Are quantum spin Hall edge modes more resilient to disorder, sample geometry and inelastic scattering than quantum Hall edge modes?

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Abstract

On the surface of 2D topological insulators, 1D quantum spin Hall (QSH) edge modes occur with Dirac-like dispersion. Unlike quantum Hall (QH) edge modes, which occur at high magnetic fields in 2D electron gases, the occurrence of QSH edge modes is due to spin–orbit scattering in the bulk of the material. These QSH edge modes are spin-dependent, and chiral-opposite spins move in opposing directions. Electronic spin has a larger decoherence and relaxation time than charge. In view of this, it is expected that QSH edge modes will be more robust to disorder and inelastic scattering than QH edge modes, which are charge-dependent and spin-unpolarized. However, we notice no such advantage accrues in QSH edge modes when subjected to the same degree of contact disorder and/or inelastic scattering in similar setups as QH edge modes. In fact we observe that QSH edge modes are more susceptible to inelastic scattering and contact disorder than QH edge modes. Furthermore, while a single disordered contact has no effect on QH edge modes, it leads to a finite charge Hall current in the case of QSH edge modes, and thus a vanishing of the pure QSH effect. For more than a single disordered contact while QH states continue to remain immune to disorder, QSH edge modes become more susceptible—the Hall resistance for the QSH effect changes sign with increasing disorder. In the case of many disordered contacts with inelastic scattering included, while quantization of Hall edge modes holds, for QSH edge modes a finite charge Hall current still flows. For QSH edge modes in the inelastic scattering regime we distinguish between two cases: with spin-flip and without spin-flip scattering. Finally, while asymmetry in sample geometry can have a deleterious effect in the QSH case, it has no impact in the QH case.

Keywords: quantum spin Hall, quantum Hall, disorder, inelastic scattering, topological insulators

(Some figures may appear in colour only in the online journal)
It is widely known that transport along edge modes in a QH setting is resilient to disorder. In view of this we test whether in a QSH bar the QSH edge modes will show the same resilience or be more resilient to the twin effects of disorder and inelastic scattering—the bane of any phenomena which rely on complete quantum coherence. The expectation is that since QSH edge modes are spin-dependent and spin has a longer relaxation time than charge, the spin Hall edge modes will be far more robust to disorder and inelastic scattering. However, contrary to expectations we see not only that there is no added advantage of QSH edge modes: compared with QH edge modes there is instead a disadvantage. We find QSH edge modes are quite susceptible to disorder and inelastic scattering while QH edge modes are not. This is of possible relevance to their use in spintronics and quantum computation applications as well as setups wherein QSH edge modes are utilized to generate Majorana fermions.

The aim of this work is to compare the quantization of Hall and longitudinal resistance seen in an ideal contacted Hall or spin Hall sample, and to investigate how this quantization is affected by disordered contacts, inelastic scattering and sample geometry. A disordered contact in contrastdirection to an ideal contact does not have a transmission probability of 1. Furthermore, as sample size increases, edge modes will be affected by inelastic scattering, in the case where inelastic scattering length $l_{\text{in}} < L$ (length of sample). However, edge modes on the upper side will not scatter to the lower side; inelastic scattering equilibrates the populations of edge states with each other on the same side of the sample. This is the case for inelastic scattering in QH samples. The situation changes in the case of QSH samples. Here we have spin-up and spin-down edge modes, and equilibration might happen at the same edge between spin-up and spin-down edge modes in effect via spin-flip scattering. In the absence of spin-flip scattering, edge modes will also equilibrate due to inelastic processes like electron–electron scattering or electron–phonon scattering; however, in this case spin-up edge modes will equilibrate only with spin-up and not spin-down, similarly for spin-down edge modes. These edge states remain in equilibrium once equilibrated [5]. In contrast to an earlier work [6], which predicted quantized values of conductance in the presence of strong disorder for topological insulator edge modes, we show that quantization of longitudinal conductance and Hall conductance is lost even when a single contact is disordered. Of course, one has to caveat the aforesaid statement since [6] considers disorder in the sample itself; however, in the cases we have considered in this work, the disorder is confined to the contacts only. The effect of random magnetic fluxes on QSH edge modes has been considered previously [7], wherein it was concluded that spin Hall edge modes are localized in their presence. Localization of QSH edge modes has also been predicted for non-magnetic disorder in the sample in [8].

Inelastic scattering, however, is not restricted to contacts but is all-pervasive and comes into picture when the sample length exceeds the inelastic scattering length. Furthermore, inelastic scattering may be accompanied by spin-flip scattering too. In the cases we consider below, the length of the sample in the ideal and single-probe disordered cases is less than the inelastic scattering length. Only when we consider the case of a sample with all probe contacts disordered, we may have length of sample exceeding the inelastic scattering length. In our work the term ‘probe’ and ‘contact’ mean the same—a metallic reservoir. Finally, we generalize the results to $N$ terminals (or probes, or contacts); this will help us in determining whether sample geometry like a difference in the number of contacts at the upper or lower edge will have any bearing on the edge modes. The study will have four parts, each with the same sections: we calculate Hall resistance $R_{\text{H}}$, longitudinal resistance $R_{\text{L}}$ and two-terminal resistance $R_{14,14}$ for each section in all the parts:

Part 1: QH edge modes: 6 terminals. a. ideal case, b. single disordered contact, c. two or more disordered contacts, d. all disordered contacts with inelastic scattering.

Part 2: QSH edge mode: 6 terminals. a. ideal case, b. single disordered contact, c. two or more disordered contacts, d. all disordered contacts with inelastic scattering (with spin-flip), e. all disordered contacts with inelastic scattering (without spin-flip).

Part 3: Generalized to $N$ terminals, the QH case. a. ideal case, b. all disordered contacts with inelastic scattering.

Part 4: Generalized to $N$ terminals, the QSH case. a. ideal case, b. all disordered contacts with inelastic scattering.

2. QH edge modes

The Landauer–Buttiker formula relating currents and voltages in a multiprobe device is [5]:

$$I_i = \sum_j (G_{ij} V_j - G_{ji} V_i) = \frac{e^2}{h} \sum_j (T_{ij} V_j - T_{ji} V_i)$$

where $V_i$ is the voltage at the $i$th terminal and $I_i$ is the current flowing from the same terminal. Here $T_{ij}$ is the transmission from the $j$th to the $i$th terminal and $G_{ij}$ is the associated conductance.

2.1. QH edge modes—ideal case

The ideal case is represented in figure 1(a). The current–voltage relations can be derived from the conductance matrix below:

$$G_{ij} = -\frac{e^2 M}{h} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$  

$M$ represents the total number of modes. In the setup shown in figure 1, $M = 1$ to avoid clutter. Substituting $t_1, t_2, t_3$ and $t_4 = 0$ and choosing reference potential $V_4 = 0$, we derive $V_1 = V_2 = V_3$ and $V_4 = V_5 = V_6 = 0$. So, the Hall resistance $R_{\text{H}} = R_{14,26} = \frac{h}{e^2 M}$ then longitudinal resistance $R_{\text{L}} = R_{14,23} = 0$, and two-terminal resistance $R_{2T} = R_{14,14} = \frac{h}{e^2 M}$. The Hall conductance is quantized...
in units of $M$ and thus the longitudinal conductance vanishes and the two-terminal conductance is also quantized in units of $M$.

2.2. QH edge modes—single disordered probe

The disordered probe case is represented in figure 1(b). We consider only a single probe to be disordered. The disorder strengths $D$’s can be written in terms of $T$, the number of transmitted edge modes, and $M$, the total number of edge modes. So, $T_i = (1 - D_i)M$ and $R_i = D_i M$. Depending on the strength of disorder, only a fraction of edge modes will be transmitted by the disordered contact and the rest would be reflected; see figure 1(b).

The current–voltage relations can be derived from the conductance matrix below:

$$
G_{ij} = -\frac{e^2}{h} \begin{pmatrix}
-M & 0 & 0 & 0 & 0 & M \\
T_2 & -T_2 & 0 & 0 & 0 & 0 \\
R_2 & T_2 & -M & 0 & 0 & M \\
0 & 0 & 0 & M & -M & 0 \\
0 & 0 & 0 & 0 & M & -M 
\end{pmatrix}. \tag{3}
$$

Here, $M$ is the total number of edge modes, $T$ is the number of transmitted edge modes into a contact and $R$ is the number of reflected edge modes. Substituting $L_i, I_i, I_5$ and $I_6 = 0$ and choosing reference potential $V_4 = 0$, we derive $V_1 = V_2 = V_3$, and $V_4 = V_5 = V_6 = 0$. So, the Hall resistance $R_{14} = R_{14,26} = \frac{h}{e^2 M}$, longitudinal resistance $R_i = R_{14,23} = 0$, and two-terminal resistance $R_{2T} = R_{14,14} = \frac{h}{e^2 M}$. We see the resistance characteristics are completely independent of the strength of disorder. Similar to the case of ideal contacts, the Hall conductance is quantized in units of $M$ and thus the longitudinal conductance vanishes and the two-terminal conductance is also quantized in units of $M$.

2.3. QH edge modes—two or more disordered probes

The case of more than a single disordered probe is an extension of that represented in figure 1(b) of a single disordered probe with all probes having finite disorder leading to non-zero reflection probability for edge modes from those probes.

Herein, we consider all the contacts to be disordered in general. The current–voltage relations can be derived from the conductance matrix below:

$$
G_{ij} = -\frac{e^2 M}{h} \begin{pmatrix}
-T_1^{11} & T_1^{12} & T_1^{13} & T_1^{14} & T_1^{15} & T_1^{16} \\
-T_2^{21} & -T_2^{22} & T_2^{23} & T_2^{24} & T_2^{25} & T_2^{26} \\
T_3^{31} & T_3^{32} & T_3^{33} & T_3^{34} & T_3^{35} & T_3^{36} \\
T_4^{41} & T_4^{42} & T_4^{43} & -T_4^{44} & T_4^{45} & T_4^{46} \\
T_5^{51} & T_5^{52} & T_5^{53} & T_5^{54} & -T_5^{55} & T_5^{56} \\
T_6^{61} & T_6^{62} & T_6^{63} & T_6^{64} & T_6^{65} & T_6^{66} 
\end{pmatrix}. \tag{4}
$$

$M$ represents the total number of modes. In the setup shown in figure 1(b), $M = 1$ to avoid clutter and only one mode is shown. In the above matrix, for example, $T_1^{15}$ is defined as the total transmission probability from contact 5 to 1 and can be written explicitly as

$$
T_{15} = (1 - D_5)D_6(1 - D_1)M + (1 - D_5)D_6D_4D_3D_2D_1D_6(1 - D_1)M + \ldots
$$

or,

$$
T_{15} = (1 - D_5)D_6(1 - D_1)M[1 + D_6D_4D_3D_2D_1D_6 + \ldots]
\frac{(1 - D_5)D_6(1 - D_1)M}{1 - D_6D_4D_3D_2D_1D_6}. \tag{5}
$$

where $D_i$ is the strength of disorder in contact $i$. An electron starting from contact 5 has the probability $1 - D_5$ to be transmitted. Since we are interested in the probability of its reaching contact 1, it has to be reflected from contact 6 with probability $D_6$, and finally it is transmitted to contact 1 with probability $1 - D_1$. However, this is the shortest of the many paths possible for an electron starting from 5 to reach 1; another path can be that of an electron starting from contact 5, with probability $1 - D_5$ to be transmitted. Since we are interested in the probability of its reaching contact 1, it has to be reflected from contact 6 with probability $D_6$, and then reflected from contact 1 with probability $D_1$. Similarly with probability $D_2$ it will be reflected from contact 2, with probability $D_3$ from contact 3, with probability $D_4$ from contact 4, with probability $D_5$ from contact 5, again get reflected with probability $D_6$ from contact 6 and finally get transmitted into contact 1. This is the second shortest path possible. Similarly one can sum over all the other paths leading to an infinite

![Figure 1](https://example.com/fig1.png)

Figure 1. Six-terminal QH bar showing QH edge modes. There are an equal number of edge modes on both sides of the sample. (a) Ideal case: contacts are reflectionless. (b) Single disordered probe: $R_i, T_i$ represent the reflection and transmission probability of edge modes from and into contact 2. To represent the effect of disorder in contact 2, an extra edge mode is shown being reflected from contact 2. (c) All disordered contacts with inelastic scattering: stars indicate equilibration of contact potentials at those places.
series, which can be summed to yield the total probability per mode for transmission from contact 5 to 1 as in equation (5). Similarly, all other transmission probabilities can be derived. Substituting $I_2, I_3, I_4$ and $I_6 = 0$, as these are voltage probes, and choosing reference potential $V_4 = 0$, we solve the above matrix and calculate the Hall, longitudinal and two-terminal resistances. The Hall resistance $R_H = R_{A26} = \frac{\hbar}{e^2 M}$, longitudinal resistance $R_L = R_{A23} = 0$, and two-terminal resistance $R_{TT} = R_{44,14} = \frac{h M^2 - 8 R_A}{e^2 M \lambda_A}$. Apart from in the two-terminal case, the QH edge mode Hall conductance remains quantized even when all contacts are disordered with inelastic scattering included.

2.4. QH edge modes—all disordered contacts with inelastic scattering

The case of QH edge modes in the presence of all disordered contacts and with inelastic scattering included has been dealt with before in [5, 9]. We can look at figure 1(c), where we consider that the length between disordered contacts is larger than the inelastic scattering length. On the occurrence of an inelastic scattering event, the edge states originating from different contacts with different energies are equilibrated to a common potential. In figure 1(c), one can see that electrons coming from contact 1 and 6 are equilibrated to potential $V'_4$. If as before contacts 1 and 4 are chosen to be the current contacts then no current flows into the other voltage probe contacts. Let’s say a current $\frac{e}{h} T_2 V'_4$ enters contact 2, while a current $\frac{e}{h} T_4 V'_6$ leaves contact 2. Since contact 2 is a voltage probe net current it has to be zero, implying $V'_2 = V'_6$. The same thing happens at contact 3 and along the lower edge, where contacts are equilibrated to $V'_4$.

Now we write the current–voltage relations in continuous fashion, eschewing our earlier method of writing it in matrix form to avoid clutter as there are not only the 6 potentials $V_1 - V_{6}$, we also have the equilibrated potentials $V'_1 - V'_6$.

\[
\begin{align*}
I_1 &= T_1 (V_1 - V'_6), \\
I_2 &= T_2 (V_2 - V'_4), \\
I_3 &= T_3 (V_3 - V'_2), \\
I_4 &= T_4 (V_4 - V'_3), \\
I_5 &= T_5 (V_5 - V'_4), \\
I_6 &= T_6 (V_6 - V'_2).
\end{align*}
\]

(6)

By assuming the net current into the voltage probe contacts 2, 3, 4 and 5 to be zero, we get the following relations between the contact potentials: $V_5 = V'_3$, $V_3 = V'_5$, $V_4 = V'_4$ and $V_6 = V'_6$. Furthermore, due to the equilibration, the net current just out of contact 2 is the sum $\frac{e}{h} (T_2 V'_2 + R_2 V_2)$, and this should be equal to $\frac{e^2}{h} M V'_2$, which is the equilibrated potential due to inelastic scattering at contact 3. Thus $\frac{e^2}{h} (T_2 V'_2 + R_2 V_2) = \frac{e^2}{h} M V'_2$ or $V'_2 = V'_2$, as $T_2 + R_2 = M$, the total number of edge modes in the system. Thus, all the upper edges are equilibrated to same potential $V'_1 = V'_2 = V'_3 = V'_6$. Similarly, for the equilibrated potentials at the lower end, we get $V'_4 = V'_5 = V'_2 = V'_6 = V'_6$. Using these relations between primed and unprimed potentials we derive the Hall resistance $R_H = R_{A26} = \frac{\hbar}{e^2 M}$, longitudinal resistance $R_L = R_{A23} = 0$ and two-terminal resistance $R_{TT} = R_{44,14} = \frac{h M^2 - 8 R_A}{e^2 M \lambda_A}$. Apart from in the two-terminal case, the QH edge mode Hall conductance remains quantized even when all contacts are disordered with inelastic scattering included.

3. QSH edge modes

QH edge modes are chiral and describe electron motion along sample boundaries. Each boundary has an integer number of states or modes, all of which carry charge in the same direction. QSH edge modes on the other hand are spin-dependent as well as chiral, with the additional quality that opposite spins move in opposite directions at one edge [12]. The Landauer–Buttiker formula relating currents and voltages in a multi-probe device has been extended to the case of QSH edge modes in [10, 11]:

\[
I_i = \sum_{j} (G_{ij} V_i - G_{ij} V_j) = \frac{e^2}{h} \sum_{j=1}^{N} (T_{ij} V_i - T_{ij} V_j),
\]

(7)

where $V_i$ is the voltage at the $i$th terminal and $I_i$ is the current flowing from the same terminal. Here $T_{ij}$ is the transmission from the $j$th to the $i$th terminal, and $G_{ij}$ is the associated conductance. $N$ denotes the number of terminals/contacts in the system. We first consider $N = 6$ and then generalize it to the $N$ terminal case as above.

3.1. QSH edge modes—ideal case

The ideal case is represented in figure 2(a). The current–voltage relations can be derived from the conductance matrix below:

\[
G_{ij} = \frac{e^2 M}{h} \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & -2 
\end{pmatrix}
\]

(8)

Substituting $I_2, I_3, I_4$ and $I_6 = 0$ and choosing the reference potential $V_4 = 0$, we derive $V_5 = V_5/2 = V_5/3$ and $V_6 = V_6/2 = V_6/3$. So, the charge Hall resistance $R_H = R_{A26} = 0$ obviously, then longitudinal resistance is $R_L = R_{A23} = \frac{\hbar}{e^2 M}$ and two-terminal resistance $R_{TT} = R_{44,14} = \frac{\hbar}{e^2 M}$. Thus, the charge Hall conductance is zero, and longitudinal conductance is quantized in units of $2M$ while two-terminal conductance is quantized in units of $\frac{2}{3} M$.

3.2. QSH edge modes—single disordered probe

This case is represented in figure 2(b), where only a single probe is disordered. The current–voltage relations can be derived from the conductance matrix below:
The case for all disordered probes is an extension of the case of single disordered probes as represented in figure 2(b). Herein we consider all the contacts to be disordered in general.

\[
G_y = -\frac{e^2}{h} \begin{pmatrix} -2M & T_2 & R_2 & 0 & 0 & M \\ T_2 & -2T_2 & T_2 & 0 & 0 & 0 \\ R_2 & T_2 & -2M & M & 0 & 0 \\ 0 & 0 & M & -2M & M & 0 \\ 0 & 0 & 0 & M & -2M & M \\ M & 0 & 0 & 0 & M & -2M \end{pmatrix}
\]

Substituting \( I_2, I_3, I_5 \) and \( I_6 = 0 \) and choosing reference potential \( V_4 = 0 \), we derive \( V_5 = V_6/2 = V_6/3 \). So, the Hall resistance \( R_H = R_{4,26} = \frac{\hbar}{2\pi M^3 + 2D_1} \), then longitudinal resistance \( R_L = R_{4,23} = \frac{\hbar}{2\pi M^3 + 2D_2} \), and two-terminal resistance \( R_{TT} = R_{4,14} = \frac{\hbar}{2\pi M^3 + 2D_2} \). Thus, all of the calculated conductances (Hall, longitudinal and two-terminal) lose their quantization and are dependent on disorder. Notice that these quantities are all influenced by disorder, in contrast to the QH case in which they are immune to disorder.

3.3. QSH edge modes—two or more disordered probes

The case for all disordered probes is an extension of the case of single disordered probes as represented in figure 2(b). Herein we consider all the contacts to be disordered in general. The current–voltage relations can be derived from the conductance matrix below:

\[
G_y = -\frac{e^2M}{h} \begin{pmatrix} -T^{11} & T^{12} & T^{13} & T^{14} & T^{15} & T^{16} \\ T^{21} & -T^{22} & T^{23} & T^{24} & T^{25} & T^{26} \\ T^{31} & T^{32} & -T^{33} & T^{34} & T^{35} & T^{36} \\ T^{41} & T^{42} & T^{43} & -T^{44} & T^{45} & T^{46} \\ T^{51} & T^{52} & T^{53} & T^{54} & -T^{55} & T^{56} \\ T^{61} & T^{62} & T^{63} & T^{64} & T^{65} & -T^{66} \end{pmatrix}
\]

where \( M \) represents the total number of modes. In the setup shown in figure 1(b), \( M = 1 \) to avoid clutter and only one mode is shown. In the above matrix, \( T^{15} \) is the total transmission probability from contact 5 to 1, in contradistinction to that for the QH case, and can be written explicitly as

\[
T^{15} = \frac{[(1 - D_2)D_6(1 - D_6) + (1 - D_3)D_3D_2(1 - D_1)]M}{1 - D_1D_2D_3D_4D_5D_6},
\]

the reason being there are two spin-polarized edge modes which are moving in opposite directions, the up-spin polarized edge mode contributes to \( T^{15} \) via the first term while the down-spin polarized edge mode contributes via the second term. So the total probability per mode for transmission from contact 5 to 1 is as defined above. Similarly, all other transmission probabilities occurring in the above matrix can be explained.

As previously noted for the QH case in section 2.3, we see that the difference between QH and spin Hall is also quite stark when it comes to more than one disordered contact. In the QH case, while the Hall and longitudinal resistances do not deviate from the ideal quantized values, for the QSH case there is a deviation. In fact, for a particular choice as in figure 3(d), the Hall current for spin Hall edge modes is not only finite but also changes sign, indicating the complete breakdown of the spin Hall effect via disorder. Furthermore, for other choices of disorder probes as shown in figures 3(b) and (c), we see that Hall, longitudinal and two-terminal resistance for the QH case is quantized, while the same quantities for the QSH case deviate from their ideal quantized values, indicating that the QSH edge modes are much more fragile than QH edge modes.

3.4. QSH edge modes—all disordered probes with inelastic scattering (with spin-flip)

The case of QSH edge modes in the presence of completely disordered contacts and with inelastic scattering included can be understood by extending the approach towards QH edge modes. We can look at figure 2(c) where we consider that the length between disordered contacts is larger than the inelastic scattering length. On the occasion of an inelastic scattering event happening, the edge states originating from different contacts with different energies are equilibrated to a common potential as in the QH case. In figure 2(c), one can see that electrons coming from contact 1 and 6 are equilibrated to potential \( V' \). If as before contacts 1 and 4 are chosen to be the current contacts, then no current flows into the other voltage

Figure 2. Six-terminal QSH bar showing QSH edge modes. These edge modes differ from their QH counterparts since these are spin-polarized and chiral not only with respect to the edges (like QH edge modes) but also with respect to spin: spin-up and spin-down edge modes move in opposite directions. (a) Ideal case: contacts are reflectionless. (b) Single disordered probe: \( R_0, T_0 \) represent the reflection and transmission probability of edge modes from and into contact 1; an extra edge mode is shown to represent the effect of disorder in contact 2. (c) All disordered contacts with inelastic scattering: stars indicate equilibration of contact potentials at those locations.
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Let’s say a current $I'$ enters contact 2. The first part $I_{TV}^e$ is the spin-up component while the second part $I_{TV}^h$ is the spin-down component, moving in exactly the opposite direction. Similarly, the current $I_e$ leaves contact 2, and since contact 2 is a voltage probe then net current has to be zero, implying $V_2 = (V'_1 + V'_2)/2$. The same thing happens at contact 3 and along the lower edge.

Now we write the current–voltage relations in continuous fashion, eschewing our earlier method of writing in matrix form to avoid clutter as there are not only the six potentials $V_i - V_0$, but also the equilibrated potentials $V'_1 - V'_6$.

\[
\begin{align*}
I_1 &= I_{TV}(2V_1 - V'_1 - V'_6), \\
I_2 &= I_{TV}(2V_2 - V'_1 - V'_2), \\
I_3 &= I_{TV}(2V_5 - V'_2 - V'_3), \\
I_4 &= I_{TV}(2V_4 - V'_3 - V'_4), \\
I_5 &= I_{TV}(2V_5 - V'_4 - V'_5), \\
I_6 &= I_{TV}(2V_6 - V'_5 - V'_6). \\
\end{align*}
\]
By assuming the net current into voltage probe contacts 2, 3, 4 and 5 to be zero we get the following relations between the contact potentials: $V_2 = (V_1 + V'_2)/2$, $V_3 = (V_2 + V'_3)/2$, $V_4 = (V_3 + V'_4)/2$, and $V_5 = (V_4 + V'_5)/2$.

Furthermore, due to the equilibration the net spin-up current out of contact 2 is the sum $\frac{e^2}{\hbar}(T_{V_2} + R_{V'_2})$ and the net spin-down current out of contact 3 is the sum $\frac{e^2}{\hbar}(T_{V_3} + R_{V'_3})$ and this should be equal to $\frac{e^2}{\hbar}2MV'_2$, which is the net current out of $V'_2$, the equilibrated potential due to inelastic scattering between contacts 2 and 3. Similarly, we can write the net spin polarized currents into and out of the equilibrated potentials. Since there are six equilibrated potentials we will have six such equations. The origin of the first equation has already been explained above. Below, we list all of them:

\[
\frac{e^2}{\hbar}(T_{V_2} + R_{V'_2}) + \frac{e^2}{\hbar}(T_{V_3} + R_{V'_3}) = \frac{e^2}{\hbar}2MV'_2
\]

\[
\frac{e^2}{\hbar}(T_{V_1} + R_{V'_1}) = \frac{e^2}{\hbar}2MV'_1
\]

\[
\frac{e^2}{\hbar}(T_{V_4} + R_{V'_4}) = \frac{e^2}{\hbar}2MV'_4
\]

\[
\frac{e^2}{\hbar}(T_{V_5} + R_{V'_5}) = \frac{e^2}{\hbar}2MV'_5
\]

\[
\frac{e^2}{\hbar}(T_{V_6} + R_{V'_6}) = \frac{e^2}{\hbar}2MV'_6
\]

Solving the above six equations gives the equilibrated potentials $V'_i$, $i = 1,\ldots,6$ in terms of the contact potentials $V_i$, $i = 1,\ldots,6$. Substituting the obtained $V'_i$, $i = 1,\ldots,6$ in equation (12), we can derive the necessary resistances. We especially consider the case wherein $D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = 0$, and we have

\[
R_{th} = R_{th,4,26} = \frac{\hbar}{2e^2M} \left( \frac{(D_1 - D_2)}{3 + 2D_1 + D_2(2 + D_1) + D_2(2 + D_1 + D_2)} \right)
\]

\[
R_{th} = R_{th,4,23} = \frac{\hbar}{2e^2M} \left( \frac{(3 + 2D_1 + D_2)}{3 + 2D_1 + D_2(2 + D_1) + D_2(2 + D_1 + D_2)} \right)
\]

\[
R_{th} = R_{th,4,14} = \frac{\hbar}{2e^2M} \left( \frac{4D_2^2 - (3 + D_1)(3 + D_2)}{(1 + D_3)(3 + 2D_1 + D_2(2 + D_1) + D_2(2 + D_1 + D_2))} \right)
\]

(14)

Here $D_1$ denotes disorder in current contacts while $D_2(D_1)$ represent disorder in contacts at the upper (lower) edge. When disorder in contacts at the upper and lower edge are unequal one sees finite charge Hall conductance and thus the pure QSH effect vanishes. This is unlike what happens in the case of QH edge modes. Not only is QH conductance resilient to disorder and inelastic scattering, it also retains its quantization and longitudinal resistance, and the voltage drop across the sample remains zero. In the case of QH edge modes, inelastic scattering has no effect on Hall quantization in the presence of all disordered contacts, regardless of whether their strengths are equal or not; however, in the case of QSH the inelastic scattering destroys the pure QSH effect in the presence of unequal disorder. Only when contact disorder at various probes are the same does the pure QSH effect reappear.

In figure 3 we plot the longitudinal and Hall resistances for the QH and QSH cases. One can see from the insets how the spin Hall case is dependent on disorder while the QH case remains untroubled by disorder.

3.5. QSH edge modes—all disordered probes with inelastic scattering (without spin-flip)

The case of QSH edge modes in the presence of completely disordered contacts and with inelastic scattering included but now without any spin-flip scattering can be understood by extending the approach of the previous section. We can look at figure 4(a), where we consider that the length between disordered contacts is larger than the inelastic scattering length. Furthermore, we now only have equilibration between the same spin edge modes and as before no equilibration occurs across the sample edges. In the event of inelastic scattering happening, the edge states originating from different contacts with different energies are equilibrated to a common potential as in the QH case. In figure 4(a), one can see that electrons with spin-up coming from contacts 1 and 6 are equilibrated to potential $V'_1$ while spin-down electrons coming from contact 2 and 3 are equilibrated to potential $V'_3$. Similarly, the other potentials $V'_i$, $i = 2,\ldots,6$ are decided for equilibration of spin-up edge modes while the potentials $V'_i$, $i = 2,\ldots,6$ are decided for equilibration of spin-down edge modes. If as before contacts 1 and 4 are chosen to be the current contacts then no current flows into the other voltage probe contacts. Let’s say a spin-up current $\frac{e^2}{\hbar}(T_{V'_2} + R_{V'_2})$ and a spin-down current $\frac{e^2}{\hbar}(T_{V'_2} + R_{V'_2})$ enters contact 2, while the current $\frac{e^2}{\hbar}2Tv'_2$ leaves contact 2, and since contact 2 is a voltage probe the net current has to be zero, implying $V_2 = (V'_1 + V'_2)/2$. The same thing happens at the other contacts.

Now we write the current–voltage relations in continuous fashion, as in the previous sections. There are not only the six potentials $V_1 - V_6$, we also have the equilibrated spin-up potentials $V'_1 - V'_6$ and the spin-down potentials $V''_1 - V''_6$.

\[
I_1 = \frac{e^2}{\hbar}T_2(V_1 - V''_1 - V'_1),
\]

\[
I_2 = \frac{e^2}{\hbar}T_2(V_2 - V''_2 - V'_2),
\]

\[
I_3 = \frac{e^2}{\hbar}T_2(V_3 - V''_3 - V'_3),
\]

\[
I_4 = \frac{e^2}{\hbar}T_2(V_4 - V''_4 - V'_4),
\]

\[
I_5 = \frac{e^2}{\hbar}T_2(V_5 - V''_5 - V'_5),
\]

\[
I_6 = \frac{e^2}{\hbar}T_2(V_6 - V''_6 - V'_6).
\]

(15)

By assuming the net current into voltage probe contacts 2, 3, 4 and 5 to be zero we get the following relations between the contact potentials: $V_2 = (V'_1 + V'_2)/2$, $V_3 = (V'_2 + V'_3)/2$, $V_4 = (V'_3 + V'_4)/2$, $V_5 = (V'_4 + V'_5)/2$, and $V_6 = (V'_5 + V'_6)/2$. 


Furthermore, due to the equilibration the net spin-up current out of contact 1 is the sum of the spin-up current out of the contact and the spin-up equilibrated potential $V'_{i}$. Similarly, the net spin-up currents out of contacts 2–6 are equilibrated to the potentials $V'_{j}$, see equation (16).

We adopt the same procedure for the down-spin currents and these are written below in equation (16). The origin of the first equation has already been explained above. Below, we list all of them:

$$\frac{e^2}{h}(T_i V_i + R_i V'_i) = \frac{e^2}{h} M V'_1$$
$$\frac{e^2}{h}(T_2 V_2 + R_2 V'_2) = \frac{e^2}{h} M V'_2$$
$$\frac{e^2}{h}(T_3 V_3 + R_3 V'_3) = \frac{e^2}{h} M V'_3$$
$$\frac{e^2}{h}(T_4 V_4 + R_4 V'_4) = \frac{e^2}{h} M V'_4$$
$$\frac{e^2}{h}(T_5 V_5 + R_5 V'_5) = \frac{e^2}{h} M V'_5$$
$$\frac{e^2}{h}(T_6 V_6 + R_6 V'_6) = \frac{e^2}{h} M V'_6.$$  

Solving the above 12 equations gives the equilibrated potentials $V'_1$ and $V'_h = 1,..6$ in terms of the contact potentials $V_i$, $i = 1,..6$. Substituting the obtained $V'_i$ and $V''_i$, $i = 1,..6$ in equation (15), we can derive the necessary resistances. We have

$$R_{H} = R_{4,26} = \frac{h}{2e^2M} \left\{ \frac{(-1 + D_3)(D_1 - D_3)}{(-3 + D_3)(-1 + D_1 - D_3)} \right\}.$$  
$$R_{L} = R_{4,23} = \frac{h}{2e^2M} \left\{ \frac{(3 - D_1)(-1 + D_3)}{(-3 + D_3)(-1 + D_1 - D_3)} \right\}.$$  
$$R_{2T} = R_{4,14} = \frac{h}{2e^2M} \left\{ \frac{(3 - D_3)(3 - D_3)}{(-3 + D_3)(-1 + D_1 - D_3)} \right\}.$$  

The case of a single disordered probe as was done for the six-terminal case can be easily calculated and the resistances (Hall, longitudinal and two-terminal) are identical to the ideal case with disorder having no impact. Importantly, the quantization of Hall resistance is independent of any asymmetry in the number of contacts at the upper and lower edge while as we will see below in the case of QSH edge modes this is not the case.

4. Generalization to $N$ terminals

4.1. N-terminal QH bar

4.1.1. Ideal case. The ideal case is represented in figure 5(a). The current–voltage relations can be derived from the current–voltage equation:

$$I_1 = -\frac{e^2M}{h}$$

| $I_1$ | $V_1$ |
|-------|-------|
| 0     | $V_{N-1}$ |
| 0     | $V_{N-1}$ |
| 0     | $V_{N-1}$ |
| 0     | $V_{N-1}$ |
| 0     | $V_{N-1}$ |
| 0     | $V_{N-1}$ |

Substituting $I_2$, $I_3$, $I_{k-1}$ and $I_{k+1},..N = 0$ and choosing reference potential $V_N = 0$, we derive $V_1 = V_2 = ... = V_{k-1}$ and $V_{k} = ... = V_{N-1} = V_{N} = 0$. So, the Hall resistance $R_H = R_{k,k+1} = \frac{h}{2e^2M}$, with $1 \leq i < k \leq N$, then longitudinal resistance $R_L = R_{k,k+1}$, with $1 \leq i < k \leq N$, and finally two-terminal resistance $R_{2T} = R_{k,k+1} = \frac{h}{2e^2M}$ with $1 \leq i,j < N$.  

Here, $D_i$ denotes disorder in current contacts while $D_0(D_1)$ represent disorder in contacts at the upper (lower) edge. When disorder in contacts at the upper and lower edge are unequal, one sees finite charge Hall conductance, and thus the pure QSH effect vanishes. This is unlike what happens in this case for QH edge modes. Not only is QH conductance resilient to disorder and inelastic scattering, it also retains its quantization and longitudinal resistance, and the voltage drop across the sample remains zero. In the case of QH edge modes the inelastic scattering has no effect on the Hall quantization in the presence of all disordered contacts, regardless of whether their strengths are equal or not; however, in the case of QSH edge modes inelastic scattering destroys the pure QSH effect in the presence of unequal disorder. Only when contact disorder at various probes is the same does the pure QSH effect reappear.

In figure 4(b) we plot the longitudinal, Hall and two-terminal resistances for the QSH case with spin-flip and in

8
contacts is larger than the inelastic scattering length. On the occasion of an inelastic scattering event happening, the edge states originating from different contacts with different energies are equilibrated to a common potential. In figure 5(b), one can see that electrons coming from contact 1 and $\mathcal{N}$ are equilibrated to potential $V_{1}^\prime$. If as before contacts 1 and $k$ are chosen to be the current contacts then no current flows into the other voltage probe contacts. Let’s say a current $\frac{e}{h}T_1V_{1}^\prime$ enters contact 2 while a current $\frac{e}{h}T_2V_{2}$ leaves contact 2. Since contact 2 is a voltage probe the net current has to be zero, implying $V_{2} = V_{1}^\prime$. The same thing happens at contact 3 and along the lower edge, where states are equilibrated to $V_{k}^\prime$.

Figure 4. The case without spin-flip scattering. (a) The case of inelastic scattering without spin-flip. Spin-up edge modes are equilibrated to the primed potentials while spin-down edge modes are equilibrated to the double-primed potentials. (b) $R_H, R_L$ and $R_{2T}$ versus disorder with inelastic and spin-flip scattering included. (c) $R_H, R_L$ and $R_{2T}$ versus disorder with inelastic but without any spin-flip scattering. Curiously, resistances are more susceptible to disorder in this case.

Figure 5. $N$-terminal QH bar showing QH edge modes. (a) QH—ideal case. (b) QH—all disordered contacts with inelastic scattering.
Now we write the current–voltage relations in continuous fashion, eschewing our earlier method of writing it in matrix form to avoid clutter as there are not only the $N$ potentials $V_1 - V_N$, we also have the equilibrated potentials $V'_1 - V'_N$.

\[
\begin{align*}
I_1 &= T_1(V_1 - V'_N), \\
I_2 &= T_2(V_2 - V'_1), \\
I_3 &= T_3(V_3 - V'_2), \\
&\vdots \\
I_{k-1} &= I_{k-1}(V_{k-1} - V'_{k-2}), \\
I_k &= T_k(V_k - V'_{k-1}), \\
I_{k+1} &= T_{k+1}(V_{k+1} - V'_{k}), \\
&\vdots \\
I_N &= T_N(V_N - V'_{N-1}).
\end{align*}
\]

By assuming the net current into the voltage probe contacts $2, 3, \ldots, k-1, k+1, \ldots, N$ to be zero, we get the following relations between the contact potentials: $V_2 = V'_3 = V'_2 \ldots, V_{k-1} = V'_{k-2},$ and $V_{k+1} = V'_k, \ldots, V_N = V'_{N-1}$. Furthermore, due to the equilibration, the net current just out of contact 2 is the same as the equilibrated potential due to the inelastic scattering between contacts 2 and 3.

Thus, \(\frac{e^2}{h}(T_2V_2 + R_2V'_2) = \frac{e^2}{h}MV'_2\), or $V_2 = V'_2$, as $T_2 + R_2 = M$, the total number of edge modes in the system. Thus, all the upper edges are equilibrated to same potential $V'_1 = V'_2 = V'_3 = V'_4 = \ldots = V'_{k-1} = V'_1$.

Similarly, for the equilibrated potentials at the lower edge we get $V'_k = V'_{k+1} = V'_1 = \ldots = V_N = 0$, as $V_N = 0$. So, the Hall resistance $R_H = R_{ik,ij} = \frac{h}{e^2M}$ with $1 < i < k < j < N$, longitudinal resistance $R_L = R_{ik,ij} = 0$ with $1 < i, j < k$ or $k < i, j < N$, and two-terminal resistance $R_{TT} = \frac{hM - R_{0}}{M^2}$. Thus, sample geometry has no role in this case.

4.2. N-terminal QSH bar

4.2.1. Ideal case. The ideal case is represented in figure 6(a). The current–voltage relations can be derived from the current–voltage equation:

\[
\begin{pmatrix}
I_1 \\
I_2 \\
I_{k-1} \\
I_k \\
I_{k+1} \\
I_{k+2} \\
I_N
\end{pmatrix}
=
\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_{k-1} \\
V_k \\
V_{k+1} \\
V_{k+2} \\
V_N
\end{pmatrix}
\]

Substituting $I_1, I_2, I_{k-1}, I_k, I_{k+1}, I_N = 0$ and choosing reference potential $V_N = 0$, we derive $V_i = (i - 1)V_2 - (i - 2)V_h$, where $1 \leq i \leq k$ with $V_2 = \frac{2k - N - 2}{k - 1}V_h$. Similarly, $V_i = -(N - i)V_h$, where $k \leq i \leq N$.

So, the Hall resistance

\[
R_H = R_{ik,ij} = \frac{h}{e^2M} \frac{(i - 1)(2k - N - 2)}{N}, \text{ with } 1 \leq i \leq k.
\]

and two-terminal resistance $R_{TT} = R_{ik,ij} = \frac{h}{e^2M} \frac{k - 1}{N}$, with $1 \leq i, j \leq N$. Surprisingly, a finite charge Hall current flows even when there is no disorder. It arises only due to asymmetry between the number of contacts at the upper and lower edge. This number asymmetry has no role as far as QH edge modes are concerned.

4.2.2. All disordered probes with inelastic scattering and spin–flip scattering. The completely disordered case with inelastic scattering is represented in figure 6(b). Here, both the voltage as well as the current probe contacts are disordered, i.e. $D_i = D_j, i,j = 1 \ldots, N$. To simplify the calculation, all the contacts are considered equally disordered. The disorder strengths $D$ can be written in terms of the number of transmitted edge modes $T$ and the total number of edge modes $M$. So, $T_i = T = (1 - D_i)M = (1 - D)M$ and $R_i = R = D M = DM$.

In figure 6(b), we consider the length between disordered contacts to be larger than the inelastic scattering length. On the occasion of an inelastic scattering event happening, the edge states originating from different contacts with different energies are equilibrated to a common potential. In figure 6(b), one can see that electrons coming from contact 1 and $N$ are equilibrated to potential $V'_1$. If as before contacts 1 and $k$ are chosen to be the current contacts then no current flows into the other voltage probe contacts. Let’s say a current $\frac{e^2}{h}(T_1V'_1 + T_2V'_2)$ enters contact 2. The first part $\frac{e^2}{h}T_1V'_1$ is the spin-up component while the second part $\frac{e^2}{h}T_2V'_2$ is the spin-down component, moving in exactly the opposite direction. Similarly, the current $\frac{e^2}{h}2T_1V'_2$ leaves contact 2, and since contact 2 is a voltage probe, the net current has to be zero, implying $V'_2 = (V'_1 + V'_2)/2$. The same thing happens at contact 3 and along the lower edge.

Now we write the current–voltage relations in continuous fashion, and as before eschewing our earlier method of writing it in matrix form to avoid clutter as there are not only the $N$ potentials $V_1 - V_N$, we also have the equilibrated potentials $V'_1 - V'_N$.

\[
I_1 = 2TV_1 - T(V'_1 + V'_2), \\
I_2 = 2TV_2 - T(V'_1 + V'_2), \\
I_3 = 2TV_3 - T(V'_1 + V'_2), \\
&\vdots \\
I_{k-1} &= 2TV_{k-1} - T(V'_{k-1} + V'_{k-2}), \\
I_k &= 2TV_k - T(V'_{k-1} + V'_k), \\
I_{k+1} &= 2TV_{k+1} - T(V'_k + V'_{k+1}), \\
&\vdots \\
I_N &= 2TV_N - T(V'_{N-1} + V'_N).
\]
Furthermore, due to the equilibration the net current just out of contact 2 is the sum
\[ \left( TV_2 + RV_1 + RV_3 + TV_5 \right) \]
and this should be equal to \( \frac{e^2}{h} 2M V_2 \), which is the equilibrated potential due to inelastic scattering between contacts 2 and 3. Similarly, the net current just out of the \( k \)th contact is
\[ \left( TV_k + RV_{k-1} + RV_{k+1} + TV_{k+2} \right) \]
and this should be equal to \( \frac{e^2}{h} 2M V_k \), which is the equilibrated potential due to inelastic scattering between contacts \( k \) and \( k + 1 \). By assuming the net current into the voltage probe contacts 2, 3, ..., \( k - 1 \), \( k + 1 \), ..., \( N \) to be zero, we get the following relations between the contact potentials:
\[ V_i' = (i - 1)V_i - (i - 2)V_i' \text{ with } 2 \leq i \leq (k - 1) \text{ and } V_i' = -(2N - 2i - 1)V_{i+N} \text{ with } k \leq i \leq N. \]

So, the Hall resistance with \( j = N - i + 2 \) is given as
\[ R_{H} = R_{h,i,j} = \frac{h}{e^2 M} \frac{2(i - 1)(2k - N - 2)}{(1 + D)N} \text{ with } 1 < i < k < j \leq N \]
and if we consider \( k = N/2 + 1 \), i.e. a symmetric sample (with an equal number of contacts at the upper and lower edge) then \( R_{H} = 0 \). So sample geometry (number asymmetry between contacts at the upper and lower edge) has a direct bearing on whether one sees a pure spin Hall effect or whether it is contaminated by a charge current. Furthermore, in the symmetric case there is no charge Hall current—it is seen only when all the probes are equally disordered, i.e. when the QSH effect is restored. If on the other hand the probes are not equally disordered, as seen in the six-terminal case, \( R_{H} \neq 0 \) even in the presence of inelastic scattering. In the case of QH edge modes, inelastic scattering restores Hall quantization in the presence of all disordered contacts regardless of whether their strengths are equal or not; however, in the case of QSH, inelastic scattering also fails to restore the pure QSH effect in the presence of unequal disorder.

Next, for longitudinal resistance
\[ R_{L} = R_{l,i,j} = \frac{h}{e^2 M} \frac{(j - i)2(1 - k + N)}{(1 + D)N} \text{ with } 2 < i, j < k \]
\[ = \frac{h}{e^2 M} \frac{2(i - j)(k - 1)}{(1 + D)N}, \text{ with } k + 1 \leq i, j \leq N, \]
and finally the two-terminal resistance
\[ R_{TT} = R_{t,i,k} = -\frac{h}{e^2 M} \frac{2(k - 1)(k - N - 1 - D(1 + k + 2N - k(N + 2)))}{(1 - D)(1 + D)N}. \]

The \( N \)-terminal results reduce to the six-terminal case of an equal number of probes at the upper and lower edge (symmetric case) in all cases, confirming the results obtained before. Furthermore, they shed light on the asymmetric case wherein probes on the upper and lower edge are unequal. Asymmetry has no role as far as QH edge modes are concerned, but in the case of QSH edge modes they have a non-trivial role, even destroying the pure QSH effect, regardless of whether there is disorder or not.

5. Conclusions

The conclusions of the work reported here shed light on QSH edge modes, shattering the myth that QSH edge modes are better as far as their ability to withstand the deleterious effects of disorder and inelastic scattering are concerned. In fact, we show them to be worse than QH edge modes. Furthermore, even sample geometry can have a negative bearing on the QSH effect while leaving the QH effect intact. Given that QSH insulators are considered to be the next big breakthrough after graphene, their complete failure to withstand disorder and sample asymmetry, not to mention inelastic scattering, means that they will have limited use in spintronics and applications.
Furthermore, since topological insulator edge modes are considered to be useful in a host of other areas ranging from topological quantum computation (braiding of majorana fermions) to searching for novel spin-dependent effects, this paper casts a long shadow of doubt with regard to their utility. Caution is the watchword and any benefits from QSH edge modes will have to be counter-balanced by the implications of our work.

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