A Note on Fractional KdV Hierarchies

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Abstract

One of the cornerstone of the theory of integrable systems of KdV type has been the remark that the $n$–GD equations are reductions of the full KP theory. In this paper we address the analogous problem for the fractional KdV theories, by suggesting candidates of the “KP–theories” lying behind them. These equations are called “KP\textsuperscript{(m)} hierarchies”, and are obtained as reductions of a bigger dynamical system, which we call the “central system”. The procedure allowing to pass from the central system to the KP\textsuperscript{(m)} equations, and then to the fractional KdV\textsuperscript{m} equations, is discussed in detail in the paper. The case of KdV\textsuperscript{2} is given as paradigmatic example.

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1 Introduction

The study of integrable systems of KdV type received, in the last few years, a new impulse from important developments in Two–Dimensional (Quantum) Field Theory. In this framework, much attention has been paid to the conformal $\mathcal{W}_N$–algebras of symmetries of these theories. In the cases studied at first, they have been shown [1, 2] to be quantum extension of the second Poisson structure of the $n$–GD (Gel’fand Dickey) theories [3, 4] associated with a classical Lie algebra $\mathfrak{g}$. The powerful method of Hamiltonian reduction has been widely applied to study and classify those hierarchies of partial differential equations and the associated $\mathcal{W}$–algebras (see, e.g., [5]). A class of new integrable hierarchies (called fractional KdV or generalized DS) has been obtained [6, 7, 8, 9, 10, 11] by means of a generalization of the Drinfel’d–Sokolov construction. Roughly speaking, fractional KdV hierarchies correspond to the case in which the value of the momentum mapping for the infinitesimal gauge action of the loop algebra $L\mathfrak{g}$ is different from the sum of the (duals of the) simple positive roots of $\mathfrak{g}$. They have been shown [8, 9, 12] to be classified by homogeneous elements in Heisenberg subalgebras of the affinization $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$.

One of the cornerstones of the theory of integrable systems of KdV type is the fact that $n$–GD theories can be obtained from the full Kadomtsev–Petviashvili (KP) hierarchy by means of a suitable reduction process [4, 13, 14]. In this paper we study the problem of the generalization of such a link to the case of fractional KdV theories. Namely we want to suggest candidates of “KP–theories” lying behind fractional KdV ones.

Our starting point is a special hierarchy of dynamical systems described by equations of Riccati type, which we call the central system (CS). They are defined on the space $\mathcal{H}$ of sequences $\{H^{(k)}\}_{k\in\mathbb{N}}$ of Laurent series of the form

$$H^{(0)} = 1, \quad H^{(k)} = z^k + \sum_{l\geq 1} H^l z^{-l} \quad (1.1)$$

by the following equations

$$\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = H^{(j+k)} + \sum_{l=1}^k H^l_j H^{(k-l)} + \sum_{l=1}^j H^l_j H^{(j-l)} \quad (1.2)$$

They arise in the framework of the bihamiltonian approach to the KdV equation, as we shall see in Section 2. One can remark that the space $\mathcal{H}$ is a subspace of the big cell of the Sato Grassmannian $Gr(H)$ [15, 13], and therefore, one can suspect that these equations are related to the linear flows of the $gl_{\infty}$–action on $Gr(H)$. This is indeed correct, since one can prove that the two types of equations are related by a Darboux transformations. The study of this connection, however, is outside the scopes of the present paper [16]. Our aim, in this paper, is to derive the fractional KdV equations from CS by a double process of reduction.
The ideas of the present approach are conveniently described by adopting a geometric point of view. We regard equations (1.2) as defining a hierarchy of vector fields $X_j$ on $\mathcal{H}$. We notice that these vector fields commute, and we consider the space

$$Q^m = \mathcal{H}/X_m$$

of the orbits of the vector field $X_m$. Since the CS flows commute, each vector field $X_j$ sends solutions of $X_m$ into different solutions, and therefore induces a flow on the space $Q_m$. We shall call the hierarchy of the reduced flows on $Q_m$ the $KP^{(m)}$ hierarchy. The reason is that for $m = 1$, one obtains in this way the usual KP hierarchy in the $1 + 1$–dimensional picture. We were led to consider different values of $m$ by the conjecture [10, 25] that $m$–fractional KdV hierarchies can be obtained by “exchanging the roles of $t_m$ and $t_1$” in the $m$–GD equations. However, we stress that the projection onto $Q_m$, which can also be thought of as a field–theoretic redefinition of flows and “independent variables”, in our geometrical scheme is more conveniently considered as a reduction of the central system.

Besides the space $Q^m$, we consider the manifold

$$Z_n := \{ H \in \mathcal{H} \mid X_n(H) = 0 \}$$

of the zeroes of $X_n$ and its submanifold

$$S_n = \{ H \in Z_n \mid H^{(n)} = z^n \},$$

where the Laurent series $H^{(n)}$ assumes the constant value $H^{(n)} = z^n$. We remark that they are invariant submanifolds for the central system. The restrictions of the vector fields $X_j$ to $S_n$ give rise to a second system of reduced equations which we call $CS_n$ equations. Finally, we combine the two reduction processes, by considering the space

$$Q^m_n = S_n/X_m$$

of the orbits of the vector field $X_m$ restricted to $S_n$. We obtain a reduction of the $CS_n$ equations which we call $KdV^m_n$ equations. Since the reduction processes commute, they can also be seen as restrictions of the $KP^{(m)}$ equations to $Q^m_n$. The relevant manifolds are shown in Figure 1.

It is our belief that, when $m$ and $n$ are coprime, the $KdV^m_n$ equations coincide with the fractional KdV equations (or generalized type I GD equations) studied in [8, 12]. This will be shown explicitly in the case of $KdV^2_3$.

Other interesting equations can be obtained by different choices of the reduction spaces. So, for example, the restrictions of $X_j$ to the intersection $S_n \cap Z_p$ of two invariant manifolds give rise to finite dimensional dynamical
systems of Bogoyavlensky–Dubrovin–Novikov type \cite{[17], [18]}. Similarly, the projection of the KP\(^{(m)}\) hierarchy into the space

\[ Q^{(m,l)} = Q^m / X_l \]  

of the orbits of the reduction of \(X_l\) to \(Q^m\), can lead to equations in two space dimensions. They are denoted KP\(^{(m,l)}\) in Figure 2, where the full reduction scheme is represented; the left arrows denote a restriction to invariant submanifolds, and the right arrows denote a projection onto orbit spaces \(Q_m\).

The main aim of this paper is to study the equations CS\(_n\), KP\(^{(m)}\) and KdV\(_m\) and their relations. The paper is organized as follows. Section 2 is a brief introduction to the central system from the point of view of the bihamiltonian approach to the KdV equations. The first fundamental properties of CS are studied in Section 3, where we describe in detail the submanifolds \(S_n\) and the quotient spaces \(Q^n\). Section 4 gives a preliminary view of the equations which can be obtained by iterating and combining the reduction processes in the simplest case \(m = 1, n = 2\) corresponding to the usual KP theory. In Section 5 we exhibit the explicit structure of the equations CS\(_n\) and KP\(^{(m)}\). Section 6 is devoted to the KdV\(_m\) equations. In particular we work out the case KdV\(_2\). Finally, in Section 7 we give the bihamiltonian interpretation of these equations along the lines explained in Section 2, and we make an explicit comparison with the approach based on the generalized DS procedure.
2 The Central System

The present approach to fractional KdV hierarchies comes from the bihamiltonian theory of KdV equation. It may be suitable, therefore, to collect in this section the main ideas of this theory.

The starting point is the relation between bihamiltonian manifolds and integrable systems clarified by a theorem of Gel’fand and Zakharevich \[19\]. Let the phase space $\mathcal{M}$ be a $(2n + 1)$-dimensional manifold, endowed with a pencil of compatible Poisson brackets

$$\{f, g\}_\lambda = \{f, g\}_1 - \lambda \{f, g\}_0.$$  \hfill (2.1)

We assume that this pencil has maximal rank everywhere on $\mathcal{M}$, and that the symplectic leaves of $\{f, g\}_\lambda$ are submanifolds of dimension $2n$ in $\mathcal{M}$. As it was noticed by Gel’fand and Zakharevich, its Casimir function is a polynomial of degree $n$ in the parameter $\lambda$

$$H(\lambda) = H_0 + 1 \lambda + \ldots + H_n \lambda^n,$$  \hfill (2.2)

and its coefficients $(H_0, H_1, \ldots, H_n)$ are in involution with respect to all the Poisson brackets of the pencil. Therefore, they verify the conservation laws

$$\frac{\partial}{\partial t_j} H_k = 0,$$  \hfill (2.3)

where the derivative of the function $H_k$ is taken along the Hamiltonian vector field associated with the function $H_j$. These vector fields define an integrable system on $\mathcal{M}$.

This bihamiltonian strategy may be formally extended to partial differential equations in one space variable $x$, with two significant differences. The Casimir functions $H(\lambda)$ now becomes a Laurent series in $\lambda$ (or in some power of $\lambda$) and

![Figure 2: The Full Reduction Schemes](image)
the involution relations \( \{ H_i, H_j \}_\lambda = 0, \ i, j = 0, \ldots, n \), are replaced by local
conservations laws (or continuity equations) of the form
\[
\frac{\partial h}{\partial t_j} = \partial_x H^{(j)}.
\] (2.4)

This fact is responsible for the appearance of a further important object of the
theory: the current densities \( H^{(j)} \). They define a new geometrical structure
associated with the points of the phase space, which evolves in time along the
orbits of the equations (2.4). The central system is the system of equations
describing the time evolution of the current densities \( H^{(j)} \).

Let us describe these features in the example of the KdV equations. From
the bihamiltonian point of view, the KdV theory may be seen as the study
of the Casimir function and the current densities associated with the Poisson pencil
\[
\dot{u} = -\frac{1}{2} v_{xxx} + 2(u + \lambda) v_x + u_x v.
\] (2.5)

This pencil can be obtained by a Marsden–Ratiu reduction of a Lie–Poisson pencil defined over the \( \mathfrak{sl}(2) \) loop algebra [20, 21]. Here, \( u \) is a point of the
phase space of the KdV theory, \( v \) is a covector, and the Poisson pencil is a
map from the cotangent space to the tangent space.

The computation of the Casimir function is comparatively easy. Let us
set \( \lambda = z^2 \), and let us introduce the Hamiltonian density \( h(z) \) of the Casimir function according to
\[
H(z) = 2z \int_{S^1} h(z) dx.
\] (2.6)

One can prove that \( h(z) \) is the unique monic Laurent series
\[
h(z) = z + \sum_{l \geq 1} h_l(x) z^{-l}
\] (2.7)

which solves the Riccati equation
\[
h_x + h^2 = u + z^2.
\] (2.8)

Thus, the role of this equation is to define the Casimir function at any point
\( u \) of the phase space.

The computation of the current \( H^{(j)} \) is, instead, a little more tricky. We
have to consider the Faà di Bruno polynomials \( h^{(k)} \) defined by the recursive
formula
\[
h^{(k+1)} = h^{(k)} + hh^{(k)},
\] (2.9)

starting from \( h^{(0)} = 1 \). Then we have to linearly combine these polynomials,
with coefficients depending on the coefficients \( h_i \) of \( h(z) \), but independent of \( z \),
\[
H^{(j)} = \sum_{k=0}^{j} c_k^j(h_1, h_2, \ldots) h^{(k)}(z)
\] (2.10)
in such a way to obtain Laurent series $H^{(j)}$ of the form
\[ H^{(j)}(z) = z^j + \sum_{l \geq 1} H_l^j z^{-l}. \tag{2.11} \]

They are the current densities we are looking for. Indeed one can prove \cite{22, 21} that, when $u$ evolves according to the KdV hierarchy, the solution $h(z)$ of the Riccati equation \eqref{2.8} evolves exactly according to the conservation laws \eqref{2.4}, with the currents \eqref{2.11}. Furthermore, the current densities presently defined evolve in time according to the central system \eqref{1.2}.

To arrive at the general form of the central system \eqref{1.2}, we have to remark that the assumption that $h(z)$ be a solution of the Riccati equation is actually inessential. Indeed, equations \eqref{2.10} and \eqref{2.11} define the current densities $H^{(j)}$ associated with any Laurent series $h(z)$ of the form \eqref{2.7}. Therefore, we can regard equations \eqref{2.4} as a system of local conservation laws for any monic Laurent series $h(z)$, whether or not it satisfies the Riccati equation \eqref{2.8}. This enlarged hierarchy is the KP theory. Indeed, equations \eqref{2.4}, considered on an arbitrary monic Laurent series $h(z)$, are (a possible form of) the celebrated KP equations \cite{22, 15, 23}.

The second remark is that also the definition \eqref{2.10} of the current densities we have used to construct the KP equations, is inessential. The central system is actually independent of the definition \eqref{2.10} of the currents $H^{(j)}$, and only rests on the particular form of their Laurent expansion \eqref{2.11}. Therefore, we can eliminate any reference to a “space variable $x$” and to a “Hamiltonian density $h(z)$”, and we can regard the central system as defining an independent family of vector fields on the space $\mathcal{H}$ of collections of such currents. From this perspective, equations \eqref{2.10} (associated with the KP equations) and the Riccati equation \eqref{2.8} (leading to the KdV theory) are simply a set of constraints, which are compatible with the central system. Other constraints are possible as well, leading to different systems of equations. Among them there are the fractional KdV systems.

### 3 The Central System and its Reductions

In this section we begin to study the central systems and its reductions. The starting point is the observation that the CS vector fields pairwise commute. To prove this property, we begin with some remarks on its structure.

Let us consider the space $\mathcal{L}$ of truncated Laurent series
\[ l(z) = \sum_{-\infty}^{N(l)} l_j z^j. \tag{3.1} \]

In this space the currents $H^{(j)}$, $j \geq 0$, introduced in Section 1 as
\[ H^{(0)} = 1, \quad H^{(k)} = z^k + \sum_{l \geq 1} H_l^k z^{-l}, \tag{3.2} \]
determine a subspace
\[ \mathcal{H}_+ := \langle H^{(0)}, H^{(1)}, \ldots \rangle, \] (3.3)
transversal to the subspace
\[ \mathcal{H}_- := \langle z^{-1}, z^{-2}, \ldots \rangle \] (3.4)
of the Laurent series of strictly negative degree. We can now regard any
solution of CS as a subspace \( \mathcal{H}_+(t) \) moving in \( \mathcal{L} \). The characteristic property
of the solutions of CS is the invariance of \( \mathcal{H}_+(t) \) with respect to the action of
the differential operators \( \frac{\partial}{\partial t} + H^{(j)} \) associated with the currents.

**Proposition 3.1** Along the flows of the central system the subspace \( \mathcal{H}_+(t) \)
satisfies the invariance relation
\[ \left( \frac{\partial}{\partial t} + H^{(j)} \right) (\mathcal{H}_+) \subset \mathcal{H}_+. \] (3.5)

**Proof.** We show that this relation completely defines CS. Since the currents
\( H^{(k)} \) form a basis of \( \mathcal{H}_+ \), the invariance property \( (3.5) \) entails the existence of
coefficients \( c_{jk} \) independent of \( z \) such that
\[ \frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = \sum_{l=0}^{j+k} c_{jk}^{(l)} H^{(l)}. \] (3.6)
These coefficients are easily determined by comparing the expansion in powers
of \( z \) of both sides of equation \( (3.6) \). The final result is equation \( (1.2) \) defining
the central system.

\[ \square \]

As an immediate consequence of the previous result, we obtain the following
compact representation of CS. We denote by \( \pi_+ \) and \( \pi_- \) the canonical projections associated with the decomposition
\[ \mathcal{L} = \mathcal{H}_+ \oplus \mathcal{H}_- \] (3.7)
of the space \( \mathcal{L} \). Then we have

**Proposition 3.2** The central system is the family of dynamical systems on
the currents \( H^{(j)} \) given by
\[ \frac{\partial}{\partial t_k} H^{(j)} = -\pi_-(H^{(j)} H^{(k)}). \] (3.8)
Proof. We apply $\pi_-$ to both sides of (3.6) and we get

$$\pi_- \left( \frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} \right) = 0.$$  

The assertion follows from the fact that $H^{(k)} = z^k + \sum_{i \geq 1} H^i z^{-i}$ so that $(\frac{\partial H^{(k)}}{\partial t_j}) \in \mathcal{H}_-$.

□

From the symmetry in $j$ and $k$ of the right-hand side of (3.8), we finally obtain the following

**Proposition 3.3** Every solution of CS satisfies the exactness condition

$$\frac{\partial}{\partial t_k} H^{(j)} = \frac{\partial}{\partial t_j} H^{(k)}.$$  

(3.9)

□

We are now ready to prove the commutativity property of the CS flows.

**Proposition 3.4** The flows of the central system pairwise commute.

**Proof.** We compute the action of the commutator $[X_j, X_k]$ of two vector fields of the hierarchy on a generic current:

$$[X_j, X_k](H^{(i)}) = \frac{\partial}{\partial t_j} \frac{\partial H^{(i)}}{\partial t_k} - \frac{\partial}{\partial t_k} \frac{\partial H^{(i)}}{\partial t_j}.$$  

(3.10)

We remark that that this quantity belongs to $\mathcal{H}_-$, thanks to the specific form of $H^{(j)}$. Then we observe that, using the exactness property (3.9), this commutator can be also written in the form

$$[X_j, X_k](H^{(i)}) = \left[ \frac{\partial}{\partial t_j} + H^{(j)} , \frac{\partial}{\partial t_k} + H^{(k)} \right] H^{(i)},$$

so that the commutator of the vector fields coincides with the commutator of the operators $\frac{\partial}{\partial t_j} + H^{(j)}$ associated with the currents. Hence the invariance property (3.3) entails that $[X_j, X_k](H^{(i)})$ belongs to the subspace $\mathcal{H}_+$ for all the $H^{(i)}$’s. But $\mathcal{H}_+ \cap \mathcal{H}_- = \{0\}$, and therefore $[X_j, X_k]$ vanishes.

□
3.1 The Invariant Submanifolds $S_n$

From the commutativity of the flows it follows that the set $Z_n$ of zeroes of the vector field $X_n$, defined by the quadratic equations

$$H^{(k+n)} - H^{(k)}H^{(n)} + \sum_{l=1}^{k} H_l^n H^{(k-l)} + \sum_{l=1}^{n} H_l^k H^{(n-l)} = 0, \quad (3.11)$$

is an invariant submanifold for CS. Moreover, on $Z_n$ we have

$$\frac{\partial H^{(n)}}{\partial t_j} = \frac{\partial H^{(j)}}{\partial t_n} = 0, \quad (3.12)$$

thanks to the exactness property (3.9). Therefore the manifold $Z_n$ is foliated by invariant submanifolds defined by the equation $H^{(n)} = \text{constant}$. Among all these leaves we choose the one given by the condition

$$H^{(n)} = z^n, \quad (3.13)$$

which is the counterpart of the choice usually considered in the reduction theory from KP to the n–GD hierarchies [13, 14].

**Definition 3.5** We call $S_n$ the submanifold of the zeroes of $X_n$, where the current $H^{(n)}$ satisfies the relation (3.13).

The equations defining this submanifold can be explicitly found by eliminating $H^{(n)}$ from (3.11) by means of the constraints (3.13). This leads to the conclusion that $S_n$ is the subset of $H$ defined by the equations

$$H^{(j+n)} = z^n H^{(j)} - \sum_{l=1}^{n} H_l^j H^{(n-l)}. \quad (3.14)$$

Therefore, it is parametrized by the first $n - 1$ currents $(H^{(1)}, \ldots, H^{(n-1)})$.

A more intrinsic characterization is provided by the following

**Proposition 3.6** The submanifold $S_n$ is the subset of $H$ given by the equation

$$z^n(H_+ \subset H_+, \quad (3.15)$$

i.e., the set of the points where the operator of multiplication by $z^n$ leaves the space $H_+$ invariant.

**Proof.** First of all, it is easily shown that the condition (3.15) is necessary. It is enough, indeed, to observe that the invariance relation

$$\left( \frac{\partial}{\partial t_n} + H^{(n)} \right)(H_+ \subset H_+) \quad (3.16)$$
characterizing the vector field $X_n$, at the points of $S_n$ boils down to the relation \((3.13)\) thanks to \((3.13)\).

To show that the condition is also sufficient, let us remark that $z^n \cdot H^{(0)} = z^n \cdot 1 = z^n$ belongs to $H_+$ and then can be developed as

$$z^n = \sum_{l=0}^{n} c_i^n H^{(l)}.$$ \hspace{1cm} \(3.17\)

By a comparison between the coefficients of the positive powers of $z$ we obtain that

$$z^n = H^{(n)}.$$ \hspace{1cm} \(3.18\)

In the same way it can be proved that

$$z^n H^{(j)} = H^{(j+n)} + \sum_{i=1}^{n} H_{ij} H^{(n-l)}.$$ \hspace{1cm} \(3.19\)

Now it is enough to write this relation in the form

$$H^{(j+n)} - z^n H^{(j)} + \sum_{i=1}^{n} H_{ij} H^{(n-l)} = 0$$ \hspace{1cm} \(3.20\)

with the condition \((3.14)\).

\[\square\]

### 3.2 The Quotient Spaces $Q_m$

Let us consider now the second process of reduction. We concentrate on a vector field of the hierarchy, say $X_m$, and we denote the corresponding time $t_m$ by

$$t_m = x,$$ \hspace{1cm} \(3.21\)

in order to point out its special role in the reduction. This amounts to convert $t_m$ into a “space variable”, in the terminology used in the theory of KP equations. It is worthwhile to remark that the projection to $Q_m$ with $m \neq 1$ formalizes the procedure that in the physics literature \cite{25} is usually described as the “exchange $x \leftrightarrow t_m$”.

**Definition 3.7** We call $Q_m$ the space of the solutions of the $m$–th flow of the central system, i.e., the space of orbits of the vector field $X_m$.

Thanks to the commutativity condition, any vector field $X_j$ of CS induces a flow, which we still denote by $X_j$, on $Q_m$. The first problem to be solved is to characterize the variety $Q_m$. 

Proposition 3.8. The quotient spaces $Q^m$ can be identified with the space of $m$–tuples $\{H^{(a)}(z)\}_{a=1,\ldots,m}$ of Laurent series of the form

$$H^{(a)}(z) = z^a + \sum_{l \geq 1} H^a_l(x)z^{-l}, \quad (3.22)$$

whose coefficients are functions of the space variable $x$.

Proof. It suffices to remark that the equations defining the vector field $X^m$ may be written in the form of recursion relations

$$H^{(j+m)} = \left(\frac{\partial}{\partial x} + H^{(m)}\right)H^{(j)} - \sum_{l=1}^{j} H^m_l H^{(j-l)} - \sum_{l=1}^{m} H^j_l H^{(m-l)}. \quad (3.23)$$

They allow to compute the currents $(H^{(m+1)}, H^{(m+2)}, \ldots)$ as differential polynomials in the first $m$ currents $(H^{(1)}, \ldots, H^{(m)})$. Explicitly, this means that the coefficients of $(H^{(m+1)}, H^{(m+2)}, \ldots)$ are polynomials in the coefficients of $(H^{(1)}, \ldots, H^{(m)})$ and their $x$–derivatives. Since the generators $(H^{(1)}, \ldots, H^{(m)})$ depend on $x$ in a completely arbitrary way, we conclude that the space of orbits of the vector field $X^m$ coincides with the space of $m$–tuples of Laurent series $(3.22)$.

□

4 Preliminary examples of reduced equations

Let us consider the simplest examples of reductions of the central system. We first consider the submanifold $S_2$. According to equation (3.14), it is parametrized by the single Laurent series $H^{(1)}$. To simplify the notations we set

$$h(z) := H^{(1)} = z + \sum_{l \geq 1} h_l z^{-l}. \quad (4.1)$$

The constraints defining $S_2$ are

$$H^{(2)} = z^2, \quad H^{(3)} = z^2 h(z) - h_1 h(z) - h_2, \quad H^{(4)} = z^4$$

$$H^{(5)} = z^2 H^{(3)} - H^3_1 h(z) - H^3_2$$

$$= (z^4 - h_1 z^2 + h_1^2 - h_3) h(z) - h_2 z^2 + h_2 h_1 - h_4,$$

and so on. To construct $CS_2$, it is sufficient to plug these constraints in the equation

$$\frac{\partial h(z)}{\partial t_j} = H^{(j+1)} - h H^{(j)} + \sum_{l=1}^{j} h_l H^{(j-l)} + H^j_1. \quad (4.3)$$
Table 1: The first CS$_2$ equations

\[
\begin{align*}
\frac{\partial h_3}{\partial t_1} &= -2h_2 \\
\frac{\partial h_4}{\partial t_1} &= -(2h_3 + h_1^2) \\
\frac{\partial h_5}{\partial t_1} &= -(2h_4 + 2h_1h_2) \\
\frac{\partial h_6}{\partial t_1} &= -(2h_5 + 2h_1h_3 + h_2^2) \\
\frac{\partial h_7}{\partial t_1} &= -(2h_6 + 2h_1h_4 + 2h_2h_3)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial h_3}{\partial t_5} &= 2h_3h_2 - 2h_6 + 2h_1h_4 - 2(h_1)^2h_2 \\
\frac{\partial h_4}{\partial t_5} &= h_3^2 + 2h_2h_4 + h_1^2h_3 - 2h_7 - h_1h_2^2 + (h_1)^4 \\
\frac{\partial h_5}{\partial t_5} &= 2h_3h_4 - 2h_1h_2 - 2h_8 + 2h_1h_3h_2 \\
\frac{\partial h_6}{\partial t_5} &= -2h_9 + 3h_1h_2^2 - 2h_1^2h_3 + h_4^2 - h_2h_1^2 + h_2h_2h_3 \\
\frac{\partial h_7}{\partial t_5} &= 2h_2h_3^2 - 2h_2h_1^2h_3 - 2h_10 + 4h_1h_3h_4 - 2h_1^2h_4
\end{align*}
\]

to get the equations displayed in Table I. It is worthwhile to display also the first CS$_3$ equations. By setting

\[
k(z) := H^{(2)} = z^2 + \sum_{l \geq 1} k_l z^{-l}
\]

along with (4.1) and using the first parametric equations of the submanifold S$_3$

\[
\begin{align*}
H^{(3)} &= z^3 \\
H^{(4)} &= z^3h - h_1k - h_2h - h_3 \\
H^{(5)} &= z^3k - k_1k - k_2h - k_3
\end{align*}
\]

we get the equations collected in Table II.

Let us now consider the quotient space $\mathcal{Q}_1$. We keep the usual notation $H^{(1)} = h$, but we recall that here $h$ must be considered as a a Laurent series $h = z + \sum_{l \geq 1} h_l(x)/z^l$, whose coefficients depend on the space variable $x$. By
Table 2: The first CS\textsubscript{3} equations

\[
\begin{align*}
\frac{\partial h_1}{\partial t_1} &= k_1 - 2 h_2 \\
\frac{\partial h_2}{\partial t_1} &= k_2 - 2 h_3 - h_1^2 \\
\frac{\partial h_3}{\partial t_1} &= k_3 - 2 h_2 h_1 - 2 h_4 \\
\frac{\partial h_4}{\partial t_1} &= k_4 - 2 h_1 h_3 - 2 h_5 - h_2^2 \\
\frac{\partial h_1}{\partial t_2} &= \frac{\partial k_1}{\partial t_1} = -h_3 - k_2 + h_1^2 \\
\frac{\partial h_2}{\partial t_2} &= \frac{\partial k_3}{\partial t_1} = -h_1 k_1 - h_4 - k_3 + h_2 h_1 \\
\frac{\partial h_3}{\partial t_2} &= \frac{\partial k_3}{\partial t_1} = -k_1 h_2 - h_1 k_2 - h_5 - k_4 + h_1 h_3 \\
\frac{\partial h_4}{\partial t_2} &= \frac{\partial k_3}{\partial t_1} = -k_1 h_3 - k_2 h_2 - h_1 k_3 - h_6 - k_5 + h_1 h_4 \\
\frac{\partial k_1}{\partial t_2} &= h_4 + h_1 k_1 - h_2 h_1 - 2 k_3 \\
\frac{\partial k_2}{\partial t_2} &= h_5 - h_2^2 - h_1 k_2 + 2 k_1 h_2 - 2 k_4 - k_1^2 \\
\frac{\partial k_3}{\partial t_2} &= h_6 - h_1 k_3 - h_2 h_3 - 2 k_1 k_2 + 2 k_1 h_3 - 2 k_5 \\
\frac{\partial k_4}{\partial t_2} &= h_7 - h_1 k_4 - h_2 h_4 - 2 k_1 k_3 + 2 h_4 k_1 - 2 k_6 - k_2^2
\end{align*}
\]

using the equations of the vector field \(X_1\), from (3.22) we obtain the relations

\[
H^{(2)} = h_x + h^2 - 2h_1 \\
H^{(3)} = \left( h_x \right)^{(2)} + h H^{(2)} - h_1 h - h_2 \\
\quad \quad = h_{xx} + 3hh_x + h^3 - 3h_1 h - 3(h_{1x} + h_2) \tag{4.6}
\]

\[\ldots\]

The KP\textsuperscript{(1)} equations, describing the projection of CS on \(Q_1\), are obtained by introducing the recursion relation (4.6) in the equation

\[
\frac{\partial h}{\partial t_j} = H^{(j+1)} - hH^{(j)} + \sum_{i=1}^{j} h_i H^{(j-i)} + H_i^j = \frac{\partial H^{(j)}}{\partial x}. \tag{4.7}
\]

Expanding these equations in powers of \(z\), one constructs the equations displayed in Table III. After a suitable coordinate change [15, 22, 23], these equa-
Table 3: The first KP(1) equations

\[
\begin{align*}
\frac{\partial h_1}{\partial t_1} &= h_1 x \\
\frac{\partial h_2}{\partial t_2} &= h_2 x \\
\frac{\partial h_3}{\partial t_1} &= h_3 x \\
\frac{\partial h_4}{\partial t_3} &= \partial_x (h_{1xx} + 3h_{2x} + 3h_3) \\
\frac{\partial h_5}{\partial t_3} &= \partial_x (h_{2xx} + 3h_{3x} + 3h_1 h_{1x} + 3h_4 + 3h_1 h_2) \\
\frac{\partial h_6}{\partial t_3} &= \partial_x (h_{3xx} + 3h_{4x} + 3h_1 h_{2x} + 3h_1 h_2 + 3h_1 h_3 + 3h_2^2 + h_1^3) \\
\end{align*}
\]

...
Table 4: The $CS_2$ equations as conservation laws.

\[
\begin{align*}
\frac{\partial h_1}{\partial t_1} &= -2h_2 \\
\frac{\partial h_2}{\partial t_1} &= -(2h_3 + h_1^2) \\
\frac{\partial h_3}{\partial t_1} &= -(2h_4 + 2h_1 h_2) \\
\frac{\partial h_4}{\partial t_1} &= -(2h_5 + 2h_1 h_3 + h_2^2) \\
\frac{\partial h_5}{\partial t_1} &= -(2h_6 + 2h_1 h_4 + 2h_2 h_3) \\
\frac{\partial h_6}{\partial t_5} &= \frac{\partial}{\partial t_1} (h_5 - 2h_1 h_3 + h_1^3) \\
\frac{\partial h_7}{\partial t_5} &= \frac{\partial}{\partial t_1} (h_6 - h_2 h_3 - h_1 h_4 + h_2 h_1^2) \\
\frac{\partial h_8}{\partial t_5} &= \frac{\partial}{\partial t_1} (h_7 - h_3 h_4 - h_1 h_5 + h_3 h_1^2) \\
\frac{\partial h_9}{\partial t_5} &= \frac{\partial}{\partial t_1} (-h_3 h_4 + h_1^2 h_4 - h_1 h_6 + h_8) \\
\frac{\partial h_{10}}{\partial t_5} &= \frac{\partial}{\partial t_1} (-h_3 h_5 - h_1 h_7 + h_9 + h_1^2 h_5)
\end{align*}
\]

They allow to identify the space $Q_2^1$ with the space of scalar functions $u \equiv 2h_1$ in one space variable $x$ (the phase space of KdV). Finally, we insert these constraints in the first component of each vector field $X_j$ of $CS_2$ (the other components can be neglected, since they give rise only to differential consequences of the previous ones). Once again, we get the KdV equations

\[
\begin{align*}
\frac{\partial u}{\partial t_3} &= \partial_x (h_3 - h_1^2) = \frac{1}{8} \partial_x(u_{xx} - 3u^2) \\
\frac{\partial u}{\partial t_5} &= \partial_x (h_5 - 2h_1 h_3 + h_1^3) = \frac{1}{32} \partial_x (u_{xxxx} - 10u u_{xx} - 5u_x^2 + 10u^3)
\end{align*}
\]

and so on. This computation clearly points out how the projection process allows to transform a family of dynamical system into a hierarchy of evolution
partial differential equation in 1 + 1 dimensions (one space–dimension and one time–dimension).

We end this section by briefly describing another example of double reduction of CS, on the intersection $S_n \cap Z_p$. In this case we obtain systems of ordinary differential equations in a finite number of fields. Let us exemplify this feature by considering the intersection $S_2 \cap Z_5$. Looking at the equations defining $X_5$ in Table III, we obtain the constraints

$$
\begin{align*}
h_6 &= h_1 h_4 + h_2 h_3 - h_1^2 h_2 \\
h_7 &= h_2 h_4 + \frac{1}{2}(h_3^2 + h_1^2 h_3 - h_1 h_2^2 - h_1^4) \\
h_8 &= h_3 h_4 + h_1 h_2 h_3 - h_1^3 h_2 \\
&\ldots
\end{align*}
$$

which are an infinite system of recurrence relations allowing to express all the $h_l$’s, for $l \geq 6$, as polynomial functions of $h_1, \ldots, h_5$. Therefore the invariant submanifold $S_2 \cap Z_5$ is 5–dimensional. The restriction of CS$_2$ to this submanifold is simply constructed by plugging the constraints (4.12) in the first five components of the vector fields $X_1$ and $X_3$. We get the equations

$$
\begin{align*}
\frac{\partial h_1}{\partial t_1} &= -2h_2 \\
\frac{\partial h_2}{\partial t_1} &= -(2h_3 + h_1^2) \\
\frac{\partial h_3}{\partial t_1} &= -(2h_4 + 2h_1 h_2) \\
\frac{\partial h_4}{\partial t_1} &= -(2h_5 + 2h_1 h_3 + h_2^2) \\
\frac{\partial h_5}{\partial t_1} &= -(4h_1 h_4 + 4h_2 h_3 - 2h_1^2 h_2) \\
\frac{\partial h_6}{\partial t_1} &= -(2h_1 h_4 - 2h_1^2 h_2 - 2h_1^2 h_4) \\
\frac{\partial h_7}{\partial t_3} &= -(2h_1 h_4 - 2h_1^2 h_2 - 2h_1^2 h_4) \\
\frac{\partial h_8}{\partial t_3} &= -(2h_1 h_4 - 2h_1^2 h_2 - 2h_1^2 h_4)
\end{align*}
$$

The equations corresponding to the vector fields $X_{2j+1}$, for $j \geq 2$, are combinations of equations (4.13). Hence the equations we have written represent the whole reduced system. After a suitable coordinates change, they coincide with the second Novikov system [17] relative to the KdV hierarchy.

5 The equations CS$_n$ and KP$^{(m)}$

In Section 3 we explained the geometric process leading to the CS$_n$ and KP$^{(m)}$ equations. In this section we shall give an algebraic definition of these equations by using the concept of Faà di Bruno basis associated with a (finite set of) monic Laurent series. We present this idea first for the CS$_n$ case. In Section 3 we showed that this is a system of ordinary differential equations in the
first \((n-1)\) currents
\[
H^{(a)} = z^a + \sum_{l \geq 1} H_l^a z^{-l},
\]  
(5.1)

for \(a = 1, \ldots, n-1\). As usual, we set \(H^{(0)} = 1\) and we consider the point \(H^{(n)} = z^n\) in the space \(L\) of all truncated Laurent series. We call stationary Faà di Bruno basis at the point \(z^n\) the basis of \(L\) defined by the iterates
\[
F^{(j+n)} = z^n \cdot F^{(j)}
\]
\[
F^{(a)} = H^{(a)} \quad \text{for } a = 0, \ldots, n-1,
\]  
(5.2)
of the initial generators \((H^{(0)}, \ldots, H^{(n-1)})\), for \(j \in \mathbb{Z}\). Let us denote with \(\mathcal{H}_+\) the subspace of \(L\) spanned by the nonnegative elements of the Faà di Bruno basis at the point \(z^n\), and with \(\mathcal{H}_-\) the subspace spanned by the negative ones. We decompose \(L\) in the direct sum
\[
L = \mathcal{H}_+ \oplus \mathcal{H}_-,
\]  
(5.3)
and we call
\[
H^{(j)} = \pi_+(z^j)
\]  
(5.4)
the projections on \(\mathcal{H}_+\) of the powers \(z^j\) with respect to this decomposition.

**Definition 5.1 (Second definition of \(CS_n\))** We call \(CS_n\) the dynamical system in the currents \((H^{(1)}, \ldots, H^{(n-1)})\) defined by
\[
\frac{\partial H^{(a)}}{\partial t_j} = -\pi_-(H^{(a)}H^{(j)}),
\]  
(5.5)
for \(a = 1, \ldots, n-1\), and \(j \in \mathbb{N}\).

We proceed in the same way for the \(KP^{(m)}\) equations. In this case \(L\) is the space of truncated Laurent series whose coefficients are functions of the space variable \(x\), and we consider the point
\[
H^{(m)} = z^m + \sum_{l \geq 1} H_l^m z^{-l}.
\]  
(5.6)
We call differential Faà di Bruno basis at the point \(H^{(m)}\) the basis of \(L\) defined by the iterates
\[
F^{(j+m)} = (\partial_x + H^{(m)}) \cdot F^{(j)}
\]
\[
F^{(a)} = H^{(a)} \quad \text{for } a = 0, \ldots, m-1,
\]  
(5.7)
of the initial generators \((H^{(0)}, \ldots, H^{(m-1)})\), for \(j \in \mathbb{Z}\). We observe that the computation of the elements \(H^{(j)}\), with \(j < 0\), of the Faà di Bruno basis does
not require any integration, since the recurrence relation (5.7) can be inverted in a purely algebraic way. As before, we decompose the space \( L \) in the form (1.3), and we introduce the projections

\[ H^{(j)} = \pi_+(z^j) \]  

(5.8)

of the powers \( z^j \). Even if we used the same notations to point out the analogy between the two situations, one has to keep in mind that they live in different spaces, and that the Faà di Bruno basis are defined in a different way.

**Definition 5.2 (Second definition of KP\((m)\))** We call KP\((m)\) the system of evolutionary partial differential equations in the currents \((H^{(1)}, \ldots, H^{(m)})\) defined by

\[
\frac{\partial H^{(a)}}{\partial t_j} = -\pi_-(H^{(a)}H^{(j)}),
\]

(5.9)

for \( a = 1, \ldots, m, \) and \( j \in \mathbb{N} \).

Now we have to show that the equations defined above coincide with the ones obtained by the reduction process.

**Proposition 5.3** The equations (5.3) and (5.9) constructed by the Faà di Bruno basis algorithm coincide with the reduced equations CS\(_n\) and KP\((m)\) defined by the reduction scheme.

**Proof.** We call \( \widehat{H}_+ \) the subspace of \( L \) spanned by the currents \( H^{(j)} \) of the central system (the same subspace that we denoted with \( H_+ \) in Section 3). We consider \( \widehat{H}_+ \) at a generic point of the submanifold \( S_n \). From (3.13), at the points of \( S_n \) the subspace \( \widehat{H}_+ \) is invariant with respect to multiplication by \( z^n \). Hence it contains all nonnegative elements of the stationary Faà di Bruno basis associated with the point \( z^n \). Therefore \( \widehat{H}_+ \) at the points of \( S_n \) coincides with the subspace \( H_+ \) appearing in the decomposition (5.3), constructed by choosing as generators the currents \((H^{(1)}, \ldots, H^{(n-1)})\) defining the point of \( S_n \). It follows that the projection (5.4) belongs to \( \widehat{H}_+ \), and therefore coincides with the corresponding current of CS evaluated at \( S_n \). The first part of the proposition then follows from Proposition 3.2.

As far as the KP\((m)\) equations are concerned, one can argue in the same way. In this case we evaluate the subspace \( \widehat{H}_+ \) at the points of a generic integral curve of the vector field \( X_m \). We use the invariance property (3.3) to conclude that at these points the subspace \( \widehat{H}_+ \) contains all nonnegative elements of the differential Faà di Bruno basis (5.7), and therefore it coincides with the subspace \( H_+ \) associated with the point \( H^{(m)} \). This observation leads to the conclusion that the projection (5.8) coincides with the current \( H^{(j)} \) of CS evaluated at the points of the integral curve of the vector field \( X_m \). This suffices to prove the coincidence between the KP\((m)\) equations obtained by projection and equations (5.9).
To enlighten the (differential) Faà di Bruno algorithm, we consider the KP\(^{(2)}\) equations. We put \(h = H^{(1)}\) and \(k = H^{(2)}\). The first elements of the Faà di Bruno basis at the point \(k\), associated with the generators \((1, h)\), are

\[
\begin{align*}
F^{(0)} &= 1 & F^{(3)} &= h_x + kh \\
F^{(1)} &= h & F^{(4)} &= k_x + k^2 \\
F^{(2)} &= k & F^{(5)} &= h_x + 2kh_x + hk_x + k^2h.
\end{align*}
\]

We combine these elements in the form

\[
\begin{align*}
H^{(1)} &= F^{(1)} & H^{(3)} &= F^{(3)} - h_1F^{(1)} - (h_2 + k_1)F^{(0)} \\
H^{(2)} &= F^{(2)} & H^{(4)} &= F^{(4)} - 2k_1F^{(1)} - 2k_2F^{(0)}
\end{align*}
\]

in order to obtain Laurent series with the asymptotic behavior \(H^{(j)} = z^j + O(z^{-1})\). They are the currents \(H^{(j)}\) written as differential polynomials of \(h\) and \(k\). To obtain the KP\(^{(2)}\) equations we can either compute the projections \(\pi_{-}(hH^{(j)})\) and \(\pi_{-}(kH^{(j)})\), or write the equations

\[
\begin{align*}
\frac{\partial}{\partial t}h &= H^{(j+1)} - hH^{(j)} + \sum_{l=1}^{j} h_lH^{(j-l)} + H_1^j \\
\frac{\partial}{\partial t}k &= \partial_x H^{(j)}.
\end{align*}
\]

In particular, the first vector field of the KP\(^{(2)}\) hierarchy is given by

\[
\begin{align*}
\frac{\partial}{\partial t_1}h &= k - h^2 + 2h_1 \\
\frac{\partial}{\partial t_1}k &= \partial_x h.
\end{align*}
\]

This computation allows to illustrate the link between KP\(^{(1)}\) and KP\(^{(2)}\). Indeed we write equations (5.13) in the form

\[
\begin{align*}
k &= \frac{\partial}{\partial t_1}h + h^2 - 2h_1 \\
\partial_x h &= \frac{\partial}{\partial t_1}k
\end{align*}
\]

and we take the \(x\)-derivative of the first equation above to get

\[
\partial_x k = \frac{\partial}{\partial t_1}h_x + 2hh_x - 2h_1x.
\]
By substitution we get
\[ \partial_x k = \frac{\partial^2}{\partial t_1^2} k + 2h \frac{\partial}{\partial t_1} k - 2 \frac{\partial}{\partial t_1} k_1. \]  
(5.15)

By this process, we can express \( h, k \), and the collection of their “space derivatives” \( (h_x, k_x, h_{xx}, k_{xx}, \ldots) \) as \( t_1 \)-differential polynomials in \( h \). As a consequence, all the currents \( H^{(j)} \) are expressed as \( t_1 \)-differential polynomials in \( h \), and the KP\(^{(2)}\)-system (5.12) reduces to the KP\(^{(1)}\) equations with \( x = t_1 \),
\[ \frac{\partial}{\partial t_j} h = \frac{\partial}{\partial t_1} H^{(j)}. \]  
(5.16)

Conversely, the KP\(^{(2)}\) hierarchy can be obtained as a differential prolongation of KP\(^{(1)}\). More precisely, starting from KP\(^{(1)}\), one has to put \( k := H^{(2)} \) and to use the definition of \( k \) and the second equation of KP\(^{(1)}\),
\[ k = h_x + h^2 - 2h_1 \]  
(5.17)
to express all the \( x \)-derivatives of \( h \) in terms of \( h, k \) and their \( t_2 \)-derivatives. At this point one simply changes the names of the variables \( x \) and \( t_2 \), by putting at first \( x \leftrightarrow t_1 \) and then \( t_2 \leftrightarrow x \). This kind of procedure is easily extended to describe the relations between the KP\(^{(1)}\) and KP\(^{(m)}\) equations, for \( m \geq 3 \). As a matter of fact, the existence of such a link between different KP\(^{(m)}\) theories, which gives rise to these mappings of flows, is extremely natural in our approach. It is a consequence of the fact that all KP\(^{(m)}\) theories are different field-theoretic pictures of the same infinite dimensional dynamical system, the central system.

6 The KdV\(_n^m\) equation

The KdV\(_n^m\) equations are the restriction of the KP\(^{(m)}\) equations to the invariant submanifold \( Q_n^m \) of the orbits of \( X_m \) which are tangent to \( S_n \) (see Figure 1). In the case \( n > m \), the space \( Q_n^m \) is characterized by the following property.

**Proposition 6.1** Let \( (H^{(1)}, \ldots, H^{(m)}) \) be a generic point of \( Q_n^m \), and let \( (H^{(1)}, \ldots, H^{(n-1)}) \) be the first \((n-1)\) currents constructed by the (differential) Faà di Bruno algorithm of Section 5. The point \( (H^{(1)}, \ldots, H^{(m)}) \) belongs to \( Q_n^m \) if and only if the Laurent series \( (\partial_x + H^{(m)}) H^{(\alpha)} \), for \( \alpha = 0, \ldots, n-1 \) admit the linear expansion
\[ (\partial_x + H^{(m)}) H^{(\alpha)} = \sum_{\beta=0}^{n-1} u_\beta^\alpha(\lambda) H^{(\beta)}. \]  
(6.1)
where the coefficients $u^\alpha_\beta(\lambda)$ depend polynomially on $\lambda = z^n$ and on the coordinates $H^1_1, \ldots, H^m_1$ of the point of $Q^n_m$.

**Proof.** We recall that the vector field $X_m$ is defined by the equation

$$
\left( \frac{\partial}{\partial t_m} + H^{(m)} \right)(H_+) \subset H_+
$$

and that $S_n$ is characterized by the property

$$
z^n : (H_+) \subset H_+.
$$

Thus, on $S_n$ the subspace $H_+$ is generated by $(H^{(1)}, \ldots, H^{(n-1)})$ and by multiplication by powers of $\lambda$. Equation (6.1) simply means that $X_m$ is tangent to $S_n$. □

To see how this condition works in practice, we will restrict from now on to the case of KdV^2_3. In the notations of Section 4, i.e., $h(z) = H^{(1)}$, $k(z) = H^{(2)}$, $\lambda = z^3$, the submanifold $Q^n_3$ is defined by the constraints

$$
(\partial_x + k(z)) \begin{bmatrix} 1 \\ h(z) \\ k(z) \end{bmatrix} = U(\lambda) \begin{bmatrix} 1 \\ h(z) \\ k(z) \end{bmatrix}.
$$

The matrix $U$ is easily computed by comparing the powers of $z$ on both sides. We obtain the *generalized Frobenius matrix*

$$
U = \begin{pmatrix}
0 & 0 & 1 \\
\lambda + h_2 + h_1 & h_1 & 0 \\
2k_2 - h_3 & \lambda + 2k_1 - h_2 & -h_1
\end{pmatrix}.
$$

Therefore the constraints (6.4) are equivalent to the pair of Riccati equations

$$
h(z)_x + h(z)k(z) = z^3 + h_1 h(z) + (h_2 + k_1) \quad \text{(6.6)}$$

$$
k(z)_x + k(z)^2 = -h_1 k(z) + h(z)(2k_1 - h_2 + z^3) - (h_3 - 2k_2). \quad \text{(6.7)}$$

**Proposition 6.2** The $Q^n_3$ constraints allow to express all the fields $\{h_j, k_j\}_{j \geq 3}$ of the $KP^{(2)}$ theory as differential polynomials of the first four components $h_1, h_2, k_1, k_2$. Therefore the space $Q^n_3$ can be identified with the space of generalized Frobenius matrices (6.3) and KdV^2_3 reduces to a hierarchy on four fields, $h_1, h_2, k_1, k_2$.  

---
Proof. The coefficient of \(z^{-1}\) of (6.6) gives
\[
h_3 = h_1^2 - h_{1x} - k_2,
\]
and, in general, the coefficients of \(z^{-i}\) of (6.7) and of \(z^{-(i+1)}\) of (6.6) can be read as a linear system in the unknowns \((k_{i+2}, h_{i+3})\) of the form
\[
\begin{bmatrix}
2 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
k_{i+2} \\
h_{i+3}
\end{bmatrix}
= \begin{bmatrix}
\alpha_i \\
\beta_i
\end{bmatrix}
\]
(6.9)
where the RHS is a \(x\)-differential polynomial in the Laurent coefficients \(\{h_l, k_m\}\) with \(l < i + 3, m < i + 2\).

\[\square\]

To obtain the KdV\(^2\) hierarchy in the form of a zero–curvature equation we consider the vector field \(X_j\) and the associated current \(H(j)\) constructed with the differential Fàa di Bruno algorithm of Section 5. Since this vector field is tangent to \(S_3\) at the points of \(Q_3^2\), there exists a matrix \(V(j)(\lambda)\), whose entries \(v(j)_{\alpha\beta}(\lambda)\) are polynomials in \(\lambda = z^3\) with coefficients depending on the Laurent coefficients \(\{h_i\}\) and \(\{k_i\}\), such that
\[
\left(\frac{\partial}{\partial t_j} + H(j)\right)H(\alpha) = \sum_{\beta=0}^{n-1} v(j)^{\alpha\beta}(\lambda) H(\beta)
\]
(6.10)
for \(\alpha = 0, 1, 2\).

**Proposition 6.3** The KdV\(^2\) hierarchy on the space of generalized Frobenius matrices (6.5) admits the zero–curvature representation
\[
\frac{\partial}{\partial t_j} \mathcal{U} - \frac{\partial}{\partial x} \mathcal{V}(j) + [\mathcal{U}, \mathcal{V}(j)] = 0.
\]
(6.11)

**Proof.** It is enough to cross differentiate equations (6.1) and (6.10) defining the matrices \(\mathcal{U}\) and \(\mathcal{V}(j)\), and to recall the result of Proposition 6.2 about the possibility of expressing the Laurent coefficients \(h_j, k_j, j \geq 3\), in terms \((h_1, h_2, k_1, k_2)\).

\[\square\]

The same procedure can be used for other fractional KdV equations. It should be compared with the approach of [8, 9].
7 On the Bihamiltonian aspects of KdV$^2_3$

To show that our KdV$^2_3$ hierarchy coincides with the fractional $\mathfrak{sl}_3^{(2)}$ KdV hierarchy of \cite{25}, it could suffice to remark that KdV$^2_3$ can be also obtained from KdV$^1_3$ (which is the classical Boussinesq hierarchy) by means of the $x \leftrightarrow t_2$ interchange, in the same way as KP$^{(2)}$ is obtained from KP$^{(1)}$. Actually, this hierarchy was first studied in \cite{7} by means of the Hamiltonian reduction suggested in \cite{20, 8}, and the existence of two compatible Hamiltonian structures was also pointed out. The theory of fractional KdV hierarchy has been further explored and generalized in a number of papers (see, e.g., \cite{8, 9, 11, 12}), by means of the study of its Lie–algebraic aspects.

In this section we will briefly address the problem of showing a direct relation between the KdV$^2_3$ treated in Section 6 and the corresponding generalized $\mathfrak{sl}_3^{(2)}$ DS one of \cite{7, 8}, assuming as a starting point its bihamiltonian aspects. Full proofs and detailed explanations will be given elsewhere \cite{27}. The reader should keep in mind the logical path followed in Section 2, where the KP$^{(1)}$ hierarchy has been derived from the Poisson pencil of KdV.

We recall that a manifold $\mathcal{M}$ is said to be bihamiltonian if it is endowed with two compatible Poisson brackets, and that a Poisson bracket can be assigned in terms of a Poisson tensor $P : T^* \mathcal{M} \to T \mathcal{M}$ as

$$\{ f, g \} = \langle df, P dg \rangle.$$  \hspace{1cm} (7.1)

We consider the Lie algebra $\mathfrak{sl}(3)$ and its loop algebra $\mathfrak{g} = L(\mathfrak{sl}(3))$, i.e., the space of $C^\infty$–maps from $S^1$ to $\mathfrak{sl}(3)$. The algebra $\mathfrak{g}$ is a bihamiltonian manifold (see, e.g., \cite{3, 20}). The Poisson structures we will consider hereinafter are:

$$\langle P_0 \rangle_M \cdot V = [A_2, V]$$  \hspace{1cm} (7.2)

$$\langle P_1 \rangle_M \cdot V = V_x + [V, M].$$  \hspace{1cm} (7.3)

Here, $M$ is a point in $\mathfrak{g}$, $V$ is a cotangent vector at $M$, $x$ is the parameter on $S^1$, and $A_2$ is the element

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ \hspace{1cm} (7.4)

in $\mathfrak{g}$. To lower the number of degrees of freedom, one seeks (following the seminal paper \cite{3}) a Poisson reduction of this bihamiltonian structure. According to the bihamiltonian variant \cite{20} of the Marsden–Ratiu reduction theorem \cite{28}, one has to consider a symplectic leaf $\mathcal{S}$ of $\langle P_0 \rangle$, the (integrable) distribution
$D = P_1(\ker P_0)$, and the intersection $E = D \cap TS$. Then the quotient manifold $N = S/E$ is still a bihamiltonian manifold. In our case, we choose $S$ to be the symplectic leaf passing through the point

$$B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

(7.5)

The submanifold $S$ has dimension 6 over $C^\infty(S^1)$, and its generic point can be parametrized as

$$S = \begin{pmatrix} p_0(x) & p_2(x) & 1 \\ q_0(x) & p_1(x) - p_0(x) & -p_2(x) \\ q_2(x) & q_1(x) & -p_1(x) \end{pmatrix}. $$

(7.6)

Then we consider the distribution $D$, which turns out to be tangent to $S$, so $E = D$. The quotient manifold $N = S/E$ is parametrized by the four fields

$$
\begin{align*}
    u_0 &= -p_0 + p_1 + p_2^2 \\
    u_1 &= q_0 + 2p_0p_2 - p_1p_2 - p_2^3 + p_2x \\
    u_2 &= q_1 - p_0p_2 + 2p_1p_2 + p_2^3 + p_2x \\
    u_3 &= q_2 + p_0x - p_2p_{2x} + q_0p_2 - q_1p_2 + p_0p_1 - p_2^4 - 2p_1p_2^2. 
\end{align*}

(7.7)

The reduced Poisson pencil $P^N_\lambda = P^N_1 - \lambda P^N_0$ on $N$ turns out to be

$$P^N_\lambda = \begin{pmatrix}
\frac{2}{3}\partial_x & -u_1 - \lambda & u_2 + \lambda & \left(\frac{1}{3}\partial_x^2 - u_0\partial_x - u_{0x}\right) \\
* & 0 & (P^N_\lambda)^{23} & (2u_1\partial_x + u_{1x} - 2u_0u_1) - \lambda (-2\partial_x + 2u_0) \\
* & * & 0 & (u_2\partial_x + u_{2x} + 2u_0u_2) - \lambda (-2\partial_x - 2u_0) \\
* & * & * & (P^N_\lambda)^{44}
\end{pmatrix},$$

(7.8)

where

$$(P^N_\lambda)^{23} = -\partial_x^2 + 3u_0\partial_x + 2u_{0x} + u_3 - 2u_0^2$$

$$(P^N_\lambda)^{44} = -\frac{2}{3}\partial_x^3 + \left(\frac{4}{3}u_{0x} + 2u_3 + \frac{2}{3}u_0^2\right)\partial_x + \frac{2}{3}u_0u_{0x} + u_{3x} + \frac{2}{3}u_{0xx}.$$
They are computed according to the standard procedure explained in [28, 20]. The reduced Poisson tensor $P_N^1$ coincides with the one given in [9], after the change of coordinates

$$u_0 = -\tilde{U}, \quad u_1 = \tilde{G}^+, \quad u_2 = \tilde{G}^-, \quad u_3 = \tilde{T} - \tilde{U}^2 + \frac{1}{2} \tilde{U}_x$$

(7.9)

and $x \mapsto -x$.

The basic remark to connect this theory with the KdV $^2_3$ treated in Section 6 is that the quotient space $N$ can be identified with the space of (generalized) Frobenius matrices

$$U = \begin{pmatrix} 0 & 0 & 1 \\ u_1 & u_0 & 0 \\ u_3 & u_2 & -u_0 \end{pmatrix}$$

(7.10)

by noticing that the constraints $p_0 = p_2 = 0$ define a submanifold $U$ of $S$ transversal to the distribution $D$. Consequently, by comparing the matrices (6.5) and (7.10) we arrive at the identification

$$u_0 = h_1, \quad u_1 = h_2 + k_1, \quad u_2 = 2k_1 - h_2, \quad u_3 = 2k_2 - h_3 = 3k_2 - h_1^2 + h_1 x,$$

(7.11)

which can be inverted in local form to yield

$$h_1 = u_0, \quad h_2 = \frac{1}{3}(2u_1 - u_2), \quad k_1 = \frac{1}{3}(u_1 + u_2), \quad k_2 = \frac{1}{3}(u_3 + u_0^2 - u_0 x).$$

(7.12)

This procedure sets up a diffeomorphism

$$\Phi : N \longrightarrow Q_3^2$$

(7.13)

between the reduced bihamiltonian manifold $N$ and the phase space of the KdV $^2_3$ theory. This diffeomorphism enjoys the following properties, which we limit ourselves to state without proofs (they will be detailed elsewhere [27]):

1. The integral on $S^1$

$$H(z) = 3z^2 \int_{S^1} k(z) \, dx,$$

(7.14)

of the pull–back of the second current of the KdV $^2_3$ theory is a Casimir function of the Poisson pencil (7.8).

2. The push–forward of the Hamiltonian vector fields on $N$ associated with this Casimir function are the KdV $^2_3$ flows (5.11) on $Q_3^2$. Hence, we are allowed to identify the two theories.
This result can be usefully compared with the KdV case briefly treated in Section 2. The KdV \((=\text{KdV}_1)\) theory is a theory in a single field \(u\), defined on the quotient space \(N^2_1\) associated with the Lie–Poisson pencil on the loop algebra of \(\mathfrak{sl}_2\) and with the matrices
\[
A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\tag{7.15}
\]

The KdV\(_3\) theory is a theory on four fields, \((u_0, u_1, u_2, u_3)\) defined on the quotient space \(N^2_3\) associated with the Lie–Poisson pencil on the loop algebra of \(\mathfrak{sl}_3\) and with the matrices
\[
A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\tag{7.16}
\]

In the first case the Casimir function of the reduced Poisson pencil is defined by the solution \(h(z)\) of the single Riccati equation
\[
h_x + h^2 = u + z^2.
\tag{7.17}
\]

In the latter case, the Casimir function is computed by solving the pair of Riccati equations
\[
h_x + kh = u_0 h + (u_1 + \lambda)
\]
\[
k_x + k^2 = -u_0 k + (u_2 + \lambda) h + u_3,
\tag{7.18}
\]
in the first two currents \(H^{(1)} = h(z), \ H^{(2)} = k(z)\). The appearance of these systems of Riccati equations is a general feature of KdV\(_m\) theories with \(m \geq 2\) which, in our opinion, deserves further attention.

When the point \(u\) evolves according to the KdV hierarchy, the solution \(h\) of the Riccati equation (7.17) evolves according to the KP\(^{(1)}\) equations
\[
\frac{\partial}{\partial t_j} h = -\pi_-(h H^{(j)})
\tag{7.19}
\]
defined in Section 2. Similarly, when the point \((u_0, u_1, u_2, u_3)\) evolves according to the KdV\(_3\) hierarchy, the solution \((h, k)\) of the Riccati system (7.18) evolves according to the KP\(^{(2)}\) equations
\[
\frac{\partial}{\partial t_j} h = -\pi_-(h H^{(j)})
\]
\[
\frac{\partial}{\partial t_j} k = -\pi_-(k H^{(j)})
\tag{7.20}
\]
In both cases the current densities $H^{(j)}$, which are constructed in a different way in the two theories evolve in time according to the central system (1.2). This completes the view of the relations connecting the central system and the fractional KdV hierarchies we discussed in the case of KdV$_2^3$.

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