GLOBAL EXISTENCE FOR DEFOCUSING CUBIC NLS
AND GROSS-PITAEVSKII EQUATIONS IN THREE
DIMENSIONAL EXTERIOR DOMAINS

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ABSTRACT. We prove global wellposedness in the energy space of the
defocusing cubic nonlinear Schrödinger and Gross-Pitaevskii equations
on the exterior of a non-trapping domain in dimension 3. The main
ingredient is a Strichartz estimate obtained combining a semi-classical
Strichartz estimate [4] with a smoothing effect on exterior domains [10].

1. Introduction

Let $\Theta \neq \emptyset$, $\Theta \subset \mathbb{R}^3$, a nontrapping obstacle with compact boundary and
let $\Omega = \mathbb{R}^3 \setminus \Theta$. In this paper we are interested in the Cauchy problem for
the cubic defocusing NLS equation (here written with Dirichlet boundary
conditions) on $\Omega$:

$$
\begin{cases}
    i \partial_t u + \Delta u = |u|^2 u, \text{ on } \mathbb{R} \times \Omega \\
    u|_{t=0} = u_0, \text{ on } \Omega \\
    u|_{\mathbb{R} \times \partial \Omega} = 0.
\end{cases}
$$

This equation appears in the nonlinear optics and more generally in propagation
of nonlinear waves. For more details on nonlinear Schrödinger equations
see for example the books of C.Sulem-P.L.Sulem [22], T.Cazenave [11] and
the references therein.

There is a wide literature on the Cauchy problem in the Euclidean space.
One of the main tools in addressing this problem is the Strichartz inequality,
which translates the dispersive property of the linear Schrödinger flow. We
refer to the work of Strichartz [23], Ginibre-Velo [14] and Keel-Tao [19].

Recently, the question of the influence of the geometry on the solution has
been studied. Let us mention the work of J.Bourgain [8] on the tori $\mathbb{T}^d$ for
d $= 2, 3$ and of N.Burq-P.Gérard-N.Tzvetkov [9], [10] on compact manifold
and exterior of non-trapping obstacles.

In recent works on superfluidity and Bose-Einstein condensates (see for
example the book of A.Aftalion [2]) the following variant of NLS (1) is studied

$$
\begin{cases}
    i \partial_t u + \Delta u = (|u|^2 - 1)u, \text{ on } \mathbb{R} \times \Omega \\
    u|_{t=0} = u_0, \text{ on } \Omega \\
    \frac{\partial u}{\partial \nu}|_{\mathbb{R} \times \partial \Omega} = 0.
\end{cases}
$$

This is called the cubic Gross-Pitaevskii equation with Neumann boundary
conditions. The main difference between the NLS (1) and the Gross-
Pitaevskii equation (2) is in their energy space. For Gross-Pitaevskii it reads

$$
E = \{ u \in H^1_{loc}(\Omega), \nabla u \in L^2(\Omega), |u|^2 - 1 \in L^2(\Omega) \}.
$$
Namely, the initial datum in the energy space, \( u_0 \in E \), is not an \( L^2(\Omega) \) function. In [6], [5], [3], [16], [17], [12] the question of existence of travelling waves and vortices is studied. We are interested here in providing a mathematical background for the study of the dynamics of these phenomena. More precisely, we are interested in showing wellposedness in the energy space. There have been previous works on the Cauchy problem for the Gross-Pitaevskii equation: P.E.Zhidkov [24], [25] in Zhidkov spaces \( X^1(\mathbb{R}) \), F.Béthuel- J.C.Saut [6] in the space of functions \( 1 + H^1(\mathbb{R}^d) \), for \( d = 2,3 \), P.Gérard in [15] in the energy space on the whole Euclidean space \( \mathbb{R}^d \), for \( d = 2,3,4 \), C.Gallo [13] in the energy space \( u_0 + H^1(\Omega) \) for exterior domains in \( d = 2 \).

For both (1) and (2) the method we use is based on a new Strichartz estimate obtained combining a smoothing effect in exterior domains [10] with a semiclassical Strichartz estimate on small intervals of time depending on the frequencies where the flow is localised [4]. In dimension 2 the smoothing effect [10] provides wellposedness for both NLS [10] and Gross-Pitaevskii [13], with all power nonlinearities. In dimension 3 the smoothing effect only provides wellposedness of subcubic nonlinearities [10], [13]. Improving the Strichartz inequality allows us to treat the cubic nonlinearity in exterior domains in dimension 3.

Let us recall the definition of an admissible pair.

**Definition 1.** A pair \((p,q)\) is called admissible in dimension 3 if \( p \geq 2 \) and

\[
\frac{2}{p} + \frac{3}{q} = \frac{3}{2}.
\]

The Strichartz inequality we obtain is the following.

**Proposition 1.1.** For \((p,q)\) an admissible couple in dimension 3 and \( \varepsilon > 0 \), there exists \( c_\varepsilon > 0 \) such that for all \( u_0 \in H^1_D(\Omega) \),

\[
\| e^{it\Delta_D} u_0 \|_{L^p(I,L^q(\Omega))} \leq c_\varepsilon \| u_0 \|_{H^1(\Omega)}.
\]

A similar result holds for the linear Schrödinger flow with Neumann boundary conditions.

Having a Strichartz inequality we obtain classically a local existence theorem for (1) by Picard iteration scheme. These also enables propagation of the regularity of the initial data. Local existence in the energy space \( H^1_0(\Omega) \) combined with the conservation of the energy (and for defocusing nonlinearity of the \( H^1_0(\Omega) \) norm) enables us to conclude that the solution to (1) is global in time.

**Theorem 1.2.** For all \( u_0 \in H^1_0(\Omega) \) there exists an unique solution

\[
u \in C(\mathbb{R}, H^1_0(\Omega)) \cap L^p_{loc}(\mathbb{R}, L^\infty(\Omega))
\]

(for every \( 2 < p < 3 \)) of equation (1). Moreover, for every \( T > 0 \) and for every bounded subset \( B \) of \( H^1_0(\Omega) \), the flow \( u_0 \mapsto u \) is Lipschitz from \( B \) to \( C([-T,T], H^1_0(\Omega)) \). For \( 1 < \sigma \leq 2 \) and \( u_0 \in H^1_0(\Omega) \cap H^\sigma(\Omega) \) we have \( u \in C([-T,T], H^1_0(\Omega) \cap H^\sigma(\Omega)) \).
For Gross-Pitaevskii equation, as \( u_0 \in E \) is not an \( L^2(\Omega) \) function, the Strichartz inequality does not apply directly. We adapt the arguments of [15] to the boundary case for the description of the structure of \( E \) and of the action of the linear Schrödinger group on \( E \). In particular we define a structure of complete metric space on \( E \). The global existence theorem for the Gross-Pitaevskii equation (2) follows by combining the latter structure with dispersive estimates (3).

**Theorem 1.3.** For all \( u_0 \in E \) there exists an unique solution

\[
u \in C(\mathbb{R}, E) \cap L^p_{\text{loc}}(\mathbb{R}, L^\infty(\Omega))\]

(for every \( 2 < p < 3 \)) of equation (2). Moreover, the following properties hold: for every bounded subset \( B \) of \( E \) there exists \( T > 0 \) such that for all \( u_0 \in B \) the flow \( u \mapsto \) \( u \) is Lipschitz from \( B \) to \( C([-T, T], E) \); we have \( u - u_L \in C(\mathbb{R}, H^1(\Omega)) \); if \( u_0 \in E \) is such that \( \Delta u_0 \in L^2(\Omega) \) and \( \partial u_0/\partial \nu = 0 \), then \( \Delta u \in C(\mathbb{R}, L^2(\Omega)) \).

**Remark 1.** After the completion of this work Blair-Smith-Sogge [7] announced an improved Strichartz inequality on boundary domains. They prove a Strichartz inequality with a loss of \( \frac{4}{3p} \) derivatives as opposed to the Strichartz inequality [4] with a loss of \( \frac{3}{2p} \) derivatives we used here. Although this may improve our Strichartz inequality (3) it does not improve the well-posedness results of Theorem 1.2 and of Theorem 1.3.

The structure of the paper is as follows: in Section 2 we show how we obtain the Strichartz estimate (3). In Section 3 we give the proof of Theorem 1.2. In Section 4 we deal with the Gross-Pitaevskii equation (1) and we give the proof of Theorem 1.3.

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## 2. Strichartz estimate in exterior domains

The idea is to combine Strichartz inequality on exterior domains [4] with the gain of \( \frac{1}{2} \) derivative from the smoothing effect [10]. Rather than using the Strichartz estimate with loss of \( \frac{4}{3p} + \epsilon \) derivatives (45) of [4], we prefer to use the Strichartz estimate without loss of derivatives (Proposition 4.13 of [4]), that holds for frequency localised initial data and small intervals of time depending on the frequency.

In order to do that here, we need to recall some of the notations and results from [4]. That is done in Subsection 2.1. In Subsection 2.2 we recall the results of N.Burq, P.Gérard and N.Tzvetkov [10] concerning the smoothing effect and Strichartz estimate away from the obstacle. Subsection 2.3 is the core of this section. We prove a new Strichartz estimate close to the obstacle by combining semiclassical Strichartz estimate and smoothing effect. In Subsection 2.4 we deduce the proof of Proposition 1.1.
2.1. Preliminaries. We recall here the classical mirror reflection that allows us to pass from a manifold with boundary to a boundaryless manifold. This method consists in taking a copy of the domain and glue it to the initial one by identifying the points of the boundary. In order for this to be a manifold we have to choose the coordinates carefully. Thus, taking normal coordinates at the boundary is like straightening a neighborhood of the boundary into a cylinder $\partial \Omega \times [0, 1)$ and gluing the two cylinders along the boundary makes a nice smooth manifold. This can be properly done using for example tubular neighborhoods (e.g. [20], pp. 468 and 74). Let $M = \Omega \times \{0\} \cup_{\partial \Omega} \Omega \times \{1\}$, where we identify $(p, 0)$ with $(p, 1)$ for $p \in \partial \Omega$.

**Lemma.** ([20]) There is a unique $C^\infty$ structure on $M$ such that $\Omega \times \{j\} \hookrightarrow M$ is $C^\infty$ and $\tilde{\chi} : U \times \{0\} \cup_{\partial \Omega} U \times \{1\} \to \partial \Omega \times (-1, 1)$ is a diffeomorphism, where $U$ is a small neighborhood of $\partial \Omega$ for which there are deformation retractions onto $\partial \Omega$.

On $M$ we define the metric $G$ induced by the new coordinates. As we have chosen coordinates in the normal direction, the metric is well defined over the boundary, its coefficients are Lipschitz in local coordinates and diagonal by blocs (no interaction between the normal and the tangent components). Moreover,

$$G(r(y)) = G(y),$$

where $r : M \to M$, $r(x, 0) = (x, 1)$, $r^2 = Id$ is the reflection with respect to the boundary $\partial \Omega$.

For the **Dirichlet** problem we introduce the space $H^1_{AS}$ of functions of $H^1(M)$ which are anti-symmetric with respect to the boundary. Let

$$H^1_{AS} = \{ v : M \to \mathbb{C}, \ v \in H^1(M), \ v(y) = -v(r(y)) \}.$$

Note that for $v \in H^1_{AS}$ the restriction $v|_{\Omega \times 0}$ is in $H^1_0(\Omega)$ and every function from $H^1_{AS}$ is obtained from a function of $H^1_0(\Omega)$. We shall prove the stability of $H^1_{AS}$ under the action of $e^{it\Delta_G}$.

By complex interpolation define $H^s_{AS}$ for $s \in [0, 1]$ and deduce its stability under the action of $e^{it\Delta_G}$. Moreover, the restriction to $\Omega$ of functions in $H^s_{AS}$ belongs to $H^s_0(\Omega)$ and vice versa. This allows us to deduce the Strichartz inequality for $e^{it\Delta_G}$ on $\Omega$ from the Strichartz inequality for $e^{it\Delta_G}$ on $M$.

Similarly, we can define for the **Neumann** problem the space $H^1_S$ of symmetric functions with respect to the boundary. This space is also stable under the action of $e^{it\Delta_G}$.

$$H^1_S = \{ v : M \to \mathbb{C}, \ v \in H^1(M), \ v(y) = v(r(y)) \}.$$

Let us prove the stability of $H^1_{AS}$ under the action of $e^{it\Delta_G}$. Let $v_0 \in H^1_{AS}$ and $v(t, y) = e^{it\Delta_G} v_0$. Then $v$ satisfies to $i\partial_t v(t, y) + \Delta_G v(t, y) = 0, \ v(0) = v_0$. Let $\tilde{v}(t, y) = v(t, r(y))$. We shall look for the equation verified by $\tilde{v}$. First note that $\tilde{v}(0) = -v_0$ and $\partial_y \tilde{v}(t, y) = \partial_y v(t, y)$. As $G$ is diagonal by blocs, having no interactions between the normal and tangent components, so is $G^{-1}$. Thus in $\Delta_G v(t, y)$ there is no crossed term. Consequently $\Delta_G(\tilde{v}(t, y)) = \Delta_G v(t, y)$. We see thus that $\tilde{v}$ satisfies to the linear Schrödinger equation with initial data $-v_0(y)$. But $-v(t, y)$ satisfies the same equations. By
uniqueness we conclude that
\[ v(t, r(y)) = -v(t, y). \]

Moreover, if \( v_0(t, y) = u_0(t, y) \) for all \( y \in \Omega \), then \( v(t, y) = u(t, y) \) for all \( t \) and for all \( y \in \Omega \), where \( u(t) = e^{itD}u_0 \).

We prepare the frequency decomposition. We begin with a partition of unity on \( M \). Since \( M \) is flat outside a compact set, let \((U_j, \kappa_j)_{j \in J}\) be a covering of the area of \( M \) where \( G \neq \mathbb{I}d \). This area is compact, so we can choose \( J \) of finite cardinal. We have \( M = \bigcup_{j \in J} U_j \cup U_{1, \infty} \cup U_{2, \infty} \), where \( U_{1, \infty} \) and \( U_{2, \infty} \) are two disjoint neighborhood of \( \infty \), diffeomorphic to \( \mathbb{R}^d \setminus B \). Let \((\chi_j)_{j \in J}, \chi_{1, \infty}, \chi_{2, \infty} : M \to [0, 1]\) be a partition of unity subordinated to the previous covering. For all \( j \in J \) let \( \tilde{\chi}_j : M \to [0, 1] \) be a \( C^\infty \) function such that \( \tilde{\chi}_j = 1 \) on the support of \( \chi_j \) and the support of \( \tilde{\chi}_j \) is contained in \( U_j \). Similarly we define \( \tilde{\chi}_{1, \infty}, \tilde{\chi}_{2, \infty} : M \to [0, 1] \). Let \( \varphi_0 \in C^\infty(\mathbb{R}^d) \) be supported in a ball centered at origin and \( \varphi \in C^\infty(\mathbb{R}^d) \) be supported in an annulus such that for all \( \lambda \in \mathbb{R}^d \)

\[ \varphi_0(\lambda) + \sum_{k \in \mathbb{N}} \varphi(2^{-k} \lambda) = 1. \]

We define a family of spectral truncations : for \( f \in C^\infty(M) \) and \( h \in (0, 1) \) let

\[ J_h f = \sum_{j \in J} (\kappa_j)^* \left( \tilde{\chi}_j \varphi(hD)(\kappa_j^{-1})^*(\chi_j f) \right) + F_{1, h, \infty} f + F_{2, h, \infty} f \]

and

\[ J_0 f = \sum_{j \in J} (\kappa_j)^* \left( \tilde{\chi}_j \varphi_0(D)(\kappa_j^{-1})^*(\chi_j f) \right) + F_{1, 0, \infty} f + F_{2, 0, \infty} f, \]

where \(^*\) denotes the usual pullback operation and \( F_{h, \infty} f = \tilde{\chi}_{\infty} \varphi(hD) \chi_{\infty} f(x) \). The following identity holds

\[ J_0 f(x) + \sum_{k=0}^\infty J_{2^{-k}} f(x) = f(x). \]

This will be useful for the Littlewood Paley theory.

We need more regularity on the coefficients of the metric than the Lipschitz regularity. Therefore we define a regularized metric \( G_h \) as follows : let \( \psi \) be a \( C^\infty_0(\mathbb{R}^d) \) radially symmetric function with \( \psi \equiv 1 \) near 0. Let

\[ G_h = \sum_{j \in J} (\kappa_j)^* \left( \tilde{\chi}_j \psi(hD)(\kappa_j^{-1})^*(\chi_j G) \right). \]

The transformation of \( G \) into \( G_h \) does not spoil the symmetry. Note also that \( G_h \) converges uniformly in \( x \) to \( G \), and thus, for \( h \) sufficiently small, \( G_h \) is positive definite. Therefore, \( G_h \) is still a metric. We present some properties of metrics \( G \) and \( G_h \).

**Lemma.** The metric \( G : M \to M_d(\mathbb{R}) \) is symmetric, positive definite and Lipschitz : there exist \( c, C, c_1 > 0 \) such that for all \( x \in M \)

\[ c\|d \leq G(x) \leq C\|d, |\partial G| \leq c_1, \]
where we have denoted by $\partial G$ the derivatives of the metric in a system of coordinates. The regularized metric $G_h$ is a $C^\infty$ function that verifies the followings: there exists $c, C > 0$ and $c_\gamma > 0$ for all $\gamma \in \mathbb{N}^d$ such that

$$c|\partial \xi| \leq G_h(\xi) \leq C|\partial \xi|, \quad |\partial^\gamma G_h(\xi)| \leq c_\gamma h^{-\alpha \max(|\gamma|-1,0)}.$$  

We present next a collection of estimates on $J_h$. There exist constants $c > 0$ such that, for all $h \in (0,1)$:

- $\|J_h\|_{L^p \to L^p} \leq c_p$, for all $1 \leq p \leq \infty$.
- $\|[J_h, \Delta G_h]\|_{L^2 \to L^2} \leq \frac{c}{h}$ and $\|[J_h, \Delta G_h]\|_{H^1 \to L^2} \leq c$.

As one may not apply two derivatives on $G$, the similar statement for $[F_h, \Delta G]$ only holds for the $H^1 \to L^2$ norm: $\|[F_h, \Delta G]\|_{H^1 \to L^2} \leq c$.

- $\|[J_h(\Delta G_h - \Delta G)]\|_{H^1 \to L^2} \leq ch^{-\frac{1}{2}}$.

We define also a spectral cut-off slightly larger than $J_h$. Let $\tilde{\varphi}$ be a $C^\infty$ function supported in an annulus such that $\tilde{\varphi} = 1$ on a neighborhood of the support of $\varphi$. We define $\tilde{J}_h$ just like $J_h$, replacing $\varphi$ by $\tilde{\varphi}$ in (4):

$$\tilde{J}_h f = \sum_{j \in J} (\kappa_j)^* \left( \chi_j \tilde{\varphi}(hD)(\kappa_j^{-1})^*(\chi_j f) \right) + \tilde{F}_{1,h,\infty} f + \tilde{F}_{2,h,\infty} f$$

Then the action of $\tilde{J}_h$ on $J_h$ and $[J_h, \Delta G_h]$ is close to identity in $L^p \to L^p$ norm, $p \geq 2$, and $L^2 \to L^2$ norm respectively.

- $\|\tilde{J}_h J_h - J_h\|_{L^p \to L^p} \leq c_N h^N$.
- $\|[J_h, \Delta G_h] - [J_h, \Delta G_h] \tilde{J}_h\|_{L^2 \to L^2} \leq c_N h^N$, for all $N \in \mathbb{N}$.

Let us recall also the Strichartz estimate we use from [4].

**Lemma.** (4.13 of [4]) For all couples $(p,q)$ admissible in dimension 3 and $I_h$ an interval of time such that $|I_h| = ch^\frac{1}{2}$, we have

$$\|J_h e^{it\Delta G_h} u_0\|_{L^p(I_h, L^q(M))} \leq c\|u_0\|_{L^2}.$$  

We prefer to go back to the estimate on $e^{it\Delta G_h}$ since the form of the Strichartz estimate for $e^{it\Delta G}$ is more difficult to handle (45) of [4] :

$$\|J_h e^{it\Delta G_h} u_0\|_{L^p(I_h, L^q(M))} \leq ch\|u_0\|_{H^1}.$$  

This is due to the fact that $\Delta G$ and $\Delta G_h$ are not both selfadjoint in the same space, because of the the volume density $\frac{1}{\sqrt{\det G(x)}}$.

**2.2. Smoothing effect and Strichartz estimate away from the obstacle.** In this section we recall two results of N.Burq, P.Gerard and N.Tzvetkov [10] on the smoothing effect for the Schrödinger flow on exterior domains and the Strichartz estimate away from the obstacle. The smoothing effect was obtained via resolvent bounds. For Strichartz estimate they used a strategy inspired by G.Staffilani and D.Tataru’s paper [26] on $C^2$ short range perturbation of the free Laplacian on $\mathbb{R}^d$. Thus, they proved that away from the obstacle the linear Schrödinger flow satisfies the usual Strichartz estimates. We present an equivalent statement on the double manifold.
Proposition. \((2.7 \text{ of } [10])\) Assume that \(\Theta \neq \emptyset\). Then for every \(T > 0\), for every \(\chi \in C_0^\infty(\mathbb{R}^d)\), \(d \geq 2\),
\[
|\chi e^{it\Delta_D} u_0|_{L^2_H H^{s+\frac{1}{2}}(\Omega)} \leq c |u_0|_{L^2_H H^s(\Omega)}, \text{ for } s \in [0, 1].
\]

Proposition. \((2.10 \text{ of } [10])\) For every \(T > 0\), for every \(\chi \in C_0^\infty(\mathbb{R}^d)\), \(\chi = 1\) close to \(\Theta\), there exists \(C > 0\) such that
\[
\|(1 - \chi) u\|_{L^p_{T} W^{s,q}(\Omega)} \leq C |u_0|_{H^s(\Omega)},
\]
where \(s \in [0, 1]\), \(u(t) = e^{it\Delta_D} u_0\) and \((p, q)\) any Strichartz admissible pair.

The proof relies on the use of the smoothing effect and the fact that \((1 - \chi)e^{it\Delta_D} u_0\) can be seen as a solution to some nonlinear Schrödinger equation on \(\mathbb{R}^d\).

Although the properties are written for the Dirichlet Laplacian, Remark 1.2 of \([10]\) ensures that the results hold for the Neumann conditions as well. From the way we constructed the double manifold and flows, we deduce that those results extend easily on the double manifold.

Proposition 2.1. Assume that \(\Theta \neq \emptyset\). Then for every \(T > 0\), for every \(\chi \in C_0^\infty(M)\),
\[
|\chi e^{it\Delta_G} u_0|_{L^2_H H^{s+\frac{1}{2}}(M)} \leq c |u_0|_{L^2_H H^s(M)}, \text{ for } s \in [0, 1].
\]

Proposition 2.2. For every \(T > 0\), for every \(\chi \in C_0^\infty(M)\), \(\chi = 1\) close to \(D\Theta\), where \(D\Theta\) represents the double of \(\Theta\), there exists \(C > 0\) such that
\[
\|(1 - \chi) e^{it\Delta_G} u_0\|_{L^p_{T} W^{s,q}(M)} \leq C |u_0|_{H^s(M)},
\]
where \(s \in [0, 1]\) and \((p, q)\) any Strichartz admissible pair.

2.3. Strichartz estimate near the obstacle. We want to combine Strichartz estimate on domains \([4]\) with smoothing effect \([10]\). For this we use the Strichartz estimate of the frequency localised linear flow, without loss of derivatives, which holds on a small interval of time (see estimate \([9]\) from Subsection 2.1).

Let \(\varphi \in C_0^\infty(\mathbb{R})\) such that \(\varphi \equiv 1\) on \([-\frac{1}{4}, \frac{1}{4}]\), \(0 \leq \varphi \leq 1\), \(\text{supp} \varphi \subset [-\frac{1}{4}, \frac{1}{4}]\) and there exists \(J \subset \mathbb{R}\) a discrete set such that \(\sum_{t_0 \in J} \varphi^2(t - t_0) = 1\) for all \(t \in \mathbb{R}\). Let \(\delta > 0\) be a small number. If we consider \(J = [-\delta, 1 + \delta] \cap ch^2 J\) then, for \(t \in [-\frac{\delta}{2}, 1 + \frac{\delta}{2}]\),
\[
\sum_{t_0 \in J} \varphi^2 \left(\frac{t - t_0}{ch^2}\right) = 1.
\]

Let us denote by \(I_h(t_0) = [t_0 - \frac{ch^2}{4}, t_0 + \frac{ch^2}{4}]\), \(I'_h(t_0) = [t_0 - \frac{ch^2}{2}, t_0 + \frac{ch^2}{2}]\), \(u_L(t) = e^{it\Delta_G} u_0\) and by
\[
v(t) = \varphi \left(\frac{t - t_0}{ch^2}\right) J_h^* \chi e^{it\Delta_G} u_0.
\]

Notice that \(v(t) = J_h^* \chi e^{it\Delta_G} u_0\) for \(t \in I_h(t_0)\) and \(\text{supp} v \subset I'_h\). We write the end-point Strichartz estimate for \(v\) on \(I'_h\). Notice that the couple \((2, 6)\) is admissible in dimension 3.
Lemma 2.3. For \( t_0 \in \mathbb{R} \), \( \tilde{\chi} \in C_0^{\infty}(M) \) such that \( \tilde{\chi}\chi = \chi \) and \( \tilde{J}_h \) a spectral cut-off slightly larger than \( J_h \) (5), we have

\[
\| \varphi \left( \frac{t-t_0}{ch^3} \right) J_h^* \chi e^{it\Delta_h u_0} \|_{L^2(I_h',L^6(M))} \leq c h^{-\frac{3}{2}} \| J_h^* \chi u_L \|_{L^2(I_h',L^2(M))} + h^2 \| \tilde{J}_h^* \tilde{\chi} u_L \|_{L^2(I_h',H^1(M))} + c \| \tilde{\chi} u_L \|_{L^2(I_h',H^1(M))},
\]

(14)

Proof. For simplicity, let us suppose that \( I_h'(t_0) = [0,T] \), where \( T = ch^\frac{3}{4} \). Then \( v(t) \) verifies, for \( t \in I_h'(t_0) \), the equation

\[
\begin{aligned}
\{ i\partial_t v + \Delta_{G_h} v &= f_1 + f_2 + f_3 \\
v_{t=0} &= 0
\end{aligned}
\]

where \( f_1 = \frac{i}{ch^2} \varphi \left( \frac{t-t_0}{ch^2} \right) J_h^* \chi u_L \), \( f_2 = \varphi \left( \frac{t-t_0}{ch^2} \right) [\Delta_{G_h}, J_h^* \chi] u_L \) and \( f_3 = \varphi \left( \frac{t-t_0}{ch^2} \right) J_h^* \chi (\Delta_{G} - \Delta_{G_h}) u_L \). By the Duhamel formula and using that \( J_h^* J_h^* = J_h^* + cN_h N \) in \( L^p \to L^p \) norm, for \( p \geq 2 \), we have

\[
v(t) = v_1(t) + v_2(t) + v_3(t) + cN_h N,
\]

where we define

\[
\begin{aligned}
v_1(t) &= \frac{1}{ch^2} f_0^t J_h^* \chi u_L(\tau) d\tau \\
v_2(t) &= -i \int_0^t J_h^* \chi (\Delta_{G_h} - \Delta_{G_h}) u_L(\tau) d\tau \\
v_3(t) &= -i \int_0^t J_h^* \chi (\Delta_{G} - \Delta_{G_h}) u_L(\tau) d\tau,
\end{aligned}
\]

By Minkowski inequality and estimate (9), we have

\[
\| v_1 \|_{L^2(I_h',L^6(M))} \leq c h^{-\frac{3}{2}} \int_0^T \left| \varphi \left( \frac{\tau}{ch^2} \right) \right| \| \chi e^{it\Delta_h u_L(\tau)} \|_{L^2(L^6(M))} d\tau \\
\leq c h^{-\frac{3}{2}} \int_0^T \left| \varphi \left( \frac{\tau}{ch^2} \right) \right| \| J_h^* \chi u_L(\tau) \|_{L^2(M)} d\tau.
\]

Using Cauchy Schwarz inequality and \( \| \varphi \left( \frac{\tau}{ch^2} \right) \|_{L^2} = c h^{\frac{3}{4}} \), we obtain

\[
\| v_1 \|_{L^2(I_h',L^6(M))} \leq c h^{-\frac{3}{2}} \| J_h^* \chi u_L \|_{L^2(I_h' \times M)}
\]

Similarly, we have \( \| v_2 \|_{L^2(I_h',L^6(M))} \leq c h^{\frac{3}{2}} \| [\Delta_{G_h}, J_h^* \chi] u_L \|_{L^2(I_h' \times M)} \). Using that \( \| [\Delta_{G_h}, J_h^* \chi] \|_{H^1 \rightarrow L^2} \leq c \) and \( \| [\Delta_{G_h}, J_h^* \chi] \|_{H^1 \rightarrow L^2} \sim \| [\Delta_{G_h}, J_h^* \chi] \|_{H^1 \rightarrow L^2} \) modulo \( ch^{-N} \), we obtain

\[
\| v_2 \|_{L^2(I_h',L^6(M))} \leq c h^{\frac{3}{2}} \| J_h^* \tilde{\chi} u_L \|_{L^2(I_h',H^1(M))}.
\]

We estimate the third term \( v_3 \) in \( L^2(I_h', L^6(M)) \) norm in a similar manner. We get : \( \| v_3 \|_{L^2(I_h',L^6(M))} \leq c h^{\frac{3}{2}} \| J_h^* \chi (\Delta_{G} - \Delta_{G_h}) u_L \|_{L^2(I_h' \times M)} \). Using the estimate \( \| J_h^* \chi (\Delta_{G} - \Delta_{G_h}) f \|_{L^2(M)} \leq c h^{-\frac{3}{2}} \| \tilde{\chi} f \|_{H^1(M)} \), we obtain

\[
\| v_3 \|_{L^2(I_h',L^6(M))} \leq c h^{\frac{3}{2}} \| \tilde{\chi} u_L \|_{L^2(I_h',H^1(M))}.
\]

Recalling that \( v(t) = v_1(t) + v_2(t) + v_3(t) \), the result follows from the triangle inequality and the sum of (15), (16) and (17). \( \square \)
We proceed to the summation over the intervals of time in order to obtain a Strichartz inequality (for the frequency localized flow) on a fixed interval of time. Let us denote by \( I = [0, 1] \) and by \( I_\delta = I + [-\delta, \delta] \), where \( \delta \) is chosen like in \( 12 \).

Lemma 2.4. Under the same notations as in Lemma \ref{lemma1}, we have

\[
\| J_h^n \chi u L \|_{L^2(I, L^6(M))} \leq \frac{c_h}{\epsilon^2} \| J_h^n \chi u L \|_{L^2(I_\delta, L^2(M))} + h \| J_h^n \chi u L \|_{L^2(I_\delta, H^1(M))} + c \| \chi u L \|_{L^2(I_\delta, H^1(M))}. 
\]

Proof. We sum the square of \( 14 \) over \( t_0 \in J \), where \( J \) was defined for the identity \( 12 \). From \( 12 \) and the definition of \( \varphi \) we deduce that the reunion of intervals \( J_h^n(t_0) \), for \( t_0 \in J \), recovers \( I_\delta \) at most twice. Thus, \( \sum_{t_0 \in J} \| f \|_{L^2(I_h^n, t_0)}^2 \leq 2 \| f \|_{L^2([-\delta, 1])}^2 \). The result follows by merely observing that \( \| J_h^n \chi u L \|_{L^2(I, L^6(M))} \leq \| J_h^n \chi u L \|_{L^2(I_\delta, L^6(M))} \).

From \( 18 \) we get the Strichartz inequality near the obstacle by means of Littlewood Paley summation.

Proposition 2.5. For every \( \epsilon > 0 \) there exists \( c_\epsilon > 0 \) such that for \( (p,q) \) admissible in dimension 3,

\[
\| \chi e^{it \Delta} u_0 \|_{L^p([0,1], W^{\frac{2}{p}, q}(M))} \leq c_\epsilon \| u_0 \|_{H^{\frac{1}{p}}(M)}.
\]

Proof. We apply a corollary of the Littlewood Paley theorem for \( p,q \geq 2 \) :

\[
\| u \|_{L^p(W^{\sigma,q})} \leq c \| S_0 u \|_{L^p(L^q)} + \left( \sum_{j=0}^{\infty} 2^{2j\sigma} \| \Delta_j u \|^2_{L^p(L^q)} \right)^{\frac{1}{2}}
\]

Here we apply it for \( (p,q) = (2,6) \) and \( h = 2^{-j}, \Delta_j = J_{2^{-j}}, \sigma = \frac{1}{4} - \epsilon > 0 \) and \( u = \chi u L \) on \( I = [0,1] \). The left hand side term reads \( \| \chi u L \|_{L^2(I, W^{\sigma,q}(M))} \).

Using \( 18 \), the parenthesis from the right hand side term is bounded by a sum \( \sum_{j=0}^{\infty} \) of terms like

\[
2^{(2-2\nu)} \| \Delta_j \chi u L \|_{L^2(I_\delta, L^2(M))}^2 + 2^{-j(1+2\nu)} \| \Delta_j \chi u L \|_{L^2(I_\delta, H^1(M))}^2 + 2^{-2j\epsilon} \| \chi u L \|_{L^2(I_\delta, H^1(M))}^2.
\]

Using the Plancherel theorem for the first two series and the geometric summation for the third (notice that \( \sum_{j=0}^{\infty} 2^{-2j\epsilon} = c_\epsilon \)), we obtain

\[
\| \chi u L \|_{L^2(I, W^{\sigma,q}(M))} \leq \| \chi u L \|_{L^2(I_\delta, H^1(M))} + \| \chi u L \|_{L^2(I_\delta, H^1(M))} + \| \chi u L \|_{L^2(I_\delta, H^1(M))}. 
\]

We apply the smoothing effect (see Proposition 2.7 of \ref{10} and the translation on the double), thus, \( \| \chi u L \|_{L^2(I, W^{\frac{1}{2}, q}(M))} \leq c \| u_0 \|_{H^{\frac{1}{2}}(M)} \).

We want to perform a complex interpolation between the previous estimate and the conservation of the \( L^2 \) norm (we used also \( 0 \leq \chi \leq 1 \) :

\[
\| \chi u L \|_{L^\infty(I, L^2(M))} \leq c \| u_0 \|_{L^2(M)}.
\]

Using a weight of \( \xi_{\frac{2}{p}} \), respectively \( 1 - \xi_{\frac{2}{p}} \), we get an estimates of Strichartz type with loss of derivatives :

\[
\| \chi u L \|_{L^p(I, W^{\xi_{\frac{2}{p}}, q}_{\frac{2}{p}}(M))} \leq c \| u_0 \|_{L^p(M)}.
\]
where \((p,q)\) satisfy \(\frac{2}{p} + \frac{3}{q} = \frac{3}{2}\), i.e. they form an admissible couple in dimension 3.

\[\square\]

2.4. **Proof of Proposition 1.1** Combining estimates (11) (Strichartz estimate near the boundary of \(\Omega\)) with (1l) (Strichartz estimate away from the boundary) for \(s \geq 1\) in dimension 3.

Proof of Proposition 1.1.

2.4. where \((p,q)\) theorem by Picard iteration scheme. These also enables propagation of the flow (see Section 2.1), we have \(e^{t\Delta G}v(0) = e^{t\Delta D}u(0)\) and uniqueness and stability at reflection over the boundary of \(\Omega\) of the linear and

\[\square\]

3. Global existence for NLS

Having a Strichartz inequality we obtain classically a local existence theorem by Picard iteration scheme. These also enables propagation of the regularity of the initial data. Local existence in the energy space \(H^1(\Omega)\) combined with the conservation of the energy (and for defocusing nonlinearity of the \(H^1(\Omega)\) norm) enables us to conclude that the solution to is global in time.

**Proof.** (of Theorem 1.2) Let us denote by \(X_T = C([-T,T],H^1_0(\Omega)) \cap L^p([-T,T],L^\infty(\Omega))\) and, for a fix \(u_0 \in B \subset H^1(\Omega)\), by \(\Phi : X_T \to X_T\) the functional

\[\Phi(u)(t) = e^{t\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^2 u(\tau) d\tau.\]

The space \(X_T\) is a complete Banach space for the following norm

\[\|u\|_{X_T} = \max_{|\tau| \leq T} \|u(t)\|_{H^1(\Omega)} + \|u\|_{L^p([-T,T],L^\infty(\Omega))}.\]

We prove that for a \(T > 0\) and \(R > 0\) small enough, \(\Phi\) is a contraction from \(B(0, R) \subset X_T\) into itself. We begin by estimating the \(H^1\) norm of \(\Phi(u)\):

\[\|\Phi(u)(t)\|_{H^1} \leq \|u_0\|_{H^1} + c T^{1-\frac{2}{p}} \|u\|_{L^p(L^\infty)}^2 \|u\|_{L^\infty(H^1)} \leq \|u_0\|_{H^1} + c T^{1-\frac{2}{p}} \|u\|_{X_T}^3.\]
We have considered $2 < p < 3$. Thus, there exists $\epsilon > 0$ such that $\epsilon < \frac{3}{2p} - \frac{1}{2}$. Therefore, by Sobolev imbedding theorem we have, for $(p, q)$ admissible in dimension $3$, that $W^{1 - \frac{1}{2} - \epsilon, \epsilon}(\Omega) \subset L^\infty(\Omega)$:

$$\|\Phi(u)\|_{L^p_tL^\infty_x(\Omega)} \leq c\|\Phi(u)\|_{L^p_tW^{1 - \frac{1}{2} - \epsilon, \epsilon}_x(\Omega)}.$$  

Using the Strichartz estimate [3] and Minkowski inequality (like in the proof of [14]), we have

$$\|\Phi(u)\|_{L^p(L^\infty)} \leq \left[ e^{u\Delta}u_0 \right]_{L^p_tW^{1 - \frac{1}{2} - \epsilon, \epsilon}_x} + \left[ \int_0^T e^{i(t-\tau)\triangle}|u|^2(\tau)u(\tau)\mathrm{d}\tau \right]_{L^p_tW^{1 - \frac{1}{2} - \epsilon, \epsilon}_x} \leq \left[ e^{u\Delta}u_0 \right]_{H^{1}} + c\int_0^T \|u|^2u(\tau)\|_{H^1(\Omega)}\mathrm{d}\tau.$$

Using that $\|u|^2(\tau)u(\tau)\|_{H^1(\Omega)} \leq c\|u(\tau)\|_{H^1(\Omega)}\|u(\tau)\|_{L^\infty_x}^2$, we obtain

$$\|\Phi(u)\|_{L^p(L^\infty)} \leq \left[ e^{u\Delta}u_0 \right]_{H^{1}} + cT^{-\frac{2}{p}}\|u\|_{L^\infty_tH^1(\Omega)}\|u\|_{L^p_tL^\infty_x(\Omega)}^2 \leq \left[ e^{u\Delta}u_0 \right]_{H^{1}} + cT^{-\frac{2}{p}}\|u\|_{X_T}^3.$$

Thus, $\|\Phi(u)\|_{X_T} \leq \left[ e^{u\Delta}u_0 \right]_{H^{1}} + cT^{-\frac{2}{p}}\|u\|_{X_T}^3$.

Consequently, there exist $T, R > 0$, depending only on $B \subset H^1_0(\Omega)$ ($u_0 \in B$), such that, for $u \in X_T$ with $\|u\|_{X_T} \leq R$, we have $\|\Phi(u)\|_{X_T} < R$.

As above, we prove that, for $u, v \in X_T$ such that $u(0) = u_0 = v(0)$,

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq cT^{-\frac{2}{p}}(\|u\|_{X_T}^2 + \|v\|_{X_T}^2)\|u - v\|_{X_T}.$$  

Choosing $T$ eventually smaller, we ensure that $\Phi$ is a contraction on the ball $B(0, R) \subset X_T$, $B(0, R) = \{ u \in X_T, \|u\|_{X_T} < R \}$. Consequently, there exists a fix point of $\Phi$, which is therefore solution to (1).

For the Lipschitz property of the flow let us consider $u, v \in B(0, R) \subset X_T$ two solutions of (1) with initial data respectively $u_0, v_0 \in B$. As above, we have

$$\|u - v\|_{X_T} \leq \left[ e^{u\Delta}u_0 \right]_{H^{1}} + cT^{-\frac{2}{p}}(\|u\|_{X_T}^2 + \|v\|_{X_T}^2)\|u - v\|_{X_T}.$$  

For $T, R > 0$ chosen before we have $cT^{-\frac{2}{p}}(\|u\|_{X_T}^2 + \|v\|_{X_T}^2) < 1$ and therefore, $\exists \tilde{c} > 0$ such that $\|u - v\|_{X_T} \leq \tilde{c}\|u_0 - v_0\|_{H^{1}}$. We conclude that the flow $u_0 \mapsto u$ is Lipschitz on $B \subset H^1_0$.

Let $\sigma \geq 1$ and suppose $u_0 \in H^\sigma(\Omega) \cap H^1_0(\Omega)$. Let us estimate $\Phi(u)$ in $Y_T = C([-T, T], H^\sigma(\Omega)) \cap L^p([-T, T], L^\infty(\Omega))$ norm:

$$\|u\|_{Y_T} = \max_{|t| \leq T}\|u(t)\|_{H^\sigma(\Omega)} + \|u\|_{L^p([-T, T], L^\infty(\Omega))}.$$  

As above, we obtain

$$\|\Phi(u)\|_{L^p_tH^\sigma} \leq \left[ e^{u\Delta}u_0 \right]_{H^\sigma} + cT^{-\frac{2}{p}}(\|u\|_{X_T}^2 + \|u\|_{X_T}^2)\|u\|_{L^p_tH^\sigma}.$$  

We have chosen $T > 0$ such that $cT^{-\frac{2}{p}}(\|u\|_{X_T}^2 + \|v\|_{X_T}^2) < 1$. Consequently, the $H^\sigma$ norm does not blow up for $|t| \leq T$:

$$\|u\|_{L^p_tH^\sigma} \leq \tilde{c}\|u_0\|_{H^\sigma}.$$  

Therefore we can conclude that regularity propagates.

The semilinear Schrödinger equation (11) has a Hamiltonian structure with gauge invariance and thus conservation laws hold for $H^2$ initial data. For
$u_0 \in H^1$ we deduce them by density: the solution of (1) constructed above satisfies, for $|t| \leq T$, to
\[
\begin{cases}
\int |u(t)|^2 dx = \int |u_0|^2 dx, \\
\int |\nabla u(t)|^2 + \frac{1}{2}|u(t)|^4 dx = \int |\nabla u_0|^2 + \frac{1}{2}|u_0|^4 dx.
\end{cases}
\]
Moreover, note that $T > 0$ depends only on $\|u_0\|_{H^1}$. Therefore, conservation of $H^1$ norm enables us to obtain, via a bootstrap argument, the global existence.

\[\square\]

## 4. Global existence for Gross-Pitaevskii

The Gross-Pitaevskii equation (2) is associated to the energy
\[
E(u) = \int_\Omega \frac{1}{2} |\nabla u|^2(x) + \frac{1}{4} (|u|^2(x) - 1)^2 dx.
\]
The main difference between the NLS (1) and the Gross–Pitaevskii equation (2) is their energy space. For Gross–Pitaevskii it reads
\[
E = \{ u \in H^1_{loc}(\Omega), \nabla u \in L^2(\Omega), |u|^2 - 1 \in L^2(\Omega) \}.
\]
Namely, the initial data in the energy space, $u_0 \in E$, is not an $L^2(\Omega)$ function. Therefore we begin this section by describing the structure of $E$ and of the action of the linear Schrödinger group on $E$ by adapting the arguments of [15] to the boundary case. Then, we give the proof of the global existence theorem for the Gross–Pitaevskii equation (2) by combining the latter structure with dispersive estimates derived in Section 2.2 and 2.3.

### 4.1. The energy space

This section is inspired from [15]. In that paper, the Cauchy problem for Gross–Pitaevskii equation is studied in the whole Euclidean space $\mathbb{R}^d$, for $d = 2, 3, 4$. In the special case of $d = 3$, $u_0 \in E$ can be expressed in an explicit form as $u_0 = c + v_0$, where $c \in \mathbb{C}$ and $v_0 \in H^1$. We show here that the same holds outside a non-trapping obstacle and give the outline of the proof. For more details we refer to [15].

We denote by $C^\infty_0(\bar{\Omega})$ the restriction to $\bar{\Omega}$ of $C^\infty_0(\mathbb{R}^3)$ and by $\dot{H}^1(\Omega)$ the completion of $C^\infty_0(\bar{\Omega})$ in the norm $\|\nabla \cdot \|_{L^2(\Omega)}$. We recall that
\[
\dot{H}^1(\Omega) = \{ u \in L^6(\Omega), \nabla u \in L^2(\Omega) \}.
\]
Moreover, we have the following approximation property.

Let $\chi \in C^\infty_0(\mathbb{R}^3)$, $\chi = 1$ on the ball of radius 1 $B(0, 1)$ and $\chi = 0$ outside $B(0, 2)$. We define $\chi_R(x) = \chi(\frac{x}{R})$. For $v \in \dot{H}^1(\Omega)$ we have $\chi_Rv \in \dot{H}^1(\Omega)$ and
\[
\chi_Rv \xrightarrow{R \to \infty} v \text{ in the } \|\nabla \cdot \|_{L^2(\Omega)} \text{ norm.}
\]
We prove the main result of this section.

**Proposition 4.1.** The energy space $E$ has the following structure
\[
E = \{ c + v, \ c \in \mathbb{C}, \ |c| = 1, \ v \in \dot{H}^1(\Omega), \ |v|^2 + 2Re(c^{-1}v) \in L^2(\Omega) \}.
\]
The space $E$ is a complete metric space with the distance function
\[
\delta_E(c+v, \tilde{c}+\tilde{v}) = |c-\tilde{c}| + \|\nabla v - \nabla \tilde{v}\|_{L^2(\Omega)} + \|v|^2 + 2Re(c\bar{v}) - |\tilde{v}|^2 - 2Re(\tilde{c}\bar{v})\|_{L^2(\Omega)}.
\]
Proof. The embedding "\( \subset \)" is obvious. For the converse we consider \( R_0 > 0 \) such that \( \mathbb{D} \subset B(R_0) \). For \( u \in E \) we define, for every \( \omega \in S^2 \) and \( R > R_0 \),
\[
U_R(\omega) = u(R\omega).
\]

Just as in the proof of Lemma 7 of [15], we show that \( U_R \) converges to \( U \) in \( L^2(S^2) \) norm and moreover \( \nabla_\omega U = 0 \). This enables us to conclude that \( U \) is a constant \( c(u) \). Since \( |u|^2 - 1 \in L^2(\Omega) \), we conclude that \( c(u) = 1 \). Let us proceed to the proof by noticing that
\[
\int_{\Omega} R^2 |\partial_R U_R|^2 d\omega \leq \int_{\Omega} R^2 |\partial_\omega U_R|^2 d\omega + \int_{\Omega} |\nabla u|^2 d\omega < \infty.
\]

By Cauchy Schwarz, \( \int_{\Omega} R^2 |\partial_R U_R|^2 d\omega \leq \int_{\Omega} R^2 |\partial_\omega U_R|^2 d\omega \) and thus \( \int_{\Omega} |\partial_\omega U_R|^2 d\omega \) satisfies the Cauchy criterion for convergence in \( L^2(S^2) \). We conclude the existence of a limit \( U \) of \( U_R \) in \( L^2(S^2) \). From (24) we deduce also that \( \int R^2 |\nabla_\omega U_R|^2 d\omega \) goes to 0 as \( R \to \infty \). Since \( \nabla_\omega U = \lim_{R \to \infty} \int R^2 |\nabla_\omega U_R|^2 d\omega \) we conclude that \( \|\nabla_\omega U\|_{L^2(S^2)} = 0 \). Thus, \( U = c \), a constant of absolute value 1.

Let us show that, if we denote by \( v = u - c \), then \( v \in H^1(\Omega) \). Notice that \( \nabla v = \nabla u \in L^2(\Omega) \). Let \( \chi \in C_0^\infty(\mathbb{R}^3) \), \( \chi = 0 \) on the ball of radius \( 1 \) \( B(0,1) \) and \( \chi = 0 \) outside \( B(0,2) \). We define \( \chi_R(x) = \chi(R^{-1}x) \). We show that \( v \) is the limit of \( \chi_R v \) in the norm \( \|\nabla \cdot \|_{L^2(\Omega)} \). As \( \chi_Rv \in H^1(\Omega) \), we obtain \( v \in H^1(\Omega) \).

Notice that we have \( v(R\omega) = -\int R^2 \partial_\omega U_R d\omega = -\int R^2 \omega \cdot (\nabla u)(\omega)d\omega \). By Cauchy Schwarz we obtain \( |v(R\omega)| \leq \frac{1}{\sqrt{R}} \left( \int R^2 \omega^2 (\omega)d\omega d\rho \right)^{\frac{1}{2}} \). Consequently,
\[
\int_{\Omega} R^2 \int_{S^2} |v(R\omega)|^2 d\omega dR \leq \int_{\Omega} \frac{1}{R} \int_{R} \int_{S^2} \omega^2 (\omega)d\omega d\rho dR.
\]
Let us denote by \( g(R) = \int R^2 \int_{S^2} \omega^2 (\omega)d\omega d\rho \). The function \( g \) is a decreasing function whose limit is 0 at \( \infty \). Then \( \int R^2 \frac{1}{R} g(R)dR < g(R')ln2 \), which goes to 0 as \( R' \) goes to \( \infty \). Consequently,
\[
\lim_{R \to \infty} \int_{R} \int_{S^2} |v(R\omega)|^2 d\omega dR = 0.
\]
This enables us to show that \( \|\nabla (v - \chi_R v)\|_{L^2(S^2)} \to 0 \) as \( R \to \infty \). We have that
\[
\nabla (v - \chi_R v) = \frac{1}{R} (\nabla \chi) R v + (1 - \chi_R) v.
\]
By writing \( v \) in polar coordinates we obtain, for \( R > R_0 \),
\[
\int \frac{1}{R^2} |(\nabla \chi) R v(x)|^2 dx \leq c \int_{R} \int_{S^2} |v(\rho\omega)|^2 d\omega d\rho \to 0
\]
as \( R \to \infty \). The other term also goes to 0 in \( L^2(\Omega) \) norm as \( R \to \infty \):
\[
\|(1 - \chi_R) v\|_{L^2(\Omega)} \leq c \|v\|_{L^2(|x|>R)} \to 0.
\]
This concludes the proof of \( v = u - c \in H^1(\Omega) \) and thus of the embedding "\( \subset \)". The completeness of the metric space \( E \) is an easy consequence of its structure. \( \square \)
We end this section by showing that $E + H^1(\Omega) \subset E$ (see also Lemma 2 of [15]).

**Lemma 4.2.** Let $u \in E$ and $w \in H^1(\Omega)$. Then $u + w \in E$ and

\[ \|u + w\|^2 - 1 \leq (\sqrt{\mathcal{E}(u)} + \|w\|_{H^1(\Omega)})(1 + \|w\|_{H^1(\Omega)}). \]

Moreover, for $\tilde{u} \in E$ and $\tilde{w} \in H^1(\Omega)$, we have

\[ \delta_{\mathcal{E}}(u + w, \tilde{u} + \tilde{w}) \leq (1 + \|w\|_{H^1} + \|\tilde{w}\|_{H^1})\delta_{\mathcal{E}}(u, \tilde{u}) + \]

\[ (1 + \sqrt{\mathcal{E}(u)} + \sqrt{\mathcal{E}(\tilde{u})} + \|w\|_{H^1} + \|\tilde{w}\|_{H^1})|w - \tilde{w}|_{H^1}. \]

**Proof.** From Proposition 4.1 we know that $u = c + v$, $c \in \mathbb{C}$, $|c| = 1$ and $v \in \dot{H}^1(\Omega)$. Then $u + w = c + (v + w)$ and $v + w \in \dot{H}^1(\Omega) + H^1(\Omega) \subset \dot{H}^1(\Omega)$. We have to show that $v + w \in F_c$ or equivalent, that $|u + w|^2 - 1 \in L^2(\Omega)$. We have

\[ |u + w|^2 - 1 = |v|^2 + 2Re(c^{-1}v) + |w|^2 + 2Re(c^{-1}w) + 2Re(\bar{v}w). \]

From Proposition 4.1 we have $|v|^2 + 2Re(c^{-1}v) \in L^2(\Omega)$ and from [22] $\|v\|^2 + 2Re(c^{-1}v)\|_{L^2(\Omega)} \leq \sqrt{\mathcal{E}(u)}$. From $w \in H^1(\Omega) \subset L^2(\Omega) \cap L^6(\Omega)$ we deduce $\|w\|^2_{L^2(\Omega)} \leq c\|w\|_{H^1(\Omega)}^2$, $\|2Re(\bar{v}w)\|_{L^2(\Omega)} \leq c\|v\|_{L^6(\Omega)}\|w\|_{H^1(\Omega)}$ and $\|2Re(cw)\|_{L^2(\Omega)} \leq c\|w\|_{H^1(\Omega)}$. Estimate (25) follows. For (26) we proceed similarly. \qed

4.2. The action of $S(t) = e^{it\Delta_N}$ on $E$. This section is devoted to defining the action of the group $S(t) = e^{it\Delta_N}$ on the energy space $E$. In view of the Neumann condition, $S(t)$ leaves constants invariant. We have to justify that $S(t)$ acts on $\dot{H}^1(\Omega)$. We begin by recalling some functional calculus facts (e.g., [21]).

The domain of $-\Delta_N$ in $L^2(\Omega)$ is $H^2_N(\Omega) = H^2(\Omega) \cap \{\partial_{\nu}v = 0\}$. For $v \in H^2_N(\Omega)$ we have $\|\sqrt{-\Delta_N}v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$. Indeed,

\[ \|\sqrt{-\Delta_N}v\|_{L^2(\Omega)}^2 = (\sqrt{-\Delta_N}v, \sqrt{-\Delta_N}v)_{L^2(\Omega)} = (v, -\Delta_Nv)_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}^2. \]

The domain of $\sqrt{-\Delta_N}$ in $L^2(\Omega)$ is $H^1(\Omega)$. For $u \in H^1(\Omega)$ we also have the identity $\|\sqrt{-\Delta_N}u\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$. Indeed, let $v \in H^2_N(\Omega)$. Then

\[ (\sqrt{-\Delta_N}u, \sqrt{-\Delta_N}v)_{L^2(\Omega)} = (u, -\Delta_Nv)_{L^2(\Omega)} = (\nabla u, \nabla v)_{L^2}. \]

From $\|\sqrt{-\Delta_N}v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$ for $v \in H^2_N(\Omega)$ we deduce the same identity for $u \in H^1(\Omega)$.

**Lemma 4.3.** Using the notations of [23], for $v \in \dot{H}^1(\Omega)$ the limit

\[ \lim_{R \to \infty} \sqrt{-\Delta_N(\chi_{\Omega}v)} \]

exists in the $L^2(\Omega)$ norm and we denote it by $\sqrt{-\Delta_N}v$. Moreover,

\[ \|\sqrt{-\Delta_N}v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}. \]

**Proof.** From [23] we have that $(\nabla(\chi_{\Omega}v))_R$ is a Cauchy sequence in the $L^2(\Omega)$ norm. As $\chi_{\Omega}v \in H^1(\Omega)$, the identity $\|\sqrt{-\Delta_N(\chi_{\Omega}v)}\|_{L^2(\Omega)} = \|\nabla(\chi_{\Omega}v)\|_{L^2(\Omega)}$ holds. Therefore, $(\sqrt{-\Delta_N(\chi_{\Omega}v)})_R$ is also a Cauchy sequence in the $L^2(\Omega)$ norm. Denoting by $\sqrt{-\Delta_N}v$ its limit, we obtain

\[ \|\sqrt{-\Delta_N}v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}. \]
Remark 2. Using the previous lemmas we can define a functional calculus \( \varphi(\sqrt{-\Delta_N}) \) on \( H^1(\Omega) \) for functions \( \varphi : [0, \infty) \to \mathbb{C} \) such that \( \lambda \mapsto \frac{\varphi(\lambda)}{\lambda} \) is continuous and bounded for \( \lambda \in [0, \infty) \). We denote by

\[
\varphi(\sqrt{-\Delta_N})v = \frac{\varphi(\sqrt{-\Delta_N})}{\sqrt{-\Delta_N}} \sqrt{-\Delta_N}v
\]

and this is well defined for \( v \in \dot{H}^1(\Omega) \) as \( \sqrt{-\Delta_N}v \in L^2(\Omega) \). An equivalent definition is : \( \varphi(\sqrt{-\Delta_N})v \) is the limit, in \( L^2(\Omega) \) norm, of \( \varphi(\sqrt{-\Delta_N})(\chi Rv) \).

An important consequence of the previous remark is the definition of \( S(t) = e^{it\Delta_N} \) on \( H^1(\Omega) \). Let \( v \in H^1(\Omega) \). We have \( S(t)v = v + (e^{it\Delta_N} - 1)v \) and each term of the sum is well defined.

Lemma 4.44. For all \( t \in \mathbb{R} \) we have \( S(t) : \dot{H}^1(\Omega) \to \dot{H}^1(\Omega) \) and moreover, for \( v \in \dot{H}^1(\Omega) \), we have

\[
(27) \quad \|S(t)v - v\|_{H^1(\Omega)} \leq c(1 + |t|^\frac{1}{2})\|\nabla v\|_{L^2(\Omega)}.
\]

Proof. By functional calculus we have that \( \frac{e^{it\Delta_N} - 1}{\sqrt{-\Delta_N}} = \varphi(-\Delta_N) \) acts on \( L^2(\Omega) \) with a norm \( \|\varphi(-\Delta_N)\|_{L^2 \to L^2} \leq \sup_{\lambda \in \sigma(-\Delta_N)}|\varphi(\lambda)| \). Here \( \varphi(\lambda) = \frac{e^{it\lambda} - 1}{\sqrt{\lambda}} \), for \( \lambda > 0 \). We have \( \|\varphi\|_{L^\infty} \leq c \min(|t|\sqrt{\lambda}, \sqrt{\lambda^{-1}}) \). Optimising on \( \lambda \) we obtain \( \|\varphi\|_{L^\infty} \leq c|t|^{\frac{1}{2}} \) and thus

\[
\|\frac{e^{it\Delta_N} - 1}{\sqrt{-\Delta_N}} \sqrt{-\Delta_N}v\|_{L^2(\Omega)} \leq c|t|^\frac{1}{2}\|\nabla v\|_{L^2(\Omega)}.
\]

We have also \( \|\sqrt{-\Delta_N}(e^{it\Delta_N} - 1)v\|_{L^2(\Omega)} \leq c\|\sqrt{-\Delta_N}v\|_{L^2(\Omega)} \leq c\|\nabla v\|_{L^2(\Omega)} \). Thus, \( S(t)v = v + (S(t) - 1)v \in \dot{H}^1(\Omega) + H^1(\Omega) \subset \dot{H}^1(\Omega) \).

From the previous lemmas we shall deduce that \( E \) is stable under the action of \( S(t) \), for all \( t \in \mathbb{R} \).

Proposition 4.5. For every \( t \in \mathbb{R} \) we have \( S(t)E \subset E \). Moreover, for every \( R > 0 \), for every \( T > 0 \), there exists \( C > 0 \) such that, for \( u_0, \tilde{u}_0 \in E \) with \( E(u_0), E(\tilde{u}_0) \leq R \), the following holds :

\[
(28) \quad \sup_{|t| \leq T} \delta_E(S(t)u_0, S(t)\tilde{u}_0) \leq C\delta_E(u_0, \tilde{u}_0).
\]

Proof. We write \( S(t)u_0 = u_0 + (S(t) - 1)u_0 \). Writing \( u_0 = c_0 + v_0 \), with \( v_0 \in \dot{H}^1(\Omega) \), we have that \( S(t)u_0 - u_0 = S(t)v_0 - v_0 \). From (27) we deduce \( (S(t) - 1)u_0 \in H^1(\Omega) \). From Lemma 1.2 we have \( S(t)u_0 = u_0 + (S(t) - 1)u_0 \in E \). Estimate (28) follows from (25), which reads in this setting :

\[
\delta_E(S(t)u_0, S(t)\tilde{u}_0) \leq c(1 + |t|^\frac{1}{2})(1 + \sqrt{E(u_0)} + \sqrt{E(\tilde{u}_0)})\delta_E(u_0, \tilde{u}_0).
\]
4.3. Strichartz inequality and energy space. As we mentioned in the beginning of Section 4.3, one of the main differences between NLS and Gross-Pitaevskii is that the initial data is not in $L^2(\Omega)$ for Gross-Pitaevskii. Therefore, it is not obvious to guess what the Strichartz inequality gives for $S(t)u_0$, when $u_0 \in E$. This is the purpose of this section. We denote by $u_L(t) = S(t)u_0$, for all $t \in \mathbb{R}$. We show in this section that for $u_0 \in E$ and $2 < p < 3$ we have $u_L \in L^p([-T,T], L^\infty(\Omega))$, for some $T > 0$. We decompose $u_L$ in its high and low frequency parts and we treat them separately.

Let $\varphi_1 \in C^\infty_0(\mathbb{R})$ such that $\varphi_1(s) = 1$ pour $|s| \leq 1$ and $\varphi_1(s) = 0$ pour $|s| \geq 2$. Let $\varphi_2 \in C^\infty(\mathbb{R})$ such that $\varphi_1 + \varphi_2 = 1$. Let $u_0 \in E$, $u_0 = c_0 + v_0$, with $c_0 \in \mathbb{C}$, $|c_0| = 1$ and $v_0 \in H^1(\Omega)$.

We denote by $v_{20} = \varphi_2(\sqrt{-\triangle_N})v_0$. From Remark 2 and Lemma 4.3 we deduce the following properties of $v_{20}$.

Lemma 4.6. Under the previous notations, we have $v_{20} \in H^1(\Omega)$ and

$$\|v_{20}\|_{H^1(\Omega)} \leq c\|\nabla v_0\|_{L^2(\Omega)}.$$ 

In view of Lemma 4.6 we can apply the Strichartz inequality (3) (in Neumann setting) to $S(t)v_{20}$.

Lemma 4.7. Let $v_2(t) = S(t)v_{20}$. For $T > 0$ and $2 < p < 3$, the following holds: $v_2 \in L^p([-T,T], L^\infty(\Omega)) \cap L^\infty([-T,T], H^1(\Omega))$ and

$$\|v_2\|_{L^p_L(\Omega)} + \|v_2\|_{L^\infty_{H^1}(\Omega)} \leq C\|\nabla v_0\|_{L^2(\Omega)}.$$ 

Proof. From Lemma 4.6 we have $v_{20} \in H^1(\Omega)$. Let $(p,q)$ be an admissible couple in dimension 3 and $\epsilon > 0$. From the Strichartz inequality (3) we deduce

$$\|v_2\|_{L^p_L(\Omega)} \leq C\|\nabla v_0\|_{L^2(\Omega)}.$$ 

For $2 < p < 3$ there exists $\epsilon > 0$ such that $W^{1,-\frac{1}{p}-\epsilon,q}(\Omega) \subset L^\infty(\Omega)$ (see the proof of [12]). Thus, $\|v_2\|_{L^p_L(\Omega)} \leq C\|\nabla v_0\|_{L^2(\Omega)}$. The estimate on $\|v_2\|_{L^\infty_{H^1}(\Omega)}$ follows from the conservation of the $H^1$ norm by the linear Schrödinger flow $e^{it\triangle_N}$. \hfill \Box

We denote by $v_{10} = v_0 - v_{20} = \varphi_1(\sqrt{-\triangle_N})v_0$ and by $v_1(t) = S(t)v_{10}$.

Lemma 4.8. For $T > 0$, there exists $C > 0$ such that we have $v_1 \in L^\infty([-T,T] \times \Omega)$ satisfying

$$\|v_1\|_{L^\infty_L} \leq C\|\nabla v_0\|_{L^2}.$$ 

Proof. In this proof we look at $v_1$ separately near the obstacle and away from the obstacle. The reason is that $v_1$ is only an $H^1(\Omega)$ function. Indeed, $\varphi_1(\sqrt{-\triangle_N}) : L^6(\Omega) \to L^6(\Omega)$ and $\varphi_1(\sqrt{-\triangle_N}) : L^2(\Omega) \to L^2(\Omega)$. As $S(t) : H^1(\Omega) \to H^1(\Omega)$ by Lemma 4.3, we obtain $v_1 \in H^1(\Omega)$.

We consider $\chi \in C^\infty_0(\mathbb{R}^3)$ such that $\chi = 1$ near $\Theta = \mathbb{C} \Omega$. Then $\chi v_1 \in L^\infty([-T,T], L^2(\Omega))$ : $\|\chi v_1(t)\|_{L^2_L(\Omega)} \leq \|\chi\|_{L^3(\Omega)}\|v_1\|_{L^2_L(L^6(\Omega))} \leq C\|\nabla v_0\|_{L^2(\Omega)}$.

Similarly, we obtain $\triangle (\chi v_1) = (\triangle \chi)v_1 + 2\nabla \chi \cdot \nabla v_1 + \chi(\triangle v_1) \in L^\infty_L(L^2(\Omega))$. Moreover, $\frac{\partial}{\partial t} (\chi v_1) = 0$ as $\chi = 1$ in the neighborhood of $\partial \Omega$. \hfill \Box
Thus, $\chi v_1 \in L^\infty \mathcal{H}^2_\mathcal{N}(\Omega)$, where $\mathcal{H}^2_\mathcal{N}(\Omega)$ is the domain of $-\Delta_N$ in $L^2(\Omega)$. As $\mathcal{H}^2_\mathcal{N}(\Omega) \subset L^\infty(\Omega)$, we obtain $\chi v_1 \in L^\infty([-T,T] \times \Omega)$.

We pass to the term $(1 - \chi)v_1$. It can be seen as a function on $\mathbb{R}^3$ in the $x$ variable extending it by 0. Since $v_1 \in L^\infty L^6(\Omega)$, we have $(1 - \chi)v_1 \in L^\infty L^6(\mathbb{R}^3)$. We show that $(1 - \chi)v_1 \in L^\infty W^{2,6}(\mathbb{R}^3)$. For that purpose, it suffices to show that $\Delta((1 - \chi)v_1) \in L^\infty(L^6(\mathbb{R}^3))$. We have

$$
(29) \quad \Delta((1 - \chi)v_1) = -(\Delta \chi)v_1 - 2\nabla \chi \cdot \nabla v_1 + (1 - \chi)(\Delta v_1).
$$

Clearly, the first and the last term of the right hand side expression are in $L^\infty(L^6(\mathbb{R}^3))$. For $\nabla \chi \cdot \nabla v_1$ we need to do finer analysis. As $\nabla v_1 \in L^\infty L^2(\Omega)$ we deduce $\nabla \chi \cdot \nabla v_1 \in L^\infty L^2(\mathbb{R}^3)$. We show that $\nabla \chi \cdot \nabla v_1 \in L^\infty W^{2,2}(\mathbb{R}^3)$. We compute

$$
\Delta(\nabla \chi \cdot \nabla v_1) = (\Delta \nabla \chi) \cdot \nabla v_1 + 2(\nabla^2 \chi) \cdot (\nabla^2 v_1) + \nabla \chi \cdot (\Delta \nabla v_1).
$$

We have $(\Delta \nabla \chi) \cdot \nabla v_1 \in L^\infty L^2(\mathbb{R}^3)$ and $\nabla \chi \cdot (\Delta \nabla v_1) \in L^\infty L^2(\mathbb{R}^3)$. The middle term, $2(\nabla^2 \chi) \cdot (\nabla^2 v_1)$ can be written as $P(x,D)(1 - \Delta)v$, with $P(x,D) = 2(\nabla^2 \chi) \cdot (\nabla^2 (1 - \Delta)^{-1})$ an pseudo-differential operator of order 0 with compact support. Its coefficients are independent of $t$. Consequently, $2(\nabla^2 \chi) \cdot (\nabla^2 v_1) \in L^\infty L^6(\mathbb{R}^3)$ and since this function is compactly supported in $x$, it belongs also to $L^\infty L^2(\mathbb{R}^3)$.

We obtain $\nabla \chi \cdot \nabla v_1 \in L^\infty W^{2,2}(\mathbb{R}^3) \subset L^\infty L^6(\mathbb{R}^3)$. Going back to (29) we deduce $(1 - \chi)v_1 \in L^\infty W^{2,6}(\mathbb{R}^3) \subset L^\infty([-T,T] \times \mathbb{R}^3)$. Taking the restriction to $\Omega$ concludes the proof. □

From the previous lemmas, we deduce easily the following.

**Proposition 4.9.** For $T > 0$ and $2 < p < 3$, there exists $C > 0$ such that, for $u_0 \in E$ and $u_L(t) = e^{it\Delta_N}u_0$, we have $u_L \in L^p([-T,T],L^\infty(\Omega))$ and

$$
(30) \quad ||u_L||_{L^p_{t}(L^\infty_{x})} \leq 1 + C||\nabla v_0||_{L^2(\Omega)}.
$$

Moreover, for $\tilde{u_0} \in E$ and $\tilde{u}_L(t) = e^{it\Delta_N}\tilde{u}_0$,

$$
(31) \quad ||u_L - \tilde{u}_L||_{L^p_{t}(L^\infty_{x})} \leq C\delta E(u_0,\tilde{u}_0).
$$

**Proof.** We write $u_L(t) = c_0 + e^{it\Delta_N}v_0 = c_0 + v_1(t) + v_2(t)$. The conclusion follows from $c_0 \in \mathbb{C}$, $v_1 \in L^p_T(L^\infty)$, $v_2 \in L^\infty_T(L^p)$ and their respective estimates. □

We close this section by collecting estimates which will be useful in the sequel. We consider $u_0,\tilde{u}_0 \in E$, $u_L(t) = S(t)u_0$ and $\tilde{u}_L(t) = S(t)\tilde{u}_0$, $w, \tilde{w} \in X_T = C([-T,T],H^2_\mathcal{N}(\Omega)) \cap L^p([-T,T],L^\infty(\Omega))$ with the associated norm $||w||_{X_T} = \max_{|t| \leq T}||w(t)||_{H^1(\Omega)} + ||w||_{L^p([-T,T],L^\infty(\Omega))}$. Let $u = u_L + w$ and $\tilde{u} = \tilde{u}_L + \tilde{w}$. We denote by

$$
\gamma = \delta E(u_0,\tilde{u}_0) + ||w - \tilde{w}||_{X_T}.
$$

As a corollary of Lemma 4.3 and 4.2 we have

$$
(32) \quad ||u_L^2 - 1||_{L^\infty_T L^2(\Omega)} \leq c(\sqrt{\mathcal{E}(u_0)} + \mathcal{E}(u_0))
$$

$$
(33) \quad ||u^2 - 1||_{L^\infty_T L^2(\Omega)} \leq (1 + \mathcal{E}(u_0))||w||_{X_T} + ||w||_{X_T}^2.
$$
As a corollary of Proposition 4.9 we have
\begin{equation}
\|u\|_{L^p_t L^\infty_x(\Omega)} \leq c(1 + \sqrt{E(u_0) + \|w\|_{X_T}})
\end{equation}
\begin{equation}
\|u - \bar{u}\|_{L^p_t L^\infty_x(\Omega)} \leq \gamma
\end{equation}
From (34), (35) and (28) we deduce
\begin{equation}
\|u^2 - |\bar{u}|^2\|_{L^p_t L^\infty_x(\Omega)} \leq \gamma(1 + \sqrt{E(u_0) + \sqrt{E(\bar{u}_0)} + \|w\|_{X_T} + \|\bar{w}\|_{X_T}})
\end{equation}
Moreover,
\begin{equation}
\|u^2 - 1\|_{L^p_t L^\infty_x(\Omega)} \leq 1 + E(u_0) + \|w\|^2_{X_T}
\end{equation}
\begin{equation}
\|u - \bar{u}\|_{L^p_t L^\infty_x(\Omega)} \leq \gamma(1 + \sqrt{E(u_0) + \sqrt{E(\bar{u}_0)} + \|w\|_{X_T} + \|\bar{w}\|_{X_T}})
\end{equation}
By simple computations we obtain
\begin{equation}
|\nabla u|_{L^p_t L^2_x(\Omega)} \leq \sqrt{E(u_0) + \|w\|_{X_T}}
\end{equation}
\begin{equation}
|\nabla u - \nabla \bar{u}|_{L^p_t L^2_x(\Omega)} \leq \gamma
\end{equation}
The estimates (32) to (40) follow from simple computations, decomposing u = u_L + w and applying Hölder and Sobolev inequalities combined with the estimates cited.

4.4. Proof of Theorem 1.3. Let u_0 \in E. In Section 4.2 we presented the action of S(t) = e^{it\Delta_N} on E. We recall the notation u_L(t) = S(t)u_0. We call the solution of (2) the solution to the Duhamel associated formula:
\begin{equation}
u(t) = u_L(t) - i \int_0^t e^{i(t-\tau)\Delta_N} F(u)\,d\tau,\end{equation}
where F(u) = (|u|^2 - 1)u. We denote by w = u - u_L and by \Phi the functional
\begin{equation}\Phi(w) = -i \int_0^t e^{i(t-\tau)\Delta_N} F(u_L + w)\,d\tau.
\end{equation}
We show the local existence of u that satisfies (11) by showing that \Phi has a fixed point \Phi(w) = w. For that purpose we define, for T > 0 and 2 < p < 3, \(X_T = C([-T,T], H^1_0(\Omega)) \cap L^p([-T,T], L^\infty(\Omega)). The space \(X_T is a complete Banach space for the following norm
\|w\|_{X_T} = \max_{|t| \leq T} \|w(t)\|_{H^1(\Omega)} + \|w\|_{L^p([-T,T], L^\infty(\Omega))}.
We prove that, for a T > 0 and R > 0 small enough, \Phi is a contraction from B(0,R) \subset X_T into itself.

**Lemma 4.10.** Using the previous notations we have, for w \in X_T, that
\|\Phi(w)\|_{X_T} \leq c\|F(u_L + w)\|_{L^1_t L^1_x H^1(\Omega)}.

**Proof.** From (11) we deduce, by Minkowski inequality, that
\|\Phi(w)\|_{L^\infty_t L^2_x(\Omega)} \leq c\|F(u_L + w)\|_{L^1_t L^2_x(\Omega)}.
As \nabla (\Phi(w)) = -i \int_0^t e^{i(t-\tau)\Delta_N} \nabla (F(u_L + w))(\tau)\,d\tau we have also
\|\nabla (\Phi(w))\|_{L^\infty_t L^2_x(\Omega)} \leq c\|\nabla (F(u_L + w))\|_{L^1_t L^2_x(\Omega)}.
We have considered $2 < p < 3$. Thus, there exists $\epsilon > 0$ such that, for $(p, q)$ an admissible couple in dimension 3, $W^{1, \frac{1}{p} - \epsilon, \frac{q}{p}}(\Omega) \subset L^\infty(\Omega)$. From the Strichartz inequality \(3\) we obtain:

$$\|\Phi(w)\|_{L^p_T L^\infty(\Omega)} \lesssim \|\Phi(w)\|_{L^p_T W^{1, \frac{1}{p} - \epsilon, \frac{q}{p}}(\Omega)} \lesssim c\|F(u_L + w)\|_{L^1_T H^1(\Omega)}.$$

We have to estimate $F(u)$ in $L^1_T H^1(\Omega)$ for $u = u_L + w, \ w \in X_T$. For the fixed point method we also need to estimate $\|F(u_L + w) - F(\tilde{u}_L + \tilde{w})\|_{L^1_T H^1(\Omega)}$.

**Proposition 4.11.** Under the conditions of Section 4.4 we have

$$\|F(u)\|_{L^1_T L^2} \leq cT^{1-\frac{1}{p}}(1 + \mathcal{E}(u_0) + \|w\|_{X_T})^2\|w\|_{X_T}$$

and

$$\|\nabla(F(u))\|_{L^1_T L^2} \leq cT^{1-\frac{2}{p}}(1 + \sqrt{\mathcal{E}(u_0)} + \|w\|_{X_T})^2$$

where we have denoted by $\gamma = \delta_{E}(u_0, \tilde{u}_0) + \|w\|_{X_T}$. Notice that, if $u_0 = \tilde{u}_0$, then we have $\gamma = \|w - \tilde{w}\|_{X_T}$.

**Proof.** The conclusions follow from estimates \(32\) to \(40\). Let us explain one of the conclusions, for example the estimate on $F(u_L + w) - F(\tilde{u}_L + \tilde{w})$. We have

$$F(u) - F(\tilde{u}) = (|u|^2 - |\tilde{u}|^2)u + (u - \tilde{u})(|u|^2 - 1).$$

We apply the Hölder inequality combined with \(36\) and \(41\) for the first term and \(35\) and \(33\) for the second one. We bound thus $\|F(u_L + w) - F(\tilde{u}_L + \tilde{w})\|_{L^1_T L^2}$. By Hölder inequality we obtain the positive power of $T$:

$$\|F(u) - F(\tilde{u})\|_{L^1_T L^2} \leq cT^{1-\frac{1}{p}}\|F(\tilde{u})\|_{L^1_T L^p}.$$

The other estimates follow similarly. \(\square\)

Combining the estimates on the nonlinear term from Proposition 4.11 with Lemma 4.10 we obtain the following

**Corollary 4.12.** Under the conditions on Lemma 4.10 we have

$$(43)\|\Phi(w)\|_{X_T} \leq cT^{1-\frac{2}{p}}(1 + \mathcal{E}(u_0) + \|w\|_{X_T})^3$$

$$(44)\|\Phi(w) - \Phi(\tilde{w})\|_{X_T} \leq cT^{1-\frac{2}{p}}(1 + \mathcal{E}(u_0) + \mathcal{E}(\tilde{u}_0) + \|w\|_{X_T} + \|\tilde{w}\|_{X_T})^2;$$

where we denoted by $\gamma = \delta_{E}(u_0, \tilde{u}_0) + \|w - \tilde{w}\|_{X_T}$.

As a consequence, we can prove the global wellposedness result from Theorem 1.3 on Gross-Pitaevskii equation (2).

**Proof.** We fix $u_0 \in B \subset E$. From estimate \(11\) we deduce that there exist $T, R > 0$, depending only on $B \subset E$ ($u_0 \in B$), such that, for $w \in X_T$ with $\|w\|_{X_T} \leq R$, we have $\|\Phi(w)\|_{X_T} < R$. \(19\)
For $\tilde{u}_0 = u_0$ estimate (14) reads

$$\|\Phi(w) - \Phi(\tilde{w})\|_{X_T} \leq cT^{-\frac{1}{2}} \left(1 + \|w\|_{X_T} + \|\tilde{w}\|_{X_T}\right)^2 \|w - \tilde{w}\|_{X_T}.$$  

As $2 < p$, choosing $T$ eventually smaller ensures that $\Phi$ is a contraction on the ball $B(0, R) \subset X_T$, $B(0, R) = \{w \in X_T, \|w\|_{X_T} < R\}$. Consequently, there exists a fixed point of $\Phi$ in $B(0, R)$, which is therefore solution to (2).

For the Lipschitz property of the flow let us consider $u, \tilde{u} \in B(0, R) \subset X_T$ two solutions of $\Phi(u - u_L) = u - u_L$, therefore of (2), with initial data respectively $u_0, \tilde{u}_0 \in B$.

From (14) we have, for $w = u - u_L$ and $\tilde{w} = \tilde{u} - \tilde{u}_L$,

$$\|w - \tilde{w}\|_{X_T} \leq cT^{1 - \frac{1}{2p}}(\delta_E(u_0, \tilde{u}_0) + \|w - \tilde{w}\|_{X_T})(1 + E(u_0) + E(\tilde{u}_0) + \|w\|_{X_T} + \|\tilde{w}\|_{X_T})^2,$$

For $T, R > 0$ chosen before we have $cT^{1 - \frac{1}{2p}}(1 + E(u_0) + E(\tilde{u}_0) + \|w\|_{X_T} + \|\tilde{w}\|_{X_T})^2 < 1$ and therefore, $\exists \epsilon > 0$ such that

$$\|w - \tilde{w}\|_{H^1} \leq \|w - \tilde{w}\|_{X_T} \leq \epsilon \delta_E(u_0, \tilde{u}_0).$$

From (20) we have $\delta_E(u(t), \tilde{u}(t)) \leq C(R, B)(\delta_E(u_0, \tilde{u}_0) + \|w - \tilde{w}\|_{L^\infty_T H^1})$.

Consequently, there exists $C > 0$ such that $\delta_E(u(t), \tilde{u}(t)) \leq C \delta_E(u_0, \tilde{u}_0)$, for all $|t| \leq T$. We conclude that the flow $u_0 \mapsto u(t)$ is Lipschitz on $B \subset C$.

The proof of the propagation of regularity from section 3.3 of [15] adapts to the framework of exterior domains using techniques similar to those of Section 4.2. Those techniques combined with the stability of $E$ by summation with an $H^1$ element (see Lemma 1.2) enables us to show that $u_0 \in E$ can be approached, in $\delta_E$ distance, by $u_0^* \in E$ such that $\triangle u_0^* \in L^2(\Omega)$. As one can prove conservation of energy $E$ for initial data $f \in E$ such that $\triangle f \in L^2(\Omega)$, from (25) we deduce that conservation of energy holds for $u_0 \in E : E(u(t)) = E(u_0)$.

Notice that $T$, the existence time for which we applied the fixed point method, depends on $E(u_0)$ and on $R$. From the conservation of energy for the solutions of (2) we have $E(u_0) = E(u(t))$ for all $|t| \leq T$. Consequently, we can apply a bootstrap argument and conclude to the extension globally in time of $u \in C(\mathbb{R}, E)$, solution of (2).

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