1. Introduction

Poly-symplectic structures arise in the geometric formulation of Classical Field Theories in the same way that symplectic structures appear in the Hamiltonian formalism of classical mechanics [18]. More precisely, poly-symplectic structures are $\mathbb{R}^k$-valued 2-form, which are closed and satisfy a nondegeneracy condition, in such a way that they coincide with usual symplectic forms when $k = 1$. Poly-symplectic geometry has been studied in recent years by several authors, including [2, 3, 21, 23, 31]; see also [16, 20, 22, 29, 33] for further connections with physics.

In recent work [19], D. Iglesias J.C. Marrero and M. Vaquero introduced a generalization of Poisson structure by considering the inverse structures of poly-symplectic
forms, analogous to the way Poisson structures are defined from symplectic forms. In this paper, we give a new viewpoint and study new aspects of the work in [19] by considering a slight variation of their definition of poly-Poisson structure. Our definition relies on the relationship between symplectic groupoids and Poisson manifolds [35, 11], but now in the setting of poly-symplectic groupoids, which are natural extensions of symplectic groupoids to poly-symplectic geometry.

Similarly to symplectic groupoids, poly-symplectic groupoids are defined by a poly-symplectic form on a Lie groupoid satisfying a compatibility condition, which says that the poly-symplectic form is multiplicative (in the sense of (2.2) below). One of the main properties of symplectic groupoids is that they are the global versions of Poisson structures (see [35, 11]), that is, the manifold of objects of a symplectic groupoid is endowed with a Poisson structure whose corresponding Lie algebroid is isomorphic to the Lie algebroid of the groupoid. Moreover, the Poisson structure is uniquely determined by the condition that the target map is a Poisson morphism. Starting with a poly-symplectic groupoid, the corresponding infinitesimal geometric structure is what we identify and call poly-Poisson structure. In other words, the poly-Poisson structures we introduce here relate to poly-symplectic groupoids exactly in the same way that Poisson structures relate to symplectic groupoids. A similar idea in the context of multi-symplectic geometry (see [9, 10]) is studied in [8].

The notion of $k$-poly-Poisson structure arising in this way is slightly less general than the one given in [19], but contains the essential examples of the theory. Moreover, for $k = 1$, the notion agrees with ordinary Poisson structures (in contrast with the more general definition of [19]). From our viewpoint to poly-Poisson structures, we will revisit some results in [19] and extend known facts about Poisson structures, e.g., concerning their underlying Lie algebroids and foliations. Also, following the description of Poisson structures as particular cases of Dirac structures [12], we discuss an analogous picture for poly-Poisson structures. In this case, however, Dirac structures are not enough, and we must consider AV-Courant algebroids and a suitable extension of AV-Dirac structures, as in [24].

Poly-symplectic manifolds $M$ equipped with symmetries given by a Lie group $G$ induce, under suitable regularity conditions, a quotient poly-Poisson structure on the manifold $M/G$. In order to find poly-symplectic groupoids integrating such quotients, we need to discuss some aspects of hamiltonian actions and Marsden-Weinstein reduction in poly-symplectic geometry, see e.g. [18, 27]. This allows us to extend some constructions in [30, 17] and [7], and show that the symmetries $G$ of an integrable poly-Poisson manifold can be lifted to hamiltonian symmetries of its integrating (source-simply-connected) poly-symplectic groupoid, and that its poly-symplectic reduction at level zero is a poly-symplectic groupoid integrating the quotient poly-Poisson structure on $M/G$.

There are several aspects of the approach to higher Poisson structures considered in this paper that we plan to pursue in future work, including the study of normal forms (see [2, 28] and the more recent work in [16]), the geometry of the corresponding higher versions of Dirac structures, and the potential connections with Field Theory.
This paper is organized as follows: In Section 2 we introduce poly-symplectic groupoids. The key result of this section, which generalizes [8, Prop. 4.1], is Proposition 2.4 where we obtain the relation between global and infinitesimal objects. Poly-Poisson structures are defined in Section 3, where we discuss their Lie algebroid structure, the underlying foliation, together with their relation with poly-symplectic groupoids via integration. Poly-Poisson structures are illustrated with some examples from [19]. At the end we give a different way to describe poly-Poisson structures related to AV-Dirac structures [24]. Section 4 is devoted to the study of symmetries of poly-Poisson structures and hamiltonian actions on poly-symplectic manifolds, see Theorem 4.1 and Prop. 4.4. Finally, applying the hamiltonian reduction, we describe integrations of quotients of an integrable poly-Poisson manifold.

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Notation: Lie algebroids will be denoted by $A \to M$, with anchor map $\rho : A \to TM$ and bracket $[,]$. For a Lie groupoid $G$ over $M$, the source and target maps will be $s, t : G \to M$, $\epsilon : M \to G$ denotes the unit map, $\text{inv} : G \to G$ is the inversion map, and the groupoid multiplication is $m : G(2) \to G$, where the space of composable arrows is $G(2) := G \times_{s,t} G = \{(g, h) \in G \times G| t(h) = s(g)\}$. The right and left translation on the groupoid are $R_g, L_g$, respectively, for $g \in G$.

For a vector space $V$, we will denote by $\oplus_k V$ the $k$-fold direct sum of $V$, or equivalently, the space $V \otimes \mathbb{R}^k$. On vector spaces we will use two different annihilator spaces. For a vector subspaces $W$ of a vector space $V$, we will denote by $\text{Ann}(W)$ the space of elements on $V^*$ vanishing on $W$. For any subspace $S$ of $\oplus_k V^*$, $S^\circ$ stands for the space of elements on $V$ which annihilate the elements of $S$, i.e $S^\circ = \{v \in V| \alpha(v) = 0 \text{ for all } \alpha \in S\}$. This notation will be used, more generally, for vector bundles $E \to M$ rather than vector spaces.

The coadjoint action $\text{Ad}^* : G \to \text{End}(g^*)$ of a Lie group $G$ on the dual of its Lie algebra $g$ induces a diagonal coadjoint action of $G$ on the product $g^*_k$, and we keep the notation $\text{Ad}^*$ to this action, i.e., $\text{Ad}^*_g(\zeta_1, \ldots, \zeta_k) = (\text{Ad}^*_g\zeta_1, \ldots, \text{Ad}^*_g\zeta_k)$.

2. Poly-symplectic groupoids

In this section we will recall the concept of poly-symplectic manifold (see e.g. [18, 19, 2]) and introduce poly-symplectic groupoids, which will guide us towards poly-Poisson structures.

2.1. Poly-symplectic structures. A $k$-poly-symplectic form on a manifold $M$ is a $\mathbb{R}^k$-valued differential form $\omega \in \Omega^2(M, \mathbb{R}^k)$ which is closed and nondegenerate, in the sense that the induced bundle map

$$\omega^\flat : TM \to T^*M \otimes \mathbb{R}^k$$

is injective ($\ker(\omega) = \{0\}$). Writing $\omega$ in terms of its components, $\omega = (\omega_1, \ldots, \omega_k)$, it is poly-symplectic if and only if each $\omega_j \in \Omega^2(M)$ is closed and

$$\cap_{j=1}^k \ker(\omega_j) = \{0\}.$$
One way to obtain examples of poly-symplectic structures is the following. Let $M$ be a manifold endowed with $k$ surjective, submersion maps $p_j : M \to M_j$, such that $\cap_{j=1}^k \ker(dp_j) = \{0\}$. If each $M_j$ is equipped with a $l_j$-poly-symplectic form $\omega_j$, then

$$\omega = (p_1^*\omega_1, \ldots, p_k^*\omega_k)$$

is an $l$-poly-symplectic form on $M$, where $l = l_1 + \ldots + l_k$. In particular, if $(M_j, \omega_j)$ is an $l_j$-poly-symplectic manifold, $j = 1, \ldots, k$, this construction endows $M := M_1 \times \cdots \times M_k$ with an $l$-poly-symplectic structure, for $l = l_1 + \ldots + l_k$. This shows that the product of $k$ symplectic manifolds naturally carries a $k$-poly-symplectic structure.

The following is a particular case of interest in classical field theory [13]:

**Example 2.1.** ($k$-covelocities on a manifold) Recall that any cotangent bundle $T^*Q$ has a canonical symplectic form $\omega_{\text{can}}$. The manifold of $k$-covelocities is the Whitney sum

$$\oplus_{(k)} T^*Q = T^*Q \oplus \cdots \oplus T^*Q,$$

which is equipped with the natural projections $pr_j : \oplus_{(k)} T^*Q \to T^*Q$. It is clear that $\cap_{j=1}^k \ker(dp_j) = \{0\}$, and

$$\omega := (pr_1^*\omega_{\text{can}}, \ldots, pr_k^*\omega_{\text{can}}) \in \Omega^2(\oplus_{(k)} T^*Q, \mathbb{R}^k)$$

is a $k$-poly-symplectic form.

Other examples of poly-symplectic structures are discussed e.g. in [13, 19, 32].

### 2.2. Multiplicative forms and poly-symplectic groupoids

We now consider poly-symplectic structures on Lie groupoids. Let $\mathcal{G}$ be a Lie groupoid over $M$.

A differential form $\theta \in \Omega^*(\mathcal{G})$ is called *multiplicative* if it satisfies

$$m^*\theta = pr_1^*\theta + pr_2^*\theta,$$

where $pr_i : \mathcal{G} \times_{s,t} \mathcal{G} \to \mathcal{G}$ are the projection maps. Note that condition (2.2) still makes sense for $\mathbb{R}^k$-valued forms $\theta = (\theta_1, \ldots, \theta_k)$, and it simply says that each component $\theta_i$ is multiplicative.

Recall that a *symplectic groupoid* is a Lie groupoid $\mathcal{G} \Rightarrow M$ endowed with a multiplicative symplectic form $\omega \in \Omega^2(\mathcal{G})$, see e.g. [11, 35]. A direct generalization leads to

**Definition 2.1.** A $k$-poly-symplectic groupoid is a Lie groupoid $\mathcal{G} \Rightarrow M$ together with a $k$-poly-symplectic form $\omega = (\omega_1, \ldots, \omega_k) \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$ satisfying (2.2). More explicitly, each $\omega_j \in \Omega^2(\mathcal{G})$ is closed, multiplicative, and $\cap_{j=1}^k \ker(\omega_j) = \{0\}$.

Suppose that $\mathcal{G}_j \Rightarrow M_j$ are $l_j$-poly-symplectic groupoids, $j = 1, \ldots, k$. As discussed in Section 2.1, we can verify that if a Lie groupoid $\mathcal{G}$ is equipped with surjective submersions $p_j : \mathcal{G} \to \mathcal{G}_j$, $j = 1, \ldots, k$, which are groupoid morphisms and satisfy $\cap_{j} \ker(dp_j) = \{0\}$, then $\omega = (p_1^*\omega_1, \ldots, p_k^*\omega_k) \in \Omega^2(\mathcal{G}, \mathbb{R}^k)$ makes $\mathcal{G}$ into an $l$-poly-symplectic groupoid, for $l = l_1 + \ldots + l_k$. Here we use the fact that the pullback of a multiplicative form by a groupoid morphism is again multiplicative. In particular, we have:

**Proposition 2.2.** The direct product of symplectic groupoids $(\mathcal{G}_j, \omega_j)$, $j = 1, \ldots, k$, naturally carries a multiplicative $k$-poly-symplectic structure given by

$$\omega = (pr_1^*\omega_1, \ldots, pr_k^*\omega_k),$$
where \( \text{pr}_j : \mathcal{G}_1 \times \ldots \times \mathcal{G}_k \to \mathcal{G}_j \) is the natural projection.

More conceptually, multiplicative poly-symplectic forms are very special cases of multiplicative forms with values in representations, as in [15]. Given a Lie groupoid \( \mathcal{G} \Rightarrow M \) and a vector bundle \( E \to M \), consider the pullback bundle \( t^*E \to \mathcal{G} \). An \( E \)-valued \( r \)-form on \( \mathcal{G} \) is an element \( \theta \in \Omega^r(\mathcal{G}, t^*E) \). If \( E \) is a representation of \( \mathcal{G} \) (see [25]), we say that \( \theta \in \Omega^r(\mathcal{G}, t^*E) \) is multiplicative if for all composable arrows \( (g, h) \in \mathcal{G} \times_{s,t} \mathcal{G} \) we have

\[
\tag{2.3}
(m^*\theta)_{(g,h)} = \text{pr}_1^*\theta + g \cdot (\text{pr}_2^*\theta),
\]

where \( m, \text{pr}_1, \text{pr}_2 \) are as in (2.2). It is clear that for the trivial bundle \( \text{Id} : \mathcal{G} \to \mathcal{G} \), we recover the notion of multiplicative \( \mathbb{R}^k \)-valued forms previously discussed.

For later use, we observe the \( E \)-valued version of the equations in [4, Lemma 3.1(i)]:

**Lemma 2.3.** If \( \theta \in \Omega^k(\mathcal{G}, t^*E) \) is multiplicative then

\[
\tag{2.4}
\epsilon^*\theta = 0, \quad \text{and} \quad \theta_g = -g \cdot (\text{inv}^*\theta_{\text{inv}(g)})
\]

for all \( g \in \mathcal{G} \).

**Proof.** Define the map \( (Id \times \text{inv}) (g) := (g, g^{-1}) \) from \( \mathcal{G} \) to \( \mathcal{G}(2) \). If we apply the pull-back of \( (Id \times \text{inv}) \) to Equation (2.3) and recall that \( \epsilon \circ t = m \circ (Id \times \text{inv}) \), we obtain:

\[
t^*\epsilon^*\theta_{\epsilon(t(g))} = (Id \times \text{inv})^* (m^*\theta)_{(g,g^{-1})} = \theta_g + (Id \times \text{inv})^* (g \cdot (\text{pr}_2^*\theta_{g^{-1}}))
\]

\[
\quad = \theta_g + g \cdot ((Id \times \text{inv})^* \text{pr}_2^*\theta_{g^{-1}}).
\]

Therefore

\[
\tag{2.5}
t^*\epsilon^*\theta_{\epsilon(t(g))} = \theta_g + g \cdot (\text{inv}^*\theta_{g^{-1}}).
\]

If in particular we fix \( g = \epsilon(m) \) for some \( m \in M \) and take the pull-back by the unit map in (2.5), we conclude that \( \epsilon^*\theta = 0 \). Using this identity and (2.5), it follows that \( \theta_g + g \cdot (\text{inv}^*\theta_{g^{-1}}) = 0 \).

### 2.3. Infinitesimal data of poly-symplectic groupoids.

It is well known that Poisson structures are the infinitesimal counterparts of symplectic groupoids, see e.g. [11, 35]. We will now discuss the infinitesimal counterpart of poly-symplectic groupoids, in the spirit of [8], which leads to a generalization of Poisson structures in poly-symplectic geometry.

Let \( A \to M \) denote the Lie algebroid of a Lie groupoid \( \mathcal{G} \Rightarrow M \), with anchor \( \rho : A \to TM \) and bracket \([\cdot, \cdot]\) on \( \Gamma(A) \). Recall from [11, 6, 4] that a closed multiplicative \( r \)-form \( \theta \) on \( \mathcal{G} \) is infinitesimally described by a bundle map (over the identity)

\[
\mu : A \to \wedge^{r-1}T^*M,
\]

satisfying the conditions

\[
\tag{2.6}
i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u), \quad \forall u, v \in A
\]

\[
\tag{2.7}
\mu([u, v]) = L_{\rho(u)}\mu(v) - i_{\rho(v)}d\mu(u), \quad \forall u, v \in \Gamma(A).
\]

The map \( \mu \) is related to \( \theta \) via

\[
\tag{2.8}
i_u \theta = t^*\mu(u),
\]
for \( u \in \Gamma(A) \), where \( uR \) denotes the right-invariant vector field on \( \mathcal{G} \) defined by \( u \). For source-simply-connected Lie groupoids, \( \mu \) and \( \theta \) completely determine one another.

It follows that a closed multiplicative \( \mathbb{R}^k \)-valued 2-form \( \omega = (\omega_1, \ldots, \omega_k) \in \Omega^2(\mathcal{G}, \mathbb{R}^k) \) infinitesimally corresponds to a bundle map

\[
(2.9) \quad \mu = (\mu_1, \ldots, \mu_k) : A \to \bigoplus_{(k)} T^* M
\]

satisfying the same equations (2.6) and (2.7), which simply means the equations are infinitesimally corresponds to a bundle map \( G \longrightarrow \text{ker}(\mu) = 0 \) in terms of the map \( \mu \) in (2.9). We will do that in the more general framework of multiplicative forms on \( \mathcal{G} \) with values in representations \( E \rightarrow M \).

The infinitesimal version of multiplicative \( E \)-valued \( r \)-forms on a Lie groupoid \( \mathcal{G} \) was studied in [15], where it is proven that (under the usual source-simply-connectedness condition on \( \mathcal{G} \)) such forms \( \theta \) are in 1-1 correspondence with pairs of maps \( (D, \mu) \),

\[
(2.10) \quad D : \Gamma(A) \to \Omega^r(M, E), \quad \mu : A \to \wedge^{r-1} T^* M \otimes E,
\]

satisfying suitable conditions (that we will not need explicitly), see [15, Sec. 2.2]. We will only need the following facts about the infinitesimal data \((D, \mu)\). First, the relation between the bundle map \( \mu \) and the multiplicative \( E \)-valued form \( \theta \) is a direct generalization of that in (2.8); indeed, using [4, Eqs. (3.1)-(3.3)], it follows that the second equation of [15, (2.4)] is equivalent to

\[
(2.11) \quad i_u \theta = t^*(\mu(u)).
\]

Second, when \( E = \mathbb{R}^k \) is the trivial representation and the multiplicative form \( \theta \) is closed, then \( D \) is determined by \( \mu \), in fact \( D = d\mu \) (see [6]); so in this case one only needs \( \mu \) for the infinitesimal description of \( \theta \).

We say that an \( r \)-form \( \theta \in \Omega^r(\mathcal{G}, t^* E) \) is non-degenerate when the map

\[
\tilde{\theta} : T \mathcal{G} \to \wedge^{r-1} T^* \mathcal{G} \otimes t^* E, \quad X \mapsto i_X \theta
\]

has trivial kernel. When \( \theta \) is multiplicative, we have the following infinitesimal description of this property.

**Proposition 2.4.** Consider \( \theta \in \Omega^r(\mathcal{G}, t^* E) \) a multiplicative \( E \)-valued \( r \)-form on a Lie groupoid \( \mathcal{G} \), and let \( \mu : A \to \wedge^{r-1} T^* M \otimes E \) be such that (2.10) holds. Then \( \theta \) is nondegenerate if and only if

\[
(2.11) \quad \ker(\mu) = \{0\}, \quad \text{and} \quad (\text{Im}(\mu))^\circ = \{0\},
\]

where \((\text{Im}(\mu))^\circ = \{X \in TM \mid i_X \mu(u) = 0 \text{ for all } u \in A\}\).

**Proof.** The proof uses the relation (2.10) and follows the same idea of [8, Prop. 4.1]. We recall the details for the reader’s convenience.

First we suppose that conditions (2.11) hold for \( \mu \) and take \( X \in T_g \mathcal{G} \) in the kernel of the multiplicative form. We get that \( dX = 0 \) because \( i_X t^*(\mu(u)) = 0 \) for all \( u \in A \) (from (2.10)), hence \( X \) is tangent to the \( t \)-fibers, which implies the existence of \( v \in A \) for which \( X = u^L_g = d_g \text{inv}(v^L_g) \). As consequence of the second equation in (2.4) and (2.10), we see that \(-g \cdot (s^*(\mu(v))) = i_v \theta_g = i_X \theta_g = 0 \) for any \( g \in \mathcal{G} \), hence \( s^*(\mu(v)) = 0 \). This shows that \( v \in \ker(\mu) \), therefore \( X = v^L_g = 0 \).
For the other direction, let \( u \in \ker(\mu) \). Then \( i_uR\theta = t^*(\mu(u)) = 0 \), which implies \( u^R = 0 \) by nondegeneracy of the form, thus the first condition in (2.11) holds. Now fixing \( X \in (\im(\mu))_m^\circ \) for \( m \in M \), (2.10) implies that \( i_u\iota_X\theta = 0 \) for all \( u \in A_m \). The splitting \( T_mG = T_mM \oplus A_m \) allows us to write \( Z_j = X_j + u_j \in T_mG, j = 1, \ldots, r - 1 \), and the multilinearity of \( \theta \) implies that

\[
i_{Z_{r-1}} \cdots i_{Z_1}i_X\theta = i_{X_{r-1}} \cdots i_{X_1}i_X\theta,
\]

because the other terms vanish from the fact that \( i_u\iota_X\theta = 0 \) for all \( u \in A_m \). Now the first condition in (2.31) implies that \( i_{Z_{r-1}} \cdots i_{Z_1}i_X\theta = 0 \) for all \( Z_j \in T_mG \), hence \( X = 0 \).

For the trivial representation \( E = M \times \mathbb{R} \) and forms of arbitrary degree \( r \), Proposition [2.4] recovers [8, Prop. 4.1]. For the trivial representation \( E = M \times \mathbb{R}^k \) and \( r = 2 \), we obtain the infinitesimal description of multiplicative \( k \)-poly-symplectic forms.

**Corollary 2.5.** Let \( G \Rightarrow M \) be a source-simply-connected groupoid. Then there is a one-to-one correspondence between multiplicative poly-symplectic forms \( \omega \in \Omega^2(G, \mathbb{R}^k) \) and bundle maps \( \mu : A \to \oplus (k)T^*M \) satisfying (2.6), (2.7) and (2.11) via the relation \( i_uR\omega = t^*(\mu(u)) \), for all \( u \in \Gamma(A) \).

Given a Lie algebroid \( A \to M \), we see from Corollary 2.5 that bundle maps \( \mu : A \to \oplus (k)T^*M \) satisfying (2.6), (2.7) and (2.11) are the infinitesimal counterparts of multiplicative poly-symplectic forms on Lie groupoids. So we refer to these objects as IM poly-symplectic forms, where “IM” stands for “infinitesimally multiplicative”. We say that two IM poly-symplectic forms \( \mu : A \to \oplus (k)T^*M \) and \( \mu' : A' \to \oplus (k)T^*M \) are equivalent if there is a Lie algebroid isomorphism \( \varphi : A \to A' \) such that \( \mu = \mu' \circ \varphi \). Under the equivalence in Corollary 2.5, they correspond to isomorphic poly-symplectic groupoids.

We will now use the infinitesimal geometry of poly-symplectic groupoids described in Corollary 2.5 to provide a new viewpoint to [19].

### 3. Poly-Poisson Structures

**3.1. Definition.** The notion of poly-Poisson structure that we now introduce is a slight modification of that in [19].

**Definition 3.1.** A \( k \)-poly-Poisson structure on a manifold \( M \) is a pair \((S, P)\), where \( S \to M \) is a vector subbundle of \( \oplus (k)T^*M \) and \( P : S \to TM \) is a vector-bundle morphism (over the identity) such that

(i) \( \iota_{P(\bar{\eta})}\bar{\gamma} = 0 \), for all \( \bar{\eta} \in S \),

(ii) \( S^\circ = \{ X \in TM | i_X\bar{\eta} = 0, \ \forall \bar{\eta} \in S \} = \{0\} \),

(iii) the space of section \( \Gamma(S) \) is closed under the bracket

\[
[\bar{\eta}, \bar{\gamma}] := \mathcal{L}_{P(\bar{\eta})}\bar{\gamma} - \mathcal{L}_{P(\bar{\gamma})}\bar{\eta} + d(i_{P(\bar{\eta})}\bar{\gamma}) = \mathcal{L}_{P(\bar{\eta})}\bar{\gamma} - i_{P(\bar{\gamma})}d\bar{\eta}, \text{ for } \bar{\gamma}, \bar{\eta} \in \Gamma(S),
\]

and the restriction of this bracket to \( \Gamma(S) \) satisfies the Jacobi identity.

We will call the triple \((M, S, P)\) a \( k \)-poly-Poisson manifold.

We observe that the bracket (3.1) is skew-symmetric (by condition (i)) and satisfies the Leibniz rule:

\[
[\bar{\eta}, f \bar{\gamma}] = f[\bar{\eta}, \bar{\gamma}] + (\mathcal{L}_{P(\bar{\eta})}f)\bar{\gamma},
\]
for all $\tilde{\eta}, \tilde{\gamma} \in \Gamma(S)$ and $f \in C^\infty(M)$. It follows that, for a poly-Poisson manifold $(M, S, P)$, the vector bundle $S \to M$ is a Lie algebroid with bracket (3.1) and anchor map $P: S \to TM$. Since for any Lie algebroid the anchor map preserves Lie brackets, we have that

$$P([\tilde{\eta}, \tilde{\gamma}]) = [P(\tilde{\eta}), P(\tilde{\gamma})], \quad \forall \tilde{\eta}, \tilde{\gamma} \in \Gamma(S).$$

**Remark 3.2.** In (iii) of Def. 3.1 assuming that $\Gamma(S)$ is closed under the bracket (3.1), we can replace the condition on the Jacobi identity by the bracket-preserving property (3.2). Indeed, if (3.2) holds and for $\tilde{\eta}, \tilde{\lambda}, \tilde{\gamma} \in \Gamma(S)$, then

$$[[\tilde{\eta}, \tilde{\gamma}], \tilde{\lambda}] + [[\tilde{\gamma}, \tilde{\tilde{\eta}}], \tilde{\lambda}] = \mathcal{L}_{[P(\tilde{\eta}), P(\tilde{\gamma})]} \tilde{\lambda} - i_{P(\tilde{\lambda})} d[\tilde{\eta}, \tilde{\gamma}] + \mathcal{L}_{P(\tilde{\gamma})} [\tilde{\eta}, \tilde{\lambda}] - i_{[P(\tilde{\eta}), P(\tilde{\lambda})]} d\tilde{\gamma}$$

$$= \mathcal{L}_{P(\tilde{\eta})} \mathcal{L}_{P(\tilde{\gamma})} \tilde{\lambda} - i_{P(\tilde{\lambda})} \mathcal{L}_{P(\tilde{\gamma})} d\tilde{\eta} + i_{P(\tilde{\lambda})} \mathcal{L}_{P(\tilde{\gamma})} d\tilde{\eta} - \mathcal{L}_{P(\tilde{\gamma})} i_{P(\tilde{\lambda})} d\tilde{\eta} - i_{[P(\tilde{\eta}), P(\tilde{\lambda})]} d\tilde{\gamma}$$

$$= \mathcal{L}_{P(\tilde{\eta})} [\tilde{\gamma}, \tilde{\lambda}] - i_{[P(\tilde{\eta}), P(\tilde{\lambda})]} d\tilde{\eta} = [\tilde{\eta}, [\tilde{\gamma}, \tilde{\lambda}]],$$

where the second equality holds by $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ and the third results from Cartan’s magic formula.

It follows from this remark that condition (iii) in Def. 3.1 is equivalent to

(iii)' the space of section $\Gamma(S)$ is closed under the bracket (3.1) and (3.2) holds.

**Remark 3.3** (Comparison with [19]). The notion of poly-Poisson structure in Def. 3.1 is slightly more restrictive than the notion introduced by Iglesias, Marrero and Vaquero in [19] Def. 3.1. The difference is that in [19] our condition (ii) in Def. 3.1, namely $S^c = \{0\}$, is replaced by the following weaker requirement:

$$\text{Im}(P) \cap S^c = \{0\}.$$  

We will refer to such objects as weak-poly-Poisson structures.

Let $(M_j, S_j, P_j)$, $j = 1, 2$, be $k$-poly-Poisson manifolds.

**Definition 3.4.** A smooth map $f: M_1 \to M_2$ is called a poly-Poisson morphism if

a) $f^* \tilde{\eta} \in S_1$ for all $\tilde{\eta} \in S_2$,

b) for every $x \in M_1$ and $\tilde{\eta} \in S_2 |_{f(x)}, \quad Tf|_x (P_1(Tf|_x \tilde{\eta})) = P_2(\tilde{\eta}).$

The following are basic examples of Def. 3.1.

**Example 3.1.** For $k = 1$, a $k$-poly-Poisson structure is simply a usual Poisson structure. Indeed, if $S$ is subbundle of $T^*M$, condition (ii) in Def. 3.1 shows that $S = T^*M$.

(Note that this is not guaranteed by the weaker condition (3.3).) Condition (i) shows that $P: T^*M \to TM$ is of the form $P = \pi^2$ for a bivector field $\pi \in \Gamma(\wedge^2 TM)$, where $\pi^2(\alpha) = i_\alpha \pi$. Finally, condition (iii) amounts to the usual integrability condition $[\pi, \pi] = 0$ (i.e., the bracket on $C^\infty(M)$ given by $(f, g) \mapsto \pi(df, dg)$ satisfies the Jacobi identity). The Lie algebroid structure on $S = T^*M$ is the usual one for Poisson manifolds [31]: the anchor is $\pi^2$ and the bracket $[\cdot, \cdot]$ on $\Omega^1(M)$ is the one such that $[df, dg] = d(\pi(df, dg))$. The notion of morphism in Def. 3.1 also recovers to the usual notion of Poisson morphism.
Example 3.2. Let \((M, \omega)\) be a \(k\)-poly-symplectic manifold, and consider the injective bundle map \(\omega^\flat : TM \to \oplus_{(k)} T^*M\). We define a subbundle \(S_\omega\) of \(\oplus_{(k)} T^*M\) and a bundle map \(P_\omega : S \to TM\) as follows:

\[
S_\omega := \text{Im}(\omega^\flat) \quad \text{and} \quad P_\omega(i_X\omega) := X \in TM.
\]

See [19] Prop. 2.3 and Example 3.3. Note that condition (ii) in Def. 3.1 is equivalent to the non-degeneracy of \(\omega\).

Moreover, given \(k\)-poly-symplectic manifolds \((M_j, \omega_j), j = 1, 2\), a diffeomorphism \(f : M_1 \to M_2\) preserves poly-Poisson structures (as in Def. 3.4) if and only if

\[
f^*\omega_2 = \omega_1.
\]

Example 3.3. Let \(Q\) be a manifold. We can always regard it as a Poisson manifold with the Poisson bracket that is identically zero. For each \(k\), we can also view \(Q\) as a \(k\)-poly-Poisson manifold, and this can be done in several ways. For example, \(S_1 = \oplus_{(k)} T^*Q\) and \(P_1 = 0\) define a poly-Poisson structure on \(Q\), and the same is true for \(S_2 = \{\alpha \oplus \ldots \oplus \alpha | \alpha \in T^*Q\} \subset \oplus_{(k)} T^*Q\) and \(P_2 = 0\), or \(S_3 = \{\alpha \oplus 0 \oplus \ldots \oplus 0 | \alpha \in T^*Q\} \subset \oplus_{(k)} T^*Q\) and \(P_3 = 0\).

Considering \(\oplus_{(k)} T^*Q\) equipped with its poly-symplectic structure (see Example 2.1), the natural projection \(\oplus_{(k)} T^*Q \to Q\) is a poly-Poisson map when \(Q\) is equipped with either one of the poly-Poisson structures \((S_i, P_i)\), for \(i = 1, 2, 3\).

Remark 3.5. It is a well-known fact in Poisson geometry that \(M\) is a Poisson manifold and \(f : M \to N\) is a surjective submersion, then there is at most one Poisson structure on \(N\) for which \(f\) is a Poisson map. Example 3.3 shows that this is not necessarily the case for \(k\)-poly-Poisson structures, for \(k \geq 2\).

On the other hand, let \(M\) be a \(k\)-poly-Poisson manifold and \(f : M \to N\) be a surjective submersion. Then if \((S_1, P_1)\) and \((S_2, P_2)\) are \(k\)-poly-Poisson structures on \(N\) for which \(f\) is a poly-Poisson map and we know that \(S_1 = S_2\), then \(P_1 = P_2\).

As explained in [19] Example 3.8, the product of \(k\)-poly-Poisson manifolds carries a natural \(k\)-poly-Poisson structure.

Example 3.4. Let \((M_j, \pi_j), j = 1, \ldots, k\), be Poisson manifolds. Let \(M = M_1 \times \cdots \times M_k\). Denote by \(S_j\) the natural inclusion of \(T^*M_j\) into \(T^*M\), and let \(S \subset \oplus_{(k)} T^*M\) be defined by \(S := S_1 \oplus \cdots \oplus S_k\). Consider the bundle map \(P : S \to TM\),

\[
P(\alpha_1, \ldots, \alpha_k) = (\pi_1^s(\alpha_1), \ldots, \pi_k^s(\alpha_k)),
\]

where \(\alpha_j \in S_j\). One may verify that \((M, S, P)\) is a \(k\)-poly-Poisson manifold directly from the definition.

In addition, let \(f_j : (M_j, \pi_j) \to (N_j, \Lambda_j)\) for \(j = 1, \ldots, k\), be \(k\) Poisson maps between the Poisson manifolds \(M_j\) and \(N_j\) respectively. From the construction above we obtain \(k\)-poly-Poisson structures \((S_M, P_M)\) and \((S_N, P_N)\) on the product manifolds \(M = \prod_{j=1}^k M_j\) and \(N = \prod_{j=1}^k N_j\), and denote by \(p_j^M : M \to M_j\), and \(p_j^N : N \to N_j\) the natural projections. The Poisson maps \(f_j\) induce a product map \(\tilde{f} = (f_1, \ldots, f_k) : M \to N\) that, as a consequence of the definition of the \(k\)-poly-Poisson manifold and the relations \(p_j^N \circ \tilde{f} = f_j \circ p_j^M\), is a poly-Poisson map.

The next example is a particular case of the direct-sum of linear Poisson structures treated in [19] Example 3.9].
Example 3.5. Let \( \mathfrak{g} \) be a Lie algebra, and let

\[
\mathfrak{g}(k) := \mathfrak{g} \times \cdots \times \mathfrak{g}, \quad \mathfrak{g}^*_k := \mathfrak{g}^* \times \cdots \times \mathfrak{g}^*.
\]

For \( u \in \mathfrak{g} \), let \( u_j \in \mathfrak{g}(k) \) denote the element \((0, \ldots, 0, u, 0, \ldots, 0)\), with \( u \) in the \( j \)-th entry. Since \( \mathfrak{g}^* \) is equipped with its Lie-Poisson structure, \( \mathfrak{g}^*_k \) naturally carries a product poly-Poisson structure, as in Example 3.4. More important to us is the following \textit{direct-sum} poly-Poisson structure \( \oplus \) : over each \( \zeta = (\zeta_1, \ldots, \zeta_k) \in \mathfrak{g}^*_k \), we define

\[
S|_\zeta := \{ (u_1, \ldots, u_k) | u \in \mathfrak{g} \} \subseteq \oplus(\zeta) \mathfrak{T}^*_\zeta \mathfrak{g}^* \cong \oplus(\zeta) \mathfrak{g},
\]

and the bundle map \( P: S \to T\mathfrak{g}^*_k \),

\[
P_\zeta(u_1, \ldots, u_k) := (\operatorname{ad}^*_u \zeta_1, \ldots, \operatorname{ad}^*_u \zeta_k) \in T\mathfrak{g}^*_k \cong \mathfrak{g}^*.
\]

We remark that \( S \) satisfies (ii) in Def. 3.1, not just (3.3).

3.2. Poly-Poisson structures and poly-symplectic groupoids. We will now justify our definition of poly-Poisson structure in Def. 3.1 in light of its relation with poly-symplectic groupoids.

Let \((M, S, P)\) be a \( k \)-poly-Poisson manifold. We saw in Section 3.1 that the vector subbundle \( S \subseteq \oplus(\mathfrak{k}) \mathfrak{T}^* M \) is a Lie algebroid, with anchor \( P : S \to \mathfrak{T} M \) and bracket \( \mathfrak{g} \).

Lemma 3.6. Let \( \mu : S \hookrightarrow \oplus(\mathfrak{k}) \mathfrak{T}^* M \) be the inclusion. Then \( \mu \) is an IM poly-symplectic form on the Lie algebroid \( S \to M \), i.e., \( \mu \) satisfies (2.6), (2.7) and (2.11).

Conversely, any IM poly-symplectic form \( \mu : A \to \oplus(\mathfrak{k}) \mathfrak{T}^* M \) is equivalent to one coming from a \( k \)-poly-Poisson structure.

Proof. Note that (2.6) is just (i) in Def. 3.1 while property (2.7) follows from (iii) in Def. 3.1. Since \( \mu \) is an inclusion, \( \ker(\mu) = \{ 0 \} \). The second condition in (2.11) is (ii) in Def. 3.1.

On the other hand, given an IM poly-symplectic form \( \mu : A \to \oplus(\mathfrak{k}) \mathfrak{T}^* M \), we define \( S = \operatorname{Im}(\mu) \). Note (from the first condition in (2.11)) that \( \mu \) is a vector-bundle isomorphism onto \( S \), and let \( P : S \to \mathfrak{T} M \) be its inverse \( S \to A \) composed with the anchor \( A \to \mathfrak{T} M \). One may directly verify from conditions (2.6), (2.7) and (2.11) that \( S \) and \( P \) define a \( k \)-poly-Poisson structure, and that \( \mu \) is equivalent to the inclusion \( S \hookrightarrow \oplus(\mathfrak{k}) \mathfrak{T}^* M \).

In short, the lemma says that a \( k \)-poly-Poisson manifold \((M, S, P)\) endows \( S \) with a Lie algebroid structure for which the inclusion \( S \hookrightarrow \oplus(\mathfrak{k}) \mathfrak{T}^* M \) is an IM poly-symplectic form, and that any IM poly-symplectic form is equivalent to one of this type.

Following Corollary 2.5, we see that poly-Poisson manifolds are the infinitesimal counterparts of poly-symplectic groupoids, as explained by the next result. For a \( k \)-poly symplectic groupoid \((G \to M, \omega)\), let \( \mu : A \to \oplus(\mathfrak{k}) \mathfrak{T}^* M \) be the bundle map determined by \( \omega \) as in Cor. 2.5. Explicitly, using the natural decomposition \( TG|_M = \mathfrak{T} M \oplus A \),

\[
\mu(u) = \omega^\flat(u)|_{\oplus(\mathfrak{k}) \mathfrak{T} M},
\]

for \( u \in A \).
Remark 3.8. Given a implies that $t$ is a poly-Poisson morphism.

Lemma 3.6. It remains to verify that condition (3.5) implies that $\iota_u \circ \omega = t^* (\mu(u))$, and that $\mu$ corresponds to a poly-Poisson structure $(S, P)$ on $M$, as described in Lemma 3.6. It remains to verify that condition (3.5) implies that $t$ is a poly-Poisson map.

Let $\alpha \in \text{Im}(\mu) = S$. Then $\alpha = \mu(u)$ for a unique $u \in A$, and $P(\alpha) = \rho(u)$. Let $(S_\omega, P_\omega)$ be the poly-Poisson structure defined by $\omega$, as in (3.4). Then (3.5) says that $t^* \alpha \in S_\omega$ and $u^R = P_\omega(t^* \alpha)$; the fact that on any Lie groupoid we have $t_u(u^R) = \rho(u)$ implies that $t_u P_\omega(t^* \alpha) = P(\alpha)$, i.e., $t$ is a poly-Poisson map. \hfill \Box

Remark 3.8. Given a $k$-poly-symplectic groupoid $(G \rightrightarrows M, \omega)$, the uniqueness of the induced poly-Poisson structure $(S, P)$ on $M$ follows from Remark 3.7. Note that $S$ is determined by $\omega$, while $P$ is completely defined from the property that $t$ is a poly-Poisson map.

We illustrate the correspondence in Theorem 3.7 with some simple examples.

Example 3.6. The $k$-poly-symplectic manifold $\oplus_{(k)} T^* Q$ of Example 2.1 is a poly-symplectic groupoid over $Q$, with respect to fibrewise addition; the source and target maps coincide with the projection $\oplus_{(k)} T^* Q \to Q$. The corresponding $k$-poly-Poisson structure on $Q$ is the trivial one, given by $S := \oplus_{(k)} T^* Q$ and $P = 0$. Note that Example 3.3 shows other poly-Poisson structures on $Q$ for which the projection $\oplus_{(k)} T^* Q \to Q$ is a poly-Poisson map, but there is only one with the bundle $S$ prescribed by Theorem 3.7.

Example 3.7. Let $(M, \omega)$ be a $k$-poly-symplectic manifold, that we view as a poly-Poisson manifold as in Example 3.2. The non-degeneracy of $\omega$ implies that the Lie algebroid $(S_\omega, P_\omega)$ is isomorphic to $TM$. Hence this poly-Poisson structure is integrated by the pair groupoid $M \times M \rightrightarrows M$, equipped with the $k$-poly-symplectic structure $t^* \omega - s^* \omega \in \Omega^2(M \times M, \mathbb{R}^k)$, where $s, t$ are the source and target maps on the pair groupoid, i.e $t(x, y) = x$ and $s(x, y) = y$.

Example 3.8. Consider Poisson manifolds $(M_j, \pi_j)$, $j = 1, \ldots, k$, and equip $M = M_1 \times \ldots \times M_k$ with the product poly-Poisson structure of Example 3.4. For each
suppose that \((G_j \rightrightarrows M_j, \omega_j)\) is a symplectic groupoid integrating \((M_j, \pi_j)\). Then the product poly-symplectic groupoid \(G = G_1 \times \ldots \times G_k\) of Prop. 2.2 integrates \(M\). Indeed, one may verify that the bundle \(S\) on \(M\) described in Example 3.3 agrees with the one prescribed by Theorem 3.7 and, as a consequence of the construction of poly-Poisson maps as products of Poisson maps in Example 3.3, the target map on \(G \rightrightarrows M\) is a poly-Poisson map.

**Example 3.9 (Lie-Poisson structures).** Let \(G\) be a Lie group and \(\mathfrak{g}\) its Lie algebra. As seen in Example 2.1, \(\oplus_k T^* G\) has a natural \(k\)-poly-symplectic structure \(\omega\).

The diagonal coadjoint action of \(G\) on \(\mathfrak{g}^*_\text{(k)}\), denoted by \(\text{Ad}_g^*\), endows \(G \times \mathfrak{g}^*_\text{(k)}\) with a groupoid structure over \(\mathfrak{g}^*_\text{(k)}\), with source and target maps given by

\[
s(g, \zeta) = \zeta, \quad t(g, \zeta) = \text{Ad}_g^* \zeta
\]

and multiplication \(m((g, \zeta), (h, \eta)) = (gh, \eta)\) if \(\text{Ad}_h^* \eta = \zeta\). Using the identification \(T^* G \cong G \times \mathfrak{g}^*\) (by right translation), we see that

\[
\oplus_k T^* G \cong G \times \mathfrak{g}^*_\text{(k)},
\]

so we may consider \(\oplus_k T^* G\) as a Lie groupoid, and its poly-symplectic structure \(\omega\) makes it into a poly-symplectic groupoid. This structure integrates the direct-sum poly-Poisson structure on \(\mathfrak{g}^*_\text{(k)}\) described in Example 3.3. Indeed, \(t\) has the Poisson maps \(G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*\) as its coordinates, so it is a poly-Poisson map. And one can check that the bundle \(S\) of the direct-sum poly-Poisson structure is the one induced by the poly-symplectic structure \(\omega\) according to Theorem 3.7.

**Remark 3.9.** More generally: following [19], there is a direct-sum poly-Poisson structure on \(A^* \oplus \ldots \oplus A^*\) where \(A^* \rightarrow M\) is endowed with the linear Poisson structures (defined on the dual bundle to the Lie algebroid \(A \rightarrow M\)). Each \(A^*\) is integrated by the symplectic groupoid \(T^* G \rightrightarrows A^*\), where \(G \rightrightarrows M\) is the groupoid integrating \(A\), and it can be similarly proved that the direct sum \(T^* G \oplus \ldots \oplus T^* G\) over \(G\) is the poly-symplectic groupoid integrating \(A^* \oplus \ldots \oplus A^*\).

### 3.3. Poly-symplectic foliation.

It is well known that any Poisson manifold has an underlying symplectic foliation which uniquely determines the Poisson structure. More generally, let \((S, P)\) be a \(k\)-poly-Poisson structure on \(M\). Since \(S\) has a Lie algebroid structure, the distribution \(D := P(S) \subseteq TM\) is integrable, and its leaves define a singular foliation on \(M\). Each leaf \(\iota: O \hookrightarrow M\) carries an \(\mathbb{R}^k\)-valued 2-form \(\omega_O\) determined by the condition

\[
\omega_O^\iota : T O \rightarrow \oplus_k T^* O, \quad P(\eta) \mapsto \iota^* \eta.
\]

The fact that the 2-form \(\omega_O\) on \(O\) is well defined follows from (i) in Def. 3.1, (ii) guarantees that it is non-degenerate and (iii) that it is closed, see [19], Sec. 3. So \((S, P)\) determines a singular foliation on \(M\) with \((k + 1)\)-poly-symplectic leaves.

A first remark on the poly-symplectic foliation of a \(k\)-poly-Poisson structure is that, in contrast with the case \(k = 1\), different \(k\)-poly-Poisson structures may correspond to the same poly-symplectic foliation, as shown in the next example.

**Example 3.10.** Let \(\omega_t\) be a smooth family of \(k\)-poly-symplectic forms on \(M\) parametrized by \(t \in \mathbb{R}\) and define the following vector subbundles of \(\oplus_k T^* (M \times \mathbb{R})\):
Poisson structures. We consider the bundle $A$ of the $(A,\omega)\in \mathfrak{pa}(T^*M)$ on each poly-Poisson structures (the proposition to the regular poly-symplectic foliation given in Example 3.10, which illustrate last claim and the poly-Poisson structure from (3.7) and (3.8) we apply the proposition is an "extension" of the weak-poly-Poisson structure. In order to poly-Poisson structure as in [19, Theorem 3.4], then the poly-Poisson structure on $M$ is a regular poly-symplectic foliation. Given a subspace $S_0 \subseteq \oplus (k)T^*M$ on each $S_j$ we define $P_j(i_{X\omega_t}\tilde{\gamma}) = X$. Observe that each $(S_j, P_j)$ is a poly-Poisson structure on $M \times \mathbb{R}$ but these three $k$-poly-Poisson structures have the same poly-symplectic foliation on $M \times \mathbb{R}$. Same conclusion holds for the weak-poly-Poisson structure given by $S_0|_{(m,t)} := \{i_{X\omega_t,0}|X \in T_mM\}$ and $P_0(i_{X\omega_t,0}) = X$ where the poly-symplectic foliation is described on Theorem 3.4 on [19].

We now discuss the possibility of defining a poly-Poisson structure from a poly-symplectic foliation. Given a subspace $D_m \subseteq T_m M$, for $m \in M$, equipped with a $(k+1)$-poly-symplectic form $\omega_m$, we consider the subspace $S_m \subseteq \oplus (k)T^*M$ given by

$$S_m := \{\tilde{\eta} \in \oplus (k)T^*M | \exists X \in D_m, \tilde{\eta}|_{\oplus (k)D_m} = i_X\omega_m\},$$

which has dimension $k(n-p) + p$, where $p$ is the dimension of $D_m$. One may verify that $S_m^0 = \{0\}$ and there is a well-defined map $P_m : S_m \to D_m \subseteq T_m M$,

$$P_m(\tilde{\eta}) = X, \quad \text{where} \quad \tilde{\eta}|_{\oplus (k)D_m} = i_X\omega_m.$$

Given now a regular poly-symplectic foliation on $M$, letting $D$ be its tangent distribution, we use the previous pointwise construction to see that (3.7) defines a subbundle $S \subseteq \oplus (k)T^*M$, satisfying $S^0 = \{0\}$, and equipped with a bundle map $P : S \to TM$. Moreover, using the fact that the $\mathbb{R}^k$-valued form defined on each leaf is closed, it follows that $(S, P)$ satisfies (iii) in Def. 3.1, so it is a poly-Poisson structure. In conclusion we have the following proposition (see [19, Sec. 3]),

**Proposition 3.10.** If $(D, \omega)$ is a regular $k$-poly-symplectic foliation on $M$ then $(S, P)$, defined pointwise by (3.7) and (3.8), is a $k$-poly-Poisson on $M$.

In particular, if the regular $k$-poly-symplectic foliation on $M$ comes from a weak-poly-Poisson structure as in [19, Theorem 3.4], then the poly-Poisson structure on the proposition is an “extension” of the weak-poly-Poisson structure. In order to illustrate last claim and the poly-Poisson structure from (3.7) and (3.8) we apply the proposition to the regular poly-symplectic foliation given in Example 3.10 which is the same for each poly-Poisson structures $(S_j, P_j)$ for $j = 1, 2, 3$ and for the weak-poly-Poisson $(S_0, P_0)$, and get the “maximal” poly-Poisson structure $(S_1, P_1)$.

### 3.4. Relation with AV-Dirac structures.

It is well known that Poisson structures on $M$ can be understood as special types of Dirac structures in the Courant algebroid $TM \oplus T^*M$ [12].

As we now see, this picture can be generalized to poly-Poisson structures. We consider the bundle $\mathcal{A} := TM \oplus (\oplus (k)T^*M)$, equipped with the $(\mathbb{R}^k$-valued) fibrewise inner product

$$\langle X \oplus \tilde{\eta}, Y \oplus \tilde{\gamma} \rangle := i_X\tilde{\gamma} + i_Y\tilde{\eta},$$

and bracket on sections of $\mathcal{A}$ given by

$$[X \oplus \tilde{\eta}, Y \oplus \tilde{\gamma}] := [X, Y] \oplus \mathcal{L}_X\tilde{\gamma} - i_Y d\tilde{\eta}. $$


For $k = 1$, this is the standard Courant algebroid $TM \oplus T^*M$. In general, this is a very particular case of the AV-Courant algebroids introduced in [24, Sec. 2] (with respect to the Lie algebroid $A = TM$ and representation on $V = M \times \mathbb{R}^k \to M$ given by the Lie derivative $\mathcal{L}_X(f_1, \ldots, f_k) = (\mathcal{L}_X f_1, \ldots, \mathcal{L}_X f_k)$ on $C^\infty(M, \mathbb{R}^k)$).

Following [24], one may consider AV-Dirac structures on any AV-Courant algebroid: these are subbundles $L \subseteq A$ which are lagrangian, i.e.,

\[ L = L^\perp, \]

with respect to the fibrewise inner product, and which are involutive with respect to the bracket $[\cdot, \cdot]$ on $\Gamma(A)$. Recall that $L$ is called isotropic if $L \subset L^\perp$.

**Example 3.11.** Any $k$-poly-symplectic structure $\omega$ on $M$ may be seen as an AV-Dirac structure in $A := TM \oplus (\oplus(k)T^*M)$ via

\[ L := \text{graph}(\omega) = \{X \oplus i_X\omega | X \in TM\}. \]

Note that this $L$ satisfies the additional condition

\[ L \cap (\oplus(k)T^*M) = \{0\}. \]

In fact, poly-symplectic structures on $M$ are in one-to-one correspondence with AV-Dirac structures which project isomorphically over $TM$ and satisfy $L \cap TM = \{0\}$ and (3.10).

Our goal now is to define, in the same way, a subbundle $L$ from a poly-Poisson structure $(S, P)$, i.e. consider

\[ L = \{P(\bar{\eta}) \oplus \bar{\eta} | \bar{\eta} \in S\}. \]

Note that $L$ is isotropic as a consequence of (i) in Definition 3.1. But, as we now see, the lagrangian condition generally fails.

**Example 3.12.** Let $g$ be a Lie algebra and consider the poly-Poisson structure on $g^*_2$ as in Example 3.5. Observe that $L$ over the point $\zeta = (0,0) \in g^*_2$ can be written as

\[ L_\zeta = \{(0,0) \oplus ((u,0), (0,u)) | u \in g\}. \]

But for any $v_1, v_2, w_1, w_2 \in g$ we have $(0,0) \oplus ((v_1, v_2), (w_1, w_2)) \in L^\perp$, hence $L$ is properly contained in $L^\perp$.

Therefore, in general, poly-Poisson structures are not AV-Dirac structures. In order to include poly-Poisson structures in the formalism of AV-Courant algebroids, one then needs to relax the lagrangian condition (3.9).

Let us consider subbundles $L \subseteq TM \oplus (\oplus(k)T^*M)$ satisfying

\[ L = L^\perp \cap (L + TM). \]

Note that (3.9) implies that (3.11) holds, but the converse is not true.

The following results characterize $k$-poly-Poisson structures as subbundles of $\mathcal{A} = TM \oplus (\oplus(k)T^*M)$:

**Proposition 3.11.** There is a one-to-one correspondence among the following:

(a) $k$-poly-Poisson structures $(S, P)$ on $M$,

(b) Involutive, isotropic subbundles $L \subseteq \mathcal{A}$ satisfying $L^\perp \cap TM = \{0\}$,

(c) Involutive subbundles $L \subseteq \mathcal{A}$ satisfying (3.11) and $L \cap TM = \{0\}$. 

Proof. Given a $k$-poly-Poisson structure $(S, P)$, we define the subbundle $L \subset A$ by
\[(3.12)\]
\[L = \{P(\bar{\eta}) \oplus \bar{\eta} | \bar{\eta} \in S\}.
\]
This bundle is isotropic by condition (i) in Def. 3.1, condition (ii) amounts to $L^\perp \cap TM = \{0\}$ while (iii) is equivalent to the involutivity of $L$. Conversely, given $L$ as in (b), the image of the natural projection $L \to \oplus(k)T^*M$ defines a vector bundle $S$ and a bundle map $P : S \to TM$ by
\[P(\bar{\eta}) = X \text{ if and only if } X \oplus \bar{\eta} \in L,
\]
in such a way that $(S, P)$ is a $k$-poly-Poisson structure. This gives the correspondence between (a) and (b).

For a $k$-poly-Poisson structure $(S, P)$ and $L$ as in (3.12), one may directly verify that (i) in Def. 3.1 implies that (3.11) holds, while (ii) implies that $L \cap TM = \{0\}$, so $L$ satisfies the properties in (c). It remains to check that given an $L$ as in (c), then it satisfies the properties described in (b). Note that (3.11) implies that $L$ is isotropic and that $L \cap TM = L^\perp \cap TM$, so that $L^\perp \cap TM = \{0\}$.

\[\square\]

Remark 3.12. For $k = 1$, the objects in (b) and (c) are just usual Dirac structures on $M$, satisfying the additional condition $L \cap TM = \{0\}$ (conditions (3.9) and (3.11) turn out to be equivalent for $k = 1$), while the objects in (a) are usual Poisson structures. So for $k = 1$ Prop. 3.11 boils down to the known characterization of Poisson structures as particular types of Dirac structures.

4. Symmetries and reduction

We now discuss poly-Poisson structures and poly-symplecti c groupoids in the presence of symmetries, with the aim of using reduction as a tool for integration of poly-Poisson manifolds, along the lines of [30, 17].

4.1. Poly-Poisson actions. An action $\varphi$ of a Lie group $G$ on a $k$-poly-Poisson manifold $(M, S, P)$ is a poly-Poisson action if for each $g \in G$ the diffeomorphism $\varphi_g : M \to M$ is a poly-Poisson morphism (Def. 3.4). In the case of $k$-poly-symplectic manifold $(M, \omega)$, this means that $\varphi_g^* \omega = \omega$, see Example 3.2.

Let us consider a poly-Poisson action $\varphi$ of a Lie group $G$ on $(M, S, P)$, and let us assume henceforth that this action is free and proper, so that we have a principal $G$-bundle:
\[(4.1)\]
\[\Pi : M \to M/G.
\]
Let $V \subseteq TM$ denote the vertical bundle defined by this action.

It is well-known that, when $k = 1$, i.e., $M$ is an ordinary Poisson manifold, $M/G$ inherits a Poisson structure for which $\Pi$ is a Poisson map. For poly-Poisson manifolds, we will need additional conditions. We call the action $\varphi$ is reducible if
\[(4.2)\]
\[\begin{cases} (a) \ S \cap \oplus_k \text{Ann}(V) \text{ has constant rank} , \\
(b) \ (S \cap \oplus_k \text{Ann}(V))^\circ \subset V.\end{cases}
\]
The projection map \((4.1)\) induces a map \(d\Pi_{(k)} : \oplus_{(k)}TM \to \Pi^* (\oplus_{(k)} T(M/G))\), and its transpose is an injective bundle map \(\Pi^* (\oplus_{(k)} T^s(M/G)) \to \oplus_{(k)} T^* M\), whose image is the subbundle \(\oplus_{(k)} \text{Ann}(V) \subseteq \oplus_{(k)} T^* M\). So we have an induced isomorphism

\[
\Pi^* (\oplus_{(k)} T^s(M/G)) \cong \oplus_{(k)} \text{Ann}(V).
\]

The next result is analogous to [19, Thm. 4.1] (but stated for our stronger notion of poly-Poisson structure).

**Theorem 4.1.** Let us consider a poly-Poisson \(G\)-action on a \(k\)-poly-Poisson manifold \((M, S, P)\) which is free and proper, and reducible. Then \(M/G\) inherits a \(k\)-poly-Poisson structure \((S_{red}, P_{red})\), where the subbundle \(S_{red} \subseteq \oplus_{(k)} T^s(M/G)\) corresponds to \(S \cap \oplus_{(k)} \text{Ann}(V)\) via \((4.3)\), and \(P_{red}\) is unique so that the quotient map \((4.1)\) is a \(k\)-poly-Poisson morphism.

**Proof.** The first condition in \((4.2)\) guarantees that \(S_{red} \subseteq \oplus_{(k)} T^s(M/G)\), defined by the condition that \(\Pi^* S_{red}\) is isomorphic to \(S \cap \oplus_{(k)} \text{Ann}(V)\) under \((4.3)\), is a vector subbundle. Note that we have a natural map \(\Pi^*(S_{red}) \to \Pi^*(T(M/G))\) given by the composition

\[
(4.4) \quad \Pi^*(S_{red}) \xrightarrow{d\Pi_{(k)}} S \cap \oplus_{(k)} \text{Ann}(V) \xrightarrow{P} TM \xrightarrow{d\Pi} \Pi^*(T(M/G)),
\]

and this defines a bundle map

\[
(4.5) \quad P_{red} : S_{red} \to T(M/G)
\]
as a consequence of the \(G\)-invariance of \((S, P)\).

To check that \((S_{red}, P_{red})\) defines a \(k\)-poly-Poisson structure on \(M/G\), one must verify that it satisfies conditions (i), (ii), and (iii) in Def. [31]. Condition (i) follows directly from the definition of \((S_{red}, P_{red})\) and the fact that this condition is satisfied by \((S, P)\). It is also routine to check that condition (iii) holds for \((S_{red}, P_{red})\), given that it holds for \((S, P)\).

As for condition (ii), it is a consequence of property (b) in \((4.2)\). Indeed, by the way \(S_{red}\) is defined, the fact that \(\bar{X} \in S_{red}\) implies that \(\bar{X} = d\Pi(X)\), for \(X \in (S \cap \oplus_{(k)} \text{Ann}(V))^o\). But then (b) in \((4.2)\) implies that \(\bar{X} = d\Pi(X) = 0\).

It is also clear from the definition of \(P_{red}\) that \(\Pi\) is a poly-Poisson map.

\[\square\]

We mention two concrete examples, discussed in [19].

**Example 4.1.**

(a) Let \(Q\) be a manifold equipped with a free and proper \(G\)-action, and let \((M = \oplus_{(k)} T^* Q, \omega)\) be the poly-symplectic manifold of Example 2.1. We keep the notation \(\text{pr}_j : M \to T^* Q\) for the natural projection onto the \(j\)-factor. The cotangent lift of the \(G\)-action on \(Q\) defines an action on \(T^* Q\), which induces a \(G\)-action on \(M\) which preserves the poly-symplectic structure (i.e., it is a poly-Poisson action), and there is a natural identification

\[
M/G \cong \oplus_{(k)} (T^* Q/G).
\]

We observe here that both conditions in \((4.2)\) hold, i.e., the \(G\)-action on \(M\) is reducible. To verify this fact, let \(V \subseteq TM\) be the vertical bundle of the \(G\)-action on \(M\), so that \(V_j = d\text{pr}_j(V) \subseteq T(T^* Q)\) is the vertical bundle of the
\( G \)-action on the \( j \)-th factor \( T^*Q \). Note that the natural projection \( T^*Q \to Q \) induces a projection of \( V_{\text{can}}^\omega \) onto \( TQ \), and one then sees that
\[
V_{\text{can}}^\omega_j \times TQ \cdots \times TQ V_{\text{can}}^\omega_k \subseteq T(T^*Q) \times TQ \cdots \times TQ T(T^*Q) = TM
\]
is a vector subbundle, that we denote by \( W \). One can now check that
\[
S_\omega \cap \oplus (k) \text{Ann}(V) = \{ i_X \omega \mid X \in W \},
\]
from where one concludes that condition (a) of (4.2) holds. From (4.6), one directly sees that
\[
(S_\omega \cap \oplus (k) \text{Ann}(V))^\circ = (V_{\text{can}}^\omega_j) \times TQ \cdots \times TQ (V_{\text{can}}^\omega_k) = V_1 \times TQ \cdots \times TQ V_k = V,
\]
showing that (b) of (4.2) also holds. So the action is reducible. As shown in [19, Ex. 4.3], the reduced poly-Poisson structure on \( \oplus (k) (T^*Q/G) \) is the one defined by direct-sum of the natural linear Poisson structure on \( T^*Q/G \) (dual to the Atiyah algebroid \( TQ/G \) of the principal bundle \( Q \to Q/G \)).

(b) In the particular case of \( Q = G \) with the action by left multiplication, as shown in [19, Ex. 4.2], the poly-Poisson reduction of \( \oplus (k) T^*G \) with respect to the lifted \( G \)-action is identified with \( g^*_k \) of Example 3.5.

4.2. Hamiltonian actions on poly-symplectic manifolds. We now consider poly-Poisson actions on poly-symplectic manifolds in the presence of moment maps.

Let \( (M, \omega) \) be a \( k \)-poly-symplectic manifold equipped with a poly-Poisson action of \( G \), denoted by \( \varphi \). Consider the diagonal coadjoint action of \( G \) on the space \( g^* \).

This action is called \textit{hamiltonian} [18, 27] if there is a \textit{moment map}, i.e., a map \( J : M \to g^*_k \) that satisfies
\[
(\text{i}) \quad J \circ \varphi_g = \text{Ad}_g^* \circ J \quad \text{and} \quad (\text{ii}) \quad i_{u_M} \omega = d(J, u).
\]
for all \( u \in g \). Here \( u_M \in \mathfrak{X}(M) \) denotes the infinitesimal generator corresponding to \( u \in g \).

Example 4.2. Let \( (M, \omega) \) be a poly-symplectic manifold such that \( \omega = -d \theta \), and assume that \( G \) acts on \( M \) preserving the 1-form \( \theta \). Then the maps \( J_1, \ldots, J_k : M \to g^* \) defined by \( \langle J_i, u \rangle = \theta_l(u_{M_l}) \), \( u \in g \), define a moment map for the action.

A particular case of this example is when \( M = \oplus (k) T^*Q \) (as in Example 2.1) and the action of \( G \) on \( M \) is the lift of an action on \( Q \), see Example 4.1(a). Here the moment map is \( \langle J(\eta), u \rangle = (\langle \eta_j, u_Q \rangle)_{j=1,\ldots,k} \).

The following observation generalizes a well-known fact in Poisson geometry. Consider \( g^*_k \) with the poly-Poisson structure of Example 3.5.

Proposition 4.2. The moment map \( J : M \to g^*_k \) of a \textit{hamiltonian} action of \( G \) on \( (M, \omega) \) is a \textit{poly-Poisson} morphism.

Proof. Denote the poly-Poisson structure on \( g^*_k \) by \( (S, P) \), as in Example 3.5. Consider \( (u_1, \ldots, u_k) \in S \), and \( Y \in T_xM \) with \( J(x) = \zeta \). By condition (ii) in (1.1) we have
\[
(J^*(u_1, \ldots, u_k))(Y) = (u_1, \ldots, u_k)(dJ(Y)) = (\langle dJ(Y), u_j \rangle)_{j=1,\ldots,k}
\]
\[
= (\langle dJ_j(Y), u \rangle)_{j=1,\ldots,k} = (i_{u_M} \omega)|_x(Y),
\]
Proposition 4.3. Comparing with (4.9), we conclude the following:

\[(4.11) \quad \text{symplectic if and only if} \quad \omega = \text{nondegenerate}, \]

i.e., there exists a (unique) closed form \(\omega\) where, for a subbundle \(W\) of \(\mathfrak{g}\), indeed, it is nondegenerate if and only if we have an equality in (4.9).

From condition (i) in (4.7) we can derive that \(dJ(u_M(x)) = u_g^*(\zeta_j)\), therefore on points \(\zeta = J(x)\) we obtain

\[dJ(P_\omega|_x(J^*(u_1, \ldots, u_k))) = (dJ_j(u_M(x))) = (u_g^*(\zeta_j)) = P|_\zeta(u_1, \ldots, u_k).\]

Let us consider a Hamiltonian \(G\)-action on a \(k\)-poly-symplectic manifold \((M, \omega)\), with moment map \(J : M \to \mathfrak{g}^*_\langle k \rangle\). Let \(\zeta \in \mathfrak{g}^*_\langle k \rangle\) be a clean value for \(J\), i.e.,

\[
\begin{align*}
J^{-1}(\zeta) & \quad \text{is a submanifold of } M, \\
\ker(d_xJ) = T_xJ^{-1}(\zeta), & \quad \text{for all } x \in J^{-1}(0).
\end{align*}
\]

The submanifold \(J^{-1}(\zeta)\) is invariant by the action of \(G_\zeta\), the isotropy group of \(\zeta\) with respect to the diagonal coadjoint action. We assume that the \(G_\zeta\)-action on \(J^{-1}(\zeta)\) is free and proper, so we can consider the reduced manifold

\[M_\zeta := J^{-1}(\zeta)/G_\zeta.\]

We let \(\Pi_\zeta : J^{-1}(\zeta) \to M_\zeta\) be the natural projection map, and \(i_\zeta : J^{-1}(\zeta) \to M\) the inclusion. We denote by \(V_\zeta \subseteq TJ^{-1}(\zeta)\) the vertical bundle with respect to the \(G_\zeta\)-action. It follows from (i) in (4.7) that \(V_\zeta = V \cap TJ^{-1}(\zeta)\), while (ii) implies that

\[\ker(i_\zeta^*\omega) \subseteq V_\zeta\]

This last condition, together with the \(G_\zeta\)-invariance of \(i_\zeta^*\omega\), implies that \(i_\zeta^*\omega\) is basic, i.e., there exists a (unique) closed form \(\omega_{\text{red}} \in \Omega^2(M_\zeta, \mathbb{R}^k)\) so that

\[\Pi_\zeta^*\omega_{\text{red}} = i_\zeta^*\omega.\]

In general, however, the form \(\omega_{\text{red}}\) fails to be poly-symplectic, as it may be degenerate; indeed, it is nondegenerate if and only if we have an equality in (4.9).

Note that (ii) in (4.7) says that

\[TJ^{-1}(\zeta) = \ker(dJ) = V^\omega,\]

where, for a subbundle \(W \subseteq TM\), we use the notation \(W^\omega = \{Y \in TM, | \omega(X, Y) = 0 \ \forall X \in W\}\). Writing \(S = \text{Im}(\omega^\theta)\), one may also check that

\[(\ker(dJ))^\omega = (V^\omega)^\omega = (S \cap \oplus_k \text{Ann}(V))^\circ,\]

from where we conclude that

\[\ker(i_\zeta^*\omega) = (TJ^{-1}(\zeta))^\omega \cap TJ^{-1}(\zeta) = (S \cap \oplus_k \text{Ann}(V))^\circ \cap TJ^{-1}(\zeta).\]

Comparing with (4.9), we conclude the following:

**Proposition 4.3.** The reduced form \(\omega_{\text{red}} \in \Omega^2(M_\zeta, \mathbb{R}^k)\) defined by (4.10) is poly-symplectic if and only if

\[(4.11) \quad (S \cap \oplus_k \text{Ann}(V))^\circ \cap TJ^{-1}(\zeta) \subseteq V_\zeta = V \cap TJ^{-1}(\zeta).\]
A similar, but not equivalent, result of the previous condition was stated on Lemma 3.16.

**Example 4.3.** Consider symplectic manifolds \((M_j, \omega_j)_{j=1,\ldots,k}\) each of them carrying a Hamiltonian action of a Lie group \(G_j\) with respective moment map \(J_j : M_j \to g_j^*\). On the product \(k\)-poly-symplectic manifold \((M, \omega)\) (see Section 2.1) there is a poly-symplectic hamiltonian action given by the product action of \(G := \prod_{j=1}^k G_j\) on \(M\) and the moment map \(J : M \to \oplus_{(k)}(\prod_{j=1}^k g_j^*)\), \(J(m) = \oplus_{j=1}^k (0, \ldots, 0, J_j(m_j), 0, \ldots, 0)\).

Let \(\zeta = \oplus_{j=1}^k (0, \ldots, 0, \zeta_j, 0, \ldots, 0) \in \oplus_{(k)}(\prod_{j=1}^k g_j^*)\) where \(\zeta_j \in g_j^*\) is a clean value for \(J_j\). Then \(J^{-1}(\zeta) = \prod_{j=1}^k J_j^{-1}(\zeta_j)\) and, assuming that each \(G_j\) acts freely and properly on \(J_j^{-1}(\zeta_j)\), then

\[
M_\zeta := J^{-1}(\zeta)/G_\zeta = \prod_{j=1}^k J_j^{-1}(\zeta_j)/G_{\zeta_j} = \prod_{j=1}^k M_{j,\zeta_j},
\]

and the reduced \(\mathbb{R}^k\)-valued 2-form on \(M_\zeta\) is the product \(k\)-poly-symplectic form defined by the reduced symplectic forms on \(M_{j,\zeta_j}\).

The moment-map reduction of Prop. 4.3 can now be compared with the quotient of poly-Poisson structures in Thm. 4.1.

Assuming that the \(G\)-action on \(M\) is free and proper, and that \(\zeta\) is a clean value of a moment map \(J : M \to g^{(k)}\), it follows that the \(G_\zeta\)-action on \(J^{-1}(\zeta)\) is also free and proper, and we have the following diagram of submersions and natural inclusions:

\[
\begin{array}{ccc}
J^{-1}(\zeta) & \xrightarrow{i_\zeta} & M \\
\downarrow_{\Pi_\zeta} & & \downarrow \Pi \\
M_\zeta & \longrightarrow & M/G
\end{array}
\]

(4.12)

**Proposition 4.4.** Let \((M, \omega)\) be a poly-symplectic manifold equipped with a hamiltonian \(G\)-action with moment map \(J : M \to g^{(k)}\). Assume that the \(G\)-action on \(M\) is free, proper and reducible (4.2). If \(\zeta \in g^{(k)}\) is a clean value for the moment map, then:

(a) The reduced manifold \(M_\zeta = J^{-1}(\zeta)/G_\zeta\) carries a natural poly-symplectic form defined by equation (4.10);

(b) The poly-symplectic manifold \(M_\zeta\) sits in \(M/G\) as a union of poly-symplectic leaves of the reduced poly-Poisson manifold on \(M/G\) (given by Thm. 4.1).

**Proof.** Note that (4.2)(b) directly implies (4.11), so the reduced form \(\omega_{\text{red}}\) on \(M_\zeta\) is indeed poly-symplectic, proving part (a).

By the moment-map condition (4.7)(ii), \(X \in \ker(dJ)\) if and only if \((i_X \omega)(u_M) = 0\) for all \(u \in g\), therefore

\[
TJ^{-1}(\zeta) = \{ X \in TM | i_X \omega \in \oplus_k \text{Ann}(V) \}
= P_\omega(S_\omega \cap \oplus_k \text{Ann}(V)) = P_\omega(d\Pi^{s}_k S_{\text{red}}).
\]
It follows from \( (4.12) \) and the construction of the reduced poly-Poisson structure, see \( (4.3) \) and \( (4.5) \), that

\[
TM_\zeta = d\Pi_\zeta(TJ^{-1}(\zeta)) = d\Pi(P_\omega(d\Pi^*_k S_{\text{red}})) = P_{\text{red}}(S_{\text{red}}).
\]

Hence \( M_\zeta \) is a union of poly-symplectic leaves in \( M/G \). It remains to check that the poly-symplectic structures (the one coming from reduction and the one induced from the poly-Poisson structure on \( M/G \)) agree.

Consider \( \bar{X} = d\Pi_\zeta(X), \bar{Y} = d\Pi_\zeta(Y) \in TM_\zeta \), with \( X, Y \) tangent to \( J^{-1}(\zeta) \), and let us compute the two 2-forms on them. For the leafwise poly-symplectic form \( \omega_L \), we have (see \( (3.6) \))

\[
\omega_L(\bar{X}, \bar{Y}) = \bar{\eta}_r(\bar{Y}) = (\Pi^*_\zeta \omega_L)(X, Y),
\]

where \( \bar{\eta}_r \) is such that \( \bar{X} = P_{\text{red}}(\bar{\eta}_r) \). Letting \( \bar{\eta} = d\Pi^*_k(\bar{\eta}_r) \in S_\omega \cap \oplus \text{Ann}(V) \), then

\[
(\Pi^*_\zeta \omega_L)(X, Y) = \bar{\eta}_r(d\Pi_\zeta(Y)) = \bar{\eta}(Y).
\]

Note that there exists a unique \( X_0 \in TM \) such that \( \bar{\eta} = i_{X_0}\omega \in \oplus \text{Ann}(V) \). By \( (4.13) \), we know that \( X_0 \in TJ^{-1}(\zeta) \). Furthermore,

\[
d\Pi_\zeta(X_0) = d\Pi(P_\omega(\bar{\eta})) = d\Pi(P_\omega(d\Pi^*_k(\bar{\eta}_r))) = P_{\text{red}}(\bar{\eta}_r) = \bar{X},
\]

so \( d\Pi_\zeta(X_0) = d\Pi_\zeta(X) \). Recalling that \( \Pi^*_\zeta \omega_{\text{red}} = i^*_\zeta \omega \), we see that

\[
(\Pi^*_\zeta \omega_{\text{red}})(X, Y) = (\Pi^*_\zeta \omega_{\text{red}})(X_0, Y) = (i_{X_0}\omega)(Y) = \bar{\eta}(Y) = (\Pi^*_\zeta \omega_L)(X, Y),
\]

showing that \( \omega_{\text{red}} = \omega_L \) on \( M_\zeta \). \( \square \)

Example 4.4.

(a) Let us consider a \( G \)-action on \( Q \) and its lift to \( M = \oplus (k) T^*Q \) as in Example \( (1.1) \)a). The action on \( M \) is hamiltonian, and using the explicit formula for the moment map in Example \( (1.2) \) one sees that its poly-symplectic reduction at \( \zeta = 0 \) is \( \oplus (k) T^*(Q/G) \), with the poly-symplectic form of Example \( (2.1) \) Proposition \( 3.4 \)b) realizes \( \oplus (k) T^*(Q/G) \) as a poly-symplectic leaf of \( M/G \).

(b) Following Example \( (4.1) \)b), in the particular case \( Q = G \) Proposition \( (4.1) \)b) implies that the poly-symplectic reduction of the lifted action on \( \oplus (k) T^*G \) at level \( \zeta \) (see \( [27 \text{ Sec. 3.3.2}] \)) is identified with the poly-symplectic leaf of \( g^*_k \) through \( \zeta \), which is the orbit of \( \zeta \) under the diagonal coadjoint action of \( G \) on \( g^*_k \) (c.f. Example \( (3.9) \)) equipped with a poly-symplectic generalization of the usual KKS symplectic form on coadjoint orbits, see \( [19 \text{ Example 2.9}] \) and \( [27 \text{ App. A.3}] \).

4.3. Reduction and integration. In this section, we show (along the lines of \( [7 \text{ 17}] \)) how passing from poly-Poisson manifolds to poly-symplectic groupoids has the effect of turning poly-Poisson actions into hamiltonian actions, and how poly-symplectic reduction can be used in the construction of poly-symplectic groupoids associated with poly-Poisson quotients.

In the remainder of this section, we will consider the following set-up:

1. A \( k \)-poly-Poisson manifold \( (M, S, P) \), such that its underlying Lie algebroid is integrable, and \( (G \rightrightarrows M, \omega) \) the source-simply connected \( k \)-poly-symplectic groupoid integrating it.

2. A poly-Poisson action \( \phi \) of the Lie group \( G \) on \( (M, S, P) \).
Since \( \varphi \) preserves the poly-Poisson structure on \( M \), the cotangent lift of \( \varphi \) induces an action \( \hat{\varphi} : G \times S \to S \) by Lie-algebroid automorphisms, which can be integrated to a poly-symplectic \( G \)-action on \( G \), denoted by
\[
\hat{\Phi} : G \times G \to G.
\]
We will now see that this action on \( G \) admits a natural moment map (as in (4.7)), so it is hamiltonian.

Let us start by recalling that any action on \( M \) induces a Hamiltonian \( G \)-action on the symplectic manifold \( T^*M \) with moment map \( J_{\text{can}} : T^*M \to \mathfrak{g}^* \) given by
\[
\langle J_{\text{can}}(\alpha), u \rangle = \langle \alpha, u_M \rangle
\]
for all \( \alpha \in T^*M \) and \( u \) in the Lie algebra \( \mathfrak{g} \) of \( G \). We have an induced map
\[
\oplus_{(k)} T^*M \to \mathfrak{g}^*_k,
\]
that we restrict to \( S \) to define
\[
J^s : S \to \mathfrak{g}^*_k.
\]
It is clear from the \( G \)-equivariance of \( J_{\text{can}} \) that \( J^s \) is \( G \)-equivariant (with respect to the diagonal coadjoint action on \( \mathfrak{g}^*_k \)).

The same proof as in [7, Lemma 3.1] shows that, viewing \( \mathfrak{g}^*_k \) as a trivial Lie algebra, \( J^s \) is a Lie-algebroid morphism. According to our sign conventions, it is more convenient to consider \( -J^s \), which is also a Lie-algebroid morphism, and integrate it to a Lie-groupoid morphism
\[
J : G \to g^*_k.
\]
Just as in [7, Prop. 3.2] one can verify that \( J \) is \( G \)-equivariant and satisfies:
\[
i_{u_G} \omega = d\langle J, u \rangle,
\]
for all \( u \in \mathfrak{g} \), where \( u_G \) is the infinitesimal generator for the action on \( G \). In other words, \( J \) is a moment map for the action \( \Phi \) on \( G \). The next result summarizes the discussion:

**Proposition 4.5.** The \( G \)-action \( \Phi \) on the poly-symplectic groupoid \((G, \omega)\) is Hamiltonian with moment map (4.15).

We now discuss the connection between integration and reduction. We assume from now on that the \( G \)-action \( \varphi \) on \( M \) is free, proper and reducible (4.2). Then the action \( \Phi \) on \( G \) is also free and proper [17, Prop. 4.4]. Let \((S_{\text{red}}, P_{\text{red}})\) be the quotient poly-Poisson structure on \( M/G \).

**Theorem 4.6.** Let \( 0 \in \mathfrak{g}^*_k \) be a clean value for the moment map (4.15). Then \( G_{\text{red}} = J^{-1}(0)/G \) is a Lie groupoid over \( M/G \), and the reduced form \( \omega_{\text{red}} \in \Omega^2(G_{\text{red}}, \mathbb{R}^k) \) makes it into a poly-symplectic groupoid integrating \((S_{\text{red}}, P_{\text{red}})\).

**Proof.** Let \( V_M \subset TM \) be the vertical bundle with respect to the action on \( M \). According to [7, Lemma 3.1] and condition (a) on (4.2) we conclude that \( (J^s)^{-1}(0) = S \cap \oplus_{(k)} \text{Ann}(V_M) \) is Lie subalgebroid of \( S \). The \( G \)-invariance allows us to construct, as in [7, Prop. 4.3], the reduced Lie algebroid \( S_{\text{red}} = (J^s)^{-1}(0)/G \) over \( M/G \). Furthermore, the reduced Lie algebroid \( S_{\text{red}} \) coincides the one defined by the reduced poly-Poisson structure of Theorem 4.1.

If \( 0 \) is a clean value for \( J \), \( J^{-1}(0) \) is Lie subgroupoid (see [7, Lemma 5.1]). Following the same lines of [7, Prop. 5.2], we see that \( G_{\text{red}} = J^{-1}(0)/G \) is a Lie groupoid over
$M/G$, whose Lie algebroid is $(S_{red}, P_{red})$, and the quotient map $\Pi_0 : J^{-1}(0) \to G_{red}$ is a groupoid morphism.

Let $\omega_{red}$ be the reduced form on $G_{red}$, characterized by $\Pi_0^* \omega_{red} = i_0^* \omega$, where $i_0$ is the natural inclusion of $J^{-1}(0)$ on $G$. The second part of [7, Prop. 5.2] allows us to conclude that $\omega_{red}$ is multiplicative.

The fact that the quotient map $\Pi_0$ and the inclusion $i_0$ are groupoid morphism yields

$$\Pi_0^*(i_{\bar{u}}R\omega_{red}) = i_{\bar{u}}R\Pi_0^* \omega_{red} = i_0^*(i_{\bar{u}}R\omega)$$

for any $\bar{u} \in S_{red}$ and $u = d\Pi_0^*(\bar{u}) \in S \cap \oplus(k)\Ann(V_M)$, where $\bar{u}^R$ and $u^R$ are the respective right-invariant vector fields on the correspondent Lie groupoid. Moreover, if $t, t_0, t_{red}$ denote the target maps on the Lie groupoids $G, J^{-1}(0)$ and $G_{red}$, respectively, we have

$$i_0^*(i_{\bar{u}}R\omega) = i_0^*(t^*u) = t_0^*d\Pi_0^*(\bar{u}) = \Pi_0^*t_{red}^*\bar{u},$$

which implies that $i_{\bar{u}}R\omega_{red} = t_{red}^*\bar{u}$. It follows from Prop. 2.4 that $\omega_{red}$ is nondegenerate, so $(G_{red}, \omega_{red})$ is a poly-symplectic groupoid, and it integrates $(S_{red}, P_{red})$. □

Theorem 4.6 is a generalization of the following example.

**Example 4.5.** In Example 3.6 we saw that $\oplus(k)T^*Q \rightharpoonup Q$ is the poly-symplectic Lie groupoid integrating the trivial $k$-poly-Poisson structure on $Q$. In this case, for a free and proper $G$-action on $Q$, the Hamiltonian action of Prop. 4.3 is the one induced by cotangent lift, see Example 4.1(a). We conclude that the poly-symplectic reduction in Theorem 4.6 is $\oplus(k)T^*(Q/G)$, as in Example 4.4, which is a presymplectic groupoid integrating the trivial $k$-poly-Poisson structure on $Q/G$.

**Example 4.6.** Recall that for a simply connected manifold $M$, the $k$-poly-symplectic manifold $(M, \omega)$, viewed as poly-Poisson manifold, is integrated by the $s$-simply connected poly-symplectic groupoid $M \times M \rightharpoonup M$ endowed with the poly-symplectic form $t^*\omega - s^*\omega$, where $t, s$ are the natural projections from $M \times M$ to $M$. If $(M, \omega)$ is equipped with a Hamiltonian-acyclic action of the Lie group $G$ and $J_0 : M \to g^*_{(k)}$ is its moment map, then the moment map $\mathfrak{g}_{(k)}^{*}$ for the Hamiltonian action on the groupoid is $J = t^*J_0 - s^*J_0$. If the action on $M$ is free, proper, reducible and $0 \in g_{(k)}^{*}$ is a clean value for $J$, then the symplectic groupoid $J^{-1}(0)/G$ over $M/G$ integrates the reduced poly-Poisson structure $(S_{red}, P_{red})$ induced by $(M, \omega)$.

The poly-symplectic groupoid $G_{red}$ in Theorem 4.6 is not necessarily the source-simply connected Lie groupoid integrating the reduced structure. This claim is illustrated on [17, Example 4.8] for the case $k = 1$.

**Remark 4.7.** Rather than assuming that $0$ is a clean value of the moment map $J$ on $G$, one can also proceed as in [7, Prop. 5.3] and consider the source-simply-connected groupoid $G_0$ integrating the Lie algebroid $(J^s)^{-1}(0)$. With the same arguments as in [7, Prop. 5.3], one can see that this Lie groupoid is equipped with a $G$-action and inherits a $G$-basic multiplicative $2$-form $\omega_0 \in \Omega^2(G_0, \mathbb{R}^k)$ from the natural map $G_0 \to G$, integrating the inclusion $(J^s)^{-1}(0) \to S$. Then $\mathcal{G}_{0, red} = G_0/G$ is a Lie groupoid over $M/G$ and $\omega_0$ reduces to a poly-symplectic form $\omega_{0, red}$ on $\mathcal{G}_{0, red}$ integrating the quotient poly-Poisson structure $(S_{red}, P_{red})$.

Finally, previous remark allows us to conclude that reduced poly-Poisson structure $(S_{red}, P_{red})$ is integrable if the Lie algebroid $(S, P)$ is also integrable.
References

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INSTITUTO DE MATEMÁTICA PURA E APLICADA, ESTRADA DONA CASTORINA 110, RIO DE JANEIRO, 22460-320, BRASIL.

E-mail address: nicolasm@impa.br