Quantum mechanics from Newton’s second law and the canonical commutation relation \([X, P] = i\)

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Abstract
Despite the fact that it has been known since the time of Heisenberg that quantum operators obey a quantum version of Newton’s laws, students are often told that derivations of quantum mechanics must necessarily follow from the Hamiltonian or Lagrangian formulations of mechanics. Here, we first derive the existing Heisenberg equations of motion from Newton’s laws and the uncertainty principle using only the equations \(F = \frac{dP}{dt}\), \(P = m\frac{dX}{dt}\), and \([X, P] = i\). Then, a new expression for the propagator is derived that makes a connection between time evolution in quantum mechanics and the motion of a classical particle under Newton’s laws. The propagator is solved for three cases where an exact solution is possible: (1) the free particle; (2) the harmonic oscillator; and (3) a constant force, or linear potential in the standard interpretation. We then show that for a general force \(F(X)\), by Taylor expanding \(X(t)\) in time, we can use this methodology to reproduce the Feynman path integral formula for the propagator. Such a picture may be useful for students as they make the transition from classical to quantum mechanics and help solidify the equivalence of the Hamiltonian, Lagrangian, and Newtonian pictures of physics in their minds.

Keywords: quantum mechanics, Newton’s laws, propagator, classical physics

1. Introduction
Typical introductory quantum mechanics classes take place after students have studied, at least to some extent, the Hamiltonian and Lagrangian formulations of classical mechanics. The role of the Hamiltonian and the Schrödinger equation are emphasized, and it is often taught that these energy-based formulations of physics are more general because they allow physics to be extended into the quantum regime. Quantum mechanics is, then, treated as a theory that
depends on the existence of Lagrangian and Hamiltonian mechanics and where Newton’s laws no longer have any applicability, outside of the occasional reference to the Ehrenfest theorem [1]. This treatment is apparent from the current standard introductory quantum mechanics textbooks [2–4].

Heisenberg, in his initial derivation of matrix mechanics, made use of correspondence between the time evolution of quantum operators and classical particles [5]. And while quantum and classical correspondence has been acknowledged since the earliest days of quantum physics [6], it seems that the Newtonian-like dynamics of quantum operators has never been used as a starting point for the development of quantum physics. The Hamiltonian and in more advanced courses, Lagrangian formulation of Feynman [7], are generally taken to be both necessary and fundamental.

We will first rederive Heisenberg picture mechanics starting from Newton’s laws plus the uncertainty principle. This is presented mainly as a tool for reinforcing the equivalence between the Newtonian and Hamiltonian formulations of physics, even within the quantum regime. On its own, however, it does not clearly exhibit the utility of quasi-Newtonian principles in quantum physics.

A Newton-like formulation of quantum mechanics is possible, which we demonstrate through the derivation of a new expression for the propagator. This expression utilizes the concept of a position operator that evolves in time in an analogous manner to the position of a Newtonian particle. The propagator is then solved for three cases where an exact solution is possible: the free particle, a harmonic oscillator, and a constant force.

Our expression emphasizes the time-evolution of the operator $X(t)$, just as in classical mechanics, the classical variable $X(t)$ evolves according to Newton’s laws. The initial value $X_0$ and subsequent derivatives $\frac{d}{dt}P$ and $\frac{d}{dt}F$ are used to build the time dependence of $X(t)$ without referencing the Hamiltonian or any energy-based formulation of mechanics. Although there have been descriptions of quantum mechanics that treat it as a classical theory with random Newtonian forces leading to a stochastic differential equation [8, 9], a Newtonian-based derivation of standard quantum physics does not appear to have been previously developed.

2. Reproducing the Heisenberg equations of motion

We will start by reproducing Heisenberg picture quantum mechanics, defined by the relation (in units where $\hbar = 1$)

$$\text{i}[H, O] = \frac{\partial}{\partial t} O$$

from the equations

$$F = \frac{dP}{dt}$$

$$P = m \frac{dX}{dt}$$

$$[X, P] = i.$$  

We can begin by finding the commutator of $[X^n, P]$ for positive $n$. Using the third equation, we can rewrite the commutator as:

$$X^n P - PX^n = X^n P - X^{n-1} PX + X^{n-1} PX - X^{n-2} PX^2 + \cdots - PX^n$$

$$= X^n [X, P] + X^{n-2} [X, P] X + \cdots + [X, P] X^{n-1}$$

$$= i n X^{n-1} = i \frac{d}{dX} X^n.$$  

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For negative powers of $X$, we can write

$$[X^{-n}, P] = X^{-n}[P, X^n]X^{-n}$$

$$= -iX^{-n}nX^{n-1}X^{-n}$$

$$= -i nX^{-n-1} = i \frac{d}{dX} X^{-n} \tag{6}$$

and in either case, it is clear that commuting a power of $X$ with $P$ results in its derivative with respect to $X$.

Starting with some arbitrary function of $X$, $O(X)$, it can be Laurent expanded as:

$$O(X) = \sum_{n=-\infty}^{\infty} C_n X^n \tag{7}$$

where the $C_n$’s are constants.

From equations (5) and (6), the commutator of $P$ with each term in the Laurent series results in the derivative of that term with respect to $X$. Thus:

$$[O(X), P] = i \frac{d}{dX} O(X). \tag{8}$$

The same argument can be used to show that for a function of momentum $O(P)$

$$[O(P), X] = -i \frac{d}{dP} O(P). \tag{9}$$

The Laurent expansion of $O$ also provides a convenient representation in which to find the time derivative of $O$. Since in quantum mechanics, the commutators $[X, \frac{d}{dt} X]$ and $[P, \frac{d}{dt} P]$ are not necessarily zero, time derivatives of powers of $X$ and $P$ must be taken term by term. Through the Laurent series, this can then be used to find the time derivative of arbitrary functions of $X$ and $P$.

Before we can define the time derivative of the Laurent series, we must first define the time derivative of $X^{-1}$, which can be found through

$$\frac{d}{dt} X^{-1} = \frac{d}{dt} (X^{-1}XX^{-1})$$

$$= 2 \frac{d}{dt} X^{-1} + X^{-1} \frac{d}{dt} X^{-1} \tag{10}$$

which implies

$$\frac{d}{dt} X^{-1} = -X^{-1} \frac{d}{dt} X^{-1} \tag{11}$$

and by the same argument

$$\frac{d}{dt} X^{-n} = -X^{-n} \left( \frac{d}{dt} X^n \right) X^{-n} \tag{12}$$

The time derivative of $X^n$ can be found term by term as

$$\frac{d}{dt} X^n = \frac{dX}{dt} X^{n-1} + \frac{dX}{dt} X^{n-2} + \cdots + X^{n-1} \frac{dX}{dt}$$

$$= \frac{P}{m} X^{n-1} + X^{n-2} \frac{P}{m} + \cdots + X^{n-1} \frac{P}{m}. \tag{13}$$

Commuting all of the $P$’s to left, this equation becomes

$$\frac{d}{dt} X^n = \frac{1}{m} \left( nPX^{n-1} + \sum_{j=1}^{n-1} [X^j, PX^{n-j}] \right) \tag{14}$$
and using the fact that $nX^{n-1} = -i[X^n, P]$, we can write this as

$$\frac{d}{dt} X^n = \frac{1}{m} \left( -iP[X^n, P] + \sum_{j=1}^{n-1} [X^j, PX^{n-j}] \right).$$  \hspace{1cm} (15)

If instead, we commute all the $P$'s to the right, we get

$$\frac{d}{dt} X^n = \frac{1}{m} \left( -i[X^n, P]P - \sum_{j=1}^{n-1} [X^j, PX^{n-j}] \right)$$  \hspace{1cm} (16)

where the minus sign on the second commutator is picked up because we have commuted the $P$'s to the opposite side.

Since both equations (15) and (16) are equal to $\frac{d}{dt} X^n$, the average of the two of them is still equal to $\frac{d}{dt} X^n$, and we can write

$$\frac{d}{dt} X^n = -\frac{i}{2m} \left( P[X^n, P] + [X^n, P]P \right) = i \left[ \frac{P^2}{2m}, X^n \right]$$  \hspace{1cm} (17)

for positive values of $n$.

For inverse powers of $X$, we can now rewrite equation (12) as

$$\frac{d}{dt} X^{-n} = -X^{-n} \left[ \frac{P^2}{2m}, X^n \right] X^{-n} = i \left[ \frac{P^2}{2m}, X^{-n} \right]$$  \hspace{1cm} (18)

and so, for an arbitrary function of $X$, via the Laurent expansion

$$\frac{d}{dt} O(X) = i \left[ \frac{P^2}{2m}, O(X) \right].$$  \hspace{1cm} (19)

There is another way of arriving at the same result that we found above which is useful when $\frac{d}{dt} X$ is a more general function of $P$, as in the relativistic case. For a velocity that is an arbitrary function of momentum $V(P) = \frac{d}{dt} X$, we can make the substitution

$$V(P) = -i \left[ X, \int V(P) dP \right]$$  \hspace{1cm} (20)

that is, $V$ is the derivative of the integral of $V(P)$ with respect to $P$. The time derivative of $X^n$ becomes

$$\frac{dX^n}{dt} = i \left[ \int V(P) dP, X \right] X^{n-1} + iX \left[ \int V(P) dP, X \right] X^{n-1} + \cdots$$  \hspace{1cm} (21)

and the time derivative of $O(X)$ is

$$\frac{d}{dt} O(X) = i \left[ \int V(P) dP, O(X) \right].$$  \hspace{1cm} (22)

It is easy to see, that for the Newtonian velocity/momentum relationship, this returns the usual $\frac{P^2}{2m}$ commutator.

This method can be employed again for finding the time derivative of $P^n$. Since the force, $F$, can be an arbitrary function of $X$, there is no simple algebraic way of taking the time derivative as in equation (17). But, by making the substitution

$$F(X) = i \left[ P, \int F dX \right]$$  \hspace{1cm} (23)
we can find the time derivative of $P^n$ by the same method that we used to get equation (21). We see then, that

$$\frac{d}{dt} O(P) = -i \left[ \int F \, dX, \, O(P) \right].$$

(24)

A function of $X$ and $P$, $O(X, P)$ can be Laurent expanded as

$$O(X, P) = \sum_{-\infty}^{\infty} C_{nmjk} X^n P^m X^j P^k \ldots$$

(25)

with an arbitrary number of alternating powers of $X$ and $P$ where the indexed coefficient is a constant and the summation is taken over each independent power $n, m, j, k$, etc. Commuting this series with $-F(X)$ and $V(P)$ gives us

$$-\sum_{-\infty}^{\infty} C_{nmjk} \left( X^n [F(X), \, P^m] X^j P^k \ldots + X^n P^m X^j [F(X), \, P^k] \ldots + \ldots \right)$$

(26)

$$+ \sum_{-\infty}^{\infty} C_{nmjk} \left( [V(P), \, X^n] P^m X^j P^k \ldots + X^n P^m [V(P), \, X^j] P^k \ldots + \ldots \right)$$

(27)

and it is clear that the sum of these two series is the full time derivative of $O(X, P)$, differentiated term-by-term, via the chain rule. Thus, for an arbitrary function $O(X, P)$, the time derivative can be written as

$$\frac{d}{dt} O(X, P) = i \left[ \int V(P) \, dP - \int F(X) \, dX, \, O(X, P) \right]$$

(28)

or specifically, in Newtonian mechanics

$$\frac{d}{dt} O(X, P) = i \left[ \frac{\dot{P}^2}{2m} - \int F(X) \, dX, \, O(X, P) \right]$$

(29)

which is exactly the Heisenberg equation of motion. Equation (29) provides a complete description of Heisenberg picture quantum mechanics and can be used to solve for the time propagator $U(t) = \exp(-iHt)$.

It is no coincidence that the integrals $\int F(X) \, dX$ and $\int V(P) \, dP$ that appear in equation (28) when added together produce the Hamiltonian. From Hamilton’s equations:

$$\frac{\partial H}{\partial X} = -\dot{P}$$

(30)

$$\frac{\partial H}{\partial P} = \dot{X}$$

(31)

and thus, for a Hamiltonian that is separable into $H(X, P) = H(X) + H(P)$ we can write

$$H(X, P) = \int \dot{X} \, dP - \int \dot{P} \, dX$$

(32)

Equation (29) is the quantum equivalent of

$$\frac{d}{dt} O(X, P) = \frac{\partial O}{\partial X} \frac{P}{m} + \frac{\partial O}{\partial P} F$$

(33)

but in a way that respects the matrix properties of the $X$ and $P$ operators.

By taking the derivative in this manner, we have reproduced Heisenberg picture quantum mechanics, that is, the fact that the time derivative of an operator is proportional to its commutator with the Hamiltonian. We have done so without resorting to energy, conserved
quantities, or even the term Hamiltonian itself. Instead, the integrals of force and velocity appeared as a way of simplifying the commutators that arose in our calculations.

This derivation, however, ultimately results in the use of the Hamiltonian, whether referred to as such or not, and does not clearly underscore the fact that the quantum operators for position and momentum evolve in time in a way that is very similar to their classical counterparts under Newton’s laws. After all, Newton’s laws do not make use of any analogous method of taking partial derivatives and typically only involve X and its derivatives, rather than general functions of X and P. In the next section, we will explore a formulation of the propagator that highlights the Newtonian-like dynamics of the operator X(t).

3. The propagator from the Newtonian dynamics of X(t)

Just as in classical mechanics, in quantum mechanics, X(t) can be written as

\[
X(t) = X_0 + \frac{1}{m} P_0 t + \frac{1}{2m} F_0 t^2 + \frac{1}{6m} dF dt^3 + \cdots
\]

the difference being that \(X_0\) and \(P_0\) are matrices that obey the canonical commutation relation.

For simplicity, we can rewrite equation (34) as

\[
X(t) = X_0 + \frac{1}{m} \int_0^t P(t) \, dt
\]

where \(P(t)\) is a matrix with a complicated time dependence determined by the force, \(F(X)\).

At any time, \(t\), there is a vector \(|X_0; t\rangle\) that is an eigenvector of \(X(t)\) with eigenvalue \(x_0\), such that

\[
X(t)|X_0; t\rangle = \left( X_0 + \frac{1}{m} \int_0^t P(t) \, dt \right) |X_0; t\rangle = x_0 |X_0; t\rangle
\]

At \(t = 0\), this eigenvector is the Dirac delta function \(|X_0; 0\rangle = \delta(X - X_0)\), but at a later time \(t\) is given by

\[
|X_0; t\rangle = U^\dagger(t)|X_0; 0\rangle
\]

since \(X(t)\) evolves according to \(X(t) = U^\dagger(t)X_0 U(t)\).

We can take the expectation value of \(X(t)\) with two different eigenvectors at two different times to find \(|X_0; 0\rangle \langle X(t)|X_0; t\rangle\) and \(|X_0; t\rangle \langle X(t)|X_0; 0\rangle\) which gives us

\[
[X_0]|X_0; t\rangle = \left( X_0 + \frac{1}{m} \int_0^t P(t) \, dt \right) U^\dagger(t)|X_0\rangle = x_0 \langle X_0|U^\dagger(t)|X_0\rangle
\]

\[
[X_0]|U(t)\left( X_0 + \frac{1}{m} \int_0^t P(t) \, dt \right)|X_0\rangle = x_0 \langle X_0|U(t)|X_0\rangle
\]

where \(|X_0\rangle\) and \(|X_0\rangle\) are taken to be the eigenvectors at \(t = 0\).

It is worth noting that if we allow \(X_0\) to act on \(|X_0\rangle\) of equation (38), we can write

\[
[X_0] \int_0^t P(t) \, dt U^\dagger(t)|X_0\rangle = (x_b - x_0) \langle X_0|U^\dagger(t)|X_0\rangle.
\]

The left hand side of the equation contains the integral of momentum with respect to time, and the right hand contains the displacement \(\Delta x = x_b - x_0\). In other words, we have written the quantum analog of the classical equation \(\Delta X = \int_0^t P \, dt\).

In principle, finding the propagator \(|X_0 U(t)|X_0\rangle\) amounts to finding the solution to equations (38) and (39). In practice, this can be difficult, although there are at least three cases that admit an exact solution. A complete differential equation for the propagator can be written with this method if and only if an exact solution for the time dependent operators \(X(t)\)
and \( P(t) \) can be found. In the three cases described in this paper, the time derivatives of \( X(t) \) and \( P(t) \) at \( t = 0 \) are at most linear in \( X_0 \) or \( P_0 \). Because of this, repeated differentiation will not cause mixtures of alternating powers of \( X_0 \) and \( P_0 \), the Taylor series in time can be written to infinite order, and the exact operators plugged into equations (38) and (39).

Since equations (38) and (39) do not include the time derivative of \( U(t) \), there is the possibility that our solution could differ from the true propagator either by a purely time dependent factor \( A(t) \) or by an additional purely time dependent term \( g(t) \) that needs to be added to it. The fact that \( U(t) \) is unitary, precludes the possibility that a purely time dependent function could be added to our solution, since this would change the magnitude of \( U(t) \) with time, and thus, \( g(t) \) must equal zero.

\( A(t) \) can be determined by the criterion that \( U(t) = \delta(x_b - x_a) \) at \( t = 0 \). Any additional time dependent factor cannot affect the amplitude of \( U(t) \), again because of unitarity. Although this does not rule out time dependent phase factors, such a factor would be the equivalent of at most shifting the potential by a time dependent, real function \( f(t) \) that is constant over all space. Such a time dependent change in phase cannot affect any measurable properties of the system. In other words, the requirement that \( U(t) \) be unitary restricts the possible solutions to physically equivalent expressions.

### 3.1. The free particle

If \( F(X) \) is zero everywhere, \( \int P(t) \, dt \) becomes \( P_0 \). It is convenient to let \( X(t) \) act to the left in equation (38) and to the right in equation (39). The operator \( P_0 \) can then be defined by its action on \( |X_a\rangle \) and \( |X_b\rangle \) as

\[
P|X_a\rangle = \int_{-\infty}^{\infty} P_0|P_0\rangle\langle P_0|X_a\rangle \, dP_0 = \int_{-\infty}^{\infty} P_0|P_0\rangle \, e^{-iP_0} \, dP_0
\]

\[
= \int_{-\infty}^{\infty} i \frac{\partial}{\partial x_a}|P_0\rangle \, e^{-iP_0} \, dP_0 = i \frac{\partial}{\partial x_a}|X_a\rangle
\]

and through the same procedure

\[
\langle X_b|P_0 = -i \frac{\partial}{\partial X_b} |X_b\rangle.
\]

Equations (38) and (39) then become

\[
x_a\langle X_b|U(t)|X_a\rangle - \frac{im}{\hbar} \frac{\partial}{\partial x_b} \langle X_b|U\dagger(t)|X_a\rangle = x_a\langle X_b|U\dagger(t)|X_a\rangle
\]

\[
x_a\langle X_b|U(t)|X_a\rangle + \frac{im}{\hbar} \frac{\partial}{\partial x_a} \langle X_b|U(t)|X_a\rangle = x_b\langle X_b|U(t)|X_a\rangle
\]

where the derivative operator has different signs in (43) and (44) because it is acting to the left and to the right, respectively.

Relabeling \( \langle X_b|U(t)|X_a\rangle \) as \( U(x_b, x_a, t) \), we can turn equation (43) into the integral equation

\[
\int \frac{dU_{\dagger}(x_b, x_a, t)}{U\dagger(x_b, x_a, t)} = \frac{i m}{\hbar} \int (x_a - x_b) \, dx_b
\]

which has the solution

\[
U\dagger(x_b, x_a, t) = A\dagger(t) \exp \left\{ -im \left( \frac{1}{2m^2} - \frac{x_b x_a + f(x_a)}{t} \right) \right\}
\]
By the same method, the solution to equation (44) is

\[ U(x_b, x_a, t) = A(t) \exp \left( \frac{im}{2t} \frac{x_b^2 - x_a x_a + f(x_b)}{2} \right). \] (47)

The solutions of equations (46) and (47) set \( f(x_a) = \frac{1}{2} x_a^2 \) and \( f(x_b) = \frac{1}{2} x_b^2 \). Furthermore, the boundary condition \( U(x_b, x_a, 0) = \delta(x_a - x_b) \) determines \( A(t) \), so that the propagator is equal to

\[ U(x_b, x_a, t) = \left( \frac{m}{2 \pi i t} \right)^{1/2} \exp \left( \frac{im}{2t} (x_b - x_a)^2 \right) \] (48)

which correctly matches the known solution.

3.2. The Harmonic oscillator propagator

To solve the propagator for the force \( F(x) = -\omega^2 mX \), we can Taylor expand \( X(t) \) to get:

\[ X(t) = X_0 + \frac{P_0}{m} t - \frac{\omega^2}{2} X_0 t^2 - \frac{\omega^2}{6} P_0 t^3 + \cdots \]
\[ = X_0 \cos(\omega t) + \frac{P_0}{m \omega^2} \sin(\omega t). \] (49)

Equations (38) and (39) then become

\[ \langle X_b | \left( X_0 \cos(\omega t) + \frac{P_0}{m \omega^2} \sin(\omega t) \right) U^\dagger(t) | X_a \rangle = x_a \langle X_b | U^\dagger | X_a \rangle \] (50)

\[ \langle X_b | U(t) \left( X_0 \cos(\omega t) + \frac{P_0}{m \omega^2} \sin(\omega t) \right) | X_a \rangle = x_b \langle X_b | U | X_a \rangle. \] (51)

Equation (50) can be turned into an integral equation, as with the free particle, yielding

\[ \int \frac{dU^\dagger(x_b, x_a, t)}{U^\dagger(x_b, x_a, t)} = im \omega \int \frac{x_b - x_a \cos(\omega t)}{\sin(\omega t)} dx_b. \] (52)

Combined with the solution to equation (51) and, once again, the condition that \( U(x_b, x_a, 0) = \delta(x_b - x_a) \), we get

\[ U(x_b, x_a, t) = \left( \frac{m \omega}{2 \pi i \sin(\omega t)} \right)^{1/2} \exp \left\{ \frac{m \omega}{2i} \frac{(x_b^2 + x_a^2) \cos(\omega t) - 2x_b x_a}{\sin(\omega t)} \right\} \] (53)

which, again, matches the known result.

3.3. The constant force propagator

If a constant force is applied to a particle, \( F(t) = F_0 \), corresponding to the potential \( \mathcal{U}(x) = -F_0 X \), then \( X(t) \) and \( P(t) \) can be solved exactly and are

\[ P(t) = P_0 + F_0 t \] (54)

\[ X(t) = X_0 + \frac{P_0}{m} t + \frac{1}{2m} F_0 t^2. \] (55)

This adds only a small amount of complexity beyond the free particle case. Equations (38) and (39) become

\[ \left( x_b + \frac{1}{2m} F_0 t^2 \right) \langle X_b | U^\dagger(t) | X_a \rangle = \frac{i}{m \omega \partial_{x_b}} \langle X_b | U^\dagger(t) | X_a \rangle = x_a \langle X_b | U^\dagger | X_a \rangle \] (56)
\[
\left( x_a + \frac{1}{2m} F \frac{\partial^2}{\partial t^2} \right) \langle x_b | U(t) | x_a \rangle + i \frac{m}{\hbar} \frac{\partial}{\partial x_a} \langle x_b | U(t) | x_a \rangle = x_b \langle x_b | U(t) | x_a \rangle. \tag{57}
\]

The solution to equations (56) and (57), using the same integral method as in the free particle case, is
\[
U(x_b, x_a, t) = \left( \frac{m}{2\pi \hbar t} \right)^{1/2} \exp \left\{ \frac{im}{2\hbar} \left( (x_b - x_a)^2 + \frac{1}{m} \frac{\partial^2}{\partial x_a^2} (x_b - x_a) \right) \right\} \tag{58}
\]
where the coefficient out front is set by the same delta function boundary condition. Again, this matches the known propagator \([10, 11]\) up to a phase factor that is constant over all space and the result is achieved in a very simple fashion, since \(X(t)\) is easily solvable for a constant force.

### 4. Connection to the path integral

Although the propagator was only solved for three particular cases where the time dependence of \(X(t)\) and \(P(t)\) could be solved exactly, this technique is, in theory, applicable to particles under the influence of any arbitrary force \(F(X)\). Although the exact differential equation for the propagator can only be written when there is an analytic solution to the time dependence of \(X(t)\), it is always possible to write an approximate solution to the propagator over a small time interval. We will show that by piecing together propagators over small intervals, we can use this technique to reproduce the Feynman path integral formula, much in the same way as it can be derived starting with the Hamiltonian formalism.

As stated in equation (34), \(X(t)\) can be Taylor expanded in terms of \(P(t)\), \(F(t)\), and further time derivatives. If we keep only the terms to second order in time, for a small time interval, \(\Delta t\), we get
\[
X(t) \approx X_0 + \frac{1}{m} P_0 \Delta t + \frac{1}{2m} F(X_0) \Delta t^2. \tag{59}
\]

Using this approximate \(X(t)\), we can write the differential equation for the propagator over a small time interval \(U(\Delta t)\) as
\[
\left( x_b + \frac{1}{2m} F(x_b) \Delta t^2 \right) \langle x_b | U(\Delta t) | x_a \rangle - \frac{i\Delta t}{m} \frac{\partial}{\partial x_b} \langle x_b | U(\Delta t) | x_a \rangle = x_b \langle x_b | U^\dagger(\Delta t) | x_a \rangle \tag{60}
\]
\[
\left( x_a + \frac{1}{2m} F(x_a) \Delta t^2 \right) \langle x_a | U(\Delta t) | x_b \rangle + \frac{i\Delta t}{m} \frac{\partial}{\partial x_a} \langle x_a | U(\Delta t) | x_b \rangle = x_b \langle x_a | U^\dagger(\Delta t) | x_b \rangle \tag{61}
\]

Equation (60) becomes the integral equation
\[
\int \frac{dU^\dagger(\Delta t)}{U(\Delta t)} = \frac{im}{\Delta t} \int \left( x_a - x_b - \frac{1}{2m} F(x_b) \Delta t^2 \right) dx_b \tag{62}
\]
which has the solution
\[
U^\dagger(\Delta t) = A^\dagger(\Delta t) \exp \left\{ \frac{im}{\Delta t} \left( x_a x_b - \frac{1}{2} x_a^2 - \frac{1}{2m} \int F(x_b) \, dx_b \Delta t^2 + f(x_a) \right) \right\} \tag{63}
\]
where \(A(\Delta t)\) is defined, as in the previous section, to be a factor that will set the boundary condition that \(U(t)\) is a delta function at \(t = 0\). Solving equation (61) in a similar manner fixes \(f(x_a)\) and we find that the propagator is
\[
U(x_b, x_a, \Delta t) = A(\Delta t) \exp \left\{ \frac{im}{2\Delta t} \left( x_b - x_a \right)^2 + \frac{1}{m} \left( \int F(x_a) \, dx_a + \int F(x_b) \, dx_b \right) \Delta t^2 \right\}. \tag{64}
\]
Since equation (64) is only valid in the limit of small $\Delta t$, to calculate a propagator that spans a larger time period, we can subdivide the time interval into $N$ smaller steps and string together several propagators over small $\Delta t$. Since only the endpoints ($x_1$ and $x_N$, corresponding to the initial and final locations) are fixed, we must integrate over all intermediate locations, and we get

$$
\langle x_N | U(t) | x_1 \rangle = A(t) \int \! dx_2 \ldots dx_{N-1} \langle x_N | U(\Delta t) | x_{N-1} \rangle \langle x_{N-1} | U(\Delta t) | x_{N-2} \rangle \ldots \langle x_2 | U(\Delta t) | x_1 \rangle
$$

$$
= A(t) \int \! dx_2 \ldots dx_{N-1} \prod_{i=2}^{N} \frac{e^{-i \pi (\xi_i - \xi_{i-1})^2 / \Delta t}}{\left( \int F(x_i) \, dx - \int F(x_{i-1}) \, dx \right) \Delta t}
$$

(65)

where all of the factors $A(\Delta t)$ have been combined into a single factor, $A(t)$ that enforces the boundary condition at $t = 0$.

Noting that $x_i - x_{i-1} = \frac{1}{m} p_i \Delta t$, where $p_i$ represents the average momentum on the interval between $x_{i-1}$ and $x_i$, and that $\int F(x) \, dx = -U(x)$, we can rewrite (65) as

$$
U(x_b, x_a, t) = A(t) \int \! dx_2 \ldots dx_{N-1} \prod_{i=2}^{N} \frac{e^{+i \pi p_i^2 / m \Delta t - i \xi_i / \Delta t}}{U(x_i) + U(x_{i-1}) \Delta t}
$$

(66)

The $\frac{1}{2} (U(x_i) + U(x_{i-1}))$ in the exponent is approximately average potential between $x_{i-1}$ and $x_i$. This is true since we are considering $\Delta t$ to be a very small time interval and will eventually take the limit as $\Delta t$ goes to zero. We can then make the substitution that $\frac{1}{2} (U(x_i) + U(x_{i-1})) = U(x_i + \frac{1}{2})$. The term that appears in the exponent, $\frac{1}{2} m p_i^2 - U(x_i + \frac{1}{2})$, is the Lagrangian, $L$. Furthermore, the product of exponentials can be turned into a sum of exponentials, leaving us with

$$
U(x_b, x_a, t) = A(t) \int \! dx_2 \ldots dx_{N-1} e^{i \sum_{i=2}^{N} L(x_i + \frac{1}{2}) - L(x_i)} \Delta t.
$$

(67)

In the limit that we subdivide into an infinite number of infinitesimal intervals, each spanning an infinitesimal $\Delta t$ we arrive at our final expression for the propagator

$$
U(x_b, x_a, t) = A(t) \int \! D(x(t)) e^{i \int L(x(t), \dot{x}(t)) \, dt}
$$

(68)

where the capital $D$ refers to a sum over all paths $x(t)$. This is exactly the expression derived by Feynman for obtaining the propagator with the path integral method [7].

5. Concluding remarks

There are still issues that are difficult to address in a Newtonian formulation of quantum physics, such as the fact that the momentum operator $P = -i \hbar \nabla$ is the canonical momentum, rather than $m\dot{v}$. This can necessitate, as in the case of the Aharonov–Bohm problem, the addition of a term whose interpretation is unclear in Newtonian mechanics to produce the standard Newtonian momentum.

The fact that the integral of force that appears in equation (29) is an indefinite integral is also confusing in the case of a delta function force, which corresponds to a discontinuous, step function potential. Without the motivation of a well defined potential energy function, it is difficult to see why the integral at every point must be defined in such a way that there is a step at the location of the force, although it may be possible hand wave an argument based on the non-locality of momentum states that the force acts on.

Despite these interpretational difficulties for certain classes of problems, this derivation of quantum mechanics provides a key connection between Hamiltonian, Lagrangian, and...
Newtonian physics in the quantum regime. Especially for students who are new to Hamiltonian and Lagrangian mechanics, it can be used to form a bridge to facilitate the transition from their old way of thinking about physics to the new, and often seemingly bizarre quantum regime.

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