From triangulated categories to module categories via localization II: calculus of fractions

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Dedicated to Idun Reiten on the occasion of her 70th birthday

Abstract

We show that the quotient of a Hom-finite triangulated category \( C \) by the kernel of the functor \( \text{Hom}_C(T, \cdot) \), where \( T \) is a rigid object, is preabelian. We further show that the class of regular morphisms in the quotient admits a calculus of left and right fractions. It follows that the Gabriel–Zisman localization of the quotient at the class of regular morphisms is abelian. We show that it is equivalent to the category of finite-dimensional modules over the opposite of the endomorphism algebra of \( T \) in \( C \).

Introduction

Let \( k \) be a field and \( C \) a skeletally small, triangulated Hom-finite \( k \)-category which is Krull–Schmidt and has Serre duality. A standard example of such a category is the bounded derived category of finite-dimensional modules over a finite-dimensional algebra of finite global dimension (see [6]). In this case, the triangulated category is obtained from the abelian category of modules by Gabriel–Zisman (or Verdier) localization of the quasi-isomorphisms in the bounded homotopy category of complexes of modules.

Here, our approach is the other way around. Given a triangulated category \( C \) as above, we are interested in gaining information about related abelian categories. We are particularly interested in the module categories over (the opposites of) endomorphism algebras of objects in \( C \). An object \( T \) in \( C \) satisfying \( \text{Ext}^1(T, T) = 0 \) is known as a rigid object. In this case, it is known [2] that the category of finite-dimensional modules over \( \text{End}(T)^{\text{op}} \) can be obtained as a Gabriel–Zisman localization of \( C \), formally inverting the class \( S \) of maps which are inverted by the functor \( \text{Hom}_C(T, \cdot) \). However, the class \( S \) does not admit a calculus of left or right fractions in the sense of Gabriel and Zisman [4, Section I.2] (see also [13, Section 3]).

If \( T \) is a cluster-tilting object then, by a result of Koenig–Zhu [12, Corollary 4.4], the additive quotient \( C/\Sigma T \), where \( \Sigma \) denotes the suspension functor of \( C \), is equivalent to \( \text{mod End}_C(T)^{\text{op}} \) (see also [8, Proposition 6.2] and [11, Section 5.1]; the case where \( C \) is 2-Calabi–Yau was proved in [11, Proposition 2.1], generalizing [3, Theorem 2.2]). However, when \( T \) is rigid, this is no longer the case in general. It is natural to consider instead the quotient \( C/\mathcal{X}_T \), where \( \mathcal{X}_T \) is the class of objects in \( C \) sent to zero by the functor \( \text{Hom}_C(T, \cdot) \), since, in the cluster-tilting case, \( \mathcal{X}_T = \text{add} \Sigma T \). However, one does not obtain the module category this way, since in general \( C/\mathcal{X}_T \) is not abelian.
Our approach here is to show first that \( \mathcal{C}/\mathcal{X}_T \) is preabelian, using some arguments generalizing those of Koenig–Zhu \([12]\). This means that, in addition to \( \mathcal{C}/\mathcal{X}_T \) being an additive category, every morphism in \( \mathcal{C}/\mathcal{X}_T \) has a kernel and a cokernel. This category, in general, possesses regular morphisms which are not isomorphisms (that is, morphisms which are both monomorphisms and epimorphisms but which do not have inverses), so it cannot be abelian in general.

However, we show that \( \mathcal{C}/\mathcal{X}_T \) does have a nice property. It is integral, that is, the pullback of any epimorphism (respectively, monomorphism), is again an epimorphism (respectively, monomorphism). This allows us to apply a result of Rump \([17, \text{p. 173}]\), which implies that \( (\mathcal{C}/\mathcal{X}_T)_\mathcal{R} \) the localization of the category \( \mathcal{C}/\mathcal{X}_T \) at the class \( \mathcal{R} \) of regular morphisms, is abelian.

We assume that \( \mathcal{C} \) is skeletally small to ensure that the localization exists. Furthermore, by the same reference, the class \( \mathcal{R} \) admits a calculus of left and right fractions.

We go on to show that the projective objects in \( (\mathcal{C}/\mathcal{X}_T)_\mathcal{R} \) are, up to isomorphism, exactly the objects induced by the objects in the additive subcategory of \( \mathcal{C} \) generated by \( T \). This implies our main result:

**Theorem.** Let \( \mathcal{C} \) be a skeletally small, Hom-finite, Krull–Schmidt triangulated category with Serre duality, containing a rigid object \( T \). Let \( \mathcal{X}_T \) denote the class of objects \( X \) in \( \mathcal{C} \) such that \( \text{Hom}_\mathcal{C}(T, X) = 0 \). Let \( \mathcal{R} \) denote the class of regular morphisms in \( \mathcal{C}/\mathcal{X}_T \). Then \( \mathcal{R} \) admits a calculus of left and right fractions. Let \( (\mathcal{C}/\mathcal{X}_T)_\mathcal{R} \) denote the localization of \( \mathcal{C}/\mathcal{X}_T \) at \( \mathcal{R} \). Then

\[
(\mathcal{C}/\mathcal{X}_T)_\mathcal{R} \simeq \text{mod End}_\mathcal{C}(T)^{\text{op}}.
\]

Let \( \mathcal{S} \) denote the class of maps in \( \mathcal{C} \) which are inverted by \( \text{Hom}_\mathcal{C}(T, -) \), and let \( \overline{\mathcal{S}} \) denote the image of this class in \( \mathcal{C}/\mathcal{X}_T \). Then \( \mathcal{R} = \overline{\mathcal{S}} \), and the localization functor \( L_{\mathcal{S}} : \mathcal{C} \to \overline{\mathcal{C}}_\mathcal{S} \) factors through \( \mathcal{C}/\mathcal{X}_T \). The main result of Buan and Marsh \([2]\) was the construction of an equivalence \( G \) from \( \overline{\mathcal{C}}_\mathcal{S} \) to \( \text{mod End}_\mathcal{C}(T)^{\text{op}} \), such that \( \text{Hom}_\mathcal{C}(T, -) = \text{GL}_\mathcal{S} \). Our theorem above can be seen as a refinement of this. It was noted in \([2]\) that \( \mathcal{S} \) does not admit a calculus of left or right fractions, thus we can observe that the advantage of passing first to the quotient \( \mathcal{C}/\mathcal{X}_T \) is that the subsequent localization does then admit such a calculus.

We note that \([15]\) contains results obtaining abelian categories as subquotients of triangulated categories; we give an explanation of the relationship between the results obtained here and those in \([15]\) in Section 6. We also remark that A. Beligiannis has recently informed us that, in a subsequent work using a different approach, he has been able to generalize our main result to the case of a functorially finite rigid subcategory.

There are interesting parallels between our approach here and the construction of the derived category of an abelian category \( \mathcal{A} \). We follow \([10]\) and \([5, \text{III.2, III.4}]\). The derived category of \( \mathcal{A} \) can be defined (following Grothendieck) as the Gabriel–Zisman localization of the category \( \mathcal{C}(\mathcal{A}) \) of complexes over \( \mathcal{A} \) at the class of quasi-isomorphisms. This class does not, in general, admit a calculus of left and right fractions. However, the more commonly used construction (due to Verdier) of the derived category involves passing first to the homotopy category \( K(\mathcal{A}) \). Then \( \mathcal{C}(\mathcal{A}) \) is a Frobenius category and \( K(\mathcal{A}) \) is the corresponding stable category, hence a quotient of \( \mathcal{C}(\mathcal{A}) \). Then the class of quasi-isomorphisms in \( K(\mathcal{A}) \) admits a calculus of left and right fractions. Localizing at this class gives rise to the derived category of \( \mathcal{A} \).

In Section 1, we set-up the context in which we work. In Section 2, we recall the definitions of semi-abelian and integral categories and some results of Rump \([17]\) (see also \([18]\)) which will be useful. In Section 3, we prove that \( \mathcal{C}/\mathcal{X}_T \) is integral. In Section 4, we recall the Gabriel–Zisman theory of localization and calculi of fractions and also how it can be applied (following \([17, \text{Section 1}]\)) to the case of the regular morphisms in an integral category. In Section 5, we apply this to \( (\mathcal{C}/\mathcal{X}_T)_\mathcal{R} \) to show that it is abelian. By classifying the projective objects in \( (\mathcal{C}/\mathcal{X}_T)_\mathcal{R} \), we deduce the main result. In Section 6, we explain the relationship of the results here to work
of Nakaoka [15]. In Section 7, we explain the relationship between our main result and the results in [2].

1. **Notation**

We first set up the context in which we work and define some notation. Let $k$ be a field and $C$ be a skeletally small, triangulated, Hom-finite, Krull–Schmidt $k$-category with suspension functor $\Sigma$. We need the skeletally small assumption to ensure that the localizations we need exist. We assume that $C$ has a Serre duality, that is, an autoequivalence $\nu: C \to C$ such that $\text{Hom}_C(X, Y) \simeq D\text{Hom}_C(Y, \nu X)$ (natural in $X$ and $Y$) for all objects $X$ and $Y$ in $C$, where $D$ denotes the duality $\text{Hom}_k(-, k)$. Let $T$ be a rigid object in $C$ and set $\Gamma = \text{End}_C(T)^{\text{op}}$.

For a full subcategory $X$ of $C$, let $X^\perp = \{ C \in C \mid \text{Ext}^1(X, C) = 0 \text{ for each } X \in X \}$, and define $^\perp X$ dually. For an object $X$ in $C$, let $\text{add}X$ denote its additive closure, and let $X^\perp = (\text{add}X)^\perp$. A rigid object $T$ is called cluster-tilting if $\text{add}T = T^\perp$. Let $X_T = (\Sigma T)^\perp$.

We also recall the triangulated version of Wakamatsu’s Lemma; see, for example, [8, Section 2]; see also [10, Lemma 2.1].

**Lemma 1.1.** Let $X$ be an extension-closed subcategory of a triangulated category $C$.

(a) Suppose that $X \to C$ is a minimal right $X$-approximation of $C$ and $\Sigma^{-1}C \to Y \to X \to C$ a completion to a triangle. Then $Y$ is in $X^\perp$, and the map $\Sigma^{-1}C \to Y$ is a left $X^\perp$-approximation of $\Sigma^{-1}C$.

(b) Suppose that $C \to X$ is a minimal left $X$-approximation of $C$ and $\Sigma^{-1}Z \to C \to X \to Z \to \Sigma C$ a completion to a triangle. Then $Z$ is in $^\perp X$, and the map $Z \to \Sigma C$ is a right $^\perp X$-approximation of $\Sigma C$.

Using this, we obtain:

**Lemma 1.2.** Let $T$ be a rigid object in $C$. Then the subcategory $X_T$ of $C$ is functorially finite.

**Proof.** This follows from combining Wakamatsu’s Lemma (Lemma 1.1) with the existence of Serre duality.

2. **Preabelian categories**

Recall that an additive category $A$ is said to be preabelian if every morphism has a kernel and a cokernel. In this section, we shall recall some of the theory of preabelian categories that we need in order to study $C/X_T$. A morphism is said to be regular (or a bimorphism) if it is both an epimorphism and a monomorphism.

According to [17, Section 1] a preabelian category is called left semi-abelian (respectively, right semi-abelian) if every morphism $f$ has a factorization of the form $ip$ where $p$ is a cokernel and $i$ is a monomorphism (respectively, where $p$ is an epimorphism and $i$ is a kernel); see [17, Section 1], where it is pointed out that in the left semi-abelian case $p$ is necessarily $\text{coim}(f) = \text{coker}(\text{ker}(f))$ and in the right semi-abelian case $i$ is necessarily $\text{im}(f) = \ker(\text{coker}(f))$. A preabelian category is said to be semi-abelian if it is both left and right semi-abelian.

We remark that pullbacks and pushouts always exist in a preabelian category. For the pullback of maps $c: B \to D$ and $d: C \to D$, we can take the kernel of the map $B \coprod C \to D$.
whose components are $c$ and $-d$, obtaining a pullback diagram:

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow b & & \downarrow c \\
C & \longrightarrow & D.
\end{array}
\]

There is a dual construction for the pushout.

We recall the following characterization of semi-abelian categories:

**Proposition 2.1** [17, Proposition 1]. Let $\mathcal{A}$ be a preabelian category. Then $\mathcal{A}$ is left semi-abelian if and only if, in any pullback diagram as above, $a$ is an epimorphism whenever $d$ is a cokernel.

A dual characterization in terms of pushout diagrams holds for right semi-abelian categories.

A preabelian category is said to be left integral provided that, in any pullback diagram as above, $a$ is an epimorphism whenever $d$ is an epimorphism. A dual definition involving pushouts is used to define right integral categories. A preabelian category which is both left and right integral is said to be integral.

The following then follows from Proposition 2.1.

**Proposition 2.2** [17, Corollary 1]. Any left integral (respectively, integral) category is left semi-abelian (respectively, semi-abelian).

We recall the following two results from [17].

**Lemma 2.3** [17, Lemma 1]. Let $\mathcal{A}$ be a preabelian category. In a pullback diagram (1), whenever $d$ is a monomorphism, $a$ is a monomorphism.

We also recall the following (which also includes a dual statement involving pushouts which we will not need here).

**Proposition 2.4** [17, Proposition 6]. Let $\mathcal{A}$ be a semi-abelian category. Then the following are equivalent.

(a) The category $\mathcal{A}$ is integral.

(b) For any pullback diagram (1), $a$ is regular whenever $d$ is regular.

Finally, we note the following, which is easy to show using the definitions.

**Lemma 2.5.** Let $h$ be a map in an additive category which is a weak cokernel of a map $g$ and an epimorphism. Then $h$ is a cokernel of $g$.

3. **Properties of $\mathcal{C}/\mathcal{X}_T$**

In this section, we consider the factor category $\mathcal{C}/\mathcal{X}_T$. The objects in $\mathcal{C}/\mathcal{X}_T$ are the same as those in $\mathcal{C}$. For objects $X, Y$ in $\mathcal{C}$, $\text{Hom}_{\mathcal{C}/\mathcal{X}_T}(X, Y)$ is given by $\text{Hom}_{\mathcal{C}}(X, Y)$ modulo morphisms factoring through $\mathcal{X}_T$. We denote the image of a morphism $f$ in $\mathcal{C}$ by $\overline{f}$. Note that, since $\mathcal{C}$ is $\mathcal{k}$-additive, so is $\mathcal{C}/\mathcal{X}_T$. 
We will show the following result, which can be regarded as a generalization of Koenig and Zhu [12, Theorem 3.3]. Our proof is inspired by the proof in [12].

**Theorem 3.1.** The factor category $\mathcal{C}/\mathcal{X}_T$ is preabelian.

In order to prove Theorem 3.1, we will need the following lemmas.

**Lemma 3.2.** Consider a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{\delta_1} & & \downarrow{\delta_2} \\
A' & \xrightarrow{\alpha'} & B'
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{\gamma} & \Sigma A \\
\downarrow{\delta_3} & & \downarrow{\Sigma\delta_1} \\
C' & \xrightarrow{\gamma'} & \Sigma A'
\end{array}
\]

in a triangulated category, where the rows are triangles.

(a) If the composition $\delta_2\alpha$ vanishes, then there are maps $\epsilon_1: C \to B'$ and $\epsilon_2: \Sigma A \to C'$, such that $\delta_3 = \beta'\epsilon_1 + \epsilon_2\gamma$.

(b) If the composition $\gamma'\delta_3$ vanishes, then there are maps $\phi_1: B \to A'$ and $\phi_2: C \to B'$ such that $\delta_2 = \phi_2\beta + \alpha'\phi_1$.

For a map $f: X \to Y$ in $\mathcal{C}$, consider the triangle $\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$.

**Lemma 3.3.**

(a) The map $f: X \to Y$ is a monomorphism if and only if $h = 0$.

(b) The map $f: X \to Y$ is an epimorphism if and only if $g = 0$.

(c) The map $f: X \to Y$ is regular if and only if $g = 0 = h$.

**Proof.** (c) follows by definition from (a) and (b). We prove only (b), the proof of (a) being dual.

We consider first the case when $Z$ is in $\mathcal{X}_T$. We then need to show that $f$ is an epimorphism in $\mathcal{C}/\mathcal{X}_T$.

Let $p: Y \to M$ be a map such that $pf = 0$. Then there is an object $U'$ in $\mathcal{X}_T$, and a commuting square:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow{p} \\
U' & \xrightarrow{q} & M
\end{array}
\]

By Lemma 1.2, a minimal right $\mathcal{X}_T$-approximation $U \to M$ exists. The map $U' \to M$ factors through $U \to M$, hence there is also a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q} & & \downarrow{p} \\
U & \xrightarrow{f'} & M
\end{array}
\]

which we extend to a commutative diagram

\[
\begin{array}{ccc}
\Sigma^{-1}Z & \xrightarrow{f} & X \\
\downarrow & & \downarrow{q} \\
\Sigma^{-1}N & \xrightarrow{f'} & U
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{p} & & \downarrow{p} \\
M & \xrightarrow{g} & N
\end{array}
\]
where the rows are triangles. By Wakamatsu’s Lemma (Lemma 1.1), we have that $\Sigma^{-1}N$ is in $\mathcal{X}_T^+$. Therefore, the map $Z \to N$ is zero, using that $Z$ is by assumption in $\mathcal{X}_T$. By commutativity, the composition $\Sigma^{-1}Z \to X \to U$ vanishes. Hence, we have by Lemma 3.2, that there are maps $v_1: Y \to U$, and $v_2: Z \to M$, such that $p = f'v_1 + v_2g$. Hence, $p$ factors through $U \amalg Z$, which is in $\mathcal{X}_T$, so we have $p = 0$.

Now consider the general case, so assume $g$ factors through an object $V$ in $\mathcal{X}_T$ and consider the induced commutative diagram:

\[
\begin{array}{ccc}
\Sigma^{-1}V & \longrightarrow & N \\
\downarrow & & \downarrow f' \\
\Sigma^{-1}Z & \longrightarrow & Y \\
\downarrow h & & \downarrow g \\
X & \longrightarrow & Z
\end{array}
\]

where the rows are triangles. Now $f' = fr$ and hence $f' = fr$. Note that since $V$ is in $\mathcal{X}_T$, we have that $f'$ is an epimorphism. It follows that $fr$, and hence $f$, is an epimorphism.

Conversely, if $f$ is an epimorphism then, since $gf = 0$ we have $gf = 0$, so $g = 0$. \hfill \Box

**Lemma 3.4.** For any map $f: X \to Y$ in $\mathcal{C}$, the map $f: X \to Y$ has a kernel and a cokernel.

**Proof.** We construct a cokernel of $f$. The construction of a kernel is dual.

Consider a minimal right add-$T$-approximation $a: T_0 \to X$. Compose this with $f$, and complete the composition $fa$ to a triangle

\[
T_0 \longrightarrow Y \xrightarrow{c} M \longrightarrow \Sigma T_0.
\]

By the octahedral axiom, there is a commutative diagram:

\[
\begin{array}{ccc}
T_0 & \longrightarrow & Y \\
\downarrow a & & \downarrow c \\
X & \longrightarrow & Z \\
\downarrow f & & \downarrow b \\
\Sigma T_0 & \longrightarrow & \Sigma X
\end{array}
\]

We claim that $c$ is a cokernel for $f$.

Consider a map $p: Y \to N$, such that $pf = 0$. Assume $pf$ factors through an object $U$ in $\mathcal{X}_T$, so there is a commuting square

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow f & & \downarrow p \\
U & \longrightarrow & N
\end{array}
\]

Extend this to a commuting diagram of triangles, and compose with the previous map of triangles, to obtain the diagram:

\[
\begin{array}{ccc}
T_0 & \longrightarrow & Y \\
\downarrow f & & \downarrow p \\
X & \longrightarrow & Z \\
\downarrow b & & \downarrow d \\
\Sigma T_0 & \longrightarrow & \Sigma X \\
\downarrow \Sigma a & & \downarrow \Sigma d \\
\Sigma U & \longrightarrow & \Sigma U.
\end{array}
\]
The composition $T_0 \rightarrow X \rightarrow U$ vanishes, since $U$ is in $\mathcal{X}_T$. Hence the composition $M \xrightarrow{b} Z \xrightarrow{d} Z' \rightarrow \Sigma U$ also vanishes. Now Lemma 3.2 implies that there exist maps $e_1 : Y \rightarrow U$ and $e_2 : M \rightarrow N$, such that $p = re_1 + e_2c$. Since $U$ is in $\mathcal{X}_T$, this implies $p = e_2c$.

This shows that $\xi$ is a weak cokernel for $f$. It is also clear that $\xi$ is an epimorphism, using the triangle (2). It then follows from Lemma 2.5 that $\xi$ is actually a cokernel for $f$.

Proof of Theorem 3.1. The additivity of $\mathcal{C}/\mathcal{X}_T$ follows directly from the additivity of $\mathcal{C}$. By Lemma 3.4, we have that, for any map $f : X \rightarrow Y$, the induced map $\underline{f}$ has both a kernel and a cokernel.

In order to show that $\mathcal{C}/\mathcal{X}_T$ is also integral, we need to study its projective objects. According to [14], an object $P$ in a preabelian category (indeed, in any category) is said to be projective if, for any epimorphism $c : B \rightarrow C$, any morphism $f : P \rightarrow C$ factors through $c$:

$$
\begin{array}{ccc}
P & \rightarrow & C \\
\downarrow^f & & \downarrow^c \\
B & \xrightarrow{\xi} & C.
\end{array}
$$

We shall use this definition. But, we note that Rump [17, p. 170] uses a different definition; the above diagram should commute only for any cokernel $c$. Such objects are referred to as quasi-projectives in [16, Definition 7.5.2] and we shall use this terminology. Note that the two notions are the same in an abelian category, as then epimorphisms and cokernels coincide. The dual objects will be referred to as quasi-injectives.

In the following three proofs, we use some arguments based on the proof of Koenig and Zhu [12, Theorem 4.3].

**Lemma 3.5.** Every object in $\text{add}T$, when regarded as an object in $\mathcal{C}/\mathcal{X}_T$, is projective.

**Proof.** Let $f : X \rightarrow Y$ be an epimorphism in $\mathcal{C}/\mathcal{X}_T$, and $\xi : T_0 \rightarrow Y$ any morphism, where $T_0$ lies in $\text{add}T$. Completing $f$ to a triangle in $\mathcal{C}$, we have the diagram:

$$
\begin{array}{ccc}
T_0 & \rightarrow & X \\
\downarrow^u & & \downarrow^f \\
\rightarrow & \rightarrow & \rightarrow \\
\Sigma X & \rightarrow & \Sigma X.
\end{array}
$$

Since $f$ is an epimorphism, by Lemma 3.3, we have that $g$ factors through $\mathcal{X}_T$. Hence $gu = 0$, so $u$ factors through $f$ and thus $\xi$ factors through $\underline{f}$ as required.

**Lemma 3.6.** The category $\mathcal{C}/\mathcal{X}_T$ has enough projectives.

**Proof.** Let $X$ be an object in $\mathcal{C}/\mathcal{X}_T$ and let $f : T_0 \rightarrow X$ be a minimal right $\text{add}T$ approximation of $X$ in $\mathcal{C}$. Complete it to a triangle:

$$
\begin{array}{ccc}
U & \rightarrow & T_0 \\
\rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow \\
\Sigma U & \rightarrow & \Sigma U.
\end{array}
$$

By Wakamatsu’s Lemma (see Lemma 1.1), $U$ lies in $T^\perp$, so $\Sigma U$ lies in $\Sigma T^\perp = \mathcal{X}_T$. Hence, by Lemma 3.3, $\xi : T_0 \rightarrow X$ is an epimorphism, as required. It now follows from Lemma 3.5 that $\mathcal{C}/\mathcal{X}_T$ has enough projectives.
Dually, it can be shown that:

**Lemma 3.7.**
(a) Every object in \( \text{add}\Sigma^2 T \), regarded as an object in \( C/X_T \), is injective.
(b) The category \( C/X_T \) has enough injectives.

We recall:

**Proposition 3.8** [17, Corollary 2]. If \( A \) is a preabelian category with enough quasi-projectives (respectively, quasi-injectives), then \( A \) is left (respectively, right) semi-abelian.

It already follows from Proposition 3.8 that \( C/X_T \) is semi-abelian, using Lemmas 3.6 and 3.7. However, a modification of the argument in the proof of this result allows us to show:

**Proposition 3.9.** Let \( A \) be a preabelian category.

1. Suppose that \( A \) has enough projectives. Then \( A \) is left integral.
2. Suppose that \( A \) has enough injectives. Then \( A \) is right integral.
3. Suppose that \( A \) has enough projectives and enough injectives. Then \( A \) is integral.

**Proof.** As remarked above, we use an approach similar to the proof of [17, Corollary 2]. For (a), suppose we are given a pullback diagram:

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^{b} & & \downarrow^{c} \\
C & \rightarrow & D
\end{array}
\]

in \( C/X_T \) with \( d \) an epimorphism. Since \( A \) has enough projectives, there is an epimorphism \( a' : P \rightarrow B \) in \( C/X_T \) where \( P \) is projective in \( A \). Since \( d \) is an epimorphism, there is a map \( b' : P \rightarrow C \) in \( C/X_T \) such that \( ca' = db' \). Since the diagram (3) is a pullback, there is a map \( \varepsilon : P \rightarrow A \) such that \( \alpha \varepsilon = a' \) and \( \beta \varepsilon = b' \). Since \( a' \) is an epimorphism, so is \( a \), and (a) follows.

The proof of (b) is dual to the proof of (a), and (c) follows from (a) and (b).

We also remark that, for a semi-abelian category, left integrality is equivalent to right integrality (see [17, Corollary p. 173]).

**Corollary 3.10.** The category \( C/X_T \) is integral.

**Proof.** This follows from Lemmas 3.5–3.7, together with Proposition 3.9.

4. Localization

Let \( \mathcal{D} \) be a category. A class \( \mathcal{R} \) of morphisms in \( \mathcal{D} \) is said to admit a calculus of right fractions [4, I.2] provided that the following holds.
(RF1) The identity morphisms of $D$ lie in $R$ and $R$ is closed under composition.

(RF2) Any diagram of the form:

$$
\begin{array}{ccc}
B & \xrightarrow{f} & D \\
\downarrow & & \downarrow \\
C & \xrightarrow{r} & D
\end{array}
$$

with $r \in R$ has a completion to a commuting square of the following form:

$$
\begin{array}{ccc}
A & \xrightarrow{r'} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{r} & D
\end{array}
$$

with $r'$ in $R$.

(RF3) If $r: Y \to Y'$ lies in $R$ and $f, f': X \to Y$ are maps such that $rf = rf'$, then there is a map $r': X' \to X$ in $R$ such that $r'f = f'r'$.

There is a dual set of axioms, (LF1-3), for left fractions. Let us assume that $D$ is skeletally small, so that the Gabriel–Zisman localization $D_R$ of $D$ at $R$ exists. In this situation, $D_R$ has a very nice description; see [4, I.2] or [13, Section 3]. The objects in $D_R$ are the same as the objects of $D$. The morphisms from $X$ to $Y$ are right fractions, of the form:

$$
X \xleftarrow{r} A \xrightarrow{f} Y
$$
denoted $[r, f]_{RF}$, up to an equivalence relation: two such fractions $[r, f]_{RF}$ and $[r', f']_{RF}$ are equivalent if there is a commutative diagram of the form:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xleftarrow{r''} & A' \\
\downarrow & & \downarrow \\
X & \xleftarrow{r'} & A' \\
\downarrow & & \downarrow \\
A & \xrightarrow{f'} & Y
\end{array}
$$

where $r''$ lies in $R$.

The composition of two right fractions $[r', f']_{RF} \circ [r, f]_{RF}$ is given by the right fraction $[r r'', f f'']_{RF}$ where $r''$, a morphism in $R$, and $f''$, a morphism in $C$, are obtained from an application of axiom (RF2) which gives rise to the following commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & Z
\end{array}
$$

The localization functor from $D$ to $D_R$ takes a morphism $f$ to $[id, f]_{RF}$. We shall denote this image by $[f]$. For $r \in R$, $[r, id]_{RF}$ is the inverse of $[r]$ (that is, the formal inverse adjoined in the localization). We shall denote it $x_r$. Thus, every morphism in $D_R$ has the form $[r, f]_{RF} = [f]x_r$, where $f$ is a morphism in $D$ and $r$ lies in $R$. Similarly, if $D$ satisfies (LF1-3), there is a dual description of $D_R$ by left fractions, and so every morphism in $D_R$ can be written in the form $[g, s]_{LF} = x_s[g]$, where $g$ is a morphism in $D$ and $s$ lies in $R$. 
According to Rump [17, p. 173], the following result holds. We include a proof for the convenience of the reader.

**Proposition 4.1** [17, p. 173]. Let $\mathcal{A}$ be a semi-abelian category. Then $\mathcal{A}$ is integral if and only if the class $\mathcal{R}$ of regular morphisms in $\mathcal{A}$ admits a calculus of right fractions and a calculus of left fractions.

**Proof.** We first note that it follows from the definitions that $\mathcal{R}$ satisfies (RF1). Let $r, f$ and $f'$ be as in (RF3) above and suppose that $rf = rf'$. Then $r(f - f') = 0$. Since $r$ is regular, it is a monomorphism, so $f - f' = 0$ and $f = f'$. Thus, we can just take $r'$ to be the identity map on $X$ and we see that (RF3) is satisfied.

We now show that $\mathcal{A}$ is left integral if and only if (RF2) holds. Suppose first that $\mathcal{A}$ is left integral and we are given a diagram:

\[
\begin{array}{ccc}
B & \\
\uparrow f & \downarrow r & \\
C & \\
\uparrow f' & \downarrow & \\
D
\end{array}
\]

with $r$ regular. Let

\[
\begin{array}{ccc}
A & \to & C \\
\downarrow b & & \downarrow f \\
B & \to & D
\end{array}
\]

be the pullback of this diagram. Since $r$ is an epimorphism and $\mathcal{A}$ is left integral, $a$ is also an epimorphism. Since $r$ is a monomorphism, $a$ is a monomorphism by Lemma 2.3. Hence $a$ is also regular and we see that (RF2) holds.

Conversely, suppose that (RF2) holds and consider a pullback diagram of the form:

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow b & & \downarrow c \\
C & \to & D
\end{array}
\]

with $d$ regular. By (RF2), there is a commuting diagram

\[
\begin{array}{ccc}
A' & \to & B \\
\downarrow b' & & \downarrow c \\
C & \to & D
\end{array}
\]

with $a'$ regular. Since the diagram (5) is a pullback, we have a map $e: A' \to A$ making the diagram:

\[
\begin{array}{ccc}
A' & \to & A \\
\downarrow e & & \downarrow a \\
C & \to & D
\end{array}
\]
commute. Since \( a' \) is regular, it is an epimorphism, so \( a \) is also an epimorphism. Again by Lemma 2.3, the fact that \( d \) is a monomorphism implies that \( a \) is also a monomorphism. Hence \( a \) is regular. By Proposition 2.4, \( \mathcal{A} \) is integral, hence left integral.

Thus, we have seen that \( \mathcal{A} \) is left integral if and only if (RF2) holds, if and only if \( \mathcal{R} \) admits a calculus of right fractions. A similar argument shows that \( \mathcal{A} \) is right integral if and only if \( \mathcal{R} \) admits a calculus of left fractions. The result is proved.

We note that the proof shows that in fact:

**Corollary 4.2.** Let \( \mathcal{A} \) be a semi-abelian category. Then \( \mathcal{A} \) is integral if and only if the class \( \mathcal{R} \) of regular morphisms in \( \mathcal{A} \) admits a calculus of right fractions (respectively, a calculus of left fractions).

**Proof.** If \( \mathcal{A} \) is integral, then it is left integral. The proof of Proposition 4.1 shows that then RF1–3 are satisfied by \( \mathcal{R} \) and conversely that if RF1–3 are satisfied, then \( \mathcal{A} \) is integral. The statement for right fractions follows, and a dual argument shows the statement for left fractions.

In the rest of this section, we assume that \( \mathcal{A} \) is skeletally small, so that localizations exist.

**Remark 4.3.** We note that, by Gabriel–Zisman \([4, 3.3, \text{Corollary 2}]\), the localization of \( \mathcal{A} \) at \( \mathcal{R} \) in the situation of Proposition 4.1 is additive, since \( \mathcal{A} \) is additive and \( \mathcal{R} \) admits a calculus of right fractions. Furthermore, the localization functor is additive. See also \([19, 10.3.11]\).

**Lemma 4.4.** Let \( \mathcal{A} \) be an integral category and let \( \mathcal{R} \) be the class of regular morphisms in \( \mathcal{A} \). Then the localization functor \( L: \mathcal{A} \to \mathcal{A}_\mathcal{R} \) is faithful.

**Proof.** By Corollary 4.2 (and recalling 2.2), \( \mathcal{R} \) admits a calculus of right fractions. Let \( f: X \to Y \) be a morphism in \( \mathcal{A} \) and suppose that \( [f] = 0 \). Then we have a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{r''} & A'' & \xrightarrow{f''} & Y \\
\downarrow{id} & & \uparrow{u} & & \downarrow{id} \\
A' & \xrightarrow{v} & A & \xrightarrow{0} & \mathcal{A}
\end{array}
\]

We see that \( r'' = u = v \) is regular, and \( fu = 0 \). Since \( u \) is an epimorphism, \( f = 0 \) as required.

We remark that, as a consequence, \([r, f]_\mathcal{R} = [g, s]_\mathcal{L} \) if and only if \([f]x_r = x_s[g] \), if and only if \([sf] = [gr] \), if and only if \( sf = gr \), as noted in \([17, \text{p. 173}]\).

We note that:

**Lemma 4.5.** Let \( \mathcal{A} \) be an integral category and let \( \mathcal{R} \) be the class of regular morphisms in \( \mathcal{A} \). Then a morphism \( f \) in \( \mathcal{A} \) is an epimorphism if and only if \([f] \) is an epimorphism. It is a monomorphism if and only if \([f] \) is a monomorphism.
Proof. Suppose first that \([f]\) is an epimorphism and \(g\) is a morphism in \(\mathcal{A}\) for which \(gf = 0\). Then \([g][f] = [gf] = 0\). Hence \([g] = 0\), since \([f]\) is an epimorphism. By Lemma 4.4, \(g = 0\), so \(f\) is an epimorphism. Conversely, suppose that \(f\) is an epimorphism and that \((x_r[g])[f] = 0\) for a morphism \(g\) in \(\mathcal{A}\) and a regular morphism \(r\) in \(\mathcal{A}\). Then \([g][f] = 0\), so \([gf] = 0\), so by Lemma 4.4, \(gf = 0\). Hence \(g = 0\), so \([g] = 0\) and therefore \(x_r[g] = 0\) as required. So \([f]\) is an epimorphism. The monomorphism case is proved similarly. The result is proved.

Lemma 4.6. Let \(\mathcal{A}\) be an integral category and let \(\mathcal{R}\) be the class of regular morphisms in \(\mathcal{A}\). Let \(f, r\) be morphisms in \(\mathcal{A}\), with \(r\) regular, and let \(c\) be a cokernel of \(f\) in \(\mathcal{A}\). Then \([c]\) is a cokernel of \([f]x_r\) in \(\mathcal{A}_R\). Similarly, if \(j\) is a kernel of \(f\) in \(\mathcal{A}\) then \([j]\) is a kernel of \(x_r[f]\) in \(\mathcal{A}_R\).

Proof. We have \([c][f]x_r = [cf]x_r = [0]x_r = 0\). Suppose that \(g\) and \(s\) are morphisms in \(\mathcal{A}\), with \(s\) regular, and \((x_s[g])([f]x_r) = 0\). Then \(x_s[gf]x_r = 0\), so \([gf] = 0\), so \(gf = 0\) in \(\mathcal{A}\). Hence \(g\) factors through \(c\), so \([g]\) factors through \([c]\), so \(x_s[g]\) factors through \([c]\). Hence \([c]\) is a weak cokernel of \([f]x_r\). Since \(c\) is an epimorphism in \(\mathcal{A}\), it follows from Lemma 4.5 that \([c]\) is an epimorphism in \(\mathcal{A}\). Therefore, by Lemma 2.5, \([c]\) is a cokernel of \([f]x_r\). The result for kernels is proved similarly.

We recall that every morphism \(f: X \to Y\) in a preabelian category \(\mathcal{A}\) has a factorization of the form:

\[
\begin{array}{c}
X \xrightarrow{u} \text{coim}(f) \xrightarrow{\tilde{f}} \text{im}(f) \xrightarrow{v} Y \\
\end{array}
\]

and \(\mathcal{A}\) is abelian if and only if \(\tilde{f}\) is an isomorphism for all morphisms \(f\) in \(\mathcal{A}\).

Lemma 4.7 [17, p. 167]. Let \(\mathcal{A}\) be a preabelian category. Then \(\mathcal{A}\) is semi-abelian if and only if \(\tilde{f}\) is regular for all morphisms \(f\) in \(\mathcal{A}\).

Proof. By definition (see Section 2), if \(\mathcal{A}\) is semi-abelian, then every morphism \(f\) has a factorization of the form \(ip\) where \(p = \text{coim}(f)\) and \(i\) is a monomorphism. Comparing this with the factorization above we see that \(i = v\tilde{f}\) and thus, since \(i\) is a monomorphism, so is \(\tilde{f}\). Dually, we see that \(\tilde{f}\) is an epimorphism, and hence regular. Conversely, suppose that for all morphisms \(f\) in \(\mathcal{A}\), \(\tilde{f}\) is regular. Then, in the factorization above, \(v\tilde{f}\) must be a monomorphism as \(v\) and \(f\) are. Dually, \(fu\) is an epimorphism and we see that \(\mathcal{A}\) is semi-abelian as required.

According to [17], we have the following theorem. Again we give details for the interested reader.

Theorem 4.8 [17, p. 173]. Let \(\mathcal{A}\) be an integral category. Then the localization \(\mathcal{A}_R\) (if it exists) of \(\mathcal{A}\) at the class of regular morphisms is an abelian category.

Proof. As we have already observed, by Gabriel and Zisman [4, 3.3, Corollary 2], \(\mathcal{A}_R\) is an additive category. By Lemma 4.6, \(\mathcal{A}_R\) is preabelian. Since \(\mathcal{A}\) is semi-abelian (by Proposition 2.2), it follows from Lemma 4.7 that in the factorization:

\[
\begin{array}{c}
X \xrightarrow{u} \text{coim}(f) \xrightarrow{\tilde{f}} \text{im}(f) \xrightarrow{v} Y \\
\end{array}
\]

of any morphism \(f\) in \(\mathcal{A}\), \(\tilde{f}\) is regular. It is easy to check that applying the localization functor to this factorization gives the corresponding factorization of \([f]\). It follows that for morphisms
of the form $\alpha = [f]$ in $A_R$, $\alpha$ is invertible. Since every morphism in $A_R$ can be obtained by composing a morphism of this form with an invertible morphism in $A_R$, it follows that $\tilde{u}$ is invertible for all morphisms $u$ in $A_R$ and hence that $A_R$ is abelian as required.

5. The localization of $C/X_T$ is equivalent to mod $\Gamma$

In this section, we show that $(C/X_T)_R$ is isomorphic to mod $\Gamma$, where as before $\Gamma = \text{End}_C(T)^{op}$.

We have seen (Corollary 3.10) that $C/X_T$ is an integral category. Since we assume $C$ is skeletally small, $C/X_T$ is also skeletally small. Applying Theorem 4.8 to the integral category $C/X_T$ we see that $(C/X_T)_R$ is abelian:

**Theorem 5.1.** Let $C$ be a skeletally small, Hom-finite, Krull–Schmidt triangulated category with Serre duality, containing a rigid object $T$. Let $X_T$ denote the class of objects $X$ in $C$ such that $\text{Hom}_C(T,X) = 0$. Then the class $R$ of regular morphisms in $C/X_T$ admits a calculus of left fractions and a calculus of right fractions. Furthermore, the localization $(C/X_T)_R$ of $C/X_T$ at the class $R$ is abelian.

**Remark 5.2.** We have seen that the localization $(C/X_T)_R$ inherits an additive structure from $C/X_T$ (see Remark 4.3). The localization $(C/X_T)_R$ also inherits a $k$-additive structure from $C/X_T$: a scalar $\lambda$ takes the fraction $[r,f]_R$ to $[r,\lambda f]_R$. It can be checked that this action is well defined and, together with the additive structure inherited from $C/X_T$, gives a $k$-additive structure on $(C/X_T)_R$ for which the localization functor is $k$-additive.

We will show that the projectives in $(C/X_T)_R$ are the objects in $\text{add}T$ and that $(C/X_T)_R$ has enough projectives. From this, it will follow that $(C/X_T)_R$ is equivalent to mod $\Gamma$.

**Lemma 5.3.** Let $A$ be an additive category and $P$ a projective object in $A$. If $r: U \to P$ is a regular morphism, then it is an isomorphism.

**Proof.** Since $r$ is an epimorphism, the identity map on $P$ factors through $r$, so there is a morphism $s: P \to U$ such that $rs = id$. Then $r(sr - id) = (rs)r - r = 0$. Since $r$ is a monomorphism, $sr - id = 0$, and it follows that $r$ is an isomorphism. □

**Lemma 5.4.** Let $A$ be a skeletally small integral category and $R$ the class of regular morphisms in $A$. Suppose that $P$ is a projective object in $A$. Then $P$, when regarded as an object in the localization $A_R$, is again a projective object.

**Proof.** Suppose $P$ is projective in $A$ and we have a diagram:

\[
\begin{array}{c}
P \\
| \quad \downarrow \quad | \\
X \\
| \quad \downarrow \quad | \\
Y
\end{array}
\]

in $A_R$ with $p, f, r$ and $s$ morphisms in $A$, $s: U \to P$ and $r$ regular morphisms and $[f]_{x_r}$ an epimorphism in $A_R$. Since $s$ is regular and $P$ is projective in $A$, $s$ is an isomorphism by Lemma 5.3. Hence $P \simeq U$ in $A$, so $U$ is also projective in $A$. By Lemma 4.5, $f$ is an epimorphism in $A$, so $p$ factors through $f$. It follows that $[p]_{x_s}$ factors through $[f]_{x_r}$ as required. □
Lemma 5.5. (a) The projectives in \((C/X_T)_R\) are exactly the objects in \(\text{add} T\).
(b) The category \((C/X_T)_R\) has enough projectives.

Proof. By Lemmas 5.4 and 3.5, the objects in \(\text{add} T\) are projective in \((C/X_T)_R\). Let \(X\) be an object in \((C/X_T)_R\). Then, by Lemma 3.6, there is an epimorphism \(T_0 \to X\) in \(C/X_T\), where \(T_0\) lies in \(\text{add} T\), and (b) follows, using Lemma 4.5. If \(X\) is projective, the identity map on \(X\) factors through \([p]\), so \([p]\) is a split epimorphism and \(X\) is isomorphic to a summand of \(T_0\), hence in \(\text{add} T\), and (a) follows.

Lemma 5.6. We have that \(\text{End}_{(C/X_T)_R}(T) \cong \text{End}_C(T)\).

Proof. The localization functor induces a morphism \(\varphi: \text{End}_{(C/X_T)_R}(T) \to \text{End}_{(C/X_T)_R}(T)\).

If \([f]_{x_T}\) is an arbitrary element of \(\text{End}_{(C/X_T)_R}(T)\) with \(f, x_T\) morphisms in \(C/X_T\) and \(x_T\) regular, then \(x_T\) is an isomorphism in \(C/X_T\) by Lemma 5.3, so \(x_T = [v^{-1}]\) and we see that \(\varphi\) is surjective. By Lemma 4.4, it is also injective, so

\(\text{End}_{(C/X_T)_R}(T) \cong \text{End}_{(C/X_T)_R}(T)\).

The result follows, since the only homomorphisms from \(T\) to objects in \(X_T\) are zero by definition.

Theorem 5.7. Let \(C\) be a skeletally small, Hom-finite, Krull–Schmidt triangulated category with Serre duality, containing a rigid object \(T\). Let \(X_T\) denote the class of objects \(X\) in \(C\) such that \(\text{Hom}_C(T, X) = 0\). Let \(R\) denote the class of regular morphisms in \(C/X_T\) and \((C/X_T)_R\) the localization of the integral category \(C/X_T\) at \(R\). Then

\((C/X_T)_R \cong \text{mod} \text{End}_C(T)^{\text{op}}\).

Proof. By Theorem 5.1, \((C/X_T)_R\) is an abelian category. By Lemma 5.5, \((C/X_T)_R\) has enough projectives, given by the objects in \(\text{add} T\). The result follows, with an equivalence being given by the functor \(\text{Hom}_{(C/X_T)_R}(T, -)\), noting that \(T\) is a projective generator for \((C/X_T)_R\).

We note that, by Remark 5.2, \((C/X_T)_R\) inherits a \(k\)-additive structure from \(C/X_T\). It is easy to see that the above equivalence preserves this structure (as well as the abelian structure).

6. Cotorsion pairs

We recall the notion of a cotorsion pair in a triangulated category, considered in Nakaoka [15]. By Nakaoka [15, 2.3] this can be defined as a pair \((U, V)\) of full additive subcategories satisfying the following:

(a) \(U^\perp = V\);
(b) \(V^\perp = U\);
(c) for any object \(C\), there is a (not necessarily unique) triangle:
\[ U \to C \to \Sigma V \to \Sigma U, \]
with \(U \in U\) and \(V \in V\).
Nakaoka points out that \((\mathcal{U}, \mathcal{V})\) is a cotorsion pair in this sense if and only if \((\mathcal{U}, \Sigma \mathcal{V})\) is a torsion theory in the sense of [8, 2.2]. This is distinct from the notion of torsion theory given in [1, Definition 2.1], since the assumption of closure under the suspension functor is omitted.

If \(C\) is a triangulated category as in Section 1 and \(T\) is a rigid object in \(C\), then \((\text{add} T, T^\perp)\) is a cotorsion pair (for example, see [2, Section 6]). One might then ask whether Theorem 3.1 can be generalized to this set-up. However, it is easy to see that this cannot be the case. Consider a triangulated category \(\mathcal{C}\), satisfying our usual assumptions. Assume that \(\mathcal{C}\) has a non-zero non-isomorphism between two indecomposables. Then this map does not have a cokernel or a triangulated category \(\mathcal{C}\) be generalized to this set-up. However, it is easy to see that this cannot be the case. Consider a cotorsion pair \((\mathcal{U}, \mathcal{V})\) such that \(\mathcal{C}/\mathcal{V}\) is not preabelian.

More interesting examples, where \(\mathcal{C}/\mathcal{V}\) is not preabelian also exist. Let \(\mathcal{C}\) be the cluster category associated to the quiver:

\[
1 \leftarrow 2 \rightarrow 3.
\]

For a vertex \(i\), let \(P_i\) (respectively, \(I_i\), \(S_i\)) denote the corresponding indecomposable projective (respectively, injective, simple) module. Let \(\mathcal{U}\) be the additive subcategory whose indecomposable objects are \(P_2, P_3\) and \(\Sigma P_3\). It is easy to check that \(\mathcal{U}^\perp\) is the additive subcategory with indecomposables given by \(P_1, P_2\) and \(S_2\) and that \((\mathcal{U}, \mathcal{U}^\perp)\) is a cotorsion pair. Note that the torsion pair \((\mathcal{U}, \Sigma \mathcal{U}^\perp)\) appears in [7].

Let \(f\) be a non-zero map from \(P_3\) to \(I_2\). Suppose that \(c: I_2 \rightarrow C\) is a cokernel of \(f\) in \(C/\mathcal{U}^\perp\). Since the only non-zero maps \(g: I_2 \rightarrow Y\) with \(Y\) indecomposable such that \(g f = 0\) have \(Y = \Sigma P_1\) or \(Y = \Sigma P_2\), it follows that \(C\) is a direct sum of copies of \(\Sigma P_1\) and \(\Sigma P_2\). Then \(d c = 0\) for any non-zero map \(d\) from \(C\) to \(\Sigma P_3\), a contradiction to the fact that \(c\) is an epimorphism.

7. The functor \(\text{Hom}_C(T, -): \mathcal{C} \rightarrow \text{mod} \Gamma\)

Let \(T\) denote a rigid object in a triangulated category \(\mathcal{C}\), where \(\mathcal{C}\) satisfies the same properties as earlier, and let \(\Gamma = \text{End}_C(T)^{op}\). We have seen that we can obtain \(\text{mod} \Gamma\) in a process consisting of two steps: first forming the preabelian factor category \(\mathcal{C}/\mathcal{X}_T\), and then localizing this category with respect to the class of regular morphisms.

In [2], we considered the functor \(H = \text{Hom}_C(T, -): \mathcal{C} \rightarrow \text{mod} \Gamma\). Let \(\mathcal{S} = \mathcal{S}_T\) be the collection of maps \(f: X \rightarrow Y\) in \(\mathcal{C}\) with the property that in the induced triangle

\[
\Sigma^{-1} Z \xrightarrow{h} X \xrightarrow{f} Y \oplus Z,
\]

both \(g\) and \(h\) factor through \(\mathcal{X}_T\). Let \(L_S: \mathcal{C} \rightarrow \mathcal{C}_S\) denote the Gabriel–Zisman localization. We proved in [2] that there is an equivalence \(G: \mathcal{C}_S \rightarrow \text{mod} \Gamma\) such that \(GL_S = H\). We also proved that a map \(s\) belongs to \(\mathcal{S}\) if and only if \(H(s)\) is an isomorphism in \(\text{mod} \Gamma\).

In this section, we point out that the functor \(H\) is actually naturally equivalent to the composition of the quotient functor \(\mathcal{C} \rightarrow \mathcal{C}/\mathcal{X}_T\) and the localization functor with respect to regular morphisms.

Consider the set of maps \(\mathcal{S}_0 = \{X \sqcup U \rightarrow X \mid U \in \mathcal{X}_T, X \in \mathcal{C}\}\) and the Gabriel–Zisman localization \(L_{\mathcal{S}_0}: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{S}_0}\). The quotient functor \(Q: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{X}_T\) inverts all maps in \(\mathcal{S}_0\), so there is a functor \(G_0: \mathcal{C}_{\mathcal{S}_0} \rightarrow \mathcal{C}/\mathcal{X}_T\), making the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{Q} & \mathcal{C}/\mathcal{X}_T \\
\downarrow_{L_{\mathcal{S}_0}} & & \downarrow_{G_0} \\
\mathcal{C}_{\mathcal{S}_0}
\end{array}
\]

The localization functor \(L_{\mathcal{S}_0}\) has the following elementary properties.
Lemma 7.1.  (a) For $U$ in $\mathcal{X}_T$, consider the projection map $\pi_X : X \amalg U \rightarrow X$. The inverse of $L_{S_0}(\pi_X)$ is $L_{S_0}(\iota_X)$, where $\iota_X : X \rightarrow X \amalg U$ is the canonical inclusion map.

(b) Let $u$ and $v$ be maps in $\mathcal{C}$ such that $v$ factors through $\mathcal{X}_T$. Then $L_{S_0}(u + v) = L_{S_0}(u)$ in $\mathcal{C}_{S_0}$.

Proof. The proof is identical to the proof of Lemma 3.5 in [2].

By construction, $G_0$ is the identity on objects. It is clear that $G_0$ is full, since $Q$ has this property.

We claim that $G_0$ is also faithful. First, note that by Lemma 7.1(a), $L_{S_0}$ is full. So let $f$ and $f'$ be maps in $\mathcal{C}$ with $G_0L_{S_0}(f) = G_0L_{S_0}(f')$. Then $H(f) = H(f')$, so $f - f'$ factors through $\mathcal{X}_T$ (by [2, Lemma 2.3]), and hence, by Lemma 7.1, we have that $L_{S_0}(f) = L_{S_0}(f' + f - f') = L_{S_0}(f')$. Hence, we have the following.

Proposition 7.2. The induced functor $G_0 : \mathcal{C}_{S_0} \rightarrow \mathcal{C}/\mathcal{X}_T$ is an isomorphism of categories.

By Lemma 3.3, a morphism $f$ in $\mathcal{C}/\mathcal{X}_T$ is regular if and only if $f$ lies in $S$. Combining this with Proposition 7.2, we see that the image $L_{S_0}(S)$ in the preabelian category $\mathcal{C}_{S_0}$ consists of exactly the regular morphisms.

Let $\mathcal{R}$ denote the regular morphisms in $\mathcal{C}/\mathcal{X}_T$. By the universal property of localization, it follows that we also get an induced isomorphism of categories

$$K : (\mathcal{C}_{S_0})_{L_{S_0}(S)} \rightarrow (\mathcal{C}/\mathcal{X}_T)_R$$

making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{C}/\mathcal{X}_T & \xrightarrow{L_{\mathcal{R}}} & (\mathcal{C}/\mathcal{X}_T)_R \\
\downarrow{G_0} & & \downarrow{H} \\
\mathcal{C}_{S_0} & \xrightarrow{L_{S_0}(S)} & (\mathcal{C}_{S_0})_{L_{S_0}(S)} \\
\end{array}
$$

It is clear that $H$ factors uniquely through $Q$, and hence by the universal property of localization, also uniquely through $L_{\mathcal{R}} : \mathcal{C}/\mathcal{X}_T \rightarrow (\mathcal{C}/\mathcal{X}_T)_R$, so we have a commutative diagram of functors:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{Q} & \mathcal{C}/\mathcal{X}_T \\
& \searrow{H} & \downarrow{H'} \\
& \mathcal{C}/\mathcal{X}_T & \xrightarrow{H'} \mod \text{End}_{\mathcal{C}}(T)^{\text{op}} \\
\end{array}
$$

Lemma 7.3. The functor $H'$ is naturally isomorphic to the functor $\text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_R}(T, -)$ which gives an equivalence between $(\mathcal{C}/\mathcal{X}_T)_R$ and $\mod \text{End}_{\mathcal{C}}(T)^{\text{op}}$ in Theorem 5.7.

Proof. First, we note that the map $\varphi_X : f \mapsto [f]$ gives an isomorphism from $H'(X) = \text{Hom}_{\mathcal{C}}(T, X)$ to $\text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_R}(T, X)$ as $\text{End}_{\mathcal{C}}(T)^{\text{op}}$-modules (arguing as in the proof of Lemma 5.6). Secondly, let $u = x_1[f] \in \text{Hom}_{(\mathcal{C}/\mathcal{X}_T)_R}(X, Y)$ be an arbitrary morphism, where $f : X \rightarrow Z$ and $r : Y \rightarrow Z$ are morphisms in $\mathcal{C}$ for some object $Z$ in $\mathcal{C}$ and $r$ is regular in $\mathcal{C}/\mathcal{X}_T$. 

Consider the diagram:

\[
\begin{array}{ccc}
H'(X) & \xrightarrow{\varphi_X} & \text{Hom}_{\mathcal{C}/\mathcal{X}_T}(T, X) \\
H'(u) & \downarrow & \text{Hom}_{\mathcal{C}/\mathcal{X}_T}(T, u) \\
H'(Y) & \xrightarrow{\varphi_Y} & \text{Hom}_{\mathcal{C}/\mathcal{X}_T}(T, Y).
\end{array}
\]

Let \( \alpha \in H'(X) = \text{Hom}_\mathcal{C}(T, X) \). Then:

\[
H'(u)(\alpha) = H'(x_r)[f](\alpha) = H'(x_r)(f \alpha) = \text{Hom}_\mathcal{C}(T, r)^{-1}(f \alpha) = g,
\]

where \( f \alpha = rg \). Hence, \( \varphi_Y(H'(u)(\alpha)) = [g] \). We also have

\[
\text{Hom}_{\mathcal{C}/\mathcal{X}_T}(T, u)(\varphi_X(\alpha)) = \text{Hom}_{\mathcal{C}/\mathcal{X}_T}(T, x_r)[f](\alpha)
\]

\[
= x_r[f][\alpha]
\]

\[
= x_r[r][g] = [g],
\]

so the diagram commutes and the lemma is proved.

It follows that \( H' \) is an equivalence. 

We also have a commutative diagram of functors:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{L_{S_0}} & \mathcal{C}_{S_0} & \xrightarrow{L'_{S_0}(s)} & (\mathcal{C}_{S_0})_{L_{S_0}(s)} & \xrightarrow{L'} & \mathcal{C}_S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{L_{S_0}} & \mathcal{C}_{S_0} & \xrightarrow{L'_{S_0}(s)} & (\mathcal{C}_{S_0})_{L_{S_0}(s)} & \xrightarrow{L'} & \mathcal{C}_S
\end{array}
\]

in which \( L' \) is an isomorphism of categories. This follows from the universal property satisfied by the localization functors involved.

Let \( G^{-1} \) denote a quasi-inverse of \( G \). Summarizing, we have:

**Proposition 7.4.** We have the following diagram of functors. The diagram commutes, apart from the rightmost square, which commutes only up to natural isomorphism.
\textbf{Proof.} We have checked above that the diagram commutes apart from the rightmost square. We recall that, for a localization functor $L$ and two functors $J, J'$ composable with it, $JL = J'L$ implies that $J = J'$, by the universal property. We have that

$$GL'L_{S_0(S)}L_{S_0} = GLS = H = H'L_RQ = H'L_RG_0L_{S_0} = H'KL_{S_0(S)}L_{S_0},$$

so $GL' = H'K$. It follows that $G^{-1}H'$ is naturally equivalent to $L'K^{-1}$.

We remark that the fact that $L'$ is an isomorphism implies that $H'$ is an equivalence of categories, by the commutativity of the right-hand square. Thus, the equivalence in Theorem 5.7 can also be derived from the fact that $G$ is an equivalence (that is, \cite[Theorem 4.3]{2}) together with the above analysis.

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