OPTIMAL RECONSTRUCTION OF A PURE QUBIT STATE
WITH LOCAL MEASUREMENTS

E. BAGAN, M. BAIG, A. MONRAS AND R. MUÑOZ-TAPIA
Grup de Física Teòrica & IFAE, Facultat de Ciències, Edifici Cn, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona) Spain

We analyse the reconstruction of an unknown pure qubit state. We derive the optimal guess that can be inferred from any set of measurements on $N$ identical copies of the system with the fidelity as a figure of merit. We study in detail the estimation process with individual von Neumann measurements and demonstrate that they are very competitive as compared to (complicated) collective measurements. We compute the expressions of the fidelity for large $N$ and show that individual measurement schemes can perform optimally in the asymptotic regime.

1 Introduction

State estimation is a fundamental issue in Quantum Information from both theoretical and practical points of view. Imagine we are asked to reconstruct the unknown state of a quantum system. This can only be done by performing measurements on an ensemble of identically prepared systems. With an infinite ensemble of copies, the state could be determined exactly. In practice, however, we have access to a limited number of copies and the state can only be determined approximately. In this context, three essential questions arise: i) what is the optimal measurement scheme?, ii) what is the best reconstructed state?, and iii) how good is the overall estimation process?.

In recent years a lot of work has been devoted to answer these questions for different settings. The optimal strategies, which provide the ultimate limits that can be achieved, have been identified in several interesting cases. However, they usually involve collective measurements (CM), i.e. a generalised measurement on all copies at the same time. These, although very interesting from the theoretical point of view, are very difficult to implement in practice. Far more interesting for experimentalists are individual von Neumann measurements for they can be readily performed in a laboratory. In this case however, fewer analytical results are known. Here, we present some theoretical results along these lines.

We focus on the problem of estimating the most basic quantum state, a pure qubit, with physically realizable von Neumann measurements. We study quantitatively tomographic inspired schemes, but also consider the most general individual measurement procedure, i.e. when depending on the previous outcomes, one allows to optimally adapt the measurement on the subsequent copy. To ease our presentation, we will loosely write "local measurements" for individual von Neumann measurements. Our aim is to investigate how good these local measurements are as compared to the collective
ones. We use the fidelity as the figure of merit quantifying the quality of the estimation process and compute the analytical expressions of the average fidelity for large $N$. Two interesting situations will be analysed which we will refer to as 2D and 3D. In 2D the qubit is only known to be a state of the equator of the Bloch sphere. This is formally equivalent to phase estimation. In 3D no prior knowledge of the qubit is assumed.

This paper is organised as follows. In next section we obtain the optimal state that can be inferred from any set of measurements. This is a general result valid for any measurement scheme (either collective or local) and for any a priori probability distribution. The 2D and 3D case with fixed (non-adaptive) local measurements is studied in section 3. The most general local scheme is presented in section 4. We conclude with a summary and outlook for further work.

2 Optimal guess

The estimation procedure goes as follows. Assume that we are given an ensemble of $N$ copies of the qubit state, which we denote by $|\vec{n}\rangle$, where $\vec{n}$ is the unique unit vector on the Bloch sphere that satisfies $|\langle \vec{n}|\vec{n}\rangle| = (1 + \vec{n} \cdot \vec{\sigma})/2$ and $\vec{\sigma}$ are the usual Pauli matrices. After performing a set of measurements on the $N$ copies of the qubit, one obtains a set of outcomes symbolically denoted by $x$. Based on $x$, an estimate for $|\vec{n}\rangle$ is guessed, $|\vec{M}(x)\rangle$. How well $|\vec{M}(x)\rangle$ approximates the signal state $|\vec{n}\rangle$ is quantified by the fidelity, defined as the overlap

$$f_n(x) \equiv |\langle \vec{n}|\vec{M}(x)\rangle|^2 = \frac{1 + \vec{n} \cdot \vec{M}(x)}{2}. \quad (1)$$

Eq. (1) is a kind of “score”: we obtain ‘1’ for a perfect determination ($\vec{M} = \vec{n}$) and ‘0’ for a completely wrong guess ($\vec{M} = -\vec{n}$). Our aim is to maximize the average fidelity, hereafter fidelity in short, over the initial probability and all possible outcomes,

$$F \equiv \langle f \rangle = \sum_x \int dn \ f_n(x) \ p_n(x), \quad (2)$$

where $p_n(x)$ is the probability of getting outcome $x$ if the signal state was $|\vec{n}\rangle$, and $dn$ is the a priori probability distribution. For a completely unknown qubit, $dn$ is the invariant measure on the two-sphere (on the unit circle in 2D). Eqs. (1) and (2) can be rewritten as

$$\vec{V}(x) = \int dn \vec{n} p_n(x). \quad (3)$$

It is obvious that the choice

$$\vec{M}(x) = \frac{\vec{V}(x)}{|\vec{V}(x)|}. \quad (4)$$

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maximizes the value of $F$, which then reads

$$F = \frac{1}{2} \left( 1 + \sum_x |\tilde{V}(x)| \right).$$  \tag{5}

Eq. 4 gives the best inferred state and Eq. 5 the maximum fidelity that can be obtained for any a priori probability and any measurement scheme specified by the conditional probabilities $p_n(x)$.

In the next sections we show how these simple results can be used to improve the estimation procedure. From now on we will only consider the 2D and 3D isotropic probability distributions given by $dn = d\phi/(2\pi)$ and $dn = \sin \theta d\theta d\phi/(4\pi)$, respectively. A fidelity with no explicit label refers to the 3D case.

3 Fixed local measurements

Before dealing with local measurements, let us recall some known results of the collective schemes. The optimal fidelity for the 2D case is

$$F_{CM}^{(2D)} = \frac{1}{2} + \frac{1}{2N+1} \sum_i \sqrt{\binom{N}{i} \binom{N}{i+1}} \xrightarrow{N \to \infty} 1 - \frac{1}{4N} + \cdots,$$  \tag{6}

whereas for the 3D case reads

$$F_{CM} = \frac{N + 1}{N + 2} \xrightarrow{N \to \infty} 1 - \frac{1}{N} + \cdots.$$  \tag{7}

These results could, in principle, also be derived from Eq. 5. Notice that $F_{CM}^{(2D)} > F_{CM} \forall N$, as it should, since in the 2D case we have more a priori information about the state than in the 3D case. These results are the absolute upper bound for any measurement scheme.

Let us now turn our attention to local measurements. Any individual von Neumann measurement is represented by two projectors $O(\pm \vec{m}) = (1 \pm \vec{m} \cdot \vec{\sigma})/2$, where $\vec{m}$ is a unit Bloch vector characterizing the measurement (in a spin system, e.g., $\vec{m}$ is the orientation of a Stern-Gerlach). Quantum state tomography tells us that, given a large number of copies, von Neumann measurements along two (three) fixed orthogonal directions, $x, y, (z)$, are sufficient to reconstruct the state.

Consider $N = 2N \ (3N)$ copies of the state $|\vec{n}\rangle$. After $N$ measurements in each axis, we obtain a set of outcomes +1 and −1 with relative frequencies $\alpha_i$ and $1 - \alpha_i$, respectively. This occurs with probability

$$p(\alpha|\vec{m}) = \prod_{i=x,y,z} \left( \binom{N}{\alpha_i} \right) \left( \frac{1 + \alpha_i}{2} \right)^{N\alpha_i} \left( \frac{1 - \alpha_i}{2} \right)^{N(1-\alpha_i)},$$  \tag{8}
where \( n_i \) are the projections of the vector \( \vec{n} \) in each direction and we have used the shorthand notation \( \alpha = \{ \alpha_i \} \). Since the expectation value of \( \vec{\sigma} \) is 
\[
\langle \vec{n} | \vec{\sigma} | \vec{n} \rangle = \vec{n}
\]
one is driven to propose a guess 
\[
M_{T,i}(\alpha) = \frac{2\alpha_i - 1}{\sqrt{\sum_j (2\alpha_j - 1)^2}}, \tag{9}
\]
where the subscript stands for tomographic. Notice the presence of a normalization factor such that \( |\vec{M}_{T}| = 1 \), therefore \( \vec{M}_{T} \) always corresponds to a physical pure state. Actually, (9) is the guess for pure states of maximum likelihood procedures.\(^{11}\) The law of large numbers ensures that \( \vec{M}_{T} \xrightarrow{N \to \infty} \langle \vec{n} | \vec{\sigma} | \vec{n} \rangle = \vec{n} \), but our main goal is to know the rate at which this limit is attained.

The asymptotic fidelity can essentially be computed by means of the following systematic approximations (see Bagan et al.\(^{5,8,12}\) for more details). First, use the central limit approximation in (8)
\[
\left( \frac{N}{\alpha N} \right) q^{\alpha N} (1-q)^{(1-\alpha) N} \approx \frac{1}{\sqrt{2\pi N q(1-q)}} \exp\left( -\frac{N (\alpha - q)^2}{2 q(1-q)} \right) + \cdots \tag{10}
\]
with \( q = (1 + n_j)/2 \). Second, transform the discrete sum into an integral using the Euler-McLaurin formula
\[
\sum_{j=1}^{N} \frac{1}{N} f(j/N) = \int_{0}^{1} dx f(x) + \frac{f(1) - f(0)}{2N} + \frac{f'(1) - f'(0)}{12N^2} + \cdots. \tag{11}
\]
The change of variables \( r_i = 2\alpha_i - 1 \), suggested by (9), proves to be useful to simplify the expressions. Finally, use saddle point techniques to evaluate the integrals. This just amounts to consider that the value of the integrals is dominated by the minimum of the exponent in (10) and to expand systematically around this point.

### 3.1 2D results.

For the tomographic guess (9), and using the techniques described above, we obtain the following asymptotic expression of the fidelity (2):
\[
F_{T}^{(2D)} = 1 - \frac{3}{8} \frac{1}{N} + \cdots \tag{12}
\]
Note that \( F_{T}^{(2D)} \) approaches unity linearly in \( 1/N \). In this sense, one may argue that the tomographic approach is qualitatively similar to the optimal collective scheme. Nevertheless, the coefficient of the first correction is a 50\% larger than the optimal one (6).

From our discussion in section 2 we know that there is a better guess that can be inferred from the same set of measurements. It is given by (4), with the outcomes labelled by \( x = \{ \alpha_x, \alpha_y \} \) and the probabilities again given by (8). The fidelity is then \( F_{OG} = 1/2(1 + \sum_{\alpha} |\vec{V}(\alpha)|) \). The analytical calculation
of the large $N$ limit is now more involved, mainly due to the presence of the modulus, but can be performed basically with the same techniques. It reads
\[
F_{\text{OG}}^{(2D)} = 1 - \frac{1}{4N} + \cdots, \tag{13}
\]
where OG stands for optimal guess. This is a remarkable result. Provided the optimal guess is used, the most basic estimation strategy, namely with local and minimal fixed von Neumann measurements, saturates asymptotically the optimal CM bound (6).

### 3.2 3D results

The same analysis can be carried out in the 3D case, i.e. when $|\vec{n}\rangle$ is a completely unknown qubit pure state. The calculations are rather more difficult, but can be done analytically till the end. For the tomographic guess \((9)\) we obtain
\[
F_T = 1 - \frac{6}{5N} + \cdots, \tag{14}
\]
whereas for the optimal guess
\[
F_{\text{OG}} = 1 - \frac{13}{12N} + \cdots. \tag{15}
\]
As expected, $F_{\text{OG}} > F_T$. Notice that, again, the first correction of the fidelity goes linearly with $1/N$, now with a coefficient very close to one. However, in contrast to the 2D case, the improvement of the optimal guess is not sufficient to saturate the CM bound for which $F = 1 - 1/N + \cdots$.

### 4 Optimal local measurements

The local measurements discussed so far were the most basic ones: fixed and minimal. We have not considered yet local schemes in full. In particular, classical communication, i.e. the possibility to adapt the orientation of the measuring devices depending on previous outcomes, was not exploited. In this section, we obtain the optimal scheme in this general setting and show explicit results for low $N$. For large $N$, we also obtain the asymptotic expression of the fidelity. Hereafter only the general case 3D will be considered.

We need first to introduce a suitable notation to include arbitrary orientations of the devices and classical communication. Consider the set of von Neumann measurements specified by the collection of Bloch vectors \{$\vec{m}_k$\}. The set of outcomes $x$ can be expressed as an $N$-digit binary number $x = i_N i_{N-1} \cdots i_2 i_1$, where $i_k (= 0, 1)$. The most general local measurement is realized when we allow $\vec{m}_{k+1}$ to depend also on the list of previous outcomes.
\( i_k i_{k-1} \cdots i_2 i_1 \equiv x_k \) (hence, \( x = x_N \)). We thus write \( \vec{m}(x_k) \) instead of \( \vec{m}_k \).

Note that \( \vec{m}(x_k) \) must satisfy the von Neumann condition

\[
\vec{m}(1x_{k-1}) = -\vec{m}(0x_{k-1}).
\]

(16)

For any set of outcomes, the optimal guess is given by (4) and (3), where now the conditional probability is

\[
p_n(x) = \prod_{k=1}^{N} \frac{1 + \vec{n} \cdot \vec{m}(x_k)}{2}
\]

(17)

and the fidelity reads

\[
F = \frac{1}{2} \left( 1 + \sum_{x=00\cdots0}^{2^{N-1}} \int d\vec{n} \prod_{k=1}^{N} \frac{1 + \vec{n} \cdot \vec{m}(x_k)}{2} \right). 
\]

(18)

The optimal scheme is the one that maximizes (18) over a set of vectors \( \{\vec{m}(x_k)\} \) with the von Neumann constraint (16).

4.1 Low \( N \) cases

\( N = 2 \). Here, there are three independent Bloch vectors: \( \vec{m}(0) \), \( \vec{m}(00) \), and \( \vec{m}(01) \) (the other three are obtained using Eq. (16)). The first vector \( \vec{m}(0) \) is arbitrary and can be fixed at will. The optimal fidelity is then obtained by maximizing (18) with respect to \( \vec{m}(00) \) and \( \vec{m}(01) \). A straightforward calculation yields the following conditions: \( \vec{m}(0) \cdot \vec{m}(00) = 0 = \vec{m}(0) \cdot \vec{m}(01) \).

Note that \( \vec{m}(00) \) and \( \vec{m}(01) \) do not need to be equal, they are only required to be orthogonal to \( \vec{m}(0) \). Substituting back in (18) one finds \( F(2) = (3 + \sqrt{2})/6 \). This is the largest value of fidelity that can be obtained with two copies and local measurements. Obviously \( (3 + \sqrt{2})/6 < F_{\text{CM}} = 3/4 \). The optimal guess is easily obtained from Eq. (11): \( \vec{M}^{(2)}(x) = [\vec{m}(x_2) + \vec{m}(x_1)]/\sqrt{2} \). This is a very gratifying result: \( \vec{M}^{(2)}(x) \) is the ‘weighted’ sum of the outcomes.

The case \( N = 3 \) is very similar. The optimal Bloch vectors, \( \vec{m}(x_1), \vec{m}(x_2), \vec{m}(x_3) \), are found to be mutually orthogonal. They can be chosen to coincide with three fixed (i.e. independent of \( x \)) directions. Thus for \( N = 3 \) (as well as for \( N = 2 \)) the optimal estimation schemes based on local measurements do not require classical communication. For each outcome \( x \) the optimal guess is \( \vec{M}^{(3)}(x) = [\vec{m}(x_3) + \vec{m}(x_2) + \vec{m}(x_1)]/\sqrt{3} \), which is a straightforward generalization of \( \vec{M}^{(2)} \). The fidelity is \( F(3) = (3 + \sqrt{3})/6 \). These results could somehow be anticipated: if \( O(\vec{m})|\vec{n}\rangle \neq 0 \) we can only be sure that \( \vec{n} \neq -\vec{m} \). It is then reasonable to explore the plane orthogonal to \( \vec{m} \) with the next copy of \( |\vec{n}\rangle \). Thus, the optimal Bloch vectors \( \vec{m}(x_k) \) tend to be mutually orthogonal.

The case \( N = 4 \) is more complex, since four mutually orthogonal vectors cannot fit onto the Bloch sphere. We do not reproduce here the explicit expressions of the optimal vectors \( \vec{m}_k \). Instead we would like to point out some properties of the solution which, in turn, will bring us insight as to how to
compute the asymptotic limit. We observe that the optimal Bloch vectors now depend on the outcomes of the previous measurements. Therefore classical communication does play a crucial role for \(N > 3\). One also sees that the third measurement probes the plane orthogonal to the vector one would guess from the first two outcomes and analogously does the fourth measurement. The fidelity in this case reads \(F^{(4)} = 0.8206\), which is just 1.5\% lower than the absolute CM bound \(F_{\text{CM}}^{(4)} = 5/6 = 0.8333\). The maximal fidelities for \(N = 5, 6\) are \(F^{(5)} = 0.8450\) and \(F^{(6)} = 0.8637\).

### 4.2 Asymptotic fidelity

We can finally compute the asymptotic fidelity of the optimal local scheme. Suppose we have performed a (large) number \(N_0\) of measurements and obtained an optimal guess \(\vec{M}_0\). It is clear that the subsequent guesses will hardly differ from \(\vec{M}_0\). It is also clear from our results of low \(N\) that the following measurements will basically probe the orthogonal plane of \(\vec{M}_0\). Hence, a good approximation to the optimal local strategy would be to consider:

- **a)** fixed measurements in the orthogonal plane to \(\vec{M}_0\) (i.e. along two orthonormal vectors \(\vec{u}, \vec{v}\) of the plane) and
- **b)** a guess of the form \(\vec{M}(x) \approx \vec{M}_0 \cos \omega + (\vec{u} \cos \tau + \vec{v} \sin \tau) \sin \omega\), where \(\omega = \lambda \sqrt{(2\alpha_u - 1)^2 + (2\alpha_v - 1)^2}\), \(\tan \tau = (2\alpha_v - 1)/(2\alpha_u - 1)\), and \(\lambda\) is a tunable parameter. Here \(\alpha_{u,v}\) are the relative frequencies of the outcomes as defined in section 3. Note that in average \(\omega\) will be small since we expect \(\alpha_{u,v} \approx 1/2\), and only terms up to order \(\omega^2\) need to be retained. The fidelity can be computed from (2) yielding

\[
F \gtrapprox 1 - (1 - \lambda)^2(1 - F_0) - \lambda^2 \frac{1 - 4(1 - F_0)}{N - N_0} + \cdots,
\]

where \(F_0\) is the optimal fidelity for the first \(N_0\) measurements and the dots stand for subleading terms in inverse powers of \(N\) and \(N_0\). If \(N_0 = N^\beta\) with \(0 < \beta < 1\), it is clear that the optimal choice is \(\lambda = 1\), and then \(F \approx 1 - 1/N\). Therefore local measurements saturate the CM bound at leading order.

### 5 Conclusions

We have obtained the optimal estimation of the a pure qubit state for any given set of measurements and any a priori probability distribution. We have focussed on local measurement schemes. For states that are known to lay on the equator of the Bloch sphere (2D case), we have explicitly shown that, rather surprisingly, the most basic scheme (local and without classical communication) saturates the CM bound. This does not happen in the 3D case, although the basic scheme yields a fidelity very close to the CM bound. We have also obtained the optimal local scheme and shown that indeed the

\[\text{In fact, it can be shown that } \beta = 1/2 \text{ is the best choice for the partitioning of } N.\]
CM bound is saturated. Furthermore, numerical analysis reveals that the CM regime is reached for values of $N$ as low as 12. Our main conclusion is that CM do not provide a significant improvement over local measurements.

Our results can be generalised to other interesting issues, such as estimating mixed states, unknown unitary operations, trace preserving maps, etc. For those, the use of local measurements is of outmost interest.

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