Group classification applications for analysis of discrete models of flow in porous media

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Abstract. This work is about examples of applications of group classifications for continuous (differential equations) and discrete models of flow in porous media. Difference schemes and discrete dynamical systems are considered as discrete models. Group classifications have been carried out on the basis of continuous groups (Lie groups) of transformations in $n$-dimensional Euclidean spaces. The example of explicit difference scheme for Buckley-Leverett equation has been demonstrated where the correspondence to one of classes of the obtained earlier classifications of discrete dynamical system has been shown. The example of fractal capillary network on the basis of Sierpinski triangle has been demonstrated for the group classification of discrete dynamical systems. The approach of continuous symmetry absence or presence proof has been demonstrated using classifications of discrete dynamical systems. Difference schemes with continuous symmetry groups for partial differential equations of parabolic type, which correspond to the equation of gas flow in porous media and Rapoport-Leas equation (generalized Buckley-Leverett equation), are considered. Examples of difference schemes for these cases are given and related to classes of the obtained earlier classifications. Applications of continuous symmetry groups for numerical solutions generations are demonstrated for these cases.

1. Introduction

Analysis using continuous groups of symmetries proved to be a powerful instrument of analysis of different types of equations. But in many cases of interest it is difficult to find continuous symmetries and use them for further research. This work represents several approaches of how to use known group classifications for different discrete models of flow in porous media as examples.

This work presents three following applications approaches for group classifications of discrete models which are discrete dynamical systems \cite{1} and difference schemes \cite{2}:

- Applications of known group classifications of discrete dynamical systems for difference schemes.
- Proof of presence or absence of continuous symmetries for some particular discrete model of interest on the basis of known group classifications.
- Numerical solutions generations using continuous symmetries of discrete models.

The used in this work definition of continuous groups (Lie groups) of transformations in $n$-dimensional Euclidean spaces can be found in \cite{3}, \cite{4}, \cite{5}, \cite{6}. The definitions and the complete
theory of continuous symmetries for discrete dynamical systems and difference schemes used in this paper can be found in [1], [7] and [2], [8], respectively.

2. Difference schemes as discrete dynamical systems

The purpose of this section is to demonstrate an example of application for group classifications of discrete dynamical systems. Let us consider Buckley-Leverett equation [9] which describes process of two-phase flow in one-dimensional porous media without taking into account capillary forces and with constant total rate of liquid

\[
\frac{\partial S}{\partial t} + \frac{V}{\phi} \frac{\partial f}{\partial x} = 0,
\]

where \( V = V_o + V_w = \text{const} \) - constant total rate of two-phase flow, \( \phi \) - constant coefficient of porosity, \( S(t,x) \) - water saturation, \( \mu_o, \mu_w = \text{const} \) - viscosities of oil and water, respectively, \( f(S) \) - fractional flow rate which is expressed as

\[
f(S) = \frac{K_w(S)/\mu_w}{K_w(S)/\mu_w + K_o(S)/\mu_o},
\]

\( K_w(S) \) - relative permeability for water and \( K_o(S) \) - relative permeability for oil. Group analysis of this type of equations has been done in the paper [10].

For the following functions of relative permeability and fractional flow rate

\[
K_w(S) = S^2, \quad K_o(S) = (1 - S^2), \quad \mu_o = \mu_w = \mu, \quad f(S) = S^2,
\]

let us construct an example of an implicit difference scheme for six spatial cells with an uniform spatial step and with a constant time step. This difference scheme is essentially a six-dimensional discrete dynamical system and can be written as

\[
\begin{align*}
S_{1+1}^n &= S_1^n - S_1^n V \phi^{-1} 2\Delta t \Delta x^{-1} \left( S_1^n - S_2^n \right), \\
S_{2+1}^n &= S_2^n - S_2^n V \phi^{-1} 2\Delta t \Delta x^{-1} \left( S_2^n - S_3^n \right), \\
S_{3+1}^n &= S_3^n - S_3^n V \phi^{-1} 2\Delta t \Delta x^{-1} \left( S_3^n - S_4^n \right), \\
S_{4+1}^n &= S_4^n - S_4^n V \phi^{-1} 2\Delta t \Delta x^{-1} \left( S_4^n - S_5^n \right), \\
S_{5+1}^n &= S_5^n - S_5^n V \phi^{-1} 2\Delta t \Delta x^{-1} \left( S_5^n - S_6^n \right), \\
S_{6+1}^n &= S_6^n - S_6^n V \phi^{-1} 2\Delta t \Delta x^{-1} \left( S_6^n - S_1^n \right),
\end{align*}
\]

where the superscript \( n \) is responsible for the time layer, the subscript is responsible for cells numbers on the spatial mesh, \( S_1 \) and \( S_6 \) are constant boundary saturations on the left and right boundaries, respectively. This discrete dynamical system has the two-parameter group of continuous symmetry for which infinitesimal operators prolonged to discrete variables have the following form

\[
\begin{align*}
X_1 &= S_2^n \frac{\partial}{\partial S_2} + S_{2+1}^n \frac{\partial}{\partial S_{2+1}^n}, \\
X_2 &= S_5^n \frac{\partial}{\partial S_5} + S_{5+1}^n \frac{\partial}{\partial S_{5+1}^n}.
\end{align*}
\]

The left and right finite-difference derivatives used for equations 2 and 5 in (4), respectively, allow one to have this symmetry group. The discrete dynamical system (4) and the two-dimensional symmetry group represented by the operators (5) refer to the case 1 of the classification of two-dimensional discrete dynamical systems [11] taking into account generalization of the classification to multidimensional cases.
3. Proof of presence or absence of continuous symmetries

This section is intended to demonstrate applications of the obtained classifications of discrete dynamical systems to search for new symmetries or, in particular, to prove their absence what is also a useful result.

It is necessary to make \( k \) iterations of resistance values on one of edges of the initial approximation when simulating flow in fractal capillary networks (as it was shown in [12]) in order to obtain a hydraulic resistance on one of edges of \( k \)th approximation of the fractal set. Iterations are performed with help of fractional-rational functions with the following conditions:

- the degree of the numerator polynomial is one greater than the degree of the denominator polynomial;
- the polynomials in the numerator and in the denominator are homogeneous functions, i.e. the total degrees of all monomials are equal to each other.

It has been shown that the mapping from the paper [1] for the mentioned above discreet model can be transformed to an one-dimensional discrete dynamical system. The similar result can be obtained for other fractal capillary networks [13] where the mapping for the considered model can written as

\[
\begin{align*}
    x_{k+1} &= K(x_k, y_k)L(x_k, y_k)^{-1}, \\
    y_{k+1} &= M(x_k, y_k)N(x_k, y_k)^{-1},
\end{align*}
\]

(7)

where \( K, L, M, N \) are polynomials satisfying the conditions described above. The discrete dynamical system given by the mapping of the form (7) admits a group of homogeneous dilations

\[
x' = xe^\alpha, \quad y' = ye^\alpha, \quad X_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}
\]

(8)

The canonical variables for the given transformation group have the form

\[
u = \ln y,
\]

(9)

where the operator \( X_1 \) has the form of \( \frac{\partial}{\partial y} \). The mapping (7) with using of the variables change (9) is transformed to the following form where \( \tilde{K}(u) = K(u^{-1}, 1) \) and so on:

\[
\begin{align*}
    u_{k+1} &= \tilde{L}(u_k)\bar{M}(u_k)\left(\tilde{N}(u_k)\bar{N}(u_k)^{-1}\right)^{-1}, \\
    v_{k+1} &= v_k + \ln \left(\bar{M}(u_k)\bar{N}(u_k)^{-1}\right).
\end{align*}
\]

(10)

Let us next consider some infinitesimal operator

\[
X_2 = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}
\]

(11)

and assume that this operator relates to some continuous symmetry group together with the symmetry (8) for the system (7). Than let us use, for example, the class 4 from the classification of two-dimensional discrete dynamical systems which has been obtained in the paper [11]. Than one finds an operator \( X_2 \) that together with \( X_1 \) is related to the class 4. For this one writes and solves the following system of equations

\[
X_2u = \xi(x, y)(-yx^{-2}) + \eta(x, y)\cdot x^{-1} = 0, \quad X_2v = \xi(x, y)\cdot 0 + \eta(x, y)\cdot y^{-1} = y.
\]

(12)
The solution of the system (12) is

\[ \xi(x, y) = xy, \quad \eta(x, y) = 0. \]  (13)

Than one uses the system of equations from [7], [11] to check whether these continuous groups of
transformations corresponding to (8) and (12) are the symmetry group for (7) or not:

\[ K \cdot M \cdot (L \cdot N)^{-1} - xy \frac{\partial}{\partial x} \left( K \cdot L^{-1} \right) - 0 = 0, \quad 0 - xy \frac{\partial}{\partial x} \left( M \cdot N^{-1} \right) - 0 = 0. \]  (14)

It arises from the second equation that \( M/N = \text{const} \) what contradicts the conditions for the
functions of the discrete dynamical system (7). Hence, the assumption of the presence of yet another
one-parameter symmetry group is not true. Therefore, the system (7) cannot be transformed to systems
of the class 4.

4. Numerical solutions generations using continuous symmetries
Method of numerical solutions generations using transformations from continuous groups of
symmetries has been presented in [14]. Knowledge of a continuous symmetry from group
classifications allows in some cases to calculate one numerical solution for certain initial and boundary
conditions using some classical methods. Than numerical solutions for the remaining set of conditions
can be obtained by transformations using the symmetry group providing these conditions are
connected by the symmetry group. Advantages of this approach are a significant increase in the
calculation speed by several orders and the possibility of accuracy control for subsequent calculations.
The main disadvantage is that one must find continuous symmetries first for an equation together with
boundary and initial conditions of interest but the wide spectrum of known group classifications can
help to overcome these difficulties.

4.1. Example of equation of gas flow in porous media
The first example is a particular case for parabolic type of partial differential equations - the equation
of gas flow in one-dimensional porous media [15] which is written as

\[ \frac{\partial (\rho(P)\varphi(P))}{\partial P} \frac{\partial P}{\partial t} - \frac{\partial}{\partial x} \left( \frac{K(P)\rho(P)\varphi(P)}{\mu(P)} \frac{\partial P}{\partial x} \right) = 0, \]  (15)

where \( \rho(P) \) - gas density, \( \varphi(P) \) - porosity, \( K(P) \) - absolute permeability, \( \mu(P) \) - gas viscosity,
\( P(t, x) \) - pressure. This equation is similar to the corresponding heat conduction equations. Different
types of heat conduction equations are well studied from the point of view of group analysis using
continuous groups of point transformations [16]. The group classification for the differential case was
carried out for the family of equations (15) and can be found in [17].

Let us consider a concrete example of equations (15) with the equation of state of ideal gas
\( \rho = \chi P \) at a constant temperature and the Klinkenberg relation for permeability [15], [18] which is
given by the formula (16):

\[ K(P) = K_1 \left( 1 + K_2 P^{-1} \right), \]  (16)

where - \( \chi = M/RT = \text{const} \), \( K_1 \) - effective liquid permeability, \( K_2 \) - the slope ratio in coordinates
\( K(P) \) and \( P^{-1} \), also let \( \varphi = \text{const} \) and \( \mu = \text{const} \). The equation (15) can be rewritten for these
coefficients as

\[ \frac{\partial P}{\partial t} - \gamma \frac{\partial}{\partial x} \left( (P + K_2) \frac{\partial P}{\partial x} \right) = 0, \quad \gamma = \frac{K_1}{\mu \varphi}. \]  (17)
The equation (17) is related to the case of class 2 on the page 148 from the known group classification [16] with respect to the variable change \( P = P + K_2 \). This equation has the four-parameter continuous group of symmetry [16], [17] with infinitesimal operators

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad X_4 = -t \frac{\partial}{\partial t} + (P + K_2) \frac{\partial}{\partial x}.
\]

(18)

The following explicit difference scheme is used for numerical calculations

\[
\left( P_k^{n+1} - P_k^n \right) / \tau^n - \gamma h^{-2} \left( (P_{k+1}^n - P_k^n)(P_k^n + K_2) - (P_k^n - P_{k-1}^n)(P_k^n + K_2) \right) = 0,
\]

(19)

where \( \Delta \) - coefficient depending on maximum time value, \( P^n \) - average pressure for \( n \)th step.

The transformations from groups of symmetries with operators \( X_1 \) and \( X_2 \) from (18) mean translations for time and spatial variables for solutions what in turn means simply moving the mesh without any changes to the difference equations. More interesting is to consider of the operators \( X_3 \) and \( X_4 \) from (18) for which the transformation groups have the form for \( X_3 \) and \( X_4 \), respectively:

\[
\bar{t} = e^\tau t, \quad \bar{x} = e^\tau x, \quad \bar{P} = P, \quad \bar{t} = e^{-\tau t}, \quad \bar{x} = x, \quad \bar{P} = (P + K_2) e^\tau - K_2
\]

(20)

Figure 1 shows the action of the symmetry group with \( X_4 \) from (20) on the numerical solution which has been obtained by solving of the corresponding system of linear equations. The specified initial pressures of the left boundary determine the parameters for generations. Verification of accuracy by the exact solution showed a coincidence of solutions and that is why they are not shown in the figure. The exact solution is presented in [19], the used initial and boundary conditions for the difference scheme are chosen with respect to the exact solution for accuracy verification.

4.2. Example of Rapoport-Leas equation
Rapoport-Leas partial differential equation describes two-phase flow in porous media with constant total liquid flow and with taking into account of capillary forces. It is also known as generalized Buckley-Leverett equation. The general form for one-dimensional porous media [20] is

\[
\frac{\partial S}{\partial t} + \frac{K}{\phi \mu_o} \frac{\partial}{\partial x} \left( K_o f \frac{dP}{dS} \frac{\partial S}{\partial x} \right) + \frac{V}{\phi} \frac{df}{dS} \frac{\partial S}{\partial x} = 0,
\]

(21)

\( K = const \) - absolute permeability, \( P_2 = P_p - P_e \) - capillary pressure, other parameters are defined in the section 2.
The group analysis of the equation (21) has been presented in [16], [17], [21]. The equation (21) has the fairly obvious symmetry group of time and space transitions for arbitrary functions of relative permeability and capillary pressure. When these functions are specified in the form [14]

$$\frac{\partial S}{\partial t} - \alpha \frac{\partial}{\partial x} \left( S^{N-1} \frac{\partial S}{\partial x} \right) - \beta S^{N-1} \frac{\partial S}{\partial x} = 0, \quad \alpha = \frac{KP}{\mu \phi}, \quad \beta = -\frac{VN}{\phi}. \tag{23}$$

The equation (23) relates to the case of class 1 from table 3 according to the classifications from [21]. This equation has the three-parameter symmetry group which has the following infinitesimal operators and the form of the transformations, respectively:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = -t \frac{\partial}{\partial t} + \frac{S}{N-1} \frac{\partial}{\partial S}, \tag{24}$$

$$\tilde{T} = e^{-\alpha t} + a_2, \quad \tilde{x} = x + a_1, \quad \tilde{S} = e^{N-1} S, \quad a_1, a_2, a_3 \in \mathbb{R}.$$

The following implicit difference scheme is used to represent the method of numerical solutions generations

$$S_k^{n+1} = S_k^n + \alpha \Delta t \Delta x^{-2} \left( \left( S_k^n \right)^{N-1} \left( S_k^{n+1} - S_k^{n+1} \right) - \left( S_k^{n+1} - S_k^{n+1} \right) \right) + \beta \Delta t \Delta x^{-1} \left( S_k^n \right)^{N-1} \left( S_k^{n+1} - S_k^{n+1} \right), \quad \Delta t = t^{n+1} - t^n = const, \quad \Delta x = x_{k+1} - x_k = const. \tag{25}$$

Figure 2 shows the results for numerical solutions generations using $X_3$. The parameters used for the calculations are presented in [14]. The particular exact solutions for the equation (23) are from [19]. It is used below to verify the numerical calculations obtained for the chosen difference scheme. The verification of accuracy showed good convergence of numerical solutions.

![Figure 2](image-url)  
Figure. 2. Profiles of saturation in time (left) and space (right) for generated solutions of (25).

The calculation time using the generation algorithm with help of continuous symmetries is three orders lower than the calculation time using classical methods for difference schemes. It has been obtained during the numerical calculations using two examples above. This advantage allows to find solutions to many problems faster and more efficiently which require a large number of multivariate calculations for similar models: uncertainty analysis, history matching, upscaling and others.
5. Conclusions
This work is intended to demonstrate different approaches how to apply group classifications to analysis of discrete models for flow in porous media. Presented approaches of analysis of difference schemes as discrete dynamical systems, proof of absence or presence of continuous symmetries and numerical solutions generations are applied to the following models correspondently: the difference scheme for Buckley-Leverett equation, the fractal capillary network model on the basis of the Sierpinski triangle and two difference schemes for the gas flow equation and Rapoport-Leas equation.

These presented approaches can help to find applications for other group classifications of different types of equations and for different processes. First two approaches can help to find a class for particular discrete model and its equation of interest. The last one is intended to find numerical solutions by orders faster than other methods. Considered discrete models can be, for example, used for macroscale models of oil and gas reservoirs and microscale models of digital core samples.

Acknowledgments
The reported study was funded by RFBR according to the research project № 16-29-15119.

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