COMBINATORIAL REMARKS
ON A CLASSICAL THEOREM OF DELIGNE

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Abstract. We examine Deligne’s classical proof of the asphericity of simplicial arrangements from the viewpoint of the combinatorics of the poset of regions of the arrangement. This turns out to be very natural. In particular, we show that an arrangement is simplicial only if it satisfies Deligne’s property on positive paths, thus answering a question posed by Paris in [20].

1. Introduction

An arrangement of hyperplanes is a finite set of affine or linear codimension 1 subspaces of $\mathbb{C}^d$. The arrangement induces a stratification of the ambient space by its hyperplanes and their intersections. The poset of strata ordered by reverse inclusion is customarily perceived as the combinatorial data of the arrangement. A famous open question in arrangement theory is the so-called $K(\pi, 1)$-problem. An arrangement is said to be $K(\pi, 1)$ if its complement in $\mathbb{C}^d$ is aspherical. It is an open question whether being $K(\pi, 1)$ is a combinatorial property in general.

A real arrangement of hyperplanes is called simplicial if the maximal regions of the stratification it induces on real space (its chambers) are cones over simplices. In his seminal paper [10], Deligne proved that the complexification of a simplicial arrangement is $K(\pi, 1)$. Deligne’s proof consists essentially in two steps. Assuming simpliciality of the arrangement, he first derives a technical property of the category of directed paths on the arrangement graph (called ‘property D’ by Paris in [20]), and then uses this property to show contractibility of the universal cover of the complement.

Edelman introduced in [14] a partial ordering of the chambers of a real arrangement as a geometric generalization of the weak order on Coxeter groups. Since general arrangements are not symmetric, this ordering depends on a choice of a ‘base chamber’, and the orders associated to different base chambers can have quite different properties. The order-theoretic properties of these posets were studied (e.g. in [7, 15, 13]), and formalized in the general framework of oriented matroids (see [6, Chapter 4, 12]).

The weak order of Coxeter groups is an example of a combinatorial Garside structure. The construction of the complex associated in [1, 7] to any Garside structure can be generalized to complexified hyperplane arrangements and leads to the construction of ‘Garside-type’ combinatorial models for the covers of complexified arrangements (see [11, Chapter 6]). These models are tiled by copies of the order complexes of the posets of regions.

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Björner, Edelman and Ziegler [5] studied the structure of the orderings of the regions of some combinatorially defined classes of real arrangements. In particular, they show that a real arrangement is simplicial if and only if the ordering of its regions with respect to any base chamber is a lattice.

We adopt this combinatorial point of view on simplicial arrangements in examining Deligne’s proof. This turns out to be a very natural way of formulating the argument. We prove that an arrangement is simplicial only if it satisfies property D, thus showing that the first part of Deligne’s theorem is indeed an equivalence. This answers a question posed by Paris (see [20, p. 168]).

Moreover, we see that the language of posets allows a very compact proof of the contractibility of Garside-type models for the universal cover of the complement of a simplicial arrangement.

We will begin by laying down the combinatorial framework and introducing the main tools for our work in Section 2. We prove that Deligne’s first step is an equivalence in Section 3. Section 4 proves contractibility of the garside-type model of the universal cover starting from the formulation of property D in terms of posets. We close the paper with Section 5, a short appendix containing some considerations on a possible weakening of the simpliciality condition.

2. Combinatorics of real arrangements

2.1. Basics. Let $A := \{H_i\}_{i=1,...,n}$ denote an arrangement of linear hyperplanes in $\mathbb{R}^d$. The complement of $A$ in $\mathbb{R}^d$ is given by a set $C(A)$ of disjoint contractible components that we call chambers of $A$. We write $C(A)$ for the set of chambers of $A$. Choose a point $p_C$ in the interior of every chamber $C$. Given two chambers $C, C' \in C(A)$, we say that an hyperplane $H \in A$ separates $C$ from $C'$ if the segment joining $p_C$ with $p_{C'}$ intersects $H$. The set of hyperplanes separating $C$ from $C'$ will be denoted by $S(C, C')$. Two chambers $C, C' \in C(A)$ are said to be adjacent if there is only one hyperplane separating them. The hyperplanes that separate a chamber $C$ from its adjacent chambers are called walls of $C$. Since all hyperplanes are linear, the arrangement is centrally symmetric with respect to the origin. Thus, if $C \in C(A)$ then $-C \in C(A)$. Linearity of the hyperplanes implies also that every chamber is a cone with the origin as apex. The base space of this cone is a convex $d$-polyhedron. If this polyhedron is a simplex for every $C \in C(A)$, then $A$ is called simplicial.

The complexification of $A$ is the arrangement $A_C$ obtained by considering the defining forms for the $H_i$’s over $\mathbb{C}$. Let $M(A) := \mathbb{C}^d \setminus \bigcup A_C$ denote the complement of $A_C$.

2.2. Partially ordered sets. In our considerations we will use some terminology and facts about the combinatorics and topology of partially ordered sets (or, as we will say from now, posets) that we briefly recall. A more detailed introduction can be found e.g. in [22] [3].

Let $P$ be a finite poset. Two elements $p, q \in P$ are called comparable if either $p \geq q$ or $p < q$. Given $p \in P$ we define the subposets $P_{\leq p} := \{q \in P \mid q \leq p\}$, $P_{\geq p} := \{q \in P \mid q \geq p\}$, $P_{< p} := P_{\leq p} \setminus \{p\}$, $P_{\geq p} := P_{\geq p} \setminus \{p\}$. Any pair of comparable elements $p < q$ of $P$ determines an interval $[p, q] := P_{\geq p} \cap P_{\leq q}$. We say that $q$ covers $p$, and write $q > p$, if there is no element between $p$ and $q$, i.e., if $[p, q] = \{p, q\}$. A
subset of $P$ consisting of pairwise comparable elements is called a chain. The length of a chain is its cardinality. If every maximal chain of $P$ has the same cardinality, the poset is called graded and admits a rank function, i.e., a function $\rho : P \to \mathbb{Z}$ that is constant on minimal elements of $P$ and such that $\rho(p) = \rho(q) + 1$ whenever $p$ covers $q$. We say that $P$ is bounded if it possesses a maximal and a minimal element (that are usually denoted by $\hat{1}$ and $\hat{0}$, respectively). The poset $P$ is called a lattice if for every pair of elements $p, q \in P$ the posets $P_{\geq p} \cap P_{\geq q}$ and $P_{\leq p} \cap P_{\leq q}$ are nonempty and bounded. In this case $p \land q := \min(p_{\geq p} \cap P_{\geq q})$ is the unique minimal upper bound, called join, of $p$ and $q$. Similarly, $p \lor q := \max(P_{\geq p} \cap P_{\geq q})$ is the unique maximal lower bound, called meet, of $p$ and $q$.

The order complex of $P$, denoted by $\Delta(P)$, is the simplicial complex given by the chains of $P$. Note that we will make no explicit distinction between an abstract simplicial complex and its geometric realization. If $P$ has a maximal element $\hat{1}$, the order complex of $P$ is clearly a cone with apex $\{ \hat{1} \}$ over the space $\Delta(P_{<\hat{1}})$ and thus, in particular, contractible.

If $v$ is a vertex of a simplicial complex $K$, the star of $v$ is the subcomplex given by all simplices that contain $v$ and their boundaries. The link of $v$ is given by all simplices of the star of $v$ that do not have $v$ as a vertex.

2.3. The arrangement graph. Let $\Gamma(A)$ denote the simple graph on the vertex set $C(A)$ where two vertices are joined by an edge if and only if the corresponding chambers are adjacent. The arrangement graph $G(A)$ is an oriented graph with the same set of vertices (i.e., $C(A)$) and a pair of opposite oriented edges between every two adjacent chambers.

The arrangement graph can be realized geometrically as the 1-skeleton of the Salvetti complex, i.e., a CW complex that is homotopy equivalent to $M(A)$ (see [21]). Thus, paths on $G(A)$ correspond naturally to topological paths in $M(A)$. The way the Salvetti complex is constructed implies that any two directed paths of minimal length with the same beginning- and endpoint are homotopic. Therefore we may write $(C \to C')$ for the equivalence class of the paths that are directed from $C$ to $C'$ and have minimal length (called positive minimal paths), and abuse terminology by referring to it as to the positive minimal path from $C$ to $C'$. In fact, two paths are homotopic in $M(A)$ if and only if they are related by a sequence of substitutions of equivalent positive minimal paths. Paths will be denoted by greek lowercase letters, and composed by concatenation.

The quotient of the free category on $G(A)$ with respect to the relation generated by identifying any two positive minimal paths with the same begin- and endpoint is the category of positive paths $G^+(A)$. Completion of $G^+(A)$ gives the arrangement groupoid $G(A)$, which is clearly an instance of the fundamental groupoid of $M(A)$. For a precise account of this construction and its significance for the modeling of the arrangement covers, see [11] Chapters 2, 4, 6].

Remark 2.1. The objects and facts of this section were already present and proved in the seminal work by Pierre Deligne [10], where positive paths are called galeries. We choose to adopt the above viewpoint because of the convenience of the notation for our purposes.

2.4. The order of regions. We now define a partial ordering of the set of regions of a real hyperplane arrangement that was introduced by Edelman [14] (see also [17], [5] for further study of this object).
Definition 2.2. Let \( \mathcal{A} \) be a real arrangement of linear hyperplanes, and fix a base chamber \( C_0 \in \mathcal{C}(\mathcal{A}) \). We define the partial order \( \mathcal{P}_{C_0}(\mathcal{A}) \) with base chamber \( C_0 \) on the set \( \mathcal{C}(\mathcal{A}) \) by setting

\[
C_1 \leq_{C_0} C_2 \text{ if and only if } S(C_0, C_1) \subseteq S(C_0, C_2).
\]

One sees that the Hasse diagram of any \( \mathcal{P}_{C_0}(\mathcal{A}) \) is given by \( \Gamma(\mathcal{A}) \) after suitable choice of the “bottom vertex”. It is natural to ask about the order-theoretic properties of this poset. For terminology and basic definitions on posets, see [3]. First of all, from the above definition it is not hard to prove the following basic fact that we remark for later reference, pointing to [6, Corollary 4.2.11] for a proof.

Remark 2.3. Let \( \mathcal{A} \) be a real linear arrangement, let \( C, C_1, C_2 \in \mathcal{C}(\mathcal{A}) \) and suppose \( C_1 <_{C} C_2 \). Then the interval \([C_1, C_2] \subset \mathcal{P}_C\) is isomorphic to \( \mathcal{P}_{C_1}(\mathcal{A}) \leq_{C_2} \). Thus, the structure of an interval is the same in all poset of regions where the interval is defined.

It is clear that, for any \( C_0 \in \mathcal{C}(\mathcal{A}) \), the poset \( \mathcal{P}_{C_0} \) is bounded by \( C_0 \) and \( -C_0 \). Moreover, the cardinality of the sets \( S(C_0, C) \) is a rank function for \( \mathcal{P}_{C_0}(\mathcal{A}) \) by [14, Proposition 1.1]. In particular, the rank of \( \mathcal{P}_{C_0} \) equals the cardinality of \( \mathcal{A} \). Thus, the following Lemma gives a ‘local’ sufficient condition for \( \mathcal{P}_{C_0}(\mathcal{A}) \) to be a lattice.

Lemma 2.4 (Lemma 2.1 of [5]). Let \( P \) be a bounded poset of finite rank such that, for any \( p, q \in P \), if \( p \) and \( q \) both cover an element \( w \) then the join \( p \lor q \) exists. Then \( P \) is a lattice.

This lemma is one of the ingredients of the proof of the following characterization of simplicial arrangements in terms of their posets of regions.

Definition 2.5. Let \( \mathcal{A} \) be a real arrangement of linear hyperplanes and let \( \mathcal{C}(\mathcal{A}) \) denote the set of its chambers. We say that \( \mathcal{A} \) satisfies the strong lattice property if \( \mathcal{P}_{C_0} \) is a lattice for every \( C_0 \in \mathcal{C}(\mathcal{A}) \).

Theorem 2.6 (Theorem 3.1 and 2.4 of [5]). A real arrangement of linear hyperplanes \( \mathcal{A} \) is simplicial if and only if it satisfies the strong lattice property.

2.5. Topology of \( \mathcal{M}(\mathcal{A}) \). Since the complement of a hyperplane arrangement in complex space is always connected, asphericity of \( \mathcal{M}(\mathcal{A}) \) is equivalent to contractibility of the universal covering space. One possible way to approach the \( K(\pi, 1) \)-problem is therefore to construct combinatorially defined complexes that model the homotopy type of the universal cover of \( \mathcal{M}(\mathcal{A}) \). This was indeed the way taken by Deligne in [10]: he considered a model for the universal cover that was obtained by gluing together many copies of the unit ball of \( \mathbb{R}^d \) respecting the stratification given by the arrangement. Later on, Paris made this point of view more explicit and formulated Deligne’s argument using a complex that, in the case of a linear arrangement, lifted to the universal cover the simplicial structure of the Salvetti complex (see [20]). On the other hand, if \( \mathcal{A} \) is the reflection arrangement of a finite irreducible Coxeter group other complexes were studied, exploiting in different ways the symmetry of this situation (see e.g. [8, 9, 2]). We want to emphasize here the construction of Bestvina [2], that was later formulated in the more general context of Garside groups by Charney, Meyer and Whittlesey [7], who described a universal cover complex for Coxeter arrangements that is tiled by order complexes of the weak Bruhat order. This construction can be seen as a specially
symmetric case of the following complex, that models the homotopy type of any complexified arrangement of linear hyperplanes and can be obtained by appropriately gluing copies of the order complexes of all posets of regions associated to the arrangement, as was shown in \cite{11}.

**Definition-Theorem 2.1** (see Section 3.2 of \cite{11}). Let $A$ be a complexified arrangement of linear hyperplanes, and fix a chamber $C_0 \in C(A)$. We define a simplicial complex $U(A)$ which vertices are all morphisms of $G(A)$ that start at $C_0$ (i.e., all equivalence classes of paths on $G(A)$ that start at $C_0$). This simplicial complex is defined by declaring a set $\{\gamma_1, \ldots, \gamma_{d+1}\}$ of paths to be a simplex if and only if there are positive minimal paths $\alpha_1, \ldots, \alpha_d$ with $\gamma_{i+1} = \gamma_i \alpha_i$ for all $i = 1, \ldots, d$ and $\alpha_1 \ldots \alpha_d$ positive minimal.

The complex $U(A)$ is homotopy equivalent to the universal cover of the complement $M(A)(A)$ of the complexification of $A$.

2.6. **Oriented matroids.** We point out that this section can be phrased purely combinatorially in terms of the oriented matroid of the arrangement (by saying “element” instead of “hyperplane” and “tope” instead of “region”). Thus, everything can be defined for arbitrary arrangements of pseudospheres, though it is not clear what the topological meaning of the constructions would be. For a comprehensive introduction and a general reference to oriented matroids, see \cite{6}.

3. **Necessity of the Strong Lattice Condition**

The first part in Deligne’s proof of asphericity of simplicial arrangements is devoted to show that, if the arrangement $\mathcal{A}$ is simplicial, the morphisms of the positive category $G^+(\mathcal{A})$ (i.e., the positive paths) can be written in a particular normal form. Though it was recently referred to as ‘the Deligne normal form’ (see e.g. \cite{2, 7}), we introduce it by rephrasing in our language Definition of \cite{20}.

**Definition 3.1.** Let $\mathcal{A}$ a real arrangement of hyperplanes and fix $C \in C(\mathcal{A})$. The arrangement satisfies Property D if for every positive path $\gamma$ starting at $\tilde{C}$ there is a chamber $C_\gamma$ such that one can write $\gamma \sim \gamma'(C' \rightarrow C)$ for a positive path $\gamma'$ if and only if $C' < C_\gamma$ in $P_C$, where $C$ is the chamber in which $\gamma$ ends.

Paris asked in \cite{20} whether there are arrangements that satisfy property D but are not simplicial. We will answer this question negatively.

First of all, we remark that, in view of Theorem 2.6 the result of the first part of Deligne’s argument can be stated as follows.

**Theorem 3.2** (Equivalent to Theorem 1.19 (iii) of \cite{11}). If the arrangement $\mathcal{A}$ satisfies the Strong Lattice Property, then it satisfies property D.

Deligne’s proof starts with the assumption of simpliciality. Our remark is that, looking at it with today’s eyes, his argument has to spend quite a lot of work in deriving some technical properties that are immediate consequences of the lattice structure of the $P_V$s. In fact, the proof can be written entirely in terms of posets of regions (see \cite{11}). From this combinatorial point of view we can answer Paris’ question as follows.
Theorem 3.3. If the real arrangement $\mathcal{A}$ satisfies property D then it satisfies the Strong Lattice Condition and is therefore simplicial.

Proof: We will argue by contraposition. Suppose that $\mathcal{A}$ does not satisfy the Strong Lattice Condition, i.e., that there is a chamber $C_0$ such that $\mathcal{P}_{C_0}$ is not a lattice. By Lemma 2.4, this is only possible if there are chambers $A, B, C$ such that $A, B$ cover $C$ in the poset $\mathcal{P}_{C_0}$ and the join $A \lor B$ does not exist in $\mathcal{P}_{C_0}$. Since the interval $[C, -C_0] = (\mathcal{P}_{C_0})_{\geq C}$ in $\mathcal{P}_{C_0}$ is isomorphic to the interval $[C, -C_0] = (\mathcal{P}_{C})_{\leq C_0}$ in $\mathcal{P}_C$ (see Remark 2.3), we may from now on consider the situation in the latter poset, that is therefore also not a lattice. In particular, the chamber $C$ cannot be simplicial (Theorem 3.1 of [5]). Still, the following lemma tells us something about the structure of $\mathcal{P}_C$ ‘near the bottom’.

Lemma 3.4 (Lemma 4.4.4 of [6], “realizable version”). Let $A, B, C, K$ be chambers of $\mathcal{A}$, and suppose that $A$ and $B$ are atoms in the interval $[C, K]$ of $\mathcal{P}_C(A)$. Then there exists a sequence of atoms $A = A_0, A_1, \ldots, A_k = B$ and a sequence of other elements $T_1, T_2, \ldots, T_k$ in $[C, K]$ such that $[C, T_i]$ is elementary and contains $A_i$ and $A_{i+1}$, for all $1 \leq i \leq k$. If the chamber $C$ is simplicial, then $k = 1$. (See Figure 1 (a))

In our setting, since the join of $A$ and $B$ does not exist, we have that, for any $K$, the minimal possible associated $k$ is at least 2. Let us choose an element $M$ among the minimal upper bounds of $A$ and $B$. We have the following situation (Figure 1 (b)): atoms $A = A_0, A_1, \ldots, A_k = B$ with the associated $T_1, T_2, \ldots, T_k$, and all
those elements are in the interval \([C, M]\). We will denote \(H_i\) the unique element of \(S(C, A_i)\).

Consider
\[
\gamma := (M \rightarrow C)(C \rightarrow A_1).
\]
Clearly, \(\gamma\) ends by both the positive paths \(A \rightarrow C \rightarrow A_1\) and \(B \rightarrow C \rightarrow A_1\). Let us show that there is no chamber \(K\) such that \(\gamma\) ends with \(K \rightarrow A_1\) and
\[
S(A_1, K) \supset S(A, A_1) \cup S(B, A_1) = \{H_0, H_1, H_k\}.
\]
Since lower intervals in lattices are closed under join, this shows that \(A\) does not satisfy property D and will therefore conclude the proof.

Indeed, for such \(K\) we would have \(\{H_0, H_k\} \subset S(C, K)\) (because neither \(H_0\) nor \(H_k\) separate \(A_1\) from \(C\)), therefore both \(A\) and \(B\) are atoms in the interval \([C, K]\).

Now, since \(H_1\) is a wall of both \(A_1\) and \(C\), we have \((K \rightarrow A_1) = (K \rightarrow C)(C \rightarrow A_1)\). Therefore, if such a \(K\) would exist, then \(\gamma\) would consist of a positive path from \(M\) to \(K\) (thus crossing \(H'\)) followed by \((K \rightarrow A_1)\) (that crosses \(H'\)).

But, by definition, \(\gamma\) does not cross \(H'\) since this hyperplane does not separate \(M\) from \(C\). This gives a contradiction: equivalent positive paths cross the same number of times every hyperplane (intuitively: \(M \rightarrow K \rightarrow A_1\) ‘turns around’ \(H'\), while \(\gamma\) does not). □

4. Contractibility of the universal cover

Let us now turn to the second part of Deligne’s proof of asphericity of simplicial arrangements, where property D is used to show contractibility of the universal cover. We want to phrase also this step in combinatorial terms, using our complex \(\mathcal{U}(\mathcal{A})\) (recall Definition 2.1).

In analogy with Deligne’s argument we have the following basic observation, that is now standard.

**Lemma 4.1** (Proposition 6.4.4 of [11]. See Proposition 2.14 of [10].) Let \(\mathcal{A}\) be an arrangement of linear hyperplanes in \(\mathbb{R}^d\). Then \(\mathcal{U}(\mathcal{A})\) is contractible if the subcomplex \(\mathcal{U}^+\) given by the vertices that correspond to positive paths is contractible.

Now, to obtain the result it suffices to prove the following statement. We will do this by using

**Theorem 4.2.** Let \(\mathcal{A}\) be an arrangement of linear hyperplanes in \(\mathbb{R}^d\). If \(\mathcal{A}\) satisfies property D, then \(\mathcal{U}^+\) is contractible.

**Proof:** Let \(\mathcal{U}^+_m\) denote the subcomplex of \(\mathcal{U}^+\) given by the vertices that correspond to paths of edge-length at most \(m\). We will show that, for any \(m > 0\), \(\mathcal{U}^+_m\) retracts onto \(\mathcal{U}^+_{m-1}\).
Indeed, let $\gamma$ represent a vertex of $U^+_m \setminus U^+_{m-1}$: it is a positive path of length $m$ that ends, say, in the chamber $C$. Its link in $U^+_m$ is spanned by all vertices indexed by positive paths $\gamma'$ such that
\[\gamma \sim \gamma'(C' \to C) \text{ and } C' \neq C,\]
where $C'$ denotes the chamber in which $\gamma'$ ends. Property D tells us that there is $C_\gamma \in C(A)$ such that $P_{C_\gamma}(A) = \Delta(I_{\gamma})$, which is contractible because $C_\gamma$ is a maximal element for $I_{\gamma}$.

Thus, the link of $\gamma$ is a contractible subcomplex of $U^+_{m-1}$, and we can then retract the star of $\gamma$ to it. Note that this process did not involve any other vertex of $U^+_m \setminus U^+_{m-1}$ and can be therefore be carried out successively for all vertices that correspond to paths of length $m$. Concatenation of the resulting retractions gives an explicit global retraction of $U^+_m$ onto $U^+_{m-1}$. $\square$

5. Appendix: the Weak Lattice Property

Speaking about the order of regions as related to asphericity of arrangements, it can not be omitted to mention a suggestive fact, obtained by collecting results of [5, 18].

**Fact 5.1.** Let $A$ be a linear arrangement of real hyperplanes. If $A$ is simplicial, supersolvable or hyperfactored then there is $C_0 \in C(A)$ such that $P_{C_0}(A)$ is a lattice.

We say that these classes of arrangements satisfy the ‘Weak Lattice Property’.

For background and definitions we refer to [5, 18, 19]. Here we may only recall that simplicial and supersolvable arrangements are the two combinatorially defined classes of arrangements that are up to now known to be $K(\pi, 1)$. Hyperfactored arrangements are a generalization of supersolvable arrangements with which the $K(\pi, 1)$ property was never refuted.

It is clear that the Weak Lattice Property does not imply asphericity of arrangements (the arrangement $A_{-2}$ of Edelman and Reiner [16] is not aspherical, but satisfies the Weak Lattice Property by [5 Theorem 3.2]). Nevertheless, it would be interesting to investigate the significance of the structure of the poset of regions for the asphericity of complexified arrangements. Some partial results in this sense can be found in [11]. We plan to expand on it in future work.

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