Abstract—In this article, we study a target–attacker–defender (TAD) differential game involving one attacker, one target, and multiple defenders. We consider two variations where (a) the attacker and the target have unlimited observation range and the defenders are visibility constrained (b) only the attacker has unlimited observation range and the remaining players are visibility constrained. We model the players’ interactions as a dynamic game with asymmetric information. Here, the visibility constraints of the players induce a visibility network, which encapsulates the visibility information during the evolution of the game. Based on this observation, we introduce network adapted feedback or implementable strategies for visibility constrained players. Using the inverse game theory approach, we obtain network adapted feedback Nash equilibrium strategies. We introduce a consistency criterion for selecting a subset (or refinement) of network adapted feedback Nash strategies, and provide an optimization-based approach for computing them. Finally, we illustrate our results with numerical experiments.

Index Terms—Dynamic games over networks, limited observations, Nash equilibrium, networked multiagent systems, target-attacker-defender (TAD) differential game.

I. INTRODUCTION

The study of networked autonomous multiagent interactions has received a lot of interest in recent years, in the areas such as surveillance, rescue and combat missions, navigation, and analysis of biological behaviors; see [1], [2], [3], [4], [5]. This article is motivated by strategic situations observed in the practical engineering applications, such as protection of critical infrastructures (e.g., aircrafts, naval ships, power grid) against attacks from incoming missiles, interceptors defending an asset against intrusions, and biological interactions involving protection of the young from predators.

The above situations are analyzed using the mathematical framework of pursuit-evasion games [6] with two or three players (groups). A two-player (group) interaction is referred to as a pursuit evasion (PE) game. Here, the objective of the pursuer is to capture the evader, which tries to avoid being captured by the pursuer. A three-player (group) interaction is referred to as a target–attacker–defender (TAD) game [7], [8]. Here, the goal of the attacker is to capture the target, which tries to evade the attacker, and the goal of the defender is to intercept the attacker before the attacker captures the target. Three-player interactions resulting in the rescue of the target by the defender have been studied recently [9]. Clearly, a TAD game is far more complex than a PE game in that the former involves two simultaneous PE-type interactions resulting in more outcomes.

A. Contributions

The existing literature on TAD games assumes that all players have unlimited sensing capabilities, which allow them to observe other players during their interactions. However, in the real world, a player (an engineered agent) has limited sensing capabilities, and can observe other players only when they are within its sensing range. For example, this situation occurs when a team of unmanned ground vehicles (UGVs), equipped with inferior sensing capabilities, must safeguard an asset from potential attacks by a well-equipped UGV with superior sensing capabilities. Further, limited sensing situations can also arise due to potential failures in the sensing equipment during interactions.

The novelty of this article lies in the study of TAD games that involve players with limited visibility capabilities. In many real-world applications in civilian or military settings, limited visibility is an important challenge to address, and has practical implications leading to questions such as 1) how would players adapt their strategies to the evolving visibility information during their interactions? and 2) under what informational assumptions can the players synthesize their implementable strategies? This article aims to address these questions by developing a general framework for analyzing TAD interactions with limited observations. We use differential game methodology, more specifically, the Game of Degree approach, for modeling interactions among the players; see [6] and [10]. We note that players’ visibility constraints induce a (dynamic) directed network, which captures the evolution of visibility information. To be deemed implementable, the strategies of the players must be adapted to this information.

To address question 1), we introduce the notion of network adapted feedback strategies or the implementable strategies for the visibility constrained players. A TAD game with limited observations can have many possible interaction configurations. In this article, we focus on two scenarios. In the first, we assume that the attacker and the target have unlimited visibility range. We assume that the (multiple) defenders have limited visibility...
range, and due to which they act as a team. We model this interaction as a nonzero-sum linear quadratic differential game. In the second, we assume that both the target and the defenders are visibility constrained, due to which they act as a team against the attacker who has an unlimited visibility range. We model this interaction as a zero-sum linear quadratic differential game. We emphasize that our choice of scenarios is canonical, leading to nonzero-sum and zero-sum dynamic game models, and using the framework developed in this article, the other interaction configurations can be studied.

To address question ii), we assume that players use feedback Nash equilibrium strategies as an outcome of their interactions. Since the visibility-constrained players cannot have complete observations, we synthesize their network adapted feedback Nash strategies using an inverse game theory approach, in Theorems 2 and 5. As an inverse problem, we obtain a plethora of implementable strategies. Then, based on the idea that when all the players can observe others, their implementable strategies must be the same as their standard feedback Nash strategies, we develop an information consistency criterion for selecting the implementable strategies. In Theorem 6, we provide an optimization-based approach for synthesizing the implementable strategies. Further, in Theorem 7, we perform sensitivity analysis of visibility radii to analyze the effect of visibility parameters on these strategies.

The rest of this article is organized as follows. Preliminaries and problem formulation are presented in Section II. We analyze the first variation of the TAD game in Section III, and the second variation in Section IV. In Section V, we introduce information consistency-based procedure for selecting the feedback gains of the visibility constrained players. In Section VI, we illustrate our results with numerical simulations. Finally, Section VII concludes this article.

B. Overview of Related Literature

TAD-type interactions were studied in [7] and [8] in the context of defending ships from an incoming torpedo using counter-weapons. A TAD-type interaction referred to as the lady, the bandits, and the bodyguards was proposed in [11]. In [12], the authors study a TAD terminal game and propose attacker strategies for evading the defender while continuing to pursue the target. In [10], the authors study the problem of defending an asset by modeling the interactions as a linear quadratic differential game, and proposed moving horizon strategies for different configurations of the target. In [13], a guidance law for defending a nonmaneuverable aircraft is proposed. In [14], [15], [16], [17], [18], and [19], the authors study the problem of defending aircrafts from an incoming homing missile using defensive missiles by considering various interaction scenarios. In [20], [21], [22], and [23], the authors study interactions where a homing missile tries to pursue an aircraft, and a defender missile aims at intercepting the attacker. In particular, they study cooperative mechanisms between the target-defender team against the attacker so that the defender can intercept the attacker before the attacker can capture the target. Role switching of attacker in TAD games was studied recently in [24]. In [25], the defender’s strategies force the attacker to retreat instead of engaging the target. In [26], the authors study the possibility of role switch as well as the cooperation between the target and defender. The recent tutorial article [27] provides a survey of PE and TAD differential games. In all the above TAD game-related works, all the players are assumed to have unlimited observations without visibility constraints.

In the context of PE games, the authors in [28], [29], and [30] analyze interactions involving players with limited sensing capabilities. In [30], the authors study a PE interaction between one evader with an unlimited observation range and multiple pursuers with limited visibility capabilities. This is the closest reference we could find related to our work. This paper differs from [30] both in scope and content as a TAD game is far more complex than a PE game. In particular, the differences with [30] are as follows. In our work, the structure of network feedback adaptive strategies is provided in a very general setting, in that the feedback gains matrices associated with the neighboring defenders are different, whereas in [30], all the feedback gain matrices of the neighboring pursuers are the same. In a TAD game, different interaction configurations can arise among the players, and we have considered these possibilities in our work in Sections III-A and IV. Further, we develop the notion of information consistency, which was not studied before in the literature, towards the refinement of network adapted feedback Nash strategies.

Notation: Throughout this article, \( \mathbb{R}^n \) denotes the set of \( n \)-dimensional real column vectors, and \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices. The symbol \( \otimes \) denotes the Kronecker product. The transpose of a vector or matrix \( E \) is denoted by \( E' \). The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is denoted by \( ||x||_2 = \sqrt{x'x} \). For any \( x \in \mathbb{R}^n \) and \( S \in \mathbb{R}^{n \times n} \), we denote the quadratic term \( x'Sx \) by \( ||x||_S^2 \), and the Frobenius norm of \( S \) by \( ||S||_F = \sqrt{\text{trace}(SS')} \). \( O_{m \times n} \) denotes the \( m \times n \) matrix with all its entries equal to zero. \( I_n \) denotes identity matrix of size \( n \), and \( e_i' \) denotes the \( i \)th column of \( I_n \). \( e_i \) denotes the \( n \times 1 \) vector with all its entries equal to 1. \( \text{col}\{e_1, \ldots, e_n\} \) denotes the single vector or matrix obtained by stacking the vectors or matrices \( e_1, \ldots, e_n \) vertically. \( \text{diag}\{e_1, \ldots, e_n\} \) denotes the block diagonal matrix obtained by taking the matrices \( e_1, \ldots, e_n \) as diagonal elements in this sequence. A directed network (or a graph) is denoted by a pair \( G := (V, \mathcal{E}) \). \( V = \{v_1, \ldots, v_n\} \) denotes the set of vertices, and \( \mathcal{E} \subseteq \{(v_i, v_j) \in V \times V | v_i, v_j \in V, i \neq j\} \) denotes the set of directed edges without self-loops. A directed edge of \( G \), from \( v_i \) to \( v_j \), is denoted by \( v_i \rightarrow v_j := (v_i, v_j) \).

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Dynamics and Interactions of the Players

We consider a team of \( n \) defenders denoted by \( D := \{d_1, d_2, \ldots, d_n\} \), the target by \( \tau \), and the attacker by \( a \). The set of players is denoted by \( P := D \cup \{\tau, a\} \). We assume that the players interact in a 2-D plane. The dynamics of each player is governed by the following single integrator dynamics:

\[
\begin{bmatrix}
\dot{x}_p(t) \\
\dot{y}_p(t)
\end{bmatrix} = \begin{bmatrix}
u_{px}(t) \\
u_{py}(t)
\end{bmatrix}
\]

(1)
where \((x_p(t), y_p(t)) \in \mathbb{R}^2\) is the position vector of the player \(p \in P\) at time \(t\), \((u_{px}(t), u_{py}(t)) \in \mathbb{R}^2\) represents the control input of player \(p\) at time \(t\), and \((x_ρ(t), y_ρ(t)) \in \mathbb{R}^2\) represents the initial position vector of player \(p\). We denote the state and control vectors of player \(p \in P\) as

\[
X_p(t) = \begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix}, \quad u_p(t) = \begin{bmatrix} u_{px}(t) \\ u_{py}(t) \end{bmatrix}.
\]

In this article, we study the following two variations of TAD-type interactions among the players.

**I.** a) The attacker and target have an unlimited visibility range. The visibility-constrained defenders (target) cooperate as a team and try to intercept the attacker. A nonsuicidal attacker tries to evade the defenders while trying to capture the target, whereas a suicidal attacker—no interest in its survival—only tries to capture the target. The target always tries to evade the attacker.

b) The attacker and target are not aware of the defenders’ visibility constraints. All defenders are aware of their own visibility constraints as well as unlimited visibility capabilities of the attacker and target. Further, defenders also know that the attacker and target are unaware of the defenders’ visibility constraints.

**II.** a) The attacker has an unlimited observation range. The target and defenders have visibility constraints and cooperate as a team. The defenders try to intercept the attacker before it captures the target, which always tries to evade the attacker.

b) The interaction is similar to **I.** b), by replacing the attacker and target with the attacker, and the defenders with defenders-target team.

Besides consideration of multiple defenders, our work is distinguished from the existing literature due to the following features in the interactions.

**F1.** The visibility-constrained players act as a team in achieving their objectives. The limited observation range of the visibility-constrained players induces a time-varying (directed) visibility network, which captures information of the state variables that are accessible to the visibility-constrained players during the interactions.

**F2.** Whenever the visibility-constrained players cannot see other players, the interaction results in a situation where the available information is asymmetric, that is, visibility-constrained players have private information about their visibility capabilities.

Due to the nature of interactions, the state space can be reduced using relative coordinates. That is, at any time instant \(t\), we denote by \(z_p(t) = X_p(t) - X_a(t)\) the displacement vector between the player \(p \in P\) and the attacker \(a\). The global state vector associated with the reduced state space is denoted by \(z(t) = \text{col}\{z_{d_1}(t), \ldots, z_{d_n}(t), z_\tau(t)\} \in \mathbb{R}^{2(n+1)}\). Using this, the dynamic interaction environment of the players can be written compactly as

\[
\dot{z}(t) = \sum_i B_{d_i} u_{d_i}(t) + B_\tau u_\tau(t) + B_a u_a(t)
\]

where \(B_{d_i} = [e_i' 0] \otimes I_2\), \(B_\tau = [0_{1 \times n} 1]' \otimes I_2\), and \(B_a = -1_{n+1} \otimes I_2\).

**B. Network Induced by Visibility Constraints**

We assume that a visibility constrained player \(p \in P\) can see a player \(q \in P\), at time \(t\), when the player \(q\) lies within player \(p\)’s observation radius \(\zeta_p > 0\), that is, when the following condition holds true:

\[
||X_p(t) - X_q(t)||_2 = ||z_p(t) - z_q(t)||_2 \leq \zeta_p.
\]

We set \(\zeta_\tau = \infty\) (\(\zeta_\tau < \infty\)) as target has unlimited (limited) visibility range in the interaction **I.** **II.** The above constraint induces a time-varying directed network \(G(t) := (P, E(t))\), where an outgoing edge \(p \in P\) indicates that a player \(q \in P\) is visible to the visibility constrained player \(p\) at time \(t\).

**C. Termination Criterion**

Let the positive real numbers \(\sigma_{d_i} < \zeta_{d_i}\) \((d_i \in D)\) and \(\sigma_a\) denote the capture radii of the defender \(d_i\) and the attacker \(a\), respectively. The interactions terminate when a defender intercepts the attacker, that is, whenever \(||X_d(t) - X_a(t)||_2 \leq \sigma_{d_i}\) holds for at least one \(d_i \in D\), or when the attacker captures the target, that is, whenever \(||X_a(t) - X_\tau(t)||_2 \leq \sigma_a\) holds.

**D. Problem Statement**

Let \(T < \infty\) denote the duration of the interaction. We assume that the duration of the interaction is large enough so that the termination criterion is satisfied during the interval \([0, T]\). We seek to determine the control strategies, which can be used by the players during the interactions **I.** and **II.**

In the next two sections, using the differential game approach [31], we analyze the interactions **I.** and **II.**
III. ANALYSIS OF INTERACTION

Recall that the visibility constrained defenders act as a team in the interaction. We denote the control input for team of defenders by \( u_d(t) = \text{col}(u_{d_1}(t), \ldots, u_{d_n}(t)) \in \mathbb{R}^{2n} \). Using this, the dynamic interaction environment of the players is written as

\[
\dot{z}(t) = B_d u_d(t) + B_r u_r(t) + B_a u_a(t)
\]

(5)

where \( B_d = [I_n \; 0_{n \times 1}]^T \otimes I_2 \). The objectives of the players are described as follows. The target maximizes its weighted distance with the attacker during the time period \([0, T]\). The attackers jointly minimize the sum of their individual weighted distances with the attacker during the interval \([0, T]\). The objective of the non-suicidal attacker is to maximize the sum of its weighted distances with the defenders while simultaneity minimizing its weighted distance with the target during the time period \([0, T]\). Whereas a suicidal attacker minimizes its weighted distance with the attacker during the time period \([0, T]\). All the players simultaneously minimize their control efforts during the time period \([0, T]\). Using their controls \( u_{d}(\cdot) \), defenders jointly minimize the following objective function subject to (5):

\[
J_d(u_d(\cdot), u_r(\cdot), u_a(\cdot)) = \frac{1}{2} \sum_{j=1}^{n} \|z_{d_j}(T)\|_{F_{d_j}}^2 + \frac{1}{2} \int_{0}^{T} \left( \sum_{j=1}^{n} \|z_{d_j}(t)\|_{Q_{d_j}}^2 + \|u_{d_j}(t)\|_{R_{d_j}}^2 \right) dt
\]

(6a)

Using the controls \( u_r(\cdot) \), the target minimizes the following objective function subject to (5):

\[
J_T(u_d(\cdot), u_r(\cdot), u_a(\cdot)) = \frac{1}{2} \|z(T)\|_{F_T}^2 + \frac{1}{2} \int_{0}^{T} \left( \|u_r(t)\|_{R_T}^2 - \|z(t)\|_{Q_T}^2 \right) dt
\]

(6b)

Using the controls \( u_a(\cdot) \), the attacker minimizes the following objective function subject to (5):

\[
J_a(u_d(\cdot), u_r(\cdot), u_a(\cdot)) = \frac{1}{2} \|z(T)\|_{F_a}^2 - \frac{\lambda}{2} \sum_{j=1}^{n} \|z_{d_j}(T)\|_{Q_{d_{adj}}}^2 + \frac{\lambda}{2} \sum_{j=1}^{n} \|z_{d_j}(T)\|_{R_{adj}}^2 + \|u_a(t)\|_{R_a}^2 \right) dt
\]

(6c)

where \( F_a = f_{ap} I_2, \; F_p = f_{pa} I_2, \; Q_a = q_{ap} I_2, \; Q_p = q_{pa} I_2, \; p \in D \cup \{\tau, a\}, \; R_T = r_{p} I_2, \; p \in \{\tau, a\} \) with \( f_{ap} > 0, \; q_{ap} > 0, \; q_{pa} > 0, \) and \( r_p > 0 \). Using these, the matrices associated with the terminal costs are given by \( F_d = \text{diag}\{f_{d_1}, \ldots, f_{d_n}, 0\} \otimes I_2 \), \( F_T = \text{diag}\{0_{2n \times 2n}, -f_{r_{\tau}} I_2\} \), and \( F_a = \text{diag}\{-\lambda f_{d_1}, \ldots, -\lambda f_{d_n}, -f_{a} I_2\} \otimes I_2 \). The matrices associated with the instantaneous costs are given by \( Q_d = \text{diag}\{q_{d_1}, \ldots, q_{d_n}, 0\} \otimes I_2 \), \( Q_T = \text{diag}\{0_{2n \times 2n}, -\gamma_r I_2\} \), and \( Q_a = \text{diag}\{-\gamma_{aq_{d_1}}, \ldots, -\gamma_{aq_{d_n}}, -\gamma_a I_2\} \). Finally, the control cost parameter \( R_d \) is given by \( R_d = \text{diag}\{r_{d_1}, \ldots, r_{d_n}\} \otimes I_2 \). The parameter \( \lambda \in (0, 1) \) in (6c) is set to \( \lambda = 1 \) for a non-suicidal (suicidal) attacker.

As reflected in the players’ objectives, both conflict and cooperation coexist from the strategic interaction of players, and necessitates the analysis using nonzero sum differential games [31]. In particular, the state dynamics and objectives of the players defined in (5)-(6) constitute a nonzero-sum linear quadratic differential game (NZLQDG); see [31] and [32]. In a differential game, the strategies or the controls used by players depend upon the information available to them during the game, also referred to as information structure. In the feedback information structure, the control of a player \( p \in \mathcal{P} \) at time \( t \in [0, T] \) is a function of time and state variable \( z(t) \), that is, \( u_p(t) = \gamma_p(t, z(t)) \), where the mapping \( \gamma_p : [0, T] \times \mathbb{R}^{2(n+1)} \to \mathbb{R}^{2} \) is a strategy of the player \( p \). In this article, we assume feedback strategies due to their robustness toward perturbations in the state variable, also referred to as strong time consistency property; see [31, Definition 5.14]. Due to linear dynamics (5) and quadratic objectives (6) we restrict to linear feedback strategies, and the set of feedback strategies of a player \( p \in \mathcal{P} \) is given by

\[
\Gamma_p := \left\{ \gamma_p : [0, T] \times \mathbb{R}^{2(n+1)} \to \mathbb{R}^{2} \right\}
\]

(7)

Since the defenders act as a team, we denote by \( \gamma_d = \text{col}(\gamma_{d_1}, \ldots, \gamma_{d_n}) \) and \( \Gamma_d = \Gamma_{d_1} \times \cdots \times \Gamma_{d_n} \) as their joint feedback strategy and joint feedback strategy set, respectively.

A. Network Adapted Feedback Information Structure

In this section, we motivate the need for modifying the information structure of the defenders due to their visibility constraints. As the attacker, target, and the team of defenders individually minimize their interrelated objectives, Nash equilibrium [31] is a natural choice for the outcome of NZLQDG. The Nash equilibrium in feedback strategies is defined as follows.

**Definition 1 (Nash equilibrium):** The strategy profile \( (\gamma_d^{*}, \gamma_r^{*}, \gamma_a^{*}) \) is a feedback Nash equilibrium (FNE) for NZLQDG if the following set of inequalities holds true:

\[
J_d(\gamma_d, \gamma_r^{*}, \gamma_a^{*}) \leq J_d(\gamma_d, \gamma_r^{*}, \gamma_a^{*}) \; \forall \gamma_d \in \Gamma_d
\]

(8a)

\[
J_f(\gamma_d^{*}, \gamma_r^{*}, \gamma_a^{*}) \leq J_f(\gamma_d^{*}, \gamma_r^{*}, \gamma_a^{*}) \; \forall \gamma_r^{*} \in \Gamma_r
\]

(8b)

\[
J_a(\gamma_d^{*}, \gamma_r^{*}, \gamma_a^{*}) \leq J_a(\gamma_d^{*}, \gamma_r^{*}, \gamma_a^{*}) \; \forall \gamma_a^{*} \in \Gamma_a
\]

(8c)

The following theorem from [32] characterizes FNE.
Theorem 1: [32, Theorem 8.3] Consider the \((n + 2)\)-player finite horizon NZLQDG described by (5)–(6). This game has, for every initial state, a linear FNE if and only if the following set of coupled Riccati differential equations (RDE) has a set of symmetric solutions \(\{P_d(t), P_\tau(t), P_\alpha(t)\}\) on \([0, T]\)

\[
\begin{align*}
\dot{P}_d(t) &= P_d(t)S_dP_d(t) + P_d(t)S_{\tau}P_d(t) + P_d(t)S_{\alpha}P_d(t) - Q_d \\
\dot{P}_\tau(t) &= P_d(t)S_dP_\tau(t) + P_\tau(t)S_dP_\tau(t) + P_\tau(t)S_{\alpha}P_\tau(t) - Q_\tau \\
\dot{P}_\alpha(t) &= P_d(t)S_dP_\alpha(t) + P_\alpha(t)S_dP_\alpha(t) + P_\alpha(t)S_{\tau}P_\alpha(t) - Q_\alpha
\end{align*}
\]

with \(P_d(T) = F_d, P_\tau(T) = F_\tau, P_\alpha(T) = F_\alpha\). Moreover, in that case there is a unique equilibrium. The FNE control actions of player \(p \in \{d, \tau, a\}\) is given by

\[
u^*_p(t) = \gamma^*_p(t, z(t)) = -R^{-1}_pB'_pP_p(t)z(t).
\]

Due to their unlimited observation range, the attacker and target can implement their FNE strategies (10) as they have access to the state information \(z(t)\) for all \(t \in [0, T]\). The FNE strategy of the defender \(d_i\) \((d_i \in D)\) can be rewritten as

\[
\begin{align*}
\text{u}^{*}_{d_i}(t) &= \left(e_n^i \otimes I_2\right)\text{u}^{*}_d(t) \\
&= -r_{d_i}^{-1}\left[\sum_{j=1}^{n+1} P^{ij}_d(t)z_{d_j}(t) + r_{d_i}^{-1}P_{d_i}^{n+1}(t)(z_{d_i}(t) - z_\tau(t)) + r_{d_i}^{-1}\sum_{j=1}^{n} P^{ij}_d(t)(z_{d_j}(t) - z_{d_j}(t))\right]
\end{align*}
\]

where matrix \(P^{ij}_d(t)\) is the \(ith\) row and \(jth\) column element (a block matrix) obtained by partitioning the matrix \(P_d(t)\) into block matrices of dimension \(2 \times 2\).

Remark 1: From (11), it is clear that strategy of the defender \(d_i\) not only depends on the visibility of the attacker and target, but also on the visibility of the other defenders \(D\{d_i\}\). This implies, the FNE strategy (10) is not implementable by the defender under limited observations. More specifically, at time \(t\), if a player \(p \in P\{d_i\}\) lies outside the visibility range of the defender \(d_i\), then the coefficient of the term \(X_{d_i}(t) - X_{d_j}(t)\) in (11) must be zero for the defender \(d_i\) to implement the FNE control (11) at time \(t\). In other words, the feedback strategies (7) of the defenders must be adapted to visibility network \(\mathcal{G}(t) := (\mathcal{P}, \mathcal{E}(t))\), induced by (4) at every time instant \(t \in [0, T]\), to be deemed implementable.

We recall that at any time \(t \in [0, T]\), an outgoing edge \(d_i \rightarrow p \in \mathcal{E}(t)\) in the visibility network \(\mathcal{G}(t) := (\mathcal{P}, \mathcal{E}(t))\) indicates that the defender \(d_i\) can see the player \(p \in \mathcal{P}\{d_i\}\) at time \(t\), whenever \(\|X_{d_i}(t) - X_p(t)\|_2 = \|z_{d_i}(t) - z_p(t)\|_2 \leq \zeta_{d_i}\). Using this, for the defender \(d_i\) \((d_i \in D)\) to indicate the visibility of a player \(q \in \{a, \tau\}\) at time \(t \in [0, T]\), we define the binary function \(\phi^*_d : [0, T] \rightarrow \mathbb{R} (q \in \{a, \tau\})\) as

\[
\phi^*_d(t) = \begin{cases}
1 & d_i \rightarrow q \in \mathcal{E}(t) \\
0 & d_i \rightarrow q \notin \mathcal{E}(t)
\end{cases}, \quad q \in \{a, \tau\}.
\]

To indicate the visibility of other defenders \(d_j \in D\{d_i\}\), we define the following binary matrix function \(\text{Ad} : [0, T] \rightarrow \mathbb{R}^{n \times n}\) with the \(ij\)th entry defined as

\[
[\text{Ad}(t)]_{ij} = \begin{cases}
1 & d_i \rightarrow d_j \in \mathcal{E}(t) \\
0 & d_i \rightarrow d_j \notin \mathcal{E}(t)
\end{cases}.
\]

Here, \(\text{Ad}(t)\) represents the out-degree adjacency matrix associated with the subnetwork of \(\mathcal{G}(t)\) as the vertex set. Due to the reduced state space, we can define the \(n \times (n + 1)\) augmented adjacency matrix as follows:

\[
\mathbf{A}(t) = \left[ \Phi_d(t) + \text{Ad}(t) \right] \Phi_\tau(t)
\]

where

\[
\Phi_d(t) = \text{diag}\{\phi^*_d(t), \phi^*_a(t), \ldots, \phi^*_a(t)\}
\]

\[
\Phi_\tau(t) = \text{col}\{\phi^*_d(t), \phi^*_a(t), \ldots, \phi^*_a(t)\}.
\]

The \(ith\) row of the matrix \(\mathbf{A}(t)\), denoted by \(\mathbf{A}(t)_{\bullet i}\), provides the information about all the players \(p \in P\{d_i\}\) who are visible to \(d_i\) at time \(t \in [0, T]\). In particular, the \(ith\) and \((n + 1)th\) elements of \(\mathbf{A}(t)_{\bullet i}\) indicate the visibility of the attacker and the target, respectively, and the \(jth\) element, with \(j \neq \{i, n + 1\}\) indicates the visibility to the defender \(d_j \in D\{d_i\}\). Using this, we define the implementable or network adapted feedback strategies for defenders as follows.

Definition 2: The set of network adapted linear feedback strategies of the defender \(d_i\) \((d_i \in D)\) is given by

\[
\Gamma^\text{Ad}_d := \{\gamma^\text{Ad}_d : [0, T] \times \mathbb{R}^{2(n+1)} \rightarrow \mathbb{R}^2\}
\]

\[
u^*_d(t) := \gamma^\text{Ad}_d(t, z(t); K_d(t)) = K_d(t)\mathcal{I}_d(t)z(t)
\]

\[
K_d(t) \in \mathbb{R}^{2(n+1)}, t \in [0, T]\}
\]

and \(\Gamma^\text{Ad} = \Gamma^\text{Ad}_d \times \cdots \times \Gamma^\text{Ad}_{d_n}\) denotes the set of defender team’s joint network adapted strategies.

Here, the matrix function \(\mathcal{I}_d(t) : [0, T] \rightarrow \mathbb{R}^{(2n+1)\times (2n+1)}\) captures state information of players in \(P\{d_i\}\) who are visible to defender \(d_i\) at time \(t\), which is defined by

\[
\mathcal{I}_d(t) := \text{diag}\{\mathbf{A}(t)_{\bullet i}\}
\]

and the gain matrix \(K_d(t)\) can be partitioned into \(n + 1\) block matrices of size \(2 \times 2\) as follows:

\[
K_d(t) = \begin{bmatrix}
K^d_{i1}(t) & \cdots & K^d_{i(n+1)}(t) & K^d_{i(n+1)+1}(t)
\end{bmatrix}
\]

The defender team’s joint network adapted linear feedback strategy given by

\[
\gamma^\text{Ad}(t, z(t); K_d(t)) = K_d(t)\mathcal{I}_d(t)z(t)
\]

\[
K_d(t) = \text{diag}\{K_{d_1}(t), \ldots, K_{d_n}(t)\}
\]
\[ I_d(t) = \text{col}\{I_{d_1}(t), \ldots, I_{d_n}(t)\}. \tag{18c} \]

**B. Network Adapted Feedback Nash Equilibrium Strategies**

In this section, we derive network adapted feedback Nash equilibrium (NAFNE) strategies of the defenders using Theorem 1.

**Remark 2:** Due to feature F2, in interaction I1, the attacker and the target being unaware of defenders’ visibility constraints becomes common knowledge of the game; see [33]. As a result, the outcome of the game is that the attacker and target will use their FNE strategies \( u^*_d(t) = \gamma^*_d(t), z(t) = -R_d^{-1}B_dP(t)z(t), \) \( p \in \{a, \tau\}, \) for all \( t \in [0, T] \) associated with the game where all the players have unlimited observation range. Further, defenders also know that attacker and target will use their FNE strategies.

When the defenders use their network adapted feedback strategy \( \gamma^a_d \), it is required that the strategy profile \( (\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \) is a FNE. However, this strategy profile cannot be a FNE for NZLQDG, with players’ objectives given by (6), unless all the defenders have unlimited observation range. To see this, we recall that in a game setting, the objectives of the players are interrelated. Further, we note that the defenders, due to lack of full state information, deviate unilaterally from using (10) while implementing \( \gamma^a_d \). From Remark 2, as the attacker and target strategies are fixed at their standard FNE strategies (10), the strategy profile \( (\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \) cannot be a Nash equilibrium. This implies, the performance indices or objectives of the players for which the strategy profile \( (\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \) is a FNE differs from the objectives (6).

In the following theorem, we use inverse game theory approach based on strategies obtained in Theorem 1; see also [30] in the context of a PE game. In particular, we construct a class of performance indices, parameterized by the gain matrices \( K_d(t), t \in [0, T] \) with respect to which the strategy profile \( (\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \) is a NAFNE.

**Theorem 2:** Consider the \((n + 2)\)-player finite horizon NZLQDG described by (5)-(6). For an arbitrary gain matrix \( K_d(t) = \text{diag}\{K_{d_1}(t), K_{d_2}(t), \ldots, K_{d_n}(t)\}, t \in [0, T], \) the strategy profile \( (\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \), with \( \gamma^a_d(t, z(t)) = -R_d^{-1}B_dP(t)z(t), \gamma^\tau_d(t, z(t)) = -R_a^{-1}B_aP(t)z(t) \) and \( \gamma^a_a(t, z(t); K_d(t)) = K_d(t)I_d(t); t \in [0, T], \) forms a NAFNE characterized by the inequalities

\[
J^d_d(\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \leq J^d_d(\tilde{\gamma}^a_d, \gamma^\tau_d, \gamma^a_a) \quad \forall \tilde{\gamma}^a_d \in \Gamma^a_d
\]

\[
J^\tau(\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \leq J^\tau(\tilde{\gamma}^a_d, \gamma^\tau_d, \gamma^a_a) \quad \forall \gamma^\tau_d \in \Gamma^\tau
\]

\[
J^a(\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \leq J^a(\tilde{\gamma}^a_d, \gamma^\tau_d, \gamma^a_a) \quad \forall \gamma^a_a \in \Gamma^a
\]

with parametric performance indices \( (J^d_d, J^\tau, J^a; K_d(\cdot)) \) given by

\[
J^d_d(u_d(\cdot), u_\tau(\cdot), u_a(\cdot)) = \frac{1}{2}||z(T)||^2_{P_d} + \frac{1}{2} \int_0^T \left(||z(t)||^2_{Q_d^M(t)} + ||u_d(t)||^2_{R_d} \right) dt
\]

\[
J^\tau(u_d(\cdot), u_\tau(\cdot), u_a(\cdot)) = \frac{1}{2}||z(T)||^2_{P_d}
\]

\[
+ \frac{1}{2} \int_0^T \left(||z(t)||^2_{Q_d^M(t)} + ||u_\tau(t)||^2_{R_\tau(t)} \right) dt \tag{19b}
\]

\[
J^a(u_d(\cdot), u_\tau(\cdot), u_a(\cdot)) = \frac{1}{2}||z(T)||^2_{P_a}
\]

\[
+ \frac{1}{2} \int_0^T \left(||z(t)||^2_{Q_d^M(t)} + ||u_a(t)||^2_{R_a(t)} \right) dt \tag{19c}
\]

where

\[
\Delta Q^d_d(t) = -P_d(t)B_dR_d^{-1}S_1(t) - S'_1(t)R_d^{-1}B_d'P_d(t)
\]

\[
\Delta Q^\tau(t) = -P_d(t)B_dR_d^{-1}S'_1(t)R_d^{-1}B_d'P_d(t) + I_d(t)K_d(t)R_dK_d(t)I_d(t)
\]

\[
\Delta Q^a(t) = -P_a(t)B_aR_a^{-1}S_1(t) - S'_1(t)R_a^{-1}B_a'P_a(t).
\]

Here, \( \Delta Q^d_d(t) = Q^d_d(t) - Q_p, p \in \{d, a, \tau\}, \) and \( P_d(t), P_a(t), P_\tau(t) \) are solutions of symmetric coupled RDE (9).

**Remark 3:** Recalling feature F2 we note that the interaction I1 is a game of asymmetric information. In the language of Bayesian games [33], this implies that the attacker’s and target’s beliefs, over defenders’ type set, would assign probability equal to one to the type where defenders’ have unlimited observations. As a result, the attacker and the target use their standard FNE strategies associated with the performance indices \( (J^d_d, J^\tau, J^a; K_d(\cdot)) \), whereas the defenders use their NAFNE strategies associated with the parametric performance indices \( (J^d_d, J^\tau, J^a; K_d(\cdot)) \).

Theorem 2 characterizes parametric performance indices \( (J^d_d, J^\tau, J^a; K_d(\cdot)) \) for which the strategy profile \( (\gamma^a_d, \gamma^\tau_d, \gamma^a_a) \) is a NAFNE. Since the choice of gain matrices \( K_d(t), t \in [0, T] \) is arbitrary, we obtain a very large class of performance indices. In Section V, we develop a consistency criterion for selecting the gain matrices.

When the attacker is nonsuicidal, the interactions inherently involve two simultaneous PE games. First one involving the attacker and the defenders’ team, and the second one involving the attacker and the target. Though we obtain the implementable strategies, through Theorem 2, it is difficult to geometrically characterize the trajectories of the players. However, when the attacker is suicidal, then the defenders are only reacting to a single PE interaction involving the attacker and the target. In the next theorem, we recover the classical result [6] that the trajectories of the attacker and the target evolve along a straight line.

**Theorem 3:** Consider the \((n + 2)\)-player finite horizon NZLQDG described by (5)-(6) with \( \lambda = 0 \). Then, the suicidal attacker and the target move on the straight line joining their locations at time \( t = 0 \). Further, the visibility constraints of the defenders have no effect on the control and state trajectories of the attacker and the target.
The (lengthy) proof of Theorem 3 can be accessed from the online supplementary version [34].

Remark 4: In Theorem 3, though the defenders’ visibility constraints have no effect on strategies of the attacker and target, they can influence the eventual outcome and the termination time of the game.

IV. ANALYSIS OF INTERACTION 12

In this section, we analyze interactions 12 where the visibly constrained defenders and target cooperate as a team against the nonsuicidal attacker. Following a similar approach developed in Section III-A, we model this interaction as a two-player zero-sum linear quadratic differential game (ZLQDG) [31] with attacker as the first player, and the team of defenders and the target as the second player. The dynamics (5) of the players can be rewritten as

\[
\dot{z}(t) = B_{dr} u_{dr}(t) + B_a u_a(t)
\]

where \( u_{dr}(t) = \text{col}(u_d(t), u_t(t)) \), \( t \in [0, T] \), \( B_{dr} = [B_d B_r] \).

The objective function minimized by the attacker and maximized by the team of defenders and the target is given by

\[
J(u_{dr}(\cdot), u_a(\cdot)) = \frac{1}{2} ||z(T)||^2_F + \frac{1}{2} \int_0^T \left( ||u_a(t)||^2_{R_a} + ||u_{dr}(t)||^2_{R_{dr}} + ||z(t)||^2_{Q_a} \right) dt
\]

where \( R_{dr} = \text{diag}\{R_{d1}, \ldots, R_{dn}, R_r\} \), \( F = \text{diag}\{-F_{ad_r}, F_{ar}\} \), and \( Q = \text{diag}\{-Q_{ad_r}, F_{ar}\} \). Let \( \gamma_{dr} = \text{col}(\gamma_d, \gamma_r) \in \Gamma_{dr} \times \Gamma_r \) represents the feedback strategy set of the defenders and target team.

The strategy profile \((\gamma_{dr}, \gamma_a)\) is a FNE for the ZLQDG if the following set of inequalities hold true:

\[
J(\gamma_{dr}, \gamma_a) \geq J(\gamma_{dr}, \gamma_a^+) \quad \forall \gamma_{dr} \in \Gamma_{dr}
\]

\[
J(\gamma_{dr}, \gamma_a) \leq J(\gamma_{dr}, \gamma_a) \quad \forall \gamma_a \in \Gamma_a.
\]

The next theorem from [32] provides conditions for the existence of FNE associated with ZLQDG.

Theorem 4: [32, Theorem 8.4] Consider the ZLQDG described by (21)-(22). This game has a FNE, denoted by \((\gamma_{dr}^*, \gamma_a^*)\), for every initial state, if and only if the following RDE has a solution:

\[
\dot{P}(t) = -Q + P(t) (S_a - S_{dr}) P(t), \quad P(T) = F
\]

where \( S_i = B_i R_i^{-1} B_i' \), \( i \in \{dr, a\} \). Moreover, if (24) has a solution, the game has a unique equilibrium. The equilibrium actions are given by

\[
u_{dr}^a(t) = \gamma_{dr}^a(t, z(t)) = -R_{dr}^{-1} B_{dr}' P(t) z(t)
\]

\[
u_a^a(t) = \gamma_a^a(t, z(t)) = R_{dr}^{-1} B_{dr}' P(t) z(t).
\]

The conditions under which the RDE (24) admits a solution follow from [32, Corollary 5.13]. The equilibrium team strategy (25b) can be decomposed as follows:

\[
u_{dr}(t) = r_{dr} \left[ \sum_{j=1}^{n+1} P^{ij}(t) \right] z_{dr}(t)
\]

where matrix \( P^{ij}(t) \) is the \( (i, j) \)-th block matrix of size \( n \times n \) obtained by partitioning the matrix \( P(t) \) into block matrices of dimension \( 2 \times 2 \).

The set of network adapted linear feedback strategies for the target \( \tau \) is given by

\[
\Gamma_{\tau}^{Ad} := \left\{ \gamma_{\tau}^{Ad} : [0, T] \times \mathbb{R}^{2\times(n+1)} \rightarrow \mathbb{R}^{2\times2n} \right\}
\]

\[
u_{\tau}(t) := \gamma_{\tau}^{Ad}(t, z(t); K_{\tau}(t)) = K_{\tau}(t) J_{\tau}(t) z(t)
\]

\[K_{\tau}(t) \in \mathbb{R}^{2\times2n+1}, \quad t \in [0, T].\]

Here, the matrix function \( J_{\tau} : [0, T] \rightarrow \mathbb{R}^{2\times2n+1} \) captures state information of players in \( P \setminus \{\tau\} \) who are visible to target at time \( t \), which is defined by

\[
J_{\tau}(t) := \Phi_{\tau}(t) \left( I_{n+1} - e_{n+1}^t \otimes I_{n+1} - e_{n+1}^t \right) \bigoplus I_2
\]

where \( \Phi_{\tau}(t) = \text{diag}\{\phi_{\tau}^1(t), \ldots, \phi_{\tau}^n(t), \phi_{\tau}^e(t)\} \) and the gain matrix \( K_{\tau} \) can be partitioned into \( n + 1 \) block matrices of size \( 2 \times 2 \) as follows:

\[
K_{\tau}(t) = \begin{bmatrix} K_{\tau}^{d1}(t) & \cdots & K_{\tau}^{dn}(t) & K_{\tau}^e(t) \end{bmatrix}.
\]

The network adapted feedback team strategy set is denoted by \( \Gamma_{Ad} := \Gamma_{Ad} \times \Gamma_{Ad} \). Following Remark 2, we require that an arbitrary network adapted feedback strategy of the defenders \( \gamma_{\tau}^{Ad}(t, z(t)) = K_{\tau}(t) J_{\tau}(t) z(t) \), when coupled with the target’s network adapted feedback strategy (28) as

\[
\gamma_{\tau}^{Ad}(t, z(t); (K_{\tau}(t), K_{\tau}(t))) = \begin{bmatrix} \gamma_{\tau}^{Ad}(t, z(t); K_{\tau}(t)) \\ \gamma_{\tau}^{Ad}(t, z(t); K_{\tau}(t)) \end{bmatrix}
\]

forms a FNE along with the attacker’s strategy (25a). However, the strategy profile \((\gamma_{dr}^{Ad}, \gamma_a^{Ad})\) cannot be a FNE for the performance indices (22) due to defenders’ visibility constraint. Similar to Theorem 2, in the next theorem we construct a clas
of performance indices, parameterized by the gain matrices \((K_d(t), K_r(t)) t \in [0, T]\) with respect to which the strategy profile \((\gamma_d(t), \gamma_a(t))\) is a NAFNE.

**Theorem 5:** Consider 2-player ZLQDG described by (21)-(22). For an arbitrary gain matrices \(K_d(t) = \text{diag}\{K_{d1}(t), \ldots, K_{dn}(t)\}\), and \(K_r(t), t \in [0, T]\) the strategy profile \((\gamma_d(t), \gamma_a(t))\) with \(\gamma_a(t, z(t)) = -R_a^{-1}B_a^TP(t)z(t)\) and \(\gamma_d(t, z(t); (K_d(t), K_r(t)))\) forms a NAFNE characterized by the inequalities

\[
J^{Ad}(\gamma_d(t), \gamma_a(t)) \geq J^{Ad}(\gamma_d(t), \gamma_a^*), \quad \forall \gamma_a \in \Gamma_a
\]

\[
J^{Ad}(\gamma_d(t), \gamma_a^*) \leq J^{Ad}(\gamma_d(t), \gamma_a(t)), \quad \forall \gamma_a \in \Gamma_a
\]

with the parametric performance index \(J^{Ad}((K_d(.), K_r(.)))\) given by

\[
J^{Ad}(u_d(t), u_a(t)) = \frac{1}{2}\|z(t)\|^2 + \frac{1}{2}\int_0^T (\|z(t)\|^2Q_n(t) + u_d(t)S_2(t)z(t) + z(t)S_2^t(t)u_a(t) + u_a(t)S_3(t)z(t) + z(t)S_3^t(t)u_a(t) + \|u_a(t)\|^2R_a - \|u_d(t)\|^2R_d) dt
\]

where

\[
\Delta Q^{Ad}(t) = P(t)B_dR_d^{-1}B_d^TP(t) + P(t)B_rR_r^{-1}B_r^TP(t) - T_d(t)K'_dR_d(t)I_d(t) - T_r(t)K'_rR_r(t)I_r(t)
\]

\[
T_d(t)K'_dR_d(t)I_d(t) - T'_r(t)K'_rR_r(t)I_r(t)
\]

\[
S_2(t) = R_dK_d(t)I_d(t) - B_d^TP(t),
\]

\[
S_3(t) = R_rK_r(t)I_r(t) - B_r^TP(t).
\]

Here, \(\Delta Q^{Ad}(t) = Q^{Ad}(t) - Q(t)\) is the solution of the RDE (24).

The proof of Theorem 5 follows along the lines of the proof of Theorem 2, and can be accessed from the online supplementary version [34].

**Remark 5**: Similar to Remark 3, the feature \(F2\) in interaction \(II\) results in a game of asymmetric information. As a result, the attacker uses its standard FNE strategies associated with the performance index \(I\), whereas the defenders and target use their NAFNE strategies associated with the parametric performance index \(J^{Ad}((K_d(.), K_r(.)))\).

## V. SYNTHESIS OF NETWORK ADAPTED FEEDBACK NASH EQUILIBRIUM STRATEGIES

The NAFNE strategies obtained from Theorem 2 (Theorem 5) are parameterized by arbitrary gain matrices \(K_d(t), (K_d(t), K_r(t)), t \in [0, T]\) leading to a plethora of implementable strategies for the visibility constrained players. To address this issue, we develop an information consistency criterion for selecting a subset, also referred to as a refinement, of NAFNE strategies. The main idea of this refinement procedure is that, whenever the information is symmetric, that is defenders in interaction \(I\) (defenders and target in interaction \(II\)) are able to see all the players, we require that the defenders’ controls (defenders’ and target’s controls), at those time instants, using NAFNE strategy must coincide with those using a standard FNE strategy. We formalize this (informational) consistency property in the following definition.

**Definition 3**: Let \(t_2 \in [0, T]\) be a time instant when the defenders in interaction \(I\) (defenders and target in interaction \(II\)) can see all the players in the game process. A NAFNE strategy, parameterized by the gain matrices \(K_d(t), (K_d(t), K_r(t)) t \in [0, T]\) is consistent, and denoted by c-NAFNE, if the control \(u_d(t_1) (u_d(t_1))\) satisfies \(u_d(t_1) = \gamma_d(t_1, z(t_1); K_d(t_1)) = \gamma_d^* t_1, z(t_1)); K_d(t_1), K_r(t_1)) = \gamma_a^* (t_1, z(t_1))).

In the next theorem, we provide a method for computing the c-NAFNE strategies. First, we introduce the following error function:

\[
\Theta_1(t) = \gamma_1||\Delta Q^{Ad}(t)||^2 + \gamma_2||\Delta Q^{Ad}(t)||^2 + \gamma_3||\Delta Q^{Ad}(t)||^2 + \gamma_4||S_1(t)||^2
\]

which is parametric in \(K_d(t)\) with \(\gamma_i \in [0, 1], i = 1, 2, 3, 4\) for the interaction \(I\), and

\[
\Theta_2(t) = \gamma_1||\Delta Q^{Ad}(t)||^2 + \gamma_2||S_2(t)||^2 + \gamma_3||S_3(t)||^2
\]

which is parametric in \((K_d(t), K_r(t))\) with \(\gamma_i \in [0, 1], i = 1, 2, 3\) for the interaction \(II\). The gradient of \(\Theta_1(t)\) with respect to \(K_d(t)\), in interaction \(I\), is given by

\[
\nabla_{K_d(t)}\Theta_1(t) = \left(\mathbf{e}_n^t \otimes \mathbf{I}_2\right) \left[-4\gamma_1R_dK_d(t)I_d(t)\Delta Q^{Ad}(t) - 4\gamma_2B_d^TP_r(t)\Delta Q^{Ad}(t) - 4\gamma_3B_r^TP_r(t)\Delta Q^{Ad}(t) + 2\gamma_4R_dS_1(t)I_d(t)\right]
\]

Further, in the interaction \(I\), the gradient of \(\Theta_2(t)\) with respect to \(K_d(t)\) is given by

\[
\nabla_{K_d(t)}\Theta_2(t) = \left(\mathbf{e}_n^t \otimes \mathbf{I}_2\right) \left[-4\gamma_1R_dK_d(t)I_d(t)\Delta Q^{Ad}(t) + 2\gamma_2R_dS_2(t)I_d(t)\right]
\]

and with respect to \(K_r(t)\) is given by

\[
\nabla_{K_r(t)}\Theta_2(t) = \left(\mathbf{e}_n^t \otimes \mathbf{I}_2\right) \left[-4\gamma_1R_rK_r(t)I_r(t)\Delta Q^{Ad}(t) + 2\gamma_2R_rS_3(t)I_r(t)\right]
\]

**Theorem 6**: In interaction \(II\), let for every \(t \in [0, T]\), \(K_d^*(t)\) be the solution of the following optimization problem

\[
K_d^*(t) = \arg \min_{K_d(t)} \Theta_1(t).
\]

Then the NAFNE strategy parameterized by \(K_d^*(t), t \in [0, T]\), that is, \(\gamma_d^*(t, z(t)); K_d^*(t), t \in [0, T]\) is a c-NAFNE strategy. Similarly, in interaction \(II\), for every \(t \in [0, T]\), \(K_d^*(t), K_r^*(t)\) be the solution of the following optimization problem

\[
(K_d^*(t), K_r^*(t)) = \arg \min_{(K_d(t), K_r(t))} \Theta_2(t).
\]
K_3(t), t \in [0, T], \) obtained from (37), result in performance indices \((J_3^{d}, J_3^{a}, J_3^{ad}; K_3^{d}(),)\), which are closer to \((J_{a}, J_{r}, J_{d}).\) The performance indices parameterized by the gain matrices \(K_3^{d}(t), t \in [0, T]\) can be referred to as best achievable performance indices; see [30] where this concept was introduced. For interaction 12, using similar arguments, it follows the performance index \((J_{3^{ad}}, (K_3^{d}(), K_3^{a}())\) parametrized by the gain matrices \((K_3^{d}(t), K_3^{a}(t)), t \in [0, T]\) is the best achievable and closer to \(J.\)

In the next theorem, we study the effect of varying visibility radii on the c-NAFNE strategies.

**Theorem 7:** Consider two TAD games with limited observations with identical problem parameters (including the initial state), and differ only in defender \(d_i\)’s visibility radius in interaction 11 (either defender \(d_i\)’s or target visibility radius in interaction 12). Let \(T_1\) and \(T_2\) represent the time instants at which there exist an outgoing edge from the defender \(d_i\) in interaction 11 (either defender \(d_i\)’s or target in interaction 12) for the first time in these two games, respectively. Then, in interaction 11 (12), the control actions of the defenders (defenders and target) using their c-NAFNE strategies in these two games are identical during the time period \([0, \min\{T_1, T_2\}].\)

**Remark 7:** As the attacker and target use their standard FNE strategies in interaction 11 their state and control trajectories are also identical in these games during the time period \([0, \min\{T_1, T_2\}].\); a similar conclusion holds true only for the attacker in the interaction 12.

**Remark 8:** The optimization problem, though well-posed, is nonconvex and can be solved numerically. Further, from (36) the computation of the gradient by the defender \(d_i\) (\(d_i \in D\)) in interaction 11 (defender \(d_i \in D\) or target \(\tau\) in interaction 12) requires joint feedback gain \(K_3(t)\) (\((K_3(t), K_3(t))\)) and joint connectivity information \(I_3(t)\) \(I_3(t).\) In the real-world implementation, this information must be shared among the defenders (defenders and target) through a protocol as a part of cooperation. Such a protocol leads to a semidecentralized implementation of c-NAFNE strategies.

**VI. ILLUSTRATIVE EXAMPLES**

In this section, we illustrate the performance of c-NAFNE strategies studied in Section V through numerical experiments. In real-world applications involving networked agents, with limited visibility, the presence of network externalities plays an important role in the synthesis of players’ Strategies. In other words, when an outgoing link from a visibility constrained player forms or breaks, then it is important to know how this would affect the strategies of other team players who are not directly connected to this player. Besides verifying the obtained theoretical results, the numerical examples are designed to illustrate the effect of network externalities. To this end, we consider a 5 player TAD game with 1 target, 1 attacker, and 3 defenders.

**1) interaction 11:** Initially, the players \(\{d_1, d_2, d_3, \tau, a\}\) are located at \((\{0, 0\}, (1, 1.5\}, (-1, 0), (0, 1), (-2, 2)\},\) respectively. The control penalty parameter values are selected as \(\{r_{d_1}, r_{d_2}, r_{d_3}, r_{\tau}, r_{a}\} = \{1, 1, 1, 2, 0.8\}.\) The interaction parameters \(\{q_{d,a}, f_{d,a}, g_{d,a}, f_{a,d}, f_{\tau}, q_{\tau}, f_{\tau,a}, f_{\tau,r}\}, d_i \in D\) in (6) are set equal to 1, and the remaining parameters are taken as \(T = 6, \sigma_p = 0.1,\) for \(p \in \{d_1, d_2, d_3, a\}.\) For implementation, we discretize the duration \([0, T]\) with a step size of \(\delta = 0.005,\) and we use the MATLAB program \(\text{fminunc}\) for solving the optimization (37) at each time step. First, we illustrate the scenario with a nonsuicidal attacker. Fig. 2(a) illustrates the trajectories of the players with complete observations when all the players use their standard FNE strategies given by (10). The game terminates at \(t = 2.66\) and results in the capture of the target by the attacker. Next, the visibility radii of the defenders are set to \(\zeta_d = 5,\) \(\zeta_{d_2} = 2.25,\) and \(\zeta_{d_3} = 1.25.\) Fig. 2(b) illustrates the trajectories using the c-NAFNE strategies from Theorem 2 and synthesized using Theorem 6. The parameters in the optimization problem (37) are set as \(\gamma_i = 0.25, i = 1, 2, 3, 4\) to indicate that the error terms in (34) are weighted equally. Fig. 3(a)-(f) illustrate the evolution of visibility network in the game. In the following discussion, we explain the presence of network externalities in detail. Whenever a new link forms (or disappears) in the network, an additional gain term is included (or deducted) from a defender’s network adapted feedback strategy. This change in the network will reflect in all the defenders’ control trajectories verifying the presence of network externalities. Consequently, structural changes in the visibility network leads to jumps in the defenders’ control trajectories; see Fig. 4(a). Further, these jumps lead to kinked state trajectories; see the labeled markers in Fig. 2(b). After \(t = 0.96,\) all players can see each other, and from Theorem 6, in the subgame starting at \(t = 0.96\) the
Visibility radii of other defenders and target set as $\zeta_1$, $\zeta_2$, and $\zeta_3$. Time instant when the edge $I_1 \rightarrow \tau$ becomes active can see all the players and the target indicates that time instant when the edge $F_{I_1 \rightarrow \tau}$ illustrates evolution of the visibility information during this period.

$d_3 \rightarrow \tau$ and $d_2 \rightarrow \tau$ are changed from 0.3 to 0.6. The dashed vertical line $t = 1$ illustrates the c-NAFNE strategies (along $y$-axis) with complete and limited observations. The game terminates at $t = 1.305$ indicating that time instant when the edge $d_3 \rightarrow \tau$ becomes active.

**Fig. 6.** Trajectories of players with visibility constrained defenders-target team, with parameters $\zeta_1 = 5$, $\zeta_2 = 3$, $\zeta_3 = 0.3$, $\zeta_5 = 2.5$ in panel (a) and with parameters $\zeta_1 = 5$, $\zeta_2 = 3$, $\zeta_3 = 0.6$, $\zeta_5 = 2.5$ in panel (b).

Fig. 7. Evolution of visibility network for the interaction illustrated in Fig. 6(a). (a) $t \in [0,0.51)$. (b) $t \in [0.51,0.895)$. (c) $t \in (0.895,0.91)$. (d) $t \in (0.91,1.2)$. (e) $t \in [1.2,3.46)$.

**Fig. 8.** Visibility radii of other defenders and target set as $\zeta_1 = 5$, $\zeta_2 = 3$, $\zeta_5 = 2.5$. Panel (a) illustrates the defenders-target team’s c-NAFNE strategies (along $y$-axis) in the game with $\zeta_3 = 0.3$. Panel (b) illustrates the defenders-target team's c-NAFNE strategies (along $y$-axis) when $\zeta_3$ is changed from 0.3 to 0.6. The dashed vertical line at $t = 1.305$ indicates that time instant when edge $d_3 \rightarrow \tau$ becomes active.

Using Theorem 6. The parameters in the optimization problem (37) are set as $\gamma_i = \frac{1}{3}$, $i = 1, 2, 3$ to indicate that the error terms in (34) are weighted equally. The game terminates at $t = 4.065$ with attacker capturing the target. Here, due to large visibility radius, the defender $d_1$ and the target $\tau$ can see all the players throughout the game process.

Next, we set the visibility radius of the target as $\zeta_5 = 2.5$, indicating that the target cannot see all the players initially. Fig. 6(a) illustrates the trajectories of the players, and Fig. 7(a)–(e) illustrate evolution of the visibility information during this interaction. Fig. 8(a) illustrates the c-NAFNE strategies (in the context of the...
Taking the time derivative of the value function associated with the cooperative defenders, we get
\[ \dot{V}_d(t, z(t)) = \frac{1}{2} \dot{z}(t) P_d(t) z(t) + \frac{1}{2} \dot{z}(t) \dot{P}_d(t) z(t) \]
\[ + \frac{1}{2} \dot{z}(t) P_d(t) \dot{z}(t). \]  
(40)

Substituting for state dynamics in (40) we get
\[ \dot{V}_d(t, z(t)) = \frac{1}{2} [B_d u_d(t) + B_r u_r(t) + B_a u_a(t)]^T P_d(t) z(t) \]
\[ + \frac{1}{2} \dot{z}(t) P_d(t) [B_d u_d(t) + B_r u_r(t) + B_a u_a(t)] \]
\[ + \frac{1}{2} \dot{z}(t) [-Q_d + P_d(t) S_d P_d(t) + P_d(t) S_r P_r(t) + P_d(t) S_a P_a(t)] \]
\[ + P_d(t) S_d P_d(t) + P_r(t) S_r P_d(t) \]
\[ + P_d(t) S_a P_a(t) z(t). \]

Using (20), terms in the above equation can be rearranged as
\[ \dot{V}_d(t, z(t)) = \frac{1}{2} ||u_d(t) - K_d(t) I_d(t) z(t)||^2 R_d \]
\[ + \frac{1}{2} ||u_r(t) + R_1^{-1} B_t P_r(t) z(t)||^2 R_d \]
\[ + \frac{1}{2} ||u_a(t) + R_1^{-1} B_a P_a(t) z(t)||^2 R_d \]
\[ - \frac{1}{2} ||z(t)||^2 Q_d(t) + \frac{1}{2} u'_d(t) S_d(t) z(t) \]
\[ + \frac{1}{2} u'_r(t) S_r(t) u_r(t) + \frac{1}{2} u'_a(t) S_a(t) u_a(t). \]

Integrating the above equation from 0 to \( T \) and rearranging terms, we get
\[ V_d(T, z(T)) = \frac{1}{2} \int_0^T \left( ||u_d(t)||^2 R_d + ||z(t)||^2 Q_d(t) \right) dt \]
\[ - u'_d(t) S_d(t) z(t) - \frac{1}{2} ||z(t)||^2 Q_d(t) \]
\[ = V_d(0, z(0)) + \frac{1}{2} \int_0^T \left( ||u_d(t) - K_d(t) I_d(t) z(t)||^2 R_d \right) dt \]
\[ + 2 \dot{z}(t) P_d(t) B_r [u_r(t) + R_1^{-1} B_t P_r(t) z(t)] \]
\[ + 2 \dot{z}(t) P_d(t) B_a [u_a(t) + R_1^{-1} B_a P_a(t) z(t)] \right) dt. \]

As \( V_d(T, z(T)) = \frac{1}{2} \dot{z}(T) P_d(T) z(T) \) and \( P_d(T) = F_d \), we thus obtain
\[ J^d_{\text{ad}}(u_d(.), u_r(.), u_a(.)) = V_d(0, z(0)) \]
\[ + \frac{1}{2} \int_0^T \left( 2 \dot{z}(t) P_d(t) B_r [u_r(t) - \gamma_d^*(t, z(t))] \right) dt \]
\[ + ||u_d(t) - \gamma_d^*(t, z(t)); K_d(t)||^2 R_d \]
\[ + 2 \dot{z}(t) P_d(t) B_a [u_a(t) - \gamma_a^*(t, z(t))] \right) dt, \]  
(41)

where \( J^d_{\text{ad}}(u_d(.), u_r(.), u_a(.)) \) is defined in (19a). Using the same approach as above we can show the following relations
\[ J^D_t(u_d(\cdot), u_r(\cdot), u_a(\cdot)) = V_t(0, z(0)) + \frac{1}{2} \int_0^T \left( |u_r(t) - \gamma^*_r(t, z(t))|^2 \right) dt + \frac{1}{2} \int_0^T \left( |u_a(t) - \gamma^*_a(t, z(t))|^2 \right) dt \]

(42)

\[ J^A_t(u_d(\cdot), u_r(\cdot), u_a(\cdot)) = V_t(0, z(0)) + \frac{1}{2} \int_0^T \left( |u_a(t) - \gamma^*_a(t, z(t))|^2 \right) dt \]

(43)

Clearly, \((\gamma^*_d, \gamma^*_r, \gamma^*_a)\) is a NAFNE of the game with performance indices (19).

**Proof of Theorem 6:** For interaction II, let \(t_1 \in [0, T]\) be a time instant in the game process when all the defenders can see all the players. From (16), this implies that the information matrices \(I_d(t_1)\) are nonsingular for all \(i = 1, 2, \ldots, n\). For the interaction given in P1, we show that the feedback gain matrix \(K_d(t_1)\) with its diagonal entries given by \(K_{d,j}(t_1) = -\left(e_n^I \otimes I_2\right)R_d^{-1}B_d^TP_d(t_1)I_{\Delta_d}^{-1}(t_1)\) for \(i = 1, 2, \ldots, n\), solves (37). To see this, with the above choice of matrices the feedback gain matrix satisfies \(K_d(t_1)I_d(t_1) = -R_d^{-1}B_d^TP_d(t_1)I_d(t_1)\). Then, using this in (20) gives \(S_1(t_1) = 0, Q^D_d(t_1) = Q_d\) for \(p \in \{d, r, a\}\). Then, from (34) and (36a), we get \(\Theta_1(t_1) = 0\) and \(\nabla_{K_d}(t_1)\Theta_1(t_1) = 0\) for all \(i = 1, 2, \ldots, n\). This implies, \(K_d(t_1)\) minimizes \(\Theta_1(t_1)\), that is, \(K_d(t_1) = K_d(t_1)\). Then, the control action at \(t_1\) using the NAFNE strategy parameterized \(K_d(t_1)\), \(t \in [0, T]\) satisfies \(\gamma^*_d(t_1, z(t_1)); K_{d,j}(t_1) = K_{d,j}(t_1)I_d(t_1)\). Then, \(K_{d,j}(t_1)I_d(t_1) = -R_d^{-1}B_d^TP_d(t_1)z(t_1) = \gamma^*_d(t_1, z(t_1))\). This implies, from Definition 3, \(\gamma^*_d(t, z(t)); K_{d,j}(t_1)\), \(t \in [0, T]\) is a c-NAFNE strategy. In interaction II, let \(t_1 \in [0, T]\) be the time instant when all the defenders and the target can see all the players. Then, from (16) and (29) we have that the matrices \(I_d(t_1)\) and \(\tau_r(t_1)\) are invertible. Then, using \(K_d(t_1)\) with its diagonal entries given by \(K_{d,j}(t_1) = \left(e_n^I \otimes I_2\right)R_d^{-1}B_d^TP_d(t_1)I_{\Delta_d}^{-1}(t_1)\) for \(i = 1, \ldots, n\), \(K_{r,n}(t_1) = R_d^{-1}B_d^TP_d(t_1)I_{\Delta_d}^{-1}(t_1)\), in (33) we get \(S_2(t_1) = 0, S_3(t_1) = 0, Q^A_d(t_1) = Q\). Then, from (35), (36b), and (36c), we get \(\Theta_2(t_1) = 0\). Then, \(K_d(t_1)\Theta_1(t_1) = 0\) for all \(i = 1, \ldots, n\) and \(\nabla_{K_d}(t_1)\Theta_2(t_1) = 0\). This implies, \((K_d(t_1), K_r(t_1))\) minimize \(\Theta_2(t_1)\), that is, \((K_{d,i}(t_1), K_{r,i}(t_1)) = (K_d(t_1), K_r(t_1))\). Using the same arguments as before we have that \(\gamma^*_d(t, z(t)); (K_d(t_1), K_r(t_1)), t \in [0, T]\) is a c-NAFNE strategy.

**Proof of Theorem 7:** In interaction II, following the network feedback information structure, the defender \(d_i\)’s information matrix (16) satisfies \(I_d(t_1) = 0\) for all \(t \in [0, \min\{T_1, T_2\}\) in both the games. Further, as all other parameters in both the games are identical, except defender \(d_i\)’s visibility radius, the joint equilibrium control actions of the defenders \(u_d(t)\) using their c-NAFNE strategies is identical in both the games for all \(t \in [0, \min\{T_1, T_2\}\) at the target in interaction II.
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