On the Matching Equations of Kinetic Energy Shaping in IDA-PBC

M. Reza J. Harandi, Hamid D. Taghirad

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ABSTRACT

Interconnection and damping assignment passivity-based control scheme has been used to stabilize many physical systems such as underactuated mechanical systems through total energy shaping. In this method, some partial differential equations (PDEs) arisen by kinetic and potential energy shaping, shall be solved analytically. Finding a suitable desired inertia matrix as the solution of nonlinear PDEs related to kinetic energy shaping is a challenging problem. In this paper, a systematic approach to solve this matching equation for systems with one degree of underactuation is proposed. A special structure for desired inertia matrix is proposed to simplify the solution of the corresponding PDE. It is shown that the proposed method is more general than that of some reported methods in the literature. In order to derive a suitable desired inertia matrix, a necessary condition is also derived. The proposed method is applied to three examples, including VTOL aircraft, pendubot and 2D SpiderCrane system.

1. Introduction

Different methods to solve Partial differential equations (PDEs) with and without boundary condition are proposed in the literature. PDEs are very common in different fields of engineering such as thermodynamic, chemical process, wave analysis, etc. In some recently developed control methodologies, the process of controller design is reduced to finding the solution of such equations. Interconnection and damping assignment passivity-based control is one of these methods that is based on the general solution of some PDEs [1]. Providing new methodologies to find the required solution of PDEs opens new horizon to design novel controllers for many applications.

Passivity-based control (PBC) is a well-known methodology which was first introduced in [2] to define a controller design for stabilization through passivity. In this method, the control objective is to stabilize the desired equilibrium point of the system which is the minimum of a preselected storage function. This method, which clearly reminiscent of standard Lyapunov procedure, is successfully applied to simple mechanical systems that can be stabilized by shaping merely the potential energy [3]. For applications that requires kinetic energy shaping, PBC may be used [4], but the structure of the system in closed–loop will be changed and the storage function of the passive map does not have the interpretation of total energy anymore, while an unnatural stable invertibility requirement is imposed to the system [5].

In order to conquer this drawback and expand the range of applications of PBC, an extended version of this method is proposed. In this version, a storage function is not fixed at first, but the desired structure of the closed-loop system such as port-controlled Hamiltonian (PCH) or Lagrangian, is selected. Then all assignable energy functions which are applicable to this structure are obtained via the solution of partial differential equations. The most popular examples of this method are the interconnection and damping assignment (IDA) [5] and the controlled Lagrangian [6]. IDA-PBC reform the closed-loop system to a Hamiltonian structure with three matrices containing interconnection between subsystems, damping term and kernel of input matrix. One of the most important advantages of port Hamiltonian modeling is that it is based on energy exchange and dissipation of the system, thus passive structure of the system is visible. Readers are referred to [7] and references therein for examples and other features of IDA-PBC.

The most difficulty of IDA-PBC, which restricts the application of this method, is solving a set of PDEs called matching equations. Especially, in the case of underactuated robots; which have fewer actuators than the system degrees of freedom (DOF); a nonlinear PDE arises for kinetic energy shaping. In order to obviate this difficulty, some solutions are reported in the literature. As representatives, consider [8] that focuses on robots with one degree of underactuation, where the inertia matrix depends only on unactuated configuration. In the proposed method the PDEs are reduced to a simple set of nonlinear ODEs that are only solvable for some 2 DOF robots. A Similar method for PDE of kinetic energy is reported in [9]. In [10] a method for transforming PDEs to ODEs is proposed. This

Corresponding author
jafari@email.kntu.ac.ir (M. Reza J. Harandi); taghirad@kntu.ac.ir (Hamid D. Taghirad)

ORCID(s):

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A necessary condition and an algorithm for simplifying the PDE of kinetic energy are proposed in Section 2. Review of IDA-PBC methodology for simple mechanical systems

In here, IDA-PBC method for underactuated mechanical systems is reviewed. The readers are referred to [9, 10] for more details. If it is assumed that the system has no natural damping, the equations of motion may be written in the PCH form as

\[
\begin{bmatrix}
\dot{q} \\
\dot{\rho}
\end{bmatrix} = \begin{bmatrix}
0_{n\times n} & I_n \\
-I_n & 0_{n\times n}
\end{bmatrix} \begin{bmatrix}
\nabla_q H \\
\nabla_\rho H
\end{bmatrix} + \begin{bmatrix}
0_{n\times m} \\
\rho
\end{bmatrix} G(q),
\]

(1)

where \( H(q, p) = 1/2 p^T M^{-1}(q) p + V(q) \) is total energy of the system, \( q, p \in \mathbb{R}^n \) are generalized position and momenta, respectively, \( M^T(q) = M(q) > 0 \) is the inertia matrix, \( V(q) \) is the potential energy and rank of \( G(q) \) is equal to \( m < n \). Suppose that the desired structure for \( H_d \) is given as follows

\[
H_d(q, p) = 1/2 p^T M_d^{-1}(q) p + V_d(q),
\]

where \( M_d(q) \) and \( V_d(q) \) represent the desired inertia matrix and potential energy function, respectively, and it is required that the desired equilibrium point \( q_e \) satisfies \( q_e = \arg \min \nabla V_d(q) \). The desired interconnection matrix is also given as follows

\[
J_d(q, p) = \begin{bmatrix}
0_{n\times n} & M^{-1}(q) M_d(q) \\
-M_d(q) M^{-1}(q) & J_2(q, p)
\end{bmatrix}
\]

Notation: \( I_n \) denotes \( n \times n \) identity matrix, \( 0_{m\times n} \) is \( m \times n \) zero matrix and \( 0_n \) is a \( n \) dimensional column vector of zeros. \( x^{(i)} \) and \( \xi^{(j)} \) with \( x \in \mathbb{R}^n, \xi \in \mathbb{R}^{m\times n} \) denote \( i \)-th and \( (i,j) \)-th element of \( x \) and \( \xi \), respectively. \( e_i \in \mathbb{R}^n \) with \( i \in \tilde{n} \) is the Euclidean basis vector where \( \tilde{n} = \{1, \ldots, n\} \). Gradient of a scalar function \( f(x) \) with \( x \in \mathbb{R}^n \) which is denoted by \( \nabla f \) is a column vector as \( \nabla f = [\frac{\partial f(x)}{\partial x^{(1)}}, \ldots, \frac{\partial f(x)}{\partial x^{(n)}}]^T \) .
in which the skew-symmetric matrix \( J_2(q, p) \) is a free design parameter. It is possible to split the control into \( u = u_{es}(q, p) + u_d(q, p) \), in which

\[
\begin{align*}
    u_{es} &= (G^T G)^{-1} G^T \left( \nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p \right) \\
    u_d &= -K_\nu G^T \nabla_p H_d
\end{align*}
\]  

(2)

with \( K_\nu > 0 \). This restricts the desired damping matrix to have the form of

\[
R_d(q) = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} \\ 0_{n \times n} & G K_\nu G^T \end{bmatrix}
\]

The closed-loop system takes the Hamiltonian form

\[
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = 
\begin{bmatrix} 0_{n \times n} & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G K_\nu G^T \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}
\]

(3)

the matching equations of the IDA-PBC can be separated into the terms that depend on the kinetic and the potential energies, i.e. the terms depend on \( p \) and terms which are independent of \( p \), respectively. This leads to

\[
\begin{align*}
    G^\perp(q) \{ \nabla_q (p^T M^{-1}(q)p) - M_d M^{-1}(q) \nabla_q (p^T M_d^{-1}(q)p) + 2 J_2 M_d^{-1} p \} &= 0_s, \\
    G^\perp(q) \{ \nabla_q V(q) - M_d M^{-1} \nabla_q V_d(q) \} &= 0_s,
\end{align*}
\]

(4a)

(4b)

where \( G^\perp \in \mathbb{R}^{k \times n} \) is left annihilator of \( G \) and \( s = n - m \). Equation (4a) is a nonlinear PDE respect to positive definite desired inertia matrix. Given \( M_d \), equation (4b) is a linear PDE with respect to the desired potential energy. Therefore, the main difficulty of these PDEs is finding analytical solution for equation (4a).

In the sequel, we focus on the PDE of kinetic energy. Note that the proposed method works for the system with one degree of underactuation. The aim is to solve PDE (4a) or propose a methodology to simplify it. Invoking [17], and by considering a special form for \( M_d \) and utilizing \( J_2(q, p) \), this PDE is transformed to some algebraic equations and a single PDE in which the unknown parameter is the unactuated diagonal parameter of \( M_d^{-1} \). Notice that \( M_d \) has a critical role to ensure \( q_s = \arg \min V_d(q) \). Regardless of the most previous researches such as [15, 18] that just focus on solving equation (4a) without directly considering PDE (4b), Here a necessary condition is proposed to restrict selection of \( M_d \) to conduce a suitable \( V_d \).

3. Main results

In this section a constructive method with respect to PDE of kinetic energy is proposed. To accomplish that, let us introduce a condition on the selection of \( M_d \) as stated in the following proposition.

**Proposition 1.** Consider PDE (4b) and assume that \( n - m = 1 \). If Hessian matrix \( \frac{\partial^2 V_d}{\partial q^2} \bigg|_{q=q^*} \) is positive definite, then the following inequality holds

\[
\left( G^\perp M_d M^{-1} \frac{\partial (G^\perp \nabla V)}{\partial q} \right)_{q=q^*} > 0. 
\]

(5)

**Proof:** Differentiate both side of PDE (4b) respect to \( q \):

\[
\frac{\partial (G^\perp \nabla V)}{\partial q} = \left( G^\perp M_d M^{-1} \frac{\partial^2 V_d}{\partial q^2} \right)^T + \frac{\partial G^\perp M_d M^{-1}}{\partial q} \nabla V_d.
\]

(6)

Note that \( \nabla V_d \big|_{q=q^*} = 0_n \). Thus, (6) at \( q = q^* \) is

\[
\frac{\partial (G^\perp \nabla V)}{\partial q} \bigg|_{q=q^*} = \left( G^\perp M_d M^{-1} \frac{\partial^2 V_d}{\partial q^2} \right)^T \bigg|_{q=q^*} = \left( \frac{\partial^2 V_d}{\partial q^2} (G^\perp M_d M^{-1})^T \right) \bigg|_{q=q^*}
\]

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To complete the proof, multiply both side of above equation from left to \( (G^\perp M_d M^{-1}) \bigg|_{q=q^e} \), and notice that arbitrary matrix \( A \) is positive definite if \( \xi^T A \xi > 0 \) for any \( \xi \neq 0 \).

As explained before, we suppose that \( m = n-1 \). Thus, with a minor loss of generality, suppose that \( G = P[I_m, 0_{m\times(n-m)}]^T \), with \( P \) a permutation matrix which results in \( G^\perp = e_k^T, k \in \tilde{n} \). Simplify PDE (4a) term by term as follows. The first term is:

\[
G^\perp(q) \nabla_q \left( p^T M^{-1}(q)p \right) = p^T \frac{\partial M^{-1}}{\partial q(k)} p
\]

In the sequel, the following notations are used

\[
M^{-1} = \frac{1}{\det M} \mathfrak{M}(q) \quad \Longrightarrow \quad \frac{\partial M^{-1}}{\partial q(k)} = \frac{1}{(\det M)^2} \mathfrak{M}(q)
\]

(7)

where \( \mathfrak{M} \in \mathbb{R}^{n \times n} \) is adjugate matrix of \( M \) and \( \mathfrak{N} \in \mathbb{R}^{n \times n} \) is matrix of nominator elements of \( \frac{\partial M^{-1}}{\partial q(k)} \). Note that in spite of most previously reported research on this topic, it is not assumed that \( M(q) \) merely depends on some specified configuration variables. Regard to second term of (4a), it is assumed that \( M_d^{-1}(q) \) has the following structure:

\[
M_d^{-1} = \begin{bmatrix}
a_1 & 0 & \ldots & b_1 & 0 & \ldots & 0 \\
0 & a_2 & \ldots & b_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
b_1 & b_2 & \ldots & a(q) & b_k & \ldots & b_{n-1} \\
0 & 0 & \ldots & b_k & a_k & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & b_{n-1} & 0 & \ldots & a_{n-1}
\end{bmatrix}
\]

(8)

where all the elements of it are zero except diagonal elements, and the \( k \)-th row and column. Notice that \( a(q) \) is the only element which is state dependent. In other words, since our aim is to solve PDE of kinetic energy as simple as possible or at least simplify it, \( a_i \)s and \( b_j \)s are considered to be constant. Notice that the most important property of this structure is that \( k \)-th row of adjugate matrix \( \mathfrak{M} \) is independent of configuration variables and \( b_j \)s.

In order to simplify second term of (4a), \( G^\perp M_d M^{-1} \) is represented as follows

\[
G^\perp M_d M^{-1} = \frac{1}{\det M \det M_d^{-1}}
\]

(9)

where \( \gamma \in \mathbb{R}^n \) is a row vector independent of \( a(q) \). This is another advantage of the selected form of (8). determinant of \( M_d^{-1} \) is

\[
\det M_d^{-1} = \phi_1 a(q) + \phi_2
\]

where \( \phi_1, \phi_2 \) are constant parameters depending on other elements of \( M_d^{-1} \). Finally, second term of (4a) may be reduced to

\[
G^\perp M_d M^{-1} \nabla_q \left( p^T M_d^{-1}(q)p \right) = \frac{p^T \sum_{i=1}^{n} \left( \gamma^{(i)} \frac{\partial M_d^{-1}}{\partial q^{(i)}} \right) p}{\det M \det M_d^{-1}}
\]

(10)

in which all elements of \( \frac{\partial M_d^{-1}}{\partial q^{(i)}} \) are zero except the \( (k,k) \) element. Notice that if \( M_d \) was selected like what is reported in previous works [10, 8] as a function of only \( q^{(k)} \), then the above equation reduces to

\[
\frac{\gamma^{(k)} p^T \frac{\partial M_d^{-1}}{\partial q^{(k)}} p}{\det M \det M_d^{-1}}
\]
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In order to simplify the last term of (4a), as reported in [10], $J_2$ is linear with respect to $p$. Therefore, $J_2$ can be parameterized in the following form

$$J_2(q, p) = \frac{1}{\det M} \begin{bmatrix} 0 & p^T \alpha_1(q) & \cdots & p^T \alpha_{n-1}(q) \\ p^T \alpha_n(q) & 0 & \cdots & p^T \alpha_{2n-2}(q) \\ \vdots & \vdots & \ddots & \vdots \\ p^T \alpha_{n^2-2n+2}(q) & p^T \alpha_{n^2-2n+3}(q) & \cdots & 0 \end{bmatrix}$$

(11)

where $\alpha_i \in \mathbb{R}^n$, $i \in \{n(n-1)\}$. Note that this form of $J_2$ is not generally skew-symmetric. However, only a row of this matrix will be determined, thus, the column corresponding to this row will be selected in such a way that $J_2$ becomes skew-symmetric. Other elements of this matrix are free design parameters. Invoking [10], $J_2$ may be rewritten as follows

$$J_2 = \frac{1}{\det M} \sum_{i=1}^{n_0} p^T \alpha_i W_i, \quad n_0 = n(n-1),$$

where $W_i$ are matrices which are set as follows

$$W_1 = W^{1,2}, W_2 = W^{1,3}, \ldots, W_{n-1} = W^{1,n}, W_n = W^{2,1}, \ldots, W_{n_0} = W^{n,n-1},$$

in which $W^{i,j}$ is a matrix such that all of its elements are zero except the $(i, j)$ element which is equal to 1. Hence, $G^\perp J_2$ can be written as follows

$$G^\perp (q) J_2(p, q) = \frac{1}{\det M} p^T J(q) A, \quad J = \begin{bmatrix} \alpha_1 & \cdots & \alpha_{n_0} \end{bmatrix} \in \mathbb{R}^{n \times n_0}, \quad A = \begin{bmatrix} (G^\perp W_1)^T & \cdots & (G^\perp W_{n_0})^T \end{bmatrix}^T \in \mathbb{R}^{n_0 \times n_0}.$$

Thus, $J A$ may be written as:

$$J A = [\alpha_{(k-1)n-k+2}, \ldots, \alpha_{(k-1)n+1}, 0_n, \alpha_{(k-1)n+2}, \ldots, \alpha_{kn-k}] \triangleq B(q) \in \mathbb{R}^{n \times n}.$$

Notice that just one of the rows of $J_2$ appears in this equation. Finally, third term in PDE (4a) is reduced to

$$G^\perp J_2(q, p) M_d^{-1}(q) p = \frac{1}{\det M} p^T B M_d^{-1} p$$

(12)

All of the terms in (4a) are quadratic with respect to $p$ and should be symmetric. Replacing (7), (10) and (12) in (4a) results in the following relation:

$$\frac{\mathbb{M}}{\det M} - \sum_{i=1}^{n} y^{(i)} \frac{\partial M_d^{-1}}{\partial q^{(i)}} + (BM_d^{-1} + M_d^{-1} B^T) = 0$$

(13)

There are $\frac{n(n+1)}{2}$ equations and $n(n-1)$ free parameters in above relation. At first, it seems that for $n \geq 3$ there is no need to calculate $\sum_{i=1}^{n} y^{(i)} \frac{\partial M_d^{-1}}{\partial q^{(i)}}$. However, invoking Lemma2 in [10], it is easy to show that rank of $(BM_d^{-1} + M_d^{-1} B^T)$ is always $n-1$. It is also shown in [17] that the number of PDEs which should be solved is $\frac{1}{6} s(s+1)(s+2)$. Therefore, equality (13) leads to $\frac{n(n+1)}{2} - 1$ algebraic equations and one PDE with respect to $a(q)$. Notice that base on the structure of $M_d^{-1}$, all the elements of the second term are zero except the $(k, k)$ element. Therefore, the $(k, k)$ element of equation (13) leads to a PDE and other elements results in simple algebraic equations. These algebraic equations are derived by some manipulation as follows:

$$\frac{1}{\det M} \begin{bmatrix} \mathbb{M}(1), \mathbb{M}(2), \ldots, \mathbb{M}(n-1), \mathbb{M}(2), \ldots, \mathbb{M}((k-1)k), \mathbb{M}(k), \mathbb{M}(n) \end{bmatrix}^T$$

$$= -\Psi[a^{(1)}_{(k-1)n-k+2}, a^{(2)}_{(k-1)n-k+2}, \ldots, a^{(n)}_{(k-1)n-k+2}, a^{(1)}_{(k-1)n-k+3}, \ldots, a^{(n)}_{(k-1)n-k+3}, \ldots, a^{(1)}_{kn-k}, \ldots, a^{(n)}_{kn-k}]^T$$

(14)
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where

\[ \Psi = [\psi_1, \ldots, \psi_{n(n-1)}] \in \mathbb{R}^{\binom{n(n+1)}{2} \times n(n-1)} \]

\[ \psi_1 = [2a_1, 0^T_{k-2}, b_1, 0^T_{n(n+1)-k-1}]^T, \quad \psi_2 = [0, a_1, 0^T_{n+k-4}, b_1, 0^T_{n(n+1)-k-1}]^T, \quad \cdots \]

\[ \psi_{n+1} = [0, a_2, 0^T_{k-3}, b_1, 0^T_{n(n+1)-k-1}]^T, \quad \cdots \quad \psi_{n(n-1)} = [0^T_{k(2n-k+1)-1}, b_n, 0^T_{n(n+1)-k(2n-k+1)-1}]^T, \]

in which \( a^{(j)} \) is the \( j \)-th element of vector \( a \). Matrix \( \Psi \) is generally full rank; therefore, equation (14) has at least one solution. The remaining PDE is given by

\[
\frac{\sum_{i=1}^{n} \frac{\partial a(q)}{\partial \dot{q}^{(i)}}}{\det M_d^{-1}} - \frac{\mathcal{M}(kk)}{\det M} - 2 \sum_{i=1}^{n-1} b_i a^{(k)}_{(k-1)n-k+1+i} = 0. \tag{15}
\]

Note that \( a, b \) should be determined such that \( \Psi \) is full rank, \( M_d^{-1} \) is positive definite and proposition 1 is satisfied.

The following statements can be verified for (13):

- If the assumption of [8] holds, i.e. \( M \) is only function of unactuated coordinate \( q^{(k)} \), the second term of (13) is reduced to \( y^{(k)} \frac{\partial M^{-1}}{\partial \dot{q}^{(i)}} \) and PDE (15) is replaced by the following ODE

\[
\frac{y^{(k)} \frac{\partial a(q^{(k)})}{\partial \dot{q}^{(i)}}}{\phi_1 a(q^{(k)}) + \phi_2} = \frac{\mathcal{M}(kk)}{\det M} + 2 \sum_{i=1}^{k-1} b_i a^{(k)}_{(k-1)n-k+1+i} = f(q^{(k)}) \tag{16}
\]

Analytic solution of this ODE is

\[
a(q^{(k)}) = \frac{\lambda e^{\phi_1 F(q^{(k)}) - \phi_2}}{\phi_1} \tag{17}
\]

where \( \lambda \) is a constant parameter and \( F = \int \frac{1}{y^{(k)}} dq^{(k)} \). Note that in the method proposed in [8], the obtained ODEs is generally a set of ODE and has analytic solution if \( n = 2 \).

- If \( \mathcal{M}(kk) \) is equal to zero, then the second term in (15) is omitted. Hence, it is easy to select \( a(q) \) with respect to free parameters. Special cases of this condition is considered in [10] where an analytic solution is proposed in which \( a(q) \) is merely a function of one of the \( q^{(i)} \). In the proposed solution, since the aim is to derive a simple solution, \( M_d \) will be considered to be a constant matrix such that Proposition 1 is satisfied. Note that if \( M \) is constant, we can choose a constant value for \( M_d \) without considering a special form for \( G(q) \). The proposed IDA-PBC in [19] for acrobot is an example of this case.

- In other cases, a PDE should be solved. One may invoke the methods proposed in [20, 15] to simplify it. Another powerful method is using Pfaffian differential equations detailed in [16, 21]. Base on this method, the corresponding Pfaffian equations to PDE (15) are

\[
\frac{\det M_d^{-1} dq^{(1)}}{\gamma^{(1)}} = \cdots = \frac{\det M_d^{-1} dq^{(n)}}{\gamma^{(n)}} = \frac{da}{\det M + \sum_{i=1}^{n-1} b_i a^{(k)}(k-1)n-k+1+i}
\]

In [16] (see also [21, ch.2]) some tips are proposed to solve Pfaffian differential equations.

In the next section, some illustrative case studies are examined to show the applicability of proposed method. Note that similar to [15, 18] we only concentrate on the matching equation related to kinetic energy.
4. Case Studies

In the following three case studies is proposed to verify the three above mentioned statements. The first example is Pendubot in which its matching equation is replaced by an ODE. The second example is VTOL aircraft where the corresponding PDE is solved easily by a constant $M_d$. The last example is 2D SpiderCrane in which its matching equation is solved systematically. Inertia matrix and potential energy of this robot are given by:

$$M = \begin{bmatrix} c_1 + 2c_3 \cos(q^{(2)}) & c_2 + c_3 \cos(q^{(2)}) \\ c_2 + c_3 \cos(q^{(2)}) & c_2 \end{bmatrix}, \quad V = c_4 g \cos(q^{(1)}) + c_5 g \cos(q^{(1)} + q^{(2)}), \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (18)$$

with $c_i$s defined in [22]. After some manipulation, the following expressions are obtained

$$\det M = c_1 c_2 - c_3^2 \cos^2(q^{(2)}), \quad \Re = \begin{bmatrix} c_2 & -c_2 - c_3 \cos(q^{(2)}) \\ -c_2 - c_3 \cos(q^{(2)}) & c_1 + c_2 + 2c_3 \cos(q^{(2)}) \end{bmatrix},$$

\begin{equation}
\mathcal{M} = \begin{bmatrix} -2c_3^2 \sin(q^{(2)}) \cos^2(q^{(2)}) & c_1 c_2 c_3 \sin(q^{(2)}) + c_3^2 \sin(q^{(2)}) \cos^2(q^{(2)}) + 2c_2 c_3^2 \sin(q^{(2)}) \cos(q^{(2)}) \\ c_1 c_2 c_3 \sin(q^{(2)}) + c_3^2 \sin(q^{(2)}) \cos^2(q^{(2)}) + 2c_2 c_3^2 \sin(q^{(2)}) \cos(q^{(2)}) & -2c_1 c_2 c_3 \sin(q^{(2)}) - 2c_3^2 \sin(q^{(2)}) \cos^2(q^{(2)}) - 2c_3^2 (c_1 + c_2) \sin(q^{(2)}) \cos(q^{(2)}) \end{bmatrix}.
\end{equation}

$$M_d^{-1} = \begin{bmatrix} a_1 & b_1 \\ b_1 & a(q^{(2)}) \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & p^T a_1 \\ p^T a_2 & 0 \end{bmatrix}, \quad \gamma^T = \begin{bmatrix} -c_2 b_1 - c_2 a_1 - c_3 a_1 \cos(q^{(2)}) \\ a_1 c_1 + a_1 c_2 + b_1 c_2 + b_1 c_3 \cos(q^{(2)}) + 2a_1 c_3 \cos(q^{(2)}) \end{bmatrix}.$$

Note that in the following $a_2$ will be determined and $a_1 = -a_2$ will be set. Equation (13) for this case is derived as follows

$$\frac{\gamma^{(2)}}{\det M_d^{-1}} \begin{bmatrix} 0 & 0 \\ \frac{\partial a}{\partial q^{(2)}} & \frac{\partial b}{\partial q^{(2)}} \end{bmatrix} = \frac{1}{\det M} \Re + \begin{bmatrix} 2a_2^{(1)} a_1 \\ a_2^{(1)} a_1 + a_2^{(1)} b_1 \\ a_2^{(1)} b_1 \\ 2a_2^{(1)} b_1 \end{bmatrix}.$$

By solving two algebraic equations, $a_2$ is obtained as follows

$$a_2^{(1)} = -\frac{\Re^{(11)}}{2a_1 \det M} = \frac{c_3^2 \sin(q^{(2)}) \cos^2(q^{(2)})}{a_1 (c_1 c_2 - c_3^2 \cos^2(q^{(2)}))},$$

$$a_2^{(2)} = -\frac{\Re^{(21)}}{a_1 \det M} - \frac{a_2^{(1)} b_1}{a_1} = \frac{c_2 c_3 \sin(q^{(2)}) + c_3^2 \sin(q^{(2)}) \cos^2(q^{(2)}) + 2c_2 c_3^2 \sin(q^{(2)}) \cos(q^{(2)})}{a_1 (c_1 c_2 - c_3^2 \cos^2(q^{(2)}))}$$

Finally, the following ODE should be solved

$$\frac{1}{a_1 a^{(2)} b_1^2 d q^{(2)}} \frac{d a}{d q^{(2)}} = \frac{1}{\gamma^{(2)} \det M} \left( \Re^{(22)} - \frac{2b_1 \Re^{(21)}}{a_1} + \frac{b_1^2 \Re^{(11)}}{a_1^2} \right).$$

(22)

This ODE is in the form of (16) and its solution is derived from (17) with

$$\phi_1 = b_1, \quad \phi_2 = -b_1^2, \quad F(q^{(2)}) = \int \frac{1}{\gamma^{(2)} \det M} \left( \Re^{(22)} - \frac{2b_1 \Re^{(21)}}{a_1} + \frac{b_1^2 \Re^{(11)}}{a_1^2} \right) d q^{(2)}.$$

For example, assume that $c_1 = 4, c_2 = 1$ and $c_3 = 1.5$. By some manipulation, $a(q^{(2)})$ is obtained as follows

$$a(q^{(2)}) = \cos(q^{(2)})^{-7/3} + (4 - 3 \cos(q^{(2)}))^{19/6} - (4 + 3 \cos(q^{(2)}))^{-7/2},$$

(23)

where $a_1 = 1, b_1 = -5, \lambda = 1$ are chosen to simplify the ODE (22) and also the necessary condition (5) is satisfied. The solution of potential energy PDE is proposed in Appendix.
4.2. VTOL Aircraft

Dynamic model of VTOL in PCH form (1) is given as follows

\[
G(q) = \begin{bmatrix} -\sin(\theta) & e \cos(\theta) \\ e \cos(\theta) & e \sin(\theta) \\ 0 & 1 \end{bmatrix}, \quad M = I, \quad V = gy, \quad q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}
\]

where, \(x\) and \(y\) denote the position of center of mass, \(\theta\) is the roll angle and \(e\) models the effect of the slopped wings. The desired equilibrium point of the system is \([x^*, y^*, 0]^T\). In [10] a controller with state-dependent \(M_d\) by defining new inputs is derived. Since the inertia matrix is constant, it is possible to solve the PDE of kinetic energy with a constant \(M_d\), represented by:

\[
M_d = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}
\]

Necessary condition (5) in this case leads to following inequality

\[
\left( g \cos(\theta)(e \cos(\theta) + f \sin(\theta) - ce) \right)_{\theta=0} > 0.
\]

A suitable choice for the matrix parameters is

\[
a = \kappa e^2, \quad b = 1, \quad c = \kappa', \quad d = 0, \quad e = e, \quad f = 0,
\]

where the constants \(\kappa, \kappa' > 0\) should be selected such that \(\kappa \kappa' > 1\). Note that \(M_d = I\) does not satisfy the necessary condition (5) which is in line with our prior knowledge that it is not possible to stabilize the system with merely potential energy shaping. Although solving the potential energy PDE (4b) is out of scope of this paper, but in this case its solution with \(\kappa = 20\) and \(\kappa' = 0.1\) is derived as follows

\[
V_d = \left( e(y - y^*) + \ln \left( e \cos(\theta) - 0.1e \right) \right)^2 + \left( \frac{1}{20e}(x - x^*) - (\theta - \theta^*) - 0.1 \text{arctanh} \left( 1.1055 \tan \left( \frac{\theta}{2} \right) \right) \right)^2
\]

\[
- 2e \ln(0.9e)(y - y^*) - \frac{g - 2e \ln(0.9e)}{ge} \ln \left( e \cos(\theta) - 0.1e \right).
\]

| Figure 1: Simulation results of proposed controller on VTOL aircraft. The aircraft moves toward its desired position with smooth states and control law. |
To simulate the response, consider the initial condition as \( q(0) = [6, -5, -1]^T \) with zero velocity while the desired position is \( q_d = [0, 0, 0]^T \). Set, \( \epsilon = 0.3 \) and \( K_v = \text{diag}(1, 0.5) \). Simulation results are illustrated in Fig. 1. As shown in Fig. 1(a), the errors converge to zero in about 20 second with an acceptable control efforts amplitude. The motion of the robot in \( X - Y \) plane is depicted in Fig. 1(a). Due to coupling of inputs, the aircraft first moves to farther position to correct its orientation and then goes to the desired position. Note that the advantage of the proposed controller in comparison with that reported in [10] is its simplicity.

4.3 2D SpiderCrane

This system consists of a load suspended from a ring which is controlled by two cables. The schematic of this system is depicted in Fig. 2. The position of the ring and the mass are denoted by \((x_r, y_r)\) and \((x, y)\), respectively, and their mass is denoted by \(M\) and \(m\), respectively. The length of the controlled cables is denoted by \(l_1\) and \(l_2\), while \(l_3\) denotes the fixed length of the cable between ring and the mass. Dynamic equation of the system is in the form (1) with following parameters

\[
q = \begin{bmatrix} x_r \\ y_r \\ \theta \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V = (M + m)g y_r - m g l_2 \cos(\theta), \quad M(q) = \begin{bmatrix} M + m & 0 & ml_3 \cos(\theta) \\ 0 & M + m & ml_3 \sin(\theta) \\ ml_3 \cos(\theta) & ml_3 \sin(\theta) & ml_3^2 \end{bmatrix}.
\]

Two IDA-PBC controller have been designed for SpiderCrane. In [23] merely the potential energy is shaped while in [24] total energy shaping method proposed in [10] is used such that first a partial feedback linearization is applied to the system and then a desired inertia matrix which is merely a function of \(\theta\) is chosen. In here, the aim is to derive a more general solution such that \(M_d\) may be set as a function of \(x_r\) and \(y_r\). Consider \(M_d^{-1}\) in the form of (8). One can easily check that necessary condition (5) is satisfied if \(b_1 ml_3 + a_2(M + m) > 0\). In order to solve matching equation (4a), the following parameters are derived

\[
\text{det} M(q) = (M + m)(M + m)ml_3^2 - (M + m)ml_3^2 l_2^2,
\]

\[
\Re = \begin{bmatrix} (M + m)ml_3^2 - ml_3^2 \sin^2(\theta) & ml_3^2 \sin(\theta) \cos(\theta) & -(M + m)ml_3 \cos(\theta) \\ ml_3^2 \sin(\theta) \cos(\theta) & (M + m)ml_3^2 - ml_3^2 \cos^2(\theta) & -(M + m)ml_3 \sin(\theta) \\ -(M + m)ml_3 \cos(\theta) & -(M + m)ml_3 \sin(\theta) & (M + m)^2 \end{bmatrix},
\]

\[
\Im = \text{det} M(q) = \begin{bmatrix} -2ml_3^2 \sin(\theta) \cos(\theta) & ml_3^2 \cos(2\theta) & (M + m)ml_3 \sin(\theta) \\ ml_3^2 \cos(2\theta) & ml_3^2 \sin(2\theta) & -(M + m)ml_3 \cos(\theta) \\ (M + m)ml_3 \sin(\theta) & -(M + m)ml_3 \cos(\theta) & 0 \end{bmatrix},
\]

\[
\gamma^T = \begin{bmatrix} -a_1 b_1 (M + m)ml_3^2 + a_2 b_1 ml_3^2 \sin^2(\theta) - a_1 b_2 ml_3^2 \sin(\theta) \cos(\theta) - a_1 a_2 (M + m)ml_3 \cos(\theta) \\ -a_2 b_1 (M + m)ml_3^2 - a_1 b_2 ml_3^2 \sin^2(\theta) - a_1 b_2 ml_3^2 \sin(\theta) \cos(\theta) - a_1 a_2 (M + m)ml_3 \cos(\theta) \\ a_2 b_1 (M + m)ml_3 \cos(\theta) + a_1 b_2 (M + m)ml_3 \sin(\theta) + a_1 a_2 (M + m) \end{bmatrix}
\]

Based on necessary condition and simplifying the corresponding matching equation, we choose \(b_1 = b_2 = 0\). By this means, the matrix \(\Psi\) in equality (14) is in the following form

\[
\Psi = \begin{bmatrix} 2a_1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & 0 \\ 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
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This matrix is full rank, hence \([a^T_5, a^T_6, I^T] \) is determined by right pseudo-inverse of \(\Psi\). The Pfaffian differential equations of PDE (15) for this system is given as follows

\[
\begin{align*}
\frac{dx}{-ml_3 \cos(\theta)} & = \frac{dy}{-ml_3 \sin(\theta)} = \frac{d\theta}{M + m} = \frac{da}{0} \\
\end{align*}
\]

The solutions to these equations are

\[
\begin{align*}
x + \frac{ml_3}{M + m} \sin(\theta) &= c_1, \\
y - \frac{ml_3}{M + m} \cos(\theta) &= c_2,
\end{align*}
\]

with \(c_1\) and \(c_2\) as free parameters. Invoking \([16] a(q)\) is

\[
a(q) = \phi\left(x + \frac{ml_3}{M + m} \sin(\theta), y - \frac{ml_3}{M + m} \cos(\theta)\right),
\]

where \(\phi\) is an arbitrary function. General form of \(V_d\) is proposed in Appendix.

5. Conclusions and Future Prospects

In this paper a systematic method to simplify the matching equation related to kinetic energy shaping for underactuated robots with one degree of underactuation was proposed. A special structure of desired inertia matrix was considered in such a way that just one of its elements depends on configuration variables. By this means, the arisen PDE can be analytically solved for robots with some properties including manipulators with inertia matrix depending on just one variable. The proposed method was successfully implemented on VTOL aircraft, pendobot and 2D SpiderCrane. Extension of this method to robots with more degrees of underactuation and also consideration of potential energy PDE are currently being examined in our research group.

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