A unique non-Landau/Fermi liquid in 2d for high \(T_c\) superconductivity

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It is shown that the main features of the high \(T_c\) phase diagram can be calculated as a function of doping in a simple, essentially unique non-Landau/Fermi liquid in 2d with quartic interactions. This depends on a single parameter \(0 < \gamma < 1\) which encodes the strength of the interaction at short distances. A new d-wave gap equation has solutions that fall under a superconducting dome, which terminates at the renormalization group fixed point. Optimal doping is estimated to occur just below \(3/2\pi^2\). The scale for \(T_c\) is set by the recently measured universal nodal Fermi velocity and lattice spacing, and is estimated to be \(120 K < T_c < 160 K\) for LaSrCuO.

There remain many fundamental unanswered questions in the theory of high \(T_c\) superconductivity in the cuprates. How can one theory interpolate between antiferromagnetic (AF) and superconducting (SC) order, since AF requires repulsive interactions and SC attractive? What is the precise mechanism that gives d-wave pairing? What is the nature of the pseudogap? The most important guide in tackling these problems is the non-Fermi liquid behavior, as emphasized early on by Anderson[1]. The reason is that if one insists on a local non-Fermi behavior, as emphasized early on by Anderson[1]. The reason is that if one insists on a local non-Fermi behavior, as emphasized early on by Anderson[1].

Consider free non-relativistic particles with energy \(\varepsilon(k)\) that is rotationally invariant, e.g. \(\varepsilon(k) = k^2/2m_\ast\).

\[H = \int_{|p|<\Lambda_c} (d^3p) \left[ \frac{1}{2} |p|^2 + (\varepsilon_F |p| + \bar{\mu})a_p^\dagger a_p + \bar{\mu}b_p^\dagger b_p \right] \]

where \(\bar{\mu} = \mu - \varepsilon_F\) is zero at zero temperature.

Our expansion around the Fermi surface is in the same spirit as in[3], but we now depart from it by demanding a consistent local effective quantum field theory that reproduces the above \(H\). Since the energy corresponds to massless particles with a linear dispersion relation, we identify an emergent Lorentz symmetry and describe \(H\) using a relativistic field theory. There are only 2 known candidates which differ in whether the lagrangian is first or second order in derivatives. The first order case requires a multi-component Dirac field, and since here a 4-fermion interaction is an irrelevant dimension 4 operator.
in 2d, it cannot lead to a non-Fermi liquid. Furthermore, Dirac fermions usually require additional properties of the Fermi surface (Dirac points) as in graphene, where the multi-components have to do with sub-lattices. The other possibility is second-order in both space and time derivatives with action

$$S = \int dt d^2 x \left( \partial_t \chi^- \partial_t \chi^+ - v_F^2 \chi^- \cdot \nabla \chi^+ \right)$$  \hspace{1cm} (2)$$

The Fermi velocity plays the role of the speed of light which just serves to convert units of space and time so it can be set equal to 1. This form for a fermionic field is very unconventional, and to a particle physicist it appears to violate the spin-statistics theorem. Since the above kinetic term is crucial to all that follows, we give the following compelling arguments in favor of it: (i) It correctly reproduces the desired effective Hamiltonian for particles and holes near the Fermi surface. (See below.) Thus the free theory is perfectly hermitian and unitary in momentum space, i.e. has no negative norm states. (ii) In the condensed matter context, spin is a flavor in this context and the Pauli-exclusion principle is built in from the fermionic nature of the \(\chi\)-fields. The issue rather has to do with unitarity. The mode expansion of the fields is

$$\chi^- (x, t) = \int \frac{(d^2 p)}{\sqrt{2\pi p}} \left( a^+_p e^{-ip \cdot x} + b_p e^{ip \cdot x} \right)$$

$$\chi^+ (x, t) = \int \frac{(d^2 p)}{\sqrt{2\pi p}} \left( -b^+_p e^{-ip \cdot x} + a_p e^{ip \cdot x} \right)$$ \hspace{1cm} (4)$$

where \(\omega_p = \sqrt{p^2}\) and \(p \cdot x = \omega_p t - p \cdot x\). The additional minus sign in the expansion of \(\chi^+\) is chosen so that the canonical quantization relations of the fields leads to the usual canonical relations in momentum space for the a's and b's. The canonical Hamiltonian of the theory is precisely (1). Introduce a unitary operator \(C\) that distinguishes particles and holes: \(CaC = a, \ CbC = -b\) where \(C^2 = 1\). Then \(\chi^+ = C(\chi^-)^\dagger\) and in terms of fields the Hamiltonian is pseudo-hermitian: \(H^\dagger = CHC\). It was understood long ago by Pauli that a pseudo-hermitian Hamiltonian gives a consistent quantum mechanics with a unitary time evolution and real eigenvalues. In the present context, pseudo-hermiticity has additional meaning with regard to the kinematics of the expansion around the Fermi surface since \(C\) distinguishes particles and holes. Conservation of the physical momentum \(k\) is only equivalent to conservation of \(p\) for processes where particles are paired with particles and holes with holes. For the study of SC, these are of course the processes we are primarily concerned with. We are thus only interested in eigenstates which are also eigenstates of \(C\), and for these \(H = H^\dagger\).

The \(SO(5)\) symmetry is easiest to see if one introduces an \(N\)-component version with fields \(\chi_{\pm,\alpha}\), \(\alpha = 1, \ldots, N\), which has \(Sp(2N)\) symmetry. For spin \(1/2\) particles, \(N = 2\) and \(Sp(4) = SO(5)\). The \(SO(5)\) has an \(SU(2)\) spin subgroup and a \(U(1)\) charge that commutes with it. The \(\pm\) indices on the fields \(\chi_{\pm}\) correspond to electric charge. One can construct an \(SO(5)\) vector of order parameters \(\vec{\Phi} = (\phi_x, \phi_y, \phi_z, \phi^+_x, \phi^+_y, \phi^+_z)\) where \(\phi\) is an electrically neutral \(SU(2)\) vector and \(\phi_{\pm}^\dagger\) are Cooper pair fields of charge \(\pm 2\) which are \(SU(2)\) spin singlets:

$$\vec{\phi} = \chi^- \vec{\sigma} \chi^+/\sqrt{2}, \quad \phi_{\pm}^\dagger = \chi_{\pm}^\dagger \chi_{\pm}^+$$ \hspace{1cm} (5)$$

The interaction can be written in a manifestly \(SO(5)\) invariant fashion: \(\mathcal{H}_{\text{int}} = -\frac{4\pi^2 g}{5} \vec{\Phi} \cdot \vec{\Phi}\).
It is important to carry out the RG directly in 2d. As usual, the RG prescription involves two energy scales, the cut-off $\Lambda_c$ and a lower running scale $\Lambda$. In many calculations $\Lambda$ is a lower cut-off. Since the coupling $g$ has units of energy in 2d, we define $g(\Lambda) = \Lambda \tilde{g}(\Lambda)$ where $\tilde{g}$ is dimensionless. The 1-loop beta function is $-\frac{dg}{d\log \Lambda} = \tilde{g} - 8\tilde{g}^2$ which has a low energy fixed point at $\tilde{g}_* = 1/8$. To understand the phase diagram as a function of doping, it is first convenient to introduce the variable $x = 1/\tilde{g}$ where the fixed point value is at $x_* = 8$. We also introduce a variable $x_0$ that encodes the strength of the coupling at the cut-off: $x_0 = 1/\tilde{g}_0$ where $g(\Lambda_c) = \Lambda_c \tilde{g}_0$. We assume that at short distances the coupling is strong, i.e. $g > g_*$. It will also be useful to define $\gamma = (x_* - x_0)/x_*$ which is a small parameter between 0 and 1. The coupling at short distances can be arbitrarily strong, where infinite coupling corresponds to $\gamma = 1$. Integrating the beta-function with this initial short-distance data gives a linear form that is specific to 2d

$$T_{pg} \equiv \frac{\Lambda}{\Lambda_c} = -\frac{1}{\gamma} \left( \frac{x}{x_*} - 1 \right)$$

and turns out to be important in connection with hole doping, which we now turn to.

In the non-linear sigma-model description at half-filling, the order parameter $\phi$ is constrained to have fixed length. As explained above this constraint follows from a constraint on the $\chi$ fields: $\chi^a_\uparrow \chi^a_\downarrow + \chi^a_\downarrow \chi^a_\uparrow = i h \Delta_c$. Then one can show $\vec{\phi} \cdot \vec{\phi} = 3h^2 \Delta_c^2/2$. Relaxing this constraint moves away from half-filling. Thus a measure of hole doping is the 1-point function $h = -i(\chi^- \chi^+)/\Lambda_c$. The overall scale of $h$ as a doping variable can be justified by reintroducing the chemical potential and noting that near the Fermi surface it couples to the above operator. Including the 1-loop order $g$ self-energy correction to the propagator and expressing everything in terms of the $x$ variables one obtains

$$h(x) = \frac{1}{\pi^2} \left( \frac{x - x_0}{x_* - x_0} \right) \left[ 1 + \frac{4}{x} \left( \frac{x - x_0}{x_* - x_0} \right) \right]$$

In Figure 1 the straight line is a plot of $\Lambda/\Lambda_c$ in eq. (6) as a function of the above $h$ for $x_0 = 0$, i.e. $\gamma = 1$. (For $\gamma \neq 1$ it is not exactly a straight line.) It crosses the $h$ axis at $h_* = 3/2\pi^2$, which is the location of the RG fixed point. Below this line, the energy scale is such that electrons described by our $\chi$ fields are strongly correlated and exhibit non-Fermi liquid behavior since the fixed point is not a free field theory. Only $\chi^- \chi^+ \neq 0$ and no symmetries are broken at this line. The region below is what is normally called the pseudogap. The scale $\Lambda$ can be associated with a pseudogap temperature $T_{pg}$.

The AF phase can be analyzed by a standard mean field analysis. One introduces an auxiliary field $s$ for the order parameter $\phi$ and derives the effective potential for constant $s$ by performing the functional integral over the $\chi$ fields. Minimizing this effective potential with respect to $s$ gives the gap equation

$$\bar{s} = -16\pi^2 g \int_0^{\Lambda_c} \frac{d\omega \, d^2k}{(2\pi)^3} \left( \frac{\bar{s}}{(\omega^2 + k^2)^2} - \frac{\bar{s}^2}{\Delta_c^2} \right)$$(8)

For positive $g$, there are solutions due to the compensating minus signs. Since $s$ has dimension 2, define $s = \bar{s} \Delta_c^2$. Then $\delta_s$ is a solution to the equation

$$\frac{\Lambda_c}{g} \frac{4}{\delta_s} \left( \frac{1}{2} \log \left( \frac{\delta_s + 1}{\delta_s - 1} \right) \right) - \tan^{-1} \left( \frac{1}{\delta_s} \right)$$

(9)

When $g$ is small enough, the solution flattens out with $\delta_s \approx 1^+$. This behavior is unphysical since the gap should be zero when $g$ is zero. The resolution of this puzzle involves regulating the infra-red divergence with the low energy cut-off $\Lambda$ and interpreting the result with the RG. Setting $s = 0$, the gap equation can be approximately re-expressed as $1/g(\Lambda) = 8/\Lambda_c$. This shows that at $g = \Lambda_c/8$, a consistent non-trivial solution is $s = 0$. We interpret this as a first-order transition where $\delta_s$ drops discontinuously to zero. In terms of $x$ this occurs at $x_{AF} = \frac{x_0}{1+\gamma}$. Since $\delta_s$ is in units of the cut-off it is meaningful to rescale it and define $\delta_{s}' = \frac{\Lambda_c}{\Lambda} \delta_s$, where the scale factor is given in terms of $x$ in eq. (6). In Figure 2 we show the solutions to the gap equation as a function of doping $h$. The Néel temperature $T_N$ is proportional to the zero temperature gap as we will describe below. The AF transition occurs at $h_{AF} = h(x_{AF}) = 3/4\pi^2$ when $\gamma = 1$.

Introducing constant auxiliary fields $q^\pm$ for the SC order parameters $\phi^\pm$ and repeating the above mean field analysis gives a gap equation for $q^2 = q^+ q^-$ that has the same form as (8) with $s^2 \to -q^2$. This leads to the expected result: for repulsive interactions (positive $g$) there are no solutions, i.e. no s-wave SC. This shows that the AF and SC phases do not compete, and the SC phase is not simply related to the AF one by the $SO(5)$ symmetry. When one goes beyond mean field and incorporates momentum dependent scattering of Cooper pairs near the Fermi surface, an attractive d-wave channel opens up. Introducing non-constant auxiliary pair fields, one can derive the momentum dependent gap equation

$$q(k) = -\int \frac{d\omega \, d^2k'}{(2\pi)^3} G(k, k') \frac{q(k')}{(\omega^2 + k'^2)^2 + q(k')^2}$$

(10)

where $q^+ = q^-$ up to a phase. The kernel $G$ is related to a particular 4-particle Green function specialized to Cooper pairs of opposite momenta $\pm k$ and $\pm k'$. In a rotationally invariant theory one can expand in circular harmonics:

$$G(k, k') = \sum_{\ell=0}^{\infty} G_\ell(k, k') \cos \ell(\theta - \theta')$$

$$q(k) = \sum_{\ell=0}^{\infty} q_\ell(k) \cos \ell\theta$$

(11)
where $k$ is the magnitude of $k$ and $\theta - \theta'$ is the angle between $k$ and $k'$. Performing a low energy momentum expansion at 1-loop one finds an attractive $\ell = 2$ channel: $G_2(k,k') = -8\pi^2 g_2 k^2 k'^2$ where $g_2 = 4g^2/25\Lambda^3$. The solution to the gap equation has the characteristic d-wave form

$$q(k) = \delta_q^2 k^2 \cos 2\theta = \delta_q^2 (k_x^2 - k_y^2)$$  \hspace{1cm} (12)

where $\delta_q$ is a constant solution to the integral equation

$$\delta_q^4 = 2g_2 \int_0^\Lambda d\omega dk^2 \left( 1 - \frac{\omega^2 + k^2}{\left(\omega^2 + k^2\right)^2 + \delta_q^4 k^4} \right)$$  \hspace{1cm} (13)

To plot the gap as a function of doping one needs to express the constant $g_2$ in terms of $x$: $g_2 = \frac{1}{2\Lambda} 25x^2 (4x^2 - 1)^2$, and one sees that it changes sign at $x = x_*$, which implies that the SC phase terminates at the RG fixed point. The other important feature of the above gap equation for $\delta_q$ is that SC turns off at a lower, non-universal value of $x_1 \approx (1 - 0.1\gamma) x_*$. Thus the non-zero solutions of the d-wave gap equation fall under a dome $x_1 < x < x_*$. Comparison with experiments shows that this dome as calculated is too narrow; however there are many effects we have neglected that could broaden it, e.g. interplane coupling, disorder, or higher order corrections. Numerical study of solutions of [13] indicates that optimal doping occurs in the tight range between 0.13 and 0.15 where the upper bound is simply $h_*$. Solutions to the d-wave gap equation $\delta_q^4 = \Lambda^4 / \Lambda_c$ are shown in Figure 1.

As in the BCS theory, one expects that the critical temperature $T_c$ is proportional to the zero temperature gap. Rather than developing a full-blown finite temperature formalism, one can study the effect of a small finite temperature by introducing a relativistic mass term in the lagrangian $-m^2 \chi^{-1} \chi^+$ where $m = \alpha T$ with $\alpha$ a dimensionless constant. The justification for this is that in the free theory this thermal perturbation reproduces the correct electronic specific heat $C_V \propto T^2 / v_F$. Requiring the overall coefficient to match the standard non-interacting result extended to 2d fixes $\alpha = \frac{\pi^{5/4}}{\sqrt{6}} \approx 1.7$. We have verified numerically that the solutions of the AF and d-wave gap equations with this additional mass term, $\omega^2 \to \omega^2 + m^2$, vanish when $m$ is too large, and the critical value $m_c = \alpha T_c$ is indeed proportional to the zero temperature gap up to constants of order 1 that depend weakly on $\gamma$. Restoring fundamental constants and the Fermi velocity $v_F$ one then finds $k_B T_c = \frac{\pi}{\alpha} v_F h^3 q / \Lambda_c$. At optimal doping the constant $c$ is in the range $.5 < c < .7$ depending on $\gamma$ and $\delta_q^4 \approx 0.11$ is largely insensitive to $\gamma$. A similar formula applies to the Néel temperature $T_N$ with $c \approx 1$. The Fermi velocity $v_F$ in our relativistic model is a fixed constant that plays the rôle of the speed of light which is universal in the sense that it is independent of the coupling and thus the doping $h$. Remarkably, at low energies a doping independent universal nodal Fermi velocity has been observed in recent years[11]. This can actually be viewed as a prediction of our theory, where the nodal direction $k$ from $(0,0)$ to $(\pi,\pi)$ shown in Figure 2 is what is important since this is the direction where the half-filling diamond is closest to the nearly circular Fermi surface just inside it. Using the results in[11] we estimate $v_F \approx 1.4 evA \approx 210 km/s$ for LaSrCuO. This leads to an estimate of $T_c$ at optimal doping. The cut-off $\Lambda_c$ should be equated with the inverse lattice spacing $a$. A practical form is

$$T_c = \frac{v_F}{a} \cdot 650K$$  \hspace{1cm} (14)

where $v_F$ is in evA and the lattice spacing in $A$ and we have set $\alpha$ to its estimated value of 1.7. With the above $v_F$ and $a = 3.8A$ for the CuO square lattice, this gives $T_c$ in the range $120K < T_c < 160K$ depending on $\gamma$. The maximum $T_c$ occurs around $\gamma = 1/4$. The simple result [14] provides some insights on how to possibly obtain a higher $T_c$: shorten the lattice spacing or increase the Fermi velocity by possibly reducing the effective electron mass $m_*$.

Many more properties can be calculated, such as anomalous corrections to the specific heat, a resistivity linear in temperature, and electron correlation functions in the pseudgap region. These additional results and a more detailed exposition will be presented in the comprehensive article[12].

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