Anisotropic spectral cut-off estimation under multiplicative measurement errors

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Abstract
We study the non-parametric estimation of an unknown density \( f \) with support on \( \mathbb{R}^d \) based on an i.i.d. sample with multiplicative measurement errors. The proposed fully-data driven procedure is based on the estimation of the Mellin transform of the density \( f \) and a regularisation of the inverse of the Mellin transform by a spectral cut-off. The upcoming bias-variance trade-off is dealt with by a data-driven anisotropic choice of the cut-off parameter. In order to discuss the bias term, we consider the Mellin-Sobolev spaces which characterize the regularity of the unknown density \( f \) through the decay of its Mellin transform. Additionally, we show minimax-optimality over Mellin-Sobolev spaces of the spectral cut-off density estimator.

Keywords: Adaptation, anisotropic density estimation, anisotropic Mellin-Sobolev spaces, inverse problem, Mellin transform, minimax theory, multiplicative measurement errors

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1. Introduction
In this work we consider the estimation of an unknown density \( f : \mathbb{R}^d \to \mathbb{R} \) of a positive random variable \( X = (X_1, \ldots, X_d) \) given independent and identically distributed (i.i.d.) copies of \( Y = UX = (X_1U_1, \ldots, X_dU_d) \), where \( X \) and \( U \) are independent of each other and \( U \) has a known density \( g : \mathbb{R}^d \to \mathbb{R} \). The density \( f_Y : \mathbb{R}^d \to \mathbb{R}_+ \) of \( Y \) is then given by
\[
f_Y(y) = [f * g](y) := \int_{\mathbb{R}^d} f(x)g(y/x)x^{-1} dx \quad \forall y \in \mathbb{R}^d,
\]
where \( x/y := (x_1/y_1, \ldots, x_d/y_d) \) and \( x^{-1} := \prod_{j=1}^d x_j^{-1} \). Here "*" denotes multiplicative convolution. The estimation of \( f \) using an i.i.d. sample \( Y_1, \ldots, Y_n \) from \( f_Y \) is thus an inverse problem called multiplicative deconvolution.

In the additive deconvolution literature the density estimation for multivariate variables based on non-parametric estimators has been studied by many authors. A kernel estimator approach was investigated by [5] with respect to \( L^2 \)-risk and by [16] for general \( L^p \)-risk. The multivariate convolution structure density model was considered by the authors [11]. The recent work [6] focuses on the study of deconvolution problems on \( \mathbb{R}^d \) and introduces a data-driven estimator based on a projection on the Laguerre basis. To the knowledge of the author, the estimation for multivariate random variables with multiplicative measurement errors has not been studied yet.

For the univariate case, the recent work of [3] should be mentioned which uses the Mellin transform to construct a density estimator under multiplicative measurement errors. The model of multiplicative measurement errors was motivated in the work of [1] as a generalisation of several models, for instance the multiplicative censoring model or the stochastic volatility model.

A summary of related work regarding the connection between the multiplicative measurement errors model and similar models can be found in [3] and [1]. In the work of [1], the authors used the Mellin transform to construct a kernel estimator for the pointwise density estimation. In their work, the authors shown that the log transformation of the observation is a special case of their estimation strategy. In fact, by applying the logarithm the model \( Y = UX \) writes as \( \log(Y) = \log(X) + \log(U) \). This naive approach allows then the usage of commonly used deconvolution
techniques to construct an estimator of the density \( \log(X) \) (see for example [13]) and which can be then transformed back to an estimator of \( f \). It is worth stressing out, that in this case the regularity assumptions are considered for the density of \( \log(X) \) instead of \( f \) directly. This provokes difficulties for the interpretation of these regularity conditions. For the global risk case, additional complications occurs using this naive approach as pointed out by [4].

In this work, we generalise the results of [3] in a similar way to the works [5] and [6] for the additive deconvolution model. To do so, we introduce a notion of the Mellin transform for multivariate random variables and show that the necessary properties of the univariate Mellin transform remain true. Exploiting the multiplication theorem, that is \( M[f g] = M[f] M[g] \) [3] introduced for the univariate case a spectral cut-off density estimator of \( f \) based on the i.i.d. sample \( Y_1, \ldots, Y_n \). Considering the multivariate case, we are analogously making use of the multiplication theorem of the Mellin transform and apply a spectral cut-off regularisation of the inversion of the Mellin-transform to define a density estimator. The accuracy of the proposed estimator is measured in terms of the global risk with respect to a weighted \( L^2_{\omega} \)-norm. We identify the underlying inverse problem using the rich theory of Mellin transform and characterise the natural regularity conditions expressed in the form of Mellin-Sobolev spaces. Here, we borrow ideas from the inverse problems community ([8]) and discuss the relation between the Mellin-Sobolev spaces and analytical properties of the density \( f \). In the regularisation step of the inverse problem, an additional tuning parameter is introduced. For this parameter we propose a model selection technique to end up with a fully data-driven estimator. We establish an oracle inequality for the fully-data driven spectral cut-off estimator under fairly mild assumptions on the error density \( g \). Moreover, we show that uniformly over Mellin-Sobolev spaces the proposed data-driven estimator is minimax-optimal by stating both an upper and lower bound for the mean weighted integrated squared error of the minimax risk of the density estimation for given an i.i.d. sample of \( Y \).

The paper is organized as follows. In Section 2 we begin with an introduction of the Mellin transform for multivariate random variables including several properties which are commonly used throughout this paper. Based on the observations \( Y_1, \ldots, Y_n \), we then introduce the spectral cut-off estimator of the density \( f \) and analyse its properties for a large class of error densities. Furthermore, we study the global behavior of the proposed estimator over the Mellin-Sobolev spaces for smooth error density. Here, we show upper and lower bounds for the weighted \( L^2_{\omega} \)-risk of our estimator implying its minimax-optimality. In Section 3 we propose a data-driven method for the choice of the cut-off parameter only depending on the sample \( Y_1, \ldots, Y_n \) based on a model selection. Finally, results of a simulation study are reported in section 4 which visualize the reasonable finite sample performance of our estimators. Proofs of theorems of Section 2 and Section 3 are postponed to the Appendix.

2. Minimax theory

In this section we introduce the Mellin transform and collect some of its properties while more detailed proof sketches are given in Appendix 4.4. Define for a weight function \( \omega : \mathbb{R}^d \to \mathbb{R} \), the corresponding weighted norm by \( \|h\|_{\omega}^2 := \int_{\mathbb{R}^d} |h(x)|^2 \omega(x) dx \) for a measurable function \( h : \mathbb{R}^d \to \mathbb{C} \). Denote by \( L^2(\mathbb{R}^d, \omega) \) the set of all complex-valued, measurable functions with finite \( \| \cdot \|_{\omega} \)-norm and by \( \langle h_1, h_2 \rangle_\omega := \int_{\mathbb{R}^d} h_1(x) h_2(x) \omega(x) dx \) for \( h_1, h_2 \in L^2(\mathbb{R}^d, \omega) \) the corresponding weighted scalar product. Similarly, define \( L^2(\Omega, \omega) := \{ h : \Omega \to \mathbb{C} : \| h \|_{\Omega, \omega}^2 := \int_{\Omega} |h(x)|^2 \omega(x) dx < \infty \} \) for any \( \Omega \subseteq \mathbb{R}^d \).

For two vectors \( u = (u_1, \ldots, u_d)^T, v = (v_1, \ldots, v_d)^T \in \mathbb{R}^d \) and a scalar \( \lambda \in \mathbb{R} \) we define the componentwise multiplication \( u v := u \cdot v := (u_1 v_1, \ldots, u_d v_d)^T \) and denote by \( \lambda u \) the usual scalar multiplication. Further, if there exists no index \( i \in [d] \) such that \( v_i = 0 \) we define the multivariate power through \( v^\lambda := \prod_{i \in [d]} v_i^\lambda \). Additionally, we define the componentwise division by \( \frac{u}{v} := u/v := (u_1/v_1, \ldots, u_d/v_d)^T \) and denote by \( |u| := \sqrt{u^T u} \).

2.1. The Mellin transform

Let \( c \in \mathbb{R}^d \). For two functions \( h_1, h_2 \in L^1(\mathbb{R}^d, x^{-c}) \) we define the multiplicative convolution \( h_1 \ast h_2 \) of \( h_1 \) and \( h_2 \) by

\[
(h_1 \ast h_2)(y) = \int_{\mathbb{R}^d} h_1(y/x) h_2(x) x^{c} dx, \quad y \in \mathbb{R}^d.
\]
In fact, one can show that the function \( h_1 * h_2 \) is well-defined, \( h_1 * h_2 = h_2 * h_1 \) and \( h_1 * h_2 \in L^1(\mathbb{R}_+^d, x^{-1}) \). A proof sketch of this and the following properties can be found in Appendix \([4.4]\). Further, if additionally \( h_1 \in L^2(\mathbb{R}_+^d, x^{-1}) \) then \( h_1 * h_2 \in L^2(\mathbb{R}_+^d, x^{-1}) \).

We will now define the Mellin transform for functions \( h_1 \in L^1(\mathbb{R}_+^d, x^{-1}) \). To do so, let \( h_1 \in L^1(\mathbb{R}_+^d, x^{-1}) \). Then, we define the Mellin transform of \( h_1 \) at the development point \( c \in \mathbb{R}^d \) as the function \( \mathcal{M}_c[h_1] : \mathbb{R}^d \to \mathbb{C} \) by

\[
\mathcal{M}_c[h_1](t) := \int_{\mathbb{R}_+^d} x^{c-1 + it} h_1(x) \, dx, \quad t \in \mathbb{R}^d.
\]

(2)

Note that for any density \( h \in L^1(\mathbb{R}_+^d, x^0) \) of a positive random variable \( Z \) the property \( h \in L^1(\mathbb{R}_+^d, x^{-1}) \) is equivalent to \( E_Z(Z^{-1}) < \infty \).

One key property of the Mellin transform, which makes it so appealing for the use of multiplicative deconvolution, is the so-called convolution theorem, that is for \( h_1, h_2 \in L^1(\mathbb{R}_+^d, x^{-1}) \) holds

\[
\mathcal{M}_c[h_1 * h_2](t) = \mathcal{M}_c[h_1](t) \mathcal{M}_c[h_2](t), \quad t \in \mathbb{R}^d.
\]

(3)

In analogy to the Fourier transform, one can define the Mellin transform for functions \( h \in L^2(\mathbb{R}_+^d, x^{-1}) \). In fact, let \( \varphi : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto (\exp(-x_1), \ldots, \exp(-x_d))^T \) and \( \varphi^{-1} : \mathbb{R}^d \to \mathbb{R}^d \) its inverse. Then as diffeomorphisms \( \varphi \) and \( \varphi^{-1} \) map Lebesgue null sets on Lebesgue null sets. Thus the isomorphism \( \Phi_c : L^2(\mathbb{R}_+^d, x^{-1}) \to L^2(\mathbb{R}^d), h \mapsto \varphi h \circ \varphi \) is well-defined for any \( c \in \mathbb{R}^d \). Furthermore, let \( \Phi_c^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}_+^d, x^{-1}) \) denote its inverse. Then for \( h \in L^2(\mathbb{R}_+^d, x^{-1}) \) we define the Mellin transform of \( h \) developed in \( c \in \mathbb{R}^d \) by

\[
\mathcal{M}_c[h](t) := (2\pi)^{d/2} \mathcal{F}[\Phi_c[h]](t), \quad t \in \mathbb{R}^d.
\]

(4)

where \( \mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is the Plancherel-Fourier transform. Due to this definition several properties of the Mellin transform can be deduced from the well-known theory of Fourier transforms. In the case \( h \in L^1(\mathbb{R}_+^d, x^{-1}) \cap L^2(\mathbb{R}_+^d, x^{-1}) \) we have

\[
\mathcal{M}_c[h](t) = \int_{\mathbb{R}_+^d} x^{c-1 + it} h(x) \, dx, \quad t \in \mathbb{R}^d,
\]

(5)

which coincides with the usual notion of Mellin transforms as considered in \([15]\) for the case \( d = 1 \).

Further, due to this construction of the operator \( \mathcal{M}_c : L^2(\mathbb{R}_+^d, x^{-1}) \to L^2(\mathbb{R}^d) \) it is an isomorphism and we denote by \( \mathcal{M}_c^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}_+^d, x^{-1}) \) its inverse. If additionally to \( H \in L^2(\mathbb{R}^d), H \in L^1(\mathbb{R}^d) \) holds then we can express the inverse Mellin transform explicitly through

\[
\mathcal{M}_c^{-1}[H](x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} x^{c-1} H(t) \, dt, \quad \text{for any} \ x \in \mathbb{R}_+^d.
\]

(6)

Furthermore, we can directly show that a Plancherel-type equation holds for the Mellin transform, that is for all \( h_1, h_2 \in L^2(\mathbb{R}_+^d, x^{-1}) \)

\[
\langle h_1, h_2 \rangle_{L^2} = \frac{1}{(2\pi)^d} \langle \mathcal{M}_c[h_1], \mathcal{M}_c[h_2] \rangle_{\mathbb{R}^d} \text{ and thus } \|h_1\|_{L^2}^2 = \frac{1}{(2\pi)^d} \|\mathcal{M}_c[h_1]\|_{\mathbb{R}^d}^2.
\]

(7)

2.2. Estimation strategy

Let us define for \( k \in \mathbb{R}_+^d \) the hyper cuboid \( Q_k := \{ x \in \mathbb{R}^d : \forall i \in \{d\} : |x_i| \leq k_i \} \). Then for \( f \in L^2(\mathbb{R}_+^d, x^{-1}) \) we have that \( \mathcal{M}_c[f] \mathbb{1}_{Q_k} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) and thus

\[
f_k(x) := \frac{1}{(2\pi)^d} \int_{Q_k} x^{c-1} \mathcal{M}_c[f](t) \, dt, \quad x \in \mathbb{R}_+^d,
\]

is an approximation of \( f \) in the \( L^2(\mathbb{R}_+^d, x^{-1}) \)-sense, that is \( \|f_k - f\|_{L^2} \to 0 \) for \( k \to \infty \) where the limit \( k \to \infty \) means that every component of \( k \) is going to infinity.
Now let us additionally assume that \( f \in L^2(\mathbb{R}^d, x^{2d-1}) \cap L^1(\mathbb{R}^d, x^{2d-1}) \) and \( g \in L^1(\mathbb{R}^d, x^{2d-1}) \). Then from the convolution theorem, one deduces \( \mathcal{M}_c[f] = \mathcal{M}_c[f] / \mathcal{M}_c[g] \). Under the mild assumption that \( \mathcal{M}_c[g](t) \neq 0 \), for any \( t \in \mathbb{R}^d \) we can rewrite the last equation as \( \mathcal{M}_c[f] = \mathcal{M}_c[f] / \mathcal{M}_c[g] \). Thus we have
\[
f_k(x) := \frac{1}{(2\pi)^d} \int_{Q_a} x^{-c-it} \frac{\mathcal{M}_c[f](t)}{\mathcal{M}_c[g](t)} dt, \quad x \in \mathbb{R}^d.
\]

Let us now consider for any \( t \in \mathbb{R}^d \) the unbiased estimator \( \hat{\mathcal{M}}_c(t) := n^{-1} \sum_{i \in [n]} Y_i^{-c-it} \) of \( \mathcal{M}_c[f](t) \). We see easily that \( |\hat{\mathcal{M}}_c(t)| \leq |\mathcal{M}_c(0)| < \infty \) almost surely. If additionally \( 1_n \mathcal{M}_c[g]^{-1} \in L^2(\mathbb{R}^d) \) then \( 1_n \mathcal{M}_c / \mathcal{M}_c[g] \in L^2(\mathbb{R}^d) \) and we can define our spectral cut-off density estimator by \( \hat{f}_k := \mathcal{M}_c^{-1}[1_n \mathcal{M}_c / \mathcal{M}_c[g]] \). More explicitly, we have
\[
\hat{f}_k(x) = \frac{1}{(2\pi)^d} \int_{Q_a} x^{-c-it} \frac{\mathcal{M}_c(t)}{\mathcal{M}_c[g](t)} dt, \quad x \in \mathbb{R}^d.
\]

Up to now, we had two minor assumptions on the error density \( g \) which we want to collect in the following assumption:
\[
\forall t \in \mathbb{R}^d : \mathcal{M}_c[g](t) \neq 0 \quad \text{and} \quad \forall k \in \mathbb{R}^d : \int_{Q_a} |\mathcal{M}_c[g](t)|^{-2} dt < \infty.
\]

**Remark 1.** The assumption \( \|f\|_{\mathcal{L}_2} < \infty \) resembles strongly the rather typical error assumption in context of deconvolution problems, compare \([13]\). Examples of multivariate density, which fullfills the assumption \( \|f\|_{\mathcal{L}_2} < \infty \) are given in Example \([5]\) and Example \([6]\). It is worth stressing out, that one can construct deconvolution estimators under a weaker assumption on the error density, that is \( \mathcal{M}_c[g](t) \neq 0 \) almost every, compare \([2]\). In particular, the weaker assumption is in that sense minimal that without it, we need addional constraints on the class of densities to ensure that the density is indeed identifiable, compare \([13]\) and \([2]\).

The following proposition shows that the proposed estimator is consistent for a suitable choice of the cut-off parameter \( k \in \mathbb{R}^d \). Its proof is postponed to Appendix \( 4.5 \).

**Proposition 1.** Let \( f \in L^2(\mathbb{R}^d, x^{2d-1}) \), \( \sigma := \mathbb{E}_f(Y^{2d-2}) < \infty \) and assume that \( \|f\|_{\mathcal{L}_2} < \infty \) holds for \( g \). Then we have for any \( k \in \mathbb{R}^d \),
\[
\mathbb{E}_f(\|f - \hat{f}_k(x)\|_{\mathcal{L}_2}^2) = \|f - f_k\|_{\mathcal{L}_2}^2 + \frac{\sigma \Delta_x(k)}{n},
\]
where \( \Delta_x(k) := \frac{1}{2\pi^d} \int_{Q_a} |\mathcal{M}_c[g](t)|^{-2} dt \). Now choosing \( k_n \) such that \( \Delta_x(k_n) n^{-1} \rightarrow 0 \) and \( k_n \rightarrow \infty \) implies the consistency of \( \hat{f}_k \).

Let us comment on the last result. For a suitable choice of the spectral cut-off parameter \( k \in \mathbb{R}^d \) we can show that the estimator is consistent in the sense of the weighted \( L^2 \) distance. The second parameter, the model parameter \( c \in \mathbb{R}^d \), is linked to the considered risk and the assumptions on the densities \( f \) and \( g \). In fact, choosing \( c = 1 \), we see that \( \sigma = 1 \) for any densities \( f, g \). In this scenario, no additional moment assumptions on the densities \( f \) and \( g \) are needed. If one wants to consider the unweighted \( L^2 \), one should set \( c = 1/2 \) which leads in the case \( d = 1 \) to the assumption that \( \mathbb{E}_f(Y^{-1}) < \infty \). As one can see, the choice of the parameter \( c \in \mathbb{R} \) is more of a modeling nature. Nevertheless, it is worth stressing out, that the authors of \([11]\) considered optimal choices of \( c \in \mathbb{R} \) for the pointwise estimation of the density. Since in the global estimation the risk itself is dependent on \( c \in \mathbb{R} \), the role of \( c \in \mathbb{R} \) is quite different from the pointwise estimation.

Up to now, the assumptions on \( f \) and \( g \) were to ensure the well-definedness of the estimator and the weighted \( L^2 \)-risk. Here, we can already see that the first summand, called bias term, in Proposition \([11]\) is decreasing if \( k \in \mathbb{R}_+ \) is increasing in any direction while the second summand, called variance term, is increasing. For a more sophisticated analysis of both terms we will consider stronger assumptions on the densities \( f \) and \( g \). Let us first start with the noise density \( g \).
2.3. Noise assumption

As already mentioned, the variance term in (10) is monotonically increasing in each component \( k_j, j \in [d] \), of \( k \). More precisely, the growth of \( \Delta_k \) is determined by the decay of the Mellin transform of \( g \) in each direction.

In the context of additive deconvolution problems (compare [9]), densities whose Fourier transform decay polynomially, like in Examples [1] and [2] are called smooth error densities. To stay in this way of speaking we say that an error density \( g \) is a smooth error density if there exists \( c_g, C_g \in \mathbb{R}_+ \) such that

\[
c_g \prod_{j=1}^d (1 + t_j^2)^{-\gamma_j/2} \leq |\mathcal{M}_t[g](t)| \leq C_g \prod_{j=1}^d (1 + t_j^2)^{-\gamma_j/2}, \quad t \in \mathbb{R}.
\]

(11)

This assumption on the error density was also considered in the works of [1] and [3]. Under this assumption, we see that \( \Delta_k(k) \leq c_g \prod_{j=1}^d k_j^{2\gamma_j+1} \) for every \( k \in \mathbb{R}_+^d \). After a more sophisticated bound of the variance term we will consider now the bias term which occurs in (10).

2.4. Regularity spaces

Let us for \( s, c \in \mathbb{R}_+^d \) define the anisotropic Mellin-Sobolev space by

\[
\mathcal{W}^p_s(\mathbb{R}^d) := \{ h \in L^2(\mathbb{R}_+, x^{2s-1}) : \| h \|_{L^p(\mathbb{R}^d)} := \sum_{j=1}^d \| (1 + t_j)^{1/2} \mathcal{M}_t[h] \|_{L^p(\mathbb{R}^d)} < \infty \}
\]

(12)

and the corresponding ellipsoids with \( L \in \mathbb{R}_+ \) by \( \mathcal{W}^p_s(L) := \{ h \in \mathcal{W}^p_s(\mathbb{R}^d) : \| h \|_{L^p(\mathbb{R}^d)} \leq L \} \). Since \( \bigcup_{j=1}^d \{ t \in \mathbb{R}^d : |t_j| > k_j \} \supset \mathcal{Q}^s_k \) we deduce from the assumption \( f \in \mathcal{W}^p_s(L) \) that

\[
\| f - f_k \|_{L^2(\mathbb{R}^d)} \leq \sum_{j=1}^d \| (1 + t_j)^{1/2} \mathcal{M}_t[f] \|_{L^2(\mathbb{R}^d)} \leq L \sum_{j=1}^d k_j^{2\gamma_j+1}.
\]

Setting \( D_s^k(L) := \{ f \in \mathcal{W}^p_s(L) : f \text{ density, } \mathbb{E}_f(X^{2c-2}) \leq L \} \), the previous discussion leads to the following statement.

Lemma 1. Let \( f \in L^2(\mathbb{R}_+, x^{2s-1}) \) and \( \mathbb{E}_g(U^{2c-2}) < \infty \). Then under the assumptions [9] and [11] holds

\[
\sup_{f \in D_s^k(L)} \mathbb{E}_g^n(f - f_k \mid X = x) \leq C(L, g, s)n^{-1/1(1+2\gamma)} \Sigma_{i=1}^{2(2\gamma+1)} \|
\]

(13)

for the choice \( k_n = (k_{1,n}, \ldots, k_{d,n}) \) with \( k_{i,n} := n^{1/1(2c+2\gamma+1)} \Sigma_{i=1}^{2(2\gamma+1)} x_i \).

Considering the rate in Lemma [1] the natural question arises if whether exists an estimator based on the sample \( Y_1, \ldots, Y_n \) with a sharper rate uniformly over \( D_s^k(L) \).

In the following paragraph we will show that such a scenario cannot occur. From this we deduce that our estimator \( f_k \) is minimax-optimal over the ellipsoids \( D_s^k(L) \) for many classes of error densities.

2.5. Lower bound

For the following part, we will need to have further assumption on the error density \( g \). In fact, we will distinguish if \( c_j \in (0, 1/2] \) or \( c_j > 1/2 \) for \( j \in [d] \). Let therefore \( \tilde{c} := (\tilde{c}_1, \ldots, \tilde{c}_d) \in \mathbb{R}_+^d \) where \( \tilde{c}_j := 2^{-1} \mathbb{E}_f(X^{1/2\gamma+1/2}) \). Let us assume that \( g \) has a bounded support, that is for all \( x \in \mathbb{R}^d \setminus \{ 0 \}^d \) : \( g(x) = 0 \) and that there exists constants \( c_g, C_g > 0 \) such that

\[
c_g \prod_{j=1}^d (1 + t_j^2)^{-\gamma_j/2} \leq |\mathcal{M}_t[g](t)| \leq C_g \prod_{j=1}^d (1 + t_j^2)^{-\gamma_j/2}, \quad t \in \mathbb{R}^d.
\]

(13)

With this additional assumption we can show the following theorem where its proof can be found in Appendix 4.5.

Theorem 1. Let \( s, \gamma \in \mathbb{N}^d, c > 0 \) and assume that [11] and [13] holds. Then there exist constants \( C_g, L_{s,d,c} > 0 \), such that for all \( L \geq L_{s,d,c}, n \in \mathbb{N} \) and for any estimator \( f \) of \( f \) based on an i.i.d. sample \( (Y_i)_{i=1}^n \),

\[
\sup_{f \in D_s^k(L)} \mathbb{E}_g^n(\| f - f_k \|_2^2) \geq c_{g,c} n^{-1/1(1+2\gamma)} \Sigma_{i=1}^{2(2\gamma+1)} x_i \gamma\).
\]

Let us shortly comment on the assumption [13]. For the case that \( c_j > 1/2 \) for \( j \in [d] \) then \( \tilde{c}_j = 1/2 \), thus we need to assume that \( \mathbb{E}_g(U^{2\gamma-2}) < \infty \) which is a mild condition. If \( c_j \leq 1/2 \) we have that \( \tilde{c}_j = 0 \). In this case, we have automatically that \( \mathbb{E}_g(U^{2\gamma-2}) < \infty \) if we assume that \( \mathbb{E}_g(U^{2(2\gamma-2)}) < \infty \), compare Proposition [1].
3. Data-driven method

Although we have shown that in certain situations the estimator \( \hat{f}_k \) in Lemma 1 is minimax-optimal, the choice of \( k_n \) is still dependent on the regularity parameter \( s \in \mathbb{R}_+^d \) of the unknown density \( f \), which is again, unknown. Therefore, we will propose a fully data-driven choice of \( k \in \mathbb{R}_+^d \) based on the sample \( Y_1, \ldots, Y_n \). For the special case of \( d = 1 \) the authors of [3] proposed a data-driven choice for the parameter \( k \in \mathbb{R}_+ \) based on a penalized contrast approach. For the multivariate case, a model selection approach has been mainly used if one considers an isotropic choice of the cut-off parameter, that is, instead of considering the estimator defined in (8) one would use for \( k \in \mathbb{R}_+ \) and \( B_k := \{x \in \mathbb{R}^d : |x| \leq k \} \) the estimator

\[
\hat{f}_k(x) = \frac{1}{(2\pi)^d} \int_{B_k} x^{-d} \frac{\mathcal{M}_f(t)}{\mathcal{M}_k[g](t)} dt, \quad x \in \mathbb{R}^d.
\]

For the family \((\hat{f}_k)_{k \in \mathbb{R}_+^d}\), a data-driven choice of the parameter \( k \in \mathbb{R}_+^d \) based on a model selection approach is possible. Although it might be tempting to use this estimator as the multivariate generalisation of the estimator presented in [3], an anisotropic estimator has the advantage that it is more flexible. In fact, if the regularity in two directions of the density differs substantial, respectively the decay of the Mellin transform of the error density, an isotropic choice of the cut-off parameter is obviously inappropiate. For the anisotropic estimator defined in (8) we propose a data-driven choice of the parameter, that is, instead of considering the estimator defined in (8) one would use for

\[
\mathcal{K}_n := \{k \in \mathbb{N}^d : \Delta_\chi(k) \leq n\}
\]

and define for \( k \in \mathbb{R}_+^d \) and \( \chi > 0 \) the penalty term

\[
\text{pen}(k) := \chi \sigma \Delta_\chi(k)n^{-1}.
\]

It can be seen that the bias \( \|f - \hat{f}_k\|_{\chi^{-1}}^2 = \|f\|_{\chi^{-1}}^2 - \|f_k\|_{\chi^{-1}}^2 \) behaves like \( -\|f_k\|_{\chi^{-1}}^2 \). Exchanging \( -\|f_k\|_{\chi^{-1}}^2 \) and \( \text{pen}(k) \) with their empirical counterparts, we define the fully data-driven model selection by

\[
\hat{k} := \arg \min_{k \in \mathcal{K}_n} (\|f_k\|_{\chi^{-1}}^2 + \text{pen}(k)), \quad \text{pen}(k) := \chi \sigma \Delta_\chi(k)n^{-1}
\]

(14)

where \( \sigma := n^{-1} \sum_{j=1}^n Y_j^{2\chi-2} \). Now, let us show that this data-driven procedure mimics the optimal choice up to a negligible term.

**Theorem 2.** Assume that \( \mathbb{E}_{f_k}(Y^{2(\chi-1)}) < \infty \), \( \|x^{2\chi-1} f_k\|_\infty < \infty \) and (11) is fulfilled. Then for \( \chi \geq 144 \) we have

\[
\mathbb{E}_{f_k} \|f - \hat{f}_k\|_{\chi^{-1}}^2 \leq 3 \inf_{k \in \mathcal{K}_n} (\|f - f_k\|_{\chi^{-1}}^2 + \text{pen}(k)) + \frac{C_2}{n}
\]

\( C_2 \) is a constant depending on \( \chi, \|f x^{2\chi-1}\|_\infty, \sigma, \mathbb{E}_{f_k}(Y^{2(\chi-1)}) \) and \( g \).

For every \( s \in \mathbb{R}_+^d \) we can see that \( k_n \), defined in Lemma 1, lies in \( \mathcal{K}_n \). Due to this, and the consideration in the minimax theory section, we can deduce the following Corollary directly whose proof is thus omitted.

**Corollary 1.** Under the assumption of Theorem 2 and the additional assumption that \( f \in \mathbb{E}^s(L) \) we get

\[
\mathbb{E}_{f_k} \|f - \hat{f}_k\|_{\chi^{-1}}^2 \leq C n^{-1/(1+2\chi)} \sum_{j=1}^n (2^{j+1}s_j)^{-1}
\]

where \( C \) is a positive constant depending on \( \chi, \|f x^{2\chi-1}\|_\infty, \sigma, \mathbb{E}_{f_k}(Y^{2(\chi-1)}) \), \( g \) and \( L \).
4. Examples and Numerical study

4.1. Examples

In this subsection, we aim to motivate the definition of the Mellin-Sobolev spaces and the noise assumption by considering various examples presented in work [1]. For the sake of readability, we will begin with the case $d = 1$ and will then consider then consider examples for $d = 2$.

Univariate case.

Example 1 (The Beta and the Log-Gamma Distribution). Consider the family $(g_b)_{b \in \mathbb{N}}$ of Beta$(1, b)$-densities given by $g_b(x) := 1_{(0, 1)}(x) b(1 - x)^{b-1}$, for $b \in \mathbb{N}$ and $x \in \mathbb{R}_+$. Obviously, we see that $g_b \in L^2_0((x^{b-1}) \cap L^1_0((x^{-1})$ for $b > 0$ and it holds

$$M_t[g_b](t) = \prod_{j=1}^{b} e^{-1/j + j + t}, \quad t \in \mathbb{R}.$$  

Considering the decay of the Mellin transform we get $c_{g,b}(1 + t^2)^{-b/2} \leq |M_t[g_b](t)| \leq C_{g,b}(1 + t^2)^{-b/2}, t \in \mathbb{R}$, where $c_{g,b}, C_{g,b} > 0$ are positive constants only depending on $g$ and $b$.

Example 2 (The Scaled Log-Gamma Distribution). Consider the family $(g_{a,\lambda})_{(a,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+}$ of sLI$(g,\alpha,\lambda)$ densities with $g_{a,\lambda}(x) = \frac{\exp(a|x|)}{\Gamma(\lambda)} x^{-\frac{1}{\lambda}} \exp(\lambda x - \lambda \mu x^\alpha)$, for $a, \lambda, x \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$. Then for $c = \lambda + 1$ holds $g_{a,\lambda} \in \mathbb{L}^2_0((x^{c-1}) \cap \mathbb{L}^1_0((x^{-1})$ and

$$M_t[g_{a,\lambda}](t) = \exp(\mu(c-1+it)(\lambda-c+1-it)^{-a}, \quad t \in \mathbb{R}.$$  

If $a = 1$ then $g_{a,\lambda}$ is the density of a Pareto distribution with parameter $e^\mu$ and $\lambda$. If $\mu = 0$ we have that $g_{0,\lambda}$ is the density of a Log-Gamma distribution. Considering the decay of the Mellin transform we get $c_{g,a,\lambda}(1 + t^2)^{-\lambda/2} \leq |M_t[g_{a,\lambda}](t)| \leq C_{g,a,\lambda}(1 + t^2)^{-\lambda/2}, t \in \mathbb{R}$, where $c_{g,a,\lambda}, C_{g,a,\lambda} > 0$ are positive constants only depending on $g$ and $c$.

Example 3 (Gamma Distribution). Consider the family $(g_d)_{d \in \mathbb{R}^+}$ of Gamma$(d, 1)$ densities with $g_d(x) = \frac{x^{d-1} \exp(-x)}{\Gamma(d)} d_\mathbb{R}(x)$ for $d, x \in \mathbb{R}^+$. Obviously, we see that $g_d \in L^2_0((x^{2d-1}) \cap \mathbb{L}^1_0((x^{-1})$ for $c > -d + 1$ and it holds

$$M_t[g_d](t) = \frac{\Gamma(c + d - 1 + it)}{\Gamma(d)}, \quad t \in \mathbb{R}.$$  

Applying the Stirling formula, compare with [1], leads to $c_{g,d}(1 + |t|)^{2d-2} \exp(-|t|) \leq |M_t[g_d](t)|^2 \leq C_{g,d}(1 + |t|)^{2d-2} \exp(-|t|), |t| \to \infty$.

Example 4 (Weibull Distribution). Consider the family $(g_m)_{m \in \mathbb{R}^+}$ of Weibull$(m, 1)$ densities $g_m(x) = m x^{m-1} \exp(-x^m) d_\mathbb{R}(x)$ for $m, x \in \mathbb{R}_+$. Obviously, we see that $M_t[g_m]$ is well-defined for $c > -m + 1$ and it holds

$$M_t[g_m](t) = \frac{(c-1+it)}{m!} \left(\frac{c-1+it}{m}\right), \quad t \in \mathbb{R}.$$  

Applying the Stirling formula one sees that $c_{m,|t|}^{(-1/m)} e^{-|t|/m} \leq |M_t[g_m](t)|^2 \leq C_{m,|t|}^{(-1/m)} e^{-|t|/m}$, for all $|t| \to \infty$.  

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Example 5.

Obviously, we see that $g$ of polynomial decay. The following table shows the resulting risk rates combining the Example 1 to Example 4.

Table 1: Comparison of different combinations of unknown densities $f$ and error densities $g$. The decay of the Mellin transform of $f$ is in both cases polynomial. For Examples 1 and 2, the decay of the Mellin transform of $g$ is polynomial and exponential for the Examples 3 and 4.

| $f$ | $g$ | $sL\Gamma_0(\alpha_2, 1)$ | $\Gamma_{(d_1)}$ | $W_{(m_1, 1)}$ |
|-----|-----|----------------|-----------------|---------------|
| Beta($\alpha_1, \beta_1$) | $\ Beta(1, \beta_1)$ | $n^{-\frac{1}{\alpha_1} - \frac{1}{\beta_1} - 1}$ | $n^{-\frac{1}{\alpha_1} - \frac{1}{\beta_1} - 1}$ | $\log(n)^{1-2\beta_1}$ | $\log(n)^{1-2\beta_1}$ |
| $sL\Gamma_0(\alpha_1, 1)$ | $n^{-\frac{1}{\alpha_1} - \frac{1}{\beta_1} - 1}$ | $n^{-\frac{1}{\alpha_1} - \frac{1}{\beta_1} - 1}$ | $\log(n)^{1-2\beta_1}$ | $\log(n)^{1-2\beta_1}$ |

We want to stress out, that there are several assumptions on the upcoming parameters of the density, for instance, $\Gamma, \Sigma$ and $\mu$.

Example 6.

We will now consider two examples of densities $f$ with $f(x) = g_1(x_1)g_2(x_2)$ since in this case, the Mellin transform $M_c[g](t) = M_c[g_1](t_1)M_c[g_2](t_2)$.

Example 5 (Bivariate Log-Normal Distribution). Consider the family $(g_{\mu, \Sigma})_{(\mu, \Sigma) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 2}}$ of LN($\mu, \Sigma$) densities where $g_{\mu, \Sigma}$ for $\Sigma \in \mathbb{R}^{2 \times 2}$, positive definite, $x \in \mathbb{R}^2$, and $\mu \in \mathbb{R}^2$ is given by $g_{\mu, \Sigma}(x) = \frac{1}{\sqrt{2\pi} \log(\Sigma)} \exp(-\frac{1}{2}(\log(x) - \mu)^T \Sigma^{-1}(\log(x) - \mu))1_{\mathbb{R}^2}(x)$ and $\log(x) = (\log(x_1), \log(x_2))^T$. Obviously, we see that $M_c[g_{\mu, \Sigma}]$ is well-defined for all $c \in \mathbb{R}^2$ and it holds

$$M_c[g_{\mu, \Sigma}](t) = \exp(\mu^T(c - 1 + it))^T \Sigma(c - 1 + it)$$

for all $t \in \mathbb{R}^2$. Then we can easily see that $|M_c[g_{\mu, \Sigma}](t)|^2 = \exp(-t^T \Sigma t) \exp(-\frac{1}{2}t^T \Sigma t + 2t_1\Sigma_{11} + t_2\Sigma_{22}), t \in \mathbb{R}$.

Example 6 (Uniform distribution on $S := \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in (1, 2)\}$). Consider the density $g(x) = \frac{1}{4} 1_{\mathbb{R}^2}(x), x \in \mathbb{R}^2$. Obviously, we see that $M_c[g]$ is well-defined for any $c \in \mathbb{R}^2$ and it holds

$$M_c[g](t) = \frac{2^{c_1 + c_2 + t_1 + t_2}}{3(c_1 + it_1)(c_2 + it_2)}$$

for all $t \in \mathbb{R}^2$. Then we can see that $c_1(1 + |t_1|^2)^{-1}(1 + |t_2|^2)^{-1} \leq |M_c[g](t)|^2 \leq c_2(1 + |t_1|^2)^{-1}(1 + |t_2|^2)^{-1}, t \in \mathbb{R}$.

The Mellin transform of the error densities in Example 5 and 6 fulfill both assumption (11). While the log normal distribution is an example for a super smooth error density, it can be seen that the density in 6 is a smooth error density with $\gamma_1 = \gamma_2 = 1$. We will now consider two examples of densities $f$ which do not factorise, that is, where there exists no $g_1, g_2$ with $g(x) = g_1(x_1)g_2(x_2)$.
4.2. Numerical simulation

Let us illustrate the performance of the estimator $\hat{f}_k$ defined in (8) and (14). We will restrict ourselves to the case $d = 2$. For a simulation study with a data-driven choice of the dimension parameter for the univariate case, we refer to [3] where they used a penalized contrast strategy. In the bivariate case, we will actually study the performance of the fully data-driven method presented in (14), while we omit the consideration of different values of $c \in \mathbb{R}^2$.

In the upcoming simulation study we will consider the densities

1. Gamma Distribution: $f_1(x) = \frac{x^{\gamma - 1}e^{-x/\theta}}{\Gamma(\gamma)}$, see Example 1 and Table 1.
2. Weibull Distribution: $f_2(x) = x^2 \exp(-x^2)\mathbb{I}_{\mathbb{R}^+}(x)$, see Example 2 and Table 2.
3. Beta Distribution: $f_3(x) = \frac{1}{B(\gamma_1, \gamma_2)} x^{\gamma_1-1}(1-x)^{\gamma_2-1}$, see Example 3 and Table 2.
4. Log-Normal Distribution: $f_4(x) = \frac{1}{\sqrt{2\pi}x} \exp(-\frac{(\log(x))^2}{2})\mathbb{I}_{(0,\infty)}(x)$, see Example 2 and Table 2.

For the error densities, we consider the univariate densities

1. Pareto Distribution: $g_1(x) = x^{-\gamma} \mathbb{I}_{(1,\infty)}(x)$, see Tables 1 and 2 and Example 2.
2. Log-Gamma Distribution: $g_2(x) = \frac{1}{\gamma_1 \Gamma(\gamma_2)} x^{-\gamma_2} \exp(-x^{\gamma_2}) \mathbb{I}_{(1,\infty)}(x)$, see Tables 1 and 2.

In the sense of (11) we see that $g_1$ has the parameter $\gamma = 1$, while $g_2$ has $\gamma = 1/2$.

Let us now consider the data-driven choice defined (8) and (14) for the case of $d = 2$. To illustrate the performance of our estimator, we consider the following three cases

1. Error densities: $f(x_1, x_2) = f_1(x_1)f_1(x_2)$ with direct observations compared to observations with $g(x_1, x_2) = g_2(x_1)g_2(x_2)$ with $c = (1/2, 1/2)^T$.
2. Anisotropic density: $f(x_1, x_2) = f_3(x_1)f_4(x_2)$ with direct observations and $c = (1/2, 1/2)^T$.
3. Dependency: $X \sim LN_{\mu, \Sigma}$ with $\mu = (\log(4), \log(4))^T$, $c = (1/2, 1/2)^T$ and direct observations. For $\Sigma$ we compare

\[
\Sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.81 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 0.81 \end{pmatrix}.
\]

4. Anisotropic error: $f(x) = f_2(x_1)f_3(x_2)$ with $U = (U_1, U_2)^T$, where $U_1 \sim sL(0,1,1)$ and $U_2 \sim sL(0,1/2,1)$, dependent, and $c = (1/2, 1/2)^T$.

For the first case, we visualise the impact of observations with measurement error compared to direct observations. The second case resembles the case when the decay of the Mellin transform of the density $f$ has significantly different behaviour in different direction. The third case shall illustrate the behaviour of the estimator when the two coordinates of $X$ are dependent while in the fourth case the decay of the Mellin transform of the density is similar but the decays of the error density are not the same. By minimising an integrated weighted squared error over a family of histogram densities with randomly drawn partitions and weights we select $\chi = 1$ (respectively $\chi = 0.3$) for the cases of direct observation (respectively contaminated data), where $\chi$ is the variance constant, see (14).

Case 1: Let us start by compare the influence of measurement errors by comparing the estimator $\hat{f}_k$ based on the copies of $X$ compared to copies of $Y$. 


Case 2: In the second case, we additionally compare the anisotropic estimator $\hat{f}_k$ with the isotropic choice, that is we define $\hat{f}_k := \hat{f}_{k\tilde{k}}$ with

$$\tilde{k} := \arg \min_{k \in \mathbb{N}, \Delta_g((k,k)) \leq \eta} -||\hat{f}_{(\hat{k}, \hat{k})}||_2^2 + \kappa \sigma \frac{\Delta_g((k,k))}{n},$$

a penalized contrast upproach which is a direct generalisation of the estimator given in [3]. Here we choose $\kappa = 5$ by a preliminary simulation study.

As one can see in Fig. 2, the anisotropic estimator seems to invest more in the approximation of the Beta distribution.
than the Log normal distribution. This leads to worse performance in the Log normal direction but to an overall satisfying result. In comparison to that, the isotropic estimator chooses in both direction the same cut-off parameter leading to a better approximation of the log normal distribution but also to a worse approximation of the beta distribution. Overall it seems that the anisotropic estimator behaves better.

Case 3: Now we consider the influence of the dependency between the coordinates of $X$. While $\Sigma_1$ resembles the case of independent coordinates, $\Sigma_2$ is not a diagonal matrix and thus the coordinates are dependent.

Case 4: We finish our simulation study by considering the case, where the decay of the Mellin transform of the density behaves similar in both direction, while the decay of the Mellin transform of the error densities differs. For $U = (U_1, U_2)^T$ we set $U_1 := \xi_1 \xi_2$ and $U_2 := \xi_2$ where $\xi_1, \xi_2 \sim \mathcal{S}L(0,1/2,1)$, compare Example 2. Then we have $U_1 \sim \mathcal{S}L(0,1,1)$ and $U_2 \sim \mathcal{S}L(0,1/2,1)$ and

$$M_{1/2}(g)(t) = (3/2 - it_1)^{-1/2}(3/2 - it(t_1 + t_2))^{-1/2}, \quad t \in \mathbb{R}^2,$$

leading that $g$ satisfies in this situation with $\gamma = (1, 1/2)$. For this case, we deduced by a preliminary simulation study the choice $\chi_1 = \chi_2 = 0.25$. For the distribution of $X = (X_1, X_2)^T$ we set for the sake of simplicity $X_1, X_2 \sim \mathcal{S}L(2,1)$, see Example 4. We compare the performance of the data-driven anisotropic estimator $\hat{f}_k$ with the performance of the data-driven isotropic estimator $\hat{f}$ introduced in (15) with the choice $\kappa = 1.02$. For both estimator we consider $c = 1/2 = (1/2, 1/2)^T$. 

In Fig. 3 we can see that although the estimator does reconstruct the general shape of the density $f$, the included dependency impede slightly the estimation.
As one can see in Fig. 4, the anisotropic estimator \( \hat{f}_k \) behaves better in the second coordinates as the isotropic estimator \( \hat{f}_k \), which is consistent with the theory since the decay of the Mellin transform of the error density in this direction is slower.

4.3. Comment

As seen by the simulation study, the anisotropic data-driven estimator \( \hat{f}_k \) behaves reasonable for the case \( d = 2 \). Compared to the an isotropic estimator, the desired flexibility of an anisotropic choice can be interpreted in the Figures 2 and 4, where an anisotropic choice is necessary due to different behaviour of the decay of the Mellin transform of the density \( f \), respectively the error density \( g \). Allthough the implementation of the estimator for the case \( d > 2 \) is possible, we do not provide a simulation study for these cases. It is worth stressing out, that for higher dimensions, the estimator does suffers under the well-known curse of dimensionality, that is, the convergence rates, and therefore the performance of the estimation strategy, is slower than in lower dimensions.
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Appendix

4.4. Preliminaries

We will now present some proof sketches for the properties of the Mellin transform stated in Section 2. We recall that for a function $h \in L^2(\mathbb{R}^d, x^{-d-1})$ we defined the Mellin transform developed in $c \in \mathbb{R}$ by

$$M_c[h](t) := (2\pi)^{d/2} \mathcal{F}[\Phi_c[h]](t), \quad t \in \mathbb{R}^d,$$

where

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), H \mapsto (t \mapsto \mathcal{F}[H](t) := \lim_{k \to \infty} (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp(-i(t,x))H(x)dt$$

is the Plancherel-Fourier transform where the limit is understood in a $L^2(\mathbb{R}^d)$ convergence sense and the function $\Phi_c$ is defined by $\Phi_c : L^2(\mathbb{R}^d, x^{-d-1}) \to L^2(\mathbb{R}^d)$, $h \mapsto \phi \ast h \circ \varphi$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto (\exp(-x_1), \ldots, \exp(-x_d))^T$.

By assuming $h \in L^2(\mathbb{R}^d, x^{-d-1}) \cap L^1(\mathbb{R}^d, x^{-1})$ we get that $\Phi_c[h] \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In that case, we know that we can write the Fourier-Plancherel transform explicitly and get for all $t \in \mathbb{R}^d$,

$$M_c[h](t) = \int_{\mathbb{R}^d} \exp(-i(t,x))\phi(x)h(x)dx = \int_{\mathbb{R}^d} \phi(x)\phi(x)h(x)dx = \int_{\mathbb{R}^d} x^{-1+it}h(x)dx$$

by single change of variables.

Since $M_c : L^2(\mathbb{R}^d, x^{-d-1}) \to L^2(\mathbb{R}^d)$ is a composition of isomorphism we see that it is invertible and its inverse $M_c^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d, x^{-d-1})$ can be expressed through $M_c^{-1}[H] = (2\pi)^{-d/2} \Phi_c^{-1}[\mathcal{F}^{-1}[H]]$ for any $H \in L^2(\mathbb{R}^d)$ where $\mathcal{F}^{-1}$ is the inverse of the Fourier-Plancherel transform. If additionally $H \in L^1(\mathbb{R}^d)$, we can express the inverse Fourier-Plancherel transform explicitly and get for any $x \in \mathbb{R}^d$,

$$M_c^{-1}[H](x) = (2\pi)^{-d} \Phi_c^{-1} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(i(t,x))H(t)dt \right)$$

Next, we are going to show a Plancherel-type equation for the Mellin transform, that is, for any $h_1, h_2 \in L^2(\mathbb{R}^d, x^{-d-1})$ holds $\langle h_1, h_2 \rangle_{L^2} = (2\pi)^{-d} \langle M_c[h_1], M_c[h_2] \rangle_{L^2}$. Again we see that by application of a change of variable and the Plancherel equation for the Fourier-Plancherel transform that

$$\langle h_1, h_2 \rangle_{L^2} = \int_{\mathbb{R}^d} x^c h_1(x) h_2(x)dx = \langle \Phi_c[h_1], \Phi_c[h_2] \rangle_{L^2}$$

$$\langle \Phi_c[h_1], \Phi_c[h_2] \rangle_{L^2} = (2\pi)^{-d} \langle M_c[h_1], M_c[h_2] \rangle_{L^2}.$$

Now let us finish this short introduction by showing the convolution theorem for the Mellin transform, that is for $h_1, h_2 \in L^1(\mathbb{R}^d, x^{-1})$ holds $M_c[(h_1 \ast h_2)] = M_c[h_1]M_c[h_2]$ where $(h_1 \ast h_2)$ denotes the multiplicative convolution of $h_1$ and $h_2$ which was given by

$$(h_1 \ast h_2)(y) = \int_{\mathbb{R}^d} h_1(x)h_2(y/x)x^{-1}dx.$$
by \( (H_1 \ast, H_2)(y) := \int_{\mathbb{R}^d} H_1(y-x)H_2(x)dx \) for any \( y \in \mathbb{R}^d \) we know from functional analysis that \( (H_1 \ast, H_2) \in L^1(\mathbb{R}^d) \).

Thus it follows that \( (h_1 \ast h_2) = \Phi^{-1}_c((\Phi_c[h_1] \ast \Phi_c[h_2])) \in L^1(\mathbb{R}^d, x^{-2}) \). The fact that \( (h_1 \ast h_2) = \Phi^{-1}_c ((\Phi_c[h_1] \ast \Phi_c[h_2])) \) follows from simple calculus. Through this representation we see that

\[
\mathcal{M}_c[(h_1 \ast h_2)] = \langle 2\pi \delta^2/\mathcal{F} \Phi_c[\phi_1] \ast \Phi_c[\phi_2] \rangle = \langle 2\pi \delta^2/\mathcal{F} \Phi_c[\phi_1] \ast \Phi_c[h_2] \rangle
\]

in analogy, we can see that if additionally \( h_1, h_2 \in L^1(\mathbb{R}^d, x^{q-1}) \) and \( h_2 \in L^2(\mathbb{R}^d, x^{2q-1}) \), then \( \Phi_c[h_1] \ast \Phi_c[h_2] \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) which implies that \( h_1 \ast h_2 \in L^2(\mathbb{R}^d, x^{2q-1}) \).

In the upcoming proofs we are in need of the following inequalities. The first inequality is due to \([10]\), the formulation can be found for example in \([10]\).

**Lemma 2 (Talagrand inequality).** Let \( X_1, \ldots, X_n \) be independent \( Z\)-valued random variables and let

\[
\hat{v}_h = n^{-1} \sum_{i=1}^n [v_h(X_i) - \mathbb{E}(v_h(X_i))]
\]

for \( v_h \) belonging to a countable class \( \{v_h, h \in \mathcal{H}\} \) of measurable functions. Then,

\[
\mathbb{E}(\sup_{h \in \mathcal{H}} |\hat{v}_h|^2 - 6\Psi^2)_\tau \leq C \left( \frac{r}{n} \exp\left( -\frac{n\Psi^2}{6r} \right) + \frac{\psi^2}{n^2} \exp\left( \frac{-K n \Psi}{\psi^2} \right) \right)
\]

with numerical constants \( K = (\sqrt{2} - 1)/(21\sqrt{2}) \) and \( C > 0 \) and where

\[
\sup_{h \in \mathcal{H}} \sup_{z \in Z} |v_h(z)| \leq \psi, \quad \mathbb{E}(\sup_{h \in \mathcal{H}} |\hat{v}_h|) \leq \Psi, \quad \sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \text{Var}(v_h(X_i)) \leq \tau.
\]

**Remark 2.** Keeping the bound \([16]\) in mind, let us specify particular choices \( K \), in fact \( K \geq \frac{1}{1008} \). The next bound is now an immediate consequence,

\[
\mathbb{E}(\sup_{h \in \mathcal{H}} |\hat{v}_h|^2 - 6\Psi^2)_\tau \leq C \left( \frac{r}{n} \exp\left( -\frac{n\Psi^2}{6r} \right) + \frac{\psi^2}{n^2} \exp\left( \frac{-K n \Psi}{1008\psi^2} \right) \right).
\]

In the sequel we will make use of the slightly simplified bounds \([17]\) rather than \([16]\).

The next inequality was proven by \([14]\). A similar formulation can be found in \([12]\) equation (1.3).

**Lemma 3 (Nagaev’s inequality).** Let \( X_1, \ldots, X_n \) be i.i.d. mean-zero random variables with \( \mathbb{E}(|X_1|^p) < \infty \) for \( p > 2 \). Then for any \( x \in \mathbb{R}_+ \) holds

\[
\mathbb{P}(\sum_{j=1}^n X_j \geq x) \leq (1 + 2p^{-1})^p \frac{n\mathbb{E}(|X_1|^p)}{x^p} + 2e^{(-a_p x^2)} \frac{n\mathbb{E}(|X_1|^p)}{n\mathbb{E}(X_1)}
\]

where \( a_p = 2e^{-p}(p + 2)^{-2} \).

### 4.5. Proofs of Section 4

**Proof of Proposition 1.** For \( k \in \mathbb{R}^d \) we see that \( f - f_k \in L^2(\mathbb{R}^d, x^{2q-1}) \) with \( M_c[f - f_k] = M_c[f]1_{\mathbb{R}^d \setminus Q_k} \). We deduce by application of the Plancherel equality that \( (f - f_k, f_k - f_k)_{L^2} = \langle M_c[f]1_{\mathbb{R}^d \setminus Q_k}, (M_c[f] - M_c[f_k])1_{\mathbb{R}^d \setminus Q_k} \rangle_{L^2} = 0 \) which implies that

\[
\mathbb{E}^{n}_{f_k} \|f - f_k\|_{L^2}^2 = \|f - f_k\|_{L^2}^2 + \mathbb{E}^{n}_{f_k} (\|f_k - f\|_{L^2}^2).
\]

Now by application of the Fubini-Tonelli theorem we interchange the integration order to get

\[
\mathbb{E}^{n}_{f_k} (\|f_k - f\|_{L^2}^2) = \frac{1}{(2\pi)^d} \int_{Q_k} \mathbb{E}^{n}_{f_k} (|\mathcal{M}_c[f_k|(t) - \mathcal{M}_c[f]|(t)|^2)\mathcal{M}_c[g]|(t)|^2 dt \leq \frac{1}{(2\pi)^d} \sigma^2_t(k).
\]

\( \square \)
Proof of Theorem[1] First we outline the main steps of the proof. Let us denote by \( I := \{ j \in \mathbb{N}^d : c_j > 1/2 \} \) the subset of indices. We will construct a family of functions in \( D^2_\varepsilon(L) \) by a perturbation of the density \( f_\nu : \mathbb{R}_+ \to \mathbb{R}_+ \) with small bumps, such that their \( L^1(\mathbb{R}_+^d, x^{2\varepsilon-1}) \)-distance and the Kullback-Leibler divergence of their induced distributions can be bounded from below and above, respectively. The claim then follows then by applying Theorem 2.5 in [18]. We use the following construction, which we present first.

Denote by \( C_c^\infty(\mathbb{R}) \) the set of all smooth functions with compact support in \( \mathbb{R} \) and let \( \psi \in C_c^\infty(\mathbb{R}) \) be a function with support in \([0, 1] \) and \( \int \psi(x) dx = 0 \). For each \( j \in \mathbb{N}^d \) and \( K_j \in \mathbb{N} \) (to be selected below) and \( k_j \in \{ 0, K_j \} \) we define the bump-functions \( \psi_{k_j, K_j}(x_j) := \psi(x_j K_j - k_j), x_j \in \mathbb{R} \) and define for \( p \in \mathbb{N}_0 := \{ z \in \mathbb{Z} : z \geq 0 \} \) the finite constant \( C_{p, \infty} := \max(||\phi^p||_\infty, l \in \{ 0, p \} \) for \( K \). Let us further define the operator \( S : C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R}) \) with \( S[f](x) = -x f^{(1)}(x) \) for all \( x \in \mathbb{R} \) and define \( S^1 := S \) and \( S^n := S \circ S^{-1} \) for \( n \in \mathbb{N}, n \geq 2 \). Now, for \( p \in \mathbb{N} \), we define the function \( \psi_{k_j, K_j}(x_j) := S^p[\psi_{k_j, K_j}](x_j) = \sum_{i=1}^p c_i \psi_{k_j, K_j}(x_j) \) for \( x_j \in \mathbb{R}_+ \) and \( c_i \geq 1 \) and let \( c_p := \sum_{i=1}^p c_i \).

For a bump-amplitude \( \delta > 0, \gamma \in \mathbb{N}^d \) and \( K := (K_1, \ldots, K_d)^T \in \mathbb{N}^d \) define

\[
    \mathcal{K} := \bigotimes_{j \in \mathbb{N}^d} \left[ 0, K_j \right] := \left[ 0, K_1 \right] \times \cdots \times \left[ 0, K_d \right] = \{ k \in \mathbb{N}^d : \forall j \in \mathbb{N}^d : k_j < K_j \}.
\]

and a vector \( \theta = (\theta_{k+1})_{k \in \mathbb{N} \in \mathbb{N}^d} \in (0, 1) \) \( \times \cdots \times \mathbb{N}^d \) \( \mathcal{K} \) we define

\[
    f_\theta(x) = f_0(x) + \delta F_{K, \gamma, \delta} \sum_{k=0}^{K_j-1} \theta_{k+1} \sum_{j \in \mathbb{N}^d} \psi_{k+K_j}(x_j),
\]

where \( F_{K, \gamma, \delta} := K^{2\gamma} \sum_{j \in \mathbb{N}^d} J^{2\gamma} \) and \( f_0(x) := \prod_{j \in \mathbb{N}^d} f_{0,j}(x_j) \) with

\[
    f_{0,j}(x) := \begin{cases} 
        \exp(-x) 1_{\mathbb{R}_+}(x_j), & \text{if } j \in I; \\
        x \exp(-x) 1_{\mathbb{R}_+}(x_j), & \text{else.}
    \end{cases}
\]

Until now, we did not give a sufficient condition to ensure that our constructed functions \( \{ f_\theta : \theta \in \Theta \} \) are in fact densities. This condition is given by the following lemma.

Lemma 4. Let \( 0 < \delta < \delta_0(\psi, \gamma) := \exp(-2d)/(\prod_{j \in \mathbb{N}^d} \psi_{d, \gamma} c_{y_j}) \). Then for all \( \theta \in \Theta \), \( f_\theta \) is a density.

Further, one can show that these densities all lie inside the ellipsoids \( D^2_\varepsilon(L) \) for \( L \) big enough. This is captured in the following lemma.

Lemma 5. Let \( s \in \mathbb{N}^d \). Then, there is \( L_{s, \gamma, \delta, \theta} > 0 \) such that \( f_\theta \) and any \( f_\nu \) as in [18] with \( \theta \in \Theta \), belong to \( D^2_\varepsilon(L_{s, \gamma, \delta, \theta}) \).

For sake of simplicity we denote for a function \( \varphi \in L^2(\mathbb{R}_+^d, x^{2\varepsilon-1}) \) the multiplicative convolution with \( g \) by \( \varphi * g := (\varphi * g) \). Futher we see that for \( \varepsilon \in (0, 2) \) holds

\[
    \tilde{f}_\theta(y) = y^{1-2\varepsilon} \int_{\mathbb{R}_+^d} \left( \sum_{j \in \mathbb{N}^d} \prod_{j \in \mathbb{N}^d} \exp(-y_j / x_j) dx \right) \geq y^{1-2\varepsilon} \int_{\mathbb{R}_+^d} \left( \sum_{j \in \mathbb{N}^d} \prod_{j \in \mathbb{N}^d} \exp(-2 / x_j) dx \right) := c_\varepsilon y^{1-2\varepsilon}
\]

where \( c_\varepsilon > 0 \) since otherwise \( g \equiv 0 \) almost everywhere. Exploiting Varshamov-Gilbert’s lemma (see [18]) in Lemma 6 we show further that there is \( M \in \mathbb{N} \) with \( M \geq 2^{1 + 1/\delta} K_j^{1/8} \) and a subset \( \{ \theta^{(0)}, \ldots, \theta^{(M)} \} \) of \( \Theta \) with \( \theta^{(0)} = (0, \ldots, 0) \) such that for all \( j, l \in \{ 0, M \} \), \( j \neq l \) the \( L^2(\mathbb{R}_+^d, x^{2\varepsilon-1}) \)-distance and the Kullback-Leibler divergence are bounded for \( K \geq K_j(\gamma, c, \psi) \).

Lemma 6. Let \( K \geq K_j(\gamma, c, \psi) \) understood componentwise. Then there exists a subset \( \{ \theta^{(0)}, \ldots, \theta^{(M)} \} \) of \( \Theta \) with \( \theta^{(0)} = (0, \ldots, 0) \) such that \( M \geq 2^{\varepsilon + 1/\delta} K_j^{1/8} \) and for all \( j, l \in \{ 0, M \} \), \( j \neq l \) holds

(i) \( ||f_{\theta^{(0)}} - f_{\theta^{(l)}}||^2_{L^2} \geq \frac{C_{\varepsilon, \delta}^2}{\sum_{j=0}^{M} K_j^{1/8}} \)

(ii) \( KL(f_{\theta^{(0)}}, f_{\theta^{(l)}}) \leq \frac{C_{\varepsilon, \delta}^2 \log(M)}{\sum_{j=0}^{M} K_j^{1/8}} \).

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where $KL$ is the Kullback-Leibler-divergence.

Selecting $K_j = \lfloor n^{1/(2s+1)} \sum_{\ell \in \mathbb{Q}} (2s+1)^{\ell} \rfloor$, it follows that for $n \geq n_{s,y}$

$$
\left( \sum_{j \in [d]} K_j^{2s+1} \right)^{-1} \geq c n^{-1/(1+0.5 \sum_{\ell \in \mathbb{Q}} (2s+1)^{\ell})}, \quad \frac{K_j^{-2s-1}}{\sum_{j \in [d]} K_j^{2s+1}} \leq n^{-1}
$$

and thus

$$
\frac{1}{M} \sum_{j=1}^{M} KL((f_0)^{\otimes n}, (f_{\delta}^{\otimes n})) = \frac{M}{n} \sum_{j=1}^{M} KL(f_0, f_{\delta}^{\otimes n}) \leq c_{\delta,y} \log(M),
$$

where $c_{\delta,y} < 1/8$ for all $\delta \leq \delta_1 (g, y, s)$ and $M \geq 2$ for $n \geq n_{s,y}$. Thereby, we can use Theorem 2.5 of [18], which in turn for any estimator $\bar{f}$ of $f$ implies

$$
\sup_{x \in \mathbb{R}^d} \mathbb{P}(\|\bar{f} - f\|_2^2 > 2n^{-1/(1+0.5 \sum_{\ell \in \mathbb{Q}} (2s+1)^{\ell})}) \geq 0.07.
$$

Note that the constant $c_{\delta,y}$ does only depend on $\psi$, $\gamma$, and $\delta$, hence it is independent of the parameters $s$, $L$, and $n$. The claim of Theorem 4 follows by using Markov’s inequality, which completes the proof.

**Proof of Lemma 5**

For any $h \in C^\infty_c(\mathbb{R})$ we can state that $\int_{-\infty}^{\infty} S(h(x)) dx = [-x h(x)]_{x=\infty}^{\infty} + \int_{-\infty}^{\infty} h(x) dx = \int_{-\infty}^{\infty} h(x) dx$ and therefore $\int_{-\infty}^{\infty} S'[h](x) dx = \int_{-\infty}^{\infty} h(x) dx$ for $p \in \mathbb{N}$. Thus for every $j \in \mathbb{N}$ we get $\int_{-\infty}^{\infty} \psi_{k,j}(x) dx = \int_{-\infty}^{\infty} \psi_{j}(x) dx = 0$ which implies that for any $\delta > 0$ and $\theta \in \Theta$ we have $\int_{[0,1]} f_{\delta}(x) dx = 1$.

Now due to the construction [18] of the functions $\psi_{k,j}$ we easily see that the function $\psi_{k,j}$ has support on $[1 + k/j, 1 + (k + 1)/j]$ which leads to $\psi_{k,j}$ and $\psi_{j}$ having disjoint supports if $k_j \neq j$. Here, we want to emphasize that $supp(S[h]) \subseteq supp(h)$ for all $h \in C^\infty_c(\mathbb{R})$. This implies that $\psi_{k,j}, \theta_1$ and $\psi_{j}, \theta_1$ have disjoint supports if $k_j \neq j$, too. For $x \in \mathbb{R}^d \setminus \bigcup_{j \in \mathbb{N}} [1, 2]$ we have $f_{\delta}(x) = \exp(-\sum_{j \in \mathbb{N}} x_j) \prod_{j \in \mathbb{N}} x_j \geq 0$. Now let us consider the case $x \in \mathbb{R}^d \setminus \bigcup_{j \in \mathbb{N}} [1, 2]$. In fact there are $k_1, \ldots, k_{d+1} \in [0, K_1], \ldots, k_{d+1} \in [0, K_d]$ such that $x \in \mathbb{R}^d \setminus \bigcup_{j \in \mathbb{N}} [1 + k_i/j, 1 + (k_i + 1)/j]$ and hence for $k_0 := (k_1, \ldots, k_{d+1})^T$

$$
f_{\theta}(x) = f_{\theta}(x) + \delta \pi_{K_{d+1}} \prod_{j \in [d]} \psi_{k_j,j}(x_j) \geq \exp(-2\delta) \pi_{K_{d+1}} \prod_{j \in [d]} \psi_{k_j,j}(x_j) \geq 0
$$

since $\|\psi_{k_j,j} \|_{\infty} \leq 2^{1/j} \psi_{k_j,j} \pi_{K_{d+1}}$ for any $k_j \in [0, K_j]$ and $j \in [d]$ and $F_{k,j} \geq 1$. Choosing $\delta \leq \delta_1(\psi, \gamma) = \exp(-2\delta)/(\int \prod_{j \in \mathbb{N}} \psi_{k_j,j}(x_j) \geq 0$ for all $x \in \mathbb{R}^d$.

**Proof of Lemma 6**

Our proof starts with the observation that $f_{\delta}(x) = \prod_{j \in \mathbb{N}} f_{\delta}(x_j)$ where $f_{\delta}(x_j) := \exp(-x_j)$ if $j \in T$ and $f_{\delta}(x_j) := x_j \exp(-x_j)$ else, for all $x \in \mathbb{R}^d$.

By the definition of the multivariate Mellin transform, compare [5], we see that $f_{\delta} \in L^2(\mathbb{R}, x^{2d-1}) \cap L^1(\mathbb{R}, x^{2d-1})$ holds for every $e \in \mathbb{R}^d$ and that for all $t \in \mathbb{R}^d$ we have

$$
\mathcal{M}[f_{\delta}](t) = \prod_{j \in \mathbb{N}} \mathcal{M}_{\psi}(f_{\delta})(t_j) = \prod_{j \in \mathbb{N}} \Gamma(c_j + it_j) \cdot \Gamma(c_j + it_j).
$$

Now by applying the Stirling formula (see also [11]) we get $|\Gamma(c_j + it_j)| \sim |t_j|^{c_j+1/2} \exp(-\pi|t_j|^2/2), |t_j| \geq 2$. Thus for every $s \in \mathbb{N}^d$ there exists $L_{s,e}$ such that $|f_{\delta}|^2_{L^2} \leq L_{s,e}$.

Next we consider $\|f_{\delta} - f_0\|_{L^2}$. Again we see that, $f_0 - f_{\theta} \in C^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}, x^{2d-1}) \cap L^1(\mathbb{R}, x^{2d-1})$ with

$$
\mathcal{M}[f_0 - f_{\theta}](t) = \delta F_{K_{d+1}}^{-1} \sum_{k \in \mathbb{R}^d} \mathcal{M}_{\psi_{k,j}}(f_0 - f_{\theta})(t_j), \quad t \in \mathbb{R}^d.
$$
Now for any fixed \( \ell \in \mathbb{Z} \) we derive from \( \psi_{k,K,y} = \mathbb{S}^k[\psi_{k,K}] \) for any \( t_j \in \mathbb{R} \) that \( \mathcal{M}_t[\psi_{k,K,y}](t) = (c_1 + i t \gamma)^{-\ell} \mathcal{M}_t[\psi_{k,K,y}](t) \). This implies that
\[
(1 + t_j)^\ell \mathcal{M}_t[f_0 - f_0(t)]^2 \leq C_{c,\gamma,\delta} \frac{\delta^2 F_{K,y,s}^{-1}}{\sum_{k \in K} \| \psi_{k,K,y+s} \|_{\ell}^2} \sum_{k \in K} \| \psi_{k,K,y+s} \|_{\ell}^2 \prod_{j \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j) \]
Now using that the inverse Mellin operator is linear and by a factorization argument we get
\[
\mathcal{M}_t^{-1} \left[ \sum_{k \in K} \theta_k \mathcal{M}_t[\psi_{k,K,y+s}](t) \right] \prod_{j \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j) = \sum_{k \in K} \theta_k \psi_{k,K,y+s}(x) \prod_{j \in [d]} \psi_{k,K,y}(x_j)
\]
which implies that due to the disjoint supports of \( \psi_{k,K,y} \) and another factorization argument,
\[
\| (1 + t_j)^\ell \mathcal{M}_t[f_0 - f_0(t)]^2 \|_{L^2} \leq C_{c,\gamma,\delta} \| \psi_{k,K,y} \|_{\ell}^2 \sum_{k \in K} \| \psi_{k,K,y+s} \|_{\ell}^2 \prod_{j \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j) \]
since \( \| \psi_{k,K,y} \|_{\ell}^2 \leq \| f_0 \|_{L^2}^2 = \| f_0 \|_{L^2}^2 + \| f_0 \|_{L^2}^2 \leq 2(\| f_0 \|_{L^2}^2 + \| f_0 \|_{L^2}^2) \leq 2C_{\gamma,\delta} + L_x =: L_{\gamma,\delta,2} \).
Now we have to consider the moment condition \( \mathbb{E}_/r(\mathcal{X}_{K,y}^2) \leq L \). In fact we have
\[
\int_{\mathbb{R}^+} \sum_{\ell \leq 2} \mathcal{M}_t[f_0 - f_0(t)]^2 d_x = \prod_{j \in [d]} \int_{\mathbb{R}^+} \sum_{\ell \leq 2} \mathcal{M}_t[\psi_{k,K,y+s}](t) = \sum_{k \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j) \prod_{j \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j)
\]
Proof of Lemma \( 6 \) (i): Using that the functions \( (\psi_{k,K,y}) \) with different index \( k_j \) have disjoint supports and a factorization argument we get
\[
\| f_0 - f_0(t)^2 \|_{L^2}^2 = \sum_{k \in K} \| \psi_{k,K,y+s} \|_{\ell}^2 \prod_{j \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j) \sum_{k \in [d]} \| \psi_{k,K,y+s} \|_{\ell}^2 \prod_{j \in [d]} \mathcal{M}_t[\psi_{k,K,y}](t_j) \]
where the last step follows if we can show that there exists a \( c_{\ell} > 0 \)
\[
\| \psi_{k,K,y} \|_{\ell}^2 \leq \int_0^\infty \psi_{k,K,y}(x)^2 x^{2 \gamma - 1} d_x \geq \frac{c_{\gamma,K}^2 \| \psi_{k,K,y} \|_{\ell}^2}{2}
\]
for \( K_j \) big enough. Here \( \rho(\theta, \theta') := \sum_{k \in K} \| \psi_{k,K,y+s} \|_{\ell}^2 \) denotes the Hamming distance.
To show \( 20 \) we observe that
\[
\| \psi_{k,K,y} \|_{\ell}^2 = \sum_{x \in \mathbb{Z} \times \mathbb{Z}} c_{k,K,y} f_{k,K}(x) \psi_{k,K}(x)^2 d_x
\]
and by defining \( \Sigma := \| \psi_{k,K,y} \|_{\ell}^2 = \int_0^\infty (x^{2 \gamma} \psi_{k,K}(x)^2) x^{2 \gamma - 1} d_x \) we can show
\[
\| \psi_{k,K,y} \|_{\ell}^2 = \Sigma + \int_0^\infty (x^{2 \gamma} \psi_{k,K}(x)^2) x^{2 \gamma - 1} d_x \geq \Sigma + c_{\gamma,K}^2 \| \psi_{k,K,y} \|_{\ell}^2 \geq \frac{c_{\gamma,K}^2 \| \psi_{k,K,y} \|_{\ell}^2}{2}
\]
as soon as $|\Sigma| \leq \frac{c_{\gamma,K}\nu^{-\nu}|d|^\nu}{\gamma}$. This is obviously true as soon as $K_j \geq K_\nu(y_j, c_j, \psi)$ and thus $\|f_\nu - f_\nu\|^2_{\gamma^2,\Gamma^2} \geq \delta^2 C_{\theta,\Theta} K^{-1}(\sum_{j \in \mathcal{J}} K_j^{2\alpha})^{-1} F(t, \theta^0)$ for $K \geq K_\nu(\psi, \gamma, \epsilon)$ which is understood in a componentwise sense.

Now let us interpret the objects $\Theta$ as vectors using the canonical bijection $T : \Theta \rightarrow [0,1]^{I_{\theta} \cap K_\nu}$. Then we have $\rho(\theta_{k+1}, \theta^0) = \bar{\rho}(T(\theta_{k+1}), T(\theta^0))$ where $\bar{\rho}(\theta, \theta^0) := \sum_{j \in \mathcal{J}} \omega_{k,j} K_j \mathbb{1}_{\{\theta, \theta^0\}}$ for any $\theta, \theta^0 \in [0,1]^{I_{\theta} \cap K_\nu}$. Using the Varshamov-Gilbert Lemma (see [13]) which states that for $\prod_{j \in \mathcal{J}} K_j \geq 8$ there exists a subset $(\theta^0, \ldots, \theta^M)$ of $[0,1]^{I_{\theta} \cap K_\nu}$ with $\theta^{(0)} = (0, \ldots, 0)$ such that $\bar{\rho}(\theta^{(j)}, \theta^{(k)}) \geq K_j/8$ for all $j, k \in [0, M]$, $j \neq k$ and $M \geq 2^{x^3} I_{\theta} \cap K_\nu$. Defining $\theta^{(0)} := T^{-1}(\theta^{(j)})$ for $j \in [0, M]$ leads to $\|f_\nu - f_\nu\|^2_{\gamma^2,\Gamma^2} \geq C_{\theta,\Theta} K^{-1}(\sum_{j \in \mathcal{J}} K_j^{2\alpha})^{-1}$.

(ii): For the second part we have $f_\nu = f_{\nu^0}$, and by using $\mathrm{KL}(f_\nu, f_\nu) \leq \chi^2(f_\nu, f_\nu) = \int_{\mathbb{R}^d} (f_\nu(x) - f_\nu(x))^2 f_\nu(x) dx$ it is sufficient to bound the $\chi^2$-divergence. We notice that since $U_0, \ldots, U_n$ are independent we can write $g(x) = \prod_{j \in \mathcal{J}} g_j(x_j)$ for $x \in \mathbb{R}^d$. Further, $f_\nu - f_\nu$ has support in $[0,2]^d$ since $f_\nu - f_\nu$ has support in $[1,2]^d$ and $g$ has support in $[0,1]^d$. In fact for $y \in \mathbb{R}^d$ with $y_0 > 2$ for $f \in \mathcal{D}$,

$$\tilde{f}_\nu(y) - f_\nu(y) = \int_{\mathbb{R}^d} (f_\nu - f_\nu)(x) x^{-1} g(y/x) dx = \delta F_{K,y}^{-1/2} \sum_{k \in K} \prod_{j \in \mathcal{J}} \int_{\mathbb{R}^d} \psi_{k,j}(x_j) g_j(y_j/x_j) x_j^1 dx_j = 0.$$ 

Next we have for any $t \in \mathbb{R}^d$ by application of assumption of Theorem [1] the convolution theorem and the fact that $M_{\psi_k}[\psi_k, x, y] = (\hat{\epsilon} + i t)^{1/2} M_{\psi_k}[\psi_k, x, y]$, we get

$$|\tilde{M}_{\psi_k}[\tilde{f}_\nu, \tilde{f}_\nu](t)| = \|F_{K,y}^{-1/2} \sum_{k \in K} \prod_{j \in \mathcal{J}} M_{\psi_k}[\psi_k, x, y] M_{\psi_k}[\psi_k, x, y] \| \leq C_{\gamma,y} \delta^2 F_{K,y}^{1/2} \sum_{k \in K} \prod_{j \in \mathcal{J}} M_{\psi_k}[\psi_k, x, y],$$

for all $t \in \mathbb{R}^d$. Applying the Parseval equality and using the disjoint supports, the factorization property and eq. 19 we get

$$\chi^2(\tilde{f}_\nu, f_\nu) = C_{\gamma} \| f_\nu - f_\nu \|^2_{\gamma^2,\Gamma^2} = C_{\gamma,y} \delta^2 F_{K,y}^{1/2} \sum_{k \in K} \prod_{j \in \mathcal{J}} \| \psi_{k,j}[\psi_k, x, y] \|^2_{\gamma^2,\Gamma^2}.$$ 

The inequality $\| \psi_{k,j}[\psi_k, x, y] \|^2_{\gamma^2,\Gamma^2}$. The fact that $\| \psi_{k,j}[\psi_k, x, y] \|^2_{\gamma^2,\Gamma^2} \leq K_j^{-1} \| \psi_{k,j} \|^2_{\gamma^2,\Gamma^2}$ implies $\chi^2(\tilde{f}_\nu, f_\nu) \leq C_{\gamma,y} \delta^2 F_{K,y}^{1/2}$. Since $M \geq 2^{x^3} I_{\theta} \cap K_\nu$ we can deduce that $\mathrm{KL}(\tilde{f}_\nu, f_\nu) \leq C_{\gamma,y} \delta^2 \log(M) K^{-1}(\sum_{j \in \mathcal{J}} K_j^{2\alpha})^{-1}$.qed

**Proof of Theorem 2.** Let $k \in \mathcal{K}_\nu$. By definition of the estimator, (8), we have $Q_{k} = \mathrm{supp}(M_{\psi_k}[f_k])$, for $k' \in \mathcal{K}_\nu$ and we can find a $K_\nu \in \mathbb{N}^2$ such that for all $k' \in \mathcal{K}_\nu$ holds $Q_{k'} \subseteq Q_{k}$. Then we have for any $k' \in \mathcal{K}_\nu$ that $\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} - \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} = \|f_k - f_k\|^2_{\gamma^2,\Gamma^2}$ implying with (14)

$$\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + \|\tilde{f}_\nu - f_k\|^2_{\gamma^2,\Gamma^2} \leq \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + \|\tilde{f}_\nu - f_k\|^2_{\gamma^2,\Gamma^2}.$$ 

Now for every $k' \in \mathcal{K}_\nu$ we have $\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} = \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + 2 \langle f_k - f_k, f_k - f_k \rangle_{\gamma^2,\Gamma^2}$ which combined with (22) implies

$$\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} - \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} \leq \|\tilde{f}_\nu - f_k\|^2_{\gamma^2,\Gamma^2} - \|f_k - f_k\|^2_{\gamma^2,\Gamma^2},$$

which with (23) gives

$$\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} \leq \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} - \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + \|\tilde{f}_\nu - f_k\|^2_{\gamma^2,\Gamma^2} + 2 \langle f_k - f_k, f_k - f_k \rangle_{\gamma^2,\Gamma^2}.$$ 

Since $\langle f_k - f_k, f_k - f_k \rangle_{\gamma^2,\Gamma^2} = \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} - \|f_k - f_k\|^2_{\gamma^2,\Gamma^2}$ we get

$$\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} \leq \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} - \|f_k - f_k\|^2_{\gamma^2,\Gamma^2} + \|\tilde{f}_\nu - f_k\|^2_{\gamma^2,\Gamma^2} + \|\tilde{f}_\nu - f_k\|^2_{\gamma^2,\Gamma^2} + 2 \langle f_k - f_k, f_k - f_k \rangle_{\gamma^2,\Gamma^2}.$$ 

We now consider the term $2 \langle f_k - f_k, f_k - f_k \rangle_{\gamma^2,\Gamma^2}$. First we remind that for any $k' \in \mathcal{K}_\nu$

$$\|f_k - f_k\|^2_{\gamma^2,\Gamma^2} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{1}_{Q_k}(t) \frac{|M_{\psi_k}f_k(t) - \tilde{M}_{\psi_k}(t)|^2}{|M_{\psi_k}(t)|^2} dt.$$
Setting \( Q' := Q_k \cup Q_k \) we have \( M_k[f_k - f_k] = M_k[f_k(1 - 1_{Q_k})] \) implying that \( \text{supp}(M_k[f_k - f_k]) \subseteq Q' \subseteq Q_k \), by definition of \( K_n \). Using that \( 2ab \leq a^2 + b^2 \) we deduce

\[
\frac{1}{2} \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} \leq \left( \frac{1}{2} \right) \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} + \frac{1}{2} \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} + 4 \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} + \frac{7}{2} \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2},
\]

implying that

\[
\langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} \leq 3 \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} + 6 \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} + 2 \text{pen}(k) + 11 \langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2} - 2 \text{pen}(k).
\]

Now since \( E_n f_k(\text{pen}(k)) = \text{pen}(k) \) and \( 6E_n f_k(\langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2}) \leq 6 \text{pen}(k) \leq \text{pen}(k) \) we get combined with \( 25 \)

\[
E_n f_k(\text{pen}(k)) \leq \text{pen}(k) \text{ and } 6E_n f_k(\langle f_k - f_k, f_k - f_k, f_k \rangle_{L^2}) \leq 6 \text{pen}(k) \leq \text{pen}(k)
\]

The two expectations on the right hand side of the last inequality can be bounded using the following Lemma.

**Lemma 7.** Assume that \( E_n f_k \left( Y_n^{(c,1)} \right) \) and \( \| f_k \|_{L^2} \) are finite. Then

(i) \( E_n f_k \left( \| f_k - f_k \|_{L^2}^2 - \text{pen}(k) \right) \leq C_{\text{finite}}(1) \),

(ii) \( E_n f_k \left( \text{pen}(k) - \text{pen}(k) \right) \leq C_{\text{finite}}(2) \).

Consequently, we have

\[
E_n f_k(\text{pen}(k)) \leq \| f_k - f_k \|_{L^2}^2 + 3 \left( \| f_k - f_k \|_{L^2}^2 + \text{pen}(k) \right) + \frac{C_{\text{finite}}}{n} \leq 3 \left( \| f_k - f_k \|_{L^2}^2 + \text{pen}(k) \right) + \frac{C_{\text{finite}}}{n}.
\]

Taking the infimum over all \( k \in K_n \) implies the claim.

**Proof of Lemma 7** We start by proving (i). Let us therefore define the set \( U := \{ h \in L^2(\mathbb{R}^d, x^{2c-1}) : \| h \|_{L^2} \leq 1 \} \).

Then for \( k' \in \mathbb{R}^d, \| f_k - f_k \|_{L^2} = \text{sup}_{h \in U} \| f_k - f_k, h \|_{L^2} \) where

\[
\langle f_k - f_k, h \rangle_{L^2} = \langle f_k - f_k, h \rangle_{L^2} \leq (2\pi)^d \int_{Q_k} \langle \tilde{M}_c(t) - E_n f_k \tilde{M}_c(t) \rangle \frac{M_k[h(-t)]}{M_k[g(t)]} dt,
\]

by application of the Plancherel equality. Now for a sequence \( (c_n)_{n \in \mathbb{N}} \) we decompose the estimator \( \tilde{M}_c(t) \) into

\[
\tilde{M}_c(t) = n^{-1} \sum_{j \in [n]} Y_j^{1+ct} I_{(0,c_n)}(Y_j^{1+ct}) + n^{-1} \sum_{j \in [n]} Y_j^{1+ct} I_{(c_n, \infty)}(Y_j^{1+ct}) =: \tilde{M}_{c,1}(t) + \tilde{M}_{c,2}(t)
\]

where \( (0, c_n) := (0, c_n)^d \). Setting

\[
\nu_{f_k,h}(t) := \frac{1}{(2\pi)^d} \int_{Q_k} \langle \tilde{M}_c(t) - E_n f_k \tilde{M}_c(t) \rangle \frac{M_k[h(-t)]}{M_k[g(t)]} dt, \quad h \in U, i \in \{1, 2\},
\]

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we can deduce that
\[
\mathbb{E}_{f_r}(\mathbb{E}(\nu_{k,l}(h)^2 - \frac{1}{12} \text{pen}(\tilde{k})) + 2 \mathbb{E}_{f_r}(\sup_{k \in \mathcal{K}} \nu_{k,1}(h)^2 - \frac{1}{24} \text{pen}(k)) + 2 \mathbb{E}^n_{f_r}(\sup_{k \in \mathcal{K}} \nu_{k,2}(h)^2)).
\] (26)

We start by bounding the first summand. To do so, we see that
\[
\sum_{k \in \mathcal{K}} \mathbb{E}_{f_r}(\sup_{k \in \mathcal{K}} \nu_{k,1}(h)^2 - \frac{1}{24} \text{pen}(k)) \leq \sum_{k \in \mathcal{K}} \mathbb{E}_{f_r}(\sup_{k \in \mathcal{K}} \nu_{k,1}(h)^2 - \frac{1}{24} \text{pen}(k)).
\]

To control each summand we apply the Talagrand inequality, see Remark [2] which can be done since there exists a dense subset of $U$. For each $k' \in \mathcal{K}_n, h \in U$ we set
\[
v_h(y) := \frac{1}{(2\pi)^d} \int_{Q_{h'}} y^{c-1} dI_{(y^{c-1})}, \quad y \in \mathbb{R}^d.
\]

So, $v_h = v_{k',h}$ in the notation of Remark [2]. Thus we need to determine the parameters $\tau, \Psi^2, \psi$. Let us begin with $\Psi$. For $h \in U$ we have $1 \geq \|h\|_{Q_{h'}} = (2\pi)^{-d} \|M_{k}[h]\|_{\mathcal{V}}$. Using the Cauchy Schwartz inequality delivers
\[
\mathbb{E}_{f_r}(\sup_{h \in U} \|v_h\|_{Q_{h'}}^2) \leq (2\pi)^{-d} \int_{Q_{h'}} \mathbb{E}_{f_r}(\|\mathcal{M}_{k}[h](t)\|_{\mathcal{V}}^2) \mathbb{E}_{f_r}(\|\mathcal{M}_{k}[g](t)\|_{\mathcal{V}}^2) dt \leq \sigma n^{-1} \Delta_{k}(k') =: \Psi^2.
\]

Now for $\tau$ we see that $\text{Var}_{f_r}(\nu_{Y}(f_{r})) \leq \mathbb{E}_{f_r}(\nu_{Y}(f_{r}))^{2} \leq \|f_{h}\|_{\mathcal{V}}^{2} \|\nu_{h}\|_{Q_{h'}}^{2}$. Further,
\[
\|\nu_{h}\|_{Q_{h'}}^{2} = (2\pi)^{-d} \int_{Q_{h'}} |\mathcal{M}_{k}[h][t]^{2}| |\mathcal{M}_{k}[g][t]^{2}| dt \leq \|\nu_{h}\|_{Q_{h'}}^2 \mathcal{M}_{k}[g]^{2} dt.
\]

Thus, we choose $\tau := \|f_{h}\|_{\mathcal{V}}^{2} \mathcal{M}_{k}[g]^{2} \|\nu_{h}\|_{Q_{h'}}^{2}$. Let us now consider $\psi^2$. We have for any $y \in \mathbb{R}_d$, $|v_h(y)|^2 = (2\pi)^{-d} \int_{Q_{h'}} y^{c-1} dI_{(y^{c-1})} \mathbb{E}_{f_r}(\|\mathcal{M}_{k}[h](t)\|_{\mathcal{V}}^2) \mathbb{E}_{f_r}(\|\mathcal{M}_{k}[g](t)\|_{\mathcal{V}}^2) dt \leq c_{\alpha} n \Delta_{k}(k)^2 =: \psi^2,
\]

since $\|h\|_{\mathcal{V}}^{2} \leq 1$ and $|y|^2 = 1$. Applying now the Talagrand inequality we get
\[
\mathbb{E}_{f_r}(\sup_{h \in U} \nu_{h}(Y)) = \mathbb{E}_{f_r}(\sup_{h \in U} \nu_{h}(Y))^{2} \leq C_{f_r} n \left(\frac{\sqrt{\Delta_{k}(h)^2}}{n} \right)^2 \mathbb{E}_{f_r}(\nu_{k,1}(h)^2) \leq C_{f_r} n \left(\frac{\sqrt{\Delta_{k}(h)^2}}{n} \right)^2 \mathbb{E}_{f_r}(\nu_{k,2}(h)^2)
\]

for the choice $c_0 := \sqrt{\log(n(d+1))}$. For $\chi \geq 144$ we can conclude that
\[
\mathbb{E}_{f_r}(\sup_{h \in U} \nu_{k,1}(h)^2 - \frac{\chi}{24} \Delta_{k}(k)^{n-1}) \leq \sum_{k \in \mathcal{K}_n} \left(\frac{\sqrt{\Delta_{k}(h)^2}}{n} \right)^2 \mathbb{E}_{f_r}(\nu_{k,2}(h)^2) \leq C_{f_r} \frac{\sqrt{\log(n)}}{c_0} \mathbb{E}_{f_r}(\nu_{k,2}(h)^2)
\]

since $|\mathcal{K}_n| \leq n^d$. For the second summand in (26) we get for any $k' \in \mathcal{K}_n$ and $h \in U$,
\[
|v_{k,2}(h)|^2 \leq (2\pi)^{-d} \int_{Q_{h'}} |\mathcal{M}_{k,2}(t) - \mathbb{E}_{f_r}(\mathcal{M}_{k,2}(t))^{2}| \mathbb{E}_{f_r}(\mathcal{M}_{k,2}(t))^{2} dt
\]

since $|\mathcal{K}_n| \leq n^d$. For the second summand in (26) we get for any $k' \in \mathcal{K}_n$ and $h \in U$,
\[
|v_{k,2}(h)|^2 \leq (2\pi)^{-d} \int_{Q_{h'}} |\mathcal{M}_{k,2}(t) - \mathbb{E}_{f_r}(\mathcal{M}_{k,2}(t))^{2}| \mathbb{E}_{f_r}(\mathcal{M}_{k,2}(t))^{2} dt
\]

Thus we have for any $u > 0$
\[
\mathbb{E}_{f_r}(\sup_{h \in U} \nu_{k,2}(h)^2) \leq \sum_{k \in \mathcal{K}_n} \Delta_{k}(k)^{n-1} \mathbb{E}_{f_r}(\nu_{k,2}(h)^2) \leq \sum_{k \in \mathcal{K}_n} \Delta_{k}(k)^{n-1} \mathbb{E}_{f_r}(\nu_{k,2}(h)^2) \leq \sum_{k \in \mathcal{K}_n} \Delta_{k}(k)^{n-1} \mathbb{E}_{f_r}(\nu_{k,2}(h)^2) \leq \sum_{k \in \mathcal{K}_n} \Delta_{k}(k)^{n-1} \mathbb{E}_{f_r}(\nu_{k,2}(h)^2) \leq \frac{C_{\mathcal{K}_n}}{c_0} \mathbb{E}_{f_r}(\nu_{k,2}(h)^2).
\]
Now under assumption (11) we have $|\mathcal{K}_n| \leq |\{ k \in \mathbb{N}^d : k^{2\gamma+1} \leq c_d n \}| \leq C_d n \log(n)^{d-1}$, compare [7]. Now choosing $u = 5$ implies

$$\mathbb{E}_f (\sup_{h \in U} \nu_{k,\widehat{\sigma}}(h))^2) \leq C_g \mathbb{E}_f (Y_{\frac{2(\gamma+1)}{\gamma+1}})^2 n^{-1}.$$ 

To finish the proof we still need to show (ii). To do so, we define the event $\Omega := \{ |\widehat{\sigma} - \sigma| \leq \sigma/2 \}$. On $\Omega$ holds $\sigma \leq 2\widehat{\sigma} \leq 3\sigma$ and we deduce

$$(\text{pen} (\hat{k}) - 2\text{pen} (\hat{\sigma}))_+ \leq \chi (\sigma - 2\hat{\sigma}) + \mathbb{I}_\Omega$$

since $\hat{k} \in \mathcal{K}_n$. Therefore we get by application of the Cauchy-Schwartz inequality and the Markov inequality

$$\mathbb{E}((\text{pen} (\hat{k}) - 2\text{pen} (\hat{\sigma}))_+) \leq C(\chi, \sigma) \mathbb{V} \text{ar}(\hat{\sigma}) = C(\chi, \sigma, \mathbb{E}(Y_{\frac{2(\gamma+1)}{\gamma+1}})^2) n^{-1}.$$ 

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