The generating solution of regular N=8 BPS black holes

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Abstract

In this paper we construct the 5 parameter generating solution of N = 8 BPS regular supergravity black holes as a five parameter solution of the N = 2 STU model. Our solution has a simpler form with respect to previous constructions already appeared in the literature and moreover, through the embedding \([SL(2)]^3 \subset SU(3, 3) \subset E_{7(7)}\) discussed in previous papers, the action of the U–duality group is well defined. This allows to reproduce via U–duality rotations any other solution, like those corresponding to R–R black holes whose microscopic description is given by intersecting D–branes.
1 Introduction

After the advent of D-branes \[1\] there has been a renewed interest in the study of supergravity
black \(p\)-branes, in particular those preserving a fraction of the original supersymmetry. This
is due to their identification with the BPS saturated non–perturbative states of superstring
theory \[2, 3\] which has promoted them from classical solutions of the low–energy theory to
solutions of the whole quantum theory. Therefore they represent an important tool in probing
the non–perturbative regime of superstring theories. Of particular interest are the regular ones,
namely those having a non–vanishing Bekenstein–Hawking entropy.

In this paper we deal with BPS static black hole solutions of \(D = 4\) supergravity preserving
1/8 of the original \(N = 8\) supersymmetry, completing a program started in \[4\] and continued
in \[5\]. Let us recall that in the context of toroidally compactified type II supergravity the
only regular black hole solutions are the 1/8 supersymmetry preserving ones. The 1/2 and
1/4 black holes, whose general form has been completely classified in \[6\], have a vanishing
horizon area. More precisely, the only 4–dimensional \textit{regular} black hole configurations are
those preserving 4 supersymmetry charges, irrespectively of their higher dimensional origin.
One of the results of \[4\] was to show that the most general 1/8 black hole solution of \(N = 8\)
supergravity can be related, through a \(U\)–duality transformation, to a solution of a suitable
consistent truncation \(N = 8 \rightarrow N = 2\) of the supergravity theory (corresponding to a Calabi–
Yau compactification). \(U\)–duality in supergravity denotes the largest global symmetry of the
field equations and Bianchi identities. We shall recall the main facts about \(U\)–duality orbits
of BPS black holes in section 2 and it will be shown that the problem of studying the most
general 1/8 BPS black hole in the \(N = 8\) theory can be reduced to that of finding the so called
\textit{generating solution}, that is the most general one modulo \(U\)–duality transformations, which is
a solution of the simpler \(N = 2\) truncation. Once this solution is found, acting on it with
the maximal compact subgroup \(H = SU(8)\) of the \(U\)–duality group one generates the general
charged black hole and then acting with the whole \(U\)–duality group \(U = E_{7(7)}\) one generates
the most general solution, namely that with fully general asymptotic values of the scalar fields.
In order to define the action of the \(N = 8\) \(U\)–duality group on a solution of the aforementioned
\(N = 2\) truncation, the embedding of the latter in the \(N = 8\) theory has to be defined in a precise
group theoretical fashion. This was done in \[4\] and \[5\] using the solvable Lie algebra formalism
\[7\]. The \(N = 2\) truncation mentioned above is called the \(STU\) model (first studied in \[8\])
and essentially is a \(N = 2\) supergravity coupled to 3 vector multiplets interacting through the
Special Kähler manifold \([SL(2, \mathbb{R})/SO(2)]^3\). It has therefore 6 real scalars (3 dilaton–like and
3 axion–like) and four vector fields. A generic static, spherically symmetric black hole solution
of this model is then characterized by 4 electric \((q_A)\) and 4 magnetic \((p^A)\) quantized charges
and a point $\phi_\infty$ in the scalar manifold representing the asymptotic behavior of the scalar fields at infinite radial distance from the center of the solution. The generating solution is obtained by fixing the $[SL(2, \mathbb{R})]^3$ duality action on a generic BPS black hole solution, which is achieved by first choosing a particular point $\phi_\infty$ of the scalar manifold and then fixing the action of the isotropy group $[SO(2)]^3$ on the solution, which amounts to imposing three suitable conditions on the quantized charges. These conditions in turn allow for three conditions on the evolving scalar fields of the solution, compatible with the field equations and the BPS requirement. The generating solution therefore will be described by 5 charges and three independent scalars. The mathematical framework in which to construct such a solution has been defined in [5], however an explicit solution was given only in a simplified case where the 3 dilatons $b_i$ and the 3 axions $a_i$ were separately set equal to a single dilaton field $b$ and a single axion field $a$. This solution, if properly gauge–fixed, would depend only on 2 parameters. The aim of the present paper is to relax these restrictions and compute the 5–parameter generating solution within the $STU$ model. Other regular solutions of the same theory, different from the generating one, have been considered in other papers [9].

As pointed out in [4] and [10], the generating solution represents a pure NS–NS black hole, that is a black hole whose 10 dimensional microscopic configuration is made only of NS branes. Indeed, the generating solution for black holes of the toroidally compactified type II string theory is the same as the one of the toroidally compactified heterotic string and it was already constructed in [11]. Our point is that, as noticed in [12], as far as the microscopic entropy counting is concerned it is difficult to count the states of a pure NS–NS configuration. Therefore it is essential to be able to reproduce R–R configurations which are those represented in terms of intersecting D–branes only. This in principle would not be a problem since $U$–duality does not make any difference between R–R and NS–NS fields and one should be able to move from a solution to any other acting via $U$–duality transformations. However, as previously pointed out, in order for this to be possible one has not only to build up the generating solution itself but also to know how the latter is embedded in the full theory.

The paper is organized as follows:

In section 2 we describe the structure of the $U$–duality orbits for 1/8 BPS black holes in terms of 5 independent invariants and characterize the generating solution as a solution depending on 5 independent charge parameters, in terms of which the 5 invariants can be expressed as independent functions on the chosen point of the moduli space at infinity.

In section 3 we explicitly construct the 5 parameters generating solution by solving both the first and second order differential equations, after performing the aforementioned 3 parameter gauge fixing. The need to check that a particular solution of the first order equations repre-
senting the BPS condition fulfill the second order field equations as well is due to a known feature of solitonic solutions in supergravity that the former equations in general don’t imply the latter.

Most of the technical aspects concerning both the derivation of the first and second order differential equations (the former representing the BPS conditions, the latter being the equations of motion) and the proper formalism needed in order to embed the STU model solution in the $N = 8$ theory properly, have already been given in [4] and [5]. Hence we shall skip most technical details and make use of the results already obtained in [5].

We end in section 4 with few comments and some concluding remarks.

2 $U$–duality orbits and the generating solution

Let us recall the main facts about the $U$–duality orbits of 1/8 BPS black holes in the $N = 8$ classical supergravity theory [13], [14]. It is well known that the equations of motion and the Bianchi identities of the $N = 8$ classical supergravity theory in 4–dimensions are invariant with respect to the $U$–duality group $E_{7(7)}$. This invariance requires the group $E_{7(7)}$ to act simultaneously on both the 70 scalar fields $\phi^\alpha$ spanning the manifold $M_{scal} = E_{7(7)}/SU(8)$ and on the vector $\vec{Q}$ consisting of the 28 electric and 28 magnetic quantized charges. The $U$–duality group acts on the scalar fields as the isometry group of $M_{scal}$ and on $\vec{Q}$ in the 56 (symplectic) representation. A static, spherically symmetric BPS black hole solution is characterized in general by the vector $\vec{Q}$ and a particular point $\phi_\infty$ on the moduli space of the theory whose 70 coordinates $\phi^\alpha_\infty$ are the values of the scalar fields at infinity ($r \to \infty$). Acting on a black hole solution $(\phi_\infty^\alpha, \vec{Q})$ by means of a $U$–duality transformation $g$ one generates a new black hole solution $(\phi_g^\alpha, \vec{Q}_g)$:

$$\forall g \in G \ \{ \phi_\infty^\alpha \rightarrow \phi_g^\alpha(\phi_0), \ \vec{Q} \rightarrow \vec{Q}_g = S(g) \cdot \vec{Q} \ \} \text{ where } S(g) \in Sp(56, \mathbb{R})$$

(2.1)

The BPS black hole solutions fill therefore $U$–duality orbits.

As far as the 1/8 BPS black holes are concerned these orbits turn out to be parameterized by 5 functions $I(\phi_\infty, \vec{Q})_I$ ($I = 1, \ldots, 5$) which are invariant under the duality transformations in (2.1). These invariants are expressed in terms of the $8 \times 8$ anti–symmetric central charge matrix $Z_{AB}(\phi_\infty, \vec{Q})$ (the antisymmetric couple $(AB)$, as $A$ and $B$ run from 1 to 8, labels the representation 28 of $SU(8)$) in the following way ([14]):

$$I_k = \text{Tr} \left( \overline{Z} Z \right)^k \ \ \ \ \ k = 1, \ldots, 4$$

$$I_5 = \text{Tr} \left( \overline{Z} Z \right)^2 - \frac{1}{4} (\text{Tr} \overline{Z} Z)^2 + \frac{1}{96} \left( \epsilon_{ABCDEFGH} Z^{AB} Z^{CD} Z^{EF} Z^{GH} + \text{c.c.} \right)$$

(2.2)
where $\overline{Z}Z$ denotes the matrix $Z^{AC}Z_{CB}$ and the convention $Z^{AB} = (Z_{AB})^*$ is adopted. Among the $I(\phi_\infty, \vec{Q})_I$ a particular role is played by the moduli–independent invariant $I_5(\vec{Q})$ that is the quartic invariant (which will be denoted in the sequel also by $P_{(4)}(\vec{Q})$ in order to refer to its group theoretical meaning) of $E_{7(7)}$ whose value is related to the entropy of the black hole $[15, 16]$. As pointed out in $[17]$, $P_{(4)}$ must be non-negative in order for the solution to be BPS ($P_{(4)} \geq 0$). For a fixed value of $I_5(\vec{Q})$ the inequivalent orbits are parameterized by the remaining four invariants $I(\phi_\infty, \vec{Q})_k$, ($k = 1, \ldots, 4$). The behavior of the scalars describing the regular solutions with fixed entropy is schematically represented in Figure 1 in which the scalar fields flow from their boundary values $\phi_\infty$ at infinity which span $M_{scal}$ (the disk) to their fixed values $\phi_{fix}$ at the horizon $r = 0$ $[14]$. It should be understood, of course, that the $\phi$ axis is a $n$–dimensional space, where $n$ is the dimension of $M_{scal}$. The invariants $I(\phi_\infty, \vec{Q})_I$ turn out to be independent functions of the quantized charges in any generic point $\phi_\infty$ of the moduli–space except for some “singular” points where the number of truly independent invariants could be less than five. This is the case, for example, of the point $\phi_\infty = \phi_{fix}$ parameterized by the fixed values of the scalar fields at the horizon. In this point the only independent invariant is the moduli–independent one, $I_5$. We will come back on this at the end of this section.

The generating solution may be characterized as the solution depending on the the minimal number of parameters sufficient to obtain all possible 5–plets of values for the 5 invariants on a particular point $\phi_\infty \neq \phi_{fix}$ of the moduli–space (a possible vacuum of the theory). From the above characterization it follows that the whole $U$–duality orbits of 1/8 BPS black hole solutions may be constructed by acting by means of $E_{7(7)}$ transformations (2.1) on the generating one. In particular, if we focus on 1/8 BPS black hole solutions having a fixed value of the entropy (proportional to the square root of $I(\vec{Q})_5$) and on a particular bosonic vacuum for the theory...
specified by a point $\phi_\infty$ in the moduli–space, by acting only on the charges of the generating solution with the $U$–duality group, it would be possible to construct the whole spectrum of 1/8 BPS solutions of the theory realized in the chosen vacuum $\phi_\infty$ (see Figure 2). Since in a particular point $\phi_\infty \neq \phi_{fix}$ on $\mathcal{M}_{scal}$ the minimum number of parameters a solution should depend on in order to reproduce all the 5–plets of values for the independent invariants is obviously five, we expect the charge vector $\vec{Q}$ of the generating solution $(\phi_\infty, \vec{Q})$ to depend on five independent charges.

In order to motivate, in brief, the result obtained in \cite{4} according to which the generating solution for 1/8 BPS black holes in the $N = 8$ theory is described within a suitable $N = 2$ truncation of the theory (the $STU$ model), let us notice that the 5 quantities in eq. (2.2) are invariant with respect to the action of $SU(8)$ on $Z_{AB}$. In particular, by means of a 48–parameter $SU(8)$ transformation, the central charge matrix can be brought to its normal form in which it is skew–symmetrized with complex eigenvalues $(Z_k), \ k = 1, \ldots, 4 \ (|Z_4| > |Z_3| \geq |Z_2| \geq |Z_1|)$. As a consequence of this rotation (see \cite{4} and \cite{5} for a detailed discussion) the central charge eigenvalues end up depending on just 6 (dynamical) scalar fields and 8 quantized charges which

Figure 2: the scalar moduli–space $\mathcal{M}_{scal}$ and the action of the $U$–duality group on a generic point $\phi_\infty$. The tower on each point of $\mathcal{M}_{scal}$ represents the 1/8 BPS black hole spectrum of the theory realized on that bosonic background. In a generic point $\phi_\infty$ it describes a “snapshot” of the $U$–duality orbit of this particular kind of solutions. Acting on the charges only, one generates all black holes having the same values of the asymptotic fields and moves in the tower at a given $\phi_\infty$. Acting on both the charges and the moduli one moves in $\mathcal{M}_{scal}$ but the solution does not change its ADM mass (this being a $U$–duality invariant quantity). Since the $U$–duality orbit is characterized by 5 invariants $I_i$, in a point $\phi_\infty$ in which $I_i$ are independent functions of the charges $\vec{Q}$ the generating solution may be characterized as a minimal set of solutions in the corresponding tower on which the five invariants assume all possible values (compatible with the BPS condition). Therefore the generating solution should depend only on five charge parameters and, by acting just on the latter by means of the $U$–duality group (“vertical action” in the figure), one is able to reconstruct all the states of the tower.
characterize the solutions of an STU model suitably embedded in the original theory. Within this truncation, the 6 scalar fields (3 dilatons \( b_i \) and 3 axions \( a_i \)) belong to 3 vector multiplets and span a manifold \( \mathcal{M}_{STU} = [SL(2, \mathbb{R})/SO(2)]^3 \), while the 4 electric charges \( q_\Lambda \) and 4 magnetic charges \( p^\Lambda (\Lambda = 0, \ldots, 3) \) transform in the \((2, 2, 2)\) of \([SL(2, \mathbb{R})]^3\). In the framework of the STU model, the central charge eigenvalues \( Z_i(a_i, b_i, \vec{Q}) \) and \( Z_i(a_i, b_i, \vec{Q}) \) \((i = 1, 2, 3)\) are, respectively, the local realization on moduli space \( \mathcal{M}_{STU} \) of the \( N = 2 \) supersymmetry algebra central charge \( Z \) and of the 3 matter central charges associated with the 3 matter vector fields (which are related to the central charge via the following relation: \( Z^i(z, \bar{z}, p, q) = h^{ij} \nabla_j^* Z(z, \bar{z}, p, q) \), \( h_{ij}^* \) being the Kähler metric on \( \mathcal{M}_{STU} \) and \( z_i = a_i + ib_i \)). On the 1/8 BPS black hole solutions these four eigenvalues are in general independent in a generic point of the moduli–space and the BPS condition reads:

\[
M_{ADM} = \lim_{r \to \infty} |Z_4(a_i, b_i, \vec{Q})| \tag{2.3}
\]

Since the \( SU(8) \) transformation used to define the STU truncation of the original theory did not affect the values of the 5 invariants in eq. (2.2), the latter are expected to assume all possible 5–plets of values on BPS solutions of this theory. From this we conclude that the generating solution for 1/8 BPS black holes in the \( N = 8 \) theory is a solution of the STU truncation as well.

In the framework of the STU model, the five invariants in eq. (2.2) are rewritten in the following form:

\[
I(\vec{Q})_k = \sum_{k'=1}^{4} |Z_{k'}|^2k \\
I(\vec{Q})_5 = \sum_{k=1}^{4} |Z_k|^4 - 2 \sum_{k_1 > k_2 = 1}^{4} |Z_{k_1}|^2 |Z_{k_2}|^2 + 4 (Z_1^* Z_2^* Z_3^* Z_4^* + Z_1 Z_2 Z_3 Z_4) \tag{2.4}
\]

However, in this model there is still a residual invariance of the above quantities represented by the 3 parameter group \([SO(2)]^3\), isotropy group of the scalar manifold \( \mathcal{M}_{STU} \) and subgroup of \( SU(8) \). It acts on the four phases \( \theta_k \) of the central charge eigenvalues \( Z_k \) leaving the overall phase \( \theta = \sum_k \theta_k \) invariant. The generating solution is obtained by fixing this gauge freedom and therefore it depends, consistently with what stated above, on 5 parameters represented by the four norms of the central charge eigenvalues \( |Z_k| \) plus the overall phase \( \theta \). These quantities are U–duality invariants as well. It can be shown indeed that the norms \( |Z_k| \) may be expressed in terms of the four invariants \( I_k \) \((k = 1, 2, 3, 4)\) while the overall phase is contained in the expression of the Pfaffian in \( I_5 \) and thus is an invariant quantity as well which is expressed in terms of all the five \( I_1 \). Indeed, see eqs (2.2) and (2.4):

\[
\frac{1}{96} (\epsilon_{ABCDEFGH} Z^{AB} Z^{CD} Z^{EF} Z^{GH} + c.c.) = 4 (Z_1^* Z_2^* Z_3^* Z_4^* + Z_1 Z_2 Z_3 Z_4) = 2 |Z_1 Z_2 Z_3 Z_4| \cos \theta \tag{2.5}
\]
The moduli independent invariant \( I \), computed in the \( STU \) model, is the quartic invariant of the \( (2, 2, 2) \) of \([SL(2, \mathbb{R})]^3\) and it is useful to express it in a form which is intrinsic to this representation. We may indeed represent the vector \( \vec{Q} = (p^\alpha, q_2) \) as a tensor \( q^{\alpha_1 \alpha_2 \alpha_3} \) where \( \alpha_i = 1, 2 \) are the indices of the 2 of each \( SL(2, \mathbb{R}) \) factor. The invariants are constructed by contracting the indices of an even number \( 2m \) of \( q^{\alpha_1 \alpha_2 \alpha_3} \) with 3\( m \) invariant matrices \( \epsilon_{\alpha_i \beta_j} \). This contraction gives zero for \( m \) odd while for \( m \) even one finds:

\[
\begin{align*}
P_{(2)}(p, q) &= q^{\alpha_1 \alpha_2 \alpha_3} q^{\beta_1 \beta_2 \beta_3} \epsilon_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} \epsilon_{\alpha_3 \beta_3} = 0 \\
P_{(4)}(p, q) &= q^{\alpha_1 \alpha_2 \alpha_3} q^{\beta_1 \beta_2 \beta_3} q^{\gamma_1 \gamma_2 \gamma_3} q^{\delta_1 \delta_2 \delta_3} \epsilon_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} \epsilon_{\gamma_1 \delta_1} \epsilon_{\gamma_2 \delta_2} \epsilon_{\gamma_3 \delta_3} \\
&= 4(p^3 q_0 + q_1 q_2)(p^1 p^2 - p^0 q_3) - (p^0 q_0 + p^1 q_1 + p^2 q_2 - p^3 q_3)^2 \\
P_{(8)}(p, q) &= c \times (P_{(4)}(p, q))^2 \\
P_{(12)}(p, q) &= c' \times (P_{(4)}(p, q))^3
\end{align*}
\]

It can be shown rigorously that the quartic invariant written above is the only independent invariant of this representation, that is any other invariant may be expressed as powers of it. The square root of \( P_{(4)} \) is proportional to the entropy of the solution and its expression is consistent with the result of \( \text{[8]} \). In terms of the 8 quantized charges the components of \( q^{\alpha_1 \alpha_2 \alpha_3} \) are:

\[
\begin{align*}
q^{1,1,1} &= p^0; \\
q^{1,2,1} &= p^1; \\
q^{1,1,2} &= p^2; \\
q^{1,2,2} &= q_0; \\
q^{2,1,2} &= q_1; \\
q^{2,1,1} &= q_2; \\
q^{2,2,2} &= q_3
\end{align*}
\]

According to the above characterization of the generating solution, it is apparent therefore that the black hole found in \( \text{[4]} \) represents just a particular (regular) solution characterized by just 2 invariant parameters (\( |Z_1| = |Z_2| = |Z_3| \) and \( |Z_4| \)), although depending on three charges. That solution was obtained by setting \( S = T = U \) and \( p^1 = p^2 = p^3 = p, q_1 = q_2 = q_3 = q, p^0 = 0 \). Therefore, acting on it by means of an \( U \)–duality transformation, it is clear from our previous discussion that it would be possible to span only a 2–parameter sub–orbit of the whole \( U \)–duality orbit. The same argument holds for all the regular 1/8 BPS black holes characterized by less than 5 independent invariant parameters, in particular for the double fixed solution found in \( \text{[8]} \). This is in fact a very particular solution in which the scalar fields do not evolve and their values at infinity coincide with their fixed ones at the horizon \( \phi_\infty = \phi_{fix}. \) At this point the matter central charges vanish \( (Z_i(\phi_{fix}, \vec{Q}) = 0, i = 1, 2, 3) \) and the central charge \( Z_4(\phi_{fix}, \vec{Q}) \) becomes proportional to \( \mathcal{I}(\vec{Q})^{1/4} \) which represents therefore the only invariant parameter characterizing the spectrum of 1/8 BPS solutions on this particular point of the moduli–space. The 5–plet of invariants \( (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5) \) on this solution have
the specific simple form \((a, a^2, a^3, a^4, a^2)\), \(a\) being a constant depending on the 8 charges and proportional to the entropy.

Summarizing, the three main facts reviewed in the present section are:

- The \(U\)--duality orbits of 1/8 BPS black hole solutions are characterized by five \(U\)--duality invariants \(I_I\).
- The generating solution \((\phi_\infty, \vec{Q})\) is characterized by a generic point on the moduli space at infinity \(\phi_\infty \neq \phi_{fix}\) (in which the five invariants are actually independent functions of the charges \(\vec{Q}\)) and by five independent charge parameters \(\vec{Q}\) such that, by varying the latter one obtains all possible combinations of values of \(I_I\) consistent with the BPS condition \((I_5 \geq 0)\).
- The generating solution of 1/8 susy preserving black holes of \(N = 8\) supergravity can be thought of as a 1/2 preserving solution of an \(STU\) model suitably embedded in the original theory \([3]\). In this framework, which is that we will deal with, the five invariants \(I_I\) can be expressed as proper combinations of the norms of the four central charges \(|Z_k|\) (the supersymmetry and the three matter ones corresponding to \(k = 4\) and \(k = 1, 2, 3\) respectively) and their overall phase \(\theta\), according to eqs. (2.4) and (2.5).

3 The 5 parameter generating solution

In the present section we shall compute the 5--parameter generating solution as a 1/2 BPS black hole solution of the \(STU\) model.

The property a BPS saturated state has of preserving a fraction of the original supersymmetries can be characterized, on a bosonic background, by requiring that the supersymmetry transformations of all the fermion fields vanish along a suitable direction in the supersymmetry parameter space (Killing spinor):

\[
\delta_\epsilon \text{fermions} = 0 \\
\gamma^0 \epsilon_A = \pm i \epsilon_{AB} \epsilon^B \quad \text{if} \quad A, B = 1, 2
\] (3.1)

The first of these equations is equivalent to a system of first order differential equations on the background fields. In particular the system of equations in the scalar fields has a fixed point, which will be denoted by \(\phi_{fix} = (b_i^{fix}, a_i^{fix})\), towards which the solution \(\phi(r) = (b_i(r), a_i(r))\) flows to at the horizon \(r \to 0\) \([13]\).

The way eq. (3.1) has been written expresses the reality condition for \(Z_4(\phi, p, q)\) and it amounts to fix one of the three \(SO(2)\) gauge symmetries of \(H\) already giving therefore a
condition on the 8 charges and the scalar fields. In general however, as pointed out in [18], one should in fact consider a more general form for the killing spinor condition

\[ \gamma^0 \epsilon_A = \pm \frac{Z_4}{|Z_4|} \epsilon_{AB} \epsilon^B \]

(3.2)

and, as noticed by G. Moore, imposing the reality of the central charge could in principle imply that some topologically non-trivial solutions are disregarded. Nevertheless studying such a special class of solutions is not among the purposes of our present investigation and therefore we shall choose the supersymmetry central charge to be real (\text{Im}(Z_4) = 0).

Let us consider the gauge fixing procedure in detail. The four central charges \( Z_k(\phi, \vec{Q}) \) of the \( STU \) model, depending on the asymptotic values of the six scalars \( \phi_\infty = (a_i^\infty, b_j^\infty) \) and 8 charges \( \vec{Q} = (p^A, q_B) \), transform under \([SL(2, \mathbb{R})]^3\) duality (2.1) as follows:

\[ \forall g \in [SL(2, \mathbb{R})]^3 \quad Z_k(\phi^g, \vec{Q}^g) = h_g \cdot Z_k(\phi, \vec{Q}) \]

\[ h_g \in SO(2)^3 \quad h_g \cdot Z_k \equiv e^{ig_k} Z_k \]

(3.3)

Hence an \([SL(2, \mathbb{R})]^3\) duality transformation on the moduli at infinity and on the quantized charges amounts to an \([SO(2)]^3\) phase transformation on the four central charges. This holds true in particular if we consider \( g \in [SO(2)]^3 \). It follows that the \([SO(2)]^3\) gauge fixing may be achieved by either imposing three suitable conditions on the phases of the central charges, or alternatively fixing the \([SO(2)]^3\) action on \( \vec{Q} \) on a chosen point \( \phi_\infty^0 \) of the moduli space at infinity. We shall pursue the latter way which amounts to impose three suitable conditions on the quantized charges. These conditions will not be derived with group theoretical arguments; it will suffice, as it has been pointed out in the last section, to show that the five invariants, computed in \( \phi_\infty \), are independent functions of the remaining five charges.

Since, on one hand, the two-fold action of a duality transformation (and in particular of a \([SO(2)]^3\) transformation) on both the quantized charges and the scalar fields is an invariance of the equations of motion, and on the other hand the charges \( \vec{Q} \) are “constants of motion” (with respect to the \( r \)-evolution), we expect that the three gauge fixing conditions on the electric and magnetic charges have a counterpart in three \( r \)-independent conditions on the fields \( \phi(r) \), such that the restricted system of scalar fields and vector fields is still a solution of the field equations. After the gauge fixing procedure therefore we would expect the solution to be described by five independent charges, three real scalar fields and the metric function \( \mathcal{U}(r) \). The evolution of the latter four fields will be described in terms of four different harmonic functions.

Let us now recall the main positions on the background fields used to derive the first order equations from equation (3.1).

\[ ^1 \text{This result was eventually used in [26] and [28].} \]
The first of eqs. (3.1) can be specialized to the supersymmetry transformations of the gravitino and gaugino within the \textit{STU} model in the following way:

\[ \delta \epsilon \psi_{A|\mu} = \nabla_\mu \epsilon_A - \frac{1}{4} T_\rho^{|\sigma} \gamma_\rho \gamma_\mu \epsilon_{AB} \epsilon^B = 0 \]

\[ \delta \epsilon \lambda^{i|A} = i \nabla_\mu \gamma_\mu \epsilon_A + G_{\rho|\sigma}^{-i} \gamma_\rho \epsilon^{AB} \epsilon^B = 0 \]

where \( i = 1, 2, 3 \) labels the three matter vector fields, \( A, B = 1, 2 \) are the \textit{SU}(2) R-symmetry indices and \( T_\rho^{|\sigma} \) and \( G_{\rho|\sigma}^{-i} \) are the graviphoton and matter field strengths respectively (the \( - \) sign stands for the anti–self dual part). According to the procedure defined in \[5\], we adopt the following ansätze for the vector fields:

\[ F^{-|A} = \frac{t^A(r)}{4\pi} E^-, \quad t^A(r) = \frac{2}{\pi} (p^A + i \ell^A(r)) \]

\[ F^A = 2 \text{Re} F^{-|A}; \quad \tilde{F}^A = -2 \text{Im} F^{-|A} \]

\[ F^A = \frac{p^A}{2r^3} \epsilon_{abc} x^a dx^b \wedge dx^c - \frac{\ell^A(r)}{r^3} \frac{e^{2\ell}}{r^3} dt \wedge \vec{x} \cdot d\vec{x} \]

\[ \tilde{F}^A = -\frac{\ell^A(r)}{2r^3} \epsilon_{abc} x^a dx^b \wedge dx^c - \frac{p^A}{r^3} \frac{e^{2\ell}}{r^3} dt \wedge \vec{x} \cdot d\vec{x} \] (3.5)

where

\[ E^- = \frac{1}{2r^3} \epsilon_{abc} x^a dx^b \wedge dx^c + \frac{ie^{2\ell}}{r^3} dt \wedge \vec{x} \cdot d\vec{x} = \]

\[ E^{-|a} dx^b \wedge dx^c + 2 E_{0^a} dt \wedge dx^a \]

\[ 4\pi = \int_{S^2_\infty} E^{-|a} dx^a \wedge dx^b \]

(3.6)

The moduli–independent quantized charges \((p^A, q_{\Sigma})\) and the moduli–dependent electric charges \(\ell_{\Sigma}(r)\) \[5\] are obtained by the following integrations:

\[ 4\pi p^A = \int_{S^2_r} F^A = \int_{S^2_\infty} F^A = 2 \text{Re} t^A \]

\[ 4\pi q_{\Sigma} = \int_{S^2_r} G_{\Sigma} = \int_{S^2_\infty} G_{\Sigma} \]

\[ 4\pi \ell^A(r) = -\int_{S^2_r} \tilde{F}^A = 2 \text{Im} t^A \] (3.7)

where \( S^2_r \) and \( S^2_\infty \) denote the spheres centered in \( r = 0 \) of radius \( r \) and \( \infty \) respectively. The expression of the moduli–dependent charges \(\ell_{\Sigma}(r)\) in terms of the scalars \((a_i, b_i)\) and the charges \((p^A, q_{\Sigma})\) is given in the appendix.
As far as the metric $g_{\mu\nu}$, the scalars $z^i = a_i + ib_i$, parameterizing $[SL(2, \mathbb{R})/SO(2)]^3$, and the Killing spinors $\epsilon_A(r)$ are concerned, the ansätze we adopt are the following:

$$ds^2 = e^{2U(r)} dt^2 - e^{-2U(r)} dx^2$$

$$z^i \equiv z^i(r)$$

$$\epsilon_A(r) = e^{f(r)} \xi_A \quad \xi_A = \text{constant}$$

$$\gamma_0 \xi_A = \pm i \epsilon_{AB} \xi_B$$ (3.8)

Substituting the above ansätze in eqs. (3.4), after some algebra, one obtains an equivalent system of first order differential equations on the background fields of the form:

$$\frac{dz^i}{dr} = \pm \left(\frac{e^{U(r)}}{4\pi r^2}\right) h^{ij'} \nabla_{j'} Z_4(z, \bar{z}, p, q)$$

$$\frac{dU}{dr} = \pm \left(\frac{e^{U(r)}}{r^2}\right) |Z_4(z, \bar{z}, p, q)|$$ (3.9)

where the supersymmetry central charge $Z_4$ has the following expression:

$$Z_4(z, \bar{z}, p, q) = -\frac{1}{4\pi} \int_{S^2} T^- = M_\Sigma p^\Sigma - L^A q_A$$

The vector $(L^A(z, \bar{z}), M_\Sigma(z, \bar{z}))$ is the covariantly holomorphic section on the symplectic bundle defined on the Special Kähler manifold $\mathcal{M}_{\text{STU}}$.

As already stressed, in order to find a proper solution we need also the equations of motion that must be satisfied together with the first order ones. The former can be derived from an $N = 2$ pure supergravity action coupled to 3 vector multiplets (see [19] for notation):

$$S = \int d^4x \sqrt{-g} \mathcal{L}$$

where

$$\mathcal{L} = R[g] + h_{ij'}(z, \bar{z}) \partial_\mu z^i \partial^\mu \bar{z}^{j'} + \left(\Im \mathcal{N}_{\Lambda\Sigma} F^A :F^{\Sigma}|\cdot\right) + \left(\Re \mathcal{N}_{\Lambda\Sigma} F^A :\bar{F}^{\Sigma}|\cdot\right)$$ (3.10)

It was shown in [3] that the Maxwell equations are automatically satisfied by the ansätze (3.3). What really matters then are the scalar and Einstein equations which should be fulfilled by our ansätze. Their explicit form have been computed in [5] and we report them in the appendix.

### 3.1 Gauge fixing conditions and invariants

The generating solution we are going to define is characterized by the following conditions on the quantized charges and on the scalar fields:

**On the charges:**

$$p^0 = 0 \quad , \quad \frac{p^1}{p^3} = \frac{p^2}{p^3} = \alpha \quad , \quad p^1 q_1 + p^2 q_2 + p^3 q_3 = 0 \quad \text{with} \quad \alpha \neq 0$$ (3.11)
On the fields:

\[ b_1 = \alpha b \, , \, b_2 = \frac{b}{\alpha} \, , \, b_3 \equiv b \]  \hspace{1cm} (3.12)

\[ \frac{a_1}{\alpha} + \alpha a_2 + a_3 = 0 \]  \hspace{1cm} (3.13)

The relation (3.13) on the axions derives from the reality condition on the supersymmetry central charge \( Z_4 \):

\[ \text{Im} Z_4 = 0 \]  \hspace{1cm} (3.14)

once the conditions (3.11) on the charges and the two conditions on the scalars in eqs (3.12) are taken into account.

One may check that the positions (3.11), (3.12) and (3.13) are indeed consistent with the field equations and with the system of first order equations (A.1). In particular they are fulfilled by the fixed point values of the scalar fields \((a_i^{\text{fix}}, b_i^{\text{fix}})\):

\[
\begin{align*}
a_1^{\text{fix}} &= -\frac{\alpha q_1}{p^3} ; & a_2^{\text{fix}} &= \frac{(\alpha q_1 + q_3)}{\alpha p^3} ; & a_3^{\text{fix}} &= -\frac{q_3}{p^3} \\
b_1^{\text{fix}} &= \alpha b^{\text{fix}} ; & b_2^{\text{fix}} &= \frac{b^{\text{fix}}}{\alpha} ; & b_3^{\text{fix}} &= b^{\text{fix}} \\
b_4^{\text{fix}} &= \frac{1}{4 (p^3)^4} P_4(p, q) \\
P_4(p, q) &= 4 (p^3)^3 \left( q_0 - \frac{q_1^2 \alpha^2}{p^3} - \frac{q_3}{p^3} (\alpha q_1 + q_3) \right)
\end{align*}
\]  \hspace{1cm} (3.15)

where \( P_4(p, q) \) was defined in eqs. (2.6).

A solution consistent with (3.11),(3.12) and (3.13) will be described by four independent fields, say \( a_1, a_2, b, \mathcal{U} \) and five independent parameters, say \( \alpha, p^3, q_0, q_1, q_3 \). As previously pointed out, in order for our solution to be BPS it is necessary that \( P_4(p, q) \geq 0 \) which provides an inequality condition on the five parameters \( \alpha, p^3, q_0, q_1, q_3 \).

To check that conditions (3.11),(3.12) and (3.13) actually fix the \([SO(2)]^3\) gauge, let us compute the five invariants \( |Z_k|, \text{tg(}\theta) \) in a suitable point of the moduli space at infinity \( \phi_\infty \) which will characterize the asymptotic behavior of our generating solution. For simplicity we shall choose:

\[
\phi_\infty = (a_i^\infty = 0 ; b_1^\infty = -\alpha , b_2^\infty = -1/\alpha , b_3^\infty = b^\infty = -1) \]  \hspace{1cm} (3.16)

It is useful to express the invariants in terms of the following quantities:

\[
x = \text{Re}(Z_1) = \sqrt{-\frac{b}{2} p^3 (a_1 - a_1^{\text{fix}})}
\]
\[ y = \text{Re}(Z_2) = \sqrt{-\frac{b}{2} p^3} \left( a_2 - a_2^{fix} \right) \]

\[ w = \text{Im}(Z_3) = \sqrt{-\frac{1}{2b}} \left( \sum_i q_i a_i - \sum_{P(ijk)} p^i a_j a_k + q_0 - b^2 p^3 \right) \]

\[ v = \text{Re}(Z_4) = \sqrt{-\frac{1}{8b^3}} \left( \sum_i q_i a_i - \sum_{P(ijk)} p^i a_j a_k + q_0 + 3b^2 p^3 \right) \] (3.17)

where \( P(ijk) \) denotes the three cyclic permutations of the indices \((ijk)\). It is straightforward to show that the five invariants have the following expressions in terms of the above quantities:

\[ |Z_1|^2 = \alpha^2 w^2 + x^2 \]

\[ |Z_2|^2 = \frac{w^2}{\alpha^2} + y^2 \]

\[ |Z_3|^2 = w^2 + \left( \frac{y\alpha + x}{\alpha} \right)^2 \]

\[ |Z_4|^2 = v^2 \]

\[ \tan(\theta) = \frac{w(x^2 + \alpha^2 y^2 + \alpha^2 w^2 + xy)}{\alpha xy(x + \alpha^2 y)} \] (3.18)

where we have used the property that \( \text{Re}(Z_1), \text{Re}(Z_2) \) and \( \text{Re}(Z_3) \) have to fulfill the same linear relation (3.13) as the axions \( a_i \). An important feature of the system (3.18) is to have a real solution in terms of \( x, y, w, v, \alpha \) for any 5plet of values of the invariants. Particular care, however, has to be taken when dealing with the case in which one or more of the matter central charges vanish. The corresponding solution of the system (3.18) is obtained by defining suitable limits of the quantities \( x, y, w, v, \alpha \) in which it can be shown that the generating solution is still regular. The case for which \( |Z_i| = 0, i = 1, 2, 3 \) is easily solved by setting \( x = y = w \to 0 \), and corresponds to the double–fixed solution.

Now let us compute the invariants in (3.18) on our solution, which is characterized by the above defined point \( \phi_\infty \) in the moduli space, eq. (3.16), and by the five independent parameters \( \alpha, p^3, q_0, q_1, q_3 \) left over after applying the conditions (3.11) on the 8 quantized charges. To show that the five invariants in \( \phi_\infty \) are actually independent functions of \( \alpha, p^3, q_0, q_1, q_3 \), we simply need to show that the quantities \( x, y, w, v, \alpha \) are. Specializing equations (3.17) to the point \( \phi_\infty \) in (3.16) one obtains:

\[ x = \sqrt{2} \alpha^2 q_1 \]

\[ y = -\sqrt{2} \frac{\alpha}{\alpha} (\alpha q_1 + q_3) \]

\[ w = \frac{1}{\sqrt{2}} (q_0 - p^3) \]

\[ v = w + 2\sqrt{2} p^3 \] (3.19)
the above system has clearly a solution in terms of \( \alpha, p^3, q_0, q_1, q_3 \) for any combinations of real values for \( \alpha, x, y, w, v \).

Therefore we have shown that by varying the five parameters \( \alpha, p^3, q_0, q_1, q_3 \), one may cover the whole spectrum of values for the five invariants in (3.18) (or equivalently \( I_i, I = 1, \ldots, 5 \) defined in section 2) describing all possible \( U \)-duality inequivalent solutions.

3.2 The solution

Implementing the conditions (3.11), (3.12) and (3.13), the system of first order equations (3.14) simplifies dramatically and reduces to the following form:

\[
\begin{align*}
\frac{db}{dr} &= \left( \frac{e^{U}}{r^2} \right) \frac{1}{\sqrt{-2b}} \left( F(a_i) - b^2 p^3 \right) \\
\frac{da_1}{dr} &= \left( \frac{e^{U}}{r^2} \right) \sqrt{-b} \sqrt{2} p^3 \left( a_1 - a_1^{fix} \right) \\
\frac{da_2}{dr} &= \left( \frac{e^{U}}{r^2} \right) \sqrt{-b} \sqrt{2} p^3 \left( a_2 - a_2^{fix} \right) \\
\frac{dU}{dr} &= \left( \frac{e^{U}}{r^2} \right) \sqrt{-b} \left( F(a_i) + 3b^2 p^3 \right) \\
F(a_i) &= \sum_i q_i a_i - \sum_i \sum_{P(ijk)} p^i a_j a_k + q_0
\end{align*}
\]

To solve the above equations it is useful to start defining the function \( h(r) = e^{U} \sqrt{-b} \) whose equation is easily solved:

\[
\begin{align*}
\frac{dh}{dr} &= \frac{h^2}{r^2} \sqrt{2} p^3 \\
h(r) &= H(r)^{-1} \\
H(r) &= A + \frac{k}{r} \\
k &= \sqrt{2} p^3
\end{align*}
\]

In what follows we shall put \( A = 1 \).

The equations for the independent axions \( a_1, a_2 \) may then be rewritten as follows:

\[
\frac{da_i}{dr} = \left( \frac{h(r)}{r^2} \right) \sqrt{2} p^3 \left( a_i - a_i^{fix} \right) \quad i = 1, 2
\]

The solution of the above equations is:

\[
\begin{align*}
\{ a_i(r) \} &= \frac{H_i(r)}{H(r)} \\
H_i(r) &= (a_i^{\infty} + k_i \frac{r}{r}) = \frac{k_i}{r} \quad i = 1, 2 \\
k_i &= k a_i^{fix}
\end{align*}
\]

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As far as the equation for $b$ is concerned, substituting the expressions of $a_i(r)$ in $F(a_i)$ one obtains an equation for $b(r)$ which is easily solved by introducing a new harmonic function $H_4(r)$. Once $b(r)$ is known, from the expression of $h(r)$ it is straightforward to derive the solution for $U(r)$. The result for the independent evolving fields $a_{1,2}(r), b(r), U(r)$ is summarized below in terms of the four different harmonic functions $H(r), H_{1,2}(r), H_4(r)$:

$$
\begin{align*}
  a_i(r) &= H_i(r) \quad H(r) \\
  b(r) &= -\sqrt{\frac{H(r)H_4(r)+mH_1(r)+nH_2(r)}{H(r)^2}} \\
  e^{U(r)} &= \left[H(r)^2(H(r)H_4(r)+mH_1(r)+nH_2(r))\right]^{-1/4} \\
  m &= -\frac{q_1}{\alpha p_3}, \quad n = \alpha^2 a^2_{fix} \\
  H_4(r) &= 1 + \frac{k_4}{r}, \quad k_4 = k b^2_{fix}
\end{align*}
$$

(3.24)

It can be shown that the above functional expressions for the fields $a_i(r), b(r)$ and $U(r)$ fulfill the field equations and therefore are a solution of the theory. Eq. (3.24), together with eq.s (3.11)-(3.13), represents our 5 parameter generating solution.

Let us consider now the near–horizon limit of the function $e^{U(r)}$:

$$
\lim_{r \to 0} e^{U(r)} = r \left(k^3 k_4\right)^{-1/4}
$$

$$
k^3 k_4 = P_4(p, q)
$$

(3.25)

Substituting this limit in the ansätze for the metric (3.8) we obtain the known result that the near horizon geometry of a regular BPS black hole is described by a Bertotti–Robinson metric:

$$
\begin{align*}
  ds^2 &= \frac{r^2}{M^2_{BR}} dt^2 - \frac{M^2_{BR}}{r^2} dr^2 - M^2_{BR} \left(\sin^2(\theta) d\phi^2 + d\theta^2\right) \\
  M^2_{BR} &= \sqrt{P_4(p, q)}
\end{align*}
$$

The entropy is proportional to the area of the horizon ($\text{Area}_H = 4\pi M^2_{BR}$) according to the Bekenstein–Hawking formula:

$$
S_{BH} = \frac{\text{Area}_H}{4G_N} = \alpha^2 \pi \sqrt{P_4(p, q)} = 2\pi \sqrt{(p^3 q_0 + p^1 q_1 p^2 q_3 - (p^1 q_1 + p^3 q_3)^2}
$$

(3.26)

where we have expressed $\alpha$ as the ratio $p^1/p^3$.

As a final remark, we would like to comment on the difference between the form of the generating solution in (3.24) and the one described in Statement 3.1 of [5], which was characterized by two double–fixed axions. The functional expression of the generating solution obviously depends on the gauge fixing procedure adopted and, for computational convenience, in the present work we have fixed the $[SO(2)]^3$ gauge using a prescription which is different from that suggested in [5].
4 Conclusions

In this paper we have constructed the generating solution for BPS saturated static black holes of $N = 8$ supergravity (that is either M–theory compactified on $T^7$, or equivalently type II string theory compactified on $T^6$). This solution preserves 4 supercharges and can be seen either as $1/8$ preserving in the context of $N = 8$ theory or as $1/2$ preserving in a type II Calabi–Yau compactification (or eventually $1/4$ supersymmetry preserving in heterotic on $T^6$ or type II on $K^3 \times T^2$). Actually, among compactification of string theory down to four dimensions, is the only regular one, modulo $U$-duality transformation, irrespectively of the original ten–dimensional theory. In order for it to act as a generating solution one has to embed it in the different $U$–duality groups ($E_7(7)$ for toroidal type II compactifications and $SO(6,22)$ for heterotic ones).

Our 5 parameter solution should be related via $U$–duality transformations to that found in [10]. While both these solutions carry only NS–NS charges, the proper group theoretical embedding of our $STU$ model generating solution in the $N = 8$ theory ([3]) allows one to obtain, in principle, the macroscopic description of pure R–R black holes which can be interpreted microscopically in terms of D–branes only [20]. There has been an intense study in giving the precise correspondence between macroscopic and microscopic black hole configurations in the last couple of years. This has been investigated both in the context of $N = 8$ compactifications (see for example [20]–[22]) and of $N = 2$ compactifications (see for example [23]–[27]). However, all these solutions were somewhat particular under one circumstance or another. What we mean is that a precise and general recipe to give this correspondence for any macroscopic configuration is still lacking. On the contrary, if we know how to transform the generating solution into a generic one, in particular to those whose microscopic interpretation is known, then we can derive the microscopic stringy description of any geometric macroscopic solution. And this could shed light even on some still not understood aspects of black hole physics as, for example, the very conceptual basis of the microscopic entropy counting (for recent work in this direction see for instance [29]). While most of the group theoretical machinery necessary in order to make the embedding has been already constructed in [3], it is our aim, in a forthcoming paper, to give some explicit examples of the use of $U$–duality transformations in moving from a given solution to other ones [30]. For example, one could find the macroscopic description corresponding to the five parameter R–R configuration of [12] which, to our knowledge, is the only pure R–R microscopic configuration depending on five parameters present in the literature.

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Appendix A: the full set of first and second order differential equations

Setting \( z^i = a_i + ib_i \), eqs. (3.3) can be rewritten in the form:

\[
\frac{da_1}{dr} = \pm \frac{\mu(r)}{r^2} \sqrt{\frac{b_1}{2b_2 b_3}} [ -b_1 q_1 + b_2 q_2 + b_3 q_3 + (-a_2 a_3 b_1 + a_3 a_3 b_2 + a_1 a_2 b_3 + b_1 b_2 b_3) p^0 + \nonumber \\
+ (-a_3 b_2 - a_2 b_3) p^1 + (a_3 b_1 - a_1 b_3) p^2 + (a_2 b_1 - a_1 b_2) p^3 ] \nonumber \\
\frac{db_1}{dr} = \pm \frac{\mu(r)}{r^2} \sqrt{-\frac{b_1}{2b_2 b_3}} [ a_1 q_1 + a_2 q_2 + a_3 q_3 + (a_1 a_2 a_3 + a_3 a_1 b_1 + b_2 a_1 b_3 - a_1 b_2 b_3) p^0 + \nonumber \\
+ (-a_2 a_3 + b_2 b_3) p^1 - (a_1 a_3 + b_1 b_3) p^2 - (a_1 a_2 + b_1 b_2) p^3 + q_0 ] \nonumber \\
\frac{da_2}{dr} = (1, 2, 3) \rightarrow (2, 1, 3) \\
\frac{db_2}{dr} = (1, 2, 3) \rightarrow (2, 1, 3) \\
\frac{da_3}{dr} = (1, 2, 3) \rightarrow (3, 2, 1) \\
\frac{db_3}{dr} = (1, 2, 3) \rightarrow (3, 2, 1) \\
\frac{dl}{dr} = \pm \frac{\mu(r)}{r^2} \frac{1}{2 \sqrt{2} (-b_1 b_2 b_3)^{1/2}} [ a_1 q_1 + a_2 q_2 + a_3 q_3 + (a_1 a_2 a_3 - a_3 b_1 b_2 - a_2 b_1 b_3 - a_1 b_2 b_3) p^0 + \nonumber \\
- (a_2 a_3 - b_2 b_3) p^1 - (a_1 a_3 - b_1 b_3) p^2 - (a_1 a_2 - b_1 b_2) p^3 + q_0 ] \nonumber \\
0 = b_1 q_1 + b_2 q_2 + b_3 q_3 + (a_2 a_3 b_1 + a_1 a_3 b_2 + a_1 a_2 b_3 - b_1 b_2 b_3) p^0 - (a_3 b_2 + a_2 b_3) p^1 \nonumber \\
- (a_3 b_1 + a_1 b_3) p^2 - (a_2 b_1 + a_1 b_2) p^3 \quad (A.1) \nonumber 
\]

The explicit form of the equations of motion for the most general case is:

Scalar equations :

\[
\left( a''_1 - 2 \frac{a'_1 b'_1}{b_1} + 2 \frac{a'_1}{r} \right) = \frac{-2 b_1 e^{2U}}{r^4} [ a_1 b_2 b_3 (p^0)^2 - \ell(r)_0^2 ] + b_2 (-b_3 p^0 p^1) + b_3 \ell(r)_0 \ell(r)_1 + \nonumber \\
+ b_1 (-2 a_2 a_3 p^0 \ell(r)_0 + a_3 p^2 \ell(r)_0 + a_2 p^0 \ell(r)_1 + a_3 p^0 \ell(r)_2 + \nonumber \\
- p^3 \ell(r)_2 + a_2 p^0 \ell(r)_3 - p^2 \ell(r)_3)] \nonumber \\
\left( b''_1 + 2 \frac{b'_1}{r} + \frac{(a'_1^2 - b'_1^2)}{b_1} \right) = \frac{-e^{2U}}{b_2 b_3 r^4} [ - (a_1^2 b_2^2 b_3^2 p^0^2) + b_1^2 b_2^2 b_3^2 p^0^2 + 2 a_1 b_2^2 b_3^2 p^0 p^1 + \nonumber \\
- b_2^2 b_3^2 p^1^2 + b_1^2 b_3^2 p^2 + b_1^2 b_2^2 p^2 + a_2^2 b_2^2 b_3^2 \ell(r)_0^2 + \nonumber \\
- b_1^2 b_2^2 b_3^2 \ell(r)_0^2 + a_3^2 b_2^2 b_3^2 (p^0^2 - \ell(r)_0^2) + a_2^2 b_2^2 b_3^2 \nonumber \\
( p^0^2 - \ell(r)_0^2 - 2 a_1 b_2^2 b_3^2 \ell(r)_0 \ell(r)_1 + b_2^2 b_3^2 \ell(r)_1^2 + \nonumber 

\]
where the quantity $S_{00}$ on the right hand side of the Einstein eqs. has the following form:

$$S_{00} = \frac{e^4 U}{4 b_1 b_2 b_3 r^4} (a_1 b_2 b_3^2 p^0 + b_1 b_2 b_3^2 q_1 a_2 a_3) + a_2 (q_0 + a_1 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + b_1 b_2 b_3^2 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + a_1 (q_0 + a_2 (-a_3 p^1) + q_2) + a_3 q_3)$$

The explicit expression of the $\ell_A(r)$ charges in terms of the quantized ones is:

$$\ell_A(r) = \left( \begin{array}{c} \ell_0 + a_1 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + b_1 b_2 b_3^2 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + a_1 (q_0 + a_2 (-a_3 p^1) + q_2) + a_3 q_3) \\ a_1 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + b_1 b_2 b_3^2 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + a_1 (q_0 + a_2 (-a_3 p^1) + q_2) + a_3 q_3) \\ a_0 a_1 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + b_1 b_2 b_3^2 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + a_1 (q_0 + a_2 (-a_3 p^1) + q_2) + a_3 q_3) \\ a_0 a_1 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + b_1 b_2 b_3^2 (a_2 a_3 p^0 - a_3 p^0 - a_2 p^2 + q_1) + a_1 (q_0 + a_2 (-a_3 p^1) + q_2) + a_3 q_3) \end{array} \right)$$

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