Effects of Normal-Distributed Measurement Error on Frequency Tuning for a Cylindrical Vibratory Gyroscopes

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ABSTRACT Cylindrical vibratory gyroscope is a kind of mode-matched gyros, and the frequency split would affect its sensitivity significantly. Frequency tuning is a major method to decrease the split. In the process of frequency tuning, the natural frequency should be measured firstly, and the measurement error is one of the decisive factor of tuning accuracy. Due to the measurement error of frequency is Normal Distribution, its effects on frequency tuning are studied by theoretical and simulation analysis from the statistical aspect. Firstly, the influence of frequency splitting on gyro sensitivity is studied, which shows the importance of reducing frequency splitting. Then the effect of different measuring accuracy on the measured value is analyzed. Thirdly, the tuning error that caused by the measurement error is researched in the process of trimming, and the tuning error is derived by theoretical analysis. Simulation and experiment results under different measurement error exemplify the value of tuning error. Finally, the optimal measurement accuracy that ensures achieving tuning accuracy is discussed.

INDEX TERMS Cylindrical vibratory gyroscope, frequency tuning, measurement errors, normal distribution.

I. INTRODUCTION
Vibratory gyroscope utilizes the Coriolis Effect that comes from vibrating resonator to measure rotation, and it is useful in avionics system, naval equipment and robotics. The resonator structure include hemisphere, disks, ring, etc [1]–[3]. The resonator vibrates in the driving mode. After the angular velocity is input, the vibration is excited in the sensing mode. The angle measurement is realized by demodulating the vibration in the sensing mode. The natural frequency difference between the driving mode and the sensing mode is defined as frequency split. In practice, no matter what structure it is, frequency split would exist due to manufacturing errors, which would decrease the performance of vibratory gyroscope [4], [5]. Therefore, a lot of researches about tuning theories and methods are presented [6], [7]. The natural frequencies of two work modes should be measured before tuning [8], and the measurement accuracy affects the tuning progress significantly. However, the relationship between measurement accuracy and tuning accuracy is indistinct, which would result in an over high measurement accuracy for a general accuracy tuning mission. Hence, this paper figures out the relationship by analyzing the effects of measurement error on tuning to select the optimal measurement accuracy.

As for tuning, three primary methods are usually employed. The first one is removing mass from resonator by laser ablation and other mechanical method, and the tuning accuracy is about 0.1-0.01Hz that depends on the amount of the removed mass [9], [10]. The second one is depositing mass on a special pad of resonator to change its natural frequency, and the tuning accuracy of this method is about 1Hz [11], [12]. Last but not the least, the electrostatic tuning can reduce the split to 6 mHz by changing the tuning voltage that applied on electrodes [13], [14]. In these tuning methods, frequencies of two modes could conduct tuning parameters
such as tuning modes, tuning positions and tuning amount. The measurement accuracy of frequency has a significant effect on the tuning accuracy, and a precise tuning accuracy is hard to achieve if the measurement accuracy is lower. Therefore, a high measurement accuracy is needed for precision tuning [15].

Up to now, frequency can be obtained by Frequency Response Analyzers [16]–[18] with the accuracy of 0.001Hz, Frequency-scanning Meter (0.1Hz) [19] and measurement circuit (0.1-0.01Hz) [20], [21]. However, Frequency Response Analyzers and Frequency-scanning Meter are usually costly, while the measurement circuit is complicated. Therefore, magnetic-acoustic hybrid method [22] and acoustic method [23], [24] that using microphone to test vibration sound wave are presented to obtain frequency, but the accuracy of this acoustic method is about 0.1Hz. These measurement methods indicate that the measurement accuracy is not only hard to improve but also costly, which means that it is not necessary to adopt a very high measurement accuracy for a general tuning accuracy. Hence, the approach to select an optimal measurement accuracy is of great importance in practice, and this paper proposed a method to select.

Sec. II in this paper introduces the basic principle of the cylindrical vibratory gyroscope, and the relationship between gyroscope sensitivity and split is analyzed. The distribution of measurement error is studied in Sec. III, and the experiment results show that it obeys Normal Distribution. Afterwards, the influence of different measuring accuracy on the measured value is analyzed in Sec. IV to judge whether the split achieves the required tuning accuracy. Sec. V of this paper mainly studies the effects of measurement error on trimming results during the tuning process. Based on the above effects, Sec. VI puts forward the optimal measurement accuracy for a certain accuracy tuning. The main novelty of this paper is studying the effects of measurement error on tuning from the statistical aspect of the frequency measurement. Furthermore, the proposed optimal measurement accuracy can be broadly used in other kinds of gyroscope.

II. BASIC PRINCIPLE AND SENSITIVITY ANALYSIS

A. BASIC PRINCIPLE

As presented in Fig. 1, the cylindrical vibratory gyroscope mainly consists of a cylindrical shell resonator, a circuit board, a base, and eight piezoelectric electrodes that pasted on the bottom plate of the resonator and separated from each other by 45°. The vibration of the gyroscope is excited and detected by the use of piezoelectric electrodes, and the wire in the structure is used for transferring vibration signal.

The Fig. 2 illustrates the working principle of the cylindrical vibratory gyroscope. When an alternative voltage is applied to the driving electrodes, for the converse-piezoelectric effect, the resonator is induced into a circle-ellipse flexural vibration in the driving mode at the frequency of the alternative voltage. The circumferential displacement \( v_d (\theta, t) \) and radial displacement function \( w_d (\theta, t) \) of driving mode can be expressed as follow [25], [26]:

\[
\begin{align*}
    v_d (\theta, t) &= v_{d, st} \eta_d \sin(\omega_d t - \beta_d) \\
    w_d (\theta, t) &= w_{d, st} \eta_d \cos(\omega_d t - \beta_d)
\end{align*}
\]

In which,

\[
\begin{align*}
    v_{d, st} &= \frac{1}{2} w_{d, st} \\
    w_{d, st} &= \frac{F^*}{k_d} \\
    \eta_d &= \frac{1}{\sqrt{(1 - v_d^2)^2 + 4\xi^2 v_d^2}} \\
    \beta_d &= \arccos\left(\frac{1 - v_d^2}{\sqrt{(1 - v_d^2)^2 + 4\xi^2 v_d^2}}\right)
\end{align*}
\]

where \( w_{d, st} \) is the static radial displacement that equal to the equivalent force \( F^* \) dividing by equivalent stiffness \( k_d^* \), and the force is caused by driving electrodes. \( \eta_d \) is the magnified coefficient of the dynamic displacement, where \( v_d = \omega_d / \omega_d \) is the frequency ratio of alternative voltage \( \omega_d \) and driving mode \( \omega_d \), and \( \xi \) is the effective damping ratio. \( \beta_d \) is the phase angle.

In order to achieve resonance, the driving frequency of alternative voltage \( \omega_d \) should be equal to the natural frequency of driving mode \( \omega_d \), which can be achieved by adjusting the alternative voltage.

While the resonator vibrates in the driving mode, and the gyroscope is rotating about its axis at an angular velocity.
\( \Omega \), the sensing mode would be excited due to the Coriolis force. The circumferential displacement \( v_s(\theta, t) \) and radial displacement function \( w_s(\theta, t) \) of sensing mode can be expressed as follow:

\[
\begin{align*}
  v_s(\theta, t) &= -v_{s, st} \eta_d \cos 2\theta \sin (\omega_s t - \beta_d + \frac{\pi}{2} - \beta_s) \\
  w_s(\theta, t) &= w_{s, st} \eta_d \sin 2\theta \sin (\omega_s t - \beta_d + \frac{\pi}{2} - \beta_s)
\end{align*}
\] (3)

In which,

\[
\begin{align*}
  v_{s, st} &= \frac{1}{2} w_{s, st} \\
  w_{s, st} &= \frac{F_c^*}{k_s^*} \\
  \eta_d &= \frac{1}{\sqrt{(1 - u_s^2)^2 + 4\xi^2 u_s^2}} \\
  \beta_s &= \arccos \left( \frac{1 - u_s^2}{\sqrt{(1 - u_s^2)^2 + 4\xi^2 u_s^2}} \right)
\end{align*}
\] (4)

The static radial displacement \( w_{s, st} \) of sensing mode equal to the equivalent Coriolis force \( F_c^* \) dividing by the equivalent stiffness \( k_s^* \) in sensing mode axis, and the equivalent Coriolis force \( F_c^* = -2m^* \Omega \times (\dot{v}_d + \dot{w}_d) \) is related to the angular velocity \( \Omega \), equivalent mass \( m^* \) and vibration velocity of driving mode. \( v_s = \omega_d / \omega_s \) is the natural frequency ratio of driving mode \( \omega_d \) and sensing mode \( \omega_s \). \( \eta_d \) and \( \beta_s \) is the magnified coefficient and phase angle separately.

In order to achieve resonance of sensing mode, the resonator should vibrate in the natural frequency of sensing mode \( \omega_s \), which requires trimming the natural frequencies of driving and sensing modes to be equal.

According to the piezoelectric effect, the electrodes would convert the vibration of sensing mode to a voltage signal, and the angular velocity of the gyroscope can be obtained by demodulating the voltage signal. The demodulated voltage \( U_\Omega \) is proportional to angular velocity \( \Omega \), and can be expressed as follow:

\[
U_\Omega = c\Omega \frac{1}{\sqrt{(1 - u_s^2)^2 + 4\xi^2 u_s^2}} |\sin \beta_s| 
\] (5)

where \( c \) is a constant that relates to material and structure of gyroscope and piezoelectric electrodes.

**B. SENSITIVITY ANALYSIS OF THE GYROSCOPE**

The sensitivity \( S_g \) of the gyroscope can be defined as the ratio of the voltage \( U_\Omega \) from the sensing electrodes to the angular velocity \( \Omega \). Hence, the sensitivity can be written as the following expression:

\[
S_g = \frac{U_\Omega}{\Omega} = c \frac{1}{\sqrt{(1 - u_s^2)^2 + 4\xi^2 u_s^2}} |\sin \beta_s| 
\] (6)

As for second-order system of this gyroscope, the relationship between quality factor \( Q \) and effective damping ratio \( \xi \) is that \( Q = 1/2\xi \), and \( |\sin \beta_s| \) can be expressed as follow:

\[
|\sin \beta_s| = \sqrt{1 - \cos^2 \beta_s} = \frac{u_s}{\sqrt{Q^2 (1 - u_s^2)^2 + u_s^2}} 
\] (7)

Therefore, the Eq. (6) can be re-written as:

\[
S_g = \frac{U_\Omega}{\Omega} = c \frac{Q^2 u_s}{Q^2 (1 - u_s^2)^2 + u_s^2} 
\] (8)

It is noteworthy that the relationship between the natural frequency \( f \) and the angular natural frequency \( \omega \) can be expressed as \( \omega = 2\pi f \). Hence, the frequency ratio \( \nu_s \) can be rewritten as follow:

\[
\nu_s = \frac{\omega_d}{\omega_s} = \frac{f_d}{f_s} 
\] (9)

The difference of two modes’ natural frequency is split \( \Delta f = |f_d - f_s| \). Hence, when the natural frequency of driving mode \( f_d \) is higher than sensing mode’s \( f_s \), the relationship of sensitivity \( S_g \) and split \( \Delta f \) can be derived as follow:

\[
S_g = c \frac{Q^2 f_d (f_d - \Delta f)^3}{Q^2 (2f_d - \Delta f)^2 + f_d^2 (f_d - \Delta f)^2} 
\] (10)

As for cylindrical vibratory gyroscopes in this study, the quality factor \( Q \) is about 10000 in atmosphere, and the natural frequency \( f \) is approximately 4000 Hz. Assuming that the effects of split on the quality factor \( Q \) could be ignored, and the \( Q \) factor is regarded as a constant. According to (10), the relationship between the sensitivity \( S_g \) and frequency split \( \Delta f \) is shown in Fig. 3. On the whole, the sensitivity decreases with the increasing of frequency split. The sensitivity decrease sharply when the split increases from 0 to 0.4Hz, and the descent rate would decrease when the split exceeds 0.4Hz. When the natural frequency of sensing mode \( f_s \) is higher than driving mode’s \( f_d \), the results is the same. Hence, in order to improve the sensitivity, it is necessary to tune the split to a small scale (usually 0.01Hz).

In addition, the bias of gyro is also affected by the frequency split. Bias refers to the output of the sensing mode of the gyro when the angular velocity input is zero. The output
mainly comes from the coupling vibration with the driving mode and the output at the node of sensing mode due to the vibration mode deflection. Due to the existence of frequency split, the sensing mode is excited to vibrate and then overlaps with the vibration mode of driving mode on the resonator to form a stable standing wave. The analysis of the stable standing wave is presented in [4]. For a vibrating gyroscope with cylindrical shell, the offset of standing wave will directly affect the bias. The results of [4] show that the stability of standing wave is affected synthetically by frequency splitting, damping coefficient and excitation force. Therefore, reducing the frequency split is also beneficial to improve the bias.

Fig. 4 is a flow chart of tuning, the value of frequency and split should be measured firstly to conduct the following tuning. On the one hand, the measurement value affects the judgment that whether the split meets the demand. On the other hand, the trimming parameters are also affected by the measurement value. Therefore, the measurement error would play an important effect on the tuning results, and the effect rules would be discussed as following parts.

III. DISTRIBUTION OF MEASUREMENT ERROR

Before analyzing the influence of measurement error on the tuning accuracy, the distribution law of frequency measurement error is studied. A cylindrical vibratory gyroscope that is driving and sensing by piezoelectric electrodes is utilized to research the distribution of measurement error, and the measurement equipment is a driving and sensing circuit (as shown in Fig. 5). The cylindrical shell resonator used in the experiment is designed and manufactured by ourselves. The diameter of the shell is about 25 mm and the height is about 18 mm. Moreover, the experiment is implemented in the temperature chamber LHS-80HC-I to decrease the influence of temperature fluctuation, and the temperature in the chamber is setting as 20 ± 0.5°C. The influence factors of measurement accuracy include test equipment and temperature. Due to the test is implemented in the temperature chamber, the measurement error mainly comes from the random error of test equipment.

Due to the stability of each resonator is different, and the natural frequency is also disturbed by the environment. Therefore, during the test, waiting for the excitation frequency to be stable firstly after the circuit board is powered on. Then, continuously collect 100 datum (sampling frequency is 20Hz), and take the average value of these 100 datum as the test result. After one test, turn off and on the excitation power again, and repeat the test procedure. After 39 times measurement, the distribution of frequencies is showed in Fig. 6.

As shown in bar graph of Fig. 6, frequencies mainly distribute around 3870.432∼3870.486 Hz and express an approximating normal distribution. The curve in Fig. 6 is the probability density curve of a normal distribution with the mean value \( \mu = 3870.461 \text{Hz} \) and standard deviation \( \sigma = 0.009 \text{Hz} \) that calculated from measurement results. The Normal Distribution fitting is tested by Kolmogorov-Smirnov Test method (Appendix) [27], [28]. As for Normal Distribution, the measurement error is regarded as the three times of standard deviation. Hence, the measurement error in this experiment is 0.027 Hz. The experimental results show that although the frequency changes linearly with the temperature, the frequency value measured by the temperature compensation method also conforms to the normal distribution [29].

As mentioned above, the frequency split \( \Delta f = |f_d - f_s| \) is an indirect variable. Therefore, the value and distribution of frequency split can be derived according to the measurement results of frequency as following. The influence factors of measurement accuracy include test equipment and temperature. Due to the measurement is implemented in the temperature chamber, the measurement error mainly come from the random error of test equipment. Therefore, the distributions
of the two normal mode frequencies are independent of each other. More specifically, assume the distribution of two work modes’ frequencies are \( f_d \sim N(\mu_d, \sigma_d^2) \) and \( f_s \sim N(\mu_s, \sigma_s^2) \) separately, which means the natural frequency of driving mode \( f_d \) obey the normal distribution with mean value \( \mu_d \) and standard deviation \( \sigma_d \), and mean value \( \mu_s \) and standard deviation \( \sigma_s \) for sensing mode. Moreover, assume that \( \mu_d > \mu_s \), then the distribution of split is \( \Delta f = f_d - f_s \sim N(\mu_d - \mu_s, \sigma_d^2 + \sigma_s^2) \). Normally, due to the measurement equipment and temperature are similar for two modes, the standard deviation should be equal and denoted by \( \sigma_f = \sigma_d = \sigma_s \).

Therefore, the distribution of split becomes \( \Delta f = f_d - f_s \sim N(\mu_d - \mu_s, 2\sigma_f^2) \), and named the \( \mu = \mu_1 - \mu_2 \) and \( \sigma = \sqrt{2}\sigma_f \), then the distribution can be marked as \( \Delta f = f_d - f_s \sim N(\mu, \sigma^2) \).

In this condition, the distribution law of frequencies and split are as follow:

\[
\begin{align*}
  f_d &\sim N(\mu_d, \sigma_d^2) \\
  f_s &\sim N(\mu_s, \sigma_s^2) \\
  \Delta f &= f_d - f_s \sim N(\mu, \sigma^2)
\end{align*}
\]

(11) (12) (13)

The effects of the above measurement errors on the tuning mainly include two aspects. On the one hand, it affects the evaluation that whether the measured value meets the required accuracy. On the other hand, it brings fluctuation in the process of tuning. These effects will be studied based on the distribution of measurement errors.

**IV. EFFECTS ON MEASUREMENT VALUE**

In the actual measurement, the measurement value is regarded as the mean value of the Normal Distribution, and the measurement error is regarded as the standard deviation. As shown in Fig. 7, it is assumed that frequency split is \( \mu \), and the measurement error is \( \pm 3\sigma \), hence the possible value of split is \( \Delta f \sim N(\mu, \sigma^2) \). The Probability Density Function \( h(\Delta f) \) and the Cumulative Density Function \( H(\Delta f) \) are separately as (14) and (15):

\[
\begin{align*}
  h(\Delta f) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\Delta f - \mu)^2}{2\sigma^2}}, \quad -\infty < \Delta f < +\infty \quad (14) \\
  H(\Delta f) &= P(t < \Delta f) \\
  &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\Delta f} e^{-\frac{(t - \mu)^2}{2\sigma^2}} dt, \quad -\infty < \Delta f < +\infty
\end{align*}
\]

(15)

The formula of calculating the probability that the split \( \Delta f \) achieves required tuning accuracy \( \varepsilon \) (\( \Delta f \in [-\varepsilon, \varepsilon] \)) is (16):

\[
\begin{align*}
  P(-\varepsilon < \Delta f < \varepsilon) &= H(\varepsilon) - H(-\varepsilon) \\
  &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{(t - \mu)^2}{2\sigma^2}} dt \quad (16)
\end{align*}
\]

Equation (16) can be expressed as (17) to simplify the integration by the transformation between the normal distribution and standardized normal distribution:

\[
\begin{align*}
  P(-\varepsilon < \Delta f < \varepsilon) &= H(\varepsilon) - H(-\varepsilon) \\
  &= \Phi \left( \frac{\varepsilon - \mu}{\sigma} \right) - \Phi \left( \frac{-\varepsilon - \mu}{\sigma} \right) \quad (17)
\end{align*}
\]

where \( \Phi(x) \) is the Cumulative Density Function of standardized normal distribution, and the value can be acquired by look-up table.

As shown in Fig. 8, the probability changes along with the variety of measurement value \( \mu \) and standard deviation \( \sigma \). It is easy to find out that there is an interval \( X \) when the standard deviation is less than one third of required accuracy \( \sigma_s < \varepsilon/3 \), which made the probability higher than 99.73%. On contrary, when the standard deviation exceed \( \varepsilon/3 \), the probability must lower than 99.73% although the measurement value is less than \( \varepsilon \). The probability of 99.73% is the requirement of engineering, therefore it is necessary to ensure the probability beyond 99.73%.

Furthermore, taking the standard deviation equal to one third of the required accuracy as boundary line, and research the effects on the probabilities under the three conditions that
the standard deviation within, equal to and exceed one third of the required accuracy. Due to the standard deviation equal to one third of measurement error, the boundary line can be described as measurement error of split equal to the required tuning accuracy.

**A. MEASUREMENT ERROR < REQUIRED ACCURACY**

As talked above, when the measurement error is less than the required accuracy \( \sigma_s < \epsilon/3 \), there is an interval \( X \) that makes the probability higher than 99.73%. Fig. 9 is a scheme of the interval, and the two curves of normal distribution are two extreme conditions. When the measurement value \( \mu = \epsilon - 3\sigma_s \), the split have a probability of 99.73% in the interval of \([\mu - 3\sigma_s, \mu + 3\sigma_s]\), corresponding to \([\epsilon - 6\sigma_s, \epsilon]\) in Fig. 9, which indicates that the split reach the border of the required accuracy \([-\epsilon, \epsilon]\). It is the same for another extreme condition that when the measurement value \( \mu = -\epsilon + 3\sigma_s \).

According to the two extreme conditions, when the measurement value is in the range of \([-\epsilon + 3\sigma_s, \epsilon - 3\sigma_s]\) (corresponding to \(X\)), the probability of split belonging to \([-\epsilon, \epsilon]\) reaches the requirement of engineering. The formula of interval length of \(X\) is showed in (18):

\[
X = 2\epsilon - 6\sigma_s
\] (18)

According to this formula, when the required accuracy \(\epsilon\) and measurement error \(3\sigma_s\) are given, the sufficient condition of achieving the required accuracy is that the measurement value \(\mu\) belongs to \([-X/2, X/2]\). For example, when the required accuracy \(\epsilon = 0.01\)Hz, and the measurement error \(3\sigma_s = 0.005\)Hz, then the desirable interval of measurement value is \(X = 0.01\)Hz. That means the measurement value should belong to \([-0.005, 0.005]\). Meanwhile, the measurement error of frequency is \(3\sigma_f = 3\sigma_s/\sqrt{2} \approx 0.003\)Hz.

On the other condition, when the measurement value exceed the interval of \(X\) \([0.005, 0.005]\), \(\mu = 0.006\)Hz for instance, the probability of achieving the required accuracy would not reach 99.73%. In this condition, the probability could not meet the requirement of engineering.

In conclusion, when the measurement error is less than the required accuracy, and the measurement value is within the interval of \([-X/2, X/2]\), the split achieves the required tuning accuracy. On contrary, when the measurement value exceeds this interval, the achieving probability unable to reach 99.73%.

**B. MEASUREMENT ERROR = REQUIRED ACCURACY**

When the measurement error is equal to the required accuracy \(\sigma_s = \epsilon/3\), the tuning accuracy can be achieved only when the measurement value \(\mu_1 = 0\). As shown in Fig. 10, when the measurement value \(\mu_1 = 0\), the split would locate in the interval of \([-\epsilon, \epsilon]\) exactly, which means achieving the required accuracy. However, once the measurement value is greater or less than zero, the split would beyond the required accuracy. Hence, the measurement value must exactly be 0. In practice, it is difficult to decrease the measurement value of split to 0, and only happened by accident.

**C. MEASUREMENT ERROR > REQUIRED ACCURACY**

With the standard deviation getting bigger \(\sigma_s > \epsilon/3\), the distribution is decentral and the measurement error beyond the required accuracy, as shown in Fig. 10. In this condition, even though the measurement value is \(\mu_2 = 0\), the probability of achieving the required accuracy is less than 99.73%. For instance, as shown in Fig. 8, when the standard deviation \(\sigma_s = 2\epsilon\), the probability of achieving the required accuracy is less than 50% when the measurement value is 0. Therefore, when the measurement error exceeds the required accuracy, the tuning has no way to be achieved with the probability of 99.73% in practice.

In conclusion, in order to achieve the required accuracy, the standard deviation should be less than one third of required accuracy, and the measurement value should locate in the interval of \([-X/2, X/2]\). This part discusses the probability of reach the required tuning accuracy under different measurement value and error, and the effects of a constant measurement error on trimming process are presented in Sec. V.

**V. EFFECTS ON TRIMMING**

This part introduces the tuning error caused by the measurement error, and the value of tuning error is derived by theoretical analysis and exemplified by simulation and experiment.
A. THEORETICAL ANALYSIS

Assume that the true values of two modes’ natural frequencies are $\mu_{d0}$ and $\mu_{s0}$ separately (the number represents the trimming times), and the measurement error is $3\sigma_f$, then the measurement value of two modes should obey the normal distributions as follow:

\[
\begin{align*}
 f_{d0} & \sim N(\mu_{d0}, \sigma_f^2) \\
 f_{s0} & \sim N(\mu_{s0}, \sigma_f^2)
\end{align*}
\]

(19) (20)

Suppose $\mu_{d0} > \mu_{s0}$, then the measurement value of split obey the following distribution:

\[
\Delta f_0 = f_{d0} - f_{s0} \sim N(\mu_{d0} - \mu_{s0}, 2\sigma_f^2)
\]

(21)

After obtaining the natural frequency, assuming that the split does not meet the demand, and the process of tuning could be discussed as following.

In practice, different trimming approaches can be employed for different gyroscopes, and natural frequencies of two modes would change separately. Therefore, in order to analyze the process of trimming, a weight coefficient $\alpha$ should be determined to imitate the tuning. As for Cylindrical Shell Vibratory Gyroscope (Ref. [10]), the natural frequency of trimming axis increases, while the other axis decreases for holes-trimming method. Assume that the weight coefficients of decreasing and increasing are $\alpha_1$ and $\alpha_2$ separately, and $\alpha_1 + \alpha_2 = 1$. Hence, in the process of forming holes in the resonator, the natural frequency of trimming axis would increase by $\Delta f_0 \cdot \alpha_2$, and the other axis would decrease by $\Delta f_0 \cdot \alpha_1$. With this prerequisite, the true value of frequency after trimming should obey the distribution as follow:

\[
\begin{align*}
 f_{d1t} &= \mu_{d0} - \alpha_1 \Delta f_0 \sim N(\alpha_2\mu_{d0} + \alpha_1\mu_{d0}, 2\alpha_1^2\sigma_f^2) \\
 f_{s1t} &= \mu_{s0} + \alpha_2 \Delta f_0 \sim N(\alpha_2\mu_{d0} + \alpha_1\mu_{d0}, 2\alpha_2^2\sigma_f^2)
\end{align*}
\]

(22) (23)

Furthermore, the true value of frequency split can be obtained by $\Delta f_{1t} = f_{d1t} - f_{s1t}$. It is worth mentioning that these two frequencies $f_{d1t}$ and $f_{s1t}$ are interrelated cause the $\Delta f_0$ is included in both frequencies. As a result, the distribution of split’s true value could not be calculated directly by the distribution parameter. It is necessary to simplify the equation to independent distribution as follow:

\[
\Delta f_{1t} = f_{d1t} - f_{s1t} = \mu_{d0} - \alpha_1 \Delta f_0 - \mu_{s0} - \alpha_2 \Delta f_0 = \mu_{d0} - \mu_{s0} - \Delta f_0 \sim N(0, 2\sigma_f^2)
\]

(24)

As show in (24), the distribution of split $\Delta f_{1t}$ is regardless of weight coefficient. The results are same when either the two frequencies increase or decrease for other trimming methods, and the relationship of two coefficients is that the difference equal to 1 instead of sum. Therefore, these results are applicable for any kind of gyroscope and trimming method.

Normally, the split can not be eliminated after once tuning, and it is a multiple tuning process. In order to continue the next tuning, the second measurement is necessary to conduct tuning. Due to the existence of measurement error, as shown in (24), the true value $\Delta f_{1t}$ after one trimming have a probability to show in the range of $[-3\sqrt{2}\sigma_f, 3\sqrt{2}\sigma_f]$. The distribution of $\Delta f_{1t}$ is normal distribution, and the mean value and standard deviation is 0 and $\sqrt{2}\sigma_f$ respectively. In fact, the true value is determined, and it is the mean value of the distribution of the next measurement result. As shown in Fig. 11, the solid line is the distribution of the next measurement split ($\Delta f_{1m}$) when the true value is $\Delta f_{1t}$, and the true value obeys the distribution of dashed line. According to above results, the distribution of the next measurement value is a normal distribution that the mean value obeys true value’s distribution and the standard deviation is one third of the measurement error.

The distribution of true value is $\Delta f_{1t} \sim N(0, 2\sigma_f^2)$, and the Probability Density Function is (25) as follow:

\[
h(\Delta f_{1t}) = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma_f} e^{-\frac{1}{2}\left(\frac{\Delta f_{1t}^2}{\sigma_f^2}\right)}, \quad -\infty < \Delta f_{1t} < +\infty
\]

(25)

Assume that the next measurement value is $\Delta f_{1m}$, then it should obey the normal distribution that mean and standard deviation is $\Delta f_{1t}$ and $\sqrt{2}\sigma_f$ separately. Therefore, the Probability Density Function of $\Delta f_{1m}$ is as (26):

\[
h(\Delta f_{1m}) = \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma_f} e^{-\frac{1}{2}\left(\frac{(\Delta f_{1m} - \Delta f_{1t})^2}{\sigma_f^2}\right)}, \quad -\infty < \Delta f_{1m} < +\infty
\]

(26)

Owing to the true value $\Delta f_{1t}$ obey normal distribution, the Probability Density Function of $\Delta f_{1m}$ is also described as (27):

\[
h(\Delta f_{1m}) = \int_{-\infty}^{\infty} h(\Delta f_{1t}) \cdot h(\Delta f_{1m}) \cdot d\Delta f_{1t}
\]

\[
= \frac{1}{4\pi\sigma_f^2} \cdot e^{-\frac{\Delta f_{1m}^2}{4\sigma_f^2}} \cdot e^{-\frac{\Delta f_{1t}^2}{8\sigma_f^2}} \cdot \sqrt{2}\sigma_f
\]
described as follow: the tuning error that caused by measurement error, and it is the following tuning and prediction is similar to the first shows.

\[\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma_f^2} \left( \Delta f_{lm} - \frac{\Delta f_{m}}{2} \right)^2} \frac{\Delta f_{lm} - \frac{\Delta f_{m}}{2}}{\sqrt{2\sigma_f}} \, d \Lambda f_{lm} = \frac{1}{\sqrt{2\pi} \cdot 2\sigma_f} e^{-\frac{1}{2} (2\sigma_f)^2 \Delta f_{lm}^2}, \quad -\infty < \Delta f_{lm} < +\infty\]  

(27)

According to the Probability Density Function (27) of measurement value, the second measurement would obey the normal distribution that mean value and standard deviation are 0 and 2\(\sigma_f\) separately, in other words as follow:

\[\Delta f_{lm} \sim N(0, 4\sigma_f^2) = N(0, 2\sigma_f^2)\]  

(28)

It is worth to mention that the distribution of \(\Delta f_{lm}\) is the prediction of second measurement, and it can not conduct the second tuning. The second tuning parameters depend on the realistic measurement value, assuming that the second measurement results are as follow:

\[f_{d1} \sim N(\mu_{d1}, \sigma_f^2)\]  

(29)

\[f_{s1} \sim N(\mu_{s1}, \sigma_f^2)\]  

(30)

Then the corresponding split is as follow:

\[\Delta f_1 = f_{d1} - f_{s1} \sim N(\mu_{d1} - \mu_{s1}, 2\sigma_f^2)\]  

(31)

Therefore, the true value \(\mu_{d1} - \mu_{s1}\) has a probability of 99.73% in \([-3\sqrt{2}\sigma_f, 3\sqrt{2}\sigma_f]\) as (24) shows, and the second measurement value \(\Delta f_1\) would be in \([-6\sigma_f, 6\sigma_f]\) as (28) shows.

Due to there is an extra measurement before second tuning, the following tuning and prediction is similar to the first time. Therefore, the interval of \([-6\sigma_f, 6\sigma_f]\) is regarded as the tuning error that caused by measurement error, and it is described as follow:

\[Y = 12\sigma_f = 6\sqrt{2}\sigma_s\]  

(32)

**B. SIMULATION AND EXPERIMENT OF TUNING**

In order to verify the distribution of split after tuning, the simulation of tuning is accomplished by a program that written in C Language. There are two parts in simulation, one is simulating the measurement, the other is simulating the trimming. The simulation of measurement is accomplished by adding a random number on the true value of frequency. The random number that obey normal distribution is regarded as measurement error, and the mean value of its distribution is zero while the standard deviation is one third of measurement error. Ten thousand random numbers that obey standardized normal distribution are generated by a library function in C Language, and the statistical results is showed in Fig. 12. It is easy to find out that the ten thousand random numbers are almost in the interval of \([-3, 3]\), which means in the range of three times standard deviation and it is coincident with perfect standardized normal distribution.

The simulation of trimming is completed by adding or subtracting the value of split multiplied by a weight coefficient \(\alpha\) to the true value. The weight coefficients depend on the gyroscope, and the cylindrical vibratory gyroscope is utilized in this simulation. Just as Ref. [10] proposed, when the trimming mass is within 0.5 mg, the variation of split is about 2 Hz for holes-trimming, and the linearity of frequency variation is well. In this condition, the sensitivity of trimmed axis is 2.9687Hz/mg for the low frequency mode, and the other sensitivity is 2.8535Hz/mg for the high frequency mode. Based on these situations, the corresponding weight coefficient can be obtained. The weight coefficient of low frequency mode is 2.9687/(2.9687 + 2.8535) = 0.51, and the coefficient of high frequency mode is 0.49.

Assume that the natural frequencies of two modes are 5000.020 Hz and 5000.000 Hz separately, and by means of the measurement and trimming methods that talked above to complete simulation. During 5000 times tuning, the measurement value of splits is showed in the left of Fig. 13 with the frequency measurement error of 0.001 Hz, and the statistical results of measurement value are showed in the right of Fig. 13.

The measurement accuracy of frequency in this simulation is 3\(\sigma_f = 0.001Hz\), hence the standard deviation of true value and measurement value for split after trimming, according to
the results of (24) and (28), are $\sqrt{2} \sigma_f = 4.71404 \times 10^{-4} \text{Hz}$ and $2\sigma_f = 6.66666 \times 10^{-4} \text{Hz}$ separately. In simulation, the statistical results of standard deviation is $4.71018 \times 10^{-4} \text{Hz}$ for true value and $6.65979 \times 10^{-4} \text{Hz}$ for measurement value separately, and the error between analytical solution and simulation is only about 0.1%.

Furthermore, the simulations under different measurement error of frequency are performed for 50 times, and the measurement errors include 0.001 Hz, 0.005 Hz and 0.01 Hz. The simulation results are showed in Fig. 14.

From the above simulation results, the tuning error of split increase with the decrease of measurement accuracy. In general, no matter how much the measurement error is, the natural frequencies of two modes have an interlaced variety, and the split decreases to about zero after first trimming and then waves around zero. It is easy to find out that the amplitude of wave is related to measurement error, and the amplitude is defined as the tuning error. When the measurement error is 0.001 Hz, the fluctuation of split is nearly a straight line parallel to horizontal axis. However, the fluctuation is obvious when the measurement accuracy is 0.01 Hz, the fluctuation amplitude of frequency split is about 0.244Hz, and when the measurement accuracy is $3\sigma_f = 0.027\text{Hz}$, the fluctuation of theoretical result is 0.108Hz. The actual fluctuation is larger than the simulation result, which is mainly due to the fact that the trimming technology accuracy will also have a greater impact during the actual tuning. In addition, the material of the real resonator is not uniform, the structure is not symmetrical and other factors will also bring some errors.

As researched in Sec. IV, the measurement value should locate in the interval of $[-X/2, X/2]$ to ensure the true value achieving required accuracy, which is related to measurement value and error. Meanwhile, the measurement error also affects the tuning error, just as the simulation results show in Sec. V. Hence, an appropriate measurement accuracy that makes the interval of $[-X/2, X/2]$ existed and the measurement value had a high probability (99.73%) to locate in is needed in tuning. The optimal measurement accuracy would be discussed in the next part by combining the two effects.

VI. THE OPTIMAL MEASUREMENT ACCURACY

In order to achieve a rapid and precision tuning, the probability that the measurement value locates in $[-X/2, X/2]$ should be at least 99.73%, and the probability can be calculated according to the combination of the two effects.

The measurement value after tuning is $f \sim N(0, 2\sigma_f^2)$ as shown in (28), then the Probability Density Function and Cumulative Density Function are as follow separately in (33) and (34):

$$h(\Delta f) = \frac{1}{\sqrt{2\pi} \sqrt{2\sigma_f}} e^{-\frac{\Delta f^2}{2\sigma_f^2}}$$  \hspace{1cm} (33)

$$H(\Delta f) = \frac{1}{\sqrt{2\pi} \sqrt{2\sigma_f}} \int_{-\infty}^{\Delta f} e^{-\frac{t^2}{2\sigma_f^2}} dt$$  \hspace{1cm} (34)

Hence, the probability that the measurement value is located in $[-X/2, X/2]$ can be calculated as follow in (35):

$$P\left(-\frac{X}{2} < \Delta f < \frac{X}{2}\right) = H\left(\frac{X}{2}\right) - H\left(-\frac{X}{2}\right) = \Phi\left(\frac{X}{2\sqrt{2\sigma_f}}\right) - \Phi\left(-\frac{X}{2\sqrt{2\sigma_f}}\right)$$  \hspace{1cm} (35)
The probability under different standard deviation $\sigma_s$ (measurement error) is showed in Fig. 16. As an example, the required accuracy $\varepsilon = 6$, and the standard deviation $\sigma_s < \varepsilon/3 = 2$, which corresponding to the Part one of Sec. IV. As discussed above, the interval that ensure the true value achieving the required accuracy is $X = 2\varepsilon - 6\sigma_s$, as (18), and the tuning error $Y = 6\sqrt{2}\sigma_s$ as (32). Then the relationship of X and Y is showed in Fig. 16.

As shown in Fig. 16:

1) When the standard deviation is small, which means the measurement accuracy is high, and the tuning error Y of split is also small while the interval of X is big. Under this condition, the probability of measurement value locating in $[-X/2, X/2]$ is nearly hundred percent, which means the required accuracy can be achieved easily.

2) Along with the increasing of standard deviation, the tuning error Y expands to exceed the length of X. In this case, the probability of split meeting the required accuracy has no way to reach 99.73%. For example, when the standard deviation is 1.0, the tuning error Y is bigger than the interval length of X. However, the probability could run up to 96.61%, which means it is also easy to achieve the required accuracy.

3) When the standard deviation is big, $\sigma_s = 1.8$ for instance, the tuning error $Y = 15.3$ while the interval $X = 1.2$. As shown in Fig. 16, though the measurement accuracy is terrible, there is still a probability of 18.6% to achieve the required accuracy.

From these analysis, the probability of achieving the required accuracy is related to the measurement accuracy directly. It is easy to achieve when the accuracy is high enough, while the probability would decrease when the measurement accuracy decrease. However, the increase of measurement accuracy brings extra cost, and the high measurement accuracy could not be reached in some times. Therefore, the measurement accuracy should be decreased as long as the probability meets the requirement of engineering. In order to meet the requirements, the following formula should be satisfied:

$$X = 2\varepsilon - 6\sigma_s = 6\sqrt{2}\sigma_s = Y$$

$$\sigma_s = \frac{\varepsilon}{3\left(1 + \sqrt{2}\right)}, \quad \text{or} \quad \varepsilon = \left(1 + \sqrt{2}\right) \cdot 3\sigma_s \quad (36)$$

Hence, the optimal measurement accuracy could be chosen by $3\sigma_s = \varepsilon/\left(1 + \sqrt{2}\right)$, which corresponding to the dashed line in Fig. 16. On this condition, the measurement split after trimming has a probability of 99.73% to locate in $[-X/2, X/2]$, and the true value also have a probability of 99.73% to achieve the required accuracy.

When the measurement error larger than $\varepsilon/\left(1 + \sqrt{2}\right)$, the probability of achieving the required accuracy would decrease from 99.73%. In order to achieve the required accuracy, more times tuning should be processed. Besides, only the tuning times reach a very high level, the required accuracy could be come true by accident while the measurement error is larger $\varepsilon/\left(1 + \sqrt{2}\right)$ by a lot.

On contrary, the probability of achieving the required accuracy would increase from 99.73% when the measurement error is less than $\varepsilon/\left(1 + \sqrt{2}\right)$. It is worth mentioning that the probability of achieving the required accuracy is 99.73% when $\sigma_s = \varepsilon/\left(3\left(1 + \sqrt{2}\right)\right)$, and there is little space to increase. In practice, the increase of measurement accuracy brings a considerable cost. Therefore, the measurement accuracy should be suitable for a certain tuning mission.

According to (36) $\varepsilon = \left(1 + \sqrt{2}\right) \cdot 3\sigma_s = \left(2 + \sqrt{2}\right) \cdot 3\sigma_f$, the tuning accuracy can be predicted under a certain measurement condition. For example, when the measurement accuracy of frequency is $3\sigma_f = 0.001$Hz, the split can be decreased to 0.0034 Hz easily without consideration of other trimming errors.

VII. CONCLUSION AND FUTURE WORK

In this paper, the effects of measurement error on frequency split tuning are studied from the statistical aspect, and the method to select an optimal measurement accuracy for a specific required accuracy is proposed according to the effects. Before discussing the effects, the basic principle of cylindrical vibratory gyroscope is introduced, and the relationship between sensitivity and frequency split is studied to prove the importance of tuning. Firstly, experiments have shown that the frequency and the split measurement value are subject to a normal distribution. Secondly, only when the measurement accuracy is better than the required tuning accuracy and the measurement value is in the interval of X, the probability of achieving the required tuning accuracy is greater than 99.73%. Thirdly, the effects on the process of trimming are discussed, which figures out the tuning error $Y = 12\sigma_f$ by theoretical and simulation analysis. Finally, a selection method of the optimal measurement accuracy for a specific required tuning accuracy is proposed. When the optimal measurement accuracy is $\varepsilon/\left(1 + \sqrt{2}\right)$, the tuning error Y would
be exactly equal to the interval $X$, and the required accuracy would be achieved easily.

In practice, in addition to the measurement accuracy, the accuracy of the trimming technology will also have a greater impact on the tuning. When removing or increasing the mass by mechanical tuning, the minimum quantity of the changed mass corresponds to the frequency adjustment accuracy that can be achieved. When using electrostatic trimming, the stiffness change caused by applied voltage will also affect the trimming accuracy. The tuning error would increase further when the trimming technology errors are considered, while the interval $x$ would not be affected. In order to further improve the tuning accuracy, the trimming technology errors will be studied in the next step.

**APPENDIX**

Kolmogorov-Smirnov Test method is a common method of hypothesis testing, which is simple in application and high in accuracy. Its basic steps are as follows:

1. Suppose there is a set of data $x_1, x_2, x_3, \ldots, x_n$, and $S_n(X)$ is the cumulative integral function of this group of data;
2. Suppose $F(X)$ is the theoretical cumulative integral function to be fitted;
3. Define the maximum distance $D = \max |F(X) - S_n(X)|$.

Generally, the test data should be close to the theoretical distribution, in other words, $D$ should be small.

Test hypothesis $H_0 : F(X) = S_n(X)$

Alternative hypothesis $H_1 : F(X) \neq S_n(X)$

For the selected significance level $\alpha$, if the $D$ value is equal to or greater than the critical value $D_{n, \alpha}$ of the schedule, i.e. $D \geq D_{n, \alpha}$, then $H_0$ will be rejected, otherwise it will be accepted. $n$ represents the size of data.

The data in Figure 6 are tested to estimate if they are in normal distribution. The average and standard deviation of the original data are 3870.461 and 0.009, respectively. Therefore, the theoretical cumulative integral function is

$$F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi} \cdot 0.009} e^{-\frac{(x-3870.461)^2}{2 \cdot 0.009^2}} dt \quad (A1)$$

In other words, it is necessary to check whether the group of data conforms to the normal distribution $N \sim (3870.461, 0.009^2)$. The test results are shown in Table 1:

**TABLE 1.** Progress of Kolmogorov-Smirnov test.

| Range            | Count | $S_n(x)$ | $F(x)$ | $D$  |
|------------------|-------|----------|--------|------|
| 3870.44000-3870.44571 | 1     | 0.02564  | 0.027169 | 0.001529 |
| 3870.44571-3870.45142 | 6     | 0.17949  | 0.093841 | 0.085649 |
| 3870.45143-3870.45714 | 5     | 0.30769  | 0.238935 | 0.068755 |
| 3870.45714-3870.46285 | 10    | 0.5641   | 0.458958 | 0.105142 |
| 3870.46285-3870.46856 | 9     | 0.79487  | 0.692733 | 0.102137 |
| 3870.46857-3870.47428 | 5     | 0.92308  | 0.866791 | 0.056289 |
| 3870.47428-3870.47999 | 3     | 1        | 0.957103 | 0.042897 |

It can be seen from the table that $D = 0.1051$. For significance $\alpha = 0.05$ and sample size $n = 39$, $D_{n, \alpha} = \frac{\sqrt{39}}{\sqrt{n}} = 0.2178 > D$ can be obtained from Table 2. Therefore, we accept the hypothesis that the sample data conforms to the normal distribution.

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