A degree sum condition on the order, the connectivity and the independence number for Hamiltonicity

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Abstract

In [Graphs Combin. 24 (2008) 469–483.], the third author and the fifth author conjectured that if $G$ is a $k$-connected graph such that $\sigma_{k+1}(G) \geq |V(G)| + \kappa(G) + (k - 2)(\alpha(G) - 1)$, then $G$ contains a Hamiltonian cycle, where $\sigma_{k+1}(G)$, $\kappa(G)$ and $\alpha(G)$ are the minimum degree sum of $k + 1$ independent vertices, the connectivity and the independence number of $G$, respectively. In

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this paper, we settle this conjecture. This is an improvement of the result obtained by Li: If \( G \) is a \( k \)-connected graph such that \( \sigma_{k+1}(G) \geq |V(G)| + (k - 1)(\alpha(G) - 1) \), then \( G \) is Hamiltonian. The degree sum condition is best possible.

1 Introduction

1.1 Degree sum condition for graphs with high connectivity to be Hamiltonian

In this paper, we consider only finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained, we refer the reader to [5].

A Hamiltonian cycle of a graph is a cycle containing all the vertices of the graph. A graph having a Hamiltonian cycle is called a Hamiltonian graph. The Hamiltonian problem has long been fundamental in graph theory. Since it is NP-complete, no easily verifiable necessary and sufficient condition seems to exist. Then instead of that, many researchers have investigated sufficient conditions for a graph to be Hamiltonian. In this paper, we deal with a degree sum type condition, which is one of the main stream of this study.

We introduce four invariants, including degree sum, which play important roles for the existence of a Hamiltonian cycle. Let \( G \) be a graph. The number of vertices of \( G \) is called its order, denoted by \( n(G) \). A set \( X \) of vertices in \( G \) is called an independent set in \( G \) if no two vertices of \( X \) are adjacent in \( G \). The independence number of \( G \) is defined by the maximum cardinality of an independent set in \( G \), denoted by \( \alpha(G) \).

For two distinct vertices \( x, y \in V(G) \), the local connectivity \( \kappa_G(x, y) \) is defined to be the maximum number of internally-disjoint paths connecting \( x \) and \( y \) in \( G \). A graph \( G \) is \( k \)-connected if \( \kappa_G(x, y) \geq k \) for any two distinct vertices \( x, y \in V(G) \). The connectivity \( \kappa(G) \) of \( G \) is the maximum value of \( k \) for which \( G \) is \( k \)-connected.

We denote by \( N_G(x) \) and \( d_G(x) \) the neighbor and the degree of a vertex \( x \) in \( G \), respectively. If \( \alpha(G) \geq k \), let

\[
\sigma_k(G) = \min \left\{ \sum_{x \in X} d_G(x) : X \text{ is an independent set in } G \text{ with } |X| = k \right\};
\]

otherwise let \( \sigma_k(G) = +\infty \). If the graph \( G \) is clear from the context, we simply write \( n, \alpha, \kappa \) and \( \sigma_k \) instead of \( n(G), \alpha(G), \kappa(G) \) and \( \sigma_k(G) \), respectively.

One of the main streams of the study of the Hamiltonian problem is, as mentioned above, to consider degree sum type sufficient conditions for graphs to have a Hamiltonian cycle. We list some of them below. (Each of the conditions is best possible in some sense.)
Theorem 1. Let $G$ be a graph of order at least three. If $G$ satisfies one of the following, then $G$ is Hamiltonian.

(i) (Dirac [7]) The minimum degree of $G$ is at least $\frac{n}{2}$.

(ii) (Ore [12]) $\sigma_2 \geq n$.

(iii) (Chvátal and Erdős [6]) $\alpha \leq \kappa$.

(iv) (Bondy [4]) $G$ is $k$-connected and $\sigma_{k+1} > \frac{(k+1)(n-1)}{2}$.

(v) (Bauer, Broersma, Veldman and Li [2]) $G$ is 2-connected and $\sigma_3 \geq n + \kappa$.

To be exact, Theorem 1 (iii) is not a degree sum type condition, but it is closely related. Bondy [3] showed that Theorem 1 (iii) implies (ii). The current research of this area is based on Theorem 1 (iii). Let us explain how to expand the research from Theorem 1 (iii): Let $G$ be a $k$-connected graph, and suppose that one wants to consider whether $G$ is Hamiltonian. If $\alpha \leq k$, then it follows from Theorem 1 (iii) that $G$ is Hamiltonian. Hence we may assume that $\alpha \geq k + 1$, that is, $G$ has an independent set of order $k + 1$. Thus, it is natural to consider a $\sigma_{k+1}$ condition for a $k$-connected graph. Bondy [4] gave a $\sigma_{k+1}$ condition of Theorem 1 (iv).

In this paper, we give a much weaker $\sigma_{k+1}$ condition than that of Theorem 1 (iv).

Theorem 2. Let $k$ be an integer with $k \geq 1$ and let $G$ be a $k$-connected graph. If

$$\sigma_{k+1} \geq n + \kappa + (k-2)(\alpha-1),$$

then $G$ is Hamiltonian.

Theorem 2 was conjectured by Ozeki and Yamashita [15], and has been proven for small integers $k$: The case $k = 2$ of Theorem 2 coincides Theorem 1 (v). The cases $k = 1$ and $k = 3$ were shown by Fraisse and Jung [8], and by Ozeki and Yamashita [15], respectively.

1.2 Best possibility of Theorem 2

In this section, we show that the $\sigma_{k+1}$ condition in Theorem 2 is best possible in some senses.

We first discuss the lower bound of the $\sigma_{k+1}$ condition. For an integer $l \geq 2$ and $l$ vertex-disjoint graphs $H_1, \ldots, H_l$, we define the graph $H_1 + \cdots + H_l$ from the union of $H_1, \ldots, H_l$ by joining every vertex of $H_i$ to every vertex of $H_{i+1}$ for $1 \leq i \leq l-1$. Fix an integer $k \geq 1$. Let $\kappa, m$ and $n$ be integers with $k \leq \kappa < m$ and $2m + 1 \leq n \leq 3m - \kappa$. Let $G_1 = K_{n-2m} + K_\kappa + K_m + K_{m-\kappa}$, where $K_i$
denotes a complete graph of order \( l \) and \( \overline{K}_l \) denotes the complement of \( K_l \). Then
\[
\alpha(G_1) = m + 1, \quad \kappa(G_1) = \kappa \quad \text{and} \quad \sigma_{k+1}(G_1) = n(G_1) + \kappa(G_1) + (k - 2)(\alpha(G_1) - 1) - 1.
\]
(Note that it follows from condition “\( n \leq 3m - \kappa \)” that \( n - 2m - 1 + \kappa < m \).) Since deleting all the vertices in \( \overline{K}_\kappa \) and those in \( \overline{K}_{m-\kappa} \) breaks \( G_1 \) into \( m + 1 \) components, we see that \( G_1 \) has no Hamiltonian cycle. Therefore, the \( \sigma_{k+1} \) condition in Theorem 2 is best possible.

We next discuss the relation between the coefficient of \( \kappa \) and that of \( \alpha - 1 \). By Theorem 1 (iii), we may assume that \( \alpha \geq \kappa + 1 \). This implies that
\[
\alpha(G_1) = \frac{n + 1}{2} \geq \frac{n + m + \kappa + (k - 2)(\alpha - 1)}{2}.
\]
for arbitrarily \( \varepsilon > 0 \). Then one may expect that the \( \sigma_{k+1} \) condition in Theorem 2 can be replaced with “\( n + (1 + \varepsilon)\kappa + (k - 2 - \varepsilon)(\alpha - 1) \)” for some \( \varepsilon > 0 \). However, the graph \( G_1 \) as defined above shows that it is not true: For any \( \varepsilon > 0 \), there exist two integers \( m \) and \( \kappa \) such that \( \varepsilon(m - \kappa) \geq 1 \). If we construct the above graph \( G_1 \) from such integers \( m \) and \( \kappa \), then we have
\[
\sigma_{k+1}(G_1) = n + \kappa + (k - 2)m - 1
= n + (1 + \varepsilon)\kappa + (k - 2 - \varepsilon)m - 1 + \varepsilon(m - \kappa)
\geq n(G_1) + (1 + \varepsilon)\kappa(G_1) + (k - 2 - \varepsilon)(\alpha(G_1) - 1),
\]
but \( G_1 \) is not Hamiltonian. This means that the coefficient 1 of \( \kappa \) and the coefficient \( k - 2 \) of \( \alpha - 1 \) are, in a sense, best possible.

1.3 Comparing Theorem 2 to other results

In this section, we compare Theorem 2 to Theorem 1 (iv) and Ota’s result (Theorem 3).

We first show that the \( \sigma_{k+1} \) condition of Theorem 2 is weaker than that of Theorem 1 (iv). Let \( G \) be a \( k \)-connected graph satisfying the \( \sigma_{k+1} \) condition of Theorem 1 (iv). Assume that \( \alpha \geq (n + 1)/2 \). Let \( X \) be an independent set of order at least \( (n + 1)/2 \). Then \( |V(G) \setminus X| \leq (n - 1)/2 \) and \( |V(G) \setminus X| \geq k \) since \( V(G) \setminus X \) is a cut set. Hence \( (n + 1)/2 \geq k + 1 \), and we can take a subset \( Y \) of \( X \) with \( |Y| = k + 1 \). Then \( N_G(Y) \subseteq V(G) \setminus X \) for \( y \in Y \), and hence \( \sum_{y \in Y} d_G(y) \leq (k + 1)|V(G) \setminus X| \leq (k + 1)(n - 1)/2 \). This contradicts the \( \sigma_{k+1} \) condition of Theorem 1 (iv). Therefore \( n/2 \geq \alpha \). Moreover, by Theorem 1 (iii), we
may assume that $\alpha \geq \kappa + 1$. Therefore, the following inequality holds:

$$
\sigma_{k+1} > \frac{(k+1)(n-1)}{2} \\
= n - 1 + \frac{(k-1)(n-1)}{2} \\
\geq n - 1 + \frac{(k-1)(2\alpha - 1)}{2} \\
\geq n - 1 + (k-1)(\alpha - 1) \\
\geq n + \kappa + (k-2)(\alpha - 1) - 1.
$$

Thus, the $\sigma_{k+1}$ condition of Theorem 1 (iv) implies that of Theorem 2.

We next compare Theorem 2 to the following Ota’s result.

**Theorem 3** (Ota [13]). Let $G$ be a 2-connected graph. If $\sigma_{l+1} \geq n + l(l - 1)$ for all integers $l$ with $l \geq \kappa$, then $G$ is Hamiltonian.

We first mention about the reason to compare Theorem 2 to Theorem 3. Li [10] proved the following theorem, which was conjectured by Li, Tian, and Xu [11]. (Harkat-Benhamadine, Li and Tian [9], and Li, Tian, and Xu [11] have already proven the case $k = 3$ and the case $k = 4$, respectively.)

**Theorem 4** (Li [10]). Let $k$ be an integer with $k \geq 1$ and let $G$ be a $k$-connected graph. If $\sigma_{k+1} \geq n + (k-1)(\alpha - 1)$, then $G$ is Hamiltonian.

In fact, Li showed Theorem 4 just as a corollary of Theorem 3. Note that Theorem 2 is, assuming Theorem 1 (iii), an improvement of Theorem 4. Therefore we should show that Theorem 2 cannot be implied by Theorem 3. (Ozeki, in his Doctoral Thesis [14], compared the relation between several theorems, including Theorem 1 (i), (ii), (iii) and (v), the case $k = 3$ of Theorems 2 and 4, and Theorem 3.)

Let $\kappa, r, k, m$ be integers such that $4 \leq r$, $3 \leq k \leq \kappa - 2$ and $m = (k+1)(r-2)+4$. Let $G_2 = K_1 + \overline{K}_\kappa + K_{\kappa+m-r} + (\overline{K}_m + K_r)$. Then $n(G_2) = 2k + 2m + 1$, $\kappa(G_2) = \kappa$ and $\alpha(G_2) = \kappa + m$. Since

$$
\kappa + k(\kappa + m) - (k+1)(\kappa + m - r + 1) = (k+1)(r-1) - m \\
= (k+1)(r-1) - (k+1)(r-2) - 4 \\
= k - 3 \\
\geq 0,
$$

it follows that

$$
\sigma_{k+1}(G_2) = \min \{ \kappa + k(\kappa + m), (k+1)(\kappa + m - r + 1) \} \\
= \kappa + k(\kappa + m) - (k-3) \\
= (2k + 2m + 1) + \kappa + (k-2)(\kappa + m - 1) \\
= n(G_2) + \kappa(G_2) + (k-2)(\alpha(G_2) - 1).
$$
Hence the assumption of Theorem 2 holds. On the other hand, for \( l = \alpha(G_2) - 1 = \kappa + m - 1 \), we have

\[
\begin{align*}
n(G_2) + l(l - 1) - \sigma_{l+1}(G_2) &= (2\kappa + 2m + 1) + (\kappa + m - 1)(\kappa + m - 2) \\
& \quad - \{\kappa(\kappa + m - r + 1) + m(\kappa + m)\} \\
& = \kappa(r - 2) - m + 3 \\
& = \kappa(r - 2) - (k + 1)(r - 2) - 4 + 3 \\
& = (\kappa - k - 1)(r - 2) - 1 \\
& \geq (r - 2) - 1 \\
& > 0.
\end{align*}
\]

Hence the assumption of Theorem 3 does not hold. These yield that for the graph \( G_2 \), we can apply Theorem 2 but cannot apply Theorem 3.

## 2 Notation and lemmas

Let \( G \) be a graph and \( H \) be a subgraph of \( G \), and let \( x \in V(G) \) and \( X \subseteq V(G) \). We denote by \( N_G(X) \) the set of vertices in \( V(G) \backslash X \) which are adjacent to some vertex in \( X \). We define \( N_H(x) = N_G(x) \cap V(H) \) and \( d_H(x) = |N_H(x)| \). Furthermore, we define \( N_H(X) = N_G(X) \cap V(H) \). If there is no fear of confusion, we often identify \( H \) with its vertex set \( V(H) \). For example, we often write \( G - H \) instead of \( G - V(H) \). For a subgraph \( H \), a path \( P \) is called an \( H \)-path if both end vertices of \( P \) are contained in \( H \) and all internal vertices are not contained in \( H \). Note that each edge of \( H \) is an \( H \)-path.

Let \( C \) be a cycle (or a path) with a fixed orientation in a graph \( G \). For \( x, y \in V(C) \), we denote by \( C[x, y] \) the path from \( x \) to \( y \) along the orientation of \( C \). The reverse sequence of \( C[x, y] \) is denoted by \( \overrightarrow{C[y, x]} \). We denote \( C[x, y] \setminus \{x, y\} \), \( C[x, y] \setminus \{x\} \) and \( C[x, y] \setminus \{y\} \) by \( C(x, y) \), \( C(x, y) \) and \( C(x, y) \), respectively. For \( x \in V(C) \), we denote the successor and the predecessor of \( x \) on \( C \) by \( x^+ \) and \( x^- \), respectively. For \( X \subseteq V(C) \), we define \( X^+ = \{x^+ : x \in X\} \) and \( X^- = \{x^- : x \in X\} \). Throughout this paper, we consider that every cycle has a fixed orientation.

In this paper, we extend the concept of insertible, introduced by Ainouche [1], which has been used for the proofs of the results on cycles.

Let \( G \) be a graph, and \( H \) be a subgraph of \( G \). Let \( X(H) = \{u \in V(G - H) : uv_1, uv_2 \in E(G) \text{ for some } v_1, v_2 \in E(H)\} \), let \( I(x; H) = \{v_1, v_2 \in E(H) : xv_1, xv_2 \in E(G)\} \) for \( x \in V(G - H) \), and let \( Y(H) = \{u \in V(G - H) : d_H(u) \geq \alpha(G)\} \).

**Lemma 1.** Let \( D \) be a cycle of a graph \( G \). Let \( k \) be a positive integer and let \( Q_1, Q_2, \ldots, Q_k \) be paths of \( G - D \) with fixed orientations such that \( V(Q_i) \cap V(Q_j) = \emptyset \) for \( 1 \leq i < j \leq k \). If the following (I) and (II) hold, then \( G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)] \) is Hamiltonian.
Proof. We can easily see that $V$ in a graph $G$ contains all vertices of $X$ such that $I_C^E$ can obtain a cycle $D_i$ such that $D_i$. We may assume $I_C^E$ contains a cycle $D^*$ such that $V(D) \cup X(D) \cap V(Q_i) \subseteq V(D^*)$. In fact, we can insert all vertices of $X(D) \cap V(Q_i)$ into $D$ by choosing the following $u_i, v_i \in V(Q_i)$ and $w_{i,1}^+ \in E(D)$ inductively. Take the first vertex $u_i$ in $X(D) \cap V(Q_i)$ along the orientation of $Q_1$, and let $v_i$ be the last vertex in $X(D) \cap V(Q_i)$ on $Q_1$ such that $I'(u_i; D) \cap I(v_i; D) \neq \emptyset$. Then we can insert all vertices of $Q_i[u_i,v_i]$ into $D$. To be exact, taking $w_{i,1}^+ \in I'(u_i; D) \cap I(v_i; D)$, $D^+_i := w_{i,1}[u_i,v_i]D[w_{i,1}^+, w_i]$ is such a cycle. By the choice of $u_i$ and $v_i$, $w_{i,1}^+ \notin I(x; D)$ for all $x \in V(Q_i - Q)[u_i,v_i]$, and $X(D) \cap V(Q_i - Q)[u_i,v_i]$ is contained in some component of $Q_i - Q[u_i,v_i]$. Moreover, note that $E(D) \setminus \{w_{i,1}^+\} \subseteq E(D^+_i)$. Hence by repeating this argument, we can obtain a cycle $D^+_i$ of $G[V(D \cup Q_i)]$ such that $V(D) \cup (X(D) \cap V(Q_i)) \subseteq V(D^+_i)$ and $E(D) \setminus \bigcup_{x \in V(Q_i)} I(x; D) \subseteq E(D^+_i)$. Then by (II), $I(x; D) \subseteq E(D^+_i)$ for all $x \in V(Q_2 \cup \cdots \cup Q_k)$. Therefore $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ contains a cycle $D^*$ such that $V(D) \cup (X(D) \cap V(Q_1 \cup Q_2 \cup \cdots \cup Q_k)) \subseteq V(D^*)$.

We choose a cycle $C$ of $G[V(D \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k)]$ containing all vertices in $V(D) \cup (X(D) \cap V(Q_1 \cup Q_2 \cup \cdots \cup Q_k))$ so that $|C|$ is as large as possible. Now, we change the “base” cycle from $D$ to $C$, and use the symbol $(\cdot)^+$ for the orientation of $C$. Suppose that $V(Q_i - C) \neq \emptyset$ for some $i$ with $i \in \{1, 2, \ldots, k\}$. We may assume that $i = 1$. Let $w$ be the last vertex in $V(Q_1 - C)$ along $Q_1$. Since $C$ contains all vertices in $(X(D) \cap V(Q_1))$, it follows from (I) that $w \in Y(Q_1(w, b_1) \cup D)$, that is, $|N_G(w) \cap V(Q_1(w, b_1) \cup D)| \geq \alpha(G)$. By the choice of $w$, we obtain $V(Q_1(w, b_1) \cup D) \subseteq V(C)$. Therefore $|N_C(w)^+ \cap \{w\}| \geq |N_G(w) \cap V(Q_1(w, b_1) \cup D)| + 1 \geq \alpha(G) + 1$. This implies that $N_C(w)^+ \cap \{w\}$ is not an independent set in $G$. Hence $wz^+ \in E(G)$ for some $z \in N_C(w)$. Then $C'' = wC[z^+, z]w$ is a cycle of $G[V(D \cup Q_1 \cup Q_2 \cup Q_k)]$ such that $V(C) \cup \{w\} \cup C''$, which contradicts the choice of $C$. Thus $V(Q_1 \cup Q_2 \cup Q_k)$ are contained in $C$, and hence $C$ is a Hamiltonian cycle of $G[V(D \cup Q_1 \cup Q_2 \cup Q_k)]$. 

In the rest of this section, we fixed the following notation. Let $C$ be a longest cycle in a graph $G$, and $H_0$ be a component of $G - C$. For $u \in N_C(H_0)$, let $u' \in N_C(H_0)$ be a vertex such that $C(u, u') \cap N_C(H_0) = \emptyset$, that is, $u'$ is the successor of $u$ in $N_C(H_0)$ along the orientation of $C$.

For $u \in N_C(H_0)$, a vertex $v \in C(u, u')$ is insertible if $v \in X(C[w', u]) \cup Y(C(v, u])$. A vertex in $C(u, u')$ is said to be non-insertible if it is not insertible.

Lemma 2. There exists a non-insertible vertex in $C(u, u')$ for $u \in N_C(H_0)$.  

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Proof. Let \( u \in N_C(H_0) \), and suppose that every vertex in \( C(u, u') \) is insertible. Let \( P \) be a \( C \)-path joining \( u \) and \( u' \) with \( V(P) \cap V(H_0) \neq \emptyset \). Let \( D = C[u', u]P[u, u'] \) and \( Q = C(u, u') \). Let \( v \in V(Q) \). Since \( v \) is insertible, it follows that \( v \in X(C[u', u]) \cup Y(C(v, u)) \). Since \( C[u', u] \) is a subpath of \( D \), we have \( v \in X(D) \cup Y(Q(v, u') \cup D) \). Hence, by Lemma 1, \( G[V(D \cup Q)] \) is Hamiltonian, which contradicts the maximality of \( C \).

Figure 1: Lemma 3

Lemma 3. Let \( u_1, u_2 \in N_C(H_0) \) with \( u_1 \neq u_2 \), and let \( x_i \) be the first non-insertible vertex along \( C(u_i, u'_i) \) for \( i \in \{1, 2\} \). Then the following hold (see Figure 7).

(i) There exists no \( C \)-path joining \( v_1 \in C(u_1, x_1) \) and \( v_2 \in C(u_2, x_2) \). In particular, \( x_1x_2 \not\in E(G) \).

(ii) If there exists a \( C \)-path joining \( v_1 \in C(u_1, x_1) \) and \( w \in C(v_1, u_2) \), then there exists no \( C \)-path joining \( v_2 \in C(u_2, x_2) \) and \( w^+ \).

(iii) If there exist a \( C \)-path joining \( v_1 \in C(u_1, x_1) \) and \( w_1 \in C(v_1, u_2) \) and a \( C \)-path joining \( v_2 \in C(u_2, x_2) \) and \( w_2 \in C[w_1, u_2] \), then there exists no \( C \)-path joining \( w_1^- \) and \( w_2^+ \).
Proof. Let $P_0$ be a $C$-path which connects $u_1$ and $u_2$, and $V(P_0) \cap V(H_0) \neq \emptyset$. We first show (i) and (ii). Suppose that the following (a) or (b) holds for some \((1)\) or \((2)\):

- For each \(i \in \{1, 2\}\), there exists a $C$-path joining $v_i \in C(u_i, x_i)$ and $w_i \in C(v_i, u_{3-i})$, then there exists no $C$-path joining $w_1^-$ and $w_2^-$.

Let $l \in \{1, 2\}$ and $v_l \in C(v_l, u_{3-l})$. We choose such vertices $v_1$ and $v_2$ so that $|C[u_1, v_1]| + |C[u_2, v_2]|$ is as small as possible. Without loss of generality, we may assume that $l = 1$ if (b) holds. Since $N_C(H_0) \cap \{v_1, v_2\} = \emptyset$, $(V(P_1) \cup V(P_2) \cup V(P_3)) \cap V(P_0) = \emptyset$. Therefore, we can define a cycle

$$D = \begin{cases} P_1[v_1, v_2]C[v_2, u_1]P_0[u_1, u_2]C[u_2, v_1] & \text{if (a) holds}, \\ P_2[v_1, w]C[w, u_2]P_0[u_2, u_1]C[u_1, v_2]P_3[v_2, w^-]C[w^-, v_1] & \text{otherwise}. \end{cases}$$

For $i \in \{1, 2\}$, let $Q_i = C(u_i, v_i)$. By Lemma 2, we can obtain the following statements (1), and by the choice of $v_1$ and $v_2$, we can obtain the following statements (2)–(5):

1. $N_G(x) \cap P_0(u_1, u_2) = \emptyset$ for $x \in V(Q_1 \cup Q_2)$.
2. $N_G(x) \cap (P_1(v_1, v_2) \cup P_2(v_1, w) \cup P_3(v_2, w^-)) = \emptyset$ for $x \in V(Q_1 \cup Q_2)$.
3. $xy \notin E(G)$ for $x \in V(Q_1)$ and $y \in V(Q_2)$.
4. $I(x; C) \cap I(y; C) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$.
5. If (b) holds, then $w^-w \notin I(x; C)$ for $x \in V(Q_1 \cup Q_2)$.

Let $a \in V(Q_i)$ for some $i \in \{1, 2\}$. Note that each vertex of $Q_i$ is insertible, that is, $a \in X(C[u_i', u_i]) \cup Y(C(a, u_i))$. We show that $a \in X(D) \cup Y(Q_i(a, v_i) \cup D)$. If $a \in X(C[u_i', u_i])$, then the statements (3) and (5) yield that $a \in X(D)$. Suppose that $a \in Y(C(a, u_i))$. By (3), $N_G(a) \cap C(a, u_i) \subseteq N_G(a) \cap Q_i(a, v_i) \cup D$. This implies that $a \in Y(Q_i(a, v_i) \cup D)$. By (1), (2) and (4), $I(x; D) \cap I(y; D) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. Thus, by Lemma 3, $G[V(D \cup Q_1 \cup Q_2)]$ is Hamiltonian, which contradicts the maximality of $C$.

By using similar argument as above, we can also show (iii) and (iv). We only prove (iii). Suppose that for some $v_1 \in C(u_1, x_1)$ and $v_2 \in C(u_2, x_2)$, there exist disjoint $C$-paths $P_1[v_1, w_1], P_2[v_2, w_2]$ and $P_3[w_1^-, w_2^+]$ with $w_1 \in C(v_1, u_2)$ and $w_2 \in C(v_2, u_1)$. We choose such $v_1$ and $v_2$ so that $|C[u_1, v_1]| + |C[u_2, v_2]|$ is as small as possible. Let $Q_i = C(u_i, v_i)$ for $i \in \{1, 2\}$. Then by Lemma 3, $xy \notin E(G)$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. By the choice of $v_1$ and $v_2$ and Lemma 3, $w_1w_1^-, w_2w_2^+ \notin I(x; C[v_1, u_1]) \cup I(y; C[v_2, u_2])$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. By
Lemma 3 (i) and (ii), $I(x; C[v_1, u_2] ∪ C[v_2, u_1]) \cap I(y; C[v_1, u_2] ∪ C[v_2, u_1]) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. Hence by applying Lemma 1 as

$$D = P_1[v_1, w_1|C[w_1, w_2]|P_2[w_2, v_2]C[v_2, u_1]P_0[u_1, u_2]|C[w_2, w_1^+]P_3[w_1^+, w_1^-]C[w_1^-, v_1],$$

$Q_1$ and $Q_2$, we see that there exists a longer cycle than $C$, a contradiction. □

3 Proof of Theorem 2

Proof of Theorem 2. The cases $k = 1$, $k = 2$ and $k = 3$ were shown by Fraisse and Jung 8, by Bauer et al. 2 and by Ozeki and Yamashita 15, respectively. Therefore, we may assume that $k \geq 4$. Let $G$ be a graph satisfying the assumption of Theorem 2. By Theorem 1 (iii), we may assume $\alpha(G) \geq \kappa(G) + 1$. Let $C$ be a longest cycle in $G$. If $C$ is a Hamiltonian cycle of $G$, then there is nothing to prove. Hence we may assume that $G - V(C) \neq \emptyset$. Let $H = G - V(C)$ and $x_0 \in V(H)$. Choose a longest cycle $C$ and $x_0$ so that

$$d_C(x_0)$$

is as large as possible.

Let $H_0$ be the component of $H$ such that $x_0 \in V(H_0)$. Let

$$N_C(H_0) = \{u_1, u_2, \ldots, u_m\}.$$

Note that $m \geq \kappa(G) \geq k$. Let

$$M_0 = \{0, 1, \ldots, m\} \text{ and } M_1 = \{1, 2, \ldots, m\}.$$

Let $u'_i$ be the vertex in $N_C(H_0)$ such that $C(u_i, u'_i) \cap N_C(H_0) = \emptyset$. By Lemma 2 there exists a non-insertible vertex in $C(u_i, u'_i)$. Let $x_i \in C(u_i, u'_i)$ be the first non-insertible vertex along the orientation of $C$ for each $i \in M_1$, and let

$$X = \{x_1, x_2, \ldots, x_m\}.$$

Note that $d_C(x_0) \leq |U| = |X|$. Let

$$D_i = C(u_i, x_i) \text{ for each } i \in M_1 \text{, and } D = \bigcup_{i \in M_1} D_i.$$

We check the degree of $x_i$ in $C$ and $H$. Since $x_i$ is non-insertible, we can see that

$$d_C(x_i) \leq |D_i| + \alpha(G) - 1 \text{ for } i \in M_1. \quad (1)$$

By the definition of $x_i$, we clearly have $N_{H_0}(x_i) = \emptyset$ for $i \in M_1$. Moreover, by Lemma 3 (i), $N_H(x_i) \cap N_H(x_j) = \emptyset$ for $i, j \in M_1$ with $i \neq j$. Thus we obtain

$$\sum_{i \in M_0} d_H(x_i) \leq |H| - 1, \quad (2)$$

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and
\[ \sum_{i \in M_1} d_H(x_i) \leq |H| - |H_0|. \tag{3} \]

We check the degree sum in \( C \) of two vertices in \( X \). Let \( i \) and \( j \) be distinct two integers in \( M_1 \). In this paragraph, we let \( C_i = C[x_i, u_j] \) and \( C_j = C[x_j, u_i] \). By Lemma \[ \text{(i)}, \] we have \( N_{C_i}(x_i) \cap N_{C_j}(x_j) = \emptyset \) and \( N_{C_j}(x_j) \cap N_{C_i}(x_i) = \emptyset \). By Lemma \[ \text{(i)}, \] \( N_{C_i}(x_i) \cup N_{C_j}(x_j) \subseteq C_i \setminus D \), \( N_{C_j}(x_j) \cup N_{C_i}(x_i) \subseteq C_j \setminus D \) and \( N_{D_i}(x_j) = N_{D_j}(x_i) = \emptyset \). Thus, we obtain
\[ d_{C_i}(x_i) + d_{C_j}(x_j) \leq |C| - \sum_{h \in M_1 \setminus \{i, j\}} |D_h| \quad \text{for} \ i, j \in M_1 \text{ with } i \neq j. \tag{4} \]

By Lemma \[ \text{(i)}, \] and since \( N_{H_0}(x_i) = \emptyset \) for \( i \in M_1 \), we obtain the following.

**Claim 1.** \( X \cup \{x_0\} \) is an independent set, and hence \( |X| \leq \alpha(G) - 1 \).

**Claim 2.** \( |X| \geq \kappa(G) + 1 \).

**Proof.** Let \( s \) and \( t \) be distinct two integers in \( M_1 \). By the inequality \[ (1), \] we have
\[ d_{C}(x_s) + d_{C}(x_t) \leq |C| - \sum_{i \in M_1 \setminus \{s, t\}} |D_i|. \]

Let \( I \) be a subset of \( M_0 \) such that \( |I| = k + 1 \) and \( \{0, s, t\} \subseteq I \). By Claim \[ (1), \] \( \{x_i : i \in I\} \) is an independent set. By the inequality \[ (1), \] we deduce
\[ \sum_{i \in I \setminus \{0, s, t\}} d_{C}(x_i) \leq \sum_{i \in I \setminus \{0, s, t\}} |D_i| + (k - 2)(\alpha(G) - 1). \]

By the inequality \[ (2), \] and the definition of \( I \), we obtain
\[ \sum_{i \in I} d_{H}(x_i) \leq |H| - 1. \]
Thus, it follows from these three inequalities that
\[ \sum_{i \in I} d_{G}(x_i) \leq n + (k - 2)(\alpha(G) - 1) - 1 + d_{C}(x_0). \]

Since \( \sigma_{k+1}(G) \geq n + \kappa(G) + (k - 2)(\alpha(G) - 1) \), we have \( |X| \geq d_{C}(x_0) \geq \kappa(G) + 1 \). \[ \square \]

Let \( S \) be a cut set with \( |S| = \kappa(G) \), and let \( V_1, V_2, \ldots, V_p \) be the components of \( G \setminus S \). By Claim \[ (2), \] we may assume that
there exists an integer \( l \) such that \( C[u_l, u_l'] \subseteq V_1 \).

By Lemma \[ (3), \] \( (i), \) we obtain
\[ d_{C}(x_l) \leq |C \cap (V_1 \cup S)| - |(\bigcup_{i \in M_1 \setminus \{l\}} D_i \cup X) \cap (V_1 \cup S)|. \tag{5} \]
By replacing the labels $x_2$ and $x_3$ if necessary, we may assume that $x_1$, $x_2$ and $x_3$ appear in this order along the orientation of $C$. In this paragraph, the indices are taken modulo 3. From now we let

$$C_i = C[x_i, u_{i+1}]$$

and

$$W_i := \{ w \in V(C_i) : w^+ \in N_{C_i}(x_i) \text{ and } w^- \in N_{C_i}(x_{i+1}) \}$$

for each $i \in \{1, 2, 3\}$, and let $W := W_1 \cup W_2 \cup W_3$ (see Figure 2). Note that $W \cap (U \cup \{x_1, x_2, x_3\}) = \emptyset$, by the definition of $C_i$ and $W_i$ and by Lemma 3 (i).

![Figure 2: The definition of $W$.](image)

**Claim 3.** $D \cup X \cup W \cup H \subseteq V_1 \cup S$. In particular, $x_0 \in V_1 \cup S$.

**Proof.** We first show that $D \cup X \cup W \subseteq V_1 \cup S$. Suppose not. Without loss of generality, we may assume that there exists an integer $h$ in $M_1 \setminus \{l\}$ such that $(D_h \cup \{x_h\} \cup (W \cap C(x_h, u'_h))) \cap V_2 \neq \emptyset$, say $v \in (D_h \cup \{x_h\} \cup (W \cap C(x_h, u'_h))) \cap V_2$. Since $v \in V_2$, it follows from Lemma 3 (i) and (ii) that

$$d_C(v) \leq |C \cap (V_2 \cup S)| - |(\bigcup_{i \in M_1 \setminus \{h\}} D_i \cup X) \cap (V_2 \cup S)|.$$

Let $I$ be a subset of $M_0 \setminus \{h\}$ such that $|I| = k$ and $\{0, I\} \subseteq I$. By Claim [□] and Lemma 3 (i) and (ii), $\{x_i : i \in I\} \cup \{v\}$ is an independent set of order $k + 1$. By the
above inequality and the inequality (5), we obtain
\[
d_C(x_i) + d_C(v) \\
\leq |C \cap (V_1 \cup V_2 \cup S)| + |C \cap S| - |( \bigcup_{i \in M_1 \setminus \{l, h\}} D_i \cup X) \cap (V_1 \cup V_2 \cup S)| \\
= |C| + |C \cap S| - |C \cap (\bigcup_{1 \leq j \leq p} V_j)| - |( \bigcup_{i \in M_1 \setminus \{l, h\}} D_i \cup X) \cap (\bigcup_{1 \leq j \leq p} V_j)| \\
\leq |C| + |C \cap S| - |( \bigcup_{i \in M_1 \setminus \{l, h\}} D_i \cup X) \cap (\bigcup_{1 \leq j \leq p} V_j)| \\
- |( \bigcup_{i \in M_1 \setminus \{l, h\}} D_i \cup X) \cap (V_1 \cup V_2 \cup S)| \\
\leq |C| + \kappa(G) - \sum_{i \in M_1 \setminus \{l, h\}} |D_i| \cap (\bigcup_{1 \leq j \leq p} V_j \cup S)| - |X \cap (\bigcup_{1 \leq j \leq p} V_j)| \\
\leq |C| + \kappa(G) - \sum_{i \in I \setminus \{0, l\}} |D_i| - |X| \\
\leq |C| + \kappa(G) - \sum_{i \in I \setminus \{0, l\}} |D_i| - d_C(x_0).
\]

On the other hand, the inequality (11) yields that
\[
\sum_{i \in I \setminus \{0, l\}} d_C(x_i) \leq \sum_{i \in I \setminus \{0, l\}} |D_i| + (k - 2)(\alpha(G) - 1).
\]

By the above two inequalities, we deduce
\[
\sum_{i \in I} d_C(x_i) + d_C(v) \leq |C| + \kappa(G) + (k - 2)(\alpha(G) - 1).
\]

Recall that \( \{x_i : i \in I\} \cup \{v\} \) is an independent set, in particular, \( x_0 \not\in \bigcup_{i \in I} N_H(x_i) \cup N_H(v) \). Since \( N_H(x_i) \cap N_H(x_j) = \emptyset \) for \( i, j \in I \) with \( i \neq j \) and \( \bigcup_{i \in I} N_H(x_i) \cap N_H(v) = \emptyset \) by Lemma 3 (i) and (ii), it follows that \( \sum_{i \in I} d_H(x_i) + d_H(v) \leq |H| - 1 \). Combining this inequality with the above inequality, we get \( \sum_{i \in I} d_G(x_i) + d_G(v) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1 \), a contradiction.

We next show that \( H - H_0 \subseteq V_1 \cup S \). Suppose not. Without loss of generality, we may assume that there exists a vertex \( y \in (H - H_0) \cap V_2 \). Let \( H_y \) be a component of \( H \) with \( y \in V(H_y) \). Note that \( H_y \neq H_0 \). Suppose that \( N_C(H_y) \cap (D_h \cup \{x_h\}) \neq \emptyset \) for some \( h \in M_1 \setminus \{l\} \). Then Lemma 3 (i) yields that
\[
d_C(y) \leq |C \cap (V_2 \cup S)| - |( \bigcup_{i \in M_1 \setminus \{l, h\}} D_i \cup X) \cap (V_2 \cup S)|.
\]

Hence, by the same argument as above, we can obtain a contradiction. Thus we may assume that \( N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset \) for all \( i \in M_1 \setminus \{l\} \). Then, since \( y \in V_2 \) and
$D_i \cup \{x_i\} \subseteq V_1$, we have

$$d_C(y) \leq |C \cap (V_2 \cup S)| - |(\bigcup_{i \in M_1} D_i \cup X) \cap (V_2 \cup S)|.$$ 

Let $I$ be a subset of $M_0$ such that $|I| = k$ and $\{0, l\} \subseteq I$. Since $x_l \in V_1$, $y \in V_2$, $H_y \neq H_0$ and $N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset$ for all $i \in M_1 \setminus \{l\}$, it follows from Claim 1 that $\{x_i : i \in I\} \cup \{y\}$ is an independent set of order $k + 1$. By the above inequality and the inequality (5), we obtain

$$d_C(x_l) + d_C(y) \leq |C| + |C \cap X| - |(\bigcup_{i \in M_1 \setminus \{l\}} D_i \cup X) \cap (V_2 \cup S)| \leq |C| + |C \cap S| - \sum_{i \in I \setminus \{0, l\}} |D_i| - d_C(x_0).$$

Therefore, by the above inequality and the inequality (1), we obtain

$$\sum_{i \in I} d_C(x_i) + d_C(y) \leq |C| + |C \cap S| + (k - 2)(\alpha(G) - 1).$$

Since $H_0 \neq H_y$ and $N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset$ for all $i \in M_1 \setminus \{l\}$, it follows that $(\bigcup_{i \in I \setminus \{l\}} N_H(x_i)) \cap V(H_y) = \emptyset$. Since $x_l \in V_1$ and $y \in V_2$, we have $N_H(x_l) \cap N_H(y) \subseteq H \cap S$. Therefore, we obtain

$$\sum_{i \in I} d_H(x_i) + d_H(y) \leq |H| + |H \cap S| - 2.$$ 

Combining the above two inequalities, $\sum_{i \in I} d_G(x_i) + d_G(y) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 2$, a contradiction.

We finally show that $H_0 \subseteq V_1 \cup S$. Suppose not. Without loss of generality, we may assume that there exists a vertex $y_0 \in H_0 \cap V_2$. Then

$$d_C(y_0) \leq |U \cap (V_2 \cup S)| + |H_0| - 1.$$ 

Since $u_l \in V_1$, we have $H_0 \cap S \neq \emptyset$. Note that by the above argument, $X \subseteq V_1 \cup S$. Therefore, by Claim 2 $|X \cap V_1| = |X| - |X \cap S| \geq \kappa(G) + 1 - (|S| - |H_0 \cap S|) \geq \kappa(G) + 1 - (\kappa(G) - 1) = 2$. Let $x_s \in X \cap V_1$ with $x_s \neq x_l$. Let $I$ be a subset of $M_1$ such that $|I| = k$ and $\{l, s\} \subseteq I$. Then $\{x_i : i \in I\} \cup \{y_0\}$ is an independent set of order $k + 1$. By Lemma 3(i), we have $N_C(x_s) \cap (U \setminus \{u_l\}) = \emptyset$ and $N_C(x_s) \cap (U \setminus \{u_s\}) = \emptyset$. Since $x_1, x_s \in V_1$, it follows that $(N_C(x_1) \cup N_C(x_s)) \cap (U \cup V_2) = \emptyset$. Therefore, we can improve the inequality (4) as follows:

$$d_C(x_l) + d_C(x_s) \leq |C| - \sum_{i \in I \setminus \{l, s\}} |D_i| - |U \cap V_2|.$$
By the inequality (1) and the inequality (3),
\[
\sum_{i \in \Gamma \setminus \{t,s\}} d_C(x_i) \leq \sum_{i \in \Gamma \setminus \{t,s\}} |D_i| + (k - 2)(\alpha(G) - 1) \quad \text{and} \quad \sum_{i \in I} d_H(x_i) \leq |H| - |H_0|.
\]
Hence, by the above four inequalities, we deduce \(d_G(y_0) + \sum_{i \in I} d_G(x_i) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1\), a contradiction.

By Claim 3, there exists an integer \(r\) such that \(C(x_r, u'_r) \cap \bigcup_{i=2}^p V_i \neq \emptyset\), say
\[v_2 \in C(x_r, u'_r) \cap \bigcup_{i=2}^p V_i.
\]
Choose \(r\) and \(v_2\) so that \(v_2 \neq u'_r\) if possible. Without loss of generality, we may assume that \(v_2 \in V_2\). Note that
\[d_G(v_2) \leq |V_2 \cup S| - 1. \quad (6)
\]

**Claim 4.** \(d_G(w) \leq d_G(x_0) \leq |X| \leq \alpha(G) - 1\) for each \(w \in W\).

**Proof.** Let \(w \in W\). Without loss of generality, we may assume that \(w \in W_1\). Then by applying Lemma 1 as \(Q_1 = D_1\), \(Q_2 = D_2\) and
\[D = x_1C[w^+, u_2]P[u_2, u_1]\overline{C}[u_1, x_2]\overline{C}[w^-, x_1],
\]
where \(P[u_2, u_1]\) is a \(C\)-path passing through some vertex of \(H_0\), we can obtain a cycle \(C'\) such that \(V(C) \setminus \{w\} \subseteq V(C')\) and \(V(C') \cap V(H_0) \neq \emptyset\) (note that (I) and (II) of Lemma 1 hold, by Lemma 3 (i) and (ii) and the definition of insertible and \(D_i\)). Note that by the maximality of \(|C|\), \(|C'| = |C|\). Note also that \(d_{C'}(w) \geq d_C(w)\). By the choice of \(C\) and \(x_0\), we have \(d_{C'}(w) \leq d_C(x_0)\), and hence by Claim 1 and the fact that \(d_C(x_0) \leq |X|\), we obtain \(d_C(w) \leq d_C(x_0) \leq |X| \leq \alpha(G) - 1\).

By Lemma 3 and Claim 3, we have
\[\sum_{i \in M_0} d_H(x_i) + \sum_{w \in W} d_H(w) \leq |H| - |\{x_0\}| = |H \cap (V_1 \cup S)| - 1. \quad (7)
\]
Moreover, by Lemma 3 and Claim 1 the following claim holds.

**Claim 5.** \(X \cup W \cup \{x_0\}\) is an independent set.

We now check the degree sum of the vertices \(x_1, x_2\) and \(x_3\) in \(C\). In this paragraph, the indices are taken modulo 3. By Lemma 3 (ii), \((N_{C_i}(x_i^-) \cup N_{C_i}(x_{i+1}^+)) \cap N_{C_i}(x_{i+2}) = \emptyset\) for \(i \in \{1, 2, 3\}\). Clearly, \(N_{C_i}(x_i^-) \cap N_{C_i}(x_{i+1}^+) = W_i\) and \(N_{C_i}(x_i^-) \cup \)
Figure 3: The definition of $L$.

$$N_C(x_{i+1})^+ \cup N_C(x_{i+2}) \subseteq C_i \cup \{u_{i+1}^+\}.$$ By Lemma 3(i), $(N_C(x_i)^- \cup N_C(x_{i+2})) \cap D_j = \emptyset$ for $i \in \{1, 2, 3\}$ and $j \in M_1$. For $i \in \{1, 2, 3\}$, let

$$L_i = \{x_j \in X \setminus \{x_{i+1}\} : N_C(x_{i+1})^+ \cap D_j \neq \emptyset\}$$

and let $L = \bigcup_{i \in \{1, 2, 3\}} L_i$ (see Figure 3).

Note that $L \cap \{x_1, x_2, x_3\} = \emptyset$ and $W \cap L = \emptyset$ by Lemma 3(i). Therefore the following inequality holds:

$$d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(x_3) \leq \left|C_i\right| + \left|W_i\right| + 1 - \sum_{j \in M_1} \left|C_i \cap D_j\right| + \left|L_i\right|$$

for $i \in \{1, 2, 3\}$. By Lemma 3(i), we have $N_C(x_i) \cap D_j = \emptyset$ for $i, j \in M_1$ with $i \neq j$, and hence

$$d_{D_i}(x_1) + d_{D_i}(x_2) + d_{D_i}(x_3) \leq \left|D_i\right|$$

for $i \in \{1, 2, 3\}$. Let $I$ be a subset of $M_0$ such that $|I| = 3 - 2$ and $I \cap \{1, 2, 3\} = \emptyset$. Let $L_I = L \cap \{x_i : i \in I\}$. Note that $|L \cap \{x_i\}| - |D_i| \leq 0$ for each $i \in M_1 \setminus \{1, 2, 3\}$. Thus, we deduce

$$d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(x_3) \leq \sum_{i=1}^{3} \left(\left|C_i\right| + \left|W_i\right| + \left|L_i\right| + 1 - \sum_{j \in M_1} \left|C_i \cap D_j\right| + \left|D_i\right|\right)$$

$$= \left|C\right| + \left|W\right| + \left|L\right| - \sum_{i \in M_1 \setminus \{1, 2, 3\}} \left|D_i\right| + 3$$

$$\leq \left|C\right| + \left|W\right| + \left|L\right| - \sum_{i \in I \setminus \{0\}} \left|D_i\right| + 3 \quad (8)$$

$$\leq \left|C\right| + \left|W\right| + 3. \quad (9)$$

Claim 6. $|W| + |L| \geq \kappa(G) - 2 \geq 1$.

Proof. Let $I$ be a subset of $M_0$ such that $|I| = k - 2$ and $I \cap \{1, 2, 3\} = \emptyset$. Suppose that $|W| + |L_I| \leq \kappa(G) - 3$. By Claim 5, $\{x_i : i \in I\} \cup \{x_1, x_2, x_3\}$ is an independent
set of order \( k + 1 \). By the inequality (8), we obtain

\[
d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |C| + \kappa(G) - \sum_{i \in I \setminus \{0\}} |D_i|.
\]

Therefore, this inequality, the inequalities (1) and (2) and Claim 4 yield that

\[
\sum_{i=1}^{3} d_G(x_i) + \sum_{i \in I} d_G(x_i) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1,
\]
a contradiction. Therefore, this inequality, the inequalities (1) and (2) and Claim 4 yield that

\[
\sum_{i=1}^{3} d_G(x_i) + \sum_{i \in I} d_G(x_i) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1,
\]
a contradiction. Therefore, \(|W| + |L| \geq |W| + |L_i| \geq \kappa(G) - 2\).

**Claim 7.** \( d_C(x_0) = |U| = |X| = \alpha(G) - 1 \). In particular, \( N_C(x_0) = U \).

**Proof.** Suppose that \( d_C(x_0) \leq \alpha(G) - 2 \). In this proof, we assume \( x_i = x_1 \) (recall that \( l \) is an integer such that \( C[u_i, u'_i] \subseteq V_1 \), see the paragraph below the proof of Claim 2). We divide the proof into two cases.

**Case 1.** \(|W| \geq k - 3\).

**Subclaim 7.1.** \(|W| \leq \kappa(G) + k - 5\).

**Proof.** Suppose that \(|W| \geq \kappa(G) + k - 4\). By Claim 3 we obtain

\[
|W \cup \{x_0, x_1, x_2, x_3\} \cap V_1| = |W \cup \{x_0, x_1, x_2, x_3\}| - |(W \cup \{x_0, x_1, x_2, x_3\}) \cap S| \\
\geq (\kappa(G) + k - 4 + 4) - \kappa(G) = k.
\]

Let \( W' \) be a subset of \((W \cup \{x_0, x_1, x_2, x_3\}) \cap V_1\) such that \(|W'| = k\) and \( x_1 \in W' \).

Since \( W' \subseteq V_1 \) and \( v_2 \in V_2 \), it follows from Claim 5 that \( W' \cup \{v_2\} \) is an independent set of order \( k + 1 \). By the inequality (5) and Claims 3 and 4 we obtain

\[
d_C(x_1) \leq |C \cap (V_1 \cup S)| - \sum_{i \in M_1 \setminus \{1\}} |(D_i \cap (V_1 \cup S)| - |X \cap (V_1 \cup S)| \\
\leq |C \cap (V_1 \cup S)| - \sum_{i \in \{2,3\}} |D_i| - |X| \\
\leq |C \cap (V_1 \cup S)| - \sum_{i \in \{2,3\}} |D_i| - d_C(w_0),
\]

where \( w_0 \in W' \setminus \{x_1, x_2, x_3\} \) (note that \(|W'| = k \geq 4\)). By the inequality (1) and Claim 4

\[
\sum_{x \in W' \setminus \{x_2, x_3\}} d_C(x) + \sum_{w \in W' \setminus \{w_0, x_1, x_2, x_3\}} d_C(w) \leq \sum_{i \in \{2,3\}} |D_i| + (k - 2)(\alpha(G) - 1).
\]

By the above two inequalities, we obtain

\[
\sum_{w \in W'} d_C(w) \leq |C \cap (V_1 \cup S)| + (k - 2)(\alpha(G) - 1).
\]
Therefore, since \( \sum_{w \in W} d_H(w) \leq |H \cap (V_1 \cup S)| - 1 \) by the inequality (7), it follows that
\[
\sum_{w \in W'} d_G(w) \leq |V_1 \cup S| + (k - 2)(\alpha(G) - 1) - 1.
\]
Summing this inequality and the inequality (9) yields that \( \sum_{w \in W'} d_G(w) + d_G(v_3) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 2 \), a contradiction.

By the assumption of Case 1, we can take a subset \( W^* \) of \( W \cup \{x_0\} \) such that \( |W^*| = k - 2 \). By Claim 5, \( W^* \cup \{x_1, x_2, x_3\} \) is independent. Moreover, by Claim 4 and the assumption that \( d_C(x_0) \leq \alpha(G) - 2 \), we have
\[
\sum_{w \in W^*} d_C(w) \leq (k - 2)(\alpha(G) - 2).
\]
By Subclaim [7], summing this inequality and the inequality (9) yields that
\[
\sum_{i=1}^{3} d_C(x_i) + \sum_{w \in W^*} d_C(w) 
\leq |C| + |W| + 3 + (k - 2)(\alpha(G) - 2)
\leq |C| + (\kappa(G) + k - 5) + 3 - (k - 2) + (k - 2)(\alpha(G) - 1)
= |C| + \kappa(G) - (\alpha(G) - 1).
\]
Therefore, since \( \sum_{i=1}^{3} d_H(x_i) + \sum_{w \in W'} d_H(w) \leq |H| - 1 \) by the inequality (7), we obtain \( \sum_{i=1}^{3} d_G(x_i) + \sum_{w \in W'} d_G(w) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1 \), a contradiction.

Case 2. \( |W| \leq k - 4 \).

By Claim 4 we can take a subset \( L^* \) of \( L \) such that \( |L^*| = k - 3 - |W| \). Let \( I = \{i : x_i \in L^*\} \). By Claim 5, \( W \cup L^* \cup \{x_0, x_1, x_2, x_3\} \) is an independent set of order \( k + 1 \). By the inequality (8), we have
\[
d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |C| + |W| + |L^*| - \sum_{i \in I} |D_i| + 3
= |C| + k - 3 - \sum_{i \in I} |D_i| + 3
\leq |C| + \kappa(G) - \sum_{i \in I} |D_i|.
\]

On the other hand, it follows from Claim 4 the assumption \( d_C(x_0) \leq \alpha - 2 \) and the inequality (11) that
\[
\sum_{w \in W \cup \{x_0\}} d_C(w) + \sum_{x \in L^*} d_C(x) \leq (|W| + 1)(\alpha(G) - 2) + \sum_{i \in I} |D_i| + |L^*|(|\alpha(G) - 1|)
= (k - 2)(\alpha(G) - 1) - |W| - 1 + \sum_{i \in I} |D_i|
\leq (k - 2)(\alpha(G) - 1) + \sum_{i \in I} |D_i| - 1.
\]
Thus, we deduce
\[
\sum_{i=1}^{3} d_C(x_i) + \sum_{w \in W \cup \{x_0\}} d_C(w) + \sum_{x \in L^*} d_C(x) \leq |C| + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1.
\]
By the inequality (7), we obtain
\[
\sum_{i=1}^{3} d_H(x_i) + \sum_{w \in W \cup \{x_0\}} d_H(w) + \sum_{x \in L^*} d_H(x) \leq |H| - 1.
\]
Summing the above two inequalities yields that \(\sum_{i=1}^{3} d_C(x_i) + \sum_{w \in W \cup \{x_0\}} d_C(w) + \sum_{x \in L^*} d_C(x) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 2\), a contradiction.

By Cases 1 and 2, we have \(d_C(x_0) \geq \alpha(G) - 1\). Since \(|U| = |X|\), it follows from Claim 4 that \(d_C(x_0) = |U| = |X| = \alpha(G) - 1\). In particular, \(N_C(x_0) = U\) because \(N_C(x_0) \subseteq N_C(H_0) = U\). This completes the proof of Claim 7.

\[\text{Claim 8. } W \subseteq X.\]

\[\text{Proof.} \text{ If } W \setminus X \neq \emptyset, \text{ then by Claim 5 we have } d_C(x_0) \leq |X| \leq \alpha(G) - 2, \text{ which contradicts Claim 7.} \]

\[\text{Claim 9. If there exist distinct two integers } s \text{ and } t \text{ in } M_1 \text{ such that } u_s \in N_C(x_t), \text{ then } N_C(x_s) \cap C[u_t, u_s] \subseteq U.\]

\[\text{Proof.} \text{ Suppose that there exists a vertex } z \in N_C(x_s) \cap C[u_t, u_s] \text{ such that } z \notin U. \text{ We show that } X \cup \{x_0, z^+\} \text{ is an independent set of order } |X| + 2. \text{ By Claim 5 we only show that } z^+ \notin X \text{ and } z^+ \notin N_C(x_i) \text{ for each } x_i \in X \cup \{x_0\}. \text{ Since } z \notin U, \text{ it follows from Lemma 3(i) that } z^+ \notin X. \text{ Suppose that } x_h \in N_C(x_h) \text{ for some } x_h \in X \cup \{x_0\}. \text{ Since } x_s \text{ is a non-insertible vertex, it follows that } x_h \neq x_s. \text{ Let } z_h \text{ be the vertex in } C(u_s, x_s) \text{ such that } z \in N_G(z_s) \text{ and } z \notin N_G(v) \text{ for all } v \in C(u_s, z_s). \text{ By Lemma 3(ii), we obtain } x_h \notin C[u'_s, z]. \text{ Therefore, } x_h \in C(z, u_s) \cup \{x_0\}. \text{ If } x_h \in C(z, u_s), \text{ then we let } z_h \text{ be the vertex in } C(u_h, x_h) \text{ such that } z^+ \in N_G(z_h) \text{ and } z^+ \notin N_G(v) \text{ for all } v \in C(u_h, z_h). \text{ We define the cycle } C^* \text{ as follows (see Figure 4):}
\]

\[
C^* = \left\{ \begin{array}{ll}
z_sC[z, x_t]C[u_s, z_h]C[z^+, u_h]x_0C[u_t, z_s] & \text{if } x_h \in C(z, u_s), \\
z_sC[z, x_t]C[u_s, z^+]x_hC[u_t, z_s] & \text{if } x_h = x_0. 
\end{array} \right.
\]

Then, by similar argument in the proof of Lemma 3 we can obtain a longer cycle than \(C\) by inserting all vertices of \(V(C \setminus C^*)\) into \(C^*\). This contradicts that \(C\) is longest. Hence \(z^+ \notin N_C(x_h) \text{ for each } x_h \in X \cup \{x_0\}. \text{ Thus, by Claim 7, } X \cup \{x_0, z^+\} \text{ is an independent set of order } |X| + 2 = \alpha(G) + 1, \text{ a contradiction.} \]

\[\square\]
We divide the rest of the proof into two cases.

**Case 1.** \( v_2 \notin U \).

Let \( Y = N_G(v_2) \cap X \), and let \( \gamma = |X| - \kappa(G) - 1 \). Note that \( |X| = \kappa(G) + \gamma + 1 \geq k + \gamma + 1 \) and \( x_l \notin Y \) since \( x_l \in V_1 \).

**Claim 10.** \( |Y| \geq \gamma + 3 \).

**Proof.** Suppose that \( |Y| \leq \gamma + 2 \). By the assumption of Case 1, we have \( x_0v_2 \notin E(G) \). Since \( |M_0| = |X| + 1 \geq k + \gamma + 2 \) and \( |Y| \leq \gamma + 2 \), there exists a subset \( I \) of \( M_0 \setminus \{i : x_i \in Y\} \) such that \( |I| = k \) and \( \{0, l\} \subseteq I \). Then \( \{x_i : i \in I\} \cup \{v_2\} \) is an independent set of order \( k + 1 \). By the inequality (5) and Claims 3 and 7, we obtain

\[
\sum_{i \in I} d_C(x_i) \leq |C \cap (V_1 \cup S)| - \sum_{i \notin \{0, l\}} |D_i| - |X|
\]

\[
= |C \cap (V_1 \cup S)| - \sum_{i \notin \{0, l\}} |D_i| - d_C(x_0).
\]

Therefore it follows from the inequality (11) that

\[
\sum_{i \in I} d_C(x_i) \leq |C \cap (V_1 \cup S)| + (k - 2)(\alpha(G) - 1).
\]

By the inequality (12), \( \sum_{i \in I} d_H(x_i) \leq |H \cap (V_1 \cup S)| - 1 \). Summing these two inequalities and the inequality (4) yields that

\[
\sum_{i \in I} d_G(x_i) + d_G(v_2) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 2,
\]

a contradiction. \( \square \)
Recall that $r$ is an integer such that $v_2 \in C(x_r, u'_r) \cap V_2$ (see the paragraph below the proof of Claim 3). In the rest of Case 1, we assume that $l = 1$. If $u'_r \neq u_1$, then let $r = 2$ and $u_3 = u'_2$; otherwise, let $r = 3$ and let $u_2$ be the vertex with $u'_2 = u_3$.

By Claim 8, we have $W \subseteq X$. Hence we obtain $Y \cup W \cup L \subseteq X \setminus \{x_1\}$. Recall that $W \cap L = \emptyset$. Therefore, by Claims 8 and 10, we obtain

$$|Y \cap (W \cup L)| = |Y| + |W| + |L| - |Y \cup (W \cup L)|$$

$$\geq \gamma + 3 + \kappa(G) - 2 - |X \setminus \{x_1\}|$$

$$= \gamma + 3 + \kappa(G) - 2 - ((\kappa(G) + \gamma + 1) - 1) = 1.$$

Hence there exists a vertex $x_h \in Y \cap (W \cup L)$, that is, $v_2 \in N_C(x_h) \setminus U$. Since $C(x_2, x_3) \cap X = \emptyset$ and $C(x_3, x_1) \cap X = \emptyset$ if $r = 3$, either $u_h \in N_C(x_1)$ and $u_h \in C(x_3, u_1)$ or $u_h \in N_C(x_2)$ and $u_h \in C(x_1, u_2)$ holds (especially, if $r = 3$ then $u_h \in N_C(x_2)$ and $u_h \in C(x_1, u_2)$ holds) (see Figure 5).

If $r = 2$ and $u_h \in N_C(x_1)$, then $v_2 \in C[u_1, u_h]$ (see Figure 5 (i)). If $r = 2$ and $u_h \in N_C(x_2)$, then $v_2 \in C[u_2, u_h]$ (see Figure 5 (ii)). If $r = 3$, then $u_h \in N_C(x_2)$ and $v_2 \in C[u_2, u_h]$ (see Figure 5 (iii)). In each case, we obtain a contradiction to Claim 9.

**Case 2.** $v_2 \in U$.

We rename $x_i \in X$ for $i \geq 1$ as follows (see Figure 6): Rename an arbitrary vertex of $X$ as $x_1$. For $i \geq 1$, we rename $x_{i+1} \in X$ so that $u_{i+1} \in N_C(x_i) \cap (U \setminus \{u_i\})$ and $|C[u_{i+1}, x_i]|$ is as small as possible. (For $x_i \in X$, let $x'_i$ and $x''_i$ be the successors of $x_i$ and $x'_i$ in $X$ along the orientation of $C$, respectively. Then by applying Claim 8 as $x_1 = x_i$, $x_2 = x'_i$ and $x_3 = x''_i$, it follows that $W \cup L \neq \emptyset$. By the definition of $x'_i, x''_i$ and Claim 8, we have $W_1 = W_2 = \emptyset$ (note that $W \cap \{x_1, x_2, x_3\} = \emptyset$). By the definitions of $x'_i, x''_i, L_1$ and $L_2$, we also have $L_1 = L_2 = \emptyset$. Thus $W_3 \cup L_3 \neq \emptyset$. By Lemma 8 (i) and since $W \cup L \subseteq X$, this implies that $N_C(x_i) \cap (U \setminus \{u_i\}) \neq \emptyset$. Let
Let \( h \) be the minimum integer such that \( x_{h+1} \in C(x_h, x_1) \). Note that this choice implies \( h \geq 2 \). We rename \( h \) vertices in \( X \) as \( \{x_1, x_2, \ldots, x_h\} \) as above, and \( m - h \) vertices in \( X \setminus \{x_1, x_2, \ldots, x_h\} \) as \( \{x_{h+1}, x_{h+2}, \ldots, x_m\} \) arbitrarily. Let \( A_1 = A_{h+1} = C[x_1, x_h] \) and \( A_i = C[x_i, x_{i-1}] \) for \( 2 \leq i \leq h \).

Let

\[
U_1 = \{u_i \in U : x_i \in X \cap V_1\}.
\]

If possible, choose \( x_1 \) so that \( A_2 \cap U_1 = \emptyset \).

![Diagram](image)

Figure 6: The choice of \( \{x_1, \ldots, x_h\} \).

We divide the proof of Case 2 according to whether \( h \leq k \) or \( h \geq k + 1 \).

**Case 2.1.** \( h \leq k \).

By the choice of \( \{x_1, \ldots, x_h\} \), we have

\[
N_{A_{i+1}}(x_i) \cap U \subseteq \{u_i\} \quad \text{for} \quad 1 \leq i \leq h. \tag{10}
\]

By Claim \[ \text{[9]} \] and \[ \text{[10]} \], we obtain

\[
N_{C \setminus A_i}(x_i) \subseteq (U \setminus (A_i \cup A_{i+1})) \cup D_i \cup \{u_i\} \quad \text{for} \quad 2 \leq i \leq h. \tag{11}
\]

By Lemma \[ \text{[3]} (i) \] and \( (ii) \), \( N_{A_i}(x_i) - N_{A_i}(x_1) = \emptyset \) for \( 2 \leq i \leq h \). By Lemma \[ \text{[3]} (i) \], we have \( N_{A_i}(x_i) - N_{A_i}(x_1) \subseteq A_i \setminus D \) for \( 3 \leq i \leq h \). Thus, it follows from \[ \text{[11]} \] that for \( 3 \leq i \leq h \)

\[
d_C(x_i) \leq (|U| - |(A_i \cup A_{i+1}) \cap U| + |D_i| + 1) + (|A_i| - |A_i \cap D| - d_{A_i}(x_1)).
\]
By Lemma 3 (i) and (10), we have $N_{A_2}(x_2) \cup N_{A_2}(x_1) \subseteq (A_2 \setminus (U \cup D)) \cup D_1 \cup \{u_1\}$. Thus, by (11), we have

$$d_C(x_2) \leq (|U| - |A_2 \cup A_3| \cap U| + |D_2| + 1) + (|A_2| - |A_2 \cap (U \cup D)| + |D_1| + 1 - d_{A_2}(x_1)).$$

Since $|A_1 \cap X| = |A_1 \cap U|$, it follows from Lemma 3 (i) that

$$d_{A_1}(x_1) \leq |A_1| - |A_1 \cap D| - |A_1 \cap X| = |A_1| - |A_1 \cap D| - |A_1 \cap U|.$$

By Claim 7, $d_C(x_0) = |U| = \alpha(G) - 1$. Thus, since $h \leq k$, we obtain

$$\sum_{0 \leq i \leq h} d_C(x_i) \leq \sum_{1 \leq i \leq h} |A_i| + h|U| - 2 \sum_{1 \leq i \leq h} |A_i \cap U| + h + \sum_{1 \leq i \leq h} |D_i| - \sum_{1 \leq i \leq h} |A_i \cap D| \leq |C| + (h - 2)|U| + h + \sum_{1 \leq i \leq h} |D_i| - |D| \leq |C| + k + (h - 2)(\alpha(G) - 1) + \sum_{1 \leq i \leq h} |D_i| - |D|.$$ 

Let $I$ be a subset of $M_0$ such that $|I| = k + 1$ and $\{0, 1, \ldots, h\} \subseteq I$. By Claim 5, $\{x_i : i \in I\}$ is an independent set of order $k + 1$. By the above inequality and the inequality (1), we have

$$\sum_{i \in I} d_C(x_i) \leq |C| + k + (k - 2)(\alpha(G) - 1)$$

By the inequality (2), $\sum_{i \in I} d_H(x_i) \leq |H| - 1$. Hence $\sum_{i \in I} d_G(x_i) \leq |G| + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1$, a contradiction.

**Case 2.2.** $h \geq k + 1$.

By Claims 9 and 7 the assumption of Case 2 and the choice of $r$ and $v_2$, we have $\bigcup_{i=1}^p V_i \subseteq U = N_C(x_0)$. Since $x_0 \in V_1 \cup S$ by Claim 8, this implies that $x_0 \in S$.

**Claim 11.** $|X \cap V_i| \leq k - 1$.

*Proof.* Suppose that $|X \cap V_i| \geq k$. Let $I$ be a subset of $M_1$ such that $|I| = k$ and $I \subseteq \{i : x_i \in X \cap V_i\}$. Then $\{x_i : i \in I\} \cup \{v_2\}$ is an independent set of order $k + 1$. Let $s$ and $t$ be integers in $I$. Since $x_s, x_t \in V_1, D \subseteq V_1 \cup S$ and $\bigcup_{i=1}^p V_i \subseteq U$, the similar argument as that of the inequality (11) implies that

$$d_C(x_s) + d_C(x_t) \leq |C \cap (V_1 \cup S)| - \sum_{i \notin \{s, t\}} |D_i|.$$

By the inequalities (11) and (7), we have

$$\sum_{i \notin \{s, t\}} d_C(x_i) \leq \sum_{i \notin \{s, t\}} |D_i| + (k - 2)(\alpha(G) - 1)$$

and

$$\sum_{i \in \{s, t\}} d_H(x_i) \leq |H \cap (V_1 \cup S)| - 1,$$

respectively. On the other hand, we obtain $d_G(v_2) \leq |V_2 \cup S| - 1$. By these four inequalities, $\sum_{i \in I} d_G(x_i) + d_G(v_2) \leq n + \kappa(G) + (k - 2)(\alpha(G) - 1) - 2$, a contradiction. Therefore $|X \cap V_i| \leq k - 1$. \qed
Recall $U_1 = \{u_i \in U : x_i \in X \cap V_1\}$. By Claim 11 we have $|U_1| \leq k - 1$. By the assumption of Case 2.2 and the choice of $x_1$, we obtain $A_2 \cap U_1 = \emptyset$, and hence we can take a subset $I$ of $\{2, 3, \ldots, h\}$ such that $|I| = k$ and $\{i : A_{i+1} \cap U_1 \neq \emptyset\} \subseteq I$. Let

$$X_I = \{x_i : i \in I\}.$$  

By Claim 5, $X_I \cup \{x_0\}$ is an independent set of order $k + 1$. Let

$$B_1 = B_{h+1} = C(u_1, u_h) \quad \text{and} \quad B_i = C(u_i, u_{i-1}) \quad \text{for} \ 2 \leq i \leq h.$$  

Then, since $|C(u_i, u'_i)| \geq 2$ for $i \in M_1 \setminus I$, the following inequality holds:

$$|C| \geq \sum_{i \in I} |B_i \cup \{u_i\}| + 2 \left(|U| - \sum_{i \in I} |(B_i \cup \{u_i\}) \cap U|\right)$$

$$= \sum_{i \in I} |B_i| + 2 \left(|U| - \sum_{i \in I} |B_i \cap U|\right) - k.$$  

If $x_i \in X_I \cap S$, then it follows from Lemma 3 (i) and Claim 9 that

$$d_C(x_i) \leq \left(|U| - |B_i \cup U| - |B_{i+1} \cap U_1|\right) + \left(|B_i| - |\{x_i\}| - |(B_i \cap U)^{+}|\right)$$

$$= |U| + |B_i| - 2|B_i \cap U| - |B_{i+1} \cap U_1| - 1.$$  

If $x_i \in X_I \cap V_1$, then, by Lemma 3 (i) and Claim 9

$$d_C(x_i) \leq \left(|U| - |B_i \cup U| - |B_{i+1} \cap U_1| - |(U \cap V_2) \setminus B_i| + |B_{i+1} \cap U_1 \cap V_2|\right)$$

$$+ \left(|B_i| - |\{x_i\}| - |(B_i \cap U)^{+}| - |U \cap V_2 \cap B_i|\right)$$

$$= |U| + |B_i| - 2|B_i \cap U| - |B_{i+1} \cap U_1| - 1 - \left(|U \cap V_2| - |B_{i+1} \cap U_1 \cap V_2|\right).$$  

Since $U \cap V_2 \neq \emptyset$, we obtain $|U \cap V_2| - |B_{i+1} \cap U_1 \cap V_2| \geq 1$ for all $i \in I$ except for at most one, and hence

$$\sum_{i \in I : x_i \in X_I \cap V_1} \left(|U \cap V_2| - |B_{i+1} \cap U_1 \cap V_2|\right) \geq |X_I \cap V_1| - 1.$$  

By the choice of $I$, we have

$$|U_1| = \sum_{i \in I} |A_{i+1} \cap U_1| = \sum_{i \in I} |B_{i+1} \cap U_1| + |\{u_i : x_i \in X_I \cap V_1\}|.$$  

On the other hand, since $x_0 \in S$, it follows from Claim 3 that

$$|U_1| = |X \cap V_1| = |X \setminus S| \geq |X| - (\kappa(G) - 1).$$  

Moreover, by Claim 7

$$d_C(x_0) = |U| = |X| = \alpha(G) - 1.$$  

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Thus, we deduce
\[
\sum_{i \in I \cup \{0\}} d_C(x_i) \leq (k + 1)|U| + \sum_{i \in I} |B_i| - 2 \sum_{i \in I} |B_i \cap U|
\]
\[
- \sum_{i \in I} |B_{i+1} \cap U_1| - k - (|X_f \cap V_1| - 1)
\]
\[
= \left( \sum_{i \in I} |B_i| + 2\left(|U| - \sum_{i \in I} |B_i \cap U|\right) - k \right) + (k - 1)|U|
\]
\[
- \left( \sum_{i \in I} |B_{i+1} \cap U_1| + \{|u_i : x_i \in X_f \cap V_1\}| \right) + 1
\]
\[
\leq |C| + (k - 1)|U| + \kappa(G) - |X|
\]
\[
= |C| + \kappa(G) + (k - 2)(\alpha(G) - 1).
\]

By the inequality (2), \( \sum_{i \in I \cup \{0\}} d_H(x_i) \leq |H| - 1 \). Hence \( \sum_{i \in I \cup \{0\}} d_G(x_i) \leq |G| + \kappa(G) + (k - 2)(\alpha(G) - 1) - 1 \), a contradiction. \( \Box \)

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