DYNAMICS OF SEMIGROUPS OF HÉNON MAPS

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Abstract. The goal of this article is two fold. Firstly, we explore the dynamics of a semigroup of polynomial automorphisms of \( \mathbb{C}^2 \), generated by a finite collection of Hénon maps. In particular, we construct the positive and negative dynamical Green’s functions \( G^+ \) and the corresponding dynamical Green’s currents \( \mu^+ \) for a semigroup \( S \), generated by a collection \( G \). Using them, we show that the positive (or the negative) Julia set of the semigroup \( S \), i.e., \( J^+_S \) (or \( J^-_S \)) is equal to the closure of the union of individual positive (or negative) Julia sets of the maps, in the semigroup \( S \). Furthermore, we prove that \( \mu^+ \) is supported on the whole of \( J^+_S \) and is also the unique positive closed \((1,1)\)-current supported on \( J^+_S \), satisfying a semi-invariance relation that depends on the generating set \( G \).

Secondly, we study the dynamics of a non-autonomous sequence of Hénon maps, say \( \{h_k\} \), contained in the semigroup \( S \). Similarly, as above, here too, we construct the non-autonomous dynamical positive and negative Green’s function and the corresponding dynamical Green’s currents. Further, we use the properties of Green’s function to conclude that the non-autonomous attracting basin of any such sequence \( \{h_k\} \), sharing a common attracting fixed point, is biholomorphic to \( \mathbb{C}^2 \).

1. Introduction

In this article, we study the dynamics of a semigroup of polynomial automorphisms of \( \mathbb{C}^2 \), generated by finitely many Hénon maps. To explain the setup, let \( G \) be a given finite collection of automorphisms of \( \mathbb{C}^2 \). We will consider the semigroup \( S \) generated by the elements of \( G \) under the composition operation, which will be denoted as

\[
S = \langle G \rangle \text{ where } G = \{H_i : 1 \leq i \leq n_0\},
\]

and \( n_0 \geq 1 \), a positive integer. Furthermore, we assume the maps \( H_i \) are Hénon maps, i.e., for every \( 1 \leq i \leq n_0 \), there exists \( m_i \geq 1 \) such that

\[
H_i = H_{i_1}^1 \circ H_{i_2}^2 \circ \cdots \circ H_{i_{m_i}}^{m_i}
\]

and \( H_{i_j}(x, y) \) is a map of the form

\[
H(x, y) = (y, p(y) - ax),
\]

where \( a \neq 0 \) and \( p \) is a polynomial of degree at least 2, for every \( 1 \leq i \leq n_0 \) and \( 1 \leq j \leq m_i \).

Recall from [26], a map of the above form (1.3) was termed as generalised Hénon map, and as Hénon if \( p(y) = y^2 + c \), classically. However, the maps of form (1.2) are more general than (1.3) and the methods to study maps of form (1.3), mostly generalises for the entire class. Hence by the terminology Hénon map we will mean maps of the form (1.2). Our interest to study the dynamics of semigroups generated by Hénon maps, is for the following facts

- Firstly, a classical result of Friedland–Milnor [17] states that these maps are essentially the only class of polynomial automorphisms of \( \mathbb{C}^2 \), exhibiting interesting (iterative) dynamics and have been studied intensively. For instance, see [21], [4], [12] etc. Further, their dynamics are known to be connected to the dynamics of polynomial maps in \( \mathbb{C} \) (see [9]) and they also extend as (bi)rational maps of \( \mathbb{P}^2 \).

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• Secondly, the study of dynamics of arbitrary families, i.e., non-iterative families of holomorphic self-maps (endomorphisms) is important from the point of view of complex function theory. In particular, the (non-autonomous) basins of attraction — see Theorem 1.1 below for definition — of a sequence of automorphisms of \( C^k \), \( k \geq 2 \), with a common attracting fixed point has lead to the construction of pathological domains in \( C^k \) (see [28, 1, 15]).

• Also, it is conjectured — follows as a consequence of a conjecture, originally due to Bedford ([3, 16]) — that a non-autonomous basin of attraction of sequences of automorphisms of \( C^2 \), that vary within an infinite (or finite) collection sharing a common uniformly attracting fixed point should be biholomorphic to \( C^2 \). The same is true for autonomous (or iterative) basins of endomorphisms with an attracting fixed point (see [28, Theorem 9.1]).

To mention here in Section 7, we answer the above for a finite collection of Hénon maps by the methods developed in this article to study the semigroup \( S \). It is stated as

**Theorem 1.1.** Let \( S \) be as in (1.1), such that the generators, \( H_i \), \( 1 \leq i \leq n_0 \), are attracting on a neighbourhood of origin, i.e., there exist \( r > 0 \) and \( 0 < \alpha < 1 \) such that

\[
\|H_i(z)\| \leq \alpha \|z\| \quad \text{for every } z \in B(0;r).
\]

Then the non-autonomous basin of attraction at the origin of every sequence \( \{h_k\} \subset S \) defined as \( \Omega_{\{h_k\}} := \{z \in C^2 : h_k \circ h_{k-1} \circ \cdots \circ h_1(z) \to 0 \text{ as } k \to \infty \} \) is biholomorphic to \( C^2 \).

Further, we study a few particular cases of an infinite collection or parametrised families of Hénon maps sharing a common attracting fixed point with ‘uniform bounds’. In particular, the following example is obtained as a consequence of Corollary 7.4 (also see Example 7.5), in comparison to Theorem 1.4 and 1.10 in [15].

**Example 1.2.** Let \( H_k(x,y) = (a_k y, a_k x + y^2) \) where \( p \) is a polynomial of degree at least 2 and \( c < |a_k| < d \) for every \( k \geq 1 \), with \( 0 < c < d < 1 \). Then the basin of attraction of the sequence \( \{H_k\} \), i.e., \( \Omega_{\{H_k\}} \) (as defined in Theorem 1.1) is biholomorphic to \( C^2 \).

• Lastly, the study of dynamics of rational semigroups on \( P^1 \) is an interesting and widely studied area. This setup was introduced by Hinkkanen–Martin, in [20], motivated by their connection to the dynamics of Kleinian groups on the Riemann sphere, observed in [18].

Our primary goal in this article is, to explore the dynamics of a semigroup of Hénon maps both in \( P^2 \) and \( C^2 \), motivated by the study of dynamics of rational semigroups in \( P^1 \). In particular, we will attempt to connect between results from iterative dynamics of Hénon maps of the form (1.2) and semigroup dynamics of rational maps in \( P^1 \) to the current setup. Later, we will generalise a few results appropriately in the setup of non-autonomous families to obtain the aforementioned applications.

Let \( X \) be a complex manifold and \( S \) be an infinite family of holomorphic self-maps of \( X \). The **Fatou set** for the family \( S \) is the largest open set of \( X \) where the family \( S \) is normal, i.e.,

\[
\mathcal{F}_S = \{z \in X : \text{there exists a neighbourhood of } z \text{ where the family } S \text{ is normal}\}.
\]

The **Julia set** \( J_S \), is the complement of the Fatou set in \( X \).

As reported earlier, the setup considering \( X = P^1 \) and \( S \), a semigroup generated by more than one rational map of degree at least 2, was introduced in [20] and later on has been explored extensively. A major difficulty in this framework — as compared to the iterative dynamics — is neither the Julia set nor the Fatou set is completely invariant, in general.

It is a classical result of Brolin [8] that says - if \( S \) is the semigroup of iterates of a (single) polynomial map \( p \) of degree at least 2, the limiting distribution of points in the preimages of a generic point \( z \in P^1 \), corresponds to the equilibrium measure of the Julia set. Further, the potential associated with this measure, i.e., the Green’s function of the Julia set can be constructed via the dynamics of \( p \). The equidistribution of the iterated preimages of a
generic point \( z \in \mathbb{P}^1 \), i.e., the limiting distribution is independent of the (generic) \( z \), was established for the iterations of rational map of \( \mathbb{P}^1 \) by Lyubich in [25]. Boyd in [7], extended Lyubich’s method and constructed an equidistributed measure supported on the Julia set of a finitely generated semigroup of rational maps (of degree at least 2) in \( \mathbb{P}^1 \). For a finitely generated semigroup of polynomials of degree at least 2, Boyd’s measure is not, in general, the equilibrium measure of its Julia set. Recently, in [24] the latter measure is interpreted as an equilibrium measure in the presence of an external field, which is given by a generalisation of the Greens function — attributed as ‘dynamical Greens function’.

To note, equidistributed measures exist for dynamics of certain meromorphic correspondences on compact connected Kähler manifolds, of appropriate intermediate degree (see [10]). However, birational maps of \( \mathbb{P}^2 \) obtained from extension of Hénon maps do not belong to the above category. Also, for iterative families of a Hénon map the Julia set is captured via the equilibrium measure of its Julia set. Recently, in [24] the latter measure is interpreted as an equilibrium measure in the presence of an external field, which is given by a generalisation of the Greens function — attributed as ‘dynamical Greens function’.

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Let us first recall a few important properties of iterations of a Hénon map \( H \). The pluri-complex Green’s function (see [22] for the definition) associated to the Julia set of iterates of \( H \) can be constructed by generalising ideas from [7], [11], [24] — both for the semigroups \( S \) and \( \mathbb{P}^d \), the support of a unique positive closed (1,1)-current, obtained by the action of \( dd^c \)-operator on the pluri-complex Green’s function of the Julia set. Furthermore, it is an equidistributed current in \( \mathbb{C}^2 \), in the sense, that it can be recovered as a limit of appropriately weighted preimages of an algebraic variety in \( \mathbb{C}^2 \) — see [4, Theorem 4.7] or [12, Corollary 6.7].

To mention here, construction of currents for non-autonomous families of Hénon maps have been done on an appropriate bounded region containing the origin, in [11], via the dynamics. In particular, if \( d_H \) is the degree of the map \( H \) then

\[
G^\pm_H(z) = \lim_{k \to \infty} \frac{\log^+ \| H^{\pm k}(z) \|}{d_H^k} \quad \text{and} \quad \mu^\pm_H(z) = \frac{1}{2\pi} dd^c(\pi \Gamma^{\pm}(G^\pm_H)),
\]

where \( \log^+ x = \max\{\log x, 0\} \) for every \( x > 0 \) and \( \| \cdot \| \) be the supremum norm in \( \mathbb{C}^2 \). Also, \( \mu^\pm_H \) are the closed positive (1,1)-(equidistributed) currents as mentioned previously. Note that the above definition holds for any norm on \( \mathbb{C}^2 \), however for the sake of convenience we will use the notation \( \| \cdot \| \) to denote the supremum norm, throughout this article.

Now, let \( S \) be the semigroup as introduced in (1.1), i.e., \( S = \langle \mathcal{G} \rangle \) where \( \mathcal{G} = \{ H_i : 1 \leq i \leq n_0 \} \) and \( H_i \) are Hénon maps of the form (1.2), and of degree \( d_i \geq 2 \) for every \( 1 \leq i \leq n \). We first generalise a few definitions and observe some basic results regarding the semigroup \( S \) in Section 2. Particularly, we note that \( S \) might have more than one generating set, however it has a unique minimal generating set.

In Section 3 we generalise the construction of positive and negative Green’s functions, i.e., the functions \( G_{\mathcal{S}}^{\pm} \) noted above, in the setup of the semigroup \( S \). To do the same, we define the total degree of the semigroup \( S \) with respect to the generating set \( \mathcal{G} \) as \( D_{\mathcal{G}} = \sum_{i=1}^{n_0} d_i \), and consider the sequence of plurisubharmonic functions \( G_k^{\pm} \) on \( \mathbb{C}^2 \) defined as

\[
G_k^{\pm}(z) = \frac{1}{D_{\mathcal{G}}^k} \sum_{h \in \mathcal{G}_k} \log^+ \| h(z) \| \quad \text{and} \quad G_k^{-}(z) = \frac{1}{D_{\mathcal{G}}^k} \sum_{h \in \mathcal{G}_k} \log^+ \| h^{-1}(z) \|, \quad (1.5)
\]

where \( \mathcal{G}_k \) denote the elements of the semigroup \( S \) of length \( k \) with respect to the generating set \( \mathcal{G} \), i.e., \( \mathcal{G}_k = \{ H_{i_1} \circ \cdots \circ H_{i_k} : 1 \leq i_j \leq n_0 \text{ and } 1 \leq j \leq k \} \). We prove that the pointwise limits of the sequences \( \{ G_k^{\pm} \} \) constructed in (1.5) exist, which is stated as

\[
\} \}
\]
Corollary 1.4. The sequences \( \{G_k^{\pm}\} \) converge pointwise to plurisubharmonic, continuous functions \( G_q^{\pm} \) on \( \mathbb{C}^2 \), respectively.

Henceforth, the functions \( G_q^{\pm} \) will be referred as the dynamical positive (or negative) Green’s function associated to the semigroup \( S \) generated by the set \( \mathcal{I} = \{H_i : 1 \leq i \leq n_0\} \).

The need to specify the generating set \( \mathcal{I} \) is important as the semigroup \( S \) may admit multiple generating sets. Also, note that the functions \( G_q^{\pm} \) satisfy the following semi-invariance relation right by the construction (1.5) and Theorem 1.3.

Corollary 1.4. \( \sum_{i=1}^{n_0} G_q^{\pm} \circ H_i(z) = D_q. G_q^{\pm}(z) \) and \( \sum_{i=1}^{n_0} G_q^{\pm} \circ H_i^{-1}(z) = D_q. G_q^{\pm}(z) \).

Thus, as consequences of the proof of Theorem 1.3 we note that the functions \( G_q^{\pm} \) admit logarithmic growth on appropriate regions, and the strong filled positive and negative Julia sets of the semigroup \( S \) are pseudoconcave sets (see Corollary 3.4 and Remark 3.5).

Next, we analyse the Julia sets \( J^S_q^{\pm} \) and the properties the dynamical Green’s \((1, 1)\)-currents associated to the functions \( G_q^{\pm} \), defined as \( \mu_q^{\pm} = \frac{1}{\pi} dd^c G_q^{\pm} \). Consequently, in Section 4, we prove the analogue to Corollary 2.1 from [20] — an important fact from the dynamics of semigroups of the rational maps on \( \mathbb{P}^1 \) — via the supports of \( \mu_q^{\pm} \).

Theorem 1.5. The positive and negative Julia sets corresponding to the dynamics of the semigroup \( S \) is equal to the closure of the union of the (positive and negative) Julia sets of the elements of \( S \) respectively, i.e., \( J^S_q^{\pm} = \bigcup_{h \in S} J^h_q^{\pm} \) and \( J^S_q^{\mp} = \bigcup_{h \in S} J^h_q^{\mp} \).

Further, the positive and the negative dynamical Green’s currents \( \mu_q^{\pm} \) are \((1, 1)\)-closed positive currents of mass 1 supported (respectively) on the Julia sets, i.e., \( \text{Supp} (\mu_q^{\pm}) = J^S_q^{\pm} \).

Thus from Theorem 1.3 and the above, \( G_q^\pm \) is actually pluripolar on the Fatou sets \( F_q^{\pm} \).

In Section 5, we study the extension of the currents \( \mu_q^{\pm} \) to \( \mathbb{P}^2 \) and prove that they are limits of (weighted) equidistributed projective varieties, in the spirit of [12, Theorem 6.2]. Also, consequently we observe the following uniqueness of \( \mu_q^{\pm} \) up to a semi-invariance property.

Corollary 1.6. The current \( \mu_q^{\pm} \) is the unique current of mass 1 supported on \( J^S_q^{\pm} \) and the current \( \mu_q^{\mp} \) is the unique current of mass 1 supported on \( J^S_q^{\mp} \) satisfying the following semi-invariance relations (respectively)

\[
\frac{1}{D_q} \sum_{i=1}^{n_0} H_i^+(\mu_q^{\pm}) = \mu_q^{\pm} \quad \text{and} \quad \frac{1}{D_q} \sum_{i=1}^{n_0} H_i^-(\mu_q^{\mp}) = \mu_q^{\mp}.
\]  

In Section 6, we consider the dynamics of a non-autonomous sequence of Hénon maps, say \( \{h_k\} \in S \) and prove that there exist plurisubharmonic and continuous dynamical (positive and negative) Green’s functions, denoted by \( G_{\{h_k\}}^{\pm} \), with logarithmic growth. Thus \( \mu_{\{h_k\}}^{\pm} = \frac{1}{\pi} dd^c (G_{\{h_k\}}^{\pm}) \), are positive (1, 1)-currents of mass 1, supported on the positive and negative Julia sets of the sequence \( \{h_k\} \). Also, we obtain the analogs to the equidistribution results, i.e., Corollaries 5.8 and 5.10 in this setup of non-autonomous dynamics. However, we will discuss them briefly as most of the ideas are similar to that realised in Section 5 and, depends upon the existence of the dynamical Green’s function with suitable growth at infinity.

Finally, we study the non-autonomous attracting basin of a sequence of Hénon maps of the form (1.2), admitting a uniformly attracting behaviour, on a neighbourhood of the origin. Further, we prove Theorem 1.1 as an application of the existence of Green’s functions \( G_{\{h_k\}}^{\pm} \) and enlist a few more applications, which follows from the technique. All of these affirmatively answers a few particular cases of the equivalent formulation of the Bedford Conjecture, in \( \mathbb{C}^2 \) for Hénon maps — as alluded to in the beginning.
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2. SOME BASIC DEFINITIONS AND PRELIMINARIES

In this section, we first observe a proposition about the generating set $\mathcal{G}$ of the semigroup $\mathcal{S}$ as in (1.1), which might not be unique, always. Recall the setup from Section 1, let $\mathcal{G} = \{H_i : i \leq i \leq n_0\}$ where $H_i$’s are Hénon maps of the form (1.2), with degree $d_i \geq 2$. The total degree of the semigroup $\mathcal{S}$ with respect to the generating set $\mathcal{G}$ is $D_{\mathcal{G}} = \sum_{i=1}^{n_0} d_i$ and $\mathcal{G}_k$ is the set of all elements of length $k$, $k \geq 1$ in the semigroup $\mathcal{S}$ with respect to $\mathcal{G}$.

**Proposition 2.1.** Let $\mathcal{S}$ be a finitely generated semigroup as in (1.1), then there exists a unique minimal set $\mathcal{G}_0$ of maps of the form (1.2) that generates $\mathcal{S}$, i.e., any set of generators $\mathcal{G}$ of $\mathcal{S}$ is a superset of $\mathcal{G}_0$.

**Proof.** For $n \geq 1$, let $\mathcal{S}(n) = \{H \in \mathcal{S} : \text{degree of } H \text{ is } n\}$. Note that $\mathcal{S}(1)$ is an empty set, as the degree of every element in the generating set $\mathcal{G}$ is at least 2. However, $\mathcal{S}(n)$ for every $n \geq 1$, need not necessarily be empty but is always a finite set. We will construct the minimal generating set $\mathcal{G}_0$ inductively, such that it terminates after finitely many steps. Let

$$A_2 := \mathcal{S}(2), A_3 := \mathcal{S}(3) \setminus \{A_2\}, A_4 := \mathcal{S}(4) \setminus \{A_2 \cup A_3\}, \ldots, A_n := \mathcal{S}(n) \setminus \left( \bigcup_{i=2}^{n-1} A_i \right).$$

Since $\mathcal{S}$ is finitely generated, there exists an $n_0 \geq 1$, such that $A_n = \emptyset$ for $n > n_0$ and $A_{n_0} \neq \emptyset$. Let

$$\mathcal{G}_0 = \bigcup_{i=2}^{n_0} A_i.$$

Note that by construction, any element in $\mathcal{G}_0$ is not generated by lower degree maps of form (1.2). Further as $A_n = \emptyset$ for every $n > n_0$, $\mathcal{G}_0$ is the minimal set generating $\mathcal{S}$. \qed

**Remark 2.2.** Thus the total degree of a semigroup $\mathcal{S}$ is dependent on the generating set $\mathcal{G}$ and is not unique, in general. Consequently, the sequence of plurisubharmonic functions $\{G_k^+\}$ defined in (1.5) and the positive and negative dynamical Green’s function is also dependent on the generating set $\mathcal{G}$ of the semigroup $\mathcal{S}$.

Next, we revisit and introduce a few important definitions (and notations) with respect to the dynamics of the semigroup $\mathcal{S}$, that are independent of the generating set $\mathcal{G}$.

- Let $\mathcal{S}^-$ denote the semigroup of maps comprising of the inverse of the maps that belong to $\mathcal{S}$ and $\mathcal{G}^-$ the inverse of the elements that belong to $\mathcal{G}$, i.e.,

$$\mathcal{G}^- = \{H_i^{-1} : i \leq i \leq n_0\} \quad \text{and} \quad \mathcal{S}^- = \langle \mathcal{G}^- \rangle.$$

- The Fatou sets of $\mathcal{S}$ and $\mathcal{S}^-$ — as stated in Section 1 — is denoted by $\mathcal{F}^+_\mathcal{S}$ and $\mathcal{F}^-_{\mathcal{S}}$ respectively. The positive and negative Julia sets are denoted by $J^+_\mathcal{S}$.

- We consider the following two alternatives for the filled positive and negative Julia sets.

  (1) The strong positive (or negative) filled Julia set is defined as the collection of all the points $z \in \mathbb{C}^2$ such that for every sequence $\{h_k\} \subset \mathcal{S}$, the sequence $\{h_k(z)\}$ (or the sequence $\{h_k^{-1}(z)\}$, respectively) is bounded, i.e.,

$$\mathcal{K}^+_\mathcal{S} = \{z \in \mathbb{C}^2 : \text{for every sequence } \{h_k\} \subset \mathcal{S} \text{ the sequence } \{h_k(z)\} \text{ is bounded}\},$$

$$\mathcal{K}^-_{\mathcal{S}} = \{z \in \mathbb{C}^2 : \text{for every sequence } \{h_k\} \subset \mathcal{S} \text{ the sequence } \{h_k^{-1}(z)\} \text{ is bounded}\}.$$
Recall from \([4]\), for \(H\) where
\[
K^+_S = \{ z \in \mathbb{C}^2 : \text{there exist } h_k \in \mathcal{G}_{n_k} \text{ such that } n_k \to \infty \text{ and } \{h_k(z)\} \text{ is bounded} \},
\]
\[
K^-_S = \{ z \in \mathbb{C}^2 : \text{there exist } \hat{h}_k \in \mathcal{G}_{n_k} \text{ such that } n_k \to \infty \text{ and } \{\hat{h}_k^{-1}(z)\} \text{ is bounded} \}.
\]
Note that \(K^+_S \subset K^+_\mathbb{C}\) and these sets are uniquely associated to the semigroup \(S\).

- Similarly as above we introduce the \textbf{weak} and \textbf{strong} escaping sets \(U^+_S\) and \(U^+_S\)
\[
U^+_S = \mathbb{C}^2 \setminus K^+_S \quad \text{and} \quad U^+_S = \mathbb{C}^2 \setminus K^-_S.
\]
Note that \(U^+_S\) is the Fatou component at infinity with respect to the dynamics of the semigroup \(S\). Similarly \(U^-_S\) is the Fatou component at infinity for \(S^-\).

- Finally, we define the \textit{cumulative positive and negative Julia sets} for the semigroup \(S\), i.e.,
\[
J^+_S = \bigcup_{h \in S} J^+_h, J^-_S = \bigcup_{h \in S} J^-_h.
\]

Observe that, either of the sets \(K^+_S\) or \(K^-_S\) or both may be empty for some semigroups \(S\), of form \([1,1]\). However, this situation does not affect the dynamics as such.

We will now explore some important properties of the sets introduced above via the filtration properties of the elements of \(S\), on appropriate domains. To discuss this in detail, let us first recall the definition of the sets \(V_R\) and \(V^\pm_R\) for some \(R > 0\), introduced in [21] (or [4]) for filtering the dynamics of (finite) compositions of generalised Hénon maps. They are
\[
V_R = \{(x,y) \in \mathbb{C}^2 : \max\{|x|,|y|\} \leq R\}, \quad \text{polydisk of radius } R
\]
\[
V^+_R = \{(x,y) \in \mathbb{C}^2 : |y| \geq \max\{|x|,R\}\}, \quad V^-_R = \{(x,y) \in \mathbb{C}^2 : |y| \geq \max\{|y|,R\}\}.
\]
Also recall the subsets \(\mathcal{G}_k\) of \(S = \langle \mathcal{G} \rangle\), for every \(k \geq 1\), defined as
\[
\mathcal{G}_k = \{H_{i_1} \circ \cdots \circ H_{i_k} : 1 \leq i_j \leq n_0 \text{ and } 1 \leq j \leq k\}, \quad \text{where } \mathcal{G}_1 = \mathcal{G} = \{H_i : 1 \leq i \leq n_0\}.
\]
We first record the dynamical behaviour of the semigroup \(S\) on \(V^\pm_R\) for an appropriate \(R > 0\).

\begin{lemma}
There exists \(R_S > 0\) such that for every \(R > R_S\),
\[
\overline{h(V^+_R)} \subset V^+_R \quad \text{and} \quad \overline{h^{-1}(V^-_R)} \subset V^-_R, \quad \text{i.e., } h^{-1} V^+_R \cap h V^-_R = \emptyset \quad \text{and} \quad V^+_R \cap h^{-1} V^-_R = \emptyset
\]
whenever \(h \in S\). Further let \(\{h_k\} \subset S\) such that \(h_k \in \mathcal{G}_k\) for every \(k \geq 1\), then there exists a sequence positive real numbers \(R_k \to \infty\) satisfying
\[
V_{R_k} \cap h_k(V^+_R) = \emptyset \quad \text{and} \quad V_{R_k} \cap h^{-1}_k(V^-_R) = \emptyset.
\]
\end{lemma}

\begin{proof}
Recall from [4], for \(R > 0\) (sufficiently large) there exists \(0 < m < M\) such that
\[
m|y|^d_1 \leq |\pi_2 \circ H_i(x,y)| < M|x|^d_1 \quad \text{on } V^+_R, \quad m|x|^d_1 \leq |\pi_1 \circ H_i^{-1}(x,y)| < M|x|^d_1 \quad \text{on } V^-_R.
\]
Recall from \([1,2]\)
\[
H_i = H^i_1 \circ H^i_2 \circ \cdots \circ H^i_{m_i},
\]
where \(H^i_j(x,y) = (y, p_j(y) - a_jx)\). Thus degree of \(\pi_1 \circ H_i < \text{degree of } \pi_2 \circ H_i\), for every \(1 \leq i \leq n_0\). Now as \(V^+_R\) is a closed subset of \(\mathbb{C}^2\) by definition, the above identity further implies that \(|\pi_1 \circ H_i(z)| < |\pi_2 \circ H_i(z)|\) and \(|\pi_2 \circ H_i(z)| > R\), for \(z \in V^+_R\), \(R > 0\), sufficiently large. In particular, there exists \(R > 0\), large enough such that
\[
H_i(V^+_R) = H_i(V^+_R) \subset \text{int}(V^+_R), \quad H_i^{-1}(V^-_R) = H_i^{-1}(V^-_R) \subset \text{int}(V^-_R) \quad \text{for every } 1 \leq i \leq n_0.
\]
Let \(d_0 = \min\{d_i : 1 \leq i \leq n_0\} \geq 2\) and \(R_S > 1\) be sufficiently large such that \(1 < R_S < m R^i_{d_0}\). Hence from \((2.1)\), for \(R_k = m R^i_{d_0}\) whenever \(k \geq 2\) and \(R_1 > R_S\). Thus the proof. \(\Box\)
The constant $R_S > 0$ obtained in Lemma 2.3 is actually independent of the generators and will be referred along, as the radius of filtration for the semigroup $S$.

Remark 2.4. Note that in the above proof we may assume $0 < m < 1 < M$, such that for every $R > R_S$ and $1 \leq i \leq n_0$

\[ m \cdot |y^{d_i}| < |\pi_2 \circ H_i(x, y)| < M \cdot |y^{d_i}| \quad \text{on } V_R^+ \], and $m \cdot |x^{d_i}| < |\pi_1 \circ H_i^{-1}(x, y)| < M \cdot |x^{d_i}| \quad \text{on } V_R^-.$

Proposition 2.5. The sets $K_S^\pm$ and $\mathcal{K}_S^\pm$ are closed subsets of $\mathbb{C}^2$ and $\mathcal{K}_S^\pm \subset K_S^\pm \subset V_R \cup V_R^\mp$ (respectively) for $R \geq R_S$.

Proof. Let $U_0 = \text{int}(V_R^+)$ and let $\{U_k\}, \{\mathcal{U}_k\}$ be the sequences of open subsets defined as

\[ U_k = \bigcap_{h \in \mathcal{G}_k} h^{-1}(U_0) \quad \text{and} \quad \mathcal{U}_k = \bigcup_{h \in \mathcal{G}_k} h^{-1}(U_0). \]

Then

\[ \overline{U}_k = \bigcap_{h \in \mathcal{G}_k} h^{-1}(\overline{U_0}) \quad \text{and} \quad \overline{\mathcal{U}}_k = \bigcup_{h \in \mathcal{G}_k} h^{-1}(\overline{U_0}). \]

Since by Lemma 2.1, $H_i(z_0) \in \text{int}(V_R^+)$ for every $1 \leq i \leq n_0$ whenever $z_0 \in V_R^+ = \overline{U_0}$, we have $\overline{U_0} \subset H_i^{-1}(\overline{U_0})$. Hence $\overline{U_0} \subset U_1 \subset \mathcal{U}_1$. Further for every $h \in \mathcal{G}_k$, $h^{-1}(U_0) \subset h^{-1} \circ H_i^{-1}(U_0)$ for every $H_i \in \mathcal{G}$, where $k \geq 1$. Thus

\[ \overline{U}_k \subset U_{k+1} \quad \text{and} \quad \overline{\mathcal{U}}_k \subset \mathcal{U}_{k+1}. \quad (2.2) \]

Let

\[ U^+ = \bigcup_{k \geq 0} U_k \quad \text{and} \quad U^- = \bigcup_{k \geq 0} \mathcal{U}_k. \]

Observe that for every $k \geq 1$, $h(U_k) \subset V_R^+$ whenever $h \in \mathcal{G}_k$. Hence $U^+ \subset U_S^+$, Now for $z \in \mathbb{C}^2 \setminus U^+$, note that $h_k(z) \in V_R \cup V_R^\mp$ for every sequence $\{h_k\} \subset \mathcal{S}$. Let $z_0 \in U_S^+ \cap (\mathbb{C}^2 \setminus U^+)$. Then there exists a sequence $\{h_k\} \subset \mathcal{S}$ such that $h_k \in \mathcal{G}_{n_k}$ with $n_k \to \infty$ as $k \to \infty$ and $h_k(z_0) \in V_R^-$ for every $k \geq 1$, i.e., $z_0 \in h_k^{-1}(V_R^-)$. Hence by Lemma 2.3, $z_0 \notin \cup_k V_R = \mathbb{C}^2$, which is a contradiction! Thus $U^+ = U_S^+$, A similar argument works for $U^- = U_S^-$.

Similarly for $z \in U^+$ there exists $h_k \in \mathcal{G}_k$ such that $h_k(z) \in V_R^+$, hence $U^+ \subset U_S^+$. Now for $z_0 \in U_S^+ \cap (\mathbb{C}^2 \setminus U^+)$, as in the above case, there exists a sequence $\{h_k\}$ such that $h_k \in \mathcal{G}_{n_k}$ with $n_k \to \infty$ as $k \to \infty$ and $h_k(z_0) \in V_R^-$ for every $k \geq 1$, i.e., $z_0 \in h_k^{-1}(V_R^-)$. Hence by Lemma 2.3, $z_0 \notin \cup_k V_R = \mathbb{C}^2$, which is a contradiction! Thus $U^+ = U_S^+$. A similar argument works for $U^- = U_S^-$.

Note that the above observations also prove that $V_R^\pm \subset U_S^\pm \subset U_S^\pm$, hence $K_S^\pm \subset K_S^\pm \subset V_R \cup V_R^\mp$ (respectively).

Further, from the proof of Proposition 2.5 we get

Corollary 2.6. The escaping sets of $\mathcal{S}$ can be further realised as

(1) $U_S^+ = \bigcup_{k \geq 1} \bigcap_{h \in \mathcal{G}_k} h^{-1}(V_R^+)$ and $U_S^- = \bigcup_{k \geq 1} \bigcap_{h \in \mathcal{G}_k} h(V_R^-)$;

(2) $U_S^+ = \bigcup_{k \geq 1} \bigcap_{h \in \mathcal{G}_k} h^{-1}(V_R^+)$ and $U_S^- = \bigcup_{k \geq 1} \bigcap_{h \in \mathcal{G}_k} h(V_R^-)$.

Remark 2.7. Note that $K_S^\pm \setminus \text{int}(K_S^\pm) = U_S^\pm \setminus U_S^\pm$ and the Julia sets $J_S^\pm \subset K_S^\pm \setminus \text{int}(K_S^\pm)$, however, they might not be equal. Thus so far we have the straightforward inclusion relation

\[ J_S^\pm \subset J_S^\pm \subset K_S^\pm \setminus \text{int}(K_S^\pm). \]

Also both $\partial K_S^\pm, \partial K_S^\pm \subset J_S^\pm$, i.e., they may be proper subsets $J_S^\pm$ unlike the iterative dynamics of Hénon maps. So it leads to the question: Is $J_S^\pm = K_S^\pm \setminus \text{int}(K_S^\pm)$ or $J_S^\pm = \partial K_S^\pm \cup \partial K_S^\pm$?
The shaded region in Figure 1 corresponds to the weak filled Julia sets $K^+_S$, with the lighter shade representing the Fatou components and the darker shades representing the Julia sets $J^+_S$ contained in them, respectively.

Finally we conclude this section, by observing the following crucial fact which will be very important for further computations

Lemma 2.8. Let $R > R_S$, where $R_S > 0$ is the radius of filtration for the semigroup $S$ and $C$ be a compact subset of $\mathbb{C}^2$.

(i) Then there exists a positive integer $k_C \geq 1$ such that $h(C) \subset V_R \cup \text{int}(V_R^+)$ for every $h \in G_k$ and $k \geq k_C$.

(ii) Then there exists a positive integer $\tilde{k}_C \geq 1$ such that $h^{-1}(C) \subset V_R \cup \text{int}(V_R^-)$ for every $h \in G_k$ and $k \geq \tilde{k}_C$.

Proof. Suppose the statement (i) is not true, i.e., there exist a sequence $\{z_n\} \subset C$ and a sequence $\{h_n\}$ with $h_n \in G_{k_n}$, $k_n \to \infty$ as $n \to \infty$ such that $h_n(z_n) \in V_R^-$. Then by Lemma 2.3, $z_n \in h_n^{-1}(V_R^-) \subset V_{R_{k_n}}$. As $R_{k_n} \to \infty$ for $n \to \infty$, $\|z_n\| \to \infty$. This contradicts the fact that $\{z_n\}$ is contained in the compact set $C$.

A similar argument works for part (ii). \qed

3. Proof of Theorem 1.3

In this section, we will first complete the proof of Theorem 1.3 and consequently observe a few important corollaries. To begin, let us recall the definition of the sequence of plurisubharmonic functions $\{G_k\}$, introduced in Section 1

$$G_k^+(z) = \frac{1}{D^k} \sum_{h \in G_k} \log^+ \|h(z)\| \quad \text{and} \quad G_k^-(z) = \frac{1}{D^k} \sum_{h \in G_k} \log^+ \|h^{-1}(z)\|,$$

where $D = D_g$ is the total degree corresponding to the generating set $\mathcal{G}$ of the semigroup $S$.

First, we note the following straightforward consequence from the results in Section 2

Lemma 3.1. Fix an $R \geq R_S$, the radius of filtration for the semigroup $S$. Then

- for a compact set $C^+ \subset U^+_S$ there exists a positive integer $N_{C^+} \geq 1$ such that $h(C^+) \subset V_R^+$ whenever $h \in G_k$, $k \geq N_{C^+}$;
- for a compact set $C^- \subset U^-_S$ there exists a positive integer $N_{C^-} \geq 1$ such that $h^{-1}(C^-) \subset V_R^-$ whenever $h \in G_k$, $k \geq N_{C^-}$. 

\[\]
Proof. By Corollary 2.6 and the proof of Proposition 2.5 there exists $N_{C^+}, N_{C^-} \geq 1$ such that

$$C^+ \subset \bigcap_{h \in \mathcal{F}_k} h^{-1}(V^+_R), \quad C^- \subset \bigcap_{h \in \mathcal{F}_l} h(V^-_R)$$

whenever $k \geq N_{C^+}$ and $l \geq N_{C^-}$. \qed

Now we are ready to present the proof of Theorem 1.3, which involves some steps.

**Proof of Theorem 1.3.** Let $R \geq R_S$ be as in Lemma 3.1.

**Step 1:** The sequence $\{G^+_k\}$ converges uniformly to a pluriharmonic function on $V^+_R$.

Suppose $(x, y) \in V^+_R$ and the constants $0 < m < 1 < M$ be as assumed in Remark 2.4. Then

$$\log m + d_i \log |y| < \log |\pi_2 \circ H_i(x, y)| < \log M + d_i \log |y| \quad (3.1)$$

for every $1 \leq i \leq n_0$. Let $h \in \mathcal{F}_k$. Then $h = H_{j_1} \circ \ldots \circ H_{j_k}$ where $1 \leq j_i \leq n_0$ for every $1 \leq i \leq k$ and the degree of $h$ (denoted by $d_h$) is given by the product $d_{j_1} \ldots d_{j_k}$. Also, recall $D = d_1 + \ldots + d_{n_0}$, the total degree of $S = (\mathcal{F})$. Now by (2.1), for $(x, y) \in V^+_R$ and $k \geq 1,

$$G^+_k(x, y) = \sum_{h \in \mathcal{F}_k} \frac{\log |\pi_2 \circ h(x, y)|}{D^k}.$$  

Hence $G^+_k$ is pluriharmonic on $V^+_R$ for every $k \geq 1$ and

$$G^+_{k+1}(x, y) = \sum_{i=1}^{n_0} \sum_{h \in \mathcal{F}_k} \frac{\log |\pi_2 \circ h(x, y)|}{D^{k+1}}.$$  

Thus from (3.1)

$$G^+_{k+1}(x, y) \leq \left(\frac{n_0}{D}\right)^{k+1} \log M + \sum_{i=1}^{n_0} \sum_{h \in \mathcal{F}_k} \frac{d_i \log |\pi_2 \circ h(x, y)|}{D^{k+1}} \leq \left(\frac{n_0}{D}\right)^{k+1} \log M + G^+_k(x, y).$$

Similarly, by using the left inequality of (3.1) we have

$$\left(\frac{n_0}{D}\right)^{k+1} \log m + G^+_k(x, y) \leq G^+_{k+1}(x, y) \leq \left(\frac{n_0}{D}\right)^{k+1} \log M + G^+_k(x, y).$$

Since $D \geq 2n_0$, the above inequality reduces to

$$|G^+_{k+1}(x, y) - G^+_k(x, y)| \leq \left(\frac{1}{2}\right)^{k+1} M_0 \quad (3.2)$$

where $M_0 = \max\{|\log m|, |\log M|\}$ and $(x, y) \in V^+_R$. Thus the sequence $\{G^+_k\}$ is uniformly Cauchy on $V^+_R$, and hence it converges uniformly to a pluriharmonic function $G^+_S$ on $V^+_R$. A similar argument on $V^-_R$, proves the same for $G^-_S$.

**Step 2:** The sequence $\{G^+_k\}$ converges uniformly to the pluriharmonic function $G^+_S$ on compact subsets of $U^+_S$, respectively.

As noted earlier similar arguments work on $U^-_S$, so we complete the proof only for $U^+_S$. Let $C$ be a compact subset of $U^+_S$. By Lemma 3.1 there exists $N_C \geq 1$ such that $h(C) \subset V^+_R$ for every $h \in \mathcal{F}_k$, $k \geq N_C$. Note that $\mathcal{F}_k$ has $n_0$ elements. Let $C_h = h(C) \subset V^+_R$ (by Lemma 3.1) for every $h \in \mathcal{F}_N$. Thus for $z \in C$

$$G^+_k(z) = \frac{1}{D^{N_C}} \sum_{h \in \mathcal{F}_k} G^+_{k-N_C}(h(z)).$$

whenever $k > N_C$. Now by Step 1, $\{G^+_{k-N_C}\}$ is convergent on every $C_h$. Hence $\{G^+_k\}$ converges uniformly on $C \subset U^+_S$ to a pluriharmonic function and this completes Step 2. Thus, $G^+_S$ are (respectively) pluriharmonic on the $U^+_S$. 


Also $G^+_g$ are pluriharmonic on $\text{int}(K_\mathcal{S}^+)$, as $G^+_g$ are identically zero in here. Next, for $z_0 \in \mathbb{C}^2$ and for $k \geq 1$, we define the following subsets of $\mathcal{S}$, dependent on $z_0$ as
\[
\mathcal{S}^b(z_0) = \{ h \in \mathcal{S} : h(z_0) \in V_R \cup V_R^c \}, \quad \mathcal{G}^b(z_0) = \mathcal{S}^b(z_0) \cap \mathcal{G}_k
\]
and
\[
\mathcal{S}^u(z_0) = \{ h \in \mathcal{S} : h(z_0) \in \text{int}(V_R^c) \}, \quad \mathcal{G}^u(z_0) = \mathcal{S}^u(z_0) \cap \mathcal{G}_k.
\]
Note that by (2.1), the following inequality about the cardinality of the sets $\mathcal{G}^b(z_0)$ and $\mathcal{G}^u(z_0)$ is immediate for every $k \geq 1$
\[
\#\mathcal{G}^b_{k+1}(z_0) \leq n_0(\#\mathcal{G}^b_k(z_0)), \quad n_0(\#\mathcal{G}^u_k(z_0)) \leq \#\mathcal{G}^u_{k+1}(z_0). \tag{3.3}
\]
Now for $z_0 \in U^+_S$, there exists $k_{z_0} \geq 1$ such that $\mathcal{G}^b_k(z_0) = \emptyset$, for $k \geq k_{z_0}$. Otherwise from Lemma 2.8 there exists $k_{z_0} \geq 1$ such that $h(z_0) \in V_R$ whenever $h \in \mathcal{G}^u_k(z_0)$ and $k \geq k_{z_0}$. Thus consider the following sequences of functions $\{G^b_k\}$ and $\{G^u_k\}$ defined on $\mathbb{C}^2$ as
\[
G^b_k(z) = \sum_{h \in \mathcal{G}^b_k(z)} \frac{\log^+ ||h(z)||}{D^k} \quad \text{and} \quad G^u_k(z) = \sum_{h \in \mathcal{G}^u_k(z)} \frac{\log^+ ||h(z)||}{D^k}. \tag{3.4}
\]

**Remark 3.2.** Note that $G_k(z) = G^b_k(z) + G^u_k(z)$ for every $z \in \mathbb{C}^2$ and $k \geq 1$. Also if $C$ is a compact subset of $\mathbb{C}^2$ then $h(C) \subset V_R$ whenever $h \in \mathcal{G}^u_k$, $k \geq k_C$, as obtained in Lemma 2.8

**Step 3:** The sequence $\{G^b_k\}$ converges uniformly to zero on every compact subset $C \subset \mathbb{C}^2$.

Since $C$ is a compact subset of $\mathbb{C}^2$ from Lemma 2.8 for $k \geq k_C$, we have
\[
\sup \{ ||h(z)|| : z \in C \text{ and } h \in \mathcal{G}^b_k(z) \} \leq R.
\]
Now, by (3.3) for every $z \in \mathbb{C}^2$, $\#\mathcal{G}^b_k(z) \leq \#\mathcal{G}^b_k = n_0^k$. Hence by the above claim for $z \in C$
\[
G^b_k(z) \leq \left( \frac{n_0}{D} \right)^k \log R \leq \left( \frac{1}{2} \right)^k \log R \to 0
\]
as $k \to \infty$. This completes Step 3.

**Step 4:** For every $z_0 \in K_\mathcal{S}^+ \setminus K_\mathcal{S}^+$ there exist a constant $\tilde{M} > 1$ and a positive integer $\ell_{z_0} \geq 1$ such that for $k \geq \ell_{z_0}$,
\[
G^b_k(z_0) - \tilde{M} \left( \frac{n_0}{D} \right)^{k+1} \leq G^b_k(z_0) \leq G^b_k(z_0) + \tilde{M} + C^a_k(z_0). \tag{3.5}
\]
Since $z_0 \in K_\mathcal{S}^+ \setminus K_\mathcal{S}^+$ there exists a sequence $\{h_n\} \subset \mathcal{S}$ such that $||h_n(z_0)|| \to \infty$ as $n \to \infty$. In particular $\mathcal{S}^u(z_0) \neq \emptyset$, i.e., by (3.3) there exist positive integers $k_{z_0}, N_{z_0} \geq 1$ such that
\[
1 \leq \#\mathcal{G}^u_{k_{z_0}}(z_0) = N_{z_0} < n_{k_{z_0}}^k.
\]
Now note that from (2.1) it follows that $H_i \circ h(z_0) \in V^+_R$ whenever $h \in \mathcal{G}^u_{k_{z_0}}$ and $1 \leq i \leq n_0$. Hence by (3.3) for every $k \geq k_{z_0}$ we have
\[
n_0^{k-k_{z_0}} N_{z_0} \leq \#\mathcal{G}^u_{k_{z_0}}(z_0) < n_0^k. \tag{3.6}
\]
As $0 < m < 1$, from (3.6), (3.4) and (3.1) it follows that for every $k \geq k_{z_0}$,
\[
G^a_{k+1}(z_0) \geq \frac{1}{D^{k+1}} \sum_{i=1}^{n_0} \sum_{h \in \mathcal{G}^u_k(z_0)} \log ||H_i \circ h(z_0)|| = \sum_{i=1}^{n_0} \sum_{h \in \mathcal{G}^u_k(z_0)} \log ||H_i \circ h(z_0)||, \quad \frac{1}{D^{k+1}} \log m = \sum_{h \in \mathcal{G}^u_k(z_0)} \log ||h(z_0)|| = \frac{n_0(\#\mathcal{G}^u_k(z_0))}{D^{k+1}} \log m
\]
\[
\geq G^u_k(z_0) + \left( \frac{n_0}{D} \right)^{k+1} \log m.
\]
Now from Lemma 2.8, there exists $k_0' \geq 1$ such that for $h \in \mathcal{G}_k^u(z_0)$, $\|h(z_0)\| \leq R$ whenever $k \geq k_0'$. Let $\ell_{z_0} = \max\{k_{z_0}, k_0'\}$ and $B = \max\{\|H_i(z)\| : z \in V_{R+1}, 1 \leq i \leq n_0\}$. Further, for every $k, l \geq 1$, we introduce the following subsets $\mathcal{G}_{k,l}^u(z_0)$ of $S$ defined as

$$
\mathcal{G}_{k,l}^u(z_0) = \{h_1 \circ h_2 : h_1 \in \mathcal{G}_k \text{ and } h_2 \in \mathcal{G}_l^u(z_0)\}.
$$

Now by (2.1), $\mathcal{G}_{k,l}^u(z_0) \subset \mathcal{G}_{k+1,l}^u(z_0)$ and $\#\mathcal{G}_{k+1,l}^u(z_0) = n_0(\#\mathcal{G}_k^u(z_0))$ for $k \geq \ell_{z_0}$. Let $h \in \mathcal{G}_{k+1,l}^u \setminus \mathcal{G}_{k,l}^u(z_0)$ then $h = H_i \circ \tilde{h}$ for some $\tilde{h} \in \mathcal{G}_{k,l}^u$ and $1 \leq i \leq n_0$. Thus by the above assumption, $|\pi_2 \circ h(z_0)| \leq B$. Since

$$
\#(\mathcal{G}_{k+1,l}^u \setminus \mathcal{G}_{k,l}^u(z_0)) \leq n_0^{k+1} - n_0(\#\mathcal{G}_k^u(z_0))
$$

we have the following inequalities (as before)

$$
G_{k+1,l}^u(z_0) = \frac{1}{D^{k+1}} \sum_{h \in \mathcal{G}_{k,l}^u} \log \|h(z_0)\| + \frac{1}{D^{k+1}} \sum_{h \in \mathcal{G}_{k+1,l}^u \setminus \mathcal{G}_{k,l}^u(z_0)} \log \|h(z_0)\|
$$

$$
\leq \sum_{i=1}^{n_0} \sum_{h \in \mathcal{G}_{k,l}^u} d_i \log \|h(z_0)\| + \frac{1}{D^{k+1}} \sum_{h \in \mathcal{G}_{k+1,l}^u \setminus \mathcal{G}_{k,l}^u(z_0)} \log \|h(z_0)\|
$$

$$
\leq G_k^u(z_0) + n_0(\#\mathcal{G}_k^u(z_0)) \cdot \log M + \frac{\#(\mathcal{G}_{k+1,l}^u \setminus \mathcal{G}_{k,l}^u(z_0))}{D^{k+1}} \log B
$$

$$
\leq G_k^u(z_0) + \frac{n_0}{D} \log \tilde{M} + \frac{n_0}{D} \log \tilde{M}
$$

where $\tilde{M} = \max\{\log M, |\log B|, |\log m|\}$. This completes the proof of Step 4.

Since $G_k^u \equiv 0$ on $\mathcal{K}_S^+$ and $G_k^h \equiv 0$ on $U_S^+$, by Steps 2, 3 and 4, $G_k^+$ converges pointwise to a non-negative function $G_k^+$ on $\mathbb{C}^2$ which is identically zero on $\mathcal{K}_S^+$ and plurisubharmonic on $U_S^+$.

**Step 5:** $G_k^+$ is non-negative, continuous and plurisubharmonic on $U_S^+$.

Let $C$ be a compact subset of $U_S^+$ and $z \in C$. By the same reasoning as in Step 5, there exist $k_z, n_z \geq 1$ such that

$$
1 \leq \#\mathcal{G}_k^u(z) = N_z \leq n_0^{k_z}.
$$

Since $\text{int}(V_R^+)$ is open, there exists $\delta_z > 0$ such that the closed ball $\overline{B(z; \delta_z)}$ is contained in $U_S^+$ and $h(B(z; \delta_z)) \subset \text{int}(V_R^+)$ for every $h \in \mathcal{G}_k^u(z)$, i.e., for every $\xi \in B(z; \delta_z)$

$$
\mathcal{G}_k^u(z) \subseteq \mathcal{G}_k^u(\xi) \quad \text{and} \quad N_z \leq \#\mathcal{G}_k^u(\xi) \leq n_0^{k_z}.
$$

Since $\overline{B(z; \delta_z)}$ is compact, by Lemma 2.8, there exists $k_z' \geq 1$, such that $h(\xi) \in V_R$ whenever $h \in \mathcal{G}_k^u(\xi)$ for every $k \geq k_z'$ and $\xi \in B(z; \delta_z)$. Hence by exactly similar argument as in the proof of Step 4, for every $k \geq \max\{k_z, k_z'\}$

$$
G_k^u(\xi) - \tilde{M} \left(\frac{n_0}{D}\right)^{k+1} \leq G_{k+1}(\xi) \leq \left(\frac{n_0}{D}\right)^{k+1} \tilde{M} + G_k^u(\xi)
$$

where $\tilde{M}$ is as chosen in Step 4. Now for a given $\epsilon > 0$, there exist a positive integer $\ell_z \geq \max\{k_z, k_z'\}$ sufficiently large, such that for every $p, q \geq \ell_z$ and for every $\xi \in \overline{B(z; \delta_z)}$

$$
\|G_p^u(\xi) - G_q^u(\xi)\| \leq \left(\frac{1}{2}\right)^{\ell_z} \tilde{M} \leq \frac{\epsilon}{2}.
$$
As $C$ is compact, there exists a finite collection \{\(z_i \in C : 1 \leq i \leq N_C\)\} such that $C \subset \bigcup_i B(z_i; \delta_z)$. Let $\tilde{C} = \bigcup_i B(z_i; \delta_z)$, then by Lemma 2.8, there exists $\ell_C \geq \max\{\ell_{z_i} : 1 \leq i \leq N_C\}$ sufficiently large, such that for every $\xi \in \tilde{C}$ and for every $p, q \geq \ell_C$

\[
\|G_p^h(\xi) - G_q^h(\xi)\| < \left(\frac{1}{2}\right)^{\ell_C - 1} \log R < \frac{\epsilon}{2}.
\]

Thus for every $\xi \in C$ and $p, q \geq \ell_C$

\[
\|G_p^+(\xi) - G_q^+(\xi)\| < \epsilon,
\]
i.e., \(\{G_k^+\}\) is uniformly Cauchy on the compact set $C$. Further as $\{G_k^+\}$ is a non-negative, subharmonic and continuous sequence of functions on $U_S^+$, so is $G_d^+$.  

**Step 6: $G_d^+$** is non-negative, non-constant, continuous and plurisubharmonic on $C^2$.

Note that if $\mathcal{K}_S^+$ is empty there is nothing to proof. So we assume $\mathcal{K}_S^+ \neq \emptyset$. By (3.2), it follows that on $V_R^+$

\[
\|G_d^+ - \log |y|\| \leq M_0 \sum_{i=0}^{\infty} \left(\frac{n_0}{D}\right)^i.
\]

Thus for $(x, y) \in V_R^+$ with $|y|$ sufficiently large, $G_d^+(x, y)$ is both positive and non-constant. Now from Step 5, $G_d^+$ is continuous on $U_S^+$ and it is identically zero on $\mathcal{K}_S^+$. Hence to establish the continuity of $G_d^+$, it is sufficient to prove $G_d^+$ is continuous on $\partial \mathcal{K}_S^+ = \partial U_S^+$.

We will prove it by contradiction, so suppose there exists a sequence \(\{z_k\} \subset U_S^+\) such that $G_d^+(z_k) > c > 0$ for every $k \geq 1$ and $z_k \to z_0 \in \partial \mathcal{K}_S^+$. Choose $k_0 \geq 1$, large enough such that

\[
c > \frac{\widetilde{M}}{2k_0} \sum_{i=1}^{\infty} \left(\frac{n_0}{D}\right)^i,
\]

where $\widetilde{M}$ is as obtained in Step 4. Also, we may assume for every $k \geq 1$,

\[
\sup \{\|h(z_k)\| : h \in \mathcal{G}_{k_0}^u(z_k)\} \leq R.
\]

**Claim:** For every $k \geq 1$, $\mathcal{G}_{k_0}^u(z_k) \neq \emptyset$.

If not, then $\mathcal{G}_{k_0}^u(z_l) = \emptyset$ for some fixed $l \geq 1$, i.e., $\mathcal{G}_{k_l}^u(z_l) = \emptyset$ for every $1 \leq k \leq k_0$. However, there exists $k_1 > k_0$ such that $\mathcal{G}_{k_1}^u(z_l) \neq \emptyset$, as $G_d^+(z_l) > 0$. Let $k_l'$ be the least of all such numbers, i.e., $\mathcal{G}_{k_l'}^u(z_l) \neq \emptyset$ for every $k \geq k_l' > k_0$. Thus for every $k \geq k_l$

\[
G_k^u(z_l) \leq \sum_{i=k_l}^{\infty} \left(\frac{n_0}{D}\right)^i \widetilde{M} < c.
\]

As $G_k^u(z_l) \to 0$, the above thus proves that $G_d^+(z_l) < c$, which contradicts the assumption on the sequence \(\{z_k\}\). Hence the claim follows.

Since $\mathcal{G}_{k_0}^u(z_k) \subset \mathcal{G}_{k_0}$ for every $k \geq 1$ and $\mathcal{G}_{k_0}$ is finite, there are only finitely many subsets of $\mathcal{G}_{k_0}$. Thus there exists a subsequence \(\{z_{k_n}\}\) of \(\{z_k\}\) such that the sets $\mathcal{G}_{k_n}^u(z_{k_n})$ are equal for every $n \geq 1$. Define the sequence \(\{h_l\}\) as $h_l = H_l \circ h$ for some $h \in \mathcal{G}_{k_0}^u(z_{k_n})$. Since $h(z_{k_n}) \in V_R^+$, $h_l(z_{k_n}) \in V_{R_l}$, where $\{R_l\}$ is as obtained in Lemma 2.3 for every $l, n \geq 1$. As $z_{k_n} \to z_0$, this implies $h_l(z_0) \in V_{R_l}$, in particular it contradicts that $z_0 \in \partial \mathcal{K}_S^+ = \partial U_S^+$.

Hence for every sequence \(\{z_k\} \subset U_S^+\) and $z_k \to z_0$, $G_d^+(z_k) \to 0$, which consequently proves $G_d^+$ is continuous on $C^2$. Finally, as $G_d^+$ coincides with its upper semicontinuous regularisation of $G_d^+$ and satisfies the sub-mean value property on $\partial \mathcal{K}_S^+$, $G_d^+$ is plurisubharmonic on $C^2$.

Similarly replicating Step 3, 4, 5 and 6 for the semigroup $S^-$ with $\mathcal{G}^-$ as the generating set, gives that $G_d^-$ is a plurisubharmonic continuous function on $C^2$. \(\square\)
Corollary 3.3. There exist constants $c_{\tilde{g}}^\pm \in \mathbb{R}$ such that for $(x, y) \in V_R^\pm$ (respectively),
\[ G_{\tilde{g}}^+(x, y) = \log |y| + O(1) \quad \text{and} \quad G_{\tilde{g}}^-(x, y) = \log |x| + O(1). \]

Proof. It follows directly from the equation (3.2), in the proof Theorem 1.3 i.e., on $V_R^+$
\[ -M_0 \frac{k}{D} n_0 \frac{i}{D} + \log |y| \leq G^+_k (x, y) \leq M_0 \frac{k}{D} n_0 \frac{i}{D} + \log |y|. \]
Similarly the analogue to (3.2) on $V_R^-$ gives the result for $G_{\tilde{g}}^-$. \qed

Corollary 3.4. The functions $G_{\tilde{g}}^+ > 0$ restricted to $U_{S^*}^\pm$ (respectively).

Proof. From Corollary 3.3 and Lemma 2.3 for $z \in U_{S^*}^+$ there exists $n_z \geq 1$ and $h^z \in \mathcal{G}_n$ such that $h^z(z) \in V_R^+$ and $G_{\tilde{g}}^+(h^z(z)) > 0$. Then from Corollary 1.4 we have $G_{\tilde{g}}^+(z) \geq \frac{1}{D^{n_z}} G_{\tilde{g}}^+(h^z(z)) > 0$. \qed

Remark 3.5. The above corollary also proves $K_{S^*}^+$ is pseudoconcave, provided it is non-empty.

4. Proof of Theorem 1.5

Recall from Remark 2.7 the cumulative (positive and negative) Julia sets are contained in the (positive and negative) Julia sets. Our goal in this section is to prove that these two sets are actually equal, by analysing the supports of the positive (1,1)-currents on $\mathbb{C}^2$ defined as
\[ \mu_{\tilde{g}}^+ = \frac{1}{2\pi} dd^c G_{\tilde{g}}^+ \quad \text{and} \quad \mu_{\tilde{g}}^- = \frac{1}{2\pi} dd^c G_{\tilde{g}}^- . \]

Note that the above fact is also true for the dynamics of semigroups of rational functions in $\mathbb{P}^1$, and is proved using Ahlfors’ covering lemma in [20], a tool not available in higher dimensions. Instead we will use the Harnack’s inequality for harmonic functions and the properties of the dynamical Green’s functions $G_{\tilde{g}}^\pm$.

Theorem 4.1. Let $\{\tilde{G}_k\}$ be the sequence of plurisubharmonic functions on $\mathbb{C}^2$ defined as
\[ \tilde{G}_k^\pm (z) = \frac{1}{D^k} \sum_{h \in \mathcal{G}_k} G_h^\pm (h^\pm (z)) = \frac{1}{D^k} \sum_{h \in \mathcal{G}_k} d_h G_{\tilde{h}}^\pm (z) . \]

Then the sequences $\{\tilde{G}_k^\pm\}$ converge uniformly to $G_{\tilde{g}}^\pm$ (respectively) on compact sets of $\mathbb{C}^2$.

Thus, we first observe Lemma 4.2 which is an extension of the proof of Theorem 1.3

Lemma 4.2. The sequences $\{G_k^\pm\}$ — as in Section 3 — converge uniformly on compact subsets of $\mathbb{C}^2$ to $G_{\tilde{g}}^\pm$, respectively.

Proof. As before, we only prove the convergence of $\{G_k^+\}$ to $G_{\tilde{g}}^+$ by generalising the proof of Lemma 8.3.4 from [26]. The convergence of $\{G_k^-\}$ will follow likewise. Let $C$ be any compact subset of $\mathbb{C}^2$, then

• If $C$ is contained in $K_{S^*}^\pm$, $G_k^\pm (z) = G_{\tilde{g}}^\pm (z)$ (as defined in the proof of Theorem 1.3). Hence by Remark 3.2 $G_k^\pm \to G_{\tilde{g}}^\pm$ uniformly on $C$;

• If $C$ is contained in $U_{S^*}^\pm$, from the proof of Step 5 of Theorem 1.3 it follows that $G_k^\pm \to G_{\tilde{g}}^\pm$ uniformly on $C$.

So we assume $C \cap U_{S^*}^\pm \neq \emptyset$ and $C \cap K_{S^*}^\pm \neq \emptyset$ and let $z \in C$. By the above facts, for a given $\epsilon > 0$ there exists $k_1 \geq 1$, such that $|G_k^\pm (z)| < \epsilon$ and $\|h(z)\| < R$ for every $h \in \mathcal{G}_k (z)$ whenever $k \geq k_1$. In particular, for $z \in C \cap K_{S^*}^\pm$ and $k \geq k_1$ $|G_k^\pm (z)| < \epsilon$. Recall the sets $\{U_k\}$ from Proposition 2.5 defined as
\[ U_k = \bigcup_{h \in \mathcal{G}_k} h^{-1} (U_0) \]
for every \( k \geq 1 \). Also from (2.2), we have \( \overline{U}_k \subset U_{k+1} \subset \overline{U}_{k+1} \) and \( U^+_S = \cup_{k=1}^\infty U_k \). Let
\[
C_k := C \cap (U_k \setminus U_{k-1})
\]
for every \( k \geq 1 \). Since \( C \cap \bigcup_{k=1}^\infty U_k^+ \neq \emptyset \) and \( \overline{U}_k \subset U_{k+1} \), \( C_k \)'s are non-empty sets for \( k \geq 1 \), sufficiently large. Also \( \cup_{k=1}^\infty C_k = C \cap \bigcup_{k=1}^\infty U_k^+ \). Now for \( k > k_1 \) and \( z \in C_k \), \( h(z) \in V_R \cup V_R^- \) whenever \( h \in \mathcal{G}_p \), \( 1 \leq p \leq k-1 \), i.e., \( \mathcal{G}_p^b(z) = \mathcal{G}_p \) for all \( z \in C_k \). Thus from Lemma 2.8 and Remark 3.2, for \( k \) large enough, \( h(z) \in V_R \) whenever \( z \in C_k \) and \( h \in \mathcal{G}_{k-1} \). Let
\[
B = \sup \{ ||H_i(z)|| : z \in \overline{V_R}, k \geq 1 \}
\]
So for \( h \in \mathcal{G}_k \), \( ||h(z)|| < B \) whenever \( z \in C_k \), \( k \) sufficiently large. Let \( l > k \) and \( z \in C_k \), then
\[
G^+_l(z) = \frac{1}{D^k} \sum_{h \in \mathcal{G}_k} G^+_{l-k}(h(z)) \leq \frac{n^0}{D^k} (\log B + \log M) \leq \frac{\tilde{M}}{2^{k-1}}
\]
where \( \tilde{M} = \max \{ |\log M|, |\log B|, |\log m| \} \) and \( m, M \) is as obtained in Remark 2.4. Now by continuity of \( G^+_k \) for \( \epsilon > 0 \) there exists a neighbourhood \( W \) of \( C \cap \partial K^*_g \) such that \( |G^+_k(z)| < \epsilon/2 \) for \( z \in W \). Further choose \( k_2 \geq k_1 \) large enough, such that \( \frac{\tilde{M}}{2^{k_2}} < \epsilon/4 \) for every \( k \geq k_2 \) and
\[
\tilde{C}_{k_2} := \bigcup_{k=k_2}^\infty C_k \subset W.
\]
Then for every \( z \in \tilde{C}_{k_2} \), \( |G^+_k(z) - G^+_g(z)| < \epsilon \) whenever \( k \geq k_2 \). Note that
\[
(C \cap \cup_{k=1}^\infty U_k^+) \setminus \tilde{C}_{k_2} \subset C \cap \cup_{k=1}^\infty U_k^-\]
and \( C \cap \cup_{k=1}^\infty U_k^-\) is a compact set contained in \( U^+_S \). Hence there \( k_3 \geq 1 \) such that for every \( z \in (C \cap \cup_{k=1}^\infty U_k^+) \setminus \tilde{C}_{k_2} \), \( |G^+_k(z) - G^+_g(z)| \leq \epsilon \). Thus \( |G^+_k - G^+_g| < \epsilon \) on \( C \) for \( k \geq \max\{k_1, k_2, k_3\} \). □

**Remark 4.3.** Note that in the proof of Theorem 1.3, we use the pointwise convergence of \( G^+_k \) to prove \( G^+_g \) is a continuous function. However, proof of Lemma 4.2 uses the continuity of \( G^+_g \) crucially, to establish the convergence is uniform on compact subsets of \( \mathbb{C}^2 \).

Now we complete

**Proof of Theorem 4.1.** We will show that for a given compact set \( C \subset \mathbb{C}^2 \) and an \( \epsilon > 0 \) there exists \( k_C \) such that \( |G_k - G_k|_C < \epsilon/2 \) for every \( k \geq k_C \), and use Lemma 4.2.

As before for the compact subset \( C \) by Lemma 2.8 and Remark 3.2, there exists \( k \geq 1 \) such that \( ||h(z)|| < R \) whenever \( h \in \mathcal{G}_k(z) \) for every \( z \in C \) and \( k \geq k_1 \). Now choose \( k \) such that \( \frac{\tilde{M}}{2^k} < \epsilon \) for \( k \geq k_2 \) where \( \tilde{M} \) as obtained in Lemma 4.2. Thus, for every \( z \in C \) with \( h \in \mathcal{G}_k(z) \) where \( k \geq k_C = \max\{k_1, k_2\} \)
\[
\frac{\log^+ ||h(z)||}{d_h} < \frac{\tilde{M}}{2^k} < \frac{\epsilon}{2}
\]

**Step 1:** For \( k \geq k_C \), \( G^+_h(z) < \frac{\epsilon}{2} \) for every \( z \in C \) and \( h \in \mathcal{G}_k(z) \), i.e., \( \left| \frac{\log^+ ||h(z)||}{d_h} - G^+_h(z) \right| < \epsilon \).

If \( z \in K^+_h \cap C \) then \( G^+_h(z) = 0 \). So suppose \( z \notin K^+_h \cap C \), i.e., \( h(z) \in V_R \). Let \( h = h_k \circ \ldots \circ h_1 \) where \( h_i \in \mathcal{G} \) for every \( 1 \leq i \leq k \). Further let \( 1 \leq l \leq k \) and \( n_z \geq 1 \), the minimum positive integers, such that
\[
w_z := h_l \circ \ldots \circ h_1 \circ h^{n_z}(z) \in V_R^+.
\]
In particular, \( h_j \circ \ldots \circ h_1 \circ h^n(z) \in V_R \), whenever \( 1 \leq n \leq n_z \) and \( 1 \leq j \leq l - 1 \). Then \( \|w_z\| = \|h_l \circ \ldots \circ h_1 \circ h^n(z)\| < B \), where \( B \) is as chosen in Lemma 4.2. If \( 1 \leq l < k \), then from (2.1)

\[
m\|w_z\|^{d_{t+1}} \leq \|h_{t+1}(w_z)\| \leq M\|w_z\|^{d_{t+1}}
\]

where \( d_{t+1} \) is the degree of \( h_{t+1} \). As \( M > 1 \), we will consider a more robust bound to the above inequality, i.e., \( M\|w_z\|^{d_{t+1}} < \|Mw_z\|^{d_{t+1}} \) and obtain the following

\[
\log^+ \|h_{i+t} \circ \ldots \circ h_{i+1}(w_z)\| < d_{i+t} \ldots d_{t+1}(\log B + \log M).
\]

for \( 1 \leq i \leq k - l \). By continuing to repeat the same argument, we get that for every \( j \geq 1 \)

\[
\log^+ \|h^{j+n_z}(z)\| < d_j^i d_k^i \ldots d_{t+1}(\log B + \log M) < d_j^i(\log B + \log M),
\]

where \( d_h = d_k \ldots d_1 \) is the degree of \( h \). Note that if \( l = k \), then the final bound on the above inequality is anyway true. Since \( d_h \geq 2^k \), for every \( j \geq 1 \),

\[
\frac{\log^+ \|h^{j+n_z}(z)\|}{d_j^{n_z}} \leq \frac{\log B + \log M}{d_h^{n_z}} \leq \frac{\log B + \log M}{d_h} \leq \frac{\tilde{M}}{2^{k-1}} < \frac{\epsilon}{2},
\]

and thus Step 1 follows by taking the limit of \( j \rightarrow \infty \) and (4.1).

Step 2: For \( h \in \mathcal{G}_h^n(z) \) and \( l \geq 2 \)

\[
\frac{\log \|h(z)\|}{d_h} - \frac{\tilde{M}}{d_h} \sum_{i=1}^{l-1} \frac{1}{d_h^i} \leq \frac{\log \|h^l(z)\|}{d_h} \leq \frac{\log \|h(z)\|}{d_h} + \frac{\tilde{M}}{d_h} \sum_{i=1}^{l-1} \frac{1}{d_h^i}. \tag{4.2}
\]

We will prove the above by induction. Let \( l = 2 \) then by (2.1)

\[
m^{1+\sum_{i=2}^k d_h(i)} \|h(z)\| \leq \|h^2(z)\| \leq M^{1+\sum_{i=2}^k d_h(i)} \|h(z)\|, \tag{4.3}
\]

where \( d(i) = d_i \ldots d_k, \ 2 \leq i \leq k \). Now, applying logarithm and dividing by \( d_h^2 \) to the right inequality of the identity (4.3), it follows that

\[
\frac{\log \|h^2(z)\|}{d_h^2} \leq \frac{\tilde{M}}{d_h} \left( \sum_{i=1}^{k} \frac{1}{d_1 \ldots d_i} \right) + \frac{\log \|h(z)\|}{d_h} \leq \frac{\tilde{M}}{d_h} \left( \sum_{i=1}^{k} \frac{1}{2^i} \right) + \frac{\log \|h(z)\|}{d_h} \leq \frac{\tilde{M}}{d_h} + \frac{\log \|h(z)\|}{d_h},
\]

as \( \tilde{M} = \max\{\|m\|, \|M\|\} \). A similar argument applied to the left inequality of (4.3) along with the above observation gives

\[
-\frac{\tilde{M}}{d_h} + \frac{\log \|h(z)\|}{d_h} \leq \frac{\log \|h^2(z)\|}{d_h^2} \leq \frac{\tilde{M}}{d_h} + \frac{\log \|h(z)\|}{d_h},
\]

which proves (4.2) for \( l = 2 \). Now, assume (4.2) is true for some \( l \geq 2, \) by above

\[
-\frac{\tilde{M}}{d_h} + \frac{\log \|h^l(z)\|}{d_h} \leq \frac{\log \|h^{l+1}(z)\|}{d_h^{l+1}} \leq \frac{\tilde{M}}{d_h} + \frac{\log \|h^l(z)\|}{d_h}.
\]

Hence dividing further by \( d_h^{l+1} \) and substituting the assumption gives

\[
\frac{\log \|h(z)\|}{d_h} - \tilde{M} \sum_{i=1}^{l} \frac{1}{d_h^i} \leq \frac{\log \|h^l(z)\|}{d_h} - \tilde{M} \sum_{i=1}^{l-1} \frac{1}{d_h^i} \leq \frac{\log \|h^{l+1}(z)\|}{d_h^{l+1}} \leq \frac{\log \|h^l(z)\|}{d_h} + \tilde{M} \sum_{i=1}^{l} \frac{1}{d_h^i},
\]

which proves the induction hypothesis and hence the Step 2.

Thus, by taking limit \( l \rightarrow \infty \) on the identity (4.2), we have

\[
\left| G_h^+(z) - \frac{\log \|h(z)\|}{d_h} \right| \leq \frac{2\tilde{M}}{d_h} \leq \frac{\tilde{M}}{2^{k-1}} < \epsilon.
\]
Hence for \( z \in C \) and \( k \geq k_C \)
\[
|\tilde{G}_k^+(z) - G_k^+(z)| \leq \sum_{h \in \mathcal{F}_k} \frac{d_h}{D^k} |G_h^+(z) - \frac{\log \|h(z)\|}{d_h}| < \epsilon. \quad \square
\]

**Corollary 4.4. Support of \( \mu_{\tilde{g}}^+ \) is contained in the cumulative Julia sets \( J_S^\pm \).**

**Proof.** Let \( \mu_k^+ = \frac{1}{2\pi} dd^c \tilde{G}_k^+ \) then from Lemma 3.6 of [4], it follows that
\[
\text{supp} (\mu_k^+) = \bigcup_{h \in \mathcal{F}_k} J_k^+.\]
Let \( S \) be any positive \((1,1)\)-form supported in the complement of \( J_S^+ \) then \( \mu_k^+(S) = 0 \) for every \( k \geq 1 \). By Theorem 4.1 and Corollary 3.6 of [9], \( \mu_k^+ \to \mu_{\tilde{g}}^+ \), in the sense of currents, i.e., \( \mu_{\tilde{g}}(S) = 0 \). Hence the proof. A similar argument works for \( \mu_{\tilde{g}}^- \).
\[
\square
\]

Finally, we are ready to complete

**Proof of Theorem 1.5.** Note that by Corollary 4.4, \( dd^c (G_{\tilde{g}}^+) = 0 \) everywhere in the complement of \( J_S^+ \). Choose any ball \( B \) contained in \( \mathbb{C}^2 \setminus J_S^+ \). As \( G_{\tilde{g}}^+ \) is continuous on \( \mathbb{P} \), by uniqueness of solution to the Dirichlet problem it follows that \( G_{\tilde{g}}^+ \) is plurisubharmonic on \( B \) and \( \mathbb{C}^2 \setminus J_S^+ \).

Now suppose \( J_S^+ \setminus \text{supp} (\mu_{\tilde{g}}^+) \neq \emptyset \) and \( z_0 \in J_S^+ \setminus \text{supp} (\mu_{\tilde{g}}^+) \). Then there exists \( r > 0 \), such that the ball of radius \( r \) at \( z_0 \), \( B(z_0; r) \subset (\text{supp} (\mu_{\tilde{g}}^+))^c \). Let \( 0 < r' < r \). Since \( z_0 \in J_S^+ \), there exists a sequence \( \{h_n\} \subset S \) that is neither locally uniformly bounded nor uniformly divergent to infinity on \( B(z_0; r') \). In particular, there exist sequences of points \( \{z_n\} \) and \( \{w_n\} \) in \( B(z_0; r') \) such that
\[
\|h_n(z_n)\| \text{ is bounded and } \|h_n(w_n)\| \to \infty
\]
as \( n \to \infty \). Note that without loss of generality we may assume, the length of \( h_n \to \infty \) as \( n \to \infty \). Now again, by Lemma 2.8 and Remark 3.2 the above may be modified further as – for \( n \) sufficiently large,
\[
h_n(z_n) \in V_r \text{ and } h_n(w_n) \in V_{r_n}^+,
\]
where \( r_n \) is a sequence of positive real numbers that diverges to infinity as \( n \to \infty \). Hence
\[
G_{\tilde{g}}^+ \circ h_n(z_n) < C_0 \text{ and } G_{\tilde{g}}^+ \circ h_n(w_n) \to \infty. \quad (4.4)
\]
Also as \( G_{\tilde{g}}^+ \) is plurisubharmonic on \( B(z_0; r) \) and plurisubharmonic on \( \mathbb{C}^2 \), by Corollary 1.4 we have \( G_{\tilde{g}}^+ \circ h \) is plurisubharmonic on \( B(z_0; r) \) for every \( h \in S \). Now by Harnack’s inequality (See Theorem 2.5, [19] Page 16), there exists \( A > 0 \), a positive constant dependent on \( z_0, r \) and \( r' \), such that for every harmonic function \( u \) on \( B(z_0; r) \)
\[
\sup_{B(z_0; r')} u(z) \leq A \inf_{B(z_0; r')} u(z).
\]
Hence \( 0 \leq G_{\tilde{g}}^+ \circ h_n(w_n) \leq AC_0 \) which contradicts (4.4). Hence \( \text{supp} (\mu_{\tilde{g}}^+) = J_S^+ = J_S^+ \).
\[
\square
\]

**Remark 4.5.** Thus by Proposition 3.2 of [9], the measure \( \mu_{\tilde{g}} := \mu_{\tilde{g}}^+ \wedge \mu_{\tilde{g}}^- \) is a probability measure compactly supported on the intersection of the positive and negative Julia sets.

**Corollary 4.6.** The Fatou component at infinity of the semigroup \( S \) and \( S^- \), i.e.,
\[
U_S^\pm = \text{int} \left( \bigcap_{h \in S} U_h^\pm \right). \quad (4.5)
\]
Proof. Let $F^+_h$ denote the Fatou set corresponding to a $h \in S$. Note that $F^+_h = U^+_h \cup F^b_h$ where $U^+_h$ is the component at infinity and $F^b_h$ are the Fatou components contained in $K^+_h$. Similarly $F^-_h = U^-_h \cup F^b_{h^{-1}}$ where $U^-_h$ is the component at infinity of $h^{-1}$ whenever $h \in S$. By Theorem 1.5, it follows that

$$F^+_S = \mathbb{C}^2 \setminus J^+_S = \text{int} \left( \bigcap_{h \in S} F^+_h \right) = \text{int} \left( \bigcap_{h \in S} (U^+_h \cup F^b_{h \pm}) \right).$$

Hence the components at infinity, corresponding to the dynamics of the semigroup $S$ and $S^-$ is given by \([4.5]\).

\[ \square \]

Remark 4.7. Also note, if $K^-_S = K^+_S$ then $K^+_h = K^+_h$ and by Theorem 5.4 from \([23]\), there exists $m, n \geq 1$ such that $h^m_h = h^n_h$. Thus $J^+_S = J^+_h$ for every $h \in S$. In particular from Theorem 5.8, it follows that $G^+_h = G^+_h$ for every $h \in S$, i.e., the Green’s function is unique.

However, the next corollary proves that the positive and negative Green’s functions obtained corresponding to the semigroup $S$ is generally non-unique (i.e., whenever $K^+_S \subsetneq K^+_S$), as a consequence of Corollary 1.4 and Corollary 3.3.

\textbf{Corollary 4.8.} If $K^+_S \subsetneq K^+_S$ then the Green’s functions $G^+_h$ are non-unique and depends on the generating set $\mathcal{G}$.

\[ \text{Proof.} \] Suppose not, i.e., let the positive Green’s function be unique corresponding to semigroup $S$. By Proposition 2.1, it follows that $S$ admits a minimal generating set $\mathcal{G}_0$. Let $\mathcal{G}_0 = \mathcal{G}_0 \cup h, h \in S \setminus \mathcal{G}_0$. Then $S = \langle \mathcal{G}_0 \rangle = \langle \mathcal{G}_h \rangle$. Then by assumption, $G^+_{\mathcal{G}_0} = G^+_{\mathcal{G}_h}$, and thus from Corollary 1.4 we have that

$$\left( D_{\mathcal{G}_0} + d_h \right) G^+_{\mathcal{G}_0}(z) = \sum_{H_i \in \mathcal{G}_0} G^+_{\mathcal{G}_0}(H_i(z)) + G^+_{\mathcal{G}_0}(h(z)) = D_{\mathcal{G}_0} G^+_{\mathcal{G}_0}(z) + G^+_{\mathcal{G}_0}(h(z))$$

where $D_{\mathcal{G}_0}$ is the total degree of the generating set $\mathcal{G}_0$ and $d_h$ is the degree of $h \in S$. Hence $G^+_{\mathcal{G}_0}(z) = d_h G^+_{\mathcal{G}_0}(h(z))$, i.e., $G^+_{\mathcal{G}_0}(z) = 0$ if $z \in K^+_h$. But from Corollary 3.4, the above implies $K^+_h \subset K^+_S$ for every $h \in S$. Suppose $K^+_S \setminus K^+_h \neq \emptyset$ and $z_0 \in K^+_S \setminus K^+_h$, then there exist sequence \(\{h_n\}\) and \(\{h_n\}\) in $S$ such that both the lengths of $h_n$ and $h_n$ go to infinity as $n \to \infty$. Further by Lemma 2.8 there exists $k_0 \geq 1$ such that for every $n \geq k_0$, $h_n(z_0) \in V^+_R$ and $h_n(z_0) \in V_R$.

Then by Corollary 3.3

$$G^+_{\mathcal{G}_0}\left(H^l_1 \circ h_{k_0}(z_0)\right) = \log |\pi_2 \circ H^l_1 \circ h_{k_0}(z_0)| + O(1),$$

for every $l \geq 1$. Hence $G^+_{\mathcal{G}_0}\left(H^l_1 \circ h_{k_0}(z_0)\right) \to \infty$ as $l \to \infty$. Fix $l_1 \geq 1$, sufficiently large such that $G^+_{\mathcal{G}_0}\left(H^l_1 \circ h_{k_0}(z_0)\right) > B$, where $B = \max\{G^+_{\mathcal{G}_0}(z) : z \in V_R\}$. Let $k_1 \geq k_0$

$$h_1 = H^l_1 \circ h_{k_0} \text{ and } h_2 = \tilde{h}_{k_1},$$

such that the degree of $h_2 = d_{h_2} > d_{h_1} = \text{degree of } h_1$. Since we have assumed that $G^+_{\mathcal{G}_0} \text{ is unique, it follows that}$

$$(d_{h_2} - d_{h_1}) G^+_{\mathcal{G}_0}(z_0) = G^+_{\mathcal{G}_0}(h_2(z_0)) - G^+_{\mathcal{G}_0}(h_1(z_0)),$$

i.e., $G^+_{\mathcal{G}_0}(z_0) < 0$, which is a contradiction! Thus $K^+_S = K^+_S$.

Now, if the negative Green’s function is unique, similar argument as above will imply $K^-_S = K^-_S$. Hence $K^-_{h_1} = K^-_{h_2}$ for every $h_1, h_2 \in S$. Now by Remark 4.7, the positive Green’s function will also be unique, i.e., $K^+_S = K^+_S$, which is again a contradiction! \(\square\)
5. Equidistributed projective currents and proof of Corollary 1.6

Recall that every polynomial map \( g : \mathbb{C}^2 \to \mathbb{C}^2 \), i.e., \( g(x, y) = (g_1(x, y), g_2(x, y)) \) where \( g_1 \) and \( g_2 \) are polynomials in \( x \) and \( y \), extends as a rational map \( \tilde{g} \) on \( \mathbb{P}^2 \). Further in the homogeneous coordinates of \( \mathbb{P}^2 \), it is defined as

\[
\tilde{g}[x : y : z] = \left[ z^d g_1 \left( \frac{x}{z}, \frac{y}{z} \right) : z^d g_2 \left( \frac{x}{z}, \frac{y}{z} \right) : z^d \right]
\]

where \( d = \max\{ \text{degree of } g_1, \text{degree of } g_2 \} \). Now for any map \( h \in \mathcal{S} \), \( h^{-1} \) is also a polynomial map. Hence both \( h \) and \( h^{-1} \) extend as rational maps on \( \mathbb{P}^2 \), in the homogeneous coordinates. Further the degree of \( \pi_2 \circ h \) is strictly greater than \( \pi_1 \circ h \) and \( \pi_2 \circ h(x, y) = y^{d_h} + \text{l.o.t.} \). Hence the indeterminancy point of the rational map \( \tilde{h} \) in \( \mathbb{P}^2 \) (for every \( h \in \mathcal{S} \)) is \( I^+ = [1 : 0 : 0] \). A similar argument gives that the indeterminancy point of \( \tilde{h}^{-1} \) is \( I^- = [0 : 1 : 0] \). Let \( \mathcal{S} \) and \( \mathcal{S}^- \) be the family of the rational maps on \( \mathbb{P}^2 \) defined as

\( \mathcal{S} = \{ h : h \in \mathcal{S} \} \) and \( \mathcal{S}^- = \{ h^{-1} : h \in \mathcal{S} \} \).

Next, we study the dynamics of the above families in \( \mathbb{P}^2 \) and generalise a few facts from \( \mathbb{12} \). Note that the line at infinity, except the point \( I^+ \), i.e., \( L^+_\infty = \{ [x : y : z] \in \mathbb{P}^2 : z = 0 \} \) \( \setminus I^+ \) is contracted to \( I^- \) by every \( h \in \mathcal{S} \). Similarly the line at infinity, except the point \( I^- \), i.e., \( L^-\infty = \{ [x : y : z] \in \mathbb{P}^2 : z = 0 \} \) \( \setminus I^- \) is contracted to \( I^+ \) by every \( h \in \mathcal{S}^- \). Also, \( V^+ \) (respectively \( V^- \)) lies in the basis of attraction of \( I^- \) (respectively \( I^+ \)) for every \( h \in \mathcal{S} \) (respectively for every \( \tilde{g} \in \mathcal{S}^- \)). Hence \( I^- \) is attracting fixed point for every \( \tilde{h} \in \mathcal{S} \) and \( I^+ \) is attracting fixed point for every \( \tilde{g} \in \mathcal{S}^- \). Thus, we define the following sets.

- \( \tilde{U}^+_\mathcal{S} = \text{int} \left( \bigcap_{h \in \mathcal{S}} \tilde{U}^+_h \right) \), where \( \tilde{U}^+_h \) is the basin of attraction of \( I^- \) for \( \tilde{h} \).
- \( \tilde{U}^-\mathcal{S} = \text{int} \left( \bigcap_{h \in \mathcal{S}} \tilde{U}^-_h \right) \), where \( \tilde{U}^-_h \) is the basin of attraction of \( I^+ \) for \( \tilde{h}^{-1} \).

**Proposition 5.1.** The sets \( \tilde{U}^+_\mathcal{S} \cap \mathbb{C}^2 = U^+_\mathcal{S} \). Also, the closure of the sets \( K^+\mathcal{S} \) in \( \mathbb{P}^2 \) is given by \( K^+\mathcal{S} = K^+\mathcal{S} \cup I^± \).

**Proof.** Let \( \tilde{U}^+_\mathcal{S} \) be the basin of attraction of \( I^± \) for the family \( \mathcal{S} \) and \( \mathcal{S}^- \) in \( \mathbb{P}^2 \) \( \setminus I^± \), i.e.,

\[
\tilde{U}^+_\mathcal{S} = \{ \tilde{z} \in \mathbb{P}^2 \setminus I^+: \exists \text{ a neighbourhood } W \text{ of } \tilde{z} \text{ such that } \tilde{h}_{n|W} \to I^- \text{ for every } \{h_n\} \subset \mathcal{S} \}
\]

and

\[
\tilde{U}^-\mathcal{S} = \{ \tilde{z} \in \mathbb{P}^2 \setminus I^- : \exists \text{ a neighbourhood } W \text{ of } \tilde{z} \text{ such that } \tilde{h}_{n|W} \to I^+ \text{ for every } \{h_n\} \subset \mathcal{S} \}.
\]

Observe that by definition, if \( (x, y) \in U^+_\mathcal{S} \), then \( [x : y : 1] \in \tilde{U}^+_\mathcal{S} \). In particular \( U^+_\mathcal{S} \subset \tilde{U}^+_\mathcal{S} \cap \mathbb{C}^2 \).

Now for any point \( \tilde{z}_0 = [x_0 : y_0 : z_0] \in L^\pm_{\infty} \), \( z_0 = 0 \) and \( |y_0| \neq 0 \). Hence for every \( h \in \mathcal{S} \), \( \tilde{h}(\tilde{z}_0) = \tilde{h}[x_0 : y_0 : 0] = [0 : 1 : 0] \) is immediate.

**Claim:** There exist open sets \( W^± \) containing \( L^\pm_{\infty} \) which is contained \( \tilde{U}^+_\mathcal{S} \), respectively.

**Case 1:** Suppose \( |x_0| < |y_0| \), then choose a neighbourhood \( W_{\tilde{z}_0} \) of \( \tilde{z}_0 \) such that \( |x| < |y| \) for every \( \tilde{z} = [x : y : z] \in W_{\tilde{z}_0} \) and \( |z| < R^{-1}|y| \) if \( z \neq 0 \), where \( R > R_S \), the radius of filtration as in Lemma 2.3. Hence for \( \tilde{z} \in W_{\tilde{z}_0} \setminus L^\pm_{\infty} \), \( \tilde{z} = [x : y : 1] \) such that \( (x, y) \in V^+_R \), i.e, \( \tilde{h}(\tilde{z}) \to [0 : 1 : 0] \) as length of \( h \) tends to infinity.

**Case 2:** Otherwise, there exists some \( \alpha > 1 \) such that \( |x_0| < \alpha |y_0| \). Note, we need to choose an appropriate neighbourhood of \( \tilde{z}_0 \) contained in \( \tilde{U}^+_\mathcal{S} \). We will do so in the light of Remark 5.3 which is a consequence of the following modification of Lemma 2.2 from \( \mathbb{4} \).
Lemma 5.2. Let $H(x,y) = (y, p(y) - ax)$ where $p$ is a polynomial of degree $d_H \geq 2$ and $a \neq 0$. Also, let $R_H > 0$ be the radius of filtration for $H$ as obtained in Lemma 2.2 of [7]. For $R > 0$ and $\alpha > 1$ we define the following sets as

$$V_{\alpha,R}^+ = \{(x,y) \in \mathbb{C}^2 : |x| \leq \alpha |y|, |y| \geq \alpha^{-1} R\} \quad V_{\alpha,R}^- = \{(x,y) \in \mathbb{C}^2 : |x| \geq \alpha |y|, |y| > \alpha^{-1} R\}$$

and

$$\bar{V}_{\alpha,R}^- = \{(x,y) \in \mathbb{C}^2 : |y| \leq \alpha |x| , |x| \geq \alpha^{-1} R\}, \quad \bar{V}_{\alpha,R}^+ = \{(x,y) \in \mathbb{C}^2 : |y| \geq \alpha |x| , |x| > \alpha^{-1} R\}.$$ 

Then there exists an $R^\alpha > \alpha R_H$ such that $H(V_{\alpha,R^\alpha}) \subset V_{R_H}^+$ and $H^{-1}(\bar{V}_{\alpha,R^\alpha}) \subset V_{R_H}^-$. 

Proof. Note that

$$V_{R_H}^+ \subset V_{\alpha,R_H}^+, \quad V_{\alpha,R_H}^- \subset V_{R_H}^- \quad V_{R_H}^- \subset \bar{V}_{\alpha,R_H}^- \quad \mbox{and} \quad \bar{V}_{\alpha,R_H}^+ \subset V_{R_H}^+.$$

Also there exists an $R_\alpha > \alpha R_H$, sufficiently large, and constant $C_1 > 0$ such that for $(x,y) \in V_{\alpha,R_\alpha}$, i.e., $|y| = \alpha^{-1} R > \alpha^{-1} R_\alpha$ and $|x| \leq R$ 

$$|\pi_2 \circ H(x,y)| > \alpha^{-d_H} C_1 R^{d_H} - |a| R > \alpha^{-1} R = |\pi_1 \circ H(x,y)|.$$

Similarly there exists an $\tilde{R}_\alpha > \alpha R_H$, sufficiently large, and constant $C_2 > 0$ such that for $(x,y) \in \bar{V}_{\alpha,\tilde{R}_\alpha}$, i.e., $|x| = \alpha^{-1} R > \alpha^{-1} \tilde{R}_\alpha$ and $|y| \leq R$ 

$$|\pi_1 \circ H^{-1}(x,y)| > \alpha^{-d_H} C_2 R^{d_H} - |a|^{-1} R > \alpha^{-1} R = |\pi_2 \circ H^{-1}(x,y)|.$$

Let $R^\alpha$ be the maximum of $R_\alpha$ and $\tilde{R}_\alpha$. Then $H(V_{\alpha,R^\alpha}) \subset V_{R^\alpha}^+$ and $H^{-1}(\bar{V}_{\alpha,R^\alpha}) \subset V_{R^\alpha}^-$. \hfill \Box

Remark 5.3. By a similar technique as in the proof of Lemma 2.3, the above further assures that $R^\alpha > \alpha R_S$, the radius of filtration of the semigroup $S$, such that $h(V_{\alpha,R^\alpha}) \subset V_{R^\alpha}^+$ and $h^{-1}(\bar{V}_{\alpha,R^\alpha}) \subset V_{R^\alpha}^-$ for every $h \in S$. 

We now choose a neighbourhood $W_{z_0}$ of $z_0$ such that $|x| < \alpha |y|$ for every $z = [x : y : z] \in W_{z_0}$ and $|z| R^\alpha < \alpha |y|$ if $z \neq 0$, where $R^\alpha$ is as obtained in Remark 5.3. Hence for $z \in W_{z_0} \setminus L_\infty^+$, $z = [x : y : 1]$ such that $(x,y) \in V_{\alpha,R^\alpha}$, i.e., $h(z) \rightarrow [0 : 1 : 0]$ as length of $h$ tends to infinity. 

By similar arguments for $h^{-1}$, $h \in S$ there exists an open set $W^-$ containing $L_\infty^-$ such that $W^- \subset \bar{U}_S^-$. Further note that for $z \in \bar{U}_S^+ \setminus L_\infty^+$, $z = [x : y : 1]$ such that $(x,y) \in \bar{U}_S^+$. Since $p^2 = C^2 \cup L_\infty^+ \cup I^\pm$, $\bar{U}_S^+ \cap C^2 = \bar{U}_S^+ \setminus L_\infty^+ = U_S^+$.

Finally, as a consequence of Corollary 4.6 and Proposition 5.5 of [12, page 28] — which implies $\bar{U}_h^+ \cap C^2 = U_h^+$ for every $h \in S$ — we can write

$$\bar{U}_S^+ \setminus L_\infty^+ = \bar{U}_S^+ \cap C^2 = U_S^+ = \int \bigcup_{h \in S} U_h^+ = \int \bigcap_{h \in S} (\bar{U}_h^+ \cap C^2) = \int \bigcap_{h \in S} (\bar{U}_h^+ \setminus L_\infty^+).$$

But $L_\infty^+ \subset W^+ \subset \bar{U}_S^+$ and $\bar{U}_S^+ \subset \bar{U}_h^+$ for every $h \in S$, i.e., $W^+$ is contained in the interior of $\left(\bigcap_{h \in S} \bar{U}_h^+\right)$. Hence the above identity reduces to

$$U_S^+ = \bar{U}_S^+ \setminus L_\infty^+ = \left(\int \bigcap_{h \in S} \bar{U}_h^+\right) \setminus L_\infty^+ = \left(\int \bigcap_{h \in S} \bar{U}_h^+\right) \cap C^2 = \bar{U}_S^+ \cap C^2. \quad (5.1)$$

Now, since $L_\infty^+ \subset \bar{U}_S^+$ it follows that $\overline{K_S^+} \subset K_S^+ \cup I^\pm$. Also, $K_h^+ \subset K_S^+$ for every $h \in S$ and by Proposition 5.8 in [12, page 29], $I^\pm \subset K_h^+$. Hence $I^\pm \subset \overline{K_S^+}$, which completes the proof. \hfill \Box
Remark 5.4. As a consequence of Proposition 5.1 it follows that the basins of attraction of \(I^\pm\) for the families \(\overline{S}\) and \(\overline{S}^-\) are \(\overline{U}_S^\pm\), respectively. Further, the closure of the positive and negative Julia sets \(\overline{J}_S^\pm\) in \(\mathbb{P}^2\), i.e., \(\overline{J}_S^\pm = \overline{J}_S^+ \cup \overline{J}_S^-\). Hence from Skoda-El-Mir extension Theorem (see [9]), the \((1,1)\)-currents \(\mu_{\tilde{g}_1}^\pm\) extends by 0 to positive closed \((1,1)\)-currents (will also be denoted by \(\mu_{\tilde{g}_1}^\pm\)) on \(\mathbb{P}^2\). Now as \(G_g^\pm\) are the logarithmic potential of \(\mu_{\tilde{g}_1}^\pm\) restricted to \(\mathbb{C}^2\) — from the observation in Example 3.7 in [12] — the functions \(g_g^\pm(z) = G_g^\pm(z) - \frac{1}{2} \log(\|z\|^2 + 1)\) are the quasi-potentials corresponding to the currents \(\mu_{\tilde{g}_1}^\pm\) on \(\mathbb{P}^2\) (respectively).

Remark 5.5. Note that the functions \(g_g^\pm(z)\) is uniformly bounded and pluriharmonic on \(V_{R_S}^+\) (respectively) from Corollary 3.3. Hence for every \(k \geq 1\), on \(U_k := \bigcap_{h \in \mathcal{G}_k} h^{-1}(V_R^+)\)
\[
G_g^+(z) - \frac{1}{2} \log(\|z\|^2 + 1) = \frac{1}{D^k} \sum_{h \in \mathcal{G}_k} G_g^+(h(z)) - \frac{1}{2} \log(\|z\|^2 + 1)\]
is pluriharmonic.

Since \(\overline{U}_S^+ = \overline{U}_S^+ \cup L_{\infty}^+\) is an open set containing \(L_{\infty}^+\), \(g_g^+(z)\) extends as a pluriharmonic function on \(\overline{U}_S^+\). A similar arguments gives \(g_g^-(z)\) extends as a pluriharmonic function on \(\overline{U}_S^-\).

Next, we prove a generalisation of Theorem 6.6 from [12] in our setup.

**Proposition 5.6.** Let \(\mathcal{V}\) be a neighbourhood of \(I^-\) and \(\{S_k\}\), a sequence of positive \((1,1)\)-closed currents of mass 1 in \(\mathbb{P}^2\) such that each \(S_k\), \(k \geq 1\), admits a quasi-potential \(u_k\), satisfying \(0 < |u_k| \leq A\) (a constant) on \(\mathcal{V}\). Then there exists \(c > 0\) such that for every \(C^2\) test \((1,1)\)-form \(\phi\) on \(\mathbb{P}^2\)
\[
|\langle \mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm, \phi \rangle| \leq \frac{ck}{2^k} \|\phi\|_{C^2}, \ i.e., \ \mu_{\tilde{g}_1}^\pm := \frac{1}{D^k} \sum_{h \in \mathcal{G}_k} h^*(S_k) \rightarrow \mu_{\tilde{g}_1}^\pm.
\]

**Proof.** Note that we may assume that \(ddc^\prime u_k = S_k - \omega_{FS}\), where \(\omega_{FS}\) is the Fubini-Study \((1,1)\)-form on \(\mathbb{P}^2\) and \(\mathcal{V}\) is a sufficiently small neighbourhood of \(I^-\) contained in \(\overline{U}_S^+\). In particular, \(\mathcal{V} \cap C^2 \subset V_{R_S}^+\). Then for \(z \in \mathcal{V} \cap V_{R_S}^+\), and by the identities in Section 4, \(g_{h_k}^+(z) \leq \widetilde{M}\)
\[
g_{h_k}^+(z) := \tilde{G}_{h_k}^+(z) - \frac{1}{2} \log(\|z\|^2 + 1) \leq \tilde{M}\]for every \(h \in \mathcal{G}_k\), \(k \geq 1\).

Thus by Remarks 5.4 and 5.5 the quasipotentials \(u_k - g_{h_k}^+, u_k - g_{k}^+\) and \(u_k - g_{k}^+\) are uniformly bounded on \(\mathcal{V}\), by \(M + A\), with \(g_{h_k}^+ \rightarrow g_{k}^+\) uniformly on \(\mathcal{V}\). Also by the proof of Lemma 5.2, let \(\alpha \geq 1\) be such that \(\text{supp}(S_k) \cap C^2 \subset V_{R_a} \cup V_{R_a}^-\), for every \(k \geq 1\). Hence \(\text{supp}(h^*(S_k)) \cap C^2 \subset V_{R_a} \cup V_{R_a}^-\) for every \(h \in \mathcal{S}\). Thus we refine \(\mathcal{V}\) again, so that \(\mathcal{V} \cap C^2 \subset V_{R_a}^+\) and by continuity
\[
\text{supp}(\mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm) \cap \mathcal{V} = \emptyset, \ \text{supp}(\mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm) \cap \mathcal{V} = \emptyset, \ \text{supp}(\frac{1}{dh} h^*(S_k) - \mu_{\tilde{g}_1}^\pm) \cap \mathcal{V} = \emptyset.
\]
for every \(h \in \mathcal{S}\) and \(k \geq 1\). Hence
\[
|\langle \mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm, \phi \rangle|_{\mathbb{P}^2} = |\langle \mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm, \phi \rangle|_{\mathbb{P}^2 \setminus \mathcal{V}} \quad \text{and} \quad |\langle \mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm, \phi \rangle|_{\mathbb{P}^2 \setminus \mathcal{V}} = |\langle \mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm, \phi \rangle|_{\mathbb{P}^2 \setminus \mathcal{V}} \quad \text{(5.2)}
\]
Also \(H_{-1}(\mathbb{P}^2 \setminus \mathcal{V}) \subset \mathbb{P}^2 \setminus \mathcal{V}\) as \(I^+\) is a super attracting fixed point for every \(h_i\), \(1 \leq i \leq n_0\). Hence the \(C^1\)-norm of every \(h^{-1}\) is bounded by \(M^k\) for some \(M > 0\), whenever \(h \in \mathcal{G}_k\). Since \(u_k - g_{h_k}^+, u_k - g_{k}^+\) and \(u_k - g_{k}^+\) are d.s.h. functions in \(\mathbb{P}^2\), by [12] Lemma 3.11] the DSH-norm (see [12] Section 3 for definition) of \(u_k - g_{h_k}^+, u_k - g_{k}^+\) and \(u_k - g_{k}^+\) are uniformly bounded for every \(k \geq 1\) and \(h \in \mathcal{S}\). Hence by [12] Lemma 3.13] there exists a constant \(C_0 \geq 0\) such that
\[
|\langle \mu_{\tilde{g}_1}^\pm - \mu_{\tilde{g}_1}^\pm, \phi \rangle|_{\mathbb{P}^2 \setminus \mathcal{V}} = D^{-k} \sum_{h \in \mathcal{G}_k} |\langle u_k - g_{h_k}^+, dd^\prime (\phi \circ h^{-1}) \rangle| \leq C_0(n_0D^{-1})(1 + \log^+ M^k)\|\phi\|_{C^2} \quad \text{(5.3)}
\]
which completes the proof. □
Remark 5.7. Further Lemma 3.11 and Lemma 3.13 of [12] also gives that for \( C_0 > 0 \), where \( C_0 \) is as obtained in the above proof of Proposition 5.6:

\[
\| \langle \mu_k - \mu_k^* \rangle_{\mathcal{F}} \|_{L^2} = D_k^{-k} \sum_{h \in \mathcal{G}_k} \| (u_k - g_k, dd^c(\phi \circ h^{-1})) \|_{L^2} \leq C_0 (e^{\log + M^4 k}) \| \phi \|_{L^2},
\]

\[
\| \langle d_k h^*(S_k) - \mu_k^* \rangle_{\mathcal{F}} \|_{L^2} = d_k^{-1} \sum_{h \in \mathcal{G}_k} \| (u_k - g_k, dd^c(\phi \circ h^{-1})) \|_{L^2} \leq C_0 d_k^{-1} (1 + \log^+ M^4 k) \| \phi \|_{L^2}.
\]

Now, as a direct consequence of the above proposition we observe the following.

**Corollary 5.8.** Let \( S \) be a closed positive \((1,1)\)-current in \( \mathbb{P}^2 \) of mass 1, such that support of \( S \) does not contain the point \([0 : 1 : 0] \) and \( h \) be the extension of \( h \) to \( \mathbb{P}^2 \), \( h \in \mathcal{S} \). Then

\[
\lim_{k \to \infty} \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*(S) \to \mu_{\mathcal{F}}^+.\]

**Proof.** Note that if \( S \) is an \((1,1)\) positive closed current of mass 1 in \( \mathbb{P}^2 \), and let \( u \) be a quasi-potential associated to \( S \), i.e., \( u \) is a quasi p.s.h function and \( dd^c u = S - \omega_{FS} \). Then \( u \) is bounded on a neighbourhood of \( I^- \) and by Proposition 5.6, the proof follows.

**Remark 5.9.** Since the analogue of Proposition 5.6 is true for the current \( \mu_{\mathcal{F}}^- \) as well, if \( S \) is a positive \((1,1)\) current of mass 1 on \( \mathbb{P}^2 \) then

\[
\lim_{k \to \infty} \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*(S) \to \mu_{\mathcal{F}}^-.
\]

Thus, we conclude the uniqueness of \( \mu_{\mathcal{F}}^\pm \) from Corollary 5.8.

**Proof of Corollary 1.6.** Let \( S \) be a positive \((1,1)\) closed current supported on \( J^+_S \) satisfying property [1.6]. Then \( S \) extends across \( I^+ \) by zero as a closed \((1,1)\) current of mass 1 on \( \mathbb{P}^2 \) that does not intersect \( I^- \). Thus by Theorem 5.8 it follows that on \( \mathbb{C}^2 \)

\[
\lim_{k \to \infty} \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*(S) \to \mu_{\mathcal{F}}^+.
\]

But from [1.6], \( \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*(S) = S \). Hence \( S = \mu_{\mathcal{F}}^+ \). A similar argument for \( \mu_{\mathcal{F}}^- \).

Finally, we end this section with the interpretation of Corollary 5.8 for algebraic varieties.

**Corollary 5.10.** Let \( S \) be an affine algebraic variety of codimension 1 in \( \mathbb{C}^2 \), then there exist non-zero constants \( c^\pm > 0 \) such that

\[
\lim_{k \to \infty} \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*[S] \to c^+ \mu_{\mathcal{F}}^+ \text{ and } \lim_{k \to \infty} \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*[S] \to c^- \mu_{\mathcal{F}}^-.
\]

**Proof.** Let \( S \) be an algebraic variety of codimension 1 in \( \mathbb{C}^2 \), i.e., \( S = \{(x, y) \in \mathbb{C}^2 : p(x, y) = 0\} \) where \( p \) is a polynomial of degree at least 1. Let \( p(x, y) = \sum_{a, b \in \mathbb{N}} p_{ab} x^a y^b \) such that \( p_{ab} = 0 \) whenever \( a \) and \( b \) is greater than some fixed positive integer. The degree of \( p \) is \( d_p = \text{max}\{a + b : p_{ab} \neq 0\} \).

**Case 1:** Let \( p(x, y) = c p_{xy} d_p + \text{l.o.t.} \) then \( S \) is a quasi-projective variety of \( \mathbb{P}^2 \) of codimension 1 and \( S \) extends to \( \mathbb{P}^2 \) as an analytic variety, that does not contain \( I^- \). Hence the current of integration of \( [S] \) is a closed positive \((1,1)\) current of finite mass, say \( c^+ \) (see [9, Page 140]).

Thus from Theorem 5.8 it follows that

\[
\lim_{k \to \infty} \frac{1}{D_k} \sum_{h \in \mathcal{G}_k} h^*[S] \to c^+ \mu_{\mathcal{F}}^+.
\]

**Case 2:** For any polynomial \( p \), a generalisation of Proposition 4.2 in [1] (or Proposition 8.6.7 in [20]) gives that there exists \( k \geq 1 \) such that \( p_k = p \circ h \) for all \( h \in \mathcal{G}_k \), as in Case 1, i.e.,

**Claim:** There exists \( k_0 \geq 1 \) such that for every \( h \in \mathcal{G}_k \) and \( k \geq k_0 \),

\[
p \circ h(x, y) = c_{p_k} h^* + \text{l.o.t.}
\]

(5.4)
For a positive integer \( i \geq 1 \), let \( \lambda_i(p) = \max\{a + ib : p_{ab} \neq 0\} \) and 
\[ \rho_i(p) = \{(a, b) : a + ib = \lambda_i(p) \text{ and } p_{ab} \neq 0\}, \]
i.e., the terms in the leading part of the polynomial \( p \) with weight \( i \). Let \( H \) be a generalised Hénon map of the form (1.3) of degree \( d_H \). We first note the following result, which is a rephrasing of Lemma 8.6.5 from [26].

**Result.** For a polynomial \( p(x, y) = \sum_{a, b \in \mathbb{N}} p_{ab} x^a y^b \) the number of elements in the leading term of \( p \circ H \) in \( i \) weight, \( i \geq 2 \), i.e., \( \sharp \rho_i(p \circ H) \) satisfies the following inequality
\[ \sharp \rho_i(p \circ H) \leq 1 + \frac{\sharp \rho_{d_H}(p) - 1}{i}. \]

So if \( H \) is a map of the form (1.2), of degree sufficiently large, it follows from the above result, that the number of leading terms in any weight \( i \), \( i \geq 2 \) of the polynomial \( p \circ H \) is 1. Now if \( H \) is a generalised Hénon map of form (1.3), then the degree of \( p \circ H \circ H(x, y) \) is \( \lambda_{d_H}(p \circ H) \) and \( \rho_{d_H}(p \circ H) = c_{p_{ab}} x^a y^b \) where \( a + d_H b = \lambda_{d_H}(p \circ H) \). Hence \( p \circ H \circ H(x, y) = c_{p_{ab}} x^a y^{\lambda_{d_H} + 1} \) o.t.

Since for very \( h \in \mathcal{G}_k \), degree of \( h \) is greater than \( 2^k \), from Lemma 8.6.6 of [26] there exists \( k_0 \geq 1 \) such that the polynomial \( p \circ h \) has the desired form (5.4) whenever \( h \in \mathcal{G}_k, k \geq k_0 \).

Thus, Case 1 applied to every polynomial \( p_n, h \in \mathcal{G}_k \), proves Corollary 5.10 for \( \mu^*_h \). \( \Box \)

6. **Green’s functions for non-autonomous sequences in \( \mathcal{S} \)**

Let \( \{h_k\} \subset \mathcal{S} \), where \( \mathcal{S} \) is the semigroup of Hénon maps as defined in (1.1). Recall that to study dynamics of the sequence \( \{h_k\} \), one needs to study the behaviour of the sequences \( \{h(k)\} \) and \( \{h^{-1}(k)\} \) defined as
\[ h(k) := h_k \circ \cdots \circ h_1 \text{ and } h^{-1}(k) = h_k^{-1} \circ \cdots \circ h_1^{-1}. \]

Now as each \( h_i, i \geq 1 \), is generated by the finitely many elements of \( \mathcal{G} \), there exist a sequence \( \{\tilde{h}_k\} \subseteq \mathcal{G} \) and a sequence \( \{n_k\} \) of positive integers such that for every \( k \geq 1 \)
\[ h(k) = h_k \circ \cdots \circ h_1 = \tilde{h}_{n_k} \circ \cdots \circ \tilde{h}_1 = \tilde{h}(n_k). \] (6.1)

Hence with abuse of notation, we will assume that \( \{h_k\} \subseteq \mathcal{G} \), i.e., the elements of the sequence \( \{h_k\} \) varies within the finite collection \( \mathcal{G} = \{h_i : 1 \leq i \leq n_0\} \). Also analogue of the (positive and negative) escaping sets and the bounded sets for the sequence \( \{h_k\} \) is defined as
\[ U^+_{\{h_k\}} = \{z \in \mathbb{C}^2 : h(k)(z) \to \infty \text{ as } k \to \infty\}, \quad U^-_{\{h_k\}} = \{z \in \mathbb{C}^2 : h^{-1}(k)(z) \to \infty \text{ as } k \to \infty\} \]
and
\[ K^+_{\{h_k\}} = \{z \in \mathbb{C}^2 : \text{bounded}\}, \quad K^-_{\{h_k\}} = \{z \in \mathbb{C}^2 : \text{bounded}\}. \]

Since \( h(k) \in \mathcal{G}_k \), by Remark 2.4 the following inequality holds for every \( (x, y) \in V_R^+, R > R_S \) (sufficiently large)
\[ m|y|^{d_k} < \|h_k(x, y)\| = |\pi_2 \circ h_k(x, y)| < M|y|^{d_k}, \] (6.2)
where \( d_k \) is the degree of \( h_k \). Also for \( (x, y) \in V_R^- \)
\[ m|x|^{d_k} < \|h_k^{-1}(x, y)\| = |\pi_1 \circ h_k^{-1}(x, y)| < M|x|^{d_k}. \] (6.3)

Also \( h(k)(V_R^+) \subset \text{int}(V_{R_k}^+) \) and \( h(k)^{-1}(V_R^-) \subset \text{int}(V_{R_k}^-) \) where \( R_k \to \infty \) as \( k \to \infty \).

**Remark 6.1.** Thus \( \text{int}(V_R^+) \subset U^+_{\{h_k\}} \). Also we enlist the following observations on the escaping and non-escaping sets, which follows from the same arguments as in Proposition 2.5.

- \( U^+_{\{h_k\}} = \bigcup_{k=0}^{\infty} h(k)^{-1}(\text{int}(V_R^+)) \) and \( U^-_{\{h_k\}} = \bigcup_{k=0}^{\infty} (h^{-1}(k))^{-1}(\text{int}(V_R^-)) \).
- \( K^+_{\{h_k\}} \) are closed subsets of \( \mathbb{C}^2 \) and \( K^-_{\{h_k\}} = \mathbb{C}^2 \setminus U^+_{\{h_k\}} \).
• $U_S^+ \subset U_{\{h_k\}}^+ \subset U_S^+$ and $K_S^+ \subset K_{\{h_k\}}^+ \subset K_S^+ \subset V_R \cup V_R^+$.

Now as in Section 2 consider the following sequences of plurisubharmonic functions on $\mathbb{C}^2$

$$G_k^+(z) = \frac{1}{d_k} \log^+ \|h(k)(z)\| \text{ and } G_k^-(z) = \frac{1}{d_k} \log^+ \|h^{-1}(k)(z)\|, \quad (6.4)$$

where $d_k = d_1 \ldots d_k$ is the degree of $h(k)$. Then, we have an analogue to Theorem 1.3 here.

**Theorem 6.2.** The sequences of functions $\{G_k^\pm\}$ converges pointwise to a plurisubharmonic continuous functions $G_{\{h_k\}}^\pm$ on $\mathbb{C}^2$, respectively. Further, $G_{\{h_k\}}^\pm$ is pluriharmonic on $U_{\{h_k\}}^\pm$ and $\text{int}(K_{\{h_k\}}^\pm)$.

The proof of the above theorem and other important results — obtained in this section — are essentially revisiting the techniques discussed through sections 3, 4 and 5 in the current non-autonomous dynamical setup. Hence the presentations will be mostly brief and sketchy. Also, note that Remark 6.4 and the definition of functions $\{G_k^\pm\}$ above, is valid for any non-autonomous sequence $\{h_k\}$ of Hénon maps, satisfying the identities (6.2) and (6.3).

**Proof.** Step 1: The sequence of functions $\{G_k^+\}$ converges uniformly on compact subsets of $V_R^+$ and the sequence of functions $\{G_k^-\}$ converges uniformly on compact subsets of $V_R^-$.

From the filtration identity (6.2) it follows that for $(x, y) \in V_R^+$

$$G_{k-1}^+(x, y) + \frac{\log m}{d_k} \leq G_k^+(x, y) \leq G_{k-1}^+(x, y) + \frac{\log M}{d_k}.$$

As $d_k \geq 2^k$ for every $k \geq 1$, we have

$$|G_{k-1}^+(x, y) - G_k^+(x, y)| \leq \frac{\hat{M}_0}{2^k},$$

where $\hat{M}_0 = \max\{\|\log m\|, \|\log M\|\}$. Thus for a given $\epsilon > 0$ there exists $m, n \geq 1$, sufficiently large, $|G_m^+ - G_n^-| \leq \epsilon$ on $V_R^+$. A similar argument works on $V_R^-$.  

Step 2: The sequence of functions $\{G_k^+\}$ converges uniformly on compact subsets of $U_{\{h_k\}}^+$ and the sequence of functions $\{G_k^-\}$ converges uniformly on compact subsets of $U_{\{h_k\}}^-$.  

Note that by Remark 6.1 for a given compact set $C \subset U_{\{h_k\}}^+$, there exists $\ell_C \geq 1$, large enough such that $h(\ell_C)(C) \subset V_{R}^+$. Thus by similar argument as above, for $(x, y) \in C$

$$|G_{k-1}^+(h(\ell_C)(x, y)) - G_k^+(h(\ell_C)(x, y))| \leq \frac{\hat{M}_0}{2^k}.$$

Now for a fixed $\ell_0 \geq 1$ and $k > 1$ consider the sequence of functions defined as

$$G_k^{\ell_0}(z) = \frac{d_{\ell_0}}{d_k + \ell_0} \log^+ \|h(k + \ell_0)h(\ell_0)^{-1}(z)\|$$

As $h(k + \ell_0)h(\ell_0)^{-1} = h_{k+\ell_0} \circ \cdots \circ h_{1+\ell_0}$, the functions $\{G_k^{\ell_0}\}$ are pluriharmonic on $V_R^+$, by the same argument as for $\{G_k^+\}$. Since $C \subset h(\ell_C)^{-1}(V_R^+)$, also by the filtration identity (6.2)

$$|G_{k-1}^{\ell_C}(h(\ell_C)(x, y)) - G_k^{\ell_C}(h(\ell_C)(x, y))| \leq \frac{\hat{M}_0}{2^k}$$

for every $(x, y) \in C$. Note that

$$G_{k+\ell_C}(x, y) = \frac{G_k^{\ell_C}(h(\ell_C)(x, y))}{d_{\ell_C}}.$$
Hence for $k \geq 1$, sufficiently large and $(x, y) \in C$
\[ |G^+_{k+\ell C-1}(x, y) - G^+_{k+\ell C}(x, y)| \leq \frac{1}{d_{\ell C}} \tilde{M}_0 .\]

Thus $\{G^+_k\}$ converges to a pluriharmonic function on $h((\ell C)^{-1}(V^+_R))$. Hence the function $G^+_{\{h_k\}}$ is pluriharmonic on $U^+_{\{h_k\}}$. A similar proof works for $G^-_{\{h_k\}}$ and $U^-_{\{h_k\}}$.

**Step 3:** Let $G^\pm_{\{h_k\}} := \lim_{k \to \infty} G^\pm_k$, the pointwise limits of $\{G^\pm_k\}$. Then both the limit functions $G^\pm_{\{h_k\}}$ are continuous and plurisubharmonic on $\mathbb{C}^2$.

To complete the above, we first prove that $G^\pm_{\{h_k\}}$ is continuous on $\mathbb{C}^2$, in particular it is continuous on $\partial K^\pm_{\{h_k\}}$. Suppose not, then there exist a point $z_0 \in \partial K^\pm_{\{h_k\}}$ and a sequence $\{z_n\} \in U^\pm_{\{h_k\}}$ such that $z_n \to z_0$ such that $G^\pm_{\{h_k\}}(z_n) > c > 0$ for every $n \geq 1$. Let $B = \max\{\log^+ ||H_i(z)|| : z \in V_R\}$ and $1 \leq i \leq n_0\}$. Also let $k_1 \geq 1$, sufficiently large, such that
\[ \frac{c}{2} > \frac{\tilde{M}}{2k_1} \]
where $\tilde{M} := \max\{\tilde{M}_0, B\}$. Further note that there exists $k_2 \geq 1$ such that $h(k)(z_n) \in V_R \cup V^+_R$ for $k \geq k_2$. If not, then there exists a subsequence $\{k_n\}$ of positive integers diverging to infinity and a sequence $\{z_{i_n}\}$ of $\{z_n\}$ such that $h(k_n)(z_{i_n}) \in V_R^-$, i.e., $z_{i_n} \in h(k_n)^{-1}(V^-_R)$. Hence $z_{i_n} \notin \text{int}(V_Rk_n)$. As $Rk_n \to \infty$, this would mean $||z_{i_n}|| \to \infty$, which is a contradiction!

**Claim:** The sequence $\{h(k)(z_n)\} \in V^+_R$ whenever $k \geq \max\{k_1, k_2\}$. Suppose not, then there exist $k_1 \geq \max\{k_1, k_2\}$ and $z_l \in \{z_n\}$ such that $h(k_l)(z_l) \in V_R$. Also from (6.2), $h(k_l)(z_l) \notin V^+_R$ for every $k \leq k_l$. Let $k > k_l$ be the minimum positive integer such that $h(k_l)(z_l) \in V^+_R$, i.e., $\|h(k_l)(z_l)\| \leq B$ and $h(k_l)(z_l) \in V^+_R$ for every $k \geq k_l$. Hence for every $k > k_l$
\[ G^\pm_k(z_l) \leq \frac{\log^+ ||h(k_l)(z_l)||}{d_{k_l}} + \tilde{M}_0 \sum_{i=k_l}^{k} \frac{1}{d_i} \leq \tilde{M} \sum_{i=k_l}^{k} \frac{1}{d_i} .\]

Since $d_i \geq 2^i$ and $k_l - 1 \geq k_1$, the above simplifies to
\[ G^\pm_k(z_l) \leq \tilde{M} \sum_{i=k_l}^{k} \frac{1}{2^i} \leq \frac{\tilde{M}}{2^{k_l-1}} \leq \frac{\tilde{M}}{2k_1} \leq c .\]

So $G^\pm_{\{h_k\}}(z_l) < c$, which is a contradiction to the assumption. Thus the claim follows.

As $z_0 \in \partial K^\pm_{\{h_k\}} \subset K^\pm_{\{h_k\}}$ (since it is closed), by [Lemma 2.8](#) and [Remark 3.2](#) there exists $k_0 \geq 1$ such that $||h(k)(z_0)|| \leq R$ for every $k \geq k_0$. Also, from the above Claim and (6.2) we may fix a $k_0 > \max\{k_0, k_1, k_2\}$ such that $h(k_0)(z_0) \in \text{int}(V^+_R)$. Since $z_n \to z_0$, by continuity of $h(k_0)$ we have $h(k_0)(z_0) \in V^+_R$, i.e., $||h(k_0)(z_0)|| \geq R + 1 > R$, which is not possible (as $z_0 \in K^\pm_{\{h_k\}}$). Thus $G^\pm_{\{h_k\}}$ is continuous on $\mathbb{C}^2$.

Now $G^\pm_{\{h_k\}}$ is pluriharmonic on $U^\pm_{\{h_k\}}$, and is identically zero, i.e, is also pluriharmonic in the interior of $K^\pm_{\{h_k\}}$ (provided it is non-empty). Also it is continuous on $\mathbb{C}^2$, hence the upper semi-continuous regularisation of $G^\pm_{\{h_k\}}$ on $\mathbb{C}^2$ matches with itself and Step 2 holds.

A similar argument will work for $G^-_{\{h_k\}}$, which completes the proof.

**Corollary 6.3.** There exist constants $c^\pm_{\{h_k\}} \in \mathbb{R}$ such that for $(x, y) \in V^+_R$ (respectively),
\[ G^\pm_{\{h_k\}}(x, y) = \log |y| + O(1) \text{ and } G^-_{\{h_k\}}(x, y) = \log |x| + O(1). \]

**Proof.** The proof is same as the proof of Corollary 3.3. $\square$
Lemma 6.4. The sequences \( \{ \mathcal{G}^\pm_k \} \) converges uniformly on compact subsets of \( \mathbb{C}^2 \) to \( \mathcal{G}^\pm(h_k) \), respectively.

Proof. The proof is again completely similar to the proof of Lemma 4.2, however, we revisit the steps briefly. Note that if \( C \subset U_{h_k}^+ \) then the uniform convergence is immediate, as the sequence \( \{ \mathcal{G}^+_k \} \) is uniformly Cauchy on \( U_{h_k}^+ \), by the proof of Theorem 6.2.

Next, let \( C \subset K_{h_k}^+ \) then \( h(k) \in \mathcal{G}^+_k(z) \) for every \( z \in C \) and \( k \geq 1 \), where \( \mathcal{G}^+_k(z) \) is as introduced in the Step 3 of the proof of Theorem 1.3 and hence by Remark 3.2 there exists a positive integer \( k_0(\mathcal{C}) \geq 1 \) such that \( h(k)(C) \subset V_R^+ \) for every \( k \geq k_0 \). Thus \( \mathcal{G}^+_k|_C \leq \frac{R}{2\pi} \), which proves the uniform convergence in this case.

Finally, let \( C \) intersects both \( K_{h_k}^+ \) and \( U_{h_k}^+ \), then the uniform convergence is immediate from above on \( C \cap K_{h_k}^+ \). i.e., for a given \( \epsilon > 0 \), there exists \( k_0 \geq 1 \) such that \( |\mathcal{G}^+_k(z) - \mathcal{G}^+_k(z)| \leq \epsilon \) for every \( k \geq k_0 \). Further by Lemma 2.8, there exists \( k_0(C) \geq 1 \) such that \( h(k)(C) \subset V_R \cup V_R^+ \) for every \( k \geq k_0(C) \). Note that by (2.1),

\[
 h(k)^{-1}(V_R^+) = h(k)^{-1}(V_R^+) \subset \text{int}(h(k+1)^{-1}(V_R^+)). \tag{6.5}
\]

Now as in the proof of Lemma 4.2 we define the following subsets of \( C \)

\[
 C_k = C \cap (h^{-1}(\text{int}(V_R^+)) \setminus h^{-1}(\text{int}(V_R^+))) \quad \text{and} \quad C_0 = C \cap \text{int}(V_R^+),
\]

i.e., \( \cup_{k=0}^\infty C_k = C \cap U_{h_k}^+ \). Since \( C \cap U_{h_k}^+ \neq \emptyset \), it follows from (6.5) that \( C_k \)’s are non-empty sets for \( k \geq 1 \), sufficiently large. Also, let \( B = \max\{||H_i(z)|| : z \in V_{R_{l+1}}, 1 \leq i \leq n_0 \} \) and \( C_k = \cup_{l=k}^\infty C_l \). Then for \( z \in C_k \), \( \mathcal{G}^+_k(z) \leq \frac{\log B}{d_k} \) whenever \( k_0(C) \leq l \leq k \). Now for \( n \geq 1 \)

\[
 \mathcal{G}^+_{k+n}(z) \leq \log B + \log M \sum_{i=1}^n \frac{1}{d_{k+i}} \leq \frac{2\widetilde{M}}{d_k} \leq \frac{\widetilde{M}}{2^{k-1}},
\]

where \( \widetilde{M} = \max\{||\log M||, ||\log m||, ||\log B||\} \). Again, by the same arguments as in proof of Lemma 4.2 i.e., by continuity of \( \mathcal{G}^+_k \) and the above, there exists \( k_1 \geq 1 \) such that

\[
 |\mathcal{G}^+_k(z) - \mathcal{G}^+_k(z)| < \epsilon
\]

whenever \( z \in \overline{C}_{k_1} \) and \( k \geq k_1 \). Now, as

\[
 (U_{h_k}^+ \cap C) \setminus \overline{C}_{k_1} \subset C \cap h(k_1 - 1)^{-1}(V_R^+),
\]

and \( C \cap h(k_1 - 1)^{-1}(V_R^+) \) is a compact set contained in \( U_{h_k}^+ \), there exists \( k_2 \geq 0 \) such that \( |\mathcal{G}^+_k(z) - \mathcal{G}^+_k(z)| \leq \epsilon \) whenever \( z \in (U_{h_k}^+ \cap C) \setminus \overline{C}_{k_1} \). Since \( C = (C \cap K_{h_k}^+) \cup \overline{C}_{k_1} \cup ((U_{h_k}^+ \cap C) \setminus \overline{C}_{k_1}) \), for \( k \geq \max\{k_0, k_1, k_2\} \), we have \( |\mathcal{G}^+_k - \mathcal{G}^+_k|_C \leq \epsilon \). \( \square \)

Theorem 6.5. For every \( k \geq 1 \), let \( \mathcal{G}^+_h(k) \) denote the Green’s function corresponding to the maps \( h(k) \) and \( h^{-1}(k) \). Then the sequence \( \{ \mathcal{G}^+_h(k) \} \) converge uniformly to \( \mathcal{G}^+_h(h_k) \), respectively, on compact subsets of \( \mathbb{C}^2 \).

Proof. This proof is again similar to the proof of Theorem 4.1. Let \( C \) be a compact subset of \( \mathbb{C}^2 \), then let \( C_1 = K_{h_k}^+ \cap C \). Since \( C_1 \) is a compact set contained in \( K_{h_k}^+ \), by Lemma 2.8 and Remark 3.2 there exists \( k_{C_1} \geq 1 \) such that \( h(k)(z) \in V_R \) for \( k \geq k_{C_1} \). Thus \( h(k) \in \mathcal{G}^+_k(z) \) for \( z \in C_1 \) and \( h(k) \in \mathcal{G}^u_k(\tilde{z}) \) for \( \tilde{z} \in C \setminus C_1 \), whenever \( k \geq k_{C_1} \).
Now by Step 1 in the proof of Theorem 4.1, there exists $k_1 \geq k_{C_1}$ such that for $k \geq k_1$, 
\[ \left| G^+_h(k)(z) - \frac{\log \|h(k)(z)\|}{d_k} \right|_{C_1} = |G^+_h(k)(z) - G^+_k(z)|_{C_1} < \epsilon/2. \]

Also as $h(k) \in \mathcal{G}_k^+(\tilde{z})$ for $\tilde{z} \in C \setminus C_1$ and $k \geq k_1$, by Step 2 in the proof of Theorem 4.1, 
\[ \frac{\log \|h(k)(\tilde{z})\|}{d_k} - \tilde{M} \sum_{i=1}^{l-1} \frac{1}{d_i} \leq \frac{\log \|h(k)^l(\tilde{z})\|}{d_k} \leq \frac{\log \|h(k)(\tilde{z})\|}{d_k} + \tilde{M} \sum_{i=1}^{l-1} \frac{1}{d_i}, \]
whenever $l \geq 2$. Hence there exists $k_2 \geq k_1$ such that for $k \geq k_2$, 
\[ \left| G^+_h(k)(z) - \frac{\log \|h(k)(z)\|}{d_k} \right|_{C \setminus C_1} = \left| G^+_h(k)(z) - G^+_k(z) \right|_{C \setminus C_1} \leq \frac{2\tilde{M}}{d_k} < \epsilon/2. \]

Finally, by Lemma 6.4, there exists $k_C \geq k_2$ such that the theorem holds. A similar argument will work for $\{G^+_h(k)\}$ and $G^+_k$. \hfill \Box

Now $G^+_h(k) \equiv 0$ in the int($K^+_h(k)$), provided it is non-empty, hence $K^+_h(k)$ are pseudoconcave subsets of $C^2$. Also, as an immediate corollary to Theorems 6.2 and 6.5, we have the initial statement of the following.

**Corollary 6.6.** The currents $\mu^+_h(k) := \frac{1}{2\pi} dd^c(G^+_h(k))$ are positive $(1,1)$ currents of mass 1, supported on $J^+_h(k) = \partial K^+_h(k)$, respectively. Also support of $\mu^+_h(k) = J^+_h(k)$ and $\mu^-_h(k) := \mu^+_h(k) \wedge \mu^-_h(k)$ is a compactly supported probability measure.

**Proof.** We only prove $\text{Supp} \mu^+_h(k) = J^+_h(k)$ here. Since $G^+_h(k)$ is non-constant on $C^2$ and attains the minimum value, i.e., zero, in the interior of neighborhood of a point $z_0 \in J^+_h(k)$, the function is strictly pluriharmonic at $z_0$. As $z_0$ is an arbitrary point on $J^+_h(k)$, the support of $\mu^+_h(k)$ is equal to $J^+_h(k)$. A similar argument will work for $\mu^-_h(k)$. Also $\mu^-_h(k)$ is a positive measure and is immediate, and it is compactly supported follows from Remark 6.1 \hfill \Box

**Remark 6.7.** Note that any subsequence of $\{h(k)\}$ neither diverges to infinity nor is it bounded on any neighbourhood of a point $z_0 \in \partial K^+_h(k)$. Thus $J^+_h(k) = \partial K^+_h(k)$ is contained in the Julia set for the dynamics of the non-autonomous family $\{h_k\}$. But note that 
\[ C^2 \setminus J^+_h(k) = \text{int}(K^+_h(k)) \cup U^+_h(k). \]
Hence by Lemma 2.8 and Remark 3.2, int($K^+_h(k)$) is contained in the Fatou set and thus the Julia set corresponding to the dynamics of $\{h_k\}$ is equal to $J^+_h(k)$.

**Remark 6.8.** Note that the two ‘crucial’ conditions required on a non-autonomous sequence of Hénon maps $\{h_k\}$ of the form (1.2), to complete the proof of Theorem 6.2 and 6.5 are
(i) The sequence $\{h_k\}$ admits a uniform radius filtration $R(h_k) > 1$ (in the above case it is the radius of filtration of the semigroup $\mathcal{S}$, generated by $\mathcal{G}$), such that for every $R > R(h_k)$
\[ h_k(V^+_R) \subset V^+_R \quad \text{and} \quad h_k^{-1}(V^-_R) \subset V^-_R. \]
there exists a sequence positive real numbers $\{R_k\}$ diverging to infinity, with $R_0 = R$, satisfying $V_{R_k} \cap h(k)(V^+_R) = \emptyset$ and $V_{R_k} \cap h^{-1}(k)(V^-_R) = \emptyset$.
\[ \text{There exist uniform constants } 0 < m < M, \text{ such that the filtration identities } (6.2) \text{ and } (6.3) \text{ are satisfied on } V^+_R, \text{ respectively}, \]
(ii) For every $R \geq R(h_k)$, there exists a uniform constant $B_R = \max\{\|h_k(z)\| : z \in V_R\} < \infty$.
The same holds in the above setup of Theorem 6.2 and 6.5 as the choices for $h_k$ are finite, for every $k \geq 1$. 

Hence we have the following analogue of of Theorem 6.2 and 6.5 in a more general setup.

Remark 6.9. Let \( \{h_k\} \) be a non-autonomous sequence of Hénon maps satisfying conditions (i) and (ii) of Remark 6.8 then

- The sequences of plurisubharmonic function \( \{G^\pm_k\} \), as defined in (6.4) converges to a plurisubharmonic continuous functions \( G^\pm_{\{h_k\}} \) on \( \mathbb{C}^2 \), respectively. Further \( G^\pm_{\{h_k\}} \) is plurisubharmonic on \( U^\pm_{\{h_k\}} \) and \( \text{int}(K^\pm_{\{h_k\}}) \), where \( U^\pm_{\{h_k\}} \) and \( \text{int}(K^\pm_{\{h_k\}}) \).

- The sequences \( \{G^\pm_{h(k)}\} \) converge uniformly to \( G^\pm_{\{h_k\}} \), respectively, on compact subsets.

Example 6.10. Let \( H_k(x, y) = (a_k y, a_k x + p(y)) \) where \( p \) is a polynomial of degree at least 2, then the sequence \( \{H_k\} \) defined as below is a sequence of Hénon maps.

\[
H_k(x, y) := H_{2k} \circ H_{2k-1}(x, y) = (y, a_{2k}a_{2k-1}x + p(y/a_{2k})) \circ (y, a_{2k}a_{2k-1}x + a_{2k}p(y)).
\]

Further, if \( 0 < c < |a_k| < d \) for every \( k \geq 1 \), the conditions (i) and (ii) in Remark 6.8 are satisfied, and by Remark 6.9 it is possible to construct the dynamical Green’s functions. However, the condition (i) in Remark 6.8 fails for \( H_k^{-1} \), if \( |a_k| \to 0 \) (see Theorem 1.4 in [15]).

Also note that the functions \( G^\pm_{\{h_k\}} \) admit logarithmic growth at infinity, and the closure of the sets \( K^\pm_{\{h_k\}} \) in \( \mathbb{P}^2 \) is \( K^\pm_{\{h_k\}} \cup I^\pm \), as defined in Section 5. Hence it is possible to generalise the results stated in Section 5 to the setup of dynamics of a non-autonomous sequence of Hénon maps \( \{h_k\} \subset S \). In particular, the analogue to Corollary 5.8 is

Corollary 6.11. Let \( S^\pm \) be two closed positive \((1, 1)\)-currents in \( \mathbb{P}^2 \) of mass 1, such that the support of \( S^+ \) does not contain the point \( [0 : 1 : 0] \) and the support of \( S^- \) does not contain the point \( [1 : 0 : 0] \). Also, let \( h^{(k)} \) denote the extension of \( h(k) \) to \( \mathbb{P}^2 \), for every \( k \geq 1 \) then

\[
\lim_{k \to \infty} \frac{1}{d_k} h^{(k)} (S^+) \to \mu^+_{\{h_k\}} \text{ and } \lim_{k \to \infty} \frac{1}{d_k} h^{-1}(k) (S^-) \to \mu^-_{\{h_k\}}.
\]

The proof is immediate from Remark 5.7 and Theorem 6.5 Also the proof of Corollary 6.11 does not generalises to general non-autonomous families of Hénon maps (observed in Remark 6.8), unlike Theorems 6.2 and 6.5. It crucially requires that \( h(k) \in \mathcal{G}_k, k \geq 1 \).

Remark 6.12. Note that as mentioned in the introduction, the above result is a more explicit version of Theorem 5.1 in [11], for Hénon maps. The latter established the existence of similar non-autonomous currents for families of horizontal maps on appropriate subdomains of \( \mathbb{C}^k \), \( k \geq 2 \) and Hénon maps of the above form are indeed known to be horizontal on a large enough polydisc at the origin in \( \mathbb{C}^2 \), by [20]. Also, the construction and the convergence properties of similar Green’s current for parametrised families of skew-product of (monic) Hénon maps — of fixed degree — over compact complex manifolds, have been studied in [27].

7. Attracting Basins of Non-Autonomous Sequences in \( S \)

Let \( S \) be a semigroup generated by finitely many Hénon maps, having an attracting behaviour, i.e., satisfy (1.4) at the origin. Then for every \( i \geq 1 \), there exist \( r > 0 \) and \( 0 < \alpha < 1 \) such that \( H_i(B(0; r)) \subset B(0; \alpha r) \). In particular, for every sequence \( \{h_k\} \subset S, h_k(z) \to 0 \) as \( n \to \infty \) for \( z \in B(0; r) \). Hence we have the following observations.

- The strong filled positive Julia set \( K^+_S \) is non-empty and contains a neighbourhood of the origin. Also, the strong filled negative Julia set \( K^-_S \) is non-empty and contains the origin.

- The basin of attraction at the origin of every \( h \in S \), say \( \Omega_h \), is a Fatou–Bieberbach domain, i.e., biholomorphic to \( \mathbb{C}^2 \) (see [28] for the proof).

- The non-autonomous basin of attraction at the origin for a sequence \( \{h_k\} \) — denoted by \( \Omega_{\{h_k\}} \), defined in statement of Theorem 1.1 — is an elliptic domain containing the origin as every \( h_k \) satisfies the uniform bound condition. (see [16], [14] for the result).
Lemma 7.1. $\partial \Omega_{\{h_k\}} \subset \partial K^+_{\{h_k\}}$ for the non-autonomous dynamical system $\{h_k\}$.

Proof. Observe that by the argument as in Remark 6.7, $\partial \Omega_{\{h_k\}} \cap U^+_{\{h_k\}} = \emptyset$. Now, if $z_0 \in \partial \Omega_{\{h_k\}} \cap \text{int}(K^+_{\{h_k\}})$, then there exists a neighbourhood of $B(z_0; \delta) \subset \text{int}(K^+_{\{h_k\}})$, i.e., the sequence $\{h(k)\}$ is locally uniformly bounded and hence normal on $B(z_0; \delta)$. Then there exists a subsequence $\{n_k\}$ such that $h(n_k)(z_0)$ does not tend to 0, however $h(n_k)(z_1) \to 0$ whenever $z_1 \in B(z_0; \delta) \cap \Omega^+_{\{h_k\}}$. Thus $\{h(k)\}$ is not normal on any neighbourhood of $z_0$. Hence by Remark 6.7, $\partial \Omega_{\{h_k\}} \subset J^+_{\{h_k\}} = \partial K^+_{\{h_k\}}$. □

Remark 7.2. Note that in the setup of iterative dynamics of Hénon maps, the boundary of any attracting basin is equal to the Julia set (see [5, Theorem 2]). In the above lemma we only show that $\partial \Omega_{\{h_k\}}$ is properly contained in the Julia set. This leads to the question: Is $J^+_{\{h_k\}} = \partial \Omega_{\{h_k\}}$ in the non-autonomous setup?

Let $\Omega_{h(k)}$ be the basin of attraction of the origin for every $h(k), k \geq 1$, as origin is an attracting fixed point, i.e., the definitions are compared as

$$\Omega_{h(k)} = \{z \in \mathbb{C}^2 : h(k)^n(z) \to 0 \text{ as } n \to \infty\} \text{ and } \Omega_{\{h_k\}} = \{z \in \mathbb{C}^2 : h(k)(z) \to 0 \text{ as } k \to \infty\}.$$

Lemma 7.3. Let $K$ be a compact set contained in $\Omega_{\{h_k\}}$ then there exists a positive integer dependent on $K$, i.e., $N_0(K) \geq 1$ such that $K \subset \Omega_{h(k)}$ for every $k \geq N_0(K)$.

Proof. Note that $\{h_k\}$ varies within a collection of finitely many Hénon maps, $\{H_i : 1 \leq i \leq n_0\}$, each admitting an attracting fixed point at the origin. Thus there exists a neighbourhood $B(0; r)$ at the origin and $0 < \alpha < 1$ such that $H_i(B(0; r)) \subset B(0; \alpha r)$ for every $1 \leq i \leq n_0$. In particular, $B(0; r)$ is contained in attracting basin of the origin for every $h, h \in \mathcal{S}$. Since $K \subset \Omega_{\{h_k\}}$ is compact, $h(k)(w) \in B(0; r)$ for every $w \in K$ and $k \geq N_0(K)$. Hence

$$G^+_{h(k)}(h(k)(w)) = 0, \text{ i.e., } G^+_{h(k)}(w) = 0$$

for every $w \in K$ and $n \geq N_0(K)$. So $K \subset \text{int}(K^+_{h(k)})$. But $h(k)(0) = 0$, hence $h(k)(\Omega_{h(k)}) = \Omega_{h(k)}$, and the above implies $h(k)(K) \subset \Omega_{h(k)}$. Thus $K \subset \Omega_{h(k)}$ for every $k \geq N_0(K)$. □

Next, we complete the proof of Theorem 1.1 by appealing to an idea used in [30].

Proof of Theorem 1.1. Let $\{K_k\}$ be an exhaustion by compacts of $\Omega_{\{h_k\}}$. Then from Lemma 7.3, there exists an increasing sequence of positive integers $\{n_k\}$ such that $K_k \subset \Omega_{h(n_k)}$. Since every $\Omega_{h(n_k)}$ is a Fatou-Bieberbach domain, our goal is to construct a sequence of biholomorphisms $\{\phi_k\}$, i.e., holomorphic maps that are both one-one and onto from $\Omega_{h(n_k)}$ to $\mathbb{C}^2$, appropriately and inductively, such that for a given summable sequence of positive real numbers $\{\rho_k\}$ the following holds

$$\|\phi_k(z) - \phi_{k+1}(z)\| < \rho_k \text{ for } z \in K_k \text{ and } \|\phi_k^{-1}(z) - \phi_{k+1}^{-1}(z)\| < \rho_k \text{ for } z \in B(0; k). \quad (7.1)$$

Basic step: Let $\phi_1 : \Omega_{h(n_1)} \to \mathbb{C}^2$ be a biholomorphism. By results in [2] Theorem 2.1] for $\delta < \rho_1/2$ there exists $F_2 \in \text{Aut}(\mathbb{C}^2)$ such that

$$\|\phi_1(z) - F_2(z)\| < \delta \text{ for } z \in K_1(r) \text{ and } \|\phi_1^{-1}(z) - F_2^{-1}(z)\| < \delta \text{ for } z \in B(0; 1 + r) \quad (7.2)$$

where $K_1(r) = \cup_{z \in K_1} (B(z; r)) \subset \Omega_{h(n_1)}$ for some $r > 0$, i.e., an $r$-neighbourhood of $K_1$, contained in $\Omega_{h(n_1)}$. Since $F_2$ is uniformly continuous on $K_1(r)$, there exists $\epsilon_0 > 0$ such that for $z, w \in K_1(r)$

$$\|F_2(z) - F_2(w)\| < \delta \text{ whenever } \|z - w\| < \epsilon_0. \quad (7.3)$$
Let $\epsilon < \min\{\epsilon_0, r, \delta\}$. Then from [30] Lemma 4, there exists a biholomorphism $\psi_2 : \Omega_{h(n_2)} \to \mathbb{C}^2$ such that
\[ \|\psi_2(z) - z\| < \epsilon \text{ for every } z \in K_1 \text{ and } \|\psi_2^{-1}(z) - z\| < \epsilon \text{ for every } z \in F_{2}^{-1}(B(0; 1)). \] (7.4)
Thus for $z \in K_1$, $\psi_2(z) \in K_1(r)$ and by (7.3), (7.4) it follows that $\|F_2 \circ \psi_2(z) - F_2(z)\| < \delta$. Hence from (7.2),
\[ \|F_2 \circ \psi_2(z) - \phi_1(z)\| < 2\delta < \rho_1 \text{ for } z \in K_1. \]
Also by (7.4), for $z \in B(0; 1)$, $\|\psi_2^{-1} \circ F_2^{-1} - F_2^{-1}(z)\| < \epsilon < \delta$. Again by (7.2),
\[ \|\psi_2^{-1} \circ F_2^{-1}(z) - \phi_1^{-1}(z)\| < 2\delta < \rho_1 \text{ for } z \in B(0; 1). \]
Thus $\phi_1$ and $\phi_2 := F_2 \circ \psi_2$ satisfies (7.1) for $k = 1$.

**Induction step:** Suppose for $N \geq 2$, and there exist biholomorphisms $\phi_k : \Omega_{h(n_k)} \to \mathbb{C}^2$ such that (7.1) is satisfied for every $1 \leq k \leq N - 1$. Our goal is to construct $\phi_{N+1}$ such that (7.1) holds for $k = N$. As before, for $\delta < \rho_N/2$, there exists $F_{N+1} \in \text{Aut}(\mathbb{C}^2)$ such that
\[ \|\phi_N(z) - F_{N+1}(z)\| < \delta \text{ for } z \in K_N(r) \text{ and } \|\phi_N^{-1}(z) - F_{N+1}^{-1}(z)\| < \delta \text{ for } z \in B(0; N + r), \]
where $K_N(r)$ is an $r$-neighbourhood of $K_N$, contained in $\Omega_{h(n_N)}$ for some $r > 0$. Since $F_{N+1}$ is uniformly continuous on $K_N(r)$, there exists $\epsilon_0 > 0$ such that for $z, w \in K_N(r)$
\[ \|F_{N+1}(z) - F_{N+1}(w)\| < \delta \text{ whenever } \|z - w\| < \epsilon_0. \] (7.5)
Let $\epsilon < \min\{\epsilon_0, r, \delta\}$. Then again by [30] Lemma 4, there exists a biholomorphism $\psi_{N+1} : \Omega_{h(n_{N+1})} \to \mathbb{C}^2$ such that
\[ \|\psi_{N+1}(z) - z\| < \epsilon \text{ for } z \in K_N \text{ and } \|\psi_{N+1}^{-1}(z) - z\| < \epsilon \text{ for } z \in F_{N+1}^{-1}(B(0; N)). \] (7.6)
Thus for $z \in K_N$, $\psi_{N+1}(z) \in K_N(r)$, and by (7.5), (7.6) it follows that
\[ \|F_{N+1} \circ \psi_{N+1}(z) - F_{N+1}(z)\| < \delta. \]
Hence $\|F_{N+1} \circ \psi_{N+1}(z) - \phi_N(z)\| < 2\delta < \rho_N$ for $z \in K_N$. Also, similarly as above, by (7.6), for $z \in B(0; N)$, $\|\psi_{N+1}^{-1} \circ F_{N+1}^{-1}(z) - F_{N+1}^{-1}(z)\| < \epsilon < \delta$ and by assumption on $F_{N+1}$,
\[ \|\psi_{N+1}^{-1} \circ F_{N+1}^{-1}(z) - \phi_N^{-1}(z)\| < 2\delta < \rho_N. \]
Thus $\phi_N$ and $\phi_{N+1} := F_{N+1} \circ \psi_{N+1}$ satisfies (7.1) for $k = N$.

As $\{\rho_k\}$ is summable, the sequences $\{\phi_k\}$ and $\{\phi_k^{-1}\}$ constructed converge on every compact subset of $\Omega_{h_k}$ and $\mathbb{C}^2$, i.e., there exist analytic limit maps $\phi : \Omega_{h_k} \to \mathbb{C}^2$ and $\tilde{\phi} : \mathbb{C}^2 \to \mathbb{C}^2$.

Since $\phi$ is a limit of biholomorphisms, either $\phi$ is one-one or $\text{Det} \, D\phi \equiv 0$ on $\Omega_{h_k}$.

Choose $A > 0$ and $k \geq 1$, sufficiently large, such that $\sum_{i=k}^{\infty} \rho_i < A/2$. Also let $K = \tilde{\phi}_k^{-1}(B(0; A))$. Then $\text{vol}(K) > 0$, and by (7.1), $B(0; A/2) \subset \phi(K) \subset B(0; 3A/2)$, i.e., $\text{vol}(\phi(K)) > \text{vol}(B(0; A/2))$. But if $\text{Det} \, D\phi \equiv 0$, then $\text{vol}(\phi(K)) = 0$, which is not true. Hence $\phi$ is one-one on $\Omega_{h_k}$.

Finally, we prove that $\phi(\Omega_{h_k}) = \mathbb{C}^2$. So first, observe that as a consequence of Theorem 5.2 in [13], $\phi_n^{-1}$ converges uniformly to $\tilde{\phi}$ on compact subsets of $\mathbb{C}^2$ and $\tilde{\phi}^{-1} = \phi$ on $\Omega_{h_k}$.

Next we claim that for every positive integer $N_0 \geq 1$, $\tilde{\phi}(B(0, N_0)) \subset \Omega_{h_k}$. Suppose not, then there exists $z_0 \in \tilde{\phi}(B(0, N_0))$ such that $G_{h_k}^{+}(z_0) > 0$. Let $w_o = \phi^{-1}(z_0) \in B(0; N_0)$ and $z_k = \phi_k^{-1}(w_o)$. Then $z_k \in \Omega_{h_k}$, $G_{h_k}^{+}(z_k) = 0$, for every $k \geq 1$ and $z_k \to z_0$ by (7.1).

But by Theorem 6.5, $G_{h_k}^{+}$ converges uniformly to $G_{h_k}^{+}$ on compact subsets of $\mathbb{C}^2$. Hence $G_{h_k}^{+}(z_0) = 0$, which is a contradiction. Thus $\tilde{\phi}(B(0, N_0)) \subset \text{int}(K_{h_k}^{+})$ with $\tilde{\phi}(0) \in \Omega_{h_k}$. Hence by Lemma 7.1, $\tilde{\phi}(\mathbb{C}^2) \subset \Omega_{h_k}$ or $\mathbb{C}^2 \subset \phi(\Omega_{h_k}) = \mathbb{C}^2$.

Also, the following is immediate from the above proof, and the Remarks 6.8 and 6.9.
Corollary 7.4. Let \( \{ H_k \} \) be a sequence of Hénon maps of form (I.4), such that it satisfy
- admits uniform filtration and bound conditions (i) and (ii) stated in Remark 6.8, and
- is (upper) uniformly attracting on a neighbourhood of origin, i.e., satisfying (1.4).

Then the basin of attraction of the sequence \( \{ H_k \} \) at the origin is biholomorphic to \( \mathbb{C}^2 \).

Further, for parametrised families of Hénon maps over compact manifolds, we have

Example 7.5. Let \( M \) be a compact complex manifold and \( \mathcal{H} : M \times \mathbb{C}^2 \to M \times \mathbb{C}^2 \) be a skew product of Hénon map parametrised over \( M \), i.e., \( \mathcal{H}(\lambda, x, y) = (\sigma(\lambda), H_\lambda(x, y)) \) such that \( \sigma \) is an (holomorphic) endomorphism of \( M \) and \( H_\lambda \) is a Hénon map of a fixed degree \( d \geq 2 \) for every \( \lambda \in M \), i.e., the family \( \{ H_\lambda \}_{\lambda \in M} \) satisfies conditions (i) and (ii) of Remark 6.8. Further, if the family \( \{ H_\lambda \}_{\lambda \in M} \) is uniformly attracting on a neighbourhood of origin, i.e, it satisfies (1.4), then for every \( p \in (p_0, 0, 0) \), the stable manifold \( \Sigma^s_H(p) \), defined as

\[
\Sigma^s_H(p) := \{ (\lambda, z) \in M \times \mathbb{C}^2 : \mathcal{H}^k(\lambda, z) \to p \text{ as } k \to \infty \},
\]

is biholomorphic to \( \mathbb{C}^2 \), provided it is non-empty. This, in fact also answers a few particular cases of Problem 38 and 39, stated in [1].

Finally, we conclude with an analytic property of the strong filled Julia set \( K^+_S \).

Proposition 7.6. Suppose there exists \( 1 \leq i \neq j \leq n_0 \) such that \( K^+_H_i \neq K^+_H_j \), then \( \Omega_{\{ h_k \}} \not\subset K^+_S \), the strong filled Julia set, for every \( \{ h_k \} \subset \mathcal{S} \).

Proof. We first claim that for every \( h \in \mathcal{S} \), \( \Omega_h \not\subset K^+_S \). Note that by definition \( K^+_S \subset K^+_h \) for every \( h \in \mathcal{S} \). If \( \Omega_h \subset K^+_S \) then \( \partial \Omega_h = J^+_h \subset K^+_S \). By assumption on \( \mathcal{S} \), there exists (at least one) \( g \in \mathcal{S} \) such that \( K^+_g \neq K^+_h \). Now from the above \( J^+_h \subset K^+_S \subset K^+_g \), i.e., \( \mu^+_h \) is a positive closed \((1,1)\) current supported on \( K^+_g \). Hence from Theorem 6.5 in [12], \( \mu^+_h = \mu^+_g \) or \( K^+_h = K^+_g \), which is a contradiction to the assumption.

Now suppose there exists a sequence \( \{ h_k \} \subset \mathcal{S} \), such that \( \Omega_{\{ h_k \}} \subset K^+_S \). Define the sequence \( h_k = h(k) \circ h_1^{-1} \), for \( k \geq 2 \). Thus \( z \in \Omega_{\{ h_k \}} \subset K^+_S \), \( h_k(z) \) is bounded, i.e., \( h(k) \circ h_1^{-1}(z) \) is bounded. Let \( F_k(z) := \| h(k) \circ h_1^{-1}(z) \| \). Then \( F_k \) is a sequence of positive pluri-subharmonic functions on \( \Omega_{\{ h_k \}} \). Further on any compact subset of \( \Omega_{\{ h_k \}} \), all \( F_k \)'s, except finitely many is bounded uniformly by \( R \), where \( R \) is the radius of filtration of the semigroup \( \mathcal{S} \) (as in Remark 2.4).

Next, let \( F(z) = \limsup F_k(z) \) for \( z \in \Omega_{\{ h_k \}} \). Hence from Theorem 2.6.3 in [22], the upper semicontinuous regularisation of \( F \), denoted by \( F^* \) of \( F \) is a bounded pluri-subharmonic function on \( \Omega_{\{ h_k \}} \). Also as \( F^*(z) = 0 \) on \( B(0; r_S) \), and the Lebesgue measure of the set \( \{ z \in \Omega_{\{ h_k \}} : F(z) \neq F^*(z) \} \) is zero, it follows that \( F^*(z) = 0 \) almost everywhere on \( B(0; r_S) \). Since \( \Omega_{\{ h_k \}} \) is a Fatou-Bieberbach domain by Theorem 1.1, it cannot admit any non-constant bounded pluri-subharmonic function. Thus \( F^* \equiv 0 \) on \( \Omega_{\{ h_k \}} \), i.e., \( h(k)(w) \to 0 \) for every \( w \in h_1^{-1}(\Omega_{\{ h_k \}}) \). Hence \( h_1^{-1}(\Omega_{\{ h_k \}}) \subset \Omega_{\{ h_k \}} \).

Let \( \mathcal{D} \subset B(0; r_S) \subset \Omega_{\{ h_k \}} \subset K^+_S \) be a relatively compact subset of an one-dimensional algebraic variety such that \( \partial \mathcal{D} \cap J^+_h = \emptyset \) and \( \mathcal{D} \cap \mu^+_h = c \neq 0 \). Thus by Corollary 1.7 of [2],

\[
S_n = \frac{1}{d_{h_1}} h_n^H([\mathcal{D}]) \to c \mu^+_h \text{ as } n \to \infty.
\]

Note \( S_n \)'s are positive \((1,1)\)-currents supported on \( h_n^{-1}(\mathcal{D}) \subset h_1^{-1}(\Omega_{\{ h_k \}}) \subset \Omega_{\{ h_k \}} \subset K^+_S \) (from the previous observation). Hence \( \mu^+_h \) is supported on \( K^+_S \), in particular \( J^+_h \subset K^+_S \), which is not possible from the claim above. \( \square \)
Remark 7.7. The above also proves that $K^+_S$ cannot contain any $(1, 1)$ positive closed current of finite mass and the positive Green’s function $G^+_S$ is unbounded on all — both autonomous and non-autonomous — basins of attraction.

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