Research Article

Fractional moments

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ABSTRACT
We evaluate the moments of some functions composed with the fractional part of $1/x$. We call them fractional moments. In particular, we obtain expressions for the fractional moments of some trigonometric functions, the Bernoulli polynomials and the functions $x^m$ and $x^m(1-x)^m$.

1. Introduction

The main purpose of this paper is the evaluation of moments of some functions composed with the fractional part of $1/x$. In fact, we will analyze the integrals

$$I_k f = \int_0^1 x^k \left\{ \frac{1}{x} \right\}^m f \left( \left\{ \frac{1}{x} \right\} \right) \, dx, \quad k = 0, 1, 2, \ldots$$

We call these integrals the fractional moments of the function $f$.

Up to our knowledge, the particular case $f(x) = x^m$ appears in the literature in different references (see [1–6]). For example, in [3, Theorem 2.1] or [2, Problem 2.22] we can see the identity

$$\int_0^1 x^k \left\{ \frac{1}{x} \right\}^m \, dx = \frac{m!}{(k+1)!} \sum_{j=1}^{\infty} \frac{(k+j)!}{(m+j)!} (\zeta(k+j+1) - 1), \quad m, k \in \mathbb{Z}^+,$$

where $\zeta$ denotes the Riemann zeta function. Moreover, for the particular case $m = k + 1$, by using the identity

$$\sum_{j=1}^{\infty} \frac{\zeta(j+1) - 1}{j+1} = 1 - \gamma,$$

with $\gamma$ being the Euler–Mascheroni constant, it is proved that

$$\int_0^1 x^k \left\{ \frac{1}{x} \right\}^{k+1} \, dx = H_{k+1} - \gamma - \sum_{j=2}^{k+1} \frac{\zeta(j)}{j},$$

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where $H_n$ is the $n$th harmonic number. In this paper, we will make a systematic analysis of this kind of integrals and we will obtain appropriated closed forms for them.

The main tool to evaluate fractional moments will be an identity relating them with an integral involving the function $\log \Gamma(x + 1)$ and the derivatives of the function $f$ (see Lemma 2.1 in the next section). From this result, we deduce the fractional moments for some functions. In particular, we analyze the fractional moments for some trigonometric functions, the Bernoulli polynomials and the functions $x^m$ and $x^m(1 - x)^m$. The integrals containing $\log \Gamma(x + 1)$ will be evaluated by using some results in [7]. In that paper, we see that the integrals

$$\int_0^1 B_n(x) \log \Gamma(x) \, dx,$$

where $B_n$ are the Bernoulli polynomials, are simpler to evaluate than the integrals

$$\int_0^1 x^n \log \Gamma(x) \, dx.$$

In fact, the latter integrals are evaluated in terms of the former ones. By this reason, to analyze the fractional moments of $x^m(1 - x)^m$ we start giving an expansion for these functions and their derivatives in terms of the Bernoulli polynomials (see Lemma 6.2 and the remark following it). We believe that this result has its own interest.

The paper is organized as follows: in Section 2 we present the lemma allowing us to obtain the fractional moments and in the rest of the paper we give some examples and applications related to trigonometric functions, Bernoulli polynomials and the functions $x^m$ and $x^m(1 - x)^m$.

All along the paper when an empty sum appears it must be taken as zero.

## 2. The main lemma

The following lemma will be the main tool to evaluate fractional moments. First, we define the sequence $\alpha_n = \zeta(n + 1)$, for $n > 0$, and $\alpha_0 = \gamma$.

**Lemma 2.1:** Let $f$ be a function having $k + 2$ derivatives in the interval $[0, 1]$. Then

$$I_k f = \frac{1}{(k + 1)!} \left( \sum_{j=0}^{k} (k - j)! (f^{(j)}(0) \alpha_{k-j} - f^{(j)}(1)(\alpha_{k-j} - 1)) \right) + \int_0^1 f^{(k+2)}(x) \log \Gamma(x + 1) \, dx,$$

$k \geq 0$.

The polygamma function will play a crucial role in the proof of the previous lemma. It is defined as the $(m + 1)$th derivative of the logarithm of the gamma function

$$\psi^{(m)}(x) = \frac{d^{m+1}}{dx^{m+1}} \log \Gamma(x), \quad m = 0, 1, 2, \ldots$$
Two facts about the polygamma function will be fundamental. Its representation as a series
\[ \psi^{(m)}(x) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}}, \]  
(2.1)
and its values at the positive integers
\[ \psi^{(m)}(n) = (-1)^{m+1} m! \left( \zeta(m + 1) - \sum_{k=1}^{n-1} \frac{1}{k^{m+1}} \right) \]
\[ = (-1)^{m+1} m! \sum_{k=n}^{\infty} \frac{1}{k^{m+1}}, \quad m, n = 1, 2, \ldots, \]  
(2.2)
and
\[ \phi^{(0)}(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}, \quad n = 1, 2, \ldots. \]  
(2.3)

**Proof of Lemma 2.1:** With the change of variable \( w = 1/x \), we have
\[ I_k f = \int_1^\infty f\left(\frac{w}{w^{k+2}}\right) \frac{dw}{w^{k+2}} = \sum_{j=1}^{\infty} \int_j^{j+1} f(w) \frac{dw}{w^{k+2}} \]
\[ = \sum_{j=1}^{\infty} \int_0^1 f(s) \frac{ds}{(j+s)^{k+2}} = \frac{(-1)^k}{(k+1)!} \int_0^1 f(s) \psi^{(k+1)}(s+1) \, ds, \]
where in the last step we have used (2.1). Now, applying integration by parts \( k + 2 \) times and taking into account that \( \log \Gamma(2) = \log \Gamma(1) = 0 \), we arrive at
\[ I_k f = \frac{(-1)^k}{(k+1)!} \left( \sum_{j=0}^{k} (-1)^j (f^{(j)}(1) \psi^{(k-j)}(2) - f^{(j)}(0) \psi^{(k-j)}(1)) \right. \]
\[ + (-1)^{k+2} \int_0^1 f^{(k+2)}(s) \log \Gamma(s+1) \, ds \right). \]
Finally, we conclude the proof applying (2.2) and (2.3).

### 3. Fractional moments for trigonometric functions

We start our examples given the fractional moments for the sine and cosine functions. The expression that we obtain for them involve the classical functions sine integral
\[ \text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt \]
and cosine integral
\[ \text{Ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt. \]

The following lemma contains the evaluation of two integrals for \( \log \Gamma(x + 1) \) with trigonometric functions.

**Lemma 3.1:** The following identities hold
\[
\int_0^1 \sin(2\pi x) \log \Gamma(x + 1) \, dx = \frac{\text{Ci}(2\pi)}{2\pi}
\]
and
\[
\int_0^1 \cos(2\pi x) \log \Gamma(x + 1) \, dx = \frac{1}{4} - \frac{\text{Si}(2\pi)}{2\pi}.
\]

**Proof:** Taking \( n = 1 \) in [8, 6.443.1 and 6.443.3], we have
\[
\int_0^1 \sin(2\pi x) \log \Gamma(x) \, dx = \frac{\log 2\pi + \gamma}{2\pi}
\]
and
\[
\int_0^1 \cos(2\pi x) \log \Gamma(x) \, dx = \frac{1}{4}.
\]

Then
\[
\int_0^1 \sin(2\pi x) \log \Gamma(x + 1) \, dx = \frac{\log 2\pi + \gamma}{2\pi} + \int_0^1 \sin(2\pi x) \log x \, dx
\]
and
\[
\int_0^1 \cos(2\pi x) \log \Gamma(x + 1) \, dx = \frac{1}{4} + \int_0^1 \cos(2\pi x) \log x \, dx.
\]

Now, applying integration by parts and the identity [8, 8.230.2]
\[
\text{Ci}(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t} \, dt,
\]
we deduce
\[
\int_0^1 \sin(2\pi x) \log x \, dx = \frac{1}{2\pi} \int_0^1 \frac{\cos(2\pi x) - 1}{x} \, dx
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos t - 1}{t} \, dt = -\frac{\log 2\pi + \gamma}{2\pi} + \frac{\text{Ci}(2\pi)}{2\pi}
\]
and the result for the integral with the sine follows. The integral with the cosine can be obtained by using integration by parts only. We have
\[
\int_0^1 \cos(2\pi x) \log x \, dx = -\frac{1}{2\pi} \int_0^1 \frac{\sin(2\pi x)}{x} \, dx = -\frac{1}{2\pi} \int_0^{2\pi} \frac{\sin t}{t} \, dt = -\frac{\text{Si}(2\pi)}{2\pi}. \]
With the notation
\[ f_s(x) = \sin(2\pi x) \quad \text{and} \quad f_c(x) = \cos(2\pi x), \]
we have
\[ f_s^{(2j)}(x) = (-1)^j(2\pi)^{2j}\sin(2\pi x), \quad f_s^{(2j+1)}(x) = (-1)^j(2\pi)^{2j+1}\cos(2\pi x), \]
\[ f_c^{(2j)}(x) = (-1)^j(2\pi)^{2j}\cos(2\pi x), \quad \text{and} \quad f_c^{(2j+1)}(x) = (-1)^{j+1}(2\pi)^{2j+1}\sin(2\pi x). \]
Moreover, \( f_s^{(2j)}(0) = f_s^{(2j)}(1) = 0, \) \( f_s^{(2j+1)}(0) = f_s^{(2j+1)}(1) = (-1)^j(2\pi)^{2j+1}, \) \( f_c^{(2j)}(0) = f_c^{(2j)}(1) = 0, \) and \( f_c^{(2j+1)}(0) = f_c^{(2j+1)}(1) = (-1)^j(2\pi)^{2j}. \)

Taking
\[ S_k = \int_0^1 x^k f_s \left( \left\lfloor \frac{1}{x} \right\rfloor \right) \, dx \quad \text{and} \quad C_k = \int_0^1 x^k f_c \left( \left\lfloor \frac{1}{x} \right\rfloor \right) \, dx \]
we have the following result.

**Theorem 3.1:** For \( n \geq 0, \) the following identities hold
\[ S_{2n} = \frac{(-1)^{n+1}(2\pi)^{2n+1}}{(2n + 1)!} \left( \sum_{j=0}^{n-1} \frac{(-1)^j(2j + 1)!}{(2\pi)^{2j+1}} + \text{Ci}(2\pi) \right), \]
\[ S_{2n+1} = \frac{(-1)^n(2\pi)^{2n+2}}{(2n + 2)!} \left( \sum_{j=0}^{n} \frac{(-1)^j(2j)!}{(2\pi)^{2j+1}} - \frac{\pi}{2} + \text{Si}(2\pi) \right), \]
\[ C_{2n} = \frac{(-1)^n(2\pi)^{2n+1}}{(2n + 1)!} \left( \sum_{j=0}^{n} \frac{(-1)^j(2j)!}{(2\pi)^{2j+1}} - \frac{\pi}{2} + \text{Si}(2\pi) \right), \]
and
\[ C_{2n+1} = \frac{(-1)^n(2\pi)^{2n+2}}{(2n + 2)!} \left( \sum_{j=0}^{n} \frac{(-1)^j(2j + 1)!}{(2\pi)^{2j+2}} + \text{Ci}(2\pi) \right). \]

**Proof:** The identities can be deduced immediately by using Lemma 2.1, 3.1, and the given properties about the derivatives of the functions \( f_s \) and \( f_c. \)

### 4. Fractional moments for Bernoulli polynomials

Now, we analyze the fractional moments for the Bernoulli polynomials \( B_n(x) \). They can be defined through its exponential generating function
\[ \frac{te^{xt} - 1}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \]
converging for \(|t| < \pi \). It is well known that \( B_n(0) = (-1)^n B_n(1) = B_n \), where \( B_n \) are the Bernoulli numbers. Remember that \( B_{2k+1} = 0 \), for \( k \geq 1 \), and \( B_1 = -1/2 \). A main tool in
our approach will be the identity \( B'_n(x) = nB_{n-1}(x) \) (Bernoulli polynomials are, in fact, a particular case of Appell’s polynomials). More generally, it is verified that

\[
B^{(k)}_n(x) = \frac{n!}{(n-k)!} B_{n-k}(x), \quad n \geq k. \tag{4.1}
\]

A crucial point to obtain a proper expression for the fractional moments of the Bernoulli polynomials is the following identity (see [7, (6.5) and (6.6)])

\[
\int_0^1 B_n(x) \log \Gamma(x) \, dx = a_n, \quad n \geq 0, \tag{4.2}
\]

where the sequence \( a_n \) is defined by

\[
a_n = \begin{cases} 
-\zeta'(-n), & n = 0, 2, 4, \ldots, \\
\frac{B_{n+1}}{n+1} \left( \frac{\zeta'(n+1)}{\zeta(n+1)} - \log(2\pi) - \gamma \right), & n = 1, 3, 5 \ldots,
\end{cases} \tag{4.3}
\]

with \( \zeta' \) being the derivative of the Riemann zeta function. Recall that \( \zeta'(0) = -\log \sqrt{2\pi} \) and

\[
\zeta'(2n) = (-1)^n \frac{(2n)! \zeta(2n+1)}{2(2\pi)^{2m}}.
\]

Moreover, we consider the sequence

\[
b_n = a_n - \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} \frac{B_{n+1-k} \log x}{k}, \quad n \geq 0.
\]

**Lemma 4.1:** For \( n \geq 0 \), it is verified that

\[
\int_0^1 B_n(x) \log \Gamma(x+1) \, dx = b_n.
\]

**Proof:** From (4.2), we have

\[
\int_0^1 B_n(x) \log \Gamma(x+1) \, dx = \int_0^1 B_n(x) \log x \, dx + a_n.
\]

Now, applying integration by parts to the first integral and using the identity

\[
B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_{m-k} x^k \tag{4.4}
\]

(which can be deduced from (4.1) by using that \( B_n(0) = B_n \)), we obtain that

\[
\int_0^1 B_n(x) \log x \, dx = -\frac{1}{n+1} \int_0^1 \frac{B_{n+1}(x) - B_{n+1}}{x} \, dx
\]

\[
= -\frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} B_{n+1-k} \int_0^1 x^{k-1} \, dx
\]
\[ = - \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k}}{k} \]

and the proof is completed. ■

Taking

\[ j_n^k = \int_0^1 x^k B_n \left( \left\{ \frac{1}{x} \right\} \right) \, dx, \]

we have the following result.

**Theorem 4.1:** For \( n \geq 1 \), it is verified that

\[ j_n^k = \frac{1}{(k+1)(n+k)} \left( \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{k-n+2j}{2j} B_{2j} + (k-n+1) \left( \frac{1}{2} - \zeta(k-n+2) \right) \right), \]

for \( k \geq n \),

\[ j_{n-1}^n = \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{B_{2j}}{2j} + \frac{1}{2} - \gamma, \]

and

\[ j_n^k = \frac{1}{k+1} \binom{n}{k} \left( \sum_{j=\lfloor (n-k+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \frac{B_{2j}}{2j} - \sum_{j=\lfloor (n-k+1)/2 \rfloor}^{n-k} \binom{n-k+1-j}{k-n} b_{n-k-2} \right), \]

for \( 0 \leq k \leq n-2 \).

**Proof:** The result follows by applying Lemmas 2.1, 4.1, and the properties of Bernoulli polynomials and Bernoulli numbers. ■

**5. Fractional moments for the functions \( x^m \)**

Considering the functions \( h_m(x) = x^m \), with \( m \geq 1 \), in this section we evaluate their fractional moments. More precisely, we calculate the integrals

\[ C_k^m = \int_0^1 x^k h_m \left( \left\{ \frac{1}{x} \right\} \right) \, dx. \]

To this end, we need the two following lemmas.

**Lemma 5.1:** For \( n \geq 0 \), it is verified that

\[ \int_0^1 x^n \log \Gamma(x+1) \, dx = -\frac{1}{(n+1)^2} + \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} a_k, \]

where the sequence \( a_k \) was defined in (4.3).
Proof: By using the identity

\[ x^n = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} B_j(x), \quad (5.1) \]

it is clear that

\[
\int_0^1 x^n \log \Gamma(x+1) \, dx = \int_0^1 x^n \log x \, dx + \int_0^1 x^n \log \Gamma(x) \, dx = -\frac{1}{(n+1)^2} + \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \int_0^1 B_j(x) \log \Gamma(x) \, dx
\]

and the result is obtained applying (4.2).

Lemma 5.2: The following identities hold

\[
\sum_{j=0}^{m} \frac{(k-j)!}{(m-j)!} = \frac{(k+1)!}{m!(k+1-m)}, \quad k \geq m
\]

and

\[
\sum_{j=0}^{k} \frac{(k-j)!}{(m-j)!} = \frac{(k+1)!}{m!(k+1-m)} \left(1 - \binom{m}{k+1}\right), \quad 0 \leq k \leq m-2.
\]

Proof: The first identity is equivalent to

\[
\sum_{j=0}^{m} \binom{k-m+j}{j} = \binom{k+1}{m}
\]

and this is a consequence of the relation

\[
\sum_{j=0}^{m} \binom{n+j}{j} = \binom{n+m+1}{m}, \quad n \geq 0,
\]

which can be proved elementary by using induction on \(m\).

The second identity is equivalent to

\[
\frac{m-k-1}{k+1} \sum_{j=0}^{k} \binom{m}{j} = \binom{m}{k+1} - 1.
\]

In this case, it is enough to check that both sides of the identity satisfy the recurrence relation

\[(m-k)a_{m+1,k} - (m+1)a_{m,k} = k+1, \quad 0 \leq k \leq m-2,
\]

and they coincide for \(m = 2\) and \(k = 0\). ■
In the proof of our result for $C_m^k$, we will apply that $h_m^{(k)}(x) = m!x^{m-k}/(m-k)!$, for $0 \leq k \leq m$, and $h_m^{(k)}(x) = 0$, for $k > m$. Moreover, $h_m^{(k)}(0) = 0$, for $k \neq m$, $h_m^{(m)}(0) = m!$, and $h_m^{(k)}(1) = m!/(m-k)!$, for $0 \leq k \leq m$.

Now, we have the following result.

**Theorem 5.1:** For $m \geq 1$, it is verified that

$$C_m^k = \frac{1}{k+1-m} - \frac{1}{(k+1)(k_m^k)} \sum_{j=k-m+1}^{k} \binom{j}{k-m} \xi(j+1), \quad k \geq m,$$

$$C_{m-1}^m = H_m - \gamma - \sum_{j=1}^{m-1} \frac{\xi(j+1)}{j+1},$$

and

$$C_m^k = \frac{1}{k+1-m} - \frac{1}{k+1} \binom{m}{k} \left( \sum_{j=1}^{k} \frac{\xi(j+1)}{j} + \gamma \right)$$

$$+ \binom{m}{k+1} \sum_{j=0}^{m-k-2} \binom{m-k-1}{j} a_j, \quad 0 \leq k \leq m-2.$$

**Proof:** The result can be deduced by applying Lemmas 2.1, 5.1, 5.2, and the given elementary properties for the derivatives of $h_m$.

**Remark 5.1:** We have to observe that a closed form for $C_m^k$, when $k \geq m$, appears in [6, Problem 1.47] and it is equivalent to the given one in our previous theorem.

Note that, taking $k = m$, we obtain that

$$C_m^m = 1 - \frac{1}{m+1} \sum_{j=1}^{m} \xi(j+1),$$

which is the result in [2, Problem 2.21] and [3, Corollary 2.2]. Moreover, the identity (1.1) and the previous theorem imply the following corollary.

**Corollary 5.1:** For $m \geq 1$, it is verified that

$$\frac{m!}{(k+1)!} \sum_{j=1}^{\infty} \frac{(k+j)!}{(m+j)!} (\xi(k+j+1) - 1) = \frac{1}{k+1-m}$$

$$- \frac{1}{(k+1)(k_m^k)} \sum_{j=k-m+1}^{k} \binom{j}{k-m} \xi(j+1), \quad k \geq m,$$

$$\sum_{j=1}^{\infty} \frac{\xi(m+j) - 1}{m+j} = H_m - \gamma - \sum_{j=1}^{m-1} \frac{\xi(j+1)}{j+1},$$
and

$$
\frac{m!}{(k+1)! \sum_{j=1}^{\infty} (k+1)! (\zeta(k+j+1) - 1) = \frac{1}{k+1-m} - \frac{1}{k+1} \binom{m}{k}
$$

$$
\times \left( \sum_{j=1}^{k} \frac{\zeta(j+1)}{(m-k+p)} + \gamma \right) + \left( \frac{m}{k+1} \right) \sum_{j=0}^{m-k-2} \binom{m-k-1}{j} a_j, \quad 0 \leq k \leq m - 2.
$$

By using the values

$$
a_0 = \log \sqrt{2\pi} \quad \text{and} \quad a_1 = -\frac{1}{4} + \frac{\zeta'(2)}{2\pi^2} + \frac{1}{3} \log \sqrt{2\pi} - \frac{\gamma}{12},
$$

we can see some particular cases. Taking $k = m - 2$ and $k = m - 3$, we have, respectively,

$$
\sum_{j=1}^{\infty} \frac{\zeta(m+j-1) - 1}{(m+j)(m+j-1)} = -\frac{1}{m} + \log \sqrt{2\pi} - \frac{\gamma}{2} - \sum_{n=2}^{m-1} \frac{\zeta(n)}{n(n+1)}, \quad m \geq 2,
$$

and

$$
\sum_{j=1}^{\infty} \frac{\zeta(m+j-2) - 1}{(m+j)(m+j-1)(m+j-2)}
$$

$$
= -\frac{1}{2m(m-1)} + \frac{\zeta'(2)}{2\pi^2} + \frac{1}{3} \log \sqrt{2\pi} - \frac{\gamma}{4} - \sum_{n=2}^{m-2} \frac{\zeta(n)}{n(n+1)(n+2)}, \quad m \geq 3.
$$

**Remark 5.2:** From the well-known Hermite identity

$$
\sum_{k=0}^{n-1} \left\lfloor \frac{x + k}{n} \right\rfloor = [nx], \quad x \in \mathbb{R},
$$

we obtain that

$$
\sum_{k=0}^{n-1} \left\{ \frac{x + k}{n} \right\} = \{nx\} + \frac{n-1}{2}, \quad x \in \mathbb{R}.
$$

Then, applying Theorem 5.1, we can deduce that

$$
\int_0^n x^k \sum_{k=0}^{n-1} \left\{ \frac{1}{x} + \frac{k}{n} \right\} \, dx = n^{k+1} \left( \frac{1}{k} + \frac{n-1 - 2\zeta(k+1)}{2(k+1)} \right), \quad n, k \geq 1,
$$

and

$$
\int_0^n \sum_{k=0}^{n-1} \left\{ \frac{1}{x} + \frac{k}{n} \right\} \, dx = n \left( \frac{n+1}{2} - \gamma \right), \quad n \geq 1.
Remark 5.3: In [3, Theorem 3.1], it is proved that

\[ \int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\}^m \left\{ \frac{y}{x} \right\}^k \, dx \, dy = \frac{C_k^m + C_m^k}{2}. \]

In that paper, the double integral is written as an infinite sum of values of the Riemann zeta function by using (1.1). With Theorem 5.1 it is possible to obtain an appropriate closed form for the integral.

6. Fractional moments for the functions \( x^m(1-x)^m \)

To finish with our examples, we study the fractional moments for the functions \( f_m(x) = x^m(1-x)^m \). To do this, we start obtaining an expression for \( f_m(x) \) in terms of Bernoulli polynomials because, as we said in the Introduction, the integrals of \( \log \Gamma(x+1) \) with polynomials behave better with the Bernoulli polynomials \( B_k(x) \) than with the usual powers \( x^k \).

In fact, in [7, Example 6.4] to evaluate the integrals

\[ \int_0^1 x^k \log \Gamma(x) \, dx \]

the authors write them in terms of Bernoulli polynomials by using (5.1), as we did in Lemma 5.1. To obtain the expansion of \( f_m(x) \) in terms of Bernoulli polynomials we need some sums of combinatorial numbers that are contained in the following lemma.

Lemma 6.1: The following identities hold

\[ \sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m}{k} = \frac{1}{(2m+1)(2m)}, \] (6.1)

\[ \sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m+k+1}{j} = 0, \quad j = 1, \ldots, m, \] (6.2)

and

\[ \sum_{k=j-m}^{m} \frac{(-1)^k}{m+k+1} \binom{m}{j} \binom{m+k+1}{j-m} = \frac{(-1)^m(1+(-1)^j)}{j} \binom{m}{j-m} \] (6.3)

for \( j = m+1, \ldots, 2m \).

Proof: To prove (6.1) we apply integration. Indeed, it is easy to check that

\[ \sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m}{k} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int_0^1 x^{m+k} \, dx \]

\[ = \int_0^1 x^m(1-x)^m \, dx \]

\[ = \frac{(\Gamma(m+1))^2}{\Gamma(2m+2)} = \frac{1}{(2m+1)(2m)}. \]
In the proof of (6.2) we use the falling factorial, which is defined by \((a)_n = a(a-1) \cdots (a-n+1)\), for \(n > 1\), and \((a)_0 = 1\). It is clear that \(\binom{m+k+1}{j}\) is a polynomial in the variable \(k\) of degree \(j\) and, for \(j \geq 1\),
\[
\binom{m+k+1}{j} \frac{1}{m+k+1} = \frac{(m+k)(m+k-1) \cdots (m+k+2-j)}{j!} = \sum_{\ell=0}^{j-1} a_{j,m,\ell}(k)\ell
\]
for some coefficients \(a_{j,m,\ell}\). From the identity
\[
\binom{m}{k}(k)\ell = \binom{m-\ell}{k-\ell}(m)\ell,
\]
we have
\[
\sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m}{k} \binom{m+k+1}{j} = \sum_{\ell=0}^{j-1} \sum_{k=\ell}^{m} (-1)^k \binom{m}{k}(k)\ell = \sum_{\ell=0}^{j-1} \sum_{k=\ell}^{m} (-1)^k \binom{m-\ell}{k-\ell}(m)\ell
\]
\[
= \sum_{\ell=0}^{j-1} (-1)^\ell a_{j,m,\ell}(m)\ell \sum_{k=0}^{m-\ell} (-1)^k \binom{m-\ell}{k} = 0,
\]
where in the last step we have applied the identity \(\sum_{k=0}^{p} (-1)^k \binom{p}{k} = 0\) with \(p = m-j+1, \ldots, m\).

The identity (6.3) is equivalent to
\[
\sum_{k=j-m}^{m} (-1)^k \binom{m}{k} \binom{m+k}{j-1} = (-1)^m(1 + (-1)^j) \binom{m}{j-m-1}.
\]

Obviously,
\[
\sum_{k=j-m}^{m} (-1)^k \binom{m}{k} \binom{m+k}{j-1} = \sum_{k=j-m-1}^{m} (-1)^k \binom{m}{k} \binom{m+k}{j-1} - (-1)^{j-m-1} \binom{m}{j-m-1}.
\]

Now, applying [9, (5.24)], we have
\[
\sum_{k=j-m-1}^{m} (-1)^k \binom{m}{k} \binom{m+k}{j-1} = (-1)^m \binom{m}{j-m-1}
\]
and
\[
\sum_{k=j-m}^{m} (-1)^k \binom{m}{k} \binom{m+k}{j-1} = (-1)^m(1 + (-1)^j) \binom{m}{j-m-1}
\]
finishing the proof.
Lemma 6.2: For each $m \geq 1$, the following identities hold

$$f_m(x) = \frac{1}{(2m + 1)(2m)} + (-1)^m \sum_{j=m+1}^{2m} \frac{1 + (-1)^j}{j} \left( \frac{m}{j - m - 1} \right) B_j(x).$$

Proof: To obtain the identity, we use Newton’s binomial identity and (5.1). We have,

$$f_m(x) = x^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^k = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^{m+k}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m+k+1}{k} B_j(x)$$

$$= \sum_{j=0}^{m} B_j(x) \sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m+k+1}{k} + \sum_{j=m+1}^{2m} B_j(x) \sum_{k=0}^{m} \frac{(-1)^k}{m+k+1} \binom{m+k+1}{k}$$

We finish the proof by applying the identities (6.1) – (6.3) in the previous lemma.

Remark 6.1: It is interesting to observe that, by using (4.1), we have

$$f_m(x) = (-1)^m \sum_{j=\max(k,m+1)}^{2m} \frac{1 + (-1)^j}{j} \left( \frac{m}{j - m - 1} \right) \frac{j!}{(j-k)!} B_{j-k}(x),$$

$$= (-1)^m k! \sum_{j=\max(k,m+1)}^{2m} \frac{1 + (-1)^j}{j} \left( \frac{m}{j - m - 1} \right) \frac{j!}{(j-k)!} B_{j-k}(x), \quad (6.4)$$

for $1 \leq k \leq 2m$.

Remark 6.2: Let $P_m$ be the shifted Legendre polynomial of degree $m$, i.e. $P_m(x) = P_m(2x - 1)$, where $P_m$ is the standard Legendre polynomial. By Rodrigues’ formula

$$P_m(x) = \frac{(-1)^m}{m!} f_m^{(m)}(x)$$

and applying (6.4) we deduce that, for $m \geq 1$,

$$P_m(x) = \frac{1}{m!} \sum_{j=m+1}^{2m} \frac{1 + (-1)^j}{j} \left( \frac{m}{j - m - 1} \right) \frac{j!}{(j-m)!} B_{j-m}(x)$$
\[
= \sum_{j=1}^{m} (1 + (-1)^{m+j}) \frac{(m + j - 1)!}{j!(j-1)!(m-j+1)!} B_j(x).
\]

The previous identity for shifted Legendre polynomials matches with the given ones in [10, Theorems 2.2 and 2.4].

To evaluate the fractional moments of the functions \( f_m \), we need the combinatorial identities contained in the next lemma.

**Lemma 6.3:** For \( m \geq 1 \), the identities

\[
\sum_{j=m}^{2m} \binom{m}{j-m} \frac{(m-n)(k-m-n)}{(k-2m)!} = \frac{m!(k-2m)!(k+1)}{(k+1-m)!}, \quad k \geq 2m,
\]

(6.5)

and

\[
\sum_{j=m}^{2m-1} \binom{m}{j-m} \frac{1}{(k-j)!} = 2m(H_{2m} - H_m)
\]

(6.6)

hold.

**Proof:** The first identity is equivalent to

\[
\sum_{n=0}^{m} \binom{m+n}{m} \binom{k-m-n}{k-2m} = \binom{k+1}{m}
\]

and it follows from [9, (5.26)]. To prove the second one we check that both sides satisfy the recurrence relation

\[
ma_{m+1} = (m+1)a_m + \frac{m}{2m+1}
\]

and they coincide for \( m = 1 \).

To complete the evaluation of the fractional moments for the functions \( f_m \), we will need some elementary facts about them. More exactly, \( f_m^{(j)}(x) = 0 \), for \( j > 2m \), \( f_m^{(j)}(0) = f_m^{(j)}(1) = 0 \), for \( 0 \leq j < m \), and, by using Leibniz rule for the derivative of a product,

\[
(-1)^j f_m^{(j)}(0) = f_m^{(j)}(1) = (-1)^m j! \binom{m}{j-m}, \quad m \leq j \leq 2m,
\]

(6.7)

Then, denoting

\[
T_k^m = \int_0^1 x^k f_m \left( \left\{ \frac{1}{x} \right\} \right) dx,
\]

\[
\delta(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}
\]

and \( p_{m,k} = \sum_{j=m}^{m} \binom{j-m}{k} \binom{k}{j} \),
we have the following result.

**Theorem 6.1:** For \( m \geq 1 \), the following identities hold

\[
\mathcal{I}_k^m = (-1)^m \left( \frac{1}{(k+1-m)(k-m)} - \frac{2}{k+1} \sum_{j=[m/2]}^{m-1} \frac{m}{(2j+1)} \frac{(2j+1-m)}{(2j+1-m)} \zeta(k-2j) \right),
\]

for \( k \geq 2m \),

\[
\mathcal{I}_{2m-1}^m = (-1)^m \left( H_{2m} - H_m - \gamma - \frac{1}{m} \sum_{j=[m/2]}^{m-2} \frac{m}{(2j+1)} \zeta(2m-1-2j) \right),
\]

\[
\mathcal{I}_k^m = \frac{(-1)^m}{k+1} \left( P_{m,k} - 2 \sum_{j=[m/2]}^{[k/2]-1} \frac{m}{(2j+1)} \zeta(k-2j) - 2\gamma \left( \frac{m}{2m-k} \right) \delta(1/2, [(k+1)/2]) \right)
\]

\[
+ (-1)^m(k+2) \sum_{j=[m/2]+1}^{m} \left( \frac{m}{2j-m-1} \right) \left( \frac{2j}{k+2} \right) \frac{b_{2j-k-2}}{j},
\]

for \( m \leq k \leq 2m-2 \), and

\[
\mathcal{I}_k^m = (-1)^m(k+2) \sum_{j=[m/2]+1}^{m} \left( \frac{m}{2j-m-1} \right) \left( \frac{2j}{k+2} \right) \frac{b_{2j-k-2}}{j},
\]

for \( 0 \leq k \leq m-1 \).

**Proof:** The result is a consequence of Lemma 2.1, (6.4), Lemmas 4.1 and 6.3. In particular, we have to use (6.5) for the case \( k \geq 2m \) and (6.6) for \( k = 2m-1 \). Moreover, the values of the derivatives of \( f_m \) at \( x = 0 \) and \( x = 1 \) given in (6.7) are also used. \( \square \)

**Remark 6.3:** Unfortunately, we could not find a closed form for the sum

\[
P_{m,k} = \sum_{j=m}^{k} \frac{m}{(j)}
\]

appearing in the case \( m \leq k \leq 2m-2 \). This remains as an open problem.
Note

1. This fact follows from (4.4) applying the identity

\[ \sum_{k=0}^{m} \binom{m+1}{k} B_k = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}, \]

which can be proved by using the exponential generating function

\[ \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k. \]

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References

[1] Furdui O. The evaluation of a class of fractional part integrals. Integral Transforms Spec Funct. 2015;26:635–641.
[2] Furdui O. Limits, series, and fractional part integrals: problems in mathematical analysis. New York: Springer; 2013. (Problem books in mathematics).
[3] Furdui O. Exotic fractional part integrals and Euler’s constant. Analysis (Munich). 2011;31:249–257.
[4] Li A, Sun Z, Qin H. Representation of a class of multiple fractional part integrals and their closed form. Integral Transforms Spec Funct. 2016;27:578–591.
[5] Sun Z, Li A, Qin H. Further results on generalized multiple fractional part integrals for complex values. J Comput Appl Math. 2016;302:186–199.
[6] Vălean CI. (Almost) impossible integrals, sums, and series. Cham: Springer; 2019. (Problem books in mathematics).
[7] Espinosa O, Moll VH. On some integrals involving the Hurwitz zeta function. I. Ramanujan J. 2002;6:159–188.
[8] Gradshteyn IS, Ryzhik IM. Table of integrals, series, and products. 7th ed. Amsterdam: Elsevier/Academic Press; 2007.
[9] Graham RL, Knuth DE, Patashnik O. Concrete mathematics. A foundation for computer science. 2nd ed. Reading (MA): Addison-Wesley Publishing Company; 1994.
[10] Navas LM, Ruiz FJ, Varona JL. Old and new identities for Bernoulli polynomials via Fourier series. Int J Math Math Sci. 2012;2012:129126, 14 pp.