Abraham and Minkowski Poynting vector controversy in axion modified electrodynamics

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The most sensitive haloscopes that search for axion dark matter through the two photon electromagnetic anomaly, convert axions into photons through the mixing of axions with a large background DC magnetic field. In this work we apply Poynting theorem to the resulting axion modified electrodynamics and identify two possible Poynting vectors, one which is similar to the Abraham Poynting vector in electrodynamics and the other to the Minkowski Poynting vector. Inherently the conversion of axions to photons is a non-conservative process with respect to the created oscillating photonic degree of freedom. We show that Minkowski Poynting theorem picks up the added non-conservative terms while the Abraham does not. The non-conservative terms may be categorised more generally as “curl forces”, which in classical physics are non-conservative and non-dissipative forces localised in space, not describable by a scalar potential and exist outside the global conservative physical equations of motion. To understand the source of energy conversion and power flow in the detection systems, we apply the two different Poynting theorems to both the resonant cavity haloscope and the broadband low-mass axion haloscope. Our calculations show that both Poynting theorems give the same sensitivity for a resonant cavity axion haloscope and the broadband low-mass axion haloscope. Hence we ask the question, can understanding which one is the correct one for axion dark matter detection, be considered under the framework of the Abraham-Minkowski controversy? In reality, this should be confirmed by experiment when the axion is detected. However, many electrodynamic experiments have ruled in favour of the Minkowski Poynting vector when considering the canonical momentum in dielectric media. In light of this, we show that the axion modified Minkowski Poynting vector should indeed be taken seriously for sensitivity calculation for low-mass axion haloscopes in the quasi static limit, and predict orders of magnitude better sensitivity than the Abraham Poynting vector equivalent.

I. Introduction

The axions is postulated to exist as a neutral spin-zero bosons to solve the strong charge-parity problem in QCD. Such a particle is predicted to couple very weakly to other known particles and has thus been postulated to be cold dark matter [1–13]. In particular, many experiments rely on the electromagnetic anomaly, which is a two photon coupling term with the axion. To gain a significant sensitivity, it is widely considered that the best way to search for the axion is when the first background photonic degrees of freedom is a large DC magnetic field, which generates a second photon that can be detected. This is the basis of the DC magnetic field axion haloscope, first proposed by Sikivie [14, 15] and pioneered experimentally by the ADMX collaboration [16–22]. Recently the scientific case that dark matter may include QCD axions or axion like particles of varied mass and photon coupling has gained momentum [23–34], leading to many new ideas and new detectors designs world-wide, which implement the principle of dark matter detection through the electromagnetic anomaly [35–75].

Inherently, the conversions of axions amongst a DC magnetic background field into a second photonic degree of freedom is a non-conservative process with respect to the second degree of freedom. In this process, the axion mixes with the background field to create a source term that drives the energy of conversion in the second photonic degree of freedom at a frequency corresponding to the axion mass. In this work, we implement Poynting theorem to understand the source of energy conversion and flow in the system when considering axion modified electrodynamics.

In standard electrodynamics, Poynting vector analysis is implemented in circuit and antenna theory to understand how the input source power is impressed into the system along with the system power flow and how it relates to the stored energy and losses [76–80]. In this work we undertake a similar analysis within the framework of axion modified electrodynamics. In other work, the Poynting vector has been implemented with versions of the stress-energy tensor to understand energy and forces in magnetic and dielectric matter. For example, forces in systems such as optical tweezers [81–84] and ion trapping [54], where the correct way to analyse these systems has been a subject of controversy (known as the Abraham–Minkowski controversy) and is still an active area of debate [85–92]. This debate has led to the general concept of “curl forces”, which are abundant in nature and cannot be described from the gradient of a scalar potential, and only exist in a localised space. For example, the force on a particle with complex electric polarizability is known not to be derivable from a scalar potential.
as its curl is non-zero [81] [82] [93]. Such forces are non-conservative and non-dissipative, and their inclusion has been described both classically, and quantum mechanically [91] [95], in particular the quantising of electrodynamics in dielectric and dispersive media [96]. Note, such non-conservative “curl forces” do not include the most well-known “curl force”, which is the magnetic Lorentz force, as it is a conservative force that can do no work [94], described by a magnetic vector potential.

With this in mind, it has become evident that it is possible to derive alternative versions of Poynting’s theorem (in fact four versions are possible) [97]. In particular the Minkowski Poynting vector \( \vec{S}_{DB} = \frac{1}{\epsilon_0 \mu_0} \vec{D} \times \vec{B} \), has been shown to be successful to account for experiments in dielectric media, where the field momentum is associated with the canonical momentum [87] [90] [92] [99], here the electric flux density, \( \vec{D} = \epsilon_0 \vec{E} + \vec{P} \), is the sum of the electric field, \( \vec{E} \), and electric polarization, \( \vec{P} \), and the magnetic flux density, \( \vec{B} = \mu_0 (\vec{H} + \vec{M}) \), is the sum of the magnetic field, \( \vec{H} \) and magnetization, \( \vec{M} \). Naturally, when the curl of the polarization is non zero (\( \nabla \times \vec{P} \neq 0 \)) the Minkowski Poynting vector will pick up this term, due to an unconventional but necessary modification to Faraday’s law [96] [97], while the Abraham Poynting vector [100] [101], \( \vec{S}_{EH} = \vec{E} \times \vec{H} \), will not. For the curl of the polarization to be non-zero, an energy input is required to separate the bound charge, this describes a permanent electret or energy harvesting material [102] and actually describes an external non-conservative force (or fictitious force) input similar to an impressed electric field (or fictitious electric field) that drives an active dipole in antenna theory or a voltage source in circuit theory. Furthermore, the electret or energy harvester may be classified as an active bound charge dipole. We may recognise this active dipole term generally as a non-conservative “curl force”, which necessarily modifies Faraday’s law, and is only present internally to the active antenna, voltage source or electret and not present globally outside the active device. As with all “curl forces”, this non-conservative term cannot be characterised by a scalar potential, on the other hand it has been recently shown to be characterised via an electric vector potential [61] [98] [102] [104].

Recently, it was also shown that there exists a similar non-conservative “curl force” term in axion modified electrodynamics [61] [104]. This occurs when the axion mixes with a DC background magnetic field, which converts the axion mass to the energy of the second photonic degree of freedom [61] [104]. In this representation the axion mixing with the DC magnetic field adds a similar term to a polarization with a non-zero curl [102] [103]. In this work we apply the Minkowski and Abraham Poynting vector equivalents to axion modified electrodynamics and compare the difference, where the former picks up the extra “curl force” term, while the latter does not.

### II. The Effective Axion Current and Charge Density

It is well known that axions modify electrodynamics through the axion two photon coupling [31] [105], which in vacuum leads to the following set of modified Maxwell’s equations,

\[
\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_0} + c g_{a\gamma\gamma} \vec{B} \cdot \nabla a
\]

\[
\nabla \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E} = \mu_0 \vec{J}_e - g_{a\gamma\gamma} \mu_0 c \epsilon_0 \left( \vec{B} \partial_t a + \nabla a \times \vec{E} \right)
\]

\[
\nabla \cdot \vec{B} = 0
\]

\[
\nabla \times \vec{E} + \partial_t \vec{B} = 0
\]

(1)

Here \( g_{a\gamma\gamma} \) is the two-photon coupling to an axion field, \( \rho_e \) is the amplitude of the axion field, \( c \) is the volume charge density and \( \vec{J}_e \) the volume current density. One common way to set up the equations of motion for the two photon interaction is to assume \( \nabla a = 0 \), so two of the three terms go to zero and only one modification to Ampere’s law remains,

\[
\nabla \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E} = \mu_0 \left( \vec{J}_e - g_{a\gamma\gamma} \epsilon_0 c \vec{B} \partial_t a \right),
\]

(2)

where the axion current is defined by,

\[
\vec{J}_a = -g_{a\gamma\gamma} \epsilon_0 c \vec{B} \partial_t a.
\]

(3)

This modification is commonly used in the calculation of sensitivity of haloscope experiments.

A more general version of the modifications as source terms can be obtained by substituting the following vector identities, \( \vec{B} \cdot \nabla a = \nabla \cdot (a \vec{B}) - a (\nabla \cdot \vec{B}) \) and \( \nabla a \times \vec{E} = \nabla \times (a \vec{E}) - a (\nabla \times \vec{E}) \) into [1]. Then, assuming to first order \( \nabla \cdot \vec{B} = 0 \) and \( \nabla \times \vec{E} = -\partial_t \vec{B} \), the modified Gauss’ and Ampere’s Laws, may be written as [31] [104].

\[
\epsilon_0 \nabla \cdot \vec{E} = \rho_e + \rho_{ab}
\]

\[
\frac{1}{\mu_0} \nabla \times \vec{B} - \epsilon_0 \partial_t \vec{E} = \vec{J}_e + \vec{J}_{ab} + \vec{J}_{ac},
\]

(4)

where

\[
\rho_{ab} = g_{a\gamma\gamma} \epsilon_0 c \nabla \cdot (a(t) \vec{B}(\vec{r}, t))
\]

\[
\vec{J}_{ab} = -g_{a\gamma\gamma} \epsilon_0 c \partial_t (a(t) \vec{B}(\vec{r}, t))
\]

\[
\vec{J}_{ac} = -g_{a\gamma\gamma} \epsilon_0 c \nabla \times (a(t) \vec{E}(\vec{r}, t))
\]

(5)

Here, \( \vec{J}_{ab} \) is similar to a polarisation current, and \( \rho_{ab} \) is similar to a bound charge, and are related through the continuity equation,

\[
\nabla \cdot \vec{J}_{ab} = -\partial_t \rho_{ab}.
\]

(6)

Furthermore, \( \vec{J}_{ac} \) is similar to a bound current, so the total axion current is thus \( \vec{J}_a = \vec{J}_{ab} + \vec{J}_{ac} \), which is a more
general form of Eqn. [3]. Note, that setting these terms to zero because $\nabla a = 0$, at the beginning of a calculation that analyses the sensitivity of an axion-photon coupled system can result in missing some parts of the solution [61, 104], as we show in the next section.

III. Axion Modified Electrodynamics

A. Time Dependent Form

Rather that write the equation of motion with modified source terms, as in [4] and [5], we may include the modifications in the definition of the fields themselves in a similar way to the auxiliary fields in matter [104]. With some rearrangement of these equations, we can show [31, 102, 106],

$$\nabla \cdot \left( \vec{E}(r, t) - g_\alpha a(t) c \vec{B}(r, t) \right) = \frac{\rho_e}{\epsilon_0}$$

$$\nabla \times \left( c \vec{B}(r, t) + g_\alpha a(t) \vec{E}(r, t) \right)$$

$$- \frac{1}{c} \partial_t \left( \vec{E}(r, t) - g_\alpha a(t) c \vec{B}(r, t) \right) = e \mu_0 J_e$$  (7)

$$\nabla \cdot c \vec{B}(r, t) = 0$$

$$\nabla \times \vec{E}(r, t) + \frac{1}{c} \partial_t c \vec{B}(r, t) = 0.$$

This has been shown to be equivalent to a perturbative transformation of the electromagnetic fields [104, 107, 108], given by,

$$c \vec{B}(r, t) \rightarrow c \vec{B}(r, t) + g_\alpha a(t) \vec{E}(r, t), \quad \text{and} \quad (8)$$

$$\vec{E}(r, t) \rightarrow \vec{E}(r, t) - g_\alpha a(t) c \vec{B}(r, t).$$  (9)

Where equations [8] and [9] in the quasi-static limit, effectively represent dual symmetry with respect to a rotation angle, $\theta(t) = g_\alpha a(t)$ where $\theta(t) < \alpha 1$ [109, 112]. Here $\theta(t)$ is an effective dynamical pseudo-scalar field, which in this case is the product of the axion pseudo-scalar field, $a(t)$, with the axion photon coupling, $g_\alpha a$. For dark matter axions, $a(t)$ is in general a large classical field, however $\theta(t)$ remains small due to the extremely weak coupling of axions to photons, i.e. $g_\alpha a < 1$. Note, there is also a duality transformation between electromagnetic potentials, where the dual 4-vector potential contains a magnetic scalar potential and an electric vector potential. Under this duality transform the electric vector potential manifests [108, 112], which potentially adds the axion induced “curl force” to the system under investigation. This is evident from eqn. [3], as the curl of $E_1^t$ has a non-zero spatial term.

Now considering the interaction includes two photons, we distinguish between a background field (denoted by subscript 0) and the generated photon field (denoted by subscript 1), which is created by the axion pseudo-scalar field mixing with the background field. To first order we may assume the background field satisfies Maxwell’s equations, so that,

$$\nabla \times \vec{B}_0 = \mu_0 e_0 \partial_t \vec{E}_0 + \mu_0 \vec{J}_e$$

$$\nabla \times \vec{E}_0 = -\partial_t \vec{B}_0$$

$$\nabla \cdot \vec{B}_0 = 0$$

$$\nabla \cdot \vec{E}_0 = e_0 \vec{J}_e$$  (10)

Note, any axion modification of the background field will end up second order with respect to the effects on the second generated photonic degree of freedom, so can be ignored [31, 106].

Thus for the generated photonic degree of freedom, we may write [4] in a similar way to how the auxiliary fields are included in Maxwell’s equations in matter, and we find axion modified electrodynamics in familiar form [104], given by,

$$\nabla \cdot \vec{D}_1 = \rho_e$$

$$\nabla \times \vec{H}_1 - \partial_t \vec{D}_1 = \vec{J}_1$$  (11)

$$\nabla \cdot \vec{B}_1(r, t) = 0$$

$$\nabla \times \vec{E}_1(r, t) + \partial_t \vec{B}_1(r, t) = 0,$$

where [8] and [9] are akin to the following constitutive relations,

$$\vec{H}_1(r, t) = \vec{B}_1 \mu_0 - \vec{M}_1 - \vec{M}_{1a}, \quad \text{and} \quad (12)$$

$$\vec{D}_1(r, t) = e_0 \vec{E}_1 + \vec{P}_1 + \vec{P}_{1a}.$$  (13)

Here, $\vec{M}_1$ and $\vec{P}_1$ are the non axion induced magnetization and polarization respectively, while the axion modifications, $\vec{M}_{1a}$ and $\vec{P}_{1a}$, are moved to redefinitions of the auxiliary fields rather than source terms, and to first order with respect to the background field are given by,

$$\vec{M}_{1a} = -g_\alpha a(t) \varepsilon_0 \vec{E}_0(r, t)$$

$$\frac{1}{\varepsilon_0} \vec{P}_{1a} = -g_\alpha a(t) c \vec{B}_0(r, t).$$  (14)

Here it is clear the the divergence of $\vec{M}_{1a}$ is non-zero, similar to what occurs at the boundaries of a permanent magnet, and the curl of $\vec{P}_{1a}$ is non zero similar to what occurs at the boundaries of a permanent electret, and by combining [13] and [10] can be calculated to be (assuming $\nabla a = 0$),

$$\nabla \cdot \vec{M}_{1a} = -g_\alpha a(t) \varepsilon_0 \nabla \cdot \vec{E}_0(r, t) = -g_\alpha a(t) c \rho_0$$

$$\frac{1}{\varepsilon_0} \nabla \times \vec{P}_{1a} = -g_\alpha a(t) c \nabla \times \vec{B}_0(r, t)$$

$$= -\frac{g_\alpha a(t)}{c} \partial_t \vec{E}_0 - g_\alpha a(t) c \rho_0 \vec{J}_e.$$  (14)

Note, if we followed the procedure to set $\nabla a = 0$ at the start of the calculation, then the axion current in [3] would be the only modification, and the general form
of the modified constitutive relations in (12) would be missed. This would be akin to falsely setting $\nabla \times \vec{P}_{a1} = 0$ and $\nabla \cdot \vec{M}_{a1} = 0$ even though they are in general non-zero in the approximation when $\nabla a$ is set to zero.

Assuming only a DC background magnetic field, $\vec{B}_0(\vec{r})$ with no background electric field ($E_0 = 0$ and $M_{a1} = 0$) and in vacuum ($\vec{M}_1 = 0$, $\vec{P}_1 = 0$ and $\vec{B}_1 = \mu_0 \vec{H}_1$), one can write the axion modified Ampere’s law from eqn. \[ (11) \] as,

$$\nabla \times \vec{B}_1 = \mu_0 \partial_t \vec{D}_1 + \mu_0 \vec{J}_e, \quad (15)$$

In contrast, Faraday’s law with respect to the $\vec{E}_1$ and $\vec{B}_1$ fields remains unchanged,

$$\nabla \times \vec{E}_1 = -\partial_t \vec{B}_1. \quad (16)$$

However, given the fact that the curl of $\vec{P}_{a1}$ in eqn. \[ (14) \] is non-zero, a modified Faraday’s law may be written in a similar fashion as to what is undertaken with an electret or impressed voltage source when the curl is non-zero \[96, 97, 102\], so by taking the curl of $\vec{D}_1$ in (12) and combing with (16) we obtain,

$$\frac{1}{\varepsilon_0} \nabla \times \vec{D}_1 = -\partial_t \vec{B}_1 - g_{a\gamma\gamma} \mu_0 c J_{e0}, \quad (17)$$

which is analogous to an electromagnetic systems in matter where the curl of the polarization is non-zero. It has been shown in such systems the fundamental electromagnetic quantities become the electric $\vec{D}$ and and magnetic $\vec{B}$ flux densities \[96, 97, 102\], which is compatible with the Minkowski Poynting vector.

In this work we apply these more general equations to low-mass axion haloscopes, which necessarily include the impressed current, $\vec{J}_{e0}$, which creates the background magnetic field $\vec{B}_0(\vec{r})$. Note, there also exist a dual symmetry with the source terms in the above equation \[ (15) \] and \[ (17) \], where an effective impressed magnetic current manifests through the axion interaction with the impressed electrical current, $\vec{J}_{e0}$, so $\vec{J}_{m1}(\vec{r}, t) = g_{a\gamma\gamma} a(t) \mu_0 c \vec{J}_{e0}(\vec{r}, t)$. The fact that this impressed magnetic current exists, does not necessitate the existence of free magnetic monopoles, in the same way bound currents and polarization currents do not need the existence of free electrons or any other free charge carrier. For example, bound magnetic monopoles exist in nature as permanent magnets consisting of bound north and south pole pairs, which can be set in motion, with a net bound magnetic current if one pole is kept stationary as the other rotates. Such a rotating magnet converts the mechanical motion to an electromotive force with non-zero curl (a curl force) \[102\]. This fact has been recognised as early as 1936 \[113\], where Schelkunoff from Bell Labs stated “It is true that there are no magnetic conductors and no magnetic conduction currents in the same sense as there are electric conductors and electric conduction currents but magnetic convection currents are just as real as electric convection currents, although the former exist only in doublets of oppositely directed currents since magnetic charges themselves are observable only in doublets.”

**B. Harmonic Phasor Form**

For harmonic solutions of the axion-Maxwell equations we write the equations in complex vector-phasor form. For example, we set $\vec{E}_1(\vec{r}, t) = \frac{1}{2} (E_1(\vec{r}) e^{-j\omega_1 t} + E_1^*(\vec{r}) e^{j\omega_1 t}) = \text{Re} (E_1(\vec{r}) e^{-j\omega_1 t})$, so we define the vector phasor (bold) and its complex conjugate by $\vec{E}_1(\vec{r}, t) = E_1(\vec{r}) e^{-j\omega_1 t}$ and $\vec{E}_1^*(\vec{r}, t) = E_1^*(\vec{r}) e^{j\omega_1 t}$, respectively. In contrast, the axion pseudo-scalar field, $a(t)$, may be written as, $a(t) = \frac{1}{2} (\tilde{a} e^{-j\omega_1 t} + \tilde{a}^* e^{j\omega_1 t}) = \text{Re} (\tilde{a} e^{-j\omega_1 t})$, and thus in phasor form, $\tilde{A} = \tilde{a} e^{-j\omega_1 t}$ and $\tilde{A}^* = \tilde{a}^* e^{j\omega_1 t}$. Thus, the phasor form of the modified Ampere’s law in \[ (15) \] becomes,

$$\frac{1}{\mu_0} \nabla \times \vec{B}_1 = \vec{J}_e - j\omega_1 \varepsilon_0 \vec{E}_1 + j\omega_1 g_{a\gamma\gamma} \varepsilon_0 c \tilde{A} \vec{B}_0 \quad (18)$$

while, the phasor form of the Faraday’s law in \[ (16) \] becomes,

$$\nabla \times \vec{E}_1 = j\omega_1 \times \vec{B}_1$$

and the phasor form of the modified Faraday’s law in \[ (17) \] becomes,

$$\frac{1}{\varepsilon_0} \nabla \times \vec{D}_1 = j\omega_1 \vec{B}_1 - g_{a\gamma\gamma} \mu_0 c \tilde{A} \vec{J}_{e0} \quad (20)$$

In the following we use these equations to calculate energy and power via Poynting theorem in a DC magnetic field axion haloscope.

**IV. Calculation of Power Generated in a DC Magnetic Field Axion Haloscope using Poynting Theorem**

We start by considering the instantaneous Poynting vector in its standard physics text book form of,

$$\vec{S}_1(t) = \frac{1}{\mu_0} \vec{E}_1(t) \times \vec{B}_1(t) =$$

$$\frac{1}{2} \left( E_1 e^{-j\omega_1 t} + E_1^* e^{j\omega_1 t} \right) \times \frac{1}{2\mu_0} \left( B_1 e^{-j\omega_1 t} + B_1^* e^{j\omega_1 t} \right)$$

$$= \frac{1}{2\mu_0} \text{Re} (E_1 \times B_1^*) + \frac{1}{2\mu_0} \text{Re} (E_1 \times B_1 e^{-j2\omega_1 t}) \quad (21)$$

which consists of a DC term, the first term on the right hand side of (21) and a high frequency term, the second term on the right hand side of (21). Note, the DC term in (21) is equivalent to the time average of the instantaneous Poynting vector.
Thus, the complex Poynting vector and its complex conjugate are defined by,

\[ S_1 = \frac{1}{2\mu_0} E_1 \times B_1^* \quad \text{and} \quad S_1^* = \frac{1}{2\mu_0} E_1^* \times B_1, \]  

(22)

respectively, where \( S_1 \) is the complex power density of the harmonic electromagnetic wave or oscillation, with the real part equal to the time averaged power density and the imaginary term equal to the reactive power, which may be inductive (magnetic energy dominates) or capacitive (electrical energy dominates).

In the case that the complex Poynting vector is only real, then the \( E_1 \) and \( B_1 \) fields are in-phase, this describes a propagating wave with distinct direction and momentum. Such a propagating wave can be generated by an antenna in the far-field limit, at distances larger than the wavelength of the emitted photon, and is a source of loss from the antenna. Another case where \( E_1 \) and \( B_1 \) are in phase is due to resistive losses, in this case the photon energy is converted to heat and destroyed, however both are effectively loss terms with respect to the antenna, the former known as radiation loss. Conversely, in the near field limit of an antenna, the Poynting vector is imaginary as \( E_1 \) and \( B_1 \) are out of phase. This represents reactive energy flow between the antenna power source and the antenna near field, which exists at sub wavelength distances from the antenna. In this case the photons do not propagate away from the antenna, and exist as quasi-static oscillating \( E_1 \) and \( B_1 \) fields with no net momentum.

A convenient and unambiguous way to calculate the real and imaginary part of the Poynting vector is through the following equations,

\[ \text{Re}(S_1) = \frac{1}{2}(S_1 + S_1^*) \quad \text{and} \quad j\text{Im}(S_1) = \frac{1}{2}(S_1 - S_1^*). \]  

(23)

We use these equations in the following to calculate the real and imaginary part of the complex Poynting vector in axion modified electrodynamics.

### A. Axion Modified Minkowski Poynting Theorem

Based on the axion modified \( \vec{D}_1 \) vector, we may calculate the complex axion modified Minkowski Poynting vector in a similar way to eqn. (22), which is given by,

\[ S_{DB} = \frac{1}{2\epsilon_0\mu_0} D_1 \times B_1^* \quad \text{and} \quad S_{DB}^* = \frac{1}{2\epsilon_0\mu_0} D_1^* \times B_1. \]  

(24)

Taking the divergence of eqn. (24) we find

\[ \nabla \cdot S_{DB} = \frac{1}{2} \nabla \cdot \left( \frac{1}{\epsilon_0} D_1 \times \frac{1}{\mu_0} B_1^* \right) = \frac{1}{2} \left( \frac{1}{\mu_0} B_1^* \cdot \frac{1}{\epsilon_0} \nabla \times D_1 - \frac{1}{\epsilon_0} D_1 \cdot \frac{1}{\mu_0} \nabla \times B_1^* \right), \]  

(25)

and

\[ \nabla \cdot S_{DB}^* = \frac{1}{2} \nabla \cdot \left( \frac{1}{\epsilon_0} D_1^* \times \frac{1}{\mu_0} B_1 \right) = \frac{1}{2} \left( \frac{1}{\mu_0} B_1 \cdot \frac{1}{\epsilon_0} \nabla \times D_1^* - \frac{1}{\epsilon_0} D_1^* \cdot \frac{1}{\mu_0} \nabla \times B_1 \right). \]  

(26)

Combining (25) and (26) with (22), (18) and (23) along with the divergence theorem (this is a standard technique in microwave engineering and circuit theory \[78, 97, 114, 115\]), after some calculation we obtain (see Appendix A for details),

\[ \int \text{Re}(S_{DB}) \cdot \hat{n} ds = \int \left( \frac{j(\omega - \omega_0)}{4} \epsilon_0 g_{e\gamma} c \vec{B}_0 \cdot (\hat{a} E_1^* - \hat{a}^* E_1) + \frac{1}{4} g_{e\gamma} c \vec{B}_0 \cdot (\hat{a} J_{e_1}^* + \hat{a}^* J_{e_1}) - \frac{1}{4} g_{e\gamma} \vec{J}_{e_0} \cdot (\hat{a}^* c B_1 + \hat{a} c B_1^*) - \frac{1}{4} (E_1 \cdot J_{e_1}^* + E_1^* \cdot J_{e_1}) \right) dV, \]  

\[ \int \text{Im}(S_{DB}) \cdot \hat{n} ds = \int \left( \frac{j \omega_0}{2} \frac{1}{\mu_0} B_1^* \cdot B_1 - \epsilon_0 E_1 \cdot E_1^* \right) + \frac{j (\omega + \omega_0) \epsilon_0 g_{e\gamma} c \vec{B}_0 \cdot (\hat{a} E_1^* + \hat{a}^* E_1) + \frac{1}{4} g_{e\gamma} c \vec{B}_0 \cdot (\hat{a} J_{e_1}^* - \hat{a}^* J_{e_1}) + \frac{1}{4} g_{e\gamma} \vec{J}_{e_0} \cdot (\hat{a}^* c B_1 - \hat{a} c B_1^*) - \frac{1}{4} (E_1 \cdot J_{e_1}^* - E_1^* \cdot J_{e_1}) \right) dV. \]  

(27)

The closed surface integral on the left hand side of (27) is the time averaged power radiated from inside to outside the haloscope volume, which for a closed system like a cavity will be zero. However, for an open system like a radio frequency antenna, real power will radiate in the far field. In contrast, the closed surface integral on the left hand side of (28) is the reactive power radiated outside the haloscope volume, which in general does not have to be zero, in a similar way to how reactive power oscillates to and from the voltage source charging and discharging a reactive capacitor in circuit theory, or the reactive power in an antenna, where energy oscillates from the antenna to the near field and then is reabsorbed by the antenna, due to the antenna’s self capacitance or inductance.

### B. Axion Modified Abraham Poynting Theorem

The complex Abraham pointing vector is basically the same as eqn. (22), for the case we are considering, with \( \vec{H}_1 = \frac{1}{\mu_0} \vec{B}_1 \). Taking the divergence of eqn. (22) we find,

\[ \nabla \cdot S_{EH} = \frac{1}{2} \nabla \cdot (E_1 \times H_1^*) = \frac{1}{2} H_1^* \cdot (\nabla \times E_1) - \frac{1}{2} E_1 \cdot (\nabla \times H_1^*) \]  

(29)
and 
\[ \nabla \cdot \mathbf{S}_{EH}^* = \frac{1}{2} \nabla \cdot (\mathbf{E}_1^* \times \mathbf{H}_1) = \\
\frac{1}{2} \mathbf{H}_1 \cdot (\nabla \times \mathbf{E}_1^*) - \frac{1}{2} \mathbf{E}_1^* \cdot (\nabla \times \mathbf{H}_1) \]  
(30)

Combining (29) and (30) with \[ 19 \], \[ 18 \] and \[ 23 \], along with the divergence theorem, we obtain (see Appendix A for details),
\[ \oint \text{Re} (\mathbf{S}_{EH}) \cdot \mathbf{n} ds = \int \left( \frac{j \omega a_0}{4} \mu_0 \mathbf{E}_1 \cdot \mathbf{H}_1 + \mu_0 \mathbf{E}_1 \cdot \mathbf{J}_{e_1} \right) dV \]
and
\[ \oint j \text{Im} (\mathbf{S}_{EH}) \cdot \mathbf{n} ds = \int \left( \frac{j \omega a_0}{4} \mu_0 \mathbf{H}_1 \cdot \mathbf{H}_1 + \epsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_1^* \right) \\
+ \frac{j \omega a_0}{4} \epsilon_0 g_{a a} c \mathbf{E}_1 \cdot (\mathbf{E}_1^* + \delta \mathbf{E}_1^*) \\
- \frac{1}{4} \left( \mathbf{E}_1 \cdot \mathbf{J}_{e_1} + \mathbf{E}_1^* \cdot \mathbf{J}_{e_1} \right) \right) dV \]  
(32)

Like before, the closed surface integral on the left hand side of (31) and (32) is the time averaged power and reactive power radiated outside the haloscope volume respectively. However the Abraham Poynting vector misses three extra terms the Minkowski Poynting vector picks up, due to the inclusion of the non-conservative and non-dissipative source term described by eqn. \[ 20 \].

C. Abraham or Minkowski Poynting Theorem in Axion Modified Electrodynamics?

Currently most calculations of haloscope detection sensitivity a priori assume the Abraham Poynting vector is valid, with the exception of one or two \[ 61 \] \[ 104 \]. However, as shown in the Minkowski-Abraham debate over the past century, this is not “clear-cut” and in a stationary dielectric media where the canonical momentum is under consideration, the Minkowski form agrees with experimental results \[ 87 \] \[ 90 \] \[ 92 \]. This maybe true in axion modified electrodynamics, as we can identify a similar guilty term due to eqn. \[ 14 \] and hence \[ 17 \]. The non zero curl, should do active work without adding dissipation and should not be ignored in calculations of axion detector sensitivity. This extra term has all the properties of a “curl Force” \[ 81 \] \[ 82 \] \[ 83 \], which adds spatial terms to the Poynting vector equations. In the following sections we compare the two ways of determining the power of photonic conversion for various axion haloscope topologies.

V. Resonant Cavity Haloscope

In this section we derive the sensitivity of an ADMX style radio frequency haloscope based on a cavity resonator \[ 14 \] \[ 22 \], with a schematic shown in Fig. [1]. First we undertake the calculation using the Abraham Poynting vector, as this is the a priori Poynting vector assumed across most of the literature, and then compare and contrast calculations using the Minkowski Poynting vector.

For a power source, \[ P_s \] delivering energy to a resonator as shown in Fig. [1] the resonance is defined when the reactive power delivered by the source is zero, and thus when tuned on resonance the circulating energy only oscillates between the electric and magnetic energy in the resonator at the cavity resonance frequency, with no energy oscillating between the cavity and the power source (which in this case is the axion mixing with a DC magnetic field). In this case the power delivered to the cavity is real. This corresponds to the real part of the Poynting theorem equations, which we use in the next section to calculate the sensitivity of the axion haloscope. Internal to the cavity resonator, this circulating energy is described by a reactive (or imaginary) Poynting vector, which causes the power in the resonator to build up, with respect to the source input power, \[ P_s \]. This build up is limited by the dissipation in the resonator and hence Q-factor. The build up of circulating power is given by \[ P_c = Q_1 P_s \], where in the steady state \[ P_c = P_d \], which is also related to the stored energy in the resonator, \[ U_1 \], by, \[ P_s = \omega_0 U_1 \].

Thus, in such a cavity resonator the electric and magnetic field are out of phase (as opposed to a propagating wave, which are in phase) and in this paper we represent the lossless electric field vector phasor as real, and the lossless magnetic field vector phasor as imaginary (and so the cross product is imaginary). Dissipative terms, whether calculated in the volume or on the surface assume Ohms law, dictating that the dissipative part of the electric field be in phase with surface or volume currents and hence the magnetic field. Thus, the electric field effectively gains an imaginary component when losses are included. However, the majority of the electric field is real, with \[ \text{Im}(\mathbf{E}_1) \sim -\text{Re}(\mathbf{E}_1)/Q_1 \]. The tangential real part of the electric field must be continuous at the cavity.
wall boundary, which for a perfect conductor is zero and sets the boundary conditions to calculate the frequency of the electromagnetic modes. Setting the reactive power to zero on resonance, allows us to calculate if there is any frequency shift of the bare cavity when excited by axions. We do find a small second order in Q-factor effect, calculated using Foster’s reactance theorem [116]. However, there is no major impact on the sensitivity calculation, for completeness this is detailed in Appendix B and predicts a different value of frequency shift depending if we use the Minkowski or Abraham Poynting theorem.

A. Cavity Dissipated Power

Both Poynting vectors have a dissipative term in the real components, listed as the final term on the right hand side of eqns. (27) and (31) and graphically shown in Fig. 4. For dissipation effects over the volume, the volume current is in phase with the the imaginary part of the electric field, and is of the form, \( J_v = σ_e E_1 \), where \( σ_e \) is the effective conductivity of the volume, which is related to the stored energy, \( \) electric field, and is of the form, \( \vec{K} = \vec{H} \times \vec{E} \), of dimensions Amps/metre, this means the closed surface integral on the left hand side of (31) should be set to zero (\( \oint \mathbf{S}_{DB} \cdot \hat{n} ds = 0 \)). In practice power taken is outside the cavity due to the coupling, which in effect loads the cavity Q-factor, this phenomena may be added after the calculation using standard techniques. We also assume that the magnetic and electric energy inside the resonator will be equal, again the effects of detuning may be added using standard techniques. Under these assumptions eqns. (31) becomes,

\[
P_s = \frac{g_{a\gamma}\gamma_\omega a c}{4} \int \vec{B}_0 \cdot (\hat{a}^* \epsilon_1 E_1 - \hat{a} \epsilon_1 E_1^* ) \, dV = P_d = \frac{\omega_a U_1}{Q_1}
\]  

(35)

Here, \( P_s \) is the axion source power and must be real, note the source power is equal to the dissipated power, and as calculated in the last section can occur over the volume and/or over the cavity surface.

For the source power to be non-zero either, \( E_1 \) or the axion scalar field, \( \hat{a} \), has to be imaginary. Since the axion scalar field is assumed to be lossless, we consider only the former to be imaginary, as has been suggested previously [106]. The general complex electric field is of the form, \( E_1 \approx (1 - j \tan δ) \text{Re}(E_1) \) in the regime where the loss angle is very small, \( δ \ll 1 \). Hence, the axion source term in the steady state becomes,

\[
P_s = \frac{g_{a\gamma}a_0 \omega_a c}{2Q_1} \int \vec{B}_0 \cdot \text{Re}(E_1(\vec{r}) ) \, dV, \tag{36}
\]

where \( a_0 = \frac{1}{2}(\hat{a} + \hat{a}^*) \) is the peak value of the scalar axion field, so \( a_0 = \sqrt{2} a_0 \). Equating (36) to \( P_d = \frac{\omega_a U_1}{Q_1} \) derived in (33) or (34) gives,

\[
U_1 = \frac{g_{a\gamma} a_0 \epsilon_0 c}{2} \int \vec{B}_0 \cdot \text{Re}(E_1) \, dV = \frac{\epsilon_0}{2} \int E_1 \cdot E_1^* \, dV, \tag{37}
\]

Now defining the form factor of the cavity haloscope as

\[
C_1 = \left( \frac{\int \vec{B}_0 \cdot \text{Re}(E_1) \, dV}{B_0^2 V_1 \int E_1 \cdot E_1^* \, dV} \right)^2, \tag{38}
\]

the axion induced circulating power may be calculated to be,

\[
P_1 = \omega_a QU_1 = g_{a\gamma}^2 a_0^2 \omega_a Q_1 \epsilon_0 c^2 B_0^2 V_1 C_1 = g_{a\gamma}^2 \rho_a Q_1 \epsilon_0 c^2 B_0^2 V_1 C_1 \frac{1}{\omega_a}, \tag{39}
\]

where \( (a_0)^2 = \frac{\rho_a}{c^2 m_a^2} \) and \( \rho_a \) is the axion dark matter density. This calculation is consistent with what has been derived previously [15, 31, 36].

C. Sensitivity from the Minkowski Poynting Vector

As before, assuming the real power inside the cavity haloscope is a closed system (\( \oint \text{Re}(S_{DB}) \cdot \vec{n} ds = 0 \)), the cavity is embedded inside a magnet \( (\mathcal{D}_{e_0} \cdot \hat{a} \mathbf{B}_1^* + \hat{a}^* \mathbf{B}_1) = 0 \) and that the axion and the resonator frequency coincide \( (\omega_1 = \omega_a) \). Then, in this case eqn. (27) becomes,

\[
P_s = \frac{1}{4} g_{a\gamma} c \vec{B}_0 \cdot (\hat{a}^* \epsilon_1^* + \hat{a} \epsilon_1 ) \, dV = P_d = \frac{\omega_a U_1}{Q_1}
\]  

(40)

Here, \( P_s \) is the axion source power and must be real. As undertaken in the Abraham Poynting technique, we assume a lossy volume current in phase with the electric field of
the form, \( J_{e_1} = \sigma_c E_1 \) where, \( \sigma_c = \frac{\epsilon_0}{2} \). Substituting the same value of \( J_{e_1} \) into (40) gives,

\[
P_a = \frac{g_{a\gamma\gamma} a_0 \omega_0}{2 \epsilon_0 c} \left( \begin{array}{c} B_0 \cdot \text{Re}(E_1(r)) \end{array} \right) dV,
\]

the same as calculated for the Abraham technique in eqn. (36), which means both eqns. (38) and (39) are calculable using both the Minkowski and Abraham Poynting vectors, and are consistent with previous sensitivity calculations for a standard ADMX style haloscope.

VI. Low-Mass Broadband Axion Haloscopes under DC Magnetic Field

For a low-mass broadband detector in the quasi-static limit, a haloscope may be inductive or capacitive and must be driven by reactive power from the source, so in the first approximation any dissipation or radiation loss can be ignored, and is thus set to zero. As before, we consider the generated electric field to be real (\( E_1 = E_1^* \)) and the out of phase magnetic field as imaginary, (\( B_1 = -B_1^* \)). Also, conduction currents will be in the same phase as the magnetic field and hence imaginary (\( J_2 = -J_2^* \)). In this case, it is clear that real part of the delivered complex Poynting vector given by (31) and (27) must be zero, and the sensitivity of the reactive low-mass broadband haloscope will be determined from the imaginary reactive power delivered by the axion interacting with the background DC magnetic field.

There has been some recent controversy in the calculation of sensitivity for low mass reactive experiments in the quasi static limit, where the majority of the publications suggest that the sensitivity to electric field is suppressed when the Compton wave length of the axion is larger than the experimental dimensions [40, 106, 117, 118]. These experiments assume that the only modification to Maxwell’s equations is due to the axion current (3), which is equivalent to assuming no boundary or spatial effects and thus setting the total derivative to zero. On the other hand, it has been shown that making these approximations too early in the calculation can lead to valid solutions being lost [61, 104, 119] due to extra spatial or surface terms, which in this case is due to the fact that \( \tilde{F}_{a1} = -g_{a\gamma\gamma} a(t) \epsilon_0 c \tilde{B}_0(\tilde{r}) \) has a non-zero curl. Based on this, more sensitive experiments have been proposed using inductive wire loop readouts [61], or capacitive parallel plate readouts [104, 119]. In the following, as an example, we compare the sensitivity of a parallel plate capacitor to low mass axions by implementing both Poynting vector theorems.

A. Capacitor under DC Magnetic Field

For a parallel plate capacitor as shown in Fig. 2, the last terms on the right hand side of eqns. (28) and (29) must be zero, since the conduction current must be zero in the lossless capacitor volume, this also means the third last term on the right hand side of (28) must be zero. Furthermore, if we assume the capacitor is embedded inside a DC magnet, the second last term in (28) must also be zero (it is possible to make use of this term to make a sensitive low-mass detector [61]). This means the equations for reactive power flowing into and out of the capacitor volume, using the Abraham and Minkowski forms, are given by,

\[
\oint \text{Im} (S_{EH}) \cdot \tilde{n} ds = j \omega_a \int \left( \frac{1}{2 \mu_0} B_1^* \cdot B_1 - \frac{\epsilon_0}{2} E_1^* \cdot E_1 \right) dV, \tag{42}
\]

and,

\[
\oint \text{Im} (S_{DB}) \cdot \tilde{n} ds = j \omega_a \int \left( \frac{1}{2 \mu_0} B_1^* \cdot B_1 - \frac{\epsilon_0}{2} E_1^* \cdot E_1 \right) dV, \tag{43}
\]

respectively.

For the capacitor in Fig. 2, the AC electric field phasor, ignoring fringe, is of the form:

\[
E_1 = \frac{\hat{q}_1}{\pi R_c^2} \hat{z}, \tag{44}
\]

where \( \hat{q}_1 \) is the complex phasor of electric charge on the capacitor plates. Following this, from Ampere’s law the AC magnetic field phasor within the capacitor volume, using the Abraham and Minkowski forms, are given by,

\[
B_1 = -j \omega_a \mu_0 \hat{q}_1 \frac{r}{\pi R_c^2} \hat{\theta}, \tag{45}
\]

Following this we may calculate the ratio of the magnetic energy density to electric energy density in the capacitor, given by,

\[
\frac{B_1 \cdot B_1^*}{\epsilon_0 \mu_0 E_1 \cdot E_1^*} = \frac{r^2 \omega_a^2}{4 c^2} = \frac{\pi^2 r^2}{\lambda_c^2}. \tag{46}
\]

where \( \lambda_c \) is the Compton wavelength of the axion. Integrating over the volume of the capacitor, allows us to calculate the ratio of magnetic to electric energy to be (ignoring fringe),

\[
\frac{\int_V B_1 \cdot B_1^* dV}{\epsilon_0 \mu_0 \int_V E_1 \cdot E_1^* dV} = \frac{R_c^2 \omega_a^2}{8 c^2} = \frac{\pi^2 R_c^2}{2 \lambda_c^2}. \tag{47}
\]
These equations highlight that at DC the parallel plate capacitor is purely capacitive, but at AC the capacitor has a small but finite inductance in the quasi static limit, when $\lambda_a > R_c$. When $\lambda_a \sim R_c$, the capacitor could become resonant, similar to a TM$_{010}$ mode in a cylindrical cavity, however this would not be in the quasi-static limit. Nevertheless, in a circuit where the direction of the electric field, $E_1$ in a capacitor is parallel to the applied DC magnetic field, $B_0$, eqn. (37) still holds for the capacitor, with an effective form factor of unity, which can be shown by substituting eqn. (44) into (38), and we can use this fact to help calculate the sensitivity of a low-mass capacitor experiment.

1. Sensitivity assuming the Abraham Poynting Vector

Assuming the Abraham Poynting vector, the reactive power delivered to and from a capacitor under DC magnetic field as shown in Fig. 2 can be calculated by substituting eqn. (37) into (42) and using (46) we find,

$$
p_a = \omega_a U_c, \text{ where } U_c = g_{a\gamma\gamma}(a_0)^2 c_0 c^2 B_0^2 V_1 \frac{\pi^2 R_c^2}{2\lambda_a^2}
$$

(49)

Thus the voltage phasor across the capacitor can be calculated from $U_c = \frac{1}{2} \mathcal{V}^* C_a$ $(C_a = \frac{\pi R_c c_0}{d_e})$ to be,

$$
\mathcal{V} = \sqrt{2} g_{a\gamma\gamma}(a_0) c B_0 d_e \left( \frac{\pi R_c}{\sqrt{2\lambda_a}} \right)^2
$$

(50)

which is consistent with an $rms$ voltage across the capacitor of

$$
\mathcal{V}_{rms} = g_{a\gamma\gamma}(a_0) c B_0 d_e \left( \frac{\pi R_c}{\sqrt{2\lambda_a}} \right)^2 = g_{a\gamma\gamma} d_e \frac{c}{\omega_a} B_0 \sqrt{\rho_a c^2} \mathcal{E} \left( \frac{\pi R_c}{\sqrt{2\lambda_a}} \right)^2,
$$

(51)

where $\langle a_0 \rangle = \frac{\sqrt{2}}{\omega} \frac{h}{m_a}$ and $\rho_a$ is the axion dark matter density. This calculation is consistent with other calculations based on just the axion current [106, 117, 118], as given by eqn. (3), however does not take into account the non-zero value of the curl of $\vec{P}_{a1}$. The calculation predicts suppressed sensitivity at low-mass, proportional to $R_c^2 / \lambda_a$.

2. Sensitivity assuming the Minkowski Poynting Vector

Assuming the Minkowski Poynting vector, the reactive power delivered to and from a capacitor under DC magnetic field as shown in Fig. 2 can be calculated by substituting eqn. (37) into (45). Note in this case the magnetic energy is insignificant so ignoring this component, gives,

$$
jP_a = \oint j \text{Im}(\mathbf{S}_{EH}) \cdot \hat{\mathbf{n}} ds = \frac{j\omega_a g_{a\gamma\gamma}(a_0) c^2}{2} \int \left( \dot{B}_0 \cdot \text{Re}(\mathbf{E}_1) \right) dV,
$$

(52)

Now, from the definition of the halo scope form factor (38), the energy stored in the capacitor (52) becomes,

$$
U_c = \frac{1}{2} g_{a\gamma\gamma}(a_0)^2 c_0 c^2 B_0^2 V_1 (\frac{\pi^2 R_c^2}{2\lambda_a^2})^2
$$

(53)

Thus the voltage phasor across the capacitor can be calculated from $U_c = \frac{1}{2} \mathcal{V}^* C_a$ to be,

$$
\mathcal{V} = \sqrt{2} g_{a\gamma\gamma}(a_0) c B_0 d_e,
$$

(54)

which is consistent with an $rms$ voltage of

$$
\mathcal{V}_{rms} = g_{a\gamma\gamma}(a_0) c B_0 d_e = g_{a\gamma\gamma} d_e \frac{c}{\omega_a} B_0 \sqrt{\rho_a c^3},
$$

(55)

which is the same as calculated previously [120]. Thus, we may conclude, from the Minkowski Poynting theorem, a sensitive low-mass experiment may be undertaken using a capacitive haloscope.

VII. Discussion and Conclusions

By applying Poynting theorem to axion modified electrodynamics, we have shown how the sensitivity of a resonant cavity and reactive broadband axion haloscope may be calculated. However, the way we apply the theorem is dependent on the type of detector. For example, Poynting vector analysis had already been undertaken to calculate the sensitivity of the MADMAX detector [12, 45]. However, MADMAX is in the regime where the Compton wavelength of the axion is much smaller than the detector size, and is thus in a different regime to the resonant and reactive halo scope discussed in this paper. The MADMAX detector converts energy at a dielectric boundary, and is assumed to be in the propagating wave (or far field) limit, where the $\mathbf{E}$ and $\mathbf{B}$ vector phasors are in phase, so the Poynting vector is real and represents the physical energy flux leaving a surface, and propagates through the halo scope [12, 44], and in principle can be made broadband.

In contrast, the resonant halo scope is generally the size of the Compton wavelength of the axion (unless higher order modes are implemented) and has an imaginary Poynting vector propagating internally within the resonator. This is because the axion induced photon energy produced within the resonator is reflected at the resonator boundaries, so the energy is localised in the
form of a standing wave. Thus, the propagating Poynting vector within the cavity is imaginary, with the electric and magnetic field within the cavity. In this work we have assumed the electric field is real, and thus the magnetic field is imaginary. However, on resonance (when $\omega_a = \omega_1$), the axion conversion process within the resonant cavity haloscope does not need to supply any reactive power, only real power. In this case the real part of the Poynting vector equation has both a source term and a dissipative term within the cavity, which are equal in the steady state, allowing the incident source power to escape the volume as heat, through the resistive losses. Meanwhile, reactive power flow oscillates between the electric and magnetic field within the cavity. The higher the Q-factor the more the circulating power builds up within the cavity, meaning the percentage of dissipation per cycle is smaller, and hence the detector sensitivity is proportional to the Q-factor. The down side is that the technique is narrow band, which requires complicated tuning mechanisms to scan for the axion of unknown mass.

On the other hand, low-mass experiments are in the quasi-static regime, where the Compton wavelength is much greater than the dimensions of the detector. In this case the sensitivity is determined by the reactive power flow within the detector created from the axion-photon conversion. For the higher frequency cavity ADMX style haloscopes the implementation of either the Minkowski or Abraham axion modified Poynting vector has no significant influence on the calculated sensitivity. In contrast, for low-mass reactive haloscopes there is a large difference in sensitivity calculated from the two Poynting theorems. The Minkowski Poynting theorem picks up the extra non-conservative terms in the equations, which can be classified as a “curl force”. For electrodynamics in matter, it seems that the Minkowski Poynting vector is the correct one to use, in the situation when the curl of the polarization is non-zero. This also should be true for axion modified electrodynamics under a DC background magnetic field, $B_0(\vec{r})$, because from eqn.(14), when $\nabla \alpha = 0$, $\nabla \times \vec{E}_1 = -g_{a\gamma\gamma}a(t)\epsilon_0\nabla \times B_0(\vec{r}) = -\frac{g_{\gamma\gamma}}{2}a(t)\vec{J}_0$, which adds a spatial “curl force” term to the axion modified electrodynamics equations, due to the impressed current in the coil of the DC magnet. In most prior calculations this term has been ignored.

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VIII. Appendix A: Derivation of Poynting Theorem Equations

In this appendix we derive equations (27), (28), (31) and (32) in the main text.

A. Axion Modified Minkowski Poynting Theorem

To derive eqn. (27) and (28), we begin with writing the divergence of the real and imaginary part of $S_{DB}$ as,

$$\nabla \cdot \text{Re}(S_{DB}) = \frac{1}{2} \nabla \cdot (S_{DB} + S_{DB}^*)$$

$$\nabla \cdot \text{Im}(S_{DB}) = \frac{1}{2} \nabla \cdot (S_{DB} - S_{DB}^*)$$

The next step is to calculate $\nabla \cdot S_{DB}$ and $\nabla \cdot S_{DB}^*$,

$$\nabla \cdot S_{DB} = \frac{1}{2} \nabla \cdot (\frac{1}{\epsilon_0} D_1 \times \frac{1}{\mu_0} B_1) = \frac{1}{2} \left( \frac{1}{\mu_0} B_1^* \cdot \frac{1}{\epsilon_0} \nabla \times D_1 - \frac{1}{\epsilon_0} D_1 \cdot \frac{1}{\mu_0} \nabla \times B_1^* \right)$$

and,

$$\nabla \cdot S_{DB}^* = \frac{1}{2} \nabla \cdot (\frac{1}{\epsilon_0} D_1^* \times \frac{1}{\mu_0} B_1) = \frac{1}{2} \left( \frac{1}{\mu_0} B_1 \cdot \frac{1}{\epsilon_0} \nabla \times D_1^* - \frac{1}{\epsilon_0} D_1^* \cdot \frac{1}{\mu_0} \nabla \times B_1 \right)$$

In harmonic form, the axion modified Ampere’s and Faraday’s law under a background DC field of $\vec{B}_0(\vec{r})$, created by an impressed electrical DC current in the magnet coil, $\vec{J}_o$, may be written as,

$$\frac{1}{\mu_0} \nabla \times B_1 = J_{1e} - j \omega_1 \epsilon_0 E_1 + j \omega_0 g_{a\gamma\gamma} \epsilon_0 c \vec{B}_0 \alpha$$

$$\frac{1}{\mu_0} \nabla \times B_1^* = J_{1e}^* + j \omega_1 \epsilon_0 E_1^* - j \omega_0 g_{a\gamma\gamma} \epsilon_0 c \vec{B}_0^*$$

$$\frac{1}{\epsilon_0} \nabla \times D_1 = j \omega_1 B_1 - g_{a\gamma\gamma} \epsilon_0 \mu_0 \vec{J}_o$$

$$\frac{1}{\epsilon_0} \nabla \times D_1^* = -j \omega_1 B_1^* - g_{a\gamma\gamma} \epsilon_0 \mu_0 \vec{J}_o^*.$$

Substituting eqns. (59) into eqns. (57) and (58) leads to

$$\nabla \cdot S_{DB} = \frac{1}{2\mu_0} B_1^* \cdot (j \omega_1 B_1 - g_{a\gamma\gamma} \epsilon_0 \mu_0 \vec{J}_o - \frac{1}{2} (E_1 - g_{a\gamma\gamma} \alpha \epsilon c \vec{B}_0^*) \cdot (J_{1e}^* + j \omega_1 \epsilon_0 E_1^* - j \omega_0 g_{a\gamma\gamma} \epsilon_0 \alpha^* c \vec{B}_0) - \frac{1}{2} (E_1 - g_{a\gamma\gamma} \alpha \epsilon c \vec{B}_0) \cdot J_{1e}^* - \frac{1}{2} E_1 \cdot J_{1e}^* + \frac{1}{2} g_{a\gamma\gamma} \alpha \epsilon \vec{B}_0 \cdot J_{1e}^* - \frac{1}{2} g_{a\gamma\gamma} \alpha \epsilon \vec{B}_0 \cdot \vec{J}_o) - \frac{1}{2} g_{a\gamma\gamma} \alpha \epsilon \vec{B}_0 \cdot \vec{J}_o^*.$$

\[(60)\]
∇·\mathbf{S}_{DB} = \frac{1}{2\mu_0} \mathbf{B}_1 \cdot (-j\omega_1 \mathbf{B}_1^* - g_{a\gamma\gamma} \tilde{a}^* e\mathbf{p}_0 \tilde{J}_e) - \frac{1}{2} (\mathbf{E}_1^* - g_{a\gamma\gamma} \tilde{a}^* e\mathbf{c}_B \tilde{B}_0) \cdot (\mathbf{J}_e - j\omega_1 e\mathbf{E}_1 + j\omega_a g_{a\gamma\gamma} e\mathbf{a}_c \tilde{B}_0) \\
= \frac{j\omega_1}{2} \left( e0\mathbf{E}_1 \cdot \mathbf{E}_1^* - \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{B}_1 \right) - \frac{j\omega_1}{2} e0g_{a\gamma\gamma} \tilde{a}^* e\mathbf{c}_B \tilde{B}_0 \cdot \mathbf{E}_1 \\
- \frac{1}{2} g_{a\gamma\gamma} e\mathbf{a}_c \tilde{B}_0 \cdot \mathbf{E}_1^* - \frac{1}{2} e\mathbf{J}_e \cdot \mathbf{J}_e + \frac{1}{2} g_{a\gamma\gamma} \tilde{a}^* e\mathbf{c}_B \tilde{B}_0 \cdot \mathbf{J}_e \\
- \frac{1}{2} g_{a\gamma\gamma} \tilde{a}^* e\mathbf{c}_B \tilde{B}_0 \cdot \tilde{J}_e \\
(61)

Now by substituting (60) and (66) into (66) we obtain,

∇·\mathbf{S}_{DB} = \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{E}_1 - e0\mathbf{E}_1 \cdot \mathbf{E}_1 \right) \\
+ \frac{1}{4} g_{a\gamma\gamma} e\mathbf{c}_B \tilde{B}_0 \cdot (\tilde{a}\mathbf{E}_1 + \tilde{a}^* \mathbf{E}_1) \\
+ \frac{1}{4} g_{a\gamma\gamma} e\mathbf{c}_B \tilde{B}_0 \cdot (\tilde{a}\mathbf{J}_e + \tilde{a}^* \mathbf{J}_e) - \frac{1}{4} g_{a\gamma\gamma} \tilde{J}_e \cdot (\tilde{a}^* e\mathbf{B}_1 + \tilde{a} e\mathbf{B}_1^*) \\
- \frac{1}{4} (\mathbf{E}_1 \cdot \mathbf{J}_e - \mathbf{E}_1^* \cdot \mathbf{J}_e) \\
(62)

Then applying the divergence theorem, we arrive at,

\int \mathbf{S}_{DB} \cdot \mathbf{n} \, ds = \int \left( \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{E}_1 - e0\mathbf{E}_1 \cdot \mathbf{E}_1 \right) \\
+ \frac{1}{4} g_{a\gamma\gamma} e\mathbf{c}_B \tilde{B}_0 \cdot (\tilde{a}\mathbf{E}_1 + \tilde{a}^* \mathbf{E}_1) \\
+ \frac{1}{4} g_{a\gamma\gamma} e\mathbf{c}_B \tilde{B}_0 \cdot (\tilde{a}\mathbf{J}_e + \tilde{a}^* \mathbf{J}_e) - \frac{1}{4} g_{a\gamma\gamma} \tilde{J}_e \cdot (\tilde{a}^* e\mathbf{B}_1 + \tilde{a} e\mathbf{B}_1^*) \\
- \frac{1}{4} (\mathbf{E}_1 \cdot \mathbf{J}_e - \mathbf{E}_1^* \cdot \mathbf{J}_e) \right) \, dV, \\
(64)

the same as eqn. (27) in the main text, and.

\int j\mathbf{S}_{DB} \cdot \mathbf{n} \, ds = \int \left( \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{E}_1 - e0\mathbf{E}_1 \cdot \mathbf{E}_1 \right) \\
+ \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{E}_1 - e0\mathbf{E}_1 \cdot \mathbf{E}_1 \right) \\
+ \frac{j\omega_a}{2} g_{a\gamma\gamma} e\mathbf{c}_B \tilde{B}_0 \cdot (\tilde{a}\mathbf{E}_1 + \tilde{a}^* \mathbf{E}_1) \\
+ \frac{j\omega_a}{2} g_{a\gamma\gamma} e\mathbf{c}_B \tilde{B}_0 \cdot (\tilde{a}\mathbf{J}_e + \tilde{a}^* \mathbf{J}_e) - \frac{1}{2} g_{a\gamma\gamma} \tilde{J}_e \cdot (\tilde{a}^* e\mathbf{B}_1 + \tilde{a} e\mathbf{B}_1^*) \\
- \frac{1}{4} (\mathbf{E}_1 \cdot \mathbf{J}_e - \mathbf{E}_1^* \cdot \mathbf{J}_e) \right) \, dV, \\
(65)

the same as eqn. (28) in the main text

B. Axion Modified Abraham Poynting Theorem

To derive eqn. (31) and (32), we begin with writing the divergence of the real and imaginary part of \mathbf{S}_{EH} as,

∇·Re(\mathbf{S}_{EH}) = \frac{1}{2} \nabla \cdot (\mathbf{E}_1 \times \frac{1}{\mu_0} \mathbf{B}_1) \\
∇·Im(\mathbf{S}_{EH}) = \frac{1}{2} \nabla \cdot (\mathbf{E}_1 \times \frac{1}{\mu_0} \mathbf{B}_1) \\
(66)

The next step is to calculate \nabla·\mathbf{S}_{EH} and \nabla·\mathbf{S}_{EH}^*,

∇·\mathbf{S}_{EH} = \frac{1}{2} \nabla \cdot (\mathbf{E}_1 \times \frac{1}{\mu_0} \mathbf{B}_1) = \frac{1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \frac{1}{\mu_0} \nabla \times \mathbf{B}_1 \right), \\
(67)

and,

∇·\mathbf{S}_{EH}^* = \frac{1}{2} \nabla \cdot (\mathbf{E}_1^* \times \frac{1}{\mu_0} \mathbf{B}_1) = \frac{1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \nabla \times \mathbf{E}_1^* - \mathbf{E}_1^* \cdot \frac{1}{\mu_0} \nabla \times \mathbf{B}_1 \right). \\
(68)

Considering the Abraham Poynting vector, under a background DC B-field of, \tilde{B}_0(\tilde{r}), created by an impressed electrical DC current in the magnet coil, \tilde{J}_e, in harmonic form, Ampere’s law is modified but Faraday’s law is not, and may be written as,

\frac{1}{\mu_0} \nabla \times \mathbf{B}_1 = \mathbf{J}_e + j\omega_1 e0\mathbf{E}_1 + j\omega_a g_{a\gamma\gamma} e0 c\mathbf{B}_0 \tilde{a} \\
\frac{1}{\mu_0} \nabla \times \mathbf{B}_1^* = \mathbf{J}_e^* + j\omega_1 e0\mathbf{E}_1^* - j\omega_a g_{a\gamma\gamma} e0 c\mathbf{B}_0 \tilde{a}^* \\
(69)

\mathbf{E}_1 = j\omega_1 \mathbf{B}_1 \\
\mathbf{E}_1^* = -j\omega_1 \mathbf{B}_1^*.

Substituting eqns. (69) into eqns. (67) and (68) leads to,

∇·\mathbf{S}_{EH} = \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{E}_1 - e0\mathbf{E}_1 \cdot \mathbf{E}_1 \right) \\
+ \frac{j\omega_a}{2} g_{a\gamma\gamma} e0 c\mathbf{B}_0 \cdot \mathbf{E}_1 - \frac{1}{2} \mathbf{E}_1 \cdot \mathbf{J}_e^* \\
(70)

and,

∇·\mathbf{S}_{EH}^* = \frac{j\omega_1}{2} \left( e0\mathbf{E}_1 \cdot \mathbf{E}_1^* - \frac{1}{\mu_0} \mathbf{B}_1 \cdot \mathbf{B}_1 \right) \\
- \frac{j\omega_a}{2} g_{a\gamma\gamma} e0 c\mathbf{B}_0 \cdot \mathbf{E}_1^* - \frac{1}{2} \mathbf{E}_1^* \cdot \mathbf{J}_e \\
(71)

Now by substituting (70) and (71) into (66) we obtain,

∇·Re(\mathbf{S}_{EH}) = \frac{j\omega_a}{4} e0 g_{a\gamma\gamma} c\mathbf{B}_0 \cdot (\tilde{a}^* \mathbf{E}_1 - \tilde{a} \mathbf{E}_1^*) \\
- \frac{1}{4} (\mathbf{E}_1 \cdot \mathbf{J}_e^* + \mathbf{E}_1^* \cdot \mathbf{J}_e), \\
(72)
and
\[
\nabla \cdot j \text{Im}(S_{EH}) = \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} B_1^* \cdot B_1 - \epsilon_0 E_1 \cdot E_1^* \right) \\
+ \frac{j\omega_a}{4} \epsilon_0 \gamma_\alpha c B_0 \cdot (\tilde{a}^* E_1 + \tilde{a} E_1^*) \\
- \frac{1}{4} (E_1 \cdot J_{e1}^* - E_1^* \cdot J_{e1})
\]
(73)

Then applying the divergence theorem, we arrive at,
\[
\int \text{Re}(S_{EH}) \cdot \hat{n} ds = \int \left( \frac{j\omega_a}{4} \epsilon_0 \gamma_\alpha c B_0 \cdot (\tilde{a}^* E_1 + \tilde{a} E_1^*) \right) \\
- \frac{1}{4} (E_1 \cdot J_{e1}^* + E_1^* \cdot J_{e1}) \right) dV
\]
and the same as eqn. (31) in the main text, and,
\[
\int j \text{Im}(S_{EH}) \cdot \hat{n} ds = \int \left( \frac{j\omega_1}{2} \left( \frac{1}{\mu_0} B_1^* \cdot B_1 - \epsilon_0 E_1 \cdot E_1^* \right) \\
+ \frac{j\omega_a}{4} \epsilon_0 \gamma_\alpha c B_0 \cdot (\tilde{a}^* E_1 + \tilde{a} E_1^*) \right) \\
- \frac{1}{4} (E_1 \cdot J_{e1}^* - E_1^* \cdot J_{e1}) \right) dV
\]
the same as eqn. (32) in the main text.

IX. Appendix B: Consideration of the Reactive Power Flow in a Resonant Haloscope

In this appendix we consider the impact of the reactive part of the Poynting vector on a resonant cavity axion haloscope. In general reactive coupling of power into a resonant cavity may be calculated by implementing Foster’s reactance theorem [115, 116, 121]. Foster showed that a lossless circuit network made of resonances and anti-resonances could be represented as a combination of inductor’s and capacitors [116] and following this Beringer and Dicke applied the theorem to high-Q microwave cavities [115, 121], allowing the calculation of the effect of reactive coupling to a cavity, based on the complex Poynting theorem [114]. In general, it has been shown that the reactive coupling network into a high-Q resonance may be either represented by an impedance, \( \chi_1 \) in series with a parallel LC circuit (of elements \( L_{p1}, C_{p1} \) and \( R_{p1} \)), or an admittance, \( B_1 \), in parallel with a series LC circuit (of elements \( L_{s1}, C_{s1} \) and \( R_{s1} \)). Applying this technique to axion-photon conversion in a resonant haloscope, leads to the following equivalent circuit shown in Fig. 3.

Applying Foster’s reactance theorem to the cavity resonator allows \( \chi_1 \) and \( B_1 \) to be calculated from the following equations [121],
\[
\chi_1 = j\omega_1 \frac{4(U_{B1} - U_{E1})}{\dot{H}^*} \quad \text{and} \quad B_1 = j\omega_1 \frac{4(U_{E1} - U_{B1})}{\dot{V}^*},
\]
(76)
where
\[
U_{B1} = \frac{1}{4\mu_0} \int \dot{B}_1^* \cdot \dot{B}_1 \ dV \quad \text{and} \quad U_{E1} = \frac{\epsilon_0}{4} \int \dot{E}_1 \cdot \dot{E}_1^* \ dV
\]
(77)

FIG. 3: Left, equivalent parallel LCR circuit representation of an axion coupling to a resonant cavity haloscope. Right, the equivalent series LCR circuit representation.

Following this procedure, the axion-photon coupling input impedances for both the axion modified Abraham Poynting Vector and the axion modified Minkowski Poynting Vector may be calculated, and is undertaken in the following sections.

1. Abraham Poynting Vector

To calculate the parameters for the parallel LCR circuit shown in Fig. 3, eqns. (76) and (32) are combined, and given that the real part of \( B_1 \) is out of phase with any conduction currents in the volume, and zero at the cavity boundary, then the series impedance becomes,
\[
\chi_1 = -j\omega_a \frac{\int \epsilon_0 \gamma_\alpha c B_0 \cdot (\tilde{a}^* + \tilde{a}) \text{Re}(E_1) \ dV}{\dot{H}^*},
\]
(78)

Then given that the energy in an LC resonator is given by
\[
\frac{1}{2} \dot{P}_1 = \frac{1}{2} \dot{H}^* L_{p1}
\]
and
\[
\text{and using the result from eqn. (37), eqn. (78) becomes,
\]
\[
\chi_1 = \omega_a \frac{\int \epsilon_0 \gamma_\alpha c B_0 \cdot \text{Re}(E_1) \ dV}{U_1} = \frac{\omega_a}{j\omega_1^2 C_{p1}}
\]
(79)

which is equivalent to a capacitance of \( C_a = C_{p1} \frac{\omega_a^2}{\omega_1^2} \). Thus, the input impedance, \( Z_p(\omega_a) \) of the parallel circuit representation can be written in normalized form as,
\[
Z_p(\omega_a) = \frac{\omega_a}{j\omega_1 Q_1} + \frac{1}{1 + j Q_1 \left( \frac{\omega_a}{\omega_1} - \frac{\omega_1}{\omega_a} \right)},
\]
(80)
where \( Q_1 = \frac{1}{\omega_1 L_{p1} C_{p1}} \). Defining the detuning as \( \delta \omega = \omega_a - \omega_1 \), when \( \delta \omega << \omega_1 \) and \( \delta \alpha = \frac{\delta \omega}{\omega_1} \), then
\[
Z_p(\delta \alpha) \approx \frac{1 - j2Q_1(\delta \alpha + \frac{1}{2Q_1})}{1 + 4Q_1^2 \delta \alpha^2},
\]
(81)

Setting the imaginary part to zero allows the calculation of the frequency shift of the resonant mode due to the axion coupling, which gives \( \frac{2\omega_1}{\omega_a} \sim -\frac{2\omega_1}{\omega_a} \), a very small frequency shift, which to first order does not affect the sensitivity of the axion haloscope and is the same order as a frequency shift due to dissipation. Ignoring this term gives the usual complex response of a resonant LCR circuit.
To calculate the parameters for the series LCR circuit shown in Fig. 3, we can use a similar procedure given in Eq. (70) and [32] to show that the parallel input admittance is given by,

\[ Y_p = \frac{\omega_a}{j\omega_1 Q_1} + \frac{1}{1 + j Q_1 \left( \frac{\omega_a}{\omega_1} - \frac{\omega_1}{\omega_a} \right)}, \]

where \( Q_1 = \frac{L_{s1}}{R_{s1}} \), so

\[ R_{s1} Y_s(\omega_a) \approx \frac{1 - j 2Q_1 (\delta_a + \frac{3}{2Q_1^2})}{1 + 4Q_1^2 \delta_a^2}, \]

which completes the dual representation of the resonant axion haloscope as either a parallel LCR in series with a capacitive coupling element or a series LCR circuit in parallel with an inductive coupling element.

### 2. Minkowski Poynting Vector

The reactive part of the Minkowski Poynting vector as written in Eqn. (28) has extra terms compared to the Abraham Poynting vector, and by following a similar process, the equivalent equation for the series impedance can be calculated to be,

\[ j \chi_{1M} = \frac{\omega_a + \omega_1 \frac{\mu_0 g_{\alpha\gamma\gamma} a c}{2} \int \vec{B}_0 \cdot \text{Re}(E_1) \, dV}{j\omega_1^2 C_{p1}} \cdot \frac{1}{U_1} + \frac{1}{4\omega_1^2 C_{p1}} g_{\alpha\gamma\gamma} \int \vec{B}_0 \cdot (\hat{a}^* J_{\alpha_1} - \hat{a} J_{\alpha_1}^*) \, dV \]

The second term is non-zero due to lossless inductive currents at the cavity surface, \( \kappa_{\alpha_1} \), which are in phase with the magnetic field, \( \vec{B}_1 \), and related by \( \kappa_{\alpha_1} = \frac{1}{\mu_0} \hat{n} \times \vec{B}_1 \), where \( \hat{n} \) is the normal to the cavity surface, and because the surface current and magnetic field are in imaginary phase, then \( \kappa_{\alpha_1}^* = -\kappa_{\alpha_1} \). Note \( J_{\alpha_1} = 0 \), over the volume, unless there is loss in the volume, which contributes to the real part of the Poynting vector not the reactive part. Next, by implementing the identity, \( \int ds \times \vec{B}_1 = \int \nabla \times \vec{B}_1 \, dV \), and from Eqn. (18), to first order we may substitute the following, \( \nabla \times \vec{B}_1 = -j\omega_1 \epsilon_0 \vec{E}_1 \) (ignoring terms second order in \( g_{\alpha\gamma\gamma} \)). Then it is straightforward to show (given \( ds = d\hat{n} \)),

\[ \vec{B}_0 \cdot \int (\hat{a}^* J_{\alpha_1} - \hat{a} J_{\alpha_1}^*) \, dV = 2a_0 \vec{B}_0 \cdot \phi_1 \, ds \]

for the resonant cavity haloscope. Therefore substituting Eqn. (86) into (85), the series impedance becomes,

\[ j \chi_{1M} = \frac{\omega_a + 2a_0}{j\omega_1^2 C_{p1}} \]

So the effective input capacitance represented in Fig. 3 becomes, \( C_a = C_{p1} \frac{\omega_1}{\omega_1^2 + \frac{3\omega_1^2}{2Q_1^2}} \), which is about a factor of three smaller than \( C_a \) for the Abraham equivalent circuit when \( \omega_2 \sim \omega_1 \). The normalised input impedance to first order in \( \delta_a \) can thus be written as,

\[ \frac{Z_p(\delta_a)}{R_{p1}} \approx \frac{1 - j 2Q_1 (\delta_a + \frac{3}{2Q_1^2})}{1 + 4Q_1^2 \delta_a^2}, \]

Setting the imaginary part to zero allows the calculation of the frequency shift of the resonant mode due to the axion coupling, which gives \( \frac{\Delta \omega_2}{\omega_2} \sim -\frac{3}{2Q_1^2} \) a very small frequency shift but a factor of 3 greater than what the Abraham Poynting vector predicts. A precision frequency measurement of the axion interacting with a microwave cavity haloscope would be needed to determine this frequency shift.

A similar calculation can be undertaken for the effective parallel inductance for the series LCR circuit representation, the end result is an inductance of \( L_a = L_{s1} \frac{\omega_1}{\omega_1^2 + \frac{3\omega_1^2}{2Q_1^2}} \) leading to similar conclusions and a normalised input admittance of,

\[ R_{s1} Y_s(\omega_a) \approx \frac{1 - j 2Q_1 (\delta_a + \frac{3}{2Q_1^2})}{1 + 4Q_1^2 \delta_a^2}, \]

which completes our analysis.