Bounded Martin’s Maximum is stronger than the Bounded Semi-proper Forcing Axiom

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Abstract

We show that if Bounded Martin’s Maximum (BMM) holds then for every $X \in V$ there is an inner model with a strong cardinal containing $X$. In particular, by [1], BMM is strictly stronger consistency-wise than the Bounded Semi-Proper Forcing Axiom (BSPFA).

1 Introduction.

Shelah has shown that the Semi-Proper Forcing Axiom (SPFA) is equivalent with Martin’s Maximum (MM). It was an open problem to decide whether the same holds true at least consistency-wise for the bounded versions of these axioms, i.e., to decide whether the Bounded Semi-Proper Forcing Axiom (BSPFA) is really or only apparently weaker than Bounded Martin’s Maximum (BMM). In this paper we shall solve this problem by showing that BMM yields the existence of inner models with strong cardinals; in fact, we shall prove:

**Theorem 1.1** Suppose that BMM holds. Then for every $X \in V$ there is an inner model with a strong cardinal containing $X$.

The key technical lemma which will give Theorem 1.1 is Lemma 2.3; this lemma is shown by designing a refined $K$-version of Jensen’s “reshaping” (the paper [2] contains such a version which is almost good enough for the present purpose).\(^a\)

By [1], BSPFA is equiconsistent with a reflecting cardinal, which lives consistency-wise between inaccessible and Mahlo cardinals. Theorem 1.1 therefore implies that BMM is consistency-wise strictly stronger than BSPFA.

Our Theorem 1.1 can also be construed as a negative result on iterating stationary preserving forcings. (Such negative results have also been proven long ago by Shelah.)

\(^a\)The author would like to thank David Asperó for a pivotal discussion about BMM.
2 The proof.

Definition 2.1 Let \( f, g \) both be functions from \( \omega_1 \) to \( \omega_1 \). We shall write \( f <^* g \) iff there is some club \( C \subset \omega_1 \) such that for all \( \nu \in C \), \( f(\nu) < g(\nu) \).

Of course, \( <^* \) is a well-founded relation on the set of all \( f: \omega_1 \to \omega_1 \). We shall prove Theorem 1.1 by showing that BMM gives an infinite \(<^*\)-descending chain of such functions unless there are inner models with strong cardinals.

In what follows, if \( X \) is a set of ordinals such that there is no inner model with a strong cardinal containing \( X \) then \( K(X) \) denotes the core model over \( X \) (i.e., with \( X \) “thrown in at the bottom”), and for ordinals \( \xi \), \( K(X)[\xi] \) denotes \( K(X) \) cut off at \( \xi \). The reader who is ignorant of the theory of \( K \) may always pretend that \( X^# \) does not exist, in which case \( K(X) = L[X] \) and \( K(X)[\xi] = L_\xi[X] \); of course, doing so only gives a proof of Theorem 1.1 where “for every \( X \in V \) there is an inner model with a strong cardinal containing \( X \)” is replaced by “for every \( X \in V \), \( X^# \) exists.”

Definition 2.2 Let \( a \subset \omega \) be such that there is no inner model with a strong cardinal containing \( a \), and assume that \( \omega_1^{K(a)} = \omega_1^V \). Suppose in fact that there are (unique) \( A \subset \omega_1 \) and \( (a_\nu: \nu < \omega_1) \) such that for all \( \nu < \omega_1 \), \( a_\nu \) is the \( K(A \cap \nu) \)-least subset of \( \omega \) which is almost disjoint from each member of \( \{a_\bar{\nu}: \bar{\nu} < \nu\} \), and \( \nu \in A \) iff \( a_\nu \cap a \) is finite.

Then we shall denote by \( f_a \) the following function: \( \text{dom}(f_a) = \omega_1 \), and for \( \nu < \omega_1 \), \( f_a(\nu) = \text{the least } \beta < \omega_1 \text{ such that } K(A \cap \nu)[|(\beta + 1)|] \models \nu \text{ is countable}. \)

In this situation, we shall say that \( f_a \) exists (or, that \( f_a \) is well-defined). If there are no \( A \), \( (a_\nu: \nu < \omega_1) \) as above then \( f_a \) does not exist.

Our key lemma is the following.

Lemma 2.3 Let \( a \subset \omega \) be such that there is no inner model with a strong cardinal containing \( a \), and assume that \( f_a \) is well-defined. There is then a stationary preserving set-generic extension of \( V \) in which there is some \( b \subset \omega \), \( a<_T b \), such that \( f_b \) is well-defined and \( f_b <^* f_a \).

Proof of Theorem 1.1 from Lemma 2.3. Suppose that BMM holds but that for some \( X \in V \), there is no inner model with a strong cardinal containing \( X \). We have shown in [2] that there is then a stationary preserving set-generic extension of \( V \) in
which there is some \( a \subset \omega \) with \( X \in H_{\omega_2} = K(a)\|\omega_2 \) (where \( \omega_2 \) denotes the \( \omega_2 \) of the extension). In this extension, thus

\[
\exists a \exists M \exists M' \ ( M \models \text{“I am the stack of } a \text{-mice projecting to } \omega,\text{”} \\
M \cap OR = \omega_1, \ M' \text{ is transitive and contains all sets} \\
\text{which are boldface definable over } M, \text{ and } M' \models f_a \text{ exists }).
\]

By BMM, the displayed statement holds in \( V \). If \( a_0, M, M' \in V \) witness this then by \( M \cap OR = \omega_1 \) and absoluteness, \( M = K(a_0)\|\omega_1 \). Moreover, \( M' \models f_a \) exists will imply that \( f_a \) really exists.

Now let \( C \) denote the cone of all reals \( b \) above \( a_0 \) in the Turing degrees for which \( f_b \) exists, i.e., \( C = \{ b \subset \omega : a_0 \leq_T b \land f_b \text{ exists } \} \). Let \( a \in C \). By Lemma 2.3, there is a stationary preserving set-generic extension of \( V \) in which there is some \( b \subset \omega \) with \( a <_T b \) and \( f_b <^* f_a \). In this extension, thus

\[
\exists b \exists M \exists M' \ ( a <_T b, \ M \models \text{“I am the stack of } b \text{-mice projecting to } \omega,\text{”} \\
M \cap OR = \omega_1, \ M' \text{ is transitive and contains all sets} \\
\text{which are boldface definable over } M, \text{ and } M' \models f_b <^* f_a).
\]

By BMM, the displayed statement holds in \( V \). If \( b, M, M' \in V \) witness this then \( M = K(b)\|\omega_1 \) and \( M' \models f_b <^* f_a \). But then \( f_b <^* f_a \) really holds true.

But this shows that \(<^* \) is not well-founded (in a strong sense: for each \( a \in C, \ <^*\{ f_b : a <_T b \land f_b <^* f_a \} \text{ is ill-founded} \). Contradiction! \( \square \) (Theorem 1.1)

Proof of Lemma 2.3. Fix \( a \subset \omega \) as in the statement of Lemma 2.3. Let us fix \( A \subset \omega_1 \), the subset of \( \omega_1 \) obtained by “decoding” \( a \). W.l.o.g., \( H_{\omega_2} = K(A)\|\omega_2 \) (cf. [2]).

Let \( P \in V \) be the set of all \((f, c)\) such that there is some \( \nu < \omega_1 \) with:

- \( f : \nu \to 2 \),
- \( c \subset \nu + 1 \) is closed,
- for all \( \bar{\nu} \leq \nu \), \( K(A \cap \bar{\nu}, f \upharpoonright \bar{\nu}) \models \bar{\nu} \) is countable,
- for all \( \bar{\nu} \in c \), \( K(A \cap \bar{\nu}, f \upharpoonright \bar{\nu})|f_a(\bar{\nu}) \models \bar{\nu} \) is countable.

If \( p = (f, c) \in P \) then we shall write \( p^f \) for \( f \) and \( p^c \) for \( c \). A condition \( q \) is stronger than \( p \) iff \( q^f \upharpoonright \text{dom}(p^f) = p^f \) and \( q^c \cap (\max(p^c) + 1) = p^c \).

3
The following is easy to verify.

**Claim 1.** (Extendability) Let \( p \in \mathbb{P} \). If \( \nu < \omega_1 \) then there is some \( q \leq p \) such that \( \text{dom}(q^\ell) \geq \nu \). Also, if \( \nu < \omega_1 \) then there is some \( q \leq p \) such that \( q^\rho \setminus \nu \neq \emptyset \).

Whereas it can be shown that \( \mathbb{P} \) is not semi-proper in general, the following does hold true.

**Claim 2.** \( \mathbb{P} \) is stationary preserving.

**Proof of the Claim.** Suppose that \( p \models \dot{C} \subset \check{\omega}_1 \) is club, and let \( S \subset \omega_1 \) be stationary. We aim to find some \( q \leq p \) with \( q \models \dot{C} \cap S \neq \emptyset \).

Let \( n_0 \in \omega \) be large enough. Let us first pick \( \pi : \dot{K}^* \rightarrow K(A) \rVert \omega_2 \) such that \( \dot{K}^* \) is countable and transitive, \( \text{crit}(\pi) \in S \), and \( \{ a, \mathbb{P}, p, \dot{C} \} \subset \text{ran}(\pi) \). Set \( \nu = \text{crit}(\pi) \), \( \mathbb{P} = \pi^{-1}(\mathbb{P}) \), and \( \dot{C} = \pi^{-1}(\dot{C}) \). Working in \( \dot{K}^* \) (a model of \( \mathbf{ZFC}^- \)), we may pick some

\[ \dot{K} \prec_{\Sigma_{n_0}} \dot{K}^* \]

such that \( \dot{K} \prec \dot{K}^* \) (i.e., the former is a strict initial segment of the latter), \( \rho_{n_0}(\dot{K}) = \nu \), and \( \{ a, \mathbb{P}, p, \dot{C} \} \subset \dot{K} \). (We may for instance let \( \dot{K} \) be the \( \Sigma_{n_0} \) hull of \( \nu \cup \{ a, \mathbb{P}, p, \dot{C} \} \) formed inside \( \dot{K}^* \).) We’ll have \( \dot{K}^* \prec K(A \cap \nu) \).

Set \( \beta = K^* \cap \text{OR} \).

**Subclaim.** \( \beta \leq f_a(\nu) \).

**Proof of the Subclaim.** Of course, \( \nu \) is uncountable in \( \dot{K}^* \), and thus \( \nu \) is uncountable in \( K(a)^{\dot{K}^*} \). But a straightforward coiteration argument yields \( K(a)^{\dot{K}^*} \prec K(a) \), i.e., \( K(a)^{\dot{K}^*} = K(a) \rVert \beta \). Therefore, \( \nu \) is uncountable in \( K(a) \rVert \beta \) and hence \( \beta \leq f_a(\nu) \). \( \square \) (Subclaim)

We shall now imitate an argument of \([3]\). Let \( (E_i : i < \nu) \in \dot{K}^* \) be an enumeration of all the sets which are club in \( \nu \) and which exist in \( \dot{K} \), and let \( E \setminus E_i \) be the diagonal intersection of \( (E_i : i < \nu) \). Notice that \( E \setminus E_i \) is bounded in \( \nu \) whenever \( i < \nu \). Let us pick an external sequence \( (\nu_n : n < \omega) \) of ordinals smaller than \( \nu \) which is cofinal in \( \nu \). Also, let \( \{ D_n : n < \omega \} \) be the set of all sets in \( \dot{K} \) which are open dense in \( \mathbb{P} \).

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\(^b\)Hint: Otherwise \( \forall a \exists b f_b <^* f_a \) would hold in the model of \([1]\) if this model is constructed by forcing over \( L \).
We now construct a sequence \((p_n; n < \omega)\) of conditions such that \(p_0 = p\), \(p_{n+1} \leq p_n\), and \(p_{n+1} \in D_n\) for \(n < \omega\). Simultaneously, we’ll construct a sequence \((\delta_n; n < \omega)\) of ordinals.

Suppose that \(p_n\) is given. Notice that, setting \(\gamma = \text{dom}(p_n^\delta)\), \(\gamma < \nu\) (as \(p_n \in \bar{K}\)). Work inside \(\bar{K}\) for a second. Using Claim 1, for all \(\delta\) with \(\gamma \leq \delta < \nu\) we may easily pick some \(p^\delta \leq p_n\) such that: \(p^\delta \in D_n\), \(\text{dom}((p^\delta)^\ell) > \text{max}(\{\nu_n, \delta\})\), and for all limit ordinals \(\lambda\) with \(\gamma \leq \lambda \leq \delta\), \((p^\delta)^\ell(\lambda) = 1\) iff \(\lambda = \delta\). There is some \(\bar{E} \in \mathcal{P}(\nu) \cap \bar{K}\) club in \(\nu\) such that for any \(\eta \in \bar{E}\), \(\delta < \eta \Rightarrow \text{dom}((p^\delta)^\ell) < \eta\).

Now working inside \(\bar{K}^*\), we may pick some \(\delta \in E\) such that \(E \setminus \bar{E} \subset \delta\). Let us set \(p_{n+1} = p^\delta\), and put \(\delta_n = \delta\). Of course, \(p_{n+1} \leq p_n\) and \(p_{n+1} \in D_n\). Moreover, \(\text{dom}((p_{n+1})^\ell) < \text{min}(E\setminus(\delta_n+1))\), so that for all limit ordinals \(\lambda \in E \cap \text{dom}((p_{n+1})^\ell)\) we have that \((p_{n+1})^\ell = 1\) iff \(\lambda = \delta_n\).

Now let us define an object \(q = (q^\ell, q^\rho)\) as follows. We set \(q^\ell = \bigcup_{n<\omega}(p_n)^\ell\) and \(q^\rho = (\bigcup_{n<\omega}(p_n)^\rho \cup \{\nu\}\big)\).

Let us verify that \(q \in \mathbb{P}\). Well, by Claim 1, \(\text{dom}(q^\ell) = \nu\) and \(q^\rho \cap \nu\) is unbounded in \(\nu\). Hence to prove that \(q \in \mathbb{P}\) boils down to having to show that

\[K(A \cap \nu, q^\ell) \models \nu \text{ is countable.}\]

However, by the construction of the \(p_n\)’s we have that

\[\{\lambda \in E \cap (\text{dom}(q^\ell) \setminus \text{dom}(p^\delta)): \lambda \text{ is a limit ordinal and } q^\ell(\lambda) = 1\} = \{\delta_n: n < \omega\},\]

which is cofinal in \(\nu\). But \(E \in \bar{K}^* = K(A \cap \nu)|\beta\), and therefore \(E \in K(A \cap \nu)|\mathcal{F}_a(\nu)\) by the above Subclaim. Therefore, \(\{\delta_n: n < \omega\} \in K(A \cap \nu)|\mathcal{F}_a(\nu)\) witnesses that \(\nu\) is countable in \(K(A \cap \nu)|\mathcal{F}_a(\nu)\), as desired.

It is now easy to see that \(q \models \bar{\nu} \in C \cap S\).

The rest is smooth. Let us confuse \(V^\mathbb{P}\) with a generic extension of \(V\). Because forcing with \(\mathbb{P}\) does not collapse \(\omega_1\), it adds a pair \(B, C\) such that \(B \subset \omega_1, C\) is a club subset of \(\omega_1\), for all \(\nu < \omega_1\),

\[K(A \cap \nu, B \cap \nu) \models \nu \text{ is countable,}\]

and for all \(\nu \in C\),

\[K(A \cap \nu, B \cap \nu)|\mathcal{F}_a(\nu) \models \nu \text{ is countable.}\]

Let us fix such a pair \((B, C)\), and let us write \(D = A \oplus B\). Let us code \(D\) down to a real in the usual way (cf. [2]). In order to do this, let us write \((a_\beta: \beta < \omega_1)\) for that sequence of subsets of \(\omega\) such that for each \(\beta < \omega_1\), \(a_\beta\) is the \(K(D \cap \beta)-\text{least}\) subset of \(\omega\) which is almost disjoint from every member of \(\{a_\beta: \tilde{\beta} < \beta\}\).
Specifically, let $\mathbb{A}$ consist of all pairs $(l(p), r(p))$, where $l(p): n \rightarrow 2$ for some $n < \omega$ and $r(p) \subseteq \omega_1$ is finite. A condition $q$ is stronger than $p$ iff $l(q)$ extends $l(p)$, $r(p)$ is a subset of $r(q)$, and for all $\beta \in r(q)$, if $\beta \in D$ then
\[ \{n \in \text{dom}(l(q)) \setminus \text{dom}(l(p)): l(q)(n) = 1\} \cap a_\beta = \emptyset. \]
The forcing $\mathbb{A}$ has the c.c.c., and forcing with $\mathbb{A}$ adds a real $b$ such that for all $\beta < \omega_1$,
\[ \beta \in D \text{ iff } b \cap a_\beta \text{ is finite.} \]
Let us now look at $f_b$. Let $C' = \{\nu \in C: K(b)||\nu \prec_{\Sigma_\omega} K(b)||\omega_2\}$. Of course, $C'$ is club in $\omega_1$. The proof of the following claim will therefore finish the proof of Theorem 1.1, as $V^{P^*_A}$ will be an extension as desired.

**Claim 3.** For all $\nu \in C'$, $f_b(\nu) < f_a(\nu)$.

**Proof** of Claim 3. By the choice of $A$, $\nu$ is uncountable in $K(A \cap \nu)||f_a(\nu)$. However, $\nu$ is countable in $K(D \cap \nu)||f_a(\nu)$. But $D$ is exactly the subset of $\omega_1$ obtained by “decoding” $b$. Therefore, we must have $f_b(\nu) < f_a(\nu)$.

\[ \square \text{ (Claim 3)} \]

\[ \square \text{ (Lemma 2.3)} \]

3 A conjecture.

We do not know how to prove the following.

**Conjecture.** If BMM holds then there is an inner model with a Woodin cardinal.

In fact, we do not even know how to get $0^\sharp$ from BMM. This is related to the problem that we do not know how to get $0^\sharp$ from the assumption that the theory of $L(\mathbb{R})$ is absolute for stationary preserving forcings (cf. [2]).

References

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