IN Variant HIlbert SCHEME RESOLUTION OF POPOV’S
SL(2)-VARIETIES II: THE NON-TORIC CASE

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Abstract. This article is a continuation of [Kub18], which proves that if a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety $E$ is toric, then it has an equivariant resolution of singularities given by an invariant Hilbert scheme $\mathcal{H}$. In this article, we consider the case where $E$ is non-toric and show that the Hilbert–Chow morphism $\gamma : \mathcal{H} \to E$ is a resolution of singularities and that $\mathcal{H}$ is isomorphic to the minimal resolution of a weighted blow-up of $E$.

Introduction

Let $E_{l,m}$ be a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety of height $l$ and degree $m$, and write $l$ as an irreducible fraction $l = p/q$. Batyrev and Haddad [BH08] showed that $E_{l,m}$ has a description as an affine categorical quotient of a hypersurface $H_{q-p}$ in $\mathbb{C}^5$ modulo an action of $\mathbb{C}^* \times \mu_m$. Also, they proved that an $SL(2)$-variety $E_{l,m}$ admits an action of $\mathbb{C}^*$ and becomes a spherical $SL(2) \times \mathbb{C}^*$-variety with respect to the Borel subgroup $B \times \mathbb{C}^*$. Further, it is shown that there is an equivariant flip diagram

$$
\begin{array}{ccc}
E_{l,m}^- & \xrightarrow{E_{l,m}^+} & E_{l,m}^+ \\
\downarrow & & \downarrow \\
E_{l,m} & & E_{l,m},
\end{array}
$$

where $E_{l,m}^-$ and $E_{l,m}^+$ are different GIT quotients of $H_{q-p}$ corresponding to some non-trivial characters, and that the varieties $E_{l,m}$, $E_{l,m}^-$, and $E_{l,m}^+$ are dominated by the weighted blow-up $E_{l,m}^\prime = Bl_\omega(E_{l,m})$ of $E_{l,m}$ with a weight $\omega$ defined by the above-mentioned $\mathbb{C}^*$-action on $E_{l,m}$. The weight $\omega$ is trivial if and only if the $SL(2)$-variety $E_{l,m}$ is toric, namely if $m = a(q-p)$ holds for some $a > 0$ (see [Gaı̇08, BH08]).

In our previous article [Kub18], we used the GIT quotient description of $E_{l,m}$ due to Batyrev and Haddad to construct the invariant Hilbert scheme $\mathcal{H} = \text{Hilb}_h \mathbb{C}^* \times \mu_m(H_{q-p})$, where $h$ is the Hilbert function of the general fibers of the quotient morphism $H_{q-p} \to H_{q-p}/(\mathbb{C}^* \times \mu_m)$, and considered the corresponding Hilbert–Chow morphism

$$
\gamma : \mathcal{H} \to H_{q-p}/(\mathbb{C}^* \times \mu_m) \cong E_{l,m},
$$

which is an isomorphism over the dense open orbit $\mathcal{U} \subset E_{l,m}$. We treated the case where $E_{l,m}$ is toric and showed that the main component $\mathcal{H}^{\text{main}} = \gamma^{-1}(\mathcal{U})$
is isomorphic to the blow-up $E'_{l,m}$ and that $\mathcal{H}$ coincides with $\mathcal{H}^{\text{main}}$ ([Kub18, Corollary 5.2 and Theorem 6.1]).

The goal of this article is to prove the following result.

**Main Theorem** (Theorem 4.6 and Corollary 9.5). If $E_{l,m}$ is non-toric, then:

(i) the main component $\mathcal{H}^{\text{main}}$ is isomorphic to the minimal resolution $\tilde{E}'_{l,m}$ of the weighted blow-up $E'_{l,m}$;

(ii) the invariant Hilbert scheme $\mathcal{H}$ coincides with the main component $\mathcal{H}^{\text{main}}$.

The problem of deciding the main component $\mathcal{H}^{\text{main}}$ and showing the smoothness of $\mathcal{H} = \mathcal{H}^{\text{main}}$ is easier in the toric case than in the non-toric case, since we have the relation $m = a(q - p)$. The non-toric case requires more intricate arguments, which is mainly because there is nothing to relate the height $l = p/q$ and the degree $m$ directly, but the essential idea for the proof is the same as in the toric case. In the following, we outline our approach for the non-toric case. First, as in the toric case, we show that the restriction $\gamma|_{\mathcal{H}^{\text{main}}}$ factors equivariantly through the weighted blow-up $E'_{l,m}$:

$$
\begin{array}{c}
\mathcal{H}^{\text{main}} \\
\downarrow \psi \\
E'_{l,m}
\end{array}
\quad
\begin{array}{c}
E_{l,m} \\
\downarrow \gamma|_{\mathcal{H}^{\text{main}}} \\
\end{array}
$$

If $E_{l,m}$ is toric then $\psi$ is an isomorphism, while if $E_{l,m}$ is non-toric then we see by an easy observation that $\psi$ is not an isomorphism. On the other hand, according to [BH08], the weighted blow-up $E'_{l,m}$ contains a family of cyclic quotient singularities $\mathbb{C}^2/\mu_b$, and therefore the natural candidate for $\mathcal{H}^{\text{main}}$ is the minimal resolution $\tilde{E}'_{l,m}$ of these quotient singularities, which is known to be described by the Hirzebruch–Jung continued fraction. So what we do next is to construct an equivariant morphism $\mathcal{H}^{\text{main}} \to \tilde{E}'_{l,m}$: we first realize $\tilde{E}'_{l,m}$ as a closed subscheme of a projective space over $E_{l,m}$ and then use Becker’s idea [Bec11 §4] of embedding an invariant Hilbert scheme to products of Grassmannians to construct a morphism $\Psi$ from $\mathcal{H}$ to the projective space such that $\Psi(\mathcal{H}^{\text{main}}) \cong \tilde{E}'_{l,m}$. Finally, we show that $\Psi|_{\mathcal{H}^{\text{main}}} : \mathcal{H}^{\text{main}} \to \tilde{E}'_{l,m}$ is an isomorphism. By the Zariski’s Main Theorem, it suffices to show that $\Psi|_{\mathcal{H}^{\text{main}}}$ is injective, and concerning that it is equivariant we are left to show the injectivity orbit-wise: we take a “representative” point from each orbit in $E'_{l,m}$ (e.g. we take a Borel-fixed point if the orbit is closed) and show that its fiber consists of one point, say $[Z] \in \mathcal{H}^{\text{main}}$. In showing the injectivity, the differences from the toric case are that the number of orbits in $E'_{l,m}$ depends
on the pair $(l, m)$ and that the degrees of generators of the ideal $I_Z$ of $Z$ can not be expressed in terms of $p, q,$ or $m$. What becomes a key here is the spherical geometry of $E_{l,m}^\prime$, which enables us to give a uniform approach independent of the pair $(l, m)$. To be more precise, the number of orbits can be read off from the colored fan of $E_{l,m}^\prime$. Also, the degrees of the generators of $I_Z$ are described by using ray generators of maximal cones contained in the fan of $E_{l,m}^\prime$ and the relations among them come from recursive relations arising from the Hirzebruch–Jung continued fraction. This is why the calculation of generators of the ideal $I_Z$ involves intricate combinatorial arguments in contrast to the toric case.

This article is organized as follows: we first summarize some general properties of invariant Hilbert schemes in §1 and of spherical varieties in §2. Afterwards, we review Popov’s classification of 3-dimensional affine normal quasihomogeneous $SL(2)$-varieties (Theorem 3.1) and the GIT quotient description due to Batyrev and Haddad (Theorems 3.3 and 3.4). In §4 we first review some facts from [Kub18], and then describe the minimal resolution $\overline{E}_{l,m}^\prime$ in terms of its colored fan. In §5 we realize $\overline{E}_{l,m}^\prime$ as a closed subscheme of a projective space over $E_{l,m}$ by using the spherical geometry of $E_{l,m}^\prime$ (Proposition 5.7). §6 is a preparation for later sections and is mainly devoted to the proof of Theorem 6.1 which requires some complicated combinatorial arguments. In §7 we construct the morphism $\Psi$ by using Theorem 6.1. In §8 we calculate ideals (Theorems 8.2 and 8.3), which will be shown to correspond to “representative” points in $\overline{E}_{l,m}^\prime$ via the isomorphism $\mathcal{H}^{main} \cong E_{l,m}^\prime$. In the last section, we give the proof of Main Theorem.

1. Generalities on the invariant Hilbert scheme

We review some generalities on the invariant Hilbert scheme introduced by Alexeev and Brion in [AB05]. For more details refer to Brion’s survey [Bri13].

Let $G$ be a reductive algebraic group. For any $G$-module $V$, we have its isotypical decomposition

$$V \cong \bigoplus_{M \in \text{Irr}(G)} \text{Hom}^G(M, V) \otimes M,$$

where $\text{Irr}(G)$ stands for the set of isomorphism classes of irreducible representations of $G$. We call the dimension of $\text{Hom}^G(M, V)$ the multiplicity of $M$ in $V$. If the multiplicity is finite for every $M \in \text{Irr}(G)$, we can define a function

$$h_V : \text{Irr}(G) \to \mathbb{Z}_{\geq 0}, \quad M \mapsto h_V(M) := \dim \text{Hom}^G(M, V),$$

which is called the Hilbert function of $V$. Let $X$ be an affine $G$-scheme of finite type, and $h$ a Hilbert function. The invariant Hilbert scheme $\text{Hilb}^G_h(X)$ associated to
the triple \((G, X, h)\) is a moduli space that parametrizes \(G\)-stable closed subschemes of \(X\) whose coordinate rings have Hilbert function \(h\). Namely, the set-theoretical description of \(\text{Hilb}^G_h(X)\) is given as follows:

\[
\text{Hilb}^G_h(X) = \left\{ Z \subseteq X : \begin{array}{c}
Z \text{ is a closed } G\text{-subscheme of } X; \\
\mathbb{C}[Z] \cong \bigoplus_{M \in \text{Irr}(G)} M^{\oplus h(M)} \text{ as } G\text{-modules}
\end{array} \right\}.
\]

We denote by \(T[Z] \text{Hilb}^G_h(X)\) the Zariski tangent space to the invariant Hilbert scheme \(\text{Hilb}^G_h(X)\) at a closed point \([Z]\). Let

\[
\pi : X \longrightarrow X//G := \text{Spec}(\mathbb{C}[X]^G)
\]

be the quotient morphism, and suppose that \(X\) is irreducible. Then by the generic flatness theorem \(\pi\) is flat over a non-empty open subset \(Y_0\) of \(X//G\). The Hilbert function of the flat locus \(\pi^{-1}(Y_0) \rightarrow Y_0\) is called the *Hilbert function of the general fibers of \(\pi\)* and denoted by \(h_X\). The associated Hilbert–Chow morphism

\[
\gamma : \text{Hilb}^G_{h_X}(X) \rightarrow X//G, \quad [Z] \mapsto Z//G
\]

is an isomorphism over \(Y_0\), and its restriction to the *main component* \(\mathcal{H}^\text{main} := \gamma^{-1}(Y_0)\) is projective and birational ([Bud10] Theorem I.1.1, [Bri13] Proposition 3.15), see also [Bec11, Ter14, LT15]).

To conclude this short section, we summarize Becker’s idea [Bec11, §4.2] of embedding an invariant Hilbert scheme into products of Grassmannians. Suppose that there is an action on \(X\) by another connected reductive algebraic group \(G'\). For any irreducible representation \(M \in \text{Irr}(G)\), there is a finite-dimensional \(G'\)-module \(F_M\) that generates \(\text{Hom}^G(M, \mathbb{C}[X])\) as \(\mathbb{C}[X]^G\)-modules. For \([Z] \in \text{Hilb}^G_{h_X}(X)\), we let

\[
f_{M,Z} : F_M \longrightarrow \text{Hom}^G(M, \mathbb{C}[Z])
\]

be the composition of the inclusion \(F_M \hookrightarrow \text{Hom}^G(M, \mathbb{C}[X])\) and the natural surjection \(\text{Hom}^G(M, \mathbb{C}[X]) \rightarrow \text{Hom}^G(M, \mathbb{C}[Z])\). Then, the quotient vector space \(F_M/\text{Ker} f_{M,Z}\) defines a point in the Grassmannian \(\text{Gr}(h_X(M), F_M^\vee)\). In this way, we obtain a \(G'\)-equivariant morphism

\[
\eta_M : \text{Hilb}^G_{h_X}(X) \rightarrow \text{Gr}(h_X(M), F_M^\vee), \quad [Z] \mapsto F_M/\text{Ker} f_{M,Z}.
\]

Furthermore, there is a finite subset \(\mathcal{M} \subset \text{Irr}(G)\) such that the morphism

\[
\gamma \times \prod_{M \in \mathcal{M}} \eta_M : \text{Hilb}^G_{h_X}(X) \longrightarrow X//G \times \prod_{M \in \mathcal{M}} \text{Gr}(h_X(M), F_M^\vee)
\]

is a closed immersion.
2. Generalities on spherical varieties

Let $G$ be a connected reductive algebraic group, and $H$ an algebraic subgroup of $G$. A normal $G$-variety is called spherical if it contains a dense orbit under a Borel subgroup of $G$. By a spherical embedding, we mean a normal $G$-variety $X$ together with an equivariant open embedding of a homogeneous spherical variety $G/H \hookrightarrow X$.

Let $X$ be a spherical embedding of $G/H$ with respect to a Borel subgroup $B$. We denote by $X(B)$ the group of characters of $B$, and by $C(G/H)(B)$ the set of rational $B$-eigenfunctions:

$$C(G/H)(B) = \{ f \in C(G/H)^* : \exists \chi_f \in \hat{X} \forall g \in B \cdot g \cdot f = \chi_f(g) f \}.$$

Consider a homomorphism $C(G/H)(B) \to \hat{X}$ defined by $f \mapsto \chi_f$, and let $\Gamma \subset X(B)$ be its image. Then, $\Gamma$ is a finitely generated free abelian group, and its rank is called the rank of $G/H$. Since $G/H$ contains a dense $B$-orbit, the kernel of the above homomorphism consists of constant functions. Therefore, we get the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow C(G/H)(B) \longrightarrow \Gamma \longrightarrow 0.$$

We see that any valuation $\nu : C(G/H)^* \to \mathbb{Q}$ of $G/H$ defines a homomorphism $C(G/H)(B) \to \mathbb{Q}$, $f \mapsto \nu(f)$, which factors through $\Gamma$. Hence it induces an element $\rho_\nu \in \mathbb{Q} := \text{Hom}(\Gamma, \mathbb{Q})$,

namely $\rho_\nu(\chi_f) = \nu(f)$. A valuation $\nu$ is called $G$-invariant if $\nu(g \cdot f) = \nu(f)$ holds for any $g \in G$, and we denote by $\mathcal{V}$ the set of $G$-invariant valuations.

**Proposition 2.1** ([LV83 7.4 Proposition]). The map $\mathcal{V} \to \mathbb{Q}$, $\nu \mapsto \rho_\nu$ is injective.

Let us denote by $\mathcal{D}(X)$ the set of $B$-stable prime divisors on $X$. We simply write $\mathcal{D}$ for $\mathcal{D}(G/H)$ and call an element of $\mathcal{D}$ a color. If $D \in \mathcal{D}(X)$ non-trivially meets the open orbit $G/H$, then we have $D \cap G/H \in \mathcal{D}$. Otherwise, $D$ is an irreducible component of the complement $X \setminus (G/H)$ and hence is $G$-stable. Therefore, each $G$-orbit $Y$ in $X$ determines two sets

$$\mathcal{B}_Y(X) := \{ \nu_D \in \mathcal{V} : D \in \mathcal{B}_Y(X) \text{ is } G\text{-stable} \}$$

and

$$\mathcal{F}_Y(X) := \{ D \cap G/H \in \mathcal{D} : D \in \mathcal{F}_Y(X) \text{ is not } G\text{-stable} \},$$

where

$$\mathcal{B}_Y(X) := \{ D \in \mathcal{D}(X) : Y \subset D \}.$$
Definition 2.2. A spherical embedding $X$ is called **simple** if it contains a unique closed $G$-orbit.

Remark 2.2.1. Any spherical embedding is covered by finitely many simple open subembeddings.

Remark 2.2.2. Let $X$ be a simple spherical embedding with a closed orbit $Y$, and set

$$(X)_0 := X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D$$

and

$$(X)_1 := G/H \setminus \bigcup_{D \in \mathcal{D}_Y(X)} D.$$ 

Then $(X)_0$ is a $B$-stable affine open subset, and we have

$$\mathbb{C}[(X)_0] = \{ f \in \mathbb{C}[(X)_1] : v(f) \geq 0 \text{ for all } v \in \mathcal{B}_Y(X) \}.$$ 

Also, we have $X = G(X)_0$. (see [Kno91] Theorems 2.1 and 2.3).

Now with the preceding notation, we see that there is a natural map

$$\varrho : \mathcal{D} \to Q, \quad D \mapsto \varrho(D) := \rho_{v_D}.$$ 

Definition 2.3. A **colored cone** is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subset Q$ and $\mathcal{F} \subset \mathcal{D}$ that satisfies the following properties:

- $\mathcal{C}$ is a cone generated by $\varrho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$;
- $\mathcal{C}^\circ \cap \mathcal{V} \neq \Phi$, where $\mathcal{C}^\circ$ stands for the relative interior of $\mathcal{C}$.

A colored cone $(\mathcal{C}, \mathcal{F})$ is called **strictly convex** if $\mathcal{C}$ is strictly convex and $0 \notin \varrho(\mathcal{F})$.

Let $Y$ be a $G$-orbit in a spherical embedding $X$, and $\mathcal{C}_Y(X) \subset Q$ the cone generated by $\varrho(\mathcal{B}_Y(X))$ and $\mathcal{B}_Y(X)$. Then, the pair $(\mathcal{C}_Y(X), \mathcal{B}_Y(X))$ is a strictly convex colored cone.

Theorem 2.4 ([LV83, 8.10 Proposition]). The map $X \mapsto (\mathcal{C}_Y(X), \mathcal{B}_Y(X))$ gives a bijective correspondence between the isomorphism classes of simple spherical embeddings $X$ with a closed orbit $Y$ and strictly convex colored cones.

We say that a pair $(\mathcal{C}_0, \mathcal{F}_0)$ is a face of a colored cone $(\mathcal{C}, \mathcal{F})$ if $\mathcal{C}_0$ is a face of $\mathcal{C}$, $\mathcal{C}_0^\circ \cap \mathcal{V} \neq \phi$, and $\mathcal{F}_0 = \mathcal{F} \cap \varrho^{-1}(\mathcal{C}_0)$.

Theorem 2.5 ([Kno91] Lemma 3.2]). Let $X$ be a spherical embedding, and $Y$ a $G$-orbit. Then, the map $Z \mapsto (\mathcal{C}_Z(X), \mathcal{B}_Z(X))$ gives a bijective correspondence between $G$-orbits whose closure contain $Y$ and faces of $(\mathcal{C}_Y(X), \mathcal{B}_Y(X))$. 

Definition 2.6. A colored fan is a non-empty finite set $\mathcal{F}$ of colored cones satisfying the following properties:

- every face of $(\mathcal{C}, \mathcal{F}) \in \mathcal{F}$ belongs to $\mathcal{F}$;
- for every $v \in \mathcal{V}$, there is at most one $(\mathcal{C}, \mathcal{F}) \in \mathcal{F}$ such that $v \in \mathcal{C}^\circ$.

A colored fan $\mathcal{F}$ is called strictly convex if $(0, \phi) \in \mathcal{F}$. This is equivalent to saying that all elements of $\mathcal{F}$ are strictly convex.

For a spherical embedding $X$, we define

$$\mathcal{F}(X) := \{ (\mathcal{C}_Y(X), \mathcal{F}_Y(X)) : Y \subset X \text{ is a } G\text{-orbit} \}.$$ 

Then, $\mathcal{F}(X)$ is a strictly convex colored fan.

Remark 2.6.1 ([Kno91]). We can give an order relation to the set of $G$-orbits by the inclusion of closures. Theorems 2.4 and 2.5 imply that $Y \mapsto (\mathcal{C}_Y(X), \mathcal{F}_Y(X))$ is an order-reversing bijection between the set of $G$-orbits and $\mathcal{F}(X)$. The open orbit corresponds to $(0, \phi)$.

Theorem 2.7 ([Kno91, Theorem 3.3]). The map $X \mapsto \mathcal{F}(X)$ gives a bijective correspondence between the isomorphism classes of spherical embeddings and strictly convex colored fans.

Definition 2.8. A spherical embedding $X$ is called toroidal if $\mathcal{F}_Y(X) = \emptyset$ for any $G$-orbit $Y$. This is equivalent to saying that no $D \in \mathcal{D}$ contains a $G$-orbit in its closure.

Remark 2.8.1 ([BP87, 3.4], see also [Per14, §3.3]). A local structure theorem for toroidal spherical embeddings implies that a toroidal spherical embedding $X$ has singularities of a toric variety with the same cones as those of $X$ and that subdividing its fan for toric varieties gives an equivariant resolution of $X$.

Equivariant birational morphisms between spherical embeddings have an implication in terms of colored fans.

Theorem 2.9 ([Kno91, Theorem 4.1]). Let $X$ and $X'$ be spherical embeddings of $G/H$. Then, the following are equivalent.

(i) An equivariant birational morphism $X \to X'$ exists.
(ii) For any $(\mathcal{C}, \mathcal{F}) \in \mathcal{F}(X)$ there exists $(\mathcal{C}', \mathcal{F}') \in \mathcal{F}(X')$ such that $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{F} \subset \mathcal{F}'$. 
In the rest of this section, we consider Weil divisors on a spherical embedding $X$. According to [Per14], any Weil divisor on $X$ is linearly equivalent to a divisor of the form

$$\delta = \sum_{D \in \mathcal{D}(X)} n_D D.$$ 

**Theorem 2.10 ([Per14 Theorem 3.2.1]).** Keep the above notation. Then, $\delta$ is Cartier if and only if for any $G$-orbit $Y$ there exists $f_Y \in \mathbb{C}(G/H)^{(B)}$ that satisfies $n_D = v_D(f_Y)$ for any $D \in \mathcal{D}_Y(X)$.

**Definition 2.11 ([Per14 Definition 3.2.2]).** Let $X$ be a spherical embedding.

(i) We denote by $\mathcal{C}(X)$ the union of all $\mathcal{C}_Y(X)$, where $Y$ runs over all $G$-orbits.

(ii) A collection $l = (l_Y)$ indexed by $G$-orbits $Y$ is called a piecewise linear function if it satisfies the following conditions:

- for each $G$-orbit $Y$, $l_Y$ is the restriction of an element of $\Gamma$ to $\mathcal{C}_Y(X)$;
- for any $G$-orbits $Y$ and $Z$ with $Z \subset Y$, we have $l_Z|_{\mathcal{C}_Y(X)} = l_Y$.

We denote by $PL(X)$ the abelian group consists of piecewise linear functions.

**Remark 2.11.1 ([Per14 Remark 3.2.3]).** An element $l \in PL(X)$ depends only on its values on maximal cones, namely cones of closed orbits in $X$.

Let $\text{Car}^B(X)$ be the group of $B$-stable Cartier divisors on a spherical embedding $X$. Then, we have a morphism

$$\text{Car}^B(X) \to PL(X), \quad \delta \mapsto l_\delta,$$

where $(l_\delta)_Y = f_Y$ with the notation as in Theorem 2.10. Set

$$\mathcal{D}_0(X) := \bigcup Y \mathcal{D}_Y(X),$$

where $Y$ runs over all $G$-orbits.

**Theorem 2.12 ([Tim11 Theorem 17.18]).** For any $B$-stable Cartier divisor

$$\delta = \sum_{D \in \mathcal{D}_0(X)} v_D(l_\delta) D + \sum_{D \in \mathcal{D}(X) \setminus \mathcal{D}_0(X)} n_D D$$

on $X$, the following properties are equivalent.

(i) The divisor $\delta$ is generated by global sections.

(ii) For any $G$-orbit $Y$, there exists $f_Y \in \mathbb{C}(G/H)^{(B)}$ that satisfies the following conditions:

- $f_Y|_{\mathcal{C}_Y(X)} = l_\delta|_{\mathcal{C}_Y(X)}$;
- $f_Y|_{\mathcal{C}(X) \setminus \mathcal{C}_Y(X)} \leq l_\delta|_{\mathcal{C}(X) \setminus \mathcal{C}_Y(X)}$;
- $v_D(f_Y) \leq n_D$ for any $D \in \mathcal{D}(X) \setminus \mathcal{D}_0(X)$. 

3. Quasihomogeneous \( SL(2) \)-varieties and their spherical geometry

In [Pop73], Popov gives a complete classification of affine normal quasihomogeneous \( SL(2) \)-varieties. Consult also the book of Kraft [Kra84].

**Theorem 3.1** ([Pop73]). Every 3-dimensional affine normal quasihomogeneous \( SL(2) \)-variety containing more than one orbit is uniquely determined by a pair of numbers \( (l,m) \in \{ \mathbb{Q} \cap (0,1] \} \times \mathbb{N} \).

We denote the corresponding variety by \( E_{l,m} \). The numbers \( l \) and \( m \) are called the height and the degree of \( E_{l,m} \), respectively. Write \( l = p/q \), where \( g.c.d.(q,p) = 1 \).

**Theorem 3.2** ([Gaï08], see also [BH08, Corollary 2.7]). An affine normal quasihomogeneous \( SL(2) \)-variety \( E_{l,m} \) is toric if and only if \( q-p \) divides \( m \).

We use the following notation for some closed subgroups of \( SL(2) \):

\[
T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}; \quad B := \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^*, u \in \mathbb{C} \right\};
\]

\[
U_n := \left\{ \begin{pmatrix} \zeta & u \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^n = 1, u \in \mathbb{C} \right\}; \quad C_n := \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^n = 1 \right\}.
\]

An \( SL(2) \)-variety \( E_{l,m} \) is smooth if and only if \( l = 1 \) (see [Pop73]). If \( l < 1 \), then \( E_{l,m} \) contains three \( SL(2) \)-orbits: the open orbit \( \mathcal{U} \), a 2-dimensional orbit \( \mathcal{D} \), and the closed orbit \( \{ O \} \). The fixed point \( O \) is a unique \( SL(2) \)-invariant singular point. Let

\[
k := g.c.d.(m,q-p), \quad a := \frac{m}{k}, \quad b := \frac{q-p}{k}.
\]

Then we have

\[
\mathcal{U} \cong SL(2)/C_m, \quad \mathcal{D} \cong SL(2)/U_{a(q+p)}.
\]

**Remark 3.2.1.** An explicit construction of the variety \( E_{l,m} \) is reduced to determine a system of generators of the following semigroup (see [Kra84], [Pan88]):

\[
M_{l,m}^+ := \{ (i,j) \in \mathbb{Z}_{\geq 0}^2 : j \leq li, m|(i-j) \}.
\]

Let \((i_1,j_1), \ldots, (i_u,j_u)\) be a system of generators of \( M_{l,m}^+ \), and consider a vector

\[
v = (X^{i_1}Y^{j_1}, \ldots, X^{i_u}Y^{j_u}) \in V(i_1 + j_1) \oplus \cdots \oplus V(i_u + j_u),
\]

where \( V(n) := \text{Sym}^n \langle X,Y \rangle \) is the irreducible \( SL(2) \)-representation of highest weight \( n \). Then, \( E_{l,m} \) is isomorphic to the closure \( SL(2) \cdot v \subset V(i_1 + j_1) \oplus \cdots \oplus V(i_u + j_u) \).

According to [BH08, §1], an affine normal quasihomogeneous \( SL(2) \)-variety \( E_{l,m} \) has a description as a categorical quotient of a hypersurface in \( \mathbb{C}^5 \). We consider \( \mathbb{C}^5 \)
as the $SL(2)$-module $V(0) \oplus V(1) \oplus V(1)$ with coordinates $X_0, X_1, X_2, X_3, X_4$, and identify $X_1, X_2, X_3, X_4$ with the coefficients of the $2 \times 2$ matrix
\[
\begin{pmatrix}
X_1 & X_3 \\
X_2 & X_4
\end{pmatrix}
\]
so that $SL(2)$ acts by left multiplication. We moreover consider actions of the following two diagonalizable groups:

\[G_0 := \{ \text{diag}(t, t^{-p}, t^{-p}, t^q) : t \in \mathbb{C}^* \} \cong \mathbb{C}^*;\]
\[G_m := \{ \text{diag}(1, \zeta^{-1}, \zeta^{-1}, \zeta, \zeta) : \zeta^m = 1 \} \cong \mu_m.\]

It is easy to see that the $SL(2)$-action on $\mathbb{C}^5$ commutes with the $G_0 \times G_m$-action.

**Theorem 3.3** ([BH08, Theorem 1.6]). Let $E_{l,m}$ be a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety of height $l = p/q$ and degree $m$. Then, $E_{l,m}$ is isomorphic to the categorical quotient of the affine hypersurface
\[
\mathbb{C}^5 \supset H_{q-p} := (X_0^{p-q} = X_1X_4 - X_2X_3)
\]
modulo the action of $G_0 \times G_m$.

**Remark 3.3.1.** According to the proof of [BH08, Theorem 1.6], the dense open orbit $U$ in $E_{l,m}$ is isomorphic to the $G_0 \times G_m$-quotient of the open subset in $H_{q-p}$ defined by the condition $X_0 \neq 0$. Also, the ring of $G_0$-invariants of $H_{q-p} \cap \{X_0 \neq 0\}$ is generated by the monomials
\[X := X_0^pX_1, \quad Y := X_0^{-q}X_3, \quad Z := X_0^pX_2, \quad W := X_0^{-q}X_4,\]
which satisfy the equation
\[
\begin{vmatrix}
X & Y \\
Z & W
\end{vmatrix} = X_0^{p-q}X_1X_4 - X_0^{p-q}X_2X_3 = 1.
\]
An $SL(2)$-variety $E_{l,m}$ has another description as an affine categorical quotient. To see this, let $H_b \subset \mathbb{C}^5$ be an affine hypersurface defined by the equation

$$Y_0^b = X_1X_4 - X_2X_3,$$

and consider the action of the group $G'_0 \times G_a$, where

$$G'_0 := \{ \text{diag}(t^k, t^{-p}, t^{-p}, t^q, t^q) : t \in \mathbb{C}^* \} \cong \mathbb{C}^*,$$

$$G_a := \{ \text{diag}(1, \zeta^{-1}, \zeta^{-1}, \zeta, \zeta) : \zeta^a = 1 \} \cong \mu_a.$$

**Theorem 3.4 ([BH08 Theorem 1.7]).** Let $E_{l,m}$ be a 3-dimensional affine normal quasihomogeneous $SL(2)$-variety of height $l = p/q$ and degree $m$. Then, $E_{l,m}$ is isomorphic to the categorical quotient of $H_b$ modulo the action of $G'_0 \times G_a$.

Let $L^-$ and $L^+$ be linearizations of the trivial line bundle over $H_b$ corresponding to the non-trivial characters

$$\chi^- : G'_0 \times G_a \to \mathbb{C}^*, \quad (t, \zeta) \mapsto t^{k-p+q}$$

and

$$\chi^+ : G'_0 \times G_a \to \mathbb{C}^*, \quad (t, \zeta) \mapsto t^{k+p-q}$$

de $G'_0 \times G_a$, respectively, and consider the following Zariski open subsets of $H_b$:

$$U^- := H_b \setminus \{ X_3 = X_4 = 0 \}, \quad U^+ := H_b \setminus \{ X_1 = X_2 = 0 \}.$$

**Theorem 3.5 ([BH08 Propositions 3.2 and 3.3]).** The subsets $H_b^{ss}(L^-)$ and $H_b^{ss}(L^+)$ of semistable points of $H_b$ with respect to the $G'_0 \times G_a$-linearized line bundles $L^-$ and $L^+$ are $U^-$ and $U^+$, respectively.

**Theorem 3.6 ([BH08 Theorem 3.4]).** With the above notation, set

$$E_{l,m}^- := H_b^{ss}(L^-)/(G'_0 \times G_a), \quad E_{l,m}^+ := H_b^{ss}(L^+)/(G'_0 \times G_a).$$

Then, the open embeddings $H_b^{ss}(L^-) \subset H_b$ and $H_b^{ss}(L^+) \subset H_b$ define natural birational morphisms $E_{l,m}^- \to E_{l,m}$ and $E_{l,m}^+ \to E_{l,m}$, and the $SL(2)$-equivariant flip

$$E_{l,m}^- \quad \longrightarrow \quad E_{l,m}^+ \quad \longrightarrow \quad E_{l,m},$$

**Remark 3.6.1 ([BH08 Remarks 3.12 and 4.2]).** Let $E_{l,m} \hookrightarrow V \cong V(i_1 + j_1) \oplus \cdots \oplus V(i_u + j_u)$ be an equivariant closed embedding (see Remark 3.2.1), and consider an action of $t \in \mathbb{C}^*$ on $V$ defined by multiplication of $(t^{i_1-j_1}, \ldots, t^{i_u-j_u})$. Then, since this $\mathbb{C}^*$-action commutes with the $SL(2)$-action, an affine variety $E_{l,m} \subset V$ remains...
stable under the $\mathbb{C}^*$-action, and this enables us to consider $E_{l,m}$ as an $SL(2) \times \mathbb{C}^*$-variety. We remark that there is another way to define the same $\mathbb{C}^*$-action on $E_{l,m}$: we consider an action of $\mathbb{C}^*$ on $H_b$ defined by the matrices
\[
\text{diag}(1, s^{-1}, s^{-1}, s, s), \quad s \in \mathbb{C}^*.
\]
Since this $\mathbb{C}^*$-action commutes with the $SL(2) \times G_0' \times G_a$-action, it descends to $E_{l,m}$, and we see that this $\mathbb{C}^*$-action coincides with the one defined above.

**Theorem 3.7 ([BH08 Proposition 4.1]).** An affine $SL(2) \times \mathbb{C}^*$-variety $E_{l,m}$ is spherical with respect to the Borel subgroup $\tilde{B} := B \times \mathbb{C}^*$.

Let $E'_{l,m} := B|_{\omega}(E_{l,m})$ be the weighted blow-up of $E_{l,m}$ with weight $\omega$ defined by the $\mathbb{C}^*$-action considered in Remark 3.6.1. Then we obtain surjective morphisms $E'_{l,m} \to E_{l,m}^-$ and $E'_{l,m} \to E_{l,m}^+$ such that the following diagram commutes:

\[
\begin{array}{c}
E'_{l,m} \\
\downarrow \\
E_{l,m}^-
\end{array}
\quad
\begin{array}{c}
E_{l,m}^+ \\
\downarrow \\
E_{l,m}
\end{array}
\quad
\begin{array}{c}
E_{l,m}^- \\
\downarrow \\
E_{l,m}
\end{array}
\quad
\begin{array}{c}
E_{l,m} \\
\end{array}
\]

**Theorem 3.8 ([BH08 §3]).** The weighted blow-up $E'_{l,m}$ contains a unique closed $SL(2) \times \mathbb{C}^*$-orbit $C$ isomorphic to $\mathbb{P}^1$. Moreover, along the closed orbit $C$, the variety $E'_{l,m}$ is locally isomorphic to $\mathbb{C} \times \mathbb{C}^2 / \mu_b$.

**Remark 3.8.1.** In view of Theorem 3.2, $E'_{l,m}$ is smooth if and only if $E_{l,m}$ is toric, and in the toric case the weight $\omega$ is trivial.

Batyrev and Haddad compute the colored cones of the simple spherical varieties $E_{l,m}$, $E_{l,m}^-$, $E_{l,m}^+$, and $E'_{l,m}$ (see [BH08 §4]). Firstly, the lattice $\Gamma$ of rational $\mathbb{B}$-eigenfunctions on $U$ is given as follows:
\[
\Gamma \equiv \{ ZW^j \in \mathbb{C}(U)^* : m|\{(i-j)\} \}.
\]
The varieties $E_{l,m}$, $E_{l,m}^-$, and $E_{l,m}^+$ contain exactly three $\mathbb{B}$-stable divisors
\[
D := (H_b \cap \{ Y_0 = 0 \})/(G_0' \times G_a),
\]
\[
S^- := (H_b \cap \{ X_4 = 0 \})/(G_0' \times G_a),
\]
and
\[
S^+ := (H_b \cap \{ X_2 = 0 \})/(G_0' \times G_a),
\]
and $E'_{l,m}$ contains one more $SL(2) \times \mathbb{C}^*$-stable divisor $D' \equiv \mathbb{P}^1 \times \mathbb{P}^1$, the exceptional divisor of the blow-up $E'_{l,m} \to E_{l,m}$. The divisors $D$, $S^-$, $S^+$, and $D'$ define lattice vectors $\rho_{vD}$, $\rho_{vS'}$, $\rho_{vS}$, $\rho_{vD'} \in \Gamma^\vee$ in the dual space $Q = \text{Hom}(\Gamma, \mathbb{Q})$, and we can consider $\{ \rho_{vS'}$, $\rho_{vS} \}$ as a $Q$-basis of $Q$. The set $\vee$ of $SL(2) \times \mathbb{C}^*$-invariant valuations
is given as $\mathcal{V} = \{x\rho_{v_S} + y\rho_{v_S}^{-} \in \mathcal{Q} : x + y \leq 0\}$, and the colored cones of $E_{l,m}$, $E^-_{l,m}$, $E^+_{l,m}$ and $E'_l,m$ are described as follows:

\[\begin{aligned}
&\mathcal{C} := \mathcal{C}(E_{l,m}) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_S}^{-}, \quad \mathcal{C}_{+} := \mathcal{C}(E_{l,m}^+) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_S}^{+}, \\
&\mathcal{C}_{-} := \mathcal{C}(E_{l,m}^-) = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_S}^{-}, \quad \mathcal{C}^- := \mathcal{C}(E_{l,m}^-) = \mathbb{Q}_{\geq 0}\rho_{v_S}^{-} \\
&\mathcal{C}' := \mathcal{C}(E_{l,m}') = \mathbb{Q}_{\geq 0}\rho_{v_D} + \mathbb{Q}_{\geq 0}\rho_{v_D}, \quad \mathcal{C}' := \mathcal{C}(E_{l,m}') = \phi.
\end{aligned}\]

4. Statement of the main result

In this section, we first review some facts from our previous article [Kub18] that hold without the toric hypothesis. Let

$\pi : H_{q-p} \longrightarrow H_{q-p}/(G_0 \times G_m) \cong E_{l,m}$

be the quotient morphism. Then, $\pi$ is flat over the open orbit $\mathfrak{U} \subset E_{l,m}$, and the Hilbert function $h := h_{H_{q-p}}$ of the general fibers of $\pi$ coincides with that of the regular representation $\mathbb{C}[G_0 \times G_m]$:

$h : \text{Irr}(G_0 \times G_m) \cong \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}_{\geq 0}, \quad (n, d) \mapsto h(n, d) = 1.$

The Hilbert–Chow morphism

$\gamma : \mathcal{H} = \text{Hilb}^{G_0 \times G_m}_{H_{q-p}}(H_{q-p}/(G_0 \times G_m) \cong E_{l,m}$

is an isomorphism over the open orbit $\mathfrak{U}$, and the main component $\mathcal{H}^\text{main}$ is the Zariski closure $\gamma^{-1}(\mathfrak{U})$.

Remark 4.0.1. For a $G_0 \times G_m$-module $V$, we denote by $V_{(n,d)}$ the weight space of weight $(n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ as in [Kub18] Remark 3.2.1.

Let $A$ be the polynomial ring $\mathbb{C}[X_0, X_1, X_2, X_3, X_4]$, and consider the following ideals of $A$:

$I_1 := (X_0^{q-p} - X_1 X_4, X_2, X_3, 1 - X_0^m X_1^m);$

$I_0 := (X_0^{q-p} - X_1 X_4, X_2, X_3, X_0^m X_1^m).$

Theorem 4.1 ([Kub18], §4). The following properties are true.

(i) The quotient rings $A/I_1$ and $A/I_0$ have Hilbert function $h$, namely we have $\dim(A/I_0_{(n,d)}) = \dim(A/I_1_{(n,d)}) = h(n,d)$ for any $(n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

(ii) The $E_{l,m}$-equivariant isomorphism $\gamma_{\gamma^{-1}(\mathfrak{U})} : \gamma^{-1}(\mathfrak{U}) \to \mathfrak{U}$ is given by sending $[I_1]$ to $\pi(x)$, where $x = (1,1,0,0,1) \in H_{q-p}$.

(iii) The closed point $[I_0]$ is contained in the singular fiber $\gamma^{-1}(O)$. 


Let $S$ be the coordinate ring of $H_{q-p}$:

$$S := \mathbb{C}[H_{q-p}] \cong A/(X_0^{q-p} - X_1X_4 + X_2X_3).$$

For any weight $(n, d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, there is a finite-dimensional $SL(2) \times \mathbb{C}^*$-module $F_{n,d}$ that generates the weight space $S_{(n,d)}$ over the invariant ring $S^{G_n \times G_m}$. By [Kub18 Lemma 4.3], we can take $F_{-p,-1} = \langle X_1, X_2 \rangle$ and $F_{q,1} = \langle X_3, X_4 \rangle$. It follows that for any closed point $[I] \in \mathcal{H}$, we have

$$s_1X_1 + s_2X_2 \in I$$

and

$$s_3X_3 + s_4X_4 \in I$$

for some $(s_1, s_2) \neq 0$ and $(s_3, s_4) \neq 0$, respectively. Therefore, we can construct the following equivariant morphisms:

$$\eta_{-p,-1} : \mathcal{H} \longrightarrow \text{Gr}(1, F_{-p,-1}^\vee) \cong \mathbb{P}^1, \quad \eta_{q,1} : \mathcal{H} \longrightarrow \text{Gr}(1, F_{q,1}^\vee) \cong \mathbb{P}^1.$$

Set

$$\psi := \gamma \times \eta_{-p,-1} \times \eta_{q,1} : \mathcal{H} \longrightarrow E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Then we have

$$\psi([I_1]) = (\pi(x), [1 : 0], [0 : 1])$$

by its construction (see [Kub18 §6]).

In what follows, we show that the restriction $\gamma |_{\mathcal{H}^{\text{main}}}$ factors through the weighted blow-up $E'_{l,m}$ and that $\psi(\mathcal{H}^{\text{main}}) \cong E'_{l,m}$. First, notice that we have the following equivariant commutative diagram:

![Diagram](https://example.com/diagram.png)

**Lemma 4.2.** We have $E'_{l,m} \cong E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$ as spherical $SL(2) \times \mathbb{C}^*$-varieties.

**Proof.** Set $E''_{l,m} = E^-_{l,m} \times_{E_{l,m}} E^+_{l,m}$. Then, $E''_{l,m}$ is a simple spherical $SL(2) \times \mathbb{C}^*$-variety with a dense orbit isomorphic to II. Let $`(\mathcal{C}, \mathcal{F})`$ be the colored cone of $E'_{l,m}$. Then we have $`\mathcal{C}` \subset `\mathcal{C}^\prime`, `\mathcal{C}^\prime` \subset `\mathcal{C}''`, $\mathcal{F}'' \subset \mathcal{F}^+$, and $\mathcal{F}'' \subset \mathcal{F}^-$ by Theorem 2.9. This implies that $\mathcal{F}'' = \phi = \mathcal{F}^\prime$. Since $`\mathcal{C}''` is generated by $\phi(\mathcal{F}''')$ and finite elements of $\mathcal{V}$, we obtain $`\mathcal{C}''` \subset \mathcal{V}$. This yields that $`\mathcal{C}^\prime` = `\mathcal{C}''`, and hence we have $(`\mathcal{C}''`, $\mathcal{F}''`) = (`\mathcal{C}^\prime`, $\mathcal{F}^\prime$). Therefore, $E''_{l,m} \cong E'_{l,m}$ by Theorem 2.4. Q.E.D.
Lemma 4.3. There are $SL(2) \times \mathbb{C}^*$-equivariant embeddings:

$$E_{l,m}^+ \hookrightarrow E_{l,m} \times \text{Gr}(1, F_{q,-p-1}^\vee) \cong E_{l,m} \times \mathbb{P}^1, \quad E_{l,m}^- \hookrightarrow E_{l,m} \times \text{Gr}(1, F_{q,1}^\vee) \cong E_{l,m} \times \mathbb{P}^1.$$  

Proof. We have the following equivariant morphism (this morphism was first constructed in the proof of [BH08 Theorem 3.10]):

$$U^+ \rightarrow \text{Gr}(1, F_{q,-p-1}^\vee) \cong \mathbb{P}^1, \quad (Y_0, X_1, X_2, X_3, X_4) \mapsto [X_1 : X_2].$$

Also, we have an equivariant morphism $U^+ \hookrightarrow E_{l,m}$ as a composition of the inclusion $U^+ \hookrightarrow H_b$ and the quotient morphism $H_b \rightarrow E_{l,m}$. Therefore, we get a $G_0' \times G_a$-invariant morphism $U^+ \rightarrow E_{l,m} \times \mathbb{P}^1$, which factors through $E_{l,m}^+$:

$$U^+ \rightarrow E_{l,m} \times \mathbb{P}^1 \xrightarrow{\alpha^+} \left[ E_{l,m} \times \mathbb{P}^1 \right]$$

Let $[T_1 : T_2]$ be the coordinate of $\mathbb{P}^1$. Then for each $i \in \{1, 2\}$, we have the following commutative diagram:

$$U^+ \cap \{ X_i \neq 0 \} = H_b \cap \{ X_i \neq 0 \} \xrightarrow{\text{Spec}} \left( \mathbb{C}[E_{l,m}] \left[ \frac{T_1}{T_2}, \frac{T_i}{T_2} \right] \right),$$

$$(H_b \cap \{ X_i \neq 0 \})/(G_0' \times G_a)$$

We see that

$$(\mathbb{C}[H_b] \chi)^{G_0' \times G_a} = \mathbb{C}[H_b]^{G_0' \times G_a} \left[ \frac{X_1}{X_i}, \frac{X_2}{X_i} \right]$$

holds as a subring of $\mathbb{C}[H_b]_{X_i}$, and therefore $\alpha^+$ is a closed immersion. Analogously, we have an equivariant morphism

$$U^- \rightarrow \text{Gr}(1, F_{q,1}^\vee) \cong \mathbb{P}^1, \quad (Y_0, X_1, X_2, X_3, X_4) \mapsto [X_3 : X_4],$$

which induces an equivariant morphism $\alpha^- : E_{l,m}^- \rightarrow E_{l,m} \times \mathbb{P}^1$. In a similar way, we see that $\alpha^-$ is a closed immersion. Q.E.D.

By Lemmas 4.2 and 4.3, we get the following equivariant closed embedding:

$$\varphi : E_{l,m}^+ \cong E_{l,m}^- \times E_{l,m} \xrightarrow{\psi} E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1.$$  

Corollary 4.4. We have $\psi(\mathcal{H}^{\text{main}}) \cong E_{l,m}^+.$

Proof. Let $x = (1, 1, 0, 0, 1) \in H_{q,-p}$. Then by the construction of $\varphi$, the $SL(2) \times \mathbb{C}^*$-orbit of $(\pi(x), [1 : 0], [0 : 1])$ is the dense open orbit in $\varphi(E_{l,m}^+)$ isomorphic to $\mathbb{U}$. Taking Theorem 4.1 (ii) and (iv) into account, we get $\psi(\mathcal{H}^{\text{main}}) = \psi(\gamma^{-1}(\mathbb{U})) \cong E_{l,m}^+.$ Q.E.D.
Summarizing, we obtain the following equivariant commutative diagram:

\[
\begin{array}{ccc}
H_{\text{main}} & \xrightarrow{\psi|_{H_{\text{main}}}} & E_{l,m}' \\
\downarrow{\gamma|_{H_{\text{main}}}} & & \downarrow{E_{l,m}} \\
H_{\text{main}} & \rightarrow & H_{\text{main}}
\end{array}
\]

We have seen in Remark 2.8.1 that every toroidal spherical variety has an equivariant resolution of singularities given by subdividing its fan for toric varieties. We apply this to the simple toroidal spherical \( SL(2) \times \mathbb{C}^* \)-variety \( E_{l,m}' \) and describe the minimal resolution of \( E_{l,m}' \) in terms of its colored fan. Firstly, we can take \( \{(2,0), (m,m)\} \) as a basis of the lattice \( \Gamma \subset \tilde{X}(\tilde{B}) \cong \mathbb{Z}^2 \). Let us denote its dual basis by \( \{u_1, u_2\} \). By virtue of [Pan91, Theorem 2] and [BH08, Proposition 2.8], we see that

\[
\rho_{vD} = -bu_1 + apu_2, \quad \rho_{vS^-} = u_1, \quad \rho_{vS^+} = u_1 + mu_2, \quad \rho_{vD'} = u_2.
\]

Therefore, \( E_{l,m}' \) has singularities of an affine toric surface defined by the following cone (see [BH08, Remark 3.12]):

\[
\sigma := \mathbb{Q}_{\geq 0}u_2 + \mathbb{Q}_{\geq 0}(-bu_1 + apu_2).
\]

Let \( \alpha \) and \( \beta \) be the quotient and the remainder of \( mp \) divided by \( q - p \), respectively, i.e.,

\[
mp = \alpha(q - p) + \beta, \tag{5}
\]

and set

\[
t := \frac{q - p - \beta}{k} = (\alpha + 1)b - ap. \tag{6}
\]

We consider the base change

\[
\begin{pmatrix}
u'_1 \\
\nu'_2
\end{pmatrix} := \begin{pmatrix}
-1 & \alpha + 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
u_1 \\
\nu_2
\end{pmatrix}
\]

to make \( \sigma \) into the normal form (see [CLS11, Proposition 10.1.1]):

\[
\sigma = \mathbb{Q}_{\geq 0}u_2' + \mathbb{Q}_{\geq 0}(bu_1' - tu_2').
\]

Therefore, the toric variety of the cone \( \sigma \) is a cyclic quotient singularity of type \( \frac{1}{b}(1,t) \), and it has a minimal resolution described by the Hirzebruch–Jung continued fraction of \( b/t \) (see [CLS11], [Ful93]):

\[
\frac{b}{t} = c_1 - \frac{1}{c_2 - \frac{1}{\ldots - \frac{1}{c_r}}}.
\]
We denote by \( \tilde{\text{Theorem 4.5}} \) and \( \tilde{\text{Theorem 4.6}} \).

The main component \( \tilde{\text{is the minimal resolution. The main result of this article is:}} \)

\[
\text{For } 2 \leq i \leq r + 1, \text{ we recursively define}
\]
\[
P_i := c_{i-1}P_{i-1} - P_{i-2}, \quad Q_i := c_{i-1}Q_{i-1} - Q_{i-2}.
\]

\textbf{Theorem 4.5 ([CLS11], Proposition 10.2.2]). With the above notation, we have:}

(i) \( P_0 < P_1 < \cdots < P_{r+1}, \ Q_0 < Q_1 < \cdots < Q_{r+1} \);

(ii) \( P_{i-1}Q_i - P_iQ_{i-1} = 1 \) for any \( 1 \leq i \leq r + 1 \);

(iii) \( \frac{b}{t} = \frac{P_{r+1}}{Q_{r+1}} < \frac{P_r}{Q_r} < \cdots < \frac{P_2}{Q_2} \).

Let
\[
\rho_i := -P_iu_1 + ((\alpha + 1)P_i - Q_i)u_2 \quad (0 \leq i \leq r + 1),
\]
and
\[
\mathcal{C}_i := Q_{i-1} + Q_i \quad (1 \leq i \leq r + 1).
\]

We denote by \( \tilde{E}'_{l,m} \) the toroidal spherical \( SL(2) \times \mathbb{C}^* \)-variety whose colored fan has \((\mathcal{C}_1, \phi), \ldots, (\mathcal{C}_{r+1}, \phi)\) as its maximal colored cones. Then,
\[
\tilde{E}'_{l,m} \longrightarrow E'_{l,m}
\]
is the minimal resolution. The main result of this article is:

\textbf{Theorem 4.6. The main component} \( \mathcal{H}^{\text{main}} \) \textit{is isomorphic to} \( \tilde{E}'_{l,m} \).

5. \textbf{First step towards the proof of Theorem 4.6}

In this section, we construct an equivariant morphism \( \tilde{E}'_{l,m} \rightarrow \mathbb{P}(V^\vee) \) defined by

\[
\mathcal{G}(E'_{l,m}) = \{ D_0, \ldots, D_{r+1}, S^+, \ S^- \},
\]
where \( \tilde{S}^+ \) (resp. \( \tilde{S}^- \)) \textit{is a non-}\( SL(2) \times \mathbb{C}^* \)-stable prime divisor on \( \tilde{E}'_{l,m} \) such that its image under the canonical birational morphism \( \tilde{E}'_{l,m} \rightarrow E_{l,m} \) is the \( \tilde{B} \)-stable divisor \( S^+ \) (resp. \( S^- \)) on \( E_{l,m} \). By definition, we have
\[
v_{D_i}(f) = \rho_{D_i}(\chi_f) = \rho_i(\chi_f)
\]
for any \( f \in \mathbb{C}(U)^\tilde{\mathcal{B}} \). We remark that we have \( \rho_0 = \rho_{vD} \), and \( \rho_{r+1} = \rho_{vD} \).

For each \( 0 \leq i \leq r + 1 \), we define
\[
e_i := (\alpha + 1 + m)P_i - Q_i, \quad l_i := (\alpha + 1)P_i - Q_i, \quad n_i := -pe_i + ql_i.
\]

The next lemma is a consequence of Theorem \[4.5\]

**Lemma 5.1.** We have:

(i) \( n_i = k(tP_i - bQ_i) \) for any \( 0 \leq i \leq r + 1 \);

(ii) \( n_i = c_{i-1}n_{i-1} - n_{i-2} \) for any \( 2 \leq i \leq r + 1 \);

(iii) \( n_0 = q - p > n_1 = q - p - \beta > n_2 > \cdots > n_{r-1} > n_r = k > n_{r+1} = 0 \).

For each \( 0 \leq i \leq r + 1 \), set
\[
\sigma_i := Z^i W^h, \quad f_i := \prod_{0 \leq j \leq i} \sigma_j.
\]

**Lemma 5.2.** With the preceding notation, the following properties are true.

(i) Let \( 0 \leq i, j \leq r + 1 \). Then we have
\[
v_{D_j}(\sigma_i) = \begin{cases} > 0 & (\text{if } i > j) \\ 0 & (\text{if } i = j) \\ < 0 & (\text{if } i < j). \end{cases}
\]

In particular, we have \( v_{D_j}(\sigma_{j+1}) = 1 \) and \( v_{D_j}(\sigma_{j-1}) = -1 \).

(ii) We have \( v_{D_j}(f_i) = v_{D_j}(f_{i-1}) \).

**Proof.** Since \( v_{D_j}(\sigma_i) = \rho_j(\chi_{\sigma_i}) = P_jQ_i - P_iQ_j \), we get (i) by Theorem \[4.5\]. Item (ii) follows from the definition of \( f_i \) and (i). Q.E.D.

Let \( \tilde{E}_i \) be the simple spherical open subvariety of \( \tilde{E}'_{l,m} \) corresponding to the colored cone \( (\mathcal{C}_i, \phi) \), and \( Y_i \) the unique closed orbit in \( \tilde{E}_i \). Then we have
\[
\mathcal{Y}(\tilde{E}_i) = \{ D_{i-1}|_{\tilde{E}_i}, D_i|_{\tilde{E}_i}, S^+|_{\tilde{E}_i}, S^-|_{\tilde{E}_i} \}, \quad \mathcal{Y}_{Y_i}(\tilde{E}_i) = \{ D_{i-1}|_{\tilde{E}_i}, D_i|_{\tilde{E}_i} \}.
\]

Let us consider the following \( SL(2) \times \mathbb{C}^* \)-stable divisor on \( \tilde{E}'_{l,m} \):
\[
\delta := \sum_{1 \leq i \leq r+1} v_{D_i}(f_i^{-1})D_i.
\]

Though the Cartierness of \( \delta \) follows immediately from the smoothness of \( \tilde{E}'_{l,m} \), we check the criterion for a Weil divisor to be Cartier given in Theorem \[2.10\] as a preparation for the proof of Lemma \[5.3\] with the notation used in Theorem \[2.10\]. we see by Lemma \[5.2\](ii) that \( f_i = f_i^{-1} \) satisfies the required condition.

**Lemma 5.3.** The Cartier divisor \( \delta \) is generated by global sections.
Proof. Taking Theorem 2.12 into account, it is enough to show the following:

(a) \( v_{D_i}(f_{i-1}^{r+1}) \leq v_{D_j}(f_{j-1}^{r+1}) \) and \( v_{D_{j-1}}(f_{j-1}^{r+1}) \leq v_{D_{j-1}}(f_{j-1}^{r+1}) \) hold for any \( 1 \leq i, j \leq r + 1 \); and
(b) \( v_{S_i}(f_{i-1}^{r+1}) \leq 0 \) and \( v_{S_j}(f_{j-1}^{r+1}) \leq 0 \) hold for any \( 1 \leq i \leq r + 1 \).

Condition (a) follows from Lemma 5.2. By a direct calculation, we have \( v_{S_i}(f_{i-1}) = \sum_{0 \leq j \leq i-1} e_j \) and \( v_{S_j}(f_{j-1}) = \sum_{0 \leq j \leq i-1} l_j \). This shows (b). Q.E.D.

Remark 5.3.1. Since \( \delta \) is \( SL(2) \times \mathbb{C}^* \)-stable, there is a linearization of the action of \( SL(2) \times \mathbb{C}^* \) with respect to the line bundle \( \mathcal{O}(\delta) \) such that the induced action on \( \Gamma(E_{l,m}^r \mathcal{O}(\delta)) \) coincides with that on the function field \( \mathcal{O}(E_{l,m}^r) \) (see [ADHL15]).

Let
\[
V := \langle (SL(2) \times \mathbb{C}^*) \cdot f_i : 1 \leq i \leq r \rangle,
\]
which is isomorphic to \( \bigoplus_{1 \leq i \leq r} V(e_0 + e_1 + \cdots + e_i) \otimes V(l_0 + l_1 + \cdots + l_i) \). Here, \( V(n) \) stands for the irreducible \( SL(2) \)-representation of highest weight \( n \). We can take
\[
\mathcal{A} := \left\{ \chi e_0 + e_1 + \cdots + e_{r-1} - e_i W^l : 1 \leq i \leq r; 0 \leq e \leq e_0 + e_1 + \cdots + e_{r-1}; 0 \leq l \leq l_0 + l_1 + \cdots + l_i \right\}
\]
as a basis of \( V \).

Lemma 5.4. The vector space \( V \) is an \( SL(2) \times \mathbb{C}^* \)-submodule of \( \Gamma(E_{l,m}^r \mathcal{O}(\delta)) \).

Proof. We show that \( f_i \in \Gamma(E_{l,m}^r \mathcal{O}(\delta)) \) holds for every \( 1 \leq i \leq r \). For any \( 1 \leq j \leq r + 1 \), we have
\[
\text{div}(f_i)|_{E_j} = v_{S_i}(f_i)S_i^+|_{E_j} + v_{S_j}(f_i)S_j^-|_{E_j} + v_{D_{j-1}}(f_i)D_{j-1}|_{E_j} + v_{D_j}(f_i)D_j|_{E_j}
\]
and
\[
\delta|_{E_j} = v_{D_{j-1}}(f_{j-1}^{r+1})D_{j-1}|_{E_j} + v_{D_j}(f_j^{r-1})D_j|_{E_j}.
\]
Thus we get \( \text{div}(f_i)|_{E_j} + \delta|_{E_j} \geq 0 \) by comparing each coefficient using the condition (a) in the proof of Lemma 5.3. Q.E.D.

Therefore, we obtain a natural equivariant morphism
\[
\Phi : E_{l,m}^r \longrightarrow E_{l,m} \times \mathbb{P}(V').
\]
We show that \( \Phi \) is a closed immersion. Recall that \( E_{l,m}^r \) is covered by simple open subembeddings \( E_1, \ldots, E_{r+1} \) and that \( E_i = (SL(2) \times \mathbb{C}^*)(E_i) \), where
\[
(E_i)_0 = E_i \setminus \bigcup_{D \in \mathcal{O}(E_i) \setminus \mathcal{O}Y(E_i)} D = E_i \setminus (S_i^+|_{E_i} \cup S_i^-|_{E_i})
\]
following the notation defined in §2. Also, we have \((\tilde{E}_i)_1 = \mathcal{U} \cap \{ZW \neq 0\}\), \((E_{l,m})_0 = E_{l,m}\), and \((E_{l,m})_1 = \mathcal{U}\). Therefore, it follows from Remark [2,2,2] that
\[
\mathbb{C}[(\tilde{E}_i)_0] = \{F \in \mathbb{C}[\mathcal{U}] : v_{D_{l-1}}(F) \geq 0, v_{D_i}(F) \geq 0\}
\]
and
\[
\mathbb{C}[E_{l,m}] = \{F \in \mathbb{C}[\mathcal{U}] : v_{D_{r+1}}(F) \geq 0\}.
\]
Let \(L\) be a subring of \(\mathbb{C}(\mathcal{U})\) defined by
\[
L = \{F \in \mathbb{C}[\mathcal{U}]_{ZW} : v_{D_{r+1}}(F) \geq 0\},
\]
and consider an open subset
\[
U_i := \text{Spec} \left( L \left[ \frac{f^\vee}{f_{l-1}^\vee} : f \in \mathcal{A} \right] \right) \quad (1 \leq i \leq r + 1)
\]
of \(E_{l,m} \times \mathbb{F}(V^\vee)\), where \(f^\vee\) denotes the dual basis of \(f\). Also, consider a homomorphism
\[
\Phi_i^\# : L \left[ \frac{f^\vee}{f_{l-1}^\vee} : f \in \mathcal{A} \right] \longrightarrow \mathbb{C}[(\tilde{E}_i)_0]
\]
defined by sending \(F \frac{f^\vee}{f_{l-1}^\vee}\), where \(F \in L\), to \(F \frac{f}{f_{l-1}}\).

**Lemma 5.5.** The homomorphism \(\Phi_i^\#\) is well-defined.

**Proof.** Let \(F = \frac{X^{e_i}Y^{l_i}Z^{w_i}}{(ZW)^{m_i}} \in L\), where \(e, e', l, l', d \geq 0\). Since \(F\) is invariant under the action of \(G_m\), we have \(e' + e - l' - l = mc\) for some \(c \in \mathbb{Z}\). Therefore, for any \(0 \leq j \leq r + 1\), we have
\[
v_{D_j}(F) = -P_j(l' + l - d) + \{(\alpha + 1)P_j - Q_j\}c.
\]
Taking \(j = r + 1\), we get \(\frac{pmc}{q-p} = \frac{apc}{b} \geq l' + l - d\) by using the equations (1), (5), and (6). This implies that \(c \geq 0\). Therefore, we have
\[
v_{D_j}(F) \geq -P_j \frac{pmc}{q-p} + \{(\alpha + 1)P_j - Q_j\}c = \frac{n_jc}{q-p} \geq 0
\]
cressing Lemma 5.1. Thus, \(L \subset \mathbb{C}[(\tilde{E}_i)_0]\). Also, we have \(f_j/f_{l-1} \in \mathbb{C}[(\tilde{E}_i)_0]\) by the condition (a) in the proof of Lemma 5.3 and hence \(f/f_{l-1} \in \mathbb{C}[(\tilde{E}_i)_0]\) for any \(f \in \mathcal{A}\). Q.E.D.

**Lemma 5.6.** The homomorphism \(\Phi_i^\#\) is surjective.

**Proof.** Let \(F = \frac{X^{e_i}Y^{l_i}Z^{w_i}}{(ZW)^{m_i}} \in \mathbb{C}[(\tilde{E}_i)_0]\), where \(e, e', l, l', d \geq 0\). As in the proof of Lemma 5.5 we can write \(e' + e = l' + l + mc\) for some \(c \in \mathbb{Z}\). By a direct calculation, we see that \(v_{D_{r+1}}(F) \geq 0\) if and only if \(\frac{apc}{b} \geq l' + l - d\). In the following, we assume
that \( v_{D_{r+1}}(F) < 0 \), since otherwise we have \( F = \Phi^i_f(F) \). Set \( F' = F/\sigma_i \). Then, as an element of the function field \( \mathbb{C}(\Omega) \), we can write \( F \) as

\[
F = F' \frac{f_i}{f_{i-1}}.
\]

(8)

We claim that the following two conditions hold: (a) \( v_{D_{r+1}}(F') > v_{D_{r+1}}(F) \); (b) \( F' \in \mathbb{C}[(\bar{E}_i)_{0}] \). Indeed, the condition (a) follows from Lemma 5.2. Also, in view of Lemma 5.2, it suffices to show that \( v_{D_{i-1}}(F) \geq 1 \) holds to get (b). Suppose otherwise, i.e., \( v_{D_{i-1}}(F) = 0 \). Then we have

\[
(l' + l - d)P_{i-1} = c \{(\alpha + 1)P_{i-1} - Q_{i-1}\}.
\]

(9)

If \( i = 1 \), then the conditions \( v_{D_0}(F) = 0 \) and \( v_{D_1}(F) \geq 0 \) imply that \( c = 0 \) and \( 0 \geq l' + l - d \), which contradicts to our assumption that \( v_{D_{r+1}}(F) < 0 \). Let \( i \geq 2 \). By (7), we have \( 0 \leq v_{D_i}(F) = c_id - v_{D_{i-1}}(F) - v_{D_{i-2}}(F) = -v_{D_{i-2}}(F) \), and hence

\[
c \{(\alpha + 1)P_{i-2} - Q_{i-2}\} \leq (l' + l - d)P_{i-2}.
\]

(10)

If \( i = 2 \), then we can show in a similar way that the hypothesis \( v_{D_1}(F) = 0 \) leads to a contradiction. If \( i > 2 \), then by (9) and (10) we have \( c_{\frac{Q_{i-1}}{P_{i-1}}} \leq c_{\frac{Q_{i-2}}{P_{i-2}}} \), and thus \( c = 0 \) concerning Theorem 4.5. In a same manner, we see that this contradicts to the assumption. Therefore, we have \( F' \in \mathbb{C}[(\bar{E}_i)_{0}] \). The conditions (a) and (b) and the equation (8) yield that there is an \( F'' \in \mathbb{C}[(\bar{E}_i)_{0}] \) with \( v_{D_{r+1}}(F'') \geq 0 \) such that \( F = F'' \left( \frac{f_i}{f_{i-1}} \right)^t \) holds for some \( t > 0 \). Thus, we get \( F = \Phi^i_f \left( F'' \left( \frac{f_i}{f_{i-1}} \right)^t \right) \). Q.E.D.

As a consequence of Lemmas 5.5 and 5.6, we obtain:

**Proposition 5.7.** The morphism \( \Phi : \bar{E}_{i,m} \rightarrow E_{i,m} \times \mathbb{P}(V') \) is a closed immersion.

6. Generators as a module over the invariant ring

For each \( n \geq 0 \), consider the following irreducible \( SL(2) \)-representations:

\[
A(n) := \text{Sym}^n \langle X_1, X_2 \rangle \cong V(n), \quad B(n) := \text{Sym}^n \langle X_3, X_4 \rangle \cong V(n).
\]

Also, define \( C(n) := \langle X_0^n \rangle \cong V(0) \) for each \( n \in \mathbb{Z} \), and set

\[
F_{m,0} := A(e_0) \otimes B(l_0), \quad F_{n,i,0} := A(e_i) \otimes B(l_i) \oplus C(n_i) \quad (1 \leq i \leq r).
\]

The goal of this section is to prove the following

**Theorem 6.1.** For any \( 0 \leq i \leq r \), the weight space \( S_{(n,0)} \) is generated by \( F_{n,i,0} \) as a module over the invariant ring \( S^{G_0 \times G_m} \).
We prepare notations and lemmas that we need for the proof of Theorem 6.1. Some of them have already appeared in [Kub18 §4].

Let \( R := \mathbb{C}[X_0, X_1, X_3] \subset A \). The polynomial ring \( R \) has a natural \( \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \)-grading defined by the \( G_0 \times G_m \)-action: \( R = \bigoplus_{(n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}} R_{(n,d)} \). Concerning that \( X_1 \) and \( X_2 \) (resp. \( X_3 \) and \( X_4 \)) have the same \( SL(2) \times \mathbb{C}^* \times G_0 \times G_m \)-weight, it suffices to determine a subspace of \( R_{(n,0)} \) that generates \( R_{(n,0)} \) over the invariant ring \( R^{G_0 \times G_m} \) in proving Theorem 6.1. For each \( c, n \in \mathbb{Z} \), we consider the vector subspaces

\[
R^c := \langle X_0^{d_0} X_1^{d_1} X_3^{d_3} \in R : d_1 - d_3 = c \rangle
\]

and

\[
R_n := \langle X_0^{d_0} X_1^{d_1} X_3^{d_3} \in R : d_0 - pd_1 + qd_3 = n \rangle
\]

of \( R \). Then we have

\[
R = \bigoplus_{c \in \mathbb{Z}} R^c = \bigoplus_{n \in \mathbb{Z}} R_n.
\]

Let \( R_n^c := R^c \cap R_n \). Then, the weight space \( R_{(n,d)} \) is described as follows:

\[
R_{(n,d)} = \bigoplus_{c \equiv d \pmod{m}} R^c_n.
\]

**Remark 6.1.1.** By the proof of [BH08, Theorem 1.6], we see that the invariant ring \( R^{G_0 \times G_m} = R_{(0,0)} \) is described as follows:

\[
R^{G_0 \times G_m} = \mathbb{C}[X_0^{pu_1 - qu_2} X_1^{u_1} X_3^{u_2} : (u_1, u_2) \in M^+_l, m].
\]

**Example 6.2.** Let \( l = p/q = 1/4 \), and \( m = 2 \). By using an algorithm described in [Pan88] for finding a system of generators of the semigroup \( M^+_l, m \), we see that \( M^+_l, m \) is minimally generated by \( (2, 0), (5, 1), \) and \( (8, 2) \). Therefore,

\[
R^{G_0 \times G_m} = \mathbb{C}[X_0^2 X_1^2, X_0 X_1^5 X_3, X_1^8 X_3^2].
\]

**Lemma 6.3 ([Kub18, Lemma 4.6]).** For any \( (n, d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \), the minimum

\[
c_{(n,d)} := \min\{c \in \mathbb{Z} : c \equiv d \pmod{m}, R^c_n \neq 0\}
\]

exists.

**Example 6.4 ([Kub18, Example 4.7]).** If \( 0 \leq n \leq q - p \), then \( c_{(n,0)} = 0 \). We have \( R^0_n = \langle X_0^n \rangle \) if \( 0 \leq n < q - p \), and \( R^0_{q-p} = \langle X_0^{q-p}, X_1 X_3 \rangle \).

We define another grading on \( R \) such that each graded component is finite-dimensional, which makes it easier to analyze the structure of the weight space \( R_{(n,d)} \). For that purpose, consider a \( \mathbb{Z} \)-linear map \( \mu : \mathbb{Z}^3 \to \mathbb{Z}^3 \) defined by

\[
(d_0, d_1, d_3) \mapsto \mu(d_0, d_1, d_3) := (d_0 - pd_1 + qd_3, d_1 - d_3, pd_1 - qd_3).
\]
We see that $\mu$ is injective. Let us denote by $\Lambda$ the image of $\mu|_{\mathbb{Z}^3_{\geq 0}}$, and define

$$R_\lambda := \langle X_0^{d_0} X_1^{d_1} X_3^{d_3} \in R : \mu(d_0, d_1, d_3) = \lambda \rangle$$

for each $\lambda \in \Lambda$. Then we have

$$R = \bigoplus_{\lambda \in \Lambda} R_\lambda.$$

Next, consider the projection $\tilde{\mu} : \mathbb{Z}^3 \to \mathbb{Z}^2$, $(n,c,\omega) \mapsto (n,c)$ to the first and the second factor. Set $\mu' := \tilde{\mu} \circ \mu$, and denote by $\Lambda'$ the image of $\mu'|_{\mathbb{Z}^3_{\geq 0}}$. Then we have

$$R = \bigoplus_{(n,c) \in \Lambda'} R_{n,c}^c, \hspace{1cm} R_n^c = \bigoplus_{\lambda \in \tilde{\mu}^{-1}(n,c)} R_\lambda.$$

**Lemma 6.5** ([Kub18 Lemmas 4.8 and 4.10]). With the preceding notation, the following properties hold.

(i) Let $\lambda = (n,c,\omega) \in \Lambda$. Then, the vector space $R_\lambda$ is spanned by

$$f_\lambda := X_0^{n+\omega} X_1^{\frac{q-c-\omega}{p-c}} X_3^{\frac{p-c-\omega}{p}}.$$

(ii) For any $\lambda$, $\lambda' \in \Lambda$, we have $f_\lambda f_{\lambda'} = f_{\lambda+\lambda'}$.

(iii) Let $(n,c) \in \Lambda'$. Then we have $\omega - \omega' \in (q-p)\mathbb{Z}$ for any $(n,c,\omega)$, $(n,c,\omega') \in \tilde{\mu}^{-1}(n,c)$.

We see that

$$\omega_{(n,c)}^\text{max} := \max\{\omega \in \mathbb{Z} : (n,c,\omega) \in \tilde{\mu}^{-1}(n,c)\}$$

and

$$\omega_{(n,c)} := \min\{\omega \in \mathbb{Z} : (n,c,\omega) \in \tilde{\mu}^{-1}(n,c)\}$$

exist for any $(n,c) \in \Lambda'$, and that the vector space $R_n^c$ is finite-dimensional.

**Lemma 6.6.** Let $(n,c) \in \Lambda'$. If $c < 0$, then $\omega_{(n,c)}^\text{max} = qc$. Otherwise, $\omega_{(n,c)}^\text{max} = pc$.

**Proof.** Let $\mu(d_0,d_1,d_3) = (n,c,\omega_{(n,c)}^\text{max})$. We claim that either $d_1 = 0$ or $d_3 = 0$ holds. Indeed, if $d_1 > 0$ and $d_3 > 0$, then we have

$$\mu(d_0 + q - p, d_1 - 1, d_3 - 1) = (n,c,\omega_{(n,c)}^\text{max} + q - p) \in \tilde{\mu}^{-1}(n,c),$$

which contradicts to the maximality of $\omega_{(n,c)}^\text{max}$. Thus, if $c < 0$ then we see that $d_1 = 0$, and therefore $\omega_{(n,c)}^\text{max} = qc$. Q.E.D.

**Lemma 6.7** ([Kub18 Lemma 4.11]). Let $(n,c,\omega) \in \Lambda$. Then, we have $n + \omega < q - p$ if and only if $\omega = \omega_{(n,c)}$.

**Corollary 6.8.** Let $(n,c), (n',c') \in \Lambda'$. Then the following properties are true.

(i) If $n = 0$, then $0 \leq \omega_{(0,c)} < q - p$. 

(ii) We have \( \omega_{(n+n', c+c')} = \omega_{(n, c)} + \omega_{(n', c')} \) if and only if \( \omega_{(n, c)} + \omega_{(n', c')} + n+n' < q-p \).

**Proof.** First, we have \( \omega_{(0, c)} < q-p \) by Lemma 6.7. Let \((d_0, d_1, d_3) \in \mathbb{Z}^3\) be such that \( \mu(d_0, d_1, d_3) = (n, c, \omega_{(n, c)}) \). Then we have \( d_0 = n + \omega_{(n, c)} \) by Lemma 6.5(i), and therefore we have \( \omega_{(0, c)} = d_0 \geq 0 \) if \( n = 0 \). Item (ii) follows from the fact that \( (n+n', c+c', \omega_{(n, c)} + \omega_{(n', c')}) \in \tilde{\mu}^{-1}(n+n', c+c') \) and Lemma 6.7. Q.E.D.

**Example 6.9.** We have \( \omega_{(n, mP_i)}^{\text{max}} = pmP_i \) by Lemma 6.6. By a direct calculation, we obtain the following:

\[
\omega_{(n, mP_i)}^{\text{max}} + n_i = \{\alpha(q-p) + \beta\}P_i + (q-p-\beta)P_i - (q-p)Q_i = \{(\alpha+1)P_i - Q_i\}(q-p).
\]

Therefore, we see that \((n_i, mP_i, n_i) \in \tilde{u}^{-1}(n_i, mP_i)\). It follows that \( \omega_{(n, mP_i)} = -n_i \), since \( n_i + (-n_i) < q-p \). Also, we can calculate that \( f_{(n_i, mP_i,\omega_{(n, mP_i)})} = X_1^q X_3^h \) by using Lemma 6.5(i).

**Definition 6.10.** For any positive integers \( m_1 \) and \( m_2 \), we denote by \( \text{Rem}[m_1, m_2] \) the remainder of \( m_1 \) divided by \( m_2 \).

**Corollary 6.11.** Let \((n, c) \in \Lambda' \), and suppose that \( n \geq 0 \) and that \( c > 0 \). Then, we have \( \omega_{(n, c)} \geq 0 \) if and only if \( \text{Rem}[pc, q-p] + n < q-p \).

**Proof.** By Lemmas 6.5(iii) and 6.6, we have \( pc = x(q-p) + \omega_{(n, c)} \) for some \( x \geq 0 \). If \( \omega_{(n, c)} \geq 0 \), then we get \( \text{Rem}[pc, q-p] + n = \omega_{(n, c)} + n < q-p \) by Lemma 6.7. Otherwise, we have \( \text{Rem}[pc, q-p] = x'(q-p) + \omega_{(n, c)} \) for some \( x' > 0 \). Therefore, \( \text{Rem}[pc, q-p] + n = x'(q-p) + \omega_{(n, c)} + n \geq q-p \), since \( \omega_{(n, c)} + n \geq 0 \) concerning Lemma 6.5(i). Q.E.D.

**Definition 6.12** ([Kub18, Definition 4.12]). For each \((n, d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \), we define:

(i) \( \Lambda_{(n, d)} := \{(n, c, \omega) \in \Lambda \mid c \equiv d \, (\text{mod } m)\} \);

(ii) \( \tilde{\lambda}_{(n, d)} := (n, c_{(n, d)}, \omega_{(n, c_{(n, d)})}) \in \Lambda_{(n, d)} \).

Using the notation defined above, we obtain different ways of expressing the weight space \( R_{(n, d)} \):

\[
R_{(n,d)} = \bigoplus_{c \equiv d \, (\text{mod } m)} R_{c} = \bigoplus_{c \equiv d \, (\text{mod } m)} \left( \bigoplus_{\Lambda \in \tilde{\mu}^{-1}(n, c)} R_{\Lambda} \right) = \bigoplus_{\Lambda \in \Lambda_{(n,d)}} R_{\Lambda}.
\]

Now since we have \( R_{(n, 0, \omega_{(n, 0)})} = \langle X_0^n \rangle \) and \( R_{(n, mP, \omega_{(n, mP)})} = \langle X_1^q X_3^h \rangle \), we see that Theorem 6.1 follows as a consequence of the next
Proposition 6.13. For any \(0 \leq i \leq r\), the weight space \(R_{n_i,0}\) is generated by \(R^0_{n_i}\) and \(R^{mp}_{n_i}\) as a module over the invariant ring \(R^{G_0 \times G_m}\).

The rest of this section is devoted mostly to the proof of Proposition 6.13. Recall that we have considered the Hirzebruch–Jung continued fraction of \(b/t\) in §4. Set \(t_1 := t\). Then we have the following equations that arise from the modified Euclidean algorithm (see [CLS11, §10] for more details):

\[
\begin{align*}
  b &= c_1 t_1 - t_2, \\
  t_1 &= c_2 t_2 - t_3, \\
  &\vdots \\
  t_{i-1} &= c_i t_i - t_{i+1}, \\
  &\vdots \\
  t_{r-1} &= c_r t_r.
\end{align*}
\]

(11)

We can easily see that the following equation holds for any \(2 \leq i \leq r\):

\[
  b - t_1 = (c_1 - 2)t_1 + (c_2 - 2)t_2 + \cdots + (c_{i-1} - 2)t_{i-1} + t_{i-1} - t_i.
\]

(12)

Since \(b = n_0/k\) and \(t_1 = t = n_1/k\), Lemma 5.1 and (11) yield that \(t_i = n_i/k\) holds for any \(1 \leq i \leq r\).

Now, let us consider the following two conditions:

\begin{enumerate}
  \item \((C1)\) \(1 < i \leq r + 1;\)
  \item \((C2)\) \(1 \leq \exists l \leq i - 1\) such that \(c_l > 2, c_{l+1} = \cdots = c_{i-1} = 2.\)
\end{enumerate}

Assume that conditions (C1) and (C2) hold, and let \(x\) be any integer such that \(0 \leq x < P_i - P_{i-1}\). The quotient of \(t_1(P_{i-1} + x)\) divided by \(b\) is always not less than \(Q_{i-1}\), since we have \(t_1 P_{i-1} = bQ_{i-1} + t_{i-1}\) with \(0 \leq t_{i-1} < b\) by Lemma 5.1. Keeping this in mind, let \(Q_{i-1} + \theta x\) (resp. \(\Theta[x]\)) be the quotient (resp. the remainder) of \(t_1(P_{i-1} + x)\) divided by \(b\), namely

\[
t_1(P_{i-1} + x) = b(Q_{i-1} + \theta x) + \Theta[x], \quad \Theta[x] = \text{Rem}[t_1(P_{i-1} + x), b].
\]

Then we have \(\Theta[x] = t_{i-1} + t_1 x - b\theta x\).

Remark 6.13.1. With the above notation and assumption, we have the following.

\begin{enumerate}
  \item Since \(t_1 < b\), we see that \(\theta x - \theta x_{-1} \in \{0, 1\}\). Furthermore, the following properties are true.
    \begin{itemize}
      \item We have \(\theta x - \theta x_{-1} = 0\) if and only if \(\Theta[x - 1] + t_1 - b < 0\). In this case, \(\Theta[x] = \Theta[x - 1] + t_1.\)
    \end{itemize}
\end{enumerate}
• We have $\theta_x - \theta_{x-1} = 1$ if and only if $\Theta[x-1] + t_1 - b \geq 0$. In this case, $\Theta[x] = \Theta[x-1] + t_1 - b$.

(ii) Since $P_i - P_{i-1} < b$, we have $\Theta[x] = \Theta[x']$ if and only if $x = x'$.

We proceed by induction on $j$. Suppose that $j = 1$. Firstly, we have

$$\Theta[0] + t_1 - b = t_{i-1} + t_1 - b$$

$$= t_1 - \{(c_1 - 2)t_1 + \cdots + (c_l - 2)t_l + (c_{i+1} - 2)t_{i+1} + \cdots + (c_{i-1} - 2)t_{i-1}\}$$

$$= t_1 - \{(c_1 - 2)t_1 + \cdots + (c_l - 2)t_l\} \leq t_i - t_1 < 0.$$

Therefore, we have $\Theta[P_1] = \Theta[1] = t_{i-1} + t_1$ by Remark 6.13.1. In the same way, we see that $\Theta[x-1] + t_1 - b \leq t_i - t_l < 0$ holds for any $P_1 < x < P_2$, and thus we get $\Theta[x] = t_{i-1} + xt_1$. Further, this yields that $M_1 = \Theta[P_2 - P_1]$.

Next, suppose that $j > 1$. Notice that we have $P_{j+1} = (c_j - 1)P_j + (P_j - P_{j-1})$. In the following, we divide the proof into three steps.

**Lemma 6.14.** Assume that the conditions (C1) and (C2) hold. If $\Theta[x] = t_{i-1} + (c_1 - 2)t_1 + \cdots + (c_{j-1} - 2)t_{j-1} + (c_j - 1)t_j$ holds for some $1 \leq j \leq l$, then we have $\Theta[x + 1] = t_{i-1} + t_j + t_{j+1}$.

**Proof.** By a direct calculation using (12), we have $\Theta[x] + t_1 - b = t_{i-1} + t_j + t_{j+1}$ holds for some $1 \leq j \leq l$, and therefore $\Theta[x + 1] = \Theta[x] + t_1 - b = t_{i-1} + t_j + t_{j+1}$ by Remark 6.13.1 (i). Q.E.D.

The next lemma is the key in proving Proposition 6.13.

**Lemma 6.15.** Assume that the conditions (C1) and (C2) hold. Then, the following properties are true for any $1 \leq j < l$.

(i) Let $P_i \leq x < P_{j+1}$, and denote by $\kappa$ (resp. $\varepsilon$) the quotient (resp. the remainder) of $x$ divided by $P_j$, i.e., $x = \kappa P_j + \varepsilon$. Then, we have $\Theta[x] = \Theta[\varepsilon] + \kappa t_j$. In particular, we have $\Theta[x] > t_{i-1}$.

(ii) Let $M_j := \max\{\Theta[x] : 0 \leq x < P_{j+1}\}$. Then, $M_j = \Theta[P_{j+1} - P_j] = t_{i-1} + b - t_j + t_{j+1}$.

**Proof.** First we remark that the following holds:

$$t_{i-1} + b - t_j + t_{j+1} = t_{i-1} + (c_1 - 1)t_1 + (c_2 - 2)t_2 + \cdots + (c_j - 2)t_j.$$
Step 1. We show by induction on \( \kappa \) that
\[
\Theta[\kappa P_j] = t_{i-1} + \kappa t_j
\]
holds for any \( 1 \leq \kappa \leq c_j - 1 \). Let \( \kappa = 1 \). By the induction hypothesis for (i), we see that
\[
\Theta[P_j - 1] = \Theta[P_{j-1} - P_{j-2} - 1] + (c_{j-1} - 1)t_{j-1}.
\]
Taking Remark 6.13.1 into account, either
\[
\Theta[P_{j-1} - P_{j-2}] = \Theta[P_{j-1} - P_{j-2} - 1] + t_1
\]
or
\[
\Theta[P_{j-1} - P_{j-2}] = \Theta[P_{j-1} - P_{j-2} - 1] + t_1 - b
\]
holds. On the other hand, we have \( \Theta[P_{j-1} - P_{j-2}] = M_{j-2} \) by the induction hypothesis for (ii), and therefore
\[
\Theta[P_{j-1} - P_{j-2}] - t_1 = t_{i-1} + (c_1 - 2)t_1 + \cdots + (c_{j-2} - 2)t_{j-2} > 0.
\]
Since \( \Theta[P_{j-1} - P_{j-2} - 1] < b \), it follows that \( \Theta[P_{j-1} - P_{j-2}] = \Theta[P_{j-1} - P_{j-2} - 1] + t_1 \). Therefore, we have
\[
\Theta[P_j - 1] = t_{i-1} + (c_1 - 2)t_1 + \cdots + (c_{j-2} - 2)t_{j-2} + (c_{j-1} - 1)t_{j-1},
\]
and hence \( \Theta[P_j] = t_{i-1} + t_j \) by Lemma 6.14. Next, let \( \kappa > 1 \). We first show that \( \Theta[\kappa P_j + \varepsilon] = \Theta[\varepsilon] + (\kappa - 1)t_j \) holds for any \( 1 < \varepsilon < P_j \). Since we have \( t_{i-1} + (\kappa - 1)t_j = \Theta[\varepsilon] \) by the induction hypothesis for Step 1, it suffices to check that \( \Theta[\varepsilon] + (\kappa - 1)t_j < b \) holds concerning Remark 6.13.1. Indeed, we have
\[
b - \{\Theta[\varepsilon] + (\kappa - 1)t_j\} \geq b - \{M_{j-1} + (c_j - 2)t_j\}
= (c_{j+1} - 2)t_{j+1} + \cdots + (c_l - 2)t_l + (t_l - t_{l+1}) - t_{i-1}
 \geq t_l + (t_l - t_{l+1}) - t_{i-1} > 0.
\]
Taking \( \varepsilon = P_j - 1 \), we obtain \( \Theta[\kappa P_j - 1] = \Theta[P_j - 1] + (\kappa - 1)t_j \). Therefore, we have \( \Theta[\kappa P_j - 1] + t_1 - b = t_{i-1} + \kappa t_j > 0 \), and hence \( \Theta[\kappa P_j] = t_{i-1} + \kappa t_j \).

Step 2. In this step, we prove that
\[
\Theta[(c_j - 1)P_j + \varepsilon] = \Theta[\varepsilon] + (c_j - 1)t_j
\]
holds for any \( 0 < \varepsilon < P_j - P_{j-1} \), which completes the proof of (i). If \( c_1 = \cdots = c_{j-1} = 2 \), then we have \( P_j - P_{j-1} = 1 \), and there is nothing to prove. Suppose otherwise. Then, as in Step 1, it is enough to show that
\[
\max \{\Theta[\varepsilon] : 0 < \varepsilon < P_j - P_{j-1}\} + (c_j - 1)t_j < b
\]
holds. Let $u = \max\{j' : 1 \leq j' \leq j - 1, c_{j'} > 2\}$. Then we have $P_j - P_{j-1} = P_{u+1} - P_{u} = (c_u - 2)P_u + (P_u - P_{u-1})$, and we see that

$$\max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_{u+1} - P_{u}\}$$

$$= \max \{(c_u - 3)t_u + M_{u-1}, (c_u - 2)t_u + \max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_u - P_{u-1}\}\}.$$  

Notice that

$$\max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_u - P_{u-1}\}$$

$$= \max \{(c_{u-1} - 3)t_{u-1} + M_{u-2}, (c_{u-1} - 2)t_{u-1} + \max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_{u-1} - P_{u-2}\}\},$$

and that

$$(c_u - 3)t_u + M_{u-1} - ((c_u - 2)t_u + (c_{u-1} - 3)t_{u-1} + M_{u-2}) = t_{u-1} - t_u > 0.$$  

These yield that

$$\max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_{u+1} - P_{u}\}$$

$$= \max \{(c_u - 3)t_u + M_{u-1}, (c_u - 2)t_u + \cdots + (c_2 - 2)t_2 + (c_1 - 2)t_1 + t_{i-1}\}$$

$$= (c_u - 3)t_u + M_{u-1}.$$  

Therefore,

$$b - \left\{\max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_j - P_{j-1}\} + (c_j - 1)t_j\right\}$$

$$= b - \{(c_u - 3)t_u + M_{u-1} + (c_j - 1)t_j\}$$

$$= t_u + (c_{u+1} - 2)t_{u+1} + \cdots + (c_j - 2)t_j + \cdots + (c_1 - 2)t_1 + t_l - t_{j+1} - t_{j-1} - (c_j - 1)t_j$$

$$\geq t_u + t_l + t_j - t_{i+1} - t_j - t_{i-1} > 0.$$  

This completes the proof of (i).

**Step 3.** In this last step, we give the proof of (ii). First, we show that $M_j = t_{i-1} + b - t_j + t_{j+1}$. Note that we have

$$M_j = \max \{M_{j-1}, \max \{\Theta[x] : P_j \leq x < P_{j+1}\}\}.$$  

Set

$$M_A = \max \{\Theta[x] : P_j \leq x < (c_j - 1)P_j\},$$

and

$$M_B = \max \{\Theta[x] : (c_j - 1)P_j \leq x < P_{j+1}\}.$$  

Then we see that

$$M_A = (c_j - 2)t_j + M_{j-1} = t_{i-1} + b - t_j + t_{j+1}$$

and that

$$M_B = (c_j - 1)t_j + \max \{\Theta[\varepsilon] : 0 \leq \varepsilon < P_j - P_{j-1}\}.$$
Therefore it follows that \( M_j = \max \{ M_{j-1}, \max \{ M_A, M_B \} \} = \max \{ M_A, M_B \} \). If \( c_1 = \cdots = c_{j-1} = 2 \), then we have \( M_B = (c_j - 1)t_j + t_{j-1} \), and hence \( M_A - M_B = b - t_{j-1} > 0 \). Thus, we get \( M_j = M_A \). Suppose that \( c_{j'} > 2 \) holds for some \( 1 \leq j' \leq j - 1 \), and take \( u \) as in Step 2. Then, we have \( M_B = (c_j - 1)t_j + (c_u - 3)t_u + M_{u-1} \). In a similar manner as above we see that \( M_A - M_B \geq t_u - t_j > 0 \), and therefore \( M_j = M_A \). Finally, we show \( M_j = \Theta[P_{j+1} - P_j] \). Since \( t_{i-1} + (c_j - 2)t_j = \Theta[(c_j - 2)P_j] \) and \( (c_j - 2)t_j + \Theta[P_j - P_{j-1}] = (c_j - 2)t_j + M_{j-1} = M_j < b \), it follows that

\[
(c_j - 2)t_j + \Theta[P_j - P_{j-1}] = \Theta[(c_j - 2)P_j + P_j - P_{j-1}] = \Theta[P_{j+1} - P_j].
\]

This completes the proof of the lemma. Q.E.D.

We can show the next lemma by following a similar way as in Lemma \[6,15\]

**Lemma 6.16.** Let \( P_i \leq x < P_{i+1} - P_i = P_i - P_{i-1} \), and denote by \( \kappa \) (resp. \( \epsilon \)) the quotient (resp. the remainder) of \( x \) divided by \( P_i \), i.e., \( x = \kappa P_i + \epsilon \). Then, we have \( \Theta[x] = \Theta[\epsilon] + \kappa t_i \). In particular, we have \( \Theta[x] > t_{i-1} \).

As an immediate consequence of Lemmas \[6.15\] and \[6.16\] we get the following.

**Corollary 6.17.** With the assumptions (C1) and (C2), we have \( \Theta[x] \geq t_{i-1} \) for any \( 0 \leq x < P_i - P_{i-1} \). Moreover, we have \( \Theta[x] = t_{i-1} \) if and only if \( x = 0 \).

**Corollary 6.18.** Let \( 1 \leq i \leq r + 1 \). Then we have \( \text{Rem}[tx, b] \leq b + t_i - t_{i-1} \) for any \( 0 < x < P_i \).

**Proof.** We have \( P_{j-1} \leq x < P_j \) for some \( 1 < j < i \). If \( c_1 = \cdots = c_{j-1} = 2 \), then \( P_j = j' \), \( Q_j = j' - 1 \), and \( t_j = (j' - 1)b \) hold for any \( 1 \leq j' \leq j \). It follows that \( x = j - 1 \) and \( \text{Rem}[t(j - 1), b] = t_{j-1} \leq t = b + t_i - t_{i-1} \). Next, suppose that we have \( c_l > 2 \) and \( c_{l+1} = \cdots = c_{j-1} = 2 \) for some \( 1 \leq l \leq j - 1 \). Then we have \( \text{Rem}[tx, b] = \Theta[x - P_{j-1}] \). By the proof of Lemma \[6.15\] we see that the following holds:

\[
\max \{ \Theta[y] : 0 \leq y < P_j - P_{j-1} = P_{i+1} - P_i \} = (c_l - 3)t_l + M_{i-1}.
\]

Therefore,

\[
b + t_i - t_{i-1} - \text{Rem}[tx, b] \geq b + t_i - t_{i-1} - \{(c_l - 3)t_l + M_{i-1}\}
\]

\[
\geq (c_{i-1} - 2)t_{i-1} - (c_l - 3)t_l > 0.
\]

Q.E.D.

**Corollary 6.19.** Let \( 1 \leq i \leq r + 1 \). Then, we have \( \text{Rem}[t P_i, b] = t_i \). Moreover, we have \( \text{Rem}[tx, b] \geq t_{i-1} \) for any \( 0 < x < P_i \).
Proof. We have seen that Rem\([tP_i, b] = t_i\) holds for any \(i\). Let \(i > 1\). As in the proof of Corollary 6.18 we have \(P_{j-1} < x < P_j\) for some \(1 < j < i\). If \(c_1 = \cdots = c_{j-1} = 2\), then we have \(P_j = j\), and hence Rem\([tx, b] = \text{Rem}[tP_j, b] = t_{j-1} \geq t_{i-1}\). Otherwise, we have Rem\([tx, b] = \Theta[x - P_{j-1}] \geq t_{j-1}\) by Corollary 6.17. Q.E.D.

Proof of Proposition 6.13 Let \(\lambda = (n_i, c, \omega) \in \Lambda_{(n_i, 0)}\), and write \(f_{\lambda} = X^{d_0}_0 X^{d_1}_1 X^{d_3}_3\). First, suppose that \(i = 0\). By Example 6.4 we have \(R_{(n_0, 0)} = R^0_{n_0} \oplus \left( \bigoplus_{c > 0} R^c_{n_0} \right)\) and \(R^0_{n_0} = \langle X_0^{n_0}, X_1 X_3 \rangle\). Therefore, it follows from Corollary 6.19 that Rem\((f_{\lambda}) = \Theta\). Next, suppose that \(1 \leq i \leq r\). By Example 6.4 we have

\[
R_{(n_i, 0)} = R^0_{n_i} \oplus \left( \bigoplus_{c \leq x \leq P_i} R^c_{n_i} \right) \oplus R_{n_i}^{mP_i} \oplus \left( \bigoplus_{c \leq x \leq P_i} R^c_{n_i} \right)
\]

We may assume that neither \(c = 0\) nor \(c = mP_i\). If \(\omega > \omega(n_i, c)\), then we have \(f_{\lambda} \in (X^{n_i}_0)\) by Lemma 6.7 since \(d_0 = n_0 + \omega\). Suppose that \(\omega = \omega(n_i, c)\). Then we have \(q - p + pd_1 - qd_3 = d_0 = n_0 + \omega < q - p\), and thus \(d_1 > 0\) and \(d_3 > 0\). Therefore, \(f_{\lambda} \in (X_1 X_3)\).

Next, suppose that \(1 \leq i \leq r\). By Example 6.4 we have

\[
R_{(n_i, 0)} = R^0_{n_i} \oplus \left( \bigoplus_{c \leq x \leq P_i} R^c_{n_i} \right) \oplus R_{n_i}^{mP_i} \oplus \left( \bigoplus_{c \leq x \leq P_i} R^c_{n_i} \right)
\]

We may assume that neither \(c = 0\) nor \(c = mP_i\). If \(\omega > \omega(n_i, c)\), then we have \(f_{\lambda} \in (X^{n_i}_0)\) as above. Thus, concerning that \(R_{(n_i, mP_i, \omega(n_i, mP_i))} = (X^{n_i}_1 X^{l_i}_3)\), we are left to show that if \(\omega = \omega(n_i, c)\) then \(f_{\lambda} \in (X^{n_i}_0, X^{n_i}_1 X^{l_i}_3)\). We first consider the case when \(0 < x < P_i\), and show that \(\omega(n_i, c) \geq 0\), which implies that \(f_{\lambda} \in (X^{n_i}_0)\). By Corollary 6.11 we have \(\omega(n_i, c) \geq 0\) if and only if Rem\([pc, q - p] + n_i < q - p\). Note that we have

\[
\text{Rem}[pc, q - p] + n_i < q - p \iff \text{Rem}[pc + n_i, q - p] \geq n_i
\]

\[
\iff \text{Rem}\left\lfloor \frac{pc + n_i}{k}, b \right\rfloor \geq t_i.
\]

We also have

\[
\text{Rem}\left\lfloor \frac{pc + n_i}{k}, b \right\rfloor = \text{Rem}[t(P_i - x), b],
\]

since we see by using equations (5) and (6) that

\[
\frac{pc + n_i}{k} = x \{(\alpha + 1)b - t\} + (tP_i - bQ_i) \equiv t(P_i - x) \pmod{b}.
\]

Therefore it follows from Corollary 6.19 that \(\omega(n_i, c) \geq 0\). Next, we consider the case when \(x > P_i\) and show that \(f_{\lambda} \in (X^{n_i}_1 X^{l_i}_3)\). Set \(\omega' = -n_i + q(c - mP_i)\), and \(\omega'' = -n_i + p(c - mP_i)\). First, suppose that \(d_1 < e_i\). Then we have \(qc - \omega(n_i, c) <
\[(q-p)e_i = n_i + qmP_i, \text{ and hence } \omega(n_i, c) > \omega'. \] It follows that \[0 \leq pc - \omega(n_i, c) < pc - \omega' = n_i + qmP_i - c(q-p). \] Therefore, all of the following are positive integers:

\[n_i + \omega' = q(c - mP_i), \quad \frac{qc - \omega'}{q-p} = \frac{n_i + qmP_i}{q-p} - c.\]

This implies that \((n_i, c, \omega') \in \tilde{\mu}^{-1}(n_i, c), \) which contradicts to the minimality of \(\omega(n_i, c). \) Next, suppose that \(d_3 < l_i. \) Then we have \(pc - \omega(n_i, c) < (q-p)l_i = n_i + pmP_i,\) and hence \(\omega(n_i, c) > \omega''.\) In a similar manner, we see that this implies \((n_i, c, \omega'') \in \tilde{\mu}^{-1}(n_i, c),\) which is a contradiction. Therefore, \(d_1 \geq e_i \) and \(d_3 \geq l_i. \) Q.E.D.

**Corollary 6.20.** We have \(\text{Rem}[pmx + n_i, q - p] = n_i + \text{Rem}[pmx, q - p]\) for any \(0 < x < P_i.\)

**Proof.** By the proof of Proposition 6.13, we see that \(\text{Rem}[pmx + n_i, q - p] \geq n_{i-1}.\) On the other hand, we have

\[
\text{Rem}[pmx + n_i, q - p] = \begin{cases} 
  n_i + \text{Rem}[pmx, q - p] & \text{(if } n_i + \text{Rem}[pmx, q - p] < q - p) \\
  n_i + \text{Rem}[pmx, q - p] - q + p & \text{(otherwise).}
\end{cases}
\]

Therefore we deduce that \(\text{Rem}[pmx + n_i, q - p] = n_i + \text{Rem}[pmx, q - p],\) since otherwise we have \(n_{i-1} \leq n_i + \text{Rem}[pmx, q - p] - q + p < n_i < n_{i-1}.\) Q.E.D.

### 7. Second step towards the proof of Theorem 4.6

In this section, we construct an equivariant morphism

\[\Psi : \mathcal{H} \longrightarrow E_{l,m} \times \mathbb{P}(V^\vee)\]

that satisfies \(\Psi(\mathcal{H}^{\text{main}}) = \Phi(E_{l,m}^\vee) \cong E_{l,m}^\vee (\text{Proposition 7.3}).\) First, we see by Theorem 6.1 that we can construct an equivariant morphism

\[\eta_{n_i,0} : \mathcal{H} \longrightarrow \text{Gr}(1, F_{n_i,0}^\vee) \cong \mathbb{P}(F_{n_i,0}^\vee)\]

for each \(0 \leq i \leq r.\) Set

\[\Delta := \gamma \times \prod_{0 \leq i \leq r} \eta_{n_i,0} : \mathcal{H} \longrightarrow E_{l,m} \times \prod_{0 \leq i \leq r} \mathbb{P}(F_{n_i,0}^\vee),\]

and let

\[\iota : \prod_{0 \leq i \leq r} \mathbb{P}(F_{n_i,0}^\vee) \longrightarrow \mathbb{P}(V^\vee)\]

be the Segre embedding, where \(V' := F_{n_0,0} \otimes F_{n_1,0} \otimes \cdots \otimes F_{n_r,0}.\) We see that \(V'\) coincides with

\[\bigoplus A(e_0) \otimes B(l_0) \otimes A(e_1) \otimes B(l_1) \otimes \cdots A(e_s) \otimes B(l_s) \otimes C(n_{j_1}) \otimes \cdots \otimes C(n_{j_u}),\]

where the sum runs over \(\{i_1, \ldots, i_s, j_1, \ldots, j_u\} = \{1, \ldots, r\}\) such that \(i_1 < \cdots < i_s\) and \(j_1 < \cdots < j_u.\)
Remark 7.0.1. As in Remark 3.2.1 we denote by $V(n)$ the irreducible $SL(2)$-representation of highest weight $n$. For any partition $n = \mu_1 + \cdots + \mu_s$, the tensor representation $V(\mu_1) \otimes \cdots \otimes V(\mu_s)$ contains an irreducible representation $V(\mu_1, \ldots, \mu_s)$ isomorphic to $V(n)$ by the Clebsch–Gordan theorem. For each $0 \leq i \leq n$, set

$$\phi_i := \frac{1}{n!} \sum_{i_1 + \cdots + i_s = i, \ 0 \leq i_1 \leq \mu_1, \ 0 \leq i_s \leq \mu_s} \left(\frac{\mu_1}{i_1} \right) \cdots \left(\frac{\mu_s}{i_s} \right) X^\mu_1 i_1 Y^{i_1} \otimes \cdots \otimes X^\mu_s i_s Y^{i_s} \in V(\mu_1) \otimes \cdots \otimes V(\mu_s).$$

Then, $\{\phi_0, \ldots, \phi_n\}$ forms a basis of $V(\mu_1, \ldots, \mu_s)$. On the other hand, we can take $\{X^{n-i} Y^i : 0 \leq i \leq n\}$ as a basis of $V(n)$, and the linear map

$$V(n) \to V(\mu_1, \ldots, \mu_s), \quad X^{n-i} Y^i \mapsto \phi_i$$

is an $SL(2)$-equivariant isomorphism.

Let us consider the submodule

$$\overline{V} := \bigoplus_{1 \leq i \leq r} A(e_0, e_1, \ldots, e_i) \otimes B(l_0, l_1, \ldots, l_i) \otimes C(n_{i+1}) \otimes \cdots \otimes C(n_r)$$

of $V'$, where $A(e_0, e_1, \ldots, e_i) \cong V(e_0, e_1, \ldots, e_i)$ (resp. $B(l_0, l_1, \ldots, l_i) \cong V(l_0, l_1, \ldots, l_i)$) stands for the irreducible representation of highest weight $e_0 + e_1 + \cdots + e_i$ (resp. $l_0 + l_1 + \cdots + l_i$) contained in $A(e_0) \otimes A(e_1) \otimes \cdots \otimes A(e_i)$ (resp. $B(l_0) \otimes B(l_1) \otimes \cdots \otimes B(l_i)$) in the sense of Remark 7.0.1 Since $V \subset \Gamma(E_{l,m}^{r}, \rho(\delta))$ coincides with

$$\bigoplus_{1 \leq i \leq r} A(e_0 + e_1 + \cdots + e_i) \otimes B(l_0 + l_1 + \cdots + l_i) \otimes C(-(n_0 + n_1 + \cdots + n_i)),$$

we see that $V \cong \overline{V}$, where the isomorphism

$$C(-(n_0 + n_1 + \cdots + n_i)) \cong C(n_{i+1} + \cdots + n_r) \cong C(n_{i+1}) \otimes \cdots \otimes C(n_r)$$

is given by multiplying $X^{n_0+n_1+\cdots+n_r}$.

Example 7.1. Let $l = p/q = 1/4$, and $m = 2$ as in Example 6.2 Then, we have $k = 1$, $a = 2$, $b = 3$, $\alpha = 0$, $\beta = 2$, and $t = 1$. Therefore, the Hirzebruch–Jung continued fraction of $b/t$ is $b/t = c_1 = 3$, and we have $P_0 = 0$, $Q_0 = -1$, $P_1 = 1$, $Q_1 = 0$, $P_2 = c_1 = 3$, and $Q_2 = 1$. Thus, we get $\rho_0 = u_2, \rho_1 = -u_1 + u_2$, and $\rho_2 = -3u_1 + 2u_2$, and the maximal cones of the colored fan of $E_{1/4,2}^{r}$ are the following:

$$\mathcal{C}_1 = Q \geq 0 \rho_0 + Q \geq 0 \rho_1, \quad \mathcal{C}_2 = Q \geq 0 \rho_1 + Q \geq 0 \rho_2.$$
Also, we have \((e_0, l_0, n_0) = (1, 1, 3), (e_1, l_1, n_1) = (3, 1, 1)\), and \((e_2, l_2, n_2) = (8, 2, 0)\). Thus we get \(f_0 = ZW, f_1 = Z^4W^2\), and \(f_2 = Z^{12}W^4\) by definition, and therefore
\[
V = \langle (SL(2) \times \mathbb{C}^*) \cdot ZW \rangle \oplus \langle (SL(2) \times \mathbb{C}^*) \cdot Z^4W^2 \rangle
\]

\[
\cong \langle X, Z \rangle \otimes \langle Y, W \rangle \oplus \langle X^4, X^3Z, X^2Z^2, XZ^3, Z^4 \rangle \otimes \langle Y^2, YW, W^2 \rangle
\]

\[
\cong V(1) \otimes V(1) \oplus V(4) \otimes V(2).
\]

We have \(V' = F_{n_0,0} \otimes F_{n_1,0}\), where
\[
F_{n_0,0} = A(1) \otimes B(1) = \langle X_1, X_2 \rangle \otimes \langle X_3, X_4 \rangle
\]
and
\[
F_{n_1,0} = A(3) \otimes B(1) \oplus C(1) = \langle X_1^3, X_1^2X_2, X_1X_2^2, X_2^3 \rangle \otimes \langle X_3, X_4 \rangle \oplus \langle X_0 \rangle.
\]

Furthermore, we have \(\tilde{V} = A(1, 3) \otimes B(1, 1) \oplus A(1) \otimes B(1)\), where \(A(1, 3)\) is a sub-representation of \(A(1) \otimes A(3)\) spanned by the following vectors:
\[
\begin{align*}
X_1 \otimes X_3, & \quad \frac{1}{4}(X_2 \otimes X_3^3 + 3X_1 \otimes X_1^2X_2), & \quad \frac{1}{2}(X_2 \otimes X_1^2X_2 + X_1 \otimes X_1X_2^3), \\
& \quad \frac{1}{4}(3X_2 \otimes X_1X_2^3 + X_1 \otimes X_2^3), & \quad X_2 \otimes X_3^3.
\end{align*}
\]

Also, \(B(1, 1)\) is a subrepresentation of \(B(1) \otimes B(1)\) spanned by the following vectors:
\[
X_3 \otimes X_3, \quad \frac{1}{2}(X_3 \otimes X_4 + X_4 \otimes X_3), \quad X_4 \otimes X_4.
\]

Now, set
\[
\Psi' := (id_{E_{l,m} \times t}) \circ \Delta : \mathcal{H} \longrightarrow E_{l,m} \times \mathbb{P}(V''),
\]
and consider the projection
\[
pr : E_{l,m} \times \mathbb{P}(V'') \longrightarrow E_{l,m} \times \mathbb{P}(\tilde{V}').
\]

**Proposition 7.2.** The restriction \(pr |_{\Psi' \mathcal{H}}\) of the rational map \(pr\) to the image of \(\Psi'\) is a morphism.

**Proof.** Let
\[
[(X_2X_4)^\vee : (X_1X_4)^\vee : (X_2X_3)^\vee : (X_1X_3)^\vee]
\]
and
\[
[(X_0^{n_i})^\vee : (X_2^{e_0}X_4^{l_i})^\vee : \cdots : (X_2^{e_0}X_1^{l_i}X_4^{l_i})^\vee : \cdots : (X_1^{e_0}X_3^{l_i})^\vee] \quad (1 \leq i \leq r)
\]
be the coordinate of \(\mathbb{P}(F_{n_0,0}^\vee)\) and \(\mathbb{P}(F_{n_1,0}^\vee)\), respectively. Suppose that there is a point \([I] \in \mathcal{H}\) such that \(pr\) is not defined at \(\Psi'([I])\). Let
\[
\eta_{n_0,0}([I]) = [t_{0,0}^{(0)} : t_{e_0,0}^{(0)} : t_{0,l_0}^{(0)} : t_{e_0,l_0}^{(0)}]
\]
and
\[
\eta_{n_1,0}([I]) = [u^{(i)} : t_{0,0}^{(i)} : \cdots : t_{e_1}^{(i)} : \cdots : t_{e_2}^{(i)}] \quad (1 \leq i \leq r).
\]
By (2), we have $s_1X_1 + s_2X_2 \in I$ for some $(s_1, s_2) \neq (0, 0)$. Since $\Psi'$ is $SL(2)$-equivariant, we may assume that $X_2 \in I$. The subrepresentation $A(e_0, e_1, \ldots, e_r) \otimes B(l_0, l_1, \ldots, l_r) \subset \tilde{V}$ contains $X_1^{e_0} \otimes X_1^{e_1} \otimes \cdots \otimes X_1^{e_r} \otimes X_3^{l_0} \otimes X_3^{l_1} \otimes \cdots \otimes X_3^{l_r}$, and therefore we have $t_{(0)}^{(0)} t_{(1)}^{(1)} \cdots t_{(r)}^{(r)} = 0$ by the assumption on the ideal $I$. Let $j = \min \{ i : i_{e_i, l_i} = 0, 0 \leq i \leq r \}$. Then we have $X_1^{e_j} X_3^{l_j} \in I$ by the construction of $\eta_{n_j, 0}$, and hence $X_1^{e_i} X_3^{l_i} \in I$ for every $i \geq j$. Next we have $s_3X_3 + s_4X_4 \in I$ for some $(s_3, s_4) \neq (0, 0)$ by (3). Namely, one of the following holds: (a) $s_3 \neq 0$, $s_4 \neq 0$; (b) $s_3 = 0$, $s_4 \neq 0$; (c) $s_3 \neq 0$, $s_4 = 0$. Suppose that we are in the case (a). Then, by multiplying $X_1^{e_j} X_3^{l_j-1}$ to $s_3 X_3 + s_4 X_4$, we get $X_1^{e_j} X_3^{l_j-1} X_4 \in I$. By continuing in this way, we finally obtain

$$X_1^{e_j-e} X_2 X_3^{l_j-1} X_4 \in I \quad (0 \leq e \leq e_j, \quad 0 \leq l \leq l_j)$$

centering $X_2 \in I$. Lastly, we pay attention to the vector

$$X_1^{e_0} \otimes X_1^{e_1} \otimes \cdots \otimes X_1^{e_j-1} \otimes X_3^{l_0} \otimes X_3^{l_1} \otimes \cdots \otimes X_3^{l_j-1} \otimes X_3^{n_j} \otimes \cdots \otimes X_3^{n_r}$$

centered in the following subrepresentation of $\tilde{V}$:

$$A(e_0, e_1, \ldots, e_{j-1}) \otimes B(l_0, l_1, \ldots, l_{j-1}) \otimes C(n_j) \otimes \cdots \otimes C(n_r).$$

Likewise, we have $t_{(0)}^{(0)} t_{(1)}^{(1)} \cdots t_{(j-1)}^{(j-1)} u^{(j)} \cdots u^{(r)} = 0$ by the assumption on $I$. This implies that $u^{(j)} \cdots u^{(r)} = 0$ by the minimality of $j$, and therefore we have $X_3^{n_j} \in I$. Thus, we get $F_{n_j, 0} \subset I$. Then it follows from Theorem 6.1 that $\dim(\mathbb{C}[H_{q-p}]/I_{(n_j, 0)}) = 0$, which contradicts to $[I] \in \mathcal{H}$. Q.E.D.

Combining the above discussion, we obtain the following equivariant morphism:

$$\Psi : \mathcal{H} \xrightarrow{\Psi} E_{l,m} \times \mathbb{P}(V') \xrightarrow{pr} E_{l,m} \times \mathbb{P}(\tilde{V}') \xrightarrow{\sim} E_{l,m} \times \mathbb{P}(V').$$

**Proposition 7.3.** We have $\Psi(\mathcal{H}^{main}) = \Phi(E_{l,m}^{\widetilde{\cdot}})$.

**Proof.** Let $y \in E_{l,m}^{\widetilde{\cdot}}$ be the fiber of $\pi(x) \in \mathbb{U} \subset E_{l,m}$ under the canonical birational morphism $E_{l,m}^{\widetilde{\cdot}} \to E_{l,m}$, where $x = (1, 1, 0, 0, 1) \in H_{q-p}$. Then, concerning Remark 3.3.1, we have $\Phi(y) = (\pi(x), v)$, where $v$ is a point in $\mathbb{P}(V')$ whose coordinates are all 0 except for the ones corresponding to the bases

$$(X_1^{e_0+e_1+\cdots+e_r} \mathsf{W}_l^{l_0+l_1+\cdots+l_r}) \quad (1 \leq i \leq r).$$

On the other hand, it follows from the definition of $I_1$ and the construction of $\eta_{n, 0}$ that

$$\eta_{n, 0}(I_1) = \langle (X_1 X_4) \rangle, \quad \eta_{n, 0}(I_1) = \langle (X_1^{e_i} X_3^{l_i}) \rangle \quad (1 \leq i \leq r).$$

Therefore, we get $\Psi([I_1]) = \Phi(y)$, and hence the proposition. Q.E.D.
Summarizing, we get the following equivariant commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\Psi} & E_{l,m} \times \mathbb{P}(V^\vee) \\
\mathcal{H}_{\text{main}} & \xrightarrow{\Psi|_{\mathcal{H}_{\text{main}}}} & \Phi(E'_{l,m}) \cong E_{l,m} \\
\mathcal{H}_{\text{main}} & \xrightarrow{\gamma|_{\mathcal{H}_{\text{main}}}} & E'_{l,m} \\
\end{array}
\]

8. Calculation of ideals

For each \(1 \leq i \leq r\), we consider the ideals

\[J_i^1 := (X_{0}^{n_i-1}, X_2, X_4, X_{0}^{n_i} - X_{1}^{e_i} X_{3}^{l_i}) + K\]

and

\[J_0^i := (X_{0}^{n_i-1}, X_2, X_4, X_{1}^{e_i} X_{3}^{l_i}) + K\]

of \(A = \mathbb{C}[X_0, X_1, X_2, X_3, X_4]\), where \(K\) is the ideal generated by elements of the form:

\[X_0^{pu_1-qq_2}X_1^{u_1}X_3^{u_2}, \quad (u_1, u_2) \in M_{l,m}^+ \setminus \{(0, 0)\}.
\]

Also, we define

\[J_{r+1}^1 := (X_{0}^{n_r}, X_2, X_4, X_{0}^{n_{r+1}} - X_{1}^{e_{r+1}} X_{3}^{l_{r+1}}) = (X_{0}^{k}, X_2, X_4, 1 - X_{1}^{aq} X_{3}^{ap}),\]

and

\[J_{r+1}^0 := (X_{0}^{n_r}, X_2, X_4, X_{1}^{e_{r+1}} X_{3}^{l_{r+1}}) = (X_{0}^{k}, X_2, X_4, X_{1}^{aq} X_{3}^{ap}).\]

We will see in §9 that every ideal of a closed point in \(\mathcal{H}_{\text{main}}\) can be described as an \(SL(2)\)-translate of \(I_1, I_0, J_1^i, J_0^i, J_{r+1}^1, \text{ or } J_{r+1}^0\).

**Remark 8.0.1.** Let us define \(F_j = f_{(0,m_j,\omega_{(0,m_j)})}\) for each \(1 \leq j \leq b - 1\). Then, \(J_1^i\) and \(J_0^i\) coincide with

\[(X_{0}^{n_i-1}, X_2, X_4, X_{0}^{n_i} - X_{1}^{e_i} X_{3}^{l_i}, F_1, \ldots, F_{b-1})\]

and

\[(X_{0}^{n_i-1}, X_2, X_4, X_{1}^{e_i} X_{3}^{l_i}, F_1, \ldots, F_{b-1}),\]

respectively.
Example 8.1. Let \( l = p/q = 1/4, \) and \( m = 2 \) as in Examples 6.2 and 7.1. Then we have \( F_1 = X_0^2X_1^2 \) and \( F_2 = X_0X_1^5X_3, \) and the ideals in consideration are described as follows:

\[
I_1 = (X_0^3 - X_1X_4, X_2, 1 - X_0^2X_1^2); \\
I_2 = (X_0^3 - X_1X_4, X_2, X_0^2X_1^2); \\
J_1^1 = (X_0^3, X_2, X_4, X_0 - X_1X_3, X_0X_1^5X_3); \\
J_1^2 = (X_0^3, X_2, X_4, X_0X_1^5X_3); \\
J_2 = (X_0, X_2, X_4, 1 - X_1^8X_3^2); \\
J_3 = (X_0, X_2, X_4, X_1^8X_3^2).
\]

Set \( \tilde{K} := K/(X_2, X_4). \) For each \( 1 \leq i \leq r + 1, \) we define

\[
\tilde{J}_0^i := J_0^i/(X_2, X_4) = (X_0^{n_i-1}, X_1^{n_i}, X_3^i) + \tilde{K} \subset R
\]

and

\[
\tilde{J}_1^i := J_1^i/(X_2, X_4) = (X_0^{n_i-1}, X_0^{n_i}, X_1^{n_i}, X_3^i) + \tilde{K} \subset R.
\]

Theorem 8.2. Let \( 1 \leq i \leq r + 1. \) Then, \( \dim(A/J_0^i(n,d)) = \dim(R/\tilde{J}_0^i(n,d)) \leq h(n,d) \) holds for any weight \( (n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}. \)

Theorem 8.3. Let \( 1 \leq i \leq r + 1. \) Then, \( \dim(A/J_1^i(n,d)) = \dim(R/\tilde{J}_1^i(n,d)) \leq h(n,d) \) holds for any weight \( (n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}. \)

The proof of Theorems 8.2 and 8.3 will be given after preparing a few lemmas.

Lemma 8.4. Let \( \lambda = (n, c, \omega) \in \Lambda(n,0). \) If \( n \geq 0, \) \( \omega \geq 0 \) and \( c > 0, \) then \( f_A \in \tilde{K}. \)

Proof. Write \( f_A = X_0^{d_0}X_1^{d_1}X_3^{d_3}. \) Then we have \( f_A = X_0^n(X_0^{d_0-n}X_1^{d_1}X_3^{d_3}) \) concerning \( d_0 = n + \omega. \) The conditions \( f_A \in R(n,0) \) and \( X_0^n \in R(n,0) \) imply that \( X_0^{d_0-n}X_1^{d_1}X_3^{d_3} \in R^{G_0 \times G_m}. \) Since \( 0 < c = d_1 - d_3, \) it follows that \( X_0^{d_0-n}X_1^{d_1}X_3^{d_3} \in \tilde{K}. \) Q.E.D.

Lemma 8.5. Let \((n, c) \in \Lambda'. \) Assume that \( 0 \leq n < q - p, \) and that \( c \geq 0. \) Then the following properties are true.

(i) We have \( \omega(0,c) + n < q - p \) if and only if \( \omega(0,c) = \omega(n,c). \)

(ii) We have \( \omega(0,c) + n \geq q - p \) if and only if \( \omega(0,c) = \omega(n,c) + q - p. \)

(iii) We have \( \omega(0,c) \geq q - p - \beta \) if and only if \( \omega(0,c+m) = \omega(0,c) - q + p + \beta. \)

(iv) We have \( \omega(0,c) < q - p - \beta \) if and only if \( \omega(0,c+m) = \omega(0,c) + \beta. \)

Proof. First of all, concerning Lemmas 6.5 and 6.7, we see that \( 0 \leq \omega(0,c) < q - p \) holds. Since we have \( (n,c,\omega(0,c)) = \mu\left(n + \omega(0,c), \frac{qc-\omega(0,c)}{q-p}, \frac{pc-\omega(0,c)}{q-p}\right) \in \Lambda, \) it follows
that \((n, c, \omega_{(0, c)}) \in \tilde{\mu}^{-1}(n, c)\). The if part is easy to check, so we prove the only if part.

(i) follows from Lemma \([6.7]\).

(ii) If \(n + \omega_{(0, c)} \geq q - p\), then we have \(\omega_{(0, c)} - \omega_{(n, c)} = x(q - p)\) for some \(x \geq 1\). If \(x > 1\), then we have \(\omega_{(0, c)} > q - p\), which is a contradiction.

(iii) Set \(\omega = \omega_{(0, c)} - q + p + \beta\). Then we get \(0 \leq \omega < q - p\). Since we have

\[
\frac{q(c + m) - \omega}{q - p} = \frac{q(c - \omega_{(0, c)})}{q - p} + \alpha + m + 1 > 0, \quad \frac{p(c + m) - \omega}{q - p} = \frac{p(c - \omega_{(0, c)})}{q - p} + \alpha + 1 > 0,
\]

and

\[
(0, c + m, \omega) = \mu\left(\frac{q(c + m) - \omega}{q - p}, \frac{p(c + m) - \omega}{q - p}\right),
\]

it follows that \((0, c + m, \omega) \in \tilde{\mu}^{-1}(0, c + m)\). Therefore, we have \(\omega = \omega_{(0, c + m)}\).

(iv) Set \(\omega' = \omega_{(0, c)} + \beta\). In a similar way we see that \((0, c + m, \omega') \in \tilde{\mu}^{-1}(0, c + m)\), and therefore we have \(\omega' = \omega_{(0, c + m)}\).

Q.E.D.

The next lemma follows from Lemmas \([5.1]\) and \([6.7]\).

**Lemma 8.6.** Let \(\lambda = (n, c, \omega) \in \Lambda\). If \(\omega > \omega_{(n, c)}\), then we have \(f_{\lambda} \in (X^*_0)\) for any \(1 \leq i \leq r + 1\).

**Lemma 8.7.** Let \((0, c) \in \Lambda'\) with \(c = mx\), and suppose that we have \(0 < x < P_i\) for some \(1 \leq i \leq r + 1\). Then, \(n_{i-1} - n_i \leq \omega_{(0, c)} \leq q - p - n_{i-1}\).

**Proof.** Concerning the proof of Corollary \([6.11]\), we have \(\omega_{(0, c)} = \text{Rem}[pc, q - p]\), which coincides with \(\text{Rem}[pc + n_i, q - p] - n_i\) by Corollary \([6.20]\). On the other hand, we have \(n_{i-1} \leq \text{Rem}[pc + n_i, q - p] = k \text{Rem}[t(P_i - x), b] \leq q - p + n_i - n_{i-1}\) by \([14]\) and Corollaries \([6.18][6.19]\) and \([6.20]\) and hence the lemma.

**Q.E.D.**

**Definition 8.8.** For each \(c \in m\mathbb{Z}_{>0}\), we define \(\lambda_c := (q - p - \omega_{(0, c)}, c, \omega_{(0, c)} - q + p)\).

**Remark 8.8.1.** By a direct calculation, we see that

\[
f_{\lambda_c} = X_1^{\frac{qc - \omega_{(0, c)}}{q - p} + 1} X_3^{\frac{pc - \omega_{(0, c)}}{q - p} + 1}.
\]

Also, by applying Lemma \([8.5]\) (ii) with \(n = q - p - \omega_{(0, c)}\), we have \(\omega_{(0, c)} - q + p = \omega_{(q - p - \omega_{(0, c)}, c)}\).

**Example 8.9.** By Example \([6.9]\) and Lemma \([8.5]\) we have \(\omega_{(0, mP_i)} = q - p - n_i\), and therefore \(\lambda_{mP_i} = (n_i, mP_i, \omega_{(n_i, mP_i)})\) and \(f_{\lambda_{mP_i}} = X_1^{n_i} X_3^{mP_i}\).

**Lemma 8.10.** With the above notation, we have \(f_{\lambda_{c'}} \in (f_{\lambda_c})\) if \(c' \geq c\).
Lemma 8.12. Let \( c, c' \in m\mathbb{Z}_{>0} \), we may assume that \( c' = c + m \). Then by (5) and Lemma 8.5 we have

\[
\begin{align*}
  f_{\lambda_{c'}} &= \begin{cases} 
    X_1^{\alpha+m+1}X_3^{q+1}f_{\lambda_c} & \text{if } \omega_{(0,c)} \geq q - p - \beta \\
    X_1^{\alpha+m}X_3^{\alpha}f_{\lambda_c} & \text{(otherwise)}.
  \end{cases}
\end{align*}
\]

Q.E.D.

Corollary 8.11. Let \( \lambda = (n, c, \omega_{(n,c)}) \in \Lambda_{(n,0)} \), and assume that \( 0 < c \) and that \( 0 \leq n < q - p \). Then we have the following.

(i) If \( \omega_{(0,c)} + n < q - p \), then \( f_{\lambda} \in \tilde{K} \).
(ii) If \( \omega_{(0,c)} + n \geq q - p \), then \( f_{\lambda} = X_0^{n+\omega_{(n,c)}}f_{\lambda_c} = X_0^{n+\omega_{(0,c)}-q+p}f_{\lambda_c} \).

Proof. Item (i) follows from Corollary 6.8(i), Lemmas 8.5(i), and 8.4. Item (ii) is a consequence of Lemma 8.5(ii) and the definition of \( \lambda_c \). Q.E.D.

Lemma 8.12. Let \( \lambda = (n, c, \omega_{(n,c)}) \in \Lambda_{(n,0)} \) with \( c = mx \). Then, the following properties are true for any \( 1 \leq i \leq r + 1 \).

(i) If \( 0 < x < P_i \) and \( 0 \leq n < n_{i-1} \), then \( f_{\lambda} \in \tilde{K} \).
(ii) If \( x = P_i \) and \( 0 \leq n < n_i \), then \( f_{\lambda} \in \tilde{K} \).
(iii) If \( x = P_i \) and \( n_i \leq n < q - p \), then \( f_{\lambda} \in (X_1^{\alpha}X_3^{\lambda}). \)
(iv) If \( x > P_i \) and \( 0 \leq n < q - p \), then \( f_{\lambda} \in (X_1^{\alpha}X_3^{\lambda}) + \tilde{K} \).
(v) If \( x > P_i \) and \( 0 \leq n < n_i \), then \( f_{\lambda} \in \tilde{J}_1 \).
(vi) Let \( x > P_i \) and \( n_i \leq n < n_{i-1} \).
   (vi-1) If \( x \) is not a multiple of \( P_i \), then \( f_{\lambda} \in \tilde{J}_1 \).
   (vi-2) If \( x \) is a multiple of \( P_i \), then \( f_{\lambda} - f_{(n,c-mP_i,\omega_{(n,c-mP_i)})} \in \tilde{J}_1 \).

Proof. (i) follows from Corollary 6.11, Lemmas 8.4 and 8.7.

(ii) We have \( \omega_{(0,mP_i)} = q - p - n_i \) by Example 8.9, and therefore \( f_{\lambda} \in \tilde{K} \) by Corollary 8.11(i).

(iii) follows from applying Corollary 8.11(ii) with \( c = mP_i \).

(iv) follows from Lemma 8.10 and Corollary 8.11.

(v) Concerning Corollary 8.11 we may assume that \( n + \omega_{(0,c)} \geq q - p \). Set \( n' = n + \omega_{(0,c)} - q + p \). Then we have \( f_{\lambda} = X_0^{n'}f_{\lambda_c} \). Also, by Lemma 8.10, \( f_{\lambda_c} \) can be written as \( f_{\lambda_c} = f_{\lambda_{n'}}f = X_1^{\alpha}X_3^{\lambda} \) with some \( f \in R_{n-n'}. \). Therefore, \( f_{\lambda} = X_0^{n'+n_i}f(X_0^{n_i}X_1^{\alpha}) \). Now, since \( X_0^{n'+n_i}f \in R_{n-n'}. \), we have \( X_0^{n'+n_i}f = f_{\lambda'} \) with some \( \lambda' = (n, c - mP_i, \omega') \in \tilde{\mu}^{-1}(n, c - mP_i) \). If \( \omega' > \omega_{(n,c-mP_i)} \), then we have \( f_{\lambda'} \in (X_1^{n'-1}), \) and hence \( f_{\lambda} \in \tilde{J}_1 \). Suppose that \( \omega' = \omega_{(n,c-mP_i)} \). If \( 0 < c - mP_i \leq mP_i \),
then we have \( f_{\lambda'} \in \tilde{K} \) by (i) and (ii), and hence \( f_{\lambda} \in \tilde{J}_1 \). If \( c - mP_i > mP_i \), we can apply the same process to \( f_{\lambda'} \), and continuing in this way we finally obtain \( f_{\lambda} \in \tilde{J}_1 \).

(vi) is an immediate consequence of the proof of (v).

Q.E.D.

**Lemma 8.13.** Let \((n, c, \omega_{(n,c)}) \in \Lambda_{(n,0)} \) with \( c = mx \). Suppose that \( P_j < x < P_i \), \( n_j \leq n < n_{j-1} \), and \( n - n_j < n_{i-1} \) hold for some \( 1 \leq j \leq r + 1 \). Then, we have \( f_{\lambda} = X_0^{n-n_j} f_{\lambda_{mP_j}} f_{\lambda'} \), where \( \lambda' = (0, c - mP_j, \omega_{(0,c-mP_j)}) \). In particular, \( f_{\lambda} \in \tilde{K} \).

**Proof.** Set \( \lambda'' = (n - n_j, 0, 0) \). Then we have \( \lambda'' + \lambda_{mP_j} + \lambda' = (n, c, \omega_{(0,c-mP_j)} - n_j) \). Also, we see by Lemma 8.7 that \( n + \omega_{(0,c-mP_j)} - n_j < n + q - p - n_{i-1} - n_j < q - p \). Therefore we have \( \omega_{(0,c-mP_j)} - n_j = \omega_{(n,c)} \), and hence \( \lambda'' + \lambda_{mP_j} + \lambda' = \lambda \). Taking Lemma 6.5(ii) into account, it follows that \( f_{\lambda} = X_0^{n-n_j} f_{\lambda_{mP_j}} f_{\lambda''} \), since \( X_0^{n-n_j} = f_{\lambda''} \).

The last statement follows from \( f_{\lambda'} \in \tilde{K} \). Q.E.D.

**Proof of Theorem 8.2.** Set \( J = \tilde{J}_0 \), and let \( J^n = J \cap R^n \). Then we see that \( J_{n,d} = \bigoplus_{c \equiv d \pmod{m}} J_{n,d}^c \), and therefore we have

\[
R_{n,d}/J_{n,d} \cong \bigoplus_{c \equiv d \pmod{m}} R_n^c/J_n^c.
\]

Recall that \( R_n^c = \bigoplus_{\omega \geq \omega_{(n,c)}} R_{(n,c,\omega)} \). Since we have \( \bigoplus_{\omega \geq \omega_{(n,c)}} R_{(n,c,\omega)} \subset J \) by Lemma 8.6, it suffices to prove that

\[
\dim \left( \bigoplus_{c \equiv d \pmod{m}} R_{(n,c,\omega_{(n,c)})}/(R_{(n,c,\omega_{(n,c)})} \cap J) \right) \leq 1 \tag{15}
\]

holds for any weight \((n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \). We divide the proof into two steps.

**Step 1.** We show that (15) holds if \( 0 \leq n < q - p \) and \( d = 0 \). Let \( \lambda = (n, c, \omega_{(n,c)}) \in \Lambda_{(n,0)} \). Note that we have \( c \geq 0 \) by Example 6.4 and recall that every \( R_{(n,c,\omega_{(n,c)})} \) is 1-dimensional, namely \( R_{(n,c,\omega_{(n,c)})} = \langle f_{(n,c,\omega_{(n,c)})} \rangle \).

**Case 1 of Step 1.** Let \( 0 \leq n < n_{i-1} \). By Lemma 8.12, we see that \( f_{\lambda} \in J \) if \( c > 0 \). This implies (15).

**Case 2 of Step 1.** Let \( n_{i-1} \leq n < q - p \). By Lemma 5.1, there is a unique integer \( 1 \leq j_1 \leq i - 1 \) such that \( n_{j_1} \leq n < n_{j_1-1} \). If \( n - n_{j_i} \geq n_{i-1} \), then we can take \( 1 \leq j_2 \leq i - 1 \) uniquely to satisfy \( n_{j_2} \leq n - n_{j_i} < n_{j_2-1} \). By continuing in this way, we get \( n - (n_{j_1} + n_{j_2} + \cdots + n_{j_{m-1}} + n_{j_m}) < n_{i-1} \) for some \( 1 \leq j_1, j_2, \ldots, j_m \leq i - 1 \).
Namely, we have
\[ n_j \leq n < n_{j-1}, \quad n - n_j \geq n_{i-1}, \]
\[ n_j \leq n - n_{j-1} < n_{j-1}, \quad n - (n_j + n_{j-1}) \geq n_{i-1}, \]
\[ \ldots \]
\[ n_{j_{m-1}} \leq n - (n_j + \cdots + n_{j_{m-2}}) < n_{j_{m-1}}, \quad n - (n_j + \cdots + n_{j_{m-1}}) \geq n_{i-1}, \]
\[ n_{j_{m}} \leq n - (n_j + \cdots + n_{j_{m-1}}) < n_{j_{m-1}}, \quad n - (n_j + \cdots + n_{j_{m-1}} + n_{j_{m}}) < n_{i-1}. \]

In the following, we show (15) by induction on \( u_n \). Set \( u = u_n \) and \( P = P_{j_1} + \cdots + P_{j_n} \).

First suppose that \( u = 1 \). Since \( j_1 < i \), we have \( P < P_i \). We show that \( f_{\lambda} \in J \) holds if \( c \neq mP \). If \( c = 0 \), then we have \( f_{\lambda} = X_0^n \) by Example 6.4 and therefore \( f_{\lambda} \in J \). If \( 0 < c < mP \), then we have \( f_{\lambda} \in \tilde{J} \) by applying Lemma 8.13 (i) with \( i = j_1 \). If \( mP < c < mP_i \), then by applying Lemma 8.13 with \( j = j_1 \) we see that \( f_{\lambda} \in \tilde{J} \). If \( c \geq mP_i \), then we have \( f_{\lambda} \in (X_1^{c_i} X_2^{j_i}) + \tilde{J} \) by Lemma 8.12 (iii), (iv). Next suppose that \( u > 1 \). If \( c = 0 \), then \( f_{\lambda} \in (X_0^{n-1}) \). If \( 0 < c < mP_{j_1} \), then we have \( f_{\lambda} \in \tilde{J} \) as above.

Suppose now that \( c > mP_j \), and set \( P' = P - P_{j_1}, n' = n - n_{j_1}, c' = c - mP_{j_1} \), and \( \lambda' = (n', c', \omega_{(n', c')}) \). Since we have \( \omega_{(n_j, mP_{j_1})} + \omega_{(n', c')} + n_{j_1} + n' = \omega_{(n', c')} + n' < q - p \) by Example 6.9, it follows from Corollary 6.8 that \( \omega_{(n, c)} = \omega_{(n_j, mP_{j_1})} + \omega_{(n', c')} \). Thus we get \( \lambda = \lambda_{mP_{j_1}} + \lambda' \), and hence \( f_{\lambda} = f_{\lambda_{mP_{j_1}}} f_{\lambda'} \) by Lemma 6.5. Now, since we have \( u_{n'} = u - 1 \), it follows from the induction hypothesis and the relation \( f_{\lambda} = f_{\lambda_{mP_{j_1}}} f_{\lambda'} \) that (15) holds.

**Step 2.** In this step, we prove that (15) holds for an arbitrary weight \( (n, d) \). Let \( \lambda = (n, c, \omega_{(n, c)}) \in \Lambda_{(n, d)} \). Set \( n' = n + \omega_{(n', c')(n, d)}, \quad c' = c - c_{(n, d)}, \) and \( \lambda' = (n', c', \omega_{(n', c')}) \in \Lambda_{(n', 0)} \). Also, let
\[ \lambda'' = \mu \left( \frac{q c_{(n, d)} - \omega_{(n, c_{(n, d)})}}{q - p}, \frac{p c_{(n, d)} - \omega_{(n, c_{(n, d)})}}{q - p} \right) = (n - n', c_{(n, d)}), \omega_{(n, c_{(n, d)})}) \in \Lambda_{(n-n', d)}. \]

Since \( n - n' = -\omega_{(n', c')(n, d)}, \) we have \( \omega_{(n', c')} + \omega_{(n', c)(n, d)} + n' + n - n' = \omega_{(n', c')} + n' < q - p \).

As in Case 2 of Step 1, we see that \( f_{\lambda} = f_{\lambda'} f_{\lambda''} \). On the other hand we have \( 0 \leq n' < q - p \) by Lemma 6.7 and therefore \( \dim R_{(n', 0)}/J_{(n', 0)} \leq 1 \) by Step 1. This yields that \( \dim R_{(n, d)}/J_{(n, d)} \leq 1 \), since \( f_{\lambda''} \in R_{(n', 0)} \). Q.E.D.

**Remark 8.13.1.** Let \( \lambda = (n, c, \omega) \in \Lambda_{(n, 0)} \), where \( 0 \leq n < q - p \). In view of the proof of Theorem 8.2, we deduce the following.

- Suppose that \( 0 \leq n \leq n_{i-1} \). Then we have \( f_{\lambda} \in \tilde{J}_0 \) if \( \lambda \neq (n, 0, \omega_{(n, 0)}) \).
- Suppose that \( n_{i-1} \leq n < q - p \). Then we have \( f_{\lambda} \in \tilde{J}_0 \) if \( \lambda \neq (n, mP, \omega_{(n, mP)}) \).
**Proof of Theorem 8.3.** Set \( J = \bar{J}_1 \). As in the proof of Theorem 8.2, we show that

\[
\dim \left( \bigoplus_{c \equiv d \pmod{m}} R_{(n,c,\omega_{(n,c)})}/(R_{(n,c,\omega_{(n,c)})} \cap J) \right) \leq 1
\]  
(16)

holds for any weight \((n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \).

**Step 1.** In this step, we show that (16) holds if \( 0 \leq n < q - p \) and \( d = 0 \). Let \( \lambda = (n, c, \omega_{(n,c)}) \in \Lambda_{(n,0)} \).

**Case 1 of Step 1.** Let \( 0 \leq n < n_i \). If \( c > 0 \), then we have \( f_i \in J \) by Lemma 8.12 (i), (ii), and (v).

**Case 2 of Step 1.** Let \( n_i \leq n < n_{i-1} \). Taking

\[
f_{(n,mP,\omega_{(n,mP)})} - f_{(n,0,\omega_{(n,0)})} = X_0^{n-n_i}X_1^{e_1}X_3^l - X_0^n = X_0^{n-n_i}(X_1^{e_1}X_3^l - X_0^{n_i}) \in J
\]

into account, it follows from Lemma 8.12 (vi) that (16) holds.

**Case 3 of Step 1.** Let \( n_{i-1} \leq n < q - p \). As in Case 2 of the proof of Theorem 8.2, we have \( n - (n_j + n_j + \cdots + n_j) < n_{i-1} \) for some \( 1 \leq j_1, j_2, \ldots, j_u \leq i - 1 \). Set \( u = u_i \), and \( P = P_j + \cdots + P_j \). Suppose that \( u = 1 \). If \( 0 \leq c < mP \), then we see that \( f_i \in J \) in a similar way. Let \( c \geq mP \). Then we can write \( f_i = f_{\lambda} \cdot f_{i,mP} \), where \( \lambda = (n - n_j, c - mP, \omega_{(n-n_j,c-mP)}) \). If \( 0 \leq n - n_j < n_i \), then we see that (16) holds by applying Lemma 8.12 (i), (ii), and (v) for \( f_{\lambda} \). If \( n_i \leq n - n_j < n_{i-1} \), then (16) follows from a similar argument to the one we used in Case 2.

**Step 2.** By arguing as in Step 2 of the proof of Theorem 8.2 we deduce that

\[
\dim R_{(n,d)}/J_{(n,d)} \leq 1 \text{ holds for any } (n,d) \in \mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}.
\]

Q.E.D.

**Corollary 8.14.** The quotient ring \( A/J_1^{r+1} \) has Hilbert function \( h \).

**Proof.** We can easily see that \( \mathcal{D} \) is the \( SL(2) \times \mathbb{C}^* \)-orbit of \( \pi(x') \), where \( x' = (0,1,0,1,0) \in H_{q-p} \). Let \( [J] \in \gamma^{-1}(\mathcal{D}) \) be such that \( \gamma([J]) = \pi(x') \). Since \( (e_{r+1}, l_{r+1} = (aq, ap) \in M^{+}_{l,m} \), we have \( X_1^{e_{r+1}}X_3^{l_{r+1}} \in G^{r+1} \times G_m \) by Remark 6.11. Then by a similar argument as in the proof of [Kub18, Lemma 4.5], we see that \( (X_0^{q-p}, X_2, X_4, 1 - X_1^{e_{r+1}}X_3^{l_{r+1}}) \subset J \). Also, concerning \( X_2, X_4 \in J \), it follows from Theorem 6.1 that \( s_1X_0^k + s_2X_1^{e_{r+1}}X_3^{l_{r+1}} \in J \) holds for some \( (s_1, s_2) \neq 0 \). Since we have \( e_{r+1} \geq e_r \) and \( l_{r+1} \geq l_r \), the condition \( 1 - X_1^{e_{r+1}}X_3^{l_{r+1}} \in J \) implies that \( s_2 = 0 \). Therefore, we get \( J_1^{r+1} \subset J \), and hence \( \dim(A/J_1^{r+1})_{(n,d)} \geq \dim(A/J_{(n,d)}) = h(n,d) = 1 \). Taking Theorem 8.3 into account, we obtain \( \dim(A/J_1^{r+1})_{(n,d)} = h(n,d) \).

Q.E.D.

**Remark 8.14.1.** By the proof of Corollary 8.14 we have \( \gamma^{-1}(\pi(x')) = \{ [J_1^{r+1}] \} \). We will see in Corollary 8.3 that \( J_i \) and \( J_0^{l_i} \) have Hilbert function \( h \) for any \( 1 \leq i \leq r + 1 \).
9. Proof of Main Theorem

We have $\Psi(\mathcal{H}^{\text{main}}) \cong \overline{E'_{l,m}}$ by Proposition 7.3. Therefore, in order to complete the proof of Theorem 4.6, we are left to show that $\Psi|_{\mathcal{H}^{\text{main}}}$ is injective. Indeed, considering the fact that $E'_{l,m}$ is normal, it follows from the Zariski’s Main Theorem that $\Psi|_{\mathcal{H}^{\text{main}}}$ being injective implies that $\Psi|_{\mathcal{H}^{\text{main}}}$ being a closed immersion.

The weighted blow-up $E'_{l,m} \cong \varphi(E'_{l,m}) \subset E_{l,m} \times \mathbb{P}^1 \times \mathbb{P}^1$ contains the following four $SL(2) \times \mathbb{C}^*$-orbits:

- $\mathcal{U} \cong (SL(2) \times \mathbb{C}^*) \cdot (\pi(x),[1:0],[0:1])$, where $x = (1,1,0,0,1) \in H_{q-p}$;
- $\mathcal{D} \cong (SL(2) \times \mathbb{C}^*) \cdot (\pi(x'),[1:0],[1:0])$, where $x' = (0,1,0,1,0) \in H_{q-p}$;
- $\mathcal{C} \cong (SL(2) \times \mathbb{C}^*) \cdot (O,[1:0],[1:0])$;
- $\mathcal{C'} \cong (SL(2) \times \mathbb{C}^*) \cdot (O,[1:0],[0:1])$.

**Lemma 9.1.** $\psi|_{\mathcal{H}^{\text{main}}} : \mathcal{H}^{\text{main}} \to E'_{l,m}$ is bijective outside the closed orbit $C$.

**Proof.** We show the bijectivity orbit-wise. Taking the construction of $\psi$ and the proof of Corollary 8.14 into account, we see that $\psi([(J^r+1)]) = (\pi(x'),[1:0],[1:0])$, and that

$$
\psi^{-1}(\psi([(J^r+1)])) = \{[J] \in \mathcal{H} : X_2, X_4 \in J, \gamma([J]) = \pi(x')\} = \{[J^r+1]\}.
$$

Thus, $\psi|_{\mathcal{H}^{\text{main}}}$ is bijective over $\mathcal{D}$. Analogously, we see that

$$
\psi^{-1}(O,[1:0],[0:1]) = \{[I] \in \mathcal{H} : X_2, X_3 \in I, \gamma([I]) = O\} = \{[I] \in \mathcal{H} : I_0 \subset I\} = \{[I_0]\}
$$

concerning Theorem 4.1. Therefore, $\psi|_{\mathcal{H}^{\text{main}}}$ is bijective over $C'$. Q.E.D.

Recall that the toroidal spherical variety $\overline{E'_{l,m}}$ corresponds to the colored fan $\overline{\mathcal{X}(E'_{l,m})}$ having $(\mathcal{C}_i,\phi)$ ($1 \leq i \leq r+1$) as its maximal colored cones. By Remark 2.6.1 each colored cone in $\overline{\mathcal{X}(E'_{l,m})}$ corresponds bijectively to an $SL(2) \times \mathbb{C}^*$-orbit in $E'_{l,m}$. The closed orbit $Y_i$ ($1 \leq i \leq r+1$) corresponds to the maximal colored cone $(\mathcal{C}_i,\phi)$. For each $1 \leq i \leq r$, we denote by $O_i$ the $SL(2) \times \mathbb{C}^*$-orbit corresponds to the colored cone $(\mathbb{Q}_{\geq 0} \rho_i,\phi)$. Let

$$
g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2),
$$

and denote by $y_i$ (resp. $y'_i$) the point of $\mathbb{P}(V^\vee)$ whose coordinates are all 0 except for the one(s) corresponding to the basis $g \cdot f_{i-1}^\vee$ (resp. the bases $g \cdot f_{i-1}^\vee$ and $g \cdot f_i^\vee$). Then we have

$$
\Phi(Y_i) = (SL(2) \times \mathbb{C}^*) \cdot (O_i, y_i)
$$
and
\[ \Phi(O_i) = (SL(2) \times \mathbb{C}^*) \cdot (O, y_i'). \]

**Proposition 9.2.** \( \Psi|_{\text{main}} \) is injective.

**Proof.** Taking Lemma 9.1 into account, it suffices to show that each of the set-theoretical fibers of \( (O, y_i) \) and \( (O, y_i') \) consists of one point. We show that \( \Psi^{-1}(O, y_i) = [J_i] \) and \( \Psi^{-1}(O, y_i') = [\{J_i\}] \) hold. Let \( [J] \in \Psi^{-1}(O, y_i) \), and write its image under \( \eta_{n_0, 0} \) and \( \eta_{n_i, 0} \) as
\[ \eta_{n_0, 0}([J]) = [t_{0,0}^{(0)} : t_{e_0,0}^{(0)} : t_{e_0,0}^{(0)}] \]
and
\[ \eta_{n_i, 0}([J]) = [t_{e_i,l}^{(i)} : t_{e_i,l}^{(i)} : \cdots : t_{e_i,l}^{(i)}] \quad (1 \leq i \leq r) \]
as in the proof of Proposition 7.2. First, it follows from \( \gamma([J]) = O \) that \( K \subset J \).

Since we have
\[ g \cdot f_i = X_0^{e_0} X_1^{e_1} \cdots X_i^{e_i} Y_i^{l_i} = X_0^{-(n_0 + n_1 + \cdots + n_{i-1})} X_0^{e_0} X_1^{e_1} \cdots X_i^{e_i} Y_i^{l_i}, \]
we see that \( g \cdot f_i \) maps to
\[ X_1^{e_1} \otimes X_1^{e_2} \otimes \cdots \otimes X_i^{e_i} \otimes X_3^{l_1} \cdots \otimes X_3^{l_i} \otimes X_0^{n_0} \otimes \cdots \otimes X_0^{n_r} \]
under the isomorphism \( V \cong \mathbb{V} \). Therefore, by the definition of \( y_i' \), we have
\[ t_{e_i,l_i}^{(i)} \cdot t_{e_i,l_i}^{(i-1)} \cdot t_{e_i,l_i}^{(i)} \cdot \cdots \cdot t_{e_i,l_i}^{(i-1)} \cdot u^{(i)} \cdot u^{(r)} = s \]
for some \( s \in \mathbb{C}^* \). Similarly, by paying attention to the basis \( g \cdot f_i' \), we have
\[ t_{e_i,l_i}^{(i)} \cdot t_{e_i,l_i}^{(i-1)} \cdot t_{e_i,l_i}^{(i)} \cdot \cdots \cdot t_{e_i,l_i}^{(i-1)} \cdot u^{(i+1)} \cdot u^{(r)} = 0. \]
By (17) and (18) we have \( t_{e_i,l_i}^{(i)} = 0 \), which implies that \( X_1^{e_1} X_3^{l_i} \in J \). Next notice that the vector
\[ Z^{e_0} X_1^{e_1} \cdots X_i^{e_i} W^{l_i} Y_i^{l_i} \]
maps to
\[ X_2^{e_0} X_4^{l_0} X_1^{e_1} \cdots X_i^{e_i} \otimes X_3^{l_1} \cdots \otimes X_3^{l_i} \otimes X_0^{n_0} \otimes \cdots \otimes X_0^{n_r} \]
under \( V \cong \mathbb{V} \subset V' \), which yields that
\[ t_{e_i,l_i}^{(i)} \cdot t_{e_i,l_i}^{(i-1)} \cdot u^{(i)} \cdot u^{(r)} = 0. \]
Comparing (17) and (19), we have \( t_{e_0,0}^{(0)} = 0 \), which implies that \( X_2^{e_0} X_4^{l_0} = X_2 X_4 \in J \).
In a similar way, we also have \( X_2 X_3, X_1 X_3 \in J \). Concerning (2) and (3), it follows that \( (X_2, X_4) \subset J \). Therefore, we get \( (X_0^{q-p}, X_2, X_4, X_1^{e_i} X_3^{l_i}) + K \subset J \). Now, suppose
that \( i = 1 \). Then we have \( J_1^1 \subset J \), since \( n_0 = q - p \). Taking Theorem 8.3 into account, it follows that \( J_0^1 = J \), and thus we get \( \Psi^{-1}(O, y_1) = \{[J_0^1]\} \). Next, suppose that \( i > 1 \).

Since the vector
\[
X^{e_0 + e_1 + \cdots + e_{i-2}} Y^{l_0 + l_1 + \cdots + l_{i-2}} = X^{(n_0 + n_1 + \cdots + n_{i-2})} Y^{e_0 + e_1 + \cdots + e_{i-2}} X^{l_0 + l_1 + \cdots + l_{i-2}}
\]
maps to
\[
X^{e_0} X^{l_0} X_1^{e_1} X_3^{l_1} \cdots X_1^{e_{i-2}} X_3^{l_{i-2}} X_0^{n_{i-1}} \cdots X_0^{n_r}
\]
under the isomorphism \( V \cong \overline{V} \subset V' \), we see that
\[
t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(i-2)}_{e_{i-2},l_{i-2}} u^{(i-1)} \cdots u^{(r)} = 0.
\] (20)

By (17) and (20), we get \( u^{(i-1)} = 0 \), and hence \( X_0^{i-1} \in J \). Summarizing, we get \( J_0^i \subset J \).

Therefore, we have \( J_0^i = J \) and \( \Psi^{-1}(O, y_i) = \{[J_0^i]\} \). Next, let \( [I] \in \Psi^{-1}(O, y_i) \), and write
\[
\eta_{n_0,0}([I]) = [t^{(0)}_{0,0} : t^{(0)}_{e_0,l_0} : t^{(0)}_{e_1,l_1}]
\]
and
\[
\eta_{n,0}([I]) = [u^{(i)} : t^{(i)}_{0,0} : \cdots : t^{(i)}_{e_1,l_1} : \cdots : t^{(i)}_{e_r,l_r}]
\] (1 \leq i \leq r).

as above. In a similar manner, we can show that \( (X_0^{n_{i-1}}, X_1, X_2, X_4) + K \subset I \). Moreover, we see that
\[
t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(i-1)}_{e_{i-1},l_{i-1}} u^{(i)} u^{(i+1)} \cdots u^{(r)} = s
\]
and
\[
t^{(0)}_{e_0,l_0} t^{(1)}_{e_1,l_1} \cdots t^{(i-1)}_{e_{i-1},l_{i-1}} t^{(i)}_{e_i,l_i} u^{(i+1)} \cdots u^{(r)} = s
\]
hold for some \( s \in \mathbb{C}^* \). Therefore, we get \( u^{(i)} = t^{(i)}_{e_i,l_i} \). Since we have already seen that \( X_2, X_4 \in I \), this implies that \( \eta_{n_i,0}([I]) = [1 : 0 : \cdots : 0 : 1] \). It follows that \( X_0^{n_i} - X_1^{e_i} X_3^{l_i} \in I \) concerning the construction of \( \eta_{n_i,0} \). As a consequence, we get \( J_i^i \subset I \), and therefore \( I = J_i^i \).

Q.E.D.

**Corollary 9.3.** The quotient rings \( A/J_i^1 \) and \( A/J_i^i \) have Hilbert function \( h \) for any \( 1 \leq i \leq r + 1 \).

**Remark 9.3.1.** Let \( \lambda = (n, c, \omega) \in \Lambda_{(n,0)} \), where \( 0 \leq n < q - p \). Taking Remark 8.13.1 and Corollary 9.3 into account, we see that the following properties are true.

- Suppose that \( 0 \leq n \leq n_{i-1} \). Then we have \( f_{i} \in \bar{J}_0^i \) if and only if \( \lambda \neq (n, 0, \omega_{(n,0)}) \).
- Suppose that \( n_{i-1} \leq n < q - p \). Then we have \( f_{i} \in \bar{J}_0^i \) if and only if \( \lambda \neq (n, mP, \omega_{(n,mP)}) \).

Let us denote by \( \mathcal{H}^\mathcal{B} \) the set of \( \bar{B} \)-fixed points of \( \mathcal{H} \).

**Corollary 9.4.** We have \( \mathcal{H}^\mathcal{B} = \{[J_0^1], \ldots, [J_r^{r+1}]\} \).
Proof. Let \( [J] \in \mathcal{H}^B \). Then, we have \( s_1X_1 + s_2X_2 \in J \) for some \((s_1, s_2) \neq 0\) by (2). Since \( J \) is stable under the action of \( \tilde{B} \), we have \( X_2 \in J \). Similarly, we have \( X_4 \in J \) by (3). Therefore, \((X_2, X_4) + K \subset J\) concerning \( \gamma([J]) = O \). By Theorem 6.1, we see that either \( X_0^{n_j} \in J \) or \( X_1^{e_j}X_3^{j} \in J \) holds for any \( 1 \leq j \leq r + 1 \), since \( h(n_j, 0) = 1 \). Let \( i = \min\{j : X_0^{n_j} \in J\} \). Then, we have \((X_0^{n_{i-1}}, X_1^{e_i}X_3^{j}) \subset J\), and hence \( J^i_j \in J \). This implies that \( J^i_j = J \), since both \( J^i_j \) and \( J \) have Hilbert function \( h \). Q.E.D.

**Corollary 9.5.** The invariant Hilbert scheme \( \mathcal{H} \) coincides with \( \mathcal{H}^{main} \).

Proof. By [Ter14, Lemma 1.6], we see that every closed subset of \( \mathcal{H} \) contains at least one fixed point for the action of \( \tilde{B} \). Therefore it follows that \( \mathcal{H} \) is connected, since Corollary 9.4 implies that every \( \tilde{B} \)-fixed point is contained in \( \mathcal{H}^{main} \). In the following, we show that \( \mathcal{H} \) is smooth. Concerning [Bri13, Proposition 3.5] and the proof of [Ter14, Lemma 1.7], it suffices to show that \( \dim \text{Hom}_S^{G_0 \times G_m}(J^i_0, A/J^i_0) = \dim \mathcal{H}^{main} = 3 \) holds for any \([J^i_0] \in \mathcal{H}^B\). Recall that we have seen in Remark 8.0.1 that

\[
J^i_j = (X_0^{n_{i-1}}, X_2, X_4, X_1^{e_i} X_3^{j}, F_1, \ldots, F_{b-1}).
\]

Let \( \phi \in \text{Hom}_S^{G_0 \times G_m}(J^i_0, A/J^i_0) \). Since \( \phi \) is \( G_0 \times G_m \)-equivariant, we have

\[
\phi(X_0^{n_{i-1}}) = \alpha_1 X_1^{e_i} X_3^{j-1}, \phi(X_2) = \alpha_2 X_1, \phi(X_4) = \alpha_3 X_3, \phi(X_1^{e_i} X_3^{j}) = \alpha_4 X_0^{n_i},
\]

\[
\phi(F_j) = \beta_j \quad (1 \leq j \leq b - 1)
\]

for some \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_j \in \mathbb{C} \). Also, since \( \phi \) is a homomorphism of \( S \)-modules, we have

\[
0 = \phi(X_0^{q-p} - X_1X_4 + X_2X_3) = \alpha_1 X_0^{q-p-n_{i-1}} X_1^{e_i} X_3^{j-1} - \alpha_3 X_1X_3 + \alpha_2 X_1X_3
\]

\[
= \begin{cases} 
(\alpha_1 + \alpha_2 - \alpha_3)X_1X_3 & (\text{if } i = 1) \\
(\alpha_2 - \alpha_3)X_1X_3 & (\text{otherwise})
\end{cases}
\]

concerning that \( X_0^{q-p-n_{i-1}} X_1^{e_i} X_3^{j-1} = X_1X_3F_{p_{i-1}} \in J^i_0 \) holds for every \( i > 1 \).

In the following, we show that \( \beta_j = 0 \) holds for any \( 1 \leq j \leq b - 1 \). Set

\[
d_1 = \frac{qmj - \omega(0, mj)}{q-p}, \quad d_3 = \frac{pmj - \omega(0, mj)}{q-p}.
\]

Then we have \( F_j = X_0^{a(0, mj)} X_1^{d_1} X_3^{d_3} \) by Lemma 6.5. Also, note that we have \( F_{p_i} = X_0^{q-p-n_i} X_1^{e_i} X_3^{l_i}, f_{\lambda_i} = X_1^{e_i} X_3^{j}, \) and \( f_{\lambda_i} = X_1^{d_{i+1}+1} X_3^{d_{i+1}-l_i} \).

**Case 1.** First suppose that \( j > p_i \). Then, taking Lemma 8.10 into account, we see that \( d_1 + 1 > e_i \) and \( d_3 + 1 > l_i \) hold. Set \( f = X_0^{a(0, mj)} X_1^{d_{i+1}+1-e_i} X_3^{d_{i+1}-l_i} \). Then,

\[
0 = \phi(X_1X_3F_j - X_1^{e_i} X_3^{j} f) = \beta_j X_1X_3 - \alpha_4 X_0^{n_i} f.
\]
Since \( X_0^n f \in R_{q-p}^{m_j-m P_i} \), it follows from Proposition \([6.13]\) that \( X_0^n f \in K \subset J_i^i \). Concerning \( X_1 X_3 \in R_{q-p}^0 \), we see that \( X_1 X_3 \notin J_i^i \), since otherwise we get \( \dim(A/J_i^i)_{(q-p,0)} = 0 \), which contradicts to Corollary \([9.3]\). Therefore, we have \( \beta_j = 0 \).

**Case 2.** Next, if \( j = P_i \) then we have
\[
0 = \phi(X_1 X_3 F_{P_i} - X_0^{q-p-n_i} X_1 X_3^i) = \beta_{P_i} X_1 X_3 - \alpha_4 X_0^{q-p},
\]
and hence \( \beta_{P_i} = 0 \), since \( X_1 X_3 \notin J_i^i \) and \( X_0^{q-p} \in J_i^i \).

**Case 3.** Lastly, we consider the case where \( 1 \leq j < P_i \). Following the same line as in Case 1, we see that the condition \( j < P_i \) implies \( d_1 < e_i \) and \( d_3 < l_i \). Set \( n = \omega_{(0,mj)} + n_i \), and \( c = m(P_i - j) \). Then we have \( n_{i-1} \leq n \leq q - p - n_{i-1} + n_i < q - p \) by Lemma \([8.7]\). Also, we see that \( X_1^{e_i-d_i} X_3^{l_i-d_3} = f_{\lambda c} \). Therefore, we have
\[
0 = \phi(X_1^{e_i-d_i} X_3^{l_i-d_3} F_j - X_0^{c_{(0,mj)}} X_1^{e_i} X_3^{l_i}) = \beta_j f_{\lambda c} - \alpha_4 X_0^n.
\]
It immediately follows from \( n_{i-1} \leq n \) that \( X_0^n \in J_i^i \), and we are left to show that \( f_{\lambda c} \notin J_i^i \). As in Case 2 of the proof of Theorem \([8.2]\) we have
\[
n - (n_{j_1} + n_{j_2} + \cdots + n_{j_{i-2}} + n_{j_{i-1}}) < n_{i-1}
\]
for some \( 1 \leq j_1, j_2, \ldots, j_{i-1} \leq i - 1 \). Set \( P = P_{j_1} + \cdots + P_{j_{i-1}} \), and \( \lambda = (n, m P, \omega_{(n,mP)}) \).

We show that \( f_{\lambda c} \notin (X_1^{e_i} X_3^{l_i}) \) by Lemma \([8.10]\). Next, we claim that \( f_{\lambda c} \notin K \). Indeed, if we have \( f_{\lambda c} \in K \), then \( f_{\lambda c} \in (F_1, \ldots, F_{b-1}) \). On the other hand, we see that for any \( 1 \leq l \leq b - 1 \) the degree of \( F_l \) with respect to \( X_0 \) is greater than \( 0 \), which contradicts to \( f_{\lambda c} \in (F_1, \ldots, F_{b-1}) \). Therefore, we have \( c = m P \) concerning the proof of Theorem \([8.2]\). It follows from Remark \([9.3.1]\) that \( f_{\lambda c} \notin J_0^i \), and thus we get \( \beta_j = 0 \).

Therefore we obtain \( \dim \text{Hom}_{S}^{G_0 \times G_m}(J_0^i, A/J_0^i) \leq 3 \), and hence the equality. Q.E.D.

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