Quantum Chaos in Vibrational Nuclei: 
the Green Function Approach

V.R. Manfredi\textsuperscript{(a)}, M. Rosa–Clot\textsuperscript{(b)(c)} 
L. Salasnich\textsuperscript{(d)} and S. Taddei\textsuperscript{(b)(c)}

\textsuperscript{(a)} Dipartimento di Fisica “G. Galilei” dell’Università di Padova, 
INFN, Sezione di Padova, 
Via Marzolo 8, I–35131 Padova, Italy 
Interdisciplinary Laboratory, SISSA, 
Strada Costiera 11, I–34014 Trieste, Italy

\textsuperscript{(b)} Dipartimento di Fisica dell’Università di Firenze, 
Largo E. Fermi 2, I–50125 Firenze, Italy

\textsuperscript{(c)} INFN, Sezione di Firenze, 
Largo E. Fermi 2, I–50125 Firenze, Italy

\textsuperscript{(d)} Departamento de Fisica Atomica, Molecular y Nuclear 
Facultad de Ciencias Fisicas, Universidad Complutense de Madrid, 
Ciudad Universitaria, E–28040 Madrid, Spain

\textsuperscript{1}This work has been partially supported by the Ministero dell’Università e della Ricerca Scientifica e Tecnologica (MURST).
Abstract

We show that the collective vibrational model of atomic nuclei displays quantum chaos. To avoid the problems related to the tunneling effects, a Green function deterministic numerical method has been used to evaluate the energy levels.
1. Introduction

In the last years many studies have been devoted to the properties of quantal systems which are chaotic in the classical limit [1].

In quantum mechanics, given the impossibility of defining the trajectories, classical concepts and methods cannot be applied. Nevertheless, many efforts have been made to establish the features of quantum systems which reflect the qualitative difference in the behaviour of their classical counterparts. Many schematic models have shown that this difference reveals itself in the properties of fluctuations in eigenvalue sequences. The spectral statistics for the systems with underlying classical chaotic behaviour agree with the predictions of the random matrix theory. By contrast, quantum analogs of classically integrable systems display the characteristics of Poisson distribution [2,3].

In atomic nuclei, the experimental data of nuclear spectroscopy suggest regular states near the yrast line and chaotic states near the neutron emission threshold, but the coexistence of regular, chaotic and collective states is not yet well understood [4,5].

In a previous paper [6] we studied the transition from order to chaos in the nuclear roto–vibrational model [7,8], while in this work we analyze in greater detail the numerical technique to obtain the energy levels and the effects of the order–chaos transition in the spectral statistics for the collective vibrational motion.
2. The model

The two–dimensional model which describes the collective vibrational motion of atomic nuclei was introduced by Bohr and Mottelson [7] and developed by Eisenberg and Greiner [8]. The Hamiltonian is given by

\[ H = \frac{1}{2} B (\ddot{x}_1^2 + 2 \dot{x}_2^2) + V(x_1, x_2), \]

(1)

where

\[ V(x_1, x_2) = \frac{1}{2} C_2 (x_1^2 + 2x_2^2) + \sqrt{\frac{2}{35}} C_3 x_1 (6x_2^2 - x_1^2) + \frac{1}{5} C_4 (x_1^2 + 2x_2^2)^2 + V_0. \]

(2)

The variables \( x_1 \) and \( x_2 \) are connected to the deformation \( \beta \) and asymmetry \( \gamma \) by the standard relations [9]

\[ x_1 = \beta \cos \gamma, \quad x_2 = \frac{\beta}{\sqrt{2}} \sin \gamma. \]

(3)

As discussed in [8], the presence of bound states in atomic nuclei leads to a value of \( C_4 > 0 \), whereas for \( C_3 \) a positive value corresponds to a prolate shape, a negative value to an oblate shape. Similarly \( C_2 \) may also be either positive or negative.

The shape of the nuclear potential \( V(x_1, x_2) \) is a function of \( C_2 \) and \( \chi = C_3^2/(C_2^2C_4) \). For \( C_2 > 0 \), and \( 0 < \chi < 56/9 \) the nucleus is spherical; for \( 56/9 < \chi < 7 \) the nucleus is spherical in the ground state (g.s. spherical) and deformed in the excited states (e.s. deformed); for \( \chi > 7 \) it is g.s. deformed and e.s. spherical; for \( C_2 < 0 \) it is g.s. deformed and \( \gamma \)-unstable in the excited states.

The chaotic behaviour of this model has been studied analytically by the authors of [9] using the criterion of negative curvature of the potential.
energy. However, this criterion has been demonstrated to be inaccurate in [10]. The authors of [9] have also obtained the energy levels by quantizing the Hamiltonian (1) for g.s. spherical nuclei. For g.s. deformed nuclei, like rare-earth, however, it is not simple to obtain the correct energy levels because the potential energy is an asymmetric triple well.

Table 1 shows the parameters of the nuclear potential (2), for the rare-earth $^{160}$Gd and $^{166}$Er according to [8]. The potential energy for these nuclei is plotted in Figure 1.

3. The GFND method

To obtain the energy levels, we use the Green function numerical diagonalization method (GFND) [11–14], whose starting point is the integral Schrödinger equation

$$\int \int K(x_1, x_2; y_1, y_2; \varepsilon) \psi_n(y_1, y_2) dy_1 dy_2 = e^{-\varepsilon E_n} \psi_n(x_1, x_2).$$  \hspace{1cm} (4)

$K(x_1, x_2; y_1, y_2; \varepsilon)$ is the Euclidean short time propagator and $\psi_n(x_1, x_2)$ are the Hamiltonian eigenfunctions. This equation can be approximated using a numerical integration rule which gives

$$\sum_{h=1}^{N} \sum_{k=1}^{N} w_{1,h} w_{2,k} K_{ijhk}^{\varepsilon} \psi_{hk}^{n} \simeq e^{-\varepsilon E_n} \psi_{ij}^{n}. \hspace{1cm} (5)$$

$\psi_{ij}^{n} \equiv \psi_n(x_{1,i}, x_{2,j}), K_{ijhk}^{\varepsilon} \equiv K(x_{1,i}, x_{2,j}, x_{1,h}, x_{2,k}; \varepsilon)$, $w_{1,i}, w_{2,i}$ are the weights associated with the integration rule (a good choice is the trapezoidal one).

The intervals of integration in Eq. (4) go from $-\infty$ to $+\infty$, but in Eq. (5) we take finite matrices and consequently finite intervals $L_1$ and $L_2$. This corresponds to confining the system in a “box” of sides $L_1$ and $L_2$. However, if
the intervals are large enough, this gives negligible corrections to the energies and to the wave functions of the bound states. Therefore, by diagonalizing the matrix $w_{1,\ell}w_{2,k} K_{ij\ell k}$, we obtain the energies and the wave functions directly.

The Euclidean short time propagator can be written in the form (we use units $B = \bar{\hbar} = 1$):

$$K(x_1, x_2, y_1, y_2; \varepsilon) = \frac{1}{2\pi \varepsilon} \exp \left\{ -\frac{1}{2\varepsilon} [(x_1 - y_1)^2 + 2(x_2 - y_2)^2] - f(x_1, x_2, y_1, y_2; \varepsilon) \right\},$$

(6)

where the first term in the exponential corresponds to the kinetic part of the Hamiltonian and the function $f(x_1, x_2, y_1, y_2; \varepsilon)$ is the potential term. Its explicit expression depends on the prescription chosen; for example the last point rule $f(x_1, x_2, y_1, y_2; \varepsilon) = \varepsilon V(x_1, x_2)$ gives a propagator which is correct up to $O(\varepsilon)$ only, while the symmetric expression $\varepsilon^2 \left[ V(x_1, x_2) + V(y_1, y_2) \right]$ is correct up to $O(\varepsilon^2)$. Moreover, a systematic expansion of the short time propagator in $\varepsilon$, $\Delta_1 = x_1 - y_1$ and $\Delta_2 = x_2 - y_2$ is also possible [13]. This expansion gives rise to an asymptotic series, which, if used within the GFND method with non–singular potentials, allows one to obtain high numerical accuracy. There follows a brief sketch of the expansion method.

Since the potential term of $K(x_1, x_2, y_1, y_2; \varepsilon)$ is weighed by the Gaussian function: $\exp \left\{ -\frac{1}{2\varepsilon} [(x_1 - y_1)^2 + 2(x_2 - y_2)^2] \right\}$, the difference $\Delta_i$ can be considered of the order $\sqrt{\varepsilon}$. As a consequence, we can expand in $\Delta_1$ and $\Delta_2$ as well as in $\varepsilon$. Therefore, following Ref. [13], let us write the short time propagator in the form

$$K(x_1, x_2, y_1, y_2; \varepsilon) = \frac{1}{2\pi \varepsilon} \exp \left\{ -\frac{1}{2\varepsilon} [(x_1 - y_1)^2 + 2(x_2 - y_2)^2] \right\}$$
\[ -\sum_{\nu=0}^{N} \sum_{\mu=0}^{2(N-\nu)} \sum_{i_1=1}^{2} \cdots \sum_{i_\mu=1}^{2} g_{\mu\nu\cdots i_\mu}^{i_1,\cdots,i_\mu}(y_1, y_2) (x_{i_1} - y_{i_1}) \cdots (x_{i_\mu} - y_{i_\mu}) \varepsilon^\nu \].

(7)

By expanding up to \( O(\varepsilon^4) \) and requiring that \( K(x_1, x_2, y_1, y_2; \varepsilon) \) satisfies the Schrödinger equation, we obtain the coefficients \( g_{\mu\nu\cdots i_\mu}^{i_1,\cdots,i_\mu} \) (see Appendix 1).

The GFND gives accurate results and is more precise than techniques based on the direct diagonalization of the Hamiltonian, especially in the case of tunneling problems (for a deeper discussion of this point see [11–13]). Therefore, it is an appropriate method of studying the potential given in Eq. (2), where we have small tunneling effects between wells.

4. Numerical results

As is well known, the classical global instability of a system is appropriately studied using Poincaré sections. Regular regions are characterised by sets of invariant intersection points; chaotic regions by points which are distributed irregularly [15].

From the quantal point of view, the energy spectrum obtained using the GFND, has quasi–degenerate levels, due to the presence of three similar potential wells, two of which are identical. Therefore, it would be necessary to separate the energy levels into three classes, corresponding to the states localized in each well. For our purpose, however, we need only take one of the two double degenerate levels, which are easily identified. Thus, we obtain approximately one hundred energy levels, which are enough for statistical significance. The energy spectrum thus obtained has been mapped into one with a quasi-uniform level density, by performing the unfolding procedure.
described in detail in reference [16]. Then, the distribution $P(s)$ of spacings between adjacent levels and the spectral rigidity $\Delta_3(L)$ [17,18] has been calculated. These spectral statistics are compared to Poisson statistics $P(s) = e^{-s}$ and $\Delta_3(L) = L/15$ of integrable systems, and to Gaussian Orthogonal Ensemble (GOE) statistics $P(s) = \frac{\pi}{2} se^{-s^2/\pi} / \pi^{1/4}$ and $\Delta_3(L) = \pi^{-2} \ln (L) - 0.0007$ of chaotic systems.

Fig. 2 shows that in $^{160}$Gd, which has a high saddle energy (5.56 MeV), there is coexistence of regular and chaotic motion for energies below the saddle energy. In this region, the system has only a few energy levels and so we cannot perform a good statistical study of quantum levels.

Fig. 3 and Fig. 4 show that in $^{160}$Gd above the saddle energy, at about 8 MeV, there is chaos. For higher energies, at about 18 MeV, there is a quasi–regular behaviour. The non–universal behaviour of $\Delta_3(L)$ for high values of $L$, not predicted by GOE, has been explained by Berry [19] using semiclassical quantization.

Fig. 5 and Fig. 6 show that in $^{166}$Er, which has a low saddle energy (1.89 MeV), for energies above the saddle energy, at about 4 MeV, there is prevalently chaotic behaviour; for higher energies, at about 15 MeV, there is a predominance of regular classical trajectories.

Conclusions

In conclusion, the collective vibrational behaviour of the above nuclei displays a quasi–chaos $\rightarrow$ quasi–order transition as a function of the energy. The mixed behaviour at very low energies cannot be easily shown by spectral
statistics because only a few energy levels are present.

For the sake of completeness we observe that other phenomenological collective models have been recently proposed, see for example [20] and references therein.

Acknowledgments

L.S. has been supported by a Fellowship from the University of Padova and is grateful to the "Ing. Aldo Gini" Foundation of Padova for partial support. V.R.M is very grateful to Prof. S. Fantoni, Director of the Interdisciplinary Laboratory, SISSA (Trieste), for his kind hospitality in his laboratory, where the final version of this paper was written.
Appendix 1

The coefficients $g_{i_1\ldots i_\mu}$ of Eq. (7) are given by (the $y_1$ and $y_2$ dependence in $g$ and $V$ is understood)

\[
\begin{align*}
g_{01} &= V^{(0,0)} \\
g_{11}^{i_1} &= \frac{1}{2} \frac{\partial V}{\partial x_{i_1}} \\
g_{21}^{i_1 i_2} &= \frac{1}{6} \frac{\partial^2 V}{\partial x_{i_1} \partial x_{i_2}} \\
g_{31}^{i_1 i_2 i_3} &= \frac{1}{24} \frac{\partial^3 V}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} \\
g_{41}^{i_1 i_2 i_3 i_4} &= \frac{1}{120} \frac{\partial^4 V}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3} \partial x_{i_4}} \\
g_{51}^{i_1 i_2 i_3 i_4 i_5} &= \frac{1}{720} \frac{\partial^5 V}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3} \partial x_{i_4} \partial x_{i_5}} \\
g_{61}^{i_1 i_2 i_3 i_4 i_5 i_6} &= \frac{1}{5040} \frac{\partial^6 V}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3} \partial x_{i_4} \partial x_{i_5} \partial x_{i_6}} \\
g_{02} &= \frac{1}{12} (V^{(0,2)} + V^{(2,0)}) \\
g_{12}^{i_1} &= \frac{1}{24} \left(V^{(1,2)} + V^{(3,0)}, V^{(0,3)} + V^{(2,1)}\right) \\
g_{22}^{i_1} &= \frac{1}{80} \left(V^{(2,2)} + V^{(4,0)}, V^{(1,3)} + V^{(3,1)}\right) \\
g_{111}^{i_1} &= \frac{1}{360} (V^{(3,2)} + V^{(5,0)}) \\
g_{121}^{i_1 i_2} &= \frac{1}{360} (V^{(2,3)} + V^{(4,1)}) \\
g_{221}^{i_1 i_2} &= \frac{1}{360} (V^{(1,4)} + V^{(3,2)}) \\
g_{322}^{i_1 i_2 i_3} &= \frac{1}{360} (V^{(0,5)} + V^{(2,3)})
\end{align*}
\]
\( g_{12}^{111} = \frac{1}{2016} (V^{(4,2)} + V^{(6,0)}) \)
\( g_{12}^{112} = g_{12}^{121} = g_{12}^{211} = g_{12}^{212} = \frac{1}{2016} (V^{(3,3)} + V^{(5,1)}) \)
\( g_{42}^{2211} = g_{42}^{2121} = g_{42}^{2112} = g_{42}^{2212} = \frac{1}{2016} (V^{(2,4)} + V^{(4,2)}) \)
\( g_{42}^{2221} = g_{42}^{2212} = g_{42}^{2122} = g_{42}^{1222} = \frac{1}{2016} (V^{(1,5)} + V^{(3,3)}) \)
\( g_{42}^{2222} = \frac{1}{2016} (V^{(0,6)} + V^{(2,4)}) \)
\( g_{03} = \frac{1}{24} (-V^{(0,1)^2} + V^{(0,4)}/10 \)
\( -V^{(1,0)^2} + V^{(2,2)}/5 + V^{(4,0)/10} \)
\( g_{13}^{1} = \frac{1}{24} (-V^{(0,1)}V^{(1,1)} + V^{(1,4)}/20 \)
\( +V^{(5,0)}/20 + V^{(3,2)}/10 - V^{(1,0)}V^{(2,0)} \)
\( g_{13}^{2} = \frac{1}{24} (-V^{(0,1)}V^{(0,2)} + V^{(0,5)}/20 \)
\( +V^{(4,1)}/20 + V^{(2,3)}/10 - V^{(1,0)}V^{(1,1)} \)
\( g_{23}^{11} = \frac{1}{10} (-V^{(1,1)^2}/9 - V^{(2,0)^2}/9 \)
\( -V^{(0,1)}V^{(2,1)}/8 \)
\( -V^{(1,0)}V^{(3,0)}/8 + V^{(2,4)}/168 + V^{(4,2)}/84 + V^{(6,0)}/168 \)
\[ g^{12}_{23} = g^{21}_{23} = \frac{1}{10} \left( -V^{(0,2)}V^{(1,1)}/9 + V^{(1,5)}/168 - V^{(1,1)}V^{(2,0)}/9 - V^{(1,0)}V^{(2,1)}/8 + V^{(3,3)}/84 - V^{(0,1)}V^{(1,2)}/8 + V^{(5,1)}/168 \right) \]

\[ g^{22}_{23} = \frac{1}{10} \left( -V^{(1,1)}^2/9 - V^{(0,2)^2}/9 - V^{(1,0)}V^{(1,2)}/8 - V^{(0,1)}V^{(0,3)}/8 + V^{(4,2)}/168 + V^{(2,4)}/84 + V^{(0,6)}/168 \right) \]

\[ g^{04} = \frac{1}{10} \left( -V^{(0,2)^2}/36 + V^{(6,0)}/672 - V^{(0,1)}V^{(0,3)}/12 + V^{(0,6)}/672 - V^{(1,1)^2}/18 - V^{(1,0)}V^{(1,2)}/12 - V^{(2,0)^2}/36 - V^{(0,1)}V^{(2,1)}/12 + V^{(2,4)}/224 - V^{(1,0)}V^{(3,0)}/12 + V^{(4,2)}/224 \right) \]

where \( V^{(n,m)} = \frac{\partial^2 V}{\partial x^n \partial y^m} \).
REFERENCES

[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer–Verlag, Berlin, 1990)

[2] A. M. Ozorio de Almeida, *Hamiltonian Systems: Chaos and Quantization*, Cambridge University Press (1990)

[3] K. Nakamura, *Quantum Chaos*, Cambridge Nonlinear Science Series (1993); *From Classical to Quantum Chaos*, SIF Conference Proceedings, vol. 41, Ed. G. F. Dell’Antonio, S. Fantoni, V. R. Manfredi (Editrice Compositori, Bologna, 1993)

[4] O. Bohigas and H. A. Weidenmüller, Ann. Rev. Nucl. Part. Sci. 38, 421 (1988)

[5] M. T. Lopez–Arias, V. R. Manfredi and L. Salasnich, Riv. Nuovo Cimento 17, 1 (1994)

[6] V. R. Manfredi and L. Salasnich, Int. J. Mod. Phys. E 4, 625 (1995)

[7] A. Bohr, B. Mottelson, *Nuclear Structure*, vol. 2 (Benjamin, London, 1975)

[8] J. M. Eisenberg, W. Greiner, *Nuclear Models*, vol. 1 (North Holland, Amsterdam, 1970)

[9] Yu. Bolotin, V. Yu. Gonchar, E. V. Inopin, V. V. Levenko, V. N. Tarasov and N. A. Chekanov, Sov. I. Part. Nucl. 20, 372 (1989)

[10] G. Benettin, R. Brambilla and L. Galgani, Physica A 87, 381 (1977)

[11] M. Rosa–Clot and S. Taddei, Phys. Rev. C 50, 627 (1994).

[12] M. Rosa–Clot and S. Taddei, Proceedings of the “V Convegno su Problemi di Fisica Nucleare Teorica”, Cortona 1993.
[13] M. Rosa–Clot and S. Taddei, Phys. Lett. A 197, 1 (1995).

[14] S. Taddei, submitted to Phys. Rev. C.

[15] H. Poincarè, *New Methods of Celestial Mechanics*, vol. 3, ch. 27 (Transl. NASA Washington DC 1967); M. Henon, Physica D 5, 412 (1982)

[16] V. R. Manfredi, Lett. Nuovo Cimento 40, 135 (1984)

[17] F. J. Dyson, M. L. Mehta: *J. Math, Phys.* 4, 701 (1963)

[18] O. Bohigas, M. J. Giannoni: Ann. Phys. (N.Y.) 89, 393 (1975)

[19] M. Berry, Proc. Roy. Soc. Lond. A 400, 229 (1985)

[20] A. A. Raduta, V. Baran, D. S. Delion, Nucl. Phys. A 588, 431 (1995)
TABLE CAPTIONS

Table 1: Numerical values of the parameters $C_2$, $C_3$, $C_4$, $V_0$ defining the nuclear potential $V$ for the two rare–earth nuclei studied.
FIGURE CAPTIONS

Figure 1: The potential energy of the rare-earth $^{160}Gd$ (a) and $^{166}Er$ (b).

Figure 2: The Poincaré section of $^{160}Gd$ for the energy $E = 4$ MeV.

Figure 3: Poincaré sections for $^{160}Gd$ with $E = 8$ MeV (below) and $E = 18$ MeV (above).

Figure 4: Spectral statistics for $^{160}Gd$ with $6 \leq E \leq 11$ MeV (below) and $15 \leq E \leq 20$ MeV (above); the solid lines are the Poisson statistics and the dashed lines are the GOE statistics.

Figure 5: Poincaré sections for $^{166}Er$ with $E = 4$ MeV (below) and $E = 15$ MeV (above).

Figure 6: Spectral statistics for $^{166}Er$ with $2 \leq E \leq 6$ MeV (below) and $13 \leq E \leq 16$ MeV (above); the solid lines are the Poisson statistics and the dashed lines are the GOE statistics.
| Nucleus | $C_2$ (MeV) | $C_3$ (MeV) | $C_4$ (MeV) | $V_0$ (MeV) |
|---------|-------------|-------------|-------------|-------------|
| $^{160}Gd$ | -100.64     | 37.57       | 668.01      | 5.56        |
| $^{166}Er$ | -73.77      | 64.96       | 1138.52     | 1.89        |

Table 1
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/nucl-th/9510012v1
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/nucl-th/9510012v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/nucl-th/9510012v1
This figure "fig2-2.png" is available in "png" format from:

http://arxiv.org/ps/nucl-th/9510012v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/nucl-th/9510012v1
This figure "fig2-3.png" is available in "png" format from:

http://arxiv.org/ps/nucl-th/9510012v1