Nonlinear Supersymmetric (Darboux) Covariance of the Ermakov-Milne-Pinney Equation

M. V. Ioffe\textsuperscript{a} * and H. J. Korsch\textsuperscript{b} †

\textsuperscript{a}Department of Theoretical Physics, Institute of Physics, University of Sankt-Petersburg, Ulyanovskaya 1, Sankt-Petersburg 198504, Russia

\textsuperscript{b}Fachbereich Physik, Universit"at Kaiserslautern, D-67653 Kaiserslautern, Germany

Abstract:
It is shown that the nonlinear Ermakov-Milne-Pinney equation $\rho'' + v(x)\rho = a/\rho^3$ obeys the property of covariance under a class of transformations of its coefficient function. This property is derived by using supersymmetric, or Darboux, transformations. The general solution of the transformed equation is expressed in terms of the solution of the original one. Both iterations of these transformations and irreducible transformations of second order in derivatives are considered to obtain the chain of mutually related Ermakov-Milne-Pinney equations. The behaviour of the Lewis invariant and the quantum number function for bound states is investigated. This construction is illustrated by the simple example of an infinite square well.

PACS: 03.65.-w; 02.30.-Hq; 03.65.Fd; 11.30.Pb

1. The interrelation between a special nonlinear differential equation, known as the Ermakov, Milne or Pinney \[\text{EMP}\] equation (EMP equation), and a corresponding linear equation has been investigated in great detail by many authors (see \cite{2}, the recent review in \cite{3} and references therein).

In classical dynamics these equations appear most often in context with time-dependent oscillators in the form

\[\ddot{\rho} + \omega^2(t)\rho = a\rho^{-3}\]  

and

\[\ddot{q} + \omega^2(t)q = 0,\]  

where the so-called Lewis invariant \[\text{[4]}\]

\[I = \frac{1}{2}\left[\frac{aq^2}{\rho^2} + (\rho \dot{q} - \dot{\rho} q)^2\right]\]  

*E-mail: m.ioffe@pobox.spbu.ru
†E-mail: korsch@physik.uni-kl.de
plays a role as a constant of motion for (2). For a recent application to pulse induced transitions see [5].

In Quantum Mechanics, the general solution of the Schrödinger equation

$$\psi'' + k^2(x) \psi = 0 , \quad k^2(x) = \frac{\hbar^2}{2m}(E - V(x))$$

(4)

can be expressed in amplitude-phase form as

$$\psi(x) = \alpha \rho(x) \sin \left( \int_{x_0}^{x} \rho^{-2}(x') \, dx' + \beta \right),$$

(5)

where $\alpha$, $\beta$ are constants and $\rho(x)$ is an arbitrary solution of

$$\rho'' + k^2(x) \rho = a \rho^{-3}.$$  

(6)

Eq. (5) directly implies the condition

$$N(E) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho^{-2}(x, E) \, dx \equiv n + 1 , \quad n = 0, 1, 2, \ldots$$

(7)

for the bound state energies $E_n$.

In reverse, the Ermakov method can be considered as a method to solve the EMP equation. Here we will discuss some aspects of the Darboux [6], or supersymmetric [7], covariance of this nonlinear differential equation. The role of Darboux transformations in the study of soliton solutions of nonlinear evolutionary equations, like KdV and KP, is well known (see, for example, [8] and references therein). Nevertheless, up to our knowledge, this approach was not used for an investigation of EMP equation.

2. We start from the standard one-dimensional Schrödinger equation ($\hbar = 1, m = 1/2$):

$$- \psi''(x, E) + \left( W^2(x) - W'(x) \right) \psi(x, E) = E \psi(x, E),$$

(8)

where for convenience the potential $V(x)$ is written in "supersymmetric form" [8] in terms of the real superpotential $W(x)$:

$$V(x) = W^2(x) - W'(x).$$

(9)

The Hamiltonian $H = -\partial^2 + V(x)$ is intertwined with another (superpartner) Hamiltonian

$$\tilde{H} = -\partial^2 + \tilde{V}(x);$$

$$\tilde{V}(x) = W^2(x) + W'(x)$$

(10)

(11)

by the component of supercharge

$$q^+ = -\partial + W(x);$$

$$H q^+ = q^+ \tilde{H}.$$  

(12)

(13)
This intertwining relation, together with the hermitian conjugated one,

\[ q^- H = \tilde{H} q^- \]
\[ q^- \equiv (q^+)\dagger = \partial + W(x), \]

seem to be the most important elements of standard supersymmetrical Quantum Mechanics (SUSY QM) \[7\] and its generalizations \[8, 9, 10, 11, 12\]. In particular, just these relations (13), \(14\) lead to the isospectrality (possibly, up to zero modes of operators \(q^\pm\)) of Hamiltonians \(H, \tilde{H}\) and to the connection between their normalized eigenfunctions, if \(E\) belongs to a discrete part of the spectrum:

\[ \tilde{\psi}(x, E) = \frac{1}{\sqrt{E}} q^- \psi(x, E) \]
\[ \psi(x, E) = \frac{1}{\sqrt{E}} q^+ \tilde{\psi}(x, E). \]

Thus, if the model with Hamiltonian \(H\) is exactly solvable (or quasi-exactly-solvable \[13\]), the model with Hamiltonian \(\tilde{H}\) is also exactly solvable (or quasi-exactly-solvable). By iteration of this procedure, from \(\tilde{V} = W^2 + W' \equiv \tilde{W}^2 - \tilde{W}' + c; c = const\) to the next potential \(\tilde{V} = \tilde{W}^2 + \tilde{W}' + c\), one can obtain a chain of isospectral (up to zero modes of \(q^\pm\)) models with known eigenvalues and eigenfunctions\[\dagger\]. In terminology of Mathematical Physics we described now schematically the well known Darboux-Crum \[14\] transformations \[8\] for Sturm-Liouville operator.

Let us denote two pairs of linearly independent solutions (may be both in the pair are not normalizable) of the stationary Schrödinger equations with Hamiltonians \(H, \tilde{H}\) by \(\psi_1(x, E), \psi_2(x, E)\) and \(\tilde{\psi}_1(x, E), \tilde{\psi}_2(x, E)\), correspondingly. By using (16), (17), one can check that their Wronskians coincide:

\[ \Lambda \equiv \psi_1' \psi_2 - \psi_1 \psi_2' = \tilde{\psi}_1' \tilde{\psi}_2 - \tilde{\psi}_1 \tilde{\psi}_2' \equiv \tilde{\Lambda} = const. \]

As was found \[15\], an arbitrary pair of linearly independent solutions \(\psi_1, \psi_2\) of linear equation (8) leads to a general solution

\[ \rho(x, E) \equiv \left[ A\psi_1^2(x, E) + B\psi_2^2(x, E) + 2C\psi_1(x, E)\psi_2(x, E) \right]^{1/2} \]

of the nonlinear EMP equation (for simplicity we normalized \(\rho\) in Eq.(1) as \(a = 1\)):

\[ -\rho''(x, E) + \left( W^2(x) - W'(x, E) - E \right) \rho(x, E) = \frac{-1}{\rho^3(x, E)}, \]

where \(A, B, C\) above are arbitrary constants, which have to satisfy the relation

\[ AB - C^2 = \frac{1}{\Lambda^2}. \]

\[\dagger\]In notations of \[10\] this is a class of so called reducible second order supertransformations with a shift \(c\).
By a similar procedure with an arbitrary pair $\tilde{\psi_1}(x,E)$, $\tilde{\psi_2}(x,E)$ of solutions of the partner Schrödinger equation with Hamiltonian (11), (13), one can construct a general solution

$$\tilde{\rho}(x,E) \equiv \left[ A\tilde{\psi}_2^2(x,E) + B\tilde{\psi_1}(x,E) + 2C\tilde{\psi}_1(x,E)\tilde{\psi}_2(x,E) \right]^{1/2}$$

(22)

of the partner EMP equation (with the partner coefficient function $(\tilde{V}(x) - E)$):

$$-\tilde{\rho}''(x,E) + \left( W^2(x) + W'(x,E) - E \right)\tilde{\rho}(x,E) = \frac{-1}{\tilde{\rho}^3(x,E)}.$$  

(23)

After substitution of (16) in (22), one obtains a general solution $\tilde{\rho}(x,E)$ of (23), which is expressed now not in terms of $\tilde{\psi}_1, \tilde{\psi}_2$, but in terms of linearly independent solutions $\psi_1, \psi_2$ of the partner Schrödinger equation (8). In such a way we found an indirect connection between solutions $\tilde{\rho}(x,E)$ and $\rho(x,E)$ of two EMP equations via direct connection of solutions of "their" Schrödinger equations. But this prognosticated construction would become much more elegant, if the general solution $\tilde{\rho}(x,E)$ of (23) could be expressed directly in terms of solution $\rho(x,E)$ of (20). Indeed, this is the case: straightforward calculations lead to the following nonlinear expression:

$$E\tilde{\rho}^2(x,E) = \left( \rho'(x,E) \right)^2 + W(x)\left( \rho^2(x,E) \right)' + W^2(x)\left( \rho(x,E) \right)^2 + \frac{1}{\rho^2(x,E)} =$$

$$= \left[ \left( \partial + W(x) \right)\rho(x,E) \right]^2 + \frac{1}{\rho^2(x,E)}.$$  

(24)

So, the nonlinear EMP equation possesses the following property, similar to the supersymmetry of Schrödinger equation (or to Darboux covariance of Sturm-Liouville operator): if $\rho(x,E)$ satisfies equation (20), its nonlinear transformation (24) satisfies the same EMP equation, however with a coefficient function transformed accordingly to (23).

By iterations one obtains again a chain of such EMP equations with related coefficient functions ("potentials"), whose general solutions can be expressed consecutively in terms of the solution $\rho(x,E)$ of the first original equation (21). This is a nonlinear analogue of Darboux-Crum reducible transformations for the Sturm-Liouville equation. In particular, for two consecutive transformations:

$$V = W^2 - W' \rightarrow \tilde{V} = W^2 + W' \equiv \bar{W}^2 - \bar{W}' + c \rightarrow \tilde{\tilde{V}} = \bar{W}^2 + \bar{W}' + c; \quad \rho \rightarrow \tilde{\rho} \rightarrow \tilde{\tilde{\rho}},$$

(25)

the solution $\tilde{\tilde{\rho}}$ of EMP equation (20) with coefficient function $\tilde{\tilde{V}}$ can be expressed in terms of original solution $\rho$:

$$E^2\tilde{\tilde{\rho}}^2 = \left[ W\left( W + \bar{W} \right) - E \right]^2 \rho^2 + \left( W + \bar{W} \right)^2 \left( \rho' \right)^2 + \left( W + \bar{W} \right)\left[ W\left( W + \bar{W} \right) - E \right] \left( \rho^2 \right)'$$

$$+ \frac{(W + \bar{W})^2}{\rho^2},$$  

(26)

2We choose here the same constants $A, B, C$ and take into account the equality (15) between Wronskians.
where \( W(x) \) and \( \tilde{W}(x) \) are connected by equality (25) and can be expressed in terms of a single function \( f(x) \):

\[
W = f - \frac{2f' - c}{4f}; \quad \tilde{W} = f + \frac{2f' - c}{4f}.
\]

The transformation (24) can be inverted:

\[
E\rho^2(x,E) = \left[ (\partial - W(x))\tilde{\rho}(x,E) \right]^2 + \frac{1}{\tilde{\rho}^2(x,E)}.
\]

Though the difference in signs in front of \( W(x) \) in (24) and (28) seems to be obvious due to the relation between (9) and (11), the direct check of (28) is not so trivial: it is useful to explore the intermediate auxiliary equality for \( \tilde{\rho} \) and \( \rho \):

\[
\tilde{\rho}\rho' - W\rho^2 = -\rho\rho' - W\rho^2. \tag{29}
\]

3. The various generalizations of standard SUSY QM transformations were investigated in the literature. In particular, SUSY transformations of higher order in derivatives were constructed in [9, 10, 11] (see also the recent papers [16]). It was shown [9], [10] for the one-dimensional Schrödinger equation, that only two “elementary” types of transformations exist: the first order ones, generated by \( q^\pm \) (see (12), (15)), and the irreducible second order transformations, which cannot be expressed as a product of two standard first order transformations. All supercharges (and supertransformations) of higher order can be represented as a number of first- and second-order steps [10]. In the context of the present paper it seems to be interesting, in addition to the construction which lead to (24) and (28), to take into account these second order supertransformations of Schrödinger operator and their possible analogues for EMP equation.

The most general form of solution of intertwining relations (13), but with second order supercharges \( q^\pm \), was found in [10]:

\[
q^+ = \partial^2 - 2f(x)\partial + b(x); \tag{30}
\]

\[
q^- = (q^+)^\dagger = \partial^2 + 2f(x)\partial + 2f'(x) + b(x); \tag{31}
\]

\[
b(x) = f^2(x) - f'(x) - \frac{f''(x)}{2f(x)} + \left( \frac{f'(x)}{2f(x)} \right)^2 + \frac{d}{4f^2(x)}; \tag{32}
\]

\[
V(x) = -2f'(x) + f^2(x) + \frac{f''(x)}{2f(x)} - \left( \frac{f'(x)}{2f(x)} \right)^2 - \frac{d}{4f^2(x)}; \tag{33}
\]

\[
\tilde{V}(x) = +2f'(x) + f^2(x) + \frac{f''(x)}{2f(x)} - \left( \frac{f'(x)}{2f(x)} \right)^2 - \frac{d}{4f^2(x)}, \tag{34}
\]

where \( f(x) \) is an arbitrary real function. For reducible transformations \( d = -c^2/4 \), it was introduced already in Eq.(27), and an irreducible situation corresponds to the case when the constant \( d \) is positive.

Substitution of

\[
\tilde{\psi}_i(x,E) = \frac{1}{\sqrt{E^2 + d}}q^-\psi_i(x,E); \quad i = 1, 2 \tag{35}
\]

The normalization factor is obtained from the corresponding relation of the algebra of Second Order SUSY Quantum Mechanics [9, 10]: \( q^+q^- = H^2 + d; \quad d > 0. \)
with the second order \(q^2\) from (31), (32), into Eq.(22) leads to new expression for \(\tilde{\rho}(x,E)\) in terms of original solutions \(\psi_1(x,E), \psi_2(x,E)\). This expression, after replacing \(\psi''/s\) via Schrödinger equation (8), again can be written in terms of function \(\rho(x,E)\) and its first derivative only:

\[
(E^2 + d)\tilde{\rho}^2(x,E) = 4f^2(x)\left(\rho'(x,E)\right)^2 + \left(2f^2(x) - f'(x) - E\right)^2 \rho^2(x,E) + 2f(x)\left(2f^2(x) - f'(x) - E\right)\left(\rho^2(x,E)\right)' + \frac{4f^2(x)}{\rho^2(x,E)}.
\]

(36)

4. One of the reasons of the rather long peculiar interest to Ermakov systems and to related EMP equations is connected with existence of the \(x\)-independent quantity (or \(t\)-independent in the context of classical harmonic oscillator instead of Schrödinger equation (8)). This invariant, so called Ermakov-Lewis invariant, is expressed in terms of an arbitrary solution \(\Psi(x,E)\) of (8) and an arbitrary solution \(\rho(x,E)\) of the related "auxiliary" equation (20):

\[
I = \frac{1}{2}\left[\left(\frac{\psi(x,E)}{\rho(x,E)}\right)^2 + \left(\rho(x,E)\psi'(x,E) - \rho'(x,E)\psi(x,E)\right)^2\right].
\]

(37)

Calculation of (37) with \(\tilde{\rho}\) and \(\tilde{\Psi}\) instead of \(\rho\) and \(\Psi\) shows that the value of Ermakov-Lewis invariant is not changed under this supersymmetric transformation and its iterations, i.e. \(I = \tilde{I}\).

The quantum number function (7) for bound states \(E_n\) transforms to an analogous function \(\tilde{N}(\tilde{E}_n)\) for the system \(\tilde{H}\). \(\tilde{N}(\tilde{E}_n)\) either coincides with \(N(E_n)\) or differs by \(\pm 1\). This difference depends on the interrelation of spectra \(H\) and \(\tilde{H}\), i.e. (12) on the asymptotic properties of superpotential \(W(x)\). For the case of second order transformations (31) this difference can be equal to \(\pm 2\) (see [10]).

5. As an illustration, we will discuss a simple example in some detail, namely the infinite square well potential

\[
V(x) = \begin{cases} -1 & |x| \leq \pi/2 \\ \infty & |x| > \pi/2 \end{cases},
\]

(38)

which can be expressed (3) in terms of the superpotential

\[
W(x) = \begin{cases} \tan x & |x| \leq \pi/2 \\ \infty & |x| > \pi/2 \end{cases}.
\]

(39)

In the following we will confine ourselves to the well region \(|x| \leq \pi/2\) unless otherwise stated. The supersymmetric partner is:

\[
\tilde{V}(x) = W^2(x) + W'(x) = -1 + 2\sec^2 x
\]

(40)

for the supersymmetric partner potential. The eigenvalues for \(V(x)\) are

\[
E_n = n(n + 2), \quad n = 0, 1, 2, \ldots,
\]

(41)

where by construction the ground state eigenvalue \(E_0\) is zero. Because of supersymmetry, the eigenvalues coincide with those of \(\tilde{V}(x)\):

\[
E_{n+1} = \tilde{E}_n, \quad n = 0, 1, 2, \ldots.
\]

(42)
According to Eq. (19) we can now write the general solution of the EMP-equation (20)
\[ \rho''(x) + k^2 \rho(x) = \rho^{-3}(x), \quad k^2 = E + 1 \] (43)
in terms of the solutions \( \psi_1(x) = \sin kx \) and \( \psi_2(x) = \cos kx \) of the Schrödinger equation:
\[ \rho^2(x) = A \sin^2 kx + B \cos^2 kx + 2C \sin kx \cos kx, \quad AB - C^2 = 1/k^2. \] (44)
Note that \( \rho(x) \) is infinite outside the potential well. Assuming the special case \( C = 0, \ A = B = 1/k \) for simplicity, we have the solution
\[ \rho^2(x) = 1/k = \text{const.} \] (45)
As a check of the Milne quantization condition (7) one obtains
\[ N(E) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho^{-2} \, dx = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} k \, dx = k = n + 1, \quad n = 0, 1, 2, \ldots \] (46)
which reproduces, of course, Eq.(41)

By means of the supersymmetric covariance we can now immediately write down the solution (24) of the EMP-equation (23) for the partner potential (40):
\[ \tilde{\rho}'(x) = (\rho' + \rho \tan kx)^2 + \rho^{-2} = k^{-1} \tan^2 kx + k = \gamma(k^2 + \tan^2 x) \] (47)
with \( \gamma = 1/kE. \)

By some elementary algebra, one can directly check that \( \tilde{\rho} \) is indeed a solution of the EMP equation (23).

From the solution (47) we can also compute the quantum number function
\[ \tilde{N}(E) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{\tilde{\rho}'^2} = \frac{1}{\pi \gamma k(1+k)} = k - 1 \] (48)
and with \( N(E) = k \) from (46) we arrive at the relation \( N(E) = \tilde{N}(E) + 1 \) as was predicted, taking into account Eq.(42).

Finally, we can also verify that the values of the invariants \( I \) and \( \tilde{I} \) agree. If \( \Psi(x) \) is any solution of the Schrödinger equation for the square well potential \( V(x) \) and \( \rho^2(x) = 1/k \) is taken from (45), we have
\[ 2I = \left( \frac{\Psi'}{\rho} \right)^2 + (\rho \Psi' - \rho' \Psi)^2 = k\Psi_0^2 + k^{-1}\Psi'^2. \] (49)
The partner wave function is \( \sqrt{E} \bar{\Psi} = \Psi' + \tan x \Psi \) with \( \sqrt{E} \bar{\Psi}' = (\tan^2 x - E) \Psi + \tan x \Psi' \). We evaluate the (constant) value of the invariant \( \tilde{I} \) at \( x = 0 \), which gives with \( \Psi(0) = \Psi_0 \) etc. and \( \tilde{\rho}_0^2 = \gamma k^2, \tilde{\rho}'_0 = 0 \):
\[ 2\tilde{I} = \left( \frac{\Psi_0}{\tilde{\rho}_0} \right)^2 + (\tilde{\rho}_0 \Psi'_0 - \tilde{\rho}'_0 \Psi_0)^2 \]
\[ = \left( \frac{\Psi_0}{\sqrt{E}} \right)^2 \frac{1}{\gamma k^2} + \gamma k^2 (-\sqrt{E} \Psi_0)^2 = k^{-1}\Psi_0'^2 + \gamma k^2 \Psi_0^2 = 2I. \] (50)
Acknowledgments.

One of the authors (M.I.) is grateful to DAAD for support of this work and to the University of Kaiserslautern, where this work was done, for warm hospitality. The work of M.I. was also partially supported by RFBR grant N 02-01-00499.

REFERENCES

1. V. Ermakov, Univ. Izv. Kiev, Series III 9 (1880) 1; E. W. Milne, Phys. Rev. 35 (1930) 863; E. Pinney, Proc. Am. Math. Soc. 1 (1950) 681.
2. H. J. Korsch, H. Laurent, J. Phys. B 14 (1981) 4213.
3. P. Espinoza, Master Thesis, Univ. de Guanajuato, arXiv:math-ph/0002005.
4. H. R. Lewis, Jr., Phys. Rev. Lett. 18 (1967) 510; H. R. Lewis, Jr., J. Math. Phys. 9 (1968) 1976.
5. K.-E. Thylwe, H. J. Korsch, J. Phys. A34 (2001) 3497; K.-E. Thylwe, H. J. Korsch, J. Phys. A35 (2002) 7507.
6. G. Darboux, Compt. Rend. 94 (1882) 1456.
7. G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin, 1996; F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. 25 (1965) 268.
8. V. B. Matveev, M. A. Salle, Darboux Transformations and Solitons, Springer-Verlag, 1981; F. Calogero, A. Degasperis, Spectral Transform and Solitons, vol.1, North-Holland Publ. Company, 1982.
9. A. A. Andrianov, M. V. Ioffe, V. P. Spiridonov, Phys. Lett. A 174 (1993) 273.
10. A. A. Andrianov, F. Cannata, J. -P. Dedonder, M. V. Ioffe, Int. J. Mod. Phys. A 10 (1995) 2683.
11. A. A. Andrianov, M. V. Ioffe, D. N. Nishnianidze, Phys. Lett. A 201 (1995) 103; A. A. Andrianov, M. V. Ioffe, D. N. Nishnianidze, J. Phys. A 32 (1999) 4641; F. Cannata, M. V. Ioffe, D. N. Nishnianidze, J. Phys. A 35 (2002) 1389.
12. A. A. Andrianov, N. V. Borisov, M. V. Ioffe, M. I. Eides, Phys. Lett. A 109 (1984) 143; A. A. Andrianov, N. V. Borisov, M. V. Ioffe, Phys. Lett. A 105 (1984) 19; A. A. Andrianov, N. V. Borisov, M. V. Ioffe, JETP Lett. 39 (1984) 93.
13. A. Turbiner, Commun. Math. Phys. 118 (1988) 467; A. Ushveridze, Sov. J. Part. Nucl. 20 (1989) 504 [Transl. from Fiz. Elem. Chast. Atom. Yad. 20 (1989) 1185].
14. M. Crum, Quat. J. Math. 6 (1955) 121.
15. C. J. Eliezer, A. Gray, SIAM J. Appl. Math. 30 (1976) 463.
16. H. Aoyama, M. Sato, T. Tanaka, Phys.Lett. B 503 (2001) 423; H. Aoyama, M. Sato, T. Tanaka, Nucl. Phys. B 619 (2001) 105; H. Aoyama, N. Nakayama, M. Sato, T. Tanaka, Phys. Lett. B 519 (2001) 260; M. Plyushchay, Int. J. Mod. Phys. A 15 (2000) 3679; S. Klishchivich, M. Plyushchay, Mod. Phys. Lett. A 14 (1999) 2739; R. Sasaki, K. Takasaki, J. Phys. A 34 (2001) 9533.