The effect of the tortoise coordinates on the tunnel effect

Tian Gui-hua\textsuperscript{a,b}, Zhao Zheng\textsuperscript{c}, Shi-kun Wang\textsuperscript{b}

a. School of Science, Beijing University of Posts And Telecommunications, Beijing 100876, China.
b. Academy of Mathematics and Systems Science, Chinese Academy of Sciences,(CAS) Beijing 100080, China.
c. Department of Physics, Beijing Normal University, Beijing 100875, China

Abstract

The tunnel process of the quantum wave from the light cone is carefully discussed. They are applied in the massive quantum particles from the Schwarzschild black hole in the Kruskal metric. The tortoise coordinates prevent one from understanding the tunnel process, and are investigated with care. Furthermore, the massive particles could come out of the black hole either by the Hawking radiation or by the tunnel effect; the tunnel effect might give more information about the black hole.

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When the potential \( V \) in some region is greater than the energy \( E \), the classical particles could not pass through the potential barrier and obtain admission into the other region with potential lower than the energy. But in quantum mechanics, the classical forbidden region could be penetrated and passed through. This case belongs to the non-relativistic quantum theory. Now, consider the relativistic case, the classical particle must move inside its light cone. The classical forbidden region for the particle is the outside of its future light cone. Similarly, the relativistic quantum wave equations are the K-G equations, etc. When the initial mass \( \mu \) is not zero for the particle, the particle’s phase velocity is greater than the light’s velocity. The wave can not be confined inside of the future light cone, the quantum particle could pass through the light cone. We show the tunnel process by an example of the K-G particle in two dimensional Minkowski space time

\[ ds^2 = -dT^2 + dZ^2. \]  

The massive K-G equation is

\[ -\frac{\partial^2 \Psi}{\partial T^2} + \frac{\partial^2 \Psi}{\partial Z^2} - \mu^2 \Psi = 0. \]  

The mode-decomposition solutions are

\[ \Psi = e^{-i\omega T+i k Z} \]  

with the relation

\[ \omega^2 = k^2 + \mu^2. \]  

From the Eq. (4), we see the phase velocity \( \frac{\omega}{k} > 1 \) whenever the initial mass \( \mu \neq 0 \).

The effect could be used in a wider range in physics. We first consider its application in the Minkowski space time. The 4-dim Minkowski space time is defined by

\[ ds^2 = -dT^2 + dZ^2 + dx^2 + dy^2, \]  

\[ \text{E-mail of Tian: hua2007@126.com, tgh-2000@263.net} \]
and consists of four parts. Part I, part II, part I', part II' correspond to $T^2 - Z^2 < 0$, $Z > 0$; $T^2 - Z^2 > 0$, $T > 0$; $T^2 - Z^2 < 0$, $Z < 0$; $T^2 - Z^2 > 0$, $T < 0$ respectively.

The Rindler metric is
\[
 ds^2 = -(1 + az)dt^2 + (1 + az)^{-1}dz^2 + dx^2 + dy^2. \tag{6}
\]
The constant spatial coordinates with $z < - \frac{1}{a}$ are also the world lines of constantly accelerated observers. The null geodesics have constant $z$ axis component $z = - \frac{1}{a}$ with its acceleration going to infinity. These null geodesics consist of the horizon of the Rindler space time.

The Rindler space time corresponds to the part I in the Minkowski space time, and the part I' is not communicable with it. The parts II and II' are the future and past of the Rindler space time respectively.

Because any classical particle can not surpass the light, classical particles in the Part II can not enter the Rindler space time later. Nevertheless, applying the above tunnel process, we get the massive particle could enter the Rindler space time from the part II. The effect is only for massive K-G particle, and is different from the Unruh effect. The massless K-G particles have not such effect of tunnel, but they have Unruh effect.

Because the Rindler space time is similar in many respects with the Schwarzschild black hole, the effect of the tunnel could also extend to the Schwarzschild black hole.

Why researchers have not noticed the effect of the tunnel for the massive K-G particles in the Rindler space time? How to extend it to the Schwarzschild black hole? The paper mainly address the problems.

The tortoise coordinates of the Rindler space time and the Schwarzschild black hole prevent one from finding the tunnel process from the black hole or part II for the massive particles.

In fact, the researcher found the Hawking Radiation or the Unruh effect for over thirty years, why have not they found the tunnel effect for the massive K-G particles yet? The reasons are the following.

Generally, even the quantum effects are considered, it is taken for granted that the physical entities in part II have no influence on the Rindler space time. The thought is regarded right until the appearance of the Hawking Radiation. This is mainly due to the form of the K-G equation in the Rindler space time containing the tortoise coordinate $t$.

The K-G equation in the Rindler metric (6) reads
\[
 - \frac{1}{1 + az} \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial z} \left[ (1 + az) \frac{\partial \psi}{\partial z} \right] - (k_1^2 + k_2^2 + \mu^2) \psi = 0. \tag{7}\]
Define the spatial tortoise coordinate $z_*$ as
\[
 z_* = \frac{1}{a} \ln(1 + az) \tag{8}
\]
which makes the horizon $1 + az = 0$ and the spatial infinity into $z_* \rightarrow -\infty$ and $z_* \rightarrow +\infty$ respectively. The Eq.(7) then becomes
\[
 \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial z_*^2} + (1 + az)(k_1^2 + k_2^2 + \mu^2) \psi = 0. \tag{9}\]
From the Eq.(9), we see that the velocity of the scalar particle cannot exceed speed limit at the horizon $z = - \frac{1}{a}$. It is the same as the light’s velocity. Therefore, it is natural to regard the Rindler space time as independent entity. This is wrong and discussed carefully in Ref[3].

In fact, the Eq.(9) gives less information at the horizon, and it can mislead one to error. From it, we obtain the wrong information that even the massive Klein-Gordon particle could not tunnel from the horizon. Whereas the tunnel process is very simple in the Minkowski
coordinates. It is the Eq.(9) that make the tunnel process impossible. The Eq.(9) is connected with the Rindler metric, or the accelerated frame. In the Rindler metric, the time coordinate \( t \) is a tortoise coordinate, and it could result in many difficult explanation of the physical process. Actually, the future and past horizons correspond to \( t \to +\infty \), \( t \to -\infty \) respectively. This is just the manifest of the tortoise property of the Rindler time coordinate \( t \). So, the process involved the horizon is difficult to describe by the Rindler coordinates.

We reinforce our viewpoint again: the physical process can be easily explained by the Minkowski metric, it is only the Rindler tortoise coordinate \( t \) that make the explanation of the physical process difficult.

The Unruh effect corresponds to the excitation of the modes whose phase velocity is equal to that of the light at the horizon\(^4\). It is really curious that the excited modes are not those whose phase velocity surpass the light.

Furthermore, the phase velocity could be greater than the light velocity even in the Rindler space time whenever the scalar field is not at the horizon.

The same situations exist for the Schwarzschild black hole. We now extend the tunnel effect to it.

The Schwarzschild black hole is described by the metric
\[
ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2d\Omega^2. \tag{10}\]
The observers \( r = r_0, \theta = \theta_0, \varphi = \varphi_0 \) have constant acceleration
\[
A = \frac{m}{r^2}(1 - \frac{2m}{r})^{-\frac{3}{2}}. \tag{11}\]
When \( r_0 \to 2m \), the time-like curve \( r = r_0, \theta = \theta_0, \varphi = \varphi_0 \) becomes the null geodesic with its proper acceleration \( A \to \infty \). The null geodesics \( r_0 = 2m, \theta = \theta_0, \varphi = \varphi_0 \) with \( 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \) consist the horizon of these observers \( r = r_0 < 2m, \theta = \theta_0, \varphi = \varphi_0 \). Of course, it is also the horizon of the black hole.

Nevertheless, it is obvious that the Schwarzschild metric as a whole is the same with that of the Rindler metric.

The Schwarzschild metric and the Rindler metric all are in the accelerated frame and have the same kind horizons for those accelerated observers.

Furthermore, the Schwarzschild metric has the complete space time, that is, the Kruskal space time. The Rindler’s completion is the Minkowski space time. The geometric properties of the Schwarzschild black hole are the same as those of the Rindler space time.

These striking similarities stimulate researcher to investigate their further connection.

Later, Hawking showed that the black hole could radiate thermal radiation with temperature, therefore he made great progress of black hole study. Unruh investigated the radiation connected with the Rindler space time, and obtained that the Rindler observers could also detect the thermal radiation with the temperature \( k \) equal to its surface gravity too. Unruh appeared first to notice the same origin of the Hawking Radiation and the Unruh effect. Actually, the two effects all originate from the horizon as the accelerated observers.

We now study the tunnel effect in the Schwarzschild black hole. The K-G equations in the Schwarzschild metric (10) is
\[
\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r^2} + \left(1 - \frac{2m}{r}\right)\left[l(l+1)\frac{1}{r^2} + \frac{2m}{r^3} + \mu^2\right]Q = 0 \tag{12}\]
where \( \mu \) is its mass. The third term in the Eq.(12) is zero at the horizon. This equation (12) has a striking characteristic: its phase velocity equals to that of light at the horizon. This has influence not only on the stable study, but also for the explanation of the tunnel effect\(^2\). Therefore, one is easily led to obtain the conclusion that the scalar field inside of the black hole \( r < 2m \) could not penetrate through the horizon and enter into the region.
$r > 2m$. This fact is truly wrong caused by the tortoise coordinates $t$, $r_*$. Rewriting the scalar perturbation equation in the Kruskal coordinates

$$ds^2 = \frac{32m^3}{r}e^{-\frac{2\pi}{m}} \left[-dT^2 + dX^2\right] + r^2d\Omega^2,$$

and by the decomposition of the variables $Q = \psi(T,X)Y_{lm'}$ where $Y_{lm'}$ are the spherical harmonic functions, we obtain

$$\frac{\partial^2 \psi}{\partial T^2} - \frac{\partial^2 \psi}{\partial X^2} + \left(\frac{32m^3}{r}e^{-\frac{2\pi}{m}}\right) \left[- \frac{T}{2mr} \frac{\partial \psi}{\partial T} - \frac{X}{2mr} \frac{\partial \psi}{\partial X} + \mu^2 + \frac{l(l+1)}{r^2}\right] \psi = 0. \tag{14}$$

By the geometric approximation at the horizon, we obtain the asymptotic form for out-going wave $\psi$

$$\psi = A' e^{i\omega T - ikX} \tag{15}$$

with the relation

$$\omega^2 = k^2 + 16m^2e^{-1} \left[\mu^2 + \frac{l(l+1)}{4m^2}\right] \tag{16}$$

under the condition $|\omega| \gg 4e^{-1}T$ at the time $T$. Therefore, the phase velocity at the horizon

$$V_p = \pm \frac{\omega}{k} \tag{17}$$

is greater than $c = 1$ of the light whenever the mass $\mu$ or $l$ is not equal to zero. This is just the tunnel effect.

The excited modes in Hawking radiation are those whose phase velocity are the light velocity. The tunnel effect is different from the Hawking radiation.

**Some comments**

It is shown that the tunnel process exists for the massive K-G particles, it can also easily extend to massive particles of non-zero spin. This may have some effect on the black hole’s formation. The massive particles could come out of the black hole either by the Hawking radiation or by the tunnel effect, it might give some information about the black hole.

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