Consistency Relations of an Extension Polycyclic Free Abelian Lattice Group by Quaternion Point Group

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Abstract. An extension of a free abelian lattice group by finite group is a torsion free crystallographic group. It expounds its symmetrical properties or known as homological invariants. One of the methods to compute its homological invariants is by determining the polycyclic presentation of the group. These polycyclic presentations are first shown to satisfy its consistency relations. Therefore, our focus is to show that this extension polycyclic free abelian lattice group by quaternion point group satisfy its consistency relations.

1. Introduction
The computation of the homological invariants of a group has been started long time ago [1]. However, researches on the computation of the homological invariants for an extension polycyclic of a free abelian lattice group by finite group are only carried out starting in 2009 [2]. The research focused on the computations of the nonabelian tensor squares for polycyclic groups with cyclic point group of order two, $C_2$ and elementary abelian 2-group point group, $C_2 \times C_2$ as the abelian extension. Later in 2014, Mat Hassim et al. [3], extended the research by finding the other homological invariant of a group which is the exterior square. In the same year, Mat Hassim et al. [4] computed the abelianizations of all extension of polycyclic group with cyclic point group of order three, $C_3$ in order to find its homological invariants. Later, Mat Hassim [5] also computed the other homological invariants for groups extended with cyclic point group of order two, three and five. Abdul Ladi et al. [6, 7] and Masri et al. [8], continue to compute the other homological invariants for the extended polycyclic group with point group, $C_2 \times C_2$.

In 2011 the nonabelian extension has been taken into consideration. Mohd Idrus [9] started to work on the dihedral point group of order eight, $D_4$, followed by Wan Mohd Fauzi et al. [10] in 2015 who worked on the same extension but with different dimension of the group. A year later, Tan et al. [11] used the symmetric group of order six, $S_3$ as the extension. All of these groups are isomorphic to one of the groups that were designed by Opengenorth et al. [12] to enable the user to construct and recognize space groups. Eick and Nickel [13] showed that the nonabelian tensor square of a polycyclic group given by a polycyclic presentation can be computed. Therefore, the
the relationship between generators in point group of order eight, polycyclic presentations and proved to satisfy its consistency relations. Based on [12], there are four groups of extension of a free abelian lattice group by quaternion point group of order eight, \(Q_8\) found and all of them are of dimension six. In 2015, Mohammad et al. [15] computed the polycyclic presentation of the fourth extension of a free abelian lattice group with point group \(Q_8\). However, there are some problems on the calculations of its homological invariants by using this polycyclic presentation. The new generator \(c\) was developed among the relationship between generators in \(G_4\) but the generator \(c\) is not well defined in here. The polycyclic presentation is stated as in the following:

\[
Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle | a^2 = cl_6, b^2 = cl_5, c^2 = bec^{-2}l_6^2, c = cl_6^{-1}, e = cl_4 \rangle \]

where \(Q_4\) is the group (with point group 8) free abelian lattice group with point group \(Q_8\) found and all of them are of dimension six. In 2015, Mohammad et al. [15] computed the polycyclic presentation of the fourth extension of a free abelian lattice group with point group \(Q_8\). However, there are some problems on the calculations of its homological invariants by using this polycyclic presentation. The new generator \(c\) was developed among the relationship between generators in \(G_4\) but the generator \(c\) is not well defined in here. The polycyclic presentation is stated as in the following:

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This presentation cannot be used in computing the homological invariants. Problem arise as the calculations taking over. Therefore, at the end of the calculations, the polycyclic presentation is concluded to be not consistent. Hence, the main motivation of this research is to show the new polycyclic presentation that satisfy its consistency relations for the fourth extension of a free abelian lattice group and Morse [14] is used throughout this research. All of these groups have been transformed into polycyclic presentations and proved to satisfy its consistency relations.

Based on [12], there are four groups of extension of a free abelian lattice group by quaternion point group of order eight, \(Q_8\) found and all of them are of dimension six. In 2015, Mohammad et al. [15] computed the polycyclic presentation of the fourth extension of a free abelian lattice group with point group \(Q_8\). However, there are some problems on the calculations of its homological invariants by using this polycyclic presentation. The new generator \(c\) was developed among the relationship between generators in \(G_4\) but the generator \(c\) is not well defined in here. The polycyclic presentation is stated as in the following:

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This group will later be shown to be isomorphic to the new \(Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle\) with the new generator \(c\) where \(Q\) is the group (with point group \(Q_8\)) while 4 indicates the number 4th group and (6) here indicates the dimension of the group.
2. Preliminary
As mentioned in Section 1, the technique developed by Blyth and Morse [14] will be used throughout this research. The polycyclic presentation of this extension of a free abelian lattice group by finite group is shown to be consistent. The following definitions are used throughout this research.

**Definition 1.** [13] Let $F_n$ be a free group on generators $g_1, \ldots, g_n$ and $R$ be a set of relations of group $F_n$. The relations of a polycyclic presentation have the form:

\[ g_i^{e_i} = g_{i+1}^{x_{i,i+1}} \cdots g_n^{x_{i,n}} \quad \text{for } i \in I, \]

\[ g_j^{-1} g_i g_j = g_{j+1}^{y_{j,j+1}} \cdots g_n^{y_{j,n}} \quad \text{for } j < i, \]

\[ g_j g_i g_j^{-1} = g_{j+1}^{z_{j,j+1}} \cdots g_n^{z_{j,n}} \quad \text{for } j < i \text{ and } j \notin I \]

for some $I \subseteq \{1, \ldots, n\}$, certain exponents $e_i \in \mathbb{N}$ for $i \in I$ and $x_{i,j}, y_{i,j}, z_{i,j,k} \in \mathbb{Z}$ for all $i, j$ and $k$.

Blyth and Morse [14] proved that if $G$ is polycyclic, then $G \otimes G$ is polycyclic. Hence, $G \otimes G$ has a consistent polycyclic presentation. By using the above definition, the presentation is then checked to satisfy all the consistency relations. It is crucial to check the polycyclic presentation is consistent in order to compute its homological invariants later.

**Definition 2.** [13] Let $G$ be a group generated by $g_1, \ldots, g_n$ and the consistency relations in $G$ can be evaluated in the polycyclic presentation of $G$ using the collection from the left as in the following:

\[ g_k (g_j g_i) = (g_k g_j) g_i \quad \text{for } k > j > i, \]

\[ (g_j^{-1}) g_i = g_j g_i^{-1} \quad \text{for } j > i, j \in I, \]

\[ g_j (g_i^{-1}) = (g_j g_i) g_i^{-1} \quad \text{for } j > i, i \in I, \]

\[ (g_i^{-1}) g_i = g_i (g_i^{-1}) \quad \text{for } i \in I, \]

\[ g_j = (g_j g_i^{-1}) g_i \quad \text{for } j > i \text{ and } i \notin I \]

for some $I \subseteq \{1, \ldots, n\}$, $e^i \in \mathbb{N}$. Then, $G$ is said to be given by a consistent polycyclic presentation.

3. Results and Discussion
In this section, the transformation of a group in (1) into a polycyclic group is shown. By using matrix form in (1), $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ where $l_1, l_2, l_3, l_4, l_5$ and $l_6$ are its lattices in which its basis matrix is the identity matrix, this group is shown to be isomorphic to a new group which is polycyclic, namely $Q_4(6)$. By taking the same generators of $l_1, l_2, l_3, l_4, l_5$ and $l_6$ in $G_4$, whereas $a_0$ is written as $a$ and $a_1$ is as $b$, meanwhile a new generator $c$ is developed based on the relationship among the generators in the group $G_4$ therefore, $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ now is transformed into $Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$. The polycyclic presentation can now vary depending on what generator $c$ is. This is to make sure that the new group $Q_4(6)$ satisfies its consistency relations aligning with Definition 2 later. By Definition 1, the polycyclic presentation of $Q_4(6)$ is presented as in the following.

Take $\gamma : G_4 \rightarrow Q_4(6)$ such that $\gamma(a_0) = a, \gamma(a_1) = b$. Let $c = a_0 l_6^{-1}$. The mapping $\gamma$, is well defined since $\gamma$ maps the generators of $G_4$ to generators of $Q_4(6)$. Now, under this mapping $\gamma$, all relations hold in $Q_4(6)$ are constructed.
Now, \( c = a_0^{2l-1} \) implies,

\[
c = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
\]

Thus, by mapping \( \gamma(a_0) = a, \ c = a_0^{2l-1} = a^{2l-1} \).

Next, \( a_0 \) can be shown to be equal to \( c_{l_6} \).

\[
a_0^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
\]

\[
c_{l_6} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Thus, by mapping \( \gamma(a_0) = a, \ a_0^2 = a^2 = c_{l_6} \).
Next, \( a_1^2 \) gives,

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= c.
\]

Thus, by mapping \( \gamma(a_1) = b, \ a_1^2 = b^2 = c. \)

Furthermore, \( c^2 \) can be written as \( l_5^{-1}l_6^{-1}. \)

\[
c^2 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
l_5^{-1}l_6^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}^{-1}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Thus, \( c^2 = l_5^{-1}l_6^{-1}. \)
For the conjugation action of each generator, the first relation is shown below:

\[ a_1^{a_0} = a_0^{-1} a_1 a_0 \]

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ 
= \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Meanwhile, \( a_1 c^{-1} l_5^{-1} \)

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This shows that, \( a_1^{a_0} = a_1 c^{-1} l_5^{-1} \). By mapping \( \gamma(a_0) = a \), \( \gamma(a_1) = b \), \( b^a = bc^{-1} l_5^{-1} \).
Next, to show $c^{a_0} = c$,

$$c^{a_0} = a_0^{-1}ca_0$$

$$= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$= c.$$

Hence, $c^{a_0} = a_0^{-1}ca_0 = c$. By mapping $\gamma(a_0) = a$, $c^{a_0} = c^a = c$.

The next calculation shows that $c^{a_1} = c$.

$$c^{a_1} = a_1^{-1}ca_1$$

$$= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$= c.$$

Thus, $c^{a_1} = a_1^{-1}ca_1 = c$. By mapping $\gamma(a_1) = b$, $c^{a_1} = c^b = c$.

Therefore, all possible relations which are formed by conjugation between each generator and power of certain exponent have been constructed. Thus, $G_4 = \langle a_0, a_1, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ has been shown to be isomorphic to the new $Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 \rangle$ with $c = a_0^2l_6^{-1}$. Hence, by collecting all possible relations that have been constructed, the new polycyclic presentation of
Proof. $Q_4(6)$ is established as:

$$Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 | a^2 = cl_6, b^2 = c, c^2 = l_5^{-1}l_6^{-1} , b^a = bc^{-1}l_5^{-1}, c^a = c, c^b = c, l_1^a = l_3, l_1^b = l_4, l_1^c = l_6 \rangle$$

$$l_1^1 = l_4, l_1^2 = l_5^{-1}, l_2^1 = l_4, l_2^2 = l_5^{-1}, l_2^c = l_2, l_2^b = l_3, l_2^a = l_4, l_2^d = l_5^{-1}, l_3^1 = l_4, l_3^2 = l_5^{-1}, l_3^c = l_3, l_3^b = l_4, l_3^a = l_5^{-1}, l_4^1 = l_4, l_4^2 = l_5^{-1}, l_4^c = l_4, l_4^b = l_5^{-1}, l_5^1 = l_4, l_5^2 = l_5^{-1}, l_5^c = l_5, l_5^b = l_5^{-1}, l_5^a = l_6, l_6^1 = l_6, l_6^2 = l_6, l_6^c = l_6, l_6^b = l_6, l_6^a = l_6.$$  

To show that the group is polycyclic, the polycyclic presentation has to be consistent. In this section, all relations as given in (2) are shown to satisfy the five consistency relations as given in Definition 2. Hence, some calculations on checking of the consistency polycyclic presentation of the group, are presented in the following theorem.

**Theorem 1.** Let $Q_4(6)$ be an extension of a free abelian lattice group of dimension six with point group $Q_8$ and has presentation as in (2), its polycyclic presentation is found to be

$$Q_4(6) = \langle a, b, c, l_1, l_2, l_3, l_4, l_5, l_6 | a^2 = cl_6, b^2 = c, c^2 = l_5^{-1}l_6^{-1} , b^a = bc^{-1}l_5^{-1}, c^a = c, c^b = c, l_1^a = l_3, l_1^b = l_4, l_1^c = l_6 \rangle$$

Then, $Q_4(6)$ is consistent.

Proof. $Q_4(6)$ is generated by $a, b, c, l_1, l_2, l_3, l_4, l_5$ and $l_6$. Referring to Definition 2, let $g_1 = a, g_2 = b, g_3 = c, g_4 = l_1, g_5 = l_2, g_6 = l_3, g_7 = l_4, g_8 = l_5$ and $g_9 = l_6$. For the first consistency relation, i.e. $g_k(g_jg_i) = (g_kg_jg_i)$ for $k > j > i$, the following relations hold:

(i) $c(ba) = (cb)a,$  
(ii) $l_1(ch) = (l_1c)b,$  
(iii) $l_1(ca) = (l_1c)a,$  
(iv) $l_1(ba) = (l_1b)a,$  
(v) $l_2(l_1c) = (l_2l_1)c,$  
(vi) $l_2(l_1b) = (l_2l_1)b,$  
(vii) $l_2(l_1a) = (l_2l_1)a,$  
(viii) $l_2(cb) = (l_2c)b,$  
(ix) $l_2(ca) = (l_2c)a,$  
(x) $l_2(ba) = (l_2b)a,$
the following relations hold:

$$(xxx) \quad l_4(l_1b) = (l_4l_1)b, \quad (xlix) \quad l_5(l_2b) = (l_5l_2)b, \quad (lxvii) \quad l_6(l_4c) = (l_6l_4)c,$$

$$(xxxii) \quad l_4(l_1a) = (l_4l_1)a, \quad (l) \quad l_5(l_2a) = (l_5l_2)a, \quad (lxviii) \quad l_6(l_4b) = (l_6l_4)b,$$

$$(xxxiii) \quad l_4(cb) = (l_4c)b, \quad (li) \quad l_5(l_1c) = (l_5l_1)c, \quad (lxix) \quad l_6(l_4a) = (l_6l_4)a,$$

$$(xxxiv) \quad l_4(ca) = (l_4c)a, \quad (lii) \quad l_5(l_1b) = (l_5l_1)b, \quad (lxx) \quad l_6(l_3l_2) = (l_6l_3)l_2,$$

$$(xxxv) \quad l_4(ba) = (l_4b)a, \quad (liii) \quad l_5(l_1a) = (l_5l_1)a, \quad (lxxi) \quad l_6(l_3l_1) = (l_6l_3)l_1,$$

$$(xxxvi) \quad l_5(l_4l_3) = (l_5l_4)l_3, \quad (liv) \quad l_5(cb) = (l_5c)b, \quad (lxxii) \quad l_6(l_3c) = (l_6l_3)c,$$

$$(xxxvii) \quad l_5(l_4l_2) = (l_5l_4)l_2, \quad (lv) \quad l_5(ca) = (l_5c)a, \quad (lxxiii) \quad l_6(l_3b) = (l_6l_3)b,$$

$$(xxxviii) \quad l_5(l_4l_1) = (l_5l_4)l_1, \quad (lvi) \quad l_5(ba) = (l_5b)a, \quad (lxxiv) \quad l_6(l_3a) = (l_6l_3)a,$$

$$(xxxix) \quad l_5(l_4c) = (l_5l_4)c, \quad (lvii) \quad l_6(l_3l_4) = (l_6l_3)l_4, \quad (lxxv) \quad l_6(l_2l_1) = (l_6l_2)l_1,$$

$$(xl) \quad l_5(l_4b) = (l_5l_4)b, \quad (lviii) \quad l_6(l_3l_3) = (l_6l_3)l_3, \quad (lxxvi) \quad l_6(l_2c) = (l_6l_2)c,$$

$$(xli) \quad l_5(l_4a) = (l_5l_4)a, \quad (lix) \quad l_6(l_3l_2) = (l_6l_3)l_2, \quad (lxxvii) \quad l_6(l_2b) = (l_6l_2)b,$$

$$(xlii) \quad l_5(l_3l_2) = (l_5l_3)l_2, \quad (lx) \quad l_6(l_3l_1) = (l_6l_3)l_1, \quad (lxxviii) \quad l_6(l_2a) = (l_6l_2)a,$$

$$(xliii) \quad l_5(l_3l_1) = (l_5l_3)l_1, \quad (lxii) \quad l_6(l_3c) = (l_6l_3)c, \quad (lxxix) \quad l_6(l_1c) = (l_6l_1)c,$$

$$(xliv) \quad l_5(l_3c) = (l_5l_3)c, \quad (lxiii) \quad l_6(l_3b) = (l_6l_3)b, \quad (lx) \quad l_6(l_1b) = (l_6l_1)b,$$

$$(xlv) \quad l_5(l_3b) = (l_5l_3)b, \quad (lxiv) \quad l_6(l_4l_3) = (l_6l_4)l_3, \quad (lxxi) \quad l_6(l_3a) = (l_6l_3)a,$$

$$(xlvi) \quad l_5(l_3a) = (l_5l_3)a, \quad (lxv) \quad l_6(l_4l_2) = (l_6l_4)l_2, \quad (lx) \quad l_6(l_3a) = (l_6l_3)a,$$

$$(xlvii) \quad l_5(l_2l_1) = (l_5l_2)l_1, \quad (lxvi) \quad l_6(l_4l_1) = (l_6l_4)l_1, \quad (lxxii) \quad l_6(l_3b) = (l_6l_3)b,$$

$$(xlviii) \quad l_5(l_2c) = (l_5l_2)c, \quad (lxvii) \quad l_6(l_4l_1) = (l_6l_4)l_1, \quad (lxxiv) \quad l_6(ba) = (l_6b)a.$$

Hence, by the relations of $Q_4(6)$:

For (i):

$$c(ba) = cabc^{-1}l_5^{-1} = acbc^{-1}l_5^{-1} = abcc^{-1}l_5^{-1} = abl_5^{-1}.$$  

$$(cb)a = bca = bac = abc^{-1}l_5^{-1}c = abl_5^{-1}c^{-1}c = abl_5^{-1}.$$  

For the second consistency relation, i.e. $(g_j^{(c)})g_i = g_j^{(c-1)}(g_jg_i)$ for $j > i, j \in I$, the following relations hold:

(i) $b^2a = b(ba)$,  
(ii) $c^2a = c(ca)b$,  
(iii) $c^2b = c(cb)$.

By the relations of $Q_4(6)$:

For (i):

$$b^2a = ca = ac.$$  

$$b(ba) = babc^{-1}l_5^{-1} = abbc^{-1}l_5^{-1} = abcc^{-1}l_5^{-1} = abl_5^{-1}c^{-1}l_5^{-1} = ac^{-1}l_5^{-1}c^{-1}l_5^{-1} = ac^{-1}l_5^{-1}c^{-1}l_5^{-1}.$$  

The third consistency relation i.e. $g_j(g_i^{(c)}) = (g_jg_i)g_i^{(c-1)}$ for $j > i, i \in I$, is satisfied given that the following relations hold:

(i) $ba^2 = (ba)a$,  
(ii) $ca^2 = (ca)a$,  
(iii) $l_1a^2 = (l_1a)a$,  
(iv) $l_2a^2 = (l_2a)a$,  
(v) $l_3a^2 = (l_3a)a$,  
(vi) $l_4a^2 = (l_4a)a$,  
(vii) $l_5a^2 = (l_5a)a$,  
(viii) $l_6a^2 = (l_6a)a$,  
(ix) $cb^2 = (cb)b$.  

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(x) \( l_1b^2 = (l_1b)b \),
(xiv) \( l_5b^2 = (l_5b)b \),
(xviii) \( l_3c^2 = (l_3c)c \),
(xi) \( l_2b^2 = (l_2b)b \),
(xv) \( l_6b^2 = (l_6b)b \),
(xix) \( l_4c^2 = (l_4c)c \),
(xii) \( l_3b^2 = (l_3b)b \),
(xvi) \( l_1c^2 = (l_1c)c \),
(xx) \( l_5c^2 = (l_5c)c \),
(xiii) \( l_4b^2 = (l_4b)b \),
(xvii) \( l_2c^2 = (l_2c)c \),
(xxi) \( l_6c^2 = (l_6c)c \).

Based on the relations of \( Q_4(6) \);
For (i);

\[
ba^2 = bcl_6.
\]

\[
(ba)a = abc^{-1}l_5^{-1}a = abc^{-1}al_5^{-1} = abac^{-1}l_5^{-1} = abac^{-1}l_5^{-1}c^{-1}l_5^{-1} = a^2bc^{-2}l_5^{-1}l_5^{-1} = a^2bc^{-2}l_5^{-1}l_5^{-1} = ablc_6l_5^{-1} = acl_6l_5^{-1} = cbcl_6 = bcl_6.
\]

Next, the relations of \( Q_4(6) \) are shown to satisfy the fourth consistency relation by proving that \( (g_i^a)g_i = g_i(g_i^a) \) for \( i \in I \). Therefore,

(i) \( (a^2)a = aa^2 \),
(ii) \( (b^2)b = bb^2 \),
(iii) \( (c^2)c = cc^2 \).

By the relations of \( Q_4(6) \),

For (i);

\[
(a^2)a = cl_6a = cacl_6 = acl_6.
\]

\[
ba^2 = acl_6.
\]

For (ii);

\[
(b^2)b = cb = bc.
\]

\[
bb^2 = bc.
\]

For (iii);

\[
(c^2)c = l_5^{-1}l_6^{-1}c = l_5^{-1}cl_6^{-1} = cl_5^{-1}l_6^{-1}.
\]

\[
cc^2 = cl_5^{-1}l_6^{-1}.
\]

Finally, for the fifth consistency relations i.e. \( g_j = (g_jg_i^{-1})g_i \) for \( j, i \notin I \), the following relations are shown to be true.

(i) \( l_2 = (l_2l_1^{-1})l_1 \),
(vi) \( l_3 = (l_3l_2^{-1})l_2 \),
(x) \( l_4 = (l_4l_3^{-1})l_3 \),
(ii) \( l_3 = (l_3l_1^{-1})l_1 \),
(vii) \( l_4 = (l_4l_2^{-1})l_2 \),
(xi) \( l_5 = (l_5l_3^{-1})l_3 \),
(iii) \( l_4 = (l_4l_1^{-1})l_1 \),
(viii) \( l_5 = (l_5l_2^{-1})l_2 \),
(xii) \( l_6 = (l_6l_3^{-1})l_3 \),
(iv) \( l_5 = (l_5l_1^{-1})l_1 \),
(ix) \( l_6 = (l_6l_2^{-1})l_2 \),
(xiii) \( l_5 = (l_5l_4^{-1})l_4 \),
(v) \( l_6 = (l_6l_1^{-1})l_1 \),
(x) \( l_4 = (l_4l_3^{-1})l_3 \),
(xiv) \( l_6 = (l_6l_4^{-1})l_4 \),
(v) \( l_6 = (l_6l_1^{-1})l_1 \),
(x) \( l_4 = (l_4l_3^{-1})l_3 \),
(xv) \( l_6 = (l_6l_5^{-1})l_5 \).

By applying the relations of \( Q_4(6) \), it is found that:
Since the presentation of $Q_4(6)$ satisfies the consistency relations given in Definition 2, therefore $Q_4(6)$ has a consistent polycyclic presentation.

4. Conclusion

In this research, the new polycyclic presentation that satisfy its consistency relations for one of the four groups of extension of a free abelian lattice group by quaternion point group is shown. This polycyclic presentation which is consistent is later will be used in finding the homological invariants of the group.

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