The Spinless Calogero-Sutherland model with twisted boundary condition

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In this work, the Calogero-Sutherland model with twisted boundary condition is studied. The ground state wavefunctions, the ground state energies, the full energy spectrum are provided in details.

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Exact solutions have provided us with important non-perturbative insights in dealing with systems of strong correlations. While there are very few exactly solvable systems available, the ones that exist have yielded many interesting results. Notable examples include electron systems with delta function interactions [1], the Hubbard model [2], the Kondo impurity spin system with linear conduction electron system [3], the Luttinger model [4], and the Anderson model [5]. These models have all played important roles in our understanding of physics in condensed matter theory. Ever since Haldane and Shastry independently introduced the exactly solvable spin chain of $1/r^2$ exchange interaction [6,7], there has been considerable activity in studying the variants of the Haldane-Shastry spin chain doped with holes, i.e. the $t$-$J$ models of long range hopping and exchange [3,14,18,21]. It is interesting that the chiral Hubbard model [10], which at half-filling and in the limit of large but finite on-site energy reduces to the Haldane-Shastry spin chain, is also exactly solvable for any filling numbers and any on-site energy. In the following, we will study in details the Calogero-Sutherland model with twisted boundary condition. The ground state wavefunctions, the ground state energies, and the full energy spectrum are provided. Since one has to deal with the cases of bosons and fermions with twisted boundary condition, the full discussion is divided into several sections as below.

I. THE GROUND STATES

A. Spinless boson gas

We first consider the CS model of boson gas defined on a closed ring of length $L$. In the presence of a flux tube that threads through the ring, the eigenenergy problem can be formulated as follows. Suppose that there are $N$ spinless bosons moving on the ring, $0 \leq x_i \leq L; i = 1,2,\cdots,N$. Then the eigenvalue problem is

$$H_{CS}\tilde{\Psi}(x_1\sigma_1,\cdots,x_i\sigma_i,\cdots,x_N\sigma_N) = E\tilde{\Psi}(x_1\sigma_1,\cdots,x_i\sigma_i,\cdots,x_N\sigma_N). \tag{1}$$
Here the Calogero-Sutherland Hamiltonian $H_{CS}$ takes the usual form

$$H_{CS} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} l(l+1)/\left[\left(\frac{L}{\pi}\right)^2 \sin^2 \left(\frac{\pi(x_i - x_j)}{L}\right)\right], \quad (2)$$

where we assume $l > 0$. The wavefunction obeys the twisted boundary condition

$$\tilde{\Psi}(x_1, \ldots, (x_i + L), \ldots, x_N) = e^{i\phi} \tilde{\Psi}(x_1, \ldots, x_i, \ldots, x_N). \quad (3)$$

Obviously, the system is invariant under the translational operation $\phi \rightarrow \phi + 2\pi$. Therefore we only need to consider the region $-\pi \leq \phi \leq \pi$. For the bosons,

$$\tilde{\Psi}(x_1, \ldots, x_i, \ldots, x_j, \ldots) = \tilde{\Psi}(x_1, \ldots, x_j, \ldots, x_i, \ldots), \quad (4)$$

i.e. the wavefunction $\tilde{\Psi}$ is symmetric under exchange of any two particles.

Let us define the region $R$ as follows: $\{R : 0 \leq x_i \leq L; i = 1, 2, \ldots, N\}$. The sub-region of the full $R$ is denoted by $R_1 : \{R_1 : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq L\}$. The wavefunction inside the region $R_1$ is denoted by $\tilde{\Psi}_1(x_1, x_2, \ldots, x_N)$. The wavefunction in other sub-regions can be obtained by using the symmetry property of bosons Eq.(4). The twisted boundary condition Eq.(3) is translated to be

$$\tilde{\Psi}_1(x_2, x_3, \ldots, x_N, L) = e^{i\phi} \tilde{\Psi}_1(0, x_2, x_3, \ldots, x_N). \quad (5)$$

The ground state of the spinless bosons should take the following form

$$\tilde{\Psi}_1^g(x_1, x_2, \ldots, x_N) = e^{\frac{i\phi}{L} \sum_{j=1}^{N} x_j} \prod_{1 \leq i < j \leq N} \left| \sin \left(\frac{\pi(x_i - x_j)}{L}\right)\right|^{l+1}. \quad (6)$$

One may check that the wavefunction satisfies the twisted boundary condition. This wavefunction is also an eigenstate of the Hamiltonian. Since the wavefunction has no zeros in the region $R_1$, it is the ground state. The eigenenergy of the state can be found to be

$$E_g(\phi) = \frac{1}{2} N \left(\frac{\phi}{L}\right)^2 + \frac{1}{6} (l+1)^2 \pi^2 N(N^2 - 1)/L^2. \quad (7)$$

Without the flux, the results reduces to those of Sutherland’s [17]. In the full space $R$, the ground state wavefunction $\tilde{\Psi}^g$ takes the simple form
\[ \tilde{\Psi}^g(x_1, x_2, \cdots, x_N) = e^{i\phi} \sum_{j=1}^{N} x_j \prod_{1 \leq i < j \leq N} | \sin(\frac{\pi(x_i - x_j)}{L}) |^{l+1}, \tag{8} \]

which is symmetric under exchange of two particles, and which satisfies the twisted boundary condition Eq.(3). In presence of the flux, there is a persistent current in the ring. The persistent current is \[ I(\phi) = -\frac{\partial E_g(\phi)}{\partial \phi} = -\frac{N}{L^2} \phi. \]

**B. Spinless fermion gas (odd \( N \))**

In this section, we discuss the spinless fermion model described by CS model in presence of magnetic flux tube. The eigenvalue problem is formulated as follows:

\[ H_{CS} \tilde{\Psi}(x_1 \sigma_1, \cdots, x_i \sigma_i, \cdots, x_N \sigma_N) = E \tilde{\Psi}(x_1 \sigma_1, \cdots, x_i \sigma_i, \cdots, x_N \sigma_N), \tag{9} \]

with the Calogero-Sutherland Hamiltonian \( H_{CS} \) as before

\[ H_{CS} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} l(l+1)/[(\frac{L}{\pi})^2 \sin^2 \left( \frac{\pi(x_i - x_j)}{L} \right)], \tag{10} \]

where the coupling constant \( l > 0 \). The wavefunction satisfies the twisted boundary condition

\[ \tilde{\Psi}(x_1, \cdots, (x_i + L), \cdots, x_N) = e^{i\phi} \tilde{\Psi}(x_1, \cdots, x_i, \cdots, x_N). \tag{11} \]

In this case, since the system is made of spinless fermions, the wavefunction is antisymmetric when exchanging two particles,

\[ \tilde{\Psi}(x_1, \cdots, x_i, \cdots, x_j, \cdots) = (-1) \tilde{\Psi}(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_N). \tag{12} \]

As before, we define the full region \( R \) to be \( \{ R : 0 \leq x_i \leq L; i = 1, 2, \cdots, N \} \). The sub-region \( R_1 \) is \( \{ R_1 : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq L \} \). The twisted boundary condition of the wavefunction \( \tilde{\Psi}(x_1, x_2, \cdots, x_N) \) defined in \( R \) is translated to a condition satisfied by the wavefunction \( \tilde{\Psi}_1(x_1, x_2, \cdots, x_N) \) defined in the region \( R_1 \) as below:
Given $\tilde{\Psi}_1$ inside $R_1$, the wavefunctions in other sub-regions of $R$ can be obtained using the anti-symmetry of the fermionic statistics. Since $N$ is odd, the prefactor $(-1)^{(N-1)}$ disappears. We propose the following wavefunction as the ground state: inside $R_1$, the ground state takes Jastrow form
\[ \tilde{\Psi}_g^1(x_1, x_2, \ldots, x_N) = e^{i\phi \sum_{j=1}^{N} x_j} \prod_{1 \leq i < j \leq N} |\sin(\frac{\pi(x_i - x_j)}{L})|^{l+1}. \] (14)
One may compute the corresponding eigenvalue of this wavefunction. It is found that
\[ E_g(\phi) = \frac{1}{2} N(\frac{\phi}{L})^2 + \frac{1}{6}(l + 1)^2 \pi^2 N(N^2 - 1)/L^2. \] (15)

In presence of the magnetic flux, there is a persistent current in the system. The persistent current is $I(\phi) = -\frac{\partial E_g(\phi)}{\partial \phi} = -\frac{N}{L^2} \phi$. Without the flux, our ground state wavefunction reduces to that of Sutherland’s. In the full region $R$, the ground state wavefunction $\tilde{\Psi}^g$ can be written in a compact way
\[ \tilde{\Psi}^g(x_1, x_2, \ldots, x_N) = e^{i\phi \sum_{j=1}^{N} x_j} \prod_{1 \leq i < j \leq N} |\sin(\frac{\pi(x_i - x_j)}{L})|^{l} \times \sin(\frac{\pi(x_i - x_j)}{L}), \] (16)
which is anti-symmetric when exchanging two fermions, and which also satisfies the twisted boundary condition Eq.(13).

C. Spinless fermion Gas (even $N$)

The eigenvalue problem is formulated as before. However, the boundary condition Eq.(13) should be taken great care of. First, let us consider the situation without flux, $\phi = 0$, and we impose periodic boundary condition on the wavefunction
\[ \tilde{\Psi}(x_1, \ldots, (x_i + L), \ldots, x_N) = \tilde{\Psi}(x_1, \ldots, x_i, \ldots, x_N). \] (17)
This PBC is translated to be a boundary condition for $\tilde{\Psi}_1$ as below:
\[ \tilde{\Psi}_1(x_2, x_3, \ldots, x_N, L) = (-1)\tilde{\Psi}_1(0, x_2, x_3, \ldots, x_N). \] (18)
With this in mind, the ground state for the system inside the region $R_1$ should take the following form:

$$\tilde{\Psi}_g^\phi(x_1, x_2, \cdots, x_N) = e^{i\frac{\pi}{L} \sum_{j=1}^N x_j} \times \prod_{1 \leq i < j \leq N} |\sin\left(\frac{\pi(x_i - x_j)}{L}\right)|^{l+1}.$$  \hspace{1cm} (19)

We can compute the eigen-energy of this wavefunction. The ground state energy is found to be

$$E_g = \frac{1}{2} N \left(\frac{\pi}{L}\right)^2 + \frac{1}{6} (l + 1)^2 \pi^2 N (N^2 - 1)/L^2.$$ \hspace{1cm} (20)

This energy is different from the bosonic case, as well as different from the fermionic case when the total number of particles is odd.

Now, let us consider the situation when there is no zero flux. Consider the case where $0 \leq \phi \leq \pi$. The twisted boundary condition Eq.(13) for the wavefunction $\tilde{\Psi}_1$ is satisfied by the following Jastrow product

$$\tilde{\Psi}_1^\phi(x_1, x_2, \cdots, x_N) = e^{i\frac{\pi}{L} \sum_{j=1}^N x_j} \times \prod_{1 \leq i < j \leq N} |\sin\left(\frac{\pi(x_i - x_j)}{L}\right)|^{l+1}. \hspace{1cm} (21)$$

This wavefunction is the ground state of the system, with the ground state energy given by

$$E_g(\phi) = \frac{1}{2} N \left(\frac{\pi - \phi}{L}\right)^2 + \frac{1}{6} (l + 1)^2 \pi^2 N (N^2 - 1)/L^2.$$ \hspace{1cm} (22)

The persistent current of the system is found to be $I(\phi) = -\partial E(\phi)/\partial \phi = -\frac{N}{L^2}(\phi - \pi) \geq 0$. If the system has a flux $-\pi \leq \phi \leq 0$, the ground state wavefunction is found to be

$$\tilde{\Psi}_1^\phi(x_1, x_2, \cdots, x_N) = e^{i\frac{(\phi + \pi)}{L} \sum_{j=1}^N x_j} \times \prod_{1 \leq i < j \leq N} |\sin\left(\frac{\pi(x_i - x_j)}{L}\right)|^{l+1}. \hspace{1cm} (23)$$

The corresponding eigen-energy is given by

$$E_g(\phi) = \frac{1}{2} N \left(\frac{\pi + \phi}{L}\right)^2 + \frac{1}{6} (l + 1)^2 \pi^2 N (N^2 - 1)/L^2.$$ \hspace{1cm} (24)

The persistent current is $I(\phi) = -\frac{N}{L^2}(\phi + \pi) \leq 0$. In the full space $R$ as defined before, the ground state wavefunction $\tilde{\Psi}^g$ can be written in a compact way. For $0 \leq \phi \leq \pi$, inside the full region $R$, the ground state wavefunction is
\[ \tilde{\Psi}(x_1, x_2, \ldots, x_N) = e^{i(\phi - \pi)} \sum_{j=1}^{N} x_j \prod_{1 \leq i < j \leq N} |\sin(\frac{\pi(x_i - x_j)}{L})\rangle^l \times \sin(\frac{\pi(x_i - x_j)}{L}). \] (25)

While for \(-\pi \leq \phi \leq 0\), inside \(R\), one has

\[ \tilde{\Psi}(x_1, x_2, \ldots, x_N) = e^{i(\phi + \pi)} \sum_{j=1}^{N} x_j \prod_{1 \leq i < j \leq N} |\sin(\frac{\pi(x_i - x_j)}{L})\rangle^l \times \sin(\frac{\pi(x_i - x_j)}{L}). \] (26)
C. Spinless fermions (even $N$)

Finally, for the spinless fermion gas of even $N$, we also find the full energy spectrum taking the following form for $0 \leq \phi \leq \pi$.

$$E = \frac{\pi^2}{6}(l + 1)^2 N(N^2 - 1)/L^2 + \frac{1}{2}(2\pi/L)^2 \epsilon$$

(29)

where the function $\epsilon$ is given by $\epsilon = \sum_{j=1}^{N}(n_j + \frac{\phi - \pi}{2\pi})^2 + (l + 1) \sum_{i>j}[n_i - n_j]$. The quantum numbers $n_j$ are non-negative integers that have the condition $n_{j+1} \geq n_j$. The ground state corresponds to all $n_i = 0$.

When the flux $-\pi \leq \phi \leq 0$, the energy excitation is given by

$$E = \frac{\pi^2}{6}(l + 1)^2 N(N^2 - 1)/L^2 + \frac{1}{2}(2\pi/L)^2 \epsilon$$

(30)

where the function $\epsilon$ is given by $\epsilon = \sum_{j=1}^{N}(n_j + \frac{\phi + \pi}{2\pi})^2 + (l + 1) \sum_{i>j}[n_i - n_j]$. The quantum numbers $n_j$ are non-negative integers, satisfying the condition $n_{j+1} \geq n_j$. The ground state is reached when all $n_i = 0$.

III. SUMMARY

In summary, we have discussed how the boundary condition affects the spinless CS model of long range interaction. The ground state wavefunctions, the ground state energies, the full energy spectrum are provided for both the fermionic gas and the bosonic gas. The exact solutions indicate that the parity effect for the persistent currents still hold for the fermionic gas, in spite of the electron-electron correlation.

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