Soft radiation in heavy-particle pair production:
all-order colour structure and two-loop anomalous dimension

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Abstract
We consider the total production cross section of heavy coloured particle pairs in hadronic collisions at the production threshold. We construct a basis in colour space that diagonalizes to all orders in perturbation theory the soft function, which appears in a new factorization formula for the combined resummation of soft gluon and Coulomb gluon effects. This extends recent results on the structure of soft anomalous dimensions and allows us to determine an analytic expression for the two-loop soft anomalous dimension at threshold for all production processes of interest.
1 Introduction

Perturbative calculations of partonic cross sections at hadron colliders often fail near the boundaries of partonic phase space due to logarithmically enhanced terms from soft gluon radiation. If it can be argued that the hadronic cross section is dominated numerically by these threshold logarithms, they should be summed to all orders in perturbation theory. This can be done in Mellin moment space [1–5] or directly in momentum space [6–8]. In either case, the theoretical basis for resummation is a factorization of the partonic hard-scattering cross section $\hat{\sigma}$ in the partonic threshold region into hard and soft contributions of the schematic form

$$\hat{\sigma} = H \otimes S$$

with a hard function $H$ and a soft function $S$. The soft function satisfies an evolution equation, whose driving term is the anomalous dimension matrix of $S$. Resummation amounts to solving this equation.

Particularly important for experiments at LHC and Tevatron are pair production processes of heavy coloured particles $H, H'$ in a collision of hadrons $N_1$ and $N_2$,

$$N_1(P_1)N_2(P_2) \rightarrow H(p_1)H'(p_2) + X.$$  (1.2)

In this case the partonic cross section contains terms of the form $[\alpha_s^n \ln^n \beta]$ (“threshold logarithms”) and $(\alpha_s/\beta)^n$ (“Coulomb singularity”), where $\beta = (1 - 4M^2/\hat{s})^{1/2}$ is the heavy particle velocity, which are enhanced near the partonic threshold $\hat{s} \approx 4M^2$, with $M$ the average heavy-particle mass. The threshold logarithms have been discussed in the past for various production processes of the form (1.2) leading to improved predictions for the top-quark production cross section at hadron colliders [9–17], production of supersymmetric coloured particles [18–20], and colour-octet scalars [21]. In two-to-two scattering the soft function is a matrix in colour space. In general this matrix depends on the kinematical invariants of the scattering process [3]; however, for resummation of threshold logarithms in the total cross section, the relevant quantity is the soft anomalous dimension at threshold. In this case, the question arises whether the resummation of logarithms is altered by the presence of Coulomb corrections, which must also be summed. This has not been addressed in the past, where the Coulomb correction is technically considered as part of the hard function $H$.

In a separate paper [22] we employ effective field theory and field redefinitions to derive an extended factorization formula of the form $\hat{\sigma} = H \otimes J \otimes S$ for the hard-scattering total cross sections related to (1.2), which implies a proof of factorization of soft gluons in the presence of Coulomb exchange. More precisely, the cross sections of the partonic subprocesses $p(k_1)p'(k_2) \rightarrow H(p_1)H'(p_2) + X$ where $pp' \in \{qq, q\bar{q}, gg, gq, g\bar{q}\}$ are expressed as

$$\hat{\sigma}_{pp'}(\hat{s}, \mu) = \sum_{i,i'} H_{ii'}(M, \mu) \int d\omega \sum_{R_\alpha} J_{R_\alpha}(E - \frac{\omega}{2}) W_{ii'}^{R_\alpha}(\omega, \mu).$$  (1.3)

Here $\hat{s}$ is the partonic centre-of-mass energy, and $E = \sqrt{\hat{s}} - 2M$. The formula applies to heavy particle pairs produced in an S-wave and is valid at the leading order in the
non-relativistic expansion. The new function \( J_{R_a} \) sums Coulomb gluon exchange related to the attractive or repulsive Coulomb force in the irreducible colour representations \( R_\alpha \) that appear in the product representation \( R \otimes R' \) of the final state particles and includes the leading Coulomb singularities \((\alpha_s/\beta)^n\). We note that Coulomb summation has been included in various forms in \([16, 17, 19]\) where, however, the factorization of Coulomb from soft gluons is put in as an assumption. A formula equivalent to eq. (1.3) for the factorization of electromagnetic effects in \( W \)-pair production at \( e^- e^+ \) colliders has been derived in \([23]\).

The purpose of the present paper is to discuss the colour decomposition of the generalized soft function \( W_{R_{ii'}}^{R_{ii'}}(\omega, \mu) \), which is the crucial ingredient for resummation, and to provide the two-loop anomalous dimensions and one-loop soft functions, which are necessary for next-to-next-to-leading logarithmic (NNLL) resummations. We shall also briefly outline the general structure of resummation and explain further NNLL effects not included in eq. (1.3).

The soft functions are vacuum expectation values of Wilson line operators that retain only information about colour and direction of massive and light-like particles in the hard process. The light-like \((n^2 = 0)\) Wilson line for an incoming particle in the representation \( r \) of \( SU(3) \) with generator \( T^{(r)a} \) is

\[
S^{(r)}_n(x) = P \exp \left[ ig_s \int_{-\infty}^0 ds \, n \cdot A^a(x + ns) T^{(r)a} \right].
\]  

(1.4)

For an outgoing heavy particle in representation \( R \) we define the time-like \((v^2 = 1)\) Wilson line

\[
S^{(R)\dagger}_v(x) = P \exp \left[ ig_s \int_{0}^{\infty} ds \, v \cdot A^a(x + vs) T^{(R)a} \right].
\]  

(1.5)

The inverse (adjoint) Wilson line operators follow from replacing path-ordering by anti path-ordering, and \( ig_s \rightarrow -ig_s \). The soft function \( W_{R_{ii'}}^{R_{ii'}}(\omega, \mu) \) in \((1.3)\) is a descendant of a more general soft function defined by

\[
\hat{W}^{(k)}_{\{ab\}}(z, \mu) = \langle 0 | T[S_{v,bk_2} S_{v,bk_1} S^\dagger_{n,ib_2} S^\dagger_{n,ib_1}](z) T[S_{n,a_1} S_{n,a_2} S^\dagger_{v,k_3} S^\dagger_{v,k_4}](0) | 0 \rangle,
\]  

(1.6)

where \( T \) and \( \overline{T} \) denote time-ordering and anti time-ordering, respectively. \( n \) and \( \bar{n} \) denote two light-like vectors satisfying \( n \cdot \bar{n} = 2 \). The superscript on the Wilson lines denoting the colour representation has been omitted. In the factorization formula \((1.3)\) we need the Fourier transform of the soft function defined according to

\[
W^{(k)}_{\{ab\}}(\omega, \mu) = \int \frac{dz_0}{4\pi} e^{i\omega z_0/2} \hat{W}^{(k)}_{\{ab\}}(z_0, \bar{\mu}, \mu).
\]  

(1.7)

In \((1.3)\) we also decomposed the colour multi-indices in a set of basis structures \( c^{(i)}_{\{a\}} \) defined below, and performed a projection on the irreducible representations \( R_\alpha \) of the final state particle pair. Thus \( W_{R_{ii'}}^{R_{ii'}}(\omega, \mu) \) is given by

\[
W_{R_{ii'}}^{R_{ii'}}(\omega, \mu) = P_{\{k\}}^{R_{ii'}}^{(i)} c^{(i)}_{\{a\}} W^{(k)}_{\{ab\}}(\omega, \mu) c^{(r)*}_{\{b\}}.
\]  

(1.8)
The remainder of the paper is concerned with the properties of the soft functions (1.6) and (1.8). We remark that the soft function that appears in (1.1) follows from contracting (1.6) with the trivial colour factor $\delta_{k_1 k_3} \delta_{k_2 k_4}$ instead of the projectors $P_{(k)}^{R_\alpha}$ on the representations of the heavy-particle pair as in (1.8). Therefore the factorization of the Coulomb gluons results in a more complicated colour structure of the soft function than the original one, $W_{ii'}$, which is simply $\sum_{R_\alpha} W_{ii'}^{R_\alpha}$ due to the completeness of the projectors. For this reason, the original factorization formula (1.1) should only be used for partonic thresholds where the relative velocities of the final state particles are relativistic, and Coulomb exchange is not enhanced. In this situation, the relevant soft function is (1.6) with two unequal four-velocity vectors $v_1$, $v_2$ for the final state particles, contracted with $\delta_{k_1 k_3} \delta_{k_2 k_4}$. The main simplifications discussed in the following hold only for the soft function at threshold ($v_1 = v_2$), and are thus applicable to the total partonic cross sections.

The paper is organized as follows. In section 2 we discuss the colour structure of the soft function. The simple kinematical structure of the soft function $W_{ii'}^{R_\alpha}$ at threshold allows us to construct a colour basis that diagonalizes the soft function to all orders of perturbation theory. We shall see that the four-particle soft function can be reduced to three-particle soft functions corresponding to a single heavy particle in the final state. In section 3, after discussing the general structure of soft-gluon and Coulomb resummation near threshold, we calculate the one-loop soft function for general representations of initial and final state particles and obtain the two-loop soft anomalous dimensions employing results from [25]. This supplies all ingredients for NNLL resummations for top-quark or sparticle pair production associated with the (leading) soft function (1.8). The two-loop soft anomalous dimension at threshold exhibits Casimir scaling as has been found explicitly at one-loop in previous examples [3, 10, 18]. Technical results related to colour, Wilson lines and anomalous dimensions are summarized in a number of appendices, including the colour bases for triplet, anti-triplet and adjoint coloured particles in the initial and final state, which covers the cases of interest.

2 All-order colour structure of the soft function

The physical picture of production of a heavy particle pair at threshold suggests that soft gluon radiation cannot resolve the two particles and couples to the total colour charge of the pair, determined by the representation $R_\alpha$ in the decomposition of the product representation $R \otimes R'$. Therefore the structure of the (leading) soft function for the production of a non-relativistic particle pair in a given representation should be that of a single particle in the same representation. This is in agreement with the result that the one-loop soft anomalous dimension at threshold is proportional to the quadratic Casimir operator of the representation of the heavy-particle pair [3, 10, 18]. In this section we show how to obtain this structure, to all orders in perturbation theory, from the expression (1.8) for the soft function. In order to accomplish this, we construct the projection operators $P_{(k)}^{R_\alpha}$ and the basis tensors $c^{(i)}$ in the definition of the soft function (1.8) from Clebsch-Gordan coefficients for the decompositions of the product representations of the initial and final state.
systems into irreducible ones. In this basis it is then easy to show that the soft function is diagonal to all orders of perturbation theory. Some technical details of this construction are relegated to appendix A. Explicit colour bases and projectors for all production processes of heavy particles in the fundamental, antifundamental and adjoint representations are provided in appendix B. This covers all production processes of squarks and gluinos and completes previous results for the colour bases of squark-antisquark and gluino pair production [18].

2.1 Notation

With respect to colour, we use a notation that does not distinguish particles and antiparticles. If, for example, \( H' \) is the anti-particle of \( H \), then this convention implies that \( H' \) transforms in the complex conjugate SU(3) representation of \( H \). Similarly, an initial-state antiquark transforms in the anti-fundamental representation. The Wilson lines \( S \) inherit the corresponding representations. It is useful to perform a decomposition of the product of the representations of the final state and initial state particles into irreducible representations:

\[
 r \otimes r' = \sum_\alpha r_\alpha, \quad R \otimes R' = \sum_{R_\alpha} R_\alpha. \tag{2.1}
\]

Examples relevant for the standard model and most extensions of the standard model are particles in the fundamental and the adjoint representation where we have the decompositions \( 3 \otimes \bar{3} = 1 + 8 \) (e.g. top-antitop and squark-antisquark production), \( 3 \otimes 3 = 3 + 6 + 15 \) (e.g. squark-gluino production) and \( 8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27 \) (gluino pair production). Generators of the SU(3) transformation in a representation \( R \) are denoted by \( T^{(R)a} \). In practice we need the generators in the fundamental and anti-fundamental representations, \( T^{(3)a}_{a_1a_2} = T^{a}_{a_1a_2}, \quad T^{(\bar{3})a}_{a_1a_2} = -T^{a}_{a_2a_1}, \) and in the adjoint, \( T^{(8)a}_{a_1a_2} = if^{a_1a_2a}a F_{a_1a_2} \). The quadratic Casimir operator for a representation \( R \) is denoted by \( (T^{(R)a}T^{(R)a})_{a_1a_2} = C_R \delta_{a_1a_2} \).

2.2 Construction of projectors on irreducible representations

We now review a general method to construct the projection operators on the irreducible representations of the heavy-particle pair appearing in the soft function (1.8) from the Clebsch-Gordan coefficients that combine two objects transforming in the representations \( R \) and \( R' \) of the group into a single object in an irreducible representation \( R_\alpha \). The Clebsch-Gordan coefficients are defined as a unitary basis transformation,

\[
e^{a_1a_2} = \sum_{R_\alpha} C^{R_\alpha}_{a a_1a_2} e^{R_\alpha}, \tag{2.2}
\]

from the basis vectors of the tensor product space \( R \otimes R' \) to basis vectors of the irreducible representations \( R_\alpha \) (see e.g. [26]). Here the \( e^{a_1a_2} \) are elements of a complex, orthonormal basis of the tensor product \( R \otimes R' \) while the \( e^{R_\alpha} \) are elements of a basis of the irreducible
representation $R$. In this equation and in the following, repeated indices are summed over. Unitarity of the basis transformation implies

$$\sum_{R_{a}} C_{\alpha a_{1} a_{2}}^{R_{a}} C_{\alpha a_{1} a_{2}}^{R_{a}} = \delta_{a_{1} a_{2}} \delta_{a_{1} a_{2}}, \quad (2.3)$$

$$C_{\alpha a_{1} a_{2}}^{R_{a}} C_{\beta a_{1} a_{2}}^{R_{a}} = \delta_{R_{a} R_{b}} \delta_{a_{1} a_{2}} \delta_{a_{1} a_{2}}. \quad (2.4)$$

A vector $V$ in the tensor-product space can be written in the two bases as $V = V_{a_{1} a_{2}} e_{a_{1} a_{2}} = \sum_{R_{a}} V_{a} e_{a}$, where the components are related by

$$V_{\alpha} = C_{\alpha a_{1} a_{2}}^{R_{a}} V_{a_{1} a_{2}}, \quad (2.5)$$

$$V_{a_{1} a_{2}} = \sum_{R_{a}} C_{\alpha a_{1} a_{2}}^{R_{a}} V_{\alpha}. \quad (2.6)$$

Consistency with the group transformations $V_{\alpha} \rightarrow U_{\alpha \beta}^{(R)} V_{\beta}$ and the corresponding transformations of $V_{a_{1} a_{2}}$ implies that the Clebsch-Gordan coefficients satisfy

$$C_{\alpha a_{1} a_{2}}^{R_{a}} U_{a_{1} b_{1}}^{(R)} U_{a_{2} b_{2}}^{(R')} = U_{a_{1} b_{1}}^{(R)} C_{\alpha a_{1} a_{2}}^{R_{a}} C_{\beta b_{2}}^{R_{a}}. \quad (2.7)$$

We can now construct the projectors $P_{R_{a}}$ on the irreducible representations from the coefficients $C_{\alpha a_{1} a_{2}}^{R_{a}}$ according to

$$P_{R_{a}}^{(a_{1} a_{2} a_{3} a_{4})} = C_{\alpha a_{1} a_{2}}^{R_{a}} C_{\alpha a_{3} a_{4}}^{R_{a}}. \quad (2.8)$$

The orthonormalization condition (2.4) implies that these are projectors satisfying

$$P_{R_{a}}^{R_{a}} P_{b_{1} b_{2} c_{1} c_{2}}^{R_{a}} = \delta_{R_{a} R_{b}} P_{a_{1} a_{2} c_{1} c_{2}}^{R_{a}}, \quad (2.9)$$

and the relation (2.3) is equivalent to the completeness relation

$$\sum_{R_{a}} P_{R_{a}}^{R_{a}} a_{1} a_{2} b_{1} b_{2} = \delta_{a_{1} a_{2}} \delta_{a_{1} a_{2}}. \quad (2.10)$$

As an example, consider final state particles in the 3 and the $\bar{3}$ representation of SU(3), e.g. top-antitop and squark-antisquark pairs. The two particles combine to a singlet or octet, and the Clebsch-Gordan coefficients read

$$C_{a_{1} a_{2}}^{(1)} = \frac{1}{\sqrt{N_{c}}} \delta_{a_{1} a_{2}}, \quad C_{a_{1} a_{2}}^{(8)} = \sqrt{2} \ T_{a_{1} a_{2}}^{a_{1} a_{2}}. \quad (2.11)$$

We obtain the familiar projectors

$$P_{(a)}^{(1)} = \frac{1}{N_{c}} \delta_{a_{1} a_{2}} \delta_{a_{3} a_{4}}, \quad P_{(a)}^{(8)} = 2 \ T_{a_{1} a_{2}}^{a_{1} a_{2}} T_{a_{3} a_{4}}^{a_{1} a_{2}}. \quad (2.12)$$
2.3 Colour basis for the hard production process

The colour structures \( c_{(a)}^{(i)} \) in the definition (1.8) describe the colour structure of the hard-scattering process. They provide a decomposition of the scattering amplitudes of the partonic process \( pp' \rightarrow HH' \) into independent basis tensors:

\[
A_{pp'\{a\}} = \sum_i c_{(a)}^{(i)} A_{pp'}^{(i)}.
\]

(2.13)

Here we use a multi-index notation \( \{a\} = a_1a_2a_3a_4 \). Repeated multi-indices are summed over all four components. We take the \( c_{(i)}^{(i)} \) as an orthonormal basis satisfying

\[
c^{(i)}_{\{a\}} c^{(j)}_{\{a\}} = \delta^{ij}.
\]

(2.14)

The colour basis structures can be chosen as invariant tensors in the representation \( r \otimes r' \otimes \overline{R} \otimes \overline{R}' \):

\[
c_{\{a\}}^{(i)} = U^{(R)}_{\alpha a_3} U^{(R')}_{\beta a_4} c_{\{b\}}^{(i)} U^{(r)}_{\gamma b_1} U^{(r')}_{\delta b_2}.
\]

(2.15)

A colour basis that is convenient for the discussion of the colour structure of the soft function can be constructed from the Clebsch-Gordan coefficients, similarly to the projectors. Consider the subset of representations that appears in both the sets \( \{r_\alpha\} \) and \( \{R_\beta\} \) in (2.1), treating multiple occurrences of equivalent representations in the decomposition as distinct. We then form pairs \( P_i = (r_\alpha, R_\beta) \) of equivalent representations \( r_\alpha \) and \( R_\beta \), where the index \( i \) enumerates the allowed combinations. For example, in case of \( 8 \otimes 8 \rightarrow 8 \otimes 8 \) we have the eight pairs

\[
P_i \in \{(1, 1), (8_S, 8_S), (8_A, 8_A), (8_S, 8_A), (10, 10), (\overline{10}, \overline{10}), (27, 27)\}.
\]

(2.16)

The 10 and \( \overline{10} \) are inequivalent representations, so the pair \((10, \overline{10})\) is not allowed. For the allowed pairs \( P_i \), the colour structures

\[
c_{\{a\}}^{(i)} = \frac{1}{\sqrt{\text{dim}(r_\alpha)}} C_{\alpha a_1 a_2}^{r_\alpha} C_{\alpha a_3 a_4}^{R_\beta*}
\]

(2.17)

form an orthonormal basis satisfying (2.14). By construction, the operators are invariant tensors under global SU(3) transformations satisfying (2.15). In appendix A we use colour conservation of the scattering amplitude to show that the basis tensors always can be chosen as in (2.17).

To illustrate this construction, we consider heavy particles in the 3 and \( \overline{3} \) produced from a quark-antiquark initial state. There are only two possibilities to combine the initial and final state representations, \( P_i = \{(1, 1), (8, 8)\} \). Using (2.11) we obtain two operators for the basis of the hard-scattering process:

\[
c_{\{a\}}^{(1)} = \frac{1}{N_c} \delta_{a_1 a_2} \delta_{a_3 a_4}, \quad c_{\{a\}}^{(2)} = \frac{2}{\sqrt{D_A}} T_{a_2 a_1}^\beta T_{a_3 a_4}^\beta
\]

(2.18)

with \( D_A = N_c^2 - 1 \). This is the same basis found to diagonalize the one-loop soft anomalous dimension at threshold [3]. A complete list of all projectors and basis tensors for the production of particles in the fundamental and adjoint is given in appendix B.
2.4 Diagonalization of the soft function

We now show that the soft function is diagonal to all orders of perturbation theory in the basis constructed in section 2.3. To achieve this, we express the components of the soft function (1.8) in terms of Wilson lines in the representations $R_\alpha$, as suggested by the physical picture of soft radiation off the total colour charge of the final state system. We first note that the Wilson lines satisfy a relation analogous to (2.7),

$$C_{R_\alpha}^{\alpha_1 a_1 \beta_1} S_{v, \alpha_1 b_1}^{(R)} S_{v, \alpha_2 b_2}^{(R')} = S_{v, \alpha_3 \beta_2}^{(R_\alpha)} C_{\beta_2 b_2}^{\alpha_3 \beta_2}.$$  (2.19)

A proof of this relation is given in appendix A. Using this identity and the completeness relation (2.3) of the Clebsch-Gordan coefficients we can write the soft function (1.6) in terms of Wilson lines for single particles in the irreducible representations $R_\alpha$:

$$\hat{W}_{\{ab\}}^{(k)}(z, \mu) = \sum_{R_\alpha, R_\beta} C_{R_\beta b_4 \alpha_3 a_4}^{R_\beta} C_{\alpha_3 k_1 k_2}^{R_\alpha} C_{\alpha_2 a_4}^{R_\alpha} C_{\alpha_1 \kappa k_3 k_4}^{R_\alpha} \times \langle 0 | T[S_{v, \beta k_3}^{R_\beta} S_{v, \alpha_1 b_2}^{R_\alpha} S_{v, \alpha_2 b_1}^{R_\alpha}](z) T[S_{n, a_1}^{R_\alpha} S_{n, a_2}^{R_\alpha} S_{v, \beta k_3}^{R_\beta}](0) | 0 \rangle.$$  (2.20)

This combination of the two Wilson lines in the representations $R$ and $R'$ to a single one in the representations $R_\alpha$ is only possible if the heavy particle pair is produced close to threshold and not for generic kinematics where Wilson lines in different directions $v_1, v_2$ appear in the soft function.

In (2.20) the Wilson lines related to the final state system are still in two different representations. This structure simplifies when we compute the components of the soft function (1.8) where the soft function is contracted with a projector on an irreducible representation. Using the definition of the projectors (2.8), we have the identity

$$C_{\kappa k_1 k_2}^{R_\beta} P_{\{k\}}^{R_\beta} C_{\lambda k_3 k_4}^{R_\alpha} = \delta_{\kappa \lambda} \delta_{R_\alpha, R_\gamma} \delta_{R_\alpha, R_\beta},$$  (2.21)

i.e. the projectors enforce that the representations $R_\beta$ and $R_\gamma$ are identical. For distinct equivalent representations such as the $8_S$ and $8_A$ representations in the decomposition of $8 \otimes 8$ we obtain a vanishing result. We can then express the components of the soft function (1.8) as

$$W_{ii'}^{R_\alpha}(\omega, \mu) = c_{\{aa\}}^{R_\alpha(i)} W_{\{aa, b\beta\}}^{R_\alpha}(\omega, \mu) c_{\{b\beta\}}^{R_\alpha(i')*},$$  (2.22)

where we have introduced the soft function for the production of a single particle in the representation $R_\alpha$

$$\hat{W}_{\{aa, b\beta\}}^{R_\alpha}(z, \mu) \equiv \langle 0 | T[S_{v, \beta k_3}^{R_\beta} S_{v, \alpha_1 b_2}^{R_\alpha} S_{v, \alpha_2 b_1}^{R_\alpha}](z) T[S_{n, a_1}^{R_\alpha} S_{n, a_2}^{R_\alpha} S_{v, \beta k_3}^{R_\beta}](0) | 0 \rangle.$$  (2.23)

Here we have extended our multi-index convention to the indices of the irreducible representations of the final state system by defining $\{aa\} = a_1 a_2 \alpha$. Note that the index $\kappa$ of the two final state Wilson lines in (2.23) is contracted, so this is analogous to the soft function appearing in the conventional treatment where Coulomb gluons are not factorized. This
function has been considered in [21] for the case of a single colour-octet scalar. The colour basis tensors of the production operators in the new notation are given by

\[ c_{\{\alpha\}}^{(i)} \equiv C_{\alpha a_1 a_2}^{R_{R_\alpha}} = \frac{1}{\sqrt{\dim(r_\alpha)}} C_{\alpha a_1 a_2}^{R_{R_\alpha}} \delta_{R_\alpha R_\beta}. \]  

(2.24)

As indicated, they are nonvanishing only if the final state representation \( R_\alpha \) is identical to both final state representations \( R_\beta \) and \( R_\beta' \) and hence, by construction of the basis tensors, equivalent to both initial state representations \( r_\alpha \) and \( r_\alpha' \) in the pairs \( P_i = (r_\alpha, R_\beta) \) and \( P_i' = (r_\alpha', R_\beta') \). Therefore the soft function is block-diagonal in the basis (2.17) with off-diagonal elements arising only if several representations in the decomposition (2.1) of the initial state system are equivalent. (A non-trivial structure does not appear if the several equivalent representations appear only in the final state since the projectors \( P^{R_\alpha} \) always project on a unique representation.) Since in the decompositions of \( 3 \otimes 3, 3 \otimes \bar{3} \) and \( 3 \otimes 8 \) no representation occurs more than once, the matrices \( W_{ii'}^{R_\alpha} \) are diagonal if at least one quark or anti-quark is present in the initial state. The only example where a non-trivial matrix structure can arise is the gluon-gluon channel.

For the example of an \( 8 \otimes 8 \) final state produced from gluon-gluon fusion, according to (2.16), only the soft functions for the two octet final states \( 8_S \) and \( 8_A \) are (potentially)
non-trivial two-by-two matrices,

\[
W^{8s} = \begin{pmatrix}
0 & \cdots & \cdots & \cdots \\
\vdots & W_{22}^{8s} & W_{23}^{8s} & \vdots \\
\vdots & W_{32}^{8s} & W_{33}^{8s} & \vdots \\
0 & \cdots & \cdots & \cdots
\end{pmatrix}, \quad W^{8A} = \begin{pmatrix}
0 & \cdots & \cdots & \cdots \\
\vdots & W_{44}^{8A} & W_{45}^{8A} & \vdots \\
\vdots & W_{54}^{8A} & W_{55}^{8A} & \vdots
\end{pmatrix}
\] (2.25)

with entries in the order of the basis elements used in (2.16). The soft functions for the 1, 10, \( \overline{10} \) and 27 representations, however, consist of a single non-vanishing element. Analogously, for the production of a 3 \( \otimes \) 3 final state from gluon-gluon fusion the singlet soft function consists of a single non-vanishing entry while the colour-octet soft function contains a potentially non-trivial two-by-two submatrix mixing \( 8_A \) and \( 8_s \) initial states.

We now use Bose symmetry of the soft function to show that the off-diagonal elements in the soft function for the production of a colour-octet state vanish, so that the matrices (2.25) and the analogous ones for a 3 \( \otimes \) 3 final state are in fact diagonal to all orders in perturbation theory. First note that, independent of the nature of the final state system, these off-diagonal matrix elements involve a combination of Wilson lines contracted with the Clebsch-Gordan coefficients \( C^{(8s)}_{\alpha a_1 a_2} \) and \( C^{(8s)}_{\alpha a_1 a_2} \) defined in (B.5):

\[
C^{(8s)*}_{\beta b_1 b_2} [S^\dagger_{n,ib_1} S^\dagger_{\bar{n},jb_2}] (z) [S_{\bar{n},a_2 j} S_{n,a_1 i}] (0) C^{(8s)}_{\alpha a_1 a_2} \propto \left[ S^\dagger_{n,ib_1} D_{b_1 b_2}^{\beta} S^\dagger_{\bar{n},jb_2} \right] (z) [S_{\bar{n},a_2 j} F_{a_2 a_1}^{\alpha} S_{n,a_1 i}] (0). \tag{2.26}
\]

Because the two incoming Wilson lines are indistinguishable, the soft function must be invariant under the exchange of their colour labels and momenta, i.e. the exchange \((n, i) \leftrightarrow (\bar{n}, j)\). This statement translates into the following equation:

\[
[S^\dagger_{n,ib_1} D_{b_1 b_2}^{\beta} S^\dagger_{\bar{n},jb_2}] (z) [S_{\bar{n},a_2 j} F_{a_2 a_1}^{\alpha} S_{n,a_1 i}] (0) = [S^\dagger_{\bar{n},jb_1} D_{b_1 b_2}^{\beta} S^\dagger_{n,ib_2}] (z) [S_{n,a_2 i} F_{a_2 a_1}^{\alpha} S_{\bar{n},a_1 j}] (0) = -[S^\dagger_{n,ib_1} D_{b_1 b_2}^{\beta} S^\dagger_{\bar{n},jb_2}] (z) [S_{\bar{n},a_2 j} F_{a_2 a_1}^{\alpha} S_{n,a_1 i}] (0) = 0. \tag{2.27}
\]

Here we have used the symmetry properties of the \( D \) and \( F \) tensors. Therefore the off-diagonal terms of the colour-octet soft function such as in (2.25) vanish. Furthermore from the explicit results in section 3 we deduce that the one-loop soft function and two-loop soft anomalous dimensions are determined by the quadratic Casimir operators, so the diagonal elements of \( W^{8s} \) and \( W^{8s} \) are all identical. This would extend to higher orders if Casimir scaling held to all orders. However, presently we do not have a proof for this.

To summarize we have shown that for all initial states relevant to hadron colliders the soft function is diagonal to all orders in a colour basis for the hard-scattering process given by (2.17). This holds independent of the nature of the final state particles, i.e. equally for
top-quarks, squarks or gluinos. As mentioned at the end of section 1, the approach used in most phenomenological applications corresponds to using the soft function $W_{ii'}$ obtained by summing up all final state representations so our results apply to this case as well. Let us briefly recall the main ingredients used in order to arrive at this result:

- The Coulomb interaction is diagonalized by the decomposition of the final state system into irreducible representations (2.1), leading to the definition of the components of the soft function (1.8) where the Wilson lines associated to the final-state particles are projected onto the irreducible representations.

- The hard-scattering amplitudes are colour conserving, eq. (2.15), which allowed us to choose the basis of colour tensors according to (2.17).

- For a heavy particle pair produced directly at threshold both particles have the same velocity, allowing to combine the two final-state Wilson lines into a single one.

- Due to Bose symmetry of the soft function there is no interference of the production from a symmetric and antisymmetric colour octet.

2.5 Examples

In this subsection we give some examples of the formalism in order to show how it is related to the colour bases used in previous computations. As an illustration of the result (2.22) for the components of the soft function, we give explicit expressions for the example of a $3 \otimes \bar{3}$ final state and a quark-antiquark or gluon-gluon initial state. In subsection 2.5.2 we compare the basis for an $8 \otimes 8$ final state produced in gluon fusion to that used previously [18].

2.5.1 Soft function for a $3 \otimes \bar{3}$ final state

For quark-antiquark initiated production of a heavy particle pair in the $3 \otimes \bar{3}$ representation, the required colour basis is given by (2.18) so the soft functions for the singlet and octet final states, $W^1_{ii'}$ and $W^8_{ii'}$, are two-by-two matrices. The only non-vanishing component of the soft function for the singlet case is identical to that in Drell-Yan production [8, 27]

$$\hat{W}^1_{11}(z, \mu) = \frac{1}{N_c} \langle 0 | \text{Tr} [\bar{T} S_n S_n](z) T [S_n S_n](0) | 0 \rangle = \hat{W}_{DY}(z, \mu).$$ (2.28)

Here we have deviated from our usual notation of using the anti-fundamental representation for anti-particles and expressed the result using only Wilson lines in the fundamental representation in order to simplify the matrix structure. Similarly, the only non-vanishing component of the colour-octet soft function is given by

$$\hat{W}^8_{22}(z, \mu) = \frac{2}{N_c^2 - 1} \langle 0 | \text{Tr} [\bar{T} S_v S_n T S_n (z) T [S_n S_n S_n S_n^+](0) | 0 \rangle. \tag{2.29}$$
where the $S_v$ are in the adjoint representation and the trace is over the fundamental representation.

For the production of a $3 \otimes \bar{3}$ final state from gluon fusion there are three possible combinations of initial and final state representations:

$$P_i \in \{(1, 1), (8_S, 8), (8_A, 8)\}.$$  \hfill (2.30)

The Clebsch-Gordan coefficients and colour-basis elements for this case are collected in appendix (B.1). Since the set of $P_i$’s has three elements, the singlet and octet soft functions are three-by-three matrices. The only non-vanishing element of the soft function for the singlet channel is, up to normalization, again given by (2.28), where now the Wilson lines $S_n$ and $S_{\bar{n}}$ are in the adjoint representation. An octet final state can be produced either from a symmetric or antisymmetric octet initial state corresponding to the basis elements $c(2)$ and $c(3)$ in (B.6). The non-vanishing diagonal elements of the soft-function matrix for the octet channel in this basis are given by

$$\hat{W}_{22}^8(z, \mu) = \frac{N_c}{(N_c^2 - 1)(N_c^2 - 4)} \langle 0 | \text{Tr} \left[ T[S_{v,ac}S_n^\dagger D^b S_{\bar{n}}(z) T[S_{\bar{n}}^\dagger D^b S_n^{\sigma_{v,cb}}(0)]0 \right] | 0 \rangle,$$  \hfill (2.31)

$$\hat{W}_{33}^8(z, \mu) = \frac{1}{N_c(N_c^2 - 1)} \langle 0 | \text{Tr} \left[ T[S_{v,be}S_n^\dagger F^{bc} S_{\bar{n}}(z) T[S_{\bar{n}}^\dagger F^{bc} S_n^{\sigma_{v,ca}}(0)]0 \right] | 0 \rangle.$$  \hfill (2.32)

Here we have used the fact that the Wilson lines in the adjoint, $S_n$, $S_{\bar{n}}$, are real to write the matrix product in a convenient form. The expressions for the octet soft functions agree precisely with [21] (up to their notation for the result in momentum space). The off-diagonal elements $W_{23}^8$ and $W_{32}^8$ involve the structure (2.26) and therefore vanish by symmetry arguments.

### 2.5.2 Colour octet states in $8 \otimes 8 \rightarrow 8 \otimes 8$

We would like to comment briefly on previous results for the basis for the gluon-induced production of heavy particles in the adjoint representation (e.g. gluinos). The complete basis for this case is given in appendix B. Here we will only need the four operators corresponding to the different combinations of $8_S$ and $8_A$ in the initial and final state (B.20). This basis differs slightly from the one constructed in [4] for dijet production where the linear combinations $\pm \sqrt{2}(c^{(3)} \pm c^{(5)})$ have been used. In [18] that basis has been shown to diagonalize the one-loop soft anomalous-dimension matrix corresponding to the soft function $W_{ii'}$ discussed below (1.8) at threshold, i.e. for $v_1 = v_2$. From the general arguments given above and from an explicit calculation we find that the one-loop soft functions for colour octet final states $W_{ii'}^{8_S}$ and $W_{ii'}^{8_A}$ are diagonal in the basis (B.20) but not in the one used in [18]. However, since the Coulomb functions $J_{8_A}$ and $J_{8_S}$ are identical, only the sum $W_{ii'}^{8_S} + W_{ii'}^{8_A}$ enters the cross section (1.3). Since the off-diagonal terms cancel in the sum, our result is consistent with the one in [18]. Similar remarks apply to the colour tensors related to the 10 and $\mathbf{10}$ where the basis used in [4,18] is appropriate for the sum $W^{10} + W^{\mathbf{10}}$ that is relevant to the cross section.
3 Ingredients for NNLL threshold resummation

The resummation of threshold logarithms proceeds by using the factorization scale independence of the total cross section to derive renormalization group equations for the hard function $H_{ii}$, which appears in eq. (1.3), and the soft function. To define the NLL, NNLL, etc. approximations, we note that near threshold the usual expansion, where $\alpha_s \ln \beta$ counts as order one, is combined with an expansion in $\beta$, such that $\alpha_s / \beta$ also counts as one. This leads to a parametric representation of the expansion of the cross section in the form

$$\hat{\sigma}^{(0)}_{pp'} = \hat{\sigma}^{(0)} \sum_{k=0} \left( \frac{\alpha_s}{\beta} \right)^k \exp \left[ \ln \beta g_0(\alpha_s \ln \beta) + g_1(\alpha_s \ln \beta) + \alpha_s g_2(\alpha_s \ln \beta) + \ldots \right]$$

$$\times \left\{ 1 \ (LL,NLL); \alpha_s, \beta \ (NNLL); \alpha_s^2, \alpha_s \beta, \beta^2 \ (NNNLL); \ldots \right\}, \tag{3.1}$$

which reproduces the standard structure [10] away from threshold for $k = 0$ and no expansion in $\beta$. Thus, in fixed orders, LL includes relative to the tree term $\hat{\sigma}^{(0)}$ all terms of the form

$$LL \quad \alpha_s \left\{ \frac{1}{\beta} \ln^2 \beta \right\}; \alpha_s^2 \left\{ \frac{1}{\beta^2} \ln^3 \beta \right\}; \ldots, \tag{3.2}$$

while NLL and NNLL further include all terms

$$NLL \quad \alpha_s \ln \beta; \alpha_s^2 \left\{ \frac{1}{\beta^2} \ln \beta, \ln^3 \beta \right\}; \ldots, \tag{3.3}$$

$$NNLL \quad \alpha_s \left\{ 1, \beta \times \ln^2 \beta \right\}; \alpha_s^2 \left\{ \frac{1}{\beta^2} \ln^2 \beta, \beta \times \ln^3 \beta \right\}; \ldots, \tag{3.3}$$

respectively. Note that while the LL approximation sums soft logarithms of the form $\alpha_s^n \log^m \beta^m = \alpha_s \log^2 \beta^2, \ldots$ with $n + 1 \leq m \leq 2n$, this does not include all terms of this form at $O(\alpha_s^2)$. Similarly, the NLL approximation sums soft logarithms of the form $(\alpha_s \log \beta)^n$ but $O(\alpha_s^2 \log \beta^2)$ terms from the interference of the one-loop hard function and leading soft logarithms are included only at NNLL. The NNLL terms proportional to $\beta$ originate from $\beta$-suppressed corrections to the hard functions and the soft gluon couplings. The former, however, vanish, since S-wave and P-wave production processes are not interfering. The $\beta$-suppressed soft corrections average to zero in the total cross section at $O(\alpha_s)$ (and probably as well in higher orders), such that no terms of the form $\alpha_s \beta \times \ln^2 \beta$ are present in the fixed-order expansion.

We would now like to explain briefly the several functions appearing in the factorization formula (1.3). We also discuss how the expansion (3.1) is generated from this expression and, starting from NNLL, additional contributions of a similar factorized form. The derivation of the factorization formula (1.3) in [22] relies on soft-collinear and potential non-relativistic effective field theory, and the fact that soft gluon interactions can be decoupled from collinear and potential fields in the leading-order effective Lagrangian using field redefinitions involving the Wilson lines (1.4) and (1.5). For the initial-state
partons, this redefinition is identical to that in the derivation of the factorization formula for the Drell-Yan process at partonic threshold [8]. The redefinitions for the final state particles are a generalization of those used for non-relativistic $W$-bosons [23] to the case of a colour Coulomb force. The soft function (1.6) collects the Wilson lines arising from these field redefinitions. The function $J_{R_{\alpha}}$ factorizes potential effects and sums Coulomb gluon exchange related to the attractive or repulsive Coulomb force in the irreducible colour representations $R_{\alpha}$. It is defined as a correlation function of non-relativistic fields and can be expressed in terms of the imaginary part of the zero-distance Coulomb Green function of the Schrödinger equation. The hard function is defined in terms of squared short-distance coefficients, $H_{ii'} \propto C^{(i)}_{pp'} C^{(i')*}_{pp'}$, analogous to the corresponding treatments of heavy-particle pair production in $e^+e^-$ collisions [28] and of the Drell-Yan process [8]. The coefficients $C^{(i)}_{pp'}(M, \mu)$ encode the contribution of hard momenta to the process $pp' \to HH'$ and are obtained from the scattering amplitude for the partonic subprocess evaluated directly at threshold [22].

For resummation at NLL accuracy, the required ingredients in eq. (1.3) are the hard function $H_{ii'}$ and the soft function $W_{ii'}^{R_{\alpha}}$ both at tree-level and the one-loop anomalous dimensions appearing in the evolution equations, with the exception of the so-called cusp anomalous dimension related to the leading logarithms that is required at two loops. We note that part of the NLL $\alpha_s^2/\beta \times \ln \beta$ term arises from the running coupling in the Coulomb potential and is correctly taken into account by choosing the scale to be $M \beta$ in $J_{R_{\alpha}}$.

For resummation at NNLL accuracy the hard and soft function are needed at one-loop level, the cusp anomalous dimension at three loops and all remaining anomalous dimensions at two loops. While most of the anomalous dimensions are known to the required order or higher from studies of deep-inelastic scattering, the Drell-Yan process or Higgs production from gluon fusion, the soft anomalous dimension for pair production of heavy coloured particles is currently available only at one-loop [3, 10, 18], despite recent progress on massive amplitudes at the two-loop level [24, 25, 29].

In addition to the soft corrections considered in this paper, further logarithmic contributions arise from higher-order terms in the effective Lagrangian or the production operators where soft gluons do not decouple after the field redefinitions. An example is the dipole interaction $\vec{x} \cdot \vec{E}$ in the potential non-relativistic QCD Lagrangian. These terms, however, can be treated as perturbations so that the entire expansion (3.1) can be constructed as a sum of terms in the factorized form $H^{(k)} \times J^{(k)} \ast W^{(k)}$ with new hard, potential and soft functions, of which eq. (1.3) constitutes only the leading term in the expansion in $\beta$. Due to the $\beta$ suppression, the first correction to the leading term arises at NNLL. Therefore further NNLL terms in addition to (1.3) may arise from two sources related to higher-dimensional terms in the expansion in $\beta$ in the factorization formula: first, corrections to the Coulomb function $J_{R_{\alpha}}$ due to subleading heavy-quark potentials contribute NNLL terms, which at $\mathcal{O}(\alpha_s^2)$ are of the form $\alpha_s^2 \log \beta$ [28]. Second, higher-dimensional soft functions with insertions of the $\vec{x} \cdot \vec{E}$ interaction potentially also contribute $\alpha_s^2 \log^2 \beta$ terms from the interference of a $\beta$-suppressed soft-gluon emission with one-Coulomb exchange. As mentioned before, there are no corrections linear in $\beta$ related to the hard function.
Finally, we mention that before the convolution with the parton distributions, the NNLL resummed cross section should be matched to a fixed-order two-loop calculation that is not yet available for the processes of interest.

In this section we continue our investigation of the (leading) soft function and provide all those ingredients required for an NNLL resummation for pair production of arbitrary coloured particles that are related to (1.3) except for the process-dependent one-loop hard functions $H_{ii'}$, i.e. we provide the one-loop soft function $W_{ii'}^{R_{a}}$ and its two-loop soft anomalous dimension. In subsection 3.1 we compute the one-loop soft function for initial-state particles in arbitrary representations $r$ and $r'$ of SU(3) and a final state system in an arbitrary representation $R_{a}$, using the reduction of the general soft function for two final state Wilson lines (1.6) to the soft function (2.23) with a single Wilson line representing the final state system, achieved in section 2. In subsection 3.2 we then obtain the two-loop soft anomalous dimension using results of [25, 29, 30]. In subsection 3.3 we relate our results to the conventions used in the NNLL treatment of the top-quark production cross section in Mellin space [11].

3.1 One-loop soft function

In this subsection we compute the one-loop term in the loop expansion of the soft function

$$W_{\{a_{a},b_{b}\}}^{R_{a}}(z_{0},\mu) = \sum_{n=0}^{\infty} \left(\frac{\alpha_{s}(\mu)}{4\pi}\right)^{n} W_{\{a_{a},b_{b}\}}^{(n)R_{a}}(z_{0},\mu).$$

The $n$-loop contribution to the soft function is obtained from the definition (2.23) by expanding each Wilson line up to order $g^{2n}$ and keeping all contributions to the soft function of order $g^{2n}$. At tree level, the soft function in position and momentum space is simply given by

$$W_{\{a_{a},b_{b}\}}^{R_{a}(0)}(z_{0},\mu) = \delta_{a_{1}b_{1}}\delta_{a_{2}b_{2}}\delta_{\alpha\beta},$$

$$W_{\{a_{a},b_{b}\}}^{R_{a}(0)}(\omega,\mu) = \delta_{a_{1}b_{1}}\delta_{a_{2}b_{2}}\delta_{\alpha\beta} \delta(\omega).$$

At the one-loop order, the expansion of the Wilson lines gives rise to real and virtual initial-initial (ii), initial-final (if) and final-final (ff) state interference diagrams. Examples for diagrams contributing to the real corrections are shown in figure 2. As an example consider the initial-final state interference diagram denoted by (if) in figure 2. This arises from the contribution to the soft function (2.23) where the Wilson lines $S_{(r)}^{(R_{a})}(0)$ and $S_{v,\beta}^{(R_{a})}(z)$ both contribute at order $g_{s}$ while all the other Wilson lines give trivial contributions. This diagram arises from the expectation value

$$\langle 0\lvert T[S_{(r)}^{(R_{a})}(z)]T[S_{(r)}^{(R_{a})}(0)]\lvert 0 \rangle_{if}$$

$$= \left(-ig_{s}T_{(R_{a})c}^{(R_{a})d}\right)\left(ig_{s}T_{(r)c}^{(r)d}\right)\int_{0}^{\infty} ds \int_{-\infty}^{0} dt \langle 0\lvert T[\nabla_{A}(v(z^{0}+s))]T[\nabla_{A}(tn)]\lvert 0 \rangle_{if}$$

$$= -4\pi\alpha_{s} T_{(R_{a})c}^{(R_{a})c} T_{(r)c}^{(r)c} (v \cdot n) \int_{-\infty}^{0} ds \int_{-\infty}^{0} dt D^{\perp}(v(z^{0} - s) - nt).$$

(3.6)
Here we have introduced the cut gluon propagator in position space

\[ D^{\mu\nu}_{ab}(x) = \langle 0 | A_\mu^a(x) A_\nu^b(0) | 0 \rangle \equiv (-g_{\mu\nu})\delta^{ab} D^+(x). \]  

(3.7)

It is convenient to evaluate the integrals arising in the soft function directly in position space using the dimensionally regularized form of the cut-gluon propagator \[ D^+(x) = \frac{\Gamma(1-\epsilon)}{4\pi^2-\epsilon} \frac{1}{[-(x_+ - i\delta)(x_- - i\delta)]^{1-\epsilon}} \] with \( x_+ = n \cdot x \) and \( x_- = \bar{n} \cdot x \), and where the last expression holds for \( x_\perp^\mu \equiv x^\mu - x_- n^\mu / 2 = 0 \). The usual Feynman propagator in position space is given by

\[ D^{\mu\nu}_{ab}(x) = (-g_{\mu\nu})\delta^{ab} D(x) = (-g_{\mu\nu})\delta^{ab} \frac{\Gamma(1-\epsilon)}{4\pi^2-\epsilon} \frac{1}{(-x^2 + i\delta)^{1-\epsilon}}. \]  

(3.9)

Proceeding in the same way for all diagrams we obtain the one-loop soft function. Setting scaleless integrals to zero, only the real corrections are non-vanishing. The complete one-loop soft function can then be written in terms of group-theory factors \( C \) and integrals \( I \) as

\[ \hat{W}^{(1)}_{\{aa,bb\}}(z_0, \mu) = -(4\pi)^2 \left[ C^{(ii)}_{\{aa,bb\}} I^{(ii)}(z_0, \mu) + C^{(if)}_{\{aa,bb\}} I^{(if)}(z_0, \mu) + C^{(ff)}_{\{aa,bb\}} I^{(ff)}(z_0, \mu) \right]. \]  

(3.10)

The group theory factors are given by

\[ C^{(ii)}_{\{aa,bb\}} = 2 T^{(r)a}_{a_1b_1} T^{(r)a}_{a_2b_2} \delta_{\alpha\beta}, \]  

(3.11)

\[ C^{(if)}_{\{aa,bb\}} = 2 \left( T^{(r)a}_{a_1b_1} \delta_{a_2b_2} + \delta_{a_1b_1} T^{(r)a}_{a_2b_2} \right) T_{\beta\alpha}^{(R_a)}, \]  

(3.12)

\[ C^{(ff)}_{\{aa,bb\}} = \delta_{a_1b_1} \delta_{a_2b_2} T_{\beta\alpha}^{(R_a)} T_{\kappa\alpha}^{(R_a)} = C_{R_a} \delta_{a_1b_1} \delta_{a_2b_2} \delta_{\beta\alpha}. \]  

(3.13)

For the initial-initial state diagrams it was used that diagrams with a soft-gluon coupling to two collinear particles in the same direction vanish. The integrals are given by

\[ I^{(ii)}(z_0, \mu) = \tilde{\mu}^2 (n \cdot \bar{n}) \int_{-\infty}^{0} ds dt D^+(z_0 v + tn - s\bar{n}) = -\frac{\Gamma(-\epsilon)}{8\pi^2} e^{z_0 \mu} \frac{1}{\epsilon^\epsilon} \left( \frac{i z_0 \mu}{2} \right)^2, \]  

(3.14)
\[
\mathcal{T}^{(f)}(z, \mu) = \tilde{\mu}^2 (v \cdot n) \int_0^0 ds dt \, D^+((z_0 - s)v - nt) = \frac{\Gamma(-\epsilon)}{2} e^{\gamma_E} \frac{i z_0 \mu}{2}, \quad (3.15)
\]
\[
\mathcal{T}^{(f)}(z, \mu) = \tilde{\mu}^2 v^2 \int_0^0 ds dt \, D^+((z_0 + s - t)) = \frac{\Gamma(-\epsilon)}{8} e^{\gamma_E} \frac{i z_0 \mu^2}{2}, \quad (3.16)
\]
Here \(\tilde{\mu}^2 = \mu^2 e^{\gamma_E} / (4\pi)\) and an infinitesimal imaginary part \(z_0 \rightarrow z_0 - i\delta\) is kept implicit.

It is useful to identify the amplitude \(A\) with a vector in colour space denoted by \(|\mathcal{A}_{pp'}\rangle\) [4,32]. More precisely the amplitude is a matrix element \(A_{pp'}\langle a|A_{pp'}\rangle\) with an orthogonal basis \(|\{a\}\rangle\). In our notation an antiparticle in representation \(r\) is described as a particle in the representation \(\bar{r}\). For incoming and outgoing particles the action of a generator acting on particle \(i\) in our conventions is given by
\[
\langle \{b\}|T_i|A\rangle = (-T_i a_{aj} a_{bj}) A_{bj a_{i}i}, \quad \text{incoming particle}
\]
\[
\langle \{b\}|T_i|A\rangle = A_{bi a_{i}i} T_i (\bar{R}_{a} a_{aj}), \quad \text{outgoing particle.}
\]
Colour conservation implies the identity
\[
\sum_i T_i |A\rangle = 0. \quad (3.18)
\]
In this notation the decomposition (2.13) reads
\[
|A_{pp'}\rangle = \sum_i |c^{(i)}\rangle A^{(i)}_{pp'}, \quad (3.19)
\]
and the components of the soft function (2.23) are expressed as \(W^{R\alpha}_{ii'} = \langle c^{(i)}|W^{R\alpha}|c^{(i)}\rangle\). Combining the results from eqs. (3.11) to (3.16) and expanding in \(\epsilon\), we find that the (unrenormalized) one-loop soft function in the colour-operator notation reads:
\[
\hat{W}^{(1)R\alpha}(L) = -((T_1 + T_2) \cdot T_3 + 2T_1 \cdot T_2) \left( \frac{2}{\epsilon^2} + 2 \frac{\epsilon}{L} + L^2 + \frac{\pi^2}{6} \right) + T_3^2 \left( \frac{2}{\epsilon} + 2L + 4 \right)
\]
\[
= (T_1^2 + T_2^2) \left( \frac{2}{\epsilon^2} + 2 \frac{\epsilon}{L} + L^2 + \frac{\pi^2}{6} \right) + 2 T_3^2 \left( \frac{1}{\epsilon} + L + 2 \right). \quad (3.20)
\]
Here we have introduced the variable [31]
\[
L = 2 \ln \left( \frac{i z_0 \mu e^{\gamma_E}}{2} \right), \quad (3.21)
\]
and used colour conservation (3.18) to arrive at the second equality.

The components \(W^{R\alpha}_{ii'}\) entering the factorization formula (1.3) can be obtained from the above result by contracting with the elements of the colour basis \(c^{(i)R\alpha}_{\{a\}}\) according to (2.22). As discussed in section 2.4, a large number of matrix elements are zero by construction and the soft matrices assume a block-diagonal form (c.f. (2.25)). Since the tree-level and
one-loop soft functions are proportional to the unit matrix in colour space and the colour tensors in our basis are given in terms of the Clebsch-Gordan coefficients (2.24), we find that the components (2.22) are diagonal due to the orthogonality of the Clebsch-Gordan coefficients:

$$\hat{W}_{ii}^{R_{\alpha}}(L, \mu) = \hat{W}_{ii}^{R_{\alpha}}(L, \mu) \delta_{ii'} \delta_{R_{\alpha}R_{\beta}}.$$  

(3.22)

In agreement with the general results of section 2.4 these elements are non-vanishing only if the final state representation $R_{\alpha}$ is identical to that in the pair $P_{i} = (r_{\alpha}, R_{\beta})$ that defines the basis element $c^{(i)}$. The diagonal elements at tree- and one-loop level are given by

$$\hat{W}_{i}^{(0)R_{\alpha}}(L, \mu) = 1,$$

(3.23)

$$\hat{W}_{i}^{(1)R_{\alpha}}(L, \mu) = (C_{r} + C_{r'}) \left( \frac{2}{\epsilon^{2}} + \frac{2}{\epsilon} L + L^{2} + \frac{\pi^{2}}{6} \right) + 2C_{R_{\alpha}} \left( \frac{1}{\epsilon} + L + 2 \right).$$

(3.24)

As shown in appendix C the Fourier transform of this result agrees with [21] for the special case of the production of a colour-octet particle from gluon fusion.

### 3.2 Renormalization group equations and anomalous dimensions

In the following we will provide the evolution equations of the hard function $H_{ii'}$ and the soft function $W_{ii'}^{R_{\alpha}}$ and determine the relevant anomalous dimensions at the two-loop level, as required for resummation at NNLL accuracy. As a result of the rewriting of the soft function in section 2, the soft anomalous-dimension matrix for pair production at threshold is identical to that of a two-to-one scattering process with two massless legs and one massive leg. Employing results from a recent analysis of constraints from soft-collinear factorization on the structure of infrared (IR) singularities of scattering amplitudes with massive particles [25], we extract an analytical expression for the two-loop soft anomalous dimension at threshold. Explicit expressions for the resummed cross section in momentum space will be given in [22], but we stress that the results given here are also applicable to the resummation in Mellin-moment space. The precise relation to the formalism in Mellin space is discussed in section 3.3.

As mentioned above, in the effective field theory treatment of the factorization formula the hard function is defined in terms of short-distance coefficients $C_{pp'}^{(i)}$ that are obtained from the components $A_{pp'}^{(i)}$ of the scattering amplitude (2.13) evaluated at threshold [22]. After renormalization of the ultraviolet divergences, the short-distance coefficients contain further IR divergences that match the ultraviolet divergences of the long-distance objects in the factorization formula. The IR-renormalized coefficients obtained by minimal subtraction of the IR-poles satisfy an evolution equation of the form

$$\frac{d}{d \ln \mu} C_{pp'}^{(i)}(M, \mu) = \Gamma_{ij}(M, \mu) C_{pp'}^{(j)}(M, \mu)$$

(3.25)

with an anomalous-dimension matrix whose form is constrained by soft and collinear factorization [25, 33–35]. Since we have shown that the (leading) soft function is diagonal
for all cases relevant to hadron-collider processes, only the diagonal elements of the hard function, \( H_{ii} \equiv H_i \), enter the formula for the production cross section (1.3). They satisfy the evolution equation

\[
\frac{d}{d \ln \mu} H_i(M, \mu) = 2 \text{Re} \Gamma_i(M, \mu) H_i(M, \mu)
\]

with \( \Gamma_{ii} \equiv \Gamma_i \). As shown in appendix D the results of [25] constrain the anomalous dimension to be of the form

\[
\Gamma_i(M, \mu) = \frac{1}{2} \gamma^\text{cusp} \left[ (C_r + C_r') \left( \ln \left( \frac{4M^2}{\mu^2} \right) - i\pi \right) + i\pi C_{R_\alpha} \right] + \gamma^V_i. \tag{3.27}
\]

Here we have introduced the coefficient \( \gamma^\text{cusp} \) by writing the cusp anomalous dimension for a massless parton in the representation \( r \) in the form \( \Gamma^r_{\text{cusp}} = C_r \gamma^\text{cusp} \) consistent with Casimir scaling which is appropriate at least up to three-loop order. The explicit one- and two-loop results for all anomalous dimensions needed in this section are collected in appendix D. The coefficient \( \gamma^\text{cusp} \) is known to three-loop order [36], and the \( \alpha_s(\alpha_s n_f)^k \) terms are known to all orders [37].

Adopting the result of [25] for the structure of the anomalous dimension matrix, at least up to the two-loop level the anomalous dimension \( \gamma^V_i \) can be written in terms of single-particle anomalous dimensions:

\[
\gamma^V_i = \gamma^r + \gamma^{r'} + \gamma^{R_\alpha}_{H,s}. \tag{3.28}
\]

The one- and two-loop anomalous-dimension coefficients \( \gamma^r \) of massless quarks, \( \gamma^q = \gamma^3 \), and gluons, \( \gamma^g = \gamma^8 \) are given in appendix A of ref. [35]. The anomalous dimension \( \gamma^{R_\alpha}_{H,s} \) is related to a massive particle in the final state representation \( R_\alpha \) in the pair \( P_i = (r'^\alpha, R_\alpha) \) defining the colour basis element \( c^{(i)} \) with index \( i \).

It should be mentioned that eqs. (3.27) and (3.28) are derived from ref. [25], where it is assumed that the two heavy particles have fixed but unequal velocities, when the poles in \( \epsilon \) of the hard amplitude are extracted. This is different from the limit we consider here, where \( \beta \to 0 \) before the limit \( \epsilon \to 0 \) and before the loop integrations are performed, which corresponds to the threshold expansion as defined in [38]. The order of limits does not commute, and by expanding in \( \beta \) first new IR divergences appear in the hard region that do not correspond to UV divergences in the soft and collinear but in the potential region of the threshold expansion. The complete result for the scale dependence (3.26) of the hard coefficient may thus contain additional terms related to the ultraviolet divergences of the higher-dimensional heavy-quark potentials and soft functions. In deriving eq. (3.27) from the \( 2 \to 1 \) process with a single particle in representation \( R_\alpha \) we implicitly set to zero the scale dependence of the hard function related to the contribution from potential divergences and the higher-dimensional soft functions. This allows us to relate the anomalous dimensions (3.27) and (3.28) directly to the one of the leading soft function \( W^{R_\alpha}_{ii'} \).

For the case of a heavy quark an analytical result for the two-loop anomalous dimension \( \gamma^Q = \gamma^3_{H,s} \) has been extracted in [25] from the anomalous dimension of the heavy-light
quark current in SCET. In order to generalize this result to arbitrary representations $R$, we observe, following [25], that $\gamma_{H,s}^R$ appears in the anomalous dimension of the HQET heavy-heavy current for a heavy particle in representation $R$,

$$\Gamma_{J_{hh}}^R = C_R \gamma_{\text{cusp}}(\beta, \alpha_s) + 2\gamma_{H,s}^R. \tag{3.29}$$

The cusp anomalous dimension $\gamma_{\text{cusp}}(\beta, \alpha)$ is a function of the cusp angle $\cosh \beta = v_1 \cdot v_2$, with $v_{1,2}$ the four-velocities of the heavy particles. The anomalous dimension $\Gamma_{J_{hh}}$ for heavy quarks is available at two-loop order [30]. For large cusp angle the massive cusp anomalous dimension is related to the cusp anomalous dimension for massless particles according to [25]

$$\gamma_{\text{cusp}}(\beta, \alpha_s) \rightarrow \gamma_{\text{cusp}}(\alpha_s) \beta + \ldots, \tag{3.30}$$

where the remainder vanishes for $\beta \to \infty$. It follows that the heavy particle soft anomalous dimension $\gamma_{H,s}^R$ can be obtained as one-half of the constant coefficient in the anomalous dimension of the heavy-heavy current (3.29) in the limit where the cusp angle goes to infinity. Since the anomalous dimension of the heavy-heavy formfactor in HQET is related to the expectation value of a Wilson line [30] the colour structure is constrained by the non-abelian exponentiation theorem [39, 40]. For a heavy particle in the representation $R$, at the two-loop level only the colour structures $C_RC_A$ and $C_RT_Fn_f$ appear. We therefore obtain the one- and two-loop anomalous dimensions for a heavy particle in an arbitrary representation from the result for a heavy quark by simple Casimir scaling,

$$\gamma_{H,s}^R = C_{\alpha} \gamma_{H,s}, \tag{3.31}$$

with $\gamma_{H,s} = \gamma^Q/C_F$. Adopting the two-loop anomalous dimension of the HQET formfactor [30] in the explicit formulation in terms of polylogarithms given in [29] we obtain

$$\gamma_{H,s}^{(0)} = -2,$$

$$\gamma_{H,s}^{(1)} = -C_A \left( \frac{98}{9} - \frac{2\pi^2}{3} + 4\zeta_3 \right) + \frac{40}{9} T_F n_f, \tag{3.32}$$

where the loop expansion of the anomalous dimensions is defined as in (D.10). The one-loop expression in (3.32) agrees with the well-known result that the one-loop soft anomalous dimension is proportional to the quadratic Casimir of the final state system [3, 10, 18]. The two-loop expression for the anomalous dimension (3.28) for heavy-particle pair production in an arbitrary colour representation is a new result. The Casimir scaling of the single-particle anomalous dimension (3.31) was also noted in the published version of [25].

The factorization scale independence of the hadronic cross section in the threshold region can be used to obtain the evolution equation of the soft function as [22]

$$d d \ln \mu \hat{W}_{r_{\alpha}}^R (L) = \left( (\Gamma_{\text{cusp}}^r + \Gamma_{\text{cusp}}^{r'}) L - 2\gamma_{H,i}^R \right) \hat{W}_{r_{\alpha}}^R (L), \tag{3.33}$$

where the anomalous dimension of the soft function is obtained from the anomalous dimension of the hard function by adding anomalous dimensions $\gamma^{\phi,r}$ entering the evolution
equations of the parton distribution function for a parton in the representation $r$ in the $x \to 1$ limit (see D.16):

$$\gamma_{W,i}^{R_{a}} = \gamma_{i}^{V} + \gamma_{r}^{φ,r} + \gamma_{s}^{φ,r}.$$  \hfill (3.34)

The anomalous dimensions $\gamma_{φ,r}$ for quarks and gluons are available up to three-loop order [36]. The one- and two-loop results are collected in appendix D. We note that eq. (3.34) is derived under the assumption that the Coulomb function $J_{R_{a}}$ is scale-independent, $dJ_{R_{a}}/d\ln \mu = 0$, which is no longer true starting from NNLL in the non-relativistic expansion. However, as remarked above this type of scale dependence is related to terms that we consistently dropped in the derivation of eqs. (3.27) and (3.28), so the two-loop soft anomalous dimension of the Wilson line considered here is unaffected by these complications.

Analogously to the anomalous dimension $\gamma_{i}^{V}$ of the hard function (3.28), at least up to the two-loop level the anomalous dimension of the soft function (3.34) can be written in terms of separate single-particle contributions

$$\gamma_{W,i}^{R_{a}} = \gamma_{H,s}^{R_{a}} + \gamma_{r}^{r} + \gamma_{s}^{r}.$$ \hfill (3.35)

with

$$\gamma_{r}^{r} = \gamma_{r}^{φ} + \gamma_{φ,r}^{r}.$$ \hfill (3.36)

Up to the two-loop level, the anomalous dimensions for quarks $\gamma_{q}^{g} = \gamma_{q}^{3}$ and gluons $\gamma_{g}^{g} = \gamma_{g}^{8}$ are related by Casimir scaling,

$$\gamma_{g}^{r} = C_{r} \gamma_{s}^{r},$$ \hfill (3.37)

where the one- and two-loop coefficients are [8]:

$$\gamma_{s}^{(0)} = 0,$$

$$\gamma_{s}^{(1)} = C_{A} \left( -\frac{404}{27} + \frac{11\pi^{2}}{18} + 14\zeta_{3} \right) + T_{F} n_{f} \left( \frac{112}{27} - \frac{2\pi^{2}}{9} \right).$$ \hfill (3.38)

The evolution equations (3.26) and (3.33) generalize the corresponding equations for the Drell-Yan process [8] and Higgs production [41] to processes with a heavy particle pair in the final state. The evolution equation for the soft function in momentum space involves distributions and can be solved in Mellin-moment space [31] or directly in momentum space [6–8] using a Laplace transform. Using the two-loop result (3.32), the anomalous dimension in the evolution equation (3.33) is known with the accuracy required for NNLL resummation.

### 3.3 Relation to the formalism in Mellin-moment space

In the applications of resummation in Mellin space [9–13,18,19] the Mellin moments of the partonic cross section with respect to the variable $\rho = 4M^{2}/\hat{s}$ are written in the form

$$\hat{\sigma}_{pp',R_{a}}^{N}(M^{2},\mu) \equiv \int_{0}^{1} d\rho \rho^{N-1} \hat{\sigma}_{pp',R_{a}}^{N}(4M^{2}/\rho,\mu)$$

$$= \hat{\sigma}_{pp',R_{a}}^{(0)}(M^{2},\mu) \hat{\sigma}_{pp',R_{a}}^{0}(M^{2},\mu) \exp \left( G_{pp',R_{a}}^{N+1}(M^{2},\mu^{2}) \right).$$ \hfill (3.39)
Here $\hat{\sigma}_{pp',R_\alpha}$ is the partonic cross section for the production of a heavy-particle final state pair in the representation $R_\alpha$, $\sigma_{pp',R_\alpha}^{(0)}$ are the Mellin moments of the Born cross section, the matching functions $g_{pp,R_\alpha}^{0}$ collect the $N$-independent corrections, and the exponent $G_{pp',R_\alpha}^{N}$ has the form

$$G_{pp',R_\alpha}^{N}(M^2, \mu^2) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \left[ \int_{\mu^2}^{4M^2(1-z)^2} dq q^2 \left( A_p(\alpha_s(q^2)) + A_{p'}(\alpha_s(q^2)) \right) \right]$$

$$+ D_{pp'\rightarrow HH'}^{R_\alpha}(\alpha_s(4M^2(1-z)^2)). \quad (3.40)$$

The coefficients $A_p$ contain the effect of collinear radiation off the incoming partons and are identical to the cusp anomalous dimension $\Gamma_{cusp}^r$. The coefficient $D$ describes soft radiation and can be written as a sum of terms related to the incoming partons and the final state system

$$D_{pp'\rightarrow HH'}^{R_\alpha}(\alpha_s) = \frac{1}{2} \left( D_p(\alpha_s) + D_{p'}(\alpha_s) \right) + D_{HH'}^{R_\alpha}(\alpha_s). \quad (3.41)$$

Applications of resummation in Mellin space often use an equivalent form of (3.39) where $N$-independent terms contained in the large-$N$ expansion of (3.40) are not included in the exponent. This implies a redefinition of the matching functions $g_{pp,R_\alpha}^{0}(M^2, \mu)$ but no change in the $A$ and $D$ coefficients.

The coefficient $D_{pp'\rightarrow HH'}^{R_\alpha}$ is not identical to the anomalous dimension of the soft function $\gamma_{W}^{R_\alpha}$ because the Mellin transform of the resummed cross section in momentum space involves the fixed-order soft function at the scale $\mu_s \sim M/N$ [8, 27] while in the Mellin-space formula (3.39) all the $N$-dependent soft corrections are exponentiated. Furthermore the form of the exponent is different in the two approaches. The relation of different forms of exponentiated expressions in Mellin space is discussed in [7, 8, 37, 42, 43]. But for heavy-particle pair production near threshold, the structure of the resummed expressions in Mellin space and momentum space is identical to that for the Drell-Yan process, where the relation between the two formalisms was obtained in eq. (71) of [8]. This then implies a relation of the coefficient (3.41) to the anomalous dimension of the soft function (3.35) given by

$$e^{2\gamma_E \nabla} \Gamma(1 + 2\nabla) D_{pp'\rightarrow HH'}^{R_\alpha} = 2 \gamma_{W,i}^{R_\alpha} + 2\nabla \ln \tilde{z}_i^{R_\alpha}(0, \mu)$$

$$- e^{2\gamma_E \nabla} \Gamma(1 + 2\nabla) - 1 \frac{\nabla}{\nabla} \left( \Gamma_{cusp}^r + \Gamma_{cusp}^{r'} \right), \quad (3.42)$$

where

$$\nabla = \frac{d}{d \ln \mu^2} = \frac{\beta(\alpha_s)}{2} \frac{\partial}{\partial \alpha_s}. \quad (3.43)$$

We also introduced the Laplace-transform of the soft function [6, 7] with respect to the variable $s = 1/(e^{\gamma_E} \mu e^{\rho/2})$

$$\tilde{z}_i^{R_\alpha}(\rho, \mu) = \int_0^\infty d\omega e^{-s\omega} \tilde{W}_i^{R_\alpha}(\omega, \mu), \quad (3.44)$$
where we have defined the \( \overline{\text{MS}} \)-renormalized soft function \( \tilde{W}^{R_{\alpha}}_i \). Since the soft function in position space \( \tilde{W}(z_0, \mu) \) depends on the arguments solely through the variable \( (iz_0 \mu e^{\gamma_E}/2) = e^{L/2} \) with \( L \) defined in (3.21), it is easy to see that the function \( \tilde{s}(\rho) \) is obtained by simply replacing \( L \to -\rho \) in the \( \overline{\text{MS}} \)-renormalized result for the soft function in position space. Expanding the relation (3.42) counting \( \nabla \sim \alpha_s \) we obtain the terms relevant to determine \( D \) at the two-loop level:

\[
D^{R_{\alpha}}_{pp'\rightarrow HH'} = 2\gamma^{R_{\alpha}}_{W,i} + 2\nabla \ln \tilde{s}^{R_{\alpha}}_i(0, \mu) - \frac{\pi^2}{3} \nabla (\Gamma^{r}_{\text{cusp}} + \Gamma^{r'}_{\text{cusp}}) + \mathcal{O}(\alpha_s^3). \tag{3.45}
\]

Using the loop expansion of the soft function (3.4), the anomalous dimensions (D.10) and the beta function,

\[
\beta(\alpha_s) = \mu \frac{\partial \alpha_s}{\partial \mu} = -2\alpha_s \sum_{n=0}^{\infty} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^{n+1} \beta_n \tag{3.46}
\]

with \( \beta_0 = 11/3 C_A - 4/3 T_F n_f \), we find up to the two-loop level

\[
D^{(0)R_{\alpha}}_{pp'\rightarrow HH'} = 2\gamma^{(0)R_{\alpha}}_{W,i}, \tag{3.47}
\]

\[
D^{(1)R_{\alpha}}_{pp'\rightarrow HH'} = 2\gamma^{(1)R_{\alpha}}_{W,i} - 2\beta_0 \left( \tilde{s}^{(1)R_{\alpha}}_i(0, \mu) - \frac{\pi^2}{6} (\Gamma^{(0)r}_{\text{cusp}} + \Gamma^{(0)r'}_{\text{cusp}}) \right) \\
= 2\gamma^{(1)R_{\alpha}}_{W,i} + \beta_0 \left( \pi^2 (C_r + C_{r'}) - 8C_{R_{\alpha}} \right), \tag{3.48}
\]

where we have used \( \Gamma^{(0)r}_{\text{cusp}} = 4C_r \) and the one-loop soft function (3.24) in the last step (with \( L = 0 \) and the \( 1/\epsilon \) poles discarded due to the \( \overline{\text{MS}} \) subtraction).

Using the decomposition (3.35) of the anomalous dimension of the soft function and applying (3.48) to Drell-Yan and Higgs production, in which case \( r = r' \), \( C_{R_{\alpha}} = 0 \) (colour singlet final state) and \( \gamma^{R_{\alpha}}_{W,i} = 2\gamma^r_s \) due to (3.32), we obtain the coefficients \( D_q \) and \( D_g \) introduced in (3.41):

\[
D_p = (4\gamma^r_s + 2\pi^2 C_r \beta_0) \left( \frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3). \tag{3.49}
\]

The result for \( D_q \) agrees with eq. (72) of [8] and the explicit expression obtained using (3.38) agrees with the one used in eq. (A.3) of [11].

Turning to the soft anomalous dimension related to the heavy particle pair, we find that the relation of the coefficients appearing in the Mellin-space approach to the soft anomalous dimension obtained in (3.32) is given by

\[
D^{(0)R_{\alpha}}_{HH'} = 2\gamma^{(0)R_{\alpha}}_{H,s}, \tag{3.50}
\]

\[
D^{(1)R_{\alpha}}_{HH'} = 2\gamma^{(1)R_{\alpha}}_{H,s} - 8\beta_0 C_{R_{\alpha}}. \tag{3.51}
\]

The one-loop coefficient \( D^{(0)R_{\alpha}}_{HH'} = -4C_{R_{\alpha}} \) is in agreement with previous results for top-quark and gluino production [3, 10, 18]. The result for the two-loop coefficient is new.
and shows that at NNLL level there is a non-trivial relation between the soft anomalous dimension and the resummation coefficients if eq. (3.39) is used for the resummed cross section. Our result for the two-loop soft anomalous dimension (3.32) allows to obtain for the first time the two-loop coefficient for soft radiation off a massive particle pair at threshold for all possible colour states of the heavy-particle system:

\[ D_{HH'}^{(1)R_\alpha} = -C_{R_\alpha} C_A \left( \frac{460}{9} - \frac{4\pi^2}{3} + 8\zeta_3 \right) + \frac{176}{9} C_{R_\alpha} T_F n_f. \]  

(3.52)

Our result (3.52) differs from the result quoted in [11] for the special case of a colour-octet final state by a term \( 8C_A [C_A (1 - \zeta_3) - \beta_0] \). However, the result in [11] was obtained from an analysis of the singularities of the two-loop massive quark form factor [44,45] in the limit of light quark masses using the unjustified – and as it turns out incorrect – assumption that the factor of proportionality of the two-loop and one-loop soft anomalous dimension matrices in the massless limit is identical to the one at threshold.

### 4 Conclusions and outlook

We have performed a detailed study of the soft function relevant to threshold resummation of production processes of heavy coloured particle pairs at hadron colliders. We have given a precise formulation of the physical picture of soft-gluon radiation coupling to the total colour charge of the heavy-particle pair. This has allowed us to construct a colour basis that diagonalizes the soft function to all orders in perturbation theory. Explicit expressions for all production processes of top quarks, squarks and gluinos have been provided. We have calculated the one-loop soft function for arbitrary colour representations of initial and final state particles and used recent new insights into soft-collinear factorization to obtain the two-loop soft anomalous dimension. This supplies the process-independent ingredients for NNLL resummation of threshold logarithms in arbitrary production processes of heavy coloured particles at hadron colliders. A complete NNLL resummation in the sense of eq. (3.1) further needs the colour-separated one-loop short-distance coefficients, as well as the summation of logarithms associated with subleading terms in \( \beta \) in the non-relativistic expansion.

In a subsequent publication [22] we will give a derivation of the factorization formula (1.3) that demonstrates the factorization of Coulomb gluon exchange from soft-gluon radiation using field redefinitions in an effective field theory. The formula thus provides a theoretically clearly defined separation of hard, Coulomb and soft effects and can be used for a combined resummation of soft and Coulomb gluons. This will be discussed for the case of squark-antisquark production, where the effect of Coulomb resummation may be of similar order as the effect of soft gluon radiation [19].

### Acknowledgements

We thank Thomas Becher, Lance Dixon, Sven Moch and Matthias Neubert for useful discussions. M.B. thanks the CERN theory group for its hospitality, while part of this
work was done. The work of M.B. is supported in part by the DFG Sonderforschungs- 
bereich/Transregio 9 “Computergestützte Theoretische Teilchenphysik”.

After completion of this work we learnt of an independent calculation [49] of the two-loop 
soft anomalous dimension (3.52) for the case of heavy-quark production ($C_{R \alpha} = C_A$) by 
a different method, which is in agreement with our result. We thank Michal Czakon for 
comparing results prior to publication.

Note added

During the review process of this paper refs. [50] appeared, in which the authors reported 
a non-vanishing result for the three-particle contributions to the two-loop soft anomalous 
dimension of amplitudes with two massive particles and any number of massless particles 
near threshold, which is also not diagonal in the colour basis discussed in the present paper. 
We emphasize that these findings are not in contradiction with the results reported here. 
The authors of refs. [50] calculate the $1/\epsilon$ poles away from threshold and then take the 
limit $\beta \to 0$, resulting in logarithms of $\beta$ in the anomalous dimension and non-zero three- 
particle correlations. In the present approach, where the expansion in $\beta$ is constructed 
directly within the non-relativistic effective field theory framework, the extra logarithms 
are related to the potential region and the three-particle correlations to higher-dimensional 
soft functions, both belonging to the possible NNLL terms mentioned in the text that arise 
from subleading heavy-quark potentials, and from $O(\beta)$ terms interfering with Coulomb 
singularities. Thus, the derivation of the diagonal colour basis and two-loop anomalous 
dimension for the soft function with two equal heavy particle velocities discussed in the 
present paper remains valid in the light of the results of refs. [50], but should not be 
expected to apply to higher-dimensional soft functions. (In fact, the emission of a soft 
gluon due to the subleading $\vec{x} \cdot \vec{E}$ interaction mentioned in section 3 leads to a change 
in the colour state of the heavy-particle pair, and the corresponding soft function must 
therefore be off-diagonal.) Finally, let us mention that while the results of refs. [50] may 
be used to determine all logarithmic terms at $O(\alpha_s^3)$, the summation of NNLL logarithms 
in higher orders in the strong coupling expansion requires an analysis of soft and potential 
divergences directly at threshold.

A Technical details on the colour structure

A.1 Construction of the colour basis

To see that the basis can always be chosen as in (2.17), we use the completeness relation (2.3) of the Clebsch-Gordan coefficients to decompose the scattering amplitude (2.13) 
according to

$$\mathcal{A}_{\{a\}} = \sum_{r, R} \mathcal{A}_{\alpha \beta} C_{\alpha 1 \alpha 2}^r C_{\beta \alpha 3 \alpha 4}^R$$

(A.1)
with coefficients

\[ \mathcal{A}_{\alpha\beta} = C^{r_a*}_{a_1 b_1} C^{R_{\beta}}_{\beta a_2 b_2} \mathcal{A}_{\{a\}}. \]  

(A.2)

The fact that the amplitude \( \mathcal{A}_{\{a\}} \) is colour conserving, i.e. satisfies an identity analogous to (2.15), and the invariance condition of the Clebsch-Gordan coefficients imply that the coefficients (A.2) are invariant tensors under transformations in \( r_\alpha \otimes \overline{R}_\beta \):

\[ \mathcal{A}_{\alpha\beta} = U^{(R_\beta)}_{\beta\gamma} U^{(r_\alpha)}_{\gamma\delta} \mathcal{A}_{\delta\gamma}. \]  

(A.3)

Hence according to Schur’s Lemma \( \mathcal{A}_{\alpha\beta} \) is non-vanishing only if \( r_\alpha \) and \( R_\beta \) are equivalent irreducible representations, i.e. belong to one of the pairs \( P_i = (r_\alpha, R_\beta) \) of equivalent representations appearing in the decompositions of the initial and final state system into irreducible representations introduced above eq. (2.16). In this case \( \mathcal{A}_{\alpha\beta} = \tilde{\mathcal{A}}^{(i)} \delta_{\alpha\beta} \) for some coefficient \( \tilde{\mathcal{A}}^{(i)} \). We have therefore derived the decomposition

\[ \mathcal{A}_{\{a\}} = \sum_{P_i} \tilde{\mathcal{A}}^{(i)} C^{r_a*}_{a_1 a_2} C^{R_{\alpha}}_{\alpha a_3 a_4} \mathcal{A}_{\{a\}} \equiv \frac{1}{\sqrt{\text{dim}(r_\alpha)}} \tilde{\mathcal{A}}^{(i)}. \]  

(A.4)

(A.5)

The identity (A.4) is therefore precisely the decomposition of the amplitude into the basis (2.17) for the coefficients.

### A.2 Identities for Wilson lines

In order to rewrite the soft function in terms of the Wilson line for a single heavy particle in a representation \( R_\alpha \) we used the identity (2.19) and its complex conjugate

\[ S^{(R')}_{v,b_2 a_2} S^{(R)*}_{v,b_1 a_1} C^{R_{\alpha*}}_{\alpha a_1 a_2} = C^{R_{\alpha*}}_{\alpha a_1 a_2} S^{(R_\alpha)}_{v,b_2 b_2}. \]  

(A.6)

To prove identity (2.19) we observe that the Wilson line (1.5) solves the differential equation

\[ (v \cdot D) S^{(R)}_v(x_0) = (v \cdot \partial - ig_s v \cdot A^a T^{(R)}_a) S^{(R)}_v(x_0) = 0. \]  

(A.7)

Then using the relation

\[ C^{R_{\alpha}}_{\alpha a_1 a_2} \left( T^{(R)}_{a_1 b_1} \delta_{a_2 b_2} + \delta_{a_1 b_1} T^{(R')}_{a_2 b_2} \right) = T^{(R_\alpha)}_{\alpha \beta} C^{R_{\alpha}}_{\beta b_1 b_2}, \]  

(A.8)

obtained from the invariance condition (2.7) for infinitesimal transformations, we see that the left-hand side of (2.19) also satisfies (A.7) for a Wilson line in the representation \( R_\alpha \), if the Wilson lines \( S^{(R)}_v \) and \( S^{(R')}_{v} \) satisfy the analogous definitions in their representations:

\[ (v \cdot D)_{\alpha\beta} C^{R_{\alpha}}_{\alpha a_1 a_2} S^{(R)}_{v,a_1 b_1} S^{(R')}_{v,a_2 b_2} \]

25
\[ C_{\alpha a_1a_2}^{R_\alpha} \left[ ((v \cdot D) S_v^{(R)})_{a_1b_1} S^{(R')}_{v,a_2b_2} + S_{v,a_1b_1}^{(R)} ((v \cdot D) S_v^{(R')})_{a_2b_2} \right] = 0. \quad (A.9) \]

Since the Wilson line (1.5) is the unique solution of the differential equation (A.7) with boundary condition \( \lim_{s \to \infty} S_v(sv) = 1 \), eq. (2.19) follows.

## B Clebsch-Gordan coefficients, colour bases and projectors for gluino and squark production

### B.1 Squark-antisquark (top-antitop) production

We collect here the Clebsch-Gordan coefficients, projection operators and basis tensors for quark-antiquark and gluon-gluon initiated production of a \( 3 \otimes \bar{3} \) final state. The projectors on the singlet and octet final state representations are independent of the production channel and have been given already in (2.12):

\begin{align*}
P_{\{a\}}^{(1)} & = \frac{1}{N_c} \delta_{a_1a_2} \delta_{a_3a_4}, \\
P_{\{a\}}^{(8)} & = 2 T^\alpha_{a_1a_2} T^\alpha_{a_4a_3},
\end{align*}

(B.1)

The indices take the values \( a_i \in \{1, 2, 3\} \).

#### B.1.1 Quark-antiquark fusion channel

For completeness we repeat the results for the quark-antiquark channel given already in (2.11) and (2.18). The Clebsch-Gordan coefficients for the two representations in the decomposition \( 3 \otimes \bar{3} = 1 + 8 \) are

\begin{align*}
C_{a_1a_2}^{(1)} & = \frac{1}{\sqrt{N_c}} \delta_{a_1a_2}, \\
C_{a a_1a_2}^{(8)} & = \sqrt{2} T^\alpha_{a_1a_2} T^\alpha_{a_4a_3},
\end{align*}

(B.2)

with \( \alpha \in \{1, \ldots, 8\} \). The basis elements for the colour structure of the hard production process corresponding to the two possible combinations \( P_i = \{(1, 1), (8, 8)\} \) are given by

\begin{align*}
c_{\{a\}}^{(1)} & = \frac{1}{N_c} \delta_{a_1a_2} \delta_{a_3a_4}, \\
c_{\{a\}}^{(2)} & = \frac{2}{\sqrt{D_A}} T^\beta_{a_1a_2} T^\beta_{a_4a_3},
\end{align*}

(B.3)

with \( D_A = N_c^2 - 1 \) and \( N_c = 3 \).
B.1.2 Gluon fusion channel

For the production of a $3 \otimes \bar{3}$ final state from gluon fusion there are three possible combinations of equivalent initial and final state representations:

$$P_i \in \{(1, 1), (8_S, 8), (8_A, 8)\}. \quad (B.4)$$

The Clebsch-Gordan coefficients for combining two particles in the adjoint into a singlet, a symmetric and an antisymmetric octet are

$$C_{a_1a_2}^{(1)} = \frac{1}{\sqrt{D_A}} \delta_{a_1a_2},$$
$$C_{aa_1a_2}^{(8_S)} = \frac{1}{2\sqrt{B_F}} D_{a_2a_1}^\alpha,$$
$$C_{aa_1a_2}^{(8_A)} = \frac{1}{\sqrt{N_c}} F_{a_2a_1}^\alpha, \quad (B.5)$$

where all indices run from 1 to 8. Here we have defined $F_{a_1a_2}^\alpha = i f_{a_1a_2}^\alpha$ in terms of the SU(3) structure constants, the symmetric invariant tensor $D_{a_1a_2}^\alpha = d_{a_1a_2}^\alpha$, and the coefficient $B_F = \frac{N_c^2 - 4}{4N_c} = \frac{5}{12}$ appearing in the relation $\text{Tr}[D^\alpha D^\beta] = 4B_F \delta_{\alpha\beta}$. From the definition $(2.17)$ we obtain the same colour basis that has been found to diagonalize the one-loop soft anomalous dimension matrix $[3]$

$$c_{\{a\}}^{(1)} = \frac{1}{\sqrt{N_cD_A}} \delta_{a_1a_2} \delta_{a_3a_4},$$
$$c_{\{a\}}^{(2)} = \frac{1}{\sqrt{2D_A B_F}} D_{a_2a_1}^\alpha T_{a_3a_4}^\alpha,$$
$$c_{\{a\}}^{(3)} = \frac{1}{\sqrt{2N_cD_A}} F_{a_2a_1}^\alpha T_{a_3a_4}^\alpha. \quad (B.6)$$

B.2 Squark-squark production

For quark-quark initiated processes $qq \to \tilde{q}\tilde{q}$ the initial- and final-state systems are either in the $\bar{3}$ or 6 representation of SU(3) since $3 \otimes 3 = \bar{3} + 6$. We denote the 6 by a symmetric double index $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \{1, 2, 3\}$. The Clebsch-Gordan coefficients are given by

$$C_{aa_1a_2}^{(3)} = \frac{1}{\sqrt{2}} \epsilon_{aa_1a_2},$$
$$C_{aa_1a_2}^{(6)} = \frac{1}{2} (\delta_{a_1a_1} \delta_{a_2a_2} + \delta_{a_1a_2} \delta_{a_2a_1}), \quad (B.7)$$

where the indices $a_i$ and the index $\alpha$ for the case of the $\bar{3}$ can take the values 1 to 3. For the sextet representation the normalization condition uses a symmetrized definition of the Kronecker-delta for the double indices:

$$\delta_{\alpha\beta}^{(6)} = \frac{1}{2} (\delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_2} \delta_{\alpha_2\beta_1}). \quad (B.8)$$
The projectors on the two representations needed for the decomposition of the Coulomb Green function are given by

\[
P^{(3)}_{\{a\}} = \frac{1}{2} (\delta_{a1a3}\delta_{a2a4} - \delta_{a1a4}\delta_{a2a3}),
\]
\[
P^{(6)}_{\{a\}} = \frac{1}{2} (\delta_{a1a3}\delta_{a2a4} + \delta_{a1a4}\delta_{a2a3}).
\]

(B.9)

Since the tensor product \(3 \otimes 3\) is of two identical representations and since the Clebsch-Gordan coefficients are real, the elements of the colour basis coincide with the projectors, up to normalization:

\[
c^{(1)}_{\{a\}} = \frac{1}{\sqrt{2N_c(N_c - 1)}} (\delta_{a1a3}\delta_{a2a4} - \delta_{a1a4}\delta_{a2a3}),
\]
\[
c^{(2)}_{\{a\}} = \frac{1}{\sqrt{2N_c(N_c + 1)}} (\delta_{a1a3}\delta_{a2a4} + \delta_{a1a4}\delta_{a2a3}).
\]

(B.10)

B.3 Gluino-squark production

For gluino-squark production \(qq \rightarrow \tilde{g}\tilde{g}\) the relevant representations appear in the decomposition \(3 \otimes 8 = 3 + \bar{6} + 15\). The Clebsch-Gordan coefficients are given by (for \(N_c = 3\))

\[
C^{(3)}_{a1a2} = \frac{1}{\sqrt{C_F}} T^{a2}_{a1},
\]
\[
C^{(6)}_{a1a2} = \frac{1}{2} (\epsilon_{a1b1} T^{a2}_{b2} + \epsilon_{a2b1} T^{a2}_{b1}),
\]
\[
C^{(15)}_{a1a2} = \frac{1}{\sqrt{2}} \left( \delta_{a1a1} T^{a2}_{a2a2} + \delta_{a2a2} T^{a2}_{a1a3} - \frac{1}{4} \delta_{a1a3} T^{a2}_{a2a1} - \frac{1}{4} \delta_{a2a3} T^{a2}_{a1a1} \right),
\]

where \(a_1 \in \{1, 2, 3\}\) and \(a_2 \in \{1, \ldots, 8\}\). For the \(\bar{6}\) we use the same double-index convention as in (B.7). For the 15 we have introduced a triple index \(\alpha = (\alpha_1\alpha_2\alpha_3)\) where the first two indices transform in the 3 and the last index transforms in the \(\bar{3}\) representation. The Clebsch-Gordan coefficient is symmetric under the exchange \(\alpha_1 \leftrightarrow \alpha_2\) and vanishes upon contracting \(\alpha_{1,2}\) with \(\alpha_3\). In the normalization of the coefficient \(C^{(15)}\) we use a Kronecker delta that has the same symmetries as the coefficient in both index triples:

\[
\delta^{(15)}_{\alpha\beta} = \frac{1}{2} \delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_2} \delta_{\alpha_3\beta_3} - \frac{1}{4} \delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_2} \delta_{\beta_1\beta_3} - \frac{1}{4} \delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_1} \delta_{\beta_2\beta_3} + (\alpha_1 \leftrightarrow \alpha_2).
\]

(B.12)

The projectors can be written as (for \(N_c = 3\))

\[
P^{(3)}_{\{a\}} = \frac{1}{C_F} (T^{a2}T^{a4})_{a1a3},
\]
\[
P^{(6)}_{\{a\}} = \frac{1}{2} \delta_{a1a3} T_{a2a4} - \frac{1}{2} (T^{a2}T^{a4})_{a1a3} - (T^{a4}T^{a2})_{a1a3},
\]
\[
P^{(15)}_{\{a\}} = \frac{1}{2} \delta_{a1a3} T_{a2a4} - \frac{1}{4} (T^{a2}T^{a4})_{a1a3} + (T^{a4}T^{a2})_{a1a3}.
\]

(B.13)
The colour basis that diagonalizes the one-loop soft function is related to the projectors by complex conjugation and a normalization factor $1/\sqrt{\dim(r_A)}$:

\begin{align*}
    C_{\{a\}}^{(1)} &= \frac{1}{\sqrt{3C_F}} (T^{a_1}T^{a_2})_{a_3a_4}, \\
    C_{\{a\}}^{(2)} &= \frac{1}{2\sqrt{6}} (\delta_{a_1a_3}\delta_{a_2a_4} - (T^{a_1}T^{a_2})_{a_3a_4} - 2(T^{a_2}T^{a_4})_{a_3a_4}), \\
    C_{\{a\}}^{(3)} &= \frac{1}{\sqrt{15}} \left( \frac{1}{2} \delta_{a_1a_3}\delta_{a_2a_4} - \frac{1}{4} (T^{a_1}T^{a_2})_{a_3a_4} + (T^{a_2}T^{a_4})_{a_3a_4} \right).
\end{align*}

### B.4 Gluino pair production

The projectors on the several final-state representations for gluino pairs appearing in the decomposition of $8 \otimes 8$ are the same for the quark-antiquark and gluon-induced processes and can be obtained from [46,47]:

\begin{align*}
    P_{\{a\}}^{(1)} &= \frac{1}{8} \delta_{a_1a_2}\delta_{a_3a_4}, \\
    P_{\{a\}}^{(8_S)} &= \frac{3}{5} D_{a_1a_2}^{\alpha} D_{a_4a_3}^{\alpha}, \\
    P_{\{a\}}^{(8_A)} &= \frac{1}{3} F_{a_1a_2}^{\alpha} F_{a_4a_3}^{\alpha}, \\
    P_{\{a\}}^{(10)} &= \frac{1}{4} \left( \delta_{a_1a_3}\delta_{a_2a_4} - \delta_{a_1a_4}\delta_{a_2a_4} - \frac{2}{3} F_{a_1a_2}^{\alpha} F_{a_4a_3}^{\alpha} + D_{a_3a_4}^{\alpha} F_{a_4a_2}^{\alpha} + F_{a_3a_1}^{\alpha} D_{a_4a_2}^{\alpha} \right), \\
    P_{\{a\}}^{(27)} &= \frac{1}{2} \left( \delta_{a_1a_3}\delta_{a_2a_4} + \delta_{a_1a_4}\delta_{a_2a_4} - \frac{1}{4} \delta_{a_1a_2}\delta_{a_3a_4} - \frac{6}{5} D_{a_2a_1}^{\alpha} D_{a_3a_4}^{\alpha} \right).
\end{align*}

#### B.4.1 Quark-antiquark fusion channel

For quark-antiquark induced processes, the pairs of equivalent combinations of initial and final state representations are

\begin{equation}
    P_i \in \{(1,1), \ (8,8_S), \ (8,8_A)\}.
\end{equation}

The Clebsch-Gordan coefficients that combine the initial state quarks into a singlet and an octet have been given in (B.2) while the coefficients for the final state appeared in (B.5). The resulting colour basis is related to (B.6) by exchanging initial and final states:

\begin{align*}
    C_{\{a\}}^{(1)} &= \frac{1}{\sqrt{N_cD_A}} \delta_{a_1a_2}\delta_{a_3a_4}, \\
    C_{\{a\}}^{(2)} &= \frac{1}{\sqrt{2D_A B_F}} T_{a_2a_1}^{\alpha} D_{a_3a_4}^{\alpha}, \\
    C_{\{a\}}^{(3)} &= \sqrt{\frac{2}{N_c D_A}} T_{a_2a_1}^{\alpha} F_{a_3a_4}^{\alpha}.
\end{align*}
B.4.2 Gluon fusion channel

For the production of two gluinos from gluon fusion, the allowed pairs of initial and final state representations are given by (2.16):

\[ P_i \in \{(1,1), (8_S, 8_S), (8_A, 8_S), (8_S, 8_A), (10,10), (\overline{10}, \overline{10}), (27, 27)\}. \]  

(B.18)

The Clebsch-Gordan coefficients for the singlet and octet representations are the same as in (B.5). Since according to (2.17) the colour basis elements for the production of a 10, \( \overline{10} \), and 27 are, up to normalization, the complex conjugate of the projectors given in (B.15), we do not need the lengthy Clebsch-Gordan coefficients for these representations. The basis consists of an operator corresponding to the production of a singlet:

\[ c^{(1)}_{\{a\}} = \frac{1}{D_A} \delta_{a_1a_2} \delta_{a_3a_4}, \]  

(B.19)

four operators corresponding to the different combinations of 8\(_S\) and 8\(_A\),

\[ c^{(2)}_{\{a\}} = \frac{1}{4B_F \sqrt{D_A}} D^{\alpha}_{a_2a_1} D^{\alpha}_{a_3a_4}, \]

\[ c^{(3)}_{\{a\}} = \frac{1}{2\sqrt{B_F N_c D_A}} F^{\alpha}_{a_2a_1} D^{\alpha}_{a_3a_4}, \]

\[ c^{(4)}_{\{a\}} = \frac{1}{N_c \sqrt{D_A}} F^{\alpha}_{a_2a_1} F^{\alpha}_{a_3a_4}, \]

\[ c^{(5)}_{\{a\}} = \frac{1}{2\sqrt{B_F N_c D_A}} D^{\alpha}_{a_2a_1} F^{\alpha}_{a_3a_4}, \]  

(B.20)

two operators corresponding to the 10 and \( \overline{10} \)

\[ c^{(6/7)}_{\{a\}} = \frac{1}{4\sqrt{10}} \left[ \delta_{a_1a_3} \delta_{a_2a_4} - \delta_{a_1a_4} \delta_{a_2a_3} - \frac{2}{3} F^{\alpha}_{a_2a_1} F^{\alpha}_{a_3a_4} \pm \left( D^{\alpha}_{a_3a_1} F^{\alpha}_{a_1a_2} + F^{\alpha}_{a_3a_1} D^{\alpha}_{a_1a_2} \right) \right], \]  

(B.21)

and one operator for the 27:

\[ c^{(8)}_{\{a\}} = \frac{1}{6\sqrt{3}} \left( \delta_{a_1a_3} \delta_{a_2a_4} + \delta_{a_1a_4} \delta_{a_2a_3} - \frac{1}{4} \delta_{a_1a_2} \delta_{a_3a_4} - \frac{6}{5} D^{\alpha}_{a_2a_1} D^{\alpha}_{a_3a_4} \right). \]  

(B.22)

C Fourier transform of the soft function

The Fourier transform of the soft function (1.7) enters the factorization formula (1.3) while in section 3.1 we calculated the one-loop soft function in position space. The momentum space result can be obtained by inserting the Fourier transforms of the basis integrals

\[ \mathcal{I}^{(ii)}(\omega, \mu) = -\frac{\Gamma(-\epsilon)}{8\pi^2} \frac{1}{\epsilon} e^{\gamma_E \epsilon} \frac{1}{\omega} \left( \frac{\omega}{\mu} \right)^{-2\epsilon} \theta(\omega), \]  

(C.1)
Here we have used the identity
\[
\alpha \quad \text{with a prefactor } \omega
\]
where we recall the prescription
\[
\frac{1}{(1 - 2\epsilon)} \omega \left( \frac{\omega}{\mu} \right)^{-2\epsilon} \theta(\omega)
\]
and \( \mathcal{I}^{(ii)}(\omega, \mu) = -1/2 \mathcal{I}^{(ii)}(\omega, \mu) \) into the result (3.10). For unstable heavy particles, the \( \omega \) integral in (1.3) extends to infinity and the integrals have to be expanded in \( \epsilon \) in the sense of modified plus-distributions [48]. The Fourier transform has been calculated using
\[
\int_{-\infty}^{\infty} \frac{dz_0}{4\pi} e^{i\omega z_0/2} \left( \frac{iz_0\mu}{2} \right)^{\alpha} = \frac{1}{\Gamma(-\alpha)} \omega \left( \frac{\omega}{\mu} \right)^{-\alpha} \theta(\omega), \quad (C.2)
\]
where we recall the prescription \( z_0 \rightarrow z_0 - i\delta \).

To compare the one-loop soft function (3.24) to the result obtained in [21] for the special case of the production of a colour octet scalar from gluon fusion, we compute the Fourier transform as a function of \( \omega = M(1 - z) \) and expand the basis integrals into plus-distributions, as appropriate if the decay width is neglected as in [21]:
\[
\mathcal{I}^{(ii)}(M(1 - z), \mu) = \frac{1}{8\pi^2 M} \left[ \delta(1 - z) \left( \frac{1}{\epsilon^2} + \frac{2}{\epsilon} \ln \left( \frac{\mu}{M} \right) + 2 \ln^2 \left( \frac{\mu}{M} \right) - \frac{\pi^2}{4} \right) \right.
- \left. \left[ \frac{1}{1 - z} \right] \left( \frac{2}{\epsilon} + 4 \ln \left( \frac{\mu}{M} \right) \right) + 4 \left[ \frac{\ln(1 - z)}{1 - z} \right]_+ \right] \theta(1 - z), \quad (C.3)
\]
\[
\mathcal{I}^{(ii)}(M(1 - z), \mu) = -\frac{1}{8\pi^2 M} \left[ \delta(1 - z) \left( \frac{1}{\epsilon} + 2 \ln \left( \frac{\mu}{M} \right) + 2 \right) - 2 \left[ \frac{1}{1 - z} \right]_+ \right] \theta(1 - z).
\]

Here we have used the identity
\[
(1 - z)^{-1 - 2\epsilon} = -\frac{1}{2\epsilon} \delta(1 - z) + \left[ \frac{1}{1 - z} \right]_+ - 2\epsilon \left[ \frac{\ln(1 - z)}{1 - z} \right]_+ \quad (C.4)
\]
for distributions on the interval \([0, 1]\).

The soft function in momentum space obtained from (3.24) is therefore given by
\[
W_i^{(1)R_o}(M(1 - z)) = \frac{2}{M} \left\{ (C_r + C_{r'}) \left[ \delta(1 - z) \left( \frac{1}{\epsilon^2} + \frac{2}{\epsilon} \ln \left( \frac{\mu}{M} \right) + 2 \ln^2 \left( \frac{\mu}{M} \right) - \frac{\pi^2}{4} \right) \right.ight.
- \left[ \frac{1}{1 - z} \right]_+ \left( \frac{2}{\epsilon} + 4 \ln \left( \frac{\mu}{M} \right) \right) + 4 \left[ \frac{\ln(1 - z)}{1 - z} \right]_+ \right]
+ \left. C_{R_o} \left[ \delta(1 - z) \left( \frac{1}{\epsilon} + 2 \ln \left( \frac{\mu}{M} \right) + 2 \right) - 2 \left[ \frac{1}{1 - z} \right]_+ \right] \right\}. \quad (C.5)
\]

This reproduces eq. (40) in [21] by setting \( C_r = C_{r'} = C_{R_o} = C_A = N_c \) and multiplying with a prefactor \( \alpha_s M/(4\pi) \) to account for our definition of \( W_i^{(1)R_o} \) as coefficient of \( \alpha_s/(4\pi) \) and the different normalization of the leading-order soft function: \( W_i^{(0)R_o}(M(1 - z)) = \delta(M(1 - z)) = 1/M \delta(1 - z) \), whereas \( \overline{S}_{S/P}^{(0)}(M(1 - z)) = \delta(1 - z) \) in [21].
D Anomalous dimensions

In [D.1] we derive the anomalous dimension (3.27) of the hard function. The explicit one- and two-loop results for the anomalous dimensions are collected in [D.2].

D.1 Anomalous-dimension matrix of the hard function

From [25] we find that the UV regularized, minimally subtracted short-distance coefficients for a general $2 \to n$ scattering process including massless and massive partons obey an evolution equation

$$\frac{d}{d\ln \mu} |C(\{k\},\{m\},\mu)| = \Gamma(\{k\},\{m\},\mu) |C(\{k\},\{m\},\mu)|,$$

where $\{k\} = \{k_1,\ldots,k_n\}$ and $\{m\} = \{m_1,\ldots,m_n\}$ denote momenta and masses of the scattered particles and we use the colour-state formalism [32]. For general scattering processes the colour structure of the matrix $\Gamma$ can be quite complicated with two- and three-parton colour correlations contributing at two-loop level.

The two-parton correlations take the form (c.f. eq.(10) of [25])

$$\Gamma(\{k\},\{m\},\mu)_{\text{2-parton}} = \sum_{(i,j)} \frac{T_i \cdot T_j}{2} \gamma_{\text{cusp}} \ln \left( \frac{\mu^2}{-s_{ij}} \right) + \sum_i \gamma_{r_i} + \sum_I \gamma_{R_i}^{H,s} H,s - \sum_{(I,J)} \frac{T_I \cdot T_J}{2} \gamma_{\text{cusp}}(\beta_{IJ}) + \sum_{I,j} \frac{T_I \cdot T_J}{2} \gamma_{\text{cusp}} \ln \left( \frac{m_{IJ}}{-s_{IJ}} \right).$$

Here indices $i$ ($I$) denote massless (massive) partons and the notation $(i,j)$ indicates unordered tuples of distinct parton indices. We also defined $s_{ij} = 2\sigma_{ij} p_i \cdot p_j + i0$ with $\sigma_{ij} = +1$ if partons $i$ and $j$ are both incoming or outgoing and $\sigma_{ij} = -1$ otherwise.

The anomalous dimensions related to light partons are collected in appendix [D.2]. The heavy-particle soft anomalous dimension $\gamma_{H,s}^{R_I}$ for arbitrary SU(3) representations is given in [332]. The cusp anomalous dimension $\gamma_{\text{cusp}}(\beta_{IJ})$ in the terms involving two massive partons is a function of the cusp angle $\beta_{IJ} = \arccosh(-s_{IJ}/2m_I m_J)$ and can be obtained from the heavy-heavy quark formfactor in HQET [329]. For large cusp angles the cusp anomalous dimension satisfies (3.30) while for small cusp angle as in pair production at threshold [25]

$$\gamma_{\text{cusp}}(\beta) \xrightarrow{\beta \to 0} -2\gamma^0/C_F \equiv \gamma_{H,s}.$$

For processes with two heavy final state particles three-parton correlations involve the colour structures

$$f^{abc} T_i^a T_j^b T_K^c,$$

while a two-loop analysis [24] and soft-collinear factorization [25] show that three-parton correlations with two light partons and one heavy parton proportional to $f^{abc} T_i^a T_j^b T_K^c$ are absent.

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In the following we evaluate (D.2) for a $2 \rightarrow 1$ process with a massive final state particle in an arbitrary SU(3) representation, corresponding to the reduction of the leading soft function obtained in section 2. As discussed in section 3 this corresponds to dropping contributions to the anomalous dimension related to higher-order potentials and higher-dimensional soft functions. For the three-parton process with a single heavy particle the structure (D.4) cannot appear. Furthermore, for a two-to-one process the structure $f^{abc} T_i T_j T_K^c$ vanishes by colour conservation alone. We therefore conclude that in order to obtain the two-loop anomalous dimension of the hard function in the factorization formula (1.3) corresponding to the (leading) soft function $W_{ii'}^{R}$ in the same equation, it is sufficient to consider the two-parton correlations (D.2), since all higher multi-particle correlations do not contribute.

The result (3.27) for the anomalous dimension of the hard coefficient in heavy particle pair production at threshold can be obtained from the general expression (D.2) using the reduction to a two-to-one process with a single particle in the representation $R_{a}$ with mass $2M = (m_{H} + m_{H'})$ and momentum $P = p_{1} + p_{2}$, derived for the soft function in section 2. (This reduction can be performed analogously for the soft function $\langle 0|S_{n}S_{n'}S_{i}^{s}|0 \rangle$ appearing in the factorization of the amplitude to which eq. (D.2) applies.) At threshold $k_{1} \cdot k_{2} = P \cdot k_{1} = 2M^{2}$ and the two-loop anomalous-dimension matrix becomes

$$
\Gamma \{ k, P \}, M, \mu = T_{1} \cdot T_{2} \gamma_{cusp} \ln \left( -\frac{\mu^{2}}{2k_{1} \cdot k_{2} + i0} \right) + \gamma' + \gamma'' + \gamma_{H,s}^{R}
$$

$$
+ \sum_{i=1,2} T_{i} \cdot T_{3} \gamma_{cusp} \ln \left( \frac{2M \mu}{2(P \cdot k_{i})} \right)
$$

$$
= \gamma_{cusp} \left[ T_{1} \cdot T_{2} \ln \left( -\frac{\mu^{2}}{4M^{2} + i0} \right) - T_{3}^{2} \ln \left( \frac{\mu}{2M} \right) \right] + \gamma' + \gamma'' + \gamma_{H,s}^{R},
$$

where we have used colour conservation $T_{1} + T_{2} = -T_{3}$. Further using the identity

$$
2T_{1} \cdot T_{2} = T_{3}^{2} - T_{1}^{2} - T_{2}^{2}
$$

leads to the result for the anomalous dimension quoted in (3.27):

$$
\Gamma \{ k, P \}, M, \mu = \frac{1}{2} \gamma_{cusp} \left[ (C_{r} + C_{r'}) \left( \ln \left( \frac{4M^{2}}{\mu^{2}} \right) - i\pi \right) + i\pi C_{R_{a}} \right] + \gamma' + \gamma'' + \gamma_{H,s}^{R}. (D.7)
$$

It is interesting to see how the same result is obtained by applying the general expression (D.2) directly to the four-particle process at threshold, omitting three parton correlations as discussed above. Using the fact that the heavy-particle cusp anomalous dimension near threshold simplifies according to (D.3) and $p_{J} \cdot k_{i} = m_{J} \sqrt{s}/2 \approx m_{J}M$ one finds

$$
\Gamma \{ k, p \}, \{ m \}, \mu \}_{2\text{-parton}} = T_{1} \cdot T_{2} \gamma_{cusp} \ln \left( -\frac{\mu^{2}}{2k_{1} \cdot k_{2} + i0} \right) + \gamma' + \gamma'' + \gamma_{H,s}^{R} + \gamma_{H,s}^{R'}
$$

$$
+ 2T_{3} \cdot T_{4} \gamma_{H,s} + \sum_{i,j=1,2} T_{i} \cdot T_{j+2} \gamma_{cusp} \ln \left( \frac{m_{J} \mu}{2p_{J} \cdot k_{i}} \right)
$$

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Using colour conservation and the analog of (D.6) for a four-particle process, this expression simplifies to

\[
\Gamma(k, p; m; \mu)_{2\text{-parton}} = \frac{1}{2} \gamma_{\text{cusp}} \left[ (T_1^2 + T_2^2) \left( \ln \left( \frac{4M^2}{\mu^2} \right) - i\pi \right) + i\pi (T_3 + T_4)^2 \right] + \gamma^r + \gamma^{\prime r} + (T_3 + T_4)^2 - T_3^2 - T_4^2 \gamma_{H,s} + \gamma_R^{H,s} + \gamma_{H,s}^{R'},
\]

(D.9)

Since for a final state pair in a representation \( R_\alpha \), \((T_3 + T_4)^2 = C_{R_\alpha} \), we obtain the same result as in the three-point calculation, provided the heavy-particle soft anomalous dimension satisfies Casimir scaling, \( \gamma_{H,s}^{R_\alpha} = C_{R_\alpha} \gamma_{s}^{H} \). This gives a second argument for Casimir scaling of the soft anomalous dimension, in addition to the derivation from the HQET formfactor given in the main text.

D.2 Explicit results for the anomalous dimensions

In this appendix we collect explicit results for the one- and two-loop anomalous dimensions that are already available in the literature. We define the expansion of the various anomalous dimensions in the strong coupling constant by

\[
\gamma = \sum_n \gamma^{(n)} \left( \frac{\alpha}{4\pi} \right)^{n+1}.
\]

(D.10)

The explicit one- and two-loop results for the cusp anomalous dimension are given by

\[
\gamma^{(0)}_{\text{cusp}} = 4,
\]

\[
\gamma^{(1)}_{\text{cusp}} = 4 \left[ \left( \frac{67}{9} - \frac{\pi^2}{3} \right) C_A - \frac{20}{9} T_F n_f \right]
\]

(D.11)

with \( T_F \delta_{ab} = \text{Tr}[T^a T^b] \) and \( T_F = 1/2 \).

The one- and two-loop anomalous-dimension coefficients \( \gamma^r \) of massless quarks, \( \gamma^q = \gamma^3 \), and gluons, \( \gamma^g = \gamma^8 \), are given by [35]

\[
\gamma^{(0)q} = -3C_F,
\]

\[
\gamma^{(1)q} = C_F^2 \left( -\frac{3}{2} + 2\pi^2 - 24\zeta_3 \right) + C_A C_F \left( -\frac{961}{54} - \frac{11\pi^2}{6} + 26\zeta_3 \right) + C_F T_F n_f \left( \frac{130}{27} + \frac{2\pi^2}{3} \right),
\]

\[
\gamma^{(0)g} = -\beta_0 = -\frac{11}{3} C_A + \frac{4}{3} T_F n_f,
\]

(D.12) (D.13) (D.14)
\[ \gamma^{(1)g} = C_A^2 \left( -\frac{692}{27} + \frac{11\pi^2}{18} + 2\zeta_3 \right) + C_A T_F n_f \left( \frac{256}{27} - \frac{2\pi^2}{9} \right) + 4C_F T_F n_f. \quad (D.15) \]

The anomalous dimensions \( \gamma^{\phi,r} \) appearing in the anomalous dimension of the soft function (3.34) are defined by the evolution equation of the parton distribution function for a parton \( p \) in the representation \( r \) in the \( x \to 1 \) limit,

\[ \frac{d}{d\ln \mu} f_{p/N}(x, \mu) = 2\gamma^{\phi,r}(\alpha_s)f_{p/N}(x, \mu) + 2 \Gamma^r_{\text{cusp}}(\alpha_s) \int_1^x \frac{dz}{z} f_{p/N}(x/z, \mu) [1 - z]_+ + \ldots. \quad (D.16) \]

Results for quarks, \( \gamma^{\phi,3} \equiv \gamma^{\phi} \), and gluons, \( \gamma^{\phi,8} \equiv \gamma^B \), are available up to the three-loop order [36]. The explicit values in the notation used here are given by [8, 41]:

\[ \gamma^{(0)}_{\phi} = 3C_F, \quad (D.17) \]
\[ \gamma^{(1)}_{\phi} = C_F^2 \left( \frac{3}{2} - 2\pi^2 + 24\zeta_3 \right) + C_A C_F \left( \frac{17}{6} + \frac{22\pi^2}{9} - 12\zeta_3 \right) - C_F T_F n_f \left( \frac{2}{3} + \frac{8\pi^2}{9} \right), \quad (D.18) \]
\[ \gamma^{(0)}_{B} = \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f; \quad (D.19) \]
\[ \gamma^{(1)}_{B} = 4C_A^2 \left( \frac{8}{3} + 3\zeta_3 \right) - \frac{16}{3} C_A T_F n_f - 4C_F T_F n_f. \quad (D.20) \]

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