ABNORMAL EXTREMALS OF LEFT-INvariant SUB-FINSLER QUASIMETRICS ON FOUR-DIMENSIONAL LIE GROUPS WITH THREE-DIMENSIONAL GENERATING DISTRIBUTIONS

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Abstract: We find the three-dimensional subspaces of four-dimensional Lie algebras which generate the algebras, as well as abnormal extremals on the connected Lie groups determined by these algebras and endowed with the left-invariant sub-Finsler quasimetrics defined by seminorms on the subspaces. Using the structure constants of Lie algebras and dual seminorms, we establish a criterion for the strict abnormality of the extremals.

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1. Introduction

As indicated in [1], the shortest arcs of a left-invariant sub-Finsler metric $d$ on a connected Lie group $G$ defined by a left-invariant bracket generating distribution $D$ and a norm $F$ on $D(e) = \mathfrak{q} \subset T_eG = \mathfrak{g}$ are solutions of some left-invariant time-optimal problem for the closed unit ball with the center zero of the normed vector space $(D(e), F)$ as the control domain. The distribution $D$ is bracket generating if and only if the subspace $\mathfrak{q}$ generates the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ by the Lie bracket $[\cdot, \cdot]$. Moreover, the statements about shortest arcs are also true for the pair $(D(e), F)$ with a seminorm $F$ satisfying $F(u) > 0$ for $0 \neq u \in D(e)$ and defining the left-invariant sub-Finsler quasimetric $d$ on $G$.

The Pontryagin Maximum Principle (PMP) [2] gives some necessary conditions for solutions to the time-optimal control problem. An extremal is an arclength parametrized curve in $G$ which satisfies the PMP.

An extremal can be normal or abnormal. Some extremals can be both normal and abnormal with respect to different covector functions in the PMP; these extremals are called nonstrictly abnormal. An abnormal extremal that is not nonstrictly abnormal is called strictly abnormal.

In this paper we solve the problem of searching abnormal extremals on an arbitrary four-dimensional connected Lie group with a left-invariant sub-Finsler quasimetric defined by a seminorm on a three-dimensional subspace $\mathfrak{q}$ of the Lie algebra of the group generating the Lie algebra by the Lie bracket $[\cdot, \cdot]$. We establish some criterion for the nonstrict abnormality of these extremals which also allows us to formulate the criterion of their strict abnormality. Since the quasimetric is left-invariant, we can assume that extremals issue from the unit of the group. Each of these abnormal extremals is some one-parametric subgroup of the Lie group. Moreover, an abnormal extremal in $(G, d)$ with a left-invariant sub-Riemannian metric $d$ is a geodesic; i.e., its sufficiently small segments are shortest arcs if and only if the extremal is nonstrictly abnormal.

A four-dimensional Lie group $G$ has abnormal extremals for all seminorms $F$ of the above form on $\mathfrak{q} \subset \mathfrak{g}$ if $\dim(\mathfrak{q}) = 3$ and $\mathfrak{q}$ generates $\mathfrak{g}$. In this paper we consider these Lie groups $G$ and $\mathfrak{q} \subset \mathfrak{g}$. By [3, 4], for every dimension $n > 2$ there are exactly two real Lie algebras that admit no $(n - 1)$-dimensional
subspace generating these Lie algebras. For the remaining Lie algebras, up to their automorphisms, we find the number of three-dimensional subspaces generating the algebras.

All main results of the paper are based on the results of [1, 2, 5–10]. In [10] we addressed a similar problem for the two-dimensional bracket generating subspaces and gave some detailed commentary on four-dimensional Lie algebras.

2. General Algebraic Results

We obviously have

**Proposition 1.** An \((n-1)\)-dimensional subspace \(q\) of a real Lie algebra \(g\) of dimension \(n > 2\) generates \(g\) (by the Lie bracket \([\cdot,\cdot]\)) if and only if \(q\) is not a subalgebra of \(g\).

**Proposition 2.** An \((n-1)\)-dimensional subspace \(s\) of an \(n\)-dimensional Lie algebra \(g\), \(n \geq 3\), is a subalgebra of \(g\) if dimension of the intersection of \(s\) with its normalizer \(N(s)\) in \(g\) is at least \(n-2\).

**Proof.** This follows from the fact that the codimension of the intersection in \(s\) is at most 1.

Propositions 1 and 2 imply

**Proposition 3.** If an \((n-1)\)-dimensional subspace \(q\) generates an \(n\)-dimensional real Lie algebra \(g\) with \(n \geq 3\), then \(\dim(q \cap N(q)) \leq n-3\) and \(q\) is not a subalgebra of \(g\). Consequently, the dimension of the intersection of \(q\) with every \((n-1)\)-dimensional subalgebra of \(g\) (respectively, with its every \((n-2)\)-dimensional ideal) is equal to \(n-2\) (respectively, \(n-3\)).

Proposition 4 follows from the proof of Theorem 4 in [4] which is based on [3].

**Proposition 4.** A real Lie algebra \(g\) of dimension \(n > 2\) has no \((n-1)\)-dimensional bracket generating subspace if and only if \(g\) is commutative or \(g\) includes a commutative \((n-1)\)-dimensional ideal \(I\) and has an element \(z\) such that \(ad\, z\) acts identically on \(I\).

The following was proved in [10]:

**Proposition 5.** A four-dimensional connected Lie group \(G\) with a Lie algebra \(g\) and a three-dimensional generating subspace \(q \subset g\) has abnormal extremals (for an arbitrary left-invariant quasimetric \(d\) on \(G\) defined by a seminorm \(F\) on \(q\)) if and only if \(q_1 = q \cap N(q) \neq \{0\}\), where \(N(q)\) is the normalizer of \(q\) in \(g\). Furthermore, \(\dim(q_1) = 1\) and every one-parameter subgroup \(g = g(t) = \exp(tX)\), with \(X \in q_1\) and \(F(X) = 1\), is an abnormal extremal for \((G, d)\); there is no other abnormal extremal with origin \(e \in G\). Moreover, an extremal \(g\) is strictly abnormal (nonstrictly abnormal) for every quasimetric \(d\) if and only if \(q_1 \subset [q_1, q]\) (respectively, \(q_1 = q \cap C(q)\), where \(C(q)\) is the centralizer of \(q\) in \(g\)).

**Proposition 6.** \(C(q) = C(g)\).

**Proof.** This is a consequence of the Jacobi identity and the fact that \(q\) generates \(g\). □

**Lemma 1.** Let \((g, [\cdot,\cdot])\) be a four-dimensional real Lie algebra, let \(q \subset g\) be a three-dimensional subspace generating \(g\) by the Lie bracket \([\cdot,\cdot]\). Then there is a unique one-dimensional subspace \(q_1 = q \cap N(q)\), where \(N(q)\) is the normalizer of \(q\) in \(g\). Furthermore, \([p, p] \not\subset q\) and \(\dim([p, p]) = 1\) for every two-dimensional subspace \(p \subset q\) such that \(p \cap q_1 = \{0\}\). In other words, for all linearly independent vectors \(e_1, e_3 \in p\) and a nonzero vector \(e_2 \in q_1\), the vectors \(e_1, e_2, e_3, e_4 = [e_1, e_3]\) constitute a basis for \(g\) such that

\[
C_{13}^1 = C_{13}^2 = C_{13}^3 = 0, \quad C_{13}^4 = 1, \quad C_{12}^4 = C_{23}^4 = 0.
\]

**Proof.** Proposition 1 and the condition that \(q\) generates \(g\) imply that \([q, q] \not\subset q\); i.e., there are some linearly independent vectors \(e_1, e_3 \in q\) such that \([e_1, e_3] \not\subset q\). Then for each vector \(e_2 \in q\), linearly independent with \(e_1\) and \(e_3\), the vectors \(e_1, e_2, e_3, e_4 = [e_1, e_3]\) constitute a basis for \(g\). If at least one of the structure constants \(C_{12}^4\) and \(C_{23}^4\) in this basis is nonzero; then, replacing \(e_2\) with \(e_2 - C_{23}^4e_1 - C_{12}^4e_3\) and again denoting the last vector by \(e_2\), we get \([p_1, q] \subset q\) for the one-dimensional subspace \(p_1 \subset q\) spanned by \(e_2\). Since \(q\) generates \(g\); therefore, \([p, p] \not\subset q\) and \(\dim([p, p]) = 1\) for every two-dimensional subspace \(p \subset q\) such that \(p \cap p_1 = \{0\}\). It follows that \(p_1 = q \cap N(q) = q_1\). □
Corollary 1. Every four-dimensional connected Lie group $G$ with a Lie algebra $\mathfrak{g}$ and a three-dimensional generating subspace $q \subset \mathfrak{g}$ has abnormal extremals (for an arbitrary left-invariant quasimetric $d$ on $G$ defined by some seminorm $F$ on $q$). If $q$ includes a one-dimensional central (respectively, noncentral) ideal of $\mathfrak{g}$, then every abnormal extremal is nonstrictly (respectively, strictly) abnormal.

Proof. The claim follows from Proposition 5 and Lemma 1. □

We say that a basis $(e_1, e_2, e_3)$ for the subspace $q$ is from Lemma 1, if $(e_1, e_2, e_3, e_4 := [e_1, e_3])$ is a basis for $\mathfrak{g}$ and (1) is satisfied.

Proposition 7. For the basis $(e_1, e_2, e_3, e_4)$ of $\mathfrak{g}$ from Lemma 1, we have

$$C^1_{24} = C^1_{12}C^2_{23} - C^2_{12}C^1_{23}, \quad C^2_{24} = 0, \quad C^3_{24} = C^3_{12}C^2_{23} - C^2_{12}C^3_{23}, \quad C^4_{24} = C^3_{23} - C^1_{12}.$$ 

Proof. Owing to (1), the Jacobi identity

$$[e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_1, e_2]] = 0$$

is equivalent to $[e_1, [e_2, e_3]] - [e_2, e_4] + [e_3, [e_1, e_2]] = 0$; i.e.,

$$0 = C^2_{23}[e_1, e_2] + C^3_{23}e_4 - [e_2, e_4] - C^1_{12}e_4 - C^2_{12}[e_2, e_3] = (C^3_{23} - C^4_{24} - C^1_{12}) e_4$$

$$+ (C^1_{12}C^2_{23} - C^2_{12}C^1_{23} - C^1_{24}) e_1 - C^2_{24}e_2 + (C^3_{12}C^2_{23} - C^2_{12}C^3_{23} - C^3_{24}) e_3.$$ 

This yields Proposition 7. □

Corollary 2. If $e_3 = [e_1, e_2]$; then $C^1_{24} = C^2_{24} = 0, C^3_{24} = C^2_{23},$ and $C^4_{24} = C^3_{23}.$

3. Criteria for the (Non)strict Abnormality of an Extremal

As indicated in [1], the arclength parametrized shortest curves of a left-invariant sub-Finsler metric $d$ on every connected Lie group $G$, defined by a left-invariant bracket generating distribution $D$ and a norm $F$ on $D(e)$, coincide with solutions to the time-optimal control problem for the system

$$\dot{g}(t) = dl_g(t)(u(t)), \quad u(t) \in U,$$ 

(2)

with a measurable control $u = u(t)$. Here $l_g(h) = gh$ and the control domain is the unit ball $U = \{u \in D(e) | F(u) \leq 1\}$, while $D$ is bracket generating if and only if the corresponding subspace $q := D(e) \subset \mathfrak{g}$ satisfies the hypotheses of Lemma 1. It is clear that every arclength parametrized shortest curve $g(t)$, $0 \leq t \leq a$, satisfies (2) and $F(u(t)) = 1$ for almost all $t \in [0, a]$.

These statements are true also for the case when $d$ is a quasimetric (respectively, $F$ is a seminorm on $D(e)$) such that $F(u) > 0$ for $0 \neq u \in D(e)$.

Each segment of every shortest curve in $(G, d)$ is a shortest path; each open ball of sufficiently small positive radius in $(G, d)$ is diffeomorphic to a domain of the Euclidean space; and each shortest path joining any point of the ball with its center lies in this ball. Therefore, according to the PMP [2] the control $u(t)$ and the corresponding trajectory $g(t)$, with $t \in [0, a]$, to be time-optimal requires the existence of some nowhere absolutely continuous covector function $\psi(t) \in T^*_{g(t)}G$ such that for almost all $t \in [0, a]$ the function $H(g(t); \psi(t); u) = \psi(t)(dl_g(t)(u))$ of $u \in U$ attains a maximum at $u(t)$; i.e.,

$$M(t) = \psi(t)(dl_g(t)(u(t))) = \max_{u \in U} \psi(t)(dl_g(t)(u)).$$ 

(3)

Moreover, $M(t) \equiv M \geq 0$ for all $t \in [a, b]$.

By an extremal we will mean a parametrized curve $g(t)$ in $G$ with a maximally admissible connected domain $\Omega \subset \mathbb{R}$ which satisfies the PMP, conditions (2), and $F(u(t)) = 1$ for a measurable function $u(t)$ almost everywhere on the maximal subset in $\Omega$. In the case $M = 0$ (respectively, $M > 0$) an extremal is called abnormal (respectively, normal). In the normal case, proportionally changing $\psi = \psi(t), t \in \mathbb{R}$, if need be, we can assume that $M = 1$.

The following proposition is an immediate consequence of Lemma 1, Corollary 1, and Proposition 5. However, we give here an independent proof because some details will be needed further to establish some criteria for strict and nonstrict abnormality of extremals from Proposition 8.
Proposition 8. Every four-dimensional connected Lie group $G$ with Lie algebra $\mathfrak{g}$ and a three-dimensional generating subspace $\mathfrak{q} \subset \mathfrak{g}$ has abnormal extremals (for an arbitrary left-invariant quasimetric $d$ on $G$ defined by a seminorm $F$ on $\mathfrak{q}$). Each abnormal extremal in $(G, d)$ is one of the two one-parameter subgroups

$$g(t) = \exp \left( \frac{st e_2}{F(se_2)} \right), \quad t \in \mathbb{R}, \ s = \pm 1,$$

or its left shift on $(G, d)$.

Proof. We can consider the covector function $\psi(t) \in T^*_g G$ from the PMP as a left-invariant 1-form on $(G, \cdot)$, and so we naturally identify the latter with the covector function $\psi(t) \in \mathfrak{g}^* = T^*_e G$.

In [5, 6], for an extremal $g(t) \in G$ with $t \in \mathbb{R}$, there are proved the relations satisfying

$$\dot{g}(t) = dl_g(t)(u(t)), \quad (\psi(t)(v))' = \psi(t)([u(t), v]), \ u(t) \in \mathfrak{q}, \ v \in \mathfrak{g}, \ F(u(t)) = 1 \quad (5)$$

for almost all $t$ in the domain.

Omitting for brevity the parameter $t$, we can write the second equation in (5) as $\psi'(v) = \psi([u, v])$. In particular, for $\psi_i := \psi(e_i)$ with $i = 1, 2, 3, 4$, we have

$$\psi_i' = \psi([u, e_i]). \quad (6)$$

Put $u = u_1 e_1 + u_2 e_2 + u_3 e_3 \in U$. We get from (6) and (1):

$$\psi_1' = \psi(-u_2[e_1, e_2] - u_3 e_4) = -u_2 \sum_{k=1}^{3} C^k_{12} \psi_k - u_3 \psi_4, \quad (7)$$

$$\psi_2' = \psi(u_1[e_1, e_2] - u_3[e_2, e_3]) = \sum_{k=1}^{3} (u_1 C^k_{12} - u_3 C^k_{23}) \psi_k, \quad (8)$$

$$\psi_3' = \psi(u_1 e_4 + u_2[e_2, e_3]) = u_1 \psi_4 + u_2 \sum_{k=1}^{3} C^k_{23} \psi_k, \quad (9)$$

$$\psi_4' = \psi(u_1[e_1, e_4] + u_2[e_2, e_4] + u_3[e_3, e_4]) = \sum_{k=1}^{4} (u_1 C^k_{14} + u_2 C^k_{24} + u_3 C^k_{34}) \psi_k. \quad (10)$$

Clearly, in the abnormal case there must be $\psi_1 = \psi_2 = \psi_3 \equiv 0$. Then (7)–(10), the condition $\psi_4 \neq 0$ and the equality $F(u) = 1$ imply that

$$u_1 = u_3 = 0, \quad u_2 = s/F(se_2), \quad s = \pm 1. \quad (11)$$

It follows from (11), Proposition 7, and (10) that the function

$$\psi_4(t) = \varphi_4 \exp \left( \frac{C^4_{24} st}{F(se_2)} \right) = \varphi_4 \exp \left( \frac{(C^3_{23} + C^1_{12}) st}{F(se_2)} \right), \quad s = \pm 1, \quad (12)$$

is a solution of (10) with the initial condition $\psi_4(0) = \varphi_4 \neq 0$. Obviously, it is possible to find $u(t)$ and $\psi(t)$ by the above formulas for all $t \in \mathbb{R}$.

Now Proposition 8 follows from (11) and the first equation in (5). □
Below $F(u_1, u_2, u_3) := F(u)$ and $F_U$ is the Minkowski supporting function of a body $U$; i.e.,

$$F_U(x, y, z) = \max_{(u_1, u_2, u_3) \in U} (xu_1 + yu_2 + zu_3).$$

**Theorem 1.** Abnormal extremal (4) of a four-dimensional connected Lie group $G$ with a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a left-invariant quasimetric $d$ defined by a seminorm $F$ on a three-dimensional subspace $\mathfrak{q}$, generating $\mathfrak{g}$, with a basis $(e_1, e_2, e_3 = [e_1, e_2])$ from Lemma 1 is nonstrictly abnormal if and only if one of the conditions is fulfilled:

$$C_{23}^1 = C_{23}^2 = 0 \quad \text{and for } s = \pm 1 \text{ there is } k(s) \in \mathbb{R} \text{ satisfying } F_U(k(s), s, 0) = 1/F(0, s, 0);$$

$$C_{23}^1 \neq 0 \quad \text{and } F_U(-C_{23}^2 s/C_{23}^1, s, 0) = 1/F(0, s, 0), \quad s = \pm 1.$$

**Proof. Necessity:** Assume that extremal (4) is nonstrictly abnormal. Then there is a real-analytic covector function $\psi(t)$ that is a solution of system (7)–(10), and $\psi(t)(u(t)) = F_U(\psi_1(t), \psi_2(t), \psi_3(t)) = 1$ for almost all $t \in \mathbb{R}$. This and (11) imply that

$$\psi_2(t) = 1/u_2, \quad F_U(\psi_1(t), 1/u_2, \psi_3(t)) = 1, \quad (13)$$

and $(\psi_1(t), 1/u_2, \psi_3(t))$, with $t \in \mathbb{R}$, are dual to $(0, u_2, 0)$. Therefore, the ranges of $\psi_1(t)$ and $\psi_3(t)$ with $t \in \mathbb{R}$, are segments (degenerating to a point if $F$ is differentiable at $(0, u_2, 0)$) because the body $U^*$ dual to $U$ is convex and bounded.

We get from (7)–(11), $e_3 = [e_1, e_2]$, and Corollary 2 that

$$\psi'_1 = -u_2 \psi_3, \quad \psi_2 = \varphi_2, \quad \psi'_3 = u_2 \sum_{k=1}^{3} C_{23}^k \psi_k, \quad \psi'_4 = u_2 \sum_{k=3}^{4} C_{23}^{k-1} \psi_k. \quad (14)$$

The first equality in (13) and (14) imply

$$\psi''_1 - u_2 C_{23}^3 \psi'_1 + u_2 C_{23}^2 \psi'_1 + u_2 C_{23}^2 = 0. \quad (15)$$

Assume that $C_{23}^2 = 0$. The general solution to (15) has the form

$$\psi_1(t) = \begin{cases} A_2 e^{C_{23}^3 u_2 t} + C_{23}^2 t/C_{23}^3 + A_1, & \text{if } C_{23}^3 \neq 0, \\
-\frac{1}{2} C_{23}^2 u_2 t^2 + A_2 t + A_1, & \text{if } C_{23}^3 = 0, \end{cases}$$

where $A_1$ and $A_2$ are arbitrary reals. Since $\psi_1(t), t \in \mathbb{R}$, is bounded, this implies that $C_{23}^2 = 0$ and $\psi_1(t) = A_1$ with $A_1 \in \mathbb{R}$. Then $\psi_3(t) = 0$ by the first equation in (14), while in view of (11) and (13) there is a real $k(s)$ such that $F_U(k(s), s, 0) = 1/F(0, s, 0)$ with $s = \pm 1$. Hence, the supporting plane of $U$ at the intersection point $(0, s/F(se_2), 0)$ by the axis $Ou_2$ is parallel to the axis $Ou_3$.

Assume now that $C_{23}^1 \neq 0$. Put $B = (C_{23}^3)^2 - 4 C_{23}^1$. Then the general solution to (15) has the form

$$\psi_1(t) = \begin{cases} A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} - C_{23}^2 / (C_{23}^1 u_2), & \lambda_{1,2} = u_2 / (C_{23}^3 \pm \sqrt{B})/2 \text{ if } B > 0, \\
(A_1 t + A_2) e^{C_{23}^1 u_2 t} - C_{23}^2 / (C_{23}^1 u_2), & \text{if } B = 0, \\
e^{C_{23}^1 u_2 t} (A_1 \cos (u_2 \sqrt{-B/2}) + A_2 \sin (u_2 \sqrt{-B/2})) - C_{23}^2 / (C_{23}^1 u_2) & \text{if } B < 0, \end{cases}$$

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where $A_1$ and $A_2$ are arbitrary reals. Since the function $\psi_1(t)$, with $t \in \mathbb{R}$, is bounded, this implies that either $\psi_1(t) = -C_{23}^2/(C_{23}u_2)$ or
\begin{equation}
\psi_1(t) = A_1 \cos(u_2 \sqrt{C_{23}^1} t) + A_2 \sin(u_2 \sqrt{C_{23}^1} t) - \frac{C_{23}^2}{C_{23}^1 u_2}.
\end{equation}
In (16), we can put $A_1 = A_2 = 0$; i.e. again
\begin{equation}
\psi_1(t) = -\frac{C_{23}^2}{C_{23}^1 u_2}.
\end{equation}
Then it is easy to check that $\psi_3(t) = 0$ and $\psi_4(t) = 0$ satisfy (14). In view of (11), the second equality in (13) is written as
\begin{equation}
F_U(-C_{23}^2 s/C_{23}^1, s, 0) = 1/F(0, s, 0).
\end{equation}
SUFFICIENCY: Let $C_{23}^1 = C_{23}^2 = 0$ and $F_U(k(s), s, 0) = 1/F(0, s, 0)$ for some $k(s) \in \mathbb{R}$. We put
\begin{equation}
\psi_1(t) = k(s)F(0, s, 0), \quad \psi_2(t) = sF(0, s, 0), \quad \psi_3(t) = 0, \quad \psi_4(t) = 0.
\end{equation}
The functions $\psi_i(t)$, with $i = 1, \ldots, 4$, satisfy (14), and (13) is valid. Consequently, abnormal extremal (4) satisfies the PMP with $M(t) \equiv 1$ (see (3)), and so (4) is nonstrictly abnormal.

Let $C_{23}^1 \neq 0$ and (17) be true. It is easy to check in view of (4) that the functions
\begin{equation}
\psi_1(t) = -sC_{23}^2 F(0, s, 0)/C_{23}^1, \quad \psi_2(t) = sF(0, s, 0), \quad \psi_3(t) = 0, \quad \psi_4(t) = 0
\end{equation}
satisfy (14), and (13) holds. Then (4) satisfies the PMP with $M(t) \equiv 1$ (see (3)), and hence (4) is nonstrictly abnormal. Theorem 1 is proved. \hfill \Box

**Theorem 2.** Abnormal extremal (4) of an $n$-dimensional connected Lie group $G$, $n \geq 4$, with a left-invariant sub-Riemannian metric $d$ defined by an inner product on an $(n - 1)$-dimensional subspace $q$ of the Lie algebra $g$ of $G$, generating $g$, is a geodesic; i.e. its sufficiently small segments are shortest curves, if and only if the extremal is nonstrictly abnormal.

**Proof.** SUFFICIENCY: This is a consequence of the statement that each normal sub-Riemannian extremal is a geodesic, which was proved in Appendix C in the memoir by Liu and Sussmann [11].

NECESSITY: A strictly abnormal sub-Riemannian extremal is not a geodesic in view of the equality $q + [q, q] = q$ and the Goch condition for abnormal geodesics [12, item 20.5.1]. \hfill \Box

**4. The Algebraic Results Including Classification**

In what follows, we will use Table 1 usually without any specification

Proposition 4 implies

**Corollary 3.** All four-dimensional real Lie algebras but $4q_1$ and $q_{4,5}^{11}$ have three-dimensional generating subspaces.

We will let $\langle \cdot \rangle$ stand for the linear span of the vectors in the parentheses.

**Proposition 9.** Let $q$ be a three-dimensional generating subspace of a four-dimensional Lie algebra $g$ such that $\{0\} \neq [\mathfrak{N}(q) \cap q, q] \cong \mathfrak{N}(q) \cap q$. Then $g = q_{4,8}^1$ or there is a basis $(e_1, e_2, e_3 := [e_1, e_2])$ for $q$ as in Lemma 1.
Table 1. Four-dimensional real Lie algebras \( \mathfrak{g} \), and \( k \) is the number of equivalence classes of three-dimensional subspaces generating \( \mathfrak{g} \) (cp. [7, 8])

| Type of a Lie algebra | \( k \) | Nonzero commutators |
|-----------------------|-------|-------------------|
| \( 4\mathfrak{g}_1 \) | 0     | –                 |
| \( \mathfrak{g}_{2,1} \oplus \mathfrak{g}_1 \) | 1     | \([E_1, E_2] = E_1\) |
| \( 2\mathfrak{g}_{2,1} \) | 2     | \([E_1, E_2] = E_1, \ [E_3, E_4] = E_3\) |
| \( \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \) | 1     | \([E_2, E_3] = E_1\) |
| \( \mathfrak{g}_{3,2} \oplus \mathfrak{g}_1 \) | 3     | \([E_2, E_3] = E_1 - E_2, \ [E_3, E_1] = E_1\) |
| \( \mathfrak{g}_{3,3} \oplus \mathfrak{g}_1 \) | 1     | \([E_2, E_3] = -E_2, \ [E_3, E_1] = E_1\) |
| \( \mathfrak{g}_{3,4} \oplus \mathfrak{g}_1, \ 0 \leq \alpha \neq 1 \) | 3, 4  | \([E_2, E_3] = E_1 - \alpha E_2, \ [E_3, E_1] = \alpha E_1 - E_2\) |
| \( \mathfrak{g}_{3,5} \oplus \mathfrak{g}_1, \ \alpha \geq 0 \) | 2     | \([E_2, E_3] = E_1 - \alpha E_2, \ [E_3, E_1] = \alpha E_1 + E_2\) |
| \( \mathfrak{g}_{3,6} \oplus \mathfrak{g}_1 \) | 5     | \([E_2, E_3] = E_1, \ [E_3, E_1] = E_2, \ [E_1, E_2] = -E_3\) |
| \( \mathfrak{g}_{3,7} \oplus \mathfrak{g}_1 \) | 2     | \([E_2, E_3] = E_1, \ [E_3, E_1] = E_2, \ [E_1, E_2] = E_3\) |
| \( \mathfrak{g}_{4,1} \) | 2     | \([E_2, E_4] = E_1, \ [E_3, E_4] = E_2\) |
| \( \mathfrak{g}_{4,2}, \ \alpha \neq 0 \) | 1, 3  | \([E_1, E_4] = \alpha E_1, \ [E_2, E_4] = E_2, \ [E_3, E_4] = E_2 + E_3\) |
| \( \mathfrak{g}_{4,3} \) | 3     | \([E_1, E_4] = E_1, \ [E_3, E_4] = E_2\) |
| \( \mathfrak{g}_{4,4} \) | 2     | \([E_1, E_4] = E_1, \ [E_2, E_4] = E_2 + E_2, \ [E_3, E_4] = E_2 + E_3\) |
| \( \mathfrak{g}_{4,5}^{\alpha, \beta} \) | 0, 1, 4 | \([E_1, E_4] = E_1, \ [E_2, E_4] = \beta E_2, \ [E_3, E_4] = \alpha E_3\) |
| \( \mathfrak{g}_{4,6}^{\alpha, \beta}, \ \alpha > 0, \ \beta \in \mathbb{R} \) | 2     | \([E_1, E_4] = \alpha E_1, \ [E_2, E_4] = \beta E_2 - E_3, \ [E_3, E_4] = E_2 + \beta E_3\) |
| \( \mathfrak{g}_{4,7} \) | 2     | \([E_1, E_4] = 2E_1, \ [E_2, E_4] = E_2, \ [E_3, E_4] = E_2 + E_3, \ [E_2, E_3] = E_1\) |
| \( \mathfrak{g}_{4,8}^{1} \) | 2     | \([E_2, E_3] = E_1, \ [E_2, E_4] = E_2, \ [E_3, E_4] = -E_3\) |
| \( \mathfrak{g}_{4,8}^{\alpha}, \ -1 < \alpha \leq 1 \) | 1, 2  | \([E_1, E_4] = (1 + \alpha) E_1, \ [E_2, E_4] = E_2, \ [E_3, E_4] = E_2 + \alpha E_3, \ [E_2, E_3] = E_1\) |
| \( \mathfrak{g}_{4,9}^{\alpha}, \ \alpha \geq 0 \) | 2     | \([E_1, E_4] = 2\alpha E_1, \ [E_2, E_4] = \alpha E_2 - E_3, \ [E_3, E_4] = E_2 + \alpha E_3, \ [E_2, E_3] = E_1\) |
| \( \mathfrak{g}_{4,10} \) | 1     | \([E_1, E_4] = E_1, \ [E_2, E_3] = E_2, \ [E_1, E_4] = -E_2, \ [E_2, E_4] = E_1\) |

**Proof.** By Lemma 1, \( \mathfrak{N}(q) \cap q = \langle e_2 \rangle \). Let \( s := [e_2, q] \). Then \( s \cap \langle e_2 \rangle = \{0\} \) and either \( \dim(s) = 1 \) or \( \dim(s) = 2 \).

1. If \( \dim(s) = 1 \) then the operator \( \text{ad}(e_2) : q \to q \) has as eigenvalues 0 of multiplicity 2 and a real \( \alpha \neq 0 \) of multiplicity 1. The corresponding eigenvectors \( e, f \in q \) exist such that \( e \parallel e_2, f \parallel e_2, [e_2, e] = 0 \), and \( [e_2, f] = \alpha f \). Putting \( e_1 = -(e + f), e_3 = \alpha f \), we see that \( e_3 = [e_1, e_2], (e_1, e_2, e_3) \) is a basis for \( q \) from Lemma 1.

2. If \( \dim(s) = 2 \) then \( \text{ad}(e_2) : s \to s \) has two (possibly coinciding) nonzero eigenvalues. The three cases are possible: (a) Eigenvalues are conjugate and purely imaginary; (b) Eigenvalues are real and
equal to $\alpha \neq \beta$; and (c) Eigenvalues are real and equal to $\alpha = \beta$.

(a) Let $e_1$ be a nonzero vector from $\mathfrak{s}$. Then vectors $e_1, e_2, e_3 := [e_1, e_2]$ constitute a basis for $\mathfrak{q}$ from Lemma 1.

(b) Let $e$ and $f$ be nonzero eigenvectors in $\mathfrak{s}$ with eigenvalues $\alpha$ and $\beta$. Putting $e_1 = -(e + f)$, we get that $e_1, e_2, e_3 := [e_1, e_2] = \alpha e + \beta f$ constitute a basis for $\mathfrak{q}$ from Lemma 1.

(c) Multiplying, if need be, the vector $e_2$ by $1/\alpha$, we can assume that $\alpha = 1$. The two subcases are possible here: There is a basis $(e, f)$ for $\mathfrak{s}$ such that $[e_2, e] = e + f$, $[e_2, f] = f$, or $[e_2, e] = e$ for every $e \in \mathfrak{s}$. In the first subcase, we put $e_1 = f - e$, $e_3 = e$ and get a basis $(e_1, e_2, e_3 = [e_1, e_2])$ from Lemma 1. In the second subcase, let $e_1$ and $e_3$ be some linearly independent vectors of $\mathfrak{s}$. Then $[e_2, e_1] = e_1$, $[e_2, e_3] = e_3$, $e_4 := [e_1, e_3] \not\in \mathfrak{q}$, and $(e_1, e_2, e_3, e_4)$ is a basis in $\mathfrak{g}$ from Lemma 1. Further, by the Jacobi identity for $e_1$, $e_2$, and $e_3$, we have

$$[e_2, e_4] = [e_2, [e_1, e_3]] = [[e_2, e_1], e_3] + [e_1, [e_2, e_3]] = [e_1, e_3] + [e_1, e_3] = 2e_4.$$ 

Thus, $e_1, e_2, e_3$, and $e_4$ are eigenvectors of $\text{ad}(e_2) : \mathfrak{g} \to \mathfrak{g}$ with eigenvalues $1, 0, 1$, and $2$. Moreover, by the Jacobi identity for $e_1, e_2,$ and $e_4$, we have

$$[e_2, [e_1, e_4]] = [[e_2, e_1], e_4] + [e_1, [e_2, e_4]] = [e_1, e_4] + 2[e_1, e_4] = 3[e_1, e_4]$$

and similarly $[e_2, [e_3, e_4]] = 3[e_3, e_4]$. By the above, 3 is not an eigenvalue of $\text{ad}(e_2) : \mathfrak{g} \to \mathfrak{g}$. Therefore $[e_1, e_4] = [e_3, e_4] = 0$. Putting $e_1 = E_2, e_2 = -E_4, e_3 = E_3, e_4 = E_1$, we see from Table 1 that $\mathfrak{g} = \mathfrak{g}_1$. □

Remark 1. All four-dimensional real Lie algebras $\mathfrak{g}$ but $4\mathfrak{g}_1; 2\mathfrak{g}_1; 2\mathfrak{g}_2, 1; 2\mathfrak{g}_3, 1; 2\mathfrak{g}_1 \oplus \mathfrak{g}_2, 1; 2\mathfrak{g}_1 \oplus \mathfrak{g}_3, 1; 2\mathfrak{g}_1 \oplus \mathfrak{g}_4, 1; 2\mathfrak{g}_1 \oplus \mathfrak{g}_5, 1$, with $-1 \leq \alpha \leq 1; \alpha \neq 0; \mathfrak{g}_{1,5}, 1$, with $-1 < \alpha < 1, \alpha \neq 0; \mathfrak{g}_{1,8}$, have two-dimensional generating subspaces [10].

Proposition 10. Let $\mathfrak{g}$ be a four-dimensional real Lie algebra for which there is a generating two-dimensional subspace. Some three-dimensional subspace $\mathfrak{q}$ generates the Lie algebra $\mathfrak{g}$ and includes no one-dimensional ideal of this algebra if and only if there is a basis $(e_1, e_2, e_3 := [e_1, e_2])$ for $\mathfrak{q}$ satisfying Lemma 1.

Proof. Sufficiency is obvious. Let us prove necessity.

In view of Proposition 9 and Remark 1, we have to consider the case when $e_2 \in \mathfrak{s} \neq \langle e_2 \rangle$. It is clear that $\text{ad}(e_2)$ isomorphically maps any two-dimensional subspace $\mathfrak{p} \subset \mathfrak{q}$ that does not contain $e_2$ on $\mathfrak{s}$. Since $\dim(\mathfrak{p} \cap \mathfrak{s}) = 1$, there is $e_1 \in \mathfrak{p}$ such that $e_1 \not\in \mathfrak{s}$ and $[e_1, e_2] \not\in \langle e_2 \rangle$. Hence $e_1, e_2, e_3 := [e_1, e_2]$ constitute a basis for $\mathfrak{q}$ satisfying Lemma 1. □

Subspaces $\mathfrak{q}_1, \mathfrak{q}_2 \subset \mathfrak{g}$ are said to be equivalent if $\mathfrak{q}_2 = \xi \mathfrak{q}_1$ for some automorphism $\xi$ of $\mathfrak{g}$.

Proposition 11. Every two generating three-dimensional subspaces of the Lie algebra $\mathfrak{g}_{1,8}$ are equivalent.

Proof. It is easy to see that $\mathfrak{g}_{1,8}$ has only one one-dimensional ideal $(E_1)$ and each two-dimensional subspace $\mathfrak{q} \subset \mathfrak{g}^1$, including $E_1$, is an ideal of $\mathfrak{g}_{1,8}$. Moreover, $\dim(\mathfrak{q}^1) = 3$. Therefore, $\dim(\mathfrak{q} \cap \mathfrak{q}^1) \geq 2$. Then, by Proposition 3, every three-dimensional generating subspace $\mathfrak{q}$ of $\mathfrak{g}_{1,8}$ includes no one-dimensional ideal.

From this, Remark 1, and the proof of Proposition 10 it is easy to deduce the hypotheses of Proposition 9. Remark 1 and the proof of Proposition 9 imply Proposition 11. □

Proposition 12. Abnormal extremal (4) (and its every left shift) of a connected Lie group $G$ with Lie algebra $\mathfrak{g}_{1,8}$ and left-invariant sub-Finsler quasimetric $d$ defined by a seminorm $F$ on the three-dimensional generating subspace $\mathfrak{q} \subset \mathfrak{g}_{1,8}$ with a basis $(e_1, e_2, e_3)$ such that $[e_2, e_k] = e_k, k = 1, 3$, is nonstrictly abnormal if and only if $F_U(0, s, 0) = 1/F(0, s, 0)$.

Proof. As follows from Proposition 11 and the proof of Proposition 9, for every three-dimensional subspace $\mathfrak{q}$, generating $\mathfrak{g}_{1,8}$, there exists a basis $(e_1, e_2, e_3)$ such that $[e_1, e_2] = -e_1$ and $[e_2, e_3] = e_3$. In view of (11), equations (7)–(10) can be rewritten as the system

$$\psi'_1 = u_2 \psi_1, \quad \psi'_2 = 0, \quad \psi'_3 = u_2 \psi_3, \quad \psi'_4 = 2u_2 \psi_4$$

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whose general solution has the form
\[
\psi_1(t) = \varphi_1 e^{u_1 t}, \quad \psi_2(t) = \varphi_2, \quad \psi_3(t) = \varphi_3 e^{u_2 t}, \quad \psi_4(t) = \varphi_4 e^{2u_2 t},
\]
where \( \varphi_i, i = 1, \ldots, 4 \), are some constants.

According to the PMP and (11), abnormal extremal (4) (and its every left shift) in \((G, d)\) is nonstrictly abnormal if and only if there are \( \varphi_1, \varphi_3 \in \mathbb{R} \) such that
\[
F_U (\varphi_1 e^{u_2 t}, sF(0, s, 0), \varphi_3 e^{u_2 t}) = 1
\]
for almost all \( t \in \mathbb{R} \). This and the boundedness condition for the body \( U^* \) dual to \( U \) imply that
\[
\varphi_1 = \varphi_3 = 0; \text{ i.e., } F_U(0, s, 0) = 1/F(0, s, 0).
\]

Further, \((\cdot, \cdot)\) denotes the inner product with the orthonormal basis \((e_1, e_2, e_3)\) from Proposition 12 or Theorem 1.

**Theorem 3.** Let \( d \) be a left-invariant sub-Riemannian metric on a connected four-dimensional Lie group \( G \) with a Lie algebra \((g, [\cdot, \cdot])\) defined by an inner product \((\cdot, \cdot)_1\) on the three-dimensional subspace \( q \subset g \) with orthonormal basis \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)\), with \( \tilde{e}_2 \| e_2 \) satisfying the conditions of Lemma 1. Abnormal extremal (4) is nonstrictly abnormal if and only if \( s = [e_2, q] \subset \langle \tilde{e}_1, \tilde{e}_3 \rangle \).

**Proof.** In view of Propositions 5 and 9, we consider the five cases: The two cases by Proposition 5, one case in Proposition 12, and the two cases in Theorem 1.

The criterion \( \langle \tilde{e}_2 \rangle = \mathcal{E}(q) \) for the nonstrict abnormality of (4) for any quasimetric (in particular, the sub-Riemannian metric) \( d \) on \( G \) is equivalent to the equality \( s = \{0\} \), which implies that \( s \subset \langle \tilde{e}_1, \tilde{e}_3 \rangle \).

Extremal (4) is strictly abnormal for every quasimetric (in particular, the sub-Riemannian metric) \( d \) on \( G \) for \( \langle \tilde{e}_2 \rangle \subset s \), which always gives \( s \not\subset \langle \tilde{e}_1, \tilde{e}_3 \rangle \).

Put \( g = g_{1.8}' \). By the last equality in Proposition 12 for \( F(u) = \sqrt{(u, u)} \), \( se_2 \) is \((\cdot, \cdot)\)-orthogonal to the tangent plane for \( dU \) at \( se_2/F(e_2) \). The plane is parallel to \( \langle \tilde{e}_1, \tilde{e}_3 \rangle \), and \( \langle \tilde{e}_1, \tilde{e}_3 \rangle = \langle e_1, e_3 \rangle = s \).

In the first case of Theorem 1, \( s = \langle e_3 \rangle \), and it follows from the corresponding equality for \( F_U \) in the case \( F(u) = \sqrt{(u, u)} \) that \( (k(s), s, 0) \) is \((\cdot, \cdot)\)-orthogonal to the tangent plane to \( dU \) at \( se_2/F(e_2) \). This plane is parallel to \( \langle \tilde{e}_1, \tilde{e}_3 \rangle \), and \( s = \langle e_3 \rangle \subset \langle \tilde{e}_1, \tilde{e}_3 \rangle \).

In the second case of Theorem 1 we get that for \( F(u) = \sqrt{(u, u)} \) the vector \( (-C_{23}^2 s/C_{23}^1, s, 0) \) and the tangent plane to \( dU \) at \( se_2/F(e_2) \) are orthogonal relative to \((\cdot, \cdot)\). This plane is parallel to the plane \( \langle \tilde{e}_1, \tilde{e}_3 \rangle \), containing \( [e_2, e_3] = C_{23}^1 e_1 + C_{23}^2 e_2 \) and \( [e_1, e_2] = e_3 \); therefore \( s = \langle \tilde{e}_1, \tilde{e}_3 \rangle \).

**Proposition 13.** If a four-dimensional real Lie algebra \( g \) has a two-dimensional generating subspace \( p_0 \) then there exists a basis \((e_1, e_2, e_3, e_4)\) for \( g \) such that \( e_1, e_2 \in p_0, [e_1, e_2] = e_3, [e_1, e_3] = e_4, \) and \( C_{23}^2 = 0 \). The three-dimensional subspace \( q_0 := p_0 \oplus [p_0, p_0] \) generates \( g \) and the basis \((e_1, e_2, e_3)\) for \( q_0 \) satisfies Lemma 1. If \( \tilde{p} \subset g \) is equivalent to \( p_0 \) then the corresponding three-dimensional subspace \( \tilde{q} = \tilde{p} \oplus [\tilde{p}, \tilde{p}] \) and \( q_0 \) are equivalent.

**Proof.** The first claim of Proposition 13 was proved in [10]. It immediately implies the second claim of Proposition 13. If \( \tilde{p} = \xi(p_0) \) for some automorphism \( \xi \) of \( g \) then \( (\xi(e_1), \xi(e_2), \xi(e_3)) \) is a basis for \( \tilde{q} \) because \( \xi(e_3) = \xi([e_1, e_2]) = [\xi(e_1), \xi(e_2)] \). Consequently, \( \tilde{q} = \xi(q_0) \).

**Proposition 14.** A three-dimensional subspace \( q \subset g \) generates \( g = 2g_{2,1} \) if and only if \( \dim(q \cap J) = 1 \) for every two-dimensional ideal \( J \) of \( g \). There are two equivalence classes of these subspaces; \( q \) belongs to the first (second) equivalence class if \( q \) includes one (includes no) one-dimensional ideal of \( g \).

**Proof.** Let \( g_{2,1}' \) and \( g_{2,1}'' \) denote the first and the second copies of the Lie algebra \( g_{2,1} \) respectively, and let \( L_1 \) and \( L_2 \) be their one-dimensional ideals. The Lie algebra \( g \) has the three two-dimensional ideals: \( g_{2,1}', g_{2,1}'' \), and \( g' = L_1 \oplus L_2 \). By Proposition 3, the intersection of \( q \) with each of the ideals is one-dimensional. The two cases are possible:

1. The intersection of \( q \) with exactly one of the Lie algebras \( g_{2,1}' \) and \( g_{2,1}'' \) is one-dimensional ideal; without loss of generality, we can assume that \( q \cap g_{2,1}' = L_2 \);
2. \( q \cap g_{2,1}'' \neq L_k, k = 1, 2 \).
It is clear that the subspaces \( q \) of the first and the second types are not equivalent. 

In the first case, there is a basis \((e_1, e_2, e_3)\) for \( q \) such that
\[
e_1 \in q \cap g_{2,1}^1, \quad e_2 \in \mathcal{L}_2, \quad e_3 = f_1 + f_2, \quad 0 \neq f_1 \in \mathcal{L}_1, \quad f_2 \in g_{2,1}^2, \quad f_2 \notin \mathcal{L}_2.
\]
Moreover, \([e_1, e_3] = f_1 = e_4, [e_2, e_3] = e_2, [e_1, e_4] = e_4\), and all other Lie brackets for the basis vectors \( e_1, e_2, e_3, \) and \( e_4 \) are zero. It follows that every two subspaces \( q \) of the first type are equivalent.

In the second case, there is a basis \((e_1, e_2, e_3)\) for \( q \) such that
\[
e_1 \in q \cap g_{2,1}^1, \quad e_2 = e_1 + f_2, \quad f_2 \in q \cap g_{2,1}^2, \quad e_3 \in q', \quad e_3 \notin \mathcal{L}_1, \quad e_3 \notin \mathcal{L}_2,
\]
and the components of \( e_2 \) at \( E_2 \) and \( E_4 \) are equal to 1. Then \( e_4 := [e_1, e_3] \in \mathcal{L}_1, [e_2, e_3] = -e_3, [e_1, e_4] = [e_2, e_4] = -e_4\), and all other Lie brackets for the basis vectors \( e_1, e_2, e_3, \) and \( e_4 \) are zero. It follows that every two subspaces \( q \) of the second type are equivalent. \( \square \)

**Proposition 15.** A three-dimensional subspace \( q \) of the Lie algebra \( g = g_{3,6} \oplus g_1 \) generates \( g \) if and only if \( q \neq g_{3,6} \) and the projection of \( q \) to \( g_{3,6} \) along \( g_1 \) is not a two-dimensional subalgebra of \( g_{3,6} \). There exist five equivalence classes of these subspaces.

**Proof.** The first claim follows from Proposition 1. According to [9], all two-dimensional Lie subalgebras of the Lie algebra \( g_{3,6} \) are equivalent to \((E_1 - E_3, E_2)\).

Let \( g_1 \subset q \). By Lemma 1, \( q_1 = g_1 \). Since \( p \cap q_1 = \{0\} \) for two-dimensional space \( p := g_3 \cap q \); therefore, \( p \) generates \( g_3 \). In [13] it was proved that there exist two equivalence classes of these \( p \) so the similar claims holds for the three-dimensional subspaces \( q \) generating \( g \) and including \( g_1 \).

Assume now that \( g_1 \notin q \); i.e., \( q \) includes no one-dimensional ideal of \( g \). It follows from Remark 1 and Proposition 10 that there is a basis \((e_1, e_2, e_3 := [e_1, e_2])\) for \( q \) satisfying Lemma 1; in particular, the two-dimensional subspace \( p_0 \) with the basis \((e_1, e_2)\) generates \( g \). In [10] it was proved that there are four equivalence classes of these spaces \( p_0 \), and the subspaces \((E_1, E_2 + E_3), (E_3, E_1 + E_4), (E_1, E_3 + E_4), (E_4 + (E_2 - E_3)/2, E_2 + E_3)\) belong to the first, the second, the third, and the fourth equivalence class, respectively. It is not hard to compute in view of Proposition 13, that for \( p_{0,i}, i = 1, \ldots, 4 \), of the \( i \)th equivalence class, the three-dimensional subspace \( q_{0,i} = p_{0,i} \oplus [p_{0,i}, p_{0,i}] \), with \( i = 1, \ldots, 4 \), has a basis \((e_1, e_2, e_3)\) from Lemma 1 such that \([e_1, e_2] = e_3\), and for \( e_4 := [e_1, e_3], \)
\[
[e_2, e_3] = -e_1, \quad [e_2, e_4] = 0, \quad [e_1, e_4] = e_3, \quad [e_3, e_4] = e_1;
\]
\[
[e_2, e_3] = -e_1, \quad [e_2, e_4] = 0, \quad [e_1, e_4] = -e_3, \quad [e_3, e_4] = -e_1;
\]
\[
[e_2, e_3] = e_1, \quad [e_2, e_4] = 0, \quad [e_1, e_4] = e_3, \quad [e_3, e_4] = -e_1;
\]
\[
[e_2, e_3] = e_2, \quad [e_2, e_4] = e_3, \quad [e_1, e_4] = 0, \quad [e_3, e_4] = e_4
\]
(in the ascending order of \( i \)). In result of the mutual permutation of \( e_1 \) and \( e_3 \) the equality \([e_1, e_2] = e_3\) and the last three (respectively, the first of) relations in their first quadruple will turn into the second quadruple (respectively, to \([e_1, e_2] = e_3\)). Thus, the subspaces \( q_{0,1} \) and \( q_{0,2} \) are equivalent.

Let us show that \( q_{0,k}, \) with \( k = 2, 3, 4 \), are pairwise not equivalent.

Let \( \xi \) be an automorphism of the Lie algebra \( g_{3,6} \oplus g_1 \) such that \( \xi(q_{0,k}) = q_{0,j} \), where \( 2 \leq k < j \leq 4 \). Since \([e_2]\) is the only one-dimensional normalizer of \( q_{0,k} \) and \( q_{0,j} \); therefore, \( \xi([e_2]) = [e_2] \). Hence, \( j \neq 4 \) because \([e_2] \subset ([e_2], q_{0,4}) \), and \([e_2], q_{0,k} = [e_1, e_3] = [e_2], [e_1, e_3]) \) for \( k = 2, 3 \). The eigenvalues of \( \text{ad}(e_2) \) on \([e_1, e_3]\) are equal to \( \pm 1 \) for \( q_{0,2} \) and \( \pm i \) for \( q_{0,3} \). Thus, these spectra are not real-homothetic, and so \( q_{0,2} \) and \( q_{0,3} \) are not equivalent. \( \square \)

**Proposition 16.** Every two three-dimensional subspaces of every four-dimensional real Lie algebra \( g \neq \mathfrak{sl}(2, \mathbb{R}) \oplus g_1 \), generating \( g \) and including the same one-dimensional ideal of this algebra, are equivalent.

**Proof.** Assume that some different three-dimensional subspaces \( q, \tilde{q} \subset g \) generate \( g \) and include a one-dimensional ideal \( \mathcal{L} \) of \( g \).
Suppose that \( \mathfrak{L} \not\subset \mathfrak{C}(\mathfrak{g}) \), \( 0 \neq e_2 \in \mathfrak{L} \). Then there are linearly independent vectors \( e_1 \) and \( e_3 \) from some two-dimensional subspace \( \mathfrak{p} \subset \mathfrak{q} \), where \( \mathfrak{p} \cap \mathfrak{L} = \{0\} \), for which the first two of the following equalities are satisfied:

\[
[e_1, e_2] = 0, \quad [e_2, e_3] = e_2, \quad [e_1, e_3] = e_4, \quad [e_2, e_4] = 0.
\]  

We define \( e_4 \) by the third equality in (18). It follows from Lemma 1 that \( e_4 \notin \mathfrak{q} \); therefore, \( (e_1, e_2, e_3, e_4) \) is a basis for \( \mathfrak{g} \). The fourth relation in (18) follows from the Jacobi identity for \( e_1, e_2, \) and \( e_3 \) and the previous equalities; i.e.,

\[
[e_2, e_4] = [e_2, [e_1, e_3]] = [[e_2, e_1], e_3] + [e_1, [e_2, e_3]] = [e_1, e_2] = 0.
\]

Equalities (18) and the Jacobi identities for \( e_1, e_2, e_4 \) and \( e_2, e_3, e_4 \) imply

\[
C_{14}^3 e_2 = [e_2, [e_1, e_4]] = [[e_2, e_1], e_4] + [e_1, [e_2, e_4]] = 0 \Rightarrow C_{14}^3 = 0;
\]

\[
C_{34}^3 e_2 = [e_2, [e_3, e_4]] = [[e_2, e_3], e_4] + [e_3, [e_2, e_4]] = [e_2, e_4] = 0 \Rightarrow C_{34}^3 = 0.
\]

Writing the Jacobi identity \( [e_3, [e_1, e_4]] = [e_1, [e_3, e_4]] \) for \( e_1, e_3, \) and \( e_4 \), considering (18) and the equalities \( C_{14}^3 = 0, C_{34}^3 = 0 \), we get

\[
-C_{14}^1 e_4 - C_{14}^2 e_2 + C_{14}^3 (C_{34}^1 e_1 + C_{34}^2 e_2 + C_{34}^3 e_4) = [e_3, [e_1, e_4]]
\]

\[
= [e_1, [e_3, e_4]] = C_{34}^4 (C_{14}^1 e_1 + C_{14}^2 e_2 + C_{14}^4 e_4).
\]

Then

\[
[e_1, e_4] = C_{14}^2 e_2 + C_{14}^4 e_4, \quad [e_3, e_4] = C_{34}^3 e_1 + C_{34}^2 e_2 + C_{34}^4 e_4;
\]

moreover,

\[
C_{34}^1 C_{14}^4 = 0, \quad C_{14}^1 C_{34}^2 = C_{14}^2 (C_{34}^4 + 1).
\]

Put \( C_{14}^4 = 0 \). As we proved, there is a basis \( (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4) \) for \( \mathfrak{g} \) such that \( \tilde{e}_1, e_2, \) and \( \tilde{e}_3 \) belong to \( \tilde{\mathfrak{q}} \) and

\[
[e_1, e_2] = 0, \quad [e_2, \tilde{e}_3] = e_2, \quad [\tilde{e}_1, \tilde{e}_3] = \tilde{e}_4, \quad [e_2, \tilde{e}_4] = 0.
\]

Put \( C_{14}^1 = 0 \). As we proved, there is a basis \( (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4) \) for \( \mathfrak{g} \) such that \( \tilde{e}_1, e_2, \) and \( \tilde{e}_3 \) belong to \( \tilde{\mathfrak{q}} \) and

\[
[e_1, e_2] = 0, \quad [e_2, \tilde{e}_3] = e_2, \quad [\tilde{e}_1, \tilde{e}_3] = \tilde{e}_4, \quad [e_2, \tilde{e}_4] = 0.
\]

Denote by \( a_i \) and \( b_i \), with \( i = 1, \ldots, 4 \), the coordinates of \( \tilde{e}_1 \) and \( \tilde{e}_3 \) for the basis \( (e_1, e_2, e_3, e_4) \). Owing to (18), we can add to \( \tilde{e}_1 \) any linear combination of \( e_2 \) and \( e_4 \), and we can add to \( \tilde{e}_3 \) an arbitrary vector collinear to \( e_2 \), preserving the first two equalities in (21). Since \( \tilde{\mathfrak{q}} \neq \mathfrak{q} \), we can assume that \( a_4 = a_2 = b_2 = 0 \) and \( b_4 \neq 0 \). Then in view of (18) and (21) we have

\[
0 = [\tilde{e}_1, e_2] = a_1 [e_1, e_2] - a_3 [e_2, e_3] = -a_3 e_2 \quad \Rightarrow \quad a_3 = 0,
\]

\[
e_2 = [e_2, \tilde{e}_3] = -b_1 [e_1, e_2] + b_3 [e_2, e_3] + b_4 [e_2, e_4] = b_3 e_2 \quad \Rightarrow \quad b_3 = 1.
\]

Adding, if need be, the vector \( (b_4 - b_1) \tilde{e}_1 \) to \( \tilde{e}_3 \), we can assume that \( \tilde{e}_1 = e_1 \) and \( \tilde{e}_3 = b_4 e_1 + e_3 + b_4 e_4 \). Granting (18), (19), and \( C_{14}^4 = 0 \), we consecutively find

\[
\tilde{e}_4 = [e_1, b_4 e_1 + e_3 + b_4 e_4] = e_4 + b_4 [e_1, e_4] = e_4 + b_4 C_{14}^4 e_2,
\]

\[
[\tilde{e}_1, \tilde{e}_4] = [e_1, e_4 + b_4 C_{14}^4 e_2] = [e_1, e_4] + b_4 C_{14}^4 [e_1, e_4] = [e_1, e_4],
\]

\[
[\tilde{e}_3, \tilde{e}_4] = [b_4 e_1 + e_3 + b_4 e_4, e_4 + b_4 C_{14}^4 e_2] = b_4 [e_1, e_4] + [e_3, e_4] - b_4 C_{14}^2 [e_2, e_3]
\]

\[
= b_4 C_{14}^2 e_2 + [e_3, e_4] - b_4 C_{14}^2 [e_2, e_3] = [e_3, e_4], \quad [e_2, e_4] = [e_2, e_4 + b_4 C_{14}^4 e_2] = [e_2, e_4] = 0.
\]

Consequently, the linear operator \( \xi \) of the Lie algebra \( \mathfrak{g} \) such that \( \xi(e_2) = e_2 \) and \( \xi(e_i) = \tilde{e}_i, \) while \( i = 1, 3, 4 \), is an automorphism of \( \mathfrak{g} \) and the equality \( \xi(q) = \tilde{q} \) holds; i.e., the subspaces \( \mathfrak{q} \) and \( \tilde{\mathfrak{q}} \) are equivalent.
Assume that $C_{14}^4 \neq 0$. Using (18)–(20), it is easy to check that $\mathfrak{g}$ is isomorphic to $2\mathfrak{g}_{2,1}$; the required isomorphism $\xi : \mathfrak{g} \to 2\mathfrak{g}_{2,1}$ is given by the formulas

$$\xi(e_1) = (-C_{14}^2/C_{14}^4)E_1 - C_{14}^4E_4, \quad \xi(e_2) = E_1,$$

$$\xi(e_3) = E_2 + E_3 - C_{14}^2E_4, \quad \xi(e_4) = (-C_{14}^2/C_{14}^4)E_1 + C_{14}^4E_3.$$

Then by Proposition 14 the subspaces $\mathfrak{q}$ and $\tilde{\mathfrak{q}}$ are equivalent.

Assume that $\mathfrak{L} \subset \mathfrak{C}(\mathfrak{g})$. At first, consider all indecomposable four-dimensional Lie algebras including a nontrivial (one-dimensional) central ideal; i.e., $\mathfrak{g}_{4,1}, \mathfrak{g}_{4,3}, \mathfrak{g}_{4,8}^{-1}$, and $\mathfrak{g}_{4,9}^0$.

Put $\mathfrak{g} = \mathfrak{g}_{4,1}$ or $\mathfrak{g} = \mathfrak{g}_{4,3}$. By Proposition 3, any three-dimensional subspace $\mathfrak{q}$, generating $\mathfrak{g}$ and containing $\mathfrak{C}(\mathfrak{g})$, has a basis $(e_1, e_2, e_3)$, where $e_1$ does not belong to the three-dimensional ideal $\mathfrak{I} = \langle E_1, E_2, E_3 \rangle$; $e_2 = E_1$ in the case $\mathfrak{g}_{4,1}$ and $e_2 = E_2$ in the case $\mathfrak{g}_{4,3}$; $e_3 \in \mathfrak{I}$ and $e_3$ does not belong to any two-dimensional ideal $\mathfrak{J} \subset \mathfrak{I}$, and components of $e_1$ in $E_4$ and $e_3$ in $E_3$ in the basis $(E_1, E_2, E_3, E_4)$ are equal to 1. Then $e_4 := [e_1, e_3] \notin \mathfrak{q}$, $[e_1, e_4] = e_2$ in the case $\mathfrak{g}_{4,1}$ and $[e_1, e_4] = -e_2 - e_4$ in the case $\mathfrak{g}_{4,3}$, while all other Lie brackets for $e_1, e_2, e_3$, and $e_4$ are zero. Consequently, every two three-dimensional subspaces, generating $\mathfrak{g}$ and including $\mathfrak{C}(\mathfrak{g})$, are equivalent.

Assume that $\mathfrak{g} = \mathfrak{g}_{4,8}^{-1}$ or $\mathfrak{g} = \mathfrak{g}_{4,9}^0$. By Proposition 3, every three-dimensional subspace $\mathfrak{q}$, generating $\mathfrak{g}$ and including $\mathfrak{C}(\mathfrak{g})$, has a basis $(e_1, e_2, e_3)$, where $e_1 \notin \mathfrak{g}'$, $e_2, e_3 \in \mathfrak{C}(\mathfrak{g})$, and in the case $\mathfrak{g}_{4,8}^{-1}$, the vector $e_3$ does not belong to two-dimensional ideals $\mathfrak{J}_1 = \langle E_1, E_2 \rangle$ and $\mathfrak{J}_2 = \langle E_1, E_3 \rangle$. Since we can add to a vector of a basis any vector, collinear to another vector of the basis, we can assume without loss of generality that $e_3(a_1E_1 + a_2E_2 + a_3E_3 + E_4)$, while

$$\mathfrak{g}_{4,8}^{-1}: e_2 = a_1a_5E_1, \quad e_3 = (a_3a_4 + a_2a_5)E_1 + a_4E_2 + a_5E_3, \quad a_4a_5 \neq 0;$$

$$\mathfrak{g}_{4,9}^0: e_2 = (a_1 + a_5^2)E_1, \quad e_3 = (a_3a_5 - a_2a_4)E_1 + a_4E_2 - a_5E_3, \quad a_1^2 + a_5^2 \neq 0.$$

Then $e_4 := [e_1, e_3] \notin \mathfrak{q}$. It is easy to check that $[e_1, e_4] = e_3$, $[e_3, e_4] = 2e_2$ in the case $\mathfrak{g}_{4,8}^{-1}$ and $[e_1, e_4] = -e_3$, $[e_3, e_4] = e_2$ in the case $\mathfrak{g}_{4,9}^0$, while all other Lie brackets for $e_1, e_2, e_3$, and $e_4$ are zero. Consequently, every two three-dimensional subspaces, generating $\mathfrak{g}$ and including $\mathfrak{C}(\mathfrak{g})$, are equivalent.

Assume now that $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathfrak{g}_1$, $\mathfrak{g}_3 \neq \mathfrak{g}_{3,1}$, and $\mathfrak{g}_3 \neq \mathfrak{s}(2, \mathbb{R})$. By Lemma 1, $q_1 = \mathfrak{g}_1$ for every three-dimensional subspace $\mathfrak{q}$, generating $\mathfrak{g}$. Since $\mathfrak{p} \cap q_1 = \{0\}$ for two-dimensional subspace $\mathfrak{p} := \mathfrak{g}_3 \cap \mathfrak{q}$, by Lemma 1 $\mathfrak{p}$ generates $\mathfrak{g}_3$. In [10] we proved that in this case $\mathfrak{g}_3 \neq \mathfrak{g}_{3,3}$ and every two of these subspaces $\mathfrak{p}$ are equivalent. Hence, every two such three-dimensional subspaces, generating $\mathfrak{g}$ and including $\mathfrak{g}_1$, are equivalent.

Assume that $\mathfrak{g} = \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$. Since $\dim \mathfrak{g}' = 1$, $\dim \mathfrak{C}(\mathfrak{g}) = 2$, and $\mathfrak{g}' \subset \mathfrak{C}(\mathfrak{g})$; any three-dimensional generating subspace $\mathfrak{q} \subset \mathfrak{g}$ does not include $\mathfrak{g}'$ and by Lemma 1 $q_1 = \mathfrak{C}(\mathfrak{g}) \cap \mathfrak{q}$. By virtue of Proposition 3, the subspace $\mathfrak{q}$ has a basis $(e_1, e_2, e_3)$, where $e_1$ ($e_2$) belongs to the two-dimensional commutative ideal $\mathfrak{J}_1 = \langle E_1, E_2 \rangle$ (respectively, $\mathfrak{J}_2 = \langle E_1, E_3 \rangle$), $e_1, e_3 \notin \mathfrak{g}'$, and $e_2 \in \mathfrak{q}_1$. Then $e_4 := [e_1, e_3] \notin \mathfrak{q}$, while all other Lie brackets for $e_1, e_2, e_3$, and $e_4$ are zero. Thus, every two three-dimensional subspaces, generating $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ are equivalent.

It remains to consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$. Since $\dim \mathfrak{g}' = 1$ and $\mathfrak{C}(\mathfrak{g}) = 2\mathfrak{g}_1$; therefore, any three-dimensional subspace $\mathfrak{q}$ of the Lie algebra $\mathfrak{g}$, generating it, does not include $\mathfrak{g}'$ and by Lemma 1 $q_1 = \mathfrak{C}(\mathfrak{g}) \cap \mathfrak{q}$. Since the subalgebras $\mathfrak{g}_{2,1}$ and $\mathfrak{C}(\mathfrak{g})$ are two-dimensional and $\mathfrak{g}' \subset \mathfrak{g}_{2,1}$, the subspaces $\mathfrak{q} \cap \mathfrak{g}_{2,1}$ and $\mathfrak{q} \cap \mathfrak{C}(\mathfrak{g})$ are one-dimensional. Then there is a basis $(e_1, e_2, e_3)$ for $\mathfrak{q}$ such that

$$e_1 \in \mathfrak{g}_{2,1}, \quad e_1 \notin \mathfrak{g}', \quad e_2 \in \mathfrak{C}(\mathfrak{g}), \quad e_3 = f_1 + f_2, \quad 0 \neq f_1 \in \mathfrak{g}', \quad f_2 \in \mathfrak{C}(\mathfrak{g}), \quad f_2 \nmid e_2.$$

Moreover, $[e_1, e_3] = e_4 = f_1$ and $[e_1, e_4] = e_4$, while all other Lie brackets for $e_1, e_2, e_3$, and $e_4$ are zero. Consequently, every two three-dimensional subspaces of the Lie algebra $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$, generating it, are equivalent. \(\square\)
Proposition 17. Assume that a four-dimensional Lie algebra \( g \) has three-dimensional generating subspaces. Then

1. If \( g \) has an infinite number of two-dimensional ideals then every two three-dimensional subspaces, generating the Lie algebra \( g \), are equivalent.

2. If \( g \) has a finite number (respectively, zero) of two-dimensional ideals and \( g \) is different from the Lie algebras \( 2g_{2,1} \) and \( g_{3,6} \oplus g_1 \), then \( g \) has a finite number \( m \), \( 0 \leq m \leq 3 \), of pairwise nonequivalent one-dimensional subspaces \( \mathcal{L}_1, \ldots, \mathcal{L}_m \). There exist \( m+1 \) equivalence classes of three-dimensional subspaces, generating \( g \); \( q \subset g \) belongs to the \( i \)th equivalence class \((i = 1, \ldots, m)\), if \( \mathcal{L}_i \subset q \); while \( q \subset g \) belongs to the \((m + 1)\)th equivalence class, if \( q \) includes no one-dimensional ideal of \( g \).

Proof. 1. As follows from Table 1 and Corollary 3, \( g \) is one of the Lie algebras \( g_{2,1} \oplus 2g_1 \); \( g_{3,1} \oplus g_1 \); \( g_{3,3} \oplus g_1 \); \( g_{4,2}^1 \); \( g_{4,8} \); \( g_{4,5}^{\alpha,1} \), with \( -1 \leq \alpha < 1 \) and \( \alpha \neq 0 \); and \( g_{4,5}^{\alpha,\alpha} \), with \( -1 < \alpha < 1 \) and \( \alpha \neq 0 \).

The Lie algebras \( g_{2,1} \oplus 2g_1 \) and \( g_{4,1} \oplus g_1 \) were considered in the proof of Proposition 16.

Put \( g = g_{3,3} \oplus g_1 \). Every two-dimensional subspace in \( g_{3,3} \) is a Lie algebra. Therefore a three-dimensional subspace \( q \subset g \) generates \( g \) if and only if \( g_{3,3} \) is a projection of \( q \) to \( g_{3,3} \) along \( g_1 \) and \( \dim(q \cap g') = 1 \). Then every two three-dimensional subspaces, generating the Lie algebra \( g \), are equivalent.

Put \( g = g_{4,1}^1 \). It is easy to see that every two-dimensional subspace \( \mathfrak{J} \subset g' \) containing \( E_2 \), is an ideal of \( g \). Then by Proposition 3 any three-dimensional subspace \( q \) of \( g \) generating it, does not contain \( E_2 \) and has a basis \((e_1, e_2, e_3)\), where \( e_1 \not\in g' \) and the component of \( e_1 \) at \( E_4 \) is equal to 1, \( e_2 \in (E_1, E_2) \), and \( e_3 \in E_3 \). It is easy to check that

\[
[e_1, e_2] = -e_2, \quad e_4 := [e_1, e_3] \not\in q, \quad [e_1, e_4] = -e_3 - 2e_4,
\]

while all other Lie brackets for \( e_1, e_2, e_3, \) and \( e_4 \) are zero. Consequently, every three-dimensional subspace, generating \( g_{4,1}^1 \), includes a one-dimensional ideal of this Lie algebra, and every two such subspaces are equivalent.

Put \( g = g_{4,5}^{\alpha,1} \) or \( g = g_{4,5}^{\alpha,\alpha} \), with \( -1 \leq \alpha < 1 \) and \( \alpha \neq 0 \). It is easy to see that the subspace \( \mathfrak{J} := (E_1, E_2) \) in the case \( g_{4,5}^{\alpha,1} \) \((\mathfrak{J} := (E_2, E_3) \) in the case \( g_{4,5}^{\alpha,\alpha} \)) and every two-dimensional subspace \( \mathfrak{J} \subset g' \), containing \( E_3 \) in the case \( g_{4,5}^{\alpha,1} \) \((E_1 \) in the case \( g_{4,5}^{\alpha,\alpha} \)), are ideals of this Lie algebra. Then by Proposition 3 any three-dimensional subspace \( q \) of the Lie algebra \( g \), generating it, does not contain \( E_3 \) in the case \( g_{4,5}^{\alpha,1} \) \((E_1 \) in the case \( g_{4,5}^{\alpha,\alpha} \)) and has a basis \((e_1, e_2, e_3)\), where \( e_1 \not\in g' \) and the component of \( e_1 \) at \( E_4 \) is equal to 1, \( e_2 \in \mathfrak{J}, e_3 = g_1 + E_3, g_1 \in \mathfrak{J} \), with \( g_1 \not\parallel e_2 \). It is easy to check that

\[
g_{4,5}^{\alpha,1}: \quad [e_1, e_2] = -e_2, \quad e_4 := [e_1, e_3] \not\in q, \quad [e_1, e_4] = -\alpha e_3 - (1 + \alpha)e_4,
\]

\[
g_{4,5}^{\alpha,\alpha}: \quad [e_1, e_2] = -\alpha e_2, \quad e_4 := [e_1, e_3] \not\in q, \quad [e_1, e_4] = -\alpha e_3 - (1 + \alpha)e_4,
\]

while all other Lie brackets for \( e_1, e_2, e_3, \) and \( e_4 \) are zero. Consequently, every three-dimensional subspace, generating \( g \), includes a one-dimensional ideal of this Lie algebra, and every two of these subspaces are equivalent.

The statement for the Lie algebra \( g_{4,1}^1 \) is proved in Proposition 11.

2. Let \( g \) differ from the Lie algebras \( 2g_{2,1} \) and \( g_{3,6} \oplus g_1 \) and have finitely many (respectively, zero) two-dimensional ideals. It follows from Table 1 that

1) \( m = 0 \) for \( g = g_{4,10} \);

2) \( m = 1 \) for decomposable Lie algebras \( g_{3,5}^0 \oplus g_1 \), with \( \alpha \geq 0 \), and \( g_{3,7} \oplus g_1 \) (the center is \( (E_4) \)) for the indecomposable Lie algebras \( g_{4,1}^1: g_{4,4}^{\alpha,\beta}, \) with \( \alpha > 0 \) and \( \beta \in \mathbb{R}; \) \( g_{4,7}^\alpha \); \( g_{4,5}^{\alpha,\alpha} \), with \( -1 \leq \alpha < 1 \); and \( g_{4,9}^\alpha \), with \( \alpha \geq 0 \) (the one-dimensional ideal is \( (E_1) \));

3) \( m = 2 \) for Lie algebras \((a)\) \( g_{3,2} \oplus g_1 \) (with the onedimensional ideals \( (E_1) \) and \( g_1 \)),

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(b) $\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1$ (with the central ideal $\mathfrak{g}_1$, the noncentral one-dimensional ideals $\langle E_1 + E_2 \rangle$ and $\langle E_1 - E_2 \rangle$ are translated into each other by an automorphism $\xi$ of the Lie algebra under consideration, where

$$\xi(E_1) = E_1, \quad \xi(E_2) = -E_2, \quad \xi(E_3) = -E_3, \quad \xi(E_4) = E_4,$$

(c) $\mathfrak{g}_{4,2}^0, \alpha \notin \{0, 1\}$ and $\mathfrak{g}_{4,3}$ (with the one-dimensional ideals $\langle E_1 \rangle$ and $\langle E_2 \rangle$);

4) $m = 3$ for the Lie algebras (a) $\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1$, with $0 < \alpha \neq 1$ (with one-dimensional ideals $\langle E_1 + E_2 \rangle$ and $\langle E_1 - E_2 \rangle$, $\mathfrak{g}_1$); (b) $\mathfrak{g}_{4,5}^\alpha, \beta$, with $-1 < \alpha < \beta < 1$, $\alpha \beta \neq 0$, or $\alpha = -1$, and $0 < \beta < 1$ (with the one-dimensional ideals $\langle E_1 \rangle$, $\langle E_2 \rangle$, and $\langle E_3 \rangle$).

In case 3(a) the ideals $\langle E_1 \rangle$ and $\mathfrak{g}_1$ are not equivalent because $\langle E_1 \rangle \subset \mathfrak{g}'$ and $\mathfrak{g}_1 \not\subset \mathfrak{g}'$. Case 3(b) is similar. In case 3(c) $\text{ad}(-E_4) : \mathfrak{g}' \rightarrow \mathfrak{g}'$ has its eigensubspaces $\langle E_1 \rangle$ and $\langle E_2 \rangle$ with different eigenvalues $\alpha$, $1$ for $\mathfrak{g}_{4,2}^0$, with $\alpha \notin \{0, 1\}$ and $1, 0$ for $\mathfrak{g}_{4,3}$. Denoting by $\mathfrak{g}$ any of these Lie algebras, by $\lambda_i$ the eigenvalues for $\langle E_i \rangle$, $i = 1, 2, 3$.

Then $\xi(E_4) = E_4 + f$, with $f \in \mathfrak{g}'$, and $\xi(\langle E_i \rangle) = \langle E_i \rangle$, $i = 1, 2$; while $\langle E_1 \rangle$ and $\langle E_2 \rangle$ are not equivalent.

A similar argument is applicable to prove part 4). In case (a), $\mathfrak{g}_1 \not\subset \mathfrak{g}'$ and $\text{ad} E_3$ has the eigenvectors $E_1 + E_2$ and $E_1 - E_2$, corresponding to the different eigenvalues $\alpha - 1$ and $\alpha + 1$. Therefore the one-dimensional ideals $\langle E_1 + E_2 \rangle$, $\langle E_1 - E_2 \rangle$, and $\mathfrak{g}_1$ are pairwise not equivalent. In case (b), $\text{ad}(-E_4)$ has eigenvectors $E_1$, $E_2$, and $E_3$, with corresponding different eigenvalues $1$, $\beta$, and $\alpha$. Thus, the one-dimensional ideals $\langle E_1 \rangle$, $\langle E_2 \rangle$, and $\langle E_3 \rangle$ are pairwise not equivalent.

According to Table 2 in the paper [9], for every one-dimensional ideal $\mathfrak{z}$ of the Lie algebra $\mathfrak{g}$ from item 2 of Proposition 17 there exists a three-dimensional subspace $\mathfrak{q}$, including $\mathfrak{z}$ and generating $\mathfrak{g}$. By Proposition 16, every two of these subspaces are equivalent. Moreover, by Lemma 1 the subspace $\mathfrak{q}$ cannot include every one-dimensional ideal of $\mathfrak{g}$ other than $\mathfrak{z}$. This implies immediately that the three-dimensional subspaces in $\mathfrak{g}$, generating $\mathfrak{g}$ and including nonequivalent one-dimensional ideals, are not equivalent themselves.

If the three-dimensional subspace $\mathfrak{q}$, generating the Lie algebra $\mathfrak{g}$ from item 2 of Proposition 17, includes no one-dimensional ideal of $\mathfrak{q}$, then by Proposition 10 and Remark 1 there is a basis $(e_1, e_2, e_3 := [e_1, e_2])$ for $\mathfrak{q}$ satisfying Lemma 1; i.e., some two-dimensional subspace with a basis $(e_1, e_2)$ generates $\mathfrak{g}$. In [10] we proved that every two of these two-dimensional subspace are equivalent. Moreover, $[e_2, e_3] = 0$ for the Lie algebras from item 2 of Proposition 17 with a three-dimensional commutative ideal; $C_{23}^1 \neq 0$ and $C_{23}^2 \neq 0$ for the Lie algebras $\mathfrak{g}_{4,7}; \mathfrak{g}_{4,9}, \alpha \geq 0; \mathfrak{g}_{4,8}, -1 \leq \alpha < 1$ and $\alpha \neq 0; \mathfrak{g}_{4,10} = 0$. This and Proposition 13 imply that every two of these three-dimensional subspaces $\mathfrak{q}$ are equivalent. \(\square\)

**Corollary 4.** Let $k$ be a number of equivalence classes of the three-dimensional subspaces, generating a four-dimensional real Lie algebra $\mathfrak{g}$.

1. If $\mathfrak{g} = \mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1$, with $0 \leq \alpha \neq 1; \text{then } k = 3 \text{ for } \alpha = 0 \text{ and } k = 4 \text{ for } \alpha \neq 0$.
2. If $\mathfrak{g} = \mathfrak{g}_{4,2}^\alpha$, with $\alpha \neq 0; \text{then } k = 1 \text{ for } \alpha = 1 \text{ and } k = 3 \text{ in other cases}$.
3. If $\mathfrak{g} = \mathfrak{g}_{4,5}^\alpha, \beta$, with $-1 < \alpha \leq \beta \leq 1, \alpha \beta \neq 0$, or $\alpha = -1, 0 < \beta \leq 1; \text{then } k = 1 \text{ for } \alpha = \beta \neq 1 \text{ or } \beta = 1, \alpha \neq 1 \text{ and } k = 4 \text{ in other cases}$.
4. If $\mathfrak{g} = \mathfrak{g}_{4,8}^\alpha$, with $-1 \leq \alpha < 1; \text{then } k = 1 \text{ for } \alpha = 1 \text{ and } k = 2 \text{ for } \alpha \neq 1$.

Using [10], Table 1, Propositions 9, 10, 13, and the proofs of Propositions 14–17, we arrive at

**Theorem 4.** Let $(\mathfrak{g}, [\cdot, \cdot])$ and $\mathfrak{q} \subset \mathfrak{g}$ be a four-dimensional Lie algebra and a three-dimensional subspace generating $\mathfrak{g}$ by the Lie bracket $[\cdot, \cdot]$.

1. If $\mathfrak{g} = 2\mathfrak{g}_{2,1}$ then the two cases are possible:

(1) $\mathfrak{q}$ includes a one-dimensional ideal of the Lie algebra $\mathfrak{g}$;
(2) \( q \) has a basis \( (e_1, e_2, e_3) \) such that \( [e_1, e_2] = 0 \) and \( 0 \neq [e_2, e_3] \parallel e_3 \).

2. If \( g \) is one of the Lie algebras \( g_{3.2} \oplus g_1; g_{3.4} \oplus g_1 \), with \( 0 \leq \alpha \neq 1 \); and \( g = g_{4.3} \), then the three cases are possible:

(1) \( \mathcal{C}(g) \subset q \);
(2) \( q \) includes a one-dimensional noncentral ideal of \( g \);
(3) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( [e_2, e_3] = 0 \).

3. If \( g \) is one of the Lie algebras \( g_{3.5} \oplus g_1 \), with \( \alpha \geq 0 \), and \( g_{4.1} \); then the two cases are possible:

(1) \( \mathcal{C}(g) \subset q \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( [e_2, e_3] = 0 \).

4. If \( g = g_{3.6} \oplus g_1 \) then the three cases are possible:

(1) \( \mathcal{C}(g) \subset q \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( 0 \neq [e_2, e_3] \parallel e_2 \);
(3) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( 0 \neq [e_2, e_3] \parallel e_1 \).

5. If \( g = g_{3.7} \oplus g_1 \) then the two cases are possible:

(1) \( \mathcal{C}(g) \subset q \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( 0 \neq [e_2, e_3] \parallel e_1 \).

6. If \( g \) is one of the Lie algebras \( g_{4.2}^\alpha \), with \( \alpha \notin \{0, 1\}; g_{4.4}; g_{4.5}^\alpha, \beta \), with \( -1 < \alpha < \beta < 1, \alpha \beta \neq 0 \) or \( \alpha = -1, 0 < \beta \leq 1; g_{4.6}^{\alpha, \beta} \), with \( \alpha > 0 \) and \( \beta \in \mathbb{R} \); then the two cases are possible:

(1) \( q \) includes a one-dimensional ideal of \( g \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( [e_2, e_3] = 0 \).

7. If \( g \) is one of the Lie algebras \( g_{4.7}; g_{4.8} \), with \( -1 < \alpha < 1 \) and \( \alpha \neq 0 \); \( g_{4.9}^\alpha \), with \( \alpha > 0 \); then the two cases are possible:

(1) \( q \) includes a one-dimensional ideal of \( g \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( C_{23}^0 \neq 0 \) and \( C_{23}^2 = 0 \).

8. If \( g = g_{4.8}^1 \) or \( g = g_{4.9}^0 \) then the two cases are possible:

(1) \( \mathcal{C}(g) \subset q \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( C_{23}^0 \neq 0 \) and \( C_{23}^2 = 0 \).

9. If \( g = g_{5.8}^0 \) then the two cases are possible:

(1) \( q \) includes a one-dimensional ideal of \( g \);
(2) \( q \) has a basis \( (e_1, e_2, e_3 = [e_1, e_2]) \) such that \( C_{23}^1 = C_{23}^0 = 0 \).

The next theorem follows from [10], Proposition 5, Corollary 1, the proofs of Propositions 16 and 17, and Theorems 1 and 4.

**Theorem 5.** Let \( G, q \subset g \), and \( d \) be respectively a four-dimensional connected Lie group with Lie algebra \( (g, [\cdot, \cdot]), \) a three-dimensional subspace, generating \( g \) by the Lie bracket \( [,] \), and an arbitrary left-invariant quasimetric \( d \) on \( G \) defined by some seminorm \( F \) on \( q \). Then

1. Every abnormal extremal of the space \((G, d)\) is nonstrictly abnormal for the Lie algebras \( g = g_{2.1} \oplus g_1 \) and \( g = g_{3.1} \oplus g_1 \) and in the cases of 2(1); 3(1); 4(1); 5(1); and 8(1) of Theorem 4.

2. Every abnormal extremal of the space \((G, d)\) is strictly abnormal for the Lie algebras \( g_{3.3} \oplus g_1; g_{4.1}; g_{4.5}^\alpha \), with \( -1 \leq \alpha < 1 \) and \( \alpha \neq 0 \); \( g = g_{4.5}^\alpha \), with \( -1 < \alpha < 1, \alpha \neq 0 \); and in the cases of 1(1); 2(2); 4(2); 6(1); 7(1); and 9(1) of Theorem 4.

3. For the Lie algebra \( g_{4.10} \) and in the cases of 1(2); 2(3); 3(2); 6(2); and 9(2) of Theorem 4, abnormal extremal (4) (and its every left shift) of the space \((G, d)\) is nonstrictly abnormal if and only if \( F_U(k(s), s, 0) = 1/F(0, s, 0) \) for some \( k(s) \in \mathbb{R} \).
4. In the cases of 4(3); 5(2); 7(2); and 8(2) of Theorem 4, abnormal extremal (4) (and its every left shift) of the space \((G, d)\) is nonstrictly abnormal if and only if \(F_U(0, s, 0) = 1/F(0, s, 0)\).

Remark 2. Unfortunately, in our paper [10], we made mistakes in the formulation of Theorem 4.
1. In item 2 of Theorem 4 in [10], the condition “the restriction of the Killing form \(k_{\mathfrak{g}_1, 6}\) to the projection of the subspace \(\mathfrak{p}\) onto \(\mathfrak{g}_{3, 6}\) is negative definite” must be replaced by the condition “the restriction of the Killing form \(k_{\mathfrak{g}_1, 6}\) to the one-dimensional subspace \(s = \mathfrak{p} \cap \mathfrak{g}_{3, 6}\) is nondegenerate.”
2. In item 3 of Theorem 4 in [10], the condition “the restriction of the Killing form \(k_{\mathfrak{g}_1, 6}\) to the projection of the subspace \(\mathfrak{p}\) onto \(\mathfrak{g}_{3, 6}\) is nondegenerate and with alternating signs” must be replaced by the condition “the restriction of the Killing form \(k_{\mathfrak{g}_1, 6}\) to \(s\) is degenerate.”

4. Conclusion

Here, in response to a request of the referee, we slightly supplement the abstract and point out the possible directions of further research.

In view of Corollary 3, all four-dimensional Lie algebras but \(4\mathfrak{g}_1\) and \(\mathfrak{g}_{4, 5}^{11}\) have three-dimensional generating subspaces. Propositions 5, 9, and 11 imply that pairs \((\mathfrak{g}, \mathfrak{q})\), where \(\mathfrak{q}\) is a three-dimensional generating subspace of a four-dimensional real Lie algebra \(\mathfrak{g}\), could be divided into the following classes (with conditions):

1. The subspace \(\mathfrak{q}\) includes a one-dimensional central ideal of Lie algebra \(\mathfrak{g}\);
2. \(\mathcal{N}(\mathfrak{q}) \cap \mathfrak{q} \subset [\mathcal{N}(\mathfrak{q}) \cap \mathfrak{q}, \mathfrak{q}]\);
3. The subspace \(\mathfrak{q}\) has a basis \((e_1, e_2, e_3)\) such that
   \[\mathcal{N}(\mathfrak{q}) \cap \mathfrak{q} = \langle e_2 \rangle, \quad e_2 \notin [e_2, \mathfrak{q}], \quad [e_1, e_2] = e_3, \quad [e_1, e_3] \notin \mathfrak{q};\]
4. \(\mathfrak{g} = \mathfrak{g}_{4, 8}^1\) and \(\mathfrak{q}\) is a unique (up to equivalence) three-dimensional generating subspace of \(\mathfrak{g}\).

Assume now that \(G\) is a connected Lie group with a Lie algebra \(\mathfrak{g}\) and a left-invariant quasimetric \(d\) defined by a seminorm \(F\) on some three-dimensional subspace \(\mathfrak{q}\) of \(\mathfrak{g}\) generating \(\mathfrak{q}\).

In consequence of Proposition 8, every abnormal extremal in \((G, d)\) is a one-parameter subgroup of \(G\) tangent to \(\mathcal{N}(\mathfrak{q}) \cap \mathfrak{q}\), or its left shift.

By Proposition 5, for each of these indicated quasimetrics \(d\) on \(G\), this extremal is nonstrictly abnormal in item (1) and strictly abnormal in item (2).

In items (3) and (4) an abnormal extremal may be nonstrictly or strictly abnormal depending on the choice of \(d\). The criteria for (non)strict abnormality are established in Theorem 1 and Proposition 12. We can say that in these situations an abnormal extremal is nonstrictly abnormal “with probability zero.”

The general criterion for the (non)strict abnormality of an extremal for sub-Riemannian metrics \(d\) of the indicated form is obtained in Theorem 3.

By Theorem 2, the strictly abnormal extremals under consideration are not geodesic, and our paper presents many examples for which the PMP is a necessary but insufficient condition for geodesics of the left-invariant quasimetrics, sub-Riemannian metrics inclusively, on Lie groups.

The nearest, even uneasy, problem might be the search for geodesics and shortest arcs of the left-invariant sub-Riemannian metrics of the indicated form on four-dimensional connected Lie groups. The classification of three-dimensional subspaces of four-dimensional Lie algebras, generating these algebras, and other results of this paper present a broad field for research in these problems.

In the paper [14] Mubarakzyanov obtained some classification of five-dimensional real Lie algebras that gives 67 their families, among which there are 40 families of indecomposable Lie algebras, 20 families of Lie algebras of the form \(\mathfrak{g}_4 \oplus \mathfrak{g}_1\), where \(\mathfrak{g}_4\) are Lie algebras from Table 1, and 7 families of Lie algebras of the form \(\mathfrak{g}_3 \oplus \mathfrak{g}_{2, 1}\), where \(\mathfrak{g}_3\) are indecomposable three-dimensional Lie algebras.

This shows that any extension of results from this paper and the paper [10] onto greater dimensions would be cumbersome enterprise even just technically, disregarding the fact that this will require also unusual ideas.

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