Bounded $\lambda$-harmonic functions in domains of $\mathbb{H}^n$ with asymptotic boundary with fractional dimension

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January 21, 2021

Abstract

The existence and nonexistence of $\lambda$-harmonic functions in unbounded domains of $\mathbb{H}^n$ are investigated. We prove that if the $((n-1)/2)$ Hausdorff measure of the asymptotic boundary of a domain $\Omega$ is zero, then there is no bounded $\lambda$-harmonic function of $\Omega$ for $\lambda \in [0, \lambda_1(\mathbb{H}^n)]$, where $\lambda_1(\mathbb{H}^n) = (n-1)^2/4$. For these domains, we have comparison principle and some maximum principle. Conversely, for any $s > (n-1)/2$, we prove the existence of domains with asymptotic boundary of dimension $s$ for which there are bounded $\lambda_1$-harmonic functions that decay exponentially at infinity.

1 Introduction

Let $\mathbb{H}^n$ be the hyperbolic space and let $\Omega$ be a domain (open connected set) in $\mathbb{H}^n$. For $\Omega \neq \mathbb{H}^n$, we say that a nontrivial $u$ is a $\lambda$-harmonic function of $\Omega$ if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies
\[
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1)

In the case $\Omega = \mathbb{H}^n$ we only require that $u \neq 0$ satisfies the equation above. Remind that this is a classical eigenvalue problem when $\Omega$ is a bounded domain. In this case $u$ is an eigenfunction associated to the eigenvalue $\lambda$ and, from the Spectral Theory, the set of eigenvalues is discrete and it is the spectrum of $-\Delta : H^1_0(\Omega) \to H^1_0(\Omega)$, where $H^1_0(\Omega)$ is the Sobolev space that is the closure of $C_0^\infty(\Omega)$ with the $L^2$ norm of the gradient. However the situation is different for unbounded domains and this characterization of the spectrum does not hold. Indeed, there exists $\lambda$ for which problem (1) has nontrivial solution but still $\lambda$ is neither an eigenvalue nor an element of the essential spectrum.

Even then, several results are known. For instance, if $\Omega$ is the whole $\mathbb{H}^n$, a class of $\lambda$-harmonic functions is obtained in [9], [10] using some Poisson integral representation. This representation is used in [5] to characterize the bounded $\lambda$-harmonic functions for any $\lambda \in \mathbb{C}$.

For a Hadamard manifold $M$ with sectional curvature bounded from above and below by negative constants, Ancona proved in [3] the existence of bounded $\lambda$-harmonic functions in $M$, that converge to zero at infinity with exponential rate, for any $\lambda \in (0, \lambda_1)$, where $\lambda_1$ is the first eigenvalue of $M$ defined by
\[
\lambda_1(M) = \inf_{v \in H^1(M)} \frac{\int_M |\nabla v|^2 \, dx}{\int_M |v|^2 \, dx}.
\]

On the other hand, he also exhibited examples of manifolds for which there is no positive $\lambda_1$-harmonic function that converges to 0 at the asymptotic boundary. This illustrates the importance of the first eigenvalue in the behavior of the $\lambda$-harmonic functions in general manifolds. The asymptotic boundary with the cone topology [6] also plays an important role in the estimates needed to the main results of [2] and [3].
The purpose of this work is to study the existence of bounded \( \lambda \)-harmonic functions in unbounded domains of \( \mathbb{H}^n \) for \( \lambda \leq \lambda_1(\mathbb{H}^n) \), where the first eigenvalue of \( \mathbb{H}^n \) is evaluated by McKeans in [11]:

\[
\lambda_1(\mathbb{H}^n) = \frac{(n-1)^2}{4}.
\]

We show that if the asymptotic boundary boundary is not “large enough”, the problem has no bounded solution. This extends our knowledge that for bounded domains (or domains with empty asymptotic boundary) there are no \( \lambda \)-harmonic functions, if \( \lambda \leq \lambda_1 \). One of our main results is the following theorem, that is not restricted to functions that converge to zero at the asymptotic boundary:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{H}^n \) such that the \((n-1)/2\) Hausdorff dimensional measure of \( \partial_\infty \Omega \), \( H^{(n-1)/2}(\partial_\infty \Omega) \) is zero. Then there is no bounded \( \lambda \)-harmonic function that vanishes on \( \partial \Omega \) for \( \lambda \in [0, \lambda_1] \).

A consequence of Theorem 1.1 is that for domains \( \Omega \) with \( H^{(n-1)/2}(\partial_\infty \Omega) \neq 0 \), there are no bounded classical solutions of

\[-\Delta u = \lambda u + f \quad \text{in} \quad \Omega,
\]

where \( f \in C(\Omega) \), satisfying \( u_1 \leq u_2 \) on \( \partial \Omega \), then \( u_1 \leq u_2 \) in \( \Omega \). In particular for \( f = 0 \), considering \( u_2 = 0 \), we conclude that if \( u_1 \leq 0 \) on \( \partial \Omega \), then \( u_1 \leq 0 \) in \( \Omega \). This is a special case of maximum principle studied, for instance, by Berestycki, Nirenberg and Varadhan [5] in bounded domains of \( \mathbb{R}^n \) for subsolutions of a large class of second order elliptic equations \( Lu = -\lambda u \). They proved this maximum principle provided \( \lambda < \lambda_1(L, \Omega) \), where \( \lambda_1(L, \Omega) \) is a sort of eigenvalue defined for nondivergent operators.

The comparison principle also implies the uniqueness of bounded solutions to the Dirichlet problem

\[
\left\{
\begin{array}{ll}
-\Delta u = \lambda u + f & \text{in} \quad \Omega \\
u = \varphi & \text{on} \quad \partial \Omega
\end{array}
\right.
\]

for \( \lambda \in [0, \lambda_1] \). Since any \( L^p(\Omega) \) \( \lambda \)-harmonic function in Hadamard manifolds is bounded for \( p \geq 2 \) (see [4]), this uniqueness result also holds in \( L^p(\Omega) \) for \( p \geq 2 \).

Besides Theorem 1.1 can be used to prove uniqueness of Green’s functions of the operator \(-\Delta - \lambda\) that are bounded outside a ball containing the singularity. Indeed suppose that \( G_1^1 \) and \( G_2^2 \) are Green’s functions of a domain \( \Omega \) with \( H^{(n-1)/2}(\partial_\infty \Omega) = 0 \) associated to the point \( x \in \Omega \). If they are bounded outside a ball centered at \( x \), then \( G_1^1 - G_2^2 \) is a bounded solution of (1). Therefore, it must be zero, proving that \( G_1^1 = G_2^2 \). This result does not hold for a general domain, because if a domain has a bounded \( \lambda \)-harmonic function \( u \) and such a \( G_1^1 \), then \( G_1^1 + c u, c \in \mathbb{R} \), is a family of Green’s functions with the same property. The behavior of Green’s function for points far way the singularity is studied in [2] for Hadamard manifolds.

Theorem 1.1 is optimal in the sense that if the asymptotic boundary has dimension larger than \((n-1)/2\), then Problem 1 may have a bounded solution. In fact, for any \( s \in \left(\frac{n-1}{2}, n-1\right) \) we prove the existence of open sets \( \Omega \subset \mathbb{H}^n \) such that the dimension of \( \partial_\infty \Omega \) is \( s \) and for which there exist bounded solutions that decay exponentially at infinity. (For \( n = 5 \) and \( s = 3 > (5-1)/2 \), we present in Section 4 a \( \lambda \)-harmonic function of a hiperannuli, that is a domain which possess an asymptotic boundary of dimension 3.) More precisely, we prove in Theorem 1.4 that any truly \( s \)-dimensional subset (see Definition 2.3) of \( \partial_\infty \mathbb{H}^n \) is the asymptotic boundary of a domain that admits a bounded \( \lambda \)-harmonic function. These two results give a good relation between existence of solution and the dimension of the asymptotic boundary.

In this work, some preliminaries are presented in Section 2. They include the concept asymptotic boundary of \( \mathbb{H}^n \), the cone topology and the Hausdorff dimension of a subset of the asymptotic boundary. Theorem 1.1 is proved in Section 3. In Section 4 we present conditions on a subset \( X \subset \partial_\infty \mathbb{H}^n \)
that guarantee the existence of an open set $\Omega$ that admits a bounded $\lambda$-harmonic function such that the asymptotic boundary of $\Omega$ is $X$. Then we build examples in which the asymptotic boundary has dimension $s$ for any $s \in \left(\frac{n-1}{2}, n-1\right)$. For the case $s = n - 1$ simple examples are exposed in Section 4.

2 Preliminaries about $\mathbb{H}^n$

2.1 The asymptotic boundary of $\mathbb{H}^n$

To define the asymptotic boundary of $\Omega$, $\partial_\infty \Omega$, we follow the ideas of Eberlein and O’Neill [6] and consider $\partial_\infty \mathbb{H}^n$ as the set of equivalence classes of geodesic rays, where we say that $\gamma_1 \sim \gamma_2$, iff $d(\gamma_1(t), \gamma_2(t))$ is bounded, where $t$ is the arclength. Then the closure of $\mathbb{H}^n$ is $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ and the cone topology is introduced by saying that any open set of $\overline{\mathbb{H}^n}$ is open in $\mathbb{H}^n$ and truncated cones $C(o, \gamma_0, \theta, R)$ are also open. For an arclength parametrized geodesic ray $\gamma_0$, with $\gamma_0(0) = o$, the truncated cone $C(o, \gamma_0, \theta, R)$ of opening $\theta$ centered at $\gamma_0$ is the union of the two following sets

$$C(o, \gamma_0, \theta, R) \cap \mathbb{H}^n = \{ \gamma(t) \mid \gamma \text{ is a geodesic, } \gamma(0) = o, |\gamma'(0)| = 1, \angle(\gamma'(0), \gamma_0'(0)) < \theta/2 \text{ and } t > R \}$$

and

$$C(o, \gamma_0, \theta, R) \cap \partial_\infty \mathbb{H}^n = \{ [\gamma] \mid \gamma(t) \in C(o, \gamma_0, \theta, R) \cap \mathbb{H}^n \text{ for large } t \}. \quad \text{The asymptotic boundary of a set } \Omega \subset \mathbb{H}^n, \partial_\infty \Omega, \text{ is the boundary with respect to the cone topology of } \Omega \text{ in } \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n \text{ minus its usual boundary.}

With this notion of asymptotic boundary, we may define the Hausdorff dimension of a subset $X \subset \partial_\infty \mathbb{H}^n$. We start with sets $X$ of measure zero. For that, remind that for a given point $o \in \mathbb{H}^n$, we may identify $\partial_\infty \mathbb{H}^n$ with $S_1(o) = \{ p \in \mathbb{H}^n \mid d(p, o) = 1 \}$. A point $[\gamma]$ in $\partial_\infty \mathbb{H}^n$ is identified with $f_o([\gamma]) \in S_1(o)$ intercepting the geodesic $\gamma$ in $[\gamma]$ that starts in $o$ with $S_1(o)$. Then $\partial_\infty \mathbb{H}^n$ can be identified with $S_1(o)$ by the function $f_o : \partial_\infty \mathbb{H}^n \to S_1(o)$. Observe that if $f_1 : \partial_\infty \mathbb{H}^n \to S_1(o_1)$ and $f_2 : \partial_\infty \mathbb{H}^n \to S_1(o_2)$ are the functions associated to the points $o_1, o_2 \in \mathbb{H}^n$ respectively, then $f_2 \circ f_1^{-1}$ is a bijection between $S_1(o_1)$ and $S_1(o_2)$. Indeed, $f_2 \circ f_1^{-1}$ is a Lipschitz function according to Proposition 1.3 of [12].

As a consequence, suppose that $X \subset \partial_\infty \mathbb{H}^n$ is such that $f_1(X) \subset S_1(o_1)$ is a set of $r$-dimensional Hausdorff measure zero. Then the $r$-dimensional Hausdorff measure of $f_2(X) \subset S_1(o_2)$ is also zero. Besides, if $Y \subset \partial_\infty \mathbb{H}^n$ is such that $f_1(X) \subset S_1(o_1)$ is a set of $r$-dimensional Hausdorff measure finite, then the $r$-dimensional Hausdorff measure of $f_2(X) \subset S_1(o_2)$ is also finite. Therefore the definitions below are well-posed.

Definition 2.1. The $r$-dimensional Hausdorff measure of $X \subset \partial_\infty \mathbb{H}^n$ is zero, denoted by $H^r(X) = 0$, if the $r$-dimensional Hausdorff measure of $f_1(X) \subset S_1(o_1)$ is zero for some (and therefore for all) identifications of $X$.

Definition 2.2. The $r$-dimensional Hausdorff measure of $X \subset \partial_\infty \mathbb{H}^n$ is finite, denoted by $H^r(X) < \infty$, if the $r$-dimensional Hausdorff measure of $f_1(X) \subset S_1(o_1)$ is finite for some (and therefore for all) identifications of $X$.

Moreover, we may define the Hausdorff dimension of a subset of $\partial_\infty \mathbb{H}^n$ as the Hausdorff dimension of $f_1(X)$ for an identification of $X$. This is well defined because the Hausdorff dimension of $X$ is $s(X) = \inf\{ r \geq 0 \mid H^r(X) = 0 \}$.

Some $r$ dimensional subsets of $\partial_\infty \mathbb{H}^n$ are truly $r$-dimensional, and for them we obtain a sharp result in the sense that they are the asymptotic boundary of a domain that admits a bounded $\lambda_1$-harmonic function iff $r > (n-1)/2$. Their precise definition is

Definition 2.3. For $(n-1)/2 < s < n-1$, a set $X$ contained in $\partial_\infty \mathbb{H}^n$ is a truly $s$-dimensional set if:

i) The Hausdorff dimension of $X$ is $s$;
ii) $H^s(X)$ is finite;
and $X$ admits an identification $f : \partial_\infty \mathbb{H}^n \to S_1(o)$ such that

iii) For any $p_0 \in f(X)$,
$$\liminf_{r \to 0} \frac{H^s(f(X) \cap B_r(p_0))}{r^s} > 0,$$
for $B_r(p_0)$ the ball centered at $p_0$ in $T_o \mathbb{H}^n$ with radius $r$;

iv) There exists $K = K(f(X)) > 0$ such that for any $p_0 \in f(X)$ and $r > 0$,
$$H^s(f(X) \cap B_r(p_0)) \leq Kr^s.$$

A totally geodesic hypersphere in $\mathbb{H}^n$ is a $(n - 1)$—dimensional totally geodesic submanifold of $\mathbb{H}^n$. A totally geodesic hyperball is a region bounded by a totally geodesic hypersphere. Given a totally geodesic hyperball $\mathcal{I}$ bounded by $\partial \mathcal{I}$, the distance with sign to $\partial \mathcal{I}$ is defined as the hyperbolic distance to $\partial \mathcal{I}$ with positive sign in $\mathcal{I}$ and negative sign in $\mathbb{H}^n \setminus \mathcal{I}$. The asymptotic boundary of a hypersphere is homeomorphic to $\mathbb{S}^{n-2}$ and therefore has Hausdorff dimension $n - 2$.

A horosphere in $\mathbb{H}^n$ is a $(n - 1)$—dimensional submanifold of $\mathbb{H}^n$ obtained as the limit set of a sequence of geodesic spheres centered along a geodesic ray $\gamma(t)$ that contain $\gamma(0)$. It may be seen as a sphere centered at $[\gamma] \in \partial_\infty \mathbb{H}^n$. A horoball $\mathcal{H}$ is the region bounded by a horosphere that contains the geodesic ray $\gamma((0, \infty))$. The distance with sign to $\partial \mathcal{H}$ is defined as the hyperbolic distance to $\partial \mathcal{H}$ with positive sign in $\mathcal{H}$ and negative sign in $\mathbb{H}^n \setminus \mathcal{H}$. The asymptotic boundary of a horosphere is only one point so that it has Hausdorff dimension 0.

### 2.2 The Poincaré ball model

The Poincaré ball model of $\mathbb{H}^n$ consists in endow the ball $B = B_1(0) \subset \mathbb{R}^n$ of radius one centered at the origin with a Riemannian metric given by
$$g_{ij}(p) = \frac{4}{(1 - |p|^2)^2} \delta_{ij},$$
where $| \cdot |$ is the euclidean distance to the origin. The advantage of this model is that the Hausdorff measure in $\partial_\infty \mathbb{H}^n$ can be seen in $\partial B$. It becomes then natural to integrate in subsets of $\partial_\infty \mathbb{H}^n$ by integrating in subsets of $\partial B$, as we will proceed in Section 3.

In this model the geodesics are part of circles that cross $\partial B$ orthogonally and diameters though the origin. The totally geodesic hyperspheres are $(n - 1)$—spheres that also intercept $\partial B$ orthogonally and the horospheres are spheres contained in $\overline{B}$ that touch $\partial B$ only at one point, called the center of the horosphere. Given $\alpha \in [-1, 1)$, we denote by $H_{z, \alpha}$ the horoball centered at $z \in \partial B$ with $\alpha z \in \partial H_{z, \alpha}$. We remark that given $z$ and $\alpha$, there is a unique horoball with the properties described above.

With the model set we are able to prove two lemmas about the distance to horo- and hyperspheres, where we use that the hyperbolic distance from $p \in B$ to the origin 0 of $B$ is
$$d(p, 0) = \ln \left( \frac{1 + |p|}{1 - |p|} \right).$$

**Lemma 2.4.** Let $0 \in B$ be the center of $B$ in the ball model. Given $z \in \partial B$ and a positive $\theta$, let $C(z, \theta, 0)$ be the cone with vertex at 0, opening angle $\theta$ with axis the ray that connects 0 to $z$. Let $I(z, \theta, 0)$ be the totally geodesic hyperball such that $I(z, \theta, 0) \cap \partial B = C(z, \theta, 0) \cap \partial B$.

There exist constants $C_1$, $C_2$ and $\theta_0$, such that the hyperbolic distance $d$ satisfies
$$\ln \left( \frac{C_1}{\theta} \right) \leq d(0, \partial I(z, \theta, 0)) \leq \ln \left( \frac{C_2}{\theta} \right),$$
for $\theta < \theta_0$. 
Proof. In this setting, \( \partial I(z, \theta, 0) \) is, if \( \theta < \pi \), part of an euclidean sphere in \( \mathbb{R}^n \) of radius \( \tan(\theta/2) \) and center outside \( B \), on the line though 0 and \( z \), at a distance \( \sqrt{\tan^2(\theta/2) + 1} = \sec(\theta/2) \) from 0. Therefore the euclidean distance from the origin 0 to \( \partial I(z, \theta, 0) \) is \( \sec(\theta/2) - \tan(\theta/2) \). From expression \( 3 \)

\[
d(0, I(z, \theta, 0)) = \ln \left( \frac{1 + \sec(\theta/2) - \tan(\theta/2)}{1 - \sec(\theta/2) + \tan(\theta/2)} \right),
\]

which behaves as stated above for \( \theta \) close to zero.

\[
\square
\]

Lemma 2.5. The hyperbolic distance (with sign) between \( x \in B \) and a horoball \( H_{x,0} \), though the origin is

\[
d(x, \partial H_{x,0}) = \ln \left( \frac{1 - |x|^2}{|z - x|^2} \right).
\]

Proof. We can suppose that \( z = e_1 = (1,0,\ldots,0) \in \partial B \) and use the representation \( x = (x_1,y) \), where \( y = (x_2, \ldots, x_n) \). Observe that there exists only one \( \alpha \) such that the horosphere \( \partial H_{x,\alpha} \) contains \( x \). This \( \alpha \) corresponds to the intersection between \( \partial H_{x,\alpha} \) and the \( x_1 \)-axis, that is given by \( (\alpha,0,\ldots,0) = \alpha e_1 \), where

\[
\alpha = \frac{x_1 - |x|^2}{1 - x_1}.
\]

Since the two horospheres \( \partial H_{x,\alpha} \) and \( \partial H_{x,0} \) are equidistant, the distance between \( x \) and \( \partial H_{x,0} \) is the same as the distance between \( \alpha e_1 \) and 0, that is given by

\[
d(p, \partial H_{x,0}) = d(\alpha e_1, 0) = \ln \left( \frac{1 + \alpha}{1 - \alpha} \right).
\]

From \( 3 \), we have

\[
\frac{1 + \alpha}{1 - \alpha} = \frac{1 - |x|^2}{1 - 2x_1 + |x|^2} = \frac{1 - |x|^2}{|z - x|^2}.
\]

Therefore,

\[
d(x, \partial H_{x,0}) = \ln \left( \frac{1 - |x|^2}{|z - x|^2} \right).
\]

\[
\square
\]

3 Non-existence results

In this section we prove the non existence of bounded \( \lambda \)-harmonic functions in domains of \( \mathbb{H}^n \) that are unbounded and have a sufficiently small asymptotic boundary. We first prove a weaker result for the case of the asymptotic boundary being a single point and then we generalize it for small sets. For this, we need the \( \lambda \)-harmonic functions associated to hyperballs.

Given a totally geodesic hyperball \( I \subset \mathbb{H}^n \), if \( u : \mathbb{H}^n \to \mathbb{R} \) is a function that depends only on the distance (with sign) \( d \) to the hypersphere \( \partial I \), then the Laplacian of \( u \) is given by

\[
\Delta u(p) = u''(d(p)) + (n - 1) \tanh(d(p))u'(d(p)).
\]

Hence a \( \lambda \)-harmonic function in \( \mathbb{H}^n \) that depends only on \( d \) satisfies

\[
h''(d) + (n - 1) \tanh(d) h'(d) = -\lambda h(d).
\]

Lemma 3.1. Given \( \lambda \in [0, \lambda_1] \), there are constants \( d_0 > 0 \) and \( C_3 > 0 \) such that the ODE \( 4 \) admits a solution \( h : (0, \infty) \to \mathbb{R} \) which is positive and decreasing in \((d_0, +\infty)\) and satisfies \( h(d) < C_3 e^{-rd} \) for all \( d > d_0 \). The exponent \( r \) is \( r = -\frac{1}{\lambda_1} + \frac{1}{2\lambda_1} \sqrt{1 - \frac{\lambda}{\lambda_1}}. \)
Proof. Consider the change of variables $t = \frac{1}{\sinh(d)}$, which brings the behaviour of a solution at infinity to the origin. With this change, (4) becomes

$$(1 + t^2)v''(t) + \left(\frac{2t^2 + 2 - n}{t}\right)v'(t) + \left(\frac{\lambda}{t^2}\right)v(t) = 0,$$  

(5)
solvable by the Fröbenius method.

The two (linearly independent, if $\lambda < \lambda_1$) solutions to (5) are

$$v_1(t) = \sum_{i=0}^{\infty} a_i t^{i+r_1} \quad \text{and} \quad v_2(t) = \sum_{i=0}^{\infty} b_i t^{i+r_2},$$

with $a_0 \neq 0$, $r_1 = r$ and $r_2 = r - (n - 1)\sqrt{1 - \lambda/\lambda_1}$.

Observe that $v_1(0) = 0$ and if we choose $a_0 > 0$, there is a positive $t_0$ such that $v_1$ is an increasing function in $(0, t_0)$. Besides, decreasing $t_0$ if necessary, we may find a constant $D > 0$, such that $v_1(t) \leq Dt^{r_1}$ in $(0, t_0)$.

Hence, taking $h(d) = v_1(1/\sinh(d))$, $h$ is a solution to (4) such that $h > 0$ in $(d_0, +\infty)$ where $d_0 = \arcsinh(1/t_0)$ and

$$h(d) = v_1 \left(\frac{1}{\sinh(d)}\right) \leq D \left(\frac{1}{\sinh(d)}\right)^{r_1} \leq C_3 e^{-r_1 d}$$
in this interval.

Definition 3.2. For a given totally geodesic hyperball $\mathcal{I}$ in $\mathbb{H}^n$, define a $\lambda$-harmonic function $w_\mathcal{I}$ in $\mathbb{H}^n$ by

$$w_\mathcal{I}(x) = \frac{h(d(x))}{h(d_0)}.$$  

The function $w_\mathcal{I}$ is a $\lambda$–harmonic function in $\mathbb{H}^n$ that attains value 1 on the hypersphere $d_0$ apart from $\partial \mathcal{I}$ and decreases exponentially to zero for $d > d_0$.

Theorem 3.3. Let $\Omega$ be a domain of $\mathbb{H}^n$ such that $\partial_\infty \Omega = \{p\}$. Then there is no nontrivial bounded $\lambda$-harmonic function that vanishes on $\partial \Omega$ for any $\lambda \in [0, \lambda_1]$.

Proof. Assume, for the sake of contradiction, that $\Omega \subset \mathbb{H}^n$ is a domain with $\partial_\infty \Omega = \{p\}$ that admits a bounded $\lambda$–harmonic function $u$. Without loss of generality, we may suppose sup $u = 1/2$. Let $o$ be some point of $\Omega$ such that $u(o) > 0$.

Let $h$ be the solution to (4) from Lemma 3.1. Since $h \to 0$ as $d \to +\infty$, there is some $d_1 > d_0$ with

$$\frac{h(d_1)}{h(d_0)} < u(o).$$

Let $\mathcal{S}$ be the totally geodesic hypersphere centered at $p$ with $d(o, \mathcal{S}) = d_1$. Now consider $\mathcal{I}$ the totally geodesic hyperball bounded by $\mathcal{S}$ that does not contain $p$ in its asymptotic boundary. Let $w_\mathcal{I}$ be the $\lambda$-harmonic function of $\mathcal{I}$ from Definition 3.2. We find a compact subset $\Omega_1 \subset \Omega \cap \mathcal{I}$ in which $u - w_\mathcal{I}$ is a positive bounded $\lambda$–harmonic function.

Take

$$O = \{x \in \Omega \cap \mathcal{I} : d(x, \mathcal{S}) > d_0\}.$$  

Observe that $O$ is a bounded non empty set, since $\Omega \cap \mathcal{I}$ is bounded and $o \in O$. Besides

$$w_\mathcal{I}(o) = \frac{h(d_1)}{h(d_0)} < u(o).$$
Furthermore, \( w_I > u \) on \( \partial O \), since \( w_I > 0 = u \) on \( \partial O \cap \Omega \) and \( w_I = 1 > u \) on \( \partial O \cap \Omega \). Therefore, there exists some domain \( \Omega_1 \subset O \) such that

\[
u - w_I > 0 \text{ in } \Omega_1
\]

and \( u - w_I = 0 \) on \( \partial \Omega_1 \). Hence \( v = u - w_I \) is a positive \( \lambda \)-harmonic function of the bounded domain \( \Omega_1 \), contradicting the fact that \( \lambda \leq \lambda_1 (\mathbb{H}^n) \).

In order to prove the stronger version of this theorem, we need to estimate \( w_I \) at a point by the size of \( \partial \infty I \).

**Lemma 3.4.** Let \( I = I(x, \theta, 0) \) be a totally geodesic hyperball in the ball model \( B \). Then the \( \lambda \)-harmonic function \( w_I : B \to \mathbb{R} \) associated to \( I \) satisfies

\[
0 < w_I(0) \leq C_4 \theta^{(n-1)/2},
\]

for any \( \theta < \theta_1 \), for some constants \( C_4 > 0 \) and \( \theta_1 > 0 \).

**Proof.** According to Lemma 3.1, \( h(d) \leq C_3 e^{-(n-1)d/2} \) for \( d > d_0 \) and, from Lemma 2.4,

\[
d = d(0, \partial I) > \ln \frac{C_1}{\theta}
\]

for \( \theta < \theta_0 \). Choose \( \theta_1 < \theta_0 \) such that \( \ln \left( \frac{C_1}{\theta_1} \right) \) > \( d_0 \). Hence, for \( \theta < \theta_1 \), both inequalities are satisfied. Therefore,

\[
w_I(0) = h(d(0)) \leq C_3 e^{-(n-1)d(0)/2} \leq C_3 e^{-(n-1)/2} \ln(C_1/\theta) = C_4 \theta^{(n-1)/2}.
\]

**Theorem 3.5.** Let \( \Omega \subset \mathbb{H}^n \) such that \( H^{(n-1)/2}(\partial \infty \Omega) = 0 \). Then there is no nontrivial bounded \( \lambda \)-harmonic function that vanishes on \( \partial \Omega \) for \( \lambda \in [0, \lambda_1] \).

**Proof.** We follow the same idea as in Theorem 3.3. Assume that there exists such a function \( u \), that we suppose satisfies \( \sup u = 1/2 \). Let \( B \) be the ball model of \( \mathbb{H}^n \). Without loss of generality, we assume that for 0 the origin of \( B \), \( 0 < u(0) \leq 1/2 \).

Let \( X' \subset \partial B \) be the asymptotic boundary of \( \Omega \) in this model. Since the \( (n - 1)/2 \)-dimensional Hausdorff measure of \( X' \) is zero, for any \( \varepsilon > 0 \) there exists a countable collection of balls in \( \partial B \), \( U_{\theta_i}(x_i) = C(x_i, \theta_i, 0) \cap \partial B \) that covers \( X' \), such that \( \theta_i < \theta_1 \) and

\[
\sum_i \theta_i^{(n-1)/2} < \varepsilon.
\]

Consider the hyperballs \( I_i = I(x_i, \theta, 0) \) and the associated \( \lambda_1 \)-harmonic function \( w_i = w_{I_i} \). Then, from Lemma 3.4

\[
0 \leq w_i(0) \leq C_4 \theta_i^{(n-1)/2}.
\]

Using (6), we get

\[
0 < \sum_i w_i(0) \leq C_4 \varepsilon.
\]

Taking \( \varepsilon \) small, it follows that

\[
0 < \sum_i w_i(0) < u(0).
\]
Let $O = V_{d_0} \cap \Omega$, where $V_{d_0} = \{ q \in \mathbb{H}^n : d(q, I) > d_0 \ \forall i \}$. Observe that $w_i > 0$ in $V_{d_0}$ and $\sum_{i=1}^N w_i(p)$ is bounded in compacts of $V_{d_0}$ uniformly in $N$, by Harnack Inequality. Hence $w = \sum_i w_i$ is well defined, positive and, from the regularity theory, $\lambda_1$-harmonic function.

Defining $v = u - w$ in $O$ we have that $v(o) > 0$ and $v < 0$ on $\partial O$ since $w > 0 = u$ on $\partial O \setminus \Omega$ and $w \geq 1 > u$ on $\partial O \cap \Omega$. Moreover $O$ is bounded. As before, we obtain a contradiction. 

4 Existence results

The purpose of this section is to show that there exist subsets of $\partial_\infty \mathbb{H}^n$ of dimension $s \in \left( \frac{n-1}{2}, n-1 \right]$, that are the asymptotic boundary of domains that admit bounded $\lambda_1$-harmonic functions. We are especially interested in the case $s \in \left( \frac{n-1}{2}, n-1 \right)$. For that, given a truly subset $X$ of $\partial_\infty \mathbb{H}^n$ of dimension $s$, we construct a $\lambda_1$-harmonic function $u : \mathbb{H}^n \to \mathbb{R}$ positive in $\Omega$ with $\partial_\infty \Omega = \overline{X}$, as stated in the next theorem.

**Theorem 4.1.** Let $X$ be a truly $s$-dimensional subset of $\partial_\infty \mathbb{H}^n$ for some $(n-1)/2 < s < n-1$. Then there is a bounded $\lambda_1$-harmonic function $u$ in $\mathbb{H}^n$, positive in some set $\Omega$ such that $\partial_\infty \Omega = \overline{X}$. Moreover, for any $p \in \Omega$

$$|u(p)| \leq M_0(d + 1) e^{-\left(s - \frac{n-1}{2}\right) d},$$

where $d = d(p,o)$ for some $o \in \mathbb{H}^n$ fixed and $M_0$ is a positive constant that depends only on $K(f(X))$, $n$ and $s$.

For proving this result we integrate $\lambda_1$-harmonic functions that are constant along horospheres. To find them, let $H \subset \mathbb{H}^n$ be a horoball and $d : \mathbb{H}^n \to \mathbb{R}$ be the distance with sign (positive in $H$) to $\partial H$. Then a $\lambda_1$-harmonic function that is constant on horospheres equidistant to $\partial H$ has the form $u_H(p) = u_H(d(p))$ and its Laplacian is

$$\Delta u_H(p) = u_H''(d(p)) + (n-1)u_H'(d(p)).$$

Therefore, $h = u_H \circ d$ satisfies

$$h''(d) + (n-1)h'(d) = -\lambda_1 h(d).$$

Hence,

$$u_H(p) = u_H(d(p)) = (A_1 + A_2 d(p)) e^{(n-1)d(p)/2}, \quad (7)$$

for constants $A_1$ and $A_2$.

The next lemma, a consequence of Lemma 2.5 presents an expression of $u_H$ in the Poincaré ball model.

**Lemma 4.2.** In the Poincaré ball model, any $\lambda_1$-harmonic function given by (7) associated to the horoball $H_{z,0}$ can be expressed by

$$u_{z,A_1,A_2}(x) = \left[ \frac{1 - |x|^2}{|z - x|^2} \right]^{(n-1)/2} \left( A_1 + A_2 \ln \left[ \frac{1 - |x|^2}{|z - x|^2} \right] \right). \quad (8)$$

The result follows from expression (7) and Lemma 2.5. We denote by $u_z$ the function above for $A_1 = 0$ and $A_2 = 1$.

**Proof of Theorem 4.1.** We divide the proof into three claims. The first one asserts the absolute integrability of $u_z$, the second provides a way to obtain the required solution $u$, and the third one treats the decay of $u$. 

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Claim 1. Let \( \Lambda \subset \partial B \), obtained as an identification of \( X \subset \partial B \), a truly \( s \)-dimensional set not necessarily satisfying condition iii) from Definition 2.3. There is \( M > 0 \) depending only on \( n \) and \( s \), such that
\[
\int_{\Lambda} |u_z(x)| \, dH^s(z) < MK(\Lambda),
\]
holds for any \( x \in B \).

For \( x \in B \), let \( \delta = 1 - |x| \) and \( k \in \mathbb{N} \) such that \( 2 < 2^k \delta \leq 4 \). Define
\[
\Lambda_i = \{ z \in \Lambda : 2^{i-1} \delta \leq |z - x| < 2^i \delta \} \quad \text{for} \quad i \in \{1, 2, \ldots, k\}.
\]
Observe that
\[
\Lambda = \bigcup_{i=1}^{k} \Lambda_i,
\]
since \( \delta \leq |z - x| \leq 2 \) for any \( z \in \partial B \). If \( z \in \Lambda_i \), we have \( |z - x| \geq 2^{i-1} \delta \) and, therefore
\[
\frac{1 - |x|^2}{|z - x|^2} = \frac{(1 + |x|)(1 - |x|)}{|z - x|^2} \leq \frac{2 \delta}{2^{2(i-1)} \delta^2} = \frac{8}{4^i \delta^3}.
\]
Hence
\[
\left[ \frac{1 - |x|^2}{|z - x|^2} \right]^{\frac{n-1}{2}} \leq \left( \frac{8}{4^i \delta^3} \right)^{\frac{n-1}{2}}.
\]
Moreover,
\[
\frac{1 - |x|^2}{|z - x|^2} = \frac{(1 + |x|)(1 - |x|)}{|z - x|^2} \geq \frac{\delta}{4}
\]
since \( |z - x| \leq 2 \). Using this inequality and that for \( a \leq \min\{1, t\} \)
\[
|\ln t| \leq \ln t - 2 \ln a,
\]
we obtain
\[
\left| \ln \left[ \frac{1 - |x|^2}{|z - x|^2} \right] \right| \leq \ln \left[ \frac{1 - |x|^2}{|z - x|^2} \right] - 2 \ln \left( \frac{\delta}{4} \right).
\]
Then, the monotonicity of \( \ln x \) and \( 10 \) imply that
\[
\left| \ln \left[ \frac{1 - |x|^2}{|z - x|^2} \right] \right| \leq \ln \left( \frac{8}{4^i \delta^3} \right) - 2 \ln \left( \frac{\delta}{4} \right) = \ln \left( \frac{128}{4^i \delta^3} \right) \leq \ln \left( \frac{128}{\delta^3} \right). \tag{11}
\]
From \( 10 \) and \( 11 \), we conclude that for \( z \in \Lambda_i \)
\[
|u_z(x)| = \left[ \frac{1 - |x|^2}{|z - x|^2} \right]^{\frac{n-1}{2}} \left| \ln \left[ \frac{1 - |x|^2}{|z - x|^2} \right] \right| \leq \left( \frac{8}{4^i \delta^3} \right)^{\frac{n-1}{2}} \ln \left( \frac{128}{\delta^3} \right). \tag{12}
\]
Now we estimate \( H^s(\Lambda_i) \). Let \( x_0 = \frac{x}{|x|} \in \partial B \). Notice that if \( z \in \Lambda_i \),
\[
|z - x_0| \leq |z - x| + |x - x_0| = |z - x| + \delta \leq 2|z - x| \leq 2^{i+1} \delta.
\]
Then \( \Lambda_i \subset B_{2^{i+1} \delta}(x_0) \cap \Lambda \) and from condition \( iv \) of Definition 2.3
\[
H^s(\Lambda_i) \leq K(\Lambda)(2^{i+1} \delta)^s. \tag{13}
\]
Using (12) and (13), we conclude
\[
\int_{\Lambda_i} |u_z(x)| dH^s(z) \leq \left( \frac{8}{2^{2i} \delta} \right)^{n-1} \ln \left( \frac{128}{\delta^3} \right) K(\Lambda)(2^{i+1} \delta)^s = K_0 2^{i(s-(n-1))} \delta^{s-\frac{n-1}{2}} \ln \left( \frac{128}{\delta^3} \right),
\]
where \(K_0 = 8^{n-1} 2^s K(\Lambda)\). Therefore,
\[
\int_{\Lambda} |u_z(x)| dH^s(z) = \sum_{i=1}^{k} \int_{\Lambda_i} |u_z(x)| dH^s(z) \leq K_0 \delta^{s-\frac{n-1}{2}} \ln \left( \frac{128}{\delta^3} \right) \sum_{i=1}^{k} 2^{i(s-(n-1))},
\]
Since \(s-(n-1) < 0\), we get \(\sum_{i=1}^{k} 2^{i(s-(n-1))} \leq 2^{(s-(n-1))} \). Thus
\[
\int_{\Lambda} |u_z(x)| dH^s(z) \leq K_1 \delta^{s-\frac{n-1}{2}} \ln \left( \frac{128}{\delta^3} \right),
\]
where \(K_1 = K_0 2^{(s-(n-1))} \frac{1}{1-2^{(s-(n-1))}}\). Using that the right-hand side is uniformly bounded for \(\delta \in (0, 1)\), we conclude the claim.

The second claim states that the integral in \(\Lambda\) of the family of functions \(u_z, z \in \Lambda\), given by (8) defines a bounded \(\lambda_1\)-harmonic function.

**Claim 2.** If \(\Lambda\) is a truly \(s\)-dimensional set and there is \(M > 0\) such that (8) holds for any \(x \in B\), then the function
\[
u(x) = \int_{\Lambda} u_z(x) dH^s(z),
\]
is a bounded \(\lambda_1\)-harmonic function, positive in some set \(\Omega\) such that \(\partial_\infty \Omega = \partial \Omega \cap \partial B = \overline{\Lambda}\).

From (8), \(u\) is well defined and bounded by \(MK(\Lambda)\). Moreover, since \(H^s(\Lambda)\) is finite and \(u_z(x)\) is \(C^\infty\) in \(z\) and \(x\), we have that \(u\) is \(C^\infty\) and it is a \(\lambda_1\)-harmonic function.

Let \(\Omega = \{x \in B : u(x) > 0\}\). First we prove that \(\partial_\infty \Omega \subset \overline{\Lambda}\). For \(z_0 \in \partial B \setminus \overline{\Lambda}\) and a positive \(r < \frac{1}{2d(z_0, \overline{\Lambda})}\) that will be chosen later, let \(V\) be the ball centered at \(z_0\) with radius \(r\). Hence,
\[
|z-x| > \frac{1}{2} d(z_0, \overline{\Lambda}) \quad \text{for any} \quad x \in V \cap B \quad \text{and} \quad z \in \Lambda.
\]
Naming \(d = d(z_0, \overline{\Lambda})\) and using that \(|x| > 1 - r\) for \(x \in V \cap B\), we have
\[
\ln \left[ \frac{1-|x|^2}{|z-x|^2} \right] < \ln(1 - (1 - r)^2) - \ln \left( \frac{d^2}{4} \right) = \ln \left( \frac{8r - r^2}{d^2} \right),
\]
for \(z \in \Lambda\). If \(r < \frac{d^2}{8}\), the right-hand side of this inequality is negative and, therefore,
\[
u_z(x) < 0 \quad \text{for} \quad x \in V \cap B \quad \text{and} \quad z \in \Lambda.
\]
Then \(u(x) < 0\) for \(x \in V \cap B\). As a consequence \(z_0 \notin \partial_\infty \Omega\) which proves that \(\partial_\infty \Omega \subset \overline{\Lambda}\).

We prove now that \(\partial_\infty \Omega \supset \overline{\Lambda}\). For \(x_0 \in \overline{\Lambda}\) let \(x = (1-\delta)x_0\), where \(0 < \delta < 1/8\) will be chosen later. Observe that \(\delta = 1 - |x|\). Consider the sets \(\Lambda_i\) defined in Claim 1 for \(i \in \{1, \ldots, k\}\), where \(2 < 2^k \delta \leq 4\).

If \(z \in \Lambda_i\) for \(i < (k-2)/2\), we have that \(|z-x| < 2^i \delta\) and the definition of \(k\) implies that \(2^{2k} \delta < 1\). Then
\[
\ln \left[ \frac{1-|x|^2}{|z-x|^2} \right] \geq \ln \left[ \frac{(1-|x|)(1+|x|)}{|z-x|^2} \right] \geq \ln \left( \frac{\delta}{2^{2k} \delta^2} \right) = \ln \left( \frac{1}{2^{2k} \delta} \right) > 0.
\]
Hence \( u_2(x) > 0 \) and therefore
\[
\int_{\Lambda_i} u_2(x) \, dH^s(z) \geq 0 \quad \text{for} \quad i < \frac{k - 2}{2}.
\] (18)

We can improve this estimation in \( \Lambda_1 \). Indeed, if \( z \in \Lambda_1 \), then \( |z - x| < 2\delta \). Using this and (17),
\[
u_2(x) = \left[ \frac{1 - |x|^2}{|z - x|^2} \right]^{\frac{n-1}{2}} \ln \left[ \frac{1 - |x|^2}{|z - x|^2} \right] \geq \left[ \frac{1}{4\delta^2} \right]^{\frac{n-1}{2}} \ln \left( \frac{1}{4\delta^2} \right).
\] (19)

Observe now that for \( z \in \Lambda \cap B_\delta(x_0) \), we get \( |z - x_0| < \delta \) and, therefore \( |z - x| \leq |z - x_0| + |x_0 - x| < 2\delta \). On the other hand \( |z - x| \geq \delta \). Hence
\[\Lambda \cap B_\delta(x_0) \subset \Lambda_1.\]

Condition (iii) of Definition 2.3 implies that there exist \( \delta_1 > 0 \) and \( D > 0 \) such that
\[H^s(\Lambda_1) \geq H^s(\Lambda \cap B_\delta(x_0)) > D\delta^s,\]
for \( \delta \leq \delta_1 \). From this and (19) we conclude, for \( \delta \leq \delta_1 \), that
\[
\int_{\Lambda_1} u_2(x) \, dH^s(z) \geq \left[ \frac{1}{4\delta^2} \right]^{\frac{n-1}{2}} \ln \left( \frac{1}{4\delta^2} \right) D\delta^s = \frac{D}{4^n - 1} \delta^s \ln \left( \frac{1}{4\delta^2} \right).\] (20)

Now we consider the integral on \( \Lambda_i \) for \( i \geq (k - 2)/2 \). For that, remind (13) from which we obtain
\[
\int_{\Lambda_i} u_2(x) \, dH^s(z) \geq -\int_{\Lambda_i} |u_2(x)| \, dH^s(z) \geq -K_0 2^{(s-(n-1))} \delta^s \ln \left( \frac{128}{\delta^2} \right).
\]

Hence, denoting \( E = \bigcup_{\frac{k-2}{2} \leq i \leq k} \Lambda_i \) and using that \( s - (n-1) < 0 \), we have
\[
\int_E u_2(x) \, dH^s(z) \geq -K_0 \delta^s \ln \left( \frac{128}{\delta^2} \right) \sum_{\frac{k-2}{2} \leq i \leq k} 2^{(s-(n-1))}
\geq -K_0 \delta^s \ln \left( \frac{128}{\delta^2} \right) \frac{2^{(s-(n-1))}}{1 - 2^{(s-(n-1))}} = -K_2 \delta^s \ln \left( \frac{128}{\delta^2} \right) 2^{(s-(n-1))},
\]
where \( K_2 = K_0 \frac{2^{(n-1)-(n-1)}}{2^{(2)(s-(n-1))}} \). Since \( 2^k > \frac{2}{\delta} \),
\[
\int_E u_2(x) \, dH^s(z) \geq -K_2 \delta^s \ln \left( \frac{128}{\delta^2} \right),\] (21)

From (18), (20) and (21), we get
\[
\int_{\Lambda} u_2(x) \, dH^s(z) \geq \frac{D}{4^n - 1} \delta^s \ln \left( \frac{1}{4\delta^2} \right) - K_2 \delta^s \ln \left( \frac{128}{\delta^2} \right),\] (22)

for \( \delta \leq \min \{1/8, \delta_1 \} \). The right-hand side is positive for \( \delta \) close to zero, since the negative term has the extra factor \( (\delta/2)^{\frac{(n-1)-(n-1)}{2}} \) that converges to zero. That is, there exists \( \delta_2 > 0 \) such that \( u((1 - \delta)x_0) > 0 \) for \( \delta \leq \delta_2 \). Then \( x_0 \in \partial_\infty \Omega \), proving the result.

The third and last claim required to prove Theorem 4.1 is about the boundedness of the function.
Claim 3. If $\Lambda$ is a truly $s$-dimension set, then the $\lambda_1$-harmonic function $u$ given by (16) is positive in some $\Omega$ such that $\partial_s\Omega = \overline{\Lambda}$ and

$$|u(x)| \leq M_0(d + 1) e^{-\left(s - \frac{n-1}{2}\right)d},$$

where $d = d(x,0)$ and $M_0$ is a positive constant that depends only on $K(\Lambda)$, $n$ and $s$.

Inequality (15) implies that

$$|u(x)| \leq K_1 \delta^{s - \frac{n-1}{2}} \ln \left(\frac{128}{\delta^3}\right),$$

for $\delta = 1 - |x|$. From (2), we have $\delta = 2/(1 + e^d)$ and then $e^{-d} \leq \delta \leq 2e^{-d}$. Therefore

$$|u(x)| \leq K_1 (2e^{-d})^{s - \frac{n-1}{2}} \ln \left(\frac{128}{(e^{-d})^3}\right),$$

proving the result.

□

Remark 4.3. For $x_0 \in \overline{\Lambda}$ and $x = (1 - \delta)x_0$, inequality (22) implies

$$u(x) \geq C \delta^{s - \frac{n-1}{2}} \ln \left(\frac{1}{\delta}\right),$$

for small $\delta$, where $C$ is some positive constant. Taking $\delta = \frac{2}{1 + e^d}$ as we did before, we conclude that

$$u(x) \geq \tilde{M} d e^{-\left(s - \frac{(n-1)}{2}\right)d},$$

where the positive constant $\tilde{M}$ depends on $K(\Lambda)$ (given by condition (iii) from definition (2.3), $x_0$, and the set $\Lambda$. From this and the previous claim, we get

$$\tilde{M} d e^{-\left(s - \frac{(n-1)}{2}\right)d} \leq u(x) \leq M_0(d + 1) e^{-\left(s - \frac{(n-1)}{2}\right)d},$$

for $x = (1 - \delta)x_0$ as $\delta$ is close to zero.

For $\Omega \subset \mathbb{H}^n$ such that $\partial_s\Omega$ has $(n-1)$ Hausdorff dimension, we consider specific subsets. For instance, if $\Omega = \mathbb{H}^n$, we can find a radially symmetric $\lambda$-harmonic function $u$ in $\Omega$ by solving an hypergeometric equation. Such a solution is bounded and has exponential decay at infinity (see [8]) for $\lambda \in (0, \lambda_1]$. Moreover $u$ does not change sign, otherwise $\lambda$ would be the first eigenvalue of some ball. Another example can be obtained by taking $u = u_1 - u_2$, where $u_1$ and $u_2$ are radially symmetric $\lambda$-harmonic functions with different points of symmetry, $o_1$ and $o_2$. If $u_1(o_1) = u(o_2) > 0$, then $u$ is positive in some geodesic hyperball $I$ and negative outside $I$.

In some special situations, we can find $\lambda_1$-harmonic with a more explicit representation. For instance, consider a $\lambda_1$-harmonic function $u$ presented in Section 4 that depends only on $d$, the distance (with sign) to some hypersphere $\partial I$. If $n = 5$, then $u(d)$ satisfies (4) with $\lambda = \lambda_1 = (5-1)^2 = 4$. As an example of such a solution, we can take

$$u(d) = \frac{\cosh d - d \sinh d}{\cosh^3 d}$$

that is an even function with two zeros $d_1$ and $-d_1$, positive in $(-d_1, d_1)$ and negative elsewhere. Then $u$ is a positive $\lambda_1$-harmonic function in $\Omega$, that is the hyperannulus bounded by the two hyperspheres that have distance $d_1$ to $\partial I$. The asymptotic boundary of $\Omega$ is homeomorphic to $S^3$ that has dimension 3.
4.1 Cantor like subsets of $\partial_{\infty}\mathbb{H}^n$

Given $(n-1)/2 < s < n-1$, $s$ is the Hausdorff dimension of $\partial_{\infty}\Omega$ for a domain $\Omega$ that admits a bounded $\lambda_1$-harmonic function. We exhibit truly $s$–dimensional sets in $\partial_{\infty}\mathbb{H}^n$ and then the result follows from Theorem [..]

Let us construct Cantor like sets in $\mathbb{S}^1 \subset \partial B$. Given integers $1 \leq l < m$, the Cantor like set $K_{l,m}$ is constructed as follows:

We introduce coordinates in $\mathbb{S}^1$ and think of $\Lambda$ (see [..]) and represents the volume of the unit $K$ if we think of measure (see [..]) and taking intervals of length $1/m$.

**Proof.**

Given $\Lambda$, let $K_0 = [0, 1]$.

Let $K_1 = \left[0, \frac{1}{m}\right] \cup \left[\frac{1}{l}, \frac{1}{l} + \frac{1}{m}\right] \cup \left[\frac{2}{l}, \frac{2}{l} + \frac{1}{m}\right] \cup \cdots \cup \left[\frac{l-1}{l}, \frac{l-1}{l} + \frac{1}{m}\right]

be a union of $l$ intervals of length $1/m$ equally distributed in $\mathbb{S}^1$. Then $K_n$ is obtained inductively from $K_{n-1}$, a union of $l^{n-1}$ intervals of length $1/m^{n-1}$, by taking from each interval of $K_{n-1}, l$ intervals of length $1/m^n$, equally distributed in the interval.

The generalized Cantor set $K_{l,m}$ is then

$$K_{l,m} = \bigcap_{n=1}^{\infty} K_n.$$ 

It is well known that $K_{l,m}$ has Hausdorff dimension $s$ for $s$ such that $\frac{1}{m} = 1$, hence $s = \frac{\ln l}{\ln m}$. Moreover, the $s$–dimensional Hausdorff measure of $K_{l,m}$ is

$$H^s(K_{l,m}) = \omega_s \lim_{n \to \infty} \sum_{i=1}^{l^n} \left(\frac{1}{m^n}\right)^s = \omega_s < \infty,$$

if we think of $K_{l,m}$ as a subset of $[0, 1]$. (The constant $\omega_s$ is used in the definition of the Hausdorff measure (see [..]) and represents the volume of the unit $s$–dimensional ball when $s$ is a positive integer.) Since we may identify $\mathbb{S}^1 \subset \partial B$ with $[0, 1]$, $H^s(K_{l,m})$ is finite as a subset of $\partial_{\infty}\mathbb{H}^n$. Besides, if instead of taking intervals of length $1/m$ of the size of the intervals in the step before, one takes intervals of length $a/m$ for $a \in (0, 1)$, then the Hausdorff dimension of the Cantor set would be

$$s = \frac{\ln l}{\ln m - \ln a}$$

and the $s$–dimensional measure would also be $\omega_s < \infty$. Therefore, for any $s \in (0, 1)$, there is a Cantor like set of Hausdorff dimension $s$ in $\mathbb{S}^1$.

**Proposition 4.4.** Let $\Lambda$ denote $K_{l,m}$ of dimension $s = \frac{\ln l}{\ln m}$. For any $z \in \Lambda$,

$$\frac{\omega_s r^s}{l^s} \leq H^s(\Lambda \cap B_r(z)) \leq w_s(2m)^s r^s$$

and therefore $\Lambda \subset \partial B$ is a truly $s$–dimensional subset of $\partial B$.

**Proof.** Given $z \in \Lambda$, and $r > 0$, take $n \in \mathbb{N}$ such that

$$\frac{1}{l^{n+1}} \leq r < \frac{1}{l^n}.$$

Observe that $B_r(z)$ contains, for some $i \in \mathbb{N}, 0 < i < n - 1$, $I_n = [i/l^n, (i+1)/l^n]$ and therefore $\Lambda \cap B_r(z)$ contains a contracted copy of $\Lambda$, $K_{n,i} = i/l^n + 1/l^n \times \Lambda$ which is part of $K_n$. Hence

$$H^s(\Lambda \cap B_r(z)) \geq H^s(\Lambda \cap I_n) = (1/l^n)^s H^s(\Lambda) \geq \frac{\omega_s r^s}{l^s}$$

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concluding the proof of the first inequality.

The second one is trivial if \( r > 1/m \). If \( r < 1/m \), take \( n \in \mathbb{N} \) such that

\[
\frac{1}{m^{n+1}} \leq r < \frac{1}{m^n}.
\]

(23)

Notice that \( \Lambda \) is contained in \( K_n \) and \( B_r(z) \) intercepts \( K_n \) at most in two consecutive subintervals of \( K_n \) of length \( 1/l^n \). Therefore,

\[
H^s(\Lambda \cap B_r(z)) \leq H^s(K_n \cap B_r(z)) \leq \frac{2H^s(\Lambda)}{l^n} \leq H^s(\Lambda)(2rm)^s.
\]

□

For any \( 0 < s < 1 \), we exhibited a truly \( s \)-dimensional subset \( \partial B \) that is, from Theorem 4.1, the boundary of a domain that admits a bounded \( \lambda_1 \)-harmonic function if \( s > (n-1)/2 \). The construction of the asymptotic boundary of a domain where a bounded eigenfunction exists ends by observing that if \( k \) is an integer such \( (n-1)/2 < k + s < n - 1 \), then a Cantor like set \( \Lambda \) of Hausdorff dimension \( k + s \) can be constructed as follows:

We may think of \( S^{n-1} \) as a subset of \( \mathbb{R}^n \), \( S^{n-1} = \{(x_1, ..., x_n)|x_1^2 + x_2^2 + ... + x_n^2 = 1\} \) and it contains \( S^1 = \{(0,0,x_{n-1},x_n)|x_{n-1}^2 + x_n^2 = 1\} \). Let \( K_{l,m} \) be the Cantor set contained in \( S^1 \) and then for \( 0 < \varepsilon < 1 \)

\[
X = \{(x_1, ..., x_k, 0, ..., 0, x_{n-1}, x_n)|\frac{1}{1-\sum_{i=1}^2 x_i^2}(x_{n-1}, x_n) \in K_{l,m}
\]

and

\[
(x_1^2 + x_2^2 + ... + x_k^2) \in [-\varepsilon, \varepsilon].
\]

It is clear that \( X \) is diffeomorphic to \( K_{l,m} \times [-\varepsilon, \varepsilon]^k \) which has Hausdorff dimension \( s + k \) since any covering \( \{A_t\}_{t \in I} \) of \( K_{l,m} \) induces a covering of \( X \) by taking \( \{A_t \times [-\varepsilon, \varepsilon]^k\}_{t \in I} \). Besides, \( X \) is also a truly \( (k + s) \)-dimensional set and therefore, it is the asymptotic boundary of a domain \( \Omega \) that admits a bounded positive \( \lambda_1 \)-harmonic function. This can be summarized in the following result

**Theorem 4.5.** Given \( s \in ((n-1)/2, n-1) \), \( s \) is the Hausdorff dimension of \( \partial \Omega \) for a domain \( \Omega \) that admits a bounded \( \lambda_1 \)-harmonic function.

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