A \textit{q}-analog of the Racah polynomials and the \textit{q}-algebra $SU_q(2)$

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The interest of such polynomials increase after the appearance of the $q$-algebras and quantum groups [9,10,12,16,24]. However, from the first attends to built the $q$-analogue of the Wigner-Racah formalism for the simplest quantum algebra $U_q(su(2))$ [13] (see also [11,14]) becomes clear that for obtaining the $q$-polynomials intimately connected with the $q$-analogues of the Racah and Clebsh Gordan coefficients, i.e., a $q$-analogue of the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and the dual Hahn polynomials $w_n^r(x(s), a, b)_q$, respectively, it is better to use a different lattice —in fact the $q$-Racah polynomials $R_{n,\gamma}^\alpha(x(s), N, \delta)_q$ introduced in [17] (see also [14]) were defined on the lattice $x(s) = q^{-s} + \delta q^{-N} q^s$ that depends not only of the variable $s$ but also on the parameters of the polynomials—, namely,

\begin{equation}
    x(s) = [s]_q [s + 1]_q,
\end{equation}

that only depends on $s$, where by $[s]_q$ we denote the $q$-numbers (in its symmetric form)

\begin{equation}
    [s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad \forall s \in \mathbb{C}.
\end{equation}

Abstract

We study some $q$-analogues of the Racah polynomials and some of their applications in the theory of representation of quantum algebras.

1 Introduction

In the paper [6] an orthogonal polynomial family that generalizes the Racah coefficients or 6$j$-symbols was introduced: the so-called Racah and $q$-Racah polynomials. These polynomials were in the top of the so-called Askey Scheme (see e.g. [14]) that contains all classical families of hypergeometric orthogonal polynomials. Some years later the same authors [7] introduced the celebrated Askey-Wilson polynomials. One of the important properties of these polynomials is that from them one can obtain all known families of hypergeometric polynomials and $q$-polynomials as particular cases or as limit cases (for a review on this see the nice survey [14]). The main tool in these two works was the hypergeometric and basic series, respectively. On the other hand, the authors of [21] (see also [20, Russian Edition]) considered the $q$-polynomials as the solution of a second order difference equation of hypergeometric-type on the non-linear lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. In particular, they show that the solution of the hypergeometric-type equation can be expressed as certain basic series and, in such a way, they recovered the results by Askey & Wilson.

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With this choice the $q$-Racah polynomials $u_{\alpha,\beta}^\sigma(x(s), a, b)_q$ are proportional to the $q$-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$. A very nice and simple approach to $6j$-symbols has been recently developed in [23].

Moreover, this connection gives the possibility to a deeper study of the Wigner-Racah formalism (or the $q$-analogue of the quantum theory of angular momentum [25, 26, 27, 28]) for the quantum algebras $U_q(su(2))$ and $U_q(su(1, 1))$ using the powerful and well-known theory of orthogonal polynomials on non-uniform lattices. On the other hand, using the $q$-analogue of the quantum theory of angular momentum [25, 26, 27, 28] we can obtain several results for the $q$-polynomials, some of which are non trivial from the point of view of the theory of orthogonal polynomials (see e.g. the nice surveys [15, 29]). In fact, in the present paper we present a detailed study of some $q$-analogs of the Racah polynomials on the lattice [1]. Finally, some comments and remarks about the $q$-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$ in order to establish which properties of the polynomials correspond to the $6j$-symbols and vice versa.

The structure of the paper is as follows: In section 2 we present some general results from the theory of orthogonal polynomials on the non-uniform lattices taken from [3, 20]. In section 2.1 a detailed discussion of the Racah polynomials $u_{\alpha,\beta}^\sigma(x(s), a, b)_q$ is presented, whereas in section 2.2 the $\tilde{u}_{\alpha,\beta}^\sigma(x(s), a, b)_q$ are considered. In particular, a relation between these families is established. In section 3 the comparative analysis of such families and the $6j$-symbols of the quantum algebra $U_q(su(2))$ is developed which gives, on one hand, some information about the Racah coefficients and, on the other hand, allows us to give a group-theoretical interpretation of the Racah polynomials on the lattice [1]. Finally, some comments and remarks about $q$-Racah polynomials and the quantum algebra $U_q(su(3))$ are included.

## 2 Some general properties of $q$-polynomials

We will start with some general properties of orthogonal hypergeometric polynomials on the non-uniform lattices [3, 20].

The hypergeometric polynomials are the polynomial solutions $P_n(x(s))_q$ of the second order linear difference equation of hypergeometric-type on the non-uniform lattice $x(s)$ (SODE)

$$
\sigma(s) \frac{\Delta}{\Delta x(s-\frac{1}{2})} \nabla y(s) + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad x(s) = c_1(q^s + q^{-s-\mu}) + c_3, \quad q^\mu = \frac{c_1}{c_2},
$$

or, equivalently

$$
A_s y(s + 1) + B_s y(s) + C_s y(s - 1) + \lambda y(s) = 0,
$$

where

$$
A_s = \frac{\sigma(s) + \tau(s) \Delta x(s-\frac{1}{2})}{\Delta x(s) \Delta x(s - \frac{1}{2})}, \quad C_s = \frac{\sigma(s)}{\nabla x(s) \Delta x(s - \frac{1}{2})}, \quad B_s = -(A_s + C_s).
$$

Notice that $x(s) = x(-s - \mu)$.

In the following we will use the following notations $P_n(s)_q := P_n(x(s))_q$ and $\sigma(-s - \mu) = \sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})$. With this notation the Eq. (3) becomes

$$
\sigma(-s - \mu) \frac{\Delta P_n(s)_q}{\Delta x(s)} - \sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} + \lambda_n \Delta x(s - \frac{1}{2}) P_n(s)_q = 0.
$$

The polynomial solutions $P_n(s)_q$ of [3] can be obtained by the following Rodrigues-type formula [20, 22]

$$
P_n(s)_q = \frac{B_n}{\rho(s)} \nabla^{(n)} \rho_n(s), \quad \nabla^{(n)} := \nabla \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)},
$$

where $x_m(s) = x(s + \frac{m}{2})$,

$$
\rho_n(s) = \rho(s + n) \prod_{m=1}^{n} \sigma(s + m),
$$

1In the exponential lattice $x(s) = c_1 q^{s+\mu} + c_3$, so $\mu = \pm \infty$, therefore instead of using $\sigma(-s - \mu)$ one should use the equivalent function $\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})$. 


and \( \rho(s) \) is a solution of the Pearson-type equation \( \Delta [\sigma(s) \rho(s)] = \tau(s) \rho(s) \Delta x(s - 1/2) \), or equivalently,

\[
\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\sigma(-s - \mu)}{\sigma(s+1)}.
\]  

(8)

Let us point out that the function \( \rho_n \) satisfy the equation \( \Delta [\sigma(s) \rho_n(s)] = \tau_n(s) \rho_n(s) \Delta x_n(s - 1/2) \), where \( \tau_n(s) \) is given by

\[
\tau_n(s) = \frac{\sigma(s + n) + \tau(s + n) \Delta x(s + n - \frac{1}{2}) - \sigma(s)}{\Delta x_{n-1}(s)} = \frac{\sigma(-s - n - \mu) - \sigma(s)}{\Delta x_n(s - \frac{1}{2})} = \tau'_n x_n(s) + \tau_n(0),
\]

(9)

being

\[
\tau'_n = -\frac{\lambda_{2n+1}}{[2n+1]_q}, \quad \tau_n(0) = \frac{\sigma(-s^*_n - n - \mu) - \sigma(s^*_n)}{x_n(s^*_n + \frac{1}{2}) - x_n(s^*_n - \frac{1}{2})},
\]

where \( s^*_n \) is the zero of the function \( x_n(s) \), i.e., \( x_n(s^*_n) = 0 \).

From (6) follows an explicit formula for the polynomials \( P_n \) follow Eq. (3.2.30)]

\[
P_n(s)_q = B_n \sum_{m=0}^{n} [m]_q! (-1)^{m+n} \sum_{i=0}^{n-m} [n-m]_q! \frac{\Delta x(s + m - \frac{n+1}{2})}{\rho(s)} \rho_n(s - n + m) \rho(s),
\]

(10)

where \([n]_q\) denotes the symmetric \( q \)-numbers and the \( q \)-factorials are given by

\[
[0]_q! := 1, \quad [n]_q! := [1]_q [2]_q \cdots [n]_q, \quad n \in \mathbb{N}.
\]

It can be shown that the most general polynomial solution of the \( q \)-hypergeometric equation corresponds to

\[
\sigma(s) = A \prod_{i=1}^{4} [s - s_i]_q = C q^{-2s} \prod_{i=1}^{4} (q^s - q^{s_i}), \quad A \cdot C \neq 0
\]

(11)

and has the form follow Eq. (49a), page 240]

\[
P_n(s)_q = D_n 4 \phi_3 \left( \frac{-q^{-n} q^{2n+1} + \sum_{i=1}^{4} s_i q^{s_i - s}}{q^{s_1 s_2 + s_3 s_4 + s_1 - s} q^{s_1 s_2 + s_3 + s_4 + s + \mu}} ; q, q \right),
\]

(12)

where the normalizing factor \( D_n \) is given by \( (x_q := q^{1/2} - q^{-1/2}) \)

\[
D_n = B_n \left( \frac{-A}{c_1 q^{\mu \sigma q}} \right)^n q^{-\frac{x_1}{2}} (3 s_1 s_2 + s_1 s_4 + s_2 - s) q^{s_1 s_2 + s_3 + s_4 + s + \mu} q^{s_1 s_2 + s_3 + s_4 + s + \mu} ; q, q, q, q
\]

The basic hypergeometric series \( r \phi_p \) are defined by follow Eq. (4.1)

\[
r \phi_p \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_p} ; q, z \right) = \sum_{k=0}^{\infty} \left( \frac{a_1 \cdots a_r}{b_1 \cdots b_p} \right) k^k \frac{\Delta_k}{(q, q)_k} \left( \frac{(-1)^k q^{k(k-1)}}{q^{k(k-1)}} \right)^{p-r+1},
\]

where \( (a; q)_k = \prod_{m=0}^{k-1} (1 - a q^m) \), is the \( q \)-analogue of the Pochhammer symbol.

In this paper we will deal with orthogonal \( q \)-polynomials and functions. It can be proven follow using the difference equation of hypergeometric-type follow Eq. (3.7.15)]

\[
\sum_{s=a}^{b-1} P_n(s)_q P_m(s)_q \rho(s) \Delta x(s - 1/2) = \delta_{nm} d_n^2, \quad s = a, a+1, \ldots, b-1.
\]

(13)

The squared norm is follow Eq. (3.7.15)]

\[
d_n^2 = (-1)^n A_n B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \Delta x_n(s - 1/2),
\]

(14)
where [20, page 66]

\[ A_{n,k} = \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left( -\frac{\lambda_{n+m}}{[n+m]_q} \right). \]  

(15)

A simple consequence of the orthogonality is the three-term recurrence relation (TTRR)

\[ x(s) P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \]

(16)

where \( \alpha_n, \beta_n \) and \( \gamma_n \) are given by

\[ \alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}. \]

(17)

being \( a_n \) and \( b_n \) the first and second coefficients in the power expansion of \( P_n \), i.e., \( P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \cdots \). Substituting \( s = a \) in (16) we find

\[ \beta_n = \frac{x(a) P_n(a)_q - \alpha_n P_{n+1}(a)_q - \gamma_n P_{n-1}(a)_q}{P_n(a)_q}, \]

(18)

which is an alternative way for finding the coefficient \( \beta_n \). Also we can use the expression [3, page 148]

\[ \beta_n = \frac{[n]_q \tau_{n-1}(0)}{\tau''_{n-1}} - \frac{[n+1]_q \tau_n(0)}{\tau''_n} + c_3([n]_q + 1 - [n+1]_q). \]

To compute \( \alpha_n \) (and \( \beta_n \)) we need the following formulas (see e.g. [3, page 147])

\[ a_n = \frac{B_n A_{n,n}}{[n]_q!}, \quad b_n = \frac{[n]_q \tau_{n-1}(0)}{\tau''_{n-1}} + c_3([n]_q - n). \]

(19)

The explicit expression of \( \lambda_n \) is [22, Eq. (52) page 232]

\[ \lambda_n = -\frac{A q^\mu}{c_1^2(q^{1/2} - q^{-1/2})^4 [n]_q [s_1 + s_2 + s_3 + s_4 + 2\mu + n - 1]_q} \]

\[ = -\frac{C q^{-n+1/2}}{c_1^2(q^{1/2} - q^{-1/2})^2 (1 - q^n) (1 - q^{s_1+s_2+s_3+s_4+2\mu+n-1})}, \]

(20)

which can be obtained equating the largest powers of \( q^s \) in \( \Delta \).

From the Rodrigues formula [19, 3, §5.6] follows that

\[ \frac{\Delta P_n(s - \frac{1}{2})_q}{\Delta x(s - \frac{1}{2})} = -\lambda_n \frac{B_n}{B_{n-1}} \bar{P}_{n-1}(s)_q, \]

(21)

where \( \bar{P}_{n-1} \) denotes the polynomial orthogonal with respect to the weight function \( \bar{\rho}(s) = \rho_1(s - \frac{1}{2}) \). On the other hand, rewriting [3] as

\[ \left( \sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) \frac{\Delta}{\Delta x(s)} P_n(s)_q = -\lambda_n P_n(s)_q, \]

it can be substituted by the following two first-order difference equations

\[ \frac{\Delta}{\Delta x(s)} P_n(s)_q = Q(s), \quad \left( \sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) Q(s) = -\lambda_n P_n(s)_q. \]

(22)

Using the fact that \( \frac{\Delta}{\Delta x(s)} P_n(s)_q \) is a polynomial of degree \( n - 1 \) on \( x(s + 1/2) \) (see [20, §3.1]) it follows that

\[ \frac{\Delta}{\Delta x(s)} P_n(s)_q = C_n Q_{n-1}(s + \frac{1}{2}), \]

where \( C_n \) is a normalizing constant. Comparison with [21] implies that \( Q(s) \) is the polynomial \( \bar{P}_{n-1} \) orthogonal with respect to the function \( \rho_1(s - \frac{1}{2}) \) and \( C_n = -\lambda_n B_n / B_{n-1} \). Therefore, the second expression in (22) becomes

\[ P_n(s)_q = \frac{B_n}{B_{n-1}} \left( \sigma(s) \frac{\nabla}{\nabla x_1(s)} + \tau(s) I \right) \bar{P}_{n-1}(s + \frac{1}{2})_q. \]

(23)
The \( q \)-polynomials satisfy the following differentiation-type formula \([19] \[5.6.1]\]

\[
\sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q^2 n} \left[ \tau_n(s) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right].
\]

Then, using the explicit expression for the coefficient \( \alpha_n \), we find

\[
\sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q^2 n} \left[ \tau_n(s) P_n(s)_q - \frac{\alpha_n \lambda_n}{[2n]_q^2} P_{n+1}(s)_q \right].
\]

From the above equation using the identity \( \Delta \frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\Delta P_n(s)_q}{\Delta x(s)} - \frac{\nabla P_n(s)_q}{\nabla x(s)} \) as well as the SODE \[5\] we find

\[
\sigma(-s - \mu) \frac{\Delta P_n(s)_q}{\Delta x(s)} = \frac{\lambda_n}{[n]_q^2 n} \left[ (\tau_n(s) - [n]_q \tau_n \Delta x(s - \frac{1}{2})) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q \right].
\]

To conclude this section we will introduce the following notation by Nikiforov and Uvarov \[20, 22\]. First we define another \( q \)-analog of the Pochhammer symbols \[20 \text{ Eq. (5.11.1)}\]

\[
(a|q)_k = \prod_{m=0}^{k-1} [a + m]_q = \frac{\Gamma_q(a + k)}{\Gamma_q(a)} = (-1)^k (q; q)_k (q^{1/2} - q^{-1/2})^{-k} q^{-\frac{k(k-1)}{4}},
\]

where \( \Gamma_q(x) \) is the \( q \)-analog of the \( \Gamma \) function introduced in \[20 \text{ Eq. (3.2.24)}\], and related to the classical \( q \)-Gamma function \( \Gamma_q \) by formula

\[
\tilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}} \Gamma_q(s) = q^{-\frac{(s-1)(s-2)}{4}} (1 - q)^{1-s} \frac{(q; q)_\infty}{(q^2; q)_\infty}, \quad 0 < q < 1.
\]

Next we define the \( q \)-hypergeometric function \( rF_p(\cdot|q, z) \)

\[
r_F \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_p \end{array} \bigg| q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_p)_k} \frac{z^k}{k!} \left[ x_q^k q^{k(k-1)} \right]^{p-r+1},
\]

where, as before, \( x_q = q^{1/2} - q^{-1/2} \), and \( (a|q)_k \) are given by \[24\]. Notice that

\[
\lim_{q \to 1} r_F \left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_p \end{array} \bigg| q, z x_q^{-r+1} \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_p)_k} \frac{z^k}{k!} = r_F \left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_p \end{array} \bigg| z \right),
\]

and

\[
_{p+1}F_p \left( \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{array} \bigg| q, t \right) \bigg|_{t = t_0} = _{p+1}F_p \left( \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{array} \bigg| q, z \right)
\]

where \( t_0 = z q^{\frac{1}{2}( \sum_{i=1}^{p+1} a_i - \sum_{i=1}^{p} b_i - 1)} \).

Using the above notation the polynomial solutions of \[34\] is \[22 \text{ Eq. (49), page 232}\]

\[
P_n(s)_q = B_n \left( \frac{A}{c_1 q^{-\frac{1}{2}} x_q^2} \right)^n (s_1 + s_2 + \mu|q)_n (s_1 + s_3 + \mu|q)_n 
\]

\[
(s_1 + s_4 + \mu|q)_n \ _4F_3 \left( \begin{array}{c} -n, 2\mu + n - 1 + \sum_{i=1}^{4} s_i, s_1 - s, s_1 + s + \mu \\ s_1 + s_2 + \mu, s_1 + s_3 + \mu, s_1 + s_4 + \mu \end{array} \bigg| q, 1 \right).
\]

### 2.1 The \( q \)-Racah polynomials

Here we will consider the \( q \)-Racah polynomials \( u_{\alpha, \beta}^{\alpha, \beta}(x(s), a, b)_q \) on the lattice \( x(s) = [s]_q[s + 1]_q \) introduced in \[2, 17, 20\]. For this lattice one has

\[
c_1 = q^\frac{1}{2} x_q^2, \quad \mu = 1, \quad c_3 = -(q^\frac{1}{2} + q^{-\frac{1}{2}}) x_q^{-2}.
\]
Let choose $\sigma$ in (11) as

$$
\sigma(s) = -\frac{q^{-2s}}{x_q^{2} q^{-n+2s}}(q^s - q^a)(q^s - q^{-b})(q^s - q^{\beta - a})(q^s - q^{b + \alpha}) = \{s - a\}_q [s + b]_q [s + a - \beta]_q [b + \alpha - s]_q,
$$
i.e., $s_1 = a, s_2 = -b, s_3 = \beta - a, s_4 = b + \alpha, C = -q^{1/2(\alpha + \beta)} x_q^{-4}, A = -1$, and let $B_n = (-1)^n/[n]_q!$. Here, as before, $x_q = q^{1/2} - q^{-1/2}$. Now from (20) we find

$$
\lambda_n = q^{1/2(\alpha + \beta + 2n + 1)} x_q^{-2}(1 - q^n)(1 - q^\alpha + \beta + n + 1) = [n]_q [n + \alpha + \beta + 1]_q.
$$

To obtain $\tau_n(s)$ we use (4). In this case $x_n(s) = \{s + n/2\}_q [s + n/2 + 1]_q$, then, choosing $s_n^* = -n/2$, we get

$$
\tau_n(s) = \tau_n^* x_n(s) + \tau_n(0), \ \ \ \tau_n^* = -[2n + \alpha + \beta + 2]_q, \ \ \ \tau_n(0) = \sigma(-n/2 - 1) - \sigma(-n/2).
$$

(32)

Taking into account that $\tau(s) = \tau_0(s)$, we obtain the corresponding function $\tau(s)$

$$
\tau(s) = -[2 + \alpha + \beta] x(s) + \sigma(-1) - \sigma(0).
$$

### 2.1.1 The orthogonality and the norm $d_n^2$

A solution of the Pearson-type difference equation (3) is

$$
\rho(s) = \frac{\Gamma_q(s + a + 1)\Gamma_q(s - a + \beta + 1)\Gamma_q(s + a + b + 1)\Gamma_q(b + \alpha - s)}{\Gamma_q(s - a + 1)\Gamma_q(s + b + 1)\Gamma_q(s + a - \beta + 1)\Gamma_q(b - s)}.
$$

Since $\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0$, then the $q$-Racah polynomials satisfy the orthogonality relation

$$
\sum_{s=a}^{b-1} u_n^{a,\beta}(x(s), a, b)_q u_n^{a,\beta}(x(s), a, b)_q \rho(s)[2s + 1]_q = 0, \quad n \neq m,
$$

with the restrictions $-\frac{1}{2} < a \leq b - 1, \alpha > -1, -1 < \beta < 2a + 1$. Let us now compute the square of the norm $d_n^2$. From (17) and (15) follow

$$
\rho_n(s) = \frac{\Gamma_q(s + n + a + 1)\Gamma_q(s + n - a + \beta + 1)\Gamma_q(s + n + a + b + 1)\Gamma_q(b + \alpha - s)}{\Gamma_q(s - a + 1)\Gamma_q(s + b + 1)\Gamma_q(s + a - \beta + 1)\Gamma_q(b - s - n)};
$$

$$
A_{\alpha,n} = [n]_q (-1)^n \frac{\Gamma_q(a + \beta + 2n + 1)}{\Gamma_q(a + \beta + n + 1)} \Rightarrow \Lambda_n := (-1)^n A_{\alpha,n} B_n^2 = \frac{\Gamma_q(a + \beta + 2n + 1)}{[n]_q ! \Gamma_q(a + \beta + n + 1)}
$$

Taking into account that $\nabla x_{n+1}(s) = [2s + n + 1]_q$, using (14), and the identity

$$
\Gamma_q(A - s) = \frac{\Gamma_q(A)(-1)^s}{(1 - A)_q}.
$$

(33)

we have

$$
d_n^2 = \Lambda_n \sum_{s=a}^{b-a-1} \frac{\Gamma_q(s + n + a + 1)\Gamma_q(s + n - a + \beta + 1)\Gamma_q(s + n + a + b + 1)\Gamma_q(b + \alpha - s)}{\Gamma_q(s - a + 1)\Gamma_q(s + b + 1)\Gamma_q(s + a - \beta + 1)\Gamma_q(b - s - n)[2s + n + 1]_q^{-1}}
$$

$$
= \Lambda_n \sum_{s=0}^{b-a-1} \frac{\Gamma_q(s + n + 2a + 1)\Gamma_q(s + n + a + \beta + 1)\Gamma_q(s + n + a + b + a + 1)\Gamma_q(b - a + \alpha - s)}{\Gamma_q(s + 1)\Gamma_q(s + b + a + 1)\Gamma_q(s + 2a - \beta + 1)\Gamma_q(b - a - s - n)[2s + 2a + n + 1]_q^{-1}}
$$

$$
= \frac{\Gamma_q(a + \beta + 2n + 1)\Gamma_q(2a + n + 1)\Gamma_q(n + \beta + 1)\Gamma_q(a + b + n + 1)\Gamma_q(b + \alpha - a)}{[n]_q ! \Gamma_q(a + \beta + n + 1)\Gamma_q(a + b + 1)\Gamma_q(2a - \beta + 1)\Gamma_q(b - a - n)} \times
$$

$$
\sum_{s=0}^{b-a-1} \frac{(n + 2a + 1, n + \beta + 1, n + a + a + b + b + 1, 1 - b + a + n|_q)_s}{(1, a + b + 1, 2a - \beta + 1, 1 - b + a - a|_q)_s}[2s + 2a + n + 1]_q.
$$
In the following we denote by $S_n$ the sum in the last expression. If we now use that $(a|q)_n = (-1)^n(q^a; q)_n q^{-\frac{1}{2}(n+2a-1)} x_n^{-n}$, as well as the identity

$$[2s + 2a + n + 1] q = q^{-s}[2a + n + 1] q \left( \frac{q^{\alpha + \frac{n+1}{2}}; q}{q^{\alpha + \frac{n+3}{2}}; q} \right),$$

we obtain

$$S_n = \sum_{s=0}^{b-a-n-1} \frac{(q^{2a+n+1}, q^{n+\beta+1}, q^{n+a+b+1}, q^{1-b+a+n}, q^{2a+b+1}, q^{1-b-a + a}, q^{2a+n+1}, q^{1-b+a+n})}{(q, q^{a+b+1}, q^{2a+\beta+1}, q^{n+a+b+1}, q^{1-b+a+n}, q^{2a+b+1}, q^{1-b-a + a}, q^{2a+n+1}, q^{1-b+a+n})} \frac{1}{[2a+n+1] q} \phi_3 \left( \begin{array}{c} 2a+n+1 \cr a, q^a, q^a/b, aq/b, aq/c, aq^{k+1} \end{array} \right) \left( \begin{array}{c} q, aq^{k+1} \cr \frac{bc}{aq/b, aq/c, q} \end{array} \right)$$

with $k = b - a - n - 1$, $a = q^{2a+n+1}, b = q^{n+\beta+1}, c = \infty$, we obtain

$$S_n = \frac{[2a+n+1] q}{[2a+2] q} \frac{(q^{2a+n+2}, q^{n+\beta}, q^{a+b+1}, q^{1-b+a+n}, q^{2a+b+1}, q^{1-b-a + a}, q^{2a+n+1}, q^{1-b+a+n})}{(q, q^{a+b+1}, q^{2a+\beta+1}, q^{n+a+b+1}, q^{1-b+a+n}, q^{2a+b+1}, q^{1-b-a + a}, q^{2a+n+1}, q^{1-b+a+n})}$$

Finally, using (33) and (27)

$$S_n = \frac{[2a+n+1] q}{[2a+2] q} \frac{\Gamma_q(a + b + 1) \Gamma_q(2a - \beta + 1) \Gamma_q(b - a + \alpha + \beta + n + 1) \Gamma_q(\alpha + \beta + n + 1) \Gamma_q(b - a + \alpha)}{\Gamma_q(n + 2a + 2) \Gamma_q(b + a - \beta - n) \Gamma_q(\alpha + \beta + 2n + 2) \Gamma_q(b - a + \alpha)}$$

thus

$$d_n^2 = \frac{\Gamma_q(\alpha + \beta + 2n + 1) \Gamma_q(2a + n + 1) \Gamma_q(\alpha + \beta + n + 1) \Gamma_q(\alpha + \beta + a + n + 1) \Gamma_q(\alpha + \beta + n + 1) \Gamma_q(\alpha + \beta + a + n + 1) \Gamma_q(\alpha + \beta + n + 1) \Gamma_q(\alpha + \beta + a + n + 1) \Gamma_q(b - a + \beta - n)}{[\alpha + \beta + 2n + 1] q \Gamma_q(n + 1) \Gamma_q(\alpha + \beta + n + 1) \Gamma_q(b - a + \alpha) \Gamma_q(b - a + \alpha)}$$

### 2.1.2 The hypergeometric representation

From formula (22) and (30) the following two equivalent hypergeometric representations hold

$$u_n^{\alpha, \beta}(x, a, b) = \frac{q^{\frac{\beta}{2}(2a + \alpha + \beta + n + 1)} (q^{a-b+1}; q)_n (q^{\beta+1}; q)_n (q^{a+b+\alpha+1}; q)_n}{x_n^{2n} (q; q)_n}$$

$$4\phi_3 \left( \begin{array}{c} q^{n}, q^{a+\beta+n+1}, q^{a-s}, q^{a+s+1} \cr q^{a+b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{array} \right)$$

and

$$u_n^{\alpha, \beta}(x, a, b) = \frac{(a-b+1|q)_n (\beta+1|q)_n (a+b+\alpha+1|q)_n}{[n] q!}$$

$$4\phi_3 \left( \begin{array}{c} -n, a+b+n+1, a-s, a+s+1 \cr a-b+1, \beta+1, a+b+\alpha+1 \end{array} \right)$$

Using the Sears transformation formula (17) Eq. (III.15)) we obtain the equivalent formulas

$$u_n^{\alpha, \beta}(x, a, b) = \frac{q^{\frac{\beta}{2}(-2b+\alpha+b+n+1)} (q^{a-b+1}; q)_n (q^{\alpha+1}; q)_n (q^{\beta-a-b+1}; q)_n}{x_n^{2n} (q; q)_n}$$

$$4\phi_3 \left( \begin{array}{c} q^{n}, q^{a+\beta+n+1}, q^{b-s}, q^{b+s+1} \cr q^{a+b+1}, q^{\alpha+1}, q^{a+b+\beta+1} \end{array} \right)$$

$$4\phi_3 \left( \begin{array}{c} q^{n}, q^{a+\beta+n+1}, q^{b-s}, q^{b+s+1} \cr q^{a+b+1}, q^{\alpha+1}, q^{a+b+\beta+1} \end{array} \right)$$
and
\[
u_n^{\alpha, \beta}(x(s), a, b)_q = \frac{(a - b + 1|q)_n (\alpha + 1|q)_n (-a - b + \beta + 1|q)_n}{[n]_q} \times\]
\[
\sum_{k=0}^{n} \frac{(-1)^k [2s + 2k - n + 1] \Gamma_q(s + k + a + 1) \Gamma_q(2s + k - n + 1)}{\Gamma_q(s + k + a + 1) \Gamma_q(2s + k - n + 1)}
\]
\[
\times \frac{\Gamma_q(s + k + a + 1) \Gamma_q(b + a + s + n - k)}{\Gamma_q(s + n + k + b + 1) \Gamma_q(s - n + k + a - \beta + 1) \Gamma_q(b - s - k)},
\]
from where follows
\[
u_n^{\alpha, \beta}(x(a), a, b)_q = \frac{(-1)^n \Gamma_q(b - a) \Gamma_q(\beta + n + 1) \Gamma_q(b + a + \alpha + n + 1)}{[n]! \Gamma_q(b - a - n) \Gamma_q(\beta + 1) \Gamma_q(b + a + \alpha + 1)},
\]
\[
u_n^{\alpha, \beta}(x(b - 1), a, b)_q = \frac{\Gamma_q(b - a) \Gamma_q(\alpha + n + 1) \Gamma_q(b + a - \beta)}{[n]! \Gamma_q(b - a - n) \Gamma_q(\alpha + 1) \Gamma_q(b + a - \beta - n)},
\]
that coincide with the values \(^{(38)}\) and \(^{(39)}\) obtained before.

From the hypergeometric representation the following symmetry property follows
\[
u_n^{\alpha, \beta}(x(s), a, b)_q = \nu_n^{-b-a+\beta+b+a+\alpha}(x(s), a, b)_q.
\]

Finally, notice that from \(^{(38)}\) (or \(^{(39)}\)) follows that \(\nu_n^{\alpha, \beta}(x(s), a, b)_q\) is a polynomial of degree \(n\) on \(x(s) = [s]_q[s + 1]_q\). In fact,
\[
(q^a-s; q)_{k}(q^{a+s+1}; q)_{k} = (-1)^{k} q^{k(a+rac{k+1}{2})} \prod_{l=0}^{k-1} \left( \frac{x(s) - c_3}{c_1} - q^{-\frac{1}{2}} (q^{a+l} + q^{-a-l-\frac{1}{2}}) \right),
\]
where \(c_1\) and \(c_3\) are given in \(^{(34)}\).

### 2.1.3 Three-term recurrence relation and differentiation formulas

To derive the coefficients of the TTRR \(^{(16)}\) we use \(^{(17)}\) and \(^{(18)}\). Using \(^{(16)}\) and \(^{(17)}\), we obtain
\[
a_n = \frac{\Gamma_q(\alpha + \beta + 2n + 1)}{[n]_q \Gamma_q(\alpha + \beta + n + 1)}, \quad \alpha_n = \frac{[n + 1]_q [\alpha + \beta + n + 1]_q}{[\alpha + \beta + 2n + 1]_q [\alpha + \beta + 2n + 2]_q}.
\]

\(^{2}\) Obviously the formulas \(^{(34)}\) and \(^{(36)}\) also give equivalent explicit formulas.
To compute \( \rho \) we have:

\[
\rho(s) = \frac{\Gamma_q(s + a + 1) \Gamma_q(s - a + \beta + 1) \Gamma_q(s + \alpha + b + 1) \Gamma_q(b + \alpha - s)}{\Gamma_q(s - a + 1) \Gamma_q(s + b + 1) \Gamma_q(s + a + \beta + 1) \Gamma_q(b - s)}
\]

\(-1 < a \leq b - 1, \alpha > -1, -1 < \beta < 2a + 1\)

To compute \( \sigma(s) \) we have:

\[
\sigma(s) = [s - a]_q [s + b]_q [s + a - \beta]_q [b + \alpha - s]_q
\]

To find \( \gamma_n \) we use \( 177 \):

\[
\gamma_n = \frac{[a + b + \alpha + n]_q [a + b - \beta - n]_q [a + n]_q [\beta + n]_q [b - a + \alpha + \beta + n]_q [b - a - n]_q}{[\alpha + \beta + 2n + 1]_q [\alpha + \beta + 2n + 1]_q [\alpha + \beta + 2n + 2]_q [\alpha + \beta + 2n + 2]_q}
\]

To compute \( \beta_n \) we use \( 188 \):

\[
\beta_n = x(a) - \alpha_n \frac{u_n^{\alpha, \beta}(x(a), a, b)_q}{u_n^{\alpha, \beta}(x(a), a, b)_q - \gamma_n u_n^{\alpha, \beta}(x(a), a, b)_q} = \frac{[a]_q [a + 1]_q - \frac{[a + \beta + n + 1]_q [a - b + n + 1]_q [\beta + n + 1]_q [a + b + \alpha + n + 1]_q}{[\alpha + \beta + 2n + 1]_q [\alpha + \beta + 2n + 2]_q}}{[\alpha + \beta + 2n + 1]_q [\alpha + \beta + 2n + 2]_q}
\]

The differentiation formulas \( 211 \) and \( 212 \) yield

\[
\frac{\Delta u_n^{\alpha, \beta}(x(s), a, b)_q}{\Delta x(s)} = [\alpha + \beta + n + 1]_q u_n^{\alpha, \beta + 1}(x(s + \frac{1}{2}), a + \frac{1}{2}, b - \frac{1}{2})_q,
\]

(42)
Before starting let us mention that from the representation (34) and the identity
\[\sigma(s)u_{n+1}^{\alpha+1,\beta+1}(x(s), a, b)_q = \sigma(s - 1)u_n^{\alpha+1,\beta+1}(x(s + \frac{1}{2}), a + \frac{1}{2}, b - \frac{1}{2})_q,\]
respectively. Finally, formulas (24) (or 25) and (26) lead to the differentiation formulas
\[
\sigma(s)\frac{\nabla u_n^{\alpha,\beta}(x(s), a, b)_q}{[2s]_q} = -\left[\frac{\alpha + \beta + n + 1}{\alpha + \beta + 2n + 2}\right][\tau_n(s)u_n^{\alpha,\beta}(x(s), a, b)_q + [n + 1]_q u_{n+1}^{\alpha,\beta}(x(s), a, b)_q],
\]
where \(\tau_n(s)\) is given in (32).

2.1.4 The duality of the Racah polynomials

In this section we will discuss the duality property of the \(q\)-Racah polynomials \(u_n^{\alpha,\beta}(x(s), a, b)_q\). We will follow [20] pages 38-39. First of all, notice that the orthogonal relation (13) for the Racah polynomials (13) is given in (32).

The next step is to identify the functions \(u_n^{\alpha,\beta}(x(s), a, b)_q\) as polynomials on some lattice \(x(n)\). Before starting let us mention that from the representation (33) and the identity
\[
(q^n; q)_k (q^{\alpha+\beta+n+1}; q)_k = \prod_{l=0}^{k-1} \left(1 + q^{\alpha+\beta+2l+1} - q^{\alpha+\beta+1+l}\right) x_q^2 x(t) + q^2 + q^{-2},
\]
where \(x(t) = [t]_q[t+1]_q = [n + \frac{\alpha+\beta}{2}]_q[n + \frac{\alpha+\beta}{2} + 1]_q\), follows that \(u_n^{\alpha,\beta}(x(s), a, b)_q\) also constitutes a polynomial of degree \(s - a\) (for \(s = a, a + 1, \ldots, b - a - 1\)) on \(x(t)\) with \(t = n + \frac{\alpha+\beta}{2}\).

Let us now define the polynomials —compare with the definition of the Racah polynomials (36)—
\[
u_k^{\alpha',\beta'}(x(t), a', b')_q = \frac{(-1)^k \gamma_q(b' - a') \gamma_q(b' + k + 1) \gamma_q(b' + a' + \alpha' + k + 1)}{[k]! \gamma_q(b' - a' - k) \gamma_q(b' + 1) \gamma_q(b' + a' + \alpha' + 1)} \times
\]
\[
\left(\begin{array}{c}
-k, a' + \beta' + k + 1, a' - t, a' + t + 1 \\
a' - b' + 1, \beta' + 1, a' + b' + \alpha' + 1
\end{array} \right) q, 1),
\]
and
\[
u_k^{\alpha,\beta}(x(s), a, b)_q = \sum_{i=0}^{k} \frac{(-1)^i \gamma_q(b - a) \gamma_q(b + k + 1) \gamma_q(b + a + \alpha + k + 1)}{[i]! \gamma_q(b - a - k) \gamma_q(b + 1) \gamma_q(b + a + \alpha + 1)} \times
\]
\[
\left(\begin{array}{c}
-k, a + \beta + k + 1, a - t, a + t + 1 \\
a - b + 1, \beta + 1, a + b + \alpha + 1
\end{array} \right) q, 1),
\]
where
\[ k = s - a, \quad t = n + \frac{\alpha + \beta}{2}, \quad a' = \frac{\alpha + \beta}{2}, \quad b' = b - a + \frac{\alpha + \beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta. \] (48)

Obviously they are polynomials of degree \( k = s - a \) on the lattice \( x(t) \) that satisfy the orthogonality property
\[ \sum_{t=0}^{b'-1} u_k^{\alpha', \beta'}(x(t), a', b')q u_n^{\alpha, \beta}(x(t), a', b')\rho'(t)\Delta x(t - 1/2) = (d_k')^2 \delta_{k,m}, \] (49)
where \( \rho'(t) \) and \( d_k' \) are the weight function \( \rho \) and the norm \( d_n \) given in table [11] with the corresponding change \( a, b, \alpha, \beta, s, n \) by \( a', b', \alpha', \beta', t, k \).

Furthermore, with the above choice (15) of the parameters of \( u_k^{\alpha', \beta'}(x(t), a', b')q \), the hypergeometric function \( _3F_3 \) in (47) coincides with the function \( _3F_3 \) in (33) and therefore the following relation between the polynomials \( u_k^{\alpha', \beta'}(x(t), a', b') \) and \( u_n^{\alpha, \beta}(x(s), a, b)q \) holds
\[ u_k^{\alpha', \beta'}(x(t), a', b')q = A(\alpha, \beta, a, b, n, s)u_n^{\alpha, \beta}(x(s), a, b)q, \] (50)
where
\[ A(\alpha, \beta, a, b, n, s) = \frac{(-1)^{s-a+n}\beta_q(b - a - n)\beta_q(s - a + \beta + 1)\beta_q(b + a + s + 1)\beta_q(n + 1)}{\beta_q(b - s)\beta_q(n + \beta + 1)\beta_q(b + a + \alpha + n + 1)\beta_q(s - a + 1)}. \]

If we now substitute (50) in (49) and make the change (15), then (49) becomes into relation (16), i.e., the polynomial set \( u_k^{\alpha', \beta'}(x(t), a', b')q \) defined by (17) (or (50)) is the dual set associated to the Racah polynomials \( u_n^{\alpha, \beta}(x(s), a, b)q \).

To conclude this study, let us show that the TTRR (16) of the polynomials \( u_k^{\alpha', \beta'}(x(t), a', b')q \) is the SODE (11) of the polynomials \( u_n^{\alpha, \beta}(x(s), a, b)q \) whereas the SODE (11) of the \( u_k^{\alpha', \beta'}(x(t), a', b')q \) becomes into the TTRR (16) of \( u_n^{\alpha, \beta}(x(s), a, b)q \) and vice versa.

Let denote by \( \varsigma(t) \) the \( \sigma \) function of the polynomial \( u_k^{\alpha', \beta'} \), then
\[ \varsigma(t) = [t - a']q[t + b']q[t + a' - \beta']q[b' + \alpha' - t]q = [n]q[n + b - a + \alpha + \beta]q[n + \alpha]q[b + a - n - \beta]q, \]
and therefore,
\[ \varsigma(-t - 1) = [\alpha + \beta + n + 1]q[b + a + \alpha + n + 1]q[b - a - n - 1]q[n + \beta + 1]q. \]
\[ \lambda_k = [k]q[\alpha' + \beta' + k + 1]q = [s - a]q[s + a + 1]q. \]

For the coefficients \( \alpha'_k, \beta'_k \) and \( \gamma'_k \), of the TTRR for the polynomials \( u_k^{\alpha', \beta'} \) we have
\[ \alpha'_k = \frac{[k + 1]q[\alpha' + \beta' + k + 1]q}{[\alpha' + \beta' + 2k + 1]q[\alpha' + \beta' + 2k + 2]q} = \frac{[s - a + 1]q[s + a + 1]q}{[2s + 1]q[2s + 2]q}, \]
\[ \gamma'_k = \frac{[b + \alpha + s]q[b + \alpha - s]q[s + \alpha - \beta]q[b + s]q[b - s]q}{[2s + 1]q[2s]q}, \]
and
\[ \beta'_k = [n + \alpha + \beta]q[n + \alpha + \beta]q[n + \alpha + \beta + 1]q + \frac{\sigma(-s - 1)}{[2s + 1]q[2s + 2]q} + \frac{\sigma(s)}{[2s + 1]q[2s]q}. \]

Also we have \( \Delta x(t) = [2t + 2]q = [2n + \alpha + \beta + 2]q \) and \( x(s) = [s]q[s + 1]q = [k + a]q[k + a + 1]q. \)

Let show that the SODE of the Racah polynomials \( u_n^{\alpha, \beta}(x(s), a, b)q \) is the TTRR of the \( u_k^{\alpha', \beta'}(x(t), a', b')q \) polynomials. First, we substitute the relation (50) in the SODE (11) of the polynomials \( u_n^{\alpha, \beta}(x(s), a, b)q \) and use that \( u_n^{\alpha, \beta}(x(s \pm 1), a, b)q \) is proportional to \( u_{n+1}^{\alpha', \beta'}(x(t), a', b')q \) (see (50)). After some simplification, and using the last formulas we obtain
\[ \alpha'_k u_{k+1}^{\alpha', \beta'}(x(t), a', b')q + \left( \beta'_k - [n]q[\alpha + \beta + n + 1]q - \frac{[\alpha + \beta]q[\alpha + \beta + 1]q}{[2s + 1]q[2s + 2]q} \right) u_k^{\alpha', \beta'}(x(t), a', b')q \]
\[ + \gamma'_k u_{k-1}^{\alpha', \beta'}(x(t), a', b')q = 0, \]
but
\[ [n]q[\alpha + \beta + n + 1]q + \frac{[\alpha + \beta]q[\alpha + \beta + 1]q}{[2s + 1]q[2s + 2]q} = [n + \alpha + \beta]q[n + \alpha + \beta + 1]q = x(t), \]
i.e., we obtain the TTRR for the polynomials \( u_k^{α', β'}(x(t), a', b')_q \).

If we now substitute (53) in the TTRR (10) for the Racah polynomials \( u_n^{α, β}(x(s), a, b)_q \), and use that \( u_n^{α, β}(x(s), a, b)_q \sim u_k^{α', β'}(x(t \pm 1), a', b')_q \), then we obtain the SODE

\[
\frac{1}{\Delta x(t)} \frac{\Delta x(t - \frac{1}{2})}{\Delta x(t + \frac{1}{2})} u_k^{α', β'}(x(t + 1), a', b')_q + \frac{\zeta(t)}{\Delta x(t) \Delta x(t - \frac{1}{2})} u_k^{α', β'}(x(t - 1), a', b')_q
\]

\[ - \left[ \frac{\zeta(t)}{\Delta x(t) \Delta x(t - \frac{1}{2})} + \frac{\zeta(t)}{\Delta x(t) \Delta x(t - \frac{1}{2})} \right] + [a]_q[a + 1]_q - [k + a]_q[k + a + 1]_q \right] u_k^{α', β'}(x(t), a', b')_q = 0.
\]

That is the SODE (11) of the \( u_k^{α', β'}(x(t), a', b')_q \) since

\[ [a]_q[a + 1]_q - [k + a]_q[k + a + 1]_q = -[k]_q[k + 2a + 1]_q = −[k]_q[k + α' + β' + 1]_q = −λ_k.
\]

### 2.2 The \( q \)-Racah polynomials \( \tilde{u}_n^{α, β}(x(s), a, b)_q \)

There is another possibility to define the \( q \)-Racah polynomials as it is suggested in [17, 20]. It corresponds to the function

\[ σ(s) = [s - a]_q[s + b]_q[s - a + β]_q[b + α + s]_q,
\]
i.e., \( A = 1, s_1 = a, s_2 = -b, s_3 = a - β, s_4 = -b - α \). With this choice we obtain a new family of polynomials \( \tilde{u}_n^{α, β}(x(s), a, b)_q \) that is orthogonal with respect to the weight function

\[ ρ(s) = \frac{Γ_q(s + a + 1)Γ_q(s + a - β + 1)}{Γ_q(s + a + b + 1)Γ_q(b + α - s)Γ_q(s - a + 1)Γ_q(s + b + 1)Γ_q(s - a + β + 1)Γ_q(b - s)}.
\]

All their characteristics can be obtained exactly in the same way as before. Moreover, they can be also obtained from the corresponding characteristics of the polynomials \( u_n^{α, β}(x(s), a, b)_q \) by changing \( α \rightarrow -2b - α, β \rightarrow 2a - β \) —and using the properties of the functions \( Γ_q(s), Γ_q(s), (a)_q n \) and \( (a; q)_n \). We will resume the main data of the polynomials \( \tilde{u}_n^{α, β}(x(s), a, b)_q \) in table 2.

#### 2.2.1 The hypergeometric representation

For the \( \tilde{u}_n^{α, β}(x(s), a, b)_q \) polynomials we have the following hypergeometric representation

\[ \tilde{u}_n^{α, β}(x(s), a, b)_q = \frac{q^{-\frac{1}{2}(4a - 2b - α + β + n + 1)}(q^{a - b + 1}; q)_n(q^{2a - β + 1}; q)_n(q^{a - b - α + 1}; q)_n}{\Gamma_q(q; q)_n} \times
\]

\[ \Phi_3\left(q^{-n}, q^{2a - 2b - α + β + n + 1}, q^{a - s}, q^{a + s + 1} \left| q, q \right\rangle \right),
\]

or, in terms of the \( q \)-hypergeometric series (52)

\[ \tilde{u}_n^{α, β}(x(s), a, b)_q = \frac{(a - b + 1)_q(n)(2a - β + 1)_q(n)(a - b - α + 1)_q(n)}{[n]_q!} \times
\]

\[ \Phi_3\left(-n, 2a - 2b + a - β + n + 1, a - s, a + s + 1 \left| q, 1 \right\rangle \right).
\]

Using the Sears transformation formula (11 Eq. (III.15)) we obtain the equivalent representation formulas

\[ \tilde{u}_n^{α, β}(x(s), a, b)_q = \frac{q^{\frac{1}{2}(2a - 2b - α - β + n)}(q^{a - b + 1}; q)_n(q^{2b - a - 1}; q)_n(q^{α - b + 1}; q)_n}{\Gamma_q(q; q)_n} \times
\]

\[ \Phi_3\left(q^{-n}, q^{2a - 2b - α - β + n + 1}, q^{b - s}, q^{b + s + 1} \left| q, q \right\rangle \right),
\]

and

\[ \tilde{u}_n^{α, β}(x(s), a, b)_q = \frac{(a - b + 1)_q(n)(2a - β + 1)_q(n)(a - b - β + 1)_q(n)}{[n]_q!} \times
\]

\[ \Phi_3\left(-n, 2a - 2b - α - β + n + 1, a - b + 1, -2b + a + 1, a - b - β + 1 \left| q, 1 \right\rangle \right).
\]
Table 2: Main data of the $q$-Racah polynomials $\tilde{w}^{\alpha,\beta}_n(x(s), a, b)_q$

| $P_n(s)$ | $\tilde{w}^{\alpha,\beta}_n(x(s), a, b)_q$, $x(s) = [s]_q[s+1]_q$ |
|----------|-------------------------------------------------|
| $(a, b)$ | $[a, b - 1]$ |

| $\rho(s)$ | $\tilde{\Gamma}_q(s + a + 1)\tilde{\Gamma}_q(s + a - \beta + 1) \over \tilde{\Gamma}_q(s + a + b + 1)\tilde{\Gamma}_q(b + a - \alpha + s)\tilde{\Gamma}_q(s - a + 1)\tilde{\Gamma}_q(s + b + 1)\tilde{\Gamma}_q(s - a + \beta + 1)\tilde{\Gamma}_q(b - s) - \frac{1}{\alpha} < a \leq b - 1, \alpha > -1, -1 < \beta < 2a + 1$ |
| $\sigma(s)$ | $[s - a]_q[s + b]_q[s - a + \beta]_q[b + \alpha + s]_q$ |
| $\tau(-s - 1)$ | $[s + a + 1]_q[b - s - 1]_q[s + a - \beta + 1]_q[b + \alpha - s - 1]_q$ |
| $\tau(s)$ | $[2a - \beta - 1]_q[b]_q[b + \alpha]_q - [2b + \alpha - 1]_q[a]_q[a - \beta]_q - [2b + \alpha - 1]_q[2a - \beta + 1]_q - [2b + \alpha - \beta - 2]_qx(s)$ |
| $\tau_n(s)$ | $-[2b - 2a + \alpha - \beta - 2n - 2]_q[x(s + \frac{n}{2}) + [a + \frac{\alpha}{2} + 1]_q[b - \frac{n}{2} + 1]_q[a + \frac{\alpha}{2} + 1 - \beta]_q[b - \frac{n}{2} + \alpha]_q - [a + \frac{n}{2}]_q[b - \frac{n}{2}]_q[a + \frac{n}{2} - \beta]_q[b - \frac{n}{2} + \alpha]_q$ |
| $\lambda_n$ | $[n]_q[2b - 2a + \alpha + \beta - n - 1]_q$ |
| $B_n$ | $1 \over [n]_q!$ |
| $d_n^a$ | $\tilde{\Gamma}_q(2a + n - \beta + 1)\tilde{\Gamma}_q(2b - 2a + \alpha + \beta - n)2b - 2a - 2n - 1 + \alpha + \beta + 1$ \over $\tilde{\Gamma}_q(n + 1)\tilde{\Gamma}_q(b - a - n)\tilde{\Gamma}_q(b - a + n + 1)\tilde{\Gamma}_q(a + \alpha - n + \beta)\tilde{\Gamma}_q(b - a + \alpha + \beta - n)\tilde{\Gamma}_q(2b + a - n)\tilde{\Gamma}_q(b - a + \alpha + \beta + n)$ |
| $\rho_n(s)$ | $\tilde{\Gamma}_q(s + a + n + 1)\tilde{\Gamma}_q(s + a + n - \beta + 1) \over \tilde{\Gamma}_q(s + a + b + 1)\tilde{\Gamma}_q(b + a - s - n)\tilde{\Gamma}_q(s + a + 1)\tilde{\Gamma}_q(s + b + 1)\tilde{\Gamma}_q(s + a + n - 1)\tilde{\Gamma}_q(b - s - n)$ |
| $a_n$ | $(-1)^{n}\tilde{\Gamma}_q[2b - 2a + \alpha + \beta - n]_q \over [n]_q\tilde{\Gamma}_q[2b - 2a + \alpha + \beta - 2n]_q$ |
| $\alpha_n$ | $-[n + 1]_q[2b - 2a + \alpha + \beta - n - 1]_q \over [2b - 2a + \alpha + \beta - 2n - 1]_q[2b - 2a + \alpha + \beta - 2n - 2]_q$ |
| $\beta_n$ | $[a]_q[a + 1]_q + [2b - 2a + \alpha + \beta - n + 1]_q[a - b + n + 1]_q[a - b + \alpha - n + 1]_q \over [2b - 2a + \alpha + \beta - 2n - 1]_q[a - b + \alpha + \beta - 2n - 2]_q$ |
| $\gamma_n$ | $[2b - 2a + \alpha + \beta - n]_q[b - a + n - 1]_q[b - a + n + 1]_q \over [2b - 2a + \alpha + \beta - 2n - 1]_q[2b - 2a + \alpha + \beta - 2n]_q$ |

Remark: From the above formulas follow that the polynomials $\tilde{w}^{\alpha,\beta}_n(x(s), a, b)_q$ are multiples of the standard $q$-Racah polynomials $R_n(\mu(q^{-1}; q^{a-b}; q^{a-b}; q^{a-b}; q^{a-b}; q^{a+b}; q)$.

Moreover, from the above hypergeometric representations follow the values

$$
\tilde{w}^{\alpha,\beta}_n(x(a), a, b)_q = \frac{(a - b + 1)q_n(2a - \beta + 1)q_n(a - b + \alpha - 1)q_n}{[n]_q!} = \frac{(q^{a-b+1}; q)_n (q^{2a-\beta-1}; q)_n (q^{a-b-\alpha+1}; q)_n}{q^{\frac{n}{2}} (4a - 2b - \alpha + n + 1) x^2_q (q; q)_n},
$$
\hspace{1cm} (55)

$$
\tilde{w}^{\alpha,\beta}_n(x(b - 1), a, b)_q = \frac{(a - b + 1)q_n(-2b - \alpha + 1)q_n(a - b - \beta + 1)q_n}{[n]_q!} = \frac{(q^{a-b+1}; q)_n (q^{-2b-\alpha+1}; q)_n (q^{-\beta+a-b+1}; q)_n}{q^{\frac{n}{2}} (2a - 4b - \alpha + n + 1) x^2_q (q; q)_n}.
$$
\hspace{1cm} (56)
Using (10) we obtain an explicit formula

\[
\overline{u}^\alpha_\beta(x(s),a,b)_q = \frac{\Gamma_q(s-a+1)\Gamma_q(s+b+1)\Gamma_q(s-a+\beta+1)\Gamma_q(s+\alpha+b+1)}{\Gamma_q(s+a+1)\Gamma_q(s+\alpha+\beta+1)}\times
\]

From this expression follows that

\[
\overline{u}^\alpha_\beta(x(s),a,b)_q = \frac{\Gamma_q(b-a)\Gamma_q(2a-\beta+n+1)\Gamma_q(b-a+\alpha)}{[n!]\Gamma_q(b-a-n)\Gamma_q(2a-\beta+1)\Gamma_q(b-a+\alpha-n)},
\]

that are in agreement with the values (18) and (19) obtained before.

From the hypergeometric representation follows the symmetry property

\[
\overline{u}^\alpha_\beta(x(s),a,b)_q = \overline{u}^{b-a+\beta,a+\alpha}(x(s),a,b)_q.
\]

### 2.2.2 The differentiation formulas

Next we use the differentiation formulas (21) and (23) to obtain

\[
\frac{\Delta \overline{u}^\alpha_\beta(x(s),a,b)_q}{\Delta x(s)} = -[2b-2a+\alpha+\beta-n-1]q \overline{u}^\alpha_\beta_{n-1}(x(s+\frac{1}{2}),a+\frac{1}{2}, b-\frac{1}{2})q,
\]

\[
[n]q[2s+1]q \overline{u}^\alpha_\beta(x(s),a,b)_q = \sigma(-s-1)\overline{u}^\alpha_\beta_n(x(s+\frac{1}{2}),a+\frac{1}{2}, b-\frac{1}{2})q
\]

respectively. Finally, the formulas (24) (or (28)) and (26) lead to the following differentiation formulas

\[
\sigma(s) \sum_{n=0}^{N-1} \overline{u}^\alpha_\beta(x(s),a,b)_q = \frac{[2b-2a+\alpha+\beta-n-1]q}{[2\beta-2a+\alpha+\beta-2n-2]q} \left[ \tau_n(s) \overline{u}^\alpha_\beta_n(x(s),a,b)_q - [n+1]q \overline{u}^\alpha_\beta_{n+1}(x(s),a,b)_q \right],
\]

\[
\sigma(-s-1) \frac{\Delta \overline{u}^\alpha_\beta(x(s),a,b)_q}{[2s+2]q} = \frac{[2b-2a+\alpha+\beta-n-1]q}{[2\beta-2a+\alpha+\beta-2n-2]q} \times
\]

\[
\left[ (\tau_n(s) + [n]q[2b-2a+\alpha+\beta-2n-2]q[2s+1]q) \overline{u}^\alpha_\beta_n(x(s),a,b)_q - [n+1]q \overline{u}^\alpha_\beta_{n+1}(x(s),a,b)_q \right],
\]

respectively, where \(\tau_n(s)\) is given in table (2).

### 2.3 The dual set to \(\overline{u}^\alpha_\beta(x(s),a,b)_q\)

To obtain the dual set to \(\overline{u}^\alpha_\beta(x(s),a,b)_q\) we use the same method as in the previous section. We start from the orthogonality relation (13) for the \(\overline{u}^\alpha_\beta(x(s),a,b)_q\) polynomials defined by (14) and write the dual relation

\[
\sum_{n=0}^{N-1} \overline{u}^\alpha_\beta(x(s),a,b)_q \overline{u}^\alpha_\beta(x(s'),a,b)_q \frac{1}{d_n^2} = \frac{1}{p(s)\Delta x(s-1/2)} \delta_{s,s'}, \quad N = b-a,
\]

\(^3\)Obviously the formulas (11) (28) also give two equivalent explicit formulas.
where $\rho$ and $d_n^\alpha$ are the weight function and the norm of the $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ given in Table 2. Furthermore, from (54) follows that the functions $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ are polynomials of degree $k = b - s + 1$ on the lattice $x(t) = [t]_q[t + 1]_q$ where $t = b - a - n + \frac{\alpha + \beta}{2} - 1$ (the proof is similar to the one presented in Section 2). To identify the dual set let us define the new set

$$
\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q = \frac{(-1)^k \Gamma_q(b' - a') \Gamma_q(b' - a' + \beta') \Gamma_q(2b' + \alpha')}{[k!] \Gamma_q(b' - a' - k) \Gamma_q(b' - a' + \beta' - k) \Gamma_q(2b' + \alpha' - k)} \times
$$

$$
4F_3 \left( \begin{array}{c}
-k, 2a' - 2b' - \alpha' - \beta' + k + 1, -b' - t, -b' + t + 1 \\
\alpha' - b' + 1, -2b' - \alpha' + 1, a' - b' - \beta' + 1 \\
\end{array} \right| q, 1 \right),
$$

(64)

where

$$
k = b - s - 1, \quad t = b - a - n + \frac{\alpha + \beta}{2} - 1, \quad a' = \frac{\alpha + \beta}{2}, \quad b' = b - a + \frac{\alpha + \beta}{2}, \quad \alpha' = 2a - \beta, \quad \beta' = \beta.
$$

(65)

Obviously they satisfy the following orthogonality relation

$$
\sum_{t=\alpha'}^{b'-1} \tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q \tilde{u}_m^{\alpha',\beta'}(x(t), a', b')_q \rho'(t) \Delta x(t - 1/2) = (d_k)^2 \delta_{k,m},
$$

(66)

where now $\rho'(t)$ and $d_k^\alpha$ are the weight function $\rho$ and the norm $d_n^\alpha$, respectively, given in Table 2 with the corresponding change of the parameters $a$, $b$, $\alpha$, $\beta$, $n$ by $a'$, $b'$, $\alpha'$, $\beta'$, $k$, $t$ (55).

Furthermore, with the above definition (65) for the parameters of $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$, the hypergeometric function $4F_3$ in (54) coincides with the function $4F_3$ in (54) and therefore the following relation between the polynomials $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')$ and $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ holds

$$
\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q = \tilde{A}(\alpha, \beta, a, b, n, s) \tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q,
$$

(67)

where

$$
\tilde{A}(\alpha, \beta, a, b, n, s) = \frac{(-1)^{b-s-1-n} \Gamma_q(b - a - n) \Gamma_q(2b + \alpha - n) \Gamma_q(b - a + \beta - n) \Gamma_q(n + 1)}{\Gamma_q(b - s) \Gamma_q(s + a + \beta + 1) \Gamma_q(s + b + \alpha + 1) \Gamma_q(s - a - 1)}
$$

To prove that the polynomials $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ are the dual set to $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ it is sufficient to substitute (57) in (56) and do the change (53) that transforms (56) into (54).

Let also mention that, as in the case of the $q$-Racah polynomials, the TTRR (16) of the polynomials $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ is the SODE (14) of the polynomials $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ whereas the SODE (14) of the $\tilde{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ becomes into the TTRR (16) of $\tilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$ and vice versa.

To conclude this section let us point out that there exist a simple relation connecting both polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and $\tilde{u}_k^{\alpha',\beta'}(x(s), a, b)_q$ (see (57) from below). We will establish it at the end of the next section.

### 3 Connection with the 6j-symbols of the q-algebra $SU_q(2)$

#### 3.1 6j-symbols of the quantum algebra $SU_q(2)$

It is known (see e.g. [20] and references therein) that the Racah coefficients $U_q(j_1 j_2 j_3; j_1 j_2 j_3)$ are used for the transition from the coupling scheme of three angular momenta $j_1, j_2, j_3$ into the following ones

$$
|j_1 j_2 j_3; j; m\rangle = \sum_{m_1, m_2, m_3, m_1} \langle j_1 m_1 j_2 m_2 | j_1 j_2 j_3 | m_1 m_2 m_3 | j ; m \rangle |j_1 m_1 j_2 m_2| j_3 m_3),
$$

to the following ones

$$
|j_1 j_2 j_3; j m\rangle = \sum_{m_1, m_2, m_3, m_3} \langle j_1 m_1 j_2 m_2 | j_1 j_2 j_3 | m_1 m_2 m_3 | j_3 m_3 \rangle |j_1 m_1 j_2 m_2| j_3 m_3),
$$
where \( \langle j_a m_a j_b m_b | j_{ab} m_{ab} \rangle \) denotes the Clebsh-Gordon Coefficients of the quantum algebra \( su_q(2) \). In fact we have that the recoupling is given by
\[
|j_1 j_2(j_{12}), j_3 : jm\rangle = \sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23})|j_1 j_2 j_3(j_{23}) : jm\rangle.
\]

The Racah coefficients \( U \) define an unitary matrix, i.e., they satisfy the orthogonality relations
\[
\sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23})U_q(j_1 j_2 j j_3; j_{12}' j_{23}) = \delta_{j_{12}, j_{12}'} ,
\]
\[
\sum_{j_{23}} U_q(j_1 j_2 j j_3; j_{12} j_{23})U_q(j_1 j_2 j j_3; j_{12} j_{23}' ) = \delta_{j_{23}, j_{23}'} .
\]

Usually instead of the Racah coefficients is more convenient to use the \( 6j \)-symbols defined by
\[
\begin{align*}
U_q(j_1 j_2 j j_3; j_{12} j_{23}) &= (-1)^{j_1 + j_2 + j_3 + j} \sqrt{[2j_{12} + 1]_q [2j_{23} + 1]_q} \left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q \\
&= \begin{cases} \left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q & \text{if } j_3 - j_2 \leq j_{23} \leq j_3 + j_2, \\
\text{and } j_1 - j_2 \leq j_{12} \leq j_1 + j_2, & \text{if } j_3 - j_2 \leq j_1 \leq j_3 + j_2,
\end{cases}
\end{align*}
\]

The \( 6j \)-symbols have the following symmetry property
\[
\begin{align*}
\left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q &= \left\{ \frac{ \{ j_3 j_2 \} q}{\{ j_1 j_2 \} q} \right\}_q ,
\end{align*}
\]

Here and without lost of generality we will suppose that \( j_1 \geq j_2 \) and \( j_3 \geq j_2 \), then for the moments \( j_{23} \) and \( j_{12} \) we have the intervals
\[
j_3 - j_2 \leq j_{23} \leq j_3 + j_2, \quad j_1 - j_2 \leq j_{12} \leq j_1 + j_2,
\]
respectively. Now, in order to avoid any other restrictions on these two momenta (caused by the so called triangle inequalities for the \( 6j \)-symbols) we will assume that the following restrictions hold
\[
|j - j_s| \leq \min(j_{12}) = j_1 - j_2, \quad |j - j_1| \leq \min(j_{23}) = j_3 - j_2.
\]

### 3.2 6j-symbols and the q-Racah polynomials \( u_n^{\alpha, \beta}(x(s), a, b)_q \)

Now we are ready to establish the connection of \( 6j \)-symbols with the \( q \)-Racah polynomials. We fix the variable \( s \) as \( s = j_{23} \) that runs on the interval \( a \leq s \leq b - 1 \) where \( a = j_3 - j_2, b = j_2 + j_3 + 1 \). Let us put
\[
(-1)^{j_1 + j_{23} + j} \sqrt{[2j_{12} + 1]_q} \left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}} u_n^{\alpha, \beta}(x(s), a, b)_q,
\]
where \( \rho(s) \) and \( d_n \) are the weight function and the norm, respectively, of the \( q \)-Racah polynomials on the lattice \( \mathbb{U} \) \( u_n^{\alpha, \beta}(x(s), a, b)_q \), and \( n = j_{12} - j_1 + j_2, \alpha = j_1 - j_2 - j_3 + j \geq 0, \beta = j_1 - j_2 + j_3 - j \geq 0. \)

To verify the above relation we use the recurrence relation [27 Eq. (5.17)]
\[
\begin{align*}
[2]_q [2j_{23} + 2]_q A_q^{-1} \left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q - & \left( [2]_q [2j_{23} + 2]_q [2]_q [2j_{23} + 1]_q [2]_q [2j_{23} + 1]_q [2]_q [2j_{23} + 1]_q \right) \\
\left( [2]_q [2j_{23} + 2]_q [2]_q [2j_{23} + 1]_q [2]_q [2j_{23} + 1]_q [2]_q [2j_{23} + 1]_q \right) - & \left( [2]_q [2j_{23} + 2]_q [2]_q [2j_{23} + 1]_q [2]_q [2j_{23} + 1]_q [2]_q [2j_{23} + 1]_q \right)
\end{align*}
\]

\[
[2]_q [2j_{23} + 1]_q \left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q + [2]_q [2j_{23} + 2]_q A_q^{+} \left\{ \frac{ \{ j_1 j_3 \} q}{\{ j_1 j_2 \} q} \right\}_q = 0 ,
\]
\]

\[4\text{Notice that this is equivalent to the following setting:}
\]
\[j_1 = (b - a - 1 + \alpha + \beta)/2, \quad j_2 = (a + b - 1)/2, \quad j_3 = (a + b - 1)/2, \]
\[j_{12} = (2n + \alpha + \beta)/2, \quad j_{23} = s, \quad j = (a + b - 1 + \alpha - \beta)/2.\]
where

\[
A_q^- = \sqrt{\frac{(j + j_2 + j_1 + 1)_q(j + j_2 - j_1)_q}{(j + j_2 + j_3 - 1)_q(j + j_2 - j_3 + 1)_q}}
\]

\[
A_q^+ = \sqrt{\frac{(j + j_2 - j_3 + 1)_q(j + j_2 + j_3 - 1)_q}{(j + j_2 - j_3 + 1)_q(j + j_2 + j_3 - 1)_q}}
\]

(73)

Notice that

\[
A_q^- = \sqrt{\sigma(j_23)\sigma(-j_23)}, \quad A_q^+ = \sqrt{\sigma(j_23 + 1)\sigma(-j_23 - 1)},
\]

where

\[
\sigma(j_23) = [j_23 - j_3 + j_2 + j_3 + 1]_q[j_23 - j_3 - j_1]_q[j_23 - j_3]_q
\]

\[
\sigma(-j_23 - 1) = [j_23 + j_3 + j_2 - 1]_q[j_23 + j_3 - j_2 + j_3 + 1]_q[j_23 + j_3 - j_2 + j_3 - 1]_q
\]

Substituting (74) in (75) and simplifying the obtained expression we get

\[
[2s]_q\sigma(-s - 1)u^{a\beta}_n(x(s - 1), a, b)_q + [2s + 2]_q\sigma(s)u^{a\beta}_n(x(s - 1), a, b)_q + \left(\lambda_n[2s]_q[2s + 1]_q[2s + 2]_q - [2s]_q\sigma(-s - 1) - [2s + 2]_q\sigma(s)\right)u^{a\beta}_n(x(s), a, b)_q = 0,
\]

which is the difference equation for the q-Racah polynomials (41). Since \(u^0(x(s), a, b)_q = 1\), (71) leads to

\[
(-1)^{j_1 + j_23 + j} \sqrt{[2j_1 + 2j_3 + 1]_q \begin{bmatrix} j_1 & j_2 & j_1 - j_2 \\ j_3 & j & j_23 \end{bmatrix}_q} = \sqrt{\frac{\rho(s)}{d^0}} \Rightarrow
\]

\[
\begin{bmatrix} j_1 & j_2 & j_1 - j_2 \\ j_3 & j & j_23 \end{bmatrix}_q \begin{bmatrix} 2j_1 - 2j_2 & 0 \\ \int_j \frac{q}{j+1} \end{bmatrix}_q = \begin{bmatrix} j_1 + j_s + j_1 \end{bmatrix}_q
\]

Furthermore, substituting the values \(s = a\) and \(s = b - 1\) in (74) and using (41) we find

\[
\begin{bmatrix} j_1 & j_2 & j_12 \\ j_3 & j & j_3 - j_2 \end{bmatrix}_q = (-1)^{j_12 + j_3 + j} \times
\]

\[
\begin{bmatrix} j_12 + j_3 - j \end{bmatrix}_q! [2j_2]_q! [j_12 + j_3 + j + 1]_q! [2j_3 - 2j_2]_q! [j_2 - j_1 + j_12]_q! \times
\]

\[
\begin{bmatrix} j_1 + j_2 - j_3 - j \end{bmatrix}_q! [j_12 + j_3 - j]_q! [j_1]_q! [j_12]_q! [j_3 - j_2 + j + 1]_q!
\]

(74)

and

\[
\begin{bmatrix} j_1 & j_2 & j_12 \\ j_3 & j & j_2 + j_3 \end{bmatrix}_q = (-1)^{j_12 + j_3 + j} \times
\]

\[
\begin{bmatrix} 2j_2]_q! [j_12 + j_3 + j]_q! [j_2 - j_1 + j_3 + j]_q! [2j_3]_q! [j_1 + j_2 + j_3 - j]_q! \times
\]

\[
\begin{bmatrix} j_1 + j_2 - j_12]_q! [j_12 - j_3 + j]_q! [j_3 - j_12 + j]_q! \times
\]

\[
[2j_3 + 1]_q! [j_12 - j_3 + j]_q! [j_12 + j_3 + j + j + 1]_q!
\]

(75)

that are in agreement with the results in [26].
The relation \((71)\) allows us to obtain several recurrence relations for the \(6j\)-symbols of the quantum algebra \(SU_q(2)\) by using the properties of the \(q\)-Racah polynomials. So, the TTRR \((16)\) gives

\[
[2j_{12}]_q \bar{A}_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{array} \right\}_q + [2j_{12} + 2]_q A_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} - 1 \\ j_3 & j & j_{23} \end{array} \right\}_q \\
- \left( [2j_{12}]_q [2j_{12} + 1]_q [2j_{12} + 2]_q \left( j_{23} \right)_q \right)_q \left( [j_{12} + j_{12} + 1] - [j_3 - j_2 - j - 1]_q \right) + [2j_{12}]_q \times \\
[j_1 - j_2 + j_{12} + 1]_q [j_1 - j_2 + 1]_q [j_1 - j_2 + j - 1]_q [j_{12} - j_3 - j + 1]_q + [2j_{12} + 2]_q \times \\
[j_{12} - j_3]_q [j_1 + j_2 + j_{12} + 1]_q [j_1 - j_2 + j + 1]_q [j_{12} - j_1 + j_2]_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0,
\]

where

\[
\bar{A}_q^- = \sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12}]_q [j_{12} - j_3 + j]_q [j_{12} - j_3 - j]_q [j_1 + j_2 + j_{12} + 1]_q \times \\
[j_{12} + j_3 + j + 1]_q [j_1 + j_2 - j_{12} + 1]_q [j_3 - j_{12} + j + 1]_q,
\]

\[
\bar{A}_q^+ = \sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12}]_q [j_{12} - j_3 + j + 1]_q [j_{12} - j_3 - j]_q [j_1 + j_2 + 1]_q \times \\
[j_{12} + j_3 + 1]_q [j_1 + j_2 + 1]_q [j_{12} - j_3 + j]_q [j_{12} - j_3 - j + 1]_q \times \\
[j_1 + j_2 + j_{12} + 1]_q [j_1 + j_2 - j_{12} + 1]_q [j_3 - j_{12} + j + 1]_q,
\]

The expressions \((12)\) and \((48)\) yield

\[
\sqrt{\sigma(j_{23} + 1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{array} \right\}_q + \sqrt{\sigma(-j_{23} - 1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q \\
= [2j_{23} + 2]_q \sqrt{[j_2 - j_1 + j_{12}]_q [j_1 - j_2 + j_{12} + 1]_q} \left\{ \begin{array}{ccc} j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} + \frac{1}{2} \end{array} \right\}_q,
\]

and

\[
\sqrt{\sigma(-j_{23} - 1)} \left\{ \begin{array}{ccc} j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} + \frac{1}{2} \end{array} \right\}_q + \sqrt{\sigma(j_{23})} \left\{ \begin{array}{ccc} j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\ j_3 & j & j_{23} - \frac{1}{2} \end{array} \right\}_q \\
= [2j_{23} + 1]_q \sqrt{[j_{12} - j_1 + j_{23}]_q [j_{12} + j_1 - j_{23} + 1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q,
\]

respectively, whereas the differentiation formulas \((44)\)–\((45)\) give

\[
[2j_{12}]_q A_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} - 1 \end{array} \right\}_q + [2j_{12}]_q A_q^+ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{array} \right\}_q \\
(\sigma(j_{23})[2j_{12} + 2]_q + [j_1 - j_2 + j_{12} + 1]_q[2j_{23}]_q A(j_{12}, j_{23}, j_1, j_2) \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0
\]

and

\[
[2j_{12} + 2]_q A_q^+ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} + 1 \end{array} \right\}_q - [2j_{23} + 2]_q A_q^+ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{array} \right\}_q \\
(2j_{12} + 2) \sigma(-j_{12} - 1) - 2[j_{23} + 2]_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} + 1 \\ j_3 & j & j_{23} \end{array} \right\}_q \\
[j_1 - j_1 + j_2]_q [2j_{12} + 1]_q [2j_{23} + 1]_q \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0,
\]

respectively, where \(A_q^-\) are given by \((48)\), \(\bar{A}_q^+\) by \((77)\) and

\[
\Lambda(j_{12}, j_{23}, j_1, j_2) = \sigma \left( \frac{-j_{12} + j_1 - j_2}{2} - 1 \right) - \sigma \left( \frac{-j_{12} + j_1 - j_2}{2} \right) - \\
[2j_{12} + 2]_q \left[ j_{23} + \frac{j_{12} - j_1 + j_2}{2} \right]_q \left[ j_{23} + \frac{j_{12} - j_1 + j_2}{2} + 1 \right].
\]
Using the hypergeometric representations \((35)\) and \((37)\), we obtain the representation of the 6\(j\)-symbols in terms of the \(q\)-hypergeometric function\(^5\) \((28)\):

\[
\begin{aligned}
\left\{ \begin{array}{c}
j_1 \ j_2 \ j_{12} \\
j_3 \ j \ j_{23}
\end{array} \right\}_q &= (-1)^{j_1 + j_2 + j_3 + j + j} \frac{[2j_2]_q!}{[j_1 - j_2 + j_3 - j - 1]_q!} \times \\
&\times \sqrt{\frac{[j_1 + j + j_{23} + 1]_q![j_1 + j - j_{23}]_q![j_1 + j + j_{23}]_q![j_1 - j + j_{23}]_q}{[j - j_1 + j_{23}]_q![j_3 - j_2 + j_{23}]_q![j_3 - j + j_{23}]_q![j_3 - j + j_{23}]_q}} \\
&\times \sqrt{\frac{[j_{12} - j_1 + j_{23}]_q![j_{12} + j_1 - j_{23}]_q![j_{12} + j_1 + j_{23}]_q}{[j_{12} - j_1 + j_2]_q![j_{12} + j_1 + j_2]_q![j_{12} + j_1 + j_2]_q}} \\
&\times 4F_3 \left( \begin{array}{c}
-2j_2, j_1 - j_2 + j_3 - j - 1, j_1 - j_2 + j_3 + j + 2 \\
j_1 - j_2 - j_1, j_1 - j_2 + j_1 + 1, 1 - j_3 + j_3 - j_3 - j - 1 \\
-2j_2, j_1 - j_2 - j_3 + j + 1, 1 - j_2 + j_3 - j
\end{array} \middle| \begin{array}{c}
q, 1
\end{array} \right),
\end{aligned}
\]

and

\[
\begin{aligned}
\left\{ \begin{array}{c}
j_1 \ j_2 \ j_{12} \\
j_3 \ j \ j_{23}
\end{array} \right\}_q &= (-1)^{j_1 + j_2 + j + j} \frac{[2j_2]_q!}{[j_1 - j_2 - j_3 + j + 1]_q!} \\
&\times \sqrt{\frac{[j_1 + j + j_{23} + 1]_q![j_1 + j - j_{23}]_q![j_1 + j + j_{23}]_q![j_1 - j + j_{23}]_q}{[j - j_1 + j_{23}]_q![j_3 - j_2 + j_{23}]_q![j_3 - j + j_{23}]_q![j_3 - j + j_{23}]_q}} \\
&\times \sqrt{\frac{[j_{12} - j_1 + j_{23}]_q![j_{12} + j_1 - j_{23}]_q![j_{12} + j_1 + j_{23}]_q}{[j_{12} - j_1 + j_2]_q![j_{12} + j_1 + j_2]_q![j_{12} + j_1 + j_2]_q}} \\
&\times 4F_3 \left( \begin{array}{c}
-2j_2, j_1 - j_2 - j_3 + j + 1, 1 - j_2 + j_3 - j_3 - j - 1 \\
j_1 - j_2 - j_1, j_1 - j_2 + j_1 + 1, 1 - j_3 + j_3 - j_3 - j - 1 \\
-2j_2, j_1 - j_2 - j_3 + j + 1, 1 - j_2 + j_3 - j
\end{array} \middle| \begin{array}{c}
q, 1
\end{array} \right).
\end{aligned}
\]

Notice that from the above representations the values \((24)\) and \((70)\) immediately follows. Notice also that the above formulas give two alternative explicit formulas for computing the 6\(j\)-symbols.

A third explicit formula follows from \((40)\):

\[
\begin{aligned}
\left\{ \begin{array}{c}
j_1 \ j_2 \ j_{12} \\
j_3 \ j \ j_{23}
\end{array} \right\}_q &= \sum_{k=0}^{j_{12} - j_1 + j_2} (-1)^{k+j_1+j_2} \frac{[2k+1]_q!}{[k]_q!} \frac{[j_{12} - j_1 + j_2]_q!}{[j_{12} - j_1 + j_2]_q!} \frac{[j_{12} + j_1 - j_2]_q!}{[j_{12} + j_1 - j_2]_q!} \frac{[j_{12} + j_1 + j_2]_q!}{[j_{12} + j_1 + j_2]_q!} \\
&\times \frac{[j_{23} + j_1 - j_2]_q!}{[j_{23} + j_1 - j_2]_q!} \frac{[j_1 + j_2 + j_3 + j + 1]_q!}{[j_1 + j_2 + j_3 + j + 1]_q!} \frac{[j_1 + j_2 + j_3 + j + 1]_q!}{[j_1 + j_2 + j_3 + j + 1]_q!} \\
&\times 4F_3 \left( \begin{array}{c}
j_1 - j_2 - j_1, j_1 - j_2 + j_1 + 1, 1 - j_3 + j_3 - j_3 - j - 1 \\
j_1 - j_2 - j_1, j_1 - j_2 + j_1 + 1, 1 - j_3 + j_3 - j_3 - j - 1 \\
j_1 - j_2 - j_1, j_1 - j_2 + j_1 + 1, 1 - j_3 + j_3 - j_3 - j - 1
\end{array} \middle| \begin{array}{c}
q, 1
\end{array} \right).
\end{aligned}
\]

To conclude this section let us point out that the orthogonality relations \((68)\) and \((70)\) lead to the orthogonality relations for the Racah polynomials \(u_{\alpha, \beta}^a(x(s), a, b)_q\) \((35)\) and their duals \(u_{\alpha', \beta'}^a(x(t), a', b')_q\), respectively, and also that the relation \((70)\) between \(q\)-Racah and dual \(q\)-Racah corresponds to the symmetry property \((70)\).

### 3.3 6\(j\)-symbols and the alternative \(q\)-Racah polynomials \(\tilde{u}_{\alpha, \beta}^a(x(s), a, b)_q\)

In this section we will make the same comparative analysis but for the alternative \(q\)-Racah polynomials \(\tilde{u}_{\alpha, \beta}^a(x(s), a, b)_q\). We again choose \(s = j_{23}\) that runs on the interval \([a, b-1]\), \(a = j_3 - j_2\), \(b = j_2 + j_3 + 1\). In this case the connection is given by formula

\[
(-1)^{j_{12} + j_3 + j} \sqrt{[2j_{12} + 1]_q} \left\{ \begin{array}{c}
j_1 \ j_2 \ j_{12} \\
j_3 \ j \ j_{23}
\end{array} \right\}_q = \sqrt{\frac{\rho(s)}{d_{\alpha, \beta}^a}} \tilde{u}_{\alpha, \beta}^a(x(s), a, b)_q,
\]

where \(\rho(s)\) and \(d_{\alpha, \beta}^a\) are the weight function and the norm, respectively, of the alternative \(q\)-Racah polynomials \(\tilde{u}_{\alpha, \beta}^a(x(s), a, b)_q\) (see Section 2.2) on the lattice \(\mathbb{I}\), and \(n = j_1 + j_2 - j_{12}, \alpha = j_1 - j_2 - j_3 + j \geq 0, \beta = j_1 - j_2 + j_3 - j \geq 0\).

\(^5\)To obtain the representation in terms of the basic hypergeometric series it is sufficient to use the relation \((29)\).
Using the above relations we see that the SODE \((1)\) for the \(\bar{\mu}^{a,\beta}_q(x(s), a, b)_q\) polynomials becomes into the recurrence relation \((72)\) as well as the TTRR \((16)\) becomes into the recurrence relation \((10)\). Evaluating \((82)\) in \(s = j_{23} = j_3 - j_2\) and \(s = j_{23} = j_2 + j_3 + 1\) and using \(68\) we recover the values \((74)\) and \((76)\), respectively. If we now put \(n = 0\), i.e., \(j_{12} = j_1 + j_2\) we obtain the value

\[
\left\{ \begin{array}{c}
j_1 & j_2 & j_1 + j_2 \\
j_3 & j & j_{23} \\
\end{array} \right\}_q = \left\{ \begin{array}{c}
j_1 & j_2 & j_1 + j_2 \\
j_3 & j & s \\
\end{array} \right\}_q
\]

\[
= (-1)^{j_1 + j_2 + j_3 + j} \sqrt{\frac{[2j_1]_q [2j_2]_q [j_1 + j_2 + j_3 + j + 1]_q [j_1 + j_2 - j_3]_q}{[2j_1 + 2j_2 + 1]_q [j_1 - j_2 + j_3]_q [j_2 - j_3 - s]_q [j_2 + j_3]_q}} \times
\sqrt{\frac{s - j_1 + j_2 + j_3]_q}{s - j_2 + j_3]_q}} \times
\sqrt{[j_1 + j + s]_q [j_1 + j + s + 1]_q [j_2 + j_3 - s]_q [j_2 + j_3 + s]_q},
\]

The expressions \((59)\) and \((60)\) yield

\[
\sqrt{\varsigma(j_{23} + 1)} \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} \\
j_3 & j & j_{23} + 1 \\
\end{array} \right\}_q - \sqrt{\varsigma(-j_{23} - 1)} \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} \\
j_3 & j & j_{23} \\
\end{array} \right\}_q
\]

\[
= [2j_{23} + 2]_q \sqrt{\sqrt{[j_1 + j_2 - j_{12}]_q [j_1 + j_2 + j_{12} + 1]_q} \left\{ \begin{array}{c}
j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\
j_3 - \frac{1}{2} & j & j_{23} + \frac{1}{2} \\
\end{array} \right\}_q},
\]

and

\[
\sqrt{\varsigma(-j_{23} - 1)} \left\{ \begin{array}{c}
j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\
j_3 - \frac{1}{2} & j & j_{23} + \frac{1}{2} \\
\end{array} \right\}_q - \sqrt{\varsigma(j_{23})} \left\{ \begin{array}{c}
j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_{12} \\
j_3 - \frac{1}{2} & j & j_{23} - \frac{1}{2} \\
\end{array} \right\}_q
\]

\[
= [2j_{23} + 1]_q \sqrt{[j_1 + j_2 - j_{12}]_q [j_1 + j_2 + j_{12} + 1]_q} \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} \\
j_3 & j & j_{23} \\
\end{array} \right\}_q,
\]

respectively, where

\[
\varsigma(j_{23}) = [j_3 - j_2]_q [j_3 + j_2 + j_3 + 1]_q [j_23 - j_1 + j_2 + j_3 + 1]_q,
\]

\[
\varsigma(-j_{23} - 1) = [j_3 + j_2 - 1]_q [j_2 + j_3 - j_{23}]_q [j_3 + j_1 - j_2]_q [j_1 + j_3]_q.
\]

The differentiation formulas \((61)-(64)\) give

\[
[2j_{12}]_q A_q^\pm \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} \\
j_3 & j & j_{23} - 1 \\
\end{array} \right\}_q - [2j_{23}]_q A_q^\mp \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} - 1 \\
j_3 & j & j_{23} \\
\end{array} \right\}_q
\]

\[
= 0
\]

and

\[
[2j_{12}]_q A_q^+ \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} \\
j_3 & j & j_{23} + 1 \\
\end{array} \right\}_q + [2j_{23} + 2]_q A_q^- \left\{ \begin{array}{c}
j_1 & j_2 & j_{12} - 1 \\
j_3 & j & j_{23} \\
\end{array} \right\}_q
\]

\[
= 0,
\]

respectively, where \(A_q^\pm\) are given by \((83)\), \(A_q^\pm\) by \((77)\) and

\[
\bar{\lambda}(j_{12}, j_{23}, j_1, j_2) = \varsigma \left( \frac{j_{12} - j_1 - j_2}{2} - 1 \right) - \varsigma \left( \frac{j_{12} - j_1 - j_2}{2} \right)
\]

\[
= \left\{ \begin{array}{c}
j_{12} + \frac{j_1 + j_2 - j_{12}}{2} \\
j_{23} + \frac{j_1 + j_2 - j_{12}}{2} + 1 \\
\end{array} \right\}_q.
\]
If we now use the hypergeometric representations \( \text{(52)} \) and \( \text{(54)} \) we obtain two new representations of the 6j-symbols in terms of the \( q \)-hypergeometric function \( \text{(28)} \)

\[
\begin{align*}
\left\{ \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_j \\
j_{j_23} \\
j_{j_3} \\
j_{j_323} \end{array} \right\}_q = (-1)^j_{12+3} + j_{123} \frac{\left[ 2j_2 \right]_q \left[ j_1 + j_2 + j_3 + j \right]_q}{\left[ j_3 + j_2 - j_1 + j \right]_q} \times \\
\sqrt{\frac{[j - j_1 + j_23]_q [j_3 - j_2 + j_23]_q}{\left[ j_1 + j + j_23 + 1 \right]_q [j_1 + j - j_23]_q [j_23 + j + j_23 + 1]_q [j_1 - j + j_23]_q}} \\
\times \frac{[j_1 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q}{[j_3 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q} \\
\times \frac{[j_3 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q}{[j_3 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q} \\
\times 4 \Gamma_3 \left( \begin{array}{c}
j_{j_12} - j_1 - j_2, j_1 - j_2 - j_3 - 1, j_3 - j_2 - j_23, j_3 - j_2 + 1 \end{array} \middle| q, 1 \right),
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_j \\
j_{j_23} \\
j_{j_3} \\
j_{j_323} \end{array} \right\}_q = (-1)^{j_1 + j_23 + j_3} \frac{\left[ 2j_2 \right]_q \left[ j_1 + j_2 + j_3 + j_3 + j \right]_q}{\left[ j_3 + j_2 - j_1 + j \right]_q} \times \\
\sqrt{\frac{[j - j_1 + j_23]_q [j_3 - j_2 + j_23]_q}{\left[ j_1 + j + j_23 + 1 \right]_q [j_1 + j - j_23]_q [j_23 + j + j_23 + 1]_q [j_1 - j + j_23]_q}} \\
\times \frac{[j_1 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q}{[j_3 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q} \\
\times \frac{[j_3 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q}{[j_3 - j_2 + j_23]_q [j_23 + j - j_1 + j]_q [j_23 + j + j_23 + 1]_q [j_23 - j + j_1]_q} \\
\times 4 \Gamma_3 \left( \begin{array}{c}
j_{j_12} - j_1 - j_2, j_1 - j_2 - j_3 - 1, j_3 - j_2 - j_23, j_3 - j_2 + 1 \end{array} \middle| q, 1 \right).
\end{align*}
\]

Notice that from the above representations the values \( \text{(71)} \) and \( \text{(74)} \) also follows. Obviously the above formulas give another two alternative explicit formulas for computing the 6j-symbols. Finally, from \( \text{(67)} \)

\[
\begin{align*}
\left\{ \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_j \\
j_{j_23} \\
j_{j_3} \\
j_{j_323} \end{array} \right\}_q = \sqrt{\frac{\left[ j_1 + j + j_23 + 1 \right]_q \left[ j_2 + j_3 + 1 \right]_q \left[ j_3 + j + j_23 + 1 \right]_q}{\left[ j_1 + j - j_23 \right]_q \left[ j_1 + j + j_23 + 1 \right]_q \left[ j_1 + j + j_23 + 1 \right]_q}} \\
\sqrt{\frac{\left[ \left[ j_23 + j_1 + j_2 + j_3 + 1 \right]_q \left[ j_23 + j + j_23 + 1 \right]_q \left[ j_23 + j + j_23 + 1 \right]_q}{\left[ j_1 + j - j_23 \right]_q \left[ j_1 + j + j_23 + 1 \right]_q \left[ j_1 + j + j_23 + 1 \right]_q}} \\
\sqrt{\frac{\left[ \left[ j_23 + j_1 + j_2 + j_3 + 1 \right]_q \left[ j_23 + j + j_23 + 1 \right]_q \left[ j_23 + j + j_23 + 1 \right]_q}{\left[ j_1 + j - j_23 \right]_q \left[ j_1 + j + j_23 + 1 \right]_q \left[ j_1 + j + j_23 + 1 \right]_q}} \\
\sum_{l=0}^{j_1 + j_2 - j_{j_23}} \left[ \left[ l+1 \right]_q \left[ j_1 + j_2 - j_23 + l + 1 \right]_q \left[ j_1 + j_2 - j_23 + l + 1 \right]_q \left[ j_1 + j_2 - j_23 + l + 1 \right]_q \left[ j_1 + j_2 - j_23 + l + 1 \right]_q \right] \\
\times \left[ \left[ j_{j_23} - j_1 - j_2 + j_23 + l + 1 \right]_q \left[ j_{j_23} - j_1 - j_2 + j_23 + l + 1 \right]_q \left[ j_{j_23} - j_1 - j_2 + j_23 + l + 1 \right]_q \right].
\end{align*}
\]

To conclude this section, let us point out that the orthogonality relations \( \text{(68)} \) and \( \text{(69)} \) lead to the orthogonality relations for the alternative Racah polynomials \( \bar{u}_n^{\alpha,\beta}(x(s), a, b)_q \) and their duals \( \bar{u}_k^{\alpha',\beta'}(x(t), a', b')_q \) \( \text{(70)} \), respectively, as well as the relation \( \text{(74)} \) between \( q \)-Racah and dual \( q \)-Racah corresponds to the symmetry property \( \text{(74)} \).

### 3.4 Connection between \( \bar{u}_k^{\alpha,\beta}(x(s), a, b)_q \) and \( u_n^{\alpha,\beta}(x(s), a, b)_q \)

Let us obtain a formula connecting the two families \( \bar{u}_k^{\alpha,\beta}(x(s), a, b)_q \) and \( u_n^{\alpha,\beta}(x(s), a, b)_q \). In fact, Eqs. \( \text{(71)} \) and \( \text{(72)} \) suggest the following relation between both Racah polynomials \( \bar{u}_k^{\alpha,\beta}(x(s), a, b)_q \) and \( u_n^{\alpha,\beta}(x(s), a, b)_q \)

\[
\bar{u}_n^{\alpha,\beta}(x(s), a, b)_q = (-1)^{s-a-n} \times \\
\frac{\bar{\Gamma}_q(s-a + \beta + 1) \bar{\Gamma}_q(b + \alpha + 1) \Gamma_q(a + b + \beta - n) \Gamma_q(a + b + \alpha + 1 + n)}{\Gamma_q(s + a - \beta + 1) \Gamma_q(b + a + 1) \Gamma_q(\beta + 1 + n) \Gamma_q(\alpha + 1 + n)} u_n^{\alpha,\beta}(x(s), a, b)_q.
\]  

(87)

To prove it is sufficient to substitute the above formula in the difference equation \( \text{(4)} \) of the \( \bar{u}_n^{\alpha,\beta}(x(s), a, b)_q \) polynomials. After some straightforward computations the resulting difference equation becomes into the corresponding difference equation for the polynomials \( u_n^{\alpha,\beta}(x(s), a, b)_q \).
Notice that from this relation follows that
\[ 4 \Phi_3 \left( \frac{a - b + n + 1, a - b - \alpha - \beta - n, a - s, a + s + 1}{a - b + 1, 2a - \beta + 1, a - b - \alpha + 1} \mid q, 1 \right) \]
\[ = \frac{(\beta + 1|q)_{s-a}(b + a + a + 1|q)_{s-a}}{(2a - \beta + 1|q)_{s-a}(a - b - \alpha + 1)_{s-a}} 4 \Phi_3 \left( \frac{-n, \alpha + \beta + n + 1, a - s, a + s + 1}{a - b + 1, \beta + 1, a + b + \alpha + 1} \mid q, 1 \right). \]

This yields to the following identity for terminating 4 \Phi_3 basic series, \( n, N - n - 1, k = 0, 1, 2, \ldots, \)
\[ 4 \Phi_3 \left( \frac{q^{n-N+1}, q^{-n-N+1}A^{-1}B^{-1}, q^{-k}, q^{-k}D}{q^{1-N}, q^{-2k}DB^{-1}, q^{1-N}A^{-1}} \mid q, q \right) \]
\[ = \frac{q^{-kN}}{A^k B^k (q^{-2k}DB^{-1}; q)_k, (q^{1-N}A^{-1}; q)_k} 4 \Phi_3 \left( \frac{q^{-n}, ABq^n, q^{-k}, q^{-k}D}{q^{1-N}, qB, q^{-2k}DA \mid q, q} \right). \]

4 Conclusions

Here we have provided a detailed study of two kind of Racah \( q \)-polynomials on the lattice \( x(s) = [s]q[s + 1] \), and also their comparative analysis with the Racah coefficients or \( 6j \)-symbols of the quantum algebra \( U_q(su(2)) \).

To conclude this paper we will briefly discuss the relation of these \( q \)-Racah polynomials with the representation theory of the quantum algebra \( U_q(su(3)) \). In [20] §5.5.3 it was shown that the transformation between two different bases \((\lambda, \mu)\) of the irreducible representation of the classical (not \( q \)) algebra \( su(3) \) corresponding to the reductions \( su(3) \supset su(2) \times u(1) \) and \( su(3) \supset u(1) \times su(2) \) of the \( su(3) \) algebra in two different subalgebras \( su(2) \) is given in terms of the Weyl coefficients that are, up to a sign (phase), the Racah coefficients of the algebra \( su(2) \). The same statement can be done in the case of the quantum algebra \( su_q(3) \) [18]: The Weyl coefficients of the transformation between two bases of the irreducible representation \((\lambda, \mu)\) corresponding to the reductions \( su_q(3) \supset su_q(2) \times u_q(1) \) and \( su_q(3) \supset u_q(1) \times su_q(2) \) of the quantum algebra \( su_q(3) \) in two different quantum subalgebras \( su_q(2) \) coincide (up to a sign) with the \( q \)-Racah coefficients of the \( su_q(2) \).

In fact, the Weyl coefficients satisfy certain difference equations that are equivalent to the differentiation formulas for the \( q \)-Racah polynomials \( u_{\alpha}^{(q)}(x(s), a, b)_q \) and \( \tilde{u}_{\alpha}^{(q)}(x(s), a, b)_q \) so, following the idea in [20] §5.5.3 we can assure that the main properties of the \( q \)-Racah polynomials are closely related with the representations of the quantum algebra \( U_q(su(3)) \). Finally, let us point out that the same assertion can be done but with the non-compact quantum algebra \( U_q(su(2,1)) \). This will be carefully done in a forthcoming paper.

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