Sharp Existence and Nonexistence Results for an Elliptic Equation Associated with Caffarelli-Kohn-Nirenberg Inequalities

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Abstract

This article establishes sharp existence and Liouville type theorems for the following nonlinear elliptic equation with singular coefficients,

\[-\text{div}(|x|^a Du) = |x|^b u^p, \ u > 0, \ \text{in} \ \mathbb{R}^N,\]

where $N \geq 3$, $p > 1$, and $b > a - 2 > -N$. In certain cases, this recovers the Euler-Lagrange equations connected with finding the best constant in Sobolev and Caffarelli-Kohn-Nirenberg inequalities. The first main result indicates that regular solutions exist if and only if the exponent $p$ is either critical or supercritical, i.e., either

\[p = \frac{N + 2 + 2b - a}{N - 2 + a} \quad \text{or} \quad p > \frac{N + 2 + 2b - a}{N - 2 + a},\]

respectively.

Similarly, the second main result provides necessary and sufficient conditions for the existence of finite energy solutions.

Keywords: Caffarelli-Kohn-Nirenberg inequalities; Lane-Emden equation; Liouville theorem; Hardy-Sobolev inequality; Hénon equation; positive solution.

MSC2010: Primary: 35B33, 35B53, 35J15, 35J75; Secondary: 35B40, 35B65.

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1 Introduction

In this paper, we consider elliptic equations which are related to the problem of finding best constants to Sobolev and Cafarelli-Kohn-Nirenberg (CKN) inequalities. Namely, we shall study the doubly weighted elliptic equation

\[
\begin{align*}
\text{div}(|x|^a Du) + |x|^b u^p &= 0 & \text{in } \Omega, \\
u > 0 & \text{ in } \Omega, 
\end{align*}
\]

(1.1)

where \( N \geq 3, \Omega \subseteq \mathbb{R}^N, p > 1, b \in \mathbb{R}, N - 2 + a > 0 \) and \( Du \) denotes the gradient of \( u \). By a positive weak solution \( u : \Omega \to \mathbb{R}_+ := (0, \infty) \) of problem (1.1), we mean that \( u \in H^1_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) and

\[
\int_{\Omega} |x|^a Du \cdot D\varphi \, dx = \int_{\Omega} \varphi |x|^b u^p \, dx \text{ for every } \varphi \in C^1_c(\Omega). 
\]

(1.2)

We say \( u \) is a positive regular solution, or just simply a positive solution, if \( u \in C^2(\Omega \setminus \{0\}) \cap C^1(\Omega) \) and it satisfies equation (1.1) pointwise everywhere in \( \Omega \setminus \{0\} \). Observe that regular solutions are weak and that basic elliptic regularity theory ensures that either type of solution is of class \( C^\infty(\Omega \setminus \{0\}) \) (see, e.g., [19]) and therefore a classical solution in the punctured domain \( \Omega \setminus \{0\} \).

Our first main result is the following sharp existence theorem, which verifies the ‘critical’ exponent \( p = (N + 2 + 2b - a)/(N - 2 + a) \) determines the region of existence for entire solutions to problem (1.1).

**Theorem 1.** Let \( \Omega = \mathbb{R}^N, N \geq 3, p > 1 \) and \( b > a - 2 > -N \). Equation (1.1) admits a positive regular solution if and only if

\[
p \geq \frac{N + 2 + 2b - a}{N - 2 + a}.
\]

In particular, if \( p = (N + 2 + 2b - a)/(N - 2 + a) \), then every radially symmetric positive solution of (1.1), up to a scaling (and a translation if \( a, b \neq 0 \)), is proportional to the function

\[
h(x) = \left(1 + |x|^{2+b-a}\right)^{\frac{N-2+a}{2+b-a}}.
\]

(1.3)

In fact, for each \( p > 1 \), we will prove that (1.1) admits no positive solution if \( N + b \leq 0 \) or \( b \leq a - 2 \). Therefore, our assumption that \( b > a - 2 > -N \) in Theorem [1] is indeed necessary.
Our strategy to proving Theorem 1 is to first combine Rellich-Pohozaev identities with the classical shooting method to arrive at a (radially symmetric) positive solution to (1.1) under the critical exponent or a supercritical exponent $p > (N + 2 + 2b - a)/(N - 2 + a)$ (see [27, 28] for related and recently developed methods applicable to more general equations and systems). Once we establish this existence result, its ‘sharpness’ and thereby Theorem 1 will be a consequence of the following Liouville theorem.

**Theorem 2 (Liouville).** Let $\Omega = \mathbb{R}^N$, $N \geq 3$, $p > 1$ and let $b > a - 2 > -N - a$. Then equation (1.1) has no positive regular solution whenever $p$ is subcritical, i.e., $p < \frac{N + 2 + 2b - a}{N - 2 + a}$.

A renowned case of (1.1) is when $\alpha = 0$, which reduces the problem to

\[
\begin{align*}
\Delta u + |x|^b u^p &= 0 & \text{in } \Omega, \\
u > 0 & \text{ in } \Omega.
\end{align*}
\]

Problem (1.4) is commonly known as the Hénon-Lane-Emden equation (or Lane-Emden equation if $b = 0$), and our main results recover and extend past existence and non-existence results for (1.4) [2, 11, 13, 15, 17, 20, 23, 29, 32, 33, 39, 40, 41, 43, 44, 45, 46]. If $\Omega = \mathbb{R}^N$, the Hénon-Lane-Emden equation has several important applications, e.g., it arises as an astrophysical model for stellar cluster formation [22]; it comprises the blow-up equation used to obtain a priori estimates for general elliptic boundary value problems [18, 36]; and it appears in geometric problems such as the Yamabe problem and the sharp Sobolev embedding (see [24, 34, 42] and the references therein for further details).

It is worth noting that Liouville theorems for weighted problems such as Theorem 2 are typically much more difficult to establish than their unweighted counterparts. The methods that apply to the unweighted case often no longer work for the weighted case, or at best, only achieves the result for a partial range of subcritical exponents. For instance, the Liouville theorem for the Lane-Emden equation can be completely proved via the method of moving planes or its variant, the method of moving spheres [5, 8, 16, 30, 31]. For the Hénon-Lane-Emden equation, however, both methods fail to cover the entire subcritical range if $b > 0$. The reason for this failure is due to the presence of the monotone increasing weight, which eliminates the ‘decay at infinity’ and comparison properties (see Chapter 8 in [9]) required in the methods. We also refer the reader to [21] and the references therein for another interesting example of a recently resolved conjecture whose previous partial results relied on moving plane methods in specific cases.
circumvent this issue in problem (1.1), the proof of Theorem 2 will not only incorporate a variant of the method of moving planes, but we shall supplement it with crucial monotonicity and stability estimates (our approach in this paper adopts several elements from \[3, 15, 38\]).

Let us discuss another motivation for studying equation (1.1), in particular, its connection with a certain family of interpolation inequalities obtained by Caffarelli, Kohn, and Nirenberg [6]. Let \( q \geq 2 \) and \( N \geq 3 \). Then there exists a positive constant \( C = C(N, a, b) \) such that for every \( u \in C_c^1(\mathbb{R}^N) \),

\[
\left( \int_{\mathbb{R}^N} |x|^a |u|^q \, dx \right)^{2/q} \leq C \int_{\mathbb{R}^N} |x|^a |Du|^2 \, dx, \tag{1.5}
\]

where

\[
a - 2 \leq 2b/q \leq a \quad \text{and} \quad \frac{N + b}{q} + 1 = \frac{N + a}{2}. \tag{1.6}
\]

A natural question is to ask what are the best constants for these Caffarelli-Kohn-Nirenberg (CKN) inequalities. Indeed, the best constants may be computed using a variational approach. Namely, we classify the minimizers to the variational problem

\[
S(\alpha, \beta) = \inf \{ E(u) \mid u \in D_{a,b,q}(\mathbb{R}^N) \},
\]

where \( E(u) = \|Du\|_{L_a^q(\mathbb{R}^N)}^2/\|u\|_{L_b^q(\mathbb{R}^N)}^2 \) and \( D_{a,b,q}(\Omega) \) represents the closure of \( C_c^\infty(\Omega) \) with respect to the weighted norm

\[
\|u\|_{D_{a,b,q}(\Omega)} = \|u\|_{L_b^q(\Omega)}^2 + \|Du\|_{L_a^q(\Omega)}^2.
\]

Here, for \( 1 < q < \infty \), the space \( L_b^q(\Omega) \) denotes the usual Lebesgue space with weighted norm

\[
\|u\|_{L_b^q(\Omega)} := \left( \int_{\Omega} |u(x)|^q |x|^a \, dx \right)^{1/q}.
\]

If \( \Omega = \mathbb{R}^N \), \( p = q - 1 \) and the conditions in (1.6) hold, then the equation in (1.1) is precisely the Euler-Lagrange equation for the energy functional \( E(\cdot) \). The resulting minimizers in \( D_{a,b,p+1}(\mathbb{R}^N) \) belong to \( H^{1}_{loc}(\mathbb{R}^N) \cap L_{loc}^\infty(\mathbb{R}^N) \) (see Proposition 4.4 in [10]) and thus are positive weak solutions to the corresponding elliptic problem. Moreover, if either

(a) \(- (N-2) < a \leq 0 \) and \( b < 0 \), or

(b) \( a > 0 \) and

\[
b \leq q \cdot \beta_{FS}(a) := q \cdot \left( \frac{N - 2 + a}{2} \right) \left( 1 - N[(N-2+a)^2 + 4(N-1)]^{-1/2} \right),
\]

\[4\]
then each minimizer, up to a scaling or a multiplication by a positive constant (and a translation if \( a, b = 0 \)), admits the form (1.3). Here the curve 
\[ b = q \cdot \beta_{FS}(a) \]
is related to the so-called Felli-Schneider curve \([12, 14]\). Remarkably, if \( a > 0 \) and \( b > q \cdot \beta_{FS}(a) \), then symmetry breaking occurs in this region as non-radial minimizers exist (see \([4, 7, 12, 14]\) for the details). The preceding variational problem for the sharp CKN inequalities motivates our notion of finite energy solutions. We say that \( u : \Omega \to \mathbb{R}_+ \) is a finite energy solution of (1.1) if it is a positive weak solution and belongs to \( \mathcal{D}_{a,b,p+1}(\Omega) \). Thus, the minimizers for the sharp CKN inequalities are finite energy solutions in \( \Omega = \mathbb{R}^N \). Indeed, analogous existence and non-existence results hold for finite energy solutions.

**Theorem 3.** Let \( \Omega = \mathbb{R}^N, N \geq 3 \) and \( q = p + 1 \geq 2 \).

(a) If \( u \in H^2_{loc}(\Omega) \) is a finite energy solution of equation (1.1), then we necessarily have
\[
q > 2, \quad b > a - 2 > -N \quad \text{and} \quad \frac{N + b}{q} + 1 = \frac{N + a}{2}.
\]  
(1.7)

(b) Suppose, in addition, that \( 2 \leq q < \frac{2N}{N-2} \). If
\[
q > 2, \quad b > a - 2 > -N \quad \text{and} \quad \frac{N + b}{q} + 1 = \frac{N + a}{2},
\]  
(1.8)

then there exists a finite energy solution of (1.1).

This paper is organized as follows. In Section 2 we arrive at several intermediate results that comprise the essential ingredients in our proofs of the main results. Particularly, some partial Liouville type nonexistence results are given but in a smaller range of subcritical exponents. Then, with the help of the method of moving planes, a monotonicity result is provided that allow us to extend the nonexistence result to the full range of subcritical exponents. A Rellich-Pohozaev identity is also provided which is important in obtaining the existence of positive solutions in the critical and supercritical range. In Section 3 we provide the proof of Theorem 2 and Section 4 contains the proofs of Theorem 1 and Theorem 3.

## 2 Preparations

Let us first discuss the notation and conventions we adopt hereafter. We denote by \( B_R(x) \subset \mathbb{R}^N \) the open ball of radius \( R > 0 \) centered at \( x \in \mathbb{R}^N \).
We denote its boundary by \( \partial B_R(x) \), and if \( x = 0 \) and \( R = 0 \), then we write the resulting \((N - 1)\)-dimensional sphere \( \partial B_1(0) \) as \( \mathbb{S}^{N-1} \) instead. The constant \( C \) in the many inequalities established below represents some universal constant that may change from line to line, or even within the same line itself.

Some of our methods will occasionally depend on writing (1.1) in polar coordinates. Namely, if \( u \) is a positive smooth solution of (1.1) in \( \mathbb{R}^N \setminus \{0\} \) and, for every non-zero \( x \), we write \( r = |x| \) and \( \theta = x/|x| \in \mathbb{S}^{N-1} \),

\[
(2.1)
\]
and \( u(x) = v(r, \theta) \). Then

\[
\text{div}(|x|^a Du) = r^a \left( \partial_r^2 v + \frac{N - 1 + a}{r} \partial_r v + \frac{1}{r^2} \Delta_\theta v \right),
\]

where \( \partial_r^k := \frac{\partial^k}{\partial r^k} \) and \( \Delta_\theta \) is the Laplace-Beltrami operator on \( \mathbb{S}^{N-1} \). It follows that \( v = v(r, \theta) \) satisfies

\[
\partial_r^2 v + \frac{N - 1 + a}{r} \partial_r v + \frac{1}{r^2} \Delta_\theta v + r^{b-a} v^p = 0, \ v > 0, \ \text{in} \ (0, \infty) \times \mathbb{S}^{N-1}. \quad (2.2)
\]

For the existence of positive solutions, our strategy is to search for radially symmetric solutions, in which case (2.2) indicates we solve the initial value problem,

\[
\begin{align*}
- \left( v''(r) + \frac{N - 1 + a}{r} v'(r) \right) &= r^{b-a} v(r)^p, \ r > 0, \\
v'(0) &= 0, \ v(0) = \beta > 0.
\end{align*}
\]

The existence of a unique local positive solution \( v(r) \in C^2((0, r_0)) \cap C^1([0, r_0)) \) for some \( r_0 > 0 \), is obtained as a fixed point to

\[
T(v) = \beta - \int_0^r \int_0^t \left( \frac{s^{N-1+b}}{t^{N-1+a}} \right) v(s)^p \, ds \, dt. \quad (2.3)
\]

The existence of such a fixed point follows simply from the Banach fixed point theorem, since \( p > 1 \) and \( b > a - 2 > -N \). Ultimately, our goal is to find a global solution, i.e., find \( \beta \) and a corresponding fixed point solution such that \( r_0 = +\infty \) as this would result in the desired positive solution of (1.1). Now to do so, we employ a shooting method and a Rellich-Pohozaev type identity (see Lemma 2.15 below) to ensure the local solution is indeed global as long as \( p \geq \frac{N+2+b-a}{N-2+a} \). The actual proof of the global existence result is deferred until Section 4 (see Lemma 5).
2.1 Nonexistence of positive weak solutions

The first lemma explains why we assume $b > a - 2 > -N$. The proof follows a similar line of arguments as those from [11] (see Theorem 2.3).

**Lemma 1.** Let $N \geq 3$, $p > 1$, $b \in \mathbb{R}$ and $a - 2 > -N$, and suppose that either $\Omega = \mathbb{R}^N$ or $\Omega = B_R(0)$ for $R > 0$. Then (1.1) does not admit any positive weak solution in $\Omega$ provided that either $N + b \leq 0$ or $b \leq a - 2$.

**Proof.** Let $p > 1$ and $a - 2 > -N$. Without loss of generality, we may assume $\Omega = B_R(0)$ for some fixed $R > 0$ and that $u$ is a positive weak solution of (1.1). Therefore, $u \in C^\infty(\Omega \setminus \{0\})$ satisfies (1.1) in $\Omega \setminus \{0\}$ pointwise. Writing $u = u(r, \theta)$ in polar coordinates so that $u(r, \theta)$ satisfies (2.2), we then set

$$U(r) = \int_{S_{N-1}} u(r, \theta) dS := \frac{1}{|S_{N-1}|} \int_{S_{N-1}} u(r, \theta) dS \quad \text{for } 0 < r < R.$$ 

By Jensen’s inequality,

$$\int_{S_{N-1}} u(r, \theta)^p dS \geq \left( \int_{S_{N-1}} u(r, \theta) dS \right)^p = U(r)^p. \quad (2.4)$$

Hence, (2.4) and (2.1) show that $U(r) > 0$ satisfies the differential inequality

$$- \left( U''(r) + \frac{N - 1 + a}{r} U'(r) \right) \geq r^{b-a} U(r)^p \quad \text{for } 0 < r < R. \quad (2.5)$$

This implies that, for $0 < r < R$,

$$- (r^{N-1+a} U'(r))' \geq r^{N-1+b} U(r)^p. \quad (2.6)$$

Since $U > 0$, we obtain $-(r^{N-1+a} U'(r))' > 0$ and thus

$$r^{N-1+a} U'(r) \to \ell \quad \text{as } r \to 0^+,$$

where $-\infty < \ell \leq \infty$. We infer that $\ell \leq 0$. Otherwise, if $0 < \ell \leq +\infty$, then we can find $\delta > 0$ and $r_\delta > 0$ such that

$$U'(r) \geq \delta r^{-(N+a-1)} \quad \text{for } 0 < r < r_\delta.$$ 

Integrating this in $(r_0, r)$ where $0 < r_0 < r < r_\delta$ yields

$$U(r) \geq \delta \int_{r_0}^r t^{-(N+a-2)} \frac{dt}{t}.$$ 

As $N - 2 + a > 0$, sending $r_0 \to 0^+$ leads to an impossibility.
Let \( \ell \leq 0 \). Therefore, we have that \( U'(r) < 0 \). Then there exists a positive constant \( c \) and \( r_1 > 0 \) such that

\[
U(r) \geq c \quad \text{for } 0 < r < r_1.
\]  

(2.7)

Choose a small \( r_0 \in (0, r_1) \). By integrating (2.6) in \( (r_0, r) \subset (r_0, r_1) \) and since \( U \) is monotone decreasing in this interval of integration, we obtain

\[
-r^{N-1+a}U'(r) \geq r_0^{N-1+a}U'(r_0) - r^{N-1+a}U'(r) \geq U(r)^p \int_{r_0}^r t^{N+b} \frac{dt}{t} \quad \text{for } 0 < r_0 < r < r_1.
\]  

(2.8)

If \( N + b \leq 0 \), then sending \( r_0 \to 0^+ \) leads to a contradiction, since the integral on the right diverges. Otherwise, if \( N + b > 0 \), we may integrate then send \( r_0 \to 0^+ \) in (2.8) to get

\[
-U'(r)U(r)^{-p} \geq (N + b)^{-1}r^{1+b-a} \quad \text{for } 0 < r < r_1.
\]

That is, for each \( \delta_0 > 0 \),

\[
(p-1)^{-1} \left( U(r)^{-p-1} \right)' \geq (N + b)^{-1}r^{1+b-a} \quad \text{for } 0 < \delta_0 < r < r_1.
\]

Integrating once again in the interval \((\delta_0, r)\) yields

\[
U(r)^{-(p-1)} \geq \frac{p-1}{N+b} \int_{\delta_0}^r t^{2+b-a} \frac{dt}{t}.
\]

Now, after sending \( \delta_0 \to 0^+ \) in the last estimate, the resulting improper integral diverges if \( b \leq a - 2 \). This is impossible due to (2.7).

Hence, in all possible cases we arrive at a contradiction. This completes the proof of the lemma. \( \square \)

In view of Lemma 1, we shall always assume \( N \geq 3 \) and \( b > a - 2 > -N \) hereafter. The next lemma partially resolves Theorem 2.

**Lemma 2.** Let \( \Omega = \mathbb{R}^N \) and \( p > 1 \). Equation (1.1) has no positive weak solution in \( \Omega \) whenever \( p \leq \frac{N+b}{N-2+a} \).

**Proof.** We prove this by contradiction. That is, let \( 1 < p \leq \frac{N+b}{N-2+a} \) and we assume \( u \) is a positive weak solution of (1.1) in \( \Omega = \mathbb{R}^N \). Choose any \( R > 0 \) and fix \( \xi \in C^\infty_c(B_2(0)) \) such that \( 0 \leq \xi \leq 1 \) and \( \xi \equiv 1 \) on \( B_1(0) \). Then set
\[ \varphi(x) = \xi(x/R)^{2p^*}, \] where \( p^* = p/(p-1) \) denotes the Hölder conjugate of \( p \).

Clearly, \( D\varphi(x) = 2p^* R^{-1} \xi(x/R)^{2p^*-1} D\xi(x/R) \) (2.9)

and

\[ \Delta\varphi(x) = \frac{2p^*}{R^2} \xi(x/R)^{2(p^*-1)} \left[ \xi(x/R)^2 \Delta\xi(x/R) + (2p^* - 1)|D\xi(x/R)|^2 \right]. \] (2.10)

Inserting this choice of test function into (1.2) and by integration by parts, we obtain

\[
\int_{\mathbb{R}^N} \varphi |x|^b u^p \, dx = \int_{\mathbb{R}^N} |x|^a u (-\Delta \varphi) \, dx - a \int_{\mathbb{R}^N} |x|^{a-2} u (x \cdot D\varphi) \, dx
\leq \int_{\mathbb{R}^N} |x|^a u |\Delta \varphi| \, dx + |a| \int_{\mathbb{R}^N} |x|^{a-2} u |x \cdot D\varphi| \, dx
=: E_1 + E_2. \] (2.11)

From (2.9) and (2.10), there exists a positive constant \( C \), independent of \( R \), such that for \( R \leq |x| < 2R \),

\[ |x \cdot D\varphi(x)| \leq C \xi(x/R)^{2p^*-1} |D\xi(x/R)| \leq C \xi(x/R)^{2(p^*-1)} \leq C \varphi(x)^{1/p}, \]

and

\[ |\Delta\varphi(x)| \leq \frac{C}{R^2} \xi(x/R)^{2p^*-1} \leq \frac{C}{R^2} \xi(x/R)^{2(p^*-1)} \leq \frac{C}{R^2} \varphi(x)^{1/p}. \]

We now apply these inequalities to estimate \( E_1 \) and \( E_2 \). Indeed, Hölder’s inequality implies

\[
E_1 \leq \int_{B_{2R}(0) \setminus B_R(0)} |x|^a u |\Delta \varphi| \, dx \leq \frac{C}{R^2} \int_{B_{2R}(0) \setminus B_R(0)} |x|^a u \varphi^{1/p} \, dx
\leq \frac{C}{R^2} \left( \int_{B_{2R}(0) \setminus B_R(0)} |x|^{(a-b/p)p^*} \varphi \, dx \right)^{1/p^*} \left( \int_{B_{2R}(0) \setminus B_R(0)} \varphi |x|^b u^p \, dx \right)^{1/p}
\leq CR^{-2 + \frac{N}{p^*} + a - \frac{b}{p}} \left( \int_{B_{2R}(0) \setminus B_R(0)} \varphi |x|^b u^p \, dx \right)^{1/p}.
\]

Likewise, we have that

\[
E_2 \leq C \int_{B_{2R}(0) \setminus B_R(0)} |x|^{a-2} u |x \cdot D\varphi| \, dx \leq \frac{C}{R^2} \int_{B_{2R}(0) \setminus B_R(0)} |x|^a u \varphi^{1/p} \, dx
\leq CR^{-2 + \frac{N}{p^*} + a - \frac{b}{p}} \left( \int_{B_{2R}(0) \setminus B_R(0)} \varphi |x|^b u^p \, dx \right)^{1/p}.
\]
Placing the estimates for $E_1$ and $E_2$ back into (2.11) yields

\[
\int_{\mathbb{R}^N} \varphi |x|^b u^p \, dx \leq CR^{-2 + \frac{N}{p^*} + a - \frac{b}{p}} \left( \int_{B_{2R}(0) \setminus B_R(0)} \varphi |x|^b u^p \, dx \right)^{1/p}, \tag{2.12}
\]

which further implies

\[
\int_{B_R(0)} |x|^b u^p \, dx \leq CR^N + p^* (a - \frac{b}{p} - 2). \tag{2.13}
\]

Now, if $1 < p < \frac{N + b}{N - 2 + a}$, then $N + p^* (a - \frac{b}{p} - 2) < 0$. Therefore, sending $R \to +\infty$ in (2.13) leads to $u \equiv 0$. If $p = \frac{N + b}{N - 2 + a}$, then the same argument shows $|x|^b u^p \in L^1(\mathbb{R}^N)$. In fact, there holds

\[-2 + \frac{N}{p^*} + a - \frac{b}{p} = 0\]

and we deduce $\| |x|^b u^p \|_{L^1(\mathbb{R}^N)} = 0$ after sending $R \to +\infty$ on the right-hand side of (2.12). This again leads to $u \equiv 0$. Nonetheless, we reach the desired contradiction in all cases.

Remark 1. (a) To prove Theorem 2, it only remains to establish the nonexistence for exponents in the range $\frac{N+b}{N-2+a} < p < \frac{N+2+2b-a}{N-2+a}$. This case will require a key monotonicity result, which we provide shortly below.

(b) The proof and thus the result of Lemma 2 extends to the class of distribution solutions. Interestingly, the Liouville theorem for distribution solutions is sharp (and Theorem 2 is no longer true) if we remove the local boundedness assumption on $u$. To see this, assume that $p > \frac{N+b}{N-2+a}$ so that $N - 2 + a > \gamma > 0$, where

\[
\gamma := \frac{2 + b - a}{p - 1}. \tag{2.14}
\]

Then, a straightforward calculation will show

\[
u(x) = \left[ \gamma (N - 2 + a - \gamma) \right]^{\frac{1}{p - 1}} |x|^{-\gamma}
\]

is a positive singular solution of (1.1).
2.2 A Pohozaev identity and a monotonicity property

Lemma 3 (Rellich-Pohozaev). For any $R > 0$, if $u \in C^2(B_R(0) \setminus \{0\}) \cap C^1(\overline{B}_R(0))$ is a classical solution of (1.1) in $\Omega = B_R(0) \setminus \{0\}$, then

$$
\left( \frac{N + b}{p + 1} - \frac{N - 2 + a}{2} \right) \int_{B_R(0)} |x|^b |u|^{p+1} \, dx = \frac{N - 2 + a}{2} \int_{\partial B_R(0)} R^a u \frac{\partial u}{\partial \nu} \, dS
$$

(2.15)

$$
+ \frac{1}{p + 1} \int_{\partial B_R(0)} R^b |u|^{p+1} (x \cdot \nu) \, dS + \frac{1}{2} \int_{\partial B_R(0)} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) |x|^a \, dS,
$$

where $\frac{\partial u}{\partial \nu} = Du \cdot \nu$ and $\nu$ denotes the outward pointing unit normal on $\partial \Omega$.

Proof. The proof is standard, but we outline the main steps for the reader’s convenience. Fix $R > 0$ and choose a suitably small $0 < \varepsilon < R$. Then multiply the equation,

$$
- \text{div}(|x|^a Du) = |x|^b u^p \text{ in } B_R(0) \setminus \{0\},
$$

(2.16)

by $x \cdot Du$ and integrate the resulting equation over $B_R(0) \setminus B_\varepsilon(0)$. After direct computations, integration by parts, then carefully sending $\varepsilon \to 0^+$, we obtain

$$
I_1 = I_2,
$$

where

$$
I_1 := \int_{B_R(0)} - (x \cdot Du) \text{div}(|x|^a Du) \, dx
$$

$$
= - \frac{N - 2 + a}{2} \int_{B_R(0)} |x|^a |Du|^2 \, dx - \frac{1}{2} \int_{\partial B_R(0)} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) |x|^a \, dS,
$$

and

$$
I_2 := \int_{B_R(0)} |x|^b u^p (x \cdot Du) \, dx
$$

$$
= - \left( \frac{N + b}{p + 1} \right) \int_{B_R(0)} |x|^b u^{p+1} \, dx + \frac{1}{p + 1} \int_{\partial B_R(0)} |x|^b u^{p+1} (x \cdot \nu) \, dS.
$$

Along a similar argument, if we multiply equation (2.16) by $u$, integrate over $B_R(0) \setminus B_\varepsilon(0)$, integrate by parts, then send $\varepsilon \to 0^+$, we get the identity

$$
\int_{B_R(0)} |x|^a |Du|^2 \, dx = \int_{B_R(0)} |x|^b u^{p+1} \, dx - \int_{\partial B_R(0)} |x|^a u \frac{\partial u}{\partial \nu} \, dS.
$$

The Pohozaev type identity (2.15) follows after inserting the last identity into $I_1 = I_2$ and carrying out the proper calculations. \qed
To address the nonexistence of positive solutions in the remaining range of exponents,

\[
\frac{N + b}{N - 2 + a} < p < \frac{N + 2 + 2b - a}{N - 2 + a},
\]

we shall require a key monotonicity property derived via a variant of the method of moving planes. Exploiting monotonicity properties to establish Liouville theorems for elliptic problems, some of which are well-known cases of equation (1.1), is now fairly standard and we refer the reader to [1, 3, 8, 26, 38] for additional examples.

Lemma 4. Let \( \Omega = \mathbb{R}^N \setminus \{0\} \) and suppose \( u \) is a positive classical solution of (1.1). If \( p \) satisfies (2.17), then \( |x|^{\gamma} u(x) \) is monotone increasing with respect to \(|x|\), where \( \gamma \) was defined in (2.14).

Proof. Let \( u \) be a positive classical solution of (1.1) in \( \Omega = \mathbb{R}^N \setminus \{0\} \) and suppose \( p \) satisfies (2.17).

Step 1. We apply an Emden-Fowler type transformation.

Let \( (r, \theta) \in (0, \infty) \times \mathbb{S}^{N-1} \) represent polar coordinates as defined in (2.1) and \( v(r, \theta) = u(x) \). By writing \( w(t, \theta) = r^{\gamma} v(r, \theta) \) where \( t = \ln r \) and recalling that \( v(r, \theta) \) satisfies (2.2), it follows that \( w(t, \theta) \) is a positive solution of

\[
-\partial^2_t w - \Lambda_1 \partial_t w + \Lambda_2 w - \Delta_\theta w = w^p \quad \text{in } \mathbb{R} \times \mathbb{S}^{N-1},
\]

where

\[
\Lambda_1 = N - 2 + a - 2\gamma \quad \text{and} \quad \Lambda_2 = \gamma(N - 2 + a - \gamma).
\]

Observe that (2.17) ensures that \( \Lambda_1 \leq 0 \) and \( \Lambda_2 > 0 \).

Step 2. Start the Method of Moving Planes.

It suffices to prove \( \partial_t w(t, \theta) > 0 \) in \( \mathbb{R} \times \mathbb{S}^{N-1} \) and we do so using the method of moving planes adapted to cylindrical domains. For \( \lambda \in \mathbb{R} \), we set \( \Sigma_\lambda = (-\infty, \lambda) \times \mathbb{S}^{N-1} \) and \( T_\lambda = \partial \Sigma_\lambda = \{\lambda\} \times \mathbb{S}^{N-1} \). For each \( t \leq \lambda \), we let \( t^\lambda = 2\lambda - t \), which represents the reflection of \( t \) across the boundary \( T_\lambda \), and

\[
w^\lambda(t, \theta) = w(t^\lambda, \theta) - w(t, \theta) \quad \text{for } (t, \theta) \in \Sigma_\lambda \cup T_\lambda.
\]

By the mean value theorem and direct calculations, the comparison function \( w^\lambda \) satisfies

\[
\partial^2_t w^\lambda + \Delta_\theta w^\lambda - \Lambda_1 \partial_t w^\lambda - \Lambda_2 w^\lambda + p\psi^p_\lambda - w^\lambda = -2\Lambda_1 \partial_t w \quad \text{in } \Sigma_\lambda,
\]
where $\psi(\lambda(t,\theta))$ lies between $w(\lambda(t,\theta))$ and $w(\lambda(t,\theta))$ and thus is non-negative in $\mathbb{R} \times S^{N-1}$. Moreover, by definition, there holds

$$w^\lambda \equiv 0 \text{ on } T_{\lambda},$$

and

$$\lim_{t \to -\infty} \inf_{\theta \in S^{N-1}} w^\lambda(t,\theta) \geq 0,$$

for any fixed $\lambda \in \mathbb{R}$. Now, since $\Lambda_2 > 0$, we fix $\varepsilon_0 > 0$ suitably small such that

$$-\Lambda_2 + p\varepsilon_0^{p-1} < 0.$$

In view of the fact that $u$ is locally bounded and that $\partial_t w = r^\gamma(\gamma u + r\partial_r u)$, we can choose $t_0 := \ln r_0$ near $-\infty$, i.e., $r_0 > 0$ sufficiently small, such that

$$0 < w < \varepsilon_0 \text{ and } \partial_t w > 0 \text{ in } \Sigma_{t_0}.$$

We assert that for all $\lambda \leq t_0$,

$$w^\lambda \geq 0 \text{ in } \Sigma_{\lambda}.$$

To see this, assume otherwise, i.e., there exists a $\lambda \leq t_0$ such that

$$\inf_{\Sigma_{\lambda}} w^\lambda < 0.$$

Thus, $w^\lambda$ attains a negative minimum in $\Sigma_{\lambda}$. Due to (2.20), this minimum must be achieved away from the boundary $T_{\lambda}$. That is, there exists a point $(\bar{t}, \bar{\theta}) \in \Sigma_{\lambda}$ such that

$$w^\lambda(\bar{t}, \bar{\theta}) = \min_{\Sigma_{\lambda}} w^\lambda < 0.$$

Of course, it follows that

$$\partial_t w^\lambda(\bar{t}, \bar{\theta}) = 0 \text{ and } \partial_t^2 w^\lambda(\bar{t}, \bar{\theta}) + \Delta_\theta w^\lambda(\bar{t}, \bar{\theta}) \geq 0.$$

From (2.19), the fact that $-\Lambda_2 + p\psi(\bar{t}, \bar{\theta})^{p-1} < 0$ due to (2.22), and $w^\lambda(\bar{t}, \bar{\theta}) < 0$, we obtain

$$0 < w^\lambda(\bar{t}, \bar{\theta}) \left[ -\Lambda_2 + p\psi(\bar{t}, \bar{\theta})^{p-1} \right]$$

$$\leq \partial_t^2 w^\lambda(\bar{t}, \bar{\theta}) + \Delta_\theta w^\lambda(\bar{t}, \bar{\theta}) - \Lambda_2 w^\lambda(\bar{t}, \bar{\theta}) + p\psi(\bar{t}, \bar{\theta})^{p-1} w^\lambda(\bar{t}, \bar{\theta})$$

$$= -2\Lambda_1 \partial_t w^\lambda(\bar{t}, \bar{\theta}) \leq 0.$$

We reach a contradiction and thus (2.24) holds.
Step 3. We show that we may continue to increase \( \lambda \) so long as \( w^\lambda \) remains non-negative. More precisely, (2.24) guarantees the value
\[
\lambda_0 := \sup\{ \lambda \in \mathbb{R} \mid w^\mu \geq 0 \text{ in } \Sigma_\mu \text{ for } \mu < \lambda \}
\]
exists and \( \lambda_0 > -\infty \). Therefore, we have two possibilities: either (a) \( \lambda_0 = +\infty \) or (b) \( \lambda_0 < +\infty \). Let us now examine each case carefully, then we show only case (a) is possible if \( p \) is subcritical.

Case (a) \( \lambda_0 = +\infty \).

In this case, we have for each \( \lambda \in \mathbb{R} \), \( w^\lambda > 0 \) in \( \Sigma_\lambda \). Combining this with (2.20) yields
\[
\partial_t w^\lambda \leq 0.
\]
And since \( w^\lambda(t, \theta) = w(2\lambda - t, \theta) - w(t, \theta) \), we get
\[
\partial_t w^\lambda = -2\partial_t w \text{ on } T_\lambda.
\] (2.26)
From (2.25) and (2.26), we deduce that \( \partial_t w^\lambda < 0 \) on \( T_\lambda \) for all \( \lambda \in \mathbb{R} \) and hence throughout \( \mathbb{R} \times \mathbb{S}^{N-1} \). So from (2.19), we get
\[
\partial^2_t w^\lambda + \Delta_\theta w^\lambda - \Lambda_1 \partial_t w^\lambda - \Lambda_2 w^\lambda = -2\Lambda_1 \partial_t w \leq 0,
\]
and by Hopf’s lemma, we conclude that \( \partial_t w < 0 \) on \( T_\lambda \). As this holds for arbitrary \( \lambda \) and because of (2.26), we arrive at \( \partial_t w > 0 \) in \( \mathbb{R} \times \mathbb{S}^{N-1} \). Hence, \( |x|^\gamma u(x) \) is monotone increasing in \( |x| \).

Case (b) \( \lambda_0 < +\infty \).

In this case, we prove that \( w^{\lambda_0} \equiv 0 \) in \( \Sigma_{\lambda_0} \), i.e., \( w \) is symmetric about the hyperplane \( t = \lambda_0 \). On the contrary, assume that \( \lambda_0 < +\infty \) and \( w^{\lambda_0} \neq 0 \) in \( \Sigma_{\lambda_0} \). By the strong maximum principle and Hopf’s lemma,
\[
w^{\lambda_0} > 0 \text{ in } \Sigma_{\lambda_0}, \text{ and } \partial_t w^{\lambda_0} < 0 \text{ on } T_{\lambda_0}.
\] (2.27)
By definition of \( \lambda_0 \), we can find a positive sequence \( \delta_n \to 0^+ \) and a corresponding bounded sequence of negative local minima \( (t_n, \theta_n) \in \Sigma_{\lambda_0 + \delta_n} \) such that
\[
w^{\lambda_0 + \delta_n}(t_n, \theta_n) = \inf_{\Sigma_{\lambda_0 + \delta_n}} w^{\lambda_0 + \delta_n} < 0.
\]
We then extract a convergent subsequence \( (t_{n_k}, \theta_{n_k}) \) converging to a point \( (\bar{t}, \bar{\theta}) \). By definition of \( t_0 \) and \( \lambda_0 \), we get that \( t_0 \leq \bar{t} \leq \lambda_0 \) and so \( (\bar{t}, \bar{\theta}) \)

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belongs to $\Sigma_{\lambda_0} \cup T_{\lambda_0}$. By continuity of the comparison function with respect to $\lambda, t$ and $\theta$, we also conclude that

$$w^{\lambda_0}(\bar{t}, \bar{\theta}) = 0.$$  

By (2.27), this implies that $(\bar{t}, \bar{\theta})$ belongs to $T_{\lambda_0}$ and, in particular, $\bar{t} = \lambda_0$.

Further, we may choose a sequence $s_{nk} \in (t_{nk}, \lambda_0 + \delta_{nk})$ such that

$$\partial_t w^{\lambda_0 + \delta_{nk}}(s_{nk}, \theta_{nk}) \geq 0.$$  

Thus, after sending $n_k \to +\infty$, we arrive at $\partial_t w^{\lambda_0}(\bar{t}, \bar{\theta}) = \partial_t w^{\lambda_0}(\lambda_0, \bar{\theta}) \geq 0$, but this contradicts with (2.27). Hence, we have proven that $w^{\lambda_0} \equiv 0$ in $\Sigma_{\lambda_0}$.

**Step 4.** We show that $\lambda_0 = +\infty$ provided that (2.17) holds. On the contrary, assume $\lambda < +\infty$, which leads to $w^{\lambda_0} \equiv 0$ in $\Sigma_{\lambda_0}$ from Step 3. Noticing (2.17) implies that $\Lambda_1 < 0$, we arrive at $\partial_t w \equiv 0$ in $\mathbb{R} \times S^{N-1}$ directly from (2.19). This means $w = w(\theta)$ depends only on $\theta$ and thus

$$u(x) = w(|x|)|x|^{-\gamma},$$  

but this contradicts the local boundedness of $u$. Hence, $\lambda_0 = +\infty$ and we deduce that $|x|^\gamma u(x)$ is monotone increasing in $|x|$.

\[\square\]

3 Proof of Theorem 2

Thanks to Lemma 2, we restrict our attention to exponents satisfying

$$\frac{N + b}{N - 2 + a} < p < \frac{N + 2 + 2b - a}{N - 2 + a}.$$  

Suppose that $u$ is a positive solution of (1.1) in $\Omega = \mathbb{R}^N$ and we set

$$V := \gamma u + (x \cdot Du).$$  

Owing to elliptic regularity once more, we have that $V \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$ and $V > 0$ in $\mathbb{R}^N$, since $|x|^\gamma u(x)$ is monotone increasing in $|x|$ (see Lemma 3).

**Step 1.** We claim that $V$ satisfies

$$\int_{\mathbb{R}^N} |x|^a D V \cdot D\phi \, dx = p \int_{\mathbb{R}^N} \phi |x|^b u^{p-1} V \, dx$$  

(3.1)
for each non-negative \( \phi \in C^1_c(\mathbb{R}^N) \). Indeed, for non-zero \( x \), direct calculations show \( \Delta(x \cdot Du) = x \cdot D(\Delta u) + 2\Delta u \), and so

\[
|x|^a \Delta(x \cdot Du) = |x|^a(x \cdot D(\Delta u)) + 2|x|^a \Delta u \\
= |x|^a(x \cdot D[-a|x|^{-2}(x \cdot Du) - |x|^{b-a}u^p]) \\
+ 2|x|^a(-a|x|^{-2}(x \cdot Du) - |x|^{b-a}u^p) \\
= -a|x|^{a-2}x_i x_j D_{ij} u - a|x|^{a-2}(x \cdot Du) + 2a|x|^{a-2}(x \cdot Du) \\
- p|x|^b u^{p-1}(x \cdot Du) - (b-a)|x|^b u^p - 2|x|^b u^p \\
= -a|x|^{a-2}x_i x_j D_{ij} u - a|x|^{a-2}(x \cdot Du) \\
- p|x|^b u^{p-1}(x \cdot Du) - (2+b-a)|x|^b u^p \\
= -D(|x|^a) \cdot D(x \cdot Du) - p|x|^b u^{p-1}(x \cdot Du) - (2+b-a)|x|^b u^p,
\]

where \( D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \), and it should be understood that the indices \( i \) and \( j \) are to be summed over the entire set, \( 1, 2, \ldots, N \). Thus, we obtain

\[
div(|x|^aD(x \cdot Du)) = |x|^a(x \cdot D(\Delta u)) + 2|x|^a \Delta u + D(|x|^a) \cdot D(x \cdot Du) \\
= - p|x|^b u^{p-1}(x \cdot Du) - \gamma(p - 1)|x|^b u^p.
\]

Hence,

\[
-div(|x|^a DV) = -div(|x|^a D(\gamma u + x \cdot Du)) \\
= - \gamma div(|x|^a Du) - div(|x|^a D(x \cdot Du)) \\
= \gamma|x|^b u^p + p|x|^b u^{p-1}(x \cdot Du) + \gamma(p - 1)|x|^b u^p \\
= \gamma p|x|^b u^p + p|x|^b u^{p-1}(x \cdot Du) = p|x|^b u^{p-1}(\gamma u + x \cdot Du) \\
= p|x|^b u^{p-1}V,
\]

and this verifies that \( V \) is a positive classical solution of

\[
-div(|x|^a DV) = p|x|^b u^{p-1}V \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{3.2}
\]

Now multiply \([3.2]\) by a fixed non-negative \( \phi \in C^1_c(\mathbb{R}^N) \), integrate over \( \Omega_\varepsilon := \mathbb{R}^N \setminus B_\varepsilon(0) \) for small \( \varepsilon \in (0, 1) \), then apply an integration by parts to get

\[
\int_{\Omega_\varepsilon} |x|^a DV \cdot D\phi \, dx - \varepsilon \int_{\partial B_\varepsilon(0)} \phi \frac{\partial V}{\partial n} \, dS = p \int_{\Omega_\varepsilon} \phi |x|^b u^{p-1}V \, dx, \tag{3.3}
\]

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where $n$ denotes the inward pointing unit normal on $\partial B_\varepsilon(0)$. Now define

$$f(\varepsilon) = \int_{\mathbb{S}^{N-1}} V(\varepsilon, \theta) \, dS.$$  

The regularity of $V$ implies $f \in C^1((0, 1)) \cap C([0, 1])$, and this allows us to select a subsequence $\varepsilon_k \to 0^+$ such that $\varepsilon_k f'(|\varepsilon_k|) \to 0^+$. Then

$$0 \leq \varepsilon_k \int_{\partial B_\varepsilon(0)} \phi \frac{\partial V}{\partial n} \, dS \leq C \|\phi\|_{L^\infty} \varepsilon_k^{a+N-1} \left| \frac{\partial}{\partial \varepsilon} \int_{\mathbb{S}^{N-1}} V(\varepsilon_k, \theta) \, dS \right| \leq C \varepsilon_k^{N-2+a} \varepsilon_k f'(|\varepsilon_k|) \to 0^+,$$

where we have applied Hopf’s lemma in the first inequality. Hence, by taking $\varepsilon = \varepsilon_k \to 0^+$ in (3.3), we deduce identity (3.1).

**Step 2.** We now prove a stability estimate for $u$: For any $R > 0$,

$$p \int_{B_R(0)} |x|^b u^{p-1} \varphi^2 \, dx \leq \int_{B_R(0)} |x|^a |D\varphi|^2 \, dx \quad \text{for every } \varphi \in C^1_c(B_R(0)).$$

Fix any $R > 0$ and choose an arbitrary function $\varphi \in C^1_c(B_R(0))$. Setting $\phi = \varphi^2/V$ in (3.1) and applying Young’s inequality, we obtain

$$0 = \int_{B_R(0)} \left[ |x|^a D\varphi \cdot D\left( \frac{\varphi^2}{V} \right) - p|x|^b u^{p-1} \varphi^2 \right] \, dx$$

$$= \int_{B_R(0)} 2|x|^a \frac{\varphi^2}{V^2} (DV \cdot D\varphi) - |x|^a \frac{\varphi^2}{V^2} |DV|^2 \, dx - p \int_{B_R(0)} |x|^b u^{p-1} \varphi^2 \, dx$$

$$\leq \int_{B_R(0)} |x|^a \frac{\varphi^2}{V^2} |DV|^2 + |D\varphi|^2 - |x|^a \frac{\varphi^2}{V^2} |DV|^2 \, dx - p \int_{B_R(0)} |x|^b u^{p-1} \varphi^2 \, dx$$

$$= \int_{B_R(0)} \left[ |x|^a |D\varphi|^2 - p|x|^b u^{p-1} \varphi^2 \right] \, dx.$$  

**Step 3.** Fix some function $\varphi \in C^1_c(\mathbb{R}^N)$. In (1.2), take the test function to be $\varphi^2 u$, which leads us to

$$\int_{\mathbb{R}^N} \left( \varphi^2 |x|^a |Du|^2 + 2|x|^a \varphi u Du \cdot D\varphi \right) \, dx = \int_{\mathbb{R}^N} \varphi^2 |x|^b u^{p+1} \, dx.$$  

By subtracting this from the stability inequality with test function $\varphi u$, i.e., from

$$p \int_{\mathbb{R}^N} \varphi^2 |x|^b u^{p+1} \, dx \leq \int_{\mathbb{R}^N} |x|^a |D(\varphi u)|^2 \, dx,$$  

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we obtain
\[(p - 1) \int_{\mathbb{R}^N} |x|^b u^{p+1} \, dx \leq \int_{\mathbb{R}^N} |x|^a |D\varphi|^2 u^2 \, dx. \tag{3.5}\]

Choose \(\xi \in C^1_c(B_2(0))\) such that \(0 \leq \xi \leq 1\) and \(\xi \equiv 1\) in \(B_1(0)\). If we set
\[\varphi(x) = \xi\left(\frac{x}{R}\right)^{\frac{p+1}{p-1}}\]
in estimate (3.5) and apply Hölder’s inequality, we get
\[
\int_{\mathbb{R}^N} \varphi^2 |x|^b u^{p+1} \, dx \leq C \int_{\mathbb{R}^N} |x|^a |D\varphi|^2 u^2 \, dx \leq \frac{C}{R^2} \int_{B_2R(0)\setminus B_R(0)} \varphi^{\frac{p+1}{p-1}} |x|^a u^2 \, dx \leq \frac{C}{R^2} \left( \int_{B_2R(0)\setminus B_R(0)} |x|^{a(p+1) - \frac{2b}{p-1}} \, dx \right)^{\frac{p-1}{p+1}} \left( \int_{B_2R(0)\setminus B_R(0)} \varphi^2 |x|^b u^{p+1} \, dx \right)^{\frac{p+1}{p-1}}.
\]

This implies that
\[
\int_{B_R(0)} |x|^b u^{p+1} \, dx \leq CR^{-2\frac{(p+1)}{p-1}} \int_{B_2R(0)\setminus B_R(0)} |x|^{a(p+1) - \frac{2b}{p-1}} \, dx \leq CR^{N + \frac{a(p+1) - 2(p+1+b)}{p-1}}.
\]

Sending \(R \to +\infty\) while noting that
\[N + \frac{a(p+1) - 2(p+1+b)}{p-1} < 0\]
whenever \(p < \frac{N+2+2b-a}{N-2+a}\), we deduce that \(||x|^b u^{p+1}\|_{L^1(\mathbb{R}^N)} = 0\), which means \(u \equiv 0\). This is impossible, and this completes the proof. \(\square\)

### 4 Proofs of Theorem 1 and Theorem 3

**Lemma 5.** Let \(N \geq 3\), \(p > 1\) and \(b > a - 2 > -N\). Then (1.1) admits a radially symmetric positive solution \(u\) in \(\Omega = \mathbb{R}^N\) provided that \(p \geq \frac{N+2+2b-a}{N-2+a}\). Moreover, if \(p = \frac{N+2+2b-a}{N-2+a}\), then \(u\) is proportional to the function \(h(x)\) in (1.3) after a suitable scaling (and a translation if \(a,b = 0\), if necessary).

**Proof.** The classification of radially symmetric positive solutions in the critical case follows from basic ODE theory; in particular, see Section 2.3 in [7].

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and the phase plane analysis of equation (2.18) in Theorem 9.1 of [37] (see also [10, 12]).

Now consider the initial value problem

\begin{align}
\begin{cases}
-v''(r) + \frac{N - 1 + a}{r}v'(r) = r^{b-a}v(r)^p, & r > 0, \\
v'(0) = 0, & v(0) = 1.
\end{cases}
\end{align}

(4.1)

Recall there is a maximal interval \((0, r_0)\) for some \(r_0 \in (0, +\infty)\) for which a unique positive solution \(v(r) \in C^2((0, r_0)) \cap C^1([0, r_0))\) to (4.1) exists. Moreover, as \(N - 1 + a > 0\) and \(-r^{N-1+a}v'(r)' > 0\) in \((0, r_0)\), integrating this differential inequality leads to \(v'(r) < 0\), i.e., \(v(r)\) is monotone decreasing in \((0, r_0)\). Thus, \(r_0 < +\infty\) or \(r_0 = +\infty\) are the only possibilities and that \(v(r) \to 0\) as \(r \to r_0\). Now, to finish the proof, it suffices to show that we must have \(r_0 = +\infty\). Assume, on the contrary, that \(r_0 < +\infty\) and therefore \(v(r_0) = 0\).

Claim. We claim that for any \(R > 0\), the resulting boundary value problem: Problem (1.1) with prescribed boundary condition

\begin{align}
u = 0 \quad \text{on} \quad \partial \Omega,
\end{align}

(4.2)

does not admit any classical solution \(u \in C^2(B_R(0) \setminus \{0\}) \cap C^1(\bar{B}_R(0))\) in \(\Omega = B_R(0) \setminus \{0\}\) whenever \(p \geq \frac{N+b+2b-a}{N-2+a}\).

If we momentarily take this claim to be true, then the fact that \(r_0 < +\infty\) and \(v(r_0) = 0\) ensure that \(u(x) := v(|x|) \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega})\) is a (radially symmetric) classical solution of the Dirichlet problem (1.1)-(4.2) with \(\Omega = B_{r_0}(0)\). But this contradicts with the claim and thus \(r_0 = +\infty\).

So it only remains to prove the claim. Choose an arbitrary \(R > 0\) and let \(u\) be a positive solution of (1.1)-(1.2) in \(\Omega = B_R(0)\). Identity (2.15) of Lemma 3 and the boundary condition imply

\begin{align}
\left(\frac{N+b}{p+1} - \frac{N-2+a}{2}\right) \int_{\Omega} |x|^b |u|^{p+1} \, dx = \frac{1}{2} \int_{\partial B_R(0)} \left(\frac{\partial u}{\partial \nu}\right)^2 (x \cdot \nu)|x|^a \, dS > 0,
\end{align}

where the positivity of the right-hand side is because \(x \cdot \nu(x) = |x| = R > 0\) on \(\partial B_R(0)\). This necessarily implies

\begin{align}
\frac{N+b}{p+1} - \frac{N-2+a}{2} > 0,
\end{align}

which is equivalent to \(p < \frac{N+2+2b-a}{N-2+a}\). This proves the claim.
Proof of Theorem 1. This follows from Lemma 5 and Theorem 2.

Proof of Theorem 3. Part (a) of Theorem 3 is a consequence of Lemma 6 below. Part (b) follows from the existence minimizers for the sharp CKN inequalities, which can be found in [7, 10, 12].

Lemma 6. Let \( N \geq 3 \), \( \Omega = \mathbb{R}^N \), \( p \geq 1 \), and \( a - 2 > -N \). If \( u \) is a finite energy solution of (1.1) and belongs to \( H^2_{\text{loc}}(\Omega) \), then the following conditions necessarily hold:

\[
p > 1, \quad b > a - 2 > -N \quad \text{and} \quad \frac{N + b}{p + 1} + 1 = \frac{N + a}{2}.
\]

(4.3) Proof. Suppose that \( u \in H^2_{\text{loc}}(\Omega) \cap D_{a,b,p+1} \) is a finite energy solution of (1.1). From Proposition 1, there holds \( b > a - 2 > -N \). Further, we necessarily have \( p > 1 \) as no positive weak solution exists if \( p = 1 \). The proof of this is similar to the one for Proposition 1 but it may also be found on page 147 of [10] or see Section 3 in [7].

The rest of the proof is of similar nature to the strategy adopted in [33] (see Proposition 3.1; see also [25]), and it relies on another Rellich-Pohozaev type identity. Fix a function \( \psi \in C^1_c(\mathbb{R}^N) \) whose support is contained in \( B_2(0) \) and \( \psi \equiv 1 \) in \( B_1(0) \). For any \( \lambda > 0 \), consider the test function \( \phi(x) = \psi(\lambda x)(x \cdot Du(x)) \). By definition of finite energy solutions, there holds

\[
\int_{\mathbb{R}^N} Du \cdot D\left[ \psi(\lambda x)(x \cdot Du) \right] |x|^a \, dx = \int_{\mathbb{R}^N} |x|^b u(x)^p \psi(\lambda x)(x \cdot Du) \, dx. \tag{4.4}
\]

Denote the left and right hand side of (4.4) by \( F_1 \) and \( F_2 \), respectively. By the product rule and the fact that \( \frac{\partial u}{\partial x_j} x_j D_{ij} u = \frac{1}{2} x \cdot D(|Du|^2) \), we get

\[
F_1 = \int_{\mathbb{R}^N} Du \cdot D\psi(\lambda x)(x \cdot Du)|x|^a \, dx + \frac{1}{2} \int_{\mathbb{R}^N} x \cdot D(|Du|^2) \psi(\lambda x)|x|^a \, dx
+ \int_{\mathbb{R}^N} |Du|^2 |x|^a \psi(\lambda x) \, dx.
\]

An integration by parts on the second term on the right-hand side yields

\[
F_1 = \int_{\mathbb{R}^N} Du \cdot D\psi(\lambda x)(x \cdot Du)|x|^a \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 (x \cdot D\psi(\lambda x))|x|^a \, dx
+ \left( 1 - \frac{N + a}{2} \right) \int_{\mathbb{R}^N} |Du|^2 |x|^a \psi(\lambda x) \, dx.
\]
By the Lebesgue dominated convergence theorem, sending $\lambda \to 0^+$ yields
\[ F_1 = -\left(\frac{N+a}{2} - 1\right) \int_{\mathbb{R}^N} |x|^a |Du|^2 \, dx. \] (4.5)

Likewise, we calculate
\[
F_2 = \int_{\mathbb{R}^N} |x|^b u(x) p(x \cdot Du) \psi(\lambda x) \, dx = \frac{1}{p+1} \int_{\mathbb{R}^N} |x|^b x \cdot D(u^{p+1}) \psi(\lambda x) \, dx
\]
\[ = -\frac{1}{p+1} \int_{\mathbb{R}^N} |x|^b u^{p+1} (x \cdot D\psi(\lambda x)) \, dx - \frac{N+b}{p+1} \int_{\mathbb{R}^N} |x|^b u^{p+1} \psi(\lambda x) \, dx, \]
and thus sending $\lambda \to 0^+$ in this results in the identity
\[ F_2 = -\frac{N+b}{1+p} \int_{\mathbb{R}^N} |x|^b u^{p+1} \, dx. \] (4.6)

In addition, by density we may set $\varphi = u$ in (1.2) to arrive at
\[
\int_{\mathbb{R}^N} |x|^a |Du|^2 \, dx = \int_{\mathbb{R}^N} |x|^b u^{p+1} \, dx.
\]

So in view of this, inserting (4.5) and (4.6) into the identity (4.4) yields
\[
\left( -\frac{N+a}{2} + 1 + \frac{N+b}{p+1} \right) \int_{\mathbb{R}^N} |x|^b u^{p+1} \, dx = 0.
\]

Hence,
\[
\frac{N+b}{p+1} + 1 = \frac{N+a}{2}.
\]

Acknowledgements: The author is supported by the Simons Foundation Collaboration Grants for Mathematicians 524335

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