GLOBAL EXISTENCE OF WEAK SOLUTIONS TO THE
THREE-DIMENSIONAL PRANDTL EQUATIONS WITH A
SPECIAL STRUCTURE

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Abstract. The global existence of weak solutions to the three space dimensional Prandtl equations is studied under some constraint on its structure. This is a continuation of our recent study on the local existence of classical solutions with the same structure condition. It reveals the sufficiency of the monotonicity condition on one component of the tangential velocity field and the favorable condition on pressure in the same direction that leads to global existence of weak solutions. This generalizes the result obtained by Xin-Zhang [14] on the two-dimensional Prandtl equations to the three-dimensional setting.

1. Introduction. Consider the initial boundary value problem for the Prandtl boundary layer equations in three space variables,

\begin{equation}
\begin{aligned}
\partial_t u + (u\partial_x + v\partial_y + w\partial_z)u + \partial_z p &= \partial_x^2 u, & \text{in } Q, \\
\partial_t v + (u\partial_x + v\partial_y + w\partial_z)v + \partial_y p &= \partial_x^2 v, & \text{in } Q, \\
\partial_x u + \partial_y v + \partial_z w &= 0, & \text{in } Q, \\
(u, v, w)|_{z=0} &= 0, & \lim_{z \to +\infty} (u, v) = (U(t,x,y), V(t,x,y)), \\
(u, v)|_{t=0} &= (u_0(x,y,z), v_0(x,y,z)),
\end{aligned}
\end{equation}

where \( Q := \{ t > 0, (x,y) \in D, z > 0 \} \) with \( D \subset \mathbb{R}^2 \), \( (U(t,x,y), V(t,x,y)) \) and \( p(t,x,y) \) are the traces of the tangential velocity field and the pressure of the Euler system.
flow on the boundary \( \{ z = 0 \} \). Note that the traces satisfy
\[
\begin{align*}
\partial_t U + U \partial_x U + V \partial_y U + \partial_z p &= 0, \\
\partial_t V + U \partial_x V + V \partial_y V + \partial_y p &= 0.
\end{align*}
\] (2)

Despite of its importance in physics, there are very few mathematical results on the Prandtl equations in three space variables. In fact, except the recent work [5] about the classical solution with special structure and those in the analytic framework [12, 3], most of the mathematical studies on this fundamental system in boundary layer theory are limited to the problem in two space dimensions, cf. [1, 4, 7, 9, 10, 13, 14] and the references therein.

Recently in [5], we obtain the local well-posedness of classical solutions to the problem (1) under some constraint on the structure of the solution, in order to avoid the appearance of secondary flow ([8]) in boundary layers. Precisely, assuming that for the Euler flow given in (2),
\[ U(t, x, y) > 0, \]
in the class of boundary layers that the direction of tangential velocity field is invariant in the normal variable \( z \), and the \( x \)-component of velocity \( u(t, x, y, z) \) is strictly increasing in \( z \), \( \partial_z u > 0 \), in [5] we have constructed a classical solution to the problem (1), and it is linearly stable with respect to any three-dimensional perturbation. In the class of this special structure, the solution of (1) takes the form:
\[
(u(t, x, y, z), k(x, y) u(t, x, y, z), w(t, x, y, z)),
\] (3)
where the function \( k(x, y) \) satisfies the following condition (H):
\[(H1)\text{ the function } k \text{ depending only on } (x, y) \text{ satisfies the Burgers equation in the domain } D,
\]
\[ k_x + kk_y = 0; \] (4)
\[(H2) \text{ the outer Euler flow } \]
\[
(U(t, x, y), k(x, y) U(t, x, y), 0, p(t, x, y))
\]
with \( U(t, x, y) > 0 \), satisfies that from the system (2),
\[
\begin{align*}
\partial_t U + U \partial_x U + kU \partial_y U + \partial_z p &= 0, \\
\partial_y p - k \partial_x p &= 0.
\end{align*}
\]

Moreover, the authors recently observed in [6] that for the shear flow \((u^s(t, z), v^s(t, z), 0)\) of the three-dimensional Prandtl equations, the special solution structure (3) is the only stable case.

Under the above assumption (3) of special solution structure, the original problem (1) of three-dimensional Prandtl equations is reduced to the following one for two unknown functions \((u, w)\):
\[
\begin{align*}
\partial_t u + (u \partial_x + ku \partial_y + w \partial_z) u - \partial^2_z u &= -\partial_x p, & \text{in } Q, \\
\partial_t w + \partial_y (ku) + \partial_z w &= 0, & \text{in } Q, \\
u|_{z=0} = w|_{z=0} = 0, & \lim_{z \to +\infty} u = U(t, x, y), \\
u|_{\partial Q_-} = u_1(t, x, y, z), & u|_{z=0} = u_0(x, y, z),
\end{align*}
\] (5)
where \( \partial Q_- = (0, \infty) \times \gamma_- \times \mathbb{R}_+ \), with
\[
\gamma_- = \{(x, y) \in \partial D \mid \langle 1, k(x, y) \rangle \cdot \vec{n}(x, y) < 0 \},
\]
and \( \vec{n}(x, y) \) being the unit outward normal vector of \( D \) at \((x, y) \in \partial D \).
For this reduced problem under the assumption that
\[ \partial_z u_0 > 0, \quad \partial_z u_1 > 0, \quad \text{for } z \geq 0, \] in the class of \( \partial_z u > 0 \), we apply the method developed by Oleinik [10] for two dimensional Prandtl equations. Precisely, by the Crocco transformation,
\[ \xi = x, \quad \eta = y, \quad \zeta = \frac{u(t, x, y, z)}{U(t, x, y)}, \quad W(t, \xi, \eta, \zeta) = \frac{\partial_z u(t, x, y, z)}{U(t, x, y)}, \]
the problem (5) becomes the following initial boundary value problem,
\[
\begin{cases}
L(W) := \partial_t W + \zeta U (\partial_\xi + k \partial_\eta) W + A \partial_\zeta W + BW - W^2 \partial_\zeta^2 W = 0, & \text{in } \Omega_T, \\
W|_{\zeta=1} = 0, \quad W \partial_\zeta W|_{\zeta=0} = \frac{p_x}{U}, \\
W|_{\Gamma_-} = W_1(t, \xi, \eta, \zeta) := \frac{\partial_\eta u}{U}, \\
W|_{t=0} = W_0(\xi, \eta, \zeta) := \frac{\partial_\eta u}{U},
\end{cases}
\]
where
\[ \Omega_T = \{(t, \xi, \eta, \zeta) | 0 < t < T, \ (\xi, \eta) \in D, \ 0 \leq \zeta < 1\}, \]
\[ \Gamma_- = \{(t, \xi, \eta, \zeta) | 0 < t < T, \ (\xi, \eta) \in \gamma_-, \ 0 \leq \zeta < 1\}, \]
and
\[ A = -\zeta(1-\zeta) \frac{U_t}{U} - (1-\zeta^2) \frac{p_x}{U}, \quad B = \frac{U_t}{U} + \zeta(U_x + kU_y) - \partial_\eta k \cdot \zeta U. \] (8)
For the problem (7) of the degenerate parabolic equation, we have already established local existence of classical solutions in [5].

As a continuation of the paper [5], the purpose of this paper is to prove the global in time existence of a weak solution to the problem (7) for data satisfying (6) and the favorable pressure condition:
\[ p_x(t, \xi, \eta) \leq 0, \quad \text{for } t > 0, \ (\xi, \eta) \in D. \] (9)
For this, we will adopt the approach introduced by Xin-Zhang in [14] for the two-dimensional Prandtl equations to the three-dimensional setting. As observed in [11], the main motivation of introducing the favorite condition on pressure is to avoid the separation of boundary layers.

For completeness, the definition of weak solutions to (7) is given as follows.

**Definition 1.1.** A function \( W(t, \xi, \eta, \zeta) \in \text{L}^\infty(0, T; \text{BV}(\Omega)) \) for some \( T > 0 \) is called a weak solution of the problem (7) in \( t < T \), if the following conditions hold:

(i) There exists a positive constant \( C \) such that
\[ C^{-1}(1-\zeta) \leq W(t, \xi, \eta, \zeta) \leq C(1-\zeta), \quad \forall (t, \xi, \eta, \zeta) \in \Omega_T. \]

(ii) \( W \) satisfies the first equation and the initial boundary conditions of (7) in the weak sense:
\[
\int_0^T \int_{\Omega} \left\{ \frac{1}{W} \left[ \psi_t + \zeta (U \psi)_{\xi} + \zeta (U k \psi)_{\eta} + (A \psi)_{\zeta} - B \psi \right] - W \psi_{\zeta \zeta} \right\} d\xi d\eta d\zeta dt
= \int_{\Omega} \frac{1}{W_0} \psi|_{t=0} d\xi d\eta d\zeta - \int_0^T \int_D \frac{p_x}{U} \psi \frac{1}{W}|_{\zeta=0} d\xi d\eta dt + \int_0^T \int_{\gamma_-} \int_0^1 \frac{\zeta U \psi}{W_1} \cdot k_n d\zeta d\xi dt,
\]
for any test function \( \psi(t, \xi, \eta, \zeta) \in \text{C}^\infty(\Omega_T) \) satisfying
\[ \psi = 0, \quad \text{at } t = T \text{ or } (\xi, \eta) \in \gamma_+; \quad \psi_{\zeta} = 0 \text{ at } \zeta = 0. \]
Theorem 1.2. For the problem
\[ \begin{array}{l}
(1, k(x, y)) \cdot \vec{n}(x, y) > 0 \},
\end{array} \]
and the function \( k_n = (1, k) \cdot \vec{n} \) with \( \vec{n} \) being the unit outward normal vector of \( \partial D \).

The main result of this paper can be stated as follows.

Theorem 1.2. For the problem (7) and any \( T > 0 \), assume that \( k \in C^2(D) \), \( U \in C^2((0, T) \times D) \) and \( p_e \in C^1((0, T) \times D) \) satisfy (9), and the initial boundary data \( W_0 \in C^1(\Omega) \), \( W_1 \in C^3(\Gamma_-) \) satisfy
\[ C_0^{-1}(1 - \zeta) \leq W_0, W_1 \leq C_0(1 - \zeta), \]
for a positive constant \( C_0 \). Then, there exists a weak solution \( W(t, \xi, \eta, \zeta) \in L^\infty(0, T; BV(\Omega)) \) to the problem (7) in the sense of Definition 1.1.

2. The proof of the main result. Following the approach introduced in [14], a viscous splitting method is used to construct a sequence of approximate solutions to the problem (7). Precisely, divide the time interval \([0, T]\) into \( n \) equal sub-intervals:
\[ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T, \quad t_{i+1} - t_i = \frac{T}{n}, \quad \text{for } 0 \leq i \leq n - 1. \]

First, in the time step \([0, t_1]\) or \([t_i, t_{i+1}]\) for an even \( i \), we construct the approximate solution by solving the following initial boundary value problem for a porous media-type equation:
\[ \begin{aligned}
\frac{1}{2} \partial_t W - W^2 \partial^2 W + A \partial_\zeta W + bW &= 0, \quad \text{in } (0, t_1) \times \Omega \text{ or } (t_i, t_{i+1}) \times \Omega, \\
W|_{t=0} &= W_0, \quad \text{or } W|_{t=t_i} = W(t_i, \xi, \eta, \zeta) \text{ (given in the previous step)}, \\
W|_{\zeta=1} &= 0, \quad WW|_{\zeta=0} = \frac{p_e}{b};
\end{aligned} \]
and in the time step \([t_i, t_{i+1}]\) for an odd \( i \), we construct the approximate solution by solving the following problem for a transport equation:
\[ \begin{aligned}
\frac{1}{2} \partial_t W + \zeta U(\partial_\zeta + k \partial_\eta)W + (B - b)W &= 0, \quad \text{in } (t_i, t_{i+1}) \times \Omega, \\
W|_{t=t_i} &= W(t_i, \xi, \eta, \zeta) \text{ (given in the previous step)}, \\
W|_{\zeta=1} &= W_1.
\end{aligned} \]

Here, the coefficient function \( b \) in (11) will be chosen later to satisfy the boundary condition \( W|_{\Gamma_-} = W_1 \) for the solution of (11). What is needed is to prove that the function \( W \), constructed in the time interval \([0, T]\), is uniformly bounded in \( n \) and has a uniform total variation with respect to the spatial variables \( \xi, \eta \) and \( \zeta \). This implies that as \( n \to \infty \), the limit function of the approximate solutions \( W \) constructed in (11)-(12) is a weak solution to the problem (7). The proof is divided into the following subsections.

2.1. Porous medium-type equation. In this subsection, consider the following problem for a porous medium-type equation
\[ \begin{aligned}
\partial_t W - W^2 \partial^2 W + A \partial_\zeta W + bW &= 0, \quad \text{for } 0 < t < T, \quad (\xi, \eta, \zeta) \in \Omega, \\
W|_{t=0} &= W_0(\xi, \eta, \zeta), \\
W|_{\zeta=0} &= \frac{p_e}{b} \leq 0, \quad W|_{\zeta=1} = 0.
\end{aligned} \]

In order to match the boundary condition \( W|_{\Gamma_-} = W_1 \), by observing that \( W_1 > 0 \) for \( 0 \leq \zeta < 1 \) and
\[ W_1 = O(1 - \zeta), \quad \text{as } \zeta \to 1, \]
we set
\[ b(t, \xi, \eta, \zeta) = -\frac{f(\xi, \eta)}{W_1} \left( \partial_\xi W_1 - (W_1)^2 \partial_\xi^2 W_1 + A \right|_{\Gamma_\zeta} \cdot \partial_\zeta W_1 \right), \tag{14} \]
where \( f(\xi, \eta) \) is a non-negative smooth function defined on the closure of the domain \( D \) satisfying \( f(\xi, \eta)|_{\Gamma_\zeta} = 1 \). By the formulations in (8):
\[ A = -(1 - \zeta) \left[ \frac{U_1}{U} + (1 + \zeta) \frac{P_\xi}{U} \right], \]
and the assumption given in Theorem 1.2, there exists a positive constant \( M_0 \) depending on the parameters of (7), such that for the function \( b(t, \xi, \eta, \zeta) \) given in (14),
\[ \|b(t, \xi, \eta, \zeta)\|_{W^{1, \infty}(\Omega_T)} \leq M_0. \tag{15} \]
The problem (13) can be viewed as a one space dimensional problem by regarding variables \( \xi \) and \( \eta \) as parameters.

Note that the equation in (13) is degenerate on the boundary \( \{ \zeta = 1 \} \). As [14], consider the following uniformly approximated parabolic problem:
\[ \begin{aligned}
& \begin{cases}
\partial_t W_\epsilon - (W_\epsilon^2 + \epsilon) \partial_\xi^2 W_\epsilon + A \partial_\xi W_\epsilon + b W_\epsilon = 0, & \text{for } t > 0, \ (\xi, \eta, \zeta) \in \Omega, \\
(W_\epsilon|_{\xi=0} = W_0 > 0, \ \\
\partial_\xi W_\epsilon|_{\xi=0} = \frac{p_\epsilon}{U} \leq 0, \ (W_\epsilon|_{\zeta=1} = 0, 
\end{cases} \\
\end{aligned} \tag{16} \]
for a positive constant \( \epsilon > 0 \). It is known that problem (16) has a unique smooth solution. After getting some uniform bounds of \( W_\epsilon \), we obtain a solution to the problem (13) by taking \( \epsilon \to 0 \) in (16). In fact, we have

**Theorem 2.1.** Under the assumption of Theorem 1.2, the problem (13) has a unique solution \( W \in BV(0, T; \Omega) \) for any fixed \( T > 0 \). Moreover, \( W \) has the following properties:

1. there exists a positive constant \( \beta \), depending on \( \|W_0\|_{L^\infty} \) and the \( L^\infty \)-norm of the parameters of (13), such that
\[ \theta_0 e^{-\beta t} \varphi(\zeta) \leq W(t, \xi, \eta, \zeta) \leq C_1 e^{M_1 t}(1 - \zeta), \tag{17} \]
where
\[ \varphi(\zeta) = e^{\frac{\pi}{2} \xi} \sin \left[ \frac{\pi}{2} (1 - \zeta) \right], \quad \theta_0 = \min_{(\xi, \eta, \zeta) \in \Omega} \frac{\varphi(\xi, \eta, \zeta)}{\varphi(\zeta)}, \]
\[ C_1 = \max \left\{ \|W_0\|_{L^\infty}, \left( \frac{||p_\epsilon||_{L^\infty}}{||U||_{L^\infty}} \right)^{\frac{1}{2}}, \sqrt{\left( \frac{p_\epsilon}{U} \right)_{L^\infty}} \right\}, \quad M_1 = \frac{||A||_{L^\infty}}{1 - \zeta} - \frac{||b||_{L^\infty}}{1 - \zeta}; \tag{18} \]

2. there exists a positive constant \( C_2 \), depending on \( \|W\|_{L^\infty}, \|\partial_\xi W_0\|_{L^\infty} \) and the \( C^1 \)-norm of the parameters of (13), such that
\[ |W_\epsilon| \leq C_2; \tag{19} \]

3. for any \( t > 0 \),
\[ \int_0^t |W_\zeta(t, \xi, \eta, \zeta)| d\zeta \leq \int_0^1 |W_{0, \zeta}(\xi, \eta, \zeta)| d\zeta + W(t, \xi, \eta, 0) - W_0(\xi, \eta, 0), \tag{20} \]
and
\[ \int_0^1 \frac{|W_\zeta(t)|}{W^2(t)} (1 - \zeta)^2 d\zeta \leq \frac{1}{\int_0^1 W_0^2 |W_0| d\zeta} \int_0^1 (1 - \zeta)^2 d\zeta + C_3 t \left( 1 + \int_0^t |W_{0, \zeta}| d\zeta \right) \]
\[ + C_3 \int_0^t \int_0^1 |W_\zeta(s)| W^2(s) (1 - \zeta)^2 d\zeta ds. \tag{21} \]
Here, the positive constant $C_3$ depends on $\|W\|_{L^\infty}$ and the $C^1$-norm of the parameters of (13). Also, $W_\eta$ and $W_i$ satisfy similar estimates as (21) by simply replacing the partial derivative with respect to $\xi$ in (21) by the partial derivatives with respect to $\eta$ and $t$, respectively.

Proof. To prove the inequality (17), we first show

$$W_\epsilon|_{\zeta=0} \geq 0 \quad \forall \epsilon > 0$$

(22) holds for the problem (16). Otherwise, by using the continuity of $W_\epsilon$ and $W_0|_{\zeta=0} > 0$, there exist $\epsilon_0 > 0$ and a point $P$ on $\{\zeta = 0\}$, such that $W_\epsilon|_{P} = 0$. That is, $W_\epsilon$ attains its minimum at $P$, which implies that $\partial_\zeta W^{2}_\epsilon|_{P} \geq 0$. But, from the boundary condition of (16) on $\zeta = 0$, we have

$$\partial_\zeta W^{2}_\epsilon|_{\zeta=0} = 2W_{e_\xi} \partial_\zeta W_\epsilon|_{\zeta=0} = \frac{2P_{e}}{U} < 0,$$

which is a contradiction. Hence, we obtain (22), and then $W|_{\zeta=0} \geq 0$ for the problem (13) by letting $\epsilon \to 0$.

Next, combining $W|_{\zeta=0} \geq 0$ with the boundary condition $WW|_{\zeta=0} = \frac{P_{e}}{U} \leq 0$, we obtain $W|_{\zeta=0} \leq 0$. Now, the rest of proof for (17) is similar to that of Lemma 4.2 in [14] so that we omit the details. Moreover, there is a constant $m > 0$ such that

$$W|_{\zeta=0} \geq m, \quad W_\epsilon|_{\zeta=0} \geq m, \quad \forall \epsilon > 0.$$

(23)

(2) We turn to the estimate (19). Consider the problem (16) for $W_{\epsilon}$, and the corresponding problem for $\partial_\zeta W_{\epsilon}$ as follows,

$$\begin{cases}
L^*(\partial_\zeta W_{\epsilon}) - 2W_{e_\xi} \partial_\zeta W_{\epsilon} \partial_\zeta^2 W_{\epsilon} + (A_{\zeta} + b) \partial_\zeta W_{\epsilon} = -\beta \epsilon W_{\epsilon}, \\
\partial_\zeta W_{\epsilon}|_{\zeta=0} = W_{0,\zeta}, \\
W_{\epsilon} \partial_\zeta W_{\epsilon}|_{\zeta=0} = \frac{P_{e}}{U}, \quad \partial_\zeta^2 W_{\epsilon}|_{\zeta=1} = 0,
\end{cases}$$

(24)

where the operator

$$L^* = \partial_\zeta - (W_{\epsilon}^2 + \epsilon) \cdot \partial_\zeta^2 + A \cdot \partial_\zeta.$$

Here, note that $\partial_\zeta^2 W_{\epsilon}|_{\zeta=1} = 0$ because $W_{\epsilon}|_{\zeta=1} = 0$ and $A|_{\zeta=1} = 0$ in the first equation of (16).

Set $V = \partial_\zeta W_{\epsilon} - \alpha \zeta$ with $\alpha$ being a constant to be chosen later. It follows that

$$L^*(V) - 2W_{\epsilon}(V + \alpha \zeta)V + (A_{\zeta} + b - 2\alpha W_{\epsilon})V$$

$$= (2\alpha^2 \zeta - b \zeta)W_{\epsilon} - \alpha A - \alpha A_{\zeta} + b) := Y.$$  

(25)

From the first step, we know that $Y$ is bounded provided that $\alpha$ is bounded. Let $V_1 = e^{-\beta t}V$ with $\beta > 0$ being sufficiently large and satisfying

$$\beta + A_{\zeta} + b - 2\alpha W_{\epsilon} \geq 1, \quad \text{or} \quad \beta \geq \|A_{\zeta} + b - 2\alpha W_{\epsilon}\|_{L^\infty} + 1.$$

From (25) and (24), we have

$$L^*(V_1) - 2W_{\epsilon}(V + \alpha \zeta)(V_1)_{\zeta} + (\beta + A_{\zeta} + b - 2\alpha W_{\epsilon})V_1 = e^{-\beta t}Y,$$

(26)

with the following initial and boundary conditions

$$V_1|_{\zeta=0} = W_{0,\zeta} - \alpha \zeta, \quad W_{\epsilon} V_1|_{\zeta=0} = \frac{P_{e}}{U}, \quad \partial_\zeta V_1|_{\zeta=1} = \alpha e^{-\beta t}.$$  

(27)
Firstly, note that for an arbitrarily fixed constant $\alpha > 0$, from (27) we have
$$\partial_\xi V_1|_{\xi=1} < 0.$$ Also, from (23) and the relation in (27):
$$W_0 V_1|_{\xi=0} = e^{-\beta t} \frac{P_x}{U} \leq 0,$$

it implies that $V_1$ does not attain its positive maximum on $\xi = 1$ or $\xi = 0$. Then, if $V_1$ attains its positive maximum in the interior of $\Omega_T$ or at $\{t = T\}$, we obtain from (26) that
$$V_1 \leq \max \{e^{-\beta t} Y\} \leq \|Y\|_{L^\infty}.$$

If $V_1$ achieves its positive maximum at $\{t = 0\}$, it follows that
$$V_1 \leq \max \{W_0, \alpha \xi\} \leq \|W_0, \xi\|_{L^\infty}.$$ Therefore, we conclude that $V_1 \leq \max \{\|Y\|_{L^\infty}, \|W_0, \xi\|_{L^\infty}\}$, which implies that
$$\partial_\xi W_\epsilon \leq \alpha + e^{\beta t} \max \{\|Y\|_{L^\infty}, \|W_0, \xi\|_{L^\infty}\}.$$ (28)

Secondly, for an arbitrarily fixed constant $\alpha < 0$, by considering the possible negative minimal points of $V_1$ on $\Omega_T$, similar to the above arguments, we have
$$V_1 \geq - \max \{\|Y\|_{L^\infty}, \|W_0, \xi\|_{L^\infty}, \|\frac{P_x}{U} \frac{1}{W_\epsilon} \|_{\xi=0} \|L^\infty\}\},$$
which implies that
$$\partial_\xi W_\epsilon \geq \alpha - e^{\beta t} \max \{\|Y\|_{L^\infty}, \|W_0, \xi\|_{L^\infty}, \|\frac{P_x}{U} \frac{1}{W_\epsilon} \|_{\xi=0} \|L^\infty\}\}. \quad (29)$$

Hence, by combining (28) with (29) and letting $\alpha \to 0$ it yields that
$$|\partial_\xi W_\epsilon| \leq e^{\beta t} \max \{\|Y\|_{L^\infty}, \|W_0, \xi\|_{L^\infty}, \|\frac{P_x}{U} \frac{1}{W_\epsilon} \|_{\xi=0} \|L^\infty\}\},$$

where $\beta \geq \|A_\xi + b\|_{L^\infty} + 1$ and $Y = -b_\xi W_\epsilon$. Thus, we obtain (19) as $\epsilon \to 0$.

(3) The proofs of (20) and (21) are similar to those given in Lemmas 4.6 and 4.7 of [14], respectively. And the proof for the uniqueness of solution to the problem (13) is similar to that of Theorem 4.1 in [14]. Thus, we omit the detail for brevity and this completes the proof of the theorem.

2.2. Transport equation. In this section, we will study the problem of transport equation (12) for $t \in (t_i, t_{i+1}]$ with $i$ being odd. That is, we consider the following problem,
$$\begin{align*}
\frac{1}{2} \partial_t W + \zeta U (\partial_\zeta + k \partial_\eta) W + b_1 W &= 0, \quad in \ (t_i, t_{i+1}] \times \Omega, \\
W|_{t=t_i} &= W(t_i, \xi, \eta, \zeta), \\
W|_{t=t_{i+1}} &= W_1,
\end{align*} \quad (30)$$

where the function
$$b_1(t, \xi, \eta, \zeta) = (B - b)(t, \xi, \eta, \zeta)$$
with functions $B$ and $b$ being given in (8) and (14), respectively.

For any fixed $(t, \xi, \eta, \zeta) \in (t_i, t_{i+1}] \times \Omega$, the characteristics of the equation (30) passing through this point are denoted by:
$$(s, \gamma_1(s; t, \xi, \eta, \zeta), \gamma_2(s; t, \xi, \eta, \zeta), \zeta), \quad s \in (t_i, t_{i+1}]$$
with $\gamma_1$ and $\gamma_2$ being determined by
\[
\begin{cases}
\gamma_1'(s; t, \xi, \eta, \zeta) = 2\zeta \cdot U\left(s, \gamma_1(s; t, \xi, \eta, \zeta), \gamma_2(s; t, \xi, \eta, \zeta)\right), \\
\gamma_2'(s; t, \xi, \eta, \zeta) = 2\zeta \cdot (kU)\left(s, \gamma_1(s; t, \xi, \eta, \zeta), \gamma_2(s; t, \xi, \eta, \zeta)\right), \\
(\gamma_1, \gamma_2)(t; t, \xi, \eta, \zeta) = (\xi, \eta).
\end{cases}
\] (32)

For simplicity of notations, in the following we will also use abbreviations $\gamma_1(s)$ and $\gamma_2(s)$ to represent $\gamma_1(s; t, \xi, \eta, \zeta)$ and $\gamma_2(s; t, \xi, \eta, \zeta)$ respectively, when without confusion.

Combining (32) with the property $k_\xi + kk_\eta = 0$, we have
\[
\frac{d}{ds}k\left(\gamma_1(s), \gamma_2(s)\right) = 0,
\]
which implies that
\[
k\left(\gamma_1(s), \gamma_2(s)\right) = k(\xi, \eta), \quad \forall s \in (t_i, t_{i+1}].
\] (33)

Then, from (32) and (33) it follows
\[
\gamma_2(s) = k(\xi, \eta) \cdot \gamma_1(s) + \eta - k(\xi, \eta)\xi,
\] (34)
and then $\gamma_1(s)$ is given by
\[
\begin{cases}
\gamma_1'(s) = 2\zeta \cdot U\left(s, \gamma_1(s), k(\xi, \eta) \cdot \gamma_1(s) + \eta - k(\xi, \eta)\xi\right), \\
\gamma_1(t) = \xi.
\end{cases}
\] (35)

Observe that $\gamma_1'(s) \geq 0$, and the projection of this characteristic on the $(\xi, \eta)$-plane is a straight line passing through $(\xi, \eta)$ with slope $k(\xi, \eta)$. Moreover, the function $k(\xi, \eta)$ remains constant along this line. Note that the solution of the problem (35) exists and is unique when the function $U(t, \xi, \eta)$ is Lipschitz in $(\xi, \eta)$.

Next, the solution $W$ of the problem (30) can be represented by
\[
W(t, \xi, \eta, \zeta) = W\left(s, \gamma_1(s), \gamma_2(s)\right) \exp\left\{-\int_s^t b_1\left(\tilde{s}, \gamma_1(\tilde{s}), \gamma_2(\tilde{s}), \zeta\right) d\tilde{s}\right\}. \] (36)

Denote by
\[
t^*(t, \xi, \eta, \zeta) := \inf\left\{\tilde{t} \in [t_i, t] : \forall s \in (\tilde{t}, t], \left(\gamma_1(s; t, \xi, \eta, \zeta), \gamma_2(s; t, \xi, \eta, \zeta)\right) \in D\right\}. \] (37)

Note that if $t^* > t_i$, then
\[
\left(\gamma_1(t^*; t, \xi, \eta, \zeta), \gamma_2(t^*; t, \xi, \eta, \zeta)\right) \in \partial D.
\]

Then, denote by
\[
Q_1 := \left\{(t, \xi, \eta, \zeta) \in (t_i, t_{i+1}] \times \Omega : t^*(t, \xi, \eta, \zeta) = t_i\right\},
\]
and
\[
Q_2 := \left\{(t, \xi, \eta, \zeta) \in (t_i, t_{i+1}] \times \Omega : t^*(t, \xi, \eta, \zeta) > t_i, \left(\gamma_1(t^*), \gamma_2(t^*)\right) \in \partial_\gamma\right\},
\]
where $\partial_\gamma$ is the closure of $\gamma_- = \{(\xi, \eta) \in \partial D : (1, k(\xi, \eta)) \cdot \bar{n}(\xi, \eta) < 0\}$ on the boundary $\partial D$.

To study the estimate of the solution to problem (30), we first give the following proposition for the representation of the solution.
Proposition 1. For the problem (30), we have
\[(t_i, t_{i+1}] \times \Omega = Q_1 \cup Q_2,\]
and for any \((t, \xi, \eta, \zeta) \in (t_i, t_{i+1}] \times \Omega,\) the solution of (30) can be represented as follows:
\[W(t, \xi, \eta, \zeta) = \begin{cases} \exp \left\{ - \int_{t_i}^{t} b_1(s, \gamma_1(s), \gamma_2(s), \zeta) ds \right\}, & \text{in } Q_1, \\ \exp \left\{ - \int_{t_i}^{t} b_1(s, \gamma_1(s), \gamma_2(s), \zeta) ds \right\}, & \text{in } Q_2. \end{cases} \tag{38}\]

Proof. It suffices to show that \((t_i, t_{i+1}] \times \Omega = Q_1 \cup Q_2,\) because the formulation (38) follows immediately from (36). We divide the proof into the following two parts.

(1) Firstly, we claim that
\[(t, \xi, \eta, 0) \in Q_1, \quad \forall t \in (t_i, t_{i+1}), \quad (\xi, \eta) \in D.\]
Indeed, from (34) and (35) it follows that \(\gamma'_1(s) = 0,\) and \((\gamma_1(s), \gamma_2(s)) \equiv (\xi, \eta)\) when \(\zeta = 0,\) which implies that \(t^*(t, \xi, \eta, 0) = t_i\) by using the definition (37).

(2) Next, we will prove that if
\[(t, \xi, \eta, \zeta) \in (t_i, t_{i+1}] \times \Omega \setminus Q_1 \quad \text{and} \quad \zeta > 0, \tag{39}\]
then
\[(t, \xi, \eta, \zeta) \in Q_2.\]
Indeed, for such point \((t, \xi, \eta, \zeta)\) satisfying (39),
\[t^*(t, \xi, \eta, \zeta) > t_i, \quad \text{and} \quad (\xi^*, \eta^*) := \left(\gamma_1(t^*), \gamma_2(t^*)\right) \in \partial D.\]
We then need to show that \(P = (\xi^*, \eta^*) \in \gamma_\prec.\)

Note that from (34) and (35),
\[\gamma'_1(s) > 0, \quad \gamma_1(s) > \xi^*, \quad \forall s \in \{t^*, t_i\}, \tag{40}\]
and
\[\gamma_2(s) = k(\xi^*, \eta^*) (\gamma_1(s) - \xi^*) + \eta^*, \quad \forall s \in \{t^*, t_i\}. \tag{41}\]
Without loss of generality, assume that in a neighborhood \(P_\delta\) of the point \(P = (\xi^*, \eta^*)\) in the \((\xi, \eta)\)-plane, the boundary of \(D\) is represented by a smooth function \(\eta = f(\xi),\) and \(\eta > f(\xi)\) when \((\xi, \eta) \in D \cap P_\delta.\) Then, the outward normal vector at the point \(\left(\xi, f(\xi)\right)\) is given by
\[\vec{n}(\xi, f(\xi)) = \frac{1}{\sqrt{1 + (f'(\xi))^2}} \left(f'(\xi), -1\right). \tag{42}\]
By the definition of \(t^*\), we have there exists a \(\epsilon_0 > 0\) such that
\[\gamma_2(s) > f(\gamma_1(s)), \quad \forall s \in \{t^*, t_i + \epsilon_0\}, \tag{43}\]
then, combining with (40) and (41), we know that there exists a constant \(\epsilon_1 > 0\) such that
\[k(\xi^*, \eta^*)(\xi - \xi^*) + \eta^* > f(\xi), \quad \forall \xi \in (\xi^*, \xi^* + \epsilon_1). \tag{44}\]
If \(P\) does not belong to \(\gamma_\prec,\) then by continuity, there exists a \(\delta_1 < \delta\) such that
\[(1, k(\xi, \eta)) \cdot \vec{n}(\xi, \eta) \geq 0, \quad \forall (\xi, \eta) \in \partial D \cap P_{\delta_1}, \tag{45}\]
which implies from (42) that there exists an \( \epsilon_2 > 0 \), satisfying
\[
\left(1, k(\xi, f(\xi))\right) \cdot (f'(\xi), 1) \geq 0, \quad \forall \xi \in (\xi^* - \epsilon_2, \xi^* + \epsilon_2).
\]
This is,
\[
f'(\xi) \geq k(\xi, f(\xi)), \quad \forall \xi \in (\xi^* - \epsilon_2, \xi^* + \epsilon_2).
\]
(44)

For a fixed \( \xi_0 \in (\xi^*, \xi^* + \epsilon_3) \) with \( \epsilon_3 \leq \min\{\epsilon_1, \epsilon_2\} \), consider the function
\[
F(\xi) = f(\xi) - f(\xi_0) - k(\xi, f(\xi))(\xi - \xi_0), \quad \xi \in [\xi^*, \xi_0].
\]
Since \( k_\xi + k_{\eta} = 0 \) from (4), it follows that
\[
F'(\xi) = f'(\xi) - k(\xi, f(\xi)) - [k_\xi(\xi, f(\xi)) + f'(\xi)k_{\eta}(\xi, f(\xi))] \cdot (\xi - \xi_0)
\]
\[
= f'(\xi) - k(\xi, f(\xi)) - \left[-k_\xi(\xi, f(\xi)) + f'(\xi)\right]k_{\eta}(\xi, f(\xi)) \cdot (\xi - \xi_0)
\]
\[
= \left[f'(\xi) - k(\xi, f(\xi))\right] \cdot \left[1 - k_\xi(\xi, f(\xi)) \cdot (\xi - \xi_0)\right].
\]
Then, from (44) and the fact that \( k_\eta \) is bounded, we have
\[
F'(\xi) \geq 0, \quad \forall \xi \in (\xi^*, \xi_0)
\]
(45)

provided that \( \epsilon_3 \) is sufficiently small. Therefore, (45) gives
\[
0 = F(\xi_0) \geq F(\xi^*) = \eta^* - f(\xi_0) - k(\xi^*, \eta^*)(\xi^* - \xi_0),
\]
which is a contradiction to (43) by letting \( \xi = \xi_0 \) in (43). Hence, we have \( P \in \gamma_- \), which means that \( (t, \xi, \eta, \zeta) \in Q_2 \), and this completes the proof of the proposition.

\[\square\]

2.3. Proof of Theorem 1.2. Based on the results obtained in the above two subsections, we will give the proof of Theorem 1.2 in this subsection. Before it, the following lemmas and propositions are needed.

**Lemma 2.2.** Let \( W \) be the approximate solution constructed by (11) and (12). Then
\[
|W(t, \xi, \eta, \zeta)| \leq \bar{C}_1 \bar{M}_1 e^{\tilde{M}_1 t}(1 - \zeta),
\]
(46)

where
\[
\bar{C}_1 = \max\left\{C_1, \frac{\|W_1\|_{L^\infty}}{1 - \zeta}\right\}, \quad \bar{M}_1 = \max\left\{M_1, \|B - b\|_{L^\infty}\right\}
\]

with positive constants \( C_1 \) and \( M_1 \) being given in (18). Moreover, there exists \( \tilde{\beta} \) depending only on \( \|W_0\|_{L^\infty}, \|W_1\|_{C^2} \) and the parameters of problem (7), such that
\[
W(t, \xi, \eta, \zeta) \geq \tilde{\beta}_0 e^{-\bar{\beta} t} \varphi(\zeta),
\]
(47)

where \( \varphi(\zeta) \) is given in (18) and \( \tilde{\beta}_0 = \min_{(t, \xi, \eta, \zeta) \in \Omega_T} \left\{ \frac{W_0(t, \xi, \eta, \zeta)}{\varphi(\zeta)}, \frac{W_1(t, \xi, \eta, \zeta)}{\varphi(\zeta)} \right\} \).

**Proof.** When \( 0 \leq t \leq t_1 \), the estimates (46) and (47) follow from (17) in Theorem 2.1 immediately. Assume that (46) holds for \( 0 \leq t \leq t_i \) with \( i \geq 1 \), and consider the case for \( t_i \leq t \leq t_{i+1} \). If \( i \) is even, we have that by virtue of (17),
\[
|W(t)| \leq \max\left\{ \frac{\|W(t_i)\|_{L^\infty}}{1 - \zeta}, \sqrt{\frac{\|B\|_{L^\infty}}{U}}\right\} e^{M_1(t-t_i)}(1 - \zeta) \leq \bar{C}_1 e^{\bar{M}_1 t}(1 - \zeta)
\]
by using the induction hypothesis. If \( i \) is odd, from (38) and (31) it follows
\[
|W(t)| \leq |W(t_i)| \cdot e^{\|B - b\|_{L^\infty}(t-t_i)} \leq \bar{C}_1 e^{\bar{M}_1 t}(1 - \zeta)
\]
by using the induction hypothesis again, or
\[ |W(t)| \leq W_1 \cdot e^{\|B-b\|_{L^\infty}(t-t_i)} \leq \tilde{C}_1 e^{\tilde{M}_1 t}(1-\zeta). \]
Thus, we conclude the estimate (46).

Next, suppose that (47) holds for \( 0 \leq t \leq t_i \), \( i \geq 1 \), and consider \( t \in [t_i, t_{i+1}] \).
Then if \( i \) is even, from (17) in Theorem 2.1, we know that there exists \( \bar{\beta} \) depending
on \( \tilde{C}_1, \|b\|_{L^\infty} \) and the parameters in the problem (7) such that
\[ W \geq \min \left\{ \frac{|W(t_i)|}{\varphi} \right\} e^{-\bar{\beta}(t-t_i)} \varphi \geq \tilde{\theta}_0 e^{-\bar{\beta} t} \varphi. \]
If \( i \) is odd, the estimate (47) is a direct consequence of the expression (36) in Proposition 1, provided that \( \bar{\beta} \geq \|B-b\|_{L^\infty} \).
Thus, we complete the proof. \( \square \)

**Remark 1.** From Lemma 2.2 and by virtue of (15), we find that there exists
a constant \( \tilde{C}_0 \), depending only on \( \|W_0\|_{L^\infty}, \|W_1\|_{C^1} \) and the parameters in the problem (7), such that
\[ \tilde{C}_0^{-1}(1-\zeta) \leq W(t, \xi, \eta, \zeta) \leq \tilde{C}_0(1-\zeta). \] (48)

Now, we study the \( L^1 \) estimate of the first order derivatives of the approximate
solution with respect to the spatial variables for obtaining the uniform estimate on
the total variation of the solution. Before it, we give the following two propositions
for the problem (12) of transport equation.

**Proposition 2.** For the problem (12), there exists a constant \( C_4 \) depending on
the domain \( D \), the constant \( \tilde{C}_0 \) given in (48), \( \|W_1\|_{C^1} \) and the \( C^1 \) estimates of the
parameter in the problem (12), such that for all \( t \in [t_i, t_{i+1}] \) and \( \zeta \in (0,1), \)
\[ \int_D \frac{|W_\xi(t)| + |W_\eta(t)|}{W^2(t)} (1-\zeta)^2 \, d\xi d\eta \leq \int_D \frac{|W_\xi(t_i)| + |W_\eta(t_i)|}{W^2(t_i)} (1-\zeta)^2 \, d\xi d\eta \]
\[ + C_4 (t-t_i) + C_4 \int_{t_i}^t \int_D \frac{|W_\xi(s)| + |W_\eta(s)|}{W^2(s)} (1-\zeta)^2 \, d\xi d\eta ds. \] (49)

**Proof.** For the problem (12), we know that \( \left( \frac{1}{W} \right) \xi \) satisfies
\[ \partial_t (\frac{1}{W}) \xi + \zeta \partial_\xi \left( U(t) \frac{1}{W} \right) \xi + \zeta \frac{W_\xi}{W} \partial_\eta \left( \frac{1}{W} \right) \xi + \zeta (kU)(\frac{1}{W}) \eta - (B-b)(\frac{1}{W}) \xi = \frac{(B-b) \xi}{W}. \] (50)
Taking (48) into account, we multiply the above equation (50) by \( (1-\zeta)^2 \text{sign} W_\xi \)
or \( (1-\zeta)^2 \frac{W_\xi}{\sqrt{W^2}}, \) and integrate the resulting equation over \( D \) with respect to \( (\xi, \eta), \)
to obtain that,
\[ \frac{d}{dt} \int_D \frac{|W_\xi|}{W^2} (1-\zeta)^2 \, d\xi d\eta + \int_D \zeta \partial_\xi \left( U(t) \frac{|W_\xi|}{W^2} \right)(1-\zeta)^2 \, d\xi d\eta \]
\[ + \int_D \zeta \partial_\eta \left( kU \frac{|W_\xi|}{W^2} \right)(1-\zeta)^2 \, d\xi d\eta - \int_D \zeta (kU) \eta \frac{|W_\xi|}{W^2} (1-\zeta)^2 \, d\xi d\eta \]
\[ \leq \|(kU)\xi\|_{L^\infty} \cdot \int_D \frac{|W_\eta|}{W^2} (1-\zeta)^2 \, d\xi d\eta + \|B-b\|_{L^\infty} \cdot \int_D \frac{|W_\xi|}{W^2} (1-\zeta)^2 \, d\xi d\eta \]
\[ + \int_D \|(B-b)\xi\| \frac{(1-\zeta)^2}{W} d\xi d\eta. \] (51)
From (12), we get that on the boundary \( \gamma_-, \)
\[ \zeta U(k_\tau \partial_\tau + k_\eta \partial_\eta) W|_{\gamma_-} = -\partial_\tau W_1 - (B-b)|_{\gamma_-} \cdot W_1, \]
which implies
\[ \zeta U k_n \partial_n W|_{\gamma_-} = -\partial_i W_1 - (B - b)|_{\gamma_-} \cdot W_1 - \zeta U|_{\gamma_-} \cdot k_r \partial_r W_1 := b_2. \] (52)

Obviously, there exist two bounded functions \( a_1(\xi, \eta) \) and \( a_2(\xi, \eta) \), defined on the boundary \( \gamma_- \), such that
\[ \partial_\xi = a_1 \partial_n + a_2 \partial_r, \quad \text{on} \ \gamma_- . \]

Thus, from (52) one has
\[ \zeta U k_n W_\xi|_{\gamma_-} = a_1 b_2 + a_2 \zeta U|_{\gamma_-} \cdot k_n \partial_r W_1 := b_3. \] (53)

Hence, it follows that by virtue of (53),
\[ \int_D \zeta \nabla (\xi, \eta) \cdot \left[ \frac{\partial |W|_\xi}{W^2} (1 - \zeta)^2 (1, k) \right] d\eta = \int_{\partial D} \zeta U \frac{|W|_\xi}{W^2} (1 - \zeta)^2 (1, k) \cdot \vec{n} dl \]
\[ \geq \int_{\gamma_-} \zeta U k_n \frac{|W|_\xi}{W^2} (1 - \zeta)^2 dl = - \int_{\gamma_-} b_3 \cdot (1 - \zeta)^2 \frac{L}{W^2} dl \]
\[ \geq -C_0^2 \|b_3\|_{L^\infty} \cdot l(\gamma_-), \]
where \( l(\gamma_-) \) is the length of \( \gamma_- \) and the positive constant \( C_0 \) is given in (10).

Next, for the last term of (51) we have by virtue of (48),
\[ \int_D \left| (B - b_\xi) \frac{(1 - \zeta)^2}{W} \right| d\xi d\eta \leq \tilde{C}_0 \| (B - b_\xi) \|_{L^\infty} \cdot S(D), \] (55)
where \( S(D) \) is the area of the domain \( D \).

Plugging (54) and (55) into (51), we get that there exists a constant \( \tilde{C}_4 \) depending on \( D, \tilde{C}_0, \| W_1 \|_{C^1} \) and the \( C^1 \) estimates of the parameter in the problem (12), such that
\[ \frac{d}{dt} \int_D \frac{|W_\xi(t)|}{W^2(t)} (1 - \zeta)^2 d\xi d\eta \leq \tilde{C}_4 + \tilde{C}_4 \int_D \frac{|W_\xi(t)| + |W_\eta(t)|}{W^2(t)} (1 - \zeta)^2 d\xi d\eta. \] (56)

Similarly, we can obtain that another constant \( \tilde{C}_4 \) exists, such that
\[ \frac{d}{dt} \int_D \frac{|W_\eta(t)|}{W^2(t)} (1 - \zeta)^2 d\xi d\eta \leq \tilde{C}_4 + \tilde{C}_4 \int_D \frac{|W_\xi(t)| + |W_\eta(t)|}{W^2(t)} (1 - \zeta)^2 d\xi d\eta. \] (57)

By letting \( C_4 = \tilde{C}_4 + \tilde{C}_4 \), we have that from (56) and (57),
\[ \frac{d}{dt} \int_D \frac{|W_\xi(t)| + |W_\eta(t)|}{W^2(t)} (1 - \zeta)^2 d\xi d\eta \leq C_4 + C_4 \int_D \frac{|W_\xi(t)| + |W_\eta(t)|}{W^2(t)} (1 - \zeta)^2 d\xi d\eta. \] (58)

Then, integrating the above inequality (58) over \( (t_i, t) \) gives the estimate (49) immediately, and we complete the proof. \( \square \)

**Proposition 3.** For the problem (12), there exists a constant \( C_5 \) depending on the domain \( D \), the constant \( \tilde{C}_0 \) given in (48), \( \| W_1 \|_{C^1} \) and the \( C^1 \) estimates of the parameter in the problem (12), such that for \( t \in [t_i, t_{i+1}] \) and \( \zeta \in (0, 1) \),
\[ \int_D |W_\zeta(t)| d\xi d\eta \leq \int_D |W_\zeta(t_i)| d\xi d\eta + C_5 (t - t_i) \]
\[ + C_5 \int_{t_i}^t \int_D \left( |W_\zeta(s)| + \frac{|W_\xi(s)| + |W_\eta(s)|}{W^2(s)} (1 - \zeta)^2 \right) d\xi d\eta ds. \] (59)
Similarly as in (54), we have
\[ W_\zeta + \zeta U(\partial_\zeta + k\partial_\eta)W_\zeta + U(W_\zeta + kW_\eta) + (B - b)W_\zeta = -(B - b)\zeta W. \] (60)

Multiplying the above equation (60) by \( \text{sign}W_\zeta \) or \( \frac{W_\zeta}{\sqrt{\|W_\zeta\|^2}} \), and integrating over \( D \) with respect to \((\xi, \eta)\), it follows that
\[
\frac{d}{dt} \int_D |W_\zeta|d\xi d\eta + \int_D \left( \zeta \left( \partial_\zeta (U|W_\zeta|) + \partial_\eta (kU|W_\zeta|) \right) \right) d\xi d\eta
\]
\[
- \int_D \zeta \left( U_\zeta + (kU)_\eta \right) \cdot |W_\zeta|d\xi d\eta
\]
\[
\leq \|U\|_{L^\infty} \cdot \int_D |W_\zeta|d\xi d\eta + \|kU\|_{L^\infty} \cdot \int_D |W_\eta|d\xi d\eta + \|B - b\|_{L^\infty} \cdot \int_D |W_\zeta|d\xi d\eta
\]
\[
+ \|(B - b)\zeta W\|_{L^\infty} \cdot S(D).
\] (61)

Similarly as in (54), we have
\[
\int_D \zeta \nabla(\xi, \eta) \cdot \left[ U|W_\zeta| \cdot (1, k) \right] d\xi d\eta \geq -\|k_nUW_{1,\zeta}\|_{L^\infty} \cdot l(\gamma_-). \] (62)

Obviously, by the bounded estimate (48) for \( W \) one has
\[
\int_D |W_\zeta|d\xi d\eta \leq \overline{C}_0^2 \int_D \frac{|W_\zeta|}{W^2} (1 - \zeta)^2 d\xi d\eta,
\] (63)
and
\[
\int_D |W_\eta|d\xi d\eta \leq \overline{C}_0^2 \int_D \frac{|W_\eta|}{W^2} (1 - \zeta)^2 d\xi d\eta.
\] (64)

Plugging (62), (63) and (64) into (61), we obtain that there exists a constant \( \overline{C}_5 \), depending on \( D, \overline{C}_0, \|W_1\|_{C^1} \), and the \( C^1 \) estimates of the parameter in problem (12), such that
\[
\frac{d}{dt} \int_D |W_\zeta(t)|d\xi d\eta \leq \overline{C}_5 + \overline{C}_5 \int_D \left[ |W_\zeta(t)| + \frac{|W_\zeta(t)| + |W_n(t)|}{W^2(t)} (1 - \zeta)^2 \right] d\xi d\eta.
\] (65)

Then, integrating the above inequality (65) over \((t_i, t)\), the estimate (59) in the proposition follows immediately. \(\square\)

Remark 2. Similar to the above proposition, one can show that \( W_t \) satisfies an estimate similar to (59) in Proposition 3 by replacing the partial derivative in \( \zeta \) by that in \( t \).

It is ready to give the \( L^3 \) estimate of the first order derivatives of the approximate solution \( W \) constructed in (11)-(12) with respect to the spatial variables.

Lemma 2.3. For any fixed \( T > 0 \), let \( W(t, \xi, \eta, \zeta) \), \( 0 \leq t \leq T \) be the approximate solution of (7) constructed by (11)-(12). Then, there exists a constant \( M > 0 \), depending on \( T \), the domain \( D \), the constant \( \overline{C}_0 \) given in (48), \( \|W_0\|_{L^\infty} \), \( \|W_1\|_{C^3} \) and the \( C^1 \) estimates of the parameter in the problem (7), such that for all \( t \in [0, T] \), we have
\[
\int_\Omega \left[ |W_\zeta| + |W_\zeta| + |W_\eta| \right] (t, \cdot) d\xi d\eta d\zeta
\]
\[
\leq M \left( 1 + e^{MT} \int_\Omega \left( |W_0,\zeta| + |W_0,\xi| + |W_0,\eta| \right) d\xi d\eta d\zeta \right).
\] (66)
Proof. The proof is divided into the following three steps.

(1) When $t \in (t_i, t_{i+1}]$ for even $i$, $W$ is determined by the initial boundary value problem (11) for a porous medium-type equation. From Theorem 2.1 and the boundedness (48) of $W$, we obtain that for $t \in (t_i, t_{i+1}]$,

$$\int_{\Omega} |W_{\xi}|(t, \cdot) d\xi d\eta d\zeta \leq \int_{\Omega} |W_{\xi}|(t_i, \cdot) d\xi d\eta d\zeta + \int_{D} \left[ W(t, \xi, \eta, 0) - W(t_i, \xi, \eta, 0) \right] d\xi d\eta. \quad (67)$$

Moreover, there exists a constant $\tilde{C}_3$, depending on the domain $D$, the constant $\tilde{C}_0$ given in (48), $\|W_1\|_{C^3}$ and the $C^1$ estimates of the parameter in problem (7), such that

$$\int_{\Omega} \left[ \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t_i, \cdot) d\xi d\eta d\zeta \leq \int_{\Omega} \left[ \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t_i, \cdot) d\xi d\eta d\zeta + \tilde{C}_3 \left( 1 + \int_{\Omega} |W_{\xi}|(t_i, \cdot) d\xi d\eta d\zeta \right) \cdot (t - t_i)$$

$$+ \tilde{C}_3 \int_{t_i}^{t} \int_{\Omega} \left[ \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (s, \cdot) d\xi d\eta d\zeta ds,$$

which implies that by using the Gronwall inequality,

$$\int_{\Omega} \left[ \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t, \cdot) d\xi d\eta d\zeta \leq e^{\tilde{C}_3(t-t_i)} \cdot \int_{\Omega} \left[ \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t_i, \cdot) d\xi d\eta d\zeta$$

$$+ \left( e^{\tilde{C}_3(t-t_i)} - 1 \right) \cdot \left( 1 + \int_{\Omega} |W_{\xi}|(t_i, \cdot) d\xi d\eta d\zeta \right). \quad (68)$$

Combining (67) with (68), it follows that

$$\int_{\Omega} \left[ |W_{\xi}| + \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t, \cdot) d\xi d\eta d\zeta \leq e^{\tilde{C}_3(t-t_i)} \cdot \int_{\Omega} \left[ |W_{\xi}| + \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t_i, \cdot) d\xi d\eta d\zeta$$

$$+ e^{\tilde{C}_3(t-t_i)} - 1 + \int_{D} \left[ W(t, \cdot) - W(t_i, \cdot) \right] d\xi d\eta d\zeta \bigg|_{\zeta=0}. \quad (69)$$

(2) When $t \in (t_i, t_{i+1}]$ for odd $i$, we obtain $W$ by the problem (12) for a transport equation. From Propositions 2, 3 and the estimate (48), it follows that there exists a constant $C_6$, depending on the domain $D$, the constant $\tilde{C}_0$ given in (48), $\|W_1\|_{C^3}$ and the $C^1$ estimates of the parameter in problem (7), such that for $t \in (t_i, t_{i+1}]$,

$$\int_{\Omega} \left[ |W_{\xi}| + \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t, \cdot) d\xi d\eta d\zeta \leq \int_{\Omega} \left[ |W_{\xi}| + \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (t_i, \cdot) d\xi d\eta d\zeta + C_6(t - t_i)$$

$$+ C_6 \int_{t_i}^{t} \int_{\Omega} \left[ |W_{\xi}| + \frac{|W_{\xi}|}{W^2}(1 - \zeta)^2 \right] (s, \cdot) d\xi d\eta d\zeta ds,$$
which implies that by using the Gronwall inequality,
\[
\int_{\Omega} \left[ |W_\xi| + \frac{|W_\eta| + |W_n|}{W_2^2} (1 - \zeta)^2 \right](t, \cdot) d\xi d\eta d\zeta \\
\leq e^{C_0(t-t_i)} \cdot \int_{\Omega} \left[ |W_\xi| + \frac{|W_\eta| + |W_n|}{W_2^2} (1 - \zeta)^2 \right](t_i, \cdot) d\xi d\eta d\zeta + e^{C_0(t-t_i)} - 1. 
\] (70)

(3) On the other hand, when \(i\) is odd we have that from Proposition 1,
\[
W(t_{i+1}, \xi, \eta, 0) = W(t_i, \xi, \eta, 0) \cdot \exp \left\{ - \int_{t_i}^{t_{i+1}} (B - b)(s, \xi, \eta, 0) ds \right\}. \tag{71}
\]

By combining (69), (70) and (71), and letting \(C_7 = \max\{\tilde{C}_3, C_6\}\), we obtain that for any \(t \in (t_i, t_{i+1}]\),
\[
\int_{\Omega} \left[ |W_\xi| + \frac{|W_\eta| + |W_n|}{W_2^2} (1 - \zeta)^2 \right](t, \cdot) d\xi d\eta d\zeta \\
\leq e^{C_7(t-t_i)} \cdot \int_{\Omega} \left[ |W_\xi| + \frac{|W_\eta| + |W_n|}{W_2^2} (1 - \zeta)^2 \right](t_i, \cdot) d\xi d\eta d\zeta \\
+ e^{C_7(t-t_i)} - 1 + G_i(t), 
\] (72)

where
\[
G_i(t) = \int_{D} \left[ W(t, \cdot) - W(t_i, \cdot) \right] \bigg|_{\zeta = 0} d\xi d\eta \\
= \int_{D} \left[ W(t, \cdot) - W(t_{i-1}, \cdot) \cdot \exp \left\{ - \int_{t_{i-1}}^{t_i} (B - b)(s, \cdot) ds \right\} \right] \bigg|_{\zeta = 0} d\xi d\eta 
\]
for even \(i\), and \(G_i(t) = 0\) for odd \(i\). Hence, it is obvious that,
\[
G_i(t) \leq 2 \| W \|_{L^\infty} \cdot S(D), \quad \forall \, t \in [0, T]. \tag{73}
\]

Then, (72) implies that by iteration,
\[
\int_{\Omega} \left[ |W_\xi| + \frac{|W_\eta| + |W_n|}{W_2^2} (1 - \zeta)^2 \right](t, \cdot) d\xi d\eta d\zeta \\
\leq e^{C_7 t} \cdot \int_{\Omega} \left[ |W_0| + \frac{|W_0,\xi| + |W_0,\eta|}{W_0^2} (1 - \zeta)^2 \right] d\xi d\eta d\zeta \\
+ e^{C_7 t} - 1 + G_i(t) + F_i(t), 
\] (74)

where
\[
F_i(t) = \sum_{j=0}^{i-1} e^{C_7(t-t_{j+1})} G_j(t_{j+1}) \\
= \sum_{0 \leq j \leq i-2, \, j: \text{odd}} e^{C_7(t-t_{j+1})} \cdot \int_{D} \left[ W(t_{j+1}, \cdot) - W(t_{j-1}, \cdot) \right] \bigg|_{\zeta = 0} d\xi d\eta \cdot \exp \left\{ - \int_{t_{j-1}}^{t_j} (B - b)(s, \cdot) ds \right\} \bigg|_{\zeta = 0} d\xi d\eta.
\]

Note that \(W_0|_{\zeta = 0} > 0\), then
\[
\int_{\Omega} \left[ |W_\xi| + \frac{|W_\eta| + |W_n|}{W_2^2} (1 - \zeta)^2 \right](t, \cdot) d\xi d\eta d\zeta \\
\leq e^{2C_7 t} \| W|_{\zeta = 0} \|_{L^\infty} \cdot S(D) \cdot e^{C_7(t-t_2)} \int_{D} \left[ W_0(\cdot) \cdot \exp \left\{ - \int_{0}^{t_1} (B - b)(s, \cdot) ds \right\} \right] \bigg|_{\zeta = 0} d\xi d\eta.
\]
By choosing a constant $C_7$ satisfying that $2C_7 \geq \left\| (B - b) \right\|_{L^\infty(D)}$, we obtain

$$I \leq \sum_{k=1}^{\lfloor \frac{t}{T} \rfloor - 1} e^{C_7 \frac{T}{n}(i - 2k)} \cdot \left\| \int_{t_{2k}}^{t_{2k+1}} \left[ 2C_7 + (B - b)(s, \xi, \eta, 0) \right] ds \right\|_{L^\infty(D)}$$

Plugging (73), (75) and (76) into (74), there exists a positive constant $C_8$, depending on $T, S(D), \left\| W \right\|_{L^\infty}$ and $C_7$, such that

$$\int_D \left[ |W| + \frac{|W| + |W_0|}{W^2}(1 - \zeta)^2 \right] (t, \cdot) d\xi d\eta \leq e^{C_7 t} \int_D \left[ |W_0| + \frac{|W_0| + |W_0|}{W_0^2}(1 - \zeta)^2 \right] d\xi d\eta + C_8,$$

from which the estimate (66) follows immediately by using (48). Thus, we complete the proof of the lemma.

We are now ready to give the proof of the existence of weak solution as follows.

**Proof of Theorem 1.2.** Denote by $W^n(t, \xi, \eta, \zeta)$ the approximate solution constructed in (11)-(12). Let $Y$ be the dual space of $H_0^2(\Omega)$, we claim that

$$\left\| \partial_t \frac{(1 - \zeta)^2}{W^n} \right\|_{L^2(0,T;Y)} \leq C_9$$

with a constant $C_9$ independent of $n$. Indeed, if $i$ is even, we have that from (11),

$$\int_{t_i}^{t_{i+1}} \left\| \partial_t \frac{(1 - \zeta)^2}{W^n} \right\|_{Y}^2 dt$$

$$= 4 \int_{t_i}^{t_{i+1}} \left\| (1 - \zeta)^2 \left[ \partial_\zeta \frac{W^n}{W^2} + A \partial_\zeta \left( \frac{1}{W^n} \right) - \frac{b}{W^n} \right] \right\|_{Y}^2 dt$$

$$= 4 \int_{t_i}^{t_{i+1}} \left( \sup_{\|\psi\|_{H^2(\Omega)} \leq 1} \int_{\Omega} \left[ \partial_\zeta \frac{W^n}{W^2} + A \partial_\zeta \left( \frac{1}{W^n} \right) - \frac{b}{W^n} \right] \left( 1 - \zeta \right)^2 \cdot \psi d\xi d\eta d\zeta \right)^2 dt$$

$$\leq C_{10} \int_{t_i}^{t_{i+1}} \left[ \left\| W^n(t, \cdot) \right\|_{L^\infty} + \left\| \frac{1 - \zeta}{W^n(t, \cdot)} \right\|_{L^\infty} \right] dt,$$

with a uniform constant $C_{10}$ in $n$, depending only on $D$, $\|A\|_{C^1}$ and $\|b\|_{L^\infty}$. 

If \( i \) is odd, it follows that by using (12),
\[
\int_{t_i}^{t_{i+1}} \left\| \partial_t \left( \frac{(1-\zeta)^2}{W^n} \right)(t, \cdot) \right\|_{Y}^2 dt
\]
\[
= 4 \int_{t_i}^{t_{i+1}} \left\| (1-\zeta)^2 \left[ -\zeta U(\partial_\xi + k \partial_\eta)(\frac{1}{W^n}) + \frac{B-b}{W^n} \right] \right\|_{Y}^2 dt
\]
\[
= 4 \int_{t_i}^{t_{i+1}} \sup_{\|\psi\|_{H^2(\Omega)} \leq 1} \int_\Omega \left[ -\zeta U(\partial_\xi + k \partial_\eta)(\frac{1}{W^n}) + \frac{B-b}{W^n} \right] (1-\zeta)^2 \cdot \psi d\xi d\eta d\zeta dt
\]
\[
\leq C_{11} \int_{t_i}^{t_{i+1}} \left\| 1 - \zeta \right\|_{L^\infty} dt,
\]
with a uniform constant \( C_{11} \) in \( n \), depending only in \( D, \) \( \|U\|_{C^1} \) and \( \|B - b\|_{L^\infty} \).

Thus, by using (48), the estimate (77) follows immediately.

Next, from (48) and Lemma 2.3, we know that
\[
\left\| \frac{(1-\zeta)^2}{W^n} \right\|_{L^\infty(0,T;W^{2,1}(\Omega))} \leq C_{12},
\]
where \( C_{12} \) is a positive constant independent of \( n \). Hence, by using the Lions-Aubin Lemma (see [2] for instance), we conclude that \( \left\{ \frac{(1-\zeta)^2}{W^n} \right\} \) is compact in \( L^2((0,T) \times \Omega) \). Therefore, we may assume that
\[
\frac{(1-\zeta)^2}{W^n} \to \frac{(1-\zeta)^2}{W^n}, \quad \text{in } L^2((0,T) \times \Omega),
\]
and then,
\[
W^n \to W, \quad \text{a.e. in } (0,T) \times \Omega.
\]

In particular,
\[
W^n \to W, \quad \text{in } L^2((0,T) \times \Omega).
\]

Thus, for any \( \psi \in C_0^\infty((0,T) \times \Omega) \), we have that
\[
\int_0^T \int_\Omega \left[ \frac{1}{W^n} \right] \psi + \zeta(U \psi)_\xi + \zeta(U \psi)_\eta - (B-b) \psi \right] d\xi d\eta d\zeta dt
\]
\[
= 2 \sum_{i=1,\text{odd}}^{T} \int_{t_i}^{t_{i+1}} \int_\Omega \left[ -\delta_\xi W^n - A(\frac{1}{W^n} \psi + \zeta(U \psi)_\xi + \zeta(U \psi)_\eta + \frac{2b - B}{W^n} \right] \psi d\xi d\eta d\zeta dt.
\]

From Theorem 2.1 and Remark 2, and by integrating by parts in the above equality (78) it yields that,
\[
\int_0^T \int_\Omega \frac{1}{W^n} \left[ \psi + \zeta(U \psi)_\xi + \zeta(U \psi)_\eta - (B-b) \psi \right] d\xi d\eta d\zeta dt
\]
\[
= 2 \sum_{i=1,\text{odd}}^{T} \int_{t_i}^{t_{i+1}} \int_\Omega \left[ W^n \psi \zeta + \frac{1}{W^n} \left( - (A \psi)_\xi + \zeta(U \psi)_\xi + \zeta(U \psi)_\eta + (2b - B) \psi \right) \right] d\xi d\eta d\zeta dt.
\]

(79)

Letting \( n \to \infty \) in (79) and noting that \( t_i - t_{i-1} = \frac{T}{n} \), we have
\[
\int_0^T \int_\Omega \frac{1}{W} \left[ \psi + \zeta(U \psi)_\xi + \zeta(U \psi)_\eta - (B-b) \psi \right] d\xi d\eta d\zeta dt
\]
\[
= \int_0^T \int_\Omega \left[ W \psi \zeta + \frac{1}{W} \left( - (A \psi)_\xi + \zeta(U \psi)_\xi + \zeta(U \psi)_\eta + (2b - B) \psi \right) \right] d\xi d\eta d\zeta dt,
\]
which implies
\[
\int_0^T \int_\Omega \left\{ \psi_t + \zeta (U \psi)_{\xi} + \zeta (U k \psi)_{\eta} + (A \psi)_{\zeta} - B \psi \right\} d\xi d\eta d\zeta dt = 0.
\]

Therefore, from (80) we know that the function \( W \) satisfies the equation of problem (7) in the sense of distribution. Moreover, we can obtain that \( W \) satisfies the estimate (66) by letting \( n \to \infty \), which implies that
\[
W \in L^\infty (0, T; BV(\Omega)).
\]

It remains to verify that \( W \) satisfies the initial boundary conditions given in (7). Firstly, it follows from Lemma 2.2 that,
\[
\lim_{\zeta \to 1} W(t, \xi, \eta, \zeta) = 0, \quad \text{a.e. in } (0, T) \times D.
\]

Then, we can verify the other initial boundary conditions in (7) for \( W \) in the sense of distribution respectively, through similar process as above to show that \( W \) satisfies the equation of (7) in the sense of distribution. For example, we have the following equality holds:
\[
\int_0^T \int_\Omega \frac{1}{W} \left[ \psi_t + \zeta (U \psi)_{\xi} + \zeta (U k \psi)_{\eta} - (B - b) \psi \right] d\xi d\eta d\zeta dt
\]
\[
= \frac{1}{2} \int_0^T \int_\Omega \left[ W \psi_{\zeta\zeta} + \frac{1}{W} \left( -(A \psi)_{\zeta} + \zeta (U \psi)_{\xi} + \zeta (U k \psi)_{\eta}
\right.
\]
\[
\left. + (2b - B) \psi \right] d\xi d\eta d\zeta dt - \frac{1}{2} \int_0^T \int_D \left. \frac{p x}{U} \frac{\psi}{W} \right|_{\zeta=0} d\xi d\eta dt,
\]
for any
\[
\psi \in C_0^\infty ((0, T) \times D \times (-1, 1)), \quad \psi_{\zeta}|_{\zeta=0} = 0.
\]

Consequently, we complete the proof of Theorem 1.2.

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