C*-ALGEBRAS OF ENDOMORPHISMS OF GROUPS WITH FINITE COKERNEL AND PARTIAL ACTIONS

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ABSTRACT. In this paper we extend the constructions of Boava and Exel to present the C*-algebra associated with an injective endomorphism of a group with finite cokernel as a partial group algebra and consequently as a partial crossed product. With this representation we present another way to study such C*-algebras, only using tools from partial crossed products.

1. INTRODUCTION

Consider an injective endomorphism \( \varphi \) of a discrete countable group \( G \) with unit \( \{e\} \) with finite cokernel i.e,

\[
\frac{|G|}{\varphi(G)} < \infty,
\]

as above, for \( H \) subgroup of \( G \), we use \( \frac{G}{\varphi(G)} \) to denote the set of left cosets of \( H \) in \( G \). Analyzing the natural representation of \( G \) and \( \varphi \) inside \( \mathcal{L}(l^2(G)) \) we construct a concrete C*-algebra \( C^*_r[\varphi] \subseteq \mathcal{L}(l^2(G)) \) and a universal one denoted by \( U[\varphi] \). Such constructions were presented by Hirshberg in [13], and were later generalized by Cuntz and Vershik in [8] and also in [26].

Using a semigroup crossed product description of \( U[\varphi] \) implies the existence of a (full corner) group crossed product description of it ([1], [5] and [16]), but it is not the only way to represent it as a crossed product: analogously to the work of G. Boava and R. Exel in [1] one can show that \( U[\varphi] \) has a partial group crossed product description, which can also be related to an inverse semigroup crossed product by [12]. We present in this paper the latter construction cited above and show the simplicity of \( U[\varphi] \), which is part of the conclusions in [13], using only the partial group crossed product description of that C*-algebra.

2. DEFINITION

We repeat the constructions of [13]. Let \( G \) be a discrete countable group with unit \( e \) and \( \varphi \) an injective endomorphism (monomorphism) of \( G \) with finite cokernel (11).

Consider the Hilbert space \( l^2(G) \) with orthonormal basis \( \xi_h \), taking every element of \( G \) to 0 apart from the element \( h \), which goes to 1. Define the following bounded operators on \( l^2(G) \):

\[
U_g(\xi_h) = \xi_{gh}
\]

and

\[
S(\xi_g) = \xi_{\varphi(g)}.
\]
The invertibility property of groups and the injectivity of the endomorphism $\varphi$ imply that the $U_g$'s are unitary operators and $S$ is an isometry respectively. Therefore we define the following $C^*$-algebra.

**Definition 2.1.** We denote $C^*_r[\varphi]$ the reduced $C^*$-algebra of $\varphi$, to be the $C^*$-subalgebra of $\mathcal{L}(l^2(G))$ generated by the above defined unitaries $\{U_g : g \in G\}$ and isometry $S$.

Inspired by the properties of the operators above:

**Definition 2.2.** We call $U[\varphi]$ the universal $C^*$-algebra generated by the unitaries $\{u_g : g \in G\}$ and one isometry $s$ such that:

(i) $u gdy = u gh$;
(ii) $s u_g = u_{\varphi(g)} s$;
(iii) $\sum_{g \in G/\varphi(G)} u_gss^*u_g^{-1} = 1$;

for all $g, h \in G$.

As the universal $C^*$-algebra above is defined using relations satisfied by the generators of the reduced one, obviously there is a canonical surjective $*$-homomorphism from $U[\varphi]$ onto $C^*_r[\varphi]$.

Note that the conditions (i) and (ii) above can be merged into the relation

$$u_gss^*u_g^{-1} = u_{\varphi(g)} s^*u_{\varphi(g)}.$$  

By (ii) we have, for $g \in G$, the obvious relations

$$u_g s^* = s^* u_{\varphi(g)}$$

and

$$u_{\varphi(g)} s s^* = s s^* u_{\varphi(g)}.$$  

Also note that in (iii) there is no ambiguity if we choose different representatives of the cosets:

$$u_{g\varphi(hs)} ss^* u_{(h\varphi(h))^{-1}} = u_g u_{\varphi(h\varphi(h))^{-1}} u_{\varphi(h)} u_g^{-1} = u_g s s^* u_{\varphi(h)} u_{\varphi(h^{-1})} u_g^{-1}$$

$$= u_g s s^* u_g^{-1}.$$  

Condition (iii) implies that $u_g ss^* u_g^{-1}$ and $u_h ss^* u_h^{-1}$ are orthogonal projections if $g^{-1} h \notin \varphi(G)$, so the multiplication can be described as:

$$u_g s s^* u_g^{-1} u_h s s^* u_h^{-1} = \begin{cases} 
  u_g s s^* u_g^{-1}, & \text{if } h \in g\varphi(G); \\
  0, & \text{otherwise.}
\end{cases}$$

This extends to the family of elements of type $u_g s^n s^n u_g^{-1}$ for any $n \in \mathbb{N}$

$$u_g s^n s^n u_g^{-1} u_h s^n s^n u_h^{-1} = \begin{cases} 
  u_g s^n s^n u_g^{-1}, & \text{if } h \in g\varphi^n(G); \\
  0, & \text{otherwise.}
\end{cases}$$
And note that, for \( g, h \in G \) and \( n \geq m \in \mathbb{N} \):
\[
 u_g s^n s^m u_{g^{-1}} u_h s^m s^m u_{h^{-1}} = u_g s^n s^m u_{g^{-1}} u_h s^m \left( \sum_{k \in \varphi^{-m}(G)} u_k s^{n-m} s^{n-m} u_{k^{-1}} \right) s^m u_{h^{-1}} = u_g s^n s^m u_{g^{-1}} \left( \sum_{k \in \varphi^{-m}(G)} u_{h \varphi^m(k)} s^n s^m u_{(h \varphi^m(k))^{-1}} \right) = \begin{cases} u_g s^n s^m u_{g^{-1}}, & \text{if } h \varphi^m(k) \in \varphi^n(G) \text{ for some } k \in \varphi^{-m}(G); \\ 0, & \text{otherwise}. \end{cases}
\]

3. Crossed product description of \( U[\varphi] \)

In this section we present a semigroup crossed product description of \( U[\varphi] \). The semigroup crossed product definition which we will use is the same as presented in Appendix A of [18], via covariant representations. In our case the semigroup implementing the action will be the semidirect product
\[
 S := G \rtimes_\varphi \mathbb{N} = \{(g, n) : g \in G, n \in \mathbb{N}\}
\]
with product
\[(g, n)(h, m) = (g \varphi^n(h), n + m).\]
We will also show that the action implemented by \( S \) can be split i.e, the semigroup crossed product by \( S \) can be seen as a semigroup crossed product by \( \mathbb{N} \). This crossed product description is a great tool to prove some properties of \( U[\varphi] \): we will show that when \( G \) is amenable this \( C^* \)-algebra is nuclear and satisfies UCT. Secondly, that description allows one to use the six-term exact sequence introduced by M. Khoshkam and G. Skandalis in [14] on \( U[\varphi] \).
Moreover, due to M. Laca [16], sometimes it is possible to see semigroup crossed products as full corners of group ones, which implies that both are Morita equivalent and therefore have the same K-groups. And in case the semigroup action is implemented by \( \mathbb{N} \), Laca’s dilation turns this \( \mathbb{N} \)-action into a \( \mathbb{Z} \)-action, which fits the requirements to use the classical Pimsner-Voiculescu exact sequence [21].

Set
\[
 \overline{G} := \lim_{\leftarrow} \left\{ \frac{G}{\varphi^m(G)} : p_{m,l+m} \right\}
\]
where
\[
p_{m,l+m} : \frac{G}{\varphi^l+m(G)} \to \frac{G}{\varphi^m(G)}
\]
is the canonical projection. We can see \( \overline{G} \) as
\[
 \overline{G} = \left\{ (g_m)_m \in \prod_{m \in \mathbb{N}} \frac{G}{\varphi^m(G)} : p_{m,l+m}(g_{l+m}) = g_m \right\},
\]
with the induced topology on the product \( \prod_{m \in \mathbb{N}} \frac{G}{\varphi^m(G)} \), where each finite set \( \frac{G}{\varphi^m(G)} \) carries the discrete topology, implying that \( \overline{G} \) is a compact space.
Furthermore, we have the map
\[ G \rightarrow \overline{G} \]
\[ g \mapsto (g)_m, \]
which is an embedding when \( \varphi \) is pure. Also set
\[ \mathcal{G} := \lim_{\rightarrow} \{ \mathcal{G}_m : \phi_{l+m,m} \} \]
where \( \mathcal{G}_m = \overline{G} \) for all \( m \in \mathbb{N} \) and \( \phi_{l+m,m} = \varphi^l \). We can see \( \mathcal{G} \) as
\[ \mathcal{G} = \bigcup_{m \in \mathbb{N}} \mathcal{G}_m / \sim \]
with \( x_l \sim y_m \) if and only if \( \varphi^m(x_l) = \varphi^l(y_m) \), \( x_l \in \mathcal{G}_l \) and \( y_m \in \mathcal{G}_m \). Note that \( \mathcal{G} \) is a locally compact set.
Denote by \( q \) the canonical projection
\[ q : \bigcup_{m \in \mathbb{N}} \mathcal{G}_m \rightarrow \mathcal{G}, \]
and \( i_m \) the embedding
\[ i_m : \overline{G} = \mathcal{G}_m \hookrightarrow \mathcal{G} \]
\[ x = x \mapsto q(x). \]
Again we have the identification
\[ \overline{G} \hookrightarrow \mathcal{G} \]
\[ x \mapsto i_{l_0}(x). \]

**Remark 3.1.** Note that if we suppose that our endomorphism \( \varphi \) is *totally normal*, i.e. all the \( \varphi^m(G) \) are normal subgroups of \( G \), then \( \overline{G} \) and \( \mathcal{G} \) will be groups; one just has to consider the componentwise multiplication in \( \overline{G} \) and
\[ i_m(x)i_l(y) = i_{l+m}(xy), \forall x, y \in \overline{G} \]
on \( \mathcal{G} \).

**Proposition 3.2.** The map
\[ \alpha : C^*(P) \rightarrow C(\overline{G}) \]
\[ u_g s^n s^m u_{g^{-1}} \mapsto p_{g \varphi^m(\overline{G})}, \]
where the latter denotes the characteristic function on the subset \( g \varphi^m(\overline{G}) \subseteq \overline{G} \), is an isomorphism.

**Proof.** It is clear that \( C^*(P) \) is the inductive limit of
\[ D_m := C^* \left( \left\{ u_g s^n s^m u_{g^{-1}} : g \in \frac{G}{\varphi^m(G)} \right\} \right) \]
with the inclusions (using (iii) of Definition 2.2)
\[ D_m \hookrightarrow D_{l+m} \]
\[ u_g s^n s^m u_{g^{-1}} \mapsto \sum_{h \in \frac{G}{\varphi^m(G)}} u_g \varphi^m(h) s^{l+m} s^{l+m} u_{\varphi^m(h) g^{-1}}. \]
Furthermore the pairwise orthogonality of the projections \( u_g s^n s^m u_{g^{-1}} \) for fixed \( m \in \mathbb{N} \) implies that
\[ \text{spec}(D_m) \cong \frac{G}{\varphi^m(G)} \]
with
\[ \text{spec}(D_{l+m}) \to \text{spec}(D_m) \]
\[ \chi \mapsto \chi|_{D_m} \]
corresponding to
\[ p_{l+m,m} : \frac{G}{\varphi^{l+m}(G)} \to \frac{G}{\varphi^m(G)} \]
\[ g\varphi^{l+m}(G) \mapsto g\varphi^m(G). \]
Therefore
\[ \text{spec}(C^*(P)) \cong \lim_{\rightarrow} \left\{ \frac{G}{\varphi^m(G)} : p_{m,l+m} \right\} = \overline{G}. \]
Thus we get the isomorphism
\[ \alpha : C^*(P) \to C(\overline{G}) \]
\[ u_g s^m s^m u_{g^{-1}} \mapsto p_{g\varphi^m(\overline{G})}. \]

**Definition 3.3.** The stabilization of \( \mathbb{U}[\varphi] \), denoted by \( \mathbb{U}^s[\varphi] \), is the inductive limit of the system \( \{ \mathbb{U}^s_m[\varphi] : \psi_{m,l+m} \} \) where, \( \forall \ m \in \mathbb{N} \), \( \mathbb{U}^s_m[\varphi] = \mathbb{U}[\varphi] \) and
\[ \psi_{m,l+m} : \mathbb{U}[\varphi] \to \mathbb{U}[\varphi] \]
\[ x \mapsto s^l s^m s^l. \]

Furthermore define \( C^*(P)^s = \lim_{\rightarrow} \{ C^*(P)^s_m : \psi_{m,l+m} \} \) with \( C^*(P)^s_m = C^*(P) \) and \( \psi_{m,l+m} \) as above.

**Proposition 3.4.** We have \( C^*(P)^s \cong C_0(G) \).

**Proof.** The maps \( \psi_{m,l+m} \), conjugated by \( \alpha \), give maps
\[ \tilde{\psi}_{m,l+m} := \alpha \circ \psi_{m,l+m} \circ \alpha^{-1} : C(\overline{G}) \to C(\overline{G}), \]
where \( \tilde{\psi}_{m,l+m}(f)(x) = f(\varphi^{-l}(x))p_{\varphi^l(\overline{G})}(x) : \)
\[ \tilde{\psi}_{m,l+m}(p_{g\varphi^m(\overline{G})})(x) = \tilde{\psi}_{m,l+m} \circ \alpha(u_g s^m s^m u_{g^{-1}})(x) \]
\[ = \alpha \circ \psi_{m,l+m}(u_g s^m s^m u_{g^{-1}})(x) \]
\[ = \alpha(u_{\varphi^{l}(g)}) s^{l+m} s^{l+m} u_{\varphi^{l}(g^{-1})}(x) \]
\[ = p_{\varphi^{l}(g)\varphi^{l+m}(\overline{G})}(x) \]
\[ = p_{g\varphi^m(\overline{G})}(\varphi^{-l}(x))p_{\varphi^l(\overline{G})}(x). \]
By the properties of inductive limits, we have an isomorphism
\[ \overline{\alpha} : C^*(P)^s \to \lim_{\rightarrow} \{ C(\overline{G}) : \tilde{\psi}_{m,l+m} \}. \]
Additionally we consider the *-homomorphisms
\[ \kappa_k : C(\overline{G}) \to C_0(G) \]
\[ f \mapsto f \circ i^{-1}_k \cdot p_{i_k(\overline{G})} \]
(where the $i$’s are as defined before Remark 3.1). These $\ast$-homomorphisms satisfy
\[ \kappa_{i+m} \circ \tilde{\psi}_{i,l+m} = \kappa_m, \]
since
\[ \kappa_{i+m} \circ \tilde{\psi}_{i,l+m}(f)(x) = \tilde{\psi}_{i,l+m}(f) \circ i_{l+m}(x)p_{i,l+m}(G)(x) \]
\[ = f(i_{l+m}(\varphi^{-1}(x)))p_{i,l+m}(G)(x) \]
\[ = f(i^{-1}(x))p_{i,m}(G)(x) \]
\[ = \kappa_m(f)(x). \]
Hence we have a $\ast$-homomorphism
\[ \lim \{ C(G) : \tilde{\psi}_{i,l+m} \} \rightarrow C_0(G). \]
This is injective as each $\kappa_k$ is, because of $\kappa_k(f) \circ i_k = f$. It is also surjective as $G = \cup_{m \in \mathbb{N}} \ast _m(G)$ and using the Stone-Weierstrass Theorem. So we have
\[ C^\ast(P) \ast \cong C_0(G). \]
\[ \square \]
Now we have all the tools to describe our C$^*$-algebra as a semigroup crossed product using $S = G \rtimes \mathbb{N}$. Consider the action
\[ \alpha : S \rightarrow \text{End}(C^\ast(P)) \]
\[ (g, n) \mapsto u_g s^n(\cdot)s^n u_{g^{-1}}. \]

**Theorem 3.5.** \[ U[\varphi] \] is isomorphic to $C^\ast(P) \rtimes \alpha S$.

**Proof.** By definition, $C^\ast(P) \rtimes \alpha S$ together with
\[ \iota_P : C^\ast(P) \rightarrow C^\ast(P) \rtimes \alpha S \]
\[ x \mapsto \iota_P(x) \]
and
\[ \iota_S : S \rightarrow \text{Isom}(C^\ast(P) \rtimes \alpha S) \]
\[ (g, n) \mapsto \iota_S(g, n) \]
satisfying
\[ \iota_P(u_g s^n x s^n u_{g^{-1}}) = \iota_S(g, n) \iota_P(x) \iota_S(g, n)^* \]
is the crossed product of $(C^\ast(P), S, \alpha)$. But note that the triple $U[\varphi]$,
\[ \pi : C^\ast(P) \rightarrow U[\varphi] \]
\[ x \mapsto x \]
and
\[ \rho : S \rightarrow \text{Isom}(U[\varphi]) \]
\[ (g, n) \mapsto u_g s^n \]
is a covariant representation of $(C^\ast(P), S, \alpha)$ because:
\[ \rho(g, n) \pi(x) \rho(g, n)^* = u_g s^n x s^n u_{g^{-1}} = \pi(\alpha(g, n)(x)). \]
Therefore there exists a $\ast$-homomorphism
\[ \Phi : C^\ast(P) \rtimes \alpha S \rightarrow U[\varphi] \]
such that $\Phi \circ \iota_P = \pi$ and $\Phi \circ \iota_S = \rho$.
In the other hand it is well known [17] that the crossed product $C^\ast(P) \rtimes \alpha S$ is generated as a C$^*$-algebra by elements of the form $\iota_S(g, n)$ because we have
\[ \iota_P(u_g s^n s^n u_{g^{-1}}) = \iota_S(g, n) \iota_S(g, n)^*. \]
But note that \( U[\varphi] \) can be viewed as the universal C*-algebra generated by the unitaries \( \{ u_g : g \in G \} \) and the isometry \( s \) changing conditions (i) and (ii) in Definition 2.2 to the equivalent one \( u_g s^n u_h s^m = u_{g \varphi^n(h)} s^{n+m} \).

Therefore we identify \( t_S(g, n) \) with \( u_g s^n \) because the first ones satisfy the condition above, which generate \( U[\varphi] \):

\[
t_S(g, n) t_S(h, m) = t_S(g \varphi^n(h) \cdot n + m)
\]

and

\[
\sum_{g \in G/\varphi(G)} t_S(g, n) t_S(g, n)^* = \sum_{g \in G/\varphi(G)} t_P(u_g s^n s^* s u_{g^{-1}})
\]

\[
= t_P \left( \sum_{g \in G/\varphi(G)} u_g s^n s^* s u_{g^{-1}} \right)
\]

\[
= t_P (1) = 1.
\]

Thus we get another \( * \)-homomorphism

\[
\Delta : U[\varphi] \to C^*(P) \rtimes_{\alpha} S
\]

\[
u_g s^n \mapsto t_S(g, n).
\]

As (2) and (3) are inverses of each other we can conclude that \( U[\varphi] \) and \( C^*(P) \rtimes_{\alpha} S \) are isomorphic.

In order to be able to apply the exact sequence presented in [14] we split the action of \( S \) presented above: we show that its semigroup crossed product is isomorphic to a semigroup crossed product implemented by \( \mathbb{N} \), where \( \mathbb{N} \) acts on a group crossed product by \( G \).

**Proposition 3.6.** The C*-algebra \( U[\varphi] \) is also isomorphic to the semigroup crossed product \( (C^*(P) \ltimes_{\omega} G) \ltimes_{\tau} \mathbb{N} \), where:

\[
\omega : G \to \text{Aut}(C^*(P))
\]

\[
g \mapsto u_g(\cdot)u_{g^{-1}}
\]

\[
\tau : \mathbb{N} \to \text{End}(C^*(P) \ltimes_{\omega} G)
\]

\[
n \mapsto s^n(\cdot)s^n
\]

such that for \( a_g \delta_g \in C^*(P) \ltimes_{\omega} G \), \( \tau_n(a_g \delta_g) = s^n a_g s^n \delta_{\varphi^n(g)} \).

**Proof.** We will show that \( C^*(P) \rtimes_{\alpha} S \) and \( (C^*(P) \ltimes_{\omega} G) \ltimes_{\tau} \mathbb{N} \) are isomorphic, by exploiting the universality of the semigroup crossed products, using two steps analogous to the first part of the proof of Theorem above. Consider \( C^*(P) \rtimes_{\alpha} S \) together with

\[
t_P : C^*(P) \to C^*(P) \rtimes_{\alpha} S
\]

\[
x \mapsto t_P(x)
\]

and

\[
t_S : S \to \text{Isom}(C^*(P) \rtimes_{\alpha} S)
\]

\[
(g, n) \mapsto t_S(g, n)
\]

satisfying

\[
t_P(u_g s^n x s^* s u_{g^{-1}}) = t_S(g, n) t_P(x) t_S(g, n)^*
\]
being the crossed product of \((C^*(P), S, \alpha)\). Analogously take \((C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N}\) with

\[ \iota_G : C^*(P) \rtimes_\omega G \to (C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N} \]

\[ a\delta_g \mapsto \iota_G(a\delta_g) \]

and

\[ \iota_N : \mathbb{N} \to \text{Isom}((C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N}) \]

\[ n \mapsto \iota_N(n) \]

satisfying

\[ \iota_B(s^n a\delta_g s^n \varphi^n(g)) = \iota_N(n) \iota_B(a\delta_g) \iota_N(n)^* \]

as the crossed product of \((C^*(P) \rtimes_\omega G, \mathbb{N}, \tau)\), where \(a\delta_g\) represents the generating elements of \((C^*(P) \rtimes_\omega G, g \in G)\.

Note that the triple \((C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N}\),

\[ \varrho : C^*(P) \to (C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N} \]

\[ a \mapsto \iota_G(a\delta_e) \]

and

\[ \sigma : S \to \text{Isom}((C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N}) \]

\[ (g, n) \mapsto \iota_G(1\delta_g) \iota_N(n) \]

is a covariant representation of \((C^*(P), S, \alpha)\):

\[ \sigma(g, n)\varrho(a)\sigma(g, n)^* = \iota_G(1\delta_g) \iota_N(n) \iota_G(a\delta_e) \iota_N(n)^* \iota_G(1\delta_g)^* \]

\[ = \iota_G(1\delta_g) \iota_G(s^n a s^n \delta_e) \iota_G(1\delta_g)^* \]

\[ = \iota_G(u_g s^n a s^n u_{g^{-1}} \delta_g) \iota_G(1\delta_{g^{-1}}) \]

\[ = \iota_G(u_g s^n a s^n u_{g^{-1}} \delta_e) \]

\[ = \varrho(u_g s^n a s^n u_{g^{-1}}) \]

Therefore we get a \(*\)-homomorphism

\[ \Phi : C^*(P) \rtimes_\alpha S \to (C^*(P) \rtimes_\omega G) \rtimes_\tau \mathbb{N} \]

such that \(\Phi \circ \iota_P = \varrho\) and \(\Phi \circ \iota_S = \sigma\).

Let us find an inverse for \(\Phi\) using the fact that the triple \((C^*(P) \rtimes_\alpha S, \varpi : C^*(P) \rtimes_\omega G \to C^*(P) \rtimes_\alpha S \]

\[ a\delta_g \mapsto \iota_P(a) \iota_S(g, 0) \]

and

\[ \vartheta : \mathbb{N} \to \text{Isom}(C^*(P) \rtimes_\alpha S) \]

\[ n \mapsto \iota_S(e, n) \]

is a covariant representation of \((C^*(P) \rtimes_\omega G, \mathbb{N}, \tau)\):

\[ \vartheta(n)\varpi(a\delta_g)\vartheta(n)^* = \iota_S(e, n) \iota_P(a) \iota_S(g, 0) \iota_S(e, n)^* \]

\[ = \iota_S(e, n) \iota_S(g, 0) \iota_P(u_{g^{-1}} a u_g) \iota_S(e, n)^* \]

\[ = \iota_S(\varphi^n(g), 0) \iota_S(e, n) \iota_P(u_{g^{-1}} a u_g) \iota_S(e, n)^* \]

\[ = \iota_S(\varphi^n(g), 0) \iota_P(s^n a u_{g^{-1}} a s^n u_g) \]

\[ = \iota_S(\varphi^n(g), 0) \iota_P(s^n a s^n u_{g^{-1}} a s^n u_g) \iota_S(\varphi^n(g), 0) \]

\[ = \iota_S(\varphi^n(\varphi^n(g), s^n a s^n \delta_{\varphi^n(g)})) \]

\[ = \iota_P(s^n a s^n) \iota_S(\varphi^n(g), 0) = \varpi(s^n a s^n \delta_{\varphi^n(g)}). \]
This implies the existence of a \(\ast\)-homomorphism
\[
(5) \quad \Delta : (C^*(P) \rtimes_{\omega} G) \rtimes_{\tau} \mathbb{N} \rightarrow C^*(P) \rtimes_{\alpha} S
\]
satisfying \(\Delta \circ \iota_G = \omega\) and \(\Delta \circ \iota_N = \vartheta\).
Straightforward calculations show that the \(\ast\)-homomorphisms (4) and (5) are inverses of each other.

**Example 3.7.** For any finite group \(G\), an injective endomorphism will be surjective and therefore the isometry \(s\) defining \(U[\varphi]\) will be a unitary (by item (iii) of Definition 2.2). Then as \(C^*(P) = C\),
\[
U[\varphi] \cong C^*(G) \rtimes_{\tau} \mathbb{N}
\]
where
\[
\tau : \mathbb{N} \rightarrow \text{End}(C^*(G))
\]
with
\[
\tau_n(\lambda u_g) = \lambda u_{\varphi^n(g)}.
\]
If one has the description of the K-theory of \(C^*(G)\) it is easy to calculate the K-groups of \(U[\varphi]\) by applying the Khoshkam-Skandalis sequence (14).

Since more results are known for group crossed products than for semigroup ones it is useful to find such a description of our \(C^\ast\)-algebra. We can do this using the minimal automorphic dilation of the semigroup crossed product system above (for more details, see Section 2 in [16]). One important requirement to use this dilation is that the semigroup must be an Ore semigroup: an Ore semigroup is a cancellative semigroup which is right-reversible i.e, it satisfies \(Ss \cap Sr \neq \emptyset\) for all \(s, r \in S\).

**Proposition 3.8.** The semidirect product \(S = G \rtimes_{\varphi} \mathbb{N}\) is an Ore semigroup.

**Proof.** Consider \((g_i, n_i) \in S\) for \(i \in \{1, 2, 3\}\). \(S\) is cancellative:
\[
(g_1, n_1)(g_3, n_3) = (g_2, n_2)(g_3, n_3)
\]
\[
\Rightarrow (g_1\varphi^{n_1}(g_3), n_1 + n_3) = (g_2\varphi^{n_2}(g_3), n_2 + n_3)
\]
\[
\Rightarrow n_1 = n_2 \text{ and } g_1\varphi^{n_1}(g_3) = g_2\varphi^{n_1}(g_3)
\]
\[
\Rightarrow g_1 = g_2
\]
\[
(g_1, n_1)(g_2, n_2) = (g_1, n_1)(g_3, n_3)
\]
\[
\Rightarrow (g_1\varphi^{n_1}(g_2), n_1 + n_2) = (g_1\varphi^{n_1}(g_3), n_1 + n_3)
\]
\[
\Rightarrow n_2 = n_3 \text{ and } \varphi^{n_1}(g_2) = \varphi^{n_1}(g_3)
\]
\[
\Rightarrow g_2 = g_3 \text{ as } \varphi \text{ is injective.}
\]

Also any two principal left ideals of \(S\) intersect:
\[
(\varphi^{n_2}(g_1^{-1}), n_2)(g_1, n_1) = (e, n_2 + n_1)
\]
\[
= (\varphi^{n_1}(g_2^{-1}), n_1)(g_2, n_2) \in S(g_1, n_1) \cap S(g_2, n_2).
\]

It follows that the semigroup \(S\) can be embedded in a group, called the enveloping group of \(S\), which we will denote as \(\text{env}(S)\), such that \(S^{-1}S = \text{env}(S)\) (Theorem 1.1.2 [16]). It also implies that \(S\) is a directed set by the relation defined by \((g, n) < (h, m)\) if \((h, m) \in S(g, n)\). Let us define a candidate for \(\text{env}(S)\).

\[
G := \lim_{\rightarrow} \{G_n : \varphi^n\}
\]
(with $G_n = G$ for all $n \in \mathbb{N}$) and with the extended endomorphism $\varphi$ construct the group

$$
\mathcal{S} := G \rtimes_{\varphi} \mathbb{Z}.
$$

Then we can define an extended action $\pi$ of $\mathcal{S}$ over $C^*(P)^s$:

$$
\pi : \mathcal{S} \to \text{Aut}(C^*(P)^s)
$$

$$(g, n) \mapsto s^x u_g s^{n+j}(\cdot) s^{n+j} u_{g^{-1}} s^j
$$

(note that we can also find $g_j$ such that $j \geq |n|$).

Moreover, consider $i : C^*(P) \to C^*(P)^s$ the canonical inclusion.

**Proposition 3.9.** The $C^*$-dynamical system $(C^*(P)^s, \mathcal{S}, \pi)$ is the minimal automorphic dilation of $(C^*(P), S, \alpha)$.

**Proof.** Since the subset of $S$ containing all elements of the type $(e, n)$ is cofinal in $S$, we need only prove that $\mathcal{S} = \text{env}(S)$ (to use Theorem 2.1.1 in [16]). For this we need to show that $S$ is a subsemigroup of $\mathcal{S}$ and $S \subseteq S^{-1}S$ [3].

First it is obvious that $S$ is a subsemigroup of the group $\mathcal{S}$ via the inclusion $(g, n) \mapsto (g_0, n)$, where $g_0 = g \in G = G_0 \hookrightarrow \mathcal{G}$.

Without loss of generality take $(g_i, j) \in \mathcal{S}$ with $i > |j|$. Then

$$(g_i, j) = (g_i, -i)(e, j + i) = (g_0, i)^{-1}(e, j + i) \in S^{-1}S.
$$

We may conclude that the following theorem holds (Theorem 2.2.1 of [16]).

**Theorem 3.10.** The $C^*$-algebra $U[\varphi]$ is also isomorphic to the full corner $\iota(1)(C^*(P)^s \rtimes_{\pi} \mathcal{S})\iota(1)$.

Let us denote the isomorphism given by the last theorem by

$$
\beta : U[\varphi] \to \iota(1)(C^*(P)^s \rtimes_{\pi} \mathcal{S})\iota(1),
$$

and by Theorem 2.2.1 in [16] we know that

$$
\beta(s^n u_{h^{-1}} f u_h s^m) = i(1)U_{h, m} \iota(f)U_{h', m} \iota(1).
$$

Note that the isomorphism above implies that $U[\varphi]$ and $C^*(P)^s \rtimes_{\pi} \mathcal{S}$ are Morita equivalent and so they have the same K-groups.

To finish our identifications:

**Theorem 3.11.** The stabilization (Definition 3.3) $U[\varphi]^s$ is isomorphic to the group crossed product $C^*(P)^s \rtimes_{\pi} \mathcal{S}$.

**Proof.** As in Theorem 2.4 in [16] we know that $\beta(u_g) = V(g_0, 0)\iota(1)$ and $\beta(s^n) = V(e_0, n)\iota(1)$, where $V$ represents $\mathcal{S}$ in the crossed product $C^*(P)^s \rtimes_{\pi} \mathcal{S}$.

Define

$$
\tilde{\gamma}_{m, l+m} : \iota(1)(C^*(P)^s \rtimes_{\pi} \mathcal{S})\iota(1) \to \iota(1)(C^*(P)^s \rtimes_{\pi} \mathcal{S})\iota(1)
$$

$$
x \mapsto V(e, l)xV(e, l)^*.
$$

Remembering from Definition 3.3 that

$$
\psi_{m, l+m} : U[\varphi] \to U[\varphi]
$$

$$
x \mapsto s^l x s^l,
$$

The isomorphism $C^*(P) \cong C(\mathcal{G})$ implemented in Proposition 3.2 implies that the projection $\iota(1) \in C^*(P)^s$ corresponds to $p_\mathcal{G} \in C(\mathcal{G})$ viewed inside $C_0(\mathcal{G})$ via $i_0$ (defined before Remark 3.1).
we can conclude that
\[
\beta \circ \psi_{m,l+m} \circ \beta^{-1} = \tilde{\gamma}_{m,l+m}
\]
which implies the existence of an isomorphism
\[
\tilde{\beta} : U[\varphi]^s \to \lim \{\iota(1)(C^*(P)^s \times \overline{S})\iota(1), \tilde{\gamma}_{m,l+m}\}.
\]
Moreover for \(k \geq 0\) set
\[
\lambda_k : \iota(1)(C^*(P)^s \times \overline{S})\iota(1) \to C^*(P)^s \times \overline{S}
\]
As
\[
\lambda_{l+m} \circ \tilde{\gamma}_{m,l+m}(z) = V^*(e, l + m)V(e, l)zV(e, l)^*V(e, l + m) = V^*(e, m)zV(e, m) = \lambda_m(z),
\]
we have a *-homomorphism
\[
\lambda : \lim \{\iota(1)(C^*(P)^s \times \overline{S})\iota(1) : \tilde{\gamma}_{m,l+m}\} \to C^*(P)^s \times \overline{S}.
\]
It is injective because each \(\lambda_k\) is. Moreover as
\[
\lambda_k(\iota(1)) = \overline{\alpha(e,-k)(\iota(1))V(e,0)} = s^k \iota(1)s^kV(e,0)
\]
is an approximate unit for \(C^*(P)^s \times \overline{S}\), for \(z \in C^*(P)^s \times \overline{S}\) we have
\[
\lim_k \lambda_k(\iota(1)V(e, k)zV^*(e, k)\iota(1)\iota(1))
= \lim_k [\lambda_k(\iota(1)V(e, k)zV^*(e, k)\iota(1)\lambda_k(\iota(1))]
= \lim_k [V^*(e, k)\iota(1)V(e, k)zV^*(e, k)\iota(1)V(e, k)] [s^k \iota(1)s^kV(e,0)]
= \lim_k s^k \iota(1)s^kV(e,0)z s^k \iota(1)s^kV(e,0) s^k \iota(1)s^kV(e,0)
= z,
\]
and so \(\lambda\) is surjective. Consequently \(U[\varphi]^s \cong C^*(P)^s \times \overline{S}\).

\[\square\]

**Example 3.12.** Consider a surjective endomorphism \(\varphi\) of a group \(G\). The surjectivity of \(\varphi\) implies that \(s\) is an isometry (by item (iii) of Definition 2.2). Moreover Proposition 3.6 together with the fact that \(C^*(P) = \mathbb{C}\) implies that
\[
U[\varphi] \cong C^*(G) \rtimes \tau \mathbb{N},
\]
where
\[
\tau : \mathbb{N} \to \text{End}(C^*(G))
\]
is defined by
\[
\tau_n(u_g) = u_{\varphi^n(g)}.
\]
Using the six-term exact sequence introduced by Khoshkam and Skandalis in [14], one can build the sequence
\[
\begin{array}{ccccccc}
K_0(C^*(G)) & \overset{1-K_0(\tau_1)}{\rightarrow} & K_0(C^*(G)) & \rightarrow & K_0(U[\varphi]) \\
\uparrow & & & & \\
K_1(U[\varphi]) & \leftarrow & K_1(C^*(G)) & \overset{1-K_1(\tau_1)}{\rightarrow} & K_1(C^*(G))
\end{array}
\]
(note that this example is very similar to Example 3.7).\[\square\]
4. Properties

The crossed product description in last section implies two nice properties of $U[\varphi]$.

**Proposition 4.1.** If $G$ is amenable then $U[\varphi]$ is nuclear.

*Proof.* $G$ being amenable implies that $S$ is amenable as well (amenability is closed under direct limits by [20] and also closed under semidirect products). But we know that $C^*(P)^s$ is nuclear because it is commutative, therefore $C^*(P)^s \rtimes \varphi S$ is nuclear by Proposition 2.1.2 in [23]. Since hereditary C*-subalgebras of nuclear C*-algebras are nuclear by Corollary 3.3 (4) in [2], we conclude that

$$U[\varphi] \cong C^*(P)^s \rtimes \varphi S \cong i(1)(C^*(P)^s \rtimes \varphi S)i(1)$$

is nuclear. □

**Proposition 4.2.** If $G$ is amenable then $U[\varphi]$ satisfies the UCT property.

*Proof.* Since $C^*(P)^s$ is commutative, $C^*(P)^s \rtimes \varphi S$ is isomorphic to a groupoid C*-algebra. When the group $G$ is amenable then $S$ also is, and the respective groupoid is also amenable. Therefore using a result by Tu ([25] Proposition 10.7), the crossed product satisfies UCT. By Morita equivalence, $U[\varphi]$ also satisfies it. □

We will now prove that our algebra $U[\varphi]$ is purely infinite and simple. We will proceed in the same way as in [4] and in many other papers: we present a particular faithful conditional expectation and a dense $*$-subalgebra of $U[\varphi]$ such that the conditional expectation of any positive element of this $*$-subalgebra can be described using a finite number of pairwise orthogonal projections.

For this purpose we will use the description in Theorem 3.10 of $U[\varphi]$ as a corner of a group crossed product. To define the conditional expectation, we require the amenability of the group $G$: therefore the group $S = G \rtimes \varphi Z$ (defined after Proposition 3.8) is also amenable (as mentioned in the proof of Proposition 4.1). This condition is necessary because we want to use the well-known result which says that there exists a canonical faithful conditional expectation on the reduced group crossed product, and the amenability of $S$ implies that both the full and the reduced group crossed products (implemented by $S$-actions) are isomorphic.

The main tool of this section is the following (proven in Proposition 5.2 of [18]).

**Proposition 4.3.** Let $\tilde{A}$ be a dense $*$-subalgebra of a unital C*-algebra $A$. Assume that $\epsilon$ is a faithful conditional expectation on $A$ such that for every $0 \neq x \in \tilde{A}$, there exist finitely many projections $f_i \in A$ with

(i) $f_i \perp f_j, \forall i \neq j$;

(ii) $f_i \sim s_i, 1$, via $\exists$ isometries $s_i \in A, \forall i$;

(iii) $\|\sum f_i \epsilon(x) f_i\| = \|\epsilon(x)\|$;

(iv) $f_i x f_i = f_i \epsilon(x) f_i \in C f_i, \forall i$.

Then $A$ is purely infinite and simple. □

Moreover in order to find these projections it is also necessary to require that the injective endomorphism $\varphi$ is pure, i.e:

$$\bigcap_{n \in \mathbb{N}} \varphi^n(G) = \{\epsilon\}.$$

\[\text{I.e.: } \exists s_i \text{ isometries such that } s_i s_i^* = f_i, \forall i\]
In order to apply the proposition above the first step is to define a conditional expectation. As mentioned before, we require that the group $G$ is amenable. Remember the isomorphism from Theorem 3.10:

$$\beta : U[\phi] \rightarrow \iota(1)(C^*(P)^s \rtimes_{\pi} \overline{S})\iota(1).$$

**Proposition 4.4.** There exists a faithful conditional expectation

$$\epsilon : U[\phi] \rightarrow \beta^{-1}(\iota(1)C^*(P)^s\iota(1))$$

\[ s^{n^*}u_{h^{-1}}fu_h's^m \mapsto \begin{cases} 
  s^{n^*}u_{h^{-1}}fu_h's^n, & \text{if } n = m \text{ and } h = h'; \\
  0, & \text{otherwise.}
\end{cases} \]

for all $h, h' \in G$ and $n, m \in \mathbb{N}$.

**Proof.** As $\overline{S}$ is amenable, the isomorphism $\beta$ of Theorem 3.10 can be expanded to include also the reduced group crossed product

$$U[\phi] \cong \iota(1)(C^*(P)^s \rtimes_{\pi} \overline{S})\iota(1) \cong \iota(1)(C^*(P)^s \rtimes_{r,\pi} \overline{S})\iota(1).$$

Let us denote the elements of $\overline{S}$ by $s$ and its identity by $e$. We will also use $\delta_s$ to denote the unitary elements implementing the action of $\overline{S}$ in the crossed product. Consider the well-known faithful conditional expectation on the reduced group crossed product:

$$E : C^*(P)^s \rtimes_{r,\pi} \overline{S} \rightarrow C^*(P)^s$$

\[ x\delta_s \mapsto \begin{cases} 
  x, & \text{if } s = e; \\
  0, & \text{otherwise.}
\end{cases} \]

Straightforward calculations show that the following is also a faithful conditional expectation:

$$\overline{E} : \iota(1)(C^*(P)^s \rtimes_{r,\pi} \overline{S})\iota(1) \rightarrow \iota(1)C^*(P)^s\iota(1)$$

\[ \iota(1)x\delta_s\iota(1) \mapsto \begin{cases} 
  \iota(1)x\iota(1), & \text{if } s = e; \\
  0, & \text{otherwise.}
\end{cases} \]

Using the isomorphism $\beta$ we can rewrite $\overline{E}$ to conclude that we have the faithful conditional expectation

$$\epsilon : U[\phi] \rightarrow \beta^{-1}(\iota(1)C^*(P)^s\iota(1))$$

\[ s^{n^*}u_{h^{-1}}fu_h's^m \mapsto \begin{cases} 
  s^{n^*}u_{h^{-1}}fu_h's^n, & \text{if } n = m \text{ and } h = h'; \\
  0, & \text{otherwise.}
\end{cases} \]

Now, to find projections to describe the image of $y \in \text{span}(Q)_+$ under the conditional expectation $\epsilon$ presented above, remember that $y$ has the form

$$y = \sum_{m,n,h,h',f} a_{(m,n,h,h',f)} s^{n^*}u_{h^{-1}}fu_h's^m$$

for $m, n \in \mathbb{N}$, $h, h' \in G$, $f \in P$ and $a_{(\_)} \neq 0$. As we have finitely many projections of $C^*(P)$ in the description of $y$, write them all as sums of (altogether $N$) mutually orthogonal projections $u_{g_i}s^M s^M u_{g_i^{-1}}$, with $g_i \in G/\phi^M(G)$, for all $1 \leq i \leq N$ and $M \in \mathbb{N}$ big enough.

**Proposition 4.5.** There are $N$ pairwise orthogonal projections $f_1, \ldots, f_N \in P$ such that
To understand this choice of $\varphi$. A sufficient condition for this to hold is that for each critical index $(m, n, h, h', f)$ which is equivalent to $p$.

As we have a finite number of critical indices, it is sufficient to take the biggest $p$.

Proof. Define

$$f_i := u_i s^p s^p u_{h_i}^{-1}$$

for some $p \in \mathbb{N}$ bigger than $M$ (in fact, we may choose $p$ as big as we want), where $g_i^{-1} h_i \in \varphi^M(G)$. This implies that the set of the $f_i$'s is orthogonal and that (i) holds. For (ii), first note that when $\delta_{m,n}\delta_{h,h'} = 1$ it is true that $\epsilon = Id$, and so (ii) is satisfied.

So let us take a look on those summands in $y$ with $\delta_{m,n}\delta_{h,h'} = 0$ (we will say that such an element has critical index $(m, n, h, h', f)$). The conditional expectation $\epsilon$ maps these summands to 0 and in order for (ii) to be satisfied we need that, for all $1 \leq i \leq N$,

$$f_i s^n u_{h_i}^{-1} f u_{h_i}s^m f_i = 0.$$

We calculate

$$f_i s^n u_{h_i}^{-1} f u_{h_i}s^m f_i = s^n u_{h_i}^{-1}(u_i s^n f_i s^n u_{h_i}^{-1}) f(u_i s^m f_i s^m u_{h_i}^{-1}) u_{h_i}s^m$$

Now, analysing only the expression between the brackets,

$$[u_i \varphi^{n}(h_i) s^{n+p} s^{n+p} u_{\varphi^{n}(h_i)} s^{m+n} s^{m+n} u_{\varphi^{m}(h_i)} h_i^{-1} u_{h_i} s^n u_{h_i}^{-1} f u_{h_i}s^m f_i]$$

This product will be zero if the two sums are mutually orthogonal, which happens if for all $g \in G / \varphi^m(G)$ and $k \in G / \varphi^n(G)$,

$$h_i \varphi^n(h_i) \varphi^{n+p}(g) \varphi^{m+n+p}(x) \neq h_i' \varphi^n(h_i) \varphi^{m+p+k}(y), \forall x, y \in G$$

which is equivalent to

$$\varphi^{m+p}(k^{-1}) \varphi^m(h_i^{-1}) h_i^{-1} h_i \varphi^n(h_i) \varphi^{n+p}(g) \neq \varphi^{m+n+p}(z), \forall z \in G.$$

A sufficient condition for this to hold is that $\varphi^m(h_i^{-1}) h_i^{-1} h_i \varphi^n(h_i) \neq \varphi^p(z), \forall z \in G,$ for each critical index $(m, n, h, h', f)$. Using the fact that $\varphi$ is pure we may choose some $p(m,n,h,h',f) \in \mathbb{N}$ such that

$$\varphi^m(h_i^{-1}) h_i^{-1} h_i \varphi^n(h_i) \notin \varphi^{p(m,n,h,h',f)}(G).$$

As we have a finite number of critical indices, it is sufficient to take the biggest $p(m,n,h,h',f)$ and call it $p$. □

To understand this choice of $p$, consider the following example.
Example 4.6. Let $G = \mathbb{Z}$ and

$$\varphi : \mathbb{Z} \to \mathbb{Z} \quad n \mapsto 3n.$$ 

Then we have $G\varphi^{-1}(G) = \{0, 1, 2\}$, $G\varphi(G) = \{0, 1, \ldots, 8\}$ and in general

$$G\varphi^n(G) = \{0, \ldots, 3^n - 1\} = \mathbb{Z}_{3^n}.$$ 

Take the following $y \in \text{span}(Q)$

$$y = 2s^2u_30(u_5s^4u_{-5})u_{2187}s^1 - 4s^7u_0(u_{10}s^4u_{-10})u_{-5}s^9 + s^8u_{20}s^4u_{-20}s^8$$

and note that in $y$ we have two terms with critical indices (the first ones).

Using the notation of the above proposition, $M = 4$ and, for the first term of $y$:

$n = 2$, $m = 1$, $h = -30$, $h' = 2187$ and $g_1 = 5$. Choosing $h_1 = 86$, it is true that $-g_1 + h_1 = -5 + 86 = 81 \in \varphi(G)$. Then:

$$\varphi^1(-86) - 2187 - 30 + \varphi^2(86) = -1701 = \varphi^5(7) \notin \varphi^6(\mathbb{Z}).$$

So $p_1 := p_{(1, -30, 2187, f)} = 6$ (or bigger). For the second term it is not hard to see that $p_2 = 1$:

$$\varphi^9(-91) + 5 - 0 + \varphi^7(91) = -1592131 \notin \varphi^1(\mathbb{Z}).$$

So one can choose any $p \geq 6$.

□

Using the description above of the faithful conditional expectation

$$\epsilon : U[\varphi] \to \beta^{-1}(\iota(1)C^*(P)^*\iota(1))$$

where $P = \{u_g s^n s^m u_{g^{-1}} : g \in G, n \in \mathbb{N}\}$, together with the dense $*$-subalgebra

$$\text{span}(Q) = \text{span}(\{s^n u_{h^{-1}} f u_h s^m : f \in P, h, h' \in G, n, m \in \mathbb{N}\}),$$

we can prove the main result of this section by applying Propositions 4.5 and 4.3 (the definition of pure infiniteness comes from [4]).

Theorem 4.7. Let $G$ be a discrete countable amenable group and $\varphi$ a pure injective endomorphism of $G$ with finite cokernel. Then the $C^*$-algebra $U[\varphi]$ is simple and purely infinite, i.e. for all non zero $x \in U[\varphi]$ there are $a, b \in U[\varphi]$ with $axb = 1$.

□

Corollary 4.8. When satisfied the conditions of the theorem above, the universal $C^*$-algebra $U[\varphi]$ is isomorphic to $C^*_{\varphi}$, as defined in Definitions 2.2 and 2.1 respectively.

□

Theorem 4.9. If the conditions of the theorem above are satisfied, the universal $C^*$-algebra $U[\varphi]$ is a Kirchberg algebra satisfying the UCT property.

□

It would be interesting to know if the conditions of the theorem above are also necessary: if we construct the $C^*$-algebra associated with some injective endomorphism of an amenable group, is it simple and purely infinite only if $\varphi$ is pure? Unfortunately we don’t answer this question here, but the next trivial example gives some idea about this direction.
Example 4.10. For some commutative group $G$ (thus amenable), consider $\varphi = id_G$. As $u_g s = s u_g$ for all $g \in G$ ($\varphi$ is trivial), our $C^*$-algebra will be commutative. Now, as $G / \varphi(G)$ has only the element $\{ e \}$, condition (iii) of Definition 2.2 implies that the isometry $s$ is a unitary. Then $U[\varphi]$ is the commutative $C^*$-algebra generated by the unitaries $\{ u_g, s : g \in G \}$, and this one is the non-simple tensor product $C^*(G) \otimes C^*(Z) = C^*(G) \otimes C(S^1)$.

Moreover using the Künneth Formula [23] we conclude that

$$K_0(U[\varphi]) = K_1(U[\varphi]) = K_0(C^*(G)) \oplus K_1(C^*(G)).$$

\[ \square \]

5. Description of $U[\varphi] \text{ via group partial crossed products}$

In [11] Boava and Exel constructed a partial group algebra isomorphic to the $C^*$-algebra $U[R]$ associated with a integral domain $R$ [7]. Consequently due to Theorem 4.4 of [11] one can define a certain partial crossed product which is isomorphic to $U[R]$. With the latter description it is proven in [11], using only tools from partial crossed products, that if $R$ is not a field then $U[R]$ is simple (which is part of the conclusion of Li [18], namely, Corollary 5.14).

In this section we will present analogous results adapted to our case, i.e., given a $C^*$-algebra $U[\varphi]$ associated with some injective endomorphism $\varphi$ of a group $G$ with unit $e$, we will show that $U[\varphi]$ can also be viewed as a partial group algebra and, consequently, as a partial crossed product. The ideas follow the ones presented in [11]. With this description we show that when $G$ is amenable we can rewrite the faithful conditional expectation $\epsilon$ presented in Proposition 4.4 in terms of the partial group crossed product. To finish we use a well known result from the theory of group partial crossed products to prove a weaker result than Theorem 4.7: if $G$ is commutative and $\varphi$ is pure then $U[\varphi]$ is simple.

We start with an introduction to partial actions, partial crossed products and partial group algebras, before presenting the right isomorphisms and descriptions of $U[\varphi]$.

Definition 5.1. A partial action $\alpha$ of a group $G$ on a $C^*$-algebra $A$ is a collection of closed two-sided ideals $\{D_g\}_{g \in G}$ of $A$ and *-isomorphisms $\alpha_g : D_{g^{-1}} \to D_g$ satisfying

(PA1) $D_e = A$;

(PA2) $\alpha_{h^{-1}}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}}$;

(PA3) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x), \forall x \in \alpha_h^{-1}(D_h \cap D_{g^{-1}})$.

Using (PA1) - (PA3) one can show that $\alpha_e = id_A, \alpha_{g^{-1}} = \alpha_g^{-1}$ and that $\alpha_{h^{-1}}(D_h \cap D_{g^{-1}}) = D_{(gh)^{-1}} \cap D_{h^{-1}}$.

Analogously, one can define a partial action of $G$ acting on a locally compact space $X$: just replace the ideals $D_g$ by open sets $X_g \subseteq X$ and the *-isomorphisms $\alpha_g$ by homeomorphisms $\theta_g : X_{g^{-1}} \to X_g$.

We call the triples $(\alpha, G, A)$ or $(\theta, G, X)$ partial dynamical systems, or partial actions when there is no possibility of misunderstanding.

Example 5.2. If $\theta$ is a partial action of $G$ on the locally compact space $X$ with $\theta_g : X_{g^{-1}} \to X_g$, one can easily construct a partial action of $G$ on the $C^*$-algebra $C_0(X)$ considering $D_g = C_0(X_g)$ and

$$\alpha_g : C_0(X_{g^{-1}}) \to C_0(X_g)$$

$$f \mapsto f \circ \theta_g^{-1}.$$
Now we want to define partial crossed products. There are three ways to realize them: one using Fell bundles (and we recommend [10]), another using enveloping C*-algebras (for details and some interesting examples look at Section 2 of [19]) and the last one as a universal object with respect to covariant pairs (see Section 3 of [22]). We use the last way in our proofs and therefore we present it.

Let us define first a particular set of representations called partial representations.

**Definition 5.3.** A partial representation \( \pi \) of a group \( G \) into a unital C*-algebra \( B \) is a map \( \pi : G \to B \) satisfying

- \((PR1)\) \( \pi(e) = 1 \);
- \((PR2)\) \( \pi(g^{-1}) = \pi(g)^* \);
- \((PR3)\) \( \pi(g)\pi(h)\pi(h^{-1}) = \pi(gh)\pi(h^{-1}) \).

Then the partial group crossed product \( A \rtimes_{\alpha} G \) is defined as the universal object with respect to a covariant pair \((\upsilon, \pi)\), which means a \( * \)-homomorphism with \( B \) being a unital C*-algebra

\[ \upsilon : A \to B \]

and a partial representation of \( G \)

\[ \pi : G \to B \]

satisfying

\[ \upsilon(\alpha_g(x)) = \pi(g)\upsilon(x)\pi(g^{-1}) \text{ for } x \in D_{g^{-1}}, \]

\[ \upsilon(x)\pi(g)\pi(g^{-1}) = \pi(g)\pi(g^{-1})\upsilon(x) \text{ for } x \in A. \]

To define a partial group algebra, consider the set \( [G] := \{ [g] : g \in G \} \) (without any operations).

**Definition 5.4.** The partial group algebra of \( G \), denoted \( C^*_p(G) \), is the universal C*-algebra generated by \( [G] \) with respect to the relations

- \((R_p1)\) \( [e] = 1 \);
- \((R_p2)\) \( [g^{-1}] = [g]^* \);
- \((R_p3)\) \( [g][h][h^{-1}] = [gh][h^{-1}] \).

The C*-algebra \( C^*_p(G) \) is universal with respect to partial representations of \( G \) (note the equivalence between relations \((R_p)\) and \((PR)\) of Definition 5.3).

In fact, one can define partial group algebras for more restricted situations, i.e., requiring that \([G]\) satisfies additional relations than the 3 relations above. Let us set \( e_g := [g][g^{-1}] \) and for our constructions consider \( \mathcal{R} \) a set of (extra) relations on \([G]\) such that every relation is of the form

\[ \sum_i \prod_j e_{g_{ij}} = 0. \]

**Definition 5.5.** The partial group algebra of \( G \) with relations \( \mathcal{R} \), denoted \( C^*_p(G, \mathcal{R}) \), is defined to be the universal C*-algebra generated by \([G]\) with relations \( R_p \cup \mathcal{R} \). This C*-algebra is universal with respect to partial representations which satisfy \( \mathcal{R} \).

An interesting fact is that the class of partial group algebras without restrictions and of the ones with extra relations of the type \((6)\) is contained in the class of partial crossed products (Definition 6.4 of [9] and Theorem 4.4 of [11] respectively). In our case the C*-algebra \( U[\varphi] \) will be isomorphic to a partial group algebra with additional relations of the form \((6)\) above, and we will show how these can be viewed as partial crossed products.
Consider the power set \( \mathcal{P}(G) \) (of \( G \)) with the topology given by identifying it with the compact set \( \{0,1\}^G \), and denote \( X_G \) the subset of \( \mathcal{P}(G) \) of the subsets \( \xi \) of \( G \) which contain \( e \in G \). Note that using the product topology of \( \{0,1\}^G \) implies that \( X_G \) is compact and Hausdorff.

Denote by \( 1_g \) the following function in \( C(X_G) \):

\[
1_g(\xi) = \begin{cases} 
1, & \text{if } g \in \xi; \\
0, & \text{otherwise.}
\end{cases}
\]

Denote \( \widehat{\mathcal{R}} \) the subset of \( C(X_G) \) given by the functions \( \sum_i \prod_j 1_{g_{ij}} \) where the relation \( \sum_i \prod_j e_{g_{ij}} = 0 \) is in \( \mathcal{R} \). The spectrum of the relations \( \mathcal{R} \) is defined to be the compact (Proposition 4.1) space

\[
\Omega_{\mathcal{R}} := \{ \xi \in X_G : f(g^{-1}\xi) = 0, \forall f \in \widehat{\mathcal{R}}, \forall g \in \xi \}.
\]

Now for \( g \in G \), consider

\[
\Omega_g := \{ \xi \in \Omega_{\mathcal{R}} : g \in \xi \}
\]

and let us define

\[
\theta_g : \Omega_{g^{-1}} \to \Omega_g \\
\xi \mapsto g\xi.
\]

Then we have defined a partial action \( \theta \) of \( G \) on \( \Omega_{\mathcal{R}} \). Turning this partial action (as in Example 5.2) into a partial action \( \alpha \) of \( G \) on \( C(\Omega_{\mathcal{R}}) \), it is well known (by Theorem 4.4 (iii) in [11]) that

\[
C^*_\alpha(G, R) \cong C(\Omega_{\mathcal{R}}) \rtimes G.
\]

Now let us find a partial group \( C^* \)-algebra description of \( U[\varphi] \). Therefore recall the set \( \mathcal{S} = G \rtimes \mathbb{Z} \) whose elements will be denoted by \((g_i, n)\) with \( g_i \in G, n \in \mathbb{Z} \). In case \( g \in G = G_0 \subseteq G \) we will use the notation \((g, n)\).

Consider the following relations \( \mathcal{R} \):

\[
\begin{align*}
(\mathcal{R}_1) & \quad [((g, 0))(0, 0)^{-1}] = 1, \forall g \in G; \\
(\mathcal{R}_2) & \quad [(e, -n)][(e, -n)^{-1}] = 1 \forall n \in \mathbb{N}; \\
(\mathcal{R}_3) & \quad \sum_{g \in \mathcal{S}} \sum_{n \in \mathbb{N}} [(g, n)][(g, n)^{-1}] = 1, \forall n \in \mathbb{N}.
\end{align*}
\]

Consider also the partial group algebra relations in this case i.e, on the group \( \mathcal{S} \):

\[
\begin{align*}
(\mathcal{R}_p1) & \quad [(e, 0)] = 1; \\
(\mathcal{R}_p2) & \quad [(g_i, n)^{-1}] = [(g_i, n)]^*, \forall n \in \mathbb{Z}, \forall g_i \in G; \\
(\mathcal{R}_p3) & \quad [(g_i, n)][(h_j, m)][(h_j, m)^{-1}] = [(g_i\varphi^n(h_j), n + m)][(h_j, m)^{-1}], \forall m, n \in \mathbb{Z}, \forall g_i, h_j \in G.
\end{align*}
\]

Define

\[
\pi : \mathcal{S} \to U[\varphi] \\
(g_i, n) \mapsto s^i u_g s^{n+i},
\]

remembering that we can always suppose \( i \geq |n| \). Note that when \( g \in G \), \( \pi(g, n) = u_g s^n \).

**Proposition 5.6.** The map \( \pi \) is a partial representation of \( \mathcal{S} \) which satisfies the relations \( \mathcal{R} \).

**Proof.** First we prove that \( \pi \) is a partial representation of \( \mathcal{S} \).

\( (\mathcal{R}_p1) \): \( \pi((e, 0)) = u_e = 1; \)
 Proposition 5.7. The (obviously) unitary elements $C(\pi)$ now we show that $U$

Let us find an inverse for $\Phi$ by using the relations which define $R(\pi)$:

\begin{align*}
\pi((\varphi^i(\pi^{\phi^i+n}(h))_{n+m})\pi((h_j, m)^{-1}) & = s^{i+j}u_{\varphi^i(g)}\varphi^i+n(h)\pi^{i+j+n+m}u_{h-1}^{s} \\
& = s^{i+j}u_{\varphi^i(g)}s^{i+j+n+m}u_{h-1}^{s} \\
& = s^{i+j}u_{\varphi^i(g)}s^{i+j+n+m}u_{h-1}^{s} \\
& = s^{i+j}u_{\varphi^i(g)}s^{i+j+n+m}u_{h-1}^{s} \\
& = \pi((g_i, n))\pi((h_j, m))\pi((h_j, m)^{-1}).
\end{align*}

Now we show that $\pi$ satisfies the extra relations $R(\pi)$:

\begin{align*}
\pi((g_0, 0))\pi((g_0, 0)^{-1}) & = u_e = 1; \\
\pi((e_n, -n))\pi((e_n, -n)^{-1}) & = \pi((e_n, -n)^{-1}) = s^n s^n = 1; \\
\sum_{g \in \mathbb{F}(G)} \pi((g, n))\pi((g, n)^{-1}) & = \sum_{g \in \mathbb{F}(G)} u_g s^n s^{-n} u_{g^{-1}} = 1. \quad \Box
\end{align*}

It follows from the universality of the partial group algebra $C_p^*(\mathbb{S}, R(\pi))$ that there exists a $*$-homomorphism

$$\Phi : C_p^*(\mathbb{S}, R(\pi)) \to \mathbb{U}[\varphi]$$

$$(g_i, n) \mapsto s^{i+n}u_{g^{-1}}.$$

Let us find an inverse for $\Phi$ by using the relations which define $\mathbb{U}[\varphi]$.

Proposition 5.7. The (obviously) unitary elements $[(g, 0)]$ and isometries $[(e, n)]$ of $C_p^*(\mathbb{S}, R(\pi))$ satisfy the relations which define $\mathbb{U}[\varphi]$.

Proof. Let us show that the elements above satisfy the relations (i) - (iii) of Definition 2.2:

(i):

$$[(g, 0)][(h, 0)] = [(g, 0)][(h, 0)][(h, 0)][(h, 0)] = [(gh, 0)][(h^{-1}, 0)][(h, 0)] = [(gh, 0)][(h^{-1}, 0)][(h, 0)] = [(gh, 0)];$$

(ii):

$$[(e, 1)][(g, 0)] = [(e, 1)][(g, 0)][(g, 0)][(g, 0)] = [(\varphi(g), 1)][(g^{-1}, 0)][(g, 0)] = [(\varphi(g), 1)][(e, -1)][(e, 1)] = [(\varphi(g), 0)][(e, 1)][(e, -1)][(e, 1)] = [(\varphi(g), 0)][(e, 1)];$$

(iii):

$$[(g, 0)][(e, 1)][(e, -1)][(g^{-1}, 0)] = [(g, 1)][(e, -1)][(g^{-1}, 0)] = [(g, 1)][(e, -1)][(g^{-1}, 0)][(g, 0)][(g^{-1}, 0)] = [(g, 1)][(g^{-1}, -1)][(g, 0)][(g^{-1}, 0)] = [(g, 1)][(g^{-1}, -1)] = [(g, 1)][(g, 1)^*],$$

and using $R_A$ we see that it satisfies condition (iii). \qed
Consequently we have a $\ast$-homomorphism
\[ \Psi : U[\varphi] \to C_p^*(\overline{S}, \mathcal{R}) \]
\[ u_g \mapsto [(g, 0)] \]
\[ s^n \mapsto [(e, n)]. \]

**Theorem 5.8.** The $C^*$-algebra $U[\varphi]$ is isomorphic to $C_p^*(\overline{S}, \mathcal{R})$.

**Proof.** We just have to show that the $\ast$-homomorphisms (8) and (9) are inverses of each other on the generators of the respective $C^*$-algebras.

\[ \Phi \circ \Psi(u_g) = \Phi([(g, 0)]) = u_g; \]
\[ \Phi \circ \Psi(s^n) = \Phi([(e, n)]) = s^n; \]
\[ \Psi \circ \Phi([(g, n)]) = \Psi(s^i u_g s^{n+i}) = [(e, -i)][(g, 0)][(e, n+i)] \]
\[ = [(e, -i)][(g, 0)][(e, n+i)][(e, -n-i)][(e, n+i)] \]
\[ = [(e, -i)][(g, n+i)][(e, -n-i)][(e, n+i)] \]
\[ = [(e, -i)][(e, i)][(e, -i)][(g, n+i)] \]
\[ = [(e, -i)][(e, i)][(e, -i)][(g, n+i)] \]
\[ = [(e, -i)][(e, i)][(e, -i)][(g, n+i)] \]
\[ = [(e, -i)][(e, i)][(e, -i)][(g, n+i)] \]
\[ = [(e, -i)][(e, i)][(e, -i)][(g, n+i)] \]
\[ = [(g, n+i)]. \]

\[ \square \]

In order to define a partial crossed product isomorphic to $C_p^*(\overline{S}, \mathcal{R})$ which by the theorem above is isomorphic to $U[\varphi]$, consider $X_{\overline{S}}$ the subset of $P(\overline{S})$ of the subsets $\xi$ of $\overline{S}$ which contain $(e, 0) \in \overline{S}$. Also $1_s \in C(X_{\overline{S}})$ is given by
\[ 1_s(\xi) = \begin{cases} 1, & s \in \xi; \\ 0, & \text{otherwise}. \end{cases} \]

and the partial group algebra relations $\mathcal{R}$ are
\[ (\mathcal{R}_1) \ e_{(g, 0)} - 1 = 0, \forall g \in G; \]
\[ (\mathcal{R}_2) \ e_{(e, -n)} - 1 = 0 \forall n \in \mathbb{N}; \]
\[ (\mathcal{R}_3) \sum_{g \in G}[e_{(g, n)} - 1 = 0, \forall n \in \mathbb{N}. \]

This implies that $\widehat{\mathcal{R}}$ is the subset of $C(X_{\overline{S}})$ consisting of the functions
\[ (\widehat{\mathcal{R}}_1) \ 1_{(g, 0)} - 1_{(e, 0)}, \forall g \in G; \]
\[ (\widehat{\mathcal{R}}_2) \ 1_{(e, -n)} - 1_{(e, 0)} \forall n \in \mathbb{N}; \]
\[ (\widehat{\mathcal{R}}_3) \sum_{g \in G}[1_{(g, n)} - 1_{(e, 0)}, \forall n \in \mathbb{N}. \]

The spectrum of the relations $\mathcal{R}$ is defined to be
\[ \Omega_{\mathcal{R}} = \{ \xi \in X_{\overline{S}} : f(g^{-1}\xi) = 0, \forall f \in \widehat{\mathcal{R}}, \forall g \in \xi \}. \]

Consider
\[ \Omega_s = \{ \xi \in \Omega_{\mathcal{R}} : s \in \xi \} \]
and define the partial action $\varpi$ of $\overline{S}$ on $\Omega_{\mathcal{R}}$ by
\[ \varpi_s : \Omega_{s^{-1}} \to \Omega_s \]
\[ \xi \mapsto s\xi. \]

Then it is well known by Theorem 5.8 and (7) respectively that
\[ U[\varphi] \cong C_p^*(\overline{S}, \mathcal{R}) \cong C(\Omega_{\mathcal{R}}) \rtimes_{\alpha} \overline{S}, \]

(11)
where
\[
\alpha_s : C(\Omega_{s-1}) \to C(\Omega_s)
\]
\[
f \mapsto f \circ \omega_{s-1}.
\]

The partial crossed product description of $U[\varphi]$ presented above together with the requirement that $G$ is amenable (which implies that $S$ is as well) makes it possible to define a certain conditional expectation as done in [10] Proposition 2.9 (as in the classical group crossed product construction the amenability of the group implies the isomorphism of both reduced and full constructions by [19], and a faithful conditional expectation exists for the reduced one). We will show that this conditional expectation is the same - modulo the isomorphism already established - as $\epsilon$ as given by Proposition 4.4. The conditional expectation of $C(\Omega_R) \rtimes_\alpha S$ is given by
\[
E : C(\Omega_R) \rtimes_\alpha S \to C(\Omega_R)
\]
\[
f \delta_s \mapsto \begin{cases} f, & \text{if } s = (e, 0); \\ 0, & \text{otherwise.} \end{cases}
\]

Identifying $C^*_p(S, R)$ with $C(\Omega_R) \rtimes_\alpha S$, $E$ becomes
\[
E : C^*_p(S, R) \to C^*(\epsilon_{(g_i, n)})
\]
\[
\text{finite } \prod_{(g_i, n) \in S} [(g_i, n)] \mapsto \begin{cases} \text{finite } \prod_{(g_i, n) \in S} [(g_i, n)], & \text{if } \prod_{(g_i, n) \in S} (g_i, n) = (e, 0); \\ 0, & \text{otherwise.} \end{cases}
\]

Using the isomorphism $\Psi$ (from [19]) and $\epsilon$ (from Proposition 4.4), we shall prove the following.

**Proposition 5.9.** $E \circ \Psi = \Psi \circ \epsilon$.

**Proof.** Let us prove the equality on the dense $*$-subalgebra of $U[\varphi]$ given by
\[
\text{span}(Q) = \text{span}\{s^n u_{h-1} f u_h s^n : f \in P, h, h' \in G, n, m \in \mathbb{N}\}.
\]

Consider $f = u_g s^k s^n u_{g^{-1}} P, h, h' \in G$, and $n, m \in \mathbb{N}$.

\[
E \circ \Psi(s^n u_{h-1} f u_h s^n) = E \circ \Psi(s^n u_{h-1} u_g s^k u_{g^{-1}} u_{h'} s^m)
\]
\[
= \epsilon([e, -n])[(h^{-1}, 0)][(g, 0)][(e, k)][(e, -k)][(g^{-1}, 0)][(h', 0)][(e, m)]
\]
\[
= \delta_{n, m} \delta_{h, h'}[(e, -n)][(h^{-1}, 0)][(g, 0)][(e, k)][(e, -k)][(g^{-1}, 0)][(h, 0)][(e, n)]
\]
\[
= \delta_{n, m} \delta_{h, h'}[(e, -n)][(h^{-1}, 0)] \Psi(f)[(h, 0)][(e, n)],
\]
while
\[
\Psi \circ \epsilon(s^n u_{h-1} f u_h s^n) = \Psi(\delta_{n, m} \delta_{h, h'} s^n u_{h-1} f u_h s^n)
\]
\[
= \delta_{n, m} \delta_{h, h'}[(e, -n)][(h^{-1}, 0)] \Psi(f)[(h, 0)][(e, n)].
\]

This shows that both conditional expectations $E$ and $\epsilon$ are the same, up to the isomorphism $\Psi$. \(\square\)

\[3\epsilon_{(g_i, n)} := [(g_i, n)][(g_i, n)^{-1}] \text{ with } (g_i, n) \in S = G \rtimes_{\varphi} \mathbb{Z}\]
5.1. Simplicity of $U[\varphi]$. To prove that $U[\varphi]$ is simple using partial crossed product theory, we suppose that $G$ is commutative. Therefore our group is amenable and the endomorphism $\varphi$ is totally normal i.e., the images of $\varphi$ are normal subgroups of $G$. This implies that the set $G$, defined in the beginning of Section 3, is a group.

We need some definitions (from [11]) concerning partial actions, as they play a role in the proof that $U[\varphi]$ is simple. Consider $(\theta, H, X)$ a partial dynamical system where $X$ is a locally compact space with $X_h$ being the open sets (Definition 3.1).

**Definition 5.10.** We say that a partial action $\theta$ is topologically free if for every $h \in H \setminus \{e\}$ the set $F_h := \{ x \in X_{h^{-1}} : \theta_h(x) = x \}$ has empty interior.

In order to define the minimality of $\theta$, we adjust the classical definition of invariance: a subset $V$ of $X$ is invariant under the partial action $(\theta, H, X)$ if $\theta_h(V \cap X_{h^{-1}}) \subseteq V \forall h \in H$.

**Definition 5.11.** The partial action $\theta$ is minimal if there are no invariant open subsets of $X$ other than $\emptyset$ and $X$.

Suited to our setting, there is a result due to Exel, Laca and Quigg (Corollary 2.9 of [11]) which says that the partial action $\vartheta$ defined in (10) is topologically free and minimal if and only if $C(\Omega_R) \rtimes_{\alpha}(S)$, as defined in (11) and (12), is simple (in fact their result applies to the reduced crossed product, but as we are assuming $G$ is commutative and thus amenable, we know that $\overline{\mathcal{S}}$ is amenable and this implies that both the full and reduced partial crossed products are isomorphic by [19] Proposition 4.2), so it is clear that we have to understand the topology of $\Omega_R$, which unfortunately is not an easy task.

To avoid difficulties we present a new set which is homeomorphic to $\Omega_R$, and for which we can easily understand the topology. Consider $G[\varphi]$ for negative integers $k$ and for $m \leq n$ both integers the canonical projection $p_{m,n} : G[\varphi^m(G)] \to G[\varphi^n(G)]$.

Using these, define

$$\tilde{G} := \lim_{\to} \left\{ \frac{G}{\varphi^n(G)} : p_{m,n} \right\}$$

$$= \left\{ (g_n \overline{\varphi}^n(G))_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} \frac{G}{\varphi^n(G)} : p_{m,n}(g_n) = g_m, \text{ if } m \leq n \right\},$$

where $\overline{\varphi}$ is the extension of $\varphi$ defined after Proposition 3.8. Note that when $n \leq 0$, $\frac{G}{\varphi^n(G)} = \{e\}$ and therefore for any element in $\tilde{G}$, the entries indexed by negative integers are $e$. Moreover, when $n > 0$, $\overline{\varphi}^n = \varphi^n$. Particularly it makes not necessary to carry the bar over $\varphi$ when denoting the elements of $\tilde{G}$, and we will also use the notation $(g_m)_{n \in \mathbb{Z}} \in \tilde{G}$. One can see $G$ inside $\tilde{G}$ through the map $g \mapsto (g \varphi^n(G))_n$, which is injective if $\varphi$ is pure.

Another fact is that the set defined above is isomorphic as a topological group to our previous defined $G$ (beginning of Section 3), because that set is exactly this one except for the negative entries of the vectors in $\tilde{G}$, which are always $e$. Therefore $\tilde{G}$ is compact.

Consider

$$\rho : \tilde{G} \to \mathcal{P}(\overline{\mathcal{S}})$$

$$(g_n \overline{\varphi}^n(G))_{n \in \mathbb{Z}} \mapsto \{(g_n \overline{\varphi}^n(h), n) : n \in \mathbb{Z}, h \in G\}.$$
Lemma 5.12. The set $\rho(\tilde{G})$ is contained in $\Omega_R$.

Proof. Take $(g_m)_m \in \tilde{G}$ and it is clear from the definition of $\tilde{G}$ that
\[
g_m = g_{m-n}\varphi^{m-n}(\bar{k}_1)
\]
and
\[
g_{m+n} = g_m\varphi^n(\bar{k}_2)
\]
for $n \in \mathbb{N}$ and $\bar{k}_1, \bar{k}_2 \in G$.

Denote $\xi := \rho((g_m)_m)$. We have to show that $f(g^{-1}\xi) = 0$ for all $g \in \xi$ and all $f \in R = \bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$. Therefore fix $g = (g_m\varphi^m(k), m) \in \xi$ for $m \in \mathbb{Z}$ and $k \in G$.

- $f = 1_{(h, 0)} - 1 \in \bar{R}_1$: Then $f(g^{-1}\xi) = 0 \iff g(h, 0) \in \xi$, which is true because $g(h, 0) = (g_m\varphi^m(kh), m) \in \xi$.
- $f = 1_{(e, -n)} - 1 \in \bar{R}_2$: Similarly $f(g^{-1}\xi) = 0 \iff g(e, -n) \in \xi$ and the latter holds as $g(e, -n) = (g_m\varphi^m(k), m - n) = (g_{m-n}\varphi^{m-n}(\bar{k}_1\varphi^n(k)), m - n) \in \xi$.
- $f = \sum_{h \in \varphi^n(G)} 1_{(h, n)} - 1 \in \bar{R}_3$: Here $f(g^{-1}\xi) = 0 \iff$ there exists only one class $h\varphi^n(G)$ such that $g(h, n) \in \xi$. But
\[
g(h, n) = (g_m\varphi^m(kh), m + n) = (g_{m+n}\varphi^n(\bar{k}_2^{-1}kh), m + n)
\]
belongs to $\xi$ if and only if $\bar{k}_2^{-1}kh \in \varphi^n(G) = \varphi^n(G)$ (as $n \in \mathbb{N}$), which is the same as requiring $h \in k^{-1}\bar{k}_2\varphi^n(G)$, and this can be true only for one class in $\frac{G}{\varphi^n(G)}$.

Proposition 5.13. $\rho : \tilde{G} \to \Omega_R$ is a homeomorphism.

Proof. If $\rho((g_m)_m) = \rho((h_m)_m)$ then $h_m = g_m\varphi^m(k_m)$ for all $m \in \mathbb{N}$, with $k_m \in G$. Then $g_m = h_m$ in $\frac{G}{\varphi^m(G)}$ for all $m \in \mathbb{N}$ and $(g_m)_m = (h_m)_m$ (note that for $m < 0$, $g_m = h_m = e$).

Now let us prove that $\rho$ is surjective. Take $\xi \in \Omega_R$ and remember that $(e, 0) \in \xi$ which, using $f_1^0 := 1_{(h, 0)} - 1 \in \bar{R}_1$, implies that $(h, 0) \in \xi \forall h \in G$. Also for each $j \in \mathbb{N}$, set $f_1^j := \sum_{h \in \varphi^j(G)} 1_{(h, j)} - 1 \in \bar{R}_3$.

As $f_2^j((e, 0)\xi) = 0$, for each $j$ there exists only one class $u_j\varphi^j(G) \in \frac{G}{\varphi^j(G)}$, such that $(u_j, j) \in \xi$. Using functions of the type $f_2^n := 1_{(0, -n)} - 1 \in \bar{R}_2$, for $n \in \mathbb{N}$, one sees that $(u_j\varphi^j(G))_{j \in \mathbb{Z}} \in \tilde{G}$. Now we prove that $\rho((u_j\varphi^j(G))_{j}) = \xi$.

By construction $(u_j, j) \in \xi$, which implies (using $f_1^j \in \bar{R}_1$ defined above) that $(u_j, j)(h, 0) = (u_j\varphi^j(h), j) \in \xi$ for all $h \in G$. Doing the same for every $j$ it follows that $\rho((u_j\varphi^j(G))_{j}) \subset \xi$.

Suppose that $h = (k, i) \in \xi \setminus \rho((u_j\varphi^j(G))_{j})$ and note that
\[
(k, i) \notin \rho((u_j\varphi^j(G))_{j}) \iff (k, i) \notin (u_i\varphi^i(G), i) \iff u_i^{-1}k \notin \varphi^i(G).
\]
Now consider the elements $g = (u_i, 0)$ and $h' = (u_i, i)$ of $\rho((u_j\varphi^j(G))_{j}) \subset \xi$. Since $u_i^{-1}k \notin \varphi^i(G)$, we have that $g^{-1}h = (u_i^{-1}k, i)$ and $g^{-1}h' = (e, i)$ are different, which implies that $f_3^i(g^{-1}\xi) \neq 0$, and this contradicts the fact that $\xi \in \Omega_R$.

Last, let us prove that $\rho$ preserves the topology. As the sets are compact and Hausdorff, it is enough to prove that $\rho^{-1}$ is continuous, which we will prove by showing that $\pi_m \circ \rho^{-1}$ is continuous for all $m \in \mathbb{Z}$ where $\pi_m : \tilde{G} \to \frac{G}{\varphi^m(G)}$ is the canonical projection.
As \( \frac{G}{\varphi_n(G)} \) is discrete we just have to show that \( \rho \circ \pi^{-1}_m(\{u_m\varphi^m(G)\}) \) is open in \( \Omega_R \) for all \( u_m\varphi^m(G) \in \frac{G}{\varphi^m(G)} \). But note that (by the proof of surjectivity above)

\[
\rho \circ \pi^{-1}_m(\{u_m\varphi^m(G)\}) = \{ \xi \in \Omega_R : (u_m, m) \in \xi \},
\]

which is open in \( \Omega_R \) (induced by the product topology in \( \{0,1\}^S \)).

Then \( \rho : \tilde{G} \to P(S) \) is a homeomorphism.

Using the proposition above, we identify \( \Omega_R \) with \( \tilde{G} \), and thus view \( \varpi \) as a partial action of \( G \) on \( \tilde{G} \). Remember that

\[
\Omega_s = \{ \xi \in \Omega_R : s \in \xi \}.
\]

Set

\[
\tilde{G}_s := \rho^{-1}(\Omega_s)
\]

and define

\[
\varpi_s : \tilde{G}_{s^{-1}} \to \tilde{G}_s.
\]

Using \( \rho \) we can conclude that for \( (g_i, n) \in \tilde{S} = G \times_\varpi \mathbb{Z} (g_i \in G_i \hookrightarrow G) \)

\[
\tilde{G}(g_i, n) = \{(h_m\varphi^m(G))_{m \in \mathbb{Z}} \in \tilde{G} : h_n\varphi^n(G) = g_i\varphi^n(G)\}
\]

(where \( h_n \) is viewed inside \( G = G_0 \subseteq G \) and

\[
\varpi(g_i, n)((h_m\varphi^m(G))_m) = (g_i\varphi^n(h_m)\varphi^{n+m}(G))_{n+m} = (g_i\varphi^n(h_{m-n})\varphi^m(G))_m.
\]

An easily proven and useful result follows.

**Lemma 5.14.** For \( (g_i, n) \in \tilde{S} \) the following holds:

(i) \( \tilde{G}(g_i, n) = \emptyset \iff g_i \notin G\varphi^n(G) \);
(ii) \( \tilde{G}(g_i, n) = \tilde{G} \iff G \subseteq g_i\varphi^n(G) \).

\( \square \)

For \( m \in \mathbb{Z} \) and a subset \( C_m \subseteq \frac{G}{\varphi^m(G)} \) (containing whole cosets) define the open set

\( V^C_m = \{(u_m\varphi^m(G))_n \in \tilde{G} : u_m\varphi^m(G) \in C_m\} \).

Clearly when \( m \leq n \) then \( V^C_m = V^C_n \) where

\[
C_n = \left\{ u\varphi^n(G) \in \frac{G}{\varphi^n(G)} : u\varphi^m(G) \in C_m \right\}.
\]

From the definition of the product topology, we know that finite intersections of open sets \( V^C_m \) form the base for the topology in \( \tilde{G} \). Since \( V^C_{m_1} \cap V^C_{m_2} = V^C_{m_1 \cap m_2} \cap V^C_{m_2} = V^C_{m_1 \cup m_2} \) for \( m \geq m_1, m_2 \), \( \{ V^C_m \} \) is already a base for the topology.

Also note that if \( C_m \neq \emptyset \) then for \( k > 0 \), \( C_{m+k} \) has at least 2 elements and therefore we can assume that if \( V^C_m \) is not empty then \( C_m \) has at least 2 elements (replacing \( V^C_m \) by \( V^C_n \) for \( n > m \) if necessary).

**Proposition 5.15.** When \( \varphi \) is a pure injective endomorphism of a commutative group \( G \), the partial action \( \varpi \) from \( \tilde{G} \) defined above is topologically free.

**Proof.** Let us show that

\[
F_{(g_i, n)} = \{ x \in \tilde{G}(g_i, n)^{-1} : \varpi(g_i, n)(x) = x \}
\]

has empty interior, for \( (g_i, n) \neq (e, 0) \).

- Case 1: \( n = 0 \). If \( g_i \notin G \) then Lemma 5.14 (i) assures that \( F_{(g_i, 0)} = \emptyset \).
Therefore suppose that \( g_i \in G \). If \( F_{(g_i,0)} \neq \emptyset \) the equation \( \varpi_{(g_i,0)}(x) = x \) implies \( g_i \in \varphi^m(G) \) for all \( m \in \mathbb{Z} \) (using the commutativity of \( G \)). As \( \varphi \) is pure we conclude that \( g_i = e \), and then \( F_{(g_i,0)} = \emptyset \) for \( g_i \neq e \).

- Case 2: Let \((g_i,n)\) with \( n \neq 0 \). Using again Lemma \(5.14\) (i) we can assume that \( g_i \in G\varphi^n(G) \). Take \( V \) a non-empty open set of \( \tilde{G}_{(g_i,n)}^{-1} \) and, if needed, shrink \( V \) so that \( V = V^C_m \) (and we can assume that \( m = ln > 0 \) for some big \( l > 0 \)). Note that we can assume that \( C_m \) has at least 2 distinct elements, say \( u_1\varphi^n(G) \neq u_2\varphi^n(G) \), which implies that \( u_2^{-1}u_1 \notin \varphi^n(G) \).

Suppose for a contradiction that \( \varpi_{(g_i,n)}(x) = x, \forall x \in V \). Then, since \( (u_j\varphi^k(G))_k \in V \) for \( j = 1,2 \), we have

\[
\varpi_{(g_i,n)}((u_j\varphi^k(G))_k) = (u_j\varphi^k(G))_k \Rightarrow \left(g_i\varphi^n(u_j)\varphi^k(G)\right)_k = (u_j\varphi^k(G))_k
\Rightarrow u_j^{-1}g_i\varphi^n(u_j) \in \varphi^k(G) \text{ for } j = 1,2
\Rightarrow \varphi^n(u_2^{-1})u_2u_1^{-1}\varphi^n(u_1) \in \varphi^k(G), \forall k \in \mathbb{Z}
\]

(again we used the commutativity of \( G \) to cancel the \( g_i \)'s). But as \( \varphi \) is pure, \( \varphi^n(u_2^{-1}u_1) = u_2^{-1}u_1 \Rightarrow \varphi^n(u_2^{-1}u_1) = u_2^{-1}u_1 \Rightarrow u_2^{-1}u_1 \in \varphi^n(G) \)

which contradicts our hypothesis. So no open set can be contained in \( F_{(g_i,n)} \), which implies that it has empty interior. \( \square \)

**Proposition 5.16.** The partial action \( \varpi \) is minimal.

**Proof.** We will show that all \( x \in \tilde{G} \) has dense orbit by showing the following: if \( V \) is a non-empty open set then there exists \((g_i,n) \in \mathcal{S} \) such that \( x \in \tilde{G}_{(g_i,n)}^{-1} \) and \( \varpi_{(g_i,n)}(x) \in V \).

Take \( x = (u_m\varphi^m(G))_m \in \mathbb{Z} \subset \tilde{G} \) and \( V = V_k^C \neq \emptyset \). Consider \( u\varphi^k(G) \in C_k \) and define \((uu_k^{-1},0)\). By Lemma \(5.14\) (ii), since \( uu_k^{-1}G = G \), it follows that \( \tilde{G}_{(uu_k^{-1},0)}^{-1} = \tilde{G} \) and therefore \( x \in \tilde{G}_{(uu_k^{-1},0)}^{-1} \).

To finish, note that

\[
\varpi_{(uu_k^{-1},0)}(x) = \varpi_{(uu_k^{-1},0)}((u_m\varphi^m(G))_m) = (uu_k^{-1}u_m\varphi^m(G))_m \in V.
\]

We can now conclude (and this result agrees with the previous obtained Theorem 1.7):

**Theorem 5.17.** If \( \varphi \) is a pure injective endomorphism with finite cokernel of some commutative discrete countable group \( G \) then the \( C^* \)-algebra \( U[\varphi] \) is simple. \( \square \)

**Corollary 5.18.** In the conditions of theorem above, we have

\[
\text{C}_r^*[\varphi] \cong U[\varphi].
\]

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