Flavor Twisted Boundary Conditions in the Breit Frame

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We use a generalization of chiral perturbation theory to account for the effects of flavor twisted boundary conditions in the Breit frame. The relevant framework for two light flavors is an SU(6)$^4$ partially quenched theory, where the extra valence quarks differ only by their boundary conditions. Focusing on the pion electromagnetic form factor, finite volume corrections are calculated at next-to-leading order in the chiral expansion and are estimated to be small on current lattices.

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Twisted Boundary Conditions and the Breit Frame. Simulations of QCD on Euclidean spacetime lattices are making progress towards quantitative understanding of strong interactions [1]. A source of systematic error in these calculations is the finite lattice volume. While observables generally depend upon the lattice volume, there is a potentially more serious effect: periodic boundary conditions on quark fields limit the available momentum modes to integer multiples of $2\pi/L$, where $L$ is the size of the lattice. This presents difficulty for the study of hadronic properties at small momentum. Periodic boundary conditions, however, are often chosen for convenience; and, a more general class of boundary conditions, twisted boundary conditions (TwBCs), see e.g. [2], are possible. A TwBC by an arbitrary twist angle $\theta$ on the matter field $\psi$, has the form: $\psi(x + L) = e^{i\theta}\psi(x)$, and consequently the matter field has kinematic momentum $p = (2\pi n + \theta)/L$, where $n$ is an integer. The ability to produce continuous hadronic momentum has motivated the study of TwBCs in lattice QCD [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

Producing continuous hadronic momenta via TwBCs does not necessarily give one the ability to probe amplitudes at continuous momentum transfer. In fact, these techniques are of no avail for flavor singlet form factors which require operator self-contractions. In a current self-contraction, the boundary condition chosen for the quark in the current does not affect the momenta of the external states. Flavor non-singlet form factors, however, can be accessed at continuous momentum transfer. A conceptually clear way to see this is to consider form factors of flavor changing currents [8]. If the current induces a change from a quark flavor satisfying periodic boundary conditions to one satisfying TwBCs, continuous momentum transfer, $\theta/L$, will be induced. There is another way to utilize TwBCs for flavor non-singlet currents [14]. For this method, imagine that the current strikes a quark of a given twist angle $\theta$, and then produces a different twist angle $\theta'$. Here the momentum transfer $(\theta' - \theta)/L$ is induced. In the infinite volume limit, these boundary conditions become irrelevant and the current thus produces momentum transfer by striking a single (non-self-contracted) quark. Furthermore one can choose $\theta' = -\theta$ to work in the Breit frame, which often simplifies the calculation of form factors. At finite volume, however, the initial- and final-state quarks are actually distinct because a given field can only have one boundary condition. Hence the lattice action must be described by a theory with an enlarged valence flavor group [14]. The purpose of this note is to utilize the effective field theory for the Breit frame implementation of TwBCs, discussed in [14], to compute volume corrections to observables, in particular the pion electromagnetic form factor.

Partially Quenched Chiral Perturbation Theory. To address the long-distance effects of TwBCs in the Breit frame, we must formulate the low-energy effective theory. From the discussion in [14], this variant of chiral perturbation theory includes additional fictitious valence quarks. These fictitious quarks only differ by their boundary conditions. We can remove the twists by a field redefinition in favor of periodic quark fields. In terms of these fields, the quark part of the partially twisted, partially quenched QCD Lagrangian appears as

$$\mathcal{L} = \overline{Q}(\not{D} + m_Q)Q,$$

where the vector $Q$ accommodates ten quark fields, $Q = (u_1, u_2, d_1, d_2, j, l, \bar{u}_1, \bar{u}_2, \bar{d}_1, \bar{d}_2)^T$. The field $Q$ transforms under the fundamental representation of the graded group SU(6)$^4$. The quark mass matrix in the isospin limit$^1$ of the valence and sea sectors is given by $m_Q = \text{diag}(m_u, m_u, m_u, m_j, m_j, m_u, m_u, m_u)$. Because we implement twisted boundary conditions only in the valence sector, we have additionally chosen to work away from the unitary point, where $m_j = m_u$. With this choice, we can track sea-quark effects with ease and account for partial quenching errors. Formulae relevant at the unitary point can easily be recovered by taking the limit $m_j \rightarrow m_u$.

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$^1$ Although we work with $m_d = m_u$, we have kept the flavor group to be SU(6)$^4$ [as opposed to SU(5)$^3$] to allow for strong isospin breaking, $m_d \neq m_u$. 
The $Q$ field is periodic and the effects of partially TWCBCs have been shuffled into the covariant derivative $\tilde{D}_\mu$, where they have the form of a uniform $U(1)$ gauge potential $B_\mu$: $\tilde{D}_\mu = D_\mu + iB_\mu$. The quark flavors are charged differently under this $U(1)$ potential, specifically the flavor matrix is given by $B_\mu = \text{diag}(B_{Ru}^u, -B_{Rd}^d, B_{Ru}^d, B_{Rd}^u, 0, 0, B_{Bd}^d, -B_{Bd}^u, 0, 0)$. Because the $U(1)$ field is uniform, the $B_\mu$ act as flavor dependent field momenta, each having the form $B_\mu^B = (\theta^B/L, 0)$, where $L$ is the spatial length of the lattice and $q$ is a flavor index. Notice there is no fourth component to $B_\mu$ because the boundary conditions are only spatially twisted. The twist angles $\theta$ can be implemented on the lattice by modifying the links, $U_j(x) \rightarrow U_j(x)e^{i\theta^B/L}$, where $j$ is a spatial index. Notice that the twist angles vanish for the sea-sector, this corresponds to the partially twisted scenario where existing gauge configurations can be post-multiplied by the uniform $U(1)$ gauge potential. We additionally must choose the ghost quarks to be degenerate with their valence counterparts, and further to satisfy the same boundary conditions as their valence counterparts. The determinant thus arises only from the sea quarks. The graded symmetry of $SU(6|4)$ hence provides a way to write down a theory corresponding to partially twisted, partially quenched QCD, see [17] for a more rigorous discussion.

The low-energy effective theory of partially quenched QCD is partially quenched chiral perturbation theory (PQChPT), which describes the dynamics of pseudo-Goldstone mesons arising from spontaneous chiral symmetry breaking. In finite volume, we restrict our analysis to the $p$-regime [18] in which the long-range fluctuations of the Goldstone modes cannot conspire to restore chiral symmetry. The effective theory is written in terms of the periodic coset field $\Sigma = \exp(2i\phi/f)$. Here the meson fields $\Phi$ are embedded non-linearly, and in our conventions the parameter $f$ is the pion decay constant, $f \sim 130\text{MeV}$.

To obtain the relevant generalization of PQChPT for Eq. 10, we take $\Sigma$ to transform as $\Sigma \rightarrow L\Sigma R^\dagger$ under a chiral transformation $(L, R) \in U(6|4)_L \otimes U(6|4)_R$ and write down the most general chirally invariant Lagrangian. Furthermore, we include the uniform gauge potential by requiring the theory be invariant under local $U(1)_V$ phase rotations, see [6]. In terms of $\Sigma$, the Lagrangian of PQChPT is

$$L = \frac{f^2}{8} \text{str} \left( \tilde{D}_\mu \Sigma \tilde{D}_\mu \Sigma^\dagger \right) - \lambda \text{str} \left( m_Q^4 \Sigma + \Sigma^\dagger m_Q \right) + \mu_0^2 \Phi_0^2,$$

where we have written down only the leading order terms in an expansion in $m_Q$ and $p^2$, where $p$ is a momentum. Here $\Phi_0$ is the flavor singlet field $\Phi_0 = \text{str}(\Sigma)/\sqrt{2}$, and the action of the covariant derivative $\tilde{D}_\mu$ is specified by

$$\tilde{D}_\mu \Sigma = \partial_\mu \Sigma + i[B_\mu, \Sigma] + iA_\mu^\dagger [\Sigma, \Sigma].$$

We have included an isovector source field $A_\mu^0$ which will be utilized below. The meson modes are contained in $\Phi$, which is a ten-by-ten matrix of fields. It has the form

$$\Phi = \begin{pmatrix} M_{vv} & M_{us} & \chi_{gv}^+ \\ M_{uv} & M_{ss} & \chi_{gs}^+ \\ \chi_{gv} & \chi_{gs} & M_{gg} \end{pmatrix}.$$  

The mesons of $M_{vv}$ ($M_{gg}$) are bosonic and are formed from a valence (ghost) quark-antiquark pair. These matrices have the form

$$M_{vv} = \begin{pmatrix} \eta_{11}^u & \eta_{12}^u & \pi_{11}^+ & \pi_{12}^+ \\ \eta_{21}^u & \eta_{22}^u & \pi_{11}^d & \pi_{12}^d \\ \pi_{11}^+ & \pi_{12}^d & \eta_{11}^d & \eta_{12}^d \\ \pi_{21}^+ & \pi_{22}^d & \eta_{21}^d & \eta_{22}^d \end{pmatrix}, \text{ and } M_{gg} = \begin{pmatrix} \tilde{\eta}_{11}^u & \tilde{\eta}_{12}^u & \tilde{\pi}_{11}^+ & \tilde{\pi}_{12}^+ \\ \tilde{\eta}_{21}^u & \tilde{\eta}_{22}^u & \tilde{\pi}_{11}^d & \tilde{\pi}_{12}^d \\ \tilde{\pi}_{11}^+ & \tilde{\pi}_{12}^d & \tilde{\eta}_{11}^d & \tilde{\eta}_{12}^d \\ \tilde{\pi}_{21}^+ & \tilde{\pi}_{22}^d & \tilde{\eta}_{21}^d & \tilde{\eta}_{22}^d \end{pmatrix}.$$  

The $\eta_{ij}^u$ ($\tilde{\eta}_{ij}^u$) mesons have quark content $\eta_{ij}^u \sim q_i\bar{q}_j$ ($\tilde{\eta}_{ij}^u \sim \tilde{q}_i\bar{q}_j$), while the $\pi_{ij}^+$ ($\tilde{\pi}_{ij}^+$) mesons have quark content $\pi_{ij}^+ \sim u_i\bar{d}_j$ ($\tilde{\pi}_{ij}^+ \sim \tilde{u}_i\bar{d}_j$). The valence-sea (sea-sea) mesons are bosonic and contained in $M_{uv}$ ($M_{ss}$) as

$$M_{sv} = \begin{pmatrix} \phi_{j_1u_1} & \phi_{j_2u_2} & \phi_{j_3d_1} & \phi_{j_4d_2} \\ \phi_{u_1v_1} & \phi_{u_2v_2} & \phi_{d_1v_1} & \phi_{d_2v_2} \\ \phi_{d_1u_1} & \phi_{d_2u_2} & \phi_{d_1d_1} & \phi_{d_2d_2} \\ \phi_{d_1u_1} & \phi_{d_2u_2} & \phi_{d_1d_1} & \phi_{d_2d_2} \end{pmatrix}, \text{ and } M_{ss} = \begin{pmatrix} \eta_{j_1} & \pi_{j_1} \\ \eta_{i_1} & \eta_{i_1} \end{pmatrix}.$$  

Mesons contained in $\chi_{gv}$ ($\chi_{gs}$) are built from ghost quark, valence antiquark (sea antiquark) pairs and are thus fermionic. These states appear as

$$\chi_{gv} = \begin{pmatrix} \phi_{u_1i_1} & \phi_{u_2i_1} & \phi_{u_3i_2} & \phi_{u_4i_2} \\ \phi_{u_1v_1} & \phi_{u_2v_2} & \phi_{u_3v_1} & \phi_{u_4v_2} \\ \phi_{d_1v_1} & \phi_{d_2v_2} & \phi_{d_1d_1} & \phi_{d_2d_2} \\ \phi_{d_1u_1} & \phi_{d_2u_2} & \phi_{d_1d_1} & \phi_{d_2d_2} \end{pmatrix}, \text{ and } \chi_{gs} = \begin{pmatrix} \phi_{i_1j_1} & \phi_{i_1l_1} \\ \phi_{i_2j_2} & \phi_{i_2l_2} \\ \phi_{d_1j_1} & \phi_{d_1l_1} \\ \phi_{d_2j_2} & \phi_{d_2l_2} \end{pmatrix}.$$
In writing the PQCD Lagrangian we have kept the flavor singlet field $\Phi_0$ as a device. Expanding $\mathcal{L}$ to tree level, one finds that mesons with quark content $\sim Q\bar{Q}$ have masses: $m_{QQ'}^2 = \frac{\Lambda^2}{16\pi^2}(m_Q + m_{Q'})$. The flavor singlet additionally acquires a mass $\mu_0^2$. Taking the limit $\mu_0 \to \infty$ (which is warranted by the strong axial anomaly), we integrate out the flavor singlet component. The resulting Goldstone manifold is $SU(6|4)$, but only the propagators of the flavored mesons have simple forms. The neutral meson propagators have both flavor connected and disconnected (hairpin) terms. The flavor neutral propagator was derived in general for the $SU(M + N|N)$ group in [17] and is identical for our case because the flavor neutral states are unaffected by twisting. We will not need the explicit form of this propagator in order to display our final results.

Further calculation at tree level shows that mesons with quark content $Q\bar{Q}$ have kinematic momentum $P_{QQ'}$ given by $P_{QQ'} = p + B_{Q} - B_{Q'}$. At finite volume, mass splittings are generated, and kinematic momenta are renormalized by infrared effects [13]. These effects were estimated to be on the order of a few percent on current lattices. As we shall calculate finite volume corrections to the pion form factor, which themselves are at the percent level, we can safely ignore mass splittings and momentum renormalization below.

**Pion Form Factor in Finite Volume.** To calculate the pion form factor on the lattice using TwBCs in the Breit frame, two separate current insertions are used. In the first (second) insertion, the current strikes the up (down) quark. Matching to this lattice setup, in our theory we have

$$\langle \pi_{21}^0(0)|J^{\mu}_{\mu}||\pi_{11}^0(0)\rangle_{\theta^+ = 0} + \langle \pi_{22}^+(0)|J^{\mu}_{\mu}||\pi_{11}^+(0)\rangle_{\theta^+ = 0} \to -\langle \pi^+(0)|J^{\mu}_{\mu}||\pi^+(0)\rangle,$$

with $p = \theta/L$ fixed. The currents $J^{1,2}_{\mu}$ correspond to the photon hitting the $u$ and $d$ quarks, respectively:

$$J^{1}_{\mu} = q_u \bar{u}_2 \gamma_{\mu} u_1$$

$$J^{2}_{\mu} = q_d \bar{d}_1 \gamma_{\mu} d_2.$$

Here $q_u$, $q_d$ are the light quark electric charges. The current insertion effectively injects momentum by changing flavors, from $u_1$ to $u_2$ in the case of $J^{1}_{\mu}$, for example. These quark-level operators must of course be matched onto the effective theory.

At leading order the matching has already been performed in Eq. (2) as we have minimally coupled the source field $A^\mu_{\mu}$. As explained in [8], the generators $\mathcal{T}^a$ can be chosen purely in the valence sector. The simplest choice is the obvious one: $(\mathcal{T}^a)_{ij} = \delta_{i3}\delta_{j1}$ corresponding to the current $J^1_{\mu}$, and $(\mathcal{T}^a)_{ij} = \delta_{i3}\delta_{j4}$ corresponding to $J^2_{\mu}$. At next-to-leading order there is only one additional term needed in the effective theory.\(^3\) This local correction to the current has the form:

$$\delta J^a_{\mu} = i\alpha_0 \tilde{D}_\nu \text{str} \left[ \mathcal{T}^a \left( [\tilde{D}_\mu \Sigma^\dagger, \tilde{D}_\nu \Sigma] - [\tilde{D}_\nu \Sigma^\dagger, \tilde{D}_\mu \Sigma] \right) \right].$$

This is just a rewriting of the current derived from the relevant term of the Gasser-Leutwyler Lagrangian [19]. As $\delta J^a_{\mu}$ will only contribute to tree level, it will not appear in the expression for the finite volume correction. It is instructive to note that this current depends only on the flavor non-singlet part of the generators. Any local terms in the effective theory arising from flavor singlet terms will be inaccessible using TwBCs, as these must appear $\alpha \text{str}(\mathcal{T}^a) = 0$.

Having written down the matching of the current in the effective theory up to next-to-leading order, we can now deduce the pion form factor at finite volume. As mentioned above, we work in the $p$-regime [18] with time treated as infinite in extent. Furthermore, we choose to calculate the form factor from the time-component of the current as is done in the actual lattice calculations, e.g. 12, 16. The loop diagrams contributing to the form factor are depicted in [13]. The hairpin contributions exactly cancel, which is not surprising given that flavor neutral mesons are electrically neutral. As a check on our calculation, we have verified that the infinite volume form factor agrees with the known $SU(4|2)$ partially quenched expression [20, 21]. Furthermore when $m_1 = m_2$, the partially quenched expression turns into the well-known infinite volume $SU(2)$ result [22]. The finite volume modification $\delta M(L) \equiv M(L) - M(L = \infty)$ to the time-component of the current matrix element is given by

$$\delta M(L) = 2E_\pi(\theta) \frac{1}{L^2} \int_0^1 dx \left[ -I_{1/2} \left( \frac{\theta}{L}, m_{2u}^2 \right) + I_{1/2} \left( 1 - 2x, \frac{\theta}{L}, m_{2u}^2 + 4x(1-x) \theta^2/L^2 \right) \right],$$

where we have set the overall charge $q_u - q_d = 1$. The pion energy is $E_\pi(\theta) = \sqrt{m_{2u}^2 + \theta^2/L^2}$, and the finite volume modification is encoded in the function $I_{1/2}(\nu, m^2)$ [13]. There is no isospin breaking term in the above expression.

\(^2\) There are three other combinations of matrix elements that lead to the same infinite volume physics. This is due to the redundancy of the spectator quarks, which are evaluated at zero twist angle. We have verified that each of the four matrix elements calculated in the effective theory yield the same answer at infinite and finite volume.

\(^3\) There are additional couplings that are exactly cancelled by wavefunction renormalization. Not surprisingly, these can be removed from the current by a field redefinition.
metric. This is in contrast to the result derived employing isovector symmetry, as well as a discrete rotational symmetry. The result we derive is quite compact, maintains lattice calculations of the pion electromagnetic form factor, and is supported in part by the U.S. Dept. of Energy, Grant No. DE-FG02-93ER-40762 (B.C.T.), and by the Schweizerischer Nationalfonds (F.-J.J.).

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FIG. 1: Finite volume shift of the pion form factor in the Breit frame. Plotted vs. $\theta$ is the relative change $\delta_L[G_\pi(Q^2) - 1]$, where $Q^2 = \theta^2 \times 0.025 \text{GeV}^2$.

as it would have to be proportional to the difference of the initial and final state energies. Additionally there is a symmetry under $\theta \rightarrow -\theta$. Both of these desirable features are characteristics of the Breit frame kinematics.

Numerically we can estimate the finite volume effect. For ease we consider the unitary point, $m^2_{\mu \nu} = m^2_\pi$, and choose $\theta$ to be non-vanishing in a single lattice direction. We consider the relative difference

$$\delta_L[G_\pi(Q^2) - 1] = \frac{\delta M(L)/2E_\pi(\theta)}{G_\pi(Q^2) - 1}, \quad (10)$$

where $Q^2 = 4\theta^2/L^2$ is the momentum transfer squared, and the form factor $G_\pi(Q^2)$ has the usual definition: $\langle \pi(-\theta)|J_4|\pi(\theta)\rangle = 2E_\pi(\theta)G_\pi(Q^2)$. In Fig. 1 we hold the lattice size $L$ fixed at 2.5 fm, and plot the relative difference $\delta_L[G_\pi(Q^2) - 1]$ as a function of $\theta$ for a few values of the pion mass. On current lattices, the volume corrections are at the percent level and become non-negligible only for light pions with small twists.

Conclusion. Above we investigate an extension of PQ$\chi$PT which is relevant for lattice calculations using TwBCs in the Breit frame. For hadrons consisting of two light quark flavors, the appropriate theory has an enlarged $SU(6/4)$ flavor group. The additional valence quarks are fictitious flavors differing only in their boundary conditions. This theory is described in [12], and we utilize it here to determine finite volume corrections for lattice calculations of the pion electromagnetic form factor. The result we derive is quite compact, maintains isospin symmetry, as well as a discrete rotational symmetry. This is in contrast to the result derived employing rest frame kinematics, which possessed both isospin breaking and cubic symmetry breaking terms from finite volume effects [13]. While numerically the finite volume effect is demonstrated to be small for either kinematics, it would be interesting to investigate nucleon isovector quantities in the Breit frame. Results for the nucleon isovector magnetic moment suggested a larger than expected volume correction arising from cubic symmetry breaking terms [12]. The Breit frame kinematics could mitigate such effects.

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