Absoluteness via Resurrection

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Abstract

The resurrection axioms are forms of forcing axioms that were introduced recently by Hamkins and Johnstone, developing on ideas of Chalons and Veličković. We introduce a stronger form of resurrection axioms (the \textit{iterated} resurrection axioms) and show that they imply generic absoluteness for the first-order theory of $H_c$ with parameters with respect to various classes of forcings. We also show that the consistency strength of these axioms is below that of a Mahlo cardinal for most forcing classes, and below that of a stationary limit of supercompact cardinals for the class of stationary set preserving posets. We also compare these results with the generic absoluteness results by Woodin and the second author.

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1 Introduction

It is a matter of fact that forcing is one of the most powerful tool to produce consistency results in set theory. Nonetheless a current theme of research in
contemporary set theory seeks for extensions \( T \) of ZFC which turn forcing into a powerful instrument to prove theorems over \( T \). This is done by showing that a statement \( \phi \) follows from \( T \) if and only if \( T \) proves that \( \phi \) is consistent by means of certain forcing notions. These types of results are known in the literature as generic absoluteness results and have the general form of a completeness theorem for some \( T \supseteq \text{ZFC} \) with respect to the semantic given by boolean valued models and first order calculus. More precisely generic absoluteness results fit in the following general framework:

Assume \( T \) is an extension of ZFC, \( \Theta \) is a family of first order formulas in the language of set theory and \( \Gamma \) is a certain class of forcing notions definable in \( T \). Then the following are equivalent for a \( \phi \in \Theta \) and \( S \supseteq T \):

1. \( S \) proves \( \phi \).
2. \( S \) proves that there exists a forcing \( P \in \Gamma \) such that \( P \) forces \( \phi \) and \( T \) jointly.
3. \( S \) proves that \( P \) forces \( \phi \) for all forcings \( P \in \Gamma \) such that \( P \) forces \( T \).

We refer the reader to the second author’s papers [12] and [13] for a detailed account on these type of results and for some motivation on the foundational role generic absoluteness can play in set theory. We say that a structure \( M \) definable in a theory \( T \) is generically invariant with respect to forcings in \( \Gamma \) and parameters in \( X \subset M \) when the above situation occurs with \( \Theta \) being the first order theory of \( M \) with parameters in \( X \). A quick overview of the main known generic absoluteness results is the following:

- Shoenfield’s absoluteness theorem is a generic absoluteness result for \( \Theta \) the family of \( \Sigma^1_2 \)-properties with real parameters, \( \Gamma \) the class of all forcings, \( T = \text{ZFC} \).

- The pioneering “modern” generic absoluteness results are Woodin’s proofs of the invariance under set forcings of the first order theory of \( L(\text{ON}^\omega) \) with real parameters in \( \text{ZFC} + \text{class many Woodin cardinals which are limit of Woodin cardinals} \) [8, Theorem 3.1.2] and of the invariance under set forcings of the family of \( \Sigma^1_2 \)-properties with real parameters in the theory \( \text{ZFC} + \text{CH} + \text{class many measurable Woodin cardinals} \) [8, Theorem 3.2.1].

Further results pin down the exact large cardinal strength of the assertion that \( L(\mathbb{R}) \) is generically invariant with respect to certain classes of forcings (among others see [10]).

- The bounded forcing axioms BFA(\( \Gamma \)) is equivalent to the statement that generic absoluteness holds for \( T = \text{ZFC} \) and \( \Theta \) the class of \( \Sigma^1_1 \)-formulas with parameters in \( \mathcal{P}(\omega_1) \), as shown in [1].
• Recently Hamkins and Johnstone [4] introduced the resurrection axioms RA(Γ) and the second author [12] showed that these axioms produce generic absoluteness for Θ the Σ₂-theory with parameters of \( H_c, T = ZFC + RA(Γ), Γ \) any of the standard classes of forcings closed under two step iterations.

• The second author introduced the forcing axiom MM+++(a natural strengthening of MM) and proved that \( L(ON^{\omega_1}) \) with parameters in \( P(\omega_1) \) is generically invariant with respect to SSP forcings for

\[
T = ZFC + MM^{+++} \text{ + there are class many superhuge cardinals.}
\]

Motivated by the latter results as well as by the work of Tsaprounis [11], we expand on Hamkins and Johnstone’s work [4] obtaining the following results:

• We introduce a new natural class of forcing axioms, the iterated resurrection axioms RA_α(Γ) as Γ ranges among various classes of forcing notions and α runs through the ordinals, and prove the following main theorem.

**Theorem.** Assume Γ is a definable class of forcings. If RA_ω(Γ) holds and \( B \in Γ \) forces RA_ω(Γ), then \( H^*_V \prec H^*_V[B] \).

This is a generic absoluteness result for \( T = ZFC + RA_ω(Γ) \) and Θ the first order theory of \( H_τ \) with parameters.

We remark now (and we shall prove later) that these new axioms strengthen the resurrection axioms RA(Γ) introduced by Hamkins and Johnstone [4], and that for all \( α \) RA_α(SSP) follows from the axiom MM^{+++} introduced in [13].

• We show that the consistency strength of the axioms RA_α(Γ) is below that of a Mahlo cardinal for all relevant Γ except for Γ = SSP, in which case our upper bound is below a stationary limit of supercompact cardinals.

Compared to the generic absoluteness result obtained in [13], the present result is weaker since it applies to \( H_τ \) instead of \( L(ON^{\omega_1}) \). On the other hand, the consistency of RA_α(Γ) is obtained from (in most cases much) weaker large cardinal hypothesis and the results are more general since they also apply to interesting choices of Γ ≠ SP, SSP. Moreover (as we shall see below) the arguments we employ to prove the consistency of RA_α(Γ) for the Γ ≠ SSP case are considerably simpler than the arguments developed in [13]. The proof of the consistency of RA_α(SSP) instead will rely on several results.
appearing in [13]: we do not know if this is really necessary to handle this case.

Section 2 introduces the definitions of these new forcing axioms with their basic properties, and proves the main theorem. Section 3 develops the necessary technical devices for the consistency proofs of the axioms RA_{\alpha}(\Gamma). These proofs are carried over in section 4 adapting the consistency proofs for the resurrection axioms introduced in [4] to this new setting of iterated resurrection.

1.1 Notation

As in common set-theoretic use, \( P_\kappa(x), \text{trcl}(x), \text{rank}(x) \) will denote respectively the set of subsets of size less than \( \kappa \), the transitive closure, the rank for a given set \( x \); CH will denote the continuum hypothesis and \( c \) the continuum itself.

We will use \( M \prec_n N \) to denote that \((M, \in)\) is a \( \Sigma_n \)-elementary substructure of \((N, \in)\). The notation \( f : A \to B \) will be improperly used to denote partial functions in \( A \times B \).

We shall follow Jech’s approach [6] to forcing via boolean valued models. The letters \( B, C \) will be used for complete boolean algebras, \( \Gamma \) will be used for definable classes of boolean algebras, such as \( \sigma \)-closed, ccc, \( < \kappa \)-cc, axiom-A, proper, SP (semiproper), SSP (stationary set preserving), presaturated towers. A definition of these properties can be found in [2], [3], [6].

We shall use \( V_B \) for the boolean valued model obtained from \( V \) and \( B \), \( \dot{x} \) for the elements (names) of \( V_B \), \( \dot{\pi} \) for the canonical name for a set \( x \in V \) in the boolean valued model \( V_B \), \( [\phi]_B \) for the truth value of the formula \( \phi \).

When convenient we shall also use the generic filters approach to forcing. The letters \( G, H \) will be used for generic filters over \( V \), \( \dot{G}_B \) will denote the canonical name for the generic filter for \( B \), \( \text{val}_G(\dot{x}) \) the valuation map on names by the generic filter \( G \), \( V[G] \) the generic extension of \( V \) by \( G \). We shall write \( H_\theta \prec H' \) to denote that

\[(H_\theta, \in) \prec (H'_\theta, \in)\]

for all generic filters \( G \) and formulas with parameters in \( H'_\theta \).

\( B/G \) will denote the quotient of a boolean algebra \( B \) by the ideal dual to the filter \( G \), \( B * C \) will denote the usual two-step iteration as in [6]. \( \text{Coll}(\alpha, \beta) \) is the Lévy collapse that generically adds a surjective function from \( \alpha \) to \( \beta \), \( \text{Add}(\alpha, \beta) \) is the \( < \alpha \)-closed poset that generically adds \( \beta \) many subsets to \( \alpha \). In general we shall feel free to confuse a partial order with its boolean completion. When we believe that this convention may generate misunderstandings we shall be explicitly more careful.

Given an elementary embedding \( j : V \to M \) with \( M \subseteq V \), we will use \( \text{crit}(j) \) to denote the critical point of \( j \), and say that \( j \) is \( \lambda \)-supercompact iff
\[ \lambda M \subseteq M, \text{ and that a cardinal } \kappa \text{ is supercompact iff for every } \lambda \text{ there exists a } \lambda\text{-supercompact elementary embedding with critical point } \kappa. \]

Our reference text for large cardinals is [7], while for forcing axioms is [3] Chapter 3.

1.2 Iterable forcing classes

In order to develop a unified treatment for the most common classes \( \Gamma \) of forcing notions, we introduce the following definitions.

**Definition 1.1.** Let \( \Gamma \) be any definable class of forcing notions closed under two step iterations and \( B, C \) be complete boolean algebras.

We say that \( B \leq_\Gamma C \) iff there is a complete homomorphism \( i : C \rightarrow B \) such that the quotient algebra \( B/i[\dot{G}_C] \) is in \( \Gamma \) with boolean value \( 1_C \).

We say that \( B \leq_\Gamma^* C \) iff there is a complete injective homomorphism with the same properties as above.

We denote by \( U^\kappa_\Gamma \) (the category forcing) the set \( \Gamma \cap H_\kappa \) ordered by \( \leq_\Gamma \).

**Definition 1.2.** The lottery sum of a family \( B \) of boolean algebras is the cartesian product \( \prod B \) equipped with componentwise operations.

**Definition 1.3.** We say that a definable class of forcings \( \Gamma \) is iterable iff it is closed under two-step iterations, lottery sums and the order \( \leq_\Gamma^* \) is closed for set-sized descending sequences of elements of \( \Gamma \).

Most notable cases for \( \Gamma \) are iterable: i.e, \( \Gamma = \text{all}, \text{locally ccc} \) (lottery sums of ccc algebras), axiom-A, proper, semiproper. The \( \leq_\Gamma^* \) -order is shown to be closed under set sized descending sequences using respectively finite supports (locally ccc), countable supports (axiom-A and proper) and revised countable supports (semiproper) to find lower bounds for descending chains.

**Definition 1.4.** If \( \Gamma \) is iterable and \( f : \kappa \rightarrow \kappa \) is a partial function, we define \( P_\kappa \) the lottery iteration of \( \Gamma \) relative to \( f \) recursively as \( P_{\alpha+1} = P_\alpha \times \dot{Q} \) where \( \dot{Q} \) is a \( P_\alpha \)-name for the lottery sum of all boolean algebras in \( \Gamma \cap H_{f(\alpha)} \), and \( P_\alpha \) for \( \alpha \) limit is obtained using the closure of \( \leq_\Gamma^* \).

The lottery iteration has been studied extensively by Hamkins [5] and has nice properties whenever \( f \) attains a sufficiently fast-growing behavior. We will employ these type of iterations in section 3.1.

1.3 Forcing axioms as density properties

The resurrection axiom, introduced by Hamkins in [4], can be concisely stated as a density property for the class partial order \( \leq_\Gamma \) defined above.
Definition 1.5. The resurrection axiom RA(\(\Gamma\)) is the assertion that the class
\[ \{ B \in \Gamma : H_c \prec H^{V^B} \} \]
is dense in (\(\Gamma, \leq \)).

The weak resurrection axiom wRA(\(\Gamma\)) is the assertion that the same class is dense in (\(\Gamma, \leq_{\text{all}}\)).

We shall reformulate in a similar way many of the common forcing axioms. We recall the standard definitions:

Definition 1.6. The bounded forcing axiom BFA(\(\Gamma\)) holds if for all \(B \in \Gamma\) and all families \(\{ D_\alpha : \alpha < \omega_1 \}\) of predense subsets of \(B\) of size at most \(\omega_1\), there is a filter \(G \subset B\) meeting all these dense sets.

The forcing axiom FA(\(\Gamma\)) holds if for all \(B \in \Gamma\) and all families \(\{ D_\alpha : \alpha < \omega_1 \}\) of predense subsets of \(B\), there is a filter \(G \subset B\) meeting all these dense sets.

The forcing axiom FA\(^++\)(\(\Gamma\)) holds if for all \(B \in \Gamma\) and all families \(\{ D_\alpha : \alpha < \omega_1 \}\) of predense subsets of \(B\) and all families \(\{ \dot{S}_\alpha : \alpha < \omega_1 \}\) of \(B\)-names for stationary subsets of \(\omega_1\), there is a filter \(G \subset B\) meeting all these dense sets and evaluating each \(\dot{S}_\alpha\) as a stationary subset of \(\omega_1\).

Theorem (Bagaria, [1]). BFA(\(\Gamma\)) is equivalent to the assertion that the class
\[ \{ B \in \Gamma : H_{\omega_2} \prec_1 V^B \} \]
is dense in (\(\Gamma, \leq_{\text{all}}\)).

We remark that the latter assertion is actually equivalent to requiring this class to coincide with the whole \(\Gamma\) (since \(\Sigma_1\)-formulas are always upwards absolute).

Under suitable large cardinal assumptions the unbounded versions of the forcing axioms can also be reformulated as density properties, but only for \(\Gamma = \text{SSP}\) (at least to our knowledge):

Theorem (Woodin, [8]). Assume there are class many Woodin cardinals. Then MM (i.e. FA(\(\text{SSP}\))) is equivalent to the assertion that the class
\[ \{ B \in \text{SSP} : B \text{ is a presaturated tower forcing} \} \]
is dense in (\(\text{SSP}, \leq_{\text{all}}\)).

Theorem (Viale, [13]). Assume there are class many Woodin cardinals. Then MM\(^++\) (i.e. FA\(^++\)(\(\text{SSP}\))) is equivalent to the assertion that the above class is dense in (\(\text{SSP}, \leq_{\text{SSP}}\)).

In this paper we shall also refer to the following strengthening of MM\(^++\), which is defined by a density property of the class SSP as follows:
Definition 1.7 (Viale, [13]). MM$^{+++}$ is the assertion that the class
\[ \{ B \in SSP : B \text{ is a strongly presaturated tower} \} \]
is dense in $(SSP, \leq_{SSP})$.

2 Iterated resurrection and absoluteness

Motivated by Hamkins and Johnstone’s [4], as well as by Tsaprounis’ [11], we introduce the following new forcing axioms:

Definition 2.1. Let $\Gamma$ be any definable class of forcing notions.

The $\alpha$-weak resurrection axiom $wRA_\alpha(\Gamma)$ is the assertion that for all $\beta < \alpha$ the class
\[ \{ B \in \Gamma : H_c \prec H^B_c \land V^B \models wRA_\beta(\Gamma) \} \]
is dense in $(\Gamma, \leq_{all})$.

The $\alpha$-resurrection axiom $RA_\alpha(\Gamma)$ is the assertion that for all $\beta < \alpha$ the class
\[ \{ B \in \Gamma : H_c \prec H^B_c \land V^B \models RA_\beta(\Gamma) \} \]
is dense in $(\Gamma, \leq_\Gamma)$.

We use $RFA^{++}_\alpha(\Gamma)$ for the assertion that for all $\beta < \alpha$ the class
\[ \{ B \in \Gamma : H_c \prec H^B_c \land V^B \models FA^{++}(\Gamma), RFA^{++}_\beta(\Gamma) \} \]
is dense in $(\Gamma, \leq_\Gamma)$.

Notice that $wRA_0(\Gamma), RA_0(\Gamma)$ holds vacuously true for any $\Gamma$, so that $wRA_1(\Gamma), RA_1(\Gamma)$, coincide to the non-iterated formulations given in [4].

These different forcing axioms defined above are connected by the following implications:

- $RFA^{++}_\alpha(\Gamma) \Rightarrow RA_\alpha(\Gamma) \Rightarrow wRA_\alpha(\Gamma)$,
- if $\beta < \alpha$, $wRA_\alpha(\Gamma) \Rightarrow wRA_\beta(\Gamma)$ (same with RA, $RFA^{++}$),
- if $\Gamma_1 \subseteq \Gamma_2$, $wRA(\Gamma_2) \Rightarrow wRA(\Gamma_1)$.

We shall be mainly interested in $RA_\alpha(\Gamma)$, even though $wRA_\alpha(\Gamma)$ will be convenient to state theorems in a concise form (thanks to its monotonous behavior), and $RFA^{++}_\alpha(\Gamma)$ will be convenient to handle the $\Gamma = SSP$ case.

Some implications can be drawn between iterated resurrection axiom and the usual forcing axioms, as shown in the following theorems.

\footnote{We refer the reader to [13 Def. 5.10] for a definition of strongly presaturated towers.}
Theorem 2.2. \( w\text{RA}_1(\Gamma) + \neg \text{CH} \) implies \( \text{BFA}(\Gamma) \).

Proof. We shall prove the formulation of \( \text{BFA}(\Gamma) \) as a density property given in section [13].

Let \( \mathbb{B} \) be any boolean algebra in \( \Gamma \), and \( \mathbb{C} \leq_{\text{all}} \mathbb{B} \) be such that \( H_\mathbb{C} \prec H_\mathbb{C}^\mathbb{C} \). Since \( \neg \text{CH} \) holds, we have that \( \mathbb{C} \geq \omega_2 \) hence in particular \( H_\omega_2 \prec H_\omega_2^\mathbb{C} \). By Lévy’s absoluteness theorem \( H_\omega_2^\mathbb{C} \prec_1 V^\mathbb{C} \). This concludes the proof.

Theorem 2.3. Assume there are class many super huge cardinals. Then \( \text{MM}^{+++} \) implies \( \text{RA}_\alpha(\text{SSP}) \) for all \( \alpha \in \text{ON} \).

Proof. We proceed by induction on \( \alpha \). For \( \alpha = 0 \) it is trivial, suppose now that \( \alpha > 0 \).

Let \( \mathcal{A} \) be the class of all super huge cardinals in \( V \). Since \( \mathcal{A} \) is a proper class, by [13, Theorem 3.5] the class \( \{ \mathbb{U}_\delta : \delta \in \mathcal{A} \} \) is predense in \( (\text{SSP}, \preceq_{\text{SSP}}) \) and \( H_\omega_2 \prec H_\omega_2^{\mathbb{U}_\delta} \) for all \( \delta \in \mathcal{A} \) since \( \mathbb{U}_\delta \) is forcing equivalent to a presaturated tower for any such \( \delta \) assuming \( \text{MM}^{+++} \) by [13, Corollary 5.18]. Moreover, by [13, Corollary 5.22] every such \( \mathbb{U}_\delta \) forces that \( \text{MM}^{+++} \) holds and there are class many super huge cardinals (since large cardinals are indestructible by small forcing). It follows by inductive hypothesis that every such \( \mathbb{U}_\delta \) forces \( \text{RA}_\beta(\text{SSP}) \) for any \( \beta < \alpha \) as well, hence \( \text{RA}_\alpha(\text{SSP}) \) holds in \( V \).

2.1 Resurrection axioms and generic absoluteness

The motivation for the iterated resurrection axioms can be found in the following result:

Theorem 2.4. Suppose \( n \in \omega \), \( \text{RA}_n(\Gamma) \) holds and \( \mathbb{B} \in \Gamma \) forces \( \text{RA}_n(\Gamma) \). Then \( H_\mathbb{C} \prec_n H_\mathbb{C}^\mathbb{B} \).

Proof. We proceed by induction on \( n \). If \( n = 0 \), the thesis follows by the fact that all transitive structures \( M \subset N \) are \( \Sigma_0 \)-elementary. Suppose now that \( n > 1 \), and fix \( G \ V\)-generic for \( \mathbb{B} \). By \( \text{RA}_n(\Gamma) \), let \( \mathbb{C} \subset V[G] \) be such that whenever \( H \) is \( V[G] \)-generic for \( \mathbb{C}, V[G \ast H] \models \text{RA}_{n-1}(\Gamma) \) and \( H_\mathbb{C} \prec H_\mathbb{C}^{V[G \ast H]} \). Then we have the following diagram:

\[
\begin{tikzcd}
H_\mathbb{C}^V \arrow[r, \Sigma_\omega] \arrow[d, \Sigma_{n-1}] & H_\mathbb{C}^{V[G \ast H]} \arrow[d, \Sigma_{n-1}] \\
H_\mathbb{C}^{V[G]} \arrow[r, \Sigma_{n-1}] & H_\mathbb{C}^{V[G]} 
\end{tikzcd}
\]

obtained by inductive hypothesis applied both on \( V \), \( V[G] \) and on \( V[G], V[G \ast H] \) since in all those classes \( \text{RA}_{n-1}(\Gamma) \) holds.

Let \( \phi = \exists x \psi(x) \) be any \( \Sigma_n \) formula with parameters in \( H_\mathbb{C}^V \). First suppose that \( \phi \) holds in \( V \), and fix \( \bar{x} \in V \) such that \( \psi(\bar{x}) \) holds. Since \( H_\mathbb{C} \prec_{n-1} H_\mathbb{C}^{V[G]} \) and \( \psi \) is \( \Pi_{n-1} \), it follows that \( \psi(\bar{x}) \) holds in \( V[G] \) hence
so does \( \phi \). Now suppose that \( \phi \) holds in \( V[G] \) as witnessed by \( \bar{x} \in V[G] \).

Since \( H^V \prec V[G] \), it follows that \( \psi(\bar{x}) \) holds in \( V[G+H] \), hence so does \( \phi \). Since \( H^V \prec H^V[G+H] \), the formula \( \phi \) holds also in \( V \) concluding the proof.

**Corollary 2.5.** Assume \( RA_\omega(\Gamma) \) holds and \( B \in \Gamma \) forces \( RA_\omega(\Gamma) \). Then \( H^c \prec H^B \).

A similar absoluteness result for \( L(\text{ON}_\omega) \) concerning SSP forcings in the MM+++ setting was previously obtained in [13]. Corollary 2.5 provides a weaker statement (since it concerns the smaller model \( H^c \)) while being more general, since it holds also for \( \Gamma \neq \text{SSP} \) and is compatible with \( \text{CH} \). It must also be noted that the consistency of \( RA_\omega(\Gamma) \) follows from much weaker large cardinal assumptions than the principle MM+++ needed to prove absoluteness for \( L(\text{ON}_\omega) \).

### 3 Uplifting cardinals and definable Menas functions

In order to measure the consistency strength of the \( \alpha \)-resurrection axiom, we need to introduce the following large cardinal notion.

**Definition 3.1.** We say that a cardinal \( \kappa \) is \((\alpha)-uplifting\footnote{In [4], a similar notation is used to denote a bounded version of \((1)-uplifting\ cards. We believe that since Hamkins’ definition of \(\alpha\)-uplifting is irrelevant to us and also marginally used in his paper, it should not be confusing its different use in this context.} \) iff it is inaccessible and for every \( \beta < \alpha, \theta > \kappa \), there exists a \((\beta)-uplifting\ cardinal \( \lambda > \theta \) such that \( V_\kappa \prec V_\lambda \).

We remark that the definition of \((0)-uplifting\ cardinal overlaps with inaccessible, hence \((1)-uplifting\ cards coincide with \(uplifting\ cards\) as defined in [4]. Note that if a cardinal is \((\alpha)-uplifting\ in \( V \) it is \((\alpha)-uplifting also in \( L \), and if \( \alpha > 0 \) it is also \( \Sigma_3\-reflecting\). Hence the least supercompact (if it exists) is below the least \((1)-uplifting\).

**Proposition 3.2.** There are class many \((\text{ON})-uplifting\ cards consistently relative a Mahlo cardinal.

**Proof.** Suppose that \( \lambda \) is a Mahlo cardinal, and let \( C = \{ \alpha < \lambda : V_\alpha \prec V_\lambda \} \) be a club on \( \lambda \), \( S = \{ \alpha \in C : \alpha \text{ is inaccessible} \} \) be stationary. We prove that every element of \( S \) is \((\alpha)-uplifting\ by induction on \( \alpha < \lambda \).

First, every element of \( S \) is \((0)-uplifting\ by definition. Suppose now that every element of \( S \) is \((\beta)-uplifting\ for every \( \beta < \alpha \), and let \( \kappa \) be in \( S \). Then for every \( \beta < \alpha, \theta > \kappa \) there is a \( \gamma \in S, \gamma > \theta \) that witnesses the \((\alpha)-uplifting\ness of \( \kappa \).
As shown in [4, Theorem 11], if there is a (1)-uplifting cardinal then there is a transitive model of ZFC + ON is Mahlo. So the existence of an (ON)-uplifting cardinal is in consistency strength strictly between the existence of a Mahlo cardinal and the scheme “ON is Mahlo”. We take these bounds to be rather close together and low in the large cardinal hierarchy.

To achieve a consistency result also for RFA++, we need to introduce the following definition.

**Definition 3.3.** \( \kappa \) is (0)-uplifting for supercompacts iff it is supercompact. For \( \alpha \geq 1 \), we say that a cardinal \( \kappa \) is \( (\alpha) \)-uplifting for supercompacts iff it is supercompact and for every \( \beta < \alpha, \theta > \kappa \), there exists a \( (\beta) \)-uplifting for supercompacts cardinal \( \lambda > \theta \) such that \( V_\kappa \prec V_\lambda \).

A bound for this large cardinal notion can be obtained in a completely similar way to Proposition 3.2:

**Proposition 3.4.** There are class many (ON)-uplifting for supercompacts cardinals consistently relative to a stationary limit of supercompact cardinals.

### 3.1 Menas functions for uplifting cardinals

As previously mentioned, to obtain the consistency results at hand we shall use a lottery iteration relative to a fast-growth function \( f : \kappa \rightarrow \kappa \) for a sufficiently large cardinal \( \kappa \). The exact notion of fast-growth we will need is given by the Menas property schema introduced in [9] and developed by Hamkins for several different cardinal notions in [4], [5]. We remark that it is possible to define Menas functions for cardinals that don’t have a Laver function, obtaining from these Menas functions many of the interesting consequences given by Laver functions. This is exactly what we will do for \( (\alpha) \)-uplifting cardinals:

**Definition 3.5.** A partial function \( f : \kappa \rightarrow \kappa \) is Menas for \( (\alpha) \)-uplifting iff \( \kappa \) is inaccessible and for all \( \beta < \alpha, \theta > \kappa \) there is a Menas for \( (\beta) \)-uplifting function \( g : \lambda \rightarrow \lambda \) with \( g(\kappa) > \theta \) and \( (V_\kappa, f) \prec (V_\lambda, g) \).

In order to prove the existence of a definable Menas function for \( (\alpha) \)-uplifting cardinals, we first need to outline some reflection properties of these cardinals.

**Lemma 3.6.** Let \( \kappa \) be an \( (\alpha) \)-uplifting cardinal with \( \alpha < \kappa \), and \( \delta < \kappa \) be an ordinal. Then \( (\delta) \) is \( (\alpha) \)-uplifting in \( V_\kappa \) iff it is \( (\alpha) \)-uplifting.

**Proof.** Let \( \phi(\alpha) \) be the statement of this theorem, i.e:

\[ \forall \kappa > \alpha \ (\alpha) \text{-uplifting } \forall \delta < \kappa \ ( (\delta) \text{ is } (\alpha) \text{-uplifting})^{V_\kappa} \iff \delta \text{ is } (\alpha) \text{-uplifting} \]

We shall prove \( \phi(\alpha) \) by induction on \( \alpha \). For \( \alpha = 0 \) it is easily verified, suppose now that \( \alpha > 0 \).

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For the forward direction, suppose that \((\delta \text{ is } (\alpha)-\text{uplifting})^{V_\kappa}\), and let \(\beta < \alpha, \theta > \delta \) be ordinals. Let \(\lambda > \theta \) be a \((\beta)-\text{uplifting} \) cardinal with \(V_\kappa \prec V_\lambda\), so that \((\delta \text{ is } (\alpha)-\text{uplifting})^{V_\lambda}\). Then there is a \(\gamma > \theta \) in \(V_\lambda\) with \(V_\delta \prec V_\gamma\) and \((\gamma \text{ is } (\beta)-\text{uplifting})^{V_\lambda}\). By inductive hypothesis applied to \(\beta < \alpha\),

\((\gamma \text{ is } (\beta)-\text{uplifting})^{V_\lambda}\) iff it is \((\beta)-\text{uplifting}\).

Thus \(\gamma\) is \((\beta)-\text{uplifting}\) concluding this part.

Conversely, suppose that \(\delta \text{ is } (\alpha)-\text{uplifting in } V\) and let \(\beta < \alpha, \theta > \delta\) be ordinals in \(V_\kappa\). Let \(\lambda > \theta\) be a \((\beta)-\text{uplifting}\) cardinal such that \(V_\kappa \prec V_\lambda\), and let \(\lambda > \gamma\) be a \((\beta)-\text{uplifting}\) cardinal such that \(V_\kappa \prec V_\gamma\). By inductive hypothesis, since \(\beta < \alpha \) and \(\gamma, \lambda\) are \((\beta)-\text{uplifting in } V\), \((\gamma \text{ is } (\beta)-\text{uplifting})^{V_\lambda}\), thus

\[ V_\lambda \models \exists \gamma > \theta \ V_\delta \prec V_\gamma \land \gamma \text{ is } (\beta)-\text{uplifting}. \]

By elementarity, \(V_\kappa\) models the same concluding the proof.

**Proposition 3.7.** If \(\kappa\) is \((\alpha)-\text{uplifting}\), then there is a Menas function for \((\alpha)-\text{uplifting on } \kappa\).

**Proof.** We shall prove by induction on \(\alpha\) that whenever \(\kappa\) is \((\alpha)-\text{uplifting}\), such a function is given by the following definition (failure of upliftingness function) relativized to \(V_\kappa\):

\[ f(\xi) = \sup \{ \gamma : V_\xi \prec V_\gamma \land \gamma \text{ is } (\beta)-\text{uplifting} \} \]

where \(\beta\) is minimum such that the supremum exists. Note that \(f(\xi)\) is undefined only if \(\xi\) is \((\text{ON})\)-uplifting in \(V_\kappa\).

If \(\kappa\) is \((0)\)-uplifting any function \(g : \kappa \to \kappa\) is Menas, in particular our \(f\) is such, thus suppose now that \(\kappa\) is \((\alpha)-\text{uplifting with } \alpha > 0\) and let \(\beta < \alpha, \theta > \kappa\) be ordinals. Let \(\lambda > \theta\) be a \((\beta)-\text{uplifting}\) cardinal such that \(V_\kappa \prec V_\lambda\), and let \(\lambda'\) be the least \((\beta)-\text{uplifting}\) cardinal bigger than \(\lambda\) such that \(V_\kappa \prec V_{\lambda'}\). Thus no \(\xi \in (\lambda, \lambda')\) can be \((\beta)-\text{uplifting in } V\) hence by Lemma 3.6, either \(V_\lambda'\) and \(\lambda'\). Then \(\kappa\) cannot be \((\beta + 1)-\text{uplifting in } V_{\lambda'}\), while again by Lemma 3.6 it is \((\beta)-\text{uplifting in } V_{\lambda'}\). Now observe that if \(\eta < \beta\) then \(\kappa\) is \((\eta + 1)-\text{uplifting in } V_{\lambda'}\) again by Lemma 3.6 in particular for any such \(\eta\)

\[ \sup \{ \gamma < \lambda' : V_\kappa \prec V_\gamma \land \gamma \text{ is } (\eta)-\text{uplifting} \} = \lambda'. \]

This gives that \(\beta\) is really the ordinal required to define \(f(\kappa)\) in \(V_{\lambda'}\).

It follows that \(f^{V_{\lambda'}}(\kappa) = \lambda\) concluding the proof.

### 4 Consistency strength

The results of this section expand on the ones already present in [4] and [13].

In the previous section we outlined the large cardinal properties we shall need
for our consistency proofs. Now we shall apply the machinery developed by Hamkins and Johnstone in their proof of the consistency of \( RA_1(\Gamma) \) for various classes of \( \Gamma \), and we will show that with minor adjustments their techniques will yield the desired consistency results for \( RA_\alpha(\Gamma) \) for all \( \Gamma \neq \text{SSP} \) when applied to lottery preparation forcings guided by suitable Menas functions. To prove the consistency of \( RA^{++}_\alpha(\text{SSP}) \) we will instead employ the technology introduced by the first author in \cite{13}.

For this reason we shall feel free to sketch most of the proofs, leaving to the reader to check the details, that can be developed in analogy to what is done in \cite{4} and \cite{13}.

4.1 Lower bounds

**Theorem 4.1.** If \( wRA_\alpha(\Gamma) \) holds for \( \Gamma = \{\text{Coll}(\omega_1, \beta) : \beta \in \text{ON}\} \) and CH fails, then \( c^V \) is \((\alpha)\)-uplifting in \( L \).

*Proof.* The proof is obtained by induction on \( \alpha \), following \cite{4}, Theorem 16] with minor adjustments. \hfill \Box

Notice than the same result will hold also for any bigger \( \Gamma \) (as \( \sigma \)-closed, axiom-A, proper, semiproper, SSP).

**Theorem 4.2.** If \( RA_\alpha(\Gamma) \) holds for \( \Gamma = \text{all, ccc} \) then \( c^V \) is \((\alpha)\)-uplifting in \( L \).

*Proof.* The proof is obtained by induction on \( \alpha \), following \cite{4}, Theorem 16] with minor adjustments. \hfill \Box

4.2 Upper bounds

**Theorem 4.3.** \( RA_\alpha(\Gamma) \) is consistent relative to an \((\alpha)\)-uplifting cardinal, for \( \Gamma \) iterable.

*Proof.* The proof follows the one of \cite{4}, Theorem 18]. We prove by induction on \( \alpha \) that \( P_\kappa \), the lottery iteration of \( \Gamma \) relative to a function \( f : \kappa \to \kappa \), forces \( RA_\alpha(\Gamma) \) whenever \( f \) is Menas for \((\alpha)\)-uplifting. By Lemma \ref{lem:existence}, the existence of such an \( f \) follows from the existence of an \((\alpha)\)-uplifting cardinal, giving the desired result.

Since \( RA_0(\Gamma) \) holds vacuously true, the thesis holds for \( \alpha = 0 \). Suppose now that \( \alpha > 0 \). Let \( \hat{Q} \in V^{P_\kappa} \) be a name for a forcing in \( \Gamma \), \( \beta < \alpha \) be an ordinal. Using the Menas property for \( f \), let \( V_\lambda, g \) be such that \( (V_\lambda, f, g(\kappa)) \geq \text{rank}(\hat{Q}) \) and \( g \) is a \((\beta)\)-uplifting Menas function on \( \lambda \). Let \( P_\lambda \) be the lottery iteration of \( \Gamma \) relative to \( g \). By induction hypothesis, \( P_\lambda \) forces \( RA_\beta(\Gamma) \) and by elementarity \( g \upharpoonright \kappa = f \) so that \( P_\kappa \) is an initial part of \( P_\lambda \).

Let \( G \) be any \( V \)-generic filter for \( P_\kappa \), \( H \) be a \( V[G] \)-generic filter for \( \text{val}_G(\hat{Q}) \). Since \( g(\kappa) \geq \text{rank}(\hat{Q}) \), \( \hat{Q} \) is one of the elements of the lottery sum
considered at stage $\kappa + 1$ so that $G \ast H$ is $V$-generic for $\mathbb{P}_{\kappa}$ $(\kappa + 1)$. Let $G'$ be $V[G \ast H]$-generic for $\mathbb{P}_{\kappa}$. Then by [4, Lemma 17], $H_{\kappa}[G] \prec H_{\kappa}[G \ast H \ast G']$. Furthermore, since $\mathbb{P}_{\kappa}$ is $<\kappa$-cc and $\mathbb{P}_{\lambda}$ is $<\lambda$-cc we have that $H_{\kappa}[G] = H_{\kappa}^{V[G]}$ and $H_{\lambda}[G \ast H \ast G'] = H_{\lambda}^{V[G \ast H \ast G']}$. Finally, $\kappa = \epsilon$ in $V[G]$ and $\lambda = \epsilon$ in $V[G \ast H \ast G']$ by standard density arguments, so that $H_{\epsilon}^{V[G]} \prec H_{\epsilon}^{V[G \ast H \ast G']}$ concluding the proof.

In the previous theorem one notable case was excluded, i.e. $\Gamma = SSP$. The best known upper bound for the consistency strength of $\text{RA}_{1}(\text{SSP})$ is given in [11, Theorem 3.1] where it is shown that in the presence of class many Woodin cardinals $\text{MM}^{++}$ is equivalent to $\text{RFA}_{1}^{++}(\text{SSP})$ (according to our terminology), although it is not clear how this result can be generalized to $\text{RA}_{\alpha}(\text{SSP})$ for $\alpha > 1$. $\text{MM}^{++}$ can be forced by a semi-proper iteration to get $\text{SPFA}$ and $\text{SP} = \text{SSP}$. For $\text{RA}_{\alpha}(\text{SSP})$ a similar idea has to be applied iteratively, in order to ensure that after each resurrection the equality $\text{SP} = \text{SSP}$ still holds. This leads to the definition of $\text{RFA}_{\alpha}^{++}(\Gamma)$ that was given at the beginning of this paper.

In order to prove a consistency upper bound for $\text{RFA}_{\alpha}^{++}(\text{SSP})$ we will change slightly the approach, replacing the lottery iteration with the category forcing $\mathbb{U}_{\kappa}^{\text{SSP}}$ introduced in [13] since we do not know how to define Menas functions for $(\alpha)$-uplifting for supercompacts. Thus, we shall first explore some properties of $\mathbb{U}_{\kappa}^{\text{SSP}}$ as a class forcing over $H_{\kappa}$.

**Definition 4.4.** A partial order $\mathbb{P}$ is nice for forcing in a structure $M$ if the forcing relation $\Vdash_{\mathbb{P}}$ is definable in $M$ and the forcing theorem holds, i.e. for every first-order formula with parameters $\phi$ and every $G$ $M$-generic for $\mathbb{P}$,

$$M[G] \models \phi \iff \exists p \in G \ M \models (p \Vdash_{\mathbb{P}} \phi)$$

Note that the definition is interesting only when $\mathbb{P} \notin M$ is a class forcing with respect to $M$. The next lemma provides a sufficient condition for being nice for forcing in $H_{\kappa}$.

**Lemma 4.5.** Let $\mathbb{P} \subseteq H_{\kappa}$ be a partial order preserving the regularity of $\kappa$ that is definable in $H_{\kappa}$. Then $\mathbb{P}$ is nice for forcing in $H_{\kappa}$.

**Proof.** First recall that if $V$ models that $\mathbb{P}$ preserves the regularity of $\kappa$ and $G$ is $V$-generic for $\mathbb{P}$, $H_{\kappa}[G] = H_{\kappa}^{V[G]}$, or equivalently for all $\tau \in V^{\mathbb{P}}$ such that $1 \Vdash \tau \in H_{\kappa}$, there is an open dense set of conditions $p$ such that there exists a $\sigma_{p} \in H_{\kappa} \cap V^{\mathbb{P}}$ with $p \Vdash \tau = \sigma_{p}$.

Since $\mathbb{P}$ is a definable class in $H_{\kappa}$, the corresponding forcing relation $\Vdash_{\kappa}$ given by formulas with parameters in $H_{\kappa} \cap V^{\mathbb{P}}$ and whose quantifiers range only over the $\mathbb{P}$-terms in $H_{\kappa}$ is clearly definable in $H_{\kappa}$. Moreover, it is easily proved by induction on $\phi$ that this relation coincides with the forcing relation as calculated in $V$, i.e. $p \Vdash \phi^{H_{\kappa}}$ iff $(p \Vdash_{\kappa} \phi)^{H_{\kappa}}$: for $\phi$ atomic this follows
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from absoluteness of $\Delta_1$ formulas, the case of propositional connectives is easily handled, while if $\phi = \exists x \psi(x)$ we have that

\[
p \Vdash \phi^\kappa \iff \left\{ q \leq p : \exists \tau \in V^p q \Vdash (\tau)^\kappa \land \tau \in H_\kappa \right\} \text{ open dense}
\]

\[
\iff \left\{ q \leq p : \exists \sigma \in H_\kappa \cap V^p q \Vdash (\sigma)^\kappa \right\} \text{ open dense}
\]

\[
\iff (p \Vdash_\kappa \phi)^\kappa
\]

since the intersection of two open dense sets is open dense. It follows that

\[
H_\kappa[G] \models \phi \iff V[G] \models \phi^\kappa \iff \exists p \in G p \Vdash \phi^\kappa \iff \exists p \in G H_\kappa \models p \Vdash_\kappa \phi
\]

concluding the proof.

We are now ready to prove the following main theorem.

**Theorem 4.6.** RFA$_{\alpha}^{++}$ (SSP) is consistent relative to an $(\alpha)$-uplifting for supercompacts cardinal.

**Proof.** We prove by induction on $\alpha$ that $U_\kappa^{\text{SSP}}$, the category forcing of height $\kappa$, forces RFA$_{\alpha}^{++}$ (SSP) whenever $\kappa$ is $(\alpha)$-uplifting for supercompacts. Since RFA$_{0}^{++}$ (SSP) holds vacuously true, the thesis holds for $\alpha = 0$.

Suppose now that $\alpha > 0$. Let $\dot{Q} \in V_{U_\kappa^{\text{SSP}}}$ be a name for a forcing in SSP, $\beta < \alpha$ be an ordinal. Since $\kappa$ is $(\alpha)$-uplifting for supercompacts, let $\lambda > \text{rank}(\dot{Q})$ be a $(\beta)$-uplifting for supercompacts cardinal such that $V_\kappa \in V_\lambda$. By induction hypothesis, $U_\lambda^{\text{SSP}}$ forces RFA$_{\beta}^{++}$ (SSP), and by [13] Theorem 3.21 it forces also MM$^{++}$.

Let $G$ be any $V$-generic filter for $U_\kappa^{\text{SSP}}$, $H$ be a $V[G]$-generic filter for $\text{val}_G(\dot{Q})$. Since $\lambda \geq \text{rank}(\dot{Q})$, $U_\kappa^{\text{SSP}} \ast \dot{Q}$ is in $U_\lambda^{\text{SSP}}$ hence by [13] Theorem 3.15, Fact 3.16 there is a $G'$ such that $G \ast H \ast G'$ is generic for $U_\lambda^{\text{SSP}}$.

By [13] Lemma 3.18, $U_\kappa^{\text{SSP}}$ preserves the regularity of $\kappa$, hence by Lemma [4.5] $U_\kappa^{\text{SSP}}$ is nice for forcing in $V_\kappa$. Then we can apply [3] Lemma 17 to obtain

\[
H_\kappa[G] \prec H_\lambda[G \ast H \ast G']
\]

Furthermore, since $U_\kappa^{\text{SSP}}$ preserves the regularity of $\kappa$ and $U_\lambda^{\text{SSP}}$ preserves the regularity of $\lambda$, we have that $H_\kappa[G] = H_\kappa^{V[G]}$ and $H_\lambda[G \ast H \ast G'] = H_\lambda^{V[G \ast H \ast G']}$. Finally, by [13] Theorem 3.21 $\kappa = \omega_2 = c$ in $V[G]$ and $\lambda = \omega_2 = c$ in $V[G \ast H \ast G']$, so that $H_\kappa^{V[G]} \prec H_\lambda^{V[G \ast H \ast G']}$ concluding the proof.

**4.3 Iterated resurrection and size of the continuum**

In [4] it is proved that wRA($\Gamma$) implies $c \leq \omega_2$ for any $\Gamma$ containing Coll$(\omega_1, \omega_2)$, and that RA(all) implies CH while RA(ccc) implies that $c$ is weakly inaccessible.

In the last subsection we showed that RA$_{\kappa}(\Gamma)$ is compatible with FA(\Gamma) (which in many cases imply $c = \omega_2$). However, the iterated resurrection axiom is also compatible with CH for most $\Gamma$. 

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Theorem 4.7. RA_α(Γ) + CH is consistent relative to RA_α(Γ), for any Γ closed under two-step iteration and such that Add(ω_1, 1) ∈ Γ.

Proof. The proof is obtained by induction on α, following [4, Theorem 6] with minor adjustments.

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