A CONNECTION BETWEEN DECOMPOSABLE ULTRAFILTERS AND POSSIBLE COFINALITIES. II

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Abstract. We use Shelah’s theory of possible cofinalities in order to solve a problem about ultrafilters.

Theorem 1. Suppose that $\lambda$ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter $D$ is $\kappa$-decomposable for all regular cardinals $\kappa$ with $\lambda' < \kappa < \lambda$. Then $D$ is either $\lambda$-decomposable, or $\lambda^+ $-decomposable.

We give applications to topological spaces and to abstract logics (Corollaries 7, 8 and Theorem 9).

If $F$ is a family of subsets of some set $I$, and $\lambda$ is an infinite cardinal, a $\lambda$-decomposition for $F$ is a function $f : I \to \lambda$ such that whenever $X \subseteq \lambda$ and $|X| < \lambda$ then $\{ i \in I | f(i) \in X \} \notin F$. The family $F$ is $\lambda$-decomposable if and only if there is a $\lambda$-decomposition for $F$. If $D$ is an ultrafilter (that is, a maximal proper filter) let us define the decomposability spectrum $K_D$ of $D$ by $K_D = \{ \lambda \geq \omega | D$ is $\lambda$-decomposable\}.

The question of the possible values the spectrum $K_D$ may take is particularly intriguing. Even the old problem from [Si] of characterizing those $\mu$ for which there is an ultrafilter $D$ such that $K_D = \{ \omega, \mu \}$ is not yet completely solved [Shr, p. 1007].

The case when $K_D$ is infinite is even more involved. [P] studied the situation in which $\lambda$ is limit and $K_D \cap \lambda$ is unbounded in $\lambda$; he found some assumptions which imply that $\lambda \in K_D$. This is not always the case; if $\mu$ is strongly compact and $\text{cf} \lambda < \mu < \lambda$ then there is an ultrafilter $D$ such that $K_D \cap \lambda$ is unbounded in $\lambda$, and $D$ is not $\lambda$-decomposable. If we are in the above situation, we have that necessarily $D$ is $\lambda^+$-decomposable (by [So, Lemma 3] and the proof of [P, Proposition 2]).

The above examples suggest the problem whether $K_D \cap \lambda$ unbounded in $\lambda$ implies that either $\lambda \in K_D$ or $\lambda^+ \in K_D$. In general, the problem

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is still open; here we solve it affirmatively in the particular case when
there is $\lambda' < \lambda$ such that $K_D$ contains all regular cardinals in the
interval $[\lambda', \lambda]$: moreover, when $\text{cf} \lambda > \omega$ it is sufficient to assume that
$\{\kappa < \lambda | \kappa^+ \in K_D \cap \lambda \}$ is stationary in $\lambda$.

We briefly review some known results on $K_D$. If $\kappa$ is regular and
$\kappa^+ \in K_D$ then $\kappa \in K_D$; and if $\kappa \in K_D$ is singular, then $\text{cf} \kappa \in K_D$.
Results from [D] imply that if there is no inner model with a measurable
Kardinal then $\text{cf} \kappa = \kappa^+ \in K_D$ is always an interval with minimum $\omega$. On the other
hand, it is trivial that $K_D = \{ \mu \}$ if and only if $\mu$ is either $\omega$ or a measurable
cardinal. Further comments and constraints on $K_D$ are given in [L4, L5]. Apparently, the problem of determining which sets
of cardinals can be represented as $K_F = \{ \lambda \geq \omega | F \text{ is } \lambda\text{-decomposable} \}$
for a filter $F$ has not been studied.

If $(\lambda_j)_{j \in J}$ are regular cardinals, the cofinality $\text{cf} \prod_{j \in J} \lambda_j$ of the product
$\prod_{j \in J} \lambda_j$ is the smallest cardinality of a set $G \subseteq \prod_{j \in J} \lambda_j$ having the
property that for every $f \in \prod_{j \in J} \lambda_j$ there is $g \in G$ such that $f(j) \leq g(j)$ for all $j \in J$.

We shall state our results in a quite general form, involving arbitrary
filters, rather than ultrafilters. In what follows, the reader interested
in ultrafilters only can always assume that $F$ is an ultrafilter.

**Proposition 2.** If $(\lambda_j)_{j \in J}$ are infinite regular cardinals, $\mu = \text{cf} \prod_{j \in J} \lambda_j$
and the filter $F$ is $\lambda_j$-decomposable for all $j \in J$, then $F$ is $\mu'$-decomposable for some $\mu'$
with $\text{sup}_{j \in J} \lambda_j \leq \mu' \leq \mu$.

**Proof.** Let $F$ be over $I$, and let $(g_\alpha)_{\alpha \in \mu}$ witness $\mu = \text{cf} \prod_{j \in J} \lambda_j$. For
every $j \in J$ let $f(j, -) : I \to \lambda_j$ be a $\lambda_j$ decomposition for $F$. For
any fixed $i \in I$, $f(-, i) \in \prod_{j \in J} \lambda_j$, thus there is $\alpha(i) \in \mu$ such that
$f(j, i) \leq g_\alpha(i)(j)$ for all $j \in J$.

Let $X$ be a subset of $\mu$ with minimal cardinality with respect to the
property that $Y = \{ i \in I | \alpha(i) \in X \} \in F$. Let $\mu' = |X|$. Thus, whenever $X' \subseteq \mu$ and $|X'| < \mu'$, we have $Y' = \{ i \in I | \alpha(i) \in X' \} \notin F$.
Define $\alpha(i) = \alpha(i)$ for $i \in Y$, and $\alpha(i) = 0$ for $i \notin Y$. Thus, $h : I \to X \cup \{0\}$.

If $|X'| < \mu'$ then $\{ i \in I | h(i) \in X' \} \subseteq Y' \cup (I \setminus Y) \notin F$ (otherwise,
since $F$ is a filter, $Y' \supseteq Y \cap Y' = Y \cap (Y' \cup (I \setminus Y)) \in F$, contradiction).
This shows that, modulo a bijection from $X \cup \{0\}$ onto $\mu'$, $h$ is a $\mu'$-decomposition for $F$.
Trivially, $\mu' \leq \mu$.

Hence, it remains to show that $\text{sup}_{j \in J} \lambda_j \leq \mu'$. Suppose to the contrary that $\mu' < \lambda_j$ for some $\bar{j} \in J$. Then $|\{ \alpha(i) \in X \}| \leq |\{ \alpha(i) \in X \}| \leq |X| = \mu' < \lambda_j$. Since $\lambda_j$ is regular, we have that
$\beta = \text{sup}_{\bar{j} \in Y} \alpha(i)(\bar{j}) < \lambda_j$. Hence, if $i \in Y$, then $f(\bar{j}, i) \leq g_\alpha(i)(\bar{j}) \leq
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\( \beta < \lambda_j \). Thus, \([0, \beta] < \lambda_j\), but \(\{i \in I | f(j, i) \in [0, \beta]\} \supseteq Y \in F\), and this contradicts the assumption that \(f(j, -)\) is a \(\lambda_j\) decomposition for \(F\). \( \square \)

Proposition 2 has not the most general form: we have results dealing with the cofinality \(\mu\) of reduced products \(\prod_E \lambda_j\), where \(E\) a filter on \(J\). We shall not need this more general version here.

Recall that an ultrafilter \(D\) is \((\mu, \lambda)\)-regular if and only if there is a family of \(\lambda\) members of \(D\) such that the intersection of any \(\mu\) members of the family is empty. We list below the properties of decomposability and regularity we shall need. Much more is known: see [L2, L5] and references there.

**Properties 3.**

(a) Every \(\lambda\)-decomposable ultrafilter is \(\text{cf} \lambda\)-decomposable.

(b) Every \(\text{cf} \lambda\)-decomposable ultrafilter is \((\lambda, \lambda)\)-regular.

(c) If \(\mu' \geq \mu\) and \(\lambda' \leq \lambda\) then every \((\mu, \lambda)\)-regular ultrafilter is \((\mu', \lambda')\)-regular.

(d) [CC, Theorem 1] [KP, Theorem 2.1] If \(\lambda\) is singular, \(D\) is a \(\lambda^+\)-decomposable ultrafilter, and \(D\) is not \(\text{cf} \lambda\)-decomposable then \(D\) is \((\lambda', \lambda^+)\)-regular for some \(\lambda' < \lambda\).

(e) [Ka, Corollary 2.4] If \(\lambda\) is singular then every \(\lambda^+\)-decomposable ultrafilter is \((\lambda, \lambda^+)\)-regular.

(f) [L1, Corollary 1.4] If \(\lambda\) is singular then every \((\lambda, \lambda)\)-regular ultrafilter is either \(\text{cf} \lambda\)-decomposable or \((\lambda', \lambda)\)-regular for some \(\lambda' < \lambda\).

(g) If \(\lambda\) is regular then an ultrafilter is \(\lambda\)-decomposable if and only if it is \((\lambda, \lambda)\)-regular.

**Theorem 4.** Suppose that \(\lambda\) is a singular cardinal, \(F\) is a filter, and either

(a) there is \(\lambda' < \lambda\) such that \(F\) is \(\kappa\)-decomposable for all regular cardinals \(\kappa\) with \(\lambda' < \kappa < \lambda\), or

(b) \(\text{cf} \lambda > \omega\) and \(S = \{\kappa < \lambda | \text{cf} \kappa^+\text{-decomposable} \}\) is stationary in \(\lambda\).

Then \(F\) is either \(\lambda\)-decomposable, or \(\lambda^+\)-decomposable.

If \(F = D\) is an ultrafilter, then \(D\) is \((\lambda, \lambda)\)-regular. Moreover, \(D\) is either (i) \(\lambda\)-decomposable, or (ii) \((\lambda', \lambda^+)\)-regular for some \(\lambda' < \lambda\), or (iii) \(\text{cf} \lambda\)-decomposable and \((\lambda, \lambda^+)\)-regular.

**Proof.** Recall from [She] that if \(\mathcal{a}\) is a set of regular cardinals, then \(\text{pcf} \mathcal{a}\) is the set of regular cardinals which can be obtained as \(\text{cf} \prod_E \mathcal{a}\), for some ultrafilter \(E\) on \(\mathcal{a}\). If \(\text{cf} \lambda = \nu > \omega\) then by [She, II, Claim 2.1] there is a sequence \((\lambda_\alpha)_{\alpha \in \nu}\) closed and unbounded in \(\lambda\) and such that, letting \(\mathcal{a} = \{\lambda_\alpha^+ | \alpha \in \nu\}\), we have \(\lambda^+ = \max \text{pcf} \mathcal{a}\). If \(\text{cf} \lambda = \omega\)
then we have \( \lambda^+ = \max \text{pcf} \ a \) for some countable \( a \) unbounded in \( \lambda \) as a consequence of [She, II, Theorem 1.5] (since \( a \) is countable, any ultrafilter over \( a \) is either principal, or extends the dual of the ideal of bounded subsets of \( a \)).

Letting \( b = a \cap [\lambda', \lambda) \) in case (a), and \( b = a \cap \{\kappa^+|\kappa \in S\} \) in case (b), we still have \( \max \text{pcf} \ b = \lambda^+ \), because \( b \) is unbounded in \( \lambda \), hence \( \max \text{pcf} \ b \geq \lambda^+ \), and because \( \max \text{pcf} \ b \leq \max \text{pcf} \ a = \lambda^+ \), since \( b \subseteq a \).

Assume, without loss of generality, that \( \lambda' > (cf \lambda)^+ \) in (a), and that \( \inf S > (cf \lambda)^+ \) in (b). Since \( |b| \leq |a| = cf \lambda \), then \( |b|^+ < \min b \), hence, by [She, II, Lemma 3.1], \( \lambda^+ = \max \text{pcf} \ b = \text{cf} \prod_{\kappa \in b} \kappa \). Then Proposition 2 implies that \( F \) is either \( \lambda \)-decomposable, or \( \lambda^+ \)-decomposable.

The last statements follow from Properties 3(a)-(e).

**Corollary 5.** If \( \lambda \) is a singular cardinal and the ultrafilter \( D \) is not \( cf \lambda \)-decomposable, then the following conditions are equivalent:

(a) There is \( \lambda' < \lambda \) such that \( D \) is \( \kappa \)-decomposable for all regular cardinals \( \kappa \) with \( \lambda' < \kappa < \lambda \).

(a') (Only in case \( cf \lambda > \omega \)) \( \{\kappa < \lambda|F^+ \text{ is } \kappa^+\text{-decomposable}\} \) is stationary in \( \lambda \).

(b) \( D \) is \( \lambda^+ \)-decomposable.

(c) There is \( \lambda' < \lambda \) such that \( D \) is \( (\lambda', \lambda^+) \)-regular.

(d) \( D \) is \( (\lambda, \lambda) \)-regular.

(e) There is \( \lambda' < \lambda \) such that \( D \) is \( (\lambda', \lambda) \)-regular.

(f) There is \( \lambda' < \lambda \) such that \( D \) is \( (\lambda'', \lambda'') \)-regular for every \( \lambda'' \) with \( \lambda' < \lambda'' < \lambda \).

**Proof.** (a) \( \Rightarrow \) (b) and (a') \( \Rightarrow \) (b) are immediate from Theorem 4 and Property 3(a). In case \( cf \lambda > \omega \), (a) \( \Rightarrow \) (a') is trivial.

(b) \( \Rightarrow \) (c) \( \Rightarrow \) (d) \( \Rightarrow \) (e) \( \Rightarrow \) (f) \( \Rightarrow \) (a) are given, respectively, by Properties 3(d)(c)(f)(c)(g).

**Corollary 6.** If \( \lambda \) is a singular cardinal, then an ultrafilter is \( (\lambda, \lambda) \)-regular if and only if it is either \( cf \lambda \)-decomposable or \( \lambda^+ \)-decomposable.

**Proof.** Immediate from Corollary 5(d)\( \Rightarrow \) (b) and Properties 3(b)-(d).

A topological space is \( [\mu, \lambda] \)-compact if and only if every open cover by \( \lambda \) many sets has a subcover by \( < \mu \) many sets. A family \( \mathcal{F} \) of topological spaces is **productively** \( [\mu, \lambda] \)-compact if and only if every (Tychonoff) product of members of \( \mathcal{F} \) is \( [\mu, \lambda] \)-compact.

**Corollary 7.** If \( \lambda \) is a singular cardinal, then a family of topological spaces is productively \( [\lambda, \lambda] \)-compact if and only if it is either productively \( [cf \lambda, cf \lambda] \)-compact or productively \( [\lambda^+, \lambda^+] \)-compact.
Proof. Immediate from Corollary 6, Property 3(g) and [L3, Theorem 3] (see also [Ca]). □

Henceforth, by a logic, we mean a regular logic in the sense of [E]. Typical examples of regular logics are infinitary logics, or extensions of first-order logic obtained by adding new quantifiers; e. g., cardinality quantifiers asserting “there are at least $\omega_\alpha$'s such that . . .”.

A logic $L$ is $[\lambda, \mu]$-compact if and only if for every pair of sets $\Gamma$ and $\Sigma$ of sentences of $L$, if $|\Sigma| \leq \lambda$ and if $\Gamma \cup \Sigma'$ has a model for every $\Sigma' \subseteq \Sigma$ with $|\Sigma| < \mu$, then $\Gamma \cup \Sigma$ has a model (see [Ma] for some history and further comments).

Corollary 8. If $\lambda$ is a singular cardinal, then a logic is $[\lambda, \lambda]$-compact if and only if it is either $[\text{cf} \lambda, \text{cf} \lambda]$-compact or $[\lambda^+, \lambda^+]$-compact.

Proof. Immediate from Corollary 6, Property 3(g) and [Ma, Theorem 1.4.4] (notice that in [Ma] in the definition of $(\lambda, \mu)$-regularity for an ultrafilter the order of $\mu$ and $\lambda$ is reversed). □

Theorem 9. Suppose that $(\lambda_i)_{i \in I}$ and $(\mu_j)_{j \in J}$ are sets of infinite cardinals. Then the following are equivalent:

(i) For every $i \in I$ there is a $(\lambda_i, \lambda_i)$-regular ultrafilter which for no $j \in J$ is $(\mu_j, \mu_j)$-regular.

(ii) There is a logic which is $[\lambda_i, \lambda_i]$-compact for every $i \in I$, and which for no $j \in J$ is $[\mu_j, \mu_j]$-compact.

(iii) For every $i \in I$ there is a $[\lambda_i, \lambda_i]$-compact logic which for no $j \in J$ is $[\mu_j, \mu_j]$-compact.

The logics in (ii) and (iii) can be chosen to be generated by at most $2^{|J|}$ cardinality quantifiers.

Proof. In the case when all the $\mu_j$'s are regular, the Theorem is proved in [L1, Theorem 4.1]. The general case follows from the above particular case, by applying Corollaries 6 and 8. □

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