Accurate estimation of sums over zeros of the Riemann zeta-function

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Abstract

We consider sums of the form \( \sum \phi(\gamma) \), where \( \phi \) is a given function, and \( \gamma \) ranges over the ordinates of nontrivial zeros of the Riemann zeta-function in a given interval. We show how the numerical estimation of such sums can be accelerated by a simple device, and give examples involving both convergent and divergent infinite sums.

1 Introduction

Let the nontrivial zeros of the Riemann zeta-function \( \zeta(s) \) be denoted by \( \rho = \beta + i\gamma \). In order of increasing height, the ordinates of the zeros in the upper half-plane are \( \gamma_1 \approx 14.13 < \gamma_2 < \gamma_3 < \cdots \).

Let \( \phi : [T_0, \infty) \mapsto [0, \infty) \) be a non-negative function on the interval \([T_0, \infty)\), for some \( T_0 \geq 1 \). Throughout this paper we assume that \( \phi(t) \) is twice continuously differentiable and satisfies the conditions \( \phi'(t) \leq 0 \) and \( \phi''(t) \geq 0 \) on \([T_0, \infty)\). These conditions imply that \( \phi(t) \) is convex on \([T_0, \infty)\).

We are interested in sums of the form \( \sum_{T_1 \leq \gamma \leq T_2} \phi(\gamma) \) and \( \sum_{T_1 < \gamma} \phi(\gamma) \), where \( T_0 \leq T_1 \leq T_2 \). Here the prime symbol (′) indicates that if \( \gamma = T_1 \) or \( \gamma = T_2 \) then the term \( \phi(\gamma) \) is given weight \( \frac{1}{2} \). If multiple zeros exist, then terms involving such zeros are weighted by their multiplicities. Sums of this form can be bounded using a lemma of Lehman [9, Lem. 1] that we state for reference. We have changed Lehman’s wording slightly, but the proof is the same. In the lemma and elsewhere, \( \vartheta \) denotes a real number in \([-1, 1] \), possibly different at each occurrence.

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Lemma 1 (Lehman). If $2\pi e \leq T \leq T_2$ and $\phi : [T, T_2] \mapsto [0, \infty)$ is monotone non-increasing on $[T, T_2]$, then

$$
\sum_{T \leq \gamma \leq T_2}^' \phi(\gamma) = \frac{1}{2\pi} \int_T^{T_2} \phi(t) \log(t/2\pi) \, dt + A\phi \left(2\phi(T) \log T + \int_T^{T_2} \phi(t) / t \, dt\right),
$$

where $A$ is an absolute constant.

Our Lemma 3 may be seen as a refinement of Lehman’s lemma, with the additional assumption that $\phi''(t) \geq 0$. Lemma 3 is stated and proved in §3. For simplicity, we outline here the case $T_2 \to \infty$, since this case has one fewer parameter and is of interest in many applications.

If the infinite sum $\sum_{T \leq \gamma \leq T_2}^' \phi(\gamma)$ converges, then the error term in Lemma 1 is $\gg \phi(T) \log T$. In Theorem 1, we express the error as $-\phi(T)Q(T) + E_2(T)$, where $Q(T) \ll \log T$ can be computed from (4)–(5), and $E_2(T)$ is generally of lower order than $\phi(T) \log T$. We state Theorem 1 here; the proof is given in §4. Note that the lower bound on $T$ is $2\pi$, not $2\pi e$ as in Lehman’s lemma. This is convenient in applications because $2\pi < \gamma_1 < 2\pi e$.

Theorem 1. Suppose that $2\pi \leq T_0 \leq T$ and $\int_T^{\infty} \phi(t) \log(t/2\pi) \, dt < \infty$. Let

$$
E(T) := \sum_{T \leq \gamma}^' \phi(\gamma) - \frac{1}{2\pi} \int_T^{\infty} \phi(t) \log(t/2\pi) \, dt.
$$

Then $E(T) = -\phi(T)Q(T) + E_2(T)$, where

$$
E_2(T) = -\int_T^{\infty} \phi'(t)Q(t) \, dt,
$$

and $Q(T) = N(T) - L(T)$ is defined by (4)–(5). Also,

$$
|E_2(T)| \leq 2(A_0 + A_1 \log T) |\phi'(T)| + (A_1 + A_2)\phi(T)/T.
$$

Here $A_0$ and $A_1$ are constants satisfying condition (10) below, and $A_2$ is a small constant which, from Lemma 2, we can take as $A_2 = 1/150$. We note that $E_2(T)$ is a continuous function of $T$, as can be seen from (2), whereas $E(T)$ has jumps at the ordinates of nontrivial zeros of $\zeta(s)$.

Disregarding the constant factors, Theorem 1 shows that

$$
E_2(T) \ll |\phi'(T)| \log T + \phi(T)/T.
$$

1In Lemma 1, $A$ is a constant such that $|Q(T)| \leq A \log T$ for all $T \geq 2\pi e$, where $Q(T)$ is as in [4]. From [3 Cor. 1], we may take $A = 0.28$. 2
For example, if $\phi(t) = t^{-c}$ for some $c > 1$, then $E(T) \ll T^{-c} \log T$, and $E_2(T) \ll T^{-(c+1)} \log T$ is smaller by a factor of order $T$.

As well as convergent sums, we also consider certain divergent sums. Theorem 2 shows that, if $\int_{T_0}^\infty t^{-1} \phi(t) \, dt < \infty$, then there exists

$$F(T_0) := \lim_{T \to \infty} \left( \sum_{0 < \gamma \leq T} \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^T \phi(t) \log(t/2\pi) \, dt \right).$$

In Theorem 3 we consider approximating $F(T_0)$ by computing a finite sum (over $\gamma \leq T$), with error term $E_2(T)$ the same as in Theorem 1.

For example, if $\phi(t) = 1/t$ and $T_0 = 2\pi$, we have $E(T) \ll T^{-1} \log T$ and $E_2(T) \ll T^{-2} \log T$. The latter bound allows us to obtain an accurate approximation to the constant $H = F(2\pi)$ that can equally well be defined, in analogy to Euler’s constant, by

$$H := \lim_{T \to \infty} \left( \sum_{0 < \gamma \leq T} \frac{1}{4\pi} \log^2(T/2\pi) \right).$$

This example is considered in detail in [4], where it is shown that

$$H = -0.0171594043070981495 + \vartheta(10^{18}).$$

The motivation for this paper was an attempt to generalise the results of [4].

In §2 we define some notation and mention some relevant results in the literature. We also state Lemma 2, which sharpens a result of Trudgian [13] and gives an almost best-possible explicit bound on $Q(t) - S(t)$. Lemma 3 in §3 covers finite sums. In §4–§5 we deduce Theorems 1–3 from Lemma 3. Thus, in a sense, Lemma 3 is the key result, but we have called it a lemma in deference to Lehman’s lemma.

## 2 Preliminaries

The Riemann-Siegel theta function $\theta(t)$ is defined for real $t$ by

$$\theta(t) := \arg \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) - \frac{t}{2} \log \pi,$$

see for example [6 §6.5]. The argument is defined so that $\theta(t)$ is continuous on $\mathbb{R}$, and $\theta(0) = 0$.  


Let $F$ denote the set of positive ordinates of zeros of $\zeta(s)$. Following Titchmarsh [11, §9.2–9.3], if $0 < T \not\in F$, then we let $N(T)$ denote the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$, and $S(T)$ denote the value of $\pi^{-1} \arg \zeta(\frac{1}{2} + iT)$ obtained by continuous variation along the straight lines joining $2, 2 + iT$, and $\frac{1}{2} + iT$, starting with the value 0. If $0 < T \in F$, we take $S(T) = \lim_{\delta \to 0} [S(T - \delta) + S(T + \delta)]/2$, and similarly for $N(T)$. This convention is the reason why we consider sums of the form $\sum_{T_1 \leq \gamma \leq T_2} \phi(t)$ instead of $\sum_{T_1 \leq \gamma \leq T_2} \phi(t)$.

By [11, Thm. 9.3], we have

$$N(T) = L(T) + Q(T),$$

$$L(T) = \frac{T}{2\pi} \left( \log \left( \frac{T}{2\pi} \right) - 1 \right) + \frac{7}{8},$$

$$S(T) = Q(T) + O(1/T).$$

From [11, Thm. 9.4], $S(T) \ll \log T$. Thus, from (5), $Q(T) \ll \log T$.

Trudgian [13, Cor. 1] gives the explicit bound $|Q(T) - S(T)| \leq 0.2/T$ for all $T \geq e$. In Lemma 2 we obtain a sharper constant, assuming that $T \geq 2\pi$. The result of Lemma 2 is close to optimal, since the proof shows that the constant 150 could at best be replaced by $48\pi \approx 150.8$.

**Lemma 2.** If $Q(t)$ and $S(t)$ are defined as above then, for all $t \geq 2\pi$,

$$|Q(t) - S(t)| \leq \frac{1}{150t}.$$ 

**Proof.** We shall assume that $t \not\in F$, since otherwise the result follows by continuity of $Q(t) - S(t)$. The Riemann-von Mangoldt formula states, in its most precise form,

$$N(t) = \theta(t)/\pi + 1 + S(t).$$

From [11], this implies that

$$Q(t) - S(t) = \frac{\theta(t)}{\pi} + 1 - L(t).$$

Now $\theta(t)$ has a well-known asymptotic expansion [7, Satz 4.2.3(c)]

$$\theta(t) \sim \frac{t}{2} \left( \log \left( \frac{t}{2\pi} \right) - 1 \right) - \frac{\pi}{8} + \sum_{j \geq 1} \frac{(1 - 2^{1-2j})|B_{2j}|}{4j(2j - 1)t^{2j-1}},$$

where $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \ldots$ are Bernoulli numbers. Thus, using (5), $Q(t) - S(t)$ has an asymptotic expansion

$$Q(t) - S(t) \sim \frac{1}{\pi} \sum_{j \geq 1} \frac{(1 - 2^{1-2j})|B_{2j}|}{4j(2j - 1)t^{2j-1}}.$$
In order to give an explicit bound on \( Q(t) - S(t) \), we use an explicit bound on the error incurred by taking the first \( k \) terms \( \tilde{T}_j(t) \), \( j = 1, \ldots, k \) in (7). From [2] (47), for all \( t > 0 \), this error is
\[
|\tilde{R}_{k+1}(t)| < (1 - 2^{1-2k})^{-1} (\pi k)^{1/2} \tilde{T}_k(t) + \frac{1}{2} e^{-\pi t}.
\] (9)

Substituting the expression for \( \tilde{T}_k(t) \) into (9) gives a bound
\[
|\tilde{R}_{k+1}(t)| < \frac{|B_{2k}|}{4(\pi k)^{1/2}(2k - 1) t^{2k-1}} + e^{-\pi t} \frac{1}{2\pi}
\]
for the error incurred by taking the first \( k \) terms in (8). Thus, for all \( k \geq 1 \) and \( t > 0 \),
\[
Q(t) - S(t) = \frac{1}{\pi} \sum_{j=1}^{k} \frac{(1 - 2^{1-2j})|B_{2j}|}{4j(2j - 1) t^{2j-1}} + \frac{\vartheta|B_{2k}|}{4(\pi k)^{1/2}(2k - 1) t^{2k-1}} + \frac{\vartheta e^{-\pi t}}{2\pi}.
\]

Taking \( k = 3 \) and using the assumption \( t \geq 2\pi \), we obtain the result. \( \Box \)

Define \( S_1(T) := \int_0^T S(t) \, dt \). We know that \( S_1(T) \ll \log T \), and that \( S_1(T) = o(\log T) \) if and only if the Lindelöf Hypothesis is true — see Titchmarsh [11, Thm. 9.9(A), Thm. 13.6(B), and Note 13.8].

Explicit bounds on \( S_1(T) \) are known [6, 12, 14, 15]. From [12, Thm 2.2],
\[
|S_1(T) - c_0| \leq A_0 + A_1 \log T \text{ for all } T \geq 168\pi,
\] (10)
where \( c_0 = S_1(168\pi) \), \( A_0 = 2.067 \), and \( A_1 = 0.059 \). However, a small computation shows that (10) also holds for \( T \in [2\pi, 168\pi] \). Hence, from now on we assume that \( T_0 \geq 2\pi \) and that (10) holds for \( T \geq T_0 \).

### 3 Finite sums

In this section we prove Lemma 3 which may be seen as a refinement of Lemma 1 if the conditions \( \varphi'(t) \leq 0, \varphi''(t) \geq 0 \) are satisfied. The proof of Lemma 3 is essentially the same as the proof of Lehman’s lemma up to equation (13), but then differs in the way that \( \int_{T_1}^{T_2} \varphi'(t)Q(t) \, dt \) is bounded.

From the discussion in [2], we may assume that the constants \( A_0, A_1, A_2 \) occurring in Lemma 3 are \( A_0 = 2.067, A_1 = 0.059, \) and \( A_2 = 1/150 < 0.007 \). The first two values could probably be improved significantly.
Lemma 3. If \( 2\pi \leq T_0 \leq T_1 \leq T_2 \) and

\[
E(T_1, T_2) := \sum_{T_1 \leq \gamma \leq T_2} \phi(\gamma) - \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log(t/2\pi) \, dt,
\]

then \( E(T_1, T_2) = \phi(T_2)Q(T_2) - \phi(T_1)Q(T_1) + E_2(T_1, T_2) \), where

\[
E_2(T_1, T_2) = -\int_{T_1}^{T_2} \phi'(t)Q(t) \, dt,
\]

and

\[
|E_2(T_1, T_2)| \leq 2(A_0 + A_1 \log T_1) |\phi'(T_1)| + (A_1 + A_2)\phi(T_1)/T_1.
\]

Proof. Assume initially that \( T_1 \notin \mathcal{F} \), \( T_2 \notin \mathcal{F} \). Using Stieltjes integrals, we see that

\[
\sum_{T_1 \leq \gamma \leq T_2} \phi(\gamma) = \int_{T_1}^{T_2} \phi(t) \, dN(t) = \int_{T_1}^{T_2} \phi(t) \, dL(t) + \int_{T_1}^{T_2} \phi(t) \, dQ(t)
\]

so

\[
E(T_1, T_2) = \int_{T_1}^{T_2} \phi(t) \, dQ(t) = \left[ \phi(t)Q(t) - \int \phi'(t)Q(t) \, dt \right]_{T_1}^{T_2}
\]

\[
= \phi(T_2)Q(T_2) - \phi(T_1)Q(T_1) - \int_{T_1}^{T_2} \phi'(t)Q(t) \, dt.
\]

This proves (11). To prove (12), note that, from (6) and Lemma 2

\[
\int_{T_1}^{T_2} \phi'(t)Q(t) \, dt = \int_{T_1}^{T_2} \phi'(t)S(t) \, dt + \vartheta A_2 \int_{T_1}^{T_2} \phi'(t)/t \, dt,
\]

and the last integral can be bounded using

\[
\left| \int_{T_1}^{T_2} \frac{\phi'(t)}{t} \, dt \right| \leq \frac{1}{T_1} \int_{T_1}^{T_2} |\phi'(t)| \, dt = \frac{\phi(T_1) - \phi(T_2)}{T_1} \leq \frac{\phi(T_1)}{T_1}.
\]

Also,

\[
\int_{T_1}^{T_2} \phi'(t)S(t) \, dt = \left[ \phi'(t)(S_1(t) - c_0) - \int \phi''(t)(S_1(t) - c_0) \, dt \right]_{T_1}^{T_2}
\]

\[
= \phi'(T_2)(S_1(T_2) - c_0) - \phi'(T_1)(S_1(T_1) - c_0) - \int_{T_1}^{T_2} \phi''(t)(S_1(t) - c_0) \, dt.
\]
Now, using $\phi'(t) \leq 0$ and $|S_1(t) - c_0| \leq A_0 + A_1 \log t$, we have

$$|\phi'(t)(S_1(t) - c_0)| \leq -(A_0 + A_1 \log t)\phi'(t)$$

for $t = T_1, T_2$. Thus

$$\left| \int_{T_1}^{T_2} \phi'(t)S(t) \, dt \right| \leq - \sum_{j=1}^{2} (A_0 + A_1 \log T_j)\phi'(T_j) + \left| \int_{T_1}^{T_2} \phi''(t)(S_1(t) - c_0) \, dt \right|. \quad (16)$$

Also, using $\phi''(t) \geq 0$, we have

$$\left| \int_{T_1}^{T_2} \phi''(t)(S_1(t) - c_0) \, dt \right| \leq A_0 \int_{T_1}^{T_2} \phi''(t) \, dt + A_1 \int_{T_1}^{T_2} \phi''(t) \log t \, dt$$

$$= A_0(\phi(T_2) - \phi(T_1)) + A_1 \left[ \phi(t) \log t - \int_{T_1}^{T_2} \frac{\phi'(t)}{t} \, dt \right]_{T_1}^{T_2}$$

$$= (A_0 + A_1 \log T_2)\phi'(T_2) - (A_0 + A_1 \log T_1)\phi'(T_1) - A_1 \int_{T_1}^{T_2} \frac{\phi'(t)}{t} \, dt. \quad (17)$$

Inserting (17) in (16) and simplifying, terms involving $T_2$ cancel, giving

$$\left| \int_{T_1}^{T_2} \phi'(t)S(t) \, dt \right| \leq -2(A_0 + A_1 \log T_1)\phi'(T_1) - A_1 \int_{T_1}^{T_2} \frac{\phi'(t)}{t} \, dt. \quad (18)$$

Combining (11) with (14), (15), and (18), gives (12). Finally, we note that (11)–(12) hold even if $T_1 \in F$ and/or $T_2 \in F$, because of the way that we defined $N(T)$ (and hence $Q(T) = N(T) - L(T)$) for $T \in F$.

Remark 1. With the assumptions and notation of Lemma 3, Lemma 1 gives the bound

$$|E(T_1, T_2)| \leq A \left( 2 \phi(T_1) \log T_1 + \int_{T_1}^{T_2} \frac{\phi(t)}{t} \, dt \right). \quad (19)$$

Our bound (12) on $E_2(T_1, T_2)$ is often smaller than the bound (19) on $E(T_1, T_2)$. We can take advantage of this if the terms $\phi(T_j)Q(T_j)$ ($j = 1, 2$) are known. Examples are given in §§4–5.
4 Convergent sums

In this section we assume that \( \sum_{T \leq \gamma} \phi(\gamma) < \infty \), or equivalently (given our conditions on \( \phi \)), that \( \int_{T}^{\infty} \phi(t) \log(t/2\pi) dt < \infty \). We first state an easy lemma, and then prove Theorem 1.

**Lemma 4.** Suppose that \( 2\pi \leq T_0 \leq T \) and \( \int_{T}^{\infty} \phi(t) \log(t/2\pi) dt < \infty \). Then

\[
\begin{align*}
\phi(t) \log t &= o(1) \quad \text{as } t \to \infty, \\
\phi'(t) \log t &= o(1) \quad \text{as } t \to \infty, \quad \text{and} \\
\int_{T}^{\infty} |\phi'(t)| \log t dt &< \infty.
\end{align*}
\]

**Proof.** For \( u \geq T \),

\[
\int_{u}^{u+1} \phi(t) \log(t/2\pi) dt \geq \phi(u+1) \log(u/2\pi).
\]

Thus \( \phi(u+1) \log(u/2\pi) = o(1) \) as \( u \to \infty \), and \( \phi(t) \log((t-1)/2\pi) = o(1) \). Since \( \log((t-1)/2\pi) \sim \log t \), (20) follows.

For (21), we have

\[
\begin{align*}
\phi(u) &\geq \phi(u) - \phi(u+1) = \int_{u}^{u+1} |\phi'(t)| dt \geq |\phi'(u+1)|,
\end{align*}
\]

so (20) implies that \( \phi'(u+1) \log u = o(1) \). Taking \( t = u+1 \), we have \( \phi'(t) \log(t-1) = o(1) \). Since \( \log(t-1) \sim \log t \), (21) follows.

Finally, from (23), we have

\[
\int_{T+1}^{\infty} |\phi'(t)| \log t dt \leq \int_{T+1}^{\infty} (t-1) \log t dt \leq \int_{T}^{\infty} \phi(t) \log(t/2\pi) dt < \infty.
\]

and (22) follows.

**Proof of Theorem 1.** We have \( \phi(t) \log t = o(1) \) by Lemma 4 and convergence of the integral in (1). Also, from Lemma 4 we have \( \int_{T}^{\infty} |\phi'(t)| \log t dt < \infty \), but \( Q(t) \ll \log t \), so \( \int_{T}^{\infty} \phi'(t)Q(t) dt \) converges absolutely. Now, Lemma \( \boxed{3} \) gives

\[
\sum'_{T \leq \gamma \leq T_2} \phi(\gamma) - \frac{1}{2\pi} \int_{T}^{T_2} \phi(t) \log(t/2\pi) dt
\]

\[
= \phi(T_2)Q(T_2) - \phi(T)Q(T) - \int_{T}^{T_2} \phi'(t)Q(t) dt. \tag{24}
\]
If we let $T_2 \to \infty$ in (24), $\phi(T_2)Q(T_2) \to 0$ and $\int_T^{T_2} \phi'(t)Q(t)\,dt$ tends to a finite limit. Thus, the right side of (24) tends to a finite limit, and the left side must tend to the same limit. This gives

$$\sum_{T \leq \gamma} \phi'(\gamma) - \frac{1}{2\pi} \int_T^{\infty} \phi(t) \log(t/2\pi)\,dt = -\phi(T)Q(T) - \int_T^{\infty} \phi'(t)Q(t)\,dt.$$  

We have proved (1)–(2) of Theorem 1. The bound (3) follows by observing that the bound (12) of Lemma 3 is independent of $T_2$, so

$$\left| \int_T^{\infty} \phi'(t)Q(t)\,dt \right| \leq 2(A_0 + A_1 \log T) |\phi'(T)| + (A_1 + A_2)\phi(T)/T.$$  

This completes the proof of Theorem 1.

**Example 1.** We consider computation of the constant

$$c_1 := \sum_{\gamma > 0} \frac{1}{\gamma^2} = 0.02310499\ldots.$$  

The approximation 0.023105 was given in [10, Lemma 2.9], where it was computed using a finite sum with (essentially) Lemma 1 to bound the tail.

Taking $\phi(t) = 1/t^2$ in Lemma 1 gives an error term

$$|E(T)| \leq A \left( A + 2 \log T \right) \frac{1}{T^2} = 0.14 + 0.56 \log T,$$

using the value $A = 0.28$ mentioned above. The corresponding error term given by Theorem 1 is

$$|E_2(T)| \leq \left( 4A_0 + A_1 + A_2 \right) \frac{4A_1 \log T}{T^3} \leq 8.334 + 0.236 \log T,$$

using the values of $A_0, A_1, A_2$ above. For example, taking $T = 1000$ (corresponding to the first 649 nontrivial zeros), we get $|E(T)| \leq 4.009 \times 10^{-6}$ and $|E_2(T)| \leq 9.965 \times 10^{-9}$, an improvement by a factor of 400. If we use $10^{10}$ zeros, as in Corollary 1, the improvement is by a factor of $3 \times 10^9$.

**Corollary 1.** We have

$$c_1 = \sum_{\gamma > 0} \frac{1}{\gamma^2} = 0.0231049931154189707889338104 + \vartheta(5 \times 10^{-28}).$$

9
Proof. This follows from Theorem 1 by an interval-arithmetic computation using the first \( n = 10^{10} \) zeros, with \( T = 3293531632.542 \cdots \in (\gamma_n, \gamma_{n+1}) \).

Remark 2. Assuming the Riemann Hypothesis (RH), there is an equivalent expression:

\[
c_1 = \frac{d^2 \log \zeta(s)/ds^2|_{s=1/2}}{2} + \frac{\pi^2}{8} + G - 4,
\]

where \( G = \beta(2) \) is Catalan’s constant 0.915965\( \cdots \). This enables us to confirm Corollary 1 without summing over any zeros of \( \zeta(s) \), but assuming RH. It is only rarely that such a closed form is known. One other example is the following — see, e.g., [5, Ch. 12]. Assuming RH, we have

\[
\sum_{\gamma > 0} \frac{1}{\gamma^2 + \frac{1}{4}} = \sum_{\rho} \Re \left( \frac{1}{\rho} \right) = 1 + \frac{C}{2} - \frac{\log 4\pi}{2} = 0.0230957 \cdots ,
\]

where \( C = 0.5772\ldots \) is Euler’s constant.

5 Divergent sums

In this section we give two theorems that apply, subject to a mild condition on \( \phi(t) \), even if \( \sum_{T \leq \gamma} \phi(\gamma) \) diverges. Theorem 2 shows the existence of a limit for the difference between a sum and the corresponding integral. Theorem 3 shows how we can accurately approximate the limit.

First we prove two lemmas that strengthen the first and third parts of Lemma 4. In Lemma 5, \( f \) is non-increasing but need not be differentiable.

Lemma 5. Suppose that, for some \( T \geq 1 \), \( f : [T, \infty] \mapsto [0, \infty) \) is non-negative and non-increasing on \( [T, \infty) \). If

\[
\int_T^\infty \frac{f(t)}{t} dt < \infty,
\]

then \( f(t) \log t = o(1) \).

Proof. Assume, by way of contradiction, that \( f(t) \log t \neq o(1) \). Thus, there exists a constant \( c > 0 \) and an unbounded increasing sequence \((t_n)_{n \geq 1}\) such that \( t_1 > T \) and

\[
f_n := f(t_n) \geq \frac{c}{\log t_n}.
\]

\footnote{The formula (25) is stated in [8 (21)] and is proved in [1, p. 13]. An almost indecipherable sketch of this result may be found in Riemann’s Nachlass.}
Moreover, by taking a subsequence of \((t_n)_{n \geq 1}\) if necessary, we can assume that \(t_{n+1} \geq t_n^2\) for all \(n \geq 1\). Thus
\[
\log \left( \frac{t_{n+1}}{t_n} \right) \geq \frac{\log t_{n+1}}{2}. \tag{28}
\]
Since \(f(t)\) is non-increasing, we have \(f(t) \geq f_{n+1}\) on \([t_n, t_{n+1}]\), and
\[
\int_{t_n}^{t_{n+1}} \frac{f(t)}{t} \, dt \geq \int_{t_n}^{t_{n+1}} \frac{f_{n+1}}{t} \, dt = f_{n+1} \log \left( \frac{t_{n+1}}{t_n} \right). \tag{27}
\]
Using \((27) - (28)\), this gives
\[
\int_{t_1}^{t_{n+1}} \frac{f(t)}{t} \, dt \geq \frac{1}{2} \sum_{k=1}^{n} f_{k+1} \log t_{k+1} \geq \frac{c}{2} \sum_{k=1}^{n} 1 = \frac{cn}{2} \to \infty.
\]
This contradicts the condition \((26)\). Thus, our assumption is false, and we must have \(f(t) \log t = o(1)\). \(\square\)

**Lemma 6.** If \(\int_{T_0}^{\infty} \frac{\phi(t)}{t} \, dt < \infty\), then \(\int_{T_0}^{\infty} \phi'(t) \log t \, dt\) is absolutely convergent.

**Proof.** For \(T \geq T_0\) we have
\[
\int_{T_0}^{T} \phi'(t) \log t \, dt = \phi(T) \log T - \phi(T_0) \log T_0 - \int_{T_0}^{T} \frac{\phi(t)}{t} \, dt. \tag{29}
\]
As \(T \to \infty\) in \((29)\), the term \(\phi(T) \log T \to 0\) by Lemma \(5\) and the integral on the right-hand side tends to a finite limit. Thus, the integral on the left-hand side tends to a finite limit. Since \(\phi'(t) \log t \leq 0\) has constant sign on \([T_0, \infty)\), the integral is absolutely convergent. \(\square\)

**Theorem 2.** Suppose that \(T_0 \geq 2\pi\), and
\[
\int_{T_0}^{\infty} \frac{\phi(t)}{t} \, dt < \infty. \tag{30}
\]
Then there exists
\[
F(T_0) := \lim_{T \to \infty} \left( \sum_{T_0 \leq \gamma \leq T} \phi'(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T} \phi(t) \log(t/2\pi) \, dt \right),
\]
and
\[
F(T_0) = -\phi(T_0)Q(T_0) - \int_{T_0}^{\infty} \phi'(t)Q(t) \, dt. \tag{31}
\]
Proof. Suppose that $T \geq T_0$. Applying Lemma 3, we have

$$\sum_{T_0 \leq \gamma \leq T}' \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T} \phi(t) \log(t/2\pi) \, dt$$

$$= \phi(T)Q(T) - \phi(T_0)Q(T_0) - \int_{T_0}^{T} \phi'(t)Q(t) \, dt. \quad (32)$$

Let $T \to \infty$ in (32). On the right-hand side, $\phi(T)Q(T) \to 0$ by Lemma 5 and the integral tends to a finite limit by Lemma 6, using $Q(t) \ll \log t$. Thus the left-hand side tends to a finite limit $F(T_0)$. This gives (31). \qed

The identity (31) is not convenient for accurately approximating $F(T_0)$ when $T_0$ is small, because $\int_{T_0}^{\infty} \phi'(t)Q(t) \, dt$ is not necessarily small. In Theorem 3 we use a finite sum (over $\gamma \leq T$) and integral to approximate $F(T_0)$. Theorem 3 has the same expression for the error term $E_2$ as Theorem 1 essentially because the bounds in both theorems are proved using Lemma 3.

**Theorem 3.** Suppose that $2\pi \leq T_0 \leq T_1$ and $\phi(t)$ satisfies (30). Let

$$F(T_0) := \lim_{T_0 \to \infty} \left( \sum_{T_0 \leq \gamma \leq T}' \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T} \phi(t) \log(t/2\pi) \, dt \right).$$

Then

$$F(T_0) = \sum_{T_0 \leq \gamma \leq T_1}' \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T_1} \phi(t) \log(t/2\pi) \, dt - \phi(T_1)Q(T_1) + E_2(T_1),$$

where $E_2(T_1) = -\int_{T_1}^{\infty} \phi'(t)Q(t) \, dt$, and

$$|E_2(T_1)| \leq 2(A_0 + A_1 \log T_1) |\phi'(T_1)| + (A_1 + A_2)\phi(T_1)/T_1.$$

Proof. We note that, from Theorem 2, the limit defining $F(T_0)$ exists. Also, from Lemmas 3, 6, $\phi(T)Q(T) = o(1)$ and $\int_{T_0}^{\infty} \phi'(t)Q(t) \, dt < \infty$. Thus, using Lemma 3 as in the proof of Theorem 1, we see that

$$\lim_{T_0 \to \infty} \left( \sum_{T_0 \leq \gamma \leq T_2}' \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T_2} \phi(t) \log(t/2\pi) \, dt \right)$$

$$= -\phi(T_1)Q(T_1) - \int_{T_1}^{\infty} \phi'(t)Q(t) \, dt$$
and \( \left| \int_{T_1}^{\infty} \phi'(t) Q(t) \, dt \right| \leq 2(A_0 + A_1 \log T_1) |\phi'(T_1)| + (A_1 + A_2) \phi(T_1)/T_1. \) 

Since

\[
F(T_0) = \lim_{T_2 \to \infty} \left( \sum_{T_1 \leq \gamma \leq T_2} \phi(\gamma) - \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log(t/2\pi) \, dt \right) + \sum_{T_0 \leq \gamma \leq T_1} \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T_1} \phi(t) \log(t/2\pi) \, dt, \]

the result follows. \( \square \)

**Example 2.** To illustrate the divergent case, we consider the example \( \phi(t) = 1/(\log(t/2\pi))^2 \). The constant \( 2\pi \) here is unimportant, but this choice simplifies some of the expressions below.

From Lemma 1, the asymptotic behaviour of \( \sum_{0<\gamma \leq T} \phi(\gamma) \) is given by

\[
\frac{1}{2\pi} \int_{c}^{T} \phi(t) \log(t/2\pi) \, dt = \text{li}(T/2\pi) - \text{li}(c/2\pi) \sim \frac{T}{2\pi \log T},
\]

where \( c \geq 2\pi e \) is an arbitrary constant, and \( \text{li}(x) \) is the logarithmic integral, defined in the usual way by a principal value integral. This motivates the definition of a constant \( c_2 \) by

\[
c_2 := \lim_{T \to \infty} \left( \sum_{0<\gamma \leq T} \phi(\gamma) - \text{li}(T/2\pi) \right), \tag{33}
\]

where the limit exists by Theorem 2.

If we use (33) to estimate \( c_2 \) then, by Theorem 3, the error is

\[
E(T) = -\phi(T)Q(T) + O(|\phi'(T)| \log T) + O(\phi(T)/T) \ll \frac{1}{\log T}. \]

Convergence is so slow that it is difficult to obtain more than two correct decimal digits. On the other hand, if we estimate \( c_2 \) using the approximation

\[
\sum_{0<\gamma \leq T} \phi(\gamma) - \text{li}(T/2\pi) - \phi(T)Q(T) \tag{34}
\]

suggested by Theorem 3, then the error is \( E_2(T) \ll (T \log^2 T)^{-1} \), smaller by a factor of order \( T \log T \). More precisely, from Theorem 3 we have

\[
|E_2(T)| \leq \frac{4(A_0 + A_1 \log T)}{T \log^3(T/2\pi)} + \frac{A_1 + A_2}{T \log^2(T/2\pi)} \leq \frac{0.302 \log(T/2\pi) + 8.702}{T \log^3(T/2\pi)}. \tag{35}
\]

13
Corollary 2. If $c_2$ is defined by \( (33) \), then

$$c_2 = -0.5276697875 + \frac{1}{6} \cdot 10^{-10}.$$  

Proof. Using the first $n = 10^9$ nontrivial zeros with $T \approx (\gamma_n + \gamma_{n+1})/2$ in \( (34) \), and the error bound \( (35) \), an interval-arithmetic computation gives the result.

To illustrate the speed of convergence, in Table 1 we give the estimates of $c_2$ obtained from \( (33) \) and \( (34) \) by summing over the first $n$ nontrivial zeros, and the error bound \( (35) \), with $T = (\gamma_n + \gamma_{n+1})/2$. The first incorrect digit in each entry is underlined.

| $n$  | estimate via \( (33) \) | estimate via \( (34) \) | $|E_2|$ bound \( (35) \) |
|------|------------------------|------------------------|------------------------|
| 10   | -0.499862599           | -0.527339083           | 1.96 $\times$ 10^{-2}  |
| $10^2$ | -0.540547244           | -0.527672383           | 8.64 $\times$ 10^{-4}  |
| $10^3$ | -0.522449744           | -0.527671734           | 4.58 $\times$ 10^{-5}  |
| $10^4$ | -0.531178464           | -0.527669804           | 2.78 $\times$ 10^{-6}  |
| $10^5$ | -0.530262600           | -0.527669777           | 1.87 $\times$ 10^{-7}  |

Table 1: Numerical estimation of $c_2$.

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