Analysis of Poisson Networks and Their Relation with Random Cellular Structures

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Abstract:

We perform a detailed analysis of the statistical properties of Poisson networks and show that the metric and topological properties of random cellular structures, can not be derived from simple models of random networks based on a poisson point distribution [1]. In particular we show that Lewis and Aboav-Wieare laws are not obeyed in these network.
1 Introduction

In the past two decades there has been a considerable interest in studying a class of non-equilibrium systems known collectively as Cellular Structures[2]. The geometrical and dynamical properties of these systems are best displayed in the familiar pattern formed by a soap froth confined between two transparent plates. Other examples include polycrystalline domains in metals, ceramics, magnetic domains and monolayers of fatty acids on surface of water. One can also mention other examples from material science, like cracking in glazes, fracture and dewetting of polymer films above their glass transition temperature [2]. Despite the diversity in systems in which cellular structures are formed, numerous experiments have shown that the long time statistical behaviour of these systems are characterized by certain universal, system independent laws. This means that topological and geometrical constraints influence the properties of these networks in a very essential way. For example, the fact that energy is associated with the length of the edges of the cells, immediately leads to the conclusion that in the two dimensional structures all the vertices are 3-valent. Combination of this result with the Euler character formula shows that the average number of sides per cell, $\langle i \rangle$ is equal to 6:

$$\langle i \rangle = 6$$ (1)

Similar considerations in 3-dimensions, where energy is associated with the area of faces of cells, proves that all vertices are 4-valent and that:

$$\langle f \rangle = \frac{12}{6 - \langle i \rangle}$$ (2)
where \( \langle f \rangle \) and \( \langle i \rangle \) are the average number of faces per cell and the average number of edges per face respectively. Besides these properties which are a direct consequence of the Euler character formula, experiments have revealed a number of very general, properties among which the most important are: i) von-Neumann [3], ii) Lewis [4] and iii) Aboav-Weare law [5,6]. Von-Neumann law refers to the rate of expansion of a single cell and is accounted for theoretically in a satisfactory way. It has also been generalized to curved surfaces where the curvature of the surfaces plays a role both in the dynamic of a single cell and in its stability [7]. The Aboav-Weare law which is statistical in nature and refers to the correlation of adjacent cells has been shown to hold in random 3-valent graphs, viewed as planar Feynman graphs of \( \phi^3 \) theory and solved by techniques of matrix models [8].

However there has been no explanation of the empirical Lewis law which states that the average area of cells has a linear relation with the number of cells, for large number of edges.

However recently there have been some attempts [1,9] to derive both Lewis law and Aboav-Weare law, from Poisson networks, i.e. networks based on a Poisson distribution of horizontal segments between a fixed set of parallel lines. It has been claimed [1] that in such networks both laws are obeyed. Also it has been claimed [1] that in one type of the 3-dimensional Poison networks, 3-dimensional analogues of Lewis and Aboav-Weare law are obeyed. In this paper we study closely these Poisson networks and show that the Lewis and Aboav-Weare law which is derived from these models contain highly non-linear terms, which is in sharp contrast with experimental data.

The structure of this paper is as follows: in section 2, we introduce the Poisson network in
general, such that we can use the result of this section for both 2-dimensional and 3-dimensional networks. In section 3 we show that Lewis and Aboav-Weare laws are not obeyed in these networks. In section 4 we show that even if one introduces an extra element of randomness into these models, namely, non-uniform density distribution, one can not obtain Lewis and Aboav-Weare law, although one can obtain better values for the moment of distributions of number of edges and faces in 2 and 3-dimensional cellular structure. Our conclusions are given in section 5.

2 Poisson networks with uniform density

The two dimensional poisson networks are generated as follows (figure 1)[1]. One takes a family of parallel lines in the y-direction in the plane. The distance between lines does not affect the topological properties of the network, although it affects the geometry. This distance is taken as uniform and equal to $d_0$. Suppose there are $N$ columns, $C_1, C_2, \ldots, C_N$, in the network. The region between two successive columns is divided into cells. The division of each column $C_\alpha$, is based on a poisson point distribution in the y-direction. An edge in the x-direction is taken through each Poisson point (P-point). In this way a 3-valent network is generated in the plane which, although different from the realistic cellular structures, is simple enough to be analysed closely for the study of various statistical-topological properties of the network. The 3-dimentional tetravalent network are generated by a similar process (figure 2)[1]. One takes an arbitrary 2-dimensional trivalent network (base network) on the xy plane (figure 2(a)) and
takes vertical planes (parallel to the z-axis) through each edge which divides the 3-dimensional space into prismatic columns. In each column one considers a Poisson distribution of uniform density in the z-direction and divides the columns into cells by planes perpendicular to the z-axis through each P-point (figure 2(b)). The height of a cell, $L$, is the distance between adjacent P-points in the associated distribution (figure 2(c)). The average cell height is unity. Consider one of the columns, say $C_{\alpha}$. The Poisson distribution of $n$ points in a segment of length $L$ is given by:

$$p(n) = \frac{L^n}{n!} e^{-L}. \quad (3)$$

Consider a particular set of columns $C_{\alpha}, C_{\beta}, \ldots, C_{\gamma}$. The probability that in a given distance $L$, there are $n_{\alpha}, n_{\beta}, \ldots, n_{\gamma}$ segments respectively in columns $C_{\alpha}, C_{\beta}, \ldots, C_{\gamma}$, is equal to

$$\Phi(L, n_{\alpha}, n_{\beta}, \ldots, n_{\gamma}) \equiv P(n_{\alpha})P(n_{\beta})\ldots P(n_{\gamma}) \quad (4)$$

Now consider a reference column which we denote by $C_{\alpha}$. The probability that there is a cell with height between $L$ and $L + dL$ in this column is $e^{-L}dL$. We denote the columns which are neighbors of this reference column by $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_k}$. The number of these neighboring columns depend on the dimensionality and type of the network. The joint probability that there is a cell of height between $L$ and $L + dL$ in the column $C_{\alpha}$ such that its neighbors $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_k}$ respectively have $n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_k}$ points in the interval $L$ is:

$$\Psi_{\alpha}(L, n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_k})dL = P(n_{\alpha_1})P(n_{\alpha_2})\ldots P(n_{\alpha_k})e^{-L}dL \quad (5)$$

clearly this distribution function depends on the individual values of the variables $n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_k}$, however in the sequel another probability distribution will be useful. that is $\Psi(L, I)dL$ were I
is defined as \( I = n_{\alpha_1} + n_{\alpha_2} + \ldots + n_{\alpha_k} \). The joint probability \( \Psi(L, I)dL \), is calculated as follows:

\[
\Psi_\alpha(L, I)dL = \sum' \Psi_\alpha(L, n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_k})dL = \frac{(kL)^I}{I!}e^{-(k+1)L}dL.
\] (6)

where \( \sum' \) means the sum over all \( n_{\alpha_i} \)'s subject to the constraint their sum be equal to \( I \). It’s important to note that the distribution \( \Psi_\alpha(L, I) \) is obtained from a sum of distribution of the form \( \Psi_\alpha(L, n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_k}) \) and is not equal to any of them. (compare with eq.(9a) of[1]).

3 statistical properties of poisson networks

3.1 Two dimensional networks

In two dimensions, every column has two neighbors, hence in eq.(6), \( k = 2 \), and the number of edges of a cell (see fig.(1)) in a reference column is equal to:

\[
i = I + 4
\] (7)

Combining eq.(6) and (7) gives the probability distribution \( g(L, i) \) of finding \( i \)-cells of height in the interval \( (L, L + dL) \):

\[
g(L, i)dL = \frac{(2L)^{(i-4)}}{(i-4)!}e^{-3L}dL.
\] (8)

Clearly this distribution function is normalized.

\[
\int_0^\infty \sum_{i=4}^\infty g(L, i)dL = 1.
\] (9)

From eq.(8) one can obtain the average height of \( i \)-cells:

\[
< L >_i = \int_0^\infty Lg(L, i)dL = \frac{i - 3}{3^2}\left(\frac{2}{3}\right)^{i-4}
\] (10)
This equation, then can be used to calculate the average area of $i$-cells. For the poisson networks considered in [1], where the width of all cells are equal to $d_0$, we find:

$$< A >_i = d_0 < L >_i = \frac{d_0(i - 3)}{3^2} \left( \frac{2}{3} \right)^{i-4}$$ (11)

Clearly this relation, due to it’s non-linear term, is in sharp contrast with Lewis law.

From eq.(8), one can obtain the total probability distribution of $i$-cells $g(i)$:

$$g(i) = \int_0^\infty g(L, i) dL = \frac{1}{2} \left( \frac{2}{3} \right)^{i-3}$$ (12)

One can also define a generating function:

$$G(q) = \sum_{i=4}^{\infty} g(i) e^{iq}$$ (13)

A simple calculation shows that:

$$G(q) = \frac{e^{4q}}{3 - 2e^q}$$ (14)

or

$$lnG(q) = 4q - ln(3 - 2e^q)$$ (15)

From which one obtains by successive differentiation various connected moment of the distribution:

$$< i > = 6$$ (16)

$$< i^2 > - < i > = 6$$ (17)

$$< i^3 > - 3 < i^2 > < i > + 2 < i >^3 = 30$$ (18)
3.2. three-dimensional networks.

Consider a 3-dimensional network and a column whose base is an $i$-cell. In eq.(6), we should now equate $k$ with $i$ and relate $I$ with the number of faces as follows:

$$I + 2 + i = F.$$  \hspace{1cm} (19)

The probability distribution of $F$-cells (cells with $F$-faces) of height between $L$ and $L + dL$, is now obtained as:

$$g_i(L, F)dL = \frac{(iL)^{F-i-2}}{(F-i-2)!}e^{-(i+1)L}dL$$  \hspace{1cm} (20)

From which one can obtain the average height of $F$-cells whose base are $i$-cells.

$$< L >_{i,F} = \int_0^\infty L g_i(L, F)dL = \frac{F - i - 1}{(i+1)^2} \left(\frac{i}{i+1}\right)^{F-i-2}$$  \hspace{1cm} (21)

The average height of $F$-cells with any base is given by

$$< L >_F = \sum_{i=4}^\infty < L >_{F,i} g(i)$$  \hspace{1cm} (22)

where $g(i)$ is the probability distribution of $i$-cells in the base. As an approximation one can set $i = < i > = 6$ in (21) and obtain:

$$< L >_F = \frac{F - 7}{7^2} \left(\frac{6}{7}\right)^{F-8}$$  \hspace{1cm} (23)

Compare with eq.(36) of [1]. The average volume of $F$-cells will then be:

$$< v >_F = S \frac{F - 7}{7^2} \left(\frac{6}{7}\right)^{F-8}$$  \hspace{1cm} (24)

where $S$ is the average area of cells in the base.
Here again, it’s seen that there is a pronounced non-linear term, which however is not as worse as the two dimensional case. From eq. (20) one can also find the total distribution of $F$-cells with $i$-cell bases:

$$g_i(F) = \frac{1}{i+1} \left( \frac{i}{i+1} \right)^{F-i-2}$$  \hspace{1cm} (25)

From this formula various moments can be calculated:

$$< F >_i = 2i + 2$$  \hspace{1cm} (26)

$$< F^2 > = 58 + 5i^2$$  \hspace{1cm} (27)

and

$$< F > = 2 < i > + 2 = 14$$  \hspace{1cm} (28)

$$< F^2 > = 58 + 5 < i^2 > = 268$$  \hspace{1cm} (29)

In the remaining part of this section we review briefly the basic steps of the analysis of ref.[1] and show that if one uses the correct form of the average $< L >_i$ as given by eq.(10), then for a poisson network, Aboav-Weare law will not be obeyed either, even approximately.

Consider an $i$-cell $a$ of length $L$ in a two dimensional poisson network(figure 3). This cell is in column $a$ and has two neighbors in this column, called $a'$ and $a''$.

There are two adjacent neighbors $b_1$ and $b_2$, which respectively distribute $n_1$ and $n_2$ points inside the cell $a$. The number of sides of $a$ is then equal to $L$

$$i = n_1 + n_2 + 4$$  \hspace{1cm} (30)
The total number of sides of the cells in column \( b_1 \) adjacent to \( a \) is

\[ J_1 = 4(n_1 + 1) + (m_1 + 1) + (m'_1 + 1) + m''_1 \]  

(31)

where the meaning of the numbers \( m_1, m'_1 \) and \( m''_1 \) are specified in fig. (3). Clearly \( im_i \) which is the total number of sides of the cells adjacent to \( a \) is:

\[ im_i = 12 + J_1 + J_2. \]  

(32)

Where the definition of \( J_2 \) is similar to that of \( J_1 \). For the average value of \( m_1, m'_1 \) and \( m''_1 \) we use:

\[ < m_1 + m'_1 > = < L_c - L_a > \]  

(33)

\[ < m'' > = \frac{< L_{cb} >}{1} = < L_{cb} - L_a > + < L_a > \]  

(34)

Where the segment \( L_{cb} \) in column \( b_1 \), which is the smallest segment containing \( L_a \), is called the covering length of \( l_a \) [1]. With the probability distribution found in [1] for \( L_c \) it is shown that the average \( < L_c - L_a > = 2 \), it then follows that:

\[ < im_i > = 12 + 4(n_1 + n_2) + < m_1 + m'_1 > + < m_2 + m'_2 > + < m'' > + < m''_1 > + 4 \]  

(35)

The number 12 comes from the average of sides of \( a' \) and \( a'' \). Combining eq.(33), (34) and (35) one finds that:

\[ < im_i > = 16 + 4i + 2 < L >_i \]  

(36)

Combining eq.(36) with (10) one obtains

\[ < im_i > = 16 + 4i + \frac{2(i-3)}{3^2} \left( \frac{2}{3} \right)^{i-4} \]  

(37)
which shows that Aboav-Weare law is not valid for such poisson networks. Clearly Poisson networks by construction do not have enough randomness to be good models of cellular structures, as far as geometrical properties (i.e. Lewis law) are concerned. One may even try to introduce more randomness, (i.e. non-uniform point distribution in different columns) to see if the non-linear terms shown in eq.(37) can be smoothed out. In the next section we examine briefly this possibility. The result is that although one can use this extra degree of freedom to obtain better agreement with experimental data for various moments of number of sides (i.e. $<i^2>$, etc.), again the Lewis and Aboav-weare law will not be valid.

4 Poisson networks with non-uniform density

A little modification of the previous formulas is necessary to study the properties of a network where the density of points (the average distance between points to be denoted by $\lambda$, in the following) in different columns are different. Instead of (1) we will have:

$$P_\alpha(n) = \left(\frac{L}{\lambda_\alpha}\right)^n \frac{1}{n!} e^{-\frac{L}{\lambda_\alpha}}$$

(38)

Instead of (4) we will have:

$$\Phi_0(L, n_1, n_2, \ldots, n_k) = P_1(n_1)P_2(n_2) \ldots P_k(n_k) = \prod_{\alpha=1}^{k} \left(\frac{L}{\lambda_\alpha}\right)^{n_\alpha} \frac{1}{n_\alpha!} e^{-\frac{L}{\lambda_\alpha}}$$

(39)

defining $\Psi(L, I) dL$ as before, we obtain:

$$\Psi_0(L, I) = \sum_{n_1 + n_2 + \ldots + n_k = I} \Phi(L, n_1, n_2, \ldots, n_k) \frac{1}{\lambda_0} e^{-\frac{L}{\lambda_0}} dL = \frac{L^I (\xi_0 - \frac{1}{\lambda_0})^I}{\lambda_0 I!} e^{-L \xi_0}$$

(40)
where $\xi_0 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_k}$. In a two dimensional network every column say $C_n$ has two neighbors, $C_{n-1}$ and $C_{n+1}$. The number of sides of $i$-cell (a cell with $i$-edges) in column $n$, is $i = 4 + I$, where $I = n_{n-1} + n_{n+1}$ and $n_{n-1} + n_{n+1}$ are the number of vertices contributed by the cells in adjacent columns. Then for the distribution of $i$-cells of length between $(L$ and $L + dL)$ in column $C_n$, we obtain:

$$g_n(L, i) = \frac{L^{i-4}(\xi_n - \frac{1}{\lambda_n})^{i-4}}{\lambda_n(i - 4)!} e^{-L\xi_n}$$

(41)

where $\xi_n = \frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n} + \frac{1}{\lambda_{n+1}}$. We will also obtain, the distribution of $i$-cells in column $C_n$:

$$g_n(i) = \int_0^\infty g_n(L, i) dL = \frac{1}{\lambda_n\xi_n} (1 - \frac{1}{\lambda_n\xi_n})^{i-4}$$

(42)

Clearly this distribution function is normalized. We can now evaluate various moments of the distribution $g_n(i)$. In all the following calculations, we use the following formulas for geometric sum:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

(43)

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1 - x)^2}$$

(44)

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1 + x)}{(1 - x)^3}$$

(45)

A simple calculation shows that:

$$< i >_n = \sum_{i=4}^{\infty} i g_n(i) = 3 + \lambda_n \xi_n = 4 + \lambda_n \left( \frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_{n+1}} \right)$$

(46)

where $< i >_n$ is the average number of $i$-cells in column $n$. For a uniform density, this number turns out to be equal to 6. Another quantity of interest is $< i^2 >_n = \sum_{i=4}^{\infty} i^2 g_n(i)$, which turns
out to be:

\[ < i^2 >_n = 2(\lambda_n\xi_n)^2 + 5\lambda_n\xi_n + 9 \]  \hspace{1cm} (47)

The variance turns out to be equal to:

\[ < i^2 >_n - < i >^2_n = (\lambda_n\xi_n)(\lambda_n\xi_n - 1) \]  \hspace{1cm} (48)

In order to find the moments \(< i >\) and \(< i^2 >\) in the whole lattice and not in a particular column \(n\), we replace the average over the cells in a network, by an average over the densities in an ensemble of networks. Thus from (46) and (47) we have:

\[ < i > = 4 + \lambda_n(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_{n+1}}) \]  \hspace{1cm} (49)

\[ < i^2 > = 2 < (\lambda_n\xi_n)^2 > + 5 < \lambda_n\xi_n > + 9 \]  \hspace{1cm} (50)

where the averages on the right hand side are performed with a suitable distribution

\[ P(\lambda_1, \lambda_2, \ldots, \lambda_N). \]

For simplicity in the following we assume that this distribution, is a symmetric distribution, i.e:

\[ P(\lambda_1, \lambda_2, \ldots, \lambda_N) = P(\lambda'_1, \lambda'_2, \ldots, \lambda'_N) \]  \hspace{1cm} (51)

where the primed indices are a permutation of the unprimed ones. Clearly this minor restriction allows our conclusions to be valid for a very large class of probability distributions. For a symmetric distribution we can simplify (49) and obtain:

\[ < i > = 6 \]  \hspace{1cm} (52)
as expected. Note that for a uniform distribution, we obtain
\[ < i^2 > = 42 \quad < i^2 > - < i >^2 = 6 \] (53)

in agreement with [1]. From (41) we also obtain the average height of \( i \)-cells in the \( n \)th column:
\[ < L >_{n,i} = \int_0^{\infty} L g_n(L, i) dL = \frac{i - 3}{\lambda_n \xi_n^2} (1 - \frac{1}{\lambda_N \xi_n})^{i-4} \] (54)

Again one sees that the non-linear behaviour persists in this average.

5 Three-Dimentional Networks

Consider a 3-dimentional laminated poisson network, based on an arbitrary 2-dimentional network, and a particular \( i \)-cell is the base. We label the column with this base by \( C_{i,0} \). The number of faces in a cell in the column above this \( i \)-cell is
\[ F = i + 2 + J \] (55)

where \( J = m_1 + m_2 + \ldots + m_i \), and \( m_k \) is the number of additional lateral faces which results from the cells in the adjacent \( k \)-column (fig. 4). Let \( f_i(J) \) be the fraction of the \( F \) cells in column \( C_{i,0} \). In formula (40), one should now replace \( k \) by \( i \), and \( I \) by \( J \) to obtain:
\[ \Psi_0(L, F, i) = \frac{L^{F-i-2} (\xi_0 - \frac{1}{\lambda_0})^{F-i-2}}{\lambda_0 J!} e^{-L \xi_0} \] (56)

where \( \xi_0 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_i} \). Here \( \lambda_0 \) is the density of P-point distribution in the column \( C_{i,0} \) and \( \lambda_\alpha \ (\alpha = 1, 2, \ldots, i) \) are the density of P-point distribution in the adjacent columns.
\( \Psi_0(L, F, i) \) is the probability of finding an \( F \)-cell in column \( C_{i,0} \), whose height is between \( L \) and \( L + dL \). Integrating over \( L \), one obtains

\[
\Psi_0(F, i) = \frac{1}{\lambda_0 \xi_0} (1 - \frac{1}{\lambda_0 \xi_0})^{F-i-2}
\]  

This is the probability of finding a \( F \)-cell in column \( C_{i,0} \). One can now find various moments of this distribution:

\[
< F >_{i,0} = \lambda_0 \xi_0 + i + 1
\]  

\[
< F^2 >_{i,0} = (i + 1)^2 + (2i + 1)(\lambda_0 \xi_0) + 2(\lambda_0 \xi_0)^2
\]  

Note that for uniform density one has \( \lambda_0 \xi_0 = i + 1 \) and the above formula reduce to

\[
< F >_{i} = 2i + 2
\]  

\[
< F^2 >_{i} = (i + 1)(5i + 4)
\]  

in agreement with[]. By the same averaging procedure as in the two dimensional case one can obtain

\[
< F >_{i} = \lambda_0 \xi_0 + i + 1
\]  

\[
< F^2 >_{i} = (i + 1)^2 + (2i + 1)(\lambda_0 \xi_0) + 2(\lambda_0 \xi_0)^2
\]  

The later averages are performed over the density distributions.
6 Conclusion

Our main conclusion is that Lewis and Aboav-Weare laws do not apply to poisson networks. This is due to the rather rigid geometrical structure of these networks. We have shown that by more randomizing these networks, (i.e. in the sense of random density distribution in different columns), one can not still obtain the above mentioned laws. In fact the only feature that these networks have in common with real cellular structures is that they are 3-valent, and the main property of real systems in which all the angles around any vertex are equal is obviously absent in these networks. It seems that the Lewis and to some extant Aboav-Weare law which are inherently geometrical in nature will not show up in those random models (i.e. Poisson) which neglect this very specific property of cellular structures.
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