Abstract. In the framework of ’t Hooft’s ”deterministic quantization” proposal, we show how to obtain from a composite system of two classical Bateman’s oscillators a quantum isotonic oscillator. In a specific range of parameters, such a system can be also interpreted as a particle in an effective magnetic field, interacting through a spin-orbit interaction term. In the limit of a large separation from the interaction region, the system can be described in terms of two irreducible elementary subsystems, corresponding to two independent quantum harmonic oscillators.

1. Introduction
Almost two decades ago ’t Hooft outlined a program that strived to provide an explanatory (as opposed to merely predictive) basis for quantum mechanics (QM) [1]. The program itself has been variably called “pre-quantization” [2], “emerged quantum mechanics” [3], “’t Hooft’s deterministic quantum theory” [4] or “’t Hooft’s quantization proposal” [5].

The key observation of ’t Hooft is that there exists a simple family of deterministic (though non-standard) dynamical systems that can be described by means of Hilbert space techniques without loosing their deterministic character. Only after enforcing certain constraints expressing information loss, one obtains genuine quantum systems. In this picture the observed degrees of freedom (observables) $|\Psi\rangle_{phys}$ are identified with equivalence classes of states spanning the primordial Hilbert space of the truly existing degrees of freedom (be-ables). Formation of equivalence classes is obtained by enforcing an information loss constraint (ILC) which selects/defines the physical states $|\Psi\rangle_{phys}$.

The enormous amount of information that would be lost in the process of “coarse graining” from Planckian-scale – $10^{19}$GeV, to our current observational scale – $10^{3}$GeV, would lead to formation of equivalence classes, in the sense that many distinct states of the Planckian-scale dynamics would go into a single state of the observational scale dynamics, producing in this way highly non local observable states. From this point of view, clearly, ordinary quantum mechanics is not a fundamental theory: quantum states are derived concepts, with a not strictly locally formulated definition. Their main role is to make statistical predictions.
As it is well known, a source of a negative attitude towards explanatory QM stems from Bell’s inequalities \[6, 7\]. Usual deterministic theories (i.e. realistic and local) satisfy Bell inequalities. Since QM violates Bell inequalities \[9, 10\], then this is usually interpreted as the non existence of a deterministic underpinning for quantum mechanics. However, assuming realism at least, we can say with this only that QM (as well as Bohmian Quantum Mechanics) is a non local theory\(^1\).

It could exist, in principle, a deterministic basis for quantum mechanics (maybe at the Planck scale), which looks non local to a human observer (at a few TeV scale). In ’t Hooft approach non locality, the distinctive character which a theory wishing to mimic quantum mechanics should possess, is achieved via the formation of equivalence classes through an information loss mechanism.

’t Hooft himself provided explicit examples of be-able systems that give rise, at a macroscopic level, to a genuine quantum behavior \([1, 11]\). This line of thoughts has been further progressed, e.g., in Refs. \([12, 13, 14, 15, 16, 17]\). For consistency of the pre-quantization scheme it is, however, important to show how an interacting composite quantum system can be obtained starting from deterministic classical subparts. Besides, whenever an interacting composite QM system is obtained from a be-able dynamics, then also its identifiable subsystems should be fully quantal. This in particular means that both the composite QM system and its identifiable subsystems must have their respective energies bounded from below. Up to now such a construction was missing in the literature. Aim of this paper is to fill this gap.

The plan of this paper is as follows: In Section 2 we present some fundamentals of ’t Hooft’s pre-quantization method. Section 3 is dedicated to non-interacting emergent systems. In particular, we discuss a simple example of a be-able dynamics that gives rise, at an emergent level, to two non-interacting QM oscillators. A non-trivial extension to interacting oscillators is presented in Section 4. There we show how two Bateman oscillators can give rise to a composite QM system — an isotonic oscillator, which, in the limit of large separation between subparts, decouples into two non-interacting QM oscillators. Conclusions are in Section 5.

\section{’t Hooft pre-quantization method}

In this section we outline ’t Hooft’s continuous-time pre-quantization method \([11]\). To this end we start with the assumption that the dynamics at the primordial deterministic level is described by the Hamiltonian

\[ H = \sum_i p_i f_i(q) + g(q). \]

Here \(f_i\) and \(g\) are functions of \(q = \{q_1, \ldots, q_n\}\). Note that the equation of motion for \(q_i\), i.e.

\[ \dot{q}_i = f_i(q), \]

is autonomous, since the \(p_i\) variables are decoupled. The above system is obviously deterministic, however its Hamiltonian is not bounded from below. Because of the autonomous character of the dynamical equation \(\dot{q}_i = f_i(q)\) one can define a formal Hilbert space \(\mathcal{H}\) spanned by the states \(|q\rangle; q \in \mathbb{R}^n\), and associate with \(p_i\) the operator \(\hat{p}_i = -i\partial/\partial q_i\). It is not difficult to see that the generator of time translations, i.e. the “Hamiltonian” operator \(H = \sum_i \hat{p}_i f_i(\hat{q}) + g(\hat{q})\) generates precisely the operatorial version of the deterministic evolution equation \(\dot{q}_i = f_i(q)\) (in Heisenberg picture),

\[ \dot{\hat{q}}_i = f_i(\hat{q}). \]

In other words, the operators \(\dot{\hat{q}}_i\) evolve deterministically even after “quantization”. This evolution is only between base vectors. From the Schrödinger picture point of view this means

\(^1\) Recently some authors have pointed out also possible non realistic aspects of QM, see Ref.\([8]\)
that the state vector evolves smoothly from one base vector to another. Because of this, there is no (non-trivial) linear superposition of the state vector in terms of base vectors and hence no interference phenomenon shows up when measurements of q-variables – the be-ables – are performed.

The Hamiltonian (1) is unbounded from below. In principle, this fact could create problems with physics: partition functions, equilibrium states, etc. all require in general bounded Hamiltonians. However, this should not disturb us too much since the Hamiltonian (1) governs (via an operatorial formulation) the be-able dynamics, not the observables dynamics. On the contrary, the emergent Hamiltonian, describing the physical observables, must be bounded from below. The lower bound is, in the ’t Hooft proposal, just an emergent property formed during the coarse graining of the be-able degrees of freedom down to the observational ones, and it will be constructed by implementing the information loss constraint (ILC).

We review now a simple mechanism showing how a lower bound for \( H \) may develop. Consider \( \rho(\hat{q}) \) to be some positive function of \( \hat{q}_i \) (but not \( \hat{p} \)) with \( [\hat{\rho}, \hat{H}] = 0 \). One then defines the splitting

\[
\hat{H} = \hat{H}_+ - \hat{H}_-,
\]

\[
\hat{H}_+ = \left( \hat{\rho} + \hat{H} \right)^2 \frac{\hat{\rho}^{-1}}{4},
\]

\[
\hat{H}_- = \left( \hat{\rho} - \hat{H} \right)^2 \frac{\hat{\rho}^{-1}}{4},
\]

where \( \hat{H}_+ \) and \( \hat{H}_- \) are positive-definite operators satisfying

\[
[H_+, H_-] = [\hat{\rho}, \hat{H}\pm] = 0.
\]

At this point we introduce the “coarse-graining” operator \( \hat{\Phi} \) that describes the loss of information that is happening during the passage from the be-able to observational scale. The operator originally considered by ’t Hooft [1, 11] is

\[
\hat{\Phi} = \hat{H}_-.
\]

The operator \( \hat{\Phi} \) is then implemented as a constraint on the Hilbert space \( \mathcal{H} \). The observed states, which we call physical states \( |\psi\rangle_{\text{phys}} \), are given by the condition

\[
\hat{\Phi}|\psi\rangle_{\text{phys}} = \hat{H}_-|\psi\rangle_{\text{phys}} = 0.
\]

This equation identifies the states that are still distinguishable at the observational scale\(^2\). The resulting physical state space, i.e. the space of observables, has the energy eigenvalues that are trivially bounded from below, thus in the Schrödinger picture the equation of motion

\[
\frac{d}{dt}|\psi_t\rangle_{\text{phys}} = -i\hat{H}_+|\psi_t\rangle_{\text{phys}},
\]

has only positive frequencies on physical states.

3. Two non-interacting QM oscillators
In this section we discuss a simple example of a be-able dynamics giving rise at an emergent level to two non interacting quantum oscillators. Our exposition closely follows the procedure presented in Ref. [12]. We start with a primordial dynamics described by two (classical) Bateman oscillators (BO’s). The system of two decoupled BO’s is described by the equations of motion (\( i = A, B \))

\[
m_i\ddot{x}_i + \gamma_i\dot{x}_i + \kappa_i x_i = 0, \quad m_i\ddot{y}_i - \gamma_i\dot{y}_i + \kappa_i y_i = 0,
\]

\(^2\) For a discussion about the Dirac classification of constraint (6) and on the gauge transformations generated by \( \hat{\Phi} \) see Ref. [14]
where \( m_i = (m_A, m_B), \gamma_i = (\gamma_A, \gamma_B) \) and \( \kappa_i = (\kappa_A, \kappa_B) \). This suggests that under appropriate boundary conditions the \( y_i \)-oscillator is the time-reversed image of the \( x_i \)-oscillator. In order to show that the Hamiltonians \( H_i \) for the systems (9) belongs to the class of \('t\) Hooft’s Hamiltonians, it is convenient to rewrite the former system in a rotated coordinate frame, i.e. \( x_{i1} = (x_i + y_i) / \sqrt{2} \), \( x_{i2} = (x_i - y_i) / \sqrt{2} \). Then the corresponding \( i \)th Hamiltonian reads

\[
H_i = \frac{1}{2m_i} (p_i^2 - p_i^2) - \frac{\gamma_i}{2m_i} (p_{i1} x_{i2} + p_{i2} x_{i1}) + \frac{1}{2} \left( \kappa_i - \frac{\gamma_i^2}{4m_i} \right) (x_{i1}^2 - x_{i2}^2).
\]

The algebraic structure for the total system \( H_T = H_A + H_B \) is the one of \( su(1,1) \otimes su(1,1) \). Indeed, from the dynamical variables \( p_{ai} \) and \( x_{ai} \) one may construct the functions

\[
J_{1i} = \frac{1}{2m_i \Omega_i} p_{i1} p_{i2} - \frac{m_i \Omega_i}{2} x_{i1} x_{i2}, \quad J_{2i} = \frac{1}{2} (p_{i1} x_{i2} + p_{i2} x_{i1}),
\]

\[
J_{3i} = \frac{1}{4m_i \Omega_i} \left( p_{i1}^2 + p_{i2}^2 + \frac{m_i \Omega_i}{4} \right) (x_{i1}^2 + x_{i2}^2).
\]

Here \( \Omega_i = \sqrt{\frac{1}{m_i} (\gamma_i - \frac{\gamma_i^2}{4m_i})} \), with \( \kappa_i > \frac{\gamma_i^2}{4m_i} \). The quadratic Casimirs for the Poisson algebra of the functions (11) are defined as \( C_i^2 = J_{3i}^2 - J_{1i}^2 - J_{2i}^2 \). The \( C_i \) explicitly read as

\[
C_i = \frac{1}{4m_i \Omega_i} \left[ (p_{i1}^2 + p_{i2}^2) + m_i^2 \Omega_i^2 (x_{i1}^2 - x_{i2}^2) \right].
\]

In terms of \( J_{2i} \) and \( C_i \) the Hamiltonians \( H_i \) can be formulated as

\[
H_i = 2 (\Omega_i C_i - \Gamma_i J_{2i}),
\]

with \( \Gamma_i = \gamma_i / 2m_i \). Eq. (13) indicates that \( H_i \) are now in the \('t\) Hooft form, with the \( C_i \) and \( J_{2i} \) playing the role of \( p_i \)'s, and the \( \Omega_i \) and \( \Gamma_i \) that of \( f(q_i) \)'s. By introducing hyperbolic coordinates, i.e. \( x_{i1} = r_i \cosh u_i, \quad x_{i2} = r_i \sinh u_i, \quad (r_i \in \mathbb{R}; u_i \in \mathbb{R}) \), one can cast \( J_{2i} \) and \( C_i \) in a particularly simple form (See Ref. [12]).

To consider the \textit{operatorial} description of this be-able dynamics, we promote all the relevant quantities, \( H_i, C_i, J_{2i} \) and \( \mathfrak{q}_i \) to operators. We can then properly speak about their commutators, and we note that not only \( \mathfrak{q}_i \) are be-ables but also (and this is specific for Bateman’s system) \( C_i \) and \( J_{3i} \) are be-ables (in fact they all commute among themselves, and, of course, with \( H_i \)).

\('t\) Hooft’s ILC can now be enforced according to prescription (4), (6), and (7). Choosing \( \hat{\rho} = 2 \Omega_i \hat{C}_i \), and taking \( \hat{C}_i > 0 \) (this can be done, because \( \hat{C}_i \) are constants of motion), the splitting reads

\[
\hat{H}_{i+} = \frac{1}{2 \Omega_i \hat{C}_i} (2 \Omega_i \hat{C}_i - \Gamma_i \hat{J}_{2i})^2, \quad \hat{H}_{i-} = \frac{1}{2 \Omega_i \hat{C}_i} \Gamma_i^2 \hat{J}_{2i}^2.
\]

The quantization emerges after one enforces ILC, i.e., \( \hat{H}_{i-} |\psi\rangle_{phys} = 0, \ i = A, B \) which defines/selects the physical states. The same constraint is equivalent to \( \hat{J}_{2i} |\psi\rangle_{phys} = 0 \), and thus allows us to get the lower bound for the Hamiltonian by projecting out the states responsible for the negative part of the spectrum. In fact this implies \( \hat{H}_i |\psi\rangle_{phys} = \hat{H}_{i+} |\psi\rangle_{phys} = 2 \Omega_i \hat{C}_i |\psi\rangle_{phys}, \) and

\[
2 \Omega_i \hat{C}_i |\psi\rangle_{phys} = \left[ \frac{1}{2m_i} \left( \hat{p}_{r_i}^2 + m_i^2 \Omega_i^2 \hat{r}_i^2 \right) - \frac{2 \hat{J}_{2i}}{m_i \Omega_i^2} \right] |\psi\rangle_{phys} = \left( \hat{p}_{r_i}^2 + \frac{m_i^2 \Omega_i^2 \hat{r}_i^2}{2m_i^2} \right) |\psi\rangle_{phys}.
\]

We have thus reproduced two independent quantum mechanical oscillators on the physical states \( |\psi\rangle_{phys} \). Each of the Hamiltonians \( H_A \) and \( H_B \) reduces independently to the Hamiltonian of a QM oscillator. Two independent BO’s give rise to two independent QM oscillators.
4. Composite quantum system

So far, we have only doubled the system already studied in Ref. [12]. Now we wish to show how the implementation of 't Hooft’s ILC on the global system dictates the form of the interaction between the emergent QM oscillators. To this end we consider together, as a single composite system, the two BO’s (10). The total be-able Hamiltonian can be written as

$$H_T = H_A + H_B = 2(\Omega_A C_A + \Omega_B C_B) - 2(\Gamma_A J_{2A} + \Gamma_B J_{2B}).$$

(15)

Since $C_i$ are constants of motion (being Casimirs of the respective $su(1,1)$ algebras), this ensures that once they are chosen to be positive (as we do from now on), they remain such at all times. Also $J_{2i}$ are constants of motion ($\{H_i, J_{2i}\} = 0$). We can therefore define new integrals of motion:

$$C = \frac{\Omega_A C_A + \Omega_B C_B}{\Omega}, \quad J = \frac{\Gamma_A J_{2A} + \Gamma_B J_{2B}}{\Gamma},$$

(16)

where $\Omega$ and $\Gamma$ are opportually defined numbers. Using the fact that $\Omega_i > 0$ and assuming that $\Omega > 0$, we may conclude that $C_A, C_B > 0 \Rightarrow C > 0$. The positivity of $C$ is guaranteed by our choice $C_i > 0$. The total Hamiltonian will be

$$H_T = 2\Omega C - 2\Gamma J.$$}

(17)

With the choice $\rho = 2\Omega C$, $H_T$ can be split à la (14) as

$$H_+ = \frac{1}{2\Omega C}(2\Omega C - \Gamma J)^2, \quad H_- = \frac{1}{2\Omega C}\Gamma^2 J^2.$$}

(18)

Given that $\rho = 2\Omega C$, then $\rho$ is a positive integral of motion. We note that $C, J$ are again be-ables because they are functions of be-ables. We impose the constraint on the observational scale in the form

$$\hat{H}_- |\psi\rangle_{\text{phys}} = \hat{J} |\psi\rangle_{\text{phys}} = 0.$$}

(19)

This implies $\hat{H}_T \approx \hat{H}_+ \approx 2\Omega \hat{C}$, (≈ indicates that operators are equal only on the physical states). Since $\hat{J} = (\Gamma_A \hat{J}_{2A} + \Gamma_B \hat{J}_{2B})/\Gamma$, the condition $\hat{J} \approx 0$ implies a relation between $\hat{J}_{2A}$ and $\hat{J}_{2B}$ which in turn implies an correlation between $\hat{H}_A$ and $\hat{H}_B$. Solving with respect to $\hat{J}_{2B}$, Eq.(19) gives $\hat{J}_{2B} \approx -\Gamma_A \hat{J}_{2A}/\Gamma_B$, which, when substituted into $\hat{H}_T = 2\Omega \hat{C} - 2\Gamma \hat{J}$, yields (using hyperbolic coordinates)

$$\hat{H}_T \approx \hat{H}_+ \approx \left(\frac{\hat{p}_{r_B}^2}{2m_A} - \frac{2\hat{J}_{2A}^2}{m_A \hat{r}_A^2} + \frac{1}{2} m_A \Omega_A^2 \hat{r}_A^2\right) + \left(\frac{\hat{p}_{r_B}^2}{2m_B} + \frac{1}{2} m_B \Omega_B^2 \hat{r}_B^2\right) - \frac{2}{m_B} \frac{\Gamma_A^2}{\Gamma_B} \frac{1}{\hat{r}_B^2} \hat{J}_{2A}^2.$$}

(20)

The emergent Hamiltonian $H_T$, Eq.(20), represents a good quantum system, since, by its very construction, it is bounded from below on the physical states. The term inside the first parenthesis is simply $2\Omega A \hat{C}_A$. Such a term is constant, because $\hat{C}_A$ is an integral of motion. The second term represents a QM oscillator, while the third corresponds to a centripetal barrier. The inverse square potential $1/r^2$ is analogous to the centrifugal contribution in polar coordinates and one may thus expect an exact solvability. The only difference here is that $r \in R$ and not merely $R^+$. The system with the Hamiltonian

$$H = \frac{N^2}{2} \frac{\hat{p}_{r_B}^2}{2} + \frac{Q^2}{2} \frac{\hat{r}_B^2}{2} + \frac{R^2 - N^2/4}{2r_B^2},$$}

(21)
and $r_B \in \mathbb{R}$, $N,Q,R \in \mathbb{R}^+$, is known as the isotonc oscillator [18]. Its spectrum can be exactly solved by purely algebraic means, since the Hamiltonian admits a shape-invariant factorization [19]. In our case the eigenvalue equation for $\hat{J}_2A$ reads $\hat{J}_2A|\Psi_{j,n}\rangle = \mu_A|\Psi_{j,n}\rangle$, and since we have

$$N^2 = \frac{1}{m_B}, \quad Q^2 = m_B\Omega_B^2, \quad R^2/N^2 = \frac{1}{4} - \left(\frac{2\Gamma_A}{\Gamma_B} \mu_A\right)^2,$$

the actual spectrum of (20) is

$$E_{n,\pm}(c, \mu_A) = \Omega_B \left(2n \mp \sqrt{\frac{1}{4} - \left(\frac{2\Gamma_A}{\Gamma_B} \mu_A\right)^2} + c\right), \quad n \in N,$$

(22)

where $c$ is a shift constant term due to the presence of $2\Omega_A\hat{c}_A$. Note that when $2\Gamma_A\mu_A/\Gamma_B \ll 1/2$, the inverse square potential in (20) can be neglected and the system reduces to that of a QM linear oscillator with a shift term $c$. This follows also directly from the spectrum (22) provided we set $\Gamma_A = 0$ and consider both signs.

A further important interpretation of the emergent system (20) can be obtained when $\mu_A$ is an imaginary number, i.e., when $\hat{J}_2A$ belongs to a non-unitary realization of the $D_j$ series. Then $R/N > 1/2$, the potential in (21) is repulsive at the origin, and the motion takes place only in the domain $r_B \geq 0$. This, in particular, allows to view $r_B$ as a radial coordinate and the inverse square potential in (20), (21) as a rotationally invariant interaction of the spin-orbit type. To see this we rewrite the interaction potential in (20) in the form

$$\hat{H}_{int} \approx \frac{2}{m_B} \frac{\Gamma_A}{\Gamma_B} \frac{1}{r_B} \frac{\partial V}{\partial r_B} (\hat{J}_B \cdot \hat{J}_A), \quad V = \log r_B.
$$

(23)

Here we have used $\hat{J}_2B \approx -\Gamma_A \hat{J}_2A/\Gamma_B$. We now formally identify $\hat{J}_B = i\hat{L}$ and $\hat{J}_A = i\hat{S}$, where $\hat{L}$ plays the role of the orbital angular momentum of the “particle” $B$ and $\hat{S}$ plays the role of its “spin”. With this, the interaction energy reads

$$H_{int} = -\frac{2}{m_B} \frac{\Gamma_A}{\Gamma_B} \frac{1}{r_B} \frac{\partial V}{\partial r_B} (\hat{L} \cdot \hat{S}) \equiv -\frac{g}{2mc^2} \frac{1}{r_B} \frac{\partial V}{\partial r_B} (\hat{L} \cdot \hat{S}),
$$

(24)

where we have identified $\Gamma_A/\Gamma_B$ with $g/4c^2$, i.e. with a quarter of the gyromagnetic factor. The function $V$ thus plays the role of the planar radial scalar electromagnetic potential. So, the emergent QM system can be alternatively interpreted also as a particle in a magnetic field, with a spin-orbital interaction term.

Far from the interaction region (large $r_A$, $r_B$), the interaction is switched off and the asymptotic emergent QM Hamiltonian is

$$\hat{H}_T \approx \hat{H}_+ \approx \left(\frac{\hat{p}_{r_A}^2}{2m_A} + \frac{1}{2}m_A\Omega_A^2 r_A^2\right) + \left(\frac{\hat{p}_{r_B}^2}{2m_B} + \frac{1}{2}m_B\Omega_B^2 r_B^2\right),
$$

(25)

The total system and its subsystems are still genuine QM systems because both the oscillators are bounded from below. Therefore, far from the interaction region we have two non-interacting QM oscillators. Let us finally mention two points. Firstly: before imposing the ILC on the global system, we have two non-interacting, independent quantum oscillators. Only the enforcement of the dissipation constraint on the global system gives rise to the interaction $J_2A \leftrightarrow J_2B$. So the constraint dictates the interaction. Secondly, the interaction term depends on the dissipations constants $\Gamma_A, \Gamma_B$. In particular the interaction can be switched off by setting $\Gamma_A \rightarrow 0$. So the dissipation plays a key rôle in the interaction.
5. Conclusions
With this paper we have tried to convey two main messages: Firstly, that quantum mechanics does not really preclude the existence of an underlying explanatory basis; and secondly, that it is not difficult to formulate a genuine quantum mechanical composite system which originates from an underlying deterministic dynamics, and yet where both the total system and its identifiable sub-parts have their respective energies bounded from below. In the present example, a be-able dynamics is described by two Bateman oscillators. On an emergent level they give rise to a composite QM system — an isotonic oscillator, which in the limit of large separation between subparts, decouples into two non-interacting QM oscillators.

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