ISOTOPIES VIS-À-VIS LEVEL-PRESERVING EMBEDDINGS

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Abstract. Various standard texts on differential topology maintain that the level-preserving map defined by the track of an isotopy of embeddings is itself an embedding. This note describes a simple counterexample to this assertion.

1. Introduction

Let $Q, M$ be differential manifolds. An embedding of $Q$ in $M$ is a smooth map $\varphi: Q \to M$ such that $\varphi(Q) \subset M$ is a submanifold and $\varphi$ a diffeomorphism onto its image. Equivalently, an embedding can be characterised as an immersion (i.e. a smooth map of rank equal to $\dim Q$) that maps $Q$ homeomorphically onto its image [2, Theorem 1.3.1].

An isotopy of embeddings is a smooth map $F: Q \times [0, 1] \to M$ with the property that for each $t \in [0, 1]$ the map $Q \to M, x \mapsto F(x, t)$ is an embedding. The track of this isotopy $F$ is the level-preserving map $G: Q \times [0, 1] \to M \times [0, 1], (x, t) \mapsto (F(x, t), t)$.

It is not difficult to see that any level-preserving embedding $G: Q \times [0, 1] \to M \times [0, 1]$, that is, any embedding $G$ satisfying $G(Q \times \{t\}) \subset M \times \{t\}$, is the track of an isotopy [3, Lemma II.4.2]. At least two of the standard texts on differential topology maintain that, conversely, the track of an isotopy is always an embedding [2, p. 178], [3, p. 34]. This claim persists in the literature, at times presented as ‘easy to prove’; see [3, p. 164] or [4, p. 207], for instance.

The aim of this brief note is to exhibit a simple counterexample to this assertion; notice that in any such counterexample the manifold $Q$ needs to be non-compact. Thus, we are going to show that the level-preserving map defined by the track of an isotopy of embeddings is not, in general, an embedding.

Remark. This observation is not new, see [1, Aufgabe 9.14], but apparently not nearly as well known as it ought to be.

2. The example

In the example we take $Q = \mathbb{R}^+$, the positive real numbers, and $M = \mathbb{R}^2$.  

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2.1. A family of bump functions. We first construct, in the standard way, a family of bump functions \( h_t : \mathbb{R} \to [0,1] \), where \( 0 < t \leq 1 \), with support in the interval \( [t/2, 3t/2] \). Start with the smooth monotonically increasing function \( f(s) \) defined by
\[
f(s) := \begin{cases} 
0 & \text{for } s \leq 0, \\
e^{-1/s} & \text{for } s > 0.
\end{cases}
\]
Then the function \( g_t : \mathbb{R} \to [0,1] \) defined by
\[
g_t(s) := \frac{f(s)}{f(s) + f(t/4 - s)}
\]
interpolates smoothly and monotonically between the value 0 for \( s \leq 0 \) and the value 1 for \( s \geq t/4 \). Finally, set
\[
h_t(s) := g_t(s - t/2) \cdot g_t(3t/2 - s).
\]
This bump function is identically 0 outside the interval \( [t/2, 3t/2] \), identically 1 on the interval \( [3t/4, 5t/4] \), and it interpolates monotonically in between. In particular, we have \( h_t(t) = 1 \).

The key to constructing the desired isotopy is the observation that the bump can be made to ‘disappear’ at \( t = 0 \), provided we restrict the domain of definition to the positive real numbers. In other words, if we take \( h_0 \equiv 0 \), then the function \( (x,t) \mapsto h_t(x) \) will be smooth on \( \mathbb{R}^+ \times [0,1] \).

2.2. The isotopy. We now construct an isotopy \( F : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^2 \) of embeddings \( \mathbb{R}^+ \to \mathbb{R}^2 \). Set
\[
F(x,t) := \begin{cases} 
\{x, h_t(x)\} & \text{for } x \in [0,2] \text{ and } t \in [0,1], \\
(x,0) & \text{for } x \in [0,2] \text{ and } t = 0, \\
\text{independent of } t \text{, as shown in Figure 1} & \text{for } x \geq 2.
\end{cases}
\]

Here it is understood that the choice for \( x \geq 2 \) is made in such a way that

\[\text{Figure 1. The image of the embedding } x \mapsto F(x,t), 0 < t \leq 1.\]
$x \mapsto F(x, t)$ is an embedding. The following three points, all of which are a simple consequence of the definition, establish that $F$ is an isotopy.

(i) The map $F$ is smooth at any point $(x, t)$ with $t > 0$ or $x > 3/2$.

(ii) The map $F$ is also smooth at any given point $(x, 0)$ with $0 < x \leq 3/2$, since for any sequence $(x_\nu, t_\nu)$ converging to $(x, 0)$ we have $F(x_\nu, t_\nu) = 0$ for $\nu$ sufficiently large.

(iii) The map $x \mapsto F(x, t)$ is an embedding $\mathbb{R}^+ \rightarrow \mathbb{R}^2$ for any $t \in [0, 1]$.

However, the track $G: (x, t) \mapsto (F(x, t), t)$ of this isotopy is not an embedding, since it is not a homeomorphism of $\mathbb{R}^+ \times [0, 1]$ onto the image of $G$. To see this, choose the unique $x_0 > 2$ such that $F(x_0, t) = (0, 1) \in \mathbb{R}^2$ for any $t \in [0, 1]$. Then $G(x_0, 0) = (0, 1, 0)$. Now let $(x_\nu)_{\nu \in \mathbb{N}}$ be a sequence of real numbers in $[0, 1]$ converging to $0$. Since, by construction, we have $h_\nu(t) = 1$ for $0 < t \leq 1$,

\[
G(x_\nu, 1, x_\nu) = (F(x_\nu, x_\nu), x_\nu) = (x_\nu, 1, x_\nu),
\]

which converges to $(0, 1, 0) = G(x_0, 0)$. But the sequence $(x_\nu, x_\nu)$ in $\mathbb{R}^+ \times [0, 1]$ does not converge to $(x_0, 0)$.

**Remark.** (1) The crucial feature of this example is that the embeddings of $\mathbb{R}^+$ in $\mathbb{R}^2$ given by $x \mapsto F(x, t)$ are not proper for $x$ near $0$. It is this improperness which allows one to make a bump ‘disappear’ smoothly in finite time. The properness or not for $x$ near $\infty$, on the other hand, is irrelevant.

(2) Another way to describe the essential characteristic of this example is to observe that the map

\[
t \mapsto \{x \mapsto F(x, t)\}
\]

from the interval $[0, 1]$ into the space of embeddings $\mathbb{R}^+ \rightarrow \mathbb{R}^2$ is not continuous at $t = 0$ when the space of smooth maps $\mathbb{R}^+ \rightarrow \mathbb{R}^2$ is equipped with the strong topology in the sense of [2] Chapter 2.1.

The recent text by Wall [6], largely based on notes from the 1960s, avoids the pitfall described in this note by working directly with a notion of *diffeotopy of $Q$ in $M$*, defined as a level-preserving embedding $Q \times [0, 1] \rightarrow M \times [0, 1]$. Remark (2) serves to indicate that for non-compact manifolds this is in some sense a more appropriate definition.

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