Periodic Scheduling and Packing Problems

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November 4, 2020

1 Introduction

An embedded processor that executes computation tasks needed for control loops is a typical example of an application that must operate periodically within the bounds of the control loops periods. Periodic scheduling problems are frequent in many applications, including avionics [3], automotive [10], software-defined radio [1], and periodic machine maintenance [11].

This paper studies relations between periodic scheduling problem and packing problem. Namely it shows equivalence of harmonic periodic scheduling problem and ruled harmonic 2D packing problem.

2 State of the art

Some basic preemptive periodic scheduling problems are pseudo-polynomial [2], while a non-preemptive periodic scheduling problem is strongly NP-hard [7] in most of the cases. In this paper we deal with non-preemptive and zero-jitter periodic scheduling.

On a single resource, [7] showed that the non-preemptive periodic scheduling with arbitrary task initial phases (i.e., each task is released at its release-date and must be finished within its period time units) and no criterion is strongly NP-hard. For the case of the harmonic period set the problem seems to be easier, since there are efficient heuristics algorithms ([5]). However, it is known by Cai [4] that 1|\text{\textit{harm}}|, p_{\text{\textit{nonharm}}}|− is NP complete in the strong sense, while periodic scheduling on parallel identical resources P|\{(\text{\textit{T}}_\text{i}, p_i)\}_{\text{\textit{harm}}}− is known to be polynomial by Korst [8].

Periodic scheduling on one machine with unit processing times was shown to be NP-complete by [1] using the reduction from the graph coloring problem. Furthermore, [6] proved that the problem is strongly NP-hard.

Relations between 1 machine case and 2D packing has been noticed in Korst [8] and Zhao as remarks, and formalized for powers of 2 periods in Lukasiewicz [9].
3 2DPacking and the 1 machine harmonic case

3.1 Problem definition

In this section we consider a set of $n$ independent jobs, $J_1, \ldots, J_n$. Each job $J_i$ is characterized by its processing time $p_i$, and its period $T_{n_i}$. Periods belong to an harmonic set $T_1, \ldots, T_r$, so that $\forall k \in \{2, \ldots, r\}, T_k = b_k T_{k-1}$ where $b_k$ is an integer. We also consider a value $w$ such that $T_1 = b_1 w$, and assume that any processing time of a job is not greater than $w$.

We denote by $B_k = b_1 \times \ldots \times b_k$. Notice that

$$T_k = B_k w$$

so that for any job $J_i$,

$$T_{n_i} = B_{n_i} w$$

Let $H = B_r$.

Any periodic schedule defines for each job $J_i$ a starting time $s_i \leq T_{n_i}$ of its first occurrence. An occurrence of $i$ will start at each time $s_i + kT_{n_i}, k \geq 0$. In this section we assume that the jobs are to be performed on a single machine.

Clearly, due to the periodicity of the jobs, if the periodic schedule has no resource conflict in the time interval $[0, wH]$, it has no conflict.

We define

$$h_i = \frac{wH}{T_{n_i}}$$

the number of values of $k$ such that $k \geq 0, s_i + kT_{n_i} < wH$.

The question whether there exists a 1 machine schedule such that no collision occur can be formulated as follows: for any integer $k, k'$ with $0 \leq k < h_i$ and $0 \leq k' < h_j$ and for any two jobs $J_i, J_j$ one of the two following conditions hold:

$$s_i + kT_{n_i} + p_i \leq s_j + k'T_{n_j}$$

$$s_j + k'T_{n_j} + p_j \leq s_i + kT_{n_i}$$

Now, we can decompose the starting times along with the harmonic periods as follows:

$$\forall J_i, \quad s_i = u_i + v_i w$$

where $u_i < w$. Suppose that $w$ is defined such that $u_i + p_i \leq w$ in each considered feasible schedule. (This property is always true if we choose $w = T_1$, but as we will see, different values might be interesting to choose). Let us notice that

$0 \leq v_i < B_{n_i}$ (i.e. $v_i < \frac{T_{n_i}}{w} \leq H$).

Notice that as job $J_i$ has period $T_{n_i}$, then an occurrence of $i$ will start at each time

$$s_i + kT_{n_i} = u_i + (v_i + kB_{n_i}) w$$

We can thus define the necessary and sufficient condition of a collision to occur in a periodic schedule:
Lemma 1 A periodic schedule induces a collision between two jobs \( J_i, J_j \) such that \( n_i \leq n_j \) if and only if the two following conditions hold:

\[
u_i < u_j + p_j \quad \text{and} \quad u_j < u_i + p_i \tag{5}\]

\[
\exists k, \quad v_j = v_i + kB_n \tag{6}\]

3.2 Mixed radix system and flip transformation

Let \( y \) be any integer. It is known that \( y < H \) can be decomposed uniquely according to the mixed radix numerical system \( b = (b_1, \ldots, b_r) \) as follows:

\[
y = y_1 + y_2 B_1 + y_3 B_2 + \ldots + y_r B_{r-1}
\]

with \( \forall i > 0, y_i < b_i \). We denote this decomposition as follows

\[
[y]_b = [y_1]_{b_1} \ldots [y_r]_{b_r} \tag{7}
\]

Notice that in the usual base decomposition (for example base 2 for binary decomposition), the components of the base vector \( b \) are all equal (to 2 for binary decomposition).

As the partial products depend on the base vector, we introduce the vector in the notation, since in the following the vector may change:

\[
B_j(b) = (b_1, \ldots, b_j)
\]

Let us generalize the transformation proposed by Lukasievicz for the binary decomposition of a number by defining two operators: Let \( Bf(b, k) \) be the operator that flips the \( k \) first component of the base vector \( b \):

\[
Bf(b, k) = (b_k, b_{k-1}, \ldots, b_1, b_{k+1}, \ldots, b_r) \tag{8}
\]

Observe that

\[
Bf(Bf(b, k), k) = b \tag{9}
\]

\[
B_k(Bf(b, k)) = B_k(b) \tag{10}
\]

Let now \( flip(y, k, b) \) be the number constructed by flipping the \( k \) first digits of the decomposition \([y]_b \) to give a number expressed with respect to base vector \( b' = Bf(b, k) \):

\[
[flip(y, k, b)]_{b'} = [y_k]_{b_k} \ldots [y_1]_{b_1} [y_{k+1}]_{b_{k+1}} \ldots [y_r]_{b_r} \tag{11}
\]

Hence we have:

\[
flip(y, k, b) = y_k + y_{k-1} b_k + \ldots + y_1 (b_2 \ldots b_k) + y_{k+1} (b_1 \ldots b_k) + \ldots + y_r (b_1 \ldots b_{r-1}) \tag{12}
\]

This flip operation has some important properties that will be used to transform the scheduling problem into an equivalent packing problem.
**Lemma 2** for any integer $y < H$,

$$flip(flip(y, k, b), k, Bf(b, k)) = y$$  \hspace{1cm} (13)

if $[y_k]_{b_k} < b_k - 1$, then

$$flip(y + B_{k-1}(b), k, b) = flip(y, k, b) + 1$$  \hspace{1cm} (14)

and if $[y_k]_{b_k} > 0$,

$$flip(x - B_{k-1}(b), k, b) = flip(y, k, b) - 1$$  \hspace{1cm} (15)

This implies that equidistant integers (with distance $B_{k-1}(b)$) after the flip transformation, become consecutive integers.

### 3.3 Periodic scheduling and ruled harmonic 2D packing definition

Now consider the following 2D packing problem associated with the original scheduling problem. We are given a rectangle $R$ of length $w$ and height $h$. For each job $J_i$ we define a rectangle $R_i$ of length $p_i$ and height $h_i$.

Assume that we want to pack rectangles $R_i$ into the big rectangle $R$. Hence we have to define a position of the lowest left point of each rectangle, with coordinates $(x_i, y_i)$ so that no collision occurs. Moreover we assume that in the packings we consider, we must have the additional property:

$$\forall i, y_i \% h_i = 0$$  \hspace{1cm} (16)

**Lemma 3** In a packing satisfying property (16) A collision between rectangles $R_i$ and $R_j$ such that $h_j \leq h_i$ occurs iff the two following condition hold:

$$y_i \leq y_j < y_i + h_i,$$  \hspace{1cm} (17)

$$x_j < x_i + p_i, \quad x_i < x_j + p_j$$  \hspace{1cm} (18)

**Proof.** obviously a collision occur if:

$$y_i \leq y_j < y_i + h_i \quad \text{or} \quad y_j \leq y_i < y_j + h_j$$  \hspace{1cm} (19)

$x_j < x_i + p_i, \quad x_i < x_j + p_j$  \hspace{1cm} (20)

Assuming $h_j \leq h_i$, $h_j$ divides $h_i$ and also divides $y_j$ and $y_i$ the condition $y_j \leq y_i < y_j + h_j$ cannot hold. ■
3.4 Equivalence of the two problems

In this section, we prove that the feasibility of a periodic schedule and 2D packing feasibility are equivalent problems. To this purpose we define, for any periodic schedule \( \sigma \) (which defines associated values \( s, u, v \)) an associated packing as follows:

\[
\forall J_i, \quad \begin{align*}
x^\sigma_i &= u_i \\
y^\sigma_i &= h_i \text{flip}(v_i, n_i, b)
\end{align*}
\] (21)

Conversely if a packing \( \pi \) is given, defining \( (x_i, y_i) \) for each job \( J_i \) then an associated schedule is defined as follows:

\[
\forall J_i, \quad \begin{align*}
u^\pi_i &= x_i \\
v^\pi_i &= \text{flip}(y_i, n_i, Bf(b, n_i))
\end{align*}
\] (22)

**Lemma 4** A feasible periodic schedule \( \sigma \) on 1 machine defines a feasible 2D packing satisfying property (16).

**Proof.** Let us consider the packing parameters defined by relation (21) associated to a schedule \( \sigma \). First observe that for any job \( J_i \), \( x^\sigma_i < w \), and by construction \( y^\sigma_i \% h_i = 0 \), so that condition (16) is met. Now, assume that a collision occurs in the 2D packing between jobs \( J_i \) and \( J_j \) such that \( h_j \leq h_i \), so that \( T_{n_i} \leq T_{n_j} \). According to lemma (3)

\[
y^\sigma_i \leq y^\sigma_j < y^\sigma_i + h_i
\]

Assume that \( y^\sigma_j = y^\sigma_i + \Delta \). Notice that as \( h_j \) divides \( h_i \), and as \( h_j \) divides \( y^\sigma_j \), it should also divide \( \Delta \)

moreover,

\[
h_j \text{flip}(v_j, n_j, b) = h_i \text{flip}(v_i, n_i, b) + \Delta
\] (23)

\[
\text{flip}(v_j, n_j, b) = \frac{h_i}{h_j} \text{flip}(v_i, n_i, b) + \frac{\Delta}{h_j}
\] (24)

with \( \frac{\Delta}{h_j} \leq \frac{h_i}{h_j} \) (25)

Now according to lemma (2)

\[
v_j = \text{flip}(\frac{y^\sigma_j}{h_j}, n_j, Bf(b, n_j))
\] (26)

\[
= \text{flip}(\frac{h_i}{h_j} \text{flip}(v_i, n_i, b) + \frac{\Delta}{h_j}, n_j, Bf(b, n_j))
\] (27)

Now the first \( n_j - n_i \) digits of \( \frac{h_i}{h_j} \text{flip}(v_i, n_i, b) \) in the base representation \( Bf(b, n_j) \) equal 0, whereas as \( \frac{\Delta}{h_j} \leq \frac{h_i}{h_j} \) its last \( n_i \) digits are equal to 0. So the flip operation can be applied separately on the two numbers:

\[
v_j = \text{flip}(\frac{h_i}{h_j} \text{flip}(v_i, n_i, b), n_j, Bf(b, n_j)) + \text{flip}(\frac{\Delta}{h_j}, n_j, Bf(b, n_j))
\] (28)
Now, the first $n_i$ digits of $\text{flip}(\frac{y_i}{h_i}, \text{flip}(v_i, n_i, b), n_j, Bf(b, n_j))$ in the base representation $b$ are equal to the first $n_i$ first digits of $v_i$ in base representation $b$ (and the last digits equal 0). Moreover the number $\text{flip}(\frac{y_i}{h_i}, n_j, Bf(b, n_j))$ has its first $n_i$ digits equal to 0 in the base representation $b$. So that it is a multiple of $B_i$.

$$v_j = v_i + \delta B_{n_i} \quad (29)$$

so that condition (9) occurs. Now, if condition (29) is met then so is condition (5). Hence, there would be a collision in the schedule, the contradiction. ■

**Lemma 5** Any 2D packing satisfying property (10) defines a feasible periodic schedule.

**Proof.** Let us consider a feasible packing $\pi$, and assume that there is a collision between two jobs $J_i$ and $J_j$ with $T_{n_i} \leq T_{n_j}$. Notice that if condition (5) is satisfied then so is condition (20). Now assume condition (6) is satisfied, so that we have

$$v_j^\pi = v_i^\pi + \delta B_{n_i} \quad (30)$$

So according to lemma 2 and the reversibility of the base flip ($Bf(b, k) = b$),

$$y_j = h_j \text{flip}(v_j^\pi, n_j, b)$$

$$= \text{flip}(\text{flip}(v_j^\pi, n_j, b), n_j, b) + \delta$$

Notice that as $y_i < H$ so that $\frac{y_i}{h_i} < B_{n_i}(b) = B_{n_i}(Bf(b, n_i))$ and thus when performing the operation $\text{flip}(\frac{y_i}{h_i}, n_i, Bf(b, n_i))$, we get a number still less than $B_{n_i}$ which has non null positions only in the $n_i$ first digits in base vector $b$.

So according to lemma 2

$$\text{flip}(\text{flip}(\frac{y_i}{h_i}, n_i, Bf(b, n_i)) + \delta B_{n_i}(b), n_j, b)$$

$$= \text{flip}(\text{flip}(\frac{y_i}{h_i}, n_i, Bf(b, n_i)), n_j, b) + \delta$$

$$= \frac{B_{n_i}(b)}{B_{n_i}(b)} \text{flip}(\text{flip}(\frac{y_i}{h_i}, n_i, Bf(b, n_i)), n_i, b) + \delta$$

Hence

$$\frac{y_j}{h_j} = \frac{B_{n_i}(b)}{B_{n_i}(b)} \times \frac{y_i}{h_i} + \delta \quad (33)$$
Now $h_i B_n_i(b) = h_j B_n_j(b) = H$, so that
\[ y_j = y_i + \delta \] (34)

Now $\delta < h_i$ since $v^r_j = v^r_i + \delta B_n_i < H = h_i B_n_i$. So condition (19) occurs, a contradiction. \[ \square \]

### 3.5 Approximation

Now we can look at the problem of minimizing $w$ such that a schedule exists. This might be interesting while considering for example a part of the cycle that is booked for time triggered traffic, while the rest is booked for event triggered communications. As done for example in the paper of Zhao, Qin and Liu, we can use our 2D packing transformation, and try to adapt approximation algorithms for the underlying

**Lemma 6** The FFDH algorithm from Coffman, Garey, Johnson algorithm for 2D strip packing with harmonic length produce packings that can be modified to satisfy property 16.

**Proof.** The FFDH algorithm basically sorts the rectangles according to their length (in non increasing order) and pack them by "shelves" or strips. A shelf is open when putting a rectangle at its base. The other rectangles (with smaller length) are then placed one upon the other until the total height is reached (or no more rectangle is to be placed). Then a new shelf is opened.

We claim that in each shelf or strip, a reordering can be done on the rectangles so that their coordinates $(x_i, y_i)$ satisfy the constraint 16, due to the harmonic nature of the periods. Consider a shelf, in which rectangles $R_1, \ldots, R_k$ are stored in this order (so that $p_{i_1} \geq \ldots \geq p_{i_k}$). Notice first that any permutation of rectangles on the shelf lead to a feasible packing. So we can sort them by non increasing height. And we can then prove that the vertical coordinate $y_i$ of these rectangles satisfy constraint 16. \[ \square \]

**Lemma 7** If we apply modified FFDH algorithm to 2D strip packing, in order to minimize $w$, we define a periodic schedule such that
\[ w^{CGJ} \leq w^{opt} + \max_i p_i \] (35)

### 3.6 parallel machine case

The decision problem is a bin packing problem. 2D bin packing algorithm can be extended to solve with approximation the problem of computing the minimum number of machines necessary to compute a set of periodic jobs.
3.7 Release dates and deadlines

Assume now that each job $J_i$ has a release date $r_i$ and a deadline $d_i$, which means that the successive occurrences of $J_i$ satisfy $s_i + kT_i \geq r_i + kT_i$ and $s_i + kT_i + p_i \leq d_i + kT_i$ so that $s_i \geq r_i$ and $s_i + p_i \leq d_i$.

Assume that $r_i$ and $d_i$ are multiples of $w$. Then it will imply that the constraint on $s_i$ can be expressed on $v_i$:

\[
\frac{r_i}{w} \leq v_i \leq \frac{d_i}{w}
\]

This will induce a constraint on the associated packing problem so that not all positions of the rectangle $R_i$ will be available. Notice that the allowed positions are not consecutive ones.

4 Conclusion

This paper has shown equivalence of the harmonic periodic scheduling problem and the ruled harmonic 2D packing problem and thus it contributed to understanding of the periodic problem complexity.

References

[1] Amotz Bar-Noy, Randeep Bhatia, Joseph (Seffi) Naor, and Baruch Schieber. Minimizing service and operation costs of periodic scheduling. Mathematics of Operations Research, 27(3):518–544, 2002.

[2] Sanjoy K Baruah, Louis E Rosier, and Rodney R Howell. Algorithms and complexity concerning the preemptive scheduling of periodic, real-time tasks on one processor. Real-time systems, 2(4):301–324, 1990.

[3] Sofiene Beji, Sardaouna Hamadou, Abdelouahed Gherbi, and John Mullins. Smt-based cost optimization approach for the integration of avionic functions in IMA and TTEthernet architectures. Proceedings of the 2014 IEEE/ACM 18th International Symposium on Distributed Simulation and Real Time Applications, pages 165–174, 2014.

[4] Yang Cai and MC Kong. Nonpreemptive scheduling of periodic tasks in uni- and multiprocessor systems. Algorithmica, 15(6):572–599, 1996.

[5] Jan Dvorak and Zdenek Hanzalek. Multi-variant time constrained FlexRay static segment scheduling. 2014 10th IEEE Workshop on Factory Communication Systems (WFCS 2014), pages 1–8, 2014.

[6] Tobias Jacobs and Salvatore Longo. A new perspective on the windows scheduling problem. arXiv preprint arXiv:1410.7237, 2014.
[7] Kevin Jeffay, Donald F Stanat, and Charles U Martel. On non-preemptive scheduling of periodic and sporadic tasks. In *IEEE real-time systems symposium*, pages 129–139, US, 1991. IEEE.

[8] Jan Korst, Emile Aarts, and Jan Karel Lenstra. Scheduling periodic tasks. *INFORMS journal on Computing*, 8(4):428–435, 1996.

[9] Martin Lukasiewycz, Michael Glaß, Jürgen Teich, and Paul Milbredt. FlexRay schedule optimization of the static segment. In *Proceedings of the 7th IEEE/ACM international conference on Hardware/software codesign and system synthesis*, pages 363–372, France, Grenoble, 2009. IEEE/ACM.

[10] Anna Minaeva, Benny Akesson, Zdeněk Hanzálek, and Dakshina Dasari. Time-triggered co-scheduling of computation and communication with jitter requirements. *IEEE Transactions on Computers*, 67(1):115–129, 2017.

[11] W.D. Wei and C.L. Liu. On a periodic maintenance problem. *Operations Research Letters*, 2(2):90–93, 1983.