Phase-Shifting Separable Haar Wavelets and Applications

Mais Alnasser and Hassan Foroosh

Abstract

This paper presents a new approach for tackling the shift-invariance problem in the discrete Haar domain, without trading off any of its desirable properties, such as compression, separability, orthogonality, and symmetry. The paper presents several key theoretical contributions. First, we derive closed form expressions for phase shifting in the Haar domain both in partially decimated and fully decimated transforms. Second, it is shown that the wavelet coefficients of the shifted signal can be computed solely by using the coefficients of the original transformed signal. Third, we derive closed-form expressions for non-integer shifts, which have not been previously reported in the literature. Fourth, we establish the complexity of the proposed phase shifting approach using the derived analytic expressions. As an application example of these results, we apply the new formulae to image rotation and interpolation, and evaluate its performance against standard methods.

Index Terms

Discrete Haar Wavelets, Separable Wavelets, Phase Shifting, Image Rotation, Image Interpolation

I. INTRODUCTION

The wavelet transform has been playing an ever increasing important role in the modeling and analysis of a wide range of problems in science and engineering. In signal and image processing, wavelets have been particularly instrumental in methods of constructing “optimal” basis that are often used in various image processing and computer vision applications, such as shape/scene description and classification [1][3], [9], [10], [14], [15], [13], [36][38], [52], [55], [58][60], [63], [79][82], [85][88], [93], [95], [105], [107], [118], [124], [125], [153], [155], [156], [158], [168], scene content modeling [89], [90], [94], [96][98], [159][163], image restoration and denoising [30][32], [46], [53], [54], [61], [64], [75], [77], [78], [106], [127][129], [131][137], [140], [141], [143], video content modeling [8], [11][13], [16], [17], [34], [144][149], [154], [157], image alignment [6], [18][20], [22][25], [28], [29], [65][70], [72], [73], [130], [138], [139], tracking and object pose estimation [114][116], [142], [151], camera motion quantification and calibration [7], [20], [39][41], [41][43], [35], [49][51], [71], [83], [84], [91], [92], [99][102], [108], and image-based rendering (IBR) [4], [5], [21], [27], [44], [47], [48], [73], [117], [150], [166], [167], to name a few. However, a major drawback restricting the use of such methods is the lack of shift-invariance. For example, in the case of de-noising, Gibbs phenomenon in the neighborhood of discontinuities is attributed to the lack of shift-invariance of the wavelet basis [53]. An image transform is shift-invariant if the total energy of the coefficients in any subband is invariant to translations of the original image. It can be thus readily verified that the fastest and the most compact formulations - i.e. the classical fully decimated real wavelet transforms - suffer from the lack of shift-invariance. Additional properties that are often desired in many applications of wavelets include separability, orthogonality and symmetry.

There has been two trends in responding to the shift-invariance requirement. The earlier literature has been focusing on modifying the classical real wavelets to enforce shift-invariance, while attempting to preserve other desired properties. This approach was rediscovered by various authors independently, and bears different names such as algorithme à trous [62], [110], [112], redundant wavelets [35] and undecimated wavelets [104] to name a few. The major drawbacks of this approach, of course, are the undesirable side-effect of overly redundant representation and the high computational cost, since each set of coefficients contains the same number of samples as the input signal. This level of redundancy essentially defeats the purpose of designing wavelets for compression and coding, which take advantage of the localization properties of wavelets as opposed to the shift-invariant Fourier basis.

In order to alleviate these side-effects, more recently a second approach has been investigated in the literature that attempts to directly construct shift-invariant wavelets. This line of research has led to a new class of wavelets with complex coefficients. Few examples are the Gabor wavelets for texture processing [113], harmonic wavelets for vibration and acoustic analysis [119], [120] and the Complex Wavelet Transform (CWT) for motion estimation [111]. In addition to shift-invariance, one particular advantage of complex wavelets is directionality that is similar to the steerable pyramids [152]. Complex wavelets prove to be useful in solving the shift-invariance problem without compromising many other properties. However, their major drawbacks are lack of speed and often also poor inversion properties. A more successful attempt in this category is perhaps the dual-tree complex wavelet transform (DT-CWT) and its variations [57], [123]. Although, DT-CWT provides a good trade-off between fully decimated wavelets and the redundant wavelet transform, it does so by trading off the compression capabilities and computational time of the classical real wavelets.

In this paper, we initiate and investigate a third line of approach to tackling the shift-invariance problem. Instead of modifying a classical wavelet or introducing a new complex wavelet, our goal is to determine in what way the wavelet coefficients in

Mais Alnasser was with the Department of Computer Science, University of Central Florida, Orlando, FL, 32816 USA at the time this project was conducted. (e-mail: nasserm@cs.ucf.edu).
Hassan Foroosh is with the Department of Computer Science, University of Central Florida, Orlando, FL, 32816 USA (e-mail: foroosh@cs.ucf.edu).
a fully decimated transform are related to those of a shifted signal. Of course such relation would be wavelet-dependent and may not be a straightforward relation as in redundant wavelets, where the shift in the input results in a shift in the output. The key idea is that as long as the relation is known, one can tackle shift-invariance, since all the coefficients of a shifted signal can be mapped to those of the original signal. On the other hand, shift-invariance is tackled without compromising speed and compression properties. Furthermore, establishing the explicit and direct relations between the coefficients of a signal and its shifted version, would allow us to perform compressed domain processing of signals or images without requiring a chain of forward and backward transforms. This is particularly of interest in applications such as data compression and progressive transmission, or more recent applications in compressed sensing \[109, 122\]. Our focus in this paper is on the standard Haar wavelet transform due to additional desirable properties of separability and symmetry.

We present a solution to phase-shift the Haar coefficients in the transform domain solely using the available coefficients of the unshifted transformed signal, which we refer to as the 0-shift signal. Our solution generalizes readily to an N-dimensional signal due to separability. We also show how our solution can be extended to non-integer phase shifts. To demonstrate the power of the proposed approach and to evaluate it, we performed extensive experiments on the problem of accurate image rotation \[164\]. The remaining of this paper is organized as follows: In the next section, we introduce the notations and briefly describe the Haar transform tree. The following two sections will then derive our expressions for describing the explicit relations between the Haar coefficients of a 0-shift and shifted signal for both fully and partially transformed signals. These results are then extended for sub-pixel shifting, followed by full evaluation and testing of the results on image rotation and interpolation problems. The paper concludes with a brief discussion and some remarks on the proposed new ideas.

II. THE HAAR TRANSFORM TREE

Let \(x(n)\) be a one-dimensional signal of size \(2^N\), where \(N\) is a positive integer. The Haar transform of \(x(n)\), namely \(H(x(n))\), has the form:

\[
H(x(n)) = \{A_0^0, d_0^0, d_1^0, ..., d_i^l, ..., d_0^{N-1}, ..., d_0^{2N-1-1}\} \tag{1}
\]

such that \(A_0^0\) is the dc value of the signal and \(d_i^l\) is the \(i^{th}\) detail coefficient at level \(l\), where \(l = 0, ..., N-1\) and \(i = 0, ..., 2^l - 1\). Transforming a signal using Haar wavelets can be easily done by successively convolving the blurred part of the signal by box and differencing filters until the signal is fully transformed (see for instance \[165\] for more details).

We choose to express the Haar transformation using a tree as in Fig. 1. The tree is constructed of \(N\) levels with \(x(n)\) residing at the leaves, i.e. the \(N^{th}\) level. The \(i^{th}\) node at level \(l\) in the tree can be made to hold the 0-shift \(i^{th}\) blur and detail coefficients, \(A_i^l\) and \(d_i^l\), respectively, where \(l = 0, ..., N-1\) and \(i = 0, ..., 2^l - 1\).

Each level in the tree corresponds to a reduction step \(k = 1, ..., N\), with the untransformed original signal corresponding to \(k = 0\). The signal is partially transformed with \(k\) reduction steps if \(0 < k < N\) and is said to be fully transformed if \(k = N\). At each reduction level \(k\), one obtains the partially transformed signal \(H^k(x(n))\). \(H^k(x(n))\) is composed of the blur coefficients at level \(k\) followed by the detail coefficients at the same level and all subsequent reduction levels that are less than \(k\) and greater than 1. That is:

\[
H^k(x(n)) = \{A_0^{N-k}, ..., A_0^{N-k}, d_0^{N-k}, ..., d_0^{N-k}, d_1^{N-k}, ..., d_0^{2N-k-1-1}\} \tag{2}
\]
Where, \( l = N - k, \ldots, N - 1 \) and \( i = 0, \ldots, 2^l \). Note that \( H^N(x(n)) = H(x(n)) \) is the fully transformed signal.

We use the tree to examine the behavior of the detail coefficients with respect to shifting. Note that we can denote \( x(i) \) as \( A_i^N \), in which case \( d_i^N = 0 \). By using this notation, \( l \) now has the range \( 0, \ldots, N \). Also, note that the blur coefficient \( A_i^l \) is related to its parent at level \( l - 1 \) by the following relation:

\[
A_i^l = A_{i/2}^{l-1} + d_{i/2}^{l-1}, \quad i \text{ is even}
\]

\[
A_{i/2}^{l-1} - d_{i/2}^{l-1}, \quad i \text{ is odd}
\]

(3)

Now, let \( D_i^l \) be the difference between the dc value at the root of the tree \( A_0^0 \) and the blur coefficient \( A_i^l \). Then

\[
A_i^l = A_0^0 + D_i^l
\]

(4)

By substituting (4) in (3), \( D_i^l \) can be computed recursively solely in terms of the detail coefficients using the following relation:

\[
D_i^l = D_{i/2}^{l-1} + d_{i/2}^{l-1}, \quad i \text{ is even}
\]

\[
D_{i/2}^{l-1} - d_{i/2}^{l-1}, \quad i \text{ is odd}
\]

\[
D_0^0 = 0
\]

(5)

It can be verified that \( D_i^l \) can be computed recursively with a complexity of \( O(l) \) for fully-transformed signals, which in itself is very cheap, or be simply tabulated for even a faster retrieval. Also, note that for partially transformed signals, a combination of (5) and (4) has to be used to evaluate \( D_i^l \):

\[
l = N - k:
\]

\[
D_i^l = A_i^l - A_0^0
\]

\[
l > N - k:
\]

\[
D_i^l = D_{i/2}^{l-1} + d_{i/2}^{l-1}, \quad i \text{ is even}
\]

\[
D_{i/2}^{l-1} - d_{i/2}^{l-1}, \quad i \text{ is odd}
\]

(6)

The complexity for the above equation is even less than that of (5) because the recursion needs to go a maximum depth of \( k \) rather than a maximum depth of \( N \). In other words, the complexity for the above equation is \( O(k - l) \).

At level \( N - k \), there are \( 2^k \) non-redundant coefficient sets each of size \( 2^{N-k} \) \([126]\), where \( k = 1, \ldots, N \). A shift \( s = 0, \ldots, 2^{N-1} \) can be one of the following possibilities:

- A shift that is divisible by \( 2^k \).
- An odd shift.
- An even shift that is not divisible by \( 2^k \).

In the following sections, we first analyze the behavior of the detail coefficients based on the above three possibilities for a fully transformed signal. We then analyze the behavior of the blur coefficients for signals that are partially transformed. The final analytic solutions that we provide are capable of evaluating the coefficients of the shifted signal solely using the original coefficients of the 0-shift signal, which is the goal of our paper.

III. SHIFTING FULLY TRANSFORMED SIGNALS

A. Shifting by a Multiple of \( 2^k \)

This is the simplest case. A shift \( s \) in the discrete domain that is equal to \( 2^k u \) is a circular shift of the 0-shift detail coefficients at level \( N - k \) by \( u \), that is,

\[
d_{i+u}^{N-k} = d_{i+u\%2^k}^{N-k}, \quad k = 1, \ldots, N
\]

(7)

where \( 0 \leq u \leq 2^{N-k} - 1 \) and \( \% \) is the mod operation. Note that for levels \( N - (k-1), N - (k-2), \ldots, N - 1 \) a shift of \( 2^k u \) of the original signal is a circular shift of the coefficients at those levels by \( 2u, 2^2 u, \ldots, 2^{k-1} u \), respectively. In other words, a shift of \( 2^k u \) of the original signal shifts the coefficients at level \( N - k \) by \( u \), while shifting the coefficients at level \( N - (k-1) \) by twice as much, and the coefficients at level \( N - (k-2) \) by four times as much and so on.
B. Shifting by an Odd Amount

By examining the tree in figure 1, we notice that:
\[ d_{\text{new}}^{N-k} = \left( (x_{1(2^i+1)} \% 2^N + ... + x_{(2^i-1)(2^i+1)} \% 2^N) \right. \]
\[ \left. - (x_{(2^i-1)(2^i+1)+1} \% 2^N + ... + x_{(2^i(2^i+1)+1)} \% 2^N) \right) / 2^k \]
(8)

In other words, \( d_{\text{new}}^{N-k} \) is the sum of the leaves shifted into its left branch minus the leaves shifted into its right branch divided by \( 2^k \). To simplify the above equation, we set the indices as follows:
\[ i_1 = 2^k i + s \]
\[ i_2 = 2^{k-1}(i+1) + s \]
\[ i_3 = 2^k(i+1) + s \]

Using the notation \( A_i^N \) for \( x_i \), \( (8) \) now becomes:
\[ d_{\text{new}}^{N-k} = \left( (A_{i_1}^N \% 2^N + ... + A_{(i_2-1)}^N \% 2^N) \right. \]
\[ \left. - (A_{i_2}^N \% 2^N + ... + A_{(i_3-1)}^N \% 2^N) \right) / 2^k \]
(9)

Substituting \( 4 \) and then \( 5 \) in \( 9 \) and canceling out the \( A_0^N \)'s, the relation for computing \( d_{\text{new}}^{N-k} \) for a shift \( s \) that is odd becomes:
\[ d_{\text{new}}^{N-k} = (D_{i_1}^{N-1} \% 2^N - 2 \sum_{m=i_2+1}^{i_3-1} D_{m}^{N-2} - 2 \sum_{m=i_2+1}^{i_3-1} D_{m}^{N-2}) / 2^k \]
where,
\[ i_1 = 2^{k-1} + [s/2] \]
\[ i_2 = 2^{k-2}(i+1) + [s/2] \]
\[ i_3 = 2^{k-1}(i+1) + [s/2] \]
(10)

Note that for \( k = 1 \), \( i_2 \) would be a non-integer value, in which case we must set \( d_{i_2}^{N-k} \) to 0.

C. Shifting by an Even Amount that is Not Divisible by \( 2^k \)

In this case, \( s \) is divisible by \( 2^t \), for \( 1 \leq t \leq k-1 \) and \( 2^t \) is the highest power of 2 by which \( s \) is divisible. This allows us to let \( s = 2^t u \), where \( 0 \leq u \leq 2^{N-t} - 1 \). This means that the coefficients at levels \( N-1, ..., N-t \) follow the first case. In other words, the 0-shift coefficients at levels \( N-1, N-2, ..., N-t \) are circularly shifted by \( 2^{t-1} u, 2^{t-2} u, ..., u \), respectively. Since \( 2^t \) is the highest power of 2 by which \( s \) is divisible, \( u \) must be odd. This allows us to treat this case as an odd shift of the blur details at level \( N-t \). In other words, at level \( N-k \), \( d_{\text{new}}^{N-k} \) can be evaluated using the following modification of equation \( 9 \):
\[ d_{\text{new}}^{N-k} = \left( (A_{i_1}^{N-t} \% 2^N - 2 \sum_{m=i_2+1}^{i_3-1} A_{m}^{N-t} \% 2^N) \right. \]
\[ \left. - (A_{i_2}^{N-t} \% 2^N + ... + A_{(i_3-1)}^{N-t} \% 2^N) \right) / 2^{k-t} \]
where,
\[ i_1 = 2^{k-t} + s/2^t \]
\[ i_2 = 2^{k-t-1}(i+1) + s/2^t \]
\[ i_3 = 2^{k-t}(i+1) + s/2^t \]
(11)
Following the same steps, the above can be rewritten as:

\[ d_{n+1}^{N-k} = (D_{n+1}^{N-k-1} + 2 \sum_{m=n+1}^{N-k-1} D_{m}^{N-k-1} \] 

\[ - 2 \sum_{m=n+1}^{N-k-1} D_{m+1}^{N-k-1} - D_{n+1}^{N-k-1} \] 

\[ - d_{n+1}^{N-k} + 2d_{n+1}^{N-k} - d_{n+1}^{N-k} / 2^{k-t} \]

where,

\[ i_1 = 2^{k-t-1} + 2^{t-1} \] 

\[ i_2 = 2^{k-t-2} (i + 1) + 2^{t-1} \] 

\[ i_3 = 2^{k-t-1} (i + 1) + 2^{t-1} \]

and,

\[ d_{i_2}^{N-k} = 0, \text{ if } i_2 \text{ is non-integer}. \]

Note that the second case is the same as the third case when \( i_1 = 0 \). That leaves us with the following formula:

\[ k > t : \]

\[ d_{n+1}^{N-k} = (D_{n+1}^{N-k-1} + 2 \sum_{m=n+1}^{N-k-1} D_{m}^{N-k-1} \]

\[ - 2 \sum_{m=n+1}^{N-k-1} D_{m+1}^{N-k-1} - D_{n+1}^{N-k-1} \]

\[ - d_{n+1}^{N-k} + 2d_{n+1}^{N-k} - d_{n+1}^{N-k} / 2^{k-t} \]

\[ k \leq t : \]

\[ d_{n+1}^{N-k} = d_{i_2}^{N-k} \]

where,

\[ i_1 = 2^{k-t-1} + 2^{t-1} \] 

\[ i_2 = 2^{k-t-2} (i + 1) + 2^{t-1} \] 

\[ i_3 = 2^{k-t-1} (i + 1) + 2^{t-1} \]

and,

\[ d_{i_2}^{N-k} = 0, \text{ if } i_2 \text{ is a non-integer} \]

The above relation can now be used to evaluate the new detail coefficients of the Haar transform at all different levels after any shift \( s = 0, \ldots, 2^L - 1 \) using only the coefficients of the 0-shift signal. The worst case complexity for evaluating \( d_{n+1}^{N-k} \) using (13) is \( O(\log(L)) \), where \( L \) is the size of the signal \( x(n) \) (see the complexity analysis section for more details).

IV. SHIFTING PARtIALLY TRANSFORMED SIGNALS

Depending on the application, the original signal might not be fully transformed. As we mentioned earlier, a signal that has \( k \) degrees of reduction has the form:

\[ H^k(x(n)) = \{ A_0^{N-k}, \ldots, A_{2^{k-1}N-1}^{N-k}, d_0^{N-k}, \ldots, d_{2^{k-1}N-1}^{N-k}, \ldots, \}

Where, \( 1 \leq k \leq N - 1, l = N - k, \ldots, N - 1 \) and \( i = 0, \ldots, 2^k \).

A signal that is partially transformed is composed of both blur coefficients and detail coefficients. Equation (13) shows how to evaluate the detail coefficients of a fully transformed shifted signal, which also applies to evaluating the detail coefficients of a partially transformed signal. In this section we show how to evaluate the blur coefficients at reduction step \( k \) for a signal that has been decomposed \( k \) times and shifted by the integer amount \( s \) in the time domain.

A. SHIFTING BY A MULTIPLE OF \( 2^k \)

Similar to evaluating the detail coefficients case, a shift \( s \) in the discrete domain that is equal to \( 2^k u \) is a circular shift of the 0-shift blur coefficients at level \( N - k \) by \( u \), that is,

\[ A_{n+1}^{N-k} = A_{n+1}^{N-k}, \quad k = 1, \ldots, N - 1 \]

where \( 0 \leq u \leq 2^{N-k} - 1 \).
B. Shifting by an Odd Amount

By examining the tree in figure [1], we notice that:

\[
A_{i_{new}}^{N-k} = \left( x_{(2k+1)s+1} \cdots x_{(2k-1)(s+1)+1} (2N) \right) + \left( x_{(2^k-1)(s+1)+1} \cdots x_{(2^k+1)(s+1)+1} (2N) \right) / 2^k
\]

In other words, \( A_{i_{new}}^{N-k} \) is the sum of the leaves shifted into its left branch plus the leaves shifted into its right branch divided by \( 2^k \). To simplify the above equation, we use only the starting and ending coefficients and we also use the notation \( A_i^N \) for \( x_i \):

\[
A_{i_{new}}^{N-k} = (A_i^N + \cdots + A_{(i_{2-1})}^N) / 2^k
\]

Where,

\[
i_1 = 2^k i + s \\
i_2 = 2^k (i + 1) + s
\]

Substituting [4] in the above, we get

\[
A_{i_{new}}^{N-k} = (A_0^0 + D_{i_1}^N + \cdots + A_0^0 + D_{(i_2-1)}^N) / 2^k
\]

The number of \( A_0^0 \)'s is equal to the number of coefficients \( A_i^N \) being summed, which is equal to \( 2^k \). We factor out \( A_0^0 \):

\[
A_{i_{new}}^{N-k} = A_0^0 + (D_{i_2}^N + \cdots + D_{(i_2-1)}^N) / 2^k
\]

Substituting [5] and simplifying, we get the analytic solution for evaluating \( A_{i_{new}}^{N-k} \) under an odd shift \( s \):

\[
A_{i_{new}}^{N-k} = A_0^0 + \sum_{m=i_1+1}^{i_2} D_{m}^{N-1} + D_{i_2}^{N-1} - d_{i_1}^{N-1} + d_{i_2}^{N-1} / 2^k
\]

where,

\[
i_1 = 2^{k-1} i + \lfloor s/2 \rfloor \\
i_2 = 2^{k-1} (i + 1) + \lfloor s/2 \rfloor
\]

C. Shifting by an Even Amount that is Not Divisible by \( 2^k \)

For a shift \( s = 2^t u \), where \( 0 \leq u \leq 2^{N-t} - 1 \) and \( t < k \), we can treat this case as an odd shift of the coefficients at level \( N - t \), which is similar to what we did in evaluating the detail coefficients under a shift \( s = 2^t u \). \( A_{i_{new}}^{N-k} \) can now be evaluated using the following equation:

\[
A_{i_{new}}^{N-k} = \left( A_{i_1}^{N-t} + \cdots + A_{(i_2-1)}^{N-t} \right) / 2^{k-t}
\]

Proceeding as we did in the odd shift case, we get the following solution:

\[
A_{i_{new}}^{N-k} = A_0^0 + \sum_{m=i_1+1}^{i_2-1} D_{m}^{N-t-1} + D_{i_2}^{N-t-1} - d_{i_1}^{N-t-1} + d_{i_2}^{N-t-1} / 2^{k-t}
\]

where,

\[
i_1 = 2^{k-t-1} i + \lfloor s/2^{t+1} \rfloor \\
i_2 = 2^{k-t-1} (i + 1) + \lfloor s/2^{t+1} \rfloor
\]
Combining the three cases, the final result becomes:

\[ k > t : \]

\[ A_{t_{\text{new}}}^{N-k} = A_0^N + (D_{i_1}^{N-t-1} + \sum_{m=i_1+1}^{i_2-1} D_{m}^{N-t-1} + D_{i_2}^{N-t-1}) \]

\[ k \leq t : \]

\[ d_{t_{\text{new}}}^{N-k} = d_{(i+s/2k)^2}^{N-k} \]

where,

\[ i_1 = 2^{k-t-1}i + \lfloor s/2^{t+1} \rfloor \]

\[ i_2 = 2^{k-t-1}(i + 1) + \lfloor s/2^{t+1} \rfloor \]

and,

\[ d_{i_2}^{N-t-1} = 0, \text{ if } i_2 \text{ is a non-integer} \] (23)

The above relation can now be used to evaluate the new blur coefficients of a partially transformed signal with \( k \) reduction steps after any shift \( s = 0, \ldots, 2^N - 1 \) using only the coefficients of the 0-shift signal. The worst case complexity for evaluating \( A_{t_{\text{new}}}^{N-k} \) using (22) is \( O(\log(L)) \), where \( L \) is the size of the signal \( x(n) \) (see the complexity analysis section for more details).

V. NON-INTEGER SHIFTING

In this section, we show how our solution can be extended to achieve non-integer shifts. Although, our model is based on up-sampling the original signal, the final relations that are derived require using only the coefficients of the original signal. Up-sampling by a factor of 2 can be modeled as adding levels to the lowest part of the transform tree and setting the detail coefficients in those levels to zero, with the lowest level being \( N - 1 \). On the other hand, shifting the up-sampled signal by an amount \( u \) is equivalent to shifting the original signal by \( \frac{u}{2} \), which is a precision of \( \frac{1}{2} \). More generally, adding \( h \) levels would enable us to obtain a precision of \( \frac{1}{2^h} \).

Let the size of the signal be \( 2^N, N' = N + h \) and \( k = 1 + h, \ldots, N + h \), where \( h \) is the number of added levels. Equation (13) can now be modified to allow for non-integer shifting by a precision of \( \frac{1}{2^h} \) as follows:

\[ k > t : \]

\[ d_{t_{\text{new}}}^{N'-k} = (D_{i_1}^{N'-t-1} + \sum_{i_1+1}^{i_2-1} D_{m}^{N'-t-1} + D_{i_2}^{N'-t-1}) \]

\[ k \leq t : \]

\[ d_{t_{\text{new}}}^{N'-k} = d_{(i+s/2k)^2}^{N'-k} \]

where,

\[ i_1 = 2^{k-t-1}i + \lfloor s/2^{t+1} \rfloor \]

\[ i_2 = 2^{k-t-2}(2i + 1) + \lfloor s/2^{t+1} \rfloor \]

\[ i_3 = 2^{k-t-1}(i + 1) + \lfloor s/2^{t+1} \rfloor \]

and,

\[ d_{i_3}^{N'-t-1} = 0, \text{ if } i_3 \text{ is a non-integer} \] (23)

On the other hand, we can verify that \( D_i^{N+h_0} = D_{i/2^{h_0}}^{N} \), where \( 0 \leq h_0 \leq h \). Using (5), we also know that:

\[ D_i^N = D_{i/2}^{N-1} + d_{i/2}^{N-1}, \text{ i is even} \]

\[ D_i^N = D_{i/2}^{N-1} - d_{i/2}^{N-1}, \text{ i is odd} \] (24)

The above result allows us to modify (23) in such a way that avoids having to up-sample the signal for non-integer shifts, saving thus memory space in actual implementation, especially that the size increases exponentially. However, We have to split the equation into two cases. The first is when \( h \geq t + 1 \), which is when the coefficients at the added levels are being used to
evaluate \(d_{new}^{N^t-k}\). The second is when \(t\) is large enough for the coefficients at the original levels of the tree to be used. This leads to the new form of the phase shifting relation for non-integer values as follows:

\[
h \geq t + 1:
\]

\[
d_{n+1}^{N^t-k} = (D^{N^t}_{1 + t/2^N^t-i^t} + 2 \sum_{i^t+1}^{i^t-t} D^{N^t}_{m/2^N^t-i^t})/2^k-t
\]

\[
h < t + 1:
\]

\[
d_{n+1}^{N^t-k} = (D^{N^t}_{1 + t/2^N^t-i^t} + 2 \sum_{i^t+1}^{i^t-t} D^{N^t}_{m/2^N^t-i^t})/2^k-t
\]

\[
k > t:
\]

\[
d_{n+1}^{N^t-k} = (D^{N^t}_{1 + t/2^N^t-i^t} + 2 \sum_{i^t+1}^{i^t-t} D^{N^t}_{m/2^N^t-i^t})/2^k-t
\]

\[
k \leq t:
\]

\[
d_{n+1}^{N^t-k} = (D^{N^t}_{1 + t/2^N^t-i^t} + 2 \sum_{i^t+1}^{i^t-t} D^{N^t}_{m/2^N^t-i^t})/2^k-t
\]

where,

\[
i_1 = 2^{k-t-1} i + \lfloor s/2^{t+1} \rfloor
\]

\[
i_2 = 2^{k-t-2} (2i + 1) + \lfloor s/2^{t+1} \rfloor
\]

\[
i_3 = 2^{k-t-1} (i + 1) + \lfloor s/2^{t+1} \rfloor
\]

and,

\[
d_{i^t/2^N^{t-1}} = 0, \text{ if } i^t \text{ is a non-integer}
\]

(25)

The worst case complexity of the above formula is \(O(\log(L + 2^h))\) (again please refer to the Complexity Analysis section for more details).

VI. N-DIMENSIONAL SHIFT

Due to separability, an N-dimensional standard Haar transform is constructed by applying the one-dimensional transform along each dimension. As a result, the above solution can also be easily generalized to N-dimensional signals by applying it along each dimension separately.

VII. COMPLEXITY ANALYSIS

In this section we explain in further detail the complexity of evaluating \(d_{n+1}^{N^t-k}\) using equation (13), \(A_{n+1}^{N^t-k}\) using equation (22), and \(d_{n+1}^{N^t-k}\) using equation (25).

By examining (13), it is easy to verify that the complexity of evaluating \(d_{n+1}^{N^t-k}\) can be expressed by the difference of the bounds of the two sums in the equation, that is \(O(i_3 - i_1)\). Substituting the values for \(i_1\) and \(i_3\), the complexity can be shown to be \(O(2^{k-t-1})\) when \(k \geq t\). When \(k \leq t\) the complexity becomes \(O(1)\). Therefore, one can determine that the worst case is when \(t = 0\), that is when the shift is odd. In that case the complexity of computing \(d_{n+1}^{N^t-k}\) becomes \(O(2^{k-1})\). Let \(L = 2^N\) be the size of the signal, then the number of the detail coefficients in a fully transformed signal is \(L - 1 = 2^N - 1\). At reduction level \(k = N\), i.e. the root, the complexity of evaluating \(d_{0+1}^{N^t-k}\) is \(O(2^{N-1}) = O(2^{N-1}/L)\) with a probability of \(1/L-1\). At the next reduction level \(k = N - 1\), the complexity is \(O(2^{N-1}/L) = O(2^{k-1}/L)\) with a probability of \(1/L-1\). Table (I) shows the complexity and by multiplying the each reduction level's probabilities in table (I) and summing them up, the average performance of the worst case for evaluating \(d_{n+1}^{N^t-k}\) is found to be \(O(\log(L))\).

By following a similar analysis and examining (22), one can find that the complexity for evaluating \(A_{n+1}^{N^t-k}\) is \(O(\log(L))\) as well. Also, by examining (25) one can find that complexity for evaluating \(d_{n+1}^{N^t-k}\) after a non-integer shift is \(O(\log(L + 2^h))\), where \(h\) is the number of levels added to achieve the shift.

VIII. EXPERIMENTAL VALIDATION

We validate our results on the problem of accurate image rotation using the decomposition of the rotation matrix described in [56], [103], [121]. The choice of this application is driven by the fact that it allows us to evaluate all aspects such as integer and non-integer shifts, and the separability property.
Reduction Level | Complexity | Number of Coefficients at k
---|---|---
$k = N$ | $O\left(\frac{L}{N}\right)$ | \(\frac{L}{N-1}\)
$k = N - 1$ | $O\left(\frac{L}{N-1}\right)$ | \(\frac{L}{N-2}\)
$k = N - 2$ | $O\left(\frac{L}{N-2}\right)$ | \(\frac{L}{N-3}\)
$k = N - 3$ | $O\left(\frac{L}{N-3}\right)$ | \(\frac{L}{N-4}\)
\vdots | \vdots | \vdots
$k = 1$ | $O\left(\frac{L}{N-1}\right)$ | \(\frac{L}{N-1}\)

**TABLE I**

**TABLE OF THE COMPLEXITY AND PROBABILITY AT EACH REDUCTION LEVEL $k$ FOR THE ONE-DIMENSIONAL DETAIL COEFFICIENT $d^{N-k}_{\text{new}}$.**

A. Image Rotation

We implement rotation as a sequence of sheers using the following factorization [56], [103], [121], [164]:

\[
R(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix} = \begin{bmatrix}
1 - \tan\left(\frac{\theta}{2}\right) \\
0 & 1
\end{bmatrix}
\times \begin{bmatrix}
1 & 0 \\
\sin(\theta) & 1
\end{bmatrix}
\times \begin{bmatrix}
1 - \tan\left(\frac{\theta}{2}\right) \\
0 & 1
\end{bmatrix}
\]

(26)

A sheer is in fact a sequence of shifts that are row-dependent, if the sheer is horizontal, and column-dependent if it is vertical. That is, each row is shifted by $\Delta x = -y \cdot \tan \frac{\theta}{2}$ in a horizontal sheer while each column is shifted by $\Delta y = x \cdot \sin \theta$ in a vertical sheer. Note that $\Delta x$ and $\Delta y$ are in general non-integer values, hence, the applicability of our phase-shifting relations derived in the previous sections. Figure (2-b) shows the application of our method to the 3-step shearing image rotation with $h = 3$. Figure (3) shows a magnified portion of the image under different $h$ values. An integer shift ($h = 0$) results in a jagged effect. This effect is eliminated, leading to higher quality results, as we increase the value of $h$. Note that visually satisfactory results are obtained even with $h = 2$.

As noted in [164], the worst scenario occurs when the errors get accumulated. Therefore, in order to quantify the performance, we computed the residual error, using an experiment similar to the one adopted in [164]. In other words, we successively rotated an input image by $\frac{\pi}{8}$ until it rotated back to its original position. Figures 4 and 5 show the results and the associated residual errors on two standard test images for our method as compared to the nearest-neighbor, bilinear, bicubic, and the sinc method. Note that the image in Figure 4, which was also used by [164], is specifically designed for capturing accumulated errors in successive rotations. We tested and compared our method extensively on many images, some of which are shown in Table II.

![Fig. 2. Original image, and the rotated one by 45 degrees using (25) with $h = 3$.](image)

IX. Conclusion

We have successfully shown that shift-invariance of the standard Haar wavelets may be tackled directly by establishing analytic relations between the Haar coefficients of a signal and its shifted version. This new line of approach has the advantage that it does not trade off the compression capability by retaining full decimation, while preserving symmetry and separability. Our approach does not yield a shift-invariant wavelet transform, but rather establishes the explicit relations that describe phase-shifting directly in the transform domain. Our experiments illustrate the validity of the underlying motivating ideas, and the high accuracy of results in practical problems.
Fig. 3. A magnified portion of the image rotated using equation (25). a. integer shift. b. non-integer shift with precision of $\frac{1}{2}$. c. non-integer shift with precision of $\frac{1}{4}$. d. non-integer shift with precision of $\frac{1}{8}$. e. non-integer shift with precision of $\frac{1}{16}$. f. non-integer shift with precision of $\frac{1}{32}$.

Fig. 4. The above images show the results of successively rotating the original image 16 times by a degree of $\frac{\pi}{8}$ for different methods including ours.

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The above images show the results of successively rotating the original image 16 times by a degree of $\frac{\pi}{8}$ for different methods including ours.

| Method      | rms error |
|-------------|-----------|
| Nearest Neighbor | 15.93     |
| Bilinear     | 15.53     |
| Bicubic      | 9.08      |
| Our Method   | 2.54      |

*Fig. 5. The above images show the results of successively rotating the original image 16 times by a degree of $\frac{\pi}{8}$ for different methods including ours.*

### TABLE II

| Method      | rms error |
|-------------|-----------|
| Nearest Neighbor | 23.2451  |
| Bilinear     | 21.9343   |
| Bicubic      | 15.2645   |
| Sinc         | 8.4349    |
| Our Method   | 3.3738    |

*Quantification and comparison of the accumulated residual error on several test images.*

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