MULTILINEAR OPERATORS: THE NATURAL EXTENSION
OF HIROTA’S BILINEAR FORMALISM

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Abstract
We introduce multilinear operators, that generalize Hirota’s bilinear $D$ operator, based on the principle of gauge invariance of the $\tau$ functions. We show that these operators can be constructed systematically using the bilinear $D$’s as building blocks. We concentrate in particular on the trilinear case and study the possible integrability of equations with one dependent variable. The 5th order equation of the Lax-hierarchy as well as Satsuma’s lowest-order gauge invariant equation are shown to have simple trilinear expressions. The formalism can be extended to an arbitrary degree of multilinearity.

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1. Introduction

The Hirota bilinear operators were introduced as an antisymmetric extension to the usual derivative [1], because of their usefulness for the computation of multisoliton solution of nonlinear evolution equations. The bilinear operator $D_x \equiv \partial_{x_1} - \partial_{x_2}$ acts on a pair of functions (the ‘dot product’) antisymmetrically:

$$D_x f \cdot g = (\partial_{x_1} - \partial_{x_2})f(x_1)g(x_2)|_{x_2=x_1=x} = f'g - g'f. \tag{1}$$

The Hirota bilinear formalism has been instrumental in the derivation of the multisoliton solutions of (integrable) nonlinear equations. A prerequisite to its application is a dependent variable transformation that converts the nonlinear equation into a quadratic “prepotential” form. This is best understood in a specific example, so let us consider the paradigmatic case of the KdV equation. Starting from

$$u_{xxx} + 6uu_x + u_t = 0, \tag{2}$$

we introduce the transformation $u = 2\partial_x^2 \log F$ and obtain (after one integration):

$$F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0. \tag{3}$$

This last equation can be written in a particularly condensed form using the Hirota $D$ operator:

$$(D_x^4 + D_xD_t)F \cdot F = 0. \tag{4}$$

The power of the bilinear formalism lies in the fact that for multi-soliton solutions the $F$’s are simple polynomials of exponentials [2]. Thus the construction of soliton solutions becomes an algebraic problem. This approach has made possible the investigation of large classes of bilinear equations and the classification of integrable cases [3]. (The integrability of these equations has been confirmed by singularity analysis [4]).

2. Gauge-invariant bilinear operators

An important observation (that has motivated the present line of research) is the relation of the “physical” variable $u = 2\partial_x^2 \log F$ (for the KdV equation) to the Hirota’s function $F$: the gauge transformation $F \rightarrow e^{px+\omega t}F$ leaves $u$ invariant. It turns out that this is a general property of bilinear equations. In fact, one can define the Hirota’s bilinear equations through the requirement of gauge invariance. We will now prove this statement.

Let us introduce a general bilinear expression and ask that it be invariant under the gauge transformation $F \rightarrow e^{\eta}F$ with $\eta = px + \omega t$:

$$\sum_{k=0}^{N} c_k(\partial^k e^{\eta} f)(\partial^{N-k} e^{\eta} g) = e^{2\eta} \sum_{k=0}^{N} c_k(\partial^k f)(\partial^{N-k} g). \tag{5}$$
Expanding the left hand side and equating the coefficients of \((\partial^k f)(\partial^{N-k} g)\) we obtain, for all \(n, m\):

\[
\sum_{k=0}^{N} \binom{k}{n} \binom{N-k}{m} p^{N-n-m} c_k = c_n \delta_{N-n-m}.
\]  

(6)

where \(\binom{k}{n}\) is the binomial coefficient. From the structure of the left hand side, we have the inequalities \(0 \leq n \leq k \leq N - m\). Taking \(m = N - n\) we find that \(k = n\) and (6) is identically satisfied. For all other values of \(m\) \((m = 0, 1, \ldots, N - n - 1)\) we must find those \(c_k\)’s that satisfy

\[
\sum_{k=0}^{N} \binom{k}{n} \binom{N-k}{m} c_k = 0.
\]  

(7)

In the special case \(m = N - n - 1\) (7) reduces to:

\[(N - n)c_n + (n + 1)c_{n+1} = 0,
\]

the solution of which is

\[c_n = (-1)^n \binom{N}{n} c_0.
\]  

(8)

This solution does, in fact, satisfy (7) for all values of \(m\): using (8) we find that the lhs of (7) is just the expression of \((1-1)^{N-n-m} c_0\) and thus equal to zero. So the only gauge invariant bilinear differential operator is (up to a scaling of \(c_0\)) \(\sum_{k=0}^{N} \binom{k}{n} \binom{N-k}{m} (\partial^k \partial^{N-k} = (\partial_1 - \partial_2)^N\), i.e. the Hirota operator \(D_{12}^N\). (The use of the indices (1,2) may appear superfluous at this stage, since in the bilinear case there are only two variables on which \(D\) operators act. However the notation will be particularly useful in the higher multilinear cases).

3. Gauge-invariant trilinear operators

The Hirota bilinear operators can be obtained on the sole requirement (admittedly a strong one) of gauge invariance. In this paper our objective is the extension of the bilinear formalism and the introduction of multilinear operators. Very few results exist in this direction. Satsuma and collaborators have introduced a particular class of trilinear equations, that can be written as a single \((3 \times 3)\) determinant [5]. A full hierarchy of equations was obtained and the richness of the solutions presented are a strong indication of their integrability. However, Satsuma’s approach offers no clue on what a trilinear operator should be. Here we will use the same principle used to find the bilinear \(D\)’s: gauge invariance.

For trilinear expressions the invariance condition writes:

\[
\sum_{k+l+m=N} c_{klm} (\partial^k e^\eta f)(\partial^l e^\eta g)(\partial^m e^\eta h) = e^{3\eta} \sum_{k+l+m=N} c_{klm} (\partial^k f)(\partial^l g)(\partial^m h).
\]  

(9)
In analogy to the bilinear case we find:

$$\sum_{k+l+m=N} c_{klm} \binom{k}{\kappa} \binom{l}{\lambda} \binom{m}{\mu} p^{k+l+m-\kappa-\lambda-\mu} = c_{\kappa\lambda\mu} \delta_{\kappa+\lambda+\mu-N}. \tag{10}$$

For $\kappa + \lambda + \mu = N$ equation (10) is identically satisfied and we are left with

$$\sum_{k+l+m=N} c_{klm} \binom{k}{\kappa} \binom{l}{\lambda} \binom{m}{\mu} = 0 \quad \text{(with } \kappa + \lambda + \mu < N). \tag{11}$$

Consider the $\kappa + \lambda + \mu = N - 1$ equations. They write:

$$(\mu + 1)c_{\kappa\lambda\mu+1} + (\kappa + 1)c_{\kappa+1\lambda\mu} + (\lambda + 1)c_{\kappa\lambda+1\mu} = 0. \tag{12}$$

We have $\frac{N(N+1)}{2}$ such equations for the $\binom{(N+1)(N+2)}{2}$, $c$'s. Thus these equations would determine the $c$'s up to $N + 1$ free coefficients, provided the rank of the system is maximal. This is indeed the case. In fact, the equations $(\kappa, \lambda, \mu = 0)$ contain for the first time $c_{\kappa\lambda\lambda}$ (which can be expressed in terms of the $(N + 1)$ $c_{\kappa\lambda\lambda}$). We can then compute successively the higher-$\mu$ terms up to the last equation $(\kappa = 0, \lambda = 0, \mu = N - 1)$ from which we can solve for $c_{00N-1}$. Thus all $c_{\kappa\lambda\mu}$ for $\mu > 0$ can be expressed in terms of the $c_{\kappa\lambda0}$. So, given that the rank is maximal, we can choose any basis for the $c$'s. A most convenient basis are the following $N + 1$ operators: $(\partial_1 - \partial_2)^n (\partial_1 - \partial_3)^{N-n}$ for $n = 0, \ldots, N$.

Thus the basic building blocks for the trilinear operators are again the Hirota bilinear $D$'s, we must just specify the indices in this case. We thus have $D_{12} \equiv \partial_{x_1} - \partial_{x_2}$, $D_{23} \equiv \partial_{x_2} - \partial_{x_3}$, $D_{31} \equiv \partial_{x_3} - \partial_{x_1}$, but, of course the three are not linearly independent: $D_{12} + D_{23} + D_{31} = 0$. Their action on a ‘triple dot product’ is analogous to the bilinear case:

$$D_{12} f \cdot g \cdot h = (\partial_{x_1} - \partial_{x_2}) f(x_1) g(x_2) h(x_3) \big|_{x_3 = x_2 = x_1 = x} = (f'g - fg') h. \tag{13}$$

The choice of a particular pair of $D$'s as the basic trilinear operators breaks the symmetry between the three coordinates $x_i$'s. It is possible to restore this symmetry by introducing a different basis for the trilinear operators, $T$ and $T^*$:

$$T = \partial_1 + j \partial_2 + j^2 \partial_3, \quad T^* = \partial_1 + j^2 \partial_2 + j \partial_3, \tag{14}$$

where $j$ is the cubic root of unity, $j = e^{2i\pi/3}$. (Note that the star in $T^*$ indicates complex conjugation for the coefficients in $T$ but not for the independent variables).

The price we have to pay for restoring this symmetry is that the operators are now more complicated. Note that $T^n T^m F \cdot F = 0$ unless $n - m \equiv 0 \pmod{3}$, which is the equivalent to the bilinear property $D_n F \cdot F = 0$ unless $n \equiv 0 \pmod{2}$.

The generalization to higher multilinear equations is straightforward. One can introduce the set of $n(n-1)/2$ operators $D_{ij}$ acting on $n$-tuple dot-products $D_{ij} f_1 \cdot f_2 \cdot \ldots \cdot f_n$. (Of course only $n - 1$ of the $D_{ij}$'s are independent, a convenient basis being
the $D_{1j}, j = 2, \ldots n$). As in the trilinear case, one can also construct “symmetric” operators:

$$M_m = \sum_{k=1}^{n-1} z_m^k \partial_k$$

(15)

where the $z_m$’s are the $(n-1)$ $n$-th roots of unity other than one.

4. Examples of multilinear equations

Multilinear operators are not just a trivial extension of the Hirota bilinear formalism. They are necessary for the description of nonlinear evolution equations that cannot be cast in a bilinear form and such equations do exist. An interesting example is the fifth-order equation of the Lax hierarchy [6]:

$$u_{xxxxx} + 10u u_{xxx} + 20u_x u_{xx} + 30u^2 u_x + u_t = 0.$$  

(16)

While this Lax-5 equation does not possess a simple bilinear expression like KdV itself, it has a trilinear form (with $u = 2\partial_x^2 \log F$):

$$(7T_x^6 + 20T_x^3T_x^3 + 27T_xT_t)F \cdot F \cdot F = 0.$$  

(17)

In the previous section we referred to Satsuma’s trilinear equations [5]. The lowest-order one

$$\begin{vmatrix}
F_{yy} & F_y & F_{xy} \\
F_y & F & F_x \\
F_{xy} & F_x & F_{xx}
\end{vmatrix} = 0,$$

(18)

(equivalent, through $F = e^w$, to the Monge-Ampère equation $w_{xy}^2 - w_{xx}w_{yy} = 0$) is gauge-invariant and can be written as:

$$(T_xT_x^*T_yT_y^* - T_x^2T_y^*)F \cdot F = 0.$$  

(19)

(The higher Satsuma equations are given by sets of equations with “dummy” independent variables and it is not clear how to implement the gauge-invariance requirement in such a situation).

The equation:

$$F^2 F_{xxxxy} - FF_y F_{xxxx} - 4FF_x F_{xxx} + 2FF_x F_{xy} + 4F_x F_y F_{xxx} - 2F_y F_{xx}$$

$-4F_x F_{xx} F_{xy} + 4F_x^2 F_{xy} + 4(F^2 F_{xxt} - FF_{xx} F_t - 2FF_x F_{xt} - 2F_x^2 F_t) = 0,$

(20)

obtained as a reduction of a self-dual Yang-Mills equations [7], can also be written as a genuinely trilinear equation:

$$(T_x^4T_y^* + 8T_x^3T_x^*T_y - 36T_x^2T_t)F \cdot F = 0.$$  

(21)

Further examples can be presented and we can, of course, construct also higher multilinear equations (quadri-, penta-, etc.) equations. Instead of dealing with specific
cases let us present here some general considerations. Let us start with a nonlinear (in $u$) equation, of order $k+1$ having the form $u_t + \partial_x P(u, u_x, \ldots, u_{kx}) = 0$. Several well known integrable equations belong to this class. It is simpler to work with the time independent part $P(u, u_x, \ldots, u_{kx}) = 0$ which only involves derivatives up to order $k$. We consider the leading part of $P$ which we assume to be weight-homogeneous in $u$ and $\partial_x$, with $u$ having the same weight as $\partial_x^2$. Then we can transform this leading part to a multilinear expression through the transformation $u = \alpha \partial_x^2 (\log F)$ and obtain generically an $k+2$-multilinear equation. The scaling factor $\alpha$ can then be chosen so as to make the lowest-derivative terms vanish, that is those that appear under the combination $(F'^{k+2} - \frac{k+2}{2} F'' F'^{k} F)$. (This is possible because these two terms have a common factor, polynomial in $\alpha$). Then an $F^2$ term can be factored out and the resulting multilinear equation is at most $k$-linear.

At order five ($k = 4$) we have three integrable equations. We should expect, in principle, these equations to have quadrilinear forms. Some unexpected cancellations, however, do occur. Thus the Sawada-Kotera [8] equation has a bilinear expression, the Lax-5 has the trilinear form we gave in (17), but the Kaup-Kuperschmidt [9] is quadrilinear. At order seven ($k = 6$) three integrable equations were known to exist. The higher Sawada-Kotera has a trilinear expression (see next section) while the 7th order equation in the Lax hierarchy is a pentilinear one. Again, for the higher Kaup-Kuperschmidt equation no extra simplification is possible and this equation is hexalinear. For all these equations the time derivative can be incorporated in the multilinear equations in a very simple way without altering the degree of multilinearity.

Thus the multilinear extension to Hirota bilinear approach has a wide range of applicability (in particular if we allow for multicomponent equations, introducing more than one dependent functions, in analogy to the bilinear case).

5. Singularity analysis of trilinear equations

All the above equations have as a common characteristic their integrability. The study of integrability is, in fact, the motivation behind the multilinear approach. The systematic classification of bilinear equations we presented in [3,4] was based on the study of multisoliton solutions and of the Painlevé property. We intend to come back to the investigation of soliton solutions for our multilinear equations in some future work. In the following paragraph we will limit ourselves to the singularity analysis of trilinear equations involving only one dependent variable, i.e. unicompotent equations. (Let us recall here that in the bilinear case the study of these simplest equations led us to conclusive results).

In order to perform the Painlevé analysis, we shall study the leading (highest-order) part of the equations with just one independent variable. This is sufficient for the computation of dominant singularities and resonances, although for the check of resonance compatibility we would need the full equation, which remains unspecified at this stage. (For example, for the analysis of an equation like (17), we would consider
only the $T^6, T^3 T^3$ terms and not $T_x T_t$, since the influence of the latter would appear only at the resonance condition.

Since the dependent function $u$ in a nonlinear equation is related to the multilinear $F$ through $u = 2\partial^2 \log F$ it is clear that a zero in $F$ induces a pole-like behaviour in $u$. Let us show how one performs the singularity analysis for trilinear equations in a concrete example: $T^2 T^2 f \cdot f \cdot f = 0$. Putting $f \sim x^n$ (for the dominant part) we find $n = 0, 1$ as the only possible behaviours. The first corresponds to a nonsingular Taylor-like expansion, which is always possible. The second behaviour $f \sim x$ corresponds to a simple zero that would give a (double) pole in $u$. The resonances in this case are obtained if we substitute $f = x(1 + \phi x^r)$ and collect terms linear in $\phi$. The result is $r = -1, 0, 1, 6$ and (after a check that no incompatibilities arise at any resonance) we conclude that this equation passes the Painlevé test. This is what one would have expected, had we looked at the nonlinear form of the equation, $u_{xx} + 3u^2 = 0$, which is just the time-independent part of KdV integrated once.

We shall not present the details of the singularity analysis of all the equations that we have studied. The results are summarized below. The notation we are using is the following: $E(n, m)$, (which is identical to $E(m, n)$), represents the expression $T^m T^n F \cdot F \cdot F$. Note that for a given $N = n + m$ there may exist several pairs of $(n, m)$ such that $E(n, m)$ is not identically zero, namely those for which $n \equiv m \pmod{3}$. The leading part of the general equation at order $N$ is then given by a linear combination of all the non-vanishing $E(n, m)$’s. In each case we give below the precise combinations that lead to equations with the Painlevé property. The nonlinear forms of the equations are obtained by the standard substitution $F = e^g$ followed by $u = 2g''$. 

$N = 2:$ \quad $E(1, 1)$

In this case we can write the result also in bilinear form:

$E(1, 1) \propto F(D^2 F \cdot F) = 2F(F'' F - F'^2) = 2e^{3g}g'' = e^{3g}u$.

$N = 3:$ \quad $E(3, 0)$

$E(3, 0) \propto F^{''''} F^2 - 3 F^{'''} F' F + 2 F'^3 = e^{3g}g^{'''} \propto e^{3g}u'$.

$N = 4:$ \quad $E(2, 2)$

Here also we can write the result in bilinear form:

$E(2, 2) \propto F(D^4 F \cdot F) = 2e^{3g}(g^{''''} + 6g'^{''}) = e^{3g}(u'' + 3u^2)$.

This, of course, is just the leading part of the KdV equation in potential form.

$N = 5:$ \quad $E(4, 1)$

$E(4, 1)/F^3 \propto u^{''''} + 6u'u$.

Note that this is the derivative of the expression obtained at $N = 4$.

$N = 6:$ \quad $AE(6, 0) + \mu E(3, 3)$. This is the first case where we have two possible $n, m$ pairs. The $\lambda, \mu$ combinations that pass the Painlevé test are the following

a) $(7E(6, 0) + 20E(3, 3))/F^3 \propto u^{''''} + 10u'u + 5u'^2 + 10u^3$.

This is the leading part of the 5th order equation in the Lax hierarchy of KdV, eq.(16), integrated once.

b) $(-2E(6, 0) + 20E(3, 3))/F^3 \propto u^{''''} + 15u''u + 15u^3$. 
This is the leading part of the Sawada-Kotera equation, integrated once.

c) \((E(6,0) - E(3,3))/F^3 \propto uu'' - u'^2 + u^3.\)

The trilinear form of this case is \(F'''(F''F - F'^2) - F'''F + 2F'''F''F - F''''''\), which can be cast in determinantal form and is a 1-dimensional member of the Satsuma family:

\[
\begin{vmatrix}
F''' & F'' & F''
F''' & F'' & F'
F'' & F' & F
\end{vmatrix}
\]

\(N = 7:\)

\(E(5,2)\)

\(E(5,2)/F^3 \propto u^{(5)} + 15u'''u + 15u''u' + 45u'u^2.\)

This is the Sawada-Kotera equation, i.e. the derivative of the expression obtained in \(N = 6b.\)

\(N = 8:\)

\(\lambda E(7,1) + \mu E(4,4)\) and we take \(u = 6g''\) instead of \(u = 2g''\) used before

a) \((4E(7,1) + 5E(4,4))/F^3 \propto u^{(6)} + 6u'''u + 10u''u' + 5u'^2 + 10u''u^2 + 10u'^2u + \frac{5}{3}u^4\)

This would correspond to a 7th order equation which, we believe, leads to a new integrable case.

b) \((4E(7,1) + 14E(4,4))/F^3 \propto u^{(6)} + 7u'''u + 7u''u' + 7u'^2 + 14u''u^2 + 7u'^2u + \frac{7}{3}u^4\)

This is the higher Sawada-Kotera equation we referred to in the previous section.

c) \((E(7,1) - E(4,4))/F^3 \propto u'''u - 3u''u' + 2u'^2 + 4u''u^2 - 3u'^2u + \frac{2}{3}u^4\)

This new equation looks like an extension of Satsuma’s equation given at \(N=6c\) above, but it cannot be written as a single determinant.

At order \(N = 9\) there are no cases passing the Painlevé test.

\(N = 10:\)

\(\lambda E(8,2) + \mu E(5,5)\) and we take \(u = 30g''\)

\((5E(8,2) + 4E(5,5))/F^3 \propto u^{(8)} + 2u^{(6)}u + 4u^{(5)}u' + 6u'''u'' + 5u'' + \frac{6}{5}u'''u^2 + 4u''u' + 2u'^2 + 2u''u^2 + 4u'''u'' - \frac{14}{15}u'' + \frac{2}{5}u'^2 + \frac{2}{15}u^5\)

This is also a new equation.

Furthermore, no integrable candidates were found at orders \(N = 11, 12.\) The singularity analysis at these higher orders becomes progressively more difficult. (Already at \(N = 12\) there exist three nonvanishing \(n, m\) combinations). It is, thus, not possible to extend our investigation to very high orders (as was done in our study of bilinear equations [4]), but we do believe that no further integrable candidates exist at higher orders. As in the bilinear case, a finite (and rather small) number of unicomponent trilinear equations possess the Painlevé property and can thus be integrable.

6. Conclusion

In the preceding paragraphs we have presented an extension of Hirota’s bilinear formalism that can encompass any degree of multilinearity. The main guide in our investigation has been the requirement that the equations be gauge-invariant. Since our primary objective is the study of integrability, we have also presented a classification of one-component trilinear equations that pass the Painlevé test. The crucial difference between the bilinear and the tri- (and multi-)linear case(s) is that now free parameters enter already at the leading part. This means that the Painlevé analysis of the higher
order unicomponent equations becomes increasingly difficult. Once the leading parts of these equations are fixed, one can study the lower-order terms that can be added without destroying the Painlevé property. Starting from a complete classification of unicomponent equations one can build up multicomponent ones following the approach we presented in [3] for the bilinear case. Another interesting direction would be the computation of the multisoliton solutions of the trilinear equations. This would furnish another check for their possible integrability. Clearly, the domain of multilinear equations is still a *terra incognita* that deserves serious study.

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