BLOCK DESIGNS WITH $\gcd(r, \lambda) = 1$ ADMITTING
FLAG TRANSITIVE AUTOMORPHISM GROUPS

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ABSTRACT. In this paper, we present a classification of 2-designs with $\gcd(r, \lambda) = 1$ admitting flag-transitive automorphism groups. If $G$ is a flag-transitive automorphism group of a non-trivial 2-design $D$ with $\gcd(r, \lambda) = 1$, then either $(D, G)$ is one of the known examples described in this paper, or $D$ has $q = p^d$ points with $p$ prime and $G$ is a subgroup of $\text{AGL}_1(q)$.

1. INTRODUCTION

Our purpose here is to announce a classification of the pairs $(D, G)$, where $D$ is a nontrivial 2-design with $\gcd(r, \lambda) = 1$ and $G$ is a group of automorphisms acting transitively on the flags of $D$. Here, a 2-design $D$ with parameters $(v, k, \lambda)$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of $v$ points and a set $\mathcal{B}$ of $b$ blocks such that each block is a $k$-subset of $\mathcal{P}$ and each two distinct points are contained in $\lambda$ blocks. A symmetric design is a 2-design with the same number of points and blocks, that is to say, $v = b$. The replication number $r$ of $D$ is the number of blocks incident with a given point. A flag of $D$ is a point-block pair $(\alpha, B)$ such that $\alpha \in B$. An automorphism group $G$ of $D$ is a group of permutation on $\mathcal{P}$ which maps blocks to blocks and preserving the incidence. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [17, 24, 33, 35]. In this paper, we give a classification of 2-designs with $\gcd(r, \lambda) = 1$.

Theorem 1.1. Suppose that $G$ is a flag-transitive automorphism group of a nontrivial 2-design $D$ with $\gcd(r, \lambda) = 1$. Then either
(a) $(D, G)$ is one of the known examples described in Sections 2 and 3 below, or
(b) $D$ has $q = p^d$ points with $p$ prime and $G$ is a subgroup of $\text{AGL}_1(q)$.

If $\lambda = 1$, then $D$ is a linear space and a result of Higman and McLaughlin [28] shows that $G$ acts primitively on the points of $D$, and using the O’Nan-Scott theorem for finite primitive permutation groups, Buckenhout, Delandtsheer and Doyen proved that $G$ is of almost simple or affine type, and in 1990, a classification of linear spaces admitting flag-transitive automorphism groups has been announced in [15]. The proof of this result was published in several papers. The proof in the affine case is due to Liebeck [37]. The almost simple case has been treated by several authors. Delandtsheer took the case where the simple socle is an alternating group [21]. She also handled the case where the group $G$ is one of the simple groups.
PSL\(_2(q)\), PSL\(_3(q)\), PSU\(_3(q)\) and \(2B_2(q)\) [20]. Kleidman solved the case where the socle is an exceptional group of Lie type [34] and he gave a proof for three of the ten families of exceptional groups and some hints for the remaining cases. The case of the sporadic groups was ruled out by Buekenhout, Delandtsheer, and Doyen [13] and Davies [19]. Finally, Saxl completed the proof in [44], where he dealt with the remaining families of exceptional type together with the classical groups of Lie type.

In general, if the replication number of a 2-design \(D\) is coprime to \(\lambda\), then Dembowsk [22, 2.3.7] proved that flag-transitive automorphism groups \(G\) of \(D\) act primitively on the point set of \(D\). In 1988, Zieschang [53] proved that such an automorphism group is of almost simple or affine type:

(a) Almost simple type: \(G\) has a nonabelian simple normal subgroup \(X\) such that \(X \leq G \leq \text{Aut}(X)\);

(b) Affine type: \(G\) is a subgroup of the affine group \(\text{AGL}_d(p)\) containing the translation group \(T\). The socle \(T\) of \(G\) is an elementary abelian \(p\)-group of order \(p^d\) with \(d \geq 1\). Moreover, \(G = TG_0\), where the point-stabilizer \(G_0\) of \(G\) is an irreducible subgroup of \(\text{GL}_d(p)\).

Since 2015, we have analysed these two possible cases and proved Theorem 1.1 in several papers. The proofs rely on the classification of finite simple groups and detailed knowledge concerning their subgroups and their permutation representations. We have not been able to resolve the case where \(G \leq \text{ATL}_1(q)\), however, in Section 4, we present some examples in this case.

**Proof of Theorem 1.1.** The proof of Theorem 1.1 appears in several papers. Suppose that \(G\) is a flag-transitive automorphism group of a non-trivial 2-design \(D\) with \(\gcd(r, \lambda) = 1\). It follows from [53, Theorem] that such an automorphism group \(G\) is of almost simple or affine type. If \(G\) is of almost simple type, then \((G, D)\) is one of the examples described in Section 2 below, namely, Example 2.1-2.9, see the main result in [1, 2, 4-7, 14, 21, 30, 32, 44, 44, 45, 47, 48, 50-52]. If \(G\) is of affine type, then by [11, 12, 37, 40], the group \(G\) is a subgroup of \(\text{ATL}_1(q)\) or \((D, G)\) is as in Example 3.1-3.9 described in Section 3 below.

### 2. Almost simple type

In this section, we provide some examples of 2-designs with \(\gcd(r, \lambda) = 1\) admitting a flag-transitive automorphism almost simple group with socle \(X\). The 2-designs in Examples 2.7 and 2.8 appear in [1, 2, 4, 50] when the socle \(X\) of \(G\) is a finite simple exceptional group of Lie type. The remaining examples arose from studying linear spaces, 2-transitive automorphism groups of 2-designs and automorphism groups of 2-designs with socle alternating groups, sporadic simple groups and finite simple classical groups of Lie type [5-7, 14, 21, 30, 32, 44, 45, 47, 48, 51, 52]. We note here that the examples of symmetric designs occur only in Examples 2.1 and 2.9.

#### 2.1. Point-hyperplane designs.

The point-hyperplane design of the projective space \(\text{PG}_{n-1}(q)\) with parameters \((q^n-1)/(q-1), (q^{n-1}-1)/(q-1), (q^{n-2}-1)/(q-1)\) for \(n \geq 3\) is a well-known example of flag-transitive symmetric design. A group \(G\) with \(\text{PSL}_n(q) \leq G \leq \text{PGL}_n(q)\) acts flag-transitively on \(\text{PG}_{n-1}(q)\). If \(n = 3\), then we have the desarguesian plane with parameters \((q^2+q+1, q+1, 1)\) which is a projective plane. The Fano plane is the unique projective plane \(\text{PG}_2(2)\) with parameters
(7, 3, 1). We remark that there is one additional example with parameters (15, 7, 3) and $G = A_7$ which we view as a projective space, see [14, 32].

2.2. Designs with projective points. Suppose that $G$ is an almost simple group with socle $X = \text{PSL}_n(q)$ with $n \geq 3$ and $H$ is a parabolic subgroup of type $P_1$. Let $B$ be a line (a 2-dimensional subspace of the vector space $V = \mathbb{F}_q^n$) in $\text{PG}_{n-1}(q)$ and $\alpha \in B$. Let also $\mathcal{P}$ be the point set of $\text{PG}_{n-1}(q)$ and $\mathcal{B} = (B \setminus \{\alpha\})^G$. Then the incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-$(\frac{q^n - 1}{q-1}, q, q-1)$ with $\text{gcd}(n-1, q-1) = 1$ admitting automorphism group $G$ with point-stabilizer $H$ such that $H \cap X \cong [q^{n-1}]:\text{SL}_{n-1}(q) \cdot (q-1)$, see [7].

2.3. Witt-Bose-Shrikhande spaces. This space is a 2-design with parameters $(2^{a-1}(2^a - 1), 2^a - 1, 1)$ which can be defined from the group $\text{PSL}_2(q)$ with $q = 2^a$ for $a \geq 3$ [14]. In this incidence structure which is denoted by $W(q)$, the points are the dihedral subgroups of $\text{PSL}_2(q)$ of order $2(q+1)$, the blocks are the involutions of $\text{PSL}_2(q)$, and a point is incident with a block precisely when the dihedral subgroup contains the involution. An almost simple group $G$ with socle $X = \text{PSL}_2(q)$ acts flag-transitively on Witt-Bose-Shrikhande space. Moreover, this space is not a symmetric design.

2.4. Hermitian unitals. The Hermitian unital $\mathcal{U}_H(q)$ with parameters $(q^3 + 1, q + 1, 1)$ is a well-known example of flag-transitive linear spaces [31]. Let $V$ be a three-dimensional vector space over the field $\text{GF}(q^2)$ with a non-degenerate Hermitian form. The Hermitian unital of order $q$ is an incidence structure whose points are $q^3 + 1$ totally isotropic 1-spaces in $V$, the blocks are the sets of $q + 1$ points lying in a non-degenerate 2-space, and the incidence is given by inclusion. This structure is not symmetric and any group $G$ with $\text{PSU}_3(q) \leq G \leq \text{PGU}_3(q)$ acts flag-transitively on the Hermitian unital design.

2.5. Unitary designs. A unitary design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with parameters $(q^3 + 1, q, q - 1)$ can be constructed from the Hermitian unital as in (2.4). The point set of $\mathcal{D}$ is the point set of the Hermitian unital $\mathcal{U}_H(q)$ and the block set $\mathcal{B}$ is $(\ell \setminus \{\gamma\})^G$, where $\ell$ is a line of $\mathcal{U}_H(q)$ and $\gamma \in \ell$, see [6, 47]. This structure is not symmetric and a group $G$ with $\text{PSU}_3(q) \leq G \leq \text{PGU}_3(q)$ is a flag-transitive automorphism group of $\mathcal{D}$. It is worth noting by [23] that there is a general construction method for 2-designs from linear spaces, and the unitary designs described here can also be obtained in this way from the Hermitian unital design.

2.6. Ree unitals. The Ree Unital $\mathcal{U}_R(q)$ is first discovered by Lüneburg [38], and this examples arose from studying flag-transitive linear spaces [31, 34]. This design has parameters $(q^3 + 1, q + 1, 1)$ with $q = 3^a \geq 27$. The points and blocks of $\mathcal{U}_R(q)$ are the Sylow 3-subgroups and the involutions of $\mathcal{G}_2(q)$, respectively, and a point is incident with a block if the block normalizes the point. This incidence structure is a linear space and any group with $\mathcal{G}_2(q) \leq G \leq \text{Aut}(\mathcal{G}_2(q))$ acts flag-transitively. This design is not symmetric. Note for $q = 3$ that the Ree Unital $\mathcal{U}_R(3)$ is isomorphic to the Witt-Bose-Shrikhande space $W(8)$ as $\mathcal{G}_2(3)'$ is isomorphic to $\text{PSL}_2(8)$. 
2.7. Ree designs. Suppose that $G$ is an almost simple group with socle $X = \mathfrak{G}_2(q)$ for $q = 3^a$ and $a \geq 3$ odd. Let $H$, $K_1$ and $K_2$ be subgroups of $G$ such that $H \cap X \cong q^3:(q - 1)$, $K_1 \cap X = q:(q - 1)$ and $K_2 \cap X \cong q^2:(q - 1)$. The coset geometries $(X, H \cap X, K_i \cap X)$ give rise to the 2-designs with parameters $v = q^3 + 1$, $b = q^{3-i}(q^3 + 1)$, $r = q^i$, $k = q^i$ and $\lambda = q^i - 1$, for $i = 1, 2$, see [1, 50]. Since $G$ is 2-transitive on the point set of this structure and $\gcd(r, \lambda) = 1$, $X$ is flag-transitive [22, 2.3.8]. Note that $H \cap K_i \cap X$ is a cyclic group of order $q - 1$ and the subgroup $K_i \cap X$ has an orbit of length $q^i$ with $i = 1, 2$, see [18, 50]. If $\mathcal{P} = \{1, \ldots, v\}$, then since $X$ is 2-transitive, [10, Proposition 4.6] gives rise to a 2-design $\mathcal{D}_i = (\mathcal{P}, B_i)$ with parameters $(q^3 + 1, q^i, q^i - 1)$, for $i = 1, 2$, which is not symmetric, and the group $G$ is flag-transitive on $\mathcal{D}_i$. An explicit construction for each of these designs is given in [18]. For $q = 27$, in [1, Table 1], we introduced base blocks for these type of designs.

2.8. Suzuki-Tits ovoid designs. These designs arose from studying block designs with flag-transitive almost simple finite exceptional groups, see [1, 50]. An ovoid in $\mathrm{PG}_3(q)$ with $q > 2$, is a set of $q^2 + 1$ points such that no three of which are collinear. If $q$ is odd, then all ovoids are elliptic quadricities, see [9, 42], while in even characteristic, there is only one known family of ovoids that are not elliptic quadrics in which $q \geq 8$ is an odd power of 2. These were discovered by Tits [46], and are now called the Suzuki-Tits ovoids since the Suzuki groups $X = \mathfrak{B}_2(q)$ naturally act on these ovoids. A Suzuki-Tits ovoid design is a 2-design with parameters $(q^2 + 1, q, q - 1)$ which is not symmetric, see [1, 2, 50]. The point set of this design is the Suzuki-Tits ovoid and the block set is the $X$-orbit $B^X$, where $B$ is the unique orbit of the subgroup $K := q:(q - 1)$ of length $q$, see [2, 50]. For $q \in \{8, 32\}$, we construct these type of designs with explicit base blocks in [1, Table 1]. There is another classical construction for these designs in geometry, by taking points as ovoids in $\mathrm{PG}_3(q)$ and blocks as pointed conics minus the distinguished points.

2.9. More examples. Table 1 illustrates eleven examples of designs. Each design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with parameters $(v, k, \lambda)$ is the unique design with flag-transitive automorphism group $G$ as in the seventh column of Table 1. The point-stabilizer and block-stabilizer of $\mathcal{D}$ are also given in the same table with appropriate references in each case, see also [7] for explicit base blocks.

3. AFFINE TYPE

In this section, we provide some examples of 2-designs with $\gcd(r, \lambda) = 1$ admitting a flag-transitive automorphism group $G$ whose socle $T$ is an elementary abelian $p$-group of order $p^d$, $d \geq 1$. All examples presented here can be read off from [15] and therein references for $\lambda = 1$, and in [11, 12, 40] for $\lambda > 1$.

Since $G$ acts point-primitively on $\mathcal{D}$, the point set of $\mathcal{D}$ can be identified with $V_d(p)$, the $d$-dimensional vector space over the prime field $\mathrm{GF}(p)$ in a way that $T$ is the translation group of $V_d(p)$ and $G = TG_0$ is a subgroup of $\mathrm{AGL}_d(p)$ with $G_0$ acting irreducibly on $V_d(p)$. For each divisor $n$ of $d$ the group $\Gamma_\mathrm{L}_n(p^{d/n})$ has a natural irreducible action on $V_n(p^{d/n})$, thus we may choose $n$ to be minimal such that $G_0 \leq \Gamma_\mathrm{L}_n(p^{d/n})$ in this action and write $q = p^{d/n}$.

To better understand some of the examples described below, some basics on $t$-spreads of vector spaces are provided. A (vectorial) $t$-spread $\mathcal{S}$ of a $h$-dimensional
Table 1. Some nontrivial 2-designs $D$ with $gcd(r, \lambda) = 1$ admitting flag-transitive and point-primitive automorphism group $G$.

| Line | $v$ | $b$ | $r$ | $k$ | $\lambda$ | $G$ | $G_a$ | $G_B$ | $\text{Aut}(D)$ | Design | References |
|------|-----|-----|-----|-----|--------|-----|-------|-------|--------------|--------|------------|
| 1    | 6   | 10  | 5   | 3   | 2      | $\text{PSL}_2(5)$ | $D_{10}$ | $S_3$  | $\text{PSL}_2(5)$ | [16, 49, 51] |
| 2    | 8   | 14  | 7   | 4   | 3      | $\text{PSL}_2(7)$ | $7:3$  | $A_4$  | $2^3: \text{PSL}_2(7)$ | [5]    |
| 3    | 28  | 36  | 9   | 7   | 2      | $\text{PSL}_2(8)$ | $D_{18}$ | $D_{14}$ | $\text{PSL}_2(8):3$ | [16, 49] |
| 4    | 10  | 15  | 9   | 6   | 5      | $\text{PSL}_2(9)$ | $3:4$  | $S_4$  | $S_6$ | [49, 51]    |
| 5    | 11  | 11  | 5   | 5   | 2      | $\text{PSL}_2(11)$ | $\text{PSL}_2(5)$ | $\text{PSL}_2(5)$ | $\text{PSL}_2(11)$ | Hadamard [3, 30, 41] |
| 6    | 12  | 22  | 11  | 6   | 5      | $\text{PSL}_2(11)$ | $A_6$  | $M_{11}$ | [48] |
| 7    | 22  | 77  | 21  | 6   | 5      | $\text{PSL}_3(4)$ | $2^3:A_6$ | $M_{22}$ | [48] |
| 8    | 22  | 77  | 21  | 6   | 5      | $\text{PSL}_3(4):2$ | $2^4:S_6$ | $M_{22}:2$ | [48] |
| 9    | 10  | 15  | 9   | 6   | 5      | $S_6$ | $S_4^2:2$ | $S_4 \times 2$ | $S_6$ | [49, 51] |
| 10   | 15  | 35  | 7   | 3   | 1      | $A_7$ | $\text{PSL}_2(7)$ | $(3 \times A_6):2$ | $A_7$ | PG(3, 2) | [16, 21, 51] |
| 11   | 15  | 35  | 7   | 3   | 1      | $A_8$ | $2^3: \text{PSL}_2(2)$ | $A_8^2:2$ | $A_8$ | PG(3, 2) | [16, 21, 51] |

Note: The subgroups $G_a$ and $G_B$ are point-stabilizer and block-stabilizer, respectively.

vector space $V$ over the field $GF(s)$, $s$ power of a prime, is a set of $t$-dimensional subspaces of $V$ partitioning $V - \{0\}$. Clearly, a $t$-spread of $V$ exists only if $t \mid h$. The incidence structure $\mathcal{A}(V, \mathcal{S})$, where the points are the vectors of $V$, the lines are the additive cosets of the elements of $\mathcal{S}$, and the incidence is the set-theoretic inclusion, is an André translation structure with associated translation group $T = \{x \rightarrow x + w : w \in V\}$ and lines of size $s^t$. In particular $\mathcal{A}(V, \mathcal{S})$ is a $2-(s^h, s^t, 1)$ design. Moreover, if $h$ is even and $t = h/2$, then $\mathcal{A}(V, \mathcal{S})$ is a translation plane of order $s^{h/2}$. If $\mathcal{S}$ and $\mathcal{S}'$ are two $t$-spreads of $V$ such that $\psi$ is an isomorphism from $\mathcal{A}(V, \mathcal{S})$ onto $\mathcal{A}(V, \mathcal{S}')$ fixing the zero vector, then $\psi \in \Gamma L(V)$ and $\mathcal{S}' = \mathcal{S}$. The converse is also true. Let $H_0 \leq \Gamma L(V)$ preserving $\mathcal{S}$, then $TH_0$ is the automorphism group of $\mathcal{A}(V, \mathcal{S})$. The subset of $\text{End}(V, +)$ preserving each component of $\mathcal{S}$ is a field called the kernel of $\mathcal{S}$ and is denoted by $K(V, \mathcal{S})$, or simply by $K$. Clearly, $K - \{0\} \leq H_0$. Each component of $\mathcal{S}$ is a vector subspace of $V$ over $K$ and $H_0 \leq \Gamma L_K(V)$. In particular, $\mathcal{A}(V, \mathcal{S})$ is a desarguesian affine space if, and only if, $\dim_K V = 1$ for each $Y \in \mathcal{S}$. When this occurs the spread $\mathcal{S}$ is called regular. More information on $t$-spreads and André translation structures can be found in [8, 22, 29, 39].

For $\lambda = 1$, the examples of the designs and their automorphism groups are given in Examples 3.1-3.3 below:

3.1. Desarguesian affine linear spaces. This design is the affine linear space $AG_n(q)$ with $n \geq 2$ and $q^n = p^d$, and $G_0 \leq \Gamma L_n(q)$. If $Z$ is the centre of $\Gamma L_n(q)$, then by the flag-transitivity of $G$, the group $ZG_0$ is transitive on the non-zero vectors of $\mathcal{D}$, and so is one of the transitive linear groups determined by Hering (see [36, Appendix 1] for a list of these). From this, it follows quite easily that if $n > 1$, then one of the following holds:

(a) $G$ is 2-transitive on $V$ (hence given in [36, Appendix 1]);
(b) $n = 2$, $q$ is 11 or 23, and $G$ is one of three soluble flag-transitive groups given in [25, Table II];
(c) $n = 2$, $q = 9, 11, 19, 29$ or 59, $G_0^{(\infty)} = 2.A_5$ (where $G_0^{(\infty)}$ is the last term in the derived series of $G_0$), and $G$ is given in [25, Table II];
(d) $n = 4$, $q = 3$ and $G_0^{(\infty)} = 2.A_5$. 
3.2. Non-Desarguesian translation affine planes. The examples for these designs with $\lambda = 1$ are:
(a) The Lineburg planes constructed in [39, Section 23]. These are affine planes of order $q^2$, where $q = 2^{2a+1} > 8$, and $\mathcal{B}_2(q) \leq G_0 \leq \operatorname{Aut}(\mathcal{B}_2(q))$;
(b) The Hering plane of order 27 constructed in [26]. Here $G_0 = \operatorname{SL}_2(13)$ and $G$ is 2-transitive on the points of $\mathcal{D}$;
(c) The nearfield plane of order 9. Here there are seven possibilities for $G$, given in [25, Section 5].

3.3. Hering spaces. These are two flag-transitive linear spaces with parameters $(3^6, 3^2, 1)$, constructed in [27]. In both cases $G_0 = \operatorname{SL}_2(13)$ and $G$ is 2-transitive on the points.

For $\lambda > 1$, the examples of the designs and their automorphism groups are given in Examples 3.4-3.9 below:

3.4. Designs where the blocks are subspaces of $\operatorname{AG}_d(p)$. The incidence structure $\mathcal{D} = (\mathcal{V}_d(p), \ell^G)$, where $\ell$ is a $u$-dimensional subspace of $\mathcal{V}_d(p)$, is a 2-design with parameters \( (p^d, p^n, \frac{p^n-1}{p^{\gcd(u,d/n)}-1}) \) provided that $\gcd(u, n, d/n) = 1$ and one of the following holds:
(a) $\gcd(d, u) < u < d/n$ and $\ell$ is contained in a 1-dimensional subspace of $\mathcal{V}_n(q)$;
(b) $d - d/n \leq u < d$ and $\gcd(u, d) < u$ and $\ell$ contains a hyperplane of $\mathcal{V}_u(q)$;
(c) $(q, n, u) = (3, 6, 3)$ and $G_0 \cong \operatorname{SL}_2(13)$.

In (b), $\ell$ is the image under a polarity $\varphi$ of $\operatorname{PG}_{d-1}(p)$ of either a block as in (a) or of a component of a $t$-spread $\mathcal{S}$ of $\mathcal{V}_d(p)$ (the set of 1-dimensional subspaces of $\mathcal{V}_n(q)$ is a regular $d/n$-spread of $\mathcal{V}_d(p)$). In (a) and (b), the possibilities for $G_0$ are as follows:
(i) $\operatorname{SL}_n(q) \leq G_0 \leq \operatorname{GL}_n(q) \cdot \langle \sigma_0 \rangle$, where $\langle \sigma_0 \rangle$ is the stabilizer in $\langle \sigma \rangle$ of $\ell$ and $\langle \sigma \rangle$ is induced by the field automorphism group of $\operatorname{GF}(q)$;
(ii) $n$ is even and $\operatorname{Sp}_n(q) \leq G_0 \leq \operatorname{GSp}_n(q) \cdot \langle \sigma_0 \rangle$;
(iii) $q = 2^{d/6}$, $n = 6$ and $G_2(q) \leq G_0 \leq G_2(q)' \cdot \langle \sigma_0 \rangle$;
(iv) $(q, n, u) = (2, 4, 3)$ and $G_0 \cong \Lambda_c$, where $c = 6, 7$, or $G_0 \cong \Lambda_8$;
(v) $(q, n, u) = (3, 4, 3)$ and either $(D_8 \circ Q_8) \cdot 5 \leq G_0 \leq (D_8 \circ Q_8) \cdot S_5$ or $2 \cdot S_5^2 \leq G_0 \leq (2 \cdot S_5^2) : 2$, or $8 \circ \operatorname{SL}_2(5) \leq G_0 \leq (8 \circ \operatorname{SL}_2(5)) : 2$;
(vi) $(q, n, u) = (3, 6, 4)$, where $\ell$ is the image under $\varphi$ of a component of any of the two 2-spreads of $\mathcal{V}_6(3)$ defining the two Hering spaces in (3.3), and $G_0 \cong \operatorname{SL}_2(13)$;
(vii) $(q, n, u) = (3, 6, 5)$ and $G_0 \cong \operatorname{SL}_2(13)$.

3.5. Designs where the blocks are union of distinct subspaces of $\operatorname{AG}_d(p)$. These are four families of $2$-$(q^n, p^n\omega, p^n\omega - 1)$ designs $\mathcal{D} = (\mathcal{V}_d(p), \ell^G)$, where $\ell$ is the union of $\omega$ cosets of a $u$-dimensional subspace $W$ of $\mathcal{V}_d(p)$:
(a) $\omega$ is a divisor of $p^{\gcd(u,d/n)} - 1$ such that $\gcd\left(\frac{p^n-1}{\omega}, p^n\omega - 1\right) = 1$ and one of the following holds:
(i) $0 \leq u < d/n$, $\operatorname{AGL}_1(q) \leq G_X^X \leq \operatorname{AGL}_1(q)$, where $X$ is any 1-dimensional subspace of $\mathcal{V}_n(q)$, and $\ell$ is any regular orbit of a Frobenius subgroup of $\operatorname{AGL}_1(q)$ of order $p^n\omega$ contained in $X$;
(ii) \( d - d/n \leq u < d \), there is a hyperplane \( Y \) of \( V_n(q) \) contained in \( W \) and \( G_Y \) acts on \( V_n(q)/Y \) inducing a subgroup of \( AGL_1(q) \) containing \( AGL_1(q) \).

The block \( \ell \) is the union of \( u - d + d/n \) cosets of \( Y \) permuted regularly by a Frobenius subgroup of \( AGL_1(q) \) of order \( p^{u - d + d/n} \).

In case (i), the translation complement \( G_0 \) is as in (i), (ii) and (iii) of (3.4), whereas in case (ii), \( G_0 \) is as in (i), (ii) and (vii) of (3.4).

(b) \( p = 2, n = 6, u = d/3, \omega \) is any divisor of \( q - 1 \) with \( \text{gcd}(q^6 - 1, q^2\omega - 1) = 1 \).

Here, \( G_2(q)'\leq G_0 \leq G_2(q)'\langle \sigma_0 \rangle \), where \( \langle \sigma_0 \rangle \) is the stabilizer in \( \langle \sigma \rangle \) of \( \ell \) and \( \langle \sigma \rangle \) is induced by the field automorphism group of \( GF(q) \), and \( W \) is any totally isotropic 3-dimensional subspace \( V_6(q) \). The group \( G_W \cap G_2(q)' = [q^5]:GL_2(q) \) preserves a non-isotropic 3-dimensional subspace \( Z \) containing \( W \) inducing \( GL_1(q) \) on \( Z/W \). The block \( \ell \) is the union of \( \omega \) cosets of \( W \) in \( Z \) permuted regularly by the subgroup of \( GL_1(q) \) of order \( \omega \);

(c) \( \ell = V_d(p) - W \), where \( W \) is a component of a \( u \)-spread \( S \) of \( V_d(p), \omega = p^d - u - 1 \) and one of the following occurs:

(i) \( S \) is a regular \( d/2 \)-spread of \( V_d(p) \), \( D \) is the complement of \( AG_2(q) \) and the following hold:

1. \( q = 3, 5, 7, 11, 23 \) and \( (q - 1) \cdot A_4 \leq G_0 \leq (q - 1) \cdot S_4 \);
2. \( q = 9 \) and either \( 2 \cdot S_5 \leq G_0 \leq (2 \cdot S_5) : Z_2 \) or \( 8 \circ SL_2(5) \leq G_0 \leq (8 \circ SL_2(5)) : 2 \);
3. \( q = 11, 19, 29, 59 \) and \( SL_2(5) \leq G_0 < (q - 1) \circ SL_2(5) \).

(ii) \( S \) is a Hall 2-spread of \( V_4(3) \), \( D \) is the complement of the Hall plane of order 9 and \( G_0 \cong 2 \cdot S_5 \);

(iii) \( S \) is the Hering 3-spread of \( V_6(3) \) and \( D \) is the 2-(3\(^6\), 702, 701) complement design of the Hering translation plane of order 3\(^3\) described in (b) of (3.2), and \( G_0 \cong SL_2(13) \).

(d) The remaining cases:

(i) \( D = (V_2(7), \ell^G) \), where \( \ell = W \cup (W + x) \) and \( \dim W = 1 \), is a 2-(7\(^2\), 14, 13) design and \( 2 \cdot S_4 \leq G_0 \leq 6 \cdot S_4 \). Here, \( \ell^G \) is the set of all pairs of parallel lines of \( AG_2(7) \);

(ii) \( D = (V_4(3), \ell^G) \), where \( \ell = W \cup (W + x) \) and \( \dim W = 2 \), is a 2-(3\(^4\), 18, 17) design, and the following hold:

1. \( G_0 \cong 2 \cdot S_5^* \) and \( \ell^G \) is the set of all pairs of parallel lines of the Hall plane of order 9;
2. \( 2 \cdot S_5^* \leq G_0 \leq (2 \cdot S_5^*) : 2 \) or \( 8 \circ SL_2(5) \leq G_0 \leq (8 \circ SL_2(5)) : 2 \) and \( \ell^G \) is either the set of all pairs of parallel lines of \( AG_2(9) \), or \( |W^{G_0}| = 40 \) and both \( W \) and \( W + x \) are two subplanes of \( AG_4(3) \) preserved by a Sylow 3-subgroup of \( G_0 \);

(iii) \( D = (V_6(3), \ell^G) \) is a 2-(3\(^6\), 2 \cdot 3\(^u\) \cdot 2 \cdot 3\(^u\) - 1) design, where \( u = 1, 2, 3, \) and \( G_0 \cong SL_2(13) \). Here, \( \ell = W \cup (W + x) \) and \( W \) is any \( u \)-dimensional subspace of \( V_6(3) \) such that either \( |W^{G_0}| = 364 \), or \( u = 2 \) and \( |W^{G_0}| = 91 \), or \( u = 3 \) and \( |W^{G_0}| = 28 \). The number of isomorphism classes of such designs is 4\( u \) for \( u = 1, 2, 3 \) and \( |W^{G_0}| = 364 \), five for \( u = 2 \) and \( |W^{G_0}| = 91 \), and one for \( u = 3 \) and \( |W^{G_0}| = 28 \);

(iv) \( D = (V_6(3), \ell^G) \) is a 2-(3\(^6\), 324, 323) design and \( G_0 \cong SL_2(13) \). Let \( W \) be any 4-dimensional subspace of \( V_6(3) \) such that \( G_{0,W} \cong Q_8 : 3 \). Then the block \( \ell = \bigcup_{i=1}^{d} (W + x_i) \) where the \( (W + x_i)'s \) are four 4-dimensional subspaces of \( AG_6(3) \) preserved by a Sylow 3-subgroup of \( G_0 \) and permuted
3.6. Designs where a block base is properly contained in a component of a transitive $t$-spread of $V_d(p)$. The group $G_0$ acts transitively on a $t$-spread $S$ of $V_d(p)$ and $D = (V_d(p), ℓ^G)$, where $ℓ \subseteq W - \{0\}$, $W \in S$. More precisely, one of the following holds:

(a) $ℓ$ is a quadrangle contained in the union of two 1-dimensional subspaces of $W$. Hence, $D$ is a $2-(q^n, 4, 3)$ design, provided that one of the following holds:

(i) $(q, n) = (3, 4)$, $S$ is a Hall 2-spread and either $(D_6 \circ Q_8) \cdot 5 \leq G_0 \leq (D_6 \circ Q_8) \cdot S_5$ or $G_0 \cong 2 \cdot S_5^-$;

(ii) $(q, n) = (9, 2)$, $S$ is a regular 2-spread of $V_4(3)$ and either $2 \cdot S_5^- \leq G_0 \leq (2 \cdot S_5^-) : 2$ or $8 \circ SL_2(5) \leq G_0 \leq (8 \circ SL_2(5)) : 2$;

(iii) $(q, n) = (3, 6)$, $S$ is one of the two 2-spreads defining the two Hering spaces in $(3.3)$, and $G_0 \cong SL_2(13)$.

In cases (i), (ii) and (iii) the quadrangle $ℓ$ is an orbit under a cyclic subgroup of order 4 of $G_W \cong SL_2(3)$. There are six of such quadrangles which are contained in $W - \{0\}$. Moreover, they are permuted transitively by $G_W$. Therefore, the 2-designs in (i), (ii) and (iii) are unique up to isomorphism, since $G_0$ acts transitively on the 2-spread $S$.

(b) $ℓ = W - \{0\}$, $D$ is a $2-(p^d, p^t - 1, p^t - 2)$ design and the following hold:

(i) $(q, n, t) = (3, 4, 2)$, $S$ is a Hall 2-spread and either $(D_6 \circ Q_8) \cdot 5 \leq G_0 \leq (D_6 \circ Q_8) \cdot S_5$ or $G_0 \cong 2 \cdot S_5^-$;

(ii) $(q, n, t) = (3, 4, 2)$, $S$ is a regular 2-spread and either $2 \cdot S_5^- \leq G_0 \leq (2 \cdot S_5^-) : 2$ or $8 \circ SL_2(5) \leq G_0 \leq (8 \circ SL_2(5)) : 2$;

(iii) $(q, n, t) = (3, 6, 3)$, $S$ is a one of the two 2-spreads defining the Hering spaces in $(3.3)$, and $G_0 \cong SL_2(13)$.

3.7. Designs where a block base is contained in at least two components of a transitive $t$-spread of $V_d(p)$. There are two examples in this case:

(a) $D = (V_4(3), ℓ^G)$, where $ℓ$ is any of the two $2\cdot SL_2(5)$-orbits on the set of non-zero vectors, is a $2-(3^4, 40, 39)$ design and $2 \cdot S_5^- \leq G_0 \leq (2 \cdot S_5^-) : 2$ or $8 \circ SL_2(5) \leq G_0 \leq (8 \circ SL_2(5)) : 2$. Here

$$ℓ = \bigcup_{i=1}^5 (W_i - \{0\}) = \bigcup_{j=1}^5 (W'_j - \{0\}),$$

where the $W_i$’s and the $W'_j$’s are components of two distinct Hall 2-spreads $S$ and $S'$ of $V_4(3)$. Moreover, $D$ is unique up to isomorphism;

(b) $D = (V_6(3), ℓ^G)$, where $G_0 \cong SL_2(13)$ and $ℓ$ is any orbit under a cyclic subgroup of order 26, is a $2-(3^6, 26, 25)$ design. There are eight pairwise non-isomorphic types of such 2-designs;

(c) $D = (V_6(3), ℓ^G)$, where $G_0 \cong SL_2(13)$ and $ℓ$ is any of the two orbits of length 52 under a subgroup of $G_0$ isomorphic to $2 \cdot (13 : 6)$, is a $2-(3^6, 52, 51)$ design. There are two non-isomorphic 2-designs. Indeed, if $S$ is the 3-spread of $V_6(3)$ yielding the Hering translation plane of order $3^3$, in one design $ℓ = (W_1 \cup W_2) - \{0\}$, where $W_1, W_2 \in S$, whereas in the other $ℓ \cap (W - \{0\}) = \{±x_W\}$ for each $W \in S - \{W_1, W_2\}$. 

transitively by a Klein subgroup of $G$. The number of isomorphism classes of such designs is three.
3.8. Designs arising from suitable subsets of $AG_2(p)$ of cardinality prime to $p$. These are $2-(p^2, \lambda + 1, \lambda)$ designs $D = (V_2(p), \ell^G)$, where $\gcd(p, \lambda + 1)=1$, and the possibilities are as follows:

(a) $p = 5$, $\lambda = 5$, 7 and $SU_3(3) \leq G_0 \leq GU_2(3)$;
(b) $p = 7$, $\lambda = 5, 7, 11, 23$ and $2 \cdot S_4^- \leq G_0 \leq 6 \cdot S_4^-$;
(c) $p = 11$, and one of the following holds:
   (i) $\lambda = 7$ and either $5 \times SL_2(3) \leq G_0 \leq 5 \times GL_2(3)$ or $G_0 \cong SL_2(5)$;
   (ii) $\lambda = 11$ and $G_0 \cong SL_2(5)$;
   (iii) $\lambda = 19$ and either $5 \times SL_2(3) \leq G_0 \leq 5 \times GL_2(3)$ or $SL_2(5) \leq G_0 \leq 5 \times SL_2(5)$;
   (iv) $\lambda = 23$ and $G_0 \cong SL_2(5)$;
   (v) $\lambda = 29$ and $5 \times SL_4(3) \leq G_0 \leq 5 \times GL_2(3)$.
(d) $p = 19$, $\lambda = 7, 11, 23, 71, 119$ and $G_0 \cong 9 \times SL_2(5)$;
(e) $p = 23$, $\lambda = 5, 7, 23, 43, 47, 65, 175, 263$ and $G_0 \cong 11 \times 2 \cdot S_4^-$;
(f) $p = 29$, $\lambda = 11, 13, 23, 41, 83, 139, 167$ and $7 \times SL_2(5) \leq G_0 \leq 28 \circ SL_2(5)$;
(g) $p = 59$, $\lambda = 7, 11, 19, 23, 119, 173, 289, 347$ and $G_0 \cong 29 \times SL_2(5)$.

A detailed description of these cases is contained in [12, 40].

3.9. Example arising from tensor product decomposition of the vector space. In this case $G_0 \cong S_3 \times PSL_3(2)$ preserves the tensor decomposition $V_6(2) = V_2(2) \otimes V_4(2)$. Hence, $D = (V_6(2), \ell^G)$, where $\ell = u \otimes PG_2(2)$, $u \neq 0$ is a copy of a Fano plane, is a $2-(2^6, 7, 2)$ design admitting $G = TG_0$ as a flag-transitive, point-primitive automorphism group.

In each case, except for (3.9), $G$ acts point-2-transitively on $D$. All examples given here do occur and the construction details for each 2-design can be read off from [12, 40].

The previous paragraphs contain an exhaustive list of 2-designs admitting a flag-transitive, point-primitive automorphism group except for the semilinear 1-dimensional case. Some of the known examples of 2-designs arising from the latter case are described in the following final paragraph. The list of examples presented below is far from being exhaustive.

4. The semilinear 1-dimensional case

As noted in the Introduction, we have so far been unable to handle completely the case where $G \leq AGL_1(p^d)$. This brief section is devoted to illustrate some examples, where $D$ is a 2-design with $p^d$ points and $\gcd(r, \lambda) = 1$ admitting a subgroup $G$ of the semilinear 1-dimensional group $AGL_1(p^d)$ as a flag-transitive and point-primitive automorphism group. For $\lambda = 1$, remarkable examples are due to Kantor and to Pauley and Bamberg. The reader is referred to [15] and to [43], respectively, for more details on these constructions.

Suppose that $\lambda > 1$. Case (a) in (3.4) or in (3.5) occurs also for $n = 1$ and $G \cong AGL_1(p^d) : \langle \sigma_0 \rangle$, where $\langle \sigma_0 \rangle$ is the stabilizer in $\langle \sigma \rangle$ of $\ell$, and $\langle \sigma \rangle$ is induced by the field automorphism group of $GF(p^d)$. Moreover, $D$ as in (3.9) admits $G = TG_0$ and $21 \leq G_0 \leq 21 : 3$ inside $GL_1(2^6)$ as a flag-transitive point primitive automorphism group. Further remarkable examples of $2-(2^6, 7, 2)$ designs involving the tensor decomposition of $V_6(2) = V_2(2) \otimes V_3(2)$ and admitting $G = TG_0$ and $21 \leq G_0 \leq 21 : 3$ as a flag-transitive, point-primitive automorphism group are the following: $D = (V_6(2), \ell_h^G)$, with $\ell_h = \{u_1 \otimes w^i \cap u_2 \otimes w^{i+k} \mid i = 1, \ldots, 7\}$, where $\langle \gamma \rangle$ is cyclic of order 7, $u_1 \neq u_2$ and $w \neq 0$, for each $h = 1, \ldots, 6$. There
are not corresponding examples of such 2-designs with \( \gcd(r, \lambda) = 1 \) admitting a non-solvable automorphism group. More details on these 2-designs are provided in [11].

If \( D \) is symmetric and admits a flag-transitive and point-primitive automorphism group \( G \) of affine type, then \( G \leq A\Gamma L_1(p^d) \) by [11]. Moreover, if \( O(G) \) denotes the maximal subgroup of \( G \) of odd order, then \( O(T) = T : \langle \bar{\omega}^i, \sigma \bar{\omega}^j \rangle \), where \( y \mid d, i \mid (p^d - 1, j p^d - 1) \) and \( \bar{\omega} : x \rightarrow \omega x, \omega \) is a generator of \( GF(p^d)^* \), and \( \sigma : x \rightarrow x^p \), acts flag-transitively, point-primitively on \( D \). Finally, \( D \) has parameters \( (p^d, \theta p^d - 1, \theta^2 p^d - 1 - \theta^i y) \), where \( \theta \) is a divisor of \( d/y \). Details on the structure of \( D \) and on the corresponding examples due to Paley, Chowla and Lehmer, are available in [11].

### Statements and Declarations

The authors confirm that this manuscript has not been published elsewhere. It is not also under consideration by another journal. They also confirm that there are no known conflicts of interest associated with this publication. The authors have no competing interests to declare that are relevant to the content of this article and they confirm that availability of data and material is not applicable. Shenglin Zhou is supported by the NNSF of China (Grant No.11871224).

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