FOLD-FORMS FOR FOUR-FOLDS

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Abstract. This paper explains an application of Gromov’s h-principle to prove the existence, on any orientable 4-manifold, of a folded symplectic form. That is, a closed 2-form which is symplectic except on a separating hypersurface where the form singularities are like the pullback of a symplectic form by a folding map. We use the h-principle for folding maps (a theorem of Eliashberg) and the h-principle for symplectic forms on open manifolds (a theorem of Gromov) to show that, for orientable even-dimensional manifolds, the existence of a stable almost complex structure is necessary and sufficient to warrant the existence of a folded symplectic form.

1. Introduction

One says that a differential problem satisfies the h-principle if any formal solution (i.e., a solution for the associated algebraic problem) is homotopic to a genuine (i.e., differential) solution. Therefore, when the h-principle holds, one may concentrate on a purely topological question in order to prove the existence of a differential solution.

Differential problems are equations, inequalities or, more generally, relations involving derivatives of maps. The following are examples of problems known to satisfy the h-principle: existence of immersions in strictly positive codimension (theorems of Whitney [31], Nash [25], Kuiper [16], Smale [27], Hirsch [14] and Poénaru [26]), existence of symplectic forms on open manifolds (theorems of Gromov [12], who built the general machinery of the h-principle as an obstruction theory for the sheaves of germs of maps) and existence of maps whose only singularities are folds (theorem of Eliashberg [6, 7]).

This paper explains an application of the h-principle to prove the existence, on any compact orientable 4-manifold, of a folded symplectic form, that is, a closed 2-form with only fold singularities as defined...
below. According to the h-principle philosophy, this proof is divided in two steps:

1. show that the h-principle holds for this problem,
2. show that a formal solution exists.

For the first step, the basic ingredients are the h-principle for maps whose only singularities are folds \[6, 7\] and the h-principle for symplectic forms on open manifolds \[12\]. This combination is a shortcut based on an idea contained in the forthcoming book by Eliashberg and Mishachev \[9\]. We thus avoid dealing with the h-principle in its generality.

Here is the flavor of Eliashberg’s result. Let \(Z\) be a hypersurface in a manifold \(M\), that is, a codimension 1 embedded submanifold (this is the meaning of hypersurface throughout this paper). A map \(f : M \to N\) between manifolds of the same dimension is called a \(Z\)-immersion (or said to fold along the submanifold \(Z\)) if it is regular (i.e., its derivative is invertible) on \(M \setminus Z\), and if near any \(p \in Z\) and near its image \(f(p)\) there are coordinates centered at those points where \(f\) becomes

\[(x_1, x_2, \ldots, x_n) \mapsto (x_1^2, x_2, \ldots, x_n) .\]

A homomorphism \(F : TM \to TN\) between tangent bundles is called a \(Z\)-monomorphism, if it is injective on \(T(M \setminus Z)\) and on \(TZ\), and if there exists a fiber involution \(\tau : T \to T\) on a tubular neighborhood \(T\) of \(Z\) whose set of fixed points is \(Z\) and such that \(F \circ d\tau = F\). The differential \(df : TM \to TN\) of a \(Z\)-immersion is a \(Z\)-monomorphism. Eliashberg \[6\] proved that, if every connected component of \(M \setminus Z\) is open, then any \(Z\)-monomorphism \(TM \to TN\) is homotopic (within \(Z\)-monomorphisms \(TM \to TN\)) to the differential of a \(Z\)-immersion. In the language of \[13\], the theorem says that, when \(M \setminus Z\) is open, \(Z\)-immersions satisfy the (everywhere \(C^0\)-dense) h-principle; a \(Z\)-monomorphism is then called a formal solution. For the present application, we require a more general statement \[7\] dealing with foliated target manifolds.

A folded symplectic form on a \(2n\)-dimensional manifold \(M\) is a closed 2-form \(\omega\) which is nondegenerate except on a hypersurface \(Z\) called the folding hypersurface where, centered at every point \(p \in Z\), there are coordinates for \(M\) adapted to \(Z\) where the form \(\omega\) becomes

\[x_1 dx_1 \land dx_2 + dx_3 \land dx_4 + \ldots + dx_{2n-1} \land dx_{2n} .\]

The pullback of a symplectic form by a \(Z\)-immersion is a folded symplectic form with folding hypersurface \(Z\).
A formal solution for the problem of existence of a folded symplectic form turns out to be a stable almost complex structure. Let $M$ be a $2n$-dimensional manifold with a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, where $\mathbb{R}^2$ denotes the trivial rank 2 real vector bundle over $M$. We will show that $M$ admits folded symplectic forms.

Here is how Gromov’s theorem comes in. We embed $M$ as level zero in $M \times \mathbb{R}$. The given stable almost complex structure on $M$ yields a complex hyperplane field on $M \times \mathbb{R}$ and hence an almost complex structure on $M \times \mathbb{R}^2$. Since this manifold is open, Gromov’s application of the h-principle [12] guarantees the existence of a symplectic form on $M \times \mathbb{R}^2$ inducing almost complex structures in the same homotopy class as the given one. Since $M \times \mathbb{R}$ sits here as a codimension one submanifold, the restriction $\omega_0$ of the symplectic form to this submanifold has maximal rank, i.e., has exactly a one-dimensional kernel at every point. Let $\mathcal{L}$ be the one-dimensional foliation determined by the kernel $L$ of $\omega_0$. The projection of $\omega_0$ to $T(M \times \mathbb{R})/\mathcal{L}$ is well-defined and nondegenerate. Suppose that we could immerse $M$ in $M \times \mathbb{R}$ in a good way, meaning that locally the composition of that immersion with the projection to the local leaf space of $\mathcal{L}$ is a $Z$-immersion, for some hypersurface $Z$ in $M$. Since this leaf space is symplectic, by pullback we would obtain a folded symplectic form on $M$. Hence, we concentrate on deforming the initial embedding at level zero into a good immersion in order to prove:

**Theorem A** Let $M$ be a $2n$-dimensional manifold with a stable almost complex structure $J$. Then $M$ admits a folded symplectic form consistent with $J$ in any degree 2 cohomology class.

The notion of consistency is explained in §2. The existence of a stable almost complex structure is a necessary condition for the existence of a folded symplectic form on an orientable manifold (see §2). Theorem A is then saying that it is also sufficient. This contrasts with the case of a (honest) symplectic form, for whose existence an almost complex structure is necessary, but only sufficient if the manifold is open [12]. The sphere $S^6$ is a trivial example (thanks to Stokes’ theorem) and $CP^2 \# CP^2 \# CP^2$ is an important example (thanks to Seiberg-Witten invariants [29]) of almost complex manifolds without any symplectic form.

To produce a formal solution for 4-manifolds is easily accomplished. Hirzebruch and Hopf [15] showed that the integral Stiefel-Whitney class $W_3$ vanishes for any compact orientable 4-manifold, or, in other words,
such manifolds always have stable almost complex structures. (This is the same reason why such manifolds are spin-c [17, Thm.D.2].) Since we are in the stable range, it is enough to add a trivial $\mathbb{R}^2$ bundle to $TM$ for this to admit a structure of complex vector bundle. All this is also true when $M$ is not compact [11, §5.7]. We thus obtain the following relevant special case of Theorem A:

**Theorem B** Let $M$ be an orientable 4-manifold. Then $M$ admits a folded symplectic form consistent with any given stable almost complex structure and in any degree 2 cohomology class.

In higher dimensions, there are plenty of orientable manifolds which have no stable almost complex structures ($S^1 \times SU(3)/SO(3)$, for instance [17]), and hence cannot have folded symplectic forms. The condition $W_3(M) = 0$ is necessary and sufficient in dimensions 6 (since the next obstruction $W_7$ vanishes for dimensional reasons) and 8 (where Massey [20] proved that $W_7$ always vanishes). According to [4, 30], until 1998 it was still not known general necessary and sufficient conditions (in terms of invariants such as characteristic classes and the cohomology ring) for the existence of a stable almost complex structure on manifolds of dimension $\geq 10$.

As for the contents of this paper: § 2 reviews folded symplectic manifolds and some folded tangent bundles associated to them; § 3 describes the application of Gromov’s theorem to guarantee a symplectic form starting from a structure of complex vector bundle; § 4 proves the existence of an isomorphism between a folded tangent bundle and a suitable complex vector bundle; § 5 describes the application of Eliashberg’s theorem to produce folded symplectic forms; § 6 contains the conclusion of the proof of Theorems A and B.

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Y. Eliashberg has pointed out that Theorem A can be alternatively deduced from the Singular h-Principle Theorem of Gromov [13, p.112] following a hint in Exercise (b) of page 113 in Gromov’s book to overcome the lack of microflexibility.
2. Folded Symplectic Manifolds

Let $M$ be an oriented manifold of dimension $2n$, and let $\omega$ be a closed 2-form on $M$. The highest wedge power $\omega^n$ is a section of the (trivial) orientation bundle $\wedge^{2n}T^*M$.

**Definition.** A folded symplectic form is a closed 2-form $\omega$ such that $\omega^n$ intersects the 0-section of $\wedge^{2n}T^*M$ transversally, and such that $i^*\omega$ has maximal rank everywhere, where $i : Z \hookrightarrow M$ is the inclusion of the zero-locus, $Z$, of $\omega^n$.

By transversality, $Z$ is a codimension-1 submanifold of $M$, called the folding hypersurface. A folded symplectic manifold is a pair $(M, \omega)$ where $\omega$ is a folded symplectic form on $M$. The folding hypersurface $Z$ of a folded symplectic manifold $(M, \omega)$ separates $M$ into the regions $M^+$ and $M^-$, where the form matches or is opposite to the given orientation, respectively. Hence, $Z$ has a co-orientation depending on $\omega$ and on the choice of orientation on $M$. (The notion of folded symplectic form extends to arbitrary even-dimensional manifolds, not necessarily orientable, but we will not deal with those in this paper.)

The Darboux theorem for folded symplectic forms states that, if $(M, \omega)$ is a folded symplectic manifold and $p$ is any point on the folding hypersurface $Z$, then there is a coordinate chart $(U, x_1, \ldots, x_{2n})$ centered at $p$ such that on $U$

$$\omega = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n}$$

and $Z \cap U = \{x_1 = 0\}$.

This follows, for instance, from a folded analogue of Moser’s trick [3].

Doubles of symplectic manifolds with $\omega$-convex [8] (or $\omega$-concave) boundary are easy examples of manifolds with folded symplectic forms. Simplest instances are the spheres $S^{2n}$, where a folded symplectic form is obtained by pulling back the standard symplectic form on $\mathbb{R}^{2n}$ via the folding map $S^{2n} \rightarrow D^{2n}$.

Starting in dimension 4, folded symplectic forms are not generic in the set of closed 2-forms. Let $M$ be a (compact) oriented 4-manifold, and let $\omega$ be a closed 2-form on $M$. If $\gamma$ is a given volume form on $M$, then $\omega \wedge \omega = f \gamma$ for some $f \in C^\infty(M)$. A generic $\omega$ [18] is never 0, has rank 2 on a (compact) codimension-1 submanifold, $Z$, and is nondegenerate elsewhere. The hypersurface $Z$ is the 0-locus of $f$. Its complement $M \setminus Z$ is the disjoint union of the sets $M^+ = \{f > 0\}$ where $\omega$ matches the given orientation and $M^- = \{f < 0\}$ where $\omega$ induces
the opposite orientation. For \( \omega \) to be folded symplectic, we would need that \( TZ \) and the rank 2 bundle over \( Z \) given by \( \ker \omega \) intersect transversally as subbundles of \( TM|_Z \). Yet generically \( \omega \) is not folded symplectic, since its restriction to \( Z \) vanishes along some codimension-2 submanifold \( C \) (a union of circles), where \( \ker \omega \) is contained in \( TZ \). Although a generic 2-form on a 3-manifold vanishes only at isolated points, here the 3-manifold already depends on the 2-form. Moreover, generically there are isolated parabolic points on those lines (circles), where the tangent space to those lines is contained in \( \ker \omega \). There is at least one continuous family of inequivalent neighborhoods of parabolic points [1, 10].

Now let \( M \) be an \( m \)-dimensional manifold with a separating hypersurface \( Z \). For instance, \( M \) could be an oriented manifold equipped with a folded symplectic form, and \( Z \) its folding hypersurface.

The complement \( M\setminus Z \) is the disjoint union of open sets \( M^+ \) and \( M^- \). Over \( Z \), the tangent bundle has a trivial line subbundle \( V \), spanned by a vector field transverse to \( Z \) pointing from \( M^- \) to \( M^+ \). The quotient \( TM/V \) is isomorphic to \( TZ \), so that \( TM|_Z \cong TZ \oplus V \).

**Definition.** The \( Z \)-tangent bundle of \( M \) is the rank \( m \) real vector bundle \( ^ZTM \) over \( M \) obtained by gluing \( TM|_{M\setminus M^-} \) to \( TM|_{M\setminus M^+} \) by the constant diagonal map \( \text{Id} \oplus (-1) : Z \to \text{GL}(TZ \oplus V) \).

There are analytic and algebraic approaches to \( ^ZTM \), which enhance its geometry [3]. From its definition it follows that:

**Lemma 1.** Let \( M \) be an \( m \)-dimensional manifold with a separating hypersurface \( Z \). Then there is an isomorphism of real vector bundles

\[
TM \oplus \mathbb{R} \cong ^ZTM \oplus \mathbb{R}.
\]

A complex structure on a vector bundle \( E \) over a manifold \( M \) is a bundle homomorphism \( J : E \to E \) such that \( J^2 = -\text{Id} \). If \( E \) is an orientable rank \( 2m \) vector bundle, the existence of a complex structure on \( E \) is equivalent to the existence of a section of the associated \( (SO(2m)/U(m)) \)-bundle. A stable complex structure on a vector bundle \( E \) over \( M \) is an equivalence class of complex structures on the vector bundles \( E \oplus \mathbb{R}^k (k \in \mathbb{Z}_0^+) \), two complex structures, \( J_1 \) on \( E \oplus \mathbb{R}^{k_1} \) and \( J_2 \) on \( E \oplus \mathbb{R}^{k_2} \), being equivalent when there exist \( m_1, m_2 \in \mathbb{Z}_0^+ \) such that \((E \oplus \mathbb{R}^{k_1}) \oplus \mathbb{C}^{m_1}, J_1 \oplus i) \) and \((E \oplus \mathbb{R}^{k_2}) \oplus \mathbb{C}^{m_2}, J_2 \oplus i) \) are isomorphic complex vector bundles. A stable almost complex structure on \( M \) is a stable complex structure on \( TM \).
The $Z$-tangent bundle for the folding hypersurface $Z$ of a folded symplectic form $\omega$ has a canonical complex structure $J_0$ consistent with $\omega$. We say that a folded symplectic form $\omega$ is consistent with a stable almost complex structure on $M$ if $(\mathbb{Z}TM \oplus \mathbb{C}, J_0 \oplus i)$ belongs to the given equivalence class of complex structures on $TM \oplus \mathbb{R}^2k$, $k \in \mathbb{Z}^+$.  

### 3. First Instance of the h-Principle

Let $M$ be a $2n$-dimensional manifold with a stable almost complex structure. The homotopy groups $\Pi_q(\text{SO}(2m)/U(m))$ are isomorphic for fixed $q$ and variable $m$ such that $q < 2m - 1$ (this is the so-called stable range [19]). Hence, if there exists a complex structure on $TM \oplus \mathbb{R}^2k$, then there exists a complex structure on $TM \oplus \mathbb{R}^2$.

Let $J$ be a complex structure on $TM \oplus \mathbb{R}^2$. Let

$$i : M \hookrightarrow M \times \mathbb{R} \quad \text{and} \quad \pi : M \times \mathbb{R} \twoheadrightarrow M \quad \text{be the embedding at level zero, and the projection to the first factor.}$$

By pullback, $i$ induces an isomorphism in cohomology.

Via the identification $T(M \times \mathbb{R}) \simeq \pi^*(TM) \oplus \mathbb{R}$, the structure $J$ induces a structure of complex vector bundle, still called $J$, on $T(M \times \mathbb{R}) \oplus \mathbb{R} \simeq \pi^*(TM) \oplus \mathbb{C}$. Then the complex subbundle

$$H_0 = T(M \times \mathbb{R}) \cap J(T(M \times \mathbb{R})) \subset T(M \times \mathbb{R}) \oplus \mathbb{R}$$

is a complex hyperplane field over $M \times \mathbb{R}$. Let $\omega_1$ be a 2-form of maximal rank in $M \times \mathbb{R}$ compatible with $J$, that is,

$$\omega_1(u,v) = g(Ju, v), \quad \forall u,v \in H_0, \quad \text{and} \quad \omega_1(u, \cdot) = 0, \quad \forall u \in H_0^\perp,$$

for some riemannian metric $g$ on $TM \times \mathbb{R}$, where $H_0^\perp$ denotes the orthocomplement of $H_0$ with respect to $g$. A regular homotopy of two 2-forms of maximal rank is a homotopy within 2-forms of maximal rank.

**Lemma 2.** Let $M$ be a manifold with a structure $J$ of complex vector bundle on $TM \oplus \mathbb{R}^2$. Then there exists in $M \times \mathbb{R}$ a closed 2-form of maximal rank in any degree 2 cohomology class, which is regularly homotopic to any 2-form of maximal rank compatible with $J$.

This is an immediate consequence of the following proposition which was originally proved by McDuff [21]. The proof below is taken from Eliashberg-Mishachev [9]. We reproduce it since this result is not as widely known as the other applications of the h-principle and since the
idea in this proof is crucial for the present paper’s strategy. The key to this proof is Gromov’s theorem \cite{12} saying that, for every degree 2 cohomology class on any open manifold, any nondegenerate 2-form is regularly homotopic to a symplectic form in that class; moreover, if two symplectic forms are regularly homotopic, then they are homotopic within symplectic forms. Recall that a manifold is \textit{open} if there are no closed manifolds (i.e., compact and without boundary) among its connected components.

\textbf{Proposition.} \cite{21} For any 2-form of maximal rank on an odd-dimensional manifold and any degree 2 cohomology class, there exists a closed 2-form of maximal rank in that class which is regularly homotopic to the given form.

\textbf{Proof.} Let $\omega_1$ be a 2-form of maximal rank on a $(2n+1)$-dimensional manifold $N$ and let $\alpha$ be a degree 2 cohomology class in $N$. By homotopy, the projection to the first factor $\pi : N \times \mathbb{R} \to N$ induces an isomorphism in cohomology.

If $N$ is orientable, then $\omega_1$ extends in a homotopically unique way compatible with orientations to a nondegenerate 2-form, $\omega_2$, in $N \times \mathbb{R}$. Gromov’s result \cite{12} cited above guarantees the existence, in the class $\pi^*\alpha$, of a homotopically unique symplectic form $\omega_3$ in $N \times \mathbb{R}$ regularly homotopic to $\omega_2$. The restriction of $\omega_3$ to the zero level $M$ is a closed 2-form of maximal rank.

If $N$ is not orientable, we replace $N \times \mathbb{R}$ in the previous argument by the total space of the real line bundle given by the kernel of $\omega_1$. $\square$

4. \textbf{Vector Bundle Isomorphism}

Let $\tilde{\omega}$ be a closed 2-form of maximal rank in $M \times \mathbb{R}$, and let $L$ be the line field on $M \times \mathbb{R}$ given by the kernel of $\tilde{\omega}$ at each point. By orientability of $M$, the line bundle $L$ is trivializable. Let $\mathcal{L}$ be the 1-dimensional foliation corresponding to $L$. Choose a complementary hyperplane field $H$ so that $T(M \times \mathbb{R}) \simeq H \oplus L$.

Let $Z_0$ be a separating hypersurface in $M$ with a coorientation. Since by Lemma \ref{lemma} we have that

$$Z_0TM \oplus \mathbb{R} \simeq TM \oplus \mathbb{R} \simeq i^*(H \oplus L),$$

the restriction $i^*H$ is stably isomorphic to $Z_0TM$. The Stiefel-Whitney classes are stable invariants, and the mod 2 reduction of the Euler class
of an orientable rank $m$ real vector bundle $E$ coincides with the $m$th Stiefel-Whitney class of $E$ (see, for instance, [24]). Therefore the Euler numbers (i.e., the evaluations of the Euler classes over the fundamental homology class) of $i^*H$ and of $\mathbb{Z}_0TM$ differ by an even integer, let us say

$$\chi(i^*H) = \chi(\mathbb{Z}_0TM) + 2k.$$ 

If two stably isomorphic orientable rank $2n$ real vector bundles over an $2n$-dimensional connected manifold have the same Euler number, then they are isomorphic. This was contained in the work of Dold and Whitney when the base is a 4-manifold [5]. In general, this follows from observing in the diagram

$$\begin{array}{ccc}
S^{2n} & \hookrightarrow & SO/\text{SO}(2n) \\
\downarrow & & \downarrow \\
M^{2n} & \Rightarrow & \text{BSO}(2n) \\
\downarrow & & \downarrow \\
& & \text{BSO}
\end{array}$$

that the fiber $SO/\text{SO}(2n)$ of $\text{BSO}(2n) \rightarrow \text{BSO}$ is $(2n - 1)$-connected, that $[M^{2n}, S^{2n}] \simeq \mathbb{Z}$ where the homotopy type is detected by the degree, and that the pullback of the Euler class to $S^{2n}$ is nontrivial (since $S^{2n} \rightarrow \text{BSO}(2n)$ is the classifying map for $TS^{2n}$).

Consider the following operation on rank $m$ real vector bundles over $m$-dimensional manifolds. If $E$ is such a bundle and $D^m$ is a small disk in the base manifold $M$, let $E^\sharp TS^m$ be the bundle obtained by gluing $E|_{M\setminus \text{Int}D^m}$ to the trivial bundle $\mathbb{R}^m$ over $D^m$ by the characteristic map of $TS^m$, i.e., by the map $S^{m-1} \rightarrow \text{SO}(m)$ which characterizes the tangent bundle of $S^m$ as the gluing over the equator of northern and southern trivial bundles [28, §18.1]. For an integer $k$, the bundle $E^\sharp kTS^m$ is built analogously by taking the $k$th power of the characteristic map of $S^m$. By counting with orientations the vanishing points of a section transverse to zero, we see that $E^\sharp kTS^m$ has Euler characteristic $\chi(E) + 2k$. We conclude that

$$i^*H \simeq \mathbb{Z}_0TM^\sharp kTS^{2n}.$$ 

For $k$ positive, let $Z$ be the union of $Z_0$ with $k$ homologically trivial spheres $S^n$ contained in the negative part of $M \setminus Z_0$ with respect to the given coorientation. For $k$ negative, define $Z$ similarly but with the spheres in the positive part of $M \setminus Z_0$. It follows from the computations in [6, §3.9] that $i^*H$ and $\mathbb{Z}_0TM$ have the same Euler number, and hence are isomorphic. It is possible to start from the empty hypersurface, in which case a coorientation is not defined. Yet the same argument
holds by taking $Z$ to be a union of spheres (as many as half of the absolute value of the difference of the Euler numbers of $TM$ and of $i^*H$) whose coorientation is determined by the sign of $k$ above. We have thus proved the following:

**Lemma 3.** Let $H$ be a coorientable hyperplane field in $M \times \mathbb{R}$ and $i : M \hookrightarrow M \times \mathbb{R}$ the inclusion at level zero. The restriction $i^*H$ is isomorphic to $ZTM$, where $Z$ is a separating hypersurface as described in the previous paragraph.

5. **Second Instance of the h-Principle**

Throughout this section, let $M$ be an $m$-dimensional manifold with a hypersurface $Z$, and let $N$ be an $(m+1)$-dimensional manifold with a 1-dimensional foliation $\mathcal{L}$. The following notions are due to Eliashberg [7].

**Definition.** A map $f : M \to N$ is a $Z$-immersion relative to $\mathcal{L}$, if near any point $p \in M \setminus Z$ there are coordinates $y_1, \ldots, y_{m+1}$ in $N$ adapted to the foliation (i.e., each leaf is a level set of the first $m$ coordinates) where the induced map to each level set of $y_{m+1}$ is regular, and if near any $p \in Z$ and near its image there are coordinates centered at those points and adapted to the foliation where $f$ becomes

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1^2, x_2, \ldots, x_m, 0).$$

In the adapted coordinates $x_i$, the hypersurface $Z$ is given by $x_1 = 0$. Loosely speaking, a $Z$-immersion relative to $\mathcal{L}$ is a $Z$-immersion to the leaf space of $\mathcal{L}$. The definition extends to higher-dimensional foliations whose codimension is equal to the dimension of $M$.

**Lemma 4.** Let $\widetilde{\omega}$ be a closed 2-form of maximal rank in $N$ whose kernel is the tangent space to the leaves of $\mathcal{L}$. If $f : M \to N$ is a $Z$-immersion relative to $\mathcal{L}$, then $f^*\widetilde{\omega}$ is a folded symplectic form on $M$ with folding hypersurface $Z$.

The reason is simply that the form $\widetilde{\omega}$ induces a symplectic form in the local leaf spaces and that the composition of $f$ with the local quotient maps is a $Z$-immersion.

**Proof.** Let $p \in M$. There is a neighborhood $\mathcal{U}$ of $f(p)$ where we have a trivialization $\mathcal{U} \simeq \mathcal{F}_\mathcal{U} \times \mathcal{L}_\mathcal{U}$, given in local coordinates centered at $f(p)$ by $(x_1, \ldots, x_{m+1}) \mapsto ((x_1, \ldots, x_m), x_{m+1})$, the set $\mathcal{F}_\mathcal{U}$ being a
leaf space (say the level zero of $x_{m+1}$), and $\mathcal{L}_U$ a typical leaf (say the level zero of $(x_1, \ldots, x_m)$). The restriction of $\tilde{\omega}$ to $\mathcal{F}_U$ is a symplectic form, $\omega_U$. The composition $g_U : f^{-1}(U) \to \mathcal{F}_U$ of $f$ with the projection to $\mathcal{F}_U$ is a $(Z \cap U)$-immersion, so that $g_U^*\omega_U$ is a folded symplectic form with folding hypersurface $Z \cap U$. The result follows from the fact that $f^*\tilde{\omega}$ on $f^{-1}(U)$ coincides with $g_U^*\omega_U$. □

We now turn to the formal analogue of a $Z$-immersion.

**Definition.** A bundle map $F : TM \to TN$ is a $Z$-monomorphism relative to $\mathcal{L}$, if $F|_{T(M \setminus Z)}$ is transverse to $\mathcal{L}$, and if each $p \in Z$ admits a neighborhood $U$ where $F|_{TU}$ is the differential of some $(Z \cap U)$-immersion relative to $\mathcal{L}$.

The following lemma is a direct consequence of Eliashberg’s result in [7, §6.3], where he extends to the case of foliations the result described in the introduction.

**Lemma 5.** Let $N = M \times \mathbb{R}$ be equipped with a decomposition $TN \simeq H \oplus L$, where $L$ is a line field, and let $\mathcal{L}$ be the corresponding 1-dimensional foliation. Let the hypersurface $Z$ be such that every connected component of $M \setminus Z$ is open. Then, for every $Z$-monomorphism $F : TM \to TN$ relative to $\mathcal{L}$, there exists a $Z$-immersion $f : M \to N$ relative to $\mathcal{L}$ whose differential $df$ is homotopic to $F$ through $Z$-monomorphisms relative to $\mathcal{L}$.

Part of the work to prove Theorem A consists in showing a (general) procedure to deform by homotopy a weaker bundle map into a $Z$-monomorphism relative to $\mathcal{L}$. The weaker map is of the following type:

**Definition.** A bundle map $F : TM \to TN \simeq H \oplus L$ is a $Z$-monomorphism relative to $L$, if $\pi_L \circ F|_{T(M \setminus Z)}$ and $\pi_L \circ F|_{TZ}$ are fiberwise injective, $\pi_L : TN \to H$ being the projection along $L$, and if there is a tubular neighborhood $T$ of $Z$ in $M$, with a fiber involution $\tau : T \to T$ whose set of fixed points is $Z$, where $F \circ d\tau = F$.

6. Conclusion of the Proof

Let $M$ be a compact $2n$-dimensional manifold with a stable almost complex structure $J$. Then $J$ is representable by a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, and any two such representatives are isomorphic, by Bott periodicity [2]. Let $N = M \times \mathbb{R}$ and denote still
by $J$ an induced structure of complex vector bundle on $TN \oplus \mathbb{R}$ as in §3.

By Lemma 2 there exists on $N$, in any degree 2 cohomology class, a closed 2-form $\tilde{\omega}$ of maximal rank compatible with $J$. Let $\tilde{\omega}$ be such a form and let $L$ be the line field given by its kernel, with associated foliation $\mathcal{L}$.

By Lemma 4 the existence of a folded symplectic form on $M$ with some folding hypersurface $Z$ is guaranteed by the existence of a $Z$-immersion $f : M \to N$ relative to $\mathcal{L}$. We will seek such a $Z$-immersion which is homotopic to the embedding at level zero $i : M \hookrightarrow N$, so that $f^* = i^*$ in cohomology. If $M$ is connected and $Z$ is nonempty, then $M \setminus Z$ is open.

By Lemma 5 in order to produce a $Z$-immersion $f$ relative to $\mathcal{L}$ for $M \setminus Z$ open, it suffices to show that there exists a $Z$-monomorphism $F : TM \to TN$ relative to $\mathcal{L}$. So that $f$ is homotopic to $i$, we search for an $F$ covering a map $M \to N$ homotopic to $i$.

By Lemma 3 we have a vector bundle isomorphism $F_0 : ZTM \to i^*H$ for some hypersurface $Z$, which may be chosen so that each connected component of $M \setminus Z$ is open.

The map $F_0$ may be translated into a fiberwise injective bundle map $F_1 : ZTM \to H$ covering the immersion $i : M \to N$. This map guarantees the existence of a (canonically unique up to homotopy) almost $Z$-monomorphism $F_2 : TM \to H \oplus L$ relative to $L$, still covering $i$, defined by the following recipe:

Choose a trivial line bundle $V$ over $Z$ spanned by a vector field on $M$ transverse to $Z$ pointing from $M^-$ to $M^+$. The quotient $ZTM/V$ is isomorphic to $TZ$, so that $ZTM|_Z \simeq TZ \oplus V$. We obtain $TM$ by gluing $ZTM|_{M \setminus M^-}$ to $ZTM|_{M \setminus M^+}$ by the constant diagonal map $\text{Id} \oplus (-1) : Z \to \text{GL}(TZ \oplus V)$. Using this recovery of $TM$ from $ZTM$, we may define $F_2$ equal to $F_1 \oplus 0$ outside a tubular neighborhood $\mathcal{T}$ of $Z$ in $M$, and on $\mathcal{T}$ set

$$F_2(u \oplus v) = F_1(u \oplus \psi v) \oplus 0 ,$$

with respect to the decomposition $ZTM|_{\mathcal{T}} \simeq \pi^*(TZ) \oplus \pi^*V$, where $\pi : \mathcal{T} \to Z$ is the tubular projection, and $\psi : \mathcal{T} \to [0,1]$ is equal to 1 outside a narrower tubular neighborhood of $Z$ and vanishes exactly over $Z$. By choosing $\psi$ symmetric with respect to an involution $\tau : \mathcal{T} \to \mathcal{T}$ whose set of fixed points is $Z$, we obtain $F_2$ invariant under $\tau$. 
For each \( p \in Z \), choose a connected neighborhood \( U \) whose image \( i(U) \) is contained in a connected trivialization \( \mathcal{N}_U \approx \mathcal{F}_U \times \mathcal{L}_U \) of the foliation \( \mathcal{L} \), the set \( \mathcal{F}_U \) being a local leaf space and \( \mathcal{L}_U \) a leaf segment. Let \( \pi_U : \mathcal{N}_U \to \mathcal{F}_U \) be the projection to the first factor. The composition \( F_{2,\mathcal{U}} = d\pi_U \circ F_{2,\mathcal{U}} : TU \to T\mathcal{F}_U \) is a \((Z \cap U)\)-monomorphism.

By [6, §2.2], the composition \( F_{2,\mathcal{U}} \) is homotopic, through \((Z \cap U)\)-monomorphisms, to the differential \( dg_U \) of a \( Z \)-immersion \( g_U : U \to \mathcal{F}_U \).

Moreover, if over a closed subset \( W \subset U \), the composition \( F_{2,\mathcal{U}} \) was already the differential of a map, then there is a homotopy which is constant on \( W \). Let \( G_t : TU \to T\mathcal{F}_U, 1 \leq t \leq 2 \), be a homotopy such that \( G_1 = dg_U \) and \( G_2 = F_{2,\mathcal{U}} \).

Choose a \((Z \cap U)\)-immersion \( \tilde{g}_U : U \to \mathcal{N}_U \) relative to \( \mathcal{L} \) such that \( \pi_U \circ \tilde{g}_U = g_U \). We can always pick a \( \tilde{g}_U \) extending a sensible preassigned lift over a closed subset \( W \) of \( U \).

By the covering homotopy property for the fibering \( TN_U \to T\mathcal{F}_U \), there is a lifted homotopy \( \tilde{G}_t : TU \to T\tilde{N}_U, 1 \leq t \leq 2 \), through \( Z \)-monomorphisms relative to \( L \) such that \( \tilde{G}_1 = d\tilde{g}_U \) and \( d\pi_U \circ \tilde{G}_t = G_t \) for all \( t \). If \( G_t \) was constant on a closed subset \( W \), then we may choose \( \tilde{G}_t \) also constant on \( W \).

Take a riemannian metric symmetric with respect to \( \tau \). For a point \( p \in Z \), choose spherical neighborhoods \( U_1 \) and \( U_2 \) in \( T \), consisting of points at a riemannian distance less than \( \varepsilon \) and \( 4\varepsilon \) from \( p \), with \( \varepsilon > 0 \) small enough for the exponential map to be injective and for the closure of \( U_2 \) to be contained in the neighborhood \( U \) above. Choose a smooth function \( \rho : U_2 \to [1, 2] \) satisfying \( \rho(q) = 2 \) if the distance from \( p \) to \( q \) is greater than \( 3\varepsilon \), and \( \rho(q) = 1 \) if the distance from \( p \) to \( q \) is less than \( 3\varepsilon \).
2ε. Define $F_3 : TM \to TN$ by

$$F_3 = \begin{cases} 
F_2 & \text{on } M \setminus \mathcal{U}_2 \\
\tilde{G}_{\rho(q)} & \text{over points } q \in \mathcal{U}_2 \setminus \mathcal{U}_1 \\
d\tilde{g}_{\mathcal{U}} & \text{on } \mathcal{U}_1.
\end{cases}$$

Then $F_3$ is a $\mathbb{Z}$-monomorphism with respect to $L$ whose restriction to $\mathcal{U}_1$ is the differential of a $(\mathbb{Z} \cap \mathcal{U}_1)$-immersion relative to $\mathcal{L}$.

Since $\mathcal{Z}$ is compact, take a subcover of $\mathcal{Z}$ in $M$ by a finite number of the $\mathcal{U}_1$'s. Apply iteratively the construction of the previous paragraph to an ordering of the $\mathcal{U}_1$'s, starting first from $F_2$ and then from its replacements $F_3$, etc. At each stage, the homotopy should be taken constant over the closure $\mathcal{W}$ of the previous $\mathcal{U}_1$'s.

We have thus concluded the proof of Theorem A in the compact case by showing the existence of a $\mathbb{Z}$-monomorphism relative to $\mathcal{L}$ covering a map homotopic to $i$.

**Remark.** If $M$ is a compact oriented 2-dimensional manifold, folded symplectic forms on $M$ are generic 2-forms. The cohomology class of a 2-form is determined by its total integral. The isomorphism classes of complex structures on $TM \oplus \mathbb{R}^2$ are determined by the Euler number, which is an even integer. By changing $\mathbb{Z}$ as in §4 any even number may be obtained as Euler number for $\mathbb{Z}TM$, thus fitting any given stable complex structure. Let $\omega$ be a 2-form which vanishes transversally on an appropriate $\mathcal{Z}$. By changing the values of $\omega$ over $M \setminus \mathcal{Z}$, any real number may be obtained as total integral of $\omega$. Hence, Theorem A holds easily (and not interestingly) for compact 2-manifolds.

For the noncompact case, a statement stronger than Theorem A is true. If a 2n-dimensional manifold $M$ is orientable, connected, not compact and $TM \oplus \mathbb{R}^2$ has a complex structure, then $M$ has an almost complex structure because it retracts to a $(2n - 1)$-dimensional cell complex [23 Thm.8.1] and $\Pi_q(SO(2n)/U(n)) \cong \Pi_q(SO(2n + 2)/U(n + 1))$ for $q \leq 2n - 2$. By Gromov’s theorem [12], $M$ admits a compatible symplectic form in any degree 2 cohomology class.

Let $E$ be a rank $2m$ oriented real bundle over $M$. The condition $W_3(E) = 0$ ensures the existence over the 3-skeleton of $M$ of a section for the associated $(SO(2m)/U(m))$-bundle. By Bott’s periodicity, $\Pi_q(SO(6)/U(3)) = 0$ for $q < 5$. Therefore, the Hirzebruch-Hopf fact [15] that $W_3(M) = 0$ for any orientable 4-manifold, asserts the existence of a stable complex structure on any such manifold.
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