Sparkling saddle loops of vector fields on surfaces

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Abstract

An orientation-preserving non-contractible separatrix loop of a hyperbolic saddle of a vector field on a two-dimensional surface may be accumulated by a separatrix of the same saddle. When the loop is unfolded, new saddle loops appear. We study the unfolding of such loops in generic one-parameter families of vector fields as a semi-local bifurcation. As a byproduct, we construct a countable family of pairwise non-equivalent germs of bifurcation diagrams that appear in locally generic one-parameter families.

1 Introduction

Suppose a vector field \( v \) on a closed two-dimensional surface \( M \) has a hyperbolic saddle \( P \) with a separatrix loop \( \gamma \) which is orientation-preserving, that is, when we travel once along \( \gamma \), the local orientation does not change. Small neighborhoods of \( \gamma \) are cylinders then. The loop splits such neighborhood into a monodromic and a non-monodromic semi-neighborhoods. In what follows, those separatrices of the saddle that are not involved in the loop will be called free. If the loop \( \gamma \) is also non-contractible, it may happen that one of the free separatrices comes into a monodromic semi-neighborhood of the loop and winds onto the loop. If we want it to be the outgoing separatrix, we have to impose that the sum of eigenvalues of our saddle be non-positive. In fact, we will assume that it is strictly negative, since zero sum of eigenvalues and the loop are not observed together in generic one-parameter families of vector fields. We will call saddles with negative sum of eigenvalues dissipative.

There are two possibilities then: either the local orientation is preserved when we travel along the free separatrix back to the saddle \( P \), or it is inverted, and we will focus on the first case. In this case a small neighborhood of the unstable manifold of the saddle is topologically a torus without a disk, also known as handle. It immediately follows from classification of surfaces that a vector field with such properties cannot be supported by the sphere, projective plane, and Klein bottle, because a handle cannot be embedded in these surfaces. On the other hand, it is not difficult to see (and we will see this below) that for all other closed connected surfaces such fields exist. We will need a shorthand notation for such fields.

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Definition 1. Let \( v \) be a \( C^\infty \)-smooth vector field on a smooth closed two-dimensional surface \( M \). We say that \( v \) belongs to the class \( \mathfrak{C}^+ (M) \) if the following two conditions hold.

1. The vector field \( v \) has a dissipative saddle \( P \) with a saddle loop which is orientation-preserving.

2. The unstable separatrix of \( P \) that is not involved in the loop winds onto it, and a small neighborhood of the unstable manifold of \( P \) is homeomorphic to a torus without a disk.

It is well-known (see, e.g., [IS] and references therein) what happens when a saddle loop of a vector field on the plane is unfolded in a generic one-parameter family. On one side of the critical parameter value a hyperbolic limit cycle is born. Moreover, if there are separatrices of other saddles that wind onto the loop, the critical value is accumulated from the other side by a sequence of parameter values that correspond to saddle connections between these saddles and the original one. These are called sparkling saddle connections.

The case of a field \( v \in \mathfrak{C}^+ (M) \) is analogous, except connections are formed by the separatrices of the same saddle, so they can be called sparkling saddle loops. It turns out that when a new loop of this type is formed, it is automatically accumulated by the free outgoing separatrix of the saddle, so unfolding this loop we get a new generation of loops and fields of class \( \mathfrak{C}^+ (M) \), etc. It is this fascinating proliferation of sparkling loops that we want to draw the reader’s attention to. We will see that the closure of the set of parameters that correspond to the presence of these loops is a Cantor set and these parameters themselves are the endpoints of the intervals of its complement.

Before we state the results, a digression on bifurcation diagrams is in order. For a family of vector fields, the bifurcation diagram is the set of parameter values that correspond to vector fields that are not structurally stable. Two bifurcation diagrams are deemed equivalent if the first can be taken into the second by a homeomorphism of the parameter space. Any closed subset of the parameter space, regardless of its dimension, can be a bifurcation diagram for a sufficiently degenerate family: take a family obtained by multiplying some Morse-Smale field by a smooth function of the parameter that vanishes exactly at the closed subset of choice. However, it is natural to ask what diagrams, or rather different germs of diagrams, can occur in locally generic families.

First, it was conjectured by V. Arnold that for the case of vector fields on the sphere there exists but a finite number of pairwise non-equivalent germs of bifurcation diagrams that may occur in locally generic \( k \)-parameter families at the critical value, for any \( k \). This conjecture was disproved when a countable family of different germs of bifurcation diagrams was found in [KS] for three-parameter families that unfold the polycycle ensemble called the lips. Then Yu. Ilyashenko realized that infinitely many germs can be observed in two-parameter families [I]; D. Filimonov and I. Schurov have another proof, yet unpublished. On the other hand, germs of generic one-parameter families of vector fields on the sphere have been classified in [IS], [IGS], and [St], and this classification yields, in particular, that these families admit only two nonempty germs of bifurcation diagrams,

\( ^1 \)Two germs of diagrams, say, at 0 are equivalent if they have equivalent representatives and the homeomorphism that realizes the equivalence of the representatives takes 0 to 0.
up to equivalence: the first consists of one point at the origin, the second is the union of the point at the origin and a sequence that monotonously converges to it.

It turns out that due to the phenomenon discussed in this paper, on most other two-dimensional surfaces this is not the case.

**Theorem A.** Let $M$ be a compact smooth two-dimensional surface other than the sphere, Klein bottle, and projective plane. Then there exists a countable family of pairwise non-equivalent germs of bifurcation diagrams each of which is realized on an open set in the space of smooth one-parameter families of vector fields on $M$.

**Remark 2.** Let us denote the germs of bifurcation diagrams that will appear in the proof of Theorem A by $K_n$, $n \in \mathbb{N}$. A representative of $K_n$ has the following structure: it is a union of a Cantor subset of the real line, of the sparkling-loops origin, and a sequence of points. The Cantor subset contains zero and lies on one side from it and the sequence intersects each inner interval of the complement to the Cantor set by exactly $n$ points and has no other points.

However, the main result of the present paper is the following theorem.

**Theorem B.** Let $v \in \mathcal{C}_+(M_1)$ and $w \in \mathcal{C}_+(M_2)$. For generic smooth families $V = \{v_\theta\}_{\theta \in [-1,1]}$, $v_0 = v$, and $W = \{w_\theta\}_{\theta \in [-1,1]}$, $w_0 = w$, there exist segments $0 \in J_1, J_2 \subset [-1,1]$ and neighborhoods $U_1 \subset M_1, U_2 \subset M_2$ of $W^u(P_{v_0})$ and $W^u(P_{w_0})$ respectively such that for the restrictions $\overline{V} = V|_{J_1 \cup J_2}$ and $\overline{W} = W|_{J_1 \cup J_2}$ we have the following.

- The families $\overline{V}$ and $\overline{W}$ are strongly equivalent; each is strongly structurally stable.
- $\text{Bif}(\overline{V})$ is a Cantor set $K$ that contains 0 and lies on one side from it. The boundaries of the intervals of $J_1 \setminus K$ correspond to vector fields with a separatrix loop for $P$. Whenever there is a separatrix loop for $P$, the free unstable separatrix winds onto it.
- $\text{Bif}(\overline{V})$ has Hausdorff dimension zero (and, therefore, zero Lebesgue measure).

The last assertion of the theorem means that there is almost no hope that two Cantor sets of this origin can have robust intersection which could potentially lead to examples of generic one-parameter families where separatrix loops coexist with non-trivial recurrent trajectories for some values of the parameter.

We prove both theorems essentially by reducing them to the results obtained by C. Boyd for families that unfold simple Cherry fields on $\mathbb{T}^2$. Sparkling saddle loops were observed in such families, but, it seems, were never regarded as an origin of the bifurcation, perhaps because irrational rotation numbers were there to blame.

### 2 Families of vector fields on surfaces

Hereinafter, “smooth” always stands for “$C^\infty$-smooth” if the contrary is not written explicitly. The space of smooth vector fields on a smooth manifold $M$ will be denoted

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2Inner intervals of the complement to a Cantor subset of the line are those that are subsets of its convex hull.
by $\text{Vect}^\infty(M)$. Our manifolds will usually be closed, connected two-dimensional surfaces. Such surface may be viewed, in a unique way, as a sphere, projective plane, or Klein bottle with zero or more handles attached. A handle is a torus with a disk removed and by attaching a handle to a surface we mean taking the smooth connected sum of the surface and the handle. In what follows, a surface with a handle is a closed, connected two-dimensional surface different from the sphere, projective plane, and Klein bottle.

A family of vector fields on a smooth manifold $M$ with base $B$ is a smooth map $V: B \to \text{Vect}^\infty(M)$, or, equivalently, a smooth vector field on $B \times M$ that is tangent to the fibers $\{b\} \times M$. We will consider one-parameter families, with base equal to some segment, usually $[0, 1], [-1, 1]$, or $[-\varepsilon, \varepsilon]$.

**Definition 3.** Two families $V = \{v_\theta\}_{\theta \in B}$ and $W = \{w_\tau\}_{\tau \in \hat{B}}$ of vector fields on homeomorphic manifolds $M$ and $\hat{M}$ are called strongly equivalent if there exists a homeomorphism $H: B \times M \to \hat{B} \times \hat{M}$ of the form

$$(\theta, x) \mapsto (h(\theta), H_\theta(x))$$

such that, for every $\theta \in B$, the map $H_\theta$ takes the phase portrait of $v_\theta$ into the one of $w_{h(\theta)}$, that is, takes trajectories to trajectories preserving the time-induced orientation. We say that the homeomorphism $H$ realizes this equivalence.

A family $V$ is strongly structurally stable if it is strongly equivalent to any family $W$ that is sufficiently close. There are other notions of equivalence and hence stability for families; see, e.g., [IKS, §1.1].

Throughout the paper we use standard dynamical systems notions and notation, for which the books [KH] and [PM] are a good reference.

### 3 Proof of Theorem A

#### 3.1 Idea of the proof

C. Boyd [B] has described the bifurcation diagrams for a particular open set of one-parameter families of vector fields on the torus. The vector fields in these families lie in a vicinity of simple Cherry fields that have exactly two singularities, namely a saddle $P$ and a sink $\Omega$. The bifurcation diagrams of these families are Cantor sets. We modify the families considered by Boyd by adding to the basin of the sink, or rather, to a neighborhood of the sink, $n$ additional Cherry cells each of which contains a saddle and a source. For every Cherry cell and for every vector field of the family, both unstable separatrices go to the sink $\Omega$ and and one stable separatrix goes to the source of the cell. The last stable separatrix leaves the neighborhood of the sink $\Omega$ and can create a separatrix connection with the unstable separatrix of the original saddle $P$. These connections add $n$ new bifurcation points into each interval of the complement to the Cantor set which was the bifurcation diagram for the original Boyd family. The bifurcation diagrams for different integers $n$ are not equivalent and, most importantly, they provide non-equivalent germs. Then it is not difficult to adapt this construction to the case of arbitrary surface with a handle.
3.2 Boyd’s families on the torus

Consider the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For a point in it, instead of writing $(x, y) + \mathbb{Z}^2$ or $[(x, y)]$, we will simply write $(x, y)$. On the circle $\Sigma = \{(x, y) \in \mathbb{T}^2 \mid x = 0\}$ there is a natural coordinate $y$ and the corresponding orientation. For a pair of points $a, b \in \Sigma$, we will denote by $(a, b) \subset \Sigma$ the open arc that starts at $a$ and goes in the positive direction until it reaches $b$. Whenever we differentiate a map from the circle to itself, it may be assumed that we take the derivative of the lift of our map to $\mathbb{R}$, and if we differentiate a family of maps in the parameter, it may be assumed that the whole family is lifted to $\mathbb{R}$.

At some point, we will need to change the coordinate on $\Sigma$. We will then assume that the new coordinate is given by some diffeomorphism $h : \mathbb{R}/\mathbb{Z} \to \Sigma$, maybe parameter-dependent, and so there is a global “circular chart”. We will also sometimes identify the points of $\Sigma$ and their vertical coordinates.

The following definition comes from [PM] and [B].

**Definition 4.** We will say that a $C^\infty$-smooth vector field $v$ on the torus $\mathbb{T}^2$ belongs to the class $\mathcal{N}$ if the following conditions hold.

1. The vector field $v$ has exactly two singular points: a hyperbolic saddle $P$ for which the sum of eigenvalues is positive and a hyperbolic sink $\Omega$.
2. The vector field $v$ is transverse to the circle $\Sigma = \{(x, y) \in \mathbb{T}^2 \mid x = 0\}$.
3. The local stable separatrices of the saddle $P$ first cross $\Sigma$ at points $a$ and $b$. One of the unstable separatrices of the saddle goes directly towards the sink $\Omega$ without intersecting $\Sigma$ and the second one first intersects $\Sigma$ at a point $c$.
4. For any point $y \in (a, b) \subset \Sigma$, the positive semi-orbit $\text{Orb}_+(y)$ goes straight to the sink $\Omega$ without re-intersecting $\Sigma$ and for the points $y \in (b, a) = \Sigma \setminus [a, b]$ the Poincaré map $F : (b, a) \to \Sigma$ is defined and is expansive: $f'(y) > \gamma > 1$ for all $y$.

**Remark 5.**

- If a hyperbolic saddle of a vector field on a surface has eigenvalues of positive sum, such saddle is called anti-dissipative or area-expansive. For the vector field of class $\mathcal{N}$ this property implies that $f'(y) \to +\infty$ as $y \to a - 0$ or $y \to b + 0$.

- The Poincaré map $f$ can be extended to the arc $[a, b]$ by setting $f([a, b]) = \{c\}$, where $c$ is the point where the unstable separatrix of $P$ first intersects $\Sigma$. Thus we get a continuous (non-strictly) monotonous map from $\Sigma$ to itself, of degree one. For this map the rotation number $\rho(f)$ is well-defined (see, [KH] Sect. 11.1, p. 392). In what follows, when we refer to the Poincaré map, we mean this extended map, and we will denote it by the same letter $f$ as the “true” Poincaré map.

- The class $\mathcal{N}$ is an open subset of the space $\text{Vect}^\infty(\mathbb{T}^2)$ of smooth vector fields on the torus.

The following theorem summarizes the results on the fields of class $\mathcal{N}$ obtained in [C], [PM], and [B]; see also [KH]. In short, the theorem says that the dynamics can be described in terms of the rotation number.

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3See also [PM] Lemma 3 at p. 184] and [Ha], [He].
Theorem 6 (C, PM, H). Let \( v \in \mathcal{N} \) and let \( f: \Sigma \to \Sigma \) be the corresponding extended Poincaré map.

1. If the rotation number \( \rho(f) \) is irrational, then
   
   (a) the free unstable separatrix of the saddle \( P \) intersects the transversal \( \Sigma \) infinitely many times, but it never intersects the closed arc \([a, b] \subset \Sigma; \)
   
   (b) the attraction basin \( W^s(\Omega) \) of the sink \( \Omega \) is dense in \( \mathbb{T}^2; \)
   
   (c) its complement \( \mathbb{T}^2 \setminus W^s(\Omega) \) is a transitive quasi-minimal set\(^4\) of zero measure that intersects \( \Sigma \) by a Cantor set of zero measure\(^5\).

2. If for the fields \( v, w \in \mathcal{N} \) the Poincaré maps have the same irrational rotation numbers, these fields are orbitally topologically equivalent.

3. If the rotation number \( \rho(f) \) of the Poincaré map for the vector field \( v \in \mathcal{N} \) is rational, then there are two possible cases:
   
   (a) either the free unstable separatrix of the saddle \( P \) intersects the transversal \( \Sigma \) finitely many times, the last intersection being at the point \( a \) or \( b \), and thus, there is a separatrix loop;
   
   (b) or the free unstable separatrix of the saddle \( P \) intersects \( \Sigma \) finitely many times and the last intersection is inside the open arc \((a, b) \subset \Sigma; \) then \( v \) is Morse-Smale.

   In both cases two fields of that type are orbitally topologically equivalent.

Note that when the rotation number is irrational, the field cannot be Morse-Smale due to the presence of a (non-trivial) quasi-minimal set, which is part of the non-wandering set (alternatively, we may argue that the rotation number can be made rational by a small perturbation).

Colin Boyd proved a strong stability result for quite specific families of vector fields of class \( \mathcal{N} \).

Theorem 7 (C. Boyd, [E]). Let \( V = \{v_\theta\}_{\theta \in [0, 1]}, \ v_\theta \in \mathcal{N}, \) be a \( C^1 \)-smooth one-parameter family of vector fields such that

\[
f_\theta(\cdot) = f_0(\cdot) + \theta,
\]

where \( f_\theta \) is the Poincaré map for the field \( v_\theta \). If \( v_0 \) is Morse-Smale, then the family \( V \) is strongly structurally stable.

Note that the family \( V \) in this theorem is very degenerate; in particular for all Poincaré maps the flat segment is the same and, as it is easy to see when looking at the asymptotic of the Poincaré map at the points \( a, b \), the ratio of eigenvalues of the saddle in this family does not depend on the parameter. Nevertheless, by the theorem, there are \( C^1 \)-families of general position which are equivalent to it, and the same can be said about \( C^\infty \)-families.

\(^4\)A quasi-minimal set is a set with a finite number of singular points such that every semi-orbit in this set not attracted to a singular point is dense in this set [KH, p. 465]. It can also be defined as the closure of a trajectory recurrent in both directions. It is often also assumed that this recurrent trajectory is nontrivial (i.e., neither a singular point nor a cycle) [ABZh, p. 83].

\(^5\)Moreover, this Cantor set has zero Hausdorff dimension [V].
3.3 Bifurcation diagrams of Boyd’s families

For convenience we will work only with $C^\infty$-families. Let us fix a $C^\infty$ family $V$ for which the assumptions and conclusion of Theorem 7 hold. The existence of such family is more or less obvious: it suffices to take a vector field $v \in \mathcal{N}$ and start rotating it in a vertical strip to the left from $\Sigma$. An explicit construction of of such family can be found in [PM] in the proof of Lemma 4 at p. 186 (see also [KH, p. 464]). We take as our $V$ a family like that, i.e., we require that outside some vertical strip that contains no singularities and is close to $\Sigma$ the vector fields of our family are exactly the same for all parameter values.

Let us also fix a small $C^\infty$-neighborhood $F$ of the family $V$ such that all families in $F$ are strongly equivalent to each other and contain only vector fields of class $\mathcal{N}$. Then bifurcation diagrams of these families are also equivalent: for every such family a parameter value is not in the bifurcation diagram if and only if the corresponding vector field is Morse-Smale, and the homeomorphism that realizes the strong equivalence of two families takes Morse-Smale fields to Morse-Smale ones

Let us look at the bifurcation diagram of the family $V$. Consider the corresponding family $F = \{f_\theta\}$ of Poincaré maps and the function $r(\theta) = \rho(f_\theta)$. By Theorem 4 if $r(\theta)$ is irrational, the corresponding vector field is not Morse-Smale, which yields that $A = \{\theta: r(\theta) \notin \mathbb{Q}/\mathbb{Z}\} \subset \text{Bif}(V)$. Furthermore, the family $F$ is monotonous in $\theta$; therefore $r$ is non-decreasing and, moreover, it is strictly increasing at points where it has irrational values, see [KH, Prop, 11.1.8-9], so a fixed irrational value is assumed at isolated points. Since $\theta \in [0,1]$ and condition $f_\theta = f_0 + \theta$ holds, any irrational value is assumed at exactly one point.

Now fix some $\theta$ such that $r(\theta)$ is rational. By Theorem 5 the unstable separatrix of $P_\theta$ intersects the arc $[a, b] \subset \Sigma$. This means that for some $k \in \mathbb{N}$ we have $f_\theta^{-1}(c) \in [a, b]$. Since $f_\theta(x)$ is non-decreasing in $x$ and strictly increasing in $\theta$, we can conclude that $f_\theta^{-1}(c)$ is also strictly increasing in $\theta$. This means that for the chosen rotation number the separatrix crosses $[a, b]$ when $\theta$ belongs to some segment. The endpoints of the segment correspond to existence of loops (the unstable separatrix comes to $a$ or $b$), whereas the interior of the segment corresponds to Morse-Smale vector fields. In the neighborhood of the endpoints the rotation number is non-constant: indeed, when $f_\theta^{-1}(c)$ goes past, say, $b$ as $\theta$ increases, it is no longer possible for the point $c$ to have a periodic orbit of period $k$, so the rotation number has to change.

Thus, we conclude that

$$\text{Bif}(V) = \{\theta: r(\theta) \notin \mathbb{Q}/\mathbb{Z}\} = \{\theta: v_\theta \text{ has a loop}\}.$$ 

Now it is clear that $\text{Bif}(V)$ is perfect and nowhere dense and hence it is a Cantor set.

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It is easy to construct examples of strongly equivalent families with non-equivalent bifurcation diagrams. E.g., imagine a family with a hyperbolic cycle and an equivalent family with a corresponding cycle that is not hyperbolic for some isolated parameter value. So, bifurcation diagram is not an invariant of topological classification of families, but it is an invariant if we consider only families that do not contain vector fields that are not structurally stable, but are orbitally topologically equivalent to structurally stable fields. Had we defined bifurcation diagram as the set of parameter values that do not have a neighborhood where all corresponding fields are equivalent, it would automatically be an invariant. However, we prefer our bifurcation diagrams reflect the lack of structural stability rather than indicate that the bifurcation is truly observed in the family.
### 3.4 Adding Cherry cells

We want to perform on the family $V$ a surgery that will add a Cherry cell into a neighborhood of the sink $\Omega$. We can define a Cherry cell as a pair of hyperbolic singularities: the first is a saddle and the second is a sink or a source that captures one of the separatrices of the saddle.

Recall that near the sink the vector fields of our family do not depend on the parameter. Consider a small flow box $\Pi_1$ near the sink, switch to the rectifying coordinates of this flow box, and for each parameter value replace the constant flow inside the box with a field that has a Cherry cell with a saddle $P_1$ and a source $A_1$. The outgoing separatrices of $P_1$ have to go to the sink $\Omega$, one incoming separatrix is captured by the source $A_1$ and the other is, in a sense, free, and has to cross the transversal $\Sigma$, see Fig. 1.

As we do this surgery, we can also make sure that, first, near the boundary of the flow box the fields of the family remain the same, and second, the Cherry cell slowly moves “downwards” as the parameter increases: that is, we want the point $d$ of intersection between the free stable separatrix of $P_1$ and $\Sigma$ to be strongly monotonous in $\theta$ in the sense that we must have $d'(\theta) < -\beta < 0$ for all $\theta$\footnote{Here we denote the coordinate of the point $d$ on the vertical circle by the same letter.}. We can, for example, in rectifying coordinates on $\Pi_1$ glue in the same Cherry cell with a shift by $\delta \cdot \theta$ perpendicular to the direction of the original constant field, where $\delta > 0$ is a small constant.

Denote by $B_1$ a small disk that is independent of the parameter, contains a neighborhood of the flow box $\Pi_1$, and for every parameter value is contained in the basin of the sink $\Omega$. Denote by $V_1$ the special family obtained from $V$ by adding one Cherry cell as described, and denote a small neighborhood of $V_1$ in the space of $C^\infty$-families by $\mathcal{F}_1$. We take $\mathcal{F}_1$ so small that for any family $W \in \mathcal{F}_1$ we have the following:

- the continuation of the saddle $P_1$ and source $A_1$ is in $B_1$ for every $\theta \in [0, 1]$;
- there exists a smoothly parameter-dependent curvilinear rectangle $\tilde{\Pi}_1(\theta)$ such that
  - $P_1(\theta), A_1(\theta) \in \tilde{\Pi}_1(\theta) \subset B_1$;
  - two opposite edges of $\tilde{\Pi}_1(\theta)$ are transverse segments and the other two edges are segments of trajectories;
  - inside $\tilde{\Pi}_1(\theta)$ the field can be replaced by a field that is smoothly equivalent to a constant one in such a way that that this yields a family of class $\mathcal{F}$;
- for the family $W$, the point of intersection between the free stable separatrix of the saddle $P_1$ and the circle $\Sigma$ strongly monotonously depends on the parameter in the same sense as above: namely, the derivative of its vertical coordinate in $\theta$ is negative\footnote{Here and below it is important that for two families of vector fields which are close the families of local stable (or unstable) manifolds of the hyperbolic continuations of some saddle are also close. Here we need them to be only $C^1$-close, but below we will need $C^3$-closeness. However, this also holds in $C^r$,
$1 \leq r < \infty$. This can be justified in the following way. Consider the family as one vector field $v_\theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta}$ on $T^2 \times [0, 1]$. For this field the local central-stable manifolds of the saddles $(P_\theta, \theta)$ are uniquely defined and}
We define special families $V_k$ and their open neighborhoods $F_k$, $k \in \mathbb{N}$, analogously, but in $F_k$ families have $k$ Cherry cells instead of one. The conditions above must hold for each Cherry cell, with disjoint $B_j \supseteq \tilde{\Pi}_j(\theta)$.

### 3.5 Required bifurcation diagrams for families of fields on $\mathbb{T}^2$

Note that, by construction of the set $F_k$, for any family $W \in F_k$ there exist (parameter-independent) neighborhoods $B_1, \ldots, B_k$ of the Cherry cells and a family $\overline{W} \in F$ such that $W$ and $\overline{W}$ coincide in restriction to $\mathbb{T}^2 \setminus \sqcup B_j$. This implies that $\text{Bif}(W) \subset \text{Bif}(\overline{W})$. Indeed, if the field $\overline{w}_\theta$ has a saddle loop, the same holds for the field $w_\theta$, and $\text{Bif}(\overline{W})$ is the closure of the set of such parameter values. Now, consider a parameter value $\theta \notin \text{Bif}(\overline{W})$. For this value the vector field $\overline{w}_\theta$ is Morse-Smale. The field $w_\theta$ cannot have cycles that intersect the disks $B_j$, so all its cycles are the same as for the field $w_\theta$. Actually, the Poincaré map being expansive implies that the field has only one repelling cycle, and it is hyperbolic. All singular points of $w_\theta$ are hyperbolic as well. The non-wandering set of $w_\theta$ contains only cycles and singularities, and there are no saddle loops for the saddle $P$. Peixoto’s theorem on structural stability yields then that the only way the field $w_\theta$ can be not structurally stable is by having a saddle connection between the saddle $P$ and some saddle $P_j$ (connections between saddles $P_i$ and $P_j$ are impossible by construction).

We want to show that in each interval of the complement to $\text{Bif}(\overline{W})$ there is exactly one parameter value that corresponds to the field with a separatrix connection between $P$ and $P_j$, for each $j$. For that we need the following proposition which we will prove below in section 3.8.
Proposition 8. Let $V = \{v_\theta\}_{\theta \in [0,1]}$, $v_\theta \in \mathcal{N}$, be the $C^\infty$-family defined above in Section 3.3, and let some $\varepsilon > 0$ be fixed. Then, if a family $W$ is sufficiently close in the $C^\infty$-topology to the family $V$, there exists a ($C^3$-smooth, at least) parameter-dependent coordinate on $\Sigma$ that coincides with the original coordinate outside $\varepsilon$-neighborhoods of the points $a, b$ such that the family of the Poincaré maps $g_\theta$ of the family $W$, when written in the new coordinate, is strongly monotonous in the parameter $\theta$:

$$\forall y \in \mathbb{R}/\mathbb{Z}, \forall \theta_0 \in [0,1], \text{ we have } \frac{\partial}{\partial \theta} \bigg|_{\theta=\theta_0} \hat{g}_\theta(y) > 0,$$

where $\hat{g}_\theta$ is $g_\theta$ written in the new coordinate.

Remark 9. Without switching to parameter-dependent coordinates this would not hold. Consider, for example, a family obtained from $V$ by precomposition with a vertical shift by $\varepsilon \theta$:

$$w_\theta(x, y) = v_\theta(x, y - \varepsilon \theta).$$

If the constant $\varepsilon > 0$ is small, this family is close to the family $V$, but the derivative of the Poincaré map $g_\theta(y) = f_\theta(y - \varepsilon \theta) + \varepsilon \theta$ in $\theta$ tends to $-\infty$ as $y \to a + \varepsilon \theta$ or $y \to b + \varepsilon \theta$, as direct calculation shows. The reason is that we multiply by $(f_\theta)'_y$, which is unbounded. However, it is clear that if we look at this family of Poincaré maps via the parameter-dependent coordinates change $(x, y) \mapsto (x, y - \varepsilon \theta)$, we will again see the family $\{f_\theta\}$ that is monotonous in the parameter.

Let us continue proving the theorem. Assume that for some fixed parameter value there is a saddle connection between the saddle $P$ and, say, the saddle $P_1$. Proposition 8 implies that in appropriate chart the iterates $\hat{g}_\theta^\circ \cdot$ of the Poincaré map are monotonous in the parameter. Take the last (if we count from the saddle) point of intersection between the unstable separatrix of $P$ that forms the connection and the transversal $\Sigma$. This point monotonously depends on the parameter $\theta$. On the other hand, by one of the properties of the class $\mathcal{F}_k$, the vertical coordinate of the first intersection point between the free stable separatrix of $P_1$ with this circle monotonously decreases with the parameter. Note that we can assume that this point is always far from the points $a, b$, so monotonicity is preserved when switching to the new chart. Therefore, the saddle connection happens at a unique point in the interval of the complement to $\text{Bif}(\overline{W})$. The same argument works for connections between $P$ and other saddles, hence the bifurcation diagram $\text{Bif}(W)$ consists of the Cantor set $K = \text{Bif}(\overline{W})$ and a countable set of points that intersects every interval of the complement to $K$ by exactly $k$ points which correspond to separatrix connections between the saddle $P$ and the saddles $P_1, \ldots, P_k$.

Since we proved this for an arbitrary $W \in \mathcal{F}_k$, we now have a countable family of open sets of families with different bifurcation diagrams that have the required structure. In the following section we adapt this construction to an arbitrary surface with a handle.

3.6 The case of arbitrary surface $M$

An arbitrary smooth closed surface $M$ with a handle can be obtained from another surface $N$ by gluing a handle: $M = N \# \mathbb{T}^2$. For the surface $N$, there exists a Morse-Smale
vector field with a hyperbolic sink and, therefore, there is also a Morse-Smale vector field \( v_N \) with a small contractible hyperbolic attracting cycle that bounds a disk with a single singularity — a hyperbolic source. Since \( v_N \) is Morse-Smale, it has a neighborhood \( U \ni v_N \) where all vector fields are orbitally topologically equivalent.

Fix some \( k \) and consider the special family \( V_k \in F_k \). Recall that its sink \( \Omega \) does not depend on the parameter, draw a small transverse circle around \( \Omega \), cut the disk \( D \ni \Omega \) bounded by the circle (the disk must be small and should not intersect the disks \( B_j \)), and denote what is left of the torus, namely \( \mathbb{T}^2 \setminus D \), by \( T_0 \). Note that by construction the restrictions of the fields of our family to the vicinity of \( \partial T_0 \) do not depend on the parameter.

Consider a constant family \( \hat{V} \) whose vector fields coincide with \( v_N \) for all values of the parameter. For the vector field \( v_N \), draw a small transverse circle around the aforementioned source which is inside the attracting cycle, cut out the disk bounded by the circle and denote the rest of the surface by \( N_0 \). Now let us smoothly glue \( T_0 \) to \( N_0 \) along the neighborhoods of the boundaries in such a way that on the resulting surface (diffeomorphic to \( M \)) we get a smooth family of vector fields \( V_{k,M} \) — the result of “gluing” the families \( V_k \) and \( \hat{V} \) together.

Note that for any family \( V_M \) that is sufficiently close to \( V_{k,M} \) there exists a family \( V_{T^2} \in F_k \) such that the restrictions of \( V_M \) and \( V_{T^2} \) on \( T_0 \) coincide. Moreover, if the family \( V_M \) is sufficiently close to \( V_{k,M} \), the restriction \( V_M|_{N_0} \) is close to the restriction \( \hat{V}|_{N_0} \), i.e., it is almost constant, and so every vector field of the restricted family coincides with the restriction to \( N_0 \) of some field from the neighborhood \( U \ni v_N \) where all vector fields are topologically equivalent.

The families \( V_M \) and \( V_{T^2} \) have the same bifurcation diagrams. Indeed, the inclusion \( \text{Bif}(V_{T^2}) \subset \text{Bif}(V_M) \) is obvious. On the other hand, if for some \( \theta_0 \) the field \( v_{T^2,\theta_0} \) is Morse-Smale, we immediately have that the field \( v_{M,\theta_0} \) is Morse-Smale as well.

Since this argument works for arbitrary \( k \in \mathbb{N} \), we get a countable family of open sets \( F_{k,M} \ni V_{k,M} \) of one-parameter families of vector fields that have the required bifurcation diagrams.

### 3.7 Germs of bifurcation diagrams

For a family from the set \( F_{k,M} \), the germs of bifurcation diagrams at parameter values that correspond to vector fields with separatrix loops for the saddle \( P \) are all equivalent and have the structure described above in Remark 2. However, for two families taken from the sets \( F_{k,M} \) and \( F_{j,M} \), \( k \neq j \), these germs are not equivalent; they are distinguished by the number of isolated points in the gaps of the Cantor set. The same applies to germs at parameter values where the rotation number of the extended Poincaré map, which can still be defined for fields of our families, is irrational. Recall, however, that the equivalence of germs is established by a germ of homeomorphism that takes the selected point to the selected point, so this gives us another family of non-equivalent germs. This proves Theorem A modulo Proposition 8.

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9Theorem A modulo Proposition 8.

\[ \text{Here we assume that } T_0 \text{ is simultaneously a subset of the torus and the surface } M \text{ and } N_0 \text{ is a subset of } M \text{ and } N. \]
3.8 Proof of Proposition 8

We wish we could simply say that, if two families of vector fields are close, they have families of Poincaré maps which are close, and, since for the family $V$ the derivative of the Poincaré map in the parameter is always positive, the same must hold for a family $W$ close to $V$. However, we deal with Poincaré maps that were extended through the singularity, and therefore we cannot reason like that. Also recall Remark 9.

For the special family $V$, the points where the stable separatrices of the saddle $P_\theta$ intersect the circle $\Sigma$ do not depend on $\theta$. Therefore, for a family $W$ close to $V$ these points change only slightly when the parameter varies, so there exists a $C^3$-smooth parameter-dependent change of coordinates on the torus, uniformly $C^1$-close to the identity and equal to the identity outside arbitrary (but chosen beforehand) neighborhoods of the points $a, b$, such that in the new coordinates these points appear to be independent of the parameter; moreover, we can assume that the derivative of the coordinate change in the parameter is uniformly small$^{10}$ Then we can assume that $\frac{\partial}{\partial \theta} \hat{f}_\theta > 1/2$.

The new coordinates provide a new “circular chart” on $\Sigma$, that is, a homeomorphism between $\Sigma$ and $S^1 = \mathbb{R}/\mathbb{Z}$. We will denote by $a, b$ the points in $S^1$ that correspond to the points of intersection with separatrices and we denote by $\hat{g}_\theta$ the Poincaré map written in the new chart. All points $y \in [a_\theta, b_\theta] \subset \Sigma$ have the same image under $g_\theta$ that coincides with the point $c_\theta$ of intersection between the local unstable separatrix of $P_\theta$ with $\Sigma$. This point depends at least finitely-smoothly on the parameter, therefore the derivative $\frac{\partial}{\partial \theta}\hat{g}_\theta$ is defined for the corresponding points of $S^1$, coincides with $\hat{c}'(\theta_0)$, and is positive for all $\theta_0$, provided that $W$ is sufficiently close to $V$ and the coordinate change is sufficiently close to the identity and has small derivative in $\theta$. For a point $y \in S^1 \setminus [a, b]$ this derivative is also defined and positive for all $\theta_0$ if $W$ is close to $V$. However, if $\frac{\partial}{\partial \theta}\hat{g}_\theta(y)$ were not continuous at the points $a$ and $b$, it could in principle turn out that for different points $y \in S^1 \setminus [a, b]$ this derivative is positive for the families in different, decreasing neighborhoods of the family $V$ and it could be possible to approximate $V$ by families that have negative derivative in the parameter at some points.

We will show that any $\theta_0$ has a neighborhood where, as $y \to b + 0$, we have the uniform convergence

$$\frac{\partial}{\partial \theta}\hat{g}_\theta(y) \Rightarrow \frac{\partial}{\partial \theta}\hat{g}_\theta(b) = \hat{c}'(\theta) > 0.$$ 

Then we will be able to take the finite cover of the compact parameter space by these neighborhoods and choose $\alpha > 0$ such that $\frac{\partial}{\partial \theta}\hat{g}_\theta(y) > 0$ for all $y \in [b, b + \alpha]$ and $\theta_0 \in [0, 1]$. Arguing analogously for the point $a$, we will then assume that $\frac{\partial}{\partial \theta}\hat{g}_\theta(y) > 0$ for $y \in [a - \alpha, a]$. In restriction to the arc $A = [b + \alpha, a - \alpha]$, the family $\hat{g}_\theta$ is a family of true Poincaré maps and therefore is close to the restriction to this arc of the family $\hat{f}_\theta$

\[\text{E.g., we can take the coordinates change} \]

\[(x, y) \mapsto (x, \varphi_1(y)y + \varphi_2(y)(y + a - a_\theta) + \varphi_3(y)(y + b - b_\theta))\]

where $a_\theta, b_\theta$ are the original parameter-dependent coordinates of the intersection points between local stable separatrices of $P$ with the transverse circle for the family $W$ and $\{\varphi_j\}$ is a partition of unity on $\Sigma$ such that the supports of $\varphi_2$ and $\varphi_3$ are contained inside $\varepsilon$-neighborhoods of the points $a, b$, respectively.
of Poincaré maps for the family $V$ (written in the new coordinates), hence for $y \in A$ for every $\theta$ the derivative in the parameter is also positive.

A chart on a transversal to a local stable or unstable separatrix of a saddle is called natural if its origin is at the point of intersection with the separatrix, for all parameter values if a family of vector fields is considered. Fix some parameter value $\theta_0$. Note that our new coordinate chart on $\Sigma$ is almost natural for the stable separatrices: the natural chart can be obtained by a shift. Further note that on $\Sigma$ a natural chart for the unstable separatrix can be obtained by a parameter-dependent shift. Take these natural charts in the upper semi-neighborhoods of $b$ and $c_\theta$. Denote by $\lambda(\theta)$ the characteristic value of the saddle $P_\theta$, i.e., the absolute value of the ratio between the negative eigenvalue and the positive one, and denote by $\Delta$ the monodromy map from the upper semi-neighborhood of $b$ to the upper semi-neighborhood of $c_\theta$ in the natural charts. According to [IKS, Lemma 5\textsuperscript{11}], there is a neighborhood of $\theta_0$ where for the monodromy map $\Delta_{\theta}$ we have
\[(\Delta_{\theta})'_\theta(y) = O(y^{\lambda(\theta)} \log y),\] where the constant that is implicitly present in the $O$-notation is independent of $\theta$.

Thus, in some semi-neighborhood $U_b$ of the point $b$ the map $\hat{g}_\theta$ can be written as a composition
\[\hat{g}_\theta(y) = L_\theta \circ \Delta_\theta(y - b),\] where the map $L_\theta$ is just a parameter-dependent shift $y \mapsto y + \hat{c}(\theta)$. By the chain rule, (1), and (2), we have, as $y \to b + 0$,
\[\frac{\partial}{\partial \theta} \hat{g}_\theta(y) = (L_\theta)'_\theta + (L_\theta)'_y \cdot (\Delta_\theta)'_\theta = \hat{c}'(\theta) + 1 \cdot (\Delta_\theta)'_\theta(y - b) = \hat{c}'(\theta) + o(1),\] where the constant in $o(1)$ is independent of $\theta$ in the aforementioned neighborhood of $\theta_0$. This gives us the required locally uniform convergence for the derivative. For the point $a$ the argument is analogous. The proof of Proposition 8 is complete.

4 Proof of Theorem B

4.1 Theorems of Boyd and Veerman

Theorem B is based on the following results of C. Boyd and J. P. P. Veerman.

Theorem 10 (Veerman, [V, Theorem 6.4]). Suppose that the family $\{f_\theta\}_{\theta \in I}$ of circle maps satisfies the following conditions:

$[1]$ for every $\theta$ the map $f_\theta$ has degree one and preserves the orientation;

$[2]$ $f_\theta(x) \in C^0$ as a function of $(\theta, x)$;

$[3]$ for every $\theta$ the map $f_\theta$ is constant on some segment $U_\theta$ that contains some point $O$ that does not depend on the parameter;

\textsuperscript{11}This lemma requires the natural charts to be $C^3$ both in $y$ and the parameter. For this reason we insisted that our coordinate change was $C^3$-smooth.
for all $x \in S^1 \setminus \partial U_\theta$ the map $f_\theta(x)$ is locally $C^1$-smooth as a function of $\theta$ and the derivative $\frac{\partial}{\partial x} f_\theta(x)$ is $C^0$ as a function of $\theta$;

(5) for any fixed $x$, $f_\theta(x)$ is non-decreasing and for $x \in \text{int}(U_\theta)$ one has $\frac{\partial}{\partial \theta} f_\theta(x) > 0$.

(6) for any $\theta$ the map $f_\theta$ expands uniformly outside the segment $U_\theta$, i.e.,

$$\forall x \notin U_\theta, \quad \frac{\partial}{\partial x} f_\theta(x) > \gamma > 1,$$

where the constant $\gamma$ is independent of $\theta$.

Then $\dim_H(\{\theta: \rho(f_\theta) \notin \mathbb{Q}/\mathbb{Z}\}) = 0$. Here $\dim_H$ is Hausdorff dimension.

Remark 11. This statement slightly differs from the one in [V]. Section 2 of [V] has no condition [6] but has the requirement that for all values of the parameter the map $f_\theta$ can be extended as a local diffeomorphism to a neighborhood of the arc $S^1 \setminus U_\theta$, the function $\log \frac{\partial}{\partial x} f_\theta(x)$ being of bounded variation. This latter requirement is, of course, not satisfied for families that we consider, but this is not a problem. In [6] this condition is utilized to prove via a Denjoy-like argument that the maps of the family have no homtervals. In the version above this immediately follows from condition [6]. The proof of this version of the theorem is contained in sections 5, 6 of [V]. Note that expansiveness (rather, a slightly weaker property of $(\gamma, m)$-expansiveness) is used in the proof substantially, whereas boundedness of the derivative in $x$ is not used at all. Moreover, as Veerman himself remarks, this version of the theorem is essentially due to Boyd [B]: Boyd’s paper contains the proof for the case of his special families, but it can be generalized to the above assumptions.

The following result is proven in [B] at pp. 44–46 as part of the proof of Boyd’s Theorem 3.

Theorem 12 (C. Boyd, [B Section 5]). Let $V$ and $W$ be two families of vector fields of class $\mathcal{N}$ and $\{f_\theta\}_{\theta \in [0,1]}$, $\{g_\tau\}_{\tau \in [0,1]}$ be the corresponding families of Poincaré maps. Suppose there exists a homeomorphism $s: [0,1] \to [0,1]$ such that for any $\theta \in [0,1]$ one has $\rho(g_{s(\theta)}) = \rho(f_\theta)$. Assume also that in both families the rotation number monotonously depends on the parameter, there are no (non-degenerate) segments in the parameter space that correspond to irrational rotation number, and that for every parameter segment where the rotation number is rational there are no interior points that correspond to fields with saddle loop.$^{13}$ Then the families $V$ and $W$ are strongly equivalent.

Remark 13. In this theorem the condition of monotonous dependence of the rotation number on the parameter can be omitted. Instead of requiring that there are no segments with irrational rotation number or interior points that correspond to the presence of loops one can require that $s$ establish a correspondence between such segments and points for the two families.

$^{12}$i.e., we can change $\theta$ until $x$ meets the boundary of $U_\theta$.

$^{13}$In particular, there must be no segments that correspond to fields with separatrix loops.
4.2 The main proposition

Speaking roughly, the following proposition claims that for a deformation of any vector field \( v \in \mathcal{C}_+(M) \) one can choose a neighborhood \( U \) (that is a torus without a disk) of the unstable manifold \( W^u(P_v) \) in such a way that the restriction of the deformation onto this neighborhood can be extended to a family of vector fields on the whole torus so that after reversing the time one gets a family of fields of class \( \mathcal{N} \) with some additional properties that will turn useful in proving our theorem.

**Proposition 14.** Let \( M \) be an arbitrary surface with a handle, \( v \in \mathcal{C}_+(M) \), and \( V = \{ v_\theta \}_{\theta \in [-1,1]} \) be a generic family of vector fields on \( M \) such that \( v_0 = v \). Then there exists a neighborhood \( U \subset M \) of the unstable manifold \( W^u(P_v) \), homeomorphic to a torus without a disk, a segment \( J \ni 0 \) in the parameter space, and a smooth embedding \( \iota : U \rightarrow \mathbb{T}^2 \) such that the family of vector fields \( \{ v_\theta \} \) can be extended to the whole torus \( \mathbb{T}^2 \) as a family \( \tilde{V} = \{ \tilde{v}_\theta \}_{\theta \in J} \) with the following properties.

1. The fields of the family \( \tilde{V} = \{ \tilde{v}_\theta \}_{\theta \in J} \) are of class \( \mathcal{N} \).
2. For the corresponding family \( F = \{ f_\theta \} \) of Poincaré maps the following holds:
   
   \( (a) \) \( \rho(f_0) = 0 \);
   
   \( (b) \) the function \( r(\theta) = \rho(f_\theta) \) assumes rational values on some non-degenerate (i.e., not equal to a point) segments; the fields of the family have saddle loops exactly for those parameter values that are endpoints of these segments;
   
   \( (c) \) there exists a finitely-smooth parameter-depending coordinate change such that after this change and, maybe, after changing the sign of the parameter one would have, for all \( \theta \) and \( y \), inequality \( \partial_y f_\theta(y) > 0 \) where \( f_\theta \) is \( f_\theta \) written in the new coordinate.

3. \( \text{Bif}(\tilde{V}) = \{ \theta : \rho(f_\theta) \notin \mathbb{Q}/\mathbb{Z} \} = \{ \theta : \tilde{v}_\theta \text{ has a loop} \} \), and \( \text{Bif}(\tilde{V}) \) is a Cantor set.

**Remark 15.** It will be clear from the proof that for a family \( V_1 \) sufficiently close to \( V \) we can take the same sets \( U, J \) and embedding \( \iota \).

We postpone the proof of this proposition until section 4.6.

4.3 Equivalence for cropped families

The genericity condition imposed on families in both Theorem B and Proposition 14 is that the saddle loop must be unfolded with non-zero speed as the parameter changes. This implies that on one side of the zero value a repelling hyperbolic cycle is born from the loop. Applying Proposition 14 to the families \( V \) and \( W \) from the statement of Theorem B we get two families \( \tilde{V} \) and \( \tilde{W} \) of vector fields on the torus (and also corresponding segments \( J_1, J_2 \) and embeddings \( \iota_1, \iota_2 \)).

We want to apply Theorem 12 to these families \( \tilde{V} \) and \( \tilde{W} \). First, note that by Proposition 14 the Poincaré maps for each family become strictly monotonous in \( \theta \) when written in some parameter-dependent chart. Since the rotation number does not depend on the
chart, this implies that the rotation number depends non-strictly monotonously on \( \theta \) in general and is strictly monotonous at points where irrational values are assumed \([Ha, Lemmas 1-3]\) (see also \([He]\) and \([KH, Prop, 11.1.8-9]\)), so there are no segments where the rotation number is irrational. By the proposition, saddle loops are present only for parameter values at the endpoints of segments where the rotation number is rational. Since the zero value of the parameter corresponds to zero rotation number and to the presence of a loop, we can change the sign of the parameter if necessary and assume that in the left semi-neighborhood of zero the rotation number equals zero. Furthermore, after applying, if necessary, a diffeomorphism of the torus that changes the orientation on the transversal \( \Sigma \), we can assume that the rotation number does not decrease with the parameter. We can now crop the parameter segments \( J_1 \) and \( J_2 \) in such a way that after that for the families \( F \) and \( G \) of Poincaré maps the rotation number will run through the same segment and the endpoints of the cropped segments will correspond to structurally stable fields. Now we can construct a homeomorphism between the cropped \( J_1 \) and \( J_2 \) as follows: for each pair of segments where the rotation number is the same, let \( s \) take the first segment onto the second one affinely. Since rotation number for both families depends on the parameter continuously and monotonously, the map \( s \) can be extended as a homeomorphism between the parameter spaces.

Theorem 12 yields the strong equivalence of the families \( \tilde{V} \) and \( \tilde{W} \). Let \( H: (\theta, z) \mapsto (h(\theta), H_\theta(z)), z \in \mathbb{T}^2 \), be a homeomorphism that realizes this equivalence. Now we can establish the equivalence of restrictions from the first statement of Theorem 13. Take a neighborhood \( T_1 \subset \iota_1(U_1) \) of the unstable manifold \( W^u(P_{\tilde{v}_0}) \) so that for \( \theta \) near zero (i.e., for \( \theta \) from some segment \( \tilde{J}_1 \subset J_1 \) that contains zero) we have

\[
H_\theta(T_1) \subset \iota_2(U_2).
\]

This is almost what is required, except in the image the neighborhood of the unstable manifold depends on the parameter.

Assume that for the cropped parameter segment \( \tilde{J}_1 \) the endpoints still correspond to structurally stable fields. Denote \( \iota_2(U_2) \) by \( T_2 \). Modify the restriction \( H|_{\tilde{J}_1 \times T_1} \) in the neighborhood of the set \( \tilde{J}_1 \times \partial T_1 \) in such a way that the modified map be a homeomorphism onto the image and for each \( \theta \in \tilde{J}_1 \) it realize the equivalence between \( \tilde{v}_\theta|_{T_1} \) and \( \tilde{w}_\theta|_{T_2} \).

Finally, we have established the first claim of the theorem, with \( \tilde{J}_1 \) playing the role of \( J_1 \), \( h(\tilde{J}_1) \) being \( J_2 \), and the neighborhoods of the unstable manifolds being equal to \( \tilde{U}_1 = \iota_1^{-1}(T_1) \) and \( \tilde{U}_2 = \iota_2^{-1}(T_2) \), respectively.

Strong structural stability of the cropped families is proved analogously using Remark 15.

### 4.4 The bifurcation diagram

By Proposition 14, for the family \( \tilde{V} \) the bifurcation diagram coincides with the set \( \{ \theta : \tilde{v}_\theta \text{ has a loop} \} \) and is a Cantor set. Moreover, Theorem 10 can be applied to the family \( \{ \tilde{f}_\theta \} \) of its Poincaré maps written in parameter-dependent coordinates. As we will see in the proof of Proposition 14 the flat interval of the map \( \tilde{f}_\theta \) does not depend on the parameter, but were it not the case, we could argue that it depends continuously on
the parameter and so condition [3] of Theorem 10 is satisfied after further cropping the
parameter space.

Theorem 10 yields that \( \dim_H(\text{Bif}(\tilde{V}|_{\tilde{J}_1})) = 0 \), if the segment \( \tilde{J}_1 \) is sufficiently small.
For the family \( V|_{\tilde{J}_1,\tilde{U}_1} \), the bifurcation diagram is the same (note that no homeomorphism
is required).

4.5 The free separatrix winds onto the loop

For a vector field of class \( \mathcal{N} \) that has a saddle loop, the free separatrix of the saddle \( P \) has
to wind onto this loop. Indeed, on the one hand, in reversed time the loop attracts every
point in its small monodromic semi-neighborhood. On the other hand, consider a small
segment \( L \subset \Sigma \) that is contained in this semi-neighborhood. Since the Poincaré map \( f \)
is expansive outside the flat interval, the images \( f^k(L) \) have to grow exponentially until
one of them intersects the flat interval and hence the free separatrix of the saddle. This
implies that the free separatrix is attracted to the loop in reversed time, that is, it winds
onto the loop. Proposition 14 yields that the same holds for the loops of the saddle \( P \) in
family \( V \). Theorem 3 is proven modulo the proof of Proposition 14.

4.6 Proof of Proposition 14

Idea of the proof

The set \( U \) from the statement of Proposition 14 is a small dissipative neighborhood
of \( W^u(P) \) for the field \( v = v_0 \). It is a torus-without-a-disk embedded into the surface \( M \).
We take \( U \) with the field \( v \) attached to it and glue to its boundary a disk that supports
a vector field with a sole repellor. This gluing gives us an embedding \( \tilde{U} \) of the surface \( U \)
into a torus \( T^2 \). After that we invert the direction of the vector field \( (\tilde{U})_* (v) \) and start
modifying it outside the image \( \tilde{U}(U) \) in order to obtain a vector field of class \( \mathcal{N} \). Then we
pushforward the rest of the family \( V \) and obtain, after the same procedure and appropri-
ate cropping, a small family of fields of class \( \mathcal{N} \) on the whole torus. Then we modify this
family to make the corresponding Poincaré map depend continuously on the parameter.
This latter property yields the desired facts about the bifurcation diagram.

Constructing \( \tilde{v}_0 \) of class \( \mathcal{N} \)

First, take a small dissipative neighborhood \( U \) of the unstable manifold \( W^u(P) \) for the
field \( v \) (we will specify later how small it should be). This neighborhood must be a torus
without a disk, the boundary circle \( \partial U \) being transverse to the vector field. Take also a
unit disk \( D \) with a field \( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \). Note that this field has a unique singularity which is
a hyperbolic source and this field is transverse to the boundary. Glue \( U \) to \( D \) along the
neighborhoods of the boundaries in such a way that the vector fields be glued as well to
form a smooth vector field on the torus \( T^2 = U \# D \). Multiply this field by \( -1 \) and denote
the resulting field on the torus by \( \tilde{w} \). Multiplying by \( -1 \) is equivalent to inverting the time.
So, the repellor on the glued-in disk becomes an attractor, which we will denote by \( \Omega \).
The separatrices of the saddle \( P \) change their directions. The embedding \( \tilde{U}_* : U \to T^2 \) that
originates from gluing is exactly the one that appears in the statement of Proposition 14.
and $U$ is the same, provided it was chosen sufficiently small. Denote by $\iota_D$ the embedding of $D$ in $\mathbb{T}^2$; denote $\iota_U(U)$ by $\tilde{U}$ and $\iota_D(D)$ by $\tilde{D}$.

The field $\tilde{w}$ has a global closed non-contractible transversal. To obtain one, first take a point at the local unstable separatrix of the saddle $P$ that is involved into the loop and draw a transverse curve through it so that both endpoints of this curve are on the boundary of $\tilde{U}$. Denote these two points by $p_1, p_2$. Since the original field on $D$ was radial, it is clear that $p_1$ and $p_2$ can be connected by a transverse curve that goes inside $\tilde{D}$ so that this curve together with the previous curve between $p_1$ and $p_2$ makes a smooth global transversal. However, one can obtain in this manner transversals which are of different homological type. In order to make the relative arrangement of $\tilde{U}$ and the transversal as simple as possible, we do the following. First, $\tilde{U}$ is cut into three parts: a neighborhood of the saddle and two strips attached to it; the strips may be viewed as neighborhoods of a segment of the loop and a segment of the free separatrix. Then the transverse segment $p_1p_2$ is taken inside the strip that is a neighborhood of the segment of the loop. Denote the closed transversal obtained from this segment by $\Sigma$.

Now take a smooth chart (that “unfolds” the torus into a rectangle) in which the circle $\Sigma$ becomes a vertical segment and the field in the neighborhood of $\Sigma$ becomes horizontal. Draw another vertical transversal $\tilde{\Sigma}$ near $\Sigma$ (see Fig. 2) so that the unstable separatrix involved into the loop cross it before it crosses $\Sigma$. In what follows we will refer to the geometry of the picture when we say “above” or “below”, etc. Denote by $O$ and $\hat{O}$ the points where $W^u_{loc}(P)$ first intersects $\Sigma$ and $\tilde{\Sigma}$ respectively.

In the lower semi-neighborhood of the point $O$ on $\Sigma$ the monodromy map $\Delta: \Sigma \to \hat{\Sigma}$ is defined. It is expanding in restriction to a sufficiently small lower semi-neighborhood $B \subset \tilde{D}$ and $\tilde{U}$.
Σ of O, because the saddle is now area-expanding. Our monodromy map can be continued to the point O by specifying that O be mapped into ̂O. Furthermore, O is accumulated from below by a sequence of intersection points of Σ and the free stable separatrix of the saddle. Take the very first point of this sequence and denote it by q; denote by ̂q the analogous point on the second transverse circle. We want the Δ-image of B to contain ̂q. It can be achieved by choosing the neighborhood U sufficiently small in the first place, because for the field v the monodromy map along the saddle loop was a strong contraction, provided that we looked at a sufficiently small monodromic semi-neighborhood of the loop. Thus, we can and will assume that the Δ-image of B has an endpoint d1 at the boundary of ̂U.

In the upper semi-neighborhood Γ ⊂ Σ of the point q the monodromy map to ̂Σ is defined too. Note, first, that it can be continued to q by letting q be taken to ̂O, the whole image of Γ lying above ̂O; and second, if Γ is sufficiently small, this map is expanding due to crossing the hyperbolic sector of the saddle, where the local monodromy map is expanding due to the saddle being area-expansive. We will assume that we have chosen U so small that the image of Γ under the monodromy map has an endpoint d2 ∈ ∂(̂U) and the monodromy map is expansive on Γ. We can do that because after choosing the segment Γ where the monodromy map is expansive, we could look at its monodromy image and crop our neighborhood ̂U to make the endpoint of the image be at the boundary of the cropped neighborhood. The set U in the preimage is also cropped. After this cropping, the monodromic semi-neighborhood of the loop contained in U does not change, only the neighborhood of the free local separatrix and the non-monodromic semi-neighborhood of the loop become narrower, so we can assume that we have chosen U to be like that.

Thus, we have an expansive monodromy Δ: B ∪ Γ → [d1, d2] ⊂ ̂Σ. Note that on the arc (O, q) (recall that according to our notation this arc goes from O upwards and then approaches q from below) the monodromy map is not defined because every orbit that starts at this arc goes to the sink Ω. So, we extend the map to (O, q) by setting Δ((O, q)) = {̂O}, as usual. Further note that Δ continues to the arc (q, O) automatically, but it does not have to be expansive on this whole arc. We want to obtain a field that belongs to the class N, and the only thing we need for that is to make the Poincaré map expansive outside its flat segment. In order to achieve that, we modify the field in between ̂Σ and Σ using the textbook method from [PM, Lemma 2, p. 183]. Namely, we take an expansive smooth map f: [q, O] → ̂Σ that, when written in our coordinates, coincides with the monodromy map Δ on B ∪ Γ (it is clear that such f exists). Denoting coordinate representations of our maps by the same letters as the maps themselves, we can set

\[
\varphi(y) = f \circ \Delta^{-1}(y),
\]

\[
\varphi_s(y) = (1 - s)y + s\varphi(y), \quad \forall s \in [0, 1],
\]

\[
H(x, y) = (x, \varphi_\sigma(x)(y)),
\]

where σ is a monotonous function that takes values in [0, 1], is equal to zero in a neighborhood of the x-coordinate x̂_Σ of ̂Σ, and is equal to one in a neighborhood of the x-coordinate x_Σ of Σ.

Finally, we set

\[
\tilde{v}_0(x, y) = DH(H^{-1}(x, y))[\tilde{w}(H^{-1}(x, y))]
\]

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(the derivative of $H$ taken at the point $H^{-1}(x, y)$ is applied to the vector of the field $\tilde{w}$ taken at the same point) in the strip between $\Sigma'$ and $\Sigma$ and set $\tilde{v}_0 = \tilde{w}$ outside this strip. It is easy to check now that for the vector field $\tilde{v}_0$ the (extended) Poincaré map from $\Sigma$ to itself is well-defined and coincides in coordinate representation with the expanding map $f$ on $[q, O]$. Hence we have obtained the Poincaré map that expands everywhere outside the segment where it is constant. It is also easy to check that we did not alter the field inside $w_U(U)$: on the union $B \cup \Gamma$ the maps $f$ and $\Delta$ coincide, therefore $\varphi_s(y) = y$ for all $s \in [0, 1]$ and $y \in \Delta(B \cup \Gamma)$, which implies that for the points $(x, y)$ in the intersection of the vertical strip in between the transverse circles and $\tilde{U}$ we have $H^{-1}(x, y) = (x, y)$ and $DH = Id$, so the field there remains horizontal.

It is clear now that the vector field $\tilde{v}_0$ is of class $\mathcal{N}$.

**Monotonicity in the parameter**

Consider once again the original family $V$ on $M$. When the parameter is close to zero, the field $v_\theta$ differs from $v_0$ only very slightly in a neighborhood of $\partial U$ and, therefore, is transverse to $\partial U$, so we can extend the fields $(v_\theta)_*(-v_\theta)$ simultaneously to obtain a smooth family $W = \{w_\theta\}_{\theta \in [-\varepsilon, \varepsilon]}$ of vector fields defined on the whole torus, with $w_0 = \tilde{v}_0$, by gluing, as above, a disk with a single repellor, but now by a parameter-dependent correspondence between the neighborhoods of the boundaries. No matter how we choose this correspondence, if we take $\varepsilon$ sufficiently small, we will have $w_\theta \in \mathcal{N}$ for all $\theta$, because $w_0 = \tilde{v}_0 \in \mathcal{N}$ and $\mathcal{N}$ is open.

Let $G$ be the family of the Poincaré maps for the family $W$, where the transverse circle is still $\Sigma$. Our goal now is to modify the family $W$ so that the modified Poincaré maps become monotone in the parameter, at least in some parameter-dependent coordinate.

In a small neighborhood of the vertical transversal $\Sigma$ the vector field can be rectified simultaneously for all parameter values to a unit horizontal field by a parameter-dependent smooth change of coordinates that leaves $\Sigma$ vertical. We switch to this parameter-dependent chart and draw another vertical transversal $\hat{\Sigma}$ near $\Sigma$. Since the chart depends on the parameter, so does $\hat{\Sigma}$ and also the strip by which $\tilde{U}$ intersects the cylinder $C$ which is in between the transverse circles. Define $O, \hat{O}, q, \hat{q}, B, \Gamma, d_1, d_2$ as above. From this moment on, the coordinates on $\Sigma$ and $\hat{\Sigma}$ will only be changed in accord, so that we could always assume that the vector field in $C$ is unit horizontal.

It is time to refer to the genericity condition. Consider a parameter-dependent coordinate on the transversal $\Sigma$ such that its origin for every parameter value coincides with the point of first intersection between $\Sigma$ and the stable separatrix that is involved in the loop when the parameter is zero. Recall that such coordinates are called natural with respect to this separatrix. The genericity condition is that the point $c(\theta)$ of the first intersection of $\Sigma$ and the unstable separatrix involved in all the loops has nonzero derivative in the parameter. Note that, when we switch to a different natural chart, the sign of the derivative does not change if two charts are co-oriented. After changing, if necessary, the sign of the parameter $\theta$, we will assume that $c'(0) > 0$ and, therefore, for $\theta$ near zero one also has $c'(\theta) > 0$.

We would like to work in a natural chart (with respect to the same stable separatrix) such that in it the point of the first intersection between $\Sigma$ and another stable separatrix
does not depend on the parameter. When we switch to this chart, inequality $c'(\theta) > 0$ still holds for $\theta$ near zero.

Now let us assume that we were farsighted and chose the neighborhood $U$ so small that in our natural chart we now have

$$\left. \frac{\partial}{\partial \theta} \right|_{\theta=0} g_\theta(y) > \delta > 0, \ \forall y \in B \cup \Gamma.$$ 

It is indeed possible to have done that. Arguing as in the proof of Proposition \S we can conclude that in the upper semi-neighborhood of the point $q$ and in the lower semi-neighborhood of $O$ the derivative under consideration is positive. Concerning the lower semi-neighborhood of the point $O$, when choosing $U$, we could assure that the image of this segment under the monodromy map from $\Sigma$ to itself have one endpoint at $\partial \tilde{U}$. For the segment $\Gamma$ we can argue analogously: take the upper semi-neighborhood of $q$ where the derivative in parameter is positive, consider the image of the upper endpoint of this semi-neighborhood under the monodromy map, and crop $\tilde{U}$ in such a way that this point be at the new boundary.

We can now modify the family. Consider the cylinder $C$ and the coordinates $x, y$ on it in which all vector fields of our family appear constant and horizontal. Let

- $\sigma(x)$ be a $C^\infty$-smooth bump-function that has support contained in $[x_\Sigma, x_{\hat{\Sigma}}]$, assumes values in $[0, 1]$, and is equal to one at the point $(x_\Sigma + x_{\hat{\Sigma}})/2$;

- $\kappa(y)$ be a $C^\infty$-smooth bump-function equal to one exactly in the $\alpha_2$-neighborhood of the segment $[d_1, d_2]$ and equal to zero outside its $\alpha$-neighborhood, where $\alpha > 0$ is a small constant that we will specify below.

Let us add to our family the vector field

$$\Phi = K \theta \cdot \sigma(x)(1 - \kappa(y)) \cdot \frac{\partial}{\partial y} + 0 \cdot \frac{\partial}{\partial x} + 0 \cdot \frac{\partial}{\partial \theta},$$

where $K > 0$ is a large constant, and see what effect it has on the derivative of the Poincaré map in the parameter.

Denote by $h_\theta(y)$ the parameter-dependent monodromy map from $\hat{\Sigma}$ to $\Sigma$ for the modified family, written in the same coordinates on $C$. For notational convenience, we will also write $h(y, \theta) = h_\theta(y)$, and likewise for all parameter-dependent functions. Due to the choice of the chart, we have $h_0 = Id$. Moreover, for $\theta$ near zero the map $h_\theta$ is close to the identity map. It is easy to check that outside the $\frac{\alpha}{2}$-neighborhood of the segment $[d_1, d_2]$ the derivative $(h_\theta)'_\theta(y, \theta)$ is positive and outside the $\alpha$-neighborhood of this segment it is separated from zero by some constant that depends on $K$.

No matter what $\alpha$ we choose, we can always crop our parameter space so that the following will hold: for any parameter value, the intersection of $\tilde{U}$ and the cylinder $C$ lies in the rectangle $[x_\Sigma, x_{\hat{\Sigma}}] \times [d_1 - \alpha/2, d_2 + \alpha/2]$. Then for $\theta$ near zero adding $\Phi$ does not change the vector fields inside $\tilde{U}$. Let us now see how the derivative of the Poincaré map in $\theta$ has been affected. The new Poincaré map $f_\theta$ can be written as a composition of the
old one (which in our coordinates coincides with the monodromy map from $\Sigma$ to $\hat{\Sigma}$) and the map $h_\theta$. By the chain rule we get

$$(f')_\theta(y, \theta) = \left(\frac{\partial}{\partial \theta} (hg_\theta(y), \theta)\right) + \left(g_\theta(y), \theta\right) \cdot (g')_\theta(y, \theta).$$

Here $B$ is close to one, $A$ is positive and large when $K$ is large and $g_\theta(y) \notin [d_1 - \alpha, d_2 + \alpha]$, and from $C$ we require that for $\theta$ near zero it be positive for $y$ that satisfy $g_\theta(y) \in [d_1 - \alpha, d_2 + \alpha]$ — this can be achieved by choosing $\alpha$ sufficiently small (for $\theta = 0$ expression $C$ is separated from zero on the $g_0$-preimage of $[d_1, d_2]$). Thus, by taking $K$ large and $\alpha$ small we assure that the $\theta$-derivative of the modified Poincaré map is above zero and is separated from it. As soon as we choose $\alpha$ and $K$, construction of the family $\tilde{V}$ whose existence is claimed in Proposition 14 is complete.

Properties of the bifurcation diagram

Vector fields of class $\mathcal{N}$ with irrational rotation number are not structurally stable, therefore $\{\theta: \rho(f_\theta) \notin \mathbb{Q}/\mathbb{Z}\} \subset \text{Bif}(\tilde{V})$. On the other hand, if a vector field of class $\mathcal{N}$ has a rational rotation number and has no saddle loop, this field is Morse-Smale and therefore is structurally stable. Thus, in order to prove the inverse inclusion, it suffices to check that every parameter value that corresponds to a field with a loop lies in the set $\{\theta: \rho(f_\theta) \notin \mathbb{Q}/\mathbb{Z}\}$. In other words, it suffices to prove that in our family the rotation number, as a function of the parameter, is non-constant in a neighborhood of any parameter value that corresponds to a saddle loop. Although we have monotonous dependence of Poincaré maps on the parameter only in some specifically chosen coordinates, we can argue exactly as in section 3.3 to establish that. Of course, we also use the fact that the coordinates of the points where the stable separatrices first intersect the transversal are independent of the parameter. This latter fact and monotonicity also yield that separatrix loops happen only for parameter values at the endpoints of segments where rotation number is rational. Thus, we have equality

$$\text{Bif}(\tilde{V}) = \{\theta: \rho(f_\theta) \notin \mathbb{Q}/\mathbb{Z}\} = \{\theta: \tilde{v}_\theta \text{ has a loop}\}.$$ 

From this we deduce that $\text{Bif}(\tilde{V})$ is perfect and nowhere dense. Since it is also obviously closed, it is a Cantor set.

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