Abstract. We show that homological stability holds for the family of Iwahori-Hecke algebras of type $B_n$, where homology is identified with the relevant Tor group. This family of algebras is related to the Coxeter groups of type $B_n$, which are groups of signed permutations. The result builds on Hepworth’s result for Iwahori-Hecke algebras of type $A_n$.

1. Introduction

1.1. Homological stability. A family of groups

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \ldots$$

is said to exhibit homological stability if for each $d$ the map

$$H_d(G_{n-1}) \rightarrow H_d(G_n)$$

is an isomorphism for $n$ sufficiently large. Prominent examples of sequences satisfying this include the symmetric groups $S_n$ ([Nak60]), automorphism groups of free groups $\text{Aut}(F_n)$ ([Hat95]), and the linear groups $\text{GL}_n(R)$ over a ring ([vdK80]). A more complete survey of such results can be found in [Wah22].

Recall that the homology of a group $G$ with coefficients in a commutative ring $R$ may be identified with $\text{Tor}_{RG}^1(1, 1)$ for $RG$ the group ring and $1$ the trivial module ($R$ with trivial action). If $A$ is any $R$-algebra with a natural choice of trivial module $1$, we may then define the algebra homology of $A$ to be $\text{Tor}_A^1(1, 1)$.

Definition 1.1. A family of algebras $A_1 \hookrightarrow A_2 \hookrightarrow \ldots$ exhibits homological stability if for each $d$ the stabilisation map

$$\text{Tor}_{A_{n-1}}^d(1, 1) \rightarrow \text{Tor}_{A_n}^d(1, 1)$$

is an isomorphism for $n$ sufficiently large.

Remark 1.2. For each $A_n$ we have picked a trivial module. We require also that the map $A_{n-1} \hookrightarrow A_n$ commutes with the action of each algebra on its trivial module. In practice, the choice of trivial module for the algebras we will consider is natural enough that they all evidently commute with the map.

This notion was first introduced by Hepworth in [Hep20], in which it is shown that:

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Theorem 1.3 ([Hep20]). The map

$$\text{Tor}^{{\mathcal{H}_n}}_{d-1}(1, \mathbb{1}) \rightarrow \text{Tor}^{{\mathcal{H}_n}}_{d}(1, \mathbb{1})$$

is an isomorphism for $2d \leq n - 1$.

The algebras $\mathcal{H}_n$ in the theorem above form the simplest family of finite rank Iwahori-Hecke algebras. Iwahori-Hecke algebras (to be defined in full later) are certain deformations of the group algebra of Coxeter groups, and are prevalent both in algebra and topology. They are used to define the HOMFLY-PT polynomial in knot theory ([Jon87]), which is a generalisation of several knot polynomial invariants. They also arise naturally in the study of the representation theory of both $\text{GL}_n(F_q)$ ([Mat99]) and Coxeter groups ([KL79]). The representation theory of Iwahori-Hecke algebras is much studied ([GP00]), and has connections with quantum groups.

The algebra $\mathcal{H}_n$ arises from deformations of the group algebra over the symmetric group $S_n$, with ground ring some commutative ring $R$. We study the next simplest finite rank family: this is $\mathcal{HB}_n$, which is a deformation of the group algebra of the Coxeter group $B_n$. The discussion of $B_n$ as a Coxeter group appears in Section 2, but it will be more productive throughout to identify it with the group of signed permutations.

Definition 1.4. Let $n \geq 1$. The group of signed permutations is the subgroup of $\text{GL}_n(\mathbb{Z})$ comprising of matrices with exactly one non-zero entry in each row and column, and with all non-zero entries $\in \{\pm 1\}$. Alternatively, it is the wreath product of $C_2$ with $S_n$.

Elements of $B_n$ may thus be thought of as permutations with an accompanying sign change in $(\mathbb{Z}/2\mathbb{Z})^n$, and the same notations and manipulations can be used as for $S_n$. This provides a convenient combinatorial viewpoint, some of which carries over to the algebras $\mathcal{HB}_n$. Following the combinatorial viewpoint as a guide, we prove the result corresponding to Theorem 1.3 for the algebras $\mathcal{HB}_n$ of type $B_n$: Theorem A. The map

$$\text{Tor}^{{\mathcal{HB}_n}}_{d-1}(1, \mathbb{1}) \rightarrow \text{Tor}^{{\mathcal{HB}_n}}_{d}(1, \mathbb{1})$$

is an isomorphism for $2d \leq n - 1$.

This is accomplished by constructing a chain complex $D^{\pm}(n)$ which is the algebraic analogue of a combinatorial complex one could use to prove homological stability for the signed permutation groups. Most of this paper is devoted to the proof of the following theorem:

Theorem B. $H_d(D^{\pm}(n)) = 0$ for $d \leq n - 2$.

As in the standard proof of homological stability for $S_n$ ([Far78]), this highly acyclic complex is then used in a spectral sequence argument to obtain Theorem A.

1.2. Recent related work. In addition to Theorem 1.3, a number of other results for algebra homology following the above definition have recently appeared in the literature. Boyd and Hepworth ([BH20]) showed homological stability for the Temperley-Lieb algebras $\text{TL}_n(a)$ and computed the stable homology to show that $H_d(\text{TL}_n(a)) = 0$ in a certain range and under certain conditions on the parameter $a \in R$ (which Randal-Williams later removed [RW21]). Sroka ([Sro22])
further investigated the homology of these algebras, using a generalisation of the Davis complex from the geometric study of Coxeter groups and showed that when \( n \) is odd the homology of \( T\lambda_n(a) \) vanishes in all degrees. Boyd, Hepworth and Patzt ([BHP21]) compared the homology of the Brauer algebras \( Br_n(\delta) \) to that of the symmetric group, and showed that they are isomorphic under some conditions. The algebras \( T\lambda_n(a) \) and \( Br_n(\delta) \) are both examples of diagram algebras and are related to the group algebra of the braid group \( B_n \) and the Iwahori-Hecke algebra \( H_n \).

\( H_n \) by contrast is a quotient of the algebra of the Artin group associated to \( B_n \), which includes \( RB_n \) as a subalgebra. The presence of the symmetric group in all these examples allows similar techniques to be used in each case—in particular, the face maps that appear in our complex \( D^{\pm}(n) \) are closely related to elements of the symmetric group.

1.3. Outline. In Section 2, we begin with an exposition of background information on Coxeter groups and Iwahori-Hecke algebras. In Section 3 we present a certain proof of homological stability for the signed permutation group \( B_n \) using the complex of signed injective words, which motivates the proof of homological stability for \( HB_n \). Following this, we construct the analogous chain complex \( D^{\pm}(n) \) of \( HB_n \)-modules. In Section 4, we study the double coset structure of the groups \( B_n \) and obtain some technical results regarding distinguished coset representatives. In Section 5, we define the filtration of \( D^{\pm}(n) \) in terms of these coset representatives. In Section 6, we identify the filtration quotients and show that they are highly connected, thus proving Theorem B. We then conclude in Section 7 via a routine spectral sequence argument (restated for algebra homology in [Hep20]).

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2. Background on Iwahori-Hecke algebras

This section summarises the essential theory of Coxeter groups and Iwahori-Hecke algebras, and sets out the conventions used in this paper. The results on Coxeter groups are standard (e.g. [Dav12]). The material on Iwahori-Hecke algebras follows the presentation in [GP00], and is again standard. Much of the following is similar to the background in [Hep20].

2.1. Coxeter groups. A Coxeter group is a group \( W \) with presentation:

\[
W = \left\langle S \mid s^2 = 1, \ (s,t)^{m_{s,t}} = (t,s)^{m_{t,s}} \forall s \neq t \in S \right\rangle
\]

where \( S \) is a finite set, \( m_{s,t} = m_{t,s} \in \mathbb{N}_{\geq 2} \cup \{\infty\} \), and \( (s,t)^m \) denotes the alternating product \( stst\ldots \) of length \( m \) starting with \( s \) (no relation is applied for \( m_{s,t} = \infty \)). The relations \( (s,t)^{m_{s,t}} = (t,s)^{m_{t,s}} \) are known as braid relations, and the pair \( (W, S) \) the Coxeter system.

Unpacking the braid relations, we see that \( s, t \in S \) commute if \( m_{s,t} = 2 \). In other cases:

\[
sts\ldots = tst\ldots
\]
where both sides have length $m_{s,t}$. By the relation $s^2 = t^2 = 1$, this is equivalent to:

$$(st)^{m_{s,t}} = 1$$

**Example.** For $S = \{s_1, s_2, \ldots, s_{n-1}\}$, set:

$$m_{s_i, s_j} = \begin{cases} 
3 & \text{if } |i - j| = 1 \\
2 & \text{if } |i - j| > 1
\end{cases}$$

Then for $W$ the associated Coxeter group, the relations are $s_i^2 = 1$, $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and $s_i$ commutes with $s_j$ for $|i - j| > 1$—this is the presentation for the symmetric group in $n$ letters $S_n$ generated by the $n - 1$ transpositions of adjacent elements. This is also known as the Coxeter group of type $A_{n-1}$ (there are $n - 1$ generators in $S$), so $W \cong A_{n-1} \cong S_n$.

In order to depict pictorially the relations in a Coxeter group, a Coxeter diagram is often used. This is a graph with vertex set $S$, and labelled edges assigned by the following rules:

- If $m_{s,t} = 2$, so that $s, t$ commute in $W$, no edge is drawn between them.
- If $m_{s,t} = 3$, an unlabelled edge is drawn.
- If $m_{s,t} > 3$, an edge is drawn with label $m_{s,t}$.

**Example.** Referring back to the previous example, the Coxeter group of type $A_{n-1} \cong S_n$ has Coxeter diagram:

![Coxeter diagram]

In particular, no edges are labelled, since the only values of $m_{s,t}$ that occur in the presentation of $A_n$ are 2 and 3.

If the graph in the Coxeter diagram is not connected, then $W$ is given by the direct product of the Coxeter groups associated to the connected components—in this case $W$ is called reducible. Irreducible Coxeter groups, corresponding to connected Coxeter diagrams, may be have finite or infinite order. A foundational result in the study of Coxeter groups is the classification of finite irreducible Coxeter groups.

**Theorem 2.1** (Classification of finite Coxeter groups, [Cox35]). Every finite irreducible Coxeter group is given by one of the following presentations:
where $A_n, B_n, D_n$ and $I_n$ are infinite families, and the remaining entries are the six exceptional groups.

In this paper, we are largely concerned with the family $B_n$, which may be identified with the group of signed permutations on $n$ letters discussed in the introduction. The usual cycle notation in $S_n$ may be extended to the group of signed permutations: for distinct $a_i$ and choice of signs $(−1)^{r_i}$, the cycle

$(-1)^{r_1}a_1 (-1)^{r_2}a_2 \ldots (-1)^{r_k}a_k$

represents the signed permutation which sends $+a_i$ (resp. $-a_i$) to $(-1)^{r_1+r_2}a_{i+1}$ (resp. $(-1)^{r_1+r_2+1}a_{i+1}$). For example, the cycle $(2 - 3)$ sends 2 to $-3$, $-2$ to 3, $-3$ to 2, and 3 to $-2$.

Note that by inspecting the Coxeter diagrams, $A_{n-1} \cong S_n$ embeds into $B_n$. We thus label the initial generator $u$, and the remaining $s_1, s_2, \ldots, s_{n-1}$. The isomorphism with the group of signed permutations is then given by associating $u$ with the cycle $(-1, 1)$, and $s_i$ with $(i \quad i+1)$. Also $B_m$ embeds in $B_n$ for $m \leq n$ by considering the signed permutations on $m$ letters inside the signed permutations on $n$ letters—in terms of generators, $B_m$ is the subgroup generated by $u, s_1, \ldots, s_{m-1}$.

There are several important results in the theory of Coxeter groups that are used in this paper. Most significant are two theorems relating to the word problem in Coxeter groups. If $w \in W$, then the length $l(w)$ is defined to be the smallest length of a word representing $w$ in the generators $S$. A word is reduced if it is a minimal representative, i.e. if $l(x_1x_2\ldots x_r) = r$. 

![Coxeter Diagrams](image-url)
Theorem 2.2 (Matsumoto’s theorem, [Mat64]). For a Coxeter system \((W, S)\), any two reduced words in \(S\) for the same element \(w\) are related by a series of transforms \(\langle s, t \rangle^{m_{s,t}} = (t, s)^{m_{s,t}}\).

From this, one can find a complete solution to the word problem.

Theorem 2.3 (The word problem, [Tit69]). For a Coxeter system \((W, S)\), any two words represent the same element if and only if they are related by a series of transforms of the types

- \(s^2 \Rightarrow 1\)
- \(\langle s, t \rangle^{m_{s,t}} \Rightarrow \langle t, s \rangle^{m_{s,t}}\)

These are known as \(M\)-moves.

2.2. Parabolic subgroups. Any subset \(J \subseteq S\) generates a subgroup \(X_J\) of \(X\). Such subgroups are known as parabolic, and in the study of Coxeter groups it is found that \(X_J\) is a Coxeter group in a natural way: it forms the Coxeter system \((X_J, J)\) with \(m_{s,t}\) as in \((X, J)\). From the diagram perspective, every subset \(J\) of the vertices generates a subgroup of \(W\) identified with the Coxeter group with diagram the the full diagram spanned by \(J\).

Parabolic subgroups enjoy many special properties in the theory of Coxeter groups. In particular, there are distinguished coset representatives for each parabolic group—define:

\[
X_J = \{ w \in W \mid l(sw) > l(w), \quad \forall s \in J \}
\]

This is the set of elements with no reduced expression beginning with a generator in \(J\). These are known as \((J, \emptyset)\)-reduced. By Matsumoto’s Theorem (2.2), to check if a word is \((J, \emptyset)\)-reduced it suffices to show that no series of braid relations produces a word beginning with a letter in \(J\). We may also consider the corresponding notion for words ending in an element of \(X_J\): it is clear that the set of elements with no reduced word ending with a letter in \(J\) is \(X_J^{-1}\).

The next theorem shows why \(X_J\) are known as the distinguished right coset representatives.

Theorem 2.4 ([Dav12, 4.3.3]). \(X_J\) is a complete set of representatives for \(W_J\backslash W\). Furthermore, for \(x \in X_J\), \(x\) is the shortest element in \(W_J x\).

Similarly, \(X_J^{-1}\) is a complete set of representatives for \(W/W_J\), and for \(x \in X_J^{-1}\), \(x\) is the shortest element in \(xW_J\).

The theory extends to double cosets. Take \(X_{JK} = X_J \cap X_K^{-1}\), so \(x \in X_{JK}\) are elements with no reduced words beginning with a letter in \(J\), or ending with a letter in \(K\). Analogous to the coset representatives, \(X_{JK}\) is a complete set of representatives for \(W_J \backslash W/W_K\), and is made up of the shortest element in each double coset. The elements of \(X_{JK}\) are called \((J, K)\)-reduced, and are the distinguished double coset representatives.

It is often easier to describe the words that are \((J, K)\)-reduced for some \(K\) than those that are merely \((J, \emptyset)\)-reduced. The Mackey decomposition relates these:

Proposition 2.5 (Mackey decomposition, [GP00, 2.1.9]). For \(J, K \subseteq S\):

\[
X_J = \bigsqcup_{d \in X_{JK}} d \cdot X_J^{K \cap JD_K}
\]

where for \(J \subseteq K \subseteq S\), \(X_J^K\) denotes the distinguished coset representatives for \(W_J \backslash W_K\), and \(g^d = d^{-1}gd\) is conjugation with the positive power on the right.
Inverting this produces the Mackey decomposition for right cosets:

\[ X_J^{-1} = \bigsqcup_{d \in X_{KJ}} (X^K_{K \cap dJ})^{-1} \cdot d \]

where \( d'g = dgd^{-1} \).

### 2.3. Iwahori-Hecke algebras

Given a Coxeter system \((W, S)\) and a commutative ring \( R \) along with a choice of parameter \( q \in \mathbb{R}^* \) then the associated **Iwahori-Hecke algebra** is the algebra \( \mathcal{H}W \) with generators \( T_s \) for \( s \in S \) and the following relations:

\[
\begin{align*}
\langle T_s, T_t \rangle^{m_{st}} &= \langle T_s, T_t \rangle^{m_{st}} & s \neq t \in S \\
(T_s + 1)(T_s - q) &= 0 & s \in S
\end{align*}
\]

where \( \langle T_s, T_t \rangle^m = T_s T_t T_s \ldots \) is again the alternating product of length \( m \) starting with \( T_s \). The relations of the first type define the group algebra of the Artin group associated to \((W, S)\) (which is a braid group for \( W = \mathfrak{S}_n \)), so \( \mathcal{H}W \) is a quotient of the Artin group algebra. If \( q = 1 \), the second relations become \( T^+_s = 1 \), in which case \( \mathcal{H}W \) is simply the group algebra of the Coxeter group \( W \).

Throughout we will fix \( R, q \), and take \( \mathcal{H}_n \) to be the Iwahori-Hecke algebra of type \( \mathcal{A}_{n-1} \cong \mathfrak{S}_n \), and \( \mathcal{H}B_n \) to be the Iwahori-Hecke algebra of type \( \mathcal{B}_n \)—the inclusion \( \mathcal{A}_{n-1} \) into \( \mathcal{B}_n \) via the tail of the Coxeter diagram induces an inclusion \( \mathcal{H}_n \leq \mathcal{H}B_n \). This is the generalisation to Iwahori-Hecke algebras of the inclusion of \( \mathfrak{S}_n \) into the group of signed permutations.

From the relation \((T_s + 1)(T_s - q) = 0\) there are two natural rank one modules of \( \mathcal{H}W \): \( R \) where the generators act by multiplication by \( q \), or multiplication by \(-1\). The first of these becomes the trivial module when \( q = 1 \), so will be denoted \( 1 \) and thought of as the trivial module for Iwahori-Hecke algebras. Then, as described earlier, the homology of the Iwahori-Hecke algebra is defined to be \( \text{Tor}_{\mathcal{H}_n}^1(1, 1) \) where the action of \( T_s \) on \( 1 \cong R \) is multiplication by \( q \).

**Remark 2.6.** The rank one module where all generators act by multiplication by \(-1\) may be thought of as the analogue for the sign representation of \( \mathfrak{S}_n \) in the Iwahori-Hecke setting. If the parameter \( q \) is \(-1\), it is equal to the trivial module \( 1 \).

We proceed with some consequences of Matsumoto’s Theorem (2.2).

**Proposition 2.7.** Let \( w \in W \) and \( w = s_1 s_2 \ldots s_t = \tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_r \) be two reduced expressions. Then \( T_{s_1} T_{s_2} \ldots T_{s_t} = T_{\tilde{s}_1} T_{\tilde{s}_2} \ldots T_{\tilde{s}_r} \) in \( \mathcal{H}W \).

**Proof.** By Matsumoto’s Theorem, the two expressions are related in \( W \) by a series of substitutions \( \langle s_i, s_j \rangle^{m_{i,j}} = \langle s_i, s_j \rangle^{m_{i,j}} \). But the relation \( \langle T_{s_1}, T_{s_2} \rangle^{m_{i,j}} = \langle T_{s_1}, T_{s_2} \rangle^{m_{i,j}} \) holds in \( \mathcal{H}W \) so the same substitutions show that \( T_{s_1} T_{s_2} \ldots T_{s_t} = T_{\tilde{s}_1} T_{\tilde{s}_2} \ldots T_{\tilde{s}_r} \).

By the above result, it is valid to define \( T_w = T_{s_1} T_{s_2} \ldots T_{s_t} \) for any \( w \in W \) where \( s_1 s_2 \ldots s_t \) is a reduced expression. The **basis theorem** asserts that these \( T_w \) form a basis for \( \mathcal{H}W \) over \( R \).

**Theorem 2.8** (Basis theorem, [GP00, 4.4.6]). The set \( \{ T_w \ : w \in W \} \) is a basis for \( \mathcal{H}W \) over \( R \). In particular, \( \mathcal{H}W \) is a free \( R \)-module.

As well as the \( R \)-module structure, \( \mathcal{H}W \) is a left and right \( \mathcal{H}_{W_J} \)-module for any subset \( J \subseteq S \) of generators.
Proposition 2.9. For \((W, S)\) a Coxeter system and \(J \subseteq S\), \(\mathcal{H}W\) is a free left \(\mathcal{H}W_J\)-module with basis \(\{T_x : x \in X_J\}\). Hence \(1 \otimes_{\mathcal{H}W_J} \mathcal{H}W\) is free with basis \(\{1 \otimes T_x : x \in X_J\}\).

Similarly, \(\mathcal{H}W\) is a free right \(\mathcal{H}W_J\)-module with basis \(\{T_x : x \in (X_J)^{-1}\}\), and \(\mathcal{H}W \otimes_{\mathcal{H}W_J} 1\) is free with basis \(\{T_x \otimes 1 : x \in (X_J)^{-1}\}\).

2.4. Some notation. We now define and discuss some notation for certain elements in Coxeter groups and Iwahori-Hecke algebras. The notation will also be introduced throughout the paper, so the reader may choose to ignore this section or return to it later.

The generators of the Coxeter group \(B_n\) are throughout labelled as \(u, s_1, s_2, \ldots, s_n\), where \(u\) is the initial vertex in the Coxeter diagram. Similarly, the generators of the associated Iwahori-Hecke algebra \(\mathcal{H}B_n\) are labelled \(U, T_1, T_2, \ldots, T_{n-1}\), with \(U = T_u\), and \(T_i = T_{s_i}\).

- For \(n \geq a > b \geq 1\), take \(s_{a,b}\) to be the element of \(S_n \subseteq B_n\) defined by:

\[
s_{a,b} = s_{a-1}s_{a-2}\ldots s_b
\]

As a permutation, this is

\[
(a \ b-1 \ a-2 \ \ldots \ b+1 \ b)
\]

the downwards cycle on \(\{a, a-1, a-2, \ldots, b\}\). Observe that \(s_{a,b}\) admits no M-moves, so by the solution the word problem it is reduced. We then define the following element of \(\mathcal{H}A_{n-1} \subseteq \mathcal{H}B_n\):

\[
T_{a,b} = T_{s_{a,b}} = T_{a-1}T_{a-2}\ldots T_b
\]

where the last equality uses Proposition 2.7.

- For \(m \leq n\), take \(u_m\) to be the element of \(B_n\) defined by:

\[
u_m = us_1s_2\ldots s_{m-1}
\]

In cycle notation, this is the product

\[
(1 \ -1)(1 \ 2 \ 3 \ \ldots \ m)
\]

so shifts \(\{1, 2, 3, \ldots, m-1\}\) up by one, and sends \(m\) to \(-1\). Again, this admits no M-moves, so is reduced, and we define:

\[
U_m = T_{u_m} = UT_1T_2\ldots T_{m-1}
\]

- For \(t \geq 0, m_1 > m_2 > \ldots > m_t\):

\[
v(m_1, m_2, \ldots, m_t) = v(\mathbf{m}) = u_{m_1}u_{m_2}\ldots u_{m_t}
\]

From the discussion of \(u_m, v(\mathbf{m})\) takes \(m_1\) to \(-1, m_2\) to \(-2\) and so on, while sending the other letters to some positive letters. We shall see (Proposition 4.3) that the \(v(\mathbf{m})\) are reduced, so take:

\[
V(\mathbf{m}) = T_{v(\mathbf{m})} = U_{m_1}U_{m_2}\ldots U_{m_t}
\]
3. THE COMPLEX OF IWAHORI-HECKE ALGEBRAS OF TYPE $B_n$

In this section, we discuss some combinatorial complexes that relate to the group $B_n$. The complex $C^\pm(n)$ defined below is the combinatorial analogue to our algebraic complex $D^\pm(n)$, and the proofs in this section motivate both our construction of $D^\pm(n)$ and the approach to Theorem B.

For $[n] = \{1, 2, \ldots, n\}$, an injective word on $[n]$ is defined to be an ordered tuple $(a_0, \ldots, a_r)$ with $a_i \in [n]$ and $r < n$ such that no $a_i$ appears more than once.

**Definition 3.1.** Let $n \geq 0$. The complex of injective words $C(n)$ is the chain complex given in degree $r$ ($-1 \leq r \leq n - 1$) by the free $R$-module with basis the set of injective words $(a_0, \ldots, a_r)$ of length $r + 1$ on $[n]$—$C(n)_{-1}$ is a copy of $R$ generated by the empty word.

The differential $\partial^r : C(n)_r \to C(n)_{r-1}$ is defined to be:

$$\partial^r(a_0, \ldots, a_n) = \sum_{j=0}^{r} (-1)^j (\hat{a}_0, \ldots, \hat{a}_j, \ldots, a_n)$$

where $\hat{a}_j$ means we omit the letter $a_j$ from the tuple.

$S_n$ acts on the letters of each word in a natural way, so $C(n)$ can be viewed as a chain complex of $R S_n$-modules. This complex is used in many proofs of homological stability for the permutation groups $S_n$, see for example [Far78], [Ker05]. In particular, the action of $S_n$ along with the following theorem is used in a Quillen spectral sequence argument:

**Theorem 3.2.** [Far78] $H_d(C(n)) = 0$ for $d \leq n - 2$.

For the group $B_n$ of signed permutations, we introduce a new complex $C^\pm(n)$ that will play an analogous role to $C(n)$. Define a signed injective word on $[n]$ to be an injective word on $[n]$ along with a choice of sign on each element—equivalently, an ordered tuple $(\pm a_0, \ldots, \pm a_r)$ with $a_i \in [n]$ such that no $a_i$ appears more than once. Then the complex of signed injective words $C^\pm(n)$ is the complex given in degree $r$ ($-1 \leq r \leq n - 1$) by the free $R$-module with basis the signed injective words of length $r + 1$ on $[n]$, and the same differentials as $C(n)$. $C^\pm(n)$ carries an action of $B_n$ (applying a signed permutation to each letter), so is a complex of $R B_n$ modules.

We now prove the corresponding result to Theorem 3.2. The proof presented, while far from geodesic, will carry over to the setting of Iwahori-Hecke algebras and motivate the next sections of this paper, which deal with the proof of B.

**Theorem 3.3.** $H_d(C^\pm(n)) = 0$ for $d \leq n - 2$.

To accomplish this, we first define a filtration $F_p(n)$ of $C^\pm(n)$ by the position of the negative elements in a given signed injective word.

**Definition 3.4.** For $0 \leq p \leq n$, let $F_p \subseteq C^\pm(n)$ be the subcomplex spanned by the words in which all negatives appear in the first $p$ letters. Thus $F_0 = C(n)$ is the complex of injective words, and

$$C(n) = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = C^\pm(n)$$

is a filtration of $C^\pm(n)$. The $F_p$ are not $R B_n$-submodules, but they are $R S_n$-submodules, since the action of $S_n$ does not change the signs of the letters.
Remark 3.5. From the algebraic perspective that we will develop later, it is more natural to count positions from the right of the word, rather than the left. Unfortunately, this does not give a filtration in the Iwahori-Hecke setting, for technical reasons.

Observe that by Theorem 3.2, we need only show that $F_p/F_{p-1}$ is highly acyclic.

Definition 3.6. For $X$ a chain complex, take the suspension $\Sigma^kX$ to be the complex with $(\Sigma^kX)_r = X_{r-k}$, and $d^*_{\Sigma^kX} = d_{X}^{-k}$.

Lemma 3.7. There is an isomorphism of chain complexes of $R\mathfrak{S}_n$ modules

$$\left( R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-p}} \Sigma^p\mathcal{C}(n-p) \right)^{2^p-1} \rightarrow F_p/F_{p-1}$$

Proof. Let $-1 \leq r \leq n-p-1$. In degree $p+r$, $F_p$ is the free $R$-module with basis all words of the form

$$(\pm x_1, \pm x_2, \ldots, \pm x_p, y_0, y_1, \ldots, y_r)$$

(that is, all negative terms occurring in the first $p$ places). Thus $F_p/F_{p-1}$ is free with basis all words of the form

$$(\pm x_1, \pm x_2, \ldots, -x_p, y_0, y_1, \ldots, y_r)$$

On such a word, the first $p-1$ face maps delete an $x_i$, meaning all negatives then lie in the first $p-1$ places (i.e., the image lies in $F_{p-1}$). The differential in the quotient is thus:

$$\partial^{p+r}(\pm x_1, \ldots, -x_p, y_0, \ldots, y_r) = (-1)^p \sum_{j=0}^{r} (-1)^j(\pm x_1, \ldots, -x_p, y_0, \ldots, \hat{y}_j, \ldots, y_r)$$

In particular, the signs of $x_1, \ldots, x_{p-1}$ are not changed by any face map. Thus for each $2^{p-1}$ choices of signs $\{v \in \{\pm 1\}^{p-1}\}$, there is a $R\mathfrak{S}_n$-subcomplex $M_v$ of $F_p/F_{p-1}$ spanned in degree $p+r$ by all words where the first $p-1$ letters have signs $v$, and:

$$F_p/F_{p-1} = \bigoplus_{v \in \{\pm 1\}^{p-1}} M_v$$

Consider now the map $\phi : R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-p}} \Sigma^p\mathcal{C}(n-p) \rightarrow M_v$ given in degree $p+r$ by

$$\sigma \otimes (y_0, \ldots, y_r) \mapsto \sigma(v_1 \cdot (n-p+1), \ldots, v_{p-1} \cdot (n-1), \ldots, y_0, \ldots, y_r)$$

This is a chain map by the above discussion of the differential in $F_p/F_{p-1}$. For each injective word $x = (x_1, \ldots, x_p)$ on $[n]$, pick $\sigma_x$ with $x = \sigma_x(n-p+1, \ldots, n)$. The $\sigma_x$ are a set of coset representatives for $\mathfrak{S}_n/\mathfrak{S}_{n-p}$, so every $\sigma \otimes (y_0, \ldots, y_r)$ may be written as $\sigma_x \otimes (y'_1, \ldots, y'_r)$ for unique $x$, $y'_i$. Then $\phi$ has inverse given by

$$\sigma(v_1 \cdot x_1, \ldots, v_{p-1} \cdot x_{p-1}, -x_p, y_0, \ldots, y_r) \mapsto \sigma_x \otimes \sigma_x^{-1}(y_0, \ldots, y_r)$$

so $M_v \cong R\mathfrak{S}_n \otimes_{R\mathfrak{S}_{n-p}} \Sigma^p\mathcal{C}(n-p)$ for all $v$. $\square$

Proof of Theorem 3.3. The theorem now follows immediately. $R\mathfrak{S}_n$ is free as a right $R\mathfrak{S}_{n-p}$ module, so the above lemma implies that:

$$H_d(F_p/F_{p-1}) = R\mathfrak{S}_n \otimes \left( H_{d-p}(\mathcal{C}(n-p)) \right)^{2^{p-1}}$$

which is 0 for $d \leq n-2$ by Theorem 3.2.
$\mathcal{C}^\pm(n)$ can now be used to produce a proof of homological stability for $B_n$—this also follows from the result of Hepworth for homological stability for families of Coxeter groups [Hep16], or Hatcher and Wahl’s result on homological stability for wreath products (Proposition 1.6 of [HW10]).

3.1. The algebraic complex of injective words. In order to translate the results of the previous discussion to the Iwahori-Hecke algebra $HB_n$, we first provide an algebraic description of the complex $\mathcal{C}^\pm(n)$. A detailed discussion of this construction for the case of the complex of injective words $\mathcal{C}(n)$ can be found in Section 4 of [Hep20].

Definition 3.8. Let the complex $\tilde{\mathcal{C}}^\pm(n)$ be given in degree $r (-1 \leq r \leq n - 1)$ by

$$\tilde{\mathcal{C}}^\pm(n)_r = R\mathcal{B}_n \otimes_{R\mathcal{B}_{n-1}} 1$$

with differential $\partial^r : R\mathcal{B}_n \otimes_{R\mathcal{B}_{n-1}} 1 \rightarrow R\mathcal{B}_n \otimes_{R\mathcal{B}_{n-1}} 1$

$$\partial^r_j(\sigma \otimes 1) = \sigma s_{n-r+j,n-r} \otimes 1$$

$$\partial^r = \sum_{j=0}^{r} (-1)^j \partial^r_j$$

where $s_{a,b} = s_{a-1}s_{a-2} \ldots s_b$ for $s_i$ the generators of $\mathcal{S}_n \subseteq B_n$ as a Coxeter group.

A signed injective word of length $r+1$ on $[n]$ may be thought of as a record of the image of $n-r, n-r+1, \ldots, n$ under a signed permutation $\sigma$. But the image of $n-r, n-r+1, \ldots, n$ under $\sigma$ is determined uniquely by the coset of $\sigma B_{n-r-1}$, whence we obtain a bijection from the set of signed injective words to the cosets $B_n/B_{n-r-1}$. By standard properties of the group algebra, $R\mathcal{B}_n \otimes_{R\mathcal{B}_{n-1}} 1$ is the free $R$-module with basis $\sigma \otimes 1$ for $\{\sigma\}$ a set of coset representatives of $B_n/B_{n-1}$, so there is a natural identification $R\mathcal{B}_n/B_{n-r-1} \cong R\mathcal{B}_n \otimes_{R\mathcal{B}_{n-1}} 1$.

To understand the face maps, we may observe that the projection $B_n/B_{n-r-1} \rightarrow B_n/B_{n-r}$ is given by forgetting the image of $n-r$, which under the above bijection corresponds to discarding the leftmost letters of an injective word. If we apply the permutation $s_{n-r+j,n-r} = (n-r+j, n-r+j-1, \ldots, n-r)$ before the projection map, then the image of $n-r+j$ is instead forgotten, so the $j^{th}$ letter of the injective word is discarded, and all preceding letters shifted up one space. This precisely the effect of deleting the $j^{th}$ element.

The next proposition simply formalises this discussion.

Proposition 3.9. The chain complex $\tilde{\mathcal{C}}^\pm(n)$ is isomorphic to $\mathcal{C}^\pm(n)$.

Proof. In degree $r$, let $\phi : \tilde{\mathcal{C}}^\pm(n) \rightarrow \mathcal{C}^\pm(n)$ be the map:

$$\sigma \otimes 1 \mapsto \sigma(n-r, n-r+1, \ldots, n)$$

$\tilde{\mathcal{C}}^\pm(n)_r$ is free with basis $\sigma \otimes 1$ for $\{\sigma\}$ coset representatives of $B_n/B_{n-r-1}$, each of which is uniquely determined by the image of $(n-r, \ldots, n)$. $\mathcal{C}^\pm(n)$ has basis the set of signed injective words of length $r+1$ on $[n]$, thus $\phi_r$ is a bijection on basis elements, so is an isomorphism.
To see that \( \phi \) is a chain map, we calculate:
\[
\partial_r^j(\phi_r(\sigma \otimes 1)) = \partial_r^j(\sigma(n - r, n - r + 1, \ldots, n)) \\
= \partial_r^j(\sigma(n - r), \sigma(n - r + 1), \ldots, \sigma(n)) \\
= (\sigma(n - r), \ldots, \sigma(n - r + j), \ldots, \sigma(n)) \\
= (\sigma(s_{n-r+j,n-r}(n-r)), \ldots, \sigma((s_{n-r+j,n-r}n))) \\
= \phi_{r-1}(\partial_r^j(\sigma \otimes 1))
\]
so \( \phi \) commutes with the face maps, and hence the differentials. \( \square \)

### 3.2. The complex \( D^\pm(n) \)

In order to apply the Quillen spectral sequence argument to obtain a proof of Theorem A, we wish to construct a complex over the Iwahori-Hecke algebra \( HB_n \) that plays the same role as \( C^\pm(n) \) does for \( RHB_n \). The algebraic description of \( C^\pm(n) \) indicates precisely what this should be.

**Definition 3.10.** Let the complex \( D^\pm(n) \) be given in degree \((-1 \leq r \leq n - 1)\) by

\[
D^\pm(n)_r = HB_n \otimes_{HB_{n-r-1}} \mathbb{1}
\]

with differential \( \partial^r : RHB_n \otimes_{RB_{n-r-1}} \mathbb{1} \rightarrow RHB_n \otimes_{RB_{n-r}} \mathbb{1} \)

\[
\partial^r(x \otimes 1) = xT_{n-r+j,n-r} \otimes 1
\]

\[
\partial^r = \sum_{j=0}^r (-1)^j q^{-j} \partial^j
\]

(where \( T_{a,b} = T_{a-1}T_{a-2} \ldots T_b \) for \( T_i \) the generators of \( H_n \subseteq HB_n \))

**Remark 3.11.** The powers of \( q \) appearing in \( \partial^r \) are necessary for this to be a chain complex; recall that the generators of \( HB_n \) act on \( \mathbb{1} \) via multiplication by \( q \), which introduces powers of \( q \) to the computations.

Proposition 3.9 shows that this is the same as the complex of signed injective words for \( q = 1 \). \( D^\pm(n) \) is very similar to other constructions occurring in the literature around homological stability for diagram algebras (cf. the induced modules \( TL_m \otimes_{TL_m} \mathbb{1} \) in [BH20] and \( Br_n \otimes_{Br_n} \mathbb{1} \) in [BHP21]). In particular, the complex \( D(n) \) of [Hep20] (whence our naming and notation) is identical to \( D^\pm(n) \) with \( H_n \) in place of \( HB_n \). The fact that the differentials are well-defined and that \( \partial^r \circ \partial^r = 0 \) are Lemma 6.4 and 6.6 in [Hep20] (which demonstrates the necessity of the powers of \( q \) in \( \partial^r \)), we do not repeat them here.

To transport the proof of Theorem 3.3 to the Iwahori-Hecke setting, two things must be accomplished. First, an algebraic description of the filtration \( F_p \) must be found—this is completed in Sections 4 and 5. Secondly, the quotients \( F_p/F_{p-1} \) must be identified, and a result corresponding to Lemma 3.7 proved—this is Section 6. Some choices made in the proof of Lemma 3.7 become less natural when looked at from the algebraic perspective, so the results in Section 6 have a slightly different form. The overarching argument, however, is morally the same, and in particular the decomposition of \( F_p/F_{p-1} \) into subcomplexes relating to \( C(n-p) \) (resp. \( D(n-p) \) of [Hep20]) still holds.
4. Coset Representatives for $\mathfrak{S}_n \backslash B_n$

Observe that if $C^\pm(n)$ is identified with $R_{B_n} \otimes R_{B_n, r-1, 1} \cong R[B_n / B_n, r-1]$, then the position of negatives is determined by the double coset $\mathfrak{S}_n \backslash B_n / B_n, r-1$. Indeed, two words have negatives appearing in the same places if and only if there is an (unsigned) permutation relating one to the other. Thus in order to determine the algebraic analogue of our filtration in the previous section, it will be instructive to examine these double cosets, and in particular the cosets $\mathfrak{S}_n \backslash B_n$.

Recall that $X_{\mathfrak{S}_n}^{B_n}$ is the set of distinguished right coset representatives for $\mathfrak{S}_n$ in $B_n$. By the Mackey decomposition (Proposition 2.5) applied to $\mathfrak{S}_n, B_{n-1} \subseteq B_n$:

\[(1) \quad X_{\mathfrak{S}_n}^{B_n} = \bigcup_{d \in \mathfrak{S}_n \mathfrak{S}_{n-1}} d \cdot X_{\mathfrak{S}_n}^{B_{n-1}} \cap B_{n-1} \]

Hence we first find the distinguished double coset representatives $d$ for $\mathfrak{S}_n \backslash B_n / B_{n-1}$, then calculate $\mathfrak{S}_n^d \cap B_{n-1}$ (where the intersection refers only to that of generating sets) to reconstruct $X_{\mathfrak{S}_n}^{B_n}$. Recall from Section 2.4 that for $m \leq n$, $u_m = us_1s_2 \ldots s_{m-1} \in B_n$ (with $u_m = u$ when $m = 1$).

**Lemma 4.1.** The distinguished double coset representatives for $\mathfrak{S}_n \backslash B_n / B_{n-1}$ are

\[X_{\mathfrak{S}_n}^{B_n} = \{1, u_n\}\]

**Proof.** We first find the distinguished representatives for $B_n / B_{n-1}$. Consider the elements:

\[s_t s_{t+1} \ldots s_{n-1}, \quad s_{t-1} s_{t-2} \ldots s_1 u s_1 s_2 \ldots s_{n-1}\]

for $t = 1, 2, \ldots, n - 1$. All elements in these two families admit no M-moves, so all are reduced and furthermore have unique reduced form (Theorem 2.3). None end in an element in $B_{n-1}$, so all are $(\emptyset, B_{n-1})$-reduced, hence they are distinguished coset representatives for $B_n / B_{n-1}$. $|B_n / B_{n-1}| = 2n$, so these are all the distinguished coset representatives (they are distinct by considering the length).

The distinguished double coset representatives for $\mathfrak{S}_n \backslash B_n / B_{n-1}$ are precisely the representatives for $B_n / B_{n-1}$ that are $(\mathfrak{S}_n, \emptyset)$-reduced, i.e., do not begin with a generator of $\mathfrak{S}_n$. This is only 1 and $us_1 s_2 \ldots s_{n-1} = u_n$. \hfill \Box

Next we show that for $d = u_n$, the term $X_{\mathfrak{S}_n}^{B_{n-1}} \cap B_{n-1}$ in Equation (1) is equal to $X_{\mathfrak{S}_n}^{B_{n-1}}$.

**Lemma 4.2.** For $s_1, s_2, \ldots s_{n-1}$ the generators of $\mathfrak{S}_n$, $u, s_1, \ldots, s_{n-2}$ the generators of $B_{n-1}$:

\[\{s_1, s_2, \ldots, s_{n-1}\}^{u_n} \cap \{u, s_1, \ldots, s_{n-2}\} = \{s_1, \ldots, s_{n-2}\}\]

where $\{x\}^g$ denotes conjugation by $g$ with the positive $g$ appearing on the left.

**Proof.** Consider first $s_i$ with $i > 1$. $s_i$ commutes with $u$ and $s_j$ for $j < i - 1$, and $u^2 = s_j^2 = 1$, so:

\[s_i^{u_n} = s_{n-1} \ldots s_2 s_1 u \cdot s_i \cdot u s_1 s_2 \ldots s_{n-1}\]

\[= s_{n-1} \ldots s_i s_i-1 \cdot s_i \cdot s_i-1 s_i \ldots s_{n-1}\]
By the braid relation:
\[ \sigma_{i-1} = s_{n-1} \cdots s_i s^2_i s_{i-1} \cdots s_{i+1} \]
\[ = s_{n-1} \cdots s_i s_{i-1} s_{i+1} \cdots u \]
\[ \sigma_{i-1} \text{ commutes with all remaining terms:} \]
\[ = s_{i-1} \cdot s_{n-1} \cdots s^2_{i+1} \cdots u \]
\[ = \sigma_{i-1} \cdot s_{n-1} \cdots s^2_{i+1} \cdots u \]

Thus \( \{ s_2, \ldots, s_{n-1} \}^{u_n} \cap \{ u, s_1, \ldots, s_{n-2} \} \) remains only to show that \( s_1^{u_n} \neq u \). But the image of 1 under \( s_1^{u_n} = s_{n-1} \cdots s_2 s_1 u \cdot s_1 \cdot us_1 s_2 \cdots s_{n-1} \) is \( \square \)

Now Equation (1) provides an inductive description of \( X_{B_{n}}^{\mathcal{B}_{n}} \).

**Proposition 4.3.** Given \( t \geq 0, n \geq m_1 > m_2 > \ldots > m_t \geq 1 \), define:
\[ v(m) = v(m_1, m_2, \ldots, m_t) = u_{m_1} u_{m_2} \cdots u_{m_t} \]
\[ = us_1 s_2 \cdots s_{m_1-1} s_{m_2} s_2 \cdots s_{m_2-1} \cdots us_1 s_2 \cdots s_{m_t-1} \]
so that \( t = 0 \) forces the empty product \( v(m) = 1 \). Then the elements \( v(m) \) are the distinguished coset representatives for \( \mathfrak{S}_{n} \setminus \mathcal{B}_{n} \):
\[ X_{\mathcal{B}_{n}}^{\mathfrak{S}_{n}} = \{ v(m) : t \geq 0, n \geq m_1 > m_2 > \ldots > m_t \geq 1 \} \]

**Proof.** From Lemma 4.1, \( X_{\mathfrak{S}_{n} \setminus \mathcal{B}_{n-1}} = \{ 1, u_n \} \). For \( d = 1, u_n \), we have seen that \( X_{\mathfrak{S}_{n-1} \setminus \mathcal{B}_{n-1}}^{\mathcal{B}_{n}} = X_{\mathfrak{S}_{n-1} \setminus \mathcal{B}_{n-1}}^{\mathcal{B}_{n-1}} \). Thus by Equation (1) (the Mackey decomposition):
\[ X_{\mathfrak{S}_{n} \setminus \mathcal{B}_{n}}^{\mathcal{B}_{n}} = X_{\mathfrak{S}_{n-1} \setminus \mathcal{B}_{n-1}}^{\mathcal{B}_{n-1}} \bigcup u_n \cdot X_{\mathfrak{S}_{n-1} \setminus \mathcal{B}_{n-1}}^{\mathcal{B}_{n-1}} \]

By induction, we may assume the result for \( n - 1 \) (for \( n = 2 \) the base case follows from the coset reps being \( u, us_1 \)). Then:
\[ X_{\mathfrak{S}_{n} \setminus \mathcal{B}_{n}}^{\mathcal{B}_{n}} = \{ v(m) : n - 1 \geq m_1 > \ldots > m_t \geq 1 \} \bigcup u_n \{ v(m) : n - 1 \geq m_1 > \ldots > m_t \geq 1 \} \]
\[ = \{ v(m) : n \geq m_1 > \ldots > m_t \geq 1 \} \]

**Remark 4.4.** In this formulation, the distinguished coset representatives have been parameterised by decreasing sequences \( m_1 > m_2 > \ldots > m_t \) in \([n]\). There are \( 2^n \) such sequences, as expected: \( |\mathfrak{S}_{n} \setminus \mathcal{B}_{n}| = 2^n n!/n! = 2^n \).

The \( v(m) \) defined above play a central role in the remainder of this proof. It is not hard to observe that the signed injective word of length \( n \) corresponding to \( v(m) \) has negative letters at precisely the places \( m_i \) when counted from the left, so the algebraic analogue of \( F_p \) will be based on the value of \( m_1 \), the leftmost negative.

We first complete our description of the distinguished representatives for \( \mathfrak{S}_{n} \setminus \mathcal{B}_{n} / \mathcal{B}_{n-r-1} \), as discussed earlier.

**Proposition 4.5.** The distinguished representatives for \( \mathfrak{S}_{n} \setminus \mathcal{B}_{n} / \mathcal{B}_{n-r-1} \) are given by \( v(m) = v(m_1, \ldots, m_t) \) for \( n \geq m_1 > \ldots > m_t \geq n-r \). Equivalently, \( v(m + n - r - 1) \) for \( r + 1 \geq m_1 > m_2 > \ldots > m_t \geq 1 \).

**Proof.** Each such \( v(m) \) is \( (\mathfrak{S}_{n}, \emptyset) \) reduced by the previous proposition. When \( m_t \geq n-r \), \( v(m) \) admits no \( M \)-moves that send an element of \( \mathcal{B}_{n-r-1} \) to the right end of the word, so they are \( (\mathfrak{S}_{n}, \mathcal{B}_{n-r-1}) \) reduced.
Conversely, any distinguished double coset representative is \((\mathfrak{S}_n, B_{n-r-1})\) reduced, so in particular \((\mathfrak{S}_n, \emptyset)\) reduced, hence has the form of the previous proposition. If \(m_t < n-r, v\) is not \((\emptyset, B_{n-r-1})\) reduced, hence \(m_t \geq n-r\). \(\square\)

Each representative of \(\mathfrak{S}_n \setminus B_n / B_{n-r-1}\) corresponds to a certain pattern of signs for injective words of length \(r + 1\). This proposition says that the natural choice of representative for words of length \(r + 1\) with negatives at the places \(m_t\) from the left is \(v(m + n - r - 1)\).

Recall again that the algebraic analogue of signed injective words of length \(r + 1\) is given by \(R\mathfrak{B}_n \otimes_{R B_{n-r-1}} 1 \cong R[\mathfrak{B}_n / B_{n-r-1}]\). The filtration of \(C^k(n)_r\) was given by considering the orbits under the action of \(\mathfrak{S}_n\); it is thus natural to try to understand how \(R[\mathfrak{B}_n / B_{n-r-1}]\) splits under the action of \(R\mathfrak{S}_n\). Again, the Mackey decomposition (Proposition 2.5) is used.

**Lemma 4.6.** The right cosets of \(B_{n-r-1}\) in \(B_n\) have distinguished representatives given by

\[(X_{B_{n-r-1}})^{-1} = \bigcup_{n \geq m_1 > m_2 > \ldots > m_t \geq n-r} (X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}})^{-1} v(m_1, \ldots, m_t)\]

**Proof.** By Proposition 4.5, \(X_{\mathfrak{S}_n \otimes \mathfrak{C}_{n-r-1}}\) is \(\{v(m) : n \geq m_1 > m_2 > \ldots > m_t \geq n-r\}\). The Mackey decomposition states that:

\[(X_{B_{n-r-1}})^{-1} = \bigcup_{d \in X_{\mathfrak{S}_n \otimes \mathfrak{C}_{B_{n-r-1}}}^{-1}} (X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}})^{-1} d\]

It only remains to calculate \((X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}})^{-1}\). Given \(s_k \in B_{n-r-1}\) \((k < n-r-1)\), and \(u_m, m \geq n-r\), we can compute

\[u_{m_1} s_k = (u s_1 \ldots s_{m_1-1}) s_k = s_{k+1} (u s_1 \ldots s_{m_1-1} = s_{k+1} u_{m_1}\]

and thus \(u_{m_1} s_k u_{m_1}^{-1} = s_{k+1}\). Continuing in this fashion, after \(l\) steps we have \((u_{m_1} \ldots u_m) s_k (u_{m_1} \ldots u_m)^{-1} = s_{k+l}\) (and \(m_{l+1} \geq n-r+l > k+l+1\), so the computation in each step is the same). Thus for \(d = v(m_1, m_2, \ldots, m_t)\), \(d s_k = s_{k+t}\). As a signed permutation, \(d u \) sends \(t + 1\) to \(- (t+1)\), so is not \(s_j\) for any \(j\). Hence we have that \(X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}} = X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}}\), as required. \(\square\)

As discussed earlier, the element \(v(m)\) corresponds to a word with negative terms at the places \(m_t\) from the left (and is furthermore the most natural choice of representative for this sign pattern from the Coxeter group point of view). Any word can then be written as an element of \(\mathfrak{S}_n\) acting on a \(v(m)\). The above lemma makes precise to what extent this choice of \(\mathfrak{S}_n\) is unique, and provides natural choices for this element in each case (a distinguished coset representative in \((X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}})^{-1}\).

Finally, we wish to extend this description to the complex \(D^\pm(n)\). The previous lemma, along with the basis theorem (Theorem 2.8) for Iwahori-Hecke algebras, provides a convenient basis from which we will define our filtration.

**Proposition 4.7.** \((D^\pm(n))_r = \mathcal{H}B_n \otimes_{\mathcal{H}B_{n-r-1}} 1\) is a free \(R\) module with basis

\[\bigcup_{n \geq m_1 > m_2 > \ldots > m_t \geq n-r} \{T_x V(m) \otimes 1 : x \in (X_{\mathfrak{S}_n}^{s_1, \ldots, s_{n-r-2+t}})^{-1}\}\]

where

\[V(m) = T_v(m) = U T_1 \ldots T_{m_t-1} U T_1 \ldots T_{m_t-1}\]
Proof. The $v(m)$ are reduced by Proposition 4.3, so equality in the second statement holds by Proposition 2.7. For the first statement, by Proposition 2.9, $(D^\pm(n))_r$ is free with basis $\{T_x \otimes 1 : x \in (X^{B_n}_{-r-1})^{-1}\}$. By Lemma 4.6, this is precisely the set of $T_{x-v(m)} \otimes 1 = T_x V(m) \otimes 1$ (again by Proposition 2.7), for $x \in (X^{S_n}_{\{a_1+1, ..., a_n-r-2+1\}})^{-1}$. □

5. The algebraic filtration

The elements $V(m)$ in Proposition 4.7 are the analogue of the $v(m)$ for the Iwahori-Hecke setting, and so correspond to natural choices of representatives for a given pattern of signs in a signed injective word. We thus define a filtration of $D^\pm(n)$ in terms of the $H_n$ span of certain $V(m) \otimes 1$, which corresponds to the $S_n$ span of a pattern of signs (recall that $H_n \leq HB_n$ is the Iwahori-Hecke algebra of type $A_{n-1} \cong S_n$). Earlier we saw that the natural representative (for the $S_n$ action) of a word of length $r + 1$ with negatives at places $m_1, m_2, \ldots, m_t$ (counted from the left) is $v(m + n - r - 1)$; a word thus has all negatives in the first $p$ places if and only if it is in the $S_n$ orbit of $v(m + n - r - 1)$ for some $m$ with $m_1 \leq p$.

Definition 5.1. Let $1 \leq p \leq n$. Define $F_p$ to be the subcomplex given in degree $r$ by the $RH_n$ span of the elements $V(m + n - r - 1) \otimes 1$ where $m_{\geq} \leq \min\{r + 1, p\}$:

$$(F_p)_r = RH_n \cdot \{V(m + n - r - 1) \otimes 1 : m_{\geq} \geq m_1 > m_2 > \ldots > m_t \geq 1\}$$

so that $F_0$ is generated over $H_n$ by $1 \otimes 1$. $(F_p)_r$ is evidently a $H_n$-submodule, and it will soon be shown that $F_p$ is a $H_n$-subcomplex.

In the filtration of $C^\pm(n)$ the first level of the filtration was isomorphic to a complex (the complex of injective words, $C(n)$) known to be highly acyclic. The analogue of this for Iwahori-Hecke algebras is the complex $D(n)$ introduced in [Hep20] (Definition 6.3). There it is shown that:

Theorem 5.2 ([Hep20]). $H_d(D(n)) = 0$ for $d \leq n - 2$.

Accordingly, we now identify $F_0$ with $D(n)$ and thus deduce that it is highly acyclic.

Proposition 5.3. $F_0$ is a subcomplex isomorphic as a chain complex of $H_n$ modules to the complex $D(n)$. In particular, $H_d(F_0) = 0$ for $d \leq n - 2$.

Proof. From the definition of $F_p$, $F_0$ is given in degree $r$ by the $H_n$ span of $1 \otimes 1$ in $HB_n \otimes HB_{n-r-1}$, $1$. For $x \in H_n$, $\partial^{(j)}_p (x \otimes 1) = xT_{n-r+j,n-r} \otimes 1 = xT_{n-r+j,n-r} (1 \otimes 1) \in F_0$, so $F_0$ is indeed a $H_n$-subcomplex.

Consider the map $F_0 \rightarrow D(n)$ given in degree $r$ by sending $1 \otimes 1$ to $1 \otimes 1$ and extending by $H_n$-linearity. To see that this is well-defined, note that the domain is a submodule of $HB_n \otimes HB_{n-r-1}$, $1$, and the image lies in $H_n \otimes H_{n-r-1}$, $1$. If $\lambda \in H_{n-r-1} \cap H_n = H_{n-r-1}$, then $x\lambda \otimes y$ is sent to the same image as $x \otimes \lambda y$, as required. The chain maps on $F_0$ are the same as on $D(n)$, so it is evidently a map of chain complexes. It remains to show that it is an isomorphism. From Theorem 1, $(F_0)_r$ is free with basis $T_x \otimes 1 : x \in (X^{\{a_1+1, ..., a_n-r-2\}})^{-1}$. But the image of this is a basis of $H_n \otimes H_{n-r-2}$ by Proposition 2.9. □

We now show that $F_p$ is indeed a filtration. It will be convenient to first calculate the product of $U_m = UT_1T_2 \ldots T_{m-1}$ with certain $T_{a,b}$ (relating to the face maps in $D^\pm(n)$).
Lemma 5.4. For \( m \geq b, a \geq b \):

\[
U_m \cdot T_{a,b} = \begin{cases} 
T_{a+1,b+1}U_m & m > a > b \\
T_{a,b+1}U_{m+1} & a > m \geq b \\
U_m & a = b
\end{cases}
\]

Proof. If \( a = b \), \( T_{a,b} = 1 \) which gives the final case. We assume \( a > b \). First suppose \( m > a \):

\[
U_m T_{a,b} = U T_1 \cdots T_{m-1} \cdot T_a \cdots T_b
\]

\[
= U T_1 \cdots T_a T_{a-1} T_{a+1} \cdots T_{m-1} \cdot T_a \cdots T_b
\]

\[
= U T_1 \cdots T_a T_{a-1} T_{a+1} \cdots T_{m-1} \cdot T_a \cdots T_b
\]

so inductively \( U_m T_{a,b} = T_a T_{a-1} \cdots T_{b+1} U_m = T_{a+1,b+1}U_m \).

Now if \( m < a \), we may write \( T_{a,b} = T_a T_{a-1} T_{m+1} T_{m,b} \). All terms in \( T_{a,m+1} \) commute with \( U_m \), since they are \( T_k, k \geq m+1 \), and \( U_m \) involves \( U,T_k \) for \( k \leq m-1 \). \( U_m T_m = U_{m+1} \) by definition, so:

\[
U_m T_{a,b} = U_m T_{a,m+1} T_n T_{m,b} = T_{a,m+1} U_m T_m T_{m,b}
\]

\[
= T_{a,m+1} U_{m+1} T_m T_{m,b} = T_{a,m+1} U_{m+1} T_{m+1}
\]

where the second last equality comes from the first case. \( \square \)

Proposition 5.5. \( F_p \) is a subcomplex of \( D^+(n) \). Furthermore, \( \partial^r_j (F_p)_r \subseteq (F_{p-1})_r \) when \( j < p \).

Proof. The face maps are \( \mathcal{H}_n \)-linear, so it suffices to evaluate them on the generators \( \{ V(\mathbf{m} + n - 1 - r) \otimes 1 : \min\{r+1,p\} \geq m_1 > \ldots > m_t \geq 1 \} \) of \( (F_p)_r \). Suppose first that \( j \leq m_1 - 1 < p \). Then \( n - r + j \leq m_1 + n - 1 - r \), and \( n - r \leq m_t + n - r \), so \( V(\mathbf{m} + n - 1 - r)T_{n-r+j,n-r} \) is contained in the subalgebra \( \mathcal{H}_B_{m_1,n-1-r} \). Proposition 4.7 says that (in particular) \( \mathcal{H}_B_{m_1,n-1-r} \) is spanned over \( \mathcal{H}_{m_1,n-1-r} \) by \( \{ V(\mathbf{m}') : m_1' \leq m_1 + n - 1 - r \} \), so \( V(\mathbf{m} + n - 1 - r)T_{n-r+j,n-r} \) is in this span. If \( V(\mathbf{m}') \) ends in a factor \( U_{m_i'} \) with \( m_i' \leq n - r \), this factor may be moved over the tensor product in \( \mathcal{H}_B_n \otimes \mathcal{H}_B_{n-1} \), which results only in a number of multiplications by \( q \). Hence we see that

\[
\partial^r_j V(\mathbf{m} + n - 1 - r) \otimes 1 = V(\mathbf{m} + n - 1 - r)T_{n-r+j,n-r} \otimes 1
\]

lies in the \( \mathcal{H}_n \) span of \( \{ V(\mathbf{m}') \otimes 1 : m_1 + n - 1 - r \geq m_1' > \ldots > m_t' \geq n - r + 1 \} \), which is equal to the set \( \{ V(\mathbf{m'} + n - 1 - (r-1)) \otimes 1 : m_1 - 1 \geq m_1' > \ldots > m_t' \geq 1 \} \). By definition of \( F_p, m_1 \leq \min\{r+1,p\} \), so \( m_i' \leq \min\{r,p-1\} \) and hence \( \partial^r_j V(\mathbf{m} + n - 1 - r) \in (F_{p-1})_{r-1} \).

Now suppose \( j \geq m_1 \), so that \( n - r + j > m_i + n - r + 1 \) for all \( i \). Then we show by induction on \( t \) that:

\[
V(\mathbf{m} + n - 1 - r)T_{n-r+j,n-r} = T_{n-r+j,n-r+i} V(\mathbf{m} + n - 1 - (r-1))
\]

When \( t = 0 \), \( V(\mathbf{m} + n - 1 - r) = V(\mathbf{m} + n - 1 - (r-1)) = 0 \) and there is nothing to show. For \( t > 0 \), by the second case of Lemma 5.4 (which we may apply since \( n - r + j > m_i - n - r + 1 \geq n - 1 \)):

\[
U_{m_i+n-1-r} T_{n-r+j,n-r} = T_{n-r+j,n-r+i} U_{m_i+n-r}
\]
so then
\[ V(m + n - 1 - r)T_{n-r+j,n-r} = V(m_1 + n - 1 - r, \ldots, m_{t-1} + n - 1 - r)U_{m_1 + n - 1 - r}T_{n-r+j,n-r} \]
\[ = V(m_1 + n - 1 - r, \ldots, m_{t-1} + n - 1 - r)T_{n-r+j,n-r+1}U_{m_1 + n - r} \]

We may apply the inductive hypothesis since \( n - r + j > m_{t-1} + n - 1 - r \geq n - r + 1 \), giving:
\[ = T_{n-r+j,n-r+1}V(m_1 + n - r, \ldots, m_{t-1} + n - r)U_{m_1 + n - r} \]

which is 
\[ T_{n-r+j,n-r+1}V(m + n - 1 - (r - 1)) \]

Thus we have shown:
\[ \partial_r V(m + n - 1 - r) \otimes 1 = T_{n-r+j,n-r+1}V(m + n - 1 - (r - 1)) \otimes 1 \]

\( r \geq j \geq m_1 \), so \( m_1 \leq \min\{r, p\} \), and hence this element lies in \((F_p)_{r-1}\), completing the proof that \( F_p \) is a filtration. Also if \( p > j \geq m_1 \) then it is in \((F_{p-1})_{r-1}\), and consequently the latter statement holds.

\[ \square \]

**Lemma 5.6.** \( F_n \) is the entire complex \( D^\pm(n) \).

*Proof.* We have \( r < n \), so \( \min\{r + 1, n\} = r + 1 \). Then in degree \( r \), \( F_n \) is the \( H_n \otimes 1 \) span of \( V(m_1 + n - 1 - r, \ldots, m_t + n - 1 - r) \) for \( r + 1 \geq m_1 > \ldots > m_t \geq 1 \). If \( \text{m} = m + n - (r + 1) \), then this is \( V(\tilde{m}_1, \ldots, \tilde{m}_t) \otimes 1 \) for \( n \geq \tilde{m}_1 > \ldots > \tilde{m}_t \geq n - r \), which spans \( (D^\pm(n))_r \) over \( H_n \) by Proposition 4.7. \( \square \)

## 6. Identifying the filtration quotients

In this section, we identify the quotients \( F_p/F_{p-1} \) and show that they are highly acyclic. This proceeds in three steps: first a description of basis elements and the chain maps in \( F_p/F_{p-1} \) is obtained, and it is shown that it breaks up as a direct sum of subcomplexes determined by the \( V(\text{m}) \). This is exactly analogous to the decomposition in Lemma 3.7, where the quotient was the direct sum of \( 2^{p-1} \) subcomplexes determined by choice of signs for the first \( p-1 \) elements. Next, each subcomplex is shown to be isomorphic to the suspension of a new complex, \( M' \). Finally, \( M' \) is related to the complex \( D(n-p) \), whence high acyclicity is obtained.

**Lemma 6.1.** Fix \( m_1 > m_2 > \ldots > m_t \geq 0 \). For \( t < k < m_1 \):
\[ T_k V(\text{m}) = V(\text{m})T_{k-t} \]

*Proof.* We proceed via induction on \( t \). If \( t = 0 \), \( V(\text{m}) = 1 \), so there is nothing to show. For \( t > 1 \), the first case of Lemma 5.4 \( (T_k = T_{k+1}, k, m_1 > k) \) shows that \( T_k U_{m_1} = U_{m_1} T_{k-1} \). We may then apply the inductive hypothesis to \( T_{k-1} \) and \( V(m_2, \ldots, m_t) \), since \( k < m_1 \) implies \( k - 1 < m_2 \). Thus:
\[ T_k V(\text{m}) = T_k U_{m_1} V(m_2, \ldots, m_t) \]
\[ = U_{m_1} T_{k-1} V(m_2, \ldots, m_t) \]
\[ = U_{m_1} V(m_2, \ldots, m_t) T_{k-1} \]
\[ = V(\text{m})T_{k-t} \]
\[ \square \]

**Proposition 6.2.** For \( r \geq p - 1 \), \( (F_p/F_{p-1})_r \) is a free \( R \) module with basis
\[ \bigcup_{p = m_1 > m_2 > \ldots > m_t \geq 1} \{ T_r V(\text{m} + n - 1 - r) \otimes 1 : x \in (X_{\{s_{1}, \ldots, s_{n-r-2+t}\}})^{-1} \} \]
where \( t \geq 1 \), and is 0 for \( r < p - 1 \). Moreover, the \( R \) span of each basis subset \( \{ T_r V(\text{m} + n - 1 - r) \otimes 1 : x \in (X_{\{s_{1}, \ldots, s_{n-r-2+t}\}})^{-1} \} \) is equal to the \( H_n \) span of \( V(\text{m} + n - 1 - r) \otimes 1 \).
Proof: The first statement is immediate from Theorem 4.7 and the definition of the filtration. For the second, the inclusion of \( R \cdot \{ T_x V(m + n - 1 - r) \otimes 1 \} \) in \( \mathcal{H}_n \cdot V(m + n - 1 - r) \otimes 1 \) is clear.

In the other direction, by the basis theorem (Theorem 2.8) for Iwahori-Hecke algebras it is enough to verify that \( T_y V(m + n - 1 - r) \otimes 1 \) is contained in the \( R \) span of this set for each \( y \in \mathfrak{S}_n \). Each \( y \) may be written as \( y = xz \) for \( x \in (X^p_{s_1+1, \ldots, s_n-r-t})^{-1}, \ z \in (s_1+t, \ldots, s_n-r-t) \), with \( x, z \) reduced so that \( T_y = T_x T_z \). Given \( T_{k+t} \), with \( k < n - r - 1 \) and \( m_1 \geq t + 1 \), we have \( k + t < n - r - 1 + t < m_1 + n - 1 - r \), so by Lemma 6.1:

\[
T_{k+t} V(m + n - 1 - r) = V(m + n - 1 - r) T_k
\]

We are working in (a quotient of) \( \mathcal{H}_B \otimes \mathcal{H}_{B_{n-r-1}} \), so \( V(m + n - 1 - r) T_k \otimes 1 = q(U_{m_1+n-1-r} \otimes 1) \), since \( k < n - r - 1 \). The element \( T_z \) above may be written as the product of such \( T_{k+t} \), so we see that

\[
T_y, V(m + n - 1 - r) \otimes 1 = q^{(z)} T_y V(m + n - 1 - r) \otimes 1
\]
lies in the \( R \) span of \( \{ T_x V(m + n - 1 - r) \otimes 1 : x \in (X_{s_1+1, \ldots, s_n-r-t})^{-1} \} \). \( \square \)

To understand the quotients \( F_p/F_{p-1} \), it is necessary find a useful description of the differentials. Throughout it will be important to remember that \( p = m_1 > \ldots > m_t \geq 1 \) implies \( p \geq t \).

**Lemma 6.3.** The face maps \( \partial^r_j \) of \( F_p/F_{p-1} \) are \( \mathcal{H}_n \)-linear and have the following effect on the elements \( V(m + n - 1 - r) \), for \( p = m_1 > \ldots > m_t \geq 1 \):

\[
\partial^r_j V(m + n - 1 - r) = \begin{cases} 0 & j < p \\ T_{n-r+j, n-r+t} V(m + n - 1 - (r-1)) & j \geq p \end{cases}
\]

Proof. \( \mathcal{H}_n \) linearity is immediate. Proposition 5.5 states that \( \partial^r_j (F_p)_r \subseteq (F_{p-1})_{r-1} \) for \( j < p \), so that \( \partial^r_j \) is 0 in the quotient. The calculation for the second case occurs in the proof of Proposition 5.5. \( \square \)

In \( C^\pm(n) \), the filtration quotients decomposed as the sum of \( 2^{p-1} \) submodules. With the above result, it is clear that this still holds in the Iwahori-Hecke setting—there is a subcomplex for each \( m \) with \( m_1 = p \).

**Lemma 6.4.** \( F_p/F_{p-1} \) is the direct sum of \( 2^{p-1} \) \( \mathcal{H}_n \)-subcomplexes \( M_m \) (for \( p = m_1 > \ldots > m_t \geq 1 \)), where \( M_m \) is given in degree \( r \) by the \( \mathcal{H}_n \) span of \( V(m + n - 1 - r) \).

Proof. Proposition 6.2 shows that \( (F_p/F_{p-1})_r \) is the direct sum of the \( R \) spans of the sets \( \{ T_x V(m + n - 1 - r) \otimes 1 : x \in (X_{s_1+1, \ldots, s_n-r-t})^{-1} \} \), for \( p = m_1 > \ldots > m_t \geq 1 \), and that this is in fact the direct sum of the \( \mathcal{H}_n \) spans of the elements \( V(m + n - 1 - r) \). It remains to show that these are subcomplexes: but this is clear from Lemma 6.3. \( \square \)

While \( F_p/F_{p-1} \) has now been decomposed into the \( M_m \), these subcomplexes are not easily compared to \( D(n-p) \). In order to clarify the situation, we now introduce new complexes \( M^i \), each constructed to be isomorphic to a suspension of \( M_{m_1, \ldots, m_t} \).

For \( m < n - t \), let \( \mathcal{H}_m \) be the subalgebra of \( \mathcal{H}_n \leq \mathcal{H}_B \) generated by

\[
\langle T_{1+t}, T_{2+t}, \ldots, T_{m+t} \rangle
\]
so that \( H^t_m \) is a reindexing of \( H_m \) by \( +t \). In particular, it is a subalgebra corresponding to a parabolic subgroup of the Coxeter group \( B_n \).

Let \( M^t \) be given in degree \(-1 \leq r \leq n - p - 1\) by

\[
M^t_r = H_n \otimes_{H^t_{n-r-p-1}} 1
\]

\[
= H_n \otimes (T_{i+1}, \ldots, T_{n-r-p-2+t}) 1
\]

\[
\partial^r_j(x \otimes y) = xT_{n-r+j,n-p-r+t} \otimes y, \quad \partial^r_j = (-1)^p q^{-p} \sum_{j=0}^{r} (-1)^j q^{-j} \partial^r_j
\]

**Lemma 6.5.** Fix \( p = m_1 > m_2 > \ldots > m_t \geq 1 \). The map

\[
\Phi : \Sigma^p M^t \rightarrow M_{m_1, \ldots, m_t}
\]

given in degree \( p + r \ (\ -1 \leq r \leq n - p - 1 \) by \( x \otimes 1 \mapsto xV(m + n - 1 - (p + r)) \otimes 1 \) is well-defined, and an isomorphism for each \( r \).

**Proof.** To show that \( \Phi \) is well-defined in degree \( p + r \), we must check that for \( T_k \in \{T_{i+1}, \ldots, T_{n-r-p-2+t}\} \), \( \Phi(xT_k \otimes y) = \Phi(x \otimes T_k y) \). By Lemma 6.1, since we have that \( t < k < n - r - p - 1 + t \leq m_1 + n - 1 - (p + r) \), we obtain:

\[
\Phi(xT_k \otimes y) = xT_k V(m + n - 1 - (p + r)) \otimes y = xV(m + n - 1 - (p + r))T_{k-t} \otimes y = q(xV(m + n - 1 - (p + r)) \otimes y)
\]

which is equal to \( xV(m + n - 1 - (p + r)) \otimes T_k y = \Phi(x \otimes T_k y) \).

By Proposition 2.9, \( (\Sigma^p M^t)_{p+r} = H_n \otimes_{H^t_{n-r-p-1}} 1 \) is free over \( R \) with basis \( \{T_x : x \in (X_{s_1, \ldots, s_{n-r-p-2+t}}^{-1})\} \). This is sent under \( \Phi \) to \( \{T_x V(m + n - 1 - (p + r)) : x \in (X_{s_1, \ldots, s_{n-r-p-2+t}}^{-1})\} \), which is an \( R \) basis of \( M_{m_1, \ldots, m_t} \) in degree \( p + r \) by Theorem 2. Hence \( \Phi \) is an isomorphism in each degree.

**Proposition 6.6.** The map \( \Phi \) is an isomorphism of chain complexes.

**Proof.** We show the following diagram commutes. This will show that the differentials in \( M^t \) are well-defined, and that \( \Phi \) is an isomorphism of chain complexes.

\[
\begin{array}{ccc}
(\Sigma^p M^t)_{p+r} & \xrightarrow{\Phi_{p+r}} & (M_m)_{p+r} \\
\downarrow \partial^r_j & & \downarrow \partial^r_j^{p+r} \\
(\Sigma^p M^t)_{p+r-1} & \xrightarrow{\Phi_{p+r-1}} & (M_m)_{p+r-1}
\end{array}
\]

\( \Phi \) is \( H_n \) linear, so it suffices to consider \( 1 \otimes 1 \). In degree \( p + r \), going clockwise we have that for \( j < 0 \):

\[
1 \otimes 1 \mapsto V(m - n - 1 - (p + r)) \otimes 1
\]

\[
\mapsto \partial^r_j V(m - n - 1 - (p + r)) \otimes 1 = 0
\]

by Lemma 4. Going anticlockwise, \( \partial^r_j = 0 \) for \( j < 0 \), so the diagram commutes for \( j < 0 \). For \( 0 \leq j \leq r \), again going clockwise:

\[
1 \otimes 1 \mapsto V(m - n - 1 - (p + r)) \otimes 1
\]
\[ \mapsto \partial^{p+r}_j V(m - n - 1 - (p + r)) \otimes 1 \]
\[ = T_{n-r+j, n-p-r+t} V(m + n - 1 - (p + r - 1)) \]
where the evaluation of \( \partial^{p+r}_j \) comes from Lemma 6.3. In the other direction:
\[ 1 \otimes 1 \mapsto T_{n-r+j, n-p-r+t} \otimes 1 \]
\[ \mapsto T_{n-r+j, n-p-r+t} V(m + n - 1 - (p + r - 1)) \]
as required. Thus the face maps defined above for \( M^t \) correspond to the face maps in \( F_p/F_{p-1} \), so are well-defined. The differential in \( F_p/F_{p-1} \) then corresponds to
\[ \sum_{j=0}^{p-1} (-1)^j q^{-j} \cdot 0 + \sum_{j=p}^{p+r} (-1)^j q^{-j} \partial^{r}_j \]
\[ = (-1)^p q^{-p} \sum_{j=0}^{r} (-1)^j q^{-j} \partial^{r}_j \]
which is the \( \partial^r \) defined for \( M^t \)—thus this is a chain map, and \( \Phi \) is an isomorphism of chain complexes. \( \square \)

Putting everything thus far together:

**Proposition 6.7.** There is an isomorphism of \( H_n \)-modules
\[ F_p/F_{p-1} \cong \bigoplus_{t=1}^{p} \left( \bigoplus_{t-1}^{p-1} \right) \Sigma^p M^t \]

**Proof.** There are exactly \( \binom{p-1}{t-1} \) submodules of the form \( M_{m_1, \ldots, m_t} \), with \( p = m_1 > \ldots > m_t \geq 1 \) for fixed \( t \), with \( 1 \leq t \leq p \). By Lemma 6.4, \( F_p/F_{p-1} \) is the direct sum over all of these submodules. Then by Proposition 6.6, each \( M_{m_1, \ldots, m_t} \) is isomorphic to the suspension of \( M^t \).

Proposition 6.7 motivates the development up to this point: with this result, it remains only to prove that \( H_d(M^t) = 0 \) for \( d \geq n-p-2 \). Note that the leading factor of \( (-1)^p q^{-p} \) in the definition of \( \partial^r \) does not affect the homology of \( M^t \), since \( q \in R^\times \), so we omit it from now.

Referring back to Lemma 3.7, \( \Sigma^p M^t \) is playing the part of \( M_n \), and \( D_n \) that of \( C_n \). The naive isomorphism to look for, in imitation of 3.7, is then between \( M^t \) and \( H_n \otimes_{H_{n-p}} D(n-p) \). But in degree \( r \), \( M^t \) is the module \( H_n \otimes_{H_{n-r-1}} 1 \), whereas \( H_n \otimes_{H_{n-p}} D(n-p) \) is
\[ H_n \otimes_{H_{n-p}} (H_{n-p} \otimes_{H_{n-p-r-1}} 1) \cong H_n \otimes_{H_{n-p-r-1}} 1 \]

In the latter, the tensor product is over \( H_{n-p-r-1} \), and in the former over \( H_{n-p-r-1} \). We thus consider instead a reindexed version of \( D(n-p) \) to account for this difference.

Let \( D^t(n-p) \) be the complex given in degree \( r \) by
\[ D^t(n-p)_r = H^t_{n-p} \otimes_{H^t_{n-p-r-1}} 1 \]
\[ \partial^t_j (x \otimes y) = xT_{n-p-r+j+t, n-p-r+t} \otimes y, \quad \partial^r = \sum_{j=0}^{r} (-1)^j q^{-j} \partial^t_j \]

This is simply \( D(n-p) \) with everything reindexed by a factor of \( +t \), so evidently \( D^t(n-p) \cong D(n-p) \), and \( D^t(n-p) \) is an \( H^t_{n-p} \) module.
Instead of $\mathcal{H}_n \otimes_{\mathcal{H}_{n-p}} \mathcal{D}(n-p)$, consider now $\mathcal{H}_n \otimes_{\mathcal{H}_{n-p}'} \mathcal{D}'(n-p)$. In degree $r$, this is the module:

$$\mathcal{H}_n \otimes_{\mathcal{H}_{n-p}'} (\mathcal{H}_{n-p} \otimes_{\mathcal{H}_{n-p-r-1}} \mathbb{1}) \cong \mathcal{H}_n \otimes_{\mathcal{H}_{n-p-r-1}} \mathbb{1}$$

the same as $\mathcal{M}'$. Unfortunately, the face maps in $\mathcal{H}_n \otimes_{\mathcal{H}_{n-p}'} \mathcal{D}'(n-p)$ involve right multiplication by $T_{n-p-r+j+t,n-p-t}$, while those in $\mathcal{M}'$ involve right multiplication by $T_{n-r+j,n-p+t}$. If we can find a degree-wise isomorphism that accounts for this difference, then we will be done: we know that $\mathcal{D}'(n-p) \cong \mathcal{D}(n-p)$ is highly acyclic, and $\mathcal{H}_n$ is free over $\mathcal{H}_{n-p}'$, so $\mathcal{H}_n \otimes_{\mathcal{H}_{n-p}'} \mathcal{D}'(n-p)$ is also highly acyclic.

As we shall see, right multiplication by certain elements of $\mathcal{H}_n$ is a well-defined isomorphism $\mathcal{H}_n \otimes_{\mathcal{H}_{n-p-r-1}} \mathbb{1} \to \mathcal{H}_n \otimes_{\mathcal{H}_{n-p-r-1}} \mathbb{1}$. Therefore we look for a sequence of elements $\xi(r) \in \mathcal{H}_n$ such that right multiplication by $\xi(r)$ in degree $r$ accounts for the difference in face maps.

Let the elements $\xi(r) \in \mathcal{H}_n$ be:

$$\xi(r) = T_{n-r-1}T_{n-r} \cdots T_{n-1} \cdot T_{n-r-2}T_{n-r-1} \cdots T_{n-2} \cdots \cdot T_{n-r-(p-t)}T_{n-r-(p-t)+1} \cdots T_{n-(p-t)}$$

the product of $(r+1)(p-t)$ generators.

**Lemma 6.8.** For $\xi(r)$ defined above

$$T_{n-r+j,n-r-p+t} \xi(r-1) = \xi(r) T_{n-p-r+j+t,n-r-p+t}$$

**Proof.** By induction on $p-t$. For $p = t$, by definition $\xi(r) = 1$ for all $r$ and there is nothing to show.

Assume $p > t$ and consider the first $r$ elements of $\xi(r-1)$. Since $p-t \geq 1$ implies $n-r-1 \geq n-r+p+t$:

$$T_{n-r+j,n-r-p+t} = T_{n-r+j,n-r}T_{n-r-1,n-r-p+t}$$

and $T_{n-r-1,n-r-p+t}$ commutes with $T_{n-r-1,n-r} \cdots T_{n-1}$, so:

$$T_{n-r+j,n-r-p+t}(T_{n-r}T_{n-r+1} \cdots T_{n-1}) = T_{n-r+j,n-r}(T_{n-r-1}T_{n-r} \cdots T_{n-1})T_{n-r-1,n-r-p+t}$$

Applying the braid relations:

$$= (T_{n-r-1}T_{n-r} \cdots T_{n-1})T_{n-r+j-1,n-r}T_{n-r-1,n-r-p+t}$$

We have shown that:

$$T_{n-r+j,n-r-p+t} \xi(r-1) = (T_{n-r-1}T_{n-r} \cdots T_{n-1})T_{n-r+j-1,n-r-p+t} \cdot T_{n-r-1}T_{n-r} \cdots T_{n-2} \cdots \cdot T_{n-r+(p-t)+1}T_{n-r-(p-t)} \cdots T_{n-(p-t)}$$

and may now apply the inductive hypothesis with $n' = n-1$, $p' = p-1$ to obtain:

$$T_{n-r+j,n-r-p+t} \xi(r-1) = T_{n-r-1}T_{n-r} \cdots T_{n-1} \cdot T_{n-r-(p-t)}T_{n-r-(p-t)+1} \cdots T_{n-(p-t)}T_{n-(p-t)} \cdot T_{n-p+r+j,n-r-p+t}$$

as required. \_\_\_
We may finally complete the argument that multiplication by \( \xi(r) \) is a well-defined isomorphism accounting for the difference in face maps.

**Proposition 6.9.** There is an isomorphism \( \mathcal{M}^t \rightarrow \mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{D}^t(n - p) \) of chain complexes of \( \mathcal{H}_n \) modules.

**Proof.** In degree \( r \), as discussed \( \mathcal{M}^t = \mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{1} \) and \( \mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{D}^t(n - p) \) is \( \mathcal{H}_n \otimes \mathcal{H}^*_n (\mathcal{H}^t_{n - p} \otimes \mathcal{H}^*_n - p - r - 1) \approx \mathcal{H}_n \otimes \mathcal{H}^*_n - p - r - 1 \). We thus define our isomorphism \( \Psi \) in degree \( r \) by:

\[
\Psi_r : x \otimes y \mapsto x \xi(r) \otimes y
\]

To show that this is well-defined, both sides of this map are \( \mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{1} \) so it suffices to show that \( xT_k \otimes y \) and \( x \otimes T_k y \) have the same image when \( 1 + t \leq k \leq n - p - r - 2 + t \). The smallest index of \( T_1 \) occurring in \( \xi(r) \) is \( n - r - p + t \), so \( \xi(r) \) commutes with \( T_k \), and

\[
\Psi(xT_k \otimes y) = xT_k \xi(r) \otimes y = x \xi(r) T_k \otimes y = x \xi(r) \otimes T_k y = \Psi(x \otimes T_k y)
\]

as required.

Next, we show that \( \Psi \) is a chain map: this is true by construction. Explicitly

\[
\Psi_{r-1}(\partial^r_j(1 \otimes 1)) = \Psi_{r-1}(T_{n-r+j,n-r} \otimes 1)
\]

\[
= T_{n-r+j,n-r-p+t} \xi(r - 1) \otimes 1
\]

\[
= \xi(r) T_{n-r+j-p+t,n-r-p+t} \otimes 1
\]

\[
= \partial^r_j(\xi(r) \otimes 1) = \partial^r_j(\Psi(1 \otimes 1))
\]

(Both \( \Psi \) and the face maps are \( \mathcal{H}_n \) linear, so only \( 1 \otimes 1 \) need be considered.) Finally, \( \Psi_r \) is an isomorphism. From the identity \( q^{-1}(T_s + 1 - q)T_s = 1 \), the generators of any Iwahori-Hecke algebra lie in the group of units. Consequently, \( \xi(r) \) (a product of the generators) is also a unit in \( \mathcal{H}_n \), so has an inverse \( \xi(r)^{-1} \). Then the map from \( \mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{D}^t(n - p) \) to \( \mathcal{M}^t \) given by \( x \otimes y \mapsto x \xi(r)^{-1} \otimes y \) is well defined for the same reason as \( \Psi \) (the generators appearing in \( \xi(r)^{-1} \) will be the same as those in \( \xi(r) \)), is \( \mathcal{H}_n \)-linear, and is a two-sided inverse to \( \Psi \). \( \square \)

We may now conclude.

**Theorem 6.10.** \( H_d(\mathcal{D}^\pm(n)) = 0 \) for \( d \geq n - 2 \).

**Proof.** By Proposition 6.6 and the discussion following it, it suffices to show that \( H_d(\mathcal{M}^t) = 0 \) for \( d \leq n - p - 2 \).

By Theorem 5.2 in [Hep20], \( H_d(\mathcal{D}(n - p)) = 0 \) for \( d \leq n - p - 2 \). \( \mathcal{D}(n - p) \cong \mathcal{D}^t(n - p) \), and \( \mathcal{H}_n \) is free as a right \( \mathcal{D}^t(n - p) \) module by Proposition 2.9, so using Proposition 6.9,

\[
H_d(\mathcal{M}^t) \cong H_d(\mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{D}^t(n - p))
\]

\[
= \mathcal{H}_n \otimes \mathcal{H}^*_n \mathcal{D}^t(n - p)
\]

is 0 for \( d \leq n - p - 2 \). \( \square \)
7. Proof of Theorem A

It is now possible to complete the second half of the proof of Theorem A, using the complex $\mathcal{D}^\pm(n)$ and Theorem 6.10. The spectral sequence argument used here is standard in proofs of homological stability (see [Wah22]). Hepworth ([Hep20]) has adapted this spectral sequence argument for the setting of homological stability of algebras; we use his method with almost no modification in our case of type $\mathcal{B}_n$ algebras.

**Lemma 7.1.** There is a homological spectral sequence $\{E^r\}_{r \geq 1}$ with:

- $E^1_{p,q}$ concentrated in horizontal degrees $p \geq -1$
- $E^1_{p,q} = \text{Tor}^\mathcal{H\mathcal{B}_n}_{n-p-1}(\mathds{1}, \mathds{1})$
- $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ the stabilisation map for $p$ even and vanishing for $p$ odd
- $E^\infty_{p,q} = 0$ in total degrees $p + q \leq n - 2$

**Proof.** The construction of such a spectral sequence for algebra homology is the content of Section 9 in [Hep20]. The same argument works in this case, with $\mathcal{H\mathcal{B}_n}$ in place of $\mathcal{H}_n$ and $\mathcal{D}^\pm(n)$ playing the role of $\mathcal{D}(n)$. In particular, we note the following:

- $\mathcal{H\mathcal{B}_n}$ is free as a right $\mathcal{H\mathcal{B}_n}_{n-p-1}$-module by Proposition 2.9.
- The analysis of $\Xi^{-1}_* \circ d^1 \circ \Xi_*$ remains the same as in [Hep20], since the differentials in $\mathcal{D}^\pm(n)$ match those in $\mathcal{D}(n)$.
- $T_{n-p+j,n-p}$ commutes with $\mathcal{H\mathcal{B}_n}_{n-p-1}$. □

The argument now concludes by a concise application of $\{E^r\}$ and the stated properties. Recall that Theorem A states that

$$\text{Tor}^\mathcal{H\mathcal{B}_n}_{n-1}(\mathds{1}, \mathds{1}) \to \text{Tor}^\mathcal{H\mathcal{B}_n}_{n}(\mathds{1}, \mathds{1})$$

is an isomorphism for $2d \leq n - 1$.

**Proof of Theorem A.** By induction on $n$, noting that $n = 1, 2$ hold trivially. Suppose then $n \geq 3$. We wish to show that the stabilisation map is an isomorphism. By Lemma 7.1:

$$\text{Tor}^\mathcal{H\mathcal{B}_n}_{n-1}(\mathds{1}, \mathds{1}) = E^1_{-1,d} \quad \text{Tor}^\mathcal{H\mathcal{B}_n}_{n}(\mathds{1}, \mathds{1}) = E^1_{0,d}$$

and $d^1 : E^1_{0,d} \to \text{Tor}^\mathcal{H\mathcal{B}_n}_{n-1}$ is the stabilisation map. Thus the statement of the theorem is equivalent to $E^2_{-1,d} = E^2_{0,d} = 0$ for $2d \leq n - 1$.

For $u \geq 1$, consider the differential $E^1_{2u,q} \to E^1_{2u-1,q}$. This is the stabilisation map, so by the inductive hypothesis is an isomorphism for $2q \leq n - 2u - 1$. The differentials $E^1_{2u-1,q} \to E^1_{2u-2,q}$ are 0 by Lemma 7.1, hence $E^2_{2u,q} = E^2_{2u-1,q} = 0$ for $2q \leq n - 2u - 1$. From this it follows that for $r \geq 2$:

$$E^r_{p,q} = 0 \quad \text{in bidegrees} \ (p, q) \text{ with } p \geq 1, \ 2q \leq n - p - 2$$

Now consider which differentials may affect the terms in the columns $p = -1, 0$. For $p = -1$, the differential $d^r$ landing in bidegree $(-1, q)$ originates in $E^r_{-1+r,q-r+1}$. If $2q \leq n - 1$, then $E^r_{-1+r,q-r+1} = 0$ by the above—this implies that $E^2_{-1,q} = E^\infty_{-1,q}$ for $2q \leq n - 1$. Similarly, for $p = 0$ the differential $d^r$ landing in bidegree $(0, q)$ originates in $E^r_{r,q-r+1}$. If $2q \leq n - 1$, then $E^r_{r,q-r+1} = 0$, so $E^2_{0,q} = E^\infty_{0,q}$.

By Lemma 7.1, $E^\infty_{-1,q} = E^\infty_{0,q} = 0$ for $2q \leq n - 1$, since $n \geq 3$. Thus $E^2_{-1,q} = E^2_{0,q} = 0$ for $2q \leq n - 2$, as required. □
References

[BH20] Rachael Boyd and Richard Hepworth. On the homology of the Temperley-Lieb algebras. arXiv:2006.04256, 2020.
[BHP21] Rachael Boyd, Richard Hepworth, and Peter Patzt. The homology of the Brauer algebras. Selecta Mathematica, 27(5):1-31, 2021.
[Cox35] Harold SM Coxeter. The complete enumeration of finite groups of the form $R^2 = (R_i R_j)^k_{ij} = 1$. Journal of the London Mathematical Society, 1(1):21–25, 1935.
[Dav12] Michael Davis. The geometry and topology of Coxeter groups. (LMS-32). In The Geometry and Topology of Coxeter Groups. (LMS-32). Princeton University Press, 2012.
[Far78] Frank D Farmer. Cellular homology for posets. Math. Japon, 23(6):79, 1978.
[GP00] Meinolf Geck and Gotz Pfeiffer. Characters of finite Coxeter groups and Iwahori-Hecke algebras. Number 21. Oxford University Press, 2000.
[Hat95] Allen Hatcher. Homological stability for automorphism groups of free groups. Commentarii Mathematici Helvetici, 70(1):39–62, 1995.
[Hep16] Richard Hepworth. Homological stability for families of Coxeter groups. Algebraic & Geometric Topology, 16(5):2779–2811, 2016.
[Hep20] Richard Hepworth. Homological stability for Iwahori-Hecke algebras. arXiv:2006.04252, 2020.
[HW10] Allen Hatcher and Nathalie Wahl. Stabilization for mapping class groups of 3-manifolds. Duke Mathematical Journal, 155(2):205–269, 2010.
[Jon87] Vaughan FR Jones. Hecke algebra representations of braid groups and link polynomials. In New Developments in the Theory of Knots, pages 20–73. World Scientific, 1987.
[Ker05] Moritz C Kerz. The complex of words and Nakaoka stability. Homology, Homotopy and Applications, 7(1):77–85, 2005.
[KL79] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. Inventiones mathematicae, 53(2):165–184, 1979.
[Mat64] Hideya Matsumoto. Générateurs et relations des groupes de Weyl généralisés. Comptes rendus hebdomadaires des séances de l’Académie des sciences, 258(13):3419, 1964.
[Mat99] Andrew Mathas. Iwahori-Hecke algebras and Schur algebras of the symmetric group, volume 15. American Mathematical Society Providence, RI, 1999.
[Nak60] Minoru Nakaoka. Decomposition theorem for homology groups of symmetric groups. Annals of Mathematics, pages 16–42, 1960.
[RW21] Oscar Randal-Williams. A remark on the homology of Temperley-Lieb algebras. https://www.dpmms.cam.ac.uk/~or257/notes/Descent.pdf, 2021.
[Sro22] Robin J. Sroka. The homology of a Temperley-Lieb algebra on an odd number of strands. arXiv:2202.08799, 2022.
[Tit69] Jacques Tits. Le probleme des mots dans les groupes de Coxeter. In Symposia Mathematica, volume 1, pages 175–185, 1969.
[vdK80] Wilberd van der Kallen. Homology stability for linear groups. Inventiones mathematicae, 60(3):269–295, 1980.
[Wah22] Nathalie Wahl. Homological stability: a tool for computations. arXiv:2203.07767, 2022.

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