Optimal Control over Multiple Input Lossy Channels

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Abstract—The performance of control systems with packet loss on the actuator communication channel are analysed. We extend the actuator communication channel to a multi-variable input channels allowing for an independent communication channel for each actuator. This allows for a wider class of system models to be used, such as a system without collocated actuators. The performance of the controller is measured in terms of a linear quadratic cost function. The optimal control law for a system operating with communication packet losses is derived under two communication protocols. In doing so we analytically prove that a control system operating over a UDP-like (User Datagram Protocol) protocol has a higher quadratic cost than a system operating under a TCP-like (Transmission Control Protocol) protocol. By modelling the communication channels independently it can be seen that certain states require more reliable channels. It is shown that the optimal control laws for each protocol create different behaviour in the expected state trajectory resulting in the differing costs.

I. INTRODUCTION

The paper derives the optimal control law for a plant that is communicating over multiple independent lossy channels. This is a generalisation on the actuator channel of the single channel modelled in [1]. This generalisation causes many of the steps used in the derivation of [1] to be non-trivial. The losses are assumed to take place within the actuator communication channel of the closed loop system. Derivation of the optimal control law is achieved by formulating the problem in a Model Predictive Control (MPC) framework. There is previous work in this area [1]–[4]. From this previous work, two protocols have been proposed for analysis, TCP-like and UDP-like. Both protocols mimic the communication protocols. The TCP-like protocol is depicted in Fig. 1 and transmits an acknowledgement signal to the controller [1], in contrast UDP-like lacks this acknowledgement link as is seen in Fig. 2. In [1]–[6], a control system with probabilistic losses in the connections between the plant and controller are modelled. In doing so, establishing control over a lossy communication channel. Using dynamic programming theory [1] derives the optimal control law and the optimal estimator for both protocols. We focus upon the control problem with packet losses only in the actuator channel with full state feedback. In [6] a system that only losses packets on the communication link between the sensors and the controller is considered. Whereas [1], [4] extends the work of [6] and consider losses on both the sensing and actuation communication channel. In [5], a comparison of different input signals is studied in the event of a packet loss in the actuation channel. In [2], [3] a system without, and with an acknowledgement link respectively are considered. These approaches analyse the performance of the controller and characterise the trade-off between the control system cost, stability, and the properties of the communication channel.

We consider a feedback system, consisting of plant, controller and communication channel, as shown in Figure 1 and Figure 2 for both protocols considered. The losses are in the controller-actuator link, the sensor-controller link is assumed to be a perfect channel. The TCP-like protocol used in control and the protocol used in communication differ in that in the control protocol an old signal is not be retransmitted. This is due to the most recently transmitted packet being the most important for the plant to receive. If traditional TCP is used it may lead to instability. There is no need for all packets to eventually arrive at the plant. This is comparable to the streaming a live video. Only packet losses over the actuation link are considered, however, this work can be extended to include sensor losses. The packet loss variable is modelled as an IID Bernoulli random variable on the actuation link. As a result, the plant either receives the optimal input for each actuator, or it receives zero. In the event of a packet loss the plant performs no actuation as is assumed in [1]–[4] and is the main focus of [5].

II. SYSTEM MODEL

We consider the plant model given by

$$X_{k+1} = AX_k + BV_kU_k + W_k,$$

where $A \in \mathbb{R}^{n \times n}$ is the dynamics matrix, $X_k \in \mathbb{R}^n$ describes the state of the plant at time step $k \in \mathbb{N}$, where $B \in \mathbb{R}^{n \times m}$ is the control matrix, $U_k \in \mathbb{R}^m$ is the control input vector at time step $k$, the process noise $W_k \in \mathbb{R}^n$ is a Gaussian distributed vector of random variables with mean $\mathbf{0} \in \mathbb{R}^n$ and covariance matrix $\Sigma_W \in S_{++}^n$, where $S_{++}^n$ is the set of $m$ by $m$ symmetric positive definite matrices, the packet losses are modelled via the diagonal matrix $V_k \in S^m_{++}$ where the $i$-th diagonal entry is an IID Bernoulli random variable with mean $\mu_i$, and $S_{++}^m$ is the set of $m$ by $m$ symmetric non-negative definite matrices. The initial state of the plant is determined by the Gaussian distributed vector of random variables $X_0$ with mean $\mathbf{0}$ and covariance matrix $\Sigma_X \in S_{++}^n$. Due to the lossy communication between the controller and the plant, the controller implements a communication protocol to monitor the state of the packets transmitted to the plant. We adopt the two protocol paradigms proposed by [1], namely a UDP-like
protocol that does not monitor the channel and a TCP-like protocol that acknowledges receipt of the packet from the controller by sending an *acknowledgement* message to the controller over an auxiliary channel. The difference between both protocol paradigms is depicted in Fig. [1] and Fig. [2] for the UDP-like and the TCP-like protocols, respectively. The information set available to the controller is determined by the choice of protocol. We define the information sets as

$$\mathcal{I}_k = \begin{cases} \mathcal{F}_k = \{X_k, v^{k-1}\}, & \text{TCP-like} \\ \mathcal{G}_k = \{X_k\}, & \text{UDP-like}, \end{cases}$$

(2)

where $v^{k-1} = \{V_0, V_1, \ldots, V_{k-1}\}$ and all sets are monotonically increasing, i.e. $\mathcal{I}_k \subseteq \mathcal{I}_{k+1}$. Under TCP-like protocols the controller has access to the realisation of the packet loss, $V_k$, when performing the estimation and utilises it in the error prediction

$$E_{k+1|\mathcal{F}_k} = X_{k+1} - \mathbb{E}[X_{k+1} | \mathcal{F}_k, V_k],$$

$$= AX_k + BV_k U_k + W_k - A\hat{X}_k - BV_k U_k,$$

$$= AE_{k|\mathcal{G}_{k-1}} + W_k.$$  

(3a)

Under UDP-like protocols the error prediction differs from TCP-like protocols due to the lack of knowledge of the realisation of $V_k$:

$$E_{k+1|\mathcal{G}_k} = X_{k+1} - \mathbb{E}[X_{k+1} | \mathcal{G}_k],$$

$$= AX_k + BV_k U_k + W_k - A\hat{X}_k - BMU_k,$$

$$= AE_{k|\mathcal{G}_{k-1}} + B(V_k - M)U_k + W_k.$$  

(3b)

It is interesting to note that the error term at time step $k+1$ depends on $U_k$. As shown in [1], this means the optimal linear control law, for the UDP-like protocol, can only be obtained when there is full state observation, $Y_k = X_k$. This is due to the separation principle between optimal estimation and optimal control being broken. Assuming perfect knowledge of the realisation $X_k$, as required for the optimal linear control law, (1) is predicted over a time horizon $N \in \mathbb{N}$

$$X_{k+1} = AX_k + BV_k U_k + W_k,$$

$$X_{k+2} = AX_{k+1} + BV_{k+1} U_{k+1} + W_{k+1},$$

$$\vdots$$

$$X_{k+N} = AX_{k+N-1} + BV_{k+N-1} U_{k+N-1} + W_{k+N-1},$$

(4)

exploiting the recursive structure yields

$$X_{k+1} = AX_k + BV_k U_k + W_k,$$

$$X_{k+2} = A^2 X_k + A B V_k U_k + B V_{k+1} U_{k+1} + A W_k + W_{k+1},$$

$$\vdots$$

$$X_{k+N} = A^N X_k + A^{N-1} B V_k U_k + \ldots + A B V_{k+N-2} U_{k+N-2} + B V_{k+N-1} U_{k+N-1} + A^{N-1} W_k + \ldots + W_{k+N-1},$$

(5)

which written in matrix form yields [6]. Re-labelling [6] as a prediction matrix equation gives

$$\chi_k = \Phi X_k + \Gamma Y_k + \Lambda \Xi,$$

(7)

where $\Phi \in \mathbb{R}^{N n \times n}$ is the dynamics over the prediction horizon, $\chi_k \in \mathbb{R}^{N n}$ is the state prediction vector, $\Gamma \in \mathbb{R}^{N n \times N n}$ is the propagation matrix for the control over the prediction horizon, $Y_k \in \mathbb{R}^{N n}$ is the realisation at time step $k$ of the control law, $\Lambda \in \mathbb{R}^{N n \times N n}$ is the propagation matrix for the process noise, $\Xi_k \in \mathbb{R}^{N n}$ is the process noise over the prediction horizon, and $V_k \in S_+^{N n}$ is a diagonal matrix with the Bernoulli random variables describing the packet losses over the prediction horizon in the diagonal. For the purposes of control, the estimate of $\chi_k$ must be computed. For both protocols this estimate is identical, due to the fact that neither protocol knows the realisation of $V_k$ before acting. The expected state trajectory is given by,

$$\hat{X}_k = \mathbb{E} \left[ \chi_k | \mathcal{I}_k \right] = \Phi X_k + \Gamma \mathbb{E}[Y_k].$$

(8)

It should be noted that for estimation, TCP-like protocols have access to the previous packet realisations, which results in (3a). However, when computing the optimal control law the system operator does not have access to future packet loss realisations, which results in (8). This is to maintain causality, to ensure the operator does not know if a packet is lost before transmitting. The operator does not know the realisation of a packet loss before actuating, but knows the packet loss realisation when updating the state estimate. Under the UDP-like protocol the packet loss is estimated for both the estimation and the optimal control problem. Stacking the error terms in the same fashion as the states
The controller minimises (11) by selecting the optimal input induced by the estimate, \( \nu \). This function is weighted with a diagonal state penalty and the diagonal matrix \( \chi \). As with Linear Quadratic Gaussian control (LQG), the cost function that the operator of the system minimises is a quadratic function of the states and the inputs to the system. This function is weighted with a diagonal state penalty matrix, \( \Omega \in S_{++}^n \), a diagonal input penalty matrix \( \Psi \in S_{++}^m \), and the diagonal matrix \( Q \in S_{++}^m \). The matrices can be time varying or fixed. Due to the noise present in the state, given in \( J \), this problem is a Stochastic model predictive control problem, [7]. The expected cost is defined as

\[
J(\mathcal{F}_k) \triangleq \mathbb{E} \left[ X_k^T Q X_k + \chi_k^T \Omega \chi_k + \gamma_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right],
\]

where the expectation in (10) is taken with respect to the distributions of \( v \) and \( \chi \). The expectation is taken sequentially as is shown in [2, Lemma 1(c)]. This lemma shows that the expectation at each time step is conditioned only on all previous time instances. This is achieved through the fact that each time instance forms a Markov chain with the others, dependent only on the previous instances (i.e. \( X_k \rightarrow X_{k+1} \rightarrow \cdots \rightarrow X_H \)). Therefore, the joint probability mass functions are split and the expectations are nested. The state, \( \mathcal{X}_k \), is re-written in terms of the estimate, \( \hat{\mathcal{X}}_k \), and the error induced by the estimate, \( E_{k|\mathcal{F}_k} \).

\[
J(\mathcal{F}_k) = \mathbb{E} \left[ X_k^T Q X_k + (\hat{\mathcal{X}}_k + E_k)^T \Omega (\hat{\mathcal{X}}_k + E_k) + \gamma_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right].
\]

The controller minimises (11) by selecting the optimal input law, \( \gamma_k \), and noting that \( \mathbb{E} \left[ E_k \mid \mathcal{F}_k \right] = 0 \) for both protocols yields

\[
J^*(\mathcal{F}_k) \triangleq \min \left\{ \mathbb{E} \left[ X_k^T Q X_k + \hat{\mathcal{X}}_k^T \Omega \hat{\mathcal{X}}_k + E_k^T \Omega E_k + \gamma_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right] \right\} \tag{12}
\]

### III. MPC OPTIMAL CONTROL

The derivation has been reduced to minimising (12), for both communication protocols. From (2) it can be seen that the first and second term on the right hand side of (12) are known realisations and can therefore be removed from the expectation. Furthermore, the first term does not depend on \( Y_k \) and can be removed from the optimisation.

\[
J^*(\mathcal{F}_k) = X_k^T Q X_k + \min \left\{ \hat{\mathcal{X}}_k^T \Omega \hat{\mathcal{X}}_k + Y_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right\} + \mathbb{E} \left[ E_k^T \Omega E_k \mid \mathcal{F}_k \right]. \tag{13}
\]

The quadratic nature of (13) means the expectation is non-trivial to compute. The \( Y_k^T v^T \Psi v Y_k \) term is the easier to evaluate due to the idempotency of \( v \). However, the quadratic error term involves second order statistics. The \( Y_k^T v^T \Psi v Y_k \) term can be simplified by using the commutation properties of diagonal matrices and the idempotency of \( v \). Due to the causality imposed on the system, \( Y_k \) can not depend on the future realisations of \( V_k \) or \( W_k \) and can only depend on the statistics and their past realisations (i.e \( \mathbb{E} \left[ Y_k \mid \mathcal{F}_k \right] = Y_k \)).

\[
\mathbb{E} \left[ Y_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right] = \mathbb{E} \left[ Y_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right] = Y_k^T v^T \Psi v Y_k \mid \mathcal{F}_k. \tag{14}
\]

Therefore (13) is equivalent to,

\[
J^*(\mathcal{F}_k) = X_k^T Q X_k + \min \left\{ \hat{\mathcal{X}}_k^T \Omega \hat{\mathcal{X}}_k + Y_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right\} + \mathbb{E} \left[ E_k^T \Omega E_k \mid \mathcal{F}_k \right]. \tag{15}
\]

The quadratic term involving the expected state trajectory can be expanded to give:

\[
J^*(\mathcal{F}_k) = X_k^T (Q + \Delta^\Phi) X_k + \min \left\{ \hat{\mathcal{X}}_k^T \Omega \hat{\mathcal{X}}_k + Y_k^T v^T \Psi v Y_k \mid \mathcal{F}_k \right\} + Y_k^T v^T (2FX_k + (\Delta^\Psi + \Psi) Y_k) \right\} + \mathbb{E} \left[ E_k^T \Omega E_k \mid \mathcal{F}_k \right], \tag{16}
\]

where \( \Delta^\Phi = \Phi^T \Omega \Phi, \Delta^\Psi = \Gamma^T \Omega \Gamma, \) and \( F = \Gamma^T \Phi \). However, evaluating the expectation of the error term, \( \mathbb{E} \left[ E_k^T \Omega E_k \mid \mathcal{F}_k \right] \), is not straightforward. It is in this step that the differences between the UDP-like and the TCP-like protocols become clear. Which leads to the first theorem.
Theorem 1: The following Theorem highlights the discrepancies arising from the protocol chosen. It is proved that:

\[ \mathbb{E} \left[ E_k^T \Omega E_k \bigg| \mathcal{F}_k \right] = \mathbb{E} \left[ \sum_i \Delta^i \Sigma \bigg| \mathcal{F}_k \right] , \quad (17a) \]

\[ \mathbb{E} \left[ E_k^T \Omega E_k \bigg| \mathcal{F}_k \right] = \mathbb{E} \left[ Y_k^T \nu (I \otimes \Delta^i) (I - \nu) Y_k \bigg| \mathcal{F}_k \right] + \text{tr} \left( \Delta^i \Sigma \right) . \quad (17b) \]

Proof: The proof is split into two sections for TCP-like and UDP-like, respectively.

1) TCP-like: The expected error in TCP-like follows from (3a) Substituting this into the left hand side of (17a) gives:

\[ \mathbb{E} \left[ E_k^T \Omega E_k \bigg| \mathcal{F}_k \right] = \mathbb{E} \left[ \sum_i \Delta^i \Sigma \bigg| \mathcal{F}_k \right] , \quad (18) \]

The term in (18) is a scalar and therefore the trace of this object is equal to itself. Additionally expectation and trace are linear operators, in view of this:

\[ \mathbb{E} \left[ E_k^T \Omega E_k \bigg| \mathcal{F}_k \right] = \text{tr} \left( \mathbb{E} \left[ \sum_i \Delta^i \Sigma \bigg| \mathcal{F}_k \right] \right) , \quad (19) \]

This completes this section of the proof.

2) UDP-like: It follows from (3b) that the error in UDP-like estimation is:

\[ E_k \bigg| \mathcal{F}_k = \Gamma (\nu - \bar{v}) Y_k \bigg| \mathcal{F}_k + \Delta \Sigma . \]

Substituting this into the left hand side of (17b) gives:

\[ \mathbb{E} \left[ E_k^T \Omega E_k \bigg| \mathcal{F}_k \right] = \mathbb{E}_v \left[ Y_k^T (\nu - \bar{v}) \Delta^i (\nu - \bar{v}) Y_k \bigg| \mathcal{F}_k \right] + \text{tr} \left( \Delta^i \Sigma \right) . \]

where the zero mean of $\Sigma$ is utilised to eliminate the cross terms. The second term is identical to the TCP-like case and therefore the same process is followed.

\[ \mathbb{E} \left[ E_k^T \Omega E_k \bigg| \mathcal{F}_k \right] = \mathbb{E}_v \left[ Y_k^T \nu \Delta^i \nu Y_k \bigg| \mathcal{F}_k \right] - \nu^T \Delta^i \nu Y_k \bigg| \mathcal{F}_k + \text{tr} \left( \Delta^i \Sigma \right) . \]

This leads to the first Lemma

Lemma 1: It is proved that:

\[ \mathbb{E}_v \left[ Y_k^T \nu \Delta^i \nu Y_k \bigg| \mathcal{F}_k \right] = Y_k^T \nu \Delta^i \nu Y_k + \nu^T \Delta^i \nu (I \circ \Delta^i) (I - \nu) Y_k . \quad (20) \]

where $I$ is the identity matrix, and $\circ$ is the element wise Hadamard product.

The expectation to be evaluated is:

\[ \mathbb{E}_v \left[ Y_k^T \nu \Delta^i \nu Y_k \bigg| \mathcal{F}_k \right] , \quad (21) \]

this term is scalar, and therefore, evaluating it is equivalent to:

\[ \mathbb{E}_v \left[ Y_k^T \nu \Delta^i \nu Y_k \right] = \mathbb{E}_v \left[ U_0 N M A \nu \nu^T U_0 U_0 + \mathbb{E}_v \left[ U_0 N M A \nu \nu^T U_0 U_0 + \cdots + \mathbb{E}_v \left[ U_{N M - 1} N M - 1 \nu \nu^T U_{N M - 1} U_{N M - 1} \right] \right] \right] \]

\[ = \sum_{i=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{i=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{j=1}^{N M} \left( N M \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \right) \]

\[ = \sum_{j=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{j=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{j=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{j=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{j=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]

\[ = \sum_{j=1}^{N M} \left( \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] + \sum_{j=1, j \neq i}^{N M} \mathbb{E}_v \left[ U_{i-1} N M A \nu \nu^T U_{i-1} U_{i-1} \right] \right) \]
B. Optimal Control for UDP-like Protocol

Combining (17b) and (16) the optimal cost function for UDP-like protocols is given by:

\[
J^*(\gamma_k) = X_k^T (Q + \Delta \Phi) X_k + \min_{\gamma_k} \left\{ \gamma_k^T (2FX_k + (\Delta^T \Sigma_{\Delta} + \mathbf{I}) (I - \bar{\nu})) \right\}. 
\]

Minimising with respect to \( \gamma_k \) results in,

\[
\frac{\partial J^*(\gamma_k)}{\partial \gamma_k} = 2\bar{\nu} (FX_k + (\Delta^T \bar{\nu} + \mathbf{I}) (I - \bar{\nu})) \gamma_k.
\]

The \((\mathbf{I} - \bar{\nu}) (I - \bar{\nu})\) term contains positive definite and positive semi-definite terms and is therefore strictly positive definite, as a result it is invertable. For all \( \nu \neq 0 \) the minimising \( \gamma_k \) is:

\[
\gamma_k^* = - (\mathbf{I} - \bar{\nu}) (I - \bar{\nu})^{-1} FX_k. 
\]

Re-labelling \((\mathbf{I} - \bar{\nu}) (I - \bar{\nu})\) as \( \mathbf{G}_{\bar{\nu}} \) and substituting \( \gamma_k^* \) into (17b) yields the optimal expected cost for the operator:

\[
J^* = X_k^T (Q + \Delta \Phi) X_k + \text{tr} \left( \Sigma_{\Delta} \Delta^T + X_k^T \mathbf{G}_{\bar{\nu}}^{-1} FX_k \right). 
\]

It is interesting to note the subtle differences between the two protocols and their corresponding optimal control laws. In the TCP-like protocol regime the optimal control law only depends on the average number of packet losses, \( \bar{M} \), and this term weights how the actuation propagates through the system (the \( \Delta^T \) term). Whereas, the UDP-like protocol control law contains an additional term. This term weights the diagonal of control law with the probability of a packet loss. From (25), it is highlighted that this term represents the variance of \( \bar{M} \). This difference arises from the differences in information about the variable \( M \). UDP-like has less information and must take the variance into account to be optimal. This difference in information leads to a difference in overall cost.

Theorem 2: Assuming that \( 0 < M \leq 1 \). For control system that experiences packet losses on the actuation channel as modelled in (11) and uses the LQG cost function

\[
J^*(\mathcal{F}_k) = X_k^T QX_k + \min_{\mathcal{F}_k} \left\{ \chi_k^T \mathbf{G}_k^{-1} \chi_k + \mathbf{Y}_k^{\mathcal{F}} \right\},
\]

It will be shown that the expected cost when communicating using a UDP-like protocol is higher than TCP-like. A system experiencing actuation packet losses will only achieve equality between the two protocols when \( M = 1 \) i.e. when the communication channel is perfect. Specifically

\[
J^*(\gamma_k) \geq J^*(\mathcal{F}_k) \quad (26)
\]

Proof: The optimal control laws for each communication protocol are

\[
\begin{align*}
\gamma_k^* \mathcal{F}_k &= - (\mathbf{I} - \bar{\nu}) (I - \bar{\nu})^{-1} FX_k, \\
\gamma_k^* \gamma_k &= - (\mathbf{I} - \bar{\nu}) (I - \bar{\nu})^{-1} FX_k + \mathbf{G}_k^{-1} \mathbf{Y}_k\mathcal{F}_k.
\end{align*}
\]

From the structure of \( \mathbf{G}_k^* \gamma_k \) it is seen that

\[
\mathbf{G}_k^* \gamma_k \succ 0 \quad \mathbf{G}_k^* \gamma_k \succ 0 \quad (27)
\]

Additionally from (24) and (25),

\[
\mathbf{G}_k^* \gamma_k - \mathbf{G}_k^* \gamma_k \gamma_k = (\mathbf{I} - \bar{\nu}) (I - \bar{\nu}) \succeq 0 \quad (28)
\]

with equivalence if and only if \( M = 1 \). Therefore, by assumption

\[
\mathbf{G}_k^* \gamma_k \succ \mathbf{G}_k^* \gamma_k.
\]

from [8, p. 228 10.53] it is implied that

\[
\mathbf{G}_k^{-1} \gamma_k \succ \mathbf{G}_k^{-1} \gamma_k.
\]
By assumption $\mathbf{M} \succ \mathbf{0}$, therefore, it follows that

$$
\begin{align*}
\tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} & \prec \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} \\
X_k^T F^T \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} F X_k & \prec X_k^T F^T \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} F X_k \\
-\Lambda_k^T F^T \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} F X_k & \succ -\Lambda_k^T F^T \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} F X_k \\
C - X_k^T F^T \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} F X_k & \succ C - X_k^T F^T \tilde{v}_k^{G^{-1}_{k|\mathcal{F}_k}} F X_k
\end{align*}
$$

where $C = X_k^T (Q + \Delta^2) X_k + \text{tr}(\Sigma_2 \Delta X)$. Therefore for any $0 < \mathbf{M} \preceq \mathbf{I}$

$$
J^*(\mathcal{F}_k) \geq J^*(\mathcal{F}_k) \tag{29}
$$

This concludes the proof.

IV. Numerical Results

The control laws derived above have resulted in two corresponding cost functions. Consider the scenario where the cost function is varied with respect to the loss parameter (at each step the optimal control law is recalculated) this is depicted in Figure 3. When looking at the expected cost as a function of $\mathbf{M}$ (Figure 3) it can be seen that both protocols converge to the same expected cost when $\mathbf{M} \rightarrow 1$, this is the normal case that standard control systems operate at i.e. no packet loss. This corresponds with the statement of Theorem 2. Contrasting this it can be seen as $\mathbf{M} \rightarrow 0$ the expected cost begins to asymptote. This is intuitive as it corresponds to open loop control. For an unstable $\mathbf{A}$ the cost tends to infinity for either protocol as $N \rightarrow \infty$. When considering the expected state trajectory ($\tilde{x}_k$) it is interesting to see that the differences in the information sets result in different expected state trajectories as shown in Fig. 4.

V. Conclusion and Discussion

It is seen, by extending [1] to multiple independent channels, that $\mathbf{M}$ changes the expected trajectory of each individual state. This means that in order to define the stability of this system the bounds presented in [1] require altering. This is a result of the change in the channel modelling we have introduced. For future work, it is conjectured that the $v_c$ presented in [1] is a sufficient condition for stability of a system. It might be able to be proved that a sufficient condition for stability may be $\min(\text{diag}(\mathbf{M})) > v_c$. Stability in this framework needs to be altered to account for the stable and unstable eigenvalues within $\mathbf{A}$. For example, a diagonal $\mathbf{A}$ would mean any stable states could have $\mathbf{M}_{i,j} \rightarrow 0$, this is below the $v_c$ presented in [1]. This allows for the case where the closed loop system is stable with elements of $\mathbf{M}$ less than $v_c$, this is shown in Fig. 4. Additionally, this work assumes that the sensory channel has perfect communication. This work could be extended such that this assumption is dropped, in doing so this would be a true extension of [1]. In other future work, this formulation could be extended to an optimisation problem such that the channel has a power limit, $\tilde{v}$, must distribute this power over the $m$ actuation channels. This problem would need to require that the system remains stable in addition to minimising the cost. We conjecture that in doing so this becomes a water filling problem for the operator.

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