Polynomial Heisenberg algebras, multiphoton coherent states and geometric phases

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Abstract

In this paper we will realize the polynomial Heisenberg algebras through the harmonic oscillator. We are going to construct then the Barut–Girardello coherent states, which coincide with the so-called multiphoton coherent states, and we will analyze the corresponding Heisenberg uncertainty relation and Wigner distribution function for some particular cases. We will show that these states are intrinsically quantum and cyclic, with a period being a fraction of the oscillator period. The associated geometric phases will be as well evaluated.

Keywords: coherent states, multiphoton coherent states, geometric phases, Polynomial Heisenberg algebras

1. Introduction

Polynomial Heisenberg algebras (PHA) have become important deformations of the oscillator (Heisenberg–Weyl) algebra, that can be realized by finite-order differential ladder operators $\mathcal{L}^\pm$ which commute with the system Hamiltonian $\mathcal{H}$ as the harmonic oscillator does, but between them giving place to an $m$th degree polynomial $P_m(\mathcal{H})$ in $\mathcal{H}$ \cite{1–9}. The PHA are realized in a natural way by the SUSY partners of the harmonic and radial oscillators. Moreover, for degrees 2 and 3 they can be straightforwardly connected with the Painlevé IV and V equations, respectively. Thus, a simple method for generating solutions to these nonlinear second-order ordinary differential equations has been recently designed \cite{9–11}.

In order to determine the energy spectrum for systems ruled by PHA, it is of the highest importance to identify the extremal states, i.e. those states which are simultaneously eigenstates of the Hamiltonian and belong to the kernel of the annihilation operator $\mathcal{L}^-$. Once they have been found, departing from each one of them a ladder of energy eigenstates and eigenvalues (either finite or infinite) will be generated through the iterated action of the creation operator $\mathcal{L}^+$. The Hamiltonian spectrum then turns out to be the union of all these ladders of eigenvalues. However, for most of the systems studied until now the number of extremal states is less than the order of the differential annihilation operator. It would be important to know if there is a Hamiltonian for which the number of extremal states coincides with the order of the ladder operators which are involved. Moreover, we feel that it is time to identify the simplest system realizing in a non-trivial way the PHA.

In this paper we will show that the system we are looking for is precisely the harmonic oscillator. Moreover, it will be seen that our algebraic method generalizes, thus it also includes, the standard approach. In order to implement our treatment, we will take the $k$th power of the standard (first-order) annihilation and creation operators $a^\pm$ as the new ladder operators generating the PHA. This will cause that the Hilbert space $\mathcal{H}$ decomposes as a direct sum of $k$ orthogonal subspaces $\{\mathcal{H}_i, i = 1, \ldots, k\}$. In each $\mathcal{H}_i$ there will be just one extremal state, and by applying repeatedly the creation operator $(a^+)^k$ onto this state we will generate a ladder of new eigenstates, which will constitute the basis for this subspace.

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Once the algebraic treatment has been completed, it will be natural to look for the Barut–Girardello coherent states of our system, as eigenstates of the annihilation operator with complex eigenvalue. We will study some important physical quantities for these states, such as the Heisenberg uncertainty relation and Wigner distribution function. We will see that these states coincide with the so-called multiphoton coherent states [12, 13] or kaleidoscope coherent states [14, 15] or crystallized Schrödinger cat states [16, 17]. Moreover, it will be shown that they are intrinsically quantum cyclic states, with a period which is equal to the fraction $1/k$ of the oscillator period. As for any state evolving cyclically one can associate a geometric phase, which depends only of the geometry of the state space (the projective Hilbert space), we will calculate then such a phase for the cyclic motion performed by our CS.

This paper is organized as follows. In section 2 the PHA will be briefly presented, while in section 3 we will realize them through the harmonic oscillator. We will describe as well the way of generating the energy spectrum and the eigenstates of the system by using PHA. In section 4 the MCS will be generated, as eigenstates of the $k$th power of the annihilation operator $a^-$, and some particular examples for these states will be analyzed, together with properties such as the Heisenberg uncertainty relation and Wigner distribution function. In the same section we will show that these states are cyclic and we will evaluate their associated geometric phase. Finally, in section 5 our conclusions will be presented.

2. Polynomial Heisenberg algebras (PHA)

Let us recall that the standard harmonic oscillator algebra, also known as Heisenberg–Weyl algebra, is generated by three operators $H, a^+, a^-$ which satisfy the following commutation relations:

$$[H, a^\pm] = \pm a^\pm,$$

(1a)

$$[a^-, a^+] = 1,$$

(1b)

where the so-called number operator $N$ is linear in $H$, i.e.

$$N = a^+ a^- = H - \frac{1}{2}.$$

(2)

On the other hand, the PHA of degree $m - 1$ are deformations of the Heisenberg–Weyl algebra of kind:

$$[H, \mathcal{L}^\pm] = \pm \omega \mathcal{L}^\pm,$$

(3a)

$$[\mathcal{L}^-, \mathcal{L}^+] \equiv N_m(H + \omega) - N_m(H) \equiv P_{m-1}(H),$$

(3b)

where the analog of the number operator

$$N_m(H) \equiv \mathcal{L}^+ \mathcal{L}^-$$

(4)

is a polynomial of degree $m$ in $H$.

It is possible to supply differential realizations of the PHA by assuming that $H$ is the following one-dimensional Schrödinger Hamiltonian

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x),$$

(5)

and $\mathcal{L}^\pm$ are differential ladder operators of order $m$ so that $N_m(H)$ is an $m$th degree polynomial in $H$, which can be factorized as

$$N_m(H) = \sum_{i=1}^{m} (H - \epsilon_i).$$

(6)

Thus, $P_{m-1}(H)$ is a polynomial of degree $m - 1$ in $H$.

The spectrum of $H$, $\text{Sp}(H)$, can be generated by studying the Kernel $K_{\mathcal{L}}$ of $\mathcal{L}^-$ as follows

$$\mathcal{L}^- \psi = 0 \quad \Rightarrow \quad \mathcal{L}^+ \mathcal{L}^- \psi = \prod_{i=1}^{m} (H - \epsilon_i) \psi = 0.$$  

(7)

Since $K_{\mathcal{L}}$ is invariant under $H$,

$$\mathcal{L}^- H \psi = (H + \omega) \mathcal{L}^- \psi = 0 \quad \forall \quad \psi \in K_{\mathcal{L}},$$

(8)

then it is natural to take as a basis of $K_{\mathcal{L}^-}$ the states $\psi_{\epsilon_i}$ such that

$$H \psi_{\epsilon_i} = \epsilon_i \psi_{\epsilon_i},$$

(9)

which are called extremal states. From them we can generate, in principle, $m$ energy ladders with spacing $\Delta E = \omega$. However, if just $s$ extremal states satisfy the boundary conditions of the problem (we call them physical extremal states and order them as $\{ \psi_{\epsilon_i}, i = 1, \ldots, s\}$), then from the iterated action of $\mathcal{L}^+$ we will obtain $s$ physical energy ladders (see an illustration in figure 1). We conclude that $\text{Sp}(H)$ can have up to $m$ infinite ladders, with a spacing $\Delta E = \omega$ between steps.

3. PHA and the harmonic oscillator

As mentioned previously, we can realize the PHA through systems ruled by one-dimensional Schrödinger Hamiltonians. In particular, let us consider the harmonic oscillator, for which we will take $V(x) = \frac{1}{2}x^2$ in equation (5). Moreover, let us construct the deformed ladder operators $a_g^-, a_g^+$ from the
standard annihilation and creation operators $a^-$, $a^+$ as follows:

$$a^-_k = (a^-)^k \quad a^+_k = (a^+)^k. \tag{10}$$

The operator set $\{H, a^-_k, a^+_k\}$ generates a PHA of degree $k - 1$, since it is fulfilled

$$[H, a^+_k] = \pm ka^-_k, \tag{11a}$$
$$[a^-_k, a^+_k] = N(H + k) - N(H), \tag{11b}$$
$$N(H) = a^+_k a^-_k = \prod_{i=1}^{k} \left(H - i + \frac{1}{2}\right). \tag{11c}$$

According to this formalism, we can identify now $k$ extremal state energies:

$$\mathcal{E}_i = i - \frac{1}{2}, \quad i = 1, \ldots, k, \tag{12}$$

so that the eigenvalues for the $i$th ladder turn out to be:

$$\mathcal{E}^i_n = \mathcal{E}_i + kn \quad n = 0, 1, \ldots, \tag{13}$$

while the associated eigenstates are

$$|\psi_{i,n}\rangle = |kn + i - 1\rangle = \frac{(i - 1)!}{(kn + i - 1)!} (a^-_k)^{i-1}. \tag{14}$$

According to the results of the previous section, the spectrum of the system becomes (see an example in figure 2)

$$\text{Sp}(H) = \{\mathcal{E}^0_0, \mathcal{E}^0_1, \ldots\} \cup \{\mathcal{E}^1_0, \mathcal{E}^1_1, \ldots\} \times \cup \ldots \cup \{\mathcal{E}^k_0, \mathcal{E}^k_1, \ldots\}, \tag{15}$$

which is nothing but the harmonic oscillator spectrum, as it should be. In conclusion, we have addressed the harmonic oscillator based on an uncommon algebraic structure, the PHA, which nonetheless is associated in a natural way to this physical system.

### 4. Multiphoton coherent states

After identifying the alternative algebraic structures underlying the harmonic oscillator, let us look for the corresponding coherent states. Although these states can be defined in several different ways, here we will build them as eigenstates of the deformed annihilation operator of equation (10), i.e.

$$a^-_k |\alpha\rangle_j = \alpha |\alpha\rangle_j, \tag{16}$$

where

$$|\alpha\rangle_j = \sum_{n=0}^{\infty} C_n |kn + j\rangle, \quad j = 0, \ldots, k - 1. \tag{17}$$

A standard calculation leads to the following normalized coherent states

$$|\alpha\rangle_j = \frac{|\alpha|^{j/k}}{\sqrt{[H_{kj}|\alpha^{j/k}\rangle]^j}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(kn + j)!}} |kn + j\rangle, \tag{18}$$

where

$$H_{kj}(x) = \sum_{m=0}^{\infty} \frac{x^{kn+j}}{(km+j)!}. \tag{19}$$

is a generalized hyperbolic function of order $k$ and kind $j$ [18].

Let us note that each coherent state $|\alpha\rangle_j$ is just a linear combination of the eigenstates of the harmonic oscillator Hamiltonian belonging to the $(j + 1)$th energy ladder, which generate the subspace $\mathcal{H}_{j+1}$. These coherent states are called MCS in the literature, since they are superpositions of Fock states whose difference of energies is always an integer multiple of $k$ [19–21]. This means that it is assumed that $k$ is the number of photons required to ‘jump’ from the state $|kn + j\rangle$ to the next one $|kn + j + 1\rangle$ in the $(j + 1)$-th energy ladder.

Let us analyze next some particular examples of these coherent states, for several values of $k$.

### 4.1. SCS ($k = 1$)

If we take $k = 1$ and $j = 0$ in equation (18), we recover the well known standard coherent states (SCS):

$$|\alpha\rangle_0 = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{20}$$

For these states it turns out that:

$$\langle x \rangle_0 = \sqrt{2} \text{Re}(\alpha), \quad \langle p \rangle_0 = \sqrt{2} \text{Im}(\alpha), \tag{21a}$$
\[
\langle x^2 \rangle_0 = \frac{1}{2} + [\sqrt{2} \text{ Re}(\alpha)]^2 , \quad (21b)
\]
\[
\langle p^2 \rangle_0 = \frac{1}{2} + [\sqrt{2} \text{ Im}(\alpha)]^2 , \quad (21c)
\]
\[
(\Delta x)_0^2 = (\Delta p)_0^2 = (\Delta x)_0 (\Delta p)_0 = \frac{1}{2} , \quad (21d)
\]
where \( x \) and \( p \) are the position and momentum operator, respectively. This implies that the SCS are minimum uncertainty states.

### 4.2. MCS \((k > 1)\)

For \( k \geq 2 \) the mean value of the position and momentum operators, their squares and the Heisenberg uncertainty relation are given by:

\[
\langle x \rangle_j = \langle p \rangle_j = 0 , \quad (22a)
\]
\[
\langle x^2 \rangle_j = |\alpha| |\alpha|_j^2 + \frac{1}{2} + \text{Re}(\alpha) \delta_{j2}, \quad (22b)
\]
\[
\langle p^2 \rangle_j = |\alpha| |\alpha|_j^2 + \frac{1}{2} - \text{Re}(\alpha) \delta_{j2}, \quad (22c)
\]
\[
(\Delta x)_j^2 (\Delta p)_j^2 = \left( |\alpha| |\alpha|_j^2 + \frac{1}{2} \right)^2 - [\text{Re}(\alpha)]^2 \delta_{j2}, \quad (22d)
\]
\[
|\alpha| |\alpha|_j^2 = \frac{|\alpha|^{2/k}}{H_{j,k}(|\alpha|^{2/k})} H_{j,k_{n+1}+j-1}(|\alpha|^{2/k}), \quad (22e)
\]
where \( \delta_{mn} \) is the Kronecker delta. In general, the MCS are not minimum uncertainty states, except for \( j = 0 \) in the limit \( \alpha \to 0 \), since then \( |\alpha|_0 \to |0| \).

Another important quantity is the mean energy value for a system in a MCS, which turns out to be

\[
\langle H \rangle_j = |\alpha| |\alpha|_j^2 + \frac{1}{2}. \quad (23)
\]

On the other hand, in order to guarantee a partial completeness of the MCS in the subspace \( \mathcal{H}_{j+1} \) to which \( |\alpha\rangle_j \) belongs, i.e.

\[
\int |\alpha\rangle_j \langle \alpha| \mu_j(\alpha) = I_{j+1} , \quad (24)
\]
with the measure \( \mu_j(\alpha) \) given by

\[
d\mu_j(\alpha) = \frac{1}{\pi} \frac{H_{j,k}(|\alpha|^{2/k})}{|\alpha|^{2/k} + 1} f_j(|\alpha|^2) d|\alpha| d\varphi , \quad (25)
\]
the function \( f_j(x) \) must satisfy

\[
\int_0^{\infty} x^{n-1} f_j(x) dx = \Gamma(kn + j + 1) . \quad (26)
\]

If equations (24)–(26) are fulfilled for any \( j = 0, \ldots, k - 1 \), then it is valid a completeness relation in the full Hilbert space \( \mathcal{H} \), since \( I_1 + \ldots + I_k = I \). This implies that any state can be decomposed in terms of the MCS.

Finally, the time evolution of a MCS becomes

\[
U(t)|\alpha\rangle_j = \exp \left( -i \left( \frac{1}{2} \right) t \right) |\alpha(t)\rangle_j, \quad \alpha(t) = \alpha \exp(-i t). \quad (27)
\]

This equation reflects clearly the cyclic nature of the MCS, which have a period given by \( \tau = 2\pi/k \) (the fraction 1/k of the harmonic oscillator period \( T = 2\pi \)).

In order to clarify better these ideas, let us discuss explicitly two particular cases of MCS, which are known in the literature with given specific names.

#### 4.2.1. Biphoto coherent states with \( k = 2 \).

For \( k = 2 \) we get:

\[
|\alpha\rangle_0 = \frac{1}{\left| H_{2,0}(|\alpha|) \right|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(2n)!}} |2n\rangle , \quad (28a)
\]
\[
|\alpha\rangle_1 = \left[ \alpha \tanh |\alpha| \right] |\alpha\rangle_0 , \quad (28b)
\]

These states are called even and odd coherent states, respectively, since only even or odd energy eigenstates of the oscillator are involved in the corresponding superposition [22, 23] (see also [24–30]).

For the states in equations (28a) and (28b), the expressions of equations (22d–23) turn out to be:

\[
|\alpha| |\alpha|_0^2 = |\alpha| \tanh |\alpha| , \quad (29a)
\]
\[
|\alpha| |\alpha|_1^2 = |\alpha| \cosh |\alpha| , \quad (29b)
\]
\[
(\Delta x)_0^2 (\Delta p)_0^2 = \left( |\alpha| \tanh |\alpha| + \frac{1}{2} \right)^2 - [\text{Re}(\alpha)]^2 , \quad (29c)
\]
\[
(\Delta x)_1^2 (\Delta p)_1^2 = \left( |\alpha| \cosh |\alpha| + \frac{1}{2} \right)^2 - [\text{Re}(\alpha)]^2 , \quad (29d)
\]
\[
\langle H \rangle_0 = |\alpha| \tanh |\alpha| + \frac{1}{2} , \quad (29e)
\]
\[
\langle H \rangle_1 = |\alpha| \cosh |\alpha| + \frac{1}{2} . \quad (29f)
\]

It is straightforward to check that the even states are minimum uncertainty states for \( \alpha = 0 \), while in general the odd states do not have this property (see figure 3).

#### 4.2.2. Triphoton coherent states with \( k = 3 \).

By making now \( k = 3 \) we have [31, 32]:

\[
|\alpha\rangle_0 = \frac{1}{\left| H_{3,0}(|\alpha|) \right|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(3n)!}} |3n\rangle , \quad (30a)
\]
\[
|\alpha\rangle_1 = \left[ \alpha \tanh |\alpha| \right] |\alpha\rangle_0 , \quad (30b)
\]
\[
|\alpha\rangle_2 = \left[ \alpha \cosh |\alpha| \right] |\alpha\rangle_0 . \quad (30c)
\]

We call these states the good, the bad and the ugly coherent states, respectively [32]; they have been also called trinity states in the literature [16, 17].

Similarly to the previous case, the Heisenberg uncertainty relation and mean energy value for these states are obtained by taking \( k = 3 \) in equations (22d–23), which
leads to:

\[ (\Delta x)(\Delta p) = \langle H \rangle = |a(\alpha)|^2 + \frac{1}{2}, \quad j = 0, 1, 2, \quad (31) \]

where

\[
|a(\alpha)|^2 = \frac{1}{H_{3,0}(\alpha^2/3)} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{(3n+2)!} \\
= |\alpha|^{2/3} \left[ e^{\pi \alpha^2/3} - 2 \sin \left( \frac{\alpha}{\sqrt{6}} + \frac{\sqrt{3} |\alpha|^{1/3}}{2} \right) \right]
\]

\[
|a(\alpha)|^2 = \frac{|\alpha|^{2/3}}{H_{3,1}(\alpha^2/3)} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n)!} \\
= |\alpha|^{2/3} \left[ e^{\pi \alpha^2/3} + 2 \cos \left( \frac{\alpha}{\sqrt{6}} + \frac{\sqrt{3} |\alpha|^{1/3}}{2} \right) \right]
\]

\[
|a(\alpha)|^2 = \frac{|\alpha|^{2/3}}{H_{3,2}(\alpha^2/3)} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n+1)!} \\
= |\alpha|^{2/3} \left[ e^{\pi \alpha^2/3} - 2 \sin \left( \frac{\alpha}{\sqrt{6}} - \frac{\sqrt{3} |\alpha|^{1/3}}{2} \right) \right]
\]

We can see that in the vicinity of \( \alpha = 0 \) the uncertainty relation for the triphoton coherent state with \( j = 0 \) achieves the lowest possible value, while for the other ones (with \( j = 1 \) and \( j = 2 \)) it just acquires a local minimum (see figure 4).

4.3. Superposition of SCS

Let us consider now the following unnormalized CS:

\[ |z\rangle = \sum_{n=0}^{\infty} z^n |n\rangle, \quad (33a) \]

\[ |z\rangle = \sum_{n=0}^{\infty} \frac{z^{kn+j}}{\sqrt{(kn+j)!}} |kn+j\rangle, \quad \alpha = z^k. \quad (33b) \]

The states of equation (33a) are the (unnormalized) SCS of section 4.1 while those of equation (33b) are the MCS of section 4.2, in which we have taken \( \alpha = z^k \). It is clear that the SCS are just a particular case of the MCS for \( k = 1, j = 0 \) (see section 4.1).

According to the completeness relationship in equation (24), and the related discussion, any state in \( \mathcal{H} \) can be decomposed in terms of the states \( |z\rangle, j = 0, \ldots, k - 1 \).

In particular, for \( k = 2 \) let us consider the SCS \( |z\rangle \) and \( |\exp(i\pi z)\rangle \), whose eigenvalues have a phase difference of \( \pi \) [13, 33, 34]. It is straightforward to verify that:

\[ |z\rangle = |z\rangle_0 + |z\rangle_1, \quad (34a) \]

\[ |\exp(i\pi z)\rangle = |z\rangle_0 - |z\rangle_1. \quad (34b) \]

From these equations we can solve now the even and odd coherent states in terms of the two standard ones, \( |z\rangle \) and \( |\exp(i\pi z)\rangle \). After normalization we get that:

\[ |z\rangle_0 = \frac{\exp(-|z|^2/2)}{\sqrt{2(1 + \exp(-2|z|^2))}} |z\rangle + |\exp(i\pi z)\rangle, \quad (35a) \]

\[ |z\rangle_1 = \frac{\exp(-|z|^2/2)}{\sqrt{2(1 - \exp(-2|z|^2))}} |z\rangle - |\exp(i\pi z)\rangle, \quad (35b) \]

which means that the states \( |z\rangle_0 \) and \( |z\rangle_1 \) are linear combinations of two SCS with opposite positions in the complex plane.
Note that expressions in equations (34a)–(35b)
are called quantum Fourier transforms [16].

We can use now the wavefunction of a normalized SCS
in order to analyze the time evolution of the even
and odd coherent states $|z\rangle_0$ and $|z\rangle_1$.

(see figure 5). Note that expressions in equations (34a)–(35b)
are called quantum Fourier transforms [16].

We can use now the wavefunction of a normalized SCS
in order to analyze the time evolution of the even
and odd coherent states $|z\rangle_0$ and $|z\rangle_1$.

(35b) become [35]:

$$
\psi^j_0(x) = \langle x|z\rangle_0 = N_0 \left[ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + i\langle p \rangle \right) + \exp \left( -\frac{1}{2}(x + \langle x \rangle)^2 - i\langle p \rangle \right) \right],
$$

(37a)

$$
\psi^j_1(x) = \langle x|z\rangle_1 = N_1 \left[ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + i\langle p \rangle \right) - \exp \left( -\frac{1}{2}(x + \langle x \rangle)^2 - i\langle p \rangle \right) \right],
$$

(37b)

where

$$
N_0 = \left( \frac{1}{\pi} \right)^{1/4} \frac{1}{\sqrt{2(1 + \exp(-\langle x \rangle^2 - \langle p \rangle^2))}},
$$

$$
N_1 = \left( \frac{1}{\pi} \right)^{1/4} \frac{1}{\sqrt{2(1 - \exp(-\langle x \rangle^2 - \langle p \rangle^2))}}.
$$

The time-dependent wavefunctions $\psi^j(x, t) = \langle x|U(t)|z\rangle_j$, $j = 0, 1$ appear by replacing in equations (37a) and (37b) $x \rightarrow (x) \cos t + \langle p \rangle \sin t$ and $\langle p \rangle \rightarrow \langle p \rangle \cos t - (x) \sin t$.

In figures 6 and 7 we illustrate the corresponding probability densities for $j = 0$ and $j = 1$, respectively, for the eigenvalue $z = 1 + i$.

It is important to stress that the coherent states $U(t)|z\rangle_0$ and $U(t)|z\rangle_1$ are cyclic, with a period $\tau = \pi$ which is one half of the oscillator period $T \equiv 2\pi$. This fact implies that both, the even and odd coherent states are intrinsically quantum, since they recover their initial form before the oscillator period has elapsed [30].

On the other hand, for $k = 3$ we consider the three SCS whose eigenvalues posses a phase difference of $2\pi/3$, i.e. $|z\rangle$, $|\exp(2\pi i/3)z\rangle$ and $|\exp(4\pi i/3)z\rangle$. First we express these states in terms of the triphoton coherent states as follows:

$$
|z\rangle = |z\rangle_0 + |z\rangle_1 + |z\rangle_2,
$$

(39a)
As before, from these expressions we can solve the eigenstates $|z_0\rangle$, $|z_1\rangle$, $|z_2\rangle$ of the annihilation operator $a_c^\dagger = (a^\dagger)^3$ in terms of the SCS $|z\rangle$, $|\exp(2\pi i/3)z\rangle$ and $|\exp(4\pi i/3)z\rangle$:

\begin{align}
|z_0\rangle &= N_0 |z\rangle + |\exp(2\pi i/3)z\rangle + |\exp(4\pi i/3)z\rangle, \tag{40a}
|z_1\rangle &= N_1 |z\rangle - \exp(\pi i/3) |\exp(2\pi i/3)z\rangle + \exp(2\pi i/3) |\exp(4\pi i/3)z\rangle, \tag{40b}
|z_2\rangle &= N_2 |z\rangle + \exp(2\pi i/3) |\exp(2\pi i/3)z\rangle + \exp(4\pi i/3) |\exp(4\pi i/3)z\rangle. \tag{40c}
\end{align}

where $N_j$, $j = 0, 1, 2$ are normalization constants. We note once again that expressions in equations (39a)-(40c) are called quantum Fourier transforms.

Note that the states $|z_0\rangle$, $|z_1\rangle$ and $|z_2\rangle$ are now linear combinations of three SCS which are placed in the vertices of an equilateral triangle in the complex plane (see figure 8).

Once again, using the wavefunction of equation (36) we can get the wavefunctions associated to the states in equations (40a)-(40c):

\begin{align}
|\psi_0^z(x)\rangle &= N_0 \left[ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle \right) 
+ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle_1 \right) 
+ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle_2 \right) \right], \tag{41a}
|\psi_1^z(x)\rangle &= N_1 \left[ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle \right)
- \exp(i\pi/3) \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle_1 \right)
+ \exp(2\pi i/3) \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle_2 \right) \right], \tag{41b}
|\psi_2^z(x)\rangle &= N_2 \left[ \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle \right) 
+ \exp(2\pi i/3) \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle_1 \right) 
+ \exp(4\pi i/3) \exp \left( -\frac{1}{2}(x - \langle x \rangle)^2 + ix \langle p \rangle_2 \right) \right]. \tag{41c}
\end{align}

where $\langle x \rangle$ and $\langle p \rangle$ are given in equation (21a) while

\begin{align}
\langle x \rangle_1 &= \sqrt{2} \text{ Re}(z \exp(2\pi i/3)), \\
\langle p \rangle_1 &= \sqrt{2} \text{ Im}(z \exp(2\pi i/3)). \tag{42a}
\end{align}

Figure 6. Probability density $|\psi_0^z(x, t)|^2$ for an even coherent state.

Figure 7. Probability density $|\psi_1^z(x, t)|^2$ for an odd coherent state.

Figure 8. Diagram showing the position in the complex plane of the standard coherent states $|z\rangle$, $|\exp(2\pi i/3)z\rangle$ and $|\exp(4\pi i/3)z\rangle$, whose superpositions give place to the triphoton coherent states $|z_0\rangle$, $|z_1\rangle$ and $|z_2\rangle$.
\[ \langle x \rangle_2 = \sqrt{2} \text{Re}(z \exp(4\pi i/3)), \]
\[ \langle p \rangle_2 = \sqrt{2} \text{Im}(z \exp(4\pi i/3)), \]  

and
\[ N_0 = \left( \frac{1}{\pi} \right)^{1/4} \frac{1}{3} \left[ \exp(|z|^2) + \exp(|z|^2 e^{2\pi i/3}) \right. \]
\[ \left. + \exp(|z|^2 e^{4\pi i/3}) \right]^{-1/2} \]
\[ = \left( \frac{1}{\pi} \right)^{1/4} \frac{1}{3} \left[ \frac{1}{3} \exp(|z|^2) \right. \]
\[ \left. - 2 \exp \left( -\frac{|z|^2}{2} \right) \right]^{-1/2}, \]  

\[ N_1 = \left( \frac{1}{\pi} \right)^{1/4} \frac{1}{3} \left[ \exp(|z|^2) + \exp(2\pi i/3) \exp(|z|^2 e^{2\pi i/3}) \right. \]
\[ \left. + \exp(4\pi i/3) \exp(|z|^2 e^{4\pi i/3}) \right]^{-1/2} \]
\[ = \left( \frac{1}{\pi} \right)^{1/4} \frac{1}{3} \left[ \frac{1}{3} \exp(|z|^2) \right. \]
\[ \left. - 2 \exp \left( -\frac{|z|^2}{2} \right) \sin \left( \frac{\pi}{6} - \frac{\sqrt{3} |z|^2}{2} \right) \right]^{-1/2}. \]

By applying now the evolution operator \( U(t) \) on the states of equations (40a)-(40c) we can obtain the corresponding time-dependent wavefunctions \( \psi_j^t(x, t) = \langle x | U(t) | z \rangle_j \), \( j = 0, 1, 2 \). Examples of the corresponding probability densities for the eigenvalue \( z = 1 + i \) are shown in figure 9 for \( j = 0 \), in figure 10 for \( j = 1 \) and in figure 11 for \( j = 2 \).

As in the previous case, the evolving coherent states \( U(t) | z \rangle_0 \), \( U(t) | z \rangle_1 \) and \( U(t) | z \rangle_2 \) are also cyclic, but with a period \( \tau = T/3 = 2\pi/3 \). This means that these states recover their initial form before the oscillator period has elapsed, i.e. the triphoton coherent states exhibit a marked quantum behavior.

For arbitrary \( k \), the process is analogous to the second and third order cases. Firstly, the SCS which are rotated subsequently by a phase \( \mu = \exp \left( \frac{2\pi i}{T} \right) \) are expanded in terms of the MCS as \( |\mu z \rangle_j = M_j |z\rangle_j \), with the coefficients of the expansion
The associated symmetry leading to the MCS $|\gamma\rangle$, $j = 0, \ldots, k - 1$. The associated symmetry in the one of a regular polygon (dashed line).

In this way we obtained the expressions given in [14, 15]:

$$
\begin{bmatrix}
|\gamma\rangle \\
|\mu\gamma\rangle \\
|\mu^2\gamma\rangle \\
\vdots \\
|\mu^{k-1}\gamma\rangle
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \mu & \mu^2 & \ldots & \mu^{k-1} \\
1 & \mu^2 & \mu^{2(2)} & \ldots & \mu^{2(k-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \mu^{k-1} & \mu^{2(k-1)} & \ldots & \mu^{(k-1)^2}
\end{bmatrix}
\begin{bmatrix}
|\gamma\rangle_0 \\
|\gamma\rangle_1 \\
|\gamma\rangle_2 \\
\vdots \\
|\gamma\rangle_{k-1}
\end{bmatrix}
$$

(44)

The SCS which are involved can be represented in the complex plane as in figure 12. In the second place, we express the MCS in terms of the SCS $|\mu\gamma\rangle = M^i_{ji}|\gamma\rangle$, where the inverse $M^{-1}_{ji}$ is calculated through standard methods

$$
\begin{bmatrix}
|\gamma\rangle_0 \\
|\gamma\rangle_1 \\
|\gamma\rangle_2 \\
\vdots \\
|\gamma\rangle_{k-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \mu^{-1} & \mu^{-2} & \ldots & \mu^{-(k-1)} \\
1 & \mu^{-2} & \mu^{-(2(2))} & \ldots & \mu^{-(2(k-1))} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \mu^{-(k-1)} & \mu^{-(2(k-1))} & \ldots & \mu^{-(k-1)^2}
\end{bmatrix}
\begin{bmatrix}
|\gamma\rangle \\
|\mu\gamma\rangle \\
|\mu^2\gamma\rangle \\
\vdots \\
|\mu^{k-1}\gamma\rangle
\end{bmatrix}
$$

(45)

In this way we obtained the expressions given in equations (35a), (35b) and equations (40a)–(40c). These decompositions guarantee to work with the SCS wave functions, which are more familiar than the ones for the MCS.

Also, the above expression allows to find the wavefunction of the MCS for arbitrary $k$ as:

$$
\psi_j(x) = N_j \sum_{s=0}^{k-1} \mu^{-sj}\psi_{2jp}(x), \quad j = 0, 1, \ldots, k - 1,
$$

(46)

where the factor $1/k$ has been absorbed in the normalization constants $N_j$.

Let us note that the MCS studied here supply an interesting representation of a quantum symmetry related with the quantum $q$-oscillator [16].

4.4. Wigner distribution function

As is well known [36], the Wigner distribution function for a system in a state $|\psi\rangle$ is given by

$$
W(q, p) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \psi | q + \frac{y}{2} | \psi \rangle \exp(iyp)dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi^*(q + y)\psi(q - y)\exp(2ipy)dy,
$$

(47)

which is a quasi-probability density since

$$
\int_{-\infty}^{\infty} W(q, p)dp = |\psi(q)|^2, \quad \int_{-\infty}^{\infty} W(q, p)dq = |\varphi(p)|^2,
$$

(48a) (48b)

but it is not a positive definite function. In fact, the Wigner function can take negative values, a property that can be used as a sign of the quantunness of a state [37]. This fact will be used now to analyze the MCS.

The Wigner function associated to the SCS is obtained by substituting the wavefunction of equation (36) in (47), leading to:

$$
W(q, p) = \frac{1}{\pi} \exp\left(-\frac{(q - \langle q \rangle)^2}{\langle q^2 \rangle + \langle p^2 \rangle}ight),
$$

(49)

where $\langle q \rangle$ and $\langle p \rangle$ are the mean values of the position and momentum, respectively (see also equation (21a)). Let us note that this Wigner function is positive definite, as it is also for the harmonic oscillator ground state. This is the reason why the SCS are considered semiclassical states: they remind a classical particle moving cyclically in the oscillator potential with the oscillator period $T = 2\pi$ (see figure 13).

On the other hand, the Wigner functions associated to the even and odd coherent states are also obtained from their
corresponding wavefunctions. We get that

\[
W_0^{(q, p)} = \frac{N_0^2}{\pi} \left\{ \exp\left(-q - (q)^2\right) \exp\left(-p - (p)^2\right) + \exp\left(-q + (q)^2\right) \exp\left(-p + (p)^2\right) + 2 \Re \left[ \exp\left(-q + i (p)^2\right)e^{-i\left(p - i (q)^2\right)} \right] \right\}
\]

\[(50a)\]

\[
W_1^{(q, p)} = \frac{N_1^2}{\pi} \left\{ \exp\left(-q - (q)^2\right) \exp\left(-p - (p)^2\right) + \exp\left(-q + (q)^2\right) \exp\left(-p + (p)^2\right) - 2 \Re \left[ \exp\left(-q + i (p)^2\right)e^{-i\left(p - i (q)^2\right)} \right] \right\}
\]

\[(50b)\]

where the first two terms in both equations correspond to the Wigner functions of the two SCS centered at \(\pm (q), (p)\), while the last one is an interference term.

As we can see in figures 14 and 15, now both Wigner functions take negative values in the region between the two Gaussian functions, where the interference term plays a fundamental role. This fact induces to consider them as intrinsically quantum states, without any classical counterpart.

Finally, the Wigner functions for the wavefunctions of equations (41a)–(41c), associated to the triphoton coherent states, turn out to be:

\[
W_0^{(q, p)} = \frac{N_0^2}{\pi} \left\{ e^{-\left(q + (q)^2\right)} e^{-\left(p - (p)^2\right)} + e^{-\left(q - (q)^2\right)} e^{-\left(p + (p)^2\right)} + 2 \Re \left[ e^{-\left(q + i (p)^2\right)} e^{-i\left(p - i (q)^2\right)} \right] \right\}
\]

\[(51a)\]

\[
W_1^{(q, p)} = \frac{N_1^2}{\pi} \left\{ e^{-\left(q + (q)^2\right)} e^{-\left(p - (p)^2\right)} + e^{-\left(q - (q)^2\right)} e^{-\left(p + (p)^2\right)} - 2 \Re \left[ e^{-\left(q + i (p)^2\right)} e^{-i\left(p - i (q)^2\right)} \right] \right\}
\]

\[(51b)\]

According to these expressions, each Wigner function \(W_j^{(q, p)}, j = 0, 1, 2\) is composed of the Wigner functions

Figure 14. Wigner distribution function \(W^{(q, p)}\) for an even coherent state, which shows two separated Gaussian distributions, placed in opposite positions in phase space. The oscillations in the middle appear from the interference between the two SCS \([z]\) and \([\exp(\pi i z)\]).

Figure 15. Wigner distribution function \(W^{(q, p)}\) for an odd coherent state, which shows two separated Gaussian distributions centered in opposite points in phase space. The oscillations in the middle appear from the interference between the two SCS \([z]\) and \([\exp(\pi i z)\]).
of three SCS centered at the points \((\langle q_1 \rangle, \langle p_1 \rangle), (\langle q_2 \rangle, \langle p_2 \rangle)\) and \((\langle q_3 \rangle, \langle p_3 \rangle)\) in phase space, together with an interference term. As can be seen in figures 16–18, the Wigner functions for the triphoton coherent states take negative values, which exhibits clearly the intrinsically quantum nature of these states.

Finally, by substituting equation (46) in (47), the Wigner function \(W_{q,p}(q, p)\) for arbitrary \(k\) turns out to be

\[
W_{q,p}(q, p) = N^2 \left\{ \sum_{i=0}^{k-1} W_{q,p}(q, p) + \sum_{s'=s=0}^{k-1} \mu^{f'(s')-s} W_{q,p}(q, p) \right\},
\]

where

\[
W_{q,p}(q, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi^{\dagger}_q(q + y) \psi_p(q - y) \exp(2i p y) dy.
\]

and \(W_{q,p}(q, p)\) is the Wigner function associated to the SCS with eigenvalue \(z\mu^s\) (equation (49)).

4.5. Geometric phase

It is well known that any state \(|\psi(t)\rangle\) evolving cyclically in the time interval \((0, \tau)\), such that \(|\psi(\tau)\rangle = e^{i\tau}|\psi(0)\rangle\), has associated a geometric phase \(\beta\) [38–41], which depends only of the geometry of the state space (the projective Hilbert space). In particular, if the system is ruled by a time-independent Hamiltonian \(H\), the geometric phase is given by [41, 42]:

\[
\beta = \varphi + \tau \langle \psi(0)|H|\psi(0)\rangle.
\]

As we saw in the previous sections, the MCS of equation (18) evolve cyclically. In fact, by taking \(t = \tau = 2\pi/k\) in equation (27) it is obtained

\[
U(2\pi/k)|\alpha\rangle = \exp \left( -i \frac{(2j + 1)\pi}{k} \right) |\alpha\rangle.
\]

Thus, the corresponding geometric phase turns out to be:

\[
\beta_j = -\frac{(2j + 1)\pi}{k} + \frac{2\pi}{k} j |\alpha| H|\alpha\rangle.
\]

If we employ the result of equation (33) we arrive at:

\[
\beta_j = \frac{2\pi}{k} (|a| \alpha_j|^2 - j).
\]
By taking now \( k = 2 \), i.e. for the \textit{even} and \textit{odd} coherent states, we obtain:

\[
\beta_0 = \pi |\alpha| \tanh(|\alpha|), \quad (58a)
\]

\[
\beta_1 = \pi (|\alpha| \coth(|\alpha|) - 1). \quad (58b)
\]

These geometric phases, as functions of \( \alpha \), are shown in figure 19. We can see that both functions start from the minimum value, \( \beta_j = 0 \), but the geometric phase for the \textit{even} states grows more quickly than the one for the \textit{odd} states as \( |\alpha| \) grows.

On the other hand, for the triphoton coherent states we arrive at:

\[
\beta_0 = \frac{2 \pi}{3} |\alpha|^{2/3} \left\{ \frac{\exp(|\alpha|^{2/3}) - 2 \exp\left(-\frac{|\alpha|^{2/3}}{2}\right) \sin\left(\frac{\pi}{6} + \frac{\sqrt{3} |\alpha|^{2/3}}{2}\right)}{\exp(|\alpha|^{2/3}) + 2 \exp\left(-\frac{|\alpha|^{2/3}}{2}\right) \cos\left(\frac{\sqrt{3} |\alpha|^{2/3}}{2}\right)} \right\}. \quad (59a)
\]

\[
\beta_1 = \frac{2 \pi}{3} |\alpha|^{2/3} \left\{ \frac{\exp(|\alpha|^{2/3}) + 2 \exp\left(-\frac{|\alpha|^{2/3}}{2}\right) \cos\left(\frac{\sqrt{3} |\alpha|^{2/3}}{2}\right)}{\exp(|\alpha|^{2/3}) - 2 \exp\left(-\frac{|\alpha|^{2/3}}{2}\right) \sin\left(\frac{\pi}{6} - \frac{\sqrt{3} |\alpha|^{2/3}}{2}\right)} \right\} - 1. \quad (59b)
\]

\[
\beta_2 = \frac{2 \pi}{3} |\alpha|^{2/3} \left\{ \frac{\exp(|\alpha|^{2/3}) - 2 \exp\left(-\frac{|\alpha|^{2/3}}{2}\right) \sin\left(\frac{\pi}{6} - \frac{\sqrt{3} |\alpha|^{2/3}}{2}\right)}{\exp(|\alpha|^{2/3}) - 2 \exp\left(-\frac{|\alpha|^{2/3}}{2}\right) \sin\left(\frac{\pi}{6} + \frac{\sqrt{3} |\alpha|^{2/3}}{2}\right)} \right\} - 2. \quad (59c)
\]
These geometric phases, as functions of $\alpha$, are shown in figure 20. Once again, we can see that the three functions start from the minimum value, $\beta_j = 0$, but the geometric phase for $|\alpha\rangle_0$ grows more quickly than those for $|\alpha\rangle_1$ and $|\alpha\rangle_2$ as $|\alpha|$ increases.

5. Conclusions

In this paper we have discussed the simplest, but very important, realization of the PHA, generated through the harmonic oscillator. In this approach, the generators are the oscillator Hamiltonian $H$ and the ladder operators $a^\dagger \equiv (a^\dagger)^2$. By analyzing carefully the algebraic structure that the operator set \{\boldsymbol{H}, a, a^\dagger\} generates, one can obtain the spectrum of $H$ in an unconventional way: it becomes decomposed in a set of $k$ infinity energy ladders, each one of which starts from one harmonic oscillator eigenstate $|\psi_j^{(k)}\rangle$ in the subspace $\mathcal{H}_j$. These so-called extremal states, whose number coincides precisely with the order to the differential operator $a^\dagger$, become all physical states and from them we can generate all the other eigenstates $|\psi_{k,n}\rangle$ by applying repeatedly $a^\dagger$.

We have built as well the corresponding coherent states, as eigenstates of the annihilation operator $a_{\epsilon}$ with complex eigenvalue $\alpha$. These states are called MCS in the literature, and they have been expressed in terms of the harmonic oscillator eigenstates $|\psi_{k,n}\rangle$ belonging to each subspace $\mathcal{H}_j$. In general, they are not minimum uncertainty states (see equation (22d) and figures 3 and 4). However, they satisfy a partial completeness relation in each subspace $\mathcal{H}_j$ (see equation (24)). The MCS turn out to be periodic, with a period $\tau = 2\pi/k$ which is a fraction of the original oscillator one ($T = 2\pi$).

The MCS were also expressed in terms of SCS $|z\rangle$ with different eigenvalues $z$, as can be seen for some particular values of the integer $k$. For $k = 2$ we expanded the even and odd coherent states in terms of two SCS with opposite eigenvalues (see equations (35a) and (35b)). On the other hand, for $k = 3$ we expressed the triphoton coherent states in terms of three SCS whose labels form an equilateral triangle in the complex plane (see equations (40a)–(40c)). In general, for a MCS $|z_j\rangle$ the $k$ SCS involved in the superposition are placed uniformly on a circle of radius $|z|$, each of them having a phase difference of $\phi = 2\pi/k$ with the eigenvalues of its neighbor ones, forming the vertices of a regular polygon [13]. This procedure allowed us to find in a simple way the expressions for the corresponding wavefunctions $\psi_j(x)$, as well as their associated Wigner distribution functions $W_j(q,p)$. Thus, figures 6, 7 and 9–11 indicate that the probability densities evolve cyclically, as equation (27) suggests, while figures 14–18 show clearly that the corresponding Wigner functions take negative values. This exhibits the intrinsically quantum nature of the MCS for $k \geq 2$, which implies that we cannot use them for semiclassical models, excepting the MCS of equation (20) whose semiclassical behavior is very well known. Moreover, due to the cyclic evolutions performed by the MCS, it was natural to calculate their associated geometric phase.

Let us note that the algebraic method addressed in this paper, and several properties of the MCS here generated, appear from the equidistant nature of the harmonic oscillator spectrum, a fact which is also observed in the so-called SUSY harmonic oscillator. Due to this property, it seems natural to try to generate the MCS for the SUSY harmonic oscillator. On the other hand, it would be important to find out if this treatment can be also applied to other systems, whose energy levels are not equally spaced, as well as to their SUSY partner Hamiltonians. Both ideas represent interesting research subjects which are currently under study [43, 44].

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