Convex geometry of Markovian Lindblad dynamics and witnessing non-Markovianity

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Abstract

We develop a theory of linear witnesses for detecting non-Markovianity, based on the geometric structure of the set of Choi states for all Markovian evolutions having Lindblad-type generators. We show that the set of all such Markovian Choi states form a convex and compact set under the small time interval approximation. Invoking geometric Hahn–Banach theorem, we construct linear witnesses to separate a given non-Markovian Choi state from the set of Markovian Choi states. We present examples of such witnesses for dephasing channel and Pauli channel in case of qubits. Furthermore, we have devised another method of detection NM of various qubit channels, by using projective measurements in the Bell basis. This can be done without the knowledge of what specific kind of channel we are dealing with. This gives us a huge operational advantage for NM detection. We further investigate the geometric structure of the Markovian Choi states to find that they do not form a polytope. This presents a platform to consider nonlinear improvement of non-Markovianity witnesses.

1 Introduction

The theory of open quantum systems provides a powerful tool to study system–environment interactions, spawning decoherence, dissipation and other irreversible phenomena [1,2]. Recently, much efforts have been devoted to characterize quantum
analogue of non-Markovian (NM) evolutions [3–19]. It has been shown from various information theoretic and thermodynamic aspects that NM can act as a powerful resource [11–13,20]. Therefore, identification of non-Markovianity is an extremely important aspect of recent studies in quantum theory. However, it still remains an onerous task to construct a theory of distinguishing NM evolutions from its Markovian counterparts, with proposals of experimentally feasible detection procedures. In most of the previous literature, non-Markovianity in terms of information backflow and divisibility breaking of a dynamical map is addressed by geometric or entropic distance-based quantities [3,4,6]. Those measures, though offer a proper quantification, are not feasible measures in an experimental scenario. However, from the theory of entanglement [21], we know that Hermitian operators are experimentally measurable and thus linear witnesses possess a much higher status from this perspective. Recently, the present authors have constructed a convex resource theory of NM [22], creating that very opportunity of experimental verification, by exploring the geometry of NM dynamics in a similar manner of entanglement detection.

The phenomena of witnessing entanglement [21,23–31] is a stepping stone in the study of quantum information, offering versatile tools for experimental detection of entangled quantum states. Witnesses are Hermitian operators and hence observables by construction, giving positive expectation value for all separable states, whereas negativity of the same signifies the existence of entanglement. In this article, we apply this methodology in open system dynamics, to construct NM witnesses from the similar footings of that of entanglement.

However, works have been done with the motivation of developing NM witnesses [32–34]. Recently, entanglement negativity has been shown as a universal non-Markovianity witness [35]. But in order to construct a proper NM detection theory, we need to have a convex and compact set of operators beholding all Markovian operations. Though channel-state duality [36,37] allows us to construct the set of operators, due to the non-convex nature of divisible operations [5,38], constructing a theory of linear witnesses is not possible in general. We overcome this difficulty by “small time interval” approximation, whence constructing the Choi states. This allows us to build a proper framework of linear witnesses for NM detection.

There are arguments on whether or not the indivisible operations exhaust all the non-Markovian operations [39–41]. It is established that all Markovian operations are divisible, whereas the converse is not proven to be true. However, even if there exist NM operations which are complete positive (CP) divisible, such operations cannot generate resource backflow and hence are not considered as resource [22]. It is to mention that, in this paper, we will develop the theory of witnesses for NM operations which are indivisible. It is also very important here to mention that we are restricted to the set of operations having Lindblad-type generators $\mathcal{L}$ [42,43] of the form $\dot{\rho} = \mathcal{L}(\rho) = \sum_{\alpha=1}^{d_1^2} \Gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho - \frac{1}{2} \rho(t) L_\alpha^\dagger L_\alpha \right)$, with $\Gamma_\alpha$s as the Lindblad coefficients, $L_\alpha$s as the Lindblad operators for a system with dimension $d_1$. For divisible operations, we have $\Gamma_\alpha \geq 0$, $\forall \alpha$. Generators of the dynamics can also be time-dependent $\mathcal{L}_t$. In that case, also the generators have the standard Lindblad form with time-dependent Hamiltonian, Lindblad operators and Lindblad coefficients. In Markovian case, $\Gamma_\alpha(t) \geq 0$, $\forall \alpha$, $t$. [8]. We discuss CP divisibility in sect. II. If
the evolution is non-Markovian in the sense that it is indivisible, the \( \Gamma_\alpha(t) < 0 \), for some \( \alpha \) at some instant of time \( t \). However, to ensure complete positivity of the total dynamics, the condition \( \int_0^T \Gamma_\alpha(t) dt \geq 0 \) (\( \forall \alpha, T \)) has to hold in general [8]. However, even this condition of non-Markovian operation has been generalized further in later work [44], which we refer for further details. It is also important to mention that in our theory, all the possible unitary evolutions have been considered. Since, in case of unitary evolutions, the system does not interact with the environment and as we will elaborate later that all unitary evolutions hold the divisibility property, they are considered here as trivially Markovian evolutions.

Before going into the main results of this work, let us mention that non-Markovianity is a property of a time parametrized family of completely positive trace-preserving maps which is known as dynamical maps. As completely positive maps are in one-to-one correspondence with their Choi states due to Choi–Jamiolkowski isomorphism [36,37], it is expected that non-Markovianity can also be characterized by means of Choi matrices. This allows us to explore Choi matrices corresponding to dynamical maps in this context. We first elucidate the properties of the set of Choi states for divisible operations. Then, we develop the theory of linear NM witnessing. We further consider the geometry of the set of Choi states, to identify the possibility of generalized nonlinear witnesses for NM detection. Then, we conclude with stating the possible implications.

2 Theory of linear witnesses for non-Markovianity detection

In this section, we develop the main theory of linear witnesses to detect non-Markovian dynamics, based on the structure of the Choi states corresponding to dynamical maps having Lindblad-type generators.

2.1 Divisible operations and structure of Choi states

Consider a general quantum channel \( \Lambda(t_B, t_A) : \rho(t_A) \rightarrow \rho(t_B) \). We construct a set \( D_C \) including all such channels having Lindblad-type generators. A channel is said to be CP divisible iff with the corresponding propagators \( \Lambda(t_3, t_2) \equiv T \exp \left( \int_{t_2}^{t_3} L(t) dt \right) \), satisfying composition law

\[
\Lambda_{\mathcal{M}}(t_3, t_1) = \Lambda_{\mathcal{M}}(t_3, t_2) \circ \Lambda_{\mathcal{M}}(t_2, t_1),
\]

is a CP map with \( t_3 > t_2 > t_1 \) \( \forall t_1, t_2, t_3 \), where \( T \) stands for time ordering product. This is known as time inhomogeneous Markovian dynamics [8]. Exploiting channel-state duality, we define a one-to-one connection between \( D_C \) and the set of corresponding Choi states \( C_A \).

It implies that the Choi state \( C_{\mathcal{M}}(t_2, t_1) = I \otimes \Lambda(t_2, t_1)|\phi \rangle \langle \phi | \) is a valid density matrix for every instant of time, with \( ||C_{\mathcal{M}}(t_2, t_1)||_1 = 1, \forall t_1, t_2 \). Here, \( |\phi \rangle \) corresponds to a maximally entangled state in \( d^2 \) dimension and \( || \cdot ||_1 = Tr[\sqrt{\cdot}^\dagger (\cdot)] \) is the trace norm. CP-divisibility breaking of a channel signifies NM backflow of information.
The set of all CP-divisible channels $\mathcal{D}_\mathcal{M} \subset \mathbb{D}_\mathcal{C}$ is then considered to be the set of all Markovian memoryless channels. We therefore define a subset $\mathcal{F}_\mathcal{M} \subset \mathcal{C}_\mathcal{M}$ as $\mathcal{F}_\mathcal{M} = \{ \mathcal{C}_\mathcal{M}(t_2, t_1) \mid ||\mathcal{C}_\mathcal{M}(t_2, t_1)||_1 = 1, \forall t_1, t_2 \},$ including the Choi states for all CP-divisible operations. At this point, one should recall the fact that $\mathcal{F}_\mathcal{M}$ is not a convex set in general. Convexity comes in a special situation, which we elaborate in the following proposition. It is also important to mention that unitary evolutions always hold the divisibility property. Therefore, the Choi states corresponding to the unitary evolutions are considered to be Markovian.

**Proposition 1** The set of Markovian Choi states (MCS) $\mathbb{F}^e_\mathcal{M} = \{ \mathcal{C}_\mathcal{M}(t+\epsilon, t) \mid ||\mathcal{C}_\mathcal{M}(t+\epsilon, t)||_1 = 1, \forall \epsilon \}$ is a convex and compact set in the limit $\epsilon \to 0$ and $\Gamma_\alpha(t)\epsilon << 1$ ($\forall \alpha)$, where $\Gamma_\alpha(t)$s are the Lindblad coefficients.

**Proof** Let $\mathcal{C}_\mathcal{M}^{(1)}(t+\epsilon, t)$ and $\mathcal{C}_\mathcal{M}^{(2)}(t+\epsilon, t)$ be two MCS corresponding to two separate Markovian operation $\Lambda_\mathcal{M}^{(1)}$ and $\Lambda_\mathcal{M}^{(2)}$ having Lindblad-type generator $\mathcal{L}_\mathcal{M}^{(1)}$ and $\mathcal{L}_\mathcal{M}^{(2)}$ with positive coefficients $\{ \Gamma_\alpha^{(1)} \}$ and $\{ \Gamma_\alpha^{(2)} \}$, respectively. Therefore, we have $\Lambda_\mathcal{M}^{(i)}(t+\epsilon, t) \equiv \mathbb{T} \exp \left( \int_{t}^{t+\epsilon} \mathcal{L}_\mathcal{M}^{(i)} dt \right)$, with $i = 1, 2$. Let us now consider the maps $\Lambda_\mathcal{M}^{(i)}(t_2, t_1)$ for a finite time interval $t_2 - t_1 = N\epsilon$, where we divide the total time interval in large number of $(N)$ snapshots of temporal width $\epsilon$. Considering the limit $\epsilon \to 0$, we use the Lie–Trotter formula [45] to get

$$
\Lambda_\mathcal{M}^{(i)}(t_2, t_1) = \mathbb{T} \exp \left( \sum_{v=1}^{N-1} \int_{t_v}^{t_{v+1}} \mathcal{L}_\mathcal{M}^{(i)} dt \right) = \lim_{N \to \infty} \left[ \prod_{v=1}^{N-1} \mathbb{T} \exp \left( \frac{\epsilon}{N} \mathcal{L}_\mathcal{M}^{(i)} dt \right) \right]^{N} \\
\approx \lim_{N \to \infty} \left[ \mathbb{I} + \epsilon \sum_{v=1}^{N-1} \mathcal{L}_\mathcal{M}^{(i)} \right]^{N} \approx \left[ \mathbb{I} + \epsilon \sum_{v=t_1}^{t_2} \mathcal{L}_\mathcal{M}^{(i)} \right].
$$

Here, we use the following identity $\int_{t_v}^{t_{v+1}} \mathcal{L}_\mathcal{M}^{(i)} dt = \epsilon \mathcal{L}_\mathcal{M}^{(i)}$, because $t_{v+1} - t_v = \epsilon$ ($\forall \nu$) and though the Lindblad operation is generally time dependent, in the very small snapshot of $\epsilon$, the Lindbladian is considered as constant at that instant of time. We also restrict ourselves to first order of the Taylor series expansion of the exponential term and later the binomial expansion, in the limit $\epsilon \to 0$ and $\Gamma_\alpha^{(i)}(t)\epsilon << 1$ ($\forall \alpha$).

$$
\Lambda_\mathcal{M}(t_2, t_1) = p \Lambda_\mathcal{M}^{(1)}(t_2, t_1) + (1 - p) \Lambda_\mathcal{M}^{(2)}(t_2, t_1) = \mathbb{I} + \epsilon [p \sum_{t_v = t_1}^{t_2} \mathcal{L}_\mathcal{M}^{(1)} + (1 - p) \sum_{t_v = t_1}^{t_2} \mathcal{L}_\mathcal{M}^{(2)}] = \mathbb{I} + \epsilon \sum_{t_v = t_1}^{t_2} \mathcal{L}_\mathcal{M},
$$

with $\mathcal{L}_\mathcal{M} = p \mathcal{L}_\mathcal{M}^{(1)} + (1 - p) \mathcal{L}_\mathcal{M}^{(2)}$ and $0 \leq p \leq 1$; we get that $\mathcal{L}_\mathcal{M}$ is also a Lindblad-type generator with positive coefficients. This shows that the map $\Lambda_\mathcal{M}(t_2, t_1)$ also belongs to the set of divisible Markovian maps. If we now consider $t_2 = t + \epsilon$, the results holds true, because it is valid for all $(t_1, t_2)$. This proves the statement $\mathbb{F}^e_\mathcal{M}$ is a convex set.

To prove $\mathbb{F}^e_\mathcal{M}$ is compact, it is enough to show $\mathbb{F}^e_\mathcal{M}$ is closed and bounded in the concerned topology. This fact is obvious from the definition of $\mathbb{F}^e_\mathcal{M}$ and continuity of trace norm. \hfill \(\square\)
In fact, as we have shown that even if we consider many snapshots of the width $\epsilon$, the proposition also holds true. But for our purpose, we consider a single snapshot at a particular time. But if need be, this proposition can be extended to a much larger time gap with many infinitesimally small snapshots.

From **Proposition 1**, it is clear that convex structure comes only under the small time interval approximation, by utilizing the structure of the Lindbladians. From an experimental perspective, it demands observation in snapshots taken at very frequent intervals in the dynamical evolution.

We must mention that quantum non-Markovianity is a dynamical property. Whether the dynamics exhibits information backflow can only be found if the specific information (e.g., entanglement, distinguishability) of the time evolved state is compared with that of a state at a previous or later time and hence a fixed time period thus comes naturally and unavoidably into the picture. Here, we are fixing the range of this snapshots in terms of a fixed $\epsilon$.

Here, we consider finite-dimensional normed linear spaces only, where all norms are topologically equivalent. In light of this fundamental fact, evidently $F^{\epsilon}_\mathcal{M}$ is also compact under Hilbert–Schmidt (HS) norm $||\cdot||_2 = \sqrt{\text{Tr}[(\cdot)^\dagger(\cdot)]}$. Thus, the structure of $F^{\epsilon}_\mathcal{M}$ leads us to the existence of non-Markovianity witnesses, by invoking the geometric Hahn–Banach separation theorem [46]. Let us specify for brevity we shall denote the Choi matrix corresponding to any operation $Q$ between the time $t$ and $t + \epsilon$ as $C_Q$ for $C_Q(t + \epsilon, t)$.

**Theorem 1** A non-Markovian Choi state can be separated from all MCS by a hyperplane.

**Proof** Let $\mathcal{C}_N$ be a non-Markovian Choi state and $F^{\epsilon}_\mathcal{M}$ denote the set of all MCS. Note that $D(\mathcal{C}_N | F^{\epsilon}_\mathcal{M}) = 0$ iff $\mathcal{C}_N \in \text{Cl}(F^{\epsilon}_\mathcal{M})$, where $\text{Cl}(\cdot)$ denotes the topological closure and $D(\cdot)$ is any metric. Since $F^{\epsilon}_\mathcal{M}$ is closed as $F^{\epsilon}_\mathcal{M}$ is compact, then $\text{Cl}(F^{\epsilon}_\mathcal{M}) = F^{\epsilon}_\mathcal{M}$. $\mathcal{C}_N$ does not belong to $F^{\epsilon}_\mathcal{M}$, and thus, $D(\mathcal{C}_N | F^{\epsilon}_\mathcal{M}) > 0$. Considering singleton set $\{\mathcal{C}_N\}$ as convex set, we always have a hyperplane [46] separating $\mathcal{C}_N$ and $F^{\epsilon}_\mathcal{M}$. $\square$

In Fig. 1, we represent the schematic diagram for **Theorem 1**. Since $F^{\epsilon}_\mathcal{M}$ is convex and compact, every non-Markovian Choi state can in principle be separated from $F^{\epsilon}_\mathcal{M}$ by some separating hyperplane.

### 2.2 Non-Markovianity witness

Let us consider the construction of NM witness using the techniques of entanglement theory.

**Definition** A Hermitian operator $W$ is said to be a NM witness if it satisfies following criteria:

1. $\text{Tr}(W\mathcal{C}_\mathcal{M}) \geq 0 \forall \mathcal{C}_\mathcal{M} \in F^{\epsilon}_\mathcal{M}$.
2. There exists at least one NM Choi state $\mathcal{C}_N$ such that $\text{Tr}(W\mathcal{C}_N) < 0$. © Springer
It is clear from the definition that a single witness cannot detect all non-Markovian Choi state. The witness will depend on the non-Markovian Choi state, which one wishes to detect.

Before going to the construction of non-Markovianity witness, let us illustrate the idea in brief. The total evolution is of course completely positive, and thus, the Choi state corresponding to a complete evolution starting from \( t = 0 \) to any later time \( t \) will always be a valid quantum state. The breaking of divisibility happens when complete positivity breaks down within the dynamics, e.g., from some intermediate time \( t \) to \( t + \epsilon \). Our purpose is to capture this breaking of divisibility via witnessing by a given Hermitian operator. For the very purpose, we take a maximally entangled state and apply a given quantum channel on one party of this bipartite system. The witnessing, i.e., measuring the \( W \) on this bipartite system, is done repeatedly at a very small time interval \( \epsilon \), and the corresponding measurement data are taken over these various time intervals. In a sense, this is similar to the determination of non-Markovianity by information backflow. Now let us discuss the construction of non-Markovianity witness.

### 2.2.1 Construction of NM witness

Let \( C_Q \) be a finite-dimensional Choi state corresponding to some operation \( Q \). \( C_Q \) being Hermitian, we have its spectral decomposition as:

\[
C_Q = \sum \lambda_i P_i,
\]

where \( P_i \)'s are orthogonal projections onto the subspace spanned by the normalized eigenvectors corresponding to the eigenvalues \( \lambda_i \). Note that \( Tr (C_Q P_j) = \lambda_j \delta_{ij} \), with \( \delta_{ij} \) being the Kronecker delta. If the operation is CP divisible, then \( C_Q \) being a valid
state has all nonnegative eigenvalues. Hence, $\text{Tr}(C_Q P_j) \geq 0 \ \forall \ j$. If the operation is NM, $\text{Tr}(C_Q P_j) < 0$ for at least one $j$, as $C_Q$ has at least one negative eigenvalue. Thus, the orthogonal projectors serve as witnesses for non-Markovian Choi state. Let us present two examples of NM witness based on our construction.

### 2.2.2 Examples of NM witness

**Qubit dephasing channel:** The Lindblad equation corresponding to such a channel is:

$$\frac{d\rho}{dt} = \gamma(t)(\sigma_i \rho \sigma_i - \rho),$$

where $\sigma_i$ ($i = x, y, z$) are the Pauli matrices.

Performing small time approximation, we get the corresponding Choi matrix:

$$
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} - \gamma(t)\epsilon \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} - \gamma(t)\epsilon & 0 & 0 & \frac{1}{2}
\end{bmatrix}
$$

The eigenvalues of the corresponding Choi state are $0, 0, \gamma(t)\epsilon$, and $1 - \gamma(t)\epsilon$, respectively.

The dynamics may not be completely known always, but there are some situations where one can estimate the numerical value of the decoherence rate, i.e., $\gamma(t)$ or its upper bound can be estimated [47]. Since $\epsilon$ can be chosen infinitesimally small, depending on the estimation of the decoherence rate one can have $|\gamma(t)\epsilon| << 1$. Therefore, for Markovian dynamics the latter of the two nonzero eigenvalues can always be considered as positive. We therefore take the orthogonal projector corresponding to the eigenvalue $\gamma(t)\epsilon$ as the witness for NM since $\gamma(t)\geq 0$ for CP-divisible operations and can be negative in case of NM operations. The witness for qubit dephasing operation is therefore $P_{\text{deph}} = |\chi\rangle\langle\chi|$, where $|\chi\rangle = (-1, 0, 0, 1)^T$. This suggests that one has to perform single projective measurement onto $|\chi\rangle$ state to detect non-Markovianity of qubit dephasing channel.

**Qubit Pauli Channel:** Consider now the Pauli channel:

$$\frac{d\rho}{dt} = \gamma_x(t)(\sigma_x \rho \sigma_x - \rho) + \gamma_y(t)(\sigma_y \rho \sigma_y - \rho) + \gamma_z(t)(\sigma_z \rho \sigma_z - \rho).$$

Small time approximation gives the eigenvalues of the corresponding Choi state as $1 - (\gamma_x(t) + \gamma_y(t) + \gamma_z(t))\epsilon$, $\gamma_x(t)\epsilon$, $\gamma_y(t)\epsilon$, $\gamma_z(t)\epsilon$. Following same logic, we can take the orthogonal projectors corresponding to the eigenvalues $\gamma_i(t)\epsilon$ (with $i = x, y, z$) as the witnesses of NM. They are, respectively, given by $P_x = |\chi_x\rangle\langle\chi_x|, P_y = |\chi_y\rangle\langle\chi_y|, P_z = |\chi_z\rangle\langle\chi_z|$, with $|\chi_x\rangle = (0, 1, 1, 0)^T$, $|\chi_y\rangle = (0, -1, 1, 0)^T$, and $|\chi_z\rangle = (-1, 0, 0, 1)^T$.

For the class of qubit dephasing and Pauli channels, the eigenvectors are independent of the channel parameters, as we can see from the example. The witness hence can wit non-Markovianity for any qubit dephasing channels and for the second example, any qubit Pauli channels. Therefore, for the examples we have shown the experimenter must have the prior information that the channel under study is dephasing or Pauli channel, to wit its non-Markovianity.

We would like to make a very important remark here. Non-Markovianity obtained from the second example that we have considered can be used to express something
known as “eternal” non-Markovian dynamics in the literature [44]. For example, let us consider a dynamics given by

\[ \frac{d \rho}{dt} = \gamma_x(t)(\sigma_x \rho \sigma_x - \rho) + \gamma_y(t)(\sigma_y \rho \sigma_y - \rho) + \gamma_z(t)(\sigma_z \rho \sigma_z - \rho), \]

where \( \gamma_x(t) = \gamma_y(t) = 1 \) and \( \gamma_z(t) = -\tanh t \).

The dynamics is non-Markovian in nature, since one of the Lindblad coefficients is always negative. But in spite of being non-Markovian in nature, it cannot be witnessed by various distance, volume or entanglement-based measures [35,44]. But using our proposed method, we can construct the witnesses of Pauli channel (as done in the previous example) and detect the non-Markovian nature of the dynamics. Thus, for identifying such kind of non-Markovian dynamics, our method of linear witnesses has a clear advantage.

3 Detecting non-Markovianity of various qubit channels by Bell measurement

In this section, we shall provide examples of non-Markovianity witness based on Bell measurement. These examples are operationally motivating, as without knowing the dynamics completely one can detect whether it is Markovian or not by measuring in the Bell basis. Two-qubit Bell states are given by

\[ |\psi_{1,2}\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \]

and

\[ |\psi_{3,4}\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \]

where \( |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

Let us recall for an operation \( Q \) between time span \( t \) and \( t + \epsilon \), we write the corresponding Choi matrix as \( C_Q = C_Q(t + \epsilon, t) \). It is to be noted that for a Markovian operation, the corresponding Choi matrix is a valid density matrix, i.e., it is Hermitian and positive, and its trace is 1. Hence, whenever we take expectation of a Markovian Choi state with respect to any Bell state, it gives a nonnegative value. This helps us to detect non-Markovianity of several qubit dynamics. Let us illustrate this fact via some examples.

**Pauli channels:** The Lindblad equation for any individual Pauli channel is given by

\[ \frac{d \rho}{dt} = \gamma_i(t)(\sigma_i \rho \sigma_i - \rho) = \mathcal{L}_i \rho \]
where \( i = x, y, z \). Here, after considering small time interval approximation, the action of the corresponding channel can be expressed as: 
\[
\Lambda^P_i (\rho) \approx (\mathbb{I} + \epsilon L^P_i) (\rho).
\]

**Qubit dephasing channel in \( \sigma_z \) basis:** The Lindblad equation of dephasing channel in \( \sigma_z \) basis is given by
\[
d\rho/dt = L^P_z (\rho) = \gamma_z (t) (\sigma_z \rho \sigma_z - \rho) = L^P_z (\rho),
\]
and hence, the corresponding channel is 
\[
\Lambda^P_z \approx (\mathbb{I} + \epsilon L^P_z).
\]

The Choi matrix corresponding to the operation is given by
\[
C^P_z = \mathbb{I} \otimes \Lambda^P_z |\psi_1\rangle \langle \psi_1|.
\]

Let us now note that
\[
Tr \left[ |\psi_2\rangle \langle \psi_2| C^P_z \right] = \gamma_z (t) \epsilon.
\]
For Markovian dynamics, the value of \( \gamma_z(t) \), i.e., decoherence rate, is always nonnegative. Therefore, for the Markovian dynamics, the expectation value is nonnegative. If the dynamics is non-Markovian, then the value of \( \gamma_z(t) \) is negative and hence it gives a negative value to the corresponding expectation. Thus, we can say that the Bell state \(|\psi_2\rangle \langle \psi_2|\) serves as the non-Markovianity witness for \( \sigma_z \) dephasing operation.

Similarly for a dephasing channel in \( \sigma_x \) basis, we have
\[
d\rho/dt = L^P_x (\rho) = \gamma_x (t) (\sigma_x \rho \sigma_x - \rho).
\]
In this case, we observe that
\[
Tr \left[ |\psi_3\rangle \langle \psi_3| C^P_x \right] = \gamma_x (t) \epsilon < 0 \quad \text{only when} \quad \text{the dynamics is non-Markovian.}
\]
Here, \( C^P_x \) stands for the Choi matrix corresponding to the \( \sigma_x \) dephasing operation. Therefore, the Bell state \(|\psi_3\rangle \langle \psi_3|\) serves as the non-Markovianity witness for \( \sigma_x \) dephasing operation.

In a similar way, for the dephasing operation in \( \sigma_y \) basis, it can be verified that the Bell state \(|\psi_4\rangle \langle \psi_4|\) serves as the non-Markovianity witness.

**Qubit depolarizing channel:** The Lindblad equation of Pauli depolarizing channel is given by,
\[
d\rho/dt = \sum_i \gamma_i (\sigma_i \rho \sigma_i - \rho),
\]
where \( i = x, y, z \). Let \( C^P_{\text{dep}} \) be the Choi matrix to the Pauli depolarizing operation. Then, we note that
\[
Tr \left[ |\psi_4\rangle \langle \psi_4| C^P_{\text{dep}} \right] = \gamma_y (t) \epsilon, \quad Tr \left[ |\psi_3\rangle \langle \psi_3| C^P_{\text{dep}} \right] = \gamma_x (t) \epsilon,
\]
and
\[
Tr \left[ |\psi_2\rangle \langle \psi_2| C^P_{\text{dep}} \right] = \gamma_z (t) \epsilon.
\]

Therefore, the dynamics is non-Markovian if any one or more of the above expectation values are negative. As mentioned earlier, this kind of non-Markovianity includes eternal non-Markovianity in the literature. Therefore, this protocol gives an operational way to detect eternal non-Markovianity via Bell measurement. Similarly, for any other combination of these three channels, Bell measurements can detect non-Markovianity.
**Amplitude damping channel:** The Lindblad equation of amplitude damping channel is given by,

\[
\frac{d\rho}{dt} = \gamma_{\text{amp}}(t)(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-\}, \rho)),
\]

where \(\sigma_+ = |1\rangle\langle 0|\) and \(\sigma_- = |0\rangle\langle 1|\). Let \(C_{\text{amp}}\) be the Choi matrix corresponding to the amplitude damping channel. Note that \(\text{Tr}[|\psi_{3,4}\rangle\langle \psi_{3,4}|C_{\text{amp}}] = \frac{\gamma_{\text{amp}}(t)}{4} \epsilon < 0\), when the dynamics is non-Markovian. Therefore, the Bell states \(|\psi_{3,4}\rangle\langle \psi_{3,4}|\) serve as non-Markovianity witness for amplitude damping channels.

**Thermal channel:** The Lindblad equation of thermal channel is given by,

\[
\frac{d\rho}{dt} = \gamma(t)(n + 1)(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-\}, \rho))
\]

\[
+\gamma(t)n(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+\}, \rho)),
\]

where \(n = \frac{1}{\exp(E/2T) - 1}\) (with \(E\) being the two-level energy gap) is known as Planck number. We note that \(\text{Tr}[|\psi_{3,4}\rangle\langle \psi_{3,4}|C_{\text{th}}] = (2n + 1)\frac{\gamma(t)}{4}\epsilon < 0\) when the dynamics is non-Markovian. Here, \(C_{\text{th}}\) stands for the corresponding Choi matrix. Therefore, \(|\psi_{3,4}\rangle\langle \psi_{3,4}|\) serves as the non-Markovianity witness for thermal channels.

Hence, it is important to note that given any unknown dynamics, if it falls under the category of dephasing, depolarizing or amplitude damping or thermal operations, one can detect its non-Markovianity without knowing the operation itself. Given the dynamics, one has to apply it on one part of the maximally entangled state \(|\psi_1\rangle\langle \psi_1|\) and then the expectation values with respect to four maximally entangled Bell states have to be calculated. If the expectation values come out to be nonnegative for all value of \(t\) and \(\epsilon\), the dynamics is Markovian. On contrary, if one or more expectation values come to be negative, the dynamics is non-Markovian.

Thus, we have shown in this section that for a substantial number of different qubit channels, Bell measurement can reveal their non-Markovianity without knowing the channels completely. This gives us a huge operational advantage for detection of non-Markovianity.

4 Alternative construction of NM witness

In the above-mentioned procedure, the construction of witness depends on the eigenvalues of the corresponding Choi state. For a large-dimensional system, computing the eigenvalues and the corresponding projectors may be difficult. Therefore, we adopt an alternative formalism [48] based on the structure of \(P^\epsilon_\mathcal{M}\).

In order to construct the witness, we first need to show the existence of nearest Markovian Choi state corresponding to some non-Markovian Choi state. Consider \(\mathcal{N}\) to be a NM operation having Choi state \(C_\mathcal{N}\). The distance between \(C_\mathcal{N}\) and \(P^\epsilon_\mathcal{M}\) is
given by $M = \inf_{C_M \in \mathcal{F}_M} D(C_N | C_M)$, where $D(\cdot | \cdot)$ is any proper metric. We now prove the following theorem.

**Theorem 2** Corresponding to any non-Markovian Choi state, there always exists a nearest Markovian Choi state.

**Proof** Fixing a non-Markovian Choi state $C_N$, we define a function $g : \mathcal{F}_M \to \mathbb{R}$ by setting, $g(C_M) = D(C_N | C_M) \forall C_M \in \mathcal{F}_M$. Clearly, $g$ is a continuous function on the set $\mathcal{F}_M$. Moreover, since $\mathcal{F}_M$ is compact, $\exists C_{M_0} \in \mathcal{F}_M$ such that $g(C_{M_0}) = \inf_{C_M \in \mathcal{F}_M} g(C_M)$. Hence, infimum is achieved by some Markovian Choi state.

Having proved the existence of nearest Markovian Choi state, now it will be interesting to ask whether or under which condition the nearest Markovian Choi state corresponding to a non-Markovian Choi state is unique. Thus, it follows the next proposition.

**Proposition 2** Let $C_N$ be a non-Markovian Choi state. Then, $C_{M^*}$ is the unique nearest Markovian Choi state if and only if for all $C_M \in \mathcal{F}_M$, $\text{Tr}[(C_N - C_{M^*})(C_M - C_{M^*})] \leq 0$.

**Proof** To prove the sufficient part, let for all Markovian Choi state $C_M$, $\text{Tr}[(C_N - C_{M^*})(C_M - C_{M^*})] \leq 0$. Considering $\text{Tr}[(C_N - C_M)^2]$, and by adding and subtracting $C_{M^*}$, we get $\text{Tr}[(C_N - C_{M^*})^2] - \text{Tr}[(C_N - C_M)^2] \geq -2\text{Tr}[(C_N - C_{M^*})(C_M - C_{M^*})]$. Using the hypothesis, we have $\text{Tr}[(C_N - C_{M^*})^2] \geq \text{Tr}[(C_N - C_M)^2]$. This shows $C_{M^*}$ are the nearest Markovian Choi state corresponding to the non-Markovian Choi state $C_N$.

To prove the necessary part, let $C_{M^*}$ be the nearest Markovian Choi state corresponding to non-Markovian Choi state $C_N$. Then, $\text{Tr}[(C_N - C_{M^*})^2] \geq \text{Tr}[(C_N - C_M)^2]$ for any Markovian Choi state $C_M \in \mathcal{F}_M$. This implies $\text{Tr}[(C_N - C_{M^*})(C_M - C_{M^*})] \leq \frac{1}{2} \text{Tr}[(C_{M^*} - C_M)^2]$. Since $\mathcal{F}_M$ is convex, let $C_M = (1 - \mu)C_Q + \mu C_{M^*}$ with $0 < \lambda < 1$, where $C_Q \in \mathcal{F}_M$. Therefore, $\text{Tr}[(C_N - C_{M^*})((1 - \mu)C_Q + \mu C_{M^*} - C_{M^*})] \leq \frac{1}{2} \text{Tr}[(1 - \mu)C_Q + \mu C_{M^*} - C_{M^*})^2]$, which gives the inequality $\text{Tr}[(C_N - C_{M^*})(C_Q - C_{M^*})] \leq \frac{1}{2} (1 - \mu)\text{Tr}[(C_{M^*} - C_Q)^2]$. Letting $\mu \to 1$, we have the result.

To prove the uniqueness of $C_{M^*}$, let $C_{M\infty}$ be another Markovian Choi state which minimizes $\text{Tr}[(C_N - C_M)^2]$. Then, $\text{Tr}[(C_N - C_{M^*})(C_{M\infty} - C_{M^*})] \leq 0$ and $\text{Tr}[(C_N - C_{M\infty})(C_M^* - C_{M\infty})] \leq 0$ together implies $C_{M^*} = C_{M\infty}$, proving the uniqueness of the nearest Markovian Choi state.

Armed with this proposition, we now prove the following theorem.

**Theorem 3** Let $C_{M^*}$ be the nearest Markovian Choi state to a non-Markovian Choi state $C_N$. Then,

$$\mathcal{W} = c_0 I + C_{M^*} - C_N,$$

where $c_0 = \text{Tr}(C_{M^*}(C_N - C_{M^*}))$ is a NM witness for $C_N$.  

\[ \square \]
Proof We verify that $\text{Tr}[W_{C_S}] = -\text{Tr}[(C_S - C_{M^*})(C_N - C_{M^*})]$ for any Choi state $C_S$. To prove $W$ is a NM witness for $C_N$, it is enough to show 1) $W$ is Hermitian and 2) $\text{Tr}[W_{C_M}] \geq 0 \forall C_M \in F_{\epsilon M}$ and $\text{Tr}[W_{C_N}] < 0$.

$W$ is Hermitian according to its definition. Since $\text{Tr}[W_{C_S}] = -\text{Tr}[(C_S - C_{M^*})(C_N - C_{M^*})]$ holds for any Choi state, it also holds for any $C_M$ in $F_{\epsilon M}$.

Since $C_{M^*}$ is the nearest Markovian Choi state for $C_N$, we have $\text{Tr}[(C_N - C_{M^*})(C_M - C_{M^*})] \leq 0$. Hence, it follows that

$$\text{Tr}[W_{C_M}] \geq 0 \forall C_M \in F_{\epsilon M}.$$ 

Using the trace preservation property of quantum operations, we get

$$\text{Tr}[W_{C_N}] = -\text{Tr}(C_N - C_{M^*})^2 < 0.$$ 

Having constructed the theory of linear witnesses to detect NM dynamics, we ask the immediate following question that whether linear witnesses are sufficient to determine all the NMCS. In the theory of entanglement detection, we know that nonlinear improvement of witnesses gives us further advantages to detect entanglement [49–51]. In the following, we discuss such possibilities for NM detection.

5 Existence of nonlinear witnesses

After constructing the structure of linear witnesses to detect the NMCS, a very legitimate question should be the following. How many linear witnesses are enough to capture all NMCS? To answer this question, we need to investigate the geometry of $F_{\epsilon M}$, i.e., precisely whether the set $F_{\epsilon M}$ forms a polytope determined by intersection of finitely many half-spaces obtained from linear witnesses. In analogy to Ref. [52], we surmise that these finitely many witnesses are tangents to $F_{\epsilon M}$. Minkowski’s theorem [53] states that every polytope in $\mathbb{R}^n$ is the convex hull of finitely many extreme points. Therefore, if there exist finitely many extreme points for a given convex and compact set, finitely many linear witnesses will suffice for separating all the entities outside that given set. The task is now to determine the number of extreme points of the set $F_{\epsilon M}$. We resolve this issue by proving the following theorem.

Theorem 4 The convex compact set of all MCS does not form a polytope.

Proof Extreme points of a convex set are those entities within the set, having no non-trivial decompositions in terms of convex combinations of other points in the given set. Since the MCS are always valid physical states, the pure states, if there any, will lie on the vertices of the set. Hence, to prove the theorem, it is enough to show that there exist uncountable many pure Choi states. Consider the set of all unitary channels: $\{U_\alpha(t_2, t_1) \mid \alpha \in \mathcal{I}\}$, where $\mathcal{I}$ is an uncountable index set. For each $\alpha$, $U_\alpha$ is a superoperator whose action is given by $U_\alpha(\rho) = U_\alpha \rho U_\alpha^\dagger$, with $U_\alpha$ being unitary operators. Unitary operations are divisible, i.e., $U_\alpha(t_3, t_1) = U_\alpha(t_3, t_2) \circ U_\alpha(t_2, t_1)$, with $t_3 \geq t_2 \geq t_1$, $\forall t_1, t_2, t_3$. 

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For a more detailed description, we consider unitary evolution $U_\alpha$ with unitary operator $U_\alpha$ and time-independent Hamiltonians $H_\alpha$, as $U_\alpha(t_2, t_1) = \exp \left(-i (t_2 - t_1) H_\alpha \right)$. Under small time interval approximation, $U_\alpha(t_1 + \epsilon, t_1) \approx I + \epsilon H_\alpha$.

It is straightforward to check that the evolution $\rho(t_1 + \epsilon) = U_\alpha(t_1 + \epsilon, t_1) \rho(t_1) U_\alpha^\dagger(t_1 + \epsilon, t_1)$ is divisible. Thus, the Choi state corresponding to some unitary operation $U_\alpha_0(t_2, t_1)$, given by $C_{U_\alpha_0}^{\ell_d} = I \otimes U_\alpha_0(t + \epsilon, t) \langle \phi | \langle \phi |$, is a pure maximally entangled state, and hence, it is an extreme point of $F^e_M$. Since $I$ is uncountable, there exist uncountably many such pure maximally entangled states for given dimensions. Therefore, there are uncountably many extreme points of $F^e_M$, and hence, it does not form a polytope.

As a consequence of Theorem 4, we surmise the importance of nonlinear witnesses, to improve upon the efficiency of its linear counterpart. Since $F^e_M$ does not form a polytope, finite number of hyperplanes will not be sufficient to detect all the NMCS. Therefore, nonlinear improvement of witnesses is necessary to detect them. In Fig. (2), we schematically justify the necessity of nonlinear witnesses.

From the above discussion, we have found that some of the pure maximally entangled states are in the set of all extreme points $\text{Ext}(\bar{F}^e_M)$ of the set $\bar{F}^e_M$. We know that they are also among the extreme points of the state space $S$. This fact tells us that $\text{Ext}(S) \cap \text{Ext}(\bar{F}^e_M)$ is non-empty. But it is also evident that not all the pure maximally entangled states are in $\text{Ext}(\bar{F}^e_M)$. This is because of the fact that the map is locally applied on one side of a bipartite maximally entangled state to construct the Choi states. It is therefore clear that the maximally entangled states generated by applying local unitaries on the other side will not be among the set of Choi states. Those states, though among the extreme points of $S$, will not be in $\bar{F}^e_M$. The most obvious open question is then whether $\text{Ext}(\bar{F}^e_M)$ is a strict subset of $\text{Ext}(S)$. However, we make the conjecture that this is not the case. The argument behind this statement is the
Fig. 3 Here, we represent the convex sets $S$ and $C_A$. Their intersection is another convex set $F^e_M$. We can also understand from the diagram that some of the extreme points of $F^e_M$ are mixed density matrices (Color figure online)

following. Since $F^e_M \subset S$, there are valid physical states not contained in $F^e_M$. We have already proved that there exist pure states not contained in $Ext(F^e_M)$. Therefore, there can be mixed Choi states having no non-trivial state decomposition in terms of the pure states in $Ext(F^e_M)$, though they always have the same in terms of pure states which are in $Ext(S)$.

It can be shown in a similar procedure of the proof of Proposition 1 that the set of all Choi states $C_A$ is also a convex set under the small time interval approximation. $F^e_M$ is of course a strict subset of $C_A$. We have shown earlier that $F^e_M$ is also a strict subset of $S$. The set $F^e_N = C_A \setminus F^e_M$ contains all the NMCS. Clearly, all the elements of $F^e_N$ are not valid quantum states, because NM operations break CP divisibility. Therefore, it is evident that $F^e_M = S \cap C_A$. In Fig. (3), we depict this discussion schematically.

6 Conclusion

In this paper, we develop a proper theory of NM witnesses, based on the convex and compact structure of the set of all MCS, under small time interval approximation. We construct an experimentally feasible framework of detecting non-Markovianity by Hermitian witnesses, in snapshots taken at different temporal regions for a given quantum evolution. Our method of NM detection has a clear advantage over any distance-based NM detection method, because we can witness eternal non-Markovianity, which cannot be detected by most of the usual distance-based measures. Furthermore, we have devised another method of detection NM of various qubit channels, by using projective measurements in the Bell basis. This can be done without the knowledge of what specific kind of channel we are dealing with. This gives us a huge operational advantage for NM detection. We further investigate the geometric structure of the set of all MCS to find that they do not form a polypolytope, which opens up the possibility to consider nonlinear improvement of NM witnesses as a future line of investigation in the field of non-Markovian open-system dynamics. Non-convexity of
the divisible maps has till date constrained the research in the theory and applications of quantum dynamical maps. This is one of the very first of such works, which opens up the possibility to utilize convex analysis in the theory and applications of open quantum systems. The theoretical construction presented in this work may therefore lead to rich possible implications, opening up a unique way to consolidate the theory of quantum non-Markovianity.

Lastly, we would like to mention one point. Quantum non-Markovianity is defined from various aspects [54]. We have considered non-Markovianity via divisibility of the corresponding dynamical map. This approach is sometimes called RHP non-Markovianity. Another aspect of non-Markovianity is from state distinguishability under the dynamical map which is commonly known as BLP non-Markovianity. It is evident that if a dynamics is Markovian with respect to RHP sense, it is also Markovian in BLP sense. Hence, BLP non-Markovian operations are contained in the set of RHP non-Markovian operations. But there can be such operations, which are not divisible, but do not show information backflow, e.g., some eternal non-Markovian operations [44]. But there is way to make the RHP and BLP measure equivalent [55]. Therefore, though our work is done in the backdrop of RHP protocol, there can be a similar approach from BLP perspective. For further studies, it will be interesting to explore the connection between our approach and other aspects of non-Markovianity.

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