Abstract

Consider a local diffeomorphism $f$ of an ultrametric Banach space over an ultrametric field, around a hyperbolic fixed point $x$. We show that, locally, the system is topologically conjugate to the linearized system. An analogous result is obtained for local diffeomorphisms of real $p$-Banach spaces (like $\ell^p$), for $p \in [0,1]$. More generally, we obtain a local linearization if $f$ is merely a local homeomorphism which is strictly differentiable at a hyperbolic fixed point $x$. Also a new global version of the Grobman-Hartman theorem is provided. It applies to Lipschitz perturbations of hyperbolic automorphisms of Banach spaces over valued fields. The local conjugacies $H$ constructed are not only homeomorphisms, but both $H$ and $H^{-1}$ are Hölder. We also study the dependence of $H$ and $H^{-1}$ on $f$ (keeping $x$ and $f'(x)$ fixed).

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1 Introduction and statement of main results

The linearization problem for formal or analytic diffeomorphisms of a complete ultrametric field $\mathbb{K}$ (or $\mathbb{K}^n$), via formal or analytic conjugacies, has

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attracted interest in non-archimedean analysis (see [17], [22] and [32]). Since an analytic linearization is not always possible, it is natural to ask whether at least a (local) topological conjugacy from the given system to its linearized version is available. In the current article, we answer this question in the affirmative (under natural hyperbolicity hypotheses).

More generally, for some of our results we can work with a valued field \((\mathbb{K}, |.|)\) whose absolute value \(|.|\) is assumed to define a non-discrete topology on \(\mathbb{K}\) (such a field is called ultrametric if \(|x + y| \leq \max\{|x|, |y|\}\) for all \(x, y \in \mathbb{K}\)). If \(E\) is a Banach space over \(\mathbb{K}\), we shall say that an automorphism \(A: E \to E\) of topological vector spaces is \textit{hyperbolic} if there exist \(A\)-invariant vector subspaces \(E_s\) and \(E_u\) of \(E\) such that \(E = E_s \oplus E_u\), and a norm \(\|\cdot\|\) on \(E\) defining its topology, such that

\[
\|x + y\| = \max\{|x|, |y|\} \quad \text{for all} \quad x \in E_s \text{ and } y \in E_u
\]

and

\[
\|A|_{E_s}\| < 1 \quad \text{and} \quad \|A^{-1}|_{E_u}\| < 1
\]

holds for the operator norms with respect to \(\|\cdot\|\) (then call \(\|\cdot\|\) \textit{adapted to} \(A\)).

Our two main theorems are versions of the local and global Grobman-Hartman theorem for \(C^1\)-diffeomorphisms of \(\mathbb{R}^n\) (see [12], [13], [15], [16], [19], [23] and [24] for these classical results and their analogues for flows). Our presentation is particularly indebted to [24]. We first discuss global conjugacies:

**Theorem A** (Global Grobman-Hartman Theorem) \(\text{Let } E \text{ be a Banach space over a valued field } (\mathbb{K}, |.|) \text{ and } A: E \to E \text{ be an automorphism of topological vector spaces which is hyperbolic. Let } \|\cdot\|: E \to [0, \infty[ \text{ be a norm adapted to } A \text{ and } g: E \to E \text{ be a bounded Lipschitz map such that}

\[
\operatorname{Lip}(g) < \|A^{-1}\|^{-1}, \quad \|A^{-1}|_{E_u}\|(1 + \operatorname{Lip}(g)) < 1, \quad \text{and} \quad \|A|_{E_s}\| + \operatorname{Lip}(g) < 1.
\]

Then there exists a unique bounded continuous map \(v: E \to E\) such that

\[
(A + g) \circ (\text{id}_E + v) = (\text{id}_E + v) \circ A.
\]

The map \(\text{id}_E + v\) is a homeomorphism from \(E\) onto \(E\), and both \(v\) and \(w := (\text{id}_E + v)^{-1} - \text{id}_E\) are Hölder. Moreover, \(w\) is the unique bounded continuous map such that

\[
A \circ (\text{id}_E + w) = (\text{id}_E + w) \circ (A + g).
\]
If \( g(0) = 0 \), then also \( v(0) = 0 \).

A Hölder exponent \( \alpha \) for \( v \) and \( w \) can be described explicitly (Remark 4.4 (a) and (b)). See also [1] for a recent discussion of the Hölder properties of \( v \) and \( w \) in the real case (if \( g(0) = 0 \)).

To obtain a local linearization, following Hartman [16], we shall only require strict differentiability of \( f \) at the fixed point. Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be Banach spaces over a valued field \((K, |.|)\), \( U \subseteq E \) be open and \( z \in U \). We recall from Bourbaki [4]: A map \( f: U \to F \) is called strictly differentiable at \( x \) if there exists a (necessarily unique) continuous linear map \( f'(x): E \to F \) such that

\[
\frac{\|f(y) - f(z) - f'(x)(y - z)\|_F}{\|y - z\|_E} \to 0
\]

(6)

if \((y, z) \in U \times U \setminus \{(u, u): u \in U\} \) tends to \((x, x)\). If we write \( f(y) = f(x) + f'(x)(y - x) + R(y) \), then \( f \) is strictly differentiable at \( x \) with derivative \( f'(x) \) if and only if \( R \) is Lipschitz on the ball \( B^E_r(x) \) for small \( r > 0 \), and

\[
\lim_{r \to 0} \text{Lip}(R|_{B^E_r(x)}) = 0
\]

(7)

(using standard notation as in [2,3]).

If \( E = F \) and \( f \) is strictly differentiable at \( x \in U \), we call \( x \) a hyperbolic fixed point of \( f \) if \( f(x) = x \) and \( f'(x): E \to E \) is a hyperbolic automorphism.

**Theorem B** (Local Grobman-Hartman Theorem) Let \((K, |.|)\) be an ultrametric field and \( E \) be an ultrametric Banach space over \((K, |.|)\). Or let \( K = \mathbb{R}, |.| \) be an absolute value on \( \mathbb{R} \) which defines the usual topology on \( \mathbb{R} \), and \( E \) be a Banach space over \((\mathbb{R}, |.|)\). Let \( P, Q \subseteq E \) be open and \( x \in P \cap Q \). Let \( f: P \to Q \) be a homeomorphism which is strictly differentiable at \( x \), with differential \( A := f'(x) \), and for which \( x \) is a hyperbolic fixed point. Then there exists an open \( 0 \)-neighbourhood \( U \subseteq E \) and a bi-Hölder homeomorphism \( H: U \to V \) onto an open subset \( V \subseteq P \), such that \( H(0) = x \) and

\[
f(H(y)) = H(A(y)) \quad \text{for all } y \in U \cap A^{-1}(U).
\]

(8)

\(^1\)Precisely this requirement on the non-linearity is also imposed in [16].
Recall that the absolute values $|.|$ on $\mathbb{R}$ defining its usual topology are precisely the $p$-th powers of the usual absolute value $|.|_{\mathbb{R}}$, i.e., $|.| = (|.|_{\mathbb{R}})^p$, with $p \in [0, 1]$. The Banach spaces $E$ over $(\mathbb{R}, |.|_{\mathbb{R}})$ are also known as real $p$-Banach spaces in the functional-analytic literature (see [20]).

To deduce Theorem B from Theorem A, we shall cut off the nonlinearity. Since suitable cut-offs only come to mind in the real and ultrametric cases, we have to restrict attention to these situations.

We also discuss the dependence of the conjugacies $id_{E} + v$ (and $id_{E} + w$) on $f$. In the global case, we obtain Lipschitz resp. Hölder continuous dependence of $v$ (resp., $w$) as elements in the space $BC(E, E)$ of bounded continuous functions, with respect to the supremum norm (Theorem [7,6]). Similar results are obtained for the local conjugacies from Theorem B; in this case, we also obtain continuous dependence of $H$ and $H^{-1}$ when considered as elements of appropriate Hölder spaces, if $\mathbb{K}$ is locally compact and $\dim(E) < \infty$ (see Proposition [7,8]). For earlier results concerning parameter dependence in the real case, the reader is referred to [18, Theorem 26].

To put the requirement of strict differentiability into context, we recall: If $\mathbb{K} = \mathbb{R}$, equipped with its usual absolute value, then $f$ as in (5) is strictly differentiable at each $x \in U$ if and only if $f$ is continuously Fréchet differentiable ([4, 2.3.3], [6, Theorem 3.8.1]). If $(\mathbb{K}, |.|)$ is arbitrary and $f$ is $C^2$ in the sense of [3], then $f$ is strictly differentiable at each $x$ [7, Proposition 3.4]. If $(\mathbb{K}, |.|)$ is a complete ultrametric field and $E$ of finite dimension, then $f$ is strictly differentiable at each $x$ if and only if $f$ is $C^3$ in the sense of [3] (see [10, Appendix C]), hence if and only if it is $C^1$ in the usual sense of finite-dimensional non-archimedean analysis (as in [28, 29]); see [8].

In the classical real case, it is known that conjugacies cannot be chosen locally Lipschitz in general (see [2], cited from [31]). In particular, they need not be $C^1$ (although the $C^1$-property – and higher differentiability properties – can be guaranteed under suitable non-resonance conditions [30]). The investigation of the possible continuity and differentiability properties of local conjugacies (e.g., differentiability at the fixed point) remains an active area of research (see [1], [14], [25], [26], [31] for some recent work). The current article provides a foundation for a later study of such refined questions also in the non-archimedean case.

The above concept of hyperbolicity is useful also for other ends. For example,
as in the real case, a stable manifold can be constructed around each hyperbolic fixed point (if \( f \) is analytic and the adapted norm is ultrametric) [11].

## 2 Preliminaries and notation

We fix some notation and compile facts and preparatory results for later use.

### 2.1 Given a metric space \((X,d)\), \(r > 0\) and \(x \in X\), we define \(B^{X}_{r}(x) := \{ y \in X : d(x,y) < r \}\) and \(\overline{B^{X}_{r}}(x) := \{ y \in X : d(x,y) \leq r \}\). As usual, a normed space \((E, \| . \|)\) over a valued field \((K, |.|)\) is called a Banach space if it is complete. If, moreover, \((K, |.|)\) is an ultrametric field and also \(\| . \|\) satisfies the ultrametric inequality \(\| x + y \| \leq \max\{\| x \|, \| y \|\}\), then \((E, \| . \|)\) is called an ultrametric Banach space (see [27] for further information). The ultrametric inequality implies that \(\| x + y \| = \| y \|\) for all \( x, y \in E \) such that \(\| x \| < \| y \|\).

### 2.2 If \( f : X \to E \) is a bounded map to a normed space \((E, \| . \|)\) over a valued field \((K, |.|)\) and \( A : E \to F \) a continuous linear map, then its operator norm is defined as \(\| A \| := \sup\{\| Ax \|_{F} / \| x \|_{E} : 0 \neq x \in E\} \in [0, \infty[\).

### 2.3 As usual, we call a map \( f : X \to Y \) between metric spaces \((X,d_{X})\) and \((Y,d_{Y})\) (globally) Hölder of exponent \(\alpha \in ]0,\infty[\) if there exists \(L \in [0, \infty[\) such that \(d_{Y}(f(x), f(y)) \leq L d_{X}(x,y)^{\alpha}\) for all \( x, y \in X \). We let Lip\(_{\alpha}(f)\) be the minimum choice of \(L\). If \( f \) is bijective and both \( f \) and \( f^{-1} \) are Hölder (of exponent \(\alpha\)), we call \( f \) a bi-Hölder homeomorphism (of exponent \(\alpha\)). Hölder maps of exponent 1 are called Lipschitz and we abbreviate Lip\((f) := \text{Lip}_{1}(f)\). Thus Lip\((A) = \| A \|\) for continuous linear maps. We write \(\text{Lip}_{\alpha}(X,Y)\) for the set of all Hölder maps \( f : X \to Y \) of exponent \(\alpha\). If \((E, \| . \|_{E})\) is a Banach space over a valued field \((K, |.|)\), then also

\[
\text{BL}_{\alpha}(X,E) := \text{Lip}_{\alpha}(X,E) \cap \text{Ban}(X,E)
\]
is a Banach space, with respect to the norm \( \|f\|_\alpha := \max\{\|f\|_\infty, \text{Lip}_\alpha(f)\} \).

If, moreover, \( X \) is compact, then \( BL_\alpha(X, E) = L_\alpha(X, E) \) and thus \( \|\cdot\|_\alpha \) makes \( L_\alpha(X, E) \) a Banach space.

**Lemma 2.4** Let \((X, d_X), (Y, d_Y)\) and \((Z, d_Z)\) be metric spaces, \( f: X \to Y \) be Hölder of exponent \( \alpha \), and \( g: Y \to Z \) be Hölder of exponent \( \beta \). Then \( g \circ f: X \to Z \) is Hölder of exponent \( \alpha \beta \), and

\[
\text{Lip}_\alpha(g \circ f) \leq \text{Lip}_\beta(g) (\text{Lip}_\alpha(f))^{\beta}.
\]

**Proof.** For \( x, y \in X \), we have

\[
d_Z(g(f(x)), g(f(y))) \leq \text{Lip}_\beta(g) d_Y(f(x), f(y))^{\beta} \leq \text{Lip}_\beta(g) \text{Lip}_\alpha(f)^{\beta} d_X(x, y)^{\alpha \beta}.
\]

**Lemma 2.5** Let \((X, d_X), (Y, d_Y)\) be metric spaces, \( \alpha \geq \beta > 0 \) and \( f: X \to Y \) be a Hölder map of exponent \( \alpha \), which is bounded in the sense that

\[
\text{spread}(f) := \sup\{d_Y(f(x), f(y)) : x, y \in X\} < \infty.
\]

Then \( f \) is also Hölder of exponent \( \beta \), and

\[
\text{Lip}_\beta(f) \leq \max\{\text{Lip}_\alpha(f), \text{spread}(f)\}. \tag{10}
\]

**Proof.** Let \( x, y \in X \). If \( d_X(x, y) \leq 1 \), then

\[
d_Y(f(x), f(y)) \leq \text{Lip}_\alpha(f) d_X(x, y)^{\alpha} \leq \text{Lip}_\alpha(f) d_X(x, y)^{\beta}. \tag{11}
\]

If \( d_X(x, y) \geq 1 \), then

\[
d_Y(f(x), f(y)) \leq \text{spread}(f) \leq \text{spread}(f) d_X(x, y)^{\beta}. \tag{12}
\]

Now (10) follows from (11) and (12). \qed

**Lemma 2.6** Let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be normed spaces over a valued field \((K, |\cdot|)\), \( h: E \to F \) be a bounded Lipschitz map and \( v: E \to E \) be a map which is Hölder of some exponent \( \alpha \in [0, 1] \). Then also the map \( h \circ (\text{id}_E + v): E \to F \) is Hölder of exponent \( \alpha \), and

\[
\text{Lip}_\alpha(h \circ (\text{id}_E + v)) \leq \max\{\text{Lip}(h)(1 + \text{Lip}_\alpha(v)), \text{spread}(h)\}.
\]

In particular, \( \text{Lip}_\alpha(h \circ (\text{id}_E + v)) \leq \max\{\text{Lip}(h)(1 + \text{Lip}_\alpha(v)), 2\|h\|_\infty\} \).
Proof. Let \( x, y \in E \). If \( \|y - x\|_E \leq 1 \), then \( \|y - x\|_E \leq \|y - x\|_E^2 \) and hence

\[
\|h(y + v(y)) - h(x + v(x))\|_F \leq \text{Lip}(h)\|y + v(y) - x - v(x)\|_E \\
\leq \text{Lip}(h)(\|y - x\|_E + \text{Lip}_\alpha(v)\|y - x\|_E^2) \\
\leq \text{Lip}(h)(1 + \text{Lip}_\alpha(v))\|y - x\|_E^2.
\]

If \( \|y - x\|_E \geq 1 \), we have \( \|h(y + v(y)) - h(x + v(x))\|_F \leq \text{spread}(h) \leq \text{spread}(h)\|y - x\|_E^2 \). The assertion follows from the preceding estimates.

Lemma 2.7 Let \((E, \|\cdot\|)\) be a normed space over a valued field \((\mathbb{K}, |.|)\), \((X, d)\) be a metric space and \( \xi : X \to \mathbb{K} \) and \( f : X \to E \) be bounded, Lipschitz maps. Then also the pointwise product \( \xi f \) is bounded and Lipschitz, with

\[
\text{Lip}(\xi f) \leq \text{Lip}(\xi)\|f\|_\infty + \|\xi\|_\infty \text{Lip}(f).
\]

Proof. \( \|\xi(y)f(y) - \xi(x)f(x)\| \leq |\xi(y) - \xi(x)| \|f(y)\| + |\xi(x)| \|f(y) - f(x)\| \). □

Lemma 2.8 Let \((E, \|\cdot\|)\) be a Banach space over a valued field \((\mathbb{K}, |.|)\) (such that \( E \neq \{0\} \)) and \( A : E \to E \) be an automorphism of topological vector spaces. Moreover, let \( v : E \to E \) be a Lipschitz map such that \( \text{Lip}(v) < \frac{1}{\|A^{-1}\|} \). Then the map \( f := A + v : E \to E \) is a homeomorphism, and \( f^{-1} : E \to E \) is Lipschitz with

\[
\text{Lip}(f^{-1}) \leq \frac{1}{\|A^{-1}\|^{-1} - \text{Lip}(v)} \quad \text{and} \quad (13)
\]

\[
\text{Lip}(f^{-1} - A^{-1}) \leq \frac{\|A^{-1}\|}{\|A^{-1}\|^{-1} - \text{Lip}(v)} \text{Lip}(v). \quad (14)
\]

If \( v \) is bounded, then also \( w := f^{-1} - A^{-1} \) is bounded, and \( \|w\|_\infty \leq \|A^{-1}\| \|v\|_\infty \).

Proof. Set \( a := \|A^{-1}\|^{-1} - \text{Lip}(v) > 0 \). By the Lipschitz Inverse Function Theorem (see [10, Theorem 5.3]), the restriction \( f_r := f|_{B_r(0)} \) is injective for each \( r > 0 \), whence \( f \) is injective. By the same theorem, the inverse map \( (f_r)^{-1} : f(B_r^E(0)) \to E \) is Lipschitz with \( \text{Lip}(f_r^{-1}) \leq a^{-1} \). Hence also \( f^{-1} : f(E) \to E \) is Lipschitz, with \( \text{Lip}(f^{-1}) \leq a^{-1} \), and thus \([13]\) holds. In particular, \( f \) is a homeomorphism onto its image. By the cited theorem, \( f(B_r^E(0)) \supseteq B_{ar}(f(0)) \) for each \( r \). Hence \( f(E) \supseteq \bigcup_{r>0} B_{ar}(f(0)) = E \).
whence $f$ is surjective. To complete the proof, write $w := f^{-1} - A^{-1}$. Then
$$w = -A^{-1} \circ f^{-1}.$$  
Hence $\operatorname{Lip}(w) \leq \operatorname{Lip}(A^{-1}) \operatorname{Lip}(v) \operatorname{Lip}(f^{-1}) = \|A^{-1}\| \operatorname{Lip}(v) \operatorname{Lip}(f^{-1})$. If we combine this estimate with (13), we obtain (14). Finally, assuming that $v$ is bounded, (15) shows that also $w$ is bounded, with $\|w\|_{\infty} \leq \|A^{-1}\| \|v\|_{\infty}$. \qed

3 Passage from one perturbation to another

In this section, we construct conjugacies from one perturbation of a given hyperbolic automorphism to another.

Lemma 3.1 Let $E \neq \{0\}$ be a Banach space over a valued field $(\mathbb{K}, |.|)$, $A: E \to E$ be a hyperbolic automorphism, and $\|\|_s$ be an adapted norm on $E$. Let $g = (g_s, g_u), h = (h_s, h_u): E \to E = E_s \oplus E_u$ be bounded Lipschitz maps such that
$$\operatorname{Lip}(h) < \|A^{-1}\|^{-1} \quad \text{and} \quad \Lambda := \max \{ \|A_2^{-1}\||(1 + \operatorname{Lip}(g_u)), \|A_1\| + \operatorname{Lip}(g_s) \} < 1,$$
with $A_1 := A|_{E_s}: E_s \to E_s$ and $A_2 := A|_{E_u}$. Then there exists a unique bounded continuous map $v: E \to E$ such that
$$(\operatorname{id}_E + v) \circ (A + h) = (A + g) \circ (\operatorname{id}_E + v).$$
It satisfies
$$\|v\|_{\infty} \leq \frac{\max \{ \|h_s\|_{\infty} + \|g_s\|_{\infty}, \|A_2^{-1}\||(h_u + g_u) \|_{\infty} \}}{1 - \Lambda}.$$  
If $g(0) = h(0) = 0$, then also $v(0) = 0$.

Proof. As a consequence of (16), $A + h: E \to E$ is a homeomorphism, whose inverse $(A + h)^{-1}$ is Lipschitz with $\operatorname{Lip}((A + h)^{-1}) \leq (\|A^{-1}\|^{-1} - \operatorname{Lip}(h))^{-1}$ (see Lemma 2.8). For a bounded continuous function $v: E \to E$, (18) is
equivalent to $A^{-1} \circ (\text{id}_E + v) \circ (A + h) = A^{-1} \circ (A + g) \circ (\text{id}_E + v)$, which in turn is equivalent to

$$v = A^{-1} \circ h + A^{-1} \circ v \circ (A + h) - A^{-1} \circ g \circ (\text{id}_E + v).$$

(20)

Let $\pi_s : E \to E_s$ and $\pi_u : E \to E_u$ be the projections onto the stable and unstable subspace of $E$, respectively. In the following, we identify a function $k : E \to E$ with the pair $(k_s, k_u)$ of its components $k_s := \pi_s \circ k$ and $k_u := \pi_u \circ k$. Then $BC(E, E) = BC(E, E_s) \oplus BC(E, E_u)$ as a Banach space (if we take the maximum norm on the right hand side). If $v = (v_s, v_u)$, then (20) holds if and only if both (21) and (22) are satisfied:

$$v_s = A_1^{-1} \circ h_s + A_1^{-1} \circ v_s \circ (A + h) - A_1^{-1} \circ g_s \circ (\text{id}_E + v)$$

(21)

$$v_u = A_2^{-1} \circ h_u + A_2^{-1} \circ v_u \circ (A + h) - A_2^{-1} \circ g_u \circ (\text{id}_E + v) =: \theta_2(v).$$

(22)

Moreover, (21) is satisfied if and only if

$$v_s = A_1 \circ v_s \circ (A + h)^{-1} - h_s \circ (A + h)^{-1} + g_s \circ (\text{id}_E + v) \circ (A + h)^{-1} =: \theta_1(v).$$

(23)

Thus (18) holds if and only if $v \in BC(E, E)$ is a fixed point of the self-map

$$\theta := (\theta_1, \theta_2) : BC(E, E) \to BC(E, E_s) \oplus BC(E, E_u) = BC(E, E)$$

of the Banach space $BC(E, E)$. We claim that $\theta$ is a contraction, with Lip($\theta$) $\leq \Lambda$. If this is true, then $\theta$ will have a unique fixed point (the unique $v$ we seek), by Banach’s Fixed Point Theorem [21, Theorem 3.4.1]. Starting the iterative approximation of $v$ with the zero-function $v_0 := 0 : E \to E$, the standard a priori estimate (see [21, Proposition 3.4.4])

$$\|v\|_{\infty} = \|v - v_0\|_{\infty} \leq \frac{\|\theta(v_0) - v_0\|_{\infty}}{1 - \Lambda} = \frac{\|\theta(v_0)\|_{\infty}}{1 - \Lambda}$$

and applying now the triangle inequality to the individual summands in $\|\theta(v_0)\|_{\infty} = \|\theta(0)\|_{\infty} = \max\{\|\theta_1(0)\|_{\infty}, \|\theta_2(0)\|_{\infty}\}$ (as in (23) and (22)), we obtain (19). If $g(0) = h(0) = 0$, we have $\theta^n(v_0)(0) = 0$ for each $n \in \mathbb{N}_0$, by a trivial induction. Hence also $v = \lim_{n \to \infty} \theta^n(v_0)$ vanishes at 0.

To establish the claim, we need only show that both $	ext{Lip}(\theta_1), \text{Lip}(\theta_2) \leq \Lambda$, because $\text{Lip}(\theta) = \max\{\text{Lip}(\theta_1), \text{Lip}(\theta_2)\}$. Given $v, w \in BC(E, E)$, we have

$$\|\theta_2(v) - \theta_2(w)\|_{\infty} \leq \|A_2^{-1} \circ (v_u - w_u) \circ (A + h)\|_{\infty} + \|A_2^{-1} \circ g_u \circ (\text{id}_E + v) - A_2^{-1} \circ g_u \circ (\text{id}_E + w)\|_{\infty}.$$
Since $\|A_2^{-1} \circ (v_u - w_u) \circ (A + h)\|_\infty \leq \|A_2^{-1}\| \cdot \|v - w\|_\infty$ and
\[
\|A_2^{-1} \circ g_u \circ (\text{id}_E + v) - A_2^{-1} \circ g_u \circ (\text{id}_E + w)\|_\infty \leq \|A_2^{-1}\| \cdot \text{Lip}(g_u) \circ \|v - w\|_\infty,
\]
we get $\|\theta_2(v) - \theta_2(w)\|_\infty \leq \|A_2^{-1}\|(1 + \text{Lip}(g_u))\|v - w\|_\infty$ and thus
\[
\text{Lip}(\theta_2) \leq \|A_2^{-1}\|(1 + \text{Lip}(g_u)) \leq \Lambda \tag{24}
\]
(using (17)). Moreover,
\[
\|\theta_1(v) - \theta_1(w)\|_\infty \leq \|A_1 \circ (v_s - w_s)\|_\infty + \|g_s \circ (\text{id}_E + v) \circ (A + h)^{-1} - g_s \circ (\text{id}_E + w) \circ (A + h)^{-1}\|_\infty.
\]
As $\|g_s \circ (\text{id}_E + v) \circ (A + h)^{-1} - g_s \circ (\text{id}_E + w) \circ (A + h)^{-1}\|_\infty \leq \text{Lip}(g_s)\|v - w\|_\infty$ and $\|A_1 \circ (v_s - w_s)\|_\infty \leq \|A_1\| \cdot \|v - w\|_\infty$, we obtain
\[
\text{Lip}(\theta_1) \leq \|A_1\| + \text{Lip}(g_s) \leq \Lambda \tag{25}
\]
(using (17) again). Thus
\[
\text{Lip}(\theta) \leq \Lambda, \tag{26}
\]
which completes the proof. \qed

**Lemma 3.2** In the situation of Lemma 3.1, assume that also
\[
\text{Lip}(g) < \|A^{-1}\|^{-1}, \tag{27}
\]
\[
\|A_2^{-1}\|(1 + \text{Lip}(h_u)) < 1, \text{ and } \|A_1\| + \text{Lip}(h_s) < 1 \tag{28}
\]
hold. Then the map $\text{id}_E + v \colon E \to E$ is a homeomorphism. Moreover, $w := (\text{id}_E + v)^{-1} - \text{id}_E \colon E \to E$ is the unique bounded continuous map such that
\[
(\text{id}_E + w) \circ (A + g) = (A + h) \circ (\text{id}_E + w). \tag{29}
\]

**Proof.** In view of (27) and (28), we can apply Lemma 3.1 with reversed roles of $g$ and $h$, and obtain a unique bounded continuous map $w : E \to E$ such that (29) holds. Then
\[
(\text{id}_E + v) \circ (\text{id}_E + w) = \text{id}_E + f,
\]

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where \( f := w + v \circ (\text{id}_E + w) \) is continuous and bounded. Now
\[
(id_E + f) \circ (A + g) = (id_E + v) \circ (id_E + w) \circ (A + g) = (id_E + v) \circ (A + h) \circ (id_E + w) = (A + g) \circ (id_E + v) \circ (id_E + w) = (A + g) \circ (id_E + f),
\]
using (29) to obtain the second equality and (18) for the third. Since also \((id_E + 0) \circ (A + g) = (A + g) \circ (id_E + 0)\), the uniqueness property in Lemma 3.1 (applied to \( g \) and \( g \) in place of \( g \) and \( h \)) shows that \( f = 0 \) and therefore \( (id_E + v) \circ (id_E + w) = id_E \). Reversing the roles of \( g \) and \( h \), the same argument gives \( (id_E + w) \circ (id_E + v) = id_E \). Thus \( id_E + v \) is invertible with \((id_E + v)^{-1} = id_E + w\). The assertions follow. \(\square\)

4 Hölder property of the conjugacies

We now show that the mappings \( v \) constructed in Section 3 are Hölder.

Lemma 4.1 Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, \(\alpha > 0\) and \((f_j)_{j \in J}\) be a net in \(L^\alpha(X, Y)\) which converges pointwise to a function \( f : X \to Y \). If
\[
\lambda := \sup \{ \text{Lip}_\alpha(f_j) : j \in J \} < \infty,
\]
then \( f \in L^\alpha(X, Y) \) and \( \text{Lip}_\alpha(f) \leq \lambda \).

Proof. Given \( x, y \in X \), we have \( d_Y(f_j(x), f_j(y)) \leq \lambda d_X(x, y)^\alpha \) for all \( j \in J \). Passing to the limit, we obtain \( d_Y(f(x), f(y)) \leq \lambda d_X(x, y)^\alpha \). \(\square\)

Lemma 4.2 In the situation of Lemma 3.1 let \( k := (A + h)^{-1} \) and assume that
\[
\text{Lip}_\alpha(h_s \circ k) + \text{Lip}(k)^\alpha (\varepsilon \|A_1\| + \max \{ \text{Lip}(g_h)(1 + \varepsilon), \text{spread}(g_h) \}) \leq \varepsilon \quad (30)
\]
and
\[
\text{Lip}_\alpha(A_2^{-1} \circ h_u) + \varepsilon \|A_2^{-1}\| \text{Lip}(A + h)^\alpha \\
+ \|A_2^{-1}\| \max \{ \text{Lip}(g_u)(1 + \varepsilon), \text{spread}(g_u) \} \leq \varepsilon \quad (31)
\]
for a given number \(\alpha \in ]0, 1[\). Then the bounded continuous map \( v : E \to E \) determined by (18) is Hölder of exponent \(\alpha\), and
\[
\text{Lip}_\alpha(v) \leq \varepsilon.
\]
Proof. We retain the notation introduced in the proof of Lemma 3.1 in particular, we shall use the contraction \( \theta = (\theta_1, \theta_2) : BC(E, E) \rightarrow BC(E, E) \) introduced there. By Lemma 4.1 the (non-empty) set

\[ Y := \{ f \in BC(E, E) \cap L_\alpha(E, E) : \text{Lip}_\alpha(f) \leq \varepsilon \} \tag{32} \]

is closed in \( BC(E, E) \), and hence a complete metric space with the metric induced by that on \( BC(E, E) \), \( d_\infty(u, w) := \| u - w \|_\infty \). We claim that \( \theta(Y) \subseteq Y \). If this is true, then the Banach Fixed Point Theorem provides a unique fixed point \( y \in Y \) for the contraction \( \theta|_Y : Y \rightarrow Y \) of \( Y, d_\infty \). Then \( y \) has to coincide with the unique fixed point \( v \in BC(E, E) \) of \( \theta \) (the map \( v \) determined by (18)), and thus \( v = y \in Y \), whence all assertions of the lemma hold. Since \( L_\alpha(E, E) = L_\alpha(E, E_s) \oplus L_\alpha(E, E_u) \) and (33)

\[ \text{Lip}_\alpha(f) = \max\{ \text{Lip}_\alpha(f_s), \text{Lip}_\alpha(f_u) \} \tag{34} \]

for \( f = (f_s, f_u) \in L_\alpha(E, E) \), to establish the claim we need only show that both \( \theta_1(v) \) and \( \theta_2(v) \) are Hölder of exponent \( \alpha \) for each \( v \in Y \), and \( \text{Lip}_\alpha(\theta_1(v)), \text{Lip}_\alpha(\theta_2(v)) \leq \varepsilon \). In view of Lemmas 2.4, 2.5 and 2.6 all three summands in (23) are Hölder of exponent \( \alpha \). Now

\[ \text{Lip}_\alpha(\theta_1(v)) \leq \text{Lip}_\alpha(A_1 \circ v_s \circ k) + \text{Lip}_\alpha(h_s \circ k) + \text{Lip}_\alpha(g_s \circ (\text{id}_E + v) \circ k), \tag{35} \]

where \( \text{Lip}_\alpha(A_1 \circ v_s \circ k) \leq \| A_1 \| \text{Lip}_\alpha(v_s) \text{Lip}(k)^\alpha \leq \varepsilon \| A_1 \| \text{Lip}(k)^\alpha \) by Lemma 2.4 and

\[ \text{Lip}_\alpha(g_s \circ (\text{id}_E + v) \circ k) \leq \text{Lip}_\alpha(g_s \circ (\text{id}_E + v)) \text{Lip}(k)^\alpha \]

\[ \leq \max\{ \text{Lip}(g_s)(1 + \text{Lip}_\alpha(v)), \text{spread}(g_s) \} \text{Lip}(k)^\alpha \]

\[ \leq \max\{ \text{Lip}(g_s)(1 + \varepsilon), \text{spread}(g_s) \} \text{Lip}(k)^\alpha \]

by Lemmas 2.4 and 2.6. To obtain an upper bound for \( \text{Lip}_\alpha(\theta_1(v)) \), we substitute the preceding estimates into (35). The upper bound so obtained is the left hand side of (30) and hence \( \leq \varepsilon \) by hypotheses. Thus \( \text{Lip}_\alpha(\theta_1(v)) \leq \varepsilon \). Similarly, Lemmas 2.4, 2.5 and 2.6 show that all three summands in (22) are Hölder of exponent \( \alpha \). Now

\[ \text{Lip}_\alpha(\theta_2(v)) \leq \text{Lip}_\alpha(A_2^{-1} \circ h_u) + \text{Lip}_\alpha(A_2^{-1} \circ v_u \circ (A + h)) + \text{Lip}_\alpha(A_2^{-1} \circ g_u \circ (\text{id}_E + v)); \tag{36} \]
here Lip\(_\alpha\)(A\(^{-1}\)ov\(_u\)o(A+h)) ≤ ∥A\(^{-1}\)∥Lip\(_\alpha\)(v\(_u\)) Lip(A+h)\(^\alpha\) ≤ ε∥A\(^{-1}\)∥Lip(A+h)\(^\alpha\) by Lemma 2.4 and

\[
\text{Lip}_\alpha(A_2^{-1} o g_u o (id_E + v)) \leq ∥A_2^{-1}∥ \max\{\text{Lip}(g_u)(1 + \text{Lip}_\alpha(v)), \text{spread}(g_u)\}
\]

by Lemmas 2.4 and 2.6. Combining (36) with the preceding estimates, we get the left hand side of (31) as an upper bound for Lip\(_\alpha\)(θ\(_2\)(v)). Hence also Lip\(_\alpha\)(θ\(_2\)(v)) ≤ ε and thus θ(v) ∈ Y, which completes the proof.

The conditions (30) and (31) describe exactly what we need in the proof, but they are somewhat elusive. They can be replaced by stronger (but more tangible) hypotheses, which we now state.

**Lemma 4.3** If g and h are as in Lemma 3.1 and

\[
\frac{\varepsilon∥A_1∥}{∥A^{-1}∥^{-1} - \text{Lip}(h)} + \max\left\{\frac{\text{Lip}(h_u)}{∥A^{-1}∥^{-1} - \text{Lip}(h)}, \text{spread}(h_u)\right\} + \max\{\text{Lip}(g_u)(1 + \varepsilon), \text{spread}(g_u)\} ≤ \varepsilon \tag{37}
\]

as well as

\[
∥A_2^{-1}∥ \max\{\text{Lip}(h_u), \text{spread}(h_u)\} + \varepsilon∥A_2^{-1}∥(∥A∥ + \text{Lip}(h))\(^\alpha\)
\]

\[
+ ∥A_2^{-1}∥ \max\{\text{Lip}(g_u)(1 + \varepsilon), \text{spread}(g_u)\} ≤ \varepsilon \tag{38}
\]

then the conditions (30) and (31) from Lemma 4.2 are satisfied. In particular, if α ∈ ]0, 1[ and ε > 0 are given and we choose δ > 0 so small that

\[
δ < ∥A^{-1}∥^{-1} \quad ∥A_2^{-1}∥(1 + δ) < 1, \quad ∥A_1∥ + δ < 1 \tag{39}
\]

\[
2∥A_2^{-1}∥δ + ε∥A_2^{-1}∥(∥A∥ + δ)\(^\alpha\) + ∥A_2^{-1}∥ \max\{δ(1 + ε), 2δ\} ≤ ε, \quad \text{and} \tag{40}
\]

\[
\frac{\varepsilon∥A_1∥}{∥A^{-1}∥^{-1} - δ} + \max\left\{\frac{δ}{∥A^{-1}∥^{-1} - δ}, 2δ\right\} + \max\{δ(1 + ε), 2δ\} ≤ \varepsilon \tag{41}
\]

then conditions (16), (17), (30) and (31) are satisfied for all bounded, Lipschitz maps g, h: E → E with

\[
\max\{∥g∥_\infty, \text{Lip}(g)\} ≤ δ \quad \text{and} \quad \max\{∥h∥_\infty, \text{Lip}(h)\} ≤ δ \tag{42}
\]
Proof. Let \( k := (A + h)^{-1} \), as in Lemma 4.2. Then
\[
\text{Lip}(k) \leq \frac{1}{\|A^{-1}\|^{-1} - \text{Lip}(h)},
\]
by (13). Next,
\[
\text{Lip}_\alpha(h \circ k) \leq \max\{\text{Lip}(h \circ k), \text{spread}(h \circ k)\}
\leq \max\left\{ \frac{\text{Lip}(h)}{\|A^{-1}\|^{-1} - \text{Lip}(h)}, \text{spread}(h) \right\},
\]
using Lemma 2.5, Lemma 2.4, and the estimate (43). We also have
\[
\text{Lip}_\alpha(A^{-1}_2 \circ h_u) \leq \|A^{-1}_2\| \text{Lip}_\alpha(h_u) \leq \|A^{-1}_2\| \max\{\text{Lip}(h_u), \text{spread}(h_u)\},
\]
using Lemmas 2.4 and 2.5. Finally, we have
\[
\varepsilon \|A^{-1}_2\| \text{Lip}(A + h) \geq \varepsilon \|A^{-1}_2\| (\|A\| + \text{Lip}(h))^\alpha.
\]
In view of (43)–(46), it is clear that (37) implies (30) and (38) implies (31). The final assertion of the lemma is now obvious, using that \( \text{spread}(f) \leq 2\|f\|_\infty \) for all bounded maps \( f \) between normed spaces. \( \square \)

Remark 4.4 (a) Note that, given \( h, g \) as in Lemma 3.1, one can always find \( \alpha \in ]0, 1[ \) and \( \varepsilon > 0 \) such that (37) and (38) (and hence also (30) and (31)) are satisfied. In fact, we have \( 1 - \|A_1\| - \text{Lip}(g_s) > 0 \) by (17) and hence also
\[
\Delta_{g,h} := 1 - \frac{\|A_1\| + \text{Lip}(g_s)}{(\|A^{-1}\|^{-1} - \text{Lip}(h))^\alpha} > 0
\]
for sufficiently small \( \alpha \in ]0, 1[ \). Instead of (37), to simplify the calculation let us impose a stronger condition by replacing the second maximum \( \max\{\text{Lip}(g_s)(1 + \varepsilon), \text{spread}(g_s)\} \) in (37) by the larger term
\[
\max\{\text{Lip}(g_s), \text{spread}(g_s)\} + \varepsilon \text{Lip}(g_s).
\]
We can then solve for \( \varepsilon \) and see that the strengthened inequality is equivalent to
\[
\varepsilon \geq \frac{\max\left\{ \frac{\text{Lip}(h_s)}{\|A^{-1}\|^{-1} - \text{Lip}(h)}, \text{spread}(h_s) \right\} + \frac{\max\{\text{Lip}(g_s), \text{spread}(g_s)\}}{(\|A^{-1}\|^{-1} - \text{Lip}(h))^\alpha}}{\Delta_{g,h}}.
\]
Also, we have $1 - \|A^{-1}\|(1 + \text{Lip}(g_u)) > 0$ by (17) and hence
\[
\delta_{g,h} := 1 - \|A^{-1}\|((\|A\| + \text{Lip}(h))^\alpha + \text{Lip}(g_u)) > 0
\] (49)
for sufficiently small $\alpha \in ]0,1[$. Likewise, replacing $\|A^{-1}\|$ times the second maximum in (38) by
\[
\|A^{-1}\| \max\{\text{Lip}(g_u), \text{spread}(g_u)\} + \varepsilon \|A^{-1}\| \text{Lip}(g_u),
\]
we obtain a stronger condition equivalent to
\[
\varepsilon \geq \frac{\|A^{-1}\| \max\{\text{Lip}(h_u), \text{spread}(h_u)\} + \max\{\text{Lip}(g_u), \text{spread}(g_u)\})}{\delta_{g,h}}. (50)
\]
Now choose $\varepsilon$ so large that both (48) and (50) hold.

(b) Given $g$ and $h$ as in Lemma 3.1 we can actually find $\alpha \in ]0,1[$ and $\varepsilon > 0$ such that (37) and (38) are satisfied simultaneously for $(g,h)$ and $(h,g)$ (i.e., with reversed roles of $h$ and $g$): Simply proceed as in (a) for both pairs, and replace the values of $\alpha$ obtained by their minimum. Then choose an $\varepsilon$ for this $\alpha$ in both cases, and replace the two values of $\varepsilon$ by their maximum.

(c) Note that we did not need to assume that $g(0) = 0$ or $h(0) = 0$ in our previous results (although, of course, this case is of primary interest).

(d) Because $\text{spread}(f) \leq 2\|f\|_\infty$, one can replace $\text{spread}(f)$ with $2\|f\|_\infty$ in (37) and (38) for $f = g_s, g_u, h_s, h_u$, and obtains simpler-looking, alternative conditions which also imply (30) and (31).

5 Proof of Theorem A

The assertions of the theorem are covered by Lemmas 3.1, 3.2 and 4.2 and Remark 4.4 (a), setting $h := 0$ there.

6 Proof of Theorem B

We give the proof in a form which can be re-used later in the study of parameter dependence. Avoiding only a trivial case, assume $E \neq \{0\}$. After
a translation, we may (and will) assume that \( x = 0 \). After shrinking \( P \), we may also assume that \( P = B_r^E(0) \) for some \( r > 0 \). Write \( f(y) = f(0) + f'(0)(y) + R(y) \); thus

\[
f(y) = A(y) + R(y) \quad \text{for all } y \in B_r^E(0),
\]

with \( A := f'(0) \). Let \( E = E_s \oplus E_u \) with respect to \( A \) and \( \|\cdot\| \) be an adapted norm on \( E \).

**6.1** If \( K \) and \( E \) are ultrametric, then also the adapted norm \( \|\cdot\| \) on \( E \) can (and will) be chosen ultrametric (see Appendix A). In this case, we define \( R_s : E \to E \) for \( s \in [0, r] \) via

\[
R_s(y) := \begin{cases} R(y) & \text{if } y \in B_s^E(0); \\ 0 & \text{else}. \end{cases}
\]  

Choose \( s \) so small that \( R|_{B_s^E(0)} \) is Lipschitz (see (1)). If \( y, z \in B_s^E(0) \), then

\[
\|R_s(z) - R_s(y)\| = \|R(z) - R(y)\| \leq \text{Lip}(R|_{B_s^E(0)})\|z - y\|.
\]

If \( y, z \in E \setminus B_s^E(0) \), then \( \|R_s(z) - R_s(y)\| = 0 \). If \( z \in B_s^E(0) \) and \( y \in E \setminus B_s^E(0) \), then \( \|z - y\| = \|y\| > \|z\| \) by \( \|\cdot\| \) and thus \( \|R_s(z) - R_s(y)\| = \|R(z)\| = \|R(z) - R(0)\| \leq \text{Lip}(R|_{B_s^E(0)})\|z - y\| \). Hence \( R_s \) is Lipschitz, with

\[
\text{Lip}(R_s) \leq \text{Lip}(R|_{B_s^E(0)})
\]  

(and in fact equality holds).

**6.2** In the real case, let \( \eta : [0, \infty[ \to [0, 1] \) be a Lipschitz function (with respect to the ordinary absolute value on \( \mathbb{R} \)) such that \( \eta|_{[0, 1]} = 1 \) and \( \eta(t) = 0 \) for \( t \geq 2 \). Then

\[
\text{Lip}(\eta) \geq 1.
\]  

For \( s \in [0, r/3] \), define

\[
\xi_s : E \to [0, 1], \quad \xi_s(y) := \eta(\|y\|/s)
\]

and

\[
R_s(y) := \begin{cases} \xi_s(y)R(y) & \text{if } y \in B_{3s}^E(0); \\ 0 & \text{else}. \end{cases}
\]  

Choose \( s \) so small that \( R|_{B_{3s}^E(0)} \) is Lipschitz. Then

\[
\text{Lip}(R_s) \leq (1 + 3\text{Lip}(\eta))\text{Lip}(R|_{B_{3s}^E(0)}),
\]

16
by the following arguments. First,

\[
\text{Lip}(R_{s|B_E^s(0)}) \leq \frac{1}{s} \text{Lip}(\eta) 3s \text{Lip}(R|_{B_E^s(0)}) + \text{Lip}(R|_{B_E^s(0)}) = (1 + 3 \text{Lip}(\eta)) \text{Lip}(R|_{B_E^s(0)})
\]

(using Lemma 2.7 for the first inequality). If \(y \in E \setminus B_E^s(0)\) and \(z \in E\), then \(\|R_s(z) - R_s(y)\| \neq 0\) implies \(z \in B_E^s(0)\). In this case, \(\|z - y\| \geq s\) and therefore \(\|R_s(z) - R_s(y)\| = \|R_s(z)\| \leq \|R(z)\| \leq \text{Lip}(R|_{B_E^s(0)})\|z\| \leq \text{Lip}(R|_{B_E^s(0)})2s \leq \text{Lip}(R|_{B_E^s(0)})2\|z - y\| \leq (1 + 3 \text{Lip}(\eta)) \text{Lip}(R|_{B_E^s(0)})\|z - y\|.

6.3 Returning to general \(\mathbb{K}\), given arbitrary \(\alpha \in ]0, 1[\) and \(\varepsilon > 0\) we choose \(\delta > 0\) so small that (39), (40) and (41) are satisfied.

6.4 In the ultrametric case, we use (7) to find \(s \in ]0, r]\) such that

\[
\text{Lip}(R|_{B_E^s(0)}) \leq \delta
\]

and \(s \leq 1\). Then \(\|R_s(y)\| \leq \text{Lip}(R_s(y))\|y\| \leq \delta s \leq \delta\) whenever \(\|R_s(y)\| \neq 0\), and hence

\[
\|R_s\|_{\infty} \leq \delta.
\]

6.5 In the real case, (7) provides \(s \in ]0, r]\) such that

\[
\text{Lip}(R|_{B_E^s(0)}) \leq \frac{\delta}{1 + 3 \text{Lip}(\eta)}
\]

and \(3s \leq 1\). Then again (57) holds.

6.6 Now set \(g := R_s\) as just selected, and \(h := 0\). Because \(\text{Lip}(g) \leq \delta\) by choice of \(s\) and \(\|g\|_{\infty} \leq \delta\) by (57), condition (42) is satisfied. Hence both \((g, h)\) and \((h, g)\) satisfy the conditions (16), (17), (30) and (31), by Lemma 4.3. Hence there are unique \(v, w \in \text{BC}(E, E)\) to which all conclusions of Lemmas 3.1, 3.2 and 4.2 apply. In particular, \(v, w \in BL_\alpha(E, E)\) with

\[
\text{Lip}_\alpha(v), \text{Lip}_\alpha(w) \leq \varepsilon,
\]

and \(\text{id}_E + v\) is a homeomorphism with inverse \(\text{id}_E + w\). Since \(h(0) = 0\) and \(g(0) = R(0) = 0\), we also have \(v(0) = 0\) and \(w(0) = 0\).
6.7 If we are only interested in a single given function \( f \), we can now complete the proof by setting \( V := B^E_s(0) \), \( U := (\text{id}_E + v)^{-1}(B^s_s(0)) \) and \( H := (\text{id}_E + v)|_U : U \to V \). Since

\[
(A + g) \circ (\text{id}_E + v) = (\text{id}_E + v) \circ A
\]

and \( R|_V = g|_V \), we then have

\[
f \circ H = f|_V \circ H = (A + g)|_V \circ (\text{id}_E + v)|_U = (\text{id}_E + v) \circ A|_U,
\]

from which (8) follows. This completes the proof.

6.8 Since our previous choice of \( U \) depends on \( v \) (and hence on \( f \)), it is unsuitable for the study of parameter dependence. To enable the latter, we need to make a different (usually smaller) choice of \( U \), which we now describe. It is helpful to observe that

\[
\omega : [0, \infty] \to [0, \infty], \quad \omega(a) := a + \varepsilon a^\alpha
\]

is a monotonically increasing bijection, such that \( \omega(a) \geq a \) (and hence \( \omega^{-1}(a) \leq a \)) for all \( a \geq 0 \). Now

\[
(id_E + v)^{-1}(B^E_t(0)) \supseteq B^E_{\omega^{-1}(t)}(0) \quad \text{for all } t > 0.
\]

In fact, given \( a > 0 \), we have \( \|y + v(y)\| \leq \|v\| + \text{Lip}_\alpha(v)\|y\|^{\alpha} \leq a + \varepsilon a^\alpha = \omega(a) \) for each \( y \in B^E_a(0) \), and thus

\[
(id_E + v)(B^E_a(0)) \subseteq B^E_{\omega(a)}(0).
\]

Hence \( B^E_a(0) \subseteq (id_E + v)^{-1}(B^E_{\omega(a)}(0)) \), entailing (62) (with \( a := \omega^{-1}(t) \)).

We now set \( U := B^E_{\omega^{-1}(s)}(0) \subseteq B^E_s(0) \). Since \( V := (id_E + v)(U) \subseteq B^E_s(0) \) by the preceding discussion, we can set \( H := (id_E + v)|_U : U \to V \) and complete the discussion as in 6.7.

7 Parameter dependence of the conjugacy

Before we can study parameter dependence of the conjugacies constructed earlier, we compile various auxiliary results. The first lemma is probably part of the folklore. See [18, Theorem 21] for the Lipschitz case; for completeness, the general proof is given in Appendix B.
Lemma 7.1 (Hölder dependence of fixed points on parameters) Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces, \(\alpha > 0\) and \(f: X \times Y \to Y\) be a mapping with the following three properties:

(a) The family \((f^y)_{y \in Y}\) of the maps \(f^y: X \to Y, f^y(x) := f(x,y)\) is uniformly Hölder of exponent \(\alpha\), in the sense that each \(f^y\) is Hölder of exponent \(\alpha\) and

\[
\mu := \sup \{ \text{Lip}_\alpha(f^y) : y \in Y \} < \infty.
\]

(b) The maps \(f_x: Y \to Y, y \mapsto f(x,y)\), with \(x \in X\), form a uniform family \((f_x)_{x \in X}\) of contractions, in the sense that each \(f_x\) is a contraction and

\[
\lambda := \sup \{ \text{Lip}(f_x) : x \in X \} < 1.
\]

(c) For each \(x \in X\), there exists a fixed point \(y_x \in Y\) for \(f_x\).

Then \(y_x\) is uniquely determined and the map \(\phi: X \to Y, \phi(x) := y_x\) is Hölder of exponent \(\alpha\), with

\[
\text{Lip}_\alpha(\phi) \leq \frac{\mu}{1 - \lambda}.
\]

Remark 7.2 Note that condition (a) of Lemma 7.1 is satisfied in particular if \(f\) is Hölder of exponent \(\alpha\) with respect to some metric \(d\) on \(X \times Y\) such that \(d((x_1,y),(x_2,y)) = d_X(x_1,x_2)\) for all \(x_1,x_2 \in X\) and \(y \in Y\). Condition (c) is satisfied whenever the metric space \((Y,d_Y)\) is complete (and \(Y \neq \emptyset\)), by Banach’s Fixed Point Theorem.

The dependence of \(w\) on \(v\) in the situation of Lemma 2.8 is considered next.

Lemma 7.3 Let \((E, \|\cdot\|)\) be a Banach space over a valued field \((K,|.|)\) (such that \(E \neq \{0\}\)), and \(0 < \lambda < 1\). Let \(A: E \to E\) be an automorphism of topological vector spaces, and \(\Omega\) be the set of all bounded, Lipschitz maps \(v: E \to E\) such that

\[
\text{Lip}(v)\|A^{-1}\| \leq \lambda.
\]

Equip \(\Omega\) with the metric given by \(d_\infty(v_1,v_2) := \|v_1 - v_2\|_\infty\). Given \(v \in \Omega\), let

\[
w_v := (A + v)^{-1} - A^{-1}.
\]

Then the map \(\phi: \Omega \to BC(E, E), v \mapsto w_v\) is Lipschitz, with

\[
\text{Lip}(\phi) \leq \frac{\|A^{-1}\|}{1 - \lambda}.
\]
Proof. Consider the map
\[ h : \Omega \times BC(E,E) \to BC(E,E), \quad h(v,u) := -A^{-1} \circ v \circ (A^{-1} + u). \]
We know from (15) that \( w_v \) satisfies
\[ w_v = -A^{-1} \circ v \circ (A^{-1} + w_v). \]
Thus \( w_v \) is a fixed point of \( h_v := h(v,.) \), and it only remains to verify the hypotheses of Lemma 7.1 for \( h_v \), with \( \mu \leq \| A^{-1} \| \) and the given \( \lambda \). Each \( h_v \) is Lipschitz, with \( \text{Lip}(h_v) \leq \| A^{-1} \| \text{Lip}(v) \leq \lambda \). Hence \( (h_v)_{v \in \Omega} \) is a uniform family of contractions. Fix \( u \in BC(E,E) \). Given \( v_1, v_2 \in \Omega \), we have
\[ \| h(v_2,u) - h(v_1,u) \|_\infty = \| A^{-1} \circ (v_2 - v_1) \circ (A^{-1} + u) \|_\infty \leq \| A^{-1} \| \| v_2 - v_1 \|_\infty. \]
Hence \( h(.,u) : \Omega \to BC(E,E) \) is Lipschitz with \( \text{Lip}(h(.,u)) \leq \| A^{-1} \| \), which completes the proof.
\[ \square \]

A linear map \( A : E \to F \) between Banach spaces over a locally compact, valued field is called a compact operator if \( A(B) \) is relatively compact in \( F \) for each bounded subset \( B \subseteq E \) (or equivalently, if \( A(B^E_1(0)) \subseteq F \) is relatively compact). Then \( A \) is continuous. As it is similar to the classical real case, we relegate the proof of the next result to the appendix (Appendix C).

Lemma 7.4 Let \( (K,d) \) be a compact metric space, \( (E,\| \|) \) be a normed space over a valued field \( (K,|.|) \), and \( \alpha > \beta > 0 \). Then \( L_\alpha(K,E) \subseteq L_\beta(K,E) \). Assume that, moreover, \( K \) is locally compact and \( E \) of finite dimension. If \( |.| \) is ultrametric, assume also that \( d \) is ultrametric. Then the inclusion map
\[ j_{\beta,\alpha} : L_\alpha(K,E) \to L_\beta(K,E), \quad f \mapsto f \]
is a compact operator.

Lemma 7.5 Let \( (K,d) \) be a compact metric space and \( X \subseteq K \) be a dense subset. Let \( (E,\| \|) \) be a finite-dimensional normed space over a valued field \( (K,|.|) \) that is locally compact, and \( \alpha > \beta > 0 \). If \( |.| \) is ultrametric, assume that also \( d \) is ultrametric. Let \( B \subseteq BL_\alpha(X,E) \) be bounded; thus
\[ \sup_{f \in B} \| f \|_\infty < \infty \quad \text{and} \quad \sup_{f \in B} \text{Lip}_\alpha(f) < \infty. \]
Then \( BC(X,E) \) and \( BL_\beta(X,E) \) induce the same topology on \( B \).

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Proof. Assume first that $X = K$. By Lemma 7.4, the closure $\overline{B} \subseteq L_\beta(K, E)$ is compact. Because the topology on $\overline{B}$ induced by $C(K, E)$ is Hausdorff and coarser than the previous compact topology, the two topologies coincide. The same then holds for the topologies on the smaller set $B$. In the general case, each $f \in BL_\alpha(X, E)$ extends (by uniform continuity) uniquely to a continuous function $\tilde{f} : K \to E$. Then $\text{Lip}_\alpha(f) = \text{Lip}_\alpha(\tilde{f})$ (as we can pass to limits in $(x, y)$ in the Hölder condition), and thus $BL_\alpha(X, E) \to BL_\alpha(K, E)$, $f \mapsto \tilde{f}$ is an isometric isomorphism. Likewise with $\beta$ in place of $\alpha$. The assertion hence follows from the result for maps on $K$, as just proved. □

Theorem 7.6 Let $E$ be a Banach space over a valued field $(\mathbb{K}, |.|)$ and $d_\infty : BC(E, E)^2 \to [0, \infty[, d_\infty(h_1, h_2) := \|h_1 - h_2\|_\infty$ be the supremum metric. Let $A : E \to E$ be a hyperbolic automorphism and $\alpha \in ]0, 1[$ as well as $\varepsilon, \delta > 0$ be such that (39)–(11) from Lemma 4.3 are satisfied. Let $\Omega$ be the set of all bounded, Lipschitz maps $g : E \to E$ such that $\max\{\|g\|_\infty, \text{Lip}(g)\} \leq \delta$. For $g \in \Omega$, let $v_g, w_g : E \to E$ be the bounded continuous maps determined by

$$(A + g) \circ (\text{id}_E + v_g) = (\text{id}_E + v_g) \circ A$$

and $w_g := (\text{id}_E + v_g)^{-1} - \text{id}_E$. Set $\sigma(g) := v_g$, $\tau(g) := w_g$. Then $\sigma$ is Lipschitz as map from $(\Omega, d_\infty)$ to $(BC(E, E), d_\infty)$, and $\tau : (\Omega, d_\infty) \to (BC(E, E), d_\infty)$ is Hölder of exponent $\alpha$.

Proof. Throughout the proof, we equip $BC(E, E)$ and $\Omega$ with the supremum metric $d_\infty$. Moreover, we give $\Omega \times BC(E, E)$ the metric $d$ defined via $d((g_1, v_1), (g_2, v_2)) := \max\{d_\infty(g_1, g_2), d_\infty(v_1, v_2)\}$. Given $g \in \Omega$, define $f(g, v) := \theta(v) = (\theta_1(v), \theta_2(v))$ for $v \in BC(E, E)$ as in (22) and (23) (applied with $h := 0$). We claim that

$$f : (\Omega \times BC(E, E), d) \to (BC(E, E), d_\infty)$$

satisfies the hypotheses of the Lipschitz case of Lemma 7.1. If this is true, then the map $\sigma : \Omega \to BC(E, E)$ taking $g \in \Omega$ to the fixed point $\sigma(g) := v_g$ of $f_g := f(g, \cdot) : BC(E, E) \to BC(E, E)$ is Lipschitz. To establish the claim, note first that condition (c) of Lemma 7.1 is satisfied by completeness of $BC(E, E)$ (see Remark 7.2). Condition (b) is satisfied since (26) and (17) show that

$$\text{Lip}(f_g) \leq \max\{\|A_2^{-1}\|(1 + \delta), \|A_1\| + \delta\}.$$
where the right hand side is \(<1\) and independent of \(g \in \Omega\). To see that the maps \(f^v := f(.,v) : \Omega \rightarrow BC(E,E)\), for \(v \in BC(E,E)\), are uniformly Lipschitz, note that

\[ f^v(g) - f^v(k) = (g_s - k_s) \circ (\text{id}_E + v) \circ A^{-1} + A_2^{-1} \circ (k_u - g_u) \circ (\text{id}_E + v) \]

for \(g,k \in BC(E,E)\) and thus

\[ d_\infty(f^v(g), f^v(k)) \leq \max \{ \|g_s - k_s\|_\infty, \|A_2^{-1}\| \|k_u - g_u\| \} \]

\[ \leq \max \{1, \|A_2^{-1}\| \} d_\infty(k, g). \]

Hence \(\text{Lip}(f^v) \leq \max \{1, \|A_2^{-1}\|\}\), for all \(v \in BC(E,E)\).

Now define \(Y\) as in (32). For fixed \(h \in \Omega\) and \(g := 0\), let \(\theta = (\theta_1, \theta_2)\) be as in (22) and (23), and recall from the proof of Lemma 4.2 that \(\theta\) restricts to a contraction \(f_h := \theta_1^Y\) of \(Y\). To see that \(\tau\) is Hölder, we need only show that the map \(f : \Omega \times Y \rightarrow Y, f(h, x) := f_h(x)\) satisfies the hypotheses of Lemma 7.1 (using the metric \(d_\infty\) on \(Y\) and \(d\) on the left hand side). By the proof of Lemma 4.2, \(Y\) is complete with respect to \(d_\infty\). Thus condition (c) of Lemma 7.1 is satisfied, and (b) can be shown as in the first part of this proof. To verify (a), let \(v \in Y\). For \(h, k \in \Omega\), the first and second components of \(f^v(h) - f^v(k)\) are given by

\[ A_1 \circ (v_s \circ (A + h)^{-1} - v_s \circ (A + k)^{-1}) + k_s \circ (A + k)^{-1} - h_s \circ (A + h)^{-1} \]

and

\[ A_2^{-1} \circ (h_u - k_u) + A_2^{-1} \circ (v_u \circ (A + h)^{-1} - v_u \circ (A + k)^{-1}) \]

respectively. The supremum norm of (63) is bounded by

\[ \|A_1\| \text{Lip}_\alpha(v_s)\|(A + h)^{-1} - (A + k)^{-1}\|_\infty ^\alpha + \|k_s - h_s\|_\infty \]

\[ + \text{Lip}_\alpha(h_s)\|(A + k)^{-1} - (A + h)^{-1}\|_\infty ^\alpha, \]

where \(\text{Lip}_\alpha(h_s) \leq \max\{\text{Lip}(h_s), 2\|h_s\|_\infty\}\) by Lemma 2.5. Thus \(\|k_s - h_s\|_\infty \leq \rho\|k_s - h_s\|_\infty^\alpha\) with \(\rho := \max\{1, 2\delta\}\) and

\[ \|(A + k)^{-1} - (A + h)^{-1}\|_\infty \leq \frac{\|A^{-1}\|}{1 - \delta\|A^{-1}\|} \|k - h\|_\infty \]

by Lemma 7.3. Hence the following is an upper bound for (63):

\[ (\|A_1\| + 2\delta) \left( \frac{\|A^{-1}\|}{1 - \delta\|A^{-1}\|} \right)^\alpha \|k - h\|_\infty ^\alpha + \rho\|k - h\|_\infty ^\alpha. \]
Likewise, the supremum norm of (64) is bounded by
\[ \|A_2^{-1}\|\rho\|h-k\|_\infty^\alpha + \|A_2^{-1}\| \operatorname{Lip}_\alpha(v_u) \left( \frac{\|A^{-1}\|}{1 - \delta\|A^{-1}\|} \right)^\alpha \|k-h\|_\infty^\alpha. \] (67)

Taking now the maximum of the bounds provided by (66) and (67), we see that
\[ \|f^v(h) - f^v(k)\|_\infty \leq M\|h-k\|_\infty^\alpha \] for \( h,k \in \Omega \), with some constant \( M \) independent of \( v, h, k \). \( \square \)

7.7 Let \( E \) be a Banach space over \( \mathbb{R} \) (equipped with an absolute value \( |.| \) equivalent to the usual one) or an ultrametric field \( (\mathbb{K}, |.|) \). Let \( A : E \to E \) be a hyperbolic automorphism, \( \| \cdot \| \) be a norm on \( E \) adapted to \( A \) and \( \alpha \in ]0,1[ \) as well as \( \varepsilon, \delta > 0 \) be such that (39)–(41) from Lemma 4.3 are satisfied. Let \( \Omega, d, \sigma : g \mapsto v_g \) and \( \tau : g \mapsto w_g \) be as in Theorem 7.6. If \( \mathbb{K} = \mathbb{R} \), fix a function \( \eta \) as in 6.2. Let \( r > 0 \) and \( \tilde{\Omega} \) be the set of all mappings \( f : B^E_r(0) \to E \) which are strictly differentiable at 0 with \( f(0) = 0 \) and \( f'(0) = A \), and such that \( R_f := f - A \) is Lipschitz and satisfies the following condition:

(a) If \( (\mathbb{K}, |.|) \) is ultrametric, assume that \( \operatorname{Lip}(R_f) \leq \delta \).

(b) If \( \mathbb{K} = \mathbb{R} \), assume that \( \operatorname{Lip}(R_f) \leq \frac{\delta}{1+3\operatorname{Lip}(\eta)} \leq \delta \).

The symbol \( d_\infty \) will also be used for the supremum metric on \( \tilde{\Omega} \). If \( (\mathbb{K}, |.|) \) is ultrametric, let \( s := r \). If \( \mathbb{K} = \mathbb{R} \), let \( s = r/3 \). Then (56) and (58), respectively, are satisfied by \( R_f \) (in place of \( R \)), for all \( f \in \tilde{\Omega} \). Define \( g_f := (R_f)_s \) as in (51) resp. (54) (cf. also 6.6).

Define
\[ \bar{\sigma}(f) := \bar{v}_f := v^{g_f} \quad \text{and} \quad \bar{\tau}(f) := \bar{w}_f := w^{g_f}. \]

Let \( \omega \) be as in (61) and define \( O := B^E_s(0), U := B^E_{s^{-1}(s)}(0) \) and \( W := B^E_{s^{-1}(s)}(0) \).

**Proposition 7.8** In the setting of 7.7, the map \( H_f := \text{id}_E + \bar{v}_f : E \to E \) is a homeomorphism such that \( H_f^{-1} = \text{id}_E + \bar{w}_f \),
\[ W \subseteq H_f(U) \subseteq O, \quad \text{(68)} \]
\[ f \circ H_f|_U = H_f \circ A|_U \quad \text{(69)} \]
and $H_f(0) = 0$. Moreover, $\tilde{\sigma}: f \mapsto \tilde{v}_f$ is Lipschitz as map from $(\tilde{\Omega}, d_\infty)$ to $(\mathcal{B}(E, E), d_\infty)$, and $\tilde{\tau}: (\tilde{\Omega}, d_\infty) \to (\mathcal{B}(E, E), d_\infty)$, $f \mapsto \tilde{w}_f$ is Hölder of exponent $\alpha$. If $\mathbb{K}$ is locally compact and $E$ finite-dimensional, then also the maps $f \mapsto \tilde{v}_f|_{\mathcal{B}^E(0)}$ and $f \mapsto \tilde{w}_f|_{\mathcal{B}^E(0)}$ from $(\tilde{\Omega}, d_\infty)$ to $(\mathcal{B}L_\beta(B^E_1(0), E), \|\cdot\|)$ are continuous, for all $\beta < \alpha$ and $t > 0$.

**Proof.** That $V := H_f(U) \subseteq B^E_1(0) = O$ was verified in 6.8. Since $\text{Lip}_\alpha(\tilde{w}_f) \leq \varepsilon$ (cf. (59)) and $\tilde{w}_f(0) = 0$, we have $(\text{id}_E + \tilde{w}_f)(B^E_1(0)) \subseteq B^E_{t+\varepsilon t^{\alpha}}(0) = B^E_{\omega(t)}(0)$ and thus $B^E_{t+\varepsilon t^{\alpha}}(0) = B^E_{\omega(t)}(0)$. Choosing $t = \omega^{-1}(\omega^{-1}(s))$, we deduce that $W \subseteq H_f(U)$ indeed.

The map $\Gamma: (\tilde{\Omega}, d_\infty) \to (\Omega, d_\infty)$ is Lipschitz with $\text{Lip}(\Gamma) \leq 1$. In fact, if $f_1, f_2 \in \tilde{\Omega}$ and $x \in E$, then $\|((\Gamma(f_1) - \Gamma(f_2))(x))\|$ equals 0 or $\|f_1(x) - f_2(x)\| \leq \xi_s(x)\|f_1(x) - f_2(x)\|$ (with $\xi_s$ as in [6.2]), and hence is bounded by $\|f_1 - f_2\|_\infty$ in either case. Thus $\tilde{\sigma} = \sigma \circ \Gamma$ is Lipschitz and $\tilde{\tau} = \tau \circ \Gamma$ is Hölder of exponent $\alpha$, by Theorem 7.6 and Lemma 2.4.

If $\mathbb{K}$ is locally compact and $E$ is finite-dimensional, given $t < 0$ let $\mathcal{B}$ be set of all $u \in BL_\alpha(B^E_1(0), E)$ such that $\text{Lip}_\alpha(u) \leq \varepsilon$ and $\|u\|_\infty \leq \varepsilon t^\alpha$. Then $\tilde{v}_f|_{\mathcal{B}^E(0)} \in \mathcal{B}$ and $\tilde{w}_f|_{\mathcal{B}^E(0)} \in \mathcal{B}$ for all $f \in \tilde{\Omega}$, as a consequence of (59).

Pick $\beta < \alpha$. Since $\mathcal{B} \subseteq BL_\alpha(B^E_1(0), E)$ is bounded, $BC(B^E_1(0), E)$ and $BL_\beta(B^E_1(0), E)$ induce the same topology on $\mathcal{B}$, by Lemma 7.5. Since $f \mapsto \tilde{v}_f$ and $f \mapsto \tilde{w}_f$ are continuous as maps to $BC(B^E_1(0), E)$ (by the preceding) and have image in $\mathcal{B}$, we deduce that they are continuous as maps to $BL_\beta(B^E_1(0), E)$. \hfill $\square$

7.9 Let $(E, \|\cdot\|)$ be a finite-dimensional Banach space over a locally compact valued field $(\mathbb{K}, |.|)$, and $U \subseteq E$ be an open subset. Recall from [10, Lemma 3.11] that a function $f: U \to E$ is strictly differentiable at each point if and only if $f$ is $C^1$ in the sense of [3], i.e., there exists a continuous map $f^{[1]}: U^{[1]} \to E$ on the open subset $U^{[1]} := \{(x, y, t) \in U \times E \times \mathbb{K}: x + ty \in U\}$ of $U \times E \times \mathbb{K}$ such that $f^{[1]}(x, y, t) = \frac{1}{t}(f(x + ty) - f(x))$ for all $(x, y, t) \in U^{[1]}$ with $t \neq 0$. We endow the space $C^1(U, E)$ with the compact-open $C^1$-topology $\mathcal{O}_{C^1}$, i.e., the initial topology with respect to the inclusion map $C^1(U, E) \to C(U, E)_{c.a.}$ and the map $C^1(U, E) \to C(U^{[1]}, E)_{c.a.}$, $f \mapsto f^{[1]}$, where the spaces on the right-hand side are equipped with the compact-open topology (see [9] for further information).

The proof of the next lemma can be found in Appendix D.
Lemma 7.10 In 7.9, let $K \subseteq U$ be a relatively compact subset. Then $f|_K : K \to E$ is Lipschitz for each $f \in C^1(U, E)$ and $C^1(U, E) \to [0, \infty[$, $f \mapsto \text{Lip}(f|_K)$ is a continuous seminorm on $(C^1(U, E), \mathcal{O}_{C^1})$.

7.11 Assume that the valued field $(\mathbb{K}, |.|)$ is locally compact and $\mathbb{K} \not= \mathbb{C}$. Let $E$ be a finite-dimensional Banach space over $\mathbb{K}$ and $P \subseteq E$ be an open $0$-neighbourhood. We give

$$C^1_{**}(P, E) := \{g \in C^1(P, E) : g(0) = 0 \text{ and } g'(0) = 0 \}$$

the topology $\mathcal{O}_{C^1}$ induced by $C^1(P, E)$. Pick $r > 0$ such that the compact closure of $K := B_r^E(0)$ is contained in $P$. Then $d_K : C^1_{**}(P, E)^2 \to [0, \infty[$, $d_K(g, h) := \|f|_K - g|_K\|_\infty$ is a continuous pseudometric on $(C^1_{**}(P, E), \mathcal{O}_{C^1})$. Hölder and Lipschitz maps between pseudometric spaces are defined as in the case of metric spaces (recalled in 2.3). Let $A, ||.||, \alpha, \varepsilon, \delta, \tilde{\Omega}, \tilde{\sigma}$ and $\tilde{\tau}$ be as in 7.7.

Proposition 7.12 In the situation of 7.11, the set

$$\Omega := \{g \in C^1_{**}(P, E) : A + g|_{B_r^E(0)} \in \tilde{\Omega}\}$$

is a $0$-neighbourhood in $(C^1_{**}(P, E), \mathcal{O}_{C^1})$. The map $\Lambda : (\Omega, d_K) \to (\tilde{\Omega}, d_\infty)$, $g \mapsto A + g|_{B_r^E(0)}$ is Lipschitz with $\text{Lip}(\Lambda) \leq 1$. The assignment $g \mapsto \text{Lip}(A + g|_{B_r^E(0)})$ defines a Lipschitz map from $(\Omega, d_K)$ to $(BC(E, E), d_\infty)$. The assignment $g \mapsto \text{Hölder}(A + g|_{B_r^E(0)})$ is Hölder of exponent $\alpha$ as a mapping $(\Omega, d_K) \to (BC(E, E), d_\infty)$. Moreover, the maps $g \mapsto \text{Lip}(A + g|_{B_r^E(0)})|_{B_r^E(0)}$ and $g \mapsto \text{Hölder}(A + g|_{B_r^E(0)})|_{B_r^E(0)}$ are continuous from $(\Omega, d_K)$ to $(BL_\beta(B_1^E(0), E), ||.||_\beta)$, for all $t > 0$ and $\beta < \alpha$.

Proof. If $\mathbb{K}$ is ultrametric, let $\rho := \delta$. If $\mathbb{K} = \mathbb{R}$, let $\rho := \frac{\delta}{1 + 3\text{Lip}(\eta)}$ (as in 7.7). By Lemma 7.10 the map $C^1_{**}(P, E) \to [0, \infty[$, $g \mapsto \text{Lip}(g|_K)$ is a continuous seminorm. Since $\Omega = \{g \in C^1_{**}(P, E) : \text{Lip}(g|_K) \leq \rho\}$, we deduce that $\Omega$ is a $0$-neighbourhood. We have $\Lambda(\Omega) \subseteq \tilde{\Omega}$ by definition of $\Omega$, and the formula $d_\infty(\Lambda(g), \Lambda(h)) = \|g|_K - h|_K\|_\infty = d_K(g, h)$ for $g, h \in \Omega$ entails that $\text{Lip}(\Lambda) \leq 1$. In view of Lemma 2.4, the remaining assertions now follow immediately from Proposition 7.8. \qed
A Existence of ultrametric adapted norms

Lemma A.1 Let $(E, \|\cdot\|)$ be an ultrametric Banach space over a valued field $(\mathbb{K}, |\cdot|)$, and $A: E \to E$ be a hyperbolic automorphism. Then there exists an ultrametric norm $\|\cdot\|$ on $E$ adapted to $E$.

Proof. We first assume that $E = E_s$; without loss of generality $E \neq \{0\}$. Let $\|\cdot\|$' be a (not necessarily ultrametric) norm on $E$ adapted to $A$. Since the norms $\|\cdot\|$ and $\|\cdot\|$' are equivalent, there exists $C \geq 1$ such that $C^{-1}\|\cdot\| \leq \|\cdot\| \leq C\|\cdot\|$'. Let $\theta := \|A\|' < 1$ be the operator norm of $A$ with respect to $\|\cdot\|$'. Choose an integer $n \geq 2$ so large that $\sigma := C^2\theta^{n-1} < 1$ and define an ultrametric norm $\|\cdot\|$ on $E$ equivalent to $\|\cdot\|$ via

$$\|x\| := \max\{\theta^{-\frac{k}{n-1}} \|A^k x\| : k = 0, \ldots, n-1\}.$$ 

The operator norm $\|A^n\|$ of $A^n$ with respect to $\|\cdot\|$ satisfies $\|A^n\| \leq C^2\|A^n\|$' $\leq C^2(\|A\|')^n = C^2\theta^n$. To see that $\|\cdot\|$ is adapted, let $x \in E$. Then $\|Ax\|$ is the maximum of $\max\{\theta^{-\frac{k}{n-1}} \|A^k x\| : k = 1, \ldots, n-1\} \leq \theta^{-\frac{1}{n-1}} \|x\|$ and $\theta^{-\frac{n}{n-1}} \|A^n x\| \leq C^2\theta^{n-1}\|x\| \leq C^2\theta^{n-1}\|x\|$'. By the preceding, the operator norm $\|A\|$ on $E$ equivalent to $\|\cdot\|$ satisfies

$$\|A\| \leq \max\{\theta^{-\frac{1}{n-1}}, \sigma\} < 1.$$ 

Hence $\|\cdot\|$ is an adapted norm on $E = E_s$.

In a general case, $E = E_s \oplus E_u$, the preceding arguments provide ultrametric norms $\|\cdot\|$ on $E_s$ adapted to $A|_{E_s}$ and $\|\cdot\|$ on $E_u$ adapted to $A^{-1}|_{E_u}$. Then $\|x + y\| := \max\{\|x\|, \|y\|\}$ for $x \in E_s$, $y \in E_u$ defines an ultrametric norm on $E$ adapted to $A$. 

B Proof of Lemma 7.1

If also $z_x$ is a fixed point of $f_x$, then $d_Y(y_x, z_x) = d_Y(f_x(y_x), f_x(z_x)) \leq \lambda d_Y(y_x, z_x)$, whence $d_Y(y_x, z_x) = 0$ and hence $z_x = y_x$. For $v \in X$ and $y \in Y$, we have $f^n_v(y) \to y_v$ as $n \to \infty$ since $d_Y(f^n_v(y), y_v) = d_Y(f^n_v(y), f^n_v(y_v)) \leq \lambda^n d_Y(y, y_v)$. In particular, $f^n_v(y_w) \to y_v$ for each $w \in X$. We claim:

$$d_Y(f^n_v(y_w), y_w) \leq \mu d_X(v, w)^\alpha \sum_{k=0}^{n-1} \lambda^k$$

for all $n \in \mathbb{N}$.
If this is true, letting \( n \to \infty \) we deduce that

\[
d_Y(y_v, y_w) \leq \mu d_X(w, v)^\alpha \sum_{k=0}^{\infty} \lambda^k = \frac{\mu}{1 - \lambda} d_X(w, v)^\alpha,
\]

as required. If \( n = 1 \), we have

\[
d_Y(f^n_v(y_w), y_w) = d_Y(f^n_v(y_w), f^1_w(y_w)) = d_Y(f^1_v, f^1_w) \leq \mu d_X(v, w)^\alpha,
\]

verifying the claim in this case. Assuming that the claim is true for some \( n \), we obtain

\[
d_Y(f^{n+1}_v(y_w), y_w) = d_Y(f^{n+1}_v(y_w), f^1_w(y_w)) \\
\leq d_Y(f^n_v(f^1_w(y_w)), f^1_v(y_w)) + d_Y(f^n_v(y_w), f^1_w(y_w)) \\
\leq \lambda d_Y(f^n_v(y_w), y_w) + \mu d_X(v, w)^\alpha \\
\leq \lambda \mu d_X(v, w)^\alpha \sum_{k=0}^{n-1} \lambda^k + \mu d_X(v, w)^\alpha \\
= \mu d_X(v, w)^\alpha \sum_{k=0}^{n} \lambda^k,
\]

as required. This induction proves the claim.

C Proof of Lemma 7.4

The first assertion is covered by Lemma 2.5. Now assume that \( K \) is locally compact (whence \( K \) is \( \mathbb{R} \) or \( \mathbb{C} \) as a topological field in the archimedean case), and assume that \( E \) is finite-dimensional. Then \( E \cong K^n \) (equipped with product topology) for some \( n \in \mathbb{N}_0 \) as a topological vector space (see Theorem 2 in [5, Chapter I, §2, no. 3]), whence \( E \) is locally compact.

In the real or complex case, define \( a := 1 \) and \( \zeta : ]0, \infty[ \to ]0, \infty[, \zeta(t) := t \). If \( |.| \) and \( d \) are ultrametric, let \( a \in K \) with \( 0 < |a| < 1 \). Define \( \zeta : ]0, \infty[ \to K \) via

\[
\zeta(t) := a^k \quad \text{if} \quad k \in \mathbb{Z} \quad \text{and} \quad |a|^{k+1} < t \leq |a|^k.
\]

(70)

Thus, in either case,

\[
|a| \cdot |\zeta(t)| < t \leq |\zeta(t)| \quad \text{for all} \quad t > 0.
\]

(71)

Let \( D := \{(x, y) \in K \times K : x \neq y\} \) and consider the map

\[
D \to K, \quad (x, y) \mapsto \zeta(d(x, y)^3).
\]

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The continuity of this map is obvious in the real and complex cases. In the ultrametric case, continuity follows from the fact that
\[ \{ (x, y) \in K \times K : d(x, y) = t \} \]
is open in \( K \times K \) for each \( t > 0 \) (cf. (9)). We equip \( C(K, E) \) with \( \| . \|_\infty \), let \( \phi_1 : L_\beta(K, E) \to C(K, E) \) be the inclusion map, and define
\[ \phi_2 : L_\beta(K, E) \to BC(D, E) \]
via \( \phi_2(f)(x, y) := \frac{f(y) - f(x)}{\zeta(d(y, x)^\beta)} \). As a consequence of (71),
\[ \| \phi_2(f) \|_\infty \leq \text{Lip}_\beta(f) \leq |a|^{-1}\| \phi_2(f) \|_\infty \]
for each \( f \in L_\beta(K, E) \), whence
\[ \phi = (\phi_1, \phi_2) : L_\beta(K, F) \to C(K, E) \times BC(D, E) \]
is a topological embedding. Moreover, \( \phi \) has closed image. To see this, suppose that \( \phi(f_n) \to (f, g) \) as \( n \to \infty \). Then
\[ g(x, y) = \lim_{n \to \infty} \phi_2(f_n)(x, y) = \lim_{n \to \infty} \frac{f_n(y) - f_n(x)}{\zeta(d(y, x)^\beta)} = \frac{f(y) - f(x)}{\zeta(d(y, x)^\beta)}, \]
extailing that \( f \) is Hölder with \( \text{Lip}_\beta(f) \leq |a|^{-1}\| g \|_\infty \), and \( g = \phi_2(f) \). Thus \( (f, g) = \phi(f) \).

Now abbreviate \( B := \{ f \in L_\alpha(K, E) : \| f \|_\alpha < 1 \} \), and let \( \overline{B} \) be the closure of \( B \) in \( C(K, E) \). Then
\[ \text{Lip}_\beta(f) \leq \max\{ \text{Lip}_\alpha(f), 2\| f \|_\infty \} \leq 2 \quad \text{for all} \quad f \in B, \]
using (23). Given \( x \in K \) and \( \varepsilon > 0 \), let \( \delta := \varepsilon^{\frac{1}{\alpha}} \). For each \( y \in B_\delta^K(x) \) and \( f \in B \), we then have \( \| f(y) - f(x) \| \leq \text{Lip}_\alpha(f) d(x, y)^\alpha \leq \delta^\alpha = \alpha \). Thus \( B \) is equicontinuous. Since, moreover, \( \{ f(x) : x \in B \} \subseteq B_\delta^E(0) \) is relatively compact for each \( x \in K \), Ascoli’s Theorem shows that \( \overline{B} \subseteq C(K, E) \) is compact. We claim that also \( \overline{\phi_2(B)} \subseteq BC(D, E) \) is compact. If this is true, then \( C := \text{im}(\phi) \cap (\overline{B} \times \overline{\phi_2(B)}) \) is compact and hence also \( \phi^{-1}(C) \) is compact. Since \( B \subseteq \phi^{-1}(C) \), this proves the lemma.

To verify the claim, let \( \varepsilon > 0 \) be given. We can choose \( \sigma > 0 \) so small that
\[ 2\sigma^{\alpha-\beta} \leq \varepsilon. \quad (72) \]
We let \( D_\sigma \) be the set of all \((x,y) \in K \times K\) such that \( \frac{\sigma}{9} \leq d(x,y) \leq 2 \). Since \( D_\sigma \) is compact, the continuous map \( \gamma: D_\sigma \to \mathbb{K}, \ (x,y) \mapsto \frac{1}{\zeta(d(x,y)\beta)} \) is uniformly continuous. Hence, there exists \( \delta > 0 \) such that 
\[
|\gamma(x,y) - \gamma(x',y')| \leq \varepsilon/3
\]
for all \((x,y),(x',y') \in D_\sigma\) such that \( d(x,x') < \delta \) and \( d(y,y') < \delta \). After shrinking \( \delta \) if necessary, we may assume that also
\[
\delta \leq \sigma/9 \quad \text{and} \quad \frac{2\delta^\alpha}{(\sigma/3)^\beta} \leq \varepsilon/3 . \tag{73}
\]
Let \((x,y),(x',y') \in D\) with \( d(x,x) < \delta \) and \( d(y,y') < \delta \). We show that
\[
\|\phi_2(f)(x',y') - \phi_2(f)(x,y)\| \leq \varepsilon, \tag{74}
\]
for all \( f \in B \). If this is true, then the function \( \phi_2(f) \) is uniformly continuous and hence has a unique continuous extension \( \psi(f): \overline{D} \to E \) to the compact closure \( \overline{D} \subseteq K \times K \). Letting \((x,y)\) and \((x',y')\) as before pass to limits in \( \overline{D} \), we deduce from (74) that also
\[
\|\psi(f)(x',y') - \psi(f)(x,y)\| \leq \varepsilon,
\]
for all \( f \in B\), \((x,y) \in \overline{D}\) and \((x',y') \in \overline{D}\) such that \( d(x,x') < \delta \) and \( d(y,y') < \delta \). Hence \( \Omega := \{\psi(f): f \in B\} \) is an equicontinuous set of functions in \( C(\overline{D},E)\). Given \((x,y) \in D\), we have \( \|\psi(f)(x,y)\| \leq \text{Lip}_\beta(f) \leq 2 \) for each \( f \in B \) (and, by continuity, this then also holds for all \((x,y) \in \overline{D}\)). Hence \( \{\psi(f)(x,y): f \in B\} \subseteq \overline{B}_2^E(0) \) and thus the equicontinuous set \( \Omega \) is also pointwise relatively compact. Hence, by Ascoli’s Theorem, \( \Omega \) is relatively compact in \( C(\overline{D},E) \). Because the restriction map
\[
C(\overline{D},E) \to BC(D,E) \quad h \mapsto h|_D
\]
is continuous linear and takes \( \Omega \) to \( B \), we deduce that also \( B \) is relatively compact, as claimed.

It only remains to verify (74). There are two cases. If \( d(y,x) < \sigma/3 \), then \( d(y',x') \leq \sigma \) (as we assume that \( d(x',x), d(y',y) < \delta \leq \sigma/9 \)) and hence
\[
\|\phi_2(f)(x',y') - \phi_2(f)(x,y)\| \leq \|\phi_2(f)(x',y')\| + \|\phi_2(f)(x,y)\| \\
\leq \frac{\|f(y') - f(x')\|}{d(y',x')^\beta} + \frac{\|f(y) - f(x)\|}{d(y,x)^\beta} \\
\leq \text{Lip}_\alpha(f)(d(y',x')^{\alpha-\beta} + d(y,x)^{\alpha-\beta}) \\
\leq 2\sigma^{\alpha-\beta} \leq \varepsilon,
\]
29
by (72). If \( d(y, x) \geq \sigma/3 \), then
\[
d(y, x) \geq d(y', x) \geq d(y', y) - d(x', x) \geq \frac{\sigma}{9}
\]
and
\[
\|\phi_2(f)(x', y') - \phi_2(f)(x, y)\|
\leq \frac{|f(y) - f(x) - f(y') + f(x')|}{\zeta(d(y, x)^3)}
+ \frac{1}{\zeta(d(y', x')^3)} - \frac{1}{\zeta(d(y, x)^3)} \leq \epsilon/3
\]
\[
\leq \frac{1}{d(y, x)^3} \text{Lip}_a(f)(d(y, y')^\alpha + d(x, x')^\alpha) + 2\epsilon/3
\]
\[
\leq \frac{2\delta^\alpha}{(\sigma/3)^2} + 2\epsilon/3 \leq \epsilon,
\]
using (73) for the final inequality.

**D Proof of Lemma 7.10**

It suffices to show that the set \( P := \{ f \in C^1(U, E) : \text{Lip}(f|_K) \leq 1 \} \) is a 0-neighbourhood in \((C^1(U, E), O_{C^1})\). After replacing \( K \) with its closure, we may assume that \( K \) is compact. Endow \( K \) with the metric \( d(x, y) := \|x - y\| \). Since \( K \) is compact, we have \( s := \text{spread}(K) < \infty \). Choose \( a \in K \) such that \( 0 < |a| < 1 \). Then

\[
L := \{(x, z, t) \in K \times \overline{B}_{|a|}^K(0) \times \overline{B}_{|a|}^E(0) : x + tz \in K\}
\]
is a compact subset of \( U^{[1]} \) and thus

\[
Q := \{ f \in C^1(U, E) : f^{[1]}(K) \subseteq B_{|a|}^E(0) \}
\]
is a 0-neighbourhood in \((C^1(U, E), O_{C^1})\). To complete the proof, we now show that \( Q \subseteq P \). Let \( f \in Q \). If \( x, y \in U \) such that \( x \neq y \), there is a unique integer \( k \in \mathbb{Z} \) such that

\[
|a|^{k+1} < \|y - x\| \leq |a|^k.
\]

Define \( t := a^k \). Then \( |t| < \frac{|a|}{|a|} \|y - x\| \leq \frac{|a|}{|a|} \) and \( \|t^{-1}(y - x)\| = \frac{1}{|t|} \|y - x\| \leq 1 \). Since, moreover, \( x + t(t^{-1}(y - x)) = x + (y - x) = y \in K \), we see that
\[(x, t^{-1}(y - x), t) \in L \text{ and hence}\]
\[
\|f(y) - f(x)\| = |t| \|t^{-1}(f(x + t(t^{-1}(y - x)))) - f(x)\| \\
= |t| \|f^{[1]}(x, t^{-1}(y - x), t)\| \leq |t| |a| \leq \|y - x\|.
\]
Thus $\text{Lip}(f|_K) \leq 1$ indeed and thus $f \in P$, showing that $Q \subseteq P$.  \qed

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**Helge Glöckner**, Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany. E-Mail: glockner@math.upb.de