Quantum discords of tripartite quantum systems

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Received: 4 January 2022 / Accepted: 10 March 2022 / Published online: 5 April 2022
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Abstract
The quantum discord of bipartite systems is one of the best-known measures of non-classical correlations and an important quantum resource. In the recent work appeared in [Phys. Rev. Lett 2020, 124:110401], the quantum discord has been generalized to multipartite systems. In this paper, we give analytic solutions of the quantum discord for tripartite states with fourteen parameters.

Keywords Quantum discord · Quantum correlations · Tripartite quantum states · Optimization on manifolds

Mathematics Subject Classification Primary: 81P40 · Secondary: 81Qxx

1 Introduction
The quantum discord usually involves with quantum entanglement and non-entangled quantum correlations in quantum systems. It measures the total non-classical correlation in a quantum system and has attracted widespread attention since its appearance. Applications of the non-entanglement quantum correlations in quantum information processings have been extensively studied, including the quantum computing scheme of DQC1 [1] and Grover search algorithm [2]. This partly explains why quantum schemes surpass classical schemes. Meanwhile, the quantum discord as a non-classical correlation is one of the important quantum resources and is ubiquitous in many areas of...
modern physics ranging from condensed matter physics, quantum optics, high-energy physics to quantum chemistry and thus can be regarded as one of the fundamental non-classical correlations besides entanglement and EPR-steerable states [3, 4].

The quantum discord is defined as the maximal difference between the quantum mutual information without and with a von Neumann projective measurement applying to one part of the bipartite system. For tripartite and larger systems, some generalizations of the discord have been proposed [5–10] and have been used in quantum information processing. It is well known that quantum discord is extremely difficult to evaluate and most exact solutions are only found for the X-type quantum states (cf. [11–14]). This paper is devoted to quantification of the quantum correlation in tripartite and larger systems to derive some exact solutions for non-X-type states, and we hope it can contribute to better understanding and more effective use of quantum states in realizing quantum information processing schemes.

The paper is organized as follows. We first introduce the generalized discord for tripartite systems [10] based on that of bipartite systems [3]. We derive analytic solutions for tripartite states with fourteen parameters. Furthermore, the quantum discord of some well-known states (such as GHZ states) is computed.

2 Generalization of quantum discord to tripartite states

For a bipartite state $\phi^{bc}$ on system $H_B \otimes H_C$, the quantum mutual information is $I(\phi^{bc}) := S_B(\phi^b) + S_C(\phi^c) - S_{BC}(\phi^{bc})$, where $S(\phi^X) = \operatorname{Tr} \phi^X \log_2(\phi^X)$ is the von Neumann entropy of the quantum state on system $X$. Set $\{\Pi_k^B\}$ to be an one-dimensional von Neumann projection operator on subsystem $B$ which satisfies $\sum_k \Pi_k^B = I$, $\Pi_k^B \Pi_{k'}^B = \delta_{kk'}$. Then, the state $\phi^{bc}$ under the measurement $\{\Pi_k^B\}$ is changed into

$$\phi^c_k = \frac{1}{p_k} \operatorname{Tr}_B (I \otimes \Pi_k^B) \phi^{bc} (I \otimes \Pi_k^B)$$

with the probability $p_k = \operatorname{Tr}(I \otimes \Pi_k^B) \phi^{bc} (I \otimes \Pi_k^B)$. For simplicity, we denote by $\Pi^X$ the measurement $\{\Pi^X_k\}$ on system $X$. The quantum conditional entropy is simply given by $S_{C|\Pi^B}(\phi^{bc}) = \sum_k p_k S(\phi^c_k)$. Then, the measurement-induced quantum mutual information is given by

$$C(\phi^{bc}) = S_C(\phi^c) - \min_{\Pi^B} S_{C|\Pi^B}(\phi^{bc})$$

By Olliver and Zurek [3], the original definition of the quantum discord $Q(\rho)$ is the difference of the quantum mutual information $I(\phi^{bc})$ and the measurement-induced quantum mutual information $C(\phi^{bc})$, i.e.,

$$Q(\phi^{bc}) = I(\phi^{bc}) - C(\phi^{bc}) = \min_{\Pi^B} [S_{C|\Pi^B}(\phi^{bc}) - S_{C|\Pi^B}(\phi^{bc})],$$

(2.1)
where $S_{C|B}(\phi^{bc}) = S_{BC}(\phi^{bc}) - S_{B}(\phi^{b})$ is the unmeasured conditional state on subsystem $C$.

For the tripartite system $H_A \otimes H_B \otimes H_C$, we consider the $BC$ composite system as the first subsystem and $A$-system as the second subsystem. The state $\rho^{abc}$ of system $H_A \otimes H_B \otimes H_C$ gives arise to a state on $BC$-subsystem after the von Neumann measurement $\{\Pi^A_j\}$ on $A$ subsystem. Namely, it takes the following form:

$$\rho^{bc}_j = \frac{1}{p^{bc}_j} \text{Tr}_A(\Pi^A_j \otimes I)\rho^{abc}(\Pi^A_j \otimes I) \quad (2.2)$$

with probability $p^{bc}_j = \text{Tr}(\Pi^A_j \otimes I)\rho^{abc}(\Pi^A_j \otimes I)$. The measured quantum mutual information of $\rho^{abc}$ is naturally given by

$$\mathcal{J}(\rho^{abc}|\Pi^A) = S_{BC}(\rho^{bc}) - S_{BC|\Pi^A}(\rho^{abc}). \quad (2.3)$$

The quantity of classical correlation of the tripartite state $\rho^{abc}$ is

$$C(\rho^{abc}) = \max_{\Pi^A} \mathcal{J}(\rho^{abc}|\Pi^A) = S_{BC}(\rho^{bc}) - \min_{\Pi^A} S_{BC|\Pi^A}(\rho^{abc}). \quad (2.4)$$

We know that the quantum mutual information $I(\rho^{abc}) = S_A(\rho^a) + S_{BC}(\rho^{bc}) - S_{ABC}(\rho^{abc})$. Similar to Eq.(2.1), the generalized quantum discord of the tripartite state $\rho^{abc}$ can be defined as

$$Q(\rho^{abc}) = I(\rho^{abc}) - C(\rho^{abc}) = \min_{\Pi^A} \{S_{BC|\Pi^A}(\rho^{abc}) - S_{BC|A}(\rho^{abc})\}, \quad (2.5)$$

where $S_{BC|A}(\rho^{abc}) = S_{ABC}(\rho^{abc}) - S_A(\rho^a)$ is the unmeasured conditional entropy on $BC$-bipartite subsystem.

In order to evaluate the quantity $\min_{\Pi^A} S_{BC|\Pi^A}(\rho^{abc})$, the multipartite measurement based on conditional operators can be constructed as follows: [15]

$$\Pi^{AB}_{jk} = \Pi^A_j \otimes \Pi^B_{k|j} \quad (2.6)$$

with the measurement ordering from $A$ to $B$. The projector $\Pi^B_{k|j}$ on subsystem $B$ is conditional measurement outcome of $A$. These projectors satisfy $\sum_k \Pi^B_{k|j} = I^B$, $\sum_j \Pi^A_j = I^A$. Then, after the measurement $\Pi^{AB}_{jk}$, the state $\rho^{abc}$ is collapsed to a state on subsystem $C$, i.e.,

$$\rho^{c}_{jk} = \frac{1}{p^{c}_{jk}} \text{Tr}_{AB}(\Pi^{AB}_{jk} \otimes I)\rho^{abc}(\Pi^{AB}_{jk} \otimes I) \quad (2.7)$$
with the probability $p_{jk}^c = \text{Tr}(\Pi_{jk}^{AB} \otimes I) \rho_{abc}^{} (\Pi_{jk}^{AB} \otimes I)$. The conditional entropy after the $AB$-bipartite measurement is

$$S_{C|\Pi^{AB}}(\rho_{abc}^{}) = \sum_{jkl} p_{jk}^c \lambda_l^{(jk)} \log_2 \lambda_l^{(jk)},$$

where $\lambda_l^{(jk)}$ are eigenvalues of state $\rho_{jk}^c$.

Let $\rho_{X}^\Pi = \sum_{\Pi} X \rho \Pi^X$ be the state after measurement $\Pi^X$. Then, for a bipartite state $\rho_{ab}^{}$, the conditional entropy on subsystem $B$ after the measurement on subsystem $A$ is

$$S_{B|\Pi^A}(\rho_{ab}^{}) = \sum_j p_j S_B(\rho_j^b). \quad (2.8)$$

By [10, Eq.(6)], the entropy of the measured system can always be decomposed as

$$S_{AB}(\rho_{\Pi_A}^{}) = S_A(\rho_{\Pi_A}^{}) + S_{B|\Pi^A}(\rho_{ab}^{}). \quad (2.9)$$

For the tripartite system, using the measurement $\Pi^{AB}$, we have

$$S_{ABC}(\rho_{\Pi^{AB}}^{}) - S_{AB}(\rho_{\Pi^{AB}}^{}) = S_{C|\Pi^{AB}}(\rho_{abc}^{}); \quad (2.10)$$

when the measurement on $A$ system is $\Pi^A$, then we have

$$S_{ABC}(\rho_{\Pi^A}^{}) - S_A(\rho_{\Pi^A}^{}) = S_{BC|\Pi^A}(\rho_{abc}^{}). \quad (2.11)$$

By Eq.(2.9), Eq.(2.10), Eq.(2.11), we have that

$$S_{BC|\Pi^A}(\rho_{abc}^{}) = S_{B|\Pi^A}(\rho_{ab}^{}) + S_{C|\Pi^{AB}}(\rho_{abc}^{}).$$

Meanwhile, $S_A(\rho_{\Pi^{AB}}^{}) = S_A(\rho_{\Pi^A}^{})$, so the generalization discord of a tripartite state can be written as [10]

$$Q(\rho) := \min_{\Pi^{AB}} [-S_{BC|A}(\rho) + S_{B|\Pi^A}(\rho) + S_{C|\Pi^{AB}}(\rho)]. \quad (2.12)$$

3 Quantum discord of non-X qubit–qutrit state

For the product states in the tripartite system, the discord has the special property that it reduces to the standard bipartite discord when only bipartite quantum correlations are present. This means $Q_{ABC}(\rho^x \otimes \rho^y) = Q_X(\rho^x)$ for $X = AB, BC$ and $AC$.
subsystem. We consider the following tripartite states

\[ \rho^{abc} = \frac{1}{8} (I_8 + a_3\sigma_3 \otimes I_4 + I_2 \otimes b_3\sigma_3 \otimes I_2 + I_4 \otimes \sum_{i} c_i\sigma_i ) + \sum_{i} r_i\sigma_i \otimes\sigma_i \otimes I_2 + \sum_{i} s_i\sigma_i \otimes I_2 \otimes\sigma_i + \sum_{i} T_i\sigma_i \otimes\sigma_i \otimes\sigma_i ), \tag{3.1} \]

where \( I_d \) represents the unit matrix of order \( d \), and \( \sigma_i (i = 1, 2, 3) \) are Pauli matrices. The parameters \( a_3, b_3, c_i, r_i, s_i, T_i \in \mathbb{R} \), and they are confined within the interval \([-1, 1]\). Its matrix has the following form:

\[
\rho = \begin{pmatrix}
* & * & 0 & 0 & * & * & * & *
* & * & 0 & 0 & * & * & * & *
0 & 0 & * & * & * & 0 & * & *
0 & * & * & * & * & 0 & * & *
0 & * & * & * & * & 0 & * & *
* & 0 & * & * & * & 0 & * & *
* & * & 0 & * & * & * & 0 & *
* & * & 0 & 0 & 0 & 0 & * & *
\end{pmatrix} . \tag{3.2} \]

Let \( \{ |j\rangle \langle j|, j = 0, 1 \} \) be the computational base, then any von Neumann measurement on system \( X \) can be written as \( \{ \Pi^X_j = V |j\rangle \langle j| V^\dagger, j = 0, 1 \} \) for some unitary matrix \( V \in \text{SU}(2) \). Any unitary matrix can be written as \( V = tI + \sqrt{-1} \sum_k y_k\sigma_k \) with \( t, y_k \in \mathbb{R} \). When the measurement \( \{ \Pi^X_j \} \) is performed locally on one part of the composite system \( Y \otimes X \), the ensemble \( \{ \rho^Y_j, p^Y_j \} \) is given by \( \rho^Y_j = \frac{1}{p^Y_j} \text{Tr}_X (I \otimes \Pi^X_j ) \rho^{YX} (I \otimes \Pi^X_j ) \) with the probability \( p^Y_j = \text{Tr}[\rho^{YX} (I \otimes \Pi^X_j )] \).

It follows from symmetry that

\[
V^\dagger\sigma_1 V = (t^2 + y_1^2 - y_2^2 - y_3^2)\sigma_1 + 2(ty_3 + y_1y_2)\sigma_2 + 2(-ty_2 + y_1y_3)\sigma_3 ,
V^\dagger\sigma_2 V = (t^2 + y_2^2 - y_3^2 - y_1^2)\sigma_1 + 2(ty_1 + y_2y_3)\sigma_3 + 2(-ty_3 + y_1y_2)\sigma_1, \tag{3.3}
V^\dagger\sigma_3 V = (t^2 + y_3^2 - y_1^2 - y_2^2)\sigma_1 + 2(ty_2 + y_1y_3)\sigma_3 + 2(-ty_1 + y_2y_3)\sigma_2 .
\]

Introduce new variables \( z^X_1 = 2(-ty_2 + y_1y_3), z^X_2 = 2(ty_1 + y_2y_3), z^X_3 = (t^2 + y_3^2 - y_1^2 - y_2^2) \), then \( (z^X_1)^2 + (z^X_2)^2 + (z^X_3)^2 = 1 \). Therefore, \( \Pi^X_j \sigma_k \Pi^X_j = (-1)^j z^X_k \Pi^X_j \) for \( j = 0, 1 \) and \( k = 1, 2, 3 \).
For the tripartite state $\rho^{abc}$, the conditional state on $BC$ subsystem after measurement $\{\Pi_{A}^i (j = 0, 1)\}$ on subsystem $A$ is

$$
\rho_j^{bc} = \frac{1}{p_j^{bc}} (\Pi_{A}^j \otimes I_2 \otimes I_2) \rho^{abc} (\Pi_{A}^j \otimes I_2 \otimes I_2)
$$

$$
= \frac{1}{p_j^{bc}} [(1 + (-1)^j a_3 z_3) I_2 \otimes I_2 + b_3 \sigma_3 \otimes I_2 + (-1)^j \sum_i r_i z_i^A \sigma_i \otimes I_2
$$

$$
+ \sum_i (c_i + (-1)^j s_i z_i^A) I_2 \otimes \sigma_i + (-1)^j \sum_i T_i z_i^A \sigma_i \otimes \sigma_i],
$$

(3.4)

where the probabilities are

$$
p_j^{bc} = \text{Tr}((\Pi_{A}^j \otimes I_2 \otimes I_2) \rho^{abc} (\Pi_{A}^j \otimes I_2 \otimes I_2)) = \frac{1}{2} [1 + (-1)^j a_3 z_3].
$$

and $\sum_i^3 (z_i^A)^2 = 1$. Therefore, the reduced state of $\rho_j^{bc}$ is

$$
\rho_j^b = \text{Tr}_C \rho_j^{bc} = \frac{1}{2(1 + (-1)^j a_3 z_3)} [(1 + (-1)^j a_3 z_3) I_2 + b_3 \sigma_3 + (-1)^j \sum_i r_i z_i^A \sigma_i]
$$

with the probability $p_j^b = p_j^{bc} = \frac{1}{2} [1 + (-1)^j a_3 z_3]$. The eigenvalues of $\rho_j^b$ are

$$
\lambda_j^\pm = \frac{1}{2(1 + (-1)^j a_3 z_3)} [1 + (-1)^j a_3 z_3 \pm \sqrt{(b_3 + (-1)^j r_3 z_3)^2 + \sum_i^2 (r_i z_i^A)^2}].
$$

(3.6)

We define the following entropy function

$$
H_\varepsilon(x) = \frac{1}{2} [(1 + \varepsilon + x) \log_2 (1 + \varepsilon + x) + (1 + \varepsilon - x) \log_2 (1 + \varepsilon - x)].
$$

(3.5)

Then, measured conditional entropy of $B$ subsystem can be obtained as [3, 4, 14, 16–18]

$$
S_{B|\Pi_{A}^i}(\rho) = - \sum_j p_j^b (\lambda_j^+ \log_2 \lambda_j^+ + \lambda_j^- \log_2 \lambda_j^-)
$$

$$
= -\frac{1}{2} [H_{a_3 z_3^A}(A_+) + H_{-a_3 z_3^A}(A_-) - 2H(a_3 z_3^A) - 2],
$$

(3.6)

where $A_\pm = \sqrt{(b_3 \pm r_3 z_3^A)^2 + \sum_i^2 (r_i z_i^A)^2}$. 

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After measurement $\Pi^B_{k|j}$ on BC system, the state $\rho^c_{jk}$ is changed to

$$
\rho^c_{jk} = \frac{1}{p'_{jk}}\left[(1 + (-1)^j a_3 z^A_3 + (-1)^j b_3 z^B_3 + (-1)^j + k \sum_i r_i z^A_i z^B_i)I_2 + \sum_i (c_i + (-1)^j s_i z^A_i + (-1)^j + k T_i z^A_i z^B_i)\sigma_i\right], (j, k = 0, 1)
$$

(3.7)

with the probability ($k = 0, 1$)

$$
p^c_{0k} = \frac{1}{2(1 + a_3 z^A_3)}(1 + \alpha_k), \quad p^c_{1k} = \frac{1}{2(1 - a_3 z^A_3)}(1 + \beta_k),
$$

(3.8)

where $\alpha_k = a_3 z^A_3 + (-1)^k (b_3 z^B_3 + \sum_i r_i z^A_i z^B_i), \beta_k = -a_3 z^A_3 + (-1)^k (b_3 z^B_3 - \sum_i r_i z^A_i z^B_i)$. The nonzero eigenvalues of $\rho^c_{jk}$ are given by

$$
\lambda^{\pm}_{0k} = \frac{1}{2(1 + \alpha_k)}(1 + \alpha_k \pm \gamma_k), \quad \lambda^{\pm}_{1k} = \frac{1}{2(1 + \beta_k)}(1 + \beta_k \pm \delta_k), \quad k = 0, 1
$$

(3.9)

where

$$
\gamma_k = \left[\sum_i (c_i + s_i z^A_i + (-1)^k T_i z^A_i z^B_i)^2\right]^{1/2},
$$

$$
\delta_k = \left[\sum_i (-c_i + s_i z^A_i + (-1)^k T_i z^A_i z^B_i)^2\right]^{1/2}.
$$

According to the fact that the eigenvalues in Eq.(3.9) are nonnegative, we have

$$\sqrt{\sum_i a_i^2} + \sqrt{\sum_i b_i^2} + \sqrt{\sum_i r_i^2} \leq 1.
$$

The entropy of $\rho^{abc}$ under the measurement $\Pi^{AB}$ is given by

$$
S_{C|\Pi^{AB}}(\rho) = -\sum_{j,k} p^c_{jk}(\lambda^{+}_{jk} \log_2 \lambda^{+}_{jk} + \lambda^{-}_{jk} \log_2 \lambda^{-}_{jk})
$$

$$
= \frac{1}{2(1 + a_3 z^A_3)}[H_{\alpha_0}(\gamma_0) + H_{\alpha_1}(\gamma_1) - 2H_{a_3 z^A_3}(\frac{\alpha_0 - \alpha_1}{2})] + \frac{1}{2(1 - a_3 z^A_3)}[H_{\beta_0}(\delta_0) + H_{\beta_1}(\delta_1) - 2H_{-a_3 z^A_3}(\frac{\beta_0 - \beta_1}{2})] + 2.
$$

(3.10)

In particular, $a_3 z^A_3 = \frac{\alpha_0 + \alpha_1}{2}$.

Let $G(z^A_1, z^A_2, z^A_3) = 1 - S_{B|\Pi^A}(\rho)$ and $F(z^A_1, z^A_2, z^A_3, z^B_1, z^B_2, z^B_3) = 2 - S_{C|\Pi^{AB}}(\rho)$, then we have the following result.
Theorem 3.1 For the non-X-states $\rho$ in Eq.(3.1) with 14 parameters, the quantum discord is given by

$$Q(\rho) = -S_{ABC}(\rho) + S_A(\rho) + \min\{S_{B|\Pi^A}(\rho) + S_{C|\Pi^{AB}}(\rho)\}$$

$$= 3 + \sum_{i=1}^{8} \lambda_i \log_2 \lambda_i - \sum_{k=1}^{2} \lambda_k^a \log_2 \lambda_k^a - \max_{i \in [0,1], \sum_i (z_i^X)^2 = 1} \{G + F\},$$ (3.11)

where $\lambda_i (i = 1, \cdots, 8)$ are the eigenvalues of $\rho^{abc}$, $\lambda_k^a = \frac{1}{2}[1 + (-1)^k a_3]$, $(k = 0, 1)$ are eigenvalues of $\rho^{abc}$ on subsystem $A$, and $X$ represents subsystem $A$, $B$.

Theorem 3.2 Let $r = \max \{|r_1|, |r_2|\}$, then $\max_{z_i^X \in [0,1], \sum_i (z_i^X)^2 = 1} \{G + F\}$ can be explicitly computed as follows.

Case 1: When $a_3 b_3 r_3 \leq 0$, $r_3^2 - r^2 \geq a_3 b_3 r_3$, and $(b_3 + r_3)(c_3 + s_3) \leq 0$, we have

$$\max_{z_i^X \in [0,1], \sum_i (z_i^X)^2 = 1} \{G + F\} = G(0, 0, 1) + F(0, 0, 1, 0, 0, 1),$$ (3.12)

where

$$G(0, 0, 1) = \frac{1}{2} [H_{a_3}(b_3 + r_3)| + H_{-a_3}(b_3 - r_3) - 2H(a_3)]$$ (3.13)

and

$$F(0, 0, 1, 0, 0, 1) = \frac{1}{2(1 + a_3)} [H_{a_0}(\gamma_0) + H_{a_1}(\gamma_1) - 2H_{a_3}(b_3 + r_3)]$$

$$+ \frac{1}{2(1 - a_3)} [H_{\beta_0}(\delta_0) + H_{\beta_1}(\delta_1) - 2H_{-a_3}(b_3 - r_3)].$$ (3.14)

In this case, the parameters are degenerated into $(k = 0, 1)$

$$a_k = a_3 + (-1)^k(b_3 + r_3), \, \gamma_k = \left[\sum_i c_i^2 + s_i^2 + T_3^2 + 2(c_3 s_3 + (-1)^k c_3 T_3 + s_3 T_3)\right]^\frac{1}{2},$$

$$\beta_k = -a_3 + (-1)^k(b_3 - r_3), \, \delta_k = \left[\sum_i c_i^2 + s_i^2 + T_3^2 + 2(-c_3 s_3 + (-1)^k s_3 T_3 - c_3 T_3)\right]^\frac{1}{2}.$$

Case 2: (1) When $b_3 = 0, c_1 s_1 \leq 0, s_1 \leq |c_1|$ and $\max \{|r_1|, |r_2|, |r_3|\} = |r_1|$, we have

$$\max_{z_i^X \in [0,1], \sum_i (z_i^X)^2 = 1} \{G + F\} = G(1, 0, 0) + F(1, 0, 1, 0, 0, 0),$$ (3.15)

where

$$G(1, 0, 0) = \frac{1}{2} [H_{a_3}(r_1) + H_{-a_3}(r_1) - 2H(a_3)]$$ (3.16)
and

\[ F(1, 0, 0, 1, 0, 0) = \frac{1}{2} [H_{r_1}(\gamma_0) + H_{r_1}(\gamma_1) + H_{r_1}(\delta_0) + H_{r_1}(\delta_1) - 4H(r_1)]. \]

(3.17)

In this case, the parameters are degenerated into \((k = 0, 1)\)

\[ \gamma_k = \left[ \sum_i c_i^2 + s_i^2 + T_i^2 + 2(c_1s_1 + (-1)^k(c_1T_1 + s_1T_1)) \right]^\frac{1}{2}, \]

\[ \delta_k = \left[ \sum_i c_i^2 + s_i^2 + T_i^2 + 2(-c_1s_1 + (-1)^k(c_1T_1 - s_1T_1)) \right]^\frac{1}{2}. \]

(2) When \(b_3 = 0, c_1s_1 \leq 0, s_1 \leq |c_1| \) and \(\max(|r_1|, |r_2|, |r_3|) = |r_2|\), we have

\[ \max_{z^k \in [0, 1], \sum_i (z^k_i)^2 = 1} \{G + F\} = G(0, 1, 0) + F(0, 1, 0, 0, 1, 0), \]

where

\[ G(0, 1, 0) = \frac{1}{2} [H_{a_3}(r_2) + H_{-a_3}(r_2) - 2H(a_3)]. \]

(3.18)

and

\[ F(0, 1, 0, 0, 1, 0) = \frac{1}{2} [H_{r_2}(\gamma_0) + H_{r_2}(\gamma_1) + H_{r_2}(\delta_0) + H_{r_2}(\delta_1) - 4H(r_2)]. \]

(3.19)

(3.20)

In this case, the parameters are degenerated into \((k = 0, 1)\)

\[ \gamma_k = \left[ \sum_i c_i^2 + s_i^2 + T_i^2 + 2(c_2s_2 + (-1)^kc_2T_2 + s_2T_2) \right]^\frac{1}{2}; \]

\[ \delta_k = \left[ \sum_i c_i^2 + s_i^2 + T_i^2 + 2(-c_2s_2 + (-1)^kc_2T_2 - s_2T_2) \right]^\frac{1}{2}. \]

Proof By definition, we have

\[ G + F = H_{a_3z^A_3}(B_+) + H_{-a_3z^A_3}(B_-) - 2H(a_3z^A_3) \]

\[ + \frac{1}{2(1 + a_3z^A_3)} [H_{a_0}(\gamma_0) + H_{a_1}(\gamma_1) - 2H_{a_3z^A_3}(\frac{\alpha_0 - \alpha_1}{2})] \]

\[ + \frac{1}{2(1 - a_3z^A_3)} [H_{\beta_0}(\delta_0) + H_{\beta_1}(\delta_1) - 2H_{-a_3z^A_3}(\frac{\beta_0 - \beta_1}{2})]. \]

(3.21)
Note that $F$ is a function of six variables and the first three are exactly the variables of $G$. Our strategy of locating the extremal points of $G + F$ is first finding the critical points $z_A^1, z_A^2, z_A^3$ of $G$ and verifying that at those points the critical points of $G$ are attainable, and then, we can find the maximal points of $G + F$.

For case 1: $a_3 b_3 r_3 \leq 0$ and $r_3^2 - r^2 \geq a_3 b_3 r_3$, by [14] we know that max $G(z_A^1, z_A^2, z_A^3) = G(0, 0, 1)$, and then, the parameters in function $F$ are degenerated into ($k = 0, 1$)

\[
\begin{align*}
\alpha_k &= a_3 + (-1)^k (b_3 z_3^B + r_3 z_3^B), \\
\beta_k &= -a_3 + (-1)^k (b_3 z_3^B - r_3 z_3^B), \\
\gamma_k &= \left\{ \sum_{i=0}^{3} c_i^2 + s_3 + T_3^2 (z_3^B)^2 + 2(c_3 s_3 + (-1)^k (s_3 T_3^2 z_3^B + c_3 T_3 z_3^B)) \right\}^{1/2}, \\
\delta_k &= \left\{ \sum_{i=0}^{3} c_i^2 + s_3 + T_3^2 (z_3^B)^2 + 2(-c_3 s_3 + (-1)^k (s_3 T_3 z_3^B - c_3 T_3 z_3^B)) \right\}^{1/2}.
\end{align*}
\]

Therefore, we have

\[
F(z_A^1, z_A^2, z_A^3, z_B^1, z_B^2, z_B^3) = F(0, 0, 1, z_3^B)
= \frac{1}{2(1 + a_3)}[H_{a_0}(\gamma_0) + H_{a_1}(\gamma_1)] + \frac{1}{2(1 - a_3)}[H_{a_0}(\gamma_0) + H_{a_1}(\gamma_1)].
\]

(3.22)

When $(b_3 + r_3)(c_3 + s_3) \leq 0$, it can be observed that $F$ is an even function for $z_B^3 \in [-1, 1]$, so we just need to consider $z_B^3 \in [0, 1]$. The derivative of $F$ on $z_3^B$ is given by

\[
\begin{align*}
\frac{\partial F}{\partial z_3^B} &= \frac{1}{4(1 + a_3)} \left\{ (b_3 + r_3) \log_2 (1 + \alpha_1)^2 [(1 + \alpha_0)^2 - \gamma_0^2] \right\} \\
&\quad + \frac{-c_3 T_3 - s_3 T_3 + T_3^2 z_3^B}{\gamma_1} \log_2 \frac{1 + \alpha_1 + \gamma_1}{1 + \alpha_1 - \gamma_1} + \frac{c_3 T_3 + s_3 T_3 + T_3^2 z_3^B}{\gamma_0} \log_2 \frac{1 + \alpha_0 + \gamma_0}{1 + \alpha_0 - \gamma_0} + \frac{1}{4(1 - a_3)} \left\{ (b_3 - r_3) \log_2 (1 + \beta_1)^2 [(1 + \beta_0)^2 - \delta_0^2] \right\} \\
&\quad + \frac{-c_3 T_3 - s_3 T_3 + T_3^2 z_3^B}{\delta_1} \log_2 \frac{1 + \beta_1 + \delta_1}{1 + \beta_1 - \delta_1} + \frac{c_3 T_3 + s_3 T_3 + T_3^2 z_3^B}{\delta_0} \log_2 \frac{1 + \beta_0 + \delta_0}{1 + \beta_0 - \delta_0}.
\end{align*}
\]

(3.23)
If $b_3 + r_3 \leq 0$ and $c_3 + s_3 \geq 0$, we have $\gamma_0 \geq \gamma_1, \alpha_1 \geq \alpha_0, \delta_0 \geq \delta_1$ and $\beta_1 \geq \beta_0$, then

$$
(b_3 + r_3) \log_2 \left( \frac{1 + \alpha_1^2}{(1 + \alpha_0^2)^2 - \gamma_0^2} \right) \geq 0; \quad (b_3 - r_3) \log_2 \left( \frac{1 + \beta_1}{(1 + \beta_0)^2 - \delta_1^2} \right) \geq 0;
$$

(3.24)

$$
\frac{-c_3 T_3 - s_3 T_3 + T_3^2 z_3^B}{\gamma_1} \log_2 \left( \frac{1 + \alpha_1 + \gamma_1}{1 + \alpha_1 - \gamma_1} \right) + \frac{c_3 T_3 + s_3 T_3 + T_3^2 z_3^B}{\gamma_0} \log_2 \left( \frac{1 + \alpha_0 + \gamma_0}{1 + \alpha_0 - \gamma_0} \right) 
$$

$$
\geq \frac{-c_3 T_3 - s_3 T_3 + T_3^2 z_3^B}{\gamma_0} \log_2 \left( \frac{1 + \alpha_1 + \gamma_1}{1 + \alpha_1 - \gamma_1} \right) + \frac{c_3 T_3 + s_3 T_3 + T_3^2 z_3^B}{\gamma_0} \log_2 \left( \frac{1 + \alpha_0 + \gamma_0}{1 + \alpha_0 - \gamma_0} \right)
$$

$$
= \frac{2T_3^2 z_3^B}{\gamma_0} \log_2 \left( \frac{1 + \alpha_1 + \gamma_1}{1 + \alpha_1 - \gamma_1} \right) \geq 0;
$$

(3.25)

$$
\frac{c_3 T_3 - s_3 T_3 + T_3^2 z_3^B}{\delta_1} \log_2 \left( \frac{1 + \beta_1 + \delta_1}{1 + \beta_1 - \delta_1} \right) + \frac{-c_3 T_3 + s_3 T_3 + T_3^2 z_3^B}{\delta_0} \log_2 \left( \frac{1 + \beta_0 + \delta_0}{1 + \beta_0 - \delta_0} \right)
$$

$$
\geq \frac{c_3 T_3 - s_3 T_3 + T_3^2 z_3^B}{\delta_0} \log_2 \left( \frac{1 + \beta_1 + \delta_1}{1 + \beta_1 - \delta_1} \right) + \frac{-c_3 T_3 + s_3 T_3 + T_3^2 z_3^B}{\delta_0} \log_2 \left( \frac{1 + \beta_0 + \delta_0}{1 + \beta_0 - \delta_0} \right)
$$

$$
= \frac{2T_3^2 z_3^B}{\delta_0} \log_2 \left( \frac{1 + \beta_1 + \delta_1}{1 + \beta_1 - \delta_1} \right) \geq 0.
$$

(3.26)

Hence, in this case we get $\frac{\partial F}{\partial z_3^B} \geq 0$ when $z_3^B \in [0, 1]$.

If $b_3 + r_3 \geq 0$ and $c_3 + s_3 \leq 0$, we also can show that $\frac{\partial F}{\partial z_3^B} \geq 0$ similarly. So $F$ is a strictly monotonically increasing function with $z_3^B \in [0, 1]$. Similarly, we can check that $F$ is a strictly monotonically increasing function with respect to $z_1^B \in [0, 1]$ or $z_2^B \in [0, 1]$ in case 2.

\section*{Theorem 3.3}

For the Werner-GHZ state $\rho_w = c|\psi\rangle\langle\psi| + (1-c)\frac{I}{8}$, where $|\psi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$, the quantum discord is

$$
Q = \frac{1}{8}(1-c)\log_2(1-c) + \frac{1+7c}{8}\log_2(1+7c) - \frac{1}{4}(1+3c)\log_2(1+3c).
$$

(3.27)

\section*{Proof}

Obviously, $\max \{G(z_1^A, z_2^A, z_3^A)\} = H(c)$. Let $\theta = cz_3^A z_3^B$, then

$$
F(z_1^A, z_2^A, z_3^A, z_1^B, z_2^B, z_3^B) = F(\theta)
$$

$$
= \frac{1}{2}[H_\theta(|c+\theta|) + H_{-\theta}(|c-\theta|) - 2H(\theta)].
$$

(3.28)

It is easy to see that $F(\theta)$ is monotonically increasing with respect to $\theta \in [0, 1]$. So $\max \{F(\theta)\} = F(\max \{|\theta|\}) = F(c)$. Figure 1 shows the behavior of the function $Q$. \qed
Next, we consider the following general tripartite state

$$\rho = \frac{1}{8}(I_8 + \sum_1^3 a_i \sigma_i \otimes I_4 + I_2 \otimes \sum_1^3 b_i \sigma_i \otimes I_2 + I_4 \otimes \sum_1^3 c_i \sigma_i + \sum_1^3 r_i \sigma_i \otimes I_2 + \sum_1^3 s_i \sigma_i \otimes I_2 \otimes \sigma_i + \sum_1^3 v_i I_2 \otimes \sigma_i \otimes \sigma_i + \sum_1^3 T_i \sigma_i \otimes \sigma_i \otimes \sigma_i).$$

(3.29)

Let $a = \sqrt{\sum_1^3 a_i^2}$ and $b = \sqrt{\sum_1^3 b_i^2}$, then we can get the quantum discord for some special cases.

**Theorem 3.4** For the general tripartite state $\rho$ in Eq.(3.29), we have the following results:

Case 1: When $a_i = v_i = T_i = 0$, $r_1 = r_2 = r_3 = r$, we have that

$$Q(\rho) = \sum_1^8 \lambda_i \log_2 \lambda_i + 4 + H_b(r) + H_{-b}(r) - H(\frac{|b + r|}{2})$$

(3.30)

$$- \frac{1}{2}[H_{b+B}(r) + H_{b-B}(r) + H_{-b+B}(r) + H_{-b-B}(r)],$$

where $B = [\sum_1^3 (s_i \frac{b_i}{b} + c_i)^2]^{\frac{1}{2}}$. 

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Case 2: When \( b_i = v_i = T_i = 0, r_1 = r_2 = r_3 = r \), we have that

\[
Q(\rho) = \sum_{i}^{8} \lambda_i \log_2 \lambda_i + 3 - H(a^2) - \frac{1}{2}[H_a(r) + H_{-a}(r) - 2H(a)]
\]

\[ - \frac{1}{2(1 + a)}[H_{a+A}(r) + H_{a-A}(r) - 2H_a(r)] \]

\[ - \frac{1}{2(1 - a)}[H_{-a+A}(r) + H_{-a-A}(r) - 2H_{-a}(r)], \]

where \( A = \left[ \sum_{i}^{3} (s_i \frac{a_i}{a} + c_i)^2 \right]^\frac{1}{2} \).

Case 3: When \( r_i = T_i = v_i = 0 \), we have that

\[
Q(\rho) = \sum_{i}^{8} \lambda_i \log_2 \lambda_i + 3 - H(a^2) - \frac{1}{2}[H_b(a) + H_{-b}(a) - 2H(a)]
\]

\[ - \frac{1}{2(1 + a)}[H_{a+A}(b) + H_{a-A}(b) - 2H_a(b)] \]

\[ - \frac{1}{2(1 - a)}[H_{b+A}(b) + H_{b-A}(b) - 2H_{b}(b)], \]

where \( A = \left[ \sum_{i}^{3} (s_i \frac{a_i}{a} + c_i)^2 \right]^\frac{1}{2} \).

Case 4: When \( a_i = c_i = s_i = T_i = 0, r_1 = r_2 = r_3 = r, v_1 = v_2 = v_3 = v \), we have that

\[
Q(\rho) = \sum_{i}^{8} \lambda_i \log_2 \lambda_i + H_b(r) + H_{-b}(r) + 4 - H(|b + r|)
\]

\[ - \frac{1}{2}[H_{b+v}(r) + H_{b-v}(r) + H_{b+v}(r)H_{b-v}(r)]. \]

Case 5: When \( r_i = T_i = s_i = c_i = 0, v_1 = v_2 = v_3 = v \), we have that

\[
Q(\rho) = \sum_{i}^{8} \lambda_i \log_2 \lambda_i + 3 - H(a^2) - \frac{1}{2}[H_b(a) + H_{-b}(a) - 2H(a)]
\]

\[ - \frac{1}{2(1 + a)}[H_{a+v}(b) + H_{a-v}(b) - 2H_a(b)] \]

\[ - \frac{1}{2(1 - a)}[H_{-a+v}(b) + H_{-a-v}(b) - 2H_{-a}(b)]. \]
Case 6: When \( b_i = s_i = c_i = T_i = 0, r_1 = r_2 = r_3 = r, v_1 = v_2 = v_3 = v \), we have that

\[
Q(\rho) = \sum_{i}^{8} \lambda_i \log_2 \lambda_i + 3 - H(a^2) - \frac{1}{2}[H_a(r) + H_{-a}(r) - 2H(a)]
\]

\[-\frac{1}{2(1 + a)}[H_{a+v}(r) + H_{a-v}(r) - 2H_a(r)]
\]

\[-\frac{1}{2(1 - a)}[H_{-a+v}(r) + H_{-a-v}(r) - 2H_{-a}(r)].
\]

(3.35)

**Proof** All cases can be shown similarly. Let’s consider case 1: \( a_i = v_i = T_i = 0, r_1 = r_2 = r_3 = r, \) max \( \{G(z_1^A, z_2^A, z_3^A)\} = G(b_1 \frac{b}{b}, b_2 \frac{b}{b}, b_3 \frac{b}{b}) \). Let \( \theta = \sum_{i}^{3} r_i z_i^B \frac{b_i}{b} \), then

\[
F(z_1^A, z_2^A, z_3^A, z_1^B, z_2^B, z_3^B) = F(b_1 \frac{b}{b}, b_2 \frac{b}{b}, b_3 \frac{b}{b}, \theta)
\]

\[= \frac{1}{2}[H_{b+B}(\theta) + H_{b-B}(\theta) + H_{-b+B}(\theta) + H_{-b-B}(\theta)] - H_b(\theta) - H_{-b}(\theta) - 2,
\]

(3.36)

where \( B = \left[ \sum_{i}^{3} (s_i^{B} \frac{b}{b} + c_i)^{2} \right]^{\frac{1}{2}} \).

The derivative of \( F \) over \( \theta \) is equal to

\[
\frac{\partial F}{\partial \theta} = \frac{1}{4} \left\{ \log_2 \frac{(1 + b + B + \theta)(1 + b - B + \theta)(1 + b - \theta)^2}{(1 + b + B - \theta)(1 + b - B - \theta)(1 + b + \theta)^2} + \log_2 \frac{(1 - b - B + \theta)(1 - b + B + \theta)(1 - b - \theta)^2}{(1 - b + B - \theta)(1 - b - B - \theta)(1 - b + \theta)^2} \right\}.
\]

(3.37)

Obviously, \( \frac{\partial F}{\partial \theta} \geq 0 \) when \( \theta \in [0, 1] \). Then, \( F(\theta) \) is a strictly increasing function and \( \max F(\theta) = F(\max\{\theta\}) \).

Let \( Y = \theta + \mu [1 - (z_1^B)^2 - (z_2^B)^2 - (z_3^B)^2] \), \( \frac{\partial Y}{\partial z_1^B} = r \frac{b_1}{b} - 2\mu z_1^B, \frac{\partial Y}{\partial z_2^B} = r \frac{b_2}{b} - 2\mu z_2^B, \frac{\partial Y}{\partial z_3^B} = r \frac{b_3}{b} - 2\mu z_3^B \).

\[
\frac{\partial Y}{\partial z_1^B} = r \frac{b_1}{b} - 2\mu z_1^B, \frac{\partial Y}{\partial z_2^B} = r \frac{b_2}{b} - 2\mu z_2^B, \frac{\partial Y}{\partial z_3^B} = r \frac{b_3}{b} - 2\mu z_3^B.
\]

Imposing \( \frac{\partial Y}{\partial z_1^B} = 0, \frac{\partial Y}{\partial z_2^B} = 0, \frac{\partial Y}{\partial z_3^B} = 0 \), we have \( z_i^B = \frac{b_i}{b} \). So \( \max\{\theta\} = r \) and \( \max F(\theta) = F(r) \), then case 1 is shown.

\( \Box \)

**Example 1** For a state in Eq.(3.1), when \( a_1 = 0, a_2 = 0, a_3 = 0.03, b_1 = 0, b_2 = 0.25, c_1 = 0.12, c_2 = 0.12, c_3 = 0.01, r_1 = 0.1, r_2 = 0.1, r_3 = -0.3, s_1 = 0.13, s_2 = 0.13, s_3 = -0.26, v_1 = 0, v_2 = 0, v_3 = 0, T_1 = -0.02, T_2 = -0.02, T_3 = -0.36 \). According to the case 1 of Theorem 3.2, we have \( Q = 0.8889 \). Figure 2 shows the behavior of the quantum discord \( Q \).

**Example 2** For a state of the case 1 in Theorem 3.4, when \( a_1 = a_2 = a_3 = 0, b_1 = 0.2, b_2 = 0.05, b_3 = 0.1, c_1 = 0.04, c_2 = 0.06, c_3 = 0.11, r_1 = r_2 = r_3 = 0.17, s_1 = 0.08, s_2 = 0.15, s_3 = 0.25, v_1 = v_2 = v_3 = T_1 = T_2 = T_3 = 0 \). Then,
Fig. 2 The behavior of the quantum discord $Q$ with respect to the parameters in Example 1. In this case, $z_A^1 = z_A^2 = z_B^1 = z_B^2 = 0$, the quantum discord $Q$ is only related to variables $z_A^3, z_B^3$. Then, $Q = 0.8889$.

Fig. 3 The behavior of $G(z_A^1, z_A^2, z_A^3)$ with the variables $z_A^1, z_A^2, z_A^3$ in Example 2, where $z_A^A = (z_A^1, z_A^2, z_A^3)$ is on a unit sphere. This is a four-dimensional figure. Among them, the intensity of light is used to indicate the magnitude of the $G$ value. The brighter the point, the greater the value of $G$ and $\max G = 0.1182$.

the quantum discord is $Q = 0.9970$. Figures 3 and 4 show the behavior of the function $G$ and $F$, respectively.
Fig. 4 The behavior of $F(0.8729, 0.2182, 0.4364, z_B^1, z_B^2, z_B^3)$ with the variables $z_B^1, z_B^2, z_B^3$ in Example 2, where $z_B = (z_B^1, z_B^2, z_B^3)$ is on a unit sphere. The brighter the point, the greater the value of $F$ and $\text{max } F = 0.1107$

4 Conclusions

Quantum discord is one of the important correlations in studying quantum systems. It is well known that the quantum discord is hard to compute explicitly, and only sporadic formulas are known, for instance, the Bell state and the X-state. Recently, important progresses are made to generalize the notion to multipartite quantum systems [10], and their explicit formulas are expectedly not easy to find. In this work, we have found explicit formulas of the quantum discord for tripartite non-X-states with 14 parameters, including some famous states such as the Werner-GHZ state.

Acknowledgements The research is supported in part by the NSFC grants 11871325, 12171303 and 12126351 and Natural Science Foundation of Hubei Province grant no. 2020CFB538 as well as Simons Foundation grant no. 523868.

Declarations

Data Availability Statement All data generated during the study are included in the article.

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