Error Suppression for Arbitrary-Size Black Box Quantum Operations

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(Dated: October 20, 2022)

Efficient suppression of errors without full error correction is crucial for applications with NISQ devices. Error mitigation allows us to suppress errors in extracting expectation values without the need for any error correction code, but its applications are limited to estimating expectation values, and cannot provide us with high-fidelity quantum operations acting on arbitrary quantum states. To address this challenge, we propose to use error filtration (EF) for gate-based quantum computation, as a practical error suppression scheme without resorting to full quantum error correction. The result is a general-purpose error suppression protocol where the resources required to suppress errors scale independently of the size of the quantum operation. We further analyze the application of EF to quantum random access memory, where EF offers a hardware-efficient error suppression.

Introduction – One major obstacle to performing meaningful computation on quantum devices is the presence of noise. Canonically, we expect that the theory of fault tolerance and Quantum Error Correction (QEC) codes will enable us to scale quantum computers up once we have enough qubits and physical error rates fall below a particular threshold [1,8]. However, being in the NISQ or early fault-tolerance era means that we are limited in the number and base quality of qubits available [3], which prevents us from performing full-fault-tolerant quantum computing. Recent work related to suppressing errors on NISQ devices has focused on error mitigation [5] – for instance: zero noise extrapolation [6,7], quasiprobability decomposition and probabilistic error cancellation [7,11], learning-based methods such as Clifford data regression [12,13] and deep learning noise prediction [14], and virtual distillation [15,17]. Such methods allow the user to suppress errors in extracting expectation values with minimal hardware overhead. The success of such methods in the near-term motivates the desire to suppress errors in quantum gates beyond expectation values.

One approach to achieve more robust quantum gates is to use error detection techniques. In the near-term, one may not want to use the full formalism of QEC. One promising alternative approach to detect errors without full QEC is Error Filtration (EF), which was first introduced as a means to stabilize quantum communication [15]. EF does not seek to mitigate errors in expectation values but rather to protect quantum information during noisy communication. In essence, EF multiplexes a single message, and then attempts to detect and discard the parts of this message in which errors have occurred. Up to post-selection, one is able to communicate a message over multiple similarly noisy channels with lower error rates than a single noisy channel. Given a single-channel error that goes as $\varepsilon$, EF is able to suppress errors in the fidelity of the communicated message by a factor $\varepsilon/T$, where $T$ is the number of channels in the multiplexing which corresponds to the effective dimension of the ancilla Hilbert space. Due to its ease-of-implementation, a successful proof-of-principle experiment was quickly carried out [19]. More recently, interest in EF has seen a revival, as EF poses an early example of a more general class of schemes communicating over a quantum superposition of trajectories. Such schemes boast a range of exotic and remarkable results, such as perfect quantum communication over zero-capacity channels [20–22]. In a separate effort, [23] also formalized aspects of EF and derived explicit EF fidelities for loss and dephasing channels. However, until now, EF has mostly been studied in the context of suppressing errors in communication, which is restricted to identity operations [24].

In this Letter we extend EF to the context of gate-based quantum computation (gate-based EF), and show that this provides a low-overhead means to suppress errors for a large class of quantum operations. The result is a general-purpose error detection protocol where the resources required to suppress errors scale independently of the size of the quantum operation. This has the appealing consequence of allowing us to leverage any small number of additional qubits available in a noisy device to suppress errors in large, complex quantum operations. To establish these results, we provide a general quantum circuit design (Fig. 1) that can filter out errors by employing the noisy operations as black boxes. We stress that EF deals with quantum operations, and provides a means to move beyond suppressing expectation values applicable in the near-term without the availability of QEC.

Gate-based EF – Suppose we are given a black box that imperfectly carries out some ideal unitary $U$. We
can model this as a CPTP map \( \mathcal{U} \) comprising Kraus operators \( K_{it}, i = 0, 1, \ldots, R \)\(^{25}\), where we set \( K_0 \) to be the Kraus operator whose normalized action most resembles \( U \). Our scheme, which we will refer to as gate-based EF, is depicted in Fig. 1. The scheme involves appending additional registers to this apparatus, which we assume, for now, are noiseless compared to \( U \)\(^{26}\).

One can think about the multiplexing in the original EF protocol as quantum communication over a superposition of trajectories\(^{20}\). Inspired by this structure, we want to create a superposition of \( T \) queries to \( \mathcal{U} \). To create a superposition of such calls to the black box, we need three ingredients. The first is entanglement with a control register of \( \log T \) qubits that allows us to make calls to \( \mathcal{U} \) in superposition, the second is a memory register, initialized with the desired input state \( |\psi\rangle \), to store the results of these calls, and the third is an active register, initialized in some given state \( |\phi\rangle \), for null calls to \( \mathcal{U} \)\(^{27}\). The active register will be discarded at the end of the protocol. Our strategy is to prepare the control register in an equal superposition state \( |+\rangle^\otimes \log T \). Conditioned on each branch of the control register being in some computational basis state, we query \( \mathcal{U} \) with the input state \( |\psi\rangle \).

Subsequently, we take a measurement on the control register and post-select on obtaining \(|+\rangle^\otimes \log T \). Finally, we trace out the active register, and are left with the desired state \( \rho_{\log T} \).

We can understand how the gate-based EF circuit of Fig. 1 suppresses errors as follows. Say that in a particular run \( i \) of the circuit, \( \mathcal{U} \) applies Kraus operator \( K_{it} \) in the \( t \)-th call, where \( i_t \in \{0, 1, \ldots, R\} \). We label this run with the vector \( \mathbf{i} = (i_1, i_2, \ldots, i_T) \), which fully describes the error configuration for that run. To construct the density matrix \( \rho_{\log T} \), we simply have to sum the associated output states over all possible configuration vectors \( \mathbf{i} \). It is useful to define the quantity \( \overline{K}_{it} = \prod_{s \neq t} K_{is} \), where \( K_{it} \) is applied in ascending order of \( t \). Then, the (unnormalized) conditional output state of Fig. 1 is

\[
\rho_{\log T} = \frac{1}{T} \mathcal{U}(|\psi\rangle \langle \psi|) + \frac{1}{T^2} \times \\
\sum_{t=1}^{T} \sum_{q \neq t} \left( K_{it} |\psi\rangle \langle \psi| K_{tq}^\dagger \langle \phi| \overline{K}_{it} \overline{K}_{tq} \langle \phi| + \text{h.c.} \right) \tag{1}
\]

The above is derived in more detail in the SM\(^{28}\).

To evaluate the scheme, we will use the infidelity

\[
(1 - F)_{\log T} \equiv 1 - \frac{\langle U|\psi\rangle \rho_{\log T} U|\psi\rangle}{\text{Tr} \rho_{\log T}}, \tag{2}
\]

where \( |U\psi\rangle = U|\psi\rangle \) is the ideal pure state we are trying to achieve. At first glance, the above infidelity looks state-dependent. However, as our calculations will show, \( (1 - F)_{\log T} \) as a function of \( (1 - F)_{U(i=\log T)} \) is independent of \( |\psi\rangle \). Since the scheme is post-selected, \( \rho_{\log T} \) is not normalized, with \( \rho_{\log T}^\dagger \equiv \text{Tr} \rho_{\log T} \) giving the success probability. Both the numerator and denominator in Eq. 2 are easily read off from Eq. 1.

Small error fidelity scaling – We are mainly interested in the case where \( U \) is already producing pretty good queries. We can formalize this notion by assuming that \( 0 < \|K_0 - U\| \leq \varepsilon < 1 \). Then we can write \( K_0 = U - \varepsilon \xi \) for some suitably normalized operator \( \xi \) with \( \|\xi\| = 1 \). The infidelity goes as \( (1 - F)_{\log T} \sim O(\varepsilon) \) for non-unitary errors, while \( (1 - F)_{\log T} \sim O(\varepsilon^2) \) for unitary errors\(^{28}\). Since our goal is to improve the infidelity from \( O(\varepsilon) \) to \( O(\varepsilon^2) \), we will focus on suppressing non-unitary errors\(^{29}\).

Here, we will be interested in channels with non-unitary errors. Plugging Eq. 1 into Eq. 2 and expanding to first order in \( \varepsilon \), we find that for any \( |\psi\rangle, |\phi\rangle \), the infidelity scales as

\[
(1 - F)_{T} \approx \frac{1}{T} (1 - F)_{0} + O(\varepsilon^2), \tag{3}
\]
where \((1 - F)_0 \sim O(\varepsilon)\) is the uncorrected infidelity from querying \(U\) without our scheme. Eq. (3) is a central result of our work. The above result is true for any \(|\phi\rangle\) as long as \(T < 1/\varepsilon\), and is derived in the Supplementary Material [28]. Crucially, this scheme uses no information about the ideal operation \(U\) other than that it is the dominant part of the channel. Under the two assumptions (error-free control qubits and \(K_0 = U - \varepsilon \xi\)), this allows us to obtain a \(1/T\) suppression of the infidelity independent of either the complexity of the black box or any knowledge of the ideal unitary to be carried out. In other words, unlike with full error correction, the resources required for Gate-based EF scale only with the amount of suppression \(T\) that we want, impervious to the complexity of the black box \(U\).

**Probability of success** – The success probability is given by

\[
P_T^{(S)} = \frac{1}{T^2} + \frac{1}{T^2} \times \sum_{i=1}^{T} \sum_{t \neq t} \left( |\psi\rangle \langle K_{i_q} K_{i_t} |\psi\rangle \langle \phi |K_{i_q} K_{i_t} |\phi\rangle + h.c. \right)
\]

(4)

Unfortunately, the presence of the \(K_{i_q}\) terms prevent us from simplifying this expression at this level of generality. By making the worst-case assumption that the application of any \(K_{i>0}\) on any step causes us to reject the output, we can obtain the lower bound

\[
P_T^{(S)} \geq 1 - T \varepsilon + O(\varepsilon^2)
\]

This assures us that as long as \(T\) is not too large, the scheme will still work with high probability, and Eq. (4) allows one to trade-off between error suppression and success probability.

Remarkably, the success probability can often be lower-bounded by a constant. Suppose for instance that the order in which different errors are applied to \(|\phi\rangle\) has no impact on the state, such that we can re-order the terms in \(K_{i_q}, K_{i_t}\). Then, applying the completeness relation, we can write Eq. (3) as

\[
P_T^{(log T)} = \frac{1}{T} + \frac{1}{T^2} \sum_{q \neq t} \sum_{i_q,i_t} \langle \psi |K_{i_q} K_{i_t} |\psi \rangle \langle \phi |K_{i_q} K_{i_t} |\phi \rangle
\]

Again, the worst-case assumption allows us to only consider \(K_0\) terms to get a lower bound, and substituting \(K_0 = U - \varepsilon \xi\) allows us to get

\[
P_T^{(log T)} \geq 1 - 4\varepsilon + \frac{\varepsilon}{T}
\]

which approaches a constant \(P_T^{(log T)} \rightarrow 4\varepsilon\) as \(T\) increases. Furthermore, under these conditions, the \(O(\varepsilon^2)\) term in Eq. (3) will also be upper bounded. In other words, in these situations, if the apparatus is \(\varepsilon\) close to the ideal unitary, the scheme will succeed with probability \(\varepsilon\)-close to unity, and the benefits persist for any value of \(T\)!

This suggests the potential of using gate-based EF as a subroutine in any algorithm that needs to make any large number of successive adaptive calls to a black box oracle. We elaborate on the bounding conditions and how to engineer them in the Supplementary Material [28].

**Minimal success criteria** – In the NISQ era it may often turn out that \(\varepsilon\) is not small. For practical purposes, we might ask under what minimal conditions can we expect any error suppression at all using gate-based EF? It turns out that using the same state \(|\phi\rangle = |\psi\rangle\) guarantees that \(F_1 \geq F_0\) as long as \(F_0 \geq 1/2\), with equality when the error channel coherently applies the wrong unitary [28]. Hence as long as one has a stochastic channel that returns states with fidelity at least \(1/2\), one will see some error suppression with gate-based EF.

Note that in most cases, this error suppression will have a significant residual error, as \(\varepsilon^2\) becomes non-negligible. Since the circuit for gate-based EF is relatively simple, one can often calculate the scaling exactly across the whole range of \(\varepsilon\) when the explicit error channel is known. In the SM [28], we provide additional examples where the error scaling and minimal success conditions can be explicitly stated, including Pauli channels and a generalization of the results obtained for dephasing in [13, 23].

**Error hierarchy and ancilla errors** – In our analysis we have assumed that all errors in the circuit come only from the applications of the apparatus. This means that the error rate characterizing all other operations in the circuit (single- and multi-qubit gates, measurements, and idling) are subject to an error probability characterized by some \(\varepsilon' \ll \varepsilon\). This error hierarchy can occur in a variety of ways. Most prominently, for many complicated operations, it will turn out that the apparatus is simply much noisier than the ancilla qubits even without error correction.

Regardless, we proceed to investigate the impact of ancilla errors on gate-based EF. We consider two possible errors on the control register: phase errors and bit errors. Phase errors turn out not to affect the fidelity scaling at all. Fortunately, phase errors can be detected by the measurements carried out at the end of the circuit, and are discarded by post-selection. Hence, phase errors do not affect the fidelity, at the cost of a decreased success probability. To leading order, the success probability is reduced by a factor \(\varepsilon^2 T \log T\) [28].

Bit errors on the other hand have the potential to ruin the final state, since bit errors change parts of the overall state being put into the apparatus. Since the likelihood of a bit error scales with \(T \log T\), this contributes a factor \(O(\text{flip} \times T \log T)\) to the infidelity. Since this is the only factor of \(O(T)\) appearing in the infidelity, we expect bit errors in the ancilla registers to be the limiting factor of gate-based EF. Due to the biased nature of error propagation in the control register, it may often suffice to use
highly biased qubits to the same effect. Fortuitously, the recent discovery of biased-noise cat codes allow us to construct a bias-preserving Toffoli gate, from which we can construct the required controlled-SWAP operations in a bias-preserving manner \[30,32\].

**Application to QRAM** – To provide an example illustrating the practical utility of our scheme, we consider its application to quantum random access memory (QRAM) \[33,34\]. Error-correcting QRAM has an extremely large hardware overhead. The base hardware overhead of QRAM is already exponentially large – in order to prepare a state on \(\log N\) address bits, QRAM requires \(O(N)\) physical qubits. In data processing applications, the relevant values of \(N\) could easily reach millions or billions, each qubit of which needs to be encoded. This problem of hardware overhead is compounded by the fact that QRAM is a non-Clifford operation, and will require special techniques such as magic state distillation, pieceable fault-tolerance \[35\], or code switching \[36\] to implement fault-tolerantly \[25\]. The detailed analysis of \[37\] corroborates this intuition by demonstrating that a fault-tolerant surface code implementation of QRAM for a memory of size \(N \sim 10^8 \sim 10^9\) would require some \(10^{10} \sim 10^{12}\) physical qubits.

In contrast to error correction, the resource overhead of gate-based EF scales independently of the size of the desired quantum operation. Furthermore, QRAM satisfies the two conditions for the optimal application of gate-based EF. First, the error hierarchy is enforced since \(N\) is generally an exponentially large number, ensuring many more errors occur in the apparatus than the ancillas. Second, one can implement QRAM in a way that satisfies the conditions for an upper-bounded failure probability \[28\]. This ensures that it remains feasible to embed a QRAM with gate-based EF into a quantum algorithm as an oracle. Thus, in the near-term, one is able to suppress the infidelity of QRAM in an extremely hardware-efficient manner. For a more complete review of QRAM and the numerical techniques used to simulate it, see \[38\].

To showcase our scheme’s hardware efficiency, we numerically simulate its application to QRAM circuits comprising up to \(2^n = 2^3 = 8\) qubits with up to \(T = 4\) ancilla qubits in Fig. 2 without ancilla errors. In the absence of ancilla errors, we see an excellent agreement with the noiseless case. However, our analysis considers more

![FIG. 2: Gate-based EF applied to QRAM. (a, c, e): Plot of \(\log(1 - F_{logT})\) as a function of \(\log T\), where \(F_{logT}\) denotes QRAM query fidelity obtained after gate-based EF with \(\log T\) control qubits. The solid lines indicate linear fits, and the fitted slopes are, for \(\log N = 1, 2, 3\): (a): \(-1.02, -0.91, -0.90\), (b): \(-1.02, -0.91, -0.82\), (c): \(-1.01, -1.04, -0.98\), with \(r^2 > 0.99\) for all fits. This demonstrates good agreement with the expected \(1/T\) suppression for low apparatus error, with the deviation of the estimated slope from \(-1\) explained by the \(O(\varepsilon^2)\) terms that become significant for higher apparatus error rates. (b, d, f): Plot of failure probabilities \(1 - P_{log(S)}\) as a function of \(\log T\). The simulated failure probability for depth 1, 2 QRAMs quickly plateau. All failure probability plots show sub-linear scaling, as expected.

Discussion – In this work, we proposed and analyzed the circuit implementation of error filtration on a qubit quantum computer. In the SM, we show that gate-based EF is isomorphic to the original error filtration setup in the noiseless case. However, our analysis considers more
that the many-controlled SWAP gate in Fig. 1 may not
erence to NISQ-era error mitigation schemes, we note
lems dealing with state preparation and sampling.
is an important step towards bridging the NISQ and
reach of low-overhead error suppression methods, which
lar, QRAM is non-Clifford. Gate-based EF extends the
structure of the desired quantum process – in particu-
circuits. Conversely, our scheme is entirely agnostic to the
ited capability in suppressing errors in non-Clifford cir-
comparison, one class of schemes that performs error
 suppression for unitary operations is the extended flag
gadget scheme [39, 40]. However, such schemes have lim-
finite error suppression, gate-based EF suppresses errors in quan-
tum gates, which extends the reach of error mitigation
problems as state preparation and sampling. As a
error suppression for unitary operations is the extended flag
and the Packard Foundation (2020-71479).

While we have characterized gate-based EF by reference to NISQ-era error mitigation schemes, we note that the many-controlled SWAP gate in Fig. 1 may not be easy to implement on NISQ devices. As such, one might regard gate-based EF as a scheme most suited for a post-NISQ era but before achieving large-scale QEC. However, this does not completely rule out the application of gate-based EF during the present NISQ era. To suppress infidelity by 1/2, one only needs a single
controlled SWAP gate. For the application to QRAM, this
be implemented using the same techniques used to
construct the QRAM [41]. One can also think about applying EF to optical quantum computers, imitating the original EF proposals where coherent quantum control is implemented using beam-splitters. Performing coherent quantum control optically may be well-suited to optical implementations of QRAM [12].

ACKNOWLEDGMENTS

We thank Seurui Chen and Ng Hui Khoon for helpful discussions. We acknowledge support from the ARO (W911NF-18-1-0020, W911NF-18-1-0212), ARO MURI (W911NF-16-1-0349, W911NF-21-1-0325), AFOSR MURI (FA9550-19-1-0399, FA9550-21-1-0209), AFRL (FA8649-21-P-0781), DoE Q-NEXT, NSF (OMA-1936118, ERC-1941583, OMA-2137642), NTT Research, and the Packard Foundation (2020-71479).

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Similar to our work, [23] proposed encoding single-qubit unitaries in a Fourier basis to similarly suppress errors. However, in our work, we show that in the gate-based context no such encoding is required, and that the resul-
tant error suppression extends to any number of qubits.

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For a consistent noise description, we must extend the map $\mathcal{U}$ to the entire system. One way to do this is by replacing the Kraus operators $K_i \rightarrow K'_i = I_A \otimes K_i$, where $I_A$ is the identity operator on the rest of the system [33]. This structure can be physically motivated by noting that $\mathcal{U}$ is not coupled to the rest of the system during its operation.

[27] Note that log is taken to base 2 in this text, such that $\log T$ refers to the number of control qubits utilized.

[28] Unitary errors are neither suppressed nor amplified in gate-based EF, because if the black box returns the same erroneous unitary $V \neq U$ every time, the final state will simply be $V|\psi\rangle$ regardless of how many control qubits we use. However, there exist well-established quantum control techniques to further suppress unitary errors [44].

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Supplemental Material: "Error Suppression for Arbitrary-Size Black Box Quantum Operations"

I. NOISE MODEL

We assume we are given an apparatus that is meant to carry out the unitary $U$ but instead carries out a CPTP channel $U(\cdot)$, with $R+1$ Kraus operators $\{K_0, K_1, ..., K_R\}$, where $K_0$ is the Kraus operator closest to $U$. Now, in our setup we append ancillas to this apparatus. Let $I_A$ be the collective identity operation on all parts of the system that aren’t part of the apparatus. Then, assuming the ancillas are noiseless, a reasonable way to extend the noise model to the rest of the system is to use Kraus operators $K_i(s)$ from the identity operation $I_A \otimes K_i, i = 0, 1, 2, ..., R$.

This assumption is justified when the cross-talk between the apparatus and the rest of the system is small. Subsequently, we will suppress the $I_A$ and $(s)$ superscript, writing $K_i$ to be understood in the above sense.

II. SMALL PARAMETER CASE

We begin by deriving the scaling behaviour of gate-based EF in the primary case of interest – that is, we are given an apparatus that is already pretty good, and everything else in the circuit is noiseless. Formally, we say that there exists a Kraus operator $K_0$ that is $\epsilon$-close to the ideal unitary $U$, such that we can write $K_0 = U - \epsilon \xi$ for some small $\epsilon$ and some suitably normalized operator $\xi$ with $\|\xi\| = 1$. Given such a form for $K_0$, it follows that $K_i \sim O(\sqrt{\epsilon})$ for $i \neq 0$. We will also assume that $1 - F_0 \sim O(\epsilon)$, where $F_0$ is the native fidelity of the apparatus prior to error filtration. Note that this assumption is not compatible with $K_0 = V$ where $V$ is some coherent unitary error, as we will see in Sec. II C, where we consider those cases separately.

A. $T = 2$ case

We will first work out the $T = 2, \log T = 1$ case. We index all the relevant density matrices and fidelities with $\log T$. The unnormalized output of Fig. S1 when $\log T = 1$ is the density matrix

$$\tilde{\rho}(1) = \frac{1}{2} U(\langle \psi | \psi \rangle) + \frac{1}{2} \sum_{i=0, j=0}^R K_i |\psi \rangle \langle \psi | K_j^\dagger \text{Tr} \left( \rho_\phi K_i^\dagger K_j \right),$$

where we note that $F_0 = \langle \psi | U^\dagger U(\langle \psi | \langle \psi |)U|\psi \rangle$. The conclusions of this section are independent of the state $|\phi \rangle$, so we can choose any $|\phi \rangle$. 

FIG. S1: General circuit for gate-based EF that makes some number $T$ calls to an apparatus that implements the noisy process $U$ as follows: (1) Prepare the intended input state $|\psi \rangle$ in one register, and the null state $|\phi \rangle$ in another. We refer to the former as the memory register and the latter as the active register. (2) Prepare $\log T$ ancillas in the control register in the equal superposition state $|+\ldots+\rangle$. (3) For each branch $|i \rangle$ in the computational basis (i.e. $|00\cdots0 \rangle$, $|00\cdots1 \rangle$, and so forth), swap the state $|\psi \rangle$ into the active register. We then run the active register through our apparatus, given by noisy process $E$, before swapping the registers back for the same branch. Do this for every branch of the superposition. (4) Finally, perform $\log T$ parallel measurements on the control register in the $X$ basis, and post-select for every measurement being +1. This corresponds to projecting the control register back onto the equal superposition state $|+\ldots+\rangle$. 

We will assume we are given an apparatus that is meant to carry out the unitary $U$ but instead carries out a CPTP channel $U(\cdot)$, with $R+1$ Kraus operators $\{K_0, K_1, ..., K_R\}$, where $K_0$ is the Kraus operator closest to $U$. Now, in our setup we append ancillas to this apparatus. Let $I_A$ be the collective identity operation on all parts of the system that aren’t part of the apparatus. Then, assuming the ancillas are noiseless, a reasonable way to extend the noise model to the rest of the system is to use Kraus operators $K_i(s)$ from the identity operation $I_A \otimes K_i, i = 0, 1, 2, ..., R$.

This assumption is justified when the cross-talk between the apparatus and the rest of the system is small. Subsequently, we will suppress the $I_A$ and $(s)$ superscript, writing $K_i$ to be understood in the above sense.
The probability of success is given by the trace of this density matrix,

\[ P_s^{(1)} = \frac{1}{2} + \frac{1}{2} \sum_{i=0,j=0}^{R} \langle \psi | K_i^j K_i | \psi \rangle \text{Tr} \left( \rho_o K_i^j K_i \right). \]  

(S3)

Recall our assumption that \( K_0 = U - \varepsilon \xi \). By the arguments in the previous section, the terms in the sum where \( i, j \neq 0 \) are \( O(\varepsilon^2) \). As such, when computing the fidelity we only have to keep terms where either \( i, j = 0 \). We first show that the probability of success is close to 1:

\[
P_s^{(1)} = \text{Tr} \hat{\rho}^{(1)} = \frac{1}{2} + \frac{1}{2} \langle \psi | K_0^0 K_0 | \psi \rangle \text{Tr}(\rho_o K_0^0 K_0) + \frac{1}{2} \sum_{i \neq 0} \left( \langle \psi | K_i^0 K_i | \psi \rangle \text{Tr}(\rho_o K_i^0 K_i) + h.c. \right) + O(\varepsilon^2)
\]

\[ = 1 - \frac{1}{2} \varepsilon \left( \langle \psi | \xi^T U | \psi \rangle + \langle \psi | U^T \xi | \psi \rangle + \text{Tr}(\rho_o \xi^T U) + \text{Tr}(\rho_o U^T \xi) \right)
\]

\[ + \frac{1}{2} \sum_{i \neq 0} \left( \langle \psi | U^T K_i | \psi \rangle \text{Tr}(\rho_o K_i^0 U) + h.c. \right) + O(\varepsilon^2)
\]

which is \( \varepsilon \)-close to 1. Similarly, working out the inner product with the ideal state,

\[
\langle \psi | U^T \hat{\rho}^{(2)} U | \psi \rangle = \frac{1}{2} F_0 + \frac{1}{2} \langle \psi | U^T K_0 | \psi \rangle \langle \psi | K_0^0 U | \psi \rangle \text{Tr}(\rho_o K_0^0 K_0)
\]

\[ + \frac{1}{2} \sum_{i=1}^{R} \left( \langle \psi | U^T K_i | \psi \rangle \langle \psi | K_0^0 U | \psi \rangle \text{Tr}(\rho_o K_i^0 K_0) + \langle \psi | U^T K_0 | \psi \rangle \langle \psi | K_i^0 U | \psi \rangle \text{Tr}(\rho_o K_0^0 K_i) \right) + O(\varepsilon^2)
\]

\[ = \frac{1}{2} F_0 + \frac{1}{2} \left( \langle \psi | U^T \xi | \psi \rangle + \langle \psi | \xi^T U | \psi \rangle + \text{Tr}(\rho_o \xi^T U) + \text{Tr}(\rho_o U^T \xi) \right)
\]

\[ + \frac{1}{2} \sum_{i=1}^{R} \left( \langle \psi | U^T K_i | \psi \rangle \text{Tr}(\rho_o K_i^0 U) + h.c. \right) + O(\varepsilon^2)
\]

Dividing and expanding the denominator in \( \varepsilon \), we see that all the remaining \( O(\varepsilon) \) terms cancel out, and we are left with

\[ (1 - F)_1 = 1 - \frac{\langle \psi | U^T \hat{\rho}^{(2)} U | \psi \rangle}{\text{Tr} \hat{\rho}^{(2)}} \]

\[ = \frac{1}{2} (1 - F)_0 + O(\varepsilon^2), \quad \text{S4}
\]

\[ \text{i.e. the infidelity is halved with a single ancilla qubit.} \]

\[ \text{S5} \]

B. General \( T \) case

Consider a particular trajectory for a state \( |\psi\rangle \) that goes through the circuit with \( \log T \) ancilla qubits. At each call of the apparatus it gets a different Kraus operator applied to it, which we write \( K_{i_1}, K_{i_2}, \ldots, K_{i_T} \). Whether this operator is applied to the state \( |\psi\rangle \) or \( |\phi\rangle \) depends on the branch of the ancilla. For the \( |t\rangle \) branch of the ancilla, \( K_{i_t} \) is applied to \( |\psi\rangle \) whereas the rest are applied to \( |\phi\rangle \). We can write this as,

\[
|t\rangle |\psi\rangle |\phi\rangle \rightarrow |t\rangle K_{i_t} |\psi\rangle K_{i_{t-1}} \ldots K_{i_{t+1}} K_{i_{t-1}} \ldots K_{i_1} |\phi\rangle.
\]

(S6)

for \( t = 0, 1, 2, \ldots, T - 1 \), with \( \langle t \rangle \) corresponding to the binary representation of \( t \). We can think of \( \mathbf{i} \) as a vector with \( T \) components \( i_j \in \{0, 1, \ldots, R\} \). The entire vector \( \mathbf{i} \) indexes one possible outcome of the circuit.

For ease of notation we will define,

\[
\overline{K}_{i_t} = K_{i_{t-1}} \ldots K_{i_{t+1}} K_{i_{t-1}} \ldots K_{i_1}.
\]

(S7)

To get the full density matrix, we must sum over all possible vectors \( i \in \{0, 1, \ldots, R\}^{\otimes T} \). This gives the density matrix after the \( T \) applications of the black box as,

\[
\sum_{i} \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} |t\rangle K_{i_t} |\psi\rangle |\overline{K}_{i_t} |\phi\rangle \right] \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} |t\rangle K_{i_t} |\psi\rangle |\overline{K}_{i_t} |\phi\rangle \right]^\dagger
\]
After measuring the control register, post-selecting on \(|+\rangle^{\otimes \log T}\) and tracing out the active register,

\[
\tilde{\rho}^{(\log T)} = \frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} \sum_{q=1}^{T} K_{i_t} |\psi\rangle \langle \psi| K_{i_t}^\dagger \text{Tr}_\phi \bar{K}_{i_t}^\dagger \bar{K}_{i_t}
\]

First we consider the terms where \(t = q\). These give

\[
\frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} K_{i_t} |\psi\rangle \langle \psi| K_{i_t}^\dagger \text{Tr}_\phi \bar{K}_{i_t}^\dagger \bar{K}_{i_t}
\]

where the sum over \(i\) can be refactored into

\[
\sum_{i} K_{i_t} |\psi\rangle \langle \psi| K_{i_t}^\dagger \text{Tr}_\phi \bar{K}_{i_t}^\dagger \bar{K}_{i_t} = \sum_{j_1=0}^{R} K_{j_1} |\psi\rangle \langle \psi| K_{j_1}^\dagger \sum_{j_2=0}^{R} \sum_{j_2} \text{Tr}_\phi K_{j_2}^\dagger \ldots K_{j_2}^\dagger K_{i_t} \ldots K_{i_t}^\dagger
\]

\[
= \sum_{j_1=0}^{R} K_{j_1} |\psi\rangle \langle \psi| K_{j_1}^\dagger = \mathcal{U}(|\psi\rangle \langle \psi|),
\]

where we have used completeness property of the Kraus channel. Hence the density matrix can be rewritten

\[
\tilde{\rho}^{(\log T)} = \frac{1}{T} \mathcal{U}(|\psi\rangle \langle \psi|) + \frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} \sum_{q \neq t} \left( K_{i_t} |\psi\rangle \langle \psi| K_{i_t}^\dagger \text{Tr}_\phi \bar{K}_{i_t}^\dagger \bar{K}_{i_t} \right)
\]

1. **General Success Probability**

The success probability is given by

\[
P_s^{(\log T)} = \text{Tr}\tilde{\rho}^{(T)} = \frac{1}{T} + \frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} \sum_{q \neq t} \left( \langle \psi| K_{i_t}^\dagger K_{i_t} |\psi\rangle \text{Tr}_\phi \bar{K}_{i_t}^\dagger \bar{K}_{i_t} \right)
\]

(S8)

In the general case, we can still get some idea of how the probability of success scales with \(T\). Generically we expect the probability to go down with each error and the number of errors to scale with \(T\), so we can immediately write down

\[
P_S^{(T)} \sim 1 - O(T)\varepsilon
\]

(S9)

Working through the above expressions more carefully, one finds that the exact scaling at small \(\varepsilon\) can be rigorously lower bounded by

\[
P_S^{(T)} \geq 1 - T\varepsilon.
\]

(S10)

As long as \(T \ll \frac{1}{\varepsilon}\), we will have \(P_S^{(T)}\) is \(\varepsilon\)-close to 1, we can proceed to calculate the infidelity. In fact, much tighter bounds can be given for \(P_S^{(T)}\) under some conditions on the action of the Kraus operators \(K_i\) on the state \(|\phi\rangle\).

2. **Success probability under special favourable conditions**

Now, suppose at least one of the following three conditions hold:

1. The Kraus operators are all mutually commuting, i.e. \([K_i, K_j] = 0\) for all \(i, j = 0, 1, \ldots, R\).
2. \(|\phi\rangle\) is stationary under the channel, i.e., \(K_i |\phi\rangle \propto |\phi\rangle\) for all \(i = 0, 1, \ldots, R\).
3. The circuit has a noise bias, and furthermore is bias preserving, such that we can choose some \(|\phi\rangle\) to be immune to the dominant error (e.g. if the circuit only has phase flips, we can choose \(|\phi\rangle = |0\rangle\)).
Under any of these conditions, we will find that

\[
\frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} \sum_{q>t} \left( \langle \psi | K_{i_q}^\dagger K_{i_t} | \psi \rangle \text{Tr} \rho_0 K_{i_q}^\dagger K_{i_t} + \text{h.c.} \right) \\
= \frac{T - 1}{T} \sum_{i,j} \langle \psi | K_{i}^\dagger K_{j} | \psi \rangle \langle \phi | K_{j}^\dagger K_{i} | \phi \rangle
\]

To make further progress, we note that not all trajectories containing Kraus operators where \( i \neq 0 \) or \( j \neq 0 \) are necessarily rejected by the protocol. Hence assuming that all such trajectories are rejected will give us a lower bound on the success probability, given by

\[
P_S^{(\log T)} \geq \frac{1}{T} + \frac{T - 1}{T} \langle \psi | K_0^\dagger K_0 | \psi \rangle \langle \phi | K_0^\dagger K_0 | \phi \rangle \\
\geq 1 - 4\varepsilon + \frac{4\varepsilon}{T} \\
\geq 1 - 4\varepsilon
\]

3. General fidelity scaling

Having established that in the small parameter case, we can get \( 1 - P_s \sim O(\varepsilon) \) either under the conditions of the previous section, or when \( T \) is not too large, we can now proceed to compute the fidelity associated with having \( \log T \) control qubits.

In terms of \( \tilde{\rho}(\log T) \), \( P_S^{\log T} \),

\[
(1 - F)_{\log T} = \frac{P_S^{\log T} - \langle U \psi | \tilde{\rho}(\log T) | U \psi \rangle}{P_S^{\log T}},
\]

where we can expand,

\[
\langle U \psi | \tilde{\rho}(\log T) | U \psi \rangle = \frac{1}{T} (1 - F)_0 + \frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} \sum_{q<q'} \left( \langle \psi | U_t^\dagger K_{i_q} | \psi \rangle \langle \psi | U_t^\dagger U_q | \psi \rangle \text{Tr} \rho_0 K_{i_q}^\dagger K_{i_t} \right)
\]

Since we are calculating quantities only to \( O(\varepsilon) \), in the numerator we only need to keep compute terms for which either all the applied Kraus operators are \( K_0 \), or all the applied Kraus operators but one are \( K_0 \) (we can loosen this assumption when dealing with the bounded probability case).

First, we can combine the two terms in the numerator by factoring out the \( \phi \) associated terms:

\[
P_S^{\log T} - \langle U \psi | \tilde{\rho}(\log T) | U \psi \rangle \\
= \frac{1}{T} (1 - F)_0 + \frac{1}{T^2} \sum_{i} \sum_{t=1}^{T} \sum_{q<q'} \left( \langle \psi | K_{i_q}^\dagger K_{i_t} | \psi \rangle - \langle \psi | U_t^\dagger K_{i_t} | \psi \rangle \langle \psi | K_{i_q}^\dagger U_q | \psi \rangle \text{Tr} \rho_0 K_{i_q}^\dagger K_{i_t} \right)
\]

Consider the terms in the brackets. For the terms in the sum where all Kraus operators applied are \( K_0 \), we have

\[
\langle \psi | K_0^\dagger K_0 | \psi \rangle - \langle \psi | U_t^\dagger K_0 | \psi \rangle \langle \psi | K_0^\dagger U | \psi \rangle \\
= 1 - \varepsilon \langle \psi | U_t^\dagger \xi | \psi \rangle - \varepsilon \langle \psi | \xi^\dagger U | \psi \rangle - 1 + \varepsilon \langle \psi | U_t^\dagger \xi | \psi \rangle + \varepsilon \langle \psi | \xi^\dagger U | \psi \rangle + O(\varepsilon^2) \\
= O(\varepsilon^3),
\]

so these terms don’t contribute.

For the terms where \( i_q = 0 \) and \( i_t \neq 0 \), we have

\[
\langle \psi | K_0^\dagger K_{i_t} | \psi \rangle - \langle \psi | U_t^\dagger K_{i_t} | \psi \rangle \langle \psi | K_0^\dagger U | \psi \rangle \\
= \langle \psi | U_t^\dagger K_{i_t} | \psi \rangle - \langle \psi | U_t^\dagger K_{i_t} | \psi \rangle + O(\varepsilon^{3/2}) \\
= O(\varepsilon^{3/2}).
\]
Now, we can neglect the $O(\varepsilon^{3/2})$ term, since multiplying by the remaining $\phi$-associated factor $O(\varepsilon^{1/2})$, it will be $O(\varepsilon^2)$. A similar cancellation occurs for the terms where $i_q \neq 0$ and $i_t = 0$. We are left with

$$P_{S_{\log T}} - \langle U\psi|\tilde{\rho}^{(\log T)}|U\psi\rangle = \frac{1}{T}(1 - F)_0 + O(\varepsilon^2)$$

Finally, since $(1 - F)_0 \sim O(\varepsilon)$ and $P_{S_{\log T}} \sim 1 - O(\varepsilon)$,

$$(1 - F)_{\log T} = \frac{P_{S_{\log T}} - \langle U\psi|\tilde{\rho}^{(\log T)}|U\psi\rangle}{P_{S_{\log T}}}$$

$$= \frac{1}{T}(1 - F)_0 + O(\varepsilon^2)$$

$$= \frac{1}{T}(1 - F)_0 + O(\varepsilon^2), \quad (S12)$$

which gives Eqn. [3] of the main text.

C. unitary errors

This calculation comes with one significant caveat. The way we have written $K_0 = U - \varepsilon \xi$ looks extremely general, and one is tempted to conclude that this error suppression works for any such Kraus channel. However, we note that the behaviour for unitary error channels, where $K_0 = V$ is a pure unitary is exceptional and must be considered separately. To see why that is the case, suppose $V = U - \varepsilon \xi$ is unitary. Then

$$VV^\dagger = (U - \varepsilon \xi)(U^\dagger - \varepsilon \xi^\dagger)$$

$$= 1 - \varepsilon (U\xi^\dagger + \xi U^\dagger) + O(\varepsilon^2).$$

In other words, for $V$ to be unitary to first order we require $U\xi^\dagger + \xi U^\dagger = 0$. Now, the base fidelity of such a channel is,

$$(1 - F)_0 = 1 - \langle \psi|U^\dagger V|\psi\rangle \langle \psi|V^\dagger U|\psi\rangle$$

$$= \varepsilon (U\xi^\dagger + \xi U^\dagger) + O(\varepsilon^2)$$

$$= O(\varepsilon^2).$$

Since $(1 - F)_0 \sim O(\varepsilon^2)$, the calculation of the previous section tells us nothing about the error suppression of such unitary error channels.

In fact, coherent unitary errors cannot be suppressed by gate-based EF. Suppose $U(\cdot) = V(\cdot)V^\dagger$. It is not difficult to see that the outcome of gate-based EF for any number of ancillas is always $\rho_{\log T} = V|\psi\rangle\langle\psi|V^\dagger$, hence $F_{\log T} = F_0$. We conclude that gate-based EF works to suppress stochastic errors only, leaving unitary errors alone. Fortunately, this still encompasses the vast majority of error channels commonly considered.

III. GUARANTEEING FIDELITY ENHANCEMENT WITH A SINGLE CONTROL QUBIT

We claim that error filtration can enhance fidelity with a single qubit as long as the base fidelity $F_0$ of the apparatus for the input state $|\psi\rangle$ satisfies $F_1 > \frac{1}{2}$. This can be done by setting $|\phi\rangle = |\psi\rangle$. The final state $\rho_1$ with a single control qubit is

$$\rho_1 = \frac{1}{2} \rho_0 + \frac{1}{2} \rho_2$$

where $\rho_0 = U(|\psi\rangle\langle\psi|)$.

The expression for the fidelity is

$$F_1 = \frac{\langle U\psi|\rho_1|U\psi\rangle}{\text{Tr}\rho_1} = \frac{F_0 + \langle U\psi|\rho_0|U\psi\rangle}{1 + \text{Tr}\rho_0^2}. \quad (S13)$$
The condition $F_1 > F_0$ then reduces to
\[ \langle U \psi | \rho_0^2 | U \psi \rangle - F_0 \text{Tr} \rho_0^2 \] (S14)

To evaluate this, let us write
\[ \rho_0 = F_0 |U \psi \rangle \langle U \psi| + (1 - F_0) \sigma, \] (S15)

where $\sigma$ is a density matrix satisfying $\langle U \psi | \sigma | U \psi \rangle = 0$. The relevant quantities become
\[ \langle U \psi | \rho_0^2 | U \psi \rangle = F_0^2 + (1 - F_0)^2 \langle U \psi | \sigma^2 | U \psi \rangle \] (S16)
\[ \text{Tr} \rho_0^2 = F_0 + (1 - F_0)^2 \text{Tr} \sigma^2 \] (S17)

Then,
\[ \langle U \psi | \rho_0^2 | U \psi \rangle - F_0 \text{Tr} \rho_0^2 = (1 - F_0) \left[ F_0^2 + (1 - F_0)^2 \left( \langle U \psi | \sigma^2 | U \psi \rangle - F_0 \text{Tr} \sigma^2 \right) \right] \] (S18)

where we have used $\langle U \psi | \sigma^2 | U \psi \rangle \geq 0, \text{Tr} \sigma^2 \leq 1$. The above condition is satisfied whenever $F_0 > 1/2$, with $F_0 = F_1$ for a coherent channel. Hence, a channel improvement is guaranteed as long as the initial fidelity is greater than $1/2$, representing a minimal condition for which gate-based EF yields some improvement.

IV. ERROR FLOOR FOR BOUNDED FAILURE PROBABILITY

As noted above the expressions do not lend themselves to easy simplification of an arbitrary $| \phi \rangle$, and we are unable to make strong statements about the behaviour of the gate-based EF circuit when $T \gg 1/\epsilon$ in the general case. However, in the case of channels which admit a $| \phi \rangle$ for which we can write $K_i | \phi \rangle \propto | \phi \rangle$, not only is the probability of failure bounded (following conditions 2. and 3. from the previous section), one can show that the state approaches a well-defined limit as $T \to \infty$. We will show this implies that the $O(\epsilon^2)$ terms do not blow up as $T \to \infty$, and represent an error floor to the scheme.

A. Relation to vacuum extension

The assumption we make for this section is closely related to the concept of a vacuum extension, introduced and defined Sec. (2c) of [S20]. The vacuum extension provides a consistent way to define a superposition of channels. As [S20] points out, the original error filtration scheme [S18] is such an example of a superposition of channels. Hence, one might take the point of view of error filtration as the operationalization of the superposition of channels to suppress errors. In gate-based EF, as demonstrated in our main results, we have loosened this demand, and do not require our scheme to strictly carry out a superposition of channels in order to succeed.

We note that any vacuum extension will yield the favourable probability scaling in gate-based EF. Furthermore, there are various physical systems in which a vacuum extension arises naturally, most notably in optical systems, where the vacuum extension of a system tends to be the actual electromagnetic vacuum. For instance, one might encode qubits with polarized light $| 0(1) \rangle = | H(V) \rangle$. Since most apparatuses act trivially on the electromagnetic vacuum, i.e. no input state, one can then define the vacuum extension to a channel using the actual electromagnetic vacuum.

For our purposes of the calculations in this section, we will not make full use of a rigorous definition of the vacuum extension. Instead, it will be sufficient for us to define a state $| \phi \rangle$ such that,
\[ K_i | \phi \rangle = \sqrt{q_\phi^{(i)}} | \phi \rangle, \] (S19)

where $q_\phi^{(i)}$ can be interpreted as the probability of the Kraus operator $K_i$ occurring given $| \phi \rangle$ as an input state to the apparatus. For convenience, We will refer to any state $| \phi \rangle$ with these properties relative to our channel of interest as a pseudo-vacuum state.
B. Mixed unitary error channels

1. Two unitaries

We first consider the simplest possible model where $K_0 = \sqrt{1-p}U$ and $K_1 = \sqrt{p}V$, where $U,V$ are unitary operators. We assume the existence of a pseudo-vacuum state $|\phi\rangle$ relative to this channel.

$$K_0|\phi\rangle = \sqrt{1-p}|\phi\rangle, \quad K_1|\phi\rangle = \sqrt{p}|\phi\rangle.$$  

(S20)

Now suppose we are given an input state $|\psi\rangle$. To be as restrictive as possible, we can pick the minimum fidelity state $|\psi\rangle$ in the input space, but this will not matter for our calculation. We can express the action of $V$ on $|\psi\rangle$ as:

$$V|\psi\rangle = e^{i\nu} \cos \theta |U\psi\rangle + \sin \theta |U\psi^\perp\rangle.$$  

(S21)

with $\nu$ chosen so that $\theta \in [0,\pi/2]$. Note that we have to keep this phase around because it affects the outcome of gate-based EF when $T > 1$. $|U\psi^\perp\rangle$ is simply the part of the state that is orthogonal to $|U\psi\rangle$.

The fidelity for this state is

$$F_0 = 1 - p + p \cos^2 \theta = 1 - p \sin^2 \theta.$$  

(S22)

Now we want to evaluate $F_\infty$, which is the fidelity approached by gate-based EF as $T \to \infty$. As $T \to \infty$, the output density matrix turns out to approach a pure state:

$$|\psi_\infty\rangle \propto (1 - p)U|\psi\rangle + pV|\psi\rangle$$  

(S23)

$$= (1 - p + pe^{i\nu} \cos \theta) |U\psi\rangle + p \sin \theta |U\psi^\perp\rangle.$$  

(S24)

To see why this is the case, consider the density matrix for arbitrary $T$. Consider a single error configuration, where for the some set of branches $I_0$ where $|I_0| = m$, the black box carries out $K_0$ and in the rest of the $T - m$ branches it carries out $K_1$. Prior to measuring the control register, this corresponds to the state

$$\frac{1}{\sqrt{T}} \sum_{t \in I_0} |t\rangle K_0|\psi\rangle K_0^{-1}K_1^{T-m}|\phi\rangle + \frac{1}{\sqrt{T}} \sum_{t \notin I_0} |t\rangle K_1|\psi\rangle K_0^{-1}K_1^{T-m-1}|\phi\rangle$$  

(S25)

$$= \frac{1}{\sqrt{T}} (1 - p)^{m/2} p^{(T-m)/2} \left( \sum_{t \in I_0} |t\rangle U|\psi\rangle + \sum_{t \notin I_0} |t\rangle V|\psi\rangle \right) |\phi\rangle.$$  

(S26)

Tracing out the vacuum register and measuring the control register, we are left with

$$\frac{1}{\sqrt{2T}} (1 - p)^{m/2} p^{(T-m)/2} (mU|\psi\rangle + (T - m)V|\psi\rangle)$$  

(S27)

We see that since the action on the pseudo-vacuum state is trivial, this state only depends on the number of times $K_0,K_1$ are applied in this particular error configuration. Thus all error configurations with the same $m,T - m$ distribution of Kraus operators coherently combine in the final density matrix, giving a $T$ choose $m$ enhancement to this state, such that the final (unnormalized) density matrix is

$$\rho_T = \sum_{m=1}^{T} \frac{1}{2T} \binom{T}{m} (1 - p)^m p^{T-m} (mU|\psi\rangle + (T - m)V|\psi\rangle) (mU|\psi\rangle + (T - m)V|\psi\rangle)^\dagger.$$  

(S28)

Now, we know from the binomial distribution that as $T \to \infty$ that when viewed as a function of $m$, $\binom{T}{m} (1 - p)^m p^{T-m}$ is sharply peaked around $m = (1 - p)T$. Thus for $T \to \infty$,

$$\rho_\infty \simeq ((1 - p)U|\psi\rangle + pV|\psi\rangle) ((1 - p)U|\psi\rangle + pV|\psi\rangle)^\dagger$$  

(S29)

which is the pure state $|\psi_\infty\rangle$. This provides a rigorous sense in which we are converting a mixture of states $U|\psi\rangle,V|\psi\rangle$ into a superposition. The trace and fidelity of this state are

$$\text{Tr}\rho_\infty = (1 - p + pe^{i\nu} \cos \theta)^2 + p^2 \sin^2 \theta,$$  

(S30)

$$\langle U\psi |\rho_\infty |U\psi\rangle = (1 - p + pe^{i\nu} \cos \theta)^2,$$  

(S31)

$$F_\infty = \frac{(1 - p + pe^{i\nu} \cos \theta)^2}{(1 - p + pe^{i\nu} \cos \theta)^2 + p^2 \sin^2 \theta}.$$  

(S32)
We are interested in two questions. First, when does this represent an improvement on the base fidelity – in other words, when is $F_0 < F_\infty$? Second, in the small parameter case, what does this mean for the $O(\varepsilon^2)$ terms?

First, if $\nu$ is known, then we can simply write down $F_0 < F_\infty$ in terms of $p, \nu$ as

$$1 - p + p \cos^2 \theta < \frac{(1 - p + pe^{i\nu} \cos \theta)^2}{(1 - p + pe^{i\nu} \cos \theta)^2 + p^2 \sin^2 \theta}.$$  \hspace{1cm} (S33)

Note that expression, unlike the single qubit control case, does not furnish a condition on $\nu$ because $\nu$ contains no information about $\nu$. It will be useful for us to pare down this requirement even more to a condition stated only in terms of $p$, since $p, 1 - p$ are easily determined.

It is instructive to consider the two extremal values $\nu = 0, \pi$. $\nu = 0$ corresponds to constructive interference in the final state and $\nu = \pi$ corresponds to destructive interference. The $\nu = 0$ case showcases how phase coherence can amplify the effects of error filtration. However, we will make the pessimistic assumption that $\nu = \pi$ in order to obtain a minimal and single-variable success condition, since

$$F(\psi_\infty) > F((1 - p - p \cos \theta)|U\psi) + p \sin \theta |U\psi^\perp) \equiv F'_\infty,$$  \hspace{1cm} (S34)

where $F(\cdot)$ refers to the fidelity of the state $(\cdot)$. With this, the pessimistic success condition is

$$1 - p + p \cos^2 \theta = F_1 < F'_\infty = \frac{(1 - p - p \cos \theta)^2}{(1 - p - p \cos \theta)^2 + p^2 \sin^2 \theta}.$$  \hspace{1cm} (S35)

To simplify, we write $\cos \theta$ in terms of $F_1$ and solve for $p$. Subject to the constraints $0 \leq p \leq 1, 1 - p \leq F_1 < 1$, this yields

$$p < \frac{1}{4}, \hspace{0.5cm} \text{or} \hspace{0.5cm} \frac{1}{4} < p < \frac{1}{2} \text{ and } 1 - p < F < \frac{1}{4p}.$$  \hspace{1cm} (S36, S37)

The first condition is easy to interpret. The inequality $p < \frac{1}{4}$ asserts that the probability of $V$ is sufficiently small that the final state resembles $|U\psi\rangle$ despite possible destructive interference due to $\nu$. The second condition provides for the possibility that the interference can be quite possibly be quite large, given both a destructive phase and a larger probability of $V$. However, the upper bound on $F$ in that case serves to bound the size of $\cos \theta$ and thus the size of the destructive interference.

In terms of $F_1$, this establishes two regions of success, although $F_1$ alone cannot yield a sufficient condition over this entire range. As we will see shortly, the first condition generalizes more readily to more complicated channels.

Finally, let us consider where the small parameter regime fits in this picture. Let $2p = \varepsilon \ll 1$ so that $K_0$ is of the form $U - \varepsilon A + O(\varepsilon^2)$. Then we can expand $F_\infty$ as

$$F_\infty = \frac{1 - \varepsilon(1 + e^{i\nu} \cos \theta) + \frac{1}{4} \varepsilon^2(1 + e^{i\nu} \cos \theta)^2}{1 - \varepsilon(1 + e^{i\nu} \cos \theta) + \frac{1}{4} \varepsilon^2[(1 + e^{i\nu} \cos \theta)^2 + \sin^2 \theta]} \equiv \frac{1}{1 - \frac{1}{4} \sin^2 \theta \varepsilon^2 + O(\varepsilon^2)}.$$  \hspace{1cm} (S38)

This reaffirms the small parameter scaling we derived earlier, as $F_{T \to \infty} \to 1 - O(\varepsilon^2)$. Furthermore, the coefficient of $\varepsilon^2$ as $T \to \infty$ approaches a $T$ independent limit and does not blow up. Hence, we can interpret it as an error floor to gate-based EF under the assumptions in this section.

2. Many unitaries, with $K_0$ perfect

Now we consider the case of a mixed unitary error channel with $K_0 \propto U$ and multiple unitary errors possible, so that

$$K_0 = \sqrt{1 - pU}, \hspace{0.5cm} K_{i\neq 0} = \sqrt{p\lambda_i}V_i.$$  \hspace{1cm} (S39)

where $\sum \lambda_i = 1$. We write the action of $V$ on $|\psi\rangle$ as

$$V_i|\psi\rangle = e^{i\theta_i} \cos \theta_i |U\psi\rangle + \sin \theta_i |U\psi^\perp\rangle.$$  \hspace{1cm} (S40)
Again, we choose $\nu$ such that $\theta \in [0, \pi/2]$. The base fidelity is

$$F_0 = 1 - p + p \sum_i \lambda_i \cos^2 \theta_i.$$  \hfill (S41)

Following the same argument as the two unitary case, we find

$$|\psi_\infty\rangle = (1 - p)U|\psi\rangle + p \sum_i \lambda_i V_i|\psi\rangle = (1 - p + p \sum_i \lambda_i e^{i\nu_i} \cos \theta_i)|U\psi\rangle + p \sum_i \lambda_i \sin \theta_i|U\psi^\perp\rangle$$  \hfill (S42)

To derive a minimal success condition in the same spirit as before, we note that the fidelity is minimized when $\nu_i = -\pi$ for all $i$. This corresponds to constructive interference between all the $|U\psi^\perp\rangle$, which is greatest when they are all parallel, i.e. $|U\psi^\perp\rangle = |U\psi^\perp\rangle$ for all $i$. Let

$$|\psi'_\infty\rangle = (1 - p - p \sum_i \lambda_i \cos \theta_i)|U\psi\rangle + p \sum_i \lambda_i \sin \theta_i|U\psi^\perp\rangle.$$  \hfill (S43)

Then

$$F(|\psi_\infty\rangle) > F(|\psi'_\infty\rangle) \equiv F'_\infty,$$  \hfill (S44)

and it will be sufficient for us to say $F_1 < F'_\infty$. We calculate $F'_\infty$:

$$F'_\infty = \frac{(1 - p - p \sum_i \lambda_i \cos \theta_i)^2}{(1 - p + p \sum_i \lambda_i \cos \theta_i)^2 + p^2 (\sum_i \lambda_i \sin \theta_i)^2}. $$  \hfill (S45)

The $F_0 < F_\infty$ condition now reads:

$$1 - p + p \sum_i \lambda_i \cos^2 \theta_i < \frac{(1 - p - p \sum_i \lambda_i \cos \theta_i)^2}{(1 - p + p \sum_i \lambda_i \cos \theta_i)^2 + p^2 (\sum_i \lambda_i \sin \theta_i)^2}. $$  \hfill (S46)

So far the above still contains $p, \theta_i, \lambda_i$. We’d like to reduce it to just $p$. To do so, we will wrestle it into the form we’ve already solved for the two unitary channel. To do so, we note the trivial bound

$$F_1 = 1 - p + p \sum_i \lambda_i \cos^2 \theta_i < F_1' = 1 - p + p \cos^2 \theta_{\text{max}}$$  \hfill (S47)

where $\cos \theta_{\text{max}}$ is the maximum possible value $\cos \theta_i$ can take. $F_\infty$ is similarly lower bounded by

$$F((1 - p - p \sum_i \lambda_i \cos \theta_i)|U\psi\rangle + p \sum_i \lambda_i \sin \theta_i|U\psi^\perp\rangle) > F((1 - p - p \cos \theta_{\text{max}})|U\psi\rangle + p \sin \theta_{\text{max}}|U\psi^\perp\rangle)$$  \hfill (S48)

where $\sin \theta_{\text{max}}$ is the minimum possible value of $\sin \theta_i$. Then our condition will be fulfilled if we have

$$1 - p + p \cos^2 \theta_{\text{max}} < \frac{(1 - p - p \cos \theta_{\text{max}})}{(1 - p - p \cos \theta_{\text{max}})^2 + p^2 \sin^2 \theta_{\text{max}}}$$  \hfill (S49)

which reduces to the two unitary case. Hence, applying the results of the previous section,

$$p < \frac{1}{4}$$  \hfill (S50)

is a minimal single parameter success condition.

3. Many unitaries, with $K_0$ imperfect

Our next generalization is to allow $K_0$’s action on $|\psi\rangle$ to have a perpendicular component $|U\psi^\perp\rangle$, such that

$$K_0|\psi\rangle = \sqrt{1 - p \cos \theta_0}|U\psi\rangle + \sqrt{1 - p \sin \theta_0}|U\psi^\perp\rangle.$$  \hfill (S51)
We assume that $\cos \theta_0 > 1/2$, otherwise one can hardly call the apparatus something that attempts to carry out the operation $U$. In particular, $\cos \theta_0 < 1/2$ would imply $F_1 < 1/2$, a situation which we will try to avoid entirely.

The action of $K_{i\neq 0}$ on $|\psi\rangle$ is defined as before. Again, we will deal with the two operator case first, for which

$$F_0 = (1 - p) \cos^2 \theta_0 + p \cos^2 \theta_1. \quad (S52)$$

To derive a condition on $\mathcal{U}$, we will bound the limiting fidelity $F_{\infty}^{(K)}$ associated with $\mathcal{U}$ with the limiting fidelity $F_{\infty}^{(L)}$ associated with a different channel $\mathcal{L}$. Consider such a Kraus channel $\mathcal{L}$ with three operators $L_0, L'_0, L_1$, whose action on $|\psi\rangle$ are defined by

$$L_0 |\psi\rangle = \sqrt{(1 - p) \cos \theta_0} |U \psi\rangle \quad (S53)$$

$$L'_0 |\psi\rangle = \sqrt{(1 - p) \sin \theta_0} |U \psi_0^\perp\rangle \quad (S54)$$

$$L_1 |\psi\rangle = K_1 |\psi\rangle \quad (S55)$$

where $L_0, L'_0$ are proportionate to unitaries, so that their action on the pseudo-vacuum extension is, by assumption,

$$L_0 |\phi\rangle = \sqrt{(1 - p) \cos \theta_0}, \quad L'_0 |\phi\rangle = \sqrt{(1 - p) \sin \theta_0}, \quad L_1 |\phi\rangle = K_1 |\phi\rangle. \quad (S56)$$

First let’s check if this channel is physical. We have

$$\text{Tr} (L_0 |\psi\rangle \langle \psi | L_0) = (1 - p) \cos^2 \theta_0 \quad (S59)$$

$$\text{Tr} (L'_0 |\psi\rangle \langle \psi | L'_0) = (1 - p) \sin^2 \theta_0 \quad (S60)$$

$$\text{Tr} (L_1 |\psi\rangle \langle \psi | L_1) = p \quad (S61)$$

which add up to 1. Hence, by defining the action of $L_i$ on the rest of the Hilbert space appropriately, we can construct a valid Kraus channel with these properties.

The fidelity of $\mathcal{L}$ on $|\psi\rangle$ is

$$F_0^{(L)} = (1 - p) \cos^2 \theta_0 + p \cos^2 \theta_1 = F_1 \quad (S62)$$

The $T \to \infty$ limits for $\mathcal{U}, \mathcal{L}$ before measurement of the control register are given by the pure unnormalized states,

$$|\psi^{(K)}\rangle = [(1 - p) \cos \theta_0 + pe^{i\nu_1} \cos \theta_1] |U \psi\rangle + (1 - p) \sin \theta_0 |U \psi_0^\perp\rangle + p \sin \theta_1 |U \psi_1^\perp\rangle \quad (S63)$$

$$|\psi^{(L)}\rangle = [(1 - p) \cos^2 \theta_0 + pe^{i\nu_1} \cos \theta_1] |U \psi\rangle + (1 - p) \sin^2 \theta_0 |U \psi_0^\perp\rangle + p \sin \theta_1 |U \psi_1^\perp\rangle$$

We observe that if $-\pi/2 < \nu < \pi/2$ implies that

$$F(|\psi^{(K)}\rangle) > F(|\psi^{(L)}\rangle) \quad (S64)$$

since $\cos \theta_0 - \cos^2 \theta_0 > \sin \theta_0 - \sin^2 \theta_0 > 0$, so $|\psi^{(L)}\rangle$ has a relatively larger component in $|U \psi\rangle$. If $\pi/2 < \nu < 3\pi/2$, then we just have to ensure $(1 - p) \cos \theta_0 > p \cos \theta_1$ for the same thing to be true, which can be ensured by $1 - p > p$.

Thus we only have to show that the condition holds for the channel specified by Kraus operators $L_i$. We have done this already, and re-using the minimal condition we found earlier with $(1 - p) \cos \theta_0 \to (1 - p) \cos^2 \theta_0$, we have the condition

$$(1 - p) \cos^2 \theta_0 > \frac{3}{4}, \quad (S65)$$

which straightforwardly generalizes to as Kraus operators $K_{i\neq 0}$ as we want.

### C. Channels with non-unitary Kraus operators

Finally, we deal with the most general case, where $K_i$ can be non-unitary and $K_0$ can be imperfect. In that case the probability of $K_i$ is state dependent. The key insight here is to realize that as $T \to \infty$, the Kraus operators have well-defined probability given by

$$q^{(i)}_\phi \equiv \langle \phi | K_i^\dagger K_i | \phi \rangle \quad (S66)$$
because at each time step, we feed the apparatus $T - 1$ copies of $|\phi\rangle$ and only a single copy of $|\psi\rangle$. Note that $K_i$ is effectively unitary on $|\phi\rangle$.

To generalize our previous results, we simply have to replace $1 - p$ with $q_{\phi}^{(0)}$ and $p\lambda_i$ with $q_{\phi}^{(i)}$ in the expression for $F_\infty$. If $q_{\phi}^{(0)} = 1 - p$ as with the unitary case, then $q_{\phi}^{(0)} > 3/4$ will suffice. Since $K_0|\psi\rangle$ has the largest component of $|U\psi\rangle$, if $q_{\phi}^{(0)} > 1 - p$ can only increase $F_\infty$. Thus our condition becomes

$$q_{\phi}^{(0)} > q_{\psi}^{(0)} > \frac{3}{4\cos^2 \theta_0}.$$  \hspace{1cm} (S67)

V. ISOMORPHISM WITH ERROR FILTRATION FOR QUANTUM COMMUNICATION

In the event we have a pseudo-vacuum state $|\phi\rangle$ available to us (see Sec. [IV A]), our scheme is explicitly isomorphic to the original error filtration proposal in the ideal case \[S18, S23\]. We will characterize this in the $T = 2$ case, with the more general $T$ following straight-forwardly. For simplicity, suppose our ideal unitary is the identity, as is the case for a quantum communication task. Let the state we want to transmit be

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle.$$ \hspace{1cm} (S68)

The effect of multiplexing in error filtration is to encode this in a W-state \[S23\]. For $T = 2$, this becomes

$$\alpha \hat{a}_0^\dagger + \beta \hat{b}_1^\dagger \mapsto \frac{1}{\sqrt{2}} \alpha \left( \hat{a}_0^\dagger + \hat{b}_0^\dagger \right) + \frac{1}{\sqrt{2}} \beta \left( \hat{a}_1^\dagger + \hat{b}_1^\dagger \right),$$ \hspace{1cm} (S69)

where $\hat{a}_i, \hat{b}_i$ refer to some kind of encoding in a photonic mode.

Explicitly writing this out in the fock basis, we have

$$\frac{1}{\sqrt{2}} \alpha |1, 0, 0, 0\rangle + \frac{1}{\sqrt{2}} \beta |0, 1, 0, 0\rangle + \frac{1}{\sqrt{2}} \alpha |0, 0, 1, 0\rangle + \frac{1}{\sqrt{2}} \beta |0, 0, 0, 1\rangle$$ \hspace{1cm} (S70)

In gate-based EF, we pair the state $|\psi\rangle$ with $|+\rangle$ and attach $|\phi\rangle$ states, which looks like

$$\frac{1}{\sqrt{2}} |0\rangle (\alpha |0\rangle + \beta |1\rangle) |\phi\rangle + \frac{1}{\sqrt{2}} |1\rangle (\alpha |0\rangle + \beta |1\rangle) |\phi\rangle$$ \hspace{1cm} (S71)

$$\mapsto \frac{1}{\sqrt{2}} \alpha |00\rangle |\phi\rangle + \frac{1}{\sqrt{2}} \alpha |10\rangle |\phi\rangle + \frac{1}{\sqrt{2}} \beta |01\rangle |\phi\rangle + \frac{1}{\sqrt{2}} \beta |11\rangle |\phi\rangle$$ \hspace{1cm} (S72)

Written in this way one can clearly see the one-to-one correspondence of kets in either scheme. Due to the principle of superposition, we can deal with each ket individually. If the error operators only act trivially on the state $|\phi\rangle$ (in gate-based EF) and the vacuum (in EF for quantum communication), then we can carefully map the effective error on each ket in gate-based EF to an error on each ket in error filtration.

More formally it was observed in Eqn. (3.1) of \[S20\], there exists a unitary operation that connects a spatially separated quantum superposition of paths into a quantum superposition of paths over an additional register:

$$U(|0\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\Omega\rangle$$ \hspace{1cm} (S73)

$$U(|1\rangle \otimes |\psi\rangle) = |\Omega\rangle \otimes |\psi\rangle.$$ \hspace{1cm} (S74)

By appending the state $|\phi\rangle$, one can embed this in an isomorphism $U'$ that maps gate-based EF states to error filtration states.

VI. ANCILLA ERRORS

A. Effect of ancilla errors

Finally, we work out the effect of ancilla errors. We will consider $X$ and $Z$ type errors, or bit flips and phase flips, on the control register and their effects on the fidelity and success probability of gate-based EF.
FIG. S2: Gate-based EF with $T = 2$, with potential control fault locations highlighted in red.

1. **Bit flip errors**

A bit flip error on the ancilla can potentially ruin the whole query. To see why this happens, we consider the $T = 2$ case, for which the circuit is depicted in Fig. S2. We consider the 3 potential fault locations highlighted in red. For simplicity, suppose the bit flip errors occur in a run where the black box happens to apply the perfect unitary $U$ each time. We will show that a bit flip in any of these indicated fault locations in general leads to the failure of the query.

1. A bit flip in location (1) leads to the state (before measurement and tracing out the active register)

$$|+\rangle \otimes U|\phi\rangle \otimes U|\psi\rangle$$

such that the final state is $U|\phi\rangle$, which in general is unrelated to the state we want. Since the control register is in the $|+\rangle$ state, an error in this location is maximally bad, as the final measurement will post-select $U|\phi\rangle$ with unit probability.

2. A bit flip in location (2) leads to a double query of each of the participating states:

$$\frac{1}{\sqrt{2}}|0\rangle|\psi\rangle U^2|\phi\rangle + \frac{1}{\sqrt{2}}|1\rangle U^2|\psi\rangle|\phi\rangle,$$

which does not contain our desired state $U|\psi\rangle$ at all.

3. A bit flip in location (3) leads to the same final state as (1).

To estimate the effect of such errors on the final scheme, suppose the black box takes a time $\tau_U$ to run, and the error rate of each control ancilla is $\varepsilon_{bf}$. We assume that $\tau_U$ is much longer than the idle time of the control ancilla between the controlled-SWAPs on different branches, as well before and after the first and final queries.

For larger values of $T$, we can assume any such error of the type described by fault locations (1), (3) have a 0 contribution to the fidelity, and do not affect the post-selection probability. To first order in $\varepsilon_C$ we expect the final infidelity to account for this as

$$(1 - F)_{\log T} \simeq \frac{1}{T} (1 - F)_0 + \tau_U \varepsilon_{bf} T \log T.$$  

(S77)
Due to the disastrous potential of bit flip errors, we emphasize that gate-based EF should only be applied when the error hierarchy in the main text is respected with regard to bit flip errors on the control register. The above expression allows us to concretely estimate the error hierarchy required to handle bit flip errors. For gate-based EF to yield an overall error suppression, we require

\[ \frac{1}{T}(1 - F_0) + \tau_U \varepsilon_{bf} T \log T < (1 - F)_0 \]  (S78)

We can expressing the error hierarchy as a ratio between \( \tau_U \varepsilon_{bf} \) and \((1 - F)_0\), we have

\[ \frac{\tau_U \varepsilon_{bf}}{(1 - F)_0} < \frac{T - 1}{T^2 \log T} \simeq \frac{1}{T \log T}. \]  (S79)

This tells us that minimally, for \( T = 2 \), we need the error rate on ancilla qubits to be less than \( 1/2 \) the base infidelity of the apparatus. Furthermore, this indicates that for a set-up with ancilla errors, there is an optimal \( T \) beyond which gate-based EF no longer yields any advantage. This optimal \( T \) is depicted for the case of QRAM in [FIG].

2. Phase flip errors

Since phase flip errors on the control register commute with any controlled-SWAP operation, a phase flip error on fault location (1), (2) or (3) will have the same effect. Let \( \varepsilon_{pf} \) be the phase flip error rate on each control qubit. The presence of a phase flip error anywhere can be commuted to the end of the circuit, where the Hadamard transforms it into a bit flip error, upon which such an error causes the output of the circuit to be rejected.

This means that to first order in \( \varepsilon, \varepsilon_{pf} \), phase flips do not affect the fidelity at all. Since phase flip errors are detected, we expect there to be a decrease in success probability, to first order given by

\[ \Delta P_{pf} \simeq \tau_U \varepsilon_{pf} T \log T \]  (S80)

In conclusion, the effect of phase flip and bit flip errors are well-characterized. Knowing the quantities \( \tau_U, \varepsilon_{bf}, \varepsilon_{pf}, F_0 \) then allows one to see whether the error hierarchy is respected and make a decision as to whether to carry out gate-based EF, as well as to what extent \( T \).

3. Memory errors

A final category of ancilla error to consider are memory errors. These are most likely to happen to the memory register while the active register is querying the apparatus. Any error during this time is deleterious to the final query, as it will result in an error on \(|\psi\rangle\) or \(U|\psi\rangle\) in the \( T - 1 \) inactive branches of the query. Such errors contribute in the same way to the fidelity and error hierarchy as bit flips.

B. Numerical expansion of error suppressed QRAM with ancilla errors

Unfortunately, to perform a full simulation of both the gate-based EF applied to QRAM with errors in both ancilla and QRAM qubits is extremely computationally taxing. Instead, to observe the qualitative performance of gate-based EF under ancilla errors, it will suffice to perform a first order expansion of the infidelity and failure probabilities.

To do so, we assume the infidelity and failure probabilities have the form

\[ (1 - F)_{\log_2 T} = (1 - N_{loc} \varepsilon')(1 - F)_{\log_2 T}^{(0)} + \varepsilon' \sum_{\eta \in \text{err. loc.}} (1 - F)_{\log_2 T}^{(1), \eta}, \]  (S81)

\[ P_{\log_2 T, \text{fail}} = (1 - N_{loc} \varepsilon') P_{\log_2 T, \text{fail}}^{(0)} + \varepsilon' \sum_{\eta \in \text{err. loc.}} P_{\log_2 T, \text{fail}}^{(1), \eta}, \]  (S82)

where \( N_{loc} \) is the number of ancilla error locations in the circuit, \( \varepsilon' \) is the probability of error per ancilla error location. The superscript \( (0) \) indicates the bare infidelity and probability of failure without ancilla errors, as calculated in the main text. The superscript \( (1), \eta \) requires a little more explanation. \( \eta \) indexes ancilla error locations in the circuit, and the superscript \( (1), \eta \) refers to the infidelity/probability of failure given an error in location \( \eta \) but nowhere else.
As such, the above expressions account for the first order expansion in ancilla errors, by discarding the possibility of errors occurring in both ancilla and QRAM at the same time. $(1 - F)_{\log_2 T}^{(1),\eta}, P_{\text{fail}}^{(1),\eta}$ are then calculated numerically.

In this manner, expanding with $X, Z$ errors in the ancillas yields the plots in Fig. S3. Note the qualitative features of these plots – (1) Ancilla $X$ errors eventually overcome the error suppression from gate-based EF, and we have an optimal $T$ for which the overall infidelity is lowest. (2) Probability of failure still appears to plateau with ancilla $X$ errors. (3) Ancilla $Z$ errors do not to first order affect the infidelity scaling. However, this comes at the cost of increased failure probabilities.