Analytical formulas for elastic neutrino-fermion scattering cross sections

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In this paper, analytical formulas for elastic neutrino-fermion scattering cross sections are presented. These are evaluated in the limit where the neutrino mass is negligible compared to the incoming momentum. The fermion mass is taken into account, however, yielding a good approximation to realistic scattering reactions. The intended level of readership is for undergraduates/graduates in the fields of relativistic quantum mechanics and quantum field theory. In addition to a detailed and pedagogical derivation of the analytical expressions, numerical results are also provided for \( f = e, n, p \).

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I. INTRODUCTION

The motivation for writing a paper on analytical formulas for elastic neutrino-fermion scattering cross sections is that i) their explicit derivation is hard to find in literature in general, ii) their derivation provides very useful insight into how particle reactions are analyzed in the framework of quantum field theory and Feynman diagram analysis, and iii) that some important equations and aspects of the analysis are often left out or sparsely treated in original textbooks. This paper should be of particular interest to students looking for a detailed review of how scattering reactions are dealt with, in addition to some important observations and techniques involved in this task. Textbooks such as Mandl&Shaw [1] provide excellent introductory texts to the topic of quantum field theory, and equips the reader with many tools to handle scattering reactions. The "dirty" details of such calculations can be hard to find, however, and are highly valuable and appreciated by students trying to understand the bits and pieces of how one arrives at a given result.

This paper is organized as follows. In Sec. II the concept of a reaction cross section is briefly reviewed to set the notation. Sec. III takes on neutrino interactions in electro-weak theory, which is the necessary framework for studying neutrino-fermion scattering. To set the notation, the relevant Feynman symbols are presented in Sec. IV. The general expressions for neutrino-fermion cross sections are derived in Sec. V and a summary is given in Sec. VI. Unless specifically stated otherwise, natural units \( \hbar = c = 1 \) will be used.

II. CROSS SECTION

The differential cross section \( d\sigma \) says something about how incoming particles are scattered in space, and the likelihood of the scattering to occur at all. More precisely, since the interpretation of \( \sigma \) is the number of particles scattered into a specific final state per unit time divided on the incident flux, the expression for \( d\sigma \) is

\[
d\sigma = \frac{w V}{v_{\text{rel}}} \times \prod_f \frac{V dp'_f}{(2\pi)^3},
\]

where \( w \) is the transition probability from \( |i\rangle \) to \( |f\rangle \) per unit time, \( v_{\text{rel}} \) is the relative velocity between incident and target particles, and \( \prod_f \frac{V dp'_f}{(2\pi)^3} \) is the number of states with momentum in the interval \((p'_f, p'_f + dp'_f)\). In the special
case of two incoming particles, the differential cross section reads

$$d\sigma = \frac{1}{4E_1E_2\sin(\pi/2)}\left(\sum_i p_i - \sum_j p_j\right)\prod_k \frac{dp_k}{(2\pi)^32E_k}|\mathcal{M}|^2,$$

(2)

where \(p_i\) are the 4-momenta of the incoming particles, \(p_j\) are the 4-momenta of the outgoing particles, and \(\mathcal{M}\) is the Feynman amplitude of the process. More on this in Sec. III.

Calculations are often simplified when considering the process from the Center-of-Mass (CM) system. Here, Eq. (2) is reduced to

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2(E_1^2 + E_2)^2}\frac{1}{|\mathcal{M}|^2}.$$

(3)

For a detailed derivation of these quantities, consider for instance Ref. [1].

### III. NEUTRINO INTERACTIONS IN ELECTRO-WEAK THEORY

In the following, we shall not consider neutrino interaction with the electromagnetic field \(A_i(x)\), since the electromagnetic transition moments of the neutrino are too weak for any significant coupling to photons. The upper limit for the neutrino magnetic moment is \(\sim 10^{-10}\mu_B\), taken from the Particle Data Group [2]. When dealing with neutrino interactions it is sufficient to consider the Lagrangian densities involving the external momenta being much smaller than the vector boson masses for the low-energy limit which often occurs in realistic scattering experiments. More specifically, consider the limit of Eq. (4) is valid for all neutrino energies, but we would like to derive an expression for the low-energy limit which often occurs in realistic scattering experiments. More specifically, consider the limit of the external momenta being much smaller than the vector boson masses \(m_W\) and \(m_Z\). This means that

$$\lim_{m_W \gg k^2} \left[ -i\frac{g_{\mu\nu} - k_{\mu} k_{\nu}/m_W^2}{k^2 - m_W^2 + i\epsilon} \right] = \frac{ig_{\mu\nu}}{m_W^2}$$

$$\lim_{m_Z \gg k^2} \left[ -i\frac{g_{\mu\nu} - k_{\mu} k_{\nu}/m_Z^2}{k^2 - m_Z^2 + i\epsilon} \right] = \frac{ig_{\mu\nu}}{m_Z^2}$$

(6)

with \(g_{\mu\nu} = \text{diag}(1,-1,-1,-1)\). Making these substitutions corresponds to the interaction transformation in Fig. 2 and thus provides us with the effective interaction Lagrangian density for neutrino interactions

$$\mathcal{L}_{1 eff} = \mathcal{L}_{1, W} + \mathcal{L}_{1, Z} = -\left(\frac{g}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2} J_W^\mu J_W^\mu - \left(\frac{g}{\sqrt{2}\cos\theta_{W}}\right)^2 \frac{1}{m_Z^2} J_Z^\mu J_Z^\mu.$$
The gauge boson masses are related through the Weinberg angle $\theta_W$ by $m_W/m_Z = |\cos \theta_W|$. $u$ and $d$ quarks can also be incorporated into this model, merely by adding extra terms to the currents in Eq. (1), thus producing

$$J_{\mu}^W = \sum_i \left[ I_1^\mu (1 - \gamma_5) \nu_i + \bar{d}_i \gamma^\mu (1 - \gamma_5) u_i \right]$$

$$J_{\mu}^Z = \frac{1}{2} \sum_i \left[ |\psi_i \gamma^\mu (g_\nu^0 - g_\nu^A \gamma_5) \nu_i + \bar{\tau}_i \gamma^\mu (g_\bar{\nu}^0 - g_\bar{\nu}^A \gamma_5) \bar{d}_i \right] + |\psi_i \gamma^\mu (g_\nu^0 - g_\nu^A \gamma_5) \nu_i + \bar{\tau}_i \gamma^\mu (g_\bar{\nu}^0 - g_\bar{\nu}^A \gamma_5) \bar{d}_i|,$$

where $d_\theta = d \cos \theta_C + s \sin \theta_C$. The Cabbibo angle $\theta_C$ has been experimentally determined to $\cos \theta_C \approx 0.98$, so we shall exclude the $s$ quark part from now on and set $d_\theta = d$. The introduction of quark terms enable us to describe processes such as $\nu_e + n \rightarrow p + e$. When adding the quark terms, we have used the general expression for the neutral current found in Ref. [3], namely

$$J_{\mu}^2 = \frac{1}{2} \sum_i |\bar{\psi}_i \gamma^\mu \left[ I_2^3 (1 - \gamma_5) - 2Q_i \sin^2 \theta_W \right] \psi_i,$$

where $i = (l, \nu_l, u, d)$, while $I_2^3$ is the belonging particle isospin and $Q_i$ is the particle charge in units of $e$.

Now, the Hamiltonian density $\mathcal{H}$ is related to its Lagrangian density $\mathcal{L}$ by $\mathcal{H} = \frac{\partial}{\partial \phi} \phi^k \mathcal{L}$, with $\phi = \phi(x)$ denoting the various fields appearing in $\mathcal{L}$, i.e. $l(x), \nu_l(x)$ et c. The second quantized expansion for Dirac fields reads

$$l(x) = \sum_{s,p} \sqrt{\frac{1}{2VE_p}} \left[ a_s(p) \bar{u}_s(p) e^{-ipx} + b_s^\dagger(p) \nu_s(p) e^{ipx} \right]$$

$$\bar{l}(x) = \sum_{s,p} \sqrt{\frac{1}{2VE_p}} \left[ b_s(p) \bar{\nu}_s(p) e^{-ipx} + a_s^\dagger(p) \bar{u}_s(p) e^{ipx} \right],$$

and similar for $\nu_l(x), u(x), d(x)$. Here, $E_p$ is the energy of the lepton with momentum $p$. $\{a_s(p), a_s^\dagger(p)\}$ and $\{b_s(p), b_s^\dagger(p)\}$ are annihilation and creation operators for particles and antiparticles, respectively, while $\{u_s(p), \nu_s(p)\}$ are Dirac spinors. Note that $\bar{l}(x) \equiv l^\dagger(x) \gamma^0$.

Constructing the interaction Hamiltonian density for a process is a matter of straight-forward calculation, which is considerably simplified by making the observation that $\mathcal{L}$ does not contain any time derivatives of the fields, i.e. $\mathcal{L} = -\mathcal{H}$. We arrive at $\mathcal{H}_{\text{eff}} = \mathcal{H}_{\text{eff}}^{W} + \mathcal{H}_{\text{eff}}^{Z}$, where $\mathcal{H}_{\text{eff}}^{W} = \frac{G_F}{\sqrt{2}} J_{\mu}^{W} J_{\nu}^{W}$, $\mathcal{H}_{\text{eff}}^{Z} = \frac{4G_F}{\sqrt{2}} J_{\mu}^{Z} J_{\nu}^{Z}$. The Fermi constant is defined as $G_F = \sqrt{2}g^2/8m_W^2$.

### IV. FEYNMAN RULES

The Feynman diagram of a process is an indispensable tool for analysis of any particle reaction, in order to distinguish between different outcomes and to equip us with an elegant manner of calculating important physical quantities, such as the cross section. The Feynman amplitude can be read from the diagram with some training and quantifies the process. For our needs, we shall only give a short overview of the symbols and algebraic equivalents used in Feynman diagrams describing weak processes to set the notation. This is seen in Fig. 2.
### External Lines

| Symbol | Factor in $\mathcal{M}$ |
|--------|--------------------------|
| Initial lepton line $l^-$ or $\nu_l$ | $u_r(p)$ |
| Initial lepton line $l^+$ or $\bar{\nu}_l$ | $\bar{u}_r(p)$ |
| Final lepton line $l^-$ or $\nu_l$ | $\bar{v}_r(p)$ |
| Final lepton line $l^+$ or $\bar{\nu}_l$ | $v_r(p)$ |

### Propagators

| Symbol | Factor in $\mathcal{M}$ |
|--------|--------------------------|
| Internal $W$ boson | $\frac{i(-g_{\alpha}k_{\beta}k_{\gamma}m_Z^2)}{k^2-m_W^2+i\epsilon}$ |
| Internal $Z$ boson | $\frac{i(-g_{\alpha}k_{\beta}k_{\gamma}m_Z^2)}{k^2-m_Z^2+i\epsilon}$ |

### Charged Current Vertex

| Symbol | Factor in $\mathcal{M}$ |
|--------|--------------------------|
| Interaction between leptons $\{l, \nu_l\}$ and $W, l \in \{e, \mu, \tau\}$ | $\frac{-ig}{2\sqrt{2}}\gamma^\alpha(1-\gamma_5)$ |

### Neutral Current Vertex

| Symbol | Factor in $\mathcal{M}$ |
|--------|--------------------------|
| Interaction between particle $i$ and $Z$, $i \in \{l, \nu_l, u, d\}, l \in \{e, \mu, \tau\}$ | $\frac{-ig}{2\cos\theta_W}\gamma^\alpha(glv - g_A^i\gamma_5)$ |

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**FIG. 2:** Algebraic expressions for Feynman symbols in standard weak theory, relevant for our needs.

### V. Neutrino-Fermion Scattering

In this section, we first give attention to neutral current neutrino-fermion scattering, i.e. mediated by the $Z^0$ boson. Then, we generalize to reactions mediated by the $W$ boson. Then it is time to say something about the coupling constants appearing in the expressions for the total cross section.
A. Neutral current scattering

To begin with, we shall consider $\nu_l f$ ($l \neq f$) scattering. This process is mediated only by the $Z^0$ boson, if we disregard the Higgs boson as a gauge particle. This is justified by the fact that the Feynman amplitude corresponding to the scattering process mediated by $H$ is of $O(m_\nu m_f/m_H^2)$ compared to the $Z^0$ diagram. The only relevant lowest order Feynman diagram for $\nu_l f$ ($l \neq f$) scattering is shown in Fig. 3.

![Fig. 3: Lowest contributing order Feynman diagrams for $\nu_l + f \rightarrow \nu_l + f$.](image)

The Feynman amplitude is given by

$$\mathcal{M} = \frac{ig}{2\cos\theta_W} \gamma^\alpha (g_{\nu}^{\mu}\gamma_5 - g_{A}^{\mu}\gamma_5) u_{\nu_l}(1)i \left( -\frac{g_{\alpha\beta} + k_{Z\alpha}k_{Z\beta}/m_Z^2}{k^2_Z - m_Z^2 + i\varepsilon} \right) u_{\nu_l}(1) \times \frac{ig\gamma^\beta}{2\cos\theta_W} (g_{f}^{\mu} - g_{A}^{\mu}\gamma_5) u_f(2) $$

Observe that we have shortened down $u_s(p_i)$ to $u(i)$.

These expressions can be simplified a great deal, when making some considerations. It is reasonable to expect $m_W^2 \gg k_W^2$ when using $m_W = 80.4 \text{ GeV}$ for the intermediate vector boson $W^\pm$. The same inequality goes for $m_Z = 91.2 \text{ GeV}$ and $m_H \geq 112 \text{ GeV}$. By applying this low-energy limit, we regain the effective propagator from Sec. III, arriving at

$$|\mathcal{M}|^2 = \frac{2iG_F}{\sqrt{2}} |\pi_{\nu_l}(1')| \gamma^\alpha (g_{\nu}^{\mu}\gamma_5 - g_{A}^{\mu}\gamma_5) u_{\nu_l}(1) \pi_f(2') |\gamma_\alpha (g_{f}^{\mu} - g_{A}^{\mu}\gamma_5)| u_f(2) .$$

We now pursue the absolute square of the Feynman amplitude, $|\mathcal{M}|^2$, which is part of the differential cross section from Sec. III. In a perfect world, the polarization, i.e. spins, of the reacting particles would be known. Unfortunately, this is the exception rather than the rule. If one does not know the initial and final spins of the neutrinos and electrons, this must be compensated for by averaging over the incoming spin states and adding the final states. This normally corresponds to an averaging factor $1/2$ for each incoming particle.

Neutrinos, however, are special cases. Although they are massive, their mass is so small that it is ordinarily set to zero in the standard model. This means that there is only one helicity state for the neutrino, since it consequently travels with the speed of light and never can be overcome in any Lorentz frame. Thus, there is only one spin state to consider for the neutrinos which leads to the substitution

$$|\mathcal{M}|^2 \rightarrow \frac{1}{2} \sum_{s_i,s_f} |\mathcal{M}|^2$$

in order to obtain the unpolarized cross-section. By exploiting the fact that $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^\dagger = \gamma^0\gamma^\mu\gamma^0$, the problem is attacked. Inserting Eq. (10) gives
From Eq. (3) we see that the differential cross-section in the CM system must be

\[ \frac{1}{2} \sum_{s,s'} |\mathcal{M}|^2 = \frac{1}{2} \sum_{s,s'} \left[ \frac{-2iG_F}{\sqrt{2}} \overline{u}_{\nu f}(1') \gamma^\alpha (g_{\nu f}^V - g_{\nu A}^V) u_{\nu f}(1) \overline{\pi}_f(2') \gamma_\alpha (g_{\nu f}^f - g_{\nu A}^f) u_f(2) \right] \]

\times \left[ \frac{2iG_F}{\sqrt{2}} \overline{u}_f(2) \gamma_\beta (g_{\nu f}^f - g_{\nu A}^f) u_f(2) \overline{u}_{\nu f}(1) \gamma_\beta (g_{\nu f}^V - g_{\nu A}^V) u_{\nu f}(1') \right] \]

We stride on by recalling the completeness relations for the Dirac spinors \( u \) and \( v \)

\[ \sum_s u_{\alpha s}(p,s) \overline{u}_{\beta s}(p,s) = (\bar{p} + m_t)_{\alpha \beta}, \quad \sum_s v_{\alpha s}(p,s) \overline{v}_{\beta s}(p,s) = (\bar{p} - m_t)_{\alpha \beta}. \]

Attaching indices on spinors and matrices provides us with

\[ \frac{1}{2} \sum_{s,s'} |\mathcal{M}|^2 = G_F^2 \sum_{s,s'} \left[ \overline{u}_{\nu f}(1') \gamma^\alpha (g_{\nu f}^V - g_{\nu A}^V) u_{\nu f}(1) \overline{\pi}_f(2') \gamma_\alpha (g_{\nu f}^f - g_{\nu A}^f) u_f(2) \right] \]

\times \left[ \overline{\pi}_f(2) \gamma_\beta (g_{\nu f}^f - g_{\nu A}^f) u_f(2) \overline{u}_{\nu f}(1) \gamma_\beta (g_{\nu f}^V - g_{\nu A}^V) u_{\nu f}(1') \right] \]

\[ = G_F^2 \text{Tr} \left\{ (\bar{p}_1 + m_{\nu f}) \gamma^\alpha (g_{\nu f}^V - g_{\nu A}^V) (\bar{p}_1 + m_{\nu f}) \gamma_\beta (g_{\nu f}^f - g_{\nu A}^f) \right\} \]

\times \text{Tr} \left\{ (\bar{p}_2 + m_f) \gamma_\alpha (g_{\nu f}^f - g_{\nu A}^f) (\bar{p}_2 + m_f) \gamma_\beta (g_{\nu f}^V - g_{\nu A}^V) \right\}. \quad (11) \]

We designate

\[ \circled{1} = \text{Tr} \left\{ (\bar{p}_1 + m_{\nu f}) \gamma^\alpha (g_{\nu f}^V - g_{\nu A}^V) (\bar{p}_1 + m_{\nu f}) \gamma_\beta (g_{\nu f}^f - g_{\nu A}^f) \right\} \]

\[ \circled{2} = \text{Tr} \left\{ (\bar{p}_2 + m_f) \gamma_\alpha (g_{\nu f}^f - g_{\nu A}^f) (\bar{p}_2 + m_f) \gamma_\beta (g_{\nu f}^V - g_{\nu A}^V) \right\}. \quad (12) \]

and consider each of the traces separately. Using the general relations for traces

\[ \text{Tr}\{A + B + \ldots\} = \text{Tr}\{A\} + \text{Tr}\{B\} + \ldots \]

\[ \text{Tr}\{\gamma^5\} = \text{Tr}\{\gamma^5\gamma^5\} = \text{Tr}\{\gamma^5\gamma^\alpha\gamma_\alpha\} = \text{Tr}\{\gamma^5\gamma^\alpha\gamma_\beta\gamma_\gamma\} = 0 \]

\[ \text{Tr}\{\gamma^\alpha\gamma^\beta\gamma_\gamma\} = -4i\epsilon^{\alpha\beta\gamma} \]

\[ \text{Tr}\{\gamma^\alpha\gamma^\beta\gamma^\gamma\} \neq 0 \text{ if } \gamma^\alpha\gamma^\beta\gamma^\gamma\text{ is an odd number of } \gamma\text{-matrices}, \quad (13) \]

where \( \epsilon^{\alpha\beta\gamma} \) is a completely antisymmetric tensor, we obtain

\[ \circled{1} = 8p_1 u_{\mu_\nu} \left[ ig_{\nu}^{\gamma_5} g_{\mu}^{\gamma_5} \epsilon^{\mu\alpha\beta} + \frac{1}{2} (g_{\mu}^{\gamma_5})^2 + (g_{\nu}^{\gamma_5})^2 \right] \eta_{\eta^{\mu_\nu} \eta^{\mu_\nu} + \eta^{\mu_\nu} \eta^{\mu_\nu} - \eta^{\mu_\nu} \eta^{\mu_\nu}} \]

\[ \circled{2} = 8p_2 u_\mu \left[ ig_{\nu}^{\gamma_5} g_{\mu}^{\gamma_5} \epsilon^{\mu\alpha\beta} + \frac{1}{2} (g_{\mu_\nu}^{\gamma_5})^2 + (g_{\nu_\nu}^{\gamma_5})^2 \right] \eta_{\eta^{\mu_\nu} \eta^{\mu_\nu} + \eta^{\mu_\nu} \eta^{\mu_\nu} - \eta^{\mu_\nu} \eta^{\mu_\nu}} + \frac{1}{2} m_f^2 (g_{\nu}^{\gamma_5} - g_{\nu}^{\gamma_5}) \eta_{\eta^{\mu_\nu} \eta^{\mu_\nu} + \eta^{\mu_\nu} \eta^{\mu_\nu} - \eta^{\mu_\nu} \eta^{\mu_\nu}} \]

(14)

in the limit \( m_{\nu f} \to 0 \). From Eq. (13) we see that the differential cross-section in the CM system must be

\[ \left( \frac{d\sigma}{d\Omega}_{\text{CM}} \right) = \frac{X}{64\pi^2 (E_1 + E_2)^2} \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1'|}. \quad (15) \]

where

\[ X = \frac{1}{2} \sum_{s,s'} |\mathcal{M}|^2 = G_F^2 \circled{1} \times \circled{2}. \]
The product of the two traces is

\[ \langle 1 \rangle \times \langle 2 \rangle = 128 g_{l}^{f} g_{V}^{V} g_{A}^{A} \left[ (p_{1} p_{2}) (p_{1}' p_{2}')(p_{1} p_{2}) - (p_{1}' p_{2}) (p_{1} p_{2}') \right] \\
+ 32 [(g_{V}^{f})^{2} + (g_{A}^{f})^{2}] [(g_{V}^{A})^{2} + (g_{A}^{A})^{2}] [(p_{1} p_{2}) (p_{1}' p_{2}')(p_{1} p_{2}) - (p_{1}' p_{2}) (p_{1} p_{2}')]. \tag{16} \]

The situation now looks like Fig. 4 in the CM frame with 4-momenta

\[ p_{1} = (E, 0, 0, E), \]
\[ p_{2} = (\sqrt{E^{2} + m_{f}^{2}}, 0, 0, -E), \]
\[ p_{1}' = E(1, \cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta), \]
\[ p_{2}' = (\sqrt{E^{2} + m_{f}^{2}}, -E \cos \phi \sin \theta, -E \sin \phi \sin \theta, -E \cos \theta), \tag{17} \]

where \( E \equiv E_{\text{CM}} \) is the CM kinetic energy of the neutrino.

![Diagram](https://example.com/diagram.png)

**FIG. 4:** \( \nu \ell f \ (l \neq f) \) scattering in the CM frame.

Since all 4-momenta now are known, insertion into Eq. (16) gives

\[ \langle 1 \rangle \times \langle 2 \rangle = 128 g_{l}^{f} g_{V}^{V} g_{A}^{A} \left[ (E \sqrt{E^{2} + m_{f}^{2}} + E^{2})^{2} - (E \sqrt{E^{2} + m_{f}^{2}} + E^{2} \cos \theta)^{2} \right] \\
+ 32 [(g_{V}^{f})^{2} + (g_{A}^{f})^{2}] [(g_{V}^{A})^{2} + (g_{A}^{A})^{2}] \left[ (E \sqrt{E^{2} + m_{f}^{2}} + E^{2})^{2} + (E \sqrt{E^{2} + m_{f}^{2}} + E^{2} \cos \theta)^{2} \right. \\
\left. - 32 [(g_{V}^{f})^{2} - (g_{A}^{f})^{2}] [(g_{V}^{A})^{2} + (g_{A}^{A})^{2}] E^{2} (1 - \cos \theta) \right]. \tag{18} \]

The total cross section for \( \nu \ell f \ (l \neq f) \) scattering is now within our grasp, and reads

\[ \sigma[\nu \ell f \ (l \neq f)] = \frac{G_{F}^{2}}{32\pi(E + \sqrt{E^{2} + m_{f}^{2}})^{2}} \int_{0}^{\pi} \langle 1 \rangle \times \langle 2 \rangle \sin \theta d\theta. \tag{19} \]

From Eq. (18), it is clear that we must evaluate three types of integrals, namely

\[ \int_{0}^{\pi} \sin \theta \ d\theta = 2, \quad \int_{0}^{\pi} \cos \theta \sin \theta \ d\theta = 0, \quad \int_{0}^{\pi} \cos^{2} \theta \sin \theta \ d\theta = \frac{2}{3}. \tag{20} \]

Note that \( s = (p_{1} + p_{2})^{2} = (E + \sqrt{E^{2} + m_{f}^{2}})^{2}. \) At this point, we make a little “cheat” and reveal the neutrino coupling constants \( g_{V}^{f} \) and \( g_{A}^{A} \) to be 1/2. We shall remedy this in Sec. VII C. Inserting our integrals Eq. (20) reduces Eq. (18) to

\[ \sigma[\nu \ell f \ (f \neq l)] = \frac{G_{F}^{2}(s - m_{f}^{2})^{2}}{4\pi s} \left[ (g_{V}^{f} + g_{V}^{A})^{2} + (g_{V}^{f} - g_{A}^{A})^{2} \left[ \frac{m_{f}^{2}}{s} + \frac{(s - m_{f}^{2})^{2}}{3s^{2}} \right] - [(g_{V}^{f})^{2} - (g_{A}^{A})^{2}] \frac{m_{f}^{2}}{s} \right]. \tag{21} \]
The cross section for antineutrino-fermion scattering, i.e. $\sigma_{l}f(f \neq l)$, is obtained from the following argumentation. In the limit $m_{\nu} \to 0, l = e, \mu, \tau$, neutrinos are always left-handed while antineutrinos will be right-handed. Thus, only a parity transformation $P$ on the neutrino vertex part of the Feynman amplitude Eq. (12) is required to obtain $\sigma_{l}f^{*}(l \neq l')$ scattering. Now, a scalar product of two axial vectors is invariant under a parity transformation, and so is the scalar product of the vector quantities as well. Recall that axial vectors $y^{\mu} = (y^{0}, y)$ transform as $y^{\mu} \xrightarrow{P} \bar{y}^{\mu} = (-y^{0}, y)$ under parity, while vector quantities $z^{\mu} = (z^{0}, z)$ transform as $z^{\mu} \xrightarrow{P} \bar{z}^{\mu} = (z^{0}, -z)$. Thus, it follows that a scalar product of one axial and vector quantity is not invariant under parity transformations, since

$$y^{\mu} z_{\mu} \xrightarrow{P} \bar{y}^{\mu} \bar{z}_{\mu} = -y^{0}z_{0} + yz = -y^{\mu}z_{\mu}$$

(22)

As a consequence, all mixed terms $g_{V}^{0} g_{A}^{l}$ will change sign. This means that the cross section for antineutrino-fermion scattering is found by making the substitutions

$$\sigma[\sigma_{l}f(l \neq f)] = \lim_{g_{V}^{0} g_{A}^{l} \to (-g_{V}^{0} g_{A}^{l})} \sigma[f(l \neq f)].$$

(23)

For $\sigma_{l}f^{*}(l \neq f)$ scattering, $g_{V}^{0} g_{A}^{l}$ will also change sign, yielding

$$\sigma[\sigma_{l}f^{*}(l \neq f)] = \sigma[f(l \neq f)].$$

(24)

**B. Charged current scattering**

Having derived the general expression for elastic neutrino-fermion scattering processes mediated by the $Z^{0}$ boson, we can now include the case $f = l$. This process is mediated by the $W$ boson in addition to $Z^{0}$, thus providing Feynman diagrams as shown in Fig. 5.

![Feynman diagrams for $\nu_{l} + l \rightarrow \nu_{l} + l$.](image)

The total Feynman amplitude in the low-energy limit $k^{2} \ll \{m_{Z}^{2}, m_{W}^{2}\}$ is now $M = M_{Z} + M_{W}$, where

$$M_{Z} = \frac{-2iG_{F}}{\sqrt{2}} u_{\nu_{l}}(1')[\gamma^{\alpha}(g_{V}^{0} - g_{A}^{l} \gamma_{5})]u_{\nu_{l}}(1)\overline{\nu}_{\nu_{l}}(2')[\gamma_{\alpha}(g_{V}^{l} - g_{A}^{l} \gamma_{5})]u_{l}(2),$$

$$M_{W} = \frac{-iG_{F}}{\sqrt{2}} \overline{\nu}_{l}(2')[\gamma^{\alpha}(1 - \gamma_{5})]u_{\nu_{l}}(1)\overline{\nu}_{\nu_{l}}(1')[\gamma_{\alpha}(1 - \gamma_{5})]u_{l}(2).$$

(25)

By using the Fierz identity

$$\overline{\nu}_{1}[\gamma^{\alpha}(1 - \gamma_{5})]u_{2}\overline{\nu}_{3}[\gamma_{\alpha}(1 - \gamma_{5})]u_{4} = \overline{\nu}_{1}[\gamma^{\alpha}(1 - \gamma_{5})]u_{3}\overline{\nu}_{4}[\gamma_{\alpha}(1 - \gamma_{5})]u_{2}$$

(26)

for arbitrary spinors $u_{i}$, $M_{W}$ obtains the same sequence of spinors as $M_{Z}$. In addition, we take advantage of the fact that $g_{V}^{0} = g_{A}^{l} = 1/2, l = e, \mu, \tau$. This leaves us with

$$M = \frac{-2iG_{F}}{\sqrt{2}} \overline{\nu}_{\nu_{l}}(1')[\gamma^{\alpha}(g_{V}^{0} - g_{A}^{l} \gamma_{5})]u_{\nu_{l}}(1)\overline{\nu}_{\nu_{l}}(2')[\gamma_{\alpha}(g_{V}^{l} + 1) - (g_{A}^{l} + 1) \gamma_{5}]u_{l}(2).$$

(27)
This is exactly the amplitude Eq. (10) with the replacements $g^V_f \rightarrow g^V_l + 1$, $g^A_A \rightarrow g^A_A + 1$. Therefore, we are immediately able to write down the total cross section for elastic $\nu_l l$ scattering, namely

$$\sigma[\nu_l l] = \lim_{s'_V \rightarrow g^V_l + 1 \atop s'_A \rightarrow g^A_A + 1} \sigma[\nu_l f (l \neq f)].$$

(28)

C. Coupling constants

The derivation of the coupling constants starts from the condition of U(1) and SU(2) gauge invariance for the standard electro-weak Lagrangian. Although an interesting exercise in itself, it is outside the scope of this paper, and we simply state the results in Tab. I.

| Particle | Axial coupling $g_A$ | Vector coupling $g_V$ |
|----------|----------------------|-----------------------|
| $\nu_l$  | $-1/2$               | $2\sin^2 \theta_W - 1/2$ |
| $\nu_l$  | $1/2$                | $1/2$                 |
| $\nu_l$  | $1/2$                | $1/2 - 4\sin^2 \theta_W/3$ |
| $u$      | $-1/2$               | $2\sin^2 \theta_W/3 - 1/2$ |

It should be mentioned that our factor $g_A^P = 2g_A^u + g_A^d$ is not exactly equal to 1/2. This is due to the axial current anomaly, which states that the conservation law for the axial current is broken. Ref. [4] gives a proper treatment of this topic. Neutron beta-decay experiments have determined the value of axial vector coupling constant for protons to $g_A^P \simeq 0.635$ (see Ref. [2]).

VI. SUMMARY

We have found that the elastic cross section for $\nu_l f$ scattering, where $l = e, \mu, \tau$ and $f \neq l$ is a fermion with vector coupling strength $g^V_f$, and axial coupling strength $g^A_A$, is given by

$$\sigma[\nu_l f (f \neq l)] = \frac{G^2_F(s - m_f^2)^2}{4\pi s} \left[ (g^V_f + g^A_A)^2 + (g^V_f - g^A_A)^2 \frac{m_f^2}{s} + \frac{(s - m_f^2)^2}{3s^2} \right] - [(g^V_f)^2 - (g^A_A)^2] \frac{m_f^2}{s},$$

(29)

where $s = (E_{CM} + \sqrt{E_{CM}^2 + m_f^2})^2 = 2E_{LAB}m_f + m_f^2$. For charged current $\nu_l l$ scattering, the cross section is obtained through

$$\sigma[\nu_l l] = \lim_{s'_V \rightarrow g^V_l + 1 \atop s'_A \rightarrow g^A_A + 1} \sigma[\nu_l f (f \neq l)].$$

(30)

Two specific limits of Eq. (29) are of particular interest. By taking $m_f \rightarrow 0$, the resulting cross section corresponds to elastic neutrino scattering on light leptons. This approximation is very good for $f = \{e, \mu\}$. In this case, Eq. (29) reduces to

$$\lim_{m_f \rightarrow 0} \sigma[\nu_l f (f \neq l)] = \frac{G^2_Fs}{3\pi} \left[ g^V_f + g^A_A + g_V g_A \right],$$

(31)
where we have dropped the superscript \( f \) for the fermions. Another interesting scenario is the limit \( E_{\text{LAB}}/m_f \ll 1 \). The physical interpretation of such a limit is neutrino scattering on heavy fermions, \( e.g. \) protons, where the resulting recoil energy of the fermion is small. One finds that

\[
\lim_{E_{\text{LAB}}/m_f \ll 1} \sigma[\nu f (f \neq l)] = \frac{G_F^2 E_{\text{LAB}}^2}{\pi} \left[ g_V^2 + 3g_A^2 \right].
\]  

(32)

This result agrees with theoretical predictions and experimental data (see \( e.g. \) Ref. [6]). Fig. 6 contains a plot of Eq. (21) for the most common fermion scattering components on Earth, \( f = e, n, p \). It is also instructive to consider the mean free path length \( l_{\text{mfp}} \) for the neutrinos in a medium. This quantity is given as

\[
l_{\text{mfp}} = \sum_f \frac{1}{N_f \sigma_f},
\]

where \( N_f \) is the number density of scattering component \( f \). Taking \( f = e, n, p \), the resulting mean free path length is shown in Fig. 7 for varying mass densities \( \rho = \sum_f N_f m_f \), assuming an electrically neutral medium with \( N_f = N \).

\[ \text{FIG. 6: The elastic neutrino-fermion scattering cross section Eq. (21) for } f = e, n, p \text{ as a function of the CM neutrino kinetic energy } E. \]

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FIG. 7: Log-plot of mean free path length $l_{\text{mfp}}$ in meters as a function of CM neutrino kinetic energy $E$ for various mass densities.