PARALLEL SPINOR FLOWS ON THREE-DIMENSIONAL CAUCHY HYPERSURFACES

ÁNGEL MURCIA* AND C. S. SHAHBAZI

Abstract. The three-dimensional parallel spinor flow is the evolution flow defined by a parallel spinor on a globally hyperbolic Lorentzian four-manifold. We prove that, despite the fact that Lorentzian metrics admitting parallel spinors are not necessarily Ricci flat, the parallel spinor flow preserves the vacuum momentum and Hamiltonian constraints and therefore the Einstein and parallel spinor flows coincide on common initial data. Using this result, we provide an initial data characterization of parallel spinors on Ricci flat Lorentzian four-manifolds, which in turn yields the first initial data characterization of Ricci-flat pp-waves. Furthermore, we explicitly solve the left-invariant parallel spinor flow on simply connected Lie groups, obtaining along the way necessary and sufficient conditions for the flow to be immortal. These are, to the best of our knowledge, the first non-trivial examples of evolution flows of parallel spinors. Finally, we use some of these examples to construct families of \(n\)-Einstein cosymplectic structures and to produce solutions to the left-invariant Ricci flow in three dimensions. This suggests the intriguing possibility of using first-order hyperbolic spinorial flows to construct special solutions of curvature flows in Riemannian signature.

1. Introduction

This paper is devoted to the study of the evolution problem posed by a parallel real and irreducible spinor defined on a globally hyperbolic Lorentzian four-manifold \((M, g)\). This problem is well-posed by the results of Leistner and Lichewski, who proved the statement in arbitrary dimension \([24, 23]\). Existence of such a parallel spinor field is obstructed since it implies \((M, g)\) to be a solution of Einstein equations with pure radiation type of energy momentum tensor \([25]\). This fact, which translates into a curvature condition on the underlying globally hyperbolic Lorentzian four-manifold, relates the study of parallel real spinors to the study of globally hyperbolic Lorentzian manifolds satisfying a given curvature condition, which has been a fundamental problem in global Lorentzian geometry since the seminal work of G. Mess \([26, 6]\).

The starting point of our work is the theory of spinorial polyforms and parabolic pairs, recently proposed in \([12]\), which allows to study first-order spinorial equations on pseudo-Riemannian manifolds in terms of differential systems for algebraically constrained polyforms, a reformulation which is especially convenient to study global geometric and topological aspects of such equations. The study of spinorial differential systems through differential forms is by now classical in the mathematics and physics literature, see for instance \([17, 3, 19, 18, 30, 31]\) and their references and citations for more details. Reference \([12]\) elaborates on these earlier works to provide a bijection between certain irreducible spinors and polyforms based on the precise description of the square of a spinor in terms of a semi-algebraic real body in the Kähler-Atiyah bundle of the underlying pseudo-Riemannian manifold. Using the formalism of parabolic pairs \([12]\), our first result (see Theorem 3.2) reformulates the evolution problem of a parallel spinor as a system of flow equations for a family of functions \(\{\beta_t\}_{t \in \mathbb{I}}\) and a family of coframes \(\{e_t^i\}_{t \in \mathbb{I}}\) on an appropriately chosen Cauchy hypersurface \(\Sigma \subset M\). This system of equations yields a generalization of \([28, \text{Theorem } 5.4]\) to the case in which the family \(\{\beta_t\}_{t \in \mathbb{I}}\) is non-trivial and defines the notion of parallel spinor flow as a family \(\{\beta_t, e_t^i\}_{t \in \mathbb{I}}\) satisfying equations (3.6) and (3.7). Using the notion of parallel spinor flow, the constraint equations of the initial value problem of a parallel spinor can be shown to be equivalent to a differential system, the parallel Cauchy differential system \([28]\), for a pair \((\varepsilon, \Theta)\), where \(\varepsilon\) is a coframe and \(\Theta\) is a symmetric two-tensor on \(\Sigma\). The Riemannian metric \(h\) induced by \((M, g)\) on \(\Sigma\) is given by the canonical metric for which \(\varepsilon = (e_u, e_l, e_n)\) becomes an orthonormal coframe, that is, \(h_{\varepsilon} = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n\). This allows us to define a natural map:

\[
\Psi : \text{Conf}(\Sigma) \to \text{Met}(\Sigma) \times \Gamma(T^*M \otimes T^*M), \quad (\varepsilon, \Theta) \mapsto (h_{\varepsilon}, \Theta),
\]

2020 MSC. Primary: 53C50 . Secondary: 58J45. Keywords: Lorentzian four-manifolds, parallel spinors, pp-waves, Cauchy hypersurfaces, initial value problem.

*Corresponding author: angel.murcia@pd.infn.it.
where $\text{Conf}(\Sigma)$ denotes the space of variables $(\epsilon, \Theta)$. The relevance of this map becomes apparent after observing that $\text{Met}(\Sigma) \times \Gamma(T^*M \odot T^*M)$ is precisely the configuration space of the vacuum Hamiltonian and momentum constraint equations [15]. This allows to define the notion of initial data admissible to both the parallel spinor flow and the vacuum Einstein flow, which immediately leads to the natural question of the compatibility of both flows when starting on common admissible data. We solve this question in the affirmative in Theorem 3.9, where we prove that a parallel spinor flow whose initial data are admissible to both problems preserves the Hamiltonian and momentum constraints and produces a Ricci flat Lorentzian four-manifold $(M, g)$. We find this result interesting because it allows us to obtain an initial data characterization of parallel spinors on a Ricci flat Lorentzian four-manifold, which, to the best of our knowledge, is the first of its type in the literature. More precisely, we prove the following result as a consequence of Theorem 3.9.

**Corollary 1.1.** An initial vacuum data $(\Sigma, h, \Theta)$ admits a Ricci flat Lorentzian development carrying a parallel spinor if and only if there exists a global orthonormal coframe $\epsilon$ on $\Sigma$ such that $(\epsilon, \Theta)$ is a parallel Cauchy pair, that is, if and only if $(\epsilon, \Theta)$ satisfies the following differential system:

$$
de_{u} = \Theta(e_{u}) \wedge e_{u}, \quad de_{l} = \Theta(e_{l}) \wedge e_{u}, \quad de_{n} = \Theta(e_{n}) \wedge e_{u}, \quad [\Theta(e_{u})] = 0 \in H^{1}(\Sigma, \mathbb{R}),$$

where $\epsilon = (e_{u}, e_{l}, e_{n})$.

By the well-known local equivalence between pp-waves [25] and Lorentzian four-manifolds admitting a parallel spinor [16], the previous corollary provides the first initial data characterization of Ricci-flat pp-waves. More precisely:

**Corollary 1.2.** An initial vacuum data $(\Sigma, h, \Theta)$ admits a pp-wave Ricci flat Lorentzian development if and only if $(\Sigma, h)$ admits a parallel Cauchy pair.

We also obtain the following corollary.

**Corollary 1.3.** A globally hyperbolic Lorentzian four-manifold $(M, g)$ admitting a parallel spinor is Ricci flat if and only if there exists a Cauchy hypersurface $\Sigma \subset M$ whose Hamiltonian constraint vanishes.

Another important feature of parallel spinorial flows is that they admit a canonical notion of left-invariance in terms of which we can define the notion of left-invariance of parallel spinors. The classification of left-invariant admissible initial data on simply connected three-dimensional Lie groups was completed in [28]. We elaborate on this result to obtain the classification of all associated left-invariant parallel spinor flows. As expected, the result strongly depends on the initial data $(\epsilon, \Theta)$ chosen. Given a parallel Cauchy pair $(\epsilon, \Theta)$ write:

$$\Theta = \Theta_{ab} e_{a} \otimes e_{b}, \quad a, b = u, l, n, \quad \epsilon = (e_{u}, e_{l}, e_{n}),$$

and define:

$$\lambda := \sqrt{\Theta_{ul}^{2} + \Theta_{ln}^{2}}, \quad \theta := \begin{pmatrix} \Theta_{ul} & \Theta_{ln} \\ \Theta_{ln} & \Theta_{nn} \end{pmatrix}, \quad T := \text{Tr}(\theta), \quad \Delta := \text{Det}(\theta).$$

In Section 4 we prove the following classification result.

**Theorem 1.4.** Let $\{\beta_{t}, \epsilon^{t}\}_{t \in \mathbb{I}}$ be a left-invariant parallel spinor on a simply-connected Lie group $G$. Denote by $\{h^{t}, \epsilon^{t}\}_{t \in \mathbb{I}}$ the family of Riemannian metrics associated to $\{\beta_{t}, \epsilon^{t}\}_{t \in \mathbb{I}}$ and by $(\epsilon, \Theta)$ its initial parallel Cauchy pair.

- If $\Theta_{ul}^{2} + \Theta_{ln}^{2} = 0$ and $\Theta_{uu} \neq 0$ then:

$$h_{t}^{t} = (1 - \Theta_{uu} B_{t}^{2}) e_{u} \otimes e_{u} + \left(\epsilon_{t}, e_{n}\right) Q \begin{pmatrix} 1 - \Theta_{uu} B_{t}^{2} & 0 \\ 0 & 1 - \Theta_{uu} B_{t}^{2} \end{pmatrix}^{2\rho_{u}} Q^{*} \left(\epsilon_{t}, e_{n}\right),$$

where $\rho_{u} = \frac{T + \sqrt{T^{2} - 4A}}{2\Theta_{uu}}$ are the eigenvalues of $\theta/\Theta_{uu}$ and $Q$ is its orthogonal diagonalization matrix. In particular, $G = \mathbb{R}^{3}$ if $\theta = 0$, $G = E(1, 1)$ if $T = 0$ and $\theta \neq 0$, $G = \mathbb{T}_{\lambda} \oplus \mathbb{R}$ if $T \neq 0$ and $\Delta = 0$ and $G = \mathbb{T}_{\lambda} \mu$ if $T = 0$ and $\Delta \neq 0$. The case $\Theta_{uu} = 0$ is obtained by taking the formal limit $\Theta_{uu} \rightarrow 0$ in the previous expressions.

- If $\Theta_{ul} = 0$ but $\Theta_{uu} \neq 0$, we have:

$$h_{t}^{t} = \left(1 - \Theta_{uu} B_{t}^{2}\right) \sin^{2}[y_{t}] + \frac{\Theta_{uu}}{\lambda} - \frac{2\Theta_{uu} B_{t}}{\lambda} \tan[y_{t}] e_{u} \otimes e_{u}$$

$$+ \frac{\Theta_{uu}}{\Theta_{uu}} - \Theta_{uu} B_{t} \sin^{2}[y_{t}] \left(1 - \Theta_{uu} B_{t}\right) - \frac{\lambda}{\Theta_{uu}} \tan[y_{t}] \left(1 - 2\Theta_{uu} B_{t}\right) e_{u} \otimes e_{u}$$

$$+ e_{l} \otimes e_{l} + \left(1 + \lambda^{2} B_{t}^{2} \sin^{2}[y_{t}] + 2 B_{t} \lambda \tan[y_{t}]\right) e_{n} \otimes e_{n},$$
where \( y_t = \lambda B_t + \text{Arctan} \left[ \frac{\Theta_{ul}}{\lambda} \right] \). In particular \( G = \tau_2 \oplus \mathbb{R} \). The case \( \Theta_{ul} \neq 0 \) but \( \Theta_{un} = 0 \) is obtained by just exchanging the subindices \( l \) and \( n \) in the previous expression.

- If \( \Theta_{ul} \Theta_{un} \neq 0 \) then:

\[
h_{ul} = \left( (1 + T B_t)^2 + \left( \tan[y_t](1 + T B_t + \frac{T}{X}) \right)^2 \right) e_u \otimes e_u - \left( \frac{T}{X} + \frac{\tan[y_t]}{X}(1 + 2T B_t + B_t(1 + T B_t)) \sec^2[y_t] \right) (\Theta_{ul} e_u \otimes e_l + \Theta_{ul} e_n \otimes e_n) + \left( 1 + \Theta^2_{ul} B_t \left( B_t \sec^2[y_t] + \frac{2\tan[y_t]}{X} \right) \right) e_l \otimes e_l + \Theta_{ul} \Theta_{ul} B_t \sec^2[y_t] \left( B_t + \frac{\sin[2y_t]}{2\tan[y_t]} \right) e_l \otimes e_n + \left( 1 + \Theta^2_{ul} B_t \left( B_t \sec^2[y_t] + \frac{2\tan[y_t]}{X} \right) \right) e_n \otimes e_n.
\]

In particular, \( G = \tau_2 \oplus \mathbb{R} \).

Furthermore, if \( \lambda = 0 \) the flow is globally defined (namely \( I = \mathbb{R} \)) if and only if \( \int_0^\infty \beta_\tau d\tau < |\Theta_{ul}^{-1}| \), whereas if \( \lambda \neq 0 \) the flow is globally defined if and only if \( |y_t| < \frac{\pi}{2} \forall t \in \mathbb{R} \).

The notation used above to label simply-connected three-dimensional Lie groups is standard and can be found in [14, Appendix A]. The previous theorem yields the families of three-dimensional Riemannian metrics that are canonically associated to left-invariant parallel spinor flows. For each such family \( \{ h_{ul} \}_{t \in I} \) we obtain a globally hyperbolic Lorentzian four-manifold carrying parallel spinors as follows:

\[(M, g) = (I \times \Sigma, -\beta^2_\tau dt^2 + h_{ul})\,.
\]

These globally hyperbolic four-manifolds may or may not be Ricci flat depending on the case considered as explained in Section 4.

The curvature of the families of metrics \( \{ h_{ul} \}_{t \in I} \) can be explicitly computed on a case by case basis and a direct inspection of the result reveals that appropriately choosing \( \Theta \) we can recover a particular example of a left-invariant Ricci flow as well as a family of \( \eta \)-Einstein cosymplectic structures. It would be interesting to explore the relation between spinorial flows in Lorentzian signature and weakly parabolic flows in Riemannian geometry in more generality and for more complicated types of spinorial equations, especially for those appearing as Killing spinor equations in four-dimensional supergravity theories. Work in this direction is in progress. On the other hand, it is well-known [25] that existence of a parallel and irreducible real spinor on \( (M, g) \) is locally equivalent to the following curvature condition:

\[
\text{Ric}^g = f u \otimes u,
\]

for a local function \( f \) and light-like parallel one-form \( u \). Therefore, the parallel spinor flow may potentially be locally equivalent to the evolution flow prescribed by equations (1.1). Such relation does not seem however to be straightforward, since the parallel spinor flow is of first order whereas the evolution flow prescribed by (1.1) is of second order. This offers an alternative point of view on the parallel spinor flow as a convenient tool to access the evolution flow prescribed by (1.1) in terms of a more transparent and easier to handle first order evolution flow which, intuitively speaking, provides a first integral of the evolution flow defined by (1.1).

The outline of this paper is as follows. In Section 2 we review the theory of parallel spinors in terms of parabolic pairs and we use it to characterize all standard Brinkmann space-times admitting parallel spinors, obtaining a global result in Proposition 2.5 that seems to be new in the literature. Section 3 is devoted to the description of parallel spinors on globally hyperbolic Lorentzian four-manifolds as parallel spinor flows on an appropriately chosen Cauchy surface. We prove the compatibility of the parallel spinor flow with the vacuum Einstein flow and we obtain an initial data characterization of parallel spinors on Ricci flat Lorentzian four-manifolds. Finally, in Section 4 we classify all left-invariant parallel spinor flows on simply connected three-dimensional Lie groups and we elaborate on some of their properties.

Acknowledgements/Funding. We would like to thank Vicente Cortés and Miguel Sánchez for very interesting discussions and comments. The work of A.M. was funded by the Spanish FPU Grant No. FPU17/04964, by the Deutscher Akademischer Austauschdienst (DAAD), through the Short-Term Research Grant No. 91791300, and by the Istituto Nazionale di Fisica Nucleare (INFN) through the INFN Call No. 23590. A.M. received additional support from the MCIU/AEI/FEDER UE grant PGC2018-095205-B-100 and the Centro de Excelencia Severo Ochoa Program grant SEV-2016-0597. The work of C.S.S. was supported by the Germany Excellence Strategy Quantum Universe - 390833306.
Author contribution. All authors contributed equally.

Conflicts of interest/Competing interests statement. The authors have no conflicts of interest to declare that are relevant to the content of this article.

Data Availability Statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

2. Parallel spinors on Lorentzian four-manifolds

In this section we first review the theory of parallel spinors on globally hyperbolic Lorentzian four-dimensional manifolds and, afterwards, we introduce the notion of standard Brinkmann space-times and study when they admit a parallel spinor.

2.1. General theory. We begin by briefly presenting the theory of parallel spinors on globally hyperbolic Lorentzian four-dimensional manifolds as developed in [12, 28], where parallel spinors were described in terms of a specific type of distribution satisfying a certain system of partial differential equations. Let \((M, g)\) be a four-dimensional space-time, i.e. a connected, oriented and time oriented four-manifold endowed with a Lorentzian metric \(g\). We assume that \((M, g)\) admits a bundle of irreducible real spinors \(S_g\). The existence of such \(S_g\) is equivalent \([20, 21, 22]\) to the existence of a spin structure \(Q_g\), in which case \(S_g\) can be conceived as the vector bundle associated to \(Q_g\) via the tautological representation given by the canonical embedding \(\text{Spin}_+(3, 1) \subset \text{Cl}(3, 1)\), where \(\text{Spin}_+(3, 1)\) stands for the connected component of the identity of the spin group in signature \((3, 1) = --++\) and \(\text{Cl}(3, 1)\) denotes the real Clifford algebra in the aforementioned signature. Therefore, we will assume that \((M, g)\) is spin and endowed with a fixed spin structure \(Q_g\). Under these conditions, the Levi-Civita connection \(\nabla^g\) on \((M, g)\) induces naturally a connection on \(S_g\), the spinorial Levi-Civita connection, which for the sake of simplicity is denoted by the same symbol.

**Definition 2.1.** A spinor \(\varepsilon\) on \((M, g, S_g)\) is a smooth section \(\varepsilon \in \Gamma(S_g)\) of \(S_g\). It is parallel if \(\nabla^g \varepsilon = 0\).

Let \(u \in \Omega^1(M)\) be a luminous one-form. We declare two one-forms \(l_1, l_2 \in \Omega^1(M)\) to be equivalent through the equivalence relation \(\sim_u\), \(l_1 \sim_u l_2\), if and only if \(l_1 = l_2 + f u\) with \(f \in C^\infty(M)\). We denote by:

\[
\Omega^1_u(M) := \frac{\Omega^1(M)}{\sim_u},
\]

the \(C^\infty(M)\)-module of equivalence classes defined by \(\sim_u\).

**Definition 2.2.** A parabolic pair \((u, [l])\) on \((M, g)\) is conformed by a nowhere vanishing null one-form \(u \in \Omega^1(M)\) and an equivalence class of one-forms \([l] \in \Omega^1_u(M)\) such that:

\[
g(l, u) = 0, \quad g(l, l) = 1,
\]

for any, and hence, for all, representatives \(l \in [l]\).

The one-form \(u\) is usually called the Dirac current of the spinor \(\varepsilon\). From [12, Theorems 4.26 and 4.32], it is possible to characterize parallel spinors on \((M, g)\) in terms of differential equations for parabolic pairs.

**Proposition 2.3.** A space-time \((M, g)\) admits a parallel spinor \(\varepsilon \in \Gamma(S_g)\) if and only if there exists a parabolic pair \((u, [l])\) on \((M, g)\) satisfying:

\[
\nabla^g u = 0, \quad \nabla^g l = \kappa \otimes u,
\]

where \(\kappa \in \Omega^1(M)\).

**Remark 2.4.** Specifically, [12, Theorem 4.26] states the equivalence between particular classes of first-order partial differential equations for \(\varepsilon\) and certain systems of partial differential equations for \((u, [l])\), of which Equations (2.1) represent their simplest case.

A parabolic pair \((u, [l])\) is parallel if Equations (2.1) are satisfied for a representative \(l \in [l]\).
2.2. Standard Brinkmann space-times. In order to illustrate the various uses of Proposition 2.3 and make contact with the existing literature, in this subsection we recover the well-known local characterization of a Lorentzian four-manifold \((M, g)\) admitting a parallel spinor, obtaining along the way the global characterization of standard Brinkmann space-times that admit a parallel spinor, which seems to be new in the literature. Recall that by definition a Brinkmann space-time [7, 25] is a Lorentzian four manifold equipped with a parallel null vector. Let \((u, [l])\) be a parallel parabolic pair on \((M, g)\), which by Proposition 2.3 is equivalent to the existence of a parallel spinor. Since \(u\) is parallel, \((M, g)\) is locally isometric to a Brinkmann space-time, whence it suffices to consider \((M, g)\) to be standard, namely \(M = \mathbb{R}^2 \times X\) in terms of an oriented two-dimensional manifold \(X\), equipped with the metric:

\[ g = H_{x_u} dx_u \otimes dx_u + dx_u \otimes \alpha_{x_u} + dx_u \otimes dx_v + q_{x_u}. \]

where \((x_u, x_v)\) denotes the Cartesian coordinates of \(\mathbb{R}^2\), and:

\[ \{H_{x_u}\}_{x_u \in \mathbb{R}}, \quad \{\alpha_{x_u}\}_{x_u \in \mathbb{R}}, \quad \{q_{x_u}\}_{x_u \in \mathbb{R}}, \]

respectively denote a family of functions, a family of one-forms and a family of complete Riemannian metrics on \(X\) parametrized by \(x_u \in \mathbb{R}\). The vector field \(\partial_{x_u}\) is null and parallel, so \(g(\partial_{x_u}) = dx_u\) is a null parallel one-form which we identify with \(u\). We will refer to a parallel spinor \(\varepsilon\) on a standard Brinkmann space-time as adapted if its Dirac current is \(u^\flat = \varepsilon\). In such case, the first equation in (2.1) is automatically satisfied and we only need to be concerned with the second equation in (2.1), namely:

\[ \nabla^q l = \kappa \otimes dx_u, \quad l \in [l], \quad \kappa \in \Omega^1(M), \]

which needs to be satisfied for a representative in \([l]\). This equation is equivalent to:

\[ dl = \kappa \wedge dx_u, \quad \mathcal{L}_l g = \kappa \otimes dx_u, \quad (2.2) \]

where \(l^\flat\) denotes the metric dual of \(l\) with respect to \(g\). Using that \(u = dx_u\) and \(g^{-1}(l, u) = l(\partial_{x_u}) = 0\), it follows that there exists a representative \(l \in [l]\) of the form:

\[ l = l^1, \]

where \(l^1\) denotes a bi-parametric family of unit-norm one-forms on \(X\) parametrized by \((x_u, x_v) \in \mathbb{R}^2\). The first equation in (2.2) is equivalent to:

\[ \kappa_u dx_u \wedge dx_v - \partial_{x_u} l^1 \wedge dx_v = -(\kappa^1 + \partial_{x_v} l^1) \wedge dx_u + dX l^1 = 0, \]

where we have written \(\kappa = \kappa_u dx_u + \kappa_v dx_v + \kappa^1\) and \(dX\) denotes the exterior derivative operator on \(X\). The general solution to the previous equation reads:

\[ \kappa_u = 0, \quad \partial_{x_u} l^1 = 0, \quad \kappa^1 = -\partial_{x_v} l^1, \quad dX l^1 = 0. \]

Therefore, since \(\partial_{x_u} l^1 = 0\), the one-form \(l^1\) is equivalent to a family \(\{l_{x_u}\}_{x_u \in \mathbb{R}}\) of unit-norm one-forms on \(X\). Recall now that the dual \(l^\flat\) of \(l\) with respect to \(g\) is given by the following expression:

\[ l^\flat = -\alpha_{x_u}(\ell^\flat_{x_u}) \partial_{x_u} + \ell^\flat_{x_u}, \]

where the symbol \(\sharp\) denotes musical isomorphism with respect to the metric \(q_{x_u}\). We use this equation to expand the Lie derivative of \(g\) along \(l^\flat\) as follows:

\[ \mathcal{L}_{l^\flat} g = dH_{x_u}(\ell^\flat_{x_u}) dx_u \otimes dx_u + dx_u \otimes (\alpha_{x_u}(\partial_{x_u} \ell^\flat_{x_u})) dx_u + \mathcal{L}^X_{\ell^\flat_{x_u}} \alpha_{x_u} - dx_u \otimes d(\alpha_{x_u}(\ell^\flat_{x_u})) + \mathcal{L}^X_{\ell^\flat_{x_u}} q_{x_u} \]

where \(\mathcal{L}^X\) denotes the Lie derivative on the surface \(X\) and where we have used:

\[ \mathcal{L}_{l^\flat} dx_u = 0, \quad \mathcal{L}_{l^\flat} dx_v = -d(\alpha_{x_u}(\ell^\flat_{x_u})) = -d(\alpha_{x_u}(\ell^\flat_{x_u})) - dX(\alpha_{x_u}(\ell^\flat_{x_u})), \]

\[ \mathcal{L}_{l^\flat} \alpha_{x_u} = \alpha_u(\partial_{x_u} \ell^\flat_{x_u}) dx_u + \mathcal{L}^X_{\ell^\flat_{x_u}} \alpha_{x_u}, \quad \mathcal{L}_{l^\flat} q_{x_u} = \mathcal{L}^X_{\ell^\flat_{x_u}} q_{x_u} + dx_u \otimes (\partial_{x_u} \ell_{x_u} - (\partial_{x_u} q_{x_u})(\ell^\flat_{x_u})). \]

Hence, the second equation in (2.2) is equivalent to:

\[ \kappa_u = \frac{1}{2}(dH_{x_u}(\ell_{x_u}) - (\partial_{x_u} \alpha_{x_u})(\ell^\flat_{x_u})), \quad \nabla^q u \ell_{x_u} = 0, \quad (2.3) \]

\[ 2\partial_{x_u} \ell_{x_u} - (\partial_{x_u} q_{x_u})(\ell^\flat_{x_u}) + \ell_{x_u}. dX \alpha_{x_u} = 0, \quad (2.4) \]

where we have used that \(dX \ell_{x_u} = 0\) and where \(\nabla^q u\) denotes the Levi-Civita connection of the Riemannian metric \(q_{x_u}\) on \(X\). The first equation in (2.3) simply determines \(\kappa_u\) in terms of the underlying metric and the given adapted parabolic pair, whereas the second one establishes that
\( \ell_{x_u} \) is a nowhere vanishing parallel one-form with respect to \( q_{x_u} \) whence we obtain a parallel global orthonormal frame \( \{ \ell_{x_u}, n_{x_u} \}_{x_u \in \mathbb{R}} \) on \( X \) by setting:

\[
n_{x_u} = \ast q_{x_u} \ell_{x_u}.
\]

In particular, \((X, q_{x_u})\) is a flat Riemann surface for every \( x_u \in \mathbb{R} \) and therefore isometric to either the euclidean plane, a flat cylinder or a flat torus. Projecting Equation (2.4) along \( \ell_{x_u} \) we obtain an identity, whereas projecting along \( n_{x_u} \) we obtain that Equation (2.4) is equivalent to:

\[
dx_{x_u} = \partial_{x_u} n_{x_u} \wedge n_{x_u} + \partial_{x_u} \ell_{x_u} \wedge \ell_{x_u}.
\]

All together, the previous discussion proves the following result.

**Proposition 2.5.** A standard Brinkmann space-time admits an adapted parallel spinor if and only if it is isometric to the following model:

\[
(M, g) = (\mathbb{R}^2 \times X, H_{x_u} dx_u \otimes dx_u + dx_u \otimes \alpha_{x_u} + dx_u \otimes dx_v + q_{x_u}),
\]

and there exists a family of parallel orthonormal coframes \( \{ \ell_{x_u}, n_{x_u} \}_{x_u \in \mathbb{R}} \) on \((X, q_{x_u})\) such that:

\[
dx_{x_u} = \partial_{x_u} n_{x_u} \wedge n_{x_u} + \partial_{x_u} \ell_{x_u} \wedge \ell_{x_u}.
\]

In particular, \( \{ q_{x_u} \}_{x_u \in \mathbb{R}} \) is a family of complete flat metrics on \( X \).

**Remark 2.6.** Note that the family of functions \( \{ H_{x_u} \}_{x_u \in \mathbb{R}} \) does not play any role regarding the existence of an adapted parallel spinor on a standard Brinkmann space-time.

By uniformization, we conclude that \( X \) is diffeomorphic to either \( \mathbb{R}^2 \), \( \mathbb{R}^2 \setminus \{ 0 \} \) or \( T^2 \). Appropriately choosing local coordinates the previous result immediately implies that a four-dimensional space-time admitting parallel spinors is locally isometric to a \textit{pp-wave} [16, 25], defined as a Brinkmann space for which the Ricci tensor takes the form (1.1). Note however that Proposition (2.5) is more general than this local equivalence since Proposition (2.5) is a global result that provides a classification of the classification of the corresponding Lorentzian metrics in terms of flow equations on \( X \).

**Example 2.7.** Let \( (M, g) \) be a simply-connected standard Brinkmann space-time admitting an adapted parallel spinor. Using the notation of Proposition (2.5), choose families of constant functions \( \{ c^1_{x_u}, c^2_{x_u}, f_{x_u} \}_{x_u \in \mathbb{R}} \) on \( X = \mathbb{R}^2 \) in terms of which we define the following families of one-forms on \( X \):

\[
\ell_{x_u} = e^{c^1_{x_u}} dy_1 + f_{x_u} dy_2, \quad n_{x_u} = e^{c^2_{x_u}} dy_2, \quad x_u \in \mathbb{R},
\]

where \((y_1, y_2)\) are the global Cartesian coordinates of \( \mathbb{R}^2 \). In particular:

\[
q_{x_u} = \ell_{x_u} \otimes \ell_{x_u} + n_{x_u} \otimes n_{x_u} = e^{2c^1_{x_u}} dy_1 \otimes dy_1 + f_{x_u} e^{c^1_{x_u}} dy_1 \otimes dy_2 + (e^{2c^2_{x_u}} + f_{x_u}^2) dy_2 \otimes dy_2.
\]

A quick computation shows that:

\[
\partial_{x_u} \ell_{x_u} = \partial_{x_u} c^1_{x_u} \ell_{x_u} + (\partial_{x_u} f_{x_u} - \partial_{x_u} e^{c^1_{x_u}} f_{x_u}) e^{-c^1_{x_u}} n_{x_u}, \quad \partial_{x_u} n_{x_u} = \partial_{x_u} c^2_{x_u} e^{-c^2_{x_u}} n_{x_u}, \quad x_u \in \mathbb{R},
\]

whence:

\[
dx_{x_u} = (\partial_{x_u} c^1_{x_u} f_{x_u} - \partial_{x_u} f_{x_u}) \ell_{x_u} \wedge dy_2.
\]

Solutions to this equation can be easily found by direct inspection. A particular solution is given by:

\[
\alpha_{x_u} = (\partial_{x_u} f_{x_u} - \partial_{x_u} c^1_{x_u} f_{x_u}) y_2 \ell_{x_u},
\]

which gives the following metric \( g \) on \( \mathbb{R}^4 \):

\[
g = H_{x_u} dx_u \otimes dx_u + (\partial_{x_u} f_{x_u} - \partial_{x_u} e^{c^1_{x_u}} f_{x_u}) y_2 dx_u \otimes (e^{c^1_{x_u}} dy_1 + f_{x_u} dy_2) + dx_u \otimes dx_u + e^{2c^1_{x_u}} dy_1 \otimes dy_1 + f_{x_u} e^{c^1_{x_u}} dy_1 \otimes dy_2 + (e^{2c^2_{x_u}} + f_{x_u}^2) dy_2 \otimes dy_2.
\]

This produces an explicit example of Lorentzian metric on \( \mathbb{R}^4 \) that admits parallel spinors and for which the \textit{crossed term} \( \alpha_{x_u} \) is non-trivial and, in particular, not closed.

On the other hand, if \( f_{x_u} = e^{c^1_{x_u}} \) we have \( dx \alpha_{x_u} = 0 \) and therefore \( \alpha_{x_u} = dx \partial_{x_u} \) for a family \( \{ \partial_{x_u} \}_{x_u \in \mathbb{R}} \) of functions on \( \mathbb{R}^2 \). Hence, in this case the metric \( g \) can be written as follows:

\[
g = (H_{x_u} - 2 \partial_{x_u} \partial_{x_u}) dx_u \otimes dx_u + dx_u \otimes d(\partial_{x_u} + x_u) + e^{2c^1_{x_u}} dy_1 \otimes dy_1 + e^{2c^1_{x_u}} dy_1 \otimes dy_2 + (e^{2c^2_{x_u}} + e^{2c^2_{x_u}}) dy_2 \otimes dy_2.
\]

(2.5)
and therefore we can write locally:
\[ g = \hat{H}_x \circ dx_u \otimes dx_u + dx_u \otimes d\hat{x}_v + e^{2\sqrt{s}}dy_1 \otimes dy_1 + e^{2\sqrt{s}^2}dy_1 \otimes dy_2 + (e^{2\sqrt{s}^2} + e^{2\sqrt{s}^2})dy_2 \otimes dy_2 , \]
where we have set \( \hat{H}_x = H_x - 2\partial_x \circ a_x \) and \( \hat{x}_v = a_x + x_v \).

**Remark 2.8.** By applying a \( x_u \)-dependent family of diffeomorphisms \( \{f_x\}_{x_u \in \mathbb{R}} \) of \( X \), we can remove the crossed-term \( dx_u \otimes dx_v \) in the metric \( g \) at least locally, obtaining a metric of the type:
\[ f^*_x g = H_x \circ f_x dx_u \otimes dx_u + dx_u \otimes d\hat{x}_v + f^*_x q_{x_u} , \]
However, in many situations it may not be convenient to implement this change of coordinates, since in general it will spoil any choice of coordinates in which the family of metrics \( \{q_{x_u}\}_{x_u \in \mathbb{R}} \) adopts a simple form.

### 3. Globally hyperbolic case

Let \((M, g)\) be a globally hyperbolic four-dimensional space-time. By the results of Bernal and Sánchez [4, 5], \((M, g)\) turns out to have the following isometry type:
\[ (M, g) = (I \times \Sigma, -\beta^2 dt \otimes dt + h_t ) , \tag{3.1} \]
where \( t \) is the canonical coordinate on the interval \( I \subset \mathbb{R} \), \( \{\beta_t\}_{t \in I} \) is a smooth family of strictly positive functions on \( \Sigma \) and \( \{h_t\}_{t \in I} \) is a family of complete Riemannian metrics on \( \Sigma \). From now on we fix the identification (3.1) and we set:
\[ \Sigma_t := \{t\} \times \Sigma \rightarrow M , \quad \Sigma := \{0\} \times \Sigma \rightarrow M , \]
and define:
\[ t_t = \beta_t dt , \]
to be the (outward-pointing) unit time-like one-form orthogonal to \( T^*\Sigma_t \) for every \( t \in \mathbb{R} \). We equip \( \Sigma \rightarrow M \) with the induced Riemannian metric:
\[ h := h_0|_{T\Sigma \times T\Sigma} , \]
and we consider \((\Sigma, h)\) to be the Cauchy hypersurface of \((M, g)\). Associated to each embedded manifold \( \Sigma_t \rightarrow M \) we have its shape operator or scalar second fundamental form \( \Theta_t \) which is defined in the usual way:
\[ \Theta_t := \nabla^g u^0|_{T\Sigma_t \times T\Sigma_t} , \]
or, equivalently:
\[ \Theta_t = -\frac{1}{2\beta_t} \partial_t h_t \in \Gamma(T^*\Sigma_t \otimes T^*\Sigma_t) . \]
We will denote the shape operator of \( \Sigma_t \subset M \) either by \( \Theta_t \) or \( \Theta^t \), depending on convenience. It can be checked that:
\[ \nabla^g \alpha|_{T\Sigma \times T\Sigma} = \nabla^{h_t} \alpha + \Theta_t(\alpha) \otimes t_t , \quad \forall \alpha \in \Omega^1(\Sigma_t) , \]
where \( \nabla^{h_t} \) is the Levi-Civita connection on \((\Sigma_t, h_t)\) and \( \Theta_t(\alpha) := \Theta_t(\alpha^{h_t}) \) is defined as the evaluation of \( \Theta_t \) on the metric dual of \( \alpha \). If \((u, [l])\) is a parabolic pair, we set:
\[ u = u^0 \circ t_t + u^\perp , \quad l = l^0 \circ t_t + l^\perp \in [l] , \]
where the superscript \( \perp \) indicates orthogonal projection to \( T^*\Sigma_t \) and we have defined:
\[ u^0_t = -g(u, t_t) , \quad l^0_t = -g(l, t_t) . \]
Using this orthogonal splitting we describe parallel spinors on a globally hyperbolic space-time in terms of tensorial flow equations on \( \Sigma \).

**Lemma 3.1.** [28, Lemma 2.6] A globally hyperbolic four-manifold \((M, g) = (\mathbb{R} \times \Sigma, -\beta^2 dt \otimes dt + h_t)\) admits a parallel parabolic pair, and hence a parallel spinor field, if and only if there exists a family of orthogonal one-forms \( \{u^1_t, l^1_t\}_{t \in I} \) on \( \Sigma \) satisfying the following equations:
\[ \partial_t u^1_t + \beta_t \Theta_t(u^1_t) = u^0_t \circ d\beta_t , \quad u^0_t \circ d\beta_t + \beta_t u^0_t \circ \Theta_t(l^1_t) + d\beta_t(l^1_t) u^1_t = 0 , \tag{3.2} \]
\[ \nabla^{h_t} u^1_t + u^0_t \Theta_t = 0 , \quad u^0_t \nabla^{h_t} l^1_t = \Theta_t(l^1_t) \otimes u^1_t , \tag{3.3} \]
as well as:
\[ (u^0_t)^2 = |u^\perp|_{h_t}^2 , \quad ||l^1_t||_{h_t}^2 = 1 , \tag{3.4} \]
In particular, $\partial_t u^0_t = \partial \beta_t(u^0_t)$ and $du^0_t + \Theta_t(u^0_t) = 0$. If these equations are satisfied, the corresponding parabolic pair $(u, [l])$ is given by:

$$u = u^0_t t + u^+_t, \quad [l] = [l^+_t].$$

where $|u^+_t|^2_{h_t} = h_t(u^0_t, u^+_t)$ and $|l^+_t|^2_{h_t} = h_t(l^+_t, l^+_t)$. The previous lemma gives the necessary and sufficient conditions for a globally hyperbolic Lorentzian four-manifold $(M, g)$ to admit a real parallel spinor field. We will consider as variables of these equations tuples of the form:

$$\{(\beta_t)_t \in I, \{h_t\}_t \in I, \{u^0_t\}_t \in I, \{u^+_t\}_t \in I, \{l^+_t\}_t \in I\}.$$  \hspace{1cm} (3.5)

These tuples contain the information about both the real parallel spinor and the underlying globally hyperbolic Lorentzian metric. The following theorem generalizes [28, Theorem 5.4] to the case in which $\{(\beta_t)_t \in I\}$ is not necessarily constant.

**Theorem 3.2.** An oriented globally hyperbolic Lorentzian four-manifold $(M, g)$ admits a parallel spinor field if and only if there exists an orientation preserving diffeomorphism identifying $M = \mathcal{I} \times \Sigma$, where $\Sigma$ is an oriented three-manifold equipped with a family of strictly positive functions $\{\beta_t\}_t \in I$ on $\Sigma$ and a family $\{e^t\}_t \in I$ of sections of $\Gamma(\Sigma)$ satisfying the following system of differential equations:

$$\partial te^u_t + d\beta_t(e^u_t)e^u_t + \beta_t \Theta_t(e^u_t) = \delta_a d\beta_t, \quad d\Theta_t = \Theta_t(e^t) \wedge e^u_t, \quad (\Theta_t(e^u_t)) + d(d\beta_t(e^u_t)) = 0,$$  \hspace{1cm} (3.6)

$$h_t e^u_t = e^u_t \wedge e^u_t + e^v_t \wedge e^v_t + e^w_t \wedge e^w_t + e^u_t \wedge e^v_t, \quad \Theta_t = -\frac{1}{2\beta_t} \partial_t h_t e^u_t.$$  \hspace{1cm} (3.7)

In this case, the globally hyperbolic metric $g$ is given by:

$$g = -x^a_t dt \otimes dt + h_t e^u_t,$$

where $t$ is the Cartesian coordinate in the splitting $\mathcal{M} = \mathbb{R} \times \Sigma$. The proof of the above theorem is provided in the following section. By Lemma 3.1, a globally hyperbolic Lorentzian four-manifold $(M, g)$ admits a real Killing spinor field if and only if there exists a Cauchy surface $\Sigma \rightarrow M$ equipped with a tuple (3.5) satisfying equations (3.2), (3.3) and (3.4). Let:

$$\{(\beta_t)_t \in I, \{h_t\}_t \in I, \{u^0_t\}_t \in I, \{u^+_t\}_t \in I, \{l^+_t\}_t \in I\},$$  \hspace{1cm} (3.8)

be such a solution and define:

$$e^u_a = \frac{u^+_a}{u^0_a}, \quad e^v_a = l^+_a.$$

Then $(e^u_a, e^v_a)$ is a family of nowhere vanishing and orthonormal one-forms on $\Sigma$, which can be canonically completed to a family of orthonormal coframes $\{e^t\}_t \in I$ by defining the family of one-forms $\{e^t_a\}_t \in I$ as follows:

$$e^t_a := *_{h_t} (e^t_a \wedge e^u_a),$$

where $*_{h_t}$ denotes the Hodge dual associated to the family of metrics $\{h_t\}_t \in I$. Plugging equations (3.9) into the first and second equations in (3.2) and manipulating the time derivative of $\{u^+_t\}_t \in I$ we obtain the first equation in (3.6) for $a = u$ and $a = l$. For $\{e^t_a\}_t \in I$ we compute as follows:

$$0 = \partial_t h_t^{-1}(e^t_a, e^t_b) + h_t^{-1}(\partial_t e^t_a, e^t_b) + h_t^{-1}(e^t_a, \partial_t e^t_b) = 2\beta_t \Theta_t(e^t_a, e^t_b)$$

$$+ h_t^{-1}(\partial_t e^t_a, e^t_b) + h_t^{-1}(e^t_a, \partial_t e^t_b) = \beta_t \Theta_t(e^t_a, e^t_b) + h_t^{-1}(\partial_t e^t_a, e^t_b) + h_t^{-1}(e^t_a, \partial_t e^t_b),$$

$$0 = \partial_t h_t^{-1}(e^t_a, e^t_b) + h_t^{-1}(\partial_t e^t_a, e^t_b) + h_t^{-1}(e^t_a, \partial_t e^t_b) = 2\beta_t \Theta_t(e^t_a, e^t_b)$$

$$+ h_t^{-1}(\partial_t e^t_a, e^t_b) + h_t^{-1}(e^t_a, \partial_t e^t_b) = \beta_t \Theta_t(e^t_a, e^t_b) + h_t^{-1}(\partial_t e^t_a, e^t_b),$$

which immediately implies the first equation in (3.6) for the remaining case $a = n$. On the other hand, by Lemma 3.1, we have:

$$\frac{d}{dt} u^0_t + \Theta_t(e^0_t) = 0,$$

whence $[\Theta_t(e^0_t)] = 0 \in H^1(\Sigma, \mathbb{R})$, which yields the second equation in (3.7). Taking the time derivative of the previous equations we obtain:

$$d\partial_t \log |u^+_t|^2 + \partial_t (\Theta_t(e^u_t)) = 0.$$
Since by Lemma 3.1 we have $\partial_t u^0_t = d\beta_t(u^\perp_t)$, the previous equation implies the first equation in (3.7). We compute:

$$\nabla^h c^0_t = \nabla^h a_0 + (c^0_t \wedge (\nabla^h c^0_t \wedge e^0_t)) + (\nabla^h c^0_t \wedge (\nabla^h c^0_t \wedge e^0_t)) = \Theta_t(c^0_t) \Theta_t.$$

The skew-symmetrization of the previous equation together with the skew-symmetrization of equations (3.3) yields the second equation in (3.6). Conversely, suppose that $\{c^t, \beta_t\}_{t \in \mathbb{R}}$ is a solution of equations (3.6) and (3.7), and set:

$$h^0_t = e^0_t \otimes e^0_t + c^0_t \otimes c^0_t + e^0_n \otimes e^0_n, \quad \Theta_t = -\frac{1}{2\beta_t} \partial_t h^0_t.$$

Since $[\Theta_t(c^0_t)] = 0$ in $H^1(\Sigma, \mathbb{R}) = 0$, there exists a smooth family of functions $\{\tilde{f}_t\}_{t \in \mathbb{R}}$ such that:

$$d\tilde{f}_t = -\Theta_t(c^0_t).$$

Taking the time-derivative of the previous expression we obtain:

$$d\partial_t \tilde{f}_t = -\partial_t (\Theta_t(c^0_t)).$$

Hence, comparing with the first equation in (3.7) we conclude:

$$d\partial_t \tilde{f}_t = d\beta_t(c^0_t) + c(t)$$

implying $\partial_t \tilde{f}_t = d\beta_t(c^0_t) + c(t)$ for a certain function $c(t)$ depending exclusively on $t$. Set $\tilde{f}_t := \tilde{f}_t - f(c)dt$. By construction we have $\partial_t \tilde{f}_t = d\beta_t(c^0_t)$. Furthermore:

$$d\partial_t \tilde{f}_t = -\partial_t \Theta_t(c^0_t).$$

Define now $u^\perp_t := e^\perp_t u^\perp_t$ and $l^\perp_t := e^\perp_t$. The fact that both $e^\perp_t$ and $e^\perp_t$ satisfy the first equation in (3.6) implies:

$$\partial_t u^\perp_t + \beta_t \Theta_t(u^\perp_t) + (\partial_t \beta_t) u^\perp_t = u^0 \partial_t \beta_t,$$

as well as:

$$u^0_0 \partial_t l^\perp_t + \beta_t u^0_0 \Theta_t(l^\perp_t) + (\partial_t \beta_t) u^\perp_t = 0,$$

where we have identified $u^0_t := e^\perp_t$. Using the fact that $\partial_t \tilde{f}_t = d\beta_t(c^0_t)$ we obtain equations (3.2). Equations (3.3) follow directly by interpreting the second equation in (3.6) as the first Cartan structure equations for the coframe $c^t$, considered as orthonormal with respect to the metric $h^0_t = e^0_t \otimes e^0_t + c^0_t \otimes c^0_t + e^0_n \otimes e^0_n$. Finally, equations (3.4) hold by construction and hence we conclude.

**Definition 3.3.** Equations (3.6) and (3.7) are the real parallel spinor flow equations. A real parallel spinor flow is a family $\{\beta_t, c^t\}_{t \in \mathbb{R}}$ of functions and coframes on $\Sigma$ satisfying the real parallel spinor flow equations.

Therefore, a globally hyperbolic Lorentzian four-manifold admits a parallel spinor if and only if it admits a Cauchy surface carrying a parallel spinor flow $\{\beta_t, c^t\}_{t \in \mathbb{R}}$. In particular, the corresponding parallel spinor $\epsilon$ can be fully reconstructed from $\{\beta_t, c^t\}_{t \in \mathbb{R}}$. We remark that for our purposes the explicit expression of the parallel spinor associated to a given parallel spinor flow $\{\beta_t, c^t\}_{t \in \mathbb{R}}$ is of no relevance. Instead, we are interested in the geometric and topological consequences associated to the existence of a parallel spinor $\epsilon$, rather than on its specific expression.

### 3.1. Admissible common initial data.

The parallel spinor flow equations pose an evolution problem whose associated constraint equations are equivalent to the constraint equations of the evolution problem posed by a parallel spinor on a globally hyperbolic Lorentzian four-manifold. Taking $\Sigma := \Sigma_0$ as the Cauchy hypersurface of $(M, g)$ and setting:

$$\epsilon := \epsilon^0, \quad \Theta := \Theta_0,$$

the restriction of the second set of equations in (3.6) and of the second equation in (3.7) to $\Sigma$ reads:

$$d\epsilon = \Theta(e_u) \wedge e_u, \quad d\epsilon_t = \Theta(e_t) \wedge e_u, \quad d\epsilon_n = \Theta(e_n) \wedge e_u,$$

$$[\Theta(e_u)] = 0 \in H^1(\Sigma, \mathbb{R}).$$

We will consider equations (3.10) and (3.11) as the constraint equations of the parallel spinor flow, whose solutions $(\epsilon, \Theta)$ are by definition the allowed initial data of the parallel spinor flow. We will refer to equations (3.10) and (3.11) as the parallel Cauchy differential system.

**Definition 3.4.** A parallel Cauchy pair $(\epsilon, \Theta)$ is a solution of the parallel Cauchy differential system.
We denote by $\text{Conf}(\Sigma)$ the configuration space of the parallel Cauchy differential system, that is, its space of variables $(\epsilon, \Theta)$, whereas we denote by $\text{Sol}(\Sigma)$ the space of parallel Cauchy pairs. Note that the function $\beta_0$ does not occur in the parallel Cauchy differential system, exactly as it happens with the initial value problem posed by the Ricci flat condition of a Lorentzian metric [13, 15].

Given a pair $(\epsilon, \Theta) \in \text{Conf}(\Sigma)$, we denote by $h_\epsilon$ the Riemannian metric on $\Sigma$ defined by:

$$h_\epsilon = c_u \otimes c_u + c_t \otimes c_t + c_n \otimes c_n,$$

where $\epsilon = (c_u, c_t, c_n)$. We say that $(\epsilon, \Theta)$ is complete if $h_\epsilon$ is a complete Riemannian metric on $\Sigma$. Denote by $\text{Met}(\Sigma) \times \Gamma(T^*\Sigma \otimes T^*\Sigma)$ the set of pairs consisting of Riemannian metrics and symmetric two-tensors on $\Sigma$. We obtain a canonical map:

$$\Psi: \text{Conf}(\Sigma) \to \text{Met}(\Sigma) \times \Gamma(T^*\Sigma \otimes T^*\Sigma), \quad (\epsilon, \Theta) \mapsto (h_\epsilon, \Theta).$$

The set $\text{Met}(\Sigma) \times \Gamma(T^*\Sigma \otimes T^*\Sigma)$ is in fact the configuration space of the constraint equations associated to the Cauchy problem posed by the Ricci flat condition on a globally hyperbolic Lorentzian four-manifold with Cauchy surface $\Sigma$, which are given by [13, 15]:

$$R_h = |\Theta h|^2 - \text{Tr}_h(\Theta)^2, \quad d\Gamma_h(\Theta) = \text{div}_h(\Theta),$$

for pairs $(h, \epsilon) \in \text{Met}(\Sigma) \times \Gamma(T^*\Sigma \otimes T^*\Sigma)$.

**Remark 3.5.** The first equation in (3.12) is usually called the Hamiltonian constraint whereas the second equation in (3.12) is usually called the momentum constraint.

Therefore, the map $\Psi$ provides a natural link between the initial value problem associated to a parallel spinor and the initial value problem associated to the Ricci-flatness condition. In particular, it allows introducing a natural notion of admissible initial data to both evolution problems.

**Definition 3.6.** A parallel Cauchy pair $(\epsilon, \Theta)$ is constrained Ricci flat if $(h_\epsilon, \Theta)$ satisfies the momentum and Hamiltonian constraints (3.12).

In references [1, 23] it was proven that the Cauchy problem posed by a parallel spinor is well-posed, implying that every parallel Cauchy pair admits a Lorentzian development carrying a parallel spinor and hence a parallel spinor flow. However, since a Lorentzian metric admitting a parallel spinor is not necessarily flat, it might not be possible to evolve a constrained Ricci flat parallel Cauchy pair $(\epsilon, \Theta)$ in such a way that both the Ricci-flatness condition and the existence of a parallel spinor are guaranteed. We will solve this problem in the affirmative in the next subsection and, in doing so, we will obtain an initial data characterization of parallel spinors on globally hyperbolic Ricci flat four-manifolds.

3.2. **Initial data characterization.** Denote by $\mathcal{P}(\Sigma)$ the set of parallel spinor flows on $\Sigma$, that is, the set of families $\{\beta_t, \epsilon_t\}_{t \in I}$ satisfying the parallel spinor flow equations (3.6) and (3.7). We have a canonical map:

$$\Phi: \mathcal{P}(\Sigma) \to \text{Lor}_0(M), \quad \{\beta_t, \epsilon_t\}_{t \in I} \mapsto (\beta_t, \epsilon_t)^t \implies g = -\beta_t^2 dt^2 + h_\epsilon,$$

from $\mathcal{P}(\Sigma)$ to the set $\text{Lor}_0(M)$ of globally hyperbolic Lorentzian metrics on $M = I \times \Sigma$. For simplicity in the exposition, we will refer to $\Phi(\{\beta_t, \epsilon_t\})$ as the globally hyperbolic metric determined by $\{\beta_t, \epsilon_t\}_{t \in I}$. Given a parallel spinor flow $\{\beta_t, \epsilon_t\}_{t \in I}$, there exists a smooth family functions $\{h_t\}_{t \in I}$ such that:

$$d\epsilon_t = -\Theta_t(e_u^t), \quad \partial_t h_t = d\beta_t(e_u^t),$$

which is unique modulo the addition of a real constant. Using this family of functions, we obtain a canonical map:

$$\Xi: \mathcal{P}(\Sigma) \to \mathcal{B}(M), \quad \{\beta_t, \epsilon_t\}_{t \in I} \mapsto \langle u = \epsilon_t^l (\beta_t dt + e_u^l), [l = e_u^l] \rangle,$$

from the set of parallel spinor flows on $\Sigma$ to the set $\mathcal{B}(M)$ of parabolic pairs on $M$ with respect to the globally hyperbolic metric defined by the given parallel spinor flow. The previous maps provide a construction which is essentially inverse to the splitting and reduction implemented at the beginning of Section 3 and which allows us to relate properties of a given parallel spinor flow to properties of its associated globally hyperbolic four-dimensional Lorentzian metric. For further reference, we introduce the Hamiltonian function of a parallel spinor flow $\{\beta_t, \epsilon_t\}_{t \in I}$ as follows:

$$H: M = \mathbb{R} \times \Sigma \to \mathbb{R}, \quad (t, p) \mapsto (R_{h_t} - |\Theta_t|_{h_t}^2 + \text{Tr}_{h_t}(|\Theta_t|^2)|_p),$$

where $h_t := h_\epsilon$ denotes the three-dimensional metric restricted to the Cauchy surface $\Sigma_t$ and $R_{h_t}$ its scalar curvature.
Proposition 3.7. Let \( \{\beta_t, \psi_t\} \in I \) be a parallel spinor flow on \( \Sigma \). The Ricci curvature of \( g = \Phi((\beta_t, \psi_t)) \) reads:

\[
\operatorname{Ric}^g = \frac{1}{2} H e^{-2f_t} u \otimes u,
\]

where \( I(\{\beta_t, \psi_t\}) = (u, [l]) \) and \( u = e^{h_t}(\beta_t dt + e_a') \).

Proof. Let \( \{\beta_t, \psi_t\} \in I \) be a parallel spinor flow on \( \Sigma \) and let \( g = \Phi((\beta_t, \psi_t)) = -\beta_t dt \otimes dt + h_t \) its associated globally hyperbolic metric on \( M = \mathbb{R} \times \Sigma \). The pair \( \{\beta_t, \psi_t\} \in I \) defines a global orthonormal coframe \((e_0, e_1, e_2, e_3)\) on \((M, g)\) given by:

\[
e_0|_{(t, p)} := \beta_t|_p dt, \quad e_1|_{(t, p)} := e_1' p, \quad e_2|_{(t, p)} := e_2' p, \quad e_3|_{(t, p)} := e_3' p.
\]

The fact that \( \{\beta_t, \psi_t\} \in I \) is a parallel spinor flow implies that the exterior derivatives of the coframe \((e_0, e_1, e_2, e_3)\) on \( M \) are prescribed as follows:

\[
d e_0 = d \log(\beta_t) \wedge e_0, \quad d e_a = (d \log(\beta_t)(e_a) e_1 + \Theta_a(e_a) - \delta_a d \log(\beta_t)) \wedge e_0 + \Theta_a(e_a) \wedge e_1,
\]

where \( a = 1, 2, 3 \). Interpreting the previous expression as the first Cartan structure equations for \( \nabla^g \) with respect to the orthonormal coframe \((e_0, e_1, e_2, e_3)\) and using repeatedly Equations (3.6) and (3.7), a tedious calculation yields Equation (3.13) and hence we conclude.

\[\square\]

Remark 3.8. It is well-known that the Ricci curvature \( \operatorname{Ric}^g \) of a Lorentzian four-manifold admitting parallel spinors is of the form \( \operatorname{Ric}^g = f u \otimes u \) for some function \( f \in C^\infty(M) \) [25]. Nonetheless, and to the best of our knowledge, Equation (3.13) is the first precise characterization of such function \( f \) in the case of globally hyperbolic Lorentzian four-manifolds.

Theorem 3.9. The parallel spinor flow preserves the vacuum momentum and Hamiltonian constraints.

Proof. Let \( \{\beta_t, \psi_t\} \in I \) be a parallel spinor flow. Taking the divergence of Equation (3.13) we obtain:

\[
d (\operatorname{He}^{-2f_t}) = 0,
\]

which can be equivalently written as follows:

\[
\mathcal{D}(\mathcal{H}) = \rho \mathcal{H},
\]

where \( \mathcal{D} = \partial_t + e_a' \) is a first-order symmetric hyperbolic differential operator and \( \rho = 2(\partial_t f_t + e_a'(f_t)) \) is a function completely determined by \( \{\beta_t, \psi_t\} \in I \). The fact that we are able to rewrite equation \( d (\operatorname{He}^{-2f_t}) = 0 \) in terms of the symmetric hyperbolic differential operator \( \mathcal{D} \) follows from the fact that \( u \) is nowhere vanishing and null, which in turn implies that the component \( u_t \) can never vanish on \( I \times \Sigma \) and is therefore of constant sign. Given such \( \mathcal{D} \) and \( \rho \) associated to \( \{\beta_t, \psi_t\} \in I \), consider now the initial value problem:

\[
\mathcal{D}(F) = \rho F, \quad F|_{\Sigma} = 0,
\]

for an arbitrary function \( F \) on \( M \). By the existence and uniqueness theorem for this type of equations, see [11, Theorem 19], every solution must be zero on a neighborhood of \( \Sigma \). Since \( \mathcal{H} \) is in particular a solution of this equation, it must vanish on a neighborhood of \( \Sigma \). Therefore, there exists a subinterval \( I' = (a, b) \subseteq I \) containing zero such that \( \mathcal{H}|_{b} = 0 \) for every \( t \in I' \). By [28, Proposition 2.24] this implies that the momentum constraint is also satisfied for every \( t \in I' \) and hence the parallel spinor flow preserves the Hamiltonian and momentum constraints on \( I' \). If \( I' = I \) we conclude, so assume that \( I' = (a, b) \subset I \) is the proper maximal subinterval of \( I \) for which the result holds. Since the parallel spinor flow \( \{\beta_t, \psi_t\} \in I \) is well-defined in \( I \), then both \( \rho \) and \( \mathcal{H} \) must be well-defined on \( I \times \Sigma \). Hence, by point-wise continuity on \( \Sigma \) we must have that \( \mathcal{H}|_{b} = 0 \) and therefore we can apply the previous argument to the initial value problem starting at \( b \in I \). Hence there exists an \( \varepsilon > 0 \) for which the result holds on \((a, b + \varepsilon)\), in contradiction with \((a, b)\) being maximal. Therefore, \( I' = I \) and we conclude.

\[\square\]

Call a triple \((\Sigma, h, \Theta)\) an initial vacuum data if \((h, \Theta)\) satisfies the Hamiltonian and momentum constraints. The previous theorem can be applied to prove an initial data characterization of parallel spinors on Ricci flat Lorentzian four-manifolds.

Corollary 3.10. An initial vacuum data \((\Sigma, h, \Theta)\) admits a Ricci flat Lorentzian development carrying a parallel spinor if and only if there exists a global orthonormal coframe \( \epsilon \) on \( \Sigma \) such that \((\epsilon, \Theta)\) is a parallel Cauchy pair.
Proof. The only if condition follows from Theorem 3.2. For the if condition, let \((\Sigma, h, \Theta)\) be an initial vacuum data. If in addition there exists a global orthonormal coframe \(\epsilon\) on \(\Sigma\) such that \((\epsilon, \Theta)\) is a parallel Cauchy pair, then the constraint equations of the initial value problem of a parallel spinor are satisfied. By [23, 24], the initial value problem is well-posed and there exists a Lorentzian development of \(\Sigma\) carrying a parallel spinor. By Theorem 3.9, this Lorentzian development satisfies the Hamiltonian and momentum constraint for every \(t \in I\), and by Equation (3.13) we conclude that this Lorentzian development is Ricci flat. \(\square\)

Additionally, we obtain the following corollary.

**Corollary 3.11.** A globally hyperbolic Lorentzian four-manifold \((M, g)\) admitting a parallel spinor is Ricci flat if and only if there exists a Cauchy hypersurface \(\Sigma \subset M\) whose Hamiltonian constraint vanishes.

3.3. **Left-invariant parallel Cauchy pairs.** Let \(\Sigma = G\) be a simply connected three-dimensional Lie group. A pair \((\epsilon, \Theta)\) \(\in\) Conf\((G)\) is said to be left-invariant if both \(\epsilon\) and \(\Theta\) are left-invariant. Given a left-invariant pair \((\epsilon, \Theta)\) \(\in\) Conf\((G)\) we write:

\[
\Theta = \Theta_{ab} e_a \otimes e_b, \quad a, b = u, l, n,
\]

where summation over repeated indices is understood. For further reference we introduce the following notation:

\[
\lambda := \sqrt{\Theta_{ul}^2 + \Theta_{un}^2}, \quad \theta := \left(\begin{array}{c}
\Theta_{ul} \\
\Theta_{ln}
\end{array}\right), \quad T := \text{Tr}(\theta), \quad \Delta := \text{Det}(\theta) = \Theta_{ul} \Theta_{nn} - \Theta_{un}^2.
\]

These will play an important role in the classification of left-invariant Cauchy pairs, which was completed in [28] and which we proceed to summarize.

**Theorem 3.12.** [28, Theorem 4.9] A connected and simply-connected Lie group \(G\) admits left-invariant parallel Cauchy pairs (respectively constrained Ricci flat parallel Cauchy pairs) if and only if \(G\) is isomorphic to one of the Lie groups listed in the table below. If that is the case, a left-invariant shape operator \(\Theta\) belongs to a Cauchy pair \((\epsilon, \Theta)\) for certain left-invariant coframe \(\epsilon\) if and only if \(\Theta\) is of the form listed below when written in terms of \(\epsilon = (e_u, e_l, e_n)\):

| \(G\) | Cauchy parallel pair | Constrained Ricci flat |
|------|----------------------|------------------------|
| \(\mathbb{R}^3\) | \(\Theta = \Theta_{uu} e_u \otimes e_u\) | \(\Theta = \Theta_{uu} e_u \otimes e_u\) |
| \(E(1,1)\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) \(i, j = l, n\), \(\Theta_{ll} = -\Theta_{nn}\) | Not allowed |
| \(\mathbb{T}_2 \oplus \mathbb{R}\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) \(i, j = l, n\), \(T \neq 0, \Delta = 0\) | Not allowed |
| \(\mathbb{T}_3, \mu\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) \(i, j = l, n\), \(T, \Delta \neq 0\) | Not allowed |

Regarding the case \(G = \mathbb{T}_3, \mu\):
If $\Theta_{ln} \neq 0$, then
$$\mu = \frac{T - \text{sign}(T)\sqrt{T^2 - 4\Delta}}{T + \text{sign}(T)\sqrt{T^2 - 4\Delta}}.$$ 

If $\Theta_{ln} = 0$ and $|\Theta_{ll}| \geq |\Theta_{nn}|$, then
$$\mu = \frac{\Theta_{nn}}{\Theta_{ll}}.$$ 

If $\Theta_{ln} = 0$ and $|\Theta_{nn}| \geq |\Theta_{ll}|$, then
$$\mu = \frac{\Theta_{ll}}{\Theta_{nn}}.$$ 

The previous theorem will be used extensively in the next section. We have the following corollary.

**Corollary 3.13.** Let $G$ be a connected and simply connected Lie group with a left-invariant Cauchy pair. Then the isomorphism type of $G$ is prescribed by $T$, $\Delta$ and $\lambda$ as follows:
- If $T = \Delta = \lambda = 0$, then $G \simeq \mathbb{R}^3$.
- If $T = \lambda = 0$ but $\Delta \neq 0$, then $G \simeq \mathbb{E}(1, 1)$.
- If $\Delta = 0$ but $\lambda^2 + T^2 \neq 0$, then $G \simeq \tau_2 \oplus \mathbb{R}$.
- If $T, \Delta \neq 0$ and $\lambda = 0$, then $G \simeq \tau_3, \mu$.

Observe that the case $\lambda \neq 0$ and $\Delta \neq 0$ is not allowed.

We are using standard notation for the groups $G$ as explained for example in [14, Appendix A].

### 4. Left-invariant parallel spinor flows

In this section we introduce the notion of left-invariant parallel spinor flow and solve it explicitly.

#### 4.1. Reformulation

Let $G$ be a simply connected three-dimensional Lie group. We say that a parallel spinor flow $\{\beta_t, e^t\}_{t \in \mathcal{T}}$ defined on $G$ is left-invariant if both $\beta_t$ and $e^t$ are left-invariant for every $t \in \mathcal{T}$. The latter condition immediately implies that $h_{e^t}$ is a left-invariant Riemannian metric and $\beta_t$ is constant for every $t \in \mathcal{T}$. Let $\{e^t\}_{t \in \mathcal{T}}$ be a family of left-invariant coframes on $G$. Any square matrix $A \in \text{Mat}(3, \mathbb{R})$ acts naturally on $\{e^t\}_{t \in \mathcal{T}}$ as follows:

$$A(e^t) := \left( \frac{\sum_{a,b} A_{ab} e^t_b}{\sum_{a,b} A_{ab} e^t_b} \right)$$

where we label the entries $A_{ab}$ of $A$ by the indices $a, b = u, l, n$. As a direct consequence of Theorem 3.2 we have the following result.

**Proposition 4.1.** A simply connected three-dimensional Lie group $G$ admits a left-invariant parallel spinor flow if and only if there exists a smooth family of non-zero constants $\{\beta_t\}_{t \in \mathcal{T}}$ and a family $\{e^t\}_{t \in \mathcal{T}}$ of left-invariant coframes on $G$ satisfying the following differential system:

$$\partial_t e^t + \beta_t \Theta_t(e^t) = 0, \quad d e^t = \Theta_t(e^t) \wedge e^t_u, \quad \partial_t (\Theta_t(e^t_u)) = 0, \quad d \Theta_t(e^t_u) = 0,$$

(4.1)

to which we will refer as the left-invariant (real) parallel spinor flow equations.

We will refer to solutions $\{\beta_t, e^t\}_{t \in \mathcal{T}}$ of the left-invariant parallel spinor flow equations as left-invariant parallel spinor flows. Given a parallel spinor flow $\{\beta_t, e^t\}_{t \in \mathcal{T}}$, we write:

$$\Theta^t = \sum_{a,b} \Theta^t_{ab} e^t_a \otimes e^t_b, \quad a, b = u, l, n,$$

in terms of uniquely defined functions $\Theta^t_{ab}$ on $\mathcal{T}$.

**Lemma 4.2.** Let $\{\beta_t, e^t\}_{t \in \mathcal{T}}$ be a left-invariant parallel spinor flow. The following equations hold:

$$\partial_t \Theta_{uu}^t = \beta_t ((\Theta_{uu}^t)^2 + (\Theta_{ul}^t)^2), \quad \partial_t \Theta_{ul}^t = \partial_t \Theta_{lu}^t = 0, \quad \partial_t \Theta_{ll}^t = \beta_t \Theta_{ll}^t \Theta_{uu}^t - \beta_t (\Theta_{ul}^t)^2,$$

$$\partial_t \Theta_{ln} = \beta_t \Theta_{ln} \Theta_{lu}^t - \beta_t \Theta_{un} \Theta_{lu}^t, \quad \partial_t \Theta_{nn} = \beta_t \Theta_{nn} \Theta_{uu}^t - \beta_t (\Theta_{ul}^t)^2,$$

$$\Theta_{ul}^t \Theta_{ul}^t + \Theta_{ln} \Theta_{ln}^t + \Theta_{nn} \Theta_{nn}^t = 0, \quad \Theta_{ln} \Theta_{ul}^t + \Theta_{nn} \Theta_{ul}^t + \Theta_{nn} \Theta_{lu}^t = 0.$$ 

In particular, $\Theta_{ul}^t = \Theta_{ul}$ and $\Theta_{nn}^t = \Theta_{nn}$ for some constants $\Theta_{ul}, \Theta_{nn} \in \mathbb{R}$. 

PARALLEL SPINOR FLOWS 13
Proof. A direct computation shows that equation $\partial_t(\Theta_t(e^t_a)) = 0$ is equivalent to:

$$\partial_t \Theta^t_{ab} = \beta_t \Theta^t_{ua} \Theta^t_{ab}.$$ 

On the other hand, equation $d\Theta_t(e^t_a) = 0$ is equivalent to:

$$\Theta^t_{ua} \Theta^t_{al} = 0, \quad \Theta^t_{ua} \Theta^t_{an} = 0.$$ 

The previous equations can be combined into the following equivalent conditions:

$$\partial_t \Theta^t_{ua} = \beta_t ( (\Theta^t_{ua})^2 + (\Theta^t_{ua})^2 ) , \quad \partial_t \Theta^t_{ul} = \partial_t \Theta^t_{an} = 0,$$

$$\Theta^t_{ul} \Theta^t_{ul} + \Theta^t_{in} \Theta^t_{ul} + \Theta^t_{il} \Theta^t_{ul} = 0 , \quad \Theta^t_{in} \Theta^t_{ul} + \Theta^t_{nn} \Theta^t_{ul} + \Theta^t_{nn} \Theta^t_{an} = 0 ,$$

which recover five of the equations in the statement. Similarly, equation $d(\Theta_t(e^t) \wedge e^t_a) = 0$ is equivalent to:

$$\Theta^t_{in} \Theta^t_{ul} = \Theta^t_{in} \Theta^t_{un} , \quad \Theta^t_{in} \Theta^t_{un} = \Theta^t_{nn} \Theta^t_{ul} ,$$

which yields the third line of equations in the statement. We take now the exterior derivative of the first equation in (4.1) and combine the result with the second equation in (4.1):

$$d(\partial_t e_a^t + \beta_t \Theta_t(e^t_a)) = \partial_t(\Theta^t_{ab} e^t_b \wedge e^t_a) + \beta_t \Theta^t_{ab} \Theta^t_{ac} e^t_c \wedge e^t_a = (\partial_t \Theta^t_{ab} \delta_{ac} - \beta_t \Theta^t_{ab} \Theta^t_{ac} ) e^t_b \wedge e^t_c = 0.$$ 

Expanding the previous equation we obtain the remaining three equations in the statement and we conclude. \hfill \Box

Remark 4.3. We will refer to the equations of Lemma 4.2 as the integrability conditions of the left-invariant parallel spinor flow.

The following observation is crucial in order to decouple the left-invariant parallel spinor flow equations.

Lemma 4.4. A pair $\{\beta_t, e^t\}_{t \in \mathbb{I}}$ is a left-invariant parallel spinor flow if and only if there exists a family of left-invariant two-tensors $\{\Theta_t\}_{t \in \mathbb{I}}$ such that the following equations are satisfied:

$$\partial_t e^t_a + \beta_t \Theta_t(e^t_a) = 0 , \quad d e^t = K_t(e^t) \wedge e^t_a , \quad \partial_t (K_t(e^t_a)) = 0 , \quad d(K_t(e^t_a)) = 0 .$$

Proof. The only if direction follows immediately from the definition of left-invariant parallel Cauchy pair by taking $\{K_t\}_{t \in \mathbb{I}} = \{\Theta_t\}_{t \in \mathbb{I}}$. For the if direction we simply compute:

$$\Theta_t = -\frac{1}{2\beta_t} \partial_t h_t = -\frac{1}{2\beta_t} ((\partial_t e^t_a) \otimes e^t_a + e^t_a \otimes (\partial_t e^t_a)) = K_t ,$$

hence equations (4.1) are satisfied and $\{\beta_t, e^t\}_{t \in \mathbb{I}}$ is a left-invariant parallel spinor flow. \hfill \Box

By the previous Lemma we promote the components of $\{\Theta_t\}_{t \in \mathbb{I}}$ with respect to the basis $\{e^t\}_{t \in \mathbb{I}}$ to be independent variables of the left-invariant parallel spinor flow equations (4.1). Within this interpretation, the variables of left-invariant parallel spinor flow equations consist of triples $\{\beta_t, e^t, \Theta_{ab}^t\}_{t \in \mathbb{I}}$, where $\{\Theta_{ab}^t\}_{t \in \mathbb{I}}$ is a family of symmetric matrices. On the other hand, the integrability conditions of Lemma 4.2 are interpreted as a system of equations for a pair $\{\beta_t, \Theta_{ab}^t\}_{t \in \mathbb{I}}$. In particular, the first equation in (4.1) is linear in the variable $e^t$ and can be conveniently rewritten as follows. For any family of coframes $\{e^t\}_{t \in \mathbb{I}}$, set $e = e^0$ and consider the unique smooth path:

$$U^t : \mathbb{I} \rightarrow GL_+ (3, \mathbb{R}) , \quad t \mapsto U^t ,$$

such that $e^t = U^t(e)$, where $GL_+(3, \mathbb{R})$ denotes the identity component in the general linear group $GL(3, \mathbb{R})$. More explicitly:

$$e^t_a = \sum_b U^t_{ab} e_t , \quad a, b = u, l, n ,$$

where $U^t_{ab} \in C^\infty(G)$ are the components of $U^t$. Plugging $e^t = U^t(e)$ in the first equation in (4.1) we obtain the following equivalent equation:

$$\partial_t U^t_{ab} + \beta_t \Theta^t_{ab} U^t_{bc} = 0 , \quad a, b, c = u, l, n , \quad (4.2)$$

with initial condition $U^0 = Id$.

A necessary condition for a solution $\{\beta_t, \Theta_{ab}^t\}_{t \in \mathbb{I}}$ of the integrability conditions to arise from an honest left-invariant parallel spinor pair is the existence of a left-invariant coframe $e$ on $\Sigma$ such that $(e, \Theta)$ is a Cauchy pair, where $\Theta = \Theta^t_{ab} e_t \otimes e_t$. Consequently we define the set $I(\Sigma)$ of admissible solutions to the integrability equations as the set of pairs $(\{\beta_t, \Theta_{ab}^t\}_{t \in \mathbb{I}}, e)$ such that $\{\beta_t, \Theta_{ab}^t\}_{t \in \mathbb{I}}$ is a solution to the integrability equations and $(e, \Theta)$ is a left-invariant parallel Cauchy pair.
Proposition 4.5. There exists a natural bijection \( \varphi : I(\Sigma) \rightarrow \mathcal{P}(\Sigma) \) which maps every pair:

\[
(\{\beta_t, \Theta^t_{ab}\}_{t \in I}, \epsilon) \in I(\Sigma),
\]

to the pair \( (\{\beta_t, e^t\} = U^t(\epsilon)\}_{t \in I} \in \mathcal{P}(\Sigma) \), where \( \{U^t\}_{t \in I} \) is the unique solution of (4.2) with initial condition \( U^0 = \text{Id} \).

Remark 4.6. The inverse of \( \varphi \) maps every left-invariant parallel spinor flow \( \{\beta_t, e^t\}_{t \in I} \) to the pair \( (\{\beta_t, \Theta^t_{ab}\}, \epsilon) \), where \( \Theta^t_{ab} \) are the components of the shape operator associated to \( \{\beta_t, e^t\}_{t \in I} \) in the basis \( \{e^t\}_{t \in I} \) and \( \epsilon = e^0 \).

Proof. Let \( (\{\beta_t, \Theta^t_{ab}\}, \epsilon) \in I(\Sigma) \) and let \( \{U^t\}_{t \in I} \) be the solution of (4.2) with initial condition \( U^0 = \text{Id} \), which exists and is unique on \( I \) by standard ODE theory [10, Theorem 5.2]. We need to prove that \( \{\beta_t, e^t = U^t(\epsilon)\}_{t \in I} \) is a left-invariant parallel spinor flow. Since \( \{U^t\}_{t \in I} \) satisfies (4.2) for the given \( \{\beta_t, \Theta^t_{ab}\} \), it follows that \( \Theta^t = \Theta^t_{ab} e^t_a \otimes e^t_b \) is the shape operator associated to \( \{\beta_t, e^t\}_{t \in I} \) whence the first equation in (4.1) is satisfied. On the other hand, the third and fourth equations in (4.1) are immediately implied by the integrability conditions satisfied by \( \{\beta_t, \Theta^t_{ab}\} \).

Regarding the second equation in (4.1), we observe that the integrability conditions contain the equation \( d(\Theta^t(\epsilon') \land e^t_a) = 0 \) and thus:

\[
de^t = \Theta^t(\epsilon') \land e^t_a + w^t,
\]

where \( \{w^t\}_{t \in I} \) is a family of triplets of closed two-forms on \( \Sigma \). Taking the time derivative of the previous equations, plugging the exterior derivative of the first equation in (4.1) and using again the integrability conditions, we obtain that \( w^t \) satisfies the following differential equation:

\[
\partial_t w^t_a = -\beta_t e^t_a \, w^t + \beta_t e^t_a \, w^t,
\]

with initial condition \( w^0 = w \). Restricting equation (4.3) to \( t = 0 \) it follows that \( w \) satisfies:

\[
de = \Theta(\epsilon) \land e_a + w.
\]

Since by assumption \( (\epsilon, \Theta) \) is left-invariant Cauchy pair, the previous equation is satisfied if and only if \( w = 0 \) whence \( w^t = 0 \) by uniqueness of solutions of the linear differential equation (4.4). Therefore, the second equation in (4.1) follows and \( \varphi \) is well-defined. The fact that \( \varphi \) is in addition a bijection follows directly by Remark 4.6 and hence we conclude.

Corollary 4.7. A pair \( \{\beta_t, e^t\}_{t \in I} \) is a parallel spinor flow if and only if \( \{\{\beta_t, \Theta^t_{ab}\}, \epsilon\} \) is an admissible solution to the integrability equations.

Therefore, solving the left-invariant parallel spinor flow is equivalent to solving the integrability conditions with initial condition \( \Theta_{ab} \) being part of a left-invariant parallel Cauchy pair \( (\epsilon, \Theta) \). We remark that \( \{\beta_t\}_{t \in I} \) is of no relevance locally since it can be eliminated through a reparametrization of time after possibly shrinking \( I \). However, regarding the long time existence of the flow as well as for applications to the construction of four-dimensional Lorentzian metrics it is convenient to keep track of \( I \), whence we maintain \( \{\beta_t\}_{t \in I} \) in the equations.

For further reference we define a quasi-diagonal left-invariant parallel spinor flow as one for which \( \lambda = \sqrt{\Theta^t_{uu} + \Theta^t_{nn}} = 0 \). Since the function \( t \to \int_0^t \beta_t d\tau \) is going to be a common occurrence in the following, we define:

\[
B_t := \int_0^t \beta_t d\tau.
\]

We distinguish now between the cases \( \lambda = 0 \) and \( \lambda \neq 0 \).

Lemma 4.8. Let \( \{\beta_t, e^t\}_{t \in I} \) be a quasi-diagonal left-invariant parallel spinor flow. Then, the only non-zero components of \( \Theta^t \) are:

\[
\Theta^t_{uu} = \frac{\Theta_{uu}}{1 - \Theta_{uu} B_t}, \quad \Theta^t_{ll} = \frac{\Theta_{ll}}{1 - \Theta_{uu} B_t}, \quad \Theta^t_{ln} = \frac{\Theta_{ln}}{1 - \Theta_{uu} B_t}, \quad \Theta^t_{nn} = \frac{\Theta_{nn}}{1 - \Theta_{uu} B_t},
\]

where \( \Theta^t \) is the shape operator associated to \( \{\beta_t, e^t\}_{t \in I} \) and \( \Theta = \Theta^0 \). Furthermore, every such \( \Theta^t \) satisfies the integrability equations with quasi-diagonal initial data.

Proof. Setting \( \Theta_{ul} = \Theta_{nn} = 0 \) in the integrability conditions we obtain the following equations:

\[
\partial_t \Theta^t_{uu} = \beta_t (\Theta^t_{uu})^2, \quad \partial_t \Theta^t_{ll} = \beta_t \Theta^t_{ll} \Theta^t_{uu}, \quad \partial_t \Theta^t_{ln} = \beta_t \Theta^t_{ln}, \quad \partial_t \Theta^t_{nn} = \beta_t \Theta^t_{nn} \Theta^t_{uu}.
\]

whose general solution is given in the statement of the lemma.
Remark 4.9. Let $\Theta_{uu} \neq 0$ and define $t_0$ to be the real number (in case it exists) with the smallest absolute value such that:

$$\int_0^{t_0} \beta_0 \, dr = \Theta_{uu}^{-1}.$$ 

Then the maximal interval on which $\Theta^t$ is defined is $I = (-\infty, t_0)$ if $\Theta_{uu} > 0$ and $I = (t_0, \infty)$ if $\Theta_{uu} < 0$. This is also the maximal interval on which the left-invariant parallel spinor flow in the quasi-diagonal case can be defined. If such $t_0$ does not exist, then $I = \mathbb{R}$.

We consider now the non-quasi-diagonal case $\lambda \neq 0$. Given a pair $\{\beta_t, \Theta_{ab}^t\}_{t \in I}$, we introduce for convenience the following function:

$$I \ni t \mapsto y_t = \lambda B_t + \text{Arctan} \left[ \frac{\Theta_{uu}}{\lambda} \right],$$

where $\Theta_{ab}$ are the components of $\Theta$ in the basis $e$.

Lemma 4.10. A pair $\{\beta_t, \Theta_{ab}^t\}_{t \in I}$ satisfies the integrability equations with non-quasi-diagonal initial value $\Theta_{ab}$ if and only if:

$$\Theta_{uu}^t = \lambda \tan [y_t] , \quad \Theta_{ul}^t = \Theta_{tu} , \quad \Theta_{un}^t = \Theta_{nu} , \quad \Theta_{ll}^t = c_{ll} \sec [y_t] - \frac{\Theta_{uu}^2}{\lambda} \tan [y_t] ,$$

$$\Theta_{nn}^t = c_{nn} \sec [y_t] - \frac{\Theta_{uu}^2}{\lambda} \tan [y_t] , \quad \Theta_{in}^t = c_{in} \sec [y_t] - \frac{\Theta_{uu}^2}{\lambda} \lambda,$$

where $c_{ll}, c_{nn}, c_{in} \in \mathbb{R}$ are real constants given by:

$$c_{ll} = \frac{\Theta_{ll}^2 + \Theta_{ul}^2 \Theta_{uu}^2}{\lambda \sqrt{\lambda^2 + \Theta_{uu}^2}} , \quad c_{nn} = \frac{\Theta_{nn}^2 + \Theta_{un}^2 \Theta_{uu}^2}{\lambda \sqrt{\lambda^2 + \Theta_{uu}^2}} , \quad c_{in} = \frac{\Theta_{in}^2 + \Theta_{ul}^2 \Theta_{uu} \Theta_{uu}^2}{\lambda \sqrt{\lambda^2 + \Theta_{uu}^2}},$$

such that the following algebraic equations are satisfied:

$$\Theta_{ta} \Theta_{ul} = \Theta_{tl} \Theta_{un} , \quad \Theta_{ta} \Theta_{un} = \Theta_{tn} \Theta_{uu} ,$$

$$\Theta_{ta} \Theta_{un} + \Theta_{ta} (\Theta_{ll} + \Theta_{uu}) = 0 , \quad \Theta_{ta} \Theta_{ul} + \Theta_{ta} (\Theta_{nn} + \Theta_{uu}) = 0,$$

where $\Theta_{ab}$, $a, b = u, l, n$, denote the entries of $\Theta_{ab}^t$ at $t = 0$.

Remark 4.11. Note that equations (4.5) form an algebraic system for the entries of the initial condition $\Theta$, therefore restricting the allowed initial data that can be used to solve the integrability conditions. This is a manifestation of the fact that the initial data of the parallel spinor flow is constrained by the parallel Cauchy equations. The latter were solved in the left-invariant case in [28], as summarized in Theorem 3.12, and its solutions can be easily verified to satisfy equations (4.5) automatically.

Proof. By Lemma 4.2 we have $\partial_t \Theta_{uu}^t = \partial_t \Theta_{uu}^t = 0$ whence $\Theta_{uu}^t = \Theta_{uu}$, $\Theta_{uu}^t = \Theta_{uu}$ for some real constants $\Theta_{uu}, \Theta_{uu} \in \mathbb{R}$. Plugging these constants into the first equation of Lemma 4.2 it becomes immediately integrable with solution:

$$\Theta_{uu}^t = \lambda \tan \left[ \frac{y_t}{\lambda} + k_t \right]$$

for a certain constant $k_t \in \mathbb{R}$. Imposing $\Theta_{uu}^0 = \Theta_{uu}$ we obtain:

$$k_t = \frac{1}{\lambda} \left( \text{Arctan} \left[ \frac{\Theta_{uu}}{\lambda} \right] + n\pi \right) , \quad n \in \mathbb{Z},$$

and the expression for $\Theta_{uu}^t$ follows. Plugging now $\Theta_{uu}^t = \lambda \tan [y_t]$ in the remaining differential equations of Lemma 4.2 they can be directly integrated, yielding the expressions in the statement after imposing $\Theta_{ab}^0 = \Theta_{ab}$. Plugging the explicit expressions for $\Theta_{uu}^t$ in the algebraic equations of Lemma 4.2, these can be equivalently reformulated as the algebraic system (4.5) for $\Theta_{ab}^t$ at $t = 0$ and we conclude. □

Remark 4.12. Let $t_- < 0$ denote the largest value for which $\lambda B_{t_-} + \text{Arctan} \left[ \frac{\Theta_{uu}}{\lambda} \right] = -\frac{\pi}{2}$ and let $t_+ > 0$ denote the smallest value for which $\lambda B_{t_+} + \text{Arctan} \left[ \frac{\Theta_{uu}}{\lambda} \right] = \frac{\pi}{2}$ (if $t_-, t_+$ or both do not exist, we take by convention $t_\pm = \pm \infty$). Then, the maximal interval of definition on which $\Theta^t$ is defined is $I = (t_-, t_+)$. 

4.2. Classification of left-invariant spinor flows. Proposition 4.2 states that $\Theta^t_{ab} = \Theta_{ab}$ and $\Theta^t_{un} = \Theta_{un}$ for constants $\Theta_{ab}, \Theta_{un} \in \mathbb{R}$. Therefore, we proceed to classify left-invariant parallel spinor flows in terms of the possible values of $\Theta_{ab}$ and $\Theta_{un}$. We begin with the classification of quasi-diagonal left-invariant parallel spinor flows, defined by the condition $\Theta_{ab} = \Theta_{un} = 0$, that is, $\lambda = 0$.

**Proposition 4.13.** Let $\{\beta_t, c^t\}_{t \in \mathbb{T}}$ be a quasi-diagonal left-invariant parallel spinor flow with initial data $(c, \Theta)$ satisfying $\Theta_{uu} \neq 0$. Define $Q$ to be the orthogonal two by two matrix diagonalizing $\theta/\Theta_{uu}$ as follows:

$$\frac{\theta}{\Theta_{uu}} = Q \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix} Q^*$$

with eigenvalues $\rho_+$ and $\rho_-$ and where $Q^*$ denotes the matrix transpose of $Q$. Then:

$$c^t_u = (1 - \Theta_{uu} B_t)c_u, \quad \begin{pmatrix} c^t_l \\ c^t_n \end{pmatrix} = Q \begin{pmatrix} 1 - [\Theta_{uu} B_t]^\rho_+ & 0 \\ 0 & [1 - \Theta_{uu} B_t]^\rho_- \end{pmatrix} Q^* \begin{pmatrix} e^t_l \\ e^t_n \end{pmatrix}$$

Conversely, for every family of functions $\{\beta_t\}_{t \in \mathbb{T}}$ the previous expression defines a parallel spinor flow on $G$. The case $\Theta_{uu} = 0$ is recovered by taking the formal limit $\Theta_{uu} \to 0$.

**Proof.** Define the function $I \ni t \to x_t := \log[1 - \Theta_{uu} B_t]$. By Proposition 4.5 and Corollary 4.7 it suffices to use the explicit expression for $\Theta_t^t$ obtained in Lemma 4.8 to solve Equation (4.2) with initial condition $U^0 = \text{Id}$ on a simply connected Lie group admitting quasi-diagonal parallel Cauchy pairs. Plugging the explicit expression of $\Theta_t^t$ in (4.2) we obtain:

$$\begin{pmatrix} \partial_t U^t_{uu} \\ \partial_t U^t_{ul} \\ \partial_t U^t_{lu} \\ \partial_t U^t_{nn} \end{pmatrix} = \frac{\partial x_t}{\Theta_{uu}} \begin{pmatrix} U^t_{uu} \Theta_{uu} \\ U^t_{ul} \Theta_{lu} + U^t_{lu} \Theta_{ul} \\ U^t_{nn} \Theta_{nn} \end{pmatrix}$$

in terms of the initial data $\Theta_{ab}$. The previous differential system can be equivalently written as follows:

$$\partial_t U^t_{ac} = \partial_t x_t U^t_{ac}, \quad \begin{pmatrix} \Omega_t \\ \partial_t \Omega_t \\ \partial_t \Omega_t \\ \partial_t \Omega_t \end{pmatrix} = \begin{pmatrix} 1 \Theta_{uu} \Theta_{lu} \\ \Theta_{lu} \Theta_{nn} \end{pmatrix} \begin{pmatrix} \Omega_t \\ \Theta_{lu} \\ \Theta_{nn} \end{pmatrix}$$

The general solution to the equations for $U^t_{ac}$ with initial condition $U^0 = \text{Id}$ is given by:

$$U^t_{uu} = 1 - \Theta_{uu} B_t, \quad U^t_{un} = U^t_{nn} = 0.$$

Consider now the diagonalization of the constant matrix occurring in the differential equations for $U^t_{ac}$:

$$\frac{1}{\Theta_{uu}} \begin{pmatrix} \Theta_{uu} & \Theta_{lu} \\ \Theta_{lu} & \Theta_{nn} \end{pmatrix} = Q \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix} Q^*,$$

where $Q$ is a two by two orthogonal matrix and $Q^*$ is its transpose. The eigenvalues are explicitly given by:

$$\rho_{\pm} = \frac{T \pm \sqrt{T^2 - 4\lambda}}{2\Theta_{uu}}.$$

We obtain:

$$Q^* \begin{pmatrix} \partial_t U^t_{ac} \\ \partial_t \Omega_t \end{pmatrix} = \partial_t x_t \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix} Q^* \begin{pmatrix} U^t_{ac} \\ \Omega_t \end{pmatrix}, \quad c = u, l, n,$$

whose general solution is given by:

$$\begin{pmatrix} U^t_{ac} \\ \Omega_t \end{pmatrix} = Q \begin{pmatrix} k^+_{ac} e^{\rho_+ x_t} \\ k^-_{ac} e^{\rho_- x_t} \end{pmatrix} = Q \begin{pmatrix} 1 - [\Theta_{uu} B_t]^\rho_+ \\ [1 - \Theta_{uu} B_t]^\rho_- \end{pmatrix}, \quad c = u, l, n,$$

for constants $k^+_c, k^-_c \in \mathbb{R}$. Imposing the initial condition $U^0 = \text{Id}$ we obtain the following expression for $k^+_c$ and $k^-_c$:

$$\begin{pmatrix} k^+_c \\ k^-_c \end{pmatrix} = Q^* \begin{pmatrix} \delta_{ac} \\ \delta_{ac} \end{pmatrix}, \quad c = u, l, n,$$

whence:

$$\begin{pmatrix} k^+_u \\ k^-_u \end{pmatrix} = 0, \quad \begin{pmatrix} k^+_l \\ k^-_l \end{pmatrix} = Q^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} Q^*_u \\ Q^*_l \end{pmatrix}, \quad \begin{pmatrix} k^+_n \\ k^-_n \end{pmatrix} = Q^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} Q^*_n \\ Q^*_n \end{pmatrix}.$$

We conclude that:

$$\begin{pmatrix} U^t_{ul} \\ U^t_{ln} \end{pmatrix} = Q \begin{pmatrix} e^{\rho_+ x_t} \\ 0 \end{pmatrix} Q^* \begin{pmatrix} 1 - [\Theta_{uu} B_t]^\rho_+ \\ [1 - \Theta_{uu} B_t]^\rho_- \end{pmatrix} Q^*$$
and the statement is proven. The converse follows by construction upon use of Lemma 4.4 and Proposition 4.5. It can be easily seen that the case $\Theta_{uu} = 0$ is obtained by taking the formal limit $\Theta_{uu} \to 0$ and we conclude.

\[ \widetilde{\text{Remark 4.14.}} \text{ The Ricci tensor of the family of Riemannian metrics } \{h_t\}_{t \in \mathcal{I}} \text{ associated to a left-invariant quasi-diagonal parallel spinor flow } \{\beta_t, \epsilon_t\}_{t \in \mathcal{I}} \text{ is given by:} \]

\[ \text{Ric}^{h_{\epsilon_t}} = -T^t \Theta_t + \frac{\mathcal{H}_t}{2} e_t \otimes e_t, \]  

where $T^t = \Theta_{t}^1 + \Theta_{t}^{1n}$.

If $\mathcal{H}_t = 0$ for every $t \in \mathcal{I}$, that is, if the parallel Cauchy pair defined by $\{\beta_t, \epsilon_t\}_{t \in \mathcal{I}}$ is constrained Ricci flat, then:

\[ \text{Ric}^{h_{\epsilon_t}} = \frac{T^t}{2} \partial_t h_{\epsilon_t}, \]  

which, after a reparametrization of the time coordinate can be brought into the form $\text{Ric}^{h_{\epsilon_t}} = -2 \partial_t h_{\epsilon_t}$ after possibly shrinking $\mathcal{I}$. Hence, this gives a particular example of a left-invariant Ricci flow on $G$.

We consider now $\Theta_{ul} \Theta_{un} = 0$ but $\Theta_{ul}^2 + \Theta_{un}^2 \neq 0$. This case necessarily corresponds to $G = \tau_2 \oplus \mathbb{R}$.

\[ \text{Proposition 4.15. Let } \{\beta_t, \epsilon_t\}_{t \in \mathcal{I}} \text{ be a left-invariant parallel spinor flow with initial parallel Cauchy pair } (\epsilon, \Theta) \text{ satisfying } \Theta_{ul} \Theta_{un} = 0 \text{ and } \lambda \neq 0. \text{ Then:} \]

- If $\Theta_{ul} = 0$ the following holds:

\[ e_{u}^t = (1 - \Theta_{uu} B_t) e_u - \Theta_{un} B_t e_n, \quad e_{n}^t = e_{u}, \]

\[ \text{where we have also used that, in this case, } \Theta_{un} = \Theta_{uu} = 0 \text{ as summarized in Theorem 3.12. Hence:} \]

\[ \Theta^t = \Theta_{un} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda \tan [y_t] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]

and Equation (4.2) reduces to:

\[ \partial_t \mathcal{U}^t + \beta_t \Theta_{un} \begin{pmatrix} \mathcal{U}_{uu} & \mathcal{U}_{ul} & \mathcal{U}_{un} \\ 0 & 0 & 0 \\ \mathcal{U}_{uu} & \mathcal{U}_{ul} & \mathcal{U}_{un} \end{pmatrix} + \lambda \beta_t \tan [y_t] \begin{pmatrix} \mathcal{U}_{uu} & \mathcal{U}_{ul} & \mathcal{U}_{un} \\ 0 & 0 & 0 \\ -\mathcal{U}_{uu} & -\mathcal{U}_{ul} & -\mathcal{U}_{un} \end{pmatrix} = 0, \]

or, equivalently:

\[ \partial_t \mathcal{U}_{uc}^t + \beta_t \lambda \tan [y_t] \mathcal{U}_{uc}^t + \beta_t \Theta_{un} \mathcal{U}_{nc}^t = 0, \quad \partial_t \mathcal{U}_{uc}^t = 0, \quad \partial_t \mathcal{U}_{nc}^t - \beta_t \lambda \tan [y_t] \mathcal{U}_{nc} + \beta_t \Theta_{un} \mathcal{U}_{uc}^t = 0. \]

The general solution to this system with initial condition $\mathcal{U}^0 = \text{Id}$ is given by:

\[ \mathcal{U}_{uu}^t = 1 - \Theta_{uu} B_t, \quad \mathcal{U}_{un}^t = -\Theta_{un} B_t, \quad \mathcal{U}_{ul}^t = \mathcal{U}_{ul}^0, \quad \mathcal{U}_{n}^t = \mathcal{U}_{nc}^0 = 0, \quad \mathcal{U}_{il}^t = 1, \]

\[ \mathcal{U}_{nn}^t = 1 + \lambda B_t \tan [y_t], \quad \mathcal{U}_{nu}^t = \frac{\Theta_{nu}}{\Theta_{nn}} - \frac{\lambda}{\Theta_{nn}} (1 - \Theta_{uu} B_t) \tan [y_t], \]

which implies the statement. The converse follows by construction upon use of Lemma 4.4 and Proposition 4.5. \(\square\)
Remark 4.16. The Ricci tensor of the family of metrics \( \{ h_\tau \} \) associated to a left-invariant parallel spinor with if \( \Theta_{ul} = 0 \) but \( \Theta_{un} \neq 0 \) reads:

\[
\text{Ric}^{h_\tau} = -\Theta_t \circ \Theta_t = \frac{H_t}{4} (h_\tau - e_n^t \otimes e_n^t).
\]

Recall that \( \nabla h_\tau \) is 0 and thus \( \{ h_\tau, e_n^t \} \) defines a family of \( \eta \)-Einstein cosymplectic structures [9, 29]. On the other hand, if \( \Theta_{un} = 0 \) but \( \Theta_{ul} \neq 0 \) the curvature of \( \{ h_\tau \} \) is given by:

\[
\text{Ric}^{h_\tau} = -\Theta_t \circ \Theta_t = \frac{H_t}{4} (h_\tau - e_n^t \otimes e_n^t).
\]

whence \( \{ h_\tau, e_n^t \} \) defines as well a family of \( \eta \)-Einstein cosymplectic structures on \( G \).

Finally we consider \( \Theta_{ul} \Theta_{un} \neq 0 \), a case that again corresponds to \( G = \nu \otimes \mathbb{R} \).

Proposition 4.17. Let \( \{ \beta_t, e^t \} \) be a left-invariant parallel spinor flow with initial parallel Cauchy pair \((e, \Theta)\) satisfying \( \Theta_{ul} \Theta_{un} \neq 0 \). Then:

\[
e_t^t = e_t + B_t (T e_t - \Theta_{ul} e_t - \Theta_{un} e_n),
\]

\[
e_t^t = \frac{\Theta_{ul}}{\alpha} (\frac{1}{\lambda} - 1 + (1 + T B_t) \tan[y_t]) e_t + \left(1 + \frac{\Theta_{ul} e_t e_n}{\lambda} \tan[y_t] \right) e_t + \frac{\Theta_{un} e_n e_n}{\lambda} \tan[y_t] e_n,
\]

\[
e_t^t = \frac{\Theta_{ul}}{\alpha} (\frac{1}{\lambda} - 1 + (1 + T B_t) \tan[y_t]) e_t + \frac{\Theta_{un} e_n e_n}{\lambda} \tan[y_t] e_t + \left(1 + \frac{\Theta_{ul} e_t e_n}{\lambda} \tan[y_t] \right) e_n,
\]

Conversely, every such family \( \{ \beta_t, e^t \} \) is a left-invariant parallel spinor flow for every \( \{ \beta_t \} \).

Proof. Assuming \( \Theta_{ul}, \Theta_{un} \neq 0 \) in Lemma 4.10 we obtain:

\[
\Theta_t^u = \lambda \tan[y_t], \quad \Theta_{ul}^t = \frac{\Theta_{ul}}{\Theta_{un}} \Theta_t^t, \quad \Theta_{un}^t = \frac{\Theta_{un}}{\Theta_{ul}} \Theta_t^t, \quad \Theta_t^u = -\frac{\Theta_{ul}}{\Theta_{un}} \Theta_t^t.
\]

Note that \( \Theta_t^u = -\Theta_{ul}^t - \Theta_{un}^t \). Hence:

\[
\Theta_t = \left( \begin{array}{ccc}
0 & \Theta_{ul}^t & \Theta_{un}^t \\
0 & 0 & 0 \\
\Theta_{un}^t & 0 & 0
\end{array} \right) - \lambda \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right),
\]

and Equation (4.2) reduces to:

\[
\frac{1}{\Theta_{ul}} \frac{\partial U^t}{\partial t} + U^t \Theta_{ul} + U^t \Theta_{un} + U^t \Theta_{uu} = 0,
\]

\[
\frac{1}{\Theta_{un}} \frac{\partial U^t}{\partial t} + \Theta_{ul} (U^t \Theta_{ul} - \lambda ^{-1} (\Theta_{ul} U^t + \Theta_{un} U^t) \tan[y_t]) = 0,
\]

\[
\frac{1}{\Theta_{ul}} \frac{\partial U^t}{\partial t} + \Theta_{un} (U^t \Theta_{un} - \lambda ^{-1} (\Theta_{ul} U^t + \Theta_{un} U^t) \tan[y_t]) = 0.
\]

The general solution to this system with initial condition \( U^t = \text{Id} \) is given by:

\[
U^t_{uu} = 1 - \Theta_{ul} B_t, \quad U^t_{ul} = -\Theta_{ul} B_t, \quad U^t_{un} = -\Theta_{un} B_t,
\]

\[
U^t_{ul} = \frac{\Theta_{ul}}{\alpha} \left( \frac{1}{\lambda} - 1 - (1 + \Theta_{ul} B_t) \tan[y_t] \right), \quad U^t_{ll} = 1 + \frac{\Theta_{ul} B_t}{\lambda} \tan[y_t], \quad U^t_{ln} = \frac{\Theta_{un} e_n e_n}{\lambda} \tan[y_t],
\]

\[
U^t_{uu} = \frac{\Theta_{ul}}{\alpha} \left( \frac{1}{\lambda} - 1 - (1 + \Theta_{un} B_t) \tan[y_t] \right), \quad U^t_{ln} = \frac{\Theta_{ul} e_n e_n}{\lambda} \tan[y_t], \quad U^t_{un} = 1 + \frac{\Theta_{ul} e_t e_n}{\lambda} \tan[y_t],
\]

and we conclude.

Remark 4.18. The three-dimensional Ricci tensor of the family of Riemannian metrics \( \{ h_\tau \} \) associated to a left invariant parallel spinor flow with \( \Theta_{ul} \Theta_{un} \neq 0 \) reads:

\[
\text{Ric}^{h_\tau} = -\Theta_t \circ \Theta_t = \frac{H_t}{4} (h_\tau - \eta_t \otimes \eta_t), \quad \eta_t = \frac{1}{\sqrt{\Theta_{ul}^t + \Theta_{un}^t}} (\Theta_{un} e_t^t - \Theta_{ul} e_n^t),
\]

Note that \( \nabla h_\tau \eta_t = 0 \), so \( \{ h_\tau, \eta_t \} \) defines a family of \( \eta \)-Einstein cosymplectic Riemannian structures on \( G \).

As a corollary to the classification of left-invariant parallel spinor flows presented in Propositions 4.13, 4.15 and 4.17 we can explicitly obtain the evolution of the Hamiltonian constraint in each case.

**Corollary 4.19.** Let \( \{ \beta_t, e^t \} \) be a left-invariant parallel spinor in \((M, g)\).

- If \( \Theta_{ul} = \Theta_{un} = 0 \), then \( H_t = \frac{H_{\tau}}{1 - \Theta_{un} B_t} \).
- If \( \Theta_{ul} = 0 \) but \( \Theta_{un} \neq 0 \) then \( H_t = \frac{\Theta_{ul} e_n e_n + \Theta_{un} e_n e_n}{\Theta_{ul} e_n e_n + \Theta_{un} e_n e_n} \sec^2 \left[ \lambda B_t + \arctan \left( \frac{\Theta_{un}}{\lambda} \right) \right] \).
- If \( \Theta_{un} = 0 \) but \( \Theta_{ul} \neq 0 \) then \( H_t = \frac{\Theta_{ul} e_n e_n + \Theta_{un} e_n e_n}{\Theta_{ul} e_n e_n + \Theta_{un} e_n e_n} \sec^2 \left[ \lambda B_t + \arctan \left( \frac{\Theta_{ul}}{\lambda} \right) \right] \).
• If $\Theta_{ul}, \Theta_{un} \neq 0$ then $H_t = \frac{\lambda^2 H_0}{X + \Theta_{un}} \sec^2 \left[ \lambda E_1 + \arctan \left( \frac{\Theta_{ul}}{X} \right) \right]$.

where $H_0$ is the Hamiltonian constraint at time $t = 0$.

Since the secant function has no zeroes, the Hamiltonian constraint vanishes for a given $t \in \mathcal{I}$, and hence for every $t \in \mathcal{I}$, if and only if it vanish at $t = 0$, consistently with Theorem 3.9. Theorem 3.12 implies that only quasi-diagonal left-invariant parallel spinor flows admit constrained Ricci flat initial data. Therefore the Hamiltonian constraint of left-invariant parallel spinor flows with $\lambda \neq 0$ is non-vanishing for every $t \in \mathcal{I}$ and such left-invariant parallel spinor flows cannot produce four-dimensional Ricci flat Lorentzian metrics.

4.3. Proof of Theorem 1.4. Theorem 1.4 follows through a direct computation by using the explicit form of the left-invariant parallel spinor flow obtained in Propositions 4.13, 4.15 and 4.17 for each of the possible cases, after using Theorem 3.12 to identify the underlying Lie group in each case.

References

[1] H. Baum, T. Leistner and A. Lischewski, Cauchy problems for Lorentzian manifolds with special holonomy, Differential Geom. Appl. 45, 43 - 66 (2016).

[2] H. Baum and A. Lischewski, Lorentzian Geometry - Holonomy, Spinors, and Cauchy Problems, in V. Cortés, K. Kröncke, J. Louis (eds.), Geometric Flows and the Geometry of Space-time, Birkhäuser, 2018.

[3] I. M. Benn and R. W. Tucker, Fermions without spinors. Commun. Math. Phys. 80, 341-362 (1983).

[4] A. N. Bernal and M. Sánchez, On Smooth Cauchy hypersurfaces and Geroch’s splitting theorem, Commun. Math. Phys. 243, 461 (2003).

[5] A. N. Bernal and M. Sánchez, Smoothness of time functions and the metric splitting of globally hyperbolic space-times, Commun. Math. Phys. 257 (2005), 43–50.

[6] F. Bonsante and A. Seppi, Anti-de Sitter geometry and Teichmüller theory, In the tradition of Thurston (ed. K. Ohshika and A. Papadopoulos), Springer Verlag, 2020.

[7] H. W. Brinkmann, Einstein spaces which are mapped conformally on each other, Mathematische Annalen volume 94, 119 - 145 (1925).

[8] Robert L. Bryant, Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor, Sémin. Congr., 4, Soc. Math. France, Paris, 2000, 53-94.

[9] B. Cappelletti-Montano, A. de Nicola and I.Yudin , A survey on cosmegy geometry, Rev. Math. Phys. 25 (2013), 1343002.

[10] E. A. Coddington and N. Levinson, Theory of ordinary differential equations, Pure & Applied Mathematics, McGraw–Hill Education, 1984.

[11] V. Cortés, K. Kröncke, J. Louis (Editors) Geometric Flows and the Geometry of Space-time, Birkhäuser, 2018.

[12] V. Cortés, C. Lazaroiu and C. S. Shahbazi, Spinors of real type as polyforms and the generalized Killing spinors, J. Geom. Phys. 201, 1351-1419 (2021).

[13] Y. Choquet-Bruhat, Théorème d’existence pour certains systèmes d’équations aux dérivées partielles non linéaires, Acta math., 88 (1952), pp. 141 - 225.

[14] M. Freibert, Cocalibrated $G_2$-structures on products of four- and three-dimensional Lie groups, Differential Geom. Appl. 31 (2013) 349-373.

[15] Y. Choquet-Bruhat, General Relativity and the Einstein Equations, Oxford Mathematical Monographs (2008).

[16] J. Ehlers and W. Kundt, Anti-de Sitter geometry and Teichmüller theory, Rev. Math. Phys. 4 (1992), no. 2, 239?250.

[17] W. Graf, Differential forms as spinors, Annales de l’IHP Physique théorique, Volume 29, (1978) Number 1.

[18] F. R. Harvey, Spinors and calibrations, Perspectives in Mathematics, 1990.

[19] H. B. Lawson and M. L. Michelsohn, Spin Geometry, Princeton Mathematical Series 38, 1990.

[20] C. Lazaroiu and C. S. Shahbazi, Real spinor bundles and real Lipschitz structures, Asian Journal of Mathematics Volume 23 (2019) Number 5.

[21] C. I. Lazaroiu and C. S. Shahbazi, Complex Lipschitz structures and bundles of complex Clifford modules, Differ. Geom. Appl. 61 (2018), 147 - 169.

[22] C. I. Lazaroiu and C. S. Shahbazi, Dirac operators on real spinor bundles of complex type, Differ. Geom. Appl. 80 (2022), 101849.

[23] T. Leistner and A. Lischewski, Hyperbolic Evolution Equations, Lorentzian Holonomy, and Riemannian Generalized Killing Spinors, J. Geom. Anal. 29, 33 - 82 (2019).

[24] A. Lischewski, The Cauchy problem for parallel spinors as first-order symmetric hyperbolic system, arXiv:1503.04946 [math.DG].

[25] M. Mars, On Local Characterization Results in Geometry and Gravitaton, In From Riemann to Differential Geometry and Relativity, Springer International Publishing 2017, pages 541–570, edited by L. Ji, A. Papadopoulos and S. Yamada.

[26] G. Mess, Lorentz spacetimes of constant curvature, Geom. Dedicata, 126:3-45, 2007.

[27] J. Milnor, Curvatures of left invariant metrics on Lie groups, Advances in Mathematics 21, 293 - 329 (1976).

[28] A. Murcia and C. S. Shahbazi, Parallel spinors on globally hyperbolic Lorentzian four-manifolds, Accepted for publication in Annals of Global Analysis and Geometry, arXiv:2011.02423 [math.DG].

[29] Z. Olszak, On almost cosymplectic manifolds, Kodai Math. J. 4 (1981), no. 2, 239?250.
[30] K. P. Tod, *All metrics admitting super-covariantly constant spinors*, Physics Letters B, Volume 121, (1983) Number 4.

[31] K. P. Tod, *More on supercovariantly constant spinors*, Classical and Quantum Gravity, Volume 12, (1995) Number 7.

Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Repubblica Italiana
Email address: angel.murcia@pd.infn.it

Departamento de Matemáticas, Universidad UNED - Madrid, Reino de España
Email address: cshahbazi@mat.uned.es

Fachbereich Mathematik, Universität Hamburg, Bundesrepublik Deutschland.
Email address: carlos.shahbazi@uni-hamburg.de