Effective Theory for Parity Conserving $QED_3$

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Abstract

We consider a higher derivative effective theory for an Abelian gauge field in three dimensions, which represents the result of integrating out heavy matter fields interacting with a classical gauge field in a parity-conserving way. We retain terms containing up to two derivatives of $F_{\mu\nu}$, but make no assumption about the strength of this field. We then quantize the gauge field,

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and compute the one-loop effective action for a constant \( F_{\mu\nu} \). The result is explicitly evaluated for the case of a constant magnetic field.

\[ I. \text{ INTRODUCTION} \]

\( QED_3 \)-like theories, namely those consisting of a charged matter field in interaction with an Abelian gauge field in 2+1 dimensions, have been the subject of intense research in recent years, because of their multiple applications to both Condensed Matter and High Energy Physics \[ \text{[1]} \]. The specific form of the action chosen for matter and gauge fields depends, of course, on the particular system one wants to represent. Although gauge invariance is almost exclusively assumed, both parity-violating \[ \text{[2]} \] and parity-conserving \[ \text{[3]} \] actions have been considered, depending on the realm of application. Given the large number of models one can construct within each of these categories, the task of identifying their common dynamical properties would be quite difficult if accomplished on a case by case basis.

The aim of this paper is to construct a unified description for a general gauge-invariant and parity-conserving \( QED_3 \)-like model, assuming massive matter fields. The combination of all the different models under a common description is made possible by using the ‘coarse graining’ provided by a low-momentum approximation. Our approach consists of constructing a ‘classical’ effective theory for the gauge field \( A_\mu \), which can be interpreted as arising after integrating out the massive matter fields \[ \text{[4]} \], keeping a finite number of derivatives acting on \( F_{\mu\nu} \). We may call the theory for \( A_\mu \) at this stage ‘classical’, since \( A_\mu \) is not yet quantized. The next step amounts to adding to this ‘classical’ effective theory the quantum effects due to virtual photons \[ \text{[5]} \], calculating the corresponding one-loop ‘quantum effective action’. We shall compute this effective action for a constant (but arbitrarily strong) \( F_{\mu\nu} \), and then evaluate it explicitly for the case of a constant magnetic field or arbitrary intensity.

The (at first sight) dangerous features introduced by the higher derivative character of the classical effective theory are dealt with by consistently cutting off the Euclidean momenta in the gauge field loops at values of the order of the mass of the matter fields, since the
higher derivative theory is reliable in that low-momentum region only. This procedure avoids integrating over regions where unphysical poles in the gauge field propagator could show up. These unphysical poles, on the other hand, are artifacts of the low-momentum approximation. We will show that, had the low-momentum approximation not been made, the loop integrals would have been regular everywhere.

The procedure of starting from a general effective theory, involving a small number of arbitrary parameters, has the welcome feature that it allows us to keep the discussion as general as possible, describing many cases at the same time (namely, fermions and/or bosons, many fields, etc), since the result can be stated in terms of general parameters, the values of which in any particular model can be found quite easily.

This paper is organized as follows: In section 2 the classical effective theory for the gauge field is constructed, while in section 3 the one-loop quantum effective action for the gauge field is defined and then calculated for a general constant $F_{\mu\nu}$, obtaining explicit results for some simplified situations. In section 4 we discuss our results and show how to avoid the use of the cutoff by improving upon the use of the low-momentum approximation.

II. THE MODEL

As partially advanced in the previous section, our three-dimensional classical effective action for the Abelian gauge field $A_\mu$ is determined by requiring it to be a general gauge-invariant and parity-conserving one. Locality is also assumed, and this is justified by assuming that no massless matter fields are present, since they may certainly introduce (branch-cut) non-analyticities at zero momentum.

Gauge-invariance plus parity-conservation imply that the dependence of the Lagrangian on $A_\mu$ can only occur either through the combination $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, or its dual $\tilde{F}_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F_{\nu\rho}$. Each term in the Lagrangian must be built from contractions of products of $\tilde{F}_\mu$, with a number of insertions of the derivative operator $\partial_\mu$. For example, if we allow up to four derivatives of the gauge field $A_\mu$, the Lagrangian can only be a linear combination
of the three following terms

\[ F_{\mu\nu}F_{\mu\nu} = 2\tilde{F}_\mu\tilde{F}_\mu, \quad F_{\mu\nu}\partial^2 F_{\mu\nu} = 2\tilde{F}_\mu\partial^2 \tilde{F}_\mu, \quad (F_{\mu\nu}F_{\mu\nu})^2 = 4(\tilde{F}_\mu\tilde{F}_\mu)^2, \]

since other contractions vanish due to de Bianchi identity \( \partial_\mu \tilde{F}_\mu = 0 \).

We will consider initially a more general gauge-invariant action of the form

\[
S_{\text{inv}} = \int d^3v \, \mathcal{L}_{\text{inv}}
\]

\[
\mathcal{L}_{\text{inv}} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{4 M^2} F_{\mu\nu} g(-\partial^2) F_{\mu\nu} + \frac{1}{4 M^2} \frac{1}{4!} (F_{\mu\nu} F_{\mu\nu})^2
\]

\[
= \frac{1}{2} \tilde{F}_\mu(1 + a_1 \frac{g(-\partial^2)}{M^2}) \tilde{F}_\mu + \frac{1}{2} \frac{a_2}{M^2} \frac{1}{4!} (\tilde{F}_\mu \tilde{F}_\mu)^2
\]

where \( a_1 \) and \( a_2 \) are dimensionless functions of the (dimensionless) ratio \( x = \frac{\tilde{F}_\mu^2}{M^2} = \frac{F_{\mu\nu} F_{\mu\nu}}{2M^4} \), \( M \) has the dimensions of a mass (note that \( A_\mu \) has the dimensions of \( M^{\frac{2}{2}} \)), and the numerical factors are chosen for convenience. The constant \( M \) along with the functions \( a_1, a_2 \) and \( g \) are inputs that completely define the model.

While this is not the most general gauge-invariant action \( 1 \), it encompasses several important cases. Indeed, we wish to interpret the invariant effective action as coming from integrating out heavy matter fields. Usually, this integration cannot be done exactly, and one is forced to use some approximation. A typical approximation is the derivative expansion \( 2 \), and as mentioned before, Eq. (2) reproduces the expansion up to four derivatives just taking \( g(-\partial^2) = -\partial^2 \) and constant values for the functions \( a_1 \) and \( a_2 \). The constant \( M \) is naturally related to the mass of the heavy particles, the precise relation and the values of \( a_1 \) and \( a_2 \) being of course model-dependent. \( 2 \)

Other approximations can be obtained using heat kernel techniques \( 3 \): for strong, slowly varying \( F_{\mu\nu} \), the result of the integration of matter fields gives \( a_1 = 0 \) and a model-dependent

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1 For example, terms of the form \((\tilde{F}_\mu(-\partial^2)\tilde{F}_\mu)^n, n > 1, \) and \((\tilde{F}_\mu\partial_\mu\tilde{F}_\rho)^2 \) are not included in Eq. (2).

2 Note that if many fields of the same kind but with different masses are present, \( M \) should be of the order of the lighter mass.
function $a_2(x)$. In the opposite limit, for weak and rapidly varying background fields, the action is again given by Eq. (2) with $a_1 = \text{const}$, $a_2 = 0$ and

$$g(-\partial^2) \propto \int_0^1 d\xi \xi^2 (1 + \frac{1 - \xi^2 (-\partial^2)}{4 M^2})^{-1/2} \approx \frac{1}{3} - \frac{1}{60} \frac{(-\partial^2)}{M^2} + ...$$

Note the non-analyticity of the form factor $g$ in the massless limit.

We now restrict the discussion to the Lagrangian of (2) with $g(-\partial^2) = -\partial^2$. In section 4 we will discuss returning to (4) with $g$ given by (3).

To do perturbative calculations in the model defined by (2) we need to fix the gauge. A particularly convenient choice is to add to $L_{\text{inv}}$ in (2) the gauge-fixing term $L_{gf}$:

$$L = L_{\text{inv}} + L_{gf}$$
$$L_{gf} = \frac{1}{2\alpha} \partial_{\mu} A_{\mu} (1 - a_1 \frac{\partial^2}{M^2}) \partial_{\nu} A_{\nu} ,$$

where $\alpha$ is a gauge-fixing parameter. In particular, note that for $\alpha = 1$ and $a_1 = \text{const}$ the gauge field propagator assumes a ‘Feynman-gauge’-like form

$$\langle A_{\mu} A_{\nu} \rangle = \frac{\delta_{\mu\nu}}{k^2 (1 + \frac{a_1}{M^2} k^2)} ,$$

but we shall keep $\alpha$ arbitrary in order to show the gauge-independence of our results.

Nothing can be concluded about the dimensionless parameters $a_1$ and $a_2$ from symmetry considerations only. However, there are some extra conditions on the signs of these coefficients. These signs are important in determining the perturbative properties of the model. A wrong inference might be drawn by requiring positivity of the Euclidean action $S_{\text{inv}}$ for all the possible field configurations: that both $a_1$ and $a_2$ must be positive. But this is not the case, for $a_1$ at least, since a positive $a_1$ corresponds to anti-screening of electric charge by vacuum polarization. A simple calculation shows that the propagator (4) in coordinate space reads:

$$G_{\mu\nu}(r) = \frac{\delta_{\mu\nu}}{2\pi r} (1 - e^{-\frac{Mr}{a_1}})$$

As the gauge field propagator in the absence of matter fields is just $\frac{\delta_{\mu\nu}}{2\pi r}$, the effect of $a_1$ in (4) is to dress the bare charge $Q$ to the effective one.
\[ Q_{\text{eff}}(r) = Q \sqrt{1 - e^{-\frac{Mr}{a_1}}} . \]  

(8)

Thus when \( r \to 0 \), the effective charge tends to 0, corresponding to anti-screening. The conclusion that the correct choice is a negative \( a_1 \) was also verified by explicitly calculating the coefficient \( a_1 \) in models containing either bosonic or fermionic fields.

The relevance of the sign of \( a_1 \) to the properties of the model is evident from (6); a negative \( a_1 \) (the realistic case) produces a pole in the Euclidean gauge field propagator, located at \( k^2 = -\frac{M^2}{a_1} \). Poles in the Euclidean propagators don’t make sense physically (namely, they correspond to particles with imaginary mass in Minkowski space). This undesirable feature is excluded in this case by the simple reason that we are dealing with an effective low-momentum theory, which we expect to make sense only up to a certain cutoff momentum. This cutoff is determined by \( M \), and in our case it is enough to take that cutoff smaller than \( \frac{M}{(-a_1)^{1/2}} \) to avoid the unphysical pole. This means in particular that loop momenta will run up to a cutoff of that order.

Knowledge of the sign of \( a_2 \), is not essential to the calculation, and we might keep it arbitrary. We will, however, assume it to be positive in order to have a stable theory to start with.

### III. THE EFFECTIVE POTENTIAL

#### A. General \( a_1 \) and \( a_2 \)

We are concerned with the calculation of the 1-loop effective action in the case of a constant \( F_{\mu \nu} \), with the aim of particularising to the constant magnetic field situation and constructing an ‘effective potential’ \( V_{\text{eff}}(F) \).

The one-loop effective action for the model is easily seen to be given by

\[ \Gamma_{\text{eff}} = S + \Gamma^{(1)} \]

(9)

where \( S = \int d^3v \mathcal{L}_{\text{inv}} \), with \( \mathcal{L}_{\text{inv}} \) as defined by (2), and
\begin{align*}
\Gamma^{(1)} &= \frac{1}{2} \text{Tr} \log \left( \frac{\delta^2 S}{\delta A_\mu \delta A_\nu} \right) - \frac{1}{2} \text{Tr} \log \left( \frac{\delta^2 S}{\delta A_\mu \delta A_\nu} \right) \bigg|_{A=0} \, .
\end{align*}

The second functional derivative of \( S \), evaluated for a constant \( F_{\mu\nu} \) yields

\begin{align*}
\frac{\delta^2 S}{\delta A_\mu(v) \delta A_\nu(w)} \bigg|_{F_{\mu\nu}=\text{const.}} =
\end{align*}

\begin{align*}
\left\{ [1 - a_1 \frac{\partial^2}{M^2} + \frac{2a_2}{4!} x^2 + \frac{a_2}{3!} x] (-\partial^2 \delta_{\mu\nu} + \partial_{\mu} \partial_{\nu}) - \frac{1 - a_1 \frac{\partial^2}{M^2}}{\alpha} \partial_{\mu} \partial_{\nu}
\right. \\
- \frac{1}{M^3} [ -4a'_1 \frac{\partial^2}{M^2} + \frac{a''_1}{3!} x^2 + \frac{4a'_1}{3!} x + \frac{2a_2}{3!} ] F_{\mu\alpha} F_{\nu\beta} \partial_{\alpha} \partial_{\beta} \right\} \delta(v-w) \, ,
\end{align*}

where \( a'_1 = \frac{da_1}{dx} \), and \( v, w \) represent the coordinates of two spacetime points.

Thus the effective potential for a constant \( F_{\mu\nu} \) becomes

\begin{align*}
V_{\text{eff}}(F) &= (\text{Vol})^{-1} \left[ S + \Gamma^{(1)} \right] \bigg|_{F=\text{constant}} \\
&= V_{cl}(F) + V^{(1)}(F) \, .
\end{align*}

Here \( V_{cl} \) denotes the part of the total effective potential that does not include quantum effects coming from \( A_\mu \) (it contains quantum effects of the matter fields, though)

\begin{align*}
V_{cl}(F) &= (\text{Vol})^{-1}[S]_{F=\text{constant}} \\
&= \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{a_2}{4M^3} \frac{1}{4!} (F_{\mu\nu} F_{\mu\nu})^2 \right] ,
\end{align*}

and

\begin{align*}
V^{(1)}(F) &= (\text{Vol})^{-1} \left[ \Gamma^{(1)}(F) \right]_{F=\text{constant}} \\
&= \frac{1}{2} \text{Tr} \log \left\{ [1 - a_1 \frac{\partial^2}{M^2} + \frac{2a'_1}{4!} x^2 + \frac{a_2}{3!} x] (-\partial^2 \delta_{\mu\nu} + \partial_{\mu} \partial_{\nu}) \\
- \frac{1}{\alpha} \partial_{\mu} \partial_{\nu} - \frac{1}{M^3} [ -4a'_1 \frac{\partial^2}{M^2} + \frac{a''_1}{3!} x^2 + \frac{4a'_1}{3!} x + \frac{2a_2}{3!} ] F_{\mu\alpha} F_{\nu\beta} \partial_{\alpha} \partial_{\beta} \right\} \\
- \frac{1}{2} \text{Tr} \log \left\{ (1 - \frac{a_1}{M^2} \frac{\partial^2}{\partial \delta_{\mu\nu}} (\partial_{\mu} \partial_{\nu} + \partial_{\mu} \partial_{\nu}) - \frac{1 - a_1}{\alpha} \frac{\partial^2}{M^2} \partial_{\mu} \partial_{\nu} \right\} . \tag{14}
\end{align*}

In what follows we shall evaluate \( V^{(1)} \), the one-loop correction to the classical potential.

It is convenient to convert first the functional trace in (14) to momentum space, and then to introduce the field-dependent projectors defined in the appendix. This procedure yields
\[ V^{(1)}(F) = \frac{1}{2} \int_\Lambda \frac{d^3k}{(2\pi)^3} \text{tr} \log [P_{\mu\nu} + \alpha_Q Q_{\mu\nu} + \alpha_R R_{\mu\nu}] , \]  

(15)

where

\[ \alpha_Q = \frac{1 + a_1 \frac{k^2}{M^2} + \frac{a_2}{3!} x + \frac{2a_2'}{4!} x^2}{1 + a_1(0) \frac{k^2}{M^2}} \]

\[ \alpha_R = \alpha_Q + \frac{\frac{2a_2}{3!} + \frac{4a_1'}{M^2} + \frac{4a_2'}{3!} x + \frac{a_2''}{3!} x^2}{M^3(1 + a_1(0) \frac{k^2}{M^2})} \left( \tilde{F}^2 - \frac{(k \cdot \tilde{F})^2}{k^2} \right) . \]

(16)

The subscript \( \Lambda \) in the momentum integration means that a Euclidean cutoff \( \Lambda \sim M \) has been introduced. This cutoff \( \Lambda \) takes care of the fact that high momentum fluctuations of the gauge field are not described by the classical effective theory, so they should not be included in the loops for the quantum effective theory derived therefrom. A precise value for \( \Lambda \) cannot be decided \textit{a priori}. It certainly should be of the order of \( M \), and one expects the final results to be insensitive to small changes in \( \Lambda \) around \( M \).

The evaluation in (15) of the trace over Lorentz indices of an involved function of an \( F_{\mu\nu} \)-dependent tensor is made possible by the use of the complete set of orthogonal projectors introduced in the Appendix, where we explain this construction and its application to the present case. The result of taking the trace over Lorentz indices may be written as

\[ V^{(1)}(F) = V_a^{(1)}(F) + V_b^{(1)}(F) \]

\[ V_a^{(1)}(F) = \frac{1}{2} \int_\Lambda \frac{d^3k}{(2\pi)^3} \log \left[ 1 + \frac{(a_1 - a_1(0)) \frac{k^2}{M^2} + \frac{a_2}{3!} x + \frac{2a_2'}{4!} x^2}{1 + a_1(0) \frac{k^2}{M^2}} \right] \]

\[ V_b^{(1)}(F) = \frac{1}{2} \int_\Lambda \frac{d^3k}{(2\pi)^3} \log \left[ 1 + \frac{(a_1 - a_1(0)) \frac{k^2}{M^2} + \frac{a_2}{3!} x + \frac{2a_2'}{4!} x^2}{1 + a_1(0) \frac{k^2}{M^2}} + \frac{(\tilde{F}^2 - \frac{(k \cdot \tilde{F})^2}{k^2})}{M^3(1 + a_1(0) \frac{k^2}{M^2})} \left( a_1 - a_1(0) \frac{k^2}{M^2} + \frac{2a_2}{3!} + \frac{a_2'}{3!} x + \frac{a_2''}{3!} x^2 \right) \right] . \]

(17)

As \( a_1 \) is negative, we redefine: \( a_1 \rightarrow -a_1 \), and shall consider \( a_1 \) positive in what follows. Before dealing with the evaluation of the integrals in (17), it is worth studying an important
aspect of the one loop correction $V^{(1)}$, namely, whether it can be negative. A negative $V^{(1)}$ is a necessary condition for the existence of instabilities, which may shift the true vacuum to a non-zero value of $F_{\mu\nu}$. It is not difficult to check that, if $a_1$ and $a_2$ are constants, the correction $V^{(1)}$ is always positive, since its sign is determined by integrals of logs of functions larger than 1 (of course we are assuming $a_2 \geq 0$). In the general case of field-dependent coefficients, requiring positivity of $V^{(1)}$ yields conditions on the coefficients and their derivatives. Sufficient (but not necessary) conditions to ensure stability are, for example

$$
\begin{align*}
  a_1(x) &\leq a_1(0) \\
  a_2(x) &= C x^\alpha, \quad C > 0, \quad \alpha > -2.
\end{align*}
$$

The above conditions are quite likely to be met in most of the situations. The first one ensures that the spurious pole does not move towards the origin when the field grows, and the second one says that the ‘interaction term’ (proportional to $a_2 x^2$) in the effective Lagrangian does not tend to zero for large $x$.

It remains then to evaluate the cutoff momentum integrals in (17). These integrals cannot be exactly evaluated as they stand. However some more explicit results may be provided under different simplifying assumptions, which we describe next.

**B. The case of constant $a_1$ and $a_2$**

This situation corresponds to assuming that $a_1$ and $a_2$ bear no dependence on the field $F_{\mu\nu}$, and thus become just constant parameters. This is the case when the classical effective action is expanded up to four derivatives of $A_\mu$. Defining the two dimensionless parameters $\gamma$ and $y$,

$$
\gamma = \frac{1}{a_1^2} \frac{\Lambda}{M}, \quad y = \frac{a_2 \hat{F}^2}{3! M^3} = \frac{a_2}{3! x}.
$$

the results of the integrations may be put as

$$
V_a^{(1)} = \frac{M^3}{3(2\pi)^2 a_1^2} \left\{ \gamma^3 \log(1+y) - 2\gamma y + \gamma^3 \log[1 - \frac{\gamma^2}{1+y}] \right\},
$$
\[(1 + y)^{\frac{3}{2}} \log\left(\frac{1 + y}{1 + y^{\frac{1}{2}} - \gamma} \right) - \gamma^3 \log(1 - \gamma^2) - \log\left(\frac{1 + \gamma}{1 - \gamma}\right) \right\} \quad (20)\]

and

\[V_{b}^{(1)} = \frac{M^3}{3(2\pi)^2 a_1^3} \left\{ -\gamma^3 \log(1 - \gamma^2) - \log\left(\frac{1 + \gamma}{1 - \gamma}\right) \right. \]
\[+ \gamma^3 \log(1 + y) - 2\gamma^3 + \gamma^3 \log\left(1 - \frac{\gamma^2}{1 + y}\right) \]
\[+ \gamma^3 \sqrt{\frac{2(1 - \gamma^2 + 3y)}{y}} \arctanh \sqrt{\frac{2y}{(1 - \gamma^2 + 3y)}} \]
\[- \frac{14}{3} \gamma y + \int_0^1 ds \left[1 + y(3 - 2s^2)\right]^{\frac{3}{2}} \log\left(\frac{1 + y(3 - 2s^2)}{(1 + y(3 - 2s^2))^{\frac{1}{2}} - \gamma}\right) \right\}. \quad (21)\]

This completes the computation of \(V^{(1)}\), where all but the last term in \(V_{b}^{(1)}\) of (21) are given in a closed analytic form.

C. The case \(a_1 = 0\)

If the quantum matter fields are charged fermions of mass \(M\), the invariant Lagrangian (resulting from integrating out the fermions) can be computed exactly for a constant \(F_{\mu\nu}\). It is given by [8]

\[\mathcal{L}_{inv} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{M^3 u^{3/2}}{8\pi^{3/2}} \int_0^\infty d\xi \xi^{-5/2} [\text{coth}[\xi] - 1] e^{-\xi} \]
\[\equiv \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \Delta \mathcal{L} \quad (22)\]

where \(u = \frac{eB}{M^2} = e \left(\frac{2\pi}{M}\right)^{1/2}\). The integral in the above expression can be readily computed and gives

\[\Delta \mathcal{L} = \frac{M^3}{8\pi} \left[\frac{4}{3} - 2u + 4(2u^3)^{1/2}\zeta(-1/2, 1/2u)\right] \quad (23)\]

where \(\zeta\) denotes the generalized Riemann zeta-function.

When \(u \ll 1\), the above expression reproduces the Schwinger DeWitt (or inverse mass) expansion of the effective action for a constant background. In the opposite limit, when
\( u \gg 1, \Delta \mathcal{L} \) is proportional to \((eB)^{3/2}\). In any case, the exact Lagrangian is of the form Eq. (4) with \( a_1 = 0 \). The function \( a_2(x) \) can be easily obtained by comparing Eqs. (2) and (22).

In this situation, the results of the integrals for \( V_a^{(1)} \) and \( V_b^{(1)} \) are combined to yield

\[
V^{(1)} = \frac{\Lambda^3}{6\pi^2} \left\{ \log[1 + \frac{x}{3!} \left( \frac{a_1^2}{2} x + a_2(x) \right)] + \rho \arctanh\rho^{-1} - 1 \right\},
\]

where

\[
\rho = \sqrt{\frac{1 + \frac{x}{3!} (3a_2 + \frac{9a_1'}{2} x + a_1''x^2)}{\frac{4}{3!} (2a_2 + 4a_1'x + a_1''x^2)}}
\]

IV. DISCUSSION

The use of a cutoff \( \Lambda \) in the momentum integrals corresponding to gauge-field propagators was necessary to avoid reaching the unphysical pole at Euclidean momentum. This pole appears because the classical effective action has been truncated to contain no more than two derivatives acting on \( F_{\mu\nu} \), leading to a polynomial with two zeroes in \( k^2 \), hence a propagator with two poles. On the other hand one may verify that, if no momentum truncation is made, the resulting propagator has only one pole, at the physical point \( k^2 = 0 \) (see Eq.(3)).

Thus a possible way out of the problem of the presence of the spurious pole would be to keep the full momentum dependence of the quadratic part \( \mathcal{L}_{\mathrm{q}} \), which then becomes non-local in coordinate space. More explicitly, instead of using the truncation \( g(-\partial^2) = -\partial^2 \) one can work with the complete form factor defined in Eq. (3). In this case, the one loop correction to the potential becomes

\[
V^{(1)} = V_a^{(1)} + V_b^{(1)}
\]

\[
V_a^{(1)} = \frac{M^3}{(2\pi)^2} \int_0^{\Lambda} dq q^2 \log \left[ \left(1 + \frac{1}{3!} \left( \frac{a_1'}{2} x^2 + a_2 x \right) \right) \right]
\]

\(^3\)Of course, this procedure may be criticized on the grounds that it treats differently the quadratic part and the interaction terms.
\[ V_b^{(1)} = V_a^{(1)} + \frac{2M^3}{(2\pi)^2} \int_0^\Lambda dq \, q^2 \left( \rho \arctanh \rho^{-1} - 1 \right) \]  
where now
\[ \rho = \sqrt{\frac{1 + a_1 g + 4a_1'g + \frac{1}{3}(a_2'^2x^3 + \frac{3}{2}a_2x^2 + 3a_2x)}{x[4a_1'g + \frac{1}{3}(a_2'^2x^2 + 4a_2x + 2a_2)]}}. \]
and it is possible to take the limit $\Lambda \to \infty$.

It is important to remember that even if the initial theory is renormalizable, the effective theory is not necessarily so, as in the case of the Fermi theory regarded as (part of) an effective theory for the renormalizable Electroweak Lagrangian. However, had we known the initial renormalizable theory, we would have a recipe to absorb the possible infinities, by means of counterterms which in some cases are non-local or non-polynomial in terms of the effective theory variables.

We would now like to check whether our results for the one-loop correction to the effective potential are very sensitive to the precise value of the cutoff $\Lambda$ or not. We would of course like to have at least some regime where the actual value of $\Lambda$ is not so relevant. This might be done by studying the derivative of $V^{(1)}$ with respect to $\Lambda$, to see if it is small. But as both $V^{(1)}$ and $\Lambda$ are dimensionful quantities, that smallness should be considered as relative to another dimensionful quantity, the natural one being the mass $M$. This amounts to considering the magnitude of $\frac{\partial v^{(1)}}{\partial \Lambda}$ where we have introduced the dimensionless potential $v^{(1)} = \frac{V^{(1)}}{M^3}$. It is easy to realize that in our results $V^{(1)} \propto \Lambda^3$ which implies: $\frac{\partial v^{(1)}}{\partial \Lambda} \propto \left( \frac{\Lambda}{M} \right)^2$.

The outcome of this simple estimate is that the results are not very sensitive to the actual value of $\Lambda$ when $\frac{\Lambda}{M} << 1$, since the derivative is small. If we want to integrate momenta which are nearer to $M$, more higher order terms in the derivative expansion should be included.

It is worth noting that the gauge-fixing independence of the results is explicit, since $\alpha$ does not appear in the expressions for $V^{(1)}$. This is a consequence of the fact that the effective action is gauge-fixing independent on-shell, and certainly a constant $F_{\mu\nu}$ satisfies the classical equations of motion.
We finally remark that the assumption of parity conservation was essential in determining the form of the classical effective theory, and we speculate that the results of performing an analogous calculation in the parity violating case may be qualitatively different, because in the latter case there will be a Chern-Simons term inducing a topological mass for the gauge field, which will strongly modify the low-momentum regime of the gauge-field propagator. To the mass $M$ one should then add the new scale set by the topological mass.

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APPENDIX A:

We construct here the three orthogonal projectors used in the calculation of the effective potential. We have written $V^{(1)}$ in Equation (13) as

$$V^{(1)}(F) = \frac{1}{2} \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \text{tr} \left\{ \alpha_P(k^2) P_{\mu\nu} + \alpha_Q(k^2) Q_{\mu\nu} + \alpha_R(k^2) R_{\mu\nu} \right\}, \quad (A1)$$

where $P$, $Q$ and $R$ verify

$$P^2 = P, \quad Q^2 = Q, \quad R^2 = R$$

$$P + Q + R = 1 \quad (A2)$$

and all the products between different projectors vanish. Using the properties (A2), we can write (A1) as

$$V^{(1)}(F) = \frac{1}{2} \text{tr} P \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \log (\alpha_P(k^2)) + \frac{1}{2} \text{tr} Q \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \log (\alpha_Q(k^2))$$

$$+ \frac{1}{2} \text{tr} R \int_{\Lambda} \frac{d^3k}{(2\pi)^3} \log (\alpha_R(k^2)). \quad (A3)$$

Taking into account that the only possible eigenvalues for $P$, $Q$ and $R$ are 0 and 1, and that they are $3 \times 3$ matrices, we may also deduce from (A2) that

$$\text{tr} P = \text{tr} Q = \text{tr} R = 1. \quad (A4)$$

To find the functions $\alpha_P$, $\alpha_Q$ and $\alpha_R$, we chose the set of projectors:

$$P_{\mu\nu} = \frac{k_{\mu}k_{\nu}}{k^2}$$

$$Q_{\mu\nu} = \frac{k^2 \tilde{F}_\mu \tilde{F}_\nu + (k \cdot \tilde{F})^2 \frac{k_{\mu}k_{\nu}}{k^2} - k \cdot \tilde{F}(k_{\mu} \tilde{F}_\nu + k_{\nu} \tilde{F}_\mu)}{k^2 \tilde{F}^2 - (k \cdot \tilde{F})^2}$$

$$R_{\mu\nu} = \frac{(k^2 \tilde{F}^2 - (k \cdot \tilde{F})^2) \delta_{\mu\nu} - \tilde{F}^2 k_{\mu}k_{\nu} - k^2 \tilde{F}_\mu \tilde{F}_\nu + k \cdot \tilde{F}(k_{\mu} \tilde{F}_\nu + k_{\nu} \tilde{F}_\mu)}{k^2 \tilde{F}^2 - (k \cdot \tilde{F})^2}. \quad (A5)$$
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