COMPLETE WAVE OPERATORS
IN NON-SELFADJOINT KATO MODEL OF
SMOOTH PERTURBATION THEORY

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Abstract. Subject of the paper deals with the perturbation theory of linear
operators acting in Hilbert space. For a certain class of perturbations the question is
considered about existence of transformation operators implementing linear similarity
of perturbed and unperturbed operators. In this context some results of complex
analysis prove to be useful as well as the relationship with the theory of operator
semi-groups.

Keywords: wave operators, scattering theory technique, Schroedinger operator

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§ 1. Introduction

The fundamental notion of relative smoothness was suggested by T.Kato [1]
within the framework of stationary scattering theory technique elaborated therein
for a certain class of not necessarily selfadjoint operators. The goal of the approach
worked out in [1] was construction of wave operators intertwining perturbed and
unperturbed operators $T + V$ and $T$. As a matter of fact the original Kato paper
deals with small perturbations of the form $V = B^* A$ with operators $A$ and $B$
smooth relatively to $T$ and $T^*$ respectively (for precise definitions see Section 2).
Besides it was assumed in [1] that spectrum of $T$ is real and purely continuous while
operator $Q_0 (\lambda) = A (T - \lambda I)^{-1} B^*$ has to be bounded uniformly for non-real $\lambda$.
Wave operators constructed in this context prove to be invertible and thus implement
similarity of initial and perturbed operators. Moreover the statement of the problem
also goes back to [1] about conditions under which the approach in question admits
a generalization to the case of arbitrary (not necessarily small) perturbations of the
class under consideration. Note that the technique of factorizing perturbation $V$
into the form $B^* A$ originated in [2].

Subsequent development of stationary scattering theory in the case of relatively
smooth perturbations basically dealt with selfadjoint operators (see [3]). As regards
applications of the technique elaborated in [1] to non-selfadjoint operators $T + V$ when perturbation $V = B^*A$ is not assumed to be small, such attempts essentially reduced (see e.g. [4],[5]) to construction of local wave operators which supply with (and require) information on the structure of perturbed operator $T + V$ in the subspace corresponding to a certain segment of continuous spectrum. One of the goals of the present paper is to formulate conditions under which stationary scattering theory approach in the framework of non-selfadjoint Kato model context admits an extension to the case of not necessarily small perturbations. Wave operators to be constructed within this approach prove to possess completeness property and thus realize the passage to spectral representation of the perturbed operator implemented with regard for (and in terms of) spectral continuity of the initial one.

Construction of local wave operators in the paper [5] was carried out under the condition that operator $(I + Q_0(\lambda))^{-1}$ is uniformly bounded in the neighborhood of the corresponding continuous spectrum segment of operator $T$. In the present paper complete wave operators intertwining $T + V$ and $T$, are going to be constructed provided that given resolvent $R_V(\lambda) = (T + V - \lambda I)^{-1}$ the operator function $Q_V(\lambda) = AR_V(\lambda)B^*$ is bounded uniformly for the values of its argument separated away from discrete spectrum of $T + V$. As regards the question how to effectively verify this condition it can be reduced to searching for a scalar analytic function, most likely an appropriate Fredholm determinant, such that all the singularities of operator function $Q_V(\lambda)$ after being multiplied by it become removable in $\mathbb{C}_\pm$. By virtue of resolvent identity

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda)B^*(I + Q_0(\lambda))^{-1}AR_0(\lambda)$$

condition suggested above is less restrictive compared to that required in [5] while the class of perturbations $V = B^*A$ considered here is wider in due turn.

As an application we shall consider one-dimensional Schrödinger operator $L = -d^2/dx^2 + V(x)$ on half-axis $\mathbb{R}_+$ with complex potential $V(x)$ and Dirichlet boundary condition at zero. Such an operator proves to be quite a simple and rather capacious model which displays a number of effects typical for perturbation theory in non-selfadjoint setting (see [6] and [7]). Besides that Schrödinger operator with complex potential is known (see [8]) to appear in the study of open quantum mechanical systems with energy dissipation.

The approach elaborated in [1] reveals the following condition

$$\int_0^{\infty} x |V(x)| \, dx < 1$$

which guarantees similarity of $L$ and selfadjoint operator $T = -d^2/dx^2$ corresponding to $V(x) \equiv 0$. One of the principal results of the present paper implies a criterion of the similarity in question valid for the class of potentials $V(x)$ possessing a finite first momentum which extends and supplements Kato sufficient condition [1].
As an unperturbed one consider closed operator $T$ with domain $D(T)$ dense in Hilbert space $\mathcal{H}$ possessing purely continuous spectrum $\sigma(T) \subset \mathbb{R}$ such that resolvent $R_0(\lambda) = (T - \lambda I)^{-1}$ is analytic in $\mathbb{C}_\pm$. In accordance with [1] closed operator $A$ is said to be smooth relative to $T$ if $AR_0(\lambda)\varphi \in \mathbb{H}^2_\pm$ for arbitrary $\varphi \in \mathcal{H}$, where $\mathbb{H}^2_\pm$ denote Hardy classes corresponding to half-planes $\mathbb{C}_\pm$, i.e.

$$\sup_{\varepsilon > 0} \int_{\mathbb{R}} ||AR_0(k \pm i\varepsilon)\varphi||^2 dk < \infty.$$ 

Moreover for almost all $k \in \mathbb{R}$ limits $AR_0(k \pm i0)\varphi := \lim_{\varepsilon \downarrow 0} AR_0(k \pm i\varepsilon)\varphi$ exist (see [9]) and besides $AR_0(k \pm i\varepsilon)\varphi \to AR_0(k \pm i0)\varphi$ in mean square sense as $\varepsilon \downarrow 0$.

Along with $T$ introduce perturbed operator $L = T + V$ where $V = B^*A$ so that $A$ is smooth relative to $T$ while $B$ is smooth relative to $T^*$. Below throughout the following requirements are assumed to be fulfilled:

(i) resolvent $R_V(\lambda)$ is meromorphic in $\mathbb{C} \setminus \sigma(T)$ so that its poles are just the eigenvalues of $L$ with multiplicities taken into account;

(ii) discrete spectrum $\sigma_d(L)$ of operator $L$ is a finite set while the complementary component $\sigma_c(L)$ of its spectrum coincides with $\sigma(T)$;

(iii) operator function $Q_V(\lambda) = AR_V(\lambda)B^*$ is uniformly bounded in $\mathbb{C}_\pm$ provided that the argument $\lambda$ is separated away from $\sigma_d(L)$.

For the sake of simplicity all the considerations will be carried out here in the situation when $A$ and $B$ are bounded operators and thus $D(L) = D(T)$. By means of contour integral define Riesz projection

$$P = -\frac{1}{2\pi i} \oint_{\Gamma} R_V(\lambda) d\lambda$$

onto the linear span $\mathcal{H}_d := P\mathcal{H}$ of root (eigen and associated) vectors of operator $L$ parallel to the subspace $\mathcal{H}_c := (I - P)\mathcal{H}$; here $\Gamma$ is appropriately oriented closed contour separating mutually complementary spectral components $\sigma_d(L)$ and $\sigma_c(L)$. Operator $L$ can be decomposed with respect to direct sum representation $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_d$ in the following (see [10]) sense

$$PD(L) \subset D(L), \quad L\mathcal{H}_c \subset \mathcal{H}_c, \quad L\mathcal{H}_d \subset \mathcal{H}_d,$$

while $\sigma(L|\mathcal{H}_c) = \sigma_c(L)$ and $\sigma(L|\mathcal{H}_d) = \sigma_d(L)$. By $R_V^*(\lambda)$ and $\mathcal{H}_c^*, \mathcal{H}_d^*$ let us denote the resolvent of operator $L^*$ and its invariant subspaces corresponding to disjoint spectral components $\sigma_d(L^*)$ and $\sigma_c(L^*) \subset \mathbb{R}$. 

§ 2. Statement of basic results
**Theorem 1.** Assume that operator $T$ has purely continuous spectrum $\sigma(T) \subset \mathbb{R}$, operator $V$ admits factorization $V = B^*A$ where $A$ and $B$ are smooth relative to $T$ and $T^*$ respectively, operator function $Q_0(\lambda) = AR_0(\lambda)B^*$ is bounded in $\mathbb{C}_+$, while operator $L = T + V$ satisfies conditions $(i) - (iii)$. Then there exist bounded stationary wave operators $W$ and $Z$ being determined by their bilinear forms:

\[
(W \varphi, \psi) = (\varphi, \psi) - \frac{1}{2\pi i} \int_{\mathbb{R}} (AR_0(k + i0)\varphi, BR_0^*(k + i0)\psi) \, dk \quad (1)
\]

for $\varphi \in \mathcal{H}_c$ and $(W \varphi, \psi) = 0$ if $\psi \in \mathcal{H}_d$;

\[
(Z \varphi, \psi) = (\varphi, \psi) + \frac{1}{2\pi i} \int_{\mathbb{R}} (AR_V(s + i0)\varphi, BR_0^*(s + i0)\psi) \, ds, \quad (2)
\]

when $\varphi \in \mathcal{H}_c$ and $(Z \varphi, \psi) = 0$ for $\varphi \in \mathcal{H}_d$. Besides $ZW = I$ and, moreover, wave operators possess completeness property $WZ = I - P$, where $P$ is a projection onto $\mathcal{H}_d = \ker Z$ parallel to $W\mathcal{H} = \mathcal{H}_c$.

The proof of Theorem 1 is given in Sections 3 and 4 and it follows the scheme of the approach elaborated in [1]. However in contrast with [1] operators $A$ and $B$ are by no means smooth relative to $L$ and $L^*$ respectively when $\sigma_d(L) \neq \emptyset$ and such a crucial circumstance proves to be of a considerable difficulty. Nevertheless the construction of wave operators can be somewhat modified to treat the setting in question due to the fact that $AR_V(\lambda)\varphi$ and $BR_0^*(\lambda)\psi \in \mathbb{H}^\pm_2$ provided $\varphi \in \mathcal{H}_c$ and $\psi \in \mathcal{H}_d^\ast$. To establish the fundamental properties of modified direct and inverse wave operators we take advantage of the following relationship of commutator type

\[
\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} R_0(k - i\varepsilon) [P, V] R_0(k + i\varepsilon) \, dk = P,
\]

where $[P, V] = PV - VP$. Content of section 5 is summarized by

**Theorem 2.** Under the assumptions of Theorem 1 direct and inverse wave operators $W$ and $Z$ determined by formulas (1) and (2) implement similarity of $T$ and $L_c = L|\mathcal{H}_c$ so that $L_c = WTZ$. If additionally $T = T^*$ then one has the following representations

\[
W = s\text{-}\lim_{t \to \infty} \exp(itL)(I - P)\exp(-itT),
\]

\[
Z = s\text{-}\lim_{t \to \infty} \exp(itT)\exp(-itL)(I - P).
\]

As an application of the approach developed here we shall consider in Section 6 non-selfadjoint Schroedinger operator

\[
L = T + V = -d^2/dx^2 + V(x)
\]
defined in the space $\mathcal{H} = L^2(\mathbb{R}_+)$ with the domain $\mathcal{D}(L) = \{ y \in \mathcal{H} : y' \text{ absolutely continuous, } y'' \in \mathcal{H}, y(0) = 0 \}$, where potential $V(x)$ is a bounded complex-valued function. Provided that $V \in L^1(\mathbb{R}_+)$ equation

$$\tag{3} - y'' + V(x)y = k^2y$$

for any $k \in \mathbb{C}_+$ possesses Jost solution $e(x, k)$ which is asymptotically equivalent to $e^{ikx}$ at infinity. In this situation spectrum of operator $L$ is known to consist of continuous and discrete components

$$\sigma_c(L) = \mathbb{R}_+, \quad \sigma_d(L) = \{ \lambda = k^2 : e(k) = 0, k \in \mathbb{C}_+ \},$$

where $e(k) := e(0, k)$. The set of eigenvalues $\sigma_d(L)$ is bounded, at most countable and its accumulation points (if any) belong to the half-axis $\mathbb{R}_+$, whereas operator $L$ has no eigenvalues embedded into continuous spectrum (see [6]). Furthermore Jost function $e(k)$ is analytic in the open half-plane $\mathbb{C}_+$ and admits extension by continuity to $\mathbb{R} \setminus \{0\}$, while its real zeroes correspond to distinguished points $\lambda = k^2$ of continuous spectrum $\sigma_c(L)$ called spectral singularities (see [2]).

**Theorem 3.** Suppose that non-selfadjoint Schrödinger operator

$$L = T + V = -d^2/dx^2 + V(x)$$

defined in $\mathcal{H} = L^2(\mathbb{R}_+)$ by Dirichlet boundary condition at zero and complex bounded potential $V(x)$ satisfying condition

$$\int_0^\infty x |V(x)| \, dx < \infty \quad \tag{4}$$

has no spectral singularities. Then $L$ admits decomposition with respect to direct sum $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_d$ generated by finite-dimensional Riesz projection $P$ onto the linear span $\mathcal{H}_d$ of eigen and associated vectors of operator $L$ parallel to its continuous spectrum invariant subspace $\mathcal{H}_c$. Furthermore direct and inverse stationary wave operators $W$ and $Z$ exist and all the conclusions of Theorems 1 and 2 hold true.

Absence of eigenvalues and spectral singularities in the context of Theorem 3 leads to a criterion for similarity of operator $L$ and selfadjoint operator $T$ and thus oblige $L$ to be spectral in the sense of Dunford. In the case when operator $L$ does not have any spectral singularities results of papers [6] and [7] enable one to produce transformation operators intertwining $L$ and $T$ under rather restrictive condition

$$\int_0^\infty (1 + x^2) |V(x)| \, dx < \infty.$$

However the corresponding procedure clarifies the relationship between construction of wave operators and generalized eigenfunction expansion problem for operator $L$.
Szoekefalvi-Nagy and Foias functional model technique was employed by the author to construct wave operators given dissipative Schrödinger operator with potential satisfying condition (4) (see [12] and also references therein). The results of the present paper in dissipative case were announced in [13] (cf. [14]).

§ 3. Completeness of wave operators

In what follows hypotheses of Theorem 1 are supposed to be fulfilled.

**Lemma 1.** Given \( \varphi \in \mathcal{H}_c \) and \( \psi \in \mathcal{H}_c^* \) vector functions \( AR_V(\lambda)\varphi \) and \( BR_V^*(\lambda)\psi \) belong to Hardy classes \( H^\pm_2 \).

In fact, provided that point \( \lambda \not\in \sigma_d(L) \) is contained inside \( \Gamma \) by virtue of resolvent identity for arbitrary \( f \in \mathcal{H} \) one has

\[
R_V(\lambda)Pf = -\frac{1}{2\pi i} \oint_{\Gamma} R_V(\lambda)R_V(\mu)f \, d\mu = R_V(\lambda)f - \frac{1}{2\pi i} \oint_{\Gamma} \frac{R_V(\mu)f}{\mu - \lambda} \, d\mu.
\]

Hence given \( \varphi = f - Pf \in \mathcal{H}_c \) vector function

\[
AR_V(\lambda)\varphi = \frac{1}{2\pi i} \oint_{\Gamma} \frac{AR_V(\mu)f}{\mu - \lambda} \, d\mu
\]

admits analytic continuation to the points of discrete spectrum \( \sigma_d(L) \). Finally making use of relation \( AR_V(\lambda)\varphi = (I - Q_V(\lambda))AR_0(\lambda)\varphi \), where \( AR_0(\lambda)\varphi \in H^\pm_2 \), and taking conditions (i) – (iii) into account we come to the required conclusion

\[
\sup_{\varepsilon > 0} \int_{\mathbb{R}} \| AR_V(k \pm i\varepsilon)\varphi \|^2 \, dk < \infty.
\]

Similarly given \( \psi \in \mathcal{H}_c^* \) vector function \( BR_V^*(\lambda)\psi \) extends analytically to the points \( \lambda \in \sigma_d(L^*) \) so that

\[
\sup_{\varepsilon > 0} \int_{\mathbb{R}} \| BR_V^*(k \pm i\varepsilon)\psi \|^2 \, dk < \infty.
\]

**Corollary.** Provided that \( \varphi \in \mathcal{H}_c \) and \( \psi \in \mathcal{H}_c^* \) boundary values \( AR_V(k \pm i0)\varphi := \lim_{\varepsilon \downarrow 0} AR_V(k \pm i\varepsilon)\varphi \) and \( BR_V^*(k \pm i0)\psi := \lim_{\varepsilon \downarrow 0} BR_V^*(k \pm i\varepsilon)\psi \) are well-defined for almost all \( k \in \mathbb{R} \) and moreover \( AR_V(k \pm i\varepsilon)\varphi \rightarrow AR_V(k \pm i0)\varphi \) and \( BR_V^*(k \pm i\varepsilon)\psi \rightarrow BR_V^*(k \pm i0)\psi \) in mean square sense as \( \varepsilon \downarrow 0 \).

According to Lemma 1 and subsequent corollary the bilinear forms (1) and (2) are bounded uniformly in \( \varphi, \psi \in \mathcal{H} \) subject to normalization condition \( \| \varphi \| = \| \psi \| = 1 \), and thus they determine properly bounded operators \( W \) and \( Z \). By
corresponding definitions one has $\mathcal{H}_d \subset \ker Z$ and $W\mathcal{H} \perp \mathcal{H}_d^*$ where $\mathcal{H}_d^* \perp \mathcal{H}_c$ and hence $W\mathcal{H} \subset \mathcal{H}_c$.

**Statement 1.** Wave operators $W$ and $Z$ possess the completeness property:

$$WZ = I - P.$$ 

**Proof.** If $\varphi \in \mathcal{H}_d \subset \ker Z$ then clearly $WZ\varphi = 0$. So one should verify that $WZ\varphi = \varphi$ for arbitrary $\varphi \in \mathcal{H}_c$. Since $W\mathcal{H} \subset \mathcal{H}_c$ one has $(WZ\varphi, \psi) = 0$ when $\psi \in \mathcal{H}_d^*$. Hence it suffices to show that $(WZ\varphi, \psi) = (\varphi, \psi)$ for $\varphi \in \mathcal{H}_c$ and $\psi \in \mathcal{H}_c^*$. To this end making use of (1) and (2) consider expression

$$((W - I)(Z - I)\varphi, \psi) = ((Z - I)\varphi, (W^* - I)\psi) =$$

$$= \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} (AR_V(s + i\varepsilon)\varphi, BR_0^*(s + i\varepsilon)(W^* - I)\psi) \, ds,$$

where the integrand takes the form

$$(AR_V(s + i\varepsilon)\varphi, BR_0^*(s + i\varepsilon)(W^* - I)\psi) = ((W - I)R_0(s - i\varepsilon)B^*AR_V(s + i\varepsilon)\varphi, \psi) =$$

$$= -\frac{1}{2\pi i} \int_{\mathbb{R}} (AR_0(k + i0)R_0(s - i\varepsilon)B^*AR_V(s + i\varepsilon)\varphi, BR_V^*(k + i0)\psi) \, dk.$$

Here $AR_0(k + i0)R_0(s - i\varepsilon)B^* = (k - s + i\varepsilon)^{-1}(Q_0(k + i0) - Q_0(s - i\varepsilon))$ while boundary values $Q_0(k + i0) := s\lim_{\varepsilon \downarrow 0} Q_0(k + i\varepsilon)$ are well-defined almost everywhere and thus

$$((W - I)(Z - I)\varphi, \psi) =$$

$$= \frac{1}{4\pi^2} \lim_{\varepsilon \downarrow 0} \left\{ \int_{\mathbb{R}} dk \int_{\mathbb{R}} (k - s + i\varepsilon)^{-1}(Q_0(k + i0)AR_V(s + i\varepsilon)\varphi, BR_V^*(k + i0)\psi) \, ds - \right.$$ 

$$- \left. \int_{\mathbb{R}} ds \int_{\mathbb{R}} (k - s + i\varepsilon)^{-1}(Q_0(s - i\varepsilon)AR_V(s + i\varepsilon)\varphi, BR_V^*(k + i0)\psi) \, dk \right\}.$$

Note that $AR_V(\lambda)\varphi$ and $BR_V^*(\lambda)\psi$ belong to $H_2^+$ by virtue of Lemma 1 and one can apply Cauchy integral formula to calculate transformations of Hilbert type:

$$\int_{\mathbb{R}} (k - s + i\varepsilon)^{-1}(Q_0(k + i0)AR_V(s + i\varepsilon)\varphi, BR_V^*(k + i0)\psi) \, ds =$$

$$= -2\pi i (AR_V(k + 2i\varepsilon)\varphi, Q_0(k + i0)^*BR_V^*(k + i0)\psi),$$

$$\int_{\mathbb{R}} (k - s + i\varepsilon)^{-1}(Q_0(s - i\varepsilon)AR_V(s + i\varepsilon)\varphi, BR_V^*(k + i0)\psi) \, dk =$$

$$= -2\pi i (Q_0(s - i\varepsilon)AR_V(s + i\varepsilon)\varphi, BR_V^*(s + i\varepsilon)\psi).$$
In order to pass to the limit in the right-hand side of equality

\[
((W - I)(Z - I)\varphi, \psi) = \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \left\{ \int_{\mathbb{R}} (AR_V(k + 2i\varepsilon)\varphi, Q_0(k + i0)^*BR_V^*(k + i0)\psi)\,dk - \int_{\mathbb{R}} (Q_0(s - i\varepsilon)AR_V(s + i\varepsilon)\varphi, BR_V^*(s + i\varepsilon)\psi)\,ds \right\},
\]

we shall use corollary subsequent to Lemma 1 and besides the existence of boundary values \(Q_0(k \pm i0)\) almost everywhere on the real axis. As a result (cf. [1]) it follows that

\[
((W - I)(Z - I)\varphi, \psi) = \frac{1}{2\pi i} \left\{ \int_{\mathbb{R}} (Q_0(k + i0)AR_V(k + i0)\varphi, BR_V(k - i0)^*\psi)\,dk - \int_{\mathbb{R}} (AR_V(s + i0)\varphi, Q_0(s - i0)^*BR_V(s - i0)^*\psi)\,ds \right\} =
\]

\[
= \frac{1}{2\pi i} \left\{ \int_{\mathbb{R}} (AR_0(k + i0)\varphi, BR_V(k - i0)^*\psi)\,dk - \int_{\mathbb{R}} (AR_V(s + i0)\varphi, BR_0(s - i0)^*\psi)\,ds \right\},
\]

since \(Q_0(\lambda)AR_V(\lambda) = AR_0(\lambda) - AR_V(\lambda)\) and \(Q_0(\lambda)^*BR_V(\lambda)^* = BR_0(\lambda)^* - BR_V(\lambda)^*\). Thus we get

\[
((W - I)(Z - I)\varphi, \psi) = ((I - W)\varphi, \psi) + ((I - Z)\varphi, \psi)
\]

and hence \((WZ\varphi, \psi) = (\varphi, \psi)\) for arbitrary \(\varphi \in \mathcal{H}_c\), \(\psi \in \mathcal{H}_c^*\).

**Corollary.** Under the above assumptions one has \(W\mathcal{H} = \mathcal{H}_c\) and \(\ker Z = \mathcal{H}_d\).

**§ 4. Inverse wave operator**

Along with projection \(P\) introduce the complementary projection \(\tilde{P} = I - P\) onto the subspace \(\mathcal{H}_c\) parallel to the linear span \(\mathcal{H}_d\) of eigen and associated vectors of operator \(L\).

**Lemma 2.** Under the hypotheses \((i)-(iii)\) operator function \(\tilde{Q}(\lambda) = AR_V(\lambda)\tilde{P}B^*\) is analytic and bounded in \(\mathbb{C}_\pm\).
Indeed, due to conditions (i)–(ii) operator function $AR_V(\lambda)\tilde{P}B^*$ being extended by the expression

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{AR_V(\zeta)\tilde{P}B^*}{\zeta - \lambda} d\zeta$$

to singular points $\lambda \in \sigma_d(L)$ becomes analytic in $\mathbb{C}_\pm$ and bounded in certain neighborhood of $\sigma_d(L)$. Provided that $\lambda \in \mathbb{C}_\pm$ is separated away from $\sigma_d(L)$ becomes analytic in $\mathbb{C}_\pm$ and bounded in certain neighborhood of $\sigma_d(L)$.

Provided that $\lambda \in \mathbb{C}_\pm$ is separated away from $\sigma_d(L)$ becomes analytic in $\mathbb{C}_\pm$ and bounded in certain neighborhood of $\sigma_d(L)$.

Hence it suffices to show that operator function $\tilde{Q}(\lambda) + AR_V(\lambda)PB^* = AR_V(\lambda)B^*$ is uniformly bounded by condition (iii). Hence it suffices to show that operator function $\tilde{Q}(\lambda) + AR_V(\lambda)PB^* = AR_V(\lambda)B^*$ is uniformly bounded when its argument $\lambda \in \mathbb{C}_\pm$ does not approach $\sigma_d(L)$. In this context according to [15] operator $P$ proves to be the sum of Riesz projections onto root subspaces corresponding to the points of discrete spectrum $\sigma_d(L)$. Given an eigenvalue $\mu$ of operator $L$ with geometric multiplicity $n$ the associated Riesz projection is of the form

$$P_\mu = \sum_{j=1}^{n} \left\{ (\cdot, g_j^{(m_j-1)})f_j^{(0)} + \ldots + (\cdot, g_j^{(0)})f_j^{(m_j-1)} \right\}.$$

Eigenvectors $f_j^{(0)}, j = 1, \ldots, n$, span the eigensubspace $\ker(L-\mu I)$ of dimension $n$, while $f_j^{(1)}, \ldots, f_j^{(m_j-1)}$ is a Jordan chain of associated vectors adjoint to $f_j^{(0)}$, so that $m = m_1 + \ldots + m_n$ is the algebraic multiplicity of $\mu$. In due turn $g_j^{(0)}, g_j^{(1)}, \ldots, g_j^{(m_j-1)}$ is the chain of eigen and associated vectors of operator $L^*$ corresponding to its eigenvalue $\overline{\mu}$ subject to normalization condition $(f_j^{(p)}, g_j^{(q)}) = 1$ where $p + q = m_j - 1$. Since $P_\mu$ reduces to the sum of rank 1 operators $\hat{P}_\mu = (\cdot, g_j^{(q)})f_j^{(p)}$ it suffices to consider operator function $AR_V(\lambda)\hat{P}_\mu B^*$ which by virtue of the estimate

$$\|AR_V(\lambda)\hat{P}_\mu B^*\| \leq \sum_{r=0}^{p} |\lambda - \mu|^{r-p-1}\|Af_j^{(r)}\| \|Bg_j^{(q)}\|$$

proves to be uniformly bounded in $\mathbb{C}_\pm$ outside some neighborhood of $\sigma_d(L)$.

**Corollary.** Boundary values $\tilde{Q}(k \pm i\varepsilon) := \lim_{\varepsilon \downarrow 0}\tilde{Q}(k \pm i\varepsilon)$ exist for almost all $k \in \mathbb{R}$.

**Statement 2.** Operator $Z$ is the left inverse to the direct wave operator $W$.

**Proof.** For arbitrary $\varphi, \psi \in \mathcal{H}$ one has

$$(ZW\varphi, \psi) = (W\varphi, Z^*\psi) = \varphi, Z^*\psi - \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} (AR_0(s + i\varepsilon)\varphi, BR_V^*(s + i\varepsilon)Z^*\psi) ds,$$
where \( Z^* \psi \in \mathcal{H}_c^* \), so that the integrand in virtue of (2) takes the form
\[
(ZR_V(s - i\varepsilon)V R_0(s + i\varepsilon)\varphi, \psi) = (\tilde{P} R_V(s - i\varepsilon)V R_0(s + i\varepsilon)\varphi, \psi) + \frac{1}{2\pi i} \int_\mathbb{R} (AR_V(k + i\varepsilon)\tilde{P} R_V(s - i\varepsilon)V R_0(s + i\varepsilon)\varphi, BR_0(k - i\varepsilon)^* \psi) \, dk.
\]
Thus we get
\[
(ZW \varphi, \psi) = (Z \varphi, \psi) - \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_\mathbb{R} (\tilde{P} R_V(s - i\varepsilon)V R_0(s + i\varepsilon)\varphi, \psi) \, ds + \frac{1}{4\pi^2} \lim_{\varepsilon \downarrow 0} \int_\mathbb{R} ds \int_\mathbb{R} (AR_V(k + i\varepsilon)\tilde{P} R_V(s - i\varepsilon)B^* A R_0(s + i\varepsilon)\varphi, BR_0(k - i\varepsilon)^* \psi) \, dk
\]
and according to resolvent identity
\[
AR_V(k + i\varepsilon)\tilde{P} R_V(s - i\varepsilon)B^* = (k - s + i\varepsilon)^{-1} (\tilde{Q}(k + i\varepsilon) - \tilde{Q}(s - i\varepsilon))
\]
let us split the above iterated integral into two summands
\[
\int_\mathbb{R} ds \int_\mathbb{R} (AR_V(k + i\varepsilon)\tilde{P} R_V(s - i\varepsilon)B^* A R_0(s + i\varepsilon)\varphi, BR_0(k - i\varepsilon)^* \psi) \, dk =
\]
\[
= \int_\mathbb{R} dk \int_\mathbb{R} (k - s + i\varepsilon)^{-1} (\tilde{A} R_0(s + i\varepsilon)\varphi, \tilde{Q}(k + i\varepsilon)^* B R_0(k - i\varepsilon)^* \psi) \, ds - \int_\mathbb{R} ds \int_\mathbb{R} (k - s + i\varepsilon)^{-1} (\tilde{Q}(s - i\varepsilon)A R_0(s + i\varepsilon)\varphi, B R_0(k - i\varepsilon)^* \psi) \, dk.
\]
Making use of Cauchy integral formula (cf. the proof of Statement 1) one can compute Hilbert-type transforms:
\[
\int_\mathbb{R} (k - s + i\varepsilon)^{-1} (\tilde{A} R_0(s + i\varepsilon)\varphi, \tilde{Q}(k + i\varepsilon)^* B R_0(k - i\varepsilon)^* \psi) \, ds =
\]
\[
= -2\pi i \left( \tilde{A} R_0(k + 2i\varepsilon)\varphi, \tilde{Q}(k + i\varepsilon)^* B R_0(k - i\varepsilon)^* \psi \right),
\]
\[
\int_\mathbb{R} (k - s + i\varepsilon)^{-1} (\tilde{Q}(s - i\varepsilon)A R_0(s + i\varepsilon)\varphi, B R_0(k - i\varepsilon)^* \psi) \, dk =
\]
\[
= -2\pi i \left( \tilde{Q}(s - i\varepsilon)A R_0(s + i\varepsilon)\varphi, B R_0(s - i\varepsilon)^* \psi \right);
\]
now passing in the expression for \((ZW \varphi, \psi)\) to the limit as \(\varepsilon \downarrow 0\) leads to the following representation
\[
(ZW \varphi, \psi) = (Z \varphi, \psi) - \frac{1}{2\pi i} \int_\mathbb{R} (AR_0(s + i\varepsilon)\varphi, B R_V(s - i\varepsilon)^* \tilde{P}^* \psi) \, ds + \frac{1}{2\pi i} \int_\mathbb{R} (\tilde{Q}(k + i\varepsilon)AR_0(k + i\varepsilon)\varphi, BR_0(k - i\varepsilon)^* \psi) \, dk - \frac{1}{2\pi i} \int_\mathbb{R} (AR_0(s + i\varepsilon)\varphi, \tilde{Q}(s - i\varepsilon)^* B R_0(s - i\varepsilon)^* \psi) \, ds.
\]
Due to equalities $\tilde{Q}(\lambda)AR_0(\lambda) = A\tilde{P}R_0(\lambda) - AR_V(\lambda)\tilde{P}$ and $\tilde{Q}(\lambda)^*BR_0(\lambda)^* = B\tilde{P}^*R_0(\lambda)^* - BR_V(\lambda)^*\tilde{P}$ for arbitrary $\varphi, \psi \in \mathcal{H}$ and almost all $k \in \mathbb{R}$ in virtue of Lemmas 1 and 2 there exist boundary values $A\tilde{P}R_0(k + i0)\varphi, B\tilde{P}^*R_0(k - i0)^*\psi \in L_2(\mathbb{R})$. This fact enables one to reduce bilinear form of operator $ZW$ to the following expression

$$(ZW\varphi, \psi) = ( (I - P)\varphi, \psi ) + \frac{1}{2\pi i} \int_\mathbb{R} ( A\tilde{P}R_0(k + i0)\varphi, BR_0(k - i0)^*\psi ) \, dk - \frac{1}{2\pi i} \int_\mathbb{R} ( AR_0(k + i0)\varphi, B\tilde{P}^*R_0(k - i0)^*\psi ) \, dk = ( (I - P)\varphi, \psi ) + \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_\mathbb{R} [( R_0(k - i\varepsilon)PVR_0(k + i\varepsilon)\varphi, \psi ) - ( R_0(k - i\varepsilon)VR_0(k + i\varepsilon)\varphi, \psi )] \, dk ,$$

where

$$R_0(k - i\varepsilon)PVR_0(k + i\varepsilon) - R_0(k - i\varepsilon)PVR_0(k + i\varepsilon) = PVR_0(k + i\varepsilon) - R_0(k - i\varepsilon)P - 2i\varepsilon R_0(k - i\varepsilon)PVR_0(k + i\varepsilon) .$$

In order to complete the proof we only need to establish the relationship

$$\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_\mathbb{R} [( PR_0(k + i\varepsilon)\varphi, \psi ) - ( R_0(k - i\varepsilon)P\varphi, \psi ) - 2i\varepsilon ( R_0(k - i\varepsilon)PVR_0(k + i\varepsilon)\varphi, \psi )] \, dk = ( P\varphi, \psi ) . \quad (5)$$

Since operator $P$ turns out to be the sum of Riesz projections $P_\mu$ corresponding to points of discrete spectrum $\mu \in \sigma_d(L)$ it suffices to verify relation (5) separately for each elementary rank 1 summand $\tilde{P}_\mu = ( \cdot, g ) f$, where $f = f_j^{(p)}$, $g = g_j^{(q)}$, $p + q = m_j - 1$ and $( f, g ) = 1$. Note that

$$\tilde{P}_\mu R_0(k + i\varepsilon)\varphi, \psi ) - ( R_0(k - i\varepsilon)\tilde{P}_\mu \varphi, \psi ) - 2i\varepsilon ( R_0(k - i\varepsilon)\tilde{P}_\mu R_0(k + i\varepsilon)\varphi, \psi ) =$$

$$= ( R_0(k + i\varepsilon)\varphi, g ) ( f, \psi ) - ( \varphi, g ) ( R_0(k - i\varepsilon)f, \psi ) - 2i\varepsilon ( R_0(k + i\varepsilon)\varphi, g ) ( R_0(k - i\varepsilon)f, \psi ) .$$

Taking representations $f = - \sum_{r=0}^p R_0(\mu)^{p-r+1}Vf_j^{(r)}$, $g = - \sum_{s=0}^q R_0^*(\mu)^{q-s+1}V^*g_j^{(s)}$ into account according to resolvent identity we get

$$( R_0(k + i\varepsilon)\varphi, g ) = - \frac{( \varphi, g )}{k - \mu + i\varepsilon} - \frac{\xi(k + i\varepsilon)}{k - \mu + i\varepsilon} ,$$

$$( R_0(k - i\varepsilon)f, \psi ) = - \frac{( f, \psi )}{k - \mu - i\varepsilon} - \frac{\eta(k - i\varepsilon)}{k - \mu - i\varepsilon} ,$$

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where \( \xi(z) = \sum_{s=0}^{q} (R_0(\mu)^{q-s} \varphi, R_0(z)^s V \gamma_j), \) \( \eta(z) = \sum_{r=0}^{p} (R_0(\mu)^{p-r} \psi) \) belong to Hardy classes \( H^\pm \) so that
\[
(\hat{P}_\mu R_0(k + i\varepsilon) \varphi, \psi) - (R_0(k - i\varepsilon) \hat{P}_\mu \varphi, \psi) - 2i\varepsilon (R_0(k - i\varepsilon) \hat{P}_\mu R_0(k + i\varepsilon) \varphi, \psi) =
\]
\[
\frac{(\varphi, g)}{k - \mu + i\varepsilon} \eta(k - i\varepsilon) - \frac{(f, \psi)}{k - \mu - i\varepsilon} \xi(k + i\varepsilon) - \frac{2i\varepsilon}{(k - \mu)^2 + \varepsilon^2} \xi(k + i\varepsilon) \eta(k - i\varepsilon).
\]

In the case when \( \text{Im} \mu > 0 \) by Cauchy integral formula one has
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{k - \mu - i\varepsilon} \xi(k + i\varepsilon) \, dk = \xi(\mu + 2i\varepsilon) \to - (\varphi, g), \quad \varepsilon \downarrow 0,
\]
while for sufficiently small \( \varepsilon > 0 \)
\[
\int_{\mathbb{R}} \frac{1}{k - \mu + i\varepsilon} \eta(k - i\varepsilon) \, dk = 0.
\]

Besides the following estimate is valid
\[
\left| \int_{\mathbb{R}} \frac{1}{(k - \mu)^2 + \varepsilon^2} \xi(k + i\varepsilon) \eta(k - i\varepsilon) \, dk \right| \leq \frac{1}{(\text{Im} \mu)^2} \left( \int_{\mathbb{R}} |\xi(k + i\varepsilon)|^2 \, dk \right)^{1/2} \left( \int_{\mathbb{R}} |\eta(k - i\varepsilon)|^2 \, dk \right)^{1/2},
\]
where the right-hand side is bounded uniformly provided \( \varepsilon > 0 \) is small enough.

Thus given eigenvalue \( \mu \in \mathbb{C}_+ \) the limiting relationship \( (6) \) is established for \( \hat{P}_\mu \) and
similarly it can be derived if \( \mu \in \sigma_d(L) \cap \mathbb{C}_- \). As regards the case of real \( \mu \in \sigma_d(L) \) one should take into account analyticity of resolvent \( R_0(\lambda) \) at such points to carry out passage to the limit as \( \varepsilon \downarrow 0 \)
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{k - \mu - i\varepsilon} \xi(k + i\varepsilon) \, dk = \xi(\mu + 2i\varepsilon) \to - (\varphi, g),
\]
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{k - \mu + i\varepsilon} \eta(k - i\varepsilon) \, dk = - \eta(\mu - 2i\varepsilon) \to (f, \psi).
\]

By the same argument we get
\[
\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \frac{1}{(k - \mu)^2 + \varepsilon^2} \xi(k + i\varepsilon) \eta(k - i\varepsilon) \, dk = (\varphi, g) (f, \psi),
\]
and so the above calculations imply relationship \( (6) \) for rank 1 operator \( \hat{P}_\mu \) in question.
§ 5. Non-stationary wave operators

Let us first verify that wave operators constructed in Theorem 1 and possessing completeness property implement similarity of operator $T$ and the restriction of operator $L$ to the subspace $\mathcal{H}_c$.

**Statement 3.** Wave operator $W$ intertwines the resolvents of perturbed and unperturbed operators $L$ and $T$: $R_V(\lambda)W = WR_0(\lambda)$.

**Proof.** To be specific we shall consider the case $\text{Im} \lambda > 0$. Given $\psi \in \mathcal{H}_d^*$ for arbitrary $\varphi \in \mathcal{H}$ one has $(WR_0(\lambda)\varphi, \psi) = 0$ and $(R_V(\lambda)W\varphi, \psi) = (W\varphi, R_V(\lambda)^*\psi)$ = 0 since $R_V(\lambda)^*\psi \in \mathcal{H}_d^*$.

If $\psi \in \mathcal{H}_c^*$ then $BR_V^*(\lambda)\psi \in \mathcal{H}^c_{\pm}$ by virtue of Lemma 1. According to (1) the following representation

$$(W-I)R_0(\lambda)\varphi, \psi) = -\frac{1}{2\pi i} \int_{\mathbb{R}} (AR_0(k+i0)R_0(\lambda)\varphi, BR_V^*(k+i0)\psi) \, dk$$

is valid and due to resolvent identity

$$AR_0(k+i0)R_0(\lambda)\varphi = (k-\lambda)^{-1}(AR_0(k+i0)\varphi - AR_0(\lambda)\varphi)$$

we hence get

$$(W-I)R_0(\lambda)\varphi, \psi) = -\frac{1}{2\pi i} \int_{\mathbb{R}} (k-\lambda)^{-1}(AR_0(k+i0)\varphi, BR_V^*(k+i0)\psi) \, dk$$

because

$$\int_{\mathbb{R}} (k-\lambda)^{-1}(AR_0(\lambda)\varphi, BR_V^*(k+i0)\psi) \, dk = 0.$$

Similarly noting that $R_V^*(\lambda)\psi \in \mathcal{H}_c^*$ one can carry out the following transformation

$$(R_V(\lambda)(W-I)\varphi, \psi) = ((W-I)\varphi, R_V(\lambda)^*\psi) =$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} (k-\lambda)^{-1}(AR_0(k+i0)\varphi, BR_V^*(\lambda)^*\psi) \, dk$$

$$\quad - \frac{1}{2\pi i} \int_{\mathbb{R}} (k-\lambda)^{-1}(AR_0(k+i0)\varphi, BR_V^*(k+i0)\psi) \, dk,$$

where the first summand on the right-hand side by virtue of Cauchy integral formula reduces to the expression

$$(AR_0(\lambda)\varphi, BR_V(\lambda)^*\psi) = (R_V(\lambda)VR_0(\lambda)\varphi, \psi) = (R_0(\lambda)\varphi, \psi) - (R_V(\lambda)\varphi, \psi).$$

Finally we get the equality

$$(R_V(\lambda)(W-I)\varphi, \psi) = (R_0(\lambda)\varphi, \psi) - (R_V(\lambda)\varphi, \psi) + ((W-I)R_0(\lambda)\varphi, \psi)$$
and thus for arbitrary vectors \( \varphi, \psi \in \mathcal{H} \) the required relationship \( (R_V(\lambda)W\varphi, \psi) = (WR_0(\lambda)\varphi, \psi) \) holds.

**Corollary.** Under hypotheses of Theorem 1 one has \( L_c = WTZ \).

In the case of selfadjoint unperturbed operator \( T \) along with unitary group \( U_0(t) := \exp(itT) \) we put into context a one-parameter operator family

\[
\exp(itL) = s\lim_{n \to \infty} (I - itL/n)^{-n}.
\]

**Lemma 3.** For arbitrary \( t \in \mathbb{R} \) exponents \( \exp(itT) \) and \( \exp(itL) \) satisfy the equation

\[
U_V(t) := \exp(itL) \tilde{P} = WU_0(t)Z.
\]

Indeed, the intertwining relation for the resolvents \( R_V(\lambda) \) and \( R_0(\lambda) \) (Statement 3) implies the equality

\[
(I - itL/n)^{-n}W = W(I - iT/n)^{-n},
\]

which by the use of completeness property for wave operators (Statement 1) can be rewritten in the form

\[
(I - itL/n)^{-n}(I - P) = W(I - iT/n)^{-n}Z.
\]

It thus follows that for arbitrary \( \varphi \in \mathcal{H} \) the limit

\[
\lim_{n \to \infty} (I - itL/n)^{-n} \tilde{P} \varphi = WU_0(t)Z \varphi
\]

exists, i.e. \( U_V(t) = WU_0(t)Z \). Remark that exponent \( \exp(itL) \) is well-defined on the whole space \( \mathcal{H} \): namely if \( \varphi \in \mathcal{H}_c \) then \( \exp(itL) \varphi = WU_0(t)Z \varphi \), while one calculates the value \( \exp(itL) \varphi \) for \( \varphi \in \mathcal{H}_d \) taking notice of the action formulas

\[
\exp(itL)f_j^{(r)} = e^{it\mu} \sum_{s=0}^{r} \frac{(it)^{r-s}}{(r-s)!} f_j^{(s)}
\]

in the Jordan basis \( \{f_j^{(0)}, f_j^{(1)}, \ldots, f_j^{(m_j-1)}\}_{j=1}^{n} \) of the root subspace corresponding to eigenvalue \( \mu \) of operator \( L \).

**Corollary.** One-parameter operator family \( U_V(t) \) possesses semigroup property \( U_V(t+s) = U_V(t)U_V(s) \) and is related to operator function \( R_V(\lambda)\tilde{P} \) by means of Laplace transform.

**Statement 4.** If additionally to the hypotheses of Theorem 1 unperturbed operator \( T \) is assumed to be selfadjoint then direct and inverse wave operators \( W \) and \( Z \) admit representations
$$W = \lim_{t \to \infty} U_V(t) U_0(-t), \quad Z = \lim_{t \to \infty} U_0(t) U_V(-t).$$

**Proof** follows the scheme of the approach suggested in [1]. Making use of Lemmas 1 and 3 and taking into account smoothness of operator $A$ relative to $T$ we come to a conclusion that for $\varphi \in \mathcal{H}$ and $\psi \in \mathcal{H}_c^*$ images of vector functions $AR_k(k+i0)\varphi$ and $BR^*_k(k+i0)\psi$ under the action of Fourier-Plancherel transform are given by the expressions $i \sqrt{2\pi} \theta(t) AU_0(-t)\varphi$ and $i \sqrt{2\pi} \theta(t) BU^*_V(-t)\psi$, where $\theta(t)$ is Heaviside step function and $U^*_V(-t) = \exp(-itL^*)\tilde{P} = U_V(t)$. Therefore by virtue of Parseval identity one has

$$\int_{\mathbb{R}} (AR_k(k+i0)\varphi, BR^*_k(k+i0)\psi) \, dk = 2\pi \int_0^\infty (AU_0(-t)\varphi, BU^*_V(-t)\psi) \, dt.$$  

Due to this fact and with regard for intertwining relationship $U_V(s)W = WU_0(s)$ and semigroup property $U_V(t+s) = U_V(t)U_V(s)$ (corollary subsequent to Lemma 3) bilinear form of wave operator $W$ is reduced to the expression

$$(W\varphi, \psi) = (WU_0(-s)\varphi, U^*_V(-s)\psi) = (U_V(s)U_0(-s)\varphi, \psi) + i \int_s^\infty (AU_0(-t)\varphi, BU^*_V(-t)\psi) \, dt$$

which implies the estimate

$$\left| (W\varphi - U_V(s)U_0(-s)\varphi, \psi) \right| \leq \left( \int_s^\infty \|AU_0(-t)\varphi\|^2 \, dt \right)^{1/2} \left( \int_0^\infty \|BU^*_V(-t)\psi\|^2 \, dt \right)^{1/2},$$

where

$$\int_0^\infty \|BU^*_V(-t)\psi\|^2 \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} \|BR^*_k(k+i0)\psi\|^2 \, dk.$$  

Given $\psi \in \mathcal{H}_c^*$ by virtue of Lemma 1 and the subsequent corollary a constant $C > 0$ exists such that the inequality

$$\left| (W\varphi - U_V(s)U_0(-s)\varphi, \psi) \right| \leq C \left( \int_s^\infty \|AU_0(-t)\varphi\|^2 \, dt \right)^{1/2} \|\psi\|$$

is valid for arbitrary $\varphi \in \mathcal{H}$. If however $\psi \in \mathcal{H}_d^*$ then $(W\varphi, \psi) = 0$ by the definition of $W$ and moreover $(U_V(s)U_0(-s)\varphi, \psi) = 0$ because $U_V(s)U_0(-s)\varphi \in \mathcal{H}_c \perp \mathcal{H}_d^*$. Thus one has

$$\|W\varphi - U_V(s)U_0(-s)\varphi\| \leq C \left( \int_s^\infty \|AU_0(-t)\varphi\|^2 \, dt \right)^{1/2},$$

where $AU_0(-t)\varphi \in L_2(\mathbb{R}_+)$ for arbitrary $\varphi \in \mathcal{H}$ and therefore wave operator $W$ coincides with the strong limit of $U_V(s)U_0(-s)$ as $s \to \infty$. 

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Similarly with Lemmas 1 and 3 taken into account and in view of smoothness of operator $B$ relative to $T^*$ the equality
\[
\int_{\mathbb{R}} (AR_V(k+i0)\varphi, BR_0(k+i0)\psi) \, dk = 2\pi \int_0^\infty (AU_V(-t)\varphi, BU_0(-t)\psi) \, dt
\]
can be established for arbitrary $\varphi \in \mathcal{H}_c$ and $\psi \in \mathcal{H}$. Further by the usage of intertwining relationship $ZU_V(s) = U_0(s)Z$ and semigroup property $U_V(t+s) = U_V(t)U_V(s)$ one gets
\[
(Z\varphi, \psi) - (U_0(s)U_V(-s)\varphi, \psi) = -i \int_s^\infty (AU_V(-t)\varphi, BU_0(-t)\psi) \, dt
\]
and hence the inequality
\[
| (Z\varphi - U_0(s)U_V(-s)\varphi, \psi) | \leq C \left( \int_s^\infty \|AU_V(-t)\varphi\|^2 \, dt \right)^{1/2} \|\psi\|
\]
is valid with a certain constant $C > 0$ which does not depend on $\varphi$ and $\psi$. Given $\varphi \in \mathcal{H}_c$ this implies the estimate
\[
\| Z\varphi - U_0(s)U_V(-s)\varphi \| \leq C \left( \int_s^\infty \|AU_V(-t)\varphi\|^2 \, dt \right)^{1/2},
\]
which extends to the whole $\mathcal{H}$ because $\mathcal{H}_d \subset \ker Z$ and $U_V(t) = WU_0(t)Z$. Finally, since $AU_V(-t)\varphi \in L_2(\mathbb{R}_+)$ for arbitrary $\varphi \in \mathcal{H}$, we arrive at the conclusion that $U_0(s)U_V(-s)$ converges strongly to $Z$ as $s \to \infty$.

§ 6. One-dimensional Schroedinger operator

Proof of Theorem 3 reduces to verification of hypotheses imposed in Theorem 1 in conformity with the Schroedinger operator $L = T + V = -d^2/dx^2 + V(x)$ acting in $\mathcal{H} = L_2(\mathbb{R}_+)$ and determined in Section 2 by Dirichlet boundary condition at zero. Henceforth bounded complex-valued potential $V(x)$ is assumed to satisfy condition (4).

**Statement 5** (see [6]). Resolvent $R_V(\lambda) = (L - \lambda I)^{-1}$ of Schroedinger operator $L$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$ and its poles coincide with eigenvalues of $L$ counting their multiplicities. Discrete spectrum $\sigma_d(L)$ is finite provided that operator $L$ has no spectral singularities, while $\sigma_c(L) = \mathbb{R}_+$.

Note that for $\lambda = k^2$, $\text{Im } k > 0$, the resolvent $R_V(\lambda)$ of operator $L$ proves to be an integral operator with the kernel
\[
R_V(x, \xi, \lambda) = \frac{1}{e(k)} \begin{cases} 
  s(x, k) e(\xi, k), & x \leq \xi \\
  e(x, k) s(\xi, k), & \xi \leq x
\end{cases}
\]
where \( s(x, k) \) is the solution of equation (3) satisfying initial conditions \( s(0, k) = 0, \quad s'(0, k) = 1 \), while Jost solution \( e(x, k) \) to equation (3) is determined by asymptotics \( e(x, k) \sim e^{ikx} \) at infinity.

**Lemma 4** (cf. [16]). For \( k \in \mathbb{C}_+ \) and \( x \in \mathbb{R}_+ \) the following inequalities are valid

\[
|s(x, k)e^{ikx}| \leq \min \left\{ x, \frac{1}{|k|} \right\} \exp \left( \int_0^x |V(\xi)| d\xi \right), \tag{6}
\]

\[
|e(x, k)e^{-ikx} - 1| \leq \exp \left( \int_x^\infty \min \left\{ \xi, \frac{1}{|k|} \right\} |V(\xi)| d\xi \right) - 1. \tag{7}
\]

In order to evaluate \( s(x, k) \) note that it proves to be a solution to integral equation

\[
s(x, k) = \frac{\sin kx}{k} + \int_0^x \frac{\sin k(x - \xi)}{k} V(\xi) s(\xi, k) \, d\xi,
\]

whose integral kernel admits the estimate

\[
\left| \frac{\sin k(x - \xi)}{k} \right| \leq \min \left\{ x, \frac{1}{|k|} \right\} \exp (\text{Im } k (x - \xi))
\]

and consequently one has

\[
|s(x, k)e^{ikx}| \leq \min \left\{ x, \frac{1}{|k|} \right\} \left( 1 + \int_0^x |V(\xi)| |s(\xi, k)e^{ik\xi}| \, d\xi \right).
\]

Estimate (6) now follows immediately by virtue of Gronwall-Bellman inequality. As regards estimation of Jost solution \( e(x, k) \) one should take into account integral equation

\[
e(x, k) = e^{ikx} - \int_x^\infty \frac{\sin k(x - \xi)}{k} V(\xi) e(\xi, k) \, d\xi
\]

of Lippmann-Schwinger type which can be solved by iterations method:

\[
e(x, k)e^{-ikx} = \sum_{n=0}^{\infty} \varepsilon^{(n)}(x, k), \quad \varepsilon^{(0)}(x, k) = 1,
\]

\[
\varepsilon^{(n)}(x, k) = \int_x^\infty \frac{e^{2ik(\xi - x)} - 1}{2ik} V(\xi) \varepsilon^{(n-1)}(\xi, k) \, d\xi.
\]

With regard for the inequality

\[
\left| \frac{e^{2ik(\xi - x)} - 1}{2ik} \right| \leq \min \left\{ \xi, \frac{1}{|k|} \right\},
\]
valid for $\xi \geq x$, induction argument applies to evaluate successive approximations
\[ |\varepsilon^{(n)}(x,k)| \leq \frac{1}{n!} \left( \int_x^\infty \min \left\{ \xi, \frac{1}{|k|} \right\} |V(\xi)| \, d\xi \right)^n \]
and thus ensures estimate (7).

Denote by $A$ and $B$ operators of multiplication by functions $a(x)$ and $b(x)$ to be chosen such that
\[ \langle a \rangle := \left( \int_0^\infty x |a(x)|^2 \, dx \right)^{1/2} < \infty, \]
\[ \langle b \rangle := \left( \int_0^\infty x |b(x)|^2 \, dx \right)^{1/2} < \infty. \]

**Statement 6.** Provided condition (4) is satisfied operator function $e(k)AR_V(k^2)B^*$ extends analytically to $\mathbb{C}_+$ and, moreover, for all $k \in \mathbb{C}_+$ the inequality holds
\[ \|e(k)AR_V(k^2)B^*\| \leq K \langle a \rangle \langle b \rangle, \quad K = \exp(\sqrt{|V|}). \]

**Proof.** By virtue of (6) and (7) integral kernel of the resolvent $R_V(\lambda)$, $\lambda = k^2$, admits the estimate
\[ |R_V(x,\xi,\lambda)| \leq \frac{K}{|e(k)|} \min\{x,\xi\}, \quad K = \exp \left( \int_0^\infty x |V(x)| \, dx \right). \]
As a consequence for arbitrary $f \in L_2(\mathbb{R}_+)$ one has
\[ \|e(k)AR_V(k^2)B^*f\|^2 \leq K^2 \int_0^\infty |a(x)|^2 \left( \int_0^\infty \min\{x,\xi\} |b(\xi)| |f(\xi)| \, d\xi \right)^2 \, dx \leq K^2 \int_0^\infty |a(x)|^2 \left( \int_0^\infty \left( \min\{x,\xi\} \right)^2 |b(\xi)|^2 \, d\xi \right) \left( \int_0^\infty |f(\xi)|^2 \, d\xi \right) \, dx \leq K^2 \langle a \rangle^2 \langle b \rangle^2 \|f\|^2. \]
Thus operator function $e(k)AR_V(k^2)B^*$ is analytic and according to the above estimate bounded in the vicinity of each pole of the resolvent $R_V(k^2)$ and therefore it has a removable singularity therein.

**Corollary (cf. [1]).** For arbitrary $k \in \mathbb{C}_+$ the inequality $\|AR_0(\lambda)A^*\| \leq \langle a \rangle^2$ is valid which implies smoothness of operator $A$ relative to $T = T^*$.

Statements 5 and 6 guarantee that Theorem 1 applies to Schrödinger operator
\[ L = T + V = -d^2/dx^2 + V(x) \]
with bounded potential $V(x)$ possessing finite first momentum \((4)\). In fact, the unperturbed operator $T = -d^2/dx^2$ has purely continuous spectrum $\sigma(T) = \mathbb{R}_+$ and according to Statement 5 conditions \((i)\) and \((ii)\) are satisfied provided that operator $L$ has no spectral singularities.

Further, polar decomposition $V = J|V|$, where $|V| = \sqrt{V^*V}$, while $J = \text{sgn} V$ is a partial isometry, produces an appropriate for our purposes factorization $V = B^*A$, so that $A = \sqrt{|V|}$ and $B = \sqrt{|V|}J^*$ are operators of multiplication by functions

$$a(x) = \sqrt{|V(x)|}, \quad b(x) = \left(\text{sign} V(x)\right) a(x)$$

respectively, where $\text{sign} z = z/|z|$ and $\text{sign} 0 = 0$. Under the condition \((4)\) by virtue of Statement 6 and the subsequent corollary operators $A$ and $B$ are smooth relative to $T$ and, moreover, operator function $Q_0(\lambda) = AR_0(\lambda)B^*$ is analytic and bounded in $\mathbb{C}_\pm$. If potential $V(x)$ decreases at infinity at an appropriate rate $Q_0(\lambda)$ takes the values in the trace class and Jost function $e(k)$ coincides (see. [17]) with Fredholm determinant

$$e(k) = \det \left( I + AR_0(k^2)B^* \right).$$

Finally, due to the fact that Jost function $e(k)$ is analytic in $\mathbb{C}_+$, continuous up to the real axis and in virtue of estimate \((7)\) separated from zero at infinity, Statement 6 implies condition \((iii)\) since and when operator $L$ has no spectral singularities. Thus Schroedinger operator $L = T + V$ under consideration satisfies all the hypotheses of Theorem 1 and therefore its conclusion applies completing the proof of Theorem 3.

**Corollary.** Under the assumptions of Theorem 3 condition $\sigma_d(L) = \emptyset$ gives a criterion of similarity of operators $L$ and $T$.

In conclusion we remark that Kato sufficient condition which ensures similarity of operators $L$ and $T$ (see Section 1) is optimal in the sense that among the potentials $V(x)$ for which first momentum exceeds the critical value $1$ there exist (see [10]) such that $\sigma_d(L) \neq \emptyset$. The estimates for the total number of eigenvalues and spectral singularities of Schroedinger operator with complex potential were obtained by the author in [13] and [18].

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