Vertex-monochromatic connectivity of strong digraphs

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Abstract

A vertex coloring of a strong digraph \( D \) is a strong vertex-monochromatic connection coloring (SVMC-coloring) if for every pair \( u, v \) of vertices in \( D \) there exists an \((u, v)\)-monochromatic path having all the internal vertices of the same color. Let \( smc_v(D) \) denote the maximum number of colors used in an SVMC-coloring of a digraph \( D \). In this paper we determine the value of \( smc_v(D) \) for the line digraph of a digraph. We also we give conditions to find the exact value of \( smc_v(T) \), where \( T \) is a tournament.

Key words: Digraphs, Vertex-monochromatic colorings.
MSC 05C15, 05C20, 05C40

1 Introduction

Caro and Yuster \[1\] introduced the concept of monochromatic connection of an edge colored graph. An edge-coloring of a graph \( G \) is a monochromatic-
connecting coloring if there exists a monochromatic path between any two vertices of $G$. The study of monochromatic connecting colorings arises from the rainbow connecting coloring problem, in which rainbow paths are considered (a path is said to be rainbow if no two edges of them are colored the same). The monochromatic connection problem has also been studied in oriented graphs [5]. An arc-coloring of a digraph $D$ is a **strongly monochromatic connecting coloring** (SMC-coloring, for short) if for every pair $u, v$ of vertices in $D$ there exists a directed $(u, v)$-monochromatic path and a directed $(v, u)$-monochromatic path. The **strong monochromatic connection number** of a strong digraph $D$, denoted by $smc(D)$, is the maximum number of colors used in an SMC-coloring of $D$. Concerning the strong monochromatic connection number of an oriented graph the following result was proved in [5].

**Theorem 1** Let $D$ be a strongly connected oriented graph of size $m$, and let $\Omega(D)$ be the minimum size of a strongly connected spanning subdigraph of $D$. Then

$$smc(D) = m - \Omega(D) + 1.$$ 

Cai, Li and Wu [8] defined the vertex-version of the monochromatic connecting coloring concept. A path in a vertex colored graph is **vertex-monochromatic** if its internal vertices are colored the same. A vertex-coloring of a graph is a **vertex-monochromatic connecting coloring** (VMC-coloring) if there is a vertex-monochromatic path joining any two vertices of the graph. This concept also can be extended to digraphs. A directed path in a vertex colored digraph is **vertex-monochromatic** if its internal vertices are colored the same. A vertex-coloring of a digraph is a **strongly vertex-monochromatic connecting coloring** (SVMC-coloring) if for every pair $u$ and $v$ of vertices in $D$ there exists a directed $(u, v)$-vertex-monochromatic path and a directed $(v, u)$-vertex-monochromatic path. The **monochromatic vertex-connecting number** of a strong digraph $D$, denoted by $smc_v(D)$, is the maximum number of colors that can be used in a strongly vertex-monochromatic connecting coloring of $D$.

For an overview of the monochromatic and rainbow connection subjects we refer the reader to [4,6,7].

In this work we study the SVMC-colorings of strong digraphs. The paper is organized as follows. In section 2 some basic definitions and notations are given. In section 3 lower and upper bounds for $smc_v(D)$ are presented. In section 4 we focus on the family of line digraphs. Finally, in section 5 we study the strong vertex-monochromatic connection number of strongly connected tournaments.
2 Definitions and Notation

All the digraphs considered in this work are simple; that is, digraphs with no parallel arcs, nor loops are considered. If \((u, v)\) is an arc of \(D\), then we use either \(uv\) or \(u \rightarrow v\) denote it. Two vertices \(u\) and \(v\) of a digraph are adjacent if \(u \rightarrow v\) or \(v \rightarrow u\). All walks, paths and cycles are to be considered directed. A digraph is connected if its underlying graph is connected. A digraph \(D\) is unilateral if, for every pair \(u, v \in V(D)\), either \(u\) is reachable from \(v\), or \(v\) is reachable from \(u\) (or both). A \(p\)-cycle is a cycle of length \(p\). The minimum integer \(p\) for which \(D\) has a \(p\)-cycle is the girth of \(D\) and it is denoted by \(g(D)\). A digraph \(D\) is strongly connected or strong if for every pair of vertices \(u, v \subseteq V(D)\), the vertex \(u\) is reachable from \(v\) and the vertex \(v\) is reachable from \(u\). Given a strong digraph \(D\), we use \(\Omega(D)\) to denote the minimum size of a strongly connected spanning subdigraph of \(D\). Let \(u, v\) be two vertices of \(D\). We say that \(u\) dominates \(v\), or \(v\) is dominated by \(u\), if \(v \in N^+(u)\). A set of vertices \(S \subset V(D)\) is a dominating set if each vertex \(v \in V(D) \setminus S\) is dominated by at least one vertex in \(S\). A set of vertices \(S\) is an absorbing set if for each vertex \(v \in V(D) \setminus S\) there exists a vertex \(u \in S\) such that \(v \in N^-(u)\). An orientation of a complete graph is a tournament. A subdigraph \(H\) is said to be absorbing subdigraph (dominating subdigraph) if the set \(V(H)\) is an absorbing set (dominating set) of \(D\).

Let \(D = (V, A)\) be a strong digraph. The subdigraph induced by a set of vertices \(S\) is denoted by \(D[S]\). Given a positive integer \(p\), let \([p] = \{1, 2, \ldots, p\}\). A vertex \(p\)-coloring of \(D\) is a surjective function \(\Gamma : V \rightarrow [p]\). For each “color” \(i \in [p]\) the set of vertices \(\Gamma^{-1}(i)\) will be called the chromatic class (of color \(i\)), and if \(|\Gamma^{-1}(i)| = 1\), the color \(i\) and the chromatic class \(\Gamma^{-1}(i)\) will be called singular. A subdigraph \(H\) of \(D\) will be called monochromatic if \(A(H)\) is contained in a chromatic class. A \(p\)-coloring \(\Gamma\) of \(D\) is an optimal coloring if \(p = smc(D)\) and \(\Gamma\) is an SMC-coloring of \(D\). For general concepts we may refer the reader to [1][2].

3 Bounds for \(smc_v(D)\)

In this section upper and lower bounds for the strong vertex-monochromatic connection number of a digraph \(D\) are given.

The next proposition is the digraph version of the bounds obtained [8] for the monochromatic vertex-connection number of a graph.

**Proposition 1** Let \(D\) be a strong digraph of order \(n\) and diameter \(d\). Then
i) \( \text{smc}_v(D) = n \) if and only if \( d \leq 2 \).

ii) If \( d \geq 3 \) then \( \text{smc}_v(D) \leq n - d + 2 \).

Proof.

i) Let \( \Gamma : V(D) \to [n] \) be an SVMC-coloring of \( D \). Let \( u, v \) be two vertices in \( D \) such that \( d(u, v) = d \). Let \( P \) be a \((u, v)\)-vertex-monochromatic path. Since \( \Gamma \) is an SVMC-coloring of \( D \) and all the vertices of \( D \) have a different color it follows that the length of \( P \) is at most two. Therefore \( d(u, v) \leq 2 \) and the result follows. If \( d \leq 2 \), the coloring that assigns to every vertex a different color is an SVMC-coloring of \( D \).

ii) Let \( u \) and \( v \) be two vertices in \( D \) such that \( d(u, v) = d \). Let \( P \) be a vertex-monochromatic path connecting \( u \) and \( v \). Observe that there are at least \( d - 2 \) vertices in \( P \) using the same color, therefore \( \text{smc}_v(D) \leq n - (d - 2) \) and the result follows.

\[ \]

The following example shows that the upper bound of item ii) of the above theorem is tight. Let \( D \) be the digraph with vertex set \( V(D) = \{v_1, v_2, \ldots, v_n\} \) and arc set \( A(D) = \{v_i v_1, v_2 v_3 : i = 3, 4, \ldots, n\} \cup \{v_i v_2\} \). Observe that \( \Omega_v(D) = 3 \), \( \text{diam}(D) = 3 \) and \( \text{smc}_v(D) = n - \text{diam}(D) + 2 = n - 1 > n - \Omega_v(D) + 1 \).

\[ 
\text{Fig. 1. smc}_v(D) = n - \text{diam}(D) + 2.
\]

**Theorem 2** Let \( D \) be a strong digraph and let \( \Gamma \) be an SVMC-coloring of \( D \). Let \( S \) be the set of singular chromatic classes of \( \Gamma \) and let \( D^* \) be the digraph induced by \( V(D) \setminus S \).

i) If for every vertex \( v \) of \( D \) there exists a vertex \( x \) such that \( d(v, x) \geq 3 \), then \( V(D^*) \) is a total absorbing set of \( D \).

ii) If for every vertex \( v \) of \( D \) there exists a vertex \( x \) such that \( d(x, v) \geq 3 \), then \( V(D^*) \) is a total dominating set of \( D \).

iii) If \( g(D) \geq 5 \), then \( D^* \) is strong, absorbing and dominating set of \( D \).

Proof. i) Let \( v \) be a vertex of \( D \). Let \( x \in V(D) \) such that \( d(v, x) \geq 3 \).
Since $\Gamma$ is an SVMC-coloring there exists a $(v, x)$-vertex-monochromatic path $P = (v, x_1, \ldots, x_r, x)$ with $r \geq 2$, such that the color of the internal vertices of $P$ belongs to a non-singular class, implying that $D^*$ is a total absorbing set.

**ii)** Let $v$ be a vertex of $D$ and let $x \in V(D)$ such that $d(x, v) \geq 3$. Let $P = (x, x_1, \ldots, x_r, v)$, $r \geq 2$, be a $(x, v)$-vertex-monochromatic path. Since the color of the internal vertices of $P$ are non-singular, then the set $V(D^*)$ is a total dominating set of $D$.

**iii)** Absorbing and dominating properties follows from items i) and ii). Let $u, v \in V(D^*)$ and suppose that there is no $(u, v)$-path in $D^*$. Since $\Gamma$ is an SVMC-coloring, there exists an $(u, v)$-vertex-monochromatic path of length 2 connecting and a $(v, u)$-vertex-monochromatic path of length at least 3 (because $g(D) \geq 5$). Suppose that $P' = (u, x_1, \ldots, x_\ell, v)$. Note that the color used in the internal vertices of $P'$ is a non-singular color. Therefore $x_i \in V(D^*)$ for $i = 1, 2, \ldots, \ell$. Since $\Gamma$ is an SVMC-coloring of $D$ and $g(D) \geq 5$, it follows that the $(x_i, x_{i-1})$-vertex-monochromatic paths are totally contained in $D^*$. Hence $D^*$ is strongly connected. ■

Let $D$ be a strong digraph and let $H$ be an absorbent, dominant and strong subdigraph of $D$. By coloring the vertices of $H$ with one single color and the remaining vertices with distinct colors, an SVMC-coloring of $D$ with $n - |V(H)| + 1$ colors is obtained. Let $\Omega_v(D)$ denote the minimum order of an absorbent, dominant and strong subdigraph of $D$. Therefore

$$smc_v(D) \geq n - \Omega_v(D) + 1. \tag{1}$$

**Theorem 3** Let $D$ be a strong digraph of order $n$ and girth $g(D) \geq 4$. Let $\Gamma$ be an SVMC-coloring of $D$ that uses $smc_v(D)$ colors. If $\ell$ is the minimum cardinality of a non-singular chromatic class of $\Gamma$, then

$$n - \Omega_v(D) + 1 \leq smc_v(D) \leq n - \Omega_v(D) + \frac{\Omega_v(D)}{\ell}.$$

**Proof.** The left hand of the inequality is a consequence of (1). Let $\Gamma$ be an SVMC-coloring of $D$ that uses $smc_v(D)$ colors. Let $a_i$ denote the number of the chromatic classes with cardinality $i$. Observe that $a_1 + a_2 + \cdots + a_r = smc_v(D)$, where $r$ is the cardinality of the largest chromatic class. Let $\ell$ be the minimum cardinality of a non-singular chromatic class. Hence

$$n = a_1 + \sum_{i=\ell}^{r} i a_i \geq \ell (a_1 + \sum_{i=\ell}^{r} i a_i) - (\ell - 1)a_i = \ell smc_v(D) - (\ell - 1)a_1.$$

Therefore,
Let $D^*$ be the subdigraph of $D$ induced by the non-singular classes of $\Gamma$. By Theorem ?? the set $V(D^*)$ is strong, absorbing and dominating. Then $a_1 + \Omega_v(D) \leq n$. Hence

$$smc_v(D) \leq \frac{n + (\ell - 1)a_1}{\ell}.$$ 

Corollary 4 Let $D$ be a strong digraph of order $n$. Then

$$n - \Omega_v(D) + 1 \leq smc_v(D) \leq n - \frac{\Omega_v(D)}{2}.$$ 

4 Line digraphs

Recall that the line digraph $L(D)$ of a digraph $D = (V, A)$ has $A$ for its vertex set and $(e, f)$ is an arc in $L(D)$ whenever the arcs $e$ and $f$ in $D$ have a vertex in common which is the head of $e$ and the tail of $f$. A digraph $D$ is called a line digraph if there exists a digraph $H$ such that $L(H)$ is isomorphic to $D$. In this section we determine the value of $smc_v(D)$ for a line digraph $D$.

Proposition 2 Let $D$ be a strong digraph and let $H$ be a spanning and strong subdigraph of $D$. If $L(H)$ is the subdigraph of $L(D)$ induced by the arcs of $H$, then $L(H)$ is a strong, absorbing and dominating subdigraph of $L(D)$.

Proof. Let $D$ be a strong digraph and let $H$ be a spanning strong subdigraph of $D$. Since $H$ is strong it follows that $L(H)$ is a strong subdigraph of $L(D)$. Furthermore, since $H$ is an spanning and strong subdigraph of $D$ for every vertex $e = (u, v)$ of $L(D)$ there are two vertices $f_1$ and $f_2$ of $L(H)$ such that $f_1 = (w_1, u)$ and $f_2 = (v, w_2)$. Therefore $f_1$ dominates the vertex $e$ and $f_2$ absorbs the vertex $e$. ■

Let $H$ be the line digraph a of digraph $D$. Let $\Gamma : V(H) \rightarrow [k]$ be an SVMC-coloring of $H$. Notice that the coloring $\Gamma$ induces a coloring $\Gamma'$ of the arcs in $D$. Let $\Gamma' : A(D) \rightarrow [k]$ the coloring that assigns to each arc $e$ in $D$ the color $\Gamma(e)$ of the vertex $e \in V(H)$. 

6
Let $D$ be a strong digraph. An ordered pair $(u, v)$ of vertices of $D$ is said to be a bad pair of $D$ if $N^+(u) = \{v\}$ and $N^-(v) = \{u\}$. Observe that if $(u, v)$ is a bad pair then $uv$ is an arc of $D$ and the pair $(v, u)$ is not a bad pair.

**Lemma 1** Let $D$ be a strong digraph and let $H$ be the line digraph of $D$. Let $\Gamma$ be an SVMC-coloring of $H$ and let $\Gamma'$ be the arc coloring of $D$ that assigns to each arc $e \in A(D)$ the color $\Gamma(e)$ of vertex $e \in V(H)$. Given two vertices $u$ and $v$ in $D$ there exists an $(v, u)$-monochromatic path in $D$ if one of the following conditions holds.

i) The ordered pair $(u, v)$ is not a bad pair.

ii) The ordered pair $(u, v)$ is a bad pair and there exists an arc $uv$ in $D$ such that $(v, w)$ is not a bad pair.

iii) The ordered pair $(u, v)$ is a bad pair and there exists an arc $wu$ in $D$ such that $(w, u)$ is not a bad pair.

iv) If the previous cases do not happen and $D$ is different from $C_3$.

**Proof.** Let $D$ be a strong digraph and let $H = L(D)$. Let $u, v$ be two vertices of $D$. Let $\Gamma$ be an SVMC-coloring of $H$ and let $\Gamma'$ be the arc coloring of $D$ induced by $\Gamma$.

i) Suppose that $(u, v)$ is not a bad pair. Assume that there exists a vertex $w \in N^-(v)$ such that $w \neq u$. Since $D$ is strong there exists a vertex $w_1 \in N^+(u)$ (it may happen that $w_1 = v$). Since $\Gamma$ is an SVMC-coloring of $H$, there exists an vertex-monochromatic path $P = (wv, vv_1, \ldots, v_{l-1}v_l, v_lu, uw_1)$ connecting the vertices $wv$ and $uw_1$ of $H$. The path $P$ induces a monochromatic path connecting the vertices $v$ and $u$ in $D$. If there exists a vertex $w \in N^+(u)$ such that $w \neq v$. Since $H$ is strong there exists a vertex $w_1 \in N^-(v)$ and there is a vertex-monochromatic path $P$ connecting the vertices $w_1v$ and $uw$ which induces a monochromatic path connecting the vertices $v$ and $u$ in $D$.

ii) Suppose that $(u, v)$ is a bad pair and there exists a vertex $w \in N^+(v)$ such that $(v, w)$ is not a bad pair. By the above item there is a $(w, v)$-monochromatic path $P$ of the same color of the arc $uv$. If $(w, u)$ is not a bad pair then there exists a $(u, w)$-monochromatic path $P'$ containing the arc $uv$ (because $(u, v)$ is a bad pair). The union of $P$ and $P'$ contains a $(v, u)$-monochromatic path. If $(w, u)$ is a bad pair, then $N^+(w) = \{u\}$ and $N^-(u) = \{w\}$. Since $(v, w)$ is not a bad pair there exists a vertex $z$ such that $z \in N^-(w) \setminus \{v\}$ or $z \in N^+(v) \setminus \{w\}$. Note that $(v, z), (u, z)$ and $(z, u)$ are not bad pairs because $N^+(v) \neq \{z\}$ and $v$ and $u$ and $z$ are not adjacent. Since $(v, z)$ and $(z, u)$ are not bad pairs there is a $(z, v)$-monochromatic path $P$ and a $(u, z)$-monochromatic path $P'$ of the same color because both paths contains the arc $uv$. The union of $P$ and $P'$ contains a $(v, u)$-monochromatic path.

iii) Suppose that the ordered pair $(u, v)$ is a bad pair and there exists an arc
wu in D such that (w, u) is not a bad pair. By item i) there exists a (u, w)-monochromatic path P that uses the arc uv and therefore of the same color of uv. If (v, w) is not a bad pair, then by item i) there exists a (w, v)-monochromatic P containing the arc uv and therefore of the same color of P. The union of P and P contains a (v, u)-monochromatic path in D. Continue assuming that (v, w) is a bad pair. Therefore there is a vertex such that either z ∈ N+(w) or z ∈ N−(u). Observe that (z, v), (v, z) and (w, z) are not bad pairs. Hence there exists a (v, z)-monochromatic path a (z, v)-monochromatic path and a (z, w)-monochromatic path in D. The union of these paths contains a (v, u)-monochromatic path.

iv) If D is isomorphic to C₃, then H is also isomorphic to C₃. Since smcᵥ(C₃) = 3 (see item i) of Proposition 1). The coloring Γ induces a coloring Γ′ of the arcs in D with three colors that is not an SMC-coloring of D.

Theorem 5 Let D be a strong directed graph different from the cycle of length 3. Then

\[ \text{scm}_v(L(D)) = \text{smc}(D). \]

Proof. Let D be a strong digraph of size m. By Theorem 1 it follows that smc(D) = m − Ω(D) + 1. Let H be a strong and spanning subdigraph of D of size Ω(D). By Proposition 2 it follows that L(H) is a strong, absorbing and dominantly connected subdigraph of L(D). By (1) it follows

\[ \text{smc}_v(L(D)) \geq |V(L(D))| - |V(H)| + 1 = m - \Omega(D) + 1 = \text{smc}(D). \]

Observe that if D is different from C₃, then every pair of vertices of D satisfies one of the items of Lemma 1. Hence, if Γ is an SVMC-coloring of L(D) it follows that the coloring Γ′ of D induced Γ is an SMC-coloring of D and therefore smcᵥ(L(D)) ≤ smc(D), and the result follows.

5 Monochromatic vertex-connecting number of tournaments

In this section a condition on Ωᵥ(T) of a strong tournament T is given in order to find the exact value of smcᵥ(T).

Theorem 6 Let T be a strong tournament of diameter d ≥ 6. If Ω(T) ≤ 2d − 6, then

\[ \text{smc}_v(T) = n - \Omega_v(T) + 1. \]

Proof. Let Γ be an SVMC-coloring of T and let u, v be two vertices of T such that d(u, v) = d ≥ 6. Let P = (u, x₁, x₂, ..., xₘ, v) be a (u, v)-vertex-monochromatic path. Since P is a (u, v)-path of T, it follows that s ≥ d − 1.
Observe that the subdigraph induced by \( \{x_1, x_2, \ldots, x_s\} \) is strong. Let \( H \) be the biggest strong sudigraph of \( T \) containing the set \( \{x_1, x_2, \ldots, x_s\} \) such that all the vertices of \( H \) are colored the same. We claim that \( V(H) \) is an absorbing and dominating set of \( T \).

**Claim 1.** \( V(H) \) is an absorbing set of \( T \). Suppose that there exists a vertex \( w \in V(T) \setminus V(H) \) such that \( x \rightarrow w \) for every vertex \( x \in V(H) \). Since \( \Gamma \) is an SVMC-coloring of \( T \) there exists a \((w, v)\)-vertex-monochromatic path \( P' = (w, y_1, y_2, \ldots, y_r, v) \). Note that \((u, x_1, w, y_1, \ldots, y_r, v)\) is a \((u, v)\)-path, hence \( r \geq d - 3 \). If the color of the internal vertices of \( P'\) is different to the color of the vertices in \( H \), then

\[
smc_v(D) \leq n - |V(H)| - (|V(P')| - 2) + 2 \leq n - (d - 1) - (d - 3) + 2 = n - 2d + 6.
\]

Combining the above inequality with (1) it follows that

\[
 n - (2d - 6) + 1 \leq n - \Omega_v(T) + 1 \leq smc_v(T) \leq n - 2d + 6,
\]

giving a contradiction. Hence, the color of the internal vertices of \( P \) is equal to the color of the vertices of \( H \). Observe that \( x_s \rightarrow y_1 \), otherwise \((u, x_1, w, y_1, x_s, v)\) is a \((u, v)\)-path of length 5 contradicting that \( d(u, v) = d \geq 6 \). Furthermore, for every \( y_i \in V(P')\), \( i = 2, \ldots, s \), it follows that \( x \rightarrow y_i \). If \( y_i \rightarrow x \) for some \( i = 2, \ldots, s \), the subdigraph induced by \( V(H) \cup \{y_1, y_2, \ldots, y_s\} \) would be a strong subdigraph of \( T \) bigger than \( H \), contradicting the election of \( H \). Therefore \( y \notin V(H) \) for every \( y \in V(P') \) and

\[
smc_v(D) \leq n - |V(H)| - (|V(P')| - 2) + 1 \leq n - (d - 1) - (d - 3) + 1 = n - 2d + 5,
\]

and using (1), a contradiction is obtained.

**Claim 2.** \( V(H) \) is a dominating set of \( T \). Suppose that there exists a vertex \( w \in V(T) \setminus V(H) \) such that \( w \rightarrow x \) for every \( x \in V(H) \). Let \( P' = (u, y_1, y_2, \ldots, y_r, w) \) be a \((u, w)\)-vertex-monochromatic path. Since \((u, y_1, y_2, \ldots, w, x_s, v)\) is a \((u, v)\)-path, it follows that \( s \geq d - 3 \). If the color of the internal vertices of \( P' \) is different to the color of the vertices of \( H \), using a similar reasoning as in the proof of Claim 1 a contradiction is obtained. Hence, the color of the internal vertices of \( P \) is equal to the color of the vertices of \( H \). Observe that \( x_s \rightarrow y_1 \), otherwise \((u, y_1, x_s, v)\) is a \((u, v)\)-path of length 4, a contradiction. Moreover, for every \( y_i \in V(P')\), \( i = 2, \ldots, s \), it follows that \( x \rightarrow y_i \), for every \( x \in V(H) \). If not, the digraph induced by \( V(H) \cup \{y_1, y_2, \ldots, y_s\} \) is a strong subdigraph of \( T \) bigger than \( H \), giving a contradiction. Therefore \( y \notin V(H) \) for every \( y \in V(P') \) and using a reasoning analogous to the proof of Claim 1 the result is followed.

Hence \( H \) is an absorbent, dominant and strong subdigraph of \( T \). Since \( \Gamma \) is an optimal SVMC-coloring of \( T \) that assign the same color to every vertex in \( H \), it follows that \( smc_v(T) = n - \Omega_v(T) + 1 \) and the result follows. \( \blacksquare \)
Acknowledgments. This research was supported by CONACyT-México, under project CB-222104.

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