2-LOCAL DERIVATIONS ON MATRIX RINGS OVER ASSOCIATIVE RINGS

SHAVKAT AYUPOV AND FARKHAD ARZIKULOV

Abstract. Let $M_n(\mathbb{R})$ be the matrix ring over an associative ring $\mathbb{R}$. In the present paper we prove that every inner 2-local derivation on the matrix ring $M_n(\mathbb{R})$ over a commutative associative ring $\mathbb{R}$ is an inner derivation. Also, we prove that, if every inner 2-local derivation on the whole $M_n(\mathbb{R})$ is an inner derivation then every inner 2-local derivation on a certain subring of $M_n(\mathbb{R})$, isomorphic to $M_2(\mathbb{R})$, is an inner derivation.

2010 Mathematics Subject Classification: Primary 16W25, 46L57; Secondary 47B47

INTRODUCTION

The present paper is devoted to 2-local derivations on associative rings. Recall that a 2-local derivation is defined as follows: given a ring $\mathbb{R}$, a map $\Delta : \mathbb{R} \to \mathbb{R}$ (not additive in general) is called a 2-local derivation if for every $x, y \in \mathbb{R}$, there exists a derivation $D_{x,y} : \mathbb{R} \to \mathbb{R}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

In 1997, P. Šemrl [5] introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space $H$. A similar description for the finite-dimensional case appeared later in [3]. In the paper [4] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. In [2] the authors suggested a new technique and have generalized the above mentioned results of [5] and [3] for arbitrary Hilbert spaces. Namely they considered 2-local derivations on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space $H$ and proved that every 2-local derivation on $B(H)$ is a derivation. In [1] we extended the above results and give a short proof of the theorem for arbitrary semi-finite von Neumann algebras.

In this article we develop an algebraic approach to the investigation of derivations and 2-local derivations on associative rings. Since we consider a sufficiently general case of associative rings we restrict our attention only on inner derivations and inner 2-local derivations. In particular, we consider the following problem: if an inner 2-local derivation on an associative ring is a derivation then is the latter derivation inner? The answer to this question is affirmative if the ring is generated by two elements (Proposition 10).

In this article we consider 2-local derivations on the matrix ring $M_n(\mathbb{R})$ over an associative ring $\mathbb{R}$. The first step of the investigation consists of proving that, in the case of a commutative associative ring $\mathbb{R}$ arbitrary inner 2-local derivation on
$M_n(\mathbb{R})$ is an inner derivation. This result extends the result of [4] to the infinite dimensional but commutative ring $\mathbb{R}$.

The second step consists of proving that if every inner 2-local derivation on $M_n(\mathbb{R})$ is an inner derivation then each inner 2-local derivation on a certain subring of the matrix ring $M_n(\mathbb{R})$, isomorphic to $M_2(\mathbb{R})$, is also an inner derivation.

1. 2-LOCAL DERIVATIONS ON MATRIX RINGS

Let $\mathbb{R}$ be a ring. Recall that a map $D : \mathbb{R} \to \mathbb{R}$ is called a derivation, if $D(x + y) = D(x) + D(y)$ and $D(xy) = D(x)y + xD(y)$ for any two elements $x, y \in \mathbb{R}$. A derivation $D$ on a ring $\mathbb{R}$ is called an inner derivation, if there exists an element $a \in \mathbb{R}$ such that

$$D(x) = ax - xa, x \in A.$$ 

A map $\Delta : \mathbb{R} \to \mathbb{R}$ is called a 2-local derivation, if for any two elements $x, y \in \mathbb{R}$ there exists a derivation $D_{x,y} : \mathbb{R} \to \mathbb{R}$ such that $\Delta(x) = D_{x,y}(x), \Delta(y) = D_{x,y}(y)$.

A map $\Delta : \mathbb{R} \to \mathbb{R}$ is called an inner 2-local derivation, if for any two elements $x, y \in \mathbb{R}$ there exists an element $a \in \mathbb{R}$ such that $\Delta(x) = ax - xa, \Delta(y) = ay - ya$.

Let $\mathbb{R}$ be an associative unital ring, $M_n(\mathbb{R}), n > 1$, be the matrix ring over the associative ring $\mathbb{R}$. Let $\bar{M}_2(\mathbb{R})$ be a subring of $M_n(\mathbb{R})$, generated by the subsets $\{e_{ii}M_n(\mathbb{R})e_{jj}\}_{ij=1}^{n}$ in $M_n(\mathbb{R})$. It is clear that $\bar{M}_2(\mathbb{R}) \cong M_2(\mathbb{R})$.

The following theorem is the main result of the paper.

**Theorem 1.** Let $\mathbb{R}$ be an associative unital ring, and let $M_n(\mathbb{R})$ be the matrix ring over $\mathbb{R}, n > 1$. Then

1) if the ring $\mathbb{R}$ is commutative then every inner 2-local derivation on the matrix ring $M_n(\mathbb{R})$ is an inner derivation,

2) if every inner 2-local derivation on the matrix ring $M_n(\mathbb{R})$ is an inner derivation then every inner 2-local derivation on its subring $\bar{M}_2(\mathbb{R})$ is an inner derivation.

First let us prove lemmata and propositions which are necessary for the proof of theorem 1.

Let $\mathbb{R}$ be an associative unital ring, and let $\{e_{ij}\}_{i,j=1}^{n}$ be the set of matrix units in $M_n(\mathbb{R})$ such that $e_{ij}$ is a $n \times n$-dimensional matrix in $M_n(\mathbb{R})$, i.e. $e_{ij} = (a_{kl})_{k,l=1}^{n}$, the $(i,j)$-th component of which is 1 (the unit of $\mathbb{R}$), i.e. $a_{ij} = 1$, and the rest components are zeros.

Let $\Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be an inner 2-local derivation. Consider the subset $\{a(ij)\}_{i,j=1}^{n} \subset M_n(\mathbb{R})$ such that

$$\Delta(e_{ij}) = a(ij)e_{ij} - e_{ij}a(ij),$$

$$\Delta(\sum_{k=1}^{n-1} e_{k,k+1}) = a(ij)\left(\sum_{k=1}^{n-1} e_{k,k+1}\right) - \left(\sum_{k=1}^{n-1} e_{k,k+1}\right)a(ij).$$

Put $a_{ij} = e_{ij}a(ji)e_{jj}$, for all pairs of different indices $i, j$ and let $\sum_{k\neq l} a_{kl}$ be the sum of all such elements.
Lemma 2. Let $\Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be an inner 2-local derivation. Then for any pair $i, j$ of different indices the following equality holds

$$\Delta(e_{ij}) = \sum_{k \neq l} a_{kl}e_{ij} - e_{ij} \sum_{k \neq l} a_{kl} + a_{ij}e_{ii} - e_{ij}a(ij)_{jj}, \quad (1)$$

where $a(ij)_{ii}$, $a(ij)_{jj}$ are components of the matrices $e_{ii}a(ij)e_{ii}$, $e_{jj}a(ij)e_{jj}$.

Proof. Let $m$ be an arbitrary index different from $i$, $j$ and let $a(ij, ik) \in M_n(\mathbb{R})$ be an element such that

$$\Delta(e_{im}) = a(ij, im)e_{im} - e_{im}a(ij, im) \quad \text{and} \quad \Delta(e_{ij}) = a(ij, im)e_{ij} - e_{ij}a(ij, im).$$

We have

$$\Delta(e_{im}) = a(ij, im)e_{im} - e_{im}a(ij, im) = a(im)e_{im} - e_{im}a(im)$$

and

$$e_{mm}a(ij, im)e_{ij} = e_{mm}a(im)e_{ij}.$$

Then

$$e_{mm}\Delta(e_{ij})e_{jj} = e_{mm}(a(ij, im)e_{ij} - e_{ij}a(ij, im))e_{jj} =$$

$$e_{mm}a(ij, im)e_{ij} - 0 = e_{mm}a(im)e_{ij} - e_{mm}e_{ij} \sum_{k \neq l} a_{kl}e_{jj} =$$

$$e_{mm}a_{mj}e_{ij} - e_{mm}e_{ij} \sum_{k \neq l} a_{kl}e_{jj} = e_{mm} \sum_{k \neq l} a_{kl}e_{jj} - e_{ij} \sum_{k \neq l} a_{kl}e_{jj}.$$

Similarly,

$$e_{mm}\Delta(e_{ij})e_{ii} = e_{mm}(a(ij, im)e_{ij} - e_{ij}a(ij, im))e_{ii} =$$

$$e_{mm}a(ij, im)e_{ij}e_{ii} - 0 = 0 - 0 = e_{mm} \sum_{k \neq l} a_{kl}e_{ij}e_{ii} - e_{mm}e_{ij} \sum_{k \neq l} a_{kl}e_{ii} =$$

$$e_{mm} \sum_{k \neq l} a_{kl}e_{ij} - e_{ij} \sum_{k \neq l} a_{kl}e_{ii}.$$

Let $a(ij, mj) \in M_n(\mathbb{R})$ be an element such that

$$\Delta(e_{mj}) = a(ij, mj)e_{mj} - e_{mj}a(ij, mj) \quad \text{and} \quad \Delta(e_{ij}) = a(ij, mj)e_{ij} - e_{ij}a(ij, mj).$$

We have

$$\Delta(e_{mj}) = a(ij, mj)e_{mj} - e_{mj}a(ij, mj) = a(mj)e_{mj} - e_{mj}a(mj).$$

and

$$e_{ij}a(ij, mj)e_{mm} = e_{ij}a(mj)e_{mm}.$$

Then

$$e_{ii}\Delta(e_{ij})e_{mm} = e_{ii}(a(ij, mj)e_{ij} - e_{ij}a(ij, mj))e_{mm} =$$

$$0 - e_{ij}a(ij, mj)e_{mm} = 0 - e_{ij}a(mj)e_{mm} = e_{ii} \sum_{k \neq l} a_{kl}e_{mm} - e_{ij} \sum_{k \neq l} a_{kl}e_{mm} =$$

$$e_{ij} \sum_{k \neq l} a_{kl}e_{mm} - e_{jj} \sum_{k \neq l} a_{kl}e_{mm}.$$

Also we have

$$e_{jj}\Delta(e_{ij})e_{mm} = e_{jj}(a(ij, mj)e_{ij} - e_{ij}a(ij, mj))e_{mm} =$$
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\[ 0 - 0 = e_{jj} \sum_{k \neq l} a_{kl} e_{ij}^* e_{mm} - e_{jj} e_{ij} \sum_{k \neq l} a_{kl} e_{mm} = \]
\[ e_{jj} (\sum_{k \neq l} a_{kl} e_{ij} - e_{ij} \sum_{k \neq l} a_{kl}) e_{mm}, \]
\[ e_{ii} \Delta (e_{ij}) e_{ii} = e_{ii} (a(ij) e_{ij} - e_{ij} a(ij)) e_{ii} = \]
\[ 0 - e_{ij} a(ij) e_{ii} = 0 - e_{ij} a(ij) e_{ii} = 0 - e_{ij} a_{jj} e_{ii} = \]
\[ e_{ii} \sum_{k \neq l} a_{kl} e_{ij} e_{ii} - e_{ij} \sum_{k \neq l} a_{kl} e_{ii} - \]
\[ e_{ii} (\sum_{k \neq l} a_{kl} e_{ij} - e_{ij} \sum_{k \neq l} a_{kl}) e_{ii}. \]

Thus
\[ \Delta (e_{ij}) = \sum_{k,l=1}^{n} e_{kk} \Delta (e_{ij}) e_{ll} = \]
\[ \sum_{k,l=1}^{n} e_{kk} (\sum_{k \neq l} a_{kl} e_{ij} - e_{ij} \sum_{k \neq l} a_{kl}) e_{ll} + e_{ii} \Delta (e_{ij}) e_{jj} = \]
\[ \sum_{k \neq l} a_{kl} e_{ij} - e_{ij} \sum_{k \neq l} a_{kl} + a(ij) e_{ij} - e_{ij} a(ij) e_{jj}. \]

The proof is complete. ∎

Consider the element \( x_o = \sum_{k=1}^{n-1} e_{k,k+1} \). Fix the indices \( i_o, j_o \). Then there exists an element \( c \in M_n(A) \) such that
\[ \Delta (e_{i_o j_o}) = e_{i_o j_o}^* - e_{i_o j_o} c \text{ and } \Delta (x_o) = c x_o - x_o c. \]

Let \( c = \sum_{i,j=1}^{n} e_{ij} \) be the Pierce decomposition of \( c \), \( a_{ii} = e_{ii} \) for any \( i \) and \( \bar{a} = \sum_{i,j=1}^{n} a_{ij} \).

Lemma 3. Let \( \Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R}) \) be an inner 2-local derivation. Let \( k, l \) be arbitrary different indices. Then, if \( b \in M_n(\mathbb{R}) \) is an element such that
\[ \Delta (x_o) = b x_o - x_o b \]
then \( c^{kk} - c^{ll} = b^{kk} - b^{ll} \), where \( c_{ii} = c^{ii} e_{ii}, b_{ii} = b^{ii} e_{ii} \), \( c^{ii}, b^{ii} \in \mathbb{R}, i = 1, 2, \ldots, n. \)

Proof. We can suppose that \( k < l \). We have
\[ \Delta (x_o) = c x_o - x_o c = b x_o - x_o b. \]

Hence
\[ e_{kk} (c x_o - x_o c) e_{k+1,k+1} = e_{kk} (b x_o - x_o b) e_{k+1,k+1} \]
and
\[ c^{kk} - c^{k+1,k+1} = b^{kk} - b^{k+1,k+1}. \]

Then for the sequence
\( (k, k+1), (k + 1, k + 2) \ldots (l - 1, l) \)
we have
\[ c^{kk} - c^{k+1,k+1} = b^{kk} - b^{k+1,k+1}, \quad c^{k,k+1} - c^{k+2,k+2} = b^{k+1,k+1} - b^{k+2,k+2}, \ldots \]
\[ e^{l-1,l-1} - e^{l} = b^{l-1,l-1} - b^{l}. \]
Hence
\[ c^{kk} - b^{kk} = c^{k+1,k+1} - b^{k+1,k+1}, \quad c^{k,k+1} - b^{k+2,k+2} = c^{k+2,k+2} - b^{k+2,k+2}, \ldots \]
Therefore \( c^{kk} - b^{kk} = c^{l} - b^{l}, \) i.e. \( c^{kk} - c^{l} = b^{kk} - b^{l}. \) The proof is complete. ▷

**Proposition 4.** If the ring \( \mathbb{R} \) is commutative then every inner 2-local derivation on \( M_n(\mathbb{R}) \) is an inner derivation.

**Proof.** Let \( \Delta : M_n(\mathbb{R}) \to M_n(\mathbb{R}) \) be an inner 2-local derivation, \( x \) be an arbitrary matrix in \( M_n(\mathbb{R}) \) and let \( d(ij) \in M_n(\mathbb{R}) \) be an element such that
\[ \Delta(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) \text{ and } \Delta(x) = d(ij)x - xd(ij) \]
and \( i \neq j. \) Then by Lemma 2
\[ \Delta(e_{ij}) = d(ij)e_{ij} - e_{ij}d(ij) = e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} + (1 - e_{ii})d(ij)e_{ij} - e_{ij}d(ij)(1 - e_{jj}) = \]
\[ a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj} + \sum_{k \neq l} a_{kl}e_{ij} - e_{ij} \sum_{k \neq l} a_{kl} \]
for all \( i, j. \)

Since \( e_{ii}d(ij)e_{ij} - e_{ij}d(ij)e_{jj} = a(ij)_{ii}e_{ij} - e_{ij}a(ij)_{jj} \) we have
\[ (1 - e_{ii})d(ij)e_{ii} = \sum_{k \neq l} a_{kl}e_{ii} - e_{ij}d(ij)(1 - e_{jj}) = e_{jj} \sum_{k \neq l} a_{kl} \]
for all different \( i \) and \( j. \)

Hence by lemma 2 we have
\[ e_{jj}\Delta(x)e_{ii} = e_{jj}(d(ij)x - xd(ij))e_{ii} = e_{jj}d(ij)(1 - e_{jj})xe_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{jj}d(ij)e_{ii} - e_{jj}xe_{ii}d(ij)e_{ii} = \]
\[ e_{jj} \sum_{k \neq l} a_{kl}xe_{ii} - e_{jj}x \sum_{k \neq l} a_{kl}e_{ii} + e_{jj}d(ij)e_{jj}xe_{ii} - e_{jj}xe_{ii}d(ij)e_{ii}. \]

Similarly
\[ e_{ii}\Delta(x)e_{jj} = \]
\[ e_{ii} \sum_{k \neq l} a_{kl}xe_{jj} - e_{ii}x \sum_{k \neq l} a_{kl}e_{jj} + e_{ii}d(ij)e_{jj}xe_{ii} - e_{ii}xe_{jj}d(ij)e_{jj}. \]

Also, we have
\[ e_{ii}\Delta(x)e_{ii} = e_{ii}(d(ij)x - xd(ij))e_{ii} = e_{ii}d(ij)(1 - e_{ii})xe_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} = \]
\[ e_{ii} \sum_{k \neq l} a_{kl}xe_{ii} - e_{ii}x \sum_{k \neq l} a_{kl}e_{ii} + e_{ii}d(ij)e_{ii}xe_{ii} - e_{ii}xe_{ii}d(ij)e_{ii} \]
and
\[ e_{jj}\Delta(x)e_{jj} = e_{jj}(d(ij)x - xd(ij))e_{jj} = \]
\[ e_{jj} \sum_{k \neq l} a_{kl}xe_{jj} - e_{jj}x \sum_{k \neq l} a_{kl}e_{jj} + e_{jj}d(ij)e_{jj}xe_{jj} - e_{jj}xe_{jj}d(ij)e_{jj}. \]

We have
Let $d(ij)e_i e_{ii} - e_i x_{ii} d(ij)e_{ii} = d(ij)x^i e_{ii} - x^i d(ij)i^i e_{ii} = d(ij)x^i e_{ii} - d(ij)x^i e_{ii} = 0$
\[c_{ii}e_i e_{ii} - e_i x_{ii} c_{ii} = c_{ii}i^i e_{ii} - x^i c_{ii} e_{ii} = c_{ii}i^i e_{ii} - c_{ii}i^i e_{ii} = 0\]
and
\[e_i d(ij)e_i e_{ii} - e_i x_{ii} d(ij)e_{ii} = c_{ii}e_i e_{ii} - e_i x_{ii} c_{ii},\]
similarly,
\[e_{jj}d(ij)e_{jj} x_{jj} - e_{jj} x_{jj} d(ij)e_{jj} = c_{jj} e_{jj} x_{jj} - e_{jj} x_{jj} c_{jj} = 0.\]
We have
\[\Delta(\sum_{i=1}^{n-1} e_{i+1}) = a(ji)(\sum_{i=1}^{n-1} e_{i+1}) - (\sum_{i=1}^{n-1} e_{i+1})a(ji)\]
by the definition. Then by lemma 3
\[a(ji)_{jj} - a(ji)^i = c_{jj} - c_{ii}.\]
Hence
\[e_{jj}d(ji)e_{jj} x_{jj} - e_{jj} x_{jj} d(ji)e_{jj} = e_{jj}a(ji) e_{jj} x_{jj} - e_{jj} x_{jj} a(ji) e_{jj} =
\]
\[a(ji)_{jj} e_{jj} x_{jj} - e_{jj} x_{jj} a(ji)^i = x_{jj}(c_{jj} - c_{ii}) e_{jj} =
\]
\[c_{jj} x_{jj} e_{jj} - x_{jj} c_{jj} e_{jj} = c_{jj} e_{jj} x_{jj} - e_{jj} x_{jj} c_{jj},\]
Similarly, we have
\[e_i d(ji)e_i x_{jj} - e_i x_{jj} d(ji)e_{jj} = e_i e_i x_{jj} - e_i x_{jj} c_{jj},\]
where $c_{kk} = e_{kk} e_{kk}, x = \sum_{k=1}^{n} x_{ik} e_{kk}, d_{kk} = \sum_{k=1}^{n} d_{kk} e_{kk}, k, l = 1, 2, \ldots, n.$
Hence
\[\Delta(x) = (\sum_{i} c_{ii})x - x(\sum_{i} c_{ii}) + (\sum_{k\neq l} a_{kl}x - x \sum_{k\neq l} a_{kl} =
\]
\[\frac{ax - xa}{a}\]
for all $x \in M_n(\mathbb{R})$. The proof is complete. □

2. Extension of derivations and 2-local derivations

**Proposition 5.** Let $M_2(\mathbb{R})$ be the matrix ring over a unital associative ring $\mathbb{R}$ and let $D : e_{11}M_2(\mathbb{R})e_{11} \to e_{11}M_2(\mathbb{R})e_{11}$ be a derivation on the subring $e_{11}M_2(\mathbb{R})e_{11}$. Then, if $\phi : e_{11}M_2(\mathbb{R})e_{11} \to e_{22}M_2(\mathbb{R})e_{22}$ is an isomorphism defined as $\phi(a) = e_{21}a e_{12}, a \in e_{11}M_2(\mathbb{R})e_{11}$ then the map defined by the following conditions

1) $\bar{D}(a) = D(a), a \in e_{11}M_2(\mathbb{R})e_{11},$
2) $\bar{D}(a) = \phi \circ D \circ \phi^{-1}(a), a \in e_{22}M_2(\mathbb{R})e_{22},$
3) $\bar{D}(e_{12}) = -e_{21},$
4) $\bar{D}(a) = D(a e_{21}) e_{12} + a e_{21} D(e_{12}), a \in e_{11}M_2(\mathbb{R})e_{22},$
5) $\bar{D}(a) = D(a e_{22}) e_{12} + a e_{22} D(e_{12}), a \in e_{22}M_2(\mathbb{R})e_{11},$
6) $\bar{D}(a) = D(e_{11}a e_{11}) + D(e_{11}a e_{12}) + D(e_{22}a e_{11}) + D(e_{22}a e_{22}), a \in M_2(\mathbb{R}),$

is a derivation.

**Proof.** For every $a \in M_2(\mathbb{R})$ the value $\bar{D}(a)$ is uniquely defined. Therefore $\bar{D}$ is a map.

It is clear that $\bar{D}$ is additive. Now we will prove that $\bar{D}(ab) = \bar{D}(a)b + a\bar{D}(b)$ for arbitrary elements $a, b \in M_2(\mathbb{R})$. 


Let \( a_1 = e_{11}ae_{11}, a_{12} = e_{11}ae_{22}, a_{21} = e_{22}ae_{11}, a_2 = e_{22}ae_{22}, b_1 = e_{11}be_{11}, \\
b_{12} = e_{11}be_{22}, b_{21} = e_{22}be_{11}, b_2 = e_{22}be_{22}, \overline{D} = \phi \circ D \circ \phi^{-1} \) for arbitrary elements \( a, b \in M_2(\mathbb{R}) \). Then we have the following Pierce decompositions of the elements \( a \) and \( b \)
\[
a = a_1 + a_{12} + a_{21} + a_2, b = b_1 + b_{12} + b_{21} + b_2.
\]
The following equalities hold
\[
\overline{D}(a_1b_1) = D(a_1b_1), \\
\overline{D}(a_1b_2) = \overline{D}(0) = 0 = D(a_1)b_2 + a_1\overline{D}(b_2), \\
D(a_1b_{12}) = D(a_1b_{12}e_{21})e_{12} + a_1b_{12}e_{21}\overline{D}(e_{12}) = \\
D(a_1)b_{12} + a_1D(b_{12}e_{21})e_{12} + a_1b_{12}e_{21}\overline{D}(e_{12}) = \\
D(a_1)b_{12} + a_1(D(b_{12}e_{21})e_{12} + b_{12}e_{21}\overline{D}(e_{12})) = D(a_1)b_{12} + a_1\overline{D}(b_{12}) = \\
\overline{D}(a_1)b_{12} + a_1D(b_{12}),
\]
\[
\overline{D}(a_{12}b_1) = \overline{D}(0) = 0 = a_1(\overline{D}(e_{21})e_{12}b_1 + e_{21}D(e_{12}b_1)) = \\
a_1\overline{D}(b_1) = D(a_1)b_1 + a_1\overline{D}(b_1) = D(a_1)b_1 + a_1D(b_1), \\
\overline{D}(a_{12}b_1) = \overline{D}(0) = 0 = (D(a_{12}b_1)e_{12} + a_{12}e_{21}\overline{D}(e_{12}))b_1 = \\
\overline{D}(a_{12}b_1) = D(a_{12}b_1) + a_{12}D(b_1) = \overline{D}(a_{12}b_1) + a_{12}\overline{D}(b_1).
\]
Also, since
\[
\overline{D}(e_{12})a_{12} + e_{12}\overline{D}(a_{12}) = e_{12}a_{12} + e_{12}(\overline{D}(e_{21})e_{21}a_{21} + e_{21}D(e_{12}a_{21})) = \\
e_{12}a_{21} - e_{12}e_{21}a_{12} + e_{12}e_{21}D(e_{12}a_{21}) = D(e_{12}a_{21})
\]
we have
\[
\overline{D}(a_{21}b_1) = \overline{D}(e_{21})e_{12}a_{21}b_1 + e_{21}D(e_{12}a_{21}b_1) = \\
- a_{21}b_1 + e_{21}(D(e_{12}a_{21})b_1 + e_{12}a_{21}D(b_1)) = \\
- a_{21}b_1 + e_{21}D(e_{12}a_{21})b_1 + a_{21}D(b_1) = \\
- a_{21}b_1 + e_{21}(D(e_{12}a_{21})b_1 + a_{21}D(b_1)) = \\
- a_{21}b_1 + a_{21}b_1 + e_{21}D(a_{21})b_1 + a_{21}D(b_1) = \\
e_{21}\overline{D}(a_{21})b_1 + a_{21}D(b_1) = D(a_{21})b_1 + a_{21}\overline{D}(b_1)
\]
by condition 5). Similarly we have
\[
\overline{D}(a_{21}b_{12}) = \overline{D}^+(a_{21}e_{12}e_{21}b_{12}) = \\
\overline{D}^+(a_{21}e_{12}e_{21}b_{12} + a_{21}e_{12}\overline{D}^+(e_{21}b_{12}) = \\
\overline{D}^+(e_{21}e_{12}a_{21}e_{21}e_{21}b_{12} + a_{21}e_{12}\overline{D}^+(e_{21}b_{12}) = \\
\phi \circ D(e_{12}a_{21})e_{21}b_{12} + a_{21}e_{12}\phi \circ D(b_{12}e_{21}) = \\
e_{21}D(e_{12}a_{21})e_{12}e_{21}b_{12} + a_{21}e_{12}D(b_{12}e_{21})e_{12} = \\
e_{21}D(e_{12}a_{21})b_{12} - a_{21}b_{12} + a_{21}b_{12} + a_{21}D(b_{12}e_{21})e_{12} = \\
e_{21}D(e_{12}a_{21})b_{12} + \overline{D}(e_{21})e_{12}a_{21}b_{12} + a_{21}b_{12}e_{21}\overline{D}(e_{12}) + a_{21}D(b_{12}e_{21})e_{12} = \\
\overline{D}(e_{21}D(e_{12}a_{21}) + \overline{D}(e_{21})e_{12}a_{21}b_{12} + a_{21}b_{12}e_{21}\overline{D}(e_{12}) + D(b_{12}e_{21})e_{12}) = \\
\overline{D}(a_{12}b_{12}) = \overline{D}(a_{12})b_{12} + a_{12}\overline{D}(b_{12}).
\]
By conditions 4) and 5) above the following equalities hold
\[
\overline{D}(a_{12}b_{12}) = \overline{D}(a_{12})b_{12} + a_{12}\overline{D}(b_{12}) = 0,
\]
By these equalities we have
\[
\bar{D}(ab) = \bar{D}((a_1 + a_{12} + a_2)(b_1 + b_{12} + b_2)) = \\
\bar{D}(a_1 b_1) + \bar{D}(a_1 b_{12}) + \bar{D}(a_1 b_2) + \bar{D}(a_{12} b_1) + \ldots \\
+ \bar{D}(a_{22} b_1) + \bar{D}(a_{22} b_2) = \bar{D}(a) \bar{b} + a \bar{D}(\bar{b}).
\]
Hence, the map \( \bar{D} \) is a derivation and it is an extension of the derivation \( D \) on the ring \( M_2(\mathcal{R}) \). The proof is complete. \( \triangleright \)

Let \( \bar{M}_m(\mathcal{R}) \) be a subring of \( M_n(\mathcal{R}) \), \( m < n \), generated by the subsets \( \{ e_{ij} M_n(\mathcal{R}) e_{jj} \}_{ij=1}^m \) in \( M_n(\mathcal{R}) \). It is clear that \( \bar{M}_m(\mathcal{R}) \cong M_m(\mathcal{R}) \).

**Proposition 6.** Let \( \mathcal{R} \) be an associative ring, and let \( \bar{M}_n(\mathcal{R}) \) be a matrix ring over \( \mathcal{R} \), \( n > 2 \). Then every derivation on \( \bar{M}_2(\mathcal{R}) \) can be extended to a derivation on \( M_n(\mathcal{R}) \).

**Proof.** By proposition 5 every derivation on \( \bar{M}_2(\mathcal{R}) \) can be extended to a derivation on \( M_3(\mathcal{R}) \). In its turn, every derivation on \( M_3(\mathcal{R}) \) can be extended to a derivation on \( M_6(\mathcal{R}) \) and so on. Thus every derivation \( \partial \) on \( M_2(\mathcal{R}) \) can be extended to a derivation \( D \) on \( M_{2^k}(\mathcal{R}) \). Suppose that \( n \leq 2^k \). Let \( e = \sum_{i=1}^n e_{ii} \) and
\[
\bar{D}(a) = eD(a)e, a \in \bar{M}_n(\mathcal{R}).
\]
Then \( \bar{D} : \bar{M}_n(\mathcal{R}) \rightarrow \bar{M}_n(\mathcal{R}) \) and \( \bar{D} \) is a derivation on \( \bar{M}_n(\mathcal{R}) \). Indeed, it is clear that \( \bar{D} \) is a linear map. At the same time, for all \( a, b \in \bar{M}_n(\mathcal{R}) \) we have
\[
eD(a)e + eD(eb) = eD(a) + eD(eb) = \bar{D}(a) \bar{b} + a \bar{D}(\bar{b}).
\]
Hence, \( \bar{D} : \bar{M}_n(\mathcal{R}) \rightarrow \bar{M}_n(\mathcal{R}) \) is a derivation. At the same time, on the subalgebra \( \bar{M}_2(\mathcal{R}) \) the derivation \( \bar{D} \) coincides with the derivation \( \partial \). Therefore, \( \bar{D} \) is an extension of \( \partial \) to \( \bar{M}_n(\mathcal{R}) \). \( \triangleright \)

Thus, in the case of the ring \( M_2(\mathcal{R}) \) for any derivation on the subring \( e_{11} M_2(\mathcal{R}) e_{11} \) we can take its extension onto the whole \( M_2(\mathcal{R}) \) defined as in proposition 5, which is also a derivation.

In proposition 7 we take the extensions of derivations defined as in proposition 5.

**Proposition 7.** Let \( M_2(\mathcal{R}) \) be the matrix ring over a unital associative ring \( \mathcal{R} \) and let \( \Delta : e_{11} M_2(\mathcal{R}) e_{11} \rightarrow e_{11} M_2(\mathcal{R}) e_{11} \) be a 2-local derivation on the subring \( e_{11} M_2(\mathcal{R}) e_{11} \). Then, if \( \phi : e_{11} M_2(\mathcal{R}) e_{11} \rightarrow e_{22} M_2(\mathcal{R}) e_{22} \) is an isomorphism defined as \( \phi(a) = e_{21} a e_{12}, a \in e_{11} M_2(\mathcal{R}) e_{11} \) then the map \( \nabla \) defined by the following conditions is a 2-local derivation:
1) \( \nabla(a) = \Delta(a) \) if \( a \in e_{11} M_2(\mathcal{R}) e_{11} \),
2) \( \nabla(a) = \phi \circ \Delta \circ \phi^{-1}(a) \) if \( a \in e_{22} M_2(\mathcal{R}) e_{22} \),
3) \( \nabla(e_{12}) = e_{12}, \nabla(e_{21}) = -e_{21} \),
4) \( \nabla(a) = \Delta(a e_{21}) e_{12} + a e_{21} \nabla(e_{12}) \) if \( a \in e_{11} M_2(\mathcal{R}) e_{22} \),
5) \( \nabla(a) = \nabla(e_{21}) e_{12} a + e_{21} \Delta(e_{12} a) \) if \( a \in e_{22} M_2(\mathcal{R}) e_{11} \),
6) \( \nabla(a) = \bar{D}(e_{11} a e_{11}) + \bar{D}(e_{11} a e_{22}) + \bar{D}(e_{22} a e_{11}) + \bar{D}(e_{22} a e_{22}) \),
\( a \in M_2(\mathcal{R}) \), where, if \( e_{11} a e_{11} \neq 0 \) then \( \bar{D} \) is the extension of the derivation \( D \) on \( e_{11} M_2(\mathcal{R}) e_{11} \) such that \( \Delta(e_{11} a e_{11}) = D(e_{11} a e_{11}) \).
if $e_{11}ae_{11} = 0$ and $e_{22}ae_{22} \neq 0$ then $\bar{D}$ is the extension of the derivation $D$ on $e_{11}M_2(\mathbb{R})e_{11}$ such that
\[ \Delta(e_{12}e_{22}ae_{22}e_{21}) = D(e_{12}e_{22}ae_{22}e_{21}), \]
if $e_{11}ae_{11} = e_{22}ae_{22} = 0$ and $e_{11}ae_{22} \neq 0$ then $\bar{D}$ is the extension of the derivation $D$ on $e_{11}M_2(\mathbb{R})e_{11}$ such that
\[ \Delta(e_{11}ae_{22}e_{21}) = D(e_{11}ae_{22}e_{21}), \]
if $e_{11}ae_{11} = e_{22}ae_{22} = 0$ and $e_{22}ae_{11} \neq 0$ then $\bar{D}$ is the extension of the derivation $D$ on $e_{11}M_2(\mathbb{R})e_{11}$ such that
\[ \Delta(e_{12}e_{22}ae_{11}) = D(e_{12}e_{22}ae_{11}). \]

**Proof.** It is clear that, if $a \in e_{11}M_2(\mathbb{R})e_{11}$ then the value $\nabla(a)$ defined in the case 1) coincides with the value $\nabla(a)$ defined in the case 6). Similarly, if $a \in e_{22}M_2(\mathbb{R})e_{22}$ then the value $\nabla(a)$ defined in the case 2) coincides with the value of $\nabla(a)$ defined in the case 6) and so on. Hence $\nabla$ is a correctly defined map.

Now we should prove that $\nabla$ is a 2-local derivation. Let $a, b$ be arbitrary elements of the algebra $M_2(\mathbb{R})$. Suppose that $e_{11}ae_{11} \neq 0$, $e_{11}be_{11} \neq 0$. Then by the definition there exists a derivation $D$ on $e_{11}M_2(\mathbb{R})e_{11}$ such that
\[ \Delta(a) = D(e_{11}ae_{11}) \text{ and } \Delta(b) = D(e_{11}be_{11}). \]
Let $\bar{D}$ be the extension of the derivation $D$ satisfying the conditions of the proposition 5. Hence
\[ \nabla(a) = \bar{D}(a) \text{ and } \nabla(b) = \bar{D}(b) \]
by the definition of the map $\nabla$.

Now suppose that $e_{11}ae_{11} = 0$, $e_{22}ae_{22} \neq 0$ and $e_{11}be_{11} \neq 0$. Then by the definition there exists a derivation $D$ on $e_{11}M_2(\mathbb{R})e_{11}$ such that
\[ \Delta(a) = D(e_{12}e_{22}ae_{22}e_{21}) \text{ and } \Delta(b) = D(e_{11}be_{11}). \]
Let $\bar{D}$ be the extension of the derivation $D$ satisfying the conditions of the proposition 5. Hence
\[ \nabla(a) = \bar{D}(a) \text{ and } \nabla(b) = \bar{D}(b) \]
and so on. In all cases there exists a derivation $\bar{D}$ such that
\[ \nabla(a) = \bar{D}(a) \text{ and } \nabla(b) = \bar{D}(b) \]
Since $a, b$ are arbitrary elements in $M_2(\mathbb{R})$ we have $\nabla$ is a 2-local derivation. ▷

**Proposition 8.** Let $\mathbb{R}$ be an associative ring, and let $M_n(\mathbb{R})$ be a matrix ring over $\mathbb{R}$, $n > 2$. Then every 2-local derivation on $M_2(\mathbb{R})$ can be extended to a 2-local derivation on $M_n(\mathbb{R})$.

**Proof.** By proposition 7 every 2-local derivation on $M_2(\mathbb{R})$ can be extended to a 2-local derivation on $M_4(\mathbb{R})$. In its turn, every 2-local derivation on $M_4(\mathbb{R})$ can be extended to a 2-local derivation on $M_8(\mathbb{R})$ and so on. Thus every 2-local derivation $\Delta$ on $M_2(\mathbb{R})$ can be extended to a 2-local derivation $\Delta$ on $M_{2^n}(\mathbb{R})$. Suppose that $n \leq 2^k$. Let $e = \sum_{i=1}^{n} e_{ii}$ and
\[ \nabla(a) = e\bar{\Delta}(a)e, a \in M_n(\mathbb{R}). \]
Then $\nabla : M_n(\mathbb{R}) \to \tilde{M}_n(\mathbb{R})$ and $\nabla$ is a 2-local derivation on $M_n(\mathbb{R})$. Indeed, it is clear that $\nabla$ is a map. At the same time, for all $a, b \in \tilde{M}_n(\mathbb{R})$ there exists a derivation $D : M_{2^k}(\mathbb{R}) \to M_{2^k}(\mathbb{R})$ such that

$$\tilde{\Delta}(a) = D(a), \tilde{\Delta}(b) = D(b).$$

Then

$$\nabla(a) = eD(a)e, \nabla(b) = eD(b)e.$$ 

By the proof of proposition 6 the map

$$\tilde{D}(a) = eD(a)e, a \in \tilde{M}_n(\mathbb{R})$$

is a derivation and

$$\nabla(a) = \tilde{D}(a), \nabla(b) = \tilde{D}(b).$$

Hence, $\nabla : M_n(\mathbb{R}) \to \tilde{M}_n(\mathbb{R})$ is a 2-local derivation.

At the same time, on the subalgebra $\tilde{M}_2(\mathbb{R})$ the 2-local derivation $\nabla$ coincides with the 2-local derivation $\Delta$. Therefore, $\nabla$ is an extension of $\Delta$ to $\tilde{M}_n(\mathbb{R})$. $
abla$

**Proposition 9.** Let $\mathbb{R}$ be an associative unital ring, and let $M_n(\mathbb{R})$, $n > 1$, be the matrix ring over $\mathbb{R}$. Then, if every inner 2-local derivation on the matrix ring $M_n(\mathbb{R})$ is an inner derivation then every inner 2-local derivation on the ring $\tilde{M}_2(\mathbb{R})$ is an inner derivation.

**Proof.** Let $\Delta$ be a 2-local derivation on $\tilde{M}_2(\mathbb{R})$. Then by proposition 8 $\Delta$ is extended to a 2-local derivation $\tilde{\Delta}$ on $M_n(\mathbb{R})$. By the condition $\tilde{\Delta}$ is an inner derivation, i.e. there exists $d \in M_n(\mathbb{R})$ such that

$$\tilde{\Delta}(a) = da - ad, a \in M_n(\mathbb{R}).$$

But $\tilde{\Delta}|_{\tilde{M}_2(\mathbb{R})} = \Delta$. Hence

$$\tilde{\Delta}(a) = da - ad \in \tilde{M}_2(\mathbb{R})$$

for all $a \in \tilde{M}_2(\mathbb{R})$, i.e.

$$(e_{11} + e_{22})(da - ad)(e_{11} + e_{22}) = da - ad,$$

and $da - ad = ca - ac$ for all $a \in \tilde{M}_2(\mathbb{R})$, where $c = (e_{11} + e_{22})d(e_{11} + e_{22})$. Since $c \in \tilde{M}_2(\mathbb{R})$, we have that $\Delta$ is an inner derivation. The proof is complete. $
abla$

**Proof of theorem 1.** Propositions 4 and 9 immediately imply theorem 1.

$
abla$

We conclude the paper by the following more general observation.

**Proposition 10.** Let $\Delta : \mathbb{R} \to \mathbb{R}$ be an inner 2-local derivation on an associative ring $\mathbb{R}$. Suppose that $\mathbb{R}$ is generated by its two elements. Then, if $\Delta$ is additive then it is an inner derivation.

**Proof.** Let $x, y$ be generators of $\mathbb{R}$, i.e. $\mathbb{R} = Alg(\{x, y\})$, where $Alg(\{x, y\})$ is an associative ring, generated by the elements $x, y$ in $\mathbb{R}$. We have that there exists $d \in \mathbb{R}$ such that

$$\Delta(x) = [d, x], \Delta(y) = [d, y],$$

where $[d, a] = da - ad$ for any $a \in \mathbb{R}$.

Hence by the additivity of $\Delta$ we have

$$\Delta(x + y) = \Delta(x) + \Delta(y) = [d, x + y].$$

Note that

$$\Delta(xy) = \Delta(x)y + x\Delta(y) = [d, x]y + x[d, y] = [d, xy],$$

$$\Delta(x^2) = \Delta(x)x + x\Delta(x) = [d, x]x + x[d, x] = [d, x^2],$$

for all $x, y \in \mathbb{R}$. Therefore, $\Delta$ is an inner derivation.
\[ \Delta(y^2) = \Delta(y)y + y\Delta(y) = [d, y]y + y[d, y] = [d, y^2], \]

Similarly
\[ \Delta(x^k) = [d, x^k], \Delta(y^m) = [d, y^m], \Delta(x^ky^m) = [d, x^ky^m] \]

and
\[ \Delta(x^ky^mx^l) = \Delta(x^ky^m)x^l + x^ky^m\Delta(x^l) = [d, x^ky^m]x^l + x^ky^m[d, x^l] = [d, x^ky^mx^l]. \]

Finally, for every polynomial \( p(x_1, x_2, \ldots, x_m) \in \mathbb{R} \), where \( x_1, x_2, \ldots, x_m \in \{x, y\} \) we have
\[ \Delta(p(x_1, x_2, \ldots, x_m)) = [d, p(x_1, x_2, \ldots, x_m)], \]

i.e. \( \Delta \) is an inner derivation on \( \mathbb{R} \).

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