Classical Lagrangians for Momentum Dependent Lorentz Violation

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Certain momentum-dependent terms in the fermion sector of the Lorentz-violating Standard Model Extension (SME) yield solvable classical lagrangians of a type not mentioned in the literature. These cases yield new relatively simple examples of Finsler and pseudo-Finsler structures. One of the cases involves antisymmetric $d$-type terms and yields a new example of a relatively simple covariant lagrangian.
I. INTRODUCTION

The Standard Model Extension (SME) is an effective field theory that has been formulated to include all possible Lorentz-violating background couplings to Standard Model fields [1]. The SME framework exhibits many of the properties of standard quantum field theories including gauge invariance, energy-momentum conservation, causality and stability (in concordant frames)[2], observer Lorentz invariance and hermiticity. In addition, numerous renormalizability properties of the theory have been established [3, 4, 5], etc, and various implications for particle theory, gravity [6] (Lorentz violation provides an alternative means of generating theories of gravity [7]) and cosmology (see [8] for a study of the relationship of Lorentz violation to cosmic microwave background data) have been discussed. There are a number of SME limits which yield exactly solvable dispersion relations [9]. Extensive calculations using the SME have led to numerous experiments which have been performed to bound the LV effects predicted by the theory. These experiments involve numerous aspects of the SM (bounds associated to electrons, photons, neutrinos, and hadrons, etc). An exhaustive list and a brief discussion of relevant experimental results are contained in a well-maintained set of data tables [10] (see also [11]).

A geometric framework in which the classical limit of the SME can be naturally formulated has recently been proposed. This framework, Finsler geometry, is a natural extension of Riemannian geometry (see section 2 for a brief introduction). Effectively employed by Randers [12] to study dynamics of charged particles coupled to a fixed background field, Finsler geometry has been used to study a number of special limits of the SME where certain background fields are allowed to couple to standard fields [13], [14]. The associated SME lagrangians lead to Finsler structures which do not, in general, define Randers-type geometries.

The recent announcement of the OPERA group [15], if confirmed, point towards the necessity of reworking the fundamental assumptions underlying the construction of at least the neutrino sector of the Standard Model. Of particular interest is the assumption of Lorentz invariance and the possible observable signatures which might be expected should Lorentz symmetry be broken. Reconciling the modified dispersion
relatable with other observed physics has proven challenging due to apparent inconsistencies including radiative processes [16] and effects on pion decay properties [17], and generic problems with single parameter modifications [18]. While it is unclear how to resolve these inconsistencies at present, it seems that some modification of the dispersion relation for neutrinos will be unavoidable if the OPERA result is verified. Attempts at evading these conventional problems with radiative decays using Finsler space have recently been proposed [19].

The purpose of this note is to investigate new limits of the SME in the context of Finsler geometry. Using a factorization of the dispersion relation [9], we study limits of the SME which involve momentum dependent couplings. These limits yield classical lagrangians and exactly solvable models, new to the literature, which define associated pseudo-Finsler structures away from a singular set (whose structure is of independent interest). We show that Wick rotation leads to Finsler geometries which are not Randers-type geometries.

The remainder of the paper is structured as follows. In the next section we provide an introduction to the required background material and establish notation. In the third section we derive classical lagrangians for new limits of the SME and discuss the associated velocity-momentum relationship. In the fourth section we define Finsler structures naturally associated to our limits of the SME. We conclude with a discussion which includes directions for future work.

II. NOTATION, CONVENTIONS AND BACKGROUND

The SME Lorentz-violating Lagrangian for a single spin-$\frac{1}{2}$ fermion is given by

$$L = \frac{i}{2} \bar{\psi} \Gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \psi - \bar{\psi} M \psi,$$

where

$$\Gamma^\nu = \gamma^\nu + c^{\mu\nu} \gamma_\mu + d^{\mu\nu} \gamma_5 \gamma_\mu + e^\nu + if^\nu \gamma_5 + \frac{1}{2} g^{\lambda\mu\nu} \sigma_{\lambda\mu\nu},$$

$$M = m + a_\mu \gamma^\mu + b_\mu \gamma_5 \gamma^\mu + \frac{1}{2} H_{\mu\nu} \sigma^{\mu\nu}.$$
parameters to be real. In addition, the parameters \( c_{\mu\nu} \) and \( d_{\mu\nu} \) can be taken to be traceless, \( H_{\mu\nu} \) antisymmetric, and \( g^{\lambda\mu\nu} \) antisymmetric in the first two indices. The parameters \( a_{\mu}, b_{\mu}, \) and \( H_{\mu\nu} \) have the dimension of mass, while the remaining parameters are dimensionless.

As mentioned above, the SME exhibits many of the properties of standard quantum field theories including gauge invariance, energy-momentum conservation, causality, stability, observer Lorentz invariance, hermiticity and power counting renormalizability. In addition, any theory that generates the SM and exhibits spontaneous Lorentz and CPT violation contains the SME as an appropriate limit [1].

The Dirac equation associated to the Lagrangian (1) is given by

\[
(i\Gamma^{\nu}\partial_{\nu} - M)\psi = 0,
\]

or, in momentum space coordinates (using \( \psi(x) = e^{-ip\cdot x}u(p) \) for now)

\[
(\Gamma^{\nu}p_{\nu} - M)\psi = 0.
\]

The Dirac operator \((\Gamma^{\nu}p_{\nu} - M)\) is a 4 × 4 matrix with complex entries. The dispersion relation characterizes the null space of the Dirac operator and is given by

\[
\text{det}(\Gamma^{\nu}p_{\nu} - M) = 0.
\]

Expression (6) describes the zeroes of a fourth order polynomial in \( p^0 \) whose coefficients depend smoothly on the Lorentz-violating parameters and on the momentum vector. While the explicit covariant form of this dispersion relation is readily available [2], the complexity of the general expression impedes the quantitative analysis required to produce meaningful physical predictions in the presence of Lorentz violation. One method for addressing this situation involves defining special limits of the SME by constraining certain combinations of the coefficients appearing in (2) and (3) to be zero. A well-studied example of such a special limit is the momentum independent \( ab \)-limit of the SME in which all coefficients except \( a \) and \( b \) in (2) and (3) are tuned to zero. The Hamiltonian for this limit can be implicitly defined from the covariant dispersion relation

\[
((p - a)^2 - m^2 + b^2)^2 - 4(b \cdot (p - a))^2 + 4m^2b^2 = 0,
\]
which lends itself to facile analysis. Calculating the implicit derivative \( u^i = -u^0 \frac{\partial u^0}{\partial p_i} \) and combining the resulting three equations and the dispersion relation into a single equation for \( L = -u \cdot p \) yields an octic polynomial in \( L \) that factors and yields directly to the remarkably simple classical particle lagrangian [14]

\[
L_{ab} = -m \sqrt{u^2} - a \cdot v \mp \sqrt{(b \cdot u)^2 - b^2 u^2}.
\]

It is interesting to note that an alternative action for this theory has recently been discovered that eliminates the need for the square roots and implements lagrange multipliers [20], but this will not be pursued in the present work.

In addition to providing for a detailed analysis of classical particle propagation properties, formula (8) suggests a framework for the classical theory within the context of Finsler geometry.

Let \( M \) be a \( C^\infty \) manifold with tangent bundle \( TM \), a Finsler structure for \( M \) is a function \( F : TM \rightarrow [0, \infty) \) satisfying

1. \( F \) is \( C^\infty \) away from the zero section of \( TM \).
2. \( F(x, \lambda u) = \lambda F(x, u) \) for all \( \lambda > 0 \).
3. The Hessian

\[
g_{ij} = \left( \frac{1}{2} F^2 \right)_{u_i u_j},
\]

where the subscripts indicate conventional differentiation, is positive definite at every point of \( TM \setminus 0 \).

As an example, to study the dynamics of relativistic electrons in a background magnetic field, Randers [12] introduced the Finsler structure

\[
F(x, u) = \sqrt{u^2} + A_i(x)u^i,
\]

where \( A \) is a magnetic vector potential. In more generality, if \( M \) is a \( C^\infty \) manifold with Riemannian structure \( r_{ij} \) and \( A \) is a one-form on \( M \), then

\[
F(x, u) = \sqrt{r_{ij}(x)u^i u^j} + A_i(x)u^i,
\]
defines a Finsler structure on $M$ which is called Randers. Randers structures are
classified by the Matsumoto torsion, an invariant constructed using derivatives of the
Finsler structure (for an introduction to Finsler geometry, see [21] or [22]).

The formal similarity between electrons moving in a background field and parti-
cles coupling to tensors which break Lorentz symmetry suggests that Finsler spaces
may provide a geometric framework for the SME. Indeed, a connection between the
modified dispersion relations arising in the SME and Finsler geometry goes back at
least to Bogoslovsky [23], [24]. Kostelecky has developed these ideas for the $ab$-limit
of the SME. More precisely, in [13] the classical lagrangian (8) is used to construct a
Finsler structure which is investigated in detail. Fixing a Riemannian metric $r_{ij}$ on
a background spacetime manifold $M$, let

$$
F_{ab} = \sqrt{y^2 + a \cdot y \pm \sqrt{b^2 y^2 - (b \cdot y)^2}},
$$

(12)

where the dot products are taken using $r_{ij}$, and $y$ is the velocity vector in $TM$. The
Finsler function $F$ is therefore $C^\infty$ and positive on $TM \setminus S$ where $S$ is comprised of the
zero section as well any other point at which $F$ vanishes. It has become customary
to refer to this space as "ab-space". Tuning $a$ to zero produces a Finsler structure on
$TM \setminus S$ which is non-Randers. Other relatively simple spaces have been constructed
similarly using $f$, $c$, and $e$, as well as certain limits of $H$.

To carry out a similar investigation for other SME limits, we begin with an inves-
tigation of the modified dispersion relation developed in [9]. To combine parameters
with similar behavior in the dispersion relation, the following definitions are imple-
mented

$$
d^i_1 = d^0^i
\quad d^i_p = d^{ij}p_j
$$

(13)

$$
H^i = H^0^i
\quad G^i = g^{0^i^j}p_j
$$

(14)

$$
h^i = 1/2\epsilon^{ijk}H^{^j^k}
\quad g^i = 1/2\epsilon^{ijk}g^{^j^k^l}p_l,
$$

(15)

and

$$
\alpha_0 = b^0 + \vec{d}_1 \cdot \vec{p}
\quad \alpha = \vec{H} - \vec{G}
\quad \vec{d}_1 = \vec{b} + \vec{d}_p + (\vec{g} - \vec{h})
\quad \vec{d}_2 = -\vec{b} - \vec{d}_p + (\vec{g} - \vec{h}).
$$

(16)
When $\vec{\delta}_1 = -\vec{\delta}_2$ and $\alpha_0 = 0$, the associated dispersion relation becomes [9]

$$p_0^2 = \vec{p}^2 + m^2 + \vec{\alpha}^2 + \vec{\delta}_2^2 \pm 2\sqrt{D_1(\vec{p})}, \quad (18)$$

where

$$D_1(\vec{p}) = (\vec{\alpha} \times \vec{p} - m\vec{\delta}_2)^2 + (\vec{\delta}_2 \cdot \vec{p})^2 + (\vec{\alpha} \cdot \vec{\delta}_2)^2, \quad (19)$$

is a non-negative quantity.

When $\vec{\delta}_1 - \vec{\delta}_2 = 0$ and $\vec{\alpha} = 0$ the associated dispersion relation becomes

$$p_0^2 = \vec{p}^2 + m^2 + \alpha_0^2 + \vec{\delta}_2^2 \pm 2\sqrt{D_2(\vec{p})}, \quad (20)$$

where

$$D_2(\vec{p}) = (\vec{\delta}_2 \times \vec{p})^2 + (\alpha_0 \vec{p} - m\vec{\delta}_2)^2, \quad (21)$$

and $D_2(\vec{p}) \geq 0$ as in the first case.

Note that in both of these special cases, the dispersion relation is symmetric under $p_0 \to -p_0$ indicating that positive and (reinterpreted) negative energy states are degenerate. In addition, for a fixed value of $p_0$, the set of solutions for $\vec{p}$ forms a deformed sphere with two sheets where the radius as a function of angle is determined by the relevant factor, $D_1(\vec{p})$, or $D_2(\vec{p})$. This simple geometric interpretation works well provided that the Lorentz-violation parameters are small relative to the momentum and mass involved. Special degeneracies may arise when the Lorentz-violating parameters become comparable to the size of the momentum or mass involved. In what follows, the variety along which the deformed spheres intersect defines a singular set along which special care must be taken when performing the associated analysis.

### III. LEGENDRE TRANSFORMATIONS

Recall, the Legendre transform is a natural map from $TM$ to $T^*M$ which maps the Lagrangian, $L$, of a classical system to its corresponding Hamiltonian, $H$. The inverse of the Legendre transformation (when it is defined) can be computed by solving

$$L = \vec{p} \cdot \vec{v} - H. \quad (22)$$
Starting with the modified dispersion relations defined by (20) and (21), we consider the $\vec{d}$-limit of the SME defined by allowing only $d^0 \neq 0$. The Hamiltonian for the SME $\vec{d}$-limit can be expressed in the simple form

$$H^2 = m^2 + \vec{p}^2 (1 \pm |\vec{d} \cdot \hat{p}|)^2.$$  \hspace{1cm} (23)

Note that $\sqrt{H^2 - m^2}$ is a homogeneous function of degree one in $\vec{p}$. This implies that

$$p^i \frac{\partial}{\partial p^i} (H^2 - m^2)^{1/2} = (H^2 - m^2)^{1/2}.  \hspace{1cm} (24)$$

Defining $v^i = \partial H / \partial p^i$ as usual yields the simple relation

$$LH = -m^2.$$  \hspace{1cm} (25)

It is interesting to note that equation (25) holds whenever $\sqrt{H^2 - m^2}$ happens to be a homogeneous function of the momentum. For example, choosing the spatial components $c^{ij}$ as the only nonvanishing coefficients produces an example that satisfies the above relation. Elimination of the momentum variables in terms of $\vec{v}$ yields a quadratic equation for the quantity

$$C_{d\pm} = \frac{H^2}{H^2 - m^2} = \left[ \sqrt{1 - (\hat{v} \cdot \vec{d})^2} \pm |\hat{v} \cdot \vec{d}| \right]^2.$$  \hspace{1cm} (26)

The classical Lagrangian then takes the form

$$L_{d\pm} = -m \sqrt{1 - \frac{\vec{v}^2}{C_{d\pm}}}.$$  \hspace{1cm} (27)

Another limit that produces a very similar dispersion relation is the case $g^{0ij} \neq 0$ with $g$ antisymmetric in the last two indices. Defining $g^{0ij} = \epsilon^{ijk} g^k$ allows the Hamiltonian to be expressed as

$$H = m^2 + \vec{p}^2 (1 \pm |\vec{g} \times \hat{p}|)^2.$$  \hspace{1cm} (28)

Calculations similar to those used to establish the $\vec{d}$-limit of the SME yield

$$C_{g\pm} = \frac{H^2}{H^2 - m^2} = \left[ \sqrt{1 - (\hat{v} \cdot \vec{g})^2} \pm |\hat{v} \times \vec{g}| \right]^2,$$  \hspace{1cm} (29)
with corresponding classical lagrangian

\[ L_{g^\pm} = -m \sqrt{1 - \frac{\vec{v}^2}{C_{g^\pm}}}. \]  

(30)

Note that this lagrangian is in some sense dual to the previous case as the roles of dot and cross products are simply interchanged.

IV. FINSLER SPACES

Conversion to Euclidean space converts the lagrangians to homogeneous, positive definite functions that can be used to define new Finsler geometries. One way to implement the transition is to use the replacement

\[ v^j \rightarrow \frac{u^j}{w^0}, \]  

(31)

for the three spatial velocity components, set the mass to unity, and implement an appropriate Wick rotation to yield the parametrization invariant Finsler structure

\[ F = \sqrt{(u^0)^2 + \frac{\vec{u}^2}{C_{\pm}(\vec{u})}}, \]  

(32)

where \( C_{\pm}(\vec{u}) \) is one of the \( C \)-functions mentioned in the previous section. For example, the \( \vec{g} \)-term yields

\[ C_{\pm}(\vec{u}) = \left[ \sqrt{1 - (\vec{u} \cdot \vec{g})^2} \pm |\vec{u} \times \vec{g}| \right]^2. \]  

(33)

Note that this function does not vanish provided \( \vec{g}^2 \neq 1 \). Restricting \( \vec{g}^2 < 1 \) (or \( \vec{g}^2 > 1 \)) suffices to maintain the positive-definite nature of \( F \) for all values of non-zero velocity generating a globally positive definite structure.

The Finsler metric is computed using the formula

\[ g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial w^i \partial w^j}. \]  

(34)

The zero-components of \( g \) are \( g_{00} = 1 \) and \( g_{0i} = 0 \), while the spatial components take the form

\[ g_{ij} = \frac{1}{C_{\pm}D} \left[ (D \mp (\vec{u} \cdot \vec{g})^2)\delta_{ij} \pm g_{i}g_{j} \right] \pm \frac{(\vec{u} \cdot \vec{g})^2}{D^3} g_{i} \cdot g_{j}. \]  

(35)
where \( D = |\hat{u} \times \vec{g}| \sqrt{1 - (\hat{u} \cdot \vec{g})^2} \) and \( g^\perp = g_i - (\hat{u} \cdot \vec{g}) \hat{u}_i \) is the component of \( \vec{g} \) perpendicular to \( \vec{u} \). Note that the metric is homogeneous of degree zero in \( \vec{u} \). Generalization of this case to higher numbers of spatial dimensions is straightforward as the cross product magnitude may be expressed in terms of dot products using

\[
(\vec{u} \times \vec{g})^2 = \vec{u}^2 \vec{g}^2 - (\vec{u} \cdot \vec{g})^2.
\] (36)

The metric for either sheet (plus or minus solutions) exhibits a singularity along lines \( \hat{u} \parallel \vec{r} \) due to the dependence of \( D \) in the denominator. This is due to the fact that \( F \) has a cusp at these lines and neither sheet is separately differentiable there. These are also precisely the points where the two sheets are degenerate. One simple approach is to delete these lines from the manifold and define the Finsler structure on \( TM \setminus S \), where \( S \) is the set of these degenerate points.

The Cartan torsion, defined by \( C_{ijk} = \frac{1}{2} \partial g_{ij}/\partial u^k \), takes the form

\[
C_{ijk} = \pm \frac{\hat{u} \cdot \vec{g}}{2|\vec{u}|D^3} \left[ \frac{3}{D^2}(\vec{g}^2 - (\hat{u} \cdot \vec{g})^4) g_i^\perp g_j^\perp g_k^\perp - (\hat{u} \cdot \vec{g})^2 w_{ijk} \right],
\] (37)

where \( w_{ijk} = \sum_{(ijk)} (\delta_{ij} g_k^\perp - \hat{u}_i \hat{u}_j g_k^\perp) \), and \((ijk)\) indicates a sum over cyclic permutations of the indices. The mean Cartan torsion \( I_k = g^{ij} C_{ijk} \) can be found by contracting the Cartan torsion with the inverse metric

\[
g^{ij} = C_{\pm} D \left[ \frac{1}{D \mp (\hat{u} \cdot \vec{g})^2} (\vec{g}^2 - \hat{u}^i \hat{u}^j + \vec{g}^i \vec{g}^j) + \frac{D}{D \pm \vec{g}^2} U^{ij} \right],
\] (38)

where

\[
U^{ij} = D \vec{g}^i \vec{g}^j \mp \hat{u} \cdot \vec{g} (g_i^\perp \hat{u}^j + g_j^\perp \hat{u}^i) + \left( D \mp (\hat{u} \cdot \vec{g})^2 \pm (1 + \frac{(\hat{u} \cdot \vec{g})^2 C_{\pm}^2}{D^2}) \vec{g}^2 \right) \hat{u}^i \hat{u}^j.
\] (39)

The Matsumoto torsion \( M_{jkl} = C_{jkl} - \frac{1}{(n+1)} \sum_{(jkl)} I_j h_{kl} \) can then be constructed using the above quantities together with an angular metric defined by \( h_{jk} = g_{jk} - F_{uj} F_{uk} \), where the subscripts again indicate differentiation. The general expression is not particularly illuminating, however, it can be shown that it does not generally vanish which proves (via the Matsumoto-Hojo theorem [25]) that this space is not isomorphic to a Randers space.
Taking $\vec{g}$ as a constant background field leads to zero curvature and Christoffel symbols and not much of interest as far as the geometry goes. The above computation can be generalized in two natural ways to generate a more interesting geometry. First, it is possible to replace the constant background field by $\vec{g}(x)$, a vector field depending on the location in $M$. The second possibility is to generalize the inner product to include a general Riemannian metric on the spatial part of the manifold. Either of these generalizations will produce non-trivial Christoffel symbols and curvature. These generalizations are more naturally handled using the covariant approach discussed in the next section.

V. COVARIANT APPROACH

A covariant expression for the lagrangian can be generated if one starts with a covariant set of nonvanishing lorentz-violating coefficients. To obtain a solvable case, it is useful to start with a covariant form for $d^{\mu\nu}$ with the special restriction of antisymmetry in the two indices. The dispersion relation can then be put into the form

$$ (p^2 - m^2 - B)^2 - 4m^2 B = 0, \quad (40) $$

where $B = d_{\mu\nu}d^\alpha_{\nu\rho}p^\rho p^\nu \equiv (d^2)_{\mu\nu}p^\mu p^\nu$ depends on the (symmetric) square of the background tensor. The derivatives with respect to $p^i$ can be taken implicitly leading to a vector equation relating the momentum and the velocity. This equation can be converted into three different covariant equations by taking the dot product with $p^\mu$, $u^\mu$, $(d^2 p)^\mu$ and $(d^2 u)^\mu$ and taking appropriate linear combinations of the resulting equations. The algebra is a bit involved, but in the end it produces an fourth-order polynomial for the square of the lagrangian when appropriate zero sets are chosen. Borrowing an observation of [14] regarding antisymmetric tensors,

$$ (d^4)_{\mu\nu} = Y^2 \eta_{\mu\nu} - 2X(d^2)_{\mu\nu}, \quad (41) $$

is used to reduce higher powers of the antisymmetric tensor. In this expression, $X = d_{\mu\nu}d^{\mu\nu}/4$, and $Y = d_{\mu\nu}\tilde{d}^{\mu\nu}/4$, with $\tilde{d}$ denoting the conventional dual of $d$. Closed form solutions for the above procedure exist, but their structure is very complicated and not particularly illuminating, therefore, only a special case is considered here.
The polynomial for the lagrangian factors when the covariant quantity $Y = d_{\mu\nu}\tilde{d}^{\mu\nu}/4$ vanishes, where $\tilde{d}^{\mu\nu}$ is the standard dual. The appropriate zero set then reduces to a quadratic equation in $L^2$ with particle lagrangian solutions

$$L_d = -\frac{m}{1 - 2X} \left[ \sqrt{u^2(1 - 2X)} + D \pm \sqrt{D} \right],$$

where $D = (d^2)_{\mu\nu}u^\mu u^\nu$ and $X = d_{\mu\nu}d^{\mu\nu}/4$ are covariant quantities constructed from the background tensor and velocity four-vector. Antiparticle solutions also exist with opposite sign for the mass. Note that $D \geq 0$ must hold for the lagrangian to be real. For example, this is the case if the time-components of $d^{\mu\nu}$ vanish and $d^{ij} = \epsilon^{ijk}d^k$. In this case a simple computation yields $D = (\vec{u} \times \vec{d})^2$ and $X = \vec{d}^2/2$. Note that this does not in fact yield the same lagrangian as equation (27) since in that case $d^0 = 0$ was imposed rather than antisymmetry. One advantage of the expression (42) is that it may be generalized to an arbitrary background spacetime metric by simply modifying the inner product that is inherent in the symbols with $\eta_{\mu\nu} \to r_{\mu\nu}(x)$. The anti-symmetric two-form can also be promoted to a position-dependent form $d_{\mu\nu}(x)$ in a natural way.

The momentum can be computed by differentiating the lagrangian yielding

$$p_\mu = \frac{m}{\sqrt{u^2(1 - 2X)} + D} \left( u_\mu \mp \frac{L}{m\sqrt{D}} (d^2)_{\mu\alpha} u^\alpha \right).$$

Note that the second term has a finite but undetermined limit as $D \to 0$. If this happens in some physical regime there can be interesting effects such as sudden jumps in the velocity as the momentum crosses through certain threshold conditions when a particular sign is chosen for $F$. This is not surprising since there are cusp points in momentum space along these directions when spin is neglected.

A Finsler structure can be defined by Wick rotation as before yielding the ”anti-symmetric d-space” finsler structure

$$F_d = \frac{1}{1 - 2X} \left( \sqrt{u^2(1 - 2X)} + D \pm \sqrt{D} \right).$$

It is interesting to note that this is not quite of simple bipartite form $F = \sqrt{u^2} \pm \sqrt{u^2 s_{jk} u^k}$, as defined in [13], but a slightly more general form allowing an additional
perturbation in the first term. This new effect is due to the additional momentum-dependence of the couplings involved. The metric can also be computed as
\[
g_{\mu\nu} = \frac{F}{\sqrt{D\sqrt{u^2(1-2X)}} + D} \left( \sqrt{D^r_{\mu\nu}} \pm F(d^2)_{\mu\nu} \right) \pm \frac{u^2}{(D(u^2(1-2X)+D))^3} X_{\mu\nu},
\]
where \( X_{\mu\nu} = D[(d^2)_{\mu\alpha}u^\alpha u_\nu + (d^2)_{\nu\alpha}u^\alpha u_\mu] - u^2(d^2)_{\mu\alpha}((d^2)_{\nu\beta}u^\alpha u^\beta). \)

VI. SUMMARY

Various limits of the SME have been constructed that lead to relatively simple Finsler structures. In this paper, we have presented new special cases involving momentum-dependent couplings that yield additional Finsler functionals with some interesting properties. Various non-covariant choices for the background fields yield solvable Legendre transformations, mainly due to the relatively simple dependence of the hamiltonian on the momentum. For these cases, the Legendre transformation may be explicitly constructed. The solvable case of antisymmetric \( d^{ij} \) suggested attempting a covariant implicit solution for this case which turned out to be successful. This example should be easily generalized to arbitrary riemanian metrics and more general antisymmetric two-forms that depend on manifold location. The full theory of connection coefficients and curvature calculations can then be applied to this new family of examples.

Momentum-dependent couplings involving the antisymmetric \( d^{\mu\nu} \)-term introduces a new generalized bipartite Finsler structure of the form \( F = \sqrt{u^2 + \delta_1} \pm \sqrt{\delta_2} \), in which both the first and second square roots are perturbed. All previously solved simple cases involving momentum independent terms have yielded \( \delta_1 = 0 \), while all previously solved momentum-dependent terms have yielded \( \delta_2 = 0 \), so this is in fact a new case worthy of future consideration. It is also likely that some clever choice of \( g^{\lambda\mu\nu} \) may also lead to a solvable structure, another case for future work.

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