We study the phase separation configurations and their rotational properties for a mixture of two interacting charged Bose-Einstein condensates subject to a magnetic field trapped in disc and Corbino geometries. We calculate the ground state energies of azimuthal and radial phase separation configurations using the Gross-Pitaevskii and the Thomas-Fermi approximations. We show that the results for experimentally relevant system parameters from both approaches are in good agreement. The immiscible mixture in both geometries with equal intracomponent interactions favors the azimuthal phase separation for any intercomponent interactions. Only an imbalance in the intracomponent interactions can result in a transition to the radial phase separation, for which the transition becomes sensitive to the shape of the trap. We present phase diagrams as function of the inter and intracomponent interactions. While the radial phase separation is widely favoured in disc geometry, the azimuthal phase separation is favoured for narrower Corbino geometries. We explore the rotational properties of the spatially separated condensates under the magnetic field, studying their angular momenta and velocity fields. The quantization of circulation breaks down for the azimuthal phase separation. In this case, the bulk region of the condensate continues to display superfluid flow behavior whereas the velocity field shows a rigid body behavior along the phase boundaries.

I. INTRODUCTION

Phase separation is a typical aspect of multispecies ultracold atomic systems with repulsive intercomponent interactions. Throughout the phase separation the components occupy non-overlapping separate spatial regions. The zero temperature mean field studies have revealed that a homogeneous mixture of two miscible Bose-Einstein Condensates (BECs) turns into an immiscible mixture displaying phase separation when intercomponent interaction exceeds the geometric mean of the intracomponent interactions [1]. This condition (with a slight shift) remains valid for relatively large perturbations from uniformity, where the shift is determined by the geometry of the trapping potential [2,3].

The static and dynamic features of phase separations in two component atomic BECs have been studied both experimentally [4,10] and theoretically [11–27]. Experiments have been carried out for mixtures with two different atomic species [8–10], different isotopes of the same atoms [6] or different hyperfine states of the same isotopes [4,5]. Theoretical studies have mainly been performed at the mean-field level for trapped atoms using the Thomas-Fermi (TF) [11–13], the Gross-Pitaevskii (GP) [16,21], or the Bogoliubov-de Gennes approaches [25,27]. These studies have focused on the transition from the miscible to the immiscible state and on the physical properties of immiscible states for both non-rotating and rotating BECs.

The phase separation configurations are determined by the difference in the strength of the intracomponent interactions and the shape of external potential [18–22,25]. For a mixture in a toroidal trap two configurations of phase separation can occur: the azimuthal phase separation (APS) and the radial phase separation (RPS) for which the components are restricted to semi-circular and concentric full circular non-overlapping annular regions, respectively (see Fig. 1). The APS is the ground state of symmetric immiscible mixtures with all equal physical parameters [19–22,25]. However, a phase transition from APS to RPS occurs by introducing an imbalance in the system [25].

In addition to the spatial separation of the density distributions, the phase separation also affects other physical properties of the condensates. The rotational properties of the APS configurations show different behaviour with respect to those of the RPS. While the circulation of the velocity field for both condensates remains quantized in an APS, it breaks down for an APS. The angular momenta of both condensates exhibit a smooth transition from quantized to continuous values as the mixture is driven through a transition from the RPS to the APS [20].

In current experiments for ultracold atomic and molecular systems, different trap geometries can be generated and the interaction strengths can be finely tuned [28]. Moreover, artificial magnetic fields for ultracold gases can make neutral atoms behave as if they are electrically charged [29,30]. The strength of this magnetic field depends on the internal structure of atoms and thus can be species selective. This allows us to consider various mixtures of equally and unequally charged superfluids [31–34]. Therefore, two condensates with equal or different rotation frequencies can be created.

In this study we consider phase separated mixtures of two interacting charged BECs subject to a weak magnetic field and trapped in disc and Corbino geometries. We analyse the conditions for the phase transition between the mentioned configurations of the immiscible phase and study their rotational properties. We explore the phase separated mixtures with different inter and intracomponent interactions and comment on the effects of charge imbalance at this level of approximations. We discuss how the shape of the trap becomes relevant when
there exists an asymmetry between the physical properties of the components.

We use both the GP and the TF approximations to investigate the ground state and the rotational properties of mixtures as a function of both inter and intracomponent interactions and the applied magnetic field. The coupled GP equations describing the system are solved using the imaginary time evolution [35]. We compare the results of the GP simulations with solutions obtained from the TF approximation and conclude that the latter works reasonably well for experimentally relevant systems.

This article is organized as follows: In the next section we define the physical properties of a mixture of two synthetically charged Bose-Einstein condensates in a harmonic trap and subject to an artificial magnetic field. We provide the equations describing the BEC mixture within the GP and the TF approximations. In Sec. [III] we consider the case with equal intracomponent interactions. The ground state of an immiscible mixture has the APS configuration for any value of the inter-component interaction and synthetic charges. We also show that for weak magnetic fields and synthetic charges considered, the resulting kinetic energy does not play a significant role in determining the phase boundary. In Sec. [IV] we present phase diagrams showing the phase separation configurations for different intra and intercomponent interactions. In Sec. [V] we analyze the rotational properties of the condensates and finally in Sec. [VI] we summarize and discuss our results.

II. MIXTURE OF TWO CHARGED SUPERFLUIDS

We consider a mixture of two charged superfluids consisting of equal number of atoms $N_1 = N_2 = N$, with the same particle masses $M$, and synthetic charges $q_1$ and $q_2$. The mixture is strongly confined along the longitudinal direction, $z$, in a harmonic potential of the form

$$V_{\text{ext}}(r) = V_\perp + V_z = \frac{1}{2} M \omega_\perp^2 (r_\perp - r_{\perp,0})^2 + \frac{1}{2} M \omega_z^2 z^2 \quad (1)$$

where $\omega_\perp \gg \omega_z$ are the trapping frequencies ($\omega_\perp = \omega_x = \omega_y$), and $r_{\perp,0}$ denotes the radial distance in the $xy$-plane. We adopt an effective two-dimensional description, for $r_{\perp,0} = 0$ this potential gives a disc geometry and for a finite $r_{\perp,0}$ it gives a Corbino geometry [36]. The system is under a uniform artificial magnetic field $B = B \hat{z}$ along the $z$-axis generated by the symmetric vector potential $A(r) = B(-y, x, 0)$ in the Coulomb gauge.

For each species $j = 1, 2$ the single-particle Hamiltonian in a two-dimensional harmonic potential can be written as

$$H_j = \frac{p_j^2}{2M} + V_\perp(r) + V_z(r), \quad (2)$$

$$= \frac{p_j^2}{2M} - \omega_j x p_y - \omega_j y p_x + \frac{1}{2} M \omega_j^2 r_\perp^2, \quad V_z(r) = \frac{1}{2} M \omega_z^2 z^2 + V_z(r),$$

where $\omega_j = q_j B / 2M$. Both species have the same particle mass but may feel different trapping sizes due to the additional term coming from minimal coupling, i.e. $\frac{1}{2} M \omega_j^2 r_\perp^2$ if they have different synthetic charges.

The two-dimensional intracomponent and intercomponent interactions, are modelled by the short-range contact interaction with coupling constants defined by

$$g_{jk} = \sqrt{8 \pi \hbar^2 a_{jk}} / M l_z. \quad (3)$$

where $j,k = \{1,2\}$ enumerate components in the mixture, $a_{jk}$ denote the three dimensional s-wave scattering lengths and $l_z = \sqrt{\hbar / M \omega_z}$.

The time evolution of the trapped interacting condensates is governed by a pair of coupled Gross-Pitaevskii equations

$$i \hbar \dot{\psi}_j(r,t) = \left[ -\nabla^2 / 2 - \Omega_j r_z^2 + \frac{1}{2} \sum_{k=1,2} U_{jk} |\psi_k(r,t)|^2 \right] \psi_j(r,t) + \sum_{k=1,2} U_{jk} |\psi_k(r,t)|^2 \psi_j(r,t). \quad (4)$$

The insets show the two-dimensional component densities and the dotted (green) line is the cross-section along which the profiles are shown. The parameter values are $U_{12} = 1.2 U_1 = 1.2 U_2$ for APS and $U_{12} = U_1 = 1.2 U_2$ for RPS with $r_0 = 8$ and $U_2 = 3000$ being common to both.

All the relevant quantities are non-dimensionalized via scaling lengths by the oscillator length $l_\perp = \sqrt{\hbar / M \omega_\perp}$, time by $1/\omega_\perp$, angular momenta by $\hbar$, and order parameters $\psi_j$ by $\sqrt{N / l_\perp}$ so that $r = r_\perp / l_\perp$, $\Omega_j = \omega_j / \omega_\perp$, $U_{jj} \equiv U_j = \sqrt{8 \pi N a_{jj}} / l_\perp$ and $U_{12} = \sqrt{8 \pi N a_{12}} / l_\perp$. Note that the information about the charge of each condensate is embedded inside the rotation frequency $\Omega_j$. Thus the effect of
charge imbalance can be interpreted as applying component-wise rotations to the condensates.

Besides the Gross-Pitaevskii approach the Thomas-Fermi approximation is also used to study the properties of Bose-Einstein condensates when the kinetic energy can be neglected compared with the interaction energies. The TF approximation provides algebraic equations to examine the system and is valid when $Na_j/l_j \gg 1$. In this study, at least one of the components is subject to a magnetic field. In order to account for the superfluid flow characteristics within the TF approximation we make the ansatz for the condensate wavefunctions 
\[ \psi_j(\mathbf{r}, t) = \phi_j(\mathbf{r}) e^{i\mu_j t} e^{-i\mu_j t} \] and only neglect the derivatives with respect to the radial coordinate. Keeping the derivative with respect to the azimuthal angle, we write the TF equations as 
\[
\left( \frac{n_j}{n_2} \right) = \frac{1}{U_1 U_2 - U^2_{12}} \left( \frac{U_2}{-U_{12}} \frac{-U_{12}}{U_1} \right) \left( \frac{\varepsilon_1}{\varepsilon_2} \right), \tag{5}
\]
where 
\[ \varepsilon_j = \mu_j - \frac{i}{2} \Omega_j^2 r^2 - \frac{i}{2} (r - r_0)^2 + \Omega_j \frac{i}{\Omega_j} \frac{\Omega_j}{\Omega_j}. \] Note that the chemical potentials $\mu_j$ adjust the total particle numbers and $n_1, n_2 \geq 0$ determine the TF radii $R_{TF}^j$.

When phase separation takes place ($U_{12}^2 > U_1 U_2$) the components mostly occupy non-overlapping spatial regions. We further simplify the TF equations in these cases by assuming either an azimuthal or a radial boundary between the components. In this way, by assuming strictly non-overlapping components the above TF equations decouple leading to the corresponding one-component TF equations $n_j = \varepsilon_j/\Omega_j$ subject to the given boundary. Next, we optimize the boundary assuming zero interface energy at the boundary. The resulting APS and RPS configurations can have very close energies and in order to accurately decide on the ground state configuration, the interface energy at the boundary between the components must be taken into account. Having obtained the TF solutions for each component we calculate the interface energy within the local density approximation in the following way.

The interface energy involves contributions from both interactions and confinement at the interface. Within the TF approximation in the immiscible phase the boundary region can be defined as the region where the condensate wave-functions overlap and recover from zero to their bulk densities. The relevant length scales for the width of the boundary are the dimensionless healing lengths of the condensates $\xi_j = \sqrt{\frac{l_j}{\Omega_j U_j}}$ as well as the penetration depths $\Lambda_j = \frac{\xi_j}{\sqrt{U_{12}/U_1 U_2 - 1}}$. \[ \xi_j = \sqrt{1/4U_j n_j} \] as well as the penetration depths $\Lambda_j = \frac{\xi_j}{\sqrt{U_{12}/U_1 U_2 - 1}}$. \[ \xi_j = \sqrt{1/4U_j n_j} \]

Depending on the values of the intra and intercomponent interactions, the shape of the condensate wavefunctions can take different forms at the boundary. \[ \frac{\xi_j}{\Lambda_j} \]

The shape of the boundary region has been studied in multiple limiting values of the above healing and penetration lengths covering most of the parameter space \[ \frac{\xi_j}{\Lambda_j} \]. The governing dimensionless parameters can be identified as $K = (\xi_j/\Lambda_j)^2 = U_{12}/\sqrt{U_1 U_2 - 1}$ (for equal masses) and $\xi_j/\xi_1 = (U_1/U_2)^{1/4}$ (at balanced bulk pressures). The parameters used in our calculations which are motivated by the experimental values fall in between the carefully studied regions. We, therefore, adopt an earlier physically motivated variational approach for the interface energy \[ \frac{\xi_j}{\Lambda_j} \].

We assume that the width of the boundary region is locally determined by the optimization of the boundary energy density. We take the kinetic and interaction energies into account. The pressure balance can be satisfied by making an ansatz for the boundary of width $b$ over which the densities vary linearly. The kinetic energy density is approximated by $n_j/(2\hbar^2)$ and $b$ is obtained by minimizing the energy density locally across the boundary as $b = 2\xi_1 \sqrt{3 \left( 1 + \sqrt{U_1/U_2} \right)}/K$. In this way, the interface energy density can be written as \[ \frac{\xi_j}{\Lambda_j} \].

Finally, the total interface energy is obtained by integrating the above energy density over the boundary region. For the angular and radial separation, we obtain the following expressions:
\[
E_{APS}^{TF} = 2 \int_{R_{TF}^<}^{R_{TF}^>} \sigma(r) dr \tag{7}
\]
\[
E_{RPS}^{TF} = 2\pi R_{TF}^> \sigma (R_{TF}^>) \tag{8}
\]
where $R_{TF}^<, R_{TF}^>$ denote the limits of the radial extend of the boundary in the APS configuration and $R_{TF}^>$ is the radius of the circular boundary in the RPS configuration (see Fig. 1).

We find that complementing the TF energy with the above interface energy gives accurate results when compared with our numerical simulations of the GP equations. We use the XMDS2 software package for imaginary time evolution simulations \[ \frac{\xi_j}{\Lambda_j} \]. We provide comparison of various energy expectation values from these approaches and obtain the phase separation configurations based on both GP and TF results in the following sections.

We study the phase separation configurations in two cases, namely the interaction-balanced case for which intracomponent interactions are taken equal, i.e. $U_1 = U_2 = U$ and the interaction-imbalanced case for which $U_1 \neq U_2$.

### III. INTERACTION-BALANCED MIXTURE

We start with a mixture of two BECs subject to a magnetic field with equal intracomponent interactions, i.e. $U_1 = U_2 = U$. This limit can be obtained by having equal number of particles in each gas, $N_1 = N_2$, and equal s-wave scattering lengths, $a_{11} = a_{22}$, or by choosing $N_1/N_2 = a_{22}/a_{11}$. In this case we observe that for both Corbino and disc geometries, the APS is energetically favourable compared to the RPS \[ \frac{\xi_j}{\Lambda_j} \] 25.

We calculate the APS and the RPS energies with the intracomponent interaction strength of $U = 5000$ for both equally and unequally charged cases. An intracomponent interaction
strength of $U = 5000$ with $N = 3 - 4 \times 10^6$ particles corresponds to an s-wave scattering length of $5 \text{mm}$, which is reasonable for the experimental setups [4, 41, 42]. The detailed energy values are given in Table II and Table III for a Corbino with $r_0 = 12$ and a disc geometry, respectively ($E_{\text{kin}}, E_{\text{pot}}, E_{\text{int}}, E_{\text{int}}^{12}$, and $E_{\text{tot}}$ stand for kinetic, potential, intracomponent, intercomponent and total energies, respectively).

We observe that the total APS energy is lower than that of the RPS for both geometries and also for equally and unequally charged cases. The results of the TF approximation are in good agreement with those of the GP approach, both qualitatively and quantitatively, as also shown in Fig. 1. The APS is favourable for weak and strong intracomponent interactions as seen from Table III. We present the APS and the RPS energies for $U = 1000$, 10000 and 15000 and also for larger rotational frequencies. For a slightly higher magnetic field or an arbitrary charge imbalance the difference between the APS and the RPS energies stays almost unchanged, since the contribution of the rotational kinetic energy is very small.

A change in the external trapping potential, i.e. a change in the shape of the trap geometry, affects the densities and is therefore qualitatively similar to changing the interactions. With equal intracomponent interactions for both components we find the APS configuration energetically favorable for all geometries considered here.

From surveying the magnitude of the energies given in Table II and Table III it is seen that the difference between the APS and RPS total energies is smaller than the difference between the interface energies in both of our approaches. When the interface energy $E_{\text{int}} = E_{\text{int}}^{12}$ is neglected, the RPS configuration has lower energy because of its larger boundary compared to that of the APS configuration. Even though the interface energy is quantitatively smaller in the TF approximation, it is qualitatively accurate. Therefore, the configuration of the phase separation is decided by the magnitude of the interface interaction. In other words, throughout a phase separation condensates with the same properties tend to minimize the contribution of boundary effects to the total energy [13]. In the interaction-balanced case, there is no physical factor between BECs to force an inequivalence in density distributions.

The energy difference between the APS and the RPS increases with increasing the intracomponent interactions which is shown in Fig. 2. We calculated the total energy for a disc geometry in Fig. 2 with $r_0 = 0$ and two Corbino geometries with $r_0 = 8$ and $r_0 = 12$ for a fixed intracomponent interaction of $U = 5000$. In all figures the APS remains the energetically advantageous configuration for a phase separation in a mixture of BECs with balanced intracomponent interactions. Again, we note that the difference in total energy between the APS and RPS is due to the difference in the interface boundary. A large boundary means a large interface energy. Thus, to obtain the RPS configuration an imbalance between the intracomponent interactions is needed.

### IV. INTERACTION-IMBALANCED MIXTURE

In the interaction-imbalanced case for which $U_1 \neq U_2$, the phase separation configuration is determined by the difference in the strength of the intracomponent interactions and the shape of the external potential [19, 22, 23]. Fig. 3 exhibits

### TABLE I. Contributions to the total energy for mixtures of equally and unequally charged BECs trapped in a Corbino geometry with an inner radius of $r_0 = 12$ and an equal intercomponent interaction $U_1 = U_2 = U = 5000$ and intercomponent interactions $U_{12} = 1.2U = 6000$. The energy values calculated by GP and TF approximations are in a good agreement. Azimuthal phase separation is favoured for both equally and unequally charged mixtures.

| $(\Omega_1, \Omega_2)$ | $E_{\text{kin}}$ | $E_{\text{pot}}$ | $E_{\text{int}}$ | $E_{\text{int}}^{12}$ | $E_{\text{tot}}$ |
|----------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| APS(GP)              | (0.01, 0.01)    | 0.091 6.668    | 13.664 0.155    | 20.578          |
| RPS(GP)              | (0.01, 0.01)    | 0.224 6.814    | 12.894 0.914    | 20.845          |
| APS(TF)              | (0.01, 0.01)    | 0.037 6.815    | 13.632 0.035    | 20.519          |
| RPS(TF)              | (0.01, 0.01)    | 0.201 6.809    | 13.639 0.200    | 20.845          |
| APS(GP)              | (0.01, 0.01)    | 0.091 6.668    | 13.664 0.155    | 20.578          |
| RPS(GP)              | (0.01, 0.01)    | 0.224 6.814    | 12.894 0.914    | 20.845          |
| APS(TF)              | (0.01, 0.01)    | 0.037 6.815    | 13.632 0.035    | 20.519          |
| RPS(TF)              | (0.01, 0.01)    | 0.201 6.809    | 13.639 0.200    | 20.845          |

### TABLE II. Contributions to the total energy for mixtures of equally and unequally charged BECs trapped in a disc geometry with equal intercomponent interaction $U_1 = U_2 = U = 5000$ and intercomponent interactions $U_{12} = 1.2U = 6000$. The energy values calculated by GP and TF approximations are in a good agreement. Azimuthal phase separation is favoured for both equally and unequally charged mixtures.

| $(\Omega_1, \Omega_2)$ | $E_{\text{kin}}$ | $E_{\text{pot}}$ | $E_{\text{int}}$ | $E_{\text{int}}^{12}$ | $E_{\text{tot}}$ |
|----------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| APS(GP)              | (0.01, 0.01)    | 0.216 36.967    | 37.570 0.869    | 75.622          |
| RPS(GP)              | (0.01, 0.01)    | 0.330 37.034    | 36.983 1.504    | 75.850          |
| APS(TF)              | (0.01, 0.01)    | 0.196 37.611    | 37.614 0.193    | 75.614          |
| RPS(TF)              | (0.01, 0.01)    | 0.333 37.702    | 37.523 0.330    | 75.889          |
| APS(GP)              | (0.01, 0.01)    | 0.216 36.967    | 37.570 0.869    | 75.621          |
| RPS(GP)              | (0.01, 0.01)    | 0.329 37.050    | 36.982 1.504    | 75.850          |
| APS(TF)              | (0.01, 0.01)    | 0.195 37.612    | 37.613 0.192    | 75.613          |
| RPS(TF)              | (0.01, 0.01)    | 0.332 37.702    | 37.523 0.330    | 75.888          |

### TABLE III. Total energies for a mixture of equally charged BECs trapped in a two dimensional harmonic trap with $U_{12} = 1.2U$ for different inner radii $r_0$, intercomponent energies $U_1 = U_2 = U$ and rotation frequencies $\Omega_1 = \Omega_2 = \Omega$ calculated by the GP approximation. The energy values calculated by TF approximations also show that azimuthal phase separation is always favoured for a mixture of BECs.

| $r_0$ | $\Omega$ | $U=1000$ | $U=10000$ | $U=5000$ |
|-------|----------|----------|-----------|----------|
|       |          | APS RPS  | APS RPS   | APS RPS  |
| 0     | 0.01     | 34.028   | 34.303    | 106.775  |
| 8     | 0.01     | 9.235    | 9.566     | 42.691   |
| 12    | 0.01     | 7.035    | 7.386     | 32.552   |
| 0     | 0.03     | 34.039   | 34.304    | 106.812  |
| 8     | 0.03     | 9.245    | 9.568     | 42.741   |
| 12    | 0.03     | 7.042    | 7.386     | 32.587   |
| 0     | 0.06     | 34.077   | 34.320    | 106.932  |
| 8     | 0.06     | 9.279    | 9.578     | 42.901   |
| 12    | 0.06     | 7.154    | 7.394     | 32.771   |
the transition from APS to RPS as a function of the imbalance in the intracomponent interactions. The total energies are calculated by the TF approximation for the intercomponent interaction $U_{12} = 7000$ and intracomponent interaction $U_2 = 5000$. For the disc geometry shown in Fig. 3 ($r_0 = 0$), the transition from APS to RPS happens as soon as a small imbalance in the intracomponent interactions is introduced. The similar transition for Corbino geometries occurs at larger differences as shown in Fig. 3 with respect to the inner radius of the Corbino, for $r_0 = 8$ and $r_0 = 12$, respectively.

By a transition from the APS to the RPS, the component with the weaker intracomponent interaction occupies the inner disc or annular region and the strongly interacting one occupies the outer annular region. In this configuration, in order to minimize the total energy, the condensate with smaller intracomponent interaction stays in a region with higher density, while the other condensate moves radially outwards reducing its density. The pressure balance in this configuration results in lower total energy despite the fact that the RPS has a larger boundary compared to the APS.

We extend the results shown in Fig. 3 by constructing a phase diagram as a function of the intercomponent interaction and the ratio of intracomponent interactions for all three trapping potentials considered. The boundaries between APS and RPS configurations are obtained by the TF approximation, and checked via the GP results. The boundaries for the miscible state are calculated analytically according to the condition $U_1 U_2 \leq U_{12}^2$. It is shown that this condition ensures miscibility against large deviations from uniformity.

The phase diagrams reveal an interplay between the boundary effects and the imbalance in the intracomponent interactions in determining the configuration of the phase separation. For a disc geometry in Fig. 3(a) the RPS is favourable in a large area of the phase diagram. Larger values of the intercomponent interactions move the boundary of the narrow APS region toward larger values of intracomponent interactions ratio. Larger intercomponent energy means larger difference between interface energies of APS and RPS (see Fig. 2). Accordingly, the region for APS becomes larger moving from disc to Corbino with $r_0 = 8$ and more so for the narrower Corbino with $r_0 = 12$. On the other hand, the shape of the external potential, particularly its width in the case of the Corbino trap, defines the extent of the boundaries between two separated condensates. Eventually, an interplay between the boundary effects and the imbalance in the intracomponent interactions determines the configurations of the phase separations.

V. ROTATIONAL PROPERTIES

The velocity field of an atomic superfluid is defined by the gradient of the condensate phase, so that it is irrotational. This restriction on the velocity field of superfluids $v$, i.e. $\nabla \times v = 0$, enforces the circulation around a closed path to be quantized. Accordingly, the velocity field of a vortex exhibits a profile with $1/r$ dependency perpendicular to the direction of applied rotational frequency (the experiments reveal only a $1/r$ dependency in the azimuthal component if the rotation is in the $z$-direction). This behaviour is completely different from a rigid body rotation which implies a linear dependency on $r$. Interestingly, for the APS scenario...
FIG. 4. Phase diagram of the phase separations as a function of the intercomponent interaction and the difference in intracomponent interaction for (a) $r_0 = 0$ disk geometry, (b) $r_0 = 8$ and (c) $r_0 = 12$ Corbino geometries. The boundaries for miscible mixture (M) has been calculated analytically according to the condition $U_1 U_2 \leq U_{12}^2$. The APS-RPS transition happens in the interaction-imbalanced cases and unlike the interaction-balanced case its boundary is sensitive to the geometry of the trap. The RPS is widely favoured in disc geometry and the APS is favoured for narrower Corbino geometries.

The quantization of the circulation breaks down \([19, 20]\) and the azimuthal component of the velocity field with linear dependency on $r$ appears at the boundary regions \([20]\). Here, we investigate the shape of the velocity field of APS configurations for both Corbino and disk geometries.

We present the density distributions of immiscible mixtures in the APS configuration for Corbino ($r_0 = 12$) and disc geometries in upper panels of Fig. 5 and Fig. 6, respectively. In the lower panels we give the corresponding velocity field profiles. We follow a closed path of fixed density on each component’s distribution starting from points E (E') in Corbino and D (D') in disc geometry.

The results for Corbino and disc geometries are qualitatively similar. The asymmetric interaction causes boundaries to curve in the density distributions, which are straight for interaction-balanced condensates (see top panel of Fig. 1). As seen from the velocity field profiles for both geometries, the condensates display typical superfluid behaviour everywhere expect at the phase boundaries. The azimuthal component of the velocity, $v_\theta$, indicated by the solid red lines exhibits $1/r$ dependency as seen between points A (A') and B (B') for both condensates. However, $v_\theta$ becomes linear in $r$ between points C and D for the Corbino and B and C for the disc, where an extra non-zero radial velocity indicated by the solid red lines appears. The radial velocities $v_r$ corresponding to each component in the mixture appear in opposite directions with similar magnitudes. Both profiles have been calculated with an imbalance in the intracomponent energies and they remain similar for the interaction-balanced case as shown in \([20]\).

As can be inferred from the plots, the average angular velocity for equally charged condensates is non-zero indicating that the condensates have non-zero average angular velocity about the central axis. In contrast, we find that in a charged-uncharged mixture the average angular velocity of each component becomes zero and the uncharged component has constant phase in the mean-field approximation.

The breakdown of the circulation quantization in the APS manifests itself in the angular momentum properties of the system, which is seen in Fig. 7. The angular momenta are calculated for a Corbino with $r_0 = 8$ and imbalanced intracomponent interactions as a function of rotation frequency/magnetic field. While the angular momentum of the RPS is quantized it becomes continuous for the APS.

In the mean field GP and TF approaches, where only density-density interactions are considered, the phases of the superfluid wave functions do not enter the interactions. The absence of these phases prevents any sort of interplay between the velocity fields of the superfluids. Therefore, at the GP and TF mean-field level it is not possible to observe an angular momentum transfer between two unequally charged superfluids \([32]\).

VI. SUMMARY AND CONCLUSION

We studied the phase separation configurations and their rotational properties for a mixture of two interacting charged Bose-Einstein condensates subject to a magnetic field trapped in two different geometries; disc and Corbino. The azimuthal phase separation and the radial phase separation are two types of phase separation configurations that occur for such mixtures.

In order to determine the phase separation configurations we calculated the ground state energies of the configurations using the Gross-Pitaevskii and the Thomas-Fermi approximations. We modified the Thomas-Fermi approximation for the phase separated scenarios, and added the contribution of the boundary effects to our Thomas-Fermi approach. We showed that the results of the modified Thomas-Fermi approximation are in good agreement with those obtained from the Gross-Pitaevskii approach, and can be used in determining the configurations of the phase separations. We obtained a phase dia-
gram exhibiting the range of these configurations as a function of the inter and intracomponent interactions.

The phase separation configurations are determined by the imbalance of the intracomponent interactions and the shape of external potential [18, 19, 22, 25]. We show that the APS is the only ground state of a mixture with all equal physical parameters, even for very large intracomponent interactions. The geometry of the trap does not play a role in such a symmetric mixture of BECs. In this case, with both components enjoying the same density distributions and intracomponent interaction strengths, the mixture tends to minimize the contribution of the boundary effects to the total energy.

We showed that a phase transition from APS to RPS occurs by introducing an imbalance in the system [25]. This transition occurs in order to minimize the total energy through which the condensate with a larger intracomponent energy moves radially outwards with lower density satisfying the pressure balance at the phase boundary. The configuration of the phase separation in this case is determined by an interplay between the interface energy and the intracomponent imbalance. While the radial phase separation is widely favoured in disc geometry, the azimuthal phase separation is favoured for narrower Corbino geometries.

We explored the rotational properties of the spatially separated condensates under the magnetic field studying their angular momenta and velocity fields. We showed that the circulation condition breaks down during the azimuthal phase separation. For charged-imbalanced mixtures, the rotational properties show a qualitative difference even in the mean-field level in that the rotation about the central axis stops for a charged-uncharged mixture. The transfer of angular momentum between the components provides an interesting area of further research. Beyond-mean-field treatments are needed for this purpose. The mean-field solutions in this work provide the starting point for such investigations.

![Diagram](image1)

**FIG. 5.** Density (upper panel) and velocity component (lower panel) profiles of the APS configuration for an interaction-imbalanced mixture in a Corbino trap with \( r_0 = 12 \), intracomponent interactions \( U_2 = 3000, U_1 = 1.2U_2 \), and intercomponent interaction \( U_{12} = 1.4U_2 \). For both condensates (left and right panels), the azimuthal component \( v_\theta \) of the velocity shows a superfluid flow in the bulk and a rigid body flow at the phase boundary (shaded area).

![Diagram](image2)

**FIG. 6.** Density (upper panel) and velocity component (lower panel) profiles of the APS configuration for an interaction-imbalanced mixture in a disc trap (\( r_0 = 0 \), intracomponent interactions \( U_2 = 3000, U_1 = 1.025U_2 \), and intercomponent interaction \( U_{12} = 1.2U_2 \). For both condensates (left and right panels), the azimuthal component of the velocity \( v_\theta \) shows a superfluid flow in the bulk and a rigid body flow at the phase boundary (shaded area).

![Diagram](image3)

**FIG. 7.** Angular momenta \( L_z^{(j)} \) of the condensates as a function of the applied rotational frequency \( \Omega \) for Corbino (\( r_0 = 8 \)) geometry. For the RPS configuration, the angular momenta are quantized. Here \( U_1 = 7000, U_2 = 5000, \) and \( U_{12} = 7000 \). For the APS configuration the angular momenta become linear in \( \Omega \) as the circulation quantization condition does not apply in this configuration. For this case \( U_1 = U_2 = 5000, \) and \( U_{12} = 7000 \).
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