Rotating fermions in two dimensions: Thomas Fermi Approach

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Abstract

Properties of confined mesoscopic systems have been extensively studied numerically over recent years. We discuss an analytical approach to the study of finite rotating fermionic systems in two dimension. We first construct the energy functional for a finite fermionic system within the Thomas-Fermi approximation in two dimensions. We show that for specific interactions the problem may be exactly solved. We derive analytical expressions for the density, the critical size as well as the ground state energy of such systems in a given angular momentum sector.
I. INTRODUCTION

The observation of Bose-Einstein Condensation (BEC) in extremely cold dilute gas of atoms has generated a lot of activity both on experimental as well as theoretical front on the properties of trapped dilute gases. There is now a renewed focus on the properties of trapped dilute gas of fermionic atoms at low temperatures. A recent effort in this direction is the experimental observation of quantum degeneracy in a dilute gas of trapped fermionic atoms [1]. Several theoretical papers have studied the properties of trapped dilute gas of fermionic atoms. Butts and Rokshar [2] have studied the momentum and spatial distribution of the noninteracting system in the Thomas-Fermi approximation. The ground state properties and the addition energy spectra of a two-dimensional interacting fermi system has been studied by Sinha et al [3]. Recently collective excitations of the system in the normal phase [4] and in the superfluid phase [5] have also been investigated. The thermodynamic properties of fermi gases including rotations has been analysed by Salasnich et al [6].

Especially relevant to this paper is the recent interest in the properties of rotating bose condensates [7] and the experimental observation of vortices in stirred BEC of $^{87}$Rb [8]. The properties of ground and low excited states of a rotating, weakly interacting Bose-Einstein condensate in a harmonic trap has been investigated recently in various limits [9,10]. Ho and Ciobanu [11] have recently examined theoretically the nature the ground state of a rotating trapped fermi gas in two and three dimensions in the noninteracting limit. They show that the density profile acquires features reflecting the underlying Landau level like energy spectrum. Properties of the spectrum of such a system had been earlier investigated by Bhaduri et al [12].

In this paper we investigate a system of rotating fermions in two dimensions confined within a parabolic potential with some model interactions. We show that within the Thomas-Fermi approximation the system may be solved exactly and discuss the consequent results. We follow closely the formalism developed by Gallego and Das Gupta [13] who developed the Thomas-Fermi method for rotating nuclei. The attractive feature of Thomas-Fermi
method is the ease with which non-trivial many-body solutions can be obtained even when they are interacting. As we show later, in some cases the results may be obtained purely by analytical methods. While this is not a full list of the exactly solvable models, they are chosen for their relevance to physical systems. The paper is organised as follows. In section II we outline the derivation of the Thomas Fermi energy for a system of interacting rotating fermions in parabolic confinement. In section III we discuss the results for three specific type of interactions between the fermions. We obtain analytic expression and present numerical results for the interaction energy, the rotational energy and the spatial density in a given angular momentum sector for a given number of particles. The last section contains a discussion of the results.

II. THOMAS FERMI ENERGY FUNCTIONAL

We derive the Thomas-Fermi(TF) energy functional for a confined two dimensional rotating fermi gas starting from the microscopic Hamiltonian. We shall follow the approach outlined by Gallego and Dasgupta [13] who derived the TF density functional for rotating nuclei, but applied to a two dimensional system.

The microscopic Hamiltonian of a two-dimensional rotating fermi gas confined in a harmonic potential is given by,

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} V(\vec{r}_i, \vec{r}_j) + \sum \frac{1}{2} m \omega_0^2 r_i^2 - \sum m \omega [\vec{r}_i \times \vec{p}_i], \] (1)

where \( V(\vec{r}_i, \vec{r}_j) \) denotes the two body interaction between fermions, \( \omega_0 \) and \( \omega \) denote the confinement and rotational frequencies respectively.

We want to minimise the energy of this system subject to the constraints

\[ \int d\vec{r} \rho(\vec{r}) = N, \] (2)
\[ \int d\vec{r} d\vec{p} f(\vec{r}, \vec{p})(\vec{r} \times \vec{p}) = L, \] (3)
where \( N \) and \( L \) are total particle number and the total Angular momentum, \( f(\vec{r}, \vec{p}) \) is the semiclassical phase space density which we shall define later. The configuration space density \( \rho \) is obtained by integrating the phase space density over the momentum space.

Using Lagrange multiplier \( \mu \) and \( \omega \) for each constraint respectively and regrouping the momentum dependent term we obtain the following expression for the energy functional

\[
E[\rho] = \int d\vec{r}d\vec{p}f(\vec{r}, \vec{p})\left[\frac{\vec{p}^2}{2m} - \omega(xp_y - yp_x)\right] + \int d\vec{r}[V(\vec{r}) - \mu \rho(\vec{r})] + \mu N + \omega L
\]

where the mean-field one body interaction \( V(r) \) is

\[
V(r) = \frac{1}{2}m\omega_0^2 r^2 + V_H(\vec{r}).
\]

The Hartree term is given by,

\[
V_H(\vec{r}) = \int d^2r V(\vec{r}, \vec{r}’)\rho(\vec{r}’).
\]

We ignore the exchange correction as it is in general subdominant compared to the Hartree term.

The semiclassical phase space density is given by

\[
f(\vec{r}, \vec{p}) = \frac{2}{(2\pi\hbar)^2} \Theta(\mu - \epsilon(\vec{p}, \vec{r}))
\]

where we have accounted for a factor of 2 due to spin degeneracy and

\[
\epsilon(\vec{p}, \vec{r}) = \frac{\vec{p}^2}{2m} + V(\vec{r}) - \omega(xp_y - yp_x)
\]

denotes the energy density in phase space. We may rewrite this by completing the square as,

\[
\epsilon(\vec{p}, \vec{r}) = \frac{1}{2m}[(p_x + m\omega y)^2 + (p_y - m\omega x)^2] + V(\vec{r}) - \frac{1}{2}m\omega^2 r^2.
\]

The shifted momentum simply indicates that the center of the Fermi sphere at any given point \( \vec{r} \) in the rotating frame is displaced from the usual \( \vec{p} = 0 \). This shift is of no relevance.
within the classical TF approximation except for the appearance of the extra centrifugal term in the energy functional.

The TF energy functional is therefore given by,

$$E[\rho] = \int d^2 r \left( \frac{\pi \hbar^2 \rho^2}{2m} + \frac{1}{2} m \Omega r^2 + V_H - \mu \rho(\vec{r}) + \mu N + \omega L \right)$$

(10)

where

$$\Omega^2 = \omega_0^2 - \omega^2 = C \omega^2$$

(11)

denotes the effective frequency in the presence of rotations. The TF equation for the spatial density is obtained by a variation of the energy functional, namely,

$$\frac{\pi \hbar^2 \rho^2}{m} + \frac{1}{2} m \Omega^2 r^2 + V_H = \mu$$

(12)

To obtain the ground state density in TF approximation one has to solve the above equation self-consistently with the boundary condition that the density vanishes beyond the classical turning point \( r_0 \), that is

$$\rho(r) = 0, \quad r \geq r_0.$$  

(13)

Note that in the absence of rotations \( \Omega = \omega_0 \) and the solution of the above equation describes the ground state in the absence of rotations which has been analysed in detail before [3].

We may remark that in general the above equation is difficult to solve analytically except in specific cases which we shall discuss later.

The angular momentum carried by the system is given by,

$$L = \int d^2 r d^2 p f(\vec{r}, \vec{p})(xp_y - yp_x) = m \omega \int d^2 r \rho(\vec{r}) r^2$$

(14)

which is simply the classical expression for the angular momentum. We may therefore identify the Lagranges multiplier \( \omega \) with the angular frequency.

We may also define the related energy functionals. The free energy of the system is given by,
\[ F = E - \mu N - \omega L \] (15)

The TF equation (12) may also be obtained just by varying the free energy. We also define the energy in the rotating frame through

\[ E = E' + \omega L. \] (16)

where \( E \) is the total energy in the static frame. With the help of eq. (12) it can be shown that irrespective of the nature of the interaction \( E' \) is given by,

\[ E' = \frac{1}{2} \mu N + \frac{1}{4} m \Omega^2 \int d^2 r \; r^2 \rho. \] (17)

This concludes the basic TF formalism applied to rotating fermionic systems. In the following section we solve the TF equation for specific cases. Note that it is necessary that we choose \( \omega \leq \omega_0 \) as otherwise the system becomes unstable.

### III. SOME EXACTLY SOLVABLE MODELS

We now discuss special cases of interacting fermi gases which may be solved exactly. These models are chosen not only to facilitate analytical calculations, but also because in some limits these approximate the effective electron-electron interaction in a two-dimensional fermionic system. We note that the effective interaction may be dependent on the device characteristics as well as the number of particles. As such we find that it is useful to work with different approximate forms of interactions. Throughout we assume the system is spin unpolarized. The partially and fully polarized system in the presence of magnetic field will be discussed in a subsequent paper.

#### A. Contact Interaction

The first and the simplest is the contact interaction. In general any short-range interaction may be written in terms of a gradient expansion. The leading term in the expansion
is always the contact interaction of the form

\[ V(\vec{r}) = V_0 \delta(\vec{r}), \]

where \( \vec{r} \) denotes the relative coordinate between any two fermions. In a gradient expansion the factor \( V_0 \) is related to the leading moment of the potential \cite{13,16}.

The free-energy is then given by,

\[ F[\rho] = \int d^2 r \left[ \frac{\hbar^2 \pi \rho^2}{2m} + \frac{1}{2} m \Omega^2 \rho(\vec{r}) + \frac{V_0}{2} \rho^2 \right] - \mu \int d^2 r \rho(\vec{r}) \] (18)

The TF equation for the spatial density then becomes

\[ \frac{\hbar^2 \pi}{m} (1 + g) \rho(\vec{r}) + \frac{1}{2} m \Omega^2 r^2 = \mu, \] (19)

where \( g = \frac{V_0 m}{\pi \hbar^2} \) which is dimensionless. Note that eq.(19) is identical in form to the case when there is no rotation. Only the effective frequency \( \Omega \) is different now. The solutions are therefore obtained easily by merely replacing the confinement frequency by the effective frequency. Since the density vanishes beyond a turning point \( r_0 \), the chemical potential is related to the turning point by,

\[ \mu = \frac{1}{2} m \Omega r_0^2. \] (20)

The turning point and hence the chemical potential are therefore determined by the total number of particles \( N \) which is fixed.

Substituting it back in the TF equation gives the expression for density

\[ \rho(r) = \frac{m^2 \Omega^2 (r_0^2 - r^2)}{2 \pi \hbar^2 (1 + g)} \] (21)

The total number of particles is then obtained by integrating the expression for density in eq.(21)

\[ N = \int_0^{r_0} d^2 r \rho(r) = \frac{m^2 \Omega^2 r_0^4}{4 \hbar^2 (1 + g)} \] (22)
and the total angular momentum is given by

\[ L = \int_0^{r_0} d^2r m \omega r^2 \rho(r) = \frac{m^3 \omega \Omega^2 r_0^6}{12\hbar^2(1 + g)} \]  

(23)

In a system where both \( N \) and \( L \) are fixed, the turning point \( r_0 \) and \( \omega \) are determined by entirely by these two constraints.

Using the expression for density and the total angular momentum \( L \) in eq.(23) we can easily evaluate the expression of the energy. Some straightforward algebra immediately yields,

\[ E = \hbar \omega_0 \sqrt{L^2 + \frac{4}{9}(1 + g)N^3} \]  

(24)

Note that when \( L = 0 \) and \( g = 0 \) this is the standard result for TF energy of the ground state of a fermionic system in two dimension in an oscillator confinement. The effect of interaction is to scale the density and hence the energy by a factor which involves \((1 + g)\) and the effect of rotations is contained in the \( L \) dependent term.

The interaction energy alone may be separated, by computing the interaction term in the density functional using the TF solution for the spatial density. We have therefore,

\[ E_{int} = \hbar \omega_0 \frac{2N^3g}{9\sqrt{L^2 + \frac{4}{9}N^3(1 + g)}}. \]  

(25)

Note that with the increase in angular momentum the effect of interaction decreases as it should.

A few general remarks are in order here. In the actual many body system, there will be in general a tower of states in a given angular momentum sector. However, as is obvious from the expression for the total energy the TF method gives the lowest energy state or the ground state in each angular momentum sector. This is easily seen by taking the limit \( L >> N \), when the energy is simply \( \hbar \omega_0 L \) which is the energy obtained without any radial excitations. These states are often referred to as Yrast states in the literature.

We briefly discuss the case with an arbitrary but very short range interaction. The leading term, as remarked before, is simply the contact term discussed above. The corrections
may be systematically worked out using the gradient expansion method $^{[15,16]}$. Below, we mention the results from the next to leading order correction. Again the corrections may be obtained analytically. The results are given as corrections to the results obtained with contact interaction alone.

The next higher order correction to the contact interaction in the gradient expansion of a short range potential is of the form $^{[16]}$,

$$V_{SR} = -V_2 \nabla^2 \delta(\vec{r}) \tag{26}$$

where $V_2$ is related to the second moment of the interaction potential. Note that in the gradient expansion only even moments appear in each term. The Thomas Fermi free energy functional is given by

$$F[\rho] = \int d^2 r \left[ \frac{\hbar^2 \pi \rho^2}{2m} + \frac{1}{2} m \Omega r^2 \rho(\vec{r}) + \frac{V_0}{2} \rho^2 - \frac{V_2}{2} \nabla^2 \rho \right] - \mu \int d^2 r \rho(\vec{r}) \tag{27}$$

Variation with respect to density gives the following self consistent equation for the density

$$-V_2 \nabla^2 \rho(\vec{r}) + \frac{\hbar^2 \pi}{m} (1 + g) \rho(\vec{r}) + \frac{1}{2} m \Omega^2 r^2 = \mu, \tag{28}$$

It can be easily seen that for $V_2 = 0$ eq. (28) reduces to eq. $^{[19]}$.

The circularly symmetric solutions of the above inhomogeneous partial differential equation can be found easily. Replacing $V_2$ with the dimensionless coupling $g'$, where $g' = \frac{V_2 m^2 \omega_0}{\pi \hbar^3}$ we can write the solution as

$$\rho = \rho_c + \rho_{cr} \tag{29}$$

where $\rho_c$ is given by eq.$^{[21]}$ with the contact term alone and $\rho_{cr}$ is the extra term which can be written in terms of modified Bessel functions

$$\rho_{cr} = \frac{2}{g' k^4} \left( 1 - \frac{\omega^2}{\omega_0^2} \right) \left[ \frac{I_0(kx)}{I_0(kx_0)} - 1 \right] \tag{30}$$

where
Here we have used the dimensionless variable \( x = \frac{r}{l_0} \) with oscillator length scale \( l_0 = \frac{\hbar}{m\omega_0} \) as the unit of length. Here after we use the subscript "c" to denote the contribution from the contact term alone.

The corrections to various other relevant quantities may be easily found with the density. The total no. of particles \( N \) can be similarly given by the following expression.

\[
N = N_c + \frac{2}{g'k^4} \left(1 - \frac{\omega^2}{\omega_0^2}\right) \left[ \frac{2I_1(kx_0)}{kI_0(kx_0)} - x_0^2 \right] \quad (32)
\]

The total angular momentum \( L \) can also be written as

\[
L = L_c + \frac{4\hbar}{g'k^4} \left(1 - \frac{\omega^2}{\omega_0^2}\right) \frac{\omega}{\omega_0} \left[ \frac{1}{k} \left[ x_0(x_0^2 + 4) \frac{I_1(kx_0)}{I_0(kx_0)} - 2x_0^2 \right] - \frac{x_0^4}{4} \right] \quad (33)
\]

The energy is again given by

\[
E = \frac{\mu N}{2} + \left[1 + \frac{\omega_0^2}{4\omega^2} \left(1 - \frac{\omega^2}{\omega_0^2}\right)\right] \omega L \quad (34)
\]

The interaction energy is given by

\[
E_{\text{int}} = E - \frac{1}{2} \left(2 + \sqrt{1 - \frac{\omega^2}{\omega_0^2}}\right) \omega L - \hbar \omega_0 \int d^2 r \rho^2. \quad (35)
\]

We note that the last term which involves the second moment of the density can be expressed in terms the modified Bessel functions.

B. Logarithmic Interaction

We now consider the case where two fermions interact via a logarithmic interaction. For quantum dot systems this is close to being realistic interaction at short distances where as the interaction is varies inversely as the distances at long distances [18]. The two dimensional TF atom with logarithmic interactions was solved earlier by Bhaduri et. al. [17] and was applied to quantum dots recently by Sinha et. al. [3] in order to explain the shell effect in
quantum dots. The extension of these results for the rotating case is quite straightforward. We give the main results below:

The TF mean field is given by

\[ V(r) = \frac{1}{2} m \Omega^2 r^2 - e_f^2 \int d^2 r' \rho(\vec{r}') \ln\left(\frac{\left|\vec{r} - \vec{r}'\right|}{a}\right), \]

where \( a \) is an arbitrary parameter with the dimension of length. In order that the interaction is repulsive it is essential that the parameter \( a \) is much larger than any other length scale in the problem. The largest length scale in the problem is the turning point and we set \( a = r_0 \). While this may seem arbitrary, in the analysis of the ground state of many fermions this choice is justified by a numerical analysis of the data on addition spectrum [3]. The effective strength of the interaction is denoted by \( e_f^2 \). Unlike in the three-dimensional Coulomb case, this parameter is not dimensionless, but has the dimensions of energy. The only change from the analysis of the ground state in the absence of rotations is in the first term where the effective frequency is different from the confinement frequency. We therefore only give essential results below and details may be looked up in Ref. [3].

The spatial density is given by the solution of the differential equation

\[ \frac{\hbar^2 \pi}{2m} \nabla^2 \rho(\vec{r}) = \pi e_f^2 \rho(\vec{r}) - 2m \Omega^2 \]

(37)

For a circularly symmetric spatial distribution, the above equation is easily solved [3]. The solutions are given by the modified Bessel function as,

\[ \rho(x) = \frac{2b^2}{l_0^4} \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} \left[ 1 - \frac{I_0(x)}{I_0(x_0)} \right], \]

(38)

where we have again introduced the dimensionless variables,

\[ x = \frac{r}{b}, \quad x_0 = \frac{r_0}{b}, \]

(39)

with \( b = \frac{\hbar^2}{2me_f^2} \) has the dimension of length with \( x_0 \) is again the classical turning point in units of \( b \).

The total particle number can then be obtained by integrating this density
\[ N = \int_0^{2\pi} \int_0^{r_0} d\mathbf{r} \rho(\mathbf{r}) = \frac{4b^4}{l_0^4} \frac{1}{(1 - \frac{\omega^2}{\omega_0^2})} \left[ \frac{x_0^2}{2} - \frac{x_0 I_1(x_0)}{I_0(x_0)} \right] \]  

(40)

where \( l_0^2 = \frac{\hbar}{m\omega_0} \)

Since \( N \) is fixed, the turning point \( x_0 \) can be determined from this equation. Similarly one can calculate the total angular momentum. After some straightforward algebra it can be written as

\[ \frac{L}{\hbar} = \frac{b^2}{l_0^2} \frac{\omega}{\omega_0} [N(x_0^2 + 4) - \frac{b^4 x_0^4 \omega^2}{l_0^4} \frac{\omega^2}{\omega_0^2} (1 - \frac{\omega^2}{\omega_0^2})] \]  

(41)

The energy is given as

\[ E = \mu \left( \frac{N}{2} + \left[ 1 + \frac{1}{4} \frac{\omega^2}{\omega_0^2} (1 - \frac{\omega^2}{\omega_0^2}) \right] \omega L \right) \]  

(42)

Therefore at a given \( N \) and \( L \) the constants \( r_0 \) and \( \omega \) are determined by the above set of equations. This is in general done numerically unlike in the case of contact interaction in order to express the energy as a function of \( N \) and \( L \). The interaction energy also can be written in a closed form and is given by the following expression.

\[ \frac{E_{\text{int}}}{\hbar \omega_0} = \frac{b^2}{l_0^2} (1 - \frac{\omega^2}{\omega_0^2}) [x_0^2 - 2N - 2 \frac{b^4}{l_0^4} \frac{\omega^2}{\omega_0^2} \frac{I_1^2(x_0)}{I_0^2(x)}] \]  

(43)

IV. DISCUSSION OF RESULTS

We have outlined an analytical procedure based on the simple Thomas-Fermi approach to the many body problem of rotating fermions in two dimensions. We have derived the density, the angular momentum as well the energy for model two body interactions. For simplicity we have chosen the contact interaction and logarithmic interaction. While these may not be the exact form of the interaction in the two dimensional systems they may under different circumstances approximately reflect the true nature of the interactions.

We further illustrate these analytical results with numerical computation of quantities of interest by taking \( N = 50 \) which is reasonably large for TF results to be valid. In Fig. 1 we have plotted the change in angular momentum as a function of rotational frequency.
measured in units of the confinement frequency. The curves represent the behaviour of $L$ with the two model interactions. While in the case of contact interaction the relationship is almost linear all the way up to 0.6, the non-linear dependence of $L$ on $\omega$ builds up much earlier in the case of logarithmic interaction. From eq. (23) and eq. (22) it is easy to see that

$$\frac{\omega^2}{1 - \frac{\omega^2}{\omega_0^2}} \propto L^2/N^3,$$

in the case of contact interaction. Hence the nonlinearity sets in only when the ratio between rotational frequency and the confinement frequency is considerable.

As seen in Figs. 2 and 3, the total and interaction energy are sensitive to the behaviour of $L$ as a function of $\omega$. We have shown the ratio of total energy with the same in the absence of rotations in Fig. 2. As expected the energy increases slowly at first, but becomes linear at high angular momentum since $\omega L$ term starts dominating. This is a precursor of the expected behaviour in the Yrast region of the rotational spectrum which we have discussed earlier. It will be instructive at this point to mention the behaviour of the rotating nuclei in the Yrast region. There at very high angular momentum sector though the excitation energy is quite high, the nucleus remains cold as most of the energy is spent in rotating the nucleus. In the Yrast region the interaction is of lesser importance as can be seen from Fig. 3.

A related question of interest that is relevant to a rotating nuclei as well as the two dimensional fermionic system is that under heavy internal stress generated by the centrifugal and Coriolis forces generated in the Yrast region (14) the density profile may drastically change. This issue has been investigated by Gallego and Das Gupta (13) and requires the consideration of a non radially symmetric density. This question as well as system quantum corrections will be addressed in a forthcoming paper (19).
FIGURES

FIG. 1. Plot of the angular momentum versus the ratio $\frac{\omega}{\omega_0}$. The dotted line is the result for contact interaction, while the solid line gives the result for logarithmic interaction. The number of particles is kept same for all the cases at 50. The angular momentum is in units of $\hbar$.

FIG. 2. Plot of the total energy $E$ (scaled by $E$ at $L = 0$) versus angular momentum $L$. Here also the solid line is for logarithmic interaction while the dotted line is for contact interaction. All other parameters are kept same as in Fig.1.

FIG. 3. Plot of the Interaction energy ($E_{int}$) again scaled by the $E_{int}$ at $L = 0$ as a function of the angular momentum ($L$). Here also the solid line is for logarithmic interaction while the dotted line is for contact interaction. All other parameters are kept same as in Fig.1.
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