Entanglement Evolution in a Five Qubit Error Correction Code

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Abstract In this paper I explore the entanglement evolution of qubits that are part of a five qubit quantum error correction code subject to various decohering environments. Specifically, I look for possible parallels between the entanglement degradation and the fidelity of the logical qubit of quantum information stored in the physical qubits. In addition, I note the possible exhibition of entanglement sudden death (ESD) due to decoherence and question whether ESD is actually a roadblock to successful quantum computation.

Keywords entanglement · quantum error correction · decoherence

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1 Introduction

It is currently accepted that the leading obstacle on the path towards practical quantum computation is the inevitability of decoherence which stems from unwanted interactions between the system of interest and its environment [1]. A consequence of decoherence is the degradation of entanglement between subsystems (such as qubits). Entanglement between qubits is thought to be necessary for proper operation of a quantum computer. Though the need to fight the effects of decoherence has long been realized, recent research has suggested that the these effects may be worse than originally thought. The coherence of a system may approach zero asymptotically due to unwanted interactions with the environment but the entanglement may completely disappear in a finite amount of time [2,3,4,5]. This comparatively sudden disappearance of entanglement is termed entanglement sudden death (ESD). Recent theoretical studies have been devoted to understanding this phenomenon in bi- and multi-partite systems [6,7,8,9,11] and there have been a number of experimental studies of this phenomenon as well [12,13,14]. Importantly, there has been a call to formulate methods of avoiding the ESD phenomenon [15] so as to ensure the viability of quantum computation.

Yet, it is not at all clear that quantum computer architects should be concerned with ESD in any way more than they are concerned with general issues of decoherence. A series of recent papers has shown that the onset of ESD has no singular effect on the accuracy of certain quantum protocols. ESD causes neither a dramatic drop in protocol accuracy, as measured for example by the fidelity, nor any substantive change in protocol behavior. Instead, for protocols such as a three qubit error correction code [16], a cluster-based single qubit rotation [17], and a decoherence free subspace [18], ESD is a non-descript byproduct of decoherence.

In this paper I continue to explore entanglement evolution in a system implementing a quantum protocol in a decohering environment. Specifically, I look at the entanglement between the qubits of a five-qubit quantum error correction (QEC) code [19,20]. In contrast to the previously explored three qubit QEC code explored in [16], the five qubit QEC code can fully protect one qubit of quantum information from all possible single qubit errors. The five physical qubits of the QEC code are usually entangled. In fact, it would not be possible to design a five qubit QEC protocol that fully protects one qubit of quantum information without entanglement between the constituent physical qubits. Based on this I address the following: how does the loss of entanglement affect the ability of the code to protect the
quantum information? To address this question I compare the degradation of entanglement as a function of decoherence strength to the fidelity of the stored logical qubit. We will see that for certain initial states the fidelity of the stored information remains high despite significant loss of entanglement and, in general, the decay of stored quantum information fidelity does not strongly correlate with the entanglement behavior. Finally, I show that the negative effects of decoherence render the QEC code useless well before a complete loss of entanglement in the system. In fact, in most cases explored below ESD does not occur before complete decoherence. This lack of correlation suggests that entanglement *per se* is not what drives this QEC protocol. Rather, most states in Hilbert space are entangled and decoherence strength to the fidelity of the stored information I will use as an accuracy measure for some suitable decay constant $\kappa$ and time $t$.

### 1.1 The Five Qubit Error Correction Code

For our study of the entanglement evolution, I start with an unencoded single qubit in the state $|\psi_u\rangle = \cos \alpha |0\rangle + e^{-i\beta} \sin \alpha |1\rangle$. I assume perfect encoding of this qubit of quantum information into five qubits whose state after encoding is $|\psi_L\rangle = \cos \alpha |0_L\rangle + e^{-i\beta} \sin \alpha |1_L\rangle$. There are a number of formulations of five qubit QEC codes that can fully protect one qubit of quantum information. Here I use the formulation of [19] with logical $|0\rangle$ and $|1\rangle$:

\[
|0_L\rangle = |00000\rangle - |01111\rangle + |10011\rangle + |11100\rangle \\
+ |01101\rangle + |01001\rangle + |10101\rangle + |11010\rangle
\]

\[
|1_L\rangle = -|11111\rangle + |10000\rangle + |01100\rangle + |00011\rangle \\
+ |11001\rangle + |10110\rangle + |01010\rangle + |00101\rangle
\]

The qubits are placed in a decohering environment where the five physical qubits are subject to decoherence of strength $\delta$. The error syndrome is determined by measuring qubits 1, 2, 4, and 5 and the appropriate recovery operation is applied. If $\delta$ is small the syndrome measurement will project the qubits into a state where, up to order $\delta^2$, at most one error has occurred. The error will be corrected by the recovery operation. The exact output state, however, will depend on the outcome of the syndrome measurement. Thus, to quantify the fidelity of the stored quantum information I will use as the final single qubit state, $\rho_f(\alpha, \beta, \delta)$, the mixed state weighted average of all 16 possible syndrome measurement outcomes (after application of the appropriate recovery operation).

### 1.2 Entanglement and Accuracy Measures

To quantify and monitor entanglement between physical qubits within the QEC code as they are subject to decoherence I use an entanglement measure known as the negativity, $N$ defined as the most negative eigenvalue of the partial transpose of the system density matrix [21]. There are a number of inequivalent forms of the negativity for any multi-qubit system: the partial transpose may be taken with respect to any single qubit, $N_j$, or the partial transpose may be taken with respect to any two qubits, $N_{j,k}$. Note that a zero value of all negativities does not guarantee separability of the state though it does mean that any entanglement that is present is not distillable.

As an accuracy measure for the single qubit of quantum information stored in the QEC code I use the fidelity,

\[
F(\alpha, \beta, \delta) = \langle \psi_u | \rho_f | \psi_u \rangle,
\]

which is a measure of how well the QEC code has protected the single qubit of logical information. Here we are especially interested in comparing the fidelity of the qubit of stored information with the amount of, and the degradation of, entanglement in the system before syndrome measurement.

### 2 Decoherence Models

In each of the next three subsections I explore different decohering environments in which the five qubits of the QEC code are placed. As we shall see, each of the environments affects the system very differently, both with respect to the fidelity of the stored information and with respect to the entanglement evolution. The three decoherence models are the independent qubit phase damping, amplitude damping, and depolarizing environments.

#### 2.1 Phase Damping

We first look at the entanglement evolution of the five qubit system with no interaction between the qubits, in an independent qubit dephasing environment. This environment is fully described by the Kraus operators

\[
K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\delta} \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\delta} \end{pmatrix},
\]

where the dephasing parameter $\delta$ can also be written in a time-dependent fashion such as $\delta = 1 - \exp(-\kappa t)$ for some suitable decay constant $\kappa$ and time $t$. 

The effect of decoherence on the fidelity of the QEC code and the entanglement between the physical qubits is shown in Fig. 1. Interestingly, the robustness of the quantum information is very much dependent on the initial state. As an example, the decoherence strength at which the fidelity will fall below .95 varies widely, .18 < δ < .64, depending on the initial state. The fidelity of initial states with low degrees of various types of entanglement decrease most quickly (states around α = π/4, β = 0) while the fidelity of those states with the highest initial entanglement (α = 0, π/2, β = 0) decreases most slowly. For stronger decoherence this difference in fidelity remains. Highly entangled initial states (α = 0, π/2) retain a high level of fidelity > .85, even in the limit δ → 1, but the fidelity of initial states with low entanglement decays all the way down to .5. Only one entanglement metric exhibits ESD: N1. This implies that the first qubit is, in general, less integrated into the logical qubit than the other qubits. More relevant for this study is the demonstration that ESD plays little, if any, role in determining the success of the QEC code as clearly there are a host of entanglement metrics that do not decay to zero until δ → 1.

While for most of the explored entanglement metrics the entanglement depends on the initial state, all states initially have the same N2 entanglement. For this metric, states that are slower to lose this entanglement have a somewhat faster decrease in fidelity. From the above there does seem to be a correlation between the entanglement degradation (for all explored entanglement measures except N2) and the fidelity of the stored quantum information. States with initially high degrees of entanglement lose their entanglement much more slowly than states with low initial entanglement and the fidelity of the quantum information for the former states remains much higher.

Similar results to those of initial states β = 0 are found for states with β = π/4. However, the initial entanglement is generally slightly higher, and all states contain some N1 entanglement. The N1 entanglement does undergo ESD but at decoherence values δ ≥ .8.

2.2 Amplitude Damping

We now turn to an independent qubit amplitude damping environment. As above, I explore the entanglement evolution of the qubits which make up the QEC code and compare it to the fidelity of the stored quantum information. The Kraus operators for this environment are:

\[ K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \delta} \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & \sqrt{\delta} \\ 0 & 0 \end{pmatrix} \]

(4)

where the (time dependent) amplitude damping strength is denoted δ.

As seen in Fig. 2, the fidelity of the quantum information stored in the QEC code remains close to one under amplitude damping only for very low decoherence strengths. At higher decoherence strengths we find a remarkable difference of behavior between states close to α = 0 and those close to α = π/2. States close to the latter point exhibit a deep loss of fidelity, which falls below .2, much lower than the lowest fidelity exhibited...
in the phase damping environment. In contrast, states close to the former point exhibit the opposite behavior. After the expected decrease in fidelity due to the decoherence, the fidelity begins to increase. For \( \alpha = 0 \) the decoherence strength where the transition occurs is \( \delta \approx .56 \). As \( \alpha \) increases so does the \( \delta \) where the transition occurs. The fidelity at \( \alpha = 0; \delta = 1 \) reaches the value of .875. Comparing this to the entanglement evolution we note that the initial entanglement and subsequent entanglement decay is the same for these two states. This clearly demonstrates the inability of entanglement degradation to indicate fidelity of the stored quantum information.

In general, the entanglement evolution under amplitude damping reflects the opposite of what we found for the phase damping environment. In the amplitude damping environment it is the initial states with higher entanglement that experience a faster initial fidelity decay. In fact, the entanglement degradation under amplitude damping looks very similar to the entanglement degradation under phase damping, albeit the degradation occurs at a faster rate. The fidelity decay behavior, however, is completely different. It is, for low \( \delta \), much more uniform with respect to the initial state, the fidelity increases as \( \delta \) increases for certain initial states, and states around \( \alpha = \pi/2 \) which have the highest fidelity under phase damping, achieve the lowest fidelity under amplitude damping. ESD is once again exhibited only for \( N_1 \) and only for very limited initial states, not the states that achieve the lowest fidelity. This demonstrates that ESD plays no role in determining the success of the QEC code.

### 2.3 Depolarizing

The final decohering environment we explore is an independent qubit depolarizing environment and, as above, I compare the entanglement evolution to the fidelity of the stored quantum information. The Kraus operators for this environment are:

\[
K_1 = \sqrt{1 - \frac{3\delta}{4}} \sigma_0; \quad K_j = \frac{\sqrt{\delta}}{2} \sigma_j,
\]

where \( \sigma_0 \) is the identity and \( \sigma_j \) are the Pauli spin operators, \( j = x, y, z \) and \( \delta \) is now the (time dependent) depolarizing strength.

In a depolarizing environment the fidelity drops approximately uniformly as a function of initial state before reaching \( F(\alpha, \beta, \delta \to 1) = .5 \). In contrast, the entanglement decay behavior is similar to that of the other environments in that the initial entanglement and the rate of decay depends on the initial state. \( N_2 \) however, does decay almost uniformly with initial state. In addition, the entanglement under depolarizing decays much more quickly than for the other decohering environments as does the fidelity. ESD is exhibited for all metrics and all initial states at \( \delta \leq .5 \), where the fidelity is \( .55 < F(\alpha, 0, \delta) < .6 \).
3 Discussion and Conclusions

The goal of a QEC code is to store quantum information in such a way such that it remains unaffected by decoherence. When the decoherence affecting the physical qubits of the QEC is weak, the syndrome measurement will generally project the (unmeasured) qubits into an almost pure state which can easily be rotated to the nearly correct pre-decohered state. For QEC codes like the 7-qubit CSS code, the syndrome measurements are done on ancilla qubits and the projection ‘restores’ mostly all entanglement that may have been destroyed by the decoherence. For stronger decoherence, quantum error correction will generally fail, meaning the syndrome measurement will project the qubits into a mostly incorrect state.

For the five qubit code there is only one unmeasured qubit after syndrome measurement and thus no entanglement remains. The above analysis, comparing the loss of entanglement to the fidelity of the stored quantum information, was thus done utilizing entanglement metrics applied to the state before syndrome measurement. Study of other QEC codes will allow for comparisons between fidelity decay and entanglement remaining after syndrome measurements and may reveal a closer parallel between entanglement and fidelity.

For the five qubit QEC, the similarity of the entanglement degradation when the qubits are in either an amplitude damping or phase damping environment, despite the complete lack of similarity for the decay of stored quantum information fidelity, demonstrates that the entanglement decay is not a good indicator of fidelity. Furthermore, we saw that ESD is exhibited only when the qubits are placed in a depolarizing environment or for a few specific states in other decohering environments. Nevertheless, fidelity under depolarizing decreases only to .5 while the fidelity in the other decohering environments in the limit of δ → 1 may be higher or lower depending on the environment and the initial state. Thus, we see that ESD has absolutely no effect on the overall success of the QEC code.

Studies such as the one presented here allow us to frame, and partially answer, the question: what is the role of entanglement in quantum information processing. While the encoding of all states into a five qubit QEC contains entanglement any parallels between the entanglement evolution and fidelity appear superficial. This implies that the entanglement is present only because most states in Hilbert space happen to be entangled and it is the large size of Hilbert space that allows for the conduction of QEC codes.

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