On the role of cross-helicity in $\beta$-plane magnetohydrodynamic turbulence

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Abstract

Magnetohydrodynamic (MHD) turbulence on a $\beta$-plane with an in-plane mean field, a system which serves as a simple model for the solar tachocline, is investigated analytically and computationally. We show that conservation of squared magnetic potential in this system leads to a global increase in the cross-helicity, which is otherwise conserved in pure MHD (for which the Rossby parameter $\beta$ is zero). We perform a closure using weak turbulence theory and show that the cross-helicity spectrum entirely determines momentum transport. We also note that perturbation theory for small Rossby parameter is impossible in weak turbulence, since it changes the topology of surfaces on which resonant interactions occur. We support our results with numerical simulations, which clearly indicate that the cross-helicity is most important in the transitional regime between Rossby and Alfvénic turbulence.
I. INTRODUCTION

Since the early 1990s, much attention has been paid to the solar tachocline, a thin layer which lies at the base of the convective zone (CZ) of the sun. The tachocline separates the core (which rigidly rotates on spheres) from the CZ (which differentially rotates on cylinders) and exhibits strong radial shear (see, for example, [1]). It is widely believed [2–4] that the tachocline is home to the so-called Ω-effect and thus is of crucial importance to the solar dynamo: the shear drags poloidal magnetic field lines originating from the core, converting them to a strong toroidal field, which is stored in the tachocline.

Considerable debate in the literature has surrounded the nature of turbulent momentum transport in the tachocline, which is of central importance to the question of why the tachocline exists in the first place. Spiegel and Zahn (1992) [5] first proposed that the turbulence would act as an eddy viscosity, which would prevent the tachocline from spreading inward due to meridional cells. Gough and McIntyre (1998) [6] instead argued that the tachocline is essentially 2D, and so turbulence would act as a negative viscosity and produce mean ordered jetst through potential vorticity (PV) mixing. How such PV mixing would oppose the downward “burrowing” the tachocline (due to meridional circulation in the CZ) remains unclear.

It is thus fruitful to study meso- and macro-scale turbulent momentum transport in the tachocline. Because the lower tachocline is stably stratified, it can be modeled as a quasitwo-dimensional, differentially rotating magnetic fluid, which motivates the consideration of a magnetohydrodynamic (MHD) model on a 2-D β-plane:

\[
\begin{align*}
\partial_t \nabla^2 \psi + \beta \partial_x \psi &= \{ \psi, \nabla^2 \psi \} - \{ A, \nabla^2 A \} + \nu \nabla^4 \psi + \tilde{f} \\
\partial_t A &= \{ \psi, A \} + \eta \nabla^2 A + \tilde{g}.
\end{align*}
\]

These incompressible equations express the dynamics of the streamfunction \( \psi \) and the magnetic potential \( A \), defined so that the velocity field is \( \mathbf{v} = (\partial_y \psi, -\partial_x \psi, 0) \) and the magnetic field is \( \mathbf{b} = (\partial_y A, -\partial_x A, 0) \). We have defined the Poisson bracket \( \{ a, b \} = \partial_x a \partial_y b - \partial_y a \partial_x b \). \( \beta \) is the so-called Rossby parameter, which captures the effect of a planetary vorticity/Coriolis force gradient, so that the solar rotation rate is locally given by \( 2\Omega \simeq (0, 0, f_0 + \beta y) \). \( \nu \)
is the fluid viscosity and \( \eta \) is the resistivity. At the moment we do not specify the forcing
functions \( \tilde{f} \) and \( \tilde{g} \), which respectively model kinetic stirring (including thermal excitation)
and injection of magnetic potential, say by pumping from above or below the tachocline. It is worth noting that we have normalized the magnetic field by \( 1/\sqrt{4\pi \rho} \), where \( \rho \) is the
density, so that it has units of velocity.

This model, introduced by Tobias et al. (2007) [7] and Diamond et al. (2007) [8], constitutes a major simplification of the physics of the tachocline, but it nevertheless features a
quite rich range of dynamics and is fundamentally interesting in its own right. Tobias et al.
studied the effect of a weak mean magnetic field aligned with the shear, so that \( A \rightarrow \tilde{A} + b_0 y \),
and showed numerically that, when the resistivity becomes sufficiently large (or the mean
field becomes sufficiently weak), the system undergoes a transition from Alfvénic turbulence
with a forward cascade to Rossby-like turbulence exhibiting an inverse cascade to large scales.
The inverse cascade is accompanied by zonal flow formation.

In particular, Tobias et al. explored the dependence of zonal flow formation on the
Zel'dovich parameter \( Z = R_m v_{A,0}^2 / \langle \tilde{v}^2 \rangle \). Here, \( R_m \) is the magnetic Reynolds number, \( v_{A,0} \)
is the Alfvén speed in the externally prescribed magnetic field, and \( \langle \tilde{v}^2 \rangle \) is the mean square
velocity fluctuation intensity. Extensive previous work [9–12] noted that for \( Z < 1 \), turbulent resistivity tracked
kinematic estimates, while for \( Z > 1 \), the turbulent resistivity was quenched. A key point of \( Z \)-induced quenching is that only a weak \( b_0 \) field is needed at
large \( R_m \). The physics underlying quenching can be understood as vortex disruption [13].

Given the previous history, it was natural to investigate \( \beta \)-plane MHD by a scan of \( Z \). Noting
that \( Z = \tau_c v_{A,0}^2 / \eta \), where \( \tau_c \) is the (fixed) forcing cell correlation time and \( \eta \) the resistivity,
Tobias et al. scanned the parameter space of \( b_0^2 \) and \( \eta \), so as to examine jet formation. Results indicate that the \((b_0^2, \eta) \) plane divides into regions above and below a line defined by
\( b_0^2 / \eta = \text{const.} \). Above the line, \( b_0^2 \) is strong, and jets are suppressed. Below the line, jets form.
Alternatively, strong \( b_0^2 \) clearly inhibits jet formation via Alfvénization, while \( \eta \) dissipates \( b_0 \), and thus removes the inhibitor. The manifest \( \eta \) dependence supports the relevance of the
Zel'dovich parameter. The \( b_0^2 / \eta = \text{const.} \) scaling was also recovered analytically in [14].

Another perspective on tachocline transport and mixing is given in [15].

Diamond et al. (2007) gave an analytic closure calculation and derived the “magnetic
Rhines scale” \( \ell_{MR} = \sqrt{b_0 / \beta} \), defined by the crossover of the Rossby frequency \( \omega_{\beta} = -\beta k_x / k^2 \)
and the Alfvén frequency $\omega_A = k_x b_0$. They argued that $\ell_{\text{MR}}$ is a critical lengthscale below which the turbulence is Alfvénized, so that the Maxwell-Reynolds stress

$$\langle \partial_x \tilde{\psi} \partial_y \tilde{\psi} \rangle - \langle \partial_x \tilde{A} \partial_y \tilde{A} \rangle \simeq \int d^2k \frac{k_x k_y}{k^2} (|\tilde{v}_k|^2 - |\tilde{b}_k|^2)$$

(3)

tends to vanish to leading order and thus there is no clear large-scale momentum transport to set up a zonal flow. Conversely, at scales larger than the magnetic Rhines scale, the turbulence is Rossby-like. Thus, it is natural to suspect that the system will undergo transition from Alfvénic to Rossby turbulence when the magnetic Rhines scale becomes smaller than typical scales of the problem.

Subsequent work [16] addressed the key question of the physics of jet quenching. One candidate is the competition between Reynolds and Maxwell stresses, which tend to cancel in the Alfvénic limit. Another is decoherence of the Reynolds (and Maxwell) stresses by magnetic fields. Results indicate that both mechanisms are at work, with decoherence setting in at $b_0$ levels below that required for Alfvénization. A major motivation for the present work is to explore the effect of finite cross-helicity on the Reynolds vs. Maxwell competition.

We note in passing that Guervilly and Hughes [17, 18] have shown in 3D that a small scale magnetic dynamo can inhibit large scale jet formation. In particular, they showed that when the magnetic Prandtl number exceeds the critical value for the onset of a small scale dynamo, the Reynolds stress coherence drops and jet formation weakens. This is a related example of magnetic-field-induced inhibition of jet formation.

However, omitted from previous studies is a reckoning with the fact that the Coriolis term explicitly breaks conservation of cross-helicity, that is the alignment of the magnetic field and velocity $H = \langle \tilde{v} \cdot \tilde{b} \rangle = -\langle \tilde{A} \nabla^2 \tilde{\psi} \rangle$. Cross-helicity is globally conserved in pure MHD turbulence (up to dissipation), but when $\beta \neq 0$ we have

$$\partial_t \langle \tilde{A} \nabla^2 \tilde{\psi} \rangle = -\beta \langle \tilde{v}_y \tilde{A} \rangle + \text{dissipation}.$$ (4)

The origin of the breaking of cross-helicity conservation is that the $\beta$ term, which (unlike a mean magnetic field term) cannot be removed by a simple change of variables, breaks symmetry in the $\hat{x}$ direction.

We can anticipate that a nonzero cross-helicity in this system may have important conse-
quences for transport. Cross-helicity is closely related to the turbulent emf [19], so that one may naturally suspect that the cross-helicity induced by differential rotation in the tachocline has consequences for the solar dynamo. However, it is well known that a dynamo cannot be supported in two-dimensions, so our primary subject of study will be momentum transport, which will generally couple to the cross-helicity in this system. In fact, we will see that the relationship between cross-helicity and momentum transport in this system is quite intimate.

The link of cross helicity to transport may be expressed in terms of the spectrum \( H_k = \langle \mathbf{v}_k \cdot \mathbf{b}_{-k} \rangle \). We then have

\[
\frac{k_x k_y}{2k^2} \text{Re} H_k = \hat{z} \cdot \langle \mathbf{v}_k \times \mathbf{b}_{-k} \rangle \tag{5}
\]

\[
\frac{k_x}{k^2} \text{Im} H_k = \langle \mathbf{v}_{y,k} \mathbf{A}_{-k} \rangle. \tag{6}
\]

Thus, the real part of the cross helicity determines the turbulent emf, and the imaginary part determines the turbulent flux of magnetic potential along the perpendicular axis. The turbulent emf is responsible for mean field evolution.

The cross-helicity may be viewed as a dual to the alignment of Elsässer populations. If we define the Elsässer modes \( z^\pm = \mathbf{v} \pm \mathbf{b} \), the utility of this basis lies in the well-known fact that pure MHD turbulence is entirely generated by counter-propagating Elsässer populations; \( z^+ \) and \( z^- \) interact with each other but not themselves. In this basis, the Maxwell-Reynolds stress, which gives rise to momentum transport, is equivalent to the alignment of counter-propagating Elsässers:

\[
\text{Re} \langle z_k^+ \cdot z_{-k}^- \rangle = |\mathbf{v}_k|^2 - |\mathbf{b}_k|^2. \tag{7}
\]

Similarly, cross-helicity is equivalent to an imbalance in Elsässer populations; if we define \( H_k = \langle \mathbf{v}_k \cdot \mathbf{b}_{-k} \rangle \), we have

\[
\text{Re} H_k = \frac{1}{4} (|z_k^+|^2 - |z_k^-|^2). \tag{8}
\]

In general, neither basis is preferred when the Rossby parameter is turned on. Instead, the eigenmodes are Rossby-Alfvén modes, whose dispersion combines those of simple Rossby waves and Alfvén waves:

\[
\omega = \frac{\omega_\beta \pm \sqrt{4\omega_A^2 + \omega_\beta^2}}{2}, \tag{9}
\]
where $\omega_\beta$ and $\omega_A$ are respectively the Rossby and Alfvén frequencies. The coupling of Rossby-Alfvén waves of different species will mix the Elsässer alignment and the cross-helicity, so that cross-helicity will play a role in all the spectral equations and help determine momentum transport.

In this work, we study the role of cross-helicity in $\beta$-plane MHD turbulence. First, we show by a simple non-perturbative calculation—arguing on the basis of conservation of mean squared magnetic potential $\langle \tilde{A}^2 \rangle$—that the total cross-helicity in this system attains a finite, stationary value. Second, we discuss a basic closure of the system within the framework of weak turbulence, and argue that for strong mean field, the stationary, real cross-helicity spectrum is in fact equivalent to the Elsässer alignment spectrum. This argument also shows that the turbulence is magnetic below the magnetic Rhines scale and kinetic above, as argued in [8]. We also discuss the difficulties of making progress analytically with the weak turbulence spectral equations. In particular, we point out that the $\beta \to 0$ limit is singular, and thus the tempting route of expanding the weak turbulence closure around small $\beta$ fails.

Next, we perform a set of simulations of this system for strong mean field and several values of $\beta$ and demonstrate that

- the simple non-perturbative calculation provides an accurate estimate of the actual stationary cross-helicity,
- transition from Alfvénic to Rossby turbulence begins when the magnetic Rhines wavenumber exceeds the forcing wavenumber: $k_{MR} = \sqrt{\beta/b_0} \gtrsim k_f^2$, and this transition is presaged by an increase in the global cross-helicity, which peaks around the critical $\beta$ where the above scales overlap.

We also present plots of the stationary spectra from simulation, which confirm the Rossby-like nature of the turbulence at large $\beta$ and the MHD-like nature at small $\beta$, and verify the relationship between the cross-helicity and the Elsässer alignment within weak turbulence.

The remainder of this paper is organized as follows. In §II, we calculate the stationary cross-helicity and discuss the weak turbulence closure of the spectral equations, including the relationship between the time-averaged cross-helicity spectrum and the time-averaged Elsässer alignment spectrum. In §III, we present simulation results, both for global mean
quantities and spectra. Finally, in §IV, we summarize our results and discuss directions for future research.

II. THEORY

A. Stationary cross-helicity

Our investigation begins with a simple, non-perturbative calculation of the cross-helicity at long times. The calculation is inspired by the Zel’dovich theorem [20]; like that earlier result, here we use the conservation of mean-squared magnetic potential to relate large-scale transport properties at long times to small-scale dissipation.

For the reader’s convenience, we first rewrite the system (1)-(2) with the mean magnetic field terms. We assume the system is forced kinetically only ($\tilde{g} = 0$):

$$\partial_t \nabla^2 \tilde{\psi} + \beta \partial_x \tilde{\psi} = \{\tilde{\psi}, \nabla^2 \tilde{\psi}\} - \{\tilde{A}, \nabla^2 \tilde{A}\} + b_0 \partial_x \nabla^2 \tilde{A} + \nu \nabla^4 \tilde{\phi} + \tilde{f}$$  (10)

$$\partial_t \tilde{A} = \{\tilde{\psi}, \tilde{A}\} + b_0 \partial_x \tilde{\psi} + \eta \nabla^2 \tilde{A}.$$  (11)

To be clear, $b_0$ represents an initial mean field, and in principle the fluctuations $\tilde{A}$ could set up a mean field which opposes $b_0$; however, in this work we primarily consider weak turbulence, with $\langle \tilde{b}_x \rangle \ll b_0$ in practice.

Multiplying the second equation by $A$, integrating over space, and taking $t \to \infty$ yields

$$\frac{1}{2} \partial_t \langle \tilde{A}^2 \rangle = b_0 \langle \tilde{A} \partial_x \tilde{\psi} \rangle - \eta \langle (\nabla \tilde{A})^2 \rangle = 0$$  (12)

(note that $\langle \cdot \rangle$ is a global average, so that all triplet terms cancel). Thus the mean flux of magnetic potential is

$$\langle \tilde{A} \partial_x \tilde{\psi} \rangle = \frac{\eta}{b_0} \langle \tilde{b}^2 \rangle.$$  (13)

This is just Zel’dovich’s theorem, which also requires that the total mean magnetic field must eventually decay in 2D. However, this decay may be very slow [21].

Now, multiplying the $A$ equation by $\nabla^2 \psi$ and vice-versa, integrating over space, and
summing the resulting equations yields

\[ \partial_t \langle \tilde{A} \nabla^2 \tilde{\psi} \rangle = -\beta \langle \tilde{A} \partial_x \tilde{\psi} \rangle + (\eta + \nu) \langle \nabla^2 \tilde{\psi} \nabla^2 \tilde{A} \rangle + \langle \tilde{A} \tilde{f} \rangle. \] (14)

We then introduce characteristic lengthscales of the velocity and magnetic field variation and \( \langle \nabla^2 \psi \nabla^2 A \rangle \simeq \frac{1}{\ell_v \ell_b} \langle v \cdot b \rangle \). Finally, we substitute (13) into the above, take the long-time limit, and obtain the estimate for the stationary cross-helicity

\[ H(t = \infty) = \frac{\ell_v \ell_b}{\eta + \nu} \left( \frac{\beta \eta}{b_0} \langle \tilde{b}^2 \rangle - \langle \tilde{A} \tilde{f} \rangle \right). \] (15)

Then, assuming that \( \langle \tilde{A} \tilde{f} \rangle \) is small, we have the simpler expression

\[ H(t = \infty) = \frac{\ell_v \ell_b}{1 + \text{Pm}} \frac{\beta \langle \tilde{b}^2 \rangle}{b_0}, \] (16)

where \( \text{Pm} \equiv \nu / \eta \) is the magnetic Prandtl number. In practice, the forcing term \( \langle \tilde{A} \tilde{f} \rangle \) will make a nonzero contribution to the total cross-helicity, which may be significant if the magnetic Reynolds number is very large. However, in the simulations performed in this work, this contribution was relatively small, \( \lesssim 15\% \) of that contributed by the \( \beta \) term. If the system is magnetically forced, there will be additional contributions from correlations with \( \tilde{g} \).

When \( \beta \) is not too strong, \( \beta \lesssim b_0 k_f^2 \), we can estimate \( \ell_v = \ell_b \simeq k_f^{-1} \). When \( \beta \) is large, this is no longer a good estimate for \( \ell_b \), as the magnetic field goes to small scales; a better estimate is then a magnetic Taylor microscale \( \sqrt{\frac{2}{\varepsilon} \langle \tilde{b}^2 \rangle} \), where \( \varepsilon \) is the energy injection rate (equivalent to the dissipation rate for stationary turbulence).

Note that this calculation depended on the magnetic field being latitudinally aligned; otherwise, the Ze’ldovich theorem is not enough to discern the flux of magnetic potential across the planetary vorticity gradient. Note also that the cross-helicity could be very large in the case of a weak mean field and strong fluctuations.
B. Weak turbulence closure of spectral equations

We have derived an estimate for the stationary mean cross-helicity; however, discerning transport properties requires studying the spectra. To this end, we seek a statistical closure which treats cross-correlations like the cross-helicity spectrum on an equal footing with autocorrelations like $\langle |\tilde{v}_k|^2 \rangle$. The simplest approach, which we adopt here, is the weak turbulence theory of Sagdeev and Galeev [22, 23], though we note that [24] took an approach (in pure MHD) based on the eddy-damped quasi-normal Markovian (EDQNM) closure theory. Formally a second-order time-dependent perturbation theory, weak turbulence assumes that the linear timescales are fast compared to nonlinear ones, so that the nonlinearity may be decomposed in terms of resonant triplet interactions between linear eigenmodes — here, Rossby-Alfvén modes. For this approach to be valid, we must assume a strong mean magnetic field. Even here, the weak turbulence description will be dubious at small parallel lengthscales, where the linear frequencies vanish.

In $k$-space, the $\beta$-plane MHD system is expressed as

$$\partial_t \psi_k + i\omega_{\beta,k} \psi_k + i\omega_{A,k} A_k = \frac{1}{2} \int d^2 k' d^2 k'' \delta(k - k' - k'') \left[ (k' \times k'') \cdot \hat{z} \right] \frac{k'^2 - k''^2}{k^2} \left( \psi_{k'} \psi_{k''} - A_{k'} A_{k''} \right),$$

$$\partial_t A_k + i\omega_{A,k} \psi_k = \frac{1}{2} \int d^2 k' d^2 k'' \delta(k - k' - k'') \left[ (k' \times k'') \cdot \hat{z} \right] \left( A_{k'} \psi_{k''} - \psi_{k'} A_{k''} \right),$$

where $\omega_{\beta,k} = -\beta k_x / k^2$ is the Rossby frequency and $\omega_{A,k} = b_0 k_x$ is the Alfvén frequency, and we have neglected forcing and dissipation terms.

The linear eigenmodes of this system are the two Rossby-Alfvén modes with frequencies

$$\omega^\pm = \frac{\omega_{\beta} \pm \Omega}{2},$$

and corresponding amplitudes

$$\phi^\pm = \frac{1}{\Omega} (\omega^\pm \psi - \omega_{A} A),$$

where $\Omega = \text{sgn}(k_x) \sqrt{4\omega_A^2 + \omega_\beta^2}$. The sign of the square root and the normalization are chosen so that the $k_x \to 0$ limit is not problematic. In the $b_0 \to \infty$ limit, these are just Elsässer modes; as $b_0 \to 0$, one mode becomes a Rossby wave and the other ceases to exist.
(the amplitude and frequency simultaneously vanish). In this sense, the Rossby turbulence limit is singular.

The classical weak turbulence theory, formulated for a single scalar field, is exceedingly well-known [22, 23]. However, its straightforward generalization to the case of multiple interacting scalar fields is rarely seen in the literature, so we present it here. Let \( \{ \phi^\alpha \} \) be a finite set of quadratically interacting linear eigenmodes described by the evolution equations

\[
\partial_t \phi^\alpha_k + i \omega^\alpha_k \phi^\alpha_k = \frac{1}{2} \sum_{\beta \gamma} \int d^2 k' d^2 k'' \delta(k - k' - k'') M_{\beta \gamma k,k',k''} \phi^\beta_{k'} \phi^\gamma_{k''},
\]

so that the linear frequency matrix \( \omega^{\alpha \beta} \) is diagonal. The coupling coefficients are assumed to obey the symmetry condition \( M^{\alpha \beta \gamma}_{k,k',k''} = M^{\alpha \gamma \beta}_{k,k'',k'} \).

We define the correlators \( C^{\alpha \alpha'}_{k} \) by

\[
\langle \phi^\alpha_k \phi^\alpha_{k'} \rangle \equiv C^{\alpha \alpha'}_{k} \delta(k + k') e^{-i(\omega^\alpha_k - \omega^\alpha_{k'}) t}.
\]

After some work, it can be shown (see Appendix A) that within the standard assumptions of weak turbulence (second-order perturbation theory, invoking both homogeneity and the random phase approximation), the correlators evolve according to the spectral equations

\[
\partial_t C^{\alpha \alpha'}_{k} = \sum_{\beta \gamma} \int d^2 k' d^2 k'' \delta(k - k' - k'') \left[ \pi |M^{\alpha \beta \gamma}_{k,k',k''}|^2 C^{\beta \gamma}_{k'} C^{\gamma \alpha}_{k''} \delta(\omega^\alpha_k - \omega^\beta_{k'} - \omega^\gamma_{k''}) \delta_{\alpha \alpha'} + M^{\alpha \beta \gamma}_{k,k',k''} M^{\beta \alpha \gamma}_{k,k',k''} C^{\alpha \alpha'}_{k} C^{\gamma \gamma}_{k''} \left( \pi \delta(\omega^\alpha_k - \omega^\beta_{k'} - \omega^\gamma_{k''}) + i \mathcal{P} \frac{1}{\omega^\alpha_k - \omega^\beta_{k'} - \omega^\gamma_{k''}} \right) \right].
\]

Here, \( \mathcal{P} \) means the Cauchy principal value is taken, and \( \delta_{\alpha \alpha'} \) is a Kronecker delta. Note that in the case of a single mode with real coupling coefficients, the principal value terms cancel and we obtain the usual Sagdeev-Galeev theory. The Dirac deltas constrain the collision integral to resonances where the mismatch frequency \( \Delta \omega = \omega^\alpha_k - \omega^\beta_{k'} - \omega^\gamma_{k''} \) vanishes.

To apply this result to our \( \beta \)-plane MHD system, where we have two scalar fields \( \psi \) and \( A \), we will will need to transform to the eigenbasis of two Rossby-Alfvén modes, + and −,
and obtain the coupling coefficients in this basis. These are

\[
M_{k,k',k''}^{\pm++} = \frac{\hat{z} \cdot (k' \times k'')}{\Omega_k} \left[ \frac{k_x^2 \omega_k^- - k_x^2 \omega_k^{++}}{k_t^2} + \frac{k^2}{k_t^2} \left( 1 - \frac{\omega_k^- \omega_k^{++}}{k_t^2 k_x^2 b_0^2} \right) \right],
\]

\[
M_{k,k',k''}^{\pm+-} = -\frac{\hat{z} \cdot (k' \times k'')}{\Omega_k} \left[ \frac{k_x^2 \omega_k^- - k_x^2 \omega_k^{+}}{k_t^2} + \frac{k^2}{k_t^2} \left( 1 - \frac{\omega_k^- \omega_k^{+}}{k_t^2 k_x^2 b_0^2} \right) \right],
\]

\[
M_{k,k',k''}^{\pm--} = \frac{\hat{z} \cdot (k' \times k'')}{\Omega_k} \left[ \frac{k_x^2 \omega_k^- - k_x^2 \omega_k^{--}}{k_t^2} + \frac{k^2}{k_t^2} \left( 1 - \frac{\omega_k^- \omega_k^{--}}{k_t^2 k_x^2 b_0^2} \right) \right].
\]

We also may express the Rossby-Alfvén correlators in the basis

\[
E^K_k = \langle |v_k|^2 \rangle, E^M_k = \langle |b_k|^2 \rangle, H_k : \]

\[
k^2 C_k^{\pm} = \frac{1}{\Omega^2} \left( \omega^2_k E^K_k + \omega^2_A E^M_k - 2 \omega_A \omega_{\pm} \text{Re} H_k \right),
\]

\[
k^2 \text{Re}(C_k^{+} e^{-i\Omega t}) = \frac{1}{\Omega^2} \left( \omega^2_A (E^K_k - E^M_k) + \omega_{\beta} \omega_A \text{Re} H_k \right),
\]

\[
k^2 \text{Im}(C_k^{+} e^{-i\Omega t}) = \frac{\omega_{\beta}}{2\Omega} \text{Im} H_k.
\]

We see that the (real) cross-correlator between Rossby-Alfvén modes of opposite sign mixes the (real) cross-helicity and the Elsässer alignment \(E^K_k - E^M_k\), and naturally oscillates at frequency \(\Omega = \omega^+ - \omega^-\).

In principle, one may proceed by using the above expressions for the coupling coefficients, the transformation from the Rossby-Alfvén basis to the physical basis, and the inverse of that transformation to transform the spectral equations Eq. 19 into the physical basis. However, the resulting equations are exceedingly complicated and not terribly enlightening by inspection.

It is tempting, then, to consider the limit \(\beta \to 0\) and expand about a pure (weak) MHD fixed point (say, one where the energy spectra are flat and the cross helicity is zero) \[25\], in order to perturbatively solve for the stationary spectra in powers of \(\beta\), in the spirit of mean-field electrodynamics. However, this approach fails. To see this, consider the Rossby-Alfvén wave-wave interactions that contribute to the collision integrals. In pure MHD, only counter-propagating Elsässer populations interact, so the only processes with nonzero amplitude are \(\pm, \mp \to \pm\). The linear frequencies may be expanded as

\[
\omega_{\pm} = \pm \omega_A + \frac{\omega_{\beta}}{2} + O(\beta^2),
\]
so the mismatch frequencies, which set the interaction time in weak turbulence, for these processes are

\[
\Delta \omega = \omega_k^\pm - \omega_{k'}^\pm - \omega_{k''}^\mp \simeq \pm 2b_0k_x'' + \frac{1}{2}(\omega_{\beta,k} - \omega_{\beta,k'} + \omega_{\beta,k''}),
\]

(27)

where we have used the momentum conservation constraint \(k' + k'' = k\). These processes present no particular issue for perturbation theory.

When \(\beta\) is nonzero, previously forbidden processes now have nonzero amplitude. First, there is \(\pm, \pm \rightarrow \pm\), with mismatch frequency

\[
\Delta \omega \simeq \frac{1}{2}(\omega_{\beta,k} - \omega_{\beta,k'} - \omega_{\beta,k''}),
\]

(28)
equivalent to an effective Rossby-wave interaction. This frequency is \(O(\beta)\), so the interaction is long-lived, \(\tau \sim 1/|\beta|\). These processes pick up a coefficient \(M^2 \sim \beta^2\), so their overall contribution to the collision integrals is \(O(\beta)\), still not necessarily problematic in perturbation theory.

However, the \(\pm, \pm \rightarrow \mp\) processes have mismatch frequency

\[
\Delta \omega \simeq \pm 2b_0k_x + \frac{1}{2}(\omega_{\beta,k} - \omega_{\beta,k'} - \omega_{\beta,k''}).
\]

(29)

One would like to make the formal expansion

\[
\delta(\Delta \omega_0 + \beta \Delta \omega_1 + \ldots) \simeq \Delta \omega_0 + \beta \Delta \omega_1 \delta'(\Delta \omega_0) + \ldots,
\]

(30)

where \(\delta'\) is the distributional derivative of the Dirac delta function, which may be handled in practice using integration by parts. This expansion is impossible for this process because the zeroth-order mismatch frequency \(\Delta \omega_0 = \pm 2b_0k_x\) is independent of the integration variable \(k'\), and so the distributional derivative is undefined — any attempt to perform the collision integral will introduce a divergent factor \(\frac{1}{|\partial \Delta \omega_0/\partial k'|^{-1}}\).

Physically, this issue originates from a change in the topology of the resonance surfaces for the \(\pm, \pm \rightarrow \mp\) processes when \(\beta\) is turned on. When \(\beta\) is zero, the resonance surface is effectively the entire plane \((k'_x, k'_y)\), as the resonance condition may be satisfied for any \(k'\) as
FIG. 1: Plots of the resonance surfaces for the Rossby-Alfvén process $-,- \rightarrow +$, with $b_0 = 1$ and $\beta = 0.2$. In (a), we set $k = (0.1, 1)$ and in (b), we set $k = (0, 1)$.

| Process          | Permitted in pure MHD? | Notes for small $\beta$                                      |
|------------------|------------------------|----------------------------------------------------------------|
| $\pm, \mp \rightarrow \pm$ | Yes                   | Effective Alfvén-Alfvén interaction                           |
| $\pm, \pm \rightarrow \pm$  | No                    | Effective Rossby-Rossby interaction, long-lived              |
| $\pm, \pm \rightarrow \mp$  | No                    | Singular, not decomposable into Rossby/Alfvén interactions   |

TABLE I: Rossby-Alfvén triplet interactions in the $\beta$-plane system.

long as the resultant wave has $k_x = 0$. When $\beta$ is nonzero, the resonance surface collapses to a pair of closed curves when $k_x \neq 0$ and a pair of lines when $k_x = 0$ — see Fig. 1.

Another interpretation of this is that all triplet Rossby-Alfvén processes except $\pm, \pm \rightarrow \mp$ may be decomposed into interactions of pure Alfvén and/or Rossby waves. The basic features of the various kinds of Rossby-Alfvén triplet interactions are summarized in Table I.

An important point, however, is that the $\beta \rightarrow 0$ singularity is an artifice of weak turbulence theory; in a real system, the singularity will be screened by a finite resonance broadening. Indeed, in §III, we will see that the transition from pure MHD to the $\beta$-plane appears to be smooth.

In any case, it is thus difficult to make direct analytic progress with the weak turbulence spectral equations. However, one realizes that in a stationary state, the time-averaged Rossby-Alfvén cross-correlator must vanish as long as $k_x \neq 0$, since it naturally oscillates at the fast frequency $\Omega$. This imposes a major constraint on the stationary, time-averaged
solutions: we must have
\[ E_k^K - E_k^M = \frac{\beta}{b_0 k^2} \text{Re} H_k. \] (31)

Thus, at the level of weak turbulence, a stationary cross-helicity is equivalent to a deviation from energy equipartition. This also determines momentum transport. Another view is that the alignment of the velocity field with the magnetic field is determined by the alignment of the Elsässer populations. Moreover, one may rearrange this to identify the left-hand side with the Lorentz force and the right-hand side with the turbulent emf and obtain:
\[ \frac{\langle \partial_t \nabla^2 \tilde{\psi} \rangle_k}{\langle \partial_t A \rangle_k} = k^2_{\text{MR}}. \] (32)

This tells us that the system organizes so that, at each scale \( k \), the ratio of the rate of production of velocity fluctuations to the rate of production of magnetic fluctuations is \( k^2_{\text{MR}}/k^2 \), an explicit form of the maxim that at lengthscales larger than \( \ell_{\text{MR}} \), the turbulence is primarily kinetic, and at smaller scales it is primarily magnetic.

The vanishing of the cross-correlator also tells us that the imaginary part of the cross-helicity vanishes, \( \text{Im} H_k = 0 \). This is equivalent to the statement that the turbulent flux of \( A \) vanishes in weak turbulence theory, and consistent with the Zel’dovich theorem for sufficiently large \( b_0 \) and/or sufficiently small fluctuations \( \langle \tilde{b}^2 \rangle \). We should not be surprised, as even a weak mean field is known to quench the turbulent resistivity [9–12].

We note that these results still do not allow much progress with the spectral equations, since the amplitude of the oscillations of the cross-correlator, still undetermined, plays a major role in the nonlinear dynamics.

III. SIMULATION RESULTS

A. Stationary mean energies and cross-helicity

We now present simulation results for this system, on a periodic box of size \( 2\pi \times 2\pi \) with a \( 512 \times 512 \) mesh. We used a pseudospectral code within the Dedalus framework [26]. The vorticity was forced in an annulus in Fourier space, centered at \( k_f = 32 \) and with a width of 8. The forcing had a fixed energy injection rate of \( \varepsilon = 10^{-2} \) and a correlation time of
\[ \tau_c = 5 \times 10^{-3}. \] All simulations had fixed \( \nu = \eta = 10^{-4} \) and \( b_0 = 2 \), and \( \beta \) was varied over a broad range. Hyperviscous terms with coefficients \( 10^{-8} \) were included in each equation to improve the stability properties. The magnetic Reynolds number for these simulations ranged from \( \text{Rm} \sim 6000 - 15000. \) After many dissipation times \( (t \gg 1/(\eta k_f^2)) \) the system reached a stationary state and the mean kinetic and magnetic energies, cross helicity, and zonal mean (kinetic) energy fraction \( zmf = \int dy(\int dx \, \tilde{v})^2 / \int dx dy \, \tilde{v}^2 \) were all recorded. A zmf exceeding \( \sim 0.5 \) signals an inverse cascade — most of the kinetic energy is at large parallel scales — and is indicative of a transition to Rossby turbulence. These results are plotted in Fig. 2; in Fig. 3, the cross-helicity is compared to the predictions of § II A.

There is a wealth of information in these plots and it is worth discussing. When \( \beta < b_0 k_f^2 \), the magnetic and kinetic energies are nearly equal, suggesting that the turbulence is Alfvénized. As \( \beta \) grows larger, this equipartition begins to break down, and the magnetic energy drops, signaling the possible generation of a Reynolds stress. When \( \beta \) is significantly greater than \( b_0 k_f^2 \), most of the energy is in the flow and concentrated at large parallel lengthscales, signaling an inverse cascade and a transition to Rossby turbulence.

Meanwhile, as \( \beta \) nears the transition point, the cross-helicity grows, peaks near \( \beta = b_0 k_f^2 \), and then drops again (that it tracks closely with the magnetic energy beyond this point appears to be coincidental to this choice of parameters). An estimate for the stationary cross-helicity using \( \ell_b = \ell_v = k_f^{-1} \) begins to break down for large \( \beta \); in this regime, the estimate \( \ell_b = \sqrt{\frac{\eta}{\epsilon} \langle \tilde{b}^2 \rangle} \) better fits the data.

Finally, we note the interesting fact that the zmf is nonmonotonic, and decreases during the transitional regime before becoming large.

### B. Flux of magnetic potential

In Fig. 4 we plot the turbulent resistivity \( \eta_T = -\langle \tilde{v}_y \tilde{A} \rangle / b_0 \) as a function of \( \beta \). It is generally very small, \( \eta_T < \eta \), having been quenched by the strong mean field. This is consistent with the weak turbulence expectation \( \text{Im} \, H_k = 0 \). It drops precipitously when \( \beta > b_0 k_f^2 \), due to the lack of magnetic activity — the rms turbulent magnetic potential \( \langle \tilde{A}^2 \rangle^{1/2} \) becomes very small.
FIG. 2: Plot of stationary turbulent energies, cross-helicity, and zmf obtained from simulation. All simulations were kinetically forced at an energy injection rate $\varepsilon = 10^{-2}$ at a scale $k_f = 32$ and had a mean field $b_0 = 2\hat{x}$. The transition from Alfvénic to Rossby turbulence, signified by an imbalance between kinetic and magnetic energies and a large zmf, appears to start around a critical value $\beta = b_0k_f^2$. This critical value, in general, will also depend on the resistivity. The transition is presaged by an increase in the mean cross helicity, which peaks around the critical value. $\beta$ is normalized in the plot by the critical value. The zmf reaches a minimum in the transitional regime, and becomes large ($>0.5$) when $\beta \gg b_0k_f^2$, signaling an inverse cascade.

C. Spectra

We now turn our attention to the spectra. We present plots of the stationary energy spectra, cross-helicity spectra (including both real and imaginary parts), and Elsässer alignment $\text{Re} P_k = \langle \tilde{v}_k^2 \rangle - \langle \tilde{b}_k^2 \rangle$, as functions of the parallel wavenumber $k_x$. The spectra are highly anisotropic; most of the interesting structure in the spectra is along $k_x$, whereas we found that the dependence on $k_y$ largely reflects the structure of the forcing and is peaked at harmonics of $k_f$. Thus, we average the spectra over $k_y$, as well as late times (specifically,
FIG. 3: We again plot the cross-helicity from simulation (red dashed line) and compare it to the prediction from § II A. We use two estimates for the lengthscales: using $\ell_b = \ell_v = k_f^{-1}$ (green circles), which ceases to be good as $\beta$ is large since the magnetic fluctuation scale becomes small, and a second estimate (blue squares) using a magnetic Taylor microscale $\ell_b = \sqrt{\eta/\varepsilon \langle \tilde{b}^2 \rangle}$.

The plots are shown in Figs. 6–8. The spectra are given at three values of $\beta$: $10^2$, $3 \times 10^3$, and $10^5$, respectively representing the small-$\beta$, transitional, and large-$\beta$ regimes.

In the Alfvénic regime represented by $\beta = 10^2$, the kinetic and magnetic energy spectra are nearly identical except at small $k_x$. At the largest parallel scales ($k_x = 0$), the kinetic energy becomes large, and the magnetic energy goes to zero; thus the (real) alignment spectrum is basically a delta function centered at $k_x = 0$. At finite scales, there is an inertial range were the energy spectra are essentially flat up to a cutoff, agreeing with weak turbulence theory for 2D MHD [25]. A small real cross-helicity is mixed over a broad range of scales — this mixing
FIG. 4: Plot of the turbulent resistivity $\eta_T = -\langle \tilde{v}_y \tilde{A} \rangle / b_0$ as a function of $\beta$.

occurs even when the total cross-helicity is zero. There is a small imaginary cross-helicity as well, concentrated at small but finite $k_x$, corresponding to the weak turbulent resistivity. In Fig. 5, we also show corresponding results for $\beta = 0$, i.e. pure MHD, and conclude that the spectra are consistent.

In the transitional regime represented by $\beta = 3 \times 10^3$, the kinetic and magnetic energy spectra no longer agree at any scale, so the alignment is finite over a broad range. The peak in the kinetic energy and alignment spectra has shifted from $k_x = 0$ to a large but finite parallel lengthscale — in the case plotted, $k_x = 1$, the smallest available wavenumber. This corresponds to a “near-zonal” flow which varies over the $\hat{x}$ direction with a long, characteristic wavelength equal to the system size. The real cross-helicity, too, is peaked at this scale, and otherwise spread over a broad range. The imaginary cross-helicity is now negligible,
FIG. 5: Stationary spectra, averaged over $k_y$, for $\beta = 0$. This is pure MHD, for the sake of comparison.

FIG. 6: Stationary spectra, averaged over $k_y$, for $\beta = 10^2$.

consistent with the claim of the previous subsection.

Finally, in this Rossby-like regime represented by $\beta = 1 \times 10^5$, the magnetic energy is negligible, so the kinetic energy and alignment spectra are equivalent. We have checked that these decay like $k_x^{-3}$, consistent with the known theory of 2D fluid turbulence [27]. Both real
and imaginary parts of the cross helicity are small as there is very little magnetic activity.

The weak turbulence relationship (Eq. 31) indeed holds to good approximation. In Fig. 9 we compare the kinetic energy spectra with $|\tilde{b}_k|^2 + \beta/(b_0 k^2) \Re H_k$ for the case $\beta = 3 \times 10^3$; the curves lie on top of one another except at $k_x = 0$. 
IV. CONCLUSIONS AND DISCUSSION

We have studied the stationary spectra of a simple model of the solar tachocline, $\beta$-plane MHD turbulence, with a strong mean field in the plane. Our simulation results show three distinct regimes: an Alfvénic regime at small $\beta$, a Rossby regime at large $\beta$, and a transitional regime near $\beta \approx b_0 k_f^2$. The transitional regime is characterized by the breakdown of equipartition between the (turbulent) kinetic and magnetic energies, and thus the breakdown of Alfvénization. Moreover, the the cross-helicity peaks in this regime, suggesting it plays a role in transition. A third interesting feature of this regime, which we did not attempt to study in detail in this work, is the appearance of “near-zonal” flows with a characteristic parallel wavelength equal to the system size.

We have shown that, in this system, the total turbulent cross-helicity builds up to a
predictable level. We used a simple but robust analytic argument based on Zel’dovich’s theorem. Moreover, we have shown, using weak turbulence theory, that the time-averaged cross-helicity spectrum is entirely equivalent to the time-averaged Maxwell-Reynolds stress differential. Thus, momentum transport, as well as the energy partition, is set by the cross-helicity. This latter result makes clear the role of cross-helicity in transition: the growth of the cross-helicity is directly linked to the breakdown of Alfvénization and the generation of momentum transport. Combining these two results yields an elementary estimate for the energy differential $$\langle \tilde{v}^2 \rangle - \langle \tilde{b}^2 \rangle \sim \left( \frac{k_{MR}}{k_0} \right)^4 \langle \tilde{b}^2 \rangle$$, whence $$\langle \tilde{b}^2 \rangle \sim \langle \tilde{v}^2 \rangle / \left(1 + k_{MR}^4 / k_0^4 \right)$$, where $$k_0$$ is a characteristic scale associated with the turbulence. A crude estimate is $$k_0 \simeq k_f$$.

The relationship between the spectra also implies that the turbulence must be primarily magnetic on small lengthscales and kinetic on large lengthscales, with the critical lengthscale separating these regimes being the magnetic Rhines scale. Ultimately, this is all a consequence of the fact that, in weak turbulence, the cross-correlator between the eigenmodes of the system must oscillate on a (fast) linear timescale. Another corollary of this fact is the vanishing of the time-averaged turbulent resistivity (in weak turbulence).

Our result relating the time-averaged spectra in weak turbulence still does not permit analytical solution to the spectral equations, since the amplitude of the cross-correlator oscillations is not easy to determine. The weak turbulence spectra equations are rendered challenging to work with, in part by the fact that turning on $$\beta$$ acts a singular perturbation, which in turn is the consequence of a class of Rossby-Alfvén wave interactions which were not admissible in pure MHD.

The present work suggests a number of avenues for future research. In the strong turbulence regime, the relationship between the spectra will break down; indeed, the very notion of linear Rossby-Alfvén modes ceases to be meaningful in strong turbulence. However, we still anticipate that cross-helicity could play an important role: the stationary cross-helicity is proportional to $$\langle \tilde{b}^2 \rangle / b_0$$, which may be very large in a strongly turbulent regime. In particular, there may be an effect on the flux of magnetic potential, which is essentially zero in weak turbulence, but has magnetic Reynolds number dependence [9–11] and intermittent spatial structure [12] when the mean field is weak. Future research should investigate the strong turbulence regime, and determine how the turbulent resistivity depends on $$\beta$$.

We have noted the general connection between cross-helicity and dynamo action, but not
addressed the turbulent emf for the present system. While the interaction of the planetary vorticity gradient and the mean field suffice to generate a cross-helicity, in the present setup, \( y \rightarrow -y \) reflection symmetry enforces \( \int d^2k \frac{k_x k_y}{k^2} \text{Re} H_k = 0 \), so there is no mean emf. However, this might change if this symmetry is broken, say by a component of the mean field along \( \hat{y} \) or an inhomogeneity (such as a shear flow or a magnetic field gradient). It would be interesting to study transport in such a system.

Of course, there is no dynamo in two dimensions, so any such study would constitute no more than a toy model for emf generation through cross-helicity. Thus, we suggest studying an extension of the system to three dimensions. Here, cross-helicity has a topological interpretation (see, for example, [19]) as a measure of the mutual linkage between flux tubes and vortex tubes, so the effect of cross helicity may be more significant than in 2D, where such an interpretation is not clearly meaningful. We also have in 3D [19]

\[
\langle (\tilde{v} \cdot \tilde{b})^2 \rangle + \langle (\tilde{v} \times \tilde{b})^2 \rangle = \langle (\tilde{v} b)^2 \rangle, \tag{33}
\]

which suggests that the cross-helicity may inhibit dynamo action.

**Appendix A: Derivation of weak turbulence spectral equations**

Begin by defining the mode amplitudes \( \hat{\phi}_k^\alpha = e^{i\omega_k^\alpha t} \phi_k^\alpha \). Equation (18) becomes

\[
\partial_t \hat{\phi}_k^\alpha = \frac{1}{2} \sum_{\beta \gamma} \sum_{k' + k'' = k} e^{i(\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma)t} M_k^{\alpha \beta \gamma} \hat{\phi}_{k'}^\beta \hat{\phi}_{k''}^\gamma. \tag{A1}
\]

We proceed via time-dependent perturbation theory, letting

\[
\hat{\phi}_k^\alpha(t) = \hat{\phi}_k^\alpha(0) + \delta \hat{\phi}_k^{\alpha,(1)}(t) + \delta \hat{\phi}_k^{\alpha,(2)}(t) + \ldots \tag{A2}
\]

Then

\[
\delta \hat{\phi}_k^{\alpha,(1)}(t) = \frac{1}{2} \sum_{k' + k'' = k} \sum_{\beta \gamma} M_k^{\alpha \beta \gamma} \hat{\phi}_{k'}^\beta \hat{\phi}_{k''}^\gamma(0) \int_0^t dt' e^{i(\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma)t'} \tag{A3}
\]
and
\[
\delta \hat{\phi}_{k}^{(2)}(t) = \frac{1}{2} \sum_{\beta\gamma\gamma'} \sum_{k'k''=k \atop q'q''=k} M_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} M_{k'k',q'q''}^{\alpha'\beta'\gamma'} \phi_{k'}^{(0)}(0) \phi_{q'}^{(0)}(0) \phi_{k''}^{(0)}(0) \phi_{q''}^{(0)}(0) \\
\times \int_{0}^{t} dt' e^{i(\omega_{k'}^{\beta} - \omega_{k''}^{\beta})t'} \int_{0}^{t'} dt'' e^{i(\omega_{q'}^{\beta} - \omega_{q''}^{\beta})t''},
\]
where we have combined terms by exchanging species indices and using the symmetry of the coupling coefficients.

We are interested in the evolution of \(C_{k}^{\alpha\alpha'}(t) \equiv \langle \hat{\phi}_{k}^{\alpha} \hat{\phi}_{k}^{\alpha'} \rangle \). Working to second order,
\[
\Delta C_{k}^{\alpha\alpha'} = C_{k}^{\alpha\alpha'}(t) - C_{k}^{\alpha\alpha'}(0) = \langle \delta \hat{\phi}_{k}^{\alpha}, \delta \hat{\phi}_{k}^{\alpha'} \rangle + \langle \hat{\phi}_{k}^{\alpha}, \hat{\phi}_{k}^{\alpha'} \rangle + \langle \hat{\phi}_{k}^{\alpha}, \delta \hat{\phi}_{k}^{\alpha'} \rangle + \langle \delta \hat{\phi}_{k}^{\alpha}, \hat{\phi}_{k}^{\alpha'} \rangle + \langle \delta \hat{\phi}_{k}^{\alpha}, \delta \hat{\phi}_{k}^{\alpha'} \rangle + \ldots,
\]
where we have anticipated that the first-order terms make no contribution.

Now,
\[
\Delta C_{k}^{\alpha\alpha' \text{(1)}} = \langle \delta \hat{\phi}_{k}^{\alpha}, \delta \hat{\phi}_{k}^{\alpha'} \rangle = \frac{1}{4} \sum_{\beta\gamma\gamma'} \sum_{k'k''=k \atop q'q''=k} M_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} M_{k'k',q'q''}^{\alpha'\beta'\gamma'} \langle \hat{\phi}_{k'}^{\beta}, \hat{\phi}_{k''}^{\beta'}(0) \hat{\phi}_{q'}^{\gamma}, \hat{\phi}_{q''}^{\gamma'}(0) \hat{\phi}_{k}^{\alpha}, \hat{\phi}_{k}^{\alpha'} \rangle \\
\times \int_{0}^{t} dt' \int_{0}^{t} dt'' e^{i(\omega_{k'}^{\beta} - \omega_{k''}^{\beta})t'} e^{i(\omega_{q'}^{\beta} - \omega_{q''}^{\beta})t''}.
\]

Next, we make the random phase approximation, which means we can apply Wick’s theorem to the four-mode functions, and assume spatial homogeneity, giving
\[
\langle \hat{\phi}_{k'}^{\beta}, \hat{\phi}_{k''}^{\beta'}(0) \hat{\phi}_{q'}^{\gamma}, \hat{\phi}_{q''}^{\gamma'}(0) \rangle = \langle \hat{\phi}_{k'}^{\beta}, \hat{\phi}_{k''}^{\beta'}(0) \rangle \langle \hat{\phi}_{k''}^{\beta'}, \hat{\phi}_{q'}^{\gamma}, \hat{\phi}_{q''}^{\gamma'}(0) \rangle \delta_{k'q'} \delta_{k''q''} + \langle \hat{\phi}_{k'}^{\beta}, \hat{\phi}_{k''}^{\beta'}(0) \rangle \langle \hat{\phi}_{k''}^{\beta'}, \hat{\phi}_{q''}^{\gamma}, \hat{\phi}_{q'}^{\gamma'}(0) \rangle \delta_{k'q''} \delta_{k''q'}
\]

and allowing us to make several simplifications.

We need to evaluate the integral
\[
I = \int_{0}^{t} dt' \int_{0}^{t} dt'' e^{i\Delta \omega t'} e^{-i\Delta \omega t''} = \frac{4e^{i(\Delta \omega - \Delta \omega')t/2} \sin(\Delta \omega t/2) \sin(\Delta \omega' t/2)}{\Delta \omega \Delta \omega'}
\]

(A8)
where $\Delta \omega = \omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma$ and $\Delta \omega' = \omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma$.

We seek the limit of this integral, in a distributional sense, as $t \to \infty$. We only keep terms linear in $t$; physically, we are interested in times $\omega^{-1} < t < \gamma^{-1}_{NL}$. The long-time limit vanishes unless $\Delta \omega = \Delta \omega' = 0$, but its value depends on whether we take the $\Delta \omega' \to \Delta \omega$ limit or $t \to \infty$ limit first. The sensible choice is the former, and we obtain

$$I(t \to \infty) \simeq 2\pi t \delta(\Delta \omega) \delta_{\Delta \omega, \Delta \omega'}.$$  
(A9)

The support of the Kronecker delta has measure zero in $(k', k'')$ space unless $\Delta \omega = \Delta \omega'$ identically, i.e. $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. Using these simplifications, we obtain

$$\Delta C^{\alpha \alpha'}_k(1) = \pi t \sum_{k' + k'' = k} \sum_{\beta \gamma} |M_{\beta \gamma}^{\alpha \alpha'}|^2 C_k^{\beta \gamma} C_{k'}^{\alpha \alpha'} \delta(\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma) \delta_{\alpha \alpha'},$$  
(A10)

and, applying similar reasoning to the other terms of $\Delta C^{\alpha \alpha'}_k$, we compute a second integral

$$\lim_{t \to \infty} \int_0^t dt' e^{i \Delta \omega t'} \int_0^{t'} dt'' e^{-i \Delta \omega t''} \simeq \left[ \frac{i t P}{\Delta \omega} + \pi t \delta(\Delta \omega) \right] \delta_{\Delta \omega, \Delta \omega'}$$  
(A11)

and find

$$\Delta C^{\alpha \alpha'}_k(2) = \{ \delta \phi_k^{(2)*} ; \phi_k^{(2)}(0) + \phi_k^{(2)}(0) \delta \phi_k^{(2)*} \}$$

$$= t \sum_{k' + k'' = k} \sum_{\beta \gamma} M_{\beta \gamma}^{\alpha \alpha'} M_{\beta \gamma}^{\alpha \alpha'} C_k^{\alpha \alpha'} C_{k'}^{\alpha \alpha'} \left( \pi \delta(\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma) + i P \frac{1}{\omega_k^\alpha - \omega_{k'}^\beta - \omega_{k''}^\gamma} \right)$$

$$+ \text{c.c.'}.$$  
(A12)

where c.c.' means the complex conjugate with $\alpha \leftrightarrow \alpha'$.

Finally, we obtain the claimed result by combining $\Delta C^{\alpha \alpha'}_k(1)$ with $\Delta C^{\alpha \alpha'}_k(2)$ and approximating

$$\partial_t C_k^{\alpha \alpha'} \simeq \frac{\Delta C_k^{\alpha \alpha'}}{t}.$$  

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[1] J. Schou, H. Antia, S. Basu, R. Bogart, R. Bush, S. Chitre, J. Christensen-Dalsgaard, M. Di Mauro, W. Dziembowski, A. Eff-Darwich, et al., The Astrophysical Journal 505, 390 (1998).
[2] P. Charbonneau, Living Reviews in Solar Physics 17, 1 (2020).
[3] M. S. Miesch, Living Reviews in Solar Physics 2, 1 (2005).
[4] D. W. Hughes, R. Rosner, and N. O. Weiss, eds., The solar tachocline (Cambridge University Press, 2007).
[5] E. Spiegel and J.-P. Zahn, Astronomy and Astrophysics 265, 106 (1992).
[6] D. Gough and M. McIntyre, Nature 394, 755 (1998).
[7] S. M. Tobias, P. H. Diamond, and D. W. Hughes, The Astrophysical Journal Letters 667, L113 (2007).
[8] P. H. Diamond, S.-I. Itoh, K. Itoh, and L. J. Silvers, in The solar tachocline, edited by D. Hughes, R. Rosner, and N. Weiss (Cambridge University Press, 2007) pp. 211–239.
[9] F. Cattaneo and S. I. Vainshtein, The Astrophysical Journal 376, L21 (1991).
[10] A. V. Gruzinov and P. H. Diamond, Physical review letters 72, 1651 (1994).
[11] A. V. Gruzinov and P. H. Diamond, Physics of Plasmas 3, 1853 (1996).
[12] X. Fan, P. H. Diamond, and L. Chacón, Physical Review E 99, 041201 (2019).
[13] J. Mak, S. D. Griffiths, and D. W. Hughes, Phys. Rev. Fluids 2, 113701 (2017).
[14] N. C. Constantinou and J. B. Parker, The Astrophysical Journal 863, 46 (2018).
[15] P. Garaud, in *Dynamics of the Sun and Stars* (Springer, 2020) pp. 207–220.

[16] C.-C. Chen and P. H. Diamond, The Astrophysical Journal **892**, 24 (2020).

[17] C. Guervilly, D. W. Hughes, and C. A. Jones, Physical Review E **91**, 041001 (2015).

[18] C. Guervilly, D. W. Hughes, and C. A. Jones, Journal of Fluid Mechanics **815**, 333 (2017).

[19] H. K. Moffatt, *Magnetic field generation in electrically conducting fluids* (Cambridge University Press, Cambridge-London-New York-Melbourne, 1978).

[20] Y. B. Zeldovich, Sov. Phys. JETP **4**, 460 (1957).

[21] A. Pouquet, Journal of Fluid Mechanics **88**, 1 (1978).

[22] R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory*, edited by T. M. O’Neil and D. L. Book (Benjamin, New York ad London, 1969).

[23] V. E. Zakharov, V. S. L’vov, and G. Falkovich, *Kolmogorov spectra of turbulence I: Wave turbulence* (Springer Science & Business Media, 2012).

[24] R. Grappin, J. Leorat, and A. Pouquet, Astronomy and Astrophysics **126**, 51 (1983).

[25] N. Tronko, S. V. Nazarenko, and S. Galtier, Physical Review E **87**, 033103 (2013).

[26] K. J. Burns, G. M. Vasil, J. S. Oishi, D. Lecoanet, and B. P. Brown, Physical Review Research **2**, 023068 (2020).

[27] R. H. Kraichnan, The Physics of Fluids **10**, 1417 (1967).

[28] J. Towns, T. Cockerill, M. Dahan, I. Foster, K. Gaither, A. Grimshaw, V. Hazlewood, S. Lathrop, D. Lifka, G. D. Peterson, *et al.*, Computing in Science & Engineering **16**, 62 (2014).