CALOGERO-MOSER AND TODA SYSTEMS FOR TWISTED AND
UNTWISTED AFFINE LIE ALGEBRAS *

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ABSTRACT

The elliptic Calogero-Moser Hamiltonian and Lax pair associated with a general simple Lie algebra $\mathcal{G}$ are shown to scale to the (affine) Toda Hamiltonian and Lax pair. The limit consists in taking the elliptic modulus $\tau$ and the Calogero-Moser couplings $m$ to infinity, while keeping fixed the combination $M = me^{i\pi\delta\tau}$ for some exponent $\delta$. Critical scaling limits arise when $1/\delta$ equals the Coxeter number or the dual Coxeter number for the untwisted and twisted Calogero-Moser systems respectively; the limit consists then of the Toda system for the affine Lie algebras $\mathcal{G}^{(1)}$ and $(\mathcal{G}^{(1)})^\vee$. The limits of the untwisted or twisted Calogero-Moser system, for $\delta$ less than these critical values, but non-zero, consists of the ordinary Toda system, while for $\delta = 0$, it consists of the trigonometric Calogero-Moser systems for the algebras $\mathcal{G}$ and $\mathcal{G}^\vee$ respectively.

* Research supported in part by the National Science Foundation under grants PHY-95-31023, PHY-94-07194 and DMS-95-05399.
I. INTRODUCTION

It is now well recognized that the low energy dynamics of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are governed effectively by integrable models. While it is not yet known which models arise in this manner, the models defined by Lie algebras are naturally expected to play a major role.

This paper is the second of a series [1] devoted to the study of twisted and untwisted elliptic Calogero-Moser systems defined by general simple Lie algebras, and of their role in Seiberg-Witten theory. In [2], on the basis of several consistency checks, Donagi and Witten had proposed that the low energy dynamics of the $SU(N)$ gauge theory with matter in the adjoint representation was described by a $SU(N)$ Hitchin systems.† This was verified in [3] by evaluating explicitly the prepotential, using the identification [4][5] of the $SU(N)$ Hitchin system with the $SU(N)$ elliptic Calogero-Moser system. The elliptic Calogero-Moser system is associated with an elliptic curve $\Sigma$ (or torus), defined in terms of the periods $2\omega_1$ and $2\omega_2$ by $\Sigma \equiv \mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})$. The modulus $\tau = \omega_2/\omega_1$ of $\Sigma$ is related to the gauge coupling of the super-Yang-Mills theory by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \quad (1.1)$$

The Calogero-Moser Lax pair of operators $L(z), M(z)$ depend on an arbitrary spectral parameter $z \in \Sigma$, and the Lax equation

$$\dot{L}(z) = [L(z), M(z)] \quad (1.2)$$

is equivalent to the Hamilton-Jacobi equations of the elliptic Calogero-Moser system. In terms of these data, the Seiberg-Witten curve is precisely the Calogero-Moser spectral curve

$$\Gamma = \{(k, z); \det(kI - L(z)) = 0\}, \quad (1.3)$$

and the Seiberg-Witten differential is $d\lambda = kdz$. The theme of this series of papers is to extend this analysis to gauge theories associated with an arbitrary simple Lie algebra $\mathcal{G}$.

In the first paper of the series [1], we had indicated that besides the usual elliptic Calogero-Moser systems defined by Lie algebras, the extension to non-simply laced algebras

† Extensive references to research on the connections between integrable models and supersymmetric Yang-Mills theory may be found in [1]. Further references to the derivation of Seiberg-Witten curves from effective field theories emerging on branes in string theory and M-theory, as well as from singularities in Calabi-Yau compactifications are in [8].
actually required the introduction of new systems, namely the twisted elliptic Calogero-Moser systems. We had also constructed explicitly Lax pairs with spectral parameters for all (twisted and untwisted) elliptic Calogero-Moser systems, except in the case of twisted $G_2$. Now the identification of which integrable model corresponds to any given gauge theory is still largely conjectural, and no direct derivation is available thus far. Rather, as in the original case of the $SU(N)$ four-dimensional gauge theory and Hitchin systems studied by Donagi and Witten, the identification of the correct integrable model is usually based on consistency checks such as limits of the theories as mass parameters tend to 0 or infinity. The goal of the present paper is to describe these limits and consistency checks in the case of elliptic Calogero-Moser systems, and explain why the twisted models are required for non-simply laced algebras.

It is well-known (see Inozemtsev [6-7]) that the elliptic Calogero-Moser system corresponding to $\mathcal{G} = A_n = SU(n+1)$

$$H = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 - \frac{1}{2} m^2 \sum_{i \neq j} \wp(x_i - x_j) \tag{1.4}$$

tends to either the Toda or the periodic Toda system

$$H = \frac{1}{2} \sum_{i=1}^{n+1} P_i^2 - \frac{1}{2} \sum_{i=1}^{n} e^{X_{i+1}-X_i}, \quad \text{(Toda)} \tag{1.5}$$

$$H = \frac{1}{2} \sum_{i=1}^{n+1} P_i^2 - \frac{1}{2} \left( \sum_{i=1}^{n} e^{X_{i+1}-X_i} + e^{X_1-X_{n+1}} \right), \quad \text{(periodic Toda)}$$

in the limit where $\omega_1 = -i\pi$, $\omega_2 \to \infty$, and

$$x_i = X_i - 2\omega_2 \delta i, \quad p_i = P_i, \quad 1 \leq i \leq n+1,$$

$$m = Me^{-i\pi \delta \tau}, \tag{1.6}$$

depending on whether $\delta < 1/(n+1)$ or $\delta = 1/(n+1)$. (Other limits are also discussed in [6], but we do not need them here). We shall be mainly interested in extensions of the critical case $\delta = 1/(n+1)$, although the subcritical case is easily treated by the same arguments.

For general Lie algebra $\mathcal{G}$, the scaling prescription (1.6) admits two distinct generalizations, depending essentially on whether the critical value $1/\delta = n+1$ is replaced by the Coxeter number* $h_\mathcal{G}$ or by the dual Coxeter number $h_\mathcal{G}^\vee$. For our purposes, $h_\mathcal{G}$ and $h_\mathcal{G}^\vee$

* Some key facts about Lie algebra theory, and in particular about the Coxeter numbers and dual Coxeter numbers are given in the Appendix §A of [1].
are most conveniently defined in the following manner. Let \( \alpha_i, 1 \leq i \leq n \), be a basis of simple roots for the Lie algebra \( \mathcal{G} \). For each root \( \alpha \), the coroot \( \alpha^\vee \) is defined by \( \alpha^\vee = \frac{2\alpha}{\alpha \cdot \alpha} \). Now expand \( \alpha \) and \( \alpha^\vee \) respectively in terms of the bases \( \{\alpha_i\} \) of simple roots and \( \{\alpha_i^\vee\} \) of simple coroots

\[
\alpha = \sum_{i=1}^{n} l_i \alpha_i, \quad \alpha^\vee = \sum_{i=1}^{n} l_i^\vee \alpha_i^\vee. \tag{1.7}
\]

Then \( h_\mathcal{G} \) and \( h_\mathcal{G}^\vee \) are defined as the following maxima when the root \( \alpha \) runs through the root system of \( \mathcal{G} \),

\[
h_\mathcal{G} = 1 + \max \sum_{i=1}^{n} l_i, \quad h_\mathcal{G}^\vee = 1 + \max \sum_{i=1}^{n} l_i^\vee. \tag{1.8}
\]

The Coxeter and dual Coxeter numbers are evidently the same when \( \mathcal{G} \) is simply laced, but otherwise \( h_\mathcal{G}^\vee \) is strictly less than \( h_\mathcal{G} \). Now it is not difficult to show that non-trivial limits can only arise when the dynamical variable \( x \) scales according to \( x = X + (2\omega_2)\nu \) for some fixed vector \( \nu \) in \( \mathbb{R}^n \) (c.f. §II below). Depending on whether \( 1/\delta \) is \( h_\mathcal{G} \) or \( h_\mathcal{G}^\vee \) (or equivalently, on whether we want the simple roots of \( \mathcal{G}^{(1)} \) or of \( (\mathcal{G}^{(1)})^\vee \) to survive in the limits), we have to make the following choices for the vector \( \nu \)

- \( x = X + 2\omega_2\delta \rho^\vee \), if \( m = Me^{-i\pi\delta \tau}, \delta \leq 1/h_\mathcal{G} \);
- \( x = X + 2\omega_2\delta^\vee \rho \), if \( m = Me^{-i\pi\delta^\vee \tau}, \delta^\vee \leq 1/h_\mathcal{G}^\vee \).

Here \( \rho \) and \( \rho^\vee \) are respectively the Weyl vector and the level vector.

More precisely, we shall show that, under the scaling rules associated with \( \delta = 1/h_\mathcal{G} \), the untwisted elliptic Calogero-Moser Hamiltonian and its Lax pair with spectral parameter recently constructed in [1] converge to the affine Toda Hamiltonian and Lax pair associated with \( \mathcal{G}^{(1)} \). For \( \delta \) less than this critical value, but non-zero, the limit consists of the ordinary Toda system for \( \mathcal{G} \), while for \( \delta = 0 \), we find the trigonometric Calogero-Moser system for \( \mathcal{G} \). Under the scaling rules associated with \( \delta = 1/h_\mathcal{G}^\vee \), when \( \mathcal{G} \) is not simply laced, the untwisted elliptic Calogero-Moser systems do not converge to a finite limit. However, the new twisted elliptic Calogero-Moser systems introduced in [1] as well as all the Lax pairs constructed there do converge, and the limit consists of the affine Toda system for the affine Lie algebra \( (\mathcal{G}^{(1)})^\vee \). For \( \delta \) less than this critical value, but non-zero, the limit consists in the ordinary Calogero-Moser system for \( \mathcal{G}^\vee \), while for \( \delta = 0 \), we find the trigonometric Calogero-Moser system for \( \mathcal{G}^\vee \).

We had mentioned earlier that the scaling limits in this paper constitute a key piece of evidence for identifying the integrable model describing the low energy effective theory of
the $\mathcal{N} = 2$ supersymmetric $\mathcal{G}$ gauge theory with a hypermultiplet in the adjoint representation of $\mathcal{G}$. A detailed discussion together with some of the underlying physics is postponed to the third paper of this series [8]. Here we note only that our results strongly suggest that the twisted Calogero-Moser systems associated with $\mathcal{G}$ are the correct integrable models. The mass of the adjoint hypermultiplet is given by the Calogero-Moser coupling constant. For simply laced $\mathcal{G}$, there is just one such coupling $m$, which is the hypermultiplet mass. For non-simply laced $\mathcal{G}$, there are two such Calogero-Moser couplings, one for long and one for short roots, $m_l$ and $m_s$ respectively, and both are proportional to the mass $m$ with known group theoretical factors. The limits established here correspond to letting $m \to \infty$ and thus to decoupling the hypermultiplet. The key issue is which scaling rule is the appropriate rule dictated by physics. Identifying the dual Coxeter number $h_{\mathcal{G}}^\vee$ with the quadratic Casimir of $\mathcal{G}$, the dependence of the gauge coupling on the mass $m$ in this limit is given by

$$\tau = \frac{i}{2\pi} h_{\mathcal{G}}^\vee \ln \frac{m^2}{M^2}, \quad (1.9)$$

in view of standard renormalization group arguments. Thus the scaling rules associated with the dual Coxeter number $h_{\mathcal{G}}^\vee$ are the appropriate ones, and with them, the twisted elliptic Calogero-Moser systems. As $m \to 0$ and $m \to \infty$, the desired limits emerge

(1) At $m = 0$, the integrable model is free, corresponding to the fact that the gauge theory acquires an $\mathcal{N} = 4$ supersymmetry, and the prepotential receives no quantum corrections.

(2) At $m = \infty$, the limit of the Calogero-Moser system is a twisted affine Toda system, which was previously argued by Martinec and Warner [9] to be associated with the $\mathcal{N} = 2$ supersymmetric gauge theory without hypermultiplets.

Further evidence may be obtained by comparing the prepotential derived in the weak coupling limit, $\tau \to +i\infty$, of the twisted elliptic Calogero-Moser systems with that predicted by one-loop calculations in the gauge theory. These calculations may be carried out using the explicit form of the Lax pairs in [1], and using the methods of [3] and [20]. As an example, this check is carried out successfully for $\mathcal{G} = D_n$ in [8].

**Calogero-Moser and Toda Systems and their Interrelation**

Let $\mathcal{G}$ be a simple finite-dimensional Lie algebra of rank $n$, and denote the set of all roots of $\mathcal{G}$ by $\mathcal{R}(\mathcal{G})$. The Toda and Calogero-Moser systems are Hamiltonian systems with $n$ complex degrees of freedom and their canonical momenta, denoted by $X_i$ and $P_i$ for
Toda and by \( x_i \) and \( p_i \) for Calogero-Moser,\(^*\) with \( i = 1, \cdots, n \). We assemble these degrees of freedom into \( n \)-dimensional vectors \( X, P, x \) and \( p \), and use the dot notation for inner products.

The Toda system associated with a finite-dimensional or affine Lie algebra \( \mathcal{K} \) is defined by the Hamiltonian

\[
H_T = \frac{1}{2} P \cdot P - \frac{1}{2} \sum_{\alpha \in R_+ (\mathcal{K})} M_{|\alpha|}^2 e^{-\alpha \cdot X},
\]

(1.10)

where \( M_{|\alpha|} \) are constants and \( R_+ (\mathcal{K}) \) is the set of simple roots of \( \mathcal{K} \).

- When \( \mathcal{K} = \mathcal{G} \) is any finite-dimensional Lie algebra, the system \( H_T \) is referred to as the ordinary (or non-periodic) Toda system associated with \( \mathcal{G} \).
- When \( \mathcal{K} \) is any of the affine Lie algebras, the system \( H_T \) is referred to as the affine (or periodic) Toda system associated with \( \mathcal{K} \).

The (untwisted) elliptic Calogero-Moser (CM) system is defined for any finite-dimensional Lie algebra \( \mathcal{G} \) by the Hamiltonian

\[
H_{CM} = \frac{1}{2} p \cdot p - \frac{1}{2} \sum_{\alpha \in R (\mathcal{G})} m_{|\alpha|}^2 \wp (\alpha \cdot x)،
\]

(1.11)

where \( \wp \) is the Weierstrass elliptic function of periods \( 2\omega_1 \) and \( 2\omega_2 \) of the underlying elliptic curve \( \Sigma \).

The twisted Calogero-Moser systems (TCM) may be defined for any finite-dimensional Lie algebra \( \mathcal{G} \) by the Hamiltonian

\[
H_{TCM} = \frac{1}{2} p \cdot p - \frac{1}{2} \sum_{\alpha \in R (\mathcal{G})} m_{|\alpha|}^2 \wp_\nu (\nu (\alpha \cdot x)).
\]

(1.12)

For simply laced \( \mathcal{G} \), we have \( \nu = 1 \) on all roots and the twisted Calogero-Moser system is identical to the untwisted one of (1.11). Henceforth, we shall assume that \( \mathcal{G} \) is non-simply laced. The root system is then a union of the set of long roots \( R_l \) and the set of short roots \( R_s \). On long roots, \( \nu = 1 \), while on short roots \( \nu (\alpha) \) equals the ratio of the length squared of the long roots to the short roots. Thus, \( \nu = 2 \) for \( \mathcal{G} = B_n, C_n, F_4 \), while \( \nu = 3 \)

\(^*\) In the case of \( A_n \), as we saw in the Introduction, it is sometimes more convenient to have \( n + 1 \) dynamical variables variables \((X_i, P_i)\) or \((x_i, p_i)\). The correct rank \( n \) is restored upon observing that the dynamical variables can be shifted by an arbitrary constant.
for $G = G_2$. The function $\varphi_\nu$ is twisted of order $\nu$ in one of the three half periods $\omega_1$, $\omega_2$ or $\omega_3 = \omega_1 + \omega_2$:

$$\varphi_\nu(u) = \sum_{\sigma=0}^{\nu-1} \varphi(u + 2\omega_a \nu).$$  \hfill (1.13)

In the sequel, it will be convenient to choose $\omega_a = \omega_1$. The trigonometric and rational Calogero-Moser systems are obtained from the (untwisted) elliptic systems (for each Lie algebra) by letting respectively one or both of the periods $2\omega_1$ and $2\omega_2$ tend to infinity

$$H^{\text{trig}}_{CM} = \frac{1}{2} \left( r - \frac{1}{2} \sum_{\alpha \in R(G)} m_{|\alpha|}^2 \frac{1}{\sinh^2(\alpha \cdot x)} \right),$$

$$H^{\text{rat}}_{CM} = \frac{1}{2} \left( r - \frac{1}{2} \sum_{\alpha \in R(G)} m_{|\alpha|}^2 \frac{1}{(\alpha \cdot x)^2} \right).$$  \hfill (1.14)

We shall now summarize the results of this paper in the form of the Theorems 1 and 2 below.† Recall that $h_G$ and $h_G^\vee$ denote respectively the Coxeter and dual Coxeter numbers of the finite-dimensional Lie algebra $G$, $\rho$ the Weyl vector, and $\rho^\vee$ the level vector of $G$. When $\text{Re}(\omega_2) \to \infty$ and $m_{|\alpha|} \to \infty$, while keeping the quantities $M_{|\alpha|}$, $X$ and $P$ fixed, we have the limits below.

**Theorem 1 : The Untwisted Cases**

The scaling behavior is governed by an exponent $\delta$ and is given by

$$M_{|\alpha|} = m_{|\alpha|} q^{\frac{1}{2} \delta},$$

$$X = x - 2 \omega_2 \delta \rho^\vee, \quad P = p,$$

$$Z = e^z e^{-i\pi \tau}.$$  \hfill (1.15)

The Hamiltonian $H_{CM}$ of the *untwisted* elliptic Calogero-Moser system for the Lie algebra $G$, converge to those of the

(a) affine (periodic) Toda system with *untwisted* affine Lie algebra $G^{(1)}$ when $\delta = 1/h_G$;

(b) ordinary (non-periodic) Toda system with Lie algebra $G$ when $\delta < 1/h_G$;

(c) trigonometric Calogero-Moser system with Lie algebra $G$ when $\delta = 0$.

The Lax pairs constructed in [1] for all untwisted, elliptic Calogero-Moser systems defined by simple Lie algebras converge to Lax pairs for the corresponding affine Toda system for $G^{(1)}$ (when $\delta = 1/h_G$), Toda system for $G$ (when $\delta < 1/h_G$), and trigonometric Calogero-Moser system for $G$ (when $\delta = 0$). (The case of $E_8$ was solved in [1] making use

† Henceforth, we shall set $\omega_1 = -i\pi$, so that $\tau = i\omega_2/\pi$, and $q = e^{2\pi i \tau} = e^{-2\omega_2}$. 

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of an extra assumption on the existence of a ±1-values cocycle. The same assumption is implied here.)

**Theorem 2 : The Twisted Cases**

The scaling behavior is governed by an exponent $\delta^\vee$ and is given by

$$M_{|\alpha|} = m_{|\alpha|} q^{\frac{1}{2} \delta^\vee}, \quad (1.16a)$$

$$X = x - 2\omega_2 \delta^\vee \rho, \quad P = p, \quad (1.16b)$$

$$Z = e^z e^{-i\pi \tau}. \quad (1.16c)$$

The Hamiltonian $H_{TCM}$ of the *twisted* elliptic Calogero-Moser system associated with a Lie algebra $\mathcal{G}$, converge to those of

(a) affine (periodic) Toda system with *twisted* affine Lie algebra $(\mathcal{G}^{(1)})^\vee$ when $\delta^\vee = 1/h^\vee_{\mathcal{G}}$.

(b) ordinary (non-periodic) Toda system with Lie algebra $\mathcal{G}^\vee$ when $\delta^\vee < 1/h^\vee_{\mathcal{G}}$;

(c) trigonometric Calogero-Moser system with Lie algebra $\mathcal{G}^\vee$ when $\delta^\vee = 0$.

The Lax pairs constructed in [1] for all twisted, elliptic Calogero-Moser systems defined by simple Lie algebras except $G_2$ converge to Lax pairs of the corresponding affine Toda system for $(\mathcal{G}^{(1)})^\vee$ (when $\delta^\vee = 1/h^\vee_{\mathcal{G}}$), Toda system for $\mathcal{G}^\vee$ (when $\delta^\vee < 1/h^\vee_{\mathcal{G}}$), and trigonometric Calogero-Moser system for $\mathcal{G}^\vee$ (when $\delta^\vee = 0$).

The Lax pairs of the Toda and Calogero-Moser systems will be presented explicitly in the subsequent sections of this paper.

The remainder of this paper is devoted to a complete proof of Theorems 1 and 2. In §II, we discuss and prove Theorem 1 : the limits of the Hamiltonians and Lax pairs for the *untwisted* Calogero-Moser systems. We discuss and prove Theorem 2 on the limits of the *twisted* Calogero-Moser systems in §III for the Hamiltonians and in §IV for the Lax pairs. For a discussion of the Lie algebra and elliptic function results we need, we refer the reader to Appendices §A and §B of [1] respectively. Other useful references on Lie algebras are [10-12]. Surveys of earlier work on integrable models associated with Lie algebras can be found in [13].

**II. UNTWISTED CALOGERO-MOSER AND (AFFINE) TODA SYSTEMS**

First we recall the expressions for Weierstrass elliptic functions in terms of Jacobi theta functions, as well as their product and series expansions which will be useful when
considering the limit $\text{Re}(\omega_2) \to \infty$. It is convenient to introduce the following modification of the standard $\vartheta_1$ function, with its product expansion

$$
\vartheta_1^*(u|\tau) \equiv 2\pi i \frac{\vartheta_1(u|\tau)}{\vartheta_1(0|\tau)} = 2 \sinh \left( \frac{u}{2} \right) \prod_{n=1}^{\infty} (1 - q^n e^u)(1 - q^n e^{-u})(1 - q^n)^{-2}.
$$

Then the Weierstrass functions $\sigma(u)$, $\zeta(u)$, and $\wp(u)$ are defined by [14]

$$
\sigma(u) = e^{-\frac{\eta_1}{\pi i} u^2} \vartheta_1^*(u|\tau)
$$

$$
\zeta(u) = \frac{\eta_1}{\pi i} u + \partial_u \log \vartheta_1^*(u|\tau)
$$

$$
\wp(u) = \frac{\eta_1}{\pi i} - \partial_u^2 \log \vartheta_1^*(u|\tau).
$$

It will be very convenient to express the elliptic function $\wp$ as a series expansion involving hyperbolic functions [6]

$$
\wp(u) = \frac{\eta_1}{\pi i} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\cosh(u - 2n\omega_2) - 1}.
$$

The series expansion has the advantage of being uniformly convergent throughout $u \in \mathbb{C}$, as long as $\text{Re}(\omega_2) > 0$. The constant $\eta_1 = \zeta(\omega_1)$ may be determined from the fact that $\wp(u) = u^2 + O(u^3)$. Henceforth, we shall neglect it since it does not affect the Hamilton-Jacobi equations of the systems.

A. The Scaling Limit of the Hamiltonian

Our first task is to derive the limit of the Hamiltonian of the Calogero-Moser system as $\text{Re}(\omega_2) \to \infty$, keeping $M_{\vert \alpha \vert}$ fixed, according to

$$
m_{\vert \alpha \vert} = M_{\vert \alpha \vert} e^{\delta \omega_2}.
$$

Here $\delta$ is a real scaling exponent with $\delta \geq 0$, to be determined later. It suffices to take the above limit of the combination $m_{\vert \alpha \vert}^2 \wp(\alpha \cdot x)$ separately for each root $\alpha \in \mathcal{R}(\mathcal{G})$. Using the series representation for $\wp$ of (2.3), we have

$$
m_{\vert \alpha \vert}^2 \wp(\alpha \cdot x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{M_{\vert \alpha \vert}^2 e^{2\delta \omega_2}}{\cosh(\alpha \cdot x - 2n\omega_2) - 1}.
$$

Since this series is uniformly convergent throughout $\mathbb{C}$, we may analyze its limit term by term in (2.5). Clearly, if $\delta = 0$, only the term $n = 0$ will survive in the limit, and we
recover the trigonometric Calogero-Moser system, which proves (c) of Theorem 1 for the Hamiltonian.

Henceforth, we assume that $\delta > 0$, so that $m_{|\alpha|} \to \infty$ as $\text{Re}(\omega_2) \to \infty$. We begin by giving a justification for the scaling behavior announced in (1.15b). The condition $P = p$ is manifest. It is clear from (2.5) that unless $x$ has a non-trivial dependence on $\omega_2$, the limit of $m_{|\alpha|}^2 \varphi(\alpha \cdot x)$ will diverge. It follows from the form of (2.5) that the only interesting $\omega_2$ dependence of $x$ is by a shift linear in $\omega_2$. Thus, we set $x = X + 2\omega_2 v$, for some vector $v$ in $\mathbb{R}^n$, and keep $X$ and $v$ fixed as $\text{Re}(\omega_2) \to \infty$.

A number of constraints on the vector $v$ result from the following considerations. The $n = 0$ term in (2.5) will diverge unless $|v \cdot \alpha| \geq \delta$ for all roots $\alpha$. To analyze this constraint in more detail, we fix a basis of simple roots $\alpha_i$, $1 \leq i \leq n$ for $\mathcal{G}$. Then any root $\alpha$ of $\mathcal{G}$ may be written as

$$\alpha = \sum_{i=1}^{n} l_i \alpha_i,$$

with all $l_i \geq 0$ for positive roots, and all $l_i \leq 0$ for negative roots. A finite limit of (2.5) requires that $|v \cdot \alpha_i| \geq \delta$ for all simple roots $\alpha_i$. Without loss of generality, we may assume that all inner products are positive, so that the constraint becomes $v \cdot \alpha_i \geq \delta$. Once this holds, the constraint $|v \cdot \alpha| \geq \delta$ will be satisfied for all roots $\alpha$ in view of (2.6). The situation where the inequality is saturated for every simple root produces the maximal number of roots surviving in the limit, and will thus result in a *maximally symmetric limit*. All other cases can be reduced to those upon considering directly the Calogero-Moser system associated with a subalgebra of $\mathcal{G}$. Henceforth, we shall only consider the *maximally symmetric limits*.

In any finite dimensional simple Lie algebra, there exists a unique vector, whose inner product with any simple root is 1. This is the *level vector* $\rho^\vee$, defined as the half sum of all positive coroots. The inner product of $\rho^\vee$ with the root $\alpha$ (and more generally with any weight of $\mathcal{G}$), defines the *level* function $l(\alpha)$ by

$$l(\alpha) \equiv \alpha \cdot \rho^\vee.$$  

(2.7)

For any $\mathcal{G}$, and any root $\alpha$, the level $l(\alpha)$ is an integer, and takes the value 1 if and only if $\alpha$ is a simple root. It is clear then that the *maximally symmetric limits* correspond to $v$ proportional to the *level vector* $\rho^\vee$, with proportionality factor $\delta$. We thus recover (1.15b), or equivalently, using (2.7)

$$\alpha \cdot x = \alpha \cdot X + 2\omega_2 \delta l(\alpha).$$  

(2.8)
The limit of the \( n = 0 \) term is now finite under the scaling (1.15b) or (2.8).

In order that contributions to (2.5) for all \( n \) have finite limits, further constraints must be imposed. By periodicity of \( \wp \), it is easy to see that the product \( \delta l(\alpha) \) must stay away from any integer value by a distance of at least \( \delta \). In other words, we must have

\[
0 < \delta \leq \delta l(\alpha) - \lfloor \delta l(\alpha) \rfloor \leq 1 - \delta,
\]

where \( \lfloor a \rfloor \) is the integer part of \( a \). The simplest way to realize this extra constraint is to require that \( \delta l(\alpha) < 1 \) for all positive roots. This will be the case throughout the paper. Then the preceding condition becomes

\[
\delta \leq \delta l(\alpha) \leq 1 - \delta \tag{2.9}
\]

for all positive roots \( \alpha \). If \( \alpha_0 \) is the highest root of \( \mathcal{G} \), and \( l_0 = l(\alpha_0) \) its level, then it suffices that the above condition be satisfied on \( \alpha_0 \):

\[
h_{\mathcal{G}} = 1 + l_0 \leq \frac{1}{\delta} \tag{2.10}
\]

Here \( h_{\mathcal{G}} = 1 + l_0 \) is the Coxeter number of \( \mathcal{G} \). The case where \( \delta > 1/h_{\mathcal{G}} \) is more complicated and will be discussed in a forthcoming publication. The evaluation of the limits below relies on the fact that, in the critical case where \( \delta = 1/h_{\mathcal{G}} \), the first inequality in (2.9) becomes an equality if and only if \( \alpha \) is a simple root, while the second inequality in (2.9) becomes an identity if and only if \( \alpha \) is the highest root \( \alpha_0 \).

**General Limit Formulas**

Since \( \wp \) is even, it suffices to consider positive roots \( \alpha \). In view of the asymptotics

\[
2\{\cosh(\alpha \cdot x - 2n\omega_2) - 1\} \to \begin{cases} e^{+\alpha \cdot X + 2\omega_2(\delta l(\alpha) - n)}, & \text{if } \delta l(\alpha) - n > 0 \\ e^{-\alpha \cdot X - 2\omega_2(\delta l(\alpha) - n)}, & \text{if } \delta l(\alpha) - n < 0, \end{cases} \tag{2.11}
\]

we have the following limiting behavior

\[
m^2_{|\alpha|}\wp(\alpha \cdot x) \to M^2_{|\alpha|} \left[ \sum_{n < \delta l(\alpha)} e^{-2\omega_2(\delta l(\alpha) - n - \delta) - \alpha \cdot X} + \sum_{n > \delta l(\alpha)} e^{-2\omega_2(n - \delta l(\alpha) - \delta) + \alpha \cdot X} \right]. \tag{2.12}
\]

This expression can be made more explicit upon the assumption that \( 0 < \delta < \delta l(\alpha) < 1 - \delta < 1 \), introduced above for positive roots \( \alpha \),

\[
m^2_{|\alpha|}\wp(\alpha \cdot x) \to M^2_{|\alpha|} \left[ \sum_{n \geq 0} e^{-2\omega_2(n + \delta l(\alpha) - \delta) - \alpha \cdot X} + \sum_{n \geq 1} e^{-2\omega_2(n - \delta l(\alpha) - \delta) + \alpha \cdot X} \right]. \tag{2.13}
\]
In the first sum, all contributions with \( n \geq 1 \) converge to zero, and may be ignored in the limit \( \text{Re}(\omega_2) \to \infty \). In the second sum, all contributions with \( n \geq 2 \) converge to zero and may be ignored as well. We are thus left with the following asymptotics

\[
m^2_{|\alpha|} \varphi(\alpha \cdot x) \rightarrow M^2_{|\alpha|} \left[ e^{-2\omega_2(\delta l(\alpha) - \delta) - \alpha \cdot X} + e^{-2\omega_2(1 - \delta l(\alpha) - \delta) + \alpha \cdot X} \right]
\] (2.14)

Depending upon the range of values for \( \delta \), this limit produces the ordinary Toda or the affine Toda system. We shall analyze these limits separately.

**Limit to the ordinary Toda system**

Since \( \delta > 0 \), the limit of the first term in (2.14) is zero for all positive roots \( \alpha \) for which \( l(\alpha) \geq 2 \). Thus, only the contributions of the simple roots \( \alpha_i, i = 1, \ldots, n \) of \( G \) survive. The limit of the second term in (2.14) vanishes for all positive roots \( \alpha \) for which \( l(\alpha) < 1/\delta - 1 \). Now, for all positive roots of \( R(G) \) to obey this inequality, it suffices that the highest root of \( G \) satisfy the inequality. But, the level of the highest root of \( G \) is related to the Coxeter number \( h_G \) of \( G \) by \( h_G = 1 + l_0 \), so that the above inequality becomes

\[
\delta < \frac{1}{h_G}.
\] (2.15)

Thus, whenever \( \delta \) satisfies (2.15), the second term on the r.h.s. of (2.14) will converge to zero for all roots of \( G \). Putting all together, for any roots, we have

\[
m^2_{|\alpha|} \varphi(\alpha \cdot x) \rightarrow \begin{cases} M^2_{|\alpha|} e^{\mp \alpha \cdot X}, & l(\alpha) = \pm 1 \\ 0, & \text{otherwise.} \end{cases}
\] (2.16)

The limit of the Hamiltonian \( H_{CM} \) for the Lie algebra \( G \) thus yields the Hamiltonian \( H_T \) of the ordinary Toda system for \( G \), as indeed announced in Theorem 1 (b).

**Limit to the affine Toda system**

From the above discussion, it is clear that the value

\[
\delta = \frac{1}{l_0 + 1} = \frac{1}{h_G},
\] (2.17)

corresponds to a critical case, for which the second term in (2.14) also survives the limit \( \text{Re}(\omega_2) \to \infty \). We have

\[
m^2_{|\alpha|} \varphi(\alpha \cdot x) \rightarrow M^2_{|\alpha|} \begin{cases} e^{\mp \alpha \cdot X}, & \text{if } l(\alpha) = \pm 1; \\ e^{\pm \alpha_0 \cdot X}, & \text{if } l(\alpha) = \pm l_0; \\ 0, & \text{otherwise.} \end{cases}
\] (2.18)
The limit of the Hamiltonian $H_{CM}$ for the Lie algebra $G$ then yields the Hamiltonian $H_T$ of the affine Toda system associated with the untwisted affine Lie algebra $G^{(1)}$, as announced in Theorem 1 (a). Here, $-\alpha_0$ plays the role of the affine simple root of $G^{(1)}$.

B. The Scaling Limit of the Lax Pair

The Lax operators $L$ and $M$ with spectral parameter $z$, for the (untwisted) Calogero-Moser systems associated with an arbitrary simple finite dimensional Lie algebra $G$ were constructed in [1]. The Lax operators are obtained starting from an $N$-dimensional representation of $G$, with weights $\{\lambda_I\}_{I=1,\ldots,N}$, which embeds $G$ into $GL(N, \mathbb{C})$ and are given as follows

$$L = P + X, \quad P = \sum_{i=1}^{n} p_i h_i,$$

$$M = D + Y, \quad D = \sum_{i=1}^{n} d_i h_i + \sum_{j=n+1}^{N} d_j \tilde{h}_j + \Delta.$$  \hspace{1cm} (2.19)

Here, $h_i, i = 1, \ldots, n$ generate the Cartan subalgebra $H_G$ of $G$, $\tilde{h}_j, j = n + 1, \ldots, N$ generate the orthogonal complement to $H_G$ in the Cartan algebra of $GL(N, \mathbb{C})$, and $\Delta$ belongs to the centralizer of $H_G$ in $GL(N, \mathbb{C})$, so that $[D, P] = 0$. Finally, $X$ and $Y$ are given by

$$X = \sum_{I,J=1; I \neq J}^{N} C_{I,J} \Phi(\alpha_{IJ} \cdot x, z) E_{IJ},$$

$$Y = \sum_{I,J=1; I \neq J}^{N} C_{I,J} \Phi'(\alpha_{IJ} \cdot x, z) E_{IJ}. \hspace{1cm} (2.20)$$

The combination $\alpha_{IJ} \equiv \lambda_I - \lambda_J$ is the weight under $G$ associated with the root $u_I - u_J$ of $GL(N, \mathbb{C})$, $C_{I,J}$ are constants, $\Phi'(x, z)$ is the $x$-derivatives of $\Phi(x, z)$, an elliptic function that will be defined below. The analysis of [1] implies that the coefficients $C_{I,J}$ vanish unless $\alpha_{IJ}$ is a root of $G$, in which case they are proportional to $m_{|\alpha|}$, and scale in the same way as $m_{|\alpha|}$ in (2.4),

$$C_{I,J} = \begin{cases} M_{|\alpha|} e^{\delta \omega_2} c_{I,J} & \text{when } \alpha_{IJ} = \alpha \in \mathcal{R}(G) \\ 0 & \text{when } \alpha_{IJ} \notin \mathcal{R}(G). \end{cases} \hspace{1cm} (2.21)$$

Here, the coefficients $c_{I,J}$ are purely group theoretical and were obtained in [1].

To construct a finite limit of the Lax pair $L, M$, we need to make the spectral parameter $z$ be dependent on $\omega_2$ as well. This is no problem, since the Lax operators reproduce
the Calogero-Moser system for all values of $z$. The scaling limit indicated in Theorem 1

$$e^z = Ze^{-\omega_2} \quad (2.22)$$

where $Z$ is held fixed, is the limit which generalizes the discussion for the special case of the algebra $A_n$ treated in [3]. Since the Lax pair $L, M$ has been expressed entirely in terms of $m|_{\alpha|} \Phi(\alpha \cdot x, z)$ and $m|_{\alpha|} \Phi'(\alpha \cdot x, z)$, the evaluation of their limits reduces to the evaluation of the limits of $m|_{\alpha|} \Phi(\alpha \cdot x, z)$ and $m|_{\alpha|} \Phi'(\alpha \cdot x, z)$. The definition of $\Phi(u, z)$ in terms of $\sigma(z)$ and its expression in terms of $\vartheta^*$-functions, are given by [15]

$$\Phi(u, z) = \frac{\sigma(z - u)}{\sigma(z) \sigma(u)} e^{u\zeta(z)} = \frac{\vartheta^*_1(z - u|\tau)}{\vartheta^*_1(z|\tau) \vartheta^*_1(u|\tau)} e^{u\partial_z \log \vartheta^*_1(z|\tau)}. \quad (2.23)$$

To evaluate the asymptotic behavior of this function, we use the product representation of (2.1). For $z$ satisfying (2.22), the right hand side of (2.1) can be replaced by $2 \sinh \frac{z}{2}$. Also in the limit of interest to us, $u$ in (2.23) is replaced by $\alpha \cdot x = \alpha \cdot X + 2\omega_2 \delta l(\alpha)$, with $|\delta l(\alpha)| \leq \delta l_0 \leq 1 - \delta$. Thus a similar approximation is valid for $\vartheta^*_1(u|\tau)$, and we have

$$\Phi(u, z) = e^{\frac{1}{2} u \coth \frac{z}{2}} \frac{\vartheta^*_1(z - u|\tau)}{4 \sinh \frac{z}{2} \sinh \frac{u}{2}}. \quad (2.24)$$

Combining the scaling limits of $x$ and $z$, we have

$$z - u = z - \alpha \cdot x = -\alpha \cdot X - \log Z - \omega_2(1 + 2\delta l(\alpha)).$$

The coefficient $1 + 2\delta l(\alpha)$ obeys $-1 < 1 + 2\delta l(\alpha) < 3$. Within this range, it suffices to retain the following asymptotic behavior of $\vartheta^*_1(z - u|\tau)$ for our purposes,

$$\vartheta^*_1(z - u|\tau) \to 2 \sinh \frac{z - u}{2} (1 - e^{-2\omega_2 - z + u}),$$

which results in the following asymptotic behavior for $\Phi(u, z)$

$$\Phi(u, z) \to \begin{cases} +e^{-\frac{1}{2} u}(1 - Z^{-1}e^{u - \omega_2}) & \text{Re}(u) \to +\infty \\ -e^{\frac{1}{2} u}(1 - Ze^{-u - \omega_2}) & \text{Re}(u) \to -\infty \end{cases} \quad (2.25)$$

As the function $\Phi(x, z)$ is not symmetric under $x \to -x$, we treat the cases of positive and negative roots separately.

**Positive Roots**

The asymptotics of (2.25) depends upon whether $1 + 2\delta l(\alpha)$ exceeds the critical value 2, resulting in three possible limiting behaviors.
(a) When $2\delta l(\alpha) < 1$, the second term in (2.25) converges to 0. Substituting in the limiting behavior (1.15) for $m_{|\alpha|}$, we obtain

$$C_{I,J} \Phi(\alpha \cdot x, z) \rightarrow M_{|\alpha|} c_{I,J} e^{-\frac{1}{2} \alpha \cdot X} e^{\delta \omega_2 (1-l(\alpha))}.$$ 

The only non-zero contributions in the limit $\omega_2 \rightarrow \infty$ arise for simple roots $\alpha$,

$$C_{I,J} \Phi(\alpha \cdot x, z) \rightarrow \begin{cases} 
M_{|\alpha|} c_{I,J} e^{-\frac{1}{2} \alpha \cdot X}, & \text{if } l(\alpha) = 1; \\
0, & \text{otherwise}.
\end{cases} \quad (2.26)$$

(b) When $2\delta l(\alpha) > 1$, only the second term in (2.25) survives and we find

$$C_{I,J} \Phi(\alpha \cdot x, z) \rightarrow -M_{|\alpha|} c_{I,J} e^{\frac{1}{2} \alpha \cdot X} e^{\omega_2 (\delta l(\alpha)+\delta-1)} Z^{-1}.$$ 

Two cases arise: for $\delta < 1/h_G$, the above quantity vanishes for all roots $\alpha$. For $\delta = (l_0 + 1)^{-1} = 1/h_G$, the right hand side will vanish for all roots, except for the highest root $\alpha_0$. Thus, it follows immediately that in this case

$$C_{I,J} \Phi(\alpha \cdot x, z) \rightarrow \begin{cases} 
-M_{|\alpha|} c_{I,J} Z^{-1} e^{\frac{1}{2} \alpha_0 \cdot X}, & \text{if } l(\alpha) = l(\alpha_0) \\
0, & \text{otherwise}.
\end{cases} \quad (2.27)$$

(c) When $2\delta l(\alpha) = 1$, both terms in (2.25) have the same asymptotic behavior as $\text{Re}(\omega_2) \rightarrow \infty$, which is proportional to $\exp\{\omega_2 (\delta - \frac{1}{2})\}$. For $\delta < \frac{1}{2}$, or equivalently $l_0 > 1$, this factor, and thus $m_{|\alpha|} \Phi(\alpha \cdot x, z)$ tends to 0. For $\delta = \frac{1}{2}$, or equivalently $l_0 = 1$ and $h_G = 2$, the simple Lie algebra must be $G = A_1 = B_1 = C_1$. The only positive root $\alpha$ (which is the highest root), yields

$$C_{I,J} \Phi(\alpha \cdot x, z) \rightarrow M_{|\alpha|} c_{I,J} (e^{-\frac{1}{2} \alpha \cdot X} - Z^{-1} e^{\frac{1}{2} \alpha \cdot X}). \quad (2.28)$$

**Negative Roots**

When $l(\alpha) < 0$, the second term in (2.25) is always negligible compared to the first. The limit of $C_{I,J} \Phi(\alpha \cdot x, z)$ then rather depends on whether $z - \alpha \cdot x$ tends to $-\infty$, $+\infty$, or remains finite. This corresponds to the three cases $2\delta l(\alpha) > -1$, $2\delta l(\alpha) < -1$, $2\delta l(\alpha) = -1$, which we examine in turn.

(a) When $2\delta l(\alpha) > -1$, the limit of $C_{I,J} \Phi(\alpha \cdot x, z)$ is given by

$$C_{I,J} \Phi(\alpha \cdot x, z) \rightarrow -M_{|\alpha|} c_{I,J} e^{\omega_2 (\delta + \delta l(\alpha))} e^{\frac{1}{2} \alpha \cdot X},$$
which admits a non-vanishing limit only when \( l(\alpha) = -1 \):

\[
C_{I,J} \Phi(\alpha \cdot x, z) \to \begin{cases} 
-M_{|\alpha|} c_{I,J} e^{\frac{1}{2} \alpha \cdot X}, & \text{if } l(\alpha) = -1 \\
0, & \text{otherwise.}
\end{cases} (2.29)
\]

(b) When \( 2\delta l(\alpha) < -1 \), we have the following limiting behavior

\[
C_{I,J} \Phi(\alpha \cdot x, z) = M_{|\alpha|} Z_{I,J} e^{\omega_2(\delta - \delta l(\alpha) - 1)} e^{-\frac{1}{2} \alpha \cdot X},
\]

which admits a non-vanishing limit only when \( \delta = 1/h_\mathcal{G} \) and \( l(\alpha) = -l_0 \):

\[
C_{I,J} \Phi(\alpha \cdot x, z) \to \begin{cases} 
M_{|\alpha|} Z_{I,J} e^{-\frac{1}{2} \alpha \cdot X}, & \text{if } l(\alpha) = -l_0 \text{ and } \delta = 1/h_\mathcal{G} \\
0, & \text{otherwise.}
\end{cases} (2.30)
\]

(c) When \( 2\delta l(\alpha) = -1 \), \( C_{I,J} \Phi(\alpha \cdot x, z) \) scales as \( \exp\{\omega_2(\delta - \frac{1}{2})\} \), which tends to 0 unless \( \delta = \frac{1}{2}, l_0 = 1, h_\mathcal{G} = 2 \) and thus \( \mathcal{G} = A_1, B_1, C_1 \). In this case, the only root \( \alpha \) yields

\[
C_{I,J} \Phi(\alpha \cdot x, z) = M_{|\alpha|} c_{I,J} (Ze^{-\frac{1}{2} \alpha \cdot X} - e^{\frac{1}{2} \alpha \cdot X}). (2.31)
\]

In summary, for Lie algebras with \( h_\mathcal{G} \geq 3 \), we have found that

\[
C_{I,J} \Phi(\alpha \cdot x, z) \to \begin{cases} 
\pm M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha \cdot X}, & \text{if } l(\alpha) = \pm 1; \\
\mp M_{|\alpha|} c_{I,J} e^{\pm \frac{1}{2} \alpha \cdot X} Z^{\pm 1}, & \text{if } l(\alpha) = \pm l_0 \text{ and } \delta = 1/h_\mathcal{G}; \\
0, & \text{otherwise.}
\end{cases} (2.32)
\]

The case \( h_\mathcal{G} = 2 \) for \( \mathcal{G} = A_1, B_1, C_1 \) may be read off from (2.28) and (2.31).

We turn now to the limit of \( C_{I,J} \Phi'(\alpha \cdot x, z) \). Replacing \( C_{I,J} \Phi(u, z) \) by its approximation (2.24), we may write

\[
C_{I,J} \Phi'(u, z) = C_{I,J} \Phi(u, z) \left[ \frac{1}{2} \coth \frac{z}{2} + \partial_u \log \vartheta_1^*(z - u|\tau) - \frac{1}{2} \coth \frac{u}{2} \right] (2.33)
\]

Thus we need only determine the limit of \( \partial_u \log \vartheta_1^*(z - u|\tau) \). It is readily seen that

\[
\partial_u \log \vartheta_1^*(z - u|\tau) \to \begin{cases} 
\pm \frac{1}{2}, & \delta |l(\alpha)| < \frac{1}{2} \\
\mp \frac{1}{2}, & \delta |l(\alpha)| = \frac{1}{2} \\
-\frac{1}{2}, & \delta l(\alpha) > \frac{1}{2} \\
\frac{1}{2}, & \delta l(\alpha) < -\frac{1}{2}.
\end{cases} (2.34)
\]

Putting all together, we arrive at

\[
\lim C_{I,J} \Phi'(\alpha \cdot x, z) = \frac{1}{2} \epsilon_{\alpha} \lim C_{I,J} \Phi(\alpha \cdot x, z),
\]

\[
\epsilon_{\alpha} = \begin{cases} 
+1 & l(\alpha) = +l_0 \text{ or } l(\alpha) = -1, \\
-1 & l(\alpha) = -l_0 \text{ or } l(\alpha) = +1,
\end{cases} (2.35)
\]

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While the derivation was carried out for $\delta |l(\alpha)| \neq \frac{1}{2}$ using (2.34), the final result (2.35) holds also for this case.

It remains to discuss the limit of the operator $D$ in (2.19). Its detailed structure was given in [1]. Here, the only information we shall need about it is that all contributions to $D$ are of the form $m_{|\alpha|}\varphi(\alpha \cdot x)$ for some set of roots $\alpha$. We note that in the expressions for the entries of $D$ derived in [2], constants independent of $\alpha \cdot x$ may be dropped, because the equations for the entries of $D$ involve only differences in $\varphi$. Thus we may ignore constants such as $\frac{h_1}{\pi i}$ in (2.3), just as in the case of limits of Hamiltonians. Then the key observation is that the coefficients $C_{I,J}$ in the Lax pair are proportional to a single power of the Calogero-Moser coupling constants $m_{|\alpha|}$, while in the Hamiltonian the analogous coefficients occur with the power 2. Now, we have already shown that in the Hamiltonian each of these contributions admits a finite limit with power 2. As $m_{|\alpha|} \to \infty$ in all cases, we see right away that $D \to 0$ in the limit.

Combining the results summarized in (2.29) with (2.32), and using the above result that $D \to 0$, we recover precisely the Lax pair with spectral parameter for the affine Toda system when $\delta = 1/h\mathcal{G}$, and the Lax pair for the ordinary Toda system when $\delta < 1/h\mathcal{G}$. The explicit forms may be derived by combining (2.19), (2.20), (2.21), (2.32) and (2.35) and we find

$$L_T = \sum_{i=1}^{n} P_i h_i + \sum_{\alpha \in \mathcal{R}_+(\mathcal{G})} M_{|\alpha|} e^{-\frac{i}{2} \alpha \cdot X} (E_\alpha - E_{-\alpha}) + M_{|\alpha_0|} e^{+\frac{i}{2} \alpha_0 \cdot X} (-Z^{-1} E_{\alpha_0} + Z E_{-\alpha_0})$$

$$M_T = -\frac{1}{2} \sum_{\alpha \in \mathcal{R}_+(\mathcal{G})} M_{|\alpha|} e^{-\frac{i}{2} \alpha \cdot X} (E_\alpha + E_{-\alpha}) + \frac{1}{2} M_{|\alpha_0|} e^{+\frac{i}{2} \alpha_0 \cdot X} (Z^{-1} E_{\alpha_0} + Z E_{-\alpha_0}).$$

(2.36)

with the following conventions. The summation is over the set $\mathcal{R}_+(\mathcal{G})$ of simple roots of $\mathcal{G}$. When $\delta = 1/h\mathcal{G}$, and $M_{|\alpha_0|} \neq 0$, we have the affine Toda system associated with the untwisted affine Lie algebra $\mathcal{G}^{(1)}$, where $-\alpha_0$ plays the role of the extra affine root. When $0 < \delta < 1/h\mathcal{G}$, and $M_{|\alpha_0|} = 0$, we have the ordinary Toda system associated with the finite-dimensional Lie algebra $\mathcal{G}$. The matrices $E_{\alpha}$ are expressed in terms of the constants $c_{I,J}$ of (2.21), and the generators $E_{I,J}$ of $GL(N,\mathbb{C})$, $I, J = 1, \ldots, N$ by

$$E_{\alpha} = \sum_{I \neq J; \alpha_{IJ} = \alpha} c_{I,J} E_{I,J}.$$  

(2.37)

The Lax equation $\dot{L}_T = [L_T, M_T]$ is equivalent to the Hamilton-Jacobi equations for the Toda Hamiltonian of (1.10).
III. TWISTED CALOGERO-MOSER AND AFFINE TODA SYSTEMS

We have established previously that the root system of each simple Lie algebra $G$ defines an elliptic Calogero-Moser system, with Hamiltonian $H_{CM}$, given in (1.11). In the limit where the Calogero-Moser coupling $m$ tends to $\infty$, the system tends to an affine Toda system associated with the untwisted affine Lie algebra $G^{(1)}$. The coupling $m$ and the modulus $\tau$ of (1.1) then scale according to

$$m = Me^{\omega_2 \delta} \quad \delta = 1/h_G,$$

(3.1)

where $h_G$ is the Coxeter number of $G$.

However, if $m$ is to correspond to the mass of a hypermultiplet in the adjoint representation for an $\mathcal{N} = 2$ supersymmetric $G$ gauge theory, then considerations based on the renormalization group behavior, on $R$-symmetry and on instanton calculus [16-19] require that the hypermultiplet decouple rather according to (1.9), or equivalently, according to the following scaling rule

$$m = Me^{\omega_2 \delta'} \quad \delta' = 1/h'_G,$$

(3.2)

where $h'_G$ is the dual Coxeter number. For simply-laced algebras (i.e. when all roots have equal length, c.f. Table 2 in [1]), we have $h_G = h'_G$. However, for non-simply laced $G$ (i.e. when $G$ has roots of unequal length), we have instead $h'_G < h_G$ and thus $\delta' > \delta$. In this case the elliptic Calogero-Moser systems (1.11) do not scale to a finite limit. Thus, the untwisted Calogero-Moser systems are not expected to be the correct integrable systems associated with $\mathcal{N} = 2$ supersymmetric Yang-Mills theories with adjoint hypermultiplet when the gauge group $G$ is non-simply laced.

This situation led us to introduce new, so-called twisted Calogero-Moser systems in [1], which are associated with non-simply laced $G$, and whose Hamiltonians are given by (1.12). We shall show in this section that these twisted Calogero-Moser systems associated with non-simply laced Lie algebras $G$ scale to a finite limit under (3.2). Furthermore, their limits are affine Toda systems associated with the affine Lie algebras $(G^{(1)})'$, that is, the dual of the untwisted affine Lie algebra $G^{(1)}$. We begin by briefly reviewing the key features of our construction. For more details, see [1].

- In the twisted Calogero-Moser Hamiltonians the short roots of $G$ are twisted by replacing $\wp(\alpha \cdot x)$ with $\wp_\nu(\alpha \cdot x)$, and where $\nu$ equals the ratio of the length of the long to the short roots. Clearly, we only need the values $\nu = 1, 2, 3$. The functions $\wp_\nu(u)$ are defined in (1.13).
The scaling law for the dynamical variables $x$ (which was previously $x = X + 2\omega_2\delta^\vee \rho$) is to be replaced by $x = \xi(X + 2\omega_2\delta^\vee \rho)$, where $\rho$ is now the Weyl vector of $G$, $\delta^\vee$ is a scaling exponent and $\xi$ is a normalization dependent parameter. When all long roots $\alpha$ are normalized so that $\alpha^2 = 2$, we have $\xi = 1$ for all Lie algebras, as indicated in Theorem 2. However, it is convenient to normalize the long roots $\alpha$ of $C_n$ to $\alpha^2 = 4$. As a result, the normalization we shall use leads to $\xi = 1$ for $G = B_n, F_4, G_2$ and $\xi = \frac{1}{2}$ for $G = C_n$.

Let $R_s(G)$ and $R_l(G)$ denote respectively the set of short roots and the set of long roots of $G$. Our choice of normalization is given in Table 2 of [1]. Note that in all cases except $G = C_n$, the long roots $\alpha$ are normalized to have $\alpha^2 = 2$, while for $C_n$, they have $\alpha^2 = 4$. The twisted elliptic Calogero-Moser Hamiltonian associated to $G$, and defined in (1.12) is then given by

$$H_{TCM} = \frac{1}{2}p \cdot p - \frac{1}{2} \sum_{\alpha \in R_s(G)} m_s^2 \varphi_{\nu}(\alpha \cdot x) - \frac{1}{2} \sum_{\alpha \in R_l(G)} m_l^2 \varphi(\alpha \cdot x)$$

(3.3)

We shall determine the limit of $H_{TCM}$ under the scaling rule (3.2), but we shall also allow for the scaling exponent $\delta^\vee < 1/h^\vee$, for the sake of completeness.

It is very useful to introduce the dual level function $l^\vee(\alpha)$, defined by

$$l^\vee(\alpha) \equiv \alpha \cdot \rho.$$  

(3.4)

This function is relevant here because the new scaling law for $x$ mentioned above naturally appeals to the dual level with $\alpha \cdot x \sim \alpha \cdot X + 2\omega_2\delta^\vee l^\vee(\alpha)$. For a systematic exposition, see [1]. In terms of the decompositions of a root $\alpha$ and its coroot $\alpha^\vee = 2\alpha/\alpha^2$ onto simple roots and co-roots with integer coefficients $l_i$ and $l_i^\vee$ (c.f. §I), we have

$$\alpha = \sum_{i=1}^{n} l_i \alpha_i, \quad \alpha^\vee = \sum_{i=1}^{n} l_i^\vee \alpha_i^\vee, \quad l_i^\vee = \frac{\alpha_i^2}{\alpha^2} l_i, \quad l^\vee(\alpha) = \sum_{i=1}^{n} \frac{\alpha_i^2}{2} l_i.$$  

(3.5)

It is suggestive to consider $l^\vee$ as a function of the coroots $\alpha^\vee$ rather than of the roots $\alpha$

$$l^\vee(\alpha^\vee) = \frac{2}{\alpha^2} l^\vee(\alpha) = \sum_{i=1}^{n} l_i^\vee.$$  

Then $l^\vee(\alpha^\vee)$ satisfies the following properties which are analogous to the properties of $l(\alpha)$ required in the evaluation of the limit of the untwisted Calogero-Moser system:

(1) The minimal value $l^\vee(\alpha^\vee) = 1$ on positive coroots is attained if and only if $\alpha$ is a simple root.
(2) The maximal value of $l^\vee(\alpha^\vee)$ on positive coroots is $l^\vee(\alpha^\vee) = h_G^\vee - 1 = l_0^\vee$, and it is attained if and only if $\alpha = \alpha_0$ is the highest root of $G$.

(3) The highest root $\alpha_0$ is always long, while its coroot $\alpha_0^\vee$ is always short.

By reasoning that parallels the discussion in §II.A, one argues that the convergence of the Hamiltonian under the scaling limit of (1.16) forces $x$ to be shifted by a function linear in $\omega_2$ and the Weyl vector for \textit{maximally symmetric limits}. The additional constraints from requiring that the $n \neq 0$ terms converge may then be simply satisfied by requiring that $0 < \delta^\vee \leq \delta^\vee l^\vee(\alpha^\vee) \leq 1 - \delta^\vee$ for all roots $\alpha$ of $G$. This condition is equivalent to $\delta^\vee \leq 1/h_G^\vee$. Assuming this condition, it is useful to recast (1) and (2) above as

$$\delta^\vee \leq \delta^\vee l^\vee(\alpha^\vee) \leq 1 - \delta^\vee,$$

with equality on the left if and only if $\alpha$ is a simple root, and equality on the right if and only if $\delta = 1/h_G^\vee$, and $\alpha$ is the longest root $\alpha_0$.

The limits of the Hamiltonian $H_{TCM}$ and of the Lax operators (to be analyzed in §IV) of the twisted Calogero-Moser systems will be taken according to

$$e^z = Z^e e^{-\omega_2}$$

$$m_{|\alpha|} = M_{|\alpha|} e^{\omega_2 \delta^\vee}$$

$$x = \xi (X + 2\omega_2 \delta^\vee \rho)$$

$$\alpha \cdot x = \xi (\alpha \cdot X + 2\omega_2 \delta^\vee l^\vee(\alpha))$$

where $Z, M_{|\alpha|}$ and $X$ are kept fixed, and the dual level $l^\vee(\alpha)$ of a root $\alpha$ was defined in (3.4). The factor $\xi$ is defined by $\xi = 1$ for $G = B_n, F_4, G_2$ and $\xi = \frac{1}{2}$ for $G = C_n$. It is necessary because only for $C_n$ is the normalization of the long roots $\alpha^2 = 4$ instead of 2. We shall discuss in detail only the case of positive roots.

In the subsections below, we shall establish that for the twisted Calogero-Moser systems associated with any (non-simply laced) Lie algebra $G$, the potential terms in the Hamiltonian $H_{TCM}$ converge to the following limits,

$$m_{|\alpha|}^2 |\varphi_\nu(\alpha \cdot x) \to M_{|\alpha|}^2 \begin{cases} e^{\mp \alpha^\vee \cdot X}, & \text{if } l^\vee(\alpha^\vee) = \pm 1; \\ e^{\pm \alpha_0^\vee \cdot X}, & \text{if } l^\vee(\alpha^\vee) = \pm l_0^\vee \text{ and } \delta^\vee = 1/h_G^\vee; \\ 0, & \text{otherwise}. \end{cases}$$

This means that, in the limit (3.7), $H_{TCM}$ converges to the Hamiltonian of a Toda system associated to a Lie algebra for which the simple roots are the coroots of $G$, augmented by the negative of the coroot $\alpha_0^\vee$ of the highest root $\alpha_0$ of $G$. The coroot $-\alpha_0^\vee$ plays the role
of the affine root for the dual affine Lie algebra \( (G^{(1)})^\vee \). Thus, proving (3.8) will indeed prove that the twisted Calogero-Moser Hamiltonian \( H_{T CM} \) for the finite dimensional Lie algebra \( G \) converges to the Toda Hamiltonian \( H_T \) for either the affine Lie algebra \( (G^{(1)})^\vee \) when \( \delta^\vee = 1/h_\vee^G \) or for the finite dimensional Lie algebra \( G^\vee \) when \( \delta^\vee < 1/h_\vee^G \), establishing Theorem 2 (a) and (b) for the Hamiltonian.

Although the arguments are essentially the same for all non-simply laced simple algebras, it is convenient to discuss separately the case of \( G = B_n, F_4 \), the case of \( G = G_2 \), and the case of \( G = C_n \), since their values of \( \nu \) and their normalizations differ.

(a) Twisted Elliptic Calogero-Moser for \( G = B_n, F_4 \)

These cases are characterized by the fact that the ratio of length of the roots is 2, and that the long roots \( \alpha \) of \( G \) are normalized to \( \alpha^2 = 2 \), so that \( \xi = 1 \) in (3.7).

We begin by analyzing the contributions of the (positive) long roots. Applying the asymptotics for \( \wp(z) \) in terms of hyperbolic functions established in (2.3), we find as in (2.12)

\[
m_l^2 \varphi(\alpha \cdot x) \to M_l^2 \left[ \sum_{n \leq \delta^\vee l^\vee(\alpha)} e^{-2\omega_2(-n+\delta^\vee l^\vee(\alpha) - \delta^\vee - \alpha \cdot X)} + \sum_{n > \delta^\vee l^\vee(\alpha)} e^{-2\omega_2(+n-\delta^\vee l^\vee(\alpha) - \delta^\vee + \alpha \cdot X)} \right].
\]

Since the long roots are normalized to \( \alpha^2 = 2 \), they satisfy \( \alpha = \alpha^\vee \), and \( l^\vee(\alpha) = l^\vee(\alpha^\vee) \). For reasons that will become completely clear when we deal with the short roots, we prefer to recast all expressions below in terms of coroots. Now we are assuming that \( \delta^\vee \leq 1/h_\vee^G \). Thus we have \( \delta^\vee \leq \delta^\vee l^\vee(\alpha^\vee) = \delta^\vee l^\vee(\alpha) \leq 1 - \delta^\vee \), and the above limit further reduces to

\[
m_l^2 \varphi(\alpha \cdot x) \to M_l^2 \left[ e^{-2\omega_2(\delta^\vee l^\vee(\alpha^\vee) - \delta^\vee) - \alpha \cdot X} + e^{-2\omega_2(1-\delta^\vee l^\vee(\alpha^\vee) - \delta^\vee + \alpha \cdot X)} \right].
\]

Clearly, the limit of the right hand side is then always finite, and non-zero only when either \( l^\vee(\alpha^\vee) = 1 \) or \( l^\vee(\alpha^\vee) + 1 = h_\vee^G \). This is the case exactly when \( \alpha \) is simple, or when \( \alpha \) is the highest root \( \alpha_0 \) of \( G \). We thus recover (3.8) for long positive roots.

We turn next to the short roots of \( G \). In view of (2.3), the function \( \varphi_2(u) \) can be expressed as (dropping irrelevant additive constants)

\[
\varphi_2(u) = 2 \sum_{n=-\infty}^{\infty} \frac{1}{\cosh(2u - 4n\omega_2) - 1} \quad (3.9)
\]
The condition $\delta^\vee \leq \delta^\vee l^\vee(\alpha^\vee) \leq 1 - \delta^\vee$, on short roots, for which $\alpha^\vee = 2\alpha$, becomes $\delta^\vee \leq 2\delta^\vee l^\vee(\alpha) \leq 1 - \delta^\vee$. The leading terms in $m_s^2 \wp_2(\alpha \cdot x)$ are thus given by

$$m_s^2 \wp_2(\alpha \cdot x) \to 4M_s^2 \left[ e^{-2\omega_2(2\delta^\vee l^\vee(\alpha)-\delta^\vee)} - 2\alpha \cdot X + e^{-2\omega_2(2-2\delta^\vee l^\vee(\alpha)-\delta^\vee)} + 2\alpha \cdot X \right]$$  \hfill (3.10)

The first term has a non-zero limit if and only if $2l^\vee(\alpha) = 1$, that is, if $l^\vee(\alpha^\vee) = 1$ and $\alpha$ is a simple short root. The exponent in the second term involves $2 - 2\delta^\vee l^\vee(\alpha) - \delta^\vee = 2 - \delta^\vee l^\vee(\alpha^\vee) - \delta^\vee \geq 1$ for all short roots of $G$, so that the second term always tends to zero in the limit (3.7). Recasting the final expression in terms of coroots, we see that the novel factors of 2 in (3.10) get nicely absorbed into the definition of coroots $\alpha^\vee = 2\alpha$ of short roots $\alpha$. Putting all together, we find that for all roots of $G$, we have formula (3.8).

From inspection of the limit of (3.9) when $\delta^\vee = 0$, it follows immediately that the limit gives then the trigonometric Calogero-Moser system for the dual finite dimensional Lie algebra $G^\vee$, thus establishing (c) of Theorem 2, for the Hamiltonian.

(b) Twisted Elliptic Calogero-Moser for $G = G_2$

The arguments for the long roots of $G_2$ are identical to those for the long roots of the twisted $F_4$ and $C_n$ cases. Since the long roots of $G_2$ have $\alpha^2 = 2$ with our normalization, they equal their coroot, and the limits may be expressed as in (3.8) as well.

Next, we concentrate on the short roots of $G_2$, and make use of the following expansion for $\wp_3(u)$, which is also an easy consequence of (2.3) (dropping irrelevant additive constants)

$$\wp_3(u) = \frac{9}{2} \sum_{n=\pm \infty}^\infty \frac{1}{\cosh(3u - 6n\omega_2) - 1} \hfill (3.11)$$

For short roots $\alpha$ of $G_2$, we have now $\alpha^\vee = 3\alpha$, and the condition $\delta^\vee \leq \delta^\vee l^\vee(\alpha^\vee) \leq 1 - \delta^\vee$ becomes $\delta^\vee \leq 3\delta^\vee l^\vee(\alpha) \leq 1 - \delta^\vee$. The leading terms in $m_s^2 \wp_3(\alpha \cdot x)$ can be written as

$$m_s^2 \wp_3(\alpha \cdot x) \to \frac{9}{4} M_s^2 \left[ e^{-2\omega_2(3\delta^\vee l^\vee(\alpha)-\delta^\vee)} - 3\alpha \cdot X + e^{-2\omega_2(3-3\delta^\vee l^\vee(\alpha)-\delta^\vee)} + 3\alpha \cdot X \right]. \hfill (3.12)$$

These lead to non-vanishing limits only when $3l^\vee(\alpha) = l^\vee(\alpha^\vee) = 1$, which means that $\alpha$ is a simple short root. Since $3 - 3\delta^\vee l^\vee(\alpha) - \delta^\vee = 3 - \delta^\vee l^\vee(\alpha^\vee) - \delta^\vee \geq 2$, the second term in (3.12) always converges to 0. Thus only simple short roots survive from the sum over all short roots in the Hamiltonian, and we again recover the result of (3.8).

From the asymptotics of (3.11) in the limit (3.6), it is clear that this limit gives the trigonometric Calogero-Moser system for $G_2^\vee = G_2$, when $\delta^\vee = 0$, establishing (c) of Theorem 2, for the Hamiltonian.
(c) Twisted Elliptic Calogero-Moser for $\mathcal{G} = C_n$

The only difference separating this case from the earlier ones is a difference of convention. Since our choice of longest roots for $C_n$ is $2e_i$, $1 \leq i \leq n$, the coroot of a long root $\alpha$ obeys $\alpha^\vee = \frac{1}{2} \delta$, while the coroot of a short root $\alpha$ obeys $\alpha^\vee = \alpha$. Thus with the Weyl vector $\rho$ still defined by the same formula $\rho = \sum_{i=1}^{n} \lambda_i$, where $\lambda_i$ are the fundamental weights, the choice of scaling of the dynamical variables $x$ is now as in (3.7) with $\xi = \frac{1}{2}$,

$$x = \frac{1}{2} (X + 2\omega_2 \delta^\vee l^\vee(\alpha)).$$

(3.13)

First, we consider the contributions of the long roots $\alpha$ of $C_n$, for which $\alpha^\vee = \frac{1}{2} \delta$. The condition $\delta^\vee \leq \delta^\vee l^\vee(\alpha^\vee) \leq 1 - \delta^\vee$ then becomes $\delta^\vee \leq \frac{1}{2} \delta^\vee l^\vee(\alpha^\vee) \leq 1 - \delta^\vee$. The expansion of $m_s^2 \varphi(\alpha \cdot x)$ is given by

$$m_s^2 \varphi(\alpha \cdot x) \rightarrow M_s^2 \left[ \sum_{\frac{1}{2} \delta^\vee l^\vee(\alpha) > n} e^{-2\omega_2(-\delta^\vee + \frac{1}{2} \delta^\vee l^\vee(\alpha) - n - \frac{1}{2} \alpha \cdot X)} + \sum_{n > \frac{1}{2} \delta^\vee l^\vee(\alpha)} e^{-2\omega_2(-\delta^\vee - \frac{1}{2} \delta^\vee l^\vee(\alpha) + n + \frac{1}{2} \alpha \cdot X)} \right],$$

(3.14)

and reduces to the following asymptotics

$$m_s^2 \varphi(\alpha \cdot x) \rightarrow M_s^2 \left[ e^{-2\omega_2(\delta^\vee \frac{1}{2} l^\vee(\alpha) - \delta^\vee) - \frac{1}{2} \alpha \cdot X} + e^{-2\omega_2(1 - \delta^\vee \frac{1}{2} l^\vee(\alpha) - \delta^\vee) + \frac{1}{2} \alpha \cdot X} \right].$$

(3.15)

Here, we have re-expressed the right hand side of the last line in terms of coroots. These two terms produce a non-vanishing limit respectively when $\alpha^\vee$ is a simple coroot with $l^\vee(\alpha^\vee) = 1$, so that $\alpha$ is a simple root, and when $\alpha^\vee$ is the highest coroot $\alpha_0^\vee$, characterized by $l^\vee(\alpha_0^\vee) = l_0^\vee = h_\mathcal{G}^\vee - 1$. We thus recover, for long roots of $C_n$, the result announced in (3.8).

Next, we consider the contribution from short roots. Using (3.9) and (3.13), the combination $m_s^2 \varphi_2(\alpha \cdot x)$ is given by

$$m_s^2 \varphi_2(\alpha \cdot x) = \sum_{n=-\infty}^{\infty} \frac{2M_s^2 e^{2\omega_2 \delta^\vee}}{\cosh(\alpha \cdot X + 2\omega_2 \delta^\vee l^\vee(\alpha) - 4n\omega_2) - 1}$$

(3.16)

and the leading terms are

$$m_s^2 \varphi_2(\alpha \cdot x) \rightarrow 4M_s^2 \left[ e^{-2\omega_2(-\delta^\vee + \delta^\vee l^\vee(\alpha)) - \alpha \cdot X} + e^{-2\omega_2(-\delta^\vee - \delta^\vee l^\vee(\alpha) + 2 + \alpha \cdot X)} \right].$$

(3.17)
The first term on the right hand side produces a non-vanishing limit exactly when \( l^\vee(\alpha) = 1 \), which means in this case that \( \alpha^\vee = \alpha \) is a simple (co)root. The second term does not contribute in the limit, since \( 2 - \delta^\vee - \delta^\vee l^\vee(\alpha) \geq 1 \). Expressing the full answer in terms of coroots, we again recover (3.8).

From the asymptotics of (3.11), it is clear that the limit (3.6) gives the trigonometric Calogero-Moser system for \( C_n^\vee = B_n \), when \( \delta^\vee = 0 \), thus establishing (c) of Theorem 2, for the Hamiltonian.

IV. LIMITS OF LAX PAIRS

To complete the proof of Theorem 2, we shall establish in this section the limits of the Lax pairs according to the scaling limit (3.7). The Lax pairs for the twisted Calogero-Moser systems were constructed explicitly in [1], and are of the form (2.19), but with a more general form for \( X \) and \( Y \)

\[
X = \sum_{I,J=1; I \neq J}^N C_{I,J} \Phi_{I,J}(\alpha_{IJ} \cdot x, z) E_{IJ}
\]

\[
Y = \sum_{I,J=1; I \neq J}^N C_{I,J} \Phi'_{I,J}(\alpha_{IJ} \cdot x, z) E_{IJ}.
\]

All other notations and conventions are as in §IIB. Explicit expressions for the constants \( C_{I,J} \) and for the elliptic functions \( \Phi_{I,J} \) were constructed in [1]. The data needed about these for the limits will be given in the subsections below. The main complication of the twisted cases is that there are now several different elliptic functions \( \Phi_{I,J} \), whose limits will have to be studied.

One general result is worth deriving right away. Just as in the case of the untwisted Calogero-Moser systems, the matrix \( D \) entering the Lax operator \( M \) in (2.19) is a sum of terms proportional to \( m_{|\alpha|} \wp(\alpha \cdot x) \). This combination is similar to the terms that enter the Calogero-Moser Hamiltonians, except that the power of \( m_{|\alpha|} \) is 1 instead of 2. As \( m_{|\alpha|} \to \infty \) in (3.7), it immediately follows that

\[
D \to 0 \quad \Delta, \ d_j \to 0, \ j = 1, \cdots, N,
\]

so that the Lax operator \( M \) reduces to \( Y \). Henceforth, we shall restrict to the study of the \( X \) and \( Y \) parts of the Calogero-Moser Lax operators.
In the subsections below, we shall establish the following limits of the entries $X$ and $Y$ of the Lax operators, as the constants $C_{I,J}$ scale with $\omega_2$ according to their expressions derived in [1] in terms of $m_{|\alpha|}$:

$$C_{I,J} = M_{|\alpha|} e^{\delta^\vee \omega_2} C_{I,J}$$  \hfill (4.3)

(Recall that $\alpha = \lambda_I - \lambda_J$). For the Lie algebras $B_n$ and $F_4$, we shall show that the entries of $X$ satisfy

$$C_{I,J} \Phi_I, J(\alpha \cdot x, z) \rightarrow \begin{cases} \pm \kappa_G M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha^\vee \cdot X}, & \text{if } l^\vee (\alpha^\vee) = \pm 1; \\
\mp \kappa_G M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha_0^\vee \cdot X} Z^\mp 1, & \text{if } l^\vee (\alpha^\vee) = \pm l_0^\vee \text{ and } \delta^\vee = 1/h^\vee_G; \\
0 & \text{otherwise}, \end{cases}$$  \hfill (4.4a)

where $\kappa_G$ are constants depending on the algebra $G$, with $\kappa_{B_n} = 1$ and $\kappa_{F_4} = 2$. In the cases of $B_n$ and $F_4$, the entries of the matrix $Y$ scale in analogy with the untwisted case

$$C_{I,J} \Phi_I, J(\alpha \cdot x, z) \rightarrow \begin{cases} -\frac{1}{2} \kappa_G M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha^\vee \cdot X}, & \text{if } l^\vee (\alpha^\vee) = \pm 1; \\
-\frac{1}{2} \kappa_G M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha_0^\vee \cdot X} Z^\mp 1, & \text{if } l^\vee (\alpha^\vee) = \pm l_0^\vee \text{ and } \delta^\vee = 1/h^\vee_G; \\
0 & \text{otherwise}. \end{cases}$$  \hfill (4.5a)

The case of $C_n$ differs from the other cases only in minor details. More precisely, the matrix $X$ scales in this case according to

$$C_{I,J} \Phi_I, J(\alpha \cdot x, z) \rightarrow \begin{cases} \pm 2 M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha^\vee \cdot X}, & \text{if } l^\vee (\alpha^\vee) = \pm 1; \\
\mp 2 M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha_0^\vee \cdot X} Z^\mp \frac{1}{2} \pm \frac{1}{2}, & \text{if } l^\vee (\alpha^\vee) = \pm l_0^\vee, \delta^\vee = 1/h^\vee_G, I < J; \\
\mp 2 M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha_0^\vee \cdot X} Z^\mp \frac{1}{2} \pm \frac{1}{2}, & \text{if } l^\vee (\alpha^\vee) = \pm l_0^\vee, \delta^\vee = 1/h^\vee_G, J < I; \\
0 & \text{otherwise}, \end{cases}$$  \hfill (4.4b)

while the matrix $Y$ scales as

$$C_{I,J} \Phi_I, J(\alpha \cdot x, z) \rightarrow \begin{cases} -2 M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha^\vee \cdot X}, & \text{if } l^\vee (\alpha^\vee) = \pm 1; \\
-2 M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha_0^\vee \cdot X} Z^\mp \frac{1}{2} \pm \frac{1}{2}, & \text{if } l^\vee (\alpha^\vee) = \pm l_0^\vee, \delta^\vee = 1/h^\vee_G, I < J; \\
-2 M_{|\alpha|} c_{I,J} e^{\mp \frac{1}{2} \alpha_0^\vee \cdot X} Z^\mp \frac{1}{2} \pm \frac{1}{2}, & \text{if } l^\vee (\alpha^\vee) = \pm l_0^\vee, \delta^\vee = 1/h^\vee_G, J < I; \\
0 & \text{otherwise}. \end{cases}$$  \hfill (4.5b)

The full expression for the Lax operators may be worked out, just as we did in (2.36) for the untwisted cases. In fact, the result is completely analogous, except that the summation is over the set $R_\ast (G^\vee)$ of simple coroots of $G$ and over the coroot $\alpha_0^\vee$ instead of over the root $\alpha_0$. When $\delta^\vee = 1/h^\vee_G$, and $M_{|\alpha_0^\vee|} \neq 0$, we have the affine Toda system associated with the dual affine Lie algebra $(G^{(1)})^\vee$, where $-\alpha_0^\vee$ plays the role of the extra affine root. When $0 < \delta^\vee < 1/h^\vee_G$, and $M_{|\alpha_0^\vee|} = 0$, we have the ordinary Toda system associated with
the finite-dimensional Lie algebra $G^\vee$. The matrices $E_{\alpha}^\vee$ are expressed in terms of the constants $c_{I,J}$ of (4.1) and (4.3) and the generators $E_{I,J}$ of $GL(N, \mathbb{C})$, $I, J = 1, \cdots, N$ just as in (2.37).

In order to derive (4.4) and (4.5), it is convenient to proceed separately for each of the non-simply laced finite-dimensional Lie algebras $G = B_n, C_n, F_4$ and $G_2$, since the structure of the Lax operators of the associated Calogero-Moser system is quite different in each case.

(a) The Limit of the Twisted $B_n$ Calogero-Moser Lax Pair

This case is relatively the simplest among the twisted Calogero-Moser cases, as its only new feature is the appearance of a new function $\Lambda(2u, z)$. The twisted Calogero-Moser Hamiltonian for $B_n$ admits a Lax pair of dimension $N = 2n$, with spectral parameter $z$ and two independent couplings $m_s$ and $m_l$, given by (2.19), (4.1) and

\[
\Phi_{IJ}(x, z) = \begin{cases} 
\Phi(x, z) & I - J \neq 0, \pm n \\
\Lambda(x, z) & I - J = \pm n
\end{cases} \quad (4.6a)
\]

\[
C_{I,J} = \begin{cases} 
m_l & I - J \neq 0, \pm n \\
m_s & I - J = \pm n
\end{cases} \quad (4.6b)
\]

The function $\Lambda(2u, z)$ is defined by

\[
\Lambda(2u, z) = \frac{\Phi(u, z)\Phi(u + \omega_1, z)}{\Phi(\omega_1, z)}. \quad (4.7)
\]

and was studied in detail in Appendix §B of [1]. We observe that the prescription (4.6) implies in particular that the long roots $\alpha = \pm e_i \pm e_j$ of $B_n$ occur only in entries of the form $\Phi_{IJ}(\alpha \cdot x) = \Phi(\alpha \cdot x)$, while the short roots $\alpha = e_i$ of $B_n$ emerge from $I - J = \pm n$, and appear only in entries of the form $\Phi_{IJ}(\alpha \cdot x) = \Lambda(2\alpha \cdot x)$. With the help of this observation and of the limits already evaluated in §IIB of this paper, it is easy to determine the limits of the Lax pair.

For the long roots, we use the asymptotics of $\Phi(u, z)$ in (2.25). Since $u = 2\omega_2 \delta^\vee l^\vee(\alpha) + \alpha \cdot X$, we need to consider three cases, according to whether $u - \omega_2$ tends to $-\infty, 0$ or $+\infty$ respectively.

(a) $2\delta^\vee l^\vee(\alpha) < 1$. In this case, as $C_{I,J}$ scales according to (4.3), the right hand side of (2.25) reduces to the first term and we have

\[
C_{I,J} \Phi(\alpha \cdot x, z) \to M_{[\alpha]} c_{I,J} e^{\omega_2 \delta^\vee (1 - l^\vee(\alpha)) - \frac{1}{2} \alpha \cdot X}. \quad (4.8)
\]
The limit (4.8) is non-zero exactly when $l^\vee(\alpha) = l^\vee(\alpha^\vee) = 1$, which means that $\alpha$ is a simple long root of $B_n$.

(b) $2\delta^\vee l^\vee(\alpha) = 1$. In this case both terms on the r.h.s. of (2.25) contribute in the limit and we have

$$C_{IJ} \Phi(\alpha \cdot x, z) \to M_{|\alpha|} c_{IJ} e^{\omega_2(\delta^\vee - \frac{1}{2}) - \frac{1}{2} \alpha \cdot X} (1 - e^{\alpha \cdot X} Z^{-1})$$

(4.9)

A non-zero limit requires that $\delta^\vee = \frac{1}{2}$ which cannot happen for $B_n$, since $\delta^\vee = 1/(2n - 1)$.

(c) $2\delta^\vee l^\vee(\alpha) > 1$. In this case, the second term on the r.h.s. in (2.25) dominates the asymptotics of $C_{IJ} \Phi(\alpha \cdot x, z)$, producing

$$C_{IJ} \Phi(\alpha \cdot x, z) \to - M_{|\alpha|} c_{IJ} e^{\omega_2(-1 + \delta^\vee + \delta^\vee l^\vee(\alpha))} e^{\frac{1}{2} \alpha \cdot X} Z^{-1}.$$  (4.10)

A non-vanishing limit arises when $\delta^\vee + \delta^\vee l^\vee(\alpha) = 1$, which means that $\alpha = \alpha_0$ is the highest root (which is a long root).

Now, in all three cases above, we are dealing with the long roots $\alpha$ of $B_n$, which are normalized so that $\alpha^2 = 2$ and thus $\alpha^\vee = \alpha$. Recasting the results of (4.8), (4.9) and (4.10) in terms of coroots, we readily recover the result of (4.4) for long roots of $B_n$.

We turn next to the short roots of $B_n$, which are of the form $\alpha = e_i$. The asymptotics of $\Phi(\omega_1, z)$ follows directly from (2.24) and we have $\Phi(\omega_1, z) \to \frac{1}{2} i$. Combining this result with the expressions for the asymptotics of $\Phi$ in (2.25) and the definition of $\Lambda(2u, z)$ in (4.7), we find

$$\Lambda(2u, z) \to \begin{cases} +2e^{-u}(1 - Z^{-2}e^{2u - 2\omega_2}) & \text{Re}(u) \to +\infty \\ -2e^{+u}(1 - Z^2e^{-2u - 2\omega_2}) & \text{Re}(u) \to -\infty \end{cases}$$

(4.11)

We still have the scaling law $u = \alpha \cdot x = \alpha \cdot X + 2\omega_2 \delta^\vee l^\vee(\alpha)$ of (3.7) and, assuming that $\alpha$ is a positive root, the asymptotics produces three cases according to whether $u - \omega_2$ tends to $-\infty, 0$ or $+\infty$ respectively.

(a) $2\delta^\vee l^\vee(\alpha) < 1$. In this case, the second term on the right hand side of (4.11) converges to 0, and we are left with the contribution of only the first term,

$$C_{IJ} \Lambda(2\alpha \cdot x, z) \to 2M_{|\alpha|} c_{IJ} e^{\omega_2(\delta^\vee - 2\delta^\vee l^\vee(\alpha))} e^{-\alpha \cdot X}.$$  (4.12)

This has a non-zero limit exactly when $2l^\vee(\alpha) = 1$, that is, when $\alpha$ is a simple short root.

(b) $2\delta^\vee l^\vee(\alpha) = 1$. As before, it is easily seen that although both terms in (4.11) contribute to the limit, a non-vanishing limit arises only when $\delta^\vee = 1$. This can only occur for the special case $B_1$. 

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(c) $2\delta^\vee l^\vee(\alpha) > 1$. In this case, the second term in (4.11) dominates the asymptotics and we have

$$C_{IJ}\Lambda(2\alpha \cdot x) \to -2M_{\alpha|c_{IJ}}e^{\omega_2(-2+\delta^\vee+4\delta^\vee l^\vee(\alpha))}e^{\alpha \cdot X}Z^{-2} \quad (4.13)$$

This always tends to 0, since $-2+\delta^\vee+4\delta^\vee l^\vee(\alpha^\vee) \geq \delta^\vee > 0$.

The coroots of short roots $\alpha$ of $B_n$ obey $\alpha^\vee = 2\alpha$. Recasting the results obtained in (4.12) and (4.13) in terms of coroots, we readily recover (4.4). Finally, (4.5) in the case of $B_n$ is derived using the derivative expressions of (2.35) for $\Phi$ and the analogous expression for $\Lambda$, or simpler still, of the asymptotic expansions (2.25) and (4.11).

(b) The Limit of the Twisted $C_n$ Calogero-Moser Lax Pair

The twisted Calogero-Moser Hamiltonian for $C_n$ admits a Lax pair of dimension $N = 2n + 2$, with spectral parameter and one independent couplings $m$ given by (2.19), (4.1) and

$$\Phi_{IJ}(\alpha_{IJ} \cdot x, z) = \Phi_2(\alpha_{IJ} \cdot x + \omega_{IJ}, z) \quad (4.14a)$$

$$C_{I,J} = \begin{cases} 
\frac{m}{\sqrt{2m}} & I, J = 1, \ldots, 2n; I - J \neq \pm n \\
\frac{1}{2m} & I = 2n + 1, J = 2n + 2; I \leftrightarrow J
\end{cases} \quad (4.14b)$$

The constants $\omega_{IJ}$ obey cocycle conditions, and are defined only up to shifts $\zeta_I$ resulting from shifts in the vector $x$. Both are given by

$$\omega_{II} = -\omega_{IJ}$$

$$\omega_{IJ} + \omega_{JK} + \omega_{KI} = 0$$

$$\omega_{IJ} \rightarrow \omega_{IJ} + \zeta_I - \zeta_J \quad (4.15)$$

The fact that the Lax equations for this Lax pair must reproduce the twisted Calogero-Moser Hamilton-Jacobi equations requires that $\omega_{IJ}$ take values amongst the half periods $-\omega_2, 0, +\omega_2$, up to shifts $\zeta_I$. A convenient solution is given by

$$\omega_{IJ} = \begin{cases} 
0 & I \neq J = 1, \ldots, 2n + 1 \\
+\omega_2 & I = 1, \ldots, 2n; J = 2n + 2 \\
-\omega_2 & J = 1, \ldots, 2n; I = 2n + 2
\end{cases} \quad (4.16)$$

Special care is needed in properly defining the normalizations of the roots $\alpha_{IJ}$. We have

$$\alpha_{IJ} = \lambda_I - \lambda_J \quad I, J = 1, \ldots, 2n + 2$$

$$\lambda_i = -\lambda_{n+i} = e_i, \quad i = 1, \ldots, n, \quad \lambda_{2n+1} = \lambda_{2n+2} = 0. \quad (4.17)$$
Thus, for \( I, J = 1, \ldots, n \), the entries \( \alpha_{IJ} \) yield the short roots of \( C_n \), while when either \( I \) or \( J \), but not both, equals \( 2n + 1 \) or \( 2n + 2 \), the entries \( \alpha_{IJ} \) yield half of the long roots of \( C_n \). The function \( \Phi_2 \) is the function \( \Lambda \) of (4.7), but for double the argument, defined by

\[
\Phi_2(u, z) = \Lambda(2u, z).
\] (4.18)

The limit is taken according to (3.7).

\[
x = \frac{1}{2}(X + 2\omega_2 \delta^\vee \rho), \quad e^z = Z^\frac{1}{2} e^{-\omega_2}
\] (4.19)

where \( \rho \) is the Weyl vector. Notice the extra factors of \( \frac{1}{2} \) related to the non-canonical normalization of the long roots of \( C_n \). The asymptotics of \( \Phi_2 \) follows directly from those of \( \Lambda \) in (4.11), but this time for the scaling limit of (4.19),

\[
\Phi_2(u, z) \to \begin{cases} 
+2e^{-u} (1 - Z^{-1} e^{2u - 2\omega_2}) & \text{Re}(u) \to +\infty \\
-2e^{+u} (1 - Ze^{-2u - 2\omega_2}) & \text{Re}(u) \to -\infty.
\end{cases} \] (4.20)

We shall assume in the subsequent discussion that the roots \( \alpha \) are positive. The case of negative roots can be treated by similar arguments. We consider first the entries related to the positive short roots of \( C_n \), given by \( \alpha = \alpha_{IJ} = \pm e_i \pm e_j, 1 \leq i \leq n \). These arise only when both indices \( I, J \) satisfy \( 1 \leq I, J \leq n \), in which case \( \omega_{IJ} = 0 \), and \( \Phi_{IJ}(\alpha \cdot x, z) = \Phi_2(\alpha \cdot x, z) \). Using the asymptotics of (4.20), we find

\[
C_{I,J} \Phi_2(\alpha \cdot x, z) \to 2M_{|\alpha|} c_{I,J} e^{-\omega_2 (\delta^\vee I^\vee(\alpha) - \delta^\vee) - \frac{1}{2} \alpha \cdot X} (1 - Z^{-1} e^{\alpha \cdot X + 2\omega_2 (\delta^\vee I^\vee(\alpha) - 1)})
\] (4.21)

For short roots \( \alpha \) of \( C_n \), we have \( \alpha^\vee = \alpha \), and thus \( \delta^\vee \leq \delta^\vee I^\vee(\alpha) \leq 1 - \delta^\vee \). As a result, the factor in parentheses in (4.21) converges to 1. The remaining factors tend to zero unless \( I^\vee(\alpha) = 1 \), i.e. \( \alpha \) is a simple short root. Recasting the result in terms of coroots, we recover (4.4) for the short roots of \( C_n \). Similarly, we find

\[
C_{I,J} \Phi'_{IJ}(\alpha \cdot x, z) \to -2M_{|\alpha|} c_{I,J} e^{-\frac{1}{2} \alpha \cdot X},
\]

as written earlier in (4.5).

Next we consider the entries related to the positive long roots \( 2e_i \) of \( C_n \). These arise either under the form \( \Phi_2(\frac{1}{2} \alpha \cdot x, z) \) (when either \( I \) or \( J \) is between 1 and \( n \), and the other index is \( n + 1 \)), or under the form \( \Phi_2(\frac{1}{2} \alpha \cdot x \pm \omega_2, z) \) (when one of the indices \( I \) or \( J \) is between 1 and \( n \), and the other index is \( n + 2 \)).
In the first case, we have $\Phi_2(u, z)$, with $u$ given by

$$u = \frac{1}{2} \alpha \cdot x = \frac{1}{4} \alpha \cdot X + \frac{1}{2} \omega_2 \delta^\vee l^\vee(\alpha).$$ \hfill (4.22)

As $\omega_2 \to \infty$, $u$ satisfies $u \to +\infty$ and $u - \omega_2 \to -\infty$, since $\delta^\vee l^\vee(\alpha) < 2$, so that only the first term on the r.h.s. in (4.20) remains in the limit, and we obtain

$$C_{I,J} \Phi_2(\frac{1}{2} \alpha \cdot x, z) \to 2M_{[\alpha]} c_{I,J} e^{\omega_2(\delta^\vee - \frac{1}{2} \delta^\vee l^\vee(\alpha))} e^{-\frac{1}{2} \alpha \cdot X}$$

$$- 2M_{[\alpha]} c_{I,J} e^{\omega_2(\delta^\vee - \delta^\vee l^\vee(\alpha^\vee))} e^{-\frac{1}{2} \alpha \cdot X}. \hfill (4.23)$$

Here, we have re-expressed the limit in terms of coroots on the second line. The limit is non-zero if and only if $\alpha$ is a simple long root for $C_n$, and we thus recover the result of (4.4). The limit for $C_{I,J} \Phi'_I, J(\frac{1}{2} \alpha \cdot x, z)$ in (4.5) follows then easily from the asymptotics $\Phi(u, z) \sim 2e^{-u}$ which apply in this case.

Next we consider the case $\Phi_2(u, z)$ with $u = \frac{1}{2} \alpha \cdot x + \omega_2$. As $\omega_2 \to \infty$, we now have $u \to +\infty$, but $u - \omega_2 \to +\infty$ as well. Thus, the limit is dominated by the second term on the r.h.s. of (4.20), and we find

$$C_{I,J} \Phi_2(\frac{1}{2} \alpha \cdot x + \omega_2, z) \to - 2M_{[\alpha]} c_{I,J} e^{\omega_2(\delta^\vee + \frac{1}{2} \delta^\vee l^\vee(\alpha^\vee - 1)) + \frac{1}{2} \alpha \cdot X Z^{-1}}$$

$$- 2M_{[\alpha]} c_{I,J} e^{\omega_2(\delta^\vee + \delta^\vee l^\vee(\alpha^\vee - 1)) + \frac{1}{2} \alpha \cdot X Z^{-1}}. \hfill (4.24)$$

This has a non-vanishing limit only when $\delta^\vee + \delta^\vee l^\vee(\alpha^\vee) - 1 = 0$, i.e., when $\alpha$ is the highest root. In this case, the longest root is $2e_1$, and it does indeed occur amongst these roots $\alpha$. Again, we recover the results of (4.4), and just as easily, of (4.5) for $C_{I,J} \Phi_2(\frac{1}{2} \alpha \cdot x + \omega_2, z)$.

Finally, to evaluate the limit of $C_{I,J} \Phi_2(\frac{1}{2} \alpha \cdot x - \omega_2, z)$, it is easiest to make use of the monodromy properties of $\Phi_2(u, z)$

$$\Phi_2(u - \omega_2, z) = \Phi_2(u + \omega_2, z) e^{-4(\omega_2 \zeta(z) - \eta_2 \omega_1)}$$

This relation becomes particularly simple when we set $\omega_1 = -i \pi$, take the limit $\omega_2 \to \infty$, and use the relation $\eta_1 \omega_2 - \eta_2 \omega_1 = \frac{1}{2} i \pi$

$$\Phi_2(u - \omega_2, z) = \Phi_2(u + \omega_2, z) Z \hfill (4.25)$$

Substituting in the limits found for $\Phi_1(\frac{1}{2} \alpha \cdot x + \omega_2, z)$ in (4.24), we conclude that the only non-vanishing limit occurs again only for the highest root $2e_1$, with the values indicated in (4.4). The derivative terms can be evaluated in the same way, leading to (4.5), and our treatment of the $C_n$ case is complete.
(c) The Limit of the Twisted $F_4$ Calogero-Moser Lax Pair

The twisted Calogero-Moser Hamiltonian for $F_4$ admits a Lax pair of dimension $N = 24$, with spectral parameter $z$ and two independent couplings $m_s$ and $m_l$, whose form is given by

$$\Phi_{\lambda \mu}(x, z) = \begin{cases} \Phi(x, z) & \lambda \cdot \mu = 0 \\ \Phi_1(x, z) & \lambda \cdot \mu = \frac{1}{2} \\ \Lambda(x, z) & \lambda \cdot \mu = -1 \end{cases}$$

(4.26a)

$$C_{\lambda, \mu} = \begin{cases} m_l & \lambda \cdot \mu = 0 \\ \frac{1}{\sqrt{2}} m_s & \lambda \cdot \mu = \frac{1}{2} \\ 0 & \lambda \cdot \mu = -\frac{1}{2} \\ \sqrt{2} m_s & \lambda \cdot \mu = -1 \end{cases}$$

(4.26b)

Here, the entries are labeled by the 24 non-zero weights $\lambda$ of the 26 of $F_4$, which are also the 24 short roots of $F_4$. The functions $\Lambda$ and $\Phi_1$ are defined respectively by (4.7) and

$$\Phi_1(u, z) = \Phi(u, z) + f(z)\Phi(u + \omega_1, z)$$

$$f(z) = -e^{\pi i c(z) + \eta_1 z},$$

so that we have simple monodromy with period $\omega_1$, given by $\Phi_1(u + \omega_1, z) = f(z)^{-1}\Phi_1(u, z)$.

The above classification of $\Phi_{\lambda \mu}(u, z)$ depending on the values of $\lambda \cdot \mu$ leads to the three possible ways in which roots of $F_4$ can arise in the Lax pair: when $\lambda \cdot \mu = 0$, $\lambda - \mu$ is a long root, of the form $\alpha = \pm e_i \pm e_j$; when $\lambda \cdot \mu = \frac{1}{2}$, $\lambda - \mu$ is a short root; when $\lambda \cdot \mu = -1$, $\lambda - \mu = 2\alpha$, where $\alpha$ is again any of the short roots. We also evaluate the limits separately in the three cases.

For the long roots $\lambda \cdot \mu = 0$, the discussion is identical to that of the case $B_n$ in \S IV (a). We conclude that the limit of $C_{\lambda \mu} \Phi_{\lambda \mu}(\alpha \cdot x)$ is non-zero only when $\alpha$ is either a simple (long) root or the highest root. Long roots $\alpha$ of $F_4$ satisfy $\alpha^\vee = \alpha$, so that

$$C_{\lambda \mu} \Phi_{\lambda \mu}(\alpha \cdot x, z) \rightarrow \begin{cases} +2M_{[\alpha]} c_{\lambda \mu} e^{\frac{1}{2} \alpha^\vee \cdot X}, & \text{if } l^\vee(\alpha^\vee) = 1; \\ -2M_{[\alpha]} c_{\lambda \mu} e^{\frac{1}{2} \alpha^\vee \cdot X Z^{-1}}, & \text{if } l^\vee(\alpha^\vee) + 1 = h_0^\vee, \end{cases}$$

reproducing (4.4) for long roots.

Next, we evaluate the limit of $C_{\lambda \mu} \Phi_{\lambda \mu}((\lambda - \mu) \cdot x, z)$ when $\lambda \cdot \mu = -1$, that is, when $\mu = -\lambda$. Denoting by $\alpha = \lambda$ the corresponding short root of $F_4$, the entry in the Lax pair is given by the function $\Lambda(2u, z)$ for $u = \alpha \cdot x = 4\alpha^\vee \cdot X + \omega_2 \delta^\vee l^\vee(\alpha^\vee)$. Its asymptotics is read off directly from (4.11). Since $\delta^\vee l^\vee(\alpha^\vee) \geq 1 - \delta^\vee$, we have $u \rightarrow +\infty$ and $u - \omega_2 \rightarrow -\infty$, and so only the first term on the r.h.s. of (4.11) contributes,

$$C_{\lambda \mu} \Phi_{\lambda \mu}(2\alpha \cdot x, z) \rightarrow 2M_{[\alpha]} c_{\lambda, -\lambda} e^{\omega_2 (\delta^\vee - \delta^\vee l^\vee(\alpha^\vee)) - \frac{1}{2} \alpha^\vee \cdot X}.$$
In the limit, only contributions from simple roots $\alpha$ satisfying $l^\vee(\alpha^\vee) = 1$ remain, and we recover (4.4).

It remains to discuss the contributions arising from short positive roots when $\lambda \cdot \mu = \frac{1}{2}$. They occur under the form $\Phi_1(u, z)$, for $u = \frac{1}{2} \alpha^\vee \cdot X + \omega_2 \delta^\vee l^\vee(\alpha^\vee)$. The limit of the function requires some care, since the leading behavior as $\omega_2 \to \infty$ cancels between the two terms in the definition of (4.27),

$$
\Phi_1(u, z) \to \mp 2Z^\mp 1 e^{\pm \frac{1}{2} u - \omega_2} \quad u \to \pm \infty.
$$

(4.30)

Since asymptotically, we have $u \sim \omega_2 \delta^\vee l^\vee(\alpha^\vee)$, we see that this contribution always converges to 0 since we always have $\frac{1}{2} \omega_2 \delta^\vee l^\vee(\alpha^\vee) < 1$. (Strictly speaking, we should also check that the terms discarded when we approximated $\theta_1^+(\frac{z}{2\omega_1})$ and $\theta_1^+(\frac{u}{2\omega_1})$ by $2 \sinh(\frac{z}{2})$ and $2 \sinh(\frac{u}{2})$ do not contribute. But this is also easily done.) Thus, the short roots $\alpha = \lambda - \mu$ with $\lambda \cdot \mu = \frac{1}{2}$ do not survive in the Toda Lax pair, completing the proof of (4.4) for this case. Again, the evaluation of derivatives leading to (4.5) is a mere routine, and the proof of Theorem 2 is complete.

ACKNOWLEDGEMENTS

We would like to thank the Institute for Theoretical Physics in Santa Barbara for its hospitality during the January 1998 workshop on Geometry and Duality, where part of this work was done.

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