ON THE INELASTIC BOLTZMANN EQUATION FOR SOFT POTENTIALS WITH DIFFUSION

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Abstract. We are concerned with the Cauchy problem of the inelastic Boltzmann equation for soft potentials, with a Laplace term representing the random background forcing. The inelastic interaction here is characterized by the non-constant restitution coefficient. We prove that under the assumption that the initial datum has bounded mass, energy and entropy, there exists a weak solution to this equation. The smoothing effect of weak solutions is also studied. In addition, it is shown the solution is unique and stable with respect to the initial datum provided that the initial datum belongs to $L^2(\mathbb{R}^3)$.

1. Introduction and results.

1.1. The model. We consider in this paper the Cauchy theory associated to the spatially homogeneous diffusively driven inelastic Boltzmann equation for soft potentials and non-constant restitution coefficient. Before we state our main results in more details, let us introduce the model in a precise way.

The inelastic Boltzmann equation is used to model the evolution of the rapid granular flows. In absence of the energy supply, it is well known that the energy of the system decreases in time because of the inelastic collisions, and the steady state is the Delta Dirac distribution. In order to keep the system out of the “freezing” state, we need to supply energy to the system. There are several ways to supply the energy to the system. We here accept a simple model for a driving mechanism, the so-called heat bath. This model is described by the following equation:

$$\frac{\partial f}{\partial t} = Q(f,f) + \Delta_v f,$$

(1.1)

where $f(t,v)$ is the density distribution function of particles, having position $x$ and velocity $v$ at time $t$. The diffusion term $\Delta_v f$ represents the effect of the heat...
Hence the post collision velocities \( v, v_\ast \) are the pre-collisional velocities and \( u, v_\ast \) represent the velocities after collisions. The relations between them are given as follows:

\[
v = v' - \frac{1 + e}{2}(u, \omega)\omega, \quad v_\ast = v_\ast' + \frac{1 + e}{2}(u, \omega)\omega, \tag{1.2}
\]

where the variable \( u = v - v_\ast \) is used for the relative velocity between the particles and \( \omega \in S^2 \). Similarly, \( u = v - v_\ast \) denotes the relative velocity after collisions and \( \hat{u} = \frac{u}{|u|} \) is the relative velocity unit vector. Let us remark that the parameter \( e \) is called the restitution coefficient which describes the inelastic effecting in the collision between particles and \( \omega \) is the impact direction. In this work it is assumed that the restitution coefficient is only a function of the impact velocity \( e = e(|u, \omega|) \) and the assumptions on the restitution coefficient \( e \) will be given below.

In the gain operator, \( J_f := J_e(|u, \omega|) \) is the Jacobian transformation from \( v \) and \( v_\ast \) to \( v' \) and \( v'_\ast \). It is not hard to see that

\[
J_e(|u, \omega|) = e(|u, \omega|) + |u, \omega| e_z(|u, \omega|) = \theta_z(|u, \omega|),
\]

where \( \theta(z) = zr(z) \), \( e_z \) and \( \theta_z \) denote the derivative of \( e \) and \( \theta \) respectively.

If we denote \( v' \) and \( v'_\ast \) by the velocities after collisions, \( v \) and \( v_\ast \) by the velocities before collisions. The equations (1.2) can be rewritten as

\[
v' = v - \frac{1 + e}{2}(u, \omega)\omega, \quad v'_\ast = v_\ast + \frac{1 + e}{2}(u, \omega)\omega.
\]

The post collision velocities \( v' \) and \( v'_\ast \) can also be expressed in another way. For distinct velocities \( v \) and \( v_\ast \), we introduce the following change of variables

\[
\sigma = \hat{u} - 2(\hat{u} \cdot \omega)\omega.
\]

It is easily checked that

\[(u, \sigma) = |u| - 2|u|(\hat{u} \cdot \omega)^2\]

which yields

\[|u \cdot \omega| = |u||\hat{u} \cdot \omega| = |u|\sqrt{1 - \frac{1}{2} \frac{\hat{u} \cdot \sigma}{|u|}}.
\]

Hence the post collision velocities \( v' \) and \( v'_\ast \) can be rewritten as

\[
v' = \frac{1}{2}(v + v_\ast) + \frac{1 - e}{4}(v - v_\ast) + \frac{1 + e}{4} |v - v_\ast| \sigma,
\]

\[
v'_\ast = \frac{1}{2}(v + v_\ast) - \frac{1 - e}{4}(v - v_\ast) - \frac{1 + e}{4} |v - v_\ast| \sigma,
\]
where
\[ e = e \left( |u| \sqrt{\frac{1 - \hat{u} \cdot \sigma}{2}} \right). \]

Using the above expression of \( Q(f, f) \), we can easily derive the following weak form of the collision operator. Letting \( \varphi(v) \) be a regular function, after the change of variables \((v', v_\ast) \rightarrow (v, v_\ast)\), we get
\[
\langle Q(f, f), \varphi \rangle = \int \int R^3 \times R^3 f(v) f(v_\ast) \int |v - v_\ast|^\beta b(\hat{u} \cdot \sigma)(\varphi(v') - \varphi(v)) d\sigma dv dv_\ast,
\]
where the angular kernel \( b(\hat{u} \cdot \sigma) = b_0(\hat{u} \cdot \omega)|\hat{u} \cdot \omega|^{-1} \). Furthermore, by symmetry argument, one gets another weak form of the collision operator that we shall use in this paper,
\[
\langle Q(f, f), \varphi \rangle = \frac{1}{2} \int \int \int R^3 \times R^3 \times S^2 f(v) f(v_\ast) |v - v_\ast|^\beta b(\hat{u} \cdot \sigma) \times [\varphi(v') + \varphi(v_\ast') - \varphi(v) - \varphi(v_\ast)] d\sigma dv dv_\ast,
\]
where \( v', v_\ast' \) are the post-collision velocities defined as above.

As in the theory of the elastic Boltzmann equation, we usually call hard potentials when \( \beta > 0 \), Maxwellian potentials when \( \beta = 0 \) and soft potentials when \( \beta < 0 \). See [23] for the more detailed classification on the parameter \( \beta \).

In this work, we only focus on equation (1.1) for soft potentials \(-4 < \beta < 0\) and the angular kernel assumed to satisfy the condition
\[
\int_{S^2} b(\hat{u} \cdot \sigma)(1 - \hat{u} \cdot \sigma)d\sigma < +\infty.
\]

After the precise introduction of the collision operator, we shall give the assumptions on the restitution coefficient, which are widely used in the theory of inelastic Boltzmann equation and can be found in [1, 2, 3, 4, 5].

A1: The mapping \( r \in R_+ \rightarrow e(r) \in (0, 1] \) is absolutely continuous.
A2: The mapping \( r \in R_+ \rightarrow \vartheta(r) = re(r) \) is strictly increasing.
A3: There exist \( \alpha > 0 \) and \( \gamma \geq 0 \) such that \( e(r) \simeq 1 - \alpha r^\gamma \) for \( r \simeq 0 \), and
\[
\lim_{r \rightarrow \infty} e(r) = e_0,
\]
where \( 0 < e_0 < 1 \).
A3’: There exist \( \alpha > 0 \) and \( \gamma \geq 0 \) such that \( e(r) \simeq 1 - \alpha r^\gamma \) for \( r \simeq 0 \), and
\[
\lim_{r \rightarrow \infty} e(r) = 0.
\]

In addition, in order to study the entropy estimate in Section 2, we also need to give the following assumption on the restitution coefficient.
A4: There exists a constant \( C > 0 \) such that \( \vartheta_e^{-1}(x) \leq Cx^2 \) for large \( x \), where \( \vartheta_e^{-1} \) is the inverse function of \( \vartheta_e \) defined in assumption A2.

We note the above assumptions (A1), (A2), (A3) or (A3’), (A4) are satisfied for the following three examples of commonly used restitution coefficients and the proof can be found in [1, 4].

**Example 1** (Constant restitution coefficient). The most documented example in the literature is the one in which \( e(r) = e \in (0, 1] \) for any \( r > 0 \).
Example 2 (Monotone decreasing). A second example of interest is the one in which the restitution coefficient $e(r)$ is a monotone decreasing function:

$$e(r) = \frac{1}{1 + ar^\eta},$$

where $a > 0$ and $\eta > 0$ are two given constants.

Example 3 (Viscoelastic hard-spheres). This is the most physically relevant model treated in this work. In this case, the restitution coefficients satisfy the implicit representation

$$e(r) + ar^\frac{1}{5} e(r)^\frac{3}{5} = 1,$$

where $a > 0$ is a suitable positive constant depending on the material viscosity.

1.2. Existing results. As far as we know, a large number of mathematical works have been done for the inelastic Boltzmann equation. Let us review the previous main results concerning to the inelastic Boltzmann equation.

For the homogeneous case, the Cauchy problem of the inelastic Boltzmann equation has been extensively studied. At the early stage of the studies, most of the works are based on the assumption that the restitution coefficient is a constant. Maxwellian model was first introduced and studied in [8], as an approximation to the inelastic hard-sphere model. For the Maxwellian model with the heat bath, the existence of steady states was obtained in [11] by fixed point arguments. The uniqueness of steady states was proved in [9] by means of moment methods, and moreover, such a solution was proved to be radially symmetric. Later on, existence, uniqueness, regularity and convergence in $L^1$ of steady states were considered in [6] by method of the Fourier transform. The Cauchy problem of the Maxwellian model with the heat bath has also been studied in [18]. For the more realistic hard sphere model, the inelastic Boltzmann equation with or without the external force was also studied. We refer the readers to [7, 10, 16, 19, 20, 21, 24] and the references therein. Recently, Alonso and Lods [1, 2, 3, 4, 5] extended previous works from the constant restitution coefficient to the non-constant restitution coefficient and obtained some interesting results. For soft potentials, up to our knowledge, the only result is due to Ernst, Trizac and Barrat [14], who studied the behavior of steady states at large velocities.

For the inhomogeneous case, the literature on the granular flows is rather limited. The well-known theories like Diperna and Lions renormalized solutions can not be extended to the inelastic Boltzmann equation due to lack of the entropy estimate. However, the theory of mild solution has been successfully extended from the elastic Boltzmann equation to the inelastic Boltzmann equation. The first study in this direction is due to Alonso [1], who proved the existence of mild solution in the hard sphere case when the initial data near vacuum. Wei and Zhang [26] studied the existence and stability to the inelastic Boltzmann equation with the external force for potentials $\beta > -2$. We also refer [13] for the study of the classical solution to the inelastic Vlasov-Poisson-Boltzmann system for potentials $\beta > -2$ in the near vacuum regime.

1.3. Notations. In what follows, we shall use some notations. We denote $L^p_k(R^3)$ be the space of measurable functions endowed with the norm

$$\|f\|_{L^p_k} = \left(\int_{R^3} |f(v)|^p (1 + |v|^2)^{\frac{p}{2}} dv \right)^{\frac{2}{p}}.$$
and

\[ H^\infty(R^3) = \bigcap_{k \geq 0} H^k(R^3), \]

where \( H^k \) is the usual Sobolev space.

Let us give the definition of the weak solutions throughout the paper by using the weak form of \( Q(f, f) \) in Section 1.

**Definition 1.1.** Let \( 0 \leq f_0 \in L^2_2(R^3), \int_{R^3} f_0 \ln f_0 dv < \infty \). For any \( T > 0 \), a nonnegative function \( f(t, v) \) belonging to \( L^\infty([0, T], L^1_2(R^3)) \cap L^1([0, T], L^3(R^3)) \) is a weak solution of the equation (1.1) if for every \( T > 0 \),

\[
\left[ \int_{R^3} f \varphi(v, t)dv \right]_0^T - \int_0^T \int_{R^3} f(\partial_t \varphi + \Delta_v \varphi)dvdt = \int_0^T \int_{R^3} Q(f, f) \varphi dvdt \quad (1.4)
\]

holds for all \( \varphi \in C^1([0, T], C^2(R^3)) \), where the integral

\[
\int_0^T \int_{R^3} Q(f, f) \varphi dvdt \quad (1.5)
\]

is defined in (1.3).

For soft potentials \(-4 < \beta < 0\), we note that \( \int_0^T \int_{R^3} Q(f, f) \varphi dvdt \) is well defined under the assumptions on \( f \) and \( \varphi \). Indeed, it is easy to follow the arguments for the elastic Boltzmann equation in [23] to derive

\[
\int_0^T \int_{R^3} Q(f, f) \varphi dvdt \leq C \int_0^T dt \int_{R^3} f f_* |v - v_*|^{\beta+2} dv_* ,
\]

where \( C \) is a constant depending on \( e \), the second derivative of \( \varphi \) and

\[
\int_{S^2} b(\hat{u} \cdot \sigma)(1 - \hat{u} \cdot \sigma)d\sigma .
\]

If \(-2 \leq \beta < 0\), the bound \( L^\infty([0, T], L^2_2(R^3)) \) is enough to give a sense to \( \int_0^T \int_{R^3} Q(f, f) \varphi dvdt \) by using the following inequality

\[
\int_0^T dt \int_{R^3} f f_* |v - v_*|^{\beta+2} dv_* \leq C(T) \|f\|_{L^\infty([0, T], L^2_2(R^3))}^2 .
\]

If \(-4 \leq \beta < -2\), thanks to the well known Hardy-Littlewood-Sobolev inequality, the integral is also well-defined in view of the inequality

\[
\int_0^T dt \int_{R^3} f f_* |v - v_*|^{\beta+2} dv_* \leq C \int_0^T \|f\|_{L^{\frac{6}{\beta+2}}}^2 dt ,
\]

and the following interpolation inequality

\[
\|f\|_{L^{\frac{8+6\gamma}{4+2}}} \leq \|f\|_{L^1}^{\frac{8+2\gamma}{4+2}} \|f\|_{L^3}^{\frac{8+2\gamma}{4+2}} .
\]

1.4. **Main results and methods.** We are now ready to state our main results in this paper. The first main result is concerned with the existence of the weak solution for equation (1.1).

**Theorem 1.2.** Assume that the initial datum \( f_0 \) satisfies

\[
f_0 \geq 0, \quad f_0 \in L^1_2(R^3), \quad \int_{R^3} f_0 \ln f_0 dv < \infty ,
\]

and \( 0 > \beta > \max\{-2 - \gamma, -4\} \),
If the restitution coefficient $e(z)$ satisfies assumptions (A1), (A2) and (A3), then there exists a weak solution for equation (1.1), such that
\[ f \in L^\infty([0, T], L^1_2(R^3)), \quad \sup_{[0, T]} \int_{R^3} f \ln f dv < C_1, \]
where the constant $C_1$ depends on $\|f_0\|_{L^1}$ and $\int_{R^3} f_0 \ln f_0 dv$.

(2) If the restitution coefficient $e(z)$ satisfies assumptions (A1), (A2), (A3') and (A4), then there exists a weak solution for equation (1.1), such that
\[ f \in L^\infty([0, T], L^1_2(R^3)), \quad \int_{R^3} f \ln f dv < C_2(1 + t)^2, \]
where the constant $C_2$ depends on $\|f_0\|_{L^1_2}$ and $\int_{R^3} f_0 \ln f_0 dv$.

Remark 1.1. From the previous works on both Boltzmann and Landau equations, it is known that $\beta = -2$ is a critical value in the theory of either the homogeneous case or the non-homogeneous case. Indeed, some important results hold only in the case of $\beta > -2$, see [13, 15, 25, 26] for a detailed study. But our first result shows that $\beta = -2 - \gamma$ is a critical value in the non-constant restitution coefficient case, and we recall the parameter $\gamma$ describes the behavior of the restitution coefficient at the origin. This fact is due to the entropy estimate in Section 2, and lack of entropy estimate is the main reason that we can not obtain existence of the weak solutions in the case of $\beta \leq -2 - \gamma$. On the other hand, it is also remarked that $\beta = -4$ is another important value in the theory of the elastic Boltzmann equation, see [23] and subsection 1.3.

The proof of Theorem 1.2 is similar to the proof of existence of weak solutions for elastic Boltzmann equation for soft potentials and based on the weak compactness in $L^1$ space and strong compactness in some $L^p$ space. For this, we need to introduce an approximate equation to the original equation (1.1), and establish uniform estimates for the approximate solutions. After passing to the limit, we can arrive at our goal. The key point of the proof is a priori entropy estimate given in Section 2, where the Hardy- Littlewood-Sobolev inequality is used to estimate the singular integral which has the following form
\[ \int_{R^3 \times R^3} f f_s |v - v_s|^{\alpha} dv d v_s \]
with some negative $\alpha$.

We next consider the smoothing effect to the weak solutions obtained in Theorem 1.2, that is, the solutions have higher regularity than the initial datum.

As we know, for the Boltzmann equation without angular cutoff, the smoothing effect holds since the Boltzmann operator behaves like a fractional Laplacian operator which yields the coercivity estimate in this case. The theory regarding the smoothing effect for the homogeneous and non-homogeneous problem is quite satisfactory and we refer to [12] for other references. We here only emphasize that for the homogeneous case, the smoothing effect is true for soft potentials when the solutions possess infinite $L^1$ moments, which is not necessary for hard and Maxwellian potentials. For the inelastic Boltzmann equation for hard potentials with the Laplace term [16], the same conclusion also holds.
Our second result shows that the smoothing effect is also true for soft potentials when $\beta$ is not too much negative. Comparing the following theorem with the previous work on the Boltzmann equation for soft potentials [12], one finds that we here do not assume infinite $L^1$ moments on the solutions.

**Theorem 1.3.** Assume initial datum $f_0$ satisfies $f_0 \geq 0$, $f_0 \in L^2_2(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} f_0 \ln f_0 dv < \infty$ and $-2 < \beta < 0$. If the restitution coefficient $e$ satisfies assumptions (A1), (A2), (A3) or (A3') and (A4). Then after arbitrary small time, the solution $f$ becomes smooth. More precisely, for any $T > 0$ and every $0 < t_0 < T$,

$$f \in L^\infty([t_0, T], H^\infty(\mathbb{R}^3)).$$

Furthermore, if $f_0 \in L^2(\mathbb{R}^3)$, then

$$\int_0^T \|f(t, \cdot)\|_{H^1}^2 dt < \infty.$$

In order to prove Theorem 1.3, we follow the main strategy introduced in [16], which has been used successfully to show the solutions of the inelastic Boltzmann equation with hard potentials become smooth after arbitrary positive time. More precisely, we establish a priori regularity estimates for the solutions in $H^k$ space in Section 3. After an inductive argument, the bounds in any $H^k$ space can be controlled only in terms of the mass and entropy of the initial datum. By modifying the initial data, the estimates established in Section 3 also hold for the solutions with smooth initial data. Moreover, the estimates in any $H^k$ space are uniform (see Section 4). Thanks to the standard convexity argument, the cluster point of the approximated solutions also satisfies the same $H^k$ estimate. From Theorem 1.2, we know that the cluster point is a weak solution of equation (1.1) and the proof of Theorem 1.3 is concluded. It is mentioned Corollary 2 is the beginning of our inductive argument.

As a corollary of Theorem 1.3, we deduce the uniqueness and stability of the solution of equation (1.1).

**Corollary 1.** Assume $-2 < \beta < 0$, the initial data and the restitution coefficient $e$ satisfy the conditions as in Theorem 1.3. Then there is a unique weak solution $f$ to equation (1.1). Moreover,

$$f \in L^\infty([0, T], L^2(\mathbb{R}^3)) \cap L^2([0, T], H^1(\mathbb{R}^3)),$$

and the solution $f$ is stable with respect to the initial datum $f_0$.

1.5. **Plan of the paper.** The paper is organized as follows. The Section 2 is devoted to the basic a priori estimates for solutions. In particular, the entropy estimate is the key point to prove the existence and smoothing effect of solutions. In order to prove the entropy estimate, we do not only use Sobolev inequality and Hardy-Littlewood-Sobolev inequality, but also make a detailed analysis on the entropy production term. As the restitution coefficient we assume here is non-constant, the entropy production term is more complicated.

In Section 3, we establish a priori regularity estimates in Sobolev spaces with the help of interpolation inequalities and Sobolev inequality. The main difficulty in proving a priori regularity estimates lies in how to estimate the collision operator term, and we can overcome it by using the estimate of the inelastic collision operator with soft potentials in $L^p$ space. The $L^p$ bound of the collision operator established by Alonso, Carneiro and Gamba [2] will play an important role.
In Section 4, we present the proof of Theorem 1.2 and Theorem 1.3, and Corollary 1 is a direct consequence of Theorem 1.3.

2. Basic a priori estimates. The goal of this section is to present basic a priori estimates to the solution of equation (1.1).

Now we are ready to give mass, momentum, energy and entropy estimates to the solution of equation (1.1), just as the elastic Boltzmann equation. According to equation (1.4), it is easy to show that the mass and the momentum of the solution are conserved with respect to time.

Lemma 2.1. Let \( f \) be a solution of (1.1). Then for any time \( t > 0 \), there hold

\[
\int_{\mathbb{R}^3} f(t, v) dv = \int_{\mathbb{R}^3} f_0 dv \quad \text{and} \quad \int_{\mathbb{R}^3} f(t, v) v dv = \int_{\mathbb{R}^3} f_0 v dv.
\]

The proof of Lemma 2.1 is obvious if we set \( \varphi = 1 \) and \( \varphi = v \) in (1.4), and notice that the right-hand side of equation (1.4) vanishes. Thus, we have the following identities related to conservation of mass and momentum:

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(t, v) dv = 0 \quad \text{and} \quad \frac{d}{dt} \int_{\mathbb{R}^3} f(t, v) v dv = 0.
\]

We next study a priori energy and entropy estimates for the solutions to equation (1.1). On one hand, these two estimates are used to prove the weak compactness in \( L^1 \) in the theory of the elastic Boltzmann equation and shall play the same role in our work. On the other hand, the extra diffusion term provides additional compactness in some \( L^p \) space, which is not only crucial to prove the existence of weak solution for soft potentials, but also essential for the smoothing effect of the solutions. The estimate on the energy can be stated as follows.

Lemma 2.2. Let \( f_0 \) be the initial datum with finite mass and energy and \( f \) be a solution of (1.1). Then for any \( t > 0 \), we have

\[
\int_{\mathbb{R}^3} f(t, v)|v|^2 dv \leq 6t \int_{\mathbb{R}^3} f_0 dv + \int_{\mathbb{R}^3} f_0 |v|^2 dv.
\]

Proof. In equation (1.4), setting \( \varphi = |v|^2 \) and using the weak form of \( Q(f, f) \), one can follow the computations in [3] to obtain

\[
\int_{\mathbb{R}^3} v^2 Q(f, f) dv = -\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v)f(t, v_*)|v - v_*|^{2+\beta} dv dv_*
\]

\[
\times \int_{S^2} \frac{1}{4} - \hat{u} \cdot \sigma \left( 1 - e^2(|\tilde{v} - v_*| \sqrt{1 - \hat{u} \cdot \sigma}) \right) b(\tilde{u} \cdot \sigma) d\sigma.
\]

If we define the function as in [3]

\[
\Psi_e(x) = 2\pi x^{2+\beta} \int_0^1 (1 - e(\sqrt{x}z)^2)b(1 - 2z^2)z^3 dz,
\]

then the energy \( \int_{\mathbb{R}^3} f(t, v)|v|^2 dv \) satisfies

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(t, v)|v|^2 dv = 6 \int_{\mathbb{R}^3} f(t, v) dv - \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v)f(t, v_*)\Psi_e(|v - v_*|^2) dv dv_*,
\]

where the constant 6 comes from the term \( \Delta_v f \). Since \( f \) is assumed to be a positive solution, and \( \Psi_e(|v - v_*|^2) \) is also non-negative from the expression above, we can
deduce that the energy dissipation term $\int_{R^3} v^2 Q(f, f) dv$ is non-positive. Then

$$\frac{d}{dt} \int_{R^3} f(t, v)|v|^2 dv \leq 6 \int_{R^3} f(t, v) dv.$$ 

Using Lemma 2.1, we get

$$\int_{R^3} f(t, v)|v|^2 dv \leq 6t \int_{R^3} f_0 dv + \int_{R^3} f_0 |v|^2 dv.$$ 

**Remark 2.1.** Indeed, one can make a detailed analysis on the function $\Psi_e(x)$ appearing in the energy dissipation term. Recalling the assumption (A3) on the restitution coefficient, it is not difficult to find

$$\Psi_e(x) \simeq C x^{2+\beta}, \quad x \to 0$$

and

$$\Psi_e(x) \simeq C x^{2+\beta}, \quad x \to \infty.$$ 

More discussions about $\Psi_e(x)$ in the hard sphere case can be found in [3].

**Remark 2.2.** Lemma 2.2 implies that the energy of the solutions to equation (1.4) grows at most linearly. It is worth mentioning that in the hard sphere and Maxwellian cases, the analysis [16, 18] shows that the energy is global bounded (in time). Indeed, the energy dissipation term $\int_{R^3} v^2 Q(f, f) dv$ can be bounded from below by the energy $\int_{R^3} f|v|^2 dv$, see [16, 18] for details. Unfortunately, this fact does not hold under our range on $\beta$, and thus we can not obtain global energy estimate for the solutions.

We last study the entropy estimate of the solutions to equation (1.4). In the classical theory of the elastic Boltzmann equation, the so-called entropy

$$H(f) = \int_{R^3} f \ln f dv$$

decay is the most fundamental fact for all kinds of potentials and plays an important role to study the existence, regularity and convergence to equilibrium. While for the inelastic Boltzmann equation for hard sphere case without the external force [4], the entropy is proven to grow at most logarithmically, and for the inelastic Boltzmann equation in the case of hard sphere with the heat bath [16], the entropy is proven to be bounded uniformly in time thanks to the diffusion term. From these works [4, 16], it is known that the entropy estimate in the inelastic case is not as good as the one in the elastic case.

In this paper, we will separate the study on the entropy estimate into two cases. In the first case, it will be shown that the entropy is still global bounded in time, like the entropy estimate in work [16], provided that the initial datum only has finite mass and entropy. In the second case, the growth of the entropy will be proven to be at most quadratic.

**Theorem 2.3.** Let $\max(-4, -2 - \gamma) < \beta < 0$. Suppose that $f_0$ is the initial datum with finite mass, energy and entropy, and $f(t, v)$ is a solution of (1.1).

(1) If the restitution coefficient $e(z)$ satisfies the assumptions (A1), (A2) and (A3), then

$$\sup_{t \geq 0} \int_{R^3} f(t, v) \ln f dv \leq C \left( \int_{R^3} f_0 \ln f_0 dv, \int_{R^3} f_0 dv \right).$$
If the restitution coefficient $e(z)$ satisfies the assumptions (A1), (A2), (A3') and (A4), then
\[
\int_{\mathbb{R}^3} f(t,v) \ln f dv \leq \int_{\mathbb{R}^3} f_0 \ln f_0 dv + C \left( \int_{\mathbb{R}^3} f_0 \ln f_0 dv, \int_{\mathbb{R}^3} f_0 |v|^2 dv \right) (1 + t)^2.
\]

Proof. Multiplying the equation (1.4) by $\ln f$ and integrating by parts, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f dv + 4 \int_{\mathbb{R}^3} \nabla \sqrt{f} |f| dv = \int_{\mathbb{R}^3} Q(f,f) \ln f dv.
\]

(2) If the restitution coefficient $e(z)$ satisfies the assumptions (A1), (A2), (A3) or (A3'), then
\[
\int_{\mathbb{R}^3} f(t,v) \ln f dv \leq \int_{\mathbb{R}^3} f_0 \ln f_0 dv + C \left( \int_{\mathbb{R}^3} f_0 \ln f_0 dv, \int_{\mathbb{R}^3} f_0 |v|^2 dv \right) (1 + t)^2.
\]

Proof. Multiplying the equation (1.4) by $\ln f$ and integrating by parts, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f dv + 4 \int_{\mathbb{R}^3} \nabla \sqrt{f} |f| dv = \int_{\mathbb{R}^3} Q(f,f) \ln f dv.
\]

On one hand, by the Sobolev embedding inequality,
\[
\int_{\mathbb{R}^3} |\nabla \sqrt{f}|^2 dv \geq C_{sob} \|f\|_{L^3}.
\]

On the other hand, to estimate the term on the right hand side of (2.1), we present the following lemma which concerns the entropy production for soft potentials. Such a lemma is given in [4] for hard sphere case and can be extended to soft potentials.

Lemma 2.4. Assume that the restitution coefficient satisfies the assumptions (A1), (A2), (A3) or (A3'). Then the entropy production can be rewritten as follows:
\[
\int_{\mathbb{R}^3} Q(f,f) \ln f dv = -D(f) + \int_{\mathbb{R}^3} |v - v_*|^2 f(v) f(v) \Phi_e(|v - v_*|) dv dv_*
\]

where
\[
D(f) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_* \left( \ln f \frac{f f_*}{f f_*} - \frac{f f_*}{f f_*} + 1 \right) |v - v_*|^2 b(\cos \theta) d\theta dv dv_*
\]

and $\Phi_e(x)$ is defined by
\[
\Phi_e(|u|) = \frac{2}{|u|^2} \int_0^{\vartheta^{-1}(|u|)} (r - \vartheta(r) \vartheta'(r)) dr.
\]

It is noticed that the first term on the right hand side of equality (2.3) is non-positive because of the elementary inequality $\ln x - x + 1 \leq 0$, so we only need to estimate the second term on the right hand of equality (2.3).

It is easy to observe that the behavior of the function $\Phi_e(x)$ at 0 and $\infty$ influences the second term on the right hand side of equality (2.3). This motivates us to study the properties of the function $\Phi_e(x)$ at 0 and $\infty$. We emphasize that some qualitative properties of $\Phi_e(x)$ have already been given in [4]. However, for our purpose, we shall introduce a lemma which describes the behavior of $\Phi_e(x)$ at 0 and $\infty$ precisely.

Lemma 2.5. (i) Suppose that the restitution coefficient $e(z)$ satisfies the assumptions (A1), (A2) and (A3). Then $\Phi_e(|u|) \approx C_1 |u|^\gamma$ for $|u| \to 0$, $\Phi_e(|u|) \leq C_2$ for $|u| \to \infty$, and $\Phi_e(|u|) \leq C_3$ for any closed region.

(ii) Assume the restitution coefficient $e(z)$ satisfies the assumptions (A1), (A2), (A3') and (A4). Then $\Phi_e(|u|) \approx C_1 |u|^\gamma$ for $|u| \to 0$, $\Phi_e(|u|) \leq C_4 |u|^2$ for $|u| \to \infty$, $\Phi_e(|u|) \leq C_5$ for any closed region.

Proof. For the first point, since $r - \vartheta(r) \vartheta'(r) \leq r$, one can easily check that
\[
\Phi_e(|u|) \leq \left( \frac{\vartheta^{-1}(|u|)}{|u|} \right)^2 \text{ for } \forall u \in \mathbb{R}^3.
\]
We begin to study the behavior of \( \Phi_e(|u|) \) at \( \infty \). It is enough to consider the behavior of \( \vartheta^{-1} \) at \( \infty \) from the above inequality. For large \(|u|\), since
\[
\lim_{|u| \to \infty} e(|u|) = e_0 > 0
\]
and \( \vartheta(|u|) = |u|e(|u|) \), it follows that
\[
\vartheta(|u|) \simeq |u| \quad \text{when} \quad |u| \to \infty,
\]
from which one has
\[
\Phi_e(|u|) \leq C_2 \quad \text{when} \quad |u| \to \infty.
\]
For small \(|u|\), since \( \vartheta^{-1}(|u|) \simeq |u| \) and \( r - \vartheta(r)\vartheta'(r) \simeq \alpha(2 + \gamma)r^{\gamma+1} \), we get
\[
\Phi_e(|u|) \simeq C_1|u|^\gamma \quad \text{when} \quad |u| \to 0.
\]
In addition, from assumption (A1) on the restitution coefficient together with the construction of the function \( \Phi_e(|u|) \), it follows that \( \Phi_e(|u|) \) is a continuous function and bounded on a closed region which concludes the proof of the first part. For the second point, arguing in a similar way as in point (i), there is nothing to prove except the behavior of \( \Phi_e(|u|) \) for small \(|u|\). In fact, this is a direct consequence of the inequality (2.4) and assumption (A4) on the restitution coefficient. We complete the proof of part (ii).

Let us come back to the proof of Theorem 2.3. We are now in a position to estimate the term \( \iint_{R^3 \times R^3} |v - v_*|^\beta \tilde{f}(v) f(v_*) \Phi_e(|v - v_*|)dv_*dv_v \) with the help of Lemma 2.5.

(1) Suppose that the restitution coefficient satisfies assumptions (A1), (A2), (A3). If \( 0 \leq \beta + \gamma \), we choose \( r \) small enough and \( R \) big enough, and split the integral
\[
\iint_{R^3 \times R^3} |v - v_*|^\beta \tilde{f}(v) f(v_*) \Phi_e(|v - v_*|)dv_*dv_v
\]
into three parts. One gets
\[
\iint_{R^3 \times R^3} |v - v_*|^\beta f f_* \Phi_e(|v - v_*|)dv_*dv_v
\]
\[
\leq C_1 \int_{|v - v_*| \leq r} |v - v_*|^\beta f f_* dv_*dv_v + C_3 \int_{r \leq |v - v_*| \leq R} |v - v_*|^\beta f f_* dv_*dv_v
\]
\[+ C_2 \int_{|v - v_*| \geq R} |v - v_*|^\beta f f_* dv_*dv_v
\]
\[
\leq C_1r^{\beta + \gamma} \iint_{R^3 \times R^3} f f_* dv_*dv_v + (C_2 + C_3r^\beta) \iint_{R^3 \times R^3} f f_* dv_*dv_v \leq C_6 \|f\|_{L_1}^2,
\]
where we have used Lemma 2.5. Gathering the above estimate and (2.1), (2.2), (2.3), we finally obtain the following differential inequality:
\[
\frac{d}{dt} \int_{R^3} f \ln f dv + C \|f\|_{L^3} \leq C_6 \|f\|_{L_1}^2.
\]
Further, using the inequality which has been proven in [16]:
\[
\left( \int_{R^3} f \ln f dv \right)^\lambda \leq C \|f\|_{L^3},
\]
where \( \lambda > 1 \) is a constant, we have
\[
\frac{d}{dt} \int_{R^3} f \ln f dv + C \left( \int_{R^3} f \ln f dv \right)^\lambda \leq C_6 \|f\|_{L_1}^2.
\]
According to the estimate $\|f(t, \cdot)\|_L^1 = \|f_0\|_L^1$ in Lemma 2.1, it yields
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f \, dv + C \left( \int_{\mathbb{R}^3} f \ln f \, dv \right)^\lambda \leq C_0 \|f_0\|_{L^1}^2.
\]

If $\int_{\mathbb{R}^3} f(t, v) \ln f(t, v) \, dv$ ever becomes greater than $(C_0^2 \|f_0\|_{L^1}^2)^{\frac{1}{2}}$, the above inequality shows that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f \, dv \leq 0.
\]

Then we conclude that
\[
\sup_{t \geq 0} \int_{\mathbb{R}^3} f \ln f \, dv \leq C \left( \int_{\mathbb{R}^3} f_0 \ln f_0 \, dv, \int_{\mathbb{R}^3} f_0 \, dv \right).
\]

If $-2 < \beta + \gamma < 0$, one can still use the above method to deal with the integral
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\beta f(v) \Phi_e(|v - v_*|) \, dv \, dv_*
\]
and we can get
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\beta f(v) \Phi_e(|v - v_*|) \, dv \, dv_* \leq C_1 \int_{|v - v_*| \leq r} \Phi_e(|v - v_*|) \, dv \, dv_* + (C_2 + C_3 r^\beta) \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \, dv \, dv_*.
\]

The main difference between this case and $\beta + \gamma \geq 0$ lies in that we shall apply the Hardy-Littlewood-Sobolev inequality to estimate the integral
\[
\int_{|v - v_*| \leq r} \Phi_e(|v - v_*|) \, dv \, dv_*.
\]

Indeed,
\[
\int_{|v - v_*| \leq r} \Phi_e(|v - v_*|) \, dv \, dv_* \leq C_{hts} \|f\|_{L^q} \|f\|_{L^2},
\]
where
\[
-\frac{\beta + \gamma}{3} + \frac{1}{q_1} + \frac{1}{q_2} = 2.
\]

If we choose $q_1 = q_2$, then
\[
\int_{|v - v_*| \leq r} \Phi_e(|v - v_*|) \, dv \, dv_* \leq C_{hts} \|f\|_{L^\frac{6}{\beta + \gamma}}^2.
\]

To estimate $\|f\|_{L^{\frac{6}{\beta + \gamma}}}$, we interpolate the $L^{\frac{6}{\beta + \gamma}}$ norm between $L^1$ norm and $L^3$ norm, so that
\[
\|f\|_{L^{\frac{6}{\beta + \gamma}}} \leq \|f\|_{L^1}^{\frac{x}{1-x}} \|f\|_{L^3}^{\frac{1-x}{1-x}}
\]
where
\[
\frac{6 + \beta + \gamma}{x} = x + \frac{1 - x}{3}.
\]

It yields $x = 1 + \frac{\beta + \gamma}{2}$. Combining (2.1), (2.2), (2.5), (2.6), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f \, dv + C \|f\|_{L^3} \leq C_7 \|f\|_{L^1}^\frac{\beta + \gamma}{2} \|f\|_{L^1}^{2 + \frac{\beta + \gamma}{2}} + C_8 \|f\|_{L^1}^2.
\]
Since $\beta + \gamma > -2$, applying the Young inequality to the first term on the right hand side of (2.7), one has
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f dv + C\|f\|_{L^3} \leq C(\epsilon)\|f\|_{L^1}^{\frac{4+\beta+\gamma}{3}} + \epsilon\|f\|_{L^3} + C_8\|f\|_{L^1}^2.
\]
In the above inequality, the second term on the right hand side can be absorbed by the second term on the left hand side if we choose $\epsilon$ small enough. Then we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f \ln f dv + C\|f\|_{L^3} \leq C(\epsilon)\|f\|_{L^1}^{\frac{4+\beta+\gamma}{3}} + C_8\|f\|_{L^1}^2.
\] (2.8)

Arguing as before, this completes the proof of the first part of Theorem 2.3.

Next we turn to part (2), where the restitution coefficient satisfies assumptions (A1), (A2), (A3') and (A4). The proof is analogous to the one in part (i). We again split the integral $\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\beta f(v)f(v_*)\Phi_e(|v - v_*|) dv dv_*$ into three parts and employ Lemma 2.5 to deal with these parts. After direct computations, we find that the bound for the term $\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - v_*|^\beta f(v)f(v_*)\Phi_e(|v - v_*|) dv dv_*$ should be replaced by $L^1_2$ norm instead of $L^1$ norm. And the energy estimate proven in Lemma 2.2 can help us obtain the desired result.

**Remark 2.3.** From the proof in Theorem 2.3, one could see that in order to estimate the integral
\[
\int_{|v - v_*| \leq \epsilon} ff_\epsilon |v - v_*|^{\beta + \gamma} dv dv_*
\]
when $\beta + \gamma < 0$, the Hardy-Littlewood-Sobolev inequality is necessary. Although the Hardy-Littlewood-Sobolev inequality is valid for $-3 < \beta + \gamma \leq -2$, the first term on the right hand side of inequality (2.7) can not be controlled by the terms on the left hand side. This is the main reason why we can not generalize our method to the case $\beta \leq -2 - \gamma$.

**Corollary 2.** Assume that the initial datum $f_0$ and the restitution coefficient satisfy the assumptions as in Theorem 2.3. Let $f$ be a solution of equation (1.4). Then there exists a time $t_0 > 0$, as small as desired, such that
\[
\|f(t_0, \cdot)\|_{L^2} \leq C\int_{\mathbb{R}^3} f_0 dv, \int_{\mathbb{R}^3} f_0 \ln f_0 dv, T).
\]

**Proof.** It suffices to consider the case when the restitution coefficient satisfies (A1), (A2) and (A3). Integrating the inequality (2.8) in time from 0 to $T$ we obtain
\[
\int_0^T \|f(t, \cdot)\|_{L^3} dt \leq C\int_{\mathbb{R}^3} f(t, \cdot) dv, \int_{\mathbb{R}^3} f_0 \ln f_0 dv, T).
\]
Thanks to Hölder’s inequality and Lemma 2.1, we have
\[
\int_0^T \|f(t, \cdot)\|_{L^2}^4 dt \leq \int_0^T \|f(t, \cdot)\|_{L^1}^2 \|f(t, \cdot)\|_{L^3}^2 dt \leq C\int_{\mathbb{R}^3} f_0 dv, \int_{\mathbb{R}^3} f_0 \ln f_0 dv, T).
\]
Thus one can find that $t_0 > 0$ as small as desired, such that
\[
\|f(t_0, \cdot)\|_{L^2} \leq C\int_{\mathbb{R}^3} f_0 dv, \int_{\mathbb{R}^3} f_0 \ln f_0 dv, T).
\]
\[ \beta = -2 \] is a critical value. In this paper, a wider range for \( \beta \) is allowed because of the asymptotic behavior of \( \Phi_e(x) \) at the origin.

3. \( L^p \) bounds and a priori regularity. In this section we study the a priori regularity of the solutions to equation (1.1). Our method is based on the a priori energy estimate to the solutions. Let us explain the strategy as follows: we first establish the \( L^2 \) estimate for the solutions to equation (1.1), and then consider the \( H^1 \) estimate. By induction, all \( H^k \) norms are studied in a similar way. Our results show that high order derivatives of the solutions can be controlled by lower order derivatives of the solutions. Thus, all \( H^k \) norms of the solutions can be expressed in terms of the \( L^1 \) and \( L^2 \) norms of the initial datum.

We start by giving a lemma which is concerned with the \( L^p \) bound of the inelastic collision operator \( Q^\pm(f,f) \). The \( L^p \) bound of \( Q^+(f,f) \) with hard potentials was first studied by Gustafsson [17], who revealed the convolution structure of \( Q^+(f,f) \). Later, Mouhot and Villani [22] revisited the work of Gustafsson under some cut-off conditions on \( b \). Recently, Alonso, Carneiro and Gamba [2] deduced the results in [22] with more precise constants which only depend on the integral condition in \( b \). The method developed for the elastic collision operator was also extended to the inelastic Boltzmann collision operator \( Q^\pm(f,f) \) for hard and soft potentials, see [2, 16]. Roughly speaking, we can make the following statements from the point of integrability: for hard and Maxwellian potentials, the \( L^p \) estimate on the collision operator behaves like the classical Young’s inequality, while in the case of soft potentials, it is similar to the well-known Hardy-Littlewood-Sobolev inequality. Here we only focus on the result for soft potentials.

**Lemma 3.1** ([2]). Let \( 1 < p, q, r < \infty \) and \(-3 < \beta < 0\). If the restitution coefficient \( e(z) \) satisfies assumptions (A1) and (A2), then the bilinear operator \( Q^\pm(f,g) \) is a bounded operator from \( L^p(R^3) \times L^q(R^3) \) to \( L^r(R^3) \):

\[
\|Q^\pm(f,g)\|_{L^r} \leq C(b, e)\|f\|_{L^p}\|g\|_{L^q},
\]

where

\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{\beta}{3} + \frac{1}{r},
\]

and \( C(b, e) \) is an explicit constant that depends on \( b(\cos \theta) \) and \( e \).

We are now in a position to study the \( L^2 \) norm of the solutions. The main tools used here are the coercivity of the diffusion term and the \( L^p \) estimate of the inelastic collision operator given in Lemma 3.1.

**Lemma 3.2.** Let \(-2 < \beta < 0\), and assume that \( f \) is a positive regular solution to equation (1.1) with the initial datum \( f_0 \in L^1 \cap L^2(R^3) \). If the restitution coefficient \( e(z) \) satisfies assumptions (A1), (A2), (A3) or (A3'), then

\[
\|f(\cdot, t)\|_{L^2} \leq C, \quad 0 \leq t < \infty,
\]

and

\[
\int_0^T \|\nabla f(t, \cdot)\|_{L^2}^2 dt \leq C_T
\]

for every \( 0 \leq T < \infty \), where the constants \( C \) and \( C_T \) depend on \( \|f_0\|_{L^1} \) and \( \|f_0\|_{L^2} \).

**Proof.** Multiplying equation (1.4) by \( f \) and integrating by parts, it follows that

\[
\frac{d}{dt} \int_{R^3} f^2 dv + \int_{R^3} |\nabla f|^2 dv = \int_{R^3} Q(f, f)f dv \leq \int_{R^3} Q^+(f, f)f dv.
\]
In order to estimate \( \int_{R^3} Q^+(f, f) f dv \), applying Hölder’s inequality and Lemma 3.1 with \((r, p) = (2, 2)\), one has
\[
\int_{R^3} Q^+(f, f) f dv \leq \|Q^+(f, f)\|_{L^2} \|f\|_{L^2} \leq C(b, e) \|f\|_{L^2} \|f\|_{L^p} \|f\|_{L^q},
\]
where \( q \) satisfies the following equation:
\[
1 + \frac{1}{2} + \frac{b}{3} = \frac{1}{2} + \frac{1}{q}.
\]
Then it yields \( q = \frac{3}{3 + \beta} \). The term \( \|f\|_{L^{3 + \beta}} \) and \( \|f\|_{L^2} \) can be estimated by the interpolated inequalities:
\[
\|f\|_{L^{3 + \beta}} \leq \|f\|_{L^1}^{1 - x_1} \|f\|_{L^6}^{x_1}, \quad \|f\|_{L^2} \leq \|f\|_{L^1}^{1 - x_2} \|f\|_{L^6}^{x_2},
\]
where \( x_1 \) and \( x_2 \) are determined from the following equations:
\[
\frac{3 + \beta}{3} = 1 - x_1 + \frac{x_1}{6}, \quad \frac{1}{2} = 1 - x_2 + \frac{x_2}{6}.
\]
Therefore, \( x_1 = -\frac{2\beta}{3} \), \( x_2 = \frac{2}{5} \) and from which we deduce that
\[
\int_{R^3} Q^+(f, f) f dv \leq C(b, e) \|f\|_{L^1}^{\frac{2}{5} + \frac{2\beta}{3}} \|f\|_{L^6}^{\frac{9}{5} - \frac{2\beta}{3}}.
\]
Combining the above estimates and applying Sobolev’s imbedding inequality, we get
\[
\frac{d}{dt} \int_{R^3} f^2 dv + \int_{R^3} |\nabla f|^2 dv \leq C \|f\|_{L^1}^{\frac{9}{5} + \frac{2\beta}{3}} \|\nabla f\|_{L^5}^{\frac{9}{5} - \frac{2\beta}{3}}.
\]
Applying the Young’s inequality to the right hand side of the above inequality, we have
\[
\frac{d}{dt} \int_{R^3} f^2 dv + \int_{R^3} |\nabla f|^2 dv \leq C(\epsilon) \|f\|_{L^{15/8}}^{\frac{9}{10} + \frac{2\beta}{3}} + \epsilon \|\nabla f\|_{L^2}^2.
\]
In the above inequality, the second term on the right hand side can be absorbed by the second term on the left hand side, it yields
\[
\frac{d}{dt} \int_{R^3} f^2 dv + C \|\nabla f\|_{L^2}^2 \leq C(\epsilon) \|f\|_{L^{15/8}}^{\frac{9}{10} + \frac{2\beta}{3}}. \tag{3.1}
\]
Using the following Sobolev imbedding inequality and interpolated inequality again,
\[
\|f\|_{L^3}^2 \leq C_{sob} \|\nabla f\|_{L^2}^2, \quad \|f\|_{L^2} \leq \|f\|_{L^1}^{\frac{2}{3}}, \|f\|_{L^6}^{\frac{2}{3}},
\]
we have
\[
\frac{d}{dt} \int_{R^3} f^2 dv + C \|\nabla f\|_{L^2}^{10} \|f\|_{L^3}^2 \leq C(\epsilon) \|f\|_{L^{15/8}}^{\frac{9}{10} + \frac{2\beta}{3}}.
\]
According to the Gronwall’s inequality together with \( \|f(t, \cdot)\|_{L^1} = \|f_0\|_{L^1} \), we obtain that the \( L^2 \) norm of \( f \) is bounded in terms of \( \|f_0\|_{L^2} \) and \( \|f_0\|_{L^1} \). Furthermore, we integrate the inequality (3.1) over time from 0 to \( T \). Then it follows that
\[
\int_0^T \|\nabla f(t, \cdot)\|_{L^2}^2 dt \leq C_T(\|f_0\|_{L^1}, \|f_0\|_{L^2}),
\]
which ends the proof.

\[\square\]

**Remark 3.1.** The bound for \( \int_0^T \|\nabla f(t, \cdot)\|_{L^2}^2 dt \) implies that for any small time \( t_1 > 0 \), \( \|\nabla f(t_1, \cdot)\|_{L^2} \leq C_T(\|f_0\|_{L^1}, \|f_0\|_{L^2}) \).

Our next aim is to establish a priori \( H^1 \) estimate for the solution. For this, let us recall the following differential rule for the inelastic collision operator \( Q(f, f) \).
Lemma 3.3. Let $f$ and $g$ be smooth and rapidly decaying functions of $v$. Then
\[ \partial^j Q^\pm(f, g) = \sum_{0 \leq l \leq j} \binom{l}{j} Q^\pm(\partial^{j-l}f, \partial^l g), \]
where $j = (j_1, \cdots, j_n)$, $l = (l_1, \cdots, l_n)$ are multi-indices and $\partial^j$ denotes the $j$ order derivative with respect to the variable $v$.

Remark 3.2. Lemma 3.3 is a consequence of the bilinearity and the translation invariance of the inelastic collision operator, namely $\tau_h Q(f, g) = Q(\tau_h f, \tau_h g)$, where the translation operator $\tau_h$ is defined by $\tau_h f(v) = f(v - h)$.

We can view this formula as a Leibniz formula for the inelastic collision operator. It is emphasized that this lemma has been first proven in [16] for the hard sphere model, and can be easily extended to the soft potentials.

At this point, we are ready to study the $H^1$ estimate to the solutions of equation (1.1).

Lemma 3.4. Let $-2 < \beta < 0$, and $f$ be a positive regular solution to equation (1.1) with the initial datum $f_0 \in L^1 \cap H^1(\mathbb{R}^3)$. If the restitution coefficient $e(z)$ satisfies assumptions (A1), (A2), (A3) or (A3'), then
\[ \|f(t)\|_{H^1} \leq C, \quad 0 \leq t < \infty, \]
and
\[ \int_0^T \|\nabla^2 f(t, \cdot)\|_{L^2}^2 dt \leq C_T, \]
for every $0 \leq T < \infty$, where the constants $C$ and $C_T$ depend on $\|f_0\|_{L^1}$ and $\|f_0\|_{H^1}$.

Proof. The proof is exactly the same as the one in Lemma 3.2. Let $\alpha \in \mathbb{R}^3$ be a multi-index such that $|\alpha| = 1$ and we denote $g = \partial^\alpha f$. Then $g$ satisfies the following equation:
\[ \partial_t g = Q(f, g) + Q(g, f) + \triangle g. \] (3.2)

Multiplying the equation satisfied by $g$ and integrating by parts, we easily compute
\[
\frac{d}{dt} \int_{\mathbb{R}^3} g^2 dv + \int_{\mathbb{R}^3} |\nabla g|^2 dv = \int_{\mathbb{R}^3} (Q(f, g) + Q(g, f)) \cdot g dv = \int_{\mathbb{R}^3} Q^+(f, g) g dv + \int_{\mathbb{R}^3} Q^+(g, f) g dv - \int_{\mathbb{R}^3} Q^-(f, g) g dv - \int_{\mathbb{R}^3} Q^-(g, f) g dv =: I_1 + I_2 + I_3 + I_4.
\]
We shall estimate the above four terms separately. For the first term, if $\beta \in (-\frac{3}{2}, 0)$, using Hölder’s inequality, it follows that
\[ I_1 \leq \|Q^+(f, g)\|_{L^2} \|g\|_{L^2}. \]
To estimate $\|Q^+(f, g)\|_{L^2}$, we apply Lemma 3.1 with $(r, p) = (2, 2)$ to obtain
\[ \|Q^+(f, g)\|_{L^2} \leq C(b, \epsilon) \|f\|_{L^6} \|g\|_{L^2}, \]
where $q$ satisfies the following equation:
\[ 1 + \frac{\beta}{3} + \frac{1}{2} = \frac{1}{2} + \frac{1}{q}. \]
Then \( q = \frac{3}{3 + 2} \), using Hölder’s inequality to estimate \( \| f \|_{L^{\frac{3}{3 + 2}}} \):

\[
\| f \|_{L^{\frac{3}{3 + 2}}} \leq \| f \|_{L^4} \| f \|_{L^2}^{1 - x_3},
\]
where \( x_3 \) satisfies the following equation:

\[
\frac{3 + \beta}{3} = \frac{x_3}{1} + \frac{1 - x_3}{2}.
\]

Gathering the above estimates, we obtain

\[
\| Q^+(f,g) \|_{L^2} \leq C(b,c) \| f \|_{L^4} \| f \|_{L^2}^{1 - x_3} \| g \|_{L^2}.
\]

Thanks to Lemmas 2.1 and 3.2, the above inequality yields

\[
I_1 \leq C \| g \|_{L^2} = C \| \nabla f \|_{L^2}^2 \leq \epsilon \| \nabla^2 f \|_{L^2}^2 + C_\epsilon \| f \|_{L^2}^2,
\]
where we have also used the elementary interpolation inequality.

If \( \beta \in (-2, -\frac{3}{2}] \), using Hölder’s inequality again, we have

\[
I_1 \leq \| Q^+(f,g) \|_{L^q} \| g \|_{L^6}.
\]

We next use Lemma 3.1 with \((r,p) = (\frac{6}{5}, 2)\) to get

\[
\| Q^+(f,g) \|_{L^2} \leq C(b,c) \| f \|_{L^4} \| g \|_{L^2}^2,
\]
where \( q \) satisfies the following equation:

\[
1 + \frac{3}{3} + \frac{5}{6} = \frac{1}{2} + \frac{1}{q}.
\]

We deduce \( q = \frac{3}{3 + 2} \) and use the following interpolated inequality to estimate \( \| f \|_{L^{\frac{3}{3 + 2}}} \):

\[
\| f \|_{L^{\frac{3}{3 + 2}}} \leq \| f \|_{L^4} \| f \|_{L^2}^{1 - x_4}.
\]

Collecting the above estimates, applying Lemmas 2.1 and 3.2, we finally get

\[
I_1 \leq C \| g \|_{L^2} \| g \|_{L^6}.
\]

By Sobolev inequality and the interpolated inequality, it follows that

\[
I_1 \leq C \| g \|_{L^2} \| \nabla g \|_{L^2} = C \| \nabla f \|_{L^2} \| \nabla^2 f \|_{L^2} = \epsilon \| \nabla^2 f \|_{L^2}^2 + C_\epsilon \| f \|_{L^2}^2.
\]

It is mentioned that the terms \( I_2, I_3, I_4 \) can be treated similarly. Hence,

\[
I_2 + I_3 + I_4 \leq \epsilon \| \nabla^2 f \|_{L^2}^2 + C_\epsilon \| f \|_{L^2}^2.
\]

In the end, gathering the above estimates for \( I_1, I_2, I_3 \) and \( I_4 \), we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^3} g^2 dv + \int_{\mathbb{R}^3} |\nabla g|^2 dv \leq \epsilon \| \nabla^2 f \|_{L^2}^2 + C_\epsilon \| f \|_{L^2}^2.
\]

Arguing as in Lemma 3.2, this completes the proof. \( \square \)

By using the above method again, we can also establish the following lemma concerned with the high-order derivatives.

**Lemma 3.5.** Let \( f \) be a positive regular solution to equation (1.1) with the initial datum \( f_0 \in L^1 \cap H^n(R^3) \), where \( n \geq 0 \). Then

\[
\| f(t, \cdot) \|_{H^n} \leq C, \quad 0 \leq t < \infty,
\]
and

\[
\int_0^T \| \nabla f(t, \cdot) \|_{H^{n+1}}^2 dt \leq C_T
\]
for every \( 0 \leq T < \infty \), where the constants \( C \) and \( C_T \) depend on \( \| f_0 \|_{L^1} \) and \( \| f_0 \|_{H^n} \).
Proof. We conclude the proof by inductive argument. The case \( n = 0 \) is already proven in Lemma 3.2. Assuming that this lemma holds true for \( n \). Then for every multi-index \( j \) with \( |j| = n + 1 \), by formal differentiation of equation (1.1), we get the following equation for high-order derivatives:

\[
\frac{d}{dt} \partial^j f = \sum_{0 \leq l \leq j} \binom{l}{j} (Q^+ (\partial^{j-l} f, \partial^l f) - Q^- (\partial^{j-l} f, \partial^l f)) + \Delta \partial^j f.
\]

The remaining steps are similar to the ones in the above lemmas, so that we omit it.

4. Existence, regularity, uniqueness and stability. In this section, we shall present the proof of our main results. Let us first consider the existence and the regularity.

Proof of Theorem 1.2 and Theorem 1.3. By truncation, we can construct a sequence of approximations of the collision kernel replacing the factor \(|v-v_*|^\beta\) by \(|v-v_*|_n = \min(|v-v_*|^\beta, n)\), where \( n \) is truncation parameter. Let \( Q_n(f,f) \) be the associated operator. Then we consider the following approximated equation:

\[
\frac{\partial f_n}{\partial t} = Q_n(f_n,f_n) + \Delta f_n, \quad f_n(0,v) = f_0(v),
\]

where the initial datum \( f_0 \) is assumed to belong to \( C_0^\infty \) and this assumption can be removed in the end of the proof. It is not difficult to follow the steps in the proof of Theorem 5.1 in [16] to prove that for each \( n \), there exists a smooth and non-negative solution \( f_n \) to the approximated equation. Then the a priori estimates proven in Section 2 also hold true for \( f_n \). More precisely, for any \( 0 \leq t \leq T < \infty \), there hold

\[
\int_{\mathbb{R}^3} (1 + |v|^2 + \ln f_n) f_n(t,v) dv \leq C_T(f_0) \quad (4.1)
\]

and

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla (\sqrt{f_n})|^2 dv dt \leq C_T(f_0). \quad (4.2)
\]

If \(-2 \leq \beta < 0\), the uniform bound in (4.1) implies the weak compactness in \( L^1 \) by Dunford-Pettis theorem. Let \( f \) be a weak cluster point of \( f_n \), and it is not hard to check \( f \) is a weak solution of equation (1.1), as in the case of the elastic Boltzmann equation.

If \(-2 - \gamma < \beta < -2\), the weak convergence in \( L^1 \) is not sufficient to pass to the limit because the weak form of the collision operator behaves as

\[
\int_0^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_n |v - v_*|^\beta + 2 dv dv_*,
\]

where \( \beta + 2 < 0 \). The bound in (4.2) allows us to obtain some extra \( L^p \) bound to overcome this difficulty. We use the following result which is based on (4.1) and (4.2): the sequence \( f_n \) is compact in \( L^2((0,T), L^p) \) for some \( 1 \leq p \leq \frac{3}{2} \). This is enough to pass the limit in the collision terms.

To study the smoothing effect, since \( f_n \) belongs to the class of smooth functions, we can make rigorous the a priori regularity in Section 3, and the uniform bounds in any \( H^k \) space also imply the smooth of \( f \).
Finally, when \( 0 \leq f_0 \in L^2 \) and \( \int_{R^3} f_0 \ln f_0 \, dv < \infty \), we regularize \( f_0 \) by truncation and convolution to obtain a sequence \( f_0^\epsilon \) defined as follows: for some \( 0 < R < \infty \), we set
\[
f_0^\epsilon := f_0 1_{|v| \leq R} * \rho_\epsilon,
\]
where \( 1_{|v| \leq R} \) stands for the characteristic function and \( \rho_\epsilon \) denotes the usual mollifiers. It is mentioned that \( f_0^\epsilon \) converges to \( f_0 \) in \( L^1 \) and the following uniform estimates hold,
\[
\int_{R^3} f_0^\epsilon (1 + |v|^2) \, dv \leq \int_{R^3} f_0 (1 + |v|^2) \, dv, \quad \int_{R^3} f_0^\epsilon \ln f_0^\epsilon \, dv \leq \int_{R^3} f_0 \ln f_0 \, dv. \tag{4.3}
\]
We shall consider the Cauchy problem
\[
\frac{\partial f^\epsilon}{\partial t} = Q(f^\epsilon, f^\epsilon) + \Delta_v f^\epsilon, \quad f^\epsilon(0, v) = f_0^\epsilon(v).
\]
Here \( f^\epsilon \) denotes the solution with the initial datum \( f_0^\epsilon \). Recalling again the a priori estimates in Section 2, we have
\[
\int_{R^3} (1 + |v|^2 + \ln f^\epsilon) f^\epsilon(t, v) \, dv \leq C_T(\|f_0^\epsilon\|_{L^1}, \int_{R^3} f_0^\epsilon \ln f_0^\epsilon \, dv)
\]
and
\[
\int_0^T \int_{R^3} |\nabla (\sqrt{f^\epsilon})|^2 \, dv \, dt \leq C_T(\|f_0^\epsilon\|_{L^1}, \int_{R^3} f_0^\epsilon \ln f_0^\epsilon \, dv).
\]
In view of the uniform estimates (4.3), we are able to prove the existence of the weak solution using previous argument.

As for the smoothing effect, it follows from Corollary 2 that
\[
\|f^\epsilon(t_0, \cdot)\|_{L^2} \leq C_T\left(\int_{R^3} f_0^\epsilon \, dv, \int_{R^3} f_0^\epsilon \ln f_0^\epsilon \, dv\right) \tag{4.4}
\]
for some \( t_0 > 0 \) as small as desired. Then we consider \( f^\epsilon(t_0, v) \) as an initial datum and apply Lemma 3.2 to obtain
\[
\|f^\epsilon(t, \cdot)\|_{L^2} \leq C(\|f^\epsilon(t_0, \cdot)\|_{L^1}, \|f^\epsilon(t_0, \cdot)\|_{L^2})
\]
for any \( t > t_0 \) and
\[
\|\nabla f^\epsilon(t_1, \cdot)\|_{L^2} \leq C_T(\|f^\epsilon(t_0, \cdot)\|_{L^1}, \|f^\epsilon(t_0, \cdot)\|_{L^2})
\]
for some \( t_1 > t_0 \). By inductive argument, the general \( H^k \) norms can be estimated using Lemma 3.4 and Lemma 3.5, more precisely,
\[
\|f^\epsilon(t, \cdot)\|_{H^k} \leq C_T(\|f^\epsilon(t_0, \cdot)\|_{L^1}, \|f^\epsilon(t_0, \cdot)\|_{L^2})
\]
for any \( t > t_k \), where \( t_k \) can be chosen arbitrarily small. Thanks to Lemma 2.1 and estimate (4.4),
\[
\|f^\epsilon(t, \cdot)\|_{H^k} \leq C_T\left(\int_{R^3} f_0^\epsilon \, dv, \int_{R^3} f_0^\epsilon \ln f_0^\epsilon \, dv\right).
\]
Using the uniform estimates (4.3) again, we finally get the following estimate
\[
\|f^\epsilon(t, \cdot)\|_{H^k} \leq C_T\left(\int_{R^3} f_0 \, dv, \int_{R^3} f_0 \ln f_0 \, dv\right)
\]
for any \( t > t_k \). The above procedures show the estimates on any \( H^k \) norm are uniform in \( \epsilon \). And this also means the cluster point \( f \), which is the solution of equation (1.1) with initial datum \( f_0 \), becomes smooth immediately. \( \square \)
We next prove the stability and uniqueness of the weak solution to equation (1.1) under the assumption that the initial datum belongs to $L^2$. We shall first establish the stability with respect to the initial datum. Then the uniqueness is a consequence of the stability by Gronwall’s inequality.

**Proof of Corollary 1.** Let $f_1$ and $f_2$ be two weak solutions to equation (1.1) with the same initial data $f_0$, i.e.,

$$f_1(0, \cdot) = f_2(0, \cdot) = f_0.$$ \(\text{Subtracting the equations satisfied by } f_1 \text{ and } f_2, \text{ we have}

$$\partial_t (f_1 - f_2) = Q(f_1 - f_2) + Q(f_1 + f_2) - \triangle (f_1 - f_2).$$

In the following, we set $D = f_1 - f_2$ and $S = f_1 + f_2$. Then the difference of the solutions satisfies the equation

$$\partial_t D = Q^+(D, S) + Q^+(S, D) - Q^-(D, S) - Q^-(S, D) + \triangle D.$$ \(\text{Multiplying the above equation by } D, \text{ and integrating by parts, we have}

$$\frac{d}{dt} \int_{\mathbb{R}^3} D^2 dv + \int_{\mathbb{R}^3} |\nabla D|^2 dv = \int_{\mathbb{R}^3} [Q^+(D, S)D + Q^+(S, D)D - Q^-(D, S)D - Q^-(S, D)D] dv.$$ \(\text{From the above equality, it is straightforward to see the terms on the right hand}

are analogous to the terms in Lemma 3.4. This fact leads us to employ the same method to estimate these terms. Arguing as in the proof of Lemma 3.4, we have

$$\int_{\mathbb{R}^3} [Q^+(D, S)D + Q^+(S, D)D - Q^-(D, S)D - Q^-(S, D)D] dv \leq C_\varepsilon \|D\|_{L^2}^2 + \varepsilon \|\nabla D\|_{L^2}^2,$$

where $C_\varepsilon$ depends on the $L^1$ norm and $L^2$ norm of $f_1$ and $f_2$. Then we can obtain the following differential inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^3} D^2 dv + \int_{\mathbb{R}^3} |\nabla D|^2 dv \leq C_\varepsilon \|D\|_{L^2}^2 + \varepsilon \|\nabla D\|_{L^2}^2.$$ \(\text{Taking } \varepsilon \text{ small enough, we have}

$$\frac{d}{dt} \int_{\mathbb{R}^3} D^2 dv + C \int_{\mathbb{R}^3} |\nabla D|^2 dv \leq C_\varepsilon \|D\|_{L^2}^2$$

which entails the stability with respect to the initial datum as well as the uniqueness of the solution. \(\square\)

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