BORCHERDS-BOZEC ALGEBRAS, ROOT MULTIPLICITIES
AND THE SCOIFIELD CONSTRUCTION

SEOK-JIN KANG∗

Abstract. Using the twisted denominator identity, we derive a closed form root multiplicity formula for all symmetrizable Borcherds-Bozec algebras and discuss its applications including the case of Monster Borcherds-Bozec algebra. In the second half of the paper, we provide the Schofield construction of symmetric Borcherds-Bozec algebras.

INTRODUCTION

The aims of this paper are to investigate the root multiplicities of Borcherds-Bozec algebras and to provide the Schofield construction of these algebras. The Borcherds-Bozec algebras (actually, their quantum deformation) arise as a natural algebraic structure relevant to the theory of perverse sheaves on the representation varieties of quivers with loops (see [3, 4, 5] for more details).

To begin with, let us briefly clarify the terms of Kac-Moody algebras, Borcherds algebras and Borcherds-Bozec algebras. The Kac-Moody algebras were introduced independently by V. G. Kac and R. V. Moody as a generalization of finite dimensional complex semi-simple Lie algebras [10, 18]. In [22], one can find J. P. Serre’s presentation of finite dimensional complex semi-simple Lie algebras by generators and relations associated with positive definite Cartan matrices. What Kac and Moody did was to remove the restriction of positive definiteness of Cartan matrices and they were able to construct a new family of (mostly infinite dimensional) Lie algebras, which may have roots with norms ≤ 0, the imaginary roots. It turned out that, together with Kac’s discovery of the Weyl-Kac character formula [11], Kac-Moody algebras (affine Lie algebras in particular) have a lot of significant applications to various branches of mathematics and mathematical physics

2010 Mathematics Subject Classification. 17B37, 17B67, 16G20.

Key words and phrases. Borcherds-Bozec algebra, root multiplicity, Monster Borcherds-Bozec algebra, quiver variety, Schofield construction.

∗ This research was supported by Ministry of Culture, Sports and Tourism, Korea Creative Content Agency in the Culture Technology Research and Development Program 2017 and Basic Science Research Program of NRF (Korea) under grant No. 2015R1D1A1A01059643.
such as number theory, combinatorics and statistical mechanics. Still, the generators of Kac-Moody algebras are positive and negative simple root vectors having positive norms.

In [1], R. E. Borcherds gave a generalized version of Kac-Moody algebras, the Borcherds algebras, associated with Borcherds-Cartan matrices. These matrices are allowed to have non-positive diagonal entries, which means the Borcherds algebras may have simple roots with norms $\leq 0$, the imaginary simple roots. Thus we have a decomposition of index set for the simple roots: $I = I^\text{re} \sqcup I^\text{im}$, where $I^\text{re} = \{ i \in I \mid a_{ii} = 2 \}$, the set of real indices, and $I^\text{im} = \{ i \in I \mid a_{ii} \leq 0 \}$, the set of imaginary indices. In the literature, the Borcherds algebras are often called the generalized Kac-Moody algebras. A special case of Borcherds algebras, the Monster Lie algebra, played an important role in Borcherds’ proof of the Moonshine Conjecture [2]. Nevertheless, the Borcherds algebras are generated by positive and negative simple root vectors even though they may have non-positive norms and multiplicities $> 1$.

The Borcherds-Bozec algebras form a further generalization of Kac-Moody algebras. Their presentations are still associated with Borcherds-Cartan matrices but the generators are higher degree positive and negative simple root vectors in the sense that their degrees are integral multiples of simple roots. Accordingly, the defining relations should be modified. In particular, when we deal with quantum deformation, certain Drinfel’d-type relations are included. We will denote by $I^\infty = (I^\text{re} \times \{1\}) \cup (I^\text{im} \times \mathbb{Z}_{>0})$ the index set for higher degree positive simple root vectors. Even though it seems more appropriate to call them the Borcherds-Bozec-Kac-Moody algebras, for simplicity, we will just use the term Borcherds-Bozec algebras.

The first step toward the understanding of an algebraic object would be to find a way of measuring its size. It can be shown that every Borcherds-Bozec algebra has a decomposition into a direct sum of homogeneous subspaces, the root spaces, and the dimensions of such subspaces are called the root multiplicities. Thus one gets naturally interested in (closed form or recursive form) the root multiplicity formulas.

In this paper, we give a closed form root multiplicity formula for all symmetrizable Borcherds-Bozec algebras. To be more precise, let $I$ be an index set and let $A = (a_{ij})_{i,j \in I}$ be a Borcherds-Cartan matrix. As we have seen before, we have a decomposition $I = I^\text{re} \sqcup I^\text{im}$. Take a finite subset $J$ of $I^\text{re}$ so that the submatrix $A_J = (a_{ij})_{i,j \in J}$ gives rise to a usual Kac-Moody algebra. We denote by $g$ the full Borcherds-Bozec algebra and $g^J$ the Kac-Moody algebra inside $g$. In [5], T. Bozec, O. Schiffmann and E. Vasserot derived a character formula for integrable highest weight $g$-modules. Combining this with the Weyl-Kac character formula, we prove the twisted denominator identity for Borcherds-Bozec
algebras (Proposition 3.1), from which we obtain a closed form root multiplicity formula (Theorem 3.2).

Our formula has the following advantages. First, one can study the structure of the Borcherds-Bozec algebra $\mathfrak{g}$ as an integrable representation of the Kac-Moody algebra $\mathfrak{g}^{(J)}_0$. Second, making an appropriate choice of $J$ can simplify the calculation and provide a deeper understanding of root multiplicities. Finally, different choices of $J$ would yield different expressions of the same quantity, which naturally give rise to combinatorial identities. We do not pursue these perspectives further in this paper. Instead, we illustrate how one can apply our root multiplicity formula in actual computation with the examples of rank 1 algebras, rank 2 algebras and the Monster Borcherds-Bozec algebra. It should be pointed out that, in the symmetric case, the root multiplicities are also given by the constant terms of 1-nilpotent Kac polynomials [5].

The second half of this paper is devoted to the Schofield construction of symmetric Borcherds-Bozec algebras (cf. [20]). Let $Q$ be a locally finite quiver with loops. Then one can associate a symmetric Borcherds-Cartan matrix $A_Q$ and hence a symmetric Borcherds-Bozec algebra $\mathfrak{g}_Q$ as well. Using the Euler characteristic of the projective variety of 1-nilpotent flags in each representation of $Q$, we define a bilinear pairing

$$\langle \ , \ \rangle : \mathcal{E} \times \text{Rep}(Q) \to \mathbb{C},$$

where $\mathcal{E}$ is the free associative algebra on the alphabet $\{S_{i,l} \mid (i,l) \in I^\infty\}$ and $\text{Rep}(Q)$ denotes the category of representations of $Q$.

Let $\mathcal{I}$ be the radical of the pairing $\langle \ , \ \rangle$ in $\mathcal{E}$. We show that $\mathcal{I}$ is both an ideal and a co-ideal of $\mathcal{E}$ under the co-multiplication $\Delta : \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ given by

$$S_{i,l} \mapsto S_{i,l} \otimes 1 + 1 \otimes S_{i,l} \quad \text{for} \quad (i,l) \in I^\infty.$$

Hence the quotient algebra $\mathcal{R} := \mathcal{E}/\mathcal{I}$ becomes a bi-algebra. Our main theorem (Theorem 7.16) states that $\mathcal{R}$ is isomorphic to the positive part of the universal enveloping algebra of $\mathfrak{g}_Q$. Moreover, we show that the Lie algebra $\mathcal{L}$ consisting of primitive elements in $\mathcal{R}$ is isomorphic to the positive part $\mathfrak{g}_Q^+$ of $\mathfrak{g}_Q$.

One of the key ingredients of our proof of the Serre relations in $\mathcal{R}$ is that there is a 1-1 correspondence between the set of positive roots of $\mathfrak{g}_Q$ and the set of dimension vectors of 1-nilpotent indecomposable representations of $Q$ over $\mathbb{C}$. The argument of proving this statement is due to O. Schiffmann. We also use Bozec’s construction of $U^+(\mathfrak{g}_Q)$ in terms of certain constructible functions on a Lagrangian subvariety of strongly semi-nilpotent representations of $Q$, which is in turn based on the theory of perverse sheaves and crystal bases for quantum Borcherds-Bozec algebras [3, 4, 15].
We would like to express our sincere gratitude to Professor Bernard Leclerc and Professor Olivier Schiffmann for many valuable discussions and suggestions. Without their help, this work would not have been materialized.

1. Borcherds-Bozec Algebras

Let $I$ be an index set possibly countably infinite. An integer-valued matrix $A = (a_{ij})_{i,j \in I}$ is called an even symmetrizable Borcherds-Cartan matrix if it satisfies the following conditions:

(i) $a_{ii} = 2, 0, -2, -4, ...$
(ii) $a_{ij} \leq 0$ for $i \neq j$
(iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$
(iv) there exists a diagonal matrix $D = \text{diag}(s_i \in \mathbb{Z}_{>0} | i \in I)$ such that $DA$ is symmetric.

Set $I^{re} = \{i \in I | a_{ii} = 2\}$, $I^{im} = \{i \in I | a_{ii} \leq 0\}$ and $I^{iso} = \{i \in I | a_{ii} = 0\}$.

A Borcherds-Cartan datum consists of:
(a) an even symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$,
(b) a free abelian group $P$, the weight lattice,
(c) $\Pi = \{\alpha_i \in P | i \in I\}$, the set of simple roots,
(d) $P^\vee := \text{Hom}(P, \mathbb{Z})$, the dual weight lattice,
(e) $\Pi^\vee = \{h_i \in P^\vee | i \in I\}$, the set of simple coroots

satisfying the following conditions

(i) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
(ii) $\Pi$ is linearly independent,
(iii) for each $i \in I$, there exists an element $\Lambda_i \in P$ such that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ for all $i, j \in I$.

We would like to mention that given a Borcherds-Cartan matrix, such a Borcherds-Cartan datum always exists. The $\Lambda_i$'s ($i \in I$) are called the fundamental weights.

We denote by $P^+ := \{\lambda \in P | \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$ the set of dominant integral weights. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the root lattice. Set $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For $\beta = \sum k_i \alpha_i \in Q_+$, we define its height to be $|\beta| := \sum k_i$. 
Set $\mathfrak{h} = \mathbb{C} \otimes P^\vee$. For $\lambda, \mu \in \mathfrak{h}^*$, we define a partial ordering by $\lambda \geq \mu$ if and only if $\lambda - \mu \in \mathbb{Q}_+$. Since $A$ is symmetrizable, there exists a non-degenerate symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \lambda) = s_i \langle h_i, \lambda \rangle \quad \text{for all } \lambda \in \mathfrak{h}^*.$$ 

Let $I^\infty := (I^\text{re} \times \{1\}) \cup (I^\text{im} \times \mathbb{Z}_{>0})$. We will often write $i$ for $(i, 1)$ ($i \in I^\text{re}$).

**Definition 1.1.** The **Borcherds-Bozec algebra** $\mathfrak{g}$ associated with a Borcherds-Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the Lie algebra over $\mathbb{C}$ generated by the elements $e_{il}, f_{il}$ ($(i, l) \in I^\infty$) and $h$ with defining relations

\begin{align*}
[&h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h}, \\
[e_{ik}, f_{jl}] &= k \delta_{ij} \delta_{kl} h_i \quad \text{for } i, j \in I, k, l \in \mathbb{Z}_{>0}, \\
[h, e_{jl}] &= l \langle h, \alpha_j \rangle e_{jl}, \quad [h, f_{jl}] = -l \langle h, \alpha_j \rangle f_{jl}, \\
(\text{ad} e_i)^{1-\alpha_{ij}}(e_{jl}) &= 0 \quad \text{for } i \in I^\text{re}, i \neq (j, l), \\
(\text{ad} f_i)^{1-\alpha_{ij}}(f_{jl}) &= 0 \quad \text{for } i \in I^\text{re}, i \neq (j, l), \\
[e_{ik}, e_{jl}] &= [f_{ik}, f_{jl}] = 0 \quad \text{for } a_{ij} = 0.
\end{align*}

(1.1)

Set $\text{deg} \ h = 0$ for $h \in \mathfrak{h}$ and $\text{deg} \ e_{il} = l \alpha_i$, $\text{deg} \ f_{il} = -l \alpha_i$ for $(i, l) \in I^\infty$. Then $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{Q}} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \langle h, \alpha \rangle x \quad \text{for all } h \in \mathfrak{h} \}.$$ 

An element $\alpha \in \mathbb{Q} \setminus \{0\}$ is called a root if $\mathfrak{g}_\alpha \neq 0$. It can be shown that $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$, which is called the root multiplicity of $\alpha$. Set $\Delta_+ = \{ \alpha \in \mathbb{Q}_+ \mid \mathfrak{g}_\alpha \neq 0 \}$ and $\Delta_- = -\Delta_+$, the set of positive and negative roots, respectively.

We say that a root $\alpha$ is real if $(\alpha, \alpha) > 0$ and imaginary if $(\alpha, \alpha) \leq 0$. We denote by $\Delta^\text{re}$ and $\Delta^\text{im}$, the set of real and imaginary roots, respectively. Set $\Delta_+^\text{re} = \Delta_+ \cap \Delta^\text{re}$, $\Delta_+^\text{im} = \Delta_+ \cap \Delta^\text{im}$.

For $i \in I^\text{re}$, we define the simple reflection $r_i \in GL(\mathfrak{h}^*)$ by

$$r_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i \quad \text{for } \lambda \in \mathfrak{h}^*.$$ 

The subgroup $W$ of $GL(\mathfrak{h}^*)$ generated by $r_i$ ($i \in I^\text{re}$) is called the Weyl group of $\mathfrak{g}$. One can easily verify that the symmetric bilinear form $(\ , \ )$ is $W$-invariant.
2. Representation theory

Let $g$ be a Borcherds-Bozec algebra, let $U(g)$ be its universal enveloping algebra and let $M$ be a $g$-module. We say that $M$ has a weight space decomposition if

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_{\mu}, \text{ where } M_{\mu} = \{m \in M \mid h \cdot m = \langle h, \mu \rangle m \text{ for all } h \in \mathfrak{h}\}.$$ 

We denote $\text{wt}(M) := \{\mu \in \mathfrak{h}^* \mid M_{\mu} \neq 0\}$.

A $g$-module $V$ is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there is a non-zero vector $v_{\lambda} \in V$ such that

(i) $V = U(g)v_{\lambda}$,
(ii) $h \cdot v_{\lambda} = \langle h, \lambda \rangle v_{\lambda}$ for all $h \in \mathfrak{h}$,
(iii) $e_{il}v_{\lambda} = 0$ for all $(i,l) \in I^\infty$.

Note that a highest weight module $V$ with highest weight $\lambda$ has a weight space decomposition $V = \bigoplus_{\mu \leq \lambda} V_{\mu}$.

For $\lambda \in \mathfrak{h}^*$, let $J(\lambda)$ be the left ideal of $U(g)$ generated by the elements $h - \langle h, \lambda \rangle 1$ for $h \in \mathfrak{h}$ and $e_{il}$ for $(i,l) \in I^\infty$. Then $M(\lambda) := U(g)/J(\lambda)$ is a highest weight $g$-module with highest weight $\lambda$, called the Verma module. It can be shown that $M(\lambda)$ has a unique maximal submodule $R(\lambda)$ and every highest weight $g$-module with highest weight $\lambda$ is a homomorphic image of $M(\lambda)$. We denote by $V(\lambda) := M(\lambda)/R(\lambda)$ the irreducible quotient.

Proposition 2.1. Let $\lambda \in P^+$ and $V(\lambda) = U(g)v_{\lambda}$ be the irreducible highest weight $g$-module. Then we have

$$f_i^{(h_i, \lambda) + 1} v_{\lambda} = 0 \text{ for } i \in I^\text{re},$$

$$f_{il} v_{\lambda} = 0 \text{ for } (i,l) \in I^\infty \text{ with } \langle h_i, \lambda \rangle = 0.$$ 

Proof. If $f_i^{(h_i, \lambda) + 1} v_{\lambda} \neq 0$ for $i \in I^\text{re}$, then $f_i^{(h_i, \lambda) + 1} v_{\lambda}$ would generate a submodule of $V(\lambda)$ with highest weight $\lambda - ((h_i, \lambda) + 1)\alpha_i < \lambda$, a contradiction.

Similarly, if $\langle h_i, \lambda \rangle = 0$, $f_{il} v_{\lambda}$ would generate a submodule with highest weight $\lambda - l\alpha_i < \lambda$, which is also a contradiction. \qed

Proposition 2.2. Let $\lambda \in P^+$ and $V(\lambda)$ be the irreducible highest weight $g$-module. If $\mu \in \text{wt}(V(\lambda))$ and $i \in I^\text{im}$, the following statements hold.

(a) $\langle h_i, \mu \rangle \in \mathbb{Z}_{\geq 0}$.
(b) If $\langle h_i, \mu \rangle = 0$, $V(\lambda)_{\mu + l\alpha_i} = 0$ for all $l \in \mathbb{Z}_{\geq 0}$.
Proof. Note that $\langle i, l \rangle \neq 0$ whenever $i \neq 1$. Let us prove (d). Suppose $\langle h_i, \mu \rangle = 0$, then $f_{il}(V(\lambda)_\mu) = 0$ for all $l \in \mathbb{Z}_{\geq 0}$.

To prove (b), set $\supp \beta := \{ j \in I | k_j > 0 \}$. If $\langle h_i, \mu \rangle = 0$, then $\langle h_i, \lambda \rangle = 0$ and $a_{ij} = 0$ for all $j \in \supp \beta$. If $i \in \supp \beta$, then $\langle h_i, \lambda \rangle = 0$, there is $j \in \supp \beta$ with $j \neq i$ such that $a_{ij} \neq 0$, a contradiction. Hence $i \notin \supp \beta$ and $\mu + l\alpha_i \notin \text{wt}(V(\lambda))$ for all $l \in \mathbb{Z}_{>0}$.

For each $(i, l) \in I^\infty$, let $g_{(i, l)}$ be the subalgebra of $g$ generated by $e_{il}$, $f_{il}$, $h_i$, which is isomorphic to the Lie algebra $sl(2, \mathbb{C})$ or the Heisenberg algebra. If $\langle h_i, \mu \rangle = 0$ and $v \in V(\lambda)_\mu$, we have $e_{il} v = 0$ by (b). Then $v$ generates a 1-dimensional $g_{(i, l)}$-module, which implies $f_{il} v = 0$. Thus (c) is proved.

Let us prove (d). Suppose $\langle h_i, \mu \rangle \leq -la_{ii}$. If $\langle h_i, \mu \rangle < -la_{ii}$, then $\langle h_i, \mu + l\alpha_i \rangle < 0$ and $\mu + l\alpha_i \notin \text{wt}(V(\lambda))$ by (a). If $\langle h_i, \mu \rangle = -la_{ii}$, write

$$\mu = \lambda - \beta = \lambda - k_i\alpha_i - \sum_{j \neq i} k_j\alpha_j.$$ 

If $i \notin \supp \beta$, we are done. If $i \in \supp \beta$ and $\mu + l\alpha_i \in \text{wt}(V(\lambda))$, then $k_i \geq l$ and

$$\langle h_i, \mu + l\alpha_i \rangle = \langle h_i, \lambda \rangle - (k_i - l)a_{ii} - \sum_{j \neq i} k_ja_{ij} = 0.$$ 

It follows that $\langle h_i, \lambda \rangle = 0$, $(k_i - l)a_{ii} = 0$ and $a_{ij} = 0$ for $j \in \supp \beta$. Since $\langle h_i, \lambda \rangle = 0$, there exists $j \in \supp \beta$ such that $j \neq i$ and $a_{ij} \neq 0$, a contradiction. Hence $\mu + l\alpha_i \notin \text{wt}(V(\lambda))$. \hfill \square

**Definition 2.3.** The category $O_{\text{int}}$ consists of $g$-modules $M$ such that

(i) $M$ has a weight space decomposition $M = \bigoplus_{\mu \in P} M_\mu$ with $\dim_{\mathbb{C}} M_\mu < \infty$ for all $\mu \in P$.

(ii) there exist finitely many weights $\lambda_1, \ldots, \lambda_s \in P$ such that

$$\text{wt}(M) \subset \bigcup_{j=1}^s (\lambda_j - Q_+).$$

(iii) if $i \in I^e$, $f_i$ is locally nilpotent on $M$.

(iv) if $i \in I^m$, we have $\langle h_i, \mu \rangle \geq 0$ for all $\mu \in \text{wt}(M)$.

(v) if $i \in I^e$ and $\langle h_i, \mu \rangle = 0$, then $f_{il}(M_\mu) = 0$ for all $l \in \mathbb{Z}_{>0}$.

(vi) if $i \in I^m$ and $\langle h_i, \mu \rangle \leq -la_{ii}$, then $e_{il}(M_\mu) = 0$ for all $l \in \mathbb{Z}_{>0}$. 
Note that in this definition, \( f_i \)'s are not necessarily locally nilpotent.

By Proposition 2.1 and Proposition 2.2, the irreducible highest weight \( \mathfrak{g} \)-module \( \mathcal{V}(\lambda) \) with \( \lambda \in P^+ \) belongs to the category \( \mathcal{O}_{\text{int}} \). More generally, let \( \mathcal{V} = U(\mathfrak{g})v_{\lambda} \) be a highest weight \( \mathfrak{g} \)-module with \( \lambda \in P^+ \) and assume that \( \mathcal{V} \) satisfies the condition (2.1). Then using the results in [12, Lemma 3.4, Lemma 3.5], one can show that \( \mathcal{V} \) lies in the category \( \mathcal{O}_{\text{int}} \).

Let \( \lambda \in P^+ \) and let \( F_\lambda \) be the set of elements of the form \( s = \sum_{k=1}^{r} s_k \alpha_{i_k} \) (\( r \geq 0 \)) such that

(i) \( i_k \in I^{\text{im}}, \ s_k \in \mathbb{Z}_{>0} \) for all \( 1 \leq k \leq r \),
(ii) \( (\alpha_{i_k}, \alpha_{i_l}) = 0 \) for all \( 1 \leq k, l \leq r \),
(iii) \( (\alpha_{i_k}, \lambda) = 0 \) for all \( 1 \leq k \leq r \).

When \( r = 0 \), we understand \( s = 0 \).

For \( s = \sum s_k \alpha_{i_k} \in F_\lambda \), we define

\[
\begin{align*}
    d_i(s) & = \begin{cases} 
        \# \{ k | i_k = i \} & \text{if } i \notin I^{\text{iso}}, \\
        \sum_{k=i} s_k & \text{if } i \in I^{\text{iso}},
    \end{cases} \\
    \epsilon(s) & = \prod_{i \notin I^{\text{iso}}} (-1)^{d_i(s)} \prod_{i \in I^{\text{iso}}} \phi(d_i(s)) \\
    & = (-1)^{\#(\text{supp}(s) \cap I^{\text{iso}})} \prod_{i \in I^{\text{iso}}} \phi(d_i(s)),
\end{align*}
\]

where \( \prod_{k=1}^{\infty} (1 - q^k) = \sum_{n \geq 0} \phi(n)q^n \).

We define

\[
S_\lambda = \sum_{s \in F_\lambda} \epsilon(s)e^{-s}.
\]

**Remark.** The subalgebra of \( \mathfrak{g} \) generated by the elements \( e_i, f_i, h \) is the Borcherds algebra associated with the given Borcherds-Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\). In this case, all \( s_k = 1 \) and hence we have

\[
\begin{align*}
    d_i(s) & = \# \{ k | i_k = i \} \text{ for all } i \in I^{\text{im}}, \\
    \epsilon(s) & = \prod_{i \in I^{\text{im}}} (-1)^{d_i(s)} = (-1)^{|s|}.
\end{align*}
\]

**Example 2.4.**

Let \( i \in I^{\text{im}} \) and \( \mathfrak{g}(i) \) be the Borcherds-Bozec algebra associated with \( A = (a_{ii}) \).
(a) If $a_{ii} = 0$, then $g(i)$ is the Lie algebra generated by $h_i, e_{il}, f_{il}$ for $l \geq 1$ with defining relations

$$[h_i, e_{il}] = [h_i, f_{il}] = 0, \quad [e_{ik}, f_{il}] = k \delta_{kl} h_i, \quad [e_{ik}, e_{il}] = [f_{ik}, f_{il}] = 0$$

for all $k, l \geq 1$. Hence $g(i)$ is the full Heisenberg algebra.

If $\langle h_i, \lambda \rangle > 0$, then $F_\lambda = \{0\}$ and $S_\lambda = 1$.

If $\langle h_i, \lambda \rangle = 0$, then $F_0 = \{l \alpha_i \mid l \geq 0\}$ and $d_i(l \alpha_i) = l$ for all $l \geq 0$. Hence

$$S_0 = \sum_{l \geq 0} \epsilon(l \alpha_i) e^{-l \alpha_i} = \sum_{l \geq 0} \phi(l) e^{-l \alpha_i} = \prod_{k=1}^{\infty} (1 - e^{-k \alpha_i}).$$

Note that $U(g^-(i)) \cong C[f_{il} \mid l \geq 1]$, which implies

$$\text{ch} U(g^-(i)) = \prod_{k=1}^{\infty} \frac{1}{1 - e^{-k \alpha_i}} = S_0^{-1}.$$

(b) If $a_{ii} < 0$, we have

$$[h_i, e_{il}] = l a_{ii} e_{il}, \quad [h_i, f_{il}] = -l a_{ii} f_{il}, \quad [e_{ik}, f_{il}] = k \delta_{kl} h_i$$

and there are no relations among $e_{il}$'s nor among $f_{il}$'s. Hence $g^+(i)$ (resp. $g^-(i)$) is the free Lie algebra on $V^+ = \bigoplus_{l \geq 1} C e_{il}$ (resp. $V^- = \bigoplus_{l \geq 1} C f_{il}$).

If $\langle h_i, \lambda \rangle > 0$, then $F_\lambda = \{0\}$ and $S_\lambda = 1$.

If $\langle h_i, \lambda \rangle = 0$, we have

$$F_0 = \{l \alpha_i \mid l \geq 0\}, \quad d_i(l \alpha_i) = \begin{cases} 0 & \text{if } l = 0, \\ 1 & \text{if } l \geq 1, \end{cases} \quad \epsilon(l \alpha_i) = \begin{cases} 1 & \text{if } l = 0, \\ -1 & \text{if } l \geq 1, \end{cases}$$

which implies

$$S_0 = 1 - (e^{-\alpha_i} + e^{-2 \alpha_i} + \cdots) = 1 - e^{-\alpha_i} \frac{1}{1 - e^{-\alpha_i}} = \frac{1 - 2 e^{-\alpha_i}}{1 - e^{-\alpha_i}}.$$

On the other hand, $U(g^-(i)) \cong C(f_{il} \mid l \geq 1)$, the free associative algebra on $V^- = \bigoplus_{l \geq 1} C f_{il}$, the tensor algebra on $V^-$. Since

$$\text{ch} V^- = \frac{e^{-\alpha_i}}{1 - e^{-\alpha_i}},$$
we have
\[
\text{ch} U(\mathfrak{g}^-) = \frac{1}{1 - \text{ch} V^+} = \frac{1 - e^{-\alpha_i}}{1 - e^{-\alpha_i}} = \frac{1}{1 - 2e^{-\alpha_i}} = S_{0},
\]
again.

Let \( \rho \) be a linear functional on \( \mathfrak{h} \) such that \( \langle h_i, \rho \rangle = 1 \) for all \( i \in I \). For each \( w \in W \), we denote by \( l(w) \) the length of \( w \) and set \( \epsilon(w) = (-1)^{l(w)} \).

**Theorem 2.5.** [5] Let \( V = U(\mathfrak{g})v_\lambda \) be a highest weight \( \mathfrak{g} \)-module with highest weight \( \lambda \in P^+ \). If \( V \) satisfies the condition (2.1), then the character of \( V \) is given by the following formula:

\[
\text{ch} V = \sum_{w \in W} \epsilon(w)e^{w(\lambda + \rho) - \rho}w(S_\lambda)
\]

\[
= \sum_{w \in W} \sum_{s \in F_0} \epsilon(w)\epsilon(s)e^{w(\lambda + \rho - s) - \rho}
\]

\[
\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha}
\]

**Proof.** The argument given in [5, Theorem 6.1] works for any highest weight \( \mathfrak{g} \)-module with highest weight \( \lambda \in P^+ \) satisfying the condition (2.1). \( \square \)

In particular, when \( \lambda = 0 \), we obtain the denominator identity

\[
\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha} = \sum_{w \in W} \epsilon(w)e^{w\rho - \rho}w(S_0)
\]

\[
= \sum_{w \in W} \sum_{s \in F_0} \epsilon(w)\epsilon(s)e^{w(\rho - s) - \rho}.
\]

In Borcherds algebra case, we have

\[
\text{ch} V = \sum_{w \in W} \sum_{s \in F_0} (-1)^{l(w) + |s|}e^{w(\lambda + \rho - s) - \rho}
\]

\[
\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha},
\]

which yields

\[
\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha} = \sum_{w \in W} \sum_{s \in F_0} (-1)^{l(w) + |s|}e^{w(\rho - s) - \rho}.
\]

Now we will show that every \( \mathfrak{g} \)-module in the category \( \mathcal{O}_{\text{int}} \) is completely reducible.
Proposition 2.6.

(a) Every highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$ satisfying (2.1) is isomorphic to $V(\lambda)$.

(b) If $V(\lambda)$ belongs to the category $\mathcal{O}_{\text{int}}$, then $\lambda \in P^+$.

(c) Every simple object in the category $\mathcal{O}_{\text{int}}$ is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.

(d) The category $\mathcal{O}_{\text{int}}$ is semisimple.

Proof. (a) follows from the character formula.

(b) If $i \in I^{\text{re}}$, then by the standard $sl(2, \mathbb{C})$-theory, $\langle h_i, \lambda \rangle \geq 0$. If $i \in I^{\text{im}}$, by the definition of $\mathcal{O}_{\text{int}}$, we have $\langle h_i, \lambda \rangle \geq 0$.

(c) Every simple object in $\mathcal{O}_{\text{int}}$ is a highest weight $\mathfrak{g}$-module because any maximal weight vector would generate a highest weight submodule. Then the statement (b) implies (c).

(d) Thanks to Theorem 2.5, the same argument in [7, Theorem 3.5.4] or [8, Theorem 3.7] would prove our claim. 

Example 2.7. Let $i \in I^{\text{im}}$ and $\mathfrak{g}(i)$ be the Borcherds-Bozec algebra associated with $A = (a_{ij})$. If $\langle h_i, \lambda \rangle = 0$, then the case is trivial; i.e., $V(\lambda) = \mathbb{C}v_\lambda$ and ch$V(\lambda) = e^\lambda$. Thus we assume that $\langle h_i, \lambda \rangle > 0$.

(a) If $a_{ii} = 0$, then $W = \{1\}$ and $S_\lambda = 1$. Hence

$$\text{ch}V(\lambda) = \frac{e^\lambda}{\prod_{k=1}^\infty (1 - e^{-k\alpha_i})^{\dim \mathfrak{g}_{k\alpha_i}}}. $$

By the denominator identity and Example 2.4 (a), we have

$$\prod_{k=1}^\infty (1 - e^{-k\alpha_i})^{\dim \mathfrak{g}_{k\alpha_i}} = S_0 = \prod_{k=1}^\infty (1 - e^{-k\alpha_i}).$$

It follows that $\dim \mathfrak{g}_{k\alpha_i} = 1$ for all $k \geq 1$ and

$$\text{ch}V(\lambda) = \frac{e^\lambda}{\prod_{k=1}^\infty (1 - e^{-k\alpha_i})}. $$
(b) If $a_i < 0$, then again $W = \{1\}$, $S_\lambda = 1$ and the denominator identity combined with Example 2.4(b) gives

$$\text{ch} V(\lambda) = \frac{e^\lambda}{\prod_{k=1}^\infty (1 - e^{-k\alpha_i})} = e^\lambda S_0^{-1} = \frac{e^\lambda(1 - e^{-\alpha_i})}{1 - 2e^{-\alpha_i}}.$$ 

In each case, we have $\text{ch} V(\lambda) = e^\lambda \text{ch} U (g_{(i)}).$ Thus when $\langle h_i, \lambda \rangle > 0$, the irreducible $g_{(i)}$-module $V(\lambda)$ is isomorphic to the Verma module $M(\lambda)$. Therefore, for each $i \in I_{\text{im}}$, every $g_{(i)}$-module in the category $O_{\text{int}}$ is isomorphic to a direct sum of 1-dimensional trivial modules and (infinite-dimensional) Verma modules. □

3. Root multiplicity formula

Recall the denominator identity (2.5):

$$\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) \dim g_{\alpha} = \sum_{w \in W} \sum_{s \in F_0} \epsilon(w) \epsilon(s) e^{w(\rho - s) - \rho},$$

where $F_0$ is the set of elements of the form $s = \sum_{k=1}^r s_k \alpha_i$ ($r \geq 0$) such that

(i) $i_k \in I_{\text{im}}$, $s_k \in \mathbb{Z}_{>0}$ for all $1 \leq k \leq r$,

(ii) $(\alpha_i, \alpha_i) = 0$ for all $1 \leq k, l \leq r$.

Take a finite subset $J \subset I^e$ and set

$$\Delta^J = \Delta \cap \left( \sum_{j \in J} \mathbb{Z} \alpha_j \right), \quad \Delta^J_\pm = \Delta_\pm \cap \Delta^J, \quad \Delta_\pm(J) = \Delta_\pm \setminus \Delta^J_\pm,$$

$$W^J = \langle r_j \mid j \in J \rangle, \quad W(J) = \{ w \in W \mid w \Delta_- \cap \Delta_+ \subset \Delta_+(J) \}.$$ 

Note that $W(J)$ is a set of right coset representatives of $W^J$ in $W$. Hence when $w = w'r_j$ with $l(w) = l(w') + 1$, $w \in W(J)$ if and only if $w' \in W(J)$ and $w'(\alpha_j) \in \Delta_+(J)$. In this way, one can construct $W(J)$ inductively.

Let

$$g_0^{(J)} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^J} g_{\alpha}, \quad g_\pm^{(J)} = \bigoplus_{\alpha \in \Delta^J_\pm} g_{\alpha}.$$ 

Then we have a twisted triangular decomposition

$$\mathfrak{g} = g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}.$$
Let \( P^+_j = \{ \lambda \in P \mid \langle h_j, \lambda \rangle \geq 0 \text{ for all } j \in J \} \) and let \( V_J(\lambda) \) be the irreducible highest weight \( \mathfrak{g}_0^{(J)} \)-module with highest weight \( \lambda \in P^+_j \). Then by the Weyl-Kac character formula [11, 12], we have

\[
\text{ch} V_J(\lambda) = \sum_{w \in W} \epsilon(w) e^{w(\lambda) - \rho} \prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha}.
\]

Using this and the denominator identity (2.5), we obtain the following twisted denominator identity for Borcherds-Bozec algebras, which will be the corner-stone for our root multiplicity formula.

**Proposition 3.1.**

\[
(3.1) \quad \prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha} = \sum_{w \in W} \sum_{s \in F_0} \epsilon(w) \epsilon(s) \text{ch} V_J(w(\rho - s) - \rho).
\]

**Proof.** The right-hand side is equal to

\[
\sum_{w \in W} \sum_{s \in F_0} \epsilon(w) \epsilon(s) \frac{\sum_{w' \in W} \epsilon(w') e^{w'(w(\rho - s) - \rho) - \rho}}{\prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha}}
\]

\[
= \sum_{w \in W} \sum_{s \in F_0} \epsilon(w) \epsilon(s) \frac{\sum_{w' \in W} \epsilon(w') e^{w'(w(\rho - s) - \rho) - \rho}}{\prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha}}
\]

\[
= \prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha} \prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha} = \prod_{\alpha \in \Delta^+_J} (1 - e^{-\alpha})^{\text{dim} \mathfrak{g}_\alpha}
\]

as desired. \( \square \)

For \( s = \sum s_k \alpha_{i_k} \in F_0 \), set

\[
s^+ := \sum_{i_k \notin \mathfrak{l}^\text{iso}} s_k \alpha_{i_k}, \quad s^0 := \sum_{i_k \in \mathfrak{l}^\text{iso}} s_k \alpha_{i_k}, \quad d(s) := \sum_{i \in \mathfrak{l}^\text{im}} d_i(s).
\]

Then we have

\[
s = s^+ + s^0, \quad d(s) = d(s^+) + d(s^0), \quad \epsilon(s) = \epsilon(s^+) \epsilon(s^0).
\]

Here, we understand \( d(0) = 0, \epsilon(0) = 1. \)
Define
\[ V^{(J)} = \bigoplus_{w \in W(J), s \in F_0, l(w) + d(s) > 0} (-1)^{l(w) + d(s^+) + 1} e(s^0) V_J(w(\rho - s) - \rho), \]
a virtual direct sum of \( g_0^{(J)} \)-modules. Then the twisted denominator identity can be written as
\[ \prod_{\alpha \in \Delta_+^{(J)}} (1 - e^{-\alpha})^{\dim g_\alpha} = 1 - \text{ch} V^{(J)}. \]

For each \( \mu \in P \), we denote the virtual dimension of \( V^{(J)}_\mu \) by
\[ d_\mu := \dim V^{(J)}_\mu = \sum_{w \in W(J), s \in F_0, l(w) + d(s) > 0} (-1)^{l(w) + d(s^+) + 1} e(s^0) \dim V_J(w(\rho - s) - \rho). \]

Set \( \text{wt}(V^{(J)}) = \{ \mu \in P \mid d_\mu \neq 0 \} \) and give an enumeration \((\mu_1, \mu_2, \ldots)\) of \( \text{wt}(V^{(J)}) \). For each \( \mu \in P \), let \( \mathcal{P}^{(J)}(\mu) \) be the set of partitions of \( \mu \) into a sum of \( \mu_i \)'s:
\[ \mathcal{P}^{(J)}(\mu) = \{ s = (s_i)_{i \geq 1} \mid s_i \in \mathbb{Z}_{\geq 0}, \sum s_i \mu_i = \mu \}. \]

For a partition \( s = (s_i) \in \mathcal{P}^{(J)}(\mu) \), we write \( |s| := \sum s_i \), \( s! := \prod s_i! \) and define the Witt partition function of \( \mu \) by
\[ W^{(J)}(\mu) = \sum_{s \in \mathcal{P}^{(J)}(\mu)} \frac{(|s| - 1)!}{s!} \prod_{\mu_i} (d_\mu)^{s_i}. \]

We now derive our root multiplicity formula for Borcherds-Bozec algebras (cf. [13]).

**Theorem 3.2.** For any root \( \alpha \in \Delta_+^{(J)} \), we have
\[ \dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W^{(J)}(\alpha/d), \]
where \( \mu \) denotes the Möbius function.
Proof. Note that $\text{wt}(V^{(J)}) \subset -\mathbb{Q}_+$. We write

$$\text{ch}(V^{(J)}) = \sum_{\mu \in \text{wt}(V^{(J)})} d_{\mu} e^{-\mu} = \sum_{i=1}^{\infty} d_{\mu_i} e^{-\mu_i}.$$ 

Then the twisted denominator identity yields

$$\prod_{\alpha \in \Delta_+(J)} (1 - e^{-\alpha})^{\dim g_\alpha} = 1 - \sum_{i=1}^{\infty} d_{\mu_i} e^{-\mu_i}.$$ 

Taking the logarithm of the left-hand side, we obtain

$$\log \left( \prod_{\alpha \in \Delta_+(J)} (1 - e^{-\alpha})^{\dim g_\alpha} \right) = \sum_{\alpha \in \Delta_+(J)} \dim g_\alpha \log(1 - e^{-\alpha})$$

On the other hand, taking the logarithm of the right-hand side, we get

$$\log \left( 1 - \sum_{i=1}^{\infty} d_{\mu_i} e^{-\mu_i} \right) = -\sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{i=1}^{\infty} d_{\mu_i} e^{-\mu_i} \right)^n$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{s_i \equiv (s_i)} \frac{n!}{s_i!} \prod (d_{\mu_i} e^{-\mu_i})^{s_i} \right)$$

$$= -\sum_{\beta} \left( \sum_{s_i \equiv (s_i)} \frac{(|s| - 1)!}{s!} \prod d_{\mu_i}^{s_i} \right) e^{-\beta} = -\sum_{\beta} W^{(J)}(\beta)e^{-\beta}.$$ 

Thus we have

$$\sum_{\alpha \in \Delta_+(J)} \sum_{k=1}^{\infty} \left( \frac{1}{k} \dim g_\alpha \right) e^{-k\alpha} = \sum_{\beta} W^{(J)}(\beta)e^{-\beta},$$ 

which yields

$$W^{(J)}(\beta) = \sum_{\beta = k\alpha} \frac{1}{k} \dim g_\alpha = \sum_{d|\beta} \frac{1}{d} \dim g_{\beta/d}.$$
Hence by M"obius inversion, we obtain
\[
\dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W^{(J)}(\alpha/d).
\]

\[\square\]

**Remark.** A natural interpretation of the twisted denominator identity would be the Euler-Poincaré principle:
\[
\sum_{k=0}^{\infty} (-1)^k \operatorname{ch}(A^k(g(J)) \otimes V(\lambda)) = \sum_{k=0}^{\infty} (-1)^k \operatorname{ch} H_k(g(J), V(\lambda)),
\]
where \(V(\lambda)\) denotes the irreducible highest weight \(g\)-module with highest weight \(\lambda \in P^+.\) We expect the following Kostant-type homology formula holds:
\[
H_k(g(J), V(\lambda)) \cong \bigoplus_{w \in W(J)} \epsilon(s^0) V_J(w(\lambda + \rho - s) - \rho).
\]

We now consider some concrete applications of the root multiplicity formula (3.6).

**Example 3.3.**

Let \(i \in I^\text{im} \) and let \(g_{(i)}\) be the Borcherds-Bozec algebra associated with \(A = (a_{ii}).\) In this case, we have
\[
\Delta_+ = \{k\alpha_i \mid k \geq 1\}, \quad F_0 = \{s = k\alpha_i \mid k \geq 0\}, \quad J = \emptyset, \quad W(J) = \{1\}.
\]

(a) If \(a_{ii} = 0,\) for \(s = k\alpha_i \ (k \geq 0),\) we have
\[
s = s^0, \quad d(s) = k, \quad \epsilon(s) = \phi(k),
\]
which implies
\[
V := V^{(J)} = \bigoplus_{k=1}^{\infty} (-\phi(k)) Cu_k,
\]
where \(u_k\) is a vector with weight \(-k\alpha_i \ (k \geq 1).\)

Let \(q = e^{-\alpha_i}\) and \(g(n) = g_{n\alpha_i}.\) Then the denominator identity yields
\[
\prod_{n=1}^{\infty} (1 - q^n)^{\dim g(n)} = 1 - \sum_{k=1}^{\infty} (-\phi(k)) q^k = \prod_{n=1}^{\infty} (1 - q^n),
\]
from which we conclude
\[
\dim g(n) = 1 \quad \text{for all} \quad n \geq 1
\]
as we have seen in Example 2.7 (a).

Let us compute \( \dim g(n) \) using the multiplicity formula (3.6), which will be much more complicated than the direct calculation in this case. But the multiplicity formula (3.6) can be applied to all Borcherds-Bozec algebras.

We identify \( \text{wt}(V) \) with \( \mathbb{Z}_{>0} = \{1, 2, \ldots\} \). Note that \( \dim V_k = -\phi(k) \) for all \( k \geq 1 \). For each \( n \geq 1 \), set

\[
\mathcal{P}(n) := \{ s = (s_i)_{i \geq 1} \mid \sum i s_i = n \}
\]

be the set of all partitions of \( n \) into a sum of positive integers. Then the Witt partition function is given by

\[
W(n) = \sum_{s \in \mathcal{P}(n)} \frac{(|s| - 1)!}{s!} \prod (-\phi(i))^{s_i}.
\]

Hence the root multiplicity can be computed as follows (and we obtain a combinatorial identity as well)

\[
\dim g(n) = \sum_{d | n} \frac{1}{d} \mu(d) \sum_{s \in \mathcal{P}(n/d)} \frac{(|s| - 1)!}{s!} \prod (-\phi(-i))^{s_i},
\]

which must be equal to 1 for all \( n \geq 1 \).

For example, for \( n = 4 \), the formula (3.6) gives

\[
\dim g(4) = W(4) - \frac{1}{2} W(2).
\]

Note that

\[
\prod_{n=1}^{\infty} (1 - q^n) = 1 - q - q^2 + q^5 + \cdots.
\]

Consider the partitions

\[
4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,
\]

\[
2 = 1 + 1.
\]

Then we have

\[
W(4) = (-\phi(1)) + (-\phi(3))(-\phi(1)) + \frac{1}{2!} (-\phi(2))^2 + \frac{2!}{2!} (-\phi(1))^2 (-\phi(2)) + \frac{3!}{4!} (-\phi(1))^4 = \frac{7}{4},
\]

\[
W(2) = (-\phi(2)) + \frac{1}{2!} (-\phi(1))^2 = \frac{3}{2},
\]
which gives
\[ \dim g(4) = \frac{7}{4} - \frac{1}{2} \times \frac{3}{2} = 1 \]
as expected.

(b) If \( a_{ii} < 0 \), for \( s = k\alpha_i \ (k \geq 1) \), we have
\[ s = s^+, \quad d(s) = 1, \quad \epsilon(s) = -1, \quad d(0) = 0, \quad \epsilon(0) = 1, \]
which implies
\[ V := V^{(J)} = \bigoplus_{k=1}^{\infty} C u_k, \]
where \( u_k \) is a vector with weight \(-k\alpha_i \ (k \geq 1)\).

Then the denominator identity is equal to
\[ \prod_{n=1}^{\infty} (1 - q^n)^{\dim g(n)} = 1 - \sum_{k=1}^{\infty} q^k = 1 - q - q^2 - \cdots. \]
Since \( \text{wt}(V) = \{1, 2, 3, \ldots\} \) and \( \dim V_k = 1 \) for all \( k \geq 1 \), we have
\[ W(n) = \sum_{s \in \mathcal{P}(n)} \frac{(|s| - 1)!}{s!} \]
and hence
\[ \dim g(n) = \sum_{d|n} \frac{1}{d} \mu(d) \sum_{s \in \mathcal{P}(n/d)} \frac{(|s| - 1)!}{s!}. \]

Let \( n = 4 \). It is easy to compute \( W(4) = 15/4, \ W(2) = 3/2 \), from which we obtain
\[ \dim g(4) = \frac{15}{4} - \frac{1}{2} \times \frac{3}{2} = 3. \]

On the other hand, recall that we have
\[ \prod_{n=1}^{\infty} (1 - q^n)^{\dim g(n)} = 1 - \sum_{k=1}^{\infty} q^k = \frac{1 - 2q}{1 - q}. \]
Let \( L = \bigoplus_{n=1}^{\infty} L_n \) be the free Lie algebra on \( k \) generators. Then
\[ \text{ch} U(L) = \frac{1}{1 - kq} = \prod_{n=1}^{\infty} (1 - q^n)^{-\dim L_n}, \]
and the \( \dim \mathcal{L}_n \) is given by the \( k \)-th Necklace polynomial of degree \( n \)

\[
\dim \mathcal{L}_n = N(k, n) := \frac{1}{n} \sum_{d|n} \mu(d)k^{n/d}.
\]

Thus

\[
chU(g) = \prod_{n=1}^{\infty} (1 - q^n)^{\dim g(n)} = \frac{1 - q}{1 - 2q}
\]

\[
= (1 - q) \prod_{n=1}^{\infty} (1 - q^n)^{-N(2,n)}
\]

\[
= (1 - q)^{-1} \prod_{n=2}^{\infty} (1 - q^n)^{-N(2,n)}.
\]

Therefore we obtain

\[
\dim g(n) = \begin{cases} 
1 & \text{if } n = 1, \\
N(2, n) & \text{if } n \geq 2.
\end{cases}
\]

For example, when \( n = 4 \), we have

\[
\dim g(4) = N(2, 4) = \frac{1}{4}(2^4 - 2^2) = 3
\]

as expected.

More generally, we obtain the following interesting combinatorial identity

\[
\sum_{d|n} \frac{1}{d} \mu(d) \sum_{s \in P(n/d)} \frac{(|s| - 1)!}{s!} = \frac{1}{n} \sum_{d|n} \mu(d)2^{n/d}.
\]

\( \square \)

**Example 3.4.**

Let \( I = \{0, 1\} \), \( A = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix} \) \( (a \geq 1) \) and let \( g \) be the Borcherds-Bozec algebra associated with \( A \).

Then \( I_{re} = \{0\} \), \( I_{im} = \{1\} \), \( W = \{1, r_0\} \) is the Weyl group and \( F_0 = \{s = k\alpha_1 | k \geq 0\} \). Moreover, we have

\[
d(0) = 1, \quad \epsilon(0) = 1, \quad d(k\alpha_1) = 1, \quad \epsilon(k\alpha_1) = -1 \quad \text{for all} \quad k \geq 1.
\]
Take $J = \{0\}$. Then $W(J) = \{1\}$ and

$$g_0^{(J)} = \langle e_0, f_0, h_0 \rangle \oplus \mathbb{C}h_1 \cong sl(2, \mathbb{C}) \oplus \mathbb{C}h_1.$$  

Note that, since $l(w) + d(s) > 0$, $s$ must have the form $s = k\alpha_1$ with $k \geq 1$. It follows that

$$V := V^{(J)} = \bigoplus_{k=1}^{\infty} V_J(-k\alpha_1),$$

where $V_j(-k\alpha_1)$ is the irreducible $sl(2, \mathbb{C})$-module with highest weight $-k\alpha_1$. Since $\langle h_0, -k\alpha_1 \rangle = ka$, we have

$$\text{wt}(V) = \bigcup_{k=1}^{\infty} \{-l\alpha_0 \mid 0 \leq l \leq ka\} = \bigcup_{k=1}^{\infty} \{(k, l) \mid 0 \leq l \leq ka\}$$

with $\dim V_{(k,l)} = 1$ for all $k, l$.

Give a total ordering on $\text{wt}(V)$ by

$$(k, l) < (p, q) \text{ if and only if } k < p \text{ or } k = p, l < q.$$  

For each weight $(m, n) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, let

$$\mathcal{P}(m, n) = \{s = (s_{ij}) \mid s_{ij} \in \mathbb{Z}_{\geq 0}, (i, j) \in \text{wt}(V), \sum s_{ij}(i, j) = (m, n)\}$$

be the set of partitions of $(m, n)$ into a sum of elements in $\text{wt}(V)$ with respect to the total ordering given above. Then we obtain

$$W(m, n) = \sum_{s \in \mathcal{P}(m, n)} \frac{(|s| - 1)!}{s!},$$

$$\dim g(m, n) = \sum_{d|m(n)} \frac{1}{d}\mu(d) \sum_{s \in \mathcal{P}(m/d, n/d)} \frac{(|s| - 1)!}{s!}. $$

For example, take $a = 2$ and $\alpha = (4, 4)$. Then the total ordering of $\text{wt}(V)$ defined above is given by

$$(1, 0), (1, 1), (1, 2), \ldots$$
$$(2, 0), (2, 1), (2, 2), (2, 3), (2, 4), \ldots$$
$$(3, 0), (3, 1), (3, 2), (3, 3), (3, 4), \ldots$$
$$(4, 0), (4, 1), (4, 2), (4, 3), (4, 4), \ldots$$

......
Consider the partitions
\[(4, 4) = (3, 4) + (1, 0) = (3, 3) + (1, 1) = (3, 2) + (1, 2) = (2, 4) + (1, 0) + (1, 0) = (2, 4) + (2, 0) = (2, 3) + (2, 1) = (2, 3) + (1, 1) + (1, 0) = (2, 2) + (2, 2) = (2, 2) + (1, 2) + (1, 0) + (1, 0) = (2, 0) + (1, 2) + (1, 2) = (1, 2) + (1, 2) + (1, 0) + (1, 0) = (1, 2) + (1, 1) + (1, 0) = (1, 1) + (1, 1) + (1, 0) + (1, 0) = (1, 2) + (1, 1) + (1, 1) + (1, 0) = (1, 1) + (1, 0) = (1, 1),
\]
which yield
\[
W(4, 4) = 1 + 1 + 1 + 1 + 2! + 1 + 1 + 2 + 1 + 2! + 2! + 2!
+ 2! + 3! + 3! + 3! + 4! = 20 + \frac{1}{4}.
\]

Next, the partitions
\[(2, 2) = (1, 2) + (1, 0) = (1, 1) + (1, 1)
\]
give
\[
W(2, 2) = 1 + 1 + \frac{1}{2!} = 2 + \frac{1}{2}.
\]

Hence we obtain
\[
\dim g(4, 4) = W(4, 4) - \frac{1}{2}W(2, 2) = 19.
\]

\section*{Example 3.5.}

Let \(I = \{0, 1\}\). \(A = \begin{pmatrix} 2 & -a \\ -a & 0 \end{pmatrix} (a \geq 1)\) and \(g\) be the Borcherds-Bozec algebra associated with \(A\).

Then \(I^\text{re} = \{0\}, I^\text{im} = \{1\}, W = \{1, r_0\}\) is the Weyl group and \(F_0 = \{s = k\alpha_1 \mid k \geq 0\}\).

Moreover, we have
\[
d(k\alpha_1) = k, \quad \epsilon(k\alpha_1) = \phi(k) \quad \text{for all} \quad k \geq 0.
\]

Take \(J = \{0\}\). Then \(W(J) = \{1\}\) and
\[
g_0^{(J)} = \langle e_0, f_0, h_0 \rangle \oplus Ch_1 \cong sl(2, \mathbb{C}) \oplus Ch_1.
\]

Hence \(s\) must have the form \(s = k\alpha_1\) with \(k \geq 1\) and
\[
V := V^{(J)} = \bigoplus_{k=1}^{\infty} (-\phi(k))V_J(-k\alpha_1),
\]
where $$\text{wt}(V) = \bigcup_{k=1}^{\infty} \{-ka_1 - la_0 \mid 0 \leq l \leq ka\} = \bigcup_{k=1}^{\infty} \{(k,l) \mid 0 \leq l \leq ka\}$$ with $$\dim V_{(k,l)} = -\phi(k)$$ for all $$k, l$$.

Therefore, we have $$W(m,n) = \sum_{s \in \mathcal{P}(m,n)} \frac{|s| - 1)!}{s!} \prod_{k=1}^{\infty} \prod_{l=1}^{ka} (-\phi(k))^{s_{kl}}$$,

$$\dim g(m,n) = \sum_{d|m,n} \frac{1}{d} \mu(d) \sum_{s \in \mathcal{P}(m/d,n/d)} \frac{|s| - 1)!}{s!} \prod_{k=1}^{\infty} \prod_{l=1}^{ka} (-\phi(k))^{s_{kl}}$$.

For example, take $$a = 2$$ and $$\alpha = (4, 4)$$. Then using the partitions in Example 3.4, we get

$$W(4, 4) = 16 + \frac{1}{4}, \quad W(2, 2) = 2 + \frac{1}{2}$$,

which gives

$$\dim g(4, 4) = W(4, 4) - \frac{1}{2} W(2, 2) = 15.$$ 

□

4. MONSTER BORCHERDS-BOZEC ALGEBRA

Let $$I$$ be an index set and let $$f : I \rightarrow \mathbb{Z}_{>0}$$ be a function such that $$f(i) = 1$$ for all $$i \in I^\text{re}$$. Set

$$\tilde{I} := \{(i,p) \mid i \in I, \ 1 \leq p \leq f(i)\} \subset I \times \mathbb{Z}_{>0}.$$ 

Let $$\tilde{A}$$ be a Bocherds-Cartan matrix indexed by $$\tilde{I}$$. Assume that for each pair $$(i,j) \in I \times I$$, there is a block submatrix of size $$f(i) \times f(j)$$ in $$\tilde{A}$$ such that all the entries in this block submatrix are the same, say, $$a_{ij}$$. Then we obtain a Bocherds-Cartan matrix $$\tilde{A} = (a_{ij})_{i,j \in I}$$ in this case, we say that $$\tilde{A}$$ is the Bocherds-Cartan matrix with charge $$f = (f(i) \mid i \in I)$$ induced from $$\tilde{A}$$. Conversely, given a Bocherds-Cartan matrix $$A = (a_{ij})_{i,j \in I}$$ and a sequence $$f = (f(i) \mid i \in I)$$, we obtain a Bocherds-Cartan matrix $$\tilde{A} = (\tilde{a}_{(i,p),(j,q)})$$ indexed by $$\tilde{I}$$ by setting

$$\tilde{a}_{(i,p),(j,q)} = a_{ij} \quad \text{for} \ i, j \in I, \ 1 \leq p \leq f(i), \ 1 \leq q \leq f(j).$$
Hence we can develop the theory of Borcherds-Bozec algebras associated with Borcherds-Cartan matrices $A$ with charge $f$ by taking the corresponding theory for the Borcherds-Bozec algebras associated with $\tilde{A}$.

Let $I = \{-1\} \cup \{1, 2, 3, \ldots\}$ and let

\[(4.1)\quad A = (a_{ij})_{i,j \in I} = (-\langle i, j \rangle)_{i,j \in I} = \begin{pmatrix} 2 & 0 & -1 & -2 & \cdots \\ 0 & -2 & -3 & -4 & \cdots \\ -1 & -3 & -4 & -5 & \cdots \\ -2 & -4 & -5 & -6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

be the Borcherds-Cartan matrix with charge $c = (c(i) \mid i \in I)$, where $c(i)$ is the $i$-th coefficient of the elliptic modular function

\[(4.2)\quad j(q) - 744 = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 196884q + 21493760q^2 + \cdots .\]

The associated Borcherds algebra $L$ is called the Monster Lie algebra and played an important role in Borcherds’ proof of the Moonshine Conjecture [1, 2]. Note that $\dim L_{\alpha_i} = c(i)$ for all $i \in I$. By setting $\deg \alpha_i = (1, i)$ for all $i \in I$, the Lie algebra $L$ has a $(\mathbb{Z} \times \mathbb{Z})$-grading

\[(4.3)\quad L = \bigoplus_{m,n \in \mathbb{Z}} L_{(m,n)} \quad \text{such that} \quad \dim L_{(m,n)} = c(mn) \quad \text{for all} \quad m, n \geq 1.\]

On the other hand, for $m, n \geq 1$, let

\[(4.4)\quad \mathcal{P}(m, n) = \{ s = (s_{ij})_{i,j \geq 1} \mid s_{ij} \in \mathbb{Z}_{\geq 0}, \sum s_{ij}(i, j) = (m, n) \}\]

be the set of all partitions of $(m, n)$ into a sum of pairs of positive integers with respect to any partial ordering. Then by the root multiplicity formula for Borcherds algebras [14], the Witt partition function is given by

\[(4.5)\quad W(m, n) = \sum_{s \in \mathcal{P}(m, n)} \frac{(|s| - 1)!}{s!} \prod_{i,j} c(i + j - 1)^{s_{ij}} \]

and we obtain

\[(4.6)\quad \dim L_{(m,n)} = \sum_{d \mid (m,n)} \frac{1}{d} \mu(d) \sum_{s \in \mathcal{P}(m/d,n/d)} \frac{(|s| - 1)!}{s!} \prod_{i,j} c(i + j - 1)^{s_{ij}}.\]
Combined with (4.3), we obtain a combinatorial identity

\[
(4.7) \quad c(mn) = \sum_{d|\gcd(m,n)} \frac{1}{d} \sum_{s\in \mathcal{P}(m/d,n/d)} \frac{(|s|-1)!}{s!} \prod_{i,j} c(i+j-1)^{s_{ij}}.
\]

In [9], it was noticed that, due to the identity (4.7), the coefficients \(c(1), c(2), c(3)\) and \(c(5)\) determine all other coefficients recursively.

We now turn to the Borcherds-Bozec algebra \(\mathcal{L}\) associated with the Borcherds-Cartan matrix \(A\) with charge \(c\) given in (4.1) and (4.2), the *Monster Borcherds-Bozec algebra*. We have

\[
I_{re} = \{-1\}, \quad I_{im} = \{1, 2, 3, \ldots\} \quad \text{and} \quad F_0 = \{0\} \cup \{l\alpha_k \mid k, l \geq 1\}.
\]

To apply our root multiplicity formula (3.6), we take \(J = \{-1\}\). Then \(W(J) = \{1\}\) and the condition \(l(w) + d(s) > 0\) implies \(s = l\alpha_k\) with \(k, l \geq 1\). Thus we have

\[
(4.9) \quad V = V^{(J)} = \bigoplus_{l=1}^{\infty} \bigoplus_{k=1}^{\infty} c(k)V_J(-l\alpha_k),
\]

where \(V_J(-l\alpha_k)\) is the irreducible \(sl(2, \mathbb{C})\)-module with highest weight \(-l\alpha_k\). By the standard \(sl(2, \mathbb{C})\)-theory, since \(\langle h_{-1}, -l\alpha_k \rangle = l(k-1)\), we have

\[
(4.10) \quad \text{wt}(V) = \bigcup_{l=1}^{\infty} \bigcup_{k=1}^{\infty}\{-l\alpha_k - j\alpha_{-1} \mid 0 \leq j \leq l(k-1)\}
\]

and

\[
\dim V(l, k, j) = c(k) \quad \text{for all} \quad k, l \geq 1, \quad 0 \leq j \leq l(k-1).
\]

Give a (lexicographic) total ordering on \(\text{wt}(V)\) by \((l, k, j) > (l', k', j')\) if and only if (i) \(l < l'\), or (ii) \(l = l', k < k'\), or (iii) \(l = l', k = k', j < j'\).

For \(\beta \in \mathbb{Q}_+\), let

\[
(4.11) \quad \mathcal{P}(\beta) = \{s = (s_{l,k,j}) \mid \sum_{l,k,j} s_{l,k,j}(l\alpha_k + j\alpha_{-1}) = \beta\}
\]

be the set of all partitions of \(\beta\) into a sum of elements in \(\text{wt}(V)\) with respect to the ordering given above. Since \(\text{mult}(l, k, j) = c(k)\) for all \(k, l \geq 1, 0 \leq j \leq l(k-1)\), the Witt partition
function is given by

\[(4.12) \quad W(\beta) = \sum_{s \in \mathcal{P}(\beta)} \frac{(|s| - 1)!}{s!} \prod_{l,k,j} c(k)^{s(l,k,j)}.\]

Therefore, for all \(\alpha \in \Delta^+\), we have

\[(4.13) \quad \dim \mathcal{L}_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{s \in \mathcal{P}(\alpha/d)} \frac{(|s| - 1)!}{s!} \prod_{l,k,j} c(k)^{s(l,k,j)}.\]

**Example 4.1.**

Let \(\alpha = 2\alpha_2 + 4\alpha_1 + 2\alpha_{-1}\). Then

\[\dim \mathcal{L}_\alpha = W(\alpha) - \frac{1}{2} W(\alpha/2).\]

Note that the ordering on \(\text{wt}(V)\) is given as follows.

\[\alpha_1, \alpha_2, \alpha_2 + \alpha_{-1}, 2\alpha_1, 2\alpha_2, 2\alpha_2 + \alpha_{-1}, 2\alpha_2 + 2\alpha_{-1}, \ldots.\]

To compute \(W(\alpha)\) and \(W(\alpha/2)\), we consider the partitions

\[
\alpha = 2\alpha_2 + 4\alpha_1 + 2\alpha_{-1} \\
= (2\alpha_2 + 2\alpha_{-1}) + 2(2\alpha_1) = (2\alpha_2 + 2\alpha_{-1}) + 4(\alpha_1) \\
= 2(2\alpha_1) + 2(\alpha_2 + \alpha_{-1}) = (2\alpha_1) + 2(\alpha_2 + \alpha_{-1}) + 2(\alpha_1) \\
= 2(\alpha_2 + \alpha_{-1}) + 4(\alpha_1),
\]

\[
\alpha/2 = \alpha_2 + 2\alpha_1 + \alpha_{-1} \\
= (2\alpha_1) + (\alpha_2 + \alpha_{-1}) = (\alpha_2 + \alpha_{-1}) + 2(\alpha_1),
\]

which yield

\[
W(\alpha) = \frac{2!}{2!} c(2) c(1)^2 + \frac{4!}{4!} c(2) c(1)^4 + \frac{3!}{2!2!} c(2)^2 c(1)^2 \\
+ \frac{4!}{2!2!} c(1) c(2)^2 c(1)^2 + \frac{5!}{2!4!} c(2)^2 c(1)^4 = \frac{5}{2} c(1)^4 c(2)^2 + c(1)^4 c(2) + 6 c(1)^3 c(2)^2 \\
+ \frac{3}{2} c(1)^2 c(2)^2 + c(1)^2 c(2),
\]

\[
W(\alpha/2) = c(1) c(2) + \frac{2!}{2!} c(2) c(1)^2 = c(1)^2 c(2) + c(1) c(2).
\]
Hence we obtain
\[
\dim \mathcal{L}_\alpha = \frac{5}{2} c(1)^4 c(2)^2 + c(1)^4 c(2) + 6 c(1)^3 c(2)^2 \\
+ \frac{3}{2} c(1)^2 c(2)^2 + \frac{1}{2} c(1)^2 c(2) - \frac{1}{2} c(1) c(2).
\]

\[\square\]

As in the case with the Monster Lie algebra, the Monster Borcherds-Bozec algebra $\mathcal{L}$ has a ($\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$)-grading by setting $\deg \alpha_i = (1, i)$ for $i \in I$. Since
\[
\deg (-l\alpha_k - j\alpha_{-1}) = -(l + j, lk - j),
\]
we have
\[
\begin{equation}
\text{wt}(V) = \bigcup_{l=1}^{\infty} \bigcup_{k=1}^{\infty} \{ (l + j, lk - j) \mid 0 \leq j \leq l(k - 1) \}
= \{(m, n) \mid m = l + j, n = lk - j, \ k, l \geq 1, \ 0 \leq j \leq l(k - 1) \}.
\end{equation}
\]

Each solution $(l, k, j)$ of the equation
\[
m = l + j, \ n = lk - j, \ 0 \leq j \leq l(k - 1)
\]
has the contribution of $c(k) = c\left(\frac{m+n-l}{l}\right)$ to the multiplicity of $(m, n)$ in $V$. Also the condition $0 \leq j = m - l - lk - n \leq l(k - 1)$ implies $1 \leq l \leq \min(m, n)$. Therefore, we obtain
\[
\text{mult}(m, n) := d(m, n) = \sum_{l=1}^{\min(m,n)} c\left(\frac{m+n-l}{l}\right),
\]
where $c(x) = 0$ if $x$ is not an integer. Using the partition set $\mathcal{P}(m, n)$ in (4.4), the Witt partition function is equal to
\[
\begin{equation}
W(m, n) = \sum_{s \in \mathcal{P}(m, n)} \frac{|s|-1!}{s!} \prod_{i,j} d(i, j)^{s_{ij}},
\end{equation}
\]
and we obtain
\[
\begin{equation}
\dim \mathcal{L}_{(m, n)} = \sum_{d|(m, n)} \frac{1}{d} \mu(d) \sum_{s \in \mathcal{P}(m/d, n/d)} \frac{|s|-1!}{s!} \prod_{i,j} d(i, j)^{s_{ij}}.
\end{equation}
\]
Example 4.2.
As an illustration, we will compute
\[ \dim \mathcal{L}(4, 2) = W(4, 2) - \frac{1}{2} W(2, 1). \]

Using the partitions
\[ (4, 2) = (3, 1) + (1, 1) = (2, 1) + (2, 1), \]
we have
\[ W(4, 2) = d(4, 2) - d(3, 1) d(1, 1) + \frac{1}{2} d(2, 1)^2. \]

It is easy to see that
\[ d(1, 1) = c(1), \quad d(2, 1) = c(2), \quad d(3, 1) = c(3), \quad d(4, 2) = c(5) + c(2). \]

It follows that
\[ W(4, 2) = c(5) + c(3) c(1) + \frac{1}{2} c(2)^2 + c(2). \]

Since \( (2, 1) \) has only one partition, we have
\[ W(2, 1) = d(2, 1) = c(2). \]

Hence we obtain
\[ \dim \mathcal{L}(4, 2) = c(5) + c(3) c(1) + \frac{1}{2} c(2)^2 + \frac{1}{2} c(2). \]

Using the identity \((4.7)\), we get a simpler expression
\[ \dim \mathcal{L}(4, 2) = c(8) + c(2). \]

□

5. Quiver varieties

Let \( Q = (I, \Omega) \) be a locally finite quiver with loops, where \( I \) is the set of vertices and \( \Omega \) is the set of arrows. For an arrow \( h \in \Omega \), we denote by \( \text{out}(h) \) (resp. \( \text{in}(h) \)) the outgoing vertex (resp. incoming vertex) of \( h \). We often write \( h : i \to j \) when \( \text{out}(h) = i, \text{in}(h) = j \).

For \( i \in I \), let \( g_i \) be the number of loops at \( i \) and for \( i \neq j \), let \( c_{ij} \) denote the number of arrows in \( \Omega \) from \( i \) to \( j \). We denote by \( \Omega(i) \) the set of loops at \( i \). Define a matrix \( A_Q = (a_{ij})_{i,j \in I} \) by

\[ a_{ij} := \begin{cases} 2 - 2g_i & \text{if } i = j, \\ -c_{ij} - c_{ji} & \text{if } i \neq j. \end{cases} \]
Then $A_Q$ is a symmetric Borcherds-Cartan matrix. Note that $I^\text{re} = \{ i \in I \mid g_i = 0 \}$, $I^\text{im} = \{ i \in I \mid g_i \geq 1 \}$. We will denote by $g_Q$ the Borcherds-Bozec algebra associated with $A_Q$.

**Definition 5.1.**

(a) A representation of $Q$ is a pair $(V, x)$, where $V = \bigoplus_{i \in I} V_i$ is an $I$-graded vector space and $x = (x_h : V_{\text{out}(h)} \to V_{\text{in}(h)})_{h \in \Omega}$ is a family of linear maps such that

(i) $V_i = 0$ for all but finitely many $i \in I$,  

(ii) dim$V_i < \infty$ for all $i \in I$.

(b) A morphism $\phi : (V, x) \to (W, y)$ of representations consists of a collection of linear maps $\phi = (\phi_i : V_i \to W_i)_{i \in I}$ such that for all $h \in \Omega$ the following diagram is commutative.

\[
\begin{array}{ccc}
V_{\text{out}(h)} & \xrightarrow{\phi_{\text{out}(h)}} & W_{\text{out}(h)} \\
\downarrow{x_h} & & \downarrow{y_h} \\
V_{\text{in}(h)} & \xrightarrow{\phi_{\text{in}(h)}} & W_{\text{in}(h)}
\end{array}
\]

Let $V = \bigoplus_{i \in I} V_i$ be a representation of $Q$. We define its dimension vector to be

\[\dim V = \sum_{i \in I} (\dim V_i) \alpha_i \in \mathbb{Q}_+\]  

For an element $\alpha \in \mathbb{Q}_+$, fix a representation $V$ with $\dim V = \alpha$ and let

\[E(\alpha) := \bigoplus_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}).\]

Then the group $G(\alpha) := \prod_{i \in I} GL(V_i)$ acts on $E(\alpha)$ by conjugation

\[(g \cdot x)_h = g_{\text{in}(h)} x_h g_{\text{out}(h)}^{-1}\]  

for $g = (g_i)_{i \in I}, x = (x_h)_{h \in \Omega}$.

We identify the set of isomorphism classes of representations of $Q$ with dimension vector $\alpha$ with the set of $G(\alpha)$-orbits in $E(\alpha)$.

**Definition 5.2.** Let $x = (x_h)_{h \in \Omega} \in E(\alpha)$.

(a) $x$ is nilpotent if there exists an $I$-graded flag

$L = (0 = L_0 \subset L_1 \subset \cdots \subset L_r = V)$
such that
\[ x_h(L_k) \subset L_{k-1} \quad \text{for all } h \in \Omega, \ 1 \leq k \leq r. \]

(b) \( x \) is 1-nilpotent if for each \( i \in I^\text{im} \), there exists a flag
\[ L(i) = (0 = L(i)_0 \subset L(i)_1 \subset \cdots \subset L(i)_t = V_i) \]
such that
\[ x_h(L(i)_k) \subset L(i)_{k-1} \quad \text{for all } h \in \Omega(i), \ 1 \leq k \leq t. \]

Set
\[ E(\alpha)^\text{nil} := \{ x \in E(\alpha) \mid x \text{ is nilpotent} \}, \quad E(\alpha)^{1\text{-nil}} := \{ x \in E(\alpha) \mid x \text{ is 1-nilpotent} \}. \]

Then \( E(\alpha)^\text{nil} \) and \( E(\alpha)^{1\text{-nil}} \) are Zariski-closed subvarieties of \( E(\alpha) \).

Let \( q \) be a power of some prime and let \( d^\text{nil}_\alpha(q) \) (respectively, \( d^{1\text{-nil}}_\alpha(q) \)) denote the number of isomorphism classes of nilpotent (respectively, 1-nilpotent) absolutely indecomposable representations of \( Q \) over \( \mathbb{F}_q \) with dimensional vector \( \alpha \).

**Proposition 5.3.** \(^{[5]}\) Let \( Q \) be a locally finite quiver with loops and let \( \alpha \in \mathbb{Q}_+ \). Then the following statements hold.

(a) There exist unique polynomials \( A^\text{nil}_\alpha(t), A^{1\text{-nil}}_\alpha(t) \in \mathbb{Z}[t] \) such that

(i) \( A^\text{nil}_\alpha(q) = d^\text{nil}_\alpha(q), \ A^{1\text{-nil}}_\alpha(q) = d^{1\text{-nil}}_\alpha(q) \) for all \( q \).

(ii) \( A^\text{nil}_\alpha(1) = A^{1\text{-nil}}_\alpha(1) \).

(b) \( \dim(g_Q)_\alpha = A^{1\text{-nil}}_\alpha(0) \).

We will show that the set of positive roots of \( g_Q \) are in 1-1 correspondence with the set of dimension vectors of 1-nilpotent indecomposable representations of \( Q \). For this purpose, we briefly recall some of important results on mixed Hodge polynomials and \( E \)-polynomials (see \(^{[6]}\) for more details).

**Definition 5.4.** Let \( X \) be a complex algebraic variety.

(a) The \( E \)-polynomial of \( X \) is defined to be
\[ E(X; x, y) := \sum_{p, q, j} (-1)^j h^p_{\mathbb{C}} q^j(X) x^p y^q, \]

where \( h^p_{\mathbb{C}} q^j(X) \) are compactly supported mixed Hodge numbers.
(b) A spreading-out of $X$ is a separated scheme $\tilde{X}$ over a finitely generated $\mathbb{Z}$-algebra $R$ together with an embedding $\varphi : R \hookrightarrow \mathbb{C}$ such that $X \cong \tilde{X}_\varphi$, the extension of scalars of $\tilde{X}$.

(c) We say that $X$ has polynomial count if there exists a polynomial $P_X(t) \in \mathbb{Z}[t]$ and a spreading-out $\tilde{X}$ such that for every homomorphism $\phi : R \to \mathbb{F}_q$, we have

$$P_X(q) = \# \{ \text{F}_q\text{-points of } \tilde{X}_\phi \}.$$ 

**Theorem 5.5.** [6] Suppose that a complex variety $X$ has polynomial count with the counting polynomial $P_X(t) \in \mathbb{Z}[t]$. Then we have

$$E(X; x, y) = P_X(xy).$$

We apply Theorem 5.5 to our setting. Given an element $\alpha \in Q_+$, let $X(\alpha)$ be the variety of isomorphism classes of 1-nilpotent indecomposable representations of $Q$ over $\mathbb{C}$ with dimension vector $\alpha$. Then Proposition 5.3 implies $X(\alpha)$ has polynomial count with the counting polynomial $A^\text{1-nil}_\alpha(t)$. By Theorem 5.5, one can see that $X(\alpha) \neq \emptyset$ if and only if $A^\text{1-nil}_\alpha(0) \neq 0$. Since $\text{dim}(g_Q)_ \alpha = A^\text{1-nil}_\alpha(0)$, we conclude:

**Proposition 5.6.** Let $Q$ be a locally finite quiver with loops and let $g_Q$ be the Borcherds-Bozec algebra associated with $Q$. Then there is a 1-1 correspondence

$$\Delta_+(g_Q) = \{ \text{positive roots of } g_Q \} \longleftrightarrow \{ \text{dimension vectors of 1-nilpotent indecomposable representations of } Q \text{ over } \mathbb{C} \}.$$

**Remark.** The above argument is due to O. Schiffmann.

### 6. Constructible Functions

Let $X$ be a complex algebraic variety. A complex-valued function $f : X \to \mathbb{C}$ is said to be constructible if (i) $\text{Im } f$ is finite, (ii) $f^{-1}(c)$ is a constructible set for all $c \in \mathbb{C}$. We define the convolution integral of $f$ over $X$ to be

$$\int_X f d\chi := \sum_{c \in \mathbb{C}} c \chi(f^{-1}(c)),$$

where $\chi$ denotes the Euler characteristic.

Let $Q = (I, \Omega)$ be a locally finite quiver with loops. For each arrow $h \in \Omega$, we define the opposite arrow $\overline{h}$ of $h$ by setting $\text{out}(\overline{h}) = \text{in}(h)$, $\text{in}(\overline{h}) = \text{out}(h)$. Let $\overline{\Omega} = \{ \overline{h} \mid h \in \Omega \}$
and $H = \Omega \cup \overline{\Omega}$. The quiver $\overline{Q} = (I, H)$ thus obtained is called the \textit{double quiver} of $Q$. Set $\varepsilon(h) := 1$ if $h \in \Omega$ and $\varepsilon(h) := -1$ if $h \in \overline{\Omega}$.

For $\alpha \in Q_{+}$, fix an $I$-graded vector space $V = \bigoplus_{i \in I} V_{i}$ with dimension vector $\alpha$. Set

\[ E(\alpha) := \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}). \]

We have a symplectic form $\omega_{\alpha}$ on $E(\alpha)$ given by

\[ \omega_{\alpha}(x, y) = \sum_{h \in H} \text{tr}(\varepsilon(h)x_{h}y_{h}). \]

Note that $\omega_{\alpha}$ is preserved under the $G(\alpha)$-action. We define the \textit{moment map} by

\[ \mu(\alpha) : E(\alpha) \rightarrow g(\alpha) = \bigoplus_{i \in I} \text{End}(V_{i}) \]

\[ x \mapsto \sum_{h \in H} \varepsilon(h)x_{h}x_{h}. \]

Here, we identify $g^{*} = g$ via trace pairing.

**Definition 6.1.**

(a) $x = (x_{h}, x_{\overline{h}})_{h \in \Omega}$ is \textit{nilpotent} if there is an $I$-graded flag

$L = (0 = L_{0} \subset L_{1} \subset \cdots \subset L_{r} = V)$

such that

\[ x_{h}(L_{k}) \subset L_{k-1}, \ x_{\overline{h}}(L_{k}) \subset L_{k-1} \quad \text{for all } h \in \Omega, \ 1 \leq k \leq r. \]

(b) $x = (x_{h}, x_{\overline{h}})_{h \in \Omega}$ is \textit{semi-nilpotent} if there is an $I$-graded flag

$L = (0 = L_{0} \subset L_{1} \subset \cdots \subset L_{r} = V)$

such that

\[ x_{h}(L_{k}) \subset L_{k-1}, \ x_{\overline{h}}(L_{k}) \subset L_{k} \quad \text{for all } h \in \Omega, \ 1 \leq k \leq r. \]

(c) $x = (x_{h}, x_{\overline{h}})_{h \in \Omega}$ is \textit{strongly semi-nilpotent} if there is an $I$-graded flag

$L = (0 = L_{0} \subset L_{1} \subset \cdots \subset L_{r} = V)$

such that

(i) $L_{k}/L_{k-1}$ is concentrated on one vertex for each $1 \leq k \leq r$,

(ii) $x_{h}(L_{k}) \subset L_{k-1}, \ x_{\overline{h}}(L_{k}) \subset L_{k}$ for all $h \in \Omega, \ 1 \leq k \leq r$. 

Set
\[ \Lambda(\alpha) := \{ x \in \mu^{-1}(0) \mid x \text{ is strongly semi-nilpotent} \}. \]

**Proposition 6.2.** \([4, 5]\) \( \Lambda(\alpha) \) is a Lagrangian subvariety of \( \mathcal{F}(\alpha) \).

For each \( \alpha \in \mathbb{Q}_+ \), let \( \mathcal{F}(\alpha) \) be the space of constructible functions on \( \Lambda(\alpha) \) which are constant on \( G(\alpha) \)-orbits and set \( \mathcal{F} := \bigoplus_{\alpha \in \mathbb{Q}_+} \mathcal{F}(\alpha) \).

Let \( x \in \Lambda(\alpha) \) and consider the constructible functions \( f \in \mathcal{F}(\beta), g \in \mathcal{F}(\gamma) \) with \( \alpha = \beta + \gamma \). We define the convolution product of \( f \) and \( g \) by
\[(f \ast g)(x) := \int_{W \subset V} f(x|W)g(x|V/W)d\chi,\]
where the integral is taken over the projective variety of all submodules \( W \) of \( V \) with dimension vector \( \beta \) (cf. \([16, 17]\)). Then \( \mathcal{F} \) becomes a \( \mathbb{Q}_+ \)-graded associative algebra under convolution product.

For \( (i,l) \in I^\infty \), the irreducible components of \( \Lambda(l\alpha_i) \) are parametrized by the compositions (resp. partitions) of \( l \) when \( i \in I^{im} \setminus I^{iso} \) (resp. \( i \in I^{iso} \)) \([3, 4]\). If \( i \in I^{re} \), \( \Lambda(\alpha_i) \) is a point. Take the irreducible component \( Z_{i,(l)} \) of \( \Lambda(l\alpha_i) \) corresponding to the trivial composition (or partition) of \( l \). Note that \( Z_{i,(l)} \) consists of those \( x \) such that \( x_{h} = 0 \) for all \( h \in \Omega(i) \). In this case, there is no restriction on the linear transformations \( x_{h} \) with \( h \in \Omega(i) \). Hence any constructible function on \( Z_{i,(l)} \) can be considered as a constructible function on the representation variety of \( Q \) with dimension vector \( l\alpha_i \) such that \( x_{h} = 0 \) for all \( h \in \Omega(i) \). This fact will be used in the proof of Proposition 7.15 without any further explanation.

Let \( \theta_{i,l} \) be the characteristic function of \( Z_{i,(l)} \) and let \( \mathcal{C} \) be the subalgebra of \( \mathcal{F} \) generated by \( \theta_{i,l} \)'s. Using the theory of perverse sheaves and crystal bases for quantum Borcherds-Bozec algebras, Bozec proved:

**Proposition 6.3.** \([3, 4]\) There is an algebra isomorphism
\[ U(\mathfrak{g}_Q^+) \xrightarrow{\sim} \mathcal{C} \]
given by \( e_{it} \mapsto \theta_{i,t} \) for \( (i,l) \in I^\infty \).

### 7. The Schofield construction

Let \( E = \{ S_{i,t} \mid (i,l) \in I^\infty \} \) be a set of variables and let \( \mathcal{E} = C(E) \) be the free associative algebra on \( E \) over \( C \). We will often write \( S_i \) for \( S_{i,1} \) (\( i \in I^{re} \)).
Definition 7.1. Let $Q = (I, \Omega)$ be a locally finite quiver with loops, $(M = \bigoplus_{i \in I} M_i, x = (x_h)_{h \in \Omega})$ be a representation of $Q$ and $w = S_{i_1,l_1} \cdots S_{i_r,l_r}$ be a word in $E$. An $I$-graded filtration

$$L = (0 = L_0 \subset L_1 \subset \cdots \subset L_r = M)$$

is called a 1-nilpotent flag in $(M, x)$ of type $w$ if

1. $\dim L_k/L_{k-1} = l_k \alpha_{i_k}$ for all $1 \leq k \leq r$,
2. $x_h = 0$ on $L_k/L_{k-1}$ for all $h \in \Omega(i_k)$, $1 \leq k \leq r$.

For a word $w = S_{i_1,l_1} \cdots S_{i_r,l_r}$ in $E$ and a representation $M$ of $Q$, we denote by $X_M(w)$ the projective variety consisting of 1-nilpotent flags in $M$ of type $w$. We define

$$\langle w, M \rangle := \chi(X_M(w)).$$

More generally, for $u = \sum_w a_w w \in \mathcal{E}$, we define

$$\langle u, M \rangle := \sum_w a_w \langle w, M \rangle = \sum_w a_w \chi(X_M(w)).$$

Set

$$\mathcal{I} := \{u \in \mathcal{E} \mid \langle u, M \rangle = 0 \text{ for all } M\}.$$

We will prove:

(a) $\mathcal{I}$ is an ideal of $\mathcal{E}$.

(b) $\mathcal{I}$ is a co-ideal of $\mathcal{E}$ under the co-multiplication given by

$$\Delta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}, \quad S_{i,l} \mapsto S_{i,l} \otimes 1 + 1 \otimes S_{i,l} \quad \text{for } (i,l) \in I^\infty.$$ 

Then the Schofield algebra $\mathcal{R} := \mathcal{E}/\mathcal{I}$ would be a bi-algebra. To prove our claims, we recall some basic facts on the Euler characteristic of algebraic varieties (see, for example, [19, 20, 21]).

Lemma 7.2. Let $X$ be a complex algebraic variety.

(a) If $X = \bigsqcup \alpha X_\alpha$ is a locally finite partition by constructible subsets, then

$$\chi(X) = \sum_\alpha \chi(X_\alpha).$$

(b) If $\pi : E \to X$ is a fibration with fiber $F$, then

$$\chi(E) = \chi(F)\chi(X).$$
(c) Suppose $C^*$ acts on $X$. If $U$ is an open subset on which $C^*$ acts almost freely; i.e., $\text{Stab}_{C^*}(x) = \{1\}$ or $C^*$ for all $x \in U$, then
\[ \chi(X) = \chi(X \setminus U). \]

(d) If $C^*$ acts on $X$, then
\[ \chi(X) = \chi(X^{C^*}), \]
where
\[ X^{C^*} = \{ x \in X \mid \lambda \cdot x = x \text{ for all } \lambda \in C^* \}. \]

**Proposition 7.3.** $\mathcal{I}$ is a two-sided ideal of $\mathcal{E}$.

**Proof.** Let $u = \sum_w a_w w \in \mathcal{I}$. We will show that $S_{i,l} u \in \mathcal{I}, u S_{i,l} \in \mathcal{I}$ for all $(i, l) \in I^\infty$.

Let $X^+_M(S_{i,l})$ be the variety of all submodules $M'$ of $M$ with dimension vector $l \alpha_i$ such that $x_h(M') = 0$ for all $h \in \Omega(i)$. For a word $w$ in $E$, we define a map
\[ \phi_w : X_M(S_{i,l} w) \to X^+_M(S_{i,l}) \]
by
\[ L = (0 = L_0 \subset L_1 \subset \cdots \subset L_r = M) \mapsto L_1. \]

Note that for all $M' \in X^+_M(S_{i,l})$, we have
\[ \phi_w^{-1}(M') \cong X_{M/M'}(w), \]
which is independent of the choice of $M'$. Thus we can find disjoint constructible subsets $C_j$ of $X^+_M(S_{i,l})$ such that
\begin{enumerate}[label=(i),itemsep=0.5em]
  
  \item $X^+_M(S_{i,l}) = \bigsqcup_j C_j$,
  
  \item for all $M'_j \in C_j$ and for all $w$ with $a_w \neq 0$, $\chi(\phi^{-1}_w(M'_j))$ is independent of the choice of $M'_j$.
\end{enumerate}

Hence for any representation $M$ of $Q$, we have
\[ \langle S_{i,l} u, M \rangle = \sum_w a_w \langle S_{i,l} w, M \rangle = \sum_w a_w \chi(X_M(S_{i,l} w)) \]
\[ = \sum_w a_w \sum_j \chi(C_j) \chi(\phi^{-1}_w(M'_j)) = \sum_j \chi(C_j) \sum_w a_w \chi(X_{M/M'_j}(w)) \]
\[ = \sum_j \chi(C_j) \sum_w a_w \langle w, M/M'_j \rangle = \sum_j \chi(C_j) \langle u, M/M'_j \rangle = 0, \]
which implies $S_{i,l} u \in \mathcal{I}$. 

Similarly, one can show that \( u S_{i,l} \in \mathcal{I} \) for all \((i,l) \in I^\infty\) using the map
\[
\psi_w : X_M(w S_{i,l}) \to X_M^{-}(S_{i,l}),
\]
where \( X_M^{-}(S_{i,l}) \) is the variety of submodules \( M'' \) of \( M \) such that (i) \( M/M'' \) has dimension vector \( l \alpha_i \), (ii) \( x_h = 0 \) on \( M/M'' \) for all \( h \in \Omega(i) \).

Let \( \mathcal{R} = \mathcal{E}/\mathcal{I} \) and set \( \deg S_{i,l} := l \alpha_i \). Since \( X_M(w) = \emptyset \) unless \( \dim M = \deg w \) and \( M \) is 1-nilpotent, \( \mathcal{I} \) is a \( \mathbb{Q}_+ \)-graded ideal, which implies \( \mathcal{R} \) is a \( \mathbb{Q}_+ \)-graded algebra.

Let \( E = \{ S_{i,l} \mid (i,l) \in I^\infty \} \), \( E' = \{ S'_{i,l} \mid (i,l) \in I^\infty \} \) and set \( \widetilde{E} = E \cup E' \). For a word \( w \) in \( \widetilde{E} \), we denote by \( w_1 \) (resp. \( w_2 \)) the word in \( E \) (resp. \( E' \)) obtained from \( w \). We will write \( [w] = w_1 w_2 \), the normally ordered word of \( w \).

**Lemma 7.4.** Let \( M \) and \( N \) be representations of \( Q \) and let \( w \) be a word in \( \widetilde{E} \). Consider \((M,N)\) as a representation of \( Q \). Then we have
\[
X_{(M,N)}(w) \cong X_{(M,N)}(w_1 w_2) \cong X_M(w_1) \times X_N(w_2).
\]
In particular,
\[
\langle w, (M,N) \rangle = \langle w_1, M \rangle \langle w_2, N \rangle.
\]

**Proof.** Note that any 1-nilpotent flag in \((M,N)\) of type \( w \) is a mixture of a 1-nilpotent flag in \( M \) of type \( w_1 \) and a 1-nilpotent flag in \( N \) of type \( w_2 \). Hence the set of 1-nilpotent flags in \((M,N)\) of type \( w \) is in 1-1 correspondence with the set of the products of a 1-nilpotent flag in \( M \) of type \( w_1 \) and a 1-nilpotent flag in \( N \) of type \( w_2 \), which yields
\[
X_{(M,N)}(w) \cong X_M(w_1) \times X_N(w_2) \cong X_{(M,N)}(w_1 w_2).
\]
It follows that
\[
\langle w, (M,N) \rangle = \chi(X_{(M,N)}(w)) = \chi(X_M(w_1) \times X_N(w_2)) = \chi(X_M(w_1)) \chi(X_N(w_2)) = \langle w_1, M \rangle \langle w_2, N \rangle,
\]
which proves our claim.

Let \( \widetilde{\mathcal{E}} = Q\langle \widetilde{E} \rangle = Q\langle E \cup E' \rangle \), \( \widetilde{\mathcal{I}} = \{ u \in \widetilde{\mathcal{E}} \mid \langle u, (M,N) \rangle = 0 \) for all \( M,N \} \) and set \( \widetilde{\mathcal{R}} = \widetilde{\mathcal{E}}/\widetilde{\mathcal{I}} \). We extend \([ \ ]\) by linearity to obtain a linear map \([ \ ] : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{R}} \). We first prove:

**Proposition 7.5.** There is an algebra isomorphism \( \widetilde{\mathcal{R}} \cong \mathcal{R} \otimes \mathcal{R} \).

**Proof.** For \( u = \sum_a a_w w \in \widetilde{E} \), we have \([u] = \sum_a a_w w_1 w_2 \). Then by Lemma 7.4, we see that \( u = [u] \) in \( \widetilde{\mathcal{R}} \). Therefore there exists a surjective algebra homomorphism
\[
\phi : \mathcal{R} \otimes \mathcal{R} \to \widetilde{\mathcal{R}}.
\]
We will prove $\phi$ is injective. For any $\beta \in \mathbb{Q}_+$, define a matrix $K_\beta$ by

$$K_\beta = (\langle w, M \rangle)_{\deg w = \beta}.$$  

By definition, $\dim \mathcal{R}_\beta = \text{rank} K_\beta$. Moreover, by Lemma 7.4 again, we have

$$K_\alpha \otimes K_\beta = (\langle u, M \rangle)_{\alpha} \otimes (\langle w, N \rangle)_{\beta} = (\langle u, M \rangle \langle w, N \rangle)_{(\alpha, \beta)} = (\langle uw, (M, N) \rangle)_{(\alpha, \beta)} = \tilde{K}_{(\alpha, \beta)}.$$  

Hence

$$\dim \tilde{R}_{(\alpha, \beta)} = \text{rank} \tilde{K}_{(\alpha, \beta)} = \text{rank}(K_\alpha \otimes K_\beta) = (\dim R_{\alpha})(\dim \mathcal{R}_\beta) = \dim(\tilde{R} \otimes \mathcal{R})_{(\alpha, \beta)},$$

which implies $\tilde{R} \cong \mathcal{R} \otimes \mathcal{R}$. □

Define the algebra homomorphisms $\Delta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ and $\delta : \mathcal{E} \to \tilde{\mathcal{E}}$ by

$$\Delta : S_{i,l} \mapsto S_{i,l} \otimes 1 + 1 \otimes S_{i,l},$$

$$\delta : S_{i,l} \mapsto S_{i,l} + S'_{i,l} \quad \text{for} \quad (i, l) \in I^\infty.$$  

Let $w = S_{i_1,l_1} \cdots S_{i_r,l_r}$ be a word in $E$. For a subset $S \subset \{1, \ldots, r\}$, let $w_S$ be the word in $\tilde{E}$ obtained from $w$ by replacing $S_{i_s,l_s}$ by $S'_{i_s,l_s}$ for $s \in S$. One can easily see that

$$\delta(w) = \sum_{S \subset \{1, \ldots, r\}} w_S.$$  

Lemma 7.6. For any word $w$ in $E$, we have

$$\langle w, M \oplus N \rangle = \langle \delta(w), (M, N) \rangle \quad \text{for all} \quad M, N.$$  

Proof. Let $w = w_1 \cdots w_r = S_{i_1,l_1} \cdots S_{i_r,l_r}$ and $S \subset \{1, \ldots, r\}$. Then $w_S$ yields a 1-nilpotent flag in $(M, N)$ of type $w_S$. Set

$$X_{(M, N)}(w_S) = \{\text{1-nilpotent flags in } (M, N) \text{ of type } w_S\}.$$  

Note that we have

$$\langle \delta(w), (M, N) \rangle = \sum_{S \subset \{1, \ldots, r\}} \langle w_S, (M, N) \rangle = \sum_{S \subset \{1, \ldots, r\}} \chi(X_{(M, N)}(w_S)).$$

(7.4) Each point of $X_{(M, N)}(w_S)$ determines a sequence of representations $L_1, \ldots, L_r$ such that $L_j \subset M$ if $j \notin S$ and $L_j \subset N$ if $j \in S$. Let $M_j$ (resp. $N_j$) be the largest submodule of $M$
(resp. \( N \)) in the list \( L_1, \ldots, L_j \) (\( 1 \leq j \leq r \)) or 0 if there is no such submodule in the list. By setting \( K_j := M_j \oplus N_j \), we obtain a 1-nilpotent flag in \( M \oplus N \) of type \( w \)

\[
K = (0 = K_0 \subset K_1 \subset \cdots \subset K_r = M \oplus N).
\]

Thus for all \( S \subset \{1, \ldots, r\} \), we see that \( X_{(M,N)}(w_S) \) is a subvariety of \( X_{M\oplus N}(w) \).

Clearly, if \( S \neq T \), \( X_{(M,N)}(w_S) \cap X_{(M,N)}(w_T) = \emptyset \). Therefore we have

\[
\bigcup_{S \subset \{1, \ldots, r\}} X_{(M,N)}(w_S) \subset X_{M\oplus N}(w),
\]

\[
\chi \left( \bigcup_{S \subset \{1, \ldots, r\}} X_{(M,N)}(w_S) \right) = \sum_{S \subset \{1, \ldots, r\}} \chi(X_{(M,N)}(w_S)).
\]

Conversely, given a 1-nilpotent flag

\[
K = (0 = K_0 \subset K_1 \subset \cdots \subset K_r = M \oplus N)
\]

of type \( w = S_{i_1}j_1 \cdots S_{i_r}j_r \) such that \( K_j = M_j \oplus N_j \) with \( M_j \subset M, N_j \subset N \), we define a subset \( S \subset \{1, \ldots, r\} \) by setting \( j \in S \) if and only if \( N_{j-1} \not\subset N_j \). We also define a sequence of submodules \( L_1, \ldots, L_r \) by setting \( L_j = N_j \) for \( j \in S \), \( L_j = M_j \) for \( j \notin S \). Thus we get a 1-nilpotent flag in \( (M,N) \) of type \( w_S \). Therefore we obtain a bijection

\[
(7.5) \quad \bigcup_{S \subset \{1, \ldots, r\}} X_{(M,N)}(w_S) \sim \left\{ \begin{array}{l}
\text{1-nilpotent flags in } M \oplus N \text{ of type } w \\
\text{such that } K_j = (K_j \cap M) \oplus (K_j \cap N)
\end{array} \right\}.
\]

Note that \( C^* \) acts on \( M \oplus N \) by

\[
\lambda \cdot (m,n) = (\lambda m, n) \quad \text{for } \lambda \in C^*, m \in M, n \in N,
\]

which induces a \( C^* \)-action on the Grassmannian variety \( Gr_d(M \oplus N) \) of \( d \)-dimensional subspaces of \( M \oplus N \). One can easily show that this \( C^* \)-action on \( Gr_d(M \oplus N) \) is almost free. Moreover, \( V \subset M \oplus N \) is a fixed point under the \( C^* \)-action if and only if \( V = (V \cap M) \oplus (V \cap N) \). By (7.4) and (7.5), we see that \( C^* \) acts on \( X_{M\oplus N} \) with

\[
(X_{M\oplus N}(w))^{C^*} = \bigcup_{S \subset \{1, \ldots, r\}} X_{(M,N)}(w_S).
\]
Therefore, by Lemma 7.2, we conclude
\[
\langle w, M \oplus N \rangle = \chi(X_{M \oplus N}(w)) = \chi((X_{M \oplus N}(w))^{C^*})
\]
\[
= \chi \left( \bigcup_{S \subset \{1, \ldots, r\}} X_{(M,N)}(w_S) \right) = \sum_{S \subset \{1, \ldots, r\}} \chi(X_{(M,N)}(w_S))
\]
\[
= \sum_{S \subset \{1, \ldots, r\}} \langle w_S, (M, N) \rangle = \langle \delta(w), (M, N) \rangle
\]
as desired. □

**Lemma 7.7.** Let \( a \in E \) and write \( \Delta(a) = \sum a_{(1)} \otimes a_{(2)} \). For a word \( w = S_{i_1, l_1} \cdots S_{i_r, l_r} \), set \( w' = S'_{i_1, l_1} \cdots S'_{i_r, l_r} \). Then the following statements hold.

(a) \( [\delta(a)] = \sum a_{(1)} a'_{(2)} \).

(b) For any representations \( M, N \) of \( Q \), we have
\[
\langle a, M \oplus N \rangle = \sum \langle a_{(1)}, M \rangle \langle a_{(2)}, N \rangle.
\]

**Proof.** We will prove (a) by induction on the length of \( a \). If \( a = S_{i, l} \), our assertion is clear. Note that
\[
\Delta(aS_{i, l}) = \Delta(a)\Delta(S_{i, l}) = (\sum a_{(1)} \otimes a_{(2)})(S_{i, l} \otimes 1 + 1 \otimes S_{i, l})
\]
\[
= \sum a_{(1)} a_{(2)} S_{i, l} + \sum a_{(1)} \otimes a_{(2)} S_{i, l}.
\]

On the other hand,
\[
[\delta(aS_{i, l})] = [\delta(a)\delta(S_{i, l})] = [(\sum a_{(1)} a'_{(2)})(S_{i, l} + S'_{i, l})]
\]
\[
= \sum a_{(1)} a'_{(2)} S_{i, l} + \sum a_{(1)} a'_{(2)} S'_{i, l} = \sum a_{(1)} a'_{(2)} S_{i, l} + \sum a_{(1)} a'_{(2)} S'_{i, l}
\]
as desired.

Thus by Lemma 7.4 and Lemma 7.6, we have
\[
\langle a, M \oplus N \rangle = \langle \delta(a), (M, N) \rangle = \langle \delta(a)_1 \delta(a)_2, (M, N) \rangle
\]
\[
= \sum \langle a_{(1)} a'_{(2)}, (M, N) \rangle = \sum \langle a_{(1)}, M \rangle \langle a_{(2)}, N \rangle,
\]
which proves (b). □

**Proposition 7.8.** \( \mathcal{I} \) is a co-ideal of \( E \). That is,
\[
\Delta(\mathcal{I}) \subset E \otimes \mathcal{I} + \mathcal{I} \otimes E.
\]
Proof. If \( a \in \mathcal{I} \), then \( \langle \delta(a), (M, N) \rangle = \langle a, M \oplus N \rangle = 0 \) for all \( M, N \), which implies \( \delta(a) \in \mathcal{I} \). Thanks to the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \overset{[\delta]}{\longrightarrow} & \mathcal{E} \\
\downarrow{\Delta} & & \downarrow{i} \\
\mathcal{E} \otimes \mathcal{E} & \longrightarrow & \mathcal{R} \otimes \mathcal{R}
\end{array}
\]

we conclude

\[ \Delta(a) \in \mathcal{E} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{E}. \]

We say that an element \( a \in \mathcal{R} \) is primitive if \( \Delta(a) = a \otimes 1 + 1 \otimes a \).

**Theorem 7.9.** Let \( \mathcal{L} = \{a \in \mathcal{R} \mid \Delta(a) = a \otimes 1 + 1 \otimes a\} \). Then the following statements hold.

(a) \( \mathcal{L} \) is a Lie bi-algebra.

(b) \( \mathcal{R} \cong U(\mathcal{L}) \).

**Proof.** Using Proposition 7.8, it is straightforward to verify (a). Since \( \mathcal{R} \) is a bi-algebra generated by its primitive elements, the statement (b) follows from [23]. \( \square \)

**Proposition 7.10.** Every primitive element vanishes on decomposable modules.

**Proof.** Note that for any representation \( M \) of \( Q \),

\[ \langle 1, M \rangle = \chi(X_M(1)) = \chi(\emptyset) = 0. \]

Hence for any primitive element \( a \in \mathcal{R} \) and representations \( M, N \) of \( Q \), we have

\[ \langle a, M \oplus N \rangle = \langle a, M \rangle \langle 1, N \rangle + \langle 1, M \rangle \langle a, N \rangle = 0, \]

which is our assertion. \( \square \)

**Corollary 7.11.** If there is no 1-nilpotent indecomposable representation of \( Q \) with dimension vector \( \alpha \), then \( \mathcal{L}_\alpha = 0 \).

**Proof.** Recall that for \( a \in \mathcal{L}_\alpha \), \( \langle a, M \rangle = 0 \) unless \( \dim M = \alpha \) and \( M \) is 1-nilpotent. Hence by Proposition 7.10, \( \langle a, M \rangle = 0 \) for all \( M \), which implies \( a = 0 \). \( \square \)
Proposition 7.12. For $i \in I^\infty$, $(j,l) \in I^\infty$ with $i \neq (j,l)$, we have

$$(\text{ad}S_i)^{1-l_{ij}}(S_{j,l}) = 0$$

in $\mathcal{L}$, where $a_{ij} = -c_{ij} - c_{ji}$ as in (5.1).

Proof. Let $u = (\text{ad}S_i)^{1-l_{ij}}(S_{j,l}) \in \mathcal{L}^{(1-l_{ij})\alpha_i + l\alpha_j}$. Since

$$\langle h_i, -la_{ij}\alpha_i + l\alpha_j \rangle = -2la_{ij} + la_{ij} = -la_{ij} \geq 0,$$

we see that

$$-la_{ij}\alpha_i + l\alpha_j + \alpha_i = (1 - la_{ij})\alpha_i + l\alpha_j \notin \Delta_+(\mathfrak{g}_Q).$$

By Proposition 5.6, there is no 1-nilpotent indecomposable representation of $Q$ with dimension vector $(1 - la_{ij})\alpha_i + l\alpha_j$. Hence by Corollary 7.11, $u = 0$. \hfill \Box

Proposition 7.13. If $a_{ij} = 0$ for some $i, j \in I$, then

$$[S_{i,k}, S_{j,l}] = 0$$

for all $k, l \geq 1$.

Proof. Let $u = [S_{i,k}, S_{j,l}] \in \mathcal{L}_{k\alpha_i + l\alpha_j}$. Note that $(\mathfrak{g}_Q)_{k\alpha_i + l\alpha_j}$ is spanned by the brackets of the form

$$[e_{i,p_1}, e_{i,p_2}, \cdots, e_{i,p_r}, e_{j,q_1}, e_{j,q_2}, \cdots, e_{j,q_s}] \quad (r, s \geq 1, \ p_m, q_n \geq 1).$$

But if $a_{ij} = 0$, then $[e_{i,p}, e_{j,q}] = 0$ for all $p, q \geq 1$, and hence all the above brackets are 0, from which we conclude $(\mathfrak{g}_Q)_{k\alpha_i + l\alpha_j} = 0$. Thus by Proposition 5.6, $k\alpha_i + l\alpha_j$ is not a dimension vector of a 1-nilpotent indecomposable representation of $Q$ and Corollary 7.11 implies $u = 0$. \hfill \Box

Combining Proposition 7.12 and Proposition 7.13, we obtain:

Theorem 7.14. (a) There is a surjective algebra homomorphism

$$\Phi : U^+(\mathfrak{g}_Q) \longrightarrow \mathcal{R}$$

given by $e_{i,l} \mapsto S_{i,l}$ for $(i, l) \in I^\infty$.

(b) There is a surjective Lie algebra homomorphism

$$\Phi_0 : \mathfrak{g}_Q^+ \longrightarrow \mathcal{L}$$

given by $e_{i,l} \mapsto S_{i,l}$ for $(i, l) \in I^\infty$.

We will prove $\Phi$ and $\Phi_0$ are isomorphisms. To this end, it suffices to prove the following proposition.

Proposition 7.15. Let $\mathcal{C}$ be the algebra generated by the characteristic functions $\theta_{i,l}$ for $(i, l) \in I^\infty$. Then there exists an algebra isomorphism

$$\mathcal{R} \xrightarrow{\sim} \mathcal{C}$$

given by $S_{i,l} \mapsto \theta_{i,l}$ for $(i, l) \in I^\infty$. 

Proof. Given a word $w = S_{i_1,l_1} \cdots S_{i_r,l_r}$ in $E$, we denote $\theta_w := \theta_{i_1,l_1} \ast \cdots \ast \theta_{i_r,l_r}$. Then for any representation $(M, x)$ of $Q$ with dimension vector $\alpha = l_1 \alpha_{i_1} + \cdots + l_r \alpha_{i_r}$, we have
\[
\theta_w(x) = \theta_{i_1,l_1} \ast \cdots \ast \theta_{i_r,l_r}(x) = \int_{L = (0 \subset L_1 \subset \cdots \subset L_r = M)} \theta_{i_1,l_1}(x|L_1) \theta_{i_2,l_2}(x|L_2/L_1) \cdots \theta_{i_r,l_r}(x|L_r/L_{r-1}) d\chi
\]
\[
= \chi \left( \left\{ L = (0 \subset L_1 \subset \cdots \subset L_r = M) \mid \begin{array}{l}
\text{(i) } \dim L_k/L_{k-1} = l_k \alpha_{i_k}, \\
\text{(ii) } x_h = 0 \text{ on } L_k/L_{k-1} \end{array} \text{ for all } h \in \Omega(i_k) \right\} \right)
\]
\[
= \chi(X_M(w)) = \langle w, M \rangle.
\]

Therefore, whenever there is a relation $\sum_w a_w w = 0$ in $\mathcal{R}$, we have the corresponding relation $\sum_w a_w \theta_w = 0$ in $\mathcal{G}$, and vice versa, which proves our claim. $\square$

Now we obtain the Schofield construction of Borcherds-Bozec algebras.

**Theorem 7.16.** The homomorphisms $\Phi$ and $\Phi_0$ are isomorphisms.

**Proof.** Our assertion follows from Proposition 6.3, Theorem 7.14 and Proposition 7.15. $\square$

**References**

[1] R. Borcherds, *Generalized Kac-Moody algebras*, J. Algebra 115 (1988), 501–512.

[2] R. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. 109 (1992), 405–444.

[3] T. Bozec, *Quivers with loops and perverse sheaves*, Math. Ann. 362 (2015), 773–797.

[4] T. Bozec, *Quivers with loops and generalized crystals*, Compositio Math. 152 (2016), 1999–2040.

[5] T. Bozec, O. Schiffmann, E. Vasserot, *On the number of points of nilpotent quiver varieties over finite fields*, arXiv:1701.01797.

[6] T. Hausel, F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties (with an appendix by N. Katz)*, Invent. Math. 174 (2008), 555–624.

[7] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Grad. Stud. Math. 42, Amer. Math. Soc., Providence, RI, 2002.

[8] K. Jeong, S.-J. Kang, M. Kashiwara, *Crystal bases for quantum generalized Kac-Moody algebras*, Proc. London Math. Soc. (3) 90 (2005), 395–438.

[9] E. Jurisich, J. Lepowsky, R. L. Wilson, *Realization of the Monster Lie algebras*, Selecta Math. New Ser. 1 (1995), 129–161.

[10] V. G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Math. USSR. Izv. 2 (1968), 1271–1311.

[11] V. G. Kac, *Infinite dimensional Lie algebras and the Dedekind eta-function*, Funct. Anal. Appl. 8 (1974), 68–70.
[12] V. G. Kac, *Infinite Dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.
[13] S.-J. Kang, *Root multiplicities of Kac-Moody algebras*, Duke Math. J. **74** (1994), 635–666.
[14] S.-J. Kang, *Generalized Kac-Moody algebras and the modular function j*, Math. Ann. **74** (1994), 373–384.
[15] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
[16] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
[17] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, Boston, 1993.
[18] R. V. Moody, *A new class of Lie algebras*, J. Algebra **10** (1968), 211–230.
[19] J. Milnor, J. Stasheff, *Characteristic Classes*, Princeton University Press, 1974.
[20] A. Schofield, *Quivers and Kac-Moody Lie algebras*, unpublished.
[21] E. Spanier, *Algebraic Topology*, Springer, 1982.
[22] J. P. Serre, *Complex Semisimple Lie Algebras*, Springer-Verlag, Berlin Heidelberg, 2001.
[23] M. E. Sweedler, *Hopf Algebras*, W. A. Benjamin, Inc., New York, 1969.

Research Institute of Computers, Information and Communication, Pusan National University, 2 Busandaehak-ro, Pusan 46241, Korea

E-mail address: soccerkang@hotmail.com