Non-asymptotic Analysis of ℓ₁-norm Support Vector Machines
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Abstract
Support Vector Machines (SVM) with ℓ₁ penalty became a standard tool in analysis of high-dimensional classification problems with sparsity constraints in many applications including bioinformatics and signal processing. Although SVM have been studied intensively in the literature, this paper has to our knowledge first non-asymptotic results on the performance of ℓ₁-SVM in identification of sparse classifiers. We show that a d-dimensional s-sparse classification vector can be (with high probability) well approximated from only $O(s \log(d))$ Gaussian trials. The methods used in the proof include concentration of measure and probability in Banach spaces.

Index Terms
Support vector machines, compressed sensing, machine learning, regression analysis, signal reconstruction, classification algorithms, functional analysis, random variables

I. INTRODUCTION

A. Support Vector Machines
Support vector machines (SVM) are a group of popular classification methods in machine learning. Their input is a set of data points $x_1, \ldots, x_m \in \mathbb{R}^d$, each equipped with a label $y_i \in \{-1, +1\}$, which assigns each of the data points to one of two groups. SVM aims for binary linear classification based on separating hyperplane between the two groups of training data, choosing a hyperplane with separating gap as large as possible.

Since their introduction by Vapnik and Chervonenkis [27], the subject of SVM was studied intensively. We will concentrate on the so-called soft margin SVM [8], which allow also for misclassification of the training data are the most used version of SVM nowadays.

In its most common form (and neglecting the bias term), the soft-margin SVM is a convex optimization program

$$\min_{\|w\|_2 \leq R} \sum_{i=1}^{m} [1 - y_i \langle x_i, w \rangle]_+ \quad \text{subject to} \quad \|w\|_2 \leq R,$$

for some tradeoff parameter $\lambda > 0$ and so called slack variables $\xi_i$. It will be more convenient for us to work with the following equivalent reformulation of (I.1)

$$\min_{w \in \mathbb{R}^d} \sum_{i=1}^{m} [1 - y_i \langle x_i, w \rangle]_+ \quad \text{subject to} \quad \|w\|_2 \leq R,$$

where $R > 0$ gives the restriction on the size of $w$. We refer to monographs [25], [28], [29] and references therein for more details on SVM and to [13, Chapter B.5] and [9, Chapter 9] for a detailed discussion on dual formulations.

B. ℓ₁-SVM
As the classical SVM (I.1) and (I.2) do not use any pre-knowledge about $w$, one typically needs to have more training data than the underlying dimension of the problem, i.e. $m \gg d$. Especially in analysis of high-dimensional data, this is usually not realistic and we typically deal with much less training data, i.e. with $m \ll d$. On the other hand, we can often assume some structural assumptions on $w$, in the most simple case that it is sparse, i.e. that most of its coordinates are zero. Motivated by the success of LASSO [26] in sparse linear regression, it was proposed in [6] that replacing the ℓ₂-norm $\|w\|_2$ in (I.2) by its ℓ₁-norm $\|w\|_1 = \sum_{j=1}^{d} |w_j|$ leads to sparse classifiers $w \in \mathbb{R}^d$. This method was further popularized in [34] by Zhu, Rosset, Hastie, and Tibshirani, who developed an algorithm that efficiently computes the whole solution path (i.e. the solutions of (I.2)}
for a wide range of parameters $R > 0$). We refer also to \[5\], \[2\], \[18\] and \[19\] for other generalizations of the concept of SVM.

Using the ideas of concentration of measure \[20\] and random constructions in Banach spaces \[21\], the performance of LASSO was analyzed in the recent area of compressed sensing \[11\], \[7\], \[3\], \[10\], \[12\].

$\ell_1$-SVM (and its variants) found numerous applications in high-dimensional data analysis, most notably in bioinformatics for gene selection and microarray classification \[30\], \[31\], \[15\]. Finally, $\ell_1$-SVM’s are closely related to other popular methods of data analysis, like elastic nets \[32\] or sparse principal components analysis \[33\].

### C. Main results

The main aim of this paper is to analyze the performance of $\ell_1$-SVM in the non-asymptotic regime. To be more specific, let us assume that the data points $x_1, \ldots, x_m \in \mathbb{R}^d$ can be separated by a hyperplane according to the given labels $y_1, \ldots, y_m \in \{-1, +1\}$, and that this hyperplane is normal to a $s$-sparse vector $a \in \mathbb{R}^d$. Hence, $\langle a, x_i \rangle > 0$ if $y_i = 1$ and $\langle a, x_i \rangle < 0$ if $y_i = -1$. We then obtain $\hat{a}$ as the minimizer of the $\ell_1$-SVM. The first main result of this paper (Theorem \[\text{II.3}\]) then shows that $\|a - \hat{a}\|_2$ is a good approximation of $a$, if the data points are i.i.d. Gaussian vectors and the number of measurements scales linearly in $s$ and logarithmically in $d$.

Later on, we introduce a modification of $\ell_1$-SVM by adding an additional $\ell_2$-constraint. It will be shown in Theorem \[\text{IV.1}\] that it still approximates the sparse classifiers with the number of measurements $m$ growing linearly in $s$ and logarithmically in $d$, but the dependence on other parameters improves. In this sense, this modification outperforms the classical $\ell_1$-SVM.

### D. Organization

The paper is organized as follows. Section \[\text{II}\] recalls the concept of $\ell_1$-Support Vector Machines of \[34\]. It includes the main result, namely Theorem \[\text{II.3}\]. It shows that the $\ell_1$-SVM allows to approximate sparse classifier $a$, where the number of measurements only increases logarithmically in the dimension $d$ as it is typical for several reconstruction algorithms from the field of compressed sensing. The two most important ingredients of its proof, Theorems \[\text{II.1}\] and \[\text{II.2}\] are also discussed in this part. The proof techniques used are based on the recent work of Plan and Vershynin \[24\], which in turn makes heavy use of classical ideas from the areas of concentration of measure and probability estimates in Banach spaces \[20\], \[21\].

Section \[\text{III}\] gives the proofs of Theorems \[\text{II.1}\] and \[\text{II.2}\]. In Section \[\text{IV}\] we discuss several extensions of our work, including a modification of $\ell_1$-SVM, which combines the $\ell_1$ and $\ell_2$ penalty.

Finally, in Section \[\text{V}\] we show numerical tests to demonstrate the convergence results of Section \[\text{II}\]. In particular, we compare different versions of SVM and 1-Bit Compressed Sensing, which was first introduced by Boufounos and Baraniuk in \[4\] and then discussed and continued in \[23\], \[24\], \[22\], \[1\], \[17\] and others.

### E. Notation

We denote by $[\lambda]_+ := \max(\lambda, 0)$ the positive part of a real number $\lambda \in \mathbb{R}$. By $\|w\|_1$, $\|w\|_2$ and $\|w\|_\infty$ we denote the $\ell_1$, $\ell_2$ and $\ell_\infty$ norm of $w \in \mathbb{R}^d$, respectively. We denote by $\mathcal{N}(\mu, \sigma^2)$ the normal (Gaussian) distribution with mean $\mu$ and variance $\sigma^2$. When $\omega_1$ and $\omega_2$ are random variables, we write $\omega_1 \sim \omega_2$ if they are equidistributed. Multivariate normal distribution is denoted by $\mathcal{N}(\mu, \Sigma)$, where $\mu \in \mathbb{R}^d$ is its mean and $\Sigma \in \mathbb{R}^{d \times d}$ is its covariance matrix. By $\log(x)$ we denote the natural logarithm of $x \in (0, \infty)$ with basis $e$. Further notation will be fixed in Section \[\text{II}\] under the name of “Standing assumptions”, once we fix the setting of our paper.

## II. $\ell_1$-NORM SUPPORT VECTOR MACHINES

In this section we give the setting of our study and the main results. Let us assume that the data points $x_1, \ldots, x_m \in \mathbb{R}^d$ are equipped with labels $y_i \in \{-1, +1\}$ in such a way that the groups $\{x_i : y_i = 1\}$ and $\{x_i : y_i = -1\}$ can indeed be separated by a sparse classifier $a$, i.e. that

$$y_i = \text{sign}(\langle x_i, a \rangle), \quad i = 1, \ldots, m \quad (\text{II.1})$$

and

$$\|a\|_0 = \# \{j : a_j \neq 0\} \leq s. \quad (\text{II.2})$$

As the classifier is usually not unique, we cannot identify $a$ exactly by any method whatsoever. Hence we are interested in a good approximation of $a$ obtained by $\ell_1$-norm SVM from a minimal number of training data. To achieve this goal, we will assume that the training points

$$x_i = r \tilde{x}_i, \quad \tilde{x}_i \sim \mathcal{N}(0, \text{Id}) \quad (\text{II.3})$$

are i.i.d. measurement vectors for some constant $r > 0$.\[\text{2}\]
To allow for more generality, we replace (II.2) by
\[ \|a\|_2 = 1, \quad \|a\|_1 \leq R. \tag{II.4} \]
Let us observe, that \( \|a\|_2 = 1 \) and \( \|a\|_0 \leq s \) implies also \( \|a\|_1 \leq \sqrt{s} \), i.e. (II.4) with \( R = \sqrt{s} \).
Furthermore, we denote by \( \hat{a} \) the minimizer of
\[ \min_{w \in \mathbb{R}^d} \sum_{i=1}^m [1 - y_i \langle x_i, w \rangle]_+ \quad \text{subject to} \quad \|w\|_1 \leq R. \tag{II.5} \]
Let us summarize the setting of our work, which we will later on refer to as “Standing assumptions” and which we will keep for the rest of this paper.

**Standing assumptions:**

(i) \( a \in \mathbb{R}^d \) is the true (nearly) sparse classifier with \( \|a\|_2 = 1 \), \( \|a\|_1 \leq R \), \( R \geq 1 \), which we want to approximate;
(ii) \( x_i = r \hat{x}_i, \quad \hat{x}_i \sim \mathcal{N}(0, \text{Id}), i = 1, \ldots, m \) are i.i.d. training data points for some constant \( r > 0 \);
(iii) \( y_i = \text{sign}(\langle x_i, a \rangle), \quad i = 1, \ldots, m \) are the labels of the data points;
(iv) \( \hat{a} \) is the minimizer of (II.5);
(v) Furthermore, we denote
\[ K = \{w \in \mathbb{R}^d \mid \|w\|_1 \leq R\}, \tag{II.6} \]
\[ f_a(w) = \frac{1}{m} \sum_{i=1}^m [1 - y_i \langle x_i, w \rangle]_+, \tag{II.7} \]
where the subindex \( a \) denotes the dependency of \( f_a \) on \( a \) (via \( y_i \)).

In order to estimate the difference between \( a \) and \( \hat{a} \) we adapt the ideas of [24]. First we observe
\[
0 \leq f_a(a) - f_a(\hat{a}) = \left( \mathbb{E} f_a(a) - \mathbb{E} f_a(\hat{a}) \right) + \left( f_a(a) - \mathbb{E} f_a(a) \right) - \left( f_a(\hat{a}) - \mathbb{E} f_a(\hat{a}) \right) \\
\leq \mathbb{E}(f_a(a) - f_a(\hat{a})) + 2 \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)|,
\]
i.e.
\[ \mathbb{E}(f_a(\hat{a}) - f_a(a)) \leq 2 \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)|. \tag{II.8} \]
Hence, it remains
- to bound the right hand side of (II.8) from above and
- to estimate the left hand side in (II.8) by the distance between \( a \) and \( \hat{a} \) from below.

We obtain the following two theorems, whose proofs are given in Section III.

**Theorem II.1.** Let \( u > 0 \). Under the “Standing assumptions” it holds
\[ \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \leq \frac{8\sqrt{8\pi} + 18rR\sqrt{2\log(2d)}}{\sqrt{m}} + u \]
with probability at least
\[ 1 - 8 \left( \exp\left( -\frac{mu^2}{32} \right) + \exp\left( -\frac{mu^2}{32r^2R^2} \right) \right). \]

**Theorem II.2.** Let the “Standing assumptions” be fulfilled and let \( w \in K \). Put
\[ c = \langle a, w \rangle, \quad c' = \sqrt{\|w\|_2^2 - \langle a, w \rangle^2} \]
and assume that \( c' > 0 \). If furthermore \( c \leq 0 \), then \( \pi \mathbb{E}(f_a(w) - f_a(a)) \) can be estimated from below by
\[ \frac{\pi}{2} + c' \sqrt{\frac{\pi}{2}} - \frac{\sqrt{2\pi}}{r}. \]
If \( c > 0 \), then \( \pi \mathbb{E}(f_a(w) - f_a(a)) \) can be estimated from below by
\[ \frac{\sqrt{\pi}}{\sqrt{2}} \int_0^{1/cr} (1 - c't)e^{\frac{3t^2}{2}} dt + \frac{c'}{c} \exp\left( -\frac{1}{2c'rt^2} \right) - \frac{\sqrt{2\pi}}{r}. \]
Combining Theorems II.1 and II.2 with (II.8) we obtain our main result.

**Theorem II.3.** Let \( d \geq 2 \), \( 0 < \varepsilon < 0.18 \), \( r > \sqrt{2\pi(0.57 - \pi\varepsilon)^{-1}} \) and \( m \geq C\varepsilon^{-2}r^2R^2\log(d) \) for some constant \( C \). Under the “Standing assumptions” it holds

\[
\frac{\|a - \hat{a}\|_2}{\langle a, \hat{a} \rangle} \leq C' \left( \varepsilon + \frac{1}{r} \right) \tag{II.9}
\]

with probability at least

\[
1 - \gamma \exp(-C''\log(d)) \tag{II.10}
\]

for some positive constants \( \gamma, C', C'' \).

**Remark II.4.**
1) If the classifier \( a \in \mathbb{R}^d \) with \( \|a\|_2 = 1 \) is \( s \)-sparse, we always have \( \|a\|_1 \leq \sqrt{s} \) and we can choose \( R = \sqrt{s} \) in Theorem II.3. The dependence of \( m \), the number of samples needed, is then linear in \( s \) and logarithmic in \( d \). Intuitively, this is the best what we can hope for. On the other hand, we leave it open, if the dependence on \( \varepsilon \) and \( r \) is optimal in Theorem II.3.

2) Theorem II.3 uses the constants \( C, C' \) and \( C'' \) only for simplicity. More explicitly we show that taking

\[
m \geq 4\varepsilon^{-2} \left( 8\sqrt{8\pi} + 19rR\sqrt{2\log(2d)} \right)^2,
\]

we get the estimate

\[
\frac{\|a - \hat{a}/\|\hat{a}\|_2}{\langle a, \hat{a}/\|\hat{a}\|_2 \rangle} \leq 2e^{1/2} \left( \pi\varepsilon + \frac{\sqrt{2\pi}}{r} \right)
\]

with probability at least

\[
1 - 8 \left( \exp \left( \frac{-r^2R^2\log(2d)}{16} \right) + \exp \left( \frac{-\log(2d)}{16} \right) \right).
\]

3) If we introduce an additional parameter \( t > 0 \) and choose \( m \geq 4\varepsilon^{-2}(8\sqrt{8\pi} + (18 + t)rR\sqrt{2\log(2d)})^2 \), nothing but the probability changes to

\[
1 - 8 \left( \exp \left( \frac{-t^2r^2R^2\log(2d)}{16} \right) + \exp \left( \frac{-t^2\log(2d)}{16} \right) \right).
\]

Hence, by fixing \( t \) large, we can increase the value of \( C'' \) and speed up the convergence of (II.10) to 1.

**Proof of Theorem II.3** To apply Theorem II.1 we choose

\[
u = \frac{rR\sqrt{2\log(2d)}}{\sqrt{m}}
\]

and

\[
m \geq 4\varepsilon^{-2}(8\sqrt{8\pi} + 19rR\sqrt{2\log(2d)})^2
\]

and we obtain the estimate

\[
\sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \leq \frac{8\sqrt{8\pi} + 19rR\sqrt{2\log(2d)}}{\sqrt{m}} + u \leq \varepsilon / 2
\]

with probability at least

\[
1 - 8 \left( \exp \left( \frac{-32u^2}{32} \right) + \exp \left( \frac{-32u^2}{32} \right) \right)
\]

\[
= 1 - 8 \left( \exp \left( \frac{-r^2R^2\log(2d)}{16} \right) + \exp \left( \frac{-\log(2d)}{16} \right) \right).
\]

Using (II.8) this already implies

\[
\mathbb{E}(f_a(\hat{a}) - f_a(a)) \leq \varepsilon \tag{II.11}
\]

with at least the same probability. Now we want to apply Theorem II.2 with \( w = \hat{a} \) to estimate the left hand side of this inequality. Therefore we first have to deal with the case \( c' = \sqrt{\|\hat{a}\|_2^2 - \langle a, \hat{a} \rangle^2} = 0 \), which only holds if \( \hat{a} = \lambda a \) for some
\(\lambda \in \mathbb{R}\). If \(\lambda > 0\), then \(\hat{a}/\|\hat{a}\|_2 = a\) and the statement of the Theorem holds trivially. If \(\lambda \leq 0\), then the condition \(f(\hat{a}) \leq f(a)\) can be rewritten as

\[
\sum_{i=1}^{m} (1 + |\lambda| \cdot |(x_i, a)|)_{+} \leq \sum_{i=1}^{m} (1 - |(x_i, a)|)_{+}.
\]

This inequality holds if, and only if, \((x_i, a) = 0\) for all \(i = 1, \ldots, m\) - and this in turn happens only with probability zero.

We may therefore assume that \(\lambda' \neq 0\) holds almost surely and we can apply Theorem [1.2] Here we distinguish the three cases \(c = (\hat{a}, a) \leq 0, 0 < c \leq 1/r\) and \(1/r < c\). First, we will show that the two cases \(c \leq 0\) and \(0 < c < 1/r\) lead to a contradiction and then, for the case \(c > 1/r\), we will prove our claim.

1. case \(c \leq 0\): Using Theorem [1.2] we get the estimate

\[
\pi E(f_a(\hat{a}) - f_a(a)) \geq \frac{\pi}{2} + \frac{c r}{2} \frac{\sqrt{2 \pi}}{r} - \frac{\sqrt{2 \pi}}{2}
\]

and (II.11) gives (with our choices for \(r\) and \(\varepsilon\)) the contradiction

\[
\frac{1}{\pi} \left( \frac{\pi}{2} - \frac{\sqrt{2 \pi}}{r} \right) \leq E(f_a(\hat{a}) - f_a(a)) \leq \varepsilon.
\]

2. case \(0 < c \leq 1/r\): As in the first case we use Theorem [1.2] in order to show a contradiction. First we get the estimate

\[
\pi E(f_a(\hat{a}) - f_a(a)) \geq \frac{\pi}{\sqrt{2}} \int_0^{1/c r} (1 - c t) e^{-\frac{t^2}{2}} dt + \frac{c'}{c} e^{\frac{r}{2 c^2 r^2}} - \frac{\sqrt{2 \pi}}{r}
\]

\[
\geq \frac{\sqrt{2 \pi}}{\sqrt{2}} \int_0^{1/c r} (1 - c t) e^{-\frac{t^2}{2}} dt - \frac{\sqrt{2 \pi}}{r}
\]

Now we consider the function

\[
g: (0, \infty) \to \mathbb{R}, \quad z \mapsto \int_0^{1/c r} (1 - z t) e^{-\frac{t^2}{2}} dt.
\]

It holds \(g(z) \geq 0\) and

\[g'(z) = - \int_0^{1/c r} t e^{-\frac{t^2}{2}} dt < 0,\]

so \(g\) is monotonic decreasing. With \(c r < 1\) this yields

\[
\pi E(f_a(\hat{a}) - f_a(a)) \geq \frac{\sqrt{2 \pi}}{\sqrt{2}} \int_0^{1/c r} (1 - c t) e^{-\frac{t^2}{2}} dt - \frac{\sqrt{2 \pi}}{r}
\]

\[
= \frac{\sqrt{2 \pi}}{\sqrt{2}} g(c r) - \frac{\sqrt{2 \pi}}{r} \geq \frac{\sqrt{2 \pi}}{\sqrt{2}} g(1) - \frac{\sqrt{2 \pi}}{r}
\]

\[
= \frac{\sqrt{2 \pi}}{\sqrt{2}} \int_0^{1} (1 - t) e^{-\frac{t^2}{2}} dt - \frac{\sqrt{2 \pi}}{r}
\]

\[
\geq 0.57 - \frac{\sqrt{2 \pi}}{r}.
\]

Again, (II.11) now gives the contradiction

\[
\frac{1}{\pi} \left( 0.57 - \frac{\sqrt{2 \pi}}{r} \right) \leq E(f_a(\hat{a}) - f_a(a)) \leq \varepsilon.
\]

We conclude that it must hold \(c' > 0\) and \(c > 1/r\) almost surely.

3. case \(1/r < c\): In this case we get the estimate

\[
\pi E(f_a(\hat{a}) - f_a(a)) \geq \frac{\sqrt{2 \pi}}{\sqrt{2}} \int_0^{1/c r} (1 - c t) e^{-\frac{t^2}{2}} dt + \frac{c'}{c} e^{\frac{r}{2 c^2 r^2}} - \frac{\sqrt{2 \pi}}{r}
\]

\[
\geq \frac{c'}{c} e^{-1/2} - \frac{\sqrt{2 \pi}}{r}.
\]

\[(II.12)\]
where we used $cr > 1$ for the last inequality. Further we get

$$
\frac{c'}{c} = \frac{\|a\|^2 - \langle a, \hat{a} \rangle^2}{\langle a, \hat{a} \rangle} = \frac{\|\hat{a}\|^2 - \langle a, \hat{a} \rangle^2}{\langle a, \hat{a} \rangle} 
$$

$$
= \sqrt{\left(\frac{\|\hat{a}\|}{\langle a, \hat{a} \rangle} - \frac{\langle a, \hat{a} \rangle}{\langle a, \hat{a} \rangle} \right)^2 + \left(\frac{\langle a, \hat{a} \rangle}{\langle a, \hat{a} \rangle} + \frac{\langle a, \hat{a} \rangle}{\langle a, \hat{a} \rangle} \right)^2}
$$

$$
= \sqrt{\frac{(2 - 2(a, \hat{a})/(\|\hat{a}\|))(2 + 2(a, \hat{a})/(\|\hat{a}\|))}{4(a, \hat{a})^2}}
$$

(II.13)

$$
= \sqrt{\frac{\|a - \hat{a}/\|\hat{a}\|^2 \cdot |a + \hat{a}/\|\hat{a}\|^2}{4(a, \hat{a})^2}}
$$

$$
\geq \frac{1}{2} \frac{\|a - \hat{a}/\|\hat{a}\|^2}{(a, \hat{a}/\|\hat{a}\|^2)}.
$$

Finally, combining (II.1), (II.2) and (II.13), we arrive at

$$
\frac{1}{\pi} \left( \frac{\|a - \hat{a}/\|\hat{a}\|^2}{(a, \hat{a}/\|\hat{a}\|^2)} \right) \left( \frac{1}{2} e^{-1/2} - \frac{\sqrt{2\pi}}{r} \right)
$$

$$
\leq \mathbb{E}(f_a(\hat{a}) - f_a(a)) \leq \varepsilon,
$$

which finishes the proof of the theorem.

III. PROOFS

The main aim of this section is to prove Theorems II.1 and II.2. Before we come to that, we shall give a number of helpful Lemmas.

A. Concentration of $f_a(w)$

In this subsection we want to show that $f_a(w)$ does not deviate uniformly far from its expected value $\mathbb{E}f_a(w)$, i.e. we want to show that

$$
\sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)|
$$

is small with high probability. Therefore we will first estimate its mean

$$
\mu := \mathbb{E} \left( \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \right) \quad (III.1)
$$

and then use a concentration inequality to prove Theorem II.1. The proof relies on standard techniques from [21] and [20] and is inspired by the analysis of 1-bit compressed sensing given in [24].

For $i = 1, \ldots, m$ let $\varepsilon_i \in \{+1, -1\}$ be i.i.d. Bernoulli variables with

$$
P(\varepsilon_i = +1) = P(\varepsilon_i = -1) = 1/2.
$$

(III.2)

Let us put

$$
\mathcal{A}_i(w) = [1 - y_i(x_i, w)]_+, \quad \mathcal{A}(w) = [1 - y(x, w)]_+,
$$

(III.3)

where $x$ is an independent copy of any of the $x_i$ and $y = \text{sign}(\langle x, a \rangle)$. Further, we will make use of the following lemmas.

**Lemma III.1.** For $m \in \mathbb{N}$, i.i.d. Bernoulli variables $\varepsilon_1, \ldots, \varepsilon_m$ according to (III.2) and any scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ it holds

$$
P \left( \sum_{i=1}^m \varepsilon_i \lambda_i + t \right) \leq 2P \left( \sum_{i=1}^m \varepsilon_i \lambda_i + t \right).
$$

(III.4)

**Proof:** First we observe

$$
P \left( \sum_{i=1}^m \varepsilon_i \lambda_i + t \right) = P \left( \sum_{\lambda_i \geq 0} \varepsilon_i \lambda_i + t \right)
$$

$$
= P \left( \sum_{\lambda_i \geq 0} \varepsilon_i \lambda_i + t \right) + P \left( \sum_{\lambda_i < 0} \varepsilon_i \lambda_i + t \right)
$$

$$
= P \left( \sum_{\lambda_i \geq 0} \varepsilon_i \lambda_i + t \right) + P \left( \sum_{\lambda_i < 0} \varepsilon_i \lambda_i + t \right).
$$
Now we can estimate the second of these two probabilities by the first one and we arrive at
\[
P\left(\sum_{i=1}^{m} \varepsilon_i \lambda_i \geq t\right) \leq 2P\left(\sum_{\lambda_i \geq 0} \varepsilon_i \lambda_i \geq t\right) \quad \text{and} \quad \sum_{\lambda_i < 0} \varepsilon_i \lambda_i \geq t
\]

Lemma III.2. 1) For Gaussian random variables \(x_1, \ldots, x_m \in \mathbb{R}^d\) according to (II.3) it holds
\[
E\left\|\frac{1}{m} \sum_{i=1}^{m} x_i\right\|_\infty \leq \frac{r}{\sqrt{m}} \sqrt{\log(2d)}.
\]
(III.5)

2) Let the i.i.d. Bernoulli variables \(\varepsilon_1, \ldots, \varepsilon_m\) be according to (III.2) and let \(u > 0\). Then it holds
\[
P\left(\left|\frac{1}{m} \sum_{i=1}^{m} \varepsilon_i\right| \geq u\right) \leq 2 \exp\left(-\frac{mu^2}{2}\right).
\]
(III.6)

3) For \(x_1, \ldots, x_m \in \mathbb{R}^d\) and \(K \subset \mathbb{R}^d\) according to (II.3) and (II.6) we denote
\[
\tilde{\mu} = E\left(\sup_{w \in K} \langle \frac{1}{m} \sum_{i=1}^{m} x_i, w \rangle\right).
\]
(III.7)

Then it holds
\[
P\left(\sup_{w \in K} \langle \frac{1}{m} \sum_{i=1}^{m} x_i, w \rangle \geq \tilde{\mu} + u\right) \leq \exp\left(-\frac{mu^2}{2\sigma^2}\right).
\]
(III.8)

Proof:
1) The statement follows from
\[
E\left\|\frac{1}{m} \sum_{i=1}^{m} x_i\right\|_\infty = \frac{r}{\sqrt{m}} E\|\tilde{x}\|_\infty
\]
with \(\tilde{x} \sim \mathcal{N}(0, \text{Id})\) and proposition 8.1 of [13]:
\[
\frac{\sqrt{\log(d)}}{4} \leq E\|\tilde{x}\|_\infty \leq \sqrt{2\log(2d)}.
\]
(III.9)

2) The estimate follows as a consequence of Hoeffding’s inequality [16].
3) Theorem 5.2 of [24] gives the estimate
\[
P\left(\sup_{w \in K} \langle \frac{1}{m} \sum_{i=1}^{m} x_i, w \rangle \geq \tilde{\mu} + u\right) \leq \exp\left(-\frac{mu^2}{2\sigma^2}\right)
\]
with
\[
\sigma^2 = \sup_{w \in K} E\left(\left\|\frac{1}{m} \sum_{i=1}^{m} x_i, w\right\|^2\right).
\]

Since the \(x_i\)'s are independent we get
\[
\frac{1}{m} \sum_{i=1}^{m} x_i = \frac{r}{\sqrt{m}} \tilde{x} \quad \text{with} \quad \tilde{x} \sim \mathcal{N}(0, \text{Id})
\]
and we end up with
\[
\sigma^2 = \sup_{w \in K} E\left(\frac{r^2}{m} \langle \tilde{x}, w \rangle^2\right) = \frac{r^2}{m} \sup_{w \in K} \|w\|^2 = \frac{r^2 R^2}{m}.
\]
(III.10)
1) Estimate of the mean $\mu$: To estimate the mean $\mu$, we first derive the following symmetrization inequality, cf. [21] Chapter 6 and [24] Lemma 5.1.

**Lemma III.3 (Symmetrization).** Let $\varepsilon_1, \ldots, \varepsilon_m$ be i.i.d. Bernoulli variables according to (III.2). Under the “Standing assumptions” it holds for $\mu$ defined by (III.1)

\[
\mu \leq 2 \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)]_+ \right|. 
\]  

(III.11)

**Proof:** Let $A_i(w)$ and $A(w)$ be according to (III.3). Let $x'_i$ and $x'$ be independent copies of $x_i$ and $x$. Then $A'_i(w)$ and $A'(w)$, generated in the same way (III.3) with $x'_i$ and $x'$ instead of $x_i$ and $x$, are independent copies of $A_i(w)$ and $A(w)$. We denote by $\mathbb{E}'$ the mean value with respect to $x'_i$ and $x'$. Using $\mathbb{E}'(A'_i(w) - \mathbb{E}'A'(w)) = 0$, we get

\[
\mu = \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} (A_i(w) - \mathbb{E}A(w)) \right|
\]

\[
= \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} (A_i(w) - \mathbb{E}A(w)) \right|
\]

\[
- \mathbb{E}'(A'_i(w) - \mathbb{E}'A'(w))
\]

\[
= \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}'(A_i(w) - A'_i(w)) \right|
\]

Applying Jensen’s inequality we further get

\[
\mu \leq \mathbb{E} \mathbb{E}' \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} (A_i(w) - A'_i(w)) \right|
\]

\[
= \mathbb{E} \mathbb{E}' \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i (A_i(w) - A'_i(w)) \right|
\]

\[
\leq 2 \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i A_i(w) \right|
\]

\[
= 2 \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)]_+ \right|
\]

as claimed.

Equipped with this tool, we deduce the following estimate for $\mu$.

**Lemma III.4.** Under the “Standing assumptions” we have

\[
\mu = \mathbb{E} \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \leq \frac{4\sqrt{8\pi} + 8rR \sqrt{2 \log(2d)}}{\sqrt{m}}.
\]

**Proof:** Using Lemma III.3 we obtain

\[
\mu = \mathbb{E} \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)|
\]

\[
\leq 2 \mathbb{E} \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)]_+ \right|
\]

\[
= 2 \int_{0}^{\infty} \mathbb{P} \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)]_+ \right| \geq t \right) dt.
\]

Now we can apply Lemma III.1 to get

\[
\mu \leq 4 \int_{0}^{\infty} \mathbb{P} \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)] \right| \geq t \right) dt
\]

\[
\leq 4 \int_{0}^{\infty} \mathbb{P} \left( \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i \right| \geq t/2 \right)
\]

\[
+ \mathbb{P} \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i y_i(x_i, w) \right| \geq t/2 \right) dt.
\]
Using the second part of Lemma III.2 we can further estimate

\[ \mu \leq \frac{4\sqrt{8\pi}}{\sqrt{m}} + 4 \int_0^\infty \mathbb{P} \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i y_i(x_i, w) \right| \geq t/2 \right) dt \]

= \frac{4\sqrt{8\pi}}{\sqrt{m}} + 8\mathbb{E} \left( \sup_{w \in K} \left| \left\langle \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i x_i, w \right\rangle \right| \right).

Using the duality \( \| \cdot \|_1^* = \| \cdot \|_\infty \) and the first part of Lemma III.2 we get

\[ \frac{4\sqrt{8\pi}}{\sqrt{m}} + 8R \mathbb{E} \left\| \frac{1}{m} \sum_{i=1}^{m} x_i \right\|_\infty \leq \frac{4\sqrt{8\pi}}{\sqrt{m}} + \frac{8rR \sqrt{2 \log(2d)}}{\sqrt{m}}. \]

2) **Concentration inequalities:** In this subsection we will estimate the probability that \( f_a(w) \) deviates anywhere on \( K \) far from its mean, i.e. the probability

\[ \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq \mu + t \right) \]

for some \( t > 0 \). First we obtain the following modified version of the second part of Lemma 5.1 of [24], cf. also [21, Chapter 6.1].

**Lemma III.5 (Deviation inequality).** Let \( \varepsilon_1, \ldots, \varepsilon_m \) be i.i.d. Bernoulli variables according to \( \text{(III.2)} \) and let the “Standing assumptions” be fulfilled. Then, for \( \mu \in \mathbb{R} \) according to \( \text{(III.1)} \) and any \( t > 0 \), it holds

\[ \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu + t \right) \leq 4\mathbb{P} \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)]_+ \right| \geq t/2 \right). \]

**Proof:** Using Markov’s inequality let us first note

\[ \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu \right) \leq \mathbb{E} \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| = \frac{1}{2}. \]

Using this inequality we get

\[ \frac{1}{2} \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu + t \right) \]

\[ \leq \left( 1 - \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu \right) \right) \cdot \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu + t \right) \]

\[ = \mathbb{P} \left( \forall w \in K : |f_a(w) - \mathbb{E} f_a(w)| < 2\mu \right) \]

\[ \cdot \mathbb{P} \left( \exists w \in K : |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu + t \right). \]

Let \( A_i \) and \( \varepsilon_i \) be again defined by \( \text{(III.2)} \), \( \text{(III.3)} \) and let \( A'_i \) be independent copies of \( A_i \). We further get

\[ \frac{1}{2} \mathbb{P} \left( \sup_{w \in K} |f_a(w) - \mathbb{E} f_a(w)| \geq 2\mu + t \right) \]

\[ \leq \mathbb{P} \left( \forall w \in K : \left| \frac{1}{m} \sum_{i=1}^{m} \left( A_i(w) - \mathbb{E} A(w) \right) \right| < 2\mu \right) \]

\[ \cdot \mathbb{P} \left( \exists w \in K : \left| \frac{1}{m} \sum_{i=1}^{m} \left( A'_i(w) - \mathbb{E} A'(w) \right) \right| \geq 2\mu + t \right) \]

\[ \leq \mathbb{P} \left( \exists w \in K : \left| \frac{1}{m} \sum_{i=1}^{m} \left( A_i(w) - \mathbb{E} A(w) \right) \right| \geq 2\mu + t \right). \]
which yields the claim. 

Combining the Lemmas [III.1] and [III.5] we deduce the following result.

**Lemma III.6.** Under the “Standing assumptions” it holds for $\mu$ and $\tilde{\mu}$ according to [III.1] and [III.7] and any $u > 0$

$$
P \left( \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \geq 2\mu + 2\tilde{\mu} + u \right) 
\leq 8 \left( \exp \left( -\frac{mu^2}{32} \right) + \exp \left( -\frac{mu^2}{32r^2R^2} \right) \right). \tag{III.13}
$$

**Proof:** Applying Lemma [III.5] and Lemma [III.1] we get

$$
P \left( \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \geq 2\mu + 2\tilde{\mu} + u \right) 
\leq 4P \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i [1 - y_i(x_i, w)]_+ \right| \geq \tilde{\mu} + u/2 \right) 
\leq 8P \left( \sup_{w \in K} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i (1 - y_i(x_i, w)) \right| \geq \tilde{\mu} + u/2 \right) 
\leq 8P \left( \left| \sum_{i=1}^{m} \varepsilon_i \right| \geq u/4 \right) 
\quad + 8P \left( \left| \frac{1}{m} \sum_{i=1}^{m} x_i, w \right| \geq \tilde{\mu} + u/4 \right).
$$

Finally, applying the second and third part of Lemma [III.2] this can be further estimated from above by

$$
\leq 8 \left( \exp \left( -\frac{mu^2}{32} \right) + \exp \left( -\frac{mu^2}{32r^2R^2} \right) \right),
$$

which finishes the proof. 

Using the two Lemmas [III.4] and [III.6] we can now prove Theorem [III.1]

**Proof of Theorem [III.1].** Lemma [III.6] yields

$$
P \left( \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \geq 2\mu + 2\tilde{\mu} + u \right) 
\leq 8 \left( \exp \left( -\frac{mu^2}{32} \right) + \exp \left( -\frac{mu^2}{32r^2R^2} \right) \right).
$$

Using Lemma [III.4] we further get

$$
\mu \leq \frac{4\sqrt{8\pi} + 8r R \sqrt{2 \log(2d)}}{\sqrt{m}}.
$$

Invoking the duality $\| \cdot \|_1 = \| \cdot \|_{\infty}$ and the first part of Lemma [III.2] we can further estimate $\tilde{\mu}$ by

$$
\tilde{\mu} = R \mathbb{E} \left\| \frac{1}{m} \sum_{i=1}^{m} x_i \right\|_{\infty} \leq \frac{r R \sqrt{2 \log(2d)}}{\sqrt{m}}.
$$

Hence, with probability at least

$$
1 - 8 \left( \exp \left( -\frac{mu^2}{32} \right) + \exp \left( -\frac{mu^2}{32r^2R^2} \right) \right) 
$$
we have
\[
    \sup_{w \in K} |f_a(w) - \mathbb{E}f_a(w)| \leq 2\mu + 2\tilde{\mu} + u \\
    \leq \frac{8\sqrt{8\pi} + 18rR\sqrt{2\log(2d)}}{\sqrt{m}} + u
\]
as claimed.

B. Estimate of the expected value

In this subsection we will estimate
\[
    \mathbb{E}(f_a(w) - f_a(a)) = \mathbb{E}[1 - y(x, w)]_+ - \mathbb{E}[1 - y(x, a)]_+
\]
for some \( w \in \mathbb{R}^d \setminus \{0\} \) with \( ||w||_1 \leq R \). We will first calculate both expected values separately and later estimate their difference. We will make use of the following statements from probability theory.

Lemma III.7. Let \( a, x \in \mathbb{R}^d \) be according to (II.4), (II.3) and let \( w \in \mathbb{R}^d \setminus \{0\} \). Then it holds

1) \( \langle x, a \rangle, \langle x, \frac{w}{||w||_2} \rangle \sim \mathcal{N}(0, r^2) \),
2) \( \text{Cov}(\langle x, a \rangle, \langle x, w \rangle) = r^2 \langle a, w \rangle \).

Proof: The first statement is well known in probability theory as the 2-stability of normal distribution. For the second statement we get
\[
    \text{Cov}(\langle x, a \rangle, \langle x, w \rangle) = \mathbb{E}(\langle x, a \rangle \langle x, w \rangle) = \sum_{i,j=1}^{d} a_i w_j \mathbb{E}(x_i x_j)
\]
\[
    = r^2 \sum_{i=1}^{d} a_i w_i = r^2 \langle a, w \rangle
\]
as claimed.

It is very well known, cf. [14, Corollary 5.2], that projections of a Gaussian random vector onto two orthogonal directions are mutually independent.

Lemma III.8. Let \( x \sim \mathcal{N}(0, \text{Id}) \) and let \( a, b \in \mathbb{R}^d \) with \( \langle a, b \rangle = 0 \). Then \( \langle x, a \rangle \) and \( \langle x, b \rangle \) are independent random variables.

Applying these two lemmas to our case we end up with the following lemma.

Lemma III.9. For \( a \in \mathbb{R}^d \) according to (II.4), \( x \sim \mathcal{N}(0, r^2 \text{Id}) \) and \( w \in \mathbb{R}^d \) we have
\[
    \langle x, w \rangle = c \langle x, a \rangle + c' Z
\]
for some \( Z \sim \mathcal{N}(0, r^2) \) independent of \( \langle x, a \rangle \) and
\[
    c = \langle a, w \rangle, \quad c' = \sqrt{||w||_2^2 - c^2}. \quad (\text{III.14})
\]

Remark III.10. Note that \( c' \) is well defined, since \( c^2 \leq ||w||_2^2 ||a||_2^2 = ||w||_2^2 \).

Proof: If \( c' = 0 \), the statement holds trivially. If \( c' \neq 0 \), we set
\[
    Z = \frac{1}{c'} (\langle x, w \rangle - c \langle x, a \rangle) = \frac{1}{c'} \sum_{i=1}^{d} x_i (w_i - ca_i).
\]
Hence, \( Z \) is indeed normally distributed with \( \mathbb{E}(Z) = 0 \) and \( \text{Var}(Z) = r^2 \). It remains to show that \( Z \) and \( \langle x, a \rangle \) are independent. We observe that
\[
    \langle a, w - ca \rangle = \langle a, w \rangle - \langle a, w \rangle ||a||_2 = 0
\]
and, finally, Lemma III.8 yields the claim.

Lemma III.11. Let \( a \in \mathbb{R}^d \) and \( f_a: \mathbb{R}^d \rightarrow \mathbb{R} \) be according to (II.4), (II.7). Then it holds

1) \( \mathbb{E}f_a(a) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} [1 - r|t|]_+ e^{-t^2} dt \),
2) \( \mathbb{E}f_a(w) = \frac{1}{2\pi} \int_{\mathbb{R}^d} [1 - cr|t| - c'r^2t^2]_+ e^{-\frac{t^2 - r^2}{2}} dt_1 dt_2 \), where \( c \) and \( c' \) are defined by (III.14).

Proof:
1) Let $\omega \sim \mathcal{N}(0,1)$ and use the first part of Lemma [III.7] to obtain
\[
\mathbb{E} f_a(a) = \mathbb{E}[1 - |\langle x, a \rangle|]_+ = \mathbb{E}[1 - r|\omega|]_+ = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [1 - rt]_+ e^{-\frac{r^2}{2}} dt.
\]

2) Using the notation of Lemma [III.9] we get
\[
\mathbb{E} f_a(w) = \mathbb{E}[1 - \text{sign}(\langle x, a \rangle)\langle x, w \rangle]_+ = \mathbb{E}[1 - c\langle x, a \rangle - c\text{sign}(\langle x, a \rangle)Z]_+ = \mathbb{E}[1 - c\langle x, a \rangle - c'Z]_+ = \frac{1}{2\pi} \int_{\mathbb{R}^2} [1 - cr|t_1| - c'rt_2]_+ e^{-\frac{r^2}{2}} dt_1 dt_2.
\]

Using this result we now can prove Theorem [II.2]

**Proof of Theorem [II.2]**: Using Lemma [III.11] we first observe
\[
-\pi \mathbb{E} f_a(a) = \frac{\sqrt{\pi}}{\sqrt{2}} \int_{\mathbb{R}} [1 - rt]_+ e^{-\frac{r^2}{2}} dt \quad (\text{III.15})
\]
\[
= -\sqrt{2\pi} \int_{0}^{\frac{1}{c'}} (1 - rt) e^{-\frac{r^2}{2}} dt \geq -\sqrt{2\pi} \frac{1}{c'}.
\]
To estimate the expected value of $f_a(w)$ we now distinguish the two cases $c \leq 0$ and $c > 0$.

1) $c \leq 0$: In that case we get
\[
\pi \mathbb{E} f_a(w) = \int_{\mathbb{R}^2} \int_{0}^{\infty} [1 - crt_1 - c'rt_2]_+ e^{-\frac{r^2}{2}} dt_1 dt_2.
\]
Since $-crt_1 \geq 0$ for $0 \leq t_1 < \infty$ we can further estimate
\[
\pi \mathbb{E} f_a(w) \geq \int_{\mathbb{R}} \int_{0}^{\infty} [1 - c'rt_2]_+ e^{-\frac{r^2}{2}} dt_1 dt_2 \geq \int_{-\infty}^{0} \int_{0}^{\infty} (1 - c'rt_2) e^{-\frac{r^2}{2}} dt_1 dt_2 = \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\frac{r^2}{2}} dt_1 dt_2 + c'r \int_{0}^{\infty} t_2 e^{-\frac{r^2}{2}} dt_1 dt_2 = \frac{\pi}{2} + c'r \frac{\sqrt{\pi}}{\sqrt{2}}.
\]
As claimed, putting both terms together, we arrive at
\[
\pi \mathbb{E} f_a(w) - f_a(a) \geq \frac{\pi}{2} + c'r \frac{\sqrt{\pi}}{\sqrt{2}} - \frac{\sqrt{2\pi}}{c'}.
\]

2) $c > 0$: First let us observe that $1 - crt_1 - c'rt_2 \geq 0$ on $[0, 1/cr] \times (-\infty, 0] \subset \mathbb{R}^2$. Hence, we get
\[
\pi \mathbb{E} f_a(w) = \int_{\mathbb{R}^2} [1 - crt_1 - c'rt_2]_+ e^{-\frac{r^2}{2}} dt_2 dt_1 \geq \int_{0}^{\frac{1}{c'}} \int_{-\infty}^{0} (1 - crt_1 - c'rt_2) e^{-\frac{r^2}{2}} dt_1 dt_2 = \int_{0}^{\frac{1}{c'}} (1 - crt)e^{-\frac{r^2}{2}} dt + c'r \int_{0}^{\frac{1}{c'}} e^{-\frac{r^2}{2}} dt \geq \frac{\sqrt{\pi}}{\sqrt{2}} \int_{0}^{\frac{1}{c'}} (1 - crt)e^{-\frac{r^2}{2}} dt + \frac{c'}{c} \exp \left( -\frac{1}{2c^2r^2} \right).
Combining this estimate with (III.15) we arrive at
\[
\pi E(f_a(w) - f_a(a)) \geq \frac{\sqrt{\pi}}{\sqrt{2}} \int_0^\frac{1}{\sqrt{r}} (1 - c \tau) e^{-\frac{\tau^2}{2}} d\tau + \frac{c'}{c} \exp \left( -\frac{1}{2c^2 r^2} \right) \frac{\sqrt{2\pi}}{r}.
\]

**IV. \ell_1$$\cdot$$SVM with additional \ell_2$$\cdot$$constraint**

A detailed inspection of the analysis done so far shows that it would be convenient if the convex body $K$ would not include vectors with large $\ell_2$-norm. For example, in (III.10) we needed to calculate $\sup_{w \in K} \|w\|_2^2 = R^2$, although the measure of the set of vectors in $K$ with $\ell_2$-norm close to $R$ is extremely small.

Therefore, we will modify the $\ell_1$$\cdot$$SVM (II.3) by adding an additional $\ell_2$-constraint, that is instead of (II.3) we consider the optimization problem

\[
\min_{w \in \mathbb{R}^d} \sum_{i=1}^m [1 - y_i(x_i, w)]_+ \text{ s.t. } \|w\|_1 \leq R \text{ and } \|w\|_2 \leq 1.
\]

The combination of $\ell_1$ and $\ell_2$ constraints is by no means new - for example, it plays a crucial role in the theory of elastic nets [32]. Furthermore, let us remark that the set

\[
\tilde{K} = \{ w \in \mathbb{R}^d \mid \|w\|_1 \leq R \text{ and } \|w\|_2 \leq 1 \}
\]

appears also in [24]. We get $\tilde{K} \subset K$ with $K$ according to (II.6). Hence, Theorem II.1 and (II.8) still remain true if we replace $K$ by $\tilde{K}$ and we obtain

\[
\sup_{w \in \tilde{K}} |f_a(w) - E f_a(w)| \leq \frac{8\sqrt{8\pi} + 18R\sqrt{2\log(2d)}}{\sqrt{m}} + u
\]

with high probability and

\[
E(f_a(\hat{a}) - f_a(a)) \leq 2 \sup_{w \in K} |f_a(\hat{a}) - f_a(a)|,
\]

where $\hat{a}$ is now the minimizer of (IV.1).

It remains to estimate the expected value $E(f_a(w) - f_a(a))$ in order to obtain an analogue of Theorem II.3 for (IV.1), which reads as follows.

**Theorem IV.1.** Let $d \geq 2, 0 < \varepsilon < 1/2, r > 2\sqrt{2\pi}(1 - 2\varepsilon)^{-1}, a \in \mathbb{R}^d$ according to (II.4), $m \geq C\varepsilon^{-2}r^2R^2\log(d)$ for some constant $C$, $x_1, \ldots, x_m \in \mathbb{R}^d$ according to (II.3) and $\hat{a} \in \mathbb{R}^d$ a minimizer of (IV.1). Then it holds

\[
\|a - \hat{a}\|_2^2 \leq \frac{C'\varepsilon}{r(1 - \exp \left( -\frac{r^2\log(d)}{16} \right))}
\]

with probability at least

\[
1 - \gamma \exp \left( -C''\log(d) \right)
\]

for some positive constants $\gamma, C', C''$.

**Remark IV.2.**

1) As for Theorem II.3 we can write down the expressions explicitly, i.e. without the constants $\gamma, C, C'$ and $C''$. That is, taking $m \geq 4\varepsilon^{-2} \left(8\sqrt{8\pi} + (18 + t)rR\sqrt{2\log(2d)}\right)$ for some $t > 0$, we get

\[
\|a - \hat{a}\|_2^2 \leq \frac{\sqrt{\pi/2} \varepsilon}{r \left(1 - \exp \left( -\frac{r^2\log(2d)}{16} \right) \right)},
\]

with probability at least

\[
1 - 8 \left( \exp \left( -\frac{r^22^22^2\log(2d)}{16} \right) + \exp \left( -\frac{\pi^22^2\log(2d)}{16} \right) \right).
\]

2) The main advantage of Theorem IV.1 compared to Theorem II.3 is that the parameter $r$ does not need to grow to infinity. Actually, (IV.3) is clearly not optimal for large $r$. Indeed, if (say) $\varepsilon < 0.2$, we can take $r = 10$, and obtain

\[
\|a! - \hat{a}\|_2^2 \leq C'\varepsilon
\]
for \( m \geq C_2^{-2}R^2\log(d) \) with high probability.

**Proof:** As in the proof of Theorem II.3 we first obtain 
\[ c' = \sqrt{2}c_0 \geq \langle a, \hat{a} \rangle > c = \langle a, \hat{a} \rangle > 0. \]
Using Lemma III.11 we get
\[
\pi \mathbb{E}( f_{\alpha}(w) - f_{\alpha}(a)) \geq \int_{0}^{\frac{1}{2} \sqrt{\frac{1}{m}} - c} \int_{\mathbb{R}} \left( (1 - cr_1 - c'rt_2) - (1 - rt_1) \right) e^{-t_2^2} dt_2 dt_1 \]
\[ = r(1 - c)\sqrt{2\pi} \int_{0}^{\frac{1}{2} \sqrt{\frac{1}{m}} - c} te^{-t^2} dt \]
with
\[ 1 - c = 1 - \langle a, \hat{a} \rangle \geq \frac{1}{2} \left( \|a\|_2^2 + \|\hat{a}\|_2^2 \right) - \langle a, \hat{a} \rangle = \frac{1}{2} \|a - \hat{a}\|_2^2. \]

The claim now follows from (IV.4) and (IV.3). \( \blacksquare \)

**V. Numerical experiments**

We performed several numerical tests to exhibit different aspects of the algorithms discussed above. In the first two parts of this section we fixed \( d = 1000 \) and set \( \hat{a} \in \mathbb{R}^d \) with 5 nonzero entries: \( \hat{a}_{10} = 1, \hat{a}_{140} = -1, \hat{a}_{234} = 0.5, \hat{a}_{360} = -0.5, \hat{a}_{780} = 0.3 \). Afterwards we normalized \( \hat{a} \) and set \( a = \hat{a}/\|\hat{a}\|_2 \) and \( R = \|a\|_1 \).

**A. Dependency on \( r \)**

We run the \( \ell_1 \)-SVM (II.5) with \( m = 200 \) and \( m = 400 \) for different values of \( r \) between zero and 1.5. The same was done for the \( \ell_1 \)-SVM with the additional \( \ell_2 \)-constraint (IV.1), which is called \( \ell_{1,2} \)-SVM in the legend of the figure. The average error of \( n = 20 \) trials between \( a \) and \( \hat{a}/\|\hat{a}\|_2 \) is plotted against \( r \). We observe that especially for small \( r \)'s the \( \ell_1 \)-SVM with \( \ell_2 \)-constraint performs much better than classical \( \ell_1 \)-SVM.

![Figure 1. Dependency on \( r \)](image)

**B. Dependency on \( m \) and comparison with 1-Bit CS**

In the second experiment, we run \( \ell_1 \)-SVM with and without the extra \( \ell_2 \)-constraint for two different values of \( r \), namely for \( r = 0.75 \) and for \( r \) depending on \( m \) as \( r = \sqrt{m}/30 \). We plotted the average error of \( n = 40 \) trials for each value. The last method used is 1-bit Compressed Sensing (24), which is given as the maximizer of
\[
\max_{w \in \mathbb{R}} \sum_{i=1}^{m} y_i \langle x_i, w \rangle \quad \text{subject to} \quad \|w\|_2 \leq 1, \|w\|_1 \leq R. \quad (V.1)
\]

Note that maximizer of (V.1) is independent of \( r \), since it is linear in \( x_i \). First, one observes that the error of \( \ell_1 \)-SVM does not converge to zero if the value of \( r = 0.75 \) is fixed. This is in a good agreement with Theorem II.3 and the error estimate II.9. This drawback disappears when \( r = \sqrt{m}/30 \) grows with \( m \), but \( \ell_1 \)-SVM still performs quite badly. The two versions of \( \ell_{1,2} \)-SVM perform essentially better than \( \ell_1 \)-SVM, and slightly better than 1-bit Compressed Sensing.
C. Dependency on $d$

In figure 3 we investigated the dependency of the error of $\ell_1$-SVM on the dimension $d$. We fixed the sparsity level $s = 5$ and for each $d$ between 100 and 3000 we draw an $s$-sparse signal $a$ and measurement vectors $x_i$ at random. Afterwards we run the $\ell_1$-SVM with the three different values $m = m_1 \log(d)$ with $m_1 = 10$, $m_2 = 20$ and $m_3 = 40$. We plotted the average errors between $a$ and $\hat{a}/\|\hat{a}\|_2$ for $n = 60$ trials.

We indeed see that to achieve the same error, the number of measurements only needs to grow logarithmically in $d$, explaining once again the success of $\ell_1$-SVM for high-dimensional classification problems.

VI. DISCUSSION

In this paper we have analyzed the performance of $\ell_1$-SVM in recovering sparse classifiers. Theorem 2 shows, that a good approximation of such a sparse classifier can be achieved with small number of learning points $m$ if the data is well spread. The geometric properties of well distributed learning points are modelled by independent Gaussian vectors with growing variance $r$ and it would be interesting to know, how $\ell_1$-SVM performs on points chosen independently from other distributions. The number of learning points needs to grow logarithmically with the underlying dimension $d$ and linearly with the sparsity of the classifier. On the other hand, the optimality of the dependence of $m$ on $\varepsilon$ and $r$ remains open. Another important question left open is the behavior of $\ell_1$-SVM in the presence of misclassifications, i.e. when there is a (small) probability that the signs $y_i \in \{-1, +1\}$ do not coincide with $\text{sign}(\langle x_i, a \rangle)$. Finally, we proposed a modification of $\ell_1$-SVM by incorporating an additional $\ell_2$-constraint.

ACKNOWLEDGMENT

We would like to thank A. Hinrichs, M. Omelka, and R. Vershynin for valuable discussions.

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