On Representations of Graphs as Two-Distance Sets

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Abstract

Let $\alpha \neq \beta$ be two positive scalars. A Euclidean representation of a simple graph $G$ in $\mathbb{R}^r$ is a mapping of the nodes of $G$ into points in $\mathbb{R}^r$ such that the squared Euclidean distance between any two points is $\alpha$ if the corresponding nodes are adjacent and $\beta$ otherwise. A Euclidean representation is spherical if the points lie on an ($r-1$)-sphere, and is $J$-spherical if this sphere has radius 1 and $\alpha = 2 < \beta$. Let $\dim_E(G)$, $\dim_S(G)$ and $\dim_J(G)$ denote, respectively, the smallest dimension $r$ for which $G$ admits a Euclidean, spherical and $J$-spherical representation.

In this paper, we extend and simplify the results of Roy [18] and Nozaki and Shinohara [17] by deriving exact simple formulas for $\dim_E(G)$ and $\dim_S(G)$ in terms of the eigenvalues of $V^TAV$, where $A$ is the adjacency matrix of $G$ and $V$ is the matrix whose columns form an orthonormal basis for the orthogonal complement of the vector of all 1’s.

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We also extend and simplify the results of Musin [16] by deriving explicit formulas for determining the $J$-spherical representation of $G$ and for determining $\dim_J(G)$ in terms of the largest eigenvalue of $\bar{A}$, the adjacency matrix of the complement graph $\bar{G}$. As a by-product, we obtain several other related results and in particular we answer a question raised by Musin in [16].

1 Introduction

Let $G$ be a simple graph, i.e., no loops and no multiple edges, on $n$ nodes. A Euclidean representation of $G$ in $\mathbb{R}^r$, the $r$-dimensional Euclidean space, is an $n$-point configuration $p^1, \ldots, p^n$ in $\mathbb{R}^r$ such that: for all $i, j = 1, \ldots, n$, we have

$$||p^i - p^j||^2 = \begin{cases} \alpha & \text{if } \{i, j\} \in E(G), \\ \beta & \text{if } \{i, j\} \notin E(G), \end{cases}$$

for two distinct positive scalars $\alpha$ and $\beta$, where $||x||^2 = x^T x$ and $E(G)$ is the set of edges of $G$. In other words, the points $p^1, \ldots, p^n$ form a two-distance set. The Euclidean representation number [18] of $G$, denoted by $\dim_E(G)$, is the smallest $r$ for which $G$ admits a Euclidean representation in $\mathbb{R}^r$. A Euclidean representation of $G$ in $\mathbb{R}^r$ is said to be a spherical representation of $G$ in $\mathbb{R}^r$ if the points $p^1, \ldots, p^n$ lie on an $(r-1)$-sphere in $\mathbb{R}^r$. Moreover, the spherical representation number of $G$, denoted by $\dim_S(G)$, is the smallest $r$ for which $G$ admits a spherical representation in $\mathbb{R}^r$. In the special case of a spherical representation of $G$, where the sphere has unit radius and $\alpha = 2 < \beta$, the spherical representation is said to be a $J$-spherical representation [16]. In the same manner, the $J$-spherical representation number of $G$, denoted by $\dim_J(G)$, is the smallest $r$ for which $G$ admits a $J$-spherical representation in $\mathbb{R}^r$. Evidently

$$\dim_E(G) \leq \dim_S(G) \leq \dim_J(G).$$

Einhorn and Schoenberg [8] gave exact formulas for $\dim_E(G)$ in terms of the multiplicities of certain roots of the discriminating polynomial defined in [13]. They [9] also determined all two-distance sets in dimensions two and three. A full classification of all maximal two-distance sets in dimension $r$ for all $r \leq 7$ is given in [15]. Recently, there has been a renewed interest in the problems of determining $\dim_E(G)$, $\dim_S(G)$ and $\dim_J(G)$ [18, 17, 16]. Roy [18] derived bounds on $\dim_E(G)$ using the multiplicities of the smallest and the
second smallest distinct eigenvalues of $A$, the adjacency matrix of $G$. He also gave exact formulas for $\dim_E(G)$ using the main angles of the graph. Nozaki and Shinohara [17] considered the problem of determining $\dim_S(G)$ and, using Roy’s results, they obtained necessary and sufficient conditions for a Euclidean representation of $G$ to be spherical. Musin [16] considered the problem of determining $\dim_J(G)$ and proved that any graph which is neither complete nor null admits a unique, up to an isometry, $J$-spherical representation. He also obtained exact formulas for $\dim_S(G)$ and $\dim_J(G)$ in terms of the multiplicities of the roots of a polynomial defined by the Cayley-Menger determinant. Finally, we should point out that a classification of all two-distance sets in dimension four is given in [20].

In this paper, we extend and simplify the results of Roy [18] and Nozaki and Shinohara [17] by deriving exact simple formulas for $\dim_E(G)$ and $\dim_S(G)$ in terms of the multiplicities of the smallest and the largest eigenvalues of the $(n - 1) \times (n - 1)$ matrix $V^TAV$, where $A$ is the adjacency matrix of $G$ and $V$, defined in (3), is the matrix whose columns form an orthonormal basis for the orthogonal complement of the vector of all 1’s. This is made possible by using projected Gram matrices for representing $n$-point configurations. As a by-product, we obtain a characterization of $(0 - 1)$ Euclidean distance matrices (Theorem 3.1).

We also extend and simplify the results of Musin [16] by deriving explicit formulas for determining the $J$-spherical representation of $G$ and for determining $\dim_J(G)$ in terms of the largest eigenvalue of $\bar{A}$, the adjacency matrix of the complement graph $\bar{G}$, and its multiplicity. This is made possible by the extensive use of the theory of Euclidean distance matrices. We also answer a question raised by Musin in [16].

The remainder of this paper is organized as follows. Section 2 presents the background we need from Euclidean matrices (EDMs), spherical EDMs, projected Gram matrices and Gale transform. In Section 3, we present some of the spectral properties of the matrix $V^TAV$ since this matrix plays a key role in determining $\dim_E(G)$ and $\dim_S(G)$. Sections 4, 5 and 6 discuss, respectively, Euclidean, spherical and $J$-spherical representation of graph $G$.

### 1.1 Notation

We collect here the notation used throughout the paper. $e_n$ and $E_n$ denote, respectively, the vector of all 1’s in $\mathbb{R}^n$ and the $n \times n$ matrix of all 1’s. The
subscript is omitted if the dimension is clear from the context. The identity matrix of order \( n \) is denoted by \( I_n \). The zero matrix or zero vector of appropriate dimension is denoted by \( 0 \). For a matrix \( A \), \( \text{diag}(A) \) denotes the vector consisting of the diagonal entries of \( A \). \( m(\lambda) \) denotes the multiplicity of eigenvalue \( \lambda \).

\( K_n \) denotes the complete graph on \( n \) nodes. The adjacency matrix of a graph \( G \) is denoted by \( A \), and the adjacency matrix of the complement graph \( \bar{G} \) is denoted by \( \bar{A} \). \( \mu_{\text{min}} \) and \( \mu_{\text{max}} \) denote, respectively, the minimum and the maximum eigenvalues of \( V^T AV \). Likewise, \( \bar{\mu}_{\text{min}} \) and \( \bar{\mu}_{\text{max}} \) denote, respectively, the minimum and the maximum eigenvalues of \( V^T \bar{A}V \). Finally, PSD and PD stand for positive semidefinite and positive definite.

2 Preliminaries

The theory of Euclidean distance matrices (EDMs) provides a natural and powerful tool for determining \( \dim_E(G) \), \( \dim_S(G) \) and \( \dim_J(G) \). In this section, we present the necessary background concerning EDMs, spherical EDMs, projected Gram matrices and Gale matrices. For a comprehensive treatment of these topics and EDMs in general, see the monograph [1].

2.1 EDMs

An \( n \times n \) matrix \( D = (d_{ij}) \) is said to be an EDM if there exist points \( p^1, \ldots, p^n \) in some Euclidean space such that

\[
d_{ij} = ||p^i - p^j||^2 \quad \text{for all} \quad i, j = 1, \ldots, n,
\]

\( p^1, \ldots, p^n \) are called the generating points of \( D \) and the dimension of their affine span is called the embedding dimension of \( D \). Let \( D \) be an EDM of embedding dimension \( r \). We always assume throughout this paper that the generating points of \( D \) are in \( \mathbb{R}^r \). Hence, the \( n \times r \) matrix

\[
P = \begin{bmatrix} (p^1)^T \\ \vdots \\ (p^n)^T \end{bmatrix}
\]

has full column rank. \( P \) is called a configuration matrix of \( D \).
Let \( e \) be the vector of all 1’s in \( \mathbb{R}^n \) and let \( E = ee^T \). The following theorem is a well-known characterization of EDMs \[19\ \[23\ \[12\ \[6\].

**Theorem 2.1.** Let \( D \) be an \( n \times n \) real symmetric matrix whose diagonal entries are all 0’s and let \( s \in \mathbb{R}^n \) such that \( e^T s = 1 \). Then \( D \) is an EDM if and only if \( D \) is negative semidefinite on \( e^\perp \), the orthogonal complement of \( e \) in \( \mathbb{R}^n \); i.e., iff
\[
B = -\frac{1}{2}(I - es^T)D(I - se^T)
\]
is positive semidefinite (PSD), in which case, the embedding dimension of \( D \) is given by rank \( (B) \).

Note that \( B \), which can be factorized as \( B = PP^T \), is the Gram matrix of the generating points of \( D \), or the Gram matrix of \( D \) for short. Moreover, \( Bs = 0 \) and hence \( P^T s = 0 \). It is well known \[12\] that if \( D \) is a nonzero EDM, then \( e \) lies in the column space of \( D \). Hence, there exists \( w \) such that
\[
Dw = e.
\] (2)

Two choices of vector \( s \) in (1) are of particular interest to us. First, \( s = e/n \). This choice fixes the origin at the centroid of the generating points of \( D \) and thus the corresponding Gram matrix satisfies \( Be = 0 \). Second, \( s = 2w \), where \( w \) is as defined in (2). This choice, as we will see in Section 6, is particularly useful when the radius of a spherical EDM is known.

Assume that \( B \), the Gram matrix of \( D \), satisfies \( Be = 0 \). Let \( V \) be the \( n \times (n - 1) \) matrix whose columns form an orthonormal basis of \( e^\perp \); i.e., \( V \) satisfies
\[
V^T e = 0 \text{ and } V^T V = I_{n-1}.
\] (3)

Hence, \( VV^T = I_n - E/n \) is the orthogonal projection on \( e^\perp \). Thus, \( -2B = VV^T DVV^T \). Let
\[
X = V^T BV = -\frac{1}{2}V^T DV,
\] (4)
and thus, \( B = VXV^T \). Moreover, it readily follows that \( B \) is PSD of rank \( r \) iff \( X \) is PSD of rank \( r \). As a result, \( X \) is called the projected Gram matrix of \( D \). Consequently, a real symmetric matrix with \( \text{diag} (D) = 0 \) is an EDM of embedding dimension \( r \) if and only if its projected Gram matrix \( X \) is PSD of rank \( r \).
It should be pointed out that $V$ as defined in (3) is not unique. One such choice of $V$ is

$$V = \begin{bmatrix} y e_{n-1}^T \\ I_{n-1} + x E_{n-1} \end{bmatrix},$$

where $y = \frac{-1}{\sqrt{n}}$ and $x = \frac{-1}{n + \sqrt{n}}$.

Another choice of $V$, which we use in the sequel and is particularly convenient when dealing with block matrices, is

$$V = \begin{bmatrix} V'_{3} & 0 & ae_{3} \\ 0 & V'_{n-3} & be_{n-3} \end{bmatrix}.$$  \hfill (5)

Here, $V'_{3}$ and $V'_{n-3}$ are, respectively, $3 \times 2$ and $(n - 3) \times (n - 4)$ matrices satisfying (3),

$$a = \left( \frac{n - 3}{3n} \right)^{1/2}, \quad b = -\left( \frac{3}{n(n - 3)} \right)^{1/2}$$

and

$$V'_{3} = \begin{bmatrix} y & y \\ 1 + x & x \\ x & 1 + x \end{bmatrix}, \quad y = \frac{-1}{\sqrt{3}} \text{ and } x = \frac{-1}{3 + \sqrt{3}}.$$ 

Gale transform \cite{10} \cite{13}, or Gale matrix, plays an important role in theory of EDMs. Let $Z$ be the $n \times (n - r - 1)$ matrix whose columns form a basis of the null space of

$$\begin{bmatrix} P^T \\ e^T \end{bmatrix},$$

where $P$ is a configuration matrix of $D$. Then $Z$ is called a Gale matrix of $D$. The following lemma establishes the relationship between Gale matrix $Z$ and the null space of the projected Gram matrix $X$.

**Lemma 2.1** (\cite{2}). Let $D$ be an $n \times n$ EDM of embedding dimension $r \leq n - 2$ and let $X$ be the projected Gram matrix of $D$. Further, let $U$ be the matrix whose columns form an orthonormal basis of the null space of $X$. Then $VU$ is a Gale matrix of $D$, where $V$ is as defined in (3).
2.2 Spherical EDMs

An EDM $D$ is said to be spherical if its generating points lie on a sphere. We denote the radius of the generating points of a spherical EDM $D$ by $\rho$ and we will refer to it as the radius of $D$. Among the many different characterizations of spherical EDMs, the ones that are relevant to this paper are given in the following theorem.

**Theorem 2.2.** Let $D$ be an $n \times n$ EDM of embedding dimension $r$ and let $Dw = e$. Let $P$ and $Z$ be, respectively, a configuration matrix and a Gale matrix of $D$ and assume that $P^T e = 0$. If $r = n - 1$, then $D$ is spherical. Otherwise, if $r \leq n - 2$, then the following statements are equivalent:

1. $D$ is spherical,
2. $DZ = 0$.
3. $\text{rank}(D) = r + 1$.
4. There exists $a \in \mathbb{R}^r$ such that
   \[ Pa = \frac{1}{2}(I - \frac{E}{n})\text{diag}(PP^T) \]
   in which case, the generating points of $D$ lie on a sphere centered at $a$ and with radius
   \[ \rho = \left( a^T a + \frac{e^T De}{2n^2} \right)^{1/2}. \]
5. $e^T w > 0$, in which case, the radius of $D$ is given by
   \[ \rho = \left( \frac{1}{2e^T w} \right)^{1/2}. \]

The equivalence between Statement 1 and 2 was proven by Alfakih and Wolkowicz [3]. The equivalence between Statement 1 and 3 was proven by Gower [12]. The equivalence between Statement 1 and 4 was proven by Tarazaga et al [21]. Finally, the equivalence between Statement 1 and 5 was proven by Gower [11, 12].

An interesting subclass of spherical EDMs is that of regular EDMs. A spherical EDM $D$ is said to be regular if the center of the sphere containing
the generating points of \( D \) coincides with the centroid of these points. Regular EDMs are characterized \[14\] as those EDMs which have \( e \) as an eigenvector. It is easy to see that an \( n \times n \) regular EDM \( D \) has radius

\[
\rho = \left( \frac{e^T D e}{2n^2} \right)^{1/2}.
\]

3 Spectral Properties of \( V^T AV \)

Cluster graphs and complete multipartite graphs play a special role in this paper. Graph \( G \) is said to be a cluster graph if it is the disjoint union of complete graphs. Note that \( K_1 \), the graph consisting of a single isolated node, is considered complete. The complement of a cluster graph \( G \) is called a complete multipartite graph. Thus, the vertices of a complete multipartite graph can be partitioned into independent sets. We often denote a complete multipartite graph by \( K_{n_1, \ldots, n_s} \), where \( n_1, \ldots, n_s \) are the sizes of its independent sets. Let \( P_3 \), or \( K_{1,2} \), denote the graph consisting of a path on 3 nodes. Then it is well known that \( G \) is a cluster graph if and only if it is \( P_3 \)-free, i.e., it has no \( P_3 \) as an induced subgraph. As a result, \( G \) is a complete multipartite graph if and only if it is \( \overline{P_3} \)-free. It should be pointed out that \( K_n \) is both a cluster graph and a complete multipartite graph, and thus the null graph, \( \overline{K_n} \), is also both a cluster graph and a complete multipartite graph.

Let \( A \) denote the adjacency matrix of \( G \) and let \( \lambda_{\min} \) and \( \lambda_{\max} \) denote, respectively, the minimum and the maximum eigenvalues of \( V^T AV \). Note that if \( A \neq 0 \), then \( \lambda_{\min} < 0 \) since \( \text{trace}(V^T AV) = -e^T Ae/n \).

**Proposition 3.1.** Let \( \lambda_n(A) \) and \( \lambda_{n-1}(A) \) denote the smallest and the second smallest eigenvalues of \( A \). Also, let \( \lambda_1(A) \) and \( \lambda_2(A) \) denote the largest and the second largest eigenvalues of \( A \). Then

\[
\lambda_1 \geq \lambda_{\max} \geq \lambda_2 \quad \text{and} \quad \lambda_{n-1} \geq \lambda_{\min} \geq \lambda_n(A).
\]

**Proof.** Let \( Q = [e/\sqrt{n} \ V] \). Then

\[
Q^T AQ = \begin{bmatrix}
\frac{e^T Ae}{n} & \frac{e^T AV}{\sqrt{n}} \\
\frac{V^T Ae}{\sqrt{n}} & V^T AV
\end{bmatrix}.
\]

The result follows from the interlacing theorem since \( Q \) is orthogonal. \( \square \)
The following proposition is an immediate consequence of the proof of Proposition 3.1.

**Proposition 3.2.** Let $A$ denote the adjacency matrix of a $k$-regular graph. Then the eigenvalues of $A$ are exactly those of $V^T AV$ in addition to the eigenvalue $k$.

**Lemma 3.1.** Let $A$ denote the adjacency matrix of graph $G$. Then

1. $\mu_{\text{max}} = -1$ if and only if $A = E - I$.

2. $\mu_{\text{max}} = 0$ if and only if $A$ is an EDM and $A \neq E - I$.

**Proof.** Clearly, $\mu_{\text{max}} \leq 0$ iff $(-V^T AV)$ is PSD iff $A$ is an EDM. Now if $A = E - I$, then obviously $\mu_{\text{max}} = -1$. On the other hand, if $\mu_{\text{max}} = -1$, then $(-V^T AV)$ is positive definite (PD) and thus $A$ is an EDM of embedding dimension $n - 1$. Assume, by way of contradiction, that $A \neq E - I$. Then at least one off-diagonal entry of $A$ is zero, and thus at least two of the generating points of $A$ coincide. Accordingly, the embedding dimension of $A$ is $\leq n - 2$, a contradiction. Therefore, $A = E - I$. Also, we conclude that if $A$ is an EDM and $A \neq E - I$, then the embedding dimension of $A$ is $\leq n - 2$, i.e., $\text{rank}(V^T AV) \leq n - 2$ and hence $\mu_{\text{max}} = 0$ since $(-V^T AV)$ is PSD.

$\blacksquare$

The following theorem is a characterization of $(0 - 1)$ EDMs.

**Theorem 3.1.** Let $A$ denote the adjacency matrix of graph $G$. Then $A$ is an EDM if and only if $G$ is a complete multipartite graph.

**Proof.** Assume that $G$ is a complete multipartite graph and assume that the nodes of $G$ are partitioned into $s$ independent sets. Then obviously $A$ is an EDM whose generating points have the property that $p^i = p^j$ if and only if nodes $i$ and $j$ belong to the same independent set.

To prove the other direction, assume that $G$ is not a complete multipartite graph. Then $G$ has $P_3$ as an induced subgraph. Wlog assume that the nodes of $P_3$ are 1, 2, and 3. Therefore, the third leading principal submatrix of $A$ is

$$
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
$$
Let $V$ be as defined in (5). Then the second leading principal submatrix of $V^TAV$ is
\[
V_3' = \frac{1}{3(1 + \sqrt{3})} \begin{bmatrix} 2 & -(1 + \sqrt{3}) \\ -(1 + \sqrt{3}) & -4 - 2\sqrt{3} \end{bmatrix},
\]
which has eigenvalues $-1$ and $1/3$. Therefore, it follows from the interlacing theorem that $\mu_{\max} \geq 1/3$ and thus $(-V^TAV)$ is not PSD. Consequently, $A$ is not an EDM.

\[\square\]

A remark is in order here. Let $G$ be a complete multipartite graph and assume that its nodes are partitioned into $s$ independent sets. Then the adjacency matrix of $G$ is an EDM embedding dimension $s - 1$. For example, the nodes of $K_n$ and $\overline{K}_n$ are, obviously, partitioned into $n$ and 1 independent sets. Consequently, the embedding dimensions of the corresponding adjacency matrices, i.e., $E - I$ and $0$, are respectively $n - 1$ and $0$ as expected.

**Theorem 3.2.** Let $G$ be a graph on $n$ nodes which is not null and let $A$ denote its adjacency matrix. Then $\mu_{\min} \leq -1$. Moreover, $\mu_{\min} = -1$ if and only if $G$ is a cluster graph.

**Proof.** Assume that $G$ is a cluster graph. If $G$ is the disjoint union of $K_2$ and $(n - 2)$ isolated nodes, then by the proof of Theorem 3.1 we have $\mu_{\min} = -1$. Otherwise, it follows from Proposition 3.1 that $\mu_{\min} = -1$ since $(-1)$ is an eigenvalue of $A$ of multiplicity at least 2.

To prove the other direction, assume that $G$ is not a cluster graph. Then $G$ has $P_3$ as an induced subgraph. Therefore, wlog, assume that the nodes of $P_3$ are 1, 2, and 3. Therefore, the third leading principal submatrix of $A$ is
\[
\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Let $V$ be as defined in (5). Then the second leading principal submatrix of $V^TAV$ is
\[
V_3' = \frac{-2}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]
Therefore, it follows from the interlacing theorem that $\mu_{\min} \leq -4/3$.

Let $G$ be a cluster graph and assume that $G \neq K_n$ and $G \neq \overline{K_n}$. Then $G$ is the disjoint union of at least two complete graphs, say $K_{n_1}$ and $K_{n_2}$, where $n_1 \geq 2$. Thus $G$ has an induced $P_3$ whose nodes are two from $K_{n_1}$ and one from $K_{n_2}$. Consequently, $G$ is not a complete multipartite graph. On the other hand, if $G = K_n$, then obviously $\mu_{\max} = \mu_{\min} = -1$ and if $G = \overline{K_n}$, then trivially $\mu_{\max} = \mu_{\min} = 0$. Hence, we have proven the following corollary.

**Corollary 3.1.** There exists no graph $G$ such that $\mu_{\max} = 0$ and $\mu_{\min} = -1$.

### 4 Euclidean Representations

Let $G$ be a simple graph on $n$ nodes which is neither complete nor null. Then $G$ admits a Euclidean representation in $\mathbb{R}^r$ iff there exist two distinct positive scalars $\alpha$ and $\beta$ such that $D = \alpha A + \beta \bar{A}$ is an EDM of embedding dimension $r$.

Wlog assume that $\alpha = 1$ and thus $0 < \beta \neq 1$. Hence, $D = \beta (E - I) + (1 - \beta)A$.

Next, we derive upper and lower bounds on $\beta$ such that $D$ is an EDM. To this end, $X$, the projected Gram matrix of $D$, is given by

$$2X = \beta I_{n-1} + (\beta - 1) V^T A V.$$  \hspace{1cm} (7)

Hence, by Theorem 2.1, $D$ is an EDM of dimension $r$ iff $X$ is PSD of rank $r$.

Recall that $\mu_{\min}$ and $\mu_{\max}$ denote, respectively, the minimum and the maximum eigenvalues of $V^T A V$. Assume that $\beta > 1$. Then, in light of Theorem 3.2, $X$ is PSD iff

$$1 < \beta < +\infty \quad \text{if } \mu_{\min} = -1,$$

$$1 < \beta \leq \frac{|\mu_{\min}|}{|\mu_{\min}| - 1} \quad \text{if } \mu_{\min} < -1.$$  

On the other hand, assume that $0 < \beta < 1$. Then, in light of Lemma 3.1 and since $G \neq K_n$, $X$ is PSD iff

$$0 < \beta < 1 \quad \text{if } \mu_{\max} = 0,$$

$$\frac{\mu_{\max}}{\mu_{\max} + 1} \leq \beta < 1 \quad \text{if } \mu_{\max} > 0.$$  

Let us define

$$\beta_l = \frac{\mu_{\max}}{\mu_{\max} + 1} \quad \text{and } \beta_u = \frac{|\mu_{\min}|}{|\mu_{\min}| - 1}. \hspace{1cm} (8)$$
Therefore, \( X \) is PSD iff
\[
\begin{align*}
\beta &\in [\beta_l, 1) \cup (1, +\infty) & \text{if } \mu_{\min} = -1 \text{ and } \mu_{\max} > 0, \\
\beta &\in (0, 1) \cup (1, \beta_u] & \text{if } \mu_{\min} < -1 \text{ and } \mu_{\max} = 0, \\
\beta &\in [\beta_l, 1) \cup (1, \beta_u] & \text{if } \mu_{\min} < -1 \text{ and } \mu_{\max} > 0.
\end{align*}
\]

(9)

Note that Corollary 3.1 rules out the case in which \( \mu_{\min} = -1 \) and \( \mu_{\max} = 0 \). Therefore, for \( G \neq K_n \) and \( G \neq \overline{K}_n \), we have
\[
\begin{align*}
\beta &\in [\beta_l, 1) \cup (1, +\infty) & \text{if } G \text{ is a cluster graph}, \\
\beta &\in (0, 1) \cup (1, \beta_u] & \text{if } G \text{ is a complete multipartite graph}, \\
\beta &\in [\beta_l, 1) \cup (1, \beta_u] & \text{otherwise}.
\end{align*}
\]

(10)

Let \( m(\mu_{\min}) \) and \( m(\mu_{\max}) \) denote the multiplicities of \( \mu_{\min} \) and \( \mu_{\max} \). Then
\[
\text{rank } (X) = \begin{cases} 
n - 1 & \text{if } \beta \neq \beta_l \text{ and } \beta \neq \beta_u, \\
n - 1 - m(\mu_{\max}) & \text{if } \beta = \beta_l, \\
n - 1 - m(\mu_{\min}) & \text{if } \beta = \beta_u.
\end{cases}
\]

(11)

Therefore, if \( G \) is a cluster graph, then \( \dim_E(G) = n - 1 - m(\mu_{\max}) \); and if \( G \) is a complete multipartite graph, then \( \dim_E(G) = n - 1 - m(\mu_{\min}) \). Otherwise,
\[
\dim_E(G) = \min\{n - 1 - m(\mu_{\max}), n - 1 - m(\mu_{\min})\}.
\]

(12)

As a result, \( n - 2 \) is an upper bound on \( \dim_E(G) \) as proved in [22]. Using a different approach, Einhorn and Schoenberg [8, 9], obtained an equivalent equation for \( \dim_E(G) \). Next, we derive their equation and we show the equivalence between the two equations. To this end, let \( D(t) = A + t\bar{A} \) and let
\[
\tilde{X}(t) = -[\mathbf{-1} \quad I_{n-1}] D(t) \begin{bmatrix} -\mathbf{1}^T_{n-1} \\ I_{n-1} \end{bmatrix}.
\]

Now \( \begin{bmatrix} -\mathbf{1}^T_{n-1} \\ I_{n-1} \end{bmatrix} = V\Phi \) for some nonsingular \( \Phi \) since the columns of \( \begin{bmatrix} -\mathbf{1}^T_{n-1} \\ I_{n-1} \end{bmatrix} \) form a basis of \( e^\perp \). Therefore, \( \tilde{X}(t) = 2\Phi^T X(t)\Phi \), where \( X(t) \) is the projected Gram matrix of \( D(t) \). Thus, \( \tilde{X}(t) \) is PSD and of rank \( r \) iff \( X(t) \) is PSD and of rank \( r \). The *discriminating polynomial* [8] of \( D(t) \) is defined as
\[
p(t) = \det(\tilde{X}(t)).
\]

(13)
Note that $\tilde{X}(1) = I_{n-1} + E_{n-1}$ is PD and thus $p(1) > 0$. Also, note that $D(t)$ is an EDM iff $\tilde{X}(t)$ is PSD. Now if all roots of $p(t)$ are $< 1$, then $\tilde{X}(t)$ is PSD for all $t \geq 1$. Otherwise, $\tilde{X}(t)$ is PSD for all $t : 1 \leq t \leq t_2$, where $t_2$ is the smallest root of $p(t)$ such that $t_2 > 1$. On the other hand, let $t_1$ be the largest root of $p(t)$ such that $0 < t_1 < 1$ if such root exists. Thus, $\tilde{X}(t)$ is PSD for all $t_1 \leq t \leq 1$ if $t_1$ exists, and for all $0 < t \leq 1$ otherwise. As a result, Einhorn and Schoenberg obtained that, if both $t_1$ and $t_2$ exist, then

$$\dim_E(G) = \min\{n - 1 - m(t_1), n - 1 - m(t_2)\},$$

where $m(t_1)$ and $m(t_2)$ are the multiplicities of $t_1$ and $t_2$. To establish the equivalence between Equations (12) and (14), note that

$$p(t) = c \det(tI_{n-1} + (t - 1)V^TAV) = c(t - 1)^{n-1} \chi(t/(1 - t)), $$

where $c$ is a constant and $\chi(\mu)$ is the characteristic polynomial of $V^TAV$.

Hence, $t_1 = \frac{\mu_{\max}}{\mu_{\max} + 1}$ and $t_2 = \frac{|\mu_{\min}|}{|\mu_{\min}| - 1}$.

Let $G$ be a $k$-regular graph, which is neither complete nor null, and let $\lambda_1(A) = k \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ be the eigenvalues of its adjacency matrix. Then Proposition 3.2 implies that $\mu_{\min} = \lambda_n(A)$ and $\mu_{\max} = \lambda_2(A)$. Hence, as was proven in [18], if $G$ is neither a cluster graph nor a complete multipartite graph, then

$$\dim_E(G) = \min\{n - 1 - m(\lambda_2(A)), n - 1 - m(\lambda_n(A))\}. \quad (15)$$

**Example 4.1.** Let $G = C_5$, the cycle on 5 nodes. Then $\mu_{\max} = (\sqrt{5} - 1)/2$ with multiplicity 2 and $\mu_{\min} = -(\sqrt{5} + 1)/2$ with multiplicity 2. Hence, $\dim_E(G) = 2$, $\beta_u = (\sqrt{5} + 3)/2$ and $\beta_l = (-\sqrt{5} + 3)/2$. Observe that $\beta_u \beta_l = 1$ as expected since the regular pentagon is the unique two-distance representation of $C_5$ in $\mathbb{R}^2$.

Let $D = A + \beta \bar{A}$ be the EDM of a Euclidean representation of $G$. Obviously, every Euclidean representation of $G$ is at the same time a Euclidean representation of the complement graph $\bar{G}$. More precisely,

$$\bar{D} = \frac{D}{\beta} = \bar{A} + \frac{1}{\beta} A \quad (16)$$
is the EDM of a Euclidean representation of $\bar{G}$. Moreover, if we let $\mu_{\min}$ and $\mu_{\max}$ denote, respectively, the minimum and the maximum eigenvalues of $V^T \bar{A} V$. Then it is immediate that
\[ \bar{\mu}_{\min} = -1 - \mu_{\max} \quad \text{and} \quad \bar{\mu}_{\max} = -1 - \mu_{\min}, \tag{17} \]
and
\[ m(\bar{\mu}_{\min}) = m(\mu_{\max}) \quad \text{and} \quad m(\bar{\mu}_{\max}) = m(\mu_{\min}). \tag{18} \]
Consequently, $\dim_E(G) = \dim_E(\bar{G})$ as expected. We end this section by noting the following well-known lower bound on $\dim_E(G)$. It is well known [5, 4] that any two-distance $n$-point configuration in $\mathbb{R}^r$ satisfies
\[ n \leq \frac{(r + 1)(r + 2)}{2}. \]
Hence, for any graph $G$, which is neither complete nor null, we have
\[ \dim_E(G) \geq \frac{1}{2}\sqrt{8n + 1} - 3. \]

5 Spherical Representations

Graph $G$ admits a spherical representation in $\mathbb{R}^r$ iff there exist two distinct positive scalars $\alpha$ and $\beta$ such that $D = \alpha A + \beta \bar{A}$ is a spherical EDM of embedding dimension $r$. Wlog, assume that $\alpha = 1$ and hence $D = (1 - \beta)A + \beta (E - I)$. Assume that $D$ is an EDM and let $G$ be a $k$-regular graph, i.e., $Ae = ke$. Then, $De = ((1 - \beta)k + \beta(n - 1))e$. Consequently, $D$ is a regular EDM with radius
\[ \rho = \left( \frac{(1 - \beta)k + \beta(n - 1)}{2n} \right)^{1/2}. \]
Now for a general graph $G$, Theorem 2.2 and equation (11) imply that if $\beta \neq \beta_l$ and $\beta \neq \beta_u$, where $\beta_l$ and $\beta_u$ are as defined in (8), then $D$ is spherical of radius
\[ \rho = \left( \frac{1}{2e^T((1 - \beta)A + \beta (E - I))^{-1}e} \right)^{1/2}, \]
since, in this case, $w = D^{-1}e$. Otherwise, if $\beta = \beta_u$ or $\beta = \beta_l$, then $D$ may or may not be spherical. To characterize the sphericity of $D$ in this case, we need the following two definitions.
**Definition 5.1.** For adjacency matrix $A$, let $U_u$ be the $(n - 1) \times m(\mu_{\min})$ matrix whose columns form an orthonormal basis for the eigenspace of $V^TAV$ associated with $\mu_{\min}$. That is, the columns of $U_u$ are orthonormal eigenvectors of $V^TAV$ corresponding to $\mu_{\min}$.

**Definition 5.2.** For adjacency matrix $A$, let $U_l$ be the $(n - 1) \times m(\mu_{\max})$ matrix whose columns form an orthonormal basis for the eigenspace of $V^TAV$ associated with $\mu_{\max}$.

The following theorem establishes a necessary and sufficient condition for EDMs $D_u = A + \beta_u \bar{A}$ and $D_l = A + \beta_l \bar{A}$ to be spherical.

**Theorem 5.1.** Let $D_u = A + \beta_u \bar{A}$, where $\beta_u$ is as given in (8). Then the EDM $D_u$ is spherical if and only if

$$AVU_u = \mu_{\min}VU_u.$$ 

Similarly, let $D_l = A + \beta_l \bar{A}$. Then the EDM $D_l$ is spherical if and only if

$$AVU_l = \mu_{\max}VU_l.$$

**Proof.** We present the proof for $D_u$. The proof for $D_l$ is similar. Now it follows from (7) that the null space of $X_u$, the projected Gram matrix of $D_u$, is given by

$$\text{null}(X_u) = \left\{ \xi \in \mathbb{R}^{n-1} : V^TAV\xi = \frac{\beta_u}{1 - \beta_u} \xi = \mu_{\min}\xi \right\}.$$

Moreover, it also follows from (7) that

$$2X_uU_u = (1 + \mu_{\min})\beta_u - \mu_{\min})U_u = 0.$$ 

Hence, the columns of $U_u$ form an orthonormal basis of the null space of $X_u$ and thus, by Lemma 2.1, $VU_u$ is a Gale matrix of $D_u$. Therefore, by Theorem 2.2, $D_u$ is spherical iff $D_uVU_u = 0$; i.e., iff

$$AVU_u = \frac{\beta_u}{1 - \beta_u}VU_u = \mu_{\min}VU_u.$$ 

The following corollaries are immediate.

\[ \square \]
Corollary 5.1. Let $G$ be a graph on $n$ nodes, $G \neq K_n$ and $G \neq \overline{K}_n$. Let $Z_l = VU_l$. If $G$ is a cluster graph, i.e., if $\mu_{\min} = -1$, then
\[
\dim_S(G) = \begin{cases} 
  n - 1 - m(\mu_{\max}) & \text{if } AZ_l = \mu_{\max}Z_l, \\
  n - 1 & \text{otherwise}.
\end{cases}
\]

Corollary 5.2. Let $G$ be a graph on $n$ nodes, $G \neq K_n$ and $G \neq \overline{K}_n$. Let $Z_u = VU_u$. If $G$ is a complete multipartite graph, i.e., if $\mu_{\max} = 0$, then
\[
\dim_S(G) = \begin{cases} 
  n - 1 - m(\mu_{\min}) & \text{if } AZ_u = \mu_{\min}Z_u, \\
  n - 1 & \text{otherwise}.
\end{cases}
\]

We should point out that, in light of (18), $m(\mu_{\min})$ in Corollary 5.1 is equal to $m(\mu_{\max})$ in Corollary 5.2. Furthermore, it is easy to see that $Z_l$ in Corollary 5.1 is equal to $Z_u$ in Corollary 5.2, and that $AZ_l = \mu_{\max}Z_l$ in Corollary 5.1 iff $AZ_u = \mu_{\min}Z_u$ in Corollary 5.2.

Corollary 5.3. Let $G$ be a graph on $n$ nodes, $G \neq K_n$ and $G \neq \overline{K}_n$. Assume that $G$ is neither a cluster graph nor a complete multipartite graph, i.e., $\mu_{\min} \neq -1$ and $\mu_{\max} \neq 0$. Let $Z_l = VU_l$ and $Z_u = VU_u$. Then
\[
\dim_S(G) = \begin{cases} 
  n - 1 & \text{if } AZ_u \neq \mu_{\min}Z_u \text{ and } AZ_l \neq \mu_{\max}Z_l, \\
  n - 1 - m(\mu_{\min}) & \text{if } AZ_u = \mu_{\min}Z_u \text{ and } AZ_l \neq \mu_{\max}Z_l, \\
  n - 1 - m(\mu_{\max}) & \text{if } AZ_u \neq \mu_{\min}Z_u \text{ and } AZ_l = \mu_{\max}Z_l.
\end{cases}
\]

Otherwise, i.e., if $AZ_u = \mu_{\min}Z_u$ and $AZ_l = \mu_{\max}Z_l$. Then
\[
\dim_S(G) = \min\{n - 1 - m(\mu_{\min}), n - 1 - m(\mu_{\max})\}.
\]

Example 5.1. Let $G = C_5$ which was considered in Example 4.1, where $\mu_{\max} = (\sqrt{5} - 1)/2$ with multiplicity 2 and $\mu_{\min} = -(\sqrt{5} + 1)/2$ with multiplicity 2. Now since $G$ is 2-regular, $D = A + \beta \bar{A}$ is a spherical EDM for all $\beta_l \leq \beta \leq \beta_u$. On the other hand, two Gale matrices of $D_l = A + \beta_l \bar{A}$ and $D_u = A + \beta_u \bar{A}$ are

\[
Z_l = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \mu_{\max} \\ -\mu_{\max} & -\mu_{\max} \\ \mu_{\max} & -1 \end{bmatrix} \quad \text{and} \quad Z_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & \mu_{\min} \\ -\mu_{\min} & -\mu_{\min} \\ \mu_{\min} & -1 \end{bmatrix}.
\]

It is easy to verify that $AZ_l = \mu_{\max}Z_l$ and $AZ_u = \mu_{\min}Z_u$. Therefore, as expected, both EDMs $D_l$ and $D_u$ are spherical with radii (squared) $\rho^2_l = 2/(5 + \sqrt{5})$, and $\rho^2_u = 2/(5 - \sqrt{5})$. As a result, $\dim_S(G) = 2$. 

16
Example 5.2. Consider the “bow tie” graph depicted in Figure 1. Then, in this case, $\mu_{\min} = -1.4$, $\mu_{\max} = 1$ and $m(\mu_{\min}) = m(\mu_{\max}) = 1$. Thus, $\beta_l = 1/2$ and $\beta_u = 7/2$. Moreover,

$$Z_l = VU_l = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad Z_u = VU_u = \frac{1}{\sqrt{20}} \begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$  

It is easy to verify that $AZ_l = \mu_{\max} Z_l$ and $AZ_u \neq \mu_{\min} Z_u$. Therefore, the EDM $D_l = A + \beta_l \bar{A}$ is spherical of radius $\rho_l = 1/\sqrt{3}$, while the EDM $D_u = A + \beta_u \bar{A}$ is not spherical. As a result, $\dim S(G) = 3$.

Obviously, every spherical representation of $G$ is at the same time a spherical representation of the complement graph $\bar{G}$. More precisely,

$$\bar{D} = \frac{D}{\beta} = \bar{A} + \frac{1}{\beta} A$$

is the EDM of a spherical representation of $\bar{G}$. Therefore, it follows from (17) and (18) that $\dim S(G) = \dim S(\bar{G})$ as expected. Moreover, if we let $\bar{\rho}$ denote the radius of $\bar{D}$, then clearly $\bar{\rho}^2 = \rho^2 / \beta$.

Now suppose that the EDM $D_u = A + \beta_u \bar{A}$ is spherical. Then $\rho$, the radius of the sphere containing its generating points, can be given explicitly in terms of $A$. To this end, let $P$ be a configuration matrix of $D_u$ such that $P^T e = 0$ and let $B_u = PP^T$. By Theorem 2.2, $\rho^2 = a^T a + e^T D_u e / (2n^2)$, where $2Pa = (I - E/n)\text{diag}(B_u)$. Now it follows from (1), since $s = e/n$, that

$$\text{diag}(B_u) = \frac{D_u e}{n} - \frac{e^T D_u e}{2n^2} e.$$  

17
Therefore,

\[ 4a^T a = (\text{diag}(B_u))^T P (P^T P)^{-2} P^T \text{diag}(B_u) = \frac{e^T D u B_u^\dagger D u e}{n^2}, \]

where \( B_u^\dagger = P (P^T P)^{-2} P^T \) is the Moore-Penrose inverse of \( B_u \). Let \( V^T A V = \mu_{\min} U U^T + W U \Lambda_u W^T \) be the spectral decomposition of \( V^T A V \). Then the projected Gram matrix of \( D_u \) is given by

\[ X_u = \frac{1}{2(\mu_{\min} + 1)} W \mu_{\min} I - \Lambda_u W^T, \]

and thus

\[ X_u^\dagger = 2(\mu_{\min} + 1) W (\mu_{\min} I - \Lambda_u)^{-1} W^T. \]

Hence,

\[ a^T a = \frac{1}{2n^2(\mu_{\min} + 1)} e^T A V W_u (\mu_{\min} I - \Lambda_u)^{-1} W^T V^T A e. \]

As a result,

\[ \rho^2 = \frac{1}{2n^2(\mu_{\min} + 1)} (e^T A V W_u (\mu_{\min} I - \Lambda_u)^{-1} W^T V^T A e + \mu_{\min} (n^2 - n) + e^T A e). \]

We end this section by noting the following well-known lower bound on \( \dim_S(G) \). It is well known [7] that any two-distance \( n \) point spherical configuration in \( \mathbb{R}^r \) satisfies

\[ n \leq \frac{r(r + 3)}{2}. \]

Hence, for any graph \( G \) we have

\[ \dim_S(G) \geq \frac{1}{2}(\sqrt{8n + 9} - 3). \]

6 J-Spherical Representations

Musin [16] proved that every graph \( G \), which is not complete or null, admits a unique, up to an isometry, \( J \)-spherical representation. Unfortunately, his proof is not constructive. In this section, we give explicit simple formulas for the \( J \)-spherical representation of \( G \) and for \( \dim_J(G) \) in terms of the largest eigenvalue of \( \bar{A} \), the adjacency matrix of the complement graph \( \bar{G} \), and its multiplicity. We also answer a question raised in [16].
Evidently, $G$ admits a $J$-spherical representation in $\mathbb{R}^r$ iff there exists a scalar $\beta > 2$ such that $D = 2A + \beta \bar{A}$ is a spherical EDM of unit radius and of embedding dimension $r$. Let $D$ be a spherical EDM of unit radius. Then Theorem 2.2 implies that $2e^T w = 1$ where $Dw = e$. Consequently, we will find it convenient, in this section, to set $s = 2w$ in Theorem 2.1 i.e., we fix the origin such that $B$, the Gram matrix of $D$, satisfies $Bw = 0$. Therefore, $B$, in this case, is given by

$$B = E - \frac{1}{2}D$$

and satisfies $Bw = 0$. \hfill (19)

Let $\beta = 2 + 2\delta$, where $\delta > 0$. Then $D = 2A + \beta \bar{A} = 2(E - I) + 2\delta \bar{A}$ and thus

$$B = I - \delta \bar{A}$$

and satisfies $Bw = 0$. \hfill (20)

As a result, $G$ admits a $J$-spherical representation in $\mathbb{R}^r$ iff there exists $\delta > 0$ such that

$$B = I - \delta \bar{A}$$

is PSD, $Bw = 0$ and rank ($B$) = $r$. \hfill (21)

Now let $\lambda_1(\bar{A}) \geq \cdots \geq \lambda_n(\bar{A})$ be the eigenvalues of $\bar{A}$. Then $B$ is PSD iff

$$\delta \leq \frac{1}{\lambda_1(\bar{A})}.$$ 

On the other hand, $Bw = 0$ is equivalent to

$$\bar{A}w = \frac{1}{\delta}w.$$ \hfill (22)

Hence, $1/\delta$ is an eigenvalue of $\bar{A}$ and thus $1/\delta \leq \lambda_1(\bar{A})$. Consequently,

$$\delta = \lambda_1(\bar{A}).$$

As a result, we have proven the following theorem.

**Theorem 6.1.** Let $G$ be a graph on $n$ nodes, which is neither complete nor null, and let $\delta = 1/\lambda_1(\bar{A})$, where $\lambda_1(\bar{A})$ is the largest eigenvalue of $\bar{A}$, the adjacency matrix of the complement graph $\bar{G}$. Then $G$ admits a unique, up to an isometry, $J$-spherical representation whose EDM is given by

$$D = 2(E - I) + 2\delta \bar{A}.$$ 

Moreover, $\dim_J(G) = n - m(\lambda_1(\bar{A}))$. 

19
Following [16], let us refer to $\alpha = 2$ as the first distance (squared) and to $\beta = 2 + 2\delta$ as the second distance (squared). The following observation is worth pointing out. It follows from the interlacing theorem and (17) that $\lambda_1(\bar{A}) \geq \bar{\mu}_{\text{max}} = |\mu_{\text{min}}| - 1$. Hence, the second distance (squared) satisfies

$$\beta \leq 2 \frac{|\mu_{\text{min}}|}{|\mu_{\text{min}}| - 1} = 2\beta_u$$

as expected. Note that the factor of 2 results from the fact that $\alpha$, the first distance (squared), is 2 instead of 1 as was the case in previous sections.

**Example 6.1.** Consider the graph $G = C_5$. Then $\lambda_1(\bar{A}) = 2$ with multiplicity 1. Thus, as was observed in [16], $\dim_J(G) = 4$.

**Example 6.2.** Let $G$ be the “bow tie” graph depicted in Figure 7 and considered in Example 5.2. Then $\lambda_1(\bar{A}) = 2$ with multiplicity 1. Thus, $\delta = 1/2$ and $\dim_J(G) = 4$.

**Example 6.3.** Let $G = K_{n_1, \ldots, n_s}$ be a complete multipartite graph, where $n = n_1 + \cdots + n_s$ and

$$n_1 = \cdots = n_k > n_{k+1} \geq \cdots \geq n_s.$$  

Then Musin [16] proved that $\dim_J(G) = n - k$. Clearly, $\bar{G}$ in this case is a cluster graph and $\lambda_1(\bar{A}) = n_1 - 1$ with multiplicity $k$.

We conclude this paper by presenting a characterization of graphs whose $J$-spherical representations have the same second distance (squared) $\beta = 2 + 2\delta$. This characterization follows as an immediate corollary of Theorem 6.1 and answers a question raised by Musin [16].

**Theorem 6.2.** Let $G_1$ and $G_2$ be two distinct graphs, which are neither complete nor null, and let $\bar{A}_1$ and $\bar{A}_2$ be, respectively, the adjacency matrices of the complement graphs $\bar{G}_1$ and $\bar{G}_2$. Then the two $J$-spherical representations of $G_1$ and $G_2$ have the same second distance (squared) if and only if $\lambda_1(\bar{A}_1) = \lambda_1(\bar{A}_2)$.

We conclude this paper with the following two examples as an illustration of Theorem 6.2.

**Example 6.4.** Musin [16] gave the following cluster graphs $G_1 = 3K_2$, $G_2 = 2K_4$, $G_3 = K_2 \cup K_8$ and $G_4 = K_1 \cup K_{16}$ as an example of graphs whose $J$-spherical representations have the same second (squared) distance of $\beta = 5/2$. It is easy to verify that for all these graphs $\lambda_1(\bar{A}) = 4$ and thus $\beta = 5/2$. 

20
Example 6.5. Consider the graphs $G_n = \overline{C_n}$ for $n \geq 4$. It is immediate that for all these graphs $\lambda_1(\overline{A}) = 2$. Hence, the $J$-spherical representations of all these graphs have the same second distance (squared) of $\beta = 3$.

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