On the superconvergence of a hybridizable discontinuous Galerkin method for the Cahn-Hilliard equation

GANG CHEN*, DAOZHI HAN†, JOHN SINGLER‡ and YANGWEN ZHANG§

Abstract

We propose a hybridizable discontinuous Galerkin (HDG) method for solving the Cahn-Hilliard equation. The temporal discretization is based on the backward Euler method and the convex-splitting of the free energy function. We establish the superconvergence of the fully discrete scheme in both $L^\infty(L^2)$ and $L^\infty(H^{-1})$ norms. Moreover, a novel discrete Sobolev inequality is established.

Keywords — Cahn-Hilliard equation, hybridizable discontinuous Galerkin method, superconvergence

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a polygonal domain with Lipschitz boundary $\partial\Omega$, and $T$ be a positive constant. We consider the following Cahn-Hilliard equation: find $(u, \phi)$ satisfying

\begin{align}
  u_t - \Delta \phi &= 0 \quad \text{in } \Omega \times (0, T], \quad (1a) \\
  -\epsilon \Delta u + \epsilon^{-1} f(u) &= \phi \quad \text{in } \Omega \times (0, T], \quad (1b) \\
  \nabla u \cdot n &= \nabla \phi \cdot n = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (1c) \\
  u(\cdot, 0) &= u^0(\cdot) \quad \text{in } \Omega, \quad (1d)
\end{align}

where $f(u) = u^3 - u$. The Cahn-Hilliard equation is a fourth order, nonlinear parabolic equation which is originally proposed by Cahn and Hilliard [8][10].
as a phenomenological model for phase separation and coarsening in a binary alloy. Since then the Cahn-Hilliard-type equation has found applications in a variety of fields, including multiphase flow [2,31], two-phase flow in porous media [20], tumor growth [46], pattern formation [47], thin films [7] and many others. Owing to its importance many works have been devoted to the design and analysis of numerical schemes for solving the Cahn-Hilliard equation. In this article, we focus on the error analysis of a hybridizable discontinuous Galerkin (HDG) method for the Cahn-Hilliard equation.

The Cahn-Hilliard equation is traditionally solved as a system of two second order equations based on finite difference methods [11,25,27], mixed or nonconforming finite element methods [6,14,16,18,19,23] or Fourier-spectral methods [33,42,43]. A critical property of the Cahn-Hilliard equation is the decaying of energy in time. Special attention in the aforementioned works is paid to the preservation of this property so that convergence and error estimates can be readily established in energy norms.

In recent years, the discontinuous Galerkin (DG) method has become popular for solving the Cahn-Hilliard equation, owing to its flexibility in handling higher order derivative, high-order accuracy, the property of local conservation which is crucial for applications in porous medium flow and transport phenomenon, high parallelizability and ease of achieving \(hp\)-adaptivity. Applications of discontinuous Galerkin finite element method to fourth order elliptic problems have been considered by Babuška and Zlámal in [4] and by Baker in [5], and more recently by Mozolevski et al in a series of works [35–38,45]. In [21], Feng and Karakashian design and analyze a discontinuous Galerkin method of interior penalty type based on the fourth order formulation of the Cahn-Hilliard equation. Optimal error estimates in various energy norms are established, see also [22]. Kay et al propose and analyze a different DG method [32] that treats the Cahn-Hilliard equation as a system of second order equations allowing a relatively larger penalty term. A fully adaptive version of the interior penalty DG is recently constructed in [3] for the Cahn-Hilliard equation with a source where optimal \(L^2\) error bound is derived, see also [24] for solving the advective Cahn-Hilliard equation. The local discontinuous Galerkin method has also been proposed for the discretization of the Cahn-Hilliard equation by writing it as a system of four first-order equations. Dong and Shu in [13] analyzed a local DG scheme for general elliptic equations including the linearized Cahn-Hilliard equation and obtained optimal error estimate in \(L^2\). Recently, the local DG method has been employed for solving a number of Cahn-Hilliard fluid models, cf. [28,30,44].

The DG method is however often criticized for the larger amount of degrees of freedom, especially in 3D, compared to continuous Galerkin method (CG). In the seminal work [12] Cockburn et al propose a hybridizable local DG numerical method (HDG) for second order elliptic problems. In a nutshell, the HDG method maps the interior degrees of freedom in each
element with those (numerical trace) on the element boundary via a local
LDG solver, which are in turn connected by the continuity of fluxes across
inter-element boundaries (transmission condition). Hence the globally
coupled degrees of freedom are those numerical traces, resulting in a significant
reduction of the number of unknowns in classical DG methods. In addition,
the HDG method has the ability to postprocess to get higher order solutions
(superconvergence), cf. [13] where a projection-based error analysis
technique of HDG is introduced.

The HDG methods have been utilized for solving many second order
elliptic and parabolic PDEs. In this contribution, we propose a HDG al-
gorithm for solving the fourth order nonlinear Cahn-Hilliard equation. The
temporal discretization can be of either convex-splitting [17,20] or backward
Euler. The primary variables are approximated by polynomials of order $k+1$
while their gradient and numerical traces are approximated by polynomi-
als of order $k$. We establish the optimal convergence rates, i.e., $k+2$ for
the primary variables and $k+1$ for the gradients, in the natural $L^\infty(L^2)$
norm and the $L^\infty(H^{-1})$ norms, respectively. Since the numerical traces
are the only globally coupled degrees of freedom, the optimal convergence
implies superconvergence for the primary variables. HDG methods using
an enhanced space for the primary variable and reduced stabilization in the
numerical fluxes have seen applications in diffusion problems [39], in Navier-
Stokes equations [41], and more recently in linear elasticity problems [40].
To the best of our knowledge, our scheme is the first superconvergent HDG
method without post-processing for the Cahn-Hilliard equation. Moreover,
we establish a novel discrete Sobolev inequality (cf. Theorem 3.15) which is
a useful tool in the numerical analysis of nonlinear problems.

The rest of the article is organized as follows. We introduce the fully
discrete HDG formulation of the Cahn-Hilliard equation in Section 2. We
gather some useful estimates in Section 3. The well-posedness of the scheme
is established in Section 4. The error estimate is furnished in Section 5.
Some supplemental estimates are provided in the appendix Section 6. We
report numerical results confirming the theoretical analysis in Section 7.

2 Fully Discretization of the Cahn-Hilliard Equa-
ton by the HDG Formulation

To introduce the fully discretization of the Cahn-Hilliard equation by the
HDG formulation. We first set some notation.

Let $T_h$ be a collection of disjoint simplexes $K$ that partition $\Omega$ and let
$\mathcal{E}_h$ be the set $\{\partial K : K \in T_h\} = \mathcal{E}_h^\circ \cup \mathcal{E}_h^\partial$ with $e \in \mathcal{E}_h^\circ$ being the interior
interelement face, and $e \in \mathcal{E}_h^\partial$ being the boundary face if the Lebesgue
measure of $e = \partial K \cap \partial \Omega$ is non-zero. We assume that $T_h$ is shape-regular
and quasi-uniform. Moreover, we set

\[(w, v)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (w, v)_K = \sum_{K \in \mathcal{T}_h} \int_K w v,\]

\[(\zeta, \rho)_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \zeta \rho.\]

For any integer \(k \geq 0\), let \(\mathcal{P}^k(K)\) denote the set of polynomials of degree at most \(k\) on the element \(K\). We introduce the following discontinuous finite element spaces:

\[V_h := \{ v_h \in [L^2(\Omega)]^d : v_h|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h\},\]

\[W_h := \{ v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h\},\]

\[W_0 := \{ v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h\},\]

\[M_h := \{ \mu_h \in L^2(\mathcal{E}_h) : \mu_h|_E \in \mathcal{P}^k(\mathcal{E}_h), \forall E \in \mathcal{E}_h\},\]

where \(L^2_0(\Omega)\) is the subspace of \(L^2(\Omega)\) of mean zero.

Since the HDG methods are based on mixed formulation, then we rewrite the system into a first order system by setting \(p + \nabla \phi = 0\) and \(q + \nabla u = 0\) in (1). The mixed formula of (1) is

\[
\begin{align*}
\mathbf{p} + \nabla \phi &= 0 & \text{in } \Omega \times (0, T], & (2a) \\
u_t + \nabla \cdot \mathbf{p} &= 0 & \text{in } \Omega \times (0, T], & (2b) \\
q + \nabla u &= 0 & \text{in } \Omega \times (0, T], & (2c) \\
\epsilon \nabla \cdot q + \epsilon^{-1} f(u) &= \phi & \text{in } \Omega \times (0, T], & (2d) \\
\mathbf{p} \cdot \mathbf{n} &= q \cdot \mathbf{n} &= 0 & \text{on } \partial \Omega \times (0, T], & (2e) \\
u(\cdot, 0) &= u^0(\cdot) & \text{in } \Omega. & (2f)
\end{align*}
\]

Now, we introduce the fully discrete HDG formulation of the Cahn-Hilliard equation.

For a fixed integer \(N\), let \(0 = t_0 < t_1 < \cdots < t_N\) be a partition of \([0, T]\), to simplify the presentation, we suppose the partition is uniform, i.e., \(T/N = \Delta t = t_n - t_{n-1}, n = 1, 2 \cdots N\). To approximate the solution of the mixed weak form (2), the HDG method seeks \((\mathbf{p}_h^n, \phi_h^n, \tilde{\phi}_h^n) \in V_h \times W_h \times M_h\) satisfying

\[
\begin{align*}
(\mathbf{p}_h^n, \mathbf{r}_1)_{\mathcal{T}_h} - (\phi_h^n, \nabla \cdot \mathbf{r}_1)_{\mathcal{T}_h} + (\tilde{\phi}_h^n, \mathbf{r}_1 \cdot \mathbf{n})_{\partial \mathcal{T}_h} &= 0, & (3a) \\
(\partial_t^+ u_h^n, w_1)_{\mathcal{T}_h} - (\mathbf{p}_h^n, \nabla w_1)_{\mathcal{T}_h} + (\tilde{\mathbf{p}}_h^n \cdot \mathbf{n}, w_1)_{\partial \mathcal{T}_h} &= 0, & (3b) \\
(\tilde{\mathbf{p}}_h^n \cdot \mathbf{n}, \mu_1)_{\partial \mathcal{T}_h} &= 0, & (3c)
\end{align*}
\]

for all \((\mathbf{r}_1, w_1, \mu_1) \in V_h \times W_h \times M_h\); and \((\mathbf{q}_h^n, u_h^n, \tilde{\mathbf{q}}_h^n) \in V_h \times W_h \times M_h\) such that

\[
(\mathbf{q}_h^n, \mathbf{r}_2)_{\mathcal{T}_h} - (u_h^n, \nabla \cdot \mathbf{r}_2)_{\mathcal{T}_h} + (\tilde{\mathbf{q}}_h^n \cdot \mathbf{n}, \mathbf{r}_2)_{\partial \mathcal{T}_h} = 0, & (3d)
\]
for all \((r_2, w_2, \mu_2) \in V_h \times W_h \times M_h\).

Here \(\partial^+_\tau u^n_h = (u^n_h - u^{n-1}_h)/\Delta t\), \(f^n(u^n_h) = (u^n_h)^3 - u^n_h\) for the fully implicit scheme and \(f^n(u^n_h) = (u^n_h)^3 - u^{n-1}_h\) for the energy-splitting scheme, and the numerical fluxes on \(\partial T_h\) are defined as

\[
\begin{align*}
\hat{p}^n_h \cdot n &= p^n_h \cdot n + h^{-1}_K (\Pi^0_K \phi^h_n - \hat{\phi}^n_h), \\
\hat{q}^n_h \cdot n &= q^n_h \cdot n + h^{-1}_K (\Pi^0_K u^n_h - \hat{u}^n_h),
\end{align*}
\]

(3g, 3h)

where \(\Pi^0_K\) is a \(L^2\) projection onto \(M_h\) such that

\[
\langle \Pi^0_K u, \mu_h \rangle_E = \langle u, \mu_h \rangle_E, \quad \forall \mu_h \in M_h \text{ and } E \in E_h.
\]

For the convenience of the analysis, we define \(A : [V_h \times W_h \times M_h]^2 \to \mathbb{R}\)

\[
A(q_h, u_h, u_h; r_h, w_h, \mu_h)
= \langle q_h, r_h \rangle_{T_h} - \langle u_h, \vartheta \cdot r_h \rangle_{T_h} + \langle \hat{u}_h, r_h \cdot n \rangle_{\partial T_h}
+ \langle \nabla \cdot q_h, w_h \rangle_{T_h} - \langle q_h \cdot n, \mu_h \rangle_{\partial T_h}
+ \langle h^{-1}_K (\Pi^0_K u_h - \hat{u}_h), \Pi^0_K w_h - \mu_h \rangle_{\partial T_h}
\]

(4)

for all \((q_h, u_h, \hat{u}_h), (r_h, w_h, \mu_h) \in V_h \times W_h \times M_h\).

Using the definition (11), we rewrite the HDG finite element formulation of the Cahn-Hilliard equation (39) as follows: for \(n = 1, 2, \cdots, N\), find \((p^n_h, \phi^n_h, \phi^n_h), (q^n_h, u^n_h, \hat{u}^n_h) \in V_h \times W_h \times M_h\) such that

\[
\begin{align*}
(\partial^+_\tau u^n_h, w_1)_{T_h} + A(p^n_h, \phi^n_h, \phi^n_h; r_1, w_1, \mu_1) &= 0, \\
(\varepsilon^{-1} f^n(u^n_h), w_2)_{T_h} + \varepsilon A(q^n_h, u^n_h, \hat{u}^n_h; r_2, w_2, \mu_2) - (\phi^n_h, w_2)_{T_h} &= 0, \\
(u^n_h, w_3)_{T_h} - (u^0, w_3)_{T_h} &= 0
\end{align*}
\]

(5a, 5b, 5c)

for all \((r_1, w_1, \mu_1), (r_2, w_2, \mu_2) \in V_h \times W_h \times M_h\) and \(w_3 \in W_h\).

3 Preliminary material

We first define the standard \(L^2\) projections \(\Pi^o_K : [L^2(\Omega)]^d \to V_h\) and \(\Pi^o_{k+1} : L^2(\Omega) \to W_h\), which satisfy

\[
\begin{align*}
(\Pi^o_K q, r_h)_K &= (q, r_h)_K, \quad \forall r_h \in [P_k(K)]^d, \\
(\Pi^o_{k+1} u, w_h)_K &= (u, w_h)_K, \quad \forall w_h \in P_{k+1}(K).
\end{align*}
\]

(6)

In the analysis, we use the following classical results:

\[
\begin{align*}
\|q - \Pi^o_K q\|_{T_h} &\leq C h^{k+1} \|q\|_{k+1, \Omega}, \quad \|u - \Pi^o_{k+1} u\|_{T_h} \leq C h^{k+2} \|u\|_{k+2, \Omega},
\end{align*}
\]

(7a)
Lemma 3.1. For any \((r_h, w_h, \mu_h) \in V_h \times W_h \times M_h\), we have

\[
\mathcal{A}(r_h, w_h, \mu_h, r_h, w_h, \mu_h) = \|r_h\|_{\mathcal{T}_h}^2 + \|h^{-1/2} (\Pi^q_k w_h - \mu_h)\|_{\mathcal{T}_h}^2.
\]

The proof of the Lemma 3.1 and the following Lemma 3.2 are very straightforward, hence we omit them.

Lemma 3.2. For all \((q_h, u_h, \hat{u}_h), (p_h, \phi_h, \hat{\phi}_h) \in V_h \times W_h \times M_h\), the operator \(\mathcal{A}\) has the following property

\[
\mathcal{A}(q_h, u_h, \hat{u}_h; p_h, \phi_h, \hat{\phi}_h) = \mathcal{A}(p_h, \phi_h, \hat{\phi}_h; q_h, -u_h, -\hat{u}_h).
\]

Lemma 3.3 (Continuity of \(\mathcal{A}\)). For all \((q_h, u_h, \hat{u}_h), (p_h, \phi_h, \hat{\phi}_h) \in V_h \times W_h \times M_h\), we have

\[
\left| \mathcal{A}(q_h, u_h, \hat{u}_h; p_h, \phi_h, \hat{\phi}_h) \right| 
\leq C \left( \|q_h\|_{\mathcal{T}_h} + \|\nabla u_h\|_{\mathcal{T}_h} + \|h^{-1/2} (\Pi^q_k u_h - \hat{u}_h)\|_{\mathcal{T}_h} \right)
\times \left( \|p_h\|_{\mathcal{T}_h} + \|\nabla \phi_h\|_{\mathcal{T}_h} + \|h^{-1/2} (\Pi^q_k \phi_h - \hat{\phi}_h)\|_{\mathcal{T}_h} \right).
\]

Proof. By the definition of \(\mathcal{A}\) in (4) and integration by parts, one gets

\[
\left| \mathcal{A}(q_h, u_h, \hat{u}_h; p_h, \phi_h, \hat{\phi}_h) \right| 
\leq \left| (q_h, p_h)_{\mathcal{T}_h} + (\nabla u_h, p_h)_{\mathcal{T}_h} - (\Pi^q_k u_h - \hat{u}_h, p_h \cdot n)_{\mathcal{T}_h} \right|
+ \left| - (q_h, \nabla \phi_h) + (q_h \cdot n, \Pi^q_k \phi_h - \hat{\phi}_h)_{\mathcal{T}_h} \right|
+ \left| (h^{-1} (\Pi^q_k u_h - \hat{u}_h), \Pi^q_k \phi_h - \hat{\phi}_h)_{\mathcal{T}_h} \right|.
\]

Then the bound (8) follows from the Cauchy-Schwarz inequality and the inverse inequality (7b). This completes the proof.

Lemma 3.4. If \((q_h, u_h, \hat{u}_h) \in V_h \times W_h \times M_h\) satisfies

\[
\mathcal{A}(q_h, u_h, \hat{u}_h; r_h, 0, 0) = 0
\]

for all \(r_h \in V_h\), then the following inequality holds

\[
\|\nabla u_h\|_{\mathcal{T}_h} + \|h^{-1/2} (u_h - \hat{u}_h)\|_{\mathcal{T}_h} \leq C \left( \|q_h\|_{\mathcal{T}_h} + \|h^{-1/2} (\Pi^q_k u_h - \hat{u}_h)\|_{\mathcal{T}_h} \right).
\]
Proof. By the definition of $A$ in \((11)\), let $r_h = \nabla u_h$ in \((11)\) and integration by parts to get

$$(q_h, \nabla u_h)_T + (\nabla u_h, \nabla u_h)_T + \langle \tilde{u}_h - u_h, \nabla u_h \cdot n \rangle_{\partial T_h} = 0.$$ 

Note that

$$(\Pi^K_0 u_h - u_h, \nabla u_h \cdot n)_{\partial T_h} = 0.$$ 

It follows from the element-wise Cauchy-Schwarz inequality and the inverse inequality \((11)\) that

$$\|\nabla u_h\|_{T_h} \leq C \left( \|q_h\|_{T_h} + \|h^{-1/2}(\Pi^K_0 u_h - \tilde{u}_h)\|_{\partial T_h} \right).$$ 

The triangle inequality gives

$$\|h^{-1/2}(u_h - \tilde{u}_h)\|_{\partial T_h} \leq \|h^{-1/2}(\Pi^K_0 u_h - \tilde{u}_h)\|_{\partial T_h} + \|h^{-1/2}(u_h - \Pi^K_0 u_h)\|_{\partial T_h} \leq C \left( \|h^{-1/2}(\Pi^K_0 u_h - \tilde{u}_h)\|_{\partial T_h} + \|\nabla u_h\|_{T_h} \right).$$ 

The desired inequality now follows by combing the last two inequalities. \(\square\)

**Definition 3.5.** For all $u_h \in W_h$ and for all $(r_h, w_h, \mu_h) \in V_h \times W_h \times M_h$, we define $(\Pi_1 V u_h, \Pi_1 W u_h, \Pi M u_h) \in V_h \times W_h \times M_h$ as following

$$A(\Pi_1 V u_h, \Pi_1 W u_h, \Pi M u_h; r_h, w_h, \mu_h) = (u_h, w_h)_{T_h}. \quad (10)$$

**Lemma 3.6 (Discrete LBB Condition of $A$).** For all $(q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h$, we have

$$\sup_{\Omega \neq (p_h, \phi_h, \psi_h) \in V_h \times W_h \times M_h} \frac{A(q_h, u_h, \tilde{u}_h; p_h, \phi_h, \psi_h)}{\|p_h\|_{T_h} + \|\nabla \phi_h\|_{T_h} + \|h^{-1/2}(\Pi^K_0 \phi_h - \psi_h)\|_{\partial T_h}} \geq C \left( \|q_h\|_{T_h} + \|\nabla u_h\|_{T_h} + \|h^{-1/2}(\Pi^K_0 u_h - \tilde{u}_h)\|_{\partial T_h} \right), \quad (11a)$$

$$\sup_{\Omega \neq (p_h, \phi_h, \psi_h) \in V_h \times W_h \times M_h} \frac{A(p_h, \phi_h, \psi_h; q_h, u_h, \tilde{u}_h)}{\|p_h\|_{T_h} + \|\nabla \phi_h\|_{T_h} + \|h^{-1/2}(\Pi^K_0 \phi_h - \psi_h)\|_{\partial T_h}} \geq C \left( \|q_h\|_{T_h} + \|\nabla u_h\|_{T_h} + \|h^{-1/2}(\Pi^K_0 u_h - \tilde{u}_h)\|_{\partial T_h} \right). \quad (11b)$$

**Proof.** Since \((11a)\) and \((11b)\) are similar, we only give a proof for \((11a)\). For all $(q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h$, we take $(p_h, \phi_h, \psi_h) = (q_h + \alpha \nabla u_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h$ to get

$$A(q_h, u_h, \tilde{u}_h; p_h, \phi_h, \psi_h) = (q_h, q_h + \alpha \nabla u_h)_{T_h} - (u_h, \nabla \cdot (q_h + \alpha \nabla u_h))_{T_h} + \langle \tilde{u}_h, (q_h + \alpha \nabla u_h) \cdot n \rangle_{\partial T_h} + (\nabla \cdot q_h, u_h)_{T_h} - (q_h \cdot n, \tilde{u}_h)_{\partial T_h}.$$
By choosing $\alpha$ such that $1 - C\alpha \leq \frac{1}{2}$ and $\alpha > 0$, we get
\[
\mathcal{A}(q_h, \widehat{u}_h; p_h, \widehat{\phi}_h) \geq C_1 \left( \|q_h\|_{T_h}^2 + \|h_K^{-1/2}(\Pi_k^0 u_h - \widehat{u}_h)\|_{\partial T_h}^2 + \|
abla u_h\|_{T_h}^2 \right).
\]
Furthermore, we have
\[
\|p_h\|_{T_h}^2 + \|h_K^{-1/2}(\Pi_k^0 \phi_h - \widehat{\phi}_h)\|_{\partial T_h}^2 + \|
abla \phi_h\|_{T_h}^2 \leq C_2 (\|q_h\|_{T_h}^2 + \|h_K^{-1/2}(\Pi_k^0 u_h - \widehat{u}_h)\|_{\partial T_h}^2 + \|
abla u_h\|_{T_h}^2).
\]

Then (11a) follows immediately.

Thanks to the discrete LBB condition: Lemma 3.6, the projection (10) in Definition 3.5 is well defined. For all $u_h \in W_h$, we define the semi-norm
\[
\|u_h\|_{-1,h}^2 := \mathcal{A}(\Pi V u_h, \Pi W u_h, \Pi M u_h; \Pi V u_h, \Pi W u_h, \Pi M u_h).
\]

Then for all $u_h \in W_h$, by Lemma 3.1 and Definition 3.5 we have
\[
\|u_h\|_{-1,h}^2 = \|\Pi V u_h\|_{T_h}^2 + \|h_K^{-1/2}(\Pi_k^0 \Pi W u_h - \Pi M u_h)\|_{T_h}^2 = (u_h, \Pi W u_h)_{T_h}. \tag{12}
\]

Next, we show that $\| \cdot \|_{-1,h}$ is a norm on the space $W_h$.

**Lemma 3.7.** $\| \cdot \|_{-1,h}$ defines a norm on the space $W_h$.

*Proof.* Thanks to (12), one only needs to show that $\|u_h\|_{-1,h} = 0$ implies $u_h = 0$ for $u_h \in \dot{V}_h$. It follows readily from (12) that
\[
\Pi V u_h = 0, \quad \Pi_k^0 \Pi W u_h - \Pi M u_h = 0.
\]

Next, the Definition 3.5 and (11) imply that for all $(r_h, w_h) \in V_h \times W_h$ we have
\[
(u_h, w_h)_{T_h} = (\Pi W u_h, \nabla \cdot r_h)_{T_h} - (\Pi M u_h, r_h \cdot n)_{\partial T_h}.
\]

Take $r_h = 0$ and $w_h = u_h$ give $u_h = 0$. This completes the proof. \qed

The following version of the piecewise Poincaré-Friedrichs inequality [?] will be used later.
Lemma 3.8. Let $v$ be a piecewise $H^1$ function with respect to the partition $T_h$. The following Poincaré-Friedrichs inequality holds

$$
\|v\|_{T_h}^2 \leq C \left( \|\nabla v\|_{T_h}^2 + |(v, 1)_{\partial T}|^2 + \sum_{E \in \mathcal{E}_h^e} |E|^{d/(1-d)} \left( \int_E \|v\|ds \right)^2 \right),
$$

where the generic constant $C$ depends only on the regularity of the partition, and $[v]$ denotes the jump of $v$ across a side $E$.

The following HDG Poincaré inequality is then a consequence of Lemma 3.8, the Cauchy-Schwarz inequality and the triangle inequality; see details in [Lemma 5].

Lemma 3.9 (The HDG Poincaré inequality). If $u_h \in W_h$, we have

$$
\|u_h\|_{T_h}^2 \leq C \left( \|\nabla u_h\|_{T_h}^2 + \|h^{-1/2}(\Pi_k^2 u_h - \Pi M u_h)\|_{\partial T_h}^2 \right).
$$

Next, we introduce an important estimation of $\Pi_W u_h$ for $u_h \in W_h$.

Lemma 3.10. For all $u_h \in W_h$, we have

$$
\|\Pi_W u_h\|_{T_h} \leq C \left( \|\Pi_V u_h\|_{T_h} + \|h^{-1/2}(\Pi_k^2 \Pi_W u_h - \Pi M u_h)\|_{\partial T_h} \right).
$$

Proof. Take $w_h = \mu_h = 0$ in Definition 3.5 and for all $r_h \in V_h$ to get

$$
\mathcal{A}(\Pi_V u_h, \Pi_W u_h, \Pi M u_h; r_h, 0, 0) = 0.
$$

By Lemma 3.3 to have

$$
\|\nabla \Pi_W u_h\|_{T_h} \leq C \left( \|\Pi_V u_h\|_{T_h} + \|h^{-1/2}(\Pi_k^2 \Pi_W u_h - \Pi M u_h)\|_{\partial T_h} \right). \tag{13}
$$

Then the desired result is obtained by Lemma 3.9.

Some properties of the negative norm $\| \cdot \|_{-1,h}$ are summarized in the following lemma.

Lemma 3.11. If $u_h \in W_h$, $(w_h, \mu_h) \in W_h \times M_h$ and the mesh is quasi-uniform, then the following estimates hold:

$$
(u_h, w_h)_{T_h} \leq C \|u_h\|_{-1,h} \left( \|\nabla w_h\|_{T_h} + \|h^{-1/2}(\Pi_k^2 w_h - \mu_h)\|_{\partial T_h} \right), \tag{14a}
$$

$$
\|u_h\|_{T_h} \leq C h^{-1} \|u_h\|_{-1,h}, \tag{14b}
$$

$$
\|u_h\|_{-1,h} \leq C \|u_h\|_{T_h}, \tag{14c}
$$

where $h$ is the smallest diameter of the mesh and $C$ is independent of the mesh.
Proof. For any \((w_h, \mu_h) \in W_h \times M_h\) and \(u_h \in W_h\), by Definition 3.5 and (1) to get

\[
(u_h, w_h)_h = \mathcal{A}(\nabla \cdot \Pi \nabla u_h, \Pi \nabla u_h, \Pi \nabla u_h; 0, w_h, \mu_h)
= (\nabla \cdot \Pi \nabla u_h, w_h)_h - (\nabla \cdot \Pi \nabla u_h, \mu_h)_{\partial \Omega} \\
+ \langle h_{K}^{-1}(\Pi_{K}^{0} \Pi \nabla u_h - \Pi \nabla u_h), \Pi_{K}^{0} \nabla w_h - \mu_h \rangle_{\partial \Omega}.
\]

By integration by parts, an inverse inequality and (12) we have

\[
(u_h, w_h)_h \leq \|\Pi \nabla u_h\|_{\partial \Omega} \|\nabla w_h\|_{\partial \Omega} + C \|h_{K}^{-1/2}(\Pi_{K}^{0} \nabla w_h - \mu_h)\|_{\partial \Omega} \\
\times \left(\|\Pi \nabla u_h\|_{\partial \Omega} + \|h_{K}^{-1/2}(\Pi_{K}^{0} \Pi \nabla u_h - \Pi \nabla u_h)\|_{\partial \Omega}\right)
\leq C \|u_h\|_{-1,h} \left(\|\nabla w_h\|_{\partial \Omega} + \|h_{K}^{-1/2}(\Pi_{K}^{0} \nabla w_h - \mu_h)\|_{\partial \Omega}\right).
\]

This proves (14a).

The estimate (14b) now follows from (14a) by letting \(w_h = u_h\) and \(\mu_h = 0\), and by the usage of an inverse inequality.

Lastly, we prove the estimate (14c). By (12) and Lemma 3.10 one gets

\[
\|u_h\|_{-1,h} \leq \|\Pi \nabla u_h\|_{\partial \Omega} \|w_h\|_{\partial \Omega} \\
\leq C \|u_h\|_{\partial \Omega} \left(\|\Pi \nabla u_h\|_{\partial \Omega} + \|h_{K}^{-1/2}(\Pi_{K}^{0} \Pi \nabla u_h - \Pi \nabla u_h)\|_{\partial \Omega}\right)
\leq C \|u_h\|_{\partial \Omega} \|u_h\|_{-1,h}.
\]

This completes our proof. \(\square\)

To derive discrete HDG Sobolev embedding properties, we need to introduce the so-called Oswald interpolation.

Lemma 3.12 (cf. [31]). There exists an interpolation operator, called Oswald interpolation, \(I_{h}^{w} : W_h \rightarrow W_h \cap H^{1}(\Omega)\), such that for any \(w_h \in W_h\),

\[
\sum_{K \in T_{h}} \|w_h - I_{h}^{w} w_h\|_{0,K}^{2} \leq C \sum_{E \in \mathcal{E}_{h}} h_{E} \|w_h\|_{0,E}^{2}, \quad (15)
\]

\[
\sum_{K \in T_{h}} |w_h - I_{h}^{w} w_h|_{1,K}^{2} \leq C \sum_{E \in \mathcal{E}_{h}} h_{E}^{-1} \|w_h\|_{0,E}^{2}, \quad (16)
\]

where \(\|w_h\|\) denotes the jump of \(w_h\) across a side \(E\).

Remark 3.13. Just from the proof of [31] Page 644, Theorem 2.1, we can easily obtain the following estimate:

\[
\sum_{K \in T_{h}} h_{K}^{1+\varepsilon} \|v_h - I_{h}^{w} v_h\|_{0,K}^{2} \leq C \sum_{E \in \mathcal{E}_{h}} h_{E}^{1+\varepsilon} \|v_h\|_{0,E}^{2}, \quad (17)
\]

where \(\varepsilon\) is any fixed real constant.
The following embedding relationships are standard (cf. [1]).

**Lemma 3.14** (Sobolev embedding). The embedding relationship

\[ W^{1,2}(\Omega) \hookrightarrow W^{0,\mu}(\Omega), \]

holds for \( \mu \) satisfying

\[
\begin{cases}
1 \leq \mu < \infty, & \text{if } d = 2, \\
1 \leq \mu \leq 6, & \text{if } d = 3.
\end{cases}
\]

(18)

Furthermore, we have

\[ W^{2,2}(\Omega) \hookrightarrow C^0(\overline{\Omega}). \]

Based on Lemma 3.14 and Lemma 3.8 we have the following discrete Sobolev embedding relation.

**Theorem 3.15** (Discrete Sobolev embedding). For \( w_h \in W_h \), it holds

\[
\|w_h\|_{0,\mu} \leq C \left( \|w_h\|_{\tau_h} + \|\nabla w_h\|_{\tau_h} + \|h^{-1/2}_E \{w_h\}\|_{E_{h}} \right),
\]

(19)

if in addition \( w_h \in \tilde{W}_h \), it holds

\[
\|w_h\|_{0,\mu} \leq C \left( \|\nabla w_h\|_{\tau_h} + \|h^{-1/2}_E \{w_h\}\|_{E_{h}} \right),
\]

(20)

where \( \mu \) satisfying (18).

**Proof.** We only give a proof for (19), since (20) is followed by (19) and Lemma 3.8. We split the proof into two cases.

**Case 1:** \( \mu \geq 2 \). First, by a triangle inequality we have

\[
\|w_h\|_{0,\mu} \leq \|I_h^c w_h\|_{0,\mu} + \|I_h^c w_h - w_h\|_{0,\mu}.
\]

For the term \( \|I_h^c w_h\|_{0,\mu} \), since \( I_h^c w_h \in H^1(\Omega) \), then by Lemma 3.14 a triangle inequality and Lemma 3.12, we have

\[
\|I_h^c w_h\|_{0,\mu} \leq C \left( \|w_h\|_{\tau_h} + \|\nabla w_h\|_{\tau_h} + \|h^{-1/2}_E \{w_h\}\|_{E_{h}} \right).
\]

Next, for the term \( \|I_h^c w_h - w_h\|_{0,\mu} \), we use an inverse inequality to get

\[
\|w_h - I_h^c w_h\|_{0,\mu} = \left( \sum_{K \in \tau_h} \|w_h - I_h^c w_h\|_{0,\mu,K}^\mu \right)^{\frac{1}{\mu}}.
\]
\[ \leq C \left( \sum_{K \in T_h} h_K^{-d \left( \frac{1}{\mu} - \frac{1}{2} \right)} \| w_h - I_h^\mu w_h \|_{0,K}^\mu \right)^{\frac{1}{\mu}} \]

\[ = \left( \sum_{K \in T_h} \left\{ h_K^{-2d \left( \frac{1}{\mu} - \frac{1}{2} \right)} \| w_h - I_h^\mu w_h \|_{0,K}^2 \right\}^{\frac{1}{2}} \right)^{\frac{1}{2}} \]

\[ \leq C \left( \sum_{K \in T_h} h_K^{-2d \left( \frac{1}{2} - \frac{1}{\mu} \right)} \| w_h - I_h^\mu w_h \|_{0,K}^2 \right)^{\frac{1}{2}}, \]

in the last inequality, we used \( \mu \geq 2 \) and the equality \( \sum_{i=1}^n |a_i|^{\mu/2} \leq (\sum_{i=1}^n |a_i|)^{\mu/2} \). By the fact \( 1 - 2d \left( \frac{1}{2} - \frac{1}{\mu} \right) \geq -1 \) and (17), we have

\[ \| w_h - I_h^\mu w_h \|_{0,\mu} \leq C \left( \sum_{E \in E_h} h_E^{-2d \left( \frac{1}{2} - \frac{1}{\mu} \right)} \| [w_h] \|_{2,E}^2 \right)^{\frac{1}{2}} \leq C \| h_E^{-1/2} [w_h] \|_{E_h^\mu}. \]

**Case 2:** \( 1 \leq \mu < 2 \). Let \( \lambda \) satisfying

\[ \frac{\mu}{2} + \frac{1}{\lambda} = 1. \]

By the Hölder’s inequality to get

\[ \| w_h \|_{0,\mu}^\mu = \int_\Omega |w_h|^\mu dx \leq \left( \int_\Omega (|w_h|^\mu)^{\frac{2}{\mu}} dx \right)^{\frac{\mu}{2}} \left( \int_\Omega 1^{\lambda} dx \right)^{\frac{1}{\lambda}} = |\Omega|^{\frac{2}{\mu}} \| w_h \|_{0,\mu}^\mu. \]

Since we already proved the inequality (19) hold for \( \mu = 2 \), then

\[ \| w_h \|_{0,\mu} \leq |\Omega|^{\frac{1}{\mu}} \| w_h \|_{0} \leq C \left( \| w_h \|_{T_h} + \| \nabla w_h \|_{T_h} + \| h^{-1/2} [w_h] \|_{E_h^\mu} \right). \]

The combination of the above theorem and a triangle inequality to get the following corollary.

**Corollary 3.16** (HDG Sobolev embedding). For \((w_h, \mu_h) \in W_h \times M_h\), it holds

\[ \| w_h \|_{0,\mu} \leq C \left( \| w_h \|_{T_h} + \| \nabla w_h \|_{T_h} + \| h^{-1/2} (\Pi_k^\mu w_h - \mu_h) \|_{\partial T_h} \right), \quad (22) \]

if in addition \( w_h \in \bar{W}_h \), it holds

\[ \| w_h \|_{0,\mu} \leq C \left( \| \nabla w_h \|_{T_h} + \| h^{-1/2} (\Pi_k^\mu w_h - \mu_h) \|_{\partial T_h} \right), \quad (23) \]

where \( \mu \) satisfying (18).
4 Existence, uniqueness and stability of the HDG formulation

In this section we establish the well-posedness and the stability of the HDG method \[^{[3]}\]. The results differ slightly between the fully implicit discretization (FI) and the convex-spliting method (CS): the CS time marching enjoys unconditionally unique solvability and stability while there is a time-step constraint in the FI scheme for uniqueness and stability. We will focus on the analysis of one method and point out the difference of the other.

**Theorem 4.1.** The energy-splitting scheme of \[^{[5]}\] admits at least one solution.

**Proof.** We take \((r_1, w_1, \mu_1) = (0, 1, 1)\) and \((r_2, w_2, \mu_2) = (0, 1, 1)\) in \[^{[3]}\] to get

\[
(u^n_h, 1)\tau_h = (u^{n-1}_h, 1)\tau_h = \cdots = (u^0_h, 1)\tau_h, \quad (\phi^n_h, 1)\tau_n = (\epsilon^{-1} f^n(u^n_h), 1)\tau_n.
\]

Define the space

\[
X = \left\{ (\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h, \mathbf{u}_h, \mathbf{u}_h) \in [V_h \times \dot{W}_h \times M_h]^2 : \mathcal{A}(\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h; r_1, 0, 0) = \mathcal{A}(\mathbf{q}_h, \mathbf{r}_h, \mathbf{u}_h; r_2, 0, 0) = 0, \forall r_1, r_2 \in V_h \right\},
\]

and the semi-norm

\[
\|(\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h, \mathbf{u}_h, \mathbf{u}_h)\|_X^2 = \left( \|\mathbf{p}_h\|_{\tau_h}^2 + \|h^{-1/2} (\Pi_K \mathbf{p}_h - \mathbf{q}_h)\|_{\partial \Omega_T h}^2 \right) + \epsilon \left( \|\mathbf{q}_h\|_{\tau_h}^2 + \|h^{-1/2} (\Pi_K \mathbf{u}_h - \mathbf{u}_h)\|_{\partial \Omega_T h}^2 \right).
\]

Lemma \[^{[4]}\] implies that \(\cdot \|_X\) defines a norm on the space \(X\). Let \((\cdot, \cdot)_X = \| \cdot \|_X^2\) and define \(G : [V_h \times \dot{W}_h \times M_h]^2 \to [V_h \times \dot{W}_h \times M_h]^2\) as follows

\[
\left( G(\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h; \mathbf{u}_h), (r_1, w_1, \mu_1, r_2, w_2, \mu_2) \right)_X = \tau \mathcal{A}(\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h; r_1, w_1, \mu_1) + (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, w_1)_{\tau_h} + c \mathcal{A}(\mathbf{q}_h, \mathbf{r}_h, \mathbf{u}_h; r_2, w_2, \mu_2) + (\epsilon^{-1} f^n(\mathbf{u}_h^n), w_2)_{\tau_h} - (\phi^n_h, w_2)_{\tau_h}.
\]

Then, we have

\[
\left( G(\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h; \mathbf{u}_h), (\mathbf{p}_h, \mathbf{q}_h, \mathbf{r}_h; \mathbf{u}_h, \mathbf{u}_h) \right)_X = \tau \left( \|\mathbf{p}_h\|_{\tau_h}^2 + \|h^{-1/2} (\Pi_K \mathbf{p}_h - \mathbf{q}_h)\|_{\partial \Omega_T h}^2 \right) - (\mathbf{u}_h^{n-1}, \epsilon^{-1} \mathbf{u}_h^n + \phi^n_h)_{\tau_h} + \epsilon \left( \|\mathbf{q}_h\|_{\tau_h}^2 + \|h^{-1/2} (\Pi_K \mathbf{u}_h^n - \mathbf{u}_h^n)\|_{\partial \Omega_T h}^2 \right) + \epsilon^2 \|\mathbf{u}_h^n\|_{0,4}^2.
\]

13
\[ \frac{\epsilon T}{2} \| (\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n) \|_X - C \| u_h^{n-1} \|^2_{T_h}, \]

where we set \( u_h^{n-1} = \overline{u}_h^{n-1} \) for simplicity. For given \( u_h^{n-1} \) and \( \| (\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n) \|_X \geq R \) and \( R \) is large enough, we have

\[ \left( G(\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n), (\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n) \right)_X > 0. \]

The function \( G \) is obviously continuous. Then \([, Lemma 2.1.4]\) implies the existence of \((\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n)\) such that \( G(\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n) = 0 \).

We define

\[ (P_h^n, \phi_h^n, \Phi_h^n, q_h^n, u_h^n, \overline{u}_h^n) = (\overline{P}_h^n, \overline{\phi}_h^n, \overline{q}_h^n, \overline{\tau}_h^n) + (\beta, \phi_h^n + \beta, q_h^n, u_h^n + \alpha, \overline{u}_h^n + \alpha), \]

where

\[ \alpha = \frac{1}{|\Omega|} (\tau U, 1)_T_h, \quad \beta = \frac{1}{|\Omega|} (\epsilon^{-1} f^n(\pi_h^n), 1)_T_h. \]

Hence, \((P_h^n, u_h^n, \overline{u}_h^n, r_h^n, \phi_h^n, \Phi_h^n)\) is the solution of \((5)\). \( \square \)

**Theorem 4.2.** The energy-splitting scheme of \((5)\) admits a unique solution.

**Proof.** Given \((p_h^{n-1}, \phi_h^{n-1}, \Phi_h^{n-1}, q_h^{n-1}, u_h^{n-1}, \overline{u}_h^{n-1})\), we let \((p_h^n, \phi_h^n, \Phi_h^n, q_h^n, u_h^n, \overline{u}_h^n)\) and \((p_h^n, \phi_h^n, \Phi_h^n, q_h^n, u_h^n, \overline{u}_h^n)\) be two solutions of \((5)\). Let

\[
\begin{align*}
P_h^n := & p_h^n - p_h^{n-1}, \quad \Phi_h^n := \Phi_h^n - \Phi_h^{n-1}, \\
Q_h^n := & q_h^n - q_h^{n-1}, \quad U_h^n := u_h^n - u_h^{n-1}, \\
& \overline{U}_h^n := \overline{u}_h^n - \overline{u}_h^{n-1}.
\end{align*}
\]

After inserting the two solutions into \((5)\), subtracting the two equations to get

\[
\begin{align*}
(\partial_t U_h^n, w_1)_T_h + \mathcal{A}(R_h^n, \Phi_h^n, \Phi_h^n, r_1, w_1, \mu_1) = 0, \quad (24a) \\
\epsilon \mathcal{A}(P_h^n, U_h^n, \overline{U}_h^n; r_2, w_2, \mu_2) - (\Phi_h^n, w_2)_T_h + (\epsilon^{-1}((u_h^n)^3 - (u_h^{n-1})^3), w_2)_T_h = 0. \quad (24b)
\end{align*}
\]

Take \((r_1, w_1, \mu_1, r_2, w_2, \mu_2) = (P_h^n, \Phi_h^n, \Phi_h^n, Q_h^n, U_h^n, \overline{U}_h^n)\) to get

\[
(\overline{U}_h^n, \Phi_h^n)_T_h + \tau \left( \| P_h^n \|^2_{T_h} + \| h_{\mathcal{K}}^{-1/2}(\Pi_k^2 \Phi_h^n - \Phi_h^n) \|^2_{T_h} \right) = 0,
\]

\[
\epsilon \left( \| Q_h^n \|^2_{T_h} + \| h_{\mathcal{K}}^{-1/2}(\Pi_k^2 U_h^n - \overline{U}_h^n) \|^2_{T_h} \right) + (\epsilon^{-1}((u_h^n)^3 - (u_h^{n-1})^3), U_h^n)_T_h - (\Phi_h^n, U_h^n)_T_h = 0.
\]

We add the above two equations to get

\[
\tau \left( \| P_h^n \|^2_{T_h} + \| h_{\mathcal{K}}^{-1/2}(\Pi_k^2 \Phi_h^n - \Phi_h^n) \|^2_{T_h} \right)
\]

14
\[ + \epsilon \left( \| P^n_h \|_{\mathcal{T}_h}^2 + \| h_K^{-1/2} (\Pi_h^n U^n_h - \hat{U}^n_h) \|_{\mathcal{T}_h}^2 \right) \]
\[ + (\epsilon^{-1} (U^n_h)^2, (u_{h,1})^2 + (u_{h,2})^2 + u_{h,1}u_{h,2})_{\mathcal{T}_h} \]
\[ = 0. \]

Since all these terms are nonnegative, combining Lemma 3.4, we obtain

\[ P^n_h = Q^n_h = 0, \quad \nabla U^n_h = \nabla \Phi^n_h = 0, \quad U^n_h = \hat{U}^n_h, \quad \Phi^n_h = \hat{\Phi}^n_h. \]

First, we take \((r_1, w_1, \mu_1) = (0, 1, 1)\) in (24a) to get

\[ (U^n_h, 1)_{\mathcal{T}_h} = (U^{n-1}_h, 1)_{\mathcal{T}_h} = \cdots = (U^0_h, 1)_{\mathcal{T}_h} = 0. \]

Then \(U^n_h = \hat{U}^n_h = 0.\)

Next, we take \((r_2, w_2, \mu_2) = (0, 1, 1)\) in (24b) to get

\[ (\Phi^n_h, 1)_{\mathcal{T}_h} = 0. \]

Therefore, \(\Phi^n_h = \hat{\Phi}^n_h = 0.\) Thus \((P^n_h, \Phi^n_h, \hat{\Phi}^n_h, Q^n_h, U^n_h, \hat{U}^n_h) = 0.\]

In the rest of the article, we focus on the fully implicit scheme, i.e., \(f^n(u^n_h) = (u^n_h)^3 - u^n_h.\) For the energy-splitting scheme, i.e. \(f^n(u^n_h) = (u^n_h)^3 - u^n_{h,-1}\) can be dealt similarly. We first establish two versions of the energy identities satisfied by the HDG scheme (5). The first version makes use of the negative norm \(\| \partial_t^+ u^n_h \|_{-1,h},\) while the second one utilizes the \(L^2\) norm \(\| P^n_h \|_{\mathcal{T}_h}.\)

We first give some useful identities.

**Lemma 4.3.** Let \(a, b\) be two real numbers, then the following three identities hold true

\[ (a - b)a = \frac{1}{2} [a^2 - b^2 + (a - b)^2], \quad (25a) \]
\[ (a^3 - a)(a - b) = \frac{1}{4} \left[ (a^2 - 1)^2 - (b - 1)^2 + (a^2 - b^2)^2 + 2(a(a - b))^2 - 2(a - b)^2 \right], \quad (25b) \]
\[ (a^3 - b)(a - b) = \frac{1}{4} \left[ (a^2 - 1)^2 - (b - 1)^2 + (a^2 - b^2)^2 + 2(a(a - b))^2 + 2(a - b)^2 \right]. \quad (25c) \]

**Lemma 4.4** (Discrete energy identity 1). Let \((p^n_h, \phi^n_h, \hat{\phi}^n_h), (q^n_h, u^n_h, \hat{u}^n_h)\) be the
solution of (5). The following energy identity holds for \( m \in [1, N] \)

\[
\frac{1}{4e} \left\| (u^n_h) - 1 \right\|_{\mathcal{V}_h}^2 + \frac{\epsilon}{2} \left\| q^n_h \right\|_{\mathcal{V}_h}^2 \\
+ \frac{\epsilon}{2} \left\| h_K^{-1/2} (\Pi_k^n u^n_h - \hat{u}^n_h) \right\|_{\partial \mathcal{V}_h}^2 + \Delta t \sum_{n=1}^m \left\| \partial_t^+ u^n_h \right\|_{-1,h}^2 \\
+ (\Delta t)^2 \sum_{n=1}^m \left( \frac{\epsilon}{2} \left\| \partial_t^+ q^n_h \right\|_{\mathcal{V}_h}^2 + \frac{\epsilon}{2} \left\| h_K^{-1/2} \partial_t^+ (\Pi_k^n u^n_h - \hat{u}^n_h) \right\|_{\partial \mathcal{V}_h}^2 \right) \\
= \frac{1}{4e} \left\| (u^n_h) - 1 \right\|_{\mathcal{V}_h}^2 + \frac{\epsilon}{2} \left\| \Pi_k^n u^n_h \right\|_{\mathcal{V}_h}^2 + \frac{\epsilon}{2} \left\| h_K^{-1/2} (\Pi_k^n u^n_h - \hat{u}^n_h) \right\|_{\partial \mathcal{V}_h}^2.
\]

\[ (5a) \]

\[ (5b) \]

**Proof.** First, we take \((r_1, w_1, \mu_1) = - (\Pi_V \partial_t^+ u^n_h, -\Pi_W \partial_t^+ u^n_h, -\Pi_M \partial_t^+ u^n_h)\) in (5a) to get

\[
(\partial_t^+ u^n_h, \Pi_W \partial_t^+ u^n_h)_{\mathcal{T}_h} = \mathcal{A}(\Pi_V \partial_t^+ u^n_h, \Pi_W \partial_t^+ u^n_h, \Pi_M \partial_t^+ u^n_h, (p^n_h, \phi^n_h, \hat{u}^n_h))_{\partial \mathcal{T}_h, n} \\
= \mathcal{A}(\Pi_V \partial_t^+ u^n_h, \Pi_W \partial_t^+ u^n_h, \Pi_M \partial_t^+ u^n_h, (p^n_h, \phi^n_h, \hat{u}^n_h))_{\partial \mathcal{T}_h, n} \quad \text{by (5b)}
\]

By (12) we have

\[
(\partial_t^+ u^n_h, \Pi_W \partial_t^+ u^n_h)_{\mathcal{T}_h} = \left\| \partial_t^+ u^n_h \right\|_{-1,h}^2.
\]

The above two equations gives

\[
\left\| \partial_t^+ u^n_h \right\|_{-1,h}^2 + \left( \partial_t^+ u^n_h, \phi^n_h \right)_{\mathcal{T}_h} = 0. \quad (27)
\]

Next, we take \((r_2, w_2, \mu_2) = (0, \partial_t^+ u^n_h, \partial_t^+ \hat{u}^n_h)\) in (5b) to get

\[
(e^{-1} f^n(u^n_h), \partial_t^+ u^n_h)_{\mathcal{T}_h} + \epsilon (\nabla : q^n_h, \partial_t^+ u^n_h)_{\mathcal{T}_h} + \epsilon (q^n_h : \nabla, \partial_t^+ u^n_h)_{\mathcal{T}_h} - \epsilon (q^n_h : \nabla, \partial_t^+ \hat{u}^n_h)_{\mathcal{T}_h} \\
+ \epsilon (h_K^{-1}(\Pi_k^n u^n_h - \hat{u}^n_h), \partial_t^+ (\Pi_k^n u^n_h - \hat{u}^n_h))_{\partial \mathcal{T}_h} + (\phi^n_h, \partial_t^+ u^n_h)_{\partial \mathcal{T}_h} = 0, \quad (28)
\]

and we also take \((r_2, w_2, \mu_2) = (q^n_h, 0, 0)\) in (5b) and apply \(\partial_t^+\) on it to get

\[
\epsilon (\partial_t^+ q^n_h, q^n_h)_{\mathcal{T}_h} + \epsilon (\partial_t^+ \hat{u}^n_h, q^n_h)_{\mathcal{T}_h} + \epsilon (\partial_t^+ \hat{u}^n_h, \nabla : q^n_h)_{\mathcal{T}_h} + \epsilon (\partial_t^+ \hat{u}^n_h, \nabla : q^n_h)_{\partial \mathcal{T}_h} = 0. \quad (29)
\]

Add (27), (28) and (29) to get

\[
\left\| \partial_t^+ u^n_h \right\|_{-1,h}^2 + \epsilon^{-1} (f^n(u^n_h), \partial_t^+ u^n_h)_{\mathcal{T}_h} + \epsilon (\partial_t^+ q^n_h, q^n_h)_{\mathcal{T}_h} \\
+ \epsilon (h_K^{-1}(\Pi_k^n u^n_h - \hat{u}^n_h), \partial_t^+ (\Pi_k^n u^n_h - \hat{u}^n_h))_{\partial \mathcal{T}_h} = 0.
\]

Then we multiply \(\Delta t\) in the last equation and use the identities in Lemma 13 to get

\[
\frac{1}{4e} \left\| (u^n_h) - 1 \right\|_{\mathcal{V}_h}^2 + \frac{\epsilon}{2} \left\| q^n_h \right\|_{\mathcal{V}_h}^2 + \frac{\epsilon}{2} \left\| h_K^{-1/2} (\Pi_k^n u^n_h - \hat{u}^n_h) \right\|_{\partial \mathcal{T}_h}^2
\]
We sum the above equation from $n = 1$ to $n = m$ to get the desired result.

Lemma 4.5 (Discrete energy identity II). Let $(p_h^n, \phi_h^n, \hat{\phi}_h^n), (q_h^n, u_h^n, \hat{u}_h^n)$ be the solution of (3). Then for any integer $m \in [1, N]$, we have the following energy identity

\[
\frac{1}{4\epsilon} (u_h^n)^2 - 1\|T_h^n\| + \frac{\epsilon}{2} \|q_h^n\|_T^2 + \frac{\epsilon}{2} \|h_K^{-1/2}(\Pi_h^n u_h^n - \hat{u}_h^n)\|^2_{\partial \Omega_h} + \Delta t \sum_{n=1}^m \left( \frac{1}{2\epsilon} \|\partial_t^+ u_h^n\|_{T_h}^2 + \frac{1}{2} \|h_K^{-1/2}(\Pi_h^n u_h^n - \hat{u}_h^n)\|^2_{\partial \Omega_h} \right)
\]

\[
+ (\Delta t)^2 \sum_{n=1}^m \left( \frac{1}{2\epsilon} \|\partial_t^+ q_h^n\|_{T_h}^2 + \frac{1}{2} \|h_K^{-1/2} T_h^+ (\Pi_h^n u_h^n - \hat{u}_h^n)\|^2_{\partial \Omega_h} \right)
\]

\[
+ (\Delta t)^2 \sum_{n=1}^m \left( \frac{1}{4\epsilon} \|\partial_t^+ (u_h^n)^2\|_{T_h}^2 + \frac{1}{2} \|h_K^{-1/2} T_h^+ (\Pi_h^n u_h^n - \hat{u}_h^n)\|^2_{\partial \Omega_h} \right)
\]

\[
= \frac{1}{4\epsilon} (u_h^n)^2 - 1\|T_h^n\| + \frac{\epsilon}{2} \|q_h^n\|_T^2 + \frac{\epsilon}{2} \|h_K^{-1/2}(\Pi_h^n u_h^n - \hat{u}_h^n)\|^2_{\partial \Omega_h}.
\]

Proof. We take $(r_1, w_1, \mu_1) = (p_h^n, \phi_h^n, \hat{\phi}_h^n)$ in (33) to get

\[
(\partial_t^+ u_h^n, \phi_h^n) + \|p_h^n\|_{\partial \Omega_h} + \|h_K^{-1/2}(\Pi_h^n \phi_h^n - \hat{\phi}_h^n)\|^2_{\partial \Omega_h} = 0.
\]

Sum the (28), (29) and (31) to give

\[
\epsilon^{-1}(f^n(u_h^n), \partial_t^+ u_h^n) + \epsilon (\partial_t^+ q_h^n, q_h^n) + \epsilon (h_K^{-1}(\Pi_h^n u_h^n - \hat{u}_h^n), \partial_t^+ (\Pi_h^n u_h^n - \hat{u}_h^n))_{\partial \Omega_h} + \|p_h^n\|_{\partial \Omega_h} + \|h_K^{-1/2}(\Pi_h^n \phi_h^n - \hat{\phi}_h^n)\|^2_{\partial \Omega_h} = 0.
\]

The energy identity now follows from applications of the identities in Lemma 4.3 and add the resulting equation from $n = 1$ to $n = m$.

The energy identities I and II in the foregoing Lemmas 4.4 and 4.5 have a negative term, i.e., $-\frac{(\Delta t)^2}{2} \|\partial_t^+ u_h^n\|_{T_h}^2$. In the next Theorem, we first bound this term and then give our stability result.
Theorem 4.6. Provided that the time-step constraint $\Delta t \leq \frac{2\epsilon}{C^2}$ with $C$ being the product of the constants in (14a) and Lemma 3.4, then for $m = 1, 2, \cdots, N$, we have the following energy bounds
\[
\|u_h^m\|_{0,4} + \|q_h^m\|_{\mathcal{T}_h} + \|\nabla u_h^m\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^b u_h^m - \hat{u}_h^m)\|_{\partial \mathcal{T}_h} + \Delta t \sum_{n=1}^m \|\nabla \phi_h^n\|_{\mathcal{T}_h}^2
\]
\[+ \Delta t \sum_{n=1}^m \|\partial_t^+ u_h^n\|_{-1,h}^2 + \Delta t \sum_{n=1}^m \left(\|\mathbf{p}_h^n\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^b \phi_h^n - \hat{\phi}_h^n)\|_{\partial \mathcal{T}_h}^2\right)\]
\[\leq \frac{1}{4\epsilon} \|(u_h^0)^2 - 1\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|q_h^0\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^b u_h^0 - \hat{u}_h^0)\|_{\partial \mathcal{T}_h}^2.
\]

Proof. By the inequality (14a) and Lemma 3.4, we have
\[
\frac{(\Delta t)^2}{2\epsilon} \|\partial_t^+ u_h^n\|_{\mathcal{T}_h}^2
\]
\[\leq \frac{(\Delta t)^2}{2\epsilon} \|\partial_t^+ u_h^n\|_{-1,h}^2 + \left(\|\partial_t^+ \nabla u_h^n\|_{\mathcal{T}_h}^2 + \|h^{-1/2}\partial_t^+ (\Pi_k^b u_h^n - \hat{u}_h^n)\|_{\partial \mathcal{T}_h}^2\right)\]
\[\leq \frac{(\Delta t)^2}{2\epsilon} \|\partial_t^+ u_h^n\|_{-1,h}^2 + \frac{(\Delta t)^2}{4\epsilon^2} \left(\frac{\epsilon}{2} \|\partial_t^+ q_h^n\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|h^{-1/2}\partial_t^+ (\Pi_k^b u_h^n - \hat{u}_h^n)\|_{\partial \mathcal{T}_h}^2\right).  
\]

Now, by the energy identity I (26) and the assumption $\Delta t \leq \frac{2\epsilon}{C^2}$, we have
\[
\frac{1}{4\epsilon} \|(u_h^m)^2 - 1\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|q_h^m\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^b u_h^m - \hat{u}_h^m)\|_{\partial \mathcal{T}_h}^2
\]
\[+ \Delta t \sum_{n=1}^m \|\partial_t^+ u_h^n\|_{-1,h}^2 + (\Delta t)^2 \sum_{n=1}^m \left(\frac{1}{4\epsilon} \|\partial_t^+ (u_h^n)^2\|_{\mathcal{T}_h}^2 + \frac{1}{2\epsilon} \|u_h^n \partial_t^+ u_h^n\|_{\mathcal{T}_h}^2\right)
\]
\[+ \frac{(\Delta t)^2}{2} \sum_{n=1}^m \left(\frac{\epsilon}{2} \|\partial_t^+ q_h^n\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|h^{-1/2}\partial_t^+ (\Pi_k^b u_h^n - \hat{u}_h^n)\|_{\partial \mathcal{T}_h}^2\right)
\]
\[\leq \frac{1}{4\epsilon} \|(u_h^0)^2 - 1\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|q_h^0\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^b u_h^0 - \hat{u}_h^0)\|_{\partial \mathcal{T}_h}^2.
\]

By the same argument, the energy identity II (30) and Lemma 3.4 to get our desired result. \qed

Remark 4.7. In the case of the energy-splitting scheme $f^n(u_h^n) = (u_h^n)^3 - u_h^{n-1}$, an energy law holds in the spirit of Lemma 4.4 where all terms are associated with the positive sign. The energy law yields unconditional energy stability (i.e. without the time-step constraint).

It is worth noting that in Theorem 4.6, the energy term $\Delta t \sum_{n=1}^N \|\phi_h^n\|_{\mathcal{T}_h}^2$ is not contained. Moreover, the HDG Poincaré inequality (23) does not apply to $\phi_h^n$ since $\phi_h^n \notin W_h$. Hence, we need a refined analysis for this term. To approximate this term, we first need to give an estimation of $\|u_h^n\|_{0,6}$. 

Lemma 4.8. If the time-step constraint $\Delta t \leq \frac{2\epsilon}{C^2}$ with $C$ being the product of the constants in (4.4) and Lemma 3.4, then for all $m = 1, 2, \ldots, N$, we have the following energy bound

$$\|u_h^n\|_{0,0} \leq C \left( \frac{1}{4\epsilon} \|(u_h^0)^2 - 1\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|q_h^0\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^0 u_h^0 - \phi_h^0)\|_{\partial \mathcal{T}_h} \right).$$

Proof. By (22) and Theorem 4.6, one has

$$\|u_h^n\|_{0,0} \leq C \left( \|u_h\|_{\mathcal{T}_h} + \|\nabla u_h^n\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^0 u_h^m - \tilde{u}_h^m)\|_{\partial \mathcal{T}_h} \right) \leq C \left( \|u_h\|_{0,4} + \|\nabla u_h^n\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^0 u_h^m - \tilde{u}_h^m)\|_{\partial \mathcal{T}_h} \right) \leq C \left( \frac{1}{4\epsilon} \|(u_h^0)^2 - 1\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|q_h^0\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^0 u_h^0 - \phi_h^0)\|_{\partial \mathcal{T}_h} \right).$$

In the following, we derive further a priori bounds for the solution to the HDG scheme (5) with the assumption $\Delta t \leq \frac{2\epsilon}{C^2}$.

Theorem 4.9. If the assumption is the same as Theorem 4.6, then

$$\Delta t \sum_{n=1}^N \|\phi_h^n\|_{\mathcal{T}_h}^2 \leq C \left( \frac{1}{4\epsilon} \|(\phi_h^0)^2 - 1\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|q_h^0\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^0 u_h^0 - \phi_h^0)\|_{\partial \mathcal{T}_h} \right).$$

Proof. We take $(r_2, w_2, \mu_2) = (0, \phi_h^n, \tilde{\phi}_h^n)$ in (5b) to get

$$\|\phi_h^n\|_{\mathcal{T}_h}^2 = \epsilon^{-1} \|(\phi_h^n)^2 - 1\|_{\mathcal{T}_h} + \epsilon \|\nabla \cdot q_h^n\|_{\mathcal{T}_h} - \epsilon \langle q_h^n, w_n \rangle_{\mathcal{T}_h} \leq C \left( \|\nabla \phi_h^n\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^0 u_h^m - \phi_h^0)\|_{\partial \mathcal{T}_h} \right),$$

The desired result now follows from Theorem 4.6 and Lemma 4.8.

Lemma 4.10. We have

$$\|\phi_h^n\|_{\mathcal{T}_h}^2 + \sum_{n=1}^m \|\phi_h^n - \phi_h^{n-1}\|_{\mathcal{T}_h}^2 + \Delta t \sum_{n=1}^m \|\partial_t u_h^n\|_{\mathcal{T}_h}^2 \leq C \left( \frac{1}{4\epsilon} \|(\phi_h^0)^2 - 1\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|q_h^0\|_{\mathcal{T}_h} + \frac{\epsilon}{2} \|h^{-1/2}(\Pi_k^0 u_h^0 - \phi_h^0)\|_{\partial \mathcal{T}_h} \right).$$
Proof. We apply $\partial_t^{+}$ on (32a) and keep (32a) unchange to get

$$
(\partial_t^{+} u^n_h, w_1)_{T_h} + A(p^n_h, \phi^n_h, \widehat{\phi}^n_h; r_1, w_1, \mu_1) = 0,
$$

(32a)

$$
\epsilon A(\partial_t^{+} p^n_h, \partial_t^{+} u^n_h, \partial_t^{+} \widehat{u}^n_h; r_2, w_2, \mu_2)
+ (\epsilon^{-1} \partial_t^{+} f^n(u^n_h), w_2)_{T_h} - (\partial_t^{+} \phi^n_h, w_2)_{T_h} = 0.
$$

(32b)

Take $(r_1, w_1, \mu_1) = -(\partial_t^{+} q^n_h, -\partial_t^{+} u^n_h, -\partial_t^{+} \widehat{u}^n_h)$ in (32a) and $(r_2, w_2, \mu_2) = (p^n_h, -\phi^n_h, -\widehat{\phi}^n_h)$ in (32b) to get

$$
\epsilon (\partial_t^{+} u^n_h, \partial_t^{+} u^n_h)_{T_h} - \epsilon A(p^n_h, \phi^n_h, \widehat{\phi}^n_h, \partial_t^{+} q^n_h, -\partial_t^{+} u^n_h, -\partial_t^{+} \widehat{u}^n_h) = 0,
$$

$$
\epsilon A(\partial_t^{+} q^n_h, \partial_t^{+} u^n_h, \partial_t^{+} \widehat{u}^n_h, p^n_h, -\phi^n_h, -\widehat{\phi}^n_h)
+ (\epsilon^{-1} \partial_t^{+} f^n(u^n_h), -\phi^n_h)_{T_h} - (\partial_t^{+} \phi^n_h, -\phi^n_h)_{T_h} = 0.
$$

We add the above two equations together to get

$$
(\partial_t^{+} \phi^n_h, \phi^n_h)_{T_h} + \epsilon \|\partial_t^{+} u^n_h\|_{T_h}^2 = \epsilon^{-1} (\partial_t^{+} f^n(u^n_h), \phi^n_h)_{T_h}.
$$

(33)

By the identity (2a) we have

$$
(\partial_t^{+} \phi^n_h, \phi^n_h)_{T_h} = \frac{1}{2\Delta t} \left( \|\phi^n_h\|_{T_h}^2 - \|\phi^{n-1}_h\|_{T_h}^2 + \|\phi^n_h - \phi^{n-1}_h\|_{T_h}^2 \right).
$$

We add (33) from $n = 1$ to $n = m$, and multiply it by $2\Delta t$ to get

$$
\|\phi^n_h\|_{T_h}^2 + \sum_{n=1}^m \|\phi^n_h - \phi^{n-1}_h\|_{T_h}^2 + 2\Delta t \epsilon \sum_{n=1}^m \|\partial_t^{+} u^n_h\|_{T_h}^2
= 2\epsilon^{-1} \Delta t \sum_{n=1}^m (\partial_t^{+} f^n(u^n_h), \phi^n_h)_{T_h} + \|\phi^0_h\|_{T_h}^2.
$$

(34)

By Hölder’s inequality, we have

$$
\Delta t \sum_{n=1}^m (\partial_t^{+} f^n(u^n_h), \phi^n_h)_{T_h}
\leq C \Delta t \sum_{n=1}^m \|\partial_t^{+} u^n_h\|_{T_h} \|\phi^n_h\|_{0.6} \left( \|u^n_h\|_{0.3}^2 + \|(u^n_h)^{-1}\|_{0.3}^2 \right)
= C \Delta t \sum_{n=1}^m \|\partial_t^{+} u^n_h\|_{T_h} \|\phi^n_h\|_{0.6} \left( \|u^n_h\|_{0.6}^2 + \|u^n_h^{-1}\|_{0.6}^2 \right).
$$

Next, by the HDG Sobolev embedding inequality in Corollary 3.16 we have

$$
\Delta t \sum_{n=1}^m (\partial_t^{+} f^n(u^n_h), \phi^n_h)_{T_h}
$$

20
\[ \leq C \Delta t \sum_{n=1}^{m} \left( \| \partial_t^+ u_h^n \|_{T_n} \| \phi_h^n \|_{0,6} + \| u_h^n \|_{T_n}^2 + \| u_h^{n-1} \|_{T_n}^2 + \| h_h^n \|_{T_n}^2 + \| q_h^n \|_{T_n}^2 + \| q_h^{n-1} \|_{T_n}^2 + \| h^{-1/2}(\Pi_k^0 u_h^n - \hat{u}_h^n) \|_{0,6}^2 \right). \]

We then use Theorem 4.6 and the Cauchy-Schwarz inequality to get
\[
\Delta t \sum_{n=1}^{m} (\partial_t^+ f^n(u_h^n), \phi_h^n)_{T_n} \leq \frac{1}{4T} \left( \frac{1}{4} \| u_h^0 \|^2 - 1 \right) \| \phi_h^0 \|_{T_n}^2 + \frac{\epsilon}{2} \| q_h^0 \|_{T_n}^2 + \frac{\epsilon}{2} \| h^{-1/2}(\Pi_k^0 u_h^n - \hat{u}_h^n) \|_{0,6}^2 + 1 \right] \times \Delta t \sum_{n=1}^{m} \| \partial_t^+ u_h^n \|_{T_n} \| \phi_h^n \|_{0,6} \leq \epsilon \Delta t \sum_{n=1}^{m} \| \partial_t^+ u_h^n \|_{T_n}^2 + C \Delta t \sum_{n=1}^{m} \| \phi_h^n \|_{0,6}^2.
\]

Together with (34) and the above inequality, we have
\[
\left\| \phi_h^m \right\|_{T_n}^2 + \sum_{n=1}^{m} \| \phi_h^n - \phi_h^{n-1} \|_{T_n}^2 + \Delta t \sum_{n=1}^{m} \| \partial_t^+ u_h^n \|_{T_n}^2 \leq C \Delta t \sum_{n=1}^{m} \| \phi_h^n \|_{0,6}^2 + C \| \phi_h^0 \|_{T_n}^2.
\]

From the HDG Sobolev embedding inequality Corollary 3.16 again, one gets
\[
\left\| \phi_h^m \right\|_{T_n}^2 + \sum_{n=1}^{m} \| \phi_h^n - \phi_h^{n-1} \|_{T_n}^2 + \Delta t \sum_{n=1}^{m} \| \partial_t^+ u_h^n \|_{T_n}^2 \leq C \Delta t \sum_{n=1}^{m} (\| p_h^n \|_{T_n}^2 + \| h^{-1/2}(\Pi_k^0 \phi_h^n - \hat{\phi}_h^n) \|_{0,6}^2 + C \| \phi_h^0 \|_{T_n}^2 + C \Delta t \sum_{n=1}^{m} \| \phi_h^n \|_{T_n}^2.
\]

Growall’s inequality and Theorem 4.6 imply the desired result.

5 Error analysis

Next, we provide a convergence analysis of the above fully implicit HDG methods for the Cahn-Hilliard equation. The strategy of the error analysis is based on [] and []. First, we give our main results. Next, we define an HDG
elliptic projection as in \([\ast]\) and \([\ast]\), which is a crucial step to prove the main result. To get the superconvergence for the solution, we need to introduce an \(H^{-1}\) estimation and in the end, we provide a rigorous error estimation for our fully implicit HDG method.

Throughout, we assume the data and the solution of \(\Pi\) are smooth enough. The same as in Section 4, we only perform an error analysis for \(\epsilon = O(1)\). The generic constant \(C\) may depend on the data of the problem but is independent of \(h\) and may change from line to line.

Given \(\Theta \in H^1(\Omega)\), the solution \((\Psi, \Phi)\) of the following system

\[
\begin{align*}
\Psi + \nabla \Phi &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \Psi &= \Theta \quad \text{in } \Omega, \\
\Psi \cdot n &= 0 \quad \text{in } \Omega, \\
\int_{\Omega} \Phi &= 0
\end{align*}
\]

We assume the following regularity holds true

\[
\begin{align*}
\|\Psi\|_1 + \|\Phi\|_2 &\leq C_{\text{reg}} \|\Theta\|_{L^2(\Omega)}, \tag{35a} \\
\|\Psi\|_2 + \|\Phi\|_3 &\leq C_{\text{reg}} \|\Theta\|_{H^1(\Omega)}. \tag{35b}
\end{align*}
\]

We notice that when \(\Omega\) is convex, \((35a)\) holds true; when \(d = 2\) and all the angles of \(\Omega\) is between \([\pi/4, \pi/2]\), \((35b)\) holds true.

### 5.1 Main Result

We can now state our main result for the fully implicit HDG method.

**Theorem 5.1.** Let \((p, \phi, q, u)\) and \((p^n_h, \phi^n_h, q^n_h, u^n_h)\) be the solution of \((2)\) and \((5)\), respectively. For all \(1 \leq n \leq N\), we have

\[
\left( \Delta t \sum_{n=1}^{N} \|q^n - q^n_h\|_{T_h} \right)^\frac{1}{2} \leq C(h^{k+1} + \Delta t),
\]

\[
\left( \Delta t \sum_{n=1}^{N} \|\phi^n - \phi^n_h\|_{T_h} \right)^\frac{1}{2} \leq C(h^{k+2} + \Delta t),
\]

\[
\|u^n - u^n_h\|_{T_h} \leq C(h^{k+2} + \Delta t),
\]

if \(k \geq 1\), we have

\[
\|\Pi^o_{k+1} u^n - u^n_h\|_{-1, T_h} \leq C(h^{k+3} + \Delta t),
\]

\[
\left( \Delta t \sum_{n=1}^{N} \|\Pi^o_k q^n - q^n_h\|_{T_h} \right)^\frac{1}{2} \leq C(h^{k+3} + \Delta t).
\]
Theorem 5.3. To the best of our knowledge, the only work for the fourth order problem [1] by HDG method; an different HDG scheme with here to investigate the Biharmonic equation and they obtained an optimal convergence rate for solution and suboptimal convergence rate for other variables. However, our HDG method get all variables achieve an optimal convergence rate. Moreover, from the view point of degrees of freedom, we obtain superconvergent convergence rate for the solution.

5.2 HDG elliptic projection

For all \( t \in [0, T] \), we define the HDG elliptic projection: find \((p_{ih}, \phi_{ih}, \hat{\phi}_{ih})\), \((q_{ih}, u_{ih}, \hat{u}_{ih})\) \(\in V_h \times W_h \times M_h\) such that

\[
\mathcal{A}(p_{ih}, \phi_{ih}, \hat{\phi}_{ih}; r_1, w_1, \mu_1) = - (\Delta \phi, w_1)_{T_h} \quad \text{and} \quad (\phi_{ih} - \phi, 1)_{T_h} = 0,
\]

\[
\mathcal{A}(q_{ih}, u_{ih}, \hat{u}_{ih}; r_2, w_2, \mu_2) = - (\Delta u, w_2)_{T_h} \quad \text{and} \quad (u_{ih} - u, 1)_{T_h} = 0
\]

for all \((r_1, w_1, \mu_1), (r_2, w_2, \mu_2) \in V_h \times W_h \times M_h\).

We have the following approximations property for the HDG elliptic projection, which is proved in Section 6:

Theorem 5.3. Let \((p, \phi, q, u)\) and \((p_{ih}, \phi_{ih}, q_{ih}, u_{ih})\) be the solution of (2) and (36), respectively. We have

\[
\|u - u_{ih}\|_{T_h} \leq Ch^{k+2}|u|_{k+2}, \tag{37a}
\]

\[
\|q - q_{ih}\|_{T_h} + \|h_K^{-1/2}(\Pi_k^0 u_{ih} - \hat{u}_{ih})\|_{\partial T_h} \leq Ch^{k+1}|u|_{k+2}, \tag{37b}
\]

\[
\|\partial_t u - \partial_t u_{ih}\|_{T_h} \leq Ch^{k+2}|\partial_t u|_{k+2}, \tag{37c}
\]

\[
\|\phi - \phi_{ih}\|_{T_h} \leq Ch^{k+2}|\phi|_{k+2}, \tag{37d}
\]

\[
\|p - p_{ih}\|_{T_h} + \|h_K^{-1/2}(\Pi_k^0 \phi_{ih} - \hat{\phi}_{ih})\|_{\partial T_h} \leq Ch^{k+1}|\phi|_{k+2}. \tag{37e}
\]

We only give a proof of (37a) and (37b), and we split the proof into three steps. To simplify the notation, we define

\[
\varepsilon^q_h := \Pi_k^0 q - q_{ih}, \quad \varepsilon^u_h := \Pi_{k+1}^0 u - u_{ih}, \quad \varepsilon^\mu_h := \Pi_k^0 \mu - \hat{\mu}_{ih}.
\]

5.2.1 Step 1: Error Equation

Lemma 5.4. For all \((r_2, w_2, \mu_2) \in V_h \times W_h \times M_h\), we have

\[
\mathcal{A}(\Pi_k^0 q, \Pi_{k+1}^0 u, \Pi_k^0 \mu; r_2, w_2, \mu_2)
\]

\[
= - (\Delta u, w_2)_{T_h} + \langle q \cdot n - \Pi_k^0 q \cdot n, \mu_2 - w_2 \rangle_{\partial T_h}
\]

\[
+ \langle h_K^{-1}(\Pi_{k+1}^0 u - u), \Pi_k^0 w_2 - \mu_2 \rangle_{\partial T_h}.
\]
Proof. By the definition of $\mathcal{A}$, we have

$$\mathcal{A}(\Pi_k^q, \Pi_{k+1}^u, \Pi_k^9; r_2, w_2, \mu_2)$$

$$= (\Pi_k^q, r_2)_{T_h} - (\Pi_{k+1}^q, \nabla \cdot r_2)_{T_h} + (\Pi_k^9 u, r_2 \cdot n)_{\partial T_h}$$

$$+ (\nabla \cdot \Pi_k^q, w_2)_{T_h} - (\Pi_k^q q, \mu_2)_{\partial T_h}$$

$$+ (h^{-1}_K (\Pi_k^9 u - \Pi_k^u), \Pi_k^9 w_2 - \mu_2)_{\partial T_h}$$

$$= (q + \nabla u, r_2)_{T_h} - (q, \nabla w_2)_{T_h} - (\Pi_k^q q, \mu_2)_{\partial T_h}$$

$$+ (h^{-1}_K (\Pi_{k+1}^q u - u), \Pi_k^9 w_2 - \mu_2)_{\partial T_h},$$

where we used the orthogonality of $\Pi_k^q$, $\Pi_{k+1}^q$, $\Pi_k^9$ in the last inequality.

Since $q = -\nabla u$ and $(q \cdot n, \mu_2)_{\partial T_h} = 0$, one gets

$$\mathcal{A}(\Pi_k^q, \Pi_{k+1}^u, \Pi_k^9; r_2, w_2, \mu_2)$$

$$= - (\Delta u, w_2)_{T_h} + (q \cdot n - \Pi_k^q q \cdot n, \mu_2 - w_2)_{\partial T_h}$$

$$+ (h^{-1}_K (\Pi_{k+1}^q u - u), \Pi_k^9 w_2 - \mu_2)_{\partial T_h}. \quad \square$$

Subtracting the equation in Lemma 5.4 by the Equation (36a) to get the following lemma.

**Lemma 5.5.** We have the error. For all $(r_2, w_2, \mu_2) \in V_h \times W_h \times M_h$, we have

$$\mathcal{A}(\varepsilon^q_h, \varepsilon^u_h, \varepsilon^9_h; r_2, w_2, \mu_2) = (q \cdot n - \Pi_k^q q \cdot n, \mu_2 - w_2)_{\partial T_h}$$

$$+ (h^{-1}_K (\Pi_{k+1}^q u - u), \Pi_k^9 w_2 - \mu_2)_{\partial T_h}. \quad (38)$$

### 5.2.2 Step 2: Energy Arguments

**Lemma 5.6.** Let $(q, u)$ and $(q_{lh}, u_{lh})$ be the solution of (1) and (36b), respectively, then we have the error estimate

$$\|q - q_{lh}\|_{T_h} + ||h^{-1/2}_K (\Pi_k^9 u_{lh} - \hat{u}_{lh})\|_{\partial T_h} \leq Ch^{k+1} |u|_{k+2}. \quad (38)$$

Proof. We take $(r_2, w_2, \mu_2) = (\varepsilon^q_h, \varepsilon^u_h, \varepsilon^9_h)$ in (38) to get

$$\|\varepsilon^q_h\|_{T_h}^2 + ||h^{-1/2}_K (\Pi_k^9 \varepsilon^u_h - \varepsilon^9_h)\|_{\partial T_h}^2$$

$$= (q \cdot n - \Pi_k^q q \cdot n, \varepsilon^9_h - \varepsilon^u_h)_{\partial T_h} + (h^{-1}_K (\Pi_{k+1}^q u - u), \Pi_k^9 \varepsilon^u_h - \varepsilon^9_h)_{\partial T_h}$$

$$\leq Ch^{k+1} |u|_{k+2} \left(\|\varepsilon^q_h\|^2_{T_h} + ||h^{-1/2}_K (\Pi_k^9 \varepsilon^u_h - \varepsilon^9_h)\|^2_{\partial T_h}\right)^{1/2},$$

where we have used the fact $\mathcal{A}(\varepsilon^q_h, \varepsilon^u_h, \varepsilon^9_h; r_2, 0, 0) = 0$ and Lemma 3.4. Then the desired result holds by the Cauchy-Schwarz inequality. \quad \square
5.2.3 Step 3: $L^2$ estimation by a dual argument

Similar to Lemma 5.4 we have the following results.

**Lemma 5.7.** For all $(r_2, w_2, \mu_2) \in Q_h \times V_h \times \widehat{V}_h$, we have the equation

$$
\mathcal{A}(\Pi^o_k \Psi, \Pi^o_{k+1} \Phi, \Pi^o_k \Phi; r_2, w_2, \mu_2) = (\Theta, w_2) + (\Psi \cdot n - \Pi^o_k \Psi \cdot n, \mu_2 - w_2)_{\partial T_h} + \langle h^{-1}_K (\Pi^o_{k+1} \Phi - \Phi), \Pi^o_2 \mu - \mu_2 \rangle_{\partial T_h},
$$

(39)

**Lemma 5.8.** Let $u$ and $u_{Ih}$ be the solution of (36a) and (36b), respectively, then we have the error estimate

$$
\|u - u_{Ih}\|_{\partial T_h} \leq C h^{k+2} |u|_{k+2}.
$$

**Proof.** We take $(r_2, w_2, \mu_2) = (\varepsilon_h^q, -\varepsilon_h^u, -\varepsilon_h^w)$ and $\Theta = -\varepsilon_h^u$ in (39) to get

$$
\|\varepsilon_h^u\|_{\partial T_h}^2 = \mathcal{A}(\Pi^o_k \Psi, \Pi^o_{k+1} \Phi, \Pi^o_k \Phi; \varepsilon_h^q, -\varepsilon_h^u, -\varepsilon_h^w) + \langle \Pi^o_k \Psi \cdot n - \Psi \cdot n, \varepsilon_h^u - \varepsilon_h^u \rangle_{\partial T_h} + \langle h^{-1}_K (\Pi^o_{k+1} \Phi - \Phi), \Pi^o_2 \varepsilon_h^u - \varepsilon_h^w \rangle_{\partial T_h}.
$$

By Lemma 5.2 the error equation (38) and (39), we have

$$
\|\varepsilon_h^u\|_{\partial T_h}^2 = \mathcal{A}(\varepsilon_h^q, \varepsilon_h^u, \varepsilon_h^w; \Pi^o_k \Psi, -\Pi^o_{k+1} \Phi, -\Pi^o_k \Phi;)
$$

$$
+ \langle \Pi^o_k \Psi \cdot n - \Psi \cdot n, \varepsilon_h^u - \varepsilon_h^u \rangle_{\partial T_h} + \langle h^{-1}_K (\Pi^o_{k+1} \Phi - \Phi), \Pi^o_2 \varepsilon_h^u - \varepsilon_h^w \rangle_{\partial T_h}
$$

$$
= -\langle \Pi^o_k q \cdot n - q \cdot n, \Pi^o_2 \Phi - \Pi^o_{k+1} \Phi \rangle_{\partial T_h}
$$

$$
- \langle h^{-1}_K (\Pi^o_{k+1} u - u), \Pi^o_2 \Pi^o_{k+1} \Phi - \Pi^o_k \Phi \rangle_{\partial T_h}
$$

$$
+ \langle \Pi^o_k \Psi \cdot n - \Psi \cdot n, \varepsilon_h^u - \varepsilon_h^u \rangle_{\partial T_h} + \langle h^{-1}_K (\Pi^o_{k+1} \Phi - \Phi), \Pi^o_2 \varepsilon_h^u - \varepsilon_h^w \rangle_{\partial T_h}
$$

$$
\leq C h^{k+2} |u|_{k+2} \|\varepsilon_h^u\|_{\partial T_h},
$$

which gives

$$
\|\varepsilon_h^u\|_{\partial T_h} \leq C h^{k+2} |u|_{k+2}.
$$

(40)

A triangle equality applied on (40) will get our final result.

5.3 $H^{-1}$ estimation

To give the $H^{-1}$ estimation, we need to introduce a Scott-Zhang type interpolation $I_h^{k+2+d}$. The definition of $I_h^{k+2+d}$ is given in Section 5 for all $(u_h, \widehat{u}_h) \in L^2(T_h) \times L^2(\partial T_h)$, $I_h^{k+2+d}(u_h, \widehat{u}_h)|_K \in P^{k+2+d}(K)$ and satisfying

$$
(I_h^{k+2+d}(u_h, \widehat{u}_h), w_h)_K = (u_h, w_h)_K \quad \text{for all } w_h \in P^{k+1}(K),
$$

(41a)

$$
(I_h^{k+2+d}(u_h, \widehat{u}_h), \mu_h)_F = (\widehat{u}_h, \mu_h)_F \quad \text{for all } \mu_h \in P^{k+2}(F), F \subset \partial K.
$$

(41b)

Moreover, we have the following estimation
Lemma 5.9. For all \((u_h, \hat{u}_h) \in W_h \times M_h\), we have
\[
\|I_h^{k+2+d}(u_h, \hat{u}_h) - u_h\|_{\mathcal{T}_h} \leq C\|h_K^{1/2}(u_h - \hat{u}_h)\|_{\partial \mathcal{T}_h},
\]
\[
\|\nabla (I_h^{k+2+d}(u_h, \hat{u}_h) - u_h)\|_{\mathcal{T}_h} \leq C\|h_K^{1/2}(u_h - \hat{u}_h)\|_{\partial \mathcal{T}_h}.
\]

(42a)
(42b)

The proof of Lemma 5.9 is founded in Section 6.

Theorem 5.10. Let \((\phi, u)\) and \((\phi_{lh}, u_{lh})\) be the solution of (1) and (36a)- (36b), respectively. Then if \(k \geq 1\), we have the following error estimates
\[
\|\Pi_{k+1}^o u - u_{lh}\|_{-1,h} \leq Ch^{k+3}|u|_{k+2},
\]
\[
\|\Pi_{k+1}^o \phi - \phi_{lh}\|_{-1,h} \leq Ch^{k+3}|\phi|_{k+2}.
\]

(43a)
(43b)

Proof. We only give a proof for (43a), the proof for (43b) is similar.

Let \(\xi_h^u = \Pi_{k+1}^o u - u_{lh}\), by Definition 3.5 and (41a) to get
\[
\|\xi_h^u\|^2_{-1,h} = (\Pi W \xi_h^u, \xi_h^u)_{\mathcal{T}_h} = (7_k^{k+2+d}(\Pi W \xi_h^u, \Pi M \xi_h^u), \xi_h^u)_{\mathcal{T}_h}.
\]

(44)

We take \((r_2, w_2, \mu_2) = (\xi_h^q, -\xi_h^u, -\xi_h^\hat{u})\) and \(\Theta = -7_k^{k+2+d}(\Pi W \xi_h^u, \Pi M \xi_h^u)\) in (39), and use (41) to get
\[
\|\xi_h^u\|^2_{-1,h} = A(\Pi_k^o \Psi, \Pi_{k+1}^o \Phi, \Pi_k^o \Phi; \xi_h^q, -\xi_h^u, -\xi_h^\hat{u})
- \langle \Pi_k^o \Psi \cdot n - \Psi \cdot n, \xi_h^u - \xi_h^\hat{u}\rangle_{\partial \mathcal{T}_h}
- \langle h_K^{-1}(\Pi_{k+1}^o \Phi - \Phi), \Pi_k^o \xi_h^u - \xi_h^\hat{u}\rangle_{\partial \mathcal{T}_h}.
\]

By the property (8) and the error equation (38) to get
\[
\|\xi_h^u\|^2_{-1,h} = -\langle \Pi_k^o q \cdot n - q \cdot n, \Pi_k^o \Phi - \Pi_{k+1}^o \Phi\rangle_{\partial \mathcal{T}_h}
- \langle h_K^{-1}(\Pi_{k+1}^o u - u), \Pi_k^o \Pi_{k+1}^o \Phi - \Pi_k^o \Phi\rangle_{\partial \mathcal{T}_h}
- \langle \Pi_k^o \Psi \cdot n - \Psi \cdot n, \xi_h^u - \xi_h^\hat{u}\rangle_{\partial \mathcal{T}_h}
- \langle h_K^{-1}(\Pi_{k+1}^o \Phi - \Phi), \Pi_k^o \xi_h^u - \xi_h^\hat{u}\rangle_{\partial \mathcal{T}_h}
\leq Ch^{k+3}(\|\Phi\|_{H^3(\Omega)} + \|\Psi\|_{H^2(\Omega)}).
\]

Then by (35a) and (35b) we have
\[
\|\xi_h^u\|^2_{-1,h} \leq Ch^{k+3}|u|_{k+2}\|7_k^{k+2+d}(\Pi W \xi_h^u, \Pi M \xi_h^u)\|_{H^1(\Omega)}.
\]

(45)

Since \(\Pi W \xi_h^u \in W_h\), then by Lemma 3.9 and (42) we have
\[
\|7_k^{k+2+d}(\Pi W \xi_h^u, \Pi M \xi_h^u)\|_{H^1(\Omega)}
\leq C \left(\|\Pi W \xi_h^u\|_{\mathcal{T}_h} + \|\nabla \Pi W \xi_h^u\|_{\mathcal{T}_h} + \|h_K^{-1/2}(\Pi W \xi_h^u - \Pi M \xi_h^u)\|_{\partial \mathcal{T}_h} \right)
\]

26
By the inequality (13) and (12) to obtain
\[ a \text{ Young's inequality}. \]
then the desired result follows by combination Lemma 4.8, Lemma 4.10 and
From the Cauchy-Schwarz inequality, one gets
5.4 Estimate
Proof. We take \((u, w, h)\) in \(V_h \times W_h \times M_h\) we have

\[
(\Delta_h u_h, w_h)_{\Gamma_h} = -A(q_h, u_h, \hat{w}_h; r_h, w_h, \mu_h),
\]
where \((q_h, \hat{w}_h) \in V_h \times M_h\) satisfying

\[
A(q_h, u_h, \hat{w}_h; r_h, 0, \mu_h) = 0
\]
for all \((r_h, \mu_h) \in V_h \times M_h\).

Lemma 5.11. Let \(u_h^n\) be the solution of (5), then for all \(n = 1, 2 \cdots, N\) we have

\[
\|\Delta_h u_h^n\|_{\Gamma_h} \leq C \left( \frac{1}{4\epsilon} \|u_h^0\|^2 - 1\|\bar{\tau}_n\| + \frac{\epsilon}{2}\|q_h^0\|_{\Gamma_h} + \frac{\epsilon}{2}\|

\leq \epsilon^{-1} f^n(u_h^n), \Delta_h u_h^n)_{\Gamma_h} - \epsilon\|\Delta_h u_h^n\|_{\Gamma_h}^2 - (\phi_h^n, \Delta_h u_h^n)_{\Gamma_h} = 0.
\]
From the Cauchy-Schwarz inequality, one gets

\[
\epsilon\|\Delta_h u_h^n\|_{\Gamma_h}^2 = \epsilon^{-1} f^n(u_h^n), \Delta_h u_h^n)_{\Gamma_h} - (\phi_h^n, \Delta_h u_h^n)_{\Gamma_h}
\leq \epsilon^{-1}(\|u_h^n\|^2_{\Gamma_h} + \|u_h^n\|_{\Gamma_h})\|\Delta_h u_h^n\|_{\Gamma_h} + \|\phi_h^n\|_{\Gamma_h}\|\Delta_h u_h^n\|_{\Gamma_h},
\]
then the desired result follows by combination Lemma 4.8, Lemma 4.10 and
a Young's inequality. \(\Box\)
Lemma 5.12 \( (\text{Gagliardo-Nirenberg inequality, cf. } [1]) \). Let \( \Omega \subset \mathbb{R}^d \) be a bounded, connected, open set with Lipschitz boundary, \( 1 \leq q, r \leq \infty, \frac{1}{m} \leq \theta \leq 1 \) and
\[
\frac{1}{p} - \frac{j}{d} = \left( \frac{1}{r} - \frac{m}{d} \right) \theta + \frac{1 - \theta}{q}.
\]
Suppose that \( v \in L^q(\Omega) \) with \( \partial^\alpha v \in L^r(\Omega) \) for all \( |\alpha| = m \). There \( \partial^\beta \psi \in L^p(\Omega) \) for all \( |\beta| = j \), and there exists a constant \( C = C(d, j, k, p, q, r, \Omega) > 0 \) such that
\[
|v|_{j,p} \leq C \left( |v|_{m,r}^{\theta} |v|_{0,q}^{1-\theta} + |v|_{0,q} \right).
\]

Lemma 5.13. For all \( w_h \in W_h \), we have the inequality
\[
\|w_h\|_\infty \leq C \left( h^{2-d/2} \|\Delta_h w_h\|_{T_h} + \|\Delta_h w_h\|_{T_h}^{\frac{d}{2}(6-d)} \|w_h\|_{0,6}^{\frac{3(4-d)}{2(6-d)}} + \|w_h\|_{0,6} \right).
\]

Proof. Consider the following continuous problem: find \( w \in H^1(\Omega) \cap L^2(\Omega) \), such that
\[-\Delta w = -\Delta_h w_h \text{ in } \Omega.\]
Since \( \Omega \) is convex, it holds the regularity estimate
\[
\|w\|_{H^2(\Omega)} \leq C_{\text{reg}} \|\Delta_h w_h\|_{L^2(\Omega)}. \tag{46}
\]
By the results in Theorem 5.3 it holds the approximation property:
\[
\|w - w_h\|_{T_h} \leq Ch^2 \|w\|_{H^2(\Omega)}. \tag{47}
\]
By a triangle inequality, we have
\[
\|w_h\|_{0,\infty} \leq \|w_h - \mathcal{P}_h w\|_{0,\infty} + \|\mathcal{P}_h w - w\|_{0,\infty} + \|w\|_{0,\infty} := R_1 + R_2 + R_3,
\]
where \( \mathcal{P}_h w \) is a polynomial of degree \( k + 1 \) and is an conforming approximation of \( w \), satisfying
\[
\|w - \mathcal{P}_h w\|_{0,p} \leq Ch^{2-d/2+d/p} \|w\|_{H^2(\Omega)}, \tag{48}
\]
where \( p \in [1, \infty] \).

Now we estimate \( \{R_i\}_{i=1}^3 \) term by term. By an inverse inequality, a triangle inequality and (48), we get
\[
R_1 \leq Ch^{-d/2} \|w_h - \mathcal{P}_h w\|_{T_h} \leq Ch^{-d/2} (\|w_h - w\|_{T_h} + \|w - \mathcal{P}_h w\|_{T_h}) \leq Ch^{2-d/2} \|w\|_{H^2(\Omega)} \leq Ch^{2-d/2} \|\Delta_h w_h\|_{T_h},
\]
28
\[ R_2 \leq Ch^{2-d/2} \|w\|_{H^2(\Omega)} \leq Ch^{2-d/2} \|\Delta_h w_h\|_{T_h}. \]

To approximate the term \( R_3 \), we need to first estimate \( \|w\|_{0,6} \) as following

\[
\|w\|_{0,6} \leq \|w - \mathcal{P}_h w\|_{0,6} + \|\mathcal{P}_h w - w_h\|_{0,6} + \|w_h\|_{0,6}
\]

by a triangle inequality

\[
\leq Ch^{2-d/3} \|w\|_{H^2(\Omega)}
\]

by (48)

\[
+ Ch^{-d/3} \|\mathcal{P}_h w - w_h\|_{T_h} + \|w_h\|_{0,6}
\]

by an inverse inequality

\[
\leq Ch^{2-d/3} \|w\|_{H^2(\Omega)} + Ch^{-d/3} \|\mathcal{P}_h w - w\|_{T_h}
\]

by a triangle inequality

\[
+ Ch^{-d/3} \|w - w_h\|_{T_h} + \|w_h\|_{0,6}
\]

by (48), (47)

\[
\leq Ch^{2-d/3} \|\Delta_h w_h\|_{T_h} + \|w_h\|_{0,6}
\]

by (46).

Using the above inequality, and Lemma 5.12 with \( j = 0 \), \( p = \infty \), \( m = 2 \), \( r = 2 \), \( q = 6 \), \( \theta = \frac{d}{2(6-d)} \), we obtain

\[
R_3 \leq C \left( \|w\|_{H^2(\Omega)} \right) \left( Ch^{2-d/3} \|\Delta_h w_h\|_{T_h} + \|w_h\|_{0,6} \right)
\]

\[
\leq C \left( Ch^{2-d/3} \|\Delta_h w_h\|_{T_h} + \|w_h\|_{0,6} \right)
\]

\[
+ C \left( h^{2-d/3} \|\Delta_h w_h\|_{T_h} + \|w_h\|_{0,6} \right)
\]

\[
\leq C \left( h^{2-d/2} \|\Delta_h w_h\|_{T_h} + \|\Delta_h w_h\|_{0,6} + \|w_h\|_{0,6} \right)
\]

Using the above estimates, we obtain

\[
\|w_h\|_{0,\infty} \leq C \left( h^{2-d/2} \|\Delta_h w_h\|_{T_h} + \|\Delta_h w_h\|_{0,6} + \|w_h\|_{0,6} \right).
\]

Utilizing the above Lemma 5.13 and Lemma 5.13 we immediately have the following result.

**Lemma 5.14** (Boundness for \( u_h^n \) in \( L^\infty \) norm). Let \( u_h^n \) be the solution of (45), then for all \( n = 1, 2, \ldots, N \) we have

\[
\|u_h^n\|_{0,\infty} \leq C \left( \frac{1}{4} \|u_h^0\|^2 - 1 \right) + \frac{\epsilon}{2} \|q_h^0\|_{T_h} + \frac{\epsilon}{2} \|h^{-1/2} (\Pi_k u_h^n - \tilde{u}_h^n)\|_{\partial T_h}.
\]

### 5.5 Proof of Theorem 5.1

To simplify the notation, we define

\[
e_h^n := p_h^n - \tilde{p}_h^n, \quad e_h^n := \phi_h^n - \tilde{\phi}_h^n, \quad e_h^n := \tilde{\phi}_h^n - \tilde{\phi}_h^n,
\]

\[
e_h^n := q_h^n - \tilde{q}_h^n, \quad e_h^n := u_h^n - u_h^n, \quad e_h^n := \tilde{u}_h^n - \tilde{u}_h^n.
\]

(49a)
Lemma 5.15. For all \((r_1, w_1, \mu_1), (r_2, w_2, \mu_2) \in V_h \times W_h \times M_h\), we have the following error equations

\[
(\partial_t^+ e_h^n, w_1)_{\mathcal{T}_h} + \mathcal{A}(e_h^n, e_h^n, \hat{e}_h^n ; r_1, w_1, \mu_1) = (\partial_t^+ u_h^n - \partial_t u^n, w_1)_{\mathcal{T}_h}, \\
(50a)
\]

\[
e\mathcal{A}(e_h^n, e_h^n, \hat{e}_h^n ; r_2, w_2, \mu_2)_{\mathcal{T}_h} = (\phi^n - \phi_{Ih}^n, w_2)_{\mathcal{T}_h} + \epsilon^{-1}(f^n(u_h^n) - f(u^n), w_2)_{\mathcal{T}_h},
\]

(50b)

Proof. We use the definition of \(\mathcal{A}\) in (4) to get

\[
(\partial_t^+ e_h^n, w_1)_{\mathcal{T}_h} + \mathcal{A}(e_h^n, e_h^n, \hat{e}_h^n ; r_1, w_1, \mu_1)
= (\partial_t^+ u_h^n, w_1)_{\mathcal{T}_h} + \mathcal{A}(p_h^n, \phi_h^n, \hat{\phi}_h^n ; r_1, w_1, \mu_1)
- (\partial_t^+ u_h^n, w_1)_{\mathcal{T}_h} - \mathcal{A}(p_h^n, \phi_h^n, \hat{\phi}_h^n ; r_1, w_1, \mu_1)
= (\partial_t^+ u_h^n, w_1)_{\mathcal{T}_h} - (\Delta \phi^n, w_1)_{\mathcal{T}_h}
= (\partial_t^+ u_h^n - \partial_t u^n, w_1)_{\mathcal{T}_h}
\]

(49a)

Moreover, we have

\[
e\mathcal{A}(e_h^n, e_h^n, \hat{e}_h^n ; r_2, w_2, \mu_2)_{\mathcal{T}_h} - (e_h^n, w_2)_{\mathcal{T}_h}
= e\mathcal{A}(q_h^n, u_h^n, \hat{u}_h^n ; r_2, w_2, \mu_2)_{\mathcal{T}_h} - (\phi_{Ih}^n, w_2)_{\mathcal{T}_h}
- e\mathcal{A}(q_h^n, u_h^n, \hat{u}_h^n ; r_2, w_2, \mu_2)_{\mathcal{T}_h} + (\phi_h^n, w_2)_{\mathcal{T}_h}
= -e(\Delta u^n, w_2)_{\mathcal{T}_h} - (\phi_{Ih}^n, w_2)_{\mathcal{T}_h} + \epsilon^{-1}(f^n(u_h^n), w_2)_{\mathcal{T}_h}
= (\phi^n - \phi_{Ih}^n, w_2)_{\mathcal{T}_h} + \epsilon^{-1}(f^n(u_h^n) - f(u^n), w_2)_{\mathcal{T}_h}
\]

(49b)

Now we complete our proof.

\]

Lemma 5.16. For all \(n = 1, 2, \ldots, N\), we have the error estimate

\[
(\partial_t^+ e_h^n, \Pi_W e_h^n)_{\mathcal{T}_h} + \epsilon \|e_h^n\|_{\mathcal{T}_h}^2 + \epsilon \|h_K^{-1/2}(\Pi_k e_h^n - e_h^n)\|_{\mathcal{T}_h}^2
+ \epsilon^{-1}(f(u^n), (\Pi_W e_h^n)_{\mathcal{T}_h}
= (\phi^n, (\Pi_k e_h^n))_{\mathcal{T}_h} + (\partial_t^+ u_h^n - \partial_t u^n, \Pi_W e_h^n)_{\mathcal{T}_h}
\]

Proof. First, we take \((r_1, w_1, \mu_1) = -(\Pi_W e_h^n, -\Pi_W e_h^n, -\Pi_M e_h^n)\) in (51a) to get

\[
(\partial_t^+ e_h^n, \Pi_W e_h^n)_{\mathcal{T}_h} + \mathcal{A}(e_h^n, e_h^n, \hat{e}_h^n ; -\Pi_W e_h^n, \Pi_W e_h^n, \Pi_M e_h^n)
= (\partial_t^+ u_h^n - \partial_t u^n, \Pi_W e_h^n)_{\mathcal{T}_h}
\]

(51)

By Lemma 3.2 and Definition 3.3 we have

\[
\mathcal{A}(e_h^n, e_h^n, \hat{e}_h^n ; -\Pi_W e_h^n, \Pi_W e_h^n, \Pi_M e_h^n)
= \mathcal{A}(\Pi_W e_h^n, \Pi_W e_h^n, \Pi_M e_h^n ; -e_h^n, e_h^n, \hat{e}_h^n)
= (e_h^n, \phi_h^n)_{\mathcal{T}_h}
\]

(52)
The equation (51) and (52) imply
\[ (\partial_t^+ e_h^n, \Pi W e_h^n)_{\mathcal{T}_h} + (e_h^n, e_h^n)_{\mathcal{T}_h} = (\partial_t^+ u_{ih}^n - \partial_t u^n, \Pi W e_h^n)_{\mathcal{T}_h}. \] (53)

Next, we take \((r_2, w_2, \mu_2) = (e_h^n, e_h^n, e_h^n)\) in (50B) to get
\[
\epsilon \| e_h^n \|^2_{\mathcal{T}_h} + \epsilon \| h^{-1/2}(\Pi^2 e_h^n - e_h^n) \|^2_{\mathcal{T}_h} + \epsilon^{-1}(f(u^n) - f^n(u_h^n), e_h^n)_{\mathcal{T}_h} + \epsilon^{-1}(f(u^n) - f^n(u_h^n), e_h^n)_{\mathcal{T}_h} = (\phi^n - \phi^n_{ih}, e_h^n)_{\mathcal{T}_h}.
\] (54)

Add the above two equations (53) and (54) to get
\[
(\partial_t^+ e_h^n, \Pi W e_h^n)_{\mathcal{T}_h} + \epsilon \| e_h^n \|^2_{\mathcal{T}_h} + \epsilon \| h^{-1/2}(\Pi^2 e_h^n - e_h^n) \|^2_{\mathcal{T}_h} + \epsilon^{-1}(f(u^n) - f^n(u_h^n), e_h^n)_{\mathcal{T}_h} = (\phi^n - \phi^n_{ih}, e_h^n)_{\mathcal{T}_h} + (\partial_t^+ u_{ih}^n - \partial_t u^n, \Pi W e_h^n)_{\mathcal{T}_h}.
\]

Lemma 5.17. We have the following identity
\[
(\partial_t^+ e_h^n, \Pi W e_h^n)_{\mathcal{T}_h} = \frac{\| e_h^n \|^2_{-1,h} - \| e_h^{n-1} \|^2_{-1,h} + (\Delta t)^2 \| \partial_t^+ e_h^n \|^2_{-1,h}}{2\Delta t}.
\]

Proof. Utilizing Definition 3.5 and (1), one has
\[
(\partial_t^+ e_h^n, \Pi W e_h^n)_{\mathcal{T}_h} = A(\Pi V \partial_t^+ e_h^n, \Pi W \partial_t^+ e_h^n, \Pi M \partial_t^+ e_h^n : 0, \Pi W e_h^n, \Pi M e_h^n) = (\nabla \cdot \Pi V \partial_t^+ e_h^n, \Pi W e_h^n)_{\mathcal{T}_h} - (\n \cdot \Pi V \partial_t^+ e_h^n, \Pi M e_h^n)_{\mathcal{T}_h} + (h^{-1/2}(\Pi^2 e_h^n - \Pi M e_h^n), \partial_t^+ (\Pi^2 e_h^n - \Pi M e_h^n))_{\mathcal{T}_h},
\] (55)

Again, by Definition 3.5 for all \((r_h, w_h, \mu_h) \in V_h \times W_h \times M_h\), we have
\[
A(\Pi V e_h^n, \Pi W e_h^n, \Pi M e_h^n : r_h, w_h, \mu_h) = (e_h^n, w_h)_{\mathcal{T}_h}.
\] (56)

Take \(r_h = \Pi V \partial_t^+ e_h^n, w_h = \mu_h = 0\) in (56) and by (1) to get
\[
(\Pi W e_h^n, \nabla \cdot \Pi V \partial_t^+ e_h^n)_{\mathcal{T}_h} - (\Pi M e_h^n, \n \cdot \Pi V \partial_t^+ e_h^n)_{\mathcal{T}_h} = (\Pi V e_h^n, \Pi V \partial_t^+ e_h^n)_{\mathcal{T}_h} + (\Pi V e_h^n, \partial_t^+ \Pi V e_h^n)_{\mathcal{T}_h}.
\] (57)

Comparing (55), (57), and utilizing Lemma 4.3, the Definition 3.5 and (12), one gets
\[
(\partial_t^+ e_h^n, \Pi W e_h^n)_{\mathcal{T}_h} = (\partial_t^+ \Pi V e_h^n, \Pi V e_h^n)_{\mathcal{T}_h}
\]

31
We use the Cauchy-Schwarz inequality, Lemma 3.10 and (12) to obtain
\[ u_{\phi}, (\Delta t + \frac{1}{C}n) \leq \parallel u_{\phi} \parallel_{2,1} - \parallel u_{\phi} \parallel_{1,1} + (\Delta t)^2 \parallel \partial_t \| \parallel u_{\phi} \parallel_{2,1} - \parallel u_{\phi} \parallel_{1,1}. \]

Lemma 5.18. We have the following estimate
\[ (\partial_t^+ u^n_{\phi} - \partial_t u^n, 1, \Pi W e_h^n) \leq C \left( \parallel \delta_t (u_{\phi} - u) \parallel_{L^2(t_n, t_{n+1}; L^2(\Omega))) + (\Delta t)^2 \parallel \delta_t u \parallel_{L^2(t_n, t_{n+1}; L^2(\Omega))) \right) \| e_h^n \parallel_{-1,1}. \]

Proof. We first show that \( e_h^n \in \bar{W}_h \). For \( n \geq 1 \), we take \( w_1 = \mu_1 = 1, r_1 = 0 \) in (12) to get \( (\partial_t^+ u^n_{\phi}, 1)_{\bar{T}_h} = 0 \). This implies
\[ (u^n_{\phi}, 1)_{\bar{T}_h} = (u^n_{\phi} - u^n, 1)_{\bar{T}_h} = (u^n, 1)_{\bar{T}_h} = (u^n - u^n, 1)_{\bar{T}_h}. \]

Therefore, for all \( n = 1, 2 \ldots, N \), it holds
\[ (e_h^n, 1)_{\bar{T}_h} = (u^n_{\phi} - u^n, 1)_{\bar{T}_h} = (u^n, 1)_{\bar{T}_h} = (u^n - u^n, 1)_{\bar{T}_h}. \]

By (14) we have \( (u_t, 1)_\Omega - (\Delta \phi, 1)_\Omega = 0 \). Integration by parts and (14) give \( (\Delta \phi, 1)_\Omega = (\nabla \phi \cdot n, 1)_\partial \Omega = 0 \), which leads to \( (u_t, 1)_\Omega = 0 \). Therefore, for all \( t \in [0, T] \), we have \( (u(t), 1)_\Omega = (u^n, 1)_\Omega \). This implies for all \( n = 1, 2 \ldots, N \), we have
\[ (e_h^n, 1)_{\bar{T}_h} = (u^n - u^n, 1)_{\bar{T}_h} = 0. \]

We use the Cauchy-Schwarz inequality, Lemma 3.10 and (12) to obtain
\[ (\partial_t^+ u^n_{\phi} - \partial_t u^n, 1, \Pi W e_h^n) \leq C \left( \parallel \delta_t u^n_{\phi} - \partial_t u^n \parallel_{\bar{T}_h} \parallel \Pi W e_h^n \parallel_{\bar{T}_h} + (\Delta t)^2 \parallel e_h^n \parallel_{-1,1} \right). \]
Next, we estimate the term \( \| \partial_t^+ u_h^n - \partial_t u^n \|_{\mathcal{T}_h} \) by a triangle inequality
\[
\| \partial_t^+ u_h^n - \partial_t u^n \|_{\mathcal{T}_h} \leq \| \partial_t^+ (u_h^n - u^n) \|_{\mathcal{T}_h} + \| \partial_t^+ u^n - \partial_t u^n \|_{\mathcal{T}_h}
\]
\[
\leq C \| \partial_t (u_h^n - u^n) \|_{L^2(t_{n-1}, t_n; L^2(\Omega))} + C(\Delta t)^2 \| \partial_t u^n \|_{L^2(t_{n-1}, t_n; L^2(\Omega))}.
\]

\[\Box\]

**Theorem 5.19** (Error estimates in \(-1\) norm). When \( k \geq 1 \), we have the error estimate
\[
\max_{1 \leq n \leq N} \| e_h^n \|_{-1,h}^2 + 2\epsilon \Delta t \sum_{n=1}^N \left( \| e_h^n \|_{h}^2 + 2\| h_K^{-1/2} (\Pi_k^0 e_h^n - e_h^n) \|_{\Omega}^2 \right)
\]
\[
+ 2\Delta t \sum_{n=1}^N \| e_h^n \|_{\Theta,4}^2 + (\Delta t)^2 \sum_{n=1}^N \| \partial_t^+ e_h^n \|_{-1,h}^2
\]
\[
\leq C \left( (\Delta t)^2 + h^{2(k+3)} \right).
\]

**Proof.** By Lemma 5.16 Lemma 5.17 and Lemma 5.18 we have
\[
\| e_h^n \|_{-1,h}^2 - \| e_h^{n-1} \|_{-1,h}^2 + (\Delta t)^2 \| \partial_t^+ e_h^n \|_{-1,h}^2
\]
\[
2\Delta t
\]
\[
+ \epsilon \| e_h^n \|_{h}^2 + \epsilon \| h_K^{-1/2} (\Pi_k^0 e_h^n - e_h^n) \|_{\Omega}^2
\]
\[
+ \epsilon^{-1} (f(u^n) - f^n(u_h^n))_{\mathcal{T}_h}
\]
\[
\leq (\phi^n - \phi_h^n, e_h^n)_{\mathcal{T}_h} + \| u_h^n \|_{-1,h}^2
\]
\[
+ \left( \| \partial_t (u_h^n - u^n) \|_{L^2(t_{n-1}, t_n; L^2(\Omega))}^2 + (\Delta t)^2 \| \partial_t u^n \|_{L^2(t_{n-1}, t_n; L^2(\Omega))}^2 \right).
\]

First, we estimate the nonlinear term \( f(u^n) - f^n(u_h^n) \) to get
\[
f(u^n) - f^n(u_h^n) = f(u^n) - f(u_h^n) + f^n(u_h^n) - f^n(u_h^n)
\]
\[
= (u^n - u_h^n)^3 - 3u_h^n(u_h^n - u^n)^2 + (3(u^n)^2 - 1)(u^n - u_h^n)
\]
\[
+ (e_h^n)^3 - 3u_h^n(e_h^n)^2 + (3(u_h^n)^2 - 1)e_h^n.
\]

We use a triangle inequality, estimate (40), (76), the fact \( \| u^n \|_{0,\infty} \leq C \) to get
\[
\| u_h^n \|_{0,\infty} \leq \| u_h^n - \Pi_k^{k+1} u^n \|_{0,\infty} + \| \Pi_k^{k+1} u^n - u^n \|_{0,\infty} + \| u^n \|_{0,\infty}
\]
\[
\leq C h^{-d/2} \| u_h^n - \Pi_k^{k+1} u^n \|_{\mathcal{T}_h} + C h^{k+2-d/2} \| u^n \|_{k+2} + \| u^n \|_{0,\infty}
\]
\[
\leq C h^{k+2-d/2} \| u^n \|_{k+2} + \| u^n \|_{0,\infty}
\]
\[
\leq C.
\]
By a triangle inequality, (60), and Lemma 5.14 we have
\[
\|e_h^n\|_{0,\infty} \leq \|u^n_I\|_{0,\infty} + \|u^n_h\|_{0,\infty} \leq C. \tag{61}
\]
Therefore, the combination of \(\|u^n\|_{0,\infty} \leq C\), (59), (60), and (61) gives
\[
(f(u^n) - f^n(u^n_h), e^n_h)_T \geq -C\|u^n - u^n_I\|_T \|e^n_h\|_T - C\|e^n_h\|^{3,0,4} + \|e^n_h\|^{4,0,4} - C\|e^n_h\|^{2,0,4}_T. \tag{62}
\]
Next, by a triangle inequality, we have
\[
(\phi^n - \phi^n_{Ih}, e^n_h)_T \leq \frac{1}{2}\|\phi^n - \phi^n_{Ih}\|^{2,0,4}_T + \frac{1}{2}\|e^n_h\|^{2,0,4}_T.
\]
We add (63) from \(n = 1\) to \(n = m\), and use (62) to get
\[
\|e^n_h\|^{3,0,4}_1 - h + \varepsilon\Delta t \sum_{n=1}^m \left(\|e^n_h\|^{2,0,4}_T + \|h^{-1/2}_K (\Pi^0_h e^n_h - e^n_h)\|^{2,0,4}_{\partial T}\right)
\]
\[
\quad + \Delta t \sum_{n=1}^m \|e^n_h\|^{3,0,4} + (\Delta t)^2 \sum_{n=1}^m \|\partial_t^2 e^n_h\|^{2,0,4}_1 \leq C\Delta t \sum_{n=1}^m \|e^n_h\|^{3,0,4} + C \varepsilon\Delta t \sum_{n=1}^m \|\phi^n - \phi^n_{Ih}\|^{2,0,4}_T + \||u^n - u^n_{Ih}\|^{2,0,4}_T \leq 0.
\]
The estimate (61) implies
\[
\|e^n_h\|^{3,0,4}_0 \leq \|e^n_h\|_{0,\infty} \|e^n_h\|^{2,0,4}_T \leq C\|e^n_h\|^{2,0,4}_T.
\]
Now we estimate \(\|e^n_h\|^{2,0,4}_T\) in detail. First, by Lemma 3.3 we have
\[
\|e^n_h\|^{2,0,4}_T = A(\Pi V e^n_h, \Pi W e^n_h; e^n_h)
\]
\[
\leq C \left(\|\Pi V e^n_h\|_T + \|\Delta e^n_h\|_T + h^{-1/2}_K (\Pi^0_h \Pi W e^n_h - \Pi^0_h e^n_h)\|_{\partial T}\right)
\]
\[
\times \left(\|e^n_h\|_T + \|\nabla e^n_h\|_T + h^{-1/2}_K (\Pi^0_h e^n_h - e^n_h)\|_{\partial T}\right).
\]
Next, from (13) and (12) one gets
\[
\|\Pi V e^n_h\|_T + \|\Delta e^n_h\|_T + h^{-1/2}_K (\Pi^0_h \Pi W e^n_h - \Pi^0_h e^n_h)\|_{\partial T}
\]
\[
\leq C(\|\Pi V e^n_h\|_T + h^{-1/2}_K (\Pi^0_h \Pi W e^n_h - \Pi^0_h e^n_h)\|_{\partial T})
\]
34
Finally, Gronwall’s inequality gives the desired result.

Moreover, by the definition of $e^\circ_h^n$ (Error estimates in Theorem 5.20 to have

$$
\| e^g_h^n \|_T h + \| \nabla e^u_h^n \|_T h + \| h^{-1/2} K (\Pi_K^n e^u_h^n - e^{\circ}_h^n) \|_T h
\leq C(\| e^g_h^n \|_T h + \| h^{-1/2} K (\Pi_K^n e^u_h^n - e^{\circ}_h^n) \|_T h).
$$

Therefore, by Young’s inequality, we have

$$
\| e^h_m \|_{-1,h}^2 + \epsilon \Delta t \sum_{n=1}^{m} \left( \| e^g_h^n \|_{-1,h}^2 + \| h^{-1/2} K (\Pi_K^n e^u_h^n - e^{\circ}_h^n) \|_T h^2 \right)
\leq C \Delta t \sum_{n=1}^{m} \left( \| \phi^n - \phi^n_{1h} \|_T h^2 + \| u^n - u^n_{1h} \|_T h^2 \right) + C \Delta t \sum_{n=1}^{m} \| e^u_h^n \|_{0,3}^2
\leq C \Delta t \sum_{n=1}^{m} \| e^u_h^n \|_{-1,h}^2 + \| e^g_h^n \|_{-1,h}^2.
$$

Moreover, by the definition of $e^g_h^n$ we obtain

$$
\| e^g_h^n \|_{-1,h}^2 = \| u^n_{1h} - u^n_{2h} \|_{-1,h} = \| u^n_{1h} - \Pi_{k+1} u^n_0 \|_{-1,h} \leq C h^{k+3}.
$$

Finally, Gronwall’s inequality gives the desired result.  

\[ \square \]

**Theorem 5.20** (Error estimates in $L^2$ norm). We have the following error estimate

for $1 \leq n \leq N$

$$
\max_{1 \leq n \leq N} \| u^n - u^n_h \|_{T h} + \Delta t \sum_{n=1}^{N} \| \phi^n - \phi^n_{1h} \|_{T h}^2 + \| u^n - u^n_{1h} \|_{T h}^2 + \Delta t) \sum_{n=1}^{N} \| \nabla \phi^n \|_{T h}^2 + \| \phi^n_{2h} - \phi^n_{0h} \|_{T h}^2
\leq C((\Delta t)^2 + h^{2(k+2)}).
$$

\[ \text{Proof.} \] First, we take $(r_1, w_1, \mu_1) = (-e^g_h^n, e^u_h^n, e^{\circ}_h^n)$ in (50a) to get

$$
( \partial_t e^u_h^n, e^u_h^n )_{T h} + A(e^p_h^n, e^\phi_h^n, e^{\circ}_h^n, -e^g_h^n, e^u_h^n, e^{\circ}_h^n ) = ( \partial_t u^n_{1h} - \partial_t u^n, e^u_h^n )_{T h}.
$$

Next, we take $(r_2, w_2, \mu_2) = (e^p_h^n, -e^\phi_h^n, -e^{\circ}_h^n)$ in (50b) to get

$$
\| e^\phi_h^n \|_{T h}^2 + \epsilon A(e^g_h^n, e^\phi_h^n, e^{\circ}_h^n, e^p_h^n, -e^g_h^n, -e^{\circ}_h^n ) = -(\phi^n - \phi^n_{1h}, e^\phi_h^n )_{T h} - \epsilon^{-1} (f^n(u^n_{2h}) - f(u^n), e^\phi_h^n )_{T h}.
$$

35
Since $\|u_h^n\|_{0,\infty}$ bounded, then we have

$$|f^n(u_h^n) - f(u^n)| \leq |(u_h^n)^3 - (u^n)^3| + |u_h^n - u^n| \leq C|u_h^n - u^n|.$$  

Next, by Lemma 4.3 to have

$$(\partial_t^+ u_h^n, e_h^n)_{T_h} = \frac{\|e_h^n\|^2_{T_h} - \|e_h^{n-1}\|^2_{T_h} + (\Delta t)^2 \|\partial_t^+ e_h^n\|^2_{T_h}}{2\Delta t}.$$  

We multiply (63) form $n=1$ to $n=m$ to get

$$\|e_h^n\|^2_{T_h} + \Delta t \sum_{n=1}^m \|e_h^{n-1}\|^2_{T_h} + (\Delta t)^2 \sum_{n=1}^m \|\partial_t^+ e_h^n\|^2_{T_h} \leq C \Delta t \sum_{n=1}^m (\|\phi^n - \phi_{I_h}^n\|^2_{T_h} + \|\partial_t^+ u_{I_h}^n - \partial_t u^n\|^2_{T_h} + |u^n - u_h^n|_{T_h}^2).$$

Thus the desired result is followed by Gronwall’s inequality and a triangle inequality.

The combination of the above two theorems and a triangle inequality will give Theorem 5.1.

## 6 Appendix

### Definition 6.1.

For every $K \in \mathcal{T}_h$, we define $\mathcal{I}_h^{k+1+d}(u_h, \hat{u}_h) \in \mathcal{P}^{k+1+d}(K)$ as follows.

1. For every vertex $A_i$ on mesh $\mathcal{T}_h$, let $N_{A_i}$ be the number of elements adjacent at $A_i$, and $\mathcal{K}_{A_i}$ denote all these elements, then $\mathcal{I}_h^{k+1+d}$ at $A_i$ is defined as

$$\mathcal{I}_h^{k+1+d}(u_h, \hat{u}_h)(A_i) = \frac{1}{N_{A_i}} \sum_{K \in \mathcal{K}_{A_i}} \hat{u}_h(A_i)|_{\partial K},$$

we note that $N_{A_i}$ is a fixed finite number since $\mathcal{T}_h$ is shape-regular.

2. If, in addition, for $d = 3$, for every edge $E$ of element $K$, there are $k+d$ interior Lagrange points on edge $E$, for any of these points $B_i$, let $N_{B_i}$ be the number of elements adjacent at $B_i$, and $\mathcal{K}_{B_i}$ denote all these elements, then $\mathcal{I}_h^{k+1+d}$ at $B_i$ is defined as

$$\mathcal{I}_h^{k+1+d}(u_h, \hat{u}_h)(B_i) = \frac{1}{N_{B_i}} \sum_{K \in \mathcal{K}_{B_i}} \hat{u}_h(B_i)|_{\partial K}.$$
Again, \( N_B \) is finite since \( \mathcal{T}_h \) is shape-regular.

(3) Since \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) \in \mathcal{P}_k^{k+1+d}(K) \), there are \((k+d_1)\) Lagrange points on every face \( F \) of \( K \), the value of \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) \) on these points are determined by
\[
\langle I_{K}^{k+1+d}(u_h, \tilde{u}_h), \hat{v}_h \rangle_F = \langle \hat{u}_h, \hat{v}_h \rangle_F \quad \text{for all } \hat{v}_h \in \mathcal{P}_k^1(F), \quad (66c)
\]
holds for all face \( F \) of \( K \).

(4) Since \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) \in \mathcal{P}_k^{k+1+d}(K) \), there are \((k+d)\) Lagrange points in every element \( K \), the value of \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) \) on these points are determined by
\[
(I_{K}^{k+1+d}(u_h, \tilde{u}_h), v_h)_K = (u_h, v_h)_K \quad \text{for all } v_h \in \mathcal{P}_k^1(K), \quad (66d)
\]

It is easy to check that the degrees of freedom of \( \mathcal{P}_k^{k+1+d}(K) \) is \((k+1+2d)\), the constrains for (66a), (66b), (66c) and (66d) are \( d + 1, (d - 1)d(k + d), (d + 1)(k + d) \), and \((k + d)\). For \( d = 3 \), there holds
\[
\left(\begin{array}{c}
k + 1 + 2d \\
d
\end{array}\right) = d + 1 + d(d - 1)\left(\begin{array}{c}
k + d \\
1
\end{array}\right) + (d + 1)\left(\begin{array}{c}
k + d \\
d - 1
\end{array}\right) + \left(\begin{array}{c}
k + d \\
d
\end{array}\right);
\]
and for \( d = 2 \), it holds
\[
\left(\begin{array}{c}
k + 1 + 2d \\
d
\end{array}\right) = d + 1 + (d + 1)\left(\begin{array}{c}
k + d \\
d - 1
\end{array}\right) + \left(\begin{array}{c}
k + d \\
d
\end{array}\right).
\]

Then the definition of \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) \) is a square system, therefore, the uniqueness and the existence of \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) \) are equivalence. In addition, it is obviously that when \( u_h = \tilde{u}_h = 0 \) we have \( I_{K}^{k+1+d}(u_h, \tilde{u}_h) = 0 \), then the operator \( I_{K}^{k+1+d} \) is well-defined. We define \( I_{h}^{k+1+d}|_K = I_{K}^{k+1+d} \), if for all \((u_h, \tilde{u}_h) \in W_h \times M_h \), we have \( I_{h}^{k+1+d}(u_h, \tilde{u}_h) \) is unique defined at every face of \( \mathcal{T}_h \) due to (66a), (66b), and (66c). Then \( I_{h}^{k+1+d}(u_h, \tilde{u}_h) \in H^1(\Omega) \).

Lemma 6.2. For all \((u_h, \tilde{u}_h) \in L^2(\mathcal{T}_h) \times L^2(\partial \mathcal{T}_h) \), we have the stability:
\[
\|I_{h}^{k+1+d}(u_h, \tilde{u}_h)\|_K \leq C \left(\|u_h\|_{S(K)} + \|h_{K}^{1/2} \tilde{u}_h\|_{\partial S(K)}\right),
\]
where \( S(K) \) is the set of all the simplex \( K^* \in \mathcal{T}_h \) such that \( K^* \) and \( K \) has at least one common node, and \( \partial S(K) \) is the set of all the faces of those simplex.

Proof. To simplify the proof, we only give a proof for \( d = 3 \), the proof of \( d = 2 \) is similar. According to (66c), we divide the Lagrange points on \( K \) of degree \( k + 1 + d \) into 4 parts, and the corresponding Lagrange basis denoted
as $\{\phi_{1,j}\}_{j=1}^{N_1}$, $\{\phi_{2,j}\}_{j=1}^{N_2}$, $\{\phi_{3,j}\}_{j=1}^{N_3}$, and $\{\phi_{4,j}\}_{j=1}^{N_4}$, which are determined by (1)(2)(3)(4) in (68), respectively. It is known that $\phi_{4,j}|_{\partial K} = 0$, since the corresponding Lagrange points are inside $K$. We also denote the dual basis of $\{\phi_{3,j}\}_{j=1}^{N_3}$ and $\{\phi_{4,j}\}_{j=1}^{N_4}$ as $\{\psi_{3,j}\}_{j=1}^{N_3}$ and $\{\psi_{4,j}\}_{j=1}^{N_4}$, respectively, such that

$$\langle \phi_{3,j}, \psi_{3,k} \rangle_F = \delta_{j,k}, \quad \langle \phi_{4,j}, \psi_{4,k} \rangle_K = \delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker delta. A result in [, Lemma 3.1] show that

$$\|\psi_{3,j}\|_{0,\infty,F} \leq Ch_F^{-(d-1)}, \quad \|\phi_{4,j}\|_{0,\infty,K} \leq Ch_K^{-d}. \quad (67)$$

We can write $T_{K}^{k+1+d}(u_h, \hat{u}_h)$ as

$$T_{K}^{k+1+d}(u_h, \hat{u}_h) = \sum_{i=1}^{N_1} a_{1,i} \phi_{1,i} + \sum_{i=1}^{N_2} a_{2,i} \phi_{2,i} + \sum_{i=1}^{N_3} a_{3,i} \phi_{3,i} + \sum_{i=1}^{N_4} a_{4,i} \phi_{4,i}, \quad (68)$$

where

$$a_{1,i} = \frac{1}{N_{A_i}} \sum_{K \in K_{A_i}} \hat{u}_h(A_i)|_{\partial K}, \quad (69a)$$

$$a_{2,i} = \frac{1}{N_{B_i}} \sum_{K \in K_{B_i}} \hat{u}_h(B_i)|_{\partial K}, \quad (69b)$$

$$a_{3,j} = \langle \hat{u}_h - \sum_{i=1}^{N_1} a_{1,i} \phi_{1,i} - \sum_{i=1}^{N_2} a_{2,i} \phi_{2,i}, \psi_{3,j} \rangle_F, \quad (69c)$$

$$a_{4,j} = (u_h - \sum_{i=1}^{N_1} a_{1,i} \phi_{1,i} - \sum_{i=1}^{N_2} a_{2,i} \phi_{2,i} - \sum_{i=1}^{N_3} a_{3,i} \phi_{3,i}, \psi_{4,j})_K, \quad (69d)$$

according to (69). By a scaling argument, one can get

$$\|\phi_{1,i}\|_{0,p,K} \leq Ch_{K}^{d/p} \|\hat{\phi}_{1,p,j}\|_{0,p,K} \leq Ch_{K}^{d/p}, \quad (70a)$$

$$\|\phi_{1,i}\|_{0,p,F} \leq Ch_{F}^{(d-1)/p} \|\hat{\phi}_{1,p,j}\|_{0,p,F} \leq Ch_{F}^{(d-1)/p}, \quad (70b)$$

for any integer $p \geq 1$. Again, by a scaling argument, for the Lagrange point $A_i$ on a face $F \subset \partial K$, and $A_i$ is also the vertex of $T_h$, one can get

$$|a_{1,i}| \leq \frac{1}{N_{A_i}} \sum_{A_i \in F} \|\hat{u}_h\|_{0,\infty,F} \leq C \sum_{A_i \in F} \|\hat{u}_h\|_{0,\infty,F} \leq C \sum_{A_i \in F} h_{F}^{-(d-1)/2} \|\hat{u}_h\|_{0,F}, \quad (71)$$

and the similar for $a_{2,i}$, for the Lagrange point $B_i$ on an edge $E \subset \partial F$, $F \subset \partial K$:

$$|a_{2,i}| \leq C \sum_{B_i \in F} h_{F}^{-(d-1)/2} \|\hat{u}_h\|_{0,F}. \quad (72)$$

38
We use (69c), (71), (72), (67), (70a), (71), (72) and a scaling argument, to get

\[
|a_{3,j}| \leq \left( \|\tilde{u}_h\|_{0,1,F} + \sum_{i=1}^{N_1} |a_{1,i}| \cdot \|\phi_{1,i}\|_{0,1,F} + \sum_{i=1}^{N_2} |a_{2,i}| \cdot \|\phi_{1,i}\|_{0,1,F} \right) \|\psi_{3,j}\|_{0,\infty,F} \\
\leq C \left( h_F^{-(d-1)/2} \|\tilde{u}_h\|_{0,F} + \sum_{A_i \in F} h_F^{-(d-1)/2} \|\tilde{u}_h\|_{0,F} + \sum_{B_i \in F} h_F^{-(d-1)/2} \|\tilde{u}_h\|_{0,F} \right).
\]

(73)

Then desired result is followed by (68), (71), (72), (73), (74) and (70a) with \(p = 2\).

Proof of Lemma 5.9. Since \(I_h^{k+2+d}(u_h, u_h) = u_h\) for every \(u_h \in V_h\), then (42d) followed by Lemma 6.2 and the fact \(I_h^{k+2+d}\) is linear, immediately, and (42b) followed by an inverse inequality.

7 Numerical Experiments

We consider two examples on unit square domains in \(\mathbb{R}^2\). In the first example we have an explicit solution of the system (1); in the second example an explicit form for the exact solution is not known.

Example 7.1. The problem data \(u^0\) and the artificial \(f\) are chosen so that the exact solution of the system (1) is given by

\[
\varepsilon = 1, \quad u = \phi = e^{-t}x^2y^2(1-x)^2(1-y)^2.
\]

We report the errors at the final time \(T = 1\) for polynomial degrees \(k = 0\) and \(k = 1\) in Tables 1 and 2 for the fully implicit scheme and Tables 3 and 4 for the energy-splitting scheme. The observed convergence rates match the theory, where \(\Delta t = h^{k+1}\).
Table 1: Example 7.1, $k = 0$ with fully implicit scheme: Errors, observed convergence orders for $u$, $\phi$ and their fluxes $q$ and $p$.

| $h/\sqrt{2}$ | 1/4     | 1/8     | 1/16    | 1/32    | 1/64    |
|-------------|---------|---------|---------|---------|---------|
| $\|q - q_h\|_{T_h}$ | 8.6745E-04 | 4.8767E-04 | 2.5058E-04 | 1.2614E-04 | 6.3177E-05 |
| order       | -       | 0.83088 | 0.96061 | 0.99025 | 0.99757 |
| $\|p - p_h\|_{T_h}$ | 8.9032E-04 | 4.9104E-04 | 2.5102E-04 | 1.2620E-04 | 6.3184E-05 |
| order       | -       | 0.85847 | 0.96803 | 0.99214 | 0.99804 |
| $\|u - u_h\|_{T_h}$ | 2.5400E-04 | 6.6287E-05 | 1.6746E-05 | 4.1975E-06 | 1.0501E-06 |
| order       | -       | 1.9380  | 1.9849  | 1.9962  | 1.9990  |
| $\|\phi - \phi_h\|_{T_h}$ | 2.5400E-04 | 6.6287E-05 | 1.6746E-05 | 4.1975E-06 | 1.0501E-06 |
| order       | -       | 1.9380  | 1.9849  | 1.9962  | 1.9990  |

Table 2: Example 7.1, $k = 1$ with fully implicit scheme: Errors, observed convergence orders for $u$, $\phi$ and their fluxes $q$ and $p$.

| $h/\sqrt{2}$ | 1/4     | 1/8     | 1/16    | 1/32    | 1/64    |
|-------------|---------|---------|---------|---------|---------|
| $\|q - q_h\|_{T_h}$ | 1.6623E-04 | 4.5233E-05 | 1.1590E-05 | 2.9202E-06 | 7.3141E-07 |
| order       | -       | 1.8778  | 1.9634  | 1.9899  | 1.9973  |
| $\|p - p_h\|_{T_h}$ | 1.6700E-04 | 4.5276E-05 | 1.1602E-05 | 2.9204E-06 | 7.3142E-07 |
| order       | -       | 1.8830  | 1.9644  | 1.9901  | 1.9974  |
| $\|u - u_h\|_{T_h}$ | 4.8698E-05 | 6.1714E-06 | 7.7349E-07 | 9.6742E-08 | 1.2094E-08 |
| order       | -       | 2.9802  | 2.9962  | 2.9992  | 2.9998  |
| $\|\phi - \phi_h\|_{T_h}$ | 4.9152E-05 | 6.1862E-06 | 7.7391E-07 | 9.6753E-08 | 1.2095E-08 |
| order       | -       | 2.9901  | 2.9988  | 2.9998  | 3.0000  |

Table 3: Example 7.1, $k = 0$ with energy-splitting scheme: Errors, observed convergence orders for $u$, $\phi$ and their fluxes $q$ and $p$.

| $h/\sqrt{2}$ | 1/4     | 1/8     | 1/16    | 1/32    | 1/64    |
|-------------|---------|---------|---------|---------|---------|
| $\|q - q_h\|_{T_h}$ | 8.6761E-04 | 4.8767E-04 | 2.5058E-04 | 1.2614E-04 | 6.3177E-05 |
| order       | -       | 0.83088 | 0.96061 | 0.99025 | 0.99757 |
| $\|p - p_h\|_{T_h}$ | 8.9032E-04 | 4.9104E-04 | 2.5102E-04 | 1.2620E-04 | 6.3184E-05 |
| order       | -       | 0.85847 | 0.96803 | 0.99214 | 0.99804 |
| $\|u - u_h\|_{T_h}$ | 2.5400E-04 | 6.6287E-05 | 1.6746E-05 | 4.1975E-06 | 1.0501E-06 |
| order       | -       | 1.9380  | 1.9849  | 1.9962  | 1.9990  |
| $\|\phi - \phi_h\|_{T_h}$ | 2.6147E-04 | 6.7626E-05 | 1.7040E-05 | 4.2683E-06 | 1.0676E-06 |
| order       | -       | 1.9510  | 1.9887  | 1.9972  | 1.9993  |
Table 4: Example 7.1, $k = 1$ with energy-splitting scheme: Errors, observed convergence orders for $u$, $\phi$ and their fluxes $q$ and $p$.

| $h/\sqrt{2}$ | 1/4   | 1/8   | 1/16  | 1/32  | 1/64  |
|-------------|-------|-------|-------|-------|-------|
| $\|q - q_h\|_{T_h}$ | 1.5809E-04 | 4.3945E-05 | 1.1415E-05 | 2.8955E-06 | 7.2935E-07 |
| order      | -     | 1.8470 | 1.9448 | 1.9790 | 1.9891 |
| $\|p - p_h\|_{T_h}$ | 1.5896E-04 | 4.3991E-05 | 1.1418E-05 | 2.8957E-06 | 7.2940E-07 |
| order      | -     | 1.8534 | 1.9459 | 1.9793 | 1.9891 |
| $\|u - u_h\|_{T_h}$ | 4.9741E-05 | 6.3026E-06 | 7.9008E-07 | 9.8850E-08 | 1.2358E-08 |
| order      | -     | 2.9804 | 2.9959 | 2.9987 | 2.9998 |
| $\|\phi - \phi_h\|_{T_h}$ | 4.9111E-05 | 6.1809E-06 | 7.7336E-07 | 9.6709E-08 | 1.2090E-08 |
| order      | -     | 2.9902 | 2.9986 | 2.9994 | 2.9998 |

Acknowledgement.

The first author’s work is supported by National Natural Science Foundation of China (NSFC) grant no. 11801063, China Postdoctoral Science Foundation project no. 2018M633339, and Key Laboratory of Numerical Simulation of Sichuan Province (Neijiang, Sichuan Province) grant no. 2017KF003. The second author acknowledges the support of a seed fund from the Material Research Center of Missouri University of Science and Technology.

References

[1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] D. M. Anderson, G. B. McFadden, and A. A. Wheeler. Diffuse-interface methods in fluid mechanics. In *Annual review of fluid mechanics, Vol. 30*, volume 30 of *Annu. Rev. Fluid Mech.*, pages 139–165. Annual Reviews, Palo Alto, CA, 1998.

[3] Andreas C. Aristotelous, Ohannes A. Karakashian, and Steven M. Wise. Adaptive, second-order in time, primitive-variable discontinuous Galerkin schemes for a Cahn-Hilliard equation with a mass source. *IMA J. Numer. Anal.*, 35(3):1167–1198, 2015.

[4] Ivo Babuška and Miloš Zlámal. Nonconforming elements in the finite element method with penalty. *SIAM J. Numer. Anal.*, 10:863–875, 1973.

[5] Garth A. Baker. Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.*, 31(137):45–59, 1977.
[6] John W. Barrett, James F. Blowey, and Harald Garcke. Finite element approximation of the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Numer. Anal.*, 37(1):286–318 (electronic), 1999.

[7] A. L. Bertozzi and M. Pugh. The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Comm. Pure Appl. Math.*, 49(2):85–123, 1996.

[8] John W. Cahn. Free energy of a nonuniform system. ii. thermodynamic basis. *The Journal of Chemical Physics*, 30(5):1121–1124, 1959.

[9] John W. Cahn and John E. Hilliard. Free energy of a nonuniform system. i. interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958.

[10] John W. Cahn and John E. Hilliard. Free energy of a nonuniform system. iii. nucleation in a twocomponent incompressible fluid. *The Journal of Chemical Physics*, 31(3):688–699, 1959.

[11] Kelong Cheng, Wenqiang Feng, Cheng Wang, and Steven M. Wise. An energy stable fourth order finite difference scheme for the cahn–hilliard equation. *Journal of Computational and Applied Mathematics*, 2018.

[12] Bernardo Cockburn, Jayadeep Gopalakrishnan, and Raytcho Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, 2009.

[13] Bernardo Cockburn, Jayadeep Gopalakrishnan, and Francisco-Javier Sayas. A projection-based error analysis of HDG methods. *Math. Comp.*, 79(271):1351–1367, 2010.

[14] Amanda E. Diegel, Cheng Wang, and Steven M. Wise. Stability and convergence of a second-order mixed finite element method for the Cahn-Hilliard equation. *IMA J. Numer. Anal.*, 36(4):1867–1897, 2016.

[15] Bo Dong and Chi-Wang Shu. Analysis of a local discontinuous Galerkin method for linear time-dependent fourth-order problems. *SIAM J. Numer. Anal.*, 47(5):3240–3268, 2009.

[16] Qiang Du and R. A. Nicolaides. Numerical analysis of a continuum model of phase transition. *SIAM J. Numer. Anal.*, 28(5):1310–1322, 1991.

[17] C. M. Elliott and A. M. Stuart. The global dynamics of discrete semilinear parabolic equations. *SIAM J. Numer. Anal.*, 30(6):1622–1663, 1993.

42
[18] Charles M. Elliott and Donald A. French. A nonconforming finite-
element method for the two-dimensional Cahn-Hilliard equation. SIAM
J. Numer. Anal., 26(4):884–903, 1989.

[19] Charles M. Elliott and Stig Larsson. Error estimates with smooth and
nonsmooth data for a finite element method for the Cahn-Hilliard equa-
tion. Math. Comp., 58(198):603–630, S33–S36, 1992.

[20] David J. Eyre. Unconditionally gradient stable time marching the
Cahn-Hilliard equation. In Computational and mathematical models
of microstructural evolution (San Francisco, CA, 1998), volume 529 of
Mater. Res. Soc. Symp. Proc., pages 39–46. MRS, Warrendale, PA,
1998.

[21] Xiaobing Feng and Ohannes A. Karakashian. Fully discrete dynamic
mesh discontinuous Galerkin methods for the Cahn-Hilliard equation of
phase transition. Math. Comp., 76(259):1093–1117 (electronic), 2007.

[22] Xiaobing Feng, Yukun Li, and Yulong Xing. Analysis of mixed interior
penalty discontinuous Galerkin methods for the Cahn-Hilliard equation
and the Hele-Shaw flow. SIAM J. Numer. Anal., 54(2):825–847, 2016.

[23] Xiaobing Feng and Andreas Prohl. Error analysis of a mixed finite el-
ement method for the Cahn-Hilliard equation. Numer. Math., 99(1):47–
84, 2004.

[24] Florian Frank, Chen Liu, Faruk O. Alpak, and Beatrice Riviere. A finite
volume/discontinuous Galerkin method for the advective Cahn-Hilliard
equation with degenerate mobility on porous domains stemming from
micro-CT imaging. Comput. Geosci., 22(2):543–563, 2018.

[25] Daisuke Furihata. A stable and conservative finite difference scheme
for the Cahn-Hilliard equation. Numer. Math., 87(4):675–699, 2001.

[26] Venkat Ganesan and Howard Brenner. A diffuse interface model of two-
phase flow in porous media. R. Soc. Lond. Proc. Ser. A Math. Phys.
Eng. Sci., 456(1996):731–803, 2000.

[27] Jing Guo, Cheng Wang, Steven M. Wise, and Xingye Yue. An $H^2$
convergence of a second-order convex-splitting, finite difference scheme
for the three-dimensional Cahn-Hilliard equation. Commun. Math. Sci.,
14(2):489–515, 2016.

[28] Ruihan Guo, Yinhua Xia, and Yan Xu. An efficient fully-discrete local
discontinuous Galerkin method for the Cahn-Hilliard-Hele-Shaw sys-
tem. J. Comput. Phys., 264:23–40, 2014.
Ruihan Guo and Yan Xu. Efficient solvers of discontinuous Galerkin discretization for the Cahn-Hilliard equations. *J. Sci. Comput.*, 58(2):380–408, 2014.

Ruihan Guo and Yan Xu. An efficient, unconditionally energy stable local discontinuous Galerkin scheme for the Cahn-Hilliard-Brinkman system. *J. Comput. Phys.*, 298:387–405, 2015.

Ohannes A. Karakashian and Frederic Pascal. Convergence of adaptive discontinuous Galerkin approximations of second-order elliptic problems. *SIAM J. Numer. Anal.*, 45(2):641–665, 2007.

David Kay, Vanessa Styles, and Endre Süli. Discontinuous Galerkin finite element approximation of the Cahn-Hilliard equation with convection. *SIAM J. Numer. Anal.*, 47(4):2660–2685, 2009.

Dong Li and Zhonghua Qiao. On second order semi-implicit Fourier spectral methods for 2D Cahn-Hilliard equations. *J. Sci. Comput.*, 70(1):301–341, 2017.

J. Lowengrub and L. Truskinovsky. Quasi-incompressible Cahn-Hilliard fluids and topological transitions. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 454(1978):2617–2654, 1998.

I. Mozolevski and P. R. Bösing. Sharp expressions for the stabilization parameters in symmetric interior-penalty discontinuous Galerkin finite element approximations of fourth-order elliptic problems. *Comput. Methods Appl. Math.*, 7(4):365–375, 2007.

Igor Mozolevski and Endre Süli. A priori error analysis for the $hp$-version of the discontinuous Galerkin finite element method for the biharmonic equation. *Comput. Methods Appl. Math.*, 3(4):596–607, 2003.

Igor Mozolevski, Endre Süli, and Paulo R. Bösing. $hp$-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation. *J. Sci. Comput.*, 30(3):465–491, 2007.

Igor Mozolevski, Endre Süli, and Paulo Rafael Bösing. Discontinuous Galerkin finite element approximation of the two-dimensional Navier-Stokes equations in stream-function formulation. *Comm. Numer. Methods Engrg.*, 23(6):447–459, 2007.

Issei Oikawa. A hybridized discontinuous Galerkin method with reduced stabilization. *J. Sci. Comput.*, 65(1):327–340, 2015.

Weifeng Qiu, Jiguang Shen, and Ke Shi. An HDG method for linear elasticity with strong symmetric stresses. *Math. Comp.*, 87(309):69–93, 2018.
[41] Weifeng Qiu and Ke Shi. A superconvergent HDG method for the incompressible Navier-Stokes equations on general polyhedral meshes. *IMA J. Numer. Anal.*, 36(4):1943–1967, 2016.

[42] Jie Shen, Jie Xu, and Jiang Yang. The scalar auxiliary variable (SAV) approach for gradient flows. *J. Comput. Phys.*, 353:407–416, 2018.

[43] Jie Shen and Xiaofeng Yang. An efficient moving mesh spectral method for the phase-field model of two-phase flows. *J. Comput. Phys.*, 228(8):2978–2992, 2009.

[44] Huailing Song and Chi-Wang Shu. Unconditional energy stability analysis of a second order implicit-explicit local discontinuous Galerkin method for the Cahn-Hilliard equation. *J. Sci. Comput.*, 73(2-3):1178–1203, 2017.

[45] Endre Süli and Igor Mozolevski. *hp*-version interior penalty DGFEMs for the biharmonic equation. *Comput. Methods Appl. Mech. Engrg.*, 196(13-16):1851–1863, 2007.

[46] S.M. Wise, J.S. Lowengrub, and V. Cristini. An adaptive multigrid algorithm for simulating solid tumor growth using mixture models. *Mathematical and Computer Modelling*, 53(12):1–20, 2011.

[47] Michael A. Zaks, Alla Podolny, Alexander A. Nepomnyashchy, and Alexander A. Golovin. Periodic stationary patterns governed by a convective Cahn-Hilliard equation. *SIAM J. Appl. Math.*, 66(2):700–720 (electronic), 2005.