Admissibility of invariant tests for means with covariates

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For a multinormal distribution with a $p$-dimensional mean vector $\theta$ and an arbitrary unknown dispersion matrix $\Sigma$, Rao ([9], [10]) proposed two tests for the problem of testing $H_0 : \theta_1 = \theta_2 = 0, \Sigma$ unspecified, versus $H_1 : \theta_1 \neq 0, \theta_2 = 0, \Sigma$ unspecified, where $\theta' = (\theta_1', \theta_2')$. These tests are referred to as Rao’s $W$-test (likelihood ratio test) and Rao’s $U$-test (union-intersection test), respectively. This work is inspired by the well-known work of Marden and Perlman [6] who claimed that Hotelling’s $T^2$-test is admissible while Rao’s $U$-test is inadmissible. Both Rao’s $U$-test and Hotelling’s $T^2$-test can be constructed by applying the union-intersection principle that incorporates the information $\theta_2 = 0$ for Rao’s $U$-test statistic but does not incorporate it for Hotelling’s $T^2$-test statistic. Rao’s $U$-test is believed to exhibit some optimal properties. Rao’s $U$-test is shown to be admissible by fully incorporating the information $\theta_2 = 0$, but Hotelling’s $T^2$-test is inadmissible.

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1. Introduction

Let \( \{ \mathbf{X}_i; 1 \leq i \leq n \} \) be independent and identically distributed random vectors (i.i.d.r.v.) with a \( p \)-variate normal distribution with mean vector \( \mathbf{\theta} \) and dispersion matrix \( \mathbf{\Sigma} \), where \( \mathbf{\Sigma} \) is assumed to be positive definite (p.d.). Partition \( \mathbf{\theta} \) and \( \mathbf{\Sigma} \) as

\[
\mathbf{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},
\]

(1.1)

where \( \theta_1 : p_1 \times 1, \theta_2 : p_2 \times 1, \Sigma_{11} : p_1 \times p_1, \Sigma_{22} : p_2 \times p_2, p_1 + p_2 = p, 0 < p_2 < p \). The problem of interest is to test

\[
H_0 : \theta_1 = 0, \theta_2 = 0, \quad \Sigma \text{ unspecified}
\]

versus

\[
H_1 : \theta_1 \neq 0, \theta_2 = 0, \quad \Sigma \text{ unspecified}.
\]

(1.2)

For every \( n (\geq 2) \), let

\[
\bar{\mathbf{X}} = n^{-1} \sum_{i=1}^{n} \mathbf{X}_i \quad \text{and} \quad \mathbf{S} = \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})',
\]

(1.3)

and express Hotelling’s \( T^2 \)-statistic as

\[
T^2 = n(n-1) \bar{\mathbf{X}}' \mathbf{S}^{-1} \bar{\mathbf{X}}.
\]

(1.4)

Partition \( \bar{\mathbf{X}} \) and \( \mathbf{S} \) similarly as in (1.1), and define

\[
\bar{\mathbf{X}}_{1:2} = \bar{\mathbf{X}}_1 - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{X}_2,
\]

(1.5)

\[
\mathbf{S}_{11:2} = \mathbf{S}_{11} - \mathbf{S}_{12} \mathbf{S}_{22}^{-1} \mathbf{S}_{21}.
\]

(1.6)

For the problem (1.2), Rao ([9], [10]) proposed two test statistics which are of the forms

\[
W = \frac{n(n-1) \bar{\mathbf{X}}_{1:2} \mathbf{S}_{11:2}^{-1} \bar{\mathbf{X}}_{1:2}}{1 + n(n-1) \bar{\mathbf{X}}_2 \mathbf{S}_{22}^{-1} \bar{\mathbf{X}}_2}
\]

(1.7)

and

\[
U = n(n-1) \bar{\mathbf{X}}_{1:2}' \mathbf{S}_{11:2}^{-1} \bar{\mathbf{X}}_{1:2}
\]

(1.8)

respectively. In the literature, these two test statistics are called Rao’s \( W \) and \( U \) statistics, respectively. The test statistics \( W \) is derived by the likelihood ratio principle. Marden and Perlman [6] showed that for problem (1.2) both Rao’s \( W \)-test and Rao’s \( U \)-test are similar and unbiased.
The invariance of problem (1.2) under a group $G$ of linear transformations, where $G$ is the group of $p \times p$ nonsingular matrices of the form

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

(1.9)

with $g_{11} : p_1 \times p_1$ and $g_{22} : p_2 \times p_2$, can be exploited so that the group $G$ acts on the sample space via $g : (\bar{X}, S) \rightarrow (g\bar{X}, gSg')$, and on the parameter space via $g : (\theta, \Sigma) \rightarrow (g\theta, g\Sigma g')$. Let $\theta_{1:2}$ and $\Sigma_{11:2}$ be defined similarly as in (1.5) and (1.6) but such that $\theta$ and $\Sigma$ replace $\bar{X}$ and $S$, respectively. Adopting the notions of Marden and Perlman [6], the maximal invariant statistic is the pair $(L(1 + M), M)$ with $L = W$ defined as in (1.7) and $M = n(n - 1)\bar{X}_2\Sigma_{22}^{-1}\bar{X}_2$, and correspondingly, the maximal invariant parameter is the pair $(\Delta_1, \Delta_2)$ with $\Delta_1 = n\theta_{1:2}'\Sigma_{11:2}^{-1}\theta_{1:2}$ and $\Delta_2 = n\theta_{2}'\Sigma_{22}^{-1}\theta_2$.

Using only $G$-invariant tests, Marden and Perlman ([6], p. 27) concluded that the problem (1.2) reduces to that of testing

$$H_0^I : \Delta = 0 \text{ versus } H_1^I : \Delta > 0$$

(1.10)

based on $(L, M)$, where $\Delta = n\theta_{1:2}'\Sigma_{11:2}^{-1}\theta_1$. Marden and Perlman [6] established the necessary and sufficient conditions for the admissibility of problem (1.10). They considered the homeomorphic transformations of $(L, M)$, and inferred that Rao’s $U$-test is inadmissible when only $G$-invariant tests are applied to problem (1.2), but the overall Hotelling $T^2$-test is admissible. However, their conclusions are against our statistical intuition. Also, their simulation studies (see Tables 4.1a-4.1c) indicate a totally different story. As such, this work attempts to clarify these phenomena.

First, note that if only the $G$-invariant tests are applied, then problem (1.2) does not reduce to that of testing problem (1.10) but should reduce to that of testing

$$H_0^{I*} : \Delta_1 = 0, \Delta_2 = 0 \text{ versus } H_1^{I*} : \Delta_1 > 0, \Delta_2 = 0.$$  
(1.11)

Moreover, problems (1.11) and (1.10) are not identical. Problem (1.11) is easily seen to imply problem (1.10), but not vice versa. Problem (1.10) can be regarded as the union of problem (1.11) and the following subproblems: (i). $H_0 : \Delta_1 = 0, \Delta_2 = 0$ versus $H_1 : \Delta_1 > 0, \Delta_2 > 0$, (ii). $H_0 : \Delta_1 = 0, \Delta_2 > 0$ versus $H_1 : \Delta_1 > 0, \Delta_2 = 0$, and (iii). $H_0 : \Delta_1 = 0, \Delta_2 > 0$ versus $H_1 : \Delta_1 > 0, \Delta_2 > 0$. Problem (1.11) provides more insight than does problem (1.10) into $\Delta_2 = 0$ both in the null hypothesis and in the alternative hypothesis. In fact, problem (1.10) is a two-dimensional testing problem in which $\Delta_2$ is the nuisance parameter, while problem (1.11) is a one-dimensional testing problem. Problem (1.11) is a subproblem.
of problem (1.10), so intuitively the optimal tests for problem (1.11) (which is equivalent to the hypothesis testing problem (1.2)) are not necessarily optimal for problem (1.10), and vice versa. Marden and Perlman [6] also clearly made this point (for details see the last three lines of page 28 of Marden and Perlman [6]). Therefore, based on the optimal criterions set up for the two-dimensional testing problem (1.10), to infer problem (1.11), which is only a one-dimensional testing problem, the conclusions drawn may provide misleading messages. To ensure the information $\mu_2 = 0$ (i.e., $\Delta_2 = 0$) being incorporated for problem (1.10), Marden and Perlman [6] further made an assumption that $M$ is an ancillary statistic. Because the density function of the ancillary statistic $M$ does not depend on the parameter $\Delta_2$ both under the null hypothesis and under the alternative hypothesis, and hence the whole statistical inference should depend on the conditional density function of $L$ given $M$, which is a noncentral $F$-type distribution, in their set up. However, for the case of Hotelling’s $T^2$-test, Marden and Perlman ([6], page 50) adopted the exponential family for the problem (1.10) [not for the problem (1.11)] set up for their statistical inference.

Let $Gl$ be the general linear group. The problem of testing $H^u_0 : \theta = 0$ versus $H^u_1 : \theta \neq 0$ is $Gl$-invariant. When only $Gl$-invariant tests are performed, this problem reduces to that of testing $H^L_{05} : \Delta^* = 0$, versus $H^L_{15} : \Delta^* > 0$, where $\Delta^* = \Delta_1 + \Delta_2$. For this $Gl$-invariant testing problem, Hotelling’s $T^2$-test is well-known to be the uniformly most powerful test (Simaika [14]), and so is admissible. Schwartz [13] employed the Birnbaum-Stein method (Birnbaum [2], Stein [15]) to study the admissibility of fully $Gl$-invariant tests in the multivariate analysis of variance setting. Problem (1.2) is not $Gl$-invariant, although it is $G$-invariant. Therefore, studying the power domination problems of Hotelling’s $T^2$-test, Rao’s $W$-test and Rao’s $U$-test for problem (1.2) via the fully $Gl$-invariant approach may not be helpful. The group $G$ is amenable and meets the conditions of the Hunt-Stein theorem (Lehmann [5]). Therefore, any minimax questions in problem (1.2) can be reduced by the group $G$.

Notably, problems (1.2) and (1.11) are the problems of testing against restricted alternatives. However, problem (1.10) is not such a problem. Therefore, by utilizing problem (1.10) to draw inferences concerning problem (1.2), we may overlook the intrinsic nature of the restricted alternative (because $\Delta_2 = 0$, which is determined directly from the basic assumption $\theta_2 = 0$) when applying results in the literature or developing new theories.

The exponential structure of the distribution of $(\bar{X}, S)$ is incorporated to generalize the Birnbaum-Stein method for problem (1.2). As a result, Section 2 presents two main results: the acceptance region of Rao’s $U$-test is convex, and Rao’s $U$-test is admissible. Section 3
applies Eaton’s [3] results to show that Hotelling’s $T^2$-test is inadmissible for problem (1.2). The discussion regarding the Rao $W$-test is given in Section 4. Some general remarks are given in the final section. The Appendix provides six lemmas to show the convexity of the acceptance region of Rao’s $U$-test.

2. Admissibility of Rao’s $U$-test

Stein [15] proved in detail that Hotelling’s $T^2$-test is admissible for the problem of testing $H^u_0: \theta = 0$ against the global alternative $H^u_1: \theta \neq 0$. This proof can also be found in Anderson’s book ([1], p. 188-190). The main purpose of this section is to incorporate the Birnbaum-Stein method to demonstrate that Rao’s $U$-test is admissible for problem (1.2). Whether the acceptance region of Rao’s $U$-test is a convex set must first be determined. Let

$$\mathcal{A}_U = \{ (\bar{X}, S) | n(n-1)\bar{X}'_1 S_{11:2}^{-1} \bar{X}_{1:2} \leq k, S \text{ is p.d.} \} \quad (2.1)$$

for a suitable $k$, and

$$B^+(S) = \begin{pmatrix} I \\ -S_{22}^{-1}S_{21} \end{pmatrix} S_{11:2}^{-1} \begin{pmatrix} I & -S_{12}S_{22}^{-1} \end{pmatrix} \quad (2.2)$$

$$= S^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & S_{22}^{-1} \end{bmatrix},$$

which is positive semi-definite (p.s.d.) and of rank $p_1$. Also let

$$B(S) = \begin{pmatrix} I \\ -S_{22}^{-1}S_{21} \end{pmatrix} (I + S_{12}S_{22}^{-1}S_{22}^{-1}S_{21})^{-1} S_{11:2} (I + S_{12}S_{22}^{-1}S_{22}^{-1}S_{21})^{-1} \begin{pmatrix} I & -S_{12}S_{22}^{-1} \end{pmatrix}.$$ \quad (2.3)

Let $C^+$ be the Moore-Penrose generalized inverse of $C$. Then,

$$B(S) = \begin{pmatrix} I & -S_{12}S_{22}^{-1} \end{pmatrix}^+ S_{11:2} \begin{pmatrix} I \\ -S_{22}^{-1}S_{21} \end{pmatrix}^+ \quad (2.4)$$

which is shown to be the Moore-Penrose generalized inverse of $B^+(S)$ in Lemma 2 of the Appendix, is easily established. For the notions related to the Moore-Penrose generalized inverse of matrices and matrix-convex (matrix-concave) functions, refer to Rao and Mitra [11] and Marshall and Olkin [7], respectively. The Appendix develops six lemmas related to the Moore-Penrose generalized inverse of matrices, matrix-convex and matrix-concave.
Anderson ([1], problem 17 of page 193) claimed that the acceptance region of Hotelling’s $T^2$-test

$$A_{T^2} = \{ (\bar{X}, S) \mid n(n-1)\bar{X}'S^{-1}\bar{X} \leq k_1 \}$$

is a convex set. \hfill (2.5)

Notably, by (2.1) and (2.2) the accepted region of Rao’s $U$-test $A_U$ can be rewritten as

$$A_U = \{ (\bar{X}, S) \mid n(n-1)\bar{X}'B^+(S)\bar{X} \leq k \},$$

where $B^+(S)$ is defined in (2.2). Let $A \succeq B$ denote that the matrix $A - B$ is p.s.d. throughout this paper. The lemmas developed in the Appendix are used to generalize Anderson’s result (2.5) to the following theorem.

**Theorem 1.** Let $S$ be the set of all $p \times p$ positive definite matrices. Then $A_U = \{ (\bar{X}, S) \mid n(n-1)\bar{X}'S^{-1}\bar{X} \leq k \}$ is convex on $R^p \times S$.

**Proof.** By (2.6), $A_U = \{ (\bar{X}, S) \mid n(n-1)\bar{X}'B^+(S)\bar{X} \leq k \}$. Let $B(S)$ be defined as in (2.3), then by Lemma 2 in the Appendix, it is the Moore-Penrose generalized inverse of $B^+(S)$. Lemma 4 shows that $B(S)$ is matrix concave on $S$, that is, $\forall \alpha \in (0, 1)$

$$B(\alpha S + (1-\alpha)T) \succeq \alpha B(S) + (1-\alpha)B(T).$$

Therefore, by Lemma 5,

$$B^+(\alpha S + (1-\alpha)T) \preceq (\alpha B(S) + (1-\alpha)B(T))^+. \hfill (2.8)$$

By the inequality (2.8) and Lemma 6,

$$(\alpha \bar{X} + (1-\alpha)\bar{Y})'B^+(\alpha S + (1-\alpha)T)(\alpha \bar{X} + (1-\alpha)\bar{Y}) \hfill (2.9)$$

$$\leq (\alpha \bar{X} + (1-\alpha)\bar{Y})'(\alpha B(S) + (1-\alpha)B(T))^+(\alpha \bar{X} + (1-\alpha)\bar{Y})$$

$$\leq \alpha X'B^+(S)X + (1-\alpha)Y'B^+(T)Y.$$

For the definitions of $\alpha$-admissible and $d$-admissible, we may refer to the page 306 of Lehmann [6].

**Remark.** For testing against global alternative, (i.e., $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$), since the Hotelling’s $T^2$-test statistic $n(n-1)\bar{X}'S^{-1}\bar{X}$ is the monotone function of $n(n-1)\bar{X}'W^{-1}\bar{X}$, where $W = S + n\bar{X}\bar{X}'$, Stein worked on the space $(\bar{X}, W)$. Theorem 5.6.6 of Anderson [1] does not require that it is necessary to work on the space $(\bar{X}, W)$. Note that for the testing $H_0 : \theta_1 = 0, \theta_2 = 0$ against the restricted alternative $H_1 : \theta_1 \neq 0, \theta_2 = 0$, the Rao’s $U$-test statistic $n(n-1)\bar{X}'B^+(S)\bar{X}$ is no longer to be a monotone function of $n(n-1)\bar{X}'B^+(W)\bar{X}$ any
more. To overcome the difficulty, we work on the space \((\bar{X}, S)\) instead. Let \(\mathcal{H}_c^* = \{ (\bar{X}, S) | n\theta\Sigma^{-1}\bar{X} - \frac{1}{2} \text{tr}(\Sigma^{-1}S) > c \} \). It is easy to note that \(\mathcal{H}_c^*\) is a half-space, and hence the assumption that the acceptance region \(A_U\) is disjoint with the half-space \(\mathcal{H}_c^*\) on the space \((\bar{X}, S)\) holds. Let

\[
\mathcal{H}_c^a = \{ (\bar{X}, S) | n\theta\Sigma^{-1}\bar{X} - \frac{1}{2} \text{tr}(\Sigma^{-1}(S + n\bar{X}X')) > c \}.
\]

Note that \((\bar{X}, S) \in \mathcal{H}_c^a\) implies that \((\bar{X}, S) \in \mathcal{H}_c^*\). Thus,

\[
\mathcal{H}_c^a \subset \mathcal{H}_c^*.
\]

And hence, the intersection of \(A_U\) and \(\mathcal{H}_c^a\) is also empty. As such, we may work the proof of Theorem 2 on the space \((\bar{X}, S)\).

**Theorem 2.** For the problem \((1.2)\), Rao’s \(U\)-test is \(\alpha\)-admissible.

**Proof.** The likelihood function of \(X_1, \cdots, X_n\) is

\[
e^{-\frac{1}{2}n\theta'\Sigma^{-1}\theta} \exp[n\theta'\Sigma^{-1}\bar{X} + \text{tr}(\frac{1}{2}\Sigma^{-1}\sum_{i=1}^n X_iX_i')].
\]

Let \(w = (w^{(1)'}, w^{(2)'})'\), where \(w^{(1)} = \Sigma^{-1}\theta\) and \(w^{(2)} = -\frac{1}{2}(\sigma^{11}, \cdots, \sigma^{1p}, \sigma^{22}, \cdots, \sigma^{pp})'\), where \((\sigma^{ij}) = \Sigma^{-1}\). Let \(w^{(1)'} = \nu' = (\nu'_1, \nu'_2)\). By Theorem 1, \(A_U\) is convex on \(R^p \times S\). Consider the other condition of theorem 5.6.5. (Anderson [1]), \(A_U\) is assumed to be disjoint with the subspace

\[
\mathcal{H}_c = \{ (\bar{X}, S) | n\nu'\bar{X} - \frac{1}{2} \text{tr}\Lambda(S + n\bar{X}X') > c \},
\]

where \(\Lambda\) is symmetric, for some \(c\).

Theorem 8 presented by Lehmann ([5], page 307) can be applied if \(w_0 + \lambda w_1 \in H_1\) can be demonstrated for \(\lambda > 0\), which can be accomplished with the following two steps: (I) \(I + \lambda \Lambda\) is p.d., and (II) \(\theta_0 + \lambda \theta \in H_1\) for \(\lambda > 0\).

(I). That \(I + \lambda \Lambda\) is p.d. is shown if \(\Lambda\) can be shown to be p.s.d., for \(\lambda > 0\). Suppose that \(\Lambda\) is not p.s.d., then by arguments similar to those in Anderson ([1], p. 189-190) it can be written

\[
\Lambda = D \begin{bmatrix}
I & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & 0
\end{bmatrix} D',
\]

where \(D\) is nonsingular. Let \(\bar{X} = (1/\gamma)X_0\) and

\[
S = (D')^{-1} \begin{bmatrix}
I & 0 & 0 \\
0 & \gamma I & 0 \\
0 & 0 & I
\end{bmatrix} D^{-1}.
\]
Then
\[
\begin{align*}
  n\nu'X - \frac{1}{2} \text{tr} \Lambda(S + nXX')
  &= \frac{n}{\gamma^2}X_0 - \frac{n}{2\gamma^2}X_0D \left[ I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] \left[ I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right]
  \left[ -I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] + \frac{1}{2} \text{tr} \left[ -I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] > c
\end{align*}
\] (2.14)

which is greater than c for sufficiently large γ. Thus, the subspace $H_c$ reduces to
\[
H_c^\gamma = \{ X_0 | \frac{n}{\gamma}\nu'X_0 - \frac{n}{2\gamma^2}X_0D \left[ I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] \left[ -I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] + \frac{1}{2} \text{tr} \left[ -I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] > c \} 
\] (2.15)

for sufficiently large γ. Obviously, $H_c^\gamma = R^p$ as γ approaches infinity. Now, let
\[
\mathcal{A}_{T2,k^*} = \{ (\bar{X}, S) | n(n-1)\bar{X}'(S + n\bar{X}\bar{X}')^{-1}\bar{X} \leq k^* \} 
\] (2.16)

with $k = k^*/(1 - k^*)$. Then, (2.16) reduces to
\[
\mathcal{A}_{T2,k^*}^\gamma = \{ X_0 | \frac{n(n-1)}{\gamma^2}X_0D \left[ I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] \left[ -I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] + \frac{1}{2} \text{tr} \left[ -I \ 0 \ 0 \\
  &\quad 0 \ -I \ 0 \\
  &\quad 0 \ 0 \ 0 \right] \leq k \}, 
\] (2.17)

for sufficiently large γ. It can be easily seen that $H_c^\gamma \cap \mathcal{A}_{T2,k^*}^\gamma \neq \emptyset$ for sufficiently large γ. Furthermore,
\[
\mathcal{A}_{T2,k^*} = \{ (\bar{X}, S) | n(n-1)\bar{X}'S^{-1}\bar{X} \leq k \} 
\] (2.18)

with $k = k^*/(1 - k^*)$. Then, (2.18) reduces to
\[
\mathcal{A}_{T2,k^*} = \{ (\bar{X}, S) | n(n-1)\bar{X}'S^{-1}\bar{X} \leq k \}
\]

Thus, if $\Lambda$ is not p.s.d., then
\[
\mathcal{A}_U \cap H_c \neq \emptyset, 
\] (2.19)
which leads to a contradiction. Therefore, $\Lambda$ is p.s.d..

To proceed with step (II), note that $I + \Lambda$ is p.d., and without loss of generality, its inverse can be denoted by $\Sigma$, so $(I + \Lambda)^{-1} = \Sigma$. Some more notation is needed. Let

$$g = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}, \quad (\theta, \Sigma) \xrightarrow{g} (g\theta, g\Sigma g')$$  \hspace{1cm} (2.20)

and write

$$\bar{\Sigma} = g\Sigma g'$$  \hspace{1cm} (2.21)

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11:2} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}.$$  

Notably, $\bar{\Sigma} = \bar{\Sigma}'$. Let $Z = g\bar{X}$ and $S_0 = gSg'$. Then,

$$Z = \begin{pmatrix} \bar{X}_1 - \Sigma_{12}\Sigma_{22}^{-1}\bar{X}_2 \\ \bar{X}_2 \end{pmatrix}$$  \hspace{1cm} (2.22)

$$S_0 = \begin{bmatrix} S_{11} - \Sigma_{12}\Sigma_{22}^{-1}S_{21} - \Sigma_{12}\Sigma_{22}^{-1}S_{21} + \Sigma_{12}\Sigma_{22}^{-1}S_{22}\Sigma_{22}^{-1}\Sigma_{21} & S_{12} - \Sigma_{12}\Sigma_{22}^{-1}S_{22} \\ S_{21} - S_{22}\Sigma_{22}^{-1}\Sigma_{21} & S_{22} \end{bmatrix}$$  \hspace{1cm} (2.23)

and

$$\text{tr} \Sigma^{-1}(S + n\bar{X}\bar{X}')$$  \hspace{1cm} (2.24)

$$= \text{tr}(g\Sigma g')^{-1}g(S + n\bar{X}\bar{X}')g'$$

$$= \text{tr} \bar{\Sigma}^{-1}(S_0 + n\bar{Z}\bar{Z}'),$$

\[ \nu'\bar{X} = \tilde{\nu}'Z, \]  

where

$$\tilde{\nu} = (g^{-1})'\nu$$

$$= \begin{pmatrix} \Sigma_{11:2}^{-1}(\theta_1 - \Sigma_{12}\Sigma_{22}^{-1}\theta_2) \\ \Sigma_{22}^{-1}\theta_2 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{pmatrix}. \]
Thus, the subspace $H^a_c$ becomes

$$H^a_c = \{ (\bar{X}, S) \mid n\tilde{\nu}'Z - \frac{1}{2} \text{tr} \tilde{\Sigma}^{-1}(S_0 + nZZ') > c \}.$$  \hfill (2.26)

Equation (2.25) indicates that the hypothesis testing problem (1.2) $H_0 : \theta_1 = 0, \theta_2 = 0$ versus $H_1 : \theta_1 \neq 0, \theta_2 = 0$ is equivalent to the hypothesis testing problem

$$H^*_0 : \nu_1 = 0, \nu_2 = 0 \text{ versus } H^*_1 : \nu_1 \neq 0, \nu_2 = 0.$$  \hfill (2.27)

And hence to show that $\theta_0 + \lambda \theta \in H_1$, $\lambda > 0$ for problem (1.2) is equivalent to showing that $\nu_0 + \lambda \tilde{\nu} \in H_1$, $\lambda > 0$ for problem (2.27). Next, step (II) is considered.

(II). To show that $\nu_0 + \lambda \tilde{\nu} \in H_1$ for $\lambda > 0$, the aim is to demonstrate that $\tilde{\nu}_1 \neq 0$ and $\tilde{\nu}_2 = 0$. If the statement that $\nu_1 \neq 0$ and $\nu_2 = 0$ is not true, then, there are three cases (i) $\tilde{\nu}_1 \neq 0$, $\tilde{\nu}_2 \neq 0$, (ii) $\tilde{\nu}_1 = 0$, $\tilde{\nu}_2 \neq 0$ and (iii) $\tilde{\nu}_1 = 0$, $\tilde{\nu}_2 = 0$. We assume that (i), (ii) and (iii) are true, and then show that those assumptions lead to contradictions. To proceed, it is enough to consider the situation that $S = \Sigma$.

Note that given $S = \Sigma$, then

$$S_0 = \begin{bmatrix} \Sigma_{11:2} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}.$$  \hfill (2.28)

Accordingly, both sets $A_U$ and $H^a_c$ are reduced to p-dimensional sets (using the notation loosely)

$$A_U = \{ Z \mid n(n - 1)Z_1'\Sigma_{11:2}^{-1}Z_1 \leq k \}$$  \hfill (2.29)

and

$$H^a_c = \{ Z \mid n\tilde{\nu}'Z - \frac{P}{2} - \frac{n}{2}Z_1'\Sigma_{11:2}^{-1}Z_1 - \frac{n}{2}Z_2'\Sigma_{22}^{-1}Z_2 > c \},$$  \hfill (2.30)

respectively. Notably, given that $S = \Sigma$, the assumption that the sets $A_U$ and $H^a_c$ are disjoint still holds.

First, (i) $\tilde{\nu}_1 \neq 0$, $\tilde{\nu}_2 \neq 0$ is assumed. In the problem of testing $H^1_0 : \nu_1 = 0$ versus $H^1_1 : \nu_1 \neq 0$, whenever $\tilde{\nu}_1 \neq 0$, then there exists a constant $c_1$ which does not depend on $Z_1$ and $\Sigma_{11:2}$ such that

$$H^a_{c_1} \cap A_{T^2,k^1_1} = \emptyset \text{ and } H^a_{c_1-\epsilon} \cap A_{T^2,k^1_1} \neq \emptyset$$  \hfill (2.31)

for any $\epsilon > 0$, where

$$H^a_{c_1} = \{ Z_1 \mid n\tilde{\nu}_1'Z_1 - \frac{p_1}{2} - \frac{p_2}{2}Z_1'\Sigma_{11:2}^{-1}Z_1 > c_1 + \frac{p_2}{2} \}.$$  \hfill (2.32)
and
\[ A_{T^2,k_1^*} = \{ Z_1 \mid n(n-1)Z_1'(\Sigma_{11:2} + nZ_1Z_1')^{-1}Z_1 \leq k_1^* \} \]
\[ = \{ Z_1 \mid n(n-1)Z_1\Sigma_{11:2}^{-1}Z_1 \leq k_1^* \}. \] (2.33)

This is equivalent to the existence of a constant \( c \) such that
\[ H_c^{\alpha*} \cap A_{T^2,k^*} = \emptyset \quad \text{and} \quad H_{c-\epsilon}^{\alpha*} \cap A_{T^2,k^*} \neq \emptyset \] (2.34)
for any \( \epsilon > 0 \). Rewrite \( H_c^{\alpha} \) as
\[ H_c^{\alpha} = \{ Z \mid n\tilde{\nu}'_1Z_1 - \frac{p}{2} - \frac{n}{2}Z_1'\Sigma_{11:2}^{-1}Z_1 + \frac{n}{2}[\tilde{\nu}'_2\Sigma_{22}\tilde{\nu}_2 - (Z_2 - \Sigma_{22}\tilde{\nu}_2)'\Sigma_{22}^{-1}(Z_2 - \Sigma_{22}\tilde{\nu}_2)] > c \}. \] (2.35)

Consider
\[ \epsilon(Z_2) = \frac{n}{2}[\tilde{\nu}'_2\Sigma_{22}\tilde{\nu}_2 - (Z_2 - \Sigma_{22}\tilde{\nu}_2)'\Sigma_{22}^{-1}(Z_2 - \Sigma_{22}\tilde{\nu}_2)] > 0. \] (2.36)

Then, from equation (2.34),
\[ H_{c-\epsilon}(Z_2) \cap A_{T^2,k^*} \neq \emptyset, \] (2.37)
where
\[ H_{c-\epsilon}(Z_2) = \{ Z_1 \mid n\tilde{\nu}'_1Z_1 - \frac{p}{2} - \frac{n}{2}Z_1'\Sigma_{11:2}^{-1}Z_1 + \frac{n}{2}[\tilde{\nu}'_2\Sigma_{22}\tilde{\nu}_2 - (Z_2 - \Sigma_{22}\tilde{\nu}_2)'\Sigma_{22}^{-1}(Z_2 - \Sigma_{22}\tilde{\nu}_2)] > c \}. \] (2.38)

Hence,
\[ H_{c-\epsilon}(Z_2) \cap (A_{T^2,k^*} \times \mathbb{R}^{p^2}) \neq \emptyset. \] (2.39)

Let
\[ B(\Sigma_{22}\tilde{\nu}_2) = \{ Z_2 \mid (Z_2 - \Sigma_{22}\tilde{\nu}_2)'\Sigma_{22}^{-1}(Z_2 - \Sigma_{22}\tilde{\nu}_2) \leq \tilde{\nu}'_2\Sigma_{22}\tilde{\nu}_2 \}. \] (2.40)
\( \Sigma_{22}\tilde{\nu}_2 \in B(\Sigma_{22}\tilde{\nu}_2) \), so \( B(\Sigma_{22}\tilde{\nu}_2) \neq \emptyset \); and \( Z_2 \in B(\Sigma_{22}\tilde{\nu}_2) \) implies that \( \epsilon(Z_2) > 0 \). Let
\[ S_c = \bigcup_{Z_2 \in B(\Sigma_{22}\tilde{\nu}_2)} H_{c-\epsilon}(Z_2). \] (2.41)

Now, (2.39) implies
\[ S_c \cap (A_{T^2,k^*} \times \mathbb{R}^{p^2}) \neq \emptyset. \] (2.42)
Notably,
\[
S_c = \bigcup_{\boldsymbol{Z}_1 \in \mathbb{B}(\Sigma_2 \nu_2)} \{ \boldsymbol{Z}_1 \mid n \nu_1' \boldsymbol{Z}_1 - \frac{p}{2} - \frac{n}{2} \boldsymbol{Z}_1' \Sigma_{11:2}^{-1} \boldsymbol{Z}_1 \\
+ \frac{n}{2} [\nu_2' \Sigma_{22} \nu_2 - (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)\Sigma_{22}^{-1} (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)] > c \}
\]

\[
= \{ \boldsymbol{Z} \mid n \nu_1' \boldsymbol{Z}_1 - \frac{p}{2} - \frac{n}{2} \left( \Sigma_{11:2}^{-1} \boldsymbol{Z}_1 + \frac{n}{2} [\nu_2' \Sigma_{22} \nu_2 - (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)\Sigma_{22}^{-1} (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)] \right) > c \}
\]

and \( \nu_2' \Sigma_{22} \nu_2 \geq (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)\Sigma_{22}^{-1} (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2) \)

\[
\subseteq \{ \boldsymbol{Z} \mid n \nu_1' \boldsymbol{Z}_1 - \frac{p}{2} - \frac{n}{2} \left( \Sigma_{11:2}^{-1} \boldsymbol{Z}_1 + \frac{n}{2} [\nu_2' \Sigma_{22} \nu_2 - (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)\Sigma_{22}^{-1} (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)] \right) > c \}
\]

\[
= \mathcal{H}_c^a.
\]

Thus,
\[
\mathcal{H}_c^a \cap (A_{T^2,k^*} \times \mathbb{R}^{p_2}) \neq \emptyset.
\]

Notably,
\[
A_{T^2,k^*} \times \mathbb{R}^{p_2} = \{ \boldsymbol{Z} \mid n(n-1)\Sigma_{11:2}^{-1} \boldsymbol{Z}_1 \leq k \} = A_U.
\]

Namely, if \( \nu_1 \neq 0 \) and \( \nu_2 \neq 0 \), then \( \mathcal{H}_c^a \cap A_U \neq \emptyset \). This implies that \( \mathcal{H}_c \cap A_U \neq \emptyset \), and leads to a contradiction.

Next, (ii) \( \nu_1 = 0, \nu_2 \neq 0 \) is assumed. Then, the set \( \mathcal{H}_c^a \) in equation (2.35) reduces to
\[
\mathcal{H}_c^a = \{ \boldsymbol{Z} \mid n\Sigma_{11:2}^{-1} \boldsymbol{Z}_1 + n(\boldsymbol{Z}_2 - \Sigma_{22} \nu_2)\Sigma_{22}^{-1} (\boldsymbol{Z}_2 - \Sigma_{22} \nu_2) < n\nu_2' \Sigma_{22} \nu_2 - p - 2c \}
\]

In passing, the condition \( n\nu_2' \Sigma_{22} \nu_2 - p - 2c > 0 \) is needed to ensure that \( \mathcal{H}_c^a \) is not an empty set. Consider \( \boldsymbol{Z}_2 = \Sigma_{22} \nu_2 \); notably, (a) \( \mathcal{H}_c^a \) is not an empty set and (b) \( \mathcal{H}_c^a \cap A_U \neq \emptyset \), which implies that \( \mathcal{H}_c \cap A_U \neq \emptyset \). This leads to a contradiction.

Finally, in case (iii) \( \nu_1 = 0 \) and \( \nu_2 = 0 \). Then, the set \( \mathcal{H}_c^a \) in equation (2.35) reduces to
\[
\mathcal{H}_c^a = \{ \boldsymbol{Z} \mid n\Sigma_{11:2}^{-1} \boldsymbol{Z}_1 + n\Sigma_{22}^{-1} \boldsymbol{Z}_2 < -p - 2c \}
\]

In this case, \( p + 2c < 0 \) is required to ensure that \( \mathcal{H}_c^a \) is not an empty set. That \( \mathcal{H}_c^a \cap A_U \neq \emptyset \) can be easily seen, so \( \mathcal{H}_c \cap A_U \neq \emptyset \). This leads to a contradiction.

The discussions of (i), (ii) and (iii) can be taken together to imply that
\[
\nu_1 \neq 0 \quad \text{and} \quad \nu_2 = 0.
\]

Therefore, Rao’s \( U \)-test satisfies the conditions of Theorem 8 of Lehmann ([5], pages 307). Marden and Perlman [6] have shown that Rao’s \( U \)-test is similar and unbiased, and the theorem follows by Corollary 2 of Lehmann ([5], page 308).
3. Inadmissibility of Hotelling’s $T^2$-test

Marden and Perlman ([6], p. 49) pointed out that, “by utilizing the exponential structure of the distribution of $(\bar{X}, S)$, the method of Stein [15] and Schwartz [13] can be applied to reveal the overall $T^2$ test is admissible for problem (1.2). Based on the logarithm of the joint density of $(\bar{X}, S)$, Marden and Perlman ([6], pages 49-50) claimed that, according to the theorem of Stein [15], the set (in our notation)

$$\left\{ (X, S) \mid \sup_{\theta \in \Theta_2} -\frac{1}{2} n(\bar{X} - \theta)' \Sigma^{-1}(\bar{X} - \theta) - \frac{1}{2} (n-1)[\text{tr } \Sigma^{-1} S/(n-1) - \ln |\Sigma^{-1} S/(n-1)|] \leq c \right\}$$

(3.1)

is an admissible acceptance region in problem (1.2) for any subset $\Theta_2 \subset \Theta_1$, where $\Theta_1 = \{ (\theta, \Sigma) \mid \theta \neq 0, \Sigma \text{ p.d.} \}$. Note that the notion in (3.1) is essentially the same as the one presented in Marden and Perlman ([6], page 50), but omits the terms for which the parameters and statistics can be separated. The method presented in (3.1) is easier to handle. However, Marden and Perlman [6] did not offer an analytical proof for their assertion. Note that the problems considered by Stein [15] and Schwartz [13] are fully $G_0$-invariant. For the $G_0$-invariant models considered by Stein [15], $\Theta_1 = \{ (\theta, \Sigma) \mid \theta \neq 0, \Sigma \text{ p.d.} \}$ for the problem of testing $H_0^\mu : \theta = 0$ against the global alternative $H_1^\mu : \theta \neq 0$. Note that $\bar{X}$ and $S$ are independent, and $\mu$ and $\Sigma$ are orthogonal. Take $\Theta_2 = \{ (\theta, \Sigma) \mid \theta \neq 0, \Sigma^{-1} \theta = 1, \Sigma \text{ p.d.} \}$, substituting the estimator $(n-1)^{-1} S$ of $\Sigma$ into $\Sigma$ and adopting the notation defined in Section 2, yields

$$\left\{ (X, S) \mid \sup_{\theta \neq 0, \Sigma^{-1} \theta = 1, \Sigma \text{ p.d.}} -\frac{1}{2} n(\bar{X} - \theta)' \Sigma^{-1}(\bar{X} - \theta) - \frac{1}{2} (n-1)[\text{tr } \Sigma^{-1} S/(n-1) - \ln |\Sigma^{-1} S/(n-1)|] \leq c \right\}$$

$$= \{ (X, S) \mid \sup_{\tilde{\nu} \neq 0, \tilde{\nu}' \tilde{S}^{-1} \tilde{\nu} = 1} n(n-1)\tilde{\nu}' \tilde{X} - \frac{1}{2} n(n-1)\tilde{\nu}' \tilde{S}^{-1} \tilde{X} \leq c + \frac{n(p+1) - p}{2} \}$$

$$= \{ (X, S) \mid n(n-1)\tilde{X}' \tilde{S}^{-1} \tilde{X} \leq k' \}$$

$$= \{ (X, S) \mid n(n-1)X' S^{-1} X \leq k' \},$$

where $\tilde{X} = (\tilde{X}_{1:2}, \tilde{X}_{2:2})$, $\tilde{S} = \text{diag}(S_{11:2}, S_{22})$ and $k' = 2c + n(p+1) - p$. The set (3.2) is equivalent to the acceptance region of Hotelling’s $T^2$-test.

Note that problem (1.2) is not $G^l$-invariant, although it is $G$-invariant. Marden and Perlman [6] transformed set (3.1) into a $G$-invariant set to work out set (3.1) when $\Theta_2 = \Theta_1$, and
reached the conclusion that the $G$-invariant set that corresponds to set (3.1) is equivalent to the acceptance region of Hotelling’s $T^2$-test. However, in their derivations (Marden and Perlman [6], p. 50) the restriction $\Delta_2 = 0$ ($\Delta_2$ defined in Section 1) had to be imposed, thus corresponding to the assumed condition that $\theta_2 = 0$ in set (3.1) was overlooked in their new $G$-invariant set. Rather than focusing only on the $G$-invariant set, this work directly determines the form of set (3.1) when $\Theta_2 = \{ (\theta, \Sigma) \mid \theta_1 \neq 0, \theta_2 = 0, \theta_1^{1:2} \Sigma^{-1}_{11:2} \theta_1^{1:2} = 1, \Sigma \text{ p.d.} \}$. Similar to arguments above, (3.1) then becomes

$$
\{ (\bar{X}, S) \mid \sup_{(\nu_{1} \neq 0, \nu_{2} = 0, \nu_{1}' S_{11:2}^{-1} \nu_{1} = 1)} n(n - 1) \nu_{2}' \bar{X} - \frac{1}{2} n(n - 1) \bar{X}' \bar{S}^{-1} \bar{X} \leq c + \frac{n(p + 1) - p}{2} \} \quad (3.2)
$$

$$
= \{ (\bar{X}, S) \mid n(n - 1)(\bar{X}_{1:2}' S_{11:2}^{-1} \bar{X}_{1:2} - \bar{X}_{2}' S_{22}^{-1} \bar{X}_{2}) \leq k' \}
$$

Notably, the set (3.3) is also a $G$-invariant set, but it is not equivalent to the acceptance region of Hotelling’s $T^2$-test

$$
\mathcal{A}_{T^2} = \{ (\bar{X}, S) \mid n(n - 1) \bar{X}' S^{-1} \bar{X} \leq k^* \}
$$

$$
= \{ (\bar{X}, S) \mid n(n - 1)(\bar{X}_{1:2}' S_{11:2}^{-1} \bar{X}_{1:2} + \bar{X}_{2}' S_{22}^{-1} \bar{X}_{2}) \leq k^* \}
$$

for a suitable $k^*$. Note that, (3.3) is obtained by using the information that $\theta_2 = 0$, but (3.2) is obtained without using that information.

Due to the fact that $\bar{X}$ and $S$ are independent, and $\mu$ and $\Sigma$ are orthogonal; based on the above discussions, an admissible acceptance region for the problem of testing $H_0^u : \theta = 0$ against the global alternative $H_1^u : \theta \neq 0$ can be simply taken as

$$
\{ (\bar{X}, S) \mid \sup_{(\nu_{1} \neq 0, \nu_{2} = 0, \nu_{1}' S_{11:2}^{-1} \nu_{1} = 1)} n(n - 1) \nu_{2}' X - \frac{1}{2} n(n - 1) \bar{X}' \bar{S}^{-1} \bar{X} \leq c \}
$$

$$
= \{ (\bar{X}, S) \mid n(n - 1) \bar{X}' S^{-1} \bar{X} \leq c \}
$$

for a suitable $c$. For problem (1.2), another admissible acceptance region is of the form

$$
\{ (\bar{X}, S) \mid \sup_{(\nu_{1} \neq 0, \nu_{2} = 0, \nu_{1}' S_{11:2}^{-1} \nu_{1} = 1)} n(n - 1) \nu_{2}' \bar{X} - \frac{1}{2} n(n - 1) \bar{X}' \bar{S}^{-1} \bar{X} \leq c \}
$$

$$
= \{ (\bar{X}, S) \mid n(n - 1) \bar{X}_{1:2}' S_{11:2}^{-1} \bar{X}_{1:2} \leq c \}
$$

which is the acceptance region of Rao’s $U$-test. Instead, based on the unproved assertion (3.1), we have provided an analytical proof that Rao’s $U$-test is admissible for problem (1.2) by using the Birnbaum-Stein method in Section 2.
Notably, for each \( \mathbf{b} \in \mathbb{R}^p \), \( T^2 \) and \( U \) can be obtained by maximizing \([n(n-1)]^{1/2} \mathbf{b}' \left( \bar{X}'_{1:2}, \bar{X}'_{2} \right)\) under the condition that \( \mathbf{b}' \tilde{\mathbf{S}}^{-1} \mathbf{b} \) is constant over the sets \( \Omega^*_1 = \{ \mathbf{b} \in \mathbb{R}^p | \mathbf{b} \neq \mathbf{0} \} \) and \( \Omega^*_2 = \{ \mathbf{b} \in \mathbb{R}^p | \mathbf{b}_1 \neq \mathbf{0}, \mathbf{b}_2 = \mathbf{0} \} \), respectively. Thus, both Hotelling’s \( T^2 \)-test statistic and Rao’s \( U \)-test statistic can be constructed by applying the union-intersection (UI) principle of Roy [12] for the problem of testing \( H^u_0 : \theta = \mathbf{0} \) against the global alternative \( H^u_1 : \theta \neq \mathbf{0} \), and the problem (1.2) of testing \( H_0 : \theta_1 = \mathbf{0}, \theta_2 = \mathbf{0} \) against the alternative \( H_1 : \theta_1 \neq \mathbf{0}, \theta_2 = \mathbf{0} \), respectively. Therefore, for problem (1.2) Rao’s test based on \( U \) may be regarded as a UI test. Although the Rao \( U \)-test statistic is constructed by incorporating the information of \( \theta_2 = \mathbf{0} \) (\( \bar{\nu}_2 = \mathbf{0} \)), the Hotelling \( T^2 \)-test statistic is not thus determined. Therefore, Hotelling’s \( T^2 \)-test may be reasonably thought to be dominated by Rao’s \( U \)-test for problem (1.2). This assertion can be numerically confirmed by the results of Tables 4.1a, 4.1b and 4.1c of Marden and Perlman [6].

The Birnbaum-Stein method fails to determine whether Hotelling’s \( T^2 \)-test is inadmissible for problem (1.2). This shortcoming is overcomed herein by applying Eaton’s [3] basic results to an essentially complete class of test functions for problem (1.2). Let \( \Phi \) be Eaton’s essentially complete class of tests, so for any test \( \varphi^* \notin \Phi \), there exists a test \( \varphi \in \Phi \) such that \( \varphi \) is at least as good as \( \varphi^* \).

**Theorem 3.** For the problem (1.2), Hotelling’s \( T^2 \)-test is inadmissible.

**Proof.** Following Eaton [3], the following is defined.

\[
\Omega_1 = \{ \Sigma^{-1} \theta | \theta_1 \neq \mathbf{0}, \theta_2 = \mathbf{0} \} \setminus \{ \mathbf{0} \}. \tag{3.5}
\]

Let \( \mathcal{V} \subseteq \mathbb{R}^p \) be the smallest closed convex cone that contains \( \Omega_1 \). Then the dual cone of \( \mathcal{V} \) is defined as

\[
\mathcal{V}^- = \{ \mathbf{w} | \mathbf{w}^\prime \mathbf{x} \leq 0, \forall \mathbf{x} \in \mathcal{V} \}. \tag{3.6}
\]

Notably, that \( \Omega_1 \) is contained in some half-space is not a necessary condition but a sufficient condition that ensures the dual cone \( \mathcal{V}^- \) is a non-empty set. Note that although \( \Sigma \) is unknown, but it is fixed. By (3.5) and (3.6), thus we have

\[
\mathcal{V} = \{ \Sigma^{-1} \theta | \theta_1 \neq \mathbf{0}, \theta_2 = \mathbf{0}, \Sigma \text{ is p.d.} \}
\]

\[
= \mathbb{R}^{p_1}, \tag{3.7}
\]

which is contained in a half-space of \( \mathbb{R}^p \). Similarly, its dual cone is

\[
\mathcal{V}^- = \{ \mathbf{w} | \mathbf{x}^\prime \mathbf{w} \leq 0, \forall \mathbf{x} \in \mathcal{V} \}
\]

\[
= \{ \tilde{\mathbf{w}} | \theta_1 \tilde{\mathbf{w}}^\prime \Sigma^{-1}_{11:2} \tilde{\mathbf{w}} \leq 0, \theta_1 \neq \mathbf{0}, \Sigma_{11:2} \text{ is p.d.} \}, \tag{3.8}
\]
where
\[
\tilde{w} = \begin{bmatrix}
I - \Sigma_{12} \Sigma_{22}^{-1} & 0 \\
0 & I
\end{bmatrix} w
\]
\[(3.9)\]
\[
= \begin{pmatrix}
w_1 - \Sigma_{12} \Sigma_{22}^{-1} w_2 \\
w_2
\end{pmatrix}
\]
\[\triangleq \begin{pmatrix}
\tilde{w}_1 \\
\tilde{w}_2
\end{pmatrix}.
\]

Therefore, \(\mathcal{V}^- = \mathbb{R}^{n_2}\), which is an unbounded set.

The acceptance region of Hotelling’s \(T^2\)-test is given by
\[
\mathcal{A}_{T^2} = \{(\bar{X}, S) \mid n(n - 1)\bar{X}'S^{-1}\bar{X} \leq t^2_{\alpha}\},
\]

where \(t^2_{\alpha}\) is the upper \(100\alpha\)% point of the null hypothesis distribution of \(T^2\) (which is linked to a F-distribution). For fixed \(S\), \(\mathcal{A}_{T^2}\) is an ellipsoidal set with origin \(0\), and is bounded, whereas \(\mathcal{V}^-\), as shown above, is still unbounded. Therefore, the proposition 2.1 of Eaton that the dual cone \(\mathcal{V}^-\) should be a subset of the acceptance region of Hotelling’s \(T^2\)-test (Eaton [3], section 4, p. 1887) is not tenable, and thus Hotelling’s \(T^2\)-test is not a member of an essentially complete class.

4. Whither Rao’s \(W\)-test?

In passing, both the Hotelling \(T^2\)-test statistic \(T^2\) and the Rao \(W\)-test statistic \(W\) can be obtained by applying the likelihood ratio principle. The Rao \(W\)-test statistic is constructed by incorporating the information that \(\theta_2 = 0\), but the Hotelling \(T^2\)-test statistic is not thus obtained. For problem (1.10), Marden and Perlman [6] adopted the generalized Bayes approach to show that Rao’s \(W\)-test is admissible when \(0 < \alpha < \alpha^*\) and is inadmissible when \(\alpha^* < \alpha < 1\). Section 1 stated that restricting problem (1.2) to \(G\)-invariant tests does not reduce it to problem (1.10), but should reduce it to problem (1.11), and that problem (1.11) and problem (1.10) differ. Thus, the optimal criteria established for problem (1.10) to draw inferences for problem (1.2) may lead to conclusions that convey misleading messages. The generalized Bayes approach of Marden and Perlman [6] can be adopted to characterize in parallel the sufficient and necessary conditions for the admissibility of the problem (1.11). A situation in which the Rao \(W\)-test can be further demonstrated to be a generalized Bayes test, and the corresponding optimality conditions for the problem (1.11) can be satisfied, can lead to completion of the task. Birnbaum [2], in the context of complete class type theorems, noted that for testing \(H_0^a : \theta = 0\) versus
a test is admissible if and only if it is a generalized Bayes test. Some admissible tests in the literature are not the Bayes tests for other hypothesis testing problems (Oosterhoff [8], p.82). For problem (1.2), the set of proper Bayes tests and their weak limits might only constitute a proper subset of an essentially complete class of tests. On the other hand, the Birnbaum-Stein method stipulated the convexity assumption for the acceptance regions of tests. However, for problem (1.2) the acceptance region of Rao’s $W$-test is a hyperbolic type set, which is no longer convex. Therefore, the Birnbaum-Stein method fails to be applicable to Rao’s $W$-test. A future study will investigate the problem of the optimality of Rao’s $W$-test.

5. Some remarks

The Hotelling $T^2$-test enjoys many optimal properties of the Neyman-Pearson hypothesis testing theory when testing against the global alternative. These include similarity, unbiasedness, power monotonicity, most stringency, uniformly most powerful invariant and alpha-admissibility etc.. However, it is still open to debate whether the Hotelling $T^2$-test is minimax. For the hypothesis testing problem (1.2) we show that the Hotelling $T^2$-test is not a member of an essentially complete class (Eaton [3]), and hence it is no longer admissible. Moreover, we adopt the Birnbaum-Stein method (Stein [15]) to demonstrate that the Rao $U$-test is admissible for the hypothesis testing problem (1.2).

Consider the hypotheses

$$H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \theta \in C \setminus \{0\},$$

where $C$ denotes a closed convex cone containing a $p$-dimensional open set. Denote the positive orthant space by $\mathcal{O}_p^+ = \{ \theta \in \mathbb{R}^p | \theta \geq 0 \}$. Notice that when $C$ is a proper set contained in a halfspace, under a suitable linear transformation the problem in (5.1) can be reduced to the problem for testing against the positive orthant space with another unknown positive definite covariance matrix. When $C$ is a specific halfspace, then it can be transformed into another halfspace by a non-singular linear transformation. Hence, without loss of generality it is sufficient to study the cases in which $C$ is the positive orthant space $\mathcal{O}_p^+$ and $C$ is the halfspace $\mathcal{H}_p^* = \{ \theta \in \mathbb{R}^p | \theta_p \geq 0 \}$. Note that the hypothesis testing problem (1.2) is a special case of the hypothesis testing problem (5.1). Therefore, we will further study whether the property of $d$-admissibility of the UIT and LRT for the problem of testing against the closed convex cone holds in the near future.

Appendix
The following six lemmas are established to prove Theorem 1.

**Lemma 1.** $B^+(S)$ is convex on $S$.

**Proof.** For any two given matrices $S$ and $T$ ($\in S$), let $\Psi^+(\alpha) = B^+(\alpha S + (1 - \alpha)T)$, where $\alpha \in (0, 1)$. Then,
\[
\frac{d}{d\alpha} \Psi^+(\alpha) = -\left\{ (\alpha S + (1 - \alpha)T)^{-1}(S - T)(\alpha S + (1 - \alpha)T)^{-1} \right. \\
- \left[ \begin{array}{cc} 0 & 0 \\ 0 & (\alpha S_{22} + (1 - \alpha)T_{22})^{-1}\left(S_{22} - T_{22}\right)(\alpha S_{22} + (1 - \alpha)T_{22})^{-1} \end{array} \right] \right\}.
\]

Write
\[
\alpha S + (1 - \alpha)T = \begin{bmatrix} \alpha S_{11} + (1 - \alpha)T_{11} & \alpha S_{12} + (1 - \alpha)T_{12} \\ \alpha S_{21} + (1 - \alpha)T_{21} & \alpha S_{22} + (1 - \alpha)T_{22} \end{bmatrix} \triangleq \begin{bmatrix} E & F \\ F' & G \end{bmatrix}
\]
and
\[
(\alpha S + (1 - \alpha)T)^{-1} = H + \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix}, \quad \text{where}
\]
\[
H = \begin{pmatrix} I \\ -G^{-1}F' \end{pmatrix} (E - FG^{-1}F')^{-1} \begin{pmatrix} I \\ -FG^{-1} \end{pmatrix}.
\]
Since $H$ is p.s.d., thus
\[
(\alpha S + (1 - \alpha)T)^{-1} \succeq \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix}.
\]
Therefore,
\[
\lambda_{\text{max}} \left[ \left( H + \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right] \leq 1, \quad (A.1)
\]
where $\lambda_{\text{max}}(A)$ denotes the largest eigenvalue of $A$. Suppose that $A$ is p.d., $B$ is p.s.d. and write $A = A^{1/2}(A^{1/2})'$, then note that $A \succeq B$ implies that $I \succeq A^{-1/2}B(A^{-1/2})'$. Thus, $1 = \lambda_{\text{max}}(I) \geq \lambda_{\text{max}}(A^{-1/2}B(A^{-1/2})') = \lambda_{\text{max}}(BA^{-1}) = \lambda_{\text{max}}(A^{-1}B)$. Note that,
\[
\frac{d^2 \Psi^+ (\alpha)}{d \alpha^2} = 2 \left\{ (\alpha S + (1-\alpha)T)^{-1} (S-T)(\alpha S + (1-\alpha)T)^{-1} (S-T)(\alpha S + (1-\alpha)T)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & (\alpha S_{22} + (1-\alpha)T_{22})^{-1} (S_{22} - T_{22})(\alpha S_{22} + (1-\alpha)T_{22})^{-1} (S_{22} - T_{22}) (\alpha S_{22} + (1-\alpha)T_{22})^{-1} \end{bmatrix} \right\}
\]

\[
= 2 \left\{ (\alpha S + (1-\alpha)T)^{-1} (S-T)(\alpha S + (1-\alpha)T)^{-1} (S-T)(\alpha S + (1-\alpha)T)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} (S-T) \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} (S-T) \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right\}
\]

\[
\geq 2 \left\{ (\alpha S + (1-\alpha)T)^{-1} (S-T)(\alpha S + (1-\alpha)T)^{-1} (S-T)(\alpha S + (1-\alpha)T)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} (\alpha S - (1-\alpha)T)^{-1} (S-T) \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right\}
\]

\[
= 2 \left\{ (\alpha S + (1-\alpha)T)^{-1} K (\alpha S + (1-\alpha)T)^{-1} - \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} K \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right\},
\]

where

\[ K = (S - T)(\alpha S + (1-\alpha)T)^{-1} (S - T). \]

Next, compare the matrices

\[
(\alpha S + (1-\alpha)T)^{-1} K (\alpha S + (1-\alpha)T)^{-1}
\]

and

\[
\begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} K \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix},
\]

that is,

\[
\left( H + \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right) K \left( H + \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right)
\]

and

\[
\begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} K \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix}.
\]

Consider the new matrix \( LL' \), where

\[
L = K^{-1/2} \left( H + \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & G^{-1} \end{bmatrix} K^{1/2}.
\]
The matrix $L$ can be rewritten as $L = P\Lambda Q'$, where $\Lambda$ denotes the diagonal matrix of eigenvalues of $L$ and $P, Q \in \mathcal{O}(p)$, the group of $p \times p$ orthogonal matrices. Thus, $LL' = P\Lambda^2 P'$. Therefore, $\lambda_{max}(LL') = \lambda_{max}(P\Lambda^2 P') = \lambda_{max}(\Lambda^2 P'P) = \lambda_{max}(\Lambda)^2 = \lambda_{max}(L)^2$. Notably,

$$\lambda_{max}(L) = \lambda_{max}\left(\frac{K^{-1/2}}{H + \left[\begin{array}{cc} 0 & 0 \\ 0 & G^{-1} \end{array}\right]}\right)^{-1}\left[\begin{array}{cc} 0 & 0 \\ 0 & G^{-1} \end{array}\right]K^{1/2}\right) \leq 1,$$

the last inequality follows from the inequality (A.1). Thus, $\lambda_{max}(LL') \leq 1$. Furthermore, notice that

$$\lambda_{max}(LL') = \lambda_{max}\left([\alpha S + (1 - \alpha)T]^{-1}(S - T)(\alpha S + (1 - \alpha)T)^{-1}(S - T)(\alpha S + (1 - \alpha)T)^{-1}\right)^{-1}\times\left[\begin{array}{cc} 0 & 0 \\ 0 & G^{-1} \end{array}\right](S - T)(\alpha S + (1 - \alpha)T)^{-1}(S - T)\left[\begin{array}{cc} 0 & 0 \\ 0 & G^{-1} \end{array}\right].$$

Thus,

$$(\alpha S + (1 - \alpha)T)^{-1}(S - T)(\alpha S + (1 - \alpha)T)^{-1}(S - T)(\alpha S + (1 - \alpha)T)^{-1} \geq \left[\begin{array}{cc} 0 & 0 \\ 0 & G^{-1} \end{array}\right](S - T)(\alpha S + (1 - \alpha)T)^{-1}(S - T)\left[\begin{array}{cc} 0 & 0 \\ 0 & G^{-1} \end{array}\right].$$

Therefore,

$$\frac{d^2 \Psi^+(\alpha)}{d\alpha^2} \geq 0$$

and hence

$B^+(S)$ is convex on $S$.

**Lemma 2.** $B(S)$ is the Moore-Penrose generalized inverse of $B^+(S)$.

**Proof.** It can be easily shown that $B(S)$ satisfies the following conditions:

(i) $B(S)B^+(S)B(S) = B(S)$,

(ii) $B^+(S)B(S)B^+(S) = B^+(S)$,

(iii) $B(S)B^+(S) = (B(S)B^+(S))'$,

(iv) $B^+(S)B(S) = (B^+(S)B(S))'$.
Lemma 3. For any two given matrices $S$ and $T \in S$, let $\Psi^+(\alpha) = B^+(\alpha S + (1 - \alpha)T)$, where $\alpha \in (0, 1)$. Then $\frac{d^2 \Psi^+(\alpha)}{d \alpha^2} \geq \frac{d^2 \Psi^+(\alpha)}{d \alpha^2}\bigg|_{\alpha=\alpha_0}$, $\forall \alpha \in (0, 1)$, where $\alpha_0$ is the stationary point of $\Psi^+(\alpha)$. Let $\Psi(\alpha)$ be the Moore-Penrose generalized inverse of $\Psi^+(\alpha)$, then $\frac{d^2 \Psi(\alpha)}{d \alpha^2} \leq \frac{d^2 \Psi^+(\alpha)}{d \alpha^2}\bigg|_{\alpha=\alpha_0}$, $\forall \alpha \in (0, 1)$.

Proof. It is easily to see that $\Psi^+(\alpha)$ is continuous and hence differentiable, $\forall \alpha \in (0, 1)$. First note that $\Psi^+(\alpha)$ is neither a linear function nor a quadratic function of $\alpha$. From the proof of Lemma 1, we note that $\Psi^+(\alpha)$ is not a monotone function of $\alpha$. By Lemma 1, $\Psi^+(\alpha)$ is convex in $\alpha$. Thus, the stationary point of $\Psi^+(\alpha)$ exists and unique. Suppose $\alpha_0$ be the stationary (critical) point, then $\Psi^+(\alpha) \succeq \Psi^+(\alpha_0) \succeq 0$, $\forall \alpha \in (0, 1)$. This implies that $\Psi^+(\alpha) \succeq (1 - \alpha)^2 \Psi^+(\alpha_0) = (\alpha^2 - 2\alpha + 1) \Psi^+(\alpha_0)$, $\forall \alpha \in (0, 1)$. Note that $\Psi^+(\alpha_0)$ is p.s.d., thus there exists a quadratic convex function $\Upsilon^+(\alpha)$ such that $\Psi^+(\alpha) \succeq \Upsilon^+(\alpha)$, $\forall \alpha \in (0, 1)$. Hence for sufficiently small $h$, there exists a quadratic convex function $\Upsilon^+_0(\alpha)$ such that $\Psi^+(\alpha) \succeq \Upsilon^+_0(\alpha)$, $\forall \alpha \in (0, 1)$, and $\Psi^+(\alpha_0) = \Upsilon^+_0(\alpha_0)$, $\Psi^+(\alpha_0 + h) = \Upsilon^+_0(\alpha_0 + h)$ and $\Psi^+(\alpha_0 + 2h) = \Upsilon^+_0(\alpha_0 + 2h)$.

Notably,

$$
\frac{d^2 \Psi^+(\alpha)}{d \alpha^2} - \frac{d^2 \Psi^+(\alpha)}{d \alpha^2}\bigg|_{\alpha=\alpha_0} = \lim_{h \to 0} \frac{\Psi^+(\alpha + 2h) - 2\Psi^+(\alpha + h) + \Psi^+(\alpha)}{h^2} - \lim_{h \to 0} \frac{\Psi^+(\alpha_0 + 2h) - 2\Psi^+(\alpha_0 + h) + \Psi^+(\alpha_0)}{h^2}
$$

$$
= \lim_{h \to 0} \frac{\Psi^+(\alpha + 2h) - 2\Psi^+(\alpha + h) + \Psi^+(\alpha)}{h^2} - \lim_{h \to 0} \frac{\Upsilon^+_0(\alpha_0 + 2h) - 2\Upsilon^+_0(\alpha_0 + h) + \Upsilon^+_0(\alpha_0)}{h^2}
$$

For any fixed $\alpha \in (0, 1)$, there exists a quadratic convex function $\Upsilon^+_0(\alpha)$ such that $\Upsilon^+_0(\alpha) \succeq \Upsilon^+_0(\alpha_0)$, and $\Upsilon^+_0(\alpha) = \Psi^+(\alpha)$, $\Upsilon^+_0(\alpha + h) = \Psi^+(\alpha + h)$ and $\Upsilon^+_0(\alpha + 2h) = \Psi^+(\alpha + 2h)$ for arbitrary small $h$. Thus,

$$
\frac{d^2 \Psi^+(\alpha)}{d \alpha^2} - \frac{d^2 \Psi^+(\alpha)}{d \alpha^2}\bigg|_{\alpha=\alpha_0} = \lim_{h \to 0} \frac{\Upsilon^+_0(\alpha + 2h) - 2\Upsilon^+_0(\alpha + h) + \Upsilon^+_0(\alpha)}{h^2} - \lim_{h \to 0} \frac{\Upsilon^+_0(\alpha_0 + 2h) - 2\Upsilon^+_0(\alpha_0 + h) + \Upsilon^+_0(\alpha_0)}{h^2}
$$

$$
= \frac{d^2 \Upsilon^+_0(\alpha)}{d \alpha^2} - \frac{d^2 \Upsilon^+_0(\alpha)}{d \alpha^2}\bigg|_{\alpha=\alpha_0} = 0, \forall \alpha \in (0, 1).
$$

Therefore,

$$
\frac{d^2 \Psi^+(\alpha)}{d \alpha^2} \geq \frac{d^2 \Psi^+(\alpha)}{d \alpha^2}\bigg|_{\alpha=\alpha_0}, \forall \alpha \in (0, 1).
$$
Similarly, $\Psi(\alpha)$ is the Moore-Penrose generalized inverse of $\Psi^+(\alpha)$, and so $\Psi(\alpha_0) \succeq \Psi(\alpha) \succeq 0$, $\forall \alpha \in (0, 1)$. This implies that $(1 + 2\alpha - \alpha^2)\Psi(\alpha_0) \succeq \Psi(\alpha)$, $\forall \alpha \in (0, 1)$. Thus, there exists a quadratic concave function $\Upsilon(\alpha)$ such that $\Upsilon(\alpha) \succeq \Psi(\alpha)$, $\forall \alpha \in (0, 1)$. Parallel arguments as in the case $\Psi^+(\alpha)$, we may also conclude that

$$\frac{d^2 \Psi(\alpha)}{d \alpha^2} \succeq \frac{d^2 \Psi(\alpha)}{d \alpha^2} |_{\alpha = \alpha_0}, \forall \alpha \in (0, 1).$$

**Lemma 4.** $B(S)$ is concave on $S$.

**Proof.** Rewrite $B^+(S)$ in (2.2) as

$$B^+(S) = \begin{pmatrix} I \\ -S_{22}^{-1}S_{21} \end{pmatrix} S_{11:2}^{-1} \begin{pmatrix} I & -S_{12}S_{22}^{-1} \end{pmatrix} = C(S)D^{-1}(S)C'(S),$$

where

$$C(S) = \begin{pmatrix} I \\ -S_{22}^{-1}S_{21} \end{pmatrix} \text{ and } D(S) = S_{11:2}.$$

Note that

1. $B(S) = C(S)(C'(S)C(S))^{-1}D(S)(C'(S)C(S))^{-1}C'(S)$
2. $B^+(S)B(S) = B(S)B^+(S) = C(S)(C'(S)C(S))^{-1}C'(S)$.

Let $\Psi^+(\alpha) = B^+(\alpha S + (1 - \alpha)T) = MN^{-1}M$, where $M = C(\alpha S + (1 - \alpha)T)$, $N = D(\alpha S + (1 - \alpha)T)$ and $\alpha \in (0, 1)$. Then, its Moore-Penrose generalized inverse is of the form

$$\Psi(\alpha) = B(\alpha S + (1 - \alpha)T) = M(M'M)^{-1}N(M'M)^{-1}M'.$$

Notably,

$$B^+(S)B(S)B^+(S) = B^+(S) \implies \Psi^+(\alpha)\Psi(\alpha)\Psi^+(\alpha) = \Psi^+(\alpha).$$

Thus,

$$\frac{d \Psi^+(\alpha)}{d \alpha} \Psi(\alpha)\Psi^+(\alpha) + \Psi^+(\alpha)\frac{d \Psi(\alpha)}{d \alpha} \Psi^+(\alpha) + \Psi^+(\alpha)\Psi(\alpha)\frac{d \Psi^+(\alpha)}{d \alpha} = \frac{d \Psi^+(\alpha)}{d \alpha} \tag{A.2}$$

and

$$\frac{d^2 \Psi^+(\alpha)}{d \alpha^2} = \frac{d^2 \Psi^+(\alpha)}{d \alpha} \Psi(\alpha)\Psi^+(\alpha) + \frac{d \Psi^+(\alpha)}{d \alpha} \frac{d \Psi(\alpha)}{d \alpha} \Psi^+(\alpha) + \frac{d \Psi^+(\alpha)}{d \alpha} - \Psi(\alpha)\frac{d \Psi^+(\alpha)}{d \alpha} \Psi(\alpha) \tag{A.3}$$

$$+ \Psi^+(\alpha)\frac{d \Psi(\alpha)}{d \alpha} \frac{d \Psi^+(\alpha)}{d \alpha} + \frac{d \Psi^+(\alpha)}{d \alpha} \Psi(\alpha)\frac{d \Psi^+(\alpha)}{d \alpha}$$

$$+ \Psi^+(\alpha)\frac{d \Psi(\alpha)}{d \alpha} \frac{d \Psi^+(\alpha)}{d \alpha} + \Psi^+(\alpha)\Psi(\alpha)\frac{d^2 \Psi^+(\alpha)}{d \alpha^2}.$$
By the results of (A.2) and (A.3), then
\[
\Psi^+(\alpha) \frac{d^2 \Psi(\alpha)}{d\alpha^2} \Psi^+(\alpha) = \frac{d^2 \Psi(\alpha)}{d\alpha^2} - \Psi^+(\alpha) \frac{d^2 \Psi^+(\alpha)}{d\alpha^2} - \Psi^+(\alpha) \frac{d^2 \Psi^+(\alpha)}{d\alpha^2} \Psi^+(\alpha) \\
- 2 \frac{d \Psi^+(\alpha)}{d\alpha} \frac{d \Psi(\alpha)}{d\alpha} - 2 \frac{d \Psi^+(\alpha)}{d\alpha} \frac{d \Psi^+(\alpha)}{d\alpha}
\]
\[
- 2 \Psi^+(\alpha) \frac{d \Psi(\alpha)}{d\alpha} \frac{d \Psi^+(\alpha)}{d\alpha}.
\]
Since \( \Psi^+(\alpha) \Psi(\alpha) = M(M'M)^{-1}M' \), thus
\[
\Psi^+(\alpha) \frac{d \Psi(\alpha)}{d\alpha} = -\frac{d \Psi^+(\alpha)}{d\alpha} \Psi(\alpha) + \frac{d}{d\alpha} [M(M'M)^{-1}M'].
\]
Therefore,
\[
\Psi^+(\alpha) \frac{d^2 \Psi(\alpha)}{d\alpha^2} \Psi^+(\alpha) = \frac{d^2 \Psi^+(\alpha)}{d\alpha^2} - \Psi^+(\alpha) \frac{d^2 \Psi^+(\alpha)}{d\alpha^2} - \Psi^+(\alpha) \frac{d^2 \Psi(\alpha)}{d\alpha^2} \Psi(\alpha) \\
+ 2 \frac{d \Psi^+(\alpha)}{d\alpha} \Psi(\alpha) \frac{d \Psi^+(\alpha)}{d\alpha} - 2 \frac{d}{d\alpha} [M(M'M)^{-1}M'] \frac{d \Psi^+(\alpha)}{d\alpha}
\]
\[
- 2 \frac{d \Psi^+(\alpha)}{d\alpha} \frac{d}{d\alpha} [M(M'M)^{-1}M'].
\]
Notably,

(i)
\[
\frac{d^2 \Psi^+(\alpha)}{d\alpha^2} = \frac{d^2 M}{d\alpha^2} N^{-1} M' + MN^{-1} \frac{d^2 M'}{d\alpha^2} + 2 \frac{d M}{d\phi} N^{-1} \frac{d M'}{d\alpha} \\
- 2 \frac{d M}{d\alpha} N^{-1} \frac{d M}{d\alpha} N^{-1} M' - 2 MN^{-1} \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} \\
+ 2 MN^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d N}{d\alpha} N^{-1} M' - MN^{-1} \frac{d^2 N}{d\alpha^2} N^{-1} M' \\
= \frac{d^2 M}{d\alpha^2} N^{-1} M' + MN^{-1} \frac{d^2 M'}{d\alpha^2} - MN^{-1} \frac{d^2 N}{d\alpha^2} N^{-1} M' \\
- 2 (MN^{-1} \frac{d N}{d\alpha}) N^{-1} (MN^{-1} \frac{d N}{d\alpha} - \frac{d M}{d\alpha})',
\]

(ii)
\[
\Psi^+(\alpha) \Psi(\alpha) \frac{d^2 \Psi^+(\alpha)}{d\alpha^2} = M(M'M)^{-1} \frac{d^2 M}{d\alpha^2} N^{-1} M' + MN^{-1} \frac{d^2 M'}{d\alpha^2} \\
+ 2 M(M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} \\
- 2 M(M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} \frac{d N}{d\alpha} N^{-1} M' - 2 MN^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d M'}{d\alpha} \\
+ 2 MN^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d N}{d\alpha} N^{-1} M' - MN^{-1} \frac{d^2 N}{d\alpha^2} N^{-1} M',
\]
(iii) 
\[
\frac{d^2 \Psi^+(\alpha)}{d\alpha^2} \Psi^{(\alpha)} \Psi^+(\alpha) = \frac{d^2 M}{d\alpha^2} N^{-1} M' + MN^{-1} \frac{d^2 M'}{d\alpha^2} M(M'M)^{-1} \\
+ 2 \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} M' \\
- 2 \frac{d M}{d\alpha} N^{-1} \frac{d N}{d\alpha} N^{-1} M' \\
- 2 MN^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} M',
\]

(iv) 
\[
\frac{d \Psi^+(\alpha)}{d\alpha} \Psi^{(\alpha)} \frac{d \Psi^+(\alpha)}{d\alpha} = \frac{d M}{d\alpha} (M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} M' + \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} \\
- \frac{d M}{d\alpha} N^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d M'}{d\alpha} \\
+ MN^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} N(M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} M' \\
+ MN^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} \frac{d M}{d\alpha} N^{-1} M' \\
- MN^{-1} \frac{d N}{d\alpha} (M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} M' \\
- MN^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d M'}{d\alpha} + MN^{-1} \frac{d N}{d\alpha} N^{-1} \frac{d N}{d\alpha} N^{-1} M',
\]

(v) 
\[
\frac{d \Psi^+(\alpha)}{d\alpha} \frac{d [M(M'M)^{-1} M']}{d\alpha} = \frac{d M}{d\alpha} N^{-1} M' \frac{d M}{d\alpha} (M'M)^{-1} M' \\
- \frac{d M}{d\alpha} N^{-1} \left( \frac{d M'}{d\alpha} M + M' \frac{d M}{d\alpha} \right) (M'M)^{-1} M' \\
+ \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} + MN^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} M' \\
- MN^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} \left( \frac{d M'}{d\alpha} M + M' \frac{d M}{d\alpha} \right) (M'M)^{-1} M' \\
+ MN^{-1} \frac{d M'}{d\alpha} M(M'M)^{-1} \frac{d M'}{d\alpha} M + \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} N^{-1} M',
\]

(vi) 
\[
\frac{d [M(M'M)^{-1} M']}{d\alpha} \frac{d \Psi^+(\alpha)}{d\alpha} = \frac{d M}{d\alpha} (M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} M' \\
- M(M'M)^{-1} \left( \frac{d M'}{d\alpha} M + M' \frac{d M}{d\alpha} \right) (M'M)^{-1} M' \frac{d M}{d\alpha} N^{-1} M' \\
+ M(M'M)^{-1} \frac{d M'}{d\alpha} \frac{d M}{d\alpha} N^{-1} M' + \frac{d M}{d\alpha} N^{-1} \frac{d M'}{d\alpha} N^{-1} M'.
\]
Thus, by the results (A.4) and (i)-(vi) and some straightforward manipulations,

\[
M'\Psi^+(\alpha)\frac{d^2\Psi^+(\alpha)}{d\alpha^2} = M'MN^{-1}d^2N\frac{dM}{d\alpha}N^{-1}M'M
\]

\[
+ 2M'M\frac{dM}{d\alpha}(M'M)^{-1}M'\frac{dM}{d\alpha}N^{-1}M'M
\]

\[
+ 2M'MN^{-1}\frac{dM}{d\alpha}M(M'M)^{-1}\frac{dM}{d\alpha}M
\]

\[
+ 2M'MN^{-1}\frac{dM}{d\alpha}M(M'M)^{-1}N(M'M)^{-1}M'\frac{dM}{d\alpha}N^{-1}M'M
\]

\[
+ 2M'MN^{-1}\frac{dM}{d\alpha}M(M'M)^{-1}M'\frac{dM}{d\alpha}
\]

\[
+ 2\frac{dM}{d\alpha}M(M'M)^{-1}M^{-1}\frac{dM}{d\alpha}N^{-1}M'M
\]

\[
- M\frac{d^2M}{d\alpha^2}N^{-1}M'M - M'MN^{-1}\frac{dM}{d\alpha}N^{-1}M'M
\]

\[
- 2\frac{dM}{d\alpha}\frac{dM}{d\alpha}N^{-1}M'M - 2M'MN^{-1}\frac{dM}{d\alpha}\frac{dM}{d\alpha}
\]

\[
- 2M'MN^{-1}\frac{dM}{d\alpha}M(M'M)^{-1}1M'\frac{dM}{d\alpha}N^{-1}M'M
\]

\[
- 2M'MN^{-1}\frac{dN}{d\alpha}(M'M)^{-1}M'\frac{dM}{d\alpha}N^{-1}M'M. \quad (A.5)
\]

Also, note that

\[
\frac{d\Psi^+(\alpha)}{d\alpha} = \frac{dM}{d\alpha}N^{-1}M' + MN^{-1}\frac{dM}{d\alpha} - MN^{-1}\frac{dN}{d\alpha}N^{-1}M'.
\]

Thus, the stationary point of \(\Psi^+(\alpha)\) satisfies the following equation

\[
\frac{dM}{d\alpha}N^{-1}M' + MN^{-1}\frac{dM}{d\alpha} = MN^{-1}\frac{dN}{d\alpha}N^{-1}M',
\]

which implies that

\[
(M'M)^{-1}\frac{dM}{d\alpha}N^{-1} + N^{-1}\frac{dM}{d\alpha}M(M'M)^{-1} = N^{-1}\frac{dN}{d\alpha}N^{-1}.
\]

Namely,

\[
M'MN^{-1}\frac{dN}{d\alpha} - M'\frac{dM}{d\alpha} = MN^{-1}\frac{dM}{d\alpha}M(M'M)^{-1}N.
\]

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Thus,
\[
(M'N^{-1}dN - M'dM)N^{-1}(M'MN^{-1}dN - M'dM)'N^{-1}(M'MN^{-1}dN - M'dM)'N^{-1}M'M.
\]

Furthermore, \( B^+(S) \) is convex on \( S \), thus \( \frac{d^2 \Psi(\alpha)}{d\alpha^2} \geq 0 \), that is,
\[
\frac{d^2 M'N^{-1}M'}{d\alpha^2} + MN^{-1}\frac{d^2 M'}{d\alpha^2} - MN^{-1}\frac{d^2 N}{d\alpha^2}N^{-1}M' \geq 2(MN^{-1}\frac{dN}{d\alpha} - \frac{dM}{d\alpha})N^{-1}(MN^{-1}\frac{dN}{d\alpha} - \frac{dM}{d\alpha})'.
\]

Substitute these results into (A.5), then
\[
M'\Psi^+(\alpha)\frac{d^2 \Psi(\alpha)}{d\alpha^2} \Psi^+(\alpha)M \bigg|_{\alpha = \alpha_0} \\
\leq 2\left[ \frac{dM'}{d\alpha}M(M'M)^{-1}M'\frac{dM}{d\alpha}N^{-1}M'M + M'MN^{-1}\frac{dM'}{d\alpha}M(M'M)^{-1}M'\frac{dM}{d\alpha} \right] \\
- 2\left[ \frac{dM'}{d\alpha} \frac{dM}{d\alpha} N^{-1}M'M + M'MN^{-1}\frac{dM'}{d\alpha} \frac{dM}{d\alpha} \right]
\leq 0.
\]

Thus,
\[
\frac{d^2 \Psi(\alpha)}{d\alpha^2} \bigg|_{\alpha = \alpha_0} \leq 0.
\]

By Lemma 3, then
\[
\frac{d^2 \Psi(\alpha)}{d\alpha^2} \leq \frac{d^2 \Psi(\alpha)}{d\alpha^2} \bigg|_{\alpha = \alpha_0} \leq 0, \quad \forall \alpha \in (0,1).
\]

Therefore, \( B(S) \) is concave on \( S \).

**Lemma 5.** Let \( A_i, i = 1, 2, \) be \( p \times p \) p.s.d. of rank \( r \) (\( r \leq p \)). Also let \( D_r = \text{diag}(d_1, \ldots, d_r) \) with elements being the non-zero eigenvalues of \( A_2A_1^+ \), where \( A_1^+ \) denotes the Moore-Penrose generalized inverse of \( A_1 \). Then there exists a nonsingular matrix \( G \) such that
\[
A_1 = G \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} G' \quad \text{and} \quad A_2 = G \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} G'.
\]

**Proof.** By Theorem A.4.1 of Anderson [1],
\[
A_1 = F \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} F',
\]
where \( F \) is a nonsingular matrix. Let \( A^* = F^{-1}A_2(F^{-1})' \), then \( A^* \) is a p.s.d. with rank \( r \). Write
\[
A^* = \begin{pmatrix} A^*_11 & A^*_12 \\ A^*_21 & A^*_22 \end{pmatrix},
\]
and take
\[
C = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix},
\]
where \( C_{11} \in Q(r) \), \( C_{22} \in Q(p-r) \) such that \( C_{22}'A^*_22 = 0 \). Thus, there exists a matrix \( C \in Q(p) \), the group of \( p \times p \) orthogonal matrices such that
\[
C'A^*C = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}
\]
that is,
\[
\begin{align*}
C'F^{-1}A_1(F^{-1})'C &= \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}, \\
C'F^{-1}A_2(F^{-1})'C &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}
\end{align*}
\]
Let \( G = FC \), thus
\[
A_1 = G \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} G'
\]
and
\[
A_2 = G \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} G'.
\]

**Lemma 6.** Let \( A \) be an \( p \times p \) p.s.d. matrix and \( x \) be a \( p \times 1 \) vector. Let \( A = BCB' \) and denotes \( A^- = (B')^+C^+B^+ \), where \( D^+ \) denotes the Moore-Penrose generalized inverse of \( D \). If \( A^- \) is a generalized inverse of \( A \), then \( f(x, A) = x'A^-x \) is convex on \( R^p \times S \).

**Proof.** Since \( f \) is continuous in \((x, A)\), it suffices to show that
\[
(x+y)'(A_1 + A_2)^-(x+y) \leq x'A_1^-x + y'A_2^-y.
\]
Write \( A = A^{1/2}(A^{1/2})' \) and take
\[
\begin{align*}
u &= (A_1^{1/2})^-x - (A_1^{1/2})'(A_1 + A_2)^-(x+y), \\
v &= (A_2^{1/2})^-y - (A_2^{1/2})'(A_1 + A_2)^-(x+y).
\end{align*}
\]
By Lemma 5, \( A_1 = G \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} G' \) and \( A_2 = G \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} G' \), where \( G \) is nonsingular and \( D_r = \text{diag}(d_1, \ldots, d_r) \) with \( d_i \) being the non-zero eigenvalues of \( A_2 A_1^+ \). Write \( A_1^{\frac{1}{2}} = G \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \) and \( A_2^{\frac{1}{2}} = G \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} \), then

\[
(A_1^{\frac{1}{2}})^- = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} G^{-1} \quad \text{and} \quad (A_2^{\frac{1}{2}})^- = \begin{bmatrix} D_r^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} G^{-1}.
\]

Notably,

\[
0 < u'u + v'v
= x'((A_1^{\frac{1}{2}})^-)'(A_1^{\frac{1}{2}})^-x - x'((A_1^{\frac{1}{2}})^-)'(A_1^{\frac{1}{2}})^-(x+y)
- (x+y)'(A_1 + A_2)^-A_1^{\frac{1}{2}}(A_1^{\frac{1}{2}})^-x + (x+y)'(A_1 + A_2)^-A_1^{\frac{1}{2}}(A_1^{\frac{1}{2}})^'(A_1 + A_2)^-(x+y)
+ y'((A_2^{\frac{1}{2}})^-)'(A_2^{\frac{1}{2}})^-y - y'((A_2^{\frac{1}{2}})^-)'(A_2^{\frac{1}{2}})^'(A_1 + A_2)^-(x+y)
- (x+y)'(A_1 + A_2)^-(A_2^{\frac{1}{2}})(A_2^{\frac{1}{2}})^-y + (x+y)'(A_1 + A_2)^-(A_2^{\frac{1}{2}}(A_2^{\frac{1}{2}})^'(A_1 + A_2)^-(x+y).
\]

Moreover,

\[
((A_1^{\frac{1}{2}})^-)'(A_1^{\frac{1}{2}})'(A_1 + A_2)^- = (G^{-1})' \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} G'(G^{-1})' \begin{bmatrix} (I_r + D_r)^{-1} & 0 \\ 0 & 0 \end{bmatrix} G^{-1}
\]

\[
= (G^{-1})' \begin{bmatrix} (I_r + D_r)^{-1} & 0 \\ 0 & 0 \end{bmatrix} G^{-1}
\]

\[
= (A_1 + A_2)^-
\]

and

\[
((A_2^{\frac{1}{2}})^-)'(A_1^{\frac{1}{2}})^- = (A_1^-)' = A_1^-.
\]

Thus,

\[
0 < u'u + v'v = x'A_1^-x + y'A_2^-y + (x+y)'(A_1 + A_2)^-(x+y),
\]

and hence the lemma follows.

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