Group Theoretical Examination of the Relativistic Wave Equations on Curved Spaces.
I. Basic Principles

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Abstract

The basic principles of generalization of the group theoretical approach to the relativistic wave equations on curved spaces are examined. The general method of the determination of wave equations from the known symmetry group of a symmetrical curved space is described. The method of obtaining the symmetrical spaces in which invariant wave equations admit the limiting passage to the relativistic wave equations on the flat (not necessarily real) spaces is explained. Starting from the equations for massless particles and from the Dirac equation in the Minkowski space, admissible real symmetrical spaces are founded. The mentioned procedure is carried out also for the complex spaces. As the basic point, the wave equations on the flat complex space are considered. It is shown that the usual Dirac equation written down in the complex variables does not lead to any curved space. By ”decomposition” of the mentioned form of the Dirac equation the equation leading to the space \( \mathbb{CP}^n \) and being alternative to the usual Dirac equation on \( \mathbb{C}^n \) is constructed.

1 Introduction

For the long time the connection of elementary particles with various classes of irreducible representations of the Minkowski space symmetry group, i.e the Poincaré group \( \mathbb{P} \) is well known. Also, it is well known, that it is possible to deduce from the fixed representation of the Poincaré group the wave equations, which represent restrictions imposed on wave functions for they to belong to the space of the appropriate irreducible representation of the Poincaré group \( \mathbb{P} \). Thus, one can deduce the wave equations for elementary particles from properties of the space-time symmetry group only, without additional suppositions.

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However, actually, this approach has a much more generality and is applicable to wave equations in symmetric curved spaces. Earlier attempts to its generalization on curved spaces in one special case were already studied — in the case of de Sitter and Anti-de Sitter spaces; it is possible to note [5] spin zero particles, [6, 7, 8] massive particles of spin 1/2 and [9, 10] particles of higher spins. The physical importance of the de Sitter space is explained by that it, unique of all real four-dimensional curved spaces (excepting the Anti-de Sitter space), has the full 10-parametric symmetry group and, consequently, is the favorite polygon in the quantum field theory in curved space-time (see [11, 12] and references therein). Invariant wave equations in the $\mathbb{R}^1 \otimes S^3$ space have also been treated in the M.Carmeli’s series of papers (see [13] and references therein).

In the present paper the basic principles of the group theoretical approach to the relativistic wave equations generalization on curved spaces (generally speaking, they are not necessarily real) are examined. The determining of these principles should precede to systematic examination of wave equations in concrete spaces. The plan of the present work is the following. In §2 the connection of wave equations with the symmetry group of space is explained without concrete definition of it. It is shown how it is possible to obtain the wave equations in symmetrical curved space via the decomposition of its symmetry group representation which describes particles with spin in the direct product of scalar and purely matrix representations. However, both curved space and the invariant wave equations on it should obey the principle of the correspondence, i.e. admit a passage in the flat space limit to the usual relativistic wave equations following from its symmetry group. In §3 it is shown how with the help of known physically important wave operators in the flat (not only real) spaces it is possible to obtain:

(a) curved symmetrical spaces in which the invariant wave equations obeys the principle of the correspondence,

(b) their symmetry groups.

(c) their matrix representations which are needed for the invariant wave equations on this spaces construction.

In §4 the method of §3 is carried out for real spaces starting from the Dirac operator and from the wave equations for massless particles of an arbitrary spin in the Minkowski space. The list of real spaces which admits the group theoretical examination of wave equations is obtained. In §5 this procedure is carried out for complex spaces, the corresponding wave equations already finding the numerous physical applications [14]: concerning the Dirac operator on complex manifolds see also [15] and references therein. It is also necessary for this to consider the wave equations on the flat space $\mathbb{C}^n$. It is shown that the usual Dirac equation written down in the complex variables does not lead to any curved space. By ”decomposition” of the mentioned form of the Dirac equation and doubling its matrix dimensionality the equation leading to the space $\mathbb{C}P^n$ and being alternative to the usual Dirac equation on space $\mathbb{C}^n$ is constructed.

The following paper will be devoted the concrete examination of wave equations in the found real irreducible spaces.
Let space have some continuous symmetry group $\mathcal{G}$. As in the case of the Minkowski space [1, 3], it is natural to connect elementary particles with the irreducible representations of the group $\mathcal{G}$. After the irreducible representation of the group $\mathcal{G}$ is fixed, the wave equations for particles, described by this representation, can be obtained unambiguously as follows [4]. Let us consider an operation of the group $\mathcal{G}$ on the configuration space; it will give us the representation of its generators as some differential operators $-\mathcal{G}$. Then we shall add to this orbital part a certain constant matrix (spin part) which corresponds to the generators of some finite-dimensional matrix representation of the group $\mathcal{G}$. It should be chosen so that the summarized generators compose the representation describing our particle. In the language of group representations it means decomposition of the irreducible representation of symmetry group into the direct product of scalar and purely matrix representations.

Such a decomposition of generators has the simple physical sense. Really, the change of the wave function under the group transformations is

$$
\psi'(x) - \psi(x) = -(\psi'(x') - \psi(x)) + (\psi'(x') - \psi(x)).
$$

The first bracket in right hand side corresponds to the change of the observation point; the orbital part of generators is responsible for it. The second bracket corresponds to the transformations of the wave function in the same point; it is obvious that such transformations will form a group. For them responsible are the spin parts of generators, which, thus, will form a representation of the group $\mathcal{G}$. Now form the operators which commute with all generators of the representation (if these operators are constructed only from the group generators, then they are called the Casimir operators). The first Schur’s lemma states that such operators have the fixed value in the irreducible representation. They will give us the wave equations.

How does the first-order wave equations for particles with the nonzero spin follow from the group $\mathcal{G}$? The Casimir operator $C_2(\mathcal{G})$ by definition is quadratic in generators. Therefore, by substituting in it the generators as the sum of orbital and spin parts we obtain

$$
C_2 = C_2^{(l)} + C_2^{(l-s)} + C_2^{(s)}.
$$

The first and third terms in the right hand side are Casimir operators of scalar and matrix representations. It is clear that they commute with the orbital and spin parts of generators separately, and, so, with full generators of representation of the appropriate spin. Therefore, the same is correct also for the second term $C_2^{(l-s)}$, which is on the first power of both the orbital and spin parts of generators. It is just (up to a constant) the wave operator, and its eigenvalues will give eigenvalues of this operator.
3 Method of admissible symmetric spaces obtaining

For the meaning of the group theoretical examination of the elementary particle in the curved space (let us designate it \( \mathcal{M} \)) it is necessary first, that it allow the limiting passage on flat space. I.e. its metric should include the parameter \( R \) which has the sense of the curvature radius. When \( R \) tends to infinity, the metric should pass to the Galilean metric on the flat space (we shall designate it \( \mathcal{M}' \)). Secondly, the principle of the correspondence for the invariant wave equations on \( \mathcal{M} \) should be satisfied i.e. they should pass in the limit \( R \to \infty \) into the some reasonable wave equations on \( \mathcal{M}' \) following from some representation of its symmetry group \( \mathcal{G}' \).

To obtain the spaces and invariant wave equations on them which obey these requirement we shall act as follows.

Let the symmetry group \( \mathcal{G}' \) of the flat space is known. Let us take now the symmetry group \( \mathcal{G} \) (not yet known) of curved space and tend \( R \) to infinity. Then the Casimir operator \( C_2(\mathcal{G}) \) will coincide within \( o(1/R) \) terms with the known operator \( C_2(\mathcal{G}') \). Further, the orbital parts of generators will coincide with the same accuracy with those for the group \( \mathcal{G}' \), which also are known. Therefore, the orbital-spin part \( C_2^{(l-s)}(\mathcal{G}) \) can be expressed through differential operators on the flat space and while unknown spin parts of generators (probably, of not all) of group \( \mathcal{G} \). The comparison of this expression with known wave equations for the flat space allow us to obtain the mentioned spin parts. Making the commutators between them allows one to obtain the rest of the generators of the group \( \mathcal{G} \) matrix representation and the commutation relations of this group. In general, this procedure can lead to the not quite unambiguous results, and for more precise definition it is necessary to make sure that the obtained curved space really supposes the limiting passage to the flat space. For it the correctness of the condition

\[
\dim \mathcal{M} = \dim \mathcal{M}'
\]

is necessary. Let us remark that any symmetrical space which is not a Lie group (we shall consider only such spaces) is possible to present as the coset space \( \mathcal{G}/\mathcal{H} \):

\[
\mathcal{M} = \mathcal{G}/\mathcal{H}, \quad \mathcal{M}' = \mathcal{G}'/\mathcal{H}',
\]

where \( \mathcal{H} \subset \mathcal{G} \) (\( \mathcal{H}' \subset \mathcal{G}' \)) is the stationary subgroup of an arbitrary point of \( \mathcal{M} \) (\( \mathcal{M}' \)). Since in the infinitesimal neighborhoods of \( \mathcal{M} \) and \( \mathcal{M}' \) should be isomorphic (in the opposite case the limiting passage from \( \mathcal{M} \) to \( \mathcal{M}' \) would not be possible) and \( \mathcal{H} \) is the maximal stationary subgroup for such spaces then

\[
\dim \mathcal{H} \leq \dim \mathcal{H}'.
\]

On the other hand, the real dimensionality of \( \mathcal{M} \) is equal to

\[
\dim \mathcal{M} = \dim \mathcal{G} - \dim \mathcal{H}
\]

and analogously for \( \mathcal{M}' \). Therefore,

\[
\dim \mathcal{G} \leq \dim \mathcal{G}'.
\]
4 Admissible real spaces

Let us consider how the above procedure may be performed for the real spaces. Taking into account the physical importance of four-dimensional space of the Lorentz signature, we shall consider just this case.

Within higher orders on $1/R$, the Casimir operator $C_2$ of the group $G$ coincides with that of the Poincaré group: $G' = \mathcal{P} \equiv \mathbb{R}^4 \rtimes SO(3,1)$ (where $\rtimes$ is the semidirect product):

$$C_2(G) \approx C_2(\mathcal{P}) = P^\mu P_\mu,$$

where $P_\mu$ are the generators of translations. Therefore

$$C_2^{(l-s)}(G) \approx 2P^{(s)}_\mu P^{(l)}_\mu.$$

Since

$$P^{(l)}_\mu \approx \partial_\mu,$$

the first-order wave operator in the Minkowski space is

$$\text{const} \cdot P^{(s)}_\mu \partial_\mu.$$  \hfill (4.1)

Let us compare this to the equations for massive and massless particles following from the Poincaré group by considering these cases separately.

1) The equations for massless particles of an arbitrary spin have the form \[4\]

$$(W_\mu + i\lambda P_\mu \text{Poinc}) \psi = 0,$$  \hfill (4.2)

where $W_\mu$ is the Pauli-Lubanski pseudovector:

$$W_\mu = \frac{1}{2} \varepsilon_{\nu\rho\sigma\mu} J^{\nu\rho} P^{\sigma}_{\text{Poinc}},$$

$\lambda$ is helicity: $\lambda = \pm s$, $s = 1/2, 1, \ldots$ Operators $P_\mu \text{Poinc}$ and $J_{\mu\nu}$ are the generators of the appropriate representation of the Poincaré group:

$$P_\mu \text{Poinc} = \partial_\mu,$$

$$J_{\mu\nu} = x^\rho (\eta_{\mu\rho} \partial_\nu - \eta_{\nu\rho} \partial_\mu) + J^{(s)}_{\mu\nu},$$  \hfill (4.3)

where $J^{(s)}_{\mu\nu}$ are the generators of the Lorentz group representation $L^{(s,0)}$ (if $\lambda = s$) or $L^{(0,s)}$ (if $\lambda = -s$), $\eta_{\mu\nu} = \text{diag}(+ - - -)$ is the Galilean metric tensor. Since for them

$$\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{(s)\rho\sigma} = \mp J^{(s)}_{\mu\nu},$$

then \[1.2\] may be simplified:

$$(s\eta^{\mu\nu} + J^{(s)\mu\nu}) \partial_\nu \psi = 0.$$  \hfill (4.4)
The set of equations (4.4) is surplus, hence its temporal component is sufficient to obtaining the form of generators $P^{(s)\mu}$. As
$$J^{(s)0} = \mp iX^{(s)i},$$
where $X^{(s)i} = \frac{1}{2}\varepsilon^{ikl}J^{(s)kl}$ are the generators of the appropriate representation of rotation group, we finally obtain:
$$(s\partial_t \pm iX^{(s)i}\partial_i)\psi = 0.$$  (4.5)
Comparing this with (4.1) it follows
$$P^{(s)0} = i\lambda/R$$
$$P^{(s)i} = X^{(s)i}/R.$$  (4.6)

The commutation relations between these generators will give us the group $T_1^R \otimes SO(3)$; the space which correspond to it doesn’t have a conventional name. This group may be supplemented up to the group $\mathbb{R}^1 \otimes SO(4)$ by the spatial rotations; it will give the Einstein space.

If $R$ in (4.6) replaced by $iR$, the generators will not compose any real group; being supplemented with rotations they will compose the group $\mathbb{R}^1 \otimes SO(3,1)$. The corresponding space is the Anti-Einstein space $\mathbb{R}^1 \otimes H^3$, where $H^3$ is the three-dimensional hyperbolic space.

2) Massive representation of spin 1/2. The comparison of the Dirac equation
$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$
with (4.1) yields
$$P^{(s)\mu} = \text{const} \cdot \gamma^\mu.$$  

Taking this together with their commutators, which are the generators of the Lorentz transformations, we shall come to the groups $SO(4,1)$ or $SO(3,2)$. Spaces supposing such symmetry groups are the de Sitter and Anti-de Sitter spaces.

So, the following real spaces are subject to examination:

1. Reducible spaces
   (a) Einstein space $\mathbb{R}^1 \times S^3$, $\mathcal{G} = \mathbb{R}^1 \otimes SO(4)$
   (b) Anti-Einstein space $\mathbb{R}^1 \otimes H^3$, $\mathcal{G} = \mathbb{R}^1 \otimes SO(3,1)$.
   (c) $\mathcal{G} = \mathbb{R}^1 \otimes SO(3)$, $\mathcal{H}$ is trivial.

2. De Sitter and Anti-de Sitter spaces.
   (a) De Sitter space $SO(4,1)/SO(3,1)$.
   (b) Anti-de Sitter space $SO(3,2)/SO(3,1)$.
5 Admissible complex spaces

Let us designate the coordinates of the $n$-dimensional complex space through $z_i$, $i = 1, \ldots, n$; the complex conjugation is designated by the over-line. The symmetry group of the flat complex space with the metric

$$ds^2 = dz_i d\bar{z}_i$$

be $T_n^c \times U(n)$ with the Hermitian generators $K_{ik}$ and $J_{ik}$ which may be constructed in the following form:

$$J_{ik} = A_{ik} - A_{ki},$$

$$K_{ik} = i(A_{ik} + A_{ki}).$$

(5.1)

Where $A_{ik}$ are the generators of the group $GL(n, \mathbb{C})$ which have the property

$$A_{ki}^\dagger = A_{ki}$$

(5.2)

and obey the commutation relations

$$[A_{ik}, A_{im}] = \delta_{kl} A_{im} - \delta_{im} A_{lk}. \quad (5.3)$$

As, according to (5.1), the Jacobian of the passage from the group $GL(n, \mathbb{C})$ generators to the group $U(n)$ generators is not equal to zero, we can, that is much more convenient, deal with the generators $A_{ik}$.

We have:

$$C_2(G') = C_2(T_n^c) = -P_i P_i^\dagger. \quad (5.4)$$

The scalar representation of the translation group be

$$P_i^{(l)} = \partial_i, \quad P_i^{(l)\dagger} = -\bar{\partial}_i,$$

where $\partial_i = \frac{\partial}{\partial z_i}$. Note that $(\partial_i)^\dagger = -\bar{\partial}_i$. Therefore

$$C_2^{(l-s)}(G) \approx P_i^{(s)} \bar{\partial}_i - P_i^{(s)\dagger} \partial_i. \quad (5.4)$$

Let us obtain the wave equations following from the group $G'$. The generators of scalar and "spinor" representations of the group $GL(n, \mathbb{C})$ are

$$A_{ik}^{(l)} = z_i \partial_k - \bar{z}_k \bar{\partial}_i$$

$$A_{ik}^{(s)} = b_i b_k^\dagger + a \delta_{ik}. \quad (5.5)$$

Where $a$ is an arbitrary real constant. It is easy to show that they obey commutation relations (5.3). Here $b_i$ are the fermionic operators of birth and annihilation:

$$\{b_i, b_k\} = \{b_i^\dagger, b_k^\dagger\} = 0, \quad \{b_i, b_k^\dagger\} = \delta_{ik}. \quad (5.6)$$
which are realized with the help of \( 2^n \)-dimensional matrices. By using them we can construct the \( \gamma \)-matrices in \( 2n \) dimensions:

\[
\gamma_k = b_k + b_k^\dagger, \quad \gamma_{n+k} = -i(b_k - b_k^\dagger).
\]

(5.7)

So \( \{\gamma_p,\gamma_q\} = 2\delta_{pq}, \ p,q = 1, \ldots, 2n \). It is possible to construct the matrix

\[
\gamma_{2n+1} = i^n(2n-1)\gamma_1\gamma_2 \ldots \gamma_{2n},
\]

commuting with all generators \( A^{(s)}_{ik} \) and having an eigenvalue \( \pm 1 \). Thus, the representation (5.3) is two-multiple reducible. Note that if the additional condition \( A^{(s)}_{ii}\psi = 0 \) is imposed, the representation of the group \( U(n) \) with weights \( 1/2(\pm 1, \pm 1, \ldots, \pm 1) \) may be obtain, where the signs in each case are chosen independently. Now, construct the operator

\[
H = b_i \partial_i.
\]

Using (5.6) it is easy to show that the operators \( H \) and \( H^\dagger \) commute with all summarized generators \( A_{ik} \) and consequently are acceptable for the construction of wave equations. Note that

\[
H - H^\dagger = \gamma_p \frac{\partial}{\partial x_p},
\]

where \( z_k = x_k + ix_{k+n} \), is the Dirac operator on \( \mathbb{R}^{2n} \). This corresponds to an embedding \( U(n) \) in \( SO(2n) \). The formulas following from (5.6)

\[
H^2 = (H^\dagger)^2 = 0
\]

\[
(H - H^\dagger)^2 = \Box \equiv \partial_i \partial^i
\]

yields that the Dirac operator is the only operator of matrix dimensionality \( 2^n \), which would be constructed from operators \( H \) and \( H^\dagger \) and leads by squaring to the Klein-Gordon operator, if we want to describe massive particles (so, the equation \( H\psi = H^\dagger\psi = 0 \) is excluded).

The Dirac operator written down in the complex variables is the conventional spinor wave operator on complex manifolds [14]. However, if we start from this operator using the method of §4, then we obtain \( 2n^2 + n \) generators \( b_i, b_i^\dagger, [b_i, b_j^\dagger], b_i b_k \) and \( b_i^\dagger b_k^\dagger \) in the contradiction with (3.2). Thus, the obtained space does not suppose the limiting passage to \( \mathbb{C}^n \). Besides, this operator, having "surplus" symmetry for our case, is not specific for the flat complex space.

One can construct the wave operator does not have these shortages, as follows. Let us construct four-multiple reducible representation

\[
A^{(s)}_{ik} = (b_ib_k^\dagger + a\delta_{ik}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(5.8)

Then the operator

\[
D = \begin{pmatrix} 0 & H^\dagger \\ H & 0 \end{pmatrix},
\]

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as well as the operator $\hat{D} = SDS^{-1}$, where

$$S = S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

will commute with new summarized generators $A_{ik}^{(l)} + A_{ik}^{(s)}$. Since $C_2^{(l)}(G') = \Box$, we have the second-order equation

$$(\Box + \alpha^2)\psi = 0.$$ 

The transition from the first-order wave operators to the second-order is carried out either with the equalities

$$D\tilde{D} = \tilde{D}D = 0$$

$$D^2 + \tilde{D}^2 = -\Box$$

or with the equality

$$D^3 = \Box D.$$ 

There are two kinds of first-order wave equations compatible with (5.9). From each of them it follows an additional wave equation:

$$D\psi = \alpha\psi \quad (\Rightarrow \tilde{D}\psi = 0),$$

$$\tilde{D}\psi = \alpha\psi \quad (\Rightarrow D\psi = 0).$$

This, together with ambiguity in a sign $\alpha$, reduces to the four-multiple ambiguity; the choice of one of these possibilities compensates the four-multiple reducibility of the representation (5.8) and leads us to the irreducible representation of group $G'$. We will choose the equation (5.10).

Comparing the equation (5.10) with (5.4) gives

$$R \cdot P_i^{(s)} = \begin{pmatrix} 0 & b_i^\dagger \\ 0 & 0 \end{pmatrix}.$$ 

The obtained generators, together with (5.8), can be closed in the Lie algebra by putting

$$A_{0i} = R \cdot P_i,$$

$$A_{00} = \begin{pmatrix} a + 1 & 0 \\ 0 & a \end{pmatrix}.$$ 

They obey the commutation relations of the group $GL(n + 1, \mathbb{C})$:

$$[A_{\mu\nu}, A_{\rho\sigma}] = \delta_{\nu\rho}A_{\mu\sigma} - \delta_{\mu\sigma}A_{\rho\nu},$$

(5.12)

It is similar to the obtaining of the Dirac equation from the Poincaré group. In this case the two-multiple ambiguity in an eigenvalue of the Dirac operator compensates the two-multiple reducibility of the Lorentz group matrix representation [4].
where $\mu, \nu, \ldots = 0, \ldots, n$.

Starting from the equation (5.11) yields the generators

\[
R \cdot \tilde{F}_i^{(s)} = \begin{pmatrix} 0 & 0 \\ b_i^* & 0 \end{pmatrix}
\]

\[
\tilde{A}_{00} = \pm \begin{pmatrix} a & 0 \\ 0 & a + 1 \end{pmatrix}
\]

\[
\tilde{A}_{ik} = A_{ik},
\]

which also obey the commutation relations (5.12). Note that two spin or representations, which we have constructed, are equivalent to each other, since

\[
\tilde{A}_{\mu\nu} = SA_{\mu\nu}S^{-1}.
\]

In general, all these considerations require a verification since the symmetry group of flat space is $U(n)$ but not $GL(n, \mathbb{C})$. However, generators of $U(n + 1)$ are constructed from generators of $GL(n + 1, \mathbb{C})$ in the same way as the generators of $U(n)$ are constructed from the generators of $GL(n, \mathbb{C})$. Therefore, if we started from $U(n)$ then we would come to the group $U(n + 1)$ and the symmetrical space $U(n + 1)/U(n)$. It, however, does not suppose the limiting passage to $\mathbb{C}^n$ because its real dimensionality calculated with the help of (3.1) equals $2n + 1$ in the contradiction with (3.2). Therefore, we demand that the operator $A_{\mu\nu}$ with the operation on an arbitrary vector of representation space give zero. Thus, we shall pass to group $SU(n + 1)$ and complex projective space

\[
\mathbb{C}P^n = SU(n + 1)/U(n)
\]

which already supposes the limiting passage to $\mathbb{C}^n$.

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