GOORENSTEIN QUOTIENT SINGULARITIES OF MONOMIAL TYPE IN DIMENSION THREE

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§1. Introduction
The purpose of this paper is to construct a crepant resolution of quotient singularities by finite subgroups of $SL(3, \mathbb{C})$ of monomial type (type (B),(C) and (D) in [5]), and prove that the Euler number of the resolution is equal to the number of conjugacy classes. This preprint contains the results which I talked at Research Institute for Mathematical Sciences of Kyoto University on 13th May, 1994.

The problem of finding a nice resolution of quotient singularities by finite subgroups of $SL(3, \mathbb{C})$ arose from mathematical physics.

Definition. (Orbifold Euler characteristic) In the physics of superstring theory, one considers the string propagation on a manifold $M$ which is a quotient by a finite subgroup of symmetries $G$. By a physical argument of string vacua of $M/G$, one concludes that the correct Euler number for the theory should be the “orbifold Euler characteristic”[2,3], defined by

$$\chi(M, G) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^{<g,h>)},$$

where the summation runs over all pairs of commuting elements of $G$, and $M^{<g,h)}$ denotes the common fixed set of $g$ and $h$. For the physicist’s interest, we only consider $M$ whose quotient space $M/G$ has trivial canonical

Conjecture I. ([2,3])
There exists a resolution of singularities $\widetilde{M}/G$ s.t. $\omega_{\widetilde{M}/G} \simeq \mathcal{O}_{\widetilde{M}/G}$, and

$$\chi(\widetilde{M}/G) = \chi(M, G).$$

This conjecture follows from its local form [4]:
Conjecture II. (local form)

Let $G \subset SL(3, \mathbb{C})$ be a finite group. Then there exists a resolution of singularities

$\sigma : \tilde{X} \rightarrow \mathbb{C}^3/G$ with $\omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$ and

$$\chi(\tilde{X}) = \sharp\{\text{conjugacy class of } G\}.$$ 

In algebraic geometry, the conjecture says that a minimal model of the quotient space by a finite subgroup of $SL(3, \mathbb{C})$ is non-singular.

Conjecture II was proved for abelian groups by Roan ([10]), and independently by Markushevich, Olshanetsky and Perelomov ([8]) by using toric method. It was also proved for 5 other groups, for which $X$ are hypersurfaces: (i) $WA_3^+, WB_3^+, WC_3^+$, where $WX^+$ denotes the positive determinant part of the Weyl group $WX$ of a root system $X$ by Bertin and Markushevich ([1]), (ii) $H_{168}$ by Markushevich ([7]), and (iii) $I_{60}$ by Roan ([11]). Recently I proved Conjecture II for trihedral groups [5]:

Definition. A trihedral group is a finite group $G = \langle H, T \rangle \subset SL(3, \mathbb{C})$, where $H \subset SL(3, \mathbb{C})$ is a finite group generated by diagonal matrices and

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Definition. Trihedral singularities are quotient singularities by trihedral groups.

Definition. A resolution of singularities $f : Y \rightarrow X$ of a normal variety $X$ with $K_X$ being $\mathbb{Q}$-Cartier is crepant if $K_Y = f^* K_X$.

Theorem 1.1[5].

Let $X = \mathbb{C}^3/G$ be a quotient space by a trihedral group $G$. Then there exists a crepant resolution of singularities

$$f : \tilde{X} \rightarrow X,$$

and

$$\chi(\tilde{X}) = \sharp\{\text{conjugacy class of } G\}.$$ 

Theorem 1.2 (Main Theorem).

The conjecture II holds for the following groups:

I. $G_1 = \langle H, S \rangle$

II. $G_2 = \langle H, H', S \rangle$

III. $G_3 = \langle H, S, T \rangle$ (r $\not\equiv 0$(mod 3))

IV. $G_4 = \langle G_3, C \rangle$

V. $G_5 = \langle C, S \rangle$
where \( H = \frac{1}{r}(0,1,-1), H' = \frac{1}{r}(1,-1,0), C = \frac{1}{3}(1,1,1) \) and
\[
S = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}.
\]

Remark 1.3. These singularities are different from trihedral singularities, but main idea for these proofs are based on the method of trihedral case. If we take any abelian group for \( H \) in Theorem 1.2, we can use the same method and the conjecture II is true.

In this paper, we prove Theorem 1.2.I in section 2, II in section 3, III in section 4 and IV and V in section 5.

§2. Proof of I

Proposition 2.1.
Let \( X = \mathbb{C}^3/G \), and \( Y = \mathbb{C}^3/G' \). Then there exists the following diagram:
\[
\begin{array}{ccccc}
\tilde{X} & \downarrow \tau \\
\tilde{Y} & \xrightarrow{\mu} & \tilde{Y}/\mathbb{Z}_2 & \downarrow \pi & \tilde{\pi} \\
\mathbb{C}^3 & \xrightarrow{\pi} & \mathbb{C}^3/G' = Y & \xrightarrow{\mu} & \mathbb{C}^3/G = X
\end{array}
\]

where \( \pi \) is a resolution of the singularity of \( Y \), and \( \tilde{\pi} \) is the induced morphism, \( \tau \) is a resolution of the singularity by \( \mathbb{Z}_2 \), and \( \tau \circ \tilde{\pi} \) is a crepant resolution of the singularity of \( X \).

Sketch of the proof. As a resolution \( \pi \) of \( Y \), we take a toric resolution, which is also crepant. Then we lift the \( \mathbb{Z}_2 \)-action on \( Y \) to its crepant resolution \( \tilde{Y} \) and form the quotient \( \tilde{Y}/\mathbb{Z}_2 \). This quotient gives in a natural way a partial resolution of the singularities of \( X \). The crepant resolution \( \tilde{X} \rightarrow \tilde{Y}/\mathbb{Z}_2 \) of the singularities of \( \tilde{Y}/\mathbb{Z}_2 \) induces a complete resolution of \( X \).

Under the action of \( \mathbb{Z}_2 \), the singularities of \( \tilde{Y}/\mathbb{Z}_2 \) lie in the union of the image of the exceptional divisor of \( \tilde{Y} \) under \( \tilde{Y} \rightarrow \tilde{Y}/\mathbb{Z}_2 \) and the locus \( C : (x = 0, y = -z) \) in \( \mathbb{C}^3 \).

In the resolution \( \tilde{Y} \) of \( Y \), the group \( \mathbb{Z}_2 \) permutes exceptional divisors. So the fixed points on the exceptional divisors consist of one point or three points.

Now, we see the proof more precisely.

At first, let \( G' \) be an abelian group which generated by diagonal matrices in \( G \). It is a normal subgroup of \( G \) in any case of Theorem 1.2.

Then we recall a toric resolution when \( G' \) is a finite abelian group in \( SL(3,\mathbb{C}) \).
Let $\mathbb{R}^3$ be a vector space, and \{\(e^i| i = 1, 2, 3\)\} the standard base. For all \(v = \frac{1}{r}(a, b, c) \in G'\), \(L = \text{the lattice generated by } e^1, e^2 \text{ and } e^3, N := L + \sum \mathbb{Z}v, \)

\[
\sigma = \left\{ \sum_{i=1}^{3} x^i e^i \in \mathbb{R}^3, \ x_i \geq 0, \forall i \right\}. 
\]

We regard $\sigma$ as a rational convex polyhedral cones in $N_{\mathbb{R}}$. The corresponding affine torus embedding $X_\sigma$ is defined as Spec$\mathbb{C}[\tilde{\sigma} \cap M]$, where $M$ is the dual lattice of $N$ and $\tilde{\sigma}$ is a dual cone of $\sigma$ in $M_{\mathbb{R}}$ defined by $\tilde{\sigma} = \{ \xi \in M_f \mathbb{R} | \xi(x) \geq 0, \forall x \in \sigma \}$. 

$\Delta := \text{the simplex in } N_{\mathbb{R}}$

\[
\left\{ \sum_{i=1}^{3} x^i e^i ; x_i \geq 0, \sum_{i=1}^{3} x_i = 1 \right\}
\]

$t : N_{\mathbb{R}} \rightarrow \mathbb{R} \quad \sum_{i=1}^{3} x^i e^i \longmapsto \sum_{i=1}^{3} x_i$

$\Phi := \left\{ \frac{1}{r}(a, b, c) \in G' \mid a + b + c = r \right\}$

**Lemma 2.2.**

$Y = \mathbb{C}^3/G'$ corresponds to $\sigma$ in $N = L + \sum_{v \in \Phi} \mathbb{Z}v$.

**Proof.** Since $Y = \text{Spec}(\mathbb{C}[x, y, z]^{G'}), x^iy^jz^k$ is $G'$-invariant if and only if $\alpha i + \beta j + \gamma k \in \mathbb{Z}$ for all $(\alpha, \beta, \gamma) \in G'$. $\Box$

**Remark 2.3.** Let $\frac{1}{r}(a, b, c) \neq (0, 0, 0)$ be an element of $G'$. There are two types.

1. $abc \neq 0$
2. $abc = 0$

We denote by $G_1$ (resp. $G_2$) the set of the elements of type (1) (resp. (2)). So $G'\{e\} = G_1 \amalg G_2$.

There are also two types in $\Phi$ as above. So we denote by $\Phi_1$ (resp. $\Phi_2$) the set of lattice points of type (1) (resp. (2)). Then $\Phi = \Phi_1 \amalg \Phi_2$.

Let $\lambda_i$ be maps from $G_i$ to $\Phi_i (i=1, 2)$.

$$\lambda_i : G_i \longrightarrow \Phi_i$$

where $\lambda_1$ maps $g = \frac{1}{r}(a, b, c) \ (a + b + c = r)$ and $g^{-1} = \frac{1}{r}(r - a, r - b, r - c)$ to a lattice point $\frac{1}{r}(a, b, c)$, and $\lambda_2$ maps $g = \frac{1}{r}(a, b, c)$ to a lattice point $\frac{1}{r}(a, b, c)$.

$$\{G_1\} \overset{2:1}{\longrightarrow} \{\Phi_1\}, \quad \{G_2\} \overset{1:1}{\longrightarrow} \{\Phi_2\}.$$

Therefore there exist a correspondence between the sets of elements of $G' - \{e\}$ and $\Phi$, which is 2:1 on $G_1$ and 1:1 on $G_2$. $\Phi$ corresponds to the exceptional divisors of a toric resolution given below.
Claim I. There exists a toric resolution of $Y$ where $\mathbb{Z}_2$ acts symmetrically on the exceptional divisors.

Proof. We can construct a unique simplicial decomposition.

Claim II. Let $X_S$ be the corresponding torus embedding, then $X_S$ is non-singular.

Proof. It is sufficient to show that the $\sigma(s)$ are basic. Let $w^1, w^2, w^3 \in \Phi \cup \{e^i\}_{i=1}^3$ which are linearly independent over $\mathbb{R}$. Assume that the simplex

$$\left\{ \sum_{i=1}^3 \alpha_i w^i | \alpha_i \geq 0, \sum_{i=1}^3 \alpha_i = 1 \right\}$$

intersects $\Phi \cup \{e^i\}_{i=1}^3$ only at $\{w^i\}_{i=1}^3$.

Lattice $N_0$ generated by $\{w^i\}_{i=1}^3$ is sublattice of $N$. If we assume $N \neq N_0$, then there exists $\beta = \beta_1 w^1 + \beta_2 w^2 + \beta_3 w^3 \in N \setminus N_0$ ($0 \leq \beta_i < 1, \beta_i \in \mathbb{R}$ and the strict inequality holds at least for one $i$.)

$$t(\beta) = \sum \beta_i t(w^i) = \sum \beta_i, 0 < \sum \beta_i < 3$$

and $t(N) \in \mathbb{Z}$, then $t(\beta) = 1$ or 2. If $t(\beta) = 2$, then we can replace it by $\beta' = \sum_{i=1}^3 (1 - \beta_i) w^i$, so we can assume that $t(\beta) = 1$.

Now, there exists an element $\beta$ in $\{\sum \alpha_i w^i | \alpha_i \geq 0, \sum \alpha_i = 1\} \cap (N - N_0)$, which is contained in $\Delta \cap N$. Since

$$N = \left\{ \bigcup_{v \in \Phi} (v \oplus L) \right\} \cup L,$$

$\Delta \cap N = \Phi \cup \{e^i\}_{i=1}^3$. From our assumption, we conclude that $\beta = w^i$ for some $i$, which contradicts $\beta \not\in N_0$. Therefore $N = N_0$. \hspace{1cm} \Box

We obtain a crepant resolution $\pi_S : X_S \to \mathbb{C}^3/G'$, because $X_S$ is non-singular.

Claim III. Let $F$ be the fixed locus on $\widetilde{Y}$ under the action of $\mathbb{Z}_2$, and $E$ be exceptional divisors of $\widetilde{Y} \to Y$. Then

$$F_0 := F \cap E = \begin{cases} 1 \text{ point} & : G' \text{ is type (I)} \\ 2 \text{ points} & : G' \text{ is type (II)} \end{cases}$$

Proof. Considering the dual graph of exceptional divisors by the toric resolution and from Remark 2.3, we can identify the two exceptional divisors by the action of $\mathbb{Z}_2$ except the central component which is a component in the center of the exceptional locus. Then there are 2 possibilities of the central component;

Type (I): one point.

Type (II): a divisor which is isomorphic to $\mathbb{P}^1$. Let $(y : z)$ be a coordinate of $\mathbb{P}^1$, then the action of $\mathbb{Z}_2$ is

$$(y : z) \mapsto (-z : -y).$$

Then the number of the fixed points are two, whose coordinates are

$$(1 : 1), (1 : -1) \hspace{1cm} \Box$$
Furthermore the $\mathbb{Z}_2$-action in the neighbourhood of a fixed point is analytically isomorphic to some linear action.

Now, we consider the resolution of the singularity of $\mathbb{C}^3$ by the group $\mathbb{Z}_2$. $F_0$ consists of 1 or 2 points, and $F = F_0 \cup C'$ where $C'$ is a strict transform of the fixed locus $C$ in $Y$ under the action of $\mathbb{Z}_2$.

**Claim IV.** Let $Z = \mathbb{C}^3/\mathbb{Z}_2$, then $\chi(\tilde{Z}) = \chi(\mathbb{C}^3, \mathbb{Z}_2) = 2$.

**Proof.** There is a representation of $\mathbb{Z}_2$ in $SL(3, \mathbb{C})$:

$$S' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The quotient singularities by $S'$ are $A_1 \times \{x-axis\}$ which are not isolated and the exceptional divisor of the resolution is $\mathbb{P}^1$-bundle. □

**Claim V.** The resolution $\tau \circ \tilde{\pi}$ is a crepant resolution.

**Lemma 2.4.**

Let $X := \mathbb{C}^3 / < G', T >$, and $f : \tilde{X} \rightarrow X$ the crepant resolution as above. Then the Euler number of $\tilde{X}$ is given by

$$\chi(\tilde{X}) = \frac{1}{2}(|G'| - k) + 2k$$

where

$$k = \begin{cases} 1 & \text{if } |G'| \equiv 1 \pmod{2} \text{ (type (I))} \\ 2 & \text{if } |G'| \equiv 0 \pmod{2} \text{ (type (II))} \end{cases}$$

**Proof.** For an abelian group $G'$, we have a toric resolution

$$\pi : \tilde{Y} \rightarrow Y = \mathbb{C}^3/G',$$

and $\chi(\tilde{Y}) = |G'|$ ([8],[11]).

By the action of $\mathbb{Z}_2$, the number of fixed points in the exceptional divisor by $\sigma$ is equal to $k$, hence

$$\chi(\tilde{Y}/\mathbb{A}_2) = \frac{1}{2}(|G'| - k) + k.$$ 

By the resolution of the fixed loci, Euler characteristic of the each exceptional locus is 2. (Claim IV)

Therefore,

$$\chi(\tilde{X}) = \frac{1}{2}(|G'| - k) + 2k. \quad \square$$
Theorem 2.5.

\[ \chi(\tilde{X}) = \#\{\text{conjugacy class of } G\}. \]

Proof.

(1) Case (I) : \(|G'| = 2m + 1, (0 < m \in \mathbb{Z})\)

For a nontrivial element \(g \in G'\), there are two conjugate elements \(g\) and \(SgS\). There are \(m\) couples of this type. There are 2 other conjugacy classes \([e]\) and \([S]\). Therefore, there are \(m + 2\) conjugacy classes in \(G\).

Then

\[ \chi(\tilde{X}) = \frac{1}{2}(|G'| - 1) + 2 = m + 2 = \#\{\text{ conjugacy class of } G\}. \]

(2) Case (II) : \(|G'| = 2m, (0 < m \in \mathbb{Z})\)

There are 2 elements in the center of \(G'\): \(e, a = \frac{1}{2}(0, 1, 1)\). The other \(2m - 2\) elements in \(G'\) are divided into \(m - 1\) conjugacy classes as in (1). There are 4 other conjugacy classes \(e, a, [S], [aS]\). Therefore, there are \(m + 3\) conjugacy classes in \(G\).

Then

\[ \chi(\tilde{X}) = \frac{1}{2}(|G'| - 2) + 4 = m + 3 = \#\{\text{ conjugacy class of } G\}. \]

§3. Proof of II

In this case, we do similarly as I. So we check the toric resolution:

Proposition 3.1. There exists a toric resolution of \(Y\) where \(\mathbb{Z}_2\) acts symmetrically on the exceptional divisors.

Proof. We show that we can construct a simplicial decomposition \(\{\sigma(s)\}_{s \in S}\) of the simplex \(\Delta\) which is \(\mathbb{Z}_2\)-invariant and the set of vertices is exactly \(\Phi \cup \{e^i\}_{i=1}^3\).

Let us consider the distance \(d\) between \(\frac{1}{r}(a, b, c)\) and \(\frac{1}{2}(a, h, h)\) \((h = (b + c)/2)\) given by

\[ d \left( \frac{1}{r}(a, b, c) \right) = \left| \frac{b}{r} - \frac{1}{h} \right| + \left| \frac{c}{r} - \frac{1}{h} \right|. \]

(1) Find lattice points \(P_i = \frac{1}{r}(a, b, c)\) whose distance \(d\) is the minimum among the points for each \(a\) in \(\Phi\) in the domain \(D = \{1/2 \geq y\}\).

(2) Make a triangle whose vertices are \(P_i\) and \(P'_i = \frac{1}{r}(a, c, b)\) symmetrically.

(3) Decompose \(D\) whose vertices are \(P_i, P'_i, (1, 0, 0)\) and \((0, 0, 1)\), into simplexes using vertices in \(\Phi\). We call this decomposition \(S_1\).

(4) By the action of \(\mathbb{Z}_2\), we obtain \(S_2\) on the other triangle. Therefore we obtain a “symmetric” resolution. \(\square\)

Next, we see the singularities in \(\tilde{Y}/\mathbb{Z}_2\):
Proposition 3.2. Let $F$ be the fixed locus on $\tilde{Y}$ under the action of $\mathbb{Z}_2$, and $E$ be exceptional divisors of $\tilde{Y} \to Y$. Then

$$F_0 := F \cap E = \{r \text{ points }\}$$

Proof. Considering the dual graph of exceptional divisors by the toric resolution and from Remark 2.3, we can identify the two exceptional divisors by the action of $\mathbb{Z}_2$ except the central components which are near central part (from $(1,0,0)$ to $\frac{1}{2}(0,1,1)$). Then there are 2 possibilities of the central component locally;

Type (I): one point.

Type (II): a divisor which is isomorphic to $\mathbb{P}^1$. From analogue of Claim III in section 2, there are $n$ points in $F_0$.

Furthermore the $\mathbb{Z}_2$-action in the neighbourhood of a fixed point is analytically isomorphic to some linear action.

Lemma 3.3.

Let $X := \mathbb{C}^3 / < G', T >$, and $f : \tilde{X} \to X$ the crepant resolution as above. Then the Euler number of $\tilde{X}$ is given by

$$\chi(\tilde{X}) = \frac{1}{2}(|G'| - r) + 2r$$

Theorem 3.4.

$$\chi(\tilde{X}) = \#\{\text{conjgacy class of } G\}.$$ 

Proof.

$|G'| = r^2, (0 < m \in \mathbb{Z})$

There are $r$ elements of type $\frac{1}{r}(a_i, h_i, h_i) =: a_i$. So for other nontrivial element $g \in G'$, there are two conjugate elements $g$ and $SgS$. There are $\frac{r^2 - r}{2}$ couples of this type. There are $r$ other conjugacy classes $[S]$ and $[a_i S]$. Therefore, there are $\frac{r^2 - r}{2} + 2r$ conjugacy classes in $G$.

Then

$$\chi(\tilde{X}) = \#\{\text{conjgacy class of } G\}.$$ 

§4. Proof of III

In this section, we assume that $r \equiv 1$ or $2 \pmod{3}$.

Proposition 4.1.

Let $X = \mathbb{C}^3 / G$, and $Y = \mathbb{C}^3 / G'$. Then there exists the following diagram:

$$\begin{array}{ccc}
\tilde{X} & \to & \tilde{Y} / S_3 \\
\downarrow \tau & & \downarrow \pi \\
\tilde{Y} & \xrightarrow{\tilde{\mu}} & \tilde{Y} / S_3 \\
\downarrow \pi & & \downarrow \tilde{\pi} \\
\mathbb{C}^3 & \xrightarrow{\mu} & \mathbb{C}^3 / G' = Y \\
& & \xrightarrow{\mu} \mathbb{C}^3 / G = X
\end{array}$$
where \( \pi \) is a resolution of the singularity of \( Y \), and \( \tilde{\pi} \) is the induced morphism, \( \tau \) is a resolution of the singularity by \( S_3 \), and \( \tau \circ \tilde{\pi} \) is a crepant resolution of the singularity of \( X \).

We do similarly as in §2.3.

\textbf{Claim I.} There exists a toric resolution of \( Y \) where \( S_3 \) acts symmetrically on the exceptional divisors.

\textbf{Proof.} We construct a simplicial decomposition with conditions for trihedral case \cite{5} and Proposition 3.1 in §3.

\textbf{Claim II.} Let \( F \) be the fixed locus on \( \tilde{Y} \) under the action of \( S_3 \), and \( E \) be exceptional divisors of \( \tilde{Y} \rightarrow Y \). Then

\[
F_0 := F \cap E = \{ \text{3r-2 points} \},
\]

where one of them are by \( S_3 \), the others are by \( \mathbb{Z}_2 \).

\textbf{Proof.} Considering the dual graph of exceptional divisors by the toric resolution and from Remark 2.3, we can identify the two exceptional divisors by the action of \( \mathbb{Z}_2 \) except the central component which is a component in the center of the exceptional locus. And the central component is a point, which fixed by the action of \( S_3 \).

\textbf{Claim III.} Let \( Z = \mathbb{C}^3 / S_3 \), then \( \chi(\tilde{Z}) = \chi(\mathbb{C}^3, S_3) = 3 \).

\textbf{Proof.} There is an equivalent representation of \( T \) in \( SL(3, \mathbb{C}) \):

\[
T' = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}
\]

The quotient singularities by \( < S, T' > \) are same as the case of Theorem 1.2 II. \( \square \)

\textbf{Lemma 4.2.}

Let \( X := \mathbb{C}^3 / < G', T > \), and \( f : \tilde{X} \rightarrow X \) the crepant resolution as above. Then the Euler number of \( \tilde{X} \) is given by

\[
\chi(\tilde{X}) = \frac{|G'| - 3(r-1) - 1}{6} + 3 + 2(r - 1) \quad (|G'| = r^2)
\]

\textbf{Proof.}

For an abelian group \( G' \), we have a toric resolution

\[
\pi : \tilde{Y} \rightarrow Y = \mathbb{C}^3 / G',
\]

and \( \chi(\tilde{Y}) = |G'| \) \cite{8,11}.

By the action of \( S_3 \), the number of fixed points in the exceptional divisor by \( \sigma \) is equal to \( r \), hence

\[
\chi(\tilde{Y} / S_3) = \frac{|G'| - 3(r-1) - 1}{6} + r.
\]
By the resolution of the fixed loci, Eular characteristic of the exceptional locus is 2
or 3. (Claim IV of §2 and Claim III of §4)
Therefore,
\[ \chi(\tilde{X}) = \frac{|G'| - 3(r - 1) - 1}{6} + 2(r - 1) + 3. \] □

**Theorem 4.3.**
\[ \chi(\tilde{X}) = \# \{ \text{conjugacy class of } G \}. \]

**Proof.**
For 3(r-1) elements of type \( \frac{1}{3} (k_i, h_i, h_i) =: a_i \), there are three conjugate elements \( a_i, Ta_iT^{-1} \) and \( T^{-1}a_iT \). For other nontrivial elements, there are six conjugate elements \( a_i, Ta_iT^{-1}, T^{-1}a_iT, Sa_iS, STa_iT^{-1}S \) and \( ST^{-1}a_iTS \). There are \( r - 1 \) conjugacy classes of type \([a_iS]\). There are 3 conjugacy classes \([e],[S],[T]\). So the number of the conjugacy classes are:
\[ (r - 1) + \frac{1}{6} \{ r^2 - 3(r - 1) - 1 \} + (r - 1) + 3. \]
Then
\[ \chi(\tilde{X}) = \# \text{conjgacy class of } G \]. □

**§5. Proof of IV**

**Proposition 5.1.**
Let \( X = \mathbb{C}^3/G \), and \( Y = \mathbb{C}^3/G' \). Then there exists the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\tau} & \tilde{X} \\
\downarrow & & \downarrow \\
\tilde{Y} & \xrightarrow{\tilde{\mu}} & \tilde{Y}/\mathfrak{S}_3 \\
\downarrow & & \downarrow \\
\mathbb{C}^3 & \xrightarrow{\pi} & \mathbb{C}^3/G' = Y \\
\end{array}
\]

where \( \pi \) is a resolution of the singularity of \( Y \), and \( \tilde{\pi} \) is the induced morphism, \( \tau \) is a resolution of the singularity by \( \mathfrak{S}_3 \), and \( \tau \circ \tilde{\pi} \) is a crepant resolution of the singularity of \( X \).

**Claim I.** There exists a toric resolution of \( Y \) where \( \mathfrak{S}_3 \) acts symmetrically on the exceptional divisors.

**Proof.** We construct a toric resolution as in §4. □

In the center of the triangle whose vertices are \((1,0,0),(0,1,1)\) and \((0,0,1)\), there exist one \( \mathbb{P}^2 \) as an exceptional component. Then it is sufficient to show the case \( G' = \langle \frac{1}{3} (1,1,1) \rangle \).
Claim II. Let $F$ be the fixed locus on $\tilde{Y}$ under the action of $\mathfrak{S}_3$, and $E$ be exceptional divisors of $\tilde{Y} \to Y$. Then

$$F_0 := F \cap E = 3 \text{ points.}$$

**Proof.** Considering the dual graph of exceptional divisors by the toric resolution, there is a possibility of the central component; a divisor which is isomorphic to $\mathbb{P}^2$. Then the number of the fixed points under the action of $\mathfrak{S}_3$ is three as in §4. □

Furthermore the $\mathfrak{S}_3$-action in the neighbourhood of a fixed point is analytically isomorphic to some linear action.

**Lemma 5.2.**

Let $X := \mathbb{C}^3 / \langle G', T \rangle$, and $f : \tilde{X} \to X$ the crepant resolution as above. Then the Euler number of $\tilde{X}$ is given by

$$\chi(\tilde{X}) = 9$$

**Proof.** For an abelian group $G'$, we have a toric resolution

$$\pi : \tilde{Y} \to Y = \mathbb{C}^3 / G',$$

and $\chi(\tilde{Y}) = |G'| = 3$ ([8],[11]).

By the action of $\mathfrak{S}_3$, the number of fixed points in the exceptional divisor by $\sigma$ is three, hence

$$\chi(\tilde{Y}/\mathfrak{A}_3) = \frac{1}{6}(|G'| - 3) + 3 = 3.$$  

By the resolution of the fixed loci, Euler characteristic of the each exceptional locus is 3. (Claim III in §4)

Therefore,

$$\chi(\tilde{X}) = \frac{1}{6}(|G'| - 3) + 3 \times 3 = 9. \quad \Box$$

**Theorem 5.3.**

$$\chi(\tilde{X}) = \sharp\{\text{conjugacy class of } G\}.$$  

**Proof.** There are nine conjugacy classes in $G$:

identity, $\frac{1}{3}(1,1,1), \frac{1}{3}(2,2,2), S, \frac{1}{3}(1,1,1)S, \frac{1}{3}(2,2,2)S, T, \frac{1}{3}(1,1,1)S, \frac{1}{3}(2,2,2). \quad \Box$

Before we think about general case, we see the case of $V$ in Theorem 1.2.

**Theorem 5.4.**

For $G = G_5$, the conjecture II holds.

**Proof.** Similarly we construct a crepant resolution:
There is an exceptional divisor which is isomorphic to $\mathbb{P}^2$ in $\tilde{Y}$. And there exist three singularities in $\tilde{Y}/\mathbb{Z}_2$, they become three $\mathbb{P}^1$-bundles in $\tilde{X}$. Then the Euler number of $X$ is 6.

On the other hands, the number of the conjugacy classes in $G$ is 6: id, $C$, $C^2$, $S$, $CS$ and $C^2S$. □

In general, we get the result:

**Theorem 5.5.**

As the normal subgroup $G'$ of $G$, we take an abelian group generated by all of the diagonal matrices in $G$. Then

$$\chi(\tilde{X}) = \frac{1}{2}(r^2 - 3r + 2) + 6r + 3,$$

and it equals to the number of the conjugacy classes in $G$.

**Proof.**

$$\chi(\tilde{Y}) = 3r^2.$$

There are 3+9(r-1) fixed points on the exceptional divisors in $\tilde{Y}$, and they turn to the 3+3(r-1) singularities in $\tilde{Y}/\mathfrak{S}_3$.

$$\chi(\tilde{Y}/\mathfrak{S}_3) = \frac{1}{6}(3r^2 - 9(r - 1) - 3) + 3(r - 1) + 3$$

Then

$$\chi(\tilde{X}) = 16\{3r^2 - 9(r - 1) - 3\} + 2 \times 3(r - 1) + 3 \times 3.$$ 

And this number coincide with the number of conjugacy classes, because $G_4 = G_3 \amalg G_3C \amalg G_3C^2$. □

Therefore, Main theorem (Theorem 1.2) is proved!

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