Boxicity and Cubicity of Asteroidal Triple free graphs

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Abstract. An axis parallel d-dimensional box is the Cartesian product $R_1 \times R_2 \times \cdots \times R_d$ where each $R_i$ is a closed interval on the real line. The boxicity of a graph $G$, denoted as box($G$), is the minimum integer $d$ such that $G$ can be represented as the intersection graph of a collection of $d$-dimensional boxes. An axis parallel unit cube in $d$-dimensional space or a $d$-cube is defined as the Cartesian product $R_1 \times R_2 \times \cdots \times R_d$ where each $R_i$ is a closed interval on the real line of the form $[a_i, a_i + 1]$. The cubicity of $G$, denoted as cub($G$), is the minimum integer $d$ such that $G$ can be represented as the intersection graph of a collection of $d$-cubes.

An independent set of three vertices is called an asteroidal triple if between each pair in the triple there exists a path which avoids the neighbourhood of the third. A graph is said to be Asteroidal Triple free (AT-free for short) if it does not contain an asteroidal triple. The class of AT-free graphs is a reasonably large one, which properly contains the class of interval graphs, trapezoid graphs, permutation graphs, cocomparability graphs etc. Let $S(m)$ denote a star graph on $m + 1$ nodes. We define claw number of a graph $G$ as the largest positive integer $k$ such that $S(k)$ is an induced subgraph of $G$ and denote it as $\psi(G)$.

Let $G$ be an AT-free graph with chromatic number $\chi(G)$ and claw number $\psi(G)$. In this paper we will show that $\text{box}(G) \leq \chi(G)$ and this bound is tight. We also show that $\text{cub}(G) \leq \text{box}(G)(\lceil \log_2 \psi(G) \rceil + 2) \leq \chi(G)(\lceil \log_2 \psi(G) \rceil + 2)$. If $G$ is an AT-free graph having girth at least 5 then $\text{box}(G) \leq 2$ and therefore $\text{cub}(G) \leq 2 \lceil \log_2 \psi(G) \rceil + 4$.

Key words: Boxicity, Cubicity, Chordal Dimension, Asteroidal Triple free Graph, Chromatic Number, Claw Number.

1 Introduction

Let $\mathcal{F}$ be a family of non-empty sets. An undirected graph $G$ is the intersection graph of $\mathcal{F}$ if there exists a one-one correspondence between the vertices of $G$ and the sets in $\mathcal{F}$ such that two vertices in $G$ are adjacent if and only if the corresponding sets have non-empty intersection. If $\mathcal{F}$ is a family of intervals on the real line, then $G$ is called an interval graph. An interval graph $G$ is said to be a unit interval graph if and only if there is some interval representation of $G$ in which all the intervals are of the same length.

Notations: Let $G(V, E)$ be a simple, finite, undirected graph on $n$ vertices. The vertex set of $G$ is denoted as $V(G)$ and the edge set of $G$ is denoted as $E(G)$. For any vertex

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\( v \in V(G) \) let \( N_G(v) = \{ w \in V(G) \mid (v, w) \in E(G) \} \) be the set of neighbors of \( v \). For each \( S \subseteq V(G) \) let \( G[S] \) denote the subgraph of \( G \) induced by the vertices in \( S \). In this paper we shall use the notation \( G \setminus S \) to denote \( G[V \setminus S] \). The \textit{girth} of a graph is the length of a shortest cycle in the graph.

Let \( G' \) be a graph such that \( V(G') = V(G) \). Then \( G' \) is a \textit{super graph} of \( G \) if \( E(G) \subseteq E(G') \). We define the \textit{intersection} of two graphs as follows: if \( G_1 \) and \( G_2 \) are two graphs such that \( V(G_1) = V(G_2) \), then the intersection of \( G_1 \) and \( G_2 \) denoted as \( G = G_1 \cap G_2 \) is a graph with \( V(G) = V(G_1) = V(G_2) \) and \( E(G) = E(G_1) \cap E(G_2) \).

1.1 Boxicity and Cubicity

A \( d \)-dimensional box is a Cartesian product \( R_1 \times R_2 \times \cdots \times R_d \) where each \( R_i \) is a closed interval of the form \([a_i, b_i]\) on the real line. A \( k \)-box representation of a graph \( G \) is a mapping of the vertices of \( G \) to \( k \)-boxes such that two vertices in \( G \) are adjacent if and only if their corresponding \( k \)-boxes have a non-empty intersection. The \textit{boxicity} of a graph \( G \), denoted as box\((G)\), is the minimum integer \( k \) such that \( G \) can be represented as the intersection graph of \( k \)-dimensional boxes. Clearly, graphs with boxicity 1 are precisely the \textit{interval graphs}.

A \( d \)-dimensional cube is a Cartesian product \( R_1 \times R_2 \times \cdots \times R_d \) where each \( R_i \) is a closed interval of the form \([a_i, a_i + 1]\) on the real line. A \( k \)-cube representation of a graph \( G \) is a mapping of the vertices of \( G \) to \( k \)-cubes such that two vertices in \( G \) are adjacent if and only if their corresponding \( k \)-cubes have a non-empty intersection. The \textit{cubicity} of \( G \) is the minimum integer \( k \) such that \( G \) has a \( k \)-cube representation. Clearly, graphs with cubicity 1 are precisely the \textit{unit interval graphs}.

Let \( G \) be a graph. Let \( I_1, I_2, \ldots, I_k \) be \( k \) interval (unit interval) graphs such that \( G = I_1 \cap I_2 \cap \cdots \cap I_k \). Then \( I_1, I_2, \ldots, I_k \) is called an \textit{interval graph representation} (\textit{unit interval graph representation}) of \( G \). The following equivalence is well known.

\textbf{Lemma 1.} (Roberts\cite{25}) The minimum \( k \) such that there exists an interval graph representation (unit interval graph representation) of \( G \) using \( k \) interval graphs (unit interval graphs) \( I_1, I_2, \ldots, I_k \) is the same as box\((G)\) (cub\((G)\)).

\textbf{Fact 1.} (Roberts\cite{25}) If \( G = G_1 \cap G_2 \cap \cdots \cap G_r \) then cub\((G)\) \( \leq \sum_{i=1}^{r} \) cub\((G_i)\).

The concept of boxicity and cubicity was introduced by F. S. Roberts\cite{25} in 1969. Boxicity finds applications in fields such as ecology and operations research: It is used as a measure of the complexity of ecological \cite{26} and social \cite{20} networks and has applications in fleet maintenance \cite{23}. Boxicity and cubicity has been investigated for various classes of graphs \cite{14,27,28,28} and has been related with other parameters such as treewidth \cite{9} and vertex cover \cite{4}. Computing the boxicity of a graph was shown to be NP-hard by Cozzens\cite{14}. This was later strengthened by Yannakakis\cite{30}, and finally by Kratochvil
who showed that deciding whether boxicity of a graph is at most 2 itself is NP-complete. Boxicity has been generalized in several ways like rectangle number \[11\], poset boxicity \[29\], grid dimension \[3\], circular dimension \[16\], boxicity of digraphs \[10\] etc. Recently Chandran et al. \[7\] showed that for any graph \(G\), \(\text{box}(G) \leq 2\chi(G^2)\) where \(G^2\) is the square of graph \(G\) and \(\chi(G)\) is the chromatic number of the graph. From this they inferred that \(\text{box}(G) \leq 2\Delta^2\), where \(\Delta\) is the maximum degree of \(G\). Very recently this result was improved by Esperet \[15\], who showed that \(\text{box}(G) \leq \Delta^2 + 2\).

Let \(n\) be the number of vertices in \(G\). In \[5\] Chandran et al. have shown that for any graph \(G\), \(\text{cub}(G) \leq \lceil(\Delta + 2)\log_2 n\rceil\). In \[6\] they have shown that for any graph \(G\), \(\text{cub}(G) \leq \lceil4(\Delta + 1)\log_2 n\rceil\).

1.2 Chordal Graph and Chordal Dimension

An undirected graph is said to be chordal if every cycle of length four or more contains a chord i.e. an edge joining two nonconsecutive vertices in the cycle. The chordal dimension of a graph \(G\) denoted as chord(\(G\)), is the minimum integer \(k\) such that \(G\) can be represented as the intersection graph of \(k\) chordal graphs. Scheinerman and Mckee \[21\] have shown that for any graph \(G\), \(\text{chord}(G) \leq \chi(G)\) and also \(\text{chord}(G) \leq \text{treewidth}(G)\) where \(\chi(G)\) is the chromatic number of \(G\). Since any interval graph is a chordal graph we have the following observation:

**Observation 1.** For any graph \(G\), \(\text{chord}(G) \leq \text{box}(G) \leq \text{cub}(G)\).

1.3 Claw Number

Let \(S(k)\) denote a star graph on \(k + 1\) vertices. (Note that \(S(k)\) is the complete bipartite graph \(K_{1,k}\)). The center of a star is that vertex which is connected to all other vertices in the star. An induced \(S(3)\) in a graph is usually known as a claw.

**Definition 1.** The claw number of a graph \(G\) is the largest positive integer \(k\) such that \(S(k)\) is an induced subgraph of \(G\) and is denoted as \(\psi(G)\).

Recently Adiga et al. \[1\] have given an almost tight bound for the cubicity of interval graphs in terms of its claw number.

**Theorem 1.** (Adiga et al. \[1\]) If \(G\) is an interval graph with claw number \(\psi(G)\) then \(\lceil\log_2 \psi(G)\rceil \leq \text{cub}(G) \leq \lceil\log_2 \psi(G)\rceil + 2\).

1.4 AT-free graphs

An independent set of three vertices is called an asteroidal triple if between every pair of vertices there is a path which avoids the neighbourhood of the third. A graph is called asteroidal triple free (AT-free for short) if it does not contain an asteroidal triple.
They form a large class of graphs since they contain interval, permutation, trapezoid, cocomparability and many other graph classes. Corneil, Olariu and Stewart have studied many structural and algorithmic properties of AT-free graphs in [12,13].

A graph is called claw-free AT-free graph if it is AT-free and does not contain $K_{1,3}$ (i.e. $S(3)$, the claw) as an induced subgraph. Kloks et al. [18] have given a characterization of claw-free AT-free graphs.

1.5 Our Results

In this paper we will show that

1. If $G$ is an AT-free graph with chromatic number $\chi(G)$ then $\text{box}(G) \leq \chi(G)$ and this bound is tight.
2. If $G$ is a claw-free AT-free graph with chromatic number $\chi(G)$ then $\text{box}(G) = \text{cub}(G) \leq \chi(G)$ and this bound is tight.
3. If $G$ is an AT-free graph having girth at least 5 then $\text{box}(G) \leq 2$ and this bound is tight. We also show that $\text{cub}(G) \leq 2 \left\lceil \log_2 \psi(G) \right\rceil + 4$.
4. If $G$ is an AT-free graph with chromatic number $\chi(G)$ and claw number $\psi(G)$ then $\text{cub}(G) \leq \text{box}(G)\left(\left\lceil \log_2 \psi(G) \right\rceil + 2\right) \leq \chi(G)\left(\left\lceil \log_2 \psi(G) \right\rceil + 2\right)$.

Remark on previous approach to boxicity and cubicity of AT-free graphs:

In [9] it has been shown that for any graph $G$, $\text{box}(G) \leq \text{treewidth}(G) + 2$. It has also been shown that if $G$ is an AT-free graph then $\text{treewidth}(G) \leq 3\Delta - 2$, hence $\text{box}(G) \leq 3\Delta$ where $\Delta$ is the maximum degree of $G$. But the result shown in this paper is much stronger. (Recall that $\chi(G) \leq \Delta + 1$ for any graph, but in general $\chi(G)$ can be much smaller.)

In [6] Chandran et al. have studied the relationship between cubicity and bandwidth of a graph. As a corollary they have also shown that if $G$ is an AT-free graph then $\text{cub}(G) \leq 3\Delta - 1$ since for AT-free graphs bandwidth is at most $3\Delta - 2$. Using the technique of [6], this upper bound cannot be improved much since $\left\lceil \frac{\Delta}{2} \right\rceil$ is a lower bound for bandwidth of any graph. In this paper we show that for any AT-free graph $G$, $\text{cub}(G) \leq \text{box}(G)\left(\left\lceil \log_2 \psi(G) \right\rceil + 2\right) \leq \chi(G)\left(\left\lceil \log_2 \psi(G) \right\rceil + 2\right)$. Clearly this result can be much stronger than that of [6] in some cases.

2 Upper bound on boxicity of AT-free graphs and cubicity of claw-free AT-free graphs

In this section we will show an upper bound on boxicity of AT-free graphs and cubicity of claw-free AT-free graphs. A triangulation of a graph $G$ is a chordal graph $H$ on the same vertex set that contains $G$ as a subgraph i.e. $V(G) = V(H)$ and $E(G) \subseteq E(H)$. $H$ is said to be a minimal triangulation of $G$ if there exists no other chordal graph $H'$. 

on the same vertex set as $G$ and $H$ such that $E(G) \subseteq E(H') \subset E(H)$. Möhring studied minimal triangulation of AT-free graphs in [22]. Parra and Scheffler have shown relations between minimal separators of a graph and its minimal triangulations in [24].

From the definition of chordal dimension and boxicity we know that for any graph $G$, $\text{chord}(G) \leq \text{box}(G)$. Now we will show that when $G$ is an AT-free graph, $\text{box}(G) \leq \text{chord}(G)$. For this we need the following theorem:

**Theorem 2.** (Möhring [22]) If $G$ is an AT-free graph then every minimal triangulation of $G$ is an interval graph.

Let $\text{chord}(G) = k$ and $G = \bigcap_{i=1}^{k} G_i$ where $G_i$ is a chordal graph for $1 \leq i \leq k$. It is easy to see that if we replace each $G_i$ by another chordal graph $G'_i$ such that $V(G_i) = V(G'_i)$ and $E(G) \subseteq E(G'_i) \subseteq E(G_i)$, we still will have $G = \bigcap_{i=1}^{k} G'_i$. It follows that there exists $G'_1, G'_2, \ldots, G'_k$ such that $G = \bigcap_{i=1}^{k} G'_i$ where each $G'_i$ is a minimal triangulation of $G$. By Theorem 2, $G'_i$ for $1 \leq i \leq k$ is an interval graph. It follows that $\text{box}(G) \leq k = \text{chord}(G)$. Thus we have the following Observation:

**Observation 2.** If $G$ is an AT-free graph then $\text{chord}(G) = \text{box}(G)$.

Scheinerman and Mckee have shown the following upper bound on chordal dimension of a graph $G$ in terms of its chromatic number $\chi(G)$.

**Theorem 3.** (Scheinerman and Mckee [21]) For any graph $G$ with chromatic number $\chi(G)$, $\text{chord}(G) \leq \chi(G)$.

Combining Observation 2 and Theorem 3 we get the following upper bound on boxicity of AT-free graphs:

**Theorem 4.** If $G$ is an AT-free graph with chromatic number $\chi(G)$ then $\text{box}(G) \leq \chi(G)$.

In general $\chi(G) \leq d + 1$, where $d$ is the degeneracy of the graph. It follows that $\text{box}(G) \leq d + 1$. Though it is known [24] that $\text{box}(G) \leq 2\chi(G^2)$ for any graph $G$, $\text{box}(G)$ need not always be less than equal to $\chi(G)$: For example it is shown in [4] that there exists bipartite graphs with boxicity $\frac{n}{2}$. It is also shown in [2] that almost all balanced bipartite graphs (with respect to a suitable probability distribution) have boxicity $\Omega(\frac{n}{\log n})$.

**Theorem 5.** (Parra and Scheffler [24]) A graph $G$ is claw-free AT-free if and only if every minimal triangulation of $G$ is a unit interval graph.

By a similar argument given for Observation 2 we get the following:

**Observation 3.** If $G$ is a claw-free AT-free graph then $\text{chord}(G) = \text{cub}(G)$.
Thus if $G$ is a claw-free AT-free graph we have $\text{chord}(G) = \text{box}(G) = \text{cub}(G)$. Combining Theorem 3 and Observation 3 we get the following upper bound on cubicity of claw-free AT-free graphs:

**Theorem 6.** If $G$ is a claw-free AT-free graph with chromatic number $\chi(G)$ then $\text{cub}(G) \leq \chi(G)$.

### 2.1 Tightness of Theorem 4 and Theorem 6

Let $G$ be a complete $k$-partite graph on $n$ vertices (We will assume that $n$ is multiple of $k$ and $n > k$). It is easy to see that this is an AT-free graph. Since the chromatic number of this graph is $k$, we have $\text{box}(G) \leq k$ by Theorem 4. But it was shown by Roberts [25] that $\text{box}(G) = k$. So the upper bound for boxicity given in Theorem 4 is tight for complete $k$-partite graphs.

Let $G = \overline{(\frac{n}{2})K_2}$, the complement of the perfect matching on $n$ vertices (We will assume that $n$ is even and $n > 3$). It is easy to see that this is a claw-free AT-free graph. Since the chromatic number of this graph is $\frac{n}{2}$, we have $\text{cub}(G) \leq \frac{n}{2}$ by Theorem 6. But it was shown by Roberts [25] that $\text{cub}(G) = \frac{n}{2}$. So the upper bound for cubicity given in Theorem 6 is tight for $(\frac{n}{2})K_2$.

### 3 Upper bound on boxicity of AT-free graphs having girth at least 5

In this section we will show an upper bound on boxicity of AT-free graphs having girth at least 5. Let $G$ be an AT-free graph having girth at least 5. Since an induced cycle of length at least 6 contains an AT, $G$ is either acyclic or all induced cycles of $G$ are of length exactly 5. Recall that diameter of a graph is the maximum of distance($u,v$) over all pairs of vertices $u, v \in V(G)$. A set of vertices $S$ of a graph $G$ is said to be dominating if every vertex in $V(G) \setminus S$ is adjacent to some vertex in $S$. A path joining vertices $x$ and $y$ is said to be a $x$-$y$ path. A pair of vertices $x, y$ is said to be a dominating pair if all $x$-$y$ paths in $G$ are dominating sets. Corneil, Olariu and Stewart have shown the following fundamental property of AT-free graphs:

**Theorem 7.** (Corneil et al. [12]) Every connected AT-free graph contains a dominating pair.

They have also proved the following theorem which we shall use to show the upper bound on boxicity:

**Theorem 8.** (Corneil et al. [12]) In every connected AT-free graph there exists dominating pair $x, y$ such that $\text{distance}(x, y) = \text{diameter}(G)$. 

Let \( x, y \) be a dominating pair in \( G \) and let \( P \) be a shortest \( x-y \) path of length equal to the diameter of \( G \). Let \( d \) be the diameter of \( G \) and \( V(P) = \{u_1, u_2, \cdots, u_d\} \) where \( x = u_1 \) and \( y = u_d \). Let \( V(P) = V(G) \setminus V(P) \).

**Lemma 2.** For each vertex \( v \in V(P) \), \( |N_G(v) \cap V(P)| = 1 \).

**Proof.** Since \( x, y \) is a dominating pair and \( P \) is a \( x-y \) path, \( V(P) \) is a dominating set. Hence for every vertex \( v \in V(P) \) we have \( |N_G(v) \cap V(P)| \geq 1 \). We will show that for each vertex \( v \in V(P) \), \( |N_G(v) \cap V(P)| \leq 1 \). If possible let \( w \in V(P) \) such that \( |N_G(w) \cap V(P)| \geq 2 \). Let \( u_i, u_j \in N_G(w) \cap V(P) \) be such that \( 1 \leq i < j \leq d \) and for all \( k, i < k < j, u_k \notin N_G(w) \). We consider the following cases

**Case 1:** When \( j \leq i + 2 \). If \( j = i + 1 \) then \( u_i-w-u_{j-1}u_i \) forms an induced cycle of length 3 in \( G \), a contradiction. Similarly if \( j = i + 2 \) then \( u_i-w-u_{j-1}u_i \) forms an induced cycle of length 4 in \( G \), a contradiction.

**Case 2:** When \( j \geq i + 3 \). Let \( P_1 \) denote the path \( u_1-u_2-\cdots-u_i \) and \( P_2 \) denote the path \( u_j-u_{j+1}-\cdots-u_d \). Clearly \( P_1wP_2 \) forms a \( x-y \) path say \( P' \) in \( G \). Now \( |V(P')| = i+1+(d-j+1) \). If \( j \geq i + 3 \) then \( |V(P')| \leq d - 1 \). Recall that \( P \) is a shortest \( x-y \) path. But \( P' \) is a shorter \( x-y \) path than \( P \), a contradiction.

Therefore for each vertex \( v \in V(P) \), \( |N_G(v) \cap V(P)| = 1 \). \( \square \)

Let \( S_i = \{v \mid v \in V(P) \text{ and } u_i \in N_G(v)\} \) for \( 1 \leq i \leq d \). From Lemma 2 it follows that \( S_1, S_2, \cdots, S_d \) is a partition of the vertex set \( V(P) \). In other words,

**Observation 4.** Let \( u \in V(P) \) and \( v \in V(P) \). Suppose \( u = u_i \) and \( v \in S_k \) where \( 1 \leq i, k \leq d \). Then \( (u, v) \notin E(G) \) if and only if \( i \neq k \).

**Lemma 3.** Let \( v \in S_i \).

1. \( |N_G(v) \cap S_i| = 0 \) where \( 1 \leq i \leq d \).
2. \( |N_G(v) \cap S_{i+1}| = 0 \) where \( 1 \leq i \leq d - 1 \).
3. \( |N_G(v) \cap S_{i+2}| \leq 1 \) where \( 1 \leq i \leq d - 2 \).
4. \( |N_G(v) \cap S_j| = 0 \) where \( i + 3 \leq j \leq d \) and \( i \geq 1 \).

**Proof(1):** If possible let \( w \in S_i \) such that \( (v, w) \in E(G) \). Now \( v-u_i-w-v \) forms an induced cycle of length 3 in \( G \), a contradiction.

**Proof(2):** If possible let \( w \in S_{i+1} \) such that \( (v, w) \in E(G) \). Then \( u_i-v-w-u_{i+1}-u_i \) forms a cycle of length 4 in \( G \), a contradiction.

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Proof(3): If possible let \( u, w \in S_{i+2} \) such that \((v, u) \in E(G) \) and \((v, w) \in E(G) \). Then \( v-w-u_{i+2}-u-v \) forms a cycle of length 4 in \( G \), a contradiction.

Proof(4): If possible let \( w \in S_j \) such that \((v, w) \in E(G) \). Since \((v, u_i) \in E(G) \) according to Lemma 2 we have \((v, u_k) \notin E(G) \) for all \( k \neq i \). Similarly since \((w, u_j) \in E(G) \) we have \((w, u_k) \notin E(G) \) for all \( k \neq j \). Since \( j \geq i + 3 \), \( u_i-v-w-u_j-u_{j-1}-u_{j-2}-\cdots-u_i \) forms an induced cycle of length at least 6 in \( G \). But \( G \) is an AT-free graph, a contradiction.

From Lemma 2 we have the following observation:

Observation 5. If \( u, v \in V(P) \), \( u \in S_i \), \( v \in S_j \) and \((u, v) \in E(G) \) then \(|j - i| = 2| \).

Lemma 4. Let \( u \in S_i \) and \( v \in S_{i+2} \) where \( 1 \leq i \leq d - 2 \). If \((u, v) \in E(G) \) then for any \( p \in S_i \setminus \{u\}, q \in S_{i+2} \setminus \{v\} \) we have \((p, q) \notin E(G) \).

Proof. Suppose not. Let \( p \in S_i \setminus \{u\} \) and \( q \in S_{i+2} \setminus \{v\} \) such that \((p, q) \in E(G) \). Since \( u, p \in S_i \), according to Lemma 3 part (1), \((u, p) \notin E(G) \). Similarly \((v, q) \notin E(G) \). According to Lemma 2 \((u, u_{i+2}) \notin E(G) \), \((p, u_{i+2}) \notin E(G) \), \((q, u_i) \notin E(G) \) and \((v, u_i) \notin E(G) \). Also we have \((u, q) \notin E(G) \) and \((v, p) \notin E(G) \) by Lemma 3 part (3). Moreover \((u_i, u_{i+2}) \notin E(G) \) since \( P \) is a shortest \( x-y \) path. Therefore \( u-u_{i}+2-p-q-u_{i+2}-v-u \) forms an induced cycle of length 6 in \( G \) and hence \( \{u, p, u_{i+2}\} \) forms an AT in \( G \), a contradiction. \( \square \)

A vertex \( v \in V(P) \) is said to be non-pendant if \( N_G(v) \cap V(P) \neq \emptyset \). Note that if \( N_G(v) \cap V(P) = \emptyset \) then \( v \) has to be a pendant vertex by Lemma 2.

Lemma 5. \( S_i \) can contain at most 2 non-pendant vertices for \( 1 \leq i \leq d \).

Proof. If \( v \in S_i \) is non-pendant then according to Observation 3, either \( N_G(v) \cap S_{i-2} \neq \emptyset \) or \( N_G(v) \cap S_{i+2} \neq \emptyset \). By Lemma 3 part (3) and Lemma 4 at most one vertex in \( S_i \) can be connected to some vertex in \( S_{i-2} \). Similarly at most one vertex in \( S_i \) can be connected to some vertex in \( S_{i+2} \). Therefore \( S_i \) can contain at most 2 non-pendant vertices. \( \square \)

Observation 6. If \( S_i \) contains two non-pendant vertices say \( u, v \) then one of the following statements is true (by Lemma 3 part (3) and Lemma 4)

1. \( N_G(u) \cap S_{i-2} = \emptyset \) and \( N_G(v) \cap S_{i+2} = \emptyset \).
2. \( N_G(u) \cap S_{i+2} = \emptyset \) and \( N_G(v) \cap S_{i-2} = \emptyset \).

3.1 Interval Graph Construction

We shall construct two interval graphs \( I_1 \) and \( I_2 \) such that \( G = I_1 \cap I_2 \). In the interval graph \( I_j \) where \( j = 1, 2 \), let \( l_j(u) \) and \( r_j(u) \) denote the left and right endpoint of the interval corresponding to vertex \( u \in V(G) \) respectively. Let \( S \) be the set of non-pendant
vertices in $V(\overline{P})$. To construct $I_1$ we map each vertex $v \in V(G)$ to an interval on the real line by the mapping:

$$g_1(v) = [i, i + 1] \quad \text{if } v \in V(P) \text{ and } v = u_i \text{ for } 1 \leq i \leq d$$

$$= [i + \frac{2j - 1}{2n}, i + \frac{2j}{2n}] \quad \text{if } v \in S_i \setminus S, \ 1 \leq i \leq d \text{ and } 1 \leq j \leq |S_i|$$

$$= [i, \frac{1}{2}, i + \frac{3}{2}] \quad \text{if } v \in S_i \cap S, \ N_G(v) \cap S_{i-2} \neq \emptyset \text{ and } N_G(v) \cap S_{i+2} \neq \emptyset$$

$$= [i + 1, i + \frac{3}{2}] \quad \text{if } v \in S_i \cap S, \ N_G(v) \cap S_{i-2} = \emptyset \text{ and } N_G(v) \cap S_{i+2} \neq \emptyset$$

$$= [i - \frac{1}{2}, i] \quad \text{if } v \in S_i \cap S, \ N_G(v) \cap S_{i-2} \neq \emptyset \text{ and } N_G(v) \cap S_{i+2} = \emptyset$$

**Lemma 6.** $I_1$ is a supergraph of $G$.

**Proof.** Let $(u, v) \in E(G)$. We shall show that $g_1(u) \cap g_1(v) \neq \emptyset$. We consider the following cases:

**Case 1:** When either $u \in V(P)$ or $v \in V(P)$. Without loss of generality we can assume that $u \in V(P)$. Let $u = u_i$ where $1 \leq i \leq d$. If $v \in V(P)$ then either $v = u_{i-1}$ or $v = u_{i+1}$. Now if $v = u_{i-1}$ then $i \in g_1(u) \cap g_1(v)$. On the other hand if $v = u_{i+1}$ then $i + 1 \in g_1(u) \cap g_1(v)$. If $v \in \overline{P}$ then according to Observation 4, $v \in S_i$. Now if $v \in S_i \setminus S$ then $i = l_1(u) < l_1(v) < r_1(u) = i + 1$ and hence $g_1(u) \cap g_1(v) \neq \emptyset$. If $v \in S_i \cap S$ then we consider the following cases: If $N_G(v) \cap S_{i-2} \neq \emptyset$ then $i \in g_1(u) \cap g_1(v)$. Otherwise if $N_G(v) \cap S_{i+2} \neq \emptyset$ then $i + 1 \in g_1(u) \cap g_1(v)$. \hfill \Box

**Case 2:** When $u, v \in V(\overline{P})$. By definition of non-pendant vertices $u, v \in S$. Let $u \in S_i$. According to Observation 5, either $v \in S_{i-2}$ or $v \in S_{i+2}$. If $v \in S_{i-2}$ then $l_1(u) = r_1(u) = i - \frac{1}{2}$. Otherwise if $v \in S_{i+2}$ then $r_1(u) = l_1(v) = i + \frac{3}{2}$. In both cases we have $g_1(u) \cap g_1(v) \neq \emptyset$.

To construct $I_2$ we map each vertex $v \in V(G)$ to an interval on the real line by the mapping:

$$g_2(v) = [1, 2] \quad \text{if } v \in V(P) \text{, } v = u_i \text{ and } i \mod 2 = 1$$

$$= [2, 3] \quad \text{if } v \in V(P) \text{, } v = u_i \text{ and } i \mod 2 = 0$$

$$= [\frac{5}{4}, \frac{7}{4}] \quad \text{if } v \in S_i \setminus S \text{ and } i \mod 2 = 1$$

$$= [\frac{9}{4}, \frac{11}{4}] \quad \text{if } v \in S_i \setminus S \text{ and } i \mod 2 = 0$$

$$= [0, 1] \quad \text{if } v \in S_i \cap S \text{ and } i \mod 2 = 1$$

$$= [3, 4] \quad \text{if } v \in S_i \cap S \text{ and } i \mod 2 = 0$$
Lemma 7. \(I_2\) is a supergraph of \(G\).

Proof. Let \((u, v) \in E(G)\). We shall show that \(g_2(u) \cap g_2(v) \neq \emptyset\). We consider the following cases:

Case 1: When either \(u \in V(P)\) or \(v \in V(P)\). Without loss of generality we can assume that \(u \in V(P)\). Let \(u = u_i\) where \(1 \leq i \leq d\). If \(v \in V(P)\) then \(2 \notin g_2(u) \cap g_2(v)\). If \(v \in V(P)\) then according to Observation 5 \(v \in S_i\). Now if \(v \in S_i \setminus S_{i-2}\) then \(l_2(u) < l_2(v) < r_2(v) < r_2(u)\) and hence \(g_2(u) \cap g_2(v) \neq \emptyset\). If \(v \in S_i \cap S_{i-2}\) we consider the following cases: If \(i \mod 2 = 1\) then \(l_2(u) = r_2(v) = 1\). On the other hand if \(i \mod 2 = 0\) then \(r_2(u) = l_2(v) = 3\). In both cases we have \(g_2(u) \cap g_2(v) \neq \emptyset\).

Case 2: When \(u, v \in V(P)\). By definition of non-pendant vertices \(u, v \in S\). Let \(u \in S_i\) and \(v \in S_j\) where \(1 \leq i, j \leq d\). According to Observation 5 \(|i - j| = 2\) which implies that \(i = j \mod 2\). Hence \(g_2(u) = g_2(v)\) and thus \(g_2(u) \cap g_2(v) \neq \emptyset\). \(\square\)

Lemma 8. For any \((u, v) \notin E(G)\) either \((u, v) \notin E(I_1)\) or \((u, v) \notin E(I_2)\).

Proof. Let \((u, v) \notin E(G)\). We consider the following cases:

Case 1: When \(u, v \in V(P)\). Let \(u = u_i\) and \(v = u_j\) where \(1 \leq i, j \leq d\). Since \((u, v) \notin E(G)\) we have \(|j - i| \geq 2\). Therefore \(|l_1(u) - l_1(v)| \geq 2\). Since in \(I_1\), the intervals corresponding to vertices in \(V(P)\) are of length 1 we have \(g_1(u) \cap g_1(v) = \emptyset\) and hence \((u, v) \notin E(I_1)\).

When \(v \in S_k \cap S\) we consider the following cases:

Subcase 2.1: When \(|k - i| \geq 2\). Now \(g_1(u) = [i, i + 1]\) and \(k - \frac{1}{2} \leq l_1(v) < r_1(v) \leq k + \frac{3}{2}\). If \(i \leq k - 2\) then \(r_1(u) \leq k - 1 < k - \frac{1}{2} \leq l_1(v)\) and hence \(g_1(u) \cap g_1(v) = \emptyset\). If \(i \geq k + 2\) then \(l_1(u) \geq k + 2 > k + \frac{3}{2} \geq r_1(v)\) and hence \(g_1(u) \cap g_1(v) = \emptyset\). Therefore \((u, v) \notin E(I_1)\).

Subcase 2.2: When \(|k - i| \leq 1\). Since \(k \neq i\) we have \(k \mod 2 \neq i \mod 2\). If \(i \mod 2 = 0\) then \(g_2(u) = [2, 3]\) and \(g_2(v) = [0, 1]\). Hence \(g_2(u) \cap g_2(v) = \emptyset\). If \(i \mod 2 = 1\) then \(g_2(u) = [1, 2]\) and \(g_2(v) = [3, 4]\). Hence \(g_2(u) \cap g_2(v) = \emptyset\). In both cases we have \((u, v) \notin E(I_2)\).

Case 3: When \(u, v \in V(P)\). We consider the following cases:

Subcase 3.1: When \(u, v \in S_i\). Let \(u \in S_i\) and \(v \in S_{i+2}\). If \(i = j\) then according to Observation 6 either \(N_G(u) \cap S_{i-2} = \emptyset\) and \(N_G(v) \cap S_{i+2} = \emptyset\) or \(N_G(u) \cap S_{i+2} = \emptyset\) and \(N_G(v) \cap S_{i-2} = \emptyset\). If \(N_G(u) \cap S_{i-2} = \emptyset\) and \(N_G(v) \cap S_{i+2} = \emptyset\) then \(r_1(v) = i < i + 1 = l_1(u)\). Hence \(g_1(u) \cap g_1(v) = \emptyset\). If \(N_G(v) \cap S_{i-2} = \emptyset\) and \(N_G(u) \cap S_{i+2} = \emptyset\) then \(r_1(u) = i < i + 1 = l_1(v)\). Hence \(g_1(u) \cap g_1(v) = \emptyset\).
If $i \neq j$ then we consider the following cases. Without loss of generality we can assume that $j > i$.

**Subcase 3.1.1:** When $(j - i) \mod 2 \neq 0$. It is easy to see that $g_2(u) \cap g_2(v) = \emptyset$.

**Subcase 3.1.2:** When $(j - i) \mod 2 = 0$. We consider the following cases:

**Subcase 3.1.2.1:** When $j = i + 2$. We will show that either $N_G(u) \cap S_{i+2} = \emptyset$ or $N_G(v) \cap S_i = \emptyset$. If possible let $N_G(u) \cap S_{i+2} \neq \emptyset$ and $N_G(v) \cap S_i \neq \emptyset$. Let $p \in S_i$ and $q \in S_{i+2}$ be such that $(u,q) \in E(G)$ and $(v,p) \in E(G)$. Since $(u,v) \notin E(G)$ we have $u \neq p$ and $v \neq q$. But then we get a contradiction to Lemma 4. Therefore either $N_G(u) \cap S_{i+2} = \emptyset$ or $N_G(v) \cap S_i = \emptyset$. If $N_G(u) \cap S_{i+2} = \emptyset$ then $r_1(u) = i < i + \frac{3}{2} = j - \frac{1}{2} \leq l_1(v)$. Therefore $g_1(u) \cap g_1(v) = \emptyset$. On the other hand if $N_G(v) \cap S_i = \emptyset$ then $r_1(u) \leq i + \frac{3}{2} < j + 1 = l_1(v)$. Therefore $g_1(u) \cap g_1(v) = \emptyset$.

**Subcase 3.1.2.2:** When $j \geq i + 4$. Then $r_1(u) \leq i + \frac{3}{2} < (i + 4) - \frac{1}{2} \leq j - \frac{1}{2} \leq l_1(v)$. Therefore $g_1(u) \cap g_1(v) = \emptyset$.

**Subcase 3.2:** When $u \notin S$ and $v \notin S$. According to the construction of $I_1$, it is easy to see that $\bigcup_{i=1}^d (S_i \setminus S)$ induces an independent set in $I_1$. Therefore $g_1(u) \cap g_1(v) = \emptyset$.

**Subcase 3.3:** When $u \notin S$ and $v \in S$. In $I_2$, $g_2(v)$ is either $[0,1]$ or $[3,4]$ and $g_2(u)$ is either $[\frac{5}{7}, \frac{7}{11}]$ or $[\frac{9}{11}, \frac{11}{14}]$. In all the four possible cases it is easy to see that $g_2(u) \cap g_2(v) = \emptyset$.

Combining Lemma 5 and 8 we have the following Theorem

**Theorem 9.** If $G$ is an AT-free graph having girth at least 5 then $\text{box}(G) \leq 2$.

### 3.2 Tightness of Theorem 9

Let $G$ be a cycle of length 5. It is easy to see that $G$ is an AT-free graph having girth at least 5. According to Theorem 9 $\text{box}(G) \leq 2$. But clearly $\text{box}(G) = 2$, since $G$ is not an interval graph. Therefore the upper bound given by Theorem 9 is tight.

### 4 Upper bound on cubicity of AT-free graphs

In this section we will show an upper bound on cubicity of AT-free graphs in terms of its boxicity and claw number. This in turn will give an upper bound in terms of chromatic number and claw number. Let $G$ be an AT-free graph with chromatic number $\chi(G)$ and claw number $\psi(G)$. We need some results shown by Parra and Scheffler [24].

For any graph $G(V,E)$ and for a given pair of nonadjacent vertices $a,b \in V$, a subset $S \subseteq V \setminus \{a,b\}$ is a $a$-$b$ vertex separator (a-$b$ separator for short) if when $S$ is removed from $G$, $a$ and $b$ belong to different connected components of $G \setminus S$. $S$ is said to be a minimal $a$-$b$ separator if no proper subset of $S$ is an $a$-$b$ separator. A separator $S$ in $G$ is said to be a minimal separator of $G$ if there exists a pair of vertices $a,b \in V(G)$ such
that $S$ is a minimal $a$-$b$ separator. It is well-known that a graph is chordal if and only if all its minimal separators induce cliques [17].

Let $S$ and $T$ be two minimal separators of $G$. $S$ is said to cross $T$ if there are two components $C, D$ of $G \setminus T$ such that $S$ intersects both $C$ and $D$. Parra and Scheffler [24] have shown that if $S$ crosses $T$ then $T$ crosses $S$ also. $S$ and $T$ are said to be parallel if they do not cross each other. Let $S_G$ denote the set of minimal separators in $G$. For $T = \{S_1, S_2, \ldots, S_k\} \subseteq S_G$, let $G_T$ denote the graph obtained by making each separator $S_i$ for $1 \leq i \leq k$ a clique. The following Theorem is proved in [24].

**Theorem 10.** (Parra and Scheffler [24])

1. Let $T = \{S_1, \ldots, S_k\}$ be a maximal set of pairwise parallel minimal separators in $G$.
   Then $H = G_T$ is a minimal triangulation of $G$ and $S_H = T$.

2. Let $H$ be a minimal triangulation of $G$. Then $S_H$ is a maximal set of pairwise parallel minimal separators in $G$ and $H = G_{S_H}$.

Let $T$ be a minimal separator of $G$. A component $C$ of $G \setminus T$ is called a full component if every vertex in $T$ is adjacent to some vertex in $C$. The following property of minimal separator is shown in [17].

**Theorem 11.** (Golumbic [17]) A separator $T$ in graph $G$ is minimal if and only if there are at least two full components in $G \setminus T$.

**Lemma 9.** Let $X$ be a minimal separator in a graph $G$ and $C, D$ be two full components in $G \setminus X$. Let $x \in X$, $c \in C$ and $d \in D$. Let $Y$ be another minimal separator of $G$ such that $c \in Y$ and $x, d \notin Y$. If $X$ is parallel to $Y$ then $x, d$ belongs to the same connected component in $G \setminus Y$.

**Proof.** Suppose $x$ and $d$ lies in different connected components in $G \setminus Y$. Since $D$ is a full component in $G \setminus X$, there exists a $x$-$d$ path say $P$ in $G[D \cup \{x\}]$. Now according to assumption, $x$ and $d$ lie in different components in $G \setminus Y$. Therefore $Y$ must contain at least one vertex from $P$. But since $x \notin Y$ and all the vertices in $P$ except $x$ lie in $D$ we have $Y \cap D \neq \emptyset$. Again $c \in Y \cap C$ and therefore $Y \cap C \neq \emptyset$. Hence $Y$ crosses $X$, a contradiction. $\Box$

**Lemma 10.** If $G$ is an AT-free graph and $H$ is a minimal triangulation of $G$ with claw number $\psi(H)$ then $\psi(H) \leq \psi(G)$.

**Proof.** Suppose $\psi(H) > \psi(G)$ and $\psi(H) = p$. An edge $(u, v) \in E(H)$ is said to be an old edge if $(u, v) \in E(G)$ and is said to be a new edge otherwise. Among all the claws of maximum size in $H$, let $U = \{s, x_1, x_2, \ldots, x_p\}$ induce the one with maximum number of old edges in it. Let $s$ be the center of the claw. Since $\psi(H) > \psi(G)$ at least one of the edges in $U$ has to be new. Without loss of generality let us assume that $(s, x_1)$ is a
new edge. Let $\mathcal{T} = \{S_1, S_2, \ldots, S_k\}$ be the collection of minimal separators of $H$. From part (2) of Theorem 10, $\mathcal{T}$ is a maximal set of pairwise parallel minimal separators of $G$ and $H = G_\mathcal{T}$. In other words if $(u, v) \in E(H) \setminus E(G)$ then there exists an $S_j \in \mathcal{T}$ such that both $u, v \in S_j$. Thus the vertices $s, x$ must belong to some minimal separator, say $X \in \mathcal{T}$ of $G$. Let $\mathcal{C}$ be the set of full components in $G \setminus X$. According to Theorem 11, $|\mathcal{C}| \geq 2$. We consider the following two cases:

**Case 1:** There exists a full component $C \in \mathcal{C}$ such that $C \cap \{x_2, x_3, \ldots, x_p\} = \emptyset$. Since $C$ is a full component of $G \setminus X$ and $s \in X$ there is at least one vertex in $C$, say $a$ such that $(s, a) \in E(G)$. Since $E(G) \subseteq E(H)$ we have $(s, a) \in E(H)$. Note that $(a, x_i) \notin E(G)$ for $2 \leq i \leq p$ because $C \cap \{s, x_1, x_2, \ldots, x_p\} = \emptyset$ by assumption and $x_i \notin X$ for $2 \leq i \leq p$ since $x_1 \in X$ and $X$ induces a clique in $H$. Then it is easy to see that $\{s, a, x_2, \ldots, x_p\}$ forms a claw of size $p$ in $H$ having more old edges than in $U$ since $(s, x_1)$ is a new edge and $(s, a)$ is an old edge. But by assumption $U$ was a maximum sized claw having maximum number of old edges in it, a contradiction.

**Case 2:** Every full component in $\mathcal{C}$ contains at least one $x_i$ where $2 \leq i \leq p$. According to Theorem 11, $|\mathcal{C}| \geq 2$ and hence there exists two full components $C, D \in \mathcal{C}$. Let $x_i \in C$ and $x_j \in D$ where $2 \leq i < j \leq p$. We will show that the triplet $\{x_1, x_i, x_j\}$ forms an AT in $G$, leading to a contradiction. Since $C$ is a full component of $G \setminus X$, $x_i \in C$ and $x_1 \in X$ there exists a $x_i$-$x_1$ path in $G[C \cup \{x_1\}]$ and this path does not intersect $N_G(x_j)$ since $x_j \in D$. Similarly since $D$ is a full component of $G \setminus X$, $x_j \in D$ and $x_1 \in X$ there exists a $x_j$-$x_1$ path in $G[D \cup \{x_1\}]$ which does not intersect $N_G(x_i)$. Now we want to show that there exists a $x_i$-$x_j$ path in $G$ which does not intersect $N_G(x_1)$. For that we need the following claim:

**Claim:**

1. There exists a $x_i$-$s$ path in $G$ that does not intersect $N_G(x_1)$.
2. There exists a $x_j$-$s$ path in $G$ that does not intersect $N_G(x_1)$.

**Proof.** We prove only part (1) since the proof of part (2) is similar. Recall that $(s, x_1)$ is a new edge by assumption. Since $\{x_1, x_2, \ldots, x_p\}$ induces an independent set in $H$ and $E(G) \subseteq E(H)$ they induce an independent set in $G$ also. If $(s, x_i) \in E(G)$ we have a $x_i$-$s$ path in $G$ that does not intersect $N_G(x_1)$ since $(s, x_1) \notin E(G)$ and $(x_1, x_i) \notin E(G)$. Therefore we can assume that $(s, x_i) \notin E(G)$. Since $(s, x_i)$ is a new edge by theorem 11 there should be a minimal separator $Y \in \mathcal{T}$ such that $s, x_i \in Y$. Clearly $X \neq Y$ since $x_i \notin X$. According to Theorem 11, $X$ and $Y$ are parallel and each separator in $\mathcal{T}$ induces a clique in $H$. Since $(x_i, x_1) \notin E(H)$, $(x_i, x_j) \notin E(H)$ and $x_i \in Y$ we have $x_1 \notin Y$ and $x_j \notin Y$. Therefore according to Lemma 11, $x_1$ and $x_j$ must belong to the same connected
component say $Q$ of $G \setminus Y$. Let $Q'$ be a full component of $G \setminus Y$ such that $Q' \neq Q$. Note that such a full component exists by Theorem 1. Now $s$ and $x_i$ must be connected in $G$ to at least one vertex in $Q'$ and therefore there is a $x_i$-$s$ path in $G(Q' \cup \{x_i, s\})$ which does not intersect $N_G(x_i)$.

Since $(s, x_1) \notin E(G)$ from the previous claim it is easy to see that there exists a $x_i$-$x_j$ path in $G$ which does not intersect $N_G(x_1)$. Therefore $\{x_1, x_i, x_j\}$ forms an asteroidal triple in $G$, a contradiction. 

**Theorem 12.** If $G$ is an AT-free graph then $\text{cub}(G) \leq \text{box}(G)(\lceil \log_2 \psi(G) \rceil + 2) \leq \chi(G)(\lceil \log_2 \psi(G) \rceil + 2)$.

**Proof.** Let $\text{box}(G) = k$ and $I_1, I_2, \ldots, I_k$ be interval graphs such that $G = \bigcap_{j=1}^{k} I_j$. It is easy to see that if we replace each $I_j$ by a chordal graph $I'_j$ such that $V(I_j) = V(I'_j)$ and $E(G) \subseteq E(I'_j) \subseteq E(I_j)$, we will have $G = \bigcap_{j=1}^{k} I'_j$. It follows that there exists chordal graphs $I'_1, I'_2, \ldots, I'_k$ such that $G = \bigcap_{j=1}^{k} I'_j$ where each $I'_j$ is a minimal triangulation of $G$. But by Theorem 2, any minimal triangulation of an AT-free graph is an interval graph. It follows that $I'_1, I'_2, \ldots, I'_k$ are interval graphs. According to Lemma 3, $\psi(I'_j) \leq \psi(G)$ for $1 \leq j \leq k$. Since $G = \bigcap_{j=1}^{k} I'_j$ we have $\text{cub}(G) \leq \sum_{j=1}^{k} \text{cub}(I'_j)$ according to Fact 1.

By Theorem 4, $\text{cub}(I'_j) \leq \lceil \log_2 \psi(I'_j) \rceil + 2$ and by Lemma 3, $\text{cub}(I'_j) \leq \lceil \log_2 \psi(G) \rceil + 2$ for $1 \leq j \leq k$. It follows that $\text{cub}(G) \leq k(\lceil \log_2 \psi(G) \rceil + 2) = \text{box}(G)(\lceil \log_2 \psi(G) \rceil + 2)$. Therefore $\text{cub}(G) \leq \text{box}(G)((\lceil \log_2 \psi(G) \rceil + 2)$. By Theorem 3 we also have $\text{cub}(G) \leq \chi(G)((\lceil \log_2 \psi(G) \rceil + 2)$.

From Theorem 3 and 12 we get the following:

**Corollary 1.** If $G$ is an AT-free graph having girth at least 5 then $\text{cub}(G) \leq 2 \lceil \log_2 \psi(G) \rceil + 4$.

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