LONGTIME DYNAMICS FOR A TYPE OF SUSPENSION BRIDGE EQUATION WITH PAST HISTORY AND TIME DELAY

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Abstract. In this paper, we investigate a suspension bridge equation with past history and time delay effects, defined in a bounded domain $\Omega$ of $\mathbb{R}^N$. Many researchers have considered the well-posedness, energy decay of solution and existence of global attractors for suspension bridge equation without memory or delay. But as far as we know, there are no results on the suspension bridge equation with both memory and time delay. The purpose of this paper is to show the existence of a global attractor which has finite fractal dimension by using the methods developed by Chueshov and Lasiecka. Result on exponential attractors is also proved. We also establish the exponential stability under some conditions. These results are extension and improvement of earlier results.

1. Introduction. In this paper, we investigate the following suspension bridge equation with past history and time delay

$$
\begin{aligned}
&u_{tt} + \alpha \Delta^2 u - \int_0^\infty \mu(s) \Delta^2 u(t-s) \, ds + ku^+ \\
&\quad + \mu_1 u_t + \mu_2 u_t(t-\tau) + f(u) = h, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
&u = \Delta u = 0, \quad (x,t) \in \partial \Omega \times \mathbb{R}^+, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
u_t(x,t) = j_0(x,t), \quad (x,t) \in \Omega \times [-\tau,0),
\end{aligned}
$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial \Omega$, $\alpha$ is a positive constant, $\mu$ is the memory kernel to be stated later, $\tau > 0$ is a time delay, $\mu_1 > 0$ and $\mu_2$ is a real number, $k$ is spring constant, $u^+ = \max\{u, 0\}$ denotes the
positive part of \( u \), the term \(-ku^+\) models a restoring force due to the cables, and the initial data \( (u_0, u_1, j_0) \) belong to a suitable function space.

From the physics point of view, the suspension bridge equations describe the transverse deflection of the roadbed in the vertical plane. The suspension bridge equations were introduced by Lazer and McKenna [13] as new problems in the field of nonlinear analysis. In the absence of memory and delay, that is when \( \mu(s) = 0 \) and \( \mu_2 = 0 \) in (1.1), the problem (1.1) has been extensively studied and many results concerning the well-posedness and the global attractors can be founded (see [1, 16, 21, 22, 26] and the references therein). An [1] established the well-posedness of the weak solution and the decay rate of the solution. Ma and Zhong [16] proved the existence of global attractors in energy space, later the authors [26] improved the results of [16] by showing the existence of strong solutions and the regularity of the global attractors. Park and Kang [21, 22] proved the existence of pullback attractor for the non autonomous suspension bridge equations and the existence of global attractors for the suspension bridge equations with nonlinear damping, respectively.

In recent years, the evolution equations with memory effects have been studied by many authors, there are many results on this aspect, we do not list the references here. Related to the suspension bridge equations with memory term, we can mention the work of Kang [10](for \( N = 2 \)) and [11], where the following equation was considered

\[
\begin{align*}
  u_{tt} + \alpha \Delta^2 u - \int_0^\infty \mu(s) \Delta^2 u(t-s)ds + ku^+ + f(u) &= h.
\end{align*}
\]

In these papers, Kang estimated the well-posedness and the long-time behavior of the suspension bridge equation when the damping mechanism is given by the memory.

On the other hand, time-delay effect often appears in many applications depending not only on the present state but also on some past occurrences. The time delay effect is sometimes unavoidable, and may be a source of instability. For example, it was shown in [20, 24] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. Nicaise and Pignotti [20] considered the wave equation with time delay

\[
\begin{align*}
  u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x,t - \tau) &= 0.
\end{align*}
\]

They obtained that the energy of the problem decays exponentially under the condition \( 0 < \mu_2 < \mu_1 \), later they extended the result to the time varying delay case in [19]. We also refer the reader to [24], where the authors obtained the same results as in [20] for the one dimension space by use of the spectral analysis approach.

Kirane and Said-Houari [12] discussed the following viscoelastic wave equation with time delay

\[
\begin{align*}
  u_{tt} - \Delta u + \int_0^t \mu(t-s) \Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(x,t - \tau) &= 0 \quad (1.2)
\end{align*}
\]

with suitable initial-boundary value conditions. They obtained the well-posedness and the energy decay of solutions for the problem (1.2) under the restriction \( 0 < \mu_2 \leq \mu_1 \). Later, Dai and Yang [7] improved the result of [12] under some weaker conditions. Moreover, Yang [25] studied the following Euler-Bernoulli viscoelastic
equation with time delay
\[ u_{tt} + \Delta^2 u - \int_0^t \mu(t-s)\Delta^2 u(s)ds + \mu_1 u_t(x, t) + \mu_2 u_t(x, t-\tau) = 0. \]

He also obtained the well-posedness and the energy decay even in the case $\mu_1 = 0$ and $|\mu_2|$ is sufficiently small. Here, we also refer the interesting paper Liu and Zhang [14], where they considered the equation (1.2) with past history and adding the source term $f(u)$ and the well-posedness and the exponential stability were obtained.

Recently, Park [23] considered the suspension bridge equation with time delay as following
\[ u_{tt} + \Delta^2 u + ku^+ + \mu_1 u_t + \mu_2 u_t(t-\tau) + f(u) = h, \quad (1.3) \]
Under some conditions, the existence of global attractor which has finite fractal dimension for (1.3) was proved. But, there are not many researches on attractors especially the exponential attractors for other delayed system, it is worth mentioning the related researches [8, 15].

But, to our best knowledge, there is no work on suspension equations with both past history and time delay. Motivated by the above-mentioned results, in the present paper, we investigate the existence of longtime dynamics for the problem. We obtain the quasi-stability of the system and the global attractor with finite fractal dimension which can be characterized by the unstable manifold of the set of stationary solution. The existence of exponential attractor is also established. We also prove the exponential decay of the system.

We set for simplicity $\alpha - \int_0^\infty \mu(s)ds = 1$. Secondly, following the framework proposed in [20], we introduce the new variable $\eta^t$ to the system with past history, namely
\[ \eta^t = \eta^t(x, s) = u(x, t) - u(x, t-s), \quad (x, s) \in \Omega \times \mathbb{R}^+, \quad t \geq 0. \]
By differentiation we obtain
\[ \eta_t^t(x, s) = -\eta_t^t(x, s) + u_t(x, t), \quad (x, s) \in \Omega \times \mathbb{R}^+, \quad t \geq 0. \quad (1.4) \]
We set for simplicity $\alpha - \int_0^\infty \mu(s)ds = 1$. Secondly, following the framework proposed in [20], we introduce the new variable
\[ z(x, \rho, t) = u_t(x, t - \rho \tau), \quad x \in \Omega, \quad \rho \in (0, 1), \quad t > 0. \]
Hence problem (1.1) can be transformed into the equivalent system
\[ \begin{cases} u_{tt} + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2 \eta^t(s)ds + ku^+ + \mu_1 u_t + \mu_2 z(x, 1, t) + f(u) = h, \quad (x, t) \in \Omega \times \mathbb{R}^+, \\ \eta_t^t = -\eta_t^t + u_t, \quad (x, t, s) \in \Omega \times (0, +\infty) \times \mathbb{R}^+, \\ \tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad (x, \rho, t) \in \Omega \times (0, 1) \times \mathbb{R}^+ \end{cases} \quad (1.5) \]
with boundary conditions
\[ u = \Delta u = 0 \text{ on } \partial \Omega \times \mathbb{R}^+, \quad \eta = \Delta \eta = 0 \text{ on } \partial \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \quad (1.6) \]
and the initial conditions
\[ u(x, 0) = u_0, \quad u_t(x, 0) = u_1, \quad \eta^0(x, 0) = 0, \quad \eta^0(x, s) = \eta_0(x, s), \quad z(x, \rho, 0) = z_0(x, \rho) \quad (1.7) \]
where
\[
\begin{aligned}
&u_0(x) = u_0(x, 0), \quad u_1(x) = \partial_t u_0(x, t)|_{t=0}, \quad x \in \Omega, \\
&\eta_0(x, s) = u_0(x, 0) - u_0(x, -s), \quad (x, s) \in \Omega \times \mathbb{R}^+, \\
&z_0(x, \rho) = j_0(x, -\rho\tau), \quad (x, \rho) \in \Omega \times (0, 1).
\end{aligned}
\]

2. Assumptions and the main results. In this section, we present some assumptions and state the main results. We first give some notations about function spaces. We denote the norm in the space \(X\) by \(\| \cdot \|_X\). As usual, we denote the scalar product in \(L^2(\Omega)\) by \((\cdot, \cdot)\) and \(L^p(\Omega)\) norm by \(\| \cdot \|_p\) respectively. In the case \(p = 2\) we write \(\| \cdot \|\) instead of \(\| \cdot \|_2\). Let
\[
H = L^2(\Omega), \quad V = H^2(\Omega) \cap H_0^1(\Omega),
\]
with the inner product and norm
\[
(u, v)_V = (\Delta u, \Delta v), \quad \|u\|_V = \|\Delta u\|,
\]
respectively. With respect to the relative displacement \(\eta\) as a new variable, we introduce the following \(\mu\)-weighted Hilbert space
\[
\mathcal{M} = L^2_{\mu}(\mathbb{R}^+; V) = \{ \eta : \mathbb{R}^+ \to V | \int_0^\infty \mu(s) \|\eta(s)\|^2_V ds < \infty \},
\]
edowed with the inner product and norm
\[
(u, v)_\mathcal{M} = \int_0^\infty \mu(s) (u(s), v(s))_V ds, \quad \|u\|_\mathcal{M}^2 = \int_0^\infty \mu(s) \|u(s)\|^2_V ds,
\]
respectively. Finally, we introduce the phase space for the trajectory solutions
\[
\mathcal{H} = V \times H \times \mathcal{M} \times L^2(\Omega \times (0, 1))
\]
equipped with the norm
\[
\|U\|_{\mathcal{H}} = \left\{ \|\Delta u\|^2 + \|v\|^2 + \|\eta\|^2_\mathcal{M} + \|z\|^2_{L^2(\Omega \times (0, 1))} \right\}^{\frac{1}{2}},
\]
for all \(U = (u, v, \eta, z) \in \mathcal{H}\). Let \(\lambda_1\) be the best constant in the Poincaré’s inequality
\[
\lambda_1\|u\|^2 \leq \|\Delta u\|^2, \quad u \in V.
\]
(2.1)
In this paper, we will omit \(x\) and \(t\) in the functions of \(x\) and \(t\) if there is no ambiguity, \(c\) and \(c_0\) are used to denote generic positive constants.

Next, we will give some assumptions used in this paper.

(H1) The nonlinear source term \(f\) is locally Lipschitz, with \(f(0) = 0\), and satisfying the growth condition
\[
|f(u) - f(v)| \leq c_0 (1 + |u|^p + |v|^p) |u - v|, \quad \forall u, v \in \mathbb{R}, \quad (2.2)
\]
where \(c_0 > 0\), and
\[
0 < p < \frac{4}{N-4} \quad \text{if } N \geq 5 \quad \text{and } p > 0 \quad \text{if } 1 \leq N \leq 4. \quad (2.3)
\]
In addition, we suppose that \(f\) satisfies the following dissipation conditions
\[
f(u)u \geq -\lambda_1 (1 - \nu) |u|^2 - m_f, \quad (2.4)
\]
\[
F(u) \geq -\frac{\lambda_1}{2} (1 - \nu) |u|^2 - m_f, \quad (2.5)
\]
for some \(\nu \in (0, 1]\) and \(m_f \geq 0\), where \(\lambda_1\) is denoted by (2.1) and \(F(u) = \int_0^u f(y) dy\).

(H2) \(h \in L^2(\Omega)\) and \(j_0 \in L^2(\Omega \times (-\tau, 0))\).

(H3) The coefficients \(\mu_1\) and \(\mu_2\) satisfy \(0 < |\mu_2| < \mu_1\).
(H4) With respect to the memory kernel, we assume that
\[ \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu \geq 0, \quad \mu(0) > 0, \] (2.6)
and there exist \( \mu_0, \delta > 0 \) such that
\[ \int_0^\infty \mu(s)ds = \mu_0 < \infty \] (2.7)
and
\[ \mu'(s) \leq -\delta \mu(s), \quad \forall s \in \mathbb{R}^+. \] (2.8)

**Remark 1.** The conditions (2.4) and (2.5) on the source term are more general, i.e., the assumption \( -m_f \leq F(u) \leq f(u)u \) considered in [10, 11, 23] as a special case.

In order to establish the main results of this paper, we first state the existence result.

**Theorem 2.1.** Assume that assumptions \((H1)-(H4)\) hold. Then we have

(i) For every \((u_0, u_1, \eta_0, z_0) \in \mathcal{H} \) and \( T > 0 \), the problem \((1.5)-(1.7)\) has a weak solution \((u, u_t, \eta, z) \in C([0, T]; \mathcal{H})\) satisfying
\[ u \in L^\infty(0, T; V), \quad u_t \in L^\infty(0, T; H), \quad \eta \in L^\infty(0, T; \mathcal{M}), \quad z \in L^\infty(0, T; L^2(\Omega \times (0, 1))). \]

(ii) Let \((u, u_t, \eta, z)\) and \((\tilde{u}, \tilde{u}_t, \tilde{\eta}, \tilde{z})\) be weak solutions of problem \((1.5)-(1.7)\) corresponding to the initial data \((u_0, u_1, \eta_0, z_0)\) and \((\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0, \tilde{z}_0)\), respectively. Then we obtain
\[ \|(u, u_t, \eta, z) - (\tilde{u}, \tilde{u}_t, \tilde{\eta}, \tilde{z})\|_{\mathcal{H}} \leq cT \|(u_0, u_1, \eta_0, z_0) - (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0, \tilde{z}_0)\|_{\mathcal{H}}, \quad t \in [0, T] \]
for some constant \( c > 0 \). In particular, the problem \((1.5)-(1.7)\) has uniqueness.

The proof can be obtained by the combining the arguments of [10, 11, 15, 23].

**Remark 2.** The well-posedness of the problem \((1.5)-(1.7)\) defines the evolution operator
\[ S(t) : \mathcal{H} \rightarrow \mathcal{H}, \quad S(t)(u_0, u_1, \eta_0, z_0) = (u(t), u_t(t), \eta^t, z(t)), \quad t \geq 0, \]
where \((u(t), u_t(t), \eta^t, z(t))\) is the weak solution corresponding to initial data \((u_0, u_1, \eta_0, z_0)\). \( S(t) \) satisfies the semigroup properties
\[ S(0) = I, \quad \text{and} \quad S(t+s) = S(t)S(s), \quad t, s \geq 0. \]
and define a nonlinear \( C_0 \)-semigroup, which is locally Lipshitz continuous on \( \mathcal{H} \). Then, the long-time dynamic of the problem \((1.5)-(1.7)\) can be studied by the continuous dynamical system \( (\mathcal{H}, S(t)) \).

In the end of this section, we shall give the main results of this paper.

**Theorem 2.2.** Assume that assumptions \((H1)-(H4)\) hold and \( 0 < k < \frac{\nu_1}{2} \). Then the dynamical system \( (\mathcal{H}, S(t)) \) corresponds to the problem \((1.5)-(1.7)\) possesses a compact global attractor \( \mathcal{A} \) with finite fractal dimensions. Moreover, it is characterized by the unstable manifold
\[ \mathcal{A} = \mathcal{M}_{+}(\mathcal{N}) \]
of the set of stationary solutions \( \mathcal{N} = \{ (u, 0, 0, 0) \in \mathcal{H} | \Delta^2 u + ku^+ + f(u) = h \} \).
Moreover, if the nonlinear source term \( f \) satisfies the further assumption
\[
f(s)s \geq F(s) - \frac{\lambda_1}{2}(1 - \nu)|u|^2, \quad \nu \in (0, 1], \forall s \in \mathbb{R},
\]
and no external force is acting, we can obtain the uniform exponential decay of the associated energy.

**Theorem 2.3.** Let (H1)-(H4) hold with \( h = 0 \) and \( m_f = 0 \), \( f \) also satisfy (2.9) and \( 0 < k < \nu \lambda_1 \). Then, for any initial data \( U \in \mathcal{H} \), the following decay estimate
\[
\|S(t)U\|_\mathcal{H} \leq Ke^{-\omega t}, \quad t \geq 0.
\]
holds for some positive constants \( K \) and \( \omega \) with \( K \) dependent of \( U \) and \( \omega \) independent of \( U \).

3. **Nonlinear dynamical systems.** In this section, for sake of completeness and further references we collect several known results in the theory of nonlinear dynamical systems. They can be found in, for instance, Babin and Vishik [3]. Below we follow more closely the book by Chueshov and Lasiecka [5].

A compact set \( A \subset \mathcal{H} \) is a global attractor for a dynamical system \( (\mathcal{H}, S(t)) \), if it is fully invariant and uniformly attracting, that is \( S(t)A = A \) for all \( t \geq 0 \), and for every bounded subset \( B \subset \mathcal{H} \),
\[
\lim_{t \to +\infty} \text{dist}_\mathcal{H}(S(t)B, A) = 0,
\]
where \( \text{dist}_\mathcal{H} \) is the Hausdorff semidistance in \( \mathcal{H} \). Given a compact set \( M \) in a metric space \( X \), the fractal dimension of \( M \) is defined by
\[
\dim^X_f = \limsup_{\varepsilon \to 0} \frac{\ln N(M, \varepsilon)}{\ln(1/\varepsilon)},
\]
where \( N(M, \varepsilon) \) is the minimal number of closed balls with radius \( \varepsilon > 0 \) which covers \( M \).

Define the unstable manifold \( \mathcal{M}_+(Y) \) emanating from the set \( Y \subset \mathcal{H} \) such that there exists a full trajectory \( \gamma = \{z(t) : t \in \mathbb{R}\} \) with the properties
\[
z(0) = z_0 \quad \text{and} \quad \lim_{t \to -\infty} \text{dist}_\mathcal{H}(z(t), Y) = 0.
\]

We recall the properties of the gradient systems. A dynamical system \( (H, S(t)) \) is called a gradient system if there exists a strict Lyapunov function for \( (H, S(t)) \) on the whole phase space \( H \), that is, (a) a continuous functional \( \Phi(z) \) such that the function \( t \to \Phi(S(t)z) \) is non-increasing for any \( z \in H \), (b) the equation \( \Phi(S(t)z) = \Phi(z) \) for all \( t > 0 \) implies that \( S(t)z = z \) for all \( t > 0 \).

Now, we give some well-known results on the existence and structure of global attractors, see for instance, Chueshov and Lasiecka [5].

**Theorem 3.1** ([5], Corollary 7.5.7). Assume the dynamical system \( (H, S(t)) \) is asymptotically smooth and gradient with Lyapunov function \( \Phi(z) \). In addition, assume that

(i) \( \Phi(z) \) is bounded from above on any bounded subset of \( H \),
(ii) The set \( \Phi_R = \{z \in H|\Phi(z) \leq R\} \) is bounded for every \( R \),
(iii) The set \( \mathcal{N} \) of stationary solution of \( (H, S(t)) \) is bounded.

Then \( (H, S(t)) \) has a compact global attractor characterized by \( \mathcal{A} = \mathcal{M}_+(\mathcal{N}) \).
Using integration by substitution, it follows from (2.1) and (2.5) that
\[ u = (u(t), u_1(t), \xi(t)), \quad z = (u_0, u_1, \xi_0) \in H, \] (3.1)
where the functions \( u \) and \( \xi \) have the regularity
\[ u \in C(\mathbb{R}^+; X) \cap C^2(\mathbb{R}^+; Y), \quad \xi \in C(\mathbb{R}^+; Z). \] (3.2)
Then \((H, S(t))\) is called quasi-stable on a set \( B \subset H \), if there exists a compact semi-norm \( n_X \) on \( X \) (i.e. if \( x_j \to 0 \) in \( X \) one has \( n_X(x_j) \to 0 \)) and nonnegative scalar functions \( a, b, c \), with \( a, c \) locally bounded in \([0, \infty)\) and \( b \in L^1(\mathbb{R}^+) \) satisfying
\[ \lim_{t \to \infty} b(t) = 0, \] such that
\[ \|S(t)z^1 - S(t)z^2\|^2_H \leq a(t) \|z^1 - z^2\|^2_H, \quad t \geq 0 \] (3.3)
and
\[ \|S(t)z^1 - S(t)z^2\|^2_H \leq b(t) \|z^1 - z^2\|^2_H + c(t) \sup_{0 < s < t} \left[ n_X(u^1(s) - u^2(s)) \right]^2 \] (3.4)
for any \( z^1, z^2 \in B \).

For quasi-stable systems, we state the following theorem, cf. [5], Prop.7.9.4 and Thm. 7.9.6.

**Theorem 3.2.** Let \((H, S(t))\) be a dynamical system given by (3.1) and satisfying (3.2) and quasi-stable on positively invariant bounded subsets of \( H \). Then \((H, S(t))\) is asymptotically smooth and its compact global attractor, if any, has finite fractal dimension.

4. Global attractor and its fractal dimension. In this section, we will use the abstract results given in Section 3 to prove Theorem 2.2. We prove that the dynamical system \((\mathcal{H}, S(t))\) is gradient and quasi-stable on bounded positively invariant sets.

First, we define the energy functional of solutions for problem (1.5)-(1.7) as
\[
E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\eta\|^2_M + \frac{\xi}{2} \int_{t-\tau}^t e^{\sigma(s-t)} \|u_t(s)\|^2 ds \\
+ \frac{1}{2} \|u^+\|^2 + \int_\Omega F(u)dx - \int_\Omega hudx,
\] (4.1)
where
\[ |\mu_2| < \xi < 2\mu_1 - |\mu_2| \quad \text{and} \quad 0 < \sigma < \frac{1}{\tau} \ln \frac{\xi}{|\mu_2|}. \] (4.2)
Using integration by substitution \( s = t - \rho \tau \), we obtain
\[
\frac{\xi}{2} \int_{t-\tau}^t e^{\sigma(s-t)} \|u_t(s)\|^2 ds = -\frac{\xi \tau}{2} \int_0^\rho e^{-\sigma \rho \tau} u_t^2(x, t - \rho \tau)dxd\rho \\
= \frac{\xi \tau}{2} \int_0^1 \int_\Omega e^{-\sigma \rho \tau} z^2(x, \rho, t)dxd\rho.
\] (4.3)
It follows from (2.1) and (2.5) that
\[
\int_\Omega F(u)dx \geq -\frac{1}{2} \|\nabla u\|^2 - m_f|\Omega|
\]
and

\[- \int_\Omega h u d x \geq - \frac{\nu}{4} \| \nabla u \|^2 - \frac{1}{\lambda_1 \nu} \| h \|^2.\]

Then, we deduce that

\[E(t) \geq \frac{1}{2} \| u_t \|^2 + \frac{\nu}{4} \| \nabla u \|^2 + \frac{1}{2} \| u^+ \|^2 + \frac{1}{2} \| \eta^+ \|^2_{\mathcal{M}}\]

\[+ \frac{\xi \tau}{2} \int_0^1 \int_\Omega e^{-\sigma \rho \tau} z^2(x, \rho, t) dx d\rho - c_{f,h}\]

\[\geq \frac{1}{2} \| u_t \|^2 + \frac{\nu}{4} \| \nabla u \|^2 + \frac{1}{2} \| u^+ \|^2 + \frac{1}{2} \| \eta^+ \|^2_{\mathcal{M}}\]

\[+ \frac{\xi \tau e^{-\sigma \tau}}{2} \int_0^1 \int_\Omega z^2(x, \rho, t) dx d\rho - c_{f,h}\]

\[\geq \frac{1}{c_1} \left( \| u_t \|^2 + \| \nabla u \|^2 + \| \eta^+ \|^2_{\mathcal{M}} + \| z \|^2_{{\mathcal{L}}^2(\Omega \times (0,1))} \right) - c_{f,h}, \tag{4.4}\]

where

\[c_{f,h} = m_f |\Omega| + \frac{1}{\lambda_1 \nu} \| h \|^2, \quad \text{and} \quad \frac{1}{c_1} = \min \left\{ \frac{\nu}{4}, \frac{\xi \tau e^{-\sigma \tau}}{2} \right\} \}

Hence, (4.4) yields that

\[\| (u, u_t, \eta, z) \|^2_{\mathcal{H}} \leq c_1 (E(t) + c_{f,h}). \tag{4.5}\]

We will prove Theorem 2.2 in two parts.

4.1. Gradient system.

**Lemma 4.1.** The dynamical system \((\mathcal{H}, S(t))\) associated to the problem (1.5)-(1.7) is gradient.

**Proof.** Let us take the functional \(\Phi\) as the energy \(E\) defined by (4.1). Then, for \(U_0 = (u_0, u_1, \eta_0, z_0) \in \mathcal{H}\), we claim that \(\Phi(S(t)U_0)\) is non-increasing. Indeed, multiplying the Eq. (1.5) by \(u_t\), we have

\[
\frac{d}{dt} \left\{ \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| \nabla u \|^2 + \frac{1}{2} \| u^+ \|^2 + \int_\Omega (F(u) - hu) dx \right\}
\]

\[+ \mu_1 \| u_t \|^2 + \mu_2 (z(1,t), u_t) + (\eta^+, u_t)_{\mathcal{M}} = 0.\]

It follows from (1.4) that

\[(\eta^+, u_t)_{\mathcal{M}} = (\eta^+, \eta^+_t + \eta^+_s)_{\mathcal{M}} = \frac{1}{2} \frac{d}{dt} \| \eta^+ \|^2_{\mathcal{M}} + \int_0^\infty \mu(s) (\eta^+(s), \eta^+_s(s))_V ds
\]

\[= \frac{1}{2} \frac{d}{dt} \| \eta^+ \|^2_{\mathcal{M}} - \frac{1}{2} \int_0^\infty \mu'(s) \| \eta^+(s) \|^2_V ds. \tag{4.6}\]

Thus, by direct calculation and Young's inequality, we get

\[E'(t) = - \mu_1 \| u_t(t) \|^2 - \mu_2 (z(1,t), u_t(t)) - \frac{\sigma \xi}{2} \int_{t-\tau}^t e^{\sigma(s-t)} \| u_t(s) \|^2 ds
\]

\[+ \frac{\xi}{2} \| u_t(t) \|^2 - \frac{\xi e^{-\sigma \tau}}{2} \| u_t(t - \tau) \|^2 + \frac{1}{2} \int_0^\infty \mu'(s) \| \eta^+(s) \|^2_V ds\]

\[\leq - (\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2}) \| u_t \|^2 - \left( \frac{\xi e^{-\sigma \tau}}{2} - \frac{\mu_2}{2} \right) \| z(1,t) \|^2
\]

\[- \frac{\sigma \xi}{2} \int_{t-\tau}^t e^{\sigma(s-t)} \| u_t(s) \|^2 ds + \frac{1}{2} \int_0^\infty \mu'(s) \| \eta^+(s) \|^2_V ds. \tag{4.7}\]
From the condition (4.2), the coefficients $\mu_1 - \frac{|\nu_2|}{2} - \frac{\xi}{2} =: c_2$ and $\frac{\xi e^{-\sigma t}}{2} - \frac{|\nu_2|}{2} =: c_3$ are positive. (2.8) and (4.7) yield that $\Phi(S(t)U_0)$ (i.e. $E(t)$) is non-increasing.

Now let us suppose that $\Phi(S(t)U_0) = \Phi(U_0)$ for all $t \geq 0$. Then the energy along the flow is constant, i.e. $E'(t) = 0$, hence

$$
\|u_t\|^2 = 0, \quad \|z(1, t)\|^2 = 0, \quad \text{and} \quad \int_0^\infty \mu'(s)\|\eta'(s)\|^2 \, ds = 0.
$$

The first two terms imply that $u(t) = u_0$ and $z(1, t) = 0$ for all $t \geq 0$. On the other hand, it follows from (2.8) that

$$
0 = -\int_0^\infty \mu'(s)\|\eta'(s)\|^2 \, ds \geq \delta \int_0^\infty \mu(s)\|\eta'(s)\|^2 \, ds = \delta\|\eta'\|_{\mathcal{H}}^2,
$$

which implies that $\eta'(s) = 0$ for all $t, s \geq 0$. Hence, $S(t)U_0 = (u_0, 0, 0, 0)$ is a stationary solution, which implies that $\Phi$ is a strict Lyapunov functional. This completes the proof. □

**Lemma 4.2.** The Lyapunov functional $\Phi$ is bounded above on any bounded subset of $\mathcal{B}$ of $\mathcal{H}$ and the set $\Phi_R = \{ U \in \mathcal{H} | \Phi(U) \leq R \}$ is bounded for any $R > 0$.

**Proof.** Since $\Phi$ is given as the energy functional $E(t)$, using the growth condition (2.2), it is easy to see that there exists $c_F > 0$ such that

$$
E(t) \leq c_F(1 + \|(u, u, \eta^t, z)\|_{\mathcal{H}}^{p+2}).
$$

Now assuming $U = (u, u, \eta^t, z)$ be any weak solution of (1.5)-(1.7) such that $\Phi(U) \leq R$, then from (4.4), we have

$$
\frac{1}{c_1}\|U(t)\|^2_{\mathcal{H}} - c_{f,h} \leq \Phi((U((t)) \leq R,
$$

which gives $\|U(t)\|^2_{\mathcal{H}} \leq c_1(R + c_{f,h})$, i.e. $\Phi_R$ is bounded. □

**Lemma 4.3.** Under the assumptions of Theorem 2.2, the set of equilibrium points $\mathcal{N}$ is bounded in $\mathcal{H}$.

**Proof.** From Lemma 4.1, let $U \in \mathcal{H}$ be a stationary solution of (1.5)-(1.7), we obtain that $U = (u, 0, 0, 0)$ and satisfies $\triangle^2 u + ku^+ + f(u) = h, \ u \in \Omega$. Then, we have

$$
\|\triangle u\|^2 = -k(u^+, u) - \int_\Omega f(u)u \, dx + \int_\Omega hudx. \quad (4.8)
$$

Observing that $|u^+| \leq |u|$, we obtain

$$
-k(u^+, u) \leq k\|u\|^2 \leq \frac{k}{\lambda_1}\|\triangle u\|^2.
$$

Also, from the assumption (2.4) and Young’s inequality, we obtain

$$
-\int_\Omega f(u)u \, dx \leq m_f|\Omega| + \lambda_1(1 - \nu)\|u\|^2 \leq m_f|\Omega| + (1 - \nu)\|\triangle u\|^2,
$$

and

$$
\int_\Omega hudx \leq \frac{\nu}{2}\|\triangle u\|^2 + \frac{1}{2\lambda_1\nu}\|h\|^2.
$$

Applying the above estimates into (4.8), we have

$$
\left(\frac{\nu}{2} - \frac{k}{\lambda_1}\right)\|\triangle u\|^2 \leq m_f|\Omega| + \frac{1}{2\lambda_1\nu}\|h\|^2.
$$

It follows from the condition $k < \frac{\nu\lambda_1}{2}$ that $\mathcal{N}$ is bounded in $\mathcal{H}$. □
4.2. Quasi-stability.

Lemma 4.4. Assume that (H1)-(H4) hold and $0 < k < \frac{\mu_{\lambda_1}}{2}$, given a bounded set $B$ of $\mathcal{H}$, let $S(t)U_0 = (u, u_t, \eta', z)$ and $S(t)\tilde{U}_0 = (\tilde{u}, \tilde{u}_t, \tilde{\eta}', \tilde{z})$ be two weak solutions of problem (1.5)-(1.7) corresponding to the initial data $U_0 = (u_0, u_1, \eta_0, z_0)$ and $\tilde{U}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0, \tilde{z}_0)$, respectively. Then

\[ \|S(t)U_0 - S(t)\tilde{U}_0\|_{\mathcal{H}}^2 \leq b_0 e^{-\gamma t} \|U_0 - \tilde{U}_0\|_{\mathcal{H}}^2 + C_B \int_0^t e^{-\gamma(t-s)} \|u(s) - \tilde{u}(s)\|_{2(p+1)}^2 ds, \]  

(4.9)

where $b_0 > 0$, $\gamma > 0$ and $C_B$ is a constant depending on the size of $B$.

Proof. We denote $w = u - \tilde{u}$, $\zeta^t = \eta^t - \tilde{\eta}^t$ and $q(x, \rho, t) = z(x, \rho, t) - \tilde{z}(x, \rho, t)$, then $(w, w_t, \zeta^t, q)$ is a weak solution of

\[
\begin{align*}
&w_{tt} + \triangle^2 w + \int_0^\infty \mu(s) \triangle^2 \zeta^t(s) ds + k(u^+ - \tilde{u}^+) \\
&+ \mu_1 w_t + \mu_2 q(x, 1, t) + f(u) - f(\tilde{u}) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \\
&\zeta_t^t = -\zeta^t + w_t, \quad (x, t) \in \Omega \times (0, +\infty) \times \mathbb{R}^+, \\
&\tau q_t(x, \rho, t) + q_{\rho}(x, \rho, t) = 0, \quad (x, \rho, t) \in \Omega \times (0, 1) \times \mathbb{R}^+,
\end{align*}
\]

(4.10)

with boundary conditions

\[ w = \triangle w = 0 \text{ on } \partial \Omega \times \mathbb{R}^+, \quad \eta = \triangle \eta = 0 \text{ on } \partial \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \]

(4.11)

and the initial conditions

\[ w(0) = u_0 - \tilde{u}_0, \quad u_t(0) = u_1 - \tilde{u}_1, \quad \zeta^t(0) = 0, \quad \zeta^0 = \eta_0 - \tilde{\eta}_0, \quad q(x, \rho, 0) = z_0(x, \rho) - \tilde{z}_0(x, \rho). \]

(4.12)

To this system, we consider the functional

\[ E_w(t) = \frac{1}{2} \|w_t\|^2 + \frac{1}{2} \|\triangle w\|^2 + \frac{1}{2} \|\zeta^t\|^2_{\mathcal{M}} + \frac{\xi}{2} \int_0^t e^{\sigma(s-t)} \|w_t(s)\|^2 dt, \]

(4.13)

where $\sigma$ and $\xi$ are as given by (4.2).

We shall divide into the following five steps to prove this Lemma.

Step 1. Proceeding as in the proof of Lemma 4.1, we obtain

\[ E'_w(t) \leq -c_2 \|w_t\|^2 - c_3 \|q(1, t)\|^2 - \frac{\sigma \xi}{2} \int_0^t e^{\sigma(s-t)} \|w_t(s)\|^2 ds \\
+ \frac{1}{2} \int_0^\infty \mu'(s) \|\zeta^t(s)\|^2 ds - k \left( u^+ - \tilde{u}^+, w_t \right) - \left( f(u) - f(\tilde{u}), w_t \right). \]

(4.14)

Noticing that $|u^+ - \tilde{u}^+| \leq |u - \tilde{u}|$ and $\|w_t\|^2 \leq \lambda \|w\|_{2(p+1)}^2$, for any $\delta_0 > 0$, we have

\[ -k \left( u^+ - \tilde{u}^+, w_t \right) \leq \frac{\lambda k^2}{\delta_0} \|w(t)\|_{2(p+1)}^2 + \frac{\delta_0}{4} \|w_t\|^2. \]

Using the generalized H"older’s inequality with $\frac{p}{2(p+1)} + \frac{1}{2(p+1)} + \frac{1}{2} = 1$, assumption (2.2) and Young’s inequality, we have

\[ -(f(u) - f(\tilde{u}), w_t) \leq c_0 \left( \|\eta\|_{2(p+1)}^p + \|u\|_{2(p+1)}^p + \|w_t\|_{2(p+1)}^p \right) \|w\|_{2(p+1)} \|w_t\| \\
\leq C(B) \|w(t)\|_{2(p+1)} \|w_t(t)\| \leq \frac{C(B)}{\delta_0} \|w(t)\|_{2(p+1)}^2 + \frac{\delta_0}{4} \|w_t(t)\|^2. \]
Hence there exists a constant $m$ different from line to line. Substituting these estimates into (4.14), we have

$$E'_w(t) \leq - (c_2 - \frac{\delta_0}{2}) ||w_t||^2 - c_3 ||q(1,t)||^2 - \frac{\sigma \xi}{2} \int_{t-\tau}^t e^{\sigma(s-t)} ||w_t(s)||^2 \, ds$$

$$+ \frac{1}{2} \int_0^\infty \mu'(s)||\zeta'(s)||^2 \, ds + \left( \frac{\lambda k^2}{\delta_0} + C(B) \right) ||w||^2_{2(p+1)}. \quad (4.15)$$

On the other hand, by the similar argument as [23], we can show that there exist positive constants $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1 E_w(t) \leq \left( ||w, w_t, \zeta', q||^2 \right)_{\mathcal{H}} \leq \alpha_2 E_w(t). \quad (4.16)$$

**Step 2.** Let us define the functional

$$G(t) = ME_w(t) + \varepsilon \varphi_w(t) + \psi_w(t),$$

where $M > 0$, $\varepsilon > 0$ are constants to be determined later and

$$\phi_w(t) = \int_{\Omega} w_t(t) w(t) \, dx, \quad \psi_w(t) = - \int_{\Omega} w_t(t) \int_0^\infty \mu(s) \zeta'(s) ds \, dx.$$ 

Then, we can show that there exist positive constants $\alpha_3$ and $\alpha_4$ such that

$$\alpha_3 E_w(t) \leq G(t) \leq \alpha_4 E_w(t). \quad (4.17)$$

Indeed, it follows from Young’s inequality and Poincaré’s that

$$|\phi_w(t)| \leq \frac{1}{2} ||w_t||^2 + \frac{1}{2\lambda_1} ||\Delta w||^2, \quad |\psi_w(t)| \leq \frac{1}{2} ||w_t||^2 + \frac{\mu_0}{2\lambda_1} ||\zeta'||^2_{\mathcal{M}}.$$ 

Hence there exists a constant $m_0 > 0$ such that

$$|G(t) - ME_w(t)| = |\varepsilon \phi_w(t) + \psi_w(t)| \leq m_0 E_w(t).$$

Then we can obtain (4.17) with $\alpha_3 = M - m_0$ and $\alpha_4 = M + m_0$ by choosing $M > 0$ such that $M - m_0 > 0$.

**Step 3.** We show the estimate of $\phi_w(t)$. Taking the derivative of $\phi_w(t)$, using Eq. (4.10), subtracting and adding $E_w(t)$, we can get

$$\phi'_w(t) = - E_w(t) + \frac{3}{2} ||w_t||^2 - \frac{1}{2} ||\Delta w||^2 + \frac{1}{2} ||\zeta'||^2_{\mathcal{M}} - \int_{\Omega} \int_0^\infty \mu(s) \zeta'(s) ds \Delta w(t) \, dx$$

$$- k \left( u^+ - \bar{u}^+, w \right) - \mu_1 (w_t, w) - \mu_2 (q(1), t, w) - (f(u) - f(\bar{u}), w) \quad (4.18)$$

As in the step 1, we can get the following estimates

$$-k \left( u^+ - \bar{u}^+, w \right) \leq k ||w||^2 \leq \frac{k}{\lambda_1} ||\Delta w||^2$$

$$- \mu_1 (w_t, w) \leq \frac{\delta_1}{4} ||\Delta w||^2 + \frac{\mu_2}{4 \delta_1 \lambda_1} ||w_t||^2,$$

$$- \mu_2 (q(1), t, w) \leq \frac{\delta_1}{4} ||\Delta w||^2 + \frac{\mu_2}{4 \delta_1 \lambda_1} ||q(1)||^2,$$

$$\int_{\Omega} \int_0^\infty \mu(s) \zeta'(s) ds \Delta w(t) \, dx \leq \frac{\delta_1}{4} ||\Delta w||^2 + \frac{\mu_0}{\delta_1} ||\zeta'||^2_{\mathcal{M}},$$

$$-(f(u) - f(\bar{u}), w) \leq C(B) ||w||_{2(p+1)} ||w|| \leq \frac{\delta_1}{4} ||\Delta w||^2 + \frac{C(B)}{\delta_1 \lambda_1} ||w||^2_{2(p+1)},$$
for any \( \delta_1 > 0 \). Inserting these estimates into (4.18), we get
\[
\phi_w(t) \leq -E_w(t) + \left( \frac{3}{2} + \frac{\mu^2}{\delta_1 \lambda_1} \right) \| w_t \|^2 - \left( \frac{1}{2} - \frac{k}{\lambda_1} - \delta_1 \right) \| \Delta w \|^2 + \left( \frac{1}{2} + \frac{\mu_0}{\delta_1} \right) \| \zeta \|^2_M + \frac{\mu_0^2}{\delta_1 \lambda_1} \| q(1, t) \|^2 + \frac{C(B)}{\delta_1 \lambda_1} \| w \|^2_{2(p+1)}.
\]  
\hspace{1cm} (4.19)

**Step 4.** We show the estimate of \( \psi_w(t) \). Taking the derivative of \( \psi_w(t) \), using Eq. (4.10), we get
\[
\psi_w = -\int w_t \int_0^\infty \mu(s) \zeta^t(s) ds dx - \int w_t \int_0^\infty \mu(s) \zeta^t_1(s) ds dx \]  
\hspace{1cm} (4.20)
\[
= \int \left( \Delta^2 w + \int_0^\infty \mu(s) \Delta^2 \zeta^t(s) ds + k \left( u^+ - \bar{u}^+ \right) + \mu_1 \omega_t \right) \int_0^\infty \mu(s) \zeta^t(s) ds dx \]  
\hspace{1cm} + \mu_2 \omega(q(1, t) + (f(u) - f(\bar{u}))) \int_0^\infty \mu(s) \zeta^t(s) ds dx - \int w_t \int_0^\infty \mu(s) \zeta^t_1(s) ds dx.
\]

Noticing the procedure used in Step 3, we can deduce the following estimates for any \( \delta_2 > 0 \)
\[
\int \Delta w(t) \int_0^\infty \mu(s) \Delta \zeta^t(s) ds dx \leq \delta_2 \| \Delta w \|^2 + \frac{\mu_0}{4 \delta_2} \| \zeta \|^2_M, \]  
\[
\int \left( \int_0^\infty \mu(s) \Delta \zeta^t(s) ds \right)^2 dx \leq \mu_0 \| \zeta \|^2_M, \]  
\[
k \int \left( u^+ - \bar{u}^+ \right) \int_0^\infty \mu(s) \zeta^t(s) ds dx \leq \delta_2 \| \Delta w \|^2 + \frac{\mu_0 k^2}{\lambda_1} \| \zeta \|^2_M, \]  
\[
\mu_1 \int w_t \int_0^\infty \mu(s) \zeta^t(s) ds dx \leq \delta_2 \| w_t \|^2 + \frac{\mu_0 \mu_1^2}{4 \delta_2 \lambda_1} \| \zeta \|^2_M, \]  
\[
\mu_2 \int q(1, t) \int_0^\infty \mu(s) \zeta^t(s) ds dx \leq \delta_2 \| q(1, t) \|^2 + \frac{\mu_0 \mu_2^2}{4 \delta_2 \lambda_1} \| \zeta \|^2_M, \]  
\[
\int (f(u) - f(\bar{u})) \int_0^\infty \mu(s) \zeta^t(s) ds dx \leq \frac{\delta_2 C(B)}{2 \lambda_1} \| w \|^2_{2(p+1)} + \frac{\mu_0}{2 \delta_2} \| \zeta \|^2_M.
\]

For the last term of (4.20), we have
\[
-\int w_t \int_0^\infty \mu(s) \zeta^t_1(s) ds dx = -\int w_t \int_0^\infty \mu(s) \left( w_t - \zeta^t_1(s) \right) ds dx \leq -\frac{\mu_0}{2} \| w_t \|^2 - \frac{\mu_0(0)}{2 \mu_0 \lambda_1} \int_0^\infty \mu'(s) \| \zeta^t(s) \|^2_V ds.
\]
Inserting the above estimates into (4.20), we get
\[
\psi_w(t) \leq -\left( \frac{\mu_0}{2} - \delta_2 \right) \| w_t \|^2 + \left( \delta_2 - \frac{\delta_2 k^2}{\lambda_1^2} \right) \| \Delta w \|^2 \]  
\hspace{1cm} + \left( \mu_0 + \frac{\mu_0 + \mu_0(0)}{4 \delta_2 \lambda_1} \right) \| \zeta \|^2_M + \delta_2 C(B) \frac{\| w \|^2_{2(p+1)}}{2 \lambda_1} \]  
\hspace{1cm} + \delta_2 \| q(1, t) \|^2 - \frac{\mu_0(0)}{2 \mu_0 \lambda_1} \int_0^\infty \mu'(s) \| \zeta^t(s) \|^2_V ds. \]  
\hspace{1cm} (4.21)

**Step 5.** First, from (2.8), we have
\[
\| \zeta \|^2_M \leq -\frac{1}{\delta} \int_0^\infty \mu'(s) \| \zeta(s) \|^2_V ds.
\]
It follows from (4.15), (4.19) and (4.21) that

\[
G'(t) \leq -\varepsilon E_w(t) - \left(\frac{\mu_0}{2} - \delta_2 + M\left(c_2 - \frac{\delta_0}{2}\right) - \left(\frac{3}{2} + \frac{\mu_1^2 + \mu_2^2}{\delta_1\lambda_1}\right)\right) \|w_t\|^2
\]

\[
- \left(Mc_3 - \frac{\mu_1^2}{\delta_1\lambda_1} \varepsilon - \delta_2\right) \|q(1,t)\|^2
\]

\[
- \left[\frac{1}{2} - \frac{k}{\lambda_1} - \delta_1\right] \varepsilon - \left(1 + \frac{k^2}{\lambda_1^2}\right)\delta_2 \|\Delta w\|^2
\]

\[
+ \left[\frac{M}{2} - \frac{1}{\delta} \left(\frac{1}{2} + \frac{\mu_0}{\delta_1}\right) \varepsilon - \frac{1}{\delta} \left(\frac{\mu_0}{\delta_2} + \frac{\mu_0(\mu_1^2 + \mu_2^2)}{4\delta_2\lambda_1}\right) - \frac{\mu(0)}{2\mu_0\lambda_1}\right]
\]

\[
\times \int_0^\infty \mu'(s)\|\zeta'(s)\|_{V'}^2 ds
\]

\[
+ \left[\frac{\lambda k^2}{\delta_0} + \frac{C(B)}{\delta_0}\right] M + \frac{C(B)}{\delta_1\lambda_1} \varepsilon + \frac{\delta_2C(B)}{2\lambda_1} \right] \|w\|^2_{2(p+1)}
\]

\[
- \frac{\sigma\xi}{2} \int_{t-\tau}^t e^{\alpha(s-t)} \|w_t(s)\|^2 ds.
\]

Now, we fix \(\delta_2 < 2\mu_0\), and then take \(\delta_1 > 0\) sufficiently small such that

\[
\frac{1}{2} - \frac{k}{\lambda_1} - \delta_1 > 0.
\]

For such \(\delta_1\), we choose \(\varepsilon > 0\) sufficiently small such that

\[
\frac{\mu_0}{2} - \left(\frac{3}{2} + \frac{\mu_1^2}{\delta_1\lambda_1}\right) \varepsilon > 0.
\]

For fixed \(\delta_1\) and \(\varepsilon\), we choose \(\delta_2\) sufficiently small such that

\[
\left(\frac{1}{2} - \frac{k}{\lambda_1} - \delta_1\right) \varepsilon - \left(1 + \frac{k^2}{\lambda_1^2}\right) \delta_2 > 0.
\]

Then we take \(M\) sufficiently large such that the coefficients of \(\int_0^\infty \mu'(s)\|\zeta'(s)\|_{V'}^2 ds\), \(\|w_t\|^2\) and \(\|q(1,t)\|^2\) are all positive. Hence, we arrive at

\[
G'(t) \leq -\varepsilon E_w(t) + C(B)\|w\|^2_{2(p+1)}.
\]

Combining the above inequality and the right hand side of (4.17), we obtain

\[
G'(t) \leq -\frac{\varepsilon}{\alpha} E_w(t) + C(B)\|w\|^2_{2(p+1)}
\]

and so

\[
G(t) \leq G(0)e^{-\frac{\varepsilon}{\alpha} t} + C(B)\int_0^t e^{-\frac{\varepsilon}{\alpha}(t-s)} \|w\|^2_{2(p+1)} ds.
\]

Using (4.17) again, we have

\[
E_w(t) \leq \frac{\alpha_4}{\alpha_3} E_w(0)e^{-\frac{\varepsilon}{\alpha} t} + C(B)\int_0^t e^{-\frac{\varepsilon}{\alpha_4}(t-s)} \|w\|^2_{2(p+1)} ds.
\]

Since

\[
\|(w, w_t, \zeta', q)\|_{\mathcal{H}_t}^2 = \|S(t)U_0 - S(t)\tilde{U}_0\|_{\mathcal{H}_t}^2, \quad t \geq 0.
\]

Hence, using (4.16), by renaming the constants, we can obtain (4.9). This completes the proof. \(\square\)
Lemma 4.5 (Quasi-stability). Under the conditions of Theorem 2.2, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$.

Proof. Since $(\mathcal{H}, S(t))$ is defined as the solution operator of problem (1.5)-(1.7), we conclude that (3.1) and (3.2) hold with $X = V, Y = H$ and $Z = M \times L^2(\Omega \times (0,1))$ by Theorem 2.2(i). Moreover, we know that the condition (3.3) also holds true from Theorem 2.2(ii).

Let $B$ be a bounded positively invariant set of $(\mathcal{H}, S(t))$. Let $S(t)U_0 = (u, u_t, \eta^t, z)$ and $S(t)\tilde{U}_0 = (\tilde{u}, \tilde{u}_t, \tilde{\eta}^t, \tilde{z})$ be the weak solution of problem (1.5)-(1.7) corresponding to the initial data $U_0 = (u_0, u_1, \eta_0, z_0) \in B$ and $\tilde{U}_0 = (\tilde{u}_0, \tilde{u}_1, \tilde{\eta}_0, \tilde{z}_0) \in B$, respectively. We consider the semi-norm given by

$$n_V(u) = \|u\|_{2(p+1)}.$$ 

By using the compact embedding $V \hookrightarrow L^{2(p+1)}$, we have that $n_V(\cdot)$ is compact in $V$. Then, from (4.9), we can see that

$$\left\|S(t)U_0 - S(t)\tilde{U}_0\right\|^2_{\mathcal{H}} \leq b(t) \left\|U_0 - \tilde{U}_0\right\|^2_{\mathcal{H}} + c(t) \sup_{0 < s < t} [n_V(u(s) - \tilde{u}(s))]^2$$

where

$$b(t) = b_0 e^{-\gamma t} \text{ and } c(t) = C_B \int_0^t e^{-\gamma(t-s)} ds, \quad t \geq 0.$$

Finally, it is easy to see that $b \in L^1\mathbb{R}^+$ with $\lim_{t \to \infty} b(t) = 0$ and $c(t)$ is locally bounded on $[0, \infty)$. So, stabilization inequality (3.4) holds. Therefore, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on bounded positively invariant set $B \subset \mathcal{H}$. \hfill \qed

Proof of Theorem 2.2. From Theorem 3.2 and Lemma 4.5, we see that the dynamical system $(\mathcal{H}, S(t))$ is asymptotically smooth. Then by using the properties of gradient system (Lemma 4.1, 4.2 4.3), we can get that all the assumptions of Theorem 3.1 are satisfied. Hence $(\mathcal{H}, S(t))$ has a compact global attractor given by $\mathcal{A} = \mathcal{M}_+(\mathcal{N})$. Moreover, the dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on the attractor $\mathcal{A}$ (Lemma 4.5), then by applying again Theorem 3.2, we know that the attractor $\mathcal{A}$ has finite fractal dimension. The proof is complete. \hfill \qed

5. Exponential attractor. In this section, we shall study the existence of exponential attractor to problem (1.5)-(1.7), and the main result is given in the following theorem.

Theorem 5.1. Assume the assumptions of Theorem 2.2 hold, the corresponding dynamical system $(\mathcal{H}, S(t))$ possesses a generalized exponential attractor. More precisely, for any given $\theta \in (0,1]$, there exists a generalized exponential attractor $\mathcal{A}_{\exp, \theta} \subset \mathcal{H}$, with finite fractal dimension in the extended space $\mathcal{H}_{-\theta}$ which is defined as the interpolation of

$$\mathcal{H}_0 := \mathcal{H}, \text{ and } \mathcal{H}_{-1} := H \times V' \times M_0 \times L^2((0,1) \times V'),$$

where $V'$ denote the topological dual of $V$ and

$$M_0 = L^2_\theta(\mathbb{R}^+; L^2) = \{\eta : \mathbb{R}^+ \to V \mid \int_0^\infty \mu(s)\|\eta(s)\|^2 ds < \infty\}.$$ 

Before proceeding, we first briefly introduce some basic theorems to exponential attractor. An exponential attractor of a dynamical system $(\mathcal{H}, S(t))$ is a compact set $\mathcal{A}_{\exp} \subset \mathcal{H}$, that enjoys three characteristic properties: (i) it has finite fractal
Proof of Theorem 5.1. Now let us take Proposition 1. Let \( L_{\text{Lyapunov functional}} \) considered in Lemma 4.1. Then we can have that for generalized exponential attractor where \( C \) an extended space \( \tilde{H} \) quasi-stable on some bounded absorbing set \( B \) exponential attractor if the dynamical system is quasi-stable.

In this paper, we consider the concept of generalized exponential attractor as presented in Chueshov and Lasiecka [5], where the set \( \mathcal{A}_{exp} \) has finite fractal dimension in an extended phase space \( \tilde{H} \supset H \). Thus we can consider the exponential attractors in weak phase spaces. In addition, we can show the existence of generalized exponential attractor if the dynamical system is quasi-stable.

To prove Theorem 5.1, we use the following abstract result given in Chueshov and Lasiecka [5] Theorem 7.9.9.

**Proposition 1.** Let \( (H, S(t)) \) be a dynamical system satisfying (3.1) and (3.2) and quasi-stable on some bounded absorbing set \( B \). In addition assume there exists an extended space \( \tilde{H} \supset H \) such that, for any \( T > 0 \),

\[
\| S(t_1)y - S(t_2)y \|_{\tilde{H}} \leq C_{BT}|t_1 - t_2|^{\gamma}, \quad t_1, t_2 \in [0, T], y \in B
\]

where \( C_{BT} > 0 \) and \( \gamma \in (0, 1] \) are constants. Then the dynamical \( (H, S(t)) \) has a generalized exponential attractor \( \mathcal{A}_{exp} \subset H \) with finite fractal dimension in \( H \).

**Proof of Theorem 5.1.** Now let us take \( B = \{ U|\Phi(U) \leq R \} \), where \( \Phi \) is the strict Lyapunov functional considered in Lemma 4.1. Then we can have that for \( R \) large \( B \) is a positively invariant bounded absorbing set, which implies that the system is quasi-stable on set \( B \).

On the other hand, for the solution \( U(t) = (u(t), u_t(t), \eta^t, z(\rho,t)) \in H \), we infer from the system (1.5)-(1.7) that

\[
(u_t(t), u_{tt}(t), \eta^t_t, z_t) \in L^2_{loc}(\mathbb{R}^+, \tilde{H}_{-1}),
\]

Then for solution \( U(t) \) with initial data \( y = U(0) \in B \), we conclude from the positive invariance of \( B \) that, for any \( T > 0 \),

\[
\int_0^T \| U_t(s) \|_{\tilde{H}_{-1}}^2 ds \leq C_{BT}^2
\]

which implies that

\[
\| S(t_1)y - S(t_2)y \|_{\tilde{H}_{-1}} \leq \int_{t_1}^{t_2} \| U_t(s) \|_{\tilde{H}_{-1}}^2 ds \leq C_{BT}|t_1 - t_2|^{\frac{\gamma}{2}},
\]

where \( C_{BT} \) is a positive constant. This shows that for any \( y \in B \) the map \( t \mapsto S(t)y \) is \( \frac{1}{2} \)-Hölder continuous in the extended phase space \( \tilde{H}_{-1} \), then Proposition 5.1 guarantees the existence of a generalized exponential attractor with finite fractal dimension in \( \tilde{H}_{-1} \).

By the same argument in [4, 9] using interpolation theorem, we can obtain the existence of exponential attractors in \( \tilde{H}_{-\theta} \) with \( \theta \in (0, 1) \). The proof is complete. \( \square \)

6. **Exponential decay.** In this section, we study the exponential decay property of solution to problem (1.5)-(1.7) with \( h = 0 \). The energy functional corresponding to this system is defined as (4.1) with \( h = 0 \).

Now, let us define a Lyapunov functional

\[
L(t) = E(t) + \varepsilon_1 \varphi(t) + \varepsilon_2 \psi(t)
\]
with
\[ \phi(t) = \int_{\Omega} u_t(t)u(t)dx, \quad \psi(t) = -\int_{\Omega} u_t(t) \int_0^\infty \mu(s)\eta'(s)dsdx \]
where \( \varepsilon_1 \) and \( \varepsilon_2 \) are two positive numbers to be determined later.

**Proof of Theorem 2.3.** To achieve our goal, we need the estimates about \( E'(t), \phi'(t) \) and \( \psi'(t) \). Some calculations are similar as the proof Lemma 4.4. Here, we only give the sketch and the different calculations.

**Step 1.** Taking the time derivative of \( \phi(t) \), using Eq. (1.5), and integrating by part and subtracting and adding \( E(t) \), we can get
\[ \phi'(t) = \int_{\Omega} (\Delta^2 u - \int_0^\infty \mu(s)\Delta^2 \eta'(s)ds - ku^+ - \mu_1 u_t - \mu_2 z(1,t) - f(u))u dx + \|u_t\|^2 \]
\[ \leq -E(t) + \frac{3}{2} \|u_t\|^2 - \left( \frac{\nu}{2} - \frac{k}{2\lambda_1} \right) \|\Delta u\|^2 + \frac{1}{2} \|\eta'\|^2_{M^2} + \frac{\xi}{2} \int_0^t \int_{\Omega} e^{\sigma(s-t)}u^2(s)dsdx \]
\[ -\mu_1 \int_{\Omega} u_t u dx - \mu_2 \int_{\Omega} z(1,t) u dx - \int_{\Omega} \int_0^\infty \mu(s)\Delta \eta'(s)\Delta u(t)dsdx, \quad (6.2) \]
where we also use
\[ -k \int_{\Omega} u^+ u dx + \frac{k}{2} \|u^+\|^2 = -\frac{k}{2} \|u^+\|^2 \leq \frac{k}{2\lambda_1} \|\Delta u\|^2, \]
and
\[ -\int_{\Omega} f(u) u dx + \int_{\Omega} F(u) dx \leq \frac{\lambda_2}{2} (1 - \nu) \|u\|^2 dx \leq \frac{1 - \nu}{2} \|\Delta u\|^2. \]
Using the similar arguments to the proof of step 3 in Lemma 4.4 for the last three terms of (6.2), we can rewrite (6.2) as
\[ \phi'(t) \leq -E(t) + \left( \frac{3}{2} + \frac{\mu_1^2}{\delta_1 \lambda_1} \right) \|u_t\|^2 - \left( \frac{\nu}{2} - \frac{k}{2\lambda_1} - \frac{3}{4} \delta_1 \right) \|\Delta u\|^2 \]
\[ + \left( \frac{1}{2} + \frac{\mu_0}{\delta_1} \right) \|\eta'\|^2_{M^2} + \frac{\mu_2}{\delta_1 \lambda_1} \|z(1,t)\|^2 + \frac{\xi}{2} \int_0^t \int_{\Omega} e^{\sigma(s-t)}u^2(s)dsdx, \quad (6.3) \]
for any \( \delta_1 > 0 \) considered in Lemma 4.4.

**Step 2.** Taking the time derivative of \( \psi(t) \), using Eq. (1.5), and integrating by part, we have
\[ \psi'(t) = \int_{\Omega} \left[ \Delta^2 u + \int_0^\infty \mu(s)\Delta^2 \eta'(s)ds + ku^+ + \mu_1 u_t + \mu_2 z(1,t) + f(u) \right] \]
\[ \times \int_0^\infty \mu(s)\eta'(s)dsdx - \int_{\Omega} u_t \int_0^\infty \mu(s)\eta'(s)dsdx. \quad (6.4) \]
Since \( E(t) \) is non-increasing via (1.7), noting (2.2) and (2.3), we obtain
\[ \int_{\Omega} f(u) \int_0^\infty \mu(s)\eta'(s)dsdx \]
\[ \leq c_0 \int_{\Omega} \left( 1 + |u|^p \right) \left| u \right| \int_0^\infty \mu(s)\eta'(s)dsdx \]
\[ \leq c_0 \left( \Omega \int_0^\infty |u|^p 2(p+1) \right) \left| u \right|_{2(p+1)} \int_0^\infty \mu(s)\eta'(s)dsdx \]
\[ \leq \delta_2 \epsilon_4 E(0) \|\Delta u\|^2 + \frac{\mu_0}{4\delta_2 \lambda_1} \|\eta'\|^2_{M^2}. \]
Lemma 4.4, we omit the details here. Hence, (6.4) can be written as
\[
\psi'(t) \leq -\left(\frac{\mu_0}{2} - \delta_2\right)\|u_t\|^2 + \left(\delta_2 + \frac{\delta_2 k^2}{\lambda_1^2} + \delta_2 c_4 E(0)^p\right)\|\Delta u\|^2
+ \delta_2\|z(1, t)\| + \mu_0 \left(1 + \frac{1}{2\delta_2} + \frac{\mu_0^2 + \mu^2 + 1}{4\delta_2 \lambda_1}\right)\|\eta\|^2_{\mathcal{M}}
\]
\[\leq -\frac{\mu(0)}{2\mu_0 \lambda_1} \int_0^\infty \mu'(s) \|\eta(s)\|^2_V ds \tag{6.5}\]
for any \(\delta_2 > 0\) considered in Lemma 4.4.

Step 3. The equivalence of \(L(t)\) and \(E(t)\). Using (2.5) with \(m_f = 0\), by the similar argument as (4.4), we have
\[E(t) \geq \frac{1}{2}\|u_t\|^2 + \frac{\nu}{2}\|\Delta u\|^2 + \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|\eta\|^2_{\mathcal{M}} + \frac{\xi \rho}{2} \int_0^1 \int_\Omega e^{-\sigma \rho \tau} z^2(x, \rho, t) dx d\rho
\]
\[\geq \frac{1}{c_5} \left(\|u_t\|^2 + \|\Delta u\|^2 + \|\eta\|^2_{\mathcal{M}} + \|z\|^2_{L^2(\Omega \times (0, 1))}\right), \tag{6.6}\]
where \(\frac{1}{c_5} = \min\{\frac{\nu}{2}, \frac{\xi \rho e^{-\sigma \rho \tau}}{2}\}\). It follows from Young’s inequality and Poincaré’s inequality that
\[|\phi(t)| \leq \frac{1}{2}\|u_t\|^2 + \frac{1}{2\lambda_1}\|\Delta u\|^2, \quad |\psi(t)| \leq \frac{1}{2}\|u_t\|^2 + \frac{\mu_0}{2\lambda_1}\|\eta\|^2_{\mathcal{M}}.
\]
Hence, from (6.6), for some suitable small \(\varepsilon_1\) and \(\varepsilon_2\), we have
\[|L(t) - E(t)| = |\varepsilon_1 \phi(t) + \varepsilon_2 \psi(t)| \leq c_6 E(t)
\]
for some positive constant \(c_6 = c_6(c_5, \varepsilon_1, \varepsilon_2) < 1\), which implies
\[\alpha_5 E(t) \leq L(t) \leq \alpha_6 E(t), \quad t \geq 0 \tag{6.7}\]
with positive constants \(\alpha_5 = 1 - c_6\) and \(\alpha_6 = 1 + c_6\).

Noticing (2.8) that
\[\|\eta\|^2_{\mathcal{M}} \leq -\frac{1}{\delta} \int_0^\infty \mu'(s) \|\eta(s)\|^2_V ds.
\]
Hence, it follows from (4.7), (6.1), (6.3) and (6.5) that
\[
L'(t) \leq -\varepsilon_1 E(t) - \left[c_2 - \varepsilon_1 \left(\frac{3}{2} + \frac{\mu_0^2}{\delta_1 \lambda_1}\right) + \varepsilon_2 \left(\frac{\mu_0}{2} - \delta_2\right)\right]\|u_t\|^2
- \left[\varepsilon_2 \left(\frac{nu}{2} - \frac{k}{2\lambda_1} - \frac{3}{4} \delta_1\right) - \varepsilon_2 \left(\delta_2 + \frac{\delta_2 k^2}{\lambda_1^2} + \delta_2 c_4 E(0)^p\right)\right]\|\Delta u\|^2
+ \frac{1}{2} - \varepsilon_6 \left(\frac{1}{2} + \frac{\mu_0}{\delta_2}\right) - \varepsilon_6 \frac{\mu_0}{\delta_2} \left(1 + \frac{1}{2\delta_2} + \frac{\mu_0^2 + \mu^2 + 1}{4\delta_2 \lambda_1}\right)\|\eta\|^2_{\mathcal{M}}
\times \int_0^\infty \mu'(s) \|\eta(s)\|^2_V ds - \left(c_2 - \varepsilon_6 \frac{\mu_1^2}{\delta_1 \lambda_1} - \varepsilon_2 \delta_2\right)\|z(1, t)\|^2
- \left(\frac{\sigma \xi}{2} - \varepsilon_6 \frac{\xi}{2}\right) \int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} u^2_\tau(s) dx ds. \tag{6.8}\]
Since $k < \nu \lambda_1$, we can take $\delta_1$ such that $\nu_2 - \frac{k}{2} - \frac{3}{4} \delta_1 > 0$. We also fix $\delta_2 < \frac{\mu_0}{2}$. Then we select $\varepsilon_1$ sufficiently small such that

$$\varepsilon_1 < \sigma, \quad c_2 - \varepsilon_1 \left( \frac{3}{2} + \frac{\mu_0^2}{\delta_1 \lambda_1} \right) > 0, \quad \frac{1}{2} - \frac{\varepsilon_1}{\delta} \left( \frac{1}{2} + \frac{\mu_0}{\delta_1} \right) > 0 \quad \text{and} \quad c_3 - \varepsilon_1 \frac{\mu_0^2}{\delta_1 \lambda_1} > 0.$$

Finally, we choose $\varepsilon_2$ sufficiently small such that all the coefficients of right hand of (6.8) are nonnegative. Hence (6.8) leads to

$$L'(t) \leq -\varepsilon_1 E(t). \quad (6.9)$$

A combination of (6.7) and (6.9) leads to $E(t) \leq \frac{\alpha_0 E(0)}{\alpha_5} e^{-\frac{\alpha_1 t}{\alpha_5}}$, which implies that (2.10) with $K = \sqrt{\frac{c_5 \alpha_0 E(0)}{\alpha_5}}$ and $w = \frac{\delta_1}{2 \alpha_5}$. The proof is complete. \hfill \Box

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