Distinguishing Black Hole Microstates using Holevo Information

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We use the Holevo information in a two-dimensional conformal field theory (CFT) of large central charge to distinguish microstates from the underlying thermal state. Holographically the CFT microstates of a thermal state are dual to black hole microstate geometries in AdS3 space. It was found recently that the holographic Holevo information shows plateau behaviors at both short and long interval regions. This indicates that the black hole microstates are indistinguishable from thermal state by measuring over a small region, but perfectly distinguishable over a region with its size comparable to the whole system. In this paper, we demonstrate that the plateaus are lifted by including the 1/c corrections from both the vacuum and non-vacuum conformal families of CFT in either canonical ensemble or microcanonical ensemble thermal state. Our results imply that the aforementioned indistinguishability and distinguishability of black hole microstate geometries from underlying black hole are spoiled by higher order Newton constant GN corrections of quantum gravity.

INTRODUCTION

The black hole information paradox lies on the fact that a pure state seems evolving into a thermal state through Hawking radiation, and thus it violates unitarity of quantum mechanics. This paradox can be partially resolved if there exists black hole microstates, which are pure states, cannot be distinguished from the underlying thermal state. This resolution however calls for a complete theory of quantum gravity which is beyond the reach at this moment. However, with the help of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence one may glimpse the answer for this quantum gravity problem from the view point of its dual CFT.

Recently, it was proposed in [2] to characterize distinguishability of the black hole microstates from its underlying thermal state by the Holevo information. One can call it in short the distinguishability of black hole microstates. The thermal state of the whole system is described by

\[ \rho = \sum_i p_i \rho_i, \quad \rho_i = |i\rangle \langle i|, \quad (1) \]

with the orthonormal microstates |i\rangle satisfying \langle i|i'\rangle = \delta_{ii'}. Note that 0 \leq p_i \leq 1, \sum_i p_i = 1. One would like to distinguish the microstates from the thermal state by performing measurements in a subsystem A, whose complement is denoted by B. The first step is to consider the relative entropy by comparing the reduced density matrix \( \rho_{A,i} = \text{tr}_B \rho_i \) of each of the microstates with the reduced density matrix \( \rho_A = \text{tr}_B \rho \) of the corresponding thermal state, i.e.,

\[ S(\rho_{A,i}||\rho_A) = \text{tr}(\rho_{A,i} \log \rho_{A,i}) - \text{tr}(\rho_{A,i} \log \rho_A). \quad (2) \]

This quantity is a well-defined divergence and characterizes the difference of the two reduced density matrices. The average relative entropy gives the Holevo information

\[ \chi_A = \sum_i p_i S(\rho_{A,i}||\rho_A) = S_A - \sum_i p_i S_{A,i}, \quad (3) \]

with entanglement entropies (EEs) \( S_A = -\text{tr}(\rho_A \log \rho_A) \), \( S_{A,i} = -\text{tr}(\rho_{A,i} \log \rho_{A,i}) \). It is just difference of the thermal state EE and the average EE of the microstates. The Holevo information \( \chi_A \) is the upper bound of the mutual information between the thermal state and any measurement inside A, which is aiming to reproduce the thermal state and to characterize the accessible information.

By construction

\[ 0 \leq \chi_A \leq S_{\text{thermal}}, \quad (4) \]

with \( S_{\text{thermal}} \) being thermal entropy of the whole system

\[ S_{\text{thermal}} = - \sum_i p_i \log p_i. \quad (5) \]

When \( \chi_A = 0 \), \( \rho_{A,i} = \rho_A \) so that the microstates are totally indistinguishable by measurements inside A. On the other hand, when \( \chi_A = S_{\text{thermal}} \), \( \rho_{A,i} \rho_A \rho_{A,i'} = 0 \) for arbitrary \( i, i' \) and thus the microstates are completely distinguishable.

To investigate the information loss paradox of black hole in Einstein gravity in the AdS3 background, i.e., the Bañados-Teitelboim-Zanelli (BTZ) black hole [3] we calculate the Holevo information in a two-dimensional (2D) CFT. When the gravity is weakly coupled, the CFT has a large central charge [4]

\[ c = \frac{3R}{2G_N}, \quad (6) \]

where \( R \) is the radius of AdS3.
with $G_N$ being the Newton constant and $R$ being the AdS radius. The $1/c$ corrections on the CFT side correspond to quantum corrections on the gravity side.

We consider a 2D large $c$ CFT in thermal state on a cylinder with spatial period $L$. For an interval $A$ with length $\ell$, one can denote the Holevo information by $\chi(\ell)$. When $\chi(\ell) = 0$, the microstates are totally indistinguishable by any measurement inside the length $\ell$ interval. On the other hand, when $\chi(\ell) = S(L)$, with $S(L)$ being the thermal entropy of the whole system, the microstates are perfectly distinguishable. The Holevo information $\chi(\ell)$ is monotonically increasing with respect to $\ell$. It is easy to see that

$$\lim_{\ell \to 0} \chi(\ell) = 0, \quad \lim_{\ell \to L} \chi(\ell) = S(L). \quad (7)$$

By using the holographic entanglement entropy (HEE) [5, 6], it was recently found in [2] that the holographic Holevo information shows plateau behaviors around $\ell \to 0$ and $\ell \to L$. It indicates that the microstates are totally indistinguishable until the interval reaches a non-vanishing critical length, and are perfectly distinguishable after the interval reaches another critical length that is shorter than length of the whole system. The HEE is only the classical gravity result, and it is expected that quantum corrections to the HEE [7, 9] would resolve both plateau and non-plateau behaviors of the holographic Holevo information. On the dual CFT side, these correspond to $1/c$ corrections. The problem has been addressed in [10] for the 2D CFT due to zero mass BTZ black hole, and in this paper we continue to investigate this problem for a general 2D large $c$ CFT. The cases we consider include the canonical ensemble thermal state with both high and low temperatures, as well as the microcanonical ensemble thermal state.

We find that the Holevo information is not vanishing as long as the region of measurement is non-vanishing, and this indicates that the black hole microstates are distinguishable from thermal state as long as the measuring region is non-vanishing. We also find the Holevo information is smaller than the thermal entropy as long as the interval is shorter than the whole system.

For calculation convenience we choose that the interval $A$ is short, i.e., $\ell/L \ll 1$, thus its complement $B$ has a length $L - \ell$ comparable to $L$. Then we have

$$S_A = S(\ell), \quad S_{A,i} = S_i(\ell), \quad \chi_A = \chi(\ell), \quad S_B = S(L - \ell), \quad S_{B,i} = S_i(L - \ell), \quad \chi_B = \chi(L - \ell). \quad (8)$$

Note that $S_{A,i} = S_{B,i}$. To get the short and long interval Holevo information $\chi_A$ and $\chi_B$, we need to calculate the short and long interval EEs of thermal state, i.e., $S_A, S_B$, and the average of the short interval EE of the microstates, i.e., $\sum_i p_i S_{A,i}$. For the short interval, as in [11, 14], we use the operator product expansion (OPE) of twist operators [11, 15, 17] to calculate the short interval expansion of the EE. For the long interval, we can still use the OPE of twist operators [18, 20].

### Canonical Ensemble Thermal State with High Temperature

For a canonical ensemble thermal state we have

$$p_i = \frac{e^{-\beta E_i}}{Z(\beta)}, \quad Z(\beta) = \sum_i e^{-\beta E_i}, \quad (9)$$

with $\beta$ being the inverse temperature. We consider high temperature limit $\beta/L \ll 1$ and omit the terms suppressed by the exponential factor $e^{-2\pi L/\beta}$. The thermal entropy is

$$S(L) = \frac{\pi c L}{3 \beta}. \quad (10)$$

which is just the entropy of BTZ black hole. Using the HEE [5, 6], one can get the holographic Holevo information [2]

$$\chi_{\text{horo}}(\ell) = \begin{cases} 0 & \ell < \frac{\beta}{2\pi} \log 2 \\ \ell > L - \frac{\beta}{2\pi} \log 2 & \\end{cases}. \quad (11)$$

The holographic Holevo information $\chi_{\text{horo}}(\ell)$ with $\frac{\beta}{2\pi} \log 2 < \ell < L - \frac{\beta}{2\pi} \log 2$ is unknown. The result is plotted in Fig. [1]. There are plateaus at both $\ell < \frac{\beta}{2\pi} \log 2$ and $\ell > L - \frac{\beta}{2\pi} \log 2$. We will resolve the plateaus in CFT.

We consider only contributions from the vacuum conformal family, and will briefly discuss the contributions from non-vacuum conformal families in the end of the paper. For the short interval $A$ we have the EE [15]

$$S_A = \frac{c}{3} \log \left( \frac{\beta}{\pi \epsilon} \sinh \frac{\pi \ell}{\beta} \right). \quad (12)$$

Though we do not calculate $S_{A,i}$ for all the pure states, using the results in [14, 21] we can get the average EE [22]

$$\sum_i p_i S_{A,i} = \frac{c}{3} \log \ell + \frac{\pi^2 \epsilon^2}{18\beta^2} - \frac{\pi^3 \epsilon^4 (\pi c L + 24 \beta)}{540 \beta^3 L} \quad (13)$$

$$+ \frac{\pi^4 \epsilon^6 (\pi^2 c^2 L^2 + 72 \pi c \beta L + 864 \beta^2)}{8505 c^3 \beta^3 L^2} + \cdots + O(\ell^{12}).$$

Combining them, we obtain the short interval Holevo information

$$\chi_A = \frac{2 \pi^3 \epsilon^4}{45 \beta^3 L} - \frac{8 \pi^4 \epsilon^6 (\pi c L + 12 \beta)}{945 \beta^3 L^2} + \cdots + O(\ell^{12}). \quad (14)$$

We find that to the order we consider it is vanishing in the thermodynamic limit [23, 24], i.e., by taking $L \to \infty$ but with $\beta, \ell$ fixed.

For the long interval $B$ we have the EE [20]

$$S_B = \frac{c}{3} \log \left( \frac{\beta}{\pi \epsilon} \sinh \frac{\pi \ell}{\beta} \right) + \frac{\pi c L}{3 \beta} - I(1 - e^{-2\pi \ell}). \quad (15)$$
The function $I(x)$ is the mutual information of two intervals on a complex plane with cross ratio $x$. The small $x$ expansion of $I(x)$ to order $x^8$ was calculated in [8, 27] and to order $x^{10}$ was calculated in [26, 27]. Combining with the fact $S_{B,i} = S_{A,i}$, we obtain the long interval Holevo information

$$\chi_B = \frac{\pi c L}{3\beta} - \frac{2\pi^3(4\pi L - 7\beta)\ell^4}{315\beta^4 L} + \frac{32\pi^5\ell^5}{3465\beta^6} + \frac{8\pi^4(32\pi^2 L^2 - 143\pi \beta L)\ell^6}{135135\beta^6 L^2} + \cdots + O(\ell^{11}, 1/c).$$

(16)

Note that $\chi_B - S(L)$ is non-vanishing in the thermodynamic limit.

We denote the results [14] and [16] as the CFT Holevo information $\chi_{\text{CFT}}(\ell)$ and $\chi_{\text{CFT}}(L - \ell)$, respectively. Note that they are only valid for $\ell \ll \beta \ll L$. They are consistent with the holographic Holevo information $\chi_{\text{holo}}$ at the leading order of large $c$, while at the sub-leading orders we see the corrections. We plot them in Fig. 1.

We see that with $1/c$ corrections both the short and long interval plateaus are resolved.

The leading $c$ of [13] is consistent with the result

$$\sum_i p_i S_{A,i} = \frac{c}{3} \log \left( \frac{\beta}{\pi \ell} \sinh \frac{\pi \ell}{\beta} \right) + O(c^0),$$

(17)

which was got in [2] by assuming that the contributions from the primary excited states dominate the average. In fact, from the result in [28], we can show that there are far more descendant states than primary states in high levels of a large $c$ CFT [21]. It is intriguing to show explicitly why primary excited states dominate the average. Supposing (17) is valid as long as $\ell < L/2$, one can get the Holevo information by Bao and Ooguri in [2]

$$\chi_{\text{BO}}(\ell) = \left\{ \begin{array}{ll} \frac{0}{\pi \ell} - \frac{\ell}{\beta} \log 2, & \ell < L/2 \\ \frac{\ell}{\beta} \log 2 - \frac{\ell}{\beta} \log 2, & \ell > L - \frac{\beta}{\pi} \log 2 \end{array} \right.$$  

(18)

It is a combination of holographic and CFT results, and is the leading order $c$ Holevo information. For comparison, we also plot $\chi_{\text{BO}}$ in Fig. 1.

**CANONICAL ENSEMBLE THERMAL STATE WITH LOW TEMPERATURE**

In low temperature, we have $\beta \gg L$. The dual gravity background is the thermal AdS and the holographic thermal entropy is vanishing

$$S_{\text{holo}}(L) = 0.$$  

(19)
From $0 \leq \chi(\ell) \leq S(L)$, we obtain
\[ \chi_{\text{holo}}(\ell) = 0. \tag{20} \]

In CFT, the above perfect indistinguishability can be lifted by taking into account the finite-size effect exponentially suppressed by the factor $q = e^{-2\pi\beta/L}$. Using the results in \cite{20} and considering only the contributions from the holomorphic sector of the vacuum conformal family, for the short interval we get
\[
\chi_A = \left[ \frac{32q^2}{15c} + \frac{24q^3}{5c} + \frac{64q^4}{5c} + O(q^5) \right] \left( \frac{\pi\ell}{L} \right)^4
+ \left[ \frac{128(c - 16)q^2}{315c^2} + \frac{32(c - 24)q^3}{35c^2} \right]
+ \frac{256(c - 40)q^4}{105c^2} + O(q^5) \left( \frac{\pi\ell}{L} \right)^6 + O(\ell^6),
\tag{21}
\]
and for the long interval we obtain
\[
\chi_B - S(L) = - \left[ \frac{32\pi\beta(\beta^2 + L^2)(4\beta^2 + L^2)q^2}{15L^5} \right]
+ O(q^3) \left( \frac{\pi\ell}{L} \right)^4 + O(\ell^6).
\tag{22}
\]

**MICROcanonical Ensemble Thermal State**

We now consider the microcanonical ensemble thermal state with fixed high energy $E$, with contributions from both the holomorphic and anti-holomorphic sectors. We have the thermal state \cite{4} with
\[
p_i = \frac{\delta(E - E_i)}{\Omega(E)}. \tag{23}
\]

At energy $E$ the number of states $\Omega(E)$ is given by Cardy's formula \cite{29} and it is an inverse Laplace transformation of canonical ensemble partition function $Z(\beta)$. Beyond the saddle point approximation of \cite{29 30}, it turns out that
\[
\Omega(E) = \sqrt{\frac{\pi cL}{6E}} I_1 \left( \sqrt{\frac{2\pi cLE}{3}} \right), \tag{24}
\]
with $L_\nu$ being modified Bessel function of the first kind. As the case of canonical ensemble thermal state with high temperature, we omit the exponentially suppressed terms of large $E$ but keep the power suppressed terms.

The Cardy's formula can be generalized to the cases of various multi-point correlation functions on a torus \cite{25 31 33}, i.e., in canonical ensemble thermal state. One can use the inverse Laplace transformation of the canonical ensemble thermal state to obtain the corresponding microcanonical ensemble one. In this way, we can derive one-point functions, and thus the short interval EE, of microcanonical ensemble thermal state from the canonical ensemble one-point functions. Similarly, we can obtain the microcanonical ensemble average short interval EE from the corresponding canonical one. Combining the short interval EE and average EE, we obtain the Holevo information
\[
\chi_A = \frac{\pi^3\ell^4}{12} \left[ \frac{\pi cL(I_3 - I_1) + 24\lambda I_2}{540\lambda^2 I_1} \right] + \cdots + O(\ell^{12}), \tag{25}
\]
with the definition $\lambda := \sqrt{\frac{\pi cL}{6E}}$ and $I_\nu$ being the short-hand notation of $I_\nu(\frac{\pi cL}{3\lambda^2})$.

For the long interval case, we use the OPE of twist operators in \cite{15 20} to obtain the following result,
\[
\chi_B - S(L) = O(\ell^{12}). \tag{26}
\]
However, we cannot get the term of order $\ell^{12}$ explicitly. It is possibly non-vanishing.

**Contributions from a Non-Identity Primary Operator**

Lastly, we consider the leading contribution to the Holevo information from a non-identity primary operator $\psi$ with normalization $\alpha_\psi$, conformal weights $(h_\psi, h_\bar{\psi})$. We have the scalding dimension $\Delta_\psi = h_\psi + h_\bar{\psi}$ and spin $s_\psi = h_\psi - h_\bar{\psi}$. For a general thermal state with density matrix \cite{1}, we use the OPE of twist operators \cite{7 15 20} and get the short and long interval Holevo information
\[
\delta_\psi \chi_A = \frac{\sqrt{\pi \Gamma(\Delta_\psi + 1)} \ell^{2\Delta_\psi} 2^{2\Delta_\psi}}{2^{2\Delta_\psi} + 2^\frac{3}{2}} \Gamma(\Delta_\psi + \frac{3}{2}) \alpha_\psi \left[ \sum_i p_i(\psi) \rho_i \right]^2 - \left( \sum_i p_i(\psi) \rho_i \right)^2 + o(\ell^{2\Delta_\psi}),
\]
\[
\delta_\psi \chi_B = \delta_\psi S(L) - \frac{\ell^{2\Delta_\psi} 2^{2\Delta_\psi}}{2^{2\Delta_\psi} + 2^\frac{3}{2}} \alpha_\psi \left[ \sum_{i \neq i'} \langle i | \psi | i' \rangle \langle i' | \bar{\psi} | i \rangle p_i \delta_n \left[ \sum_{j=1}^{n-1} \left( \frac{p_j}{p_i} / p_i \right)^j \right] + o(\ell^{2\Delta_\psi}). \tag{27}
\]
These forms are general and can be applied to both canonical ensemble and microcanonical ensemble thermal
states. The results however are not universal in the sense that they depend on the structure constants, so that we cannot evaluate their explicit forms without knowing the details of the theory.

**DISCUSSION**

For concluding the paper, we would like to mention the implication of the almost vanishing short interval Holevo information to our recent finding of non-geometric states in [34]. As shown in [34] some special descendant states are non-geometric, which indicates that they cannot be locally like thermal. The ensemble average for obtaining the Holevo information is over all states including those non-geometric descendant states. However, we see the resultant leading order $c$ short interval Holevo information is still consistent with thermality. Using the results in [28] we can show there are far more descendant states than primary ones at high levels in a large $c$ CFT [21]. This indicates that the contributions from the non-geometric descendant states are suppressed. It is intriguing to show this explicitly.

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Supplementary materials

Canonical ensemble thermal state with high temperature

With contributions from only the vacuum conformal family, the EE of one short interval in a general state $\rho$ can be written as \[ 14, 21 \]

\[
S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + 2 [\ell^2 a_T(T)_{\rho} + \ell^4 a_{TT}(T)_{\rho}^2 + \ell^6 a_{TTTT}(T)_{\rho}^3 \\
+ \ell^8 (a_{A\rho}A(T)_{\rho}^2 + a_{TTA}(T)_{\rho}\rho + a_{TTTT}(T)_{\rho}) \\
+ \ell^{10} (a_{TTA}(T)_{\rho}\rho + a_{TTTT}(T)_{\rho}\rho) \\
+ a_{TTTT}(T)_{\rho}^5 + O(\ell^{12})].
\] (28)

with the coefficients

\[
\begin{aligned}
    a_T &= -\frac{1}{6}, \\
    a_{TT} &= \frac{1}{30c}, \\
    a_{TTT} &= -\frac{4}{315c^2}, \\
    a_{A\rho} &= -\frac{1}{126c(5c + 22)}, \\
    a_{TTA} &= \frac{1}{315c^2}, \\
    a_{TTTT} &= \frac{c + 8}{630c^3}, \\
    a_{TTA\rho} &= -\frac{16}{693c^2(5c + 22)}, \\
    a_{TTTT\rho} &= -\frac{16(c + 5)}{3465c^4}.
\end{aligned}
\] (29)

Here $T$ is the stress tensor, and $A = (TT) - \frac{1}{10}\partial^2 T$. The density matrix $\rho$ can be either a thermal state, or any individual pure state, and in fact it can be any state that is translational invariant. We have included the contributions from both the holomorphic and anti-holomorphic sectors, and it is applied to the states in which the contributions from the holomorphic and anti-holomorphic sectors are the same. Otherwise, we can just write the holomorphic and anti-holomorphic contributions separately.

In high temperature limit we omit the exponentially suppressed terms and get

\[
(T)_{\beta} = -\frac{\pi^2c}{6\beta^2}, \quad (A)_{\beta} = \frac{\pi^4c(5c + 22)}{180\beta^4},
\] (30)

which are just one-point functions on a cylinder with infinite space and temporal period $\beta$. Using (28) and (30) we get the EE

\[
S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + \frac{\pi^2c\ell^2}{180\beta^2} - \frac{\pi^4c\ell^4}{540\beta^4} + \frac{\pi^6c\ell^6}{850\beta^6} - \frac{\pi^8c\ell^8}{113400\beta^8} \\
+ \frac{\pi^{10}c\ell^{10}}{1403325\beta^{15}} + O(\ell^{12}),
\] (31)

which is consistent with the exact result \[ 12 \].

To calculate the average EE, we first calculate the average products of one-point functions.
where \( r \) is an arbitrary integer and \( X_i \) can be either the one-point function of an operator or a product of the one-point functions. We also have

\[
\sum_i p_i \langle \mathcal{X} \mathcal{Y} \rangle_{\rho_i} = \frac{1}{L} \int_{-L/2}^{L/2} dx \langle \mathcal{X}(x) \mathcal{Y}(0) \rangle_{\beta},
\]

(35)

with \( \mathcal{X} = T, \mathcal{A}, \mathcal{Y} = T, \mathcal{A} \). This follows from the fact that both \( T \) and \( \mathcal{A} \) are KdV currents that commute with each other and we can choose the states \( | i \rangle \) as the common eigenstates of their zero modes. Explicitly, we derive (35) as follows. On a torus \( T \) with spatial period \( L \) and temporal period \( \beta \) there are two-point function

\[
\langle \mathcal{X}(x) \mathcal{Y}(0) \rangle_T = \frac{1}{Z(\beta)} \sum_{i,i'} \delta_{\mathcal{X},\mathcal{Y}}(s_i-s_{i'}) e^{2\pi i (s_{i'}-s_i) \langle i | \mathcal{X} \rangle \langle i' | \mathcal{Y} \rangle | i \rangle | i' \rangle. \]

(36)

For bosonic \( \mathcal{X}, \mathcal{Y} \) we require that \( s_i \) is an integer for \( \langle i | \mathcal{X} \rangle | i' \rangle \) being non-vanishing. Then we get

\[
\frac{1}{L} \int_{-L/2}^{L/2} dx \langle \mathcal{X}(x) \mathcal{Y}(0) \rangle_T = \frac{1}{Z(\beta)} \sum_{i,i'} \delta_{\mathcal{X},\mathcal{Y}} \delta_{s_i,s_{i'}} \langle i | \mathcal{X} \rangle \langle i' | \mathcal{Y} \rangle | i \rangle | i' \rangle. \]

(37)

For \( \mathcal{X}, \mathcal{Y} \) being operators in the vacuum conformal family, we require that \( | i \rangle \) and \( | i' \rangle \) are in the same conformal family. The delta function \( \delta_{s_i,s_{i'}} \) further requires that \( | i \rangle \) and \( | i' \rangle \) are at the same level, and so only the zero modes of \( \mathcal{X}, \mathcal{Y} \) contribute to \( \langle i | \mathcal{X} \rangle \langle i' | \mathcal{Y} \rangle | i \rangle | i' \rangle \). For \( \mathcal{X}, \mathcal{Y} \) being KdV currents, the states \( | i \rangle \) can be organized as common eigenstates of their zero modes. Then we have \( | i \rangle = | i' \rangle \). Omitting the exponentially suppressed terms in high temperature limit, we have \( \langle \mathcal{X}(x) \mathcal{X}(0) \rangle_T = \langle \mathcal{X}(x) \mathcal{X}(0) \rangle_{\beta} \). We finally arrive at (35).

By omitting the exponentially suppressed terms and by an analytical continuation, in evaluating (35) we use the integral

\[
\frac{1}{L} \int_{-L/2}^{L/2} \frac{dx}{\sinh \frac{\pi x}{\beta}} = \frac{\beta}{L} \frac{\Gamma(\frac{c}{2})\Gamma(1-\frac{c}{4})}{\pi^{\frac{c}{2}}}. \]

(38)

As consistency checks, we get the same \( \sum_i p_i T_i \), \( \sum_i p_i T_i A_i \), from (34) and (35). Using (28) and (42), we get the average EE (13)

\[
\sum_i p_i S_{A,i} = \frac{c}{3} \log \frac{\ell}{\epsilon} + \frac{\pi^2 \ell^2}{18 \beta^2} - \frac{\pi^3 \ell^4 (\pi c L + 24 \beta)}{540 \beta^4 L} + \frac{\pi^4 \ell^6 (\pi^2 c^2 L^2 + 72 \pi c \beta L + 864 \beta^3)}{8005 c^3 \beta^8 L^2}
\]

\[
+ \frac{\pi^5 \ell^8 [-7 \pi^3 c^3 L^3 - 2160 \pi^2 c^2 \beta L^2 + 1120 \pi c (c-28) \beta^2 L - 8040 (c + 8) \beta^3]}{793800 c^2 \beta^8 L^3}
\]

\[
+ \frac{\pi^6 \ell^{10} [7 \pi^4 c^4 L^4 + 8592 \pi^3 c^3 \beta L^3 - 672 \pi^2 c^2 (c - 100) \beta^2 L^2 + 2903040 \pi c \beta^3 L + 29030400 (c + 5) \beta^4]}{9823275 c^3 \beta^{10} L^4} + O(\ell^{12}).
\]

(39)

and then the short interval Holevo information (14)

\[
\chi_A = \frac{2 \pi \ell^4}{45 \beta^3 L} - \frac{8 \pi^4 \ell^6 (\pi c L + 12 \beta)}{945 \beta^5 L^2} + \frac{2 \pi^5 \ell^8 [27 \pi^2 c^2 L^2 - 14 \pi c (c-28) \beta L + 1008 (c + 8) \beta^2]}{19845 c^3 \beta^7 L^3}
\]

\[
+ \frac{16 \pi^6 \ell^{10} [-179 \pi^3 c^3 L^3 + 140 \pi^2 c^2 (c - 100) \beta L^2 - 60480 \pi c \beta^2 L^2 - 604800 (c + 5) \beta^3]}{3274425 c^3 \beta^9 L^4} + O(\ell^{12}).
\]

(40)

The mutual information of two intervals with cross ratio \( x \) on a complex plane can be organized by orders of large \( c \) as

\[
I(x) = I_L(x) + I_{NL}(x) + \cdots.
\]

(41)

The leading part of the mutual information is universal (7)

\[
I_L(x) = \begin{cases} 
 0 & \frac{c}{3} \log \frac{x}{1-x} & x < 1/2 \\
 1/2 & x > 1/2
\end{cases}.
\]

(42)
The next-to-leading part of the mutual information satisfies $I_{NL}(x) = I_{NL}(1 - x)$. With contributions of only the vacuum conformal family, we have

$$I_{NL}(x) = \frac{x^4}{630} + \frac{2x^5}{693} + \frac{15x^6}{4004} + \frac{x^7}{234} + \frac{167x^8}{36036} + \frac{69422x^9}{14549535} + \frac{122x^{10}}{24871} + O(x^{11}).$$  

(43)

We then get the long interval Holevo information

$$\chi_B = \frac{\pi c L}{3\beta} - \frac{2\pi^3 L^4 (4\pi L - 7\beta)}{315\beta^3 L^4} + \frac{32\pi^5 L^6}{3465\beta^5 L^6} + \frac{8\pi^4 L^2 (32\pi^2 L^2 - 143\pi\beta L)}{135135\beta^3 L^2} - \frac{32\pi^7 \ell^7}{27027\beta^7} + \frac{2\pi^5 L^4 (-109116\pi^3 L^3 + 19305\pi^2 L^2 - 1001\pi\beta^2 L^2)}{14189175\beta^8 L^3} + \frac{2596928\pi^3 \ell^9}{26189163\beta^9} + \frac{16\pi^6 \ell^{10} (-369060974\pi^4 L^4 - 751621\pi^3 \beta L^3 + 587860\pi^2 \beta^2 L^2)}{13749310675\beta^{10} L^4} + O(\ell^{11}, 1/c).$$  

(44)

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**Canonical ensemble thermal state with low temperature**

For the canonical ensemble thermal state with low temperature, we only consider the contributions of the holomorphic vacuum conformal family. The CFT is on a torus $\mathcal{T}$ with spatial period $L$ and temporal period $\beta$. In low temptation limit $L \ll \beta$, and to get non-vanishing corrections to the Holevo information we have to include the exponentially suppressed terms by $q = e^{-\frac{2\pi \ell}{L}} \ll 1$.

The holomorphic part of the partition function is

$$Z(q) = q^{-\frac{\pi}{2\ell}} [1 + q^2 + q^3 + 2q^4 + O(q^5)].$$  

(45)

Similar to [34], the average products of one-point functions for the stress tensor $T$ can be written as

$$\sum_i \langle T \rangle_{\rho_i} = \left( \frac{2\pi}{L} \right)^2 \frac{2\pi (q\partial_q)^r}{Z(q)}.$$  

(46)

We get the results

$$\sum_i \langle T \rangle_{\rho_i} = \left( \frac{2\pi}{L} \right)^2 \frac{2\pi (q\partial_q)^r}{Z(q)},$$  

(47)

from which we get the short interval EE

$$S_A = \frac{c}{6} \log \frac{\ell}{\ell} + \left( -\frac{c}{36} + \frac{4q^2}{3} + 2q^3 + 4q^4 + O(q^5) \right) \left( \frac{\pi \ell}{L} \right)^2 + \left( -\frac{c}{1080} + \frac{4q^2}{45} + \frac{2q^3}{15} + \frac{4(c - 8)q^4}{15c} \right) \left( \frac{\pi \ell}{L} \right)^4 + O(q^5),$$  

(48)

and average EE

$$\sum_i p_i S_{A,i} = \frac{c}{6} \log \frac{\ell}{\ell} + \left( -\frac{c}{36} + \frac{4q^2}{3} + 2q^3 + 4q^4 + O(q^5) \right) \left( \frac{\pi \ell}{L} \right)^2 + \left( -\frac{c}{1080} + \frac{4(c - 24)q^2}{45c} \right) \left( \frac{\pi \ell}{L} \right)^4 + \frac{2(c - 36)q^3}{15c} + \frac{4(c - 56)q^4}{15c} + O(q^5) \left( \frac{\pi \ell}{L} \right)^4 + \left( -\frac{c}{17010} + \frac{8(c - 48c + 768)q^2}{945c^2} \right) \left( \frac{\pi \ell}{L} \right)^6 + O(q^5).$$  

(49)

Then we get the short interval Holevo information [21].
The low temperature long interval EE has been calculated in \[20\]

\[
S_B = \left( \frac{4\pi^2 \beta}{L} + 1 \right) q^2 + O(q^3) + \frac{c}{6} \log \frac{\ell}{\epsilon} + \left[ -\frac{c}{1080} + \frac{4(c-24)q^2}{45c} - \frac{32\pi\beta L^2}{15L^5} \right] q^2 + O(q^3) \left( \frac{\pi \ell}{L} \right)^4 + O(\ell^6).
\]

Noting the thermal entropy

\[
S(L) = \left( \frac{4\pi^2 \beta}{L} + 1 \right) q^2 + O(q^3),
\]

we get the long interval Holevo information \[22\].

**Microcanonical ensemble thermal state**

In \[24\] the density of states at fixed energy \(E\) is defined as

\[
\Omega(E) = \sum_i \delta(E - E_i).
\]

The energy \(E\) can be written in terms of the scaling dimension as \(E = \frac{\pi c L}{6\beta} (\Delta - \frac{\pi^2}{6L})\). For the ground state \(\Delta = 0\), and so \(E = -\frac{\pi c L}{6\beta}\). In a unitary CFT \(\Delta \geq 0\), and so \(E \geq -\frac{\pi c L}{6\beta}\). The canonical ensemble partition function can be written as

\[
Z(\beta) = \sum_i e^{-\beta E_i} = \int_{-\pi c L/6\beta}^{+\infty} dE e^{-\beta E} \Omega(E).
\]

Then one can use the inverse Laplace transformation to get the density of states

\[
\Omega(E) = \frac{1}{2\pi i} \int_{-\pi c L/6\beta}^{+\infty} d\beta e^{\beta E} Z(\beta).
\]

We omit the exponentially suppressed term by higher energy, or equivalently high temperature, and have \(Z(\beta) = e^{-\frac{\pi c L}{6\beta}}\). Beyond the saddle point approximation in \[29, 30\], the integral \[54\] leads to \[24\].

\[
S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + \frac{\pi^2 c \ell^2}{18\lambda^2} - \frac{\pi^4 c \ell^4}{540\lambda^4} + \frac{\pi^6 c \ell^6}{8505\lambda^6} - \frac{\pi^8 c \ell^8 [5(c + 8)I_2^2 - 2(5c + 22)I_4I_3 + (5c + 22)I_4^2]}{2041200\lambda^8 I_1^2} + \frac{\pi^{10} c \ell^{10} [5(c + 5)I_2^2 - 2(5c + 22)I_4I_3 + (5c + 22)I_4^2]}{4209975\lambda^{10} I_1^2} + O(\ell^{12}).
\]

As what have been done in \[23, 31–33\], for other general canonical ensemble average in the form

\[
\mathcal{O}(\beta) = \frac{1}{Z(\beta)} \sum_i e^{-\beta E_i} \mathcal{O}_i,
\]

we can also do an inverse Laplace transformation and get the microcanonical ensemble average

\[
\mathcal{O}(E) = \frac{1}{\Omega(E)} \frac{1}{2\pi i} \int_{-\pi c L/6\beta}^{+\infty} d\beta e^{\beta E} \mathcal{O}(\beta) Z(\beta).
\]

Note that \(\mathcal{O}_i\) can be any quantity defined for the pure state \(|i\rangle\), e.g., a one-point function, a product of one-point functions, and the EE. For the case that the canonical average \(\mathcal{O}(\beta)\) is a polynomial of \(\beta\), we get the microcanonical average \(\mathcal{O}(E)\) from \(\mathcal{O}(\beta)\) by the substitute

\[
\beta^{-k} \rightarrow \left( \frac{\pi c L}{6E} \right)^{-k/2} I_{k-1} \left( \sqrt{\frac{2\pi c L E}{3}} \right) I_k \left( \sqrt{\frac{2\pi c L E}{3}} \right).
\]

It is convenient to define the effective length scale

\[
\lambda := \sqrt{\frac{\pi c L}{6E}},
\]

and the substitute \[57\] becomes

\[
\beta^{-k} \rightarrow \lambda^{-k} I_{k-1} \left( \frac{\pi c L}{3\lambda} \right) I_k \left( \frac{\pi c L}{3\lambda} \right).
\]

In the following, we just use the shorthand notation \(I_\nu\) for \(I_\nu \left( \frac{\pi c L}{3\lambda} \right)\).

Using the substitute \[59\], we can get the microcanonical ensemble one-point functions from the canonical ensemble ones

\[
\langle T \rangle_E = -\frac{\pi^2 c^2}{18\lambda^2}, \quad \langle A \rangle_E = \frac{\pi^4 c^4(5c + 22)}{180\lambda^4 I_1},
\]

and then we get the short interval EE.
Note that the result is valid with the exponentially suppressed terms of high energy being omitted and the power suppressed terms being kept. From the average EE in high temperature canonical ensemble thermal state \( \langle X \rangle_E \) [39], we get the average EE in high energy microcanonical ensemble thermal state

\[
\sum_i p_i S_{A,i} = \frac{c}{3} \log \frac{\ell}{\epsilon} + \frac{\pi^2 c^2}{18 \lambda^2} - \frac{\pi^4 \epsilon^6 (c \lambda L I_3 + 24 I_2)}{540 \lambda I L I_1} + \frac{\pi^4 \epsilon^6 (\pi^2 c^2 L^2 I_5 + 72 \pi c \lambda L I_4 + 864 \lambda^2 I_3)}{8505 c \lambda^5 L^2 I_1}
\]

\[
+ \frac{\pi^5 \epsilon^8 [-7 \pi^3 c^3 L^3 I_7 - 2160 \pi^2 c^2 \lambda^2 L^2 I_6 + 1120 \pi c (c - 28) \lambda^2 L I_5 - 80640 (c + 8) \lambda^3 I_4]}{793800 \epsilon c^2 \lambda^8 L^3 I_1}
\]

\[
+ \frac{\pi^6 \epsilon^{10}}{9823275 c^6 \lambda^{10} \lambda L I_1} [7 \pi^4 c^4 L^4 I_9 + 8592 \pi^3 c^3 \lambda^3 L I_8 - 672 \pi^2 c^2 (c - 100) \lambda^2 L^2 I_7
\]

\[
+ 2903040 \pi c \lambda^3 L I_6 + 29030400 (c + 5) \lambda^4 I_5] + O(\epsilon^{12}).
\] (62)

Then we get the short interval Holevo information [25]

\[
\chi_A = \frac{\pi^3 \epsilon^4 [c \lambda L (I_3 - I_4) + 24 I_2]}{540 \lambda ^4 L I_1} - \frac{\pi^4 \epsilon^6 [\pi^2 c^2 L^2 (I_5 - I_4) + 72 \pi c \lambda L I_4 + 864 \lambda^2 I_3]}{8505 c \lambda^5 L^2 I_1}
\]

\[
+ \frac{\pi^5 \epsilon^8}{14288400 c^2 \lambda^2 L^3} \left\{ -7 \pi^3 c^3 L^3 [c + 8] I_7^2 - (5c + 22) I_7 (2I_7 - 3I_3) - 181 I_7 I_7 \right\} + 38880 \pi^2 c^2 \lambda^2 L^2 I_1 I_6
\]

\[
+ 1451520 (c + 8) \lambda^3 I_1 I_4 - 20160 \pi c (c - 28) \lambda^2 L I_1 I_6 \}
\]

\[
+ \frac{\pi^6 \epsilon^{10}}{29409825 c^6 \lambda^{10} \lambda L^4 I_1} \left\{ 7 \pi^4 c^4 L^4 [c + 5] I_9^2 - (5c + 22) I_9 (2I_9 - I_3) - 3I_9 I_9 \right\} - 25776 \pi^3 c^3 \lambda L^3 I_1 I_8
\]

\[
+ 20160 \pi^2 (c - 100) c^2 \lambda^2 L^2 I_7 I_7 - 87091200 (c + 5) \lambda^4 I_5 I_5 - 87091200 \pi c \lambda^3 L I_1 I_6 \} + O(\epsilon^{12}).
\] (63)

As a byproduct in the paper, we can show that the reduced density matrix of the high energy microcanonical ensemble thermal state \( \rho_{A,E} \) equals the reduced density matrix of the high temperature canonical ensemble thermal state \( \rho_{A,\beta} \) in the thermodynamic limit, or equivalently high temperature limit. The difference of the two reduced density matrices are power suppressed. We stress that this result does not depend on the large \( c \) limit and applies to any 2D CFT.

In the first step, we identify the energy expectation values of the two states, and so we have \( \lambda = \beta \). To make it more concrete, in the following we will show that the EE's of the two states are the same up to power corrections

\[
S_{A,E} - S_{A,\beta} = O\left(\frac{\lambda}{L}, \frac{\epsilon}{L}\right),
\] (64)

and the relative entropies of the two reduced density matrices are also power suppressed

\[
S(\rho_{A,E} \parallel \rho_{A,\beta}) = O\left(\frac{\lambda}{L}, \frac{\epsilon}{L}\right),
\]

\[
S(\rho_{A,\beta} \parallel \rho_{A,E}) = O\left(\frac{\lambda}{L}, \frac{\epsilon}{L}\right).
\] (65)

Using the modular transformation of one-point functions on a torus, one can calculate the average one-point function of a general quasiprimary operator \( \mathcal{X} \) with scaling dimension \( \Delta \mathcal{X} \) and spin \( s \mathcal{X} \) in the microcanonical ensemble thermal state as [28]

\[
\langle \mathcal{X} \rangle_E \approx \frac{\langle Y | X | Y \rangle}{Y} \left( \frac{6L}{\pi c} \right)^{\frac{3}{4}} (-E_Y)^{-\frac{3}{4}} \epsilon^{-\frac{3}{4}} h \left( \sqrt{\frac{3}{4}} - \sqrt{\epsilon L} \right) \sqrt{\epsilon L},
\] (66)

with \( Y \) being a quasiprimary operator with the lowest scaling dimension that satisfies \( \langle Y | Y \rangle \neq 0 \). Note that \( E_Y > -\frac{\pi c}{6L} \), and it is assumed that \( E_Y < 0 \). In the derivation of (66) the saddle point approximation has been used and the power suppressed terms by large \( E \) has been omitted. As a consistency check of the normalization of (66), we can see that for the identity operator \( Y = 1 \), we have \( Y = 1 \), and the right hand side of (66) is one.

When \( \mathcal{X} \) is in a non-vacuum conformal family, we have \( E_Y > -\frac{\pi c}{6L} \) and the one point function \( \langle \mathcal{X} \rangle_E \) is exponentially suppressed, and thus can be omitted. When \( \mathcal{X} \) is in the vacuum conformal family, \( Y \) is the identity operator, or in other words the state \( |Y \rangle = |Y \rangle \langle 0 | \rangle \) is the ground state \( |0 \rangle \). For this case, with loss of generality we choose \( \mathcal{X} \) to be holomorphic, and so \( \Delta \mathcal{X} = s \mathcal{X} = h \mathcal{X} \) is an integer. Noting that \( \langle 0 | X | 0 \rangle = \langle X \rangle_L \), \( E_0 = -\frac{\pi c}{6L} \), \( E = \frac{\pi c}{6L} h \mathcal{X} \), we use (66) and get

\[
\langle \mathcal{X} \rangle_E \approx \langle X \rangle_L \left( \frac{L}{\lambda} \right)^{h \mathcal{X}}.
\] (67)

For the high temperature canonical ensemble thermal state, we also omit the exponentially suppressed terms.
When \( \mathcal{X} \) is in a non-vacuum conformal family we have \( \langle \mathcal{X} \rangle_{\beta} = 0 \). When \( \mathcal{X} \) is in the holomorphic vacuum conformal family, we have

\[
\langle \mathcal{X} \rangle_{\beta} = \langle \mathcal{X} \rangle_L \left( \frac{L}{\lambda} \right)^{b_{\kappa}}. \tag{68}
\]

Since we have identified \( \lambda = \beta \), we get that for all quasiprimary operators

\[
\langle \mathcal{X} \rangle_E = \langle \mathcal{X} \rangle_{\beta} + O \left( \frac{\lambda}{L} \right). \tag{69}
\]

The equivalence (69) is exact under the thermodynamic limit in [23, 24], i.e., \( E \to \infty \) and \( L \to \infty \) with \( E/L \) being finite. The equality of the two reduced density matrices are expected to be valid for general \( \ell, \lambda = \beta \) as long as the thermodynamic limit is taken \( \ell/L \to 0, \lambda/L \to 0 \).

Using OPE of twist operators, one can write the EE and relative entropy as sums of products of one-point functions, and the coefficients of the products are universal and do not depend on parameters of the state [14]. Then we use (69) and get the relations (64), (65).

It is interesting to compare directly the high energy microcanonical ensemble thermal state \( S_{A,E} \) [61], in which the exponentially suppressed terms are omitted and but the power suppressed terms are kept, and the high temperature canonical ensemble thermal state \( S_{A,\beta} = \frac{c}{2} \log(\frac{\beta}{\pi} \sinh \frac{\beta}{T}) \), in which the exponentially suppressed terms are omitted and there are no power suppressed terms. We plot them in Fig. 2 and their difference in Fig. 3. We can see the EEs of the two states are very close as long as \( \ell < \beta \), and the large difference at \( \ell > \beta \) can be attributed to the breaking down of the short interval expansion in [61].

To get the long interval Holevo information, we need to calculate the long interval EE in the microcanonical ensemble thermal state. The relevant states \( |i\rangle \) are at the same energy and are the common eigenstates of the zero modes of \( T \) and \( \mathcal{A} \), but they are not necessarily the eigenstates of the zero modes of level six quasiprimary operators \( B \) and \( D \), whose definitions can be found in [17, 25]. We use the OPE of twist operators for a long interval [18, 26] and get the partition function

\[
\text{tr}_{B\rho_B^n} = \left( \frac{\ell}{e} \right)^{-4b_{\kappa}} \frac{1}{\Omega^n} \left\{ 1 + \frac{2}{\Omega} \sum_i \left[ \ell^2 b_{TT}(T)_{\rho_i} + \ell^4 b_{TTT}(T)_{\rho_i}^2 + \ell^6 b_{TTTT}(T)_{\rho_i}^3 \right] \right.
+ \ell^8 (b_{AA}(A)_{\rho_i} + b_{TTA}(T)_{\rho_i}(A)_{\rho_i} + b_{TTTT}(T)_{\rho_i}^4) \right.
+ \ell^{10} (b_{TTAA}(T)_{\rho_i}(A)_{\rho_i} + b_{TTTTA}(T)_{\rho_i}(A)_{\rho_i} + b_{TTTTTT}(T)_{\rho_i}^5) \right.
+ \ell^{12} (b_{BB}(B)_{\rho_i}^2 + b_{DD}(D)_{\rho_i}^2 + b_{TTAD}(T)_{\rho_i}(A)_{\rho_i} + b_{TTDD}(T)_{\rho_i}(D)_{\rho_i} + b_{TTAAA}(A)_{\rho_i}^3 + b_{TTTBB}(T)_{\rho_i}^3(B)_{\rho_i} + b_{TTTTD}(T)_{\rho_i}^3(D)_{\rho_i} + b_{TTTTA}(T)_{\rho_i}^2(A)_{\rho_i}^2 + b_{TTTTTT}(T)_{\rho_i}^6 \right) \left. + \frac{2\ell^{12}}{\Omega} \sum_{i \neq i'} \left[ \ell^2 b_{BB}(i|B|i')(i'|B|i) + b_{DD}(i|D|i')(i'|D|i) + O(\ell^{14}, (n-1)^2) \right] \right\}. \tag{70}
\]
The conformal weight of the twist operators is $h_{σ} = \frac{c(σ^2-1)}{24n}$. There are contributions from both the holomorphic and anti-holomorphic sectors of the vacuum conformal family. We have restricted the sum $\sum_{i}'$ to the states of the fixed energy $E$, and $Ω$ is the total number of such states $Ω = \sum_{i}' 1$. The coefficients $b_{X_{1} \cdots X_{k}}$ are defined from the OPE coefficients $d_{X_{1} \cdots X_{k}}^{c}$ of the quasiprimary operators $X_{1}^{c} \cdots X_{k}^{c}$ in the n-fold CFT [11], and their explicit forms are not important to us. We have also used the results in [21] and omitted some order $(n - 1)^2$ terms in (70) in the $n \rightarrow 1$ limit. The omitted terms are relevant to the Rényi entropy but are irrelevant to the EE. Then the long interval EE can be written as

$$S_B = \log Ω + \frac{1}{Ω} \sum_{i}' S_{A,i} + \frac{2^{\ell 12}}{Ω} \sum_{i \neq i'} (a_{BB} \langle i|B|i'\rangle \langle i'|B|i \rangle + a_{DD} \langle i|D|i'\rangle \langle i'|D|i \rangle) + O(\ell^4), \quad (71)$$

with the coefficients [21]

$$a_{BB} = -\frac{25}{123552c(7c + 29)}, \quad (72)$$

$$a_{DD} = -\frac{70c + 29}{18018c(2c - 1)(5c + 22)(7c + 68)}.$$

We note that the thermal entropy is $S(L) = \log Ω$ and get the long interval Holevo information [20]. Unfortunately, we do not know how to calculate $\sum_{i \neq i'} \langle i|B|i'\rangle \langle i'|B|i \rangle$ or $\sum_{i \neq i'} \langle i|D|i'\rangle \langle i'|D|i \rangle$, and so we cannot evaluate the order $\ell^2$ part of the long interval Holevo information.

**Contributions from a non-identity primary operator**

Similar to what we have done for the contributions to the Holevo information from the vacuum conformal family, we can use the OPE of twist operators [7] [15] [20], and get the leading contributions from a non-identity primary operator $ψ$ [27].

For the canonical ensemble thermal state in the high temperature limit, we can further write the results with the exponentially suppressed terms omitted as

$$\delta ψ_{CA} = \frac{\sqrt{π}Γ(Δ_{ψ} + 1)}{2^{2Δ_{ψ} + 2Γ(Δ_{ψ} + \frac{3}{2})}} \frac{1}{α_{ψ}} Z(β) \sum_{i} e^{-βE_{i}} \langle ψ_{i}\rangle_{ψ}^2 + o(\ell^{2Δ_{ψ}}),$$

$$\delta ψ_{CB} = -\frac{\ell^{2Δ_{ψ}}}{2^{2Δ_{ψ} + 1}} \frac{1}{α_{ψ}} Z(β) \sum_{i \neq i'} \langle i|ψ|i'\rangle \langle i'|ψ|i \rangle e^{-βE_{i}} \partial_{α} \left[ \frac{n-1}{(\sin \frac{π}{n} 2Δ_{ψ})^{n=1}} + o(\ell^{2Δ_{ψ}}) \right]. \quad (73)$$

For the canonical ensemble thermal state in the low temperature limit, we get

$$\delta ψ_{CA} = \frac{\sqrt{π}Γ(Δ_{ψ} + 1)}{2^{2Δ_{ψ} + 2Γ(Δ_{ψ} + \frac{3}{2})}} \frac{1}{α_{ψ}} Z(β) \sum_{i} e^{-βE_{i}} \langle ψ_{i}\rangle_{ψ}^2 + o(\ell^{2Δ_{ψ}}),$$

$$\delta ψ_{CB} = \delta ψ S(L) - \left( \frac{πℓ}{L} \right)^{2Δ_{ψ}} \frac{1}{α_{ψ}} Z(β) \sum_{j=1}^{n-1} \frac{q^{jΔ_{ψ}}}{(\sin \frac{π}{n} 2Δ_{ψ})^{j=1}} + o(\ell^{2Δ_{ψ}}). \quad (74)$$

Here $ψ'$ is the primary operator with least conformal dimension that satisfies $\langle ψ'|ψ|ψ'\rangle \neq 0$. Note also that

$$\delta ψ S(L) = \left( \frac{2πΔ_{ψ} β}{L} + 1 \right) q^{Δ_{ψ}} + o(q^{Δ_{ψ}}). \quad (75)$$

For the microcanonical ensemble thermal state, we get

$$\delta ψ_{CA} = \frac{\sqrt{π}Γ(Δ_{ψ} + 1)}{2^{2Δ_{ψ} + 2Γ(Δ_{ψ} + \frac{3}{2})}} \frac{1}{α_{ψ}} Z(β) \sum_{i} e^{-βE_{i}} \langle ψ_{i}\rangle_{ψ}^2 - \frac{1}{Ω^2} \left( \frac{1}{Ω} \sum_{i} \langle ψ_{i}\rangle_{ψ}^2 \right)^2 + o(\ell^{2Δ_{ψ}}),$$

$$\delta ψ_{CB} = -\frac{\sqrt{π}Γ(Δ_{ψ} + 1)}{2^{2Δ_{ψ} + 2Γ(Δ_{ψ} + \frac{3}{2})}} \frac{1}{α_{ψ}} Z(β) \sum_{i \neq i'} e^{-βE_{i}} \langle i|ψ|i'\rangle \langle i'|ψ|i \rangle + o(\ell^{2Δ_{ψ}}). \quad (76)$$

These results are not universal, and we cannot evaluate them without knowing details of the CFT.