Space-Efficient Las Vegas Algorithms for K-SUM
(Preliminary Version)

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Abstract

Using hashing techniques, this paper develops a family of space-efficient Las Vegas randomized algorithms for $k$-SUM problems. This family includes an algorithm that can solve 3-SUM in $O(n^2)$ time and $O(\sqrt{n})$ space. It also establishes a new time-space upper bound for SUBSET-SUM, which can be solved by a Las Vegas algorithm in $O^*(2^{(1-\sqrt{8/9})n})$ time and $O^*(2^{\beta n})$ space, for any $\beta \in [0, \frac{3}{16}]$.

1 Introduction

The $k$-SUM problem on $n$ numbers can be formulated as follows: Given $k$ sets $S_1, S_2, \ldots, S_k$ with $n$ integers each and a target $t$, find $a_1, a_2, \ldots, a_k$ such that for all $i$, $a_i \in S_i$ and $\sum_{i=1}^{k} a_i = t$. Note that one common variant of the problem has only a single set $S$ from which all elements in the solution are chosen from, but the two are easily reducible to each other. The $k$-SUM problem can be trivially solved in $O(n^k)$ arithmetic operations by trying all possibilities, and a more sophisticated solution runs in $O(n^{\lceil k/2 \rceil} \log n)$ time. How much faster can it be solved? This turns out to be a fundamental question, as the complexity of $k$-SUM is related to the complexity of a number of other problems.

Gajentaan and Overmars classified many problems from computational geometry as “3SUM-hard” (i.e. there exists a $o(n^2)$ reduction from 3-SUM to the problem in question) in order to indirectly demonstrate their difficulty. Finding a subquadratic algorithm for any problem in this class of problems would immediately produce a subquadratic algorithm for 3-SUM. One example of such a problem is 3-POINTS-ON-LINE: Given a set of points in the plane, are there three collinear points? To reduce 3-SUM to this problem, map each $x \in S$ (using the single-set variation of 3-SUM) to the point $(x, x^3)$, with the idea that $a_1 + a_2 + a_3 = 0$ if and only if the points $(a_1, a_1^3), (a_2, a_2^3)$, and $(a_3, a_3^3)$ are collinear.
\textit{k-SUM} is also fundamentally connected to several NP-hard problems. Patrascu and Williams \cite{Patrascu2012} show that solving \( k\)-SUM over \( n \) numbers in \( O(n^{o(k)}) \) time would imply that 3-SAT with \( n \) variables can be solved in \( O(2^{o(n)}) \) time. Schroeppel and Shamir \cite{Schroeppel1977} have shown how the \textsc{Subset-Sum} problem can be reduced to an (exponentially-sized) \( k\)-SUM problem. Therefore, more efficient \( k\)-SUM algorithms can be used to derive faster \textsc{Subset-Sum} algorithms. The \textsc{Subset-Sum} problem on \( n \) numbers can be formulated as follows: Given a set \( S \) of \( n \) integers and a target \( t \), find a subset \( S' \subseteq S \) such that \( \sum_{a \in S'} a = t \). They then provide a space-efficient 4-SUM algorithm to yield a time and space efficient \textsc{Subset-Sum} algorithm. Schroeppel and Shamir also established a time-space tradeoff theorem for \textsc{Subset-Sum} algorithms that allowed them to provide a time/space upper bound of \( T \cdot S^2 = O^*(2^n) \) given that \( T \geq O^*(2^n/2) \). This paper will prove a parallel tradeoff result for \( k\)-SUM algorithms, and then use their \textsc{Subset-Sum} to \( k\)-SUM reduction to find an improved time-space upper bound for \textsc{Subset-Sum}.

\textbf{Our Results}

The best known algorithm for 3-SUM takes \( O(n^2) \) time, but also requires \( O(n) \) space (to hold a sorted array of numbers). Can we use significantly less space and obtain the same running time? This paper also investigates the time-space tradeoffs for the general \( k\)-SUM problem. Given some fixed time budget \( S \), we wish to solve \( k\)-SUM in time \( T \) and space \( S \) where \( T \) is minimized.

We use hashing techniques to lower the space requirement for 3-SUM:

\textbf{Theorem 1.1.} \( 3\)-SUM on \( n \) numbers can be solved by a Las Vegas algorithm\footnote{Recall that algorithms are Las Vegas randomized if they always give correct results, but may take additional running time depending on the random numbers generated (but not depending on the choice of input).} in time \( O(n^2) \) and space \( O(\sqrt{n}) \).

These techniques also help lower the space requirements for the general \( k\)-SUM problem on \( n \) numbers, albeit at the cost of some running time increase.

\textbf{Theorem 1.2.} Let \( \delta \leq 1 \). We will not define \( f(x) \) here, but it is a function from \( \mathbb{Z}^+ \to \mathbb{Z}^+ \), and \( f(x) \leq x - \sqrt{x} + 1 \). \( k\)-SUM on \( n \) numbers can be solved in \( \tilde{O}(n^{k-\delta(k-1)} + n^{k-\delta(k-1)+f(k)-1}) \) time and \( O(n^\delta) \) space by a Las Vegas algorithm.

The bound on \( f(x) \) implies the following corollary when we let \( \delta = 1 \):

\textbf{Corollary 1.3.} \( k\)-SUM on \( n \) numbers can be solved in \( \tilde{O}(n^{k-\sqrt{k}+1}) \) time and \( O(n) \) space by a Las Vegas algorithm.

Here are a few sample values of \( f \): \( f(3) = 2 \), \( f(4) = 2 \), \( f(10) = 7 \), and \( f(100) = 90 \). Substituting these values into Theorem 1.2 yields:

\textbf{Corollary 1.4.} Let \( \delta \leq 1 \). Then 3-SUM on \( n \) numbers can be solved in \( \tilde{O}(n^{3-2\sqrt{k}} + n^2) \) time and \( O(n^\delta) \) space by a Las Vegas algorithm.
Corollary 1.5. Let $\delta \leq 1$. Then 4-SUM on $n$ numbers can be solved in $\tilde{O}(n^{3-3\delta} + n^{3-\delta})$ time and $O(n^3)$ space by a Las Vegas algorithm.

Corollary 1.6. Let $\delta \leq 1$. Then 10-SUM on $n$ numbers can be solved in $\tilde{O}(n^{10-9\delta} + n^{9-2\delta})$ time and $O(n^3)$ space by a Las Vegas algorithm.

Corollary 1.7. Let $\delta \leq 1$. Then 100-SUM on $n$ numbers can be solved in $\tilde{O}(n^{100-99\delta} + n^{99-9\delta})$ time and $O(n^3)$ space by a Las Vegas algorithm.

These space-efficient algorithms also imply new time/space upper-bounds for SUBSET-SUM:

Theorem 1.8. There is a Las Vegas algorithm for SUBSET-SUM on $n$ numbers that runs in $O^*(2^{(1-\sqrt{\frac{7}{8}})n})$ time and $O^*(2^{3n})$ space, for $\beta \in [0, \frac{9}{32}]$.

This improves the tradeoff of Schroeppel and Shamir when $S$ is sufficiently small. For example, when $S = O^*(2^{0.1n})$, Schroeppel and Shamir obtain $T = O^*(2^{0.8n})$ while we obtain $T = O^*(2^{0.702n})$.

2 Preliminaries

This section covers notation, basic $k$-SUM algorithms, and hashing.

2.1 Notation

Suppression of polylogarithmic factors from polynomial functions is indicated with $\tilde{O}$. Suppression of polynomial factors from exponential functions is indicated with $O^*$.

The following definition is also useful for discussing merging the sets of $k$-SUM problems:

**Definition.** When $S$ and $T$ are sets, the set $S + T$, also called the Minkowski sum of $S$ and $T$, is defined as $\{ s + t \mid s \in S, t \in T \}$.

2.2 Basic $k$-SUM Algorithms

We present several standard algorithms for $k$-SUM on $n$ numbers for $k \leq 4$. All of them are based around the following solution to 2-SUM that requires the input sets to be sorted:

**Lemma 2.1.** Given a 2-SUM problem on $n$ numbers where the elements of $S_1$ can be accessed in nondecreasing order and the elements of $S_2$ can be accessed in nonincreasing order, where $T(n)$ is the time to access the next element of either $S_1$ or $S_2$, a solution can be found in $O(n \cdot T(n))$ time and $O(1)$ space.

**Proof.** Let $s_1$ denote an element of $S_1$ and $s_2$ denote an element of $S_2$. Begin by setting $s_1$ to the smallest element of $S_1$ and setting $s_2$ to the largest element of $S_2$. If $s_1 + s_2 = t$, then $s_1$ and $s_2$ form a solution; return it. If $s_1 + s_2 < t$, then...
advance $s_1$ to the next element of $S_1$. Otherwise, if $s_1 + s_2 > t$, then advance $s_2$ to the next element of $S_2$. Repeat this process until a solution is found or one of the sets is empty, in which case there is no solution.

**Correctness:** Notice the algorithm processes elements of $S_1$ from smallest to largest elements of $S_2$ from largest to smallest. $s_1$ only advances when it could not sum to $t$ with any element left to be considered in $S_2$. This occurs because $s_1 + s_2 < t$ implies that the sum of $s_1$ with any element left in $S_2$ is strictly less than $t$. Similarly, $s_2$ only advances when it could not sum to $t$ with any element left to be considered in $S_1$.

If the algorithm exhausts either set $S_1$ or $S_2$, then that set has no elements that could appear in a solution. Hence, there are no solutions.

**Running Time:** Each comparison with $t$ and element access removes one element to consider from either $S_1$ or $S_2$, so the algorithm requires $O(n \cdot T(n))$ time at most.

**Memory Usage:** This algorithm only requires space to store counters and compute sums, which does not depend on $n$.

This completes the proof.

The algorithms for 2-SUM, 3-SUM, and 4-SUM are just reductions to the constrained 2-SUM problem required by Lemma 2.1

**Theorem 2.2.** 2-SUM on $n$ numbers can be solved in $O(n \log n)$ time and $O(n)$ space.

*Proof.* Sort the elements of $S_1$ and $S_2$ into arrays, and run the algorithm from Lemma 2.1. Sorting requires $O(n \log n)$ time and $O(n)$ space, and note that element access can be done in constant time.

**Theorem 2.3.** 3-SUM on $n$ numbers can be solved in $O(n^2)$ time and $O(n)$ space.

*Proof.* Sort the elements of $S_1$ and $S_2$ into arrays. For each element $s_3 \in S_3$, use the algorithm from Lemma 2.1 to search for $t - s_3$.

Sorting requires $O(n \log n)$ time and $O(n)$ space. Invoking the algorithm from Lemma 2.1 $n$ times requires $O(n^2)$ time and $O(1)$ space (element access can be done in constant time).

Schroeppel and Shamir\[5\] devised the following 4-SUM algorithm:

**Theorem 2.4.** 4-SUM on $n$ numbers can be solved in $O(n^2 \log n)$ time and $O(n)$ space.

*Proof.* The key data structure is a priority queue that supports inserting, deleting, and extracting the minimum in logarithmic time per operation and takes linear space (this is possible with a heap-based priority queue). One priority queue, $PQ_1$, processes the elements of $S_1 + S_2$ in non-decreasing order while
another priority queue, $PQ_2$, processes the elements of $S_3 + S_4$ in non-increasing order.

To do this for $S_1 + S_2$, first sort $S_2$ in non-decreasing order. For every $i = 1, 2, \ldots, |S_1|$, enqueue the pair $(i, 1)$. The priority of any pair $(i, j)$ is be $S_1[i] + S_2[j]$ (the sum of the $i$th element of $S_1$ and the $j$th element of the sorted $S_2$, both of which are one-indexed). Whenever the pair $(i, j)$ is deleted, where $j < |S_2|$, immediately insert the pair $(i, j + 1)$. Since $S_2$ is sorted in non-decreasing order, any pair will be inserted before the minimum priority in the queue is larger than the pair’s priority. The elements of $S_1 + S_2$ are therefore extracted in order of non-decreasing priority, which is to say in order of non-decreasing value.

$S_3 + S_4$ is handled similarly, except $S_4$ is sorted in non-increasing order and the priority queue is used to extract the maximum priority element.

These priority queues reduce the problem to the form found in Lemma 2.1, but each set now has $n^2$ elements. Accessing elements takes $O(\log n)$ time, so the final running time is $O(n^2 \log n)$.

The priority queues each use linear memory and only ever contain a linear number of elements, so the total memory usage is $O(n)$.

### 2.3 Hash Functions

**Definition.** A family of hash functions $H = \{h : U \rightarrow [m]\}$ is said to be universal if for every $x, y \in U$, if $x \neq y$ then $Pr_{h \in H}[h(x) = h(y)] \leq \frac{1}{m}$.

**Definition.** Given a family of hash functions $H = \{h : U \rightarrow [m]\}$, and some set $S \subset U$, let the bucket of $h$ with value $v$ be $h^{-1}(\{v\})$ (i.e. all elements with hash value $v$). Also, define $B_h(x) := h^{-1}(\{h(x)\})$ (the bucket of $h$ with value $h(x)$).

The following universal family of hash functions, $H_1$, was first introduced by Dietzfelbinger [2] and applied to 3-SUM by Baran, Demaine and Patrascu [1]. It can be used on the elements of the input sets of $k$-SUM so that only a subset of them need to be considered at once, saving memory.

**Definition.** Given a word size $w$, a hash length $s$, and an odd integer $a$, let the hash function $h_a : U \rightarrow [2^s]$ be defined as $h_a(x) := \lfloor \frac{ax}{2^w} \mod 2^w \rfloor$. In C notation, this hash can be expressed as $(a * x) >> (w - s)$.

Define the family of hash functions $H_1 := \{h_a | a \in [2^w], a \text{ odd}\}$.

### 3 Almost Linear Hashing

This section covers certain useful properties of $H_1$, which will be used to construct Las Vegas algorithms for $k$-SUM. Baran, Demaine, and Patrascu [1] gave the following two lemmas when applying $H_1$ to 3-SUM:

**Lemma 3.1.** The family of hash functions $H_1$ satisfies almost-linearity, in that for all $x, y \in U$, $h(x + y) \in \{h(x)\} \oplus \{h(y)\} \oplus \{0, 1\}$ ($\oplus$ is addition modulo $2^s$).
Proof. Multiplying by $a$ is linear, and dropping the low-order bits can only influence the result by 1 due to losing the carry.

This next lemma applies to any universal family of hash functions, and hence to $H_1$ as well:

**Lemma 3.2.** Given any universal family of hash functions $H = \{ h : U \to [m] \}$ and some set $S \subset U$ of size $n$, the expected number of elements $x \in S$ with $|B_h(x)| \geq t$ is at most $2n^t - 2m/n + 2$.

Proof. Pick $x \in S, y \in S \setminus \{x\}$ randomly and let $p_h = \Pr_x[|B_h(x)| \geq t]$ and $q_h = \Pr_{x,y}[h(x) = h(y)]$. It suffices to show that $p_h \leq \frac{2n^t - 2m/n + 1}{2m/n + 2}$.

Let $S_h = \{ x \in S \mid |B_h(x)| < t \}$. Note $|S_h| = (1 - p_h)n$. Notice that:

$$\Pr[h(x) = h(y) \mid x \notin S_h] \geq \frac{t - 1}{n}$$

On the other hand, if $x \in S_h$ then also $y \in S_h$. By convexity of the square function, the collision probability of elements of $S_h$ is minimized when the same number of elements of $S_h$ hash to any value. In this case:

$$|B_h(x)| \geq \left\lfloor \frac{|S_h|}{m} \right\rfloor \geq (1 - p_h)\frac{n}{m} - 1$$

Hence:

$$\Pr[h(x) = h(y) \mid x \in S_h] \geq \frac{(1 - p_h)n/m - 2}{n}$$

Combining yields the following:

$$q_h \geq p_h \frac{t - 1}{n} + (1 - p_h)\frac{(1 - p_h)n/m - 2}{n}$$

$$\geq \frac{1}{n}(p_h(t - 1) + (1 - 2p_h)\frac{n}{m} - 2(1 - p_h))$$

$$\geq \frac{1}{n}(p_h(t - 2\frac{n}{m} + 2) + \frac{n}{m} - 2)$$

By universality, $E[q_h] \leq \frac{1}{m}$. The above inequality simplifies into:

$$p_h(t - 2\frac{n}{m} + 2) + \frac{n}{m} - 2 \leq \frac{n}{m}$$

$$p_h \leq \frac{2}{t - 2n/m + 2}$$

This completes the proof.

Lemma 3.1 guarantees that if $(k - 1)$ sets have their hash buckets fixed, any solution that uses elements from those buckets could only have its last element in one of $k$ buckets of the last set. Hence, hashing can be used to shrink the problem size with some limited growth in the number of cases. It is worth noting that this hash works best on 3-SUM, since for larger $k$ applying the hash tends to increase the running time of the algorithm. It turns out that for large enough $m$, large buckets can be completely avoided by simply inspecting a constant number of hashes (in expectation).
Corollary 3.3. Consider a universal family of hash functions \( H = \{ h : U \to [m] \} \), a set \( S \subseteq U \) of size \( n \), where \( m \leq \sqrt{n} \), and an arbitrary constant \( c \geq 1 \). Then:
\[
Pr_{h \in H} [\forall x \in S : |B(x)| \leq (c + 2)\frac{n}{m}] \geq 1 - \frac{2}{c^2}
\]

Proof. Let \( t = (c + 2)\frac{n}{m} \). Let \( b(h) \) be the number of elements \( x \in S \) with \( |B_h(x)| \geq t \). Applying Lemma 3.2 yields that \( E[b(h)] \leq \frac{2n}{c(n/m) + 2} \leq \frac{2n}{c} \). Applying a Markov bound yields \( Pr_h[b(h) \geq cm] \leq \frac{2}{c^2} \). However, if \( b(h) < cm \) then in fact \( b(h) = 0 \), since \( b(h) \) counts the number of elements in buckets of \( h \) with at least \( (c + 2)\frac{n}{m} \) elements (\( m \leq \sqrt{n} \) implies \( \frac{n}{m} \geq m \)). Hence, \( Pr_h[\forall x : |B(x)| \leq (c + 2)\frac{n}{m}] \geq 1 - \frac{2}{c^2} \). This completes the proof. \( \square \)

4 Las Vegas Algorithms for \( k \)-SUM

This section uses the hashing results to derive space-efficient Las Vegas algorithms for \( k \)-SUM problems. Specifically, we demonstrate how to reduce the space usage of \( k \)-SUM algorithms using Corollary 3.3. We use that result to derive a family of linear-space Las Vegas algorithms for \( k \)-SUM. We then reapply that result to derive a set of sublinear-space Las Vegas algorithms that we will use later to establish new time-space upper bounds for \( \text{SUBSET-SUM} \) algorithms.

Theorem 4.1. Let \( A \) be a Las Vegas algorithm that solves \( k \)-SUM \((k \geq 3)\) on \( n \) numbers in \( T(n) \) time and \( S(n) \) space where \( T(n), S(n) \in \text{poly}(n) \), and let \( \delta \leq 1 \) be an arbitrary constant. Then there is a Las Vegas algorithm \( A' \) that solves \( k \)-SUM on \( n \) numbers in \( O(n^{k-\delta(k-1)} + n^{k-\delta(k-1)-1}T(n^\delta)) \) time and \( O(n^\delta + S(n^\delta)) \) space.

This theorem allows us to reduce the space usage of a \( k \)-SUM algorithm by a factor of \( \delta \) at the cost of shrinking the gap between the running time and \( O(n^k) \).

Proof. The key idea is that to use hashing to reduce the size of each set by a square root factor at each step. However, storing any of the intermediate sets of this computation defeats the purpose of hashing any further. To avoid this, we first determine all hash functions and values to shrink each set to the desired size, and then compute the final sets in one step.

\( A' \) will recursively construct a list \( L \) whose elements are of the form \((h, v_1, v_2, \ldots, v_k)\), i.e. a hash function followed by \( k \) hash values (one for each \( S_i \)). At any step, define the active set of \( S_i \) to be \( \tilde{S}_i = \{ s \in S_i : h(s) = v_i \forall (h(v_1, v_2, \ldots, v_k) \in L) \} \). Each element appended to \( L \) reduces the size of all active sets, so elements can be repeatedly appended until the active sets are only \( O(n^\delta) \) in size, at which point it is safe to invoke \( A \). To handle the possibility that \( \delta \) is not a perfect power of \( \frac{1}{2} \), define the function \( s(x) := \max(\frac{1}{2}, x, \delta) \). Step \( i \) of the algorithm will reduce the size of all active sets from \( O(n^{s(i)}) \) to \( O(n^{s(i+1)}) \).
The recursive helper function $R$ will construct $L$ and then invoke $A$. It has access to all sets $S_i$ and does the following given a partially constructed $L$:

1. Let $\ell := |L|$. If $s(\ell) = \delta$, then compute $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k$ and call $A$ on them. Otherwise, the active sets $S_1, S_2, \ldots, S_k$, are guaranteed to each contain at most $(k + 2)^2 n^{s(\ell)}$ elements.

2. Let $V_\ell := (k + 2)n^{s(\ell) - s(\ell + 1)}$. Pick a random hash function $h \in H_1$ that maps to $V_\ell$ values. For each $S_i$ and possible hash value $v \in [V_\ell]$, iterate through all elements of $S_i$, consider only the ones in $\tilde{S}_i$, and count how many hash to the current $v$. If any count exceeds $(k + 2)^2 n^{s(\ell + 1)}$ elements, pick another hash and try again.

3. For each $(v_1, v_2, \ldots, v_{k-1}) \in [V_\ell]^{k-1}$ and $j = 0, 1, \ldots, k - 1$, let $v_k$ equal $h(t)$, less the sum of all already selected $v_i$’s, less $j$ (mod $V_\ell$). Call $R$ on $L$ appended with $(h, v_1, v_2, \ldots, v_k)$.

Algorithm $A'$ calls $R$ with $L = \emptyset$.

**Correctness:** We first prove the size guarantee made when calling $R$. $A'$ initially calls $R$ with $\ell = 0$ and sets of size $n \leq (k + 2)^2 n^{s(0)}$. $R$ ensures that the hash it has chosen creates buckets that are no larger than $(k + 2)^2 n^{s(\ell + 1)}$ in size, so it may safely append an additional element to $L$ before making a recursive call to itself.

We also want to show that if a solution exists, we will find it. Due to the almost linearity property of $H_1$, we know that a call to $R$ where each element of the solution is in an active set will in turn make some recursive call where the elements are still in active sets. Since our first call to $R$ is made with an empty $L$ (and hence with all elements in active sets), we know that any elements of a solution will begin in active sets and hence will be found by the algorithm.

**Running Time:** Checking that the buckets of a randomly-selected hash function are not too large takes $O(n^{1 + s(\ell) - s(\ell + 1)})$ time since the algorithm needs to perform a linear scan for each hash value $v \in [V_\ell]$. We apply Corollary 3.3 with $c = k$, so we know the chance of a hash failing over a specific $S_i$ is at most $\frac{1}{2}$; the chance of it failing over any $S_i$, by a union bound, is at most $\frac{1}{2}$. Since $k \geq 3$, the expected number of hashes the algorithm needs to pick and check is at most three. Hence our expected time checking for hashes during a single call to $R$, not including recursive subcalls, is $O(n^{1 + s(\ell) - s(\ell + 1)})$.

There is a single call where $\ell = 0$. Each recursive level of $R$ makes $O(n^{(k - 1)(s(\ell) - s(\ell + 1))})$ calls to the level below it. Hence, there are $O(n^{(k - 1)(1 - s(\ell))})$ calls to $R$ for a given $\ell$ (all the terms cancel) The total expected time checking for hashes during all calls with a given $\ell$ is therefore $O(n^{(k - 1)(1 - s(\ell)) + (1 + s(\ell) - s(\ell + 1))})$.

\[
(k - 1)(1 - s(\ell)) + (1 + s(\ell) - s(\ell + 1)) = k - s(\ell)(k - 2) - s(\ell + 1) \\
\leq k - s(\ell + 1)(k - 1) \\
\leq k - \delta(k - 1)
\]
Hence, the total expected time checking for hashes during all calls with a given \( \ell \) is also \( O(n^{k-\delta(k-1)}) \). Since the algorithm only searches for hash functions for \( \ell \in \{0,1,\ldots,\lceil \log_2 \frac{k}{\delta} \rceil - 1 \} \), the total expected running time checking for hashes overall is \( O(n^{k-\delta(k-1)}) \).

When \( s(\ell) = \delta \), we need to compute all \( S_i \). From our previously-derived formula, we know that there are only \( O(n^{(k-1)(1-\delta)}) \) calls where this occurs. Computing all \( S_i \) only requires a linear scan of each \( S_i \), so we can do this in time \( O(n^{k-\delta(k-1)}) \).

Finally, we invoke \( A \) \( O(n^{(k-1)(1-\delta)}) \) times on sets of size at most \( (k+2)^2n^\delta \), so in total we use \( O(n^{(k-1)(1-\delta)}T(n^\delta)) \) time making calls to \( A \).

The total time taken is hence \( O(n^{k-\delta(k-1)} + n^{k-\delta(k-1)-1}T(n^\delta)) \).

**Memory Usage:** Notice that \( L \) contains at most \( \lceil \log_2 \frac{k}{\delta} \rceil \) elements of size \( (k+1) \) each, so it takes \( O(1) \) space. The space needed to check the selected hash is also \( O(1) \), since we compute a count for only a single hash value at a time.

Invoking \( A \) on sets of size at most \( (k+2)^2n^\delta \) requires only \( O(n^\delta + S(n^\delta)) \) space (to store the inputs along with the space needed by \( A \)).

This completes the proof. \( \square \)

This family of hash functions does particularly well when applied to 3-SUM.

When applied to the basic \( O(n^2) \) time, \( O(n) \) algorithm, the space-usage decreases without any running-time cost:

**Theorem 4.4** 3-SUM on \( n \) numbers can be solved by a Las Vegas algorithm in time \( O(n^2) \) and space \( O(\sqrt{n}) \).

**Proof.** From Theorem 2.3 we know 3-SUM can be solved in \( T(n) = O(n^2) \) time and \( S(n) = O(n) \) space. We apply Theorem 4.1 with \( \delta = 0.5 \), which yields a Las Vegas algorithm that solves 3-SUM in \( O(n^{3-0.5(2)} + n^{3-0.5(2)-1/2}n^{0.5/2}) \), or \( O(n^5) \) time and \( O(\sqrt{n}) \) space. \( \square \)

Theorem 4.1 also yields a family of linear-space Las Vegas algorithms for \( k \)-SUM problems, via the following intermediate corollary:

**Corollary 4.2.** Let \( A \) be a Las Vegas algorithm that solves \( k_1 \)-SUM \((k_1 \geq 3)\) on \( n \) numbers in \( \tilde{O}(n^\alpha) \) time and \( O(n) \) space for some constant \( \alpha \). Then there is a Las Vegas algorithm \( A' \) that solves \((k_1 \cdot k_2)\)-SUM on \( n \) numbers in \( \tilde{O}(n^{k_1k_2-k_1+k_1+\alpha/k_2}) \) time and \( O(n) \) space.

**Proof.** Apply Theorem 4.1 to \( A \), choosing \( \delta = \frac{1}{k_2} \). Hence, there is a Las Vegas algorithm \( A'' \) that solves \( k_1 \)-SUM on \( n \) numbers in \( \tilde{O}(n^{k_1-k_1/k_2} + n^{k_1-k_1/k_2+\alpha/k_2}) \) time and \( O(n^{1/k_2}) \) space.

However, a \((k_1 \cdot k_2)\)-SUM problem on \( n \) numbers can be converted to a \( k_1 \)-SUM problem on \( n^{k_2} \) numbers. For \( i \in \{1,2,\ldots,k_1\} \), we let \( S'_i = \sum_{j=1}^{k_2} S_{(i-1)k_2+j} \) (the Minkowski sum of a block of \( k_2 \) sets), and we run \( A'' \) on the sets \( S'_i \) with the same target \( t \). Note that we do not actually store all elements of the sets
$S'_i$, but rather compute them on demand in constant time as $A''$ requires them, in order to avoid using too much memory.

Our algorithm $A'$ is to call $A''$ on the sets $S'_i$. Since these are $n^{k_2}$ in size, the algorithm $A'$ takes $\tilde{O}(n^{k_1k_2-k_1+1} + n^{k_1k_2-k_1+1+\alpha-k_2})$ time and $O(n)$ space, as desired.

This completes the proof. \(\square\)

Corollary 4.2 can be used to find a linear-space algorithm for $k$, given that it factors into $k_1$ and $k_2$ and that we already have an algorithm for $k_1$-SUM that runs in linear space. If $k$ does not factor nicely, it is possible to brute-force over one set to reduce to $(k-1)$-SUM:

**Lemma 4.3.** Let $A$ be an algorithm that solves $k$-SUM ($k \geq 3$) on $n$ numbers in $\tilde{O}(n^\alpha)$ time and $O(n)$ space. Then there is an algorithm $A'$ that solves $(k+1)$-SUM on $n$ numbers in $\tilde{O}(n^{\alpha+1})$ time and $O(n)$ space.

**Proof.** The algorithm $A'$ is to guess one element $s \in S_{k+1}$ of the solution and then to run $A$ on $S_1, \ldots, S_k$ for the remaining elements, which now need to sum to $t - s$. \(\square\)

We now construct a function $f(k)$ such that we can produce a Las Vegas algorithm that can solve $k$-SUM on $n$ numbers in $\tilde{O}(n^f(k))$ time and $O(n)$ space.

**Definition.** Let $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$. Let $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, and $f(4) = 2$. For $k > 4$, let:

$$f(k) = \min_{k_1, k_2} \left\{ k_1k_2 - k_1 - k_2 + 1 + \max(f(k_1), k_2) \right\} f(k-1) + 1$$

**Corollary 4.4.** $k$-SUM on $n$ numbers can be solved in $\tilde{O}(n^f(k))$ time and $O(n)$ space by a Las Vegas algorithm.

**Proof.** The base cases are covered by Theorem 2.2, Theorem 2.3, and Theorem 2.4 (and $k=1$ is trivial). For all other $k$, we either get an algorithm from Corollary 4.2 or Lemma 4.3. \(\square\)

Table 1 shows the first few values of $f(k)$. The difference between $k$ and $f(k)$ is important because higher differences will permit better time/space trade-offs. Due to the construction of $f(k)$, this value $k - f(k)$ is nondecreasing in $k$ (it is a maximum of its previous value and the result of applying Corollary 4.2).

The values of $k$ where $k - f(k)$ first increases to a new value $v$ ($k = 8, 15, 24, 32, 40, 54, \ldots$) occur when $k$ factors evenly into $(v+1) \cdot f(v+1)$ (e.g. $15 = 5 \cdot 3$) or when $k$ factors evenly into $(v+2) \cdot (f(v+2) - 1)$ (e.g. $32 = 8 \cdot 4$), whichever is smaller (the latter case occurs if $f(v+2) = f(v+1)$).

$f(x)$ has the following (coarse) upper bound:

**Lemma 4.5.** For all $x \geq 2$, $f(x) \leq x - \sqrt{x} + 1$. 

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Table 1: Time Complexity Upper Bounds for Linear-Space $k$-SUM Algorithms

| $k$ | $f(k)$ | $k - f(k)$ |
|-----|--------|------------|
| 2   | 1      | 1          |
| 3   | 2      | 1          |
| 4   | 2      | 1          |
| 5   | 3      | 2          |
| 6   | 4      | 2          |
| 7   | 5      | 2          |
| 8   | 5      | 3          |
| 9   | 6      | 3          |
| 10  | 7      | 3          |
| 11  | 8      | 3          |
| 12  | 9      | 3          |
| 13  | 10     | 3          |
| 14  | 11     | 3          |
| 15  | 11     | 4          |

Proof. Notice from Table 1 that this is true for $x \in [2, 8]$. We will now prove it for $x \geq 9$.

Let $y^2$ be the largest perfect square that is at most $x$. By the definition of $f$, we know that $f(x) \leq f(y^2 - 1) + (x - y^2 + 1)$. Hence, it suffices to show that $f(y^2 - 1) - y^2 \leq -\sqrt{x}$.

Since $x \geq 9$, $y + 1 \geq 4$. Since $k - f(k)$ is nondecreasing in $k$, $(y+1) - f(y+1) \geq 2$. Simplifying yields $(y - 1) \geq f(y + 1)$.

Notice that $y^2 - 1$ factors into $(y + 1) \cdot (y - 1)$. By our definition of $f$:

$$f(y^2 - 1) \leq (y^2 - 1) - (y + 1) - (y - 1) + 1 + \max(f(y + 1), y - 1)$$
$$\leq y^2 - y - 1$$

By our choice of $y$, though, this implies that:

$$f(y^2 - 1) - y^2 \leq -y - 1 \leq -\sqrt{x}$$

This completes the proof. 

Corollary 1.3. $k$-SUM on $n$ numbers can be solved in $\tilde{O}(n^{k-\sqrt{k}+2})$ time and $O(n)$ space by a Las Vegas algorithm.

Proof. This is a direct consequence of Lemma 4.5 combined with Corollary 4.4.

Applying Corollary 1.1 once more to this linear-space family yields sublinear algorithms:
Theorem 1.2. Let $\delta \leq 1$. Then $k$-SUM on $n$ numbers can be solved in $\tilde{O}(n^{k-\delta(k-1)} + n^{k-\delta(k-1)+\delta f(k)-1})$ time and $O(n^\delta)$ space by a Las Vegas algorithm.

Proof. Corollary 4.4 states that there is a Las Vegas algorithm for $k$-SUM that runs in $\tilde{O}(n^{f(k)})$ time and $O(n)$ space. Applying Corollary 4.1 then yields the desired result. \hfill \Box

5 SUBSET-SUM Time-Space Tradeoffs

Schroeppel and Shamir\cite{5} provided the following reduction from SUBSET-SUM to $k$-SUM:

Theorem 5.1. Let $A$ be an algorithm that solves $k$-SUM on $n$ numbers in $\tilde{O}(n^{\alpha k})$ time and $\tilde{O}(n^{\beta k})$ space for some constants $\alpha$ and $\beta$. Then SUBSET-SUM on $n$ numbers can be solved in $O^*(2^{\alpha n})$ time and $O^*(2^{\beta n})$ space.

Proof. Consider the following algorithm $A'$:

1. Given a set $S$ with $n$ elements, divide it into $k$ sets $S_1, S_2, \ldots, S_k$ of $\frac{n}{k}$ elements each. For each set $S_i$, compute the set $T_i := \{ \sum_{s \in S'_i} s \mid S'_i \subseteq S_i \}$.
   Run $A$ on $T_1, T_2, \ldots, T_k, t$.

Correctness: If there is some solution, the sum of its elements in any $S_i$ will wind up in some $T_i$, and hence $A$ will be able to find a solution that sums to $t$. Note that it is possible to backtrack and recover the original elements used to generate the elements of the $k$-SUM solution.

Running Time: We call $A$ on sets of size at most $2^{\frac{k}{2}}$, so $A'$ takes $O^*(2^{\alpha n})$ time.

Memory Usage: We call $A$ on sets of size at most $2^{\frac{k}{2}}$, so $A'$ takes $O^*(n^{\beta n})$ space.

This completes the proof. \hfill \Box

They also proved a theorem regarding SUBSET-SUM (as well as other problems in a specific class of NP-hard problems) that allowed trading increased running time in return for reduced space. Here is a $k$-SUM analogue of that result, which allows further improvement our space-time upper bound on $k$-SUM (and via Theorem 5.1 SUBSET-SUM as well):

Theorem 5.2. Let $A$ be an algorithm that solves $k$-SUM on $n$ numbers in $T = O(n^{\alpha k})$ time and $S = O(n^{\beta k})$ space for some constants $\alpha$ and $\beta$. Then $k$-SUM on $n$ numbers can be solved in any time/space combination along the tradeoff curve $T' \cdot S' = \tilde{O}(n^k)$, $\Omega(n^{\alpha k}) \leq T' \leq \tilde{O}(n^k)$.

Proof. Consider the following algorithm $A_\gamma$ ($0 \leq \gamma \leq 1$):
1. Divide each $S_i$ into $n^{1-\gamma}$ regions of consecutive elements of size $n^\gamma$.

2. For each way to choose exactly one region from each $S_i$, run $A$ on that choice.

In particular, when $\gamma = 0$, $A_\gamma$ is just a brute-force search, while when $\gamma = 1$, $A_\gamma$ reduces to algorithm $A$.

**Correctness:** We exhaustively search every combination of regions, and we know each element in a solution must appear in some region.

**Running Time:** Algorithm $A$ is called $n^{k(1-\gamma)}$ times on problems of size $O(n^\gamma)$, so $A_{\gamma}$ uses $\tilde{O}(n^{k(1-\gamma)+\alpha\gamma k})$ time.

**Memory Usage:** Algorithm $A$ is called on problems of size $O(n^\gamma)$, so $A_{\gamma}$ uses $\tilde{O}(n^{\beta\gamma k})$ space.

Notice that:

$$T' \cdot S'^{\frac{1}{\gamma\sqrt{\beta}}} = \tilde{O}(n^{k(1-\gamma)+\alpha\gamma k \frac{n^{\beta\gamma k}}{n}}) = \tilde{O}(n^{k-\gamma k+\alpha\gamma k+\gamma k-\alpha\gamma k}) = \tilde{O}(n^k).$$

This completes the proof. \qed

We have a family of space-efficient Las Vegas algorithms for $k$-SUM from Corollary 1.2. Applying Theorem 5.2 followed by Theorem 5.1 yields a piecewise upper-bound curve for SUBSET-SUM algorithms (due to the fact that $k$ must be integer). To better understand the behavior of this curve, we formulate it as a tradeoff between the exponents of $T$ and $S$, as follows:

**Theorem 1.8.** There is a Las Vegas algorithm for SUBSET-SUM on $n$ numbers that runs in $O^*(2^{(1-\sqrt{\beta})n})$ time and $O^*(2^{\beta n})$ space, for $\beta \in [0, \frac{9}{32}]$.

We first prove a lemma:

**Lemma 5.3.** Given a constant $\beta \in [0, \frac{1}{\sqrt{\gamma}}]$, there exists a $k$ such that $k$-SUM on $n$ numbers is solved by a Las Vegas algorithm that runs in $T = \tilde{O}(n^{1-\gamma\sqrt{\beta}k})$ time and $S = \tilde{O}(n^{\beta k})$ space, where $\gamma = \sqrt{\frac{8}{9}} \approx 0.942809$.

**Proof.** Notice that $T \cdot S^{\frac{1}{\gamma \sqrt{\beta}}} = \tilde{O}(n^k)$, so by Theorem 5.2 it suffices to show that there is a Las Vegas algorithm for $k$-SUM on $n$ numbers that runs in time $T'$ and space $S'$ where $T' \leq T$ and $T' \cdot S'^{\frac{1}{\gamma \sqrt{\beta}}} = \tilde{O}(n^k)$. For the remainder of the proof, we will let $\omega$ denote $\gamma \sqrt{\beta}$.

If $\omega \leq 2$ then we are already done, since by Theorem 2.1 we have a solution to 4-SUM on $n$ numbers with $T' = \tilde{O}(n^2)$ and $S' = \tilde{O}(n)$ and our range for $\beta$ implies that $\gamma \sqrt{\beta} \leq 0.5$. Hence we may assume that $\omega > 2$ for the remainder of the proof.
By Corollary 1.2, $k$-SUM on $n$ numbers can be solved in $T'' = \tilde{O}(n^{k-\delta(k-1)} + n^{k-\delta(k-1)-f(k)-1})$ time and $S'' = \tilde{O}(n^\delta)$ space by a Las Vegas algorithm. Choose $k = \lceil \omega \rceil + 1$ and $\delta = \frac{1}{\omega-1}$. Since $\omega > 2$, we know that $k \geq 4$ and so $k - 1 - f(k) \geq 1$. Hence the running time $T''$ is in $\tilde{O}(n^{k-\delta-1})$.

$\gamma$ should be chosen to guarantee that $\gamma \sqrt{\beta k} \leq \delta + 1$. Equivalently, $\gamma$ should be chosen such that:

$$
\begin{align*}
\gamma \sqrt{\beta k} &\leq \delta + 1 \\
\gamma^2 k &\leq \omega(\delta + 1) \\
\gamma^2 k &\leq \omega \frac{\omega}{\omega - 1} \\
\gamma^2 &\leq \frac{\omega^2}{(\omega - 1)k} \\
\gamma^2 &\leq \frac{\omega^2}{(\omega - 1)(\omega + 2)}
\end{align*}
$$

The right-hand side is minimized when $\frac{(\omega-1)(\omega+2)}{\omega^2}$ is maximized. Taking the derivative shows that this occurs when $\omega = 4$, so it is safe to pick $\gamma = \sqrt{\frac{8}{9}}$.

This completes the proof. \hfill \Box

Applying Theorem 5.1 to Lemma 5.3 directly yields Theorem 1.8.

The following graph demonstrates the previous best-known trade-off curve, found by Schroeppel and Shamir [5] (labeled as basic 4-SUM) along with the piecewise upper-bound obtainable from Corollary 1.2. The graph also includes the time-space tradeoff as given by Theorem 1.8.
6 Conclusion

An interesting open problem is whether there exists a deterministic algorithm that runs in the same time for the $k$-SUM problem on $n$ numbers. It might be easier to consider the $k$-XOR problem, which is identical except that the elements are vectors from $\mathbb{F}_2^n$ instead of integers. For this variant there is a simple linear universal family of hash functions, and so it seems possible that there might be a way to derandomize the hash selection process.

Another interesting question is whether the function $f(k)$ can be further improved. The fact that there exists a $\tilde{O}(n^2)$ time and $O(n)$ space algorithm for 4-SUM does not match the pattern found in the rest of the table, suggesting that there might be more efficient algorithms for other $k$ as well.

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