Deciphering and generalizing Demiański–Janis–Newman algorithm

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Abstract

Demiański have shown that the Janis–Newman algorithm can be generalized in order to include a NUT charge and another parameter \( c \), in addition to the angular momentum, in the case of vanishing cosmological constant. Moreover he proved that only a NUT charge can be added for non-vanishing cosmological constant. But despite the fact that he obtains the form of the coordinate transformation, he did not explain how to perform the complexification on the metric function, and the procedure does not follow directly from usual Janis–Newman rules. The goal of our paper is threefold: explaining the hidden assumptions of Demiański analysis, generalizing the computations to topological horizons (spherical and hyperbolic) and to charged solutions, and finally explaining how to perform the complexification of the function. These different results open the door to applications on (gauged) supergravity since they allow for a systematic application of the Demiański–Janis–Newman algorithm.

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1 Introduction

Black holes are important objects in any theory of gravity for the insight they provide into the quantum gravity realm. For this reason it is a key step, in any theory, to obtain all possible black holes solutions. Usual black holes in four dimensions are characterized by a set of conserved charges, such as the mass, the electric charge or the angular momentum. As the complexity of the solutions increases with the number of charges, it is interesting to obtain new solutions from existing ones, where new charges are added to the previous ones.

Such procedure is done through solution generating algorithm, and in this paper we will focus on the Janis–Newman (JN) algorithm [1–3]. Most of the applications of this algorithm have been dedicated to obtaining rotating solutions without cosmological constant from static ones [4–8], but it has been shown by Demiański (and partially by Newman) that other parameters can be added [9, 10], even in the presence of a cosmological constant. Among these new parameters figures the NUT charge. A major playground for this modified Demiański–Janis–Newman (DJN) algorithm would be (gauged) supergravity where many interesting solutions remain to be discovered.

The (D)JN algorithm relies on a complex coordinate transformation. The original formulation uses the Newman–Penrose tetrad formalism, but a simpler prescription that can be applied directly to the metric has been proposed by Giampieri [14, 15], and all our computations are done using this last approach.

In order to find the most general transformation Demiański starts with a metric ansatz (with one unknown metric) and two arbitrary \( \theta \)-dependent functions in the complex transformation. After applying the JN algorithm he is able to solve exactly Einstein equations and therefore to find the form of the allowed complex transformation. In particular he finds that they depend on three parameters (including angular momentum and NUT charge) for vanishing cosmological constant, and only on the NUT charge for non-vanishing cosmological constant. Moreover since the stationary metric functions are obtained from Einstein equations there is no ambiguity at all in the algorithm and this approach might be useful where we don’t know precisely all the rules of the algorithm (such as in higher dimensions [16]).

Demiański’s paper is short and results are extremely condensed. A first goal of our paper is to explain in more details how he performs his computations. In particular we uncover a hidden assumption on the form of the metric function. A generalization of this hypothesis leads to other possible solutions that may be interesting to explore (this also explains why his metric function is erroneous).

One of the obvious generalization is the inclusion of a gauge field which is needed to obtain (electrically) charged solutions. It appears that the analysis is left unchanged, the Maxwell equations being also integrable within our ansatz. This solution was already found in [12] but we demonstrate how to perform the full computation using the DJN algorithm,

\footnote{Demiański’s metric has been generalized in [11–13].}
having in mind the possible generalization to other cases. This computation has been made possible by the recent discovery of the way to apply the JN algorithm to gauge fields [15] (see also [3, 17] for other approaches).

Another major improvement of the DJN algorithm that results of our analysis is the generalization of all formula to topological horizons. In particular all existing formula can be straightforwardly generalized to the case of hyperbolic horizons, and we prove all formula by solving explicitly Einstein equations. Topological horizons are of particular interest in supergravity models since asymptotically AdS black holes can possess non-spherical horizons.

We also comment the group properties that some of the DJN transformations possess. This observation can be useful for chaining several transformations or to add parameters to solutions that already contain some of the parameters (for example adding a rotation to a solution that already contains a NUT charge).

A long standing issue of Demiański’s paper is that it is not obvious how to obtain the stationary function by complexifying the static function. If there is no way to obtain the function by complexification it would imply that the more general transformation are useless because they could not be generalized to other cases (except if one is willing to solve Einstein equations, which is not the goal of a solution generating technique). We demonstrate that the complexification can be achieved by a complexification of the mass

\[ \Lambda = 0 : \quad m \rightarrow m + in, \quad \Lambda \neq 0 : \quad m \rightarrow m + in \left( 1 - \frac{4\Lambda}{3} n^2 \right), \tag{1.1} \]

\[ n \text{ being the NUT charge, establishing that Demiański’s transformations can be interpreted as an extension of the usual JN algorithm. Such complex combination is quite natural from the point of view of Plebański–Demiański solution [18, 19]. Complexification of parameters in the context of a solution generating technique was also done by Quevedo [20, 21]. A result (often quoted) from Demiański’s analysis is the impossibility to find Kerr–AdS from the DJN algorithm and it is often quoted as a no-go theorem. But this outcome relies on the assumption that no parameter already present in the static metric is complexified, which may not be justified.} \]

In most of the paper we focus on metrics with only one unknown function since Einstein–Maxwell equations are the most easily solvable in this case. Looking at more general cases, it appears that the same terms are present for the functions of the complex transformation, and we claim that these transformations will be the same in general (this is also motivated by the solution [12, 13]).

We end the introduction by describing our ansatz. We consider the most general static metric for which \((\theta, \phi)\)-section are 2-dimensional maximally symmetric spaces (it can be the sphere \(S^2\) or the hyperboloid \(H^2\), the extension to the plane \(R^2\) being easy) and with only radial functions. Similarly the gauge field contains only one unknown radial function and it is purely electric. The DJN algorithm generates a stationary metric coupled to a gauge field for a total of six unknown functions (with only five being independent). We provide several formula in \((u, r)\) and \((t, r)\) coordinates that should be suitable for any application of the DJN algorithm. Similar formula for subcases have been obtained in [6, 22–24]. All these computations are gathered in a Mathematica file (available on demand) which includes the computations of Einstein–Maxwell equations. We insist on the fact that all these results can also be derived from the tetrad formalism.

The paper is organized as follows. In section 2 we set up our ansatz for the static metric and gauge field. Then in section 3 we apply the the DJN algorithm on the previous ansatz. In particular we explain how to generalize it to topological horizons. In section 4 we solve

Footnotes:

1We do not treat the case of flat horizon but this could be obtained from some easy reparametrization.

2We stress that at this stage these formula do not satisfy Einstein equations, they are just proxy to simplify later computations.
Einstein–Maxwell equations when there is one unknown function in the metric. In section 5 we make a brief comment on the group formed by some of the JN transformations. Finally in 6 we explain how to perform the complexification of the metric function in order to recover the results obtained from Einstein–Maxwell equations. We provide two appendices. In appendix A we recall original Demiański’s solution, while in appendix B we generalize the hidden assumption made in Demiański’s paper.

2 Setting up the ansatz

Einstein–Maxwell gravity with cosmological constant $\Lambda$ reads [25, chap. 22]

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4} F^2 \right),$$

where $\kappa^2 = 8\pi G$ is the Einstein coupling constant, $g$ is the metric with Ricci scalar $R$ and $F = dA$ is the field strength of the Maxwell field. In the rest of this paper we will set $\kappa = 1$. Spacetime signature is mostly plus.

The associated equations of motion are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 2T_{\mu\nu}, \quad \nabla_\mu F^{\mu\nu} = 0,$$

where the stress–energy tensor for the electromagnetic field is

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^\rho - \frac{1}{4} g_{\mu\nu} F^2.$$

The static electromagnetic one-form is taken to be

$$A(r) = f_A(r) \, dt.$$ (2.4)

This ansatz is purely electric since only the time component is non-zero.

The static metric ansatz in coordinates $(t, r, \theta, \phi)$ reads

$$ds^2 = -f_t(r) \, dt^2 + f_r(r) \, dr^2 + f_\Omega(r) \, d\Omega^2.$$ (2.5)

One of the functions is redundant since we are free to redefine the radial coordinate. The $(\theta, \phi)$ sections correspond to 2-dimensional maximally symmetric spaces, which are the sphere $S^2$, the euclidean plane $\mathbb{R}^2$ and the hyperboloid $H^2$ respectively for positive, vanishing and negative curvature [26]. Defining $\kappa$ as the sign of the surface curvature, the uniform metric $d\Omega^2$ is given by

$$d\Omega^2 = d\theta^2 + H(\theta)^2 \, d\phi^2$$

with

$$H(\theta) = \begin{cases} 
\sin \theta & \kappa = 1, \\
1 & \kappa = 0, \\
\sinh \theta & \kappa = -1. 
\end{cases}$$ (2.7)

In the rest of the paper we focus on $\kappa = \pm 1$. In this case $H(\theta)$ may be defined by the differential equation

$$H'' + \kappa H = 0, \quad H(0) = 0, \quad H'(0) = 1,$$

and it satisfies trigonometric-like identities, for example

$$H'^2 + \kappa H^2 = 1.$$ (2.9)
Introducing the null coordinates $u$ through the change of coordinates

$$dt = du + \sqrt{\frac{fr}{ft}} dr,$$

(2.10)

the static metric (2.5) becomes

$$ds^2 = -ft du^2 - 2\sqrt{ftfr} dudr + f_{\Omega} (d\theta^2 + H^2 d\phi^2),$$

(2.11)

while the gauge field (2.4) is found to be

$$A = f_A \left( du + \sqrt{\frac{fr}{ft}} dr \right).$$

(2.12)

Since the component $A_r$ depends only on $r$ it can be removed by a gauge transformation [15] such that

$$A = f_A du.$$

(2.13)

### 3 Generalized Janis–Newman algorithm

In this section we apply the Janis–Newman algorithm to the ansatz of the previous section. Using arbitrary functions for the complex transformation and for the functions inside the metric, we obtain a very general ansatz; then we will solve Einstein–Maxwell equations in the next section in order to find their forms. We will directly use Giampieri’s prescription [14, 15] in order to avoid the introduction of tetrads and the computation of the contravariant components of the metric and of the gauge field. Reviews and simpler applications can be found in [3, 6, 15, 22, sec. 5.4].

#### 3.1 Janis–Newman transformation

The Janis–Newman algorithm can be summarized as the following sequence of steps:

1. Start with a seed metric in $(u, r)$ coordinates.
2. Let the coordinates $u$ and $r$ becoming complex.
3. Replace the functions inside the metric by other functions depending on $r$ and its conjugate.
4. Make a change of coordinates $(r, u) \rightarrow (r', u')$, the new coordinates being real.
5. Apply Giampieri’s ansatz to recover a real metric.

The complex change of coordinates is given by

$$r = r' + i F(\theta), \quad u = u' + i G(\theta).$$

(3.1)

where $u', r' \in \mathbb{R}$, and $F(\theta)$ and $G(\theta)$ are two arbitrary functions

$$F(\theta) = -a \cos \theta, \quad G(\theta) = a \cos \theta,$$

(3.2)

Similar transformations have been studied by Talbot [27].

In his paper [10] Demiański considers functions that depend on $\theta$ and $\phi$, but he drops the $\phi$-dependence at an intermediate step; for this reason we will ignore it.
but here they are kept general and the most general transformation will be determined by Einstein equations.

As given by
\[ dr = dr' + i F'(\theta) d\theta, \quad du = du' + i G'(\theta) d\theta \] (3.3)
(the prime on \( F \) and \( G \) denoting the differentiation with respect to \( \theta \)), the differentials of the coordinates are complex which is not coherent with having a complex metric. The (generalized) Giampieri’s ansatz consists in the replacement
\[ i d\theta = H(\theta) d\phi. \] (3.4)

It is possible to show [14, 15, 28] that this ansatz is justified by comparison with the tetrad formalism. As a consequence the transformation of the differentials are
\[ dr = dr' + F'(\theta) H(\theta) d\phi, \quad du = du' + G'(\theta) H(\theta) d\phi. \] (3.5)

Finally the four functions
\[ f_i = \{f_t, f_r, f_\Omega, f_A\} \] (3.6)
are transformed to
\[ \tilde{f}_i = \{\tilde{f}_t, \tilde{f}_r, \tilde{f}_\Omega, \tilde{f}_A\}. \] (3.7)

There are only two conditions that we impose on these functions
\[ \tilde{f}_i = \tilde{f}_i(r, F(\theta)), \quad \tilde{f}_i(r, 0) = f_i(r). \] (3.8)

The first relation means that the dependence in \( \theta \) is solely contained in the functional dependence of \( F(\theta) \). On the other hand we do not try to get the functions \( \tilde{f}_i \) from the complexification of the static functions [10].

As a consequence the \( \theta \)-derivative of \( \tilde{f}_i \) reads
\[ \partial_\theta \tilde{f}_i = F' \partial_F \tilde{f}_i, \] (3.9)
such that it is sufficient to obtain the dependence of \( \tilde{f}_i \) in terms of \( F \).

### 3.2 Metric

Applying the transformations (3.1) and (3.5) and replacing the functions, the resulting stationary metric in Eddington–Finkelstein coordinates is
\[ ds^2 = -\tilde{f}_i (du + \alpha dr + \omega H d\phi)^2 + 2\beta dr d\phi + \tilde{f}_\Omega (d\theta^2 + \sigma^2 H^2 d\phi^2) \] (3.10)
where we defined the quantities
\[ \omega = G' + \sqrt{\frac{f_r}{f_t}} F', \quad \sigma^2 = 1 + \frac{f_r}{f_\Omega} F'^2, \quad \alpha = \sqrt{\frac{f_r}{f_t}}, \quad \beta = \tilde{f}_r F' H. \] (3.11)

The transformation
\[ du = dt - g(r) dr, \quad d\phi = d\phi' - h(r) dr. \] (3.12)
can be used to set the coefficient \( g_{rr} \) and \( g_{r\phi} \) to zero and to cast the metric in Boyer–Lindquist (BL) coordinates. The solution to these two conditions is
\[ g(r) = \frac{\sqrt{(f_t f_r)^{-1} f_\Omega - F' G'}}{\Delta}, \quad h(r) = \frac{F'}{H(\theta) \Delta}. \] (3.13)

---

6This assumption is not explicit in Demiański’s paper [10].
with
\[ \Delta = \frac{\tilde{f}_\Omega}{f_r} + F'^2 = \frac{\tilde{f}_\Omega}{f_r} \sigma^2. \]  
(3.14)

We stress that the functions \( g \) and \( h \) cannot depend on \( \theta \), otherwise the change of variables (3.12) is not integrable. It is thus necessary to check for given functions \( \tilde{f}_i, F \) and \( G \) that all the \( \theta \)-dependence cancels.

Finally the metric in \((t, r)\) coordinates can be written
\[ ds^2 = -\tilde{f}_t (du + \omega H d\phi)^2 + \frac{\tilde{f}_\Omega}{\Delta} dr^2 + \tilde{f}_\Omega (d\theta^2 + \sigma^2 H^2 d\phi^2). \]  
(3.15)

### 3.3 Gauge field

Applying the DJN transformations (3.5) to the gauge field (2.13)
\[ A = f_A du \]  
(3.16)
gives \(^7\)
\[ A = \tilde{f}_A (du + G'H d\phi) \]  
(3.17)

Using the explicit formula (3.13), the previous expression becomes in Boyer–Lindquist coordinates
\[ A = \tilde{f}_A \left( dt - \frac{\tilde{f}_\Omega}{\sqrt{\tilde{f}_t f_r}} \Delta dr + G'H d\phi \right). \]  
(3.18)

Here the function
\[ A_r = -\frac{\tilde{f}_A \tilde{f}_\Omega}{\sqrt{\tilde{f}_t f_r}} \]  
(3.19)
may depend on \( \theta \) in which case it would not be possible to remove it by a gauge transformation \(^8\).

### 4 Charged topological Demiański’s solution

In this section we solve Einstein–Maxwell equations (2.2) for the system
\[ f_t = f, \quad f_r = f^{-1}, \quad f_\Omega = r^2. \]  
(4.1)

First the static solution is recalled for later comparison – it corresponds to the static limit of the stationary solution. The stationary solution is derived in \((u, r)\) coordinates in order to avoid the question of the validity of the Boyer–Lindquist transformation.

#### 4.1 Static case

Consider the static metric (2.5) and gauge field (2.4).

Only the \((t)\) component of Maxwell equations is non trivial
\[ 2f'_A + r f''_A = 0, \]  
(4.2)
and its solution is
\[ f_A(r) = \frac{q}{r}. \]  
(4.3)

\(^7\)This may also be derived from the tetrad formalism [3, 15, 17].

\(^8\)In several examples where BL coordinates exist, \( A_r \) depends only on \( r \). This seems to be the generic case.
where $q$ is a constant of integration that is interpreted as the charge (we set the additional constant to zero since it can be removed by a gauge transformation).

The only relevant Einstein equation is
\[ \frac{q^2}{r^2} - \kappa + r^2 \Lambda + f + rf' = 0 \] (4.4)
whose solution reads
\[ f(r) = \kappa - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda}{3} r^2, \] (4.5)
m being a constant of integration that is identified to the mass.

We stress that we are just looking to solutions of Einstein equations and we are not concerned with regularity (in particular it is well-known that only $\kappa = 1$ is well-defined for $\Lambda = 0$).

### 4.2 Stationary case
#### 4.2.1 Simplifying the equations
The component $(r\theta)$ gives the equation
\[ F' \left( G'' + \frac{H'}{H} G' \right) = 2FF' \] (4.6)
which depends only on $\theta$ and it allows to solve for $G$ in terms of $F$. If $F' \neq 0$ it implies the equation $(rr)$ which is
\[ G'' + \frac{H'}{H} G' = \pm 2F. \] (4.7)
If $F' = 0$ this last equation should be used instead and the sign can be absorbed into $F$ since it is an arbitrary constant. As a result the equation in both cases is
\[ G'' + \frac{H'}{H} G' = 2F. \] (4.8)

The $r$-component of the Maxwell equation can be integrated to
\[ \tilde{f} A = \frac{qr}{r^2 + F^2} + \alpha \frac{r^2 - F^2}{r^2 + F^2}. \] (4.9)
We can remove the constant $\alpha$ by matching with the static case in the limit $F \to 0$, but we can also get this result from the $\theta$-equation
\[ \alpha F' = 0. \] (4.10)

The $(tr)$ equation contains only $r$-derivative of $\tilde{f}$ and it can be integrated to\footnote{In [10] the last term of $\tilde{f}$ is missing [20], as can be compared with other references on (A)dS–Taub–NUT, see for example [26].}
\[ \tilde{f} = \kappa - \frac{2mv - q^2 + 2F(\kappa F + K)}{r^2 + F^2} - \frac{\Lambda}{3} (r^2 + F^2) - \frac{4\Lambda}{3} F^2 + \frac{8\Lambda}{3} \frac{F^4}{r^2 + F^2} \] (4.11)
where again $m$ is a constant of integration interpreted as the mass. The function $K$ is defined by
\[ 2K = F'' + \frac{H'}{H} F'. \] (4.12)
This implies the equations \((r\phi)\) and \((\theta\theta)\).

As explained in section 3.1 the \(\theta\)-dependence should be contain in \(F(\theta)\) only. The second term of the function \(\tilde{f}\) contains some lonely \(\theta\) from the \(H(\theta)\) in the function \(K\): this means that they should be compensated by the \(F\), and we therefore ask that the sum \(\kappa F + K\) be constant\(^\text{10}\)

\[
\kappa F' + K' = 0 \implies \kappa F + K = \kappa n. \tag{4.13}
\]

The parameter \(n\) is interpreted as the NUT charge.

The components \((t\theta)\) and \((\theta\phi)\) give the same equation

\[
\Lambda F' = 0. \tag{4.14}
\]

Finally one can check that the last three equations \((tt), (t\phi)\) and \((\phi\phi)\) are satisfied.

Let’s summarize the equations

\[
2F = G'' + \frac{H'}{H} G', \tag{4.15a}
\]

\[
\kappa n = \kappa F + K, \tag{4.15b}
\]

\[
0 = \Lambda F' \tag{4.15c}
\]

and the function \(\tilde{f}\)

\[
\tilde{f} = \kappa - \frac{2mr - q^2 + 2F(\kappa F + K)}{r^2 + F^2} = \frac{\Lambda}{3} (r^2 + F^2) - \frac{4\Lambda}{3} F^2 + \frac{8\Lambda}{3} \frac{F^4}{r^2 + F^2}. \tag{4.15d}
\]

We also defined

\[
2K = F'' + \frac{H'}{H} F'. \tag{4.15e}
\]

As explained in the introduction, a major issue of \(\)Demiański\(')s approach is the impossibility to obtain – at least in a direct manner – the stationary \(\tilde{f}\) function (4.15d) as a complexification of the static \(f\) function (4.5). Not being able to reproduce the stationary function from the static one is equivalent to a failure because it would not be possible to apply the algorithm to other cases. This is one of the reason explaining why applications of the JN algorithm have been limited to adding a rotation parameter. We address this question in section 6 and show how to recover \(\tilde{f}\) from \(f\).

The case \(\Lambda = 0\) and \(\Lambda \neq 0\) are really different and we consider them separately.

### 4.2.2 Solution for \(\Lambda \neq 0\)

Equation (4.15c) implies that \(F' = 0\) and then

\[
F(\theta) = n \tag{4.16}
\]

by compatibility with (4.15b) and since \(K(\theta) = 0\).

Solution to (4.15a) is

\[
G(\theta) = c_1 - 2\kappa n \ln H(\theta) + c_2 \ln \frac{H(\theta/2)}{H'(\theta/2)} \tag{4.17}
\]

where \(c_1\) and \(c_2\) are two constants of integration. Since only \(G'\) appears in the metric we can set \(c_1 = 0\). On the other hand the constant \(c_2\) can be removed by the transformation

\[
du = du' - c_2 \, d\phi. \tag{4.18}
\]

\(^{10}\)In appendix B we relax this last assumption by allowing non-constant \(\kappa F + K\). In this context the equations and the function \(f\) are modified and this provides an explanation for the error in \(f\) of Demiański’s paper [10].
We summarize the solution to the system (4.15)

\[ F(\theta) = n, \quad G(\theta) = -2\kappa n \ln H(\theta). \] (4.19)

The function \( \tilde{f} \) then takes the form

\[ \tilde{f} = \kappa - \frac{2mr - q^2 + 2\kappa n^2}{r^2 + n^2} - \frac{\Lambda}{3} \left(r^2 + 5n^2\right) + \frac{8\Lambda}{3} \frac{n^4}{r^2 + n^2} \] (4.20a)

\[ = \kappa - \frac{2mr - q^2 + 2\kappa n^2}{r^2 + n^2} - \frac{\Lambda}{3} \frac{r^4 + 6n^2 - 3a^4}{r^2 + n^2}. \] (4.20b)

The transformation to BL coordinates is well defined (and \( h = 0 \))

\[ g = \frac{r^2 + n^2}{\Delta}, \quad \Delta = \kappa r^2 - 2mr + q^2 + \Lambda n^2 - \frac{\Lambda}{3} r^4 - n^2(\kappa + 2\Lambda r^2). \] (4.21)

As noted by Demiański the only parameters that appear are the mass and the NUT charge, and it is not possible to add an angular momentum for non-vanishing cosmological constant \( \Lambda \). As a consequence the JN algorithm cannot provide a derivation of (A)dS–Kerr–Newman.

### 4.2.3 Solution for \( \Lambda = 0 \)

The solution to the differential equation (4.15b) is

\[ F(\theta) = n - a H'(\theta) + \kappa c \left(1 + H'(\theta) \ln \frac{H(\theta/2)}{H'(\theta/2)}\right), \] (4.22)

where \( a \) and \( c \) denote two constants of integration.

We solve the equation (4.15a) for \( G \)

\[ G(\theta) = c_1 + \kappa a H'(\theta) - c H'(\theta) \ln \frac{H(\theta/2)}{H'(\theta/2)} - 2\kappa n \ln H(\theta) \]

\[ + (a + c_2) \ln \frac{H(\theta/2)}{H'(\theta/2)} \] (4.23)

and \( c_1, c_2 \) are constants of integration. Again since only \( G' \) appears in the metric we can set \( c_1 = 0 \). We can also remove the last term with the transformation

\[ du = du' - (c_2 + a)d\phi. \] (4.24)

We arrive at

\[ F(\theta) = n - a H'(\theta) + \kappa c \left(1 + H'(\theta) \ln \frac{H(\theta/2)}{H'(\theta/2)}\right), \] (4.25a)

\[ G(\theta) = \kappa a H'(\theta) - \kappa c H'(\theta) \ln \frac{H(\theta/2)}{H'(\theta/2)} - 2\kappa n \ln H(\theta). \] (4.25b)

The full expression for \( \tilde{f} \) is not very illuminating and we do not give it.

The Boyer–Lindquist transformation is well defined only for \( c = 0 \), in which case

\[ g = \frac{r^2 + a^2 + n^2}{\Delta}, \quad h = \frac{\kappa a}{\Delta}, \quad \Delta = \kappa r^2 - 2mr + q^2 - \kappa n^2 + \kappa a^2. \] (4.26)

\( ^{11} \)In [29] Leigh et al. generalized Geroch’s solution generating technique and also found that only the mass and the NUT charge appear when \( \Lambda \neq 0 \). We would like to thank D. Klemm for this remark.
The function $\tilde{f}$ reads $[26, \text{sec. 2.2}]
\begin{align*}
\tilde{f} &= \kappa - \frac{2mr - q^2}{\rho^2} + \frac{\kappa n(aH')}{\rho^2}, \\
\rho^2 &= r^2 + (n - aH')^2.
\end{align*}
(4.27)

The constant $a$ corresponds to the angular momentum (and one recognizes the usual JN algorithm), while $c$ is not easy to interpret $[3, 30, \text{sec. 5.3}].$

This solution was already found in $[12]$ for the case $\kappa = 1$ by solving directly Einstein–Maxwell equations, starting with a metric ansatz of Demiański’s form. In our case we wish to show that the same solution can be obtained by applying Demiański’s method on all the quantities, including the gauge field.

## 5 Group properties

In this section we want to show that (some of) DJN transformations form a group.

After a first transformation
\begin{align*}
r &= r' + iF_1, \quad u = u' + iG_1
\end{align*}
(5.1)
one obtains the metric
\begin{align*}
ds^2 &= -\tilde{f}_1^{(1)}(du + HG_1' d\phi)^2 + \tilde{f}_1^{(1)}(d\theta^2 + H^2 d\phi^2) \\
&\quad - 2\sqrt{\tilde{f}_1^{(1)} \tilde{r}_1^{(1)}} (du + G_1' H d\phi)(dr + F_1' H d\phi)
\end{align*}
(5.2)
where
\begin{align*}
\tilde{f}_1^{(1)} &= \tilde{f}_1^{(1)}(r, F_1).
\end{align*}
(5.3)

Applying a second transformation
\begin{align*}
r &= r' + iF_2, \quad u = u' + iG_2
\end{align*}
(5.4)
the previous metric becomes
\begin{align*}
ds^2 &= -\tilde{f}_1^{(1,2)}(du + HG_1' + G_2' d\phi)^2 + \tilde{f}_1^{(1,2)}(d\theta^2 + H^2 d\phi^2) \\
&\quad - 2\sqrt{\tilde{f}_1^{(1,2)} \tilde{f}_2^{(1,2)}} (du + (G_1' + G_2')H d\phi)(dr + (F_1' + F_2')H d\phi)
\end{align*}
(5.5)
where this time
\begin{align*}
\tilde{f}_1^{(1,2)} &= \tilde{f}_1^{(1,2)}(r, F_1, F_2).
\end{align*}
(5.6)
As for the first transformation we only ask for the following conditions
\begin{align*}
\tilde{f}_1^{(1,2)}(r, F_1, 0) &= \tilde{f}_1^{(1)}(r, F_1), \\
\tilde{f}_1^{(1,2)}(r, F_1, F_2) &= \tilde{f}_2^{(1,1)}(r, F_2, F_1).
\end{align*}
(5.7)

In one word a zero transformation should just give back the old metric, and the two transformations should commute.

Looking at the expression of the metric, it is obvious that the DJN transformations which are such that
\begin{align*}
\tilde{f}_1^{(1,2)}(r, F_1, F_2) &= \tilde{f}_1^{(1)}(r, F_1 + F_2)
\end{align*}
(5.8)
form an (Abelian) group if the functions $F$ and $G$ are linear in the parameters (i.e. the group is additive). This last condition means that we can decompose them on a basis of generators $\{F_A(\theta)\}$ and $\{G_M(\theta)\}$, where $A$ and $M$ are (different) indices, such that
\begin{align*}
F(\theta) &= f^A F_A(\theta), \\
G(\theta) &= g^M G_M(\theta),
\end{align*}
(5.9)
\( f^A \) and \( g^M \) being the parameters of the transformations. It is possible that \( f^A = g^M \) and \( F_A \propto G_M \) for some \( A \) and \( M \) (as we obtained in section 4) which means that the corresponding parameters \( f^A \) and \( g^M \) are not independent.

These transformations form a group because composing two transformations \((F_1, G_1)\) and \((F_2, G_2)\) gives a third transformation \((F_3, G_3)\) according to
\[
F_3 = F_1 + F_2, \quad G_3 = G_1 + G_2
\] (5.10)
with the parameters combining linearly. Moreover there an identity \((0, 0)\) and also an inverse \((-F, -G)\).

All this structure implies that we can first add one parameter, and later another (say first the NUT charge, and then an angular momentum). Said another way this group preserves Einstein equations when the seed metric is a known (stationary) solution. But note that it may be very difficult to do it as soon as one begins to replace the \( F \) in the functions by their expression, because it obscures the original function – in one word we can not find \( \tilde{f}_i(r, F) \) from \( \tilde{f}_i(r, \theta) \).

Another point worth to mention is that not all DJN transformation are in this group since it may happen that the condition (5.7) is not satisfied. Such an example in provided in 5d where the function \( f_\Omega(r) = r^2 \) is successively transformed as [16]
\[
r^2 \rightarrow |r|^2 = r^2 + a^2 \cos^2 \theta \rightarrow |r|^2 + a^2 \cos^2 \theta = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta,
\] (5.11)
where the two transformations have been
\[
F_1 = a \cos \theta, \quad F_2 = b \sin \theta,
\] (5.12)
and
\[
\tilde{f}_\Omega^{(1,2)} = r^2 + F_1^2 + F_2^2.
\] (5.13)
The condition (5.7) is clearly not satisfied. These group properties may explain why the JN algorithm is not working for \( d > 5 \), or maybe give a clue to solve this problem.

## 6 Finding the complexification

At the end of section 4.2.1, we mentioned the issue of finding the complexification of the stationary function from the static one. This is a key step if one wishes to apply the algorithm – with the new parameters \( n \) and \( c \) – in other cases. It appears that this procedure requires also the complexification of the mass parameter
\[
m = m' + in.
\] (6.1)
As strange as this may appear we found that this property is shared by other systems that will be discussed in a future work.

In what follows we ignore the electric charge since it does not modify the discussion.

### 6.1 \( \Lambda = 0 \)

The static Schwarzschild function (4.5)
\[
f = \kappa - \frac{2m}{r}
\] (6.2)
is complexified as
\[
\tilde{f} = \kappa - \left( \frac{m}{r} + \frac{\overline{m}}{\overline{r}} \right) = \kappa - \frac{2 \text{Re}(m\overline{r})}{|r|^2}.
\] (6.3)
Performing the transformation
\[ m = m' + i\kappa n, \quad r = r' + iF \] (6.4)
gives
\[ \tilde{f} = \kappa - \frac{2mr + 2\kappa nF}{r^2 + F^2} \] (6.5)
which corresponds to the correct function (4.15d).

6.2 \( \Lambda \neq 0 \)

The procedure is less straightforward in this case and we only give some preliminary steps towards the solution.

The static Schwarzschild function (4.5)
\[ f = \kappa - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \] (6.6)
is complexified as
\[ \tilde{f} = \kappa - \frac{2 \text{Re}(m\bar{r})}{|r|^2} - \frac{\Lambda}{3} |r|^2. \] (6.7)

The complexification of the mass parameter is \(^{12}\)
\[ m = m' + in \left( \kappa - \frac{4\Lambda}{3} n^2 \right), \quad r = r' + in. \] (6.8)

Moreover comparing the imaginary part of \(m\) with the previous case (6.4) suggests the replacement of the curvature sign \(^{13}\) (only in the one appearing in \(f\), not the one in (6.8))
\[ \kappa \rightarrow \kappa - \frac{4\Lambda}{3} n^2. \] (6.9)

Presented in another way, the algorithm is to first perform the transformation (6.4) followed by the above replacement for \(\kappa\) everywhere
\[ m = m' + i\kappa n, \quad \kappa \rightarrow \kappa - \frac{4\Lambda}{3} n^2. \] (6.10)

One can notice that the limit \(\Lambda \rightarrow 0\) agrees with the previous section (upon replacing \(n\) by \(F\)).

Inserting these transformations into \(\tilde{f}\) gives the result
\[ \tilde{f} = \kappa - \frac{2mr + 2\kappa n^2}{r^2 + n^2} - \frac{\Lambda}{3} \left( r^2 + 5n^2 \right) + \frac{8\Lambda}{3} \frac{n^4}{r^2 + n^2} \] (6.11)
and we retrieve (4.20).

---

\(^{12}\)The imaginary part of the new mass term appears in other contexts [18, 21, 31, 32]. In particular this corresponds to a condition of regularity in Euclidean signature.

\(^{13}\)Notice that AdS–Taub–NUT (for \(\kappa = -1, m = 0\)) is supersymmetric for \(n = \pm 1/(2g)\) where \(g^2 = -\Lambda/3\) [26, tab. 1].
7 Conclusion and discussion

In this paper we generalize Demiański analysis of the JN algorithm. Our main result consists in the proof that Demiański’s solution [10] can be recovered by a direct application of the JN prescription, and not only by solving Einstein equations. As a consequence it becomes possible to perform Demiański’s transformations on other seed metrics with and without cosmological constants.

Furthermore we showed that this solution can be generalized to include an electric charge and also to possess a topological horizon. Hence the final metric contains (for vanishing cosmological constant) four of the six Plebański–Demiański parameters [19] along with Demiański’s parameter. Having complex parameters is a key step for adding a magnetic charge and for getting supergravity solutions, and we reserve this for later work. It is intriguing that one could get all Plebański–Demiański parameters but the acceleration.

Demiański derived his transformation by solving Einstein equations in the case where there is only one unknown function. We were not able to solve analytically Einstein–Maxwell equations for a greater number of unknown functions, but looking at the equations indicate that the same structure persists in these more general cases. Therefore we claim that the transformations obtained in section 4 are the most general ones under the assumptions of our paper.

As it is necessary to shift the curvature sign of the 2-dimensional \((\theta, \phi)\) space, a guess would be to allow a similar shift of the cosmological constant in order to get more general solutions (in particular for non-vanishing cosmological constant).

Another advantage of Demiański’s approach resides in the fact that it can be used to find the form of the transformation where there are ambiguities. In particular this approach could be used to generalize the results of [16].

Finally Drake and Szekeres obtained general results on the solutions that can be derived from the original JN algorithm [6]. In their work they invert the Boyer–Lindquist equations in order to get the metric functions in terms of the BL functions, which are then obtained from Einstein equations \(^{14}\). This is another way to bypass the complexification and obtain general information on its properties, and it would be very interesting to extend their analysis with the more general Demiański’s transformation.

Even if in practice this kind of solution generating technique does not provide so many new solutions, it can help to understand better the underlying theory (which can be general relativity, modified gravities or even supergravity) [29] and understanding better the DJN algorithm may shed light on the structure of gravitational solutions.

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\(^{14}\)This idea has been generalized in [24] to include a third function.
A Original Demiański’s solution

In this section we recall the Demiański’s result, amended to fix some errors. They follow from section 4 with $\kappa = 1$ and $q = 0$. This gives also the opportunity to present prettier formulas.

The equations are

\[\begin{align*}
2F &= G'' + \cot \theta G', \\
n &= F + K, \\
0 &= \Lambda F'
\end{align*}\]  

(A.1a)  

(A.1b)  

(A.1c)

with

\[2K = F'' + \cot \theta F'.\]  

(A.2)

The function $\tilde{f}$ is

\[\tilde{f} = 1 - \frac{2mr - q^2 + 2F(F + K)}{r^2 + F^2} - \frac{\Lambda}{3} (r^2 + F^2) - \frac{4\Lambda}{3} F^2 + \frac{8\Lambda}{3} \frac{F^4}{r^2 + F^2}.\]  

(A.3)

The two solutions for $F$ and $G$ are

- $\Lambda \neq 0$
  \[F(\theta) = n, \quad G(\theta) = -2n \ln \sin(\theta).\]  
  (A.4)

- $\Lambda = 0$
  \[\begin{align*}
  F &= n - a \cos \theta + c \left(1 + \cos \theta \ln \tan \frac{\theta}{2}\right), \\
  G &= a \cos \theta - 2n \ln \sin \theta - c \cos \theta \ln \tan \frac{\theta}{2}
  \end{align*}\]  
  (A.5a)  
  (A.5b)

B Relaxing assumptions

In section 4.2.1 we obtained the equation (4.15b)

\[\kappa F + K = \kappa n, \quad 2K = F'' + \frac{H'}{H} F'\]  

(B.1)

by asking that the function (4.15d)

\[\tilde{f} = \kappa - \frac{2mr - q^2 + 2F(\kappa F + K)}{r^2 + F^2} - \frac{\Lambda}{3} (r^2 + F^2) - \frac{4\Lambda}{3} F^2 + \frac{8\Lambda}{3} \frac{F^4}{r^2 + F^2}\]  

(B.2)

depends on $\theta$ through $F(\theta)$.

A more general assumption would be that $\kappa F + K$ is some function $\chi$ of

\[\kappa F + K = \kappa \chi(F).\]  

(B.3)

The $(t\theta)$-component gives the equation

\[4\Lambda F^2 F' = F'' \partial_F \chi.\]  

(B.4)

If $F' = 0$ or $\Lambda = 0$ we found that

\[\partial_F \chi = 0 \implies \chi = n\]  

(B.5)
which reduces to the case studied in section 4.2.1.

On the other hand if $F' \neq 0$ then the previous equation becomes
\[
\partial_F \chi = 4\Lambda F^2
\]  
which can be integrated to
\[
\chi(F) = n + \frac{4}{3} \Lambda F^3
\] 
(notice that the limit $\Lambda \to 0$ is coherent). Plugging this function into equation (B.3) one obtains
\[
\kappa F + K = \kappa \left( n + \frac{4}{3} \Lambda F^3 \right). 
\] 
This differential equation is non-linear and we were not able to find an analytical solution.

Nonetheless by inserting the expression of $\chi$ in $\tilde{f}$ we see that the last term is killed
\[
\tilde{f} = \kappa - \frac{2mr - q^2 + 2\kappa nF}{r^2 + F^2} - \frac{\Lambda}{3} (r^2 + F^2) - \frac{4\Lambda}{3} F^2.
\] 
One can recognize the function given by Demiański [10]. Then this function is valid at the condition that equation (4.15b) is modified to (B.8), but in this case the solution is not the general (A)dS–Taub–NUT anymore.

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