An answer to the Whitehead’s asphericity question

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Abstract

The Whitehead asphericity problem, regarded as a problem of combinatorial group theory, asks whether any subpresentation of an aspherical group presentation is also aspherical. We give a positive answer to this question by proving that if $P = (x, r)$ is an aspherical presentation of the trivial group, and $r_0 \in r$ a fixed relation, then $P_1 = (x, r_1)$ is aspherical where $r_1 = r \setminus \{r_0\}$.

1 Introduction

A 2-dimensional CW-complex $K$ is called aspherical if $\pi_2(K) = 0$. The Whitehead asphericity problem (WAP for short), raised as a question in [44], asks whether any subcomplex of an aspherical 2-complex is also aspherical. The question can be formulated in group theoretic terms since every group presentation $P$ has a geometric realisation as a 2-dimensional CW-complex $K(P)$ and so $P$ is called aspherical if $K(P)$ is aspherical. A useful review of this question is in [42].

The purpose of the present paper is to prove that if $P = (x, r)$ is an aspherical presentation of the trivial group and $r_0 \in r$ is a fixed relation, then the subpresentation $P_1 = (x, r_1)$ where $r_1 = r \setminus \{r_0\}$ is again aspherical. This in fact implies that WAP has always a positive answer since in Theorem 1 of [32] Ivanov proves that if the WAP is false, then there is an aspherical presentation $P = (A, R \cup \{z\})$ of the trivial group where the alphabet $A$ is countable and $z \in A$ such that $P_1 = (A, R)$ is not aspherical.

An immediate implication of our result and that of Bestvina-Brady [2] is that the conjecture of Eilenberg and Ganea [11] is false. This conjecture states that if a discrete group $G$ has cohomological dimension 2, then it has a 2-dimensional Eilenberg-MacLane space $K(G, 1)$.

There is a large corpus of results which are related to ours and is mostly contained in [4], [5], [8], [10], [14], [18], [19], [20], [22], [23], [24], [25], [26], [30], [32], [16] and [43].
In the first part of our paper we will make use of the review paper [3] of Brown and Huebschmann which contains several key results about aspherical group presentations one of which is proposition 14 that gives sufficient and necessary conditions under which a group presentation $\mathcal{P} = (x, r)$ is aspherical. It turns out that the asphericity of $\mathcal{P}$ is encoded in the structure of the free crossed module $(H/P, \hat{F}, \delta)$ that is associated to $\mathcal{P}$. To be precise we state below proposition 14.

**Proposition 1.1. (Proposition 14 of [3])** Let $K(\mathcal{P})$ be the geometric realisation of a group presentation $\mathcal{P} = (x, r)$ and let $G$ be the group given by $\mathcal{P}$. The following are equivalent.

(i) The 2-complex $K(\mathcal{P})$ is aspherical.

(ii) The module $\pi$ of identities for $\mathcal{P}$ is zero.

(iii) The relation module $N(\mathcal{P})$ of $\mathcal{P}$ is a free left $\mathbb{Z}G$ module on the images of the relators $r \in r$.

(iv) Any identity $Y$-sequence for $\mathcal{P}$ is Peiffer equivalent to the empty sequence.

The last condition is of a particular interest to us. By definition, a $Y$-sequence for $\mathcal{P}$ is a finite (possibly empty) sequence of the form $((u^1r_1)^{\varepsilon_1}, ..., (u^n r_n)^{\varepsilon_n})$ where $r \in r$, $u$ is a word from the free group $F$ over $x$ and $\varepsilon = \pm 1$. A $Y$-sequence $((u^1r_1)^{\varepsilon_1}, ..., (u^n r_n)^{\varepsilon_n})$ is called an identity $Y$-sequence if it is either empty or if $\prod_{i=1}^n u_i r_i^{\varepsilon_i} u_i^{-1} = 1$ in $F$. The definition of Peiffer equivalence is based on Peiffer operations on $Y$-sequences and reads as follows.

(i) An **elementary Peiffer exchange** replaces an adjacent pair $((ur)^{\varepsilon}, (vs)^{\delta})$ in a $Y$-sequence by either $((ur^{\varepsilon})(vs)^{\delta}, (ur)^{\varepsilon})$, or by $((vs)^{\delta}, (ur^{\varepsilon})(vs)^{\delta})$.

(ii) A **Peiffer deletion** deletes an adjacent pair $((ur)^{\varepsilon}, (ur)^{-\varepsilon})$ in a $Y$-sequence.

(iii) A **Peiffer insertion** is the inverse of the Peiffer deletion.

The equivalence relation on the set of $Y$-sequences generated by the above operations is called **Peiffer equivalence**. We recall from [3] what does it mean for an identity $Y$-sequence $((u^1r_1)^{\varepsilon_1}, ..., (u^n r_n)^{\varepsilon_n})$ to have the primary identity property. This means that the indices $1, 2, ..., n$ are grouped into pairs $(i, j)$ such that $r_i = r_j$, $\varepsilon_i = -\varepsilon_j$ and $u_i = u_j$ modulo $N$ where $N$ is the normal subgroup of $\hat{F}$ generated by $r$. Proposition 16 of [3] shows that every such sequence is Peiffer equivalent to the empty sequence. Given an identity $Y$-sequence $d$ which is equivalent to the empty sequence 1, we would be interested to know what kind of insertions $((ur)^{\varepsilon}, (ur)^{-\varepsilon})$ are used along the way of transforming $d$ to 1. It is obvious that keeping track of that information is vital to tackle the Whitehead problem.

The aim of Section 3 of the present paper is to offer an alternative way in dealing with the asphericity of a group presentation $\mathcal{P} = (x, r)$ by considering a new crossed module $(\mathcal{G}(\Upsilon), \hat{F}, \hat{\theta})$ over the free group $\hat{F}$ on $x$ where $\mathcal{G}(\Upsilon)$ is the group generated by the symbols $((ur)^{\varepsilon}, (ur)^{-\varepsilon})$ subject to relations $((ur)^{\varepsilon}(vs)^{\delta} = (ur^{\varepsilon-\varepsilon}(vs)^{\delta}(ur)^{\varepsilon})$, the action of $\hat{F}$ on $\mathcal{G}(\Upsilon)$ and the map $\hat{\theta}$ are defined in the obvious fashion. The advantage of working with $\mathcal{G}(\Upsilon)$ is that unlike to
In $G(\Upsilon)$ the images of insertions ($u^r, (u^r)^{-}\varepsilon$) do not cancel out and this enables us to express the asphericity in terms of such insertions. This is realized by considering the kernel $\tilde{\Pi}$ of $\tilde{\theta}$ which is the analogue of the module $\pi$ of identities for $\mathcal{P}$ in the standard theory and is not trivial when $\mathcal{P}$ is aspherical. We call $\tilde{\Pi}$ the generalized module of identities for $\mathcal{P}$.

To prove our results we apply techniques from the theory of semigroup actions and to this end we use concepts like the universal enveloping group $G(S)$ of a given semigroup $S$, the dominion of a subsemigroup $U$ of a semigroup $S$ and the tensor product of semigroup actions. These concepts are explained, with references, in Section 2.

## 2 Monoid actions

For the benefit of the reader not familiar with monoid actions we will list below some basic notions and results that are used in the paper. For further results on the subject the reader may consult the monograph [27]. Given $S$ a monoid with identity element $1$ and $X$ a nonempty set, we say that $X$ is a left $S$-system if there is an action $(s, x) \mapsto sx$ from $S \times X$ into $X$ with the properties

\[
(st)x = s(tx) \text{ for all } s, t \in S \text{ and } x \in X, \\
1x = x \text{ for all } x \in X.
\]

Right $S$-systems are defined analogously in the obvious way. Given $S$ and $T$ (not necessarily different) monoids, we say that $X$ is an $(S, T)$-bisystem if it is a left $S$-system, a right $T$-system, and if

\[
(sx)t = sxt \text{ for all } s \in S, t \in T \text{ and } x \in X.
\]

If $X$ and $Y$ are both left $S$-systems, then an $S$-morphism or $S$-map is a map $\phi : X \to Y$ such that

\[
\phi(sx) = s\phi(x) \text{ for all } s \in S \text{ and } x \in X.
\]

Morphisms of right $S$-systems and of $(S, T)$-bisystems are defined in an analogue way. If we are given a left $T$-system $X$ and a right $S$-system $Y$, then we can give the cartesian product $X \times Y$ the structure of an $(T, S)$-bisystem by setting

\[
t(x, y) = (tx, y) \text{ and } (x, y)s = (x, ys).
\]

Let now $A$ be an $(T, U)$-bisystem, $B$ an $(U, S)$-bisystem and $C$ an $(T, S)$-bisystem. As explained above, we can give to $A \times B$ the structure of an $(T, S)$-bisystem. With this in mind we say that a $(T, S)$-map $\beta : A \times B \to C$ is a bimap if

\[
\beta(au, b) = \beta(a, ub) \text{ for all } a \in A, b \in B \text{ and } u \in U.
\]

A pair $(A \otimes_U B, \psi)$ consisting of a $(T, S)$-bisystem $A \otimes_U B$ and a bimap $\psi : A \times B \to A \otimes_U B$ will be called a tensor product of $A$ and $B$ over $U$ if for every $(T, S)$-bisystem $C$ and every
bimap \( \beta : A \times B \to C \), there exists a unique \((T, S)\)-map \( \bar{\beta} : A \otimes_U B \to C \) such that the diagram

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\psi} & A \otimes_U B \\
\downarrow{\beta} & & \downarrow{\bar{\beta}} \\
C & & C
\end{array}
\]

commutes. It is proved in [27] that \( A \otimes_U B \) exists and is unique up to isomorphism. The existence theorem reveals that \( A \otimes_U B = (A \times B)/\tau \) where \( \tau \) is the equivalence on \( A \times B \) generated by the relation

\[
T = \{((au, b), (a, ub)) : a \in A, b \in B, u \in U\}.
\]

The equivalence class of a pair \((a, b)\) is usually denoted by \( a \otimes_U b \). To us is of interest the situation when \( A = S = B \) where \( S \) is a monoid and \( U \) is a submonoid of \( S \). Here \( A \) is clearly regarded as an \((S, U)\)-bisystem with \( U \) acting on the right on \( A \) by multiplication, and \( B \) as an \((U, S)\)-bisystem where \( U \) acts on the left on \( B \) by multiplication.

Another concept that is important to our approach is that of the dominion which is defined in [31] from Isbell. By definition, if \( U \) is a submonoid of a monoid \( S \), then the dominion \( \text{Dom}_S(U) \) consists of all the elements \( d \in S \) having the property that for every monoid \( T \) and every pair of monoid homomorphisms \( f, g : S \to T \) that coincide in \( U \), it follows that \( f(d) = g(d) \). Related to dominions there is the well known zigzag theorem of Isbell. We will present here the Stenstrom version of it (theorem 8.3.3 of [27]) which reads. Let \( U \) be a submonoid of a monoid \( S \) and let \( d \in S \). Then, \( d \in \text{Dom}_S(U) \) if and only if \( d \otimes_U 1 = 1 \otimes_U d \) in the tensor product \( A = S \otimes_U S \). We mention here that this result holds true if \( S \) turns out to be a group and \( U \) a subgroup, both regarded as monoids. A key result (theorem 8.3.6 of [27]) that is used in the next section is the fact that any inverse semigroup \( U \) is absolutely closed in the sense that for every semigroup \( S \) containing \( U \) as a subsemigroup, \( \text{Dom}_S(U) = U \). It is obvious that groups are absolutely closed as special cases of inverse monoids (see [28]).

### 3 Peiffer operations and monoid actions

Before we explain how monoid actions are used to deal with the Peiffer operations on \( Y \)-sequences, we will introduce several monoids.

The first one is the monoid \( \Upsilon \) defined by the monoid presentation \( \mathcal{M} = \langle Y \cup Y^{-1}, P \rangle \) where \( Y^{-1} \) is the set of group inverses of the elements of \( Y \) and \( P \) consists of all pairs \((ab, \theta(a)ba)\) where \( a, b \in Y \cup Y^{-1} \).

The second one is the group \( G(\Upsilon) \) given by the group presentation \( (Y \cup Y^{-1}, \hat{P}) \) where \( \hat{P} \) is the set of all words \( ab\iota(a)i(\theta(a)ba) \) where \( \iota(c) \) we denote the inverse of \( c \) in the free group over \( Y \cup Y^{-1} \). Before we introduce the next two monoids and the respective monoid actions, we stop to explain that \( \Upsilon \) and \( G(\Upsilon) \) are special cases of a more general situation. If a monoid \( S \) is given by the monoid presentation \( \mathcal{M} = \langle X, R \rangle \), then its universal enveloping
group \( \mathcal{G}(S) \) (see [1] and [2]) is defined to be the group given by the group presentation 
\((X, \hat{R})\) where \( \hat{R} \) consists of all words \( u \circ v \) whenever \( (u, v) \in R \) where \( \iota(v) \) is the inverse of \( v \) in the free group over \( X \). We let for future use \( \sigma : FM(X) \to S \) the respective canoni
cal homomorphism where \( FM(X) \) is the free monoid on \( X \). It is easy to see that 
there is a monoid homomorphism \( \mu_S : S \to \mathcal{G}(S) \) which satisfies the following universal 
property. For every group \( G \) and monoid homomorphism \( f : S \to G \), there is a unique group 

homomorphism \( \hat{f} : \mathcal{G}(S) \to G \) such that \( \hat{f} \mu_S = f \). This universal property is an indication 
of an adjoint situation. Specifically, the functor \( \mathcal{G} : \text{Mon} \to \text{Grp} \) which maps every monoid to 
its universal group, is a left adjoint to the forgetful functor \( \hat{U} : \text{Grp} \to \text{Mon} \). This ensures 
that \( \mathcal{G}(S) \) is an invariant of the presentation of \( S \).

The third monoid we consider is the submonoid \( \mathcal{U} \) of \( \Upsilon \), having the same unit as \( \Upsilon \), and 
generated from all the elements of the form \( \sigma(a) \sigma(a^{-1}) \) with \( a \in Y \cup \Upsilon^{-1} \). This monoid, 
acting on the left and on the right on \( \Upsilon \) by the multiplication in \( \Upsilon \).

The last monoid considered is the subgroup \( \hat{\mathcal{U}} \) of \( \mathcal{G}(\Upsilon) \) generated by \( \mu(\mathcal{U}) \). Similarly to 
avove, \( \hat{\mathcal{U}} \) acts on \( \mathcal{G}(\Upsilon) \) by multiplication.

Given \( \alpha = (a_1, \ldots, a_n) \) an \( Y \)-sequence over the group presentation \( \mathcal{P} = (x, r) \), then perfo
rming an elementary Peiffer operation on \( \alpha \) can be interpreted in a simple way in terms of the 
monoids \( \Upsilon \) and \( \mathcal{U} \). In what follows we will denote by \( \sigma(\alpha) \) the element \( \sigma(a_1) \cdots \sigma(a_n) \in \Upsilon \). If \( \beta = (b_1, \ldots, b_n) \) is obtained from \( \alpha = (a_1, \ldots, a_n) \) by performing an elementary Peiffer ex
change, then from the definition of \( \Upsilon \), \( \sigma(\alpha) = \sigma(\beta) \), therefore an elementary Peiffer exchange 
or a finite sequence of such has no effect on the element \( \sigma(a_1) \cdots \sigma(a_n) \in \Upsilon \). Before we see 
the effect that a Peiffer insertion in \( \alpha \) has on \( \sigma(\alpha) \) we need the first claim of the following.

**Lemma 3.1.** The elements of \( \mathcal{U} \) are central in \( \Upsilon \) and those of \( \hat{\mathcal{U}} \) are central in \( \mathcal{G}(\Upsilon) \).

**Proof.** We see that for every \( a \) and \( b \in Y \cup \Upsilon^{-1} \), \( \sigma(a) \sigma(a^{-1}) \sigma(b) = \sigma(b) \sigma(a) \sigma(a^{-1}) \). Indeed,

\[
\sigma(a) \sigma(a^{-1}) \sigma(b) = \theta(a) \theta(a^{-1}) \sigma(b) (\sigma(a) \sigma(a^{-1})) = \sigma(b) \sigma(a) \sigma(a^{-1}).
\]

Since elements \( \sigma(b) \) and \( \sigma(a) \sigma(a^{-1}) \) are generators of \( \Upsilon \) and \( \mathcal{U} \) respectively, then the first 
claim holds true. The second claim follows easily. \( \square \)

If we insert \((a, a^{-1})\) at some point in \( \alpha = (a_1, \ldots, a_n) \) to obtain \( \alpha' = (a_1, \ldots, a, a^{-1}, \ldots, a_n) \), 
then from lemma [3.1]

\[
\sigma(\alpha') = \sigma(\alpha) \cdot (\sigma(a) \sigma(a^{-1})),
\]

which means that inserting \((a, a^{-1})\) inside a \( Y \)-sequence \( \alpha \) has the same effect as multiplying 
the corresponding \( \sigma(\alpha) \) in \( \Upsilon \) by the element \( \sigma(a) \sigma(a^{-1}) \) of \( \mathcal{U} \). For the converse, it is obvious 
that any word \( \beta \in FM(Y \cup \Upsilon^{-1}) \) representing \( \sigma(\alpha) \cdot (\sigma(a) \sigma(a^{-1})) \) is Peiffer equivalent to \( \alpha \). 
Of course the deletion has the obvious interpretation in our semigroup theoretic terms as the 
inverse of the above process. We retain the same names for our semigroup operations, that 
is insertion for multiplication by \( \sigma(a) \sigma(a^{-1}) \) and deletion for its inverse. Related to these 
operations on the elements of \( \Upsilon \) we make the following definition.


Definition 3.2. We denote by $\sim_u$ the equivalence relation in $\Upsilon$ generated by all pairs $(\sigma(\alpha), \sigma(\alpha) \cdot \sigma(a)\sigma(a^{-1}))$ where $\alpha \in FM(Y \cup Y^{-1})$ and $a \in Y \cup Y^{-1}$. We say that two elements $\sigma(a_1) \cdots \sigma(a_n)$ and $\sigma(b_1) \cdots \sigma(b_m)$ where $m, n \geq 0$ are Peiffer equivalent in $\Upsilon$ if they fall in the same $\sim_u$-class.

From what we said before it is obvious that two $\Upsilon$-sequences $\alpha$ and $\beta$ are Peiffer equivalent in the usual sense if and only if $\sigma(\alpha) \sim_u \sigma(\beta)$. For this reason we decided to make the following convention. If $\alpha = (a_1, \ldots, a_n)$ is a $\Upsilon$-sequence (resp. an identity $\Upsilon$-sequence), then its image in $\Upsilon$, $\sigma(\alpha)$ will again be called a $\Upsilon$-sequence (resp. an identity $\Upsilon$-sequence). In the future instead of working directly with an $\Upsilon$-sequence $\alpha$, we will work with its image $\sigma(\alpha)$.

We note that it should be mentioned that the study of $\sim_u$ might be as hard as the study of Peiffer operations on $\Upsilon$-sequences, and at this point it seems we have not made any progress at all. In fact this definition will become useful later in this section and yet we have to prove a few more things before we utilize it.

The process of inserting and deleting generators of $\mathcal{U}$ in an element of $\Upsilon$ is related to the following new concept. Given $U$ a submonoid of a monoid $S$ and $d \in S$, then we say that $d$ belongs to the weak dominion of $U$, shortly written as $d \in WDom_{S}(U)$, if for every group $G$ and every monoid homomorphisms $f, g : S \rightarrow G$ such that $f(u) = g(u)$ for every $u \in U$, then $f(d) = g(d)$. An analogue of the Stenström version of Isbell’s theorem for weak dominion holds true. The proof of the if part of its analogue is similar to that of Isbell theorem apart from some minor differences that reflect the fact that we are working with $WDom$ rather than $Dom$ and that will become clear along the proof, while the converse relies on the universal property of $\mu : S \rightarrow G(S)$.

Proposition 3.3. Let $S$ be a monoid, $U$ a submonoid and let $\hat{U}$ be the subgroup of $G(S)$ generated by elements $\mu(u)$ with $u \in U$. Then $d \in WDom_{S}(U)$ if and only if $\mu(d) \in U$.

Proof. The set $\hat{A} = G(S) \otimes_{G} G(S)$ has an obvious $(G(S), G(S))$-bisystem structure. The free abelian group $\mathbb{Z}\hat{A}$ on $\hat{A}$ inherits a $(G(S), G(S))$-bisystem structure if we define

$$g \cdot \sum z_i(g_i \otimes_{G} h_i) = \sum z_i(g g_i \otimes_{G} h_i) \text{ and } \left(\sum z_i(g_i \otimes_{G} h_i)\right) \cdot g = \sum z_i(g_i \otimes_{G} h_i g).$$

The set $G(S) \times \mathbb{Z}\hat{A}$ becomes a group by defining

$$(g, \sum z_i g_i \otimes_{G} h_i) \cdot (g', \sum z'_i g'_i \otimes_{G} h'_i) = (g g', \sum z_i g_i \otimes_{G} h_i g' + \sum z'_i g'_i \otimes_{G} h'_i).$$

The associativity is proved easily. The unit element is $(1, 0)$ and for every $(g, \sum z_i g_i \otimes_{G} h_i)$ its inverse is the element $(g^{-1}, - \sum z_i g^{-1} g_i \otimes_{G} h_i g^{-1})$. Let us now define

$$\beta : S \rightarrow G(S) \times \mathbb{Z}\hat{A} \text{ by } s \mapsto (\mu(s), 0),$$

which is clearly a monoid homomorphism, and

$$\gamma : S \rightarrow G(S) \times \mathbb{Z}\hat{A} \text{ by } s \mapsto (\mu(s), \mu(s) \otimes_{G} 1 - 1 \otimes_{G} \mu(s)),$$
which is again seen to be a monoid homomorphism. These two coincide on $U$ since for every $u \in U$

$$\gamma(u) = (\mu(u), \mu(u) \otimes_{\tilde{U}} 1 - 1 \otimes_{\tilde{U}} \mu(u)) = (\mu(u), 0) = \beta(u).$$

The last equality and the assumption that $d \in WDom_{\lambda}(U)$ imply that $\beta(d) = \gamma(d)$, therefore

$$(\mu(d), 0) = (\mu(d), \mu(d) \otimes_{\tilde{U}} 1 - 1 \otimes_{\tilde{U}} \mu(d)),$$

which shows that $\mu(d) \otimes_{\tilde{U}} 1 = 1 \otimes_{\tilde{U}} \mu(d)$ in the tensor product $G(S) \otimes G(S)$ and therefore theorem 8.3.3. [27], applied for monoids $G(S)$ and $\tilde{U}$, implies that $\mu(d) \in Dom_{G(S)}(\tilde{U})$. But $Dom_{G(S)}(\tilde{U}) = \tilde{U}$ as from theorem 8.3.6, [27] every inverse semigroup is absolutely closed, whence $\mu(d) \in \tilde{U}$.

Conversely, suppose that $\mu(d) \in \tilde{U}$ and we want to show that $d \in WDom_{\lambda}(U)$. Let $G$ be a group and $f, g : S \to G$ two monoid homomorphisms that coincide in $\tilde{U}$, therefore the group homomorphisms $\tilde{f}, \tilde{g} : G(S) \to G$ of the universal property of $\mu$ coincide in $\tilde{U}$ which, from our assumption, implies that $\tilde{f}(\mu(d)) = \tilde{g}(\mu(d))$, and then $f(d) = g(d)$ proving that $d \in WDom_{\lambda}(U)$. \quad \Box

Given a presentation $\mathcal{P} = (x, r)$ for a group $G$, we consider the following crossed module. If $G(\Upsilon)$ is the universal group associated with $\mathcal{P}$ and $\tilde{F}$ is the free group on $x$, then we define

$$\tilde{\theta} : G(\Upsilon) \to \tilde{F}$$

by $\mu \sigma(w_r)^e \mapsto wr^e u^{-1}$.

An action of $\tilde{F}$ on $G(\Upsilon)$ is given by $v(\mu \sigma(w_r)^e) = \mu \sigma(vw_r)^e$ for every $v \in \tilde{F}$ and every generator $\mu \sigma(w_r)^e$ of $G(\Upsilon)$. It is easy to check that the triple $(G(\Upsilon), \tilde{F}, \tilde{\theta})$ is a crossed module over $\tilde{F}$. The elements of Ker($\tilde{\theta}$) are central, therefore Ker($\tilde{\theta}$) is an abelian subgroup of $G(\Upsilon)$ on which $G$ acts on the left by the rule

$$g(\mu \sigma(a_1, ..., a_n) \mu \sigma(b_1, ..., b_m)) = \mu \sigma(w a_1, ..., w a_n) \mu \sigma(w b_1, ..., w b_m),$$

where $w$ is a word in $\tilde{F}$ representing $g$. With this action Ker($\tilde{\theta}$) becomes a left $G$-module which we call the generalized module of identities for $\mathcal{P}$ and is denoted by $\tilde{\Upsilon}$. Also we note that $\tilde{\Upsilon}$ is a sub $G$-module of $\tilde{\Pi}$. The module of identities $\pi$ for $\mathcal{P}$ is obtained from $\tilde{\Upsilon}$ by factoring out $\tilde{\Pi}$. In terms of $\tilde{\Pi}$ and $\tilde{\Upsilon}$ we prove the following analogue of theorem 3.1 of [38].

**Theorem 3.4.** The following assertions are equivalent.

(i) The presentation $\mathcal{P} = (x, r)$ is aspherical.

(ii) For every identity $\Upsilon$-sequence $d$, $d \in WDom_{\Upsilon}(\Pi)$.

(iii) $\tilde{\Pi} = \tilde{\Upsilon}$.

**Proof.** (i) $\Rightarrow$ (ii) Let $d = \sigma(a_1) \cdots \sigma(a_n) \in \Upsilon$ be any identity $\Upsilon$-sequence and as such it has to be Peiffer equivalent to 1. We proceed by showing that $d \in WDom_{\Upsilon}(\Pi)$. Let $G$ be any group and $f, g : \Upsilon \to G$ two monoid homomorphisms that coincide in $\Upsilon$ and we want
to show that \( f(d) = g(d) \). The proof will be done by induction on the minimal number \( h(d) \) of insertions and deletions needed to transform \( d = \sigma(a_1) \cdots \sigma(a_n) \) to 1. If \( h(d) = 1 \), then \( d \in \mathfrak{U} \) and \( f(d) = g(d) \). Suppose that \( h(d) = n > 1 \) and let \( \tau \) be the first operation performed on \( d \) in a series of operations of minimal length. After \( \tau \) is performed on \( d \), it is obtained an element \( d' \) with \( h(d') = n - 1 \). By induction hypothesis, \( f(d') = g(d') \) and we want to prove that \( f(d) = g(d) \). There are two possible cases for \( \tau \). First, \( \tau \) is an insertion and let \( u = \sigma(a)\sigma(a^{-1}) \in \mathfrak{U} \) be the element inserted. It follows that \( f(d') = f(d)f(u) \) and \( g(d') = g(d)g(u) \), but \( f(u) = g(u) \), therefore from cancellation law in the group \( G \) we get \( f(d) = g(d) \). Second, \( \tau \) is a deletion and let \( u = \sigma(a)\sigma(a^{-1}) \in \mathfrak{U} \) be the element deleted, that is \( d = d'u \). It follows immediately from the assumptions that \( f(d) = g(d) \) proving that \( d \in \text{WDom}_\mathfrak{T}(\mathfrak{U}) \).

\[(ii) \Rightarrow (iii) \] Let \( \tilde{d} \in \tilde{\mathfrak{U}} \). We may assume without loss of generality that no \( \iota(\mu \sigma(u_r)^\epsilon) \) is represented in \( d \) for if there is any such occurrence, we can multiply \( d \) by \( \mu \sigma((u_r)^\epsilon(\mu r)^{-\epsilon}) \) to obtain in return \( \tilde{d} \) where \( \mu \sigma(u_r)^\epsilon \) is now replaced by \( \mu \sigma((u_r)^{-\epsilon}) \). It is obvious that if \( \tilde{d} \in \mathfrak{U} \), then \( \tilde{d} \in \hat{\mathfrak{U}} \) and conversely. Let now \( d \) be any preimage of \( \tilde{d} \) under \( \mu \). It is clear that \( d \) is an identity \( Y \)-sequence and as such \( d \in \text{WDom}_\mathfrak{T}(\mathfrak{U}) \). Then proposition 3.3 implies that \( \tilde{d} = \mu(d) \in \hat{\mathfrak{U}} \).

\[(iii) \Rightarrow (i) \] Assume that \( \tilde{\mathfrak{U}} = \hat{\mathfrak{U}} \) and we want to show that any identity \( Y \)-sequence \( d \) is Peiffer equivalent to 1. From the assumption for \( d \) we have that \( \mu(d) \in \hat{\mathfrak{U}} \) and then proposition 3.3 implies that \( d \in \text{WDom}_\mathfrak{T}(\mathfrak{U}) \). Consider the group \( H/P \) as a quotient of \( \mathcal{G}(\Upsilon) \) obtained by identifying \( \iota(\mu \sigma(u_r)) \) with \( \mu \sigma((u_r)^{-1}) \) and let \( \nu : \mathcal{G}(\Upsilon) \to H/P \) be the respective quotient morphism. Writing \( \tau \) for the zero morphism from \( \Upsilon \) to \( H/P \), we see that \( \tau \) and the composition \( \nu \mu \) coincide in \( \mathfrak{U} \), therefore since \( d \in \text{WDom}_\mathfrak{T}(\mathfrak{U}) \), it follows that \( \nu \mu(d) = 1 \) in \( H/P \). The asphericity of \( \mathcal{P} \) now follows from theorem 2.7, p.71 of [17].

Before we prove our next result we recall the definition of the relation module \( \mathcal{N}(\mathcal{P}) \). Given \( \mathcal{P} = (x, r) \) a presentation for a group \( G \), we let \( \alpha : \hat{F} \to G \) and \( \beta : N \to N/[N, N] \) be the canonical homomorphisms where \( N \) is the normal closure of \( r \) in \( \hat{F} \) and \([N, N]\) its commutator subgroup. There is a well defined \( G \)-action on \( \mathcal{N}(\mathcal{P}) = N/[N, N] \) given by

\[
w^\alpha \cdot s^\beta = (w^{-1}sw)^\beta
\]

for every \( w \in \hat{F} \) and \( s \in N \). This action extends to an action of \( \mathbb{Z}G \) over \( \mathcal{N}(\mathcal{P}) \) by setting

\[
(w_1^\alpha \pm w_2^\alpha) \cdot s^\beta = (w_1^{-1}sw_1w_2^{-1}w_2^\pm^1w_2^\beta).
\]

When \( \mathcal{P} \) is aspherical, the basis of \( \mathcal{N}(\mathcal{P}) \) as a free \( \mathbb{Z}G \) module is the set of elements \( r^\beta \) with \( r \in r \).

**Proposition 3.5.** If \( \mathcal{P} \) is aspherical, then \( \hat{\mathfrak{U}} \) is a free \( G \)-module with bases equipotent to the set \( r \).

**Proof.** The result follows if we show that \( \hat{\mathfrak{U}} \cong \mathcal{N}(\mathcal{P}) \) as \( G \)-modules. For this we define

\[
\Omega : \mathcal{N}(\mathcal{P}) \to \hat{\mathfrak{U}}
\]

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on free generators by \( r^\beta \mapsto \mu\sigma(rr^{-1}) \) which is clearly well defined and a surjective morphism of \( G \)-modules. Now we prove that \( \Omega \) is injective. Let

\[
\xi = \sum_{i=1}^{n} u_i^\alpha \cdot r_i^\beta - \sum_{j=n+1}^{m} v_j^\alpha \cdot r_j^\beta \in \text{Ker}(\Omega),
\]

which means that

\[
\prod_{i=1}^{n} \mu\sigma(u_i r_i(u_i r_i)^{-1})u \left( \prod_{j=n+1}^{m} \mu\sigma(v_j r_j(v_j r_j)^{-1}) \right) = 1. \tag{1}
\]

To prove that \( \xi = 0 \) we will proceed as follows. Define

\[
\gamma : FM(Y \cup Y^{-1}) \to \mathcal{N}(\mathcal{P})
\]

on free generators as follows

\[
(u^\alpha r^\beta)^e \mapsto u^\alpha \cdot r^\beta.
\]

It is easy to see that \( \gamma \) is compatible with the defining relations of \( \Upsilon \), hence there is \( g : \Upsilon \to \mathcal{N}(\mathcal{P}) \) and then the universal property of \( \mu \) implies the existence of \( \hat{g} : \mathcal{G}(\Upsilon) \to \mathcal{N}(\mathcal{P}) \) such that \( \hat{g}\mu = g \). If we apply now \( \hat{g} \) on both sides of (1) obtain

\[
2 \cdot \sum_{i=1}^{n} u_i^\alpha \cdot r_i^\beta - 2 \cdot \sum_{j=n+1}^{m} v_j^\alpha \cdot r_j^\beta = 0,
\]

proving that \( \xi = 0 \).}

4 Proof of the main theorem

The proof of our main theorem is heavily based on two papers. The first one is [36] where McGlashan et al extended the Squier complex of a monoid presentation to a 3-complex and obtained a short exact sequence involving data from this complex. This sequence will be crucial in the proof of our theorem. The second one is [40] where Pride realizes the second homotopy group associated with a group presentation as the first homotopy group of a certain extension of the Squire complex arising from that presentation. For the sake of completeness we have added below a number of sections which tend to explain the material that is used in our proofs. Section 4.1 gives some basic material about rewriting systems since they are used in the construction of our complexes and in our proofs. In Section 4.2 we explain in some details how the Squier complex of a monoid presentation is defined and the cellular chain complex associated with it. Further in section 4.3 we give the definition of the extended Squier complex as it appears in [36] and some of the homological consequences that will be used in our proofs. Section 4.4 shows how the 0 and the 1-skeleton of the Squier complex is well ordered, and in the case when the rewriting system is complete, it shows how these
well orders induce another well order in the set of all 2-cells of the extended 3-complex. This new well order will be used further in section 4.6. Section 4.5 is about the Knuth-Bendix completion procedure since it is used to give a new and shorter proof of the key result of [36] regarding the short exact sequence we mentioned above. This proof is given in section 4.6. Section 4.7 is devoted to introducing the Pride complex associated with a group presentation and to explain ideas and results from [40] since we make extensive use of them in our proofs.

Finally, it is important to mention that theorem 6.6 of [33] is vital in the proof of key lemma 4.14.

4.1 Some basic concepts from rewriting systems

A rewriting system is a pair $\mathcal{P} = (x, r)$ where $x$ is a non empty set and $r$ is a set of rules $r = (r_{+1}, r_{-1}) \in F \times F$ where $F$ is the free monoid on $x$. Related with $r$ there is the so called the one single step reduction of words

$$
\rightarrow_r = \{(ur_{+1}v, ur_{-1}v)| r \in r \text{ and } u, v \in F\}.
$$

The reflexive and transitive closure of $\rightarrow_r$ is denoted by $\rightarrow^*_r$, and the reflexive, transitive and symmetric closure is denoted by $\leftrightarrow^*_r$ and is also known as the Thue congruence generated by $r$. The quotient $F/\leftrightarrow^*_r$ forms a monoid $S$ whose elements are the congruence classes $\bar{u}$ of words $u \in F$, and the multiplication is given by $\bar{u} \cdot \bar{v} = \bar{uv}$. We say that the monoid $S$ is given by $\mathcal{P}$, or that $\mathcal{P}$ is a presentation for $S$.

A rewriting system $\mathcal{P} = (x, r)$ is noetherian if there is no infinite chain

$$
w \rightarrow_r w' \rightarrow_r \ldots
$$

and is confluent if whenever we have $w \rightarrow^*_r w_1$ and $w \rightarrow^*_r w_2$, then there is $z \in F$ such that $w_1 \rightarrow^*_r z$ and $w_2 \rightarrow^*_r z$. A rewriting system $\mathcal{P} = (x, r)$ is complete if it is both noetherian and confluent.

Let $\mathcal{P} = (x, r)$ be a presentation for a monoid $S$. The natural epimorphism

$$
F \rightarrow S \text{ such that } w \mapsto \bar{w},
$$

where $F$ is the free monoid on $x$, extends linearly to a ring epimorphism

$$
\mathbb{Z}F \rightarrow \mathbb{Z}S
$$

of the corresponding integral monoid rings. The kernel of this epimorphism is denoted by $J$ which as an abelian group is generated by all

$$
u(r_{+1} - r_{-1})v \text{ where } u, v \in F \text{ and } r \in r.
$$

As a $(\mathbb{Z}F, \mathbb{Z}F)$-bimodule $J$ is generated by all $r_{+1} - r_{-1}$.
4.2 The Squier complex of a monoid presentation

The material included in this section is taken from [36] (see also [35]). At the end of the section we give shortly the respective terminology used in [33] which differs slightly from ours. The reason we explain this terminology is the use of theorem 6.6 of [33] in the proof of our key lemma 4.14.

For every rewriting system \( \mathcal{P} = (x, r) \) we can define its graph of derivations \( \Gamma(\mathcal{P}) \) whose vertices are the elements of \( F \), and the edges are all quadruples \( e = (w, r, \varepsilon, w') \) where \( w, w' \in F, \varepsilon = \pm 1, r \in r \), with initial, terminal and inverse functions 

\[
\iota_e = wr_\varepsilon w', \tau_e = wr_{-\varepsilon} w' \text{ and } e^{-1} = (w, r, -\varepsilon, w').
\]

The edge \( e \) is called positive if \( \varepsilon = 1 \). We can think of \( \Gamma(\mathcal{P}) \) as a one dimensional cw-complex with 0-cells all the elements of \( F \) and with 1-cells all positive edges. We note here that \( e^{-1} = (w, r, -1, w') \) is not a new edge attached to the complex, but is defined to mean the topological inverse of the attaching map of \( e = (w, r, 1, w') \). A path \( p \) of length \( n \) in \( \Gamma(\mathcal{P}) \) is a sequence of edges \( p = e_1 \ldots e_i e_{i+1} \ldots e_n \) where \( \tau e_i = \iota e_{i+1} \) for \( 1 \leq i \leq n - 1 \). It is called positive if the edges are positive, and is called closed if \( \iota e_1 = \tau e_n \).

There is a natural two-sided action of \( F \) on \( \Gamma(\mathcal{P}) \). The action on vertices is given by the multiplication of \( F \), and the action of \( z, z' \in F \) on edges \( e = (w, r, \varepsilon, w') \) is given by

\[
z.e.z' = (zw, r, \varepsilon, w'z'),
\]

and sometimes is called translation. This action extends to paths in the obvious way.

Note that there is a 1-1 correspondence between the elements of \( S \) given by \( \mathcal{P} \) and the connected components of \( \Gamma(\mathcal{P}) \) since \( u \leftrightarrow^* v \) if and only if there is a path in \( \Gamma(\mathcal{P}) \) connecting \( u \) with \( v \). Also note that the generators of \( J \) as an abelian group are the elements \( ue - \tau e \) where \( e \) is a positive edge.

We say that two positive edges \( e_1 \) and \( e_2 \) are disjoint if they can be written in the form

\[
e_1 = f_1 \iota f_2, f_2 = \iota f_1 f_2
\]

where \( f_1, f_2 \) are positive edges. We say that an edge \( e \) is left reduced (resp. right reduced) if it cannot be written in the form \( u.f \) (resp. \( f.u \)) for some non-empty word \( u \in F \) and an edge \( f \). A pair of positive edges with the same initial forms a critical pair it either

1. One of the pair is both left and right reduced (a critical pair of inclusion type), or
2. One of the pair is left reduced but not right reduced, and the other is right reduced but not left reduced (a critical pair of overlapping type).

We say that a critical pair \( (e_1, e_2) \) is resolvable if there are positive paths (a resolution of the critical pair) from \( \tau e_1 \) and \( \tau e_2 \) to a common vertex. It is well known [37] that, when the
system $\mathcal{P} = (x, r)$ is noetherian and if all the critical pairs are resolvable, then the system is confluent.

The Squier complex $D(\mathcal{P})$ associated with $\mathcal{P}$ is a combinatorial 2-complex with 1-skeleton $\Gamma(\mathcal{P})$, to which, for each pair of positive edges $e, f$ a 2-cell $[e, f]$ is attached along the closed path

$$\partial[e, f] = (e.\ell f)(\tau e.f)(e.\tau f)^{-1}(\tau e.f)^{-1}.$$

Sometimes we refer the 2-cell $[e, f]$ as a square 2-cell. The two-sided action of $F$ on $\Gamma(\mathcal{P})$ extends to the 2-cells by

$$w.[e, f].w' = [w.e, f.w']$$

where $w, w' \in F, e, f$ are positive edges.

We have the chain complex

$$C(D) : C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

where $C_0, C_1, C_2$ and $C_3$ are the free abelian groups generated by all 0-cells, positive edges, 2-cells, and 3-cells respectively. The boundary maps are given by

$$\partial_1 e = \ell e - \tau e$$

where $e$ is a positive edge,

$$\partial_2 [e, f] = e.(\ell f - \tau f) - (\ell e - \tau e).f$$

where $e, f$ are positive edges.

In the paper [33] of Otto and Kobayashi, a monoid presentation is denoted by $(\Sigma, R)$ and the rewriting rules of $R$ are denoted by $r \rightarrow \ell$. The edges of the graph of derivations in [33] are denoted by $(x, u, v, y)$ where $x, y \in \Sigma^*$ and $(u \rightarrow v) \in E = R \cup R^{-1}$. In [33] it is considered the set of closed paths

$$D = \{e_1 x u_2 \circ v_1 x e_2 \circ e_1^{-1} x v_2 \circ u_1 x e_2^{-1}|e_1 = (u_1, v_1) \in R, e_2 = (u_2, v_2) \in R, x \in \Sigma^*\}.$$

It is important to observe that each circuit of $D$ is in fact the boundary of a square 2-cell as the following shows

$$e_1 x u_2 \circ v_1 x e_2 \circ e_1^{-1} x v_2 \circ u_1 x e_2^{-1} = \partial[(1, r_1, 1, 1), (x, r_2, 1, 1)],$$

where $r_1 = e_1$ and $r_2 = e_2$. The free $\mathbb{Z}\Sigma^*$ bi-module $\mathbb{Z}\Sigma^* \cdot R \cdot \mathbb{Z}\Sigma^*$ considered in [33] is the abelian group $C_1$ of our complex $C(D)$, and the maps $\partial_1$ are the same in both papers. On the other hand, the free $\mathbb{Z}\Sigma^*$ bi-module $\mathbb{Z}\Sigma^* \cdot D \cdot \mathbb{Z}\Sigma^*$ of [33] is the abelian group $C_2$ of $C(D)$, and the maps $\partial_2$ are the same in both papers. Finally, the exact sequence of theorem 6.6 of [33] in our notations will be

$$C_2 \xrightarrow{\partial_2} J.R.\mathbb{Z}\Sigma^* + \mathbb{Z}\Sigma^* . R.J \xrightarrow{\partial_1} J^2 \xrightarrow{\partial_2} 0.$$

The interpretation of the exactness in the middle of the above sequence is that $Ker\partial_1 \cap (J.R.\mathbb{Z}\Sigma^* + \mathbb{Z}\Sigma^* . R.J) = Im\partial_2$. 

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4.3 The extended Squier complex

Assume now that $p$ is a set of closed paths in $D(P)$. In [36] the complex $D(P)$ has been extended to a 3-complex $(D, p)$ in the following way. We add to $D(P)$ additional 2-cells $[u, p, v]$ attached along the closed path

$$\partial[u, p, v] = u.p.v$$

where $u, v \in F$, and $p \in p$.

The construction is then completed by adding 3-cells as follows. For each positive edge $f$ and each 2-cell $\sigma$ with $\partial\sigma = e_1^\delta \ldots e_n^\delta$, 3-cells $[f, \sigma]$ and $[\sigma, f]$ are attached to the 2-skeleton by mapping their boundaries to respectively:

1. the 2-cells $\iota f.\sigma, \tau f.\sigma$ together with 2-cells $[f, e_i]$ for $1 \leq i \leq n$,
2. the 2-cells $\sigma.\iota f, \sigma.\tau f$ together with 2-cells $[e_i, f]$ for $1 \leq i \leq n$.

The 2-sided action of $F$ on the 2-skeleton extends naturally to the 3-cells. For $[f, \sigma], [\sigma, f]$ and $u, v \in F$,

$$u.[f, \sigma].v = [u.f, \sigma.v]$$

and

$$u.[\sigma, f].v = [u.\sigma, f.v].$$

The complex $C(D)$ now extends to

$$C(D, p) : 0 \longrightarrow C_3^p \overset{\partial_3}{\longrightarrow} C_2^p \oplus C_2 \overset{\partial_2}{\longrightarrow} C_1 \overset{\partial_1}{\longrightarrow} C_0 \longrightarrow 0$$

where $C_3^p$ is the free abelian group generated by the set of all 3-cells, and $C_2^p$ is the free abelian group generated by the set of all newly added 2-cells $\sigma = [u, p, v]$. The boundary map $\partial_2$ restricted to $C_2$ is $\partial_2$, and for every $[u, p, v]$ where $p \in p$ with $\partial p = f_1^\delta \ldots f_n^\delta$, it is defined

$$\tilde{\partial}_2[u, p, v] = \sum_{i=1}^n \delta_i u.f_i.v.$$

Finally, the definition of $\tilde{\partial}_3$ is done in the following way. For every positive edge $f$ and every 2-cell $\sigma$ with $\tilde{\partial}_2\sigma = \sum_{i=1}^n \varepsilon_i e_i$, we have

$$\tilde{\partial}_3[f, \sigma] = (\iota f - \tau f).\sigma + \sum_{i=1}^n \varepsilon_i [f, e_i],$$

and

$$\tilde{\partial}_3[\sigma, f] = \sigma.(\iota f - \tau f) - \sum_{i=1}^n \varepsilon_i [e_i, f].$$

The definition of the 2-cells $[u, p, v]$ where $u, v \in F, p \in p$ suggests that $C_2^p$ can be regarded as a free $(ZF, ZF)$-bimodule with basis

$$\hat{p} = \{ [1, p, 1] | p \in p \}.$$
This enables us to define a \((ZF, ZF)\)-homomorphism

\[ \varphi : C_2 \oplus C_p^2 \to \mathbb{Z}S p \mathbb{Z}S \]

by mapping \(C_2\) to 0, and every 2-cell \([u, p, v]\) to \(\bar{u}.p.\bar{v}\). The kernel of \(\varphi\) is denoted by \(K^p\).

It is shown in [36] that \(K^p = C_2 + J.p.ZF + ZF.p.J\).

Also it is shown that \(B_2(\mathcal{D}, p) \subseteq K^p\) and that the restriction of \(\tilde{\partial}_2\) on \(K^p\) sends \(K^p\) onto \(B_1(\mathcal{D})\), therefore we have the complex

\[
0 \to B_2(\mathcal{D}, p) \xrightarrow{\text{incl.}} K^p \xrightarrow{\tilde{\partial}_2} B_1(\mathcal{D}) \to 0
\]

(4)

It is proved in Proposition 14 of [36] that when \(p\) is a homology trivializer, then the sequence (4) is exact. We will give a new proof in section 4.6 for the exactness of (4). Since the proof uses the so called Knuth-Bendix completion procedure, we will explain this procedure in some details in section 4.5. Before doing that we will introduce in the next section some useful orders in the skeleta of \(\mathcal{D}(\mathcal{P})\).

### 4.4 Ordering the Squier complex

As before \(\mathcal{P} = (x, r)\) is a rewriting system and \(\mathcal{D}(\mathcal{P})\) its Squier complex. Assume in addition that for every \((r_{+1}, r_{-1}) \in r\), \(r_{+1} \neq r_{-1}\). Let \(\triangleright\) be a well ordering on \(x\). The corresponding length-lexicographical ordering on \(F\) is defined as follows. For \(u, v \in F\), we write \(u >_{\text{lex}} v\) if and only if \(|u| > |v|\), or \(|u| = |v|, u = au', v = bv'\) where \(a, b \in x, u', v' \in F\), and one of the following holds:

(i) \(a \triangleright b\),

(ii) \(a = b, u' >_{\text{lex}} v'\).

It turns out that \(>_{\text{lex}}\) is a well ordering on \(F\) (see [3]). We can always assume that \(>_{\text{lex}}\) is compatible with \(r\) in the sense that \(r_{+1} >_{\text{lex}} r_{-1}\), for if there are rules \((r_{+1}, r_{-1})\) satisfying the opposite, we can exchange \(r_{+1}\) with \(r_{-1}\). Well orderings in \(F\) that are compatible with \(r\) are usually called reduction well ordering and are the basis to start the Knuth-Bendix completion procedure.

So far we have defined a reduction well order on the 0-skeleton of \(\mathcal{D}(\mathcal{P})\) which will be denoted by \(\prec_0\). This order induces a noetherian (well founded) partial order in the 1-skeleton of \(\mathcal{D}(\mathcal{P})\) in the following way. For \(e = (u, r, +1, v)\) and \(f = (u', r', +1, v')\) positive edges in \(\mathcal{D}(\mathcal{P})\), we define \(e \prec_1 f\) if and only if \(ie = if\), and one of the following occurs:

(i) \(v'\) is a proper suffix of \(v\), or

(ii) \(v = v'\) and \(|r_{+1}| < |r'_{+1}|\), or

(iii) \(v = v', r_{+1} = r'_{+1}\) and \(r_{-1} \prec_0 r'_{-1}\).
It turns out that ≺₁ is a partial order and that it is well founded.

Further, assume that \( \mathcal{P} = (x, r) \) is confluent, so that all the critical pairs of positive edges resolve. In that case, we attach to \( \mathcal{D}(\mathcal{P}) \) 2-cells \( p \) by choosing resolutions for every critical pair of positive edges \((e, f)\) in the following way. If \( p_e, p_f \) are positive paths from \( \tau e \) and \( \tau f \) respectively to a common vertex, then the boundary of the 2-cell \( \sigma \) corresponding to \((e, f)\) is

\[
\partial \sigma = e p_e p_f^{-1} f^{-1}.
\]

Also we attach 2-cells \( u.\sigma.v \) for every \( u, v \in F \) along the loop \( u.\partial \sigma.v \). As it is explained in the section 4.3, this new 2-complex extends to a 3-complex denoted there by \( (\mathcal{D}, p) \). It is important to mention that every 2-cell of \( (\mathcal{D}, p) \), including the square 2-cells, is uniquely determined by the pair \((e, f)\) of edges meeting its maximal vertex \( w = \iota_e = \iota_f \) (according to \( \prec_0 \)). For this reason, we write the 2-cell as \([w; (e, f)]\). Now we extend the orders \( \prec_0 \) and \( \prec_1 \) to the 2-skeleton of the 3-complex \( (\mathcal{D}, p) \) as follows. For every two 2-cells \([w; (e, f)]\) and \([w'; (e', f')]\) we say that \([w; (e, f)] \prec_2 [w'; (e', f')]\) if and only if:

(i) \( w \prec_0 w' \); or
(ii) \( w = w' \) and \( e \prec_1 e' \); or
(iii) \( f = f' \) and \( e \prec_1 e' \).

This is a well founded total order in the set of all 2-cells of \( (\mathcal{D}, p) \).

Under the current assumptions, similarly to 2-cells, every 3-cell is uniquely determined by three positive edges \( e_1 \prec_1 e_2 \prec_1 e_3 \) with initial the maximal vertex \( w \) of the 3-cell, where either \( e_1 \) is disjoint from \( e_2 \) and \( e_3 \), or \( e_3 \) is disjoint from \( e_1 \) and \( e_2 \). For this reason we write the 3-cell by \([w; (e_1, e_2, e_3)]\). By (2) and (3) we see that

\[
\partial_3 [w; (e_1, e_2, e_3)] = [w; (e_2, e_3)] - [w; (e_1, e_3)] + [w; (e_1, e_2)] + \varsigma
\]

where \( \varsigma \) is a 2-chain made up of 2-cells all of which have maximal vertices less than \( w \). Also note that the maximal 2-cell represented in \( \partial_3 [w; (e_1, e_2, e_3)] \) is \([w; (e_2, e_3)]\).

### 4.5 The Knuth-Bendix completion procedure

The Knuth-Bendix procedure \( \mathcal{P} \), produces a complete system out of any given system and equivalent to it. Given a rewriting system \( \mathcal{P} = (x, r) \) and a reduction well order \( \triangleright \) on \( F \) that is compatible with \( r \) (there is always such one as explained in section 4.1), one can produce a complete rewriting system \( \mathcal{P}^\infty \) that is equivalent to \( \mathcal{P} \) in the following way. Put \( r_0 = r \). For each non-resolvable pair of edges \((e, f)\) in \( \mathcal{D}(\mathcal{P}) \) we chose positive path \( p_e, p_f \) from \( \tau e \) and \( \tau f \) respectively to distinct irreducibles. Let \( r_1 \) be the set of rules obtained from \( r \) by adding for each such critical pair \((e, f)\) the rule \( (\tau p_e, \tau p_f) \) if \( \tau p_e \triangleright \tau p_f \), otherwise adding the rule \( \tau p_f \triangleright \tau p_e \). It is clear that \( \mathcal{P}_1 = (x, r_1) \) is equivalent to \( \mathcal{P} = (x, r) \) and that \( r \subseteq r_1 \) where the inclusion is strict if \( \mathcal{P} \) is not complete. Assume by induction that we have defined a sequence of equivalent rewriting systems

\[
\mathcal{P} = (x, r_0), ... , \mathcal{P}_{n-1} = (x, r_{n-1}), \mathcal{P}_n = (x, r_n),
\]

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and consequently, an increasing sequence of complexes

\[ D(P) \subseteq \cdots \subseteq D(P_{n-1}) \subseteq D(P_n), \]

where \( P_n = (x, r_n) \) is obtained from \( P_{n-1} = (x, r_{n-1}) \) by resolving all the non-resolvable critical pairs of \( D(P_{n-1}) \). Put \( r_\infty = \bigcup_{n \geq 0} r_n \) and let \( P_\infty = (x, r_\infty) \) be the resulting rewriting system. The corresponding complex \( D(P_\infty) \) will be latter denoted by \( D^\infty \). The rewriting system \( P_\infty = (x, r_\infty) \) is obviously equivalent to \( P \) and it is complete since it is compatible with the order \( \succ \) on \( F \) and for every non-resolvable pair \((e, f)\) of edges found in some \( D(P_n) \), there is an edge \( g \) in \( D(P_{n+1}) \) connecting the endpoints of the positive paths \( p_e \) and \( p_f \) of \( D(P_n) \).

4.6 A shorter proof for the exactness of (4)

The proof that is provided below is valid in the special case when each 2-cell from \( p \) arises from the resolution of a critical pair. The proof goes through the following stages. The first stage is the same as that of [36] and for this reason is not presented here in full. In this stage it is proved that (4) is exact in the special case when the monoid presentation \( M = \langle x, r \rangle \) from which \( D \) is defined, is complete, and the set \( p \) of homology trivializers is obtained by choosing resolutions of critical pairs of \( r \). The proof is roughly as follows. Using [5], it is shown that every 2-cycle \( \xi \in K^p \) is homologous to a 2-cycle \( \bar{\xi} \in K^p \) that is obtained from \( \xi \) by replacing the maximal 2-cell \( \sigma \) represented in \( \xi \) by a 2-chain made up of lesser 2-cells than \( \sigma \). Then we proceed by Noetherian induction.

In the second stage, differently from the general case that is considered in [36], we assume that we have a monoid presentation \( M = \langle x, r \rangle \) (not necessarily complete) and that \( H_1(D) \) of the corresponding Squier complex \( D \) is trivialized by adding 2-cells \( p \) arising from the resolution of certain critical pairs. Also, the same as in [36], we assume that \( r \) is compatible with a length-lexicographic order in the free monoid \( F \) on \( x \). Using the Knuth-Bendix procedure, we obtain a new presentation \( M^\infty = \langle x, r^\infty \rangle \) with \( r \subseteq r^\infty \) and where \( r^\infty \) is compatible with the order on \( F \). The Squier complex \( D^\infty \) has trivializer \( p^\infty \) obtained by choosing resolution of all critical pairs of \( r^\infty \) and as a consequence \( p \subseteq p^\infty \). From the special case of the first stage, we have the exactness of

\[
\begin{array}{c}
0 \longrightarrow B_2(D^\infty, p^\infty) \overset{incl.}{\longrightarrow} K^p \overset{\delta^\infty}{\longrightarrow} B_1(D^\infty) \longrightarrow 0,
\end{array}
\]

where \( K^p = C_2(D^\infty) + Jp^\infty . ZF + ZF . p^\infty . J \). We will use this and the fact that \( p \subseteq p^\infty \) to prove in a shorter way the exactness of (4).

We begin by pointing out that \((D, p)\) is a subcomplex of \((D^\infty, p^\infty)\), therefore for \( i = 1, 2, 3 \), we have that \( C_i(D, p) \subseteq C_i(D^\infty, p^\infty) \). We will define for \( i = 1, 2, 3 \), retractions \( \hat{\rho}_i : C_i(D^\infty, p^\infty) \to C_i(D, p) \).

First, for every positive edge \( e \) from \((D^\infty, p^\infty)\) not belonging to \((D, p)\), we chose a path \( \rho(e) = e_1 \cdots e_n \) in \((D, p)\) connecting \( e \) with \( \tau e \) where every \( \varepsilon_i = \pm 1 \). Relative to this choice we define

\[
\hat{\rho}_1 : C_1(D^\infty, p^\infty) \to C_1(D, p)
\]
by
\[ e \mapsto \sum_i e_i, \]
whenever \( e \) is from \( (\mathcal{D}^\infty, p^\infty) \) not belonging to \( (\mathcal{D}, p) \), and for positive edges \( e \) from \( (\mathcal{D}, p) \) we define
\[ \rho_1(e) = e. \]
Thus \( \rho_1 \) is a retraction. Before we define a second retraction \( \rho_2 : C_2(\mathcal{D}^\infty, p^\infty) \to C_2(\mathcal{D}, p) \), we prove the following.

**Lemma 4.1.** For every path \( \rho = f_1^{\beta_1} \cdots f_n^{\beta_n} \) in \( (\mathcal{D}, p) \) where every \( \beta_j = \pm 1 \), we have that
\[ \partial_1(\beta_1 f_1 + \cdots + \beta_n f_n) = \iota(\rho) - \tau(\rho). \]

**Proof.** The proof will be done by induction on \( n \). For \( n = 1 \),
\[ \partial_1(\beta_1 f_1) = \beta_1(\iota f_1 - \tau f_1), \]
therefore, depending on the sign of \( \beta_1 \), we have that \( \partial_1(\beta_1 f_1) = \iota(f_1^{\beta_1}) - \tau(f_1^{\beta_1}) \). For the inductive step, we write
\[ \rho = f_1^{\beta_1} \cdots f_n^{\beta_n} \cdot f_{n+1}^{\beta_{n+1}} = \rho_1 \cdot f_{n+1}^{\beta_{n+1}}. \]
From the assumption for \( \rho_1 \) we have
\[ \partial_1(\beta_1 f_1 + \cdots + \beta_n f_n) = \iota(\rho_1) - \tau(\rho_1) = \iota(\rho) - \iota(\rho_1), \]
and then
\[
\partial_1(\beta_1 f_1 + \cdots + \beta_n f_n + \beta_{n+1} f_{n+1}) = \partial_1(\beta_1 f_1 + \cdots + \beta_n f_n) + \partial_1(\beta_{n+1} f_{n+1}) \\
= \iota(\rho) - \iota(\rho_1) + \iota(f_{n+1}^{\beta_{n+1}}) - \tau(f_{n+1}^{\beta_{n+1}}) \\
= \iota(\rho) - \tau(\rho_1) + \iota(\rho) - \tau(\rho). \]

\[ \square \]

Now we define \( \rho_2 \) in the following way. If \( z = \sum_j \delta_j f_j \in Z_1(\mathcal{D}^\infty, p^\infty) \) is a 1-cycle where at least one of \( f_j \) is from \( (\mathcal{D}^\infty, p^\infty) \) not belonging to \( (\mathcal{D}, p) \), we have the 1-chain
\[ \rho_1 \left( \sum_j \delta_j f_j \right) \]
in $C_1(\mathcal{D}, p)$. Let us show that $\hat{\rho}_1 \left( \sum_j \delta_j f_j \right)$ is in fact a 1-cycle in $Z_1(\mathcal{D}, p)$. Indeed,

$$\partial_1 \hat{\rho}_1 \left( \sum_j \delta_j f_j \right) = \sum_j \delta_j \partial_1 \hat{\rho}_1 (f_j) = \sum_j \delta_j (\tau f_j - \tau f_j) = \partial_1^\infty \left( \sum_j \delta_j f_j \right) = \partial_1^\infty (z) = 0.$$ 

Since $p$ is a homology trivializer, then for the 1-cycle $\hat{\rho}_1 \left( \sum_j \delta_j f_j \right)$ there is a 2-chain $\varsigma_z \in C_2(\mathcal{D}, p)$ such that

$$\tilde{\partial}_2 (\varsigma_z) = \hat{\rho}_1 \left( \sum_j \delta_j f_j \right). \quad (6)$$

We can apply the above for every 2-cell $\sigma \in (\mathcal{D}^\infty, p^\infty)$ not in $(\mathcal{D}, p)$ by taking $z = \tilde{\partial}_2^\infty (\sigma)$ and writing $\varsigma_\sigma$ instead of $\varsigma_z$. With these notations (6) takes the form

$$\tilde{\partial}_2 (\varsigma_\sigma) = \hat{\rho}_1 (\tilde{\partial}_2^\infty (\sigma)). \quad (7)$$

We define

$$\hat{\rho}_2 : C_2(\mathcal{D}^\infty, p^\infty) \to C_2(\mathcal{D}, p)$$

by

$$\hat{\rho}_2(\sigma) = \sigma$$

for every 2-cell $\sigma$ in $(\mathcal{D}, p)$, and for every other 2-cell $\sigma$ we define

$$\hat{\rho}_2(\sigma) = \varsigma_\sigma.$$

We will explain how this works for 2-cells $[e, f]$ with $\hat{\rho}_1 (e) = \sum_i \alpha_i e_i$ and $\hat{\rho}_1 (f) = \sum_j \beta_j f_j$ where at least one of the sums has more than one term (the corresponding edge is not in $(\mathcal{D}, p)$). In this case we have

$$\hat{\rho}_1 \tilde{\partial}_2^\infty ([e, f]) = \hat{\rho}_1 (e (\tau f - \tau f) - (\tau e - \tau e).f) = \sum_i \alpha_i e_i \cdot \sum_j \beta_j (\tau f_j - \tau f_j) - \sum_i \alpha_i (\tau e_i - \tau e_i) \cdot \sum_j \beta_j f_j \quad \text{(lemma 4.1)}$$

$$= \sum_{i,j} \alpha_i \beta_j \tilde{\partial}_2 [e_i, f_j] = \tilde{\partial}_2 \left( \sum_{i,j} \alpha_i \beta_j [e_i, f_j] \right),$$

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therefore the 2-chain $\varsigma_{[e,f]}$ in this case can be chosen to be $\sum_{i,j} \alpha_i \beta_j [e_i, f_j]$. Again $\hat{\rho}_2$ as defined above is a retraction.

Finally, we define 

$$\hat{\rho}_3 : C_3(\mathcal{D}^\infty, p^\infty) \to C_3(\mathcal{D}, p)$$

as follows. If $e$ is any edge with $\hat{\rho}_1(e) = \sum_i \varepsilon_i e_i$ and $\sigma$ a 2-cell in $(\mathcal{D}^\infty, p^\infty)$ such that $\hat{\rho}_2(\sigma) = \sum_j \mu_j \sigma_j$, then we define

$$\hat{\rho}_3([e, \sigma]) = \sum_{i,j} \varepsilon_i \mu_j [e_i, \sigma_j] \quad \text{and} \quad \hat{\rho}_3([\sigma, e]) = \sum_{i,j} \varepsilon_i \mu_j [\sigma_j, e_i].$$

It is obvious that $\hat{\rho}_3$ is a retraction.

**Lemma 4.2.** The following hold true:

(i) $\hat{\rho}_1 \tilde{\partial}_2^\infty = \tilde{\partial}_2 \hat{\rho}_2$.

(ii) $\hat{\rho}_2 \tilde{\partial}_3^\infty = \tilde{\partial}_3 \hat{\rho}_3$.

**Proof.** (i) If $\sigma \in C_2(\mathcal{D}, p)$, then

$$\hat{\rho}_1 \tilde{\partial}_2^\infty(\sigma) = \hat{\rho}_1 \tilde{\partial}_2(\sigma) = \tilde{\partial}_2(\sigma) = \tilde{\partial}_2 \hat{\rho}_2(\sigma).$$

Assume now that $\sigma$ is a 2-cell not in $C_2(\mathcal{D}, p)$, then from the definition of $\hat{\rho}_2$ and from (7) we have

$$\tilde{\partial}_2 \hat{\rho}_2(\sigma) = \tilde{\partial}_2(\varsigma_\sigma) = \hat{\rho}_1(\tilde{\partial}_2^\infty(\sigma)).$$

(ii) Let $[e, \sigma]$ be a 3-cell where $\sigma$ is any 2-cell in $(\mathcal{D}^\infty, p^\infty)$ with $\hat{\rho}_2(\sigma) = \sum_j \mu_j \sigma_j$, and $e$ is an edge with $\hat{\rho}_1(e) = \sum_i \varepsilon_i e_i$. Assume also that $\tilde{\partial}_2^\infty(\sigma) = \sum_k \delta_k f_k$ where for each $k$, $\hat{\rho}_1(f_k) = \sum_s \beta_{ks} g_{ks}$. It follows that from (i) that

$$\tilde{\partial}_2 \left( \sum_j \mu_j \sigma_j \right) = \tilde{\partial}_2(\hat{\rho}_2(\sigma)) = \hat{\rho}_1 \tilde{\partial}_2^\infty(\sigma) = \sum_k \delta_k \sum_s \beta_{ks} g_{ks}. \quad (8)$$

If for each $j$ we let $\mathcal{T}_{\sigma_j}$ be the set of terms of $\tilde{\partial}_2(\sigma_j)$, we see from (8) that for each $k$ and each $s$, there is some $j$ and some $\alpha_j x_j \in \mathcal{T}_{\sigma_j}$, such that $\mu_j \alpha_j x_j = \delta_k \beta_{ks} g_{ks}$. This implies that for
each edge $e_i$, we have that $\delta_k \beta_{ks}[e_i, g_{ks}] = \mu_j \alpha_j[e_i, x_j]$. Further we see that

$$\hat{\rho}_2 \hat{\partial}^3_3([e, \sigma]) = \hat{\rho}_2 \left( \mu \cdot (e - \tau e) \cdot \sigma + \sum_k \delta_k[e, f_k] \right)$$

$$= \sum_j \mu_j (\sum_i \varepsilon_i (\mu e_i - \tau e_i)) \cdot \sigma_j + \sum_k \delta_k \left( \sum_i \varepsilon_i \sum_s \beta_{ks}[e_i, g_{ks}] \right)$$

$$= \sum_j \varepsilon_i \left( \sum_j \mu_j (\mu e_i - \tau e_i) \cdot \sigma_j + \sum_i \delta_k \sum_s \beta_{ks}[e_i, g_{ks}] \right)$$

$$= \sum_j \varepsilon_i \left( \mu_j (\mu e_i - \tau e_i) \cdot \sigma_j + \sum_j \mu_j \sum_{\alpha_j} \alpha_j[e_i, x_j] \right)$$

$$= \sum_i \varepsilon_i \mu_j \left( \mu e_i - \tau e_i \cdot \sigma_j + \sum_{\alpha_j} \alpha_j[e_i, x_j] \right) = \hat{\partial}^3_3 \left( \sum_{i,j} \mu_j[e_i, \sigma_j] \right)$$

(by (2))

$$= \hat{\partial}_3 \hat{\rho}_3([e, \sigma]).$$

The proof for the 3-cell $[\sigma, e]$ is similar to the above and is omitted here. \qed

**Proposition 4.3.** (Proposition 14, [36]) If $p$ is a homology trivializer obtained by choosing resolutions of certain critical pairs, then the sequence (4) is exact.

**Proof.** From the special case of proposition we have the short exact sequence

$$0 \rightarrow B_2(D^\infty, p^\infty) \xrightarrow{incl.} K^p \xrightarrow{\hat{\partial}^3_2} B_1(D^\infty) \rightarrow 0 \quad (9)$$

Let $\xi \in Ker \hat{\partial}_2$. Since $K^p \subseteq K^{p^\infty}$ and the restriction of $\hat{\partial}^3_2$ on $K^p$ is $\hat{\partial}_2$, then $\xi \in Ker \hat{\partial}^3_2$ and the exactness of (9) implies the existence of a 3-chain $w \in C_3(D^\infty, p^\infty)$ such that $\hat{\partial}^3_3(w) = \xi$. It follows from lemma 4.2 that

$$\xi = \hat{\rho}_2(\xi) = \hat{\rho}_2(\hat{\partial}^3_3(w)) = \hat{\partial}_3 \hat{\rho}_3(w),$$

which shows that $\xi \in B_2(D, p)$ and hence the exactness of (4). \qed

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4.7 The Pride complex $D(P)^*$ associated with a group presentation $P$

In this section we will explain several results of Pride in [40] where it is proved that the homotopical property FDT for groups is equivalent to the homological property $FP_3$. In order to achieve this, associated to any group presentation $\hat{P} = (x, r)$, Pride considers two crossed modules. The first one is the free crossed module $(\Sigma, \hat{F}, \partial)$ associated to $\hat{P} = (x, r)$. To define the second, he constructs first a complex $D(M)^*$ arising from the monoid presentation

$$M = \langle x, x^{-1} : R = 1(R \in r), x^\varepsilon x^{-\varepsilon} = 1(x \in x, \varepsilon = \pm 1) \rangle$$

of the same group given by $\hat{P} = (x, r)$. In other words, the group is now realized as the quotient of the free monoid $F$ on $x \cup x^{-1}$ by the smallest congruence generated by the relations giving $M$. We will give below some necessary details on this complex. We first mention that $D(M)^*$ is an extension of the usual Squier complex $D(M)$ arising from $M$. This complex is called in [12] the Pride complex. We emphasize here that the definition of $D(M)$ in [40] requires the attachment of 2-cells $[e^\varepsilon, f^\delta]$ where $e, f$ are positive edges and $\varepsilon, \delta = \pm 1$. But as it is observed in the latter paper [36] (see Remark 8 there), since we are interested in the homotopy and homology of the complex, the attachment of 2-cells $[e^\varepsilon, f^\delta]$ is unnecessary in the presence of $[e, f]$ because the boundary of $[e^\varepsilon, f^\delta]$ is a cyclic permutation of $(\partial[e, f])^{\pm 1}$, hence it is null homotopic. So we assume in what follows that $D(M)$ is that one described in section 4.2.

To complete the construction of $D(M)^*$ we need to add to $D(M)$ certain extra 2-cells along the closed paths

$$t = (1, x^\varepsilon x^{-\varepsilon}, 1, x^\varepsilon) \circ (x^\varepsilon, x^{-\varepsilon} x^\varepsilon, 1, 1)^{-1},$$

where $x \in x$ and $\varepsilon = \pm 1$. The attaching of these two cells is done for every overlap of two trivial edges as depicted below

$$\begin{align*}
&x^\varepsilon x^{-\varepsilon} x^\varepsilon \\
&(1, x^\varepsilon x^{-\varepsilon}, 1, x^\varepsilon) \left\{ \begin{array}{c} (x^\varepsilon, x^{-\varepsilon} x^\varepsilon, 1, 1) \\
(1, x^\varepsilon x^{-\varepsilon}, 1, x^\varepsilon) \end{array} \right. \\
&x^\varepsilon
\end{align*}$$

The attached 2-cell has boundary made of the following edges

$$A = (1, x^\varepsilon x^{-\varepsilon}, 1, x^\varepsilon) \quad \text{and} \quad B = (x^\varepsilon, x^{-\varepsilon} x^\varepsilon, 1, 1).$$

Together with such 2-cells, there are added to the complex all their ”translates”

$$\begin{align*}
&ux^\varepsilon x^{-\varepsilon} x^\varepsilon v \\
&(u, x^\varepsilon x^{-\varepsilon}, 1, x^\varepsilon v) \left\{ \begin{array}{c} (ux^\varepsilon, x^{-\varepsilon} x^\varepsilon, 1, v) \\
ux^\varepsilon v \end{array} \right.
\end{align*}$$

In our paper the Pride complex $D(M)^*$ is denoted throughout by $(D, t)$. 21
If $P$ is a path in $(\mathcal{D}, t)$ with $\iota(P) = W$ and $\tau(P) = Z$, there are defined in [40], $T_W$ and $T_Z$ to be arbitrary trivial paths from $W^*$ to $W$ and from $Z^*$ to $Z$ where $W^*$ and $Z^*$ are the unique reduced words freely equivalent to $W$ and $Z$ respectively. Then the path $T_W PT_Z^{-1}$ is denoted by $P^*$. The notation is not ambiguous since any two parallel trivial paths in $(\mathcal{D}, t)$ are homotopic.

Pride has defined in [40] an $\hat{F}$-crossed module $\Sigma^*$ out of $(\mathcal{D}, t)$ in the following way. The elements of $\Sigma^*$ are the homotopy classes $\langle P \rangle$ where $P$ is a path in the 1-skeleton of $(\mathcal{D}, t)$ such that $\tau(P)$ is the empty word and $\iota(P)$ is a freely reduced word from $\hat{F}$. He then defines a (non commutative) operation $+$ on $\Sigma^*$ by

$$\langle P_1 \rangle + \langle P_2 \rangle = \langle (P_1 + P_2)^* \rangle$$

and an action of the free group $\hat{F}$ on $x \cup x^{-1}$ on $\Sigma^*$ by

$$[W] \langle P \rangle = \langle (WPW^{-1})^* \rangle.$$  

Also he defines

$$\partial^*: \Sigma^* \rightarrow \hat{F}$$

by

$$\partial^*(\langle P \rangle) = [\iota(P)].$$

It is proved in [40] that the triple $(\Sigma^*, \hat{F}, \partial^*)$ is a crossed module. Further, using the fact that $\Sigma$ is the free crossed module over $r$, it is proved that

$$\eta: \Sigma \rightarrow \Sigma^*$$

defined by $r \mapsto \langle (1, r, 1, 1) \rangle$ is an isomorphism of crossed modules. The inverse $\psi: \Sigma^* \rightarrow \Sigma$ of $\eta$ is the map defined in the following way. It is first defined a map $\psi_0$ from the set of edges of $(\mathcal{D}, t)$ to $\Sigma$ as follows. Every trivial edge is mapped to 0, and every edge $(u, r, \varepsilon, v)$ is mapped to $\langle [u] r \rangle^\varepsilon$ where $[u]$ is the element of $\hat{F}$ represented by $u$. It is proved that this map extends to paths of $(\mathcal{D}, t)$ and it sends the boundaries of the defining 2-cells of $(\mathcal{D}, t)$ to 0. We thus have a morphism $\psi: \Sigma^* \rightarrow \Sigma$ given by

$$\langle P \rangle \mapsto \psi_0(P)$$

which is proved to be the inverse of $\eta$. By restriction it is obtained an isomorphism between $\text{Ker}\partial$ and $\text{Ker}\partial^*$. But $\text{Ker}\partial$ is itself isomorphic to $\pi_2(\mathcal{P})$, the second homotopy module of the standard complex associated with $\mathcal{P}$, and $\text{Ker}\partial^*$ on the other hand, is isomorphic to $\pi_1(\mathcal{D}, t, 1)$, the first homotopy group of the connected component of $(\mathcal{D}, t)$ at 1. Recollecting, we have the following isomorphisms

$$\pi_2(\mathcal{P}) = \text{Ker}\partial \frac{\eta}{\psi} \text{Ker}\partial^* = \pi_1(\mathcal{D}, t, 1). \quad (10)$$

The fundamental group $\pi_1(\mathcal{D}, t, 1)$ is abelian being isomorphic to $\pi_2(\mathcal{P})$ and therefore isomorphic to its abelianization $H_1(\mathcal{D}, t, 1)$. The role of the isomorphism between the two groups will be played by the well known Hurewicz homomorphism $h: \pi_1(\mathcal{D}, t, 1) \rightarrow H_1(\mathcal{D}, t, 1)$ which sends the homotopy class of a loop to the homology class of the corresponding 1-cycle. In our proofs in the following sections, we will identify the homotopy class of any loop $\xi$ with $h(\xi)$ without further comment.
4.8 A characterization of the asphericity in terms of the Pride complex

Assume now we are given a presentation $P = (x,r)$ of a group $G$. The new presentation $\hat{P} = (x,r \cup r^{-1})$ where $r^{-1} = \{r^{-1} : r \in r\}$, is still giving $G$. The free crossed module $(\Sigma, \hat{F}, \hat{\partial})$ of $[40]$ arising from $\hat{P}$ is in fact isomorphic to our crossed module $(\hat{G}(\Upsilon), \hat{F}, \hat{\theta})$. Indeed, there is a morphism of crossed modules $\alpha : \Sigma \rightarrow \hat{G}(\Upsilon)$ induced by the map $r_1 \mapsto \mu \sigma(r_1)$, whose inverse is $\beta : \hat{G}(\Upsilon) \rightarrow \Sigma$ defined by $\mu \sigma((u_1 r_1) ) \mapsto u_1 r_1$. So there is no loss of generality if we identify $u_1 r_1 \in \Sigma$ with $\mu \sigma((u_1 r_1)) \in \hat{G}(\Upsilon)$. The isomorphism $\Sigma \cong \hat{G}(\Upsilon)$ means in particular that $\text{Ker} \hat{\partial} \cong \text{Ker} \hat{\theta} = \Pi$.

We have on the other hand the monoid presentation of $G$

$$\mathcal{M} = \langle x, x^{-1} : s \rangle$$

where

$$s = \{(r_1, 1) : r \in r, \varepsilon = \pm 1\} \cup \{(x^n x^{-\varepsilon}, 1) : x \in x, \varepsilon = \pm 1\}.$$ 

Related to $\mathcal{M}$ we have the Pride complex $(\mathcal{D}, \mathcal{t})$. Being aspherical for $P$ means in virtue of theorem 3.4 and of isomorphisms in (10) that $H_1(\mathcal{D}, \mathcal{t})$ is trivialized as an abelian group by the homology classes of all 1-cycles corresponding to $\eta(\hat{U})$. This section is devoted to proving that the asphericity of $P$ is equivalent to $H_1(\mathcal{D}, \mathcal{p}) = 0$ where $\mathcal{p} = \mathcal{q} \cup \mathcal{t}$ and $\mathcal{q}$ is the set of 1-cycles corresponding to $\eta(\mu \sigma((r_1 r_1)^{-1} ))$ with $r \in r$.

For every two paths of positive length $A = e_1^{\delta_1} \circ \cdots \circ e_n^{\varepsilon_n}$ and $B = f_1^{\delta_1} \circ \cdots \circ f_m^{\varepsilon_m}$ in $(\mathcal{D}, \mathcal{t})$ we have two parallel paths:

$$A \cdot B \circ \tau A \cdot B \text{ and } tA \cdot B \circ A \cdot \tau B.$$ 

In what follows we use the notation $C \sim D$ to mean that two parallel paths $C$ and $D$ are homotopic to each other.

**Lemma 4.4.** For every two paths $A = e_1^{\delta_1} \circ \cdots \circ e_n^{\varepsilon_n}$ and $B = f_1^{\delta_1} \circ \cdots \circ f_m^{\varepsilon_m}$ as above, $(A \cdot B \circ \tau A \cdot B) \sim (tA \cdot B \circ A \cdot \tau B)$.

**Proof.** The proof is done by induction on the maximum of $n$ and $m$. If $m = n = 1$, then it follows that

$$A \cdot B \circ \tau A \cdot B = e_1^{\delta_1} \cdot t f_1^{\delta_1} \circ \tau e_1^{\delta_1} \cdot f_1^{\delta_1} \sim \mu e_1^{\delta_1} \cdot f_1^{\delta_1} \circ e_1^{\delta_1} \cdot \tau f_1^{\delta_1} = tA \cdot B \circ A \cdot \tau B.$$ 

Indeed, if $\varepsilon_1 = \delta_1 = 1$, then this is an immediate consequence of the 2-cell $[e_1, f_1]$. If $\varepsilon_1 = \delta_1 = -1$, then, since

$$e_1 \cdot t f_1 \circ \tau e_1 \cdot f_1 \sim \mu e_1 \cdot f_1 \circ \tau e_1 \cdot e_1 \cdot \tau f_1,$$ 

(11)

it follows by taking inverses that

$$\mu e_1^{\delta_1} \cdot f_1^{\delta_1} \circ e_1^{\delta_1} \cdot \tau f_1^{\delta_1} = \tau e_1 \cdot f_1^{-1} \circ e_1^{-1} \cdot \tau f_1$$

$$\sim e_1^{-1} \cdot \tau f_1 \circ \mu e_1 \cdot f_1^{-1}$$

$$= e_1^{\delta_1} \cdot t f_1 \circ \tau e_1^{\delta_1} \cdot f_1^{\delta_1}.$$ 

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In the case when \( \varepsilon_1 = 1 \) and \( \delta_1 = -1 \), after composing on the left of (11) by \( \iota_1 e_1 f_1^{-1} \) we obtain

\[
\iota_1 e_1 f_1^{-1} \circ e_1 \cdot f_1 \circ \tau e_1 f_1 \sim \iota_1 e_1 f_1^{-1} \circ e_1 \cdot \tau f_1 = e_1 f_1,
\]

and then after composing the above on the right by \( \tau e_1 f_1^{-1} \) we get

\[
\iota_1 e_1 f_1^{-1} \circ e_1 \cdot f_1 \sim e_1 \cdot \tau f_1 \circ \tau e_1 f_1^{-1},
\]

which is the same as

\[
\iota_1 e_1 f_1^{-1} \circ e_1 \cdot \tau f_1 \sim \iota_1 e_1 f_1^{-1} \circ \tau e_1 f_1^{-1},
\]

The proof for the case when \( \varepsilon_1 = -1 \) and \( \delta_1 = 1 \) is symmetric to the above and is omitted.

For the inductive step, let (for instance) \( B \) be the path of maximal length \( m > 1 \). For the path \( B' = f_1^{\delta_1} \circ \cdots \circ f_m^{\delta_m} \) we know by induction that

\[
A.\iota B' \circ \tau A. B' \sim \iota A. B' \circ A. \tau B'.
\]

Again, by induction for \( A \) and \( f_m^{\delta_m} \) we have that

\[
A.\iota f_m^{\delta_m} \circ \tau A. f_m^{\delta_m} \sim \iota A. f_m^{\delta_m} \circ A. \tau f_m^{\delta_m}.
\]

It follows that

\[
A.\iota B \circ \tau A. B = A.\iota B \circ \tau A. B' \circ \tau A. f_m^{\delta_m}
\]

\[
\sim \iota A. B' \circ \tau A. B' \circ \tau A. f_m^{\delta_m}
\]

\[
\sim \iota A. B' \circ \iota A. f_m^{\delta_m} \circ \tau A. f_m^{\delta_m}
\]

\[
= \iota A. B \circ A. \tau B.
\]

There is a similar proof when \( A \) is of maximal length.

For every \( u \in \hat{F} \), regarded as an element of the free monoid \( F \) on \( x \cup x^{-1} \), and for every \( r \in r \), we see that

\[
\eta(\mu \sigma(u \cdot r)) = u \eta(\mu \sigma(r)) = u \langle (1, r, 1, 1) \rangle
\]

\[
= \langle (u, r, 1, u^{-1})^* \rangle
\]

\[
= \langle T_{u \cdot r \cdot u^{-1}} \circ (u, r, 1, u^{-1}) \circ T^{-1}_{u \cdot u^{-1}} \rangle.
\]

The path \( T_{u \cdot r \cdot u^{-1}} \circ (u, r, 1, u^{-1}) \circ T^{-1}_{u \cdot u^{-1}} \) is a composition of \( T_{u \cdot r \cdot u^{-1}} \), which is a trivial path from the freely reduced word \( (u \cdot r \cdot u^{-1})^* \) to \( u \cdot r \cdot u^{-1} \) followed by the edge \( (u, r, 1, u^{-1}) \) and then by the inverse of the trivial path \( T_{u \cdot u^{-1}} \) from \( u \cdot u^{-1} \) to 1. Similarly to the above we have that

\[
\eta(\mu \sigma((u \cdot r)^{-1})) = u \eta(\mu \sigma((r^{-1})) = u \langle (1, r^{-1}, 1, 1) \rangle
\]

\[
= \langle (u, r^{-1}, 1, u^{-1})^* \rangle
\]

\[
= \langle T_{u \cdot r^{-1} \cdot u^{-1}} \circ (u, r^{-1}, 1, u^{-1}) \circ T^{-1}_{u \cdot u^{-1}} \rangle.
\]
Then we have
\[
\eta(\mu(\sigma^r(u^{-1}r^{-1}))) = \eta(\mu(\sigma^r)) + \eta(\mu(\sigma^r)^{-1})
\]
\[
= (\{T_{uru^{-1}} \circ (u,r,1,u^{-1}) \circ T_{uu^{-1}}^{-1}) + (T_{ur^{-1}u^{-1}} \circ (u,r^{-1},1,u^{-1}) \circ T_{uu^{-1}}^{-1}\})^*)
\]
\[
= (T_{(ur^{-1}u^{-1})^*} \circ (ur^{-1}u^{-1})^* \circ (u,r,1,u^{-1}) \circ (ur^{-1}u^{-1})^*)
\]
\[
\circ T_{uu^{-1}}^{-1} \circ (ur^{-1}u^{-1})^* \circ T_{ur^{-1}u^{-1}} \circ (u,r^{-1},1,u^{-1}) \circ T_{uu^{-1}}^{-1}.
\]

Now we define two closed paths in \((D,t)\). First we let
\[
P(r,u) = T_{(ur^{-1}u^{-1})} \circ (u,r,1,u^{-1}ur^{-1}u^{-1})
\]
\[
\circ (T_{uu^{-1}}^{-1} \circ (ur^{-1}u^{-1}) \circ (u,r^{-1},1,u^{-1}) \circ T_{uu^{-1}},
\]
and second
\[
Q(r,u) = T_{ur^{-1}u^{-1}} \circ (u,r,1,r^{-1}u^{-1}) \circ (u,r^{-1},1,u^{-1}) \circ T_{uu^{-1}}^{-1}.
\]

**Proposition 4.5.** The presentation \(\mathcal{P}\) is aspherical if and only if \(\pi_1(D,t,1)\) is generated as an abelian group by the homotopy classes of loops \(Q(r,u)\) with \(r \in \mathcal{R}\) and \(u \in \mathcal{F}\).

**Proof.** First note that the presentation \(\mathcal{P}\) is aspherical if and only if the set of all \(\eta(\mu(\sigma^r(u^{-1}r^{-1})))\) generates \(\pi_1(D,t,1)\). The claim follows directly if we prove that for each \(r \in \mathcal{R}\) and \(u \in \mathcal{F}\),
\[
\eta(\mu(\sigma^r(u^{-1}r^{-1}))) = \langle P(r,u) \rangle
\]
and \(P(r,u) \sim Q(r,u)\).

Let us prove first that \(\eta(\mu(\sigma^r(u^{-1}r^{-1}))) = \langle P(r,u) \rangle\). Consider the following paths in \((D,t)\). First, we let
\[
a = (ur^{-1}u^{-1})^* \cdot T_{ur^{-1}u^{-1}},
\]
\[
b = T_{ur^{-1}u^{-1}} \cdot (ur^{-1}u^{-1}),
\]
\[
c = T_{ur^{-1}u^{-1}} \cdot (ur^{-1}u^{-1})^*,
\]
\[
d = (ur^{-1}u^{-1})^* \cdot T_{ur^{-1}u^{-1}},
\]
and observe from lemma 3.4 that
\[
a \circ b \sim c
\]
(12)

Second, we let
\[
e = (u,r,1,u^{-1}(ur^{-1}u^{-1})^*),
\]
\[
f = (uu^{-1}) \cdot T_{uu^{-1}}^{-1},
\]
\[
g = (u,r,1,u^{-1}(ur^{-1}u^{-1})),
\]
where again from lemma 3.4 we have that \(c \circ g \sim e \circ f\). This implies that
\[
c^{-1} \sim g \circ f^{-1} \circ e^{-1}.
\]
(13)
And finally, let
\[ y = T_{uu-1}^{-1} \cdot (ur^{-1}u^{-1})^* \]
\[ f = (uu^{-1}) \cdot T_{ur^{-1}u^{-1}} \]
\[ x = T_{uu-1}^{-1} \cdot (ur^{-1}u^{-1}) \]
\[ z = 1 \cdot T_{ur^{-1}u^{-1}}, \]
where as before \( f \circ x \sim y \circ z. \) This on the other hand implies that
\[ f^{-1} \sim x \circ z^{-1} \circ y^{-1}. \] (14)

Further we write \( \ell = T_{(ur^{-1})^*(ar^{-1}u^{-1})}, \) \( \ell_1 = T_{(ur^{-1})(ar^{-1}u^{-1})}, k = (u, r^{-1}, 1, u^{-1}) \) and \( h = T_{uu-1}. \) With the above abbreviations we have
\[ \ell \sim \ell_1 \circ b^{-1} \circ a^{-1} \] (\( \| \) trivial parallel paths are \( \sim \))
\[ \sim \ell_1 \circ c^{-1} \circ d^{-1} \] (from (12))
\[ \sim \ell_1 \circ (g \circ f^{-1} \circ e^{-1}) \circ d^{-1} \] (from (13))
\[ \sim \ell_1 \circ g \circ (x \circ z^{-1} \circ y^{-1}) \circ e^{-1} \circ d^{-1} \] (from (14)).

It follows that
\[ \eta(\mu\sigma(u r^{-1} u^{-1})) = \langle \ell \circ (d \circ e \circ y \circ z \circ k \circ h) \rangle \]
\[ = \langle (\ell_1 \circ g \circ (x \circ z^{-1} \circ y^{-1}) \circ e^{-1} \circ d^{-1}) \circ (d \circ e \circ y \circ z \circ k \circ h) \rangle \]
\[ = \langle \ell_1 \circ g \circ x \circ k \circ h \rangle \]
\[ = \langle P(r, u) \rangle. \]

Secondly, we prove that \( P(r, u) \sim Q(r, u). \) Indeed, if we consider paths
\[ A = (u, r, 1, u^{-1}r^{-1}u^{-1}) \]
\[ B = (ur) \cdot T_{u^{-1}u}^{-1} \cdot (r^{-1}u^{-1}) \]
\[ C = u \cdot T_{u^{-1}u}^{-1} \cdot (r^{-1}u^{-1}) \]
\[ D = (u, r, 1, r^{-1}u^{-1}) \]
which from lemma 4.4 satisfy \( A \circ C \sim B \circ D, \) then we have
\[ P(r, u) = T_{(ur^{-1})(ar^{-1}u^{-1})} \circ (u, r, 1, u^{-1}r^{-1}u^{-1}) \circ (T_{uu^{-1}}^{-1} \cdot ur^{-1}u^{-1}) \circ (u, r^{-1}, 1, u^{-1}) \circ T_{uu^{-1}}^{-1} \]
\[ \sim T_{(ur^{-1})(ar^{-1}u^{-1})} \circ A \circ (u \cdot T_{u^{-1}u}^{-1} \cdot r^{-1}u^{-1}) \circ (u, r^{-1}, 1, u^{-1}) \circ T_{uu^{-1}}^{-1} \]
\[ = T_{(ur^{-1})(ar^{-1}u^{-1})} \circ A \circ C \circ (u, r^{-1}, 1, u^{-1}) \circ T_{uu^{-1}}^{-1} \]
\[ \sim T_{(ur^{-1})(ar^{-1}u^{-1})} \circ B \circ D \circ (u, r^{-1}, 1, u^{-1}) \circ T_{uu^{-1}}^{-1} \]
\[ \sim T_{uu^{-1}u^{-1}} \circ B^{-1} \circ B \circ D \circ (u, r^{-1}, 1, u^{-1}) \circ T_{uu^{-1}}^{-1} \]
\[ = T_{uu^{-1}u^{-1}} \circ D \circ (u, r^{-1}, 1, u^{-1}) \circ T_{uu^{-1}}^{-1} \]
\[ = Q(r, u). \]

This concludes the proof.
Passing to homology we have the following

**Proposition 4.6.** The presentation $\mathcal{P}$ is aspherical if and only if $H_1(D, t, 1)$ is generated as an abelian group by the homology classes of 1-cycles corresponding to loops $Q(r, u)$ with $r \in r$ and $u \in F$.

**Definition 4.7.** In $(D, t)$ we let $q$ be the set of closed paths $Q(r, 1)$ with $r \in r$.

If we attach to $(D, t)$ 2-cells $\sigma$ along the closed paths $u.Q(r, 1), v$ with $u, v \in F$ and 3-cells $[e, \sigma]$ and $[\sigma, e]$ for every 2-cell $\sigma$ and each positive edge $e$, then we obtain a new 3-complex $(D, q \cup t)$. The asphericity of $\mathcal{P}$ is encoded in the homology of $(D, q \cup t)$ as the following shows.

**Theorem 4.8.** The presentation $\mathcal{P}$ is aspherical if and only if $H_1(D, q \cup t) = 0$.

To prove the theorem we first note the following two lemmas.

**Lemma 4.9.** For every $\varsigma \in Z_1(D, t)$ and every $u, v \in F$ such that $\overline{u} = \overline{v}$, $\varsigma \cdot u + B_1(D, t) = \varsigma \cdot v + B_1(D, t)$.

**Proof.** It is enough to prove that for every positive edge $f$, we have $\varsigma \cdot \iota f + B_1(D, t) = \varsigma \cdot \tau f + B_1(D, t)$. From Lemma 4.1 of [39] it follows that $\varsigma \cdot (\iota f - \tau f) \in B_1(D)$. But $B_1(D) \subseteq B_1(D, t)$ and we are done.

If $u = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \in F$ is any word of length $|u| = n \in \mathbb{N}$, then a trivial path from 1 to $uu^{-1}$ is the following

$$T_{uu^{-1}} = (1, x_1^{\varepsilon_1}x_1^{-\varepsilon_1}, 1, 1)^{-1} \circ \cdots \circ (x_1^{\varepsilon_1} \ldots x_{|u|-1}^{\varepsilon_{|u|-1}}, x_{|u|}^{\varepsilon_{|u|}}, x_{|u|}^{-\varepsilon_{|u|}}, 1, x_{|u|-1}^{\varepsilon_{|u|-1}} \ldots x_1^{-\varepsilon_1}, 1)^{-1}.$$  

We write for short

$$t_u^{(1)} = (1, x_1^{\varepsilon_1}x_1^{-\varepsilon_1}, 1, 1),$$

$$\cdots$$

$$t_u^{(|u|)} = (x_1^{\varepsilon_1} \ldots x_{|u|-1}^{\varepsilon_{|u|-1}}, x_{|u|}^{\varepsilon_{|u|}}, x_{|u|}^{-\varepsilon_{|u|}}, 1, x_{|u|-1}^{\varepsilon_{|u|-1}} \ldots x_1^{-\varepsilon_1}, 1)^{-1},$$

and let

$$\tau_{uu^{-1}} = t_u^{(1)} + \cdots + t_u^{(|u|)}.$$

**Definition 4.10.** For every $r \in r$ and $u \in F^*$, we let

$$q(r, u) = (u, r, 1, r^{-1}u^{-1}) + (u, r^{-1}, 1, u^{-1}) + \tau_{uu^{-1}} - \tau_{urr^{-1}u^{-1}},$$

be the 1-cycle that corresponds to the closed path $Q(r, u)$. When $u = 1$, we let

$$q(r, 1) = (1, r, 1, r^{-1}) + (1, r^{-1}, 1, 1) - \tau_{rr^{-1}},$$

be the 1-cycle corresponding to $Q(r, 1)$.
Lemma 4.11. For every $r \in R$ and $u \in F$, $u.q(r, 1).u^{-1} + B_1(D, t) = q(r, u) + B_1(D, t)$.

Proof. First note that $T_{rr}^{-1} \circ T_{rr}^{-1} \sim u. T_{rr}^{-1}u^{-1}$ since any two trivial paths with the same end points are homotopic with each other. For the corresponding 1-chains we have that

$$\tau_{uu^{-1}} - \tau_{rr^{-1}u^{-1}} + B_1(D, t) = -u. \tau_{rr^{-1}u^{-1}} + B_1(D, t).$$

It follows now that

$$u.q(r, 1).u^{-1} + B_1(D, t) = (u, r, 1, r^{-1}u^{-1}) + (u, r^{-1}, 1, u^{-1}) - u. \tau_{rr^{-1}u^{-1}} + B_1(D, t)$$

$$= (u, r, 1, r^{-1}u^{-1}) + (u, r^{-1}, 1, u^{-1}) + \tau_{uu^{-1}} - \tau_{rr^{-1}u^{-1}} + B_1(D, t)$$

$$= u.q(r, 1).u^{-1} + B_1(D, t),$$

proving the claim. ∎

Proof. (of theorem 4.8) If $H_1(D, q \cup t) = 0$, then $H_1(D, q \cup t, 1) = 0$ which means that the homology classes of the loops $u.Q(r, 1).v$ trivialize $H_1(D, t, 1)$. We claim that every 1-cycle corresponding to a loop $u.Q(r, 1).v$ which sits inside $(D, t, 1)$ is in fact homologous to the 1-cycle corresponding to the loop $Q(r, u)$. Indeed, since $u.Q(r, 1).v$ is a loop in $(D, t, 1)$, then $v = u^{-1}$. It follows from lemma 4.9 and lemma 4.11 that

$$u.q(r, 1).v + B_1(D, t) = u.q(r, 1).u^{-1} + B_1(D, t)$$

$$= q(r, u) + B_1(D, t),$$

which proves our claim. As a consequence of this we have that the homology classes of 1-cycles $q(r, u)$ trivialize $H_1(D, t, 1)$, and then from proposition 4.6 we get the asphericity of $P$.

Conversely, if $P$ is aspherical, then from proposition 4.6 and lemma 4.11 $H_1(D, t, 1)$ is generated as an abelian group by the homology classes of 1-cycles $u.q(r, 1).u^{-1}$. But from 4.10 the homology group $H_1(D, t, w)$ of the connected component $(D, t, w)$ is isomorphic to $H_1(D, t, 1)$, where the isomorphism $\phi_w : H_1(D, t, 1) \to H_1(D, t, w)$ maps each homology class of some 1-cycle $\xi$ to the homology class of $\xi \cdot w$. This shows that the set of the homology classes of 1-cycles $u.q(r, 1).u^{-1}w$ trivialize $H_1(D, t)$. We prove that this set equals to the set of homology classes of 1-cycles $u.q(r, 1).v$ where $u, v \in F$. Indeed, for every $u, v \in F$ and every $q(r, 1)$, if we take $w = uv$, we get that $u.q(r, 1).u^{-1}uv + B_1(D, t)$ is a generator of $H_1(D, t)$. But from lemma 4.9 $u.q(r, 1).u^{-1}uv + B_1(D, t) = u.q(r, 1).v + B_1(D, t)$, hence $u.q(r, 1).v + B_1(D, t)$ is a generator of $H_1(D, t)$. For the converse, it is obvious that any generator $u.q(r, 1).u^{-1}w + B_1(D, t)$ is of the form $u.q(r, 1).v + B_1(D, t)$ with $v = u^{-1}w$. ∎

Remark 4.12. The Squier complex $D$ of the monoid presentation $M = \langle x, x^{-1} : s \rangle$ of $G$ has an important property. As the theorem 4.8 shows, the homology trivializers of $H_1(D)$ are classes of 1-cycles corresponding to loops from $q \cup t$ and each one of them arises from the resolution of a critical pair. Indeed, if $r \in R$ has the reduced word form $r = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ in
\(\hat{F}\), then considering \(x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}\) as a word from \(F\), we see that the loop \(Q(r, 1)\) is obtained by resolving the following overlapping pair of edges

\[
((1, r, 1, r^{-1}), (x_1^{\epsilon_1} \cdots x_n^{\epsilon_n-1}, x_n^{\epsilon_n} x_n^{\epsilon_n-1}, 1, x_n^{\epsilon_n-1} \cdots x_1^{\epsilon_1})).
\]

On the other hand, if \(t = (1, x^\epsilon x^{-\epsilon}, 1, x^\epsilon) \circ (x^\epsilon, x^{-\epsilon} x^\epsilon, 1, 1)^{-1}\) is a loop of \(t\), then it arises from the resolution of the overlapping pair

\[
((1, x^\epsilon x^{-\epsilon}, 1, x^\epsilon), (x^\epsilon, x^{-\epsilon} x^\epsilon, 1, 1)).
\]

The importance of this remark stands at the fact that when the given presentation \(P\) is aspherical, then the sequence \([\mathcal{A}]\) that is associated with the complex \((\mathcal{D}, q \cup t)\) is exact.

### 4.9 A preliminary result

Let \(P = (x, r)\) be an aspherical group presentation and \(P_1 = (x, r_1)\) a subpresentation of the first where \(r_1 = r \setminus \{r_0\}\) and \(r_0 \in r\) is a fixed relation. We denote by \(\mathcal{Y}_1, \mathcal{U}_1\) monoids associated with \(P_1\) and by \(G(\mathcal{Y}_1)\) and \(\mathcal{U}_1\) their respective groups and let \(\hat{\theta}_1\) be the morphism of the crossed module \(G(\mathcal{Y}_1)\) whose kernel is denoted by \(\tilde{\Pi}_1\). Also we consider \(\mathfrak{A}_1\) the subgroup of \(\mathcal{U}\) generated by all \(\mu (b a^{-1})\) where \(a \in \mathcal{Y}_1 \cup \mathcal{Y}_1^{-1}\). Finally note that the monomorphism \(\varphi : \mathcal{Y}_1 \to \mathcal{Y}\) induced by the map \(\sigma_1(a) \to \sigma(a)\) induces a homomorphism \(\varphi : G(\mathcal{Y}_1) \to G(\mathcal{Y})\).

These data fit into a commutative diagram as depicted below.

\[
\begin{array}{ccc}
G(\mathcal{Y}_1) & \xrightarrow{\hat{\phi}} & G(\mathcal{Y}) \\
\downarrow{\hat{\theta}_1} & & \downarrow{\hat{\theta}'} \\
F & & \\
\end{array}
\]

The following will be useful in the proof of our main theorem.

**Proposition 4.13.** If \(P\) is aspherical, then \(\varphi(\tilde{\Pi}_1) = \mathfrak{A}_1\).

**Proof.** Let \(\tilde{d} = \mu_1 \sigma_1(a_1 \cdots a_n) \in \tilde{\Pi}_1\) where as before no \(a_i\) is equal to any \(i(\mu_1 \sigma_1(u) a)\) and assume that

\[
\varphi(\tilde{d}) = (\mu \sigma(b_1 b_1^{-1}) \cdots \mu \sigma(b_s b_s^{-1})) \cdot (i(\mu \sigma(b_s b_{s+1}^{-1})) \cdots i(\mu \sigma(b_r b_r^{-1})))
\]

\[
(\mu \sigma(c_1 c_1^{-1}) \cdots \mu \sigma(c_i c_i^{-1})) \cdot (i(\mu \sigma(d_1 d_1^{-1})) \cdots i(\mu \sigma(d_k d_k^{-1}))),
\]

where the first half involves elements from \(\mathcal{Y}_1 \cup \mathcal{Y}_1^{-1}\) and the second one is

\[
\mu \sigma(C)i(\mu \sigma(D))
\]

with

\[
C = c_1 c_1^{-1} \cdots c_i c_i^{-1} \quad \text{and} \quad D = d_1 d_1^{-1} \cdots d_k d_k^{-1},
\]

29
where \( C \) and \( D \) involve only elements of the form \((u_r^\varepsilon)\) with \( \varepsilon = \pm 1 \). Recalling from above that in \( G(\Upsilon) \) we have
\[
\mu_\sigma((a_1 \cdots a_n) \cdot ((b_{s+1} b_{s+1}^{-1}) \cdots (b_s b_s^{-1})) \cdot ((d_1 d_1^{-1}) \cdots (d_k d_k^{-1}))) = \mu_\sigma(((b_1 b_1^{-1}) \cdots (b_s b_s^{-1})) \cdot ((c_1 c_1^{-1}) \cdots (c_k c_k^{-1}))),
\]
we can apply \( \hat{g} \) defined in proposition 3.5 on both sides and get
\[
g\sigma((a_1 \cdots a_n) \cdot ((b_{s+1} b_{s+1}^{-1}) \cdots (b_s b_s^{-1})) \cdot ((d_1 d_1^{-1}) \cdots (d_k d_k^{-1}))) = g\sigma(((b_1 b_1^{-1}) \cdots (b_s b_s^{-1})) \cdot ((c_1 c_1^{-1}) \cdots (c_k c_k^{-1}))).
\]
If we now write each \( c_i = (u_i r_0)^{\varepsilon_i} \) and each \( d_j = (v_j r_0)^{\delta_j} \) where \( \varepsilon_i \) and \( \delta_j = \pm 1 \), while we write each \( a_\ell = (u_\ell r_\ell)^{\gamma_\ell} \) and each \( b_p = (v_p r_p)^{\xi_p} \) where all \( r_\ell \) and \( r_p \) belong to \( r_1 \) and \( \gamma_\ell, \xi_p = \pm 1 \), then the definition of \( g \) yields
\[
(w_1^\alpha r_1^\beta + \cdots + w_n^\alpha r_n^\beta) + (2\eta_{s+1}^\alpha \cdot \rho_{s+1}^\beta + \cdots + 2\eta_s^\alpha \cdot \rho_s^\beta) + (2v_1^\alpha + \cdots + 2v_k^\alpha) \cdot r_0^\beta = (2\eta_{s+1}^\alpha \cdot \rho_{s+1}^\beta + \cdots + 2\eta_s^\alpha \cdot \rho_s^\beta) + (2u_1^\alpha + \cdots + 2u_t^\alpha) \cdot r_0^\beta
\]
The freeness of \( \mathcal{N}(\mathcal{P}) \) on the set of elements \( r^\beta \) implies in particular that
\[
(2v_1^\alpha + \cdots + 2v_k^\alpha) \cdot r_0^\beta = (2u_1^\alpha + \cdots + 2u_t^\alpha) \cdot r_0^\beta
\]
from which we see that \( k = t \), and after a rearrangement of terms \( u_i^\alpha = v_i^\alpha \) for \( i = 1, \ldots, k \).
The easily verified fact that in \( G(\Upsilon) \), \( \mu_\sigma(a a^{-1}) = \mu_\sigma(a^{-1} a) \) and the fact that if \( u^\alpha = v^\alpha \), then for every \( s \in r \), \( \mu_\sigma((v^s)^\delta (v^s)^{-\delta}) = \mu_\sigma((u^s)^\delta (u^s)^{-\delta}) \), imply easily that
\[
\mu_\sigma((v^s r_0)^\delta (v^s r_0)^{-\delta}) = \mu_\sigma((u^r_0)^{\varepsilon} (u^r_0)^{-\varepsilon}).
\]
If we apply the latter to pairs \( (c_i, d_i) \) for which \( u_i^\alpha = v_i^\alpha \), we get that \( \mu_\sigma(C) \nu(\mu_\sigma(D)) = 1 \) which shows that \( \hat{\varphi}(d) \in \mathfrak{A}_1 \).  

\section{4.10 The proof}

Throughout this section we assume that \( \mathcal{P} = (x, r) \) is an aspherical presentation of the trivial group. Consider now a sub presentation \( \mathcal{P}_1 = (x, r_1) \) of \( \mathcal{P} \) where \( r_1 = r \setminus \{r_0\} \). For each of the above group presentations, we have a monoid presentation of the same group, namely
\[
\mathcal{M} = \langle x, x^{-1} : s \rangle
\]
is a monoid presentation of the trivial group, where
\[
s = \{(r^\varepsilon, 1) : r \in r, \varepsilon = \pm 1\} \cup \{(x^\varepsilon x^{-\varepsilon}, 1) : x \in x, \varepsilon = \pm 1\},
\]
and
\[
\mathcal{M}_1 = \langle x, x^{-1} : s_1 \rangle
\]
is a monoid presentation of the group given by $\mathcal{P}_1$, where
\[ s_1 = \{(r_1^e, 1) : r_1 \in \mathbf{r}_1, \varepsilon = \pm 1\} \cup \{(x^f x^{-e}, 1) : x \in \mathbf{x}, \varepsilon = \pm 1\} .\]
Related to $\mathcal{M}$ we have defined two 2-complexes. The first one is the usual Squier complex $\mathcal{D}$, and the second one is its extension $(\mathcal{D}, \mathbf{t})$, and similarly we have two 2-complexes arising from $\mathcal{M}_1$, $\mathcal{D}_1$ and its extension $(\mathcal{D}_1, \mathbf{t})$. Further, $(\mathcal{D}, \mathbf{t})$ has been extended to a 3-complex $(\mathcal{D}, \mathbf{q} \cup \mathbf{t})$ by adding first 2-cells arising from $Q(r, 1)$ and their translates, and then adding all the 3-cells $[e, \sigma]$ or $[\sigma, e]$ for every 2-cell $\sigma$ and every positive edge $e$. We write for short $(\mathcal{D}, \mathbf{q} \cup \mathbf{t})$ by $(\mathcal{D}, \mathbf{p})$ where $\mathbf{p} = \mathbf{q} \cup \mathbf{t}$. Likewise, $(\mathcal{D}_1, \mathbf{t})$ extends to a 3-complex $(\mathcal{D}_1, \mathbf{p}_1)$ where $\mathbf{p}_1 = \mathbf{q}_1 \cup \mathbf{t}$ and $\mathbf{q}_1$ is the set of 2-cells arising from $Q(r_1, 1)$ with $r_1 \in \mathbf{r}_1$. But $(\mathcal{D}_1, \mathbf{p}_1)$ is a subcomplex of $(\mathcal{D}, \mathbf{p})$, therefore we have the following exact sequence of abelian groups
\[ \ldots \rightarrow H_2(\mathcal{D}, \mathbf{p}) \rightarrow H_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1)) \rightarrow H_1(\mathcal{D}, \mathbf{p}_1) \rightarrow H_1(\mathcal{D}, \mathbf{p}) \rightarrow \ldots. \]
We know from theorem 4.8 that $H_1(\mathcal{D}, \mathbf{p}) = 0$, so if we prove that $H_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1)) = 0$, then the exactness of the above sequence will imply that $H_1(\mathcal{D}_1, \mathbf{p}_1) = 0$ and we are done. Before we proceed with the proof, we explain how the boundary maps for the corresponding quotient complex are defined. For this we consider the commutative diagram
\[
\begin{array}{ccc}
C_3(\mathcal{D}, \mathbf{p}) & \xrightarrow{\delta_3} & C_2(\mathcal{D}, \mathbf{p}) \xrightarrow{\delta_2} C_1(\mathcal{D}, \mathbf{p}) \\
\mu_3 & & \mu_2 \downarrow \mu_1 \\
C_3(\mathcal{D}, \mathbf{p})/C_3(\mathcal{D}_1, \mathbf{p}_1) & \xrightarrow{\delta_3} & C_2(\mathcal{D}, \mathbf{p})/C_2(\mathcal{D}_1, \mathbf{p}_1) \xrightarrow{\delta_2} C_1(\mathcal{D}, \mathbf{p})/C_1(\mathcal{D}_1, \mathbf{p}_1)
\end{array}
\]
where $\mu_i$ for $i = 1, 2, 3$ are the canonical epimorphisms. Then, for $i = 2, 3$ and for every $\sigma \in C_i(\mathcal{D}, \mathbf{p})$ we have
\[ \hat{\delta}_i(\mu_i \sigma) = \mu_{i-1} \hat{\delta}_i(\sigma). \]
We write $\text{Im}(\hat{\delta}_3) = B_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1))$, and similarly $\text{Im}(\hat{\delta}_2) = B_1((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1))$. Also we let $Z_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1)) = \text{Ker}(\hat{\delta}_2)$ and then
\[ H_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1)) = Z_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1))/B_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1)). \]
We note that
\[ C_2(\mathcal{D}, \mathbf{p})/C_2(\mathcal{D}_1, \mathbf{p}_1) \cong \mu_2(C_2(\mathcal{D})) \oplus C_2^{q_0}, \]
where $\mu_2(C_2(\mathcal{D}))$ is the free abelian group generated by all 2-cells $[e, f]$ where at least one of the edges $e$ or $f$ arises from $r_0$, and $C_2^{q_0}$ is the free abelian group on 2-cells $u.q_0.v$ where $q_0$ is the 2-cell attached along $Q(r_0, 1)$. We can thus regard $Z_2((\mathcal{D}, \mathbf{p}), (\mathcal{D}_1, \mathbf{p}_1))$ as a subgroup of $\mu_2(C_2(\mathcal{D})) \oplus C_2^{q_0}$. Now we let
\[ \varphi_{rel} : \mu_2(C_2(\mathcal{D})) \oplus C_2^{q_0} \rightarrow ZG.q_0.ZG \]
be the $(ZF, ZF)$-homomorphism defined by mapping $\mu_2(C_2(\mathcal{D}))$ to 0, and every 2-cell $u.q_0.v$ to $\tilde{u}.q_0.\tilde{v}$. Denote the kernel of $\varphi_{rel}$ by $K^{q_0}_{rel}$. By the same argument as that of [36] we see that
\[ K^{q_0}_{rel} = \mu_2(C_2(\mathcal{D})) + J.q_0.ZF + ZF.q_0.J. \]
Latter we will make use of the fact that $K_{rel}^q$ can be regarded as a sub group of $K^p$.

Next we show that $B_2((D, p), (D_1, p_1)) \subseteq K_{rel}^q$ and that the restriction of $\hat{\partial}_2$ on $K_{rel}^q$ sends $K_{rel}^q$ onto the subgroup $B_1(D, D_1)$ of $C_1(D, p)/C_1(D_1, p_1)$ defined by

$$B_1(D, D_1) = \left\{ \beta + C_1(D_1) | \beta \in \tilde{\partial}_2(C_2(D)) \right\}.$$  

To see that $B_2((D, p), (D_1, p_1)) \subseteq K_{rel}^q$, we must prove that for every 3-cell $[f, \sigma]$ or $[\sigma, f]$,

$$\hat{\partial}_3 ([f, \sigma] + C_3(D_1, p_1)) \in K_{rel}^q$$

and similarly,

$$\hat{\partial}_3 ([\sigma, f] + C_3(D_1, p_1)) \in K_{rel}^q.$$  

We prove the second for convenience. Let $[\sigma, f] \notin C_3(D_1, p_1)$. If $\sigma \in F.q_0.F$ or $f$ arises from $r_0$, then clearly

$$\hat{\partial}_3 ([\sigma, f] + C_3(D_1, p_1)) = \left( \sigma.(i.f - \tau.f) - \sum_i \varepsilon_i [e_i, f] \right) + C_2(D_1, p_1) \in K_{rel}^q.$$  

Otherwise, if $\sigma \notin q_0$ and $f$ arises from $r_1$, then $\sigma = [g, h]$ where at least $g$ or $h$ arises from $r_0$. Again we see that $\hat{\partial}_3 ([\sigma, f] + C_3(D_1, p_1)) \in K_{rel}^q$.

Next we prove that the restriction of $\hat{\partial}_2$ on $K_{rel}^q$ sends $K_{rel}^q$ onto $B_1(D, D_1)$. Indeed, since for every $(i.f - \tau.f).\sigma_0 \in J.q_0.F$

$$(i.f - \tau.f).\sigma_0 = \hat{\partial}_3 [f, \sigma_0] - \sum_i \varepsilon_i [f, e_i],$$

then we can derive that

$$\hat{\partial}_2 ((i.f - \tau.f).\sigma_0) = -\hat{\partial}_2 \left( \sum_i \varepsilon_i [f, e_i] \right)$$

$$= -\sum_i \varepsilon_i \hat{\partial}_2 [f, e_i] + C_1(D_1) \in B_1(D, D_1).$$

In a symmetric way one can show that $\hat{\partial}_2 (\sigma_0.(i.f - \tau.f)) \in B_1(D, D_1)$. Finally, if $[e, f] \in \mu_2(C_2(D))$, then

$$\hat{\partial}_2 [e, f] = \hat{\partial}_2 [e, f] + C_1(D_1) \in B_1(D, D_1).$$

This also shows that $\hat{\partial}_2$ is onto.

Therefore we have the complex

$$\begin{array}{ccccccc}
0 & \longrightarrow & B_2((D, p), (D_1, p_1)) & \xrightarrow{incl.} & K_{rel}^q & \xrightarrow{\hat{\partial}_2} & B_1(D, D_1) & \longrightarrow & 0.
\end{array}$$  

which is exact on the left and on the right.
Lemma 4.14. The complex (16) is exact.

Proof. For this we consider the commutative diagram

\[ \begin{array}{ccccccc}
0 & \longrightarrow & B_2(D, p) & \stackrel{\text{incl.}}{\longrightarrow} & K^p & \stackrel{\hat{\partial}_2}{\longrightarrow} & B_1(D) & \longrightarrow & 0 \\
\mu_2 & & \downarrow \mu_2 & & \downarrow \mu_1 & & \downarrow \mu_1 & & \\
0 & \longrightarrow & B_2((D, p), (D_1, p_1)) & \stackrel{\text{incl.}}{\longrightarrow} & K_{rel}^{q_0} & \stackrel{\hat{\partial}_2}{\longrightarrow} & B_1(D, D_1) & \longrightarrow & 0
\end{array} \]

The top row is exact from proposition 4.3 and from remark 4.12, and \( \mu_1, \mu_2 \) are the restrictions of the epimorphisms of (15). Let \( \xi = \sum_i z_i \mu_2(\sigma_i) \in \text{Ker} \hat{\partial}_2 \). We recall that \( \xi \) can be regarded as an element of \( K^p \) with no terms arising from \( t \) or square 2-cells \( [e, f] \) with both \( e \) and \( f \) in \( D_1 \). Further we have that

\[
0 = \hat{\partial}_2 \left( \sum_i z_i \mu_2(\sigma_i) \right) = \sum_i z_i \hat{\partial}_2 \mu_2(\sigma_i) = \sum_i z_i \mu_1 \hat{\partial}_2(\sigma_i) = \mu_1 \hat{\partial}_2 \left( \sum_i z_i \sigma_i \right),
\]

which implies that \( \hat{\partial}_2 \left( \sum_i z_i \sigma_i \right) \in C_1(D_1) \), and so \( \hat{\partial}_2 \left( \sum_i z_i \sigma_i \right) \) is a 1-cycle in \( Z_1(D_1) \). It follows that

\[
\hat{\partial}_2 \left( \sum_i z_i \sigma_i \right) \in \text{Ker} \hat{\partial}_1 \cap (J_s F + Z F J_s J).
\]

We note that each term from \( J_s F + Z F J_s J \) that is represented in \( \hat{\partial}_2 \left( \sum_i z_i \sigma_i \right) \) arises either from a 2-cell \( [e, f] \) where at least one of \( e \) or \( f \) is a positive edges that belongs to \( D_1 \), or arises from an element of the form \( j q_0 v \) or \( u q_0 j \) with \( u, v \in F \) and \( j \in J \). Theorem 6.6 of \cite{33} implies that there is a 2-chain \( \sum_j k_j \beta_j \in C_2(D_1) \) such that \( \hat{\partial}_2 \left( \sum_i z_i \sigma_i \right) = \hat{\partial}_2 \left( \sum_j k_j \beta_j \right) \) and then we have the 2-cycle \( \tilde{\xi} = \sum_i z_i \sigma_i - \sum_j k_j \beta_j \in K^p \). It follows that \( \tilde{\xi} \) is a 2-boundary since the top row is exact, and has the property that

\[
\mu_2(\tilde{\xi}) = \mu_2 \left( \sum_i z_i \sigma_i \right) - \mu_2 \left( \sum_j k_j \beta_j \right) = \mu_2 \left( \sum_i z_i \sigma_i \right) \quad \text{(since each} \beta_j \in C_2(D_1) \text{)}
\]

\[
= \sum_i z_i \mu_2(\sigma_i) = \xi,
\]

hence for some \( w \in C_3(D, p) \) we have that

\[
\xi = \mu_2(\tilde{\xi}) = \mu_2(\hat{\partial}_3(w)) = \hat{\partial}_3 \mu_3(w).
\]
This proves that $\xi$ is a relative 2-boundary and as a consequence the exactness of the bottom row.

Further we note that $B_1(\mathcal{D}, \mathcal{D}_1)$ embeds in $\text{Im}(\hat{\partial}_2)$. Indeed, any element $\hat{\partial}_2(\xi) + C_1(\mathcal{D}_1)$ of $B_1(\mathcal{D}, \mathcal{D}_1)$ where $\xi$ is a 2-chain from $C_2(\mathcal{D})$ is in $\text{Im}(\hat{\partial}_2)$ since $C_2(\mathcal{D}) \leq C_2(\mathcal{D}, p)$ and $C_1(\mathcal{D}_1, p_1) = C_1(\mathcal{D}_1)$.

Finally, consider the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & B_2((\mathcal{D}, p), (\mathcal{D}_1, p_1)) \\
& \downarrow{\iota} & \downarrow{\iota} \\
0 & \longrightarrow & Z_2((\mathcal{D}, p), (\mathcal{D}_1, p_1))
\end{array}
\]

\[
\begin{array}{ccc}
& \longrightarrow & K_{\text{rel}}^q \longrightarrow \partial \longrightarrow B_1(\mathcal{D}, \mathcal{D}_1) \longrightarrow 0 \\
& \downarrow{\iota} & \downarrow{\iota} \\
& \longrightarrow \mu_2(C_2(\mathcal{D})) \oplus C_2^q \partial \longrightarrow \text{Im}(\hat{\partial}_2) \longrightarrow 0
\end{array}
\]

where the top row is exact from lemma 11.11 and the bottom one is also exact where $\text{Im}(\hat{\partial}_2) \leq C_1(\mathcal{D}, p)/C_1(\mathcal{D}_1, p_1)$. From the Snake Lemma we get the exact sequence

\[
0 \longrightarrow H_2((\mathcal{D}, p), (\mathcal{D}_1, p_1)) \longrightarrow \mathbb{Z}G, q_0. \mathbb{Z}G \xrightarrow{\nu} \text{Im}(\hat{\partial}_2)/B_1(\mathcal{D}, \mathcal{D}_1) \longrightarrow 0, \tag{17}
\]

where $\nu(q_0) = \hat{\partial}_2([1, q_0, 1]) + B_1(\mathcal{D}, \mathcal{D}_1)$. Since $G$ is the trivial group, we have that for every $[u, q_0, v]$,

$$\hat{\partial}_2([u, q_0, v]) + B_1(\mathcal{D}, \mathcal{D}_1) = \hat{\partial}_2([1, q_0, 1]) + B_1(\mathcal{D}, \mathcal{D}_1).$$

This follows easily if we prove that for every positive edge $e$, and $v \in F$ we have that

$$\hat{\partial}_2([ue, q_0, v]) + B_1(\mathcal{D}, \mathcal{D}_1) = \hat{\partial}_2([ve, q_0, v]) + B_1(\mathcal{D}, \mathcal{D}_1),$$

and similarly, for every positive edge $f$ and $u \in F$,

$$\hat{\partial}_2([u, q_0, vf]) + B_1(\mathcal{D}, \mathcal{D}_1) = \hat{\partial}_2([u, q_0, vf]) + B_1(\mathcal{D}, \mathcal{D}_1).$$

We prove the first claim for convenience. Since

$$(ue - ve).q_0.v + C_2(\mathcal{D}_1, p_1) = \left(\hat{\partial}_2[e, q_0] - \sum_i \varepsilon_i[e, e_i]\right) + C_2(\mathcal{D}_1, p_1),$$

where $\partial q_0 = e_1^{e_1} \ldots e_n^{e_n}$, then

$$\hat{\partial}_2((ue - ve).q_0.v) + C_1(\mathcal{D}_1) = -\hat{\partial}_2 \left(\sum_i \varepsilon_i[e, e_i]\right) + C_1(\mathcal{D}_1).$$

But

$$\hat{\partial}_2 \left(\sum_i \varepsilon_i[e, e_i]\right) + C_1(\mathcal{D}_1) \in B_1(\mathcal{D}, \mathcal{D}_1),$$

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consequently
\[
\left(\hat{\partial}_2([ue, q_0, v]) - \hat{\partial}_2([\tau e, q_0, v])\right) + B_1(D, D_1) = \left(\hat{\partial}_2((ue - \tau e)q_0, v) + C_1(D_1)\right) + B_1(D, D_1)
\]
\[
= -\left(\hat{\partial}_2\left(\sum_i \varepsilon_i e_i\right) + C_1(D_1)\right) + B_1(D, D_1)
\]
\[
= B_1(D, D_1),
\]
which proves the first claim.

An obvious consequence of (15) is that \(\text{Im}(\hat{\partial}_2)/B_1(D, D_1)\) is a cyclic group with generator \(\hat{\partial}_2([1, q_0, 1]) + B_1(D, D_1)\). The key to proving our main theorem is that \(\text{Im}(\hat{\partial}_2)/B_1(D, D_1)\) is infinite cyclic. Before that, we need to do some preparatory work.

If we let \(G_1\) be the group given by \(\mathcal{P}_1\), then for every \(g \in G_1\), we let \((D_1, t, g)\) be the connected component of \((D_1, t)\) corresponding to \(g\), and let \(H_1(D_1, t, g)\) be the corresponding homology group. The homology group \(H_1(D_1, t)\) decomposes as a direct sum
\[
H_1(D_1, t) = \oplus_{g \in G_1} H_1(D_1, t, g).
\]

Any 1-cycle \(\zeta\) now decomposes uniquely as
\[
\zeta = \zeta_{g_1} + \cdots + \zeta_{g_n},
\]
where \(\zeta_{g_i} \in Z_1(D_1, t, g_i)\), and \(\zeta + B_1(D_1, t)\) writes uniquely as
\[
\zeta + B_1(D_1, t) = (\zeta_{g_1} + B_1(D_1, t, g_1)) + \cdots + (\zeta_{g_n} + B_1(D_1, t, g_n)).
\]

From \([40]\) we know that each \(H_1(D_1, t, g_i)\) is isomorphic to \(H_1(D_1, t, 1)\) where the isomorphism
\[
\theta_{u_i} : H_1(D_1, t, g_i) \to H_1(D_1, t, 1)
\]
is defined by
\[
\zeta_{g_i} + B_1(D_1, t, g_i) \mapsto \zeta_{g_i} \cdot u_i^{-1} + B_1(D_1, t, 1)
\]
where \(u_i\) is any vertex in \((D_1, t, g_i)\).

We let \(\psi_1 : H_1(D_1, t, 1) \to \widehat{\Pi}_1\) and \(\eta : \widehat{\Pi} \to H_1(D, t)\) be the isomorphism of \([40]\). With these notations the following holds true.

**Lemma 4.15.** For every \(\zeta \in Z_1(D_1, t, 1)\), \(\hat{\psi}_1(\zeta + B_1(D_1, t, 1)) = \psi(\zeta + B_1(D, t))\).

**Proof.** This follows easily from the definitions of \(\psi, \psi_1\) and \(\hat{\psi}\). Indeed, assume that
\[
\zeta = \sum_{i \in I} z_i(u_i, s_i, 1, v_i),
\]
and let
\[
J = \{i \in I : s_i = r_i^{\epsilon_i} \text{ where } r_i^{\epsilon_i} \in r_1^{\pm 1}\}.
\]
Then from the definitions of $\psi_1$ and $\psi$ we have that

$$\psi_{1,0}(\varsigma) = \prod_{j \in J} \mu_1 \sigma_1^{(u_j s_j)^{z_j}} \quad \text{and} \quad \psi_0(\varsigma) = \prod_{j \in J} \mu \sigma^{(u_j s_j)^{z_j}} \quad (19)$$

where the exponential notations of the right hand sides mean that if $z_j < 0$, then $\mu_1 \sigma_1^{(u_j s_j)^{z_j}} = \nu (\mu_1 \sigma_1^{(u_j s_j)})^{\lfloor z_j \rfloor}$ and likewise, $\mu \sigma^{(u_j s_j)^{z_j}} = \nu (\mu \sigma^{(u_j s_j)})^{\lfloor z_j \rfloor}$. We used the definitions of $\psi_{1,0}$ and $\psi_0$ by regarding $\varsigma$ as a sum of 1-cycles arising from loops in $(D_1, t, 1)$. This is always possible due to lemma 5.1 of [33]. Further we have that

$$\hat{\varphi} \psi_1(\varsigma + B_1(D_1, t, 1)) = \hat{\varphi}(\psi_{1,0}(\varsigma)) \quad \text{(from the definition of } \psi_1)$$

$$= \hat{\varphi} \left( \prod_{j \in J} \mu_1 \sigma_1^{(u_j s_j)^{z_j}} \right) \quad \text{(from } (19))$$

$$= \prod_{j \in J} \mu \sigma^{(u_j s_j)^{z_j}} \quad \text{(from the definition of } \hat{\varphi})$$

$$= \psi_0(\varsigma) \quad \text{(from } (19))$$

$$= \psi(\varsigma + B_1(D, t)) \quad \text{(from the definition of } \psi),$$

proving the lemma. \(\square\)

With the decomposition $H_1(D_1, t) = \bigoplus_{g_i \in G_1} H_1(D_1, t, g_i)$, consider the following sequence of homomorphisms

$$\bigoplus_{g_i \in G_1} H_1(D_1, t, g_i) \xrightarrow{\otimes \theta_{u_i}} \bigoplus_{g_i \in G_1} H_1(D_1, t, 1) \xrightarrow{\otimes \psi_1} \bigoplus_{g_i \in G_1} \tilde{\Pi} \xrightarrow{\otimes \hat{\varphi}} \bigoplus_{g_i \in G_1} \tilde{\Pi} \xrightarrow{\gamma} H_1(D, t),$$

where

$$\gamma \left( \sum_{g_i} d_{g_i} \right) = \sum_{g_i} \eta(d_{g_i}),$$

and write for short $\chi = \gamma(\otimes \hat{\varphi})(\otimes \psi_1)(\otimes \theta_{u_i})$.

**Lemma 4.16.** For any element $\varsigma + B_1(D_1, t) \in H_1(D_1, t)$, we have $\chi(\varsigma + B_1(D_1, t)) = \varsigma + B_1(D, t)$.

**Proof.** If $\varsigma + B_1(D_1, t)$ is expressed as

$$\varsigma + B_1(D_1, t) = \sum_{g_i} (\varsigma_{g_i} + B_1(D_1, t, g_i)),$$

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then we have
\[
\chi(\varsigma + B_1(D_1, t)) = (\gamma(\mp\hat{\varphi})(\oplus\psi_1)(\oplus\theta_u)) \left( \sum_{g_i}(\varsigma_{g_i} + B_1(D_1, t, g_i)) \right)
\]
\[
= (\gamma(\mp\hat{\varphi})) \left( \sum_{g_i}(\varsigma_{g_i} \cdot u_i^{-1} + B_1(D_1, t, 1)) \right)
\]
\[
= \gamma \left( \sum_{g_i} \hat{\varphi}\psi_1(\varsigma_{g_i} \cdot u_i^{-1} + B_1(D_1, t, 1)) \right)
\]
\[
= \gamma \left( \sum_{g_i} \psi(\varsigma_{g_i} \cdot u_i^{-1} + B_1(D, t)) \right) \quad \text{(by lemma 4.15)}
\]
\[
= \eta(\varsigma_{g_i} \cdot u_i^{-1} + B_1(D, t))
\]
\[
= \sum_{g_i} \varsigma_{g_i} \cdot u_i^{-1} + B_1(D, t)
\]
\[
= \sum_{g_i} \varsigma_{g_i} + B_1(D, t) \quad \text{(by lemma 4.9)}
\]
\[
= \varsigma + B_1(D, t).
\]

\[\blacksquare\]

**Theorem 4.17.** The subpresentation \( P_1 \) is aspherical.

**Proof.** We prove first that \( \text{Im}(\hat{\partial}_2)/B_1(D_1, D_1) \) is infinite cyclic. If we assume the contrary, then there is \( z \in \mathbb{Z} \) and a 2-chain \( \xi \in C_2(D) \) such that

\[ z\hat{\partial}_2([1, q_0, 1]) + C_1(D_1) = \hat{\partial}_2(\xi) + C_1(D_1). \]

It follows that \( \varsigma = z\hat{\partial}_2([1, q_0, 1]) - \hat{\partial}_2(\xi) \) is a 1-cycle in \( C_1(D_1) \) and therefore \( \varsigma \in Z_1(D_1, t) \).

Now we see that

\[
\chi(\varsigma + B_1(D_1, t)) = \varsigma + B_1(D, t) \quad \text{(from lemma 4.16)}
\]
\[
= (z\hat{\partial}_2([1, q_0, 1]) - \hat{\partial}_2(\xi)) + B_1(D, t)
\]
\[
= z\hat{\partial}_2([1, q_0, 1]) + B_1(D, t)
\]
\[
= z\eta(1) + B_1(D, t).
\]

But from proposition 4.13 it follows that

\[
((\mp\hat{\varphi})(\oplus\psi_1)(\oplus\theta_u))(\varsigma + B_1(D_1, t)) = \sum_{g_i} v_{g_i},
\]
where $v_{g_i} \in \hat{A}_1$, say $v_{g_i} = \mu \sigma(u_i r_i (u_i r_i)^{-1}) z_i$ where $r_i \in r_1$, $u_i \in F$ and $z_i \in \mathbb{Z}$. Now we have

$$\chi(\varsigma + B_1(D_1, t)) = \gamma \left( \sum_{g_i} v_{g_i} \right)$$

$$= \gamma \left( \sum_{g_i} \mu \sigma(u_i r_i (u_i r_i)^{-1}) z_i \right)$$

$$= \sum_{g_i} \eta(\mu \sigma(u_i r_i (u_i r_i)^{-1}) z_i)$$

$$= \sum_{g_i} z_i q(r_i, u_i) + B_1(D, t)) \quad \text{(Proposition 4.5)}.$$

Recollecting, we have in $H_1(D, t)$ the equality

$$zq(r_0, 1) + B_1(D, t) = \sum_{g_i} z_i q(r_i, u_i) + B_1(D, t).$$

In $\tilde{\Pi} = \hat{U}$ this equality translates to

$$\mu \sigma(r_0 r_0^{-1}) z = \prod_{g_i} \mu \sigma(u_i r_i (u_i r_i)^{-1}) z_i$$

which from proposition 3.5 is impossible since each $r_i \neq r_0$. So it remains that $\text{Im}(\hat{\partial}_2)/B_1(D, D_1)$ is infinite cyclic, and as a result it is isomorphic to $\mathbb{Z}G. q_0. \mathbb{Z}G$ where the isomorphism sends $q_0$ to $\hat{\partial}_2(1, q_0, 1) + B_1(D, D_1)$ which is the free generator of $\text{Im}(\hat{\partial}_2)/B_1(D, D_1)$. But this map is the map $\nu$ of (17), therefore $H_2((D, p), (D_1, p_1)) = 0$ as desired. \hfill \square

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