Development Of Mathematical Model Related To Bumps On The Road

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Abstract. Road bumps are one of the traffic calming tool installed on the roads to control the speed of the vehicles, to avoid accidents and help pedestrians to cross the road. Road bumps should be set up keeping in the mind the various factors such as location type, traffic type, population density, and others. Work was done on the various types of road bumps mainly conical-shaped road bumps and usage of the Fourier series which is effective for periodic shaped bumps. In this paper, we are going to use the Laplace transform which is effective not only on the periodic type of the road bumps but also on the complex structures of road bumps and will study it on the hollow rectangular shaped road bumps. We will find the parameters especially effective distance between the two consecutive road bumps and displacement caused to the vehicle which would help in the designing of the road bumps so that vehicle could move safely over the bumps and least damage could be caused to it and travellers inside it.

1. Introduction

Roads Bumps are the protuberance or the hollow portion on the road placed to regulate the speed of the approaching vehicles \cite{1}. They are very useful in the prevention of accidents and to make the vehicles slow in speed. They are very effective in enforcing speed limits and preventing over speeding of the vehicles. They are installed to encourage the motorists to drive carefully and at a reduced speed. They have prevented a number of accidents which ultimately saved a number of lives and damage to the vehicles. They prevent the vehicle going beyond a certain limit and travel within the given range.

There are various factors which affects the sitting of the Road bumps. The crucial factors are the area type which is residential or commercial type area, population density, types of vehicles, availability of infrastructure and others. In the Residential area, the speed should be greatly reduced as compared to the non residential area as the latter area there may be others vehicles as emergency vehicles where speed shouldn’t be decreased so much. Road bumps are mainly kept around the parking lots, around schools, residential local streets.

Road bumps are customizable according to the need. Road Bumps should be modeled in such a way that it should not harm the passengers sitting inside the vehicle and cause no or least damage to the vehicle \cite{2}. While constructing the road bump, it should be visible from a distance to prevent accidents due to oversight. If the visibility at a place is poor, the road signs can be provided to help drivers take precautionary actions. There should be low friction between the vehicle tyres and the bumps. Parameters such as bump height, bump width and bump material can be changed to achieve at the desired system functions. The vehicle should have a suspension system to provide safety to the vehicle and passengers. Flat surface road bumps with lengthy base has an advantage over the curved surface with short base of not getting wore at the top easily but with a disadvantage that curved shaped bumps reduce the speed to the greater extent compared to the former.

The literature has done a work on the modeling of the road bumps considering the conical shaped bumps only and using the fourier series which is very effective in determining the effective distance between the two consecutive bumps and to design them. So, there is a chance to choose another mathematics tool which is as effective as fourier and other geometric shape other than the usual conical shape.
In our work we are going to use Laplace Transform on the hollow rectangular shaped bumps and will compare the results with the conical shaped given already in the literature.

2. The Vehicle Interaction Model
The following notations are used in the model:
- \( \nu \) linear velocity at which vehicles are moving over the bumps
- \( f \) frequency at which vehicles are moving over the bumps
- \( \lambda \) distance between two consecutive points on the road (wavelength)
- \( T \) period, which is the time taken to complete a cycle
- \( h(t) \) height of undulation of the bump at a particular time
- \( t_n \) time of oscillations
- \( y \) vertical displacement of body due to undulation
- \( k \) spring constant (stiffness)
- \( I_f \) isolation factor
- \( c_x = \gamma \) damping force
- \( k_x \) spring force
- \( d \) linear displacement
- \( L \) general Laplace Transform notation
- \( \omega^2 \) natural frequency (rad/s)
- \( s \) complex Laplace operator
- \( S, U, n \) and constants
- \( R \) Laplace Operator

3. Mathematical Formulation of Model
The drawing of the road bump profile over the surface of the road gives a great information about the study of the parameters involved and helps in the initialization of the modeling of the bumps. The geometry of the hollow rectangular shaped bump is shown in the figure 1. This geometry is in effect of the literature work done on the road bumps concerning bump geometry.

![Figure 1: Geometry of hollow rectangular shaped road bump](image)

Following conclusions can be drawn from the figure 1. The vehicle passing over the bumps will have two movements, one is below the surface of the road due to hollow design of the bump and other is the movement over the surface of the road. Therefore, the complete motion of the vehicle over the bumps would be examined in two stages. In the first stage, the motion of the vehicle would be over the bumps and in the second stage, motion would be considered over the road surface between the two consecutive bumps. With the help of mathematical tool i.e. Laplace transform, we will derive the governing equations for the given system by analysing the periodicity of the model.

For the first stage, \( f_1(t) = h \), where \( t \) ranges from 0 to \( t_1 \). In considering the second stage, \( f_2(t) = 0 \), where \( t_1 < t < T \) where \( T \) is the time period of motion.
The Laplace transform of a periodic function \( f(t) \) with period \( T \) is given as

\[
L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) dt
\]  

(1)

The Further analysis of the equation (1) gives

\[
L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^{t_1} e^{-st} h dt + \frac{1}{1 - e^{-Ts}} \int_{t_1}^T e^{-st} 0 dt = \frac{h(1 - e^{-st_1})}{s(1 - e^{-Ts})}
\]  

(2)

Laplace Transform of a periodic function is represented by the equation(1). Therefore, it is used to find out the inverse Laplace transform of the equation(2).

\[
L^{-1}\left[ \frac{h(1 - e^{-st_1})}{s(1 - e^{-Ts})} \right] = f(t) = \frac{h}{s} \left[ R - e^{-(L)s} \right] = \frac{h}{s} \left[ \sum_{n=0}^{\infty} R - e^{-(L)s} \right] = h \left[ \sum_{n=0}^{\infty} \frac{R}{s} - \frac{e^{-(L)s}}{s} \right]
\]  

(3)

Equation(4) gives the inverse of Laplace transform. Now, we will establish the relationship between the components of the suspension system of the vehicle that are suspension spring and the damper(shock absorber). The schematic form of the suspension system is shown in figure(3).

\[
L^{-1}\left[ \sum_{n=0}^{\infty} (H(U) - H(t - nT - t_1)) \right] = L^{-1}(F(s))
\]

(4)

The free body diagram for the given suspension system is shown in figure(3).
The below equation(5) is derived with the help of forces acting in the system from the free body diagram.

\[-k(y - F) - c(\dot{y} - \dot{F}) = m\ddot{y}\]  

By further evaluation of the the equation(5), we have

\[m\ddot{y} + cy + ky = c\dot{F} + kF\]  

which represents the relationship between the spring force(kx) and the damping force(cx). From equation(2), we have

\[L[F(t)] = \frac{h}{s} \left[1 - e^{s(1)}(1 - e^{-Ts})\right] = F(s) = \frac{h}{s} \left[\sum_{n=0}^{\infty} R - e^{-(Ls)}\right]\]  

\[L[F'(t)] = s\bar{F} - F_0, F_0 = F(0)\]  

\[L[F(t)] = \bar{F}, \dot{F} = F'\]  

\[L[F'(t)] = h\left[\sum_{n=0}^{\infty} R - e^{-(Ls)}\right] - F_0\]  

The road is assumed to be smooth and there are no other sources of vibration present on the road except the bump. Therefore, at t=0, F(0)=0 because no displacement in the vehicle. Equation(8) becomes

\[L[F'(t)] = h\left[\sum_{n=0}^{\infty} R - e^{-(Ls)}\right]\]  

Substituting for \(\bar{F}\) and F in equation of motion, dividing through by m, and putting \(w^2 = k/m, \gamma = c/2m\), we have

\[\ddot{y} + 2\gamma \dot{y} + \omega^2 y = (2\gamma h + \omega^2 h/2) \sum_{n=0}^{\infty} [R - e^{-(Ls)}]\]  

Note that \(\omega^2\) is called natural frequency and \(\gamma\) is called the damping factor. But \(L[\ddot{y}] = S^2\bar{y} - Sy_0 - y_1, L[\dot{y}] = S\bar{y} - y_0,\) and \(L[y] = \bar{y}\). Then substituting in equation (9) to get

\[(S^2\bar{y} - Sy_0 - y_1) + 2\gamma (Sy_0 - y_0) + \omega^2 y = (2\gamma h + \omega^2 h/2) \sum_{n=0}^{\infty} (R - e^{-(Ls)})\]  

From the boundary conditions of the system at t=0, \(y(0)=y_0=0\). Knowing that \(y_0=0\) and \(y_1=0\), we may substitute into the above equation to obtain \((S^2\bar{y} - 0 - 0) + 2\gamma (0) + \omega^2 \bar{y} = (2\gamma h + \omega^2 h/2) \sum_{n=0}^{\infty} (R - e^{-(Ls)})\). This reduces to

\[S^2\bar{y} + 2\gamma S\bar{y} + \omega^2 \bar{y} = (2\gamma h + \omega^2 h/2) \sum_{n=0}^{\infty} (R - e^{-(Ls)})\]  

Therefore, the terms involving the variable y which is for the vertical displacement are omitted.

\[\bar{y} = 2\gamma h \sum_{n=0}^{\infty} \left(\frac{R}{M+N} + \frac{e^{-(Ls)}}{M+N}\right) + \omega^2 h \sum_{n=0}^{\infty} (BR - Be^{-(Ls)})\]  

Equation (12) gives an expression for \(\bar{y}\) where, \(B = \frac{1}{M+N}\). By using partial fraction method to split the equation, we have \(B = \frac{1}{M+N}\). This could further be stated as:

\[B = \frac{1}{\omega^2 S} - \frac{s}{M+N} - \frac{2\gamma}{M+N}\]  

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The relationship between the frequency(f) of the system and the Time Period(T) is given by \( f = \frac{1}{T} \). There would be no oscillation in the system when the distance between two consecutive points on the road(wavelength) becomes equal to the angular velocity(\( \gamma = \omega \)) which implies that the square of the wavelength becomes equal to the square of the angular velocity(\( \gamma^2 = \omega^2 \)). This makes the frequency of the system equal to zero. So, from the given relation between f and T, at \( f=0 \), T will be equal to \( T=\frac{1}{\gamma} \). However, \( y=0 \), and if angular velocity is greater than the wavelength, it implies that \( \omega^2 - \gamma^2 > 0 \). Now, let \( \omega^2 = \gamma^2 + a^2 \). Then, from the equation (15),

\[
y = 2\gamma h \sum_{n=0}^{\infty} \left( \frac{R}{M+N} e^{-Ls} + \omega^2 h \sum_{n=0}^{\infty} \frac{BR}{M} e^{-Ls} \right)
\]

where, \( B=\frac{1}{S(M+N)} \). Therefore

\[
y = 2\gamma h \sum_{n=0}^{\infty} \left( \frac{R}{M+N} e^{-Ls} + \omega^2 h \sum_{n=0}^{\infty} \frac{DR}{M} e^{-Ls} \right)
\]

where, \( D=\frac{1}{S[P]} \). By using partial fraction for \( D = \frac{1}{S[P]} = E + \frac{GS+N}{P} \), we obtain \( D = \frac{1}{P} - \frac{S}{P} e^{-Ls} - \frac{2\gamma}{P} e^{-Ls} \). Therefore,

\[
y = 2\gamma h \frac{aR}{\alpha} + \omega^2 h \sum_{n=0}^{\infty} \frac{SR}{P} e^{-Ls} + \frac{2\gamma R}{P} e^{-Ls} + \frac{e^{-Ls}}{S} - \frac{2\gamma e^{-Ls}}{P} \]

Transforming the above equation back to the original function f(t) from the laplace transform function f(s) by applying the inverse Laplace transform. For simplicity, let \( R=ae^{-nTS} \) and \( P=(S + \gamma)^2 + a^2 \), then

\[
y = \left( \frac{2\gamma h}{a} \sum_{n=0}^{\infty} \frac{RU}{P} e^{-Ls} + \frac{\omega^2 h}{B} \sum_{n=0}^{\infty} \frac{RU}{P} e^{-Ls} \right)
\]

By further simplification, let \( U=(t-nT) \) and \( K=(T-nT-t_1) \)

\[
y = \left( \frac{2\gamma h}{a} \sum_{n=0}^{\infty} (HU e^{-nU} \alpha U - HKe^{-nK} \alpha K) + \frac{\omega^2 h}{B} \sum_{n=0}^{\infty} (HU (1 - e^{-nU} \alpha U - 3HKe^{-nU})) \right)
\]

As we know, linear velocity is defined as the distance travelled divided by time taken. So, from figure 2, it can be seen that \( v = \frac{d}{t_1} \), where d is the width of the road bump in the direction of road and it is assumed that linear velocity at which the vehicle is moving over the bump is taken as constant. Further it is already stated that \( T=\frac{1}{f} \) where T is the Time period and \( f \) is the frequency. Therefore, equation(20) will take the form of

\[
y = \left( \frac{2\gamma h}{a} \sum_{n=0}^{\infty} (HZ e^{-nZ} \alpha Z - HY e^{-nY} \alpha Y) + \frac{\omega^2 h}{B} \sum_{n=0}^{\infty} (HZ (1 - e^{-nZ} \alpha Z - 3e^{-nZ}) + HY (1 - e^{-nY} \alpha Y - 3e^{-nY})) \right)
\]
The equation (21) gives the displacement of the vehicle as it passes over road bumps.

4. Analysis of Effective Distances and Isolation Factor

The relationship between the velocity, frequency and wavelength is used to calculate the effective distance between the two consecutive road bumps in the literature work [5]. From this relation, velocity and frequency are the factors which are to determined to calculate the wavelength and isolation factor is one such element which is used to to have satisfactory value of the frequency the system should possess while passing over the bumps. To determine the best suitable frequency from the isolation factor, a graph should be plotted between the isolation factor and the frequency to interpret their suitable values. The isolation factor of 80 percent is considered best. The isolation factor is usually used to determine the frequency of the system. In our work, it is assumed that the velocity is constant while the vehicle is moving over the bump. The velocity may not be constant for every vehicle moving over the bumps which will formulate another problem not discussed in this mathematical model [6]. We will extend the literature work by rederiving the isolation factor by considering number of factors into the isolation factor like road bump displacement, system displacement and other relation factors and to have control over these factors to get the desired values of the isolation factor.

The ratio of the system displacement to the support displacement is considered here as the isolation factor. The system displacement is given in the expression (21) and displacement of the support is given in the equation (3). Thus, the mathematical statement for the isolation factor is given by: 

\[ I_f = \frac{System}{Road} = \frac{x(t)}{\bar{f}(t)} = \text{equation}(21) \]

As stated earlier, 

\[ f(t) = h(\sum_{n=0}^{\infty}(HZe^{-\gamma Z}SinaY - HYe^{-\gamma Y}sinaY)) \]

By changing \( T = \frac{1}{v} \) and \( t_1 = \frac{d}{v} \). The equation can be rewritten, giving us \( f(t) = h(\sum_{n=0}^{\infty}(HZ - HY)) \). But if \( Y = (t - \frac{1}{v}) \) and \( Z = (t-n/f) \), then

\[ I_f = \left( \frac{2\pi}{a} \sum_{n=0}^{\infty}(HZe^{-\gamma Z}SinaY - HYe^{-\gamma Y}sinaY) \right) + \frac{2\pi}{a} \sum_{n=0}^{\infty}(HZ(1-e^{-\gamma Z}CosY - 3e^{-\gamma Z}) + HY(1-e^{-\gamma Y}CosY - 3e^{-\gamma Y}sinaY)) \]

Here the isolation factor is function of the frequency with which vehicles are moving over the bumps by keeping all the other variables involved in the equation (24) fixed. So, \( I_f = F(f) \). From the isolation factor, we will evaluate the frequency and frequency will be used to determine the effective distance between the road bumps by using the relation \( \frac{z}{v} - \frac{d}{v} \) as clear from the figure(1) where \( d \) is the width of the the road bump and \( v \) is the velocity of the vehicle which is kept constant and \( f \) is the frequency [7].

5. Case Study

We will consider a Hypothetical model of the road bump on the part of the this study to provide evidence in the support of our mathematical model. The hypothetical model is a road inside the Lovely Professional University, Punjab, India where pedestrians are students crossing the road over the crosswalk and vehicles are also moving over the same road. In this, Road bump will be installed in order to reduce the speed of the vehicles and prevent any tragedy. A technical group will take the responsibility of the designing, construction, installation and maintenance of the road bump by taking help of the various parameters wrapped in this study.
Figure 4 represents the graph between Laplace Transform index and its inverse taken on the y axis and the undulation of the bump taken on the x axis [8]. The values of the Laplace transform and its inverse is collected from the equation(2) and equation(4) respectively to plot it against the undulation of the bump. With the increase in the height of the bump varying from 0.18m to 2m, the slope of the curve comes to be negative [9]. A comparable graph is observed in figure 5 when the Laplace Transform index and its inverse is plotted against the time with a difference that instead of gradual fall of the curve, the curve becomes negative and then zero alternatively between t=5s and t=20s and at end between t=25s and t=30s, it remains zero.

The vertical displacement of the vehicle given by equation(21) is tested with the following values of various factors : H = 0.1m, Z = 20, γ = 0.001, a = 0.05, T = 0.2s, t = 10s, Y = 4, n = 0, 5, 10, 15, 20, and 25. Let $A = (HZe^{-\gamma Z \sin a Z} - J Y e^{-\gamma Y \sin a J}$. Different values of $A$ can be obtained by putting different values to n. For n = 0, $J = (t - nT) = 0$, we have $A_0 = -0.109$. When n=5 and J=9, $A_5 = -0.001$. Computation of n=10,15,20 and 25 are obtained as $A_{10}$, $A_{15}$, $A_{20}$ and $A_{25}$ having values of -0.077, -0.884 and -0.731 respectively. Since $A = \sum_{n=0}^{25} A_n$, therefore the sum of all $A_n$=2.93m. Also, let

$$D = \sum_{n=0}^{25} \left( HZ(1 - e^{-\gamma Z \cos a Z} - 3e^{-\gamma Z} + HY(1 - e^{\gamma Y \cos a J} - 3e^{-\gamma Y \sin a J}) \right)$$

Also $J = (t - nT)$. By using the values of the variables earlier introduced i.e. n = 0, J = 10, $D_0$ to $D_{25}$ could also be computed. Thus, when n = 0, 10, 15, 20, and 25, with J = 10, 9, 8, 7, 6, and 5, the corresponding values of $D_0$, $D_5$, $D_{10}$, $D_{15}$, $D_{20}$, and $D_{25}$ are -45.892, -45.89, -45.89, -45.89, -45.89, and -45.89 respectively. Thus, $D = \sum_{n=0}^{25} D_n = -229.46$. It follows that $x = \frac{2h\gamma^2}{a} A + \frac{h^2}{B} D$. Assuming that $B = 0.640$, $\omega = 0.3 \text{rads}^{-1}$, and $h = 0.2m$, then, $x = -6.35m$. The displacement response of the vehicle is in the opposite direction as it moves over the bump due to negative value of x obtained [10].
6. Conclusion

The work was performed to establish some relationship between the surface of the road bump and the tire that rides over this surface so as to draw significant characteristics of this interaction. This model is able to find the effective distances between the two consecutive road bumps which in turn helps in the lowering of the speed of the vehicles, minimizing the noise pollution which affects the local residents residing near the road where the road bump is installed, and to have least impact on the vehicles. Previous work was done by taking the conical shaped road bump as compared to the hollow shaped rectangular bump shown here. Big advantage of using Laplace transform is that it can be applied to the bumps with complex shape whereas Fourier series is used for the Periodic shaped bumps. Complex shaped bumps are generally not periodic in nature so it can be modeled using Laplace transform. Hollow rectangular bumps are considered here which seems to have least impact on the vehicles and ultimately on the passengers travelling inside it. With this, vehicle life gets increased and avoids any injury caused due to the impact. It ensures the safety of the travellers. Parameters can be evaluated using the complex shape of the bumps which is effectively handled with the Laplace transform as compared to the literature approach of evaluating the parameters using the simple shapes. Engineers in the industry would get a benefit from this analysis by observing all the dimensions of the road bumps and will design the suitable road bump by considering all the factors involved.

The paper is able to present the mathematical modeling of the road bump-vehicle interaction problem with an extension to the given work already done in the literature. This paper opens up a wide range of possibilities where future work can be performed by the researchers. If 2 degree of freedom system is considered then it would bring new insights into the existing model. The properties of the tire is not taken into the account and only the suspension system of the vehicle and its interaction with the bump is considered. If the wheel is taken into account then new problem could be developed which would be an addition to this existing model. This would eventually help in deeper understanding of the interaction problem.

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