THE MAY-MILGRAM FILTRATION AND $\mathcal{E}_k$-CELLS

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Abstract. We describe an $\mathcal{E}_k$-cell structure on the free $\mathcal{E}_{k+1}$-algebra on a point, and more generally describe how the May-Milgram filtration of $\Omega^m \Sigma^m S^k$ lifts to a filtration of the free $\mathcal{E}_{k+m}$-algebra on a point by iterated pushouts of free $\mathcal{E}_k$-algebras.

1. Introduction

Cell structures on topological spaces have many uses, and operadic cell structures play similar roles in the study of algebras over an operad; they appear when defining model category structures on categories of algebras over operads, and in the case of the little $k$-cubes operad $\mathcal{E}_k$ they have found applications to homological stability [KM18, GKRW18a, GKRW18b].

Despite the usefulness of such $\mathcal{E}_k$-cell structures, few have been described explicitly. In this paper we give an $\mathcal{E}_k$-cell decomposition of an $\mathcal{E}_k$-algebra weakly equivalent to the free $\mathcal{E}_{k+1}$-algebra on a point. We generalize this to a filtration of the free $\mathcal{E}_{k+m}$-algebra on a point by $\mathcal{E}_k$-algebras, and explain how this can be thought of as a lift of the May-Milgram filtration of the iterated based loop space $\Omega^m \Sigma^m S^k$.

We shall state our results without giving definitions (which appear in Section 2), with the exception of that of a cell attachment in the category $\mathcal{Alg}_{E_k}(\text{Top})$. This definition uses the free $\mathcal{E}_k$-algebra functor $E_k = F_{E_k}$, which is left adjoint to the forgetful functor $U_{E_k} : \mathcal{Alg}_{E_k}(\text{Top}) \to \text{Top}$ sending an $\mathcal{E}_k$-algebra to its underlying space. Let $A$ be an $\mathcal{E}_k$-algebra in $\text{Top}$, and let $e : \partial D^d \to U_{E_k}(A)$ be a map of topological spaces. To attach a $d$-dimensional $\mathcal{E}_k$-cell to $A$, we take the adjoint map $E_k(\partial D^d) \to A$ of $e$ and take the pushout of the following diagram in $\mathcal{Alg}_{E_k}(\text{Top})$:

$$
\begin{array}{ccc}
E_k(\partial D^d) & \longrightarrow & A \\
\downarrow & & \downarrow \\
E_k(D^d) & \longrightarrow & A \cup_{E_k} D^d.
\end{array}
$$

To make this homotopy invariant we need to require that $A$ is cofibrant, or derive the cell-attachment construction.

An $\mathcal{E}_k$-algebra is called cellular if it is built by iterated $\mathcal{E}_k$-cell attachments (such algebras are always cofibrant). Giving an $\mathcal{E}_k$-cell structure on $A$ is giving a weak equivalence between $A$ and a cellular $\mathcal{E}_k$-algebra. Note that the following colimit is also a homotopy colimit:

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Theorem 1.1. $E_{k+1}(*)$ is weakly equivalent as an $\mathcal{E}_k$-algebra to a cellular $\mathcal{E}_k$-algebra with exactly one cell in dimensions divisible by $k$ and no other cells. That is, it is weakly equivalent to the colimit $\operatorname{colim}_{r \in \mathbb{N}} A_r$ of algebras $A_r$ obtained by setting $A_{-1} = \emptyset$ and taking iterated pushouts in $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$

$$
\begin{array}{c}
E_k(\partial D^k) \longrightarrow A_r \\
\downarrow \\
E_k(D^k) \longrightarrow A_{r+1}.
\end{array}
$$

An $\mathcal{E}_k$-cell structure on $A$ induces an ordinary cell structure on its $k$-fold delooping $B^k A$ (see Remark 2.6). The one induced on $B^k E_{k+1}(*) \cong \Omega \Sigma S^k$ by the $\mathcal{E}_k$-cell structure of Theorem 1.1 is that coming from the James construction [Jam55]. The filtration coming from the James construction was generalized by May and Milgram [May72, Mil66] to a filtration on $\Omega^m \Sigma^m S^k$, and we will construct a filtration of $E_{k+m}(*)$ by $\mathcal{E}_k$-algebras which deloops to the May-Milgram filtration on $B^k E_{k+m}(*) \cong \Omega^m \Sigma^m S^k$.

To state this result precisely, we need to introduce some notation. Let $\mathbb{I}$ denote the open interval $(0,1)$, $F_r(\mathbb{I}^n)$ the space of ordered configurations of $r$ points in $\mathbb{I}^n$, and $C_r(\mathbb{I}^n) := F_r(\mathbb{I}^n)/\mathcal{S}_r$ the space of unordered configurations of $r$ points in $\mathbb{I}^n$. Furthermore, let $\phi_{m,r}$ denote the vector bundle $F_r(\mathbb{I}^n) \times_{\mathcal{S}_r} \mathbb{R}^{r-1} \rightarrow C_r(\mathbb{I}^n)$ with $\mathbb{R}^{r-1}$ the representation of the symmetric group $\mathcal{S}_r$ given by the orthogonal complement to the trivial representation in the permutation representation with its usual metric. This vector bundle inherits a Riemannian metric. For $E \rightarrow B$ a vector bundle with a Riemannian metric, let $D(E)$ denote its unit disk bundle, $S(E)$ denote its unit sphere bundle and $kE$ denote its $k$-fold Whitney sum.

Theorem 1.2. $E_{k+m}(*)$ is weakly equivalent as an $\mathcal{E}_k$-algebra to the colimit $\operatorname{colim}_{r \in \mathbb{N}} A_r$ of algebras $A_r$ obtained by setting $A_{-1} = \emptyset$ and taking iterated pushouts in $\mathsf{Alg}_{\mathcal{E}_k}(\mathsf{Top})$

$$
\begin{array}{c}
E_k(S(k\phi_{m,r+1})) \longrightarrow A_r \\
\downarrow \\
E_k(D(k\phi_{m,r+1})) \longrightarrow A_{r+1}.
\end{array}
$$

Thus the homotopy cofiber of $A_r \rightarrow A_{r+1}$ is the free $\mathcal{E}_k$-algebra on the Thom space of $k\phi_{m,r+1}$ viewed as a based space. These Thom spaces and their corresponding Thom spectra are well-studied, e.g. being related to Brown-Gitler spectra when $k = 2$ [CMM78, CCKN83].

When $k = 1$, $C_r(\mathbb{I}^k)$ is contractible and the vector bundle $k\phi_{k,r+1}$ has dimension $kr$. Thus the sphere bundle is homotopy equivalent to $\partial D^k$ and hence Theorem 1.1 is a consequence of Theorem 1.2. In Proposition 4.5 we give configuration space models for the $\mathcal{E}_k$-algebras $A_r$ and in Theorem 5.2, we prove that the $k$-fold delooping of $A_r$ is homotopy equivalent to the $r$th stage of the May-Milgram filtration of $\Omega^m \Sigma^m S^k$.

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Contents

1. Introduction 1
2. Recollections of homotopy theory for algebras over an operad

We shall work in the setting of [GKRW18a] and use similar notation when possible.

Assumption 2.1. $S$ is a simplicially enriched complete and cocomplete category with closed symmetric monoidal structure such that the tensor product $\otimes$ commutes with sifted colimits.

Assumption 2.2. $S$ comes equipped with a cofibrantly generated model structure which is both simplicial and monoidal, and such that the monoidal unit $\mathbb{1}$ is cofibrant.

The first version of [GKRW18a] required that homotopy equivalences are weak equivalences but this is in fact always the case by Proposition 9.5.16 of [Hir03].

When $G$ is a symmetric monoidal category, then we may endow the category $S^G$ of functors $G \to S$ with the Day convolution tensor product; this will also be symmetric monoidal. Similarly, the category $S_*$ of pointed objects in $S$ with smash product inherits these properties.

Example 2.3. The examples of $S$ most relevant to this paper are: (i) the category $sSet$ of simplicial sets with the Quillen model structure and cartesian product, and (ii) the category $Top$ of compactly generated weakly Hausdorff spaces with the Quillen model structure and cartesian product (see [Str09] for more details about the point-set topology).

Let $FB_\infty$ denote the category of (possibly empty) finite sets and bijections, then the objects of the category $(S^G)^{FB_\infty}$ of functors $FB_\infty \to S^G$ are called symmetric sequences. In addition to the Day convolution tensor product, $(S^G)^{FB_\infty}$ admits a composition product $\circ$ (which is rarely symmetric); for $\mathcal{X}, \mathcal{Y} \in (S^G)^{FB_\infty}$ the evaluation of the composition product $\mathcal{X} \circ \mathcal{Y}$ on the set $\{1, 2, \ldots, r\}$ is given by

$$
\mathcal{X} \circ \mathcal{Y}(r) = \bigsqcup_{d \geq 0} \mathcal{X}(d) \otimes_{\mathcal{S}_d} \left( \bigsqcup_{k_1 + \cdots + k_d = r} \mathcal{S}_r \times_{\mathcal{S}_{k_1} \times \cdots \times \mathcal{S}_{k_d}} \mathcal{Y}(k_1) \otimes \cdots \otimes \mathcal{Y}(k_d) \right).
$$

A (symmetric) operad is a unital monoid with respect to this composition product. An $O$-algebra $A$ is an object $A \in C$ with a left $O$-module structure on $A$ considered as a symmetric sequence concentrated in cardinality 0. Equivalently we can use the associated monad on $C$, for which we also use the notation $O$,

$$
O(X) := \bigsqcup_{r \geq 0} O(r) \otimes_{\mathcal{S}_r} X^\otimes r,
$$

and define an $O$-algebra to be an algebra over this monad. The category $Alg_O(S^G)$ of $O$-algebras is both complete and cocomplete.
A free $\mathcal{O}$-algebra is one of the form $\mathcal{O}(X)$ with $\mathcal{O}$-algebra structure maps induced by the monad multiplication and unit. We use the notation $F^\mathcal{O}: S^G \to \text{Alg}_\mathcal{O}(S^G)$ for the free $\mathcal{O}$-algebra functor, which is the left adjoint to the forgetful functor $U^\mathcal{O}: \text{Alg}_\mathcal{O}(S^G) \to S^G$ sending an algebra to its the underlying object. Note $\mathcal{O} = U^\mathcal{O}F^\mathcal{O}$.

Any $\mathcal{O}$-algebra admits a canonical presentation as a reflexive coequalizer of free $\mathcal{O}$-algebras:

$$F^\mathcal{O}(\mathcal{O}(U^\mathcal{O}(A))) \rightrightarrows F^\mathcal{O}(U^\mathcal{O}(A)) \to A,$$

the top map coming from the $\mathcal{O}$-algebra structure map $\mathcal{O}(U^\mathcal{O}(A)) \to U^\mathcal{O}(A)$, and the bottom map coming from the natural transformation $F^\mathcal{O}\mathcal{O} \to F^\mathcal{O}$ induced by the monad multiplication. Thus free $\mathcal{O}$-algebras generate the category of $\mathcal{O}$-algebras under sifted colimits, and the category of right $\mathcal{O}$-module functors $C \to D$ preserving sifted colimits is equivalent to the category of functors $\text{Alg}_\mathcal{O}(C) \to D$ preserving sifted colimits; one constructs the latter from the former using the canonical presentation, and one constructs the former from the latter by evaluating on free $\mathcal{O}$-algebras.

For $X \in S^G$ and $g \in G$, the evaluation $X \mapsto X(g) \in S$ has a left adjoint; given $Y \in S$ we denote its image under this left adjoint by $Y^g$. Given a map $\partial D^d \to U^\mathcal{O}(A)(g)$ (where $\partial D^d$ stands for $\partial D^d \otimes 1$, the copowering of $\partial D^d$ with the monoidal unit), we obtain by adjunction first a map $\partial D^g,d \to U^\mathcal{O}(A)$ and then a map $F^\mathcal{O}(\partial D^g,d) \to A$. An $\mathcal{O}$-cell attachment is defined to be the following pushout in $\text{Alg}_\mathcal{O}(S^G)$

$$
\begin{array}{ccc}
F^\mathcal{O}(\partial D^g,d) & \longrightarrow & A \\
\downarrow & & \\
F^\mathcal{O}(D^g,d) & \longrightarrow & A \cup^\mathcal{O} D^g,d.
\end{array}
$$

Explicitly this pushout may be constructed as the following reflexive coequalizer

$$F^\mathcal{O}(\mathcal{O}(U^\mathcal{O}(A)) \cup D^g,d) \rightrightarrows F^\mathcal{O}(U^\mathcal{O}(A) \cup D^g,d) \to A \cup^\mathcal{O} D^g,d.$$
forgetful functor $\mathcal{U}^O : \text{Alg}_{\mathcal{O}}(S^\mathcal{G}) \to S^\mathcal{G}$ preserves (trivial) cofibrations, cf. Lemma 9.5 of [GKRW18a]. When $\mathcal{O}$ is a $\Sigma$-cofibrant operad in simplicial sets, the projective model structure exists in the settings of Example 2.3, cf. Section 9.2 of [GKRW18a].

When $\mathcal{U}^O(\mathcal{A}) \in S^\mathcal{G}$ is cofibrant, we may use the monadic bar resolution to find an explicit cofibrant replacement of $\mathcal{A}$ and thus compute derived functors.

**Definition 2.5.** The monadic bar resolution is the augmented simplicial object $B_\ast(\mathcal{F}^O, \mathcal{O}, \mathcal{A})$ with $p$-simplices given by $\mathcal{F}^O(\mathcal{O}^p(\mathcal{U}^O(\mathcal{A})))$ for $p \geq 0$ and $\mathcal{A}$ for $p = -1$. The face maps and augmentation are induced by the monad multiplication and the $\mathcal{O}$-algebra structure on $\mathcal{A}$, and the degeneracies by the unit of the monad.

This is a special case of the two-sided monadic bar construction, which is used throughout the paper. It takes as input a monad $T$, a right $T$-functor $F$ and a $T$-algebra $\mathcal{A}$ with underlying object $A$, and has $p$-simplices given by $B_\ast(F, T, A) = F(T^p(A))$. The face maps and degeneracy maps are similar to above, for details see e.g. Section 9 of [May72].

Let $|\cdot|$ denote the (thin) geometric realization, and introduce the notation $B(\mathcal{F}^O, \mathcal{O}, \mathcal{A}) := |B_\ast(\mathcal{F}^O, \mathcal{O}, \mathcal{A})|$. Note that here we take geometric realization in the categories of $\mathcal{O}$-algebras, but $\mathcal{U}^O$ commutes with geometric realization by Section 8.3.3 of [GKRW18a]. The augmentation induces a map $B(\mathcal{F}^O, \mathcal{O}, \mathcal{A}) \to \mathcal{A}$, which is always a weak equivalence using an extra degeneracy argument. It is a free simplicial resolution in the sense of Definition 8.16 of [GKRW18a] when the bar construction is Reedy cofibrant. Because $\mathcal{O}$ is $\Sigma$-cofibrant, this is the case when $\mathcal{U}^O(\mathcal{A})$ is cofibrant, using the Reedy cofibrancy criterion of Lemma 9.14 of [GKRW18a].

**Remark 2.6.** In [KM18], $\mathcal{O}$-algebra cell attachments were defined using partial algebras. The formula in Definition 3.1 of [KM18], written in our notation, is $|\{p\} \mapsto \mathcal{F}^O(\mathcal{O}^p(\mathcal{A}) \cup D^d)|$. This may be obtained by inserting $B(\mathcal{F}^O, \mathcal{O}, \mathcal{A})$ into the underived formula for cell attachment. We explained above that this gives derived cell attachment when $\mathcal{U}^O(\mathcal{A})$ is cofibrant, but in $\text{Top}$ and $\text{sSet}$ this assumption is unnecessary. In $\text{sSet}$, every object is cofibrant. In $\text{Top}$, we use that geometric realization sends levelwise weak equivalences between proper simplicial spaces to weak equivalences, even if the simplicial spaces are not levelwise cofibrant. This allows us to cofibrantly replace $\mathcal{A}$ in the category of $\mathcal{O}$-algebras (which will be cofibrant in topological spaces because $\mathcal{O}$ is $\Sigma$-cofibrant). Thus results of [KM18] apply: in particular, Proposition 6.12 of [KM18] implies that an $\mathcal{E}_k$-cell structure on an $\mathcal{E}_k$-algebra in topological spaces deloops to an ordinary cell structure on the $k$-fold delooping.

## 3. Rank completion

We shall define a rank completion filtration in the case that we are working in a category of functors $S^\mathcal{G}$ where $\mathcal{G}$ has a notion of rank, and the operad $\mathcal{O}$ and $\mathcal{O}$-algebra $\mathcal{A}$ satisfy mild conditions. Later in this paper, the rank function will be used to keep track of the number of points in a configuration, and though we shall not use this, $\mathcal{G}$ can be used to record group actions on configurations.

**Assumption 3.1.** $\mathcal{G}$ is a symmetric monoidal groupoid equipped with strong monoidal functor $\kappa : \mathcal{G} \to \mathbb{N}$, which we call a rank functor.

Let $\mathcal{G}_{\leq r}$ denote the full subcategory on $\mathcal{G}$ on those objects $g$ such that $\kappa(g) \leq r$, and $\mathcal{G}_r$ denote the full subcategory on objects $g$ such that $\kappa(g) = r$. Precomposition
gives restriction functors \((\leq r)^*\) and \((r)^*\) participating in adjunctions
\[
\begin{array}{ccc}
\mathcal{S}^{G\leq r} & \xrightarrow{\{\leq r\}_*} & \mathcal{S}^G, \\
\mathcal{S}^{G_r} & \xrightarrow{(r)_*} & \mathcal{S}^G
\end{array}
\]

There are further relative restriction and extension functors between \(\mathcal{S}^{G_r}, \mathcal{S}^{G\leq r}\) for different \(r\), participating in analogous adjunctions. The functors \((\leq r)^*\) and \((r)^*\) are themselves left adjoints; though we will not use their right adjoints, we will use that \((\leq r)^*\) and \((r)^*\) commute with colimits.

It follows from the formula for Day convolution that \(\mathcal{S}^{G\leq r}\) inherits a symmetric monoidal tensor product, an alternative expression for which is given by \(X \otimes Y = (\leq r)^*((\leq r)_*(X) \otimes (\leq r)_*(Y))\). This makes visible that \((\leq r)^*\) is strong monoidal and simplicial. In particular, the functor \((\leq r)^*\) takes \(\mathcal{O}\)-algebras in \(\mathcal{S}^G\) to \(\mathcal{O}\)-algebras in \(\mathcal{S}^{G\leq r}\). Its left adjoint \((\leq r)_*\) in general does not. However, we may use the canonical presentation of \(\mathcal{O}\)-algebras explained in the previous section to construct a left adjoint \((\leq r)_*^{alg} : \text{Alg}_\mathcal{O}(\mathcal{S}^{G\leq r}) \to \text{Alg}_\mathcal{O}(\mathcal{S}^G)\) to \((\leq r)^* : \text{Alg}_\mathcal{O}(\mathcal{S}^G) \to \text{Alg}_\mathcal{O}(\mathcal{S}^{G\leq r})\).

Explicitly it is the following reflexive coequalizer
\[
\begin{array}{ccc}
\mathcal{F}^\mathcal{O}((\leq r)_*^{alg}(U^\mathcal{O}(A))) & \xrightarrow{\mathcal{F}^\mathcal{O}((\leq r)_*^{alg}(\mathcal{U}^\mathcal{O}(A))))} & \mathcal{F}^\mathcal{O}((\leq r)_*^{alg}(\mathcal{U}^\mathcal{O}(A))) \\
\end{array}
\]

It is defined uniquely up to isomorphism by demanding that \((\leq r)_*^{alg} F^\mathcal{O}(X) = F^\mathcal{O}((\leq r)_*^{alg}(X))\) and that it preserves sifted colimits.

**Definition 3.2.** We define the \(r\)th rank completion functor \(T_r : \text{Alg}_\mathcal{O}(\mathcal{S}^G) \to \text{Alg}_\mathcal{O}(\mathcal{S}^G)\) to be \((\leq r)_*^{alg}((\leq r)^*)\).

This functor underlies the monad associated to the adjunction \((\leq r)_*^{alg} \dashv (\leq r)^*\) and has a right adjoint. The counit gives a natural transformation \(\mathcal{T}_r \Rightarrow \text{id}\), and the commutative diagram of groupoids
\[
\begin{array}{cccc}
G & \xrightarrow{\mathcal{T}_0} & \mathcal{T}_1 & \xrightarrow{\mathcal{T}_2} \\
G_{\leq 0} & \xrightarrow{\mathcal{id}} & G_{\leq 1} & \xrightarrow{\mathcal{id}} \cdots
\end{array}
\]
gives rise to a tower of natural transformations of functors \(\text{Alg}_\mathcal{O}(\mathcal{S}^G) \to \text{Alg}_\mathcal{O}(\mathcal{S}^G)\)

\[
\begin{array}{cccc}
\mathcal{T}_0 & \xrightarrow{\mathcal{id}} & \mathcal{T}_1 & \xrightarrow{\mathcal{id}} \cdots
\end{array}
\]

Since colimits are computed objectwise and the map \((g)^*\mathcal{T}_r(A) \to (g)^*A\) is the identity as soon as \(r \geq \kappa(g)\), the natural transformation \(\text{colim}_{r \in \mathbb{N}} \mathcal{T}_r \Rightarrow \text{id}\) is a natural isomorphism.

The functor \((\leq r)^*\) obviously preserves fibrations and weak equivalences, so \((\leq r)_*^{alg}\) is a left Quillen functor. However, \((\leq r)^*\) also preserves cofibrations, as these are retracts of iterated pushouts along free \(\mathcal{O}\)-algebra maps, which are preserved by \((\leq r)^*\). Hence \(\mathcal{T}_r = (\leq r)_*^{alg}(\leq r)^*\) preserves trivial cofibrations between cofibrant objects, and thus admits a left derived functor by precomposition with a functorial cofibrant replacement. Moreover, as explained above, when \(\mathcal{U}^\mathcal{O}(A)\) is cofibrant we may use a monadic bar resolution to cofibrantly replace \(A\).
composition of two left adjoints, $T_r$ commutes with geometric realization. Thus we obtain the following formula for $T_r^L(A)$:

$$T_r^L(A) = B(F^O(\leq r)_*, O, (\leq r)^*U^O(A)) = B(F^O(\leq r)_*, (\leq r)^*, O, U^O(A)),$$

the latter equality following from the fact that $(\leq r)^*$ commutes with $O$.

We next restrict our attention to a setting where the underlying object of $F^O(X)$ agrees with $X$ in rank $\leq r$ up to homotopy, for those $X$ which are concentrated in rank $r$. To see when this occurs, note that for any operad $O$ and $X$ concentrated in rank $r$, $O(X)$ is isomorphic to $O(0)$ in rank 0 and $O(1) \otimes X$ in rank $r$. Hence the following assumption suffices:

**Assumption 3.3.** $O$ is a non-unitary operad in simplicial sets, i.e. $O(0) = \emptyset$, and $O(1) \simeq *$. 

**Definition 3.4.** We say $X \in S^O$ is reduced if it is concentrated in rank $> 0$, that is, $X(g)$ is initial when $\kappa(g) = 0$.

The horizontal maps in the following proposition are obtained from the identity morphisms of $(r + 1)^*U^O T_r^L(A)$ and $(r + 1)^*U^O T_{r+1}^L(A)$ respectively, the vertical maps from the natural transformation $T_r \Rightarrow T_{r+1}$.

**Proposition 3.5.** For reduced $A$ there is a homotopy cocartesian square in $\text{Alg}_O(S^O)$

$$
\begin{array}{ccc}
F^O((r + 1)_*(r + 1)^*U^O T_r^L(A)) & \longrightarrow & T_r^L(A) \\
\downarrow & & \downarrow \\
F^O((r + 1)_*(r + 1)^*U^O T_{r+1}^L(A)) & \longrightarrow & T_{r+1}^L(A),
\end{array}
$$

where we remark $(r + 1)^*U^O T_{r+1}^L(A) \cong (r + 1)^*U^O(A)$.

**Proof.** Because $T_r^L$ is obtained from $T_r$ by precomposition with a functorial cofibrant replacement, we may assume $A$ is cofibrant in $\text{Alg}_O(S^O)$ and hence in $S^O$. We may then use $B(F^O, O, U^O(A))$ as a cofibrant replacement for $A$ (geometric realization taking place in $\text{Alg}_O(S^O)$). The functors in the diagram commute with geometric realization; this is only non-obvious for $U^O$, and proven in Section 8.3.3 of [GKRW18a]. Thus we have that the square in the proposition is isomorphic to the geometric realization of a square of simplicial objects which are Reedy cofibrant by another application of Lemma 9.14 of [GKRW18a]. Geometric realization of Reedy cofibrant simplicial objects is a homotopy colimit, and thus commutes with homotopy pushouts. In particular, a levelwise homotopy cocartesian diagram of Reedy cofibrant simplicial objects geometrically realizes to a homotopy cocartesian diagram.

Hence it suffices to prove the statement in the case that $A$ is a free $O$-algebra $F^O(X)$ on a cofibrant object $X$. Because the $p$-simplices in the monadic bar resolution are reduced if $A$ is, we may assume that $X$ is reduced. In this case, the diagram becomes

$$
\begin{array}{ccc}
F^O((r + 1)_*(r + 1)^*O(\leq r)_*(\leq r)^*X)) & \longrightarrow & F^O((\leq r)_*(\leq r)^*X) \\
\downarrow & & \downarrow \\
F^O((r + 1)_*(r + 1)^*O(X)) & \longrightarrow & F^O((\leq r + 1)_*(\leq r + 1)^*X),
\end{array}
$$
where the horizontal maps are obtained from the adjoints of the identity maps of \((r + 1)^*\mathcal{U}^O((\leq r)_*((\leq r)^*X))\) and \((r + 1)^*\mathcal{U}^O((\leq r)_*(\leq r + 1)^*X))\) respectively.

This diagram is obtained by applying \((\leq r + 1)_*O\) to a diagram in \(\text{Alg}_{\mathcal{O}}(\mathcal{S}^G)\), a functor which preserves homotopy cocommutative squares as it is a left Quillen functor. Hence it suffices to prove that the following is homotopy cocartesian in \(\text{Alg}_{\mathcal{O}}(\mathcal{S}^G(\leq r + 1))\):

\[
\begin{array}{ccc}
\mathcal{F}^O((r + 1)_*(r + 1)^*\mathcal{O}((\leq r)_*(\leq r)^*X)) & \longrightarrow & \mathcal{F}^O((\leq r)_*(\leq r)^*X) \\
\downarrow & & \downarrow \\
\mathcal{F}^O((r + 1)_*(r + 1)^*\mathcal{O}(X)) & \longrightarrow & \mathcal{F}^O((\leq r + 1)_*(\leq r + 1)^*X),
\end{array}
\]

where \(\mathcal{F}^O\) now denotes the free algebra functor \(\mathcal{S}_n^G \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{S}^G_n)\), and \((\leq r)_*\) and \((r + 1)_*\) now denote the left adjoints to \(\mathcal{S}_{\leq r + 1} \rightarrow \mathcal{S}_{\leq r}\) and \(\mathcal{S}_{\leq r + 1} \rightarrow \mathcal{S}_{\leq r + 1}\) respectively.

It suffices to compare the homotopy cofibers of the vertical maps: both are obtained by applying the left Quillen functor \(\mathcal{F}^O\) to a map (the horizontal maps are not), so the homotopy cofibers are given by applying \(\mathcal{F}^O: \mathcal{S}^G \rightarrow \text{Alg}_{\mathcal{O}}(\mathcal{S}^G)\) to the homotopy cofiber of this map. The homotopy cofiber of the right vertical map is thus \(\mathcal{F}^O((r + 1)_*(r + 1)^*X(X_+)).\) Similarly, the homotopy cofiber of the left vertical map is \(\mathcal{F}^O\) applied to the evaluation at \((r + 1)\) of the homotopy cofiber of the map

\[
\bigsqcup_{n \geq 1} \mathcal{O}(n) \otimes S_n \left( \bigsqcup_{1 \leq r_1, \ldots, r_n \leq r} (r_1)_*(r_1)^*X \otimes \cdots \otimes (r_n)_*(r_n)^*X \right)
\]

\[
\bigsqcup_{n \geq 1} \mathcal{O}(n) \otimes S_n \left( \bigsqcup_{1 \leq r_1, \ldots, r_n \leq r + 1} (r_1)_*(r_1)^*X \otimes \cdots \otimes (r_n)_*(r_n)^*X \right).
\]

Since \(\mathcal{O}\) is non-unitary and \(X\) is reduced, this is an inclusion missing only the term \(\mathcal{O}(1) \otimes (r + 1)_*(r + 1)^*X(r + 1).\) Thus its homotopy cofiber is \(\mathcal{F}^O(\mathcal{O}(1) \otimes (r + 1)_*(r + 1)^*X(X_+)).\)

The map

\[
\mathcal{F}^O(\mathcal{O}(1) \otimes (r + 1)_*(r + 1)^*X(X_+)) \longrightarrow \mathcal{F}^O((r + 1)_*(r + 1)^*X(X_+))
\]
on homotopy cofibers is the adjoint to the identity map on \(\mathcal{U}^O\mathcal{F}^O((r + 1)_*(r + 1)^*X(X_+)).\) On underlying objects this is the map

\[
\mathcal{O}(1) \otimes \mathcal{O}(1) \otimes (r + 1)_*(r + 1)^*X(X_+) \longrightarrow \mathcal{O}(1) \otimes (r + 1)_*(r + 1)^*X(X_+)
\]
induced by the composition of the monoid structure on \(\mathcal{O}(1)\) induced by the operad structure. Since \(\mathcal{O}(1) \simeq \ast\), this is a weak equivalence. \(\square\)

We thus get a sequence of maps

\[
T^1_{\mathcal{O}}(A) \longrightarrow T^1_{\mathcal{O}}(A) \longrightarrow T^2_{\mathcal{O}}(A) \longrightarrow \cdots
\]
whose homotopy colimit is naturally weakly equivalent to \(A\) and whose homotopy cofibers we understand. This is the rank completion filtration.

When we can make sense of homology, e.g. in one of the settings mentioned in Section 10.1 of \([\text{GKRW18a}]\), we get a corresponding spectral sequence converging conditionally to the homology of \(\mathcal{U}^O(A)\). The \(E^1\)-page will be rather unwieldy, and we believe the following spectral sequence may be more useful:
Remark 3.6. Let $(-)_+$ denote the monad whose underlying functor takes the coproduct with the terminal object (so that algebras over it are pointed objects). As $\mathcal{O}$ is a non-unitary operad in simplicial sets, cf. Assumption 3.3, there is a canonical map of monads from $\mathcal{O}$ to $(-)_+$ which can be viewed as an augmentation of $\mathcal{O}$. This augmentation is given on $X \in S^G$ by the map $\mathcal{O}(X) = \bigcup_{n \geq 1} \mathcal{O}(n) \otimes_{\mathfrak{S}_n} X^\otimes n \to X_+$ which on the summand $\mathcal{O}(1) \otimes X$ is the map $\mathcal{O}(1) \otimes X \to \ast \otimes X = X$ and on the summands $\mathcal{O}(n) \otimes_{\mathfrak{S}_n} X^\otimes n$ for $n \geq 2$ is the unique map to the terminal object.

Taking indecomposables with respect to this augmentation, we obtain the $\mathcal{O}$-indecomposables functor $Q^G: \text{Alg}_\mathcal{O}(S^G) \to S^G$ determined uniquely up to isomorphism by demanding that $Q^G F^\mathcal{O} \cong (-)_+$ and that $Q^G$ commutes with sifted colimits. Applying its left-derived functor $Q^G_+$ to the diagram in the previous proposition, we get that if $A$ is reduced there is a homotopy cocartesian square in $S^G_+$:

$$
\begin{array}{ccl}
\overset{(r+1)_+}{(r+1)_+(r+1)^*U^\mathcal{O} T_r^G(A)_+} & \longrightarrow & Q^G_+ (T_r^G(A)) \\
& \downarrow & \downarrow \\
\overset{(r+1)_+}{(r+1)_+(r+1)^*U^\mathcal{O} T_{r+1}^G(A)_+} & \longrightarrow & Q^G_+ (T_{r+1}^G(A)).
\end{array}
$$

When we can make sense of homology, we define $\mathcal{O}$-homology by $H^G_{d,g}(A) := \tilde{H}_d((g)^*Q^G_+(A))$. The result of the previous discussion is a conditionally convergent spectral sequence (suppressing the filtration degree, so in particular the $p$ in $E^1_{p,q}$ refers to rank)

$$
E^1_{p,q} = H_{p+q}((p)^*T^G_r(A), (p)^*T^G_{r-1}(A)) \implies H^G_{p,p+q}(A),
$$

where it may be helpful to recall that $(p)^*T^G_r(A) \cong (p)^*A$.

4. An $\mathcal{E}_k$-algebraic analogue of the May-Milgram filtration

To deduce our results, we specialize the results of the previous section to $\mathcal{O} = \mathcal{E}_k$, the non-unital little $k$-cubes operad. Recall that $I$ denotes the open interval $(0,1)$, and let $\text{Emb}_{\text{rect}}(\bigsqcup_n I^n, I^k)$ denote the space of ordered $n$-tuples of rectilinear embeddings $I^k \to I^k$ with disjoint image (that is, they are a composition of translation and dilation by positive real numbers in each of the $k$ directions).

Definition 4.1. The non-unital little $k$-cubes operad $\mathcal{E}_k$ has topological space $\mathcal{E}_k(n)$ of $n$-ary operations given by

$$
\mathcal{E}_k(n) := \begin{cases} 
\emptyset & \text{if } n = 0, \\
\text{Emb}_{\text{rect}}(\bigsqcup_n I^n, I^k) & \text{if } n > 0,
\end{cases}
$$

with symmetric group $\mathfrak{S}_n$ permuting the $n$-tuples. The unit in $\mathcal{E}_k(1)$ is the identity map $I^k \to I^k$, and composition is induced by composition of embeddings.

$\mathcal{E}_k$-algebras are algebras over this operad, and we shall adopt the shorter notation $E_k$ for the free $\mathcal{E}_k$-algebra functor $F^\mathcal{E}_k$. (If $S = \text{Top}$, as a consequence of our conventions these are algebras over the operad $|\text{Sing}(\mathcal{E}_k)|$ in topological spaces.)

We shall take $G = N$, with $\kappa: N \to N$ the identity functor. Let

$$
\mathcal{U}_{\mathcal{E}_k}^{k+m}: \text{Alg}_{\mathcal{E}_k+m}(S^N) \longrightarrow \text{Alg}_{\mathcal{E}_k}(S^N)
$$

denote the forgetful functor induced by the map of operads $\mathcal{E}_k \to \mathcal{E}_{k+m}$ given by sending a cube $e: I^k \to I^k$ to $e \times \text{id}_m: I^k \times I^m \to I^k \times I^m$. For the sake of brevity we will often write $\mathcal{U}$ for $\mathcal{U}_{\mathcal{E}_k}^{k+m}$ in this and subsequent sections.
Figure 1. An element of \( F_8(I^k \times I^m) \) for \( k = 1, m = 1 \) (suppressing the labels on the point for the sake of clarity) which is in \( F_8^{[r]}(I^k \times I^m) \) when \( r \geq 4 \), but not when \( r < 4 \).

We are interested in free algebras on a point, which we will consider concentrated in rank 1. In this section we will more generally study \( E_k^{+m}(X) \) for \( X \in S^N \) satisfying a similar condition:

**Assumption 4.2.** \( X \in S^N \) is concentrated in rank 1, i.e. \( X(g) \) is initial unless \( g = 1 \) (so in particular reduced), and \( X \) is cofibrant.

We will give an elementary geometric model for the \( E_k^{+m} \)-algebra \( T_L^r(U E_k^{+m}(X)) \), and use Proposition 3.5 to describe \( U E_k^{+m}(X) \) up to weak equivalence as a colimit of iterated pushouts along maps of free \( E_k^{+m} \)-algebras. The following definition was mentioned in the introduction:

**Definition 4.3.** For a manifold \( M \) and \( n \geq 1 \), the topological space \( F_n(M) \) of ordered configurations of \( n \) points in \( M \) is given by \( \{(m_1, \ldots, m_n) \mid m_i \neq m_j \text{ if } i \neq j\} \subset M^n \). For \( n = 0 \) we define \( F_0(M) = \emptyset \).

We choose to define \( F_0(M) \) to be empty since we work with non-unital \( \mathcal{E}_k \)-algebras. For \( n > 0 \), the topological space \( F_n(M) \) is homeomorphic to the space of embeddings of the set \( \{1, \ldots, n\} \) into \( M \), and precomposition by permutations of \( \{1, \ldots, n\} \) defines a \( S_n \)-action on the space \( F_n(M) \). Thus, these topological spaces assemble to a symmetric sequence \( F(M) \in \text{Top}^{FB\infty} \). For \( M = \mathbb{I}^k \times \mathbb{I}^m \), this is a left \( \mathcal{E}_{k+m} \)-module, where the action is given by composition of embeddings. As a consequence, when we apply \( F(\mathbb{I}^k \times \mathbb{I}^m) \) to an object \( X \in S^N \), the resulting object \( F(\mathbb{I}^k \times \mathbb{I}^m)(X) \in S^N \) comes endowed with an \( \mathcal{E}_{k+m} \)-algebra structure, and we obtain a functor \( F(\mathbb{I}^k \times \mathbb{I}^m) : S^N \to \text{Alg}_{\mathcal{E}_{k+m}}(S^N) \).

**Definition 4.4.** We let \( F_n^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) \) denote the subspace of \( F_n(\mathbb{I}^k \times \mathbb{I}^m) \) consisting of ordered configurations \( \eta = (m_1, \ldots, m_n) \) such that for all \( x \in \mathbb{I}^k \) the intersection \( \eta \cap \{(x) \times \mathbb{I}^m\} \) has cardinality at most \( r \).

Since the condition defining \( F_n^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) \) is invariant under the \( S_n \)-action, these topological spaces may be assembled into a symmetric sequence \( F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) \subset F(\mathbb{I}^k \times \mathbb{I}^m) \). The left \( \mathcal{E}_{k+m} \)-module structure on \( F(\mathbb{I}^k \times \mathbb{I}^m) \) does not restrict. However, using the map of operads \( \mathcal{E}_k \to \mathcal{E}_{k+m} \) induced by the inclusion \( \mathbb{I}^k \to \mathbb{I}^k \times \mathbb{I}^m \) on
the first $k$ coordinates, we get a left $\mathcal{E}_k$-module structure on $F(\mathbb{I}^k \times \mathbb{I}^m)$ which does restrict and application of this symmetric sequence gives a functor $F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m) : \mathbb{S}^n \rightarrow \text{Alg}_{\mathcal{E}_k}(\mathbb{S}^n)$.

As we assumed that $X$ is cofibrant, we can use a monadic bar resolution to give an explicit formula for $T^k_r(\mathcal{U}(\mathbb{E}_{k+m}(X))) \in \text{Alg}_{\mathcal{E}_k}(\mathbb{S}^n)$:

$T^k_r(\mathcal{U}(\mathbb{E}_{k+m}(X))) = B(\mathbb{E}_k(\leq r)_*, \mathbb{E}_k, (\leq r)^* \mathcal{U}(\mathbb{E}_{k+m}(X)))$.

We take this specific model for the domain of the map in the following proposition:

**Proposition 4.5.** There are weak equivalences $\alpha_r : T^k_r(\mathcal{U}(\mathbb{E}_{k+m}(X))) \rightarrow F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X)$ of $\mathcal{E}_k$-algebras, which fit into commutative diagrams for $r \geq 0$

$$
\begin{array}{c}
T^k_r(\mathcal{U}(\mathbb{E}_{k+m}(X))) \\
\downarrow \alpha_r \\
F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X)
\end{array} \quad \begin{array}{c}
T^k_{r+1}(\mathcal{U}(\mathbb{E}_{k+m}(X))) \\
\downarrow \alpha_{r+1}
\end{array} \quad \begin{array}{c}
F^{[r+1]}(\mathbb{I}^k \times \mathbb{I}^m)(X)
\end{array}

(3)

Let us start by defining the maps:

**Lemma 4.6.** There are maps $\alpha_r : T^k_r(\mathcal{U}(\mathbb{E}_{k+m}(X))) \rightarrow F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)(X)$ of $\mathcal{E}_k$-algebras making Diagram (3) commute.

**Proof.** The map $\mathcal{E}_{k+m} \rightarrow F(\mathbb{I}^k \times \mathbb{I}^m)$ which sends a cube to its center is a homotopy equivalence of left $\mathcal{E}_{k+m}$-modules in symmetric sequences, so we have an induced weak equivalence $\mathbb{E}_{k+m}(X) \rightarrow F(\mathbb{I}^k \times \mathbb{I}^m)(X)$ of $\mathcal{E}_{k+m}$-algebras. To define $\alpha_r$, we first insert this weak equivalence into the right entry of the bar construction

$$
\begin{array}{c}
\mu \cdot \mathbb{E}_k(\leq r)_*, \mathbb{E}_k, (\leq r)^* \mathcal{U}(\mathbb{E}_{k+m}(X)) \\
\downarrow \sim \\
\mu \cdot \mathbb{E}_k(\leq r)_*, \mathbb{E}_k, (\leq r)^* \mathcal{U}(\mathbb{I}^k \times \mathbb{I}^m)(X))
\end{array}
$$

The assumption that $X$ is concentrated in rank 1 gives us an isomorphism $(\leq r)^* \mathcal{U}(\mathbb{I}^k \times \mathbb{I}^m)(X) \cong \mathcal{U}((\leq r)^* F(\mathbb{I}^k \times \mathbb{I}^m))(\mathbb{I}^k \times \mathbb{I}^m)$.

Here $(\leq r)^* F(\mathbb{I}^k \times \mathbb{I}^m)$ is an object in the truncated symmetric sequence category (functors from the category of possibly empty finite sets of cardinality $\leq r$ into $\text{sSet}$ with tensor product the restriction of the composition product) and $(\leq r)^* F(\mathbb{I}^k \times \mathbb{I}^m)$ is the functor given by tensoring with $(\leq r)^* F(\mathbb{I}^k \times \mathbb{I}^m)$.

Because $\otimes$ commutes with colimits in each variable and geometric realization, the target is obtained by applying the symmetric sequence

$$
\begin{array}{c}
\mu \cdot \mathbb{E}_k(\leq r)_*, \mathbb{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}) \\
\end{array}
$$

in $\mathbb{Top}$ to $X$ (as always, via Sing and the simplicial copowering). We define a map of left $\mathcal{E}_k$-modules in symmetric sequences in $\mathbb{Top}$

$$
\alpha_r : \mu \cdot \mathbb{E}_k(\leq r)_*, \mathbb{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}) \rightarrow F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)
$$

by describing an augmentation from $\mu \cdot \mathbb{E}_k(\leq r)_*, \mathbb{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})$ to $F^{[r]}(\mathbb{I}^k \times \mathbb{I}^m)$. The 0-simplices of the former are given by

$$
\mathbb{E}_k(\leq r)_*, (\leq r)^* F(\mathbb{I}^{k+m}) = \mathbb{E}_k(n) \times_{\mathcal{S}_n} \bigsqcup_{1 \leq k_1, \ldots, k_n \leq r} F^k(\mathbb{I}^k \times \mathbb{I}^m)
$$
and given a collection of embeddings $e_i : \mathbb{I}^k \rightarrow \mathbb{I}^k$ and configurations $\xi_i \in F_{k_i}(\mathbb{I}^k \times \mathbb{I}^m)$, we may take the union of the images $(e_i \times \text{id}_{\mathbb{I}^m})(\xi_i)$ in $\mathbb{I}^k \times \mathbb{I}^m$ and obtain an ordered configuration of $k_1 + \ldots + k_m$ points such that no subset $\{x\} \times \mathbb{I}^m$ contains more than $r$ points. This map is easily seen to be compatible with the left $\mathcal{E}_k$-module structures. That the diagram commutes is clear from the definition. □

We next prove that each $a_r$ is a weak homotopy equivalence, using a microfibration argument.

**Definition 4.7.** A map $\pi : E \rightarrow B$ of topological spaces is a *microfibration* if for each $i \geq 0$ and commutative diagram

$$
\begin{array}{ccc}
D^i \times \{0\} & \xrightarrow{h} & E \\
\downarrow & & \downarrow \pi \\
D^i \times [0,1] & \xrightarrow{H} & B,
\end{array}
$$

there exists an $\epsilon > 0$ and a partial lift $\tilde{H} : D^i \times [0,\epsilon] \rightarrow E$, i.e. $\pi \circ \tilde{H} = H|_{D^i \times [0,\epsilon]}$ and $\tilde{H}|_{D^i \times \{0\}} = h$.

**Lemma 4.8** (Lemma 2.2 of [Wei05]). If $\pi : E \rightarrow B$ is a microfibration with weakly contractible fibers, then $\pi$ is a weak homotopy equivalence.

Our strategy is to prove that $a_r$ is a microfibration with weakly contractible fibers. To do this, we use the following lemma in point-set topology.

**Lemma 4.9.** Let $X_{\bullet}$ be a levelwise Hausdorff simplicial space. Let

$$X_1, \bullet \subset X_2, \bullet \subset \cdots \subset X_{\bullet}$$

be an $\mathbb{N}_{>0}$-indexed sequence of simplicial subspaces such that: (i) $X_{s,p} \subset X_p$ is compact for all $s, p$, and (ii) each point $x \in X_p$ has an open neighborhood contained in some $X_{s,p}$. If $C$ is compact, then any continuous map $C \rightarrow |X_{\bullet}|$ factors as $C \rightarrow |X_s, \bullet| \rightarrow |X_{\bullet}|$ for some $s$.

**Proof.** The strategy is to first identify $|X_{\bullet}|$ with the sequential colimit $\text{colim}_s |X_{s, \bullet}|$ and then show that this particular sequential colimit commutes with maps out of the compact space $C$.

The inclusions $X_{s,p} \rightarrow X_p$ induce a continuous bijection $\text{colim}_s X_{s,p} \rightarrow X_p$. To show it is a homeomorphism we need to prove it is open: $V \subset \text{colim}_s X_{s,p}$ being open means that all $V \cap X_{s,p}$ are open, and by the hypothesis for all $x \in V$, $V$ contains an open neighborhood of $x$ in $X_p$, which means it is open in $X_p$. Since colimits of simplicial spaces are computed levelwise, $\text{colim}_s X_{s, \bullet} \rightarrow X_{\bullet}$ is an isomorphism of simplicial spaces. Since geometric realization commutes with filtered colimits (it has a right adjoint when working with compactly generated weakly Hausdorff spaces), the canonical map $\text{colim}_s |X_{s,\bullet}| \rightarrow |X_{\bullet}|$ is a homeomorphism.

In CGWH spaces, maps out of a compact space commute with sequential colimits of closed inclusions by Lemma 3.6 of [Str09]. Thus we shall verify that each map $|X_{s,\bullet}| \rightarrow |X_{s+1,\bullet}|$ is a closed inclusion, using its description as a colimit of the maps of skeleta:

$$
\begin{array}{ccc}
\text{sk}_0|X_{s,\bullet}| & \longrightarrow & \text{sk}_1|X_{s,\bullet}| \\
\downarrow & & \downarrow \\
\text{sk}_0|X_{s+1,\bullet}| & \longrightarrow & \text{sk}_1|X_{s+1,\bullet}|
\end{array}
\longrightarrow \cdots
$$
We claim all maps in this diagram are closed inclusions. All maps are clearly continuous injections and a continuous injection between compact Hausdorff spaces is always a closed inclusion, so it suffices to prove that each space is compact Hausdorff. They are compact because each \( \text{sk}_p[X_{s,*}] \) is a quotient of the compact space \( \bigsqcup_{k \leq p} \Delta^k \times X_{s,k} \). They are Hausdorff because we may freely add degeneracies to write \( \text{sk}_p[X_{s,*}] \) as the geometric realization of a levelwise Hausdorff simplicial space and apply Theorem 1.1 of \([\text{dSP13}]\). Furthermore, from the construction it is clear each square is a pullback square. The result then follows from the following result about CGWH spaces, Lemma 3.9 of \([\text{Str09}]\): given a commutative diagram

\[
\begin{array}{c}
A_0 \longrightarrow A_1 \longrightarrow \cdots \\
\downarrow f_0 \downarrow f_1 \\
B_0 \longrightarrow B_1 \longrightarrow \cdots
\end{array}
\]

with all maps closed inclusions and all squares pullbacks, the induced map \( \text{colim}_s A_s \rightarrow \text{colim}_s B_s \) is also a closed inclusion. \( \square \)

**Lemma 4.10.** The map \( a_r : [B_\bullet(\mathcal{E}_k(\leq r)_*), \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})] \rightarrow F[r](\mathbb{I}^k \times \mathbb{I}^m) \) is a microfibration.

**Proof.** Fixing a cardinality \( n \), we need to prove that the component \( a_r(n) \) is a microfibration. Suppose we are given a commutative diagram

\[
\begin{array}{c}
D^i \times \{0\} \longrightarrow [B_\bullet(\mathcal{E}_k(\leq r)_*), \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m})](n) \\
\downarrow \\
D^i \times [0,1] \rightarrow F[r](\mathbb{I}^k \times \mathbb{I}^m).
\end{array}
\]

Since \( D^i \times [0,1] \) is compact, there exists a \( \delta > 0 \) such that \( H \) factors over the compact subspace of configurations \( \xi \) where

(a) the points in \( \xi \) have distance \( \geq \delta \) from each other,
(b) for all closed cubes \( C \subset \mathbb{R}^k \) with equal sides of length \( < \delta \), the set \( C \times \mathbb{I}^m \) contains at most \( r \) points of \( \xi \).

Let us abbreviate \( B_\bullet(\mathcal{E}_k(\leq r)_*), \mathcal{E}_k, (\leq r)^* F(\mathbb{I}^{k+m}))(n) \) by \( X_p \), and by \( X_p^\delta \) the subspace of \( X_p \) of elements whose image under \( a_r(n) \) satisfies (a) and (b).

Let \( \rho_p : X_p^\delta \rightarrow (0, \infty) \) be the minimum of the distances from the points in the image \( \xi \) to the boundaries of the images of the cubes. Then \( X_p^\delta \) is a simplicial space with a sequence of continuous functions \( \rho_p : X_p^\delta \rightarrow (0, \infty) \) such that \( \rho_{p+1} \circ s_i = \rho_p \) and \( \rho_{p-1} \circ d_i \geq \rho_p \). For each integer \( s \geq 1 \), the subspaces \( X_{s,p}^\delta := \rho_p^{-1}([1/s, \infty)) \subset X_p^\delta \) assemble to a simplicial space \( X_s^\delta \).

This satisfies the hypotheses of Lemma 4.9 (condition (i) of that lemma is the reason we use \( X_s^\delta \) instead of \( X_\bullet \), and uses that only the interiors of cubes need to be disjoint, not their closures). Hence the map \( h \) factors over some stage \([X_{s,*}\bullet] \) with \( \delta \geq \frac{1}{s} \) such that for all \( d \in D^i \), the configuration \( H(d,0) \) is given by a \( \xi \) which satisfies the properties

(a) the points in \( \xi \) have distance \( \geq \frac{1}{s} \) from each other,
(b) for all closed cubes \( C \subset \mathbb{R}^k \) with equal sides of length \( \frac{1}{s} \), the set \( C \times \mathbb{I}^m \) contains at most \( r \) points of \( \xi \),
(c) the points in $\xi$ have distance $\geq \frac{1}{s}$ to the boundaries of the images of the cubes in $h(d)$.

By continuity of the map $H$, there is an $\epsilon > 0$ such that for all $d \in D^i$ and $t \in [0, \epsilon]$, the configuration $H(d, t)$ is within distance $\frac{1}{3\epsilon}$ of $H(d, 0)$. The partial lift is given by the map which assigns to $(d, t) \in D^i \times [0, \epsilon]$ the element of $|X_{s, \bullet}|$ represented by configuration $H(d, t)$ inside the cubes coming from the unique non-degenerate representative of $h(d)$. To see this is well-defined, note that (c) implies a point in $H(d, t)$ remains within the same cubes of $h(d)$ as the corresponding point in $H(d, 0)$.

To see it is continuous, note that the movement of the points in the configuration can be described by recording their displacements by an element $\Delta(d, t)$ of $([-\frac{1}{3s}, \frac{1}{3s}]^{k+m})^n$ (each $\Delta(d, 0)$ equals 0). That is, $\Delta(d, t)$ is defined by $H(d, t) = H(d, 0) + \Delta(d, t)$.

There is a simplicial map

$$\left([\frac{1}{3s}, \frac{1}{3s}]^{k+m}\right)^n \times X_{s, \bullet}^{1/s} \to X_{\bullet}$$

obtained by applying the displacement to the configuration. This is continuous, and well-defined because whenever we move points in the configuration of an element of $X_{s, \bullet}^{1/s}$ at most $\frac{1}{3s}$ in any of the directions, they do not (a) collide with each other, (b) have more than $r$ points in a subset $\{x\} \times \mathbb{I}^m$, and (c) cross boundaries of cubes. That the lift is continuous then follows by observing that it can be realized as a composition of continuous maps

$$D^i \times [0, \epsilon] \xrightarrow{\Delta \times \epsilon} \left([\frac{1}{3s}, \frac{1}{3s}]^{k+m}\right)^n \times |X_{s, \bullet}^{1/s}| \xrightarrow{\epsilon} \left([\frac{1}{3s}, \frac{1}{3s}]^{k+m}\right)^n \times X_{s, \bullet}^{1/s} \to |X_{\bullet}|.$$ 

We now prove that the fibers of $a_r(n)$ are weakly contractible, so Lemma 4.8 implies that $a_r(n)$ is a weak homotopy equivalence.

**Lemma 4.11.** The fibers of the map $a_r\colon |B_{\bullet}(\mathcal{E}_k(\leq r)_s, \mathcal{E}_k, (\leq r)^*F(\mathbb{I}^{k+m})| \to F[r^1](\mathbb{I}^k \times \mathbb{I}^m)$ are weakly contractible.

**Proof.** Fix a configuration $\xi \in F_{r^1}(\mathbb{I}^k \times \mathbb{I}^m)$. As above, the fiber $\epsilon^{-1}(\xi)$ is given by the geometric realization of the subsimplicial space of $X_{\bullet}$ with underlying configuration $\xi$. Call this simplicial space $X_{\bullet}(\xi)$. Another application of Lemma 9.14 of [GKRW18a] tells us this is Reedy cofibrant.

Let $\xi' \in \text{Sym}_p(\mathbb{I}^k) \coloneqq (\mathbb{I}^k)^n/\mathcal{S}_n$ be the configuration with multiplicities obtained by projecting $\xi$ onto $\mathbb{I}^k$. The $p$-simplices of $X_{\bullet}(\xi)$ are given by $p + 1$ levels of nested $k$-dimensional cubes such that all points of $\xi'$ are contained in an innermost cube and all cubes except the outermost ones contain at most $r$ points of $\xi'$ counted with multiplicity. Let $(e_1, e_2, \ldots, e_l) \in \mathcal{E}_k(l)$ be a collection of cubes such that every cube $e_i$ contains exactly one point of $\xi'$ (counted without multiplicity) and every point of $\xi'$ (counted without multiplicity) is in one of the cubes $e_i$. Let $X_{\bullet}(\xi, e)$ denote the subsimplicial space of $X_{\bullet}(\xi)$ where we require that if a cube contains a point of $\xi'$, then it contains the corresponding $e_i$. This is also Reedy cofibrant. Thus, since the inclusion $X_{\bullet}(\xi, e) \hookrightarrow X_{\bullet}(\xi)$ induces a levelwise homotopy
equivariantly isomorphic to \( k\phi \) degeneracy as an augmented simplicial space by adding a point in degree \(-h\). Hence, it induces a weak equivalence on geometric realizations. View a cofibrant object is a weak equivalence by Lemma 9.1 of [GKRW18a], and geometric realization preserves weak equivalences between Reedy cofibrant simplicial objects.

\[ \Box \]

**Proof of Proposition 4.5.** By combining Lemmas 4.8, 4.10, and 4.11, we see that the map \( a_r: [B^n, (\leq r) \times, \{\leq r\}^* F(\mathbb{I}^k \times \mathbb{I}^m)] \to F[\mathbb{I}^{k+1}] \) is a weak equivalence. Since \( a_r \) is a map of symmetric sequences and all of the symmetric group actions are free, it is a weak equivalence of symmetric sequences. The result follows because applying a weak equivalence between \( \Sigma \)-cofibrant symmetric sequences in \( \text{sSet} \) to a cofibrant object is a weak equivalence by Lemma 9.1 of [GKRW18a], and geometric realization preserves weak equivalences between Reedy cofibrant simplicial objects.

\[ \Box \]

The inclusion \( \mathbb{I}^k \times F_{r+1}(\mathbb{I}^m) \hookrightarrow F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m) \) given by \( (x, \xi) \mapsto x \times \xi \), i.e., sending each point \( m_i \in \xi \) to \( x \times m_i \), has image given by the complement of \( F[\mathbb{I}^{k+1}] \mathbb{I}^k \times \mathbb{I}^m \) in \( F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m) \). Let \( \varphi_{m,r} \) denote the trivial vector bundle over \( \mathbb{I}^k \times F_{r}(\mathbb{I}^m) \) given by \( \mathbb{I}^k \times F_{r}(\mathbb{I}^m) \times \mathbb{R}^{r-1} \to \mathbb{I}^k \times F_{r}(\mathbb{I}^m) \), with \( \Sigma_r \) acting diagonally and with \( \mathbb{R}^{r-1} \) the orthogonal complement to the trivial representation in the permutation representation with its usual metric. The vector bundle \( \varphi_{m,r} \) can be thought of as the \( \Sigma_r \)-equivariant analogue of \( \phi_{m,r} \) from the introduction.

**Lemma 4.12.** The normal bundle of \( \mathbb{I}^k \times F_{r+1}(\mathbb{I}^m) \) in \( F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m) \) is \( \Sigma_{r+1} \)-equivariantly isomorphic to \( k\varphi_{m,r+1} \).

**Proof.** The normal bundle to \( \mathbb{I}^k \times F_{r+1}(\mathbb{I}^m) \) is the orthogonal complement in \( T(F_{r+1}(\mathbb{I}^k \times \mathbb{I}^m)) \) to the tangent bundle \( T(\mathbb{I}^k \times F_{r+1}(\mathbb{I}^m)) \). The former is the restriction of \( T(\mathbb{I}^k \times \mathbb{I}^m) \oplus r+1 \) to \( (\mathbb{I}^k \times \mathbb{I}^m) \oplus r+1 \), and the latter is the restriction of \( T\mathbb{I}^k \oplus (T\mathbb{I}^m) \oplus r+1 \). The inclusion is the diagonal on the first term and the identity on the second, and equivariant for the \( \Sigma_{r+1} \)-action. Thus the normal bundle is \( \Sigma_{r+1} \)-equivariantly isomorphic to the restriction of the orthogonal complement of the diagonal \( T\mathbb{I}^k \subset (T\mathbb{I}^k) \oplus r+1 \). This is isomorphic to a \( k \)-fold Whitney sum of the trivial \( \mathbb{R}^r \)-bundle, with \( \Sigma_{r+1} \)-action given by the standard representation. \[ \Box \]
The vector bundle $\varphi_{m,r+1}$ inherits a Riemannian metric, and we let $S(k\varphi_{m,r+1})$ be the sphere bundle with fiber over $(x,\xi) \in \mathbb{I}^k \times \mathbb{F}_{r+1}^{[m]}$ those vectors of length $\frac{1}{2}d(y,\partial\mathbb{I}^k)$. This bounds a disk bundle $D(k\varphi_{m,r+1})$, and both are clearly isomorphic to the unit sphere and disk bundles. Using the exponential map, we obtain the horizontal maps in the following commutative diagram of $E_{r+1}$-spaces:

$$
\begin{array}{ccc}
S(k\varphi_{m,r+1}) & \longrightarrow & F^{[r]}_{r+1}([k] \times [m]) \\
\downarrow & & \downarrow \\
D(k\varphi_{m,r+1}) & \longrightarrow & F^{[r+1]}_{r+1}([k] \times [m]) = F_{r+1}([k] \times [m]).
\end{array}
$$

**Proposition 4.13.** For all $r \geq 0$, there is a zigzag of homotopy cocartesian squares

$$
\begin{array}{c}
E_k((r+1)_*S(k\varphi_{m,r+1}) \otimes \mathcal{G}_{r+1} X \otimes r+1) \\
\downarrow \\
E_k((r+1)_*D(k\varphi_{m,r+1}) \otimes \mathcal{G}_{r+1} X \otimes r+1)
\end{array} \longrightarrow \cdots \xleftarrow{\simeq} \cdots \longrightarrow \begin{array}{c}
\mathcal{T}^L_r(U(E_{k+m})(X)) \\
\downarrow \\
\mathcal{T}^L_{r+1}(U(E_{k+m})(X)).
\end{array}
$$

**Proof.** The above result implies that for cofibrant $X$, we have a homotopy cocartesian square

$$
\begin{array}{ccc}
S(k\varphi_{m,r+1}) \otimes \mathcal{G}_{r+1} X \otimes r+1 & \longrightarrow & (r+1)^*U^E_k F^{[r]}([k] \times [m])(X) \\
\downarrow & & \downarrow \\
D(k\varphi_{m,r+1}) \otimes \mathcal{G}_{r+1} X \otimes r+1 & \longrightarrow & (r+1)^*U^E_k F^{[r+1]}([k] \times [m])(X).
\end{array}
$$

By Proposition 4.5 we have a commutative diagram with horizontal maps weak equivalences

$$
\begin{array}{c}
(r+1)^*U^E_k F^{[r]}([k] \times [m])(X) \xleftarrow{\simeq_{\alpha_r}} (r+1)^*U^E_k \mathcal{T}^L_r(U(E_{k+m})(X)) \\
\downarrow \\
(r+1)^*U^E_k F^{[r+1]}([k] \times [m])(X) \xleftarrow{\simeq_{\alpha_{r+1}}} (r+1)^*U^E_k \mathcal{T}^L_{r+1}(U(E_{k+m})(X)).
\end{array}
$$

Since applying $E_k$ and $(r+1)_*$ preserves homotopy cocartesian squares, doing so gives us the left and middle squares. For the right square, specialize Proposition 3.5 to $\mathcal{O} = \mathcal{E}_k$ and $A = \mathcal{T}^L_{r+1}(U(E_{k+m})(X))$ to obtain a homotopy cocartesian square

$$
\begin{array}{c}
E_k((r+1)_*(r+1)^*U^E_k \mathcal{T}^L_r(U(E_{k+m})(X))) \\
\downarrow \\
E_k((r+1)_*(r+1)^*U^E_k \mathcal{T}^L_{r+1}(U(E_{k+m})(X))) \longrightarrow \mathcal{T}^L_{r+1}(U(E_{k+m})(X)).
\end{array}
$$

We now deduce Theorem 1.2 from this by taking $S = \text{Top}$ and $X = (1)^*(\cdot)$; we need to resolve the issue that Proposition 4.13 only provides zigzags.
Proof of Theorem 1.2. We start with an elementary homotopy-theoretic observation. Given a commutative diagram of topological spaces
\[
\begin{array}{ccc}
S(k\phi_{m,r+1}) & \longrightarrow & X \\
\downarrow & & \downarrow \cong \\
D(k\phi_{m,r+1}) & \longrightarrow & Y \\
\end{array}
\]
with decorated arrows weak equivalences, we can find maps \(S(k\phi_{m,r+1}) \to X'\) and \(D(k\phi_{m,r+1}) \to Y'\) such that in the following diagram
\[
\begin{array}{ccc}
S(k\phi_{m,r+1}) & \longrightarrow & X \\
\downarrow & & \downarrow \cong \\
D(k\phi_{m,r+1}) & \longrightarrow & Y \\
\end{array}
\]
the outer square commutes and the triangles commute up to homotopy. To prove this, first homotope \(S(k\phi_{m,r+1}) \to X\) until a lift exists (which is possible since the domain has the homotopy type of a CW-complex). Because \(S(k\phi_{m,r+1}) \to D(k\phi_{m,r+1})\) admits the structure of a NDR-pair, we may extend this to a homotopy of commutative diagrams. At this point it suffices to find a lift in the commutative diagram
\[
\begin{array}{ccc}
S(k\phi_{m,r+1}) & \longrightarrow & X' \\
\downarrow & & \downarrow \cong \\
D(k\phi_{m,r+1}) & \longrightarrow & Y' \\
\end{array}
\]
which exists as \(D(k\phi_{m,r+1}), S(k\phi_{m,r+1})\) is homotopy equivalent to a CW pair.

Given this observation, we prove by induction over \(r\) that we may construct \(A_r \simeq T_r^{e_k}(UE_k(m))\) by iterated pushouts along free algebras, obtaining in the process maps between the \(A_r\) satisfying \(\text{colim } A_r = \text{hocolim } A_r \simeq FE_{k+m}(*)\).

The initial case is \(A_{-1} = \emptyset\). For the induction step, let us assume we have produced the \(A_r\) as in the statement of Theorem 1.2 with a weak equivalence \(A_r \to T_r(UE_k(m))\). Using the observation in the diagram of Proposition 4.13, we may assume we have a homotopy cocartesian commutative diagram
\[
\begin{array}{ccc}
S(k\phi_{m,r+1}) & \longrightarrow & (r + 1)^*UE_{k+m}(*) \\
\downarrow & & \downarrow \\
D(k\phi_{m,r+1}) & \longrightarrow & (r + 1)^*T_r(UE_k(m)) \\
\end{array}
\]
Applying the observation again to lift along \(a_r\), we may assume we have a homotopy cocartesian commutative diagram
\[
\begin{array}{ccc}
S(k\phi_{m,r+1}) & \longrightarrow & (r + 1)^*UE_{k+m}(*) \\
\downarrow & & \downarrow \\
D(k\phi_{m,r+1}) & \longrightarrow & (r + 1)^*T_r(UE_k(m)) \\
\end{array}
\]

Following this pattern, we construct the May-Milgram filtration and \(\varepsilon_k\)-cells.
We take adjoints and define $A_{r+1}$ as the pushout fitting in a commutative diagram

$$
\begin{array}{ccc}
E_k((r+1)_* S(k \phi_{m,r+1})) & \rightarrow & A_r \\
\downarrow & & \downarrow \\
E_k((r+1)_* D(k \phi_{m,r+1})) & \rightarrow & A_{r+1} \\
\end{array}
\rightarrow
\begin{array}{c}
T_r^L(UE_{k+m}(\ast)) \\
T_{r+1}^L(UE_{k+m}(\ast)).
\end{array}
$$

The outer and left squares are homotopy cocartesian, the former by construction and the latter as a homotopy pushout. Thus the right square is also homotopy cocartesian, and hence the map $A_{r+1} \rightarrow T_{r+1}^L(UF^{k+m}(\ast))$ is a weak equivalence.

**Remark 4.14.** It is plausible that Theorem 1.2 may be deduced from results analogous to those in [GKRW18a]. One would need an applicable CW approximation theorem for $E_k$-algebras in $\text{Top}^+$, as well as verify that one may desuspend the identification of $\Sigma^k Q^k_L(F^{k+m}(\ast))$ with the $k$-fold bar construction with respect to the canonical augmentation.

5. Relation to the May-Milgram filtration

We now explain the relationship between the results in the previous section and the May-Milgram filtration on $\Omega^m \Sigma^m S^k$.

**Definition 5.1.** Given a based topological space $(X, x_0)$, let $C(M; X)$ be the quotient of $\bigsqcup_{n \geq 0} F_n(M) \times \sigma_n X^n$ by the relation that $(m_1, \ldots, m_n; x_1, \ldots, x_n)$ is equivalent to $(m_1, \ldots, m_{n-1}; x_1, \ldots, x_{n-1})$ if $x_n = x_0$. We call this the configuration space of unordered points in $M$ with labels in $X$.

When $X = S^0$, we recover ordinary unordered configuration spaces and drop $X$ from the notation. Work of Milgram and May implies that $\Omega^m \Sigma^m X$ has the weak homotopy type of $C(I^m; X)$ when $X$ is connected [Mil66, May72]. The $r$th stage of the May-Milgram filtration of $C(\Omega^m; X) \simeq \Omega^m \Sigma^m X$ is defined to be the image of $F_r(I^m) \times \sigma_r X^r$ in $C(I^m; X)$.

In this paper, we use only the case $X = S^k$. In that case, $\Omega^m \Sigma^m S^k$ is weakly equivalent to the $k$-fold delooping of $C(I^k \times I^m) = F(I^k \times I^m)(\ast) \simeq E_{k+m}(\ast)$. Let us denote the $E_k$-algebra $F^{[r]}(I^k \times I^m)(\ast)$ by $C^{[r]}(I^k \times I^m)$.

**Theorem 5.2.** The $k$-fold delooping of $C^{[r]}(I^k \times I^m)$ is homotopy equivalent to the $r$th stage in the May-Milgram filtration of $\Omega^m \Sigma^m S^k$.

To prove this, we need to consider a generalization of $C^{[r]}(I^k \times I^m)$ where points can vanish if they enter certain regions.

**Definition 5.3.** Let $M$ be a manifold and $N \subset M$ a subspace. Let $C^{[r]}(M \times I^m)$ denote the subspace of $C(M \times I^m)$ of configurations $\xi$ where $\xi \cap \{x\} \times I^m$ has cardinality $\leq r$ for all $x \in M$. Let $C^{[r]}((M, N) \times I^m)$ be the quotient of $C^{[r]}(M \times I^m)$ by the equivalence relation that $\xi \sim \xi'$ if $\xi \cap \{(M \setminus N) \times I^m\} = \xi' \cap \{(M \setminus N) \times I^m\}$.

We drop the superscript for $r = \infty$ and drop the $- \times I^m$ for $m = 0$. There are two configuration space models for $\Omega^m \Sigma^m S^k = \Omega^m S^{k+m}$. The first is a special case of May’s approximation theorem from [May72], building on the work of Milgram in [Mil66], and the second is a specialization of Proposition 2 of [Böd87].
Theorem 5.4 (May). For $k > 0$, $C(I^m; S^k)$ is weakly homotopy equivalent to $\Omega^m \Sigma^m S^k$.

Theorem 5.5 (Bödigheimer). For $k > 0$, $C((R^k, R^k \setminus I^k) \times I^m)$ is weakly homotopy equivalent to $\Omega^m \Sigma^m S^k$.

We will relate these two models of $\Omega^m \Sigma^m S^k$, and compare filtrations of these spaces. Let $M_r(C(I^m; S^k))$ denote the image of $F_r(I^m) \times (S^k)^r$ in $C(I^m; S^k)$, the $r$th stage of the May-Milgram filtration. The topological space $C((R^k, R^k \setminus I^k) \times I^m)$ is filtered by the $C[\ell](R^k, R^k \setminus I^k) \times I^m)$. From now on, we view $S^k$ as $R^k/(R^k \setminus I^k)$ with base point given by the image of $R^k \setminus I^k$. Define a map $\rho$ by

$$\rho: C(I^m; S^k) \rightarrow C((R^k, R^k \setminus I^k) \times I^m)$$

$$((m_1, x_1), \ldots, (m_r, x_r)) \mapsto (x_1 \times m_1, \ldots, x_r \times m_r),$$

where $m_i \in I^m$ and $x_i \in R^k/(R^k \setminus I^k) = S^k$. This inclusion has image consisting of those configurations with at most one point in each fiber of $R^k \times I^m \rightarrow I^m$. We denote its restrictions by $\rho_r: M_r(C(I^m; S^k)) \rightarrow C[\ell](R^k, R^k \setminus I^k) \times I^m)$.

Lemma 5.6. The maps $\rho$ and $\rho_r$ are homotopy equivalences.

Proof. The strategy is to scale the configurations so that in each fiber of $R^k \times I^m \rightarrow I^m$ all but at most one point is pushed into $R^k \setminus I^k$. To do so, we pick a continuous function $\eta: C((R^k, R^k \setminus I^k) \times I^m) \rightarrow (0, \infty)$ with the property that for all $\xi \in C((R^k, R^k \setminus I^k) \times I^m)$ and $x \in I^m$, there is at most one point in $\xi$ that is within distance $\eta(\xi)$ of $0 \times x \in R^k \times I^m$. Let $\phi^R_t: R^k \rightarrow R^k$ be a continuous family of maps, depending on $t \in [0, 1]$ and $R > 0$, such that:

- $\phi^R_0 = \text{id}$,
- $\phi^R_t|_{(\phi^R)^{-1}(3R)}$ a homeomorphism onto its image,
- $\phi^R_t(R^k \setminus I^k) \subset R^k \setminus I^k$ and $\phi^R_t(y) \in R^k \setminus I^k$ if $||y|| > R$.

Then we define

$$H: [0, 1] \times C((R^k, R^k \setminus I^k) \times I^m) \rightarrow C((R^k, R^k \setminus I^k) \times I^m)$$

$$(t, \xi) \mapsto (\text{id} \times \phi^R_t(\xi), \xi),$$

where the subscript $\ast$ means induced map on configuration spaces. For $t = 1$, all but at most one point in each fiber are pushed into $R^k \setminus I^k$ (where these points vanish). The map $H_1$ gives a homotopy inverse to the inclusion and the homotopy $H_t$ respects the filtrations. \hfill \Box

Thus $C[\ell](R^k, R^k \setminus I^k) \times I^m)$ is homotopy equivalent to the $r$th stage of the May-Milgram filtration. We claim that the $k$-fold delooping of $C[\ell](R^k \times I^m)$ is $C[\ell](R^k, R^k \setminus I^k) \times I^m)$. To show this, we need to compute the intermediate deloopings $B^j C[\ell](I^k \times I^m)$. Though one can define $B^j A$ for an $E_j$-algebras as the two-sided bar construction $B(S^j, E_j, A)$ [May72], it will be more convenient for us to strictify our $E_j$-algebras to associative monoids and iterate the classical bar construction. This gives a weakly equivalent answer (see e.g. Section 13.8.4 of [GKRW18a]). Given a topological monoid $M$, let $BM$ denote the bar construction $||B_\ast(M, M, \ast)||$ with $||-||$ the thick geometric realization. Using a construction reminiscent of Moore loop spaces, we define strict monoid structures on topological spaces homotopy equivalent to $C[\ell](I^k \times I^m)$ and its intermediate deloopings.
We start by defining the augmented semi-simplicial topological space $\mathbb{X}_\bullet$: its topological space of $p$-simplices

$$X_p \subset C^{[r]} \left( (I^{k-j} \times \mathbb{R}^j, I^{k-j} \times (\mathbb{R}^j \setminus \mathbb{R}^j)) \times \mathbb{I}^m \right)$$

is the subspace of $(\xi; t_0, \ldots, t_p)$ such that $t_0 \leq t_1 \leq \ldots \leq t_p$ and

$$\xi \cap \left( I^{k-j} \times \{t_0, \ldots, t_p\} \times [-1/2, 1/2]^{j-1} \times \mathbb{I}^m \right) = \emptyset.$$
Thus it is augmented over $X_{-1} = C^{|r|} \left( (I^{k-j} \times \mathbb{R}^j, I^{k-j} \times (\mathbb{R}^j \setminus \mathcal{U})) \times \mathbb{I}^m \right)$. The face map $d_i : X_p \to X_{p-1}$ is given by forgetting $t_i$ and the augmentation $X_0 \to X_{-1}$ is given by forgetting $t_0$.

We denote the map $||X_t|| \to X_{-1}$ by $\epsilon$. To show this is a weak equivalence, we prove it is a microfibration with weakly contractible fibers and invoke Lemma 4.8. For a configuration $\xi \in C^{|r|} \left( (I^{k-j} \times \mathbb{R}^j, I^{k-j} \times (\mathbb{R}^j \setminus \mathcal{U})) \times \mathbb{I}^m \right)$, let $S_\xi \subseteq \mathbb{I}$ be the subspace of $t \in I$ such that

$$
\xi \cap \left( I^{k-j} \times \{t\} \times [-1/2, 1/2]^{j-1} \times \mathbb{I}^m \right) = \emptyset.
$$

The fiber $\epsilon^{-1}(\xi)$ is the thick geometric realization of a semi-simplicial space with space of $p$-simplices homotopy equivalent to the set of order preserving-maps from $\{0, \ldots, p\}$ to $\pi_0(S_\xi)$, which is a simplex of dimension $\#(\pi_0(S_\xi)) - 1$. Since levelwise weak equivalences induce weak equivalences on thick geometric realizations (see e.g. Theorem 2.2 of [ERW17]), the fibers of $\epsilon$ are weakly contractible.

The proof that $\epsilon$ is a microfibration is similar to that of Lemma 4.10. The key fact is that if

$$
\xi \cap \left( I^{k-j} \times \{t_0, \ldots, t_p\} \times [-1/2, 1/2]^{j-1} \times \mathbb{I}^m \right) = \emptyset,
$$

the same will be true for nearby configurations (this is why we use $[-1/2, 1/2]$ instead of $I$, otherwise new points could appear and hit the forbidden regions immediately).

We will next construct a semi-simplicial map $f_* : X_* \to B_*(\ast, M^{|r|}(k \times m, j-1), \ast)$ and prove its thick geometric realization is a weak equivalence. The map

$$
f_p : X_p \to B_p(\ast, M^{|r|}(k \times m, j-1), \ast) = \left( M^{|r|}(k \times m, j-1) \right)^p
$$

is defined as follows: given $s \in (0, \infty)$ and $\xi \in C^{|r|} \left( (I^{k-j} \times \mathbb{R}^j, I^{k-j} \times (\mathbb{R}^j \setminus \mathcal{U})) \times \mathbb{I}^m \right)$, let $s \cdot \xi$ be the configuration given by multiplying the last $j-1$ of the $\mathbb{R}^j$-coordinates of points of $\xi$ by $s$. Let $\xi_i := (2 \cdot \xi) \cap \left( I^{k-j} \times \{t_i\} \times [-1/2, 1/2]^{j-1} \times \mathbb{I}^m \right)$ and set

$$
f_p(\xi, t_0, \ldots, t_p) := ((\xi_1, t_1 - t_0), \ldots, (\xi_p, t_p - t_{p-1})).
$$

Since we are taking thick geometric realizations, to prove $||f_*||$ is a weak equivalence, it suffices to prove each $f_p$ is a weak homotopy equivalence.

To prove this, we give a levelwise homotopy inverse $g_p : B_p(\ast, M^{|r|}(k \times m, j-1), \ast) \to X_p$ to $f_p$. Represent elements of $B_p(\ast, M^{|r|}(k \times m, j-1), \ast) = \left( M^{|r|}(k \times m, j-1) \right)^p$ by $p$-tuples $((\eta_1, u_1), \ldots, (\eta_p, u_p))$ with $\eta_i$ an equivalence class of configurations and $u_i \in [0, \infty)$. Let $u := u_1 + \ldots + u_p + 1$, $w_0 := 0$ and $w_i := w_{i-1} + u_{i}/u$. Let $w_i'$ be obtained from $\eta_i$ by dividing the $(k-j+1)$st coordinate of all points by $u$ and let $w := \bigcup \{w_i + w_i'\}$. Here adding a number to a configuration means we add the number to the $(k-j+1)$st coordinate of every point in the configuration. Then we define $g_p$ as

$$
g_p((\eta_1, u_1), \ldots, (\eta_p, u_p)) := (w_0, w_0, \ldots, w_p).
$$

Scaling configurations, scaling the numbers $(t_0, \ldots, t_p)$, and pushing configurations with $(k-j+1)$st coordinate outside the interval $(t_0, t_p)$ off into the regions where they vanish gives a homotopy from $g_p \circ f_p$ to the identity map. Scaling configurations and the parameters $w_i$, gives a homotopy from $f_p \circ g_p$ to the identity map. □
Proof of Theorem 5.2. By iterating Proposition 5.8 $k$ times and using the fact that
\[
C^{[r]}(\langle I^{k-j} \times \mathbb{R}, \mathbb{R}^{k-j} \times (\mathbb{R}^j \setminus I^j) \rangle \times \mathbb{I}^m) \xrightarrow{\simeq} M^{[r]}(k \times m, j),
\]
one sees that $B^k C^{[r]}(\langle I \times \mathbb{I}^m \rangle) \simeq C^{[r]}(\langle \mathbb{R}^k, (\mathbb{R}^k \setminus I^k) \times \mathbb{I}^m \rangle)$. Lemma 5.6 shows that $C^{[r]}(\langle \mathbb{R}^k, (\mathbb{R}^k \setminus I^k) \times \mathbb{I}^m \rangle)$ is homotopy equivalent to the $r$th stage of the May-Milgram filtration of $\Omega^n \Sigma^n S^k$.

Remark 5.9. Snaith showed that the May-Milgram filtration stably splits [Sna74]. However, its lift to a filtration of $E_{k+m}(\ast)$ of $E_k$-algebras does not split after taking suspension spectra. Such a splitting would imply that $C_2(\mathbb{I}^2 \times I^1) \simeq \mathbb{R}P^2$ stably splits off a copy of $F_2^{[1]}(\mathbb{I}^2 \times I^1)/G_2 \simeq \mathbb{R}P^1$.

However, this filtration does split after stabilizing in a different manner. Recall $Q_L^{\infty}$ denotes the derived indecomposables functor of Remark 3.6. Basterra-Mandell showed that derived indecomposables can be considered as stabilization of an algebra over an operad [BM05], and derived indecomposables of an $E_k$-algebra may be computed using its $k$-fold bar construction, see [BM11] or Chapter 13 of [GKRW18a]. Thus the induced filtration of the stabilization $Q_L^{\infty}(\Sigma^\infty E_{k+m}(\ast)_+)$ agrees with the suspension spectra of the May-Milgram filtration and hence splits by the work of Snaith [Sna74].

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