs in a class $\mathcal{F}$ and the arrangeability.

**Theorem 7** Let $\mathcal{F}$ be a class of graphs. The following statements are equivalent:

(i) There is a constant $C_F$ such that every $G \in \mathcal{F}$ and every neighborhood system $\Sigma$ for $G$ satisfies $\text{mad}(\Sigma) \leq C_F \cdot \rho(\Sigma)$.

(ii) There is a function $f_F$ such that every $G \in \mathcal{F}$ and every neighborhood system $\Sigma$ for $G$ satisfies $\text{mad}(\Sigma) \leq f_F(\rho(\Sigma))$.

(iii) The supremum of $\text{mad}(\Sigma)$ over all graphs in $\mathcal{F}$ and all neighborhood systems $\Sigma$ for $G$ with $\rho(\Sigma) = 2$ is finite.

(iv) $\mathcal{F}$ has bounded arrangeability.

The chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial. We now prove the implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i).

**Proof of (iii) $\Rightarrow$ (iv).** Assume first that the maximum average degree of the graphs in $\mathcal{F}$ is not bounded. Then we can find subgraphs of graphs in $\mathcal{F}$ with arbitrarily large minimum degree. It follows from Lemma 2 that we can find 1-subdivisions of graphs with arbitrarily large chromatic number (and hence, minimum degree) as subgraphs of graphs of $\mathcal{F}$. The corresponding neighborhood systems $\Sigma$ with $\rho(\Sigma) = 2$ witness that $\text{mad}(\Sigma)$ is unbounded.

We may now assume that there exists a constant $d_0$ such that every graph of $\mathcal{F}$ has maximum average degree at most $d_0$. By Dvořák [8, Theorem 9], every graph $G$ with maximum average degree at most $d$ and arrangeability more than $4d^2(4d+5)$ contains the 1-subdivision of a graph $H$ with minimum degree at least $d$ as a subgraph. Hence, if $\mathcal{F}$ contains graphs with arbitrarily large arrangeability, then we can again find a neighborhood system $\Sigma$ with $\rho(\Sigma) = 2$ and $\text{mad}(\Sigma) \geq d$ for every $d \geq d_0$, contradicting (iii).

**Proof of (iv) $\Rightarrow$ (i).** Take the linear ordering $\prec$ of the vertices of $G \in \mathcal{F}$ witnessing that $G$ has arrangeability at most $k$. Note that each vertex of $G$ has at most $k+1$ neighbors preceding it according to $\prec$, as otherwise the latest such neighbor would contradict the $k$-arrangeability of $G$.

Consider a subgraph $H$ of $G_\Sigma$, and let $v$ be its last vertex in the ordering. We want to find an upper bound on the number of neighbors $u$ of $v$ in $H$. For each such vertex $u$ there...
is a vertex \( w \) in \( G \) with \( u, v \in \Sigma(w) \). Since \( vwu \) is a path in \( G \) and \( u \prec v \), by the definition of arrangeability there are at most \( k \) choices of \( u \) with \( v \prec w \). Moreover, there are only \( k + 1 \) choices of \( w \) with \( w \prec v \) by the previous paragraph, and for each choice of \( w \) there are at most \( |\Sigma(w)| \leq \rho(\Sigma) \) choices for \( u \). Therefore, \( v \) has at most \( k + (k + 1) \rho(\Sigma) \leq (2k + 1) \rho(\Sigma) \) neighbors in \( H \). It follows that \( G^\Sigma \) is \((2k + 1) \rho(\Sigma))-degenerate, and thus \( \mad(\Sigma) \leq (4k + 2) \rho(\Sigma) \). \( \square \)

4 Choosability

In this section, we consider the list variant of \( \Sigma \)-coloring. The star choice number of a graph \( G \), denoted \( ch_s(G) \), is the minimum \( k \) such that if every vertex of \( G \) is given a list \( L(v) \) of size at least \( k \), \( G \) has a star coloring in which every vertex is assigned a color from its list.

**Theorem 8** Let \( F \) be a class of graphs. The following propositions are equivalent:

(i) \( F \) has bounded star choice number.

(ii) \( F \) has bounded arrangeability.

(iii) There is a constant \( C_F \) such that every \( G \in F \) and every neighborhood system \( \Sigma \) for \( G \) satisfies \( ch(\Sigma) \leq C_F \cdot \rho(\Sigma) \).

(iv) There is a function \( f_F \) such that every \( G \in F \) and every neighborhood system \( \Sigma \) for \( G \) satisfies \( ch(\Sigma) \leq f_F(\rho(\Sigma)) \).

(v) The supremum of \( ch(\Sigma) \), over all \( G \in F \) and all neighborhood systems \( \Sigma \) for \( G \) with \( \rho(\Sigma) = 2 \), is finite.

We use the following result of Kang [13]. A \( t \)-improper coloring of a graph \( H \) is an assignment of colors to its vertices such that each color induces a subgraph of maximum degree at most \( t \).

**Theorem 9** [13, Theorem 6] For every \( k, t \geq 0 \), there exists \( D > 0 \) with the following property. For every graph \( H \) of minimum degree at least \( D \), there exists an assignment \( L \) of lists of size \( k \) to the vertices of \( H \) such that \( H \) has no \( t \)-improper coloring from \( L \).

Let us first prove the equivalence of (i) and (ii), which is interesting in its own right.

**Lemma 10** A class of graphs \( F \) has bounded star choice number if and only if it has bounded arrangeability.

**Proof.** Consider a graph \( G \in F \). First, we show that if \( G \) has arrangeability at most \( k \), then its star choice number is at most \( (k + 2)^2 \). Let \( \prec \) be an ordering of the vertices of \( G \) witnessing its arrangeability. Let \( L \) be any assignment of lists of length \( (k + 2)^2 \) to the vertices of \( G \). For each \( v \in V(G) \), let \( P(v) \) denote the set of vertices \( u \prec v \) such that either \( u \) is adjacent to \( v \) or there exists a path \( uuv \) with \( u \prec v \). Recall that each vertex of \( G \) has at most \( k + 1 \) neighbors preceding it in \( \prec \), thus \( P(v) \) contains at most \( k + 1 \) neighbors of \( v \), and at most \( (k + 1)^2 \) vertices \( u \) such that there exists a path \( uuv \) with \( u \prec w \prec v \). By arrangeability, \( P(v) \) contains at most \( k \) vertices \( u \) such that there exists a path \( uuv \) with \( u \prec v \prec w \). Consequently, \( |P(v)| \leq (k + 1)^2 + (k + 1) + k < |L(v)| - 1 \).
Therefore, we can L-color \( G \) greedily so that each \( v \in V(G) \) has a color \( c(v) \) different from the colors of the elements of \( P(v) \). Clearly, \( c \) is a proper \( L \)-coloring of \( G \). Consider a path \( P \subseteq G \) on four vertices. Let \( u \) be the earliest vertex of \( P \) according to \( \prec \). Note that \( P \) contains a path on three vertices starting with \( u \). Let \( uvw \) be such a path. We have \( u \prec v \) and \( u \prec w \), and thus \( u \in P(v) \). Consequently, \( u, v \) and \( w \) have distinct colors, and \( P \) is not bichromatic. It follows that \( c \) is a star coloring of \( G \). Since the choice of the lists \( L \) was arbitrary, the star choice number of \( G \) is at most \((k + 2)^2\).

Suppose now that \( F \) has star choice number at most \( k \) and let \( D \) be the constant from Theorem 9 for \( t = k \). As shown in the proof of the implication (iii) \( \Rightarrow \) (iv) in Theorem 9, in order to prove that \( F \) has bounded arrangeability, it suffices to prove that for every graph \( H \) of minimum degree \( D \), the 1-subdivision of \( H \) does not appear as a subgraph of any \( G \in F \). Suppose on the contrary that there exists a graph \( H \) of minimum degree \( D \) and \( G \in F \) containing the 1-subdivision \( H^* \) of \( H \) as a subgraph. Let \( L \) be the list assignment for \( H \) from Theorem 9 for \( t = k \). Let \( L^* \) be the list assignment for \( H^* \) matching \( L \) on the vertices of \( H \) and assigning to all vertices of \( H^* \) of degree two the list \( \{1, \ldots, k\} \). Since the star choice number of \( G \) is at most \( k \), there exists a star coloring \( c \) of \( H^* \) from \( L^* \). By Observation 4 there is an orientation \( \overrightarrow{H^*} \) of \( H^* \) such that \((c, \overrightarrow{H^*}) \) is a colored in-orientation of \( H^* \). Since all vertices of \( H^* \) of degree two have the same list, the maximum out-degree of \( \overrightarrow{H^*} \) is at most \( k \). Thus, each vertex \( v \in V(H) \) has at most \( k \) neighbors \( u \in V(H) \) such that the vertex \( w \) subdividing the edge \( uv \) in \( H^* \) satisfies \( \overrightarrow{vw}, \overrightarrow{uw} \in \overrightarrow{H^*} \). By the definition of a colored in-orientation, each \( v \in V(H) \) has at most \( k \) neighbors \( u \in V(H) \) such that \( c(u) = c(v) \). Consequently, \( c \) induces a \( k \)-improper coloring of \( H \) from the lists \( L \), contradicting Theorem 9.

\( \Box \)

For the implication (ii) \( \Rightarrow \) (iii), note that by Theorem 9 there exists a constant \( C'_{\mathcal{F}} \) such that for each \( G \in \mathcal{F} \) and each neighborhood system \( \Sigma \) for \( G \), the graph \( G_{\Sigma} \) has maximum average degree at most \( C'_{\mathcal{F}} \cdot \rho(\Sigma) \). Hence, \( G_{\Sigma} \) has choice number at most \( C'_{\mathcal{F}} \cdot \rho(\Sigma) + 1 \leq (C'_{\mathcal{F}} + 1) \cdot \rho(\Sigma) \). Therefore, (iii) holds with \( C_{\mathcal{F}} = C'_{\mathcal{F}} + 1 \).

The implications (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) are trivial.

For the implication (v) \( \Rightarrow \) (ii), let \( k \) satisfy \( ch(\Sigma) \leq k \) for all \( G \in \mathcal{F} \) and all neighborhood systems \( \Sigma \) for \( G \) with \( \rho(\Sigma) = 2 \). Let \( D \) be the constant of Theorem 9 with \( t = 0 \). Then for all \( G \in \mathcal{F} \) and all neighborhood systems \( \Sigma \) for \( G \) with \( \rho(\Sigma) = 2 \) we have that \( G_{\Sigma} \) is 2-degenerate and so \( mad(\Sigma) \leq 2D \). By Theorem 7 it follows that \( \mathcal{F} \) has bounded arrangeability. This concludes the proof of Theorem 8.

It is interesting to note that in general for a class \( \mathcal{F} \), it is not equivalent to be \( \sigma \)-bounded and to have \( ch(\Sigma) \) bounded by a function of \( \rho(\Sigma) \). To prove this, it is enough by Theorems 1 and 8 to construct a family of graphs with bounded star chromatic number and unbounded arrangeability. For \( n \geq 3 \), let \( H_n \) be the 1-subdivision of the complete bipartite graph \( K_{n,n} \). Note that every \( H_n \) has star chromatic number at most three. On the other hand, \( H_n \) is a 1-subdivision of a graph with minimum degree \( n \), and thus its arrangeability is at least \((n - 1)/2\) (see the beginning of Section 2 in [8] for an argument showing this). Thus, the class \( \{H_n\}_{n \geq 3} \) has bounded star chromatic number and unbounded arrangeability.
5 Cliques

Given a graph $G = (V, E)$ and a neighborhood system $\Sigma$ for $G$, a $\Sigma$-clique is a set $C$ of elements of $V$ such for that any pair $u, v \in C$ there is a vertex $w \in V$ such that $u, v \in \Sigma(w)$. The maximum size of a $\Sigma$-clique is denoted by $\omega(\Sigma)$. Note that $\omega(\Sigma)$ is precisely $\omega(G_\Sigma)$, the clique number of $G_\Sigma$, so we trivially have $\rho(\Sigma) \leq \omega(\Sigma) \leq \chi(\Sigma)$.

In the case where $\Sigma(v) = N(v)$ for every vertex $v$ of $G$, the parameter $\omega(\Sigma)$ has been studied for several graph classes, such as planar graphs [6, 11] and line-graphs [5, 10].

As previously, it is natural to investigate classes $\mathcal{F}$ for which there is a function $f$, such that for any $G \in \mathcal{F}$ and any neighborhood system $\Sigma$ for $G$, we have $\omega(\Sigma) \leq f(\rho(\Sigma))$.

Let $\mathcal{F}$ be a class of graphs that contains as subgraphs of graphs in $\mathcal{F}$ the 1-subdivisions of arbitrarily large cliques. Then $\omega$ cannot be bounded by a function of $\rho$ in $\mathcal{F}$. In the following, we will show that the converse also holds.

**Theorem 11** Let $\mathcal{F}$ be a class of graphs, and denote by $\overline{\mathcal{F}}$ the class containing all subgraphs of graphs in $\mathcal{F}$. The following statements are equivalent:

(i) The supremum of $\omega(\Sigma)$ over all graphs in $\mathcal{F}$ and all neighborhood systems $\Sigma$ for $G$ with $\rho(\Sigma) = 2$ is finite.

(ii) There is a constant $C_\mathcal{F}$ such that every $G \in \mathcal{F}$ and every neighborhood system $\Sigma$ for $G$ satisfies $\omega(\Sigma) \leq C_\mathcal{F} \cdot \rho(\Sigma)$.

(iii) There is a function $f_\mathcal{F}$ such that every $G \in \mathcal{F}$ and every neighborhood system $\Sigma$ for $G$ satisfies $\omega(\Sigma) \leq f_\mathcal{F}(\rho(\Sigma))$.

(iv) $\overline{\mathcal{F}}$ does not contain 1-subdivisions of arbitrarily large cliques.

The implication (ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (iv) corresponds to the remark above. We now prove the implications (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii).

**Proof of (iv) $\Rightarrow$ (i).** Fix some integer $n$. If (i) does not hold, then there is a graph $G \in \mathcal{F}$ and a neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$, such that there is a $\Sigma$-clique $C$ of size at least $3n$. Consider the graph $H$ with vertex set $C$, in which two vertices $u, v \in C$ are adjacent if and only if there exists a vertex $w \notin C$ with $\Sigma(w) = \{u, v\}$. $H$ has at least $\binom{3n}{2} - 3n$ edges so by Turán’s theorem $H$ contains a clique of size $(3n)^2/(2 \times 3n + 3n) = n$. It follows that $G$ contains (as a subgraph) the 1-subdivision of a clique of size $n$. $\square$

**Proof of (i) $\Rightarrow$ (ii).** A $\Sigma$-clique can be seen as a hypergraph in which every pair of vertices is contained in a hyperedge (i.e. a hypergraph with strong stability number 1). We call such a hypergraph a full hypergraph. Given a hypergraph $\mathcal{H}$, a subhypergraph of $\mathcal{H}$ on a set $X \subseteq V(\mathcal{H})$ is a hypergraph with the vertex set $X$ whose hyperedge set is a subset of $\{X \cap e | e \in E(\mathcal{H})\}$. The rank $r(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the maximum size of a hyperedge of $\mathcal{H}$.

To prove (i) $\Rightarrow$ (ii), it is enough to prove the following:

(*) For every $r, n \geq 2$, every full hypergraph of rank at most $r$ having at least $4rn^2 + 2$ vertices contains a full subhypergraph with rank two on a set of at least $n$ vertices.

Consider a full hypergraph $\mathcal{H}$ of rank at most $r$, having $N \geq 4rn^2 + 2$ vertices. For each pair $u, v \in V(\mathcal{H})$ choose a hyperedge $e_{u,v}$ containing $u$ and $v$. Given a subset $X \subseteq V(\mathcal{H})$, we say that two vertices $u, v \in V(\mathcal{H})$ form a bad $X$-pair if $u, v \in X$ and $|X \cap e_{u,v}| > 2$. 8
We now consider a subset $X \subset V(H)$ taken uniformly at random among all subsets of $V(H)$ of size $2n$. Observe that choosing $X$ so that two given vertices $u$ and $v$ form a bad $X$-pair is the same as choosing a vertex $w \in e_{u,v}$ distinct from $u$ and $v$, and then $2n - 3$ vertices in $V(H)$ distinct from $u$, $v$, and $w$. Let us remark that if $|X \cap e_{u,v}| > 3$, then there are several possible choices of $w$. It follows that the probability that $u$ and $v$ form a bad $X$-pair is at most
\[
p = \frac{(r-2)\binom{N-3}{2n-3}}{\binom{N}{2n}} < \frac{8rn^3}{N(N-1)(N-2)}.
\]
Therefore, the expectation of the number of bad $X$-pairs is at most $\binom{N}{2}p < \frac{4rn^3}{N-2}$.

It follows that there exists a set $X \subset V(H)$ of size $2n$, such that less than $\frac{4rn^3}{N-2}$ pairs of vertices of $V(H)$ are bad $X$-pairs. Since $N \geq 4rn^2 + 2$, less than $n$ pairs of vertices from $X$ are bad $X$-pairs. Consider a graph $H$ with vertex set $X$, and edges $uv$ if and only if $u, v$ do not form a bad $X$-pair. This graph has $2n$ vertices and at most $n$ non-edges, so by Turán’s theorem it contains a clique of size $(2n)^2/(2n + 2n) = n$. This clique corresponds to a subset $Y \subseteq X$ of size $n$, such that no pair of vertices from $Y$ is a bad $Y$-pair. Hence, there exists a full subhypergraph of $\mathcal{H}$ of rank two on $Y$, which concludes the proof. \(\square\)

It is interesting to note that (*) can also be derived directly from a result of Ding, Seymour, and Winkler [7]. For the sake of completeness, we include this alternative proof.

The strong stability number $\alpha(H)$ of a hypergraph $H$ is the maximum size of a set $S \subset V(H)$ such that no hyperedge of $H$ contains more than one vertex of $S$. A fractional covering of $H$ is a map $\phi : E(H) \to [0,1]$ such that for any vertex $v$, $\sum_{e \ni v} \phi(e) \geq 1$. The fractional covering number $\kappa^*(H)$ of $H$ is the infimum of $\sum_{e \in E(H)} \phi(e)$ over all fractional coverings of $H$. Let $\lambda(H)$ denote the maximum size of a set $S \subseteq V(H)$ such that for all distinct $u, v \in S$, there exists an hyperedge $e_{u,v} \in E(H)$ such that $e_{u,v} \cap S = \{u,v\}$. Ding, Seymour, and Winkler proved the following (see the end of the proof of (5.7) in [7] where the statement is in the dual setting).

\textbf{Lemma 12 ([7])} For any hypergraph $H$, $\kappa^*(H) \leq \frac{2\lambda(H)}{\omega(H)} \left(\frac{\lambda(H) + \alpha(H)}{\alpha(H)}\right)^2$.

\textbf{Corollary 13} For any $r \geq 2$ and $n \geq 1$, every full hypergraph of rank at most $r$ having at least $\frac{27}{8}r(n+1)^2$ vertices contains a full subhypergraph with rank at most two having at least $n$ vertices.

\textbf{Proof.} Consider a full hypergraph $H$ of rank at most $r$, having $N \geq \frac{27}{8}r(n+1)^2$ vertices. Observe that $\kappa^*(H) \geq N/r$, so Lemma 12 with $\alpha(H) = 1$ gives $\frac{2\lambda(H)}{\omega(H)}(\lambda(H)+1)^2 \geq N/r$. It follows that $\lambda(H) \geq n$. For any set $S$ witnessing this inequality, there exists a full subhypergraph of $H$ on $S$ with rank two, which concludes the proof. \(\square\)

A class that is $\sigma$-bounded has the property that $\omega(\Sigma)$ is bounded by a function of $\rho(\Sigma)$, but the converse does not hold in general. Take a class $\mathcal{F}$ of triangle-free graphs with arbitrarily large chromatic numbers, and consider the class $\mathcal{F}^*$ consisting of the 1-subdivisions of the graphs in $\mathcal{F}$. The class $\mathcal{F}^*$ does not contain any graph containing the 1-subdivision of a clique of size more than two. On the other hand, $\mathcal{F}^*$ is trivially not $\sigma$-bounded.
6 Bounded expansion

The notion of a class of bounded expansion was introduced by Nešetřil and Ossona de Mendez in \cite{15, 16, 17, 18}. Examples of such classes include minor-closed classes, topological minor-closed classes, classes locally excluding a minor, and classes of graphs that can be drawn in the plane with a bounded number of crossings per edge. It was recently proved that first-order properties can be decided in linear time in classes of bounded expansion \cite{9}. Additionally, such classes have deep connections with the existence of finite homomorphism dualities \cite{18}.

For more about this topic, the reader is referred to the survey book of Nešetřil and Ossona de Mendez \cite{20}.

A graph $H$ is a shallow topological minor of $G$ at depth $d$ if $G$ contains a $(\leq 2d)$-subdivision of $H$ (i.e. a graph obtained from $H$ by subdividing each edge at most $2d$ times) as a subgraph. For any $d \geq 0$, let $\nabla_d(G)$ be defined as the maximum of $|E(H)|/|V(H)|$, over all shallow topological minors $H$ of $G$ at depth $d$. Note that $\nabla_0(G) = \text{mad}(G)/2$ and the function $d \mapsto \nabla_d(G)$ is monotone. If there is a function $f$ such that for every $d \geq 0$, every graph $G \in \mathcal{F}$ satisfies $\nabla_d(G) \leq f(d)$, then $\mathcal{F}$ is said to have bounded expansion.

Let $G$ be a simple undirected graph. For an integer $d \geq 1$ and a vertex $v \in V(G)$, let $N^d(v)$ denote the set of vertices at distance at most $d$ from $v$ in $G$, excluding $v$ itself. We say that $\Sigma$ is a $d$-neighborhood system if $\Sigma(v) \subseteq N^d(v)$ for all $v \in V(G)$. A class of graphs $\mathcal{F}$ is said to be $\sigma$-bounded at depth $d$ if there exists a function $f$ such that for any $G \in \mathcal{F}$ and any $d$-neighborhood system $\Sigma$ for $G$, we have $\chi(\Sigma) \leq f(\rho(\Sigma))$. Note that a class is $\sigma$-bounded at depth $1$ precisely if it is $\sigma$-bounded.

The following two propositions are equivalent.

(i) There exists a constant $c$, such that every $G \in \mathcal{F}$ and every $d$-neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$ satisfies $\chi(\Sigma) \leq c$.

(ii) $\mathcal{F}$ is $\sigma$-bounded at depth $d$.

Proof of (i) $\Rightarrow$ (ii). Assume that for any $G \in \mathcal{F}$ and any $d$-neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$, we have $\chi(\Sigma) \leq k$. Let $\ell = \left(\frac{\rho(\Sigma)}{2}\right)$. For each vertex $v$ of $G$, let $\ell' = \left(\frac{\rho(\Sigma)}{2}\right)$, let $\Sigma_1(v), \ldots, \Sigma_{\ell'}(v)$ be all pairs of elements of $\Sigma(v)$, and let $\Sigma_{\ell'+1}(v) = \ldots = \Sigma_{\ell}(v) = \emptyset$. For $1 \leq i \leq \ell$, let $c_i$ be a $\Sigma_i$-coloring using at most $k$ colors. Then the coloring $c$ defined by $c(v) = (c_1(v), \ldots, c_{\ell}(v))$ is a $\Sigma$-coloring using at most $k(\rho(\Sigma)/2)$ colors.

We only use the trivial direction (ii) $\Rightarrow$ (i) to prove the following result.

**Lemma 14** If a class $\mathcal{F}$ of graphs is $\sigma$-bounded at depth $d$ for every integer $d \geq 1$, then $\mathcal{F}$ has bounded expansion.

**Proof.** Assume that there is a function $f$ such that for any integer $d \geq 1$, any graph $G \in \mathcal{F}$ and any $d$-neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$, we have $\chi(\Sigma) \leq f(d)$. Let $g_2$ be the function of Lemma \cite{12}.

We will prove that for any $d \geq 1$ and any $G \in \mathcal{F}$, $\nabla_d(G) < g_2(2d+1)+1 = a$. Assume for the sake of contradiction that it is not the case. Then there exists a graph $G \in \mathcal{F}$ and an integer $d \geq 1$ such that $\nabla_d(G) \geq a$. It follows that $G$ has a subgraph that is a $(\leq 2d)$-subdivision of a graph $H$ of minimum degree at least $a$. Note that $H$ has a subgraph that is
the 1-subdivision of a graph $H'$ with $\chi(H') = f(2d + 1) + 1$. Hence, $G$ has a subgraph that is a $(\leq 4d + 1)$-subdivision of $H'$. Let $u_1, \ldots, u_t$ be the images of the vertices $v_1, \ldots, v_t$ of $H'$ in $G$. For any edge $v_i v_j$ in $H'$, select a vertex $u_{i,j}$ in $G$ lying in the middle of the path between $u_i$ and $u_j$ corresponding to the image of the edge $v_i v_j$ in $G$. Note that $u_{i,j}$ is at distance at most $2d + 1$ from $u_i$ and $u_j$, and all $u_{i,j}$ are pairwise distinct. Set $\Sigma(u_{i,j}) = \{u_i, u_j\}$ for all $i, j$, and $\Sigma(v) = \emptyset$ for all the other vertices. Observe that $\Sigma$ is a $(2d + 1)$-neighborhood system, $\rho(\Sigma) = 2$, and yet $\chi(\Sigma) > f(2d + 1)$, which contradicts the hypothesis.

Unfortunately, it turns out that the converse of Lemma 14 is not true. Consider the graph $S_n$ consisting of a star with $\binom{n}{2} + n$ leaves $\{v_{i,j} \mid 1 \leq i < j \leq n\} \cup \{v_i \mid 1 \leq i \leq n\}$. For any $i < j$, set $\Sigma(v_{i,j}) = \{v_i, v_j\}$, and set $\Sigma(v) = \emptyset$ for all other vertices. Note that $\Sigma$ is a 2-neighborhood system and $\rho(\Sigma) = 2$, but $\chi(\Sigma) \geq n$. Hence, the family $\mathcal{F} = \{S_n \mid n \geq 1\}$ is not even $\sigma$-bounded at depth 2 (while it clearly has bounded expansion).

A way to circumvent this would be to add a notion of complexity to that of a neighborhood system $\Sigma$. A realizer $R$ for a $d$-neighborhood system $\Sigma$ is a set of paths of length at most $2d$ between all pairs $u, v$ such that $u \in \Sigma(v)$. Given such a realizer $R$, we define $\lambda(R)$ as the maximum over all vertices $u$ of $G$ of the number of paths of $R$ containing $u$. The complexity $\lambda(G, \Sigma)$ is the minimum of $\lambda(R)$ over all realizers of $\Sigma$.

If we only allow $d$-neighborhood systems $\Sigma$ with $\lambda(G, \Sigma)$ bounded by a fixed function of $d$, then it can be shown that having bounded expansion is equivalent to being $\sigma$-bounded at each depth. We omit the details, since the proof is very close from that showing that the immersion and topological minor resolutions are equivalent (see [20, Section 5.8]).

Nešetřil and Ossona de Mendez [19] call a class $\mathcal{F}$ of graphs somewhere dense if there exists an integer $d$, such that the set of shallow topological minors at depth $d$ of graphs of $\mathcal{F}$ is the set of all graphs. Otherwise $\mathcal{F}$ is nowhere dense.

If a graph $G$ contains a $(\leq 2d - 1)$-subdivision of a clique on $k$ vertices, then as above we can find a $d$-neighborhood system $\Sigma$ for $G$ with $\rho(\Sigma) = 2$, and $\omega(\Sigma) \geq k$. This remark has the following direct consequence.

**Lemma 15** If there is a function $f$, such that for any $d \geq 1$, for any $G \in \mathcal{F}$ and for any $d$-neighborhood system $\Sigma$ for $G$, we have $\omega(\Sigma) \leq f(\rho(\Sigma), d)$, then $\mathcal{F}$ is nowhere-dense.

Again, we note that if we require that the complexity of the neighborhood systems $\Sigma$ we consider is bounded by a function of the depth, the two properties in Lemma 15 are indeed equivalent.

### 7 Remarks

If we add to the definition of $\Sigma$-coloring the condition that the coloring must be a proper coloring of $G$ (which is the way it was defined in [3]), then the same proofs work. This shows that the two definitions of $\Sigma$-coloring are equivalent: a class is $\sigma$-bounded according to one definition if and only if it is $\sigma$-bounded according to the other (and the same holds for the list versions).
A class of graphs for which the chromatic number is bounded by a function of the clique number is said to be $\chi$-bounded. Such classes have been extensively studied and many deep conjectures remain open about structural aspects of graphs in these classes. It is interesting to note that in our setting, classes of graphs for which $\chi(\Sigma)$ is bounded by a function of $\omega(\Sigma)$, are much easier to understand: they correspond precisely to $\sigma$-bounded classes. Indeed if a class is $\sigma$-bounded, then $\chi(\Sigma)$ is clearly bounded by a function of $\omega(\Sigma)$. Conversely if in $\mathcal{F}$, $\chi(\Sigma)$ is bounded by a function of $\omega(\Sigma)$, then $\mathcal{F}$ cannot contain (as subgraphs of graphs in the class) 1-subdivisions of arbitrarily large cliques (if it did, we could also find as subgraphs of graphs in $\mathcal{F}$ 1-subdivisions of graphs from a class that is not $\chi$-bounded). Then by Theorem 11, $\omega(\Sigma)$ is bounded by a function of $\rho(\Sigma)$ in $\mathcal{F}$, and consequently $\chi(\Sigma)$ is also bounded by a function of $\rho(\Sigma)$. It follows that $\mathcal{F}$ is $\sigma$-bounded.

Acknowledgement The two authors are grateful to Omid Amini for his kind suggestions and remarks.

References

[1] M.O. Albertson, G.G. Chappel, H.A. Kierstead, A. Kündgen and R. Ramamurthi, Coloring with no 2-colored $P_4$’s, Electron. J. Combin. 11 (2004), #R26.

[2] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), 125–134.

[3] O. Amini, L. Esperet, and J. van den Heuvel, A Unified Approach to Distance-Two Colouring of Graphs on Surfaces, to appear in Combinatorica (2013).

[4] O.V. Borodin, Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of $1$-planar graphs (in Russian), Metody Diskret. Analyz. 41 (1984), 12–26.

[5] F.R.K. Chung, A. Gyárfás, Z. Tuza, and W.T. Trotter, The maximum number of edges in $2K_2$-free graphs of bounded degree, Discrete Math. 81 (1990), 129–135.

[6] N. Cohen and J. van den Heuvel, An exact bound on the clique number of the square of a planar graph, In preparation.

[7] G. Ding, P. Seymour, and P. Winkler, Bounding the vertex cover number of a hypergraph, Combinatorica 14 (1994), 23–34.

[8] Z. Dvořák, On forbidden subdivision characterizations of graph classes, European J. Combin. 29 (2008), 1321–1332.

[9] Z. Dvořák, D. Kráľ, and R. Thomas, Deciding first-order properties for sparse graphs, in: Proc. 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS’10), 133–142.

[10] F.J. Faudree, R.H. Shelp, A. Gyárfás, and Zs. Tuza, The strong chromatic index of graphs, Ars Combin. 29B (1990), 205–211.
[11] P. Hell and K. Seyffarth, *Largest planar graphs of diameter two and fixed maximum degree*, Discrete Math. **111** (1993), 313–322.

[12] T.R. Jensen and B. Toft, *Graph Coloring Problems*, John-Wiley & Sons, New York, 1995.

[13] R.J. Kang, *Improper choosability and Property B*, J. Graph Theory **73**(3) (2013), 342–353.

[14] J. Nešetřil and P. Ossona de Mendez, *Colorings and homomorphisms of minor closed classes*. Discrete & Computational Geometry, The Goodman-Pollack Festschrift, volume 25 of Algorithms and Combinatorics (2003), 651–664.

[15] J. Nešetřil and P. Ossona de Mendez, *Linear time low tree-width partitions and algorithmic consequences*, in: Proc. 38th ACM Symposium on Theory of Computing (STOC’06), 391–400.

[16] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion I. Decompositions*, European J. Combin. **29** (2008), 760–776.

[17] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion II. Algorithmic aspects*, European J. Combin. **29** (2008), 777–791.

[18] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion III. Restricted graph homomorphism dualities*, European J. Combin. **29** (2008), 1012–1024.

[19] J. Nešetřil and P. Ossona de Mendez, *On nowhere dense graphs*, European J. Combin. **32** (2011), 600–617.

[20] J. Nešetřil and P. Ossona de Mendez, *Sparsity – Graphs, Structures, and Algorithms*, Algorithms and Combinatorics Vol. 28, Springer, 2012.

[21] N. Robertson and P. Seymour, *Graph minors XVI. Excluding a non-planar graph*, J. Combin. Theory Ser. B **81** (2003), 43–76.

[22] G. Wegner, *Graphs with given diameter and a coloring problem*, Technical Report, University of Dortmund, 1977.