Equations of Motion and Energy-Momentum 1-Forms for the Coupled Gravitational, Maxwell and Dirac Fields

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Abstract

A theory where the gravitational, Maxwell and Dirac fields (mathematically represented as particular sections of a convenient Clifford bundle) are supposed fields in Faraday’s sense living in Minkowski spacetime is presented. In our theory there exist a genuine energy-momentum tensor for the gravitational field and a genuine energy-momentum conservation law for the system of the interacting gravitational, Maxwell and Dirac fields. Moreover, the energy-momentum tensors of the Maxwell and Dirac fields are symmetric, and it is shown that the equations of motion for the gravitational potentials is equivalent to Einstein equation of General Relativity (where the second member is the sum of the energy-momentum tensors of the Maxwell, Dirac and interaction Maxwell-Dirac fields) defined in an effective Lorentzian spacetime, whose use is eventually no more than a question of mathematical convenience.

1 Introduction

In this paper we present a theory where the gravitational, Maxwell and Dirac fields are interpreted as fields in the Faraday sense living and interacting in Minkowski spacetime structure \((M, \mathcal{g}, D, \tau_{\mathcal{g}}, \uparrow_{e_0})\) (see Appendix A). The Lagrangian density of these fields are postulated and their energy-momentum tensors are evaluated. All fields in our theory are mathematically described by sections of a particular and convenient Clifford bundle \(\mathcal{C}(M, g)\) (see Appendix A) which is used as a mathematical tool. In particular the gravitational field is represented by its gravitational potentials \(\mathcal{g}^a, a = 0, 1, 2, 3\). It is very important to emphasize here that in our theory we have a genuine

\footnote{Natural units are used in this paper.}
energy-momentum conservation law for the interacting system of the gravitational, Maxwell and Dirac fields. Moreover, the energy-momentum tensor of the Dirac field in the presence of the gravitational field is symmetric. It is also very important to emphasize that the formulation of our theory does not use at any time any connection defined in $M$. However, we may interpret the structure $(M, g, D, \tau g, \uparrow e_0)$ (where $D$ is the Levi-Civita connection of $g = \eta_{ab} g^a \otimes g^b$, $	au g = g^0 g^1 g^2 g^3 \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{Cl}(M, g)$ defines a positive orientation for $M$ and $\uparrow e_0$ defines a time orientation, given by the global vector field $e_0$) as a Lorentzian spacetime representing a gravitational field generated by the matter energy-momentum tensor as in General Relativity theory. This statement is proved by showing (see details, e.g., in [14]) that the equation for the gravitational potentials $g^a$ generated by the energy-momentum tensor of the Dirac and Maxwell fields (and their interaction) is equivalent to Einstein equation in General Relativity theory. This result is particularly since it permit us to conclude that the energy-momentum tensor of the Dirac field in our theory is symmetrical (see Appendix B). Also, with the introduction of the structure $(M, g, D, \tau g, \uparrow e_0)$ in our theory it is possible to encode the energy-momentum 1-form fields for the gravitational field coming from the awful Eqs. (11) and 16 in a simple and nice formula as given by Eq. (23). The paper has three sections and three appendices. In Section 1 we present the Lagrangian densities for the coupled gravitational, Maxwell and Dirac fields. In Section 2 we present the energy-momentum 1-forms for the gravitational, Maxwell and Dirac fields and the energy-momentum 1-forms for the interaction between the Maxwell and Dirac field. Section 3 present our conclusions. Appendix A presents the notations we used and recall some results important for the intelligibility of the paper. As already said above the detailed evaluation of the energy-momentum 1-forms for the Dirac field is given in Appendix B. Finally in Appendix C we use the nice formula Eq. (23) to evaluate the energy of the Schwarzschild gravitational field for a star of mass $M$ and radius greater than its Schwarzschild radius.

2 Lagrangian Densities and Equations of Motion for the Coupled Gravitational, Maxwell and Dirac Fields

The Lagrangian density for the coupled gravitational, Dirac and Maxwell fields is:

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_M + \mathcal{L}_D + \mathcal{L}_{FD} = \mathcal{L}_c + \mathcal{L}_m.$$  (1)

With $g^a \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{Cl}(M, g)$, $a = 0, 1, 2, 3$ we have

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2See Appendix A for notations used in this paper.
\[\mathcal{L}_g : (g^a, dg^a) \mapsto \mathcal{L}_g (g^a, dg^a) \in \sec \wedge^1 T^* M.\]

\[\mathcal{L}_g (g^a, dg^a) = -\frac{1}{2} dg^a \wedge_g dg^a + \frac{1}{2} \delta g^a \wedge_g \delta g^a + \frac{1}{4} (dg^a \wedge g^a) \wedge_g (dg^b \wedge g^b),\]

(2)

Also, with \(F \in \sec \wedge^2 T^* M \mapsto \sec C(\ell, M, g)\)

\[\mathcal{L}_M : F \mapsto \mathcal{L}_F (F) \in \sec \wedge^4 T^* M,\]

\[\mathcal{L}_M (F) = -\frac{1}{2} F \wedge_g F,\]

(3)

and with \(\psi \in C(\ell, M, g)\) a representative in the Clifford bundle of a Dirac-Hestenes spinor field (once a spin frame is fixed)

\[\mathcal{L}_D (g^k, \psi, \tilde{\psi}, g^k \partial_{\tau_k} \psi, g^k \partial_{\tau_k} \tilde{\psi}) \in \sec \wedge^4 T^* M,\]

\[\mathcal{L}_D (g^k, \psi, \tilde{\psi}, g^k \partial_{\tau_k} \psi, g^k \partial_{\tau_k} \tilde{\psi}) = \left\{ \frac{1}{2} \left( \{ g^k \partial_{\tau_k} \tilde{\psi}, g^0 g^2 g^1 \} g^k \cdot \tilde{\psi} - \frac{4}{5} g^k \tilde{\psi} L(g_k) g^0 g^2 g^1 \cdot \tilde{\psi} \right) + \psi \cdot (g^k \partial_{\tau_k} \psi) g^0 g^2 g^1 + \frac{1}{5} \psi \cdot (g^k L(g_k) g^0 g^2 g^1 + m \psi \cdot \tilde{\psi}) \right\} \tau_g\]

(4)

where the symbol \(L(g_k)\) is defined in the Appendix B (see Eqs. (50) and (60)) and \(m\) is the mass of the fermion field.

The interaction Lagrangian density between the Dirac and Maxwell field is

\[\mathcal{L}_{FD} : (\psi, \tilde{\psi}, g^0, A) \mapsto \mathcal{L}_{FD} (\psi, \tilde{\psi}, g^0, A) \in \sec \wedge^4 T^* M,\]

\[\mathcal{L}_{FD} (\psi, \tilde{\psi}, g^0, A) = e \tilde{\psi} g^0 \psi \wedge_g A\]

(5)

where \(e\) is the charge of the fermion field and \(A \in \sec \wedge^1 T^* M \mapsto \sec C(\ell, M, g)\) is the electromagnetic potential such that \(F := dA \in \sec \wedge^2 T^* M \mapsto \sec C(\ell, M, g)\).

In our theory it is supposed that at least one of \(g^a\) is not closed, i.e., \(dg^a \neq 0\), for some \(a = 0, 1, 2, 3\). Putting \(F^d = dg^d\) the equation of motion for the gravitational potentials are obtained from the variational principle. We have

\[\delta \int \mathcal{L}_g = \int \delta \mathcal{L}_g = \int \delta g^d \wedge \left( \frac{\delta \mathcal{L}_g}{\delta g^d} + \frac{\delta \mathcal{L}_m}{\delta g^d} \right),\]

(6)

where

\[\star_g \sum_d = \frac{\delta \mathcal{L}_g}{\delta g^d} = - \left( \frac{\partial \mathcal{L}_g}{\partial \bar{g}^d} + d \left( \frac{\partial \mathcal{L}_g}{\partial d \bar{g}^d} \right) \right)\]

(7)
is the Euler-Lagrange functional and 

\[ \star_{g} T_{d} = - \star_{g} T_{d} = \frac{\partial L_{m}}{\partial g_{a}} = \frac{D_{d}}{g} + \frac{M}{g} + \frac{MD}{g} \]  

will be called the energy momentum 3-forms of the matter fields of the matter fields. One can show that the equations of motion for the gravitational potentials coming from \( \star_{g} \sum_{d} \) are:

\[ -d \star S_{d} - * t_{d} = \star T_{d} = - * T_{d}, \]  

with

\[ * t_{d} := \frac{\partial L_{g}}{\partial g_{d}} = \frac{1}{2} \left[ (g_{d} + \varepsilon g_{a}) \wedge d g_{a} - d g_{a} \wedge (g_{d} \wedge * g_{a}) \right] + \frac{1}{2} d \left( g_{d} \wedge * g_{a} \right) \wedge * d \wedge g_{a} + \frac{1}{2} d g_{d} \wedge * (d g_{a} \wedge g_{a}) \] 

\[ - \frac{1}{4} d g_{a} \wedge g_{a} \wedge \left[ g_{d} \wedge * (d g_{c} \wedge g_{c}) \right] - \frac{1}{4} \left[ g_{d} \wedge (d g_{c} \wedge g_{c}) \right] \wedge * (d g_{a} \wedge g_{a}). \]  

Moreover, putting \( \mathcal{F}_{a} := d g_{a} \), it is of course, \( d \mathcal{F}_{a} = 0 \) and the field equations (Eq. (10)) can be written as

\[ d \star S_{d} = - \star T_{d} - * t_{d} = \star b_{d}. \]  

where

\[ b_{d} = d \left[ (g_{d} \wedge * g_{a}) \wedge * d \wedge g_{a} - \frac{1}{2} g_{d} \wedge * (\mathcal{F}_{a} \wedge g_{a}) \right]. \]  

So, we have

\( a) \ d \mathcal{F}_{d} = 0, \quad (b) \ \delta \mathcal{F}_{d} = - \left( \frac{m}{g} T_{d} + t_{d} \right), \)

\[ t_{d} = (t_{d} + b_{d}). \]  

Also, it is very much important to recall that introducing the Levi-Civita connection of \( g = \eta_{a b} \tilde{g}_{a} \otimes \tilde{g}_{b} \) into the game one can show with some algebra (details, e.g., in [14]) that

\[ -d \star S_{d} - * t_{d} = \star G_{d} \]  

\[ ^{3} \text{We suppose that } L_{m} \text{ does not depend explicitly on the } dg_{a}, \]

\[ ^{4} \text{Details may be found, e.g., in [14]} \]
where \( G_d := G_{dk}^k \in \sec \Lambda^1 T^*M \hookrightarrow \sec \mathcal{C}l(M, g) \) are the Einstein 1-form fields, with

\[
G_{dk} = R_{dk} - \frac{1}{2} \eta_{dk} R
\]  

(18)

where \( R_{dk} \) are the components of the Ricci tensor and \( R \) is the scalar curvature in the structure \((M, g, D, \tau_g, \uparrow e_0)\).

With this result we immediately infer from Eq.(10) that writing \( m T_d = m T_{dk}^k \in \sec \Lambda^1 T^*M \hookrightarrow \sec \mathcal{C}l(M, g) \) it is

\[
m T_{dk} = m T_{kd}.
\]  

(19)

and of course we must also have:

\[
M T_{dk} = M T_{kd}, \quad D T_{dk} = D T_{kd}, \quad M D T_{dk} = M D T_{kd}.
\]  

(20)

However it is not the case that in general \( t_{dk} = t_{kd} \). See Section 3.1.

Remark 1 It is crucial to emphasize here that the introduction of a Lorentzian spacetime structure \((M, g, D, \tau_g, \uparrow e_0)\) to get Eq.(20) is to be viewed as simple a mathematical aid, no fundamental ontology is given to that Lorentzian structure. Indeed, it has been shown in details, e.g., in [9, 11, 13] that our theory of the gravitational field may be interpreted as generating spacetime structures with general connections where curvature torsion and non-metricity tensors may be non-null.

Also, we recall that the equations of motion for the Dirac and Maxwell fields are respectively (see, e.g., [14] for details of the derivation)

\[
g^a D_{\tau a} \psi g^2 g^1 - m \psi g^0 + e A \psi = 0
\]  

(21)

and

\[
dF = 0, \quad \delta F = - J_e, \quad J_e = e \psi g^0 \tilde{\psi}.
\]  

(22)

3 Energy-Momentum 1-Forms Fields for the Gravitational, Maxwell and Dirac Fields

3.1 Gravitational Energy-Momentum 1-Forms

Despite the very awful formula for \( t_d \) coming from Eqs.(11) and (16) it has been shown in [13] that it can be coded in a nice simple formula once we introduce as an auxiliary mathematical device the Levi-Civita connection of \( g \) and the Dirac operator \( \partial = g^D D_d \) acting on sections of \( \mathcal{C}l(M, g) \). Indeed, it is:
\[ t^d = \frac{1}{2} R g^d + \partial \cdot \partial \ g^d + d \delta g^d \]  

(23)

where \( \partial \cdot \partial \) is the covariant D'Alembertian\[14].

**Remark 2** It is very important to observe that the objects \( t_{da} = \eta_{ac} \partial_{di} t^c \eta^{d} \) are components of a legitimate gravitational energy-momentum tensor tensor field \( t = t_{da} g^d \otimes g^a \in \sec T^2_0 M \). Also it is worth to take into account that

\[ t^{da} - t^{ad} = (\partial \cdot \partial \ g^d)_{g^a} - (\partial \cdot \partial \ g^a)_{g^d} + (d \delta g^d)_{g^a} - (d \delta g^a)_{g^d} \]  

(24)

i.e., the energy-momentum tensor of the gravitational field is in general not symmetric. As observed in \[14\] this is important in order to have a total angular momentum conservation law for the system consisting of the gravitational plus the matter fields. In Appendix C we present \( t^{1a} \) for the Schwarzschild solution of Einstein equation in order to show that it is a viable quantity to really describe the energy momentum tensor of the gravitational field. In that example it is clear that \( t^{12} \neq t^{21} \).

### 3.2 Maxwell Energy-Momentum 1-forms

We recall moreover that the energy-momentum 1-forms \( \star T^a (= - \star T^a) \) for the Maxwell field is

\[ \star T^a = - \frac{\partial L_M}{\partial g^a} = \star \left( \frac{1}{2} \theta_a (F : F) + (\theta_a \cdot F) \cdot F \right) \]  

(25)

\[ = \star \left( \frac{1}{2} F_{g a} \hat{F} \right) = \left( \frac{1}{2} F_{g a} \hat{F} \right) \cdot g \]  

(26)

and writing \( T^a = T_{ab} g^b \) we get

\[ T_{ab} = T_a \cdot g_b = - \eta^{cd} F_{ac} F_{bd} + \frac{1}{4} F_{cd} F_{cd} \eta_{ab} = T_b \cdot g_a = T_{ba} \]  

(27)

### 3.3 Maxwell-Dirac Interaction Energy-Momentum 1-forms

The energy-momentum 1-forms \( T^a \) are trivially calculated. We have

\[ T^a = \frac{\partial L_{MD}}{\partial \tilde{g}^a} = e A_a \tilde{\psi} \tilde{g}^0 \psi, \]  

\[ \tilde{T}^a = \frac{1}{2} \left( T^a \cdot g_b + T^b \cdot g_a \right) = \frac{1}{2} e \langle A_a \tilde{\psi} \tilde{g}^0 \psi g_b + A_b \tilde{\psi} g_a \tilde{g}^0 \psi \rangle. \]  

(28)

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5See, e.g., Section 9.9 of [14] for details of the derivation.
3.4 Dirac Energy-Momentum 1-forms

The calculation of the Dirac energy-momentum 1-forms is trick and is presented in Appendix B. We found

$$\mathcal{T}_k = \langle D_{\kappa\bar{\kappa}} \bar{\psi} g^2 g^0 g^1 \bar{\psi} + \psi D_{\kappa\bar{\kappa}} \psi g^0 g^2 g^1 \rangle_1 \quad (29)$$

and

$$\mathcal{T}_{mk} = \frac{1}{2} \langle \bar{\psi} g_{(m} D_{\kappa\bar{\kappa})} \psi g^2 g^1 g^0 - D_{(\kappa \bar{\kappa})} \bar{\psi} g_{k)} g^2 g^1 g^0 \psi \rangle_0 \quad (30)$$

Also, it is worth to emphasize that in our theory we have a genuine conservation law for the energy-momentum of the matter plus the gravitational field. Indeed it follows from Eq.(15b) that

$$\delta_g \left( \mathcal{T}_d + t_d \right) = 0. \quad (31)$$

Finally, it is worth to emphasize that since our spacetime manifold is parallelizable it is possible to defined a legitimate energy-momentum covector for the matter plus the gravitational field if

$$P = P_d g^d,$$

$$P_d = \int g \left( \mathcal{T}_d + t_d \right). \quad (32)$$

4 Conclusions

In this paper we present a coherent relativistic theory of the gravitational, Maxwell and Dirac fields in interaction. In our theory field equations and the corresponding energy-momentum tensors of the fields are obtained from the variational principle through postulated Lagrangian densities for those fields and their interactions. All fields are intended as fields in Faraday’s sense living in a Minkowski spacetime structure. The energy-momentum tensors for the Maxwell and Dirac fields are symmetric and it is recalled that the equations satisfied by the gravitational potentials are equivalent to Einstein equation of General Relativity in an effective Lorentzian spacetime structure \((M, g, D, \tau, \mathbf{e})\) which differently from the case of General Relativity is not supposed to have any ontology, it is used in the paper only as a tool to obtain an important mathematical result need for the construction of the energy-momentum tensor of the Dirac field and to obtain a short formula (Eq.(23)) for the energy-momentum of the gravitational field whose derivation from the gravitational Lagrangian density produces a somewhat awful (but of course, correct) formula (see Eq.(19) and Eq.(16)). Moreover, the viability of our formula for really representing the

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6See a detailed discussion about conservation laws and conditions for existence of an energy-momentum covector (not a covector field) in [15].
energy-momentum of the gravitational field is shown by explicitly evaluating it for the Schwarzschild field of a star of mass $M$ and radius much greater than its Schwarzschild radius.

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A Notations and Recall of Some Results

In this paper $M$ designs a 4-dimensional manifold diffeomorphic to $\mathbb{R}^4$ whose elements are called events. If $\{x^\mu\}, \mu = 0, 1, 2, 3$ are global coordinates for $M$, $\{e_\mu\}, e_\mu = \frac{\partial}{\partial x^\mu} \in \sec TM$ are global smooth vector fields and we denote denote by $\{\theta^\mu = dx^\mu\} \in \sec \bigwedge^1 T_* M$ its dual basis. We can introduce in $M$ several different metric fields, in particular an euclidean metric field

$$\hat{g}_E = \delta_{\mu\nu} \theta^\mu \otimes \theta^\nu \in \sec T^0_2 M$$

and also a Lorentzian metric field

$$\hat{g} = \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu \in \sec T^0_2 M$$

of signature $7-2$.

We denoted by $\hat{g}_E, \hat{g} \in \sec T^0_2 M$ metrics on the cotangent bundle such that

$$\hat{g}_E = \delta^{\mu\nu} e_\mu \otimes e_\nu, \quad \hat{g} = \eta^{\mu\nu} e_\mu \otimes e_\nu.$$  \hspace{1cm} (34)

Moreover, we denote by $\hat{g}$ the extensor field

$$\hat{g} : \sec \bigwedge^1 T^* M \to \sec \bigwedge^1 T_* M,$$

This means that the matrix with entries $\eta_{\mu\nu}$ is the diagonal matrix $(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. Also if $\eta^{\mu\nu} \eta_{\nu\sigma} = \delta^\mu_\sigma$, then the matrix with entries $(\eta^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$.
such that for \( a, b \in \sec \bigwedge^1 T^* M \) it is\(^8\)
\[
\tilde{\mathbf{g}}(a) \bullet b := \mathbf{g}(a, b) := a \bullet b.
\] (35)

Of course we can introduce in the structure \((M, \tilde{\mathbf{g}}_E)\) [respectively \((M, \mathbf{g}_E)\)] the Clifford bundle \(\mathbf{Cl}(M, \mathbf{g}_E)\) [respectively \(\mathbf{Cl}(M, \tilde{\mathbf{g}}_E)\)] and of course, we have that \(\bigwedge T^* M\), the bundle of exterior forms is such that\(^9\) \(\bigwedge T^* M = \sum_{r=0}^4 \bigwedge^r T^* M \hookrightarrow \mathbf{Cl}(M, \tilde{\mathbf{g}}_E)\) [respectively \(\bigwedge T^* M = \sum_{r=0}^4 \bigwedge^r T^* M \hookrightarrow \mathbf{Cl}(M, \mathbf{g}_E)\)]

Following the ideas presented in \(^3\) the gravitational field generated by an energy-momentum tensor \(\mathbf{T} \in \sec T_2 M\) is represented by a gauge extensor (deformation extensor)\(^10\)
\[
\mathbf{h} : \sec \bigwedge^1 T^* M \rightarrow \sec \bigwedge^1 T^* M
\] (36)
such that putting \(\theta^a := \delta^a_\mu dx^\mu, (a = 0, 1, 2, 3)\) it is.
\[
\mathbf{h}(\theta^a) = \mathbf{g}^a.
\] (37)

The set \(\{\mathbf{g}^a\}\) are called gravitational potentials. We introduce in \(M\) the field \(\mathbf{g} \in \sec T_2^0 M\) according to the definition
\[
\mathbf{g} = \eta_{ab} \mathbf{g}^a \otimes \mathbf{g}^b.
\] (38)

If \(\{\mathbf{e}_a\} \in \sec TM\) is the dual basis of \(\{\mathbf{g}^a\}\) we define a field \(\mathbf{g} \in \sec T_2^0 M\) such that
\[
\mathbf{g} = \eta^{ab} \mathbf{e}_a \otimes \mathbf{e}_b.
\] (39)

**A.0.1 The Clifford Bundle of Differential forms \(\mathbf{Cl}(M, \mathbf{g})\)**

Since of course, the structure \((M, \mathbf{g})\) is parallelizable we can present the Clifford bundle of differential forms as the vector bundle\(^11\)
\[
\mathbf{Cl}(M, \mathbf{g}) = P_{\text{Spin}^+_{1,3}}(M, \mathbf{g}) \times_{\text{Ad}'} \mathbb{R}_{1,3}, \quad \text{where } P_{\text{Spin}^+_{1,3}}(M, \mathbf{g})
\]
\(\mathbb{R}_{1,3} \cong \mathbb{H}(2)\) is the so called spacetime algebra. We recall that \(\bigwedge T^* M = \sum_{r=0}^4 \bigwedge^r T^* M \hookrightarrow \mathbf{Cl}(M, \mathbf{g})\).

Given the structure \((M, \tilde{\mathbf{g}}_E)\) with \(\bigwedge T^* M \hookrightarrow \mathbf{Cl}(M, \tilde{\mathbf{g}}_E)\) we denoted by \(\mathbf{g} := \mathbf{h}^1 \tilde{\mathbf{g}}_E \mathbf{h}\) the extensor field
\[
\mathbf{g} : \sec \bigwedge^1 T^* M \rightarrow \sec \bigwedge^1 T^* M
\] (40)
such that for \(a, b \in \sec \bigwedge^1 T^* M\) it is
\[
\mathbf{g}(a, b) := \mathbf{g}(a) \bullet b = \mathbf{h}^1 \tilde{\mathbf{g}} \mathbf{h}(a) \bullet (a) = \tilde{\mathbf{g}} \mathbf{h}(a) \bullet \mathbf{h}(a)
\]

\(^8\)We define for \(a, b \in \sec \bigwedge^1 T^* M, \tilde{\mathbf{g}}_E(a, b) := a \bullet b.
\(^9\)\(\mathbf{Cl}(M, \mathbf{g}_E)\) has been called in \(^2\) the canonical algebra.
\(^10\)Details in \(^1\).
\(^11\)The \(\mathbf{h}\) extensor field produces a plastic distortion of the Lorentz vacuum (which is defined as the Minkowski spacetime structure). Details in \(^2\).
\(^12\)A general section of \(\mathbf{Cl}(M, \mathbf{g})\) is a sum of nonhomogeneous differential forms, called multi-form fields or Clifford fields.
Also, given the structure \((M, \tilde{g})\) with \(\bigwedge T^* M \hookrightarrow \mathcal{C}(M, \tilde{g})\) we may denote by \(g := \h^\dagger \h\) the extensor field

\[
g : \sec \bigwedge T^* M \to \sec \bigwedge T^* M
\]

such that for \(a, b \in \sec \bigwedge T^* M\) it is

\[
g(a) \bullet b = \h(a) \bullet \h(a) = g(a, b) := a \cdot b.
\] (41)

The above relations are essential for the formalism used in [3] where a Lagrangian formalism for the \(h\) field is developed. Unfortunately to grasp that theory it is first necessary to have a working knowledge of the (non trivial) mathematical theory of extensor fields and extensor functionals. So in this paper we present the gravitational theory formulated through the gravitational potentials \(g^m\) (which is a relatively simple theory) for which the Lagrangian density given by Eq.(2) is postulated.

With \(a, b \in \sec \bigwedge T^* M \hookrightarrow \mathcal{C}(M, g)\) we have the fundamental relation

\[
ab + ba = 2g(a, b)
\] (42)

and moreover

\[
a \cdot b = \frac{1}{2}(ab + ba), \quad a \wedge b = \frac{1}{2}(ab - ba).
\] (43)

A general section of \(\mathcal{C}(M, g)\) is written as a sum of nonhomogeneous differential forms, i.e.,

\[
C = \sum J C^J g^J = \sum J C^J g^J,
\]

where the symbol \(J\) denotes collective indices. Recall, e.g., that

\[
g^J = 1, g^0, \ldots, g^0 j_1 j_2 j_3 j_4 = g^0 \wedge g^0 \wedge g^0 \wedge g^0,
\]

\[
g^J = 1, g^1, \ldots, g^1 j_2 \cdots j_4 = g^1 \wedge g^2 \wedge g^3 \wedge g^4.
\] (45)

The scalar product (\(\cdot\)) and the exterior product extend to all sections of \(\mathcal{C}(M, g)\) and here we distinguish the scalar product from the operations of left and right contractions. We have for for any \(X, Y \in \sec \mathcal{C}(M, g)\)

\[
X \cdot Y = \langle \bar{X}Y \rangle_0 = \langle X\bar{Y} \rangle_0 = Y \cdot X.
\] (46)

---

13 In this paper the Clifford product is denoted by juxtaposition of symbols. A detailed explanation of all symbols and identities need for the derivations in this paper can be found in [14].

14 The concept of the Lie derivative of spinor fields is a subtle one, with many non equivalent definitions. See a sample of the bibliography in [7]. In particular it is even possible [1] to give a meaning to a statement one find in physical textbooks, like, e.g., [17, 5] that under diffeomorphisms spinor fields transform as scalars, but we will not comment more on that here.
and for arbitrary multiforms \( X,Y,Z \in \sec \mathcal{C}(M,g) \) the left and right contractions of \( X \) and \( Y \) are the mappings \( \gamma_g(m,\mathcal{C}) \times \sec \mathcal{C}(M,g) \to \sec \mathcal{C}(M,g) \), \( \gamma_g(m,\mathcal{C}) \times \sec \mathcal{C}(M,g) \to \sec \mathcal{C}(M,g) \) such that

\[
(X \cdot Y) \cdot Z = Y \cdot (X \wedge Z), \\
(X \cdot Y) \cdot Z = X \cdot (Z \wedge Y). 
\]  

(47)

### A.1 Spin-Clifford Bundle and Dirac-Hestenes Spinor Fields

In [12, 10] Dirac-Hestenes spinor fields living in a structure \((M,g)\) are sections of the spin-Clifford bundle \(\mathcal{C}(M,g) = \times_{\text{Spin}^c_1} (M,g) \times \mathbb{R}^4_1,3\) and one can show that once we fix a spin coframe a Dirac-Hestenes spinor field \(\Psi \in \sec \mathcal{C}(M,g)\) has a representative \(\psi \in \sec \mathcal{C}_0(M,g)\), i.e., an even section of the Clifford bundle \(\mathcal{C}(M,g)\). A covariant Dirac spinor field \(\psi\) used by physicists is a section of the bundle \(\mathcal{C}(M,g)\). Details of the above theory may be found in [12, 10]. Below we give a dictionary that one can use to immediately translate results of the standard matrix formalism in the language of the Clifford bundle formalism and vice-versa. This dictionary will help the reader to compare the result we found for the energy-momentum tensor of the Dirac field in the presence of a gravitational field with other results on that subject that he may find in the literature.

\[
\gamma_a \psi \leftrightarrow g_a \psi g_0, \\
i \psi \leftrightarrow \psi g_2 g_1, \\
i \gamma_5 \psi \leftrightarrow \psi g_3 = \psi g_3 g_0, \\
\tilde{\psi} = \psi^\dagger \gamma^0 \leftrightarrow \tilde{\psi}, \\
\psi^\dagger \leftrightarrow g_0 \psi g_0, \\
\psi^* \leftrightarrow -\gamma_2 \psi \gamma_2. 
\]  

(48)

where \(\gamma_a, a = 0,1,2,3\) are Dirac matrices in standard representation, \(\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3\) and \(i = \sqrt{-1}\).

**Remark 3** Note that \(\gamma_a, a = 1,4\) and the operations and \(\dagger\) are for each \(x \in M\) mappings \(\mathbb{C}^4 \to \mathbb{C}^4\). Then they are represented in the Clifford bundle formalism by extensor fields which maps \(\mathcal{C}_0(M,\eta) \to \mathcal{C}_0(M,\eta)\). Thus, to the operator \(\gamma_a\) there corresponds an extensor field, call it \(g_a: \mathcal{C}_0(M,\eta) \to \mathcal{C}_0(M,\eta)\) such that \(g_a \psi = g_a \psi g_0\).

**Remark 4** Recall that the structure \((M,\hat{g}, \hat{D}, \tau_{\hat{g}}, \uparrow_{e_0})\) is Minkowski spacetime when \(\hat{D}\) is the Levi-Civita connection of \(\hat{g}\), \(\tau_{\hat{g}} = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3\) defines a positive orientation for \(M\) and \(\uparrow_{e_0}\) defines a time orientation (given by the global vector field \(e_0\)). Also the structure \((M,g, D, \tau_g, \uparrow_{e_0})\) is a Lorentzian spacetime when \(D\) is the Levi-Civita connection of \(g\), \(\tau_g = \theta^0 \theta^1 \theta^2 \theta^3 \in \sec \bigwedge^4 T^* M \leftrightarrow \sec \mathcal{C}(M,g)\)
defines a positive orientation for $M$ and $\uparrow e_0$ defines a time orientation (given by the global vector field $e_0$).

### A.2 The Lie Derivative of Clifford and Spinor Fields

In [7] we give a geometrical motivated definition for the Lie derivative of spinor fields in the direction of an arbitrary smooth vector field $\xi \in \sec TM$ which we called the spinor Lie derivative and denoted $\bar{\mathcal{L}}_{\xi}$. Let $C \in \sec \Cl(M, g)$ (Eq. (44)) Then,

\[ \bar{\mathcal{L}}_{\xi} C := \partial_{\xi} C + \frac{1}{4} [S(\xi), C]. \]  

Also, the Lie derivative of a Dirac-Hestenes spinor field $\Psi \in \sec \Cl^{(s)}(M, g)$ is

\[ \bar{\mathcal{L}}_{\xi} \Psi := \partial_{\xi} \Psi + \frac{1}{4} S(\xi) \Psi. \]  

The spinor Lie derivative of a representative $\psi \in \sec \Cl(M, g)$ of a Dirac-Hestenes spinor field $\Psi \in \sec \Cl^{(s)}(M, g)$ is denoted $\mathcal{L}_{\xi} \psi$ and we have

\[ \mathcal{L}_{\xi} \psi := \partial_{\xi} \psi + \frac{1}{4} S(\xi) \psi. \]

In Eqs. (49), (50) and (51) $\partial_{\xi}$ denotes the Pfaff derivative and with $\xi = g(\xi, \cdot)$

\[ S(\xi) = L(\xi) + dg \]

with

\[ L(\xi) = \frac{1}{2} (c_{\text{a}k\text{l}} + c_{\text{k}\text{a}l} + c_{\text{l}\text{a}k}) \xi^\text{a} g^\text{k} \wedge g^\text{l} \]  

with $c_{\text{a}b}^\text{c}$ the structure coefficients of the basis $\{e_a\}$ of $TM$ dual of the basis $\{g^a\}$ of $\wedge^1 T^* M$. i.e.,

\[ [e_a, e_b] = c_{ab}^c e_c, \quad dg^a = -\frac{1}{2} c_{ab}^c g^a \wedge g^b. \]  

**Remark 5** Our definition of spinor Lie derivative [7] can be extended also for some cotensor fields, in particular, $\bar{\mathcal{L}}_{\xi} g = 0$ the Lie derivative of the field $g$ is null. This result is very important for the objective of this paper, where a variation $\delta g^a$ is defined as

\[ \delta g^a = -\bar{\mathcal{L}}_{\xi} g^a \]  

for appropriate vector fields $\xi$ (see below).

**Remark 6** It is very important to recall that there are several non equivalent definitions for the Lie derivative of spinor fields. Relevant references are given in [7]. Here we comment that our spinor Lie derivative of Clifford and spinor fields is obtained obtaining through the introduction of a spinor mapping $\mathcal{h}$ which gives
a spinor image of Clifford and spinor fields between points \( x' = hx \) and \( x \) (where \( h : M \rightarrow M \) is a diffeomorphism generated by an arbitrary differentiable vector field \( \xi \)). It is very important to emphasize here that if \( \Psi \in \sec C^t \rightarrow (M, g) = P^t(M, g) \times_t \mathbb{R}^0_{1,3} \) then its image \( h\Psi \) is also a section of \( C^t \rightarrow (M, g) = P^t(M, g) \times_t \mathbb{R}^0_{1,3} \).

The map \( h \) is different from the pullback map and for the case of Clifford fields it coincides with the pullback mapping only when the vector field \( \xi \) is a Killing vector field in the structure \( (M, g) \).

In particular it is important to emphasize here that \( \xi \) is odd one since in particular \( F_\xi g = 0 \), i.e., the Lie derivative of the metrical field is null, which means that when varying the gravitational potentials the field \( g \) does not change.

We also recall here that is even possible [1] to give a meaning to a statement found in physical textbooks, e.g., [14, 15, 17] that spinor fields transform under the pullback \( h^* \) mapping as scalar functions. Briefly, this is to be understood in the following way. Let \( g \) be a metric field in \( M \) and \( g' = h^* g \) the pullback metric under a mapping \( h : M \rightarrow M \). If \( \Psi \in \sec C^t \rightarrow (M, g) = P^t(M, g) \times_t \mathbb{R}^0_{1,3} \) then \( \Psi' = h^* \Psi \in \sec C^t \rightarrow (M', g') = P^t(M', g') \times_t \mathbb{R}^0_{1,3} \) is such that \( \Psi'(x) = \Psi(hx) \). This definition, a mathematical legitimate one seems to us odd one since in particular \( (M, g) \) and \( (M', g') \) are supposed in General Relativity to describe the same gravitational field even if \( M' = M \) (diffeomorphism invariance of the theory).

## B The Energy-Momentum 1-Forms for the Dirac Field in the Presence of a Gravitational Field

Let \( \mathcal{F} : \sec \wedge^k T^* M \rightarrow \sec \wedge^t T^* M, X \rightarrow \mathcal{F}(X) \) be a differentiable multiform function of a multiform variable \( X \). We recall that the directional derivative of \( \mathcal{F} \) in the direction of \( W \in \sec \wedge^t T^* M \) is denoted \( W \cdot \partial_X \mathcal{F} \) and we have

\[
W \cdot \partial_X \mathcal{F}(X) = \lim_{t \to 0} \frac{\mathcal{F}(X + t(W)X) - \mathcal{F}(X)}{t}.
\]  

(56)

Moreover, the multiform derivative \( \partial_X \mathcal{F} \) is defined by

\[
\partial_X \mathcal{F}(X) = \sum_{j} \frac{1}{\nu(J)} g^J \partial_{g^J} \mathcal{F}(X) = \sum_{J} \frac{1}{\nu(J)} g^J \partial_{g^J} \mathcal{F}(X),
\]  

(57)

where the symbols \( g^J \) and \( g^J \) are defined in Eq.(14) and \( \nu(J) = 0, 1, 2, \ldots \) for \( J = \emptyset, j_1, j_2, j_3, j_4, \ldots \) where all indices \( j_1, j_2, j_3, j_4 \) run from 0 to 3.

---

15If the reader needs details in order to follow the calculations in this Appendix (which needs many “tricks of the trade” of the Clifford bundle formalism) he can consult Chapters 2 and 7 of [13] and [7].

16In [13] we also use the notation \( \partial_X \mathcal{F}(X) = \mathcal{F}'(X) \).
Now, let the Dirac Lagrangian in interaction with the gravitational field be given by

\[ \mathcal{L}_D : \sec \Lambda^1 T^*M \times (\sec \Lambda^\Delta T^*M)^2 \times (\sec \Lambda^\nabla T^*M)^2 \to \sec \Lambda^3 T^*M, \]

\[ (g^k, \psi, \tilde{\psi}, \tilde{g}^k \partial_{\tilde{k}} \tilde{\psi}, \tilde{g}^k \partial_{\tilde{k}} \tilde{\psi}) \mapsto \mathcal{L}_D(g^k, \psi, \tilde{\psi}, \tilde{g}^k \partial_{\tilde{k}} \tilde{\psi}, \tilde{g}^k \partial_{\tilde{k}} \tilde{\psi}) \]

\[ \mathcal{L}_D = \left\{ \begin{array}{l}
(g^k \partial_{\tilde{k}} \tilde{\psi} \tilde{g}^2 \tilde{g}^1) \tilde{g}^0 \cdot \tilde{\psi} - \frac{1}{2} g^k \tilde{\psi} L(g) g^0 g^2 g^1 \cdot \tilde{\psi} \\
+ \psi \cdot (g^k \partial_{\tilde{k}} \psi g^0 g^2 g^1) + \frac{1}{4} \psi \cdot (g^k L(g) \psi g^0 g^2 g^1 + m \psi \cdot \tilde{\psi}) \end{array} \right\} \tau_g \quad (58) \]

where

\[ \sec \Lambda^\Delta T^*M = \sec(\Lambda^0 T^*M + \Lambda^2 T^*M + \Lambda^1 T^*M), \]

\[ \sec \Lambda^\nabla T^*M = \sec(\Lambda^1 T^*M + \Lambda^3 T^*) \]

and recalling Eq.(53) it is

\[ L(g) := \frac{1}{2} (c_{rks} + c_{kr s} + c_{r sk}) g^k \wedge g^l. \quad (60) \]

Now, define \( \mathcal{L}_D = \mathcal{L}_D = \mathcal{L}_D \tau_g \) with

\[ \mathcal{L}_D : \sec \Lambda^1 T^*M \times (\sec \Lambda^\Delta T^*M)^2 \times (\sec \Lambda^\nabla T^*M)^2 \to \sec \Lambda^0 T^*M. \quad (61) \]

The variation of \( \mathcal{L}_D \) induced by the lifting in the spin structure bundle of the differentiable vector field \( \xi = e\alpha \) is defined by

\[ \delta \mathcal{L}_D := \sum a \delta g^k \wedge \frac{\partial \mathcal{L}_D}{\partial g^k} = \delta g^k \wedge \frac{\partial \mathcal{L}_D}{\partial g^k}. \quad (62) \]

On the other hand the variation of \( \Sigma_D \) induced by an arbitrary variation \( g^k \mapsto g^k + \delta^k \) \((\delta^k \in \sec \Lambda^1 T^*M \hookrightarrow \sec \text{Cl}(M, g))\) is given by the directional derivative \( \delta^k \cdot \partial_{\delta^k} \Sigma_D \), i.e.,

\[ \delta \Sigma_D := \delta^k \cdot \partial_{\delta^k} \Sigma_D = \delta^k \cdot (\partial_{\delta^k} \Sigma_D) \quad (63) \]

where we have used the fact that any \( F : \sec \Lambda^\Delta T^*M \ni X \mapsto F(X) \in \sec \Lambda^0 T^*M \) it is \[ 14 \]

\[ X \cdot \delta_X F = X \cdot (\partial_X F) \quad (64) \]

So, taking \( \delta^k = \delta g^k \) it is

\[ \delta \Sigma_D := \delta g^k \cdot \partial_{\delta^k} \Sigma_D = \delta g^k \cdot (\partial_{\delta^k} \Sigma_D) = \delta g^k \cdot (\partial_{\delta^k} \Sigma_D)_1 \quad (65) \]

and since as it is easy to show \( \delta \tau_g = - \mathcal{E} \tau_g = 0 \) we can write

\[ \delta \mathcal{L}_D = \delta \left( \Sigma_D \tau_g \right) = (\delta \Sigma_D) \tau_g + \Sigma_D \delta \tau_g = (\delta \Sigma_D) \tau_g + * \delta \Sigma_D. \quad (66) \]

From Eq.(66) we get

\[ \delta \mathcal{L}_D = \delta g^k \wedge \frac{\partial \mathcal{L}_D}{\partial g^k} = (\delta g^k \cdot (\partial_{\delta^k} \Sigma_D)) \tau_g = \delta g^k \wedge * (\partial_{\delta^k} \Sigma_D)_1 \quad (67) \]
and since by definition the 1-forms of energy-momentum \( D \mathcal{T}_k = D T_{km} g^m \) of the Dirac field in the presence of the gravitational field are defined by

\[
\star \frac{D}{g} \mathcal{T}_k = \frac{\partial L_D}{\partial g^k}. \tag{68}
\]

We get using Eq.(67) the notable relation

\[
\frac{D}{g} \mathcal{T}_k = \langle \partial_{g_k} \mathcal{L}_D \rangle_1
\]

that

\[
\frac{D}{g} \mathcal{T}_k = \langle D_{c_k} \tilde{\psi} g^2 g^1 g^0 + \psi D_{c_k} \psi g^0 g^2 g^1 \rangle_1 \tag{70}
\]

where

\[
D_{c_k} \psi := \partial_{c_k} \psi + \frac{1}{4} L(g_k) \psi. \tag{71}
\]

Moreover, recalling as observed in Section 2 that by Einstein equation (for the system gravitational plus Dirac field) is \( \star \frac{D}{g} g \mathcal{T}_k = - \frac{D}{g} g \mathcal{T}_k \) and \( g \mathcal{T}_k = G_{km} g^m \) with \( G_{km} = G_{mk} \) it follows that \( \mathcal{T}_{km} = \mathcal{T}_{mk} \), i.e., \( \mathcal{T}_k \cdot g_m = \mathcal{T}_m \cdot g_k \). Observe that since for any, \( A, B \in \sec \mathcal{C}(M, g) \) it is \( \langle AB \rangle_r = (-1)^{\langle c_m \rangle} (\langle B \rangle \langle A \rangle)_r \) we can write

\[
\langle g_m \psi D_{c_k} \psi g^2 g^1 g^0 \rangle_0 = \langle (\langle g_m \psi \rangle_1 + \langle g_m \psi \rangle_3) D_{c_k} \psi g^2 g^1 g^0 \rangle_0
\]

\[
= \langle (\langle \psi g_m \rangle_1 + \langle \psi g_m \rangle_3) D_{c_k} \psi g^2 g^1 g^0 \rangle_0 = \langle \psi g_m D_{c_k} \psi g^2 g^1 g^0 \rangle_0 \tag{72}
\]

Also,

\[
\langle D_{c_k} \tilde{\psi} g^2 g^1 g^0 \psi g_m \rangle_0 = \langle D_{c_k} \tilde{\psi} g^2 g^1 g^0 g_m \psi \rangle_0
\]

\[
= - \langle D_{c_k} \tilde{\psi} g_m \psi g^2 g^1 g^0 \psi \rangle_0 \tag{73}
\]

So, using the above results we get

\[
\frac{D}{g} \mathcal{T}_{mk} = \frac{1}{2} \left( \frac{D}{g} \mathcal{T}_k \cdot g_m + \frac{D}{g} \mathcal{T}_m \cdot g_k \right) = \frac{1}{2} \langle \psi g_m D_{c_k} \psi g^2 g^1 g^0 - D_{c_k} \tilde{\psi} g_m \psi g^2 g^1 g^0 \rangle_0. \tag{74}
\]

**Remark 7** Using the dictionary (Appendix A) between the standard matrix formalism used by physicists for dealing with (covariant) Dirac spinor fields and the formalism of this paper where these objects are represented (once we fix a spin frame) by an even section \( \psi \) of the Clifford bundle \( \mathcal{C}(M, g) \) we immediately verify that Eq.(74) coincides, e.g. with the result reported in [5].
C  Energy-Momentum of the Gravitational Field for the Schwarzschild Field

The Schwarzschild solution $g$ (of Einstein equation) for a star of mass $m$ with radius $R$ greater than the Schwarzschild radius can be written in polar coordinates covering the region of interest as

$$g = g_{\mu\nu}dx^\mu \otimes dx^\nu,$$

$$g = \left(1 - \frac{2m}{r}\right)dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1}dr \otimes dr - r^2d\theta \otimes d\theta - (r^2\sin^2 \theta)d\varphi \otimes d\varphi. \quad (75)$$

Here according to the theory presented above the gravitational potentials $g^a$, $a = 0, 1, 2, 3$ are:

$$g^0 = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}dt; \quad g^1 = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}dr; \quad g^2 = r d\theta; \quad g^3 = r \sin \theta d\varphi. \quad (76)$$

Using the nice formula Eq.(23) we will evaluate the energy-momentum 1-forms of the Schwarzschild field.

Since the scalar curvature $R = 0$ outside the star we have

$$t^0 = \partial \cdot \partial g^0 + d\delta g^0 = \frac{M^2}{\sqrt{1 - \frac{2M}{r}r^4}}dx^0 = \frac{M^2}{(1 - \frac{2M}{r})r^4}g^0, \quad (77)$$

$$t^1 = \partial \cdot \partial g^1 + d\delta g^1 = 0,$$

$$t^2 = \partial \cdot \partial g^2 + d\delta g^2 = \frac{\cot(\theta)}{r^2}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}g^1 - \frac{2M}{r^3}g^2,$$

$$t^3 = \partial \cdot \partial g^3 + d\delta g^3 = \frac{-M + r + M \cos(2\theta)}{r^3}csc^2(\theta)g^3. \quad (78)$$

With a simple calculus we see that

$$0 = g^2 \cdot t^1 = t^{21} \neq t^{12} = g^1 \cdot t^2 = -\frac{\cot(\theta)}{r^2}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}. \quad (79)$$

Now it is easy to evaluate the energy of the Schwarzschild gravitational field outside the star. We have taking into account the convention used for the definition of the energy-momentum 3-forms of the fields (Eq.(8)) and the equation of motion for the gravitational potentials (Eq.(10)) that we must define the energy of the field.

**17**We denote this region by $\Omega$. 

17
\[ E := - \int_{\Omega} t^0 g^0 dV = - \int_0^{2\pi} \int_0^\pi \int_R^\infty \frac{M^2}{r^2 (1 - 2Mr)^2} \sin \theta \, dr \, d\theta \, d\phi = 4\pi M \left[ 1 - \frac{1}{(1 - 2M/R)^2} \right]. \] (79)

For sun’s like stars \( \frac{2M}{R} \approx 5.10^6 \). For such cases we have to first order in \( \frac{2M}{R} \) that

\[ E = -4\pi M^2 \frac{2}{R}. \] (80)

**Remark 8** From Eq. (80), we see that the energy of the gravitational field in the exterior of the star is negative. The idea that the energy of the gravitational field is negative is an old one. It appears, e.g., in the Tryon paper [16] which suggested that the universe appears from nothing through a vacuum fluctuation and also it is essential for the inflationary cosmology [6]. And indeed if we supposed that the spatial part of our universe is closed, e.g., is \( S^3 \) we immediately get from Eq. (13) and Stokes theorem that since \( \partial S^3 = \emptyset \) it is

\[ \int_{S^3} * (T_0 + t_0) = - \int_{S^3} d * F_0 = - \int_{\partial S^3} * F_0 = 0, \] (81)

which shows that the total gravitational energy of this universe is null, i.e., the energy-momentum of the gravitational field is negative.

And, all the momentum components \( P^i := - \int_{\Omega} t^i dV = 0 \) are trivially zero.

\[ \text{\footnote{Recall that \( S^3 \) is a parallelizable manifold for which a small modification of our theory works as well.}} \]