The Minc-type bound and the eigenvalue inclusion sets of the general product of tensors

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Abstract

In this paper, we give the Minc-type bound for spectral radius of nonnegative tensors. We also present the bounds for the spectral radius and the eigenvalue inclusion sets of the general product of tensors.

Keywords: General product of tensors, Spectral radius, Minc-type bound, Eigenvalue inclusion set

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1. Introduction

Let $\mathbb{C}^{[m,n]} \ (\mathbb{R}_{+}^{[m,n]})$ be the set of order $m$ dimension $n$ tensors (nonnegative tensors) over complex number field $\mathbb{C}$ (real number field $\mathbb{R}$). For $\mathbf{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]}$, denote

$$r_i(\mathbf{A}) = \sum_{i_2, \ldots , i_m = 1}^{n} |a_{i_2i_3\ldots i_m}|, \ r(\mathbf{A}) = \min_{i \in [n]} r_i(\mathbf{A}), \ R(\mathbf{A}) = \max_{i \in [n]} r_i(\mathbf{A}),$$

where $[n] = \{1, 2, \ldots , n\}$.

For $\mathbf{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $x = (x_1, \ldots , x_n)^T \in \mathbb{C}^n$, $\mathbf{A}x^{m-1}$ is a column vector of dimension $n$, whose the $i$th component is

$$(\mathbf{A}x^{m-1})_i = \sum_{i_2, \ldots , i_m = 1}^{n} a_{i_1i_2\ldots i_m} x_{i_2} \cdots x_{i_m}, \ i \in [n].$$

In 2005, Qi \cite{Qi} and Lim \cite{Lim} proposed the concept of eigenvalue of tensors, inde-
pendently. For \( \mathcal{A} \in \mathbb{C}^{[m,n]} \), if there exists a number \( \lambda \in \mathbb{C} \) and a nonzero vector \( x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n \) such that
\[
\mathcal{A}x^{m-1} = \lambda x^{m-1},
\]
then \( \lambda \) is called the eigenvalue of \( \mathcal{A} \) and \( x \) is called the eigenvector of \( \mathcal{A} \) corresponding to \( \lambda \), where \( x^{m-1} = (x_1^{m-1}, \ldots, x_n^{m-1})^T \). Let \( \sigma(\mathcal{A}) \) denote the set of eigenvalues of \( \mathcal{A} \). The spectral radius of \( \mathcal{A} \) is defined as
\[
\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.
\]

In 2013, Shao [3] introduced a general product of tensors as follows: Let \( \mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]} \) (\( m \geq 2 \)) and \( \mathcal{B} = (b_{i_1i_2\ldots i_k}) \in \mathbb{C}^{[k,n]} \) (\( k \geq 1 \)), then \( \mathcal{A}\mathcal{B} = (c_{i_1\alpha_1\ldots \alpha_{m-1}}) \) is an order \((m-1)(k-1)+1\) dimension \( n \) tensor with entries:
\[
c_{i_1\alpha_1\ldots \alpha_{m-1}} = \sum_{i_2\ldots i_m=1}^{n} a_{i_1i_2\ldots i_m} b_{i_2\alpha_1} \cdots b_{i_m\alpha_{m-1}}, \quad i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1},
\]
where \([n]^{k-1} = [n] \times \cdots \times [n]^{k-1}\). Clearly, when \( \mathcal{B} = x \in \mathbb{C}^{[1,n]} \), we have \( \mathcal{A}x = \mathcal{A}x^{m-1} \).

In this paper, we use \( \mathcal{A}x \) instead of \( \mathcal{A}x^{m-1} \).

A tensor \( \mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}_+^{[m,n]} \) is called weakly irreducible [7], if for any nonempty proper subset \( I \subset [n] \), there exist \( i_1, i_2, \ldots, i_m \) satisfied
\[
a_{i_1i_2\ldots i_m} > 0, \quad \text{where} \ i_1 \in I, \ i_j \in [n] \setminus I, \ j \in \{2, \ldots, m\}.
\]

In 1988, Minc [10] gave the bounds for spectral radius of nonnegative matrices as follows: Let \( A = (a_{ij}) \in \mathbb{R}_+^{[2,n]} \), and \( r_i(A) \neq 0, \ i = 1, \ldots, n \). Then
\[
\min_{i \in [n]} \left( \frac{1}{r_i(A)} \sum_{j=1}^{n} a_{ij} r_j(A) \right) \leq \rho(A) \leq \max_{i \in [n]} \left( \frac{1}{r_i(A)} \sum_{j=1}^{n} a_{ij} r_j(A) \right), \quad (1)
\]
and since \( r_i(A^2) = \sum_{j=1}^{n} a_{ij} r_j(A) \), the inequality (1) could be written by
\[
\min_{i \in [n]} \frac{r_i(A^2)}{r_i(A)} \leq \rho(A) \leq \max_{i \in [n]} \frac{r_i(A^2)}{r_i(A)}.
\]

In [4], Chang et al. gave the Perron-Frobenius theorem for nonnegative irre-
ducible tensor $\mathcal{A}$, and the Collatz-Wielandt Theorem of $\mathcal{A}$ is given as follows:

$$\max_{x \in \mathbb{R}^n_+} \min_{i \in [n]} \frac{(Ax)_i}{x_i^{m-1}} = \rho(\mathcal{A}) = \min_{x \in \mathbb{R}^n_+} \max_{i \in [n]} \frac{(Ax)_i}{x_i^{m-1}}. \quad (2)$$

In [5], Friedland et al. also showed the Collatz-Wielandt Theorem for nonnegative weakly irreducible tensors. There are still some other results on spectral radius of tensors, see [4, 9, 11].

In [1], Qi gave the Geršgorin-type eigenvalue inclusion sets for real symmetric tensors. In [12], Bu et al. gave the Brualdi-type eigenvalue inclusion sets of tensors via digraph.

Inspired by the above, in this paper, we give the Minc-type bound for spectral radius of nonnegative tensors. Further, the generalized Collatz-Wielandt Theorem for nonnegative weakly irreducible tensors is showed. We also present the bound for the spectral radius and the eigenvalue inclusion sets of the general product of tensors.

2. Preliminaries

In this section, we give some lemmas.

Lemma 2.1. [3] For $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]}$ and an invertible diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$, $\mathcal{B} = D^{-(m-1)} \mathcal{A} D$ is an order $m$ dimension $n$ tensor with entries

$$b_{i_1i_2\ldots i_m} = d_{i_1}^{-(m-1)} a_{i_1i_2\ldots i_m} d_{i_2} \cdots d_{i_m}.$$

In this case, $\mathcal{A}$ and $\mathcal{B}$ are called diagonal similar and $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.

Lemma 2.2. [3] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_+$ be a weakly irreducible tensor. Then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, and there exists a unique positive eigenvector corresponding to $\rho(\mathcal{A})$ up to a multiplicative constant.

Lemma 2.3. [4] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_+$. Then

$$r(\mathcal{A}) \leq \rho(\mathcal{A}) \leq R(\mathcal{A}).$$

Lemma 2.4. Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $\mathcal{B} = (b_{i_1i_2\ldots i_k}) \in \mathbb{C}^{[k,n]}$, $m \geq 2$, $k \geq 1$. Then

$$r_i(AB) \leq r_i(A)(R(B))^{m-1}, i \in [n].$$
(2) If $A$ and $B$ are nonnegative tensors,

$$r_{i}(AB) = \sum_{i_{2},...,i_{m}=1}^{n} a_{i_{2}...i_{m}} r_{i_{2}}(B) \cdots r_{i_{m}}(B), i \in [n].$$

**Proof.** By general tensor product, we know $AB = (c_{i_{1}\alpha_1...\alpha_{m-1}})$ is an order $(m - 1)(n - 1) + 1$ dimension $n$ tensor, where

$$c_{i_{1}\alpha_1...\alpha_{m-1}} = \sum_{i_{2},...,i_{m}=1}^{n} a_{i_{2}...i_{m}} b_{i_{2}\alpha_{1}} \cdots b_{i_{m}\alpha_{m-1}}, \alpha_j \in [n]^{k-1}, j \in [m - 1].$$

Then, for all $i \in [n]$, we have

$$r_{i}(AB) = \sum_{\alpha_1,...,\alpha_{m-1} \in [n]^{k-1}} |c_{i_{1}\alpha_1...\alpha_{m-1}}|$$

$$= \sum_{\alpha_1,...,\alpha_{m-1} \in [n]^{k-1}} \left| \sum_{i_{2},...,i_{m}=1}^{n} a_{i_{2}...i_{m}} b_{i_{2}\alpha_{1}} \cdots b_{i_{m}\alpha_{m-1}} \right|$$

$$\leq \sum_{i_{2},...,i_{m}=1}^{n} \left( \sum_{\alpha_1,...,\alpha_{m-1} \in [n]^{k-1}} |a_{i_{2}...i_{m}}| |b_{i_{2}\alpha_{1}}| \cdots |b_{i_{m}\alpha_{m-1}}| \right)$$

$$= \left( \sum_{i_{2},...,i_{m}=1}^{n} |a_{i_{2}...i_{m}}| \left( \sum_{\alpha_1,...,\alpha_{m-1} \in [n]^{k-1}} |b_{i_{2}\alpha_{1}}| \cdots |b_{i_{m}\alpha_{m-1}}| \right) \right).$$

Since

$$\sum_{\alpha_1,...,\alpha_{m-1} \in [n]^{k-1}} |b_{i_{2}\alpha_{1}}| \cdots |b_{i_{m}\alpha_{m-1}}| = \left( \sum_{\alpha_{1} \in [n]^{k-1}} |b_{i_{2}\alpha_{1}}| \right) \left( \sum_{\alpha_{2} \in [n]^{k-1}} |b_{i_{3}\alpha_{2}}| \right)$$

$$\cdots \left( \sum_{\alpha_{k-1} \in [n]^{k-1}} |b_{i_{m}\alpha_{m-1}}| \right)$$

$$= r_{i_{2}}(B) r_{i_{3}}(B) \cdots r_{i_{m}}(B)$$

$$\leq (R(B))^{m-1},$$
we have

\[ r_i(AB) \leq \sum_{i_2,\ldots,i_m=1}^{n} |a_{i_2\ldots i_m}| r_{i_2}(B) r_{i_3}(B) \cdots r_{i_m}(B) \]

\[ \leq (R(B))^{m-1} \sum_{i_2,\ldots,i_m=1}^{n} |a_{i_2\ldots i_m}| \]

\[ = r_i(A)(R(B))^{m-1}. \]

So (1) of this lemma holds.

When \( \mathcal{A} \) and \( \mathcal{B} \) are nonnegative tensors, clearly, we obtain

\[ r_i(AB) = \sum_{i_2,\ldots,i_m=1}^{n} a_{i_2\ldots i_m} r_{i_2}(B) \cdots r_{i_m}(B), \]

Thus we prove (2).

Corollary 2.5. Let \( \mathcal{A} \in \mathbb{C}^{[m,n]} \). Then

\[ R(\mathcal{A}^k) \leq (R(\mathcal{A}))^{\mu_k}. \]

where \( \mu_k = \begin{cases} \frac{(m-1)^k-1}{m-2}, & m > 2, \\ k, & m = 2. \end{cases} \)

3. The Minc-type bound for spectral radius of nonnegative tensors

In this section, we give the bounds for spectral radius of nonnegative tensors and the generalized Collatz-Wielandt Theorem of nonnegative weakly irreducible tensors.

Theorem 3.1. Let \( \mathcal{A} \in \mathbb{R}_+^{[m,n]}, \mathcal{B} \in \mathbb{R}_+^{[k,n]} \), and \( r_i(\mathcal{B}) \neq 0, i \in [n] \). Then

\[ \min_{i \in [n]} \frac{r_i(\mathcal{A}B)}{(r_i(\mathcal{B}))^{m-1}} \leq \rho(\mathcal{A}) \leq \max_{i \in [n]} \frac{r_i(\mathcal{A}B)}{(r_i(\mathcal{B}))^{m-1}}. \]

Proof. Let \( D = \text{diag}(r_1(\mathcal{B}), r_2(\mathcal{B}), \ldots, r_n(\mathcal{B})) \) be an invertible diagonal matrix. By Lemma 2.1 \( D^{-(m-1)}AD \) and \( \mathcal{A} \) are diagonal similar, then \( \rho(\mathcal{A}) = \rho(D^{-(m-1)}AD) \).

From Lemma 2.3, we get

\[ r(D^{-(m-1)}AD) \leq \rho(\mathcal{A}) = \rho(D^{-(m-1)}AD) \leq R(D^{-(m-1)}AD), \]
Using the general product of tensors, we have

\[
(D^{-(m-1)}AD)_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m} (r_{i_1} (B))^{-(m-1)} r_{i_2} (B) \cdots r_{i_m} (B),
\]

then

\[
r_i (D^{-(m-1)}AD) = \frac{\sum_{i_2, \ldots, i_m = 1}^{n} a_{i_2 \ldots i_m} r_{i_2} (B) \cdots r_{i_m} (B)}{(r_i (B))^{m-1}}.
\]

Thus

\[
\min_{i \in [n]} \frac{\sum_{i_2, \ldots, i_m = 1}^{n} a_{i_2 \ldots i_m} \prod_{j=2}^{m} r_{i_j} (B)}{(r_i (B))^{m-1}} \leq \rho (A) \leq \max_{i \in [n]} \frac{\sum_{i_2, \ldots, i_m = 1}^{n} a_{i_2 \ldots i_m} \prod_{j=2}^{m} r_{i_j} (B)}{(r_i (B))^{m-1}}. \quad (3)
\]

By Lemma 2.4, it yields

\[
r_i (AB) = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i_2 \ldots i_m} r_{i_2} (B) \cdots r_{i_m} (B),
\]

combining inequality (3), we have

\[
\min_{i \in [n]} \frac{r_i (AB)}{(r_i (B))^{m-1}} \leq \rho (A) \leq \max_{i \in [n]} \frac{r_i (AB)}{(r_i (B))^{m-1}}.
\]

Remark: In Theorem 3.1, we consider two cases as follows.

Case 1. When \( k = 1 \), let \( B = x = (x_1, \ldots, x_n) \in \mathbb{R}_{++}^n \) ( \( \mathbb{R}_{++} \) is the set of positive numbers), \( r_i (Ax) \) and \( r_i (x) \) be written by \( (Ax)_i \) and \( x_i \), respectively. Thus we obtain

\[
\min_{i \in [n]} \frac{(Ax)_i}{x_i^{m-1}} \leq \rho (A) \leq \max_{i \in [n]} \frac{(Ax)_i}{x_i^{m-1}},
\]

which is given in \( \Box \).
Case 2. When $k = 2$, let $B$ be an identity matrix, then

$$r(A) \leq \rho(A) \leq R(A),$$

which is also given in [6].

If we set $B = A$ in Theorem 3.1, we obtain the following theorem, which extends the Minc-type bound for spectral radius to tensors.

**Theorem 3.2.** Let $A = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_+$. If $r_i(A) \neq 0$, for all $i \in [n]$, then

$$\min_{i \in [n]} \frac{r_i(A^2)}{(r_i(A))^{m-1}} \leq \rho(A) \leq \max_{i \in [n]} \frac{r_i(A^2)}{(r_i(A))^{m-1}}.$$

**Example 3.3.** Let $A \in \mathbb{R}^{[3,2]}_+$, where $a_{111} = 3$, $a_{112} = 1$, $a_{121} = 2$, $a_{122} = 1$, $a_{211} = 0$, $a_{212} = 4$, $a_{221} = 2$, $a_{222} = 3$. Then

$$r_1(A) = 7, \quad r_2(A) = 9, \quad r_1(A^2) = 417, \quad r_2(A^2) = 621,$$

$$\frac{r_1(A^2)}{(r_1(A))^2} = \frac{417}{49}, \quad \frac{r_2(A^2)}{(r_2(A))^2} = \frac{621}{81}.$$

From Lemma 2.3, we have

$$7 \leq \rho(A) \leq 9.$$

From Theorem 3.2, we have

$$7.6667 \approx \frac{621}{81} \leq \rho(A) \leq \frac{417}{49} \approx 8.5102.$$

This example shows that the bound in Theorem 3.2 is better than the bound in Lemma 2.3 for some tensors.

We take $B = A^k$ in Theorem 3.1, we obtain the following theorem.

**Theorem 3.4.** Let $A = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_+$. If $r_i(A) \neq 0$, for all $i \in [n]$, then

$$\min_{i \in [n]} \frac{r_i(A^{k+1})}{(r_i(A^k))^{m-1}} \leq \rho(A) \leq \max_{i \in [n]} \frac{r_i(A^{k+1})}{(r_i(A^k))^{m-1}}.$$

**Remark:** Let $A = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}_+$ and $r_i(A) \neq 0$, for all $i \in [n]$. It follows from Lemma 2.4 that $r_i(A^2) = \sum_{i_2, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} r_i(A) \cdots r_i(A)$. Since $r_i(A) \neq 0$, we have
\( r_i(A^2) \neq 0 \). Similarly, we obtain \( r_i(A^3) \neq 0, \cdots, r_i(A^k) \neq 0 \), for all \( i \in [n] \). Thus, in Theorem 3.4 \( (r_i(A^k))^{m-1} \neq 0 \), for all \( i \in [n] \).

Next we give the generalized Collatz-Wielandt Theorem of nonnegative weakly irreducible tensors.

**Theorem 3.5.** Let \( A \in \mathbb{R}^{[m,n]}_+ \) be a weakly irreducible tensor. Then

\[
\max_{B \in \mathbb{R}^{[k,n]}_+, r_i(B) \neq 0} \min_{i \in [n]} \frac{r_i(AB)}{(r_i(B))^{m-1}} = \rho(A) = \min_{B \in \mathbb{R}^{[k,n]}_+, r_i(B) \neq 0} \max_{i \in [n]} \frac{r_i(AB)}{(r_i(B))^{m-1}},
\]

where \( k \) is any fixed positive integer.

**Proof.** Since \( A \) is a nonnegative irreducible tensor, by Lemma 2.2, \( \rho(A) \) is an eigenvalue of \( A \) and \( x = (x_1, \ldots, x_1) \in \mathbb{R}^{n++} \) is the eigenvector corresponding to \( \rho(A) \). Then

\[
\rho(A) = \frac{\sum_{i_2, \ldots, i_m=1}^{n} a_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}}{x_i^{m-1}}, \text{ for all } i \in [n].
\]

For any positive integer \( k \), there exists a tensor \( B \in \mathbb{R}^{[k,n]}_+ \), such that \( r_i(B) = x_i \), thus

\[
\rho(A) = \frac{\sum_{i_2, \ldots, i_m=1}^{n} a_{i_2 \cdots i_m} r_i(B) \cdots r_i(B)}{(r_i(B))^{m-1}} = \frac{r_i(AB)}{(r_i(B))^{m-1}},
\]

by Theorem 3.1 we know

\[
\min_{i \in [n]} \frac{r_i(AB)}{(r_i(B))^{m-1}} \leq \rho(A) \leq \max_{i \in [n]} \frac{r_i(AB)}{(r_i(B))^{m-1}}.
\]

Then

\[
\max_{B \in \mathbb{R}^{[k,n]}_+, r_i(B) \neq 0} \min_{i \in [n]} \frac{r_i(AB)}{(r_i(B))^{m-1}} = \rho(A) = \min_{B \in \mathbb{R}^{[k,n]}_+, r_i(B) \neq 0} \max_{i \in [n]} \frac{r_i(AB)}{(r_i(B))^{m-1}}.
\]

**Remark:** In Theorem 3.5, when \( B = x \in \mathbb{C}^{[1,n]} \), we obtain Collatz-Wielandt Theorem for nonnegative weakly irreducible tensors, which is showed in equality (1).
4. The eigenvalue inclusion sets of the general product of tensors

In this section, the bounds for spectral radius and the eigenvalue inclusion sets of the general product of two tensors are discussed.

**Theorem 4.1.** Let $A \in \mathbb{R}_{+}^{[m,n]}$, $B \in \mathbb{R}_{+}^{[k,n]}$. Then

$$r(A)(r(B))^{m-1} \leq \rho(AB) \leq R(A)(R(B))^{m-1}.$$ 

**Proof.** Let $A = (a_{i_1i_2 \ldots i_m}) \in \mathbb{R}_{+}^{[m,n]}$, $B \in \mathbb{R}_{+}^{[k,n]}$. By Lemma 2.3 we have

$$r_i(AB) = \sum_{i_2, \ldots, i_m=1}^{n} a_{ii_2 \ldots i_m} r_{i_2}(B) \cdots r_{i_m}(B).$$

Clearly, $r_i(AB) \geq r_i(A)(r(B))^{m-1}$, for all $i \in [n]$. Let $r_j(AB) = r_j(AB)$, $j \in [n]$. Then we get

$$r(AB) = r_j(AB) \geq r_j(A)(r(B))^{m-1} \geq r(A)(r(B))^{m-1}.$$

It follows from Lemma 2.3 that $\rho(AB) \geq r(AB)$. So we obtain

$$\rho(AB) \geq r(A)(r(B))^{m-1}.$$

Similarly, we have $\rho(AB) \leq R(AB) \leq R(A)(R(B))^{m-1}$. Thus

$$r(A)(r(B))^{m-1} \leq \rho(AB) \leq R(A)(R(B))^{m-1}.$$

\[\square\]

**Remark :** In Theorem 4.1, when $B$ is an identity matrix, we also obtain $r(A) \leq \rho(A) \leq R(A)$ (see [6]).

From Theorem 4.1 and Corollary 2.5, we have the following results.

**Corollary 4.2.** Let $A \in \mathbb{C}^{[m,n]}$. Then

1. $\rho(A^k) \leq (R(A))^{\mu_k}$;
2. If $A$ is a nonnegative tensor, we have

$$r(A))^{\mu_k} \leq \rho(A^k) \leq (R(A))^{\mu_k},$$
where $\mu_k = \begin{cases} \frac{(m-1)^k-1}{m-2}, & m > 2, \\ \frac{k}{k}, & m = 2. \end{cases}$

In [1], Qi gave the Geršgorin-type eigenvalue inclusion sets of tensors, i.e.,

$$\sigma(A) \subseteq \bigcup_{i \in [n]} \{ z \in \mathbb{C} : |z - a_{i_{1}...i_{m}}| \leq r_i(A) - |a_{i_{1}...i_{m}}| \}.$$  (4)

For $\mathcal{A} \in \mathbb{C}^{[m,n]}$ and $\mathcal{B} \in \mathbb{C}^{[k,n]}$. By Lemma 2.4, it yields

$$r_i(\mathcal{A}\mathcal{B}) \leq r_i(\mathcal{A})(R(\mathcal{B}))^{m-1}.$$  (5)

Denote by $c_{i_{1}...i_{k}}$ the diagonal elements of tensor $\mathcal{A}\mathcal{B}$, then

$$c_{i_{1}...i_{k}} = \sum_{i_2,...,i_m=1}^{n} a_{i_{1}i_{2}...i_{m}} b_{i_{2}...i_{k}} \cdots b_{i_{m}i_{1}...i_{k}}, \ i \in [n].$$  (6)

Combining (4), (5), (6), we obtain the eigenvalue inclusion sets for $\mathcal{A}\mathcal{B}$ as follows.

**Theorem 4.3.** Let $\mathcal{A} = (a_{i_1i_2...i_m}) \in \mathbb{C}^{[m,n]}$ and $\mathcal{B} = (b_{i_1i_2...i_k}) \in \mathbb{C}^{[k,n]}$. Then

$$\sigma(\mathcal{A}\mathcal{B}) \subseteq G = \bigcup_{i \in [n]} \{ z \in \mathbb{C} : |z - c_{i_{1}...i_{k}}| \leq r_i(\mathcal{A})(R(\mathcal{B}))^{m-1} - |c_{i_{1}...i_{k}}| \},$$

where $c_{i_{1}...i_{k}} = \sum_{i_2,...,i_m=1}^{n} a_{i_{1}i_{2}...i_{m}} b_{i_{2}...i_{k}} \cdots b_{i_{m}i_{1}...i_{k}}$.

For a tensor $\mathcal{A} = (a_{i_{1}...i_{m}}) \in \mathbb{C}^{[m,n]}$, we associate with $\mathcal{A}$ a directed graph $\Gamma_{\mathcal{A}}$ as follows, $\Gamma_{\mathcal{A}} = (V(\mathcal{A}), E(\mathcal{A}))$, where $V(\mathcal{A}) = \{1, \ldots, n\}$ is vertex set and $E(\mathcal{A}) = \{(i,j) : a_{i_2...i_{m}} \neq 0, j \in \{i_2, \ldots, i_m\} \neq \{i, \ldots, i\}\}$ is arc set(see [8], [5]). If for each vertex $i \in V(\mathcal{A})$, there exists a circuit $\gamma$, such that $i$ belong to $\gamma$, then $\Gamma_{\mathcal{A}}$ is called weakly connected. We denote the set of circuits of $\Gamma_{\mathcal{A}}$ by $C(\mathcal{A})$.

In [12], Bu et al. gave the Brualdi-type eigenvalue inclusion sets via digraph as follows: Let $\mathcal{A} = (a_{i_{1}i_{2}...i_{m}}) \in \mathbb{C}^{[m,n]}$. If $\Gamma_{\mathcal{A}}$ is weakly connected, then

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\gamma \in C(\mathcal{A})} \{ z \in \mathbb{C} : \prod_{i \in \gamma} |z - a_{i_{1}...i_{k}}| \leq \prod_{i \in \gamma} (r_i(\mathcal{A}) - |c_{i_{1}...i_{k}}|) \}. $$  (7)

Next we give another eigenvalue inclusion sets for tensor $\mathcal{A}\mathcal{B}$.  

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Theorem 4.4. Let $A = (a_{i_1i_2...i_m}) \in \mathbb{C}^{[m,n]}$, $B = (b_{i_1i_2...i_k}) \in \mathbb{C}^{[k,n]}$ and $\Gamma_{AB}$ be weakly connected. Then

$$\sigma(AB) \subseteq B = \bigcup_{\gamma \in C(AB)} \{ z \in \mathbb{C} : \prod_{i \in \gamma} |z - c_{i...i}| \leq \prod_{i \in \gamma} (r_i(A)(R(B))^{m-1} - |c_{i...i}|) \} \subseteq G,$$

where $c_{i...i} = \sum_{i_2,...,i_m=1}^{n} a_{i_2...i_m} b_{i_2i_3...i_k} b_{i_3i_4...i_k} \ldots b_{i_mi_1...i_k}$, $G$ is given in Theorem 4.3.

Proof. Since $A = (a_{i_1i_2...i_m}) \in \mathbb{C}^{[m,n]}$, $B = (b_{i_1i_2...i_k}) \in \mathbb{C}^{[k,n]}$ and $\Gamma_{AB}$ is weakly connected. Then combining (5), (6), (7), we obtain

$$\sigma(AB) \subseteq B = \bigcup_{\gamma \in C(AB)} \{ z \in \mathbb{C} : \prod_{i \in \gamma} |z - c_{i...i}| \leq \prod_{i \in \gamma} (r_i(A)(R(B))^{m-1} - |c_{i...i}|) \}.$$

For any $z \in B$, if $z \notin G$, then

$$|z - c_{i...i}| \leq r_i(A)(R(B))^{m-1} - |c_{i...i}|, \quad \text{for all } i \in [n].$$

Thus

$$\prod_{i \in \gamma} |z - c_{i...i}| > \prod_{i \in \gamma} (r_i(A)(R(B))^{m-1} - |c_{i...i}|), \quad \text{for all } \gamma \in C(A),$$

this is a contradiction to $z \in B$. Therefore $z \in G$, i.e., $B \subseteq G$. \qed

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