LOCAL GEOMETRY OF RANDOM GEODESICS ON HYPERBOLIC SURFACES

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ABSTRACT. It is shown that the tessellation of a compact, hyperbolic surface induced by a typical long geodesic segment, when properly scaled, looks locally like a Poisson line process. This implies that the global statistics of the tessellation – for instance, the fraction of triangles – approach those of the limiting Poisson line process.

1. MAIN RESULTS: INTERSECTION STATISTICS OF RANDOM GEODESICS

1.1. Local Statistics. Any finite geodesic segment $\gamma$ on a closed hyperbolic surface $S$ partitions $S$ into a finite number of non-overlapping geodesic polygons of various shapes and sizes, whose vertices are the self-intersection points of $\gamma$. If a geodesic segment $\gamma$ of length $T$ is chosen by selecting its initial tangent vector $X$ at random, according to (normalized) Liouville measure on the unit tangent bundle $T^1S$, then with probability 1, as $T \to \infty$ the maximal diameter of a polygon in the induced partition will converge to 0, and hence the number of polygons in the partition will become large. The goal of this paper is to elucidate some of the statistical properties of this random polygonal partition for large $T$. Our main result will be a local geometric description of the partition: roughly, this will assert that in a neighborhood of any point $x \in S$ the partition will, in the large-$T$ limit, look as if it were induced by a Poisson line process \[25\], \[26\]. We will also show that this result has implications for the global statistics of the partition: for instance, it will imply that with probability $\approx 1$ the fraction of polygons in the partition that are triangles will stabilize near a non-random limiting value $\tau_3 > 0$.

Definition 1.1. A Poisson line process $\mathcal{L}$ of intensity $\lambda > 0$ is a random collection $\mathcal{L} = \{L_n\}_{n \in \mathbb{Z}}$ of lines in $\mathbb{R}^2$ constructed as follows. Let $\{(R_n, \Theta_n)\}_{n \in \mathbb{Z}}$ be the points of a Poisson point process of intensity $\lambda/\pi$ on the infinite strip $\mathbb{R} \times [0, \pi)$. For each $n \in \mathbb{Z}$ let $L_n$ be the

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1A long segment of a random geodesic ray doesn’t quite induce a tessellation, as there will be two faces [triangles, quadrilaterals, or whatever] that contain the two ends of the geodesic segment. We ignore these, however, since they will not influence statistics when the length of the geodesic segment is large.

2The ordering of the points doesn’t really matter, but for definiteness take $\cdots < R_{-1} < 0 < R_0 < R_1 < \cdots$. The assumption that $\{(R_n, \Theta_n)\}_{n \in \mathbb{Z}}$ is a Poisson point process of intensity $\lambda/\pi$ is equivalent to the assumption that $\{R_n\}_{n \in \mathbb{R}}$ is a Poisson point process of intensity $\lambda$ on $\mathbb{R}$ and that $\{\Theta_n\}_{n \in \mathbb{Z}}$ is an independent sequence of i.i.d. random variables with uniform distribution on $[0, \pi]$. 

1\textsuperscript{arXiv:1708.09830v2 [math.GT] 12 Mar 2019}
line
\begin{equation}
L_n := \{(x, y) \in \mathbb{R}^2 : R_n = x \cos \Theta_n + y \sin \Theta_n\}.
\end{equation}

Observe that the mapping \((1)\) of points \((r, \theta)\) to lines is a bijection from the strip \(\mathbb{R} \times [0, \pi)\) to the space of all lines in \(\mathbb{R}^2\). For any convex region \(\Omega \subset \mathbb{R}^2\), call the restriction to \(\Omega\) of a Poisson line process a Poisson line process in \(\Omega\). It is not difficult to show (see Lemma 2.4 below) that, with probability one, if \(\Omega\) is a bounded domain with piecewise smooth boundary then the Poisson line process in \(\Omega\) will consist of only finitely many line segments, and that at most two line segments will intersect at any point of \(\Omega\). For any realization of the process, the line segments will uniquely determine (and be determined by) their intersection points with \(\partial \Omega\), grouped in (unordered) pairs.

Fix a point \(x\) on the surface \(S\), and consider a small disc \(D(x, r)\) on \(S\) of radius \(r\) centered at \(x\). A geodesic ray started at a randomly chosen point of \(S\) in a random direction will (with probability one) eventually enter \(D(x, r)\) at a time roughly of order \(1/r\), by a simple application of Birkhoff’s theorem. Thus, if we wish to study the local intersection statistics of a random geodesic ray of (large) length \(T\) in a neighborhood of \(x\), we should focus on the intersections \(A_T\) of the geodesic with disks \(D(x, \alpha/T)\) of radii proportional to \(1/T\). Such an intersection will consist of a finite collection of geodesic arcs (possibly empty) that cross \(D(x, \alpha/T)\). For \(T\) sufficiently large, the exponential map, scaled by \(T^{-1}\), will map the ball \(B(0, \alpha)\) in the tangent space \(T_x S\) diffeomorphically onto \(D(x, \alpha/T)\), and so the set \(A_T\) will pull back to a finite collection of smooth curves in \(B(0, \alpha)\) each with endpoints on \(\partial B(0, \alpha)\); as \(T\) becomes large, these curves will approximate line segments in \(B(0, \alpha)\). Henceforth, we will identify these collections; thus, we will, when convenient, view a collection of geodesic segments in \(D(x, \alpha/T)\) as a collection of curves in \(B(0, \alpha)\). With this convention, we now formulate our main result as follows.

**Theorem 1.** Fix \(x \in S\) and \(\alpha > 0\), and let \(A_T\) be the intersection of a random geodesic of length \(T\) with the ball \(D(x; \alpha T^{-1})\) of radius \(\alpha T^{-1}\) centered at \(x\). As \(T \to \infty\), the random collection of geodesic arcs \(A_T\) converges in distribution to a Poisson line process in \(B(0; \alpha)\) of intensity \(\kappa_g\), where
\[
\kappa_g = \frac{1}{\text{area}(S)} = \frac{1}{2\pi(2g - 2)},
\]
and \(g\) is the genus of the surface \(S\).

Because the elements of the random processes here live in somewhat unusual spaces (finite unions of geodesic segments), we now elaborate on the meaning of convergence in distribution. In general, we say that a sequence of random elements of a complete metric space \(X\) converge in distribution if their distributions (the induced probability measures on \(X\)) converge weakly. The usual definition of weak convergence is this \([2]\): if \(\mu_n, \mu\) are Borel probability measures on a complete metric space \(X\), then \(\mu_n \to \mu\) weakly if for every bounded, continuous function \(f : X \to \mathbb{R}\),
\begin{equation}
\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu.
\end{equation}

\(^3\) except possibly one or two geodesic segments which have one endpoint on the boundary of the disk \(D(x, \alpha/T)\) and the other in the interior. This will occur with probability on the order \(1/T\), so this event can be ignored.
Here we take $X = \bigcup_{n=0}^{\infty} X_n$, where $X_n$ is the set of all collections $F = \{\{y_i, z_i\}\}_{i \leq n}$ of $n$ unordered pairs $y_i, z_i \in \partial B(0; \alpha)$, each of which determines a chord of $\partial B(0; \alpha)$. For any two unordered pairs $\{y, z\}, \{y', z'\}$, set
\[
d(\{y, z\}, \{y', z'\}) = \min(d(y, y') + d(z, z') + d(y, z') + d(z, y'));
\]
this defines a metric on $X_1$ that is equivalent to the Hausdorff metric on the corresponding space of chords. Now for any two elements $F, F' \in X$, define
\[
d(F, F') = \min_{\pi \in S_n} d(\{y_{\pi(i)}, z_{\pi(i)}\}) \quad \text{if } F, F' \in X_n,
\]
\[
= \infty \quad \text{otherwise},
\]
where $S_n$ is the set of permutations of $[n]$. Henceforth, we will refer to the space $X$ as configuration space (the dependence on the parameter $\alpha > 0$ will be suppressed).

The proof of Theorem 1 will also show that the limiting Poisson line processes in neighborhoods of distinct points of $S$ are independent.

Theorem 2. Fix two distinct points $x, x' \in S$ and $\alpha > 0$, and let $A_T$ and $A'_T$ be the intersections of a random geodesic of length $T$ with the balls $B(x; \alpha T^{-1})$ and $B(x'; \alpha T^{-1})$, respectively. Then as $T \to \infty$, the random sets $A_T$ and $A'_T$ converge jointly in distribution to a pair of independent Poisson line processes in $B(0; \alpha)$, both of intensity $\kappa_g$.

1.2. Global Statistics. Theorems 1-2 describe the “local” appearance of the tessellation of the surface $S$ induced by a long random geodesic. The tessellation $T_L$ will consist of geodesic polygons, typically of diameter $L^{-1}$, since the $O(L^2)$ self-intersections will subdivide the length $L$ geodesic segment into sub-segments of length $O(L^{-1})$. Thus, it is natural to look at the statistics of the scaled tessellation $L T_L$, which we shall view as consisting of a random number of triangles, quadrilaterals, etc., each with its own set of side-lengths and interior angles.

The empirical frequencies of triangles, quadrilaterals, etc. and the empirical distribution of side-length and interior-angle sets in a Poisson line process of intensity $\lambda$ on the ball $B(0; \alpha)$ of radius $\alpha$ converge as $\alpha \to \infty$. (These results are evidently due to R. E. Miles; proofs are given in section 2 below.) Theorem 1 asserts that when $L$ is large, then for any point $x \in S$ the statistics of the polygonal partition in $B(x; \alpha^{-1} L)$ induced by a random geodesic segment of length $L$ should approach those of a Poisson line process. From this observation we will deduce the following assertion regarding global statistics.

Theorem 3. Let $T_L$ be the tessellation of $S$ induced by a random geodesic of length $L$. Then with probability approaching 1 as $L \to \infty$, the empirical frequencies of triangles, quadrilaterals, etc. and the empirical distribution of side-length and interior-angle sets approach the corresponding theoretical frequencies for a Poisson line process.

Plan of the paper. The proofs of Theorems 1-2 will occupy most of the paper. The strategy will be to reduce the problem to a corresponding counting problem in symbolic dynamics. Preliminaries on Poisson line processes will be collected in section 2, and preliminaries on symbolic dynamics for the geodesic flow in section 3. Section 4 will be devoted to heuristics and a reformulation of the problem; the proofs of Theorems 1-2 will then be carried out in sections 5, 6, 7, and 8. Theorem 3 will be proved in section 9.
section 10, we give a short list of conjectures, questions, and possible extensions of our main results.

2. PRELIMINARIES: POISSON LINE PROCESSES

The Poisson line process and its generalizations have a voluminous literature, with notable early contributions by Miles [25], [26]. See [31] for an extended discussion and further pointers to the literature. In this section we will record some basic facts about these processes. These are mostly known – some of them are stated as theorems in [25] without proofs – but proofs are not easy to track down, so we shall provide proof sketches in Appendix A.

2.1. Statistics of a Poisson line process.

Lemma 2.1. A Poisson line process of constant intensity \( \lambda \) is both rotationally and translationally invariant, that is, if \( A \) is any isometry of \( \mathbb{R}^2 \) then the configuration \( \{ AL_n \}_{n \in \mathbb{Z}} \) has the same joint distribution as the configuration \( \{ L_n \}_{n \in \mathbb{Z}} \).

Remark 2.2. This result is stated without proof in [25]. A proof of the corresponding fact for the intensity measure can be found in [28], and another in [31], ch. 8. A short, elementary proof is given in Appendix A. The following corollary, which is stated without proof as Theorem 2 in [25], follows easily from isometry-invariance.

Corollary 2.3. Let \( L \) be a Poisson line process of intensity \( \lambda > 0 \). For any fixed line \( \ell \) in \( \mathbb{R}^2 \), the point process of intersections of \( \ell \) with lines in \( L \) is a Poisson point process of intensity \( 2\lambda/\pi \).

Lemma 2.4. Let \( L \) be a Poisson line process of intensity \( \lambda > 0 \), and for each point \( x \in \mathbb{R}^2 \) and each real \( r > 0 \) let \( N(B(x; r)) \) be the number of lines in \( L \) that intersect the ball \( B(x; r) \) of radius \( r \) centered at \( x \). Then the random variable \( N(B(x; r)) \) has the Poisson distribution with mean \( 2\lambda r \). Consequently, with probability one, for any compact set \( K \subseteq \mathbb{R}^2 \) the set of lines \( L_n \) in \( L \) that intersect \( K \) is finite.

Proof. Without loss of generality, take \( K = B(0; R) \) to be the closed ball of radius \( R \) centered at the origin. Then the line \( L_n \) intersects \( K \) if and only if \( |R_n| \leq R \). Since a Poisson point process on \( \mathbb{R} \) of constant intensity has at most finitely many points in any finite interval, the result follows.

The next result characterizes the Poisson line process (see also Proposition 2.10 below). Fix a bounded, convex region \( D \subset \mathbb{R}^2 \) with \( C^\infty \) boundary \( \Gamma = \partial D \), and let \( A, B \) be non-intersecting closed arcs on \( \Gamma \). For any line process \( L \), let

\[
N_{\{A,B\}} = \# \text{ lines that cross both } A \text{ and } B.
\]

For any angle \( \theta \in [-\pi/2, \pi/2] \), the set of lines that intersect both \( A \) and \( B \) and meet the \( x \)-axis at angle \( \theta + \pi/2 \) constitute an infinite strip that intersects the line \( \{ re^{i\theta} \}_{r \in \mathbb{R}} \) in an interval; see Figure 1 below. Let \( \psi(\theta) = \psi_{A,B}(\theta) \) be the length of this interval, and define

\[
\beta_{A,B} = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \psi(\theta) d\theta.
\]
Proposition 2.5. A line process $L$ in $D$ is a Poisson line process of rate $\lambda > 0$ if and only if

(i) for any two non-intersecting arcs $A, B \subset \Gamma$, the random variable $N_{\{A,B\}}$ has the Poisson distribution with mean $\lambda \beta_{A,B}$, and

(ii) for any finite collection $\{(A_i, B_i)\}_{i \leq m}$ of non-intersecting boundary arc pairs, the random variables $N_{\{A_i, B_i\}}$ are mutually independent.

See Appendix $\text{A}$ for the proof of the forward implication, along with that of the following corollary. The converse implication in Proposition 2.5 will follow from Proposition 2.10 in section 2.3 below.

Corollary 2.6. Let $D \subset \mathbb{R}^2$ be a compact, convex region, and let $L$ be a Poisson line process with intensity $\lambda$. The number $V(D)$ of intersection points (vertices) of $L$ in $D$ has expectation

$$EV(D) = \lambda^2 |D|/\pi$$

where $|D|$ is the Lebesgue measure of $D$.

2.2. Ergodic theorem for Poisson line processes. The configuration space $\mathcal{C}$ in which a Poisson line process takes values is the set of all countable, locally finite collections of lines in $\mathbb{R}^2$. This space has a natural metric topology, specifically, the weak topology generated by the Hausdorff topologies on the restrictions to balls in $\mathbb{R}^2$. Moreover, $\mathcal{C}$ admits an action (by translations) of $\mathbb{R}^2$. Denote by $\nu_\lambda$ the distribution of the Poisson line process with intensity $\lambda$. By Lemma 2.1, the measure $\nu_\lambda$ is translation-invariant.

Proposition 2.7. The probability measure $\nu_\lambda$ is mixing (and therefore ergodic) with respect to the translational action of $\mathbb{R}^2$ on $\mathcal{C}$.

Remark 2.8. Ergodicity of the measure $\nu_\lambda$ is asserted in Miles’ papers [25], [26], and proved in his unpublished Ph. D. dissertation. We have been unable to locate a proof in the published literature, so we have provided one in the Appendix.

Corollary 2.9. Let $\Phi_{n,k}$ be the fraction of $k$–gons, $F_n$ (for “faces”) the total number of polygons, and $V_n$ (for “vertices”) the number of intersection points in the tessellation of the square $[-n,n]^2$ induced by a Poisson line process $L$ of intensity $\lambda$. There exist constants $\phi_k > 0$ such that with probability 1,

$$\lim_{n \to \infty} F_n/(2n)^2 = \lambda^2 / \pi,$$

$$\lim_{n \to \infty} V_n/(2n)^2 = \lambda^2 / \pi,$$

$$\lim_{n \to \infty} \Phi_{n,k} = \phi_k.$$
Integral formulas for the quantities $\phi_k$ are given in [7].

The ergodic theorem can also be used to prove that a variety of other statistical properties stabilize in large squares. Consider, for example, the number $N_n(A, B, C)$ of triangles contained in $[-n, n]^2$ whose side lengths $\alpha, \beta, \gamma$ lie in the intervals $A, B, C$; then as $n \to \infty$,

$$N_n(A, B, C)/(2n)^2 \longrightarrow Ef(L)1_{G(A,B,C)}(L)$$

where $G(A, B, C)$ is the event that the polygon containing the origin is a triangle with side lengths in $A, B, C$.

2.3. Weak convergence to a Poisson line process. For any unordered pair $\{A, B\}$ of non-overlapping boundary arcs of the disk $B(0, \alpha)$, let $L_{\{A,B\}}$ be the set of lines in $\mathbb{R}^2$ that intersect both $A$ and $B$. This set can be identified with the set of point pairs $\{x, y\}$ where $x \in A$ and $y \in B$. This allows us to view any random collection of unordered point pairs $\{x, y\}$ as a line process in $B(0, \alpha)$, even when the collection consists of endpoints of arcs across $B(0, \alpha)$ that are not line segments (in particular, when they are pullbacks of geodesic arcs to the tangent space). For any line process $L$ in $B(0, \alpha)$ let $N_{\{A,B\}}$ be the cardinality of $L \cap L_{\{A,B\}}$ (cf. equation (3)).

**Proposition 2.10.** Let $L_n$ be a sequence of line processes in $B(0; \alpha)$, and let $\mu_n$ be the distribution of $L_n$ (i.e., the probability measure on $\mathcal{X}$ induced by $L_n$). In order that $\mu_n \to \mu$ weakly, where $\mu$ is the law of a rate-$\lambda$ Poisson line process, it suffices that the following condition holds. For any finite collection $\{\{A_i, B_i\}\}_{i \leq m}$ of unordered pairs of non-overlapping boundary arcs of $B(0; \alpha)$ such that the sets $L_{\{A_i,B_i\}}$ are pairwise disjoint, the joint distribution of the counts $N_{\{A_i,B_i\}}$ under $\mu_n$ converges to the joint distribution under $\mu$, that is, for any choice of nonnegative integers $k_i$,

$$\lim_{n \to \infty} \mu_n\{N_{\{A_i,B_i\}} = k_i \forall i \leq m\} = \prod_{i=1}^m \frac{(\lambda_{A_i,B_i})^{k_i}}{k_i!} e^{-\lambda_{A_i,B_i}}.$$

**Proof Sketch.** Recall that the configuration space $\mathcal{X}$ is the disjoint union of the sets $\mathcal{X}_k$, where $\mathcal{X}_k$ is the set of all finite sets $F = \{\{x_i, y_i\}\}_{1 \leq i \leq k}$ consisting of $k$ unordered pairs of points on $\partial B(0, \alpha)$. Since each set $\mathcal{X}_k$ is both open and closed in $\mathcal{X}$, to prove weak convergence $\mu_n \to \mu$ it suffices to establish the convergence (2) for every continuous function $f$ supported by just one of the sets $\mathcal{X}_k$.

For each $k$, the space $\mathcal{X}_k$ is a quotient of $(\partial B(0, \alpha)^2)^k$ with the usual topology, and so every continuous function $f : \mathcal{X}_k \to \mathbb{R}$ can be uniformly approximated by “step functions”, that is, functions $g$ of configurations $F = \{\{x_i, y_i\}\}_{1 \leq i \leq k}$ that depend only on the counts $N_{A_i,B_i}$ for arcs $A_i, B_i$ in some partition of $\partial B(0, \alpha)$. If (8) holds, then it follows by linearity of expectations that for any such step function $g$,

$$\lim_{n \to \infty} \int g \, d\mu_n = \int g \, d\mu,$$

and hence (2) follows. \qed

2.4. The “law of small numbers”. A elementary theorem of discrete probability theory states that for large $n$, the Binomial–$(n, \lambda/n)$ distribution is closely approximated by the Poisson distribution with mean $\lambda$. Following is a generalization that we will find useful.
Proposition 2.11. Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with success parameters $EX_i = p_i$. Let $\alpha = \max_i p_i$ and $\beta = \sum_i p_i$. Then there is a constant $C < \infty$ not depending on $p_1, p_2, \ldots, p_n$ such that
\[
\sum_{k=0}^{\infty} |P\left(\sum_i X_i = k\right) - \frac{\beta^k}{k!} e^{-\lambda}| \leq C\alpha.
\]
See [22] for a proof. The important feature of the proposition for us is not the explicit bound, but the fact that the closeness of the approximation depends only on $\max p_i$.

A similar result holds for multinomial variables.

Proposition 2.12. Let $X_1, X_2, \ldots, X_n$ be independent random variables each taking values in the finite set $\{0, 1, 2, \ldots, K\} = \{0\} \cup [K]$, and for each pair $i, j$ set $p_{i,j} = P\{X_i = j\}$. Let $\alpha = \max_{j \geq 1} \max_i p_{i,j}$ and $\beta_j = \sum_i p_{i,j}$, and for each $j$ define
\[
T_j = \sum_{i=1}^{n} 1\{X_i = j\}.
\]
Then there is a function $C_K(\alpha)$ satisfying $\lim_{\alpha \downarrow 0} C(\alpha) = 0$ such that
\[
\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_K=0}^{\infty} |P\{T_j = m_j \forall j \in [K]\} - \prod_{j=1}^{K} \beta_j^{m_j} e^{-\beta_j} / m_j!| \leq C(\alpha).
\]

3. Preliminaries: Symbolic Dynamics

3.1. Shifts and suspension flows. The geodesic flow on the unit tangent bundle $T^1 S$ of a closed hyperbolic surface $S$ has a concrete representation as a suspension flow over a shift of finite type. In describing this representation, we shall follow (for the most part) the terminology and notation of [1], [29], and [21]. Let $A$ be a finite alphabet and $F$ a finite set of finite words on the alphabet $A$, and define $\Sigma = \Sigma_F$ to be the set of doubly infinite sequences $\omega = (\omega_n)_{n \in \mathbb{Z}}$ such that no element of $F$ occurs as a subword of $\omega$. The sequence space $\Sigma$ is given the metric $d(\omega, \sigma) = \exp\{-n(\omega, \sigma)\}$ where $n(\omega, \sigma)$ is the minimum nonnegative integer $n$ such that $\omega_j \neq \sigma_j$ for $j = \pm n$. The forward shift $\sigma : \Sigma \to \Sigma$ is known as a (two-sided) shift of finite type\footnote{Bowen [3] requires that the elements of the set $F$ all be of length 2. However, any shift of finite type can be “recoded” to give a shift of finite type obeying Bowen’s convention, by replacing the original alphabet $A$ by $A^m$, where $m$ is the length of the longest word in $F$, and then replacing each sequence $\omega$ by the sequence $\bar{\omega}$ whose entries are the successive length-$m$ subwords of $\omega$. In Series’ [29] symbolic dynamics for the geodesic flow, the alphabet $A$ is the set of natural generators for the fundamental group $\pi_1(S)$ of the surface $S$, and the forbidden subwords $F$ are gotten from the relators of $\pi_1(S)$.}.

Associated with any two-sided shift of finite type are two one-sided shifts $\sigma : \Sigma^+ \to \Sigma^+$ and $\sigma : \Sigma^- \to \Sigma^-$. The spaces $\Sigma^\pm$ consist of all one-sided sequences $\omega^+, \omega^-$ that can be obtained from two-sided sequences $\omega \in \Sigma$ by the rule
\[
\omega^+ = \omega_0\omega_1\omega_2 \cdots \quad \text{and} \quad \omega^- = \omega_{-1}\omega_{-2}\omega_{-3} \cdots.
\]
In general, the spaces $\Sigma^+$ and $\Sigma^-$ need not be the same (because in $\Sigma^-$ the restrictions on allowable transitions are determined not by $F$, but by the set $F^R$ gotten by reversing all
words in $F$, and, in particular, for Series’ [29] symbolic dynamics they will be different. However, both $(\Sigma^+, \sigma)$ and $(\Sigma^-, \sigma)$ are of finite type. For any integer $m \geq 0$ and $\omega \in \Sigma$, denote by $\Sigma^+_m(\omega)$ and $\Sigma^-_m(\omega)$ the cylinder sets in $\Sigma^\pm$ consisting of those sequences that match $\omega^+$ and $\omega^-$, respectively, in the first $m$ coordinates.

For any continuous function $F : \Sigma \to (0, \infty)$ on $\Sigma$, define the suspension space $\Sigma_F$ by

$$\Sigma_F := \{ (\omega, t) : \omega \in \Sigma \text{ and } 0 \leq t \leq F(\omega) \},$$

with points $(\omega, F(\omega))$ and $(\sigma \omega, 0)$ identified. The suspension flow with height function $F$ is the flow $\phi_t$ on $\Sigma_F$ whose orbits proceed up vertical fibers $F_\omega := \{ (\omega, s) : 0 \leq s \leq F(\omega) \}$ at speed 1, and upon reaching the ceiling at $(\omega, F(\omega))$ jumps instantaneously to $(\sigma \omega, 0)$. Clearly, an orbit of the suspension flow that goes through a point $(\omega, 0)$ is periodic if and only if $\omega$ is a periodic sequence.

There is a bijective correspondence between invariant probability measures $\mu^*$ for the flow $\phi_t$ and shift-invariant measures $\mu$ on $\Sigma$. This correspondence can be specified as follows: for any continuous function $g : \Sigma_F \to \mathbb{R}$,

$$\int g \, d\mu^* = \int \int_0^{F(\omega)} g(\omega, s) \, ds \, d\mu(\omega) / \int_\Sigma F \, d\mu.$$

If $\mu$ is ergodic for the shift $(\Sigma, \sigma)$ then $\mu^*$ is ergodic for the flow $(\Sigma_F, \phi_t)$; and if $\mu$ is mixing for the shift then $\mu^*$ is mixing for the flow provided that the height function $F$ is not cohomologous to a function $F'$ that takes values in $a + b\mathbb{Z}$ for some $b > 0$ and $a \in \mathbb{R}$. By Birkhoff’s theorem, for any ergodic $\mu$

$$\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} F \circ \sigma^j = \int_\Sigma d\mu \text{ almost surely};$$

thus, under $\mu^*$, almost every orbit makes roughly $T / \int F \, d\mu$ visits to the base $\Sigma \times \{0\}$ by time $T$, when $T$ is large.

3.2. Symbolic dynamics for the geodesic flow. Any closed, hyperbolic surface $S$ has as its universal covering space the Poincaré disk $\mathbb{D}$, and so $S$ can be represented as $\mathbb{D} / \pi_1(S)$, where $\pi_1(S)$ is the fundamental group of $S$. For any such $S$ there is a compact fundamental polygon $P$ whose boundary consists of $4g$ paired geodesic segments which, when glued in pairs, turn $P$ into $S$. We will henceforth identify $S$ with $\mathcal{P}$, and geodesics in $S$ with their lifts to geodesics in $\mathbb{D}$; however, we will assume that the base point $(t = 0)$ of such a geodesic is located in $\mathcal{P}$.

**Proposition 3.1.** (Series [29]) For any closed, hyperbolic surface $S$, there exist a topologically mixing shift $(\Sigma, \sigma)$ of finite type, a suspension flow $(\Sigma_F, \phi_t)$ over the shift, and surjective, Hölder-continuous mappings $\xi^\pm : \Sigma^\pm \to \partial \mathbb{D}$, and $\pi : \Sigma_F \to T^1 S$ such that $\pi$ is a semi-conjugacy with the geodesic flow $\gamma_t$ on $T^1 S$, i.e.,

$$\pi \circ \phi_t = \gamma_t \circ \pi \quad \text{for all } t \in \mathbb{R},$$

and such that the following additional properties hold.

(A) $\pi(\Sigma \times \{0\})$ is the set of inward-pointing tangent vectors based at points $p \in \partial \mathcal{P}$.

(B) The endpoints on $\partial \mathbb{D}$ of the (lifted) geodesic $\pi(\phi_t(\omega, 0))_{t \in \mathbb{R}}$ are $\xi^\pm(\omega^\pm)$. 


(C) \( F(\omega) \) is the time taken by this geodesic line to cross the fundamental polygon \( P \).

Furthermore, the maps \( \xi_{\pm} \) project cylinder sets \( \Sigma_{\pm}^{m}(\omega) \) onto closed arcs \( J_{m}^{\pm}(\omega_{\pm}) \) in such a way that for appropriate constants \( C < \infty \) and \( 0 < \beta_{1} < \beta_{2} < 1 \) independent of \( m \) and \( \omega \),

1. \( \text{the lengths of } J_{m}^{\pm}(\omega_{\pm}) \text{ are between } C\beta_{1}^{m} \text{ and } C\beta_{2}^{2m} \), and
2. \( \text{distinct arcs } J_{m}^{\pm}(\omega_{\pm}) \text{ and } J_{m}^{\pm}(y_{\pm}) \text{ of the same generation } m \text{ have disjoint interiors (and similarly when } + \text{ is replaced by } -) \).

Consequently, the semi-conjugacy \( \pi \) fails to be one-to-one only for geodesics whose lifts to \( \mathbb{D} \) have at least one endpoint that is an endpoint of some arc \( J_{m}^{\pm}(\omega) \).

See [29], especially Th. 3.1, and also [5]. The last point implies that the set of geodesics where the semi-conjugacy fails to be bijective is of first category, and has Liouville measure zero.

3.3. **Regenerative representation of Gibbs states.** The semi-conjugacy \((10)\) ensures that the (normalized) Liouville measure pulls back to an invariant measure \( \lambda^{*} \) for the flow. Because the mappings \( \xi_{\pm} \) in Proposition 3.1 are Hölder continuous, the height function \( F \) pulls back to a Hölder continuous function on \( \Sigma \) (which at the risk of ambiguity we shall also denote by \( F : \Sigma \to (0, \infty) \)), and since the geodesic flow is topologically mixing, this function \( F \) is non-lattice in the sense of [20] (that is, it is not cohomologous to any function \( G : \Sigma \to \mathbb{R} \) whose image is a coset of a proper closed subgroup of \( \mathbb{R} \)). See [20], Th. 8 and Cor. 11.1 for details.

The geodesic flow on a compact surface of constant negative curvature has the property the maximum-entropy invariant probability measure coincides with the normalized Liouville measure. Consequently, the pullback \( \lambda^{*} \) of the Liouville measure is the maximum-entropy invariant measure for the suspension flow on the suspension space \( \Sigma_{F} \). But it is generally true that, for any suspension flow with Hölder continuous height function \( H \) over a shift of finite type and maximum-entropy invariant measure \( \mu^{*} \), the corresponding shift-invariant measure \( \mu \) on sequence space (cf. equation (9)) is a Gibbs state in the sense of [4], ch. 1, and furthermore the potential function for this Gibbs state is \( -\delta H \), where \( \delta \) is the topological entropy of the flow [20]. The upshot is that the shift-invariant probability measure \( \lambda \) on \( \Sigma \) corresponding to the Liouville measure \( \lambda^{*} \) is a Gibbs state with a Hölder continuous potential function \( -F \).

Gibbs states with Hölder continuous potentials enjoy strong exponential mixing properties (e.g., the “exponential cluster property” 1.26 in [4], ch. 1). We shall make use of an even stronger property, the regenerative representation of a Gibbs state established in [17] (cf. also [9]). This representation is most usefully described in terms of the stationary process governed by the Gibbs state. Let \( \mu \) be a Gibbs state with Hölder continuous potential function \( f : \Sigma \to \mathbb{R} \), where \( \sigma : \Sigma \to \Sigma \) is a topologically mixing shift of finite type, and let \( X_{n} : \Sigma \to \mathcal{A} \) be the coordinate projections on \( \Sigma \), for \( n \in \mathbb{Z} \). The sequence \((X_{n})_{n \in \mathbb{Z}}\), viewed as a stochastic process on the probability space \((\Sigma, \mu)\), is a stationary process that we will henceforth call a Gibbs process.

---

5The topological entropy of the geodesic flow on a compact surface with constant negative curvature is 1.

6In some of the older probability literature, Gibbs processes are called chains with complete connections.
The regenerative representation relates the class of Gibbs processes to another class of stationary processes, called list processes (the term used by \cite{17}). A list process is a stationary, positive-recurrent Markov chain $(Z_n)_{n \in \mathbb{Z}}$ with state space $\cup_{k \geq 1} A^k$ and stationary distribution $\nu$ that obeys the following transition rules: first,

$$P(Z_{n+1} = (\omega_1, \omega_2, \ldots, \omega_m) | Z_n = (\omega'_1, \omega'_2, \ldots, \omega'_k)) = 0$$

unless either $m = 1$ or $m = k + 1$ and $\omega_i = \omega'_i$ for each $1 \leq i \leq k$; and second, for every letter $\omega_1$ and every word $\omega'_1 \omega'_2 \cdots \omega'_m$,

$$P(Z_{n+1} = \omega_1 | Z_{n+1} \in A^1 \text{ and } Z_n = (\omega'_1, \omega'_2, \ldots, \omega'_m)) = \nu((\omega_1))/\nu(A^1).$$

Thus, the process $(Z_n)_{n \in \mathbb{Z}}$ evolves by either adding one letter to the end of the list or erasing the entire list and beginning from scratch. Furthermore, by (12), at any time when the list is erased, the new 1-letter word chosen to begin the next list is independent of the past history of the entire process.

For any list process define the regeneration times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ by

$$\tau_1 = \min\{n \geq 1 : Z_n \in A^1\};$$

$$\tau_{m+1} = \min\{n \geq 1 + \tau_m : Z_n \in A^1\}.$$ 

By condition (12), the random variables $\tau_{m+1} - \tau_m$ are independent, and for $m \geq 1$ are identically distributed, as are the excursions

$$(Z_{\tau_{m+1}}, Z_{\tau_{m+2}}, \ldots, Z_{\tau_{m+1}}).$$

Denote by $\pi : \cup_{k \geq 1} A^k \to A$ the projection onto the last letter.

**Proposition 3.2.** If $(X_n)_{n \in \mathbb{Z}}$ is a Gibbs process then there is a list process $(Z_n)_{n \in \mathbb{Z}}$ such that the projected process $(\pi(Z_n))_{n \in \mathbb{Z}}$ has the same joint distribution as the Gibbs process $(X_n)_{n \in \mathbb{Z}}$. Thus, the random sequence obtained by concatenating the successive excursions $W_m := Z_{\tau_m}$, i.e.,

$$W_1 \cdot W_2 \cdot W_3 \cdots,$$

has the same distribution as the sequence $(X_n)_{n \geq 0}$. Moreover, the list process can be chosen in such a way that the excursion lengths $\tau_{m+1} - \tau_m$ satisfy

$$P(\tau_{m+1} - \tau_m \geq n) \leq C\alpha^n$$

for some $0 < \alpha < 1$ and $C < \infty$ not depending on either $m$ or $n$.

See \cite{17}, Th. 1, or \cite{9}, Th. 4.1. (The former article uses the (older) term chain with complete connections for a Gibbs process, and a different (but equivalent) definition than that given in \cite{4}. The hypothesis of Hölder continuity of the potential function $f$ is equivalent to the hypothesis in Th. 1 of \cite{17} that the sequence $\gamma_m$ decay exponentially.)

## 4. Heuristics and Proof Strategy

### 4.1. Liouville measure, shrinking targets, and Poisson heuristics

Theorem \cite{1} concerns the local intersection statistics of a long segment (length $T$) of a random geodesic with the ball $D(x, \alpha T^{-1})$ of radius $\alpha T^{-1}$ centered at a fixed point $x \in S$. This intersection – call it $A_T$ – will consist of a finite number (possibly 0) of geodesic segments, each of which crosses the ball and intersects its boundary in two points $y, y'$. These points $y, y'$ are identified, via the
exponential mapping scaled by $T^{-1}$, with points $z, z'$ on the boundary of the ball $B(0, \alpha T^{-1})$ of radius $\alpha$ in the tangent space at $x$, and so $A_T$ can be identified with a random element of the configuration space $\mathcal{X}$. Let

$$\mu_T^* = \text{distribution of } A_T.$$ 

Our objective is to show that as $T \to \infty$ the probability measures $\mu_T^*$ converge weakly to the law $\nu$ of a Poisson line process. We begin by explaining why $T$ is the right time-scale on which to look for crossings of the ball $D(x; \alpha T^{-1})$.

For large $T$, the hyperbolic area of $D(x; \alpha T^{-1})$ is well-approximated by its Euclidean area $\pi \alpha^2 T^{-2}$, and so the (normalized) Liouville measure of the set of all tangent vectors based at points in $D(x; \alpha T^{-1})$ is approximately

$$\frac{\pi \alpha^2}{T^2 |S|},$$

where $|S|$ is the hyperbolic area of $S$. Any geodesic segment that crosses $D(x; \alpha T^{-1})$ will have length on the order of the diameter $2 \alpha T^{-1}$. Thus, Birkhoff’s ergodic theorem suggests that the time needed by a random geodesic ray to first reach $D(x; \alpha T^{-1})$ should be of order $T$. Moreover, the same calculation shows that if a geodesic segment of length 1 were thrown randomly onto $S$ then the probability that it would intersect $D(x; \alpha T^{-1})$ would be of order $T^{-1}$.

Imagine now that this random geodesic ray were broken into adjacent geodesic segments of length 1. If these segments were independent random segments distributed uniformly on the surface $S$ then the number of these segments intersecting the ball $B(0; \alpha)$ would be distributed as the number of successes in a repeated success-failure experiment, and hence would follow the binomial distribution with parameters $n = T$ (assuming for now that the length $T$ is an integer) and $p = \theta T^{-1}$, for a suitable constant $\theta > 0$. As $T \to \infty$, this binomial distribution approaches the Poisson distribution with mean $\theta$.

Of course it is not true that the successive length $-1$ segments are independent. Nevertheless, across long time intervals they are approximately independent (in a sense we will not try to make precise), as the geodesic flow is mixing. If we could show that it is highly unlikely for a geodesic to return to the ball $B(x; \alpha T^{-1})$ a short time after visiting, given that it does in fact reach the ball before time $T$, then the approximate independence across long time intervals should imply that the total number of crossings by time $T$ is approximately distributed as a Poisson random variable.

4.2. **Proof Strategy.** Unfortunately, it is difficult to make this heuristic argument rigorous, because mixing of the geodesic flow (or even the sharper quantitative estimates of the mixing rate in [27] and [11]) is not enough to justify treating the success-failure indicators associated with the $T$ length $-1$ segments as approximately independent Bernoulli random variables. The difficulty is that as $T$ increases, the target ball $B(x; \alpha T^{-1})$ — and hence the probability of hitting it by a length $-1$ geodesic segment — shrinks, and so there are different mixing problems at every scale $T$.

---

7The identification is via the geodesic coordinate system based at $x$. Geodesic coordinates map balls $B(0; \alpha)$ in the tangent space to topological balls in $S$; as $T \to \infty$ these topological balls look more and more like metric balls. We will ignore the distinction, and act as if the images are metric balls in $S$. 

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The problem of understanding whether a general measure-preserving flow $\phi_t$ on a probability space $X$ hits a sequence of shrinking targets $A_t$ ($\mu(A_t) \to 0$) can be viewed as a generalization of the classical Borel-Cantelli lemma to the continuous time, dependent setting. Much of the dynamical systems literature has focused on discrete time systems (see [8] for example, for results on discrete-time hyperbolic systems with specified mixing behaviors).

Starting with Sullivan [32] there has been extensive interest in flows on homogeneous spaces, in particular for flows with hyperbolic behavior. In [32], the state space $X$ is the unit-tangent bundle of a non-compact finite volume hyperbolic manifold, and the shrinking targets are cusp neighborhoods, and the flow is the geodesic flow.

Subsequently, Kleinbock-Margulis [16] generalized these results to cusp excursions for diagonal flows on finite-volume homogeneous spaces, using representation theory to derive exponential mixing results for these flows, and a subtle argument to approximate the indicator functions of the cusp neighborhoods by appropriately smooth functions.

This approach breaks down for shrinking sets in the compact part of the space, in particular for shrinking balls. Dolgopyat [12] has obtained results for general hyperbolic systems, and Maucourant [24] for geodesic flows on hyperbolic manifolds, but it does not seem to be possible to use these directly to implement our strategies.

Because mixing problems are generally easier to handle in discrete-time systems than in continuous time, we shall use the symbolic dynamics outlined in section 3 to translate the weak convergence problem to one involving the Gibbs state $\lambda$ corresponding to the pullback $\lambda^*$ of Liouville measure to the suspension space $\Sigma_F$. Recall that the number of base crossings by a $\lambda^*$-generic orbit by time $T$ is, by Birkhoff’s ergodic theorem, roughly $T/E_\lambda F$. Given that the expected number of visits to the ball $B(x; \alpha T^{-1})$ by time $T$ is of order 1, it follows that the expected number of visits in a time interval of length $\varepsilon T$ can be made almost negligible by taking $\varepsilon$ small. Therefore, instead of choosing an initial tangent vector at random and then following the geodesic with that initial tangent for time $T$, we may choose an element $\omega \in \Sigma$ at random according to $\lambda$ and then follow the orbit of the suspension flow starting at $(\omega, 0)$ through

$$n := n(T) = \left\lfloor \frac{T}{E_\lambda F} \right\rfloor$$

base crossings. The pushforward of this segment of the suspension flow consists of $n$ geodesic arcs crossing the fundamental polygon $P$, by Proposition [3.1]. Denote by $I_n(x, \omega)$ the intersection of these with the disk $D(x; \alpha T^{-1})$; this intersection consists of finitely many geodesic segments that cross $D(x; \alpha T^{-1})$. Let $L_n(x, \omega)$ be the pullback of $I_n(x, \omega)$ to the tangent space $T_x S$ by the exponential mapping, scaled by the usual factor $T$. We shall view the random collection $L_n = L_n(x, \cdot)$ of (undirected) arcs (for $\omega$ chosen randomly according to $\lambda$) as a line process (cf. the discussion preceding Proposition 2.10). Our objective is to

---

8If $x$ lies on one of the bounding sides of the fundamental polygon then it is possible that one or two of the geodesic segments in $I_n(x; \omega)$ will not completely cross $D(x; \alpha T^{-1})$. However, if $\omega$ is chosen at random according to $\lambda$ then the chance that $I_n(x; \omega)$ contains an incomplete crossing tends to 0 as $n \to \infty$, because the Liouville measure of the set of tangent vectors whose associated geodesic rays enter or exit $D(x; \alpha T^{-1})$ before the first crossing of $P$ is vanishingly small.
prove that, for any fixed \( x \in \mathcal{S} \), the sequence of line processes \( L_n \) converges in law to a Poisson line process on \( B(0, \alpha) \). For this we will use the criterion of Proposition 2.10.

For any pair \( A, B \) of non-overlapping boundary arcs of \( \partial B(0, \alpha) \), define \( L_{A,B} \) to be the set of oriented line segments from \( A \) to \( B \), and let \( N_{A,B}(\omega) \) be the number of oriented geodesic segments in the collection \( I_n(x; \omega) \) that cross the target disk \( D(x; \alpha T^{-1}) \) from arc \( A \) to arc \( B \). The counts \( N_{A,B} \) depend on \( n = [T/E\lambda F] \), but to reduce notational clutter we shall suppress this dependence. Observe that the number of undirected crossings \( N_{\{A,B\}} \) (cf. equation (3)) is given by

\[
N_{\{A,B\}} = N_{A,B} + N_{B,A},
\]

and consequently \( EN_{\{A,B\}} = EN_{A,B} + EN_{B,A} \). Since the sum of independent Poisson random variables is Poisson, to prove that in the \( n \to \infty \) limit the random variable \( N_{\{A,B\}} \) becomes Poisson, it suffices to show that the directed crossing counts \( N_{A,B} \) become Poisson. Thus, our objective now is to prove the following assertion, which, by Proposition 2.10, will imply Theorem 1.

**Proposition 4.1.** For any finite collection \( \{(A_i, B_i)\}_{i \leq r} \) of pairs of non-overlapping closed boundary arcs of \( B(0, \alpha) \) such that the sets \( L_{A_i, B_i} \) are pairwise disjoint, and for any choice of nonnegative integers \( k_i \),

\[
\lim_{n \to \infty} \lambda \{ \omega : N_{A_i, B_i}(\omega) = k_i \ \forall \ i \} = \prod_{i=1}^{r} \left( \frac{\kappa_g \beta_{A_i, B_i}}{k_i!} \right)^{k_i} e^{-\kappa_g \beta_{A_i, B_i}/2}
\]

where \( \beta_{A,B} \) is defined by equation (4) and \( \kappa_g = (2\pi(2g - 2))^{-1} \).

Note that for fixed \( A, B \) the constants \( \beta_{A,B} \) vary linearly with \( \alpha \), because the function \( \psi = \psi_{A,B} \) in (4) is proportional to \( \alpha \). See Figure 1.

The proof of Proposition 4.1 will be accomplished in four stages, as follows.

First, we will show (in section 5) that the set \( \Sigma(A, B; T) \) of all pairs \( (\omega^-, \omega^+) \in \Sigma^- \times \Sigma^+ \) such that the hyperbolic geodesic with ideal endpoints \( \pi(\omega^-), \pi(\omega^+) \) crosses the disk \( D(0; \alpha T^{-1}) \) through the boundary arcs \( A, B \) on \( \partial D(0; \alpha T^{-1}) \) (in this order) has \( \lambda \)-measure satisfying

\[
\lim_{T \to \infty} T \lambda(\Sigma(A, B; T)) = \frac{1}{2} \kappa_g \alpha \beta_{A,B} E\lambda F.
\]

Second, in section 6, we will show that the set \( \Sigma(A, B; T) \) can be represented approximately as a finite union of cylinder sets \( \Sigma^- \times \Sigma^+ \). This will be done in such a way that the lengths of the words defining the cylinder sets satisfy \( m = (\log n)^2 = C' (\log T)^2 \). It will then follow that \( G_n(A, B, k) \) is (approximately) the set of all sequences \( \omega \in \Sigma \) whose first \( n \) letters contain exactly \( k \) occurrences of one of the length-2m sub-words

\[
y_m y_{m-1}^- \cdots y_1^- y_1^+ y_2^+ \cdots y_m^+
\]

obtained by concatenating the words \( \omega^-, \omega^+ \) that define the cylinder sets \( \Sigma^- \times \Sigma^+ \).

---

\( ^{9} \) Recall that \( B(0, \alpha) \) is identified with the disk \( D(x, \alpha T^{-1}) \) by \( v \mapsto \exp(Tv) \), and that boundary arcs of \( B(0; \alpha) \) are identified with corresponding boundary arcs of \( D(x, \alpha T^{-1}) \).
Next, we will prove (in section 7) that, with \( \lambda \)-probability converging to 1 as \( T \to \infty \), no block of \((\log T)^3\) consecutive letters in \( \omega_1 \omega_2 \cdots \omega_n \) contains more than one occurrence of one of the “magic subwords” \((17)\). This will be accomplished by showing that the conditional Liouville measure of the set of tangent vectors to geodesic rays that start in \( D(x; \alpha T^{-1}) \) and return to \( D(x; \alpha T^{-1}) \) within time \((\log T)^3\) is vanishingly small as \( T \to \infty \).

Finally, in section 8 we will use the results of steps 1, 2, and 3 to show that the number \( N_{A,B} \) of crossings through arcs \( A, B \) on \( \partial D(x; \alpha T^{-1}) \) equals (with high probability) the number of length-\((\log T)^2\) blocks that contain one of the magic subwords, and that these occurrence events are independent small-probability events. Furthermore, we will show that for distinct pairs \((A_i, B_i)\) of boundary arcs the counts \( N_{A_i, B_i} \) are (approximately) independent, and so \((15)\) will follow.

The strategy just outlined is easily adapted to Theorem 2. Fix distinct points \( x, x' \in S \). For any pair \( A, B \) of non-overlapping boundary arcs of \( \partial B(0, \alpha) \), denote by \( N_{A,B}(\omega) \) and \( N'_{A,B}(\omega) \) the numbers of geodesic arcs in the collections \( I_n(x) \) and \( I_n(x') \), respectively, that cross the target disks \( D(x; \alpha T^{-1}) \) and \( D(x'; \alpha T^{-1}) \) from arc \( A \) to arc \( B \). To prove Theorem 2 it suffices to prove the following.

**Proposition 4.2.** For any finite collections \( \{(A_i, B_i)\}_{1 \leq i \leq r} \) and \( \{(A'_i, B'_i)\}_{1 \leq i \leq r'} \) and any choice of nonnegative integers \( k_i, k'_i \),

\[
\lim_{n \to \infty} \lambda \left\{ \omega : N_{A_i, B_i}(\omega) = k_i \text{ and } N'_{A_i', B_i'}(\omega) = k'_i \right\} = \left( \prod_{i=1}^r \frac{1}{k_i!} \left( \kappa_\theta \beta_{A_i, B_i}/2 \right)^{k_i} e^{-\kappa_\theta \beta_{A_i, B_i}/2} \right) \left( \prod_{i=1}^{r'} \frac{\kappa_\theta \beta_{A'_i, B'_i}/2}{k'_i!} e^{-\kappa_\theta \beta_{A'_i, B'_i}/2} \right).
\]

5. **Measure of the Crossing Sets**

Fix \( x \in \mathcal{P} \) and \( \alpha > 0 \). To facilitate geometric arguments, we shall assume henceforth (without loss of generality) that the Poincaré disk \( \mathbb{D} \) has been parametrized in such a way that the point \( x \in \mathcal{P} (= S) \) is situated at the center \( x = 0 \) of the Poincaré disk \( \mathbb{D} \); this makes geodesics of \( \mathbb{D} \) through \( x \) straight Euclidean line segments. Let \( A, B \) be any two disjoint closed arcs, each with nonempty interior, on the boundary of the ball \( B(0, \alpha) \) in the unit tangent space \( T^1S_x \). Recall that we have agreed to identify the disk \( D(x, \alpha T^{-1}) \) in \( S (= \mathcal{P}) \) with the ball \( B(0, \alpha) \) via the exponential mapping (scaled by the factor \( T^{-1} \)), so the arcs \( A, B \) are identified with arcs in \( \partial D(x, \alpha T^{-1}) \), also denoted by \( A \) and \( B \), whose arc-lengths are approximately proportional to \( \alpha T^{-1} \). Let \( \gamma \) be a (directed) hyperbolic geodesic that intersects the arcs \( A \) and \( B \), in this order; then the ideal endpoints \( \xi^-, \xi^+ \) of \( \gamma \) are distinct points on \( \partial \mathbb{D} \), and (except with at most countably many exceptions) correspond uniquely to pairs of sequences \( \omega^\pm \) in \( \Sigma^\pm \) via the projection \( \pi \) (cf. Proposition 3.1). Define \( \Sigma(A, B; T) \subset \Sigma \) to be the set of all such pairs \( \omega^\pm \), viewed as doubly infinite sequences \( \omega^- \omega^+ \).

**Proposition 5.1.** For all \( A, B, \) and \( \alpha > 0 \),

\[
\lim_{T \to \infty} T \lambda(\Sigma(A, B; T)) = \frac{1}{2} \kappa_\theta \beta_{A,B} E_\lambda F
\]

where \( \beta_{A,B} \) is as defined by equation (4).
Proof. For ease of exposition we shall assume that \( x \) is located in the interior of \( \mathcal{P} \), so a geodesic crossing of \( \mathcal{P} \) that enters \( D(x; \alpha T^{-1}) \) will have well-defined entry and exit points on \( \partial D(x; \alpha T^{-1}) \). The case where \( x \in \partial \mathcal{P} \) can be handled in much the same way, but accommodations must be made for entries and exits of \( D(x; \alpha T^{-1}) \) on successive crossings of \( \mathcal{P} \).

Recall that the measure \( \lambda \) is related to the pullback \( \lambda^* \) of the normalized Liouville measure \( L \) by equation (9), where \( F \) is the height function for the suspension flow. By Proposition 3.1(A), each base crossing of the suspension flow corresponds to one crossing of the fundamental polygon \( \mathcal{P} \) by the geodesic flow. Thus, \( \Sigma(A, B; T) \) is the set of all sequences \( \omega \in \Sigma \) such that the fiber \( \mathcal{F}_\omega = \{(\omega, s)\}_{0 \leq s < F(\omega)} \) of the suspension space \( \Sigma_F \) over \( \omega \) projects to a directed geodesic segment across \( \mathcal{P} \) that enters \( \partial D(x; \alpha T^{-1}) \) through arc \( A \) and exits through arc \( B \). Hence, for each \( \omega \in \Sigma(A, B; T) \), there exist (unique) times \( 0 < s_A(\omega) < s_B(\omega) < F(\omega) \) such that the projection of the segment \( ((\omega, s))_{s_A(\omega) < s < s_B(\omega)} \) coincides with a geodesic segment from arc \( A \) to arc \( B \).

Denote by \( \Upsilon(A, B; T) \) the set of all \( u \in T^1 S \) that are tangents to geodesic segments from arc \( A \) to arc \( B \). Clearly, this set coincides with the projections of those points \( (\omega, s) \in \Sigma_F \) such that \( \omega \in \Sigma(A, B; T) \) and \( s_A(\omega) < s < s_B(\omega) \). Consequently, by equation (9),

\[
\lambda(\Sigma(A, B; T)) = \int_{\Sigma(A, B; T)} \frac{s_B - s_A}{s_B - s_A} d\lambda
\]

\[
= \int_{\pi^{-1}\Upsilon(A, B; T)} \frac{1}{s_B(\omega) - s_A(\omega)} d\lambda^*(\omega, s) \times \int_{\Sigma} F d\lambda
\]

\[
= \int_{\Upsilon(A, B; T)} \frac{1}{\tau(u)} dL(u) \times \int_{\Sigma} F d\lambda,
\]

where for each \( u \in \Upsilon(A, B; T) \) the length of the geodesic segment from \( A \) to \( B \) on which \( u \) lies is \( \tau(u) \).

Now we exploit the special property of the Liouville measure \( L \), specifically, that locally \( L \) looks like the product of normalized hyperbolic area with the Haar measure on the circle. For large \( T \), the exponential mapping \( v \mapsto \exp\{v/T\} \) maps the ball \( B(0, \alpha) \) in the tangent space \( T_x S \) onto \( D(x; \alpha T^{-1}) \) nearly isometrically (after scaling by the factor \( T^{-1} \)), so hyperbolic area on \( D(x; \alpha T^{-1}) \) is nearly identical with the pushforward of Lebesgue measure on \( B(0, \alpha) \), scaled by \( T^{-2} \). Furthermore, the inverse images of geodesic segments across \( D(x; \alpha T^{-1}) \) are nearly straight line segments crossing \( B(0, \alpha) \); those that cross from arc \( A \) to arc \( B \) in \( \partial D(x; \alpha T^{-1}) \) will pull back to straight line segments from arc \( A \) to arc \( B \) in \( \partial B(0, \alpha) \). These can be parametrized by the angle at which they meet the \( x \)-axis, as in Figure 1 for each angle \( \theta \), the integral of \( 1/\lambda \) over the region in \( B(0, \alpha) \) swept out by line segments crossing from arc \( A \) to arc \( B \) at angle \( \theta \) is \( \psi(\theta) \), as in Figure 1 (where the convex region is now \( B(0, \alpha) \)). Therefore, as \( T \to \infty \),

\[
\int_{\Upsilon(A, B; T)} \frac{1}{\tau(u)} dL(u) \sim T^{-1} \frac{1}{2\pi \text{area}(S)} \int_{-\pi/2}^{\pi/2} \psi(\theta) d\theta = T^{-1} \kappa g_{\alpha A, B}/2.
\]

\[\square\]
Proposition 5.2. For any two distinct points \( x, x' \in S \) and each \( \alpha > 0 \),
\[
\lim_{T \to \infty} T \lambda(H(x, x'; \alpha T^{-1})) = 0.
\]

Proof. For ease of exposition, assume that \( x \) lies in the interior of \( \mathcal{P} \), and that \( T \) is sufficiently large that the disks \( D(x; 2\alpha T^{-1}) \) and \( D(x'; 2\alpha T^{-1}) \) are contained in the interior of \( \mathcal{P} \). Then for any \( \omega \) such that the fiber \( \mathcal{F}_\omega \) projects to a geodesic segment that enters \( D(x; 2\alpha T^{-1}) \) there will be unique times \( 0 < s_0(\omega) < s_1(\omega) < F(\omega) \) of entry and exit; for those \( \omega \) such that the projection of \( \mathcal{F}_\omega \) enters the smaller disk \( D(x; \alpha T^{-1}) \), the sojourn time \( s_1(\omega) - s_0(\omega) \) will be at least \( \alpha T \). Let \( \lambda \) denote Lebesgue measure.

Denote by \( \Upsilon(x, x'; \alpha T^{-1}) \) the set of all tangent vectors \( u \in T^1S \) such that the geodesic ray with initial tangent vector \( u \) enters \( D(x'; \alpha T^{-1}) \) before exiting the fundamental polygon. Since \( x \) and \( x' \) are distinct points of \( S \), the disks \( D(x; \alpha T^{-1}) \) and \( D(x'; \alpha T^{-1}) \) are separated by at least \( \text{dist}(x, x')/2 \) (for large \( T \)), so there is a constant \( C = C(x, x', \alpha) < \infty \) such that for every point \( y \in D(x, \alpha T^{-1}) \) the set of angles \( \theta \) such that \( (y, \theta) \in \Upsilon(x, x'; \alpha T^{-1}) \) has Lebesgue measure less than \( CT^{-1} \). Now
\[
\lambda(H(x, x'; \alpha T^{-1})) = \int_{H(x,x';\alpha T^{-1})} \frac{s_1 - s_0}{s_1 - s_0} d\lambda = \int_{\pi^{-1}\Upsilon(x,x';\alpha T^{-1})} \frac{1}{s_1(\omega) - s_0(\omega)} d\lambda(\omega, s) \times E_{\lambda}F \\
= \int_{\Upsilon(x,x';\alpha T^{-1})} \frac{1}{\tau(u)} dL(u) \times E_{\lambda}F \\
\leq T^{-1}L(\Upsilon(x,x';\alpha T^{-1})) E_{\lambda}F
\]
where \( \tau(u) \) is the crossing time of \( D(x; 2\alpha T^{-1}) \) by the geodesic with initial tangent vector \( u \). Using once again the fact that (normalized) Liouville measure is the product of normalized hyperbolic area with Lebesgue angular measure, we see that for a suitable constant \( C' < \infty \),
\[
\lambda(H(x, x'; \alpha T^{-1})) \leq C'(\alpha T^{-1})^2 \times CT^{-1};
\]
thus, \( \lambda(H(x, x'; \alpha T^{-1})) = O(T^{-2}). \)

6. Decomposition of the Events \( N_{A,B} = k \)

In this section we show that the events \( \{\omega : N_{A,B}(\omega) = k\} \) can be approximated by sets consisting of those sequences \( \omega \in \Sigma \) whose first \( n \) letters contain exactly \( k \) occurrence of certain “magic subwords” each of length \( m = (\log n)^2(\approx \log T)^2 \). Recall that \( N_{A,B}(\omega) \) is the number of crossings of the disk \( D(\omega, \alpha T^{-1}) \) from boundary arc \( A \) to boundary arc \( B \) by the geodesic segment of length \( \approx T \) with symbolic representative \( \omega \) (see section 4).
As in section 5, assume that the Poincaré disk is parametrized in such a way that the point \( x \in \mathcal{P}(= S) \) is situated at the center \( x = 0 \) of \( \mathbb{D} \). Any hyperbolic geodesic in \( \mathbb{D} \) that passes through 0 is a Euclidean line segment whose endpoints on the boundary circle \( \partial \mathbb{D} \) are antipodal; consequently, any hyperbolic geodesic that intersects the ball \( D(0; \alpha T^{-1}) \) must lie within distance \( C\alpha T^{-1} \) of a line segment through 0 (in the Hausdorff metric based on Euclidean distance in \( \mathbb{D} \)). It follows that for any \( \xi^- \in \partial \mathbb{D} \), the hyperbolic geodesics with ideal endpoint \( \xi^- \) that intersects the ball \( D(0; \alpha T^{-1}) \) are precisely those hyperbolic geodesics whose second ideal endpoint \( \xi^+ \) lies in the arc of length \( 2C\alpha T^{-1} \) centered at the antipode of \( \xi^- \).

Now let \( A, B \) be any two disjoint closed arcs, each with nonempty interior, on the boundary of the ball \( B(0, \alpha) \) in the unit tangent space \( T_x S \). Recall that we have agreed to identify the disk \( D(x, \alpha T^{-1}) \) in \( S(= \mathcal{P}) \) with the ball \( B(0, \alpha) \) via the exponential mapping (scaled by the factor \( T^{-1} \)), so the arcs \( A, B \) are identified with arcs in \( \partial D(x, \alpha T^{-1}) \) whose arc-lengths are proportional to \( T^{-1} \). Let \( \gamma \) be a hyperbolic geodesic that intersects the interiors of both \( A \) and \( B \) at distances \( \geq \delta T^{-1} \) from their endpoints. Let \( \xi^-, \xi^+ \) be the ideal endpoints of \( \gamma \). Then for a suitable constant \( C > 0 \) not depending on \( T \), the geodesic \( \gamma' \) with ideal endpoints \( \zeta^+, \zeta^- \) satisfying

\[
d(\xi^+, \zeta^+) < C\delta T^{-1} \quad \text{and} \quad d(\xi^-, \zeta^-) < C\delta T^{-1}
\]

will also pass through \( A \) and \( B \). Similarly if \( \tilde{\gamma} \) is a hyperbolic geodesic that does not pass within distance \( \delta T^{-1} \) of \( A \) (respectively, \( B \)) then any hyperbolic geodesic \( \tilde{\gamma}' \) whose ideal endpoints are within distance \( C\delta T^{-1} \) of the corresponding ideal endpoints of \( \tilde{\gamma} \) will not intersect \( A \) (respectively, \( B \)).

By Proposition 3.1 (D), the boundary arcs \( J_{m}^{\pm}(\omega^{\pm}) \) corresponding to cylinder sets \( \Sigma^{\pm}(\omega^{\pm}) \) have lengths less than \( C\beta_{m}^{n} \), for some \( \beta_2 < 1 \). If

\[
m = (\log n)^2 (\approx (\log T)^2),
\]

then every arc \( J_{m}^{\pm}(\omega^{\pm}) \) will have length smaller than \( n^{-C'\log n} \). Therefore, for each pair \( J_{m}^{-}(\omega^{-}), J_{m}^{+}(\omega^{+}) \) of such arcs, one of the following will hold:
(i) every hyperbolic geodesic with ideal endpoints in $J_m^-(\omega^-), J_m^+(\omega^+)$ will pass through the arcs $A, B$ of $\partial D(x, \alpha T^{-1})$;

(ii) no hyperbolic geodesic with ideal endpoints in $J_m^-(\omega^-), J_m^+(\omega^+)$ will pass through both arcs $A, B$; or

(iii) every hyperbolic geodesic with ideal endpoints in $J_m^-(\omega^-), J_m^+(\omega^+)$ will pass within distance $C''n^{-C''\log n}$ of one of the endpoints of $A$ or $B$.

**Proposition 6.1.** For each pair $A, B$ of non-overlapping closed arcs of $\partial B(0, \alpha)$ and each $T \geq 1$ there exist sets $J_1 \subset J_2$ of pairs $J_m^+(\omega^\pm)$ such that

(A) each pair $J_m^+(\omega^\pm)$ in $J_1$ is of type (i);

(B) each pair $J_m^+(\omega^\pm)$ not in $J_2$ is of type (ii); and

(C) the set of all unit tangent vectors $x \in T^1 S$ such that the geodesic with initial tangent vector $x$ lifts to a hyperbolic geodesic with endpoints in $J_m^-(\omega^-), J_m^+(\omega^+)$, for some pair $J_m^-(\omega^-), J_m^+(\omega^+)$ in $J_2 \setminus J_1$, has Liouville measure less than $o(n^{-r})$ for all $r > 0$.

**Proof.** Define $J_1$ to be the set of all pairs of type (i), and define $J_2$ to be the complement of the set of all pairs of type (ii). What must be proved is assertion (iii).

Every hyperbolic geodesic with ideal endpoints in one of the bad pairs (those in $J_2 \setminus J_1$) must pass within distance $C''n^{-C''\log n}$ of one of the four endpoints of arcs $A, B$. Now the Liouville measure of the set of unit tangent vectors with basepoint at distance less than $C''n^{-C''\log n}$ from one of these four endpoints is of order $n^{-2C\log n}$, hence, for a geodesic ray on $S$ with basepoint chosen randomly according to normalized Liouville measure there is probability $o(n^{-r})$ that will pass within this distance of one of the four endpoints by time $T$.

**Definition 6.2.** Given arcs $A, B$ as in Proposition 6.1 and $T \geq 1$, define the magic subwords for the triple $(A, B; T)$ to be those words of length $2m$ obtained by concatenating the first $m$ letters of $\omega^-$ with the first $m$ letters of $\omega^+$ for some pair $J_m^+(\omega^\pm)$ in the collection $J_1$.

**Corollary 6.3.** The symmetric difference between the sets $\{\omega : N_{A,B} = k\}$ and the set of $\omega \in \Sigma$ with exactly $k$ occurrences of one of the magic subwords in the segment $\omega_1 \omega_2 \cdots \omega_n$ has $\lambda-$measure $o(n^{-r})$ for all $r > 0$.

**Remark 6.4.** The set of magic subwords for a particular value of $T$ will in general have no clear relationship to the magic subwords for a different value of $T$.

**Proposition 6.5.** For each $T$ let $\mathcal{M} = \mathcal{M}_T$ be the set of magic subwords for a fixed pair $A, B$ of boundary arcs and fixed $\alpha > 0$. Then

$$\lim_{T \to \infty} T \lambda\{\omega : (\omega_m \omega_{m+1} \cdots \omega_m) \in \mathcal{M}\} = \frac{1}{2} \kappa_{\gamma} \beta_{A,B} E \lambda F$$

where $\beta_{A,B}$ is defined by equation (4).

**Proof.** This follows directly from Propositions 6.1 and 5.1.

\[ \Box \]

7. NO QUICK ENTRIES OR RE-ENTRIES

**Proposition 7.1.** Let $\gamma$ be a geodesic ray whose initial tangent vector is chosen at random according to normalized Liouville measure. For any $\kappa < \infty$ the probability that the $\gamma$ enters the ball...
$D(x, \alpha T^{-1})$ before time $(\log T)\kappa$ converges to 0 as $T \to \infty$. Similarly, the probability that between time 0 and time $T$ the geodesic $\gamma$ enters the ball $D(x, \alpha T^{-1})$ and then re-enters within time $(\log T)\kappa$ converges to 0 as $T \to \infty$.

**Proof.** We first estimate the Liouville measure of the set $H = H_T$ of unit tangent vectors $v = (y, \theta)$ such that the geodesic ray $\gamma$ with initial tangent vector $v$ enters the disk $D(x; \alpha T^{-1})$ before time $(\log T)\kappa$. For this, we lift to the universal cover, viewed as the Poincaré disk $\mathbb{D}$ with the lift $\tilde{x}$ of $x$ located at the center $\tilde{x} = 0$ of the disk. Let $\tilde{\gamma}$ be the lift of the geodesic $\gamma \in F_T$; then for some deck transformation $g$ satisfying $d_H(\tilde{x}, g\tilde{x}) \leq (\log T)\kappa + 2\text{diam}(\mathcal{P})$, the geodesic $\tilde{\gamma}$ must enter the disk $D(g\tilde{x}, \alpha T^{-1})$.

Consider first the deck transformation $g = \text{id}$. Fix $\tilde{y} \in \mathcal{P}$, and let $A_{\text{id}}(\tilde{y})$ be the set of all angles $\theta$ such that the geodesic ray $\tilde{\gamma}$ based at $\tilde{v} = (\tilde{y}, \theta)$ enters $D(\tilde{x}, \alpha T^{-1})$. For any $\epsilon > 0$ there exists $C < \infty$ such that if $\tilde{y}$ is at distance more than $\epsilon$ from $\tilde{x}$ then $m(A_{\text{id}}(\tilde{y})) < C\alpha T^{-1}$. Since Liouville measure $dL(\tilde{y}, \theta)$ is locally the product of hyperbolic area $dA(\tilde{y})$ with Lebesgue measure $dm(\theta)$, it follows that the Liouville measure of the set $\{(\tilde{y}, \theta) : \theta \in A_{\text{id}}(\tilde{y})\}$ is less than $C\alpha T^{-1}$ plus the area of the ball $D(\tilde{x}, \epsilon)$. Since $\epsilon > 0$ can be made arbitrarily small, it follows that the Liouville measure of $\{(\tilde{y}, \theta) : \theta \in A_{\text{id}}(\tilde{y})\}$ tends to 0 as $T \to \infty$.

Next, fix a deck transformation $g \neq \text{id}$ and a point $\tilde{y} \in \mathcal{P}$, and consider the set $A_g(\tilde{y})$ of angles $\theta$ such that the geodesic ray $\tilde{\gamma}$ based at $\tilde{v} = (\tilde{y}, \theta)$ enters the disk $D(g\tilde{x}, \alpha T^{-1})$. Because geodesics separate exponentially, the “visibility angle” of the disk $D(g\tilde{x}, \alpha T^{-1})$ as viewed from $\tilde{y}$ decays exponentially in the distance $d_H(\tilde{x}, g\tilde{x})$, in particular, if $m$ is Lebesgue measure on the circle, then

$$m(A_g(\tilde{y})) \leq C\alpha T^{-1} \exp\left\{-d_H(\tilde{x}, g\tilde{x})\right\}$$

for a constant $C < \infty$ that does not depend on either $T$ or the choice of the point $\tilde{y} \in \mathcal{P}$. Now by a theorem of Huber [14] (cf. also Margulis [23]), the number of deck transformations $g$ satisfying $d_H(\tilde{x}, g\tilde{x}) \leq \tau$ grows like $e^\tau$; consequently,

$$\sum_{g : d_H(\tilde{x}, g\tilde{x}) \leq (\log T)\kappa} m(A_g(\tilde{y})) \leq C'\alpha T^{-1}(\log T)\kappa$$

where $C' < \infty$ does not depend on either $T$ or $\tilde{y}$. Using again the local product structure of Liouville measure, we conclude that

$$L(H_T) \leq C''\alpha T^{-1}(\log T)\kappa \to 0.$$  

This proves the first assertion of the proposition.

The proof of the second assertion is similar. Let $H'_T$ be the set of unit tangent vectors $(y, \theta)$ such that $y \in D(x, \alpha T^{-1})$ and such that the geodesic ray $\gamma$ with initial tangent vector $(y, \theta)$, after exiting $D(x, \alpha T^{-1})$, re-enters the disk $D(x; 2\alpha T^{-1})$ before time $(\log T)\kappa$. By a minor variation of the argument above,

$$L(H'_T) \leq C''''\alpha^2 T^{-2}(\log T)\kappa.$$  

Now let $H''_T$ be the set of unit tangent vectors $v$ such that the geodesic ray $\gamma$ with initial tangent vector $v$ enters $D(x, \alpha T^{-1})$ at some time $t \leq T$ and subsequently re-enters before time $t + (\log T)\kappa$. Fix $v \in H''_T$, let $\gamma$ be the corresponding geodesic ray, and let $k$ be the
smallest integer such that $k\alpha T^{-1} \geq t$. Since geodesics travel at unit speed, $\gamma$ must be located in the ball $D(x, 2\alpha T^{-1})$ at time $k\alpha T^{-1}$, and so at this time the tangent vector to $\gamma$ must lie in the set $H'_T$. Since the Liouville measure is invariant under the geodesic flow, it follows that

$$L(H''_T) \leq \sum_{k=1}^{\left\lceil T^2/\alpha \right\rceil} L(H'_T) \leq C'''T^{-1}(\log T)^\kappa.$$  

\[ \square \]

**Corollary 7.2.** If $\omega \in \Sigma$ is chosen randomly according to $\lambda$, then the probability that the initial segment $\omega_1\omega_2\cdots\omega_{\lceil \log T \rceil}$ contains a magic subword converges to 0 as $T \to \infty$. Similarly, the probability that the segment $\omega_1\omega_2\cdots\omega_n$ contains magic subwords separated by fewer than $(\log n)^\kappa$ letters converges to zero as $n \to \infty$.  

\[ \square \]

8. PROOF OF PROPOSITIONS 4.1–4.2

*Proof of (15) for $r = 1$. Consider first the case $r = 1$. In this case we are given a single pair $(A, B)$ of non-overlapping boundary arcs of $\partial B(0, \alpha)$; we must show that for any integer $k \geq 0$,

$$\lim_{n \to \infty} \lambda \{\omega \in \Sigma : N_{A,B}(\omega) = k\} = \frac{\kappa g \beta_{A,B}}{2} e^{\frac{\kappa g \beta_{A,B}}{2}},$$  

where $\beta_{A,B}$ is defined by equation (4). Recall that $N_{A,B}(\omega)$ is the number of geodesic segments in the collection $I_n(\omega)$ that cross the target disk $D(x; 2\alpha T^{-1})$ from arc $A$ to arc $B$. By Corollary 6.3, $N_{A,B}(\omega)$ is well-approximated by the number $N'_{A,B}$ of magic subwords in the word $\omega_1\omega_2\cdots\omega_n$; in particular, for any $k \geq 0$, the symmetric difference between the events $\{N_{A,B} = k\}$ and $\{N'_{A,B} = k\}$ has $\lambda$-measure tending to 0. Consequently, it suffices to prove that (22) holds when $N_{A,B}$ is replaced by $N'_{A,B}$.

Recall (sec. 3.3) that any Gibbs process is the natural projection of a list process. Thus, on some probability space there exists a sequence $W_1, W_2, W_3, \ldots$ of independent random words, of random lengths $\tau_i$, such that the infinite sequence obtained by concatenating $W_1, W_2, W_3, \ldots$ has distribution $\lambda$, that is, for any Borel subset $B$ of $\Sigma^+$,

$$P\{W_1 \cdot W_2 \cdot W_3 \cdots \in B\} = \lambda(B).$$

All but the first word $W_1$ have the same distribution, and the lengths $\tau_i$ have exponentially decaying tails (cf. inequality (13)). Since the magic subwords are of length $\lceil \log n \rceil^2$, any occurrence of one will typically straddle a large number of consecutive words in the sequence $W_i$. Thus, to enumerate occurrences of magic subwords, we shall break the sequence $\{W_i\}_{i \geq 1}$ into blocks of length $m = \lceil \log n \rceil^3$, and count magic subwords block by block. Set

$$\bar{W}_1 = W_1 W_2 \cdots W_m,$$

$$\bar{W}_2 = W_{m+1} W_{m+2} \cdots W_{2m},$$

$$\bar{W}_3 = W_{2m+1} W_{2m+2} \cdots W_{3m},$$

...
etc., and denote by \( \tilde{\tau}_k = \sum_{i=mk-k+1}^{mk} \tau_i \) the length (in letters) of the word \( \tilde{W}_k \).

**Claim 1.** For each \( C > E\tau_2 \), there exists \( \Lambda(C) > 0 \) such that for any integer \( k \geq 1 \)

\[
P\left\{ \sum_{i=1}^{k} \tau_i \geq Ck \right\} \leq e^{-k\Lambda(C)},
\]

and for all sufficiently large \( C < \infty \),

\[
\lim_{n \to \infty} P\{ \max_{k \leq n} \tilde{\tau}_k \geq Cm \} = 0.
\]

The function \( C \mapsto \Lambda(C) \) is convex and satisfies \( \lim \inf_{C \to \infty} \Lambda(C)/C > 0 \).

**Proof of Claim 1.** These estimates follow from the exponential tail decay property \([13]\) by standard results in the elementary large deviations theory, in particular, Cramér’s theorem (cf. \([10]\), sec. 2.2) for sums of independent, identically distributed random variables with exponentially decaying tails. The block lengths \( \tilde{\tau}_k \) are gotten by summing the lengths \( \tau_i \) of their \( m \) constituent words \( W_i \); for all but the first block \( \tilde{W}_1 \), these lengths are i.i.d. and satisfy \([13]\). Hence, Cramér’s theorem guarantees\([10]\) the existence of a convex rate function \( C \mapsto \Lambda(C) \) and constants \( C' = C'(C) < \infty \) such that inequality \( (23) \) holds for all \( k \geq 1 \). Applying this inequality with \( k = m = \lceil \log n \rceil^3 \) yields

\[
P\left\{ \sum_{i=2}^{m+1} \tau_i \geq Cm \right\} \leq e^{-m\Lambda(C)} = n^{-3\Lambda(C)}.
\]

Cramér’s theorem also implies that \( \Lambda(C) \) grows at least linearly in \( C \), so by taking \( C \) sufficiently large we can ensure that \( \Lambda(C) \geq 2/3 \), which makes the probability above smaller than \( n^{-2} \). Since there are only \( n \) blocks, it follows that the probability that \( \tilde{\tau}_k \geq Cm \) for one of them is smaller than \( n^{-1} \). \( \square \)

**Claim 2.** The probability that a magic subword occurs in the concatenation of the first two blocks \( \tilde{W}_1 \tilde{W}_2 \) converges to 0 as \( T \to \infty \).

**Proof of Claim 2.** The event that one of the first two blocks has length \( \geq C\lceil \log n \rceil^3 \) can be ignored, by Claim 1. On the complementary event, an occurrence of a magic subword in \( \tilde{W}_1 \tilde{W}_2 \) would require that the magic subword occurs in the first \( 2C\lceil \log n \rceil^3 \) letters. By Corollary 7.2, the probability of this event tends to 0 as \( T \to \infty \). \( \square \)

It follows from Claim 1 and Corollary 7.2 that with probability tending to 1 as \( n \to \infty \), no block \( \tilde{W}_k \) among the first \( n \) will contain more than one magic subword. On this event, then, the number \( \hat{N}_{A,B} \) of magic subwords that occur in the first \( n \) letters can be obtained by counting the number of blocks \( \tilde{W}_k \) that contain magic subwords and then adding the number of magic subwords that straddle two consecutive blocks.

**Claim 3.** As \( n \to \infty \), the probability that a magic subword straddles two consecutive blocks \( \tilde{W}_k, \tilde{W}_{k+1} \) among the first \( n/\lceil \log n \rceil^3 \) blocks converges to 0.

---

\( ^{10} \)The length of the initial block has a different distribution than the subsequent blocks, because the first excursion of the list process has a different law than the rest. However, the length of the first excursion also has an exponentially decaying tail, by Proposition 3.2, so the upper bounds given by Cramér’s theorem still apply.
Proof of Claim 3. A magic subword, since it has length $\lfloor \log n \rfloor^2$, can only straddle consecutive blocks $\hat{W}_k, \hat{W}_{k+1}$ if it begins in one of the last $\lfloor \log n \rfloor^2$ word $W_i$ of the $m = \lfloor \log n \rfloor^3$ words that constitute $\hat{W}_k$. The words $W_i$ are i.i.d. (except for $W_1$, and by Claim 2 we can ignore the possibility that a magic subword begins in $\hat{W}_1\hat{W}_2$), so the probability that a magic subword begins in $W_i$ does not depend on $i$. Since only $\lfloor \log n \rfloor^2$ of the $\lfloor \log n \rfloor^3$ words in each block $\hat{W}_k$ would produce straddles, it follows that the expected number of magic subwords in $\hat{W}_1\hat{W}_2 \cdots W_{n/m}$ is at least $\lfloor \log n \rfloor$ times the probability that a magic subword straddles two consecutive blocks. Therefore, the claim will follow if we can show that the expected number of magic subwords in $\hat{W}_1\hat{W}_2 \cdots \hat{W}_{n/m}$ remains bounded as $T \to \infty$. Denote the number of such magic subwords by $N''_{A,B}$.

The number of letters in the concatenation $\hat{W}_1\hat{W}_2 \cdots \hat{W}_{n/m}$ is $\sum_{i=1}^n \tau_i$, which by Claim 1 obeys the large deviation bound (23). Fix $K < \infty$, and let $G$ be the event that $\sum_{i=1}^n \tau_i \leq nK$. On this event, $N''_{A,B}$ is bounded by the number of magic subwords in the first $nK$ letters of the concatenation $W_1W_2 \cdots$. Since the concatenation $W_1W_2 \cdots$ is, by Proposition 3.2, a version of the Gibbs process associated with the Gibbs state $\lambda$, which by shift-invariance is stationary, it follows that the expected number of magic subwords in the first $nK$ letters is $nK \times$ the probability that a magic subword begins at the very first letter of $W_1W_2 \cdots$. But by Proposition 5.1, this probability is asymptotic to $T^{-1} \alpha \beta_{A,B} E_A F$; thus, for large $T$,

$$EN''_{A,B} 1_G \leq nKT^{-1} \alpha \beta_{A,B} E_A F = K \alpha \beta_{A,B}.$$ 

It remains to bound the contribution to the expectation from the complementary event $G^c$. For this, we use the large deviation bound (23). On the event that $\sum_{i=1}^n \tau_i \leq n(K + k)$, the count $N''_{A,B}$ cannot be more than $n(K + k)$; hence,

$$EN''_{A,B} 1_{G^c} \leq \sum_{k=1}^{\infty} n(K + k)e^{-nK(K+k)}.$$ 

Since $\Lambda(C)$ grows at least linearly in $C$, this sum remains bounded provided $K$ is sufficiently large. \hfill \square

Recall that $N'_{A,B}$ is the number of magic subwords in the first $n$ letters of the sequence $\hat{W}_1\hat{W}_2 \cdots$ obtained by concatenating the words in the regenerative representation. The blocks $\hat{W}_k$ are independent, and except for the first all have the same distribution, with common mean length $mE_T$. Let $N'_{A,B}$ be the number of magic subwords in the segment $\hat{W}_2\hat{W}_3 \cdots \hat{W}_\nu$, where $\nu = \nu(n) = n/[mE_T]$. By the central limit theorem, with probability approaching 1 the length $\sum_{i=1}^{n/\nu} \tau_i$ of the segment $\hat{W}_2\hat{W}_3 \cdots \hat{W}_\nu$ differs by no more than $\sqrt{n \log n}$ from $n$, and by the same argument as in the proof of Claim 3, the probability that a magic subword occur within the stretch of $2\sqrt{n \log n}$ letters surrounding the $n$th letter converges to 0. Thus, as $T \to \infty$,

$$P\{N'_{A,B} \neq N'_{A,B} \} \to 0.$$ 

By Claim 1 and Corollary 7.2 with probability approaching 1 no block $\hat{W}_k$ will contain more than 1 magic subword, and by Claim 3 no magic subword will straddle two blocks.
Therefore, with probability $\rightarrow 1$,

$$N'_{A,B} = N^*_{A,B} = \sum_{k=2}^{\nu} Y(\tilde{W}_k),$$

where $Y(\tilde{W}_k)$ is the indicator of the event that the block $\tilde{W}_k$ contains a magic subword. These indicators are independent, identically distributed Bernoulli random variables; by Proposition 6.5,

$$EY(\tilde{W}_k) \sim T^{-1/2} \kappa g \beta \lambda F$$

and so

$$E \sum_{k=2}^{\nu} Y(\tilde{W}_k) \rightarrow \frac{1}{2} \kappa g \beta_{A,B}.$$ 

Now Proposition 2.11 implies that for any integer $J \geq 0$,

$$P \left\{ \sum_{k=2}^{\nu} Y(\tilde{W}_k) = J \right\} \rightarrow \frac{(\kappa g \beta_{A,B}/2)^J}{J!} e^{-\kappa g \beta_{A,B}/2},$$

proving (22). □

Proof of (15) for $r \geq 1$. (Sketch) In general we are given $r \geq 1$ pairs $(A_i, B_i)$ of non-overlapping boundary arcs of $\partial B(0, \alpha)$; we must show that the counts $N_{A_i, B_i}$ converge jointly to independent Poissons with means $\alpha \beta_{A_i, B_i}$, respectively. The key to this is that the sets $\mathcal{M}_i$ of magic words for the different pairs $(A_i, B_i)$ are pairwise disjoint, because the arcs $A_i, B_i$ are non-overlapping (a geodesic segment crossing of $D(x, \alpha T^{-1})$ has unique entrance and exit points on $\partial D(x, \alpha T^{-1})$, so at most one of the pairs $(A_i, B_i)$ can contain these).

By the same argument as in the case $r = 1$, the counts $N_{A_i, B_i}$ can be replaced by the sums

$$N^*_{A_i, B_i} = \sum_{k=2}^{\nu} Y_i(\tilde{W}_k)$$

where $Y_i(\tilde{W}_k)$ is the indicator of the event that the block $\tilde{W}_k$ contains a magic subword for the pair $A_i, B_i$. Since the sets $\mathcal{M}_i$ of magic subwords are non-overlapping, the vector of these sums follows a multinomial distribution; hence, by Proposition 2.12 the vector

$$(N^*_{A_i, B_i})_{1 \leq i \leq r}$$

converges in distribution to the product of $r$ Poisson distributions, with means $\frac{1}{2} \kappa g \alpha \beta_{A_i, B_i}$. □

Proof of Proposition 4.2. The argument is virtually the same as that for the case $r \geq 2$ of Proposition 4.1, the only new wrinkle is that the sets $\mathcal{M}_i$ and $\mathcal{M}'_i$ of magic words for the pairs $(A_i, B_i)$ and $(A'_i, B'_i)$ need not be disjoint, because it is possible for a geodesic segment across the fundamental polygon $\mathcal{P}$ to enter both $D(x, \alpha T^{-1})$ and $D(x', \alpha T^{-1})$. However, Proposition 5.2 implies that the expected number of such double-hits in the first $n$ crossings of $\mathcal{P}$ converges to 0 as $T \rightarrow \infty$, and consequently the probability that there is even one double-hit tends to zero. Thus, the magic subwords for pairs $A'_i, B'_i$ that also occur as
magic subwords for pairs $A_i, B_i$ can be deleted without affecting the counts (at least with probability $→ 1$ as $T → ∞$), and so the counts $N_{A_i, B_i}$ and $N'_{A'_i, B'_i}$ may be replaced by

$$N^*_{A_i, B_i} = \sum_{k=2}^{\nu} Y_i(\tilde{W}_k) \quad \text{and}$$

$$N^{**}_{A'_i, B'_i} = \sum_{k=2}^{\nu} Y'_i(\tilde{W}_k)$$

where $Y_i(\tilde{W}_k)$ and $Y'_i$ are the indicators of the events that the block $\tilde{W}_k$ contains a magic subword for the appropriate pair (with deletions of any duplicates). Since the adjusted sets of magic subwords are non-overlapping, the vector of these counts $N^*_{A_i, B_i}$ and $N^{**}_{A'_i, B'_i}$ follows a multinomial distribution, and so the convergence (18) holds, by Proposition 2.12.

9. Global Statistics

In this section we show how Theorem 3, which describes the “global” statistics of the tessellation $T_T$ induced by a random geodesic segment of length $T$, follows from the “local” description provided by Theorem 1 and the ergodicity of the Poisson line process with respect to translations. Theorem 1 and Proposition 2.7 (cf. also Corollary 2.9) imply that locally – in balls $D(x; \alpha T^{-1})$, where $\alpha$ is large – the empirical distributions of polygons, their angles and side lengths (after scaling by $T$) stabilize as $T → ∞$. Since this is true in neighborhoods of all points $x ∈ S$, it is natural to expect that these empirical distributions also converge globally. To prove this, we must show that in those small regions of $S$ where empirical distributions behave atypically the counts are not so large as to disturb the global averages. The key is the following proposition, which limits the numbers of polygons, edges, and vertices in $T_T$.

**Proposition 9.1.** Let $f = f_T$, $v = v_T$, and $e = e_T$ be the number of polygons, vertices and edges in the tessellation $T_T$. With probability one, as $T → ∞$,

$$\lim_{T \to ∞} v_T/T^2 = \kappa_g/\pi,$$

(25)

$$\lim_{T \to ∞} e_T/T^2 = 2\kappa_g/\pi, \quad \text{and}$$

(26)

$$\lim_{T \to ∞} f_T/T^2 = \kappa_g/\pi.$$  

(27)

Moreover, there exists a (nonrandom) constant $C = C_S < ∞$ such that for every tessellation $T_T$ induced by a geodesic segment of length $T$,

$$v_T + e_T + f_T ≤ CT^2.$$  

(28)

For the proof we will need to know that multiple intersection points (points of $S$ that a geodesic ray passes through more than twice) do not occur in typical geodesics. We have the following:

**Lemma 9.2.** For almost every unit tangent vector $v ∈ T^1 S$, there are no multiple intersection points on the geodesic ray $(γ_t(v))_{t ≥ 0}$.  

24
Proof. Suppose \( v \in T^1 S \) gives rise to triple intersection. Let \( \gamma \) denote a lift of the geodesic ray \( (\gamma_t(\cdot))_{t \geq 0} \) to the universal cover \( \mathbb{H}^2 \), we have that there must be deck transformations \( A, B \) so that the geodesic rays \( A\gamma \) and \( B\gamma \) have a triple intersection. In [13], it is shown that the set of such geodesics is a positive codimension subvariety for any fixed \( A, B \), and therefore, a set of measure 0. Taking the (countable) union over all possible pairs \( A, B \), we have our result. \( \square \)

Proof of Proposition 9.1. The number \( v_T \) of vertices is the number of self-intersections of the random geodesic segment \( \gamma_T := (\gamma_t(\cdot))_{0 \leq t \leq T} \) (unless one counts the beginning and end points of \( \gamma_T \) as vertices, in which case the count is increased by 2). It is an easy consequence of Birkhoff’s ergodic theorem (see [21], sec. 2.3 for the argument, but beware that [21] seems to be off by a factor of 4 in his calculation of the limit) that the number of self-intersections satisfies (25). Following is a brief resume of the argument.

Fix \( \epsilon > 0 \) small, and partition the segment \( \gamma_T \) into non-overlapping geodesic segments \( \gamma^i_T \) of length \( \epsilon \) (if necessary, extend or delete the last segment; this will not change the self-intersection count by more than \( O(T) \)). If \( \epsilon \) is smaller than the injectivity radius then (29)

\[
v_T = \sum_{i \neq j} \mathbf{1}(\gamma^i_T \cap \gamma^j_T \neq \emptyset)
\]

is the number of pairs \((i,j)\) such that \( \gamma^i_T \) and \( \gamma^j_T \) cross. Birkhoff’s theorem implies that for each \( i \), the fraction of indices \( j \) such that \( \gamma^j_T \) crosses \( \gamma^i_T \) converges, as \( T \to \infty \), to the normalized Liouville measure of that region \( R_\epsilon \) of \( T^1 S \) where the geodesic flow will produce a ray that crosses \( \gamma^i_T \) by time \( \epsilon \). This implies that the limit on the left side of (25) exists. To calculate the limit, let \( \epsilon \to 0 \); if \( \epsilon > 0 \) is small, then for each angle \( \theta \) the set of points \( x \in S \) such that \( (x, \theta) \in R_\epsilon \) is approximately a rhombus of side \( \epsilon \) with interior angle \( \theta \).

Integrating the area of this rhombus over \( \theta \), one obtains a sharp asymptotic approximation to the normalized Liouville measure of \( R_\epsilon \): \[
L(R_\epsilon) \sim 2\epsilon^2 \int_0^\pi \sin \theta \, d\theta / (2\pi \text{area}(S)) = 2\epsilon^2 \kappa_g / \pi.
\]

Since the number of terms in the sum (29) is \( \frac{1}{2} [T/\epsilon^2] \), it follows that \( v_T/T^2 \to \kappa_g / \pi \).

The limiting relations (26) and (27) follow easily from (25). With probability one, the geodesic segment \( \gamma_T \) has no multiple intersection points, by Lemma 9.2. Consequently, as one traverses the segment \( \gamma_T \) from beginning to end, one visits each vertex twice, and immediately following each such visit encounters a new edge of \( T_T \) (except for the initial edge), so \( v_T = 2vT \pm 2 \), and hence (26) follows from (25). Finally, by Euler’s formula, \( v - e + f = -\chi(S) \), and therefore (27) follows from (25)–(26).

No geodesic ray can intersect itself before time \( \rho \), where \( \rho \) is the injectivity radius of \( S \), so for every geodesic segment \( \gamma_T \) of length \( T \) the corresponding tessellation must satisfy \( v_T \leq T^2 / \rho^2 \). The inequality (28) now follows by Euler’s formula and the relation \( e = v \pm 2 \). \( \square \)

Proof of Theorem 3. We will prove only the assertion concerning the empirical frequencies of \( k \)-gons in the induced tessellation. Similar arguments can be used to prove that the
empirical distributions of scaled side-lengths, interior angles, etc. converge to the corresponding theoretical frequencies in a Poisson line process. Denote by $T_T$ the tessellation of the surface $S$ induced by a random geodesic segment of length $T$.

We first give a heuristic argument that explains how Theorem 1, Corollary 2.9, and Proposition 9.1 together imply the convergence of empirical frequencies. Suppose that, for large $T$, the surface $S$ could be partitioned into non-overlapping regions $R_i$ each nearly isometric, by the scaled exponential mapping from the tangent space at its center $x_i$, to a square of side $\alpha T^{-1}$. (Of course this is not possible, because it would violate the fact that $S$ has non-zero scalar curvature.) The hyperbolic area of $R_i$ would be $\sim \alpha^2/T^2$, and so the number of squares $R_i$ in the partition would be $\sim T^2/(\alpha^2 \kappa_g)$.

Assume that $\alpha$ is sufficiently large that with probability at least $1 - \epsilon$, the absolute errors in the limiting relations (5), (6), and (7) (for some fixed $k$) of Corollary 2.9 are less than $\epsilon$. By Theorem 1, for any point $x \in S$ and any $\alpha$, the restriction of the geodesic tessellation $T_T$ to the disk $D(x, 2\alpha T^{-1})$, when pulled back to the ball $B(0, 2\alpha)$ of the tangent space $T_x S$, converges in distribution, as a line process, to the Poisson line process of intensity $\kappa_g$. Since this holds for every $x$, it follows that for all sufficiently large $T$, with probability at least $1 - 2\epsilon$, in all but a fraction $\epsilon$ of the regions $R_i$ the counts $V_T(R_i)$ and $F_T(R_i)$ of vertices and faces in the regions $R_i$ (in the tessellation $T_T$) and the fractions $\Phi_{k,T}(R_i)$ of $k$-gons will satisfy

$$
(30) \quad |V_T(R_i)/\alpha^2 - \kappa_g^2/\pi| < 2\epsilon, \\
(31) \quad |F_T(R_i)/\alpha^2 - \kappa_g^2/\pi| < 2\epsilon, \quad \text{and} \\
(32) \quad |\Phi_{k,T}(R_i) - \phi_k| < 2\epsilon.
$$

Call the regions $R_i$ where these inequalities hold good, and the others bad. Since all but and area of size $\epsilon \times \text{area}(S)$ is covered by good squares $R_i$, relations (31) and (27) imply that the total number of faces of $T_T$ in the bad squares satisfies

$$
\sum_{i \text{ bad}} F_T(R_i) \leq 4\epsilon T^2 \times \text{area}(S).
$$

Consequently, regardless of how skewed the empirical distribution of faces in the bad regions might be, it cannot affect the overall fraction of $k$-gons by more than $8\epsilon$. Since $\epsilon > 0$ can be made arbitrarily small, it follows from (32) that

$$
\lim_{T \to \infty} \Phi_{k,T}(S) = \phi_k.
$$

To provide a rigorous argument, we must explain how the partition into “squares” $R_i$ can be modified. Fix $\delta > 0$ small, and let $\Delta$ be a triangulation of $S$ whose triangles $\tau$ all (a) have diameters less than $\delta$ and (b) have geodesic edges. If $\delta > 0$ is sufficiently small, the triangles of $\Delta$ will all be contained in coordinate patches nearly isometric, by the exponential mapping, to disks $B(0, 2\delta)$ in the tangent space $T S_{x_\tau}$, where $x_\tau$ is a distinguished point in the interior of $\tau$. In each such ball $B(0, 2\delta)$, use an orthogonal coordinate system to foliate $B(0, 2\delta)$ by lines parallel to the coordinate axes, and then use the exponential mapping to project these foliations to foliations of the triangles $\tau$; call these foliations $F_x(\tau)$ and $F_y(\tau)$. If $\delta > 0$ is sufficiently small then the curves in $F_x(\tau)$ will cross curves in $F_y(\tau)$ at angles $\theta \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon]$, where $\epsilon > 0$ is small.
The foliations $\mathcal{F}_x(\tau)$ and $\mathcal{F}_y(\tau)$ can now be used as guidelines to partition $\tau$ into regions $R_i(\tau)$ whose boundaries are segments of curves in one or the other of the foliations. In particular, each boundary $\partial R_i(\tau)$ should consist of four segments, two from $\mathcal{F}_x(\tau)$ and two from $\mathcal{F}_y(\tau)$, and each should be of length $\sim \alpha T^{-1}$; thus, for large $T$ each region $R_i(\tau)$ will be nearly a “parallelogram” (more precisely, the image of a parallelogram in the tangent space $TS_{x_i(\tau)}$ at a central point $x_i(\tau) \in R_i(\tau)$) whose interior angles are within $\epsilon$ of $\pi/2$. The collection of all regions $R_i(\tau)$, where $\tau$ ranges over the triangulation $\Delta$, is nearly a partition of $S$ into rhombi; only at distances $O(\alpha T^{-1})$ of the boundaries $\partial\tau$ are there overlaps. The total area in these boundary neighborhoods is $o(1)$ as $T \to \infty$.

Corollary 2.9, as stated, applies only to squares. However, any rhombus $R$ whose interior angles are within $\epsilon$ of $\pi/2$ can be bracketed by squares $S_- \subset R \subset S_+$ in such a way that the area of $S_+ \setminus S_-$ is at most $C\epsilon \text{area}(S_+)$, for some $C < \infty$ not depending on $\epsilon$. Since Corollary 2.9 applies for each of the bracketing squares, it now follows as in the heuristic argument above that with probability $\geq 1 - C'\epsilon$, in all but a fraction $C\epsilon$ of the regions $R_i(\tau)$ the inequalities (30), (31), and (32) will hold. The limiting relation (33) now follows as before. □

10. Extensions, Generalizations, and Speculations

A. Surfaces of variable negative curvature. Our main results, Theorems 1–3, extend routinely to compact surfaces equipped with Riemannian metrics of negative sectional curvature; the normalizing constant $\kappa_g$ should then be changed to $1/(2\pi \text{area}(S))$. The arguments given above for hyperbolic surfaces mostly carry over with little change. (See [21] for an explanation of how Series’ symbolic dynamics for the geodesic flow can be extended to variable negative curvature. The key calculation, in Proposition 5.1, also works in variable curvature, with appropriate modification of constants, as it uses only the fact that the Liouville measure is locally the product of area measure with Lebesgue angular measure.)

B. Finite-area hyperbolic surfaces with cusps. We expect also that Theorems 1–3 extend to finite-area hyperbolic surfaces with cusps. For this, however, genuinely new arguments would seem to be needed, as our analysis for the compact case relies heavily on the symbolic dynamics of Proposition 3.1 and the regenerative representation of Gibbs states (Proposition 3.2). The geodesic flow on the modular surface has its own very interesting symbolic dynamics (cf. for example [30] and [1]), but this uses a countably infinite alphabet (the natural numbers) rather than a finite alphabet. At present there seems to be no analogue of the regenerative representation theorem (Proposition 3.2) for Gibbs states on sequence spaces with infinite alphabets.

C. Tessellations by closed geodesics. It is known that statistical regularities of “random” geodesics (where the initial tangent vector is chosen from the maximal-entropy invariant measure for the geodesic flow) mimic those of typical long closed geodesics. This correspondence holds for first-order statistics (cf. [3]), but also for second-order statistics (i.e., “fluctuations”: see [18], [19], [21]). Thus, it should be expected that Theorems 1–3 have analogues for long closed geodesics. In particular, we conjecture the following.
Conjecture 1. Let $S$ be a closed hyperbolic surface, and let $x \in S$ be a fixed point on $S$. From among all closed geodesics of length $\leq T$ choose one — call it $\gamma_T$ — at random, and let $A_T$ be the intersection of $\gamma_T$ with the ball $D(x; \alpha T^{-1})$. Then as $T \to \infty$ the random collection of arcs $A_T$ converge in distribution to a Poisson line process on $B(0; \alpha)$ of intensity $\kappa_g$.

We do not expect that this will be true on a surface of variable negative curvature, because the maximal-entropy invariant measure for the geodesic flow coincides with the Liouville measure only in constant curvature.

D. Tessellations by several closed geodesics. Given Conjecture[1], it is natural to expect that if two (or more) closed geodesics $\gamma_T, \gamma_T'$ are chosen at random from among all closed geodesics of length $\leq T$, the resulting tessellations should be independent. Thus, the intersections of these tessellations with a ball $D(x, \alpha T^{-1})$ should converge jointly in law to independent Poisson line processes of intensity $\kappa_g$.

APPENDIX A. POISSON LINE PROCESSES

Proof of Lemma[2,1] Rotational invariance is obvious, since the angles $\Theta_n$ are uniformly distributed, so it suffices to establish invariance by translations along the $x$–axis. To accomplish this, we will exhibit a sequence $\mathcal{L}_m$ of line processes that converge pointwise to a Poisson line process $\mathcal{L}$, and show by elementary means that each $\mathcal{L}_m$ is translationally invariant.

Let $\{(R_n, \Theta_n)\}_{n \in \mathbb{Z}}$ and $\{\Theta_n\}_{n \in \mathbb{Z}}$ be the Poisson point process used in the construction (1) of $\mathcal{L}$. For each $m = 3, 5, 7, \ldots$, let $A_m = \{k \pi / m\}_{0 \leq k < m}$ (the restriction to odd $m$ prevents $\pi / 2$ from occurring in $A_m$). For each $n \geq 1$, let $\Theta_n^m = [m \Theta_n] / m$ be the nearest point in $A_m$ less than $\Theta_n$. By construction, for each $m$ the random variables $\Theta_n^m$ are independent and identically distributed, with the uniform distribution on the finite set $A_m$. Now define $\mathcal{L}_m$ to be the line process constructed in the same manner as $\mathcal{L}$, but using the discrete random variables $\Theta_n^m$ instead of the continuous random variables $\Theta_n$. Clearly, as $m \to \infty$ the sequence $\mathcal{L}_m$ of line processes converges to $\mathcal{L}$.

It remains to show that each of the line processes $\mathcal{L}_m$ is invariant by translations along the $x$–axis. For this, observe that for each $\theta_k \in A_m$ the thinned process $R_{m,k}$ consisting of those $R_n$ such that $\Theta_n^m = \theta_k$ is itself a Poisson point process on $\mathbb{R}$ of intensity $\lambda / m$, and that these thinned Poisson point processes are mutually independent[11] Consequently, the line process $\mathcal{L}_m$ is the superposition of $m$ independent line processes $\mathcal{L}_{m,k}$, with $k = 1, 2, \ldots, m$, where $\mathcal{L}_{m, k}$ is the subset of all lines in $\mathcal{L}_m$ that meet the $x$–axis at angle $\pi / 2 - \theta_k$. Since the constituent processes $\mathcal{L}_{m, k}$ are independent, it suffices to show that for each $k$ the line process $\mathcal{L}_{m, k}$ translation-invariant. But this is elementary: the points where the lines in $\mathcal{L}_{m, k}$ meet the $x$–axis form a Poisson point process on the real line, and Poisson point processes on the real line of constant intensity are translation-invariant.

□

Proof of Corollary[2,3] By rotational invariance, it suffices to show this for the $x$–axis. Let $\mathcal{L}_m$ and $\mathcal{L}_m^k$ be as in the proof of Lemma[2,1], then by an easy calculation, the point process

[11]The thinning and superposition laws are elementary properties of Poisson point processes. The thinning law follows from the superposition property; see Kingman[15] for a proof of the latter.
of intersections of the lines in $\mathcal{L}_m^k$ with the $x$–axis is a Poisson point process of intensity $(\lambda/m)\sin\theta_k$. Summing over $k$ and then letting $m \to \infty$, one arrives at the desired conclusion. □

**Proof of Proposition 2.5.** The hypothesis that $\Gamma$ encloses a strictly convex region guarantees that if a line intersects both $A$ and $B$ then it meets each in at most one point. Denote by $L_{\{A,B\}}$ the set of all lines that intersect both $A$ and $B$. If $A$ and $B$ are partitioned into non-overlapping sub-arcs $A_i$ and $B_j$, then $L_{\{A,B\}}$ is the disjoint union $\bigcup_{i,j} L_{\{A_i,B_j\}}$. Since the sets $L_{\{A_i,B_j\}}$ are piecewise disjoint, the corresponding regions of the strip $\mathbb{R} \times [0, \pi)$ (in the standard parametrization) are non-overlapping, and so, by a defining property of the Poisson point process $\{(R_n, \Theta_n)\}_{n \in \mathbb{Z}}$, the counts $N_{\{A,B\}}$ are independent Poisson random variables. Since the sum of independent Poisson random variables is Poisson, to finish the proof it suffices to show that for arcs $A, B$ of length $< \epsilon$ the random variables $N_{\{A,B\}}$ are Poisson, with means $\lambda\beta_{A,B}$.

If $\epsilon > 0$ is sufficiently small then any line $L$ that intersects two boundary arcs $A, B$ of length $\leq \epsilon$ must intersect the two straight line segments $\tilde{A}, \tilde{B}$ connecting the endpoints of $A$ and $B$, respectively; conversely, any line that intersects both $\tilde{A}, \tilde{B}$ will intersect both $A, B$. Therefore, we may assume that the arcs $A, B, A_i, B_j$ are straight line segments of length $\leq \epsilon$. Because Poisson line processes are rotationally invariant, we may further assume that $A$ is the interval $[\pm \epsilon/2, \epsilon/2] \times \{0\}$.

We now resort once again to the discretization technique used in the proof of Lemma 2.1. For each $m = 3, 5, 7, \ldots$, let $N_{\{A,B\}}^m$ be the number of lines in the line process $\mathcal{L}_m$ that cross the segments $A, B$. Clearly, $N_{\{A,B\}}^m \to N_{\{A,B\}}$ as $m \to \infty$, so it suffices to show that for each $m$ the random variable $N_{\{A,B\}}^m$ has a Poisson distribution with mean $\mu_m \to \lambda\beta_{A,B}$.

Recall that the line process $\mathcal{L}_m$ is a superposition of $m$ independent line processes $\mathcal{L}_m^k$, and that for each $k$ the lines in $\mathcal{L}_m^k$ all meet the $x$–axis at a fixed angle $|\pi/2 - \theta_k|$. Hence, $N_{\{A,B\}}^m = \sum_k N_{\{A,B\}}^{m,k}$, where $N_{\{A,B\}}^{m,k}$ is the number of lines in $\mathcal{L}_m^k$ that cross both $A$ and $B$. The random variables $N_{\{A,B\}}^{m,k}$ are independent; thus, to show that $N_{\{A,B\}}^m$ has a Poisson distribution it suffices to show that each $N_{\{A,B\}}^{m,k}$ is Poisson. By construction, the lines in $\mathcal{L}_m^k$ meet the line $(s \cos \theta_k, s \sin \theta_k)_{s \in \mathbb{R}}$ at the points of a Poisson point process of intensity $\lambda/m$; consequently, they meet the $x$–axis at the points of a Poisson point process of intensity $\lambda|\cos \theta_k|/m$. Now a line that meets the $x$–axis at angle $\theta_k$ will cross both $A = [\pm \epsilon/2, \epsilon/2] \times \{0\}$ and $B$ if and only if its point of intersection with the $x$–axis lies in the $\theta_k$–shadow $J_k$ of $B$ on $A$. Therefore, $N_{\{A,B\}}^{m,k}$ has the Poisson distribution with mean $\lambda|J_k \cos \theta_k|/m = \lambda|\cos \theta_k|/m = \lambda\beta_{A,B}$. 

29
\[ \lambda \psi_{A,B}(\theta_k). \] It follows that \( N_{A,B}^m \) has the Poisson distribution with mean
\[
EN_{A,B}^m = m^{-1} \sum_{k=0}^{m-1} \lambda |J_k \cos \theta_k|
\]
\[
= m^{-1} \sum_{k=0}^{m-1} \lambda \psi_{A,B}(\theta_k)
\]
\[
\longrightarrow \frac{\lambda}{\pi} \int_{-\pi/2}^{\pi/2} \psi_{A,B}(\theta) \, d\theta.
\]

**Proof of Corollary 2.6** It suffices to prove this for disks of small radius, because by the translation-invariance of \( L \),
\[
EV(D) \sim \frac{1}{\pi \varrho^2} \int_D EV(B(x, \varrho)) \, dx = EV(B(0, \varrho))|D|/(\pi \varrho^2)
\]
as \( \varrho \to 0 \). Let \( \gamma \) be a chord of \( B(0, \varrho) \), and \( H_{\gamma} \) the event that \( \gamma \in L \cap B(0, \varrho) \). Conditional on \( H_{\gamma} \), the number of intersection points on \( \gamma \) is Poisson with mean \( 2\lambda |\gamma|/\pi \), by Corollary 2.3 and Proposition 2.5. Therefore,
\[
EV(B(0, \varrho)) = \frac{1}{2} \frac{2\lambda}{\pi} E \left( \sum_{\gamma \in L \cap B(0, \varrho)} |\gamma| \right) = \frac{\lambda}{\pi} E \Lambda(L \cap B(0, \varrho)).
\]
(The factor of 1/2 accounts for the fact that each intersection point lies on two chords.)

The expectation \( E\Lambda(L \cap B(\cup, \varrho)) \) is easily evaluated using the standard construction of the Poisson line process (Definition 1.1). The lines of \( L \) that cross \( B(0, \varrho) \) are precisely those corresponding to points \( R_n \) such that \( -\varrho < R_n < \varrho \). For any such \( R_n \), the length of the chord \( \gamma = \gamma_n \) is \( |\gamma_n| = 2\sqrt{\rho^2 - R_n^2} \). Therefore,
\[
E\Lambda(L \cap B(0, \varrho)) = \lambda \int_{\varrho}^{\varrho} 2\sqrt{\rho^2 - r^2} \, dr = \lambda \pi \rho^2.
\]

**Proof of Proposition 2.7** Let \( L \) be the Poisson line process with intensity \( \lambda \), and denote by \( \tau_z \) the translation by \( z \in \mathbb{R}^2 \). It suffices to prove that for any two bounded, continuous functions \( f, g : \mathcal{C} \to \mathbb{R} \),
\[
\lim_{|z| \to \infty} E f(L) g(\tau_z L) = E f(L) E g(L).
\]

---

\[ \text{[12]} \] The event \( H_\gamma \) has probability 0, but it is the limit of the positive-probability events that \( L \) has a line which intersects small boundary arcs centered at the endpoints of \( \gamma \). The conditional distribution of \( L \) given \( H_\gamma \) can be interpreted as the limit of the conditional distributions given these approximating events. The independence assertion of Proposition 2.5 guarantees that, conditional on \( H_\gamma \), the distribution of \( L \cap B(0, \varrho) \) is the same as the unconditional distribution of \( (L \cap B(0, \varrho)) \cup \{\gamma\} \).
Since the Poisson line process is rotationally invariant, it suffices to consider only translations $\tau_z$ for $z = (x, 0)$ on the $x$-axis. Moreover, since continuous functions that depend only on the restrictions of configurations to balls are dense in the space of all bounded, continuous functions, it suffices to establish (34) for functions $f, g$ that depend only on configurational restrictions to the ball of radius $r > 0$ centered at the origin.

To prove (34), we will show that on some probability space there are Poisson line processes $\mathcal{L}, \mathcal{L}', \mathcal{L}''$, each with intensity $\lambda$, such that

(a) the line processes $\mathcal{L}'$ and $\mathcal{L}''$ are independent;
(b) $f(\mathcal{L}) = f(\mathcal{L}')$ with probability one; and
(c) $g(\tau_z \mathcal{L}) = g(\tau_z \mathcal{L}'')$ with probability $\to 1$ as $|z| \to \infty$.

It will then follow, by translation invariance, that

$$|Ef(\mathcal{L})g(\tau_z \mathcal{L}) - Ef(\mathcal{L})g(\mathcal{L})| = |Ef(\mathcal{L})g(\tau_z \mathcal{L}) - Ef(\mathcal{L}')g(\mathcal{L}'')|$$

$$= |Ef(\mathcal{L})g(\tau_z \mathcal{L}) - Ef(\mathcal{L}')g(\tau_z \mathcal{L}'')|$$

$$\leq 2||f||_\infty ||g||_\infty P\{g(\tau_z \mathcal{L}) \neq g(\tau_z \mathcal{L}'')\} \to 0.$$ 

The line processes $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ can be built on any probability space that supports independent Poisson point processes $\{R'_n\}_{n \in \mathbb{Z}}$ and $\{R''_n\}_{n \in \mathbb{Z}}$ on $\mathbb{R}$ of intensity $\lambda$, and independent sequences $\{\Theta'_n\}_{n \in \mathbb{Z}}$ and $\{\Theta''_n\}_{n \in \mathbb{Z}}$ of random variables uniformly distributed on the interval $[-\pi, \pi]$. Let $\mathcal{L}'$ be the line process obtained by using the “standard construction” (that is, the construction explained in Definition 1.1) with the point process $\{R'_n\}_{n \in \mathbb{Z}}$ and the accompanying uniform random variables $\{\Theta'_n\}_{n \in \mathbb{Z}}$, and let $\mathcal{L}''$ be the line process obtained by the standard construction using the point process $\{R''_n\}_{n \in \mathbb{Z}}$ and the random variables $\{\Theta''_n\}_{n \in \mathbb{Z}}$. Clearly, $\mathcal{L}'$ and $\mathcal{L}''$ are independent.

The line process $\mathcal{L}$ is constructed by splicing the marked Poisson point processes $\mathcal{R}' = \{(R'_n, \Theta'_n)\}_{n \in \mathbb{Z}}$ and $\mathcal{R}'' = \{(R''_n, \Theta''_n)\}_{n \in \mathbb{Z}}$ as follows: in the interval $(-r, r)$, use the marked points of $\{(R'_n, \Theta'_n)\}_{n \in \mathbb{Z}}$; but in $\mathbb{R} \setminus (-r, r)$, use the marked points of $\{(R''_n, \Theta''_n)\}_{n \in \mathbb{Z}}$. Thus, the resulting marked point process $\mathcal{R} = \{(R_n, \Theta_n)\}_{n \in \mathbb{Z}}$ consists of (i) all pairs $(R'_n, \Theta'_n)$ such that $-r < R'_n < r$, and (ii) all pairs $(R''_n, \Theta''_n)$ such that $R''_n \notin (-r, r)$. By standard results in the elementary theory of Poisson processes, the marked point process $\mathcal{R}$ has the same distribution as $\mathcal{R}'$ and $\mathcal{R}''$, in particular, $\{R_n\}_{n \in \mathbb{Z}}$ is a rate-$\lambda$ Poisson point process on $\mathbb{R}$, and the random variables $\{\Theta_n\}_{n \in \mathbb{Z}}$ are independent and uniformly distributed on $[-\pi, \pi]$.

Let $\mathcal{L}$ be the Poisson line process constructed using $\mathcal{R}$.

It remains to show that the Poisson line processes $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ satisfy properties (b) and (c) above. Observe first that in the standard construction (Definition 1.1), only those pairs $(R_n, \Theta_n)$ such that $R_n \in (-r, r)$ will produce lines that intersect the ball $B(0, r)$ of radius $r$ centered at the origin. Consequently, the restrictions of $\mathcal{L}$ and $\mathcal{L}'$ to $B(0, r)$ are equal; since $f$ depends only on the configuration in $B(0, r)$, it follows that $f(\mathcal{L}) = f(\mathcal{L}')$.

Next, consider the configurational restrictions of $\mathcal{L}$ and $\mathcal{L}''$ to the ball $B((x, 0), r)$ for $x \gg 2r$. In the standard construction, a pair $(R_n, \Theta_n)$ such that $R_n \in (-r, r)$ will produce a line of $\mathcal{L}$ that intersects $B((x, 0), r)$ only if $|\tan \Theta_n| \leq r/(x - 2r)$. The probability that there is such a pair, in either $\mathcal{R}$ or $\mathcal{R}''$, tends to zero as $x \to \infty$; hence, with probability $\to 1$, the restrictions of $\mathcal{L}$ and $\mathcal{L}''$ agree in $B((x, 0), r)$, and on this event $g(\mathcal{L}) = g(\mathcal{L}'')$. \qed
Proof of Corollary 2.9. The number of lines in a Poisson line process $L$ that intersect a given line segment of length $m$ has the Poisson distribution with mean $C\lambda m$, where $C$ is a finite positive constant not depending on either $m$ or $\lambda$. Consequently, the probability that the number of polygons in the induced tessellation of the plane intersecting one of the four sides of $[-n, n]^2$ exceeds $n^{3/2}$ is exponentially small in $n$.

Given a line configuration $L$, let $1/f(L)$ be the area of the polygon containing the origin in the induced tessellation. (This is well-defined and positive with probability 1.) Let $A_n^-$ be the union of all polygons of the tessellation that lie entirely in the open square $(-n, n)^2$, and let $A_n^+$ be the union of the polygons that intersect $[-n, n]^2$. Then

$$\int_{A_n^-} f(\tau_z L) \, dz \quad \text{and} \quad \int_{A_n^+} f(\tau_z L) \, dz$$

count the number of polygons in $A_n^-$ and $A_n^+$, respectively; since the difference between these is less than $n^{3/2}$, except with exponentially small probability, it follows that except with small probability

$$|F_n - \int_{[-n, n]^2} f(\tau_z L) \, dz| \leq n^{3/2}.$$

Hence, by the multi-parameter ergodic theorem (see, for example, [6]), $F_n/n^2 \to Ef(L)$ almost surely.

The proof of the assertion regarding empirical frequencies of $k$-gons is similar. If $G_k$ is the event that the polygon containing the origin is a $k$-gon, then the total number of $k$-gons in the region $A_n^\pm$ is

$$\int_{A_n^\pm} (f1_{G_k})(\tau_z L) \, dz.$$

Hence, the ergodic theorem implies that the number of $k$-gons divided by $n^2$ converges to $E(f1_{G_k}(L))$, and it follows that the fraction of $k$-gons converges to

$$\phi_k = \frac{E(f1_{G_k}(L))}{Ef(L)}.$$

Now consider the number of vertices $V_n$. Because there is probability 0 that three distinct lines of a Poisson line process meet at a point, all interior vertices are shared by exactly 4 edges, and each edge is incident to two vertices; thus, since the number of vertices on the boundary of the square is $O(n^{3/2})$, we have $\mathcal{E}_n = 2V_n + O(n^{3/2})$. By Euler’s formula, $V_n - \mathcal{E}_n + F_n = 1$, so $V_n = F_n + O(n^{3/2})$; hence,

$$\lim_{n \to \infty} V_n/n^2 = \lim_{n \to \infty} N_n/n^2.$$

The value of the limit is determined by Corollary 2.6, which implies that $EV_n = 4\lambda^2 n^2/\pi$. □

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