On the Holroyd-Talbot Conjecture
for Sparse Graphs

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Abstract

Given a graph $G$, let $\mu(G)$ denote the size of the smallest maximal independent set of $G$. A family of sets is called a star if some element is in every set of the family. A split vertex has degree at least 3. Holroyd and Talbot conjectured the following Erdős-Ko-Rado-type statement about intersecting families of independent sets of graphs: if $1 \leq r \leq \mu(G)/2$ then there is an intersecting family of independent $r$-sets of maximum size that is a star. In this paper we prove similar statements for sparse graphs on $n$ vertices: roughly, for graphs of bounded average degree with $r \leq O(n^{1/3})$, for graphs of bounded degree with $r \leq O(n^{1/2})$, and for trees having a bounded number of split vertices with $r \leq O(n^{1/2})$.

1 Introduction

For $0 \leq r \leq n$, let $\binom{[n]}{r}$ denote the family of $r$-element subsets ($r$-sets) of $[n] = \{1, 2, \ldots, n\}$. For any family $\mathcal{F}$ of sets, define the shorthand $\cap \mathcal{F} = \cap_{S \in \mathcal{F}} S$.

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If $\cap F \neq \emptyset$, we say that $F$ is a star; in this case, any $x \in \cap F$ is called a center. The family $F_x = \{S \in F \mid x \in S\}$ is called the full star of $F$ at $x$. Furthermore, we define the notation $F^r = \{S \in F \mid |S| = r\}$. The family $F$ is intersecting if every pair of its members intersects.

Erdős, Ko, and Rado [11] proved the following classical theorem of central importance in extremal set theory.

**Theorem 1. (Erdős-Ko-Rado, 1961)** If $F \subseteq \binom{[n]}{r}$ is intersecting for $r \leq n/2$, then $|F| \leq \binom{n-1}{r-1}$. Moreover, if $r < n/2$, equality holds if and only if $F = \binom{[n]}{r}_x$ for some $x \in [n]$.

Hilton and Milner [16] proved the following stronger stability result.

**Theorem 2. (Hilton-Milner, 1967)** If $F \subseteq \binom{[n]}{r}$ is intersecting for $r \leq n/2$, and $F$ is not a star, then $|F| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$.

For a graph $G$, let $\mathcal{I}(G)$ denote the family of independent sets of $G$. We write $s_r(v) = |\mathcal{I}_v(G)|$ when $G$ is understood. Let $F \subseteq \mathcal{I}(G)$ be an intersecting subfamily of maximum size. We say that $G$ is $r$-EKR if some $v$ satisfies $s_r(v) = |F|$, and strictly $r$-EKR if every such $F$ equals $\mathcal{I}_v(G)$ for some $v$.

Write $\alpha(G)$ for the independence number of $G$. Let $\mu(G)$ denote the size of a smallest maximal independent set of $G$. Equivalently, $\mu(G)$ is the size of the smallest independent dominating set of $G$. Holroyd and Talbot [18] made the following conjecture.

**Conjecture 3. (Holroyd-Talbot, 2005)** For any graph $G$, if $1 \leq r \leq \mu(G)/2$ then $G$ is $r$-EKR.

Of course, this conjecture is true for the empty graph by Theorem 1. While not explicitly stated in graph-theoretic terms, earlier results by Berge [2], Deza and Frankl [10], and Bollobás and Leader [4] support the conjecture. For example, the case of $G$ equal to a disjoint union of $k$ complete graphs of sizes
n_1 \leq \cdots \leq n_r \text{ was verified (in fact for all } r \leq \alpha(G) \text{)} in [4,10] for the uniform case \(2 \leq n_1 = \cdots = n_k\), in [17] for the non-uniform case \(2 \leq n_1 \leq \cdots \leq n_k\), and in [3] for the general case. The cases of } G \text{ being a power of either a path [17] or a cycle [22], or a special chain (essentially, a path of complete graphs of increasing size) or the disjoint union of two special chains [19], were both verified for all } r \leq \alpha(G) \text{ as well. The conjecture has been proven for } \mu(G) \text{ sufficiently large in terms of } r \text{ [5], and also for various graph classes, for example, disjoint unions of complete graphs, paths, and cycles containing at least one isolated vertex [7,17], disjoint unions of complete multipartite graphs containing at least one isolated vertex [8], disjoint unions of length-2 paths [14], chordal graphs containing an isolated vertex [19], and others. In fact, for the cases of complete graphs and cycles just mentioned, [7] extends the range of } r \text{ beyond } \mu(G)/2 \text{ to } \alpha(G)/2. \text{ One can observe, for example, that the complete } k\text{-partite graph } G = K_{n_1,\ldots,n_k} \text{ is } r\text{-EKR for all } r \leq \alpha(G)/2, \text{ because every independent set is contained in some part. However, } G \text{ is not } r\text{-EKR for } \alpha(G)/2 < r \leq \alpha(G).

For vertices } u \text{ and } v \text{ in a graph } G, \text{ we use the notations } \deg_G(u) \text{ and } \dist_G(u,v) \text{ for the degree of vertex } u \text{ and the distance between } u \text{ and } v \text{ in } G, \text{ respectively; we may omit the subscript if the context is clear.}

2 \quad \text{Results}

Here we prove the following theorem.

\textbf{Theorem 4. Let } r \text{ and } d \text{ be positive integers. Suppose that } G \text{ is a graph on } n > \frac{27}{8}dr^2 \text{ vertices, having maximum degree less than } d. \text{ Then } G \text{ is } r\text{-EKR.}

We can expand the class of graphs beyond bounded degree to bounded average degree at the cost of reducing the range of } r \text{ from } O(n^{1/2}) \text{ to } O(n^{1/3}), \text{ as follows.}
**Theorem 5.** Given a positive integer \( r \), let \( c \geq e/36 \) be a constant. Suppose that \( G \) is a graph on \( n > 18cr^3 \) vertices, having at most \( cn \) edges. Then \( G \) is \( r \)-EKR.

It is likely that a quadratic bound on \( n \) is possible for Theorem 5 as well. Note that the case \( c = 1 \) in Theorem 5 is especially relevant for trees. In this case, we can retrieve a quadratic lower bound for \( n \) for one special class of trees.

A *split vertex* in a graph is a vertex of degree at least three. A *spider* is a tree with exactly one split vertex. For a spider \( S \) with split vertex \( w \) and leaves \( v_1, \ldots, v_k \), we write \( S = S(\ell_1, \ldots, \ell_k) \), where \( \ell_i = \text{dist}(w, v_i) \). The notation is written in *spider order* when the following conditions hold:

- if \( \ell_i \) and \( \ell_j \) are both odd and \( \ell_i < \ell_j \) then \( i < j \);
- if \( \ell_i \) and \( \ell_j \) are both even and \( \ell_i < \ell_j \) then \( i > j \); and
- if \( \ell_i \) is odd and \( \ell_j \) is even then \( i < j \).

Notice that, since every independent set of \( S(1, 1, \ldots, 1) \) is a subset of its leaves, Conjecture 3 is true for \( S(1, 1, \ldots, 1) \). In an attempt to prove the Holroyd-Talbot conjecture for spiders by induction, the authors of [20] proved the following result.

**Theorem 6.** (Hurlbert-Kamat, 2022) Suppose that \( S = S(\ell_1, \ldots, \ell_k) \) is a spider written in spider order. Let \( w \) be the split vertex of \( S \), for each \( i \) let \( u_i \) be any vertex on the \( wv_i \)-path, and suppose that \( r \leq \alpha(S) \). Then

1. \( s_r(w) \leq s_r(v_i) \) for all \( i \),
2. \( s_r(u_i) \leq s_r(v_i) \) for all \( i \), and
3. \( s_r(v_j) \leq s_r(v_i) \) for all \( i < j \).
Estrugo and Pastine [12] call a tree $T$ $r$-HK if $s_r(v)$ is maximized at a leaf of $T$ (and HK if $r$-HK for all $r \leq \alpha(T)$). It is proved in [19] that every tree is $r$-HK for $r \leq 4$, but Baber [1], Borg [6], and Feghali, Johnson, and Thomas [13] each found counterexamples when $r \geq 5$. However, parts 1 and 2 of Theorem 6 together imply that every spider $S$ is HK. Theorem 5 shows that spiders are $r$-EKR for $r < \left(\frac{n}{18}\right)^{1/3}$. Unfortunately, $\mu/2$ for spiders is roughly $n/6$, so there remains a big gap. Our next theorem shrinks that gap somewhat.

**Theorem 7.** Let $S = S(\ell_1, \ldots, \ell_k)$ be a spider on $n$ vertices, with split vertex $w$ and leaves $v_1, \ldots, v_k$. Suppose that $r \leq \sqrt{n \ln 2} - (\ln 2)/2$. Then $S$ is $r$-EKR.

We note that every spider $S$ has $\alpha(S) = 1 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n \leq 2$, $\alpha(S) = 2 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n = 3$, and $\alpha(S) \geq (n - 1)/2 > \sqrt{n \ln 2} - (\ln 2)/2$ for $n \geq 4$. In other words, the hypothesis of Theorem 7 implies that $r \leq \alpha(S)$ for all $n$.

Finally, we prove the following similar result for more general trees.

**Theorem 8.** Let $T$ be a tree on $n$ vertices, with exactly $s > 1$ split vertices. Suppose that $1 < s < r/2$ and $r \leq \sqrt{n \ln c} - (\ln c)/2$, where $c = 2 - 2s/r$. Then $T$ is $r$-EKR.

### 3 Technical Lemmas

**Proposition 9.** If $0 \leq x \leq 2k/(k+1)^2$ for some $k \geq 1$, then $e^{-x} < 1 - \left(\frac{k}{k+1}\right)x$.

**Proof.** Let $0 \leq x \leq 2k/(k+1)^2$ for some $k \geq 1$. Then $|x| < 1$, and so $e^{-x} = \sum_{i \geq 0}(-x)^i/i! < 1 - x + x^2/2$. Also, $(k+1)x < 2$, which implies that $x^2/2 < x/(k+1) = [1 - k/(k+1)]x$. Thus $e^{-x} < 1 - x + x^2/2 < 1 - \left(\frac{k}{k+1}\right)x$. 

**Corollary 10.** If $0 \leq y \leq 2k^2/(k+1)^3$ for some $k \geq 1$, then $1 - y > e^{-\left(\frac{k+1}{k}\right)y}$.

**Proof.** Set $x = \left(\frac{k+1}{k}\right)y$ and apply Proposition 9. 

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Lemma 11. If \( r \geq 2, d \geq 2, \) and \( n \geq \frac{8r}{27}dr^2 \), then
\[
\prod_{i=1}^{r-1} \left(1 - \frac{r+1d}{n}\right) > \frac{e}{n}.
\]

Proof. We begin with
\[
\prod_{i=1}^{r-1} \left(1 - \frac{r+1d}{n}\right) \geq 1 - \sum_{i=1}^{r-1} \frac{r+1d}{n} = 1 - \frac{r(r-1) + d(r)}{n} = 1 - \frac{(d+2)(r)}{n}.
\]

Since \( d \geq 2, \) and by using Corollary 10 with \( y = dr^2/n \) and \( k = 2, \) we have
\[
1 - \frac{(d+2)(r)}{n} > 1 - \frac{dr^2}{n} > e^{-3dr^2/2n} > e^{-4/9} > .64.
\]

In addition, we calculate
\[
\frac{r}{n} \leq \frac{8}{27dr} \leq \frac{2}{27} < .08,
\]

which completes the proof. \( \square \)

Claim 12. Let \( G \) be a graph with \( n \) vertices and maximum degree less than \( d. \) Then every vertex \( v \) satisfies
\[
s_r(v) \geq \frac{1}{(r-1)!}(n-d)(n-2d)\cdots(n-(r-1)d).
\]

Proof. Let \( W_0 \) be the set of vertices of \( G, \) and set \( w_0 = v. \) For each \( 0 < i < r, \) choose \( w_i \in W_i, \) where \( W_{i+1} = W_i - N[w_i]. \) Then by induction we have \( |W_i| \geq n-id \) for each such \( i. \) The resulting set \( \{w_0, \ldots, w_{r-1}\} \) is independent in \( G \) and there are at least \( \prod_{0 \leq i < r} (n-id) \) ways to choose such sets, ignoring replication. Accounting for replication, we obtain the result. \( \square \)

Lemma 13. Let \( H \) be a graph with at least \( m = n(1-1/3r) \) vertices and maximum degree less than \( d. \) Suppose that \( 1/3r + rd/n \leq 2k^2/(k+1)^3 \) for some
Then every vertex $v$ satisfies

$$s_r(v) \geq \frac{n^{r-1}}{(r-1)!} e^{-(r-1)2k/(k+1)^2}.$$

**Proof.** We use Claim 12 and Corollary 10 with $y = 1/3r + rd/n$ to obtain

$$s_r(v) \geq \frac{1}{(r-1)!} \prod_{0 < i < r} (m - id) \geq \frac{1}{(r-1)!} \prod_{0 < i < r} \left(1 - \frac{1}{3r} - \frac{id}{n}\right) \geq \frac{n^{r-1}}{(r-1)!} \prod_{0 < i < r} \left[1 - \left(\frac{1}{3r} + \frac{rd}{n}\right)\right] \geq \frac{n^{r-1}}{(r-1)!} \prod_{0 < i < r} e^{-\left(\frac{k+1}{3r} + \frac{rd}{n}\right)} \geq \frac{n^{r-1}}{(r-1)!} e^{-\left(r-1\right)(\frac{k+1}{3r} + \frac{rd}{n})} \geq \frac{n^{r-1}}{(r-1)!} e^{-\left(r-1\right)2k/(k+1)^2}.$$

\[\Box\]

## 4 Proof of Theorem 4

We use the following result of Frankl [15]. For $F \subseteq \binom{[n]}{r}$, define $F_x = F - F_x$.

**Theorem 14. (Frankl, 2020)** If $F \subseteq \binom{[n]}{r}$ is intersecting and $r < n/72$, then there is some $x$ such that $|F_x| \leq \binom{n-3}{r-3}$.

**Proof of Theorem 7.** The result is trivial for $r = 1$ or $d = 1$, so we assume $r \geq 2$ and $d \geq 2$. Let $x$ be as in Theorem 14 and select $E \in F_x$, which we may assume to be nonempty. Via the same counting method as in Claim 12 we have at least

$$\frac{1}{(r-1)!} (n - r - d)(n - r - 2d) \cdots (n - r - (r-1)d) \geq \binom{n-3}{r-3}.$$ (1)

r-sets $F \in I_r(x)$ with $F \cap E = \emptyset$. Since $F$ is intersecting, these sets are not in
Therefore, using Theorem 14 and the bound in (1), we have

\[ |\mathcal{F}| = |\mathcal{F}_x| + |\overline{\mathcal{F}_x}| \]
\[ \leq |\mathcal{I}_r(x)| - \frac{(n - r - d) \cdots (n - r - (r - 1)d)}{(r - 1)!} + \binom{n - 3}{r - 2}. \]

This upper bound is at most \(|\mathcal{I}_r(x)|\) precisely when

\[ \binom{n - 3}{r - 2} \leq \frac{1}{(r - 1)!} \prod_{i=1}^{r-1} (n - r - id), \]

which we rewrite as

\[ \prod_{i=1}^{r-1} (n - r - id) \geq (r - 1)! \binom{n - 3}{r - 2} = (r - 1) \prod_{i=1}^{r-2} (n - 2 - i). \]

This inequality will follow from showing that

\[ \prod_{i=1}^{r-1} (n - r - id) \geq r n^{r-2}, \]

which holds by Lemma 11 and which completes the proof.

5 Proof of Theorem 5

The result is trivial for \( r = 1 \), so we may assume that \( r \geq 2 \). Let \( V_0 \) be the set of vertices of \( G \). For each \( i \geq 0 \), choose \( v_i \in V(G_i) \) such that \( \deg_{G_i}(v_i) \geq 3cr \), where \( G_{i+1} = G_i - v_i \). Let \( t \) be minimum such that \( \Delta(G_t) < 3cr \). The number of edges removed in this process is at least \( 3tcr \), which must be at most the number of edges of \( G \); thus \( t \leq n/3r \). Hence \( V(G_t) = n - t \geq n(1 - 1/3r) \).

Now we set \( d = 3cr, k = 4r - 7 \geq 1 \), and calculate that

\[ (k + 3) + \left( \frac{3k + 1}{k^2} \right) \leq k + 7 = 4r, \]
so that \((k+1)^3 \leq 4k^2r\), which implies that

\[
\frac{1}{3r} + \frac{rd}{n} < \frac{1}{3r} + \frac{3cr^2}{18r^3} = \frac{1}{2r} \leq \frac{2k^2}{(k+1)^3}.
\]

This allows the use of Lemma 13 with \(H = G_t\), \(m = n(1 - 1/3r)\), and \(d = 3cr\). We obtain that each vertex \(v\) of \(G_t\) has \(s_r(v)\) at least

\[
\frac{n^{r-1}}{(r-1)!}e^{-(r-1)2k/(k+1)^2}.
\]

Now we use Theorem 2 to show that any intersecting family \(F\) of independent \(r\)-sets that is not a star has size less than \((2)\). First, we note the combinatorial identity

\[
\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1 = 1 + \binom{n-2}{r-2} + \binom{n-3}{r-2} + \cdots + \binom{n-r-1}{r-2}.
\]

Second, we observe the inequality \(r^2/n \leq e^{-(r-1)(8r-14)/(4r-6)} = e^{-(r-1)2k/(k+1)^2}\), because \(c \geq e/36\) and \((4r-6) > (r-1)(8r-14)\) (since \(r \geq 2\)).

Finally, if \(F\) is as above, then we have

\[
|F| < r\left(\frac{n-2}{r-2}\right) = \frac{r(r-1)}{n-1}\left(\frac{n-1}{r-1}\right) < \frac{n^{r-1}}{n} \cdot \frac{n^{r-1}}{(r-1)!} < \frac{n^{r-1}}{(r-1)!}e^{-(r-1)2k/(k+1)^2}.
\]

This finishes the proof.

### 6 Proof of Theorem 7

**Lemma 15.** Let \(S = S(\ell_1, \ldots, \ell_k)\) be a spider on \(n\) vertices and let \(v\) be a leaf of \(S\). Suppose that \(r \leq \alpha(S)\). Then

\[
s_r(v) \geq \binom{n-r-1}{r-1} + \binom{n-k-r-2}{r-2}.
\]
Proof. Let $S = S(\ell_1, \ldots, \ell_k)$, in spider order. We may assume that $v = v_k$ and then use Theorem 6 for the other leaves. For $S(1, 1, \ldots, 1)$ we have $s_r(v) = \binom{n-r}{r-1}$ and $k = n - 1$, so that $\binom{n-k-r-2}{r-2} = 0$ and $\binom{n-2}{r-1} \geq \binom{n-r-1}{r-1}$. Thus we may assume that $\ell_k \geq 2$, implying that $v$ and $w$ are not adjacent.

We first count the number of independent $r$-sets containing $v$ that do not contain the split vertex $w$. The number of such sets is

$$|\mathcal{I}_r^c(S - w)| = |\mathcal{I}_r^c(\cup_{i=1}^k P_{\ell_i})|,$$

where $P_{\ell_i}$ denotes the path on $\ell_i$ vertices.

Next we add edges to the disjoint union of paths, joining the many paths together to form one long path, thus reducing the number of independent $r$-sets that contain $v$ but not $w$. For each $1 \leq i \leq k$, let $u_i$ be the neighbor of $w$ on the $wv_i$-path in $S$; that is, the endpoint of the $i$th path of $S - w$ that is different from $v_i$. Now, for each $1 \leq i < k$, add the edge $v_i u_{i+1}$. Finally, remove $v$ and its unique neighbor, resulting in the graph $P_m$, for $m = n - 3$. This results in the inequality

$$|\mathcal{I}_r^c(\cup_{i=1}^k P_{\ell_i})| \geq |\mathcal{I}_r^{r-1}(P_m)|.$$

We relabel the vertices of $P_m$ as $x_1, \ldots, x_m$, in order. Observe that $\{x_{a_1}, x_{a_1+a_2}, \ldots, x_{a_1+\ldots+a_r-1}\}$ is independent in $P_m$ if and only if

$$\sum_{i=1}^r a_i = m, \ a_1 \geq 1, \ a_i \geq 2 \text{ for } 1 < i < r, \text{ and } a_r = m - a_{r-1} \geq 0. \quad (3)$$

Set $b_1 = a_1 - 1$, $b_i = a_i - 2$ for $1 < i < r$, and $b_r = a_r$. Then system (3) can be rewritten as

$$\sum_{i=1}^r b_i = m - 2r + 3 = n - 2r, \text{ with } b_i \geq 0, \text{ for all } 1 \leq i \leq r. \quad (4)$$
It is well known that the number of integer solutions to system (4) equals
\[
\binom{n - 2r + r - 1}{r - 1} = \binom{n - r - 1}{r - 1}.
\]

Second, we count the number of independent \(r\)-sets containing \(v\) that also contain the split vertex \(w\). The number of such sets equals
\[
|I_r^{-1}(S - N[w])| = |I_r^{-1}(\cup_{i=1}^k P_{\ell_i - 1})|.
\]

As above, we add edges to the disjoint union of paths, to reduce the number of independent \(r\)-sets that contain \(v\) and \(w\). For each \(1 \leq i \leq k\), let \(u'_i\) be the neighbor of \(u_i\) other than \(w\) on the \(wv_i\)-path in \(S\). Now, for each \(1 \leq i < k\), add the edge \(v_i u'_{i+1}\). Finally, remove \(v\) and its unique neighbor, resulting in the graph \(P_{m'}\), for \(m' = n - 3 - k\). This results in the inequality
\[
|I_r^{-1}(\cup_{i=1}^k P_{\ell_i - 1})| \geq |I_r^{-2}(P_{m'})|.
\]

Counting via the same method as above, we obtain
\[
|I_r^{-2}(P_{m'})| = \binom{n - k - r - 2}{r - 2}
\]
such sets, which completes the proof.

**Proof of Theorem 7**. It is easy to check that \(r \leq \sqrt{n \ln 2 - (\ln 2)/2}\) implies that \(r^2 \leq (n - r) \ln 2\). We use this in the calculations below.

Using Lemma 15 with Theorem 2, as in the proof of Theorem 5, the result will follow from proving the inequality
\[
\binom{n - 1}{r - 1} < 2 \binom{n - r - 1}{r - 1}.
\]
To accomplish this, we denote \( m^t = m!/(m-t)! \) and calculate the ratio

\[
\frac{(n-1)}{(r-1)} = \frac{(n-r-1)}{r-1} \leq \frac{(n-r+1)^{r-1}}{(n-2r+1)^{r-1}}
\]

\[
= \left( \frac{n-2r+1}{n-r+1} \right)^{-(r-1)} = \left( 1 - \frac{r}{n-r+1} \right)^{-(r-1)}
\]

\[
\leq e^{r(r-1)/(n-r+1)} = e^{r^2/(n-r)}\quad (6)
\]

\[
\leq e^{ln2} = 2,
\]

which finishes the proof.

7 Proof of Theorem 8

**Lemma 16.** Let \( T \) be a tree on \( n \) vertices with exactly \( s > 1 \) split vertices, and let \( v \) be a leaf of \( T \). Suppose that \( r \leq \alpha(T) \). Then

\[
s_r(v) \geq \left( \frac{n-r-s}{r-1} \right) + 1.
\]

**Proof.** Let \( W \) denote the set of split vertices of \( T \). We need only count the number of independent \( r \)-sets containing \( v \) that do not contain any split vertex. The number of such sets equals

\[
|\mathcal{I}_r^t(T-W)| > |\mathcal{I}_r^t(P_{n-s})| = |\mathcal{I}_r^{r-1}(P_{n-s-2})| = \left( \frac{n-r-s}{r-1} \right).
\]

as in the proof of Lemma 15.

The strict inequality comes from the existence of at least one independent \( r \)-set of \( T-W \) that is not independent in \( P_{n-s} \) because of the joining of the many paths that create \( P_{n-s} \). For example, let \( P' \) and \( P'' \) be two paths in \( T-W \) that are consecutive in \( P_{n-s} \), with endpoints \( u' \in P' \) and \( u'' \in P'' \) such that \( u' \) is adjacent to \( u'' \) in \( P_{n-s} \). Let \( A \in \mathcal{I}_r^t(P_{n-s}) \), define \( a' \) to be the vertex

\[
\]
in $A$ that is closest to $u'$, $a''$ to be the vertex in $A - \{a'\}$ that is closest to $u''$, and $A' = (A - \{a', a''\}) \cup \{u', u''\}$. Then $A' \in I_r^c(T - W) - I_r^c(P_{n-s})$. 

Proof of Theorem 8. As in the proof of Theorem 7, we use Lemma 16 and Theorem 2, which reduces the proof to certifying the inequality

\[
\frac{n - 1}{r - 1} \leq \left( \frac{n - r - 1}{r - 1} \right) + \left( \frac{n - r - s}{r - 1} \right). \tag{7}
\]

Suppose that $1 < s < r/2$ and $r \leq \sqrt{n \ln c - (\ln c)/2}$, where $c = 2 - 2s/r$. Let $a = \frac{r^2}{\ln r} + r$ and $b = \frac{(r+2)^3}{2(r+1)} + r + s - 1$. By rearranging the given condition on $r$, we obtain $n \geq a + \frac{\ln c}{4} > a$. Now let $d = 2(r+1)\ln c$ so that we have

\[
d(a - b) = 2(r+1)r^2 - (\ln c) \left[ (r+2)^3 + 2(r+1)(s-1) \right] \\
> 2(r+1)r^2 - (\ln 2) \left[ (r+2)^3 + (r+1)(2s-2) \right] \\
> 2(r+1)r^2 - 0.7 \left[ (r+2)^3 + (r+1)(r-2) \right] \\
= 2r^3 + 2r^2 - 0.7 \left( r^3 + 7r^2 + 11r + 6 \right) \\
= 1.3r^3 - 2.9r^2 - 7.7r - 4.2 \\
> 0
\]

since $r \geq 5$. Because $a - b > 0$ and $n > a$, we have $n > b$, which is equivalent to

\[
\frac{r + 1}{n - r - s + 1} < \frac{2(r+1)^2}{(r+2)^3}. \tag{8}
\]

Next, we derive the following estimates, using Inequality 8 to access Corollary...
with \( y = (r + 1)/(n - r - s + 1) \) and \( k = r + 1 \).

\[
\frac{(n-r-1) + (n-r-s)}{(n-r-1)^{s-1}} = 1 + \frac{(n-2r+1)}{(n-r-1)^{s-1}} \geq 1 + \left( \frac{n-2r-s+2}{n-r-1-s+2} \right)^{s-1} \\
> 1 + \left( \frac{n-2r-s}{n-r-s+1} \right)^{s} = 1 + \left( 1 - \frac{r+1}{n-r-s+1} \right)^{s} \\
> 1 + e^{-\left( \frac{r+1}{n-r-s+1} \right)^{s}} > 1 + e^{-\left( \frac{2(r+1)}{(r+2)} \right)^{s}} \\
= 1 + e^{-\left( \frac{2(r+1)}{(r+2)} \right)^{s}} > 1 + e^{-2s/r} > 2 - 2s/r.
\]

The assumption that \( s < r/2 \) makes the final result greater than 1. Finally, we follow Inequality (14), since \( r \leq \sqrt{n \ln c - (\ln c)/2} \) implies that \( r \leq \sqrt{n \ln 2 - (\ln 2)/2} \), and calculate the ratio

\[
\left( \frac{n-1}{r-1} \right) / \left( \frac{n-r-1}{r-1} \right) < e^{-2/(n-r)} \leq e^{\ln(2-2s/r)} = 2 - 2s/r,
\]

which finishes the proof.

### 8 Questions and Remarks

It is clear that improving the orders of magnitude in the upper bound on \( r \) in our results will require techniques other than comparison to the Hilton-Milner bounds. To that end, the specificity of spider structure and the knowledge of the location of their biggest stars begs for a proof that they are \( r\text{-EKR} \) for \( r \leq \mu/2 \) (or possibly \( r \leq \alpha \)).

Along these lines, consider the family \( \mathcal{T} \) of all trees having no vertex of degree 2. The authors of [20] conjecture that every tree in \( \mathcal{T} \) is HK. Naturally, we believe that such trees are \( r\text{-EKR} \) for all \( r \leq \mu(T) \) as well. As a first step in this direction, for \( i \in \{1, 2, 3\} \), let \( T_i(h) \) be a complete binary tree of depth \( h \) (i.e. having \( 2^{h+1} - 1 \) vertices), with root vertex \( v_i \). Note that \( v_i \) is the unique degree-
2 vertex in \( T_i(h) \). Now define the tree \( T(h) \) by 
\[
V(T(h)) = \{w\} \cup \bigcup_{i=1}^{3} V(T_i(h)),
\]
with \( w \) adjacent to each \( v_i \). Then \( T(h) \in \mathcal{T} \).

**Problem 17.** Show that \( T(h) \) is \( r \)-EKR for all \( r \leq \mu(T(h))/2 \).

Finally, we say that a family \( \mathcal{F} \) of sets is \textbf{EKR} if it has the property that if \( \mathcal{H} \) is an intersecting subfamily of \( \mathcal{F} \) then there is some element \( x \) such that \( |\mathcal{H}| \leq |\mathcal{F}_x| \), and that a graph \( G \) is \textbf{EKR} if \( \mathcal{I}(G) \) is \textbf{EKR}. We observe that the non-uniform case — considering \( \mathcal{I}(G) \) instead of \( \mathcal{I}^r(G) \) — has yet to be studied specifically for graphs. Of course, this is a special case of Chvátal’s conjecture (see [9]) that every subset-closed family \( \mathcal{F} \) of sets is \textbf{EKR}. For example, by a result of [21], every graph with an isolated vertex is \textbf{EKR}. Also, powers of paths or cycles (resp. special chains) are \textbf{EKR} by the results of [17, 22] (resp. [19]) for fixed \( r \) because we can use the same star center for each \( r \). The same holds for disjoint unions of complete graphs because the star center is either an isolated vertex, if it exists, or a vertex in a smallest component. Any vertex-transitive graph \( G \) that is \( r \)-EKR for all \( r \leq \alpha(G) \) would also be \textbf{EKR}. It is conjectured in [14] that if \( G \) is a disjoint union of length-2 paths then it is \( r \)-EKR for all \( \mu(G)/2 < r \leq \alpha(G)/2 \). It may also be true for all \( r \leq \alpha(G) \), which would imply that \( G \) is \textbf{EKR} because the largest star is always centered on a leaf, and all leaves look alike. Additionally, if one could prove that every spider \( S \) is \( r \)-EKR for all \( r \leq \alpha(S) \) then it would follow from Theorem [6] that spiders are \textbf{EKR}. Of course, while complete \( k \)-partite graphs \( G \) are not \( r \)-EKR for \( \alpha(G)/2 < r \leq \alpha(G) \), that does not mean that they are not \textbf{EKR}.

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