Randomized Triangle Algorithms for Convex Hull Membership

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Abstract
We present randomized versions of the triangle algorithm introduced in [9]. The triangle algorithm tests membership of a distinguished point \( p \in \mathbb{R}^m \) in the convex hull of a given set \( S \) of \( n \) points in \( \mathbb{R}^m \). Given any iterate \( p' \in \text{conv}(S) \), it searches for a pivot, a point \( v \in S \) so that \( d(p', v) \geq d(p, v) \). It replaces \( p' \) with the point on the line segment \( p'v \) closest to \( p \) and repeats this process. If a pivot does not exist, \( p' \) certifies that \( p \not\in \text{conv}(S) \). Here we propose two random variations of the triangle algorithm that allow relaxed steps so as to take more effective steps possible in subsequent iterations. One is inspired by the chaos game known to result in the Sierpinski triangle. The incentive is that randomized iterates together with a property of Sierpinski triangle would result in effective pivots. Bounds on their expected complexity coincides with those of the deterministic version derived in [9].

Keywords: Convex Hull, Linear Programming, Approximation Algorithms, Randomized Algorithms, Triangle Algorithm, Chaos Game, Sierpinski Triangle.

1 Introduction

Given a finite set \( S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m \), and a distinguished point \( p \in \mathbb{R}^m \), the convex hull membership problem (or convex hull decision problem) is to test if \( p \in \text{conv}(S) \), the convex hull of \( S \). Given a desired tolerance \( \varepsilon \in (0, 1) \), we call a point \( p_\varepsilon \in \text{conv}(S) \) an \( \varepsilon \)-approximate solution if \( d(p_\varepsilon, p) \leq \varepsilon R \), where \( R = \max\{d(p, v_i) : i = 1, \ldots, n\} \). The convex hull membership problem is the most basic of the convex hull problems, see [8] for general convex hull problems. Nevertheless, it is a fundamental problem in computational geometry and linear programming and finds applications in statistics, approximation theory, and machine learning. Problems related to the convex hull membership include, computing the distance from a point to the convex hull of a finite point set, support vector machines (SVM), approximating functions as convex combinations of other functions, see e.g. Clarkson [2] and Zhang [16], and [9]. From the theoretical point of view the problem is solvable in polynomial time via the pioneering algorithm of Khachiyan [14], or Karmarkar [13]. For large-scale problems greedy algorithms are preferable to polynomial-time algorithms. The best known such algorithms are, Frank-Wolfe algorithm [4], Gilbert’s algorithm [7], and sparse greedy approximation. For connections between these and analysis see Clarkson [2], Gärtner and Jaggi [5].

A recent algorithm for the convex hull membership problem is the triangle algorithm [9]. It can either compute an \( \varepsilon \)-approximate solution, or when \( p \not\in \text{conv}(S) \) a separating hyperplane and a point that approximates the distance from \( p \) to \( \text{conv}(S) \) to within a factor of 2. Based on preliminary experiments, the triangle algorithm performs quite well on reasonably large size problem, see [15]. It can also be applied to solving linear systems, see [11] and [6] (for experimental results). Additionally, it can be applied to linear programming, see [9]. Some variations of the triangle algorithm are given in [11] and [12]. The performance of the triangle algorithm is quite fast in detecting the cases when \( p \) is not near a boundary point of \( \text{conv}(S) \). When \( p \) is a near-boundary point of \( \text{conv}(S) \) the triangle algorithm may experience zig-zagging in achieving high accuracy approximations. In [9] we have described several strategies to remedy this, such as adding new auxiliary points to \( S \). In this article we propose two randomized versions of the triangle algorithm. The randomized algorithms are also applicable to solving linear systems and linear programming.
The article is organized as follows. In Section 2 we review the triangle algorithm, its relevant properties as well as bounds on its worst-case time complexities. In Section 3 we describe a randomized version, called Greedy-Randomized Triangle Algorithm. In Section 4 we describe a second randomized triangle algorithm inspired by the chaos game, see Barnsley [1] and Devaney [3], known to give rise to the well-known Sierpinski triangle. We call this algorithm Sierpinski-Randomized Triangle Algorithm. We conclude with some remarks.

2 Review of The Triangle Algorithm

Here we review the terminology and some results from [9]. The Euclidean distance is denoted by \( d(\cdot, \cdot) \).

**Definition 1.** Given \( p' \in \text{conv}(S) \), we say \( v \in S \) is a pivot relative to \( p \) at \( p' \) (or \( p \)-pivot, or simply pivot) if \( d(p', v) \geq d(p, v) \) (see Figure 1).

**Definition 2.** Given \( p' \in \text{conv}(S) \), we say \( v \in S \) is a strict pivot relative to \( p \) at \( p' \) (or strict \( p \)-pivot, or simply strict pivot) if \( \theta = \angle p'pv \geq \pi/2 \) (see Figure 1).

**Definition 3.** We call a point \( p' \in \text{conv}(S) \) a \( p \)-witness (or simply a witness) if \( d(p', v_i) < d(p, v_i) \), for all \( i = 1, \ldots, n \).

A witness has the property that the orthogonal bisecting hyperplane to the line \( pp' \) separates \( p \) from \( \text{conv}(S) \). Furthermore,

\[
\frac{1}{2} \leq d(p, p') \leq d(p, \text{conv}(S)) \leq d(p, p').
\]

**Theorem 1. (Distance Duality [9])** \( p \in \text{conv}(S) \) if and only if given any \( p' \in \text{conv}(S) \), there exists a pivot.

**Theorem 2. (Strict Distance Duality [9])** Assume \( p \notin S \). Then \( p \in \text{conv}(S) \) if and only if given any \( p' \in \text{conv}(S) \), there exists a strict pivot.

**Definition 4.** Given three points \( p, p', v \in \mathbb{R}^m \) such that \( d(p', v) \geq d(p, v) \). Let \( \text{nearest}(p; p'v) \) be the nearest point to \( p \) on the line segment joining \( p' \) to \( v \). Specifically, let

\[
\alpha = \frac{(p - p')^T(v - p')}{d^2(v, p')}.
\]

Then

\[
\text{nearest}(p; p'v) = \begin{cases} 
(1 - \alpha)p' + \alpha v, & \text{if } \alpha \in [0, 1]; \\
v, & \text{otherwise}.
\end{cases}
\]

**Remark 1.** By squaring the distances we have

\[
d(p', v) \geq d(p, v) \iff p'Tp' - p'Tp \geq 2v^T(p' - p). \tag{3}
\]

Thus to search for a pivot does not require taking square-roots. Neither does the computation of \( \text{nearest}(p; p'v) \). It requires \( O(mn) \) arithmetic operations.

The triangle algorithm is summarized in the box.

**Theorem 3.** ([9]) Given \( \varepsilon \in (0, 1) \), if \( p \in \text{conv}(S) \), the number of arithmetic operations of the triangle algorithm to compute \( p_{\varepsilon} \) so that \( d(p, p_{\varepsilon}) \leq \varepsilon R \) is

\[
O\left(\frac{mn}{\varepsilon^2}\right).
\]

**Theorem 4.** ([9]) Assume \( p \) lies in the relative interior of \( \text{conv}(S) \). Let \( \rho \) be the supremum of radii of the balls centered at \( p \) in this relative interior. Given \( \varepsilon \in (0, 1) \), suppose the triangle algorithm uses a strict pivot in each iteration. The number of arithmetic operations to compute \( p_{\varepsilon} \in \text{conv}(S) \) so that \( d(p_{\varepsilon}, p) < \varepsilon R \) is

\[
O\left(\frac{mn\left(\frac{R}{\rho}\right)^2}{\varepsilon^2}\ln \frac{1}{\varepsilon}\right).
\]
Triangle Algorithm
Input: $S = \{v_1, \ldots, v_n\}$, $p$, $\varepsilon \in (0, 1)$
Output: $p' \in \text{conv}(S)$, either $d(p, p') \leq \varepsilon R$, or $p'$ is a Witness
$p' = \arg\min\{d(p, v) : v \in S\}$;
while $(d(p, p') > \varepsilon R)$ do
  if no pivot exists then
    Output $p'$ is a Witness and halt;
  else
    given a pivot $v$, set $p' = \text{nearest}(p; p'v)$;
  end
end
Output $p'$;

Figure 1: An example of an iterate, a strict pivot, and $p'' = \text{nearest}(p; p'v)$.

3 Greedy-Randomized Triangle Algorithm

In this section we describe a randomized algorithm we call Greedy-Randomized Triangle Algorithm. It is designed to avoid possible zig-zagging in the triangle algorithm. Given an iterate $p'$, it computes a pivot $v$, if it exists. Then it randomly selects the new iterate as the midpoint of $p'$ and $v$, or $\text{nearest}(p; p'v)$. It records the closest known point to $p$ as $p_*$, the current incumbent candidate, and updates it whenever necessary.

Figure 2 describes a case where given an iterate $p'$ and pivot $v_1$, we can get closer to $p$ by selecting the closest point $p''$ on $p'v_1$. However, selecting instead $p'''$, the midpoint of $p'$ and $v_1$, we create the chance to select a better approximation using $p'''$ as iterate.

From properties of the triangle algorithm reviewed in the previous section we have,

Theorem 5. If $p \in \text{conv}(S)$, bound on the expected number of arithmetic operations of the Greedy-Randomized Triangle Algorithm to compute an $\varepsilon$-approximate solution is

$$O\left(\frac{mn}{\varepsilon^2}\right).$$

Moreover, if it is known that $p$ is the center of ball of radius $\rho$ contained in the relative interior of $\text{conv}(S)$, and if each times it computes a pivot for an iterate the pivot is a strict pivot, then bound on the expected number of arithmetic operations to compute an $\varepsilon$-approximate solution is

$$O\left(\frac{mn\left(\frac{R}{\rho}\right)^2 \ln \frac{1}{\varepsilon}}{\varepsilon}\right).$$

\[ \square \]
Greedy-Randomized Triangle Algorithm

**Input:** $S = \{v_1, \ldots, v_n\}$, $p, \varepsilon \in (0, 1)$

**Output:** $p' \in \text{conv}(S)$, either $d(p, p') \leq \varepsilon R$, or $p'$ is a Witness

$p' = \arg\min\{d(p, v) : v \in S\}$, $p_* = p'$;

while $(d(p, p_*) > \varepsilon R)$ do

  if no pivot exists at $p'$ then
    Output $p'$ is a Witness and halt;
  else
    given a pivot $v$; randomly set $p' = (p' + v) / 2$, or $p' = \text{nearest}(p; p'v)$;
    Update $p_*$;
  end
end

Output $p' = p_*$;

---

**Figure 2**: An example where $p''' = (p' + v_1)/2$ is a better iterate than $p'' = \text{nearest}(p; p'v_1)$ for the next iteration.
4 A Randomized Triangle Algorithm Based on The Chaos Game

As described by Devaney [3]:

The chaos game and its multitude of variations provides a wonderful opportunity to combine elementary ideas from geometry, linear algebra, probability, and topology with some quite contemporary mathematics. The easiest chaos game to understand is played as follows. Start with three points at the vertices of an equilateral triangle. Color one vertex red, one green, and one blue. Take a die and color two sides red, two sides green, and two sides blue. Then pick any point whatsoever in the triangle, this is the seed. Now roll the die. Depending upon which color comes up, move the seed half the distance to the similarly colored vertex. Then repeat this procedure, each time moving the previous point half the distance to the vertex whose color turns up when the die is rolled. After a dozen rolls, start marking where these points land.

Devaney goes on to say, when this process is repeated thousands of times, the pattern that emerges is one of the most famous fractals of all, the Sierpinski triangle. The Sierpinski triangle consists of three self-similar pieces, each of which is exactly one half the size of the original triangle in terms of the lengths of the sides.

Figure 3: The Sierpinski Triangle.

4.1 The Sierpinski-Randomized Triangle Algorithm

Consider the convex hull problem for the very simple case where \( S \) consists of three points as the vertices of an equilateral triangle and \( p \) is a point inside the triangle. We make the following claim on the Sierpinski triangle, see Figure 3 which is visually evident and provable from its topological properties. We refer to the convex hull of the dots as enclosing Sierpinski triangle.

**Proposition 1.** Given any dense subset of the Sierpinski triangle, \( \Sigma \), no matter where \( p \) is located inside the enclosing Sierpinski triangle, and no matter which of the three vertices is chosen as \( v \), we can select a Sierpinski dot, say \( p' \), for which the line segment \( p'v \) either contains \( p \), or comes as close to it as desired.

The above gives an incentive to state a randomized triangle algorithm based on its generalization. First, consider the following generalization of the chaos game.

**Definition 5.** (General Chaos Game) Given a set of points \( S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m \), let \( \Sigma(S) \) correspond to the dots generated via the following generalization of Sierpinski chaos game: Start with a seed \( p' \in \text{conv}(S) \), and with probability \( 1/n \) randomly select \( v \in S \), then record \( (p' + v)/2 \) as a new point and place it in \( \Sigma(S) \). Replace \( p' \) with \( (p' + v)/2 \) and repeat the process indefinitely.

The following hypothesis gives the incentive to define another randomized triangle algorithm, what we call the **Sierpinski-Randomized Triangle Algorithm**.
**Hypothesis 1.** Suppose \( p \in conv(S) \). Given \( \varepsilon \in (0, 1) \), \( v \in S \), there exists \( p' \in \Sigma(S) \) such that

(i) \( v \) is a \( p \)-pivot with respect to \( p' \) (i.e. \( d(p', v) \geq d(p, v) \)),
(ii) If \( p'' = \text{nearest}(p; p'v) \), then \( d(p, p'') \leq \varepsilon d(p, v) \).

Regardless of the validity of the above hypothesis, we prove that bounds on the expected complexity of the Sierpinski-Randomized Triangle Algorithm is no worse than bounds on the worst-case complexity of the triangle algorithm itself. The algorithm is inspired by the chaos game, however it keeps track of the current incumbent candidate, \( p_* \), the closest known point to \( p \).

Given an iterate \( p' \), it randomly (with equal probability) selects \( v \in S \cup \{p_*\} \). If \( v \) is a pivot, it randomly either replaces \( p' \) with \((p' + v)/2\), or with \( \text{nearest}(p; p'v) \). Otherwise, if \( v \neq p_* \), the next iterate is \((p' + v)/2\), or else \( v = p_* \) and not a pivot. In this case \( p_* \) will be taken to be the iterate and the algorithm searches for a pivot \( v' \) at \( p_* \). When such a pivot exists, the next iterate will be \( \text{nearest}(p; p_*v') \). Except for this case, the other cases take \( O(m + n) \) operations.

**Sierpinski-Randomized Triangle Algorithm**

**Input:** \( S = \{v_1, \ldots, v_n\} \), \( p \), \( \varepsilon \)

**Output:** \( p' \in conv(S) \), either \( d(p, p') \leq \varepsilon R \), or \( p' \) a **Witness**

\( p' = \text{argmin}\{d(p, v) : v \in S\} \), \( p_* = p' \);

**while** \((d(p, p') > \varepsilon R)\) **do**

randomly select \( v \in S \cup \{p_*\} \);

- if \( v \) is a \( p \)-pivot at \( p' \) then
  - at random set \( p' = (p' + v)/2 \), or \( p' = \text{nearest}(p; p'v) \);

else

  if \( v = p_* \) then
  no \( p \)-pivot exists at \( p_* \) then
  \( p' = p_* \), **Output** \( p' \) a **Witness** and halt;

else

  given a \( p \)-pivot \( v' \) at \( p_* \), \( p' = \text{nearest}(p; p_*v') \), \( p_* = p' \);

else

  \( p' = (p' + v)/2 \);

end

**end**

**Update** \( p_* \);

**end**

**Output** \( p' \);

**Lemma 1.** The expected number of arithmetic operations in each iteration of the Sierpinski-Randomized Triangle Algorithm is \( O(m + n) \).

**Proof.** The probability that in each iteration the randomly selected \( v \) coincides with \( p_* \) is \( 1/(n + 1) \). Then if \( v = p_* \) is not a \( p \)-pivot then \( p_* \) becomes the new iterate and the number of operations to compute a pivot \( v' \) at \( p_* \) is \( O(mn) \). If the randomly selected \( v \) is not \( p_* \), the number of operations to get the next iterate \( p' \) is \( O(m + n) \). Thus the expected number or operations in each iteration is

\[
\frac{n}{n + 1} O(m + n) + \frac{1}{n + 1} O(mn) = O(m + n).
\]

**Theorem 6.** If \( p \in conv(S) \), bound on the expected number of arithmetic operations to compute an \( \varepsilon \)-approximate solution is

\[
O\left(\frac{mn}{\varepsilon^2}\right).
\]
Moreover, if it is known that \( p \) is the center of ball of radius \( \rho \) contained in the relative interior of \( \text{conv}(S) \), and if each times it computes a pivot \( v' \) for \( p_* \) it is a strict pivot, bound on the expected number of arithmetic operations to compute an \( \varepsilon \)-approximate solution is

\[
O\left( mn \left( \frac{R}{\rho} \right)^2 \ln \frac{1}{\varepsilon} \right).
\]

Proof. The expected number of times a random \( v \) is selected before it equals \( p_* \) is \((n + 1)\). When \( p_* \) is not a pivot at the current iterate \( p' \), \( p' \) is replaced with \( p_* \) and a pivot \( v' \) is computed. Applying the existence results on pivot and strict pivot, Theorems 2 and Theorem 3 as well as the complexity bounds on the triangle algorithm, Theorems 3 and 4 the proof follows.

Suppose we consider a relaxed version of the above algorithm where each time \( p_* \) is selected and is not a pivot at the current iterate, thus becoming a new iterate, we select \( v' \) randomly and not necessarily as a pivot at \( p_* \), thus economizing in computation. Referring to this as the Relaxed Sierpinski-Randomized Triangle Algorithm we have.

**Theorem 7.** If \( p \in \text{conv}(S) \), bound on the expected number of arithmetic operations of the Relaxed Sierpinski-Randomized Triangle Algorithm to compute an \( \varepsilon \)-approximate solution is

\[
O\left( \frac{mn^2}{\varepsilon^2} \right).
\]

Moreover, if it is known that \( p \) is the center of ball of radius \( \rho \) contained in the relative interior of \( \text{conv}(S) \), and if each times it computes a pivot \( v' \) for \( p_* \) it is a strict pivot, bound on the expected number of arithmetic operations to compute an \( \varepsilon \)-approximate solution is

\[
O\left( mn^2 \left( \frac{R}{\rho} \right)^2 \ln \frac{1}{\varepsilon} \right).
\]

Proof. Each iteration takes \( O(m + n) \) operations. Given an iterate \( p' \), the probability that a randomly selected \( v \) in \( S \cup \{p_*\} \) is a \( p \)-pivot (strict \( p \)-pivot) at \( p' \) is \(1/(n + 1)\). This is because the Voronoi cell of \( p \) with respect to the two-point set \( \{p, p'\} \) must contain a \( p \)-point in \( S \cup \{p_*\} \) (otherwise, \( p \notin \text{conv}(S) \)). Thus the probability that at an iterate \( p_* \) is randomly selected and that at \( p_* \) a pivot (strict pivot) is randomly selected is \(1/n(n + 1)\). From these and analogous arguments as in the previous theorem, the expected complexities follow.

There is yet another relaxation: we treat \( p_* \) as any other point in \( S \), that is if an iterate \( p' \) selects \( p_* \) randomly, we do not jump to \( p_* \) as the next iterate. The expected complexity of this may remain to be the same as the relaxed version analyzed above.

**Concluding Remarks.** In this article we have described randomized versions of the triangle algorithm. Based on our previous theoretical and experimental results, see [9], [15] and [6], the triangle algorithm appears to be a promising algorithm with wide range of applications. The randomized algorithms suggest variations that could help its performance in practice or in the worst-case. Both allow exploring the the convex hull from different view points, thus increasing the chance to get better and better approximations to \( p \) by choosing more effective pivots. Also, in the Sierpinski-Randomized Triangle Algorithm as \( p_* \) gets close to \( p \), the chances are good that when an iterate \( p' \) randomly selects \( v = p_* \) that \( p_* \) is actually a pivot at \( p' \). Hence with probability \( 1/2 \) the next iteration will get closer to \( p \). To check if \( p_* \) is a pivot at \( p' \) takes \( O(m + n) \) time as opposed to \( O(mn) \) time. Additionally, it is likely that the randomized algorithms will help improve the performance of the triangle algorithm as a function of \( \varepsilon \). Some theoretical questions thus arise. Computational experimentations are needed to assess practical values. We plan to do so in future work.

**Acknowledgements.** I like to thank Mike Saks for a discussion regarding randomization.
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