Pricing TARNs Using a Finite Difference Method

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Target accumulation redemption notes (TARNs) have become very popular products among Asian foreign exchange investors due to their flexibility to be structured to suit any foreign exchange outlook. TARN payoff is path dependent, and typically practitioners use the Monte Carlo method to evaluate TARN prices. This article describes a finite difference scheme for pricing a TARN option. Key steps in the proposed scheme involve tracking multiple one-dimensional finite difference solutions, applying jump conditions at each cash flow exchange date, and a cubic spline interpolation of results after each jump. Since a finite difference scheme for TARNs has significantly different features from a typical finite difference scheme for options with a path-independent payoff, we give a step-by-step description on the implementation of the scheme, which is not available in the literature. The advantages of the proposed finite difference scheme over the Monte Carlo method are illustrated by examples using three different knockout types. In the case of constant or time-dependent volatility models (where Monte Carlo requires simulation at cash flow dates only), the finite difference method can be faster than the Monte Carlo method by an order of magnitude while achieving the same accuracy in price. The finite difference method can be even more efficient in the case of a local volatility model because the Monte Carlo method requires a significantly larger number of time steps. In terms of robust and accurate estimation of Greeks, the advantage of the finite difference method is even more pronounced.

PATH-DEPENDENT OPTIONS HAVE PAYOFFS DEPENDING ON THE TRAJECTORY FOLLOWED BY ONE OR MORE OF THE UNDERLYING PROCESSES. THE MOST STRAIGHTFORWARD AND EASIEST TO IMPLEMENT NUMERICAL SOLUTION FOR PRICING PATH-DEPENDENT OPTIONS IS BASED ON THE MONTE CARLO METHOD. IN THE CONTEXT OF PRICING PATH-DEPENDENT OPTIONS BY SOLVING PARTIAL DIFFERENTIAL EQUATIONS (PDE), TWO ADDITIONAL CHALLENGES MAY EMERGE DUE TO THE PRESENCE OF PATH DEPENDENCY. FIRST, THE DEPENDENCY MAY INTRODUCE NEW DIMENSIONS TO THE PARTIAL DIFFERENTIAL EQUATION. SECOND, IT MAY CAUSE THE RESULTING EQUATION TO BE MUCH MORE DIFFICULT TO SOLVE BECAUSE OF THE LACK OF DIFFUSION IN THE ADDITIONAL DIMENSIONS. FOR SOME DETAILED DISCUSSIONS, SEE TAVELLA AND RANDALL [2000], ZVAN ET AL. [1998], AND WILMOTT [2000b].

The nature of the path-dependent option pricing problem largely depends on whether we have a continuous or discrete sampling for the path. In general, a continuous sampling model of path dependency introduces additional convection terms in PDE, whereas for a discretely sampled path-dependent option, the convection terms are replaced by jump conditions. There are many successful attempts in pricing discretely sampled path-dependent options by the PDE approach using a lattice-based method (e.g., binomial and trinomial trees such as used in Ritchken et al. [1993], Hull and White...
[1993], Barraquand and Pudet [1996], and Forsyth et al. [2002] or a similarly finite volume or finite element method (Forsyth et al. [1999]; Zvan et al. [2001]). Most of these studies consider Asian or lookback options. The convergence study by Forsyth et al. [2002] comparing tree-based and PDE methods showed that, in the case of discrete path sampling, PDE methods are superior in terms of number of computational steps and in their flexibility of handling more complex path-dependent features and easier adapting to alternative stochastic processes. Also, in the case when transition probability density of the underlying asset or its moments is known between discrete sampling dates, pricing path dependent options can be accomplished by a recently developed method based on Gauss–Hermite quadratures; for details, see Luo and Shevchenko [2014].

A target accumulation redemption note (TARN) provides a capped sum of payments over a period with the possibility of early termination (knockout) determined by the target level imposed on the accumulated amount. A certain amount of payment (e.g., spot value minus the strike) is made on a series of cash flow dates (referred to as fixing dates) until the target level is breached. The payoff function of a TARN is path dependent in that the payment on a fixing date depends on the spot value of the asset as well as on the accumulated payment amount up to the fixing date. Typically, commercial software solutions for pricing a TARN are based on the Monte Carlo method. This article presents a finite difference scheme as an alternative to the Monte Carlo method to evaluate TARNs. The focus is on the step-by-step implementation of the finite difference scheme, which is not readily available in the literature, and on the comparison of performance of the proposed scheme relative to the Monte Carlo method. We are not aware of any finite difference scheme published in the literature, although a general outline of a PDE approach to pricing TARNs can be found in Piterbarg [2004]. Note that it should be possible to develop a tree-based method for pricing TARNs somewhat similarly to pricing discrete Asian options, but it goes beyond the purpose of this article and will be done elsewhere.

Without losing generality, we assume the underlying asset is the foreign exchange (FX) rate. The definitions of TARN options with three different knockout types and some key notations are introduced in the following section. FX rate models are described, a finite difference scheme for TARNs is presented, and numerical results for both the finite difference and Monte Carlo methods are given in the subsequent sections. We conclude in the final section.

**TARN PAYOFF DEFINITION**

There are different versions of TARN products used in FX trading. For simplicity, here we consider one specific form of TARN. The presented finite difference scheme can easily be adapted to other more general forms of TARNs, as discussed later. Denote the FX rate at time \( t \) as \( S(t) \) and other notation as follows: \( t_0 \) is today’s date; \( K \) is the number of fixing dates (cash flow dates); \( t_1, t_2, \ldots, t_K \) are fixing dates; \( X \) is strike; \( U \) is the target accrual level; \( S(t_1), S(t_2), \ldots, S(t_K) \) are FX rate values at fixing dates \( t_1, t_2, \ldots, t_K \); \( A(t) \) is accumulated amount at time \( t \); and all amounts are per unit of notional foreign amount. On each fixing date \( t_k \), there is a cash flow payment

\[
\tilde{C}_k = \beta(S(t_k) - X) \times 1_{\{X \leq U\}} 
\]

where \( \beta \) is a strategy on foreign currency (\( \beta = 1 \) corresponds to buy and \( \beta = -1 \) corresponds to sell), subject to the target level \( U \) is not breached by the accumulated amount \( A(t_k) \). If the target level \( U \) is breached before or on the last fixing date, denote \( t_K \) as the first fixing date when the target is breached; that is,

\[
\tilde{K} = \min\{k : A(t_k) \geq U\}, \quad k = 1, 2, \ldots, K
\]

Otherwise, set \( \tilde{K} = K \). The actual payment on the fixing date \( t_k \leq t_K \) can be written as

\[
\tilde{C}_k(S(t_k), A(t_{k-1})) = \tilde{C}_k \times 1_{\{X \leq U\}} + W_k \times 1_{\{X > U\}}
\]

and \( C_k = 0 \) for \( t_k > t_K \). Here, \( A(t_{k-1}) \) is the accumulation amount immediately after the fixing date \( t_{k-1} \), and \( W_k \) is the weight depending on the type of the knockout when the target level \( U \) is breached. The accumulated amount \( A(t) \) is a piece-wise constant function \( A(t) = A(t_{k-1}) \), \( t_{k-1} \leq t < t_k \), with

\[
A(t_k) = A(t_{k-1}) + C_k(S(t_k), A(t_{k-1}))
\]

There are three knockout types used in practice.
1. **Full gain.** When the target is breached on a fixing date \( t_k \), the cash flow payment on that date is allowed. This essentially permits the breach of the target once, and the total payment may exceed the target for full gain knockout.

2. **No gain.** When the target is breached, the entire payment on that date is disallowed. The total payment will never reach the target for no gain knockout.

3. **Part gain.** When the target is breached on a fixing date \( t_k \), part of the payment on that date is allowed, such that the target is met exactly.

Formally, it can be represented by the following definition of the weight:

\[
W_k = \begin{cases} 
1, & \text{if knockout type = full gain;} \\
0, & \text{if knockout type = no gain;} \\
\frac{\beta \times (S(t_k) - X)}{U - A(t_{t_k-1})}, & \text{if knockout type = part gain.}
\end{cases}
\]  

The present value (discounted value) of the TARN payoff in domestic currency for FX realization \( S = (S(t_1), S(t_2), \ldots, S(t_K)) \) is then

\[
P(S) = \sum_{k=1}^{K} C_k \left( S(t_k), A(t_{t_k-1}) \right) \frac{1}{B_k(t_0, t_k)}, \quad A(t_0) = 0 \tag{6}
\]

where \( [B_k(t_0, t_k)]^{-1} \) is domestic discounting factor from the fixing date \( t_0 \) to \( t_k \).

Other forms of TARNs used in trading include modifications of cash flow payments (see Equation (1)) and the accumulated amount rule (Equation (4)). In the present study, the cash flow payment on each fixing date is the same as the increment in the accumulated amount, both are represented by \( C_k \left( S(t_k), A(t_{t_k-1}) \right) \). In other forms of TARNs, the two quantities can differ, but this should cause no additional difficulties for the finite difference method presented here, as will be further discussed later in this article.

**FX MODEL**

Under the standard no-arbitrage option pricing methodology, today's fair price of a TARN is calculated as the expectation of payoff (Equation (6)) under the risk-neutral process. Specifically, we consider the risk-neutral process,

\[
\frac{dS(t)}{S(t)} = (r_f - r_d)dt + \sigma dW
\]

where \( r_f \) and \( r_d \) are domestic and foreign local (instantaneous) interest rates, \( \sigma \) is the local (instantaneous) volatility, and \( W \) is the standard Brownian motion. The expectation can be calculated using Monte Carlo by simulating the risk-neutral process (Equation (7)) many times and averaging the payoff realizations; or by solving corresponding PDE via the finite difference method. Here, the local interest rates can be constant or functions of time \( r_f = r_f(t), r_d = r_d(t) \); and volatility can be constant, a function of time \( \sigma = \sigma(t) \), or a function of time and FX rate, \( \sigma = \sigma(S(t), t) \). The last case corresponds to the local volatility model that can be calibrated to match observed implied volatility surface; see, for example, Wilmott [2000a].

**FINITE DIFFERENCE NUMERICAL SCHEME**

Let \( V(S(t, A)) \) be the value of TARN for spot rate \( S \) and accumulated amount \( A \) at time \( t \). Because the path-dependent quantity \( A \) is monitored discretely, there are no new diffusion terms, and the standard option pricing PDE is still valid between fixing dates:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 (S(t)) \frac{\partial^2 V}{\partial S^2} + (r_f(t) - r_d(t)) S(t) \frac{\partial V}{\partial S} - r_f(t)V = 0
\]

Typically, the PDE solution for option pricing requires final conditions (the payoff) and boundary conditions (e.g., at zero or at a barrier). For discretely sampled path-dependent options, additional jump conditions apply. Unlike in the case of path-independent options where the payoff at expiry is known a priori and typically the final condition is of the Dirichlet type with the value of the payoff, in the case of the TARN option, the final payoff is not known a priori. The expiry time is simply the last fixing time, and the final payoff depends on the path of the underlying up to the expiry time. Immediately after the final payoff, the option is worthless, and we can set the final condition to zero at \( T \):
\[ V(S, T, A) = 0 \]

where \( T = t_k \) is the last monitoring time. Applying a proper jump condition from \( T^- \) to \( T^+ \) will give us a more informative final condition at \( T^- \), where \( T^- \) is the time infinitesimally before the last monitoring time \( t_k = T \). Unfortunately, any single solution of Equation (8) based on a given final condition at \( T^- \) will not lead to the correct answer to the TARN option pricing, even if we know the final jump amount. We need multiple solutions to Equation (8) with different final payoffs or jumps. Across any fixing date, there is a discontinuous but predictable jump in the accumulated amount. In such a case, the no-arbitrage principle dictates that there must be a proper jump condition imposed on the path-dependent option values. The jump value \( C_j \) given by Equation (4) is the cash flow to the TARN owner, thus

\[
V(S, t_k, A(t_k)) = V(S, t_k, A(t_k) + C_j(S, A(t_k))) + C_j(S, A(t_k))
\]

Finally, the PDE solution will give us today’s TARN price: \( V(S(t_0), t_n, A(t_0)) \).

**Jump Condition Application**

Let us introduce an auxiliary finite grid \( A(t_k) = A_1 < A_2 < \cdots < A_J = U \) to track the accumulated amount \( A \), where \( J \) is the total number of nodes in the accumulated amount coordinate. The upper limit \( U \) is needed because the accumulated amount cannot exceed the target \( U \). For each \( A_j \), we associate a continuous finite difference solution to the one-dimensional PDE from Equation (8). For finite difference solution, at every jump we let \( A \) be one of the grid points \( A_j, 1 \leq j \leq J \). Because \( A \) is always known at each jump to be one of the fixed nodal point values, there is no need to continuously track the actual evolution of the accumulated amount \( A \) during the entire finite difference solving process.

Denote finite difference grid points in the \( S \) variable as \( S_0, S_1, \ldots, S_M \), where \( M \) is the total number of nodes in the \( S \) coordinate. For any \( S = S_m, m = 1, \ldots, M \), substituting \( A(t_k) \) with \( A_j, j = 1, \ldots, J \) in Equation (9), we get the following:

\[
V(S_m, t_k^+, A_j) = V(S_m, t_k^-, A_j) + C_j(S_m, A_j) + C_j(S_m, A_j)
\]

where \( t_k^- \) denotes the time infinitesimally before the monitoring time \( t_k \). In Equation (10), we have let the accumulated amount before the \( k \)-th payment at \( t_k \) to be one of the grid point \( A_j \). Equation (10) describes a forward jump from \( t_k^- \) to \( t_k^+ \).

Because backward time marching is carried out for the finite difference solution of PDE (8) associated with a fixed node point \( A_j \), intuitively the jump should be applied backwards from \( t_k^+ \) to \( t_k^- \). That is, in finite difference solution the value of \( A \) at \( t_k^- \) is known to be one of the grid point \( A_j \), and after a backward jump from \( t_k^+ \) to \( t_k^- \), the value of \( A \) changes from \( A_j \) to \( A_j \). This backward jump can be expressed as

\[
V(S_m, t_k^-, A_j) = V(S_m, t_k^+, A_j) + C_j(S_m, A_j)
\]

In both Equations (10) and (11), \( C_j(S, A) \) is calculated according to Equation (3). Exhibit 1 illustrates the application of the jump condition, as found in Equation (10).

**Tracking Finite Difference Solutions**

The idea is tracking \( J \) finite difference (FD) solutions corresponding to the \( J \) grid points for the auxiliary variable, the accumulated amount. For each fixed accumulated amount \( A_j, 1 \leq j \leq J \), we start solving PDE by the finite difference scheme with the final condition \( V(S, T, A) = 0 \) and a final jump condition from \( t_k^+ = T \) to \( t_k^- = t_k \). The implementation of this idea is not straightforward, because at each sampling time, the jump condition as found in Equation (11) has to be applied and the accumulated amount after each jump changes accordingly and falls off the grid points \( A_j, 1 \leq j \leq J \). Not only the accumulated amount changes, the amount of changes differ for different grid points in the underlying space. As shown in Equation (11), because \( A_j \) is not a constant, the solution \( V(S_m, t_k, A_j) \) obtained after the jump does not correspond to any grid point in the auxiliary variable space. Worse still, the set of values \( V(S_m, t_k, A_j), m = 1, \ldots, M \) does not correspond to any
continuous finite difference solution of the one-dimensional PDE—it does not satisfy the PDE because the value $A_j$ is scattered all over the place, not associated with any unique value. This is because for the option value to satisfy the one-dimensional PDE, it requires a unique accumulated amount at any time—for consistency one cannot have different accumulated amounts at the same time.

For the $M$ grid points in $S$ space, $V(S_m,t_k,A_j)$, $m = 1,\ldots,M$ correspond to $M$ different scenarios of payoffs. Before the jump, the $M$ values $V(S_m,t_k,A_j)$ are related to each other through the PDE, because they are all associated with the same accumulated amount $A_j$. The connection between the $M$ values is broken after the jump.

**Reversal of the jump direction.** Intuitively, jump conditions should be applied through Equation (11). That is, as the backward marching is performed for each of the $J$ solutions corresponding to $A_j$, $1 \leq j \leq J$, at any crossing of sampling time from $t_k$ to $t_{k-1}$, $A_j$ jumps to $A_{j-1}$ and the solution $V(S_m,t_k,A_j)$ jumps to $V(S_m,t_{k-1},A_{j-1})$ according to Equation (11). We can then interpolate from $V(S_m,t_k,A_j)$ to obtain $V(S_m,t_{k-1},A_{j-1})$ and continue time marching backwards until the next sampling date.

Unfortunately, the intuitive application of jump conditions as described in the previous paragraph is problematic in two important ways. First, from Equation (11), it is possible to get $A_{j-1} < A(t_k)$, which is invalid (meaningless) and out of the range of the auxiliary variable space. Second, because $A(t_k) \leq A_j \leq U$ and (11) for
$A_j$ is a decreasing function, the target $U$ will never be exceeded by any of the jump according to Equation (11), thus there is no way to apply the different knockout conditions for the different knockout types as specified in Equation (5). In other words, applying jump condition using (11) cannot get the correct answers to any of the knockout types of TARN. Essentially, applying (11) artificially restricts the boundary for the auxiliary variable to be within the target, instead of letting the underlying process breach the target. Another minor issue is that the jump condition (11) is implicit in $A_j^+$, that is, strictly speaking the jump amount $C_i(S_m, A_j)$ is not known before the backward jump.

The remedy to these problems is actually quite simple—we reverse the direction of the jump. Jump condition (9) is true for any values of the auxiliary variable in the range $A(t_k) \leq A \leq U$; that is, we do not have to use $A_j$, a grid point, on the right-hand side as in Equation (11). Instead, we could use Equation (10) to have the value of $A$ at $t_k$ be one of the grid point $A_j$. Now Equation (10) is explicit in $A_j$. Since grid point $A_j$ satisfies $A(t_k) \leq A \leq U$ and $A_j^+ \geq A_j$, after the jump from $A_j$ to $A_j^+, A_j^+$ will never be negative. Moreover, $A_j^+$ may now exceed the target $U$, allowing the knockout conditions to be imposed. The knockout condition is implied in the calculation of $C_i(S_m, A_j)$ in Equation (10), using Equation (3). Specifically, we have

$$C_i(S_m, A_j) = \tilde{C}_k \times (1_{A_j^+ \leq S_m} + W_1 \times 1_{S_m < A_j^+}) \tag{12}$$

Equation (10) gives the desired solutions $V(S_m, t_k, A_j)$ at the grid points $A_j$, $j = 1, \ldots, J$, given $V(S_m, t_k, A_j^+)$ and $V(S_m, t_k, A_j^-)$, a single tri-diagonal system of equations is solved once for obtaining all the $J$ values $V(S_m, t_k, A_j^+)$, $j = 1, \ldots, J$.

If we perform this interpolation for all the $M$ grid points in spot $S$ and apply jump condition (10), we will have $M \times J$ new values $V(S_m, t_k, A_j)$, $m = 1, \ldots, M$, $j = 1, \ldots, J$. For a fixed $j$, the $M$ new values $V(S_m, t_k, A_j)$ correspond to the PDE solution associated with grid point $A_j$. Given $V(S_m, t_k, A_j)$ for each fixed $A_j$, we can now continue time marching backwards until the next sampling time. The whole algorithm can be summarized as follows:

1. Apply zero final condition at $t = t_k$ for all the $J$ solutions to Equation (8) corresponding to $A_j$, $1 \leq j \leq J$.
2. Apply the jump condition (10) to obtain $A_j^+$ for each of the $J$ solutions at each of the $M$ grid points in spot, beginning with $k = K$ ($t_k = 1$) for the first jump.
3. Perform cubic spline interpolation from points $V(S_m, t_k, A_j)$, $j = 1, \ldots, J$ to new points $V(S_m, t_{k+1}, A_j^+)$ by forming a smooth function from the $J$ values $V(S_m, t_{k+1}, A_j)$ with each spot grid point $S_m$.
4. Apply the jump condition (10); that is, calculate $V(S_m, t_{k+1}, A_j^+)$ from $V(S_m, t_k, A_j^+)$. The knockout condition (a boundary condition in variable $A_j$)

As shown in a convergence study by Forsyth et al. [2002], it is possible for a numerical algorithm for discretely sampled path-dependent option pricing to be non-convergent (or convergent to an incorrect answer) if the interpolation scheme is selected inappropriately. All the previous studies of numerical PDE solution for path-dependent (Asian or lookback) options used either a linear or a quadratic interpolation in applying the jump conditions. In our experience, a better choice is the cubic spline interpolation (Press et al. [1992]). This procedure assumes the $J$ values, $V(S_m, t_{k+1}, A_j)$, $j = 1, \ldots, J$, form a smooth function in the auxiliary variable space and the cubic spline interpolation has a much higher order of accuracy than linear or quadratic interpolation. The error of cubic spline is $O(h^4)$, where $h$ is the size for the spacing of the interpolating variable, assuming a uniform spacing. In our case, $h = \delta A = U/(J - 1)$. Natural boundary conditions are imposed at the two ends $A_1 = 0$ and $A_J = U$, that is, we assume zero second derivative of the spline function at the two ends. For each fixed asset value $S_m$, a single tri-diagonal system of equations is solved once for obtaining all the $J$ values $V(S_m, t_k, A_j^+)$, $j = 1, \ldots, J$. For each fixed $j$, the $M$ new values $V(S_m, t_k, A_j)$ correspond to the PDE solution associated with grid point $A_j$. Given $V(S_m, t_k, A_j)$ for each fixed $A_j$, we can now continue time marching backwards until the next sampling time. The whole algorithm can be summarized as follows:
is implied by the calculation of $C_v(S_n,A_j)$ using Equation (12).

5. Perform the finite difference time marching backwards for each of the $J$ solutions $V(S_n, A_j)$, $j = 1, \ldots, J$, corresponding to the $J$ grid points in the auxiliary variable, until a sampling time is encountered. This gives solution $V(S_n, A_j)$.

6. Repeat steps 2 to 5 until $k = 1$.

7. Take the single solution $V(S_n, A_j)$ to do final time marching until $t = t_0$, and take $V(S_n, A_j)$ as the final solution of the TARN option.

As indicated in Step 1, at the final fixing time $t = t_K$, the zero-value final condition is applied at $T$, and the following jump condition is applied before taking any PDE solving steps:

$$V(S_n, T^-, A_j) = 0 + C_v(S_n, A_j)$$

(13)

For each set of $V(S_n, T^-, A_j)$ with fixed $j$, we begin tracking a finite difference solution through backward time marching.

In Step 7, only a single solution is needed between the first sampling time $t_0$ and the spot date—there is no more need to track all $J$ solutions, because there are no more jump conditions to be applied. For good accuracy, we require that the current spot value $S(t_k)$ be one of the grid points in $S = (S_1, S_2, \ldots, S_M)$.

**Extension to other TARN products.** The payoff structure with the three knockout types of TARNs considered in this study is typical in FX trading, but there are other types with different payoff structures and knockout types. The extension of the present FD method to other TARN types is straightforward. For example, suppose there are extra payments $C_v^j$ at each fixing date $t_k$ and this extra payment does not count in the knockout condition (3) but will also get knockout by the same knockout condition (3)—that is,

$$C_k^j(S_n, A(t_k)) - C_k^j \times 1_{(S_n,t_k) \leq \tilde{C}_k^j>0} + W_s \times A_{(S_n,t_k) \leq \tilde{C}_k^j>0}$$

(14)

$$P^j(S) = \sum_{k=1}^K \left( C_k^j(S(t_k), A(t_k)) + C_k^j(S(t_k), A(t_k)) \right)_n$$

(15)

where $\tilde{C}_k^j$ is the extra payment when the target is not breached. In this case, the only change in the finite difference scheme is to replace the price jump condition (10) with a new condition

$$V(S_n, A_j) = V(S_n, A_j) + C_v^j(S_n, A_j) + C_v^j(S_n, A_j)$$

(16)

There is no other change required in dealing with the auxiliary variable $A_j$, because the extra payment does not contribute to the monitored accumulated amount $A$ and the knockout condition remains the same.

**Boundary Condition**

Typically, a finite difference solution is sought within a rectangular domain $(0 \leq t \leq T, 0 \leq S_{\text{min}} \leq S \leq S_{\text{max}})$, where both $S_{\text{min}}$ and $S_{\text{max}}$ are chosen to be sufficiently far away from the spot price of the underlying asset, for example, three standard deviations from the spot. To ensure a unique solution, boundary conditions are required at $S_{\text{min}}$ and $S_{\text{max}}$. There are different ways of imposing proper boundary conditions that are numerically equivalent. A rather general and robust boundary condition at both $S_{\text{min}}$ and $S_{\text{max}}$ is

$$\frac{\partial^2 V}{\partial S^2}(S_{\text{min}}, t) = 0, \quad \frac{\partial^2 V}{\partial S^2}(S_{\text{max}}, t) = 0$$

which is particularly useful because it is independent of the contract being valued, provided the option has a payoff that is at most linear in the underlying for small and large values of $S$ (almost all common contracts have this property). Other boundary conditions work equally well. For example, for a call option, the following boundary condition can be applied:

$$V(S_{\text{min}}, t) = 0, \quad \frac{\partial V}{\partial S}(S_{\text{max}}, t) = 1$$

And for a put option, we have

$$V(S_{\text{max}}, t) = 0, \quad \frac{\partial V}{\partial S}(S_{\text{min}}, t) = -1$$

Some detailed discussions on various suitable boundary conditions can be found in Wilmott [2000b].
Log-Transform

It is a common practice to re-write Equation (8) in terms of $x = \ln(S)$ before finite difference discretization:

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial V}{\partial x} - rV = 0 \quad (17)
$$

where $\nu = r(t)$, $r(t) \sigma^2 / 2$. Equation (17) is slightly simpler than (8); that is, if volatility and interest rates are constant, then the coefficients of all derivatives in Equation (17) are all constant.

Discretization for Uniform Grid

Unlike barrier options, pricing the discretely monitored TARN option can always rely on uniform grids. This is because there are at most two critical points to be “pinned” to grid points—the spot and the strike, provided we make the far boundaries flexible. Because the only requirement for far boundaries is that they are sufficiently far from spot, these boundaries can certainly be extended a bit further to accommodate uniform grids with the two critical points (spot and strike) pre-determined. When the spot and the strike are almost the same, uniform grids tied to both the spot and the strike may have too large a number of nodes; in this case we chose to tie the strike only, and perform a one-off final interpolation to obtain the price corresponding to the spot.

Denote the option price at time step $n$ and grid point $S_i$ as $V_i^n$, $n = 0, 1, 2, \ldots, N$. For a uniform grid, $\Delta x = x_i - x_{i-1} = \Delta x$ is a constant, and we obtain the following finite difference approximation with second-order accuracy:

$$
\frac{\partial V}{\partial x}(x_i, t_i) = \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x} + O(\Delta x^2) \quad (18)
$$

$$
\frac{\partial^2 V}{\partial x^2}(x_i, t_i) = \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta x)^2} + O(\Delta x^2) \quad (19)
$$

The $\theta$-scheme. Define the following differential operator $F(V_i^n, x, \sigma, \nu, r_i)$ as follows:

$$
F(V_i^n, x, \sigma, \nu, r_i) \equiv \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial V}{\partial x} - r_i V \quad (20)
$$

and the associated finite difference operator $F_i^n$ as

$$
F_i^n \equiv \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2}(x_i, t_i) + \nu \frac{\partial V}{\partial x}(x_i, t_i) - r_i(t_i)V_i^n \quad (21)
$$

where the first and second derivatives are approximated by finite difference as discussed earlier. Then the $\theta$-scheme can be expressed as

$$
\frac{V_i^{n+1} - V_i^n}{\Delta t} + \theta F_i^{n+1} + (1 - \theta)F_i^n = 0 \quad (22)
$$

where $0 \leq \theta \leq 1$. Special values of $\theta = 0, \theta = 0.5$, and $\theta = 1$ correspond to fully explicit, Crank–Nicholson, and fully implicit schemes, respectively.

Calculating Greeks

Calculation of the Greeks (derivatives of the price with respect to initial asset value $S(t_0)$, volatility and interest rates) using finite difference method can be done in the usual way by perturbing the relevant parameter and re-calculating price. Then, one can use a two-point central difference approximation for the first derivatives and a three-point central difference for the second derivatives. For delta and gamma sensitivities (1st and 2nd derivatives with respect to $S(t_0)$), however, the finite difference solution already yields second-order accurate values easily obtained from Equations (18) and (19) without re-calculating price. In addition to the commonly used finite difference approximation of derivatives for the Greeks, there are other special treatments available. For examples, sensitivities with respect to volatility and interest rates can be found by differentiating pricing PDE (Equation (17)) with respect to the relevant parameter and solving the obtained new PDE; for details, see Tavella and Randall [2000].

NUMERICAL EXAMPLES

In this section we present comparison of the finite difference and Monte Carlo methods in the case of basic model with constant volatility. In this case, the number of time steps for Monte Carlo simulated paths is the same as the number of fixing dates. In the cases of basic or term structure models, simulations between
fixing dates are not required because transition density between fixing dates is known in closed form (it is just a lognormal density). For the local volatility model, simulations between fixing dates are required that will increase computations proportionally to the number of time steps.

In the examples, we consider all three types of knockout as described earlier; each knockout type has four cases with four different targets, so the total number of numerical examples is 12. The other inputs common to all the examples are spot \( S(0) = 1.05 \), strike \( X = 1.0 \), volatility \( \sigma = 0.2 \), interest rates \( r_f = r_f = 0 \), fixing dates are every 30 days, and we assume 20 fixing dates.

Results are summarized in Exhibit 2. As shown, the computing time for Monte Carlo estimates based on \( N_{\text{sim}} = 200,000 \) simulated paths is very close to that for the finite difference method with mesh \( 500 \times 100 \times 500 \) (500 points for asset \( S \) coordinate, 100 points for accumulated amount coordinate \( A \), and 500 steps for time). We use the sum of standard vanilla options with maturities at fixed dates as a control variate error reduction method that allows us to reduce the Monte Carlo standard error significantly.

In Exhibit 2, the Monte Carlo standard error is compared with the estimated relative error of the finite difference solution. Ideally, relative error should be computed as the relative difference between the numerical solution and the exact solution, for both Monte Carlo and finite difference methods. Unfortunately in the case of TARN options, a closed-form solution cannot be found except in limiting cases of one fixing date or very large target level. Nevertheless, the standard error in Monte Carlo and the estimated relative error in finite difference are both very good approximations of the exact relative error. In the case of finite difference, we estimate the relative error by using solution of the refined grids in spot, accumulated amount spaces as well as in time. Specifically, we double the number of grid cells in all three dimensions for the refined calculation; that is, using grids \( 1,000 \times 200 \times 1,000 \) for spot, accumulated amount, and time, and using this refined solution in place of the exact solution in estimating the relative error.

As shown in the Appendix, because the \( \theta \)-scheme is second order in accuracy in both spot space and time, and the cubic spline interpolation in the accumulated amount is of the order \( \mathcal{O}(h^4) \), using the solution of the refined grids in estimating the true relative error of the coarser grids is valid and well justified.

As shown in Exhibit 2, the accuracy of finite difference solution is significantly better than that of the Monte Carlo in all 12 test cases. On average, the Monte Carlo standard error is about 0.1%, while the finite difference relative error is about 0.02%. That is,

## Exhibit 2

**Finite Difference (FD) vs. Monte Carlo (MC) Results for TARN Price**

| Target     | MC    | FD    | Diff % | Stderr MC % | MC Sec | Err FD % | FD Sec |
|------------|-------|-------|--------|-------------|--------|----------|--------|
| No Gain    |       |       |        |             |        |          |        |
| 0.3        | 0.1955| 0.1955| 0.0000%| 0.10%       | 1.31   | 0.045%   | 1.12   |
| 0.5        | 0.3288| 0.3286| 0.0609%| 0.10%       | 1.32   | 0.001%   | 1.13   |
| 0.7        | 0.4507| 0.4505| 0.0443%| 0.10%       | 1.32   | -0.018%  | 1.13   |
| 0.9        | 0.5633| 0.5633| 0.0600%| 0.10%       | 1.32   | 0.015%   | 1.14   |
| Part Gain  |       |       |        |             |        |          |        |
| 0.3        | 0.2446| 0.2445| 0.041% | 0.08%       | 1.32   | 0.016%   | 1.12   |
| 0.5        | 0.3819| 0.3818| 0.0262%| 0.09%       | 1.33   | 0.005%   | 1.13   |
| 0.7        | 0.5063| 0.5061| 0.0395%| 0.10%       | 1.32   | 0.038%   | 1.13   |
| 0.9        | 0.6203| 0.6200| 0.0483%| 0.10%       | 1.32   | 0.010%   | 1.13   |
| Full Gain  |       |       |        |             |        |          |        |
| 0.3        | 0.2978| 0.2978| 0.036% | 0.08%       | 1.32   | 0.039%   | 1.12   |
| 0.5        | 0.4389| 0.4386| 0.0684%| 0.09%       | 1.33   | 0.001%   | 1.12   |
| 0.7        | 0.5646| 0.5644| 0.0354%| 0.10%       | 1.33   | 0.015%   | 1.13   |
| 0.9        | 0.6792| 0.6790| 0.0295%| 0.10%       | 1.32   | 0.012%   | 1.13   |

Notes: The notional amount is one unit of foreign currency. The column “diff %” shows the relative difference between results of FD and MC. The computing time is for a desktop computer with Intel Core i5-2400 @3.10GHz and 4 Gb RAM.
the Monte Carlo relative error is five times as large as the finite difference relative error, thus on average, Monte Carlo computing time should increase by a factor of 25 to achieve the same accuracy as finite difference because the Monte Carlo standard error is proportional to $1/\sqrt{N_{\text{mc}}}$. Note that quoted Monte Carlo relative error is computed from the standard error of the estimate, that is, it should be at least doubled for a more realistic error estimate. To improve the accuracy of Monte Carlo estimates, in our numerical example, we use the sum of vanilla options with maturities at the fixing dates as a control variate error reduction technique. Monte Carlo efficiency can also be improved by the use of other error reduction techniques, such as importance sampling as described in Piterbarg [2004], but it might be difficult to implement this for more general models such as the local volatility model, and we did not pursue this further.

These numerical results clearly demonstrate that the use of finite difference will be even more beneficial (in terms of accuracy) in the case of the local volatility model where the Monte Carlo method will require simulations for extra time slices between fixing dates. We expect that the impact in efficiency will be more pronounced in the calculation of Greeks where even small errors in price, such as 0.1%, may lead to 10%–100% error in second derivatives (e.g., gamma or vanna).

CONCLUSIONS

We have implemented a finite difference scheme for evaluating TARN options. Numerical results show that a finite difference scheme is more efficient in pricing TARNs than the Monte Carlo counterpart, even for basic models where the volatility is constant or piecewise constant between fixing dates. For a volatility surface model, the computing time in the Monte Carlo method will increase in proportion to the number of time steps in the surface model, while the finite difference scheme presented here remains essentially the same in terms of computing time. In the numerical examples, only price was considered. It is expected that if the Greeks are considered in the comparison between finite difference and Monte Carlo models, the advantage of finite difference will be much more significant. Even a small error in price such as 0.1% may lead to a large error (10%–100%) in second derivatives (e.g., gamma or vanna). Thus, pricing a TARN and its Greeks by the proposed finite difference scheme provides a significant practical advantage over the commonly used Monte Carlo method.

We presented detailed descriptions of the numerical steps required in the finite difference scheme, so that readers can easily follow the procedures to implement their own, and re-produce the results if desired. The TARN structure considered in this study is simple. However, implementation of the finite difference method can be easily extended to a more generalized accumulation rule and TARN parameters varying across fixing dates. In the case when transition probability density or its moments are known for the underlying asset between fixing dates (e.g., the case of constant or time dependent volatility), pricing TARN can also be accomplished by an even faster recently developed method based on Gauss-Hermite quadratures on a cubic spline interpolation to calculate the required option price expectations in a backward time-stepping algorithm; for details, see Luo and Shevchenko [2014]. Finally we would like to note that it should be possible to develop tree-based method for pricing TARN somewhat similarly to pricing discrete Asian options but it goes beyond the purpose of this article and will be done somewhere else.

APPENDIX

ESTIMATION OF NUMERICAL ERROR

Denote as follows: $V$ is the exact solution, $\tilde{V}$ is the numerical solution of the coarser grids (e.g., $500 \times 100 \times 500$ for spot, accumulated amount, and time), $\tilde{V}^*$ is the numerical solution of the refined grids doubled in each direction (e.g., $1,000 \times 200 \times 1,000$), and $\delta = V - \tilde{V}$ and $\delta^* = \tilde{V} - \tilde{V}^*$ are the absolute numerical errors of the two grids, respectively. Then the relative difference between the numerical solutions of the coarse grids and refined grids is

$$\hat{\varepsilon} = \frac{|\tilde{V} - \tilde{V}^*|}{\tilde{V}^*} = \frac{\delta - \delta^*}{\tilde{V}^*}$$

and the true relative difference between the numerical solution of the coarse grids and the exact solution is as follows:

$$\varepsilon = \frac{|V - \tilde{V}|}{V} = \frac{\delta}{V}$$

It is easy to show that $\hat{\varepsilon}$ is a very good approximation of $\varepsilon$. Due to the second-order accuracy in both time and space, and
fourth-order accuracy in the accumulated amount cubic spline interpolation, \( \delta^* \) can be estimated as \( \delta^* \equiv 2^{-8} \delta \). Thus,

\[
\varepsilon = \frac{\delta - \delta^*}{V^*} \equiv \frac{\delta + 2^{-8} \delta}{V + 2^{-8} \delta} = \frac{256 \delta + 1}{256 \delta + V} \equiv \frac{\delta}{V}
\]

where the last approximation sign is due to \( |\delta| \ll V \). Depending on the signs of the absolute errors, \( \varepsilon \) could be slightly overestimating or slightly underestimating the true relative error, \( \varepsilon \). Thus, using relative error between solutions of coarser grids and the refined grids (with number of grids in all dimensions doubled) as an estimate of the true relative error is well justified for a numerical scheme with second-order accuracy.

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