NS$^5$–branes in IIA supergravity and gravitational anomalies

M. Cariglia$^{*\dagger}$ and K. Lechner$^{\dagger}$

*DAMTP, Centre for Mathematical Sciences, Cambridge University
Wilberforce Road, Cambridge CB3 OWA, UK

$^{\dagger}$Dipartimento di Fisica, Università degli Studi di Padova,
and
Istituto Nazionale di Fisica Nucleare, Sezione di Padova,
Via F. Marzolo, 8, 35131 Padova, Italia

Abstract

We construct a gravitational–anomaly–free effective action for the coupled system of IIA $D = 10$ dynamical supergravity interacting with an NS$^5$–brane. The NS$^5$–brane is considered as elementary in that the associated current is a $\delta$–function supported on its worldvolume. Our approach is based on a Chern–kernel which encodes the singularities of the three–form field strength near the brane in an $SO(4)$–invariant way and provides a solution for its Bianchi identity in terms of a two–form potential. A dimensional reduction of the recently constructed anomaly–free effective action for an elementary $M^5$–brane in $D = 11$ is seen to reproduce our ten–dimensional action. The Chern–kernel approach provides in particular a concrete realization of the anomaly cancellation mechanism envisaged by Witten.

PACS: 11.10.Kk, 11.30.Cp; Keywords: 5–branes, Chern–kernels, anomalies.
1 Introduction

It is a well known fact that IIA string theory admits 1–branes and 5–branes as elementary excitations, and that its low energy limit is given by IIA supergravity. The bosonic sector of IIA supergravity is given by the metric $g_{\mu \nu}$, the dilaton $\Phi$, a one–form $A_1$, a two–form $A_2$ and a three–form $A_3$, while the bosonic fields on the $NS$–5 brane are the coordinates $x^\mu(\sigma)$, a scalar $a_0(\sigma)$ and a chiral two–form $a_2(\sigma)$. The chiral fields on the 5–brane lead to gravitational anomalies whose polynomial has been calculated in [1]; it is given by

$$2\pi \left( X_8 + \frac{1}{24}\chi_4 \chi_4 \right), \quad (1.1)$$

where $X_8 = \frac{1}{192(2\pi)^8}(trR^4 - \frac{1}{4}(trR^2)^2)$ is the $SO(1,9)$ target space anomaly polynomial and $\chi_4$ is the Euler characteristic of the $SO(4)$–normal bundle of the brane. We will use the following notation for descent equations: $\chi_4 = d\chi_3$, $\delta\chi_3 = d\chi_2$, and similarly for all other polynomials.

While the target space anomaly can be cancelled through a standard Green–Schwarz mechanism the normal anomaly requires an inflow mechanism, relying on the basic equation

$$dH_3 = gJ_4, \quad (1.2)$$

where $H_3$ is the curvature associated to $A_2$, $g$ the charge of the brane and the four–form $J_4$ is the current associated to the elementary brane, i.e. a Dirac $\delta$–function with support on the brane. For $g = 0$ one recovers pure ten–dimensional bosonic IIA supergravity and eq. (1.2) allows to introduce a potential $A_2$ according to $H_3 = dA_2$. When $g \neq 0$ instead, the introduction of a potential becomes problematic because the r.h.s. of (1.2) is different from zero. On the other hand, a consistent definition of a potential for $H_3$ becomes of fundamental importance if one wants to write an action and, consequently, if one considers the issue of anomaly cancellation by some sort of inflow mechanism.

Usually the system IIA supergravity + $NS$–5–brane has only been considered in the framework of the $\sigma$–model, where target space fields act as sources for the fields living on the brane (direct coupling), but there is no influence of the brane on target space fields (back coupling): they satisfy the equations of motion of pure supergravity. It is clear that the modified Bianchi identity (1.2) can not be the unique back coupling equation, because it induces also modifications to the Bianchi identities and equations of motion for the other bosonic supergravity fields; principal aim of the present paper is to present these modified equations and to write down the corresponding action in compatibility with anomaly cancellation. This requires in particular to solve the problem of a potential $A_2$ for $g \neq 0$.

The issue of anomaly cancellation for IIA $NS5$–branes has been addressed in two related ways up to now. The author of [1] considers an “almost $\delta$–like” source by arguing that the $NS5$–brane $\delta$–like current $J_4$ should be replaced by a cohomologically equivalent...
closed form \( \tilde{J}_4 \) with support in a small neighborhood of the brane. In order to perform the pullback on the 5–brane worldvolume \( M_6 \) of equation (1.2) he exploits the property that there exists a regular cohomological representative of \( J_4 \) such that its pullback is \( \chi_4 \),

\[
\tilde{J}_4^{(0)} = \chi_4, \tag{1.3}
\]
where with the subscript \((0)\) we indicate the pullback of a form on \( M_6 \). Then he argues that \( \chi_4 \) should be considered as the “finite part” of the \( \delta \)-function in \( J_4 \), and that (1.3) “should be taken as part of the definition of the brane”. This leads in particular to the pullback equation

\[
dH_3^{(0)} = g\chi_4. \tag{1.4}
\]
He proposes then the following counterterm in the action to cancel the normal bundle anomaly

\[
\frac{\pi}{12g} \int_{M_6} H_3^{(0)} \chi_3. \tag{1.5}
\]
Unfortunately for an elementary brane, with a \( \delta \)-like support, this counterterm is ill–defined because then \( J_4 \) as well as \( H_3 \) do not admit pullbacks on \( M_6 \). In particular, for its variation under \( SO(4) \) one would formally obtain

\[
\delta \int_{M_6} H_3^{(0)} \chi_3 = - \int_{M_6} dH_3^{(0)} \chi_2 = - g \int_{M_6} J_4^{(0)} \chi_2 = - g \int_{M_6} J_4 J_4 \chi_2,
\]
but, since locally we have \( J_4 = d^4 u \delta^4(u) \) where the \( u^r \) are normal coordinates, the product \( J_4 J_4 \) is meaningless, as is the above counterterm. To obtain the desired anomaly cancelling term one has to enforce, once more, a cohomological relation: \( J_4 J_4 \sim J_4 \chi_4 \) \[2, 3\]. Notice however that in the framework of \[2\] using (1.4) one can introduce a pullback potential according to \( H_3^{(0)} = dA_2^{(0)} + g\chi_3 \) and rewrite (1.5) in the form

\[
\frac{\pi}{12g} \int_{M_6} A_2^{(0)} \chi_4,
\]
which cancels still the normal bundle anomaly if one enforces the transformation law

\[
\delta A_2^{(0)} = - g\chi_2. \tag{1.6}
\]
This second form of the counterterm may be well defined even for an elementary brane, but in this case one has to face the problem of how to introduce a target space form \( A_2 \) which gives rise to (1.6) in a consistent way; in particular, for continuity reasons, one must now have also a non trivial target space transformation law: \( \delta A_2 \neq 0 \). This is one of the main problems solved in the present paper.

The aim of the paper \[4\] instead was a derivation of the counterterm (1.5) from the eleven–dimensional M5–brane effective action proposed in \[5\]. Since the eleven–dimensional 5–brane current \( \tilde{J}_5 \) used in this paper was a smooth one, the dimensional reduction led again to a smooth current \( \tilde{J}_4 \) and, once more, to (1.3).
The main reason why we insist on a $\delta$–like current stems from electric–magnetic duality: if one wants that 1–branes and 5–branes can consistently coexist one has to impose not only a Dirac quantization condition for their charges but also to demand that both their currents carry a $\delta$–like support; this is explained, for example, in [1]. Moreover, solitonic solutions of eleven– or ten–dimensional supergravity lead to $\delta$–like supports even for solitonic 5–branes [2]. The requirement of existence of an effective action for an elementary 5–brane is also in line with the fact that the derivation of its gravitational anomaly (1.1) can be based on a local $\sigma$–model action where, by definition, the current is strictly a $\delta$–function [3].

With these motivations in mind we present in this paper an action describing the interaction between $IIA$ supergravity and an $NS5$–brane, which copes consistently with equation (1.2) where $J_4$ is considered as a $\delta$–function on the 5–brane worldvolume, see below for a precise definition. As anticipated above the first problem one has to solve is the introduction of a two–form potential $A_2$. Since $J_4$ is a closed form the first step demands to find a three–form $K_3$ such that

$$dK_3 = J_4,$$

because this would then allow to define $A_2$ according to

$$H_3 = dA_2 + gK_3.$$

Since we want $A_2$ to be a regular field near the 5–brane, i.e. the pullback $A_2^{(0)}$ has to be well defined, the singularities which are necessarily present in $H_3$ have to be carried entirely by $K_3$. We will see that this implies that $K_3$ decomposes into a sum

$$K_3 = \chi_3 + \omega_3,$$

where $\chi_3$ represents a target–space form whose pullback on $M_6$ is regular and coincides with the Chern–Simons form associated to the Euler form. On the other hand, the three–form $\omega_3$ – a Chern–kernel – exhibits a singular and invariant behaviour near the 5–brane. Our construction of the effective action will be based on this decomposition of $K_3$.

As a check of our result we perform a dimensional reduction of the effective action for an elementary $\delta$–like $M5$–brane in $d = 11$, which has been recently constructed in [4], and which is based on a Chern–kernel, too. This action cancels the normal $SO(5)$–bundle anomaly and relies on an invariant three–form potential $A_3$, while the ten–dimensional two–form $A_2$ carries an anomalous transformation law; so the anomaly cancellation mechanisms look rather different in ten and eleven dimensions. An interesting aspect of the reduction regards the way in which the eleven–dimensional mechanism gives rise to the ten–dimensional one. Eventually the reduced action agrees with the ten–dimensional one, we have constructed independently.

An alternative approach for the solution of (1.2) is based on Dirac–branes, i.e. unphysical surfaces whose boundary is the 5–brane. Although being a standard approach,
in the present case it is appropriate for what concerns the equations of motion but it fails
at the level of the action, due the presence of the anomaly cancelling counterterm; we
explain the reason for this failure in section 3. In some sense the Chern–kernel approach
represents a refinement of the Dirac–brane one.

In order to consider the 5–brane as a $\delta$–like source one has necessarily to work with
distribution valued $p$–forms, so called “$p$–currents” [10, 11]. Differentials of forms will
always be differentials in the sense of distributions [1]. We will always suppose to deal with
topologically trivial spaces, unless otherwise stated, so a closed $p$–form is the differential
of a $(p – 1)$–form.

The rest of the paper is organized as follows: in section 2 we give a brief selfcontained
account of the eleven–dimensional effective $M5$–brane action, in section 3 we construct
the effective action for IIA dynamical bosonic supergravity interacting with a bosonic
$NS5$–brane and eventually, in section 4, we consider the issue of dimensional reduction
of the eleven–dimensional theory. Section 5 is devoted to concluding remarks.

2 The $M5$–brane in eleven dimensions

We present here the basic ingredients of the effective action for an elementary $M5$–brane,
constructed recently in [9]; for more details we refer the reader to this reference.

2.1 Poincarè–duality and push–forward

In the presence of a brane one can define differential forms which live on the brane and
differential forms which live on the target space. The pullback operation associates an
$n$–form on the brane worldvolume to an $n$–form on the target space; the push–forward
operation, instead, associates a form (properly speaking, a current) on the target space
to a form on the worldvolume. Since we will use this operation frequently we give here
its definition and state its basic properties [10, 12]. We begin by recalling the definition
of the Poincarè–dual of a $(D – p)$–manifold, open or closed, with worldvolume $M_{D−p}$: it
is the $p$–current $J_p$ – the $\delta$–function on $M_{D−p}$ – defined through

$$\int_{R^D} \Phi_{D−p} J_p = \int_{M_{D−p}} \Phi^{(0)}_{D−p}$$

for every smooth target space $(D – p)$–form, where $\Phi^{(0)}_{D−p}$ indicates its pullback.

Given an $n$–form $h_n$ on $M_{D−p}$ instead, we can define its push–forward to an $(n + p)$–
form on target space, which we indicate with “$h_n J_p$”. It is defined through

$$\int_{R^D} \Phi_{D−p−n} (h_n J_p) = \int_{M_{D−p}} \Phi^{(0)}_{D−p−n} h_n,$$  \hspace{1cm} (2.1)$$

3In our notation, when Leibnitz’ rule is valid, the differential will act from the right.
again for every target space form $\Phi_{D-p-n}$. Notice, however, that the product notation $h_nJ_p$ is formal because this target space form can not really be factorized. An explicit component expression, following from (2.1), is indeed

$$h_nJ_p = \frac{1}{(n+p)!(D-n-p)!} dx^{\mu_1} \cdots dx^{\mu_n+p} \varepsilon_{\mu_1 \cdots \mu_n+p \mu_1 \cdots \mu_{D-n-p}}$$

$$\cdot \int_{M_{D-p}} E^{\nu_1} \cdots E^{\nu_{D-n-p}} h_n \delta^D(x - x(\sigma)),$$

where $E^\nu = dx^\nu(\sigma)$, and $x^\nu(\sigma)$ parametrizes the worldvolume $M_{D-p}$. For $n = 0$, $h_n = 1$ one obtains a component expression for $J_p$. The product notation is nevertheless convenient because the definition (2.1) implies that Leibnitz’s rule for differentiation holds true, $d(h_nJ_p) = h_n dJ_p + (-)^p dh_nJ_p$. This is the basic property of the push–forward that we will use throughout this paper. In practice the definition of the push–forward implies that the “product” between the Poincaré–dual of a brane, $J_p$, and a form $h_n$ on that brane yields a well defined target space current, while this is clearly not true for products of generic target–space–forms and worldvolume–forms. Another property that we will use frequently is that

$$\Phi J_p = \Phi^{(0)} J_p,$$

whenever the target space form $\Phi$ admits pullback on $M_{D-p}$. In the rest of the paper these definitions and properties are always understood.

### 2.2 Bianchi identities and Chern–kernels

The bosonic fields of $d = 11$ supergravity are the metric $g_{\overline{\mu} \overline{\nu}}(x)$, $(\overline{\mu} = 0, \cdots, 10)$ and a three–form potential $B_3$; the $M5$–brane fields are the coordinates $x^{\overline{\mu}}(\sigma)$ and the chiral two–form $b_2(\sigma)$. We set a bar on eleven–dimensional quantities, to distinguish them from their ten–dimensional counterparts of next sections.

The fundamental equation that describes the back coupling of the brane to supergravity is

$$dH_4 = \overline{\gamma} J_5,$$  

(2.2)

where $J_5$ is the $\delta$–function on the 5–brane worldvolume $M_6$, i.e. its Poincaré–dual, and $\overline{\gamma}$ is the 5–brane’s charge. This equation should be regarded as the Bianchi identity for the curvature $H_4$ whose solution amounts to the introduction of a potential $B_3$.

Introducing a set of normal coordinates $x^{\overline{\mu}} \leftrightarrow (\sigma^i, y^a)$, $(i = 0, \ldots, 5, a = 1, \ldots, 5)$ where the $\sigma^i$ are local coordinates on the brane and the $y^a$ parametrize its normal $SO(5)$–bundle, the current can be rewritten locally as

$$J_5 = \frac{1}{5!} \varepsilon^{a_1 \cdots a_5} dy^{a_1} \cdots dy^{a_5} \delta^5(y).$$  

(2.3)

On the brane there lives an $SO(5)$–connection $A^{ab}(\sigma)$ that is obtained from the Riemannian connection by embedding the brane in the target–space. One introduces an arbitrary
target–space extension $A^{ab}(\sigma, y)$ of this connection subject to the boundary conditions

$$A^{ab}(\sigma, 0) = A^{ab}(\sigma), \quad \lim_{|y|\to\infty} A^{ab}(\sigma, y) = 0. \quad (2.4)$$

This extension ensures the correct fall–off at infinity of the eleven–dimensional Chern–kernel, which is the target–space $SO(5)$–invariant four–form

$$K_4 = \frac{1}{16(2\pi)^2} \varepsilon^{a_1\ldots a_5} \hat{y}^{a_1} K^{a_2 a_3} K^{a_4 a_5}, \quad (2.5)$$

with

$$\hat{y}^a = y^a/\sqrt{y^2}, \quad K^{ab} = F^{ab} + D\hat{y}^a D\hat{y}^b, \quad D\hat{y}^a = d\hat{y}^a + \hat{y}^b A_{ba}, \quad (2.6)$$

and $F = dA + AA$ is the extended $SO(5)$–curvature. $K_4$ has two important properties: the first is that as a distribution it satisfies

$$dK_4 = J_5, \quad \Rightarrow \quad H_4 = dB_3 + \mathcal{P} K_4, \quad (2.7)$$

allowing the introduction of a potential $B_3$ that is regular on the brane, and the second is that its (singular) behaviour near the 5–brane is universal, see below. Since $B_3^{(0)}$ is a well defined field the curvature of the chiral two–form potential on the brane can be defined in a standard way as

$$\mathcal{H}_3 = dB_2 + B_3^{(0)}, \quad (2.8)$$

in compatibility with the gauge transformations

$$\begin{cases} 
\delta B_3 = d\Phi_2 \\
\delta b_2 = -\Phi_2^{(0)}.
\end{cases} \quad (2.9)$$

A central role in the construction of the action is played by the identity

$$K_4 K_4 = \frac{1}{4} df_7, \quad (2.10)$$

$$f_7 \equiv P_7 + Y_7, \quad (2.11)$$

where $P_7$ is the Chern–Simons form associated to the second Pontrjagin form of the normal $SO(5)$–bundle,

$$P_8 = \frac{1}{8(2\pi)^4} \left((tr F^2)^2 - 2(tr F^4)\right) = dP_7,$$

and $Y_7$ is an $SO(5)$–invariant form (see [9]). Notice that due to the distributional nature of $K_4$ Leibnitz’ rule does not hold: the eight–form $K_4 K_4$ is closed even if $dK_4 \neq 0$.

### 2.3 Effective action

The effective action for the system dynamical supergravity $+ M5$–brane is the sum of a local classical part and of a quantum part:

$$\Gamma = \frac{1}{G} (S_{kin} + S_{wz}) + \Gamma_q. \quad (2.12)$$
The quantum contribution $\Gamma_q$ carries the gravitational anomaly associated to the polynomial $[1]$, 

\[ 2\pi \left( X_8 + \frac{1}{24} P_8 \right), \]

where $X_8$ is the $SO(1,10)$ target–space anomaly polynomial, formally identical to the ten–dimensional one given in the introduction, and $P_8$ is defined above. $\mathcal{G}$ is the eleven–dimensional Newton’s constant, related to $g$ by

\[ 2\pi \mathcal{G} = \frac{g^3}{4}. \]  

(2.13)

$S_{\text{kin}}$ collects the kinetic terms for the spacetime metric, for $B_3$, for the coordinates $x^{\mathcal{F}}(\sigma)$ and for $b_2(\sigma)$, the latter being written in the PST–approach $[14],$

\[ S_{\text{kin}} = \int_{M_{11}} d^{11}x \sqrt{g} \left( R - \frac{1}{2} H_4 \ast H_4 - \mathcal{G} \int_{M_6} d^6\sigma \sqrt{g_6} \left( \mathcal{L}(\tilde{h}) + \frac{1}{4} \tilde{h}^{ij} h_{ij} \right) \right). \]  

(2.14)

The kinetic terms for $b_2$ produce a generalized self–duality condition for $\tilde{h}_3$ which involves the Born-Infeld lagrangian

\[ \mathcal{L}(\tilde{h}) = \sqrt{\det \left( \delta^j_i + i\tilde{h}^j_i \right)}, \]  

(2.15)

with $h_{ij} = v^k \tilde{h}_{ijk}$, $\tilde{h}_{ij} = v^k \left( *\tilde{h} \right)_{ijk}$, $v_k = \partial_k \alpha / \sqrt{-(\partial \alpha)^2}$, and $\alpha(\sigma)$ is a non propagating scalar auxiliary field. $g_6$ is minus the determinant of the induced metric on the 5–brane.

The Wess–Zumino term is written as the integral of an eleven–form $S_{\text{wz}} = \int_{M_{11}} L_{11}$, with

\[ L_{11} = \frac{1}{6} B_3 d B_3 d B_3 - \frac{\mathcal{G}}{2} d b_2 B_3^{(0)} J_5 + \frac{\mathcal{G}}{2} B_3 d B_3 K_4 + \frac{\mathcal{G}}{2} B_3 K_4 K_4 + \frac{\mathcal{G}}{24} K_4 f_7 + \frac{2\pi \mathcal{G}}{\mathcal{G}} X_7 H_4. \]  

(2.16)

Since under an $SO(5)$–transformation we have $\delta f_7 = d P_6$, $\delta X_7 = d X_6$ it is immediately checked that $\frac{1}{\mathcal{G}} S_{\text{wz}}$ cancels the gravitational anomaly carried by $\Gamma_q$, thanks to (2.13).

The classical action $S_{\text{kin}} + S_{\text{wz}}$ allows to derive the classical equations of motion for $B_3$ and $b_2$; for a convenient gauge–fixing of the PST–symmetries the latter one amounts to

\[ h_{ij} = -2 \frac{\delta \mathcal{L}}{\delta h_{ij}} \equiv V_{ij}. \]  

(2.17)

To write the action above one had to introduce a Chern–kernel $K_4$, whose definition required two additional structures: normal coordinates and an extension of the $SO(5)$–connection. Eventually one has to make sure that the effective action does not depend on the particular choice of these additional structures. For a different choice of these structures we get a different Chern–kernel $K'_4$ which still satisfies $dK'_4 = J_5 = dK_4$. This

\[ 4 \text{In our notations the 5–brane tension is given } T_5 = \frac{\mathcal{G}}{4}. \]
means that \( K_4 \) and \( K'_4 \) differ by an exact form. The requirement of independence of \( H_4 \) of the additional structures leads to the (finite) transformations

\[
K'_4 = K_4 + dQ_3, \quad (2.18)
\]

\[
B'_3 = B_3 - \overline{g}Q_3, \quad (2.19)
\]

which imply

\[
f'_7 = f_7 + 8K_4Q_3 + 4Q_3dQ_3 + dQ_6, \quad K'_4K'_4 = \frac{1}{4}d f'_7, \quad (2.20)
\]

where

\[
Q_3^{(0)} = 0 = Q_6^{(0)}.
\]

These last identities follow from the fact that the behaviour of \( K_4 \) near the 5–brane is universal, i.e. independent of the choice of normal coordinates and of the extension of the \( SO(5) \)–connection, and they ensure that \( B_3^{(0)} \) and \( \overline{h}_3 \) are independent of these structures, too.

The formula for \( L_{11} \) reported above is completely fixed by the requirement of invariance under \((2.18)\)–\((2.20)\).

### 3 IIA NS5–brane effective action

The bosonic target space fields for IIA supergravity are a three-form \( A_3 \), a two-form \( A_2 \), a one-form \( A_1 \), the dilaton \( \Phi \) and the metric \( g_{\mu\nu} \), while on the bosonic NS5–brane there live the coordinates \( x^{\mu}(\sigma) \), a two-form \( a_2(\sigma) \) and a scalar \( a_0(\sigma) \).

In order to discuss the effective action for the system as a first step one has to understand how the direct/back–coupling mechanism works, i.e. one has to find the coupled Bianchi identities and equations of motions for the fields and to introduce potentials.

#### 3.1 The Chern–kernel

The elementary \( NS5 \)–brane in ten dimensions, with charge \( g \), is described by a current \( J_4 \) that is the ten–dimensional analog of \( J_5 \), and the normal bundle is now an \( SO(4) \) one. The basic Bianchi identity for the curvatures is the analog of eq. \((2.2)\):

\[
dH_3 = gJ_4, \quad (3.1)
\]

and again one has to solve the problem of defining a potential \( A_2 \), or equivalently of finding an appropriate Chern–kernel. The basic difference w.r.t. the eleven–dimensional case is that for \( J_4 \) there exists no invariant three–form \( K_3 \) such that \( dK_3 = J_4 \). This difference is due to the fact that \( J_5 \) is an odd current while \( J_4 \) is an even one and that the Euler characteristic of an odd bundle is zero while the one of an even bundle is non vanishing.
To find an appropriate form $K_3$ as in the previous case we introduce a set of ten-dimensional normal coordinates $(\sigma^i, u^r)$, $r = 1, \ldots, 4$, such that the 5–brane stays at $\vec{u} = 0$. $J_4$ reads then

$$J_4 = \frac{1}{4!} \varepsilon^{r_1 \cdots r_4} du^{r_1} \cdots du^{r_4} \delta^4(u),$$

(3.2)

whatever is the particular set of normal coordinates used. We also pick up an arbitrary target–space extension $W^{rs}(\sigma, u)$ of the normal bundle $SO(4)$–connection $W^{rs}(\sigma)$, with the same requirements as in eq. (2.4), and we call the corresponding extended $SO(4)$–curvature $T = dW + WW$. In terms of these extended objects and of “hatted” coordinates $\hat{u}^r = u^r/\sqrt{u^2}$ one can define the following target–space three–forms:

$$\chi_3 = \frac{1}{8(2\pi)^2} \varepsilon^{r_1 \cdots r_4} \left( W^{r_1 r_2} dW^{r_3 r_4} + \frac{2}{3} W^{r_1 r_2} (WW)^{r_3 r_4} \right),$$

(3.3)

$$\omega_3 = -\frac{1}{2(2\pi)^2} \varepsilon^{r_1 \cdots r_4} \hat{u}^{r_1} D\hat{u}^{r_2} \left( T^{r_3 r_4} + \frac{2}{3} D\hat{u}^{r_3} D\hat{u}^{r_4} \right).$$

(3.4)

The form $\chi_3$ is a Chern–Simons form for an extended Euler–characteristic, $d\chi_3 = \chi_4$,

$$\chi_4 = \frac{1}{8(2\pi)^2} \varepsilon^{r_1 \cdots r_4} T^{r_1 r_2} T^{r_3 r_4},$$

and its pullback on the 5–brane is regular and coincides of course with the Euler Chern–Simons form on the brane. The $SO(4)$–invariant three–form $\omega_3$ instead represents the Chern–kernel associated to $J_4$ [11], that by definition satisfies

$$d\omega_3 = J_4 - \chi_4,$$

(3.5)

as can be verified explicitly. The $J_4$–contribution in this formula comes entirely from the “pure Coulomb–form” $C_3$,

$$C_3 = -\frac{1}{3(2\pi)^2} \varepsilon^{r_1 \cdots r_4} \hat{u}^{r_1} d\hat{u}^{r_2} d\hat{u}^{r_3} d\hat{u}^{r_4}, \quad dC_3 = J_4.$$

(3.6)

As well as $K_4$, the form $\omega_3$ exhibits an invariant singular behaviour near the 5–brane, although its pullback does not exist. Eq. (3.5) provides in particular a realization of the Thom isomorphism [2], i.e. the cohomological equivalence of $J_4$ and $\chi_4$, the latter having as pullback on $M_6$ indeed the Euler–form.

Thanks to (3.5) we can choose for the three–form $K_3$ the combination

$$K_3 = \omega_3 + \chi_3, \quad dK_3 = J_4.$$

(3.7)

As in the eleven–dimensional case, the singular part of $K_3$ ($\omega_3$) is invariant under $SO(4)$–transformations but now $K_3$ has also a regular contribution ($\chi_3$) which transforms under $SO(4)$ as

$$\delta K_3 = d\chi_2.$$

(3.8)
We can now introduce a potential two–form via

\[ H_3 = dA_2 + gK_3, \]  

(3.9)

and we require \( A_2 \) to be regular on the brane. It is important to realize that the introduction of a potential according to the above equation does not represent the most general solution of (3.1): it is the most general solution subject to the boundary conditions represented by the universal singular behaviour near the 5–brane exhibited by \( K_3 \); this is the physical information we add. Notice also that \( H_3 \) does not admit pullback on \( M_6 \), but in the Chern–kernel approach there is no need to define this pullback, what is needed is the separation of \( H_3 \) into a regular part, \( dA_2 + g\chi_3 \), and a singular one, \( g\omega_3 \).

\( SO(4) \)-invariance of \( H_3 \) implies the anomalous transformation law

\[ \delta A_2 = -g\chi_2, \]  

(3.10)

which realizes in particular the pullback transformation law (1.6) anticipated in the introduction.

In this case as well as in the previous one we have introduced additional structures, normal coordinates and an extended \( SO(4) \)-connection. Under change of these we have \( K_3 \to K'_3 \), \( dK'_3 = J_4 = dK_3 \). This means that \( K_3 \) and \( K'_3 \) differ by a closed form, and invariance of \( H_3 \) leads to the transformations

\[ K'_3 = K_3 + dQ_2 \]  

(3.11)

\[ A'_2 = A_2 - gQ_2 \]  

(3.12)

\[ Q^{(0)}_2 = 0. \]  

(3.13)

An explicit expression for \( Q_2 \) as well as a proof of the last equation are reported in Appendix A. This equation asserts that \( Q_2 \) is a regular target–space form, whose pullback on the brane is well defined and equal to zero; this is again a consequence of the fact that the Chern–kernel has a universal singular behaviour near the 5–brane: apart from being \( SO(4) \)-invariant this behaviour is independent of the choice of normal coordinates and of the extension of the connection. As a consequence \( A_2^{(0)} \) is not only regular but also independent of the additional structures. The transformations (3.11) and (3.12) should not be confused with those under \( SO(4) \)-rotations (3.8) and (3.10) of the normal bundle; in particular \( \chi_2^{(0)} \neq 0 \), opposite to \( Q_2^{(0)} = 0 \).

The currents \( J_5 \) and \( J_4 \) exhaust all even and odd dimensional currents. For a general \( J_n \) one can introduce a form \( K_{n-1} \) as

\[ J_n = dK_{n-1}, \quad K_{n-1} = \omega_{n-1} + \chi_{n-1}, \quad d\omega_{n-1} = J_n - \chi_n, \]

where \( \omega_{n-1} \) is an invariant Chern–kernel, and the Euler characteristic of an odd bundle is zero by definition.
3.2 Bianchi identities

Now we are ready to discuss the remaining Bianchi identities. For IIA *pure* supergravity they and the resulting potentials are given by

\[
\begin{align*}
  dH_1 &= 0 \\
  dR_2 &= 0 \\
  dH_3 &= 0 \\
  dR_4 &= H_3 R_2
\end{align*}
\]  \quad \rightarrow \quad
\begin{align*}
  H_1 &= d\Phi \\
  R_2 &= dA_1 \\
  H_3 &= dA_2 \\
  R_4 &= dA_3 + H_3 A_1.
\end{align*}
\] (3.14)

When the \( NS 5 \)-brane is present, the Bianchi identity for \( H_3 \) gets modified and in order to keep the field algebra closed w.r.t. differentiation the Bianchi identity for \( R_4 \) also receives a contribution of order \( g \). The result is

\[
\begin{align*}
  dH_1 &= 0 \\
  dR_2 &= 0 \\
  dH_3 &= g J_4 \\
  dR_4 &= H_3 R_2 - gh_1 J_4 \\
  dh_1 &= R_2^{(0)} \\
  dh_3 &= (R_4 - H_3 h_1)^{(0)},
\end{align*}
\] (3.15)-(3.20)

where we added the Bianchi identities for the brane potentials \( a_0 \) and \( a_2 \).

Eqs. (3.15), (3.16) imply that the pullbacks of \( H_3 \) and \( R_4 \) on the brane are ill–defined. However, in the Bianchi identity (3.20) only the sum \( R_4 - H_3 h_1 \), which is a closed form, is required to have pullback, and we will see in a moment that this combination is indeed regular. A part from this we should also mention that the product \( h_1 H_3 \) does not really define a target space form since \( h_1 \) is only a field on the brane; these formal problems are solved below.

Eqs. (3.15), (3.16) and (3.18) can be easily solved defining \( H_1 = d\Phi \), \( R_2 = dA_1 \), \( h_1 = da_0 + A_1^{(0)} \), while the solution of eq. (3.18) requires some caution. One would be led to write formally \( R_4 = d\tilde{A}_3 + H_3 A_1 - g a_0 J_4 \), but this choice is not a good one since \( \tilde{A}_3 \) would not admit pullback. This can be seen considering the transformation law \( \delta A_1 = d\Gamma \), \( \delta a_0 = -\Gamma^{(0)} \) which requires \( \delta \tilde{A}_3 = -H_3 \Gamma \), and this does not admit pullback. On the other hand, the spurious \( \delta \)-like singularities in the term \( a_0 J_4 \) should be cancelled by the singularities present in \( \tilde{A}_3 \).

A regular potential \( A_3 \) can be introduced considering the alternative formal solution of eq. (3.18) \( R_4 = dA_3 + H_3 h_1 \), which does not exhibit any \( \delta \)-like singularity. But this solution has a different problem: \( h_1 \) is only a field living on the brane, and not a target–space one, so the product \( H_3 h_1 \) is not well defined. Luckily there is a solution for this problem: promote \( h_1 \) to a target–space form \( \hat{h}_1 \) by choosing an arbitrary extension \( \hat{a}_0(x) \) of \( a_0(\sigma) \) such that

\[
\hat{a}_0(\sigma, 0) = a_0(\sigma), \quad \lim_{|u| \to \infty} \hat{a}_0(\sigma, u) = 0,
\] (3.21)
and by defining
\[ \hat{h}_1(x) = d\hat{a}_0(x) + A_1(x), \]
with the properties
\[ \hat{h}_1^{(0)} = h_1, \quad d\hat{h}_1 = R_2. \]
This allows to make use of the product \( H_3 \hat{h}_1 \) as a well defined target space form. However, for consistency the resulting \( R_4 \) should be independent of the choice of the extension \( \hat{a}_0(x) \).

Under a change of extension \( \hat{a}_0 \) we have indeed
\[ \hat{a}_0'(x) = \hat{a}_0(x) + \Lambda(x), \quad \Lambda(0) = 0, \quad (3.22) \]
where the pullback of \( \Lambda \) is zero due to eq. (3.21). This leads to
\[ \left( H_3 \hat{h}_1 \right)' = H_3 \hat{h}_1 + H_3 d\Lambda = H_3 \hat{h}_1 + d \left( H_3 \Lambda \right), \quad (3.23) \]
and \( R_4 \) remains unchanged if one chooses
\[ A'_3 = A_3 - H_3 \Lambda, \quad (3.24) \]
which has the remarkable property of leaving the pullback of \( A_3 \) invariant \[ ] thanks to eq. (3.22). This is consistent with the requirement that \( A_3^{(0)} \) is regular and independent of the chosen extension. We stress that having potentials whose pullbacks are regular is an important part of our construction.

According to our definition of \( \hat{h}_1 \) the r.h.s. of eq. (3.20) has now to be replaced by \( (R_4 - H_3 \hat{h}_1)^{(0)} = dA_3^{(0)} \), which is regular and closed; the solution of this Bianchi identity amounts then simply to \( h_3 = da_2 + A_3^{(0)} \). We can now collect the solutions of our Bianchi identities in terms of potentials,
\[
\begin{align*}
H_1 &= d\Phi \\
R_2 &= dA_1 \\
H_3 &= dA_2 + gK_3 \\
R_4 &= dA_3 + H_3 \hat{h}_1 \\
h_1 &= da_0 + A_1^{(0)} \\
h_3 &= da_2 + A_3^{(0)}.
\end{align*}
\]

From the definition of \( h_3 \) we deduce again the need of a three–form potential with a regular pullback.

\[ ^5 \] The part of \( H_3 \) which does not admit pullback is \( g\omega_3 \); the “singularity” of \( \omega_3 \) is due to the fact that as one approaches the 5–brane \( u^r \to 0, \omega_3^{(0)} \) while remaining finite would depend on the direction along which \( u^r \) goes to zero. Thus also \( H_3 \) remains finite, and \( H_3 \Lambda \) goes to zero as \( u^r \to 0 \) thanks to (3.22).
3.3 Equations of motion

The equations of motion for pure IIA supergravity are

\[
\begin{align*}
    d \left( e^{-2\Phi} \ast H_1 \right) &= \frac{1}{4} R_4 \ast R_4 + \frac{3}{4} R_2 \ast R_2 - \frac{1}{2} H_3 \left( e^{-2\Phi} \ast H_3 \right) \\
    d \ast R_2 &= -H_3 \ast R_4 \\
    d \left( e^{-2\Phi} \ast H_3 \right) &= \frac{1}{2} R_4 R_4 - R_2 \ast R_4 \\
    d \ast R_4 &= H_3 R_4,
\end{align*}
\]

(3.31)

where consistency requires the right hand sides of these equations to be closed forms, as implied by the Bianchi identities (3.14). In the presence of a 5–brane, where the Bianchi identities change to (3.15)–(3.20), we have to change the equations of motion as well to keep the r.h.s. closed. As in the case of the Bianchi identities the new terms have to be supported on the 5–brane, i.e. to be proportional to \( J_4 \). Our proposals for the new equations of \( A_1 \), \( A_2 \) and \( A_3 \) are

\[
\begin{align*}
    d \ast R_2 &= -H_3 \ast R_4 + g J_4 \ast \mathcal{H}_1 \\
    d \left( e^{-2\Phi} \ast H_3 \right) &= \frac{1}{2} R_4 R_4 - R_2 \ast R_4 + gh_1 h_3 J_4 + \frac{2\pi G}{g} \left( X_8 + \frac{1}{24} \chi_4 J_4 \right) \\
    d \ast R_4 &= H_3 R_4 - gh_3 J_4,
\end{align*}
\]

(3.35)

(3.36)

(3.37)

while for the fields on the brane, \( a_2 \) and \( a_0 \), we propose

\[
\begin{align*}
    h_{ij} &= -2 \frac{\delta \mathcal{L}}{\delta h_{ij}} \\
    d \ast \mathcal{H}_1 &= (\ast R_4 - H_3 \hat{h}_3)^{(0)}.
\end{align*}
\]

(3.38)

(3.39)

Before justifying the various new terms in these equations we present the Bianchi identity and the equation of motion for \( A_3 \), (3.18) and (3.37), in a different but equivalent way. Notice first that the r.h.s. of (3.37) is now closed, because \( d(H_3 R_4) = d(H_3 dA_3) = g J_4 dA_3 = g J_4 dh_3 \). Setting \( R_6 = \ast R_4 \) we can use this equation to introduce a dual potential \( A_5 \). The dynamics of \( A_3 \) can then be represented equivalently through the duality invariant system

\[
\begin{align*}
    R_4 &= d A_3 + H_3 \hat{h}_1 \\
    R_6 &= d A_5 + H_3 \hat{h}_3 \\
    R_6 &= \ast R_4,
\end{align*}
\]

(3.40)

(3.41)

(3.42)

where we have extended the brane field \( a_2 \) to a target space form \( \hat{a}_2 \), in exactly the same way as \( a_0 \) above, and we defined the target space three–form

\[
\hat{h}_3 \equiv d \hat{a}_2 + A_3.
\]

Substituting (3.42) in (3.41) and applying the differential one gets indeed back (3.37).
In analogy to $A_3$ we require also the dual potential $A_5$ to be regular near the 5–brane, i.e. $A_5^{(0)}$ to be well–defined. The self–consistency of this requirement proves in the same way as in the case of $A_3$. Its basic consequences, due to (3.41) and (3.42), are that

$$d(*R_4 - H_3\hat{h}_3) = 0, \quad (*R_4 - H_3\hat{h}_3)^{(0)} \quad \text{well defined.} \quad (3.43)$$

One should notice that, although the statement that $A_3$ and $A_5$ admit pullback is duality symmetric, in the sense that the two potentials are considered as equivalent, there is an intrinsic asymmetry between $A_3$ and $A_5$ in IIA–supergravity, because there is no formulation in terms of $A_5$ only. Our formulation is in terms of $A_3$ only, so the requirement that $A_3$ admits pullback is of kinematical nature, while the requirement that also $A_5$ admits pullback is a dynamical constraint, because it asserts that only those solutions of the $A_3$–equation of motion (eq. (3.37)) are allowed, for which the combination $*R_4 - H_3\hat{h}_3$ admits pullback. In conclusion, the consistency of our system of equations of motion needs as supplementary condition the second assertion in (3.43).

Keeping this in mind we consider the equation for $A_1$, (3.36). We added on its r.h.s. a term involving (the six–dimensional Hodge dual of) a one–form on the brane, $\mathcal{H}_1$. Due to (3.43) we have

$$d(H_3 * R_4) = d(H_3(*R_4 - H_3\hat{h}_3)) = gJ_4(*R_4 - H_3\hat{h}_3)^{(0)},$$

from which it is clear that the term $gJ_4 \cdot \mathcal{H}_1$ is needed to make the r.h.s. of (3.36) a closed form, thanks to the $a_0$–equation (3.39). This last equation is well–defined, again thanks to (3.43), and it represents the $a_0$–equation of motion since at the linearized level we expect

$$\mathcal{H}_1 = h_1 + \cdots.$$  

The precise relation between $\mathcal{H}_1$ and $h_1$ will emerge below when we write the action.

The $A_2$–equation (3.30) contains a term supported on $M_6$, $gh_1h_3J_4$, which is needed to keep its r.h.s. a closed form, as can be seen using the other Bianchi identities and equations of motion. Also here, due to the presence of singularities, one is not allowed to use Leibnitz’s rule: the computation $d(\frac{1}{2}R_4 R_4) = R_4 dR_4 = R_4(H_3R_2 - gh_1J_4)$ makes no sense because $R_4$ does not admit pullback. Rather one has first to evaluate $R_4 R_4$ using the definition of $R_4$ in (3.28), which gives $R_4 R_4 = dA_3 dA_3 + 2dA_3 H_3 \hat{h}_1$, and then

$$d(\frac{1}{2}R_4 R_4) = dA_3(H_3R_2 - gh_1J_4) = R_4 H_3 R_2 - gdh_3h_1J_4;$$

the rest of the computation is straightforward. The terms proportional to the ten–dimensional Newton’s constant $G$ are closed by themselves and are required for anomaly cancellation.

---

6There is clearly a lagrangian formulation for IIA which involves both potentials, a la PST, in analogy to the IIB–case.

7The following computation has to be performed with some caution since a straightforward application of Leibnitz’s rule, $d(H_3*R_4) = H_3 d*R_4 + dH_3 * R_4 = -gH_3b_3J_4 + gJ_4 * R_4$, leads to a meaningless result because the pullback of $H_3$ does not exist.
The equation of motion for the chiral two–form on the brane, eq. (3.38), is known from previous work [16]. It is governed by formally the same Born–Infeld lagrangian $\mathcal{L}(\tilde{h})$ as the one of the $M5$–brane in (2.15). The differences are that here we have $h_3 = da_2 + A_3^{(0)}$ and that this time all six–dimensional indices in $\mathcal{L}(\tilde{h})$, in $\tilde{h}_{ij}$ and in $h_{ij}$ are contracted with the effective metric

$$
g_{\text{eff},ij} = e^{-\frac{i}{2}\Phi} \left[ g_{ij} - e^{2\Phi} h_i h_j \right], \quad (3.44)$$

$$
g^{ij}_{\text{eff}} = e^{rac{i}{2}\Phi} \left[ g^{ij} + \frac{e^{2\Phi} h^i h^j}{1 - e^{2\Phi} h^2} \right], \quad (3.45)$$

where $h_i$ are the components of the 5–brane one–form $h_1 = da_0 + A_1^{(0)}$, and $h^2 = h_i h_j g^{ij}$. Apart from this we have again the definitions $h_{ij} = v^k h_{ijk}$, $\tilde{h}_{ij} = v^k (h)_{ijk}$ etc. as in the eleven–dimensional case.

These equations of motion fix the classical action modulo terms that are independent of $A_1$, $A_2$, $A_3$ and $a_3$, while the equations of motion for $g_{\mu\nu}$ and $\Phi$, which are rather complicated, are obtained varying the resulting action. This action should fix also the dependence of $\mathcal{H}_1$ on $h_1$ and the other fields.

### 3.4 Effective action

The resulting effective action $\Gamma$ splits in a classical action and a quantum part $\Gamma_q$

$$\Gamma = \frac{1}{G} (S_{\text{kin}} + S_{\text{wz}}) + \Gamma_q, \quad (3.46)$$

where $\Gamma_q$ carries the anomaly associated to (1.11), and the action $S_{\text{kin}} + S_{\text{wz}}$ should give rise to the equations of motion of the previous section. Once the system of Bianchi identities and equations of motion is consistent the reconstruction of the action becomes a purely technical point. Apart from invariance requirements under standard gauge transformations for the potentials (as in pure supergravity) in the case at hand one has to keep in mind that, due to our definition of $H_3$ in terms of $K_3$, the action has also to be invariant under changes of $K_3$, i.e. under the transformations (3.11)–(3.13). We give now the action and explain then how it produces our equations of motion. The kinetic part is given by

$$S_{\text{kin}} = -\frac{1}{2} \int_{M_{10}} \left[ R_4 \ast R_4 + R_2 \ast R_2 + e^{-2\Phi} (H_3 \ast H_3 + 8 H_1 \ast H_1) \right] + \int_{M_{10}} d^{10}x \sqrt{g} R e^{-2\Phi} - g \int_{M_6} d^6\sigma \sqrt{g_{\text{eff}}} \left( \mathcal{L}(\tilde{h}) + \frac{1}{4} \tilde{h}_{ij} h_{ij} \right), \quad (3.47)$$

and the Wess–Zumino term reads

$$S_{\text{wz}} = -\frac{1}{2} \int_{M_{10}} A_3 d A_3 H_3 - \frac{g}{2} \int_{M_6} da_2 A_3 + \frac{2\pi G}{g} \left( \int_{M_{10}} H_3 X_7 + \frac{1}{24} \int_{M_6} A_2^{(0)} \chi_4 \right). \quad (3.48)$$

In the last term of the kinetic action all indices are contracted with the effective metric given above, but apart from this the dependence of $S_{\text{kin}} + S_{\text{wz}}$ on the field $a_2$ is entirely
fixed by the PST–symmetries. After a convenient gauge–fixing of these symmetries the
\( a_2 \)–equation of motion following from this action is indeed \( (3.38) \), see \[14\].

In deriving the \( A_3 \)–equation, the unique non trivial point is the evaluation of the
variation of the last term in \( S_{kin} \) under a generic variation of this field:

\[
\delta \int_{M_6} d^6 \sigma \sqrt{g_{eff}} \left( L(\tilde{h}) + \frac{1}{4} \tilde{h}^{ij} h_{ij} \right) = \int_{M_6} \left( \frac{1}{2} h_3 + v(h_2 - V_2) \right) \delta A_3^{(0)} = \frac{1}{2} \int_{M_6} h_3 \delta A_3^{(0)},
\]

where \( V_{ij} \equiv -2 \frac{\delta L}{\delta \tilde{h}_{ij}} \), and we used the (gauge–fixed)
\( a_2 \)–equation \( (3.38) \). The variation of
the remaining terms is standard, and the resulting equation of motion is \( (3.36) \).

The derivation of the \( A_2 \)–equation is straightforward since this field does not appear
in the last term of \( S_{kin} \).

To derive the equations for \( A_1 \) and \( a_0 \), which show up in the last term of \( S_{kin} \) only in
the combination \( h_1 \), we define the 5–brane field

\[
\mathcal{H}_i \equiv -\frac{1}{\sqrt{g}} \frac{\delta \int_{M_6} d^6 \sigma \sqrt{g_{eff}} \left( L(\tilde{h}) + \frac{1}{4} \tilde{h}^{ij} h_{ij} \right)}{\delta h^i}.
\]

This means that under a generic variation of \( h_1 \) we have

\[
\delta \int_{M_6} d^6 \sigma \sqrt{g_{eff}} \left( L(\tilde{h}) + \frac{1}{4} \tilde{h}^{ij} h_{ij} \right) = - \int_{M_6} d^6 \sigma \sqrt{g} \delta h^i \mathcal{H}_i = \int_{M_6} \delta h_1 * \mathcal{H}_1, \tag{3.49}
\]

where the *-operation refers to the induced metric \( g_{ij} \). For a generic variation of \( A_1 \) this
computation gives

\[
\int_{M_6} \delta h_1 * \mathcal{H}_1 = \int_{M_6} \delta A_1^{(0)} * \mathcal{H}_1 = \int_{M_{10}} \delta A_1 J_4 * \mathcal{H}_1,
\]

which leads eventually to \( (3.35) \). For a generic variation of \( a_0 \) instead we have

\[
\int_{M_6} \delta h_1 * \mathcal{H}_1 = \int_{M_6} \delta a_0 d(*\mathcal{H}_1). \tag{3.50}
\]

To this one has to add the variation of the first term in \( S_{kin} \), which depends on the
extended field \( \hat{a}_0 \). Under a variation of this field we have

\[
\delta \left( -\frac{1}{2} \int_{M_{10}} R_4 * R_4 \right) = \int_{M_{10}} d \delta \hat{a}_0 H_3 * R_4 = \int_{M_{10}} d \delta \hat{a}_0 H_3 \left( *R_4 - H_3 \tilde{h}_3 \right)
= g \int_{M_6} \left( *R_4 - H_3 \tilde{h}_3 \right)^{(0)} \delta a_0,
\]

where we used again \( (3.43) \). Despite the presence of the extended field \( \hat{a}_0 \) in the definition
of \( R_4 \), this variation is supported entirely on \( M_6 \) and depends only on \( \delta \hat{a}_0 \); this is clearly
a consequence of the independence of \( R_4 \) of the chosen extension. Adding this variation
to \( (3.50) \), multiplied by \(-g\), one obtains the equation for \( a_0 \). Finally, from the above
definition of \( \mathcal{H}_1 \) it is easy to see that at first order in the fields it reproduces \( h_1 \).

A further important point is that the classical action is invariant under \( (3.11)-(3.13) \)
because \( A_2 \) appears either in the combination \( H_3 = dA_2 + gK_3 \), or through its pullback
\( A_2^{(0)} \) in the anomaly cancelling term \( \int A_2^{(0)} \chi_4 \), and both are invariant. The action is clearly also invariant under standard gauge transformations of the potentials.

Under \( SO(1,9) \)– and \( SO(4) \)–rotations the Wess–Zumino action transforms as (see (3.10))

\[
\delta \left( \frac{1}{G} S_{wz} \right) = -2\pi \int_{M_6} \left( X_6 + \frac{1}{24} \chi_4 \chi_2 \right),
\]

which cancels against \( \delta \Gamma_q \).

It is worthwhile to notice that to obtain the correct equation of motion for \( A_2 \) it would have been sufficient to introduce the \( SO(1,9) \)–invariant term \( \int A_2 X_7 \) instead of \( \int H_3 X_7 \), but this term alone would spoil the invariance (3.11)–(3.13). This forces the introduction of the term \( g \int K_3 X_7 \), that is not invariant under \( SO(1,9) \) and cancels the target–space anomaly: invariance requirements and anomaly cancellation are closely related.

### 3.5 Dirac–branes

Since the \( NS5 \)–brane is electromagnetically dual to the \( NS \)–string one can ask if the 5–brane dynamics can be treated in a way analogous to the magnetic monopole in four dimensions which is, in turn, dual to the electric charge. A necessary ingredient for a lagrangian description of a magnetic monopole is the Dirac–string, i.e. a two–dimensional surface whose boundary is the monopole worldline, an ingredient which finally should result unobservable. This setup can be generalized to a generic \( p \)–brane, where the Dirac–string becomes a Dirac–\((p+1)\)–brane. In this subsection we illustrate briefly the corresponding setup for the \( NS5 \)–brane, and explain where it eventually fails in this case, due to the particular dynamics of the \( NS5 \)–brane.

With this respect the Chern–kernel approach can be considered as equivalent to a Dirac–brane approach plus a canonical superselection rule on the allowed curvatures \( H_3 \).

A Dirac–brane associated to the \( NS5 \)–brane is a seven–manifold \( M_7 \) whose boundary is \( M_6, \partial M_7 = M_6 \). Calling \( C_3 \) the Poincarè–dual of \( M_7 \) this means

\[
dC_3 = J_4.
\]

Then one can give an alternative solution of equation (3.17), in terms of a potential \( B_2 \),

\[
H_3 = dB_2 + gC_3.
\]

Up to here this solution and the previous one, \( H_3 = dA_2 + gK_3 \), are equivalent because they correspond to a redefinition of the two–form potential. Both solutions parametrize the most general solution of \( dH_3 = gJ_4 \): take a particular solution, \( gK_3 \) or \( gC_3 \), and add the most general solution of the associated homogeneous equation. However, taking the solution (3.27) in terms of \( A_2 \) we imposed implicitly the “canonical superselection rule” that \( A_2 \) is regular near the 5–brane or, equivalently, that the singularities of \( H_3 \) near
the 5–brane are the ones of the Chern–kernel. On the other hand, the solution (3.52) is completely general, and allows for example also a configuration of the kind \( H_3 = gC_3 \), which can clearly not be realized in the Chern–kernel approach. In principle one could use also (3.52) and impose the constraint that \( H_3 \) behaves in the vicinity of the 5–brane as \( K_3 \); this would amount to a very complicated constraint on \( B_2 \), while it is extremely simple when imposed on \( A_2 \).

The question remains if one can do also without imposing any constraint on \( H_3 \), using the most general solution (3.52) in terms of a Dirac–brane. The basic problem one encounters in this case is that \( B_2 \) can not admit pullback on \( M_6 \). To see this we observe that the Dirac–brane is not unique; under a change of Dirac–brane \( M_7 \) goes into a new \( M'_7 \) whose boundary is still \( M_6 \), so that \( M'_7 - M_7 \) is boundaryless and there exists a 7–brane \( M_8 \) such that \( \partial M_8 = M'_7 - M_7 \). In terms of the corresponding Poincaré–duals we have \( C'_3 - C_3 = dC_2 \), and to have a Dirac–brane independent \( H_3 \) we must require

\[
B'_2 = B_2 - gC_2,
\]

which is analogous to (3.12). This time, however, \( C_2 \) is a \( \delta \)–function on a manifold \( M_8 \) which contains as submanifold \( M_6 \) and this implies that \( C_2^{(0)} \) does not exist. This means that a “by–hand–requirement”, that \( B_2 \) admits pullback on \( M_6 \), is inconsistent. This would still not imply an inconsistency at the level of equations of motion \( 8 \), but it would at the level of the action. Indeed, in the Wess–Zumino action \( \frac{1}{G} S_{wz} \) there is a term, the normal–anomaly–cancelling term \( \frac{\pi}{12g} \int_{M_6} A_2 \chi_4 \), in which the two–form does not appear in the combination \( H_3 \). This should now be replaced by \( \frac{\pi}{12g} \int_{M_6} B_2 \chi_4 \). But this term has two problems: first, \( B_2^{(0)} \) is not defined and second, \( B_2 \) depends on the Dirac–brane according to (3.53). To cope with the second problem – Dirac–brane–dependence – one could replace this term by the formal expression

\[
\frac{\pi}{12g} \int_{M_6} H_3 \chi_3,
\]

where \( H_3^{(0)} = (dB_2 + gC_3)^{(0)} \), since it gives rise to the same equation of motion for the two–form. This would transform under \( SO(4) \), again formally, as

\[
\delta \left( \frac{\pi}{12g} \int_{M_6} H_3 \chi_3 \right) = -\frac{\pi}{12} \int_{M_6} J_4 \chi_2.
\]

But the pullbacks \( H_3^{(0)}, C_3^{(0)} \) as well as \( J_4^{(0)} \) are ill–defined, and one is back to the situation described in the introduction, where one must invoke cohomological representatives.

In conclusion, the physical content of the Chern–kernel approach is represented by the universal prescription for the singular behaviour of the invariant curvature near the brane (superselection rule); with this prescription one can write a consistent set of Bianchi

---

8There is, however, a problem related with the transformation (3.24), since now the singularities of \( H_3 \) could be even of the \( \delta \)–type and in this case the pullback of \( H_3 \Lambda \) is not defined.
identities and equations of motion, and an action which cancels the gravitational anomalies. The Dirac–brane approach, instead, furnishes a framework which does not specify the allowed singularities for the invariant curvature; it introduces, moreover, intermediate unphysical \( \delta \)-like singularities along the Dirac–brane. As a consequence it does not allow to write a well–defined action.

In the next section we will perform a Kaluza–Klein dimensional reduction of the system \( M5 \)-brane + dynamical \( D = 11 \) supergravity of section 2 down to ten dimensions. The result is the system \( NS5 \)-brane + dynamical \( IIA \) supergravity, and we expect to obtain our Chern–kernel formulation of the theory, because we are starting from a theory written, in turn, in the Chern–kernel formalism.

4 Dimensional reduction

We perform in this section the dimensional reduction of the \( D = 11 \) \( M5 \)-brane–action of section 2, down to ten dimensions, compactifying say the coordinate \( x^{10} \) on a circle of radius \( R \). The main motivation is clearly a consistency check of the ten dimensional action constructed independently in the previous section for the \( NS5 \)-brane: the reduction process should reproduce this action, modulo local terms. The second motivation is related with the fact that in the present case the reduction shows up a new (a priori problematic) feature w.r.t. the reduction of pure sourceless \( D = 11 \) supergravity, due to the presence of the \( 5 \)-brane.

We will indicate eleven–dimensional quantities with a bar and ten–dimensional ones without bar; e.g. \( x^\mu = (x^\mu, x^{10}) \), where \( \mu = (0, \cdots, 9) \). The relation between ten– and eleven–dimensional Newton’s constants and magnetic charges is standard and reads

\[
G = \frac{G}{2\pi R}, \quad g = \frac{g}{2\pi R}.
\]

Using the notations of section 2 for \( D = 11 \) fields and the ones of section 3 for \( D = 10 \) fields, we remember that the target space fields decompose as, \( g_{\mu\nu} \rightarrow (g_{\mu\nu}, A_1, \Phi) \) and \( B_3 \rightarrow (A_3, A_2) \). The eleventh coordinate on the \( M5 \)-brane becomes the scalar field of the \( NS5 \)-brane while the eleven–dimensional chiral two–form is identified directly with the ten–dimensional one:

\[
x^{10}(\sigma) = a_0(\sigma)
\]
\[
b_2(\sigma) = a_2(\sigma).
\]

We recall first the standard decompositions of the eleven–dimensional fields in the case of pure supergravity. The three–form decomposes as

\[
B_3 = A_3 + dx^{10} A_2,
\]
while the metric is reduced according to

\[ E_{\mu}^{\pi} = \begin{pmatrix} e^{-\frac{2}{3}\Phi} E_{\mu} & e^{\frac{2}{3}\Phi} A_{\mu} \\ 0 & e^{\frac{2}{3}\Phi} \end{pmatrix} \rightarrow g_{\mu\nu} = \begin{pmatrix} e^{-\frac{2}{3}\Phi} (g_{\mu\nu} - e^{2\Phi} A_{\mu} A_{\nu}) & -e^{\frac{4}{3}\Phi} A_{\mu} \\ -e^{\frac{4}{3}\Phi} A_{\mu} & -e^{\frac{4}{3}\Phi} \end{pmatrix}. \]  

(4.3)

In particular the eleventh component of the elfbein reads

\[ E^{10} \equiv dx^{10} E_{\mu}^{\pi} = e^{\frac{2}{3}\Phi} (dx^{10} + A_1), \]

where the gauge transformation of \( A_1 \) amounts to a \( D = 11 \) diffeomorphism of \( x^{10} \), such that the combination \( dx^{10} + A_1 \) is invariant.

The fields \( A_1, A_2, A_3, g_{\mu\nu} \) and \( \Phi \) are assumed to be periodic in \( x^{10} \) with period \( 2\pi R \). \( D = 10, IIA \) supergravity is obtained if one keeps only the zero modes of their Fourier expansion or, equivalently, if one assumes all those fields to be independent of \( x^{10} \). In this case (4.2) leads to the \textit{invariant} decomposition (here we define \( H_4 \equiv dB_3 \))

\[ H_4 = R_4 + (dx^{10} + A_1)H_3, \]  

(4.4)

where the \( x^{10} \)-independent curvatures \( R_4 \equiv dA_3 + dA_2 A_1 \) and \( H_3 \equiv dA_2 \) are the sourceless counterparts of the corresponding ten dimensional curvatures of section 3, i.e. with \( g = 0 = a_0 \).

In the coupled case, however, the same procedure can not be applied in a straightforward way since we have \( dH_4 = \varpi J_5 \) and the current \( J_5 \), even if in principle it admits a decomposition like (4.2), depends on \textit{all} eleven coordinates \( x^{\pi} \) in a non trivial way and it is, in particular, intrinsically non periodic in \( x^{10} \). Since \( dK_4 = J_5 \), the Chern–kernel inherits the same problematic features from the current.

In this section we will show how one can overcome these difficulties and get the relation (4.4) also in the coupled case – in the limit \( R \rightarrow 0 \) – where \( R_4 \) and \( H_3 \) are replaced with their coupled expressions in section 3. The validity of the decomposition (4.4) is indeed the fundamental ingredient of the reduction process, in the free as well as in the coupled case.

### 4.1 The reduced geometry in the presence of an M5–brane

As we will see, in this case the parametrization of the reduced metric (4.3) can be kept unchanged, while the reduction of \( B_3 \) will be more complicated then (4.2).

The first problem which arises in the presence of an M5–brane is that it is inconsistent to restrict \( x^{10} \) to a compact interval since \( x^{10}(\sigma) \) is identified with the 5–brane field \( a_0(\sigma) \) which is clearly unconstrained. A first suggestion to overcome this difficulty would be to consider as compact field the shifted variable \( x^{10} - \tilde{a}_0(\sigma) \), but this is not a target space field. As we will see below the right choice for the compact variable is

\[ x^{10} - \tilde{a}_0(x) \in [-\pi R, \pi R], \]  

(4.5)
where $\hat{a}_0(x)$ is a ten–dimensional extension of $x^{10}(\sigma)$, which we identify with the homonymous ten–dimensional field of section 3. Notice that independence of a $D = 11$ field of the variable $x^{10} - \hat{a}_0(x)$ is equivalent to independence of $x^{10}$.

We can now address the reduction of the current $J_5$. If we indicate with $J_4(x)$ the ten–dimensional current of the 5–brane, i.e. the Poincaré–dual of the surface $x^\mu(\sigma)$ w.r.t. ten dimensions, then we can rewrite the eleven–dimensional current as

\[ J_5 = (dx^{10} - d\hat{a}_0)\delta(x^{10} - \hat{a}_0)J_4, \]

which is independent on the chosen extension $\hat{a}_0$ since $J_4$ projects it down to $x^{10}(\sigma)$. We are interested in a configuration in which $D = 11$ Sugra reduces to $D = 10$, IIA Sugra, corresponding to the case in which all eleven–dimensional fields are taken to be independent of $x^{10}$; the obstacle to this procedure introduced by $J_5$ is that, despite the dependence of $J_4$ on only $x^\mu$, $J_5$ depends also on $x^{10}$ through the $\delta$–function. This difficulty can be solved by noting that in the limit $R \to 0$ we can replace

\[ \delta(x^{10} - \hat{a}_0) \to \frac{1}{2\pi R}, \quad (4.6) \]

and

\[ J_5 = d\left(\frac{x^{10} - \hat{a}_0}{2\pi R}\right)J_4, \quad (4.7) \]

which has now the desired structure and allows a consistent reduction of the equation $dH_4 = \pi J_5$.

The last problem one has to face is that in the eleven–dimensional effective action the current $J_5$ enters also indirectly, through the four–form $K_4$, which is expressed in terms of normal coordinates. So we have to determine the relation between the compact variable $x^{10} - \hat{a}_0$ and the $SO(5)$–normal coordinates $y^a$. The choice of the eleventh coordinate as the compact one is of course conventional and corresponds more generally to a vector field $V^\mu \partial_\mu$; in our case $V^\mu = (0, \ldots, 0, 1)$. On the 5–brane this vector field decomposes in a tangent and in a normal component, meaning that the choice of a compact variable identifies also a normal vector on the 5–brane (the normal component of $V^\mu$), call it $V^a(\sigma)$. Thus the normal group $SO(5)$ of the $M5$–brane reduces to its subgroup $SO(4)$ which leaves $V^a$ invariant. Conventionally, through an $SO(5)$–rotation, we can choose $V^a = (0, 0, 0, 0, 1)$ which means that we single out the fifth normal coordinate $y^5 = V^a y^a$ as an $SO(4)$–invariant one. We can then introduce ten–dimensional $SO(4)$–normal coordinates $u^r$ ($r = 1, \ldots, 4$) according to $y^a = (y^r \equiv u^r, y^5)$, and the ten–dimensional NS5–brane current $J_4$ can then be expressed for $R \to 0$ as in (3.2),

\[ J_4 = \frac{1}{4!} \varepsilon^{r_1 \cdots r_4} du^{r_1} \cdots du^{r_4} \delta^4(u), \quad J_5 = dy^5 \delta(y^5)J_4. \]

We can now decompose $K_4$ using ten–dimensional normal coordinates. The $SO(5)$–connection $A^{ab}$ is decomposed according to $A^{ab} = (A^{rs} \equiv W^{rs}, A^{r5} \equiv L^r)$, where $W^{rs}$ is
the $SO(4)$–connection and $L^r$ is an $SO(4)$–covariant vector. With these definitions the Chern–kernel in equation (2.3) allows the following fundamental decomposition, whose derivation is lengthy but straightforward:

$$K_4 = \frac{1}{2} \chi_4 \hat{y}^5 - \frac{1}{2} \omega_3 d\hat{y}^5 + dG_3,$$

where $\chi_4$ is the $SO(4)$ Euler–characteristic and $\omega_3$ is the Chern–kernel associated to $J_4$, both defined in section 3. $G_3$ is an $SO(4)$–invariant three–form whose explicit expression is given in Appendix B; it involves $L^r$, $u^r$, $y^5$ and their $SO(4)$–covariant derivatives. We remember also that

$$\hat{y}^5 = \frac{y^5}{\sqrt{(y^5)^2 + |\vec{u}|^2}}.$$

As a check of the above decomposition we note that, using the defining relation for the $SO(4)$–Chern–kernel $d\omega_3 = J_4 - \chi_4$, the differential of the r.h.s. of (1.8) reduces to

$$dK_4 = \frac{1}{2} J_4 d\hat{y}^5;$$

but $J_4$ sets $\vec{u} = 0$ and $\hat{y}^5$ gets replaced with the sign function $\epsilon(y^5)$; eventually $d\hat{y}^5 \rightarrow d\epsilon(y^5) = 2dy^5 \delta(y^5)$ and the r.h.s. of (1.9) amounts just to $J_5$.

Until now we have only rewritten $K_4$ in the uncompactified theory. At the compactified level $dK_4$ has to match moreover (1.7); comparing this with (1.9) allows to relate eventually the compact variable to the normal coordinates

$$\hat{y}^5 = \frac{1}{\pi R} \left( x^{10} - \tilde{a}_0 \right).$$

This relation is in particular consistent with the fact that $\hat{y}^5$ varies in the interval $[-1, 1]$. It is understood that in the above decomposition for $K_4$ the variable $\hat{y}^5$ is meant to be substituted by this expression, even if we maintain as notation the symbol $\hat{y}^5$. We recall that, as the replacement (1.7) is valid only for $R \rightarrow 0$, also the above identification for $\hat{y}^5$ is valid only in the same limit. In terms of normal coordinates independence of $x^{10}$ amounts then to independence of $y^5$, or equivalently of $\hat{y}^5$.

The structure of (1.8) allows now to determine the reduction of $B_3$ in the presence of a 5–brane, which maintains (1.4). It is indeed easy to see that the reduction

$$B_3 = A_3 - \pi R \hat{y}^5 (dA_2 + g\chi_3) - \overline{y}G_3,$$

leads to

$$H_4 = dB_3 + \overline{y}K_4 = R_4 + (dx^{10} + A_1)H_3,$$

where $H_3$ and $R_4$ are defined in (3.27) and (3.28) respectively. The fields $A_3$ and $A_2$ defined in this way, as well as the (extended) $SO(4)$–connection $W_{rs}$ showing up in $K_3$, are eventually taken to be independent of $x^{10}$; this ensures $x^{10}$–independence of $H_3$ and
$R_4$, as in the sourceless case. For pure Sugra, $\hat{a}_0 = 0 = \overline{g}$, (4.11) reduces to the standard decomposition (4.2) modulo a gauge transformation.

For what concerns invariances we observe that $B_3$ was invariant under $SO(5)$ and so it is under $SO(4)$; since also $G_3$ is invariant under $SO(4)$ the fields $A_3$ and $dA_2 + g\chi_3$ have to be invariant under this group separately, due to the presence of the factor $\pi R g^5$. This determines the expected anomalous transformation law for $A_2$, $\delta A_2 = -g\chi_2$, from the eleven–dimensional point of view.

The last ingredient we need is the reduction of the pullback $B_3^{(0)}$. The second term in (4.11) has vanishing pullback due to (4.10). From the explicit expression of $G_3$ in the appendix (5.17), one sees that it has actually a finite pullback, but that it depends on the direction along which it is taken. However, due to the fact that $G_3$ is multiplied by $\overline{g} = 2\pi R g$, its pullback gives no contribution for $R \rightarrow 0$\(^9\). Thus in this limit we have

$$B_3^{(0)} = A_3^{(0)}, \quad \mathcal{T}_3 = h_3 = da_2 + A_3^{(0)}.$$  \hspace{1cm} (4.13)

4.2 Reduction of the action

We are now able to perform the dimensional reduction of the classical eleven–dimensional effective action of section two, see equations (2.14) (kinetic terms) and (2.16) (Wess–Zumino). The first should reduce to $2\pi R$ times the ten–dimensional kinetic terms in (3.47), and the integral $\int_{M_{11}} L_{11}$ should reduce to $2\pi R$ times the ten–dimensional Wess–Zumino action given in (3.48). We perform the reduction of the kinetic and Wess–Zumino terms separately.

4.2.1 Kinetic terms

The reduction of the Einstein–Hilbert term in (2.14) is standard and requires only to insert the decomposition (4.3)

$$\int_{M_{11}} d^{11}x \sqrt{g} R = 2\pi R \left( \int_{M_{10}} d^{10}x e^{-2\Phi} \sqrt{g} R - \frac{1}{2} \int_{M_{10}} \left( R_2 * R_2 + 8 e^{-2\Phi} H_1 * H_1 \right) \right).$$

The reduction of the Born–Infeld action for the chiral two–form in (2.14) has been anticipated above, see also [16]. We have $\mathcal{T}_3 = h_3$, and if one introduces the decomposition (4.3) in the eleven–dimensional induced metric one obtains easily the ten–dimensional effective induced metric: $\overline{g}_{ij} \equiv \partial_i x^p \partial_j x^q g_{pq} = g_{\text{eff},ij}$; the result is, by construction, the Born–Infeld lagrangian of (3.47).

Due to the relation (4.12), which holds formally also in sourceless supergravity, the reduction of the kinetic term for the three–form is now standard, i.e. like in the case of $D = 11$ Sugra $\rightarrow D = 10$, $IIA$ Sugra:

$$\int_{M_{11}} H_4 * H_4 = 2\pi R \int_{M_{10}} \left( R_4 * R_4 + e^{-2\Phi} H_3 * H_3 \right).$$

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)

\hspace{1cm} (4.13)
The kinetic terms obtained this way, divided by $2\pi R$, coincide with the ones of (3.47).

### 4.2.2 Wess–Zumino term

The reduction of the Wess–Zumino form $L_{11}$ is a little bit more complicated. In principle one has to substitute the above expressions for $B_3$ and $K_4$, (4.11) and (4.8), together with an analogous reduction for $f_7$ in $L_{11}$ and to perform the computation. Actually one can take advantage from the invariance of the eleven–dimensional Wess–Zumino action under (2.18)–(2.20) to simplify this computation. Its details are given in the appendix, here we quote the result. Modulo a closed form one gets

$$
\frac{1}{2\pi R} L_{11} = -\frac{1}{2} A_3 dA_3 H_3 \frac{dy^5}{2} - \frac{g}{2} da_2 A_3^{(0)} J_5 + \frac{2\pi G}{g} H_3 X_7 \frac{dy^5}{2} + \frac{g^2}{24} A_2 \chi_4 J_4 \frac{d(y^5)^3}{2}. 
$$

The integral over the eleventh coordinate is trivial, $\int \frac{dy^5}{2} = \int \frac{d(y^5)^3}{2} = 1$, and $\frac{1}{2\pi R} \int_{M_{11}} L_{11}$ coincides with the ten–dimensional Wess–Zumino action in (3.48), because thanks to the relations (2.13) and (4.1) we have $g^2 = 2\pi G = 2\pi G$.

### 5 Concluding remarks

The construction of the effective action for the interacting system $NS5$–brane/$IIA$–supergravity presented in this paper is based on a consistent solution of the magnetic equation (Bianchi–identity) $dH_3 = gJ_4$, in terms of a two–form $A_2$ and a Chern–kernel. In principle duality allows to circumvent this problem by introducing a dual potential $A_6$ in which case this Bianchi–identity would become an equation of motion. However, there exists no formulation of $IIA$–supergravity in terms of only $A_6$, therefore one has necessarily to solve the magnetic equation. Form this point of view the $5$–brane is really a dual object. The magnetic equation allows essentially for two classes of solutions, Dirac–branes or Chern–kernels, the latter being in some sense a subclass of the former. As a matter of fact you have to choose a Chern–kernel solution whenever the anomaly polynomial contains an Euler–form, the eleven–dimensional $M5$–brane being an exception.

So the case considered in this paper is a prototype for a rather general situation: a dual or selfdual ($p$ or $D$)–brane with a normal bundle Euler–form in the anomaly polynomial, see e.g. [3, 8, 17, 18]. The Chern–kernel approach presented in this paper has general validity and can be extended to all these cases: it leads to well defined potentials and to a consistent anomaly cancelling classical action. In particular, in the case of intersecting $D$–branes the inflow mechanism is based at present on the cohomological identification [3]

$$
J_{M_1} J_{M_2} \sim J_{M_1 \cap M_2} \chi[N_{12}],
$$

where $\chi[N_{12}]$ indicates the Euler characteristic of the intersection of the normal bundles of the $D$–brane worldvolumes $M_1$ and $M_2$, and $J_{M_i}$ denote the Poincaré duals of $M_i$. From a local point of view (in the sense of pointwise) this formula has no meaning since the
product of currents $J_{M_1}J_{M_2}$ is not defined, not even in the sense of distributions. Also in this case the Chern–kernel approach allows to overcome this difficulty and to realize a local cancellation mechanism which does not make use of the identification (5.1), in analogy with the $IIA\; NS5$–brane where in our Chern–kernel approach the analogous relation $J_4J_4 \sim J_4\chi_4$ is never enforced.

We have seen that even currents, like $J_4$, can be written as the differential of an odd form, like $K_3$, that transforms anomalously under the normal bundle group, while odd currents like $J_5$ can be written as the differential of an invariant form, $K_4$. In the first case the potential is non–invariant while in the latter it is. In $D = 10$ even currents correspond to even brane worldvolumes where anomalies can potentially appear explaining the appearance of anomalous forms $K_{2n+1}$, realizing the inflow cancellation mechanism. Form this point of view the appearance of an invariant Chern–kernel $K_4$ in the case of the $M5$–brane seems rather strange. What characterizes eventually the exceptionality of this case is that eleven–dimensional supergravity has a Wess–Zumino term which is cubic in the three–form potential and that the normal bundle anomaly of the $M5$–brane does not factorize; moreover the Euler characteristics of the normal bundle is zero. In the reduction process from eleven to ten dimensions we have shown how those different features are related in a non trivial way, in particular the appearance of the non–invariant forms $K_3$ and $A_2$ from the invariant ones $K_4$ and $B_3$.

Acknowledgements. It is a pleasure for the authors to thank P.A. Marchetti for very useful discussions. This work is supported in part by the European Community’s Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime.

Appendix A: properties of the three–form $K_3$

In this section we calculate the explicit expression of the form $Q_2$ in equation (3.11) and prove equation (3.13), i.e. $Q_2^{(0)} = 0$.

Starting from the definition of $K_3$, (3.3), (3.4) and (3.7), one can split $K_3$ in two terms: the Coulomb–like form $C_3$, satisfying $dC_3 = J_4$, and the differential of a two–form $\Phi_2$. An explicit calculation yields

$$K_3 = C_3 + d\Phi_2$$

$$\Phi_2(\hat{u}, W) = \frac{1}{4(2\pi)^2} \varepsilon^{r_1\ldots r_4} \hat{u}^r_1 (2d\hat{u}^r_2 + (\hat{u}W)^r_2) W^{r_3r_4}$$

$$C_3(\hat{u}) = \frac{1}{3(2\pi)^2} \varepsilon^{r_1\ldots r_4} \hat{u}^r_1 \hat{u}^r_2 \hat{u}^r_3 \hat{u}^r_4,$$

where, in particular, the form $C_3$ is independent of $W$. 

25
We remember the anomalous transformation law of $K_3$ under $SO(4)$ rotations, implied by the presence of the Chern–Simons form $\chi_3$ in $K_3$. Under $SO(4)$ we have

$$\hat{u}^{rs}(\Lambda) = \Lambda^{rs}u^s, \quad W'(\Lambda) = \Lambda W\Lambda^T - \Lambda d\Lambda^T,$$

(5.5)

where $\Lambda$ is an $SO(4)$–matrix, and $K_3$ transforms as

$$K_3(\hat{u}', W') = K_3(\hat{u}, W) + d\chi_2(W, \Lambda),$$

(5.6)

where $d\chi_2(W, \Lambda)$ is proportional to $d[ tr(\Lambda^{-1}dW)] + \frac{1}{3} tr(\Lambda^{-1}d\Lambda)^3$.

As stated in the text the form $K_3$ changes under a change of normal coordinates and under a change of the extension of the $SO(4)$–connection. In both cases we have $K' - K_3 = dQ_2$. We consider first a change of normal coordinates. This amounts to a transformation $u \to u'(\sigma, u)$ such that, (see \[6\])

$$u'^{(r)}(\sigma, 0) = 0, \quad \frac{\partial u'^{(r)}}{\partial u^s} \bigg|_{u=0} = \delta^{rs}.$$

(5.7)

Since $\hat{u}'^{rs} \hat{u}'^{rs} = 1$, there exists a matrix $\Lambda(\sigma, u) \in SO(4)$ such that

$$\hat{u}'^{rs} = \Lambda^{rs}u^s,$$

(5.8)

and (5.7) implies that for $\vec{u} \to 0$ we have the crucial property

$$\Lambda^{rs} \sim \delta^{rs} + o(\vec{u}) \Rightarrow \Lambda^{(0)} = 1.$$

(5.9)

For the new normal coordinate system we have, thanks to (5.6) and (5.2),

$$K'_3 = K_3(\hat{u}', W) = K_3(\hat{u}, W'(\Lambda^{-1})) + d\chi_2(W'(\Lambda^{-1}), \Lambda)$$

(5.10)

$$= C_3(\hat{u}) + d\Phi_2(\hat{u}, W'(\Lambda^{-1})) + d\chi_2(W'(\Lambda^{-1}), \Lambda).$$

(5.11)

In the difference $K'_3 - K_3$ the Coulomb form cancels out and we are left with

$$K'_3 - K_3 = d \left[ \Phi_2(\hat{u}, W'(\Lambda^{-1})) - \Phi_2(\hat{u}, W) + \chi_2(W'(\Lambda^{-1}), \Lambda) \right] = dQ_2,$$

(5.12)

which gives an explicit formula for $Q_2$. Using (5.3), and setting $\Delta W \equiv W'(\Lambda^{-1}) - W$, we can evaluate the difference

$$\Phi_2(\hat{u}, W'(\Lambda^{-1})) - \Phi_2(\hat{u}, W)$$

(5.13)

$$= -\frac{1}{4(2\pi)^2} \epsilon^{r_1...r_4}u^r_1 \left( 2\hat{u}^{r_2}W^{r_3}r_4 + (\hat{u}W)'^{(r_2)}W^{r_3}r_4 + \left( \hat{u}W'(\Lambda^{-1}) \right)^{(r_2)}W^{r_3}r_4 \right).$$

Thanks to (5.9) we have now $\Delta W^{(0)} = 0$, and $\chi_2^{(0)}(W'(\Lambda^{-1}), \Lambda) = \chi_2(W^{(0)}, 1) = 0$. This implies also that $Q_2^{(0)} = 0$.

Under a change of extension of $W$ instead we have

$$W' = W + \Delta W, \quad \Delta W^{(0)} = 0,$$

(5.14)
since the value of $W$ on the brane is fixed. Accordingly $K_3$ transforms as

$$K_3' = K_3(\hat{u}, W') = C_3(\hat{u}) + d\Phi_2(\hat{u}, W'), \quad (5.15)$$

so that

$$K_3' - K_3 = d[\Phi_2(\hat{u}, W') - \Phi_2(\hat{u}, W)] = dQ_2. \quad (5.16)$$

Thus $Q_2$ coincides formally with (5.13) and $Q_2^{(0)}$ vanishes again, due to $\Delta W^{(0)} = 0$.

### Appendix B: the form $G_3$

The decomposition of the invariant four–form $K_4$ (4.8) involves the $SO(4)$–invariant three–form

$$G_3 = \frac{1}{10(2\pi)^2} \varepsilon^{r_1\ldots r_4} \hat{u}^{r_1} \left[ 4\rho^2 y^5 D\hat{u}^{r_2} D\hat{u}^{r_3} D\hat{u}^{r_4} + 4\rho (T^{r_2 r_3} + \rho^2 D\hat{u}^{r_2} D\hat{u}^{r_3}) L^{r_4} - 4\rho^2 y^5 D\hat{u}^{r_2} L^{r_3} L^{r_4} - \frac{4}{3} \rho^3 L^{r_2} L^{r_3} L^{r_4} \right], \quad (5.17)$$

where

$$\hat{u}^r = \frac{u^r}{|\vec{u}|}, \quad (5.18)$$

$$\rho = \frac{|\vec{u}|}{\sqrt{|\vec{u}|^2 + (y^5)^2}}, \quad (5.19)$$

and $D\hat{u}^r = d\hat{u}^r + \hat{u}^s W^{sr}$ indicates the covariant derivative with respect to $SO(4)$. Since the ten–dimensional 5–brane is defined by $u^r = 0$ we have $\rho J_4 = 0$, and therefore also $G_3 J_4 = 0$ or, equivalently, the pullback of $G_3$ on the ten–dimensional 5–brane vanishes.

### Appendix C: reduction of $L_{11}$

To reduce the eleven–dimensional Wess–Zumino action to ten dimensions we have to evaluate the expression of $L_{11}$ in (2.16) under the substitutions (4.11) for $B_3$ and (4.8) for $K_4$, where eventually the forms $A_3$, $A_2$, $\chi_3$ and $\omega_3$ are considered as ten–dimensional fields, i.e. independent of the eleventh coordinate.

Before performing the above substitutions it is convenient to eliminate the form $G_3$ from $K_4$ and $B_3$, taking advantage from the invariance of $L_{11}$ under the transformations (2.18)–(2.20) and choosing $Q_3 = -G_3$. These transformations leave $L_{11}$ invariant if $Q_3$ has vanishing pullback on the eleven–dimensional 5–brane; in the reduced geometry it is sufficient that it has vanishing pullback on the ten–dimensional 5–brane, as does $G_3$ (see appendix B). The new objects simplify as

$$K_4' = K_4 - dG_3 = \frac{1}{2} \chi_4 y^5 - \frac{1}{2} \omega_3 d\hat{y}^5 \quad (5.20)$$

$$B_3' = B_3 + \bar{y}G_3 = A_3 - \pi R\hat{y}^5 (dA_2 + g\chi_3). \quad (5.21)$$

27
For what concerns the transformed seven–form $f'_7$ it is sufficient to determine it modulo a closed form $df_6$, since it would modify the Wess–Zumino action by a term which is local on the 5–brane, $\int_{M_1} df_6 K_4 = - \int_{M_6} f_6$. This means that it is sufficient to find a solution of the equation

$$\frac{1}{4} df'_7 = K'_4 K'_4 = \frac{1}{4} (\hat{g}^5)^2 \chi_4 \chi_4 + \frac{1}{2} \hat{g}^5 \delta (\hat{g}^5)^2 \chi_4,$$

for $f'_7$. A convenient solution is

$$f'_7 = \left((\hat{g}^5)^2 \chi_4 - \omega_3 d(\hat{g}^5)^2\right) \chi_3.$$  \hfill (5.22)

Modulo a closed form we can therefore rewrite the eleven–form in (2.16) by replacing $B_3 \rightarrow B'_3, K_4 \rightarrow K'_4, f_7 \rightarrow f'_7,

$$L_{11} = \frac{1}{6} B'_3 d B'_3 d B'_3 - \frac{\check{g}}{2} d a_2 A'_3(0) J_5 + \frac{\check{g}}{2} B'_3 d B'_3 K'_4 +$$
$$+ \frac{\check{g}^2}{2} B'_3 K'_4 K'_4 + \frac{\check{g}^3}{24} K'_4 f'_7 + \frac{2 \pi G}{\check{g}} \hat{X}_7 H_4. \hfill (5.23)$$

The substitution of (5.20)–(5.22) is now a mere exercise, and the result can be divided into powers of $A_3$: the term $A_3 d A_3 d A_3$ vanishes of course because it is an eleven–form in $D = 10$; the terms quadratic in $A_3$ lead easily to the first term in (1.14), while the linear terms drop out apart from the term $da_2 A'_3(0) J_5$. The interesting terms are the ones without $A_3$, since they are related to anomalies and we evaluate them separately. The first, third and fourth terms in (5.23) produce a contribution

$$\frac{\check{g}^2 (\hat{g}^5)^3}{48} \left(\hat{g}^5 \omega_3 \chi_4 \chi_4 - 2 \pi R (d A_2 + g \chi_3) \chi_4 J_4\right), \hfill (5.24)$$

which is invariant under $SO(4)$, as are the three terms from which it comes. The fifth term instead, which carries the anomaly, becomes

$$\frac{\check{g}^3}{24} K'_4 f'_7 = \frac{\check{g}^3 (\hat{g}^5)^3}{48} (\chi_3 \chi_4 J_4 - \omega_3 \chi_4 \chi_4). \hfill (5.25)$$

We can check that it carries still the correct $SO(4)$–anomaly (the second term is invariant):

$$\delta \left(\frac{\check{g}^2}{24} \int_{M_1} K'_4 f'_7\right) = - \frac{\check{g}^2}{24} \int_{M_1} \chi_2 \chi_4 J_4 \frac{d(\hat{g}^5)^3}{2}. \hfill (5.26)$$

This formula maintains its meaning both from the eleven– and ten–dimensional points of view. In $D = 11$ $\hat{g}^5$ is given by $\hat{g}^5 = \sqrt{(\hat{g}^5)^2 + \bar{a}^2}$ which multiplied by $J_4$ reduces to the

\[ \text{We remind that the } D = 11 \text{ theory carries an } SO(5) \text{–anomaly associated to } P_8(A). \text{ Under dimensional reduction we have } A^{ab} = (A^a \equiv W^a, A^b \equiv \check{L}'), \text{ and one has the decomposition } P_8(A) = \chi_4(W) \chi_4(W) + dl_7, \text{ where } l_7 \text{ is some } SO(4) \text{–invariant form. For the Chern–Simons forms one has } P_7 = \chi_3 \chi_4 + l_7 + dl_6, \text{ implying that the } SO(5) \text{–anomaly reduces to the canonical } SO(4) \text{–anomaly } \int \chi_2 \chi_4, \text{ modulo a trivial cocycle. Our specific choice of } f'_7 \text{ amounts to the subtraction of a local counterterm which restores the canonical form of the anomaly, as seen from (5.24).} \]
sign of $y^5$, hence $J_4 \frac{d(y^5)}{2} = J_4 dy^5 \delta(y^5) = J_5$, and (5.26) reduces to $-\frac{g^2}{4} \int_{M_6} \chi_2 \chi_4$. From the ten–dimensional point of view we have to take $\chi_4$ and $\chi_2$ to be independent of $\hat{y}^5$ and we can integrate $\int_{-1}^{1} \frac{d(y^5)}{2} = 1$, and one obtains the same anomaly as before.

Summing up (5.24) and (5.25) only the $A_2$–term survives and, apart from a closed form, one gets as anomaly cancelling term

$$2\pi R \left( \frac{g^2}{24} A_2 \chi_4 J_4 \frac{d(y^5)}{2} \right),$$

which amounts to the last term in (4.14).

The reduction of the last term in (5.23), which carries the target sp ace anomaly, is standard. We can decompose the $SO(1,10)$ Chern–Simons form as $X_7 = X_7 + Z_7 + (dx^{10} + A_1)Z_6$, where $X_7$, $Z_7$ and $Z_6$ are independent of $x^{10}$; $X_7$ is the $SO(1,9)$ Chern–Simons form, and $Z_7$ and $Z_6$ are $SO(1,9)$–invariant. Recalling also the decomposition $H_4 = R_4 + (dx^{10} + A_1)H_3$ one gets

$$X_7 H_4 = (H_3 X_7 + [H_3 Z_7 + R_4 Z_6]) dx^{10}.$$ 

The term $H_3 Z_7 + R_4 Z_6$ is local and invariant and corresponds to a non minimal contribution to the ten–dimensional effective action, while $H_3 X_7 dx^{10} = \pi RH_3 X_7 \hat{y}^5$ produces the third term in (4.14).

References

[1] E. Witten, J. Geom. Phys. 22 (1997) 103.

[2] R. Bott and L. W. Tu, Differential forms in algebraic geometry, Springer Verlag 1978.

[3] Y. K. Cheung and Z. Yin, Nucl. Phys. B517 (1998) 69.

[4] K. Becker and M. Becker, Nucl. Phys. B577 (2000) 156.

[5] D. Freed, J.A. Harvey, R. Minasian and G. Moore, Adv. Theor. Math. Phys. 2 (1998) 601.

[6] K. Lechner and P.A. Marchetti, JHEP 01 (2001) 003.

[7] R. Gueven, Phys. Lett. B276 (1992) 49.

[8] K. Lechner and M. Tonin, Nucl. Phys. B475 (1996) 545.

[9] K. Lechner, P.A. Marchetti and M. Tonin, Phys. Lett. B524 (2002) 199.

\[11^1\]This decomposition holds again modulo a closed form, see the footnote on page 27.
[10] G. de Rham, *Differentiable manifolds, Forms, Currents, Harmonic Forms*, Springer Verlag 1984.

[11] F. R. Harvey and H. B. Lawson Jr., *A theory of characteristic currents associated with a singular connection*, Asterisque 213 (1993), Sociétè Mathématique de France.

[12] F. Morgan, *Geometric Measure Theory*, Academic Press, San Diego, California (1988).

[13] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, Phys. Rev. Lett. **78** (1997) 4332.

[14] P. Pasti, D. Sorokin and M. Tonin, Phys. Lett. **B398** (1997) 41.

[15] G. Dall’Agata, K. Lechner and M. Tonin, JHEP, **9807** (1998) 017.

[16] I. Bandos, A. Nurmagambetov and D. Sorokin, Nucl. Phys. **B586** (2000) 115.

[17] P. Brax and J. Mourad, Phys. Lett. **B416** (1998) 295; J. Mourad, Nucl. Phys. **B512** (1998) 199.

[18] C. Scrucca and M. Serone, Nucl. Phys. **B556** (1999) 197.