LINEAR DETERMINANTAL REPRESENTATIONS OF SMOOTH PLANE CUBICS OVER FINITE FIELDS

YASUHIRO ISHITSUKA

Abstract. In this note, we study linear determinantal representations of smooth plane cubics over finite fields. We give an explicit formula of linear determinantal representations corresponding to rational points. Using Schoof’s formula, we count the number of projective equivalence classes of smooth plane cubics over a finite field admitting prescribed number of equivalence classes of linear determinantal representations. As an application, we determine isomorphism classes of smooth plane cubics over a finite field with 0, 1 or 2 equivalence classes of linear determinantal representations.

1. Introduction

Let $k$ be a field, and

$$F(X, Y, Z) = a_{000} X^3 + a_{001} X^2 Y + a_{002} X^2 Z + a_{011} XY^2 + a_{012} XY Z + a_{022} XZ^2 + a_{111} Y^3 + a_{112} Y^2 Z + a_{122} YZ^2 + a_{222} Z^3$$

a ternary cubic form with coefficients in $k$ defining a smooth plane cubic $C \subset \mathbb{P}^2$. We say that the cubic $C$ admits a linear determinantal representation over $k$ if there are a nonzero constant $0 \neq \lambda \in k$ and three square matrices $M_0, M_1, M_2 \in \text{Mat}_3(k)$ of size 3 satisfying $F(X, Y, Z) = \lambda \cdot \det(M)$, where we put $M := XM_0 + YM_1 + ZM_2$. We say that two linear determinantal representations $M, M'$ of $C$ are equivalent if there are invertible matrices $A, B \in \text{GL}_3(k)$ such that $M' = AMB$.

Studying linear determinantal representations of smooth plane cubics is a classical topic in linear algebra and algebraic geometry (for example, see [Vin89], [Dol12]). Recently, they appear in the study of the derived category of smooth plane cubics ([Gal14], [BP15]), and have been studied from arithmetic viewpoints ([FN14], [II14], [Ish15]).

In this note, we investigate linear determinantal representations of smooth plane cubics over finite fields. Let $F_q$ be a finite field with $q$ elements. First, we prove the following bijection. Recall that any smooth plane cubic over $F_q$ has a $F_q$-rational point ([Lan55, Theorem 3]).

**Theorem 1.1** (See Proposition 2.2 and Theorem 4.1). Let $C \subset \mathbb{P}^2$ be a smooth plane cubic over $F_q$. Fix an $F_q$-rational point $P_0 \in C(F_q)$. There is a natural bijection between the following two sets:

- the set of equivalence classes of linear determinantal representations of $C$ over $F_q$,
- and

- the set $C(F_q) \setminus \{P_0\}$ of $F_q$-rational points on $C$ different from $P_0$.

We also calculate a representative of the equivalence class of linear determinantal representations corresponding to each $F_q$-rational point $P \in C(F_q) \setminus \{P_0\}$ (for a precise statement, see Theorem 4.1). In fact, these results are valid for smooth plane cubics with rational points over arbitrary fields.

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Let $\text{Cub}_q(n)$ be the number of projective equivalence classes of smooth plane cubics over $\mathbb{F}_q$ with exactly $n$ equivalence classes of linear determinantal representations. We compute $\text{Cub}_q(n)$ for $0 \leq n \leq 2$.

**Theorem 1.2** (See Corollary 5.2 and Section 6).

1. For $2 \leq q \leq 4$, we have $\text{Cub}_q(0) = 1$; otherwise, $\text{Cub}_q(0) = 0$.
2. For $2 \leq q \leq 5$, we have $\text{Cub}_q(1) = 1$; otherwise, $\text{Cub}_q(1) = 0$.
3. For $q = 2, 3, 5, 7$, we have $\text{Cub}_q(2) = 2$. For $q = 4$, we have $\text{Cub}_q(2) = 4$. Otherwise, $\text{Cub}_q(2) = 0$.

**Table 1.** The number of projective equivalence classes of smooth plane cubics over finite fields admitting prescribed number of equivalence classes of linear determinantal representations.

|       | $\mathbb{F}_2$ | $\mathbb{F}_3$ | $\mathbb{F}_4$ | $\mathbb{F}_5$ | $\mathbb{F}_7$ | $\mathbb{F}_q \ (q \geq 8)$ |
|-------|---------------|---------------|---------------|---------------|---------------|----------------------------|
| $\text{Cub}_q(0)$ | 1             | 1             | 1             | 0             | 0             | 0                          |
| $\text{Cub}_q(1)$ | 1             | 1             | 1             | 1             | 0             | 0                          |
| $\text{Cub}_q(2)$ | 2             | 2             | 4             | 2             | 2             | 0                          |

For each equivalence class in this table, we give examples of smooth plane cubics and their linear determinantal representations. In particular, we determine all projective equivalence classes of smooth plane cubics over finite fields which admit at most two equivalence classes of linear determinantal representations. See Table 4 to Table 11.

The outline of this paper is as follows. In Section 2, we recall the notion of linear determinantal representations of smooth plane curves and its relation to a class of line bundles. In Section 3, we describe an algorithm to compute a representative of linear determinantal representations corresponding to a line bundle. Then we perform this algorithm to smooth plane cubics with rational points, and obtain an explicit formula of linear determinantal representations in Section 4. In Section 5, we recall Schoof’s formula counting the number of projective equivalence classes of smooth plane cubics over finite fields with prescribed number of rational points. Then we apply it to count the number of projective equivalence classes of smooth plane cubics over finite fields admitting prescribed number of equivalence classes of linear determinantal representations. Finally, in Section 6, we determine smooth plane cubics over finite fields admitting at most two equivalence classes of linear determinantal representations.

2. **Linear determinantal representations of smooth plane cubics with rational points**

Let $k$ be a field, and $F(X, Y, Z) \in k[X, Y, Z]$ a homogeneous polynomial with coefficients in $k$ of degree $d \geq 1$ defining a smooth plane curve $C \subset \mathbb{P}^2$. Its degree is $d$, and its genus is $g = (d - 1)(d - 2)/2$. We fix projective coordinates $X, Y, Z$ of $\mathbb{P}^2$.

A linear determinantal representation of $C$ over $k$ is a square matrix $M$ of size $d$ with entries in $k$-linear forms in three variables $X, Y, Z$ which satisfies $F(X, Y, Z) = \lambda \cdot \det(M)$ for some $\lambda \in k^\times$. Two linear determinantal representations $M, M'$ are said to be equivalent if there exist two invertible matrices $A, B \in \text{GL}_d(k)$ with $M' = AMB$. We denote by $\text{LDR}(C)$ the set of equivalence classes of linear determinantal representations of $C$ over $k$.

The following theorem gives an interpretation of linear determinantal representations of $C$ in terms of non-effective line bundles on $C$. It is well known at least when $k$ is an algebraically closed field of characteristic zero.

**Theorem 2.1** (see [Bea00, Proposition 3.1], [Ish15, Proposition 2.2]). There is a natural bijection between the following two sets:
• the set LDR$(C)$ of equivalence classes of linear determinantal representations of $C$ over $k$, and
• the set of isomorphism classes of non-effective line bundles on $C$ of degree $g - 1$.

**Proof.** We briefly recall the proof because it is used to prove the correctness of the algorithm in Section 3. See also [Bea00], [Ish14], [Ish15] for details.

We take a non-effective line bundle $\mathcal{L}$ of degree $g - 1$ on $C$. Let $\iota: C \hookrightarrow \mathbb{P}^2$ be the given embedding. We denote the homogeneous coordinate ring of $\mathbb{P}^2$ by

\[
R := \Gamma_*(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)).
\]

The graded $R$-module $N = \Gamma_*(\mathbb{P}^2, \iota_*\mathcal{L}) \cong \Gamma_*(C, \mathcal{L})$ has a minimal free resolution of the form

\[
0 \longrightarrow R(-2) \otimes_k W_1 \xrightarrow{\tilde{M}} R(-1) \otimes_k W_0 \longrightarrow N \longrightarrow 0,
\]

where $W_0, W_1$ are $d$-dimensional $k$-vector spaces [Bea00, Proposition 3.1]. The homomorphism $\tilde{M}$ can be expressed by a square matrix $M$ of size $d$ with coefficients in $k$-linear forms in three variables $X, Y, Z$. We can check $M$ gives a linear determinantal representation of $C$, and its equivalence class depends only on the isomorphism class of the line bundle $\mathcal{L}$.

Conversely, we take a linear determinantal representation $M$ of $C$. This matrix gives an injective homomorphism

\[
\tilde{M}: R(-2)^{\oplus d} \rightarrow R(-1)^{\oplus d}.
\]

We denote by $N$ the cokernel of $\tilde{M}$. We can show that the coherent sheaf associated to $N$ is written as $\iota_*\mathcal{L}$ for a non-effective line bundle $\mathcal{L}$ of degree $g - 1$ on $C$. The isomorphism class of $\mathcal{L}$ depends only on the equivalence class of $M$. By construction, these two maps are inverses to each other. \(\square\)

Assume that $d = 3$, i.e., $C$ is a smooth plane cubic over $k$. We shall study the relation between the Picard group Pic$(C)$ and the group Jac$(C)(k)$ of $k$-rational points on the Jacobian variety Jac$(C)$ of $C$. In general, there can be a difference which is measured by the relative Brauer group (for example, see [CK12 Theorem 2.1], [Ish15 Example 6.9]). However, when $C$ has a $k$-rational point, the difference vanishes.

**Proposition 2.2.** Let $C$ be a smooth plane cubic over $k$ with a $k$-rational point $P_0 \in C(k)$. There is a natural bijection between the following two sets:

• the set LDR$(C)$ of equivalence classes of linear determinantal representations of $C$ over $k$, and
• the set $C(k) \setminus \{P_0\}$ of $k$-rational points on $C$ different from $P_0$.

**Proof.** There is an exact sequence

\[
0 \longrightarrow \text{Pic}(C) \longrightarrow \text{Pic}_{C/k}(k) \longrightarrow \text{Br}(k) \xrightarrow{s} \text{Br}(C),
\]

where $s$ is the pullback morphism associated to the structure morphism $C \rightarrow \text{Spec}(k)$ ([CK12 Theorem 2.1]). Since $C$ has a $k$-rational point, the homomorphism $s$ is injective. Hence we have two isomorphisms

\[
\text{Pic}(C) \xrightarrow{\sim} \text{Pic}_{C/k}(k), \quad \text{Pic}^0(C) \xrightarrow{\sim} \text{Jac}(C)(k).
\]

Then the morphism

\[
\iota_{P_0}: C \rightarrow \text{Jac}(C)
\]

\[
P \mapsto P - P_0
\]

gives an isomorphism. The only effective line bundle on $C$ of degree 0 is the trivial bundle $\mathcal{O}_C = \iota_{P_0}(P_0)$. Thus, by Theorem 2.1 and the bijection $\iota_{P_0}$, we have the desired bijection. \(\square\)
3. An Algorithm to Obtain Linear Determinantal Representations of Smooth Plane Curves

Let us make the bijection in Theorem 2.1 explicit. In this section, we shall give an algorithm to obtain a linear determinantal representation of a smooth plane curve $C$ of degree $d$ and genus $g = (d-1)(d-2)/2$ over an arbitrary field $k$.

**Algorithm 3.1.**

**Input:** a defining equation $F(X, Y, Z)$ of $C \subset \mathbb{P}^2$ with respect to fixed projective coordinates $X, Y, Z$, and a $k$-rational non-effective divisor $D$ of degree $g - 1$.

**Output:** a linear determinantal representation of $C$ over $k$ corresponding to $D$.

**Step 1 (Global Section):** Compute a $k$-basis $\{v_0, v_1, v_2\}$ of the 3-dimensional $k$-vector space $H^0(C, \mathcal{O}_C(D)(1))$.

**Step 2 (First Syzygy):** Compute a $k$-basis $\{e_0, e_1, e_2\}$ of the 3-dimensional $k$-vector space $\ker (H^0(C, \mathcal{O}_C(1)) \otimes_k H^0(C, \mathcal{O}_C(D)(1)) \to H^0(C, \mathcal{O}_C(D)(2)))$.

**Step 3 (Output Matrix):** Write the $k$-basis $\{e_0, e_1, e_2\}$ as $e_i = \sum_j l_{i,j}(X, Y, Z) \otimes v_j$, where $l_{i,j}(X, Y, Z) \in H^0(C, \mathcal{O}_C(1))$ are $k$-linear forms. Output the matrix $M = (l_{i,j}(X, Y, Z))$.

**Proof** (Proof of the correctness of Algorithm 3.1). Recall the short exact sequence (2.1)

$$
0 \longrightarrow R(-2) \otimes_k W_1 \longrightarrow \widetilde{M} \longrightarrow R(-1) \otimes_k W_0 \longrightarrow N \longrightarrow 0,
$$

where $W_0, W_1$ are 3-dimensional $k$-vector spaces. Since $R_0 = k$ and $N = \Gamma_4(C, \mathcal{O}_C(D))$ is the graded $R$-module corresponding to $\mathcal{O}_C(D)$, the degree 1 part of this sequence gives $W_0 = N_1 = \Gamma(C, \mathcal{O}_C(D)(1))$.

The degree 2 part gives a short exact sequence

$$
0 \longrightarrow W_1 \longrightarrow \widetilde{M} \longrightarrow R_1 \otimes_k W_0 \longrightarrow N_2 \longrightarrow 0.
$$

Thus we have

$$W_1 = \ker (H^0(C, \mathcal{O}_C(1)) \otimes_k H^0(C, \mathcal{O}_C(D)(1)) \to H^0(C, \mathcal{O}_C(D)(2)))$$

The morphism $\widetilde{M}$ is the canonical embedding $W_1 \to R_1 \otimes_k W_0$. Hence it is represented by the matrix $M = (l_{i,j}(X, Y, Z))$. \hfill $\square$

4. An Explicit Formula on Linear Determinantal Representations of Smooth Plane Cubics with Rational Points

We apply Algorithm 3.1 to a smooth plane cubic (i.e., $d = 3$) with a $k$-rational point. Note that, by changing projective coordinates, we may assume that the smooth plane cubic $C$ over $k$ has a $k$-rational point $P_0 = [1 : 0 : 0]$, and the tangent line of $C$ at $P_0$ is $Z = 0$.

**Theorem 4.1.** Let $C \subset \mathbb{P}^2$ be a smooth plane cubic over an arbitrary field $k$ with a $k$-rational point $P_0 = [1 : 0 : 0]$. Assume that the tangent line of $C$ at $P_0$ is the line $l = (Z = 0)$. We have the following formula for the equivalence class of linear determinantal representations of $C$ over $k$ corresponding to a point $P = [s : t : u] \in C(k) \setminus \{P_0\}$ via Proposition 2.2.
Case 1: If \( u \neq 0 \), the equivalence class of linear determinantal representations of \( C \) corresponding to \( P \) is given by
\[
M_P = \begin{pmatrix}
0 & Z & -Y \\
-uX \cdot tZ & -u^2X - (Q(t, u) + su)Z & L_2(X, Y, Z)
\end{pmatrix},
\]
where we denote
\[
L_1(X, Y, Z) := u^2a_{011}X + u^2a_{111}Y + u(a_{111}t + a_{112}u)Z,
\]
\[
L_2(X, Y, Z) := u(a_{011}t + a_{012}u)X + (a_{111}t^2 + a_{112}tu + a_{122}u^2)Z,
\]
and we denote
\[
Q(Y, Z) := a_{011}Y^2 + a_{012}YZ + a_{222}Z^2.
\]

Case 2: If \( u = 0 \), the equivalence class of linear determinantal representations of \( C \) corresponding to \( P \) is given by
\[
M_P = \begin{pmatrix}
0 & Z & -Y \\
(a_{011}X + a_{111}Y) & a_{011}Y & L_1(X, Y, Z) \\
& -Y & L_2(X, Y, Z)
\end{pmatrix},
\]
where we denote
\[
\tilde{L}_1(X, Y, Z) := a_{111}X + (a_{012}a_{111} - a_{011}a_{112})Y,
\]
\[
\tilde{L}_2(X, Y, Z) := (a_{222}a_{111} - a_{011}a_{122})Y - a_{011}a_{222}Z.
\]

We shall prove Theorem 4.1 by performing Algorithm 3.1 as follows.

4.1. Preparation. By the condition of Theorem 4.1 we may assume that \( a_{000} = a_{001} = 0 \) and \( a_{002} = 1 \). Thus we can take a defining equation of the given cubic \( C \subset \mathbb{P}^2 \) as
\[
ZX^2 + Q(Y, Z)X + C(Y, Z) = 0,
\]
where \( Q(Y, Z) \) is a binary quadratic form defined in the statement of Case 1 of Theorem 4.1 and we denote
\[
C(Y, Z) := a_{111}Y^3 + a_{112}Y^2Z + a_{122}YZ^2 + a_{222}Z^3.
\]
The divisor \( l \cap C \) on \( C \) can be written as \( 2P_0 + R \), where
\[
R = [a_{111} : -a_{011} : 0].
\]
Note that \( R \) may or may not be equal to \( P_0 = [1 : 0 : 0] \).

Take a point \( P = [s : t : u] \in C(k) \setminus \{P_0\} \). The line \( m = PP_0 \) is defined by
\[
m(Y, Z) := uY - tZ.
\]
The divisor \( m \cap C \) on \( C \) is \( P + P_0 + S \), where
\[
S = [Q(t, u) + su : -tu : -u^2] \in C(k).
\]
Since \( P - P_0 = \text{div}(m) - 2P_0 - S \), the \( k \)-vector space \( W_0 = \Gamma(C, \mathcal{O}_C(P - P_0)) \) is isomorphic to the \( k \)-vector space
\[
V = \left\{ \begin{array}{c}
qu(X, Y, Z) \\
q(X, Y, Z) \in \Gamma(X, \mathcal{O}_C(2)) \\
\text{div} q(X, Y, Z) - 2P_0 - S \geq 0
\end{array} \right\}
\]
via the isomorphism \( W_0 \to V; f \mapsto fm \). Consider a \( k \)-basis \( \{X^2, XY, Y^2, XZ, YZ, Z^2\} \) of \( \Gamma(C, \mathcal{O}_C(2)) \). The first two elements \( X^2, XY \) have order 0, 1 at \( P_0 \in C(k) \), and the other elements \( XZ, Y^2, YZ, Z^2 \) have order not less than 2 at \( P_0 \in C(k) \). Hence for a quadratic form \( q \in V \), we can write the quadratic form \( q(X, Y, Z) \) as
\[
q(X, Y, Z) = b_{02}XZ + b_{11}Y^2 + b_{12}YZ + b_{22}Z^2
\]
for some constants \( b_{02}, b_{11}, b_{12}, b_{22} \in k \) and \( q \) vanishes at \( S \). We divide the proof of Theorem 4.1 into two cases described in the statement: \( u \neq 0 \) and \( u = 0 \).
4.2. **Proof of Case 1:** when \( u \neq 0 \). In this case, we see that \( l \neq m \) and \( S \neq P_0 \). When 
\[
div q(X, Y, Z) - 2P_0 - S \geq 0,
\]
we have 
\[
u^2(-b_{02}(Q(t, u) + su) + b_{11}t^2 + b_{12}tu + b_{22}u^2) = 0.
\]
We can take a \( k \)-basis of \( W_0 \) as 
\[
v_0 := (u^2ZX + (Q(t, u) + su)Z^2)/m, \\
v_1 := Y, \\
v_2 := Z.
\]
Next we compute a \( k \)-basis of the first syzygy module 
\[
W_1 = \text{Ker} \left( \Gamma(C, \mathcal{O}_C(1)) \otimes_k W_0 \rightarrow \Gamma(C, \mathcal{O}_C(2)) \right).
\]
We find 
\[
e_0 = Z \otimes v_1 - Y \otimes v_2, \\
e_1 = (uY - tZ) \otimes v_0 - (u^2X + (Q(t, u) + su)Z) \otimes v_2, \\
e_2 = (uX - sZ) \otimes v_0 + L_1(X, Y, Z) \otimes v_1 + L_2(X, Y, Z) \otimes v_2
\]
form a \( k \)-basis of the first syzygy module \( W_1 \), where \( L_1(X, Y, Z), L_2(X, Y, Z) \) are linear forms defined in the statement of Theorem 4.1. The corresponding determinantal representation is 
\[
(M) \quad M_P = \begin{pmatrix}
0 & Z & -Y \\
uY - tZ & 0 & -u^2X - (Q(t, u) + su)Z \\
uX - sZ & L_1(X, Y, Z) & L_2(X, Y, Z)
\end{pmatrix}.
\]
We may check that \( \det(M_P) = -u^3f \). This proves Case 1 of Theorem 4.1.

4.3. **Proof of Case 2:** when \( u = 0 \). In this case, \( S = P_0 = [1 : 0 : 0] \) and \( l = m \). We can take a \( k \)-basis of \( W_0 \) as 
\[
v_0 := -(XZ + a_{011}Y^2 + a_{012}YZ + a_{022}Z^2)/Z, \\
v_1 := Y, \\
v_2 := Z.
\]
Next we compute a \( k \)-basis of the first syzygy module \( W_1 \). We find 
\[
e_0 = Z \otimes v_1 - Y \otimes v_2, \\
e_1 = Z \otimes v_0 + a_{011}Y \otimes v_1 + (X + a_{012}Y + a_{022}Z) \otimes v_2, \\
e_2 = (a_{011}X + a_{111}Y) \otimes v_0 + \tilde{L}_1(X, Y, Z) \otimes v_1 + \tilde{L}_2(X, Y, Z) \otimes v_2
\]
form a \( k \)-basis of \( W_1 \), where \( \tilde{L}_1(X, Y, Z), \tilde{L}_2(X, Y, Z) \) are \( k \)-linear forms defined in the statement of Theorem 4.1. The corresponding linear determinantal representation is 
\[
(M) \quad M_P = \begin{pmatrix}
0 & Z & -Y \\
a_{011}X + a_{111}Y & a_{011}Y & X + a_{012}Y + a_{022}Z \\
a_{011}X + a_{111}Y & \tilde{L}_1(X, Y, Z) & \tilde{L}_2(X, Y, Z)
\end{pmatrix}.
\]
We may check that \( \det(M_P) = a_{011}f \). This proves Case 2 of Theorem 4.1. \( \square \)

**Remark 4.2.** Let \( k \) be a field of characteristic not equal to 2 nor 3, and 
\[
E: (Y^2Z - X^3 - aXZ^2 - bZ^3 = 0) \subset \mathbb{P}^2
\]
an elliptic curve over \( k \) with origin \( P_0 = [0 : 1 : 0] \) defined by a Weierstrass equation. Let 
\( P = [\lambda : \mu : 1] \in E(k) \) be a \( k \)-rational point on an affine part of \( E \). Galinat gave in [Gal14].
Lemma 2.9] a representative of linear determinantal representations of $E$ over $k$ corresponding to the divisor $P - P_0$ of degree 0 as

$$M'_P := \begin{pmatrix} X - \lambda Z & 0 & -Y - \mu Z \\ \mu Z - Y & X + \lambda Z & (a + \lambda^2)Z \\ 0 & Z & -X \end{pmatrix}.$$ 

Theorem 4.1 gives an essentially same representative of linear determinantal representation in this case; actually, we can transform $M_P$ into $M'_P$ by changing coordinates and elementary transformation. When $k$ is algebraically closed, Vinnikov [Vin89] gave other representatives.

Remark 4.3. Let $k$ be a field of characteristic not equal to 2 nor 3, and

$$C: (X^3 + Y^3 + Z^3 + \lambda XYZ = 0) \subset \mathbb{P}^2$$

a smooth plane cubic over $k$ defined by Hesse’s normal form. Let $P = [a_0 : a_1 : a_2] \in C(k)$ be a $k$-rational point with $a_0a_1a_2 \neq 0$. In [BP15 Theorem A], Buchweitz and Pavlov showed that the Moore matrix

$$M''_P := \begin{pmatrix} a_0X & a_1Z & a_2Y \\ a_1Y & a_2X & a_0Z \\ a_2Z & a_0Y & a_1X \end{pmatrix}$$

gives a linear determinantal representation of $C$ over $k$ corresponding to the divisor $3P - H$ of degree 0, where $H$ is a hyperplane section of $C$. Note that, when $k$ is not algebraically closed, there can be a linear determinantal representation of $C$ over $k$ which is not equivalent to any Moore matrices. Also the Moore matrices of two distinct $k$-rational points $P, P' \in C(k)$ can give equivalent linear determinantal representations of $C$ over $k$. These are explained by the fact that the homomorphism

$$C = \text{Pic}^1(C) \rightarrow \text{Pic}^3(C) \cong \text{Pic}^0(C)$$

$$P \mapsto 3P \mapsto 3P - H$$

is not an isomorphism in general.

Remark 4.4. To compute the Cassels–Tate pairing on the 3-Selmer groups of elliptic curve, Fisher and Newton [FN14] considered linear determinantal representations when $k$ is a number field, and $C$ is locally soluble but has no $k$-rational point.

5. A COUNTING ON SMOOTH PLANE CUBICS OVER FINITE FIELDS

Let $p$ be a prime number, and $m \geq 1$ a positive integer. Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements. We recall Schoof’s formula on the number of the projective equivalence classes of smooth plane cubics over $\mathbb{F}_q$ with prescribed number of $\mathbb{F}_q$-rational points. Here, two smooth plane cubics $C, C' \subset \mathbb{P}^2$ over $\mathbb{F}_q$ are said to be projectively equivalent if there exists an isomorphism $\mathbb{P}^2 \cong \mathbb{P}^2$ over $\mathbb{F}_q$ that induces an isomorphism $C \cong C'$.

Theorem 5.1 ([Sch87 Theorem 5.2]). For an integer $n \in \mathbb{Z}$, the number of projective equivalence classes of smooth plane cubics $C$ over $\mathbb{F}_q$ with $\#C(\mathbb{F}_q) = n$ is

$$\#E_q(n) + \#E_{q,3}(n) + 3\#E_{q,3,3}(n) - \varepsilon_q(q + 1 - n).$$

Here, we use the following notation which is slightly different from [Sch87]. For reader’s convenience, we recall the definition and formulas for the terms appearing in (5.1).

- For an integer $a \in \mathbb{Z}$ and a prime number $p$, $(a/p)$ denotes the Jacobi symbol.
- For a negative integer $\Delta \in \mathbb{Z}_{<0}$ with $\Delta \equiv 0, 1 \pmod{4}$, Kronecker’s class number $H(\Delta)$ is defined to be the number of $\text{SL}_2(\mathbb{Z})$-orbits of positive definite integral binary quadratic forms

$$\{ f(U, V) = au^2 + bUV + cV^2 \in \mathbb{Z}[U, V] \mid a > 0, b^2 - 4ac = \Delta \}$$
with discriminant \( \Delta \). Here \( \gamma = \left( \begin{array}{c} p \\ r q \\ s \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) acts on \( f(U, V) \) as

\[
(\gamma \circ f)(U, V) = a(pU + rV)^2 + b(pU + rV)(qU + sV) + c(qU + sV)^2.
\]

- Let \( E_q(n) \) denote the set of isomorphism classes of elliptic curves over \( \mathbb{F}_q \) with \( \#E(\mathbb{F}_q) = n \). (In [Sch87], Schoof used \( N(q + 1 - n) \) instead of \( \#E_q(n) \).) From [Sch87] Theorem 4.6, we have the following formula.
  - If \( t^2 > 4q \), we have \( \#E_q(q + 1 - t) = 0 \).
  - If \( t^2 \leq 4q \) and \( p \not| t \), we have \( \#E_q(q + 1 - t) = H(t^2 - 4q) \).
  - If \( t^2 \leq 4q \), \( t \equiv 0 \) (mod \( p \)) and \( m \equiv 1 \) (mod 2), the case is divided into three cases:
    * If \( t = 0 \), we have \( \#E_q(q + 1 - t) = H(-4p) \).
    * If \( (t^2, p) = (2q, 2) \) or \( (3q, 3) \), we have \( \#E_q(q + 1 - t) = 1 \).
    * Otherwise, we have \( \#E_q(q + 1 - t) = 0 \).
  - If \( t^2 \leq 4q \), \( t \equiv 0 \) (mod \( p \)) and \( m \equiv 0 \) (mod 2), the case is divided into four cases:
    * If \( t = 0 \), we have \( \#E_q(q + 1 - t) = 1 - (-4/p) \).
    * If \( t^2 = q \), we have \( \#E_q(q + 1 - t) = 1 - (-3/p) \).
    * If \( t^2 = 4q \), we have \( \#E_q(q + 1 - t) = \frac{1}{12}(p + 6 - 4(-3/p) - 3(-4/p)) \).
    * Otherwise, we have \( \#E_q(q + 1 - t) = 0 \).

- Let \( E_{q,3}(n) \) denote the set of isomorphism classes of elliptic curves \( E \in E_q(n) \) with non-trivial 3-torsion points. (In [Sch87], Schoof used \( N_3(q+1-n) \) instead of \( E_{q,3}(n) \).) It is easily described as

\[
E_{q,3}(n) = \begin{cases} E_q(n) & \text{if } (3 \not| n) \\ \emptyset & \text{if } (3 \mid n). \end{cases}
\]

- Let \( E_{q,3,3}(n) \) denote the set of isomorphism classes of elliptic curves \( E \in E_q(n) \) with \( E(\mathbb{F}_q)[3] \cong (\mathbb{Z}/3\mathbb{Z})^2 \).

(In [Sch87], Schoof used \( N_{3\times3}(q+1-n) \) instead of \( E_{q,3,3}(n) \).) From [Sch87] Theorem 4.9, we have the following formula.
  - We assume that the following four conditions are satisfied: \( q \equiv 1 \) (mod 3), \( t^2 \leq 4q \), \( p \not| t \) and \( t \equiv q + 1 \) (mod 9). Then we have

\[
\#E_{q,3,3}(q + 1 - t) = H \left( \frac{1}{9}(t^2 - 4q) \right).
\]
  - We assume that the following three conditions are satisfied: \( 2 \mid m, p \not| 3 \) and \( t = 2 \cdot (p/3)^{m/2} \cdot p^{m/2} \). Then we have \( \#E_{q,3,3}(q + 1 - t) = \#E_q(q + 1 - t) \).
  - Otherwise, we have \( \#E_{q,3,3}(q + 1 - t) = 0 \).

- We set \( t_0 \in \mathbb{Z} \cup \{ \infty \} \) as follows.
  1. If \( q \not\equiv 1 \) (mod 3), then we set \( t_0 := \infty \). Note that, in this case, we always have \( t \not= t_0 \).
  2. If \( p \equiv 0 \) (mod 3) but \( q \equiv 1 \) (mod 3), we set \( t_0 := 2 \cdot (p/3)^{m/2} \cdot p^{m/2} \) (note that, in this case, \( m \) is even).
  3. If \( p \equiv 1 \) (mod 3), \( t_0 \) is the unique integer satisfying \( t \equiv q + 1 \) (mod 9), \( p \not| t \) and \( t^2 + 3x^2 = 4q \) for some integer \( x \in \mathbb{Z} \).

- We set \( t_1 \in \mathbb{Z} \cup \{ \infty \} \) as follows.
  1. If \( q \not\equiv 1 \) or \( 4 \) (mod 12), then we set \( t_1 := \infty \). Note that, in this case, we always have \( t \not= t_1 \).
If \( p \not\equiv 1 \pmod{4} \) but \( q \equiv 1 \) or \( 4 \pmod{12} \), we set \( t_1 := 2 \cdot (p/3)^{m/2} \cdot p^{m/2} \) (note that, in this case, \( m \) is even).

If \( p \equiv 1 \pmod{4} \) and \( q \equiv 1 \) or \( 4 \pmod{12} \), \( t_1 \) is the integer satisfying \( t \equiv q + 1 \pmod{9} \), \( p \nmid t \) and \( t^2 + 4x^2 = 4q \) for some integer \( x \in \mathbb{Z} \).

- We define a function \( \varepsilon_q(t) \) as follows:

\[
\varepsilon_q(t) :=
\begin{cases}
2 & (t \in \{t_0, t_1\}, \text{ but } t_0 \neq t_1) \\
3 & (t = t_0 = t_1 \text{ and } p = 2) \\
4 & (t = t_0 = t_1 \text{ and } p \neq 2) \\
0 & (\text{otherwise})
\end{cases}
\]

By Proposition 2.2 and Theorem 5.1, we have the following corollary.

**Corollary 5.2.** With the above notation, the number \( \text{Cub}_q(n) \) of projective equivalence classes of smooth plane cubics \( C \) over \( \mathbb{F}_q \) with \( \# \text{LDR}(C) = n \) is

\[
\text{Cub}_q(n) = \#E_q(n + 1) + \#E_{q,3}(n + 1) + 3\#E_{q,3,3}(n + 1) - \varepsilon_q(q - n).
\]

**Proof.** By Proposition 2.2, we have \( \# \text{LDR}(C) = \#C(\mathbb{F}_q) - 1 \) for a smooth plane cubic \( C \) over \( \mathbb{F}_q \). Using this and Theorem 5.1, we have the desired result. \( \square \)

**Remark 5.3.** The following table summarizes the values of \( \text{Cub}_q(n) \) for small \( n \).

| \( q \) | \( F_2 \) | \( F_3 \) | \( F_4 \) | \( F_5 \) | \( F_7 \) | \( F_q \) (\( q \geq 8 \)) |
|---|---|---|---|---|---|---|
| \( \text{Cub}_q(0) \) | 1 | 1 | 1 | 0 | 0 | 0 |
| \( \text{Cub}_q(1) \) | 2 | 2 | 4 | 2 | 2 | 0 |

To check this, Table 5.3 is helpful.

**Table 3.** The numbers appearing in the formula (5.2) for \( 0 \leq n \leq 2 \) and \( 2 \leq q \leq 7 \).

| \( F_2 \) | \# \( E_q(1) \) | \# \( E_q(2) \) | \# \( E_q(3) \) | \# \( E_{q,3}(1) \) | \# \( E_{q,3}(2) \) | \# \( E_{q,3,3}(3) \) |
|---|---|---|---|---|---|---|
| \( F_3 \) | 1 | 1 | 1 | 0 | 0 | 1 |
| \( F_4 \) | 1 | 1 | 2 | 0 | 0 | 2 |
| \( F_5 \) | 0 | 1 | 1 | 0 | 0 | 1 |
| \( F_7 \) | 0 | 0 | 1 | 0 | 0 | 1 |

| \( F_2 \) | \# \( E_{q,3,3}(1) \) | \# \( E_{q,3,3}(2) \) | \# \( E_{q,3,3,3}(3) \) | \( t_0 \) | \( t_1 \) | \( \varepsilon_q(q) \) | \( \varepsilon_q(q - 1) \) | \( \varepsilon_q(q - 2) \) |
|---|---|---|---|---|---|---|---|---|
| \( F_3 \) | 0 | 0 | 0 | \( \infty \) | \( \infty \) | 0 | 0 | 0 |
| \( F_4 \) | 0 | 0 | 0 | \( -4 \) | \( -4 \) | 0 | 0 | 0 |
| \( F_5 \) | 0 | 0 | 0 | \( \infty \) | \( \infty \) | 0 | 0 | 0 |
| \( F_7 \) | 0 | 0 | 0 | \( -1 \) | \( \infty \) | 0 | 0 | 0 |

**Remark 5.4.** For the values of \( H(\Delta) \) for \( -200 \leq \Delta < 0 \), see [Sch87, Table I]. We also note that [Sch87, Proposition 2.2] gives a simple formula relating Kronecker’s class numbers and the class numbers of complex quadratic orders. For small \( q \) and \( n \), we can find a table of the values of (5.1) in [Sch87].
6. Cubics admitting at most two equivalence classes of linear determinantal representations

In this section, we count the number of projective equivalence classes of smooth plane cubics over finite fields admitting at most two equivalence classes of linear determinantal representations.

Let \( p \) be a prime number, and \( m \geq 1 \) a positive integer. Let \( \mathbb{F}_q \) be a finite field with \( q = p^m \) elements. Let \( \omega \in \mathbb{F}_4 \) be an element satisfying \( \omega^2 + \omega + 1 = 0 \).

**Theorem 6.1.**

1. If \( q > 4 \), there are no smooth plane cubics over \( \mathbb{F}_q \) which do not admit linear determinantal representations over \( \mathbb{F}_q \).
2. If \( q \leq 4 \), there exists only one projective equivalence class of smooth plane cubics over \( \mathbb{F}_q \) admitting no linear determinantal representations over \( \mathbb{F}_q \). For explicit representatives of these curves, see Table 4.

**Proof.** The assertion follows from Corollary 5.2. Here we give another proof of (1) which do not use Corollary 5.2. Let \( C \) be a smooth plane cubic over \( \mathbb{F}_q \). By the Hasse–Weil bound, we have
\[
\#C(\mathbb{F}_q) \geq q + 1 - 2\sqrt{q} = (\sqrt{q} - 1)^2.
\]

If \( q > 4 \), we have \( \sqrt{q} > 2 \) and
\[
\#C(\mathbb{F}_q) > (2 - 1)^2 = 1.
\]

Hence \( C \) has at least two \( \mathbb{F}_q \)-rational points. By Proposition 2.2, \( C \) admits a linear determinantal representation over \( \mathbb{F}_q \). \( \Box \)

Next, we determine the smooth plane cubics over finite fields which admit 1 or 2 equivalence classes of linear determinantal representations.

**Theorem 6.2.**

1. If \( q > 5 \), there are no smooth plane cubics over \( \mathbb{F}_q \) admitting a unique equivalence class of linear determinantal representations over \( \mathbb{F}_q \).
2. If \( q \leq 5 \), there exists only one projective equivalence class of smooth plane cubics over \( \mathbb{F}_q \) admitting a unique equivalence class of linear determinantal representations over \( \mathbb{F}_q \). For explicit representatives of these curves, see Table 5.

**Theorem 6.3.**

1. If \( q > 7 \), there are no smooth plane cubics over \( \mathbb{F}_q \) admitting exactly two equivalence classes of linear determinantal representations over \( \mathbb{F}_q \).
2. If \( q = 2, 3, 5, 7 \), there exist 2 projective equivalence classes of smooth plane cubics over \( \mathbb{F}_q \) admitting exactly two equivalence classes of linear determinantal representations over \( \mathbb{F}_q \).
3. If \( q = 4 \), there exist 4 projective equivalence classes of smooth plane cubics over \( \mathbb{F}_q \) admitting exactly two equivalence classes of linear determinantal representations over \( \mathbb{F}_q \).

For explicit representatives of the curves in (2) and (3), see Table 7 to Table 11.

The proofs of Theorem 6.2 and Theorem 6.3 are omitted because they are similar to the proof of Theorem 6.1.

7. Tables of smooth plane cubics

Let us show examples of smooth plane cubics corresponding to cells in Table 2, i.e., smooth plane cubics over finite fields admitting at most two equivalence classes of linear determinantal representations. Moreover, using Theorem 4.1, we give a representative of each equivalence class of linear determinantal representations of each curve.
Table 4 is a summary of smooth plane cubics over finite fields admitting no linear determinantal representations.

Table 5 is a summary of smooth plane cubics over finite fields admitting a unique equivalence class of linear determinantal representations.

Note that, for these curves in Table 5, each linear determinantal representation is equivalent to a symmetric determinantal representation. For example, in the case of the smooth plane cubic $X^2Z + X Y Z + Y^3 + Y^2Z + Y Z^2$ over $\mathbb{F}_2$, we transform

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X & Y + Z & X + Z
\end{pmatrix}
= 
\begin{pmatrix}
Y & 0 & X \\
0 & Z & Y \\
X & Y & X + Y + Z
\end{pmatrix}.
$$

In fact, symmetric determinantal representations of $C$ are bijective to $\text{Pic}^0(C)[2] \setminus \{0\}$ (see [III4 Proposition 4.2]), and $\text{Pic}^0(C)[2] \cong \mathbb{Z} / 2\mathbb{Z}$ for the cubics $C$ in Table 5. By changing the basis $\{e_0, e_1, e_2\}$, we have Table 6 of symmetric determinantal representations.

Table 7 to Table 11 give summaries of smooth plane cubics over finite fields admitting exactly two equivalence classes of linear determinantal representations.

### Table 4. Smooth plane cubics over finite fields admitting no linear determinantal representations.

| $\mathbb{F}_q$ | $F(X, Y, Z)$ | $C(\mathbb{F}_q)$ | $\#\text{LDR}(C)$ |
|---------------|--------------|-------------------|-----------------|
| $\mathbb{F}_2$ | $X^2Z + XZ^2 + Y^3 + Y^2Z + Z^2$ | $[1 : 0 : 0]$ (flex) | 0 |
| $\mathbb{F}_3$ | $X^2Z + Y^3 - YZ^2 + Z^3$ | $[1 : 0 : 0]$ (flex) | 0 |
| $\mathbb{F}_4$ | $X^2Z + XZ^2 + Y^3 + \omega Z^3$ | $[1 : 0 : 0]$ (flex) | 0 |

### Table 5. Smooth plane cubics over finite fields admitting a unique equivalence class of linear determinantal representations.

| $\mathbb{F}_q$ | $F(X, Y, Z)$ | $C(\mathbb{F}_q)$ | $\#\text{LDR}(C)$ | Linear determinantal representations |
|---------------|--------------|-------------------|-----------------|-----------------------------------|
| $\mathbb{F}_2$ | $X^2Z + X YZ + Y^3 + Y^2Z + YZ^2$ | $[1 : 0 : 0]$ (flex), [0 : 0 : 1] | 1 | $\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X & Y + Z & X + Z
\end{pmatrix}$ |
| $\mathbb{F}_3$ | $X^2Z - Y^3 + YZ^2$ | $[1 : 0 : 0]$ (flex), [0 : 0 : 1] | 1 | $\begin{pmatrix}
0 & Z & -Y \\
Y & 0 & -X \\
X & -Y + Z & Z
\end{pmatrix}$ |
| $\mathbb{F}_4$ | $X^2Z + \omega X YZ + Y^3 + Y^2Z + \omega YZ^2$ | $[1 : 0 : 0]$ (flex), [0 : 0 : 1] | 1 | $\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X & Y + Z & \omega X + \omega Z
\end{pmatrix}$ |
| $\mathbb{F}_5$ | $X^2Z + Y^3 + 2Y Z^2$ | $[1 : 0 : 0]$ (flex), [0 : 0 : 1] | 1 | $\begin{pmatrix}
0 & Z & -Y \\
Y & 0 & -X \\
X & Y & 2Z
\end{pmatrix}$ |
Table 6. Examples of symmetric determinantal representations for smooth plane cubics in Table 5.

| \( F_q \) | \( F(X, Y, Z) \) | Symmetric determinantal representations |
|----------|-----------------|-----------------------------------------|
| \( F_2 \) | \( X^2 Z + X Y Z + Y^3 + Y^2 Z + Y Z^2 \) | \( \begin{pmatrix} Y & 0 & X \\ 0 & Z & Y \\ X & Y & X + Y + Z \end{pmatrix} \) |
| \( F_3 \) | \( X^2 Z - Y^3 + Y^2 Z + Y Z^2 \) | \( \begin{pmatrix} Y & 0 & -X \\ 0 & -Z & Y \\ -X & Y & -Y - Z \end{pmatrix} \) |
| \( F_4 \) | \( X^2 Z + \omega X Y Z + Y^3 + Y^2 Z + \omega Y Z^2 \) | \( \begin{pmatrix} Y & 0 & X \\ 0 & Z & Y \\ X & Y & \omega X + Y + \omega Z \end{pmatrix} \) |
| \( F_5 \) | \( X^2 Z + Y^3 + 2 Y Z^2 \) | \( \begin{pmatrix} -Y & 0 & X \\ 0 & -Z & Y \\ X & Y & 2 Z \end{pmatrix} \) |

Table 7. Smooth plane cubics over \( F_2 \) admitting exactly two equivalence classes of linear determinantal representations.

| \( F_q \) | \( F(X, Y, Z) \) | \( C(F_q) \) | \#LDR(\( C \)) | Linear determinantal representations |
|----------|-----------------|-----------------|-----------------|-----------------------------------------|
| \( F_2 \) | \( X^2 Z + X Y^2 + Y Z^2 \) | \([1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\) | 2 | \( \begin{pmatrix} 0 & Z & Y \\ Z & Y & X \\ X & 0 & Y \end{pmatrix}, \begin{pmatrix} 0 & Z & Y \\ Y & 0 & X \\ X & X & Z \end{pmatrix} \) |
| \( F_2 \) | \( X^2 Z + X Z^2 + Y^3 \) | \([1 : 0 : 0]\) (flex), \([1 : 0 : 1]\) (flex), \([0 : 0 : 1]\) (flex) | 2 | \( \begin{pmatrix} 0 & Z & Y \\ Y & 0 & X \\ X + Z & Y & 0 \end{pmatrix}, \begin{pmatrix} 0 & Z & Y \\ Y & 0 & X \\ X & Y & 0 \end{pmatrix} \) |
**Table 8.** Smooth plane cubics over $\mathbb{F}_3$ admitting exactly two equivalence classes of linear determinantal representations.

| $\mathbb{F}_q$ | $F(X, Y, Z)$ | $C(\mathbb{F}_q)$ | $\#\text{LDR}(C)$ | Linear determinantal representations |
|---------------|--------------|-----------------|------------------|------------------------------------|
| $\mathbb{F}_3$ | $X^2Z + XY^2 + YZ^2 + 2XYZ$ | $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ | 2 | \[
\begin{pmatrix}
0 & Z & -Y \\
Z & Y & X - Y \\
X & 0 & -Y
\end{pmatrix}, \\
\begin{pmatrix}
0 & Z & -Y \\
Y & 0 & -X \\
X & X & -X + Z
\end{pmatrix}
\]

| | $X^2Z - XZ^2 - XYZ - Y^3$ | $[1 : 0 : 0]$ (flex), $[1 : 0 : 1]$ (flex), $[0 : 0 : 1]$ (flex) | 2 | \[
\begin{pmatrix}
0 & Z & -Y \\
Y & 0 & -X + Z \\
X & -Y & -X
\end{pmatrix}, \\
\begin{pmatrix}
0 & Z & -Y \\
X - Z & -Y & -X \\
0 & Z & -Y
\end{pmatrix}
\]

**Table 9.** Smooth plane cubics over $\mathbb{F}_4$ admitting exactly two equivalence classes of linear determinantal representations.

| $\mathbb{F}_q$ | $F(X, Y, Z)$ | $C(\mathbb{F}_q)$ | $\#\text{LDR}(C)$ | Linear determinantal representations |
|---------------|--------------|-----------------|------------------|------------------------------------|
| $\mathbb{F}_4$ | $X^2Z + XY^2 + \omega YZ^2$ | $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ | 2 | \[
\begin{pmatrix}
0 & Z & Y \\
Z & Y & X \\
X & 0 & \omega Y
\end{pmatrix}, \\
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X & X & \omega Z
\end{pmatrix}
\]

| | $X^2Z + XY^2 + (\omega + 1)YZ^2$ | $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ | 2 | \[
\begin{pmatrix}
0 & Z & Y \\
Z & Y & X \\
X & 0 & (\omega + 1)Y
\end{pmatrix}, \\
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X & X & (\omega + 1)Z
\end{pmatrix}
\]

| | $X^2Z + XZ^2 + \omega Y^3$ | $[1 : 0 : 0]$ (flex), $[1 : 0 : 1]$ (flex), $[0 : 0 : 1]$ (flex) | 2 | \[
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X & Z & \omega Y
\end{pmatrix}, \\
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X + Z \\
X & \omega Y & 0
\end{pmatrix}
\]

| | $X^2Z + XZ^2 + (\omega + 1)Y^3$ | $[1 : 0 : 0]$ (flex), $[1 : 0 : 1]$ (flex), $[0 : 0 : 1]$ (flex) | 2 | \[
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X \\
X + Z & (\omega + 1)Y & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & Z & Y \\
Y & 0 & X + Z \\
X & (\omega + 1)Y & 0
\end{pmatrix}
\]
Table 10. Smooth plane cubics over $\mathbb{F}_5$ admitting exactly two equivalence classes of linear determinantal representations.

| $\mathbb{F}_q$ | $F(X, Y, Z)$ | $C(\mathbb{F}_q)$ | #LDR(C) | Linear determinantal representations |
|---------------|---------------|-------------------|---------|-------------------------------------|
| $\mathbb{F}_5$ | $X^2Z + XY^2 + YZ^2 - 2XYZ$ | $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ | 2       | $\begin{pmatrix} 0 & Z & -Y \\ Y & X & -2Y \\ X & 0 & -Y \end{pmatrix}, \begin{pmatrix} 0 & Z & -Y \\ Y & 0 & -X \\ X & X & -2X + Z \end{pmatrix}$ |
|               | $X^2Z - XZ^2 - 2XYZ - Y^3$ | $[1 : 0 : 0]$ (flex), $[1 : 0 : 1]$ (flex), $[0 : 0 : 1]$ (flex) | 2       | $\begin{pmatrix} 0 & Z & -Y \\ Y & 0 & -X \\ X - Z & -Y & -2X \end{pmatrix}, \begin{pmatrix} 0 & Z & -Y \\ Y & 0 & -X + Z \\ X & -Y & -2X \end{pmatrix}$ |

Table 11. Smooth plane cubics over $\mathbb{F}_7$ admitting exactly two equivalence classes of linear determinantal representations.

| $\mathbb{F}_q$ | $F(X, Y, Z)$ | $C(\mathbb{F}_q)$ | #LDR(C) | Linear determinantal representations |
|---------------|---------------|-------------------|---------|-------------------------------------|
| $\mathbb{F}_7$ | $X^2Z + XY^2 + 3YZ^2$ | $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ | 2       | $\begin{pmatrix} 0 & Z & -Y \\ Y & X & -3Y \\ X & 0 & -Y \end{pmatrix}$ |
|               | $X^2Z - XZ^2 + 3Y^3$ | $[1 : 0 : 0]$ (flex), $[1 : 0 : 1]$ (flex), $[0 : 0 : 1]$ (flex) | 2       | $\begin{pmatrix} 0 & Z & -Y \\ Y & 0 & -X \\ X - Z & 3Y & 0 \end{pmatrix}, \begin{pmatrix} 0 & Z & -Y \\ Y & 0 & -X + Z \\ X & 3Y & 0 \end{pmatrix}$ |
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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: yasu-ishi@math.kyoto-u.ac.jp