SINGULAR MULTIPLE INTEGRALS AND NONLINEAR POTENTIALS

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Abstract. We prove sharp partial regularity criteria of nonlinear potential theoretic nature for the Lebesgue-Serrin-Marcellini extension of nonhomogeneous singular multiple integrals featuring $(p, q)$-growth conditions.

1. Introduction

We provide optimal partial regularity criteria for relaxed minimizers of nonhomogeneous, singular multiple integrals of the form
\begin{equation}
W^{1,p}(\Omega, \mathbb{R}^N) \ni w \mapsto \mathcal{I}(w; \Omega) := \int_{\Omega} [F(Dw) - f \cdot w] \, dx,
\end{equation}
i.e., local minimizers of the Lebesgue-Serrin-Marcellini extension of $\mathcal{I}(\cdot)$:
\begin{equation}
\mathcal{I}(w; \Omega) := \inf_{j \to \infty} \left\{ \liminf_{j \to \infty} \mathcal{I}(w_j; \Omega) : \{ w_j \}_{j \in \mathbb{N}} \subset W^{1,q}_{loc}(\Omega, \mathbb{R}^N) \ni w_j \to w \text{ in } W^{1,p}(\Omega, \mathbb{R}^N) \right\},
\end{equation}
using tools from Nonlinear Potential Theory, thus completing the analysis started in [23] for degenerate functionals. More precisely, we prove almost everywhere gradient continuity for local minimizers of (1.2) under sharp assumptions on the external datum $f$, thus completing the analysis started in [23] for degenerate functionals. More precisely, we prove almost everywhere gradient continuity for local minimizers of the Lebesgue-Serrin-Marcellini extension of $\mathcal{I}(\cdot)$:
\begin{equation}
\mathcal{I}(w; \Omega) := \inf_{j \to \infty} \left\{ \liminf_{j \to \infty} \mathcal{I}(w_j; \Omega) : \{ w_j \}_{j \in \mathbb{N}} \subset W^{1,q}_{loc}(\Omega, \mathbb{R}^N) \ni w_j \to w \text{ in } W^{1,p}(\Omega, \mathbb{R}^N) \right\},
\end{equation}
using tools from Nonlinear Potential Theory, thus completing the analysis started in [23] for degenerate functionals. More precisely, we prove almost everywhere gradient continuity for local minimizers of $\mathcal{I}(\cdot)$ under sharp assumptions on the external datum $f$. Here, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with Lipschitz boundary, $n \geq 2$, and $F: \mathbb{R}^{N \times n} \to \mathbb{R}$ is a strictly quasiconvex integrand, verifying so-called $(p, q)$-growth conditions according to Marcellini’s terminology [60]:
\begin{equation}
|z|^p \lesssim F(z) \lesssim |z|^q, \quad 1 < p \leq q;
\end{equation}
the singular behavior of $F(\cdot)$ around zero being encoded in the requirement $p \in (1, 2)$. Let us recall that $F(\cdot)$ is quasiconvex when
\begin{equation}
\int_{B_1(0)} F(z + D\varphi) \, dx \geq F(z) \quad \text{for all } z \in \mathbb{R}^{N \times n}, \quad \varphi \in C_c^\infty(B_1(0), \mathbb{R}^N),
\end{equation}
therefore the three main aspects of (1.3)–(1.2) we are interested in are the presence of a nontrivial forcing term $f$, the $(p, q)$-growth conditions in (1.3) and the quasiconvexity (1.4) of the integrand $F(\cdot)$. Let us briefly discuss some classical and recent results on these ingredients as each of them is currently object of intense investigation.

The problem of determining the best conditions to impose on $f$ in order to prove gradient continuity for minima is classical and received a considerable attention in the past decades. To better understand this issue, let us introduce the Lorentz space $L(n, 1)$, defined by
\begin{equation}
w \in L(n, 1) \iff \|w\|_{L(n, 1)} := \int_0^\infty |\{ x \in \mathbb{R}^n : |w(x)| > t \}|^{1/n} \, dt < \infty.
\end{equation}
A related deep result of Stein [74] states that
\begin{equation}
w \in W^{1,n} \quad \text{and} \quad Dw \in L(n, 1) \implies w \text{ is continuous},
\end{equation}
so (1.5) and the immersions $L^{n+\varepsilon} \hookrightarrow L(n, 1) \hookrightarrow L^n$ for all $\varepsilon > 0$, lead to the borderline characterization of $L(n, 1)$ as the limiting space with respect to the Sobolev embedding theorem. A linear PDE interpretation of Stein’s theorem relying on the combination of (1.5) with standard Calderón-Zygmund theory prescribes that
\begin{equation}
-\Delta u = f \in L(n, 1) \implies Du \text{ is continuous},
\end{equation}
which turns out to be sharp, in the light of Cianchi’s counterexample [19]. Surprisingly enough, the same conclusion holds in a way more general setting than the linear one. It is indeed true for uniformly elliptic operators [31, 30, 21, 30, 52, 53, 55, 60, 67, 68], systems of differential forms [73], fully nonlinear elliptic equations [9, 27], general nonuniformly elliptic functionals [8, 24], and it also holds at the level of partial regularity for

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\end{footnotesize}
systems of the $p$-Laplacian type without Uhlenbeck’s structure \cite{16, 15}. The key point consists in the possibility of gaining local control on the gradient of solutions via the truncated Riesz potential of $f$, that is

\begin{equation}
I_{\psi}(x_0, \phi) := \int_0^\infty \left( \frac{\int_{B_r(x_0)} |f| \, dx}{\int_{\mathbb{R}^n} |x-y|^m \, dy} \right) \, d\sigma \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^m} \, dy,
\end{equation}

which is a standard aspect of linear equations and a remarkable feature of nonlinear ones, cf. Kuusi & Mingione’s seminal works \cite{54, 55}. On the other hand, in \cite{8, 23} the gradient of minima is dominated via a nonlinear Wolff type potential, first introduced by Havin & Max’ya \cite{43}, defined as:

\begin{equation}
I_{\psi,m}(x_0, \phi) := \int_0^\infty \left( \sigma^m \int_{B_r(x_0)} |f|^m \, dx \right)^{1/m} \frac{d\sigma}{\sigma}, \quad m > 1,
\end{equation}

sharing the same homogeneity - and therefore analogous mapping properties on function spaces - as the linear potential in \cite{16}. All the aforementioned results crucially rely on the strong ellipticity of the operators involved, while in \cite{16} the integrand $F(\cdot)$ is only quasiconvex. This notion was first introduced by Morrey \cite{65} and it turns out to be a natural condition in the multidimensional Calculus of Variations. Indeed, under polynomial growth conditions on the integrand $F(\cdot)$, quasiconvexity is a necessary and sufficient condition for sequential weak lower semicontinuity in $W^{1,p}$. \cite{16, 15, 17, 18}

A peculiar characteristic of quasiconvexity is that it is a purely nonlocal concept \cite{18, 55} in the sense that there is no condition involving only $F(\cdot)$ and a finite number of its derivatives, which is equivalent to quasiconvexity. Moreover, minimizers and critical points of quasiconvex functionals have a very different behavior. Precisely, a classical result of Evans \cite{56} states that minima are regular outside a negligible "singular" set, while Müller & Šverák \cite{55} proved that critical points, i.e. solutions to the associated Euler-Lagrange system, may be everywhere discontinuous. This is coherent with the theory of elliptic systems: well-known counterexamples \cite{61, 70} show that solutions might develop singularities, therefore in the genuine vectorial setting the best one could hope for is partial regularity. The matter of almost everywhere regularity for minimizers of quasiconvex integrals was first treated by Evans \cite{56} in the case of quadratic functionals, and, after that, it received lots of attention over the years. Subsequently, partial regularity for integral with standard $p$-growth was obtained in \cite{16, 17, 50} exploiting Evans’ blow up method, while in \cite{17} a unified approach to the partial regularity for degenerate or singular quasiconvex integrals was proposed via the $p$-harmonic approximation and in \cite{19} was derived an upper bound on the Hausdorff dimension of the singular set of minima of quasiconvex functionals. We refer to \cite{9, 13, 23, 28, 31, 10, 22, 15, 47, 67, 76, 72} and references therein for a non-exhaustive list of remarkable contributions in more general settings. The other main feature of the class of integrands considered in this paper is their $(p,q)$-growth conditions. This nomenclature was introduced by Marcellini in the fundamental papers \cite{60, 63} within the framework of nonlinear elasticity. In fact, a basic model describing the behavior of compressible materials subject to deformations is given by

\begin{equation}
W^{1,p}(\Omega, \mathbb{R}^n) \ni w \mapsto H(w; \Omega) := \int_{\Omega} \left[ |Dw|^p + \sqrt{1 + |\det(Dw)|^2} - f \cdot w \right] \, dx,
\end{equation}

for some $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, see \cite{45, 50, 63, 64}. A natural phenomenon in compressible elasticity is cavitation, i.e. the possible formation of cavities (holes) in elastic bodies after stretch, corresponding to the development of singularities in equilibrium solutions (minima) of $F(\cdot)$. Functional $\tilde{H}(\cdot)$ is quasiconvex in the sense of \cite{16, 59} Chapter 5, however in general it is not $W^{1,p}$-quasiconvex unless $p \geq n$, \cite{5} Theorem 4.1, while its Lebesgue-Serrin-Marcellini extension $\tilde{H}(\cdot)$ is $W^{1,p}$-quasiconvex provided that $p > n - 1$. \cite{71} Lemma 7.6. This means that the approach by relaxation based on the extension of the ambient space proposed in \cite{60} fits the analysis of cavitation better than the pointwise one of \cite{61}, as it allows dealing with discontinuous maps thus describing the possible formations of cavities. We also point out that the integrand $H(z) := |z|^p + \sqrt{1 + |\det(z)|^2}$ in \cite{16} verifies

\[ |z|^p \leq H(z) \lesssim 1 + |z|^n, \]

that is \cite{16} with $q = n$. This was the first main reason behind the investigation of variational integrals with $(p,q)$-growth: starting with \cite{60} for questions of semicontinuity and \cite{61, 62} about regularity, since then such class of functionals received lots of attention - with no pretence of completeness we mention the everywhere regularity results in \cite{8, 10, 24, 26, 35, 13, 31, 52}, the partial regularity proven in \cite{18, 22, 23, 29, 69} for general systems and manifold constrained problems with special structure and refer to \cite{59} for a reasonable survey. The aforementioned results hold for strictly convex variational integrals. In the quasiconvex setting partial regularity has been obtained by Schmidt \cite{70, 72} for homogeneous - $f \equiv 0$ in \cite{16} - functionals with $(p,q)$-growth and for their Lebesgue-Serrin-Marcellini extension \cite{71}, while \cite{23} contains sharp partial regularity criteria in terms of a nontrivial forcing term $f$ for relaxed minimizers of degenerate integrals of the form \cite{16}. The standard notion of relaxed local minimizers \cite{71} reads as:

\footnote{\text{1.e.: } \quad \text{holds for all } \varphi \in W^{1,p}_0(B_1(0), \mathbb{R}^n).}
Definition 1. Let \( p \in (1, \infty) \). A function \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) is a local minimizer of \( I_m \) on \( \Omega \) with \( f \in W^{1,p}(\Omega, \mathbb{R}^N)^* \) if and only if every \( x_0 \in \Omega \) admits a neighborhood \( B \Subset \Omega \) so that \( \mathcal{F}(u; B) < \infty \) and \( \mathcal{F}(u; B) \leq \mathcal{F}(w; B) \) for all \( w \in W^{1,p}(B, \mathbb{R}^N) \) so that \( \text{supp}(u - w) \in B \).

Such definition can be immediately adapted to local minimizers of functional \( I_m \). Let us point out that when considering \( I_m \) we will assume with no loss of generality that \( f \) is defined on the whole \( \mathbb{R}^n \), which is always possible if we set \( f \equiv 0 \) in \( \mathbb{R}^n \setminus \Omega \). For this reason, when stating that \( f \) belongs to a certain function space, we shall often avoid to specify the underlying domain. Further details about the notation employed can be found in \( \text{[14]} \) below. The main result of our paper is the following

Theorem 1. Under assumptions \( 2.12, 2.14, 2.16 \) and \( 2.22 \), let \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) be a local minimizer of \( I_m \). Suppose that
\[
\lim_{\varepsilon \to 0} I_{m,\varepsilon}^f(x, \varrho) = 0 \quad \text{locally uniformly in } x \in \Omega.
\]
Then there exists an open "regular" set \( \Omega_0 \subset \Omega \) of full \( n \)-dimensional Lebesgue measure with \( |\Omega \setminus \Omega_0| = 0 \) such that \( V_p(Du) \) and \( Du \) are continuous on \( \Omega_0 \). In particular, the regular set \( \Omega_0 \) can be characterized as
\[
\Omega_0 := \left\{ x_0 \in \Omega : \exists M \geq M(x_0) \in (0, \infty), \varepsilon \equiv \varepsilon(\text{data}, M), \varepsilon \equiv \varepsilon(\text{data}, M, f(\cdot)) \in (0, \min\{d_{x_0}, 1\}) \text{ such that } |(V_p(Du))_{B_{\varepsilon}(x_0)}| < M \right\}.
\]

Theorem 1 comes as a consequence of a fine establishment between the Lebesgue points of \( Du \) and \( V_p(Du) \) and the pointwise behavior of the Wolff potential \( I_{m,\varepsilon}^f(x) \).

Theorem 2. Under assumptions \( 2.12, 2.14, 2.16 \) and \( 2.22 \), let \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) be a local minimizer of \( I_m \), \( x_0 \in \Omega \) be a point such that
\[
I_{m,\varepsilon}^f(x_0, 1) < \infty
\]
and \( M \equiv M(x_0) \) be a positive constant. There exist \( \varepsilon \equiv \varepsilon(\text{data}, M) \in (0, 1) \) and \( \tilde{\varepsilon} \equiv \tilde{\varepsilon}(\text{data}, M, f(\cdot)) \in (0, \min\{1, d_{x_0}\}) \) such that if
\[
|\tilde{\varepsilon}(u; B_{\varepsilon}(x_0)) + I_{m,\varepsilon}^f(x_0, \varrho) \frac{\varepsilon^q}{\varepsilon^{qN}} + I_{m,\varepsilon}^f(x_0, \varrho) \frac{\varepsilon^q}{\varepsilon^{qN}} + M^{(2-p)/p}I_{m,\varepsilon}^f(x_0, \varrho) < \varepsilon,
\]
for some \( \varrho \in (0, \tilde{\varepsilon}) \), then
\[
\lim_{\varepsilon \to 0} (V_p(Du))_{B_{\varepsilon}(x_0)} = V_p(Du(x_0)), \quad \lim_{\varepsilon \to 0} (Du)_{B_{\varepsilon}(x_0)} = Du(x_0)
\]
and
\[
|V_p(Du(x_0)) - (V_p(Du))_{B_{\varepsilon}(x_0)}| \leq c\mathcal{N}(x_0; \sigma)
\]
\[
|Du(x_0) - (Du)_{B_{\varepsilon}(x_0)}| \leq c\mathcal{N}(x_0; \sigma)^{2/p} + c[(Du)_{B_{\varepsilon}(x_0)}]^{(2-p)/2}\mathcal{N}(x_0; \sigma),
\]
for all \( \sigma \in (0, \varrho) \), where \( c \equiv c(\text{data}, M) \) and
\[
\mathcal{N}(x_0; \sigma) \approx \tilde{\varepsilon}(u; B_{\varepsilon}(x_0)) + I_{m,\varepsilon}^f(x_0, \varrho) \frac{\varepsilon^q}{\varepsilon^{qN}} + I_{m,\varepsilon}^f(x_0, \varrho) \frac{\varepsilon^q}{\varepsilon^{qN}} + [(V_p(Du))_{B_{\varepsilon}(x_0)}]^{(2-p)/p}I_{m,\varepsilon}^f(x_0, \varrho),
\]
up to constants depending on \( (\text{data}, M) \). In particular, \( x_0 \in \Omega \) satisfying \( 1.9 \) is a Lebesgue point of \( V_p(Du) \) and \( Du \) if and only if it verifies \( 1.10 \).

Conditions \( 1.8 \) or \( 1.9 \) can be guaranteed once prescribed the membership of \( f \) to a proper function space, as stated in the following optimal function space criterion.

Theorem 3. Under assumptions \( 2.12, 2.14, 2.16 \) and \( 2.22 \), let \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) be a local minimizer of \( I_m \). There exists an open set \( \Omega_0 \subset \Omega \) of full \( n \)-dimensional Lebesgue measure such that \( f \in L^q(\Omega, \mathbb{R}^N) \) yields that \( Du, V_p(Du) \) are continuous on \( \Omega_0 \), while if \( f \in L^d(\Omega, \mathbb{R}^N) \) for some \( d > n \), then \( Du, V_p(Du) \in C^{0,\alpha}_{loc}(\Omega_0, \mathbb{R}^{N \times n}) \) with \( \alpha \equiv \alpha(n, N, p, d) \).

We stress that Theorems \( 1.3 \) hold for general strictly \( W^{1,p} \)-quasiconvex functionals\( I_m \) as in \( 1.11 \), or for functionals that coincide with their Lebesgue-Serrin-Marchielli extension. Moreover, our results also apply to relaxed local minimizer of the functional \( \mathcal{F}(\cdot) \) in \( 1.7 \) with the choice \( n = q = 2 \) and \( p \in (8/5, 2] \), cf. \( 23 \) Section 2.4; we refer to \( 24 \) for further discussions and more examples. We finally remark that the nonlinear potential theory for singular nonhomogeneous equations or systems of the \( p \)-Laplacian type is a very recent achievement.
In fact, after Duzaar & Mingione’s breakthrough \cite{32} on pointwise potential estimates with \( p \in (2 - 1/n, \infty) \), lots of efforts have been devoted to the extension of such result to all \( p \in (1, 2) \): in \cite{67} Nguyen & Phuc decreased the lower bound on \( p \) switching from \( p > 2 - 1/n \) to \( p > \frac{2n-2}{n+2} \), later on Dong & Zhu \cite{30} and Nguyen & Phuc \cite{65} (singular equations with measure data) and Byun & Youn \cite{10} (general subquadratic systems) eventually covered the full range \( p \in (1, 2) \). In this respect, our paper fits such line of research as we provide pointwise bounds on the gradient oscillation of minima of \( (2.2) \) that hold almost everywhere subject to the validity of a smallness condition on the excess functional that naturally involves also the potential \( I_{\text{min}}(\cdot) \). The combined effect of the singular character of the integrand \( F(\cdot) \) that unavoidably burdens potentials, and of its \((p, q)\)-growth leading to nonhomogeneous estimates, forces us to develop some delicate iteration schemes, relying on refined approximation methods \cite{53, 12} and on potential theoretic arguments \cite{23, 55, 50}, that simultaneously allow a uniform control on the size of the excess functional and of the gradient average during iterations and preserves the rough regularity information available on \( f \), i.e. the mere finiteness of the related Wolff type potential.

## 2. Preliminaries

In this section we record the notation employed throughout the paper, describe the structural assumptions governing the ingredients appearing in \cite{111} and collect certain basic results that will be helpful later on.

### 2.1. Notation

In this paper, \( \Omega \subset \mathbb{R}^n \) will always be an open, bounded domain with Lipschitz-regular boundary, and \( n \geq 2 \). We denote by \( c \) a general constant larger than one depending on the main parameters governing the problem. We will still denote by \( c \) distinct occurrences of constant \( c \) from line to line. Specific occurrences will be marked with symbols \( c, \tilde{c} \) or the like. Significant dependencies on certain parameters will be outlined by putting them in parentheses, i.e. \( c \equiv c(n, p) \) means that \( c \) depends on \( n \) and \( p \). By \( B_r(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \} \) we indicate the open ball with center in \( x_0 \) and radius \( r > 0 \); we shall avoid denoting the center when this is clear from the context, i.e., \( B \equiv B_r \equiv B_r(x_0) \); this happens in particular with concentric balls. For \( x_0 \in \Omega \), it is \( d_{x_0} := \text{dist}(x_0, \partial \Omega) \) and with \( z_1, z_2 \in \mathbb{R}^{\ast n} \), \( s \geq 0 \) we set \( D_s(z_1, z_2) := (s^2 + |z_1|^2 + |z_2|^2) \). Given a measurable set \( B \subset \mathbb{R}^n \) with bounded positive Lebesgue measure \( |B| \in (0, \infty) \), and a measurable map \( g : B \to \mathbb{R}^k \), \( k \geq 1 \), we set

\[
(g)_{B} := \frac{1}{|B|} \int_B g(x) \, dx.
\]

A useful feature of the average is its almost minimality, i.e.:

\[ (2.1) \quad \left( \frac{1}{|B|} \int_B |g - (g)_{B}| \, dx \right)^{1/t} \leq 2 \left( \frac{1}{|B|} \int_B |g - z| \, dx \right)^{1/t} \quad \text{for all} \quad z \in \mathbb{R}^k, \ t \geq 1. \]

For \( t \geq 1, s \geq 0, q \geq p > 1 \), we shorten:

\[ (2.2) \quad \mathfrak{I}_t(s; \mathcal{B}) := \left( \frac{1}{|B|} \int_B |g(x)|^t \, dx \right)^{1/t}, \quad \mathfrak{R}(s) := s + s^{q/p} \]

and define

\[
\mathfrak{I}_{(q > p)} := \begin{cases} 1 & \text{if } q > p \\ 0 & \text{if } q = p \end{cases}, \quad \mathfrak{I}_{(q \geq 2)} := \begin{cases} 1 & \text{if } q \geq 2 \\ 0 & \text{if } 1 < q < 2 \end{cases}
\]

Finally, if \( t > 1 \) is any number, its conjugate will be denoted by \( t' := t/(t - 1) \) and its Sobolev exponent as \( t^* := nt/(n - t) \) when \( t < n \) or any number larger than one for \( t \geq n \). To streamline the notation, we gather together the main parameters governing our problem in the shorthand data := \((n, N, \lambda, \Lambda, p, q, \omega(\cdot), F(\cdot), m)\), we refer to Section \ref{section_2} for more details on such quantities.

### 2.2. Tools for \( p \)-Laplacian type problems

The vector field \( V_{s,p} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \), defined as

\[ V_{s,p}(z) := (s^2 + |z|^2)^{(p-2)/4} z, \quad p \in (1, \infty) \quad \text{and} \quad s \geq 0 \]

for all \( z \in \mathbb{R}^{n \times n} \), which encodes the scaling features of the \( p \)-Laplacian operator, is a useful tool for handling \( p \)-Laplacian type problems. If \( s = 0 \), we simply write \( V_{0,p}(\cdot) \equiv V_p(\cdot) \). Let us premise that although most of the properties of the vector field \( V_{s,p}(\cdot) \) that we are going to list below hold for all \( p \in (1, \infty) \), from now on, we
shall always assume that \( p \in (1, 2) \). It is well-known that
\[
\begin{align*}
&\frac{|V_{s,p}(z_1 + z_2)|^2}{|z_1 + z_2|} \lesssim \frac{|V_{s,p}(z_1)|^2 + |V_{s,p}(z_2)|^2}{|z_1| + |z_2|} \\
&\frac{|V_{s,p}(z_1)|^2}{|z_1|} - \frac{|V_{s,p}(z_1) - V_{s,p}(z_2)|}{|z_1| - |z_2|} \approx (s^2 + |z_1|^2 + |z_2|^2)^{(p-2)/2} |z_1 - z_2| \\
&\frac{|V_{s,p}(z_1)|^2}{|z_1|} \approx \frac{|V_{s,p}(z_1)|^2 + |V_{s,p}(z_2)|^2}{|z_1|} \\
&|V_{s,p}(z_1)| \leq \max\{|z_1|, |z_2|^{p/2}\} \\
&|V_{s,p}(z_2)| \approx \min\{|z_1|, |z_2|^{p/2}\}
\end{align*}
\]
(2.3)
for all \( k > 0 \). - of course to avoid trivialities, above \( |z_1 + z_2|, |z_1| \) are supposed to be positive - and that whenever \( t > -1, s \in [0, 1] \) and \( z_1, z_2 \in \mathbb{R}^{N+n} \) verify \( s + |z_1| + |z_2| > 0 \), then
\[
\int_0^1 \left( s^2 + |z_1 + y(z_2 - z_1)|^2 \right)^{\frac{p}{2}} dy \approx (s^2 + |z_1|^2 + |z_2|^2)^{\frac{p}{2}}.
\]
(2.4)
As useful consequences of (2.3) we have
\[
|z_1 - z_2|^p \lesssim |V_{s,p}(z_1) - V_{s,p}(z_2)|^2 + |V_{s,p}(z_1) - V_{s,p}(z_2)|^p(|z_1| + s)^{(p-2)/2}.
\]
(2.5)
It is also worth recalling a Poincaré-type inequality involving the vector field \( V_{s,p}() \): with \( p \in (1, 2) \), \( B_p(x_0) \in \Omega \) and \( w \in W^{1,p}(B_p(x_0), \mathbb{R}^N) \) it is
\[
\int_{B_p(x_0)} \left| V_{s,p} \left( \frac{w - (w)_{B_p(x_0)}}{p} \right) \right|^p \, dx \leq c \left( \int_{B_p(x_0)} |V_{s,p}(Dw)|^2 \, dx \right)^{p/2},
\]
(2.6)
with \( p^* := 2n/(n - p) \) and \( c \equiv c(n, N, p) \), see [35] Lemma 8. For \( B_p(x_0) \in \Omega, w \in W^{1,p}(B_p(x_0), \mathbb{R}^N) \) and \( z_0 \in \mathbb{R}^N \), we define the excess functional by
\[
\mathfrak{F}(w, z_0; B_p(x_0)) := \left( \int_{B_p(x_0)} |V_p(Dw) - z_0|^2 \, dx \right)^{1/2}
\]
and further introduce the auxiliary integral
\[
\mathfrak{F}(w, z_0; B_p(x_0)) := \left( \int_{B_p(x_0)} |V_{s|_{|z_0|,p}}(Dw - z_0)|^2 \, dx \right)^{1/2}.
\]
(2.7)
If \( z_0 = (V_p(Dw))_{B_p(x_0)} \) we shall abbreviate \( \mathfrak{F}(w, V_p(Dw))_{B_p(x_0)}; B_p(x_0)) \equiv \mathfrak{F}(w; B_p(x_0)) \). Let us point out that combining (2.1) with [35] (2.6) it holds that
\[
\mathfrak{F}(w; B_p(x_0)) \approx \mathfrak{F}(w, V_p(Dw))_{B_p(x_0)}; B_p(x_0)),
\]
while if \( z_0 = (V_p)^{-1}((V_p(Dw))_{B_p(x_0)}) - \) recall that \( V_p() \) is an isomorphism of \( \mathbb{R}^{N+n} \) - via (2.3) we have
\[
\mathfrak{F}(w; B_p(x_0)) \approx \mathfrak{F}(w, z_0; B_p(x_0)).
\]
(2.8)
In all the above displays, the constants implicit in \( ^n \approx, \lesssim^n \) depend on \( (n, N, p, t) \) - the dependency on \( t \) accounts for the exponent appearing in (2.4). To the scopes of this paper, it is fundamental to record some well-known scaling features of the excess functional.

Lemma 2.1. Let \( 1 < p < 2 \) be a number, \( B_p(x_0) \subset \mathbb{R}^n \) be a ball, and \( w \in W^{1,p}(B_p(x_0), \mathbb{R}^N) \) be any function. With \( \nu \in (0, 1) \) it holds that
\[
\mathfrak{F}(w; B_{\nu p}(x_0)) \leq \frac{2}{\nu^{1/2}} \mathfrak{F}(w; B_p(x_0))
\]
(2.9)
\[
|(V_p(Dw))_{B_{\nu p}(x_0)}| \leq \frac{1}{\nu^{1/2}} \mathfrak{F}(w; B_p(x_0)) + |(V_p(Dw))_{B_p(x_0)}|
\]
\[
|(V_p(Dw))_{B_{\nu p}(x_0)} - (V_p(Dw))_{B_p(x_0)}| \leq \frac{1}{\nu^{1/2}} \mathfrak{F}(w; B_p(x_0)).
\]
Moreover, if for \( \sigma \leq g \) there is \( k \in \mathbb{N} \cup \{0\} \) satisfying \( \nu^{n+1} g < \sigma \leq \nu^n g \), then
\[
(2.10) \quad \exists h(w; B_{\nu^{n+1}g}(x_0)) \leq \frac{2}{\nu^{n+1/2}} \Lambda \leq \frac{2^2}{\nu^n} \exists h(w; B_{\nu^n g}(x_0))
\]
\[
[(V_p(Dw))_{B_{\nu^{n+1}g}(x_0)}] \leq [(V_p(Dw))_{B_{\nu^n g}(x_0)}] + \frac{1}{\nu^{n+1/2}} h(w; B_{\nu^n g}(x_0)) \leq [(V_p(Dw))_{B_{\nu^n g}(x_0)}] + \frac{2^2}{\nu^n} \exists h(w; B_{\nu^n g}(x_0)),
\]
and, whenever \( c_x \geq 1 \) is an absolute constant it is
\[
[(V_p(Dw))_{B_{\nu^n g}(x_0)}] + c_x \exists h(w; B_{\nu^n g}(x_0)) \leq \frac{2^2}{\nu^n} \left( [(V_p(Dw))_{B_{\nu^n g}(x_0)}] + c_x \exists h(w; B_{\nu^n g}(x_0)) \right)
\]
\[
(2.11) \quad \leq \frac{q}{p} \left( [(V_p(Dw))_{B_{\nu^n g}(x_0)}] + c_x \exists h(w; B_{\nu^n g}(x_0)) \right).
\]

We conclude this section with a classical iteration lemma, [39, Lemma 6.1].

**Lemma 2.2.** Let \( h : [\theta_0, \theta_1] \to \mathbb{R} \) be a non-negative and bounded function, and let \( \theta \in (0,1) \), \( A, B, \gamma_1, \gamma_2 \geq 0 \) be numbers. Assume that \( h(t) \leq \theta h(s) + A(s-t)^{-\gamma_1} + B(s-t)^{-\gamma_2} \) holds for all \( \theta_0 \leq t < s \leq \theta_1 \). Then the following inequality holds \( h(\theta_1) \leq c(\theta, \gamma_1, \gamma_2) [A(\theta_1 - \theta_0)^{-\gamma_1} + B(\theta_1 - \theta_0)^{-\gamma_2}] \).

### 2.3. Structural assumptions.

We assume that \( F : \mathbb{R}^{n \times n} \to \mathbb{R} \) is an integrand verifying:
\[
(2.12) \quad \left\{ \begin{array}{l}
F \in C^{1,0}_{\text{loc}}(\mathbb{R}^{n \times n}) \cap C^{1}_{\text{loc}}(\mathbb{R}^{n \times n} \setminus \{0\}) \\
\Lambda^{-1} |\nu|^{p} \leq F(\nu) \leq \Lambda (1 + |\nu|^{p})
\end{array} \right.
\]

for all \( \nu \in \mathbb{R}^{n \times n} \), with \( \Lambda \geq 1 \) being a positive, absolute constant and exponents \((p, q)\) satisfying:
\[
(2.13) \quad 1 < p < 2 \quad \text{and} \quad \frac{q}{p} < 1 + \frac{1}{2n}.
\]

It is fundamental that \( F(\cdot) \) is strictly degenerate quasiconvex, in the sense that whenever \( B \in \Omega \) is a ball it holds that
\[
(2.14) \quad \int_B \left[ F(\nu + \nu B) - F(\nu) \right] \, d\nu \geq \lambda \int_B (|\nu|^{2} + |\nu B|^{2}) \, d\nu \quad \text{for all} \quad \nu \in \mathbb{R}^{n \times n}, \quad \nu \in C_{c}^{\infty}(B, \mathbb{R}^{n})
\]

where \( \lambda \) is a positive, absolute constant. As a consequence, for all \( \nu \in \mathbb{R}^{n \times n} \setminus \{0\} \), \( \xi \in \mathbb{R}^{n} \), \( \zeta \in \mathbb{R}^{n} \) it holds that
\[
(2.15) \quad \partial^2 F(z)(\xi \otimes \zeta, \nu \otimes \zeta) \geq 2\lambda |\nu|^{p-2} |\xi|^{2} |\zeta|^{2},
\]

see [39, Chapter 5] or [71, Lemma 7.14]. Moreover, as a minimal requirement on the second derivatives of \( F(\cdot) \), we need to prescribe their behavior near the origin. Precisely, we need that
\[
(2.16) \quad \frac{|\partial^2 F(z) - \partial^2 F(0)|}{|\nu|^{p-2}} \to 0 \quad \text{uniformly as} \quad |\nu| \to 0.
\]

The by-product of (2.16) is summarized in the following lemma, that collects results from [1, 55, 72].

**Lemma 2.3.** Let \( F : \mathbb{R}^{n \times n} \to \mathbb{R} \) be an integrand verifying (2.12), (2.13), and (2.15). Then,

- there exists a positive constant \( c \equiv c(n, \Lambda, q) \) such that
\[
(2.17) \quad |\partial F(z)| \leq c (1 + |\nu|^{q-1}) \quad \text{for all} \quad z \in \mathbb{R}^{n \times n};
\]
- whenever \( z_0 \in \mathbb{R}^{n \times n} \) verifies \( |z_0| \leq L + 1 \) for some positive constant \( L \), it is
\[
(2.18) \quad \left\{ \begin{array}{l}
|F(z_0 + z) - F(z_0) - \partial F(z_0) z| \leq c \left( |V_{z_0,1,p}(z)|^{2} + |V_{z_0,1,q}(z)|^{2} \right) \\
|\partial F(z_0 + z) - \partial F(z_0)| \leq c |z|^{-1} \left( |V_{z_0,1,p}(z)|^{2} + |V_{z_0,1,q}(z)|^{2} \right)
\end{array} \right.
\]

for all \( z \in \mathbb{R}^{n \times n} \), with \( c \equiv c(n, \Lambda, p, q, F(\cdot), L) \);
- there exists a concave modulus of continuity \( \omega : (0, \infty) \to (0, \infty) \) with \( \lim_{s \to 0} \omega(s) = 0 \) such that
\[
(2.19) \quad |z| \leq \omega(s) \quad \Rightarrow \quad |\partial F(z) - \partial F(0)| - |z|^{p-2} \leq s |z|^{p-1};
\]
- whenever \( L > 0 \) is a positive constant and \( z \in \mathbb{R}^{n \times n} \setminus \{0\} \) it is:
\[
(2.20) \quad |\partial^2 F(z)| \leq c |z|^{p-2},
\]

for \( c \equiv c(n, p, F(\cdot), L) \).
for all positive constants $L$ and vectors $z_1, z_2 \in \mathbb{R}^{N \times n}$ so that $0 < |z_1| \leq L$, $0 < |z_1| \leq 2L$ it holds that

\[
(2.21) \quad |\partial^2 F(z_1) - \partial^2 F(z_2)| \leq \left( \frac{|z_1|^2 + |z_2|^2}{|z_1|^2|z_2|^2} \right)^{(2-p)/2} \mu_L \left( \frac{|z_1 - z_2|^2}{|z_1|^2 + |z_2|^2} \right),
\]

where $\mu_L : (0, \infty) \to (0, \infty)$ is a nondecreasing modulus of continuity with $s \mapsto \mu_L(s)^2$ concave, depending on $F(\cdot)$ and on $L$.

Finally, the forcing term $f : \Omega \to \mathbb{R}^N$ displayed in (1.4) is such that

\[
(2.22) \quad f \in L^m(\Omega, \mathbb{R}^N) \quad \text{with} \quad n > m > (p')' > 1
\]

which, together with (2.23) yields:

\[
(2.23) \quad 1 < m' < p^* < \infty \quad \text{and} \quad f \in W^{1,p}(\Omega, \mathbb{R}^N)^*.
\]

Assumption (2.22) should be interpreted as a minimal integrability requirement on the forcing term $f$, in the sense that it must at least belong to some intermediate Lebesgue space between $L^{p^*}$ and $L^N$. The motivation behind this choice is twofold: the lower bound $m > (p')'$ assures that $f \in (W^{1,p})^*$, so that the linear functional $w \mapsto \int f \cdot w \, dx$ is continuous on $W^{1,p}$; the upper bound $m < n$ reminds that all in all the foregoing estimates $f$ should appear raised to a power strictly less than $n$ - this will eventually contribute to the construction of the Wolff potential $\mathbf{W}^n(\cdot)$, which is well-behaved with respect to the embedding in Lorentz spaces exactly when $m < n$, cf. [32] Section 2.3.

2.4. Harmonic approximation lemmas. This section is devoted to a quick overview of the main features of $\mathcal{A}$-harmonic maps and of $p$-harmonic maps. Let $\mathcal{A}$ be a constant bilinear form on $\mathbb{R}^{N \times n}$, elliptic in the sense of Legendre-Hadamard i.e., satisfying

\[
(2.24) \quad |\mathcal{A}| \leq H \quad \text{and} \quad \mathcal{A}(\xi \otimes \zeta, \xi \otimes \zeta) \geq H^{-1}|\xi|^2|\zeta|^2,
\]

for all $\zeta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^N$, with $H \geq 1$ being an absolute constant. An $\mathcal{A}$-harmonic map on an open set $\Omega \subset \mathbb{R}^n$ is a function $h \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that

\[
\int_{\Omega} \mathcal{A}(Dh, D\varphi) \, dx = 0 \quad \text{for all} \quad \varphi \in C^\infty_c(\Omega, \mathbb{R}^N).
\]

In [17,31] we find that $\mathcal{A}$-harmonic maps have good regularity features; in fact for $B_\sigma(x_0) \Subset \Omega$ with $\sigma \in (0,1]$ it holds that

\[
(2.25) \quad \|Dh\|_{L^\infty(B_{\sigma^2/2}(x_0))} + \|D^2h\|_{L^\infty(B_{\sigma^2/2}(x_0))} \leq c \int_{B_{\sigma}(x_0)} |Dh| \, dx,
\]

for $c \equiv c(n, N, H, d)$. We record an $\mathcal{A}$-harmonic approximation result from [33, Lemma 4], see also [31, Lemma 6].

**Lemma 2.4.** Let $\mathcal{A}$ be a bilinear form on $\mathbb{R}^{N \times n}$ verifying (2.24). $B_{\sigma}(x_0) \Subset \Omega$ be a ball and $p > 1$ be a number. For any $\varepsilon > 0$ there exists $\delta \equiv \delta(n, N, H, p, \varepsilon) \in (0,1]$ such that if $v \in W^{1,p}(B_{\sigma}(x_0), \mathbb{R}^N)$ with $\mathcal{J}(V_{1,p}(Dv); B_{\sigma}(x_0)) \leq \sigma \leq 1$ is approximately $\mathcal{A}$-harmonic in the sense that

\[
\left| \int_{B_{\sigma}(x_0)} \mathcal{A}(Dv, D\varphi) \, dx \right| \leq \sigma \delta \|D\varphi\|_{L^\infty(B_{\sigma}(x_0))} \quad \text{for all} \quad \varphi \in C^\infty_c(B_{\sigma}(x_0), \mathbb{R}^N),
\]

then there exists an $\mathcal{A}$-harmonic map $h \in W^{1,p}(B_{\varepsilon}(x_0), \mathbb{R}^N)$ such that

\[
\int_{B_{\varepsilon}(x_0)} |V_{1,p}(Dh)|^2 \, dx \leq c \quad \text{and} \quad \int_{B_{\varepsilon}(x_0)} \left| V_{1,p} \left( \frac{v - \sigma h}{\varepsilon} \right) \right|^2 \, dx \leq c \sigma^2 \varepsilon,
\]

for $c \equiv c(n, N, p)$.

We further recall the definition of $p$-harmonic map, i.e. a function $h \in W^{1,p}(\Omega, \mathbb{R}^N)$ satisfying

\[
\int_\Omega |D|^{p-2} Dh, D\varphi | \, dx = 0 \quad \text{for all} \quad \varphi \in C^\infty_c(\Omega, \mathbb{R}^N).
\]

According to the regularity theory contained in [17,78], whenever $B_\sigma(x_0) \Subset B_r(x_0) \Subset \Omega$ are concentric balls, it is

\[
(2.26) \quad \|Dh\|_{L^\infty(B_{r/2}(x_0))} \leq c' \left( \int_{B_{\sigma}(x_0)} |Dh|^p \, dx \right)^{1/p} \quad \text{and} \quad \mathfrak{g}(h; B_\sigma(x_0)) \leq c'' \left( \frac{\sigma}{r} \right) \alpha \mathfrak{g}(h; B_r(x_0)),
\]

with $c', c'' \equiv c'(n, N, p)$ and $\alpha \equiv \alpha(n, N, p) \in (0,1)$. As a "singular" variant of Lemma 2.4 we have the following $p$-harmonic approximation lemma from [34, Lemma 1].
Lemma 2.5. Let $p \in (1, \infty)$ be a number and $B_p(x_0) \subseteq \Omega$ be any ball. For all $\varepsilon > 0$, there exists $\delta = \delta(n, N, p, \varepsilon) \in (0, 1]$ such that if $v \in W^{1,p}(B_p(x_0), \mathbb{R}^N)$ with $\mathcal{J}_p(Dv; B_p(x_0)) \leq 1$ is approximately $p$-harmonic in the sense that

$$
(2.27) \quad \left| \int_{B_p(x_0)} |Du|^p - 2Du, D\varphi| \, dx \right| \leq \delta \|D\varphi\|_{L^{\infty}(B_p(x_0))} \quad \text{for all } \varphi \in C_c^\infty(B_p(x_0), \mathbb{R}^N),
$$

then there exists a $p$-harmonic map $h \in W^{1,p}(B_p(x_0), \mathbb{R}^N)$ such that

$$
\mathcal{J}_p(Dh; B_p(x_0)) \leq 1 \quad \text{and} \quad \int_{B_p(x_0)} \frac{|v - h|}{\varepsilon^p} \, dx \leq c \varepsilon^p,
$$

with $c \equiv c(n, N, p)$.

2.5. On the Lebesgue-Serrin-Marcellini extension. Let $\mathcal{B}_1$ be the family of all open subsets of $\Omega$ and $B \in \mathcal{B}_1$. For $1 < p \leq q < \infty$, an integrand $F \in C(\mathbb{R}^{N \times n})$, and maps $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ and $w \in W^{1,p}(\Omega, \mathbb{R}^N)$, the Lebesgue-Serrin-Marcellini extension of a functional of type (1.1), i.e.:

$$
\mathcal{F}(w; B) := \int_B [F(Dw) - f \cdot w] \, dx := \mathcal{J}_0(w; B) - \int_B f \cdot w \, dx,
$$

is defined as

$$
\tilde{\mathcal{F}}(w; B) := \inf_{\{w_j\}_{j \in \mathbb{N}} \subseteq W^{1,p}_{loc}(B, \mathbb{R}^N) \cap W^{1,p}(B, \mathbb{R}^N)} \liminf_{j \to \infty} \mathcal{F}(w_j; B),
$$

with

$$
C(w; B) := \left\{\{w_j\}_{j \in \mathbb{N}} \subseteq W^{1,p}_{loc}(B, \mathbb{R}^N) \cap W^{1,p}(B, \mathbb{R}^N) : w_j \rightharpoonup w \text{ weakly in } W^{1,p}(B, \mathbb{R}^N)\right\}.
$$

By density of smooth maps in $W^{1,p}(B, \mathbb{R}^N)$ it is $C(w; B) \neq \{\emptyset\}$ and since in particular $f \in W^{1,p}(B, \mathbb{R}^N)^*$ we can rewrite

$$
(2.28) \quad \tilde{\mathcal{F}}(w; B) = \mathcal{J}_0(w; B) - \int_B f \cdot w \, dx,
$$

therefore while describing the relevant features of relaxation we shall refer to the "bulk" component of $\tilde{\mathcal{F}}(\cdot)$, i.e. $\mathcal{J}_0(\cdot)$. A first crucial observation is that $\mathcal{J}_0(\cdot)$ cannot be represented as an integral. In fact, a deep result from [15,37] states that with $F(z) \leq (1 + |z|^p)$ and $1 < p \leq q < np/(n-1)$, each $w \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that $\mathcal{J}_0(w; \Omega) < \infty$ uniquely determines a finite outer Radon measure $\mu_w$ verifying

$$
(2.29) \quad \tilde{\mathcal{J}}_0(w; \cdot) = \mu_w|_{\mathcal{B}_1} \quad \text{and} \quad \frac{d\mu_w}{dx^n} = QF(Du).
$$

Here $QF(\cdot)$ denotes the quasiconvex envelope of $F(\cdot)$, see [36, Section 5.3]. Moreover, a $W^{1,p}$-coercivity condition like $|z|^p \leq F(z)$ assures sequential lower semicontinuity of $\mathcal{J}_0(\cdot; \Omega)$ with respect to the weak topology of $W^{1,p}(\Omega, \mathbb{R}^N)$, cf. [37,71]. In [35] it is proven that $W^{1,p}$-quasiconvexity is necessary for this semicontinuity property. However, the results of [35] hold for integral functionals, while, in the light of (2.29), $\mathcal{J}_0(\cdot)$ cannot be represented as an integral. Despite the measure representation [37,71], the arguments developed in [35] can be adapted to prove that $\mathcal{J}_0(\cdot)$ features the proper notion of $W^{1,p}$-quasiconvexity, i.e.: $\mathcal{J}_0(\ell + \varphi; B) \geq \mathcal{J}_0(\ell; B)$ holds for all $B \in \mathcal{B}_1$, $\varphi \in W^{1,p}(B, \mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \subset B$ and any affine function $\ell(x) := v_0 + \langle z_0, x - x_0 \rangle$, cf. [71].

Other remarkable properties of $\mathcal{J}_0(\cdot)$ such as additivity and extremality conditions can be found in [71,72]. Now, if $F(\cdot)$ is a continuous and $W^{1,p}$-coercive integrand and $f \in W^{1,p}(\Omega, \mathbb{R}^N)^*$, the weak sequential lower semicontinuity of $\mathcal{J}_0(\cdot)$ in $W^{1,p}(\Omega, \mathbb{R}^N)$ and direct methods assure that once fixed a boundary datum $u_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that $\mathcal{J}_0(u_0; \Omega) < \infty$ - recall (2.28) - there exists a local minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of (1.2) in the sense of Definition [4]. If in addition $F \in C_c^\infty(\mathbb{R}^{N \times n})$ with (1.3), (2.12) and (2.22) in force and exponents $(p, q)$ satisfying $1 < p \leq q < \min \{np/(n-1), p+1\}$, then any local minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of (1.2) verifies by minimality the integral identity

$$
(2.30) \quad 0 = \int_{\Omega} \left[ (\partial F(Du), D\varphi) - f \cdot \varphi \right] \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N),
$$

see [23, Section 2.7] and [71, Section 7.1].

3Recall that we always work under the assumption that $\Omega$ is an open, bounded domain with Lipschitz boundary.
3. Caccioppoli inequality

We start by recording a variation obtained in [70, Lemmas 6.3 -6.5] and [71, Lemmas 4.4 and 4.6] of the extension result from [37], which will be crucial for constructing comparison maps for minima of [12].

**Lemma 3.1.** Let $0 < \tau_1 < \tau_2$ be two numbers and $B_{\tau_2} \subseteq \Omega$ be a ball. There exists a bounded, linear smoothing operator $T_{\tau_1, \tau_2} : W^{1,1}(\Omega, \mathbb{R}^N) \to W^{1,1}(\Omega, \mathbb{R}^N)$ defined as

$$W^{1,1}(\Omega, \mathbb{R}^N) \ni w \mapsto T_{\tau_1, \tau_2}[w](x) := \int_{B_r(0)} w(x + \vartheta(x)y) \, dy,$$

where it is $\vartheta(x) := \frac{1}{\tau_1} \max \left\{ \min \left\{ |x - \tau_1, \tau_2 - |x| \right\}, 0 \right\}$. If $w \in W^{1,p}(\Omega, \mathbb{R}^N)$ for some $p \geq 1$, the map $T_{\tau_1, \tau_2}[w]$ has the following properties:

(i.) $T_{\tau_1, \tau_2}[w] \in W^{1,p}(\Omega, \mathbb{R}^N)$;

(ii.) $w = T_{\tau_1, \tau_2}[w]$ almost everywhere on $(\Omega \setminus B_{\tau_2}) \cup B_{\tau_1}$;

(iii.) $T_{\tau_1, \tau_2}[w] \in w + W^{1,p}_0(B_{\tau_2} \setminus B_{\tau_1}, \mathbb{R}^N)$;

(iv.) $|D{T}_{\tau_1, \tau_2}[w]| \leq c(n)T_{\tau_1, \tau_2}[Dw]$ almost everywhere in $\Omega$.

Furthermore,

$$\|T_{\tau_1, \tau_2}[w]\|_{L^p(B_{\tau_2} \setminus B_{\tau_1})} \leq c\|w\|_{L^p(B_{\tau_2} \setminus B_{\tau_1})},$$

$$\|DT_{\tau_1, \tau_2}[w]\|_{L^p(B_{\tau_2} \setminus B_{\tau_1})} \leq c\|Dw\|_{L^p(B_{\tau_2} \setminus B_{\tau_1})} \quad \text{for } \tau_1 \leq \varepsilon \leq (\tau_1 + \tau_2)/2$$

and

$$\|DT_{\tau_1, \tau_2}[w]\|_{L^p(B_{\tau_2} \setminus B_{\tau_1})} \leq c\|Dw\|_{L^p(B_{\tau_2} \setminus B_{\tau_1})} \quad \text{for } (\tau_1 + \tau_2)/2 \leq \varepsilon \leq \tau_2,$$

for $c \equiv c(n, p)$. Finally, let $\mathcal{H} \subset \mathbb{R}$ be a set with zero Lebesgue measure. There are

$$\tilde{\tau}_1 \in \left( \tau_1, \frac{2\tau_1 + \tau_2}{3} \right) \setminus \mathcal{H}, \quad \tilde{\tau}_2 \in \left( \tau_1 + \frac{2\tau_1 + \tau_2}{3}, \tau_2 \right) \setminus \mathcal{H} \quad \text{verifying } (\tau_2 - \tau_1) \approx (\tilde{\tau}_2 - \tilde{\tau}_1)$$

up to absolute constants, such that if $1 \leq p \leq 2$, $s \geq 0$ and $\frac{2}{3} \leq d \leq d < \frac{2s}{\tau_2 - \tau_1}$, it is

$$\|\mathcal{V}_{\varepsilon,p}(DT_{\tilde{\tau}_1, \tilde{\tau}_2}[w])\|_{L^s(B_{\tau_2} \setminus B_{\tau_1})} \leq \frac{c}{(\tau_2 - \tau_1)^n(\frac{s}{3} - \frac{2}{3})}\|\mathcal{V}_{\varepsilon,p}(Dw)\|_{L^2(B_{\tau_2} \setminus B_{\tau_1})},$$

for $c \equiv c(n, p, d)$. Clearly, operator $T_{\tilde{\tau}_1, \tilde{\tau}_2}$ satisfies properties (i.)-(iv.) and (3.1) with $\tilde{\tau}_1, \tilde{\tau}_2$ substituting $\tau_1, \tau_2$.

In the next lemma we derive a preliminary version of Caccioppoli inequality that will be eventually adjusted depending on the singular/nonsingular behavior of $\tilde{\mathcal{F}}(\cdot)$.

**Lemma 3.2.** Assume (24.12)-(24.11), (21.10) and (22.22), let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of (12), $B_{\rho}(x_0) \subseteq \Omega$ be any ball, $\rho/2 \leq \tau_1 < \tau_2 \leq \rho$ be parameters and $\ell(x) := v_0 + (x_0, x - x_0)$ be an affine function with $x_0 \in \{ z \in \mathbb{R}^N \setminus \Omega : |z| \leq (80000(M + 1))^{2/p} \}$ for some positive constant $M$, and $v_0 \in \mathbb{R}^N$. Then

$$\tilde{\mathcal{F}}(u, x_0; B_{\rho/2}(x_0))^2 \leq c\mathcal{R}\left( \int_{B_{\rho}(x_0)} |V_{\varepsilon,p} \left( \frac{u - \ell}{\rho} \right)|^p \, dx \right) + c\mathcal{E}_{(\rho,p)} \mathcal{F}(u, x_0; B_{\rho}(x_0))^{2/p}$$

$$+ c\left[ \rho^m \int_{B_{\rho}(x_0)} |f|^m \, dx \right]^{2/m} + |x_0|^2 \rho^{2-p} \int_{B_{\rho}(x_0)} |f|^m \, dx \right]^{2/m}$$

holds with $c \equiv c(\text{data}, M)$ and $\mathcal{R}(\cdot)$ being defined in (22).

**Proof.** With $\tau_1, \tau_2$ as in the statement, $\tilde{\tau}_1, \tilde{\tau}_2$ as in (22.10), let $\eta \in C^1_c(B_{\tilde{\tau}_2}(x_0))$ be a cut-off function satisfying

$$\mathbb{1}_{B_{\tilde{\tau}_1}(x_0)} \leq \eta \leq \mathbb{1}_{B_{\tilde{\tau}_2}(x_0)}, \quad |D\eta| \leq \frac{1}{(\tilde{\tau}_2 - \tilde{\tau}_1)},$$

set $S(x_0) := B_{\tilde{\tau}_2}(x_0) \setminus B_{\tau_2}(x_0)$, $\tilde{S}(x_0) := B_{\tilde{\tau}_2}(x_0) \setminus B_{\tilde{\tau}_1}(x_0)$, $u(x) := u(x) - \ell(x)$ and introduce the comparison maps

$$\varphi_1(x) := T_{\tilde{\tau}_1, \tilde{\tau}_2}[(1 - \eta)u](x), \quad \varphi_2(x) := u(x) - \varphi_1(x).$$

By Lemma 3.1 (ii.)-(iii.) it is

$$\varphi_1 \equiv 0 \text{ on } B_{\tilde{\tau}_1}(x_0), \quad \varphi_2 \in W^{1,p}_0(B_{\tilde{\tau}_2}(x_0), \mathbb{R}^N), \quad \varphi_2 \equiv u \text{ on } B_{\tilde{\tau}_1}(x_0), \quad Du = D\varphi_1 + D\varphi_2.$$

\[\text{The negligible set } \mathcal{H} \text{ can be defined as in } 22 \text{ Lemma 3.1 or in } 41 \text{ Lemma 7.13}.\]
Moreover, by \(\text{(2.1)}\), \(\text{(2.2)}\) and Lemma 3.1, we see that the construction developed in \(\text{(2.1)}\) Lemma 3.1, \([7]\) Lemma 7.13 applies to our setting as well and renders:

\[
c \int_{B_{\tau_2}(x_0)} \left| V_{[\tau_0],p}(D\varphi_1) \right|^2 \, dx \leq \int_{\tilde{S}(x_0)} \left[ F(Du - D\varphi_1) - F(Du) \right] \, dx
\]

\[
+ \int_{\tilde{S}(x_0)} \left[ F(z_0 + D\varphi_1) - F(z_0) \right] \, dx
\]

\[
+ \int_{B_{\tau_2}(x_0)} f \cdot \varphi_2 \, dx =: \left| \text{I} \right| + \left| \text{II} \right| + \left| \text{III} \right|
\]

with \(c \equiv c(n, N, p, q, \lambda, \Lambda)\). Before proceeding further, let us notice that

\[
D_{[\tau_0]}(z_1, z_2)^{(q-\nu)/2} \leq \left( D_{[\tau_0]}(z_1, z_2)^{(q-2)/2} D_0(z_1, z_2) \right)^{(q-\nu)/p} + |z_0|^{q-p},
\]

for all \(z_0, z_1, z_2 \in \mathbb{R}^{N \times n}\), where \(D_{[\tau_0]}(\cdot)\) has been defined in Section 3.1 and the constants implicit in "\(\approx\)" depend on \((n, N, p, q)\), cf. \([7]\) page 256. We then rearrange:

\[
\left| \text{I} \right| + \left| \text{II} \right| = \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ \partial F(z_0) - \partial F(z_0 + Du - sD\varphi_1) \right] \, ds \right) \, D\varphi_1 \, dx
\]

\[
+ \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ \partial F(z_0 + sD\varphi_1) - \partial F(z_0) \right] \, ds \right) \, D\varphi_1 \, dx =: \left| \text{I} \right| + \left| \text{II} \right|
\]

and estimate (keep in mind the upper bound on \(|z_0|\)),

\[
\left| \text{I} \right| \leq c \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ \frac{|V_{[\tau_0],p}(Du - sD\varphi_1)|^2}{|Du - sD\varphi_1|} + \frac{|V_{[\tau_0],p}(Du - sD\varphi_1)|^2}{|Du - sD\varphi_1|} \right] \, ds \right) \, |D\varphi_1| \, dx
\]

\[
\leq c \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ \frac{|V_{[\tau_0],p}(Du)|^2}{|Du|} + \frac{|V_{[\tau_0],p}(Du)|^2}{|Du|} \right] \, ds \right) \, |D\varphi_1| \, dx
\]

\[
+ c \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ |z_0|^2 + |Du|^2 + |D\varphi_1|^2 \right] (q-2)/2 \, ds \right) \left( |Du| + |D\varphi_1| \right) \, |D\varphi_1| \, dx
\]

\[
\leq c \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ |V_{[\tau_0],p}(Du)|^2 + |V_{[\tau_0],p}(Du)|^2 \right] \, ds \right) \, |D\varphi_1|^2 \, dx
\]

\[
+ c \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ |z_0|^2 + |Du|^2 + |D\varphi_1|^2 \right] (q-2)/2 \, ds \right) \left( |Du| + |D\varphi_1| \right) \, |D\varphi_1| \, dx
\]

\[
\leq c \int_{\tilde{S}(x_0)} \left( 1 + |z_0|^{q-p} \right) \left( |V_{[\tau_0],p}(Du)|^2 + |V_{[\tau_0],p}(Du)|^2 \right) \, dx
\]

\[
+ c \int_{\tilde{S}(x_0)} \left( \int_0^1 \left[ \frac{2^{q-2}}{p} \left[ |Du|^{2q} \right] + |D\varphi_1|^{2q/p} \right] \, ds \right) \, |D\varphi_1|^2 \, dx
\]

\[
\leq c \int_{\tilde{S}(x_0)} \left( 1 + |z_0|^{q-p} \right) \left( |V_{[\tau_0],p}(Du)|^2 + |V_{[\tau_0],p}(Du)|^2 \right) \, dx
\]

\[
+ c \int_{\tilde{S}(x_0)} \left[ |V_{[\tau_0],p}(Du)|^{2q/p} \left[ |V_{[\tau_0],p}(Du)| \right] + \left[ |V_{[\tau_0],p}(Du)| \right]^{2q/p} \right] \, dx
\]

\[
\leq c \int_{\tilde{S}(x_0)} \left[ |V_{[\tau_0],p}(Du)|^2 \right] \, dx
\]

\[
+ c \left( \int_{\tilde{S}(x_0)} \left| V_{[\tau_0],p}(Du) \right|^2 \, dx \right)^{2q/p-2p} \left( \int_{\tilde{S}(x_0)} \left| V_{[\tau_0],p}(Du) \right|^{2q/p} \, dx \right)^{2p/(2q-2p)}
\]
and the upper bound imposed on the size of $c$ with $\square$ for $n 
 c \left( \frac{n}{q} \right)$.

where we also used (2.18) with $L \equiv 80000(M+1)$, and exploited that by (2.13) it is max $\{2q/p, 2p/(3p-2q)\} < 2n/(n-1)$, min $\{2q/p, 2p/(3p-2q)\} \geq 2/p$ and $c \equiv c(n,N,\lambda,p,q,M)$. In a totally similar way we bound:

$$\begin{align*}
|\text{[II']}| &\leq c \int_{\tilde{S}(\tau_2)} |V_{|z_0|,p}(D\varphi_1)|^2 + (|z_0|^2 + |D\varphi_1|^2)^{2(q-p)/\sqrt{p}}|D\varphi_1|^2 \, dx \\
\text{[III]} &\leq c \int_{\tilde{S}(\tau_2)} |V_{|z_0|,p}(D\varphi_1)|^2 + |V_{|z_0|,p}(u \tau - \tau_1)|^2 \, dx \\
&\quad + c \int_{\tilde{S}(\tau_2)} |V_{|z_0|,p}(D\varphi_1)|^{2q/p} \, dx \\
&\leq c \int_{\tilde{S}(\tau_2)} |V_{|z_0|,p}(D\varphi_1)|^2 + |V_{|z_0|,p}(u \tau - \tau_1)|^2 \, dx \\
&\quad + c \int_{\tilde{S}(\tau_2)} |V_{|z_0|,p}(D\varphi_1)|^{2q/p} \, dx
\end{align*}$$

for $c \equiv c(n,N,\lambda,p,q,M)$. Concerning term (III), we use (2.22), (2.23), (3.5) with $s = |z_0|$, $z_1 = 0$, $z_2 = D\varphi_2$ to estimate

$$\begin{align*}
|\text{[III]}| &\leq c |B_{z_2}(x_0)| \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{1/m} \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |D\varphi_2|^p \, dx \right)^{1/p} \\
&\leq c |B_{z_2}(x_0)| \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{1/m} \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |V_{|z_0|,p}(D\varphi_2)|^2 \, dx \right)^{1/p} \\
&\quad + |B_{z_2}(x_0)| \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{1/m} \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |V_{|z_0|,p}(D\varphi_2)|^2 |z_0|^{(2-p)/2} \, dx \right)^{1/p} \\
\leq \frac{1}{4} \int_{B_{z_2}(x_0)} |V_{|z_0|,p}(D\varphi_2)|^2 \, dx + c |B_{z_2}(x_0)| \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{2/m} \\
&\quad + c |B_{z_2}(x_0)||z_0|^{2-p} \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{2/m},
\end{align*}$$

with $c \equiv c(n,N,p,m)$. Merging the content of the previous displays, reabsorbing terms and recalling (3.5), (4.2) and the upper bound imposed on the size of $|z_0|$, we obtain

$$\begin{align*}
\int_{B_{z_1}(x_0)} |V_{|z_0|,p}(D\varphi_1)|^2 \, dx &\leq c \int_{B_{z_2}(x_0) \cap B_{\tau_1}(x_0)} |V_{|z_0|,p}(D\varphi_1)|^2 + |V_{|z_0|,p}(u \tau - \tau_1)|^2 \, dx \\
&\quad + c \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0) \cap B_{\tau_1}(x_0)} |f^m| \, dx \right)^{\frac{p}{m-p+1}} \\
&\quad + c |B_{z_2}(x_0)| \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{\frac{p}{m-p+1}} + |z_0|^{2-p} \left( \frac{\tau_2}{m} \int_{B_{z_2}(x_0)} |f^m| \, dx \right)^{\frac{p}{m-p+1}}
\end{align*}$$

for $c \equiv c(\text{data}, M)$. We then sum to both sides of the above inequality the quantity $c \int_{B_{z_1}(x_0)} |V_{|z_0|,p}(D\varphi_1)|^2 \, dx$ and use Lemma 4.2 to conclude with (5.4).
4. The nonsingular regime

Let us prove the approximate $A$-harmonic character of minima of $\tilde{\psi}$ within the nonsingular scenario.

**Lemma 4.1.** Under assumptions \([12],[14],[10]\) and \([22]\), let \(u \in W^{1,p}(\Omega, \mathbb{R}^N)\) be a local minimizer of \([12]\), \(B_\varepsilon(x_0) \Subset \Omega\) be a ball and \(z_0 \in \mathbb{R}^{N \times n} \setminus \{0\}\) such that \(\tilde{\psi}(z_0, B_\varepsilon(x_0)) > 0\). Then

\[
\left| \int_{B_\varepsilon(x_0)} \frac{\partial^2 F(z_0)}{\partial z^p} \left[ \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \right] D\varphi \, dx \right| \leq \frac{c|z_0|^{(2-p)/2} \|D\varphi\|_{L^\infty(B_\varepsilon(x_0))}}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \left( g^m \int_{B_\varepsilon(x_0)} |f|^m \, dx \right)^{1/m} \]

\[
+ c \left[ \frac{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))^2}{|z_0|^{p}} + \frac{1}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))^2} \right] \left( \frac{2(\kappa - 1)/2}{\|D\varphi\|_{L^\infty(B_\varepsilon(x_0))}} \right) \|D\varphi\|_{L^\infty(B_\varepsilon(x_0))},
\]

where \(\kappa := (q-1)/p\) if \(q \geq 2\) and \(\kappa := 1/p\) when \(1 < q < 2\), and \(c \equiv c(\text{data}, M)\).

**Proof.** Let \(\varphi \in C_0^\infty(B_\varepsilon(x_0), \mathbb{R}^N)\) be a map and, for the ease of reading, let us shorten \(B_\varepsilon(x_0) \equiv B_\varepsilon\), \(\tilde{\psi}(u, z_0, B_\varepsilon(x_0)) \equiv \tilde{\psi}_0(u)\), \(\ell(x) := v_0 + \langle x, z_0 - x_0 \rangle\) for \(v_0 \in \mathbb{R}^N\), \(u := -\ell \) and \(\|D\varphi\|_{L^\infty(B_\varepsilon(x_0))} \equiv \|D\varphi\|_{\infty}\). Set

\[
B^- := B_\varepsilon \cap \{|Du| \leq |z_0|\}, \quad B^+ := B_\varepsilon \cap \{|Du| > |z_0|\}
\]

and bound

\[
g := \int_{B_\varepsilon} \frac{\partial^2 F(z_0)}{\partial z^p} (Du - z_0, D\varphi) \, dx
\]

\[
\leq \int_{B^-} \frac{1}{|B^-|} \int_{B^-} \frac{\|\partial^2 F(z_0)\|_{L^\infty}}{B_\varepsilon} \left( \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \right) D\varphi \, dx + \int_{B_\varepsilon(x_0)} \int_{B^-} \frac{f \cdot \varphi}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \, dx
\]

\[
+ \frac{1}{|B_\varepsilon|} \int_{B^+} \frac{\|\partial^2 F(z_0)\|_{L^\infty}}{B_\varepsilon} \left( \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \right) \left( \frac{|z_0|^{2(\kappa - 1)/2}}{\|D\varphi\|_{L^\infty(B_\varepsilon(x_0))}} \right) \frac{|D\varphi|}{\|D\varphi\|_{L^\infty(B_\varepsilon(x_0))}} \, dx
\]

where we also used that \(\int_{B_\varepsilon} \partial F(z_0, D\varphi) \, dx = 0\). We then estimate

\[
g_1 \leq \frac{\|D\varphi\|_{L^\infty}}{\|B_\varepsilon\|} \left( \int_{B^-} \int_{B^-} \frac{1}{|B^-|} \frac{\|\partial^2 F(z_0)\|_{L^\infty}}{B_\varepsilon} \left( \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \right) D\varphi \, dx \right) \frac{|Du|}{|B_\varepsilon|} \, dx
\]

\[
\leq c \frac{\|D\varphi\|_{L^\infty}}{\|B_\varepsilon\|} \int_{B^-} \frac{\|\tilde{\psi}_0\|_{L^\infty}}{B_\varepsilon} \left( \frac{|z_0|^{(2-p)/2} \tilde{\psi}_0(u)}{|Du|^2} \right) \left( \int_{B^-} \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \, dx \right) \frac{|Du|}{|B_\varepsilon|} \, dx
\]

\[
\leq c \frac{\|\tilde{\psi}_0\|_{L^\infty}}{\|B_\varepsilon\|} \frac{|z_0|^{(2-p)/2}}{\|\tilde{\psi}_0\|_{L^\infty}} \frac{|\tilde{\psi}_0(u)|}{\|\tilde{\psi}(u, z_0; B_\varepsilon(x_0))\|_{L^\infty}} \left( \int_{B^-} \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \, dx \right) \frac{|Du|}{|B_\varepsilon|} \, dx
\]

\[
\leq c \frac{\|\tilde{\psi}_0\|_{L^\infty}}{\|B_\varepsilon\|} \frac{|z_0|^{(2-p)/2}}{\|\tilde{\psi}_0\|_{L^\infty}} \frac{|\tilde{\psi}_0(u)|}{\|\tilde{\psi}(u, z_0; B_\varepsilon(x_0))\|_{L^\infty}} \left( \int_{B^-} \frac{|z_0|^{(p-2)/2} (Du - z_0)}{\tilde{\psi}(u, z_0; B_\varepsilon(x_0))} \, dx \right) \frac{|Du|}{|B_\varepsilon|} \, dx
\]

for \(c \equiv c(n, N, p, F(\cdot), M)\). We remark that the convergence of the singular integral in the previous display can be justified as in \([9]\) Section 4. Moreover, when applying \([24]\) above we chose \(M \equiv 80000(M + 1)\) and consequently denote \(\mu_M(\cdot)\) as \(\mu_M(\cdot)\). Concerning term \(g_2\), notice that

\[
|z| \leq 2|V_{z_0,n}(z)|^{2/p} \quad \text{for all} \quad z \in \mathbb{R}^{N \times n} \cap \{z \geq |z_0|\},
\]

so we have

\[
g_2 \leq c \frac{\|D\varphi\|_{L^\infty}}{\|B_\varepsilon\|} \int_{B^-} \left( \frac{|z_0|^{p-2} + \frac{|V_{z_0,n}(Du)|^2}{|Du|^2} + \frac{|V_{z_0,n}(Du)|^2}{|Du|^2}}{\|D\varphi\|_{L^\infty}} \right) \frac{|Du|}{|B_\varepsilon|} \, dx
\]
Merging the content of the previous displays, dividing both sides of the resulting inequality by \((4.3)\), there is no loss of generality in assuming that \(1\) otherwise \((4.5)\) would be trivially true because of \((2.9)\).

We then define \((4.7)\). Now we are ready to prove a one-scale decay result valid in the nonsingular case. To this end, a fundamental observation is that under proper smallness conditions, local minimizers of \((1.2)\) are approximately \(\mathcal{A}\)-harmonic in the sense of \((2.24)\) for a suitable choice of the bilinear form \(\mathcal{A}\).

**Proposition 4.1.** Under hypotheses \((2.12)-(2.14), (2.16)\) and \((2.22)\) and let \(u \in W^{1,p}(\Omega, \mathbb{R}^N)\) be a local minimizer of \((1.2)\) satisfying

\[
|V_p(Du)|_{B_\rho(x_0)} \leq 40000(M + 1)
\]

for some \(M > 0\) on a ball \(B_\rho(x_0) \subseteq \Omega\). Then, for any \(\beta \in (0,1)\) there are \(\tau \equiv \tau_\beta(\text{data}, M, \beta) \in (0,1/16)\) and \(\varepsilon_0, \varepsilon_1 \equiv \varepsilon_0, \varepsilon_1(\text{data}, M, \beta) \in (0,1)\) such that if the smallness conditions

\[
\gamma(u; B_\rho(x_0)) < \varepsilon_0 |V_p(Du)|_{B_\rho(x_0)}
\]

and

\[
\left( \rho^m \int_{B_\rho(x_0)} |f|^m \ dx \right)^{1/m} \leq \varepsilon_1 \gamma(u; B_\rho(x_0)) |V_p(Du)|_{B_\rho(x_0)}^{(p-2)/p}
\]

are verified on \(B_\rho(x_0)\), it holds that

\[
\gamma(u; B_\rho(x_0)) \leq \tau^2 \gamma(u; B_\rho(x_0)).
\]

**Proof.** For the transparency of the exposition, let us introduce some abbreviations. As all balls considered here will be concentric to \(B_\rho(x_0)\), we shall omit denoting the center, given any ball \(B_\rho(x_0) \subseteq B_\rho(x_0)\) we shortend \((V_p(Du))_{B_\rho(x_0)} \equiv (V_p(Du))_{x_0}\) and for all \(\varphi \in C^\infty_0(B_\rho, \mathbb{R}^N)\) we denote \(\|D\varphi\|_{L^\infty(B_\rho)} \equiv \|D\varphi\|_{\infty}\). In the light of \((4.3)\), there is no loss of generality in assuming that

\[
|V_p(Du)|_e > 0 \quad \text{and} \quad \gamma(u; B_\rho) > 0,
\]

otherwise \((4.3)\) would be trivially true because of \((2.9)\). Being \(V_p(\cdot)\) an isomorphism of \(\mathbb{R}^N\), we can find \(\tilde{z} \in \mathbb{R}^N \setminus \{0\}\) such that \(V_p(\tilde{z}) = (V_p(Du))_{e}\) and

\[
|\tilde{z}| \geq \delta, \quad \gamma(u; B_\rho) \geq \gamma(u; B_\rho) \geq \delta^2, \quad |\tilde{z}| = |V_p(Du)|_e^{2/p}.
\]

We then define \(u_0(x) := |\tilde{z}|^{-1} (u(x) - (u)_p - (\tilde{z}, x - x_0))\) and, motivated by \((4.7)\), we apply Lemma \((21)\) to get

\[
\left\| \frac{\partial^2 F(\tilde{z})}{|\tilde{z}|^{p-2}}(Du_0, D\varphi) \right\| \leq \begin{cases} 
C \gamma(u; B_\rho) \left( \frac{\tilde{\gamma}(u; B_\rho)}{|\tilde{z}|^{p-2}} \right) + M \left( \frac{\tilde{\gamma}(u; B_\rho)^2}{|\tilde{z}|^{p}} \right) \|D\varphi\|_{\infty} \leq \|D\varphi\|_{\infty} \\
+ C |\tilde{z}|^{1-p} \left( \rho^m \int_{B_\rho} |f|^m \ dx \right)^{1/m} \|D\varphi\|_{\infty}
\end{cases}
\]

where we used that \(|z_0| \leq (M + 1)\). In the above display, \(\kappa\) is defined as in the statement and \(c \equiv c(\text{data}, M)\). Trivially, we also get

\[
g_3 \leq 4 \|D\varphi\|_{\infty} \left( \rho^m \int_{B_\rho} |f|^m \ dx \right)^{1/m}.
\]

Merging the content of the previous displays, dividing both sides of the resulting inequality by \(|z_0|^{(p-2)/2} \tilde{\gamma}(u)\) and recalling that by \((2.13)\) it is \(2\kappa > 1\) we obtain \((1.1)\) and the proof is complete. \(\square\)

Now we are ready to prove a one-scale decay result valid in the nonsingular case. To this end, a fundamental observation is that under proper smallness conditions, local minimizers of \((1.2)\) are approximately \(\mathcal{A}\)-harmonic in the sense of \((2.24)\) for a suitable choice of the bilinear form \(\mathcal{A}\).
with \( \tilde{c} \equiv \tilde{c}(\text{data}, M) \). Moreover, it holds that

\[
\left( \int_{B_\varrho} |V_{1,\nu}(Du_0)|^2 \, dx \right)^{1/2} = \frac{\tilde{c}(u; B_\varrho)}{|\nu|^{1/p}} \leq \mu_M(\tilde{c}_0) \leq \tilde{c}_0, \leq c_\nu, \forall \varrho.
\]

where \( c_\nu \equiv c_\nu(n, N, p) \) is the constant from the upper bound in (2.3). Now notice that by (1.2), (1.7), (2.20) with \( L = (4000(M + 1))^{2/p} \) and (2.15) we see that the bilinear form \( A := \partial^2 F(\tilde{z})|z|^2 - p \) satisfies (2.24) for some \( H \equiv H(n, \lambda, p, F(\cdot), M) \geq 1 \). We then set \( \sigma := c_\nu \tilde{c}(u; B_\varrho)/(|V_{1,\nu}(Du_0)|) \), let \( \varepsilon \in (0, 1] \) be any number to be determined later on and, recalling that by (2.3.1), if \( 2\kappa > 1 \), we assume the following smallness conditions:

\[
\max\{\sigma, c_\nu\} \varepsilon_0 < \frac{\delta}{2^{10}} \quad \text{and} \quad \mu_M(\tilde{c}_0) + \varepsilon_{n-1} + \varepsilon_1 \leq \frac{1}{2^{10}},
\]

where \( \delta \equiv \delta(n, N, \sigma, \varepsilon) \in (0, 1] \) is the small parameter given by Lemma 2.4 further restrictions on the size of the various parameters appearing in (4.3) will be imposed later on. The choice in (4.3) requires in particular that \( \varepsilon_1 \in (0, 1/3] \), fixes dependency \( \varepsilon_0 \equiv \varepsilon_0(\text{data}, M, \varepsilon) \) and ultimately gives that

\[
\int_{B_\nu} |V_{1,\nu}(Du_0)|^2 \, dx \leq \varepsilon^2 \leq 1 \quad \text{and} \quad \int_{B_\nu} A(Du_0, D\varphi) \, dx \leq \|D\varphi\|_\infty \quad \text{for all} \quad \varphi \in C^c(\partial B_\nu, \partial \mathbb{R}^N),
\]

so Lemma 2.4 applies: there exists a \( A \)-harmonic map \( h \in W^{1,p}(B_\nu, \mathbb{R}^N) \) such that

\[
\int_{B_{2\sigma}} |V_{1,\nu}(Dh)|^2 \, dx \leq c \quad \text{and} \quad \int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{u_0 - \sigma \bar{h}}{\bar{\sigma}} \right) \right|^2 \, dx \leq c\sigma^2 \varepsilon,
\]

for \( c \equiv c(n, N, p) \). Let \( \tau \in (0, 2^{10}) \) be a small number whose size will be determined later on and estimate by (2.3.7, 1.9, 2.23, 1.1) with \( |\nu| = 1 \) and the mean value theorem:

\[
\int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{u_0 - \sigma(h(x_0) + (Dh(x_0), 0))}{2\tau \bar{\sigma}} \right) \right|^2 \, dx \leq c \int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{\sigma(h - h(x_0) - (Dh(x_0), 0))}{2\tau \bar{\sigma}} \right) \right|^2 \, dx + c \int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{u_0 - \sigma \bar{h}}{2\tau \bar{\sigma}} \right) \right|^2 \, dx
\]

\[
\leq c \int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{h - h(x_0) - (Dh(x_0), 0)}{2\tau \bar{\sigma}} \right) \right|^2 \, dx \leq c \int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{u_0 - \sigma \bar{h}}{2\tau \bar{\sigma}} \right) \right|^2 \, dx.
\]

with \( c \equiv c(n, N, p) \). In the above display, we fix \( \varepsilon := \tau^{n+4} \) thus getting

\[
\int_{B_{2\sigma}} \left| V_{1,\nu} \left( \frac{u_0 - \sigma(h(x_0) + (Dh(x_0), 0))}{2\tau \bar{\sigma}} \right) \right|^2 \, dx \leq c\sigma^2 \tau,
\]

for \( c \equiv c(n, N, p) \), which yields that

\[
\int_{B_{2\tau}} |V_{|\tilde{z}|,\nu}(S(x))|^2 \, dx \text{ := } \int_{B_{2\tau}} \left| V_{|\tilde{z}|,\nu} \left( \frac{u - (u_{\nu}) - \tilde{z} - \tilde{z} \cdot (h(x_0) + (Dh(x_0), x - x_0))}{2\tau \bar{\sigma}} \right) \right|^2 \, dx
\]

\[
\leq c \int_{B_{2\tau}} \left| V_{|\tilde{z}|,\nu} \left( \frac{u_0 - \sigma(h(x_0) + (Dh(x_0), x - x_0))}{2\tau \bar{\sigma}} \right) \right|^2 \, dx
\]

\[
\leq c \int_{B_{2\tau}} \left| V_{|\tilde{z}|,\nu} \left( \frac{\tilde{c}(u; B_\nu)}{|V_{1,\nu}(Du_0)|} \right)^2 \leq c \tau^2 |\tilde{z}|^2 \tilde{c}(u; B_\nu)^2,
\]

with \( c \equiv c(n, N, p) \). Now, notice that by (2.23) and (1.7), it is \( |Dh(x_0)| \leq c(n, N, p) \), so recalling (1.8) and reducing further (with respect to (1.8)) the size of \( \varepsilon_0 \) in such a way that \( \sigma \leq \varepsilon_0 \leq \min\{2^{10}, \tau^2\} \) we obtain

\[
|\tilde{z}|(1 - \sigma) \leq |\tilde{z} + \sigma| |Dh(x_0)| \leq |\tilde{z}|(1 + \sigma) \quad \implies \quad \frac{1}{2} |\tilde{z}| \leq |\tilde{z} + \sigma| |Dh(x_0)| \leq \frac{3}{2} |\tilde{z}|
\]

We can then estimate

\[
|V_{|\tilde{z}|,\nu}(S(x))|^2 - |V_{|\tilde{z}+\sigma|,\nu}(S(x_0))|^2
\]
\[ \leq c|S(x)|^2 \leq \sup_{t \in [\varpi(1-\epsilon \nu), \varpi(1+\epsilon \nu)]} \left( t^2 + |S(x)|^2 \right)^{(p-3)/2} \left| |z| - |\bar{z} + \sigma|\bar{z}|Dh(x_0)| \right| \]

\[ \leq c|\varpi||Dh(x_0)|||S(x)|^2 \left( |\varpi(1-\epsilon \nu) + |S(x)|^2 \right)^{(p-3)/2} \]

\[ \leq \frac{c|\varpi||V_{\varpi,p}(S(x))|^2}{(1-\epsilon \nu)^{3-p}(|S(x)|^2 + |z|^2)^{1/2}} \leq \epsilon |V_{\varpi,p}(S(x))|^2, \]

for \( c \equiv c(n, N, p) \). This and (4.11) imply that

\[ (4.13) \quad \int_{B_{2r_0}^c} |V_{\varpi,\epsilon + |z||Dh(x_0)|, \varpi}(S(x))|^2 \, dx \leq c \int_{B_{2r_0}^c} |V_{\varpi,p}(S(x))|^2 \, dx \leq c\tau^2 \tilde{g}(u; B_{2r_0}), \]

with \( c \equiv c(n, p, \varpi) \). Next, notice that

\[ \tilde{g}(u, \bar{z} + \sigma|z|Dh(x_0); B_{2r_0})^2 \leq c \int_{B_{2r_0}} |V_{\varpi,p}(Du - \bar{z} - \sigma|z|Dh(x_0))|^2 \, dx \]

\[ \leq c \int_{B_{2r_0}} |V_{\varpi,p}(Du - \bar{z})|^2 \, dx + c \int_{B_{2r_0}} |V_{\varpi,p}(\sigma|z|Dh(x_0))|^2 \, dx \]

\[ \leq c\tau^{-n} \tilde{g}(u; B_{2r_0})^2 + c\varpi^2 |z|^p \leq c\tau^{-n} \tilde{g}(u; B_{2r_0})^2, \]

for \( c \equiv c(n, p, \varpi) \). At this stage, keeping in mind (4.12) we apply Caccioppoli inequality (4.3) to bound

\[ \tilde{g}(u, \bar{z} + \sigma|z|Dh(x_0); B_{2r_0})^2 \leq c \int_{B_{2r_0}} \left| V_{\varpi,\epsilon + |z||Dh(x_0)|, \varpi}(S(x)) \right|^2 \, dx \]

\[ + c\varpi |\varpi + \sigma|z||Dh(x_0)|D_{2r_0}^2 + c \left( \tau \varpi \right)^m \int_{B_{2r_0}} |f|^m \, dx \]

\[ = c \left( (I) + (II) + (III) \right), \]

with \( c \equiv c(n, \varpi, p, q, M) \). We continue estimating:

(I) \[ \leq c \int \left( \tau^2 \tilde{g}(u; B_{2r_0})^2 \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2 + c\varpi^2 |z|^p \tilde{g}(u; B_{2r_0})^2 \]

\[ \leq c \int \tilde{g}(u; B_{2r_0})^2 + cM^2 |z|^p \tilde{g}(u; B_{2r_0})^2 \]

\[ \leq c \int \tilde{g}(u; B_{2r_0})^2 \left( 1 + M^2 |z|^p \tilde{g}(u; B_{2r_0})^2 \right) \leq c \tau^2 \tilde{g}(u; B_{2r_0})^2, \]

for \( c \equiv c(\text{data}, M) \),

(II) \[ \leq c \int \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0})^{2m/p} \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2 \]

\[ \leq c \int \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0}) \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2 \]

\[ \leq c \int \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0}) \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2, \]

with \( c \equiv c(n, \varpi, p, \varphi, M) \) and

(III) \[ \leq c \int \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0}) \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2 + c \tau^2 |z|^{2-m} \left( \varpi^m \int_{B_{2r_0}} |f|^m \, dx \right)^{2/m} \]

\[ \leq c \int \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0}) \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2 \]

\[ + c \tau^2 \left| \int_{B_{2r_0}} \tilde{g}(u; B_{2r_0})^2 \tilde{g}(u; B_{2r_0})^2 \right|^{2(m-p-1)/p} \]

\[ \leq c \int \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0})^{2(m-p-1)/p} \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2 + c \tau^2 |z|^{2-m} \tilde{g}(u; B_{2r_0})^2 \]

\[ \leq c \left( \tau^{-m/p} \tilde{g}(u; B_{2r_0})^{2(m-p-1)/p} \right)^{2(m-p-1)/p} \tilde{g}(u; B_{2r_0})^2, \]

for \( c \equiv c(n, \varphi, m) \). Merging the previous four displays we obtain

\[ (4.15) \quad \tilde{g}(u, \bar{z} + \sigma|z||Dh(x_0); B_{2r_0})^2 \leq c\tau^2 \tilde{g}(u; B_{2r_0})^2, \]
where \( c \equiv c(\text{data}, M) \) and we set

\[
\mathcal{T} := \tau^2 + \mathbb{E}_{(q>p)} \tau^{-nq/p} (2q/p - \varepsilon_1)^2 |B| \left( \frac{2(n-m)}{m} \right) + \varepsilon_1^2 \tau^{2(n-m)/m}.
\]

Finally, let us observe that by triangular inequality it is

\[
|V_p(Du) - V_p(\bar{z} + \sigma|z|Dh(x_0))|^2 \leq \varepsilon_1^2 \left( |Du|^2 + |\bar{z} + \sigma|z|Dh(x_0)|^2 \right) (2-\beta)^2 \quad \text{for } c \equiv c(n, N, p),
\]

therefore

\[
\mathfrak{G}(u; B_{r^2})^2 \leq c \mathfrak{G}(u, V_p(\bar{z} + \sigma|z|Dh(x_0)); B_{r^2})^2 \leq c \mathfrak{G}(u, \bar{z} + \sigma|z|Dh(x_0)); B_{r^2})^2 \leq c \mathfrak{G}(u; B_{r^2})^2,
\]

looking at the explicit expression of \( \mathcal{T} \), we let \( \beta \in (0, 1) \) be any number, first fix \( \tau \in (0, 2^{-10}) \), then reduce further the size of \( \varepsilon_1 \in (0, 1) \) and finally restrict \( \varepsilon_1 \) in such a way that

\[
c \max \left\{ \tau^{2(1-\beta)}, \tau^{\alpha/4} \right\} \leq \frac{1}{2^{10}}
\]

\[
c \tau^2 \varepsilon_0 + c \mathbb{E}_{(q>p)} \tau^{-nq/p} \varepsilon_1^2 \left( \frac{2(n-m)}{m} \right) \leq \frac{\varepsilon_1^2}{2^{10}}
\]

\[
c \max \left\{ \tau^2 \varepsilon_0, \varepsilon_1^2 \right\} \leq \frac{\varepsilon_1^2}{2^{10}},
\]

where \( \alpha \equiv \alpha(n, N, p) \) is the same exponent appearing in (2.20). This way, we determine dependencies: \( \tau, \varepsilon_0, \varepsilon_1 \equiv \tau, \varepsilon_0, \varepsilon_1(\text{data}, M, \beta) \). Plugging the above restrictions in (1.17) we get (2.16) and the proof is complete.

Let us look at what happens when the complementary condition to (1.1) is in force.

**Proposition 4.2.** In the setting of Proposition 4.1 assume

\[
\varepsilon_1 \mathfrak{G}(u; B_{r^2}) \leq \left( \frac{\varepsilon_1^2}{2^{10}} \right) \left( \frac{\varepsilon_1^2}{2^{10}} \right)^{1/m}
\]

instead of (1.1). Then, for all \( \tau \in (0, 1) \) it is

\[
\mathfrak{G}(u; B_{r^2}) \leq c_0 \left( \frac{\varepsilon_1^2}{2^{10}} \right)^{1/m}
\]

where \( c_0 := 2 \varepsilon_1^2 \tau^{-n/2} \) and \( \varepsilon_1 \equiv \varepsilon_1(\text{data}, M) \in (0, 1) \) is the same determined in (1.15).

**Proof.** Inequality (2.20) is a direct consequence of (2.19) and (4.19).

## 5. The singular regime

We start by proving that local minimizers of (1.2) are approximately \( p \)-harmonic within the singular scenario.

**Lemma 5.1.** Under assumptions (2.12), (2.14), (2.16) and (2.22) let \( B_{r^2}(x_0) \subseteq \Omega \) be any ball and \( u \in W^{1,p} (\Omega, \mathbb{R}^N) \) be a local minimizer of (1.2). Then

\[
\left( \frac{\varepsilon_1^2}{2^{10}} \right) \left( \frac{\varepsilon_1^2}{2^{10}} \right)^{1/m} \quad \text{for all } \varphi \in C_c^\infty (B_{r^2}(x_0), \mathbb{R}^N) \text{ and any } s \in (0, \infty), \text{ with } c \equiv c(n, N, \Lambda, q).
\]

**Proof.** With the same abbreviations used in Lemma 4.1 let \( \varphi \in C_c^\infty (B_{r^2}, \mathbb{R}^N) \) be any smooth map. We use (2.30) to control

\[
\int_{B_{r^2}} |Du|^{p-2} (Du, D\varphi) \, dx = \int_{B_{r^2}} \left( \frac{\partial F(Du)}{Du} - \frac{\partial F(0)}{Du} - |Du|^{p-2} Du, D\varphi \right) - f \cdot \varphi \, dx \leq \int_{B_{r^2}} \left( \frac{\partial F(Du)}{Du} - \frac{\partial F(0)}{Du} - |Du|^{p-2} Du, D\varphi \right) \, dx + \int_{B_{r^2}} |f| \, dx =: g_1 + g_2.
\]
We fix \( s \in (0, \infty) \), notice that
\[
|B_x \cap \{|Du| > \omega(s)\}| \leq \left( \frac{3_p(Du; B_x)}{\omega(s)} \right)^p
\]
and then bound
\[
\begin{align*}
g_1 & \leq \frac{1}{|B_x|} \int_{B_x \cap \{|Du| > \omega(s)\}} |(\partial F(Du) - \partial F(0) - |Du|^{p-2} Du,D\varphi)| \ dx \\
& \quad + \frac{1}{|B_x|} \int_{B_x \cap \{|Du| \leq \omega(s)\}} |(\partial F(Du) - \partial F(0) - |Du|^{p-2} Du,D\varphi)| \ dx
\end{align*}
\]
\[
\begin{align*}
\|D\varphi\|_{\infty} & \leq s \|D\varphi\|_{\infty} \left( s \frac{3_p(Du; B_x)}{\omega(s)} \right)^{p-1} \left( \frac{3_p(Du; B_x)}{\omega(s)} \right) \\
& \quad + c \|D\varphi\|_{\infty} \left( \frac{|B_x \cap \{|Du| > \omega(s)\}|}{|B_x|} \right)^{1/p} \left( \frac{|B_x \cap \{|Du| > \omega(s)\}|}{|B_x|} \right) \\
& \quad + c \|D\varphi\|_{\infty} \left( \frac{|B_x \cap \{|Du| > \omega(s)\}|}{|B_x|} \right)^{(p-1)/(p)} \left( \frac{|B_x \cap \{|Du| > \omega(s)\}|}{|B_x|} \right)
\end{align*}
\]
\[
\begin{align*}
g_2 & \leq 4 \|D\varphi\|_{\infty} \left( \frac{\int_{B_x \cap \{|Du| > \omega(s)\}} |f|^m \ dx}{|B_x|} \right)^{1/m}
\end{align*}
\]
Merging the content of the two previous display we obtain \( g_1 \) and the proof is complete. \( \square \)

Next, a one-scale decay estimate for the excess functional valid in the singular regime.

**Proposition 5.1.** Under hypotheses \(2.12, 2.14, 2.16\) and \(2.22\), let \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) be a local minimizer of \( \mathcal{E} \) satisfying \( \mathcal{E} \) on a ball \( B_{\rho}(x_0) \subseteq \Omega \) for some positive constant \( M \). Then, for any \( \gamma \in (0, \alpha), \chi \in (0, 1] \) there are \( \theta \equiv \theta(\text{data}, \chi, \gamma, M) \in (0, 2^{-10}), \varepsilon_i \equiv \varepsilon, \varepsilon_i, (\text{data}, \gamma, M), i \in \{2, 3\} \) such that if the smallness conditions
\[
|V(Du)_{B_{\rho}(x_0)}| \leq \mathcal{F}(u; B_{\rho}(x_0)), \quad \mathcal{F}(u; B_{\rho}(x_0)) < \varepsilon_2, \quad \left( \int_{B_{\rho}(x_0)} |f|^m \ dx \right)^{1/m} < \varepsilon_3
\]
hold on \( B_{\rho}(x_0) \), then
\[
\mathcal{F}(u; B_{\rho}(x_0)) \leq \theta^\gamma \mathcal{F}(u; B_{\rho}(x_0)) + c_1 \mathcal{R} \left( \int_{B_{\rho}(x_0)} |f|^m \ dx \right)^{1/m} \mathcal{F}(u; B_{\rho}(x_0))^p/\gamma,
\]
with \( c_1 \equiv c_1(\text{data}, \chi, \gamma, M) \). Here, \( \alpha \equiv \alpha(n, M, p) \) is the exponent appearing in \( \mathcal{E} \) and \( \mathcal{R}(\cdot) \) has been defined in \( 2.22 \).

**Proof.** Let us premise that the same abbreviations appearing in Proposition \( \text{[11]} \) will be adopted also here. By triangular inequality and \( \text{(3.3)} \), we have that
\[
\mathcal{F}(u; B_{\rho}(x_0)) \leq 2 \mathcal{F}(u; B_{\rho}(x_0))^2 + 2|V(Du)_{B_{\rho}}|^2 \leq 2 \left( 1 + \frac{1}{\chi^2} \right) \mathcal{F}(u; B_{\rho})^2 =: c_2 \mathcal{F}(u; B_{\rho})^2.
\]
With this estimate at hand, \( \text{[11]} \) becomes
\[
\left| \int_{B_{\rho}} |Du|^{p-2} Du,D\varphi| \ dx \right| \leq 4 \|D\varphi\|_{\infty} \left( \int_{B_{\rho}} |f|^m \ dx \right)^{1/m} + \mathcal{F}(u; B_{\rho})^{2(p-1)/p} (p-1)/p
\]
\[
\begin{align*}
& \quad + c \gamma \mathcal{F}(u; B_{\rho})^{(p-1)/(p)} + \mathcal{F}(u; B_{\rho})^{(p-1)/p}
\end{align*}
\]
for all \( \varphi \in C_c^\infty(B_{\rho}(\mathbb{R}^N)), s \in (0, \infty) \), with \( c \equiv c(n, \Lambda, q) \). Notice that there is no loss of generality in assuming \( \mathcal{F}(u; B_{\rho}) > 0 \), otherwise \( \text{(5.4)} \) would trivially be true by means of \( \text{(2.9)} \). We then set
\[
\psi := c_2^{1/p} \mathcal{F}(u; B_{\rho})^{2/p} + \left( \frac{1}{\varepsilon_4} \right)^{(p-1)/m} \left( \int_{B_{\rho}} |f|^m \ dx \right)^{1-1/m}, \quad u_0 := \frac{u}{\psi}
\]
and divide both sides of (5.9) by $\psi^{-1}$ to get
\[
\int_{B_{\rho}} |D(u_0)|^{p-2} Du_0, D\phi) \, dx \leq \left( \varepsilon_3 + s \right) \|D\phi\|_{\infty} + cc_1^{1/p} \|D\phi\|_{\infty} \left( \omega(s)^{-1} + \omega(s)^{-p} + \omega(s)^{-p-1} \right)^{\varepsilon_2/2} \|D\phi\|_{\infty},
\]
(5.7)
for $c \equiv c(n, N, \Lambda, q)$. Now as a direct consequence of (5.8) we obtain
\[
\int_{B_{\rho}} |V|^{p} \, dx \leq \left( \varepsilon_3 + s \right) \|D\phi\|_{\infty} + \frac{cc_1^{1/p}}{4} \left( \frac{4\varepsilon_3}{\varepsilon_4} \right)^{1/(p-1)} \frac{1}{4} \leq 1,
\]
(5.9)
for $c \equiv c(n, N, p)$. In (5.8), $c'$ is the constant appearing in the Lipschitz bound (2.26). We point out that the choices in (5.5) fix dependencies $\varepsilon_3, \varepsilon_4, s \equiv s(n, N, p, \varepsilon)$ and $\varepsilon_2 \equiv \varepsilon_2(n, N, p, \omega(\cdot), \varepsilon, \chi)$. Further restrictions on the size of these parameters will be imposed in a few lines. Next, for $\theta \in (0, 2^{-10})$, we exploit the isomorphism properties of $V_p(\cdot)$ to determine $z_{22\rho} \in \mathbb{R}^{N \times n}$ such that $V_p(z_{22\rho}) = (V_p(Dh))z_{22\rho}$ and estimate via (2.23, 2.24, 5.10) and (5.9),
\[
\int_{B_{2\rho}} \left| V|z_{22\rho}| \right|^{p} \frac{u_0 - (h)_{2\rho} - (z_{22\rho} - x)\theta}{2\theta} \, dx \leq \varepsilon_3 \|V|z_{22\rho}|\|_{\infty} \frac{u_0 - h}{\theta} \, p \leq c\varepsilon^p,
\]
(5.10)
for $c \equiv c(n, N, p)$. Scaling back to $u$ in the previous display we obtain
\[
\int_{B_{2\rho}} \left| V|z_{22\rho}| \right|^{p} \frac{u - \psi((h)_{2\rho} + (z_{22\rho} - x)\theta)}{2\theta} \, dx \leq \varepsilon_3 \|V|z_{22\rho}|\|_{\infty} \frac{u - h}{\theta} \, p \leq c\varepsilon^p \left( \theta^{-n} - \varepsilon^p + \theta^{2n} \right),
\]
(5.11)
for $c \equiv c(n, N, p)$. Notice that
\[
\left\{ \begin{array}{l}
|z_{22\rho}| = |(V_p(D\phi))z_{22\rho}| \leq \psi \leq c^{1/p} \varepsilon_2/2 \leq 1, \\
\psi < c_1^{1/p} \varepsilon_2/2 \leq 1,
\end{array} \right.
\]
(5.12)
so we can bound
\[
\left\{ \begin{array}{l}
\bar{\mathfrak{S}}(u; B_0)^2 \leq 4 \int_{B_{2\rho}} \left| V_p(D\phi) - V_p(z_{22\rho}) \right|^2 \, dx \leq 4 \left\| \mathfrak{S}(u, z_{22\rho}; B_0) \right\|^2 \leq 1, \\
\end{array} \right.
\]
and divide both sides of (5.13) by $\psi^{-1}$ to get
\[
\left\{ \begin{array}{l}
\left\| \mathfrak{S}(u; B_0) \right\|^2 \leq \frac{c\varepsilon^p}{\left( \theta \right)^m} \int_{B_{2\rho}} \left| f \right|^{2m} \, dx + \frac{c\varepsilon^{2-p}}{\left( \theta \right)^m} \int_{B_{2\rho}} \left| f \right|^{2m} \, dx \leq \frac{c\varepsilon^p}{\left( \theta \right)^m} + \frac{c\varepsilon^{2-p}}{\left( \theta \right)^m} \mathfrak{S}(u, z_{22\rho}; B_0)^{2/p} \end{array} \right.
\]
\( c^2 \leq c_{\text{M}} \left( \int \left( \vartheta - n p - \varepsilon p + \vartheta^2 n \right)^{2/2m} \right) + c_{\text{M}} \left( \int \left( \vartheta \right)^m \left| f \right|^m \, dx \right)^{2/m} \\
+ c_{\text{M}} \left( \int \left( \vartheta \right)^m \left| f \right|^m \, dx \right)^{2/m} =: \text{(I) + (II) + (III) + (IV)},
\)

with \( c \equiv c(\text{data}, M) \). We continue estimating

\[
\begin{align*}
\text{(I)} & \leq c_{\text{M}} \left( \int \left( \vartheta - n p - \varepsilon p + \vartheta^2 n \right)^{2/2m} \right) + c_{\text{M}} \left( \int \left( \vartheta \right)^m \left| f \right|^m \, dx \right)^{2/m} \\
& \leq c_{\text{M}} \left( \int \left( \vartheta^2 n \right)^{2/2m} \right) + c_{\text{M}} \left( \int \left( \vartheta \right)^m \left| f \right|^m \, dx \right)^{2/m}
\end{align*}
\]

for \( c \equiv c(\text{data}, M) \). Moreover, by Young inequality with conjugate exponents \( \left( \frac{p}{2}, \frac{p}{2(p-1)} \right) \) we have

\[
\begin{align*}
\text{(II) + (III)} & \leq c_{\text{M}} \left( \int \left( \vartheta^2 n \right)^{2/2m} \right) + c_{\text{M}} \left( \int \left( \vartheta \right)^m \left| f \right|^m \, dx \right)^{2/m}
\end{align*}
\]

for \( c \equiv c(\text{data}, M) \). Finally, we control

\[
\begin{align*}
\text{(IV)} & \leq c_{\text{M}} \left( \int \left( \vartheta^2 n \right)^{2/2m} \right) + c_{\text{M}} \left( \int \left( \vartheta \right)^m \left| f \right|^m \, dx \right)^{2/m}
\end{align*}
\]

\( c \equiv c(\text{data}, M) \). Setting

\[
\mathcal{T}_1 := \int \left( \vartheta - n p - \varepsilon p + \vartheta^2 n \right)^{2/2m} \left( \vartheta \right)^m \left| f \right|^m \, dx
\]

and merging the content of all the previous displays we end up with

\[
(5.13) \quad \mathcal{T}(u; B; \varepsilon) \leq c_{\text{M}} \mathcal{T}(u; B; \varepsilon)^2 + c_{\text{M}} \left[ \left( \vartheta \right)^m \left| f \right|^m \, dx \right]^{1/m} \, (p/(p-1))
\]

for \( c \equiv c(\text{data}, M) \). We then reduce the size of the various parameter appearing in the definition of \( \mathcal{T}_1 \) to get

\[
0 < \theta - n p - \varepsilon p + \theta^2 n \leq \frac{\theta^2 n}{2m}
\]

thus fixing dependencies \( \varepsilon, \varepsilon_1 \equiv \varepsilon, \varepsilon_2 (n, N, p, q, n, \theta) \), and

\[
(5.13) \quad \mathcal{T}(u; B; \varepsilon) \leq c_{\text{M}} \mathcal{T}(u; B; \varepsilon)^2 + c_{\text{M}} \left[ \left( \vartheta \right)^m \left| f \right|^m \, dx \right]^{1/m} \, (p/(p-1))
\]

with \( c \equiv c(\text{data}, M) \). Finally, we pick any \( \gamma \in (0, \alpha) \), with \( \alpha \equiv \alpha(n, N, p) \) being the exponent in \( 2m \) and select \( \theta \in (0, \theta_0) \) so small that

\[
(5.16) \quad c_{\text{M}} \left( \vartheta \right)^m \left| f \right|^m \, dx \leq \frac{1}{2m} \Rightarrow \theta \equiv \theta(\text{data}, \varepsilon, \gamma, M).
\]

so with this choice and \( 5.15 \) we obtain \( 5.4 \) and the proof is complete. \( \square \)
6. Excess decay and the Regular set

In this section we prove that the excess functional $\mathcal{E}(\cdot)$ decays on a certain subset of $\Omega$ provided the boundedness of the potential $I_{\delta}(\cdot)$. Precisely, with $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ being a local minimizer of (1.2), we set

$$R_u := \left\{ x_0 \in \Omega : \exists M \equiv M(x_0) \in (0, \infty), \bar{\rho} \equiv \bar{\rho}(\text{data}, M, f(\cdot)) < \min\{d_{\bar{x}_0}, 1\}, \bar{\epsilon} \equiv \bar{\epsilon}(\text{data}, M) \right\}$$

(6.1)

such that $|(V_n(Du))_{B_{\bar{r}_n}(x_0)}| \leq M$ and $\mathcal{E}(u; B_{\bar{r}_n}(x_0)) < \bar{\epsilon}$ for some $\rho \in (0, \bar{\rho}]$.

According to the discussion in [23, Section 5.1], the set $R_u$ is well defined and open, with full $n$-dimensional Lebesgue measure. In particular, given any point $x_0 \in R_u$, there exists an open neighborhood $B(x_0) \subset R_u$ of $x_0$ and a radius $\delta_{x_0} \in (0, \bar{\rho}]$ such that

$$|(V_n(Du))_{B_{\delta_{x_0}}(x_0)}| < M \quad \text{and} \quad \mathcal{E}(u; B_{\delta_{x_0}}(x_0)) < \bar{\epsilon} \quad \text{for all} \quad x \in B(x_0).$$

We stress that for a given point $x_0 \in R_u$, all the radii considered from now on will be implicitly assumed to be less than $\min\{d_{\bar{x}_0}, 1\}$. Next, for $x_0 \in R_u$ verifying conditions (6.1) for some $\bar{\rho} > 0$, and parameters $\bar{\epsilon}, \bar{\rho}$ still to be fixed, we set $\nu := 2^{-2}$, choose $\gamma = \alpha/2$ in (6.2), $\beta = \beta$ in (6.3), define $\alpha_0 := \gamma$ and let $\chi = \bar{\epsilon}_0$ in (6.4). This eventually fixes the dependency of all the parameters appearing in Propositions 6.3, 6.4 and 6.5 on (data, $M$). We then define parameters

$$\bar{\epsilon} := \frac{\varepsilon \chi \sqrt{\bar{m}(\tau, \rho)}}{2^{\nu} \rho^{\nu - \rho}}, \quad H := 2^{8(n + 10)} c_3 \max\{\tau^{\rho}, \varepsilon, \chi \},$$

(6.3)

and constants $c_2 := 4(c_0 + c_1), c_3 := c_2 \max_{\delta \in \{\nu, \tau, \theta\}}(1 - \delta^{\nu})^{-1}, c_4 := \left(\frac{\varepsilon \chi \sqrt{\bar{m}(\tau, \rho)}}{2^{\nu} \rho^{\nu - \rho}}\right)^{2^{\nu - 1}}$, introduce the balanced composite excess functional

$$0, \bar{\rho}] \ni s \mapsto \mathcal{E}(x_0; s) := \mathcal{E}(u; B_s(x_0)) + |(V_n(Du))_{B_s(x_0)}| \leq \mathcal{E}(u; B_{\bar{r}_n}(x_0))$$

(6.4)

and assume that

$$\mathcal{I}_{\delta}(x_0, 1) < \infty.$$ 

(6.5)

Notice that, up to extend $f \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, the above position makes sense. Moreover, in (6.5) the finiteness of $\mathcal{I}_{\delta}(x_0, \cdot)$ is assumed to hold at radius one, but of course we can suppose that it holds at any positive radius. By (6.5) and the absolute continuity of Lebesgue integral, we can find $\hat{\rho} \equiv \hat{\rho}(\text{data}, f(\cdot), M) \in (0, \min\{1, d_{\bar{x}_0}\})$ such that

$$c_4 \mathcal{I}_{\delta}(x_0, s) \leq \frac{\hat{\rho}}{(s^{\nu - \rho} - \rho^{\nu - \rho})^{2/3}} + c_4 M^{2 - \rho/\nu} \mathcal{I}_{\delta}(x_0, s) < \bar{\epsilon}, \quad c_4 := \left(\frac{2^{8(n + 10)} c_3 \sqrt{\bar{m}(\tau, \rho)}}{2^{\nu} \rho^{\nu - \rho}}\right)^{2^{\nu - 1}}$$

(6.6)

for all $s \in (0, \hat{\rho}]$, so if $\delta \in \{\nu, \tau, \theta\}$ it is

$$\max_{s \in \{\nu, \tau, \theta\}} \delta_{x_0}(s) = \frac{\delta}{(\delta^{\nu})^m} \int_{B_{\delta}(x_0)} |f|^m \, dx \leq \frac{2^{8} \mathcal{I}_{\delta}(x_0, s)}{(\delta^{\nu})^m} \leq \frac{2^{8} \mathcal{I}_{\delta}(x_0, s)}{(\delta^{\nu})^m}$$

(6.7)

Recalling that $\nu \geq \max\{\tau, \theta\}$, by (6.7) and routine interpolation arguments we obtain that

$$\left(\frac{\delta}{\sqrt{\delta^{\nu}}} \right)^{1/m} \leq \frac{2^{8} \mathcal{I}_{\delta}(x_0, s)}{(\delta^{\nu})^m} \leq \frac{2^{8} \mathcal{I}_{\delta}(x_0, s)}{(\delta^{\nu})^m}$$

(6.8)

for all $0 < \delta \leq s/4$, which, together with (6.6) yields:

$$\lim_{\delta \to 0} \left(\frac{\delta}{\sqrt{\delta^{\nu}}} \right)^{1/m} = 0.$$ 

(6.9)
We refer to [24] Section 5.2 for more details on this matter. In (6.11), we pick \( \bar{\varepsilon} = \varepsilon \), \( \bar{\theta} = \theta \), thus determining a ball \( B_{\bar{\theta}}(x_0) \) with \( \bar{\theta} \in (0, \bar{\theta}] \) on which

\[
(V_\varepsilon(Du))_{B_{\bar{\theta}}(x_0)} < M \quad \text{and} \quad \mathcal{G}(u; B_{\bar{\theta}}(x_0)) < \varepsilon
\]

hold true. Now we are ready to prove the main result of this section.

**Theorem 4.** Under assumptions (2.12)–(2.13), (2.10), (2.22) and (6.5), let \( u \in W^{1,p}(\Omega; \mathbb{R}^N) \) be a local minimizer of (1.2), \( x_0 \in \mathcal{B}_u \) be a point and \( M = M(x_0) > 0 \) be the corresponding constant in (6.1). There are \( \bar{\varepsilon} = \varepsilon(\text{data}, M) \in (0, 1) \) and \( \bar{\theta} = \theta(\text{data}, M; f)(\cdot) \) as in (6.3) and (6.6) respectively, such that if \( \bar{\varepsilon} = \varepsilon \) and \( \bar{\theta} = \theta \) in (6.1), then for all balls \( B_{\bar{\theta}}(x_0) \subset B_{\bar{\theta}}(x_0) \) it holds

\[
(V_\varepsilon(Du))_{B_{\bar{\theta}}(x_0)} < \bar{S}(1 + M)
\]

and

\[
\mathcal{G}(u; B_{\bar{\theta}}(x_0)) \leq c_5 \left( \frac{\varepsilon}{\theta} \right)^{\alpha_0} \mathcal{G}(u; B_{\bar{\theta}}(x_0)) + c_6 \sup_{\sigma \leq \varepsilon^4} \mathcal{R} \left( \sigma \int_{B_{\bar{\theta}}(x_0)} |f|^m \, dx \right)^{1/m} \mathcal{H}\left(\frac{\varepsilon}{\theta/4}\right)
\]

\[
(6.13)
\]

with \( c_5, c_6 \equiv c_5, c_6(\text{data}, M) \) and \( \alpha_0 \equiv \alpha_0(n, N, p) \in (0, 1) \).

**Proof.** For the ease of reading, we split the proof into five steps.

**Step 1: decay estimates at the first scale.** For \( j \in \mathbb{N} \cup \{0\}, \nu \) defined as above and \( \tau, \theta \) from Propositions 4.1, 5.1 respectively, we introduce the following notation: \( \tau_j := \tau^j, \theta_j := \theta^j, \nu_j := \nu^j \) with \( \nu_0 = 0 = \nu_0 = 1 \), \( \tau_1 : = \nu_0 \) and, for \( s > 0 \) we set:

\[
\mathcal{G}(s) := \mathcal{G}(u; B_{\tau_j}(x_0)), \quad V(s) := |(V_\bar{\varepsilon}(Du))_{B_{\tau_j}(x_0)}|,
\]

\[
\mathcal{C}(s) := \mathcal{C}(x_0; s), \quad \mathcal{G}(s) := \left( s \int_{B_{\tau_j}(x_0)} |f|^m \, dx \right)^{1/m}
\]

\[
\mathcal{F}_a := \sup_{s \leq \varepsilon^4} \mathcal{G}(s), \quad \mathcal{R}_a := \sup_{s \leq \varepsilon^4} \mathcal{R}(\mathcal{G}(s)).
\]

We then estimate

\[
(6.14)
\]

With the content of display (6.14) at hand, we can start iterations.

**Step 2: maximal iteration chains.** Let us recall from [24] Section 12.4 the definition of maximal iteration chains. Given any nonempty set of indices \( J_0 \subset \mathbb{N} \cup \{0\} \), for \( \kappa \in \mathbb{N} \) the maximal iteration chain of length \( \kappa \) starting at \( \iota \) is defined as:

\[
C_\kappa^\iota := \{ j \in \mathbb{N} \cup \{0\} : \iota \leq j \leq \iota + \kappa, \ \iota \in J_0, \ \iota + \kappa + 1 \in J_0, \ j \notin J_0 \text{ if } j \gt \iota \},
\]

i.e., \( C_\kappa^\iota = \{ \iota, \iota + 1, \ldots, \iota + \kappa \} \) and all its elements lie outside \( J_0 \) except \( \iota \), which belongs to \( J_0 \). Furthermore, \( C_\kappa^\iota \) is maximal, in the sense that it cannot be properly contained in any other set of the same kind. Similarly, the infinite maximal chain starting at \( \iota \) is given by

\[
C_\kappa^\infty := \{ j \in \mathbb{N} \cup \{0\} : \iota \leq j < \infty, \ \iota \in J_0, \ j \notin J_0 \text{ if } j \gt \iota \}.
\]

We then look at two different alternatives:

\[
(6.15)
\]

with \( \mathcal{F}_a \) and \( r_1 \) defined at the beginning of Step 1, \( H \) is the constant in (6.12) and \( \varepsilon_0 \) is the same parameter appearing in (6.4).
Step 3: large composite excess at the first scale. In order to repeatedly apply (6.14), (6.16) and (6.18) while keeping under control the various parameters involved and avoiding any blow-up of the bounding constants, let us prepare the set-up for the Blocks and Chains technique introduced in [23 Section 5.2]. We assume that (6.15) holds and, with \( \nu \) as in Step 1, we consider the set of indices

\[
\mathcal{I}_{d,1}^\nu := \left\{ j \in \mathbb{N} \cup \{0\} : \mathcal{C}(\nu_j r_1) > \left( \frac{H \delta_1}{\varepsilon_0} \right)^{\frac{\nu}{\nu-1}} \right\}.
\]

Notice that \( \mathcal{I}_{d,1}^\nu \neq \emptyset \) by (6.15). We then look at two possibilities:

(i) there is at least one maximal iteration chain \( C^\nu_{n,t} \) starting at \( t \in \mathcal{I}_{d,1}^\nu \) for some \( \kappa \leq \infty; \)

(ii) \( \mathcal{I}_{d} \equiv \mathbb{N} \cup \{0\} \)

We first examine occurrence (i) at its worst: we assume that there are (countably) infinitely many finite maximal iteration chains \( \{C^{\nu_{d,j}}_{n} \}_{d \in \mathbb{N}} \) corresponding to the discrete sequences \( \{t_d\}_{d \in \mathbb{N}}, \{\kappa_d\}_{d \in \mathbb{N}} \subset \mathbb{N} \). By maximality it is easy to see that \( C^{\nu_{d,1}}_{n,1} \cap C^{\nu_{d,2}}_{n,2} = \emptyset \) for \( d_1 \neq d_2 \) and

\[
i_{d+1} \geq t_d + \kappa_d + 1 \implies \{i_d\}_{d \in \mathbb{N}} \text{ is increasing and } t_d \to \infty.
\]

By (6.16) we can split the reference interval \((0, r_1]\) into the union of disjoint blocks as \( (0, r_1] = \bigcup_{d \in \mathbb{N}, \kappa \leq \infty} B_d \), where it is \( B_0 := I_0 \cup I_1 \cup K, B_2 := I_2 \cup I_{d+1} \cup K_{d+1} \) for \( d \in \mathbb{N} \), with

\[
I_0 := (\nu_{d,1} r_1, r_1], \quad K_d := (\nu_{d+k,1}+1 r_1, \nu_{d+1}+1 r_1]
\]

and we shall implicitly identify \( I_0 \equiv I_2^\nu \). By construction, the intervals described above are disjoint and only \( I_2^\nu \) may be empty. The very definition of maximal iteration chains for the choice of \( \mathcal{I}_{d,1}^\nu \) made above yields that

\[
\mathcal{C}(\nu_j r_1) > \left( \frac{H \delta_1}{\varepsilon_0} \right)^{\frac{\nu}{\nu-1}} \quad \text{for all } d \in \mathbb{N}
\]

so if \( \varsigma \in K_d \) we can find \( j_k \in \{t_d + 1, \cdots, t_d + \kappa_d\} \) such that \( \nu_{j_k+1} r_1 < \varsigma \leq \nu_{j_k} r_1 \) and

\[
\mathcal{C}(\varsigma) \leq 2^{n+2} \mathcal{C}(\nu_{j_k} r_1) \leq 2^{n+2} \left( \frac{H \delta_1}{\varepsilon_0} \right)^{\frac{\nu}{\nu-1}}.
\]

On the other hand, if \( \varsigma \in I_0 \) or \( \varsigma \in I_2^\nu, d \in \mathbb{N} \), it is possible to determine \( j_k \in \{0, \cdots, t_d - 1\} \) or \( j_k \in \{t_d + \kappa_d + 1, \cdots, t_d+1 - 1\} \) verifying \( \nu_{j_k+1} r_1 < \varsigma \leq \nu_{j_k} r_1 \) and

\[
\mathcal{C}(\varsigma) \geq 2^{n+2} \mathcal{C}(\nu_{j_k} r_1) \geq \left( \frac{H \delta_1}{\varepsilon_0} \right)^{\frac{\nu}{\nu-1}} .
\]

Next, if \( I_2^\nu = \emptyset \) so \( B_d = I_{d+1}^\nu \cup K_{d+1} \), it turns out that the adjacent blocks \( B_{d-1} \cup B_d \) contain two consecutive chains. In fact, in this case it is \( t_{d+1} = t_d + \kappa_d + 1 \), therefore \( B_{d-1} \cup B_d = I_{d-1}^\nu \cup I_d^\nu \cup K_d \cup I_{d+1}^\nu \cup K_{d+1} \) and if in particular \( \varsigma \in K_d \cup I_{d+1}^\nu \cup K_{d+1} \), there is \( j_k \in \{t_d + 1, \cdots, t_d+1 + \kappa_d+1\} \) such that

\[
\mathcal{C}(\varsigma) \leq 2^{n+2} \mathcal{C}(\nu_{j_k} r_1) \leq 2^{n+2} \left( \frac{H \delta_1}{\varepsilon_0} \right)^{\frac{\nu}{\nu-1}} \quad \text{if } j_k \neq t_{d+1}
\]

so in any case we have that

\[
\mathcal{C}(\varsigma) \leq 2^{n+4} \left( \frac{H \delta_1}{\varepsilon_0} \right)^{\frac{\nu}{\nu-1}} \quad \text{for all } \varsigma \in K_d \cup B_d.
\]

We then consider two occurrences:

\[
\varepsilon_0 V(r_1) \leq \mathcal{F}(r_1) \quad \text{or} \quad \varepsilon_0 V(r_1) > \mathcal{F}(r_1)
\]

assume that (6.21) holds and introduce a second set of indices defined as

\[
\mathcal{J}_1 := \left\{ j \in \mathbb{N} \cup \{0\} : \varepsilon_0 V(\theta_j r_1) \leq \mathcal{F}(\theta_j r_1) \right\},
\]

which is nonempty given that \( 0 \in \mathcal{J}_1 \) by (6.21).
Step 3.1: the singular regime is stable. In this case

\begin{equation}
\|f_i\|_\infty \equiv \mathbb{N} \cup \{0\},
\end{equation}

so we can ignore the presence of blocks \(\{B_k\}_{k \in \mathbb{N}_+} \cap (0,\infty)\) and proceed in a more standard way, cf.  [33, 72]. By (6.14), (6.16) and (6.21), we see that Proposition 5.1 applies and gives

\begin{align}
\mathfrak{F}(\theta, r_i) &\leq \theta^{\alpha \phi} \mathfrak{F}(\theta, r_i) + c_1 \mathcal{R} \big( \mathfrak{F}(\theta, r_i) \big)^{\frac{\rho}{m+p-1}} \leq \frac{c_2}{\theta_0} \left( \theta^{\alpha \phi} \mathfrak{F}(\theta, r_i) + c_1 \mathcal{R} \big( \mathfrak{F}(\theta, r_i) \big)^{\frac{\rho}{m+p-1}} \right) \\
V(\theta, r_i) &\leq \frac{c_3}{\theta_0} \left( \theta^{\alpha \phi} \mathfrak{F}(\theta, r_i) + c_1 \mathcal{R} \big( \mathfrak{F}(\theta, r_i) \big)^{\frac{\rho}{m+p-1}} \right) \\
\mathfrak{F}(\theta, r_i) &\leq \frac{c_4}{\theta_0} \left( \theta^{\alpha \phi} \mathfrak{F}(\theta, r_i) + c_1 \mathcal{R} \big( \mathfrak{F}(\theta, r_i) \big)^{\frac{\rho}{m+p-1}} \right)
\end{align}

(6.23)

where we also used that by (6.22), it is \(\frac{\rho}{m+p-1} > 1\). Let us fix \(j \in \mathbb{N}\) and assume that

\begin{equation}
\mathfrak{F}(\theta, r_i) \leq c_2 \quad \text{for all } i \in \{0, \cdots, j\}.
\end{equation}

As a consequence of (6.22) and (6.23), we have

\begin{equation}
V(\theta, r_i) \leq \mathfrak{F}(\theta, r_i) \leq \frac{c_2}{\theta_0} \leq \frac{c_3}{\theta_0} \leq \frac{c_4}{\theta_0} \leq 1.
\end{equation}

(6.25)

Thanks to (6.21)–(6.25) we can apply (6.1) at the \(\theta, r_i\)-scale to get

\begin{align}
\mathfrak{F}(\theta_{j+1}, r_i) &\leq \theta^{\alpha \phi} \mathfrak{F}(\theta_{j}, r_i) + c_1 \mathcal{R} \big( \mathfrak{F}(\theta_{j}, r_i) \big)^{\frac{\rho}{m+p-1}} \\
&\leq \theta^{\alpha \phi} \mathfrak{F}(\theta_{j}, r_i) + c_1 \sum_{i=0}^{j} \theta^{\alpha \phi} \mathfrak{F}(\theta_{j}, r_i) \mathcal{R} \big( \mathfrak{F}(\theta_{j}, r_i) \big)^{\frac{\rho}{m+p-1}} \\
&\leq \theta^{\alpha \phi} \mathfrak{F}(\theta_{j}, r_i) + c_1 \mathcal{R} \big( \mathfrak{F}(\theta_{j}, r_i) \big)^{\frac{\rho}{m+p-1}} \leq \frac{c_2}{\theta_0} \leq 1.
\end{align}

(6.26)

and, via (6.22), (6.26), (6.14), (6.10),

\begin{equation}
V(\theta_{j+1}, r_i) \leq \frac{\mathfrak{F}(\theta_{j+1}, r_i)}{\theta_0} \leq \frac{c_2}{\theta_0} \leq 1.
\end{equation}

(6.27)

The arbitrariness of \(j \in \mathbb{N}\) and (6.23)–(6.27) hold for all \(j \in \mathbb{N} \cup \{0\}\); in particular it is

\begin{equation}
V(\theta_{j+1}, r_i) \leq \frac{\mathfrak{F}(\theta_{j+1}, r_i)}{\theta_0} \leq \frac{c_2}{\theta_0} \leq 1.
\end{equation}

(6.28)

for all \(j \in \mathbb{N} \cup \{0\}\). Standard interpolative arguments and (6.14) then yield that whenever \(\zeta \in (0, r_1]\) there is \(j_\zeta \in \mathcal{J}_1\) such that \(\theta_{j_\zeta}, r_1 < \zeta \leq \theta_{j_\zeta}, r_1,\)

\begin{equation}
\mathfrak{F}(\zeta) \leq \frac{2^{4+n}}{\theta^{n+2}} \bigg( \frac{\zeta}{\theta} \bigg)^{\alpha \phi} \mathfrak{F}(\theta) + \frac{2c_3}{\theta^{n+2}} \mathcal{R} \big( \mathfrak{F}(\theta) \big)^{\frac{\rho}{m+p-1}}, \quad V(\zeta) \leq \frac{3}{2}
\end{equation}

(6.29)

and

\begin{equation}
V(\zeta) \leq |V(\zeta) - V(\theta_{j_\zeta}, r_1)) + V(\theta_{j_\zeta}, r_1)| \leq \frac{\mathfrak{F}(\theta_{j_\zeta}, r_1)}{\theta_0} \leq \frac{1}{\theta_0} \left( 2^{n+1} \mathfrak{F}(\theta) + c_3 \mathcal{R} \big( \mathfrak{F}(\theta) \big)^{\frac{\rho}{m+p-1}} \right).
\end{equation}

(6.30)

Finally, if \(\zeta \in (r_1, \theta]\) via (6.14) and (2.10) it is

\begin{equation}
\mathfrak{F}(\zeta) \leq 2^{4+n} \bigg( \frac{\zeta}{\theta} \bigg)^{\alpha \phi} \mathfrak{F}(\theta) \quad \text{and} \quad V(\zeta) \leq 2^n \mathfrak{F}(\theta) + V(\theta) < 1 + M
\end{equation}

(6.31)

Merging the content of the three previous displays we obtain

\begin{equation}
\mathfrak{F}(\zeta) \leq \frac{2^{4+n}}{\theta^{n+2}} \bigg( \frac{\zeta}{\theta} \bigg)^{\alpha \phi} \mathfrak{F}(\theta) + \frac{2c_3}{\theta^{n+2}} \mathcal{R} \big( \mathfrak{F}(\theta) \big)^{\frac{\rho}{m+p-1}}, \quad V(\zeta) \leq 2 + M, \quad V(\zeta) \leq \Phi_1, \quad \text{for all } \zeta \in (0, \theta].
\end{equation}

(6.32)
Step 3.2: first change of scale. If (6.22) does not hold, there exists \( j_1 \in \mathbb{N} \) such that
\[
j_1 := \min\{j \in \mathbb{N} : \varepsilon_0 V(\theta_j r_1) > \mathfrak{F}(\theta_j r_1)\},
\]
and by (6.21) it is \( j_1 \geq 1 \). We can rephrase the minimality character of \( j_1 \) as
\[
\varepsilon_0 V(\theta_j r_1) \leq \mathfrak{F}(\theta_j r_1) \quad \text{for all} \quad j \in \{0, \ldots, j_1 - 1\} \quad \text{and} \quad \varepsilon_0 V(\theta_j r_1) > \mathfrak{F}(\theta_j r_1),
\]
therefore by (6.30) and (6.23) we deduce that (6.20)–(6.30) hold for all \( j \in \{0, \ldots, j_1 - 1\} \). In particular, it is
\[
\mathfrak{F}(\theta_j r_1) \leq \theta^\alpha_0 j_1 \mathfrak{F}(\tau_1 r_1) + c_3 \mathfrak{R}_2^{\frac{p}{p - 1}}, \quad V(\theta_j r_1) \leq \frac{3}{2} \quad \text{for all} \quad j \in \{0, \ldots, j_1 - 1\},
\]
with \( \mathfrak{H}_1 \) being defined in (6.30). We set \( r_2 := \theta_j r_1 \) and introduce a new set of indices
\[
g_2 := \left\{ j \in \mathbb{N} \cup \{0\} : \varepsilon_0 V(\tau_j r_2) > \mathfrak{F}(\tau_j r_2) \right\},
\]
which is nonempty given that \( 0 \in g_2 \) because of (6.33) and the very definition of \( r_2 \).

Step 3.3: the nonsingular regime is stable. Let us assume that
\[
\forall j \in \mathbb{N} \cup \{0\} \implies \varepsilon_0 V(\tau_j r_2) > \mathfrak{F}(\tau_j r_2) \quad \text{for all} \quad j \in \mathbb{N} \cup \{0\}.
\]
By induction, we want to show that
\[
V(\tau_j r_2) \leq 2 \quad \text{for all} \quad j \in \mathbb{N} \cup \{0\}, \quad \mathfrak{H}_2 := c_3 \left( \mathfrak{H}(\varrho) + V(\varrho) + \mathfrak{K} \left( \mathfrak{T}_{m}(x_0, \varrho) \right)^{\frac{p}{p - 1}} \right)
\]
and \( c_3' \equiv c_3'(\text{data}, M) \) has been defined at the beginning of Section 6. For \( j = 0 \), by (6.31), (6.32), (6.33) and Propositions 4.11 and 4.3, we obtain
\[
\mathfrak{H}_1 \leq 2^{-j} \mathfrak{H}_2
\]
by definition, (6.36) is proven for \( j = 0 \). Next, let us fix \( j \in \mathbb{N} \) and assume the validity of (6.36) for all \( i \in \{0, \ldots, j\} \). In particular it holds that
\[
\mathfrak{H}(\tau_i r_2) \leq \tau^\alpha_0 \mathfrak{H}(\tau_i r_2) + c_0 V(\tau_i r_2)^{\frac{2-p}{p}} \mathcal{E}(\tau_i r_2)
\]
for all \( i \in \{0, \ldots, j\} \), therefore we estimate using the discrete Fubini theorem and Young inequality with conjugate exponents
\[
V(\tau_{j+1} r_2) \leq V(\tau_j r_2) + \sum_{i=0}^{j} V(\tau_{i+1} r_2) - V(\tau_i r_2) \leq \mathfrak{H}_1 + \frac{1}{\tau^{\eta/2}} \sum_{i=0}^{j} \mathfrak{H}(\tau_i r_2) \leq \mathfrak{H}_1 + \frac{1}{\tau^{\eta/2}} \sum_{i=0}^{j} \tau^{\eta_0(i+1)} \mathfrak{H}(\tau_i r_2) + c_0 \sum_{i=0}^{j} \tau^{\eta_0(i-k)} V(\tau_i r_2)^{\frac{2-p}{p}} \mathcal{E}(\tau_i r_2)
\]
and
\[
V(\tau_j r_2) \leq 2 \quad \text{for all} \quad j \in \mathbb{N} \cup \{0\}.
\]
in all, we can conclude with (6.29) to confirm the validity of estimates (6.29)-(6.30), and when if

\[ j \]

(6.41)

Inequalities (6.40)-(6.42) prove the validity of the induction step, so by the arbitrariety of (6.42) we still get

\[ j = 0 \]

(6.3)

\[ \tau \leq \theta \]

(6.6)

\[ V(\tau_{j+1}r_2) \leq V(r_2) + 2V(r_2) + \frac{2^{n+2}}{\tau \theta^2} \frac{1}{m} \left( r_1 \right)^2 \left( r_1 \right)^2 \leq \frac{3}{2} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} \leq 2. \]

We can then combine (6.35), (6.40)-(6.41) and Proposition 11.11.2 to get

\[ \tau_{\alpha(j+1)} \leq \tau_{\alpha(j+2)} \]

(6.42)

Inequalities (6.40)-(6.42) prove the validity of the induction step, so by the arbitrariety of \( j \in N \) we can conclude that (6.30) holds for all \( j \in N \cup \{0\} \) and, once established this, we can refine (6.30) as

(6.43)

Next, if \( \epsilon \in (0, r_2) \) there is \( \delta \in N \cup \{0\} \) such that \( \tau_{\epsilon+1}r_2 < \epsilon \leq \tau_j r_2 \) and, via (2.9), (2.10), (6.6), (6.9), (6.11), (6.10) and (6.40), we have

(6.44)

while if \( \epsilon \in (r_2, r_1) \) we can find \( \delta \in \{0, \cdots, j_1 - 1\} \) verifying \( \theta_{\delta+1}r_1 < \epsilon \leq \theta_j r_1 \), so as done before we can confirm the validity of estimates (6.29)-(6.30), and when \( \epsilon \in (r_1, \theta_2) \) the bounds in (6.31) trivially hold true. All in all, we can conclude with

(6.45)

and (6.34) assures that \( j_2 \geq 1 \). The minimality of \( j_2 \) renders that

\[ \epsilon_0 V(\tau_j r_2) > \bar{\mathcal{F}}(\tau_j r_2) \]

(6.46)

\[ \epsilon_0 V(\tau_j r_2) \leq \mathcal{F}(\tau_j r_2) \]
Set \( r_3 := \tau_j r_2 \) and notice that we can repeat the same procedure described in Step 3.3 a finite number of times for getting

\[
(6.46) \quad \tilde{\theta}(\tau_j r_2) \leq \tau^{\alpha_j} \tilde{\theta}(r_2) + c_0 \mathcal{Q}_2^{(2-p)/p} \delta_3, \quad V(\tau_j r_2) \leq 2(M + 1), \quad V(\tau_j r_2) \leq \mathcal{Q}_2,
\]

for all \( j \in \{0, \ldots, j_2 - 1\} \). Next we prove that

\[
(6.47) \quad r_3 \text{ cannot belong to } I_0 \text{ or to } I_0^d \text{ for all } d \in \mathbb{N}.
\]

By contradiction, assume that (6.47) does not hold. We would then have

\[
(6.48) \quad V(\tau_j r_2) \leq \tilde{\theta}(r_3) + \frac{\epsilon_0}{\tau^{n/2} \mathcal{Q}_2(r_2)} \quad \text{for all } j \in \{0, \ldots, j_2 - 1\}.
\]

We then get

\[
V(\tau_j r_2) \leq \tilde{\theta}(r_3) + \frac{\epsilon_0}{\tau^{n/2} \mathcal{Q}_2(r_2)} \quad \text{for all } j \in \{0, \ldots, j_2 - 1\}.
\]

This means that if \( \varsigma \in I_0 \) we can find \( j_2 \in \{0, \ldots, j_1 - 1\} \) such that either \( \theta_{j_1} r_1 < \varsigma \leq \theta_{j_1} r_1 \) or \( \tau_{j_1} r_2 < \varsigma \leq \tau_{j_1} r_2 \) but in any case the estimates in (6.44) are valid. Next, we observe that

\[
(6.50) \quad V(\tau_j r_1) \leq 2^n \tilde{\theta}(\nu_{j_1} r_1) + \frac{c_0}{(\tau \theta)^{1+n/2}} \tilde{\theta}(\nu_{j_1} r_1) + \frac{c_0}{(\tau \theta)^{n/2}} \tilde{\theta}(v) + \frac{c_0}{(\tau \theta)^{n/2}} \tilde{\theta}(v) + 3(1 + M)
\]

and, concerning the average,

\[
(6.51) \quad \text{for all } j \in \{0, \ldots, j_2 - 1\}.
\]

and, using also the definition of \( c'_4 \) we get

\[
(6.52) \quad V(\tau_j r_1) \leq 2^n \tilde{\theta}(\nu_{j_1} r_1) + 2 \mathcal{Q}_2.
\]
we can replicate the whole procedure developed in (6.5) for all
for any (6.58)

Finally, if \( \varsigma \in K_1 \), by (6.18), (6.9) and (6.6) we can conclude that

Finally, if \( \varsigma \in K_1 \), by (6.18), (6.9) and (6.6) we can conclude that

Combining estimates (6.44), (6.53) and (6.54) we can conclude that for all \( \varsigma \in B_0 \) we obtain

Step 3.5: the general block \( B_d \). Here we prove that in the regularity perspective, each block \( B_d \) acts independently for all \( d \in \mathbb{N} \) (the case \( d = 0 \) is contained in Step 3.4). Recalling the definition of \( B_d \) given in Step 3, we immediately notice that if \( I_d^2 = \{ \emptyset \} \), we can conclude with (6.20) and (6.54). Next, define quantities

assume \( I_d^2 \neq \{ \emptyset \} \) and observe that

which, by means of (6.9), (6.8) and (6.3), yields:

Therefore, up to make the following substitutions:

we can replicate the whole procedure developed in Step 3.1-Step 3.4 to obtain

for all \( j \in \mathbb{N} \cup \{0\} \) in case of stability of the singular regime or

for any \( j \in \mathbb{N} \cup \{0\} \) if the nonsingular regime is stable. In any case, for all \( \varsigma \in (0, r_d] \) it is

\(^6\) Of course parameters \( j_1, j_2 \) appearing in the previous display do not necessarily coincide with those in Step 3.2 and in Step 3.4 respectively.
On the other hand, keeping in mind that by (6.14) the nonsingular regime cannot end in $I_0$, we obtain that $I_0^2 \subseteq (r_3,d_r, r_2,d_r) \cup (r_2,d_r, r_3,d_r)$, thus whenever $\varsigma \in I_0^2$ as in Step 3.4 we assure the validity of (6.59). Since by (6.58), (6.59), (6.60) and (6.31) it holds:

$$\tag{6.60} \left\{ \begin{array}{l} \mathcal{G} (\nu_{d_1+1}, \varsigma) \leq \left( \frac{2^{2n} H c_3}{\varepsilon_0(\tau \theta)^{n/2}} \right)^{\tau_0 - 1} + \frac{2^{n+6} c_3}{(\tau \theta)^{n/2}} 2^{(2-n)/p} \mathcal{G}_{2d,r_1} \\
V (\nu_{d_1+1}, r_1) \leq 6, \quad V (\nu_{d_1+1}, r_1) \leq 6 \mathcal{G}_{2d,r_1} \end{array} \right. $$

for $\varsigma \in I_{d_1+1}$ we obtain thanks to (6.60):

$$\tag{6.61} \left\{ \begin{array}{l} \mathcal{G} (\varsigma) \leq \left( \frac{2^{2n} H c_3}{\varepsilon_0(\tau \theta)^{n/2}} \right)^{\tau_0 - 1} + \frac{2^{n+6} c_3}{(\tau \theta)^{n/2}} 2^{(2-n)/p} \mathcal{G}_{2d,r_1} \\
V (\varsigma) \leq 7, \quad V (\varsigma) \leq 7 \mathcal{G}_{2d,r_1} \end{array} \right. $$

Finally, if $\varsigma \in K_{d+1}$ we directly have (6.18) and (6.61). To summarize, we have just proven the validity of estimates (6.61) for all $\varsigma \in B_d$, $d \in \mathbb{N}$.

**Step 3.6: a finite number of finite iteration chains.** Assume now that there is only a finite number, say $e_\varsigma \in \mathbb{N}$, of finite iteration chains, $\{C^d_{e_{\varsigma}}\}_{d \in \{1,\ldots,e_{\varsigma}-1\}}$. Such chains determine blocks $\{B_d\}_{d \in \{0,\ldots,e_{\varsigma}-1\}}$, on which estimates (6.59) and (6.61) apply and, being only $e_{\varsigma}$ chains, by definition it follows that $\{ j \in \mathbb{N} : j \geq \nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 \} \subseteq \mathcal{G}_0$. Then, for all $\varsigma \in (0,r_1) \setminus \bigcup_{d \in \{0,\ldots,e_{\varsigma}-1\}} B_d \equiv (0, \nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 + r_1)$ there is $j \geq \nu(\varsigma) + \kappa_{e_{\varsigma}} + 1$ such that $\nu(\varsigma) + r_1 < \varsigma \leq \nu(\varsigma) + r_1$ and

$$\tag{6.62} \mathcal{E}(\nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 + r_1) \leq \mathcal{E}(\nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 + r_1) \leq 2^{2+n} \left( \frac{H \mathcal{G}_{\varsigma}}{\varepsilon_0} \right)^{\tau_0 - 1} \leq \frac{2^{2+n} (H \mathcal{G}_{\varsigma})^{\tau_0 - 1}}{\varepsilon_0 (\tau \theta)^{npq}},$$

which means that (6.59) holds and, via (6.62) we also see that the same argument leading to (6.17) works in this case as well and renders that the nonsingular regime is stable over the whole $(0, \nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 + r_1)$. With these last informations at hand we gain that (6.58) (with $\nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 + r_1$ instead of $r_2,d$) holds, and as a consequence (6.59) is satisfied for all $\varsigma \in (0, \nu(\varsigma) + \kappa_{e_{\varsigma}} + 1 + r_1)$. To summarize, we have just proven that (6.59) and (6.61) hold for all $\varsigma \in (0, r_1)$.

**Step 3.7: an infinite iteration chain.** We describe the presence of an infinite iteration chains by introducing a number $e_{\varsigma} \in \mathbb{N}$ - assume $e_{\varsigma} \geq 2$ for the moment - and corresponding sets of integers $\{t_1, \ldots, t_{e_{\varsigma}}\} \subseteq \mathbb{N}$ and $\kappa_{e_{\varsigma}} = \infty$, determining $e_{\varsigma} - 1$ finite iteration chains $\{C^d_{e_{\varsigma}}\}_{d \in \{1,\ldots,e_{\varsigma}-1\}}$ and one infinite iteration chains $C_{e_{\varsigma}}^{\infty}$ that must be unique by maximality. On each of blocks $\{B_d\}_{d \in \{0,\ldots,e_{\varsigma}-2\}}$ determined by chains $\{C^d_{e_{\varsigma}}\}_{d \in \{1,\ldots,e_{\varsigma}-1\}}$ estimates (6.59) or (6.61) hold true. Concerning the last chain $C_{e_{\varsigma}}^{\infty}$, it generates the last block $B_{e_{\varsigma}-1} = I_{e_{\varsigma}-2} \cup I_{e_{\varsigma}} \cup K_{e_{\varsigma}}$ with $K_{e_{\varsigma}} = (0, \nu(\varsigma) + 1 + r_1)$. On intervals $I_{e_{\varsigma}-2} - I_{e_{\varsigma}}$, (6.60)-(6.61) are verified, while on $K_{e_{\varsigma}}$ we can simply conclude by means of (6.54). On the other hand, if $e_{\varsigma} = 1$, there is only one block $B_0 = I_0 \cup I_1 \cup K_1 \equiv (0, r_1)$, on which (6.44) and (6.59)-(6.61) holds, therefore we can conclude with (6.59) also in this case.

**Step 3.8: occurrence (ii).** Since $\mathcal{G}_0 \equiv \mathbb{N} \cup \{0\}$, inequality (6.19) is satisfied by all $\varsigma \in (0, r_1)$, so the validity of (6.17) is now extended to the full interval $(0, r_1]$ and this guarantees the stability of the nonsingular regime. Therefore we can proceed as done in Step 3.1-Step 3.3 to get (6.44).

**Step 4: small composite excess at the first scale.** This time, the set $\mathcal{G}_0 \subseteq \mathbb{N} \cup \{0\}$ is defined as

$$\mathcal{G}_0 := \left\{ j \in \mathbb{N} \cup \{0\} : \mathcal{C} (\nu_{d_1+j}, r_1) \leq \left( \frac{H \mathcal{G}_{\varsigma}}{\varepsilon_0} \right)^{\tau_0 - 1} \right\}_{d_1+j \neq 2} \neq \{0\}. $$

We immediately notice that if $\mathcal{G}_0 \equiv \mathbb{N} \cup \{0\}$, then (6.18) holds for all $\varsigma \in (0, r_1)$ so this, (6.54) and (6.31) give the result. We then look at the case in which there exist infinitely many finite iteration chains $\{C^d_{e_{\varsigma}}\}_{d \in \mathbb{N}}$ with $\{t_d\}_{d \in \mathbb{N}}$, $\{K_d\}_{d \in \mathbb{N}}$ as in (6.10), determining intervals $I_0 := (\nu_{d_1+j_1}, r_1)$, $K_d := (\nu_{d_1+j_1}, \nu_{d_1+j_1}+r_1)$ and blocks $B_0 := I_0 \cup K_1 \cup K_2^d$, $B_d := I_d^2 \cup K_d \cup K_{d+1}^2$ such that $(0, r_1) \equiv \bigcup_{d \in \mathbb{N}} B_d$ by (6.10). As done in Step 3, we readily observe that

$$\tag{6.63} \{0, \ldots, t_1\}, \{t_d + \kappa_d + 1, \ldots, t_{d+1}\} \subset \mathcal{G}_0 \quad \text{and} \quad \{t_d + 1, \ldots, t_d + \kappa_d\} \subset C_{e_{\varsigma}}^{\infty}, \quad (d \in \mathbb{N}). $$


therefore we have

\[
\begin{align*}
\{ \zeta \in I_0 & \text{ or } \zeta \in I_2^2 \implies \mathcal{C}(\zeta) \leq 2^{2+n} \left( \frac{H \delta_k}{\varepsilon_0} \right)^{\frac{2}{p-1}} \\
\zeta \in K_3^j & \implies \mathcal{C}(\zeta) > \frac{1}{2^{2+n}} \left( \frac{H \delta_k}{\varepsilon_0} \right)^{\frac{2}{p-1}}.
\end{align*}
\] (6.64)

Notice that \( I_2^2 \) cannot be empty otherwise \( \zeta_{d+1} = \kappa_d + I_d \) and this is not possible by means of (6.63), while if \( K_{d+1}^j = \{ \emptyset \} \) (i.e. if \( \kappa_{d+1} = 1 \)), we can exploit (6.64) and that \( \zeta_d \in \mathcal{G}_d \) for all \( d \in \mathbb{N} \) to derive

\[
\zeta \in \mathcal{G}_0 \text{ with } K_{d+1}^1 = \{ \emptyset \} \implies \mathcal{C}(\zeta) \leq 2^{2+n} \left( \frac{H \delta_k}{\varepsilon_0} \right)^{\frac{2}{p-1}}.
\] (6.65)

in other words, whenever \( K_{d+1}^j = \{ \emptyset \} \) there is nothing to prove on the related block \( B_d \) and (6.65) and (6.64) immediately follow. Now, given a general block \( B_d \) with \( K_{d+1}^1 \neq \{ \emptyset \} \) and \( d \in \mathbb{N} \cup \{ 0 \} \), if \( \zeta \in \mathcal{G}_0 \) or \( \zeta \in I_2^2 \) estimate (6.64) holds. Next, observe that

\[
\mathcal{C}(\nu_{d+1+r_1}) \leq 2^{2+n} \left( \frac{H \delta_k}{\varepsilon_0} \right)^{\frac{2}{p-1}} \quad \text{and} \quad \delta(\nu_{d+1+r_1}) < \frac{\varepsilon_0 (\tau \theta)^{npq}}{2^{npq}}, \quad V(\nu_{d+1+r_1}) \leq 1, \quad V(\nu_{d+1+r_1}) \leq \mathcal{M},
\] (6.66)

therefore, setting this time \( r_d := \nu_{d+1+r_1} \) we can plug in the substitutions in (6.59) and apply the content of Step 3.5 (with \( I_2^2 \) replaced by \( K_{d+1}^j \)) to get (6.59) and, recalling that \( K_{d+1} \) differs from \( K_{d+1}^j \) only by one scale, we recover also (6.61). Merging (6.61), (6.59), and (6.61) we can conclude that (6.61) holds for all \( \zeta \in B_d, d \in \mathbb{N} \cup \{ 0 \} \).

**Step 4.1: a finite number of finite iteration chains.** Let us assume now that there is a finite number, say \( e_* \in \mathbb{N} \) of finite iteration chains \( \{ C_{d}^e \}_{d \in \{ 1, \ldots, e_* \}} \) and corresponding blocks \( \{ B_d \}_{d \in \{ 0, \ldots, e_*-1 \}} \). On every block \( B_d, d \in \mathbb{N} \cup \{ 0 \} \), estimates (6.61) apply. Notice that (0, \( r_1 \}) \setminus \bigcup_{d=0}^{e_*-1} B_d = \{ 0, \nu_{e_*+\kappa_{e_*+1}+r_1} \} \) and, since the last finite iteration chain is \( C_{e_*}^{e_*} \), it follows that \( \{ j \in \mathbb{N} : j > \nu_{e_*+\kappa_{e_*+1}+1} \} \subseteq \mathcal{G}_0 \), therefore for all \( \zeta \in \{ 0, \nu_{e_*+\kappa_{e_*+1}+1} \} \) the bound in (6.66) is verified, so we confirm again the validity of (6.61).

**Step 4.2: an infinite iteration chain.** In this case, for \( e_* \in \mathbb{N} \) (assume for the moment that \( e_* \geq 2 \)) we can find infinite set of integers \( \{ \nu_{e_*+\kappa_{e_*-1}+1} \} \subseteq \mathbb{N} \) and \( \mathbb{N} \) and \( \kappa_{e_*} = \infty \), thus determining \( e_* \geq 1 \) finite iteration chains \( \{ C_{d}^e \}_{d \in \{ 1, \ldots, e_* \}} \) and one infinite iteration chain \( C_{\infty}^{e_*} \), that is unique by maximality. Chains \( \{ C_{d}^e \}_{d \in \{ 1, \ldots, e_*-1 \}} \) determine blocks \( \{ B_d \}_{d \in \{ 1, \ldots, e_*-2 \}} \) on which the content of Step 4 applies and (6.61) holds true, while the presence of \( C_{\infty}^{e_*} \) results into \( B_{e_*-1} = \mathcal{G}_0 \setminus \{ 0, \nu_{e_*+\kappa_{e_*+1}+r_1} \} \). If \( \zeta \in \mathcal{G}_{e_*-1} \) we directly have (6.64) which in particular implies the validity of (6.66) with \( d = e_* - 1 \), so we can reproduce the content of Step 3.5 with \( r_d := \nu_{e_*+\kappa_{e_*+1}+1} \) and eventually arrive at (6.61). Finally if \( e_* = 1 \), there is only the infinite iteration chain, thus \( \{ 0, r_1 \} = \mathcal{G}_0 \cup \{ 0, \nu_{e_*+\kappa_{e_*+1}+1} \} \). On \( I_0 \) the bound in (6.66) is in force, this in turn yields (6.66) so, proceeding as in Step 3.5 we obtain (6.61).

**Step 5: conclusions.** Collecting estimates (6.62), (6.63), (6.59), and (6.61), and setting

\[
c_5 := 2^{4n+12}/(\tau \theta)^{1+n/2}, \quad c_6 := \left( \frac{1 \cdot \varepsilon_0 (\tau \theta)^{npq}}{2^n} \right)^{\frac{2}{p-1}},
\]

we obtain (6.62)-(6.63) and the proof is complete.

For later use, let us record a couple of consequences of Theorem 4 that come along the lines of [23, Proposition 5.1 and Corollary 5.1].

**Corollary 6.1.** Assume (2.12), (2.14), and (2.22), let \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) be a local minimizer of (1.2), \( x_0 \in \mathcal{R}_u \) be a point, \( M \equiv M(x_0) > 0 \) be the constant in (6.61), \( \tilde{\varepsilon} \equiv \tilde{\varepsilon}(\cal{D}, M) \) and \( \tilde{\vartheta} \equiv \tilde{\vartheta}(\cal{D}, M, f(\cdot)) \) be as in (6.51), and (6.60) respectively.

- **If**

\[
I_{\tilde{\varepsilon},m}(x, \sigma) \to 0 \quad \text{locally uniformly in} \quad x \in \Omega,
\]

then if \( \tilde{\varepsilon} \equiv \tilde{\varepsilon}(\tilde{\varepsilon}) \) and \( \tilde{\vartheta} \equiv \tilde{\vartheta}(\tilde{\vartheta}) \) in (6.61), there is an open neighborhood \( B(x_0) \subseteq \mathcal{R}_u \) and a positive radius \( \vartheta_{x_0} \equiv \vartheta_{x_0}(\cal{D}, M, f(\cdot)) \in (0, \tilde{\vartheta}) \) such that

\[
|\langle V_p(Du) \rangle_{B_\vartheta(x_0)}| < 8(1 + M)
\] (6.67)

\[
|\langle V_p(Du) \rangle_{\vartheta_{x_0}(x, \sigma)}| \leq c_8 \left( \mathcal{C}(x; \sigma) + \tilde{\mathcal{R}} \left( \mathcal{I}_{\tilde{\varepsilon},m}(x, \sigma) \right)^{\frac{2}{p-1}} \right)
\] (6.68)
and
\[
\mathfrak{g}(u; B_r(x)) \leq c_7 \left( \begin{array}{c}
\frac{\sigma}{\epsilon} \\
\epsilon
\end{array} \right)^{\alpha_0} \mathfrak{g}(u; B_r(x)) + c_8 \sup_{s \leq \epsilon/4} \mathfrak{R} \left( s^m \int_{B_s(x)} |f|^m \, dx \right)^{1/m} \left( \epsilon \right)^{\frac{\alpha}{p-p+1}}.
\]

(6.69)

\[ (6.70) \]

\[
\mathfrak{g}(u; B_r(x)) \leq c_7 \left( \begin{array}{c}
\frac{\sigma}{\epsilon} \\
\epsilon
\end{array} \right)^{\alpha_0} \mathfrak{g}(u; B_r(x)) + c_8 \sup_{s \leq \epsilon/4} \mathfrak{R} \left( s^m \int_{B_s(x)} |f|^m \, dx \right)^{1/m} \left( \epsilon \right)^{\frac{\alpha}{p-p+1}}.
\]

(6.71)

hold for all \( x \in B(x_0), \) \( 0 < \sigma \leq \epsilon \leq \|x_0\|, \) where \( c_7 := c_52^{12\epsilon_0} \epsilon_0^{\frac{2n}{n+2}} \) and \( c_8 := c_62^{16\epsilon} \epsilon_0^{\frac{2n}{n+2}}. \)

- If \( \|x\| \) is in force instead of \( \|x_0\|, \) then the following "restricted" versions of (6.68)–(6.69) hold:

\[
\mathfrak{g}(u; B_{e}(x_0)) \leq c_6 \mathfrak{f}(x_0; \epsilon) + c_8 \mathfrak{F}(x_0; \epsilon)^{1/m} \left( \epsilon \right)^{\frac{\alpha}{p-p+1}}.
\]

(6.72)

with \( c_9 := 2^p(c_7 + c_8), c_9 \equiv c_9(\text{data}, M). \)

We conclude this section with an almost everywhere VMO result. To do so, we need some preliminaries. Assume \( \|x\| \) let \( x_0 \in \mathcal{R}_a \) be any point, \( M \equiv M(x_0) \) be the positive constant in (6.1). With \( \bar{\epsilon}, \bar{\sigma} \) still to be determined, we introduce constants:

\[
H_1 := \max \left\{ 2^{24n}Hc_9, \frac{2^{24n}H}{\epsilon_1(\bar{\sigma})^{6n}} \right\}, \quad H_2 := \left( \frac{2^{36n}c_9H}{\epsilon_1(\bar{\epsilon})^{20n}} \right)^{\frac{1}{p}}.
\]

(6.73)

and fix

\[ \bar{\epsilon} := \frac{\bar{\epsilon}}{2^{20}c_9} \]

(6.74)

\[ H \equiv H(\text{data}, M), \]

\[ \text{where } \bar{\epsilon} \equiv \bar{\epsilon}(\text{data}, M), \]

\[ \text{and } \mathcal{I}_1(\cdot, \cdot) \in L^\infty(\Omega) \text{ and } \left( s^m \int_{B_s(x)} |f|^m \, dx \right)^{1/m} \rightarrow 0 \text{ locally uniformly in } x \in \Omega. \]

By means of (6.67), we determine a threshold radius \( \bar{\sigma} := \bar{\sigma}(\text{data}, M, f(\cdot)) \in (0, \bar{\sigma}) \) such that

\[
c_{10}\mathfrak{R}(1, \mathfrak{I}_1(\cdot, s)) \leq c_{10}M^{(p-2)/p}(1, \mathfrak{I}_1(\cdot, s)) < \varepsilon_*, \quad c_{10} := \left( \frac{2^{4n\epsilon_0}c_9H_2}{\epsilon_1(\bar{\epsilon})^{20n}p} \right)^{\frac{1}{p}}.
\]

(6.76)

for all \( s \leq \bar{\sigma}, x \in B_{d_{x_0}(x)}, \) which implies via (6.8) that

\[
\left\{ \begin{array}{l}
c_8 \mathfrak{R} \left( \sup_{s \leq \epsilon/4} \left( s^m \int_{B_s(x)} |f|^m \, dx \right)^{1/m} \right)^{\frac{1}{m}} \leq \frac{1}{2^4},
\end{array} \right.
\]

(6.77)

for all \( x \in B_{d_{x_0}(x_0)}, \) and recalling the definition of \( c_9 \) given in Corollary 6.1, we have \( c_{10} > c_4 \) and by (6.73) that it is \( H_2 > H, \) so the choice made in (6.76) immediately implies the validity of (6.0) on \( B_{d_{x_0}(x_0)}. \) Now we are ready to prove:
Proposition 6.1. Under assumptions (6.67), let \( u \in W^{1,p}(\Omega, \mathbb{R}^n) \) be a local minimizer of (1.2). There exists an open set \( \Omega_u \subset \Omega \) of full n-dimensional Lebesgue measure such that \( Du \in \text{VMO}_{\text{loc}}(\Omega_u, \mathbb{R}^{n \times n}) \) which can be characterized as

\[
\Omega_u := \left\{ x_0 \in \Omega : \exists M \equiv M(x_0) > 0 : \left| (V_p(Du))_{B_\varepsilon(x_0)} \right| < M \quad \text{and} \quad \mathfrak{f}(u; B_\varepsilon(x_0)) < \varepsilon \ast \quad \text{for some} \quad \varepsilon \equiv (0, \varepsilon) \right\},
\]

with \( \varepsilon \ast \equiv \varepsilon^* \) \((\text{data}, M)\) as in (6.70) and \( \varepsilon_* \equiv \varepsilon_*(\text{data}, M, f(\cdot)) \) defined by (6.76)–(6.77). In particular, for all \( x_0 \in \Omega_u \) there is an open neighborhood \( B(x_0) \subset \Omega_u \) such that

\[
(6.78) \quad \lim_{\varepsilon \to 0} \mathfrak{f}(u; B_\varepsilon(x)) = 0 \quad \text{uniformly for all} \quad x \in B(x_0).
\]

Proof. In the light of the discussion at the beginning of Section 6, the ideal candidate for \( \Omega_u \) is set \( \mathcal{R}_u \) as in (6.1) with \( \varepsilon \equiv \varepsilon_* \) and \( \tilde{\varepsilon} \equiv \varepsilon_* \); in fact, we already know that it is an open set of full n-dimensional Lebesgue measure so we only need to prove the VMO-result. We take \( x_0 \in \mathcal{R}_u \) with \( \varepsilon \equiv \varepsilon_* \), \( \tilde{\varepsilon} \equiv \varepsilon_* \) in (6.1) and observe that (6.67) allows applying the first part of Corollary 6.1 so there exists an open neighborhood \( B(x_0) \subset \mathcal{R}_u \) and a positive radius \( \varepsilon_{\Omega_{\varepsilon}} \equiv \varepsilon_{\Omega_{\varepsilon}}(\text{data}, M, f(\cdot)) \) such that (6.68)–(6.69) are verified for all \( x \in B(x_0) \) and any \( 0 < \sigma \leq \varepsilon \leq \varepsilon_{\Omega_{\varepsilon}} \). Of course, we can always assure that \( B(x_0) \subset B_{\varepsilon_{\Omega_{\varepsilon}}}(x_0) \). Fixing an arbitrary \( r \in (0, 1) \), by (6.76) we can find a radius \( \varepsilon'' \equiv \varepsilon''(\text{data}, M, f(\cdot)) \in (0, \varepsilon_{\Omega_{\varepsilon}}) \) satisfying

\[
(6.79) \quad c_8 \sup_{x \in B_r(x)} \left( \frac{\int_{B_{\varepsilon''}(x)} |f|^m \, dx}{m} \right)^{1/m} + c_8 (2 + M)^{(2-p)/p} \sup_{x \in B_{\varepsilon''}(x)} \left( \int_{B_{\varepsilon''}(x)} |f|^m \, dx \right)^{1/m} \leq r^{2/4}.
\]

Moreover, via (6.68) with \( \sigma \equiv \tilde{\varepsilon} \) and \( \varepsilon \equiv \varepsilon_{\Omega_{\varepsilon}} \), (6.70), (6.77) and (6.72) with \( \varepsilon \equiv \varepsilon_* \), \( \tilde{\varepsilon} \equiv \varepsilon_* \) we obtain

\[
\mathfrak{f}(u; B_{\varepsilon''}(x)) \leq c_7 \left( \frac{\tilde{\varepsilon}}{\varepsilon_{\Omega_{\varepsilon}}} \right)^{\alpha_0} \mathfrak{f}(u; B_{\varepsilon_{\Omega_{\varepsilon}}}(x)) + c_8 \sup_{x \in B_{\varepsilon_{\Omega_{\varepsilon}}}} \frac{\tilde{\varepsilon}}{\varepsilon_{\Omega_{\varepsilon}}} \left( \int_{B_{\varepsilon''}(x)} |f|^m \, dx \right)^{1/m},
\]

\[
+ c_9 \left( \mathfrak{f}(x; \varepsilon_{\Omega_{\varepsilon}}) + \left( \sum_{l=1}^{m} \left( \int_{B_{\varepsilon''}(x)} |f|^m \, dx \right)^{\frac{1}{m}} \right)^{-1/m} \int_{B_{\varepsilon''}(x)} |f|^m \, dx \right)^{1/m}
\]

\[
(6.80) \quad \leq c_7 \varepsilon_* + \frac{1}{2^{1/0}} \leq 1,
\]

Finally we pick \( \sigma \equiv \sigma_{\varepsilon}(\text{data}, M, f(\cdot)) \) \((0, \varepsilon'')\) small enough that

\[
(6.81) \quad c_9 (\sigma_{\varepsilon}/\varepsilon'')^{\alpha_0} \leq r/2.
\]

Plugging (6.79)–(6.81) in (6.69) with \( \sigma \equiv \sigma_{\varepsilon} \) and \( \varepsilon \equiv \varepsilon'' \) we obtain that

\[
(6.82) \quad \varepsilon \leq \sigma_{\varepsilon} \quad \implies \quad \mathfrak{f}(u; B_{\varepsilon}(x)) \leq r \quad \text{for all} \quad x \in B(x_0).
\]

The arbitrariness of \( r \), (6.68) and a standard covering argument eventually lead to (6.78) and the proof is complete. \( \Box \)

Remark 6.1. Let us list some relevant observations.

- Replacing (6.67) with (6.5) in Proposition 6.1 we obtain that whenever \( x_0 \in \Omega \) verifies the conditions in (6.1) with \( \varepsilon \equiv \varepsilon_* \) and \( \tilde{\varepsilon} \equiv \varepsilon_* \) it holds that

\[
(6.82) \quad \lim_{\varepsilon \to 0} \mathfrak{f}(u; B_\varepsilon(x_0)) = 0.
\]

- Corollary 6.1 and Proposition 6.1 remain valid if \( M \) is replaced by \( 8(1 + M) \), without affecting the magnitude of the bounding constants appearing in the various estimates as they are all derived in correspondence of larger values than \( M \).

- Corollary 6.1 guarantees in particular that once (6.11) - with \( \varepsilon \), \( \tilde{\varepsilon} \) as in (6.3) and (6.4) respectively - is verified for a certain \( \varepsilon \in (0, \tilde{\varepsilon}) \), then the Morrey type decay estimates (6.68) and (6.71) are satisfied for all scales smaller than \( \varepsilon \). This will allow us to work on all scales smaller that \( \varepsilon \).

7. Borderline Gradient Continuity

This final section is devoted to the proof of the partial gradient continuity for minima of (1.2). Let \( x_0 \in \mathcal{R}_u \) be any point, \( M \equiv M(x_0) > 0, \varepsilon, \tilde{\varepsilon} \) be the parameters appearing in (6.1), still to be fixed as functions of \((\text{data}, M)\) and \((\text{data}, M, f(\cdot))\) respectively. We assume the validity of (6.5) at \( x_0 \), define the smallness threshold

\[
(7.1) \quad \varepsilon := \frac{\varepsilon_*}{2^p c_9 \max \{H_1, H_2\}} \quad \implies \quad \varepsilon' \equiv \varepsilon'(\text{data}, M)
\]

and determine the radius \( \varepsilon' \equiv \varepsilon'(\text{data}, M, f(\cdot)) \in (0, \varepsilon_*] \) so small that

\[
(7.2) \quad c_{11} \mathfrak{f} \left( \mathfrak{I}_{m, i}(x_0, s) \right)^{\frac{p}{m(2-p)/m}} + c_{11} M^{(2-p)/m} \mathfrak{I}_{m, i}(x_0, s) < \varepsilon', \quad c_{11} := \left( \frac{c_{10}^2 2^{16npq} \max \{H_1, H_2\}}{(17)^{16npq}} \right)^{\frac{2p}{m(2-p)/m}}.
\]
for all $s \in (0, \varrho')$, which yields
\begin{align}
\sup_{s \leq s'/4} \bar{R} \left( \sigma \int_{B_{s}(x_0)} |f|^m \, dx \right)^{1/m} \equiv \frac{\int_{B_{s}(x_0)} |f|^m \, dx}{\sigma} \leq \varepsilon',
\end{align}
(7.3)
\[ + M^{(2-p)/p} \sup_{s \leq s'/4} \left( \sigma \int_{B_{s}(x_0)} |f|^m \, dx \right)^{1/m} \leq \varepsilon', \]
where $M$ is the subquadratic counterpart of $[23, \text{Lemma 6.1}]$, both inspired by $[55, \text{Lemma 6.1}]$.

\textbf{Remark 6.1} In mind, unless otherwise specified we shall work within the setting designed at the beginning of the
\begin{align}
\text{lemma that is the subquadratic counterpart of } [23, \text{Lemma 6.1}], \text{both inspired by } [55, \text{Lemma 6.1}].
\end{align}

\textbf{Proof.}

We shall strengthen (6.5) by assuming (6.7), thus (6.8) and Proposition 6.11 will be at hand. Now, with $H \equiv H(\text{data}, M)$ as in (6.3) and $H_1, H_2 \equiv H_1, H_2(\text{data}, M)$ being defined in (6.7), we slightly modify the definition of the composite excess functional given in (6.4) and consider its "unbalanced" version:
\begin{align}
(0, \varrho) \ni s \mapsto \mathcal{E}(x_0; s) := H\bar{\mathcal{F}}(u; B_{s}(x_0)) + \|V_p(Du)\|_{H_{s}(x_0)},
\end{align}
and, for $s \in (0, \varrho)$, introduce the nonhomogeneous excess functional:
\begin{align}
\mathfrak{N}(x_0; s) := H_1\bar{\mathcal{F}}(u; B_{s}(x_0)) + c_{12}H_2 \left( \int_{I_{m}(x_0, s)} |\tau\theta|^{2/(2-p)} \right)^{1/(1-p)} + \|V_p(Du)\|_{H_{s}(x_0)}(1-\varrho)^{-m/(1-p)} \right)^{1/(1-p)}).
\end{align}

where $\mathfrak{N}(\cdot)$ is defined in (6.4) and $c_{12} := \left( 2^{1/(2-p)} \right)^{1/(1-p)}$. Notice that by (6.5) and (6.8), we have
\begin{align}
\lim_{\varrho \to 0} \mathfrak{N}(u; B_{\varrho}(x_0)) = 0 \implies \lim_{\varrho \to 0} \mathfrak{N}(x_0; \varrho) = 0.
\end{align}

For the ease of exposition, we shall adopt some abbreviations. With $\tau \equiv \tau(\text{data}, M)$ being the parameter determined in Propositions 4.1-4.2, $j \in \mathbb{N}\setminus\{-1, 0\}$ and $s \in (0, \varrho)$ set $\sigma_j := \tau_{j+1}^{-1}, \sigma_{j+1} := \sigma$ and $B_j := B_{s_j}(x_0)$. From now on we will mostly employ the shorthands described in Step 1 of the proof of Theorem 4 and, with Remark 6.1 in mind, unless otherwise specified we shall work within the setting designed at the beginning of Section 7.

\textbf{7.1. An inductive lemma.} The key tool for proving our sharp partial continuity result is an inductive technical lemma that is the subquadratic counterpart of [23, Lemma 6.1], both inspired by [55, Lemma 6.1].

\textbf{Lemma 7.1.} Let $x_0 \in \mathcal{R}_0$ be a point with $M \equiv M(x_0)$ being the positive constant in (6.1), $\gamma$ be a positive number and assume that $\tilde{\varepsilon} \equiv \tilde{\varepsilon}'$, $\tilde{\varrho} \equiv \tilde{\varrho}'$ in (6.1) with $\varepsilon' \equiv \varepsilon'(\text{data}, M)$, $\varrho' \equiv \varrho'(\text{data}, M, f(\cdot))$ defined in (7.1)-(7.2); that
\begin{align}
\mathfrak{N}(x_0; \sigma) \leq 2\gamma \quad \text{for some } \sigma \in (0, \varrho]
\end{align}
and that, for integers $k \geq i \geq 0$ inequalities
\begin{align}
\mathcal{E}_H(\sigma_i) \leq \gamma, \quad \mathcal{E}_H(\sigma_{i+1}) \geq \frac{\gamma}{16} \quad \text{for all } j \in \{i, \cdots, k\},
\end{align}
\begin{align}
\mathcal{E}_H(\sigma_i) \leq \frac{\gamma}{16}\quad \text{are verified. Then the following holds:}
\end{align}
\begin{align}
\mathcal{E}_H(\sigma_{i+1}) \geq \gamma, \quad \sum_{j=i}^{k+1} \mathfrak{N}(\sigma_j) \leq \frac{\gamma}{2H}
\end{align}
and
\begin{align}
\sum_{j=i}^{k+1} \mathfrak{N}(\sigma_j) \leq \frac{4\mathfrak{N}(\sigma_i)}{3} + \frac{2^{2}\gamma^{(2-p)/p}}{\sigma_{i+1}^{-2}} \sum_{j=i}^{k} \mathfrak{N}(\sigma_j),
\end{align}
where $H, \varepsilon_1 \equiv H, \varepsilon_1(\text{data}, M)$ defined in (6.3) and in Propositions 4.1-4.2 respectively.

\textbf{Proof.} Our preliminary observation is that $x_0 \in \mathcal{R}_0$ with $\tilde{\varepsilon}' \equiv \tilde{\varrho}'$ and $\tilde{\varrho} \equiv \tilde{\varrho}'$ guarantees the validity of (6.3) and of (6.7). A straightforward computation shows that
\begin{align}
\|V(\sigma_j) - V(\sigma_{j+1})\| \leq \frac{\mathfrak{N}(\sigma_j)}{\tau^{m/2}} \leq \mathcal{E}_H(\sigma_j) \leq \frac{\mathfrak{N}(\sigma_j)}{\tau^{m/2}H} \leq \frac{\gamma}{2H}.
\end{align}

Next, let us prove that under (7.10) the singular regime cannot be in force, i.e.:
\begin{align}
\varepsilon_0 V(\sigma_j) \leq \mathfrak{N}(\sigma_j) \quad \text{cannot hold for all } j \in \{i, \cdots, k\}.
\end{align}
By contradiction, we assume that
\begin{align}
\varepsilon_0 V(\sigma_j) \leq \mathfrak{N}(\sigma_j) \quad \text{holds true}
\end{align}
and estimate via Young inequality with conjugate exponents \( \left( \frac{1}{p'}, \frac{1}{p''} \right) \),

\[
H \delta(x_{j+1}) \leq c_7 H^{5(n+2)} \delta(\sigma) + c_8 H \sup_{s \leq \sigma/4} \mathfrak{R}(\mathfrak{S}(s))^{\frac{p}{p'-1}} \\
+ c_8 H \left( \mathfrak{C}(\sigma) + \mathfrak{R} \left( I_{m,x_0,\sigma}^{\frac{p}{p''}} \right) \right)^{\frac{p}{p'-1}} \sup_{s \leq \sigma/4} \mathfrak{S}(s) \\
\leq H(c_7 + c_8) \delta(\sigma) + c_8 H \sup_{s \leq \sigma/4} \mathfrak{R}(\mathfrak{S}(s))^{\frac{p}{p'-1}} \\
+ c_8 H \mathfrak{R} \left( I_{m,x_0,\sigma}^{\frac{p}{p''}} \right) \sup_{s \leq \sigma/4} \mathfrak{S}(s) + c_8 H V(\sigma)^{(2-p)/p} \sup_{s \leq \sigma/4} \mathfrak{S}(\sigma) \\
\leq \Omega(x_0; \sigma) \left( \frac{H(c_7 + c_8)}{H_1} + \frac{H c_8}{c_1 H_2} \left( \frac{2^{8nq}}{\tau^4 n^4} \right)^{\frac{p}{p'-1}} + \frac{2 H c_8}{c_1 H_2} \left( \frac{2^{8n}}{\tau^4 n^4} \right) \right) \\
\leq \gamma \left( \frac{2 H(c_7 + c_8)}{H_1} + \frac{2 H c_8}{c_1 H_2} \left( \frac{2^{8nq}}{\tau^4 n^4} \right)^{\frac{p}{p'-1}} + \frac{2 H c_8}{c_1 H_2} \left( \frac{2^{8n}}{\tau^4 n^4} \right) \right) \leq \frac{\gamma}{\epsilon_1}.
\]

(7.13)

Furthermore, we have

\[
V(\sigma_j) \leq \frac{\delta(\sigma_j)}{\epsilon_0} \leq \frac{\mathcal{C}_H(\sigma_j)}{\epsilon_0 H} \leq \frac{1}{\epsilon_0 H} \leq \frac{\gamma}{2H}.
\]

(7.14)

and so

\[
\mathcal{C}_H(\sigma_{j+1}) \leq |V(\sigma_{j+1}) - V(\sigma_j)| + V(\sigma_j) + H \delta(\sigma_{j+1}) \leq \frac{3\gamma}{2H} \leq \frac{\gamma}{16}
\]

in contradiction with \( \mathcal{C}_H \) and (7.11) is verified. Next, we prove the validity of

\[
V(\sigma_{j+1}) \leq \frac{\delta(\sigma_{j+1})}{4} + 2\gamma(2-p)/p \mathfrak{G}(\sigma_j) \leq \frac{\gamma}{16}.
\]

(7.15)

In the light of (7.11), we have to consider only two possibilities: either (7.11) holds and, given (7.7), the bound imposed on the size of \( \gamma \), via (6.3) and (6.8) we directly have (7.15); or (4.19) is satisfied and

\[
\delta(\sigma_{j+1}) \leq \frac{2}{\tau^{n/2}} \delta(\sigma_j) \leq \frac{2 V(\sigma_j)^{(2-p)/p} \mathfrak{G}(\sigma_j)}{\epsilon_1 T^{n/2}}
\]

(7.16)

and (7.15) follows in any case. Before proceeding further, notice that

\[
\sum_{j=1}^{k+1} \delta(\sigma_j) \leq 1 \frac{2}{\epsilon_1 T^{n/2}} \left( I_{m,x_0,\sigma}^{\frac{p}{p''}} \right) \leq \Omega(x_0; \sigma) \frac{2}{H_2} \leq \frac{2\gamma}{H_2}.
\]

(7.17)

Summing (7.15) for \( j \in \{i, \ldots, k\} \) we obtain

\[
\sum_{j=i}^{k+1} \delta(\sigma_j) \leq \frac{1}{4} \sum_{j=i}^{k} \delta(\sigma_j) + 2\gamma(2-p)/p \sum_{j=i}^{k} \mathfrak{G}(\sigma_j)
\]

(7.18)

Adding on both sides of the previous inequality \( \delta(\sigma_i) \) and reabsorbing terms, we get (7.13). We continue estimating in (7.15):

\[
\sum_{j=i}^{k+1} \delta(\sigma_j) \leq 4 \mathcal{C}_H(\sigma_j) + \frac{2\gamma}{3\epsilon_1 T^{n/2} H_2} \mathfrak{G}(\sigma_j) \leq \frac{1}{3H} + \frac{2\gamma}{3\epsilon_1 T^{n/2} H_2} \mathfrak{G}(\sigma_j) \leq \frac{5\gamma}{12H},
\]

(7.19)

which implies (7.8). Finally, we estimate

\[
V(\sigma_{k+1}) \leq |V(\sigma_{k+1}) - V(\sigma_k)| + V(\sigma_k) \leq \frac{3\gamma}{4} + \frac{k}{4} \sum_{j=i}^{k} |V(\sigma_{j+1}) - V(\sigma_j)|
\]

(7.20)

and, combining the content of the above display with (7.9) we obtain (7.8) and the proof is complete.
7.2. Oscillation estimates for large gradients. For some $\sigma \in (0, \varepsilon)$ we consider the case in which
\begin{equation}
(7.17) \quad \gamma/8 := V(\sigma) > \frac{\eta_1(x_0; \sigma)}{16} \implies \eta(x_0; \sigma) \leq 2\gamma
\end{equation}
where we used that by (7.9), (7.10) and (7.11), estimates (7.10) of Corollary 6.1 are available. Recall also Remark 6.1. We proceed with two technical lemmas, eventually leading to quantitative oscillation estimates for nonzero gradients.

**Lemma 7.2.** Assume (7.17). Then
\begin{equation}
(7.18) \quad \sum_{j=0}^{\infty} \tilde{\sigma}(j) \leq \frac{\eta_0(x_0; \sigma)}{H} \quad \text{and} \quad \gamma/16 \leq V(\sigma_j) \leq \gamma \quad \text{for all} \quad j \in \mathbb{N} \cup \{0\},
\end{equation}
for any $\sigma \in (0, \varepsilon)$ with $H \equiv H(\text{data}, M)$ defined in (8.3).2.

**Proof.** Let us first prove that
\begin{equation}
(7.19) \quad V(\sigma_j) \geq \frac{V(\sigma_0)}{2} \quad \text{for all} \quad j \in \mathbb{N} \cup \{0\}.
\end{equation}

Notice that
\begin{equation}
V(\sigma_j) \geq V(\sigma_0) - |V(\sigma_j) - V(\sigma_0)| \geq V(\sigma_0) - \tilde{\sigma}(j) \leq \frac{V(\sigma_0)}{2}.
\end{equation}

By contradiction, we assume that there is a finite exit time $\mathbf{V}_\gamma$ such that
\begin{equation}
(7.20) \quad V(\sigma_j) < \frac{V(\sigma_0)}{2} \quad \text{and} \quad V(\sigma_j) \geq \frac{V(\sigma_0)}{2} \quad \text{for all} \quad j \in \{0, \ldots, J - 1\}.
\end{equation}

Let us preliminary observe that
\begin{equation}
(7.21) \quad V(\sigma_j) \geq \frac{V(\sigma_0)}{2} \quad \text{for all} \quad j \in \{0, \ldots, J - 1\} \implies \mathcal{E}_H(\sigma_j) \leq \gamma \quad \text{for all} \quad j \in \{0, \ldots, J - 1\}.
\end{equation}

To show the validity of implication (7.21), we proceed by induction. By direct calculation, we see that
\begin{equation}
(7.22) \quad \mathcal{E}_H(\sigma_0) \leq \frac{V(\sigma_0)}{2} \leq \frac{\gamma}{2H} \quad \text{for all} \quad j \in \{0, \ldots, J - 1\}.
\end{equation}

We then fix an arbitrary $k \in \{0, \ldots, J - 2\}$, assume that $\mathcal{E}_H(\sigma_j) \leq \gamma$ holds for all $j \in \{0, \ldots, k\}$ and notice that (7.20) and (7.11) yield that $\mathcal{E}_H(\sigma_{j+1}) \geq \gamma/16$ for all $j \in \{0, \ldots, k\}$, therefore keeping (7.17) in mind, we deduce that the assumptions of Lemma 7.1 are verified with $i = 0$ and $k$ being the number used here so $\mathcal{E}_H(\sigma_{k+1}) \leq \gamma$. Implication (7.21) then follows by the arbitrariness of $k \in \{0, \ldots, J - 2\}$. By (7.20) and (7.21) now we know that $\mathcal{E}_H(\sigma_j) \leq \gamma$ for all $j \in \{0, \ldots, J - 1\}$ and $\mathcal{E}_H(\sigma_{j+1}) \geq \gamma/16$ for all $j \in \{0, \ldots, J - 2\}$, thus via (7.18) we can apply again Lemma 7.1 with $i = 0$ and $k = J - 2$ to get
\begin{equation}
(7.23) \quad \sum_{j=0}^{J-1} \tilde{\sigma}(j) \leq \frac{\gamma}{2H} \leq \frac{4V(\sigma_0)}{H},
\end{equation}
so we can bound
\begin{equation}
(7.24) \quad |V(\sigma_j) - V(\sigma_0)| \leq \sum_{j=0}^{J-1} |V(\sigma_{j+1}) - V(\sigma_j)| \leq \frac{\gamma}{2H} \leq \frac{4V(\sigma_0)}{H} \leq \frac{V(\sigma_0)}{4},
\end{equation}
for concluding:
\begin{equation}
(7.25) \quad V(\sigma_j) \geq V(\sigma_0) - |V(\sigma_j) - V(\sigma_0)| \geq \frac{3V(\sigma_0)}{4},
\end{equation}
in contradiction with (7.20). This and the arbitrariness of $J \geq 2$ yield validity of (7.19), which in turn implies the left-hand side of inequality (7.18) and, applying (7.21) for all $j \in \mathbb{N} \cup \{0\}$ we derive the full chain of inequalities in (7.18).2. We only need to verify (7.18).1. Using (7.11), (7.18)2 and (7.22) we apply Lemma 7.1 with $i = 0$ and for every integer $k$ to have
\begin{equation}
(7.26) \quad \sum_{j=0}^k \tilde{\sigma}(j) \leq \frac{4\tilde{\sigma}(\sigma_0)}{3} + \frac{2^{3+4n} \zeta(2(p)/p)}{3^2 H^{4n}} \sum_{j=0}^{\infty} \tilde{\sigma}(j) \leq \frac{8\tilde{\sigma}(\sigma_0)}{3H^{4n}} + \frac{2^{3+4n} \zeta(2(p)/p)}{3^2 H^{4n}}
\end{equation}
rounded 8\|x_0; \sigma) + \frac{2^{10n}V(x_0)^{(p-2)/p}1_{I_{n/m}}(x_0, \sigma)}{3\pi^{1/2}H_1}
\leq \frac{2^{10n}V(x_0)^{(p-2)/p}1_{I_{n/m}}(x_0, \sigma)}{3\pi^{1/2}H_1} + \frac{2^{12n}\overline{\mathcal{S}}(\sigma_1)^{(2-p)/p}1_{I_{n/m}}(x_0, \sigma)}{3\pi^{1/2}H_1}
\leq \overline{\mathcal{S}}(x_0; \sigma)\left(\frac{8}{3\pi^{n/2}H_1} + \frac{2^{10n}}{3\pi^{1/2}H_1} + \frac{2^{12n}}{3\pi^{1/2}H_1}\right)
\leq \overline{\mathcal{S}}(x_0; \sigma)\frac{6}{\pi H_1}
\leq \frac{\overline{\mathcal{S}}(x_0; \sigma)}{H_1},

where we also used Young inequality with conjugate exponents \(\left(\frac{p}{p-2}, \frac{p}{2}\right)\) and the proof is complete. \(\square\)

Lemma 7.3. Whenever \(\sigma \in (0, \rho]\) is such that (7.14) holds, the limits in (1.11) exist and inequalities

\[
|Du(x_0) - (Du)_{n(x_0)}| \leq c\mathcal{S}(x_0; \sigma)^{2/p} + c|\mathcal{D}(Du)_{n(x_0)}|^{(2-p)/2}\mathcal{S}(x_0; \sigma)
\]

hold true for a constant \(c \equiv c(\mathbf{data}, M)\).

Proof. We start by showing that \(\{(V_p(Du))_{B_j}\}_{j \in \mathbb{N}\cup\{0\}}\) is a Cauchy sequence. In fact, fixed integers \(0 \leq i \leq k - 1\) we bound

\[
|\mathcal{D}x_0; \sigma) + \frac{2^{10n}V(x_0)^{(p-2)/p}1_{I_{n/m}}(x_0, \sigma)}{3\pi^{1/2}H_1}
\leq \frac{2^{10n}V(x_0)^{(p-2)/p}1_{I_{n/m}}(x_0, \sigma)}{3\pi^{1/2}H_1} + \frac{2^{12n}\overline{\mathcal{S}}(\sigma_1)^{(2-p)/p}1_{I_{n/m}}(x_0, \sigma)}{3\pi^{1/2}H_1}
\leq \overline{\mathcal{S}}(x_0; \sigma)\left(\frac{8}{3\pi^{n/2}H_1} + \frac{2^{10n}}{3\pi^{1/2}H_1} + \frac{2^{12n}}{3\pi^{1/2}H_1}\right)
\leq \overline{\mathcal{S}}(x_0; \sigma)\frac{6}{\pi H_1}
\leq \frac{\overline{\mathcal{S}}(x_0; \sigma)}{H_1},
\]

where we also used Young inequality with conjugate exponents \(\left(\frac{p}{p-2}, \frac{p}{2}\right)\) and the proof is complete. \(\square\)

Lemma 7.3. Whenever \(\sigma \in (0, \rho]\) is such that (7.14) holds, the limits in (1.11) exist and inequalities

\[
|Du(x_0) - (Du)_{n(x_0)}| \leq c\mathcal{S}(x_0; \sigma)^{2/p} + c|\mathcal{D}(Du)_{n(x_0)}|^{(2-p)/2}\mathcal{S}(x_0; \sigma)
\]

hold true for a constant \(c \equiv c(\mathbf{data}, M)\).

Proof. We start by showing that \(\{(V_p(Du))_{B_j}\}_{j \in \mathbb{N}\cup\{0\}}\) is a Cauchy sequence. In fact, fixed integers \(0 \leq i \leq k - 1\) we bound

\[
|V_p(Du))_{B_k} - (V_p(Du))_{B_j}| \leq \sum_{j=1}^{k-1} |V_p(Du))_{B_{j+1}} - (V_p(Du))_{B_j}|
\leq \frac{1}{\pi^{n/2}} \sum_{j=1}^{k-1} \overline{\mathcal{S}}(\sigma_j) \leq \frac{1}{\pi^{n/2}} \sum_{j=1}^{\infty} \overline{\mathcal{S}}(\sigma_j) \leq c\mathcal{S}(x_0; \sigma),
\]

and

\[
|(V_p(Du))_{B_0} - (V_p(Du))_{B_{-1}}| \leq \frac{1}{\pi^{n/2}} \sum_{j=1}^{k-1} \overline{\mathcal{S}}(\sigma_j) \leq c\mathcal{S}(x_0; \sigma)
\]

with \(c \equiv c(\mathbf{data}, M)\), therefore there exists \(\ell_V \in \mathbb{R}^{N \times n}\) such that

\[
\lim_{j \to \infty} (V_p(Du))_{B_j} = \ell_V.
\]

Sending \(k \to \infty\) in (7.20) we obtain

\[
|\ell_V - (V_p(Du))_{B_i}| \leq c\mathcal{S}(x_0; \sigma) \quad \text{for all} \quad i \in \mathbb{N}\cup\{0\}.
\]

Now, given any \(s \in (0, \rho] - \) and since we are interested in \(s \to 0\) we can assume \(s \leq \sigma_0\) - there is \(j_s \in \mathbb{N}\cup\{0\}\) such that \(j_s < s \leq j_{s+1}\), and

\[
\lim_{s \to 0} |\ell_V - (V_p(Du))_{B_{j_s(x_0)}}| \leq \lim_{j_s \to \infty} |\ell_V - (V_p(Du))_{B_{j_s}}| + \lim_{j_s \to \infty} |(V_p(Du))_{B_{j_s(x_0)}} - (V_p(Du))_{B_{j_s}}|
\leq \frac{1}{\pi^{n/2}} \sum_{j_s=1}^{\infty} \overline{\mathcal{S}}(\sigma_{j_s}) \leq 0,
\]

and the first limit in (7.11) equals \(\ell_V\), which defines the precise representative of \(V_p(Du)\) at \(x_0\), i.e.: \(\ell_V = (V_p(Du))(x_0)\). Next, notice that whenever \(B \subseteq \Omega\) is a ball, by (2.1) and (5.5) it is

\[
\overline{\mathcal{S}}(u; B) \approx \left(\int_B |V_p(Du) - V_p((Du)_{B_i})|^2 \, dx\right)^{1/2},
\]

with constants implicit in "\(\approx\)" depending only on \(p\), so for any given \(j \in \mathbb{N}\cup\{0\}\) it is

\[
|(Du)_{B_j}| \leq \overline{\mathcal{S}}(V_p(Du); B_j)^{2/p} \leq c\overline{\mathcal{S}}(\sigma_j)^{2/p} + cV(\sigma_j)^{2/p}
\]

with \(c \equiv c(p)\), while for \(j = -1\) via Hölder and Young inequalities with conjugate exponents \(\left(\frac{2p}{p-2}, \frac{2p}{p}\right)\) we have

\[
|(Du)_{B_{-1}}| \leq \overline{\mathcal{S}}(V_p(Du); B_0)^{2/p} + \left(\int_{B_0} |Du - (Du)_{B_{-1}}|^p \, dx\right)^{1/p}
\leq \frac{c\overline{\mathcal{S}}(\sigma_{-1})^{2/p}}{\pi^{n/p}} + cV(\sigma_0)^{2/p} + c\left(\int_{B_0} |V_p(Du) - V_p((Du)_{B_{-1}})|^2 \, dx\right)^{1/p}
\]

with \(c \equiv c(p)\).
\begin{equation}
\frac{1}{2\sigma_1} V(\sigma_1) \leq c \mathcal{F}(\sigma_1)^{2/\rho} + cV(\sigma_0)^{2/\rho},
\end{equation}

for \( c \equiv c(\text{data}, M) \) and using this time Young inequality with conjugate exponents \((\frac{2}{s}, \frac{2}{p'})\) we get

\begin{align*}
V(\sigma_0) &\leq \mathfrak{N}(Du; B) \mathfrak{R}(\mathfrak{N}(Du; B))^{1/2} + c((Du)_B^{1/2}) + c((Du)_B^{1/2})^{1/2} \\
&\leq c((Du)_B^{1/2})^{1/2} + c\left(\mathfrak{N}(Du; B) - \mathfrak{N}(Du; B)\right) \leq \left(\mathfrak{N}(Du; B) - \mathfrak{N}(Du; B)\right)^{1/2} \leq c(p)\mathfrak{N}(u; B)
\end{align*}

and then estimate for integers \( 0 \leq i \leq k - 1 \):

\begin{align*}
&|((Du)_B)_i - (Du)_B| \leq \sum_{j=i}^{k-1} |((Du)_B)_{j+1} - (Du)_B| \\
&\leq c \sum_{j=i}^{k-1} \mathfrak{N}(Du; B)_{j+1} - \mathfrak{N}(Du; B)_j |((Du)_B)_{j+1} - (Du)_B|^{(2-p)/2} \\
&\leq c \left(\mathfrak{N}(x_0; \sigma)^{(2-p)/\rho} + \gamma^{(2-p)/\rho}\right) \sum_{j=i}^{k-1} ((Du)_B)_{j+1} - ((Du)_B)_j \\
&\leq c \left(\mathfrak{N}(x_0; \sigma)^{(2-p)/\rho} + \gamma^{(2-p)/\rho}\right) \sum_{j=i}^{k-1} ((Du)_B)_{j+1} - ((Du)_B)_j
\end{align*}

for \( c \equiv c(\text{data}, M) \). Recalling (7.65), we get that \( \{(Du)_B\}_{i \in \mathbb{N} \cup \{0\}} \) is a Cauchy sequence and there exists \( \ell \in \mathbb{R}^{N \times n} \) such that \( \lim_{i \to \infty} (Du)_{B_i} = \ell \). A standard interpolative argument analogous to that leading to (7.29) allows concluding that \( \ell \) defines the precise representative of \( Du \) at \( x_0 \), i.e. \( Du(x_0) = \ell \) and this assures the validity of the second limit in (7.14). Combining this last information with (7.26) and (7.28) we get that \( \mathfrak{F}_V(V(x_0)) = V(x_0) \) so via (7.29) we eventually recover the first limit in (7.14). Finally, merging (7.27), (7.28) and recalling that \( V_p(Du(x_0)) = V_p(Du(x_0)) \) we obtain (7.29). The proof is complete.

7.3. Oscillation estimates for small gradients. In this section we look at what happens when the complementary condition to (7.17) holds, i.e. when for \( \sigma \in (0, \epsilon] \) it is

\begin{equation}
\frac{1}{8} \geq V(x_0) = \frac{\mathfrak{N}(x_0; \sigma)}{16} \geq V(\sigma_0) \implies \mathfrak{N}(x_0; \sigma) = 2\gamma.
\end{equation}

Let us observe that to avoid trivialities, we can suppose \( \gamma > 0 \), and that there is no loss of generality in assuming that (7.36) actually holds for all \( s \in (0, \sigma] \). In fact, if for some \( s \in (0, \sigma] \) the opposite inequality to (7.36), i.e. (7.17) holds, then we can directly conclude with (7.14) and (7.29). The validity of (7.36) for
all $s \in (0, \sigma]$, \eqref{7.38} and \eqref{7.39} yield that $\lim_{s \to 0} (V_p(Du))_{B_s(x_0)} = 0$, therefore keeping in mind that also $\lim_{s \to 0} V(s) = 0$ and that
\begin{equation}
\| (Du)_{B_s(x_0)} \| \leq c(p) \left( \tilde{\f}(u; B_s(x_0))^{2/p} + V(s)^{2/p} \right)
\end{equation}
we can conclude that $\lim_{s \to 0} (Du)_{B_s(x_0)} = 0$ and the existence of the two limits in \eqref{7.36} is proven. Next, we show the validity of \eqref{7.37} also in the case in which \eqref{7.36} is in force. Let us prove by induction that
\begin{equation}
\mathcal{E}_H(\sigma_j) \leq \gamma \quad \text{for all } j \in \mathbb{N} \cup \{0\}.
\end{equation}
A direct computation renders:
\begin{equation}
\mathcal{E}_H(\sigma_0) \leq \frac{\mathcal{M}(x_0; \sigma)}{16} + \frac{2H\mathcal{S}(\sigma_0)}{\tau^\alpha/2} \leq \frac{\mathcal{M}(x_0; \sigma)}{16} + \frac{2H}{\tau^\alpha/2H_1} \leq \gamma.
\end{equation}
Then, we assume by contradiction that \{$j \in \mathbb{N} \cup \{0\} : \mathcal{E}_H(\sigma_{j+1}) > \gamma \} \neq \{0\}$, define $l := \min \{j \in \mathbb{N} \cup \{0\} : \mathcal{E}_H(\sigma_j) > \gamma\}$, i.e. the smallest integer minus one for which \eqref{7.38} fails, introduce the set $\tilde{\gamma}_l := \{ j \in \mathbb{N} \cup \{0\} : \mathcal{E}_H(\sigma_j) \leq \gamma/4, j < l + 1 \}$ and set $\chi := \max \{ \tilde{\gamma}_l \}$. Notice that by \eqref{7.39} it is $\tilde{\gamma}_l \neq \{0\}$, by definition $\mathcal{E}_H(\sigma_\chi) \leq \gamma/4$ and for $j \in \{ \chi, \ldots, l \}$ we have $\gamma \geq \mathcal{E}_H(\sigma_{j+1}) \geq \gamma/4 > \gamma/16$, therefore, recalling also \eqref{7.38}, we can apply Lemma 7.4 with $i \equiv \chi$ and $k \equiv l$ to conclude that $\mathcal{E}_H(\sigma_{j+1}) \leq \gamma$ in contradiction with the definition of $l$. This means that \{$j \in \mathbb{N} \cup \{0\} : \mathcal{E}_H(\sigma_{j+1}) > \gamma \} = \{0\}$ and \eqref{7.37} holds true. Next, we take any $s \in (0, \sigma_0]$, determine $j_s \in \mathbb{N} \cup \{0\}$ such that $\sigma_{j_s+1} < s \leq \sigma_{j_s}$, and estimate
\begin{equation}
V(s) \leq V(\sigma_{j_s}) + \frac{3\mathcal{S}(\sigma_{j_s})}{\tau^\alpha/2} \leq \frac{3 \mathcal{M}(x_0; \sigma_0)}{16} + \frac{3 \mathcal{M}(x_0; \sigma_0)}{\tau^\alpha/2H_1} \leq \gamma \leq \mathcal{M}(x_0; \sigma).
\end{equation}
Moreover, if $s \in (\sigma_0, \sigma_1]$ we directly obtain
\begin{equation}
V(s) \leq V(\sigma_0) + |V(\sigma_0) - V(s)| \leq V(\sigma_0) + \frac{3 \mathcal{S}(\sigma_{1-})}{\tau^\alpha} \leq \gamma + \frac{3 \mathcal{M}(x_0; \sigma_0)}{\tau^\alpha H_1} \leq \gamma = \mathcal{M}(x_0; \sigma),
\end{equation}
so in any case it is $V_{s,\sigma} \leq \mathcal{M}(x_0; \sigma)$, which in turn implies that $V(\sigma_{1-}) - V(s) \leq 2\mathcal{M}(x_0; \sigma)$ and \eqref{7.39}, can now be derived by sending $s \to 0$ in the previous inequality and recalling \eqref{6.11}. Concerning \eqref{7.29}$_2$, we use \eqref{7.31}-\eqref{7.32} and \eqref{7.33} to deduce that $\| (Du)_{B_s} \| \leq c\gamma^{2/p}$ for all $j \in \mathbb{N} \cup \{-1, 0\}$. This, the same interpolation argument exploited before and standard manipulations eventually render that $\| (Du)_{B_{\sigma_1}} - (Du)_{B_{\sigma_0}} \| \leq c\gamma^{2/p}$, which, together with \eqref{7.30} and \eqref{6.11} yield \eqref{7.29}$_2$ by sending $s \to 0$. In conclusion, we have just proven the following lemma.

\begin{lemma}
Assume that \eqref{7.36} holds for some $\sigma \in (0, \varrho]$. Then the limits in \eqref{6.11} exist and the bounds in \eqref{7.29}$_2$ are verified.
\end{lemma}

7.4. Sharp partial gradient continuity and proof of Theorem 12
Let us complete the proof of Theorem 12 started in Sections 7.2-7.3.

\begin{proof}[Proof of Theorem 12]
Let $x_0 \in \mathcal{R}_\alpha$ be a point satisfying \eqref{1.0}, $M \equiv M(x_0) > 0, \varepsilon (0, 1), \tilde{\varrho} \in (0, \min\{1, d_{x_0}\})$ be the corresponding parameters in \eqref{6.1} with $\varepsilon, \tilde{\varrho}$ to be determined. We define $\varepsilon := 2^{-1+\varepsilon}$ and suitably reduce the threshold radius to determine $\tilde{\varrho} \equiv \varrho (0, \varrho')$ in such a way that inequality \eqref{7.7} holds with $\varepsilon' = \varepsilon$ replacing $\varepsilon'$ for all $s \in (0, \varrho]$. Setting $\varepsilon \equiv \varepsilon/2$ and $\tilde{\varrho} \equiv \varrho$ in \eqref{6.10} we see that both \eqref{6.10} and the assumptions in force in Sections 7.2-7.3 are satisfied, therefore the existence of the limits in \eqref{6.11} follows from Lemmas 7.3-7.4 while the (almost) pointwise oscillation estimates in \eqref{6.12} are exactly those appearing in \eqref{7.25}. We are only left with the proof of the assertion on the Lebesgue points of $V_p(Du)$ and $Du$. Let us first assume that $x_0$ verifies both \eqref{1.0} and \eqref{6.10} with the just fixed parameters $\varepsilon \equiv \varepsilon (data, M)$ and $\tilde{\varrho} \equiv \varrho (data, M, f(\cdot))$. This choice assures that \eqref{6.11}, \eqref{6.70} and \eqref{6.82} are available, and this in particular assures that $x_0$ is a Lebesgue point of $V_p(Du)$. Moreover, with $\sigma \in (0, \varrho]$, recalling \eqref{6.11}$_2$, we bound by means of \eqref{7.25}, \eqref{6.30}, \eqref{6.70}, \eqref{6.82} and \eqref{6.52},
\begin{equation}
\left( \int_{B_{\sigma_0}(x_0)} \left| Du - (Du)_{B_{\sigma_0}(x_0)} \right|^p dx \right)^{1/p} \leq \frac{c}{\rho(x_0)} \left( \int_{B_{\sigma_0}(x_0)} \left| V_p(Du) - V_p((Du)_{B_{\sigma_0}(x_0)}) \right|^p \, dx \right)^{1/p} + \frac{c}{\rho(x_0)} \left( \int_{B_{\sigma_0}(x_0)} \left| V_p(Du) - V_p((Du)_{B_{\sigma_0}(x_0)}) \right|^2 \, dx \right)^{1/p} \leq \frac{c\tilde{\varrho}(u; B_{\sigma_0}(x_0))^{2/p} + c((Du)_{B_{\sigma_0}(x_0))^{(2-p)/2}}} \leq \frac{c\tilde{\varrho}(u; B_{\sigma_0}(x_0))^{2/p} + c(1 + M)^{(2-p)/p}} \equiv \tilde{\varrho}(u; B_{\sigma_0}(x_0)) \to 0
\end{equation}
with $c \equiv c(n, N, p)$ and $x_0$ is a Lebesgue point of $Du$ as well. On the other hand, if $x_0$ is a Lebesgue point of $V_p(Du)$ we know that $\tilde{\varrho}(u; B_{\sigma_0}(x_0)) \to 0$ and that \eqref{6.11}$_2$ exists, therefore, recalling that \eqref{1.0} is in force, we can
fix $\theta$ so small that \((1.10)\) holds and set $M := 2 \limsup_{\sigma \to 0} |(V_{\theta}(Du))_{B_\sigma(x_0)}| + 1$ to verify also \((1.10)\). Finally, if $x_0$ is a Lebesgue point of $Du$, then

\[
\left( \int_{B_{\sigma}(x_0)} |Du - (Du)_{B_\sigma(x_0)}|^p \, dx \right)^{1/p} \to 0, \quad \limsup_{\sigma \to 0} |(Du)_{B_\sigma(x_0)}| < \infty
\]

and \((1.11)\) exists. Since

\[
|V_{\theta}(Du)| \leq c_{p}(Du; B_\sigma(x_0))^{p/2} \leq c \left( \int_{B_{\sigma}(x_0)} |Du - (Du)_{B_\sigma(x_0)}|^p \, dx \right)^{1/2} + c((Du)_{B_\sigma(x_0)})^{p/2},
\]

with $c \equiv c(p)$ and, via triangular inequality,

\[
\bar{g}(u; B_\sigma(x_0)) \leq c \left( \int_{B_{\sigma}(x_0)} |V_{\theta}(Du)| \, dx \right)^{1/2} \leq c \left( \int_{B_{\sigma}(x_0)} |Du| + |(Du)_{B_\sigma(x_0)}| (p-2)|Du - (Du)_{B_\sigma(x_0)}|^2 \, dx \right)^{1/2}
\]

\[
\leq c \left( \int_{B_{\sigma}(x_0)} |Du - (Du)_{B_\sigma(x_0)}|^p \, dx \right)^{1/2} \to 0,
\]

for $c \equiv c(n, N, p)$, keeping \((1.10)\) and \((1.12)\) in mind we can choose $\theta$ so small that \((1.10)\) holds true, and setting $M := c + 2c \limsup_{\sigma \to 0} |(Du)_{B_\sigma(x_0)}|^{p/2}$ where $c \equiv c(p)$ is the constant appearing in \((7.41)\), we obtain also \((1.10)\) and the proof is complete.

Next, we prove Theorem 1.

**Proof of Theorem 1.** Since our results are local in nature, we can assume that \((1.5)\) holds globally in $\Omega$ - notice that being \((1.5)\) in force, we can always assume the validity of \((6.75)\). Let $R_\varepsilon$ be the set defined in \((6.1)\) with $\varepsilon \equiv \varepsilon$, $\tilde{\varepsilon} \equiv \tilde{\varepsilon}$ and $\hat{\varepsilon}$, $\tilde{\hat{\varepsilon}}$ defined in the proof of Theorem 2. The discussion at the beginning of Section 6 see also \([23]\) Section 5.1, yields that $R_\varepsilon$ is an open set of full $n$-dimensional Lebesgue measure and $|\Omega \setminus R_\varepsilon| = 0$ therefore given any $x_0 \in R_\varepsilon$ with the specifics described before there is an open neighborhood $B(x_0)$ of $x_0$ and a positive radius $\varrho_{x_0} \in (0, \tilde{\varrho})$ such that $|V_{\theta}(Du))_{B_{\varrho_{x_0}}(x_0)}| < M$ and $\bar{g}(u; B_{\varrho_{x_0}}(x_0)) < \bar{\varepsilon}$. Given \((1.5)\) and our choice of $\tilde{\varepsilon}$, $\varepsilon$ we see that \((1.11)\) holds on $B(x_0)$, Corollary 6.1, Theorem 2 and Proposition 6.1 apply, the limits in \((1.11)\) exist and define the precise representative of $V_{\theta}(Du)$ and of $Du$ at all $x \in B(x_0)$. With these informations at hand, we aim to prove that the limits in \((1.11)\) are uniform in the sense that the continuous maps $B(x_0) \ni x \mapsto (V_{\theta}(Du))_{B_{\varrho_{x_0}}(x)}$, $B(x_0) \ni x \mapsto (Du)_{B_{\varrho_{x_0}}(x)}$ with $s \in (0, \varrho_{x_0}]$ uniformly converge to $V_{\theta}(Du(x))$ and to $Du(x)$ respectively as $\sigma \to 0$ thus yielding that $V_{\theta}(Du)$ and $Du$ are continuous on $B(x_0)$. This is a consequence of the two inequalities in \((1.12)\) as their right-hand side uniformly converges to zero by means of \((1.8)\), \((6.75)\), \((6.68)\) and \((5.37)\). The proof is complete.

**7.5. Optimal function space criteria and proof of Theorem 3.** This final section is devoted to the proof of Theorem 2. Once noticed that

\[
\begin{cases}
  f \in L(n,1) \implies I^1_{\omega^m}(x_0, s) \to s \to 0 \quad \text{uniformly in } x_0 \in \Omega \\
  f \in L^d, \quad d > n \implies I^1_{\omega^m}(x_0, s) \leq \frac{d^{1-n/d} \|f\|_{L^d}}{(d-n)\omega_n^{1/d}}
\end{cases}
\]

cf. \([23]\) Section 2.3 and \([23]\) Section 6.5] respectively, keeping in mind \((6.68)\), the proof goes exactly as in \([23]\) Proof of Theorem 2.

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