A BK INEQUALITY FOR RANDOM MATCHINGS

ANDRÁS MÉSZÁROS

Abstract. Let $G = (S, T, E)$ be a bipartite graph. For a matching $M$ of $G$, let $V(M)$ be the set of vertices covered by $M$, and let $B(M)$ be the symmetric difference of $V(M)$ and $S$. We prove that if $M$ is a uniform random matching of $G$, then $B(M)$ satisfies the BK inequality for increasing events.

1. Introduction

Let $V$ be a finite set. We will consider random subsets of $V$. Let $A$ and $B$ be upward closed subsets of $2^V$, in other words, let $A$ and $B$ be increasing events. Let $A □ B$ be the event that $A$ and $B$ both occur disjointly, more formally, we define

$$A □ B = \left\{ A \cup B \mid A \in A, B \in B, A \cap B = \emptyset \right\}.$$ 

Let $G = (S, T, E)$ be a bipartite graph, and let $V = S \cup T$. Let $\mathcal{M}$ be the set of matchings in $G$. For a matching $M \in \mathcal{M}$, let $V(M)$ be the set of vertices covered by $M$, and let

$$B(M) = V(M) \Delta S,$$

where $\Delta$ denotes the symmetric difference. Note that we have $|B(M)| = |S|$ for any matching $M$.

Our main result is the following.

Theorem 1. Let $M$ be a uniform random element of $\mathcal{M}$. Then $B(M)$ satisfies the BK inequality for increasing events, that is, if $A$ and $B$ are upward closed subsets of $2^V$, then

$$P(B(M) \in A □ B) \leq P(B(M) \in A)P(B(M) \in B).$$

For a random subset with independent marginals, the BK inequality was proved by van den Berg and Kesten [4]. Later, van den Berg and Jonsson proved that it also holds for a uniform random $k$ element subset [3]. There is an extension of the notion $A □ B$ for arbitrary events, see Subsection 2.1. With this definition, the BK inequality holds for all events in the case of a random subset with independent marginals. This was conjectured by van den Berg and Kesten [4], and proved by Reimer [1]. See also the paper of van den Berg and Gandolfi [2] for further results.

We say that an event $A$ depends only on $V_0 \subseteq V$, if for any $A, B \subseteq V$ the conditions $A \cap V_0 = B \cap V_0$ and $A \in A$ imply that $B \in A$. Note that if $A$ and $B$ are increasing events depending on disjoint subsets of $V$, then $A □ B = A \cap B$. Thus, Theorem 1 has the following corollary.

Corollary 2. Let $B(M)$ be like above, then $B(M)$ has negative associations, which means the following. Let $A$ and $B$ be events depending on disjoint subsets of $V$. If $A$ and $B$ are both increasing or both decreasing, then

$$P(B(M) \in A \cap B) \leq P(B(M) \in A)P(B(M) \in B).$$
If \( \mathcal{A} \) is increasing and \( \mathcal{B} \) is decreasing, then
\[
P(\mathcal{B}(M) \in \mathcal{A} \cap \mathcal{B}) \geq P(\mathcal{B}(M) \in \mathcal{A})P(\mathcal{B}(M) \in \mathcal{B}).
\]

Now we give a few extensions of Theorem 1. Assume that every edge \( e \) of \( G \) has positive weight \( w(e) \). For a matching \( M \), we define the weight of \( M \) as \( w(M) = \prod_{e \in M} w(e) \). Let \( M \) be a random matching, where the probability of a matching is proportional to its weight. We have the following extension of Theorem 1.

**Theorem 3.** Let \( M \) be like above. Then \( \mathcal{B}(M) \) satisfies the BK inequality for increasing events, that is, if \( \mathcal{A} \) and \( \mathcal{B} \) are upward closed subsets of \( 2^V \), then
\[
P(\mathcal{B}(M) \in \mathcal{A} \square \mathcal{B}) \leq P(\mathcal{B}(M) \in \mathcal{A})P(\mathcal{B}(M) \in \mathcal{B}).
\]

Furthermore, let \( V_+ \) and \( V_- \) be disjoint subsets of \( V \). Let \( M' \) have the same distribution as \( M \) conditioned on the event that \( V_+ \subseteq \mathcal{B}(M) \) and \( V_- \cap \mathcal{B}(M) = \emptyset \). Let \( V' = V \setminus (V_+ \cup V_-) \), and let \( \mathcal{B}'(M') = \mathcal{B}(M') \cap V' \). Clearly, \( \mathcal{B}'(M') \) is a random subset of \( V' \).

**Theorem 4.** The random subset \( \mathcal{B}'(M') \) satisfies the BK inequality for increasing events.

This has the following corollary.

**Corollary 5.** Let \( M \) be like above. Then for any subset \( X \) and \( Y \) of \( V \), we have
\[
P(X \subseteq \mathcal{B}(M))P(Y \subseteq \mathcal{B}(M)) \geq P(X \cap Y \subseteq \mathcal{B}(M))P(X \cup Y \subseteq \mathcal{B}(M)).
\]

We can also deduce the following theorem from Theorem 3.

**Theorem 6.** Let \( M \) be uniform random maximum size matching. Then the random subset \( \mathcal{B}(M) \) satisfies the BK inequality for increasing events.

**Acknowledgements.** The author is grateful to Péter Csikvári and Miklós Abért for their comments. The author was partially supported by the ERC Consolidator Grant 648017.

2. The proofs

2.1. The definition of \( \mathcal{A} \square \mathcal{B} \) for arbitrary events. Let us recall how to extend the definition of \( \mathcal{A} \square \mathcal{B} \) to arbitrary events. A subset \( C \) of \( V \) is in \( \mathcal{A} \square \mathcal{B} \) if and only if there are disjoint subsets \( V_A \) and \( V_B \) of \( V \) such that
\[
\{D \subseteq V | D \cap V_A = C \cap V_A\} \subseteq \mathcal{A}
\]
and
\[
\{D \subseteq V | D \cap V_B = C \cap V_B\} \subseteq \mathcal{B}.
\]
If \( \mathcal{A} \) and \( \mathcal{B} \) are increasing, then this definition indeed coincides with our earlier definition.
2.2. The proof of Theorem 1. Our proof will use several ideas of Berg and Jonas-
son [3].

Let $I$ be the set of tuples $(W, K, L, R)$, where $W$ is a subset of $V$, $K$ and $L$ are perfect matchings in the induced subgraph $G[W]$, $R$ is a subgraph of $G[V \setminus W]$ consisting of vertex disjoint paths.

Fix a linear ordering of the edges of $G$. Consider an $i = (W, K, L, R) \in I$. Then $R$ is the vertex disjoint union of the paths $P_1, P_2, \ldots, P_k$, where we list the paths in increasing order of their lowest edge. We can write $P_j$ as the union of the matchings $M_{j,0}$ and $M_{j,1}$, this decomposition is unique once we assume that $M_{j,0}$ contains the lowest edge of $P_j$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_k) \in \{0, 1\}^k$, we define the matchings

$$C_{i,\omega} = K \cup \bigcup_{j=1}^k M_{j,\omega_j} \quad \text{and} \quad D_{i,\omega} = L \cup \bigcup_{j=1}^k M_{j,1-\omega_j}.$$ 

Moreover, we define

$$Y_i^C = \{C_{i,\omega} \mid \omega \in \{0, 1\}^k\},$$

$$Y_i^D = \{D_{i,\omega} \mid \omega \in \{0, 1\}^k\},$$

and

$$X_i = \{(C_{i,\omega}, D_{i,\omega}) \mid \omega \in \{0, 1\}^k\}.$$ 

Let $H_j$ be the set of endpoints of the paths $P_1, P_2, \ldots, P_k$. Let $V(R)$ be the vertex set of $R$. Let $B_i = (V(R) \setminus \Delta S) \setminus H_i$. Let $v_{j,0}$ and $v_{j,1}$ be the two endpoints of $P_j$. If we choose the indices in the right way, then we get that

$$B(C_{i,\omega}) = B_i \cup \{v_{j,\omega_j} \mid j = 1, 2, \ldots, k\},$$

and

$$B(D_{i,\omega}) = B_i \cup \{v_{j,1-\omega_j} \mid j = 1, 2, \ldots, k\}.$$ 

This immediately implies that

$$\{B(C_{i,\omega}) \mid \omega \in \{0, 1\}^k\} = \{B(D_{i,\omega}) \mid \omega \in \{0, 1\}^k\} =$$

$$\{B_i \cup H \mid H \subseteq H_i \text{ and } |H \cap \{v_{j,0}, v_{j,1}\}| = 1 \text{ for all } j = 1, 2, \ldots, k\}.$$ 

Let $U = \{v_{j,1} \mid j = 1, 2, \ldots, k\}$. We define the map $\tau_i : \mathcal{M} \to 2^U$ by $\tau_i(M) = B(M) \cap U$. It is clear from what is written above that the appropriate restriction of $\tau_i$ gives a bijection from $Y_i^C$ to $2^U$, and also from $Y_i^D$ to $2^U$. Moreover,

$$X_i = \{(C, D) \in Y_i^C \times Y_i^D \mid \tau_i(C) = U \setminus \tau_i(D)\}.$$ 

Lemma 7. The sets $(X_i)_{i \in I}$ give a partition of $\mathcal{M} \times \mathcal{M}$.

Proof. Let $(C, D) \in \mathcal{M} \times \mathcal{M}$. Consider the multi-graph $C \cup D$, it is a vertex disjoint union of cycles and paths. Let $R$ be the union of paths, and let $Q$ be the union of cycles. Let $W$ be the vertices covered by the cycles. Let $i = (W, C \cap Q, D \cap Q, R)$. One can easily prove that $i$ is the unique element of $I$ such that $(C, D) \in X_i$. □

Given a subset $\mathcal{F}$ of $2^V$, we define $\mathcal{M}_F$ as $\{M \in \mathcal{M} \mid B(M) \in \mathcal{F}\}$. The statement of Theorem 1 is equivalent to the statement

$$|\mathcal{M}_{AB} \times \mathcal{M}| \leq |\mathcal{M}_A \times \mathcal{M}_B|.$$ 

From Lemma 7, it follows that it is enough to prove that for any $i \in I$, we have

$$|(\mathcal{M}_{AB} \times \mathcal{M}) \cap X_i| \leq |(\mathcal{M}_A \times \mathcal{M}_B) \cap X_i|.$$

1In our terminology, a path must have at least 1 edge.
For a subset \( \mathcal{F} \) of \( 2^V \) and \( i \in I \), we define \( \mathcal{F}^i = \{ \tau_i(C) | C \in Y^C \cap \mathcal{M}_i \} \). From \( \{1\} \) it follows that \( \mathcal{F}^i = \{ \tau_i(D) | D \in Y^D \cap \mathcal{M}_i \} \). (Note that, even for an increasing \( \mathcal{F} \) it might happen that \( \mathcal{F}^i \) is not increasing.) For a subset \( \mathcal{J} \) of \( 2^U \), we define \( \overline{\mathcal{J}} = \{ U \setminus J | J \in \mathcal{J} \} \).

Then

\[
\begin{align*}
(4) \quad |(\mathcal{M}_A \times \mathcal{M}_B) \cap X_i| \\
&= |\{(C, D) \in Y^C \times Y^D | \tau_i(C) \in \mathcal{A}^i, \tau_i(D) \in \mathcal{B}^i, \tau_i(C) = U \setminus \tau_i(D)\}| \\
&= |\{(A, B) \in 2^U \times 2^U | A \in \mathcal{A}^i, B \in \mathcal{B}^i, A = U \setminus B\}| \\
&= |\mathcal{A}^i \cap \overline{\mathcal{B}}^i|.
\end{align*}
\]

Similarly,

\[
(5) \quad |(\mathcal{M}_{A \Box B} \times \mathcal{M}) \cap X_i| = |(\mathcal{A} \Box \mathcal{B})^i|.
\]

**Lemma 8.** We have \((A \Box B)^i \subseteq A^i \Box B^i\).

**Proof.** Let \( F \in (A \Box B)^i \), then \( F = \tau_i(C) \) for some \( C \in Y^C \) such that \( B(C) \in A \Box B \). Since \( A \) and \( B \) are upward closed, there are disjoint sets \( V_A \in A \) and \( V_B \in B \) such that \( B(C) = V_A \cup V_B \). We define

\[
U_A = \{ v_{j,1}, \{ v_{j,0}, v_{j,1} \} \cap V_A \neq \emptyset, j \in \{1, 2, \ldots, k\} \}
\]

and

\[
U_B = \{ v_{j,1}, \{ v_{j,0}, v_{j,1} \} \cap V_B \neq \emptyset, j \in \{1, 2, \ldots, k\} \}.
\]

Since \( V_A \) and \( V_B \) are disjoint and \( |B(C) \cap \{ v_{j,0}, v_{j,1} \}| = 1 \) for all \( j \), we obtain that \( U_A \) and \( U_B \) are disjoint.

Moreover, if for some \( C' \in Y^C \), we have \( \tau_i(C) \cap U_A = \tau_i(C') \cap U_A \), then \( V_A \subseteq B(C') \), consequently \( B(C') \in A \) and \( \tau_i(C') \in \mathcal{A}^i \). The analogous statement is true for \( V_B \) and \( U_B \). Therefore, the pair \( U_A, U_B \) witnesses that \( F = \tau_i(C) \in A^i \Box B^i \). \(\square\)

Recall the following theorem of Reimer \([1]\). See also \([3]\).

**Theorem 9 (Reimer).** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be subsets of \( 2^U \), where \( U \) is a finite set. Then

\[ |\mathcal{X} \Box \mathcal{Y}| \leq |\mathcal{X} \cap \mathcal{Y}|. \]

Combining Theorem \([9]\) with Equations \([11]\) and \([5]\) and Lemma \([8]\), we obtain that

\[ |(\mathcal{M}_{A \Box B} \times \mathcal{M}) \cap X_i| = |(\mathcal{A} \Box \mathcal{B})^i| \leq |A^i \Box B^i| \leq |A^i \cap \overline{B}^i| = |(\mathcal{M}_A \times \mathcal{M}_B) \cap X_i|. \]

This proves Inequality \([3]\).

2.3. The proof of Theorem \([3]\) Consider an \( i \in I \). Observe that \( w(C) \cdot w(D) \) is the same for any \( (C, D) \in X_i \). Thus, it is again enough to prove Inequality \([3]\), so the whole proof goes through.
2.4. The proof of Theorem \[4\] We define
\[\mathcal{M}' = \{M \in \mathcal{M} \mid V_+ \subseteq B(M), V_- \cap B(M) = \emptyset\}\].
Recall that for \(i = (W, K, L, R) \in I\), we defined \(H_i\) as the endpoints of the paths in \(R\), and \(B_i\) as \(B_i = ((W \cup V(R)) \Delta S) \setminus H_i\). Now we define
\[I' = \{i \in I \mid V_+ \subseteq B_i, V_- \cap (B_i \cup H_i) = \emptyset\}\].
Using the following lemma, the proof of Theorem \[1\] can be repeated again.

**Lemma 10.** The sets \((X_i)_{i \in I'}\) give a partition of \(\mathcal{M}' \times \mathcal{M}'\).

**Proof.** The proof is almost identical to that of Lemma \[7\] \(\square\)

2.5. The proof Corollary \[5\]. Let \(X_0 = X \setminus Y\) and \(Y_0 = Y \setminus X\). Clearly the events \(X_0 \subseteq B(M)\) and \(Y_0 \subseteq B(M)\) depend on disjoint sets. Theorem \[4\] gives us
\[P(X_0 \subseteq B(M) \mid X \cap Y \subseteq B(M))P(Y_0 \subseteq B(M) \mid X \cap Y \subseteq B(M)) \geq P(X_0 \subseteq B(M), Y_0 \subseteq B(M) \mid X \cap Y \subseteq B(M)),\]
and this is equivalent with the statement of the corollary.

2.6. The proof Theorem \[6\]. Let \(t > 0\), and set all the edge weights to be equal to \(t\). Let \(M_t\) be the corresponding random matching. By Theorem \[3\] if \(A\) and \(B\) are increasing events, then
\[P(B(M_t) \in A \Delta B) \leq P(B(M_t) \in A)P(B(M_t) \in B)\].
Observe that
\[\lim_{t \to \infty} P(B(M_t) \in A) = P(B(M) \in A), \quad \lim_{t \to \infty} P(B(M_t) \in B) = P(B(M) \in B)\]
and \(\lim_{t \to \infty} P(B(M_t) \in A \Delta B) = P(B(M) \in A \Delta B)\).
Thus, the statement follows.

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Central European University, Budapest and
Alfred Rényi Institute of Mathematics, Budapest
E-mail address: Meszaros_Andras@phd.ceu.edu