Anyons Quantum Transport and Noise Away from Equilibrium

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The quantum transport of anyons in one space dimension is investigated. After establishing some universal features of non-equilibrium systems in contact with two heat reservoirs in a generalized Gibbs state, the abelian anyon solution of the Tomonaga–Luttinger model possessing axial-vector duality is focused upon. In this context a non-equilibrium representation of the physical observables is constructed, which is the basic tool for a systematic study of the anyon particle and heat transport. The associated Lorenz number is determined and the deviation from the standard Wiedemann–Franz law induced by the interaction and the anyon statistics is explicitly described. The quantum fluctuations generated by the electric and helical currents are investigated and the dependence of the relative noise power on the statistical parameter is established.

1. Introduction

Quantum statistics play a fundamental role in the particle and heat transport in non-equilibrium quantum systems. In the present paper we pursue further the study of this feature, focussing on the impact of generalized braid statistics.[1–4] More precisely, we consider abelian anyons, whose free dynamics in 1 + 1 space-time dimensions is described by the Lagrangian density

$$\mathcal{L}_0 = i \psi_{\alpha}^* \left( \partial_\tau - \partial_x \right) \psi_{\alpha} + i \nu_{\alpha} \left( \partial_\tau + \partial_x \right) \psi_{\alpha}^*$$

Here \(\psi_{\alpha}(t, x)\) are complex fields obeying anyon equal-time exchange relations\([5]\) \((\alpha \neq \beta)\)

$$\psi_{\alpha}^*(t, x_1) \psi_{\beta}^*(t, x_2) = e^{i \epsilon (x_1 - x_2) \kappa} \psi_{\beta}(t, x_1) \psi_{\alpha}^*(t, x_2)$$

where \(\epsilon(x)\) is the sign function and \(\kappa > 0\) is the so-called statistical parameter, which interpolates between bosons (\(\kappa\) - even integer) and fermions (\(\kappa\) - odd integer). The parameter \(\kappa\) plays a central role in our investigation, being devoted to a systematic study of \(\kappa\)-dependence of the anyon quantum transport and the noise generated away from equilibrium.

The interaction, which successfully describes\([6,7]\) the universal features of a large class of 1D systems exhibiting gapless excitations with linear spectrum, is fixed by

$$\mathcal{L}_I = -\frac{\pi g_+}{2} \left( \psi_{\alpha}^* \psi_{\beta} + \psi_{\beta}^* \psi_{\alpha} \right)^2$$

where \(g_\pm \in \mathbb{R}\) are the coupling constants. The use of the normalization factor \(\pi\) in Equation (3) simplifies some basic equations in what follows and is introduced for convenience.

The total Lagrangian \(\mathcal{L}_0 + \mathcal{L}_I\) does not involve dimensional parameters and is scale invariant. Combined with Equation (2) it defines the dynamics of anyons\([8,9]\) Tomonaga–Luttinger (TL) liquid\([10–12]\) with \(U(1) \times \tilde{U}(1)\) symmetry, where the \(U(1)\)-vector and \(\tilde{U}(1)\)-axial transformations are defined by

$$\psi_{\alpha}(t, x) \mapsto e^{i \alpha} \psi_{\alpha}(t, x), \quad \alpha \in [0, 2\pi)$$

and

$$\psi_{\alpha}(t, x) \mapsto e^{i (-1)^{\tilde{\alpha}} \beta} \psi_{\alpha}(t, x), \quad \tilde{\alpha} \in [0, 2\pi)$$

respectively. The relative conserved currents describe the electric and helical transport, respectively.

The main subject of our investigation is the quantum transport induced by connecting the system \(\mathcal{L}_0 + \mathcal{L}_I\) via the gates \(G_i\) with two heat reservoirs \(R_i\) as shown in Figure 1. Each of them is described by a generalized Gibbs ensemble (GGE) with (inverse) temperatures \(\beta_i \geq 0\) \((i = 1, 2)\) and the chemical potentials \(\mu_i\). The interaction of the anyons emitted and absorbed by the two heat reservoirs \(R_i\) is described by Equation (3). This interaction drives the system away from equilibrium. The capacity of \(R_i\) is assumed large enough so that the emission-absorption processes do not change their parameters.

Systems with the structure shown in Figure 1 in one space dimension are subject of intense studies. They are successfully applied for describing the transport properties of quantum...
wire junctions\textsuperscript{[13–21]} and fractional quantum Hall samples, where abelian anyons with filling factors $\kappa = 1/(2n + 1)$ with $n = 1, 2, \ldots$ are propagated.\textsuperscript{[22]} Recently there is also the exciting possibility to conceive in laboratory the setting in Figure 1 by ultracold Bose gases.\textsuperscript{[23–25]} The remarkable control over the interactions and the geometry of the samples in such experiments allow to explore a variety of fundamental aspects of many-body quantum physics. This is also the case for the anyon systems considered in this paper.

At the theoretical side the equilibrium fermionic version of the TL model with $\kappa = 1$ is widely investigated\textsuperscript{[30–35]} in the literature. There has been also great interest\textsuperscript{[30–35]} in the behavior of the model away from equilibrium. These more recent studies focus essentially on the charge transport induced by the difference $\mu_1 - \mu_2$. However, the combined effect of the two chemical potentials $\mu$ and $\tilde{\mu}$ with generic statistical parameter $\kappa$ is poorly studied. Filling this gap is among our main goals below. In this respect we demonstrate that the presence of both $\mu$ and $\tilde{\mu}$ makes explicit the axial-vector duality that characterizes the TL model (1, 3). Let us recall in this respect that this duality is broken in the presence of energy preserving impurities, which necessarily violate\textsuperscript{[35]} at least one among the factors of the $U(1) \times \tilde{U}(1)$ symmetry. We show that the non-trivial interplay between $\mu$ and $\tilde{\mu}$ generates persistent charge and heat currents at equilibrium. We also establish the relevant impact of $\mu$ and $\tilde{\mu}$ on the entropy production in the system away from equilibrium.

Another related aspect, addressed in the paper, concerns the quantum fluctuations generated by the electric and helical currents. Such fluctuations produce noise which spoils the propagation of the anyon excitations. It is known\textsuperscript{[36–38]} however that the current fluctuations carry also useful information, providing the experimental basis of noise spectroscopy. In this respect we exhibit the dependence of the noise on the statistical parameter $\kappa$ in explicit form.

The paper is organized as follows. In the next section we focus on the universal properties of non-equilibrium quantum systems in contact with two GGE heat reservoirs. In Section 3 we summarize the operator solution of the TL model (1, 3) and construct in detail a specific representation of this solution, which describes the system connected with the two heat reservoirs. On this basis we investigate in Section 4 the anyon particle and heat transport and evaluate the associated Lorenz number. The impact of the interaction $g_\kappa$ and the statistical parameter $\kappa$ on the Wiedemann–Franz law is a central point of this section. We investigate here also the mean value of the entropy production in the presence of all chemical potentials characterizing the GGE. Section 5 is devoted to the quantum fluctuations of the electric and helical currents and the impact of $\kappa$ on the generated noise. Finally, Section 6 collects our conclusions.

2. Basic Features of Quantum Transport between Two GGE Reservoirs

Before discussing the specific anyon case, we would like to establish some general properties of the quantum transport between two GGE reservoirs as shown in Figure 1. For simplicity we assume in what follows that the total energy $E$ and the charges $\{Q, \tilde{Q}\}$ are conserved, which means that the incoming and outgoing energy and charge flows through the gates $G_i$ compensate each other. We show below that under this realistic physical assumption, the system posses remarkable universal features, which do not depend on the nature of the interaction and are therefore of great relevance. In order to establish them, we introduce the energy and charge densities $\theta_i$ and $j_i, \tilde{j}_i$, satisfying

\begin{align}
\mathcal{H} &= \int_{G_1} dx \theta_i(t, x) \\
Q &= \int_{G_1} dx j_i(t, x), \quad \tilde{Q} = \int_{G_1} dx \tilde{j}_i(t, x)
\end{align}

Let $\theta_i$ and $j_i, \tilde{j}_i$ be the corresponding local conserved currents, which obey the continuity equations

\begin{align}
\partial_t \theta_i(t, x) - \partial_x j_i(t, x) &= 0 \\
\partial_t j_i(t, x) - \partial_x \tilde{j}_i(t, x) &= \partial_t \tilde{j}_i(t, x) = 0
\end{align}

Combining Equations (6) and (7) with (8) and (9) one finds

\begin{align}
\mathcal{H} = 0 \Rightarrow \theta_i(t, G_1) = \tilde{\theta}_i(t, G_2) \\
Q = 0 \Rightarrow j_i(t, G_1) = \tilde{j}_i(t, G_2) \\
\tilde{Q} = 0 \Rightarrow \tilde{j}_i(t, G_1) = \tilde{j}_i(t, G_2)
\end{align}

which is the expected behavior in the gates $G_i$ for any $t$.

Let us consider now the heat current $q_i$ flowing through $G_i$. Since the values of the chemical potentials in $G_i$ are $\mu_i$ and $\tilde{\mu}_i$, one has, following the rules of non-equilibrium thermodynamics,\textsuperscript{[19]}

\begin{align}
q_i(t, G_i) &= \tilde{\theta}_i(t, G_i) - \mu_j_i(t, G_i) - \tilde{\mu}_j_i(t, G_i)
\end{align}

From Equation (13) it follows that for $\mu_1 \neq \mu_2$ and/or $\tilde{\mu}_1 \neq \tilde{\mu}_2$ the heat flow through $G_1$ differs from that through $G_2$. In fact,

\begin{align}
\hat{Q} &\equiv q_i(t, G_1) - q_i(t, G_2) \\
&= (\mu_2 - \mu_1)j_i(t, G_1) + (\tilde{\mu}_2 - \tilde{\mu}_1)\tilde{j}_i(t, G_1) \neq 0
\end{align}

At this point we use that the total energy of the system has three components - heat energy and two different types of chemical energies associated with the charges $Q$ and $\tilde{Q}$. Since the total energy is conserved, Equation (14) implies that heat energy
be converted in one or two types of chemical energies and vice versa. This process depends on the state \( \Psi \in H \) of the system, namely on the expectation value

\[
\langle \tilde{Q} \rangle_\nu \equiv \langle \Psi, \tilde{Q}\Psi \rangle
\]

(15)

where \( \langle \cdot, \cdot \rangle \) is the scalar product in the state space \( H \). Chemical energy is converted to heat energy if \( \langle \tilde{Q} \rangle_\nu > 0 \). The opposite process takes place for \( \langle \tilde{Q} \rangle_\nu < 0 \) and energy transmutation is absent only if \( \langle \tilde{Q} \rangle_\nu = 0 \). It is worth stressing that there is no dissipation in the energy conversion.

Therefore, one concludes that the observed mean values of the heat flow through the gates \( G_i \) and \( G_j \) are in general different.

Notice that we adopt here only the value of the heat current in the gates \( G_i \). The point is that the heat current in the interaction domain \( \mathbb{R} \) in Figure 1 is not known, because the temperature and the chemical potentials are not determined in this region. In order to introduce the concept of local temperature \( \beta(x) \) and chemical potentials \( \mu(x) \) and \( \bar{\mu}(x) \) in a point \( x \in \mathbb{R} \) one needs further model dependent assumptions\(^{[41]}\) which are not needed for our construction.

Let us discuss now the choice of the state \( \Psi \). In this paper we consider steady states which are generated by the GGE states of the heat reservoirs and are invariant under time translations, implying that the expectation value \( \langle \mathcal{O}(t, x) \rangle_\nu \) of any observable \( \mathcal{O} \) is actually \( t \)-independent. In particular, even if \( \tilde{Q} \) depends on \( t \), its expectation value \( \langle \tilde{Q} \rangle_\nu \) does not.

Concerning the action of the time reversal operation on \( \Psi \), we first observe that for steady states, describing the nonequilibrium system in Figure 1, there is a nontrivial energy exchange between the reservoirs \( R \), leading to

\[
\langle \tilde{Q}(t, x) \rangle_\nu \equiv \langle \Psi, \tilde{Q}(t, x)\Psi \rangle \neq 0
\]

(17)

Now we recall that\(^{[42]}\)

\[
T \theta_\nu(t, x) T^{-1} = -\theta_\nu(-t, x)
\]

(18)

where \( T \) is the anti-unitary operator implementing the time reversal in the Hilbert space \( H \). Taking the expectation value of Equation (18) one has

\[
\langle T \theta_\nu(t, x) T^{-1} \rangle_\nu = -\langle \theta_\nu(-t, x) \rangle_\nu
\]

(19)

which, combined with the fact that \( \langle \theta_\nu(t, x) \rangle_\nu \) is \( t \)-independent, implies that

\[
T\Psi \neq \Psi
\]

(20)

Therefore time reversal is spontaneously broken in any state \( \Psi \) obeying Equation (17). This genuine quantum field theory phenomenon\(^{[43]}\) is the origin of the non-trivial entropy production

\[
\langle S \rangle_\nu = \beta_\nu \langle q_\nu(t, G_1) \rangle_\nu - \beta_\nu \langle q_\nu(t, G_2) \rangle_\nu
\]

(21)

which takes place even for systems with time reversal invariant dynamics without dissipation. The source for the spontaneous \( T \)-breaking in the above setting is the coupling with the two reservoirs in Figure 1, which have different temperatures and chemical potentials.

Summarizing, the physical consequences of the energy and charge conservation in systems, schematically represented in Figure 1, are:

1. conversion of heat to chemical energy or vice versa;
2. non-trivial entropy production.

These features do not depend on the dynamics being therefore universal. What follows is an illustration of 1) and 2) and their impact on the particle and heat transport in the anyon TL model defined by Equations (1) and (3).

3. Anyon Luttinger Liquid

In this section, following\(^{[8,18]}\) we first briefly summarize the anyon operator solution of the TL model (1, 3). Afterward we provide a new Hilbert space representation of this solution, which is induced by the nonequilibrium steady state describing the system connected with the two GGE reservoirs as shown in Figure 1.

3.1. Operator Solution

The classical equations of motion of the TL model read

\[
i(\partial_\nu - \partial_x)\psi_1 = \pi g_1 \psi_1 + \pi g_2 \bar{\psi}_1
\]

(22)

\[
i(\partial_\nu + \partial_x)\psi_2 = \pi g_1 \psi_2 - \pi g_2 \bar{\psi}_2
\]

(23)

where

\[
\psi_1 = (\psi_1^* \psi_1 + \psi_2^* \psi_2), \quad \bar{\psi}_1 = (\psi_1^* \psi_1 - \psi_2^* \psi_2)
\]

(24)

are the charge densities generating the \( U(1) \otimes \tilde{U}(1) \) conserved charges. It is well known that the operator solution of (22, 23) is obtained via bosonization (see ref. [6]) in terms of the chiral scalar fields \( \varphi_\vartheta \) and \( \bar{\varphi}_\vartheta \) satisfying

\[
(\partial_\nu + \nu \partial_x)\varphi_\vartheta (vt - x) = 0, \quad (\partial_\nu - \nu \partial_x)\bar{\varphi}_\vartheta (vt + x) = 0
\]

(25)

where \( \nu \) is a velocity specified later on. Referring for the details to ref. [18], the anyon operator solution is given by

\[
\psi_1(t, x) = \eta \cdot e^{\sqrt{\pi} [\varphi_\vartheta (vt - x) + \varphi_\vartheta (vt + x)]}
\]

(26)

\[
\psi_2(t, x) = \eta \cdot e^{\sqrt{\pi} [\varphi_\vartheta (vt - x) + \varphi_\vartheta (vt + x)]}
\]

(27)

Here \( \eta \) is a Klein factor, whose explicit form is not relevant for what follows. The parameters \( \sigma, \tau \in \mathbb{R} \) are determined below and...
\[ \zeta_{x} \zeta_{x} = \kappa , \quad \zeta_{x} = \tau \pm \sigma \]  

For the charge densities and relative currents one has

\[ j_{t}(t, x) = \frac{-1}{2\sqrt{\pi} \zeta_{x}} [ (\partial \varphi_{x}) (vt - x) + (\partial \varphi_{x}) (vt + x) ] \]  
\[ j_{\tilde{t}}(t, x) = \frac{-1}{2\sqrt{\pi} \zeta_{x}} [ (\partial \varphi_{x}) (vt - x) - (\partial \varphi_{x}) (vt + x) ] \]  

and

\[ j_{s}(t, x) = \frac{\nu}{2\sqrt{\pi} \zeta_{x}} [ (\partial \varphi_{x}) (vt - x) - (\partial \varphi_{x}) (vt + x) ] \]  
\[ j_{\tilde{s}}(t, x) = \frac{\nu}{2\sqrt{\pi} \zeta_{x}} [ (\partial \varphi_{x}) (vt - x) + (\partial \varphi_{x}) (vt + x) ] \]  

Because of Equation (25), these densities and currents satisfy the conservation law (9). Plugging Equations (29) and (30) in the equations of motion (22) and (23) and using Equation (28) one finds finally

\[ \zeta_{x}^{2} = \kappa \left( \frac{\kappa + g_{x}}{\kappa - g_{x}} \right)^{1/2} \]  
\[ \nu = \frac{1}{\kappa} \sqrt{(\kappa + g_{x})(\kappa - g_{x})} \]  

where the positive roots are taken in the right hand side. Equations (33) and (34) determine the parameters \( \sigma \) and \( \tau \) and the velocity \( \nu \) of the interacting anyons in terms of the coupling constants \( g_{x} \) and the statistical parameter \( \kappa \). Notice that the velocity \( \nu \) of the interacting anyons differs from the free velocity \( v_{0} \) (according to our conventions (1) \( |v_{0}| = 1 \)) and depends on \( \kappa \) and \( g_{x} \) as well. We assume in what follows that \( \{ \kappa, g_{x} \} \), defining the abelian anyon Luttinger liquid, belong to the domain

\[ D = \{ \kappa > 0, \kappa > -g_{x} \} \]  

which ensures that \( \sigma, \tau, \) and \( \nu \) are real and finite.

In conclusion, we observe that the above anyon solution of the TL model for generic \( \kappa > 0 \) generates for \( \kappa = 1 \) the fermionic solution, usually described in the literature\(^{[26-29]} \) in terms of the parameters \( \{ g_{x}, g_{\tilde{x}}, K \} \) related to \( \{ g_{x}, g_{\tilde{x}}, \zeta_{x} \} \) as follows:

\[ g_{x} = \frac{1}{2}(g_{x} + g_{\tilde{x}}), \quad g_{\tilde{x}} = \frac{1}{2}(g_{x} - g_{\tilde{x}}) \]  
\[ K = \zeta_{x}^{2} \prod_{k=1}^{\infty} = \zeta_{x}^{2} \prod_{k=1}^{\infty} \]  

\[ 3.2. \text{Representation Implementing the GGE Reservoirs} \]

Our goal now is to construct a representation of the chiral fields (25), which implements the GGE reservoirs \( R_{i} \) in the operator solution (26)–(32). For this purpose we use the standard decomposition

\[ \varphi_{\lambda}(\xi) = \int_{0}^{\pi} dk \frac{\sqrt{\Delta(k)}}{\pi^{2}} [ a^{*}(k)e^{ik\xi} + a(k)e^{-ik\xi} ] \]  
\[ \varphi_{\lambda}(\xi) = \int_{-\pi}^{\pi} dk \frac{\sqrt{\Delta(k)}}{\pi^{2}} [ a^{*}(k)e^{-ik\xi} + a(k)e^{ik\xi} ] \]  

where

\[ |k|\Delta(k) = 1 \]  

and choose suitable representations of the two canonical commutation algebras \( \mathcal{A}_{\pm} = \{ a(k), a^{*}(k) : k \geq 0 \} \). Since the origin of the right moving field \( \lambda \) is the reservoir \( R_{i} \), we take for \( \mathcal{A}_{+} \) the Gibbs representation at temperature \( \beta_{1} \). For analogous reason we adopt \( \mathcal{A}_{-} \) in the Gibbs representation with temperature \( \beta_{2} \). More explicitly, consider the Bose distribution

\[ d_{i}(k) = \frac{e^{-\beta_{i}|k|\lambda}}{1 - e^{-\beta_{i}|k|\lambda}}, \quad \lambda < 0, \quad i = 1, 2 \]  

Then

\[ \langle a^{*}(p)a(k) \rangle = \begin{cases} d_{i}(k) 2\pi \delta(k - p), & p, k > 0 \\ d_{i}(k) 2\pi \delta(k - p), & p, k < 0 \end{cases} \]  

The bosonic chemical potential \( \lambda < 0 \) allows to avoid the infrared singularity at \( k = 0 \) in (41). The limit \( \lambda \rightarrow 0^{+} \) exists\(^{[8]} \) and is performed in the correlation functions of the TL observables.

Once we have the temperatures \( \beta_{i} \) via the Gibbs representation of \( \mathcal{A}_{\pm} \), we have to introduce the chemical potentials \( \mu_{i} \) and \( \tilde{\mu}_{i} \). At this point we generalise away from equilibrium the strategy of\(^{[8]} \) performing the shifts

\[ \varphi_{\lambda}(\xi) \mapsto \varphi_{\lambda}(\xi) - \frac{1}{\nu \sqrt{\pi}} \left( \frac{\mu_{1} + \tilde{\mu}_{1}}{\zeta_{x}} \right) \xi \]  
\[ \varphi_{\lambda}(\xi) \mapsto \varphi_{\lambda}(\xi) - \frac{1}{\nu \sqrt{\pi}} \left( \frac{\mu_{2} + \tilde{\mu}_{2}}{\zeta_{x}} \right) \xi \]  

where \( \zeta_{x} \) and \( \nu \) are given by Equations (33) and (34). The form of Equations (43) and (44) respects the equations of motion (25) and is fixed by requiring that at equilibrium

\[ \beta_{1} = \beta_{2} \equiv \beta, \quad \mu_{1} = \mu_{2} \equiv \mu, \quad \tilde{\mu}_{1} = \tilde{\mu}_{2} \equiv \tilde{\mu} \]  

the correlation functions of \( \psi_{i} \) satisfy the Kubo–Martin–Schwinger (KMS) condition\(^{[14]} \). Since the latter is a basic condition in our construction, it is instructive to discuss the issue in detail.

Introducing the notation \( t_{ij} \equiv t_{ij} - t_{ij}, \quad x_{ij} \equiv x_{ij} - x_{ij} \) and taking into account that \( \langle \eta^{*}\eta \rangle = 1 \) one finds in the limit \( \lambda_{s} \rightarrow 0^{+} \)
\[ \langle \psi_i^+(t_1, x_1) \psi_i(t_2, x_2) \rangle = e^{-iF(t_1, x_1; t_2, x_2)} \]

\[
\begin{align*}
&\times \left[ \frac{\beta_1}{\pi} \sinh \left( \frac{\pi}{\beta_1} (vt_1 - x_1) - i\varepsilon \right) \right]^{-r_1^2} \\
&\times \left[ \frac{\beta_2}{\pi} \sinh \left( \frac{\pi}{\beta_2} (vt_2 + x_2) - i\varepsilon \right) \right]^{-r_2^2} \\
&\text{where } \varepsilon \rightarrow 0^+ \text{ and the phase factor } F \text{ is given by } \\
F(t, x) = -\frac{\sigma}{\nu} \left( \frac{\mu_1 + \tilde{\mu}}{\zeta} \right) (vt - x) - \frac{\tau}{\nu} \left( \frac{\mu_2 + \tilde{\mu}}{\zeta} \right) (vt + x)
\end{align*}
\] (46)

The correlation function \( \langle \psi_i^+(t_1, x_1) \psi_i(t_2, x_2) \rangle \) is obtained by performing \( \sigma \rightarrow \tau \) in Equation (46).

**4. Anyon Quantum Transport**

In this section we derive and study the mean values of the charge and heat currents flowing in the system shown in Figure 1. The invariance under space-time translations implies that these mean values are both \( x \) and \( t \)-independent.

**4.1. Electric and Helical Transport**

Taking into account Equations (43) and (44), one gets for the expectation value of the electric (vector) current (31)

\[ \langle j_x \rangle = \frac{\mu_2 - \mu_1}{2\pi\zeta^2} \left( \frac{\tilde{\mu}_1 + \tilde{\mu}_2}{\zeta_+} \right) \] (53)

By means of Equations (28) and (33) one can reconstruct the explicit dependence on the statistical parameter \( \kappa \). One finds

\[ \langle j_x \rangle_{eq} = \frac{-\tilde{\mu}_2}{\pi\kappa} \] (55)

which precisely coincides with the persistent current discovered in ref. [8] and proportional to \( \kappa^{-1} \). This current has a simple physical origin. The point is that at equilibrium the chemical potential \( \tilde{\mu} \) can be equivalently implemented \(^8\) by coupling the TL model (1, 3) with a constant magnetic field \( h = \tilde{\mu} \).

An expression similar to Equation (54) holds also for the helical (axial) current

\[ \langle j_x \rangle = \frac{1}{2\pi\kappa} \left[ (\mu_2 - \mu_1) \sqrt{\kappa + g_+} - (\tilde{\mu}_1 + \tilde{\mu}_2) \right] \] (54)

It is worth mentioning that Equations (54) and (56) are related by the transformation

\[ \mu_i \leftrightarrow \tilde{\mu}_i, \quad g_+ \leftrightarrow g_- \] (57)

which implements the axial-vector duality in the model and confirms the deep interplay between helical and electric transport in the Luttinger liquid. We observe also that the currents (53) and (56) depend on the chemical potentials, but not on the temperatures. Therefore there is no thermo-electric effect at the level of
mean values, which agrees with the general prediction\textsuperscript{45,46} from non-equilibrium conformal field theory (CFT). We stress however that the quantum fluctuations of these currents, derived in Section 5 below, are instead temperature depend.

4.2. Heat Transport

In the basis of the chiral scalar fields the energy density and current, satisfying (8), are given by

\[ \langle \theta_x \rangle = \frac{\nu^2}{4} : \left[ (\partial \varphi_x) (\partial \varphi_x) (vt + x) + (\partial \varphi_x) (\partial \varphi_x) (vt - x) \right] : \] (58)

and

\[ \langle \theta_x \rangle = \frac{\nu^2}{4} : \left[ (\partial \varphi_x) (\partial \varphi_x) (vt + x) - (\partial \varphi_x) (\partial \varphi_x) (vt - x) \right] : \] (59)

where: \( : \cdots : \) is the normal product in the oscillator algebras \( A_\pm \). In what follows we need the mean value of Equation (59). By means of Equations (41) and (42) one finds

\[ \langle \theta_x \rangle = \frac{\pi \nu^2}{12} \left( \frac{1}{\beta^2_x} - \frac{1}{\beta^2_t} \right) + \frac{1}{4\pi} \left[ \left( \frac{\mu_x}{\xi_+} - \tilde{\mu}_x \right)^2 - \left( \frac{\mu_t}{\xi_+} + \tilde{\mu}_t \right)^2 \right] \] (60)

Taking the expectation value of Equation (13), we are ready at this point to derive the mean heat currents flowing through the gates \( G_i \) in Figure 1. Combining Equations (53), (56), and (60) one obtains in the gate \( G_1 \)

\[ \langle q_x(G_1) \rangle = \frac{\pi \nu^2}{12} \left( \frac{1}{\beta^2_x} - \frac{1}{\beta^2_t} \right) + \frac{1}{2\pi \xi_+ \xi_-} \left( \mu_x \tilde{\mu}_2 + \tilde{\mu}_x \mu_2 + \mu_t \tilde{\mu}_1 - \mu_t \tilde{\mu}_2 \right) + \frac{1}{4\pi \xi_+} \left( \mu_1 - \mu_2 \right)^2 + \frac{1}{4\pi \xi_-} \left( \tilde{\mu}_1 - \tilde{\mu}_2 \right)^2 \] (61)

Analogously, in the gate \( G_2 \) one has

\[ \langle q_x(G_2) \rangle = \frac{\pi \nu^2}{12} \left( \frac{1}{\beta^2_x} - \frac{1}{\beta^2_t} \right) + \frac{1}{2\pi \xi_+ \xi_-} \left( \mu_x \tilde{\mu}_2 + \tilde{\mu}_x \mu_2 + \mu_t \tilde{\mu}_1 - \mu_t \tilde{\mu}_2 \right) - \frac{1}{4\pi \xi_+} \left( \mu_1 - \mu_2 \right)^2 + \frac{1}{4\pi \xi_-} \left( \tilde{\mu}_1 - \tilde{\mu}_2 \right)^2 \] (62)

In analogy with the electric current (54), restricting Equations (61) and (62) at equilibrium (45), one obtains the persistent heat current

\[ \langle q_x(G_1) \rangle_{eq} = \langle q_x(G_2) \rangle_{eq} = \frac{\mu \tilde{\mu}}{\pi \xi_+ \xi_-} \] (63)

driven exclusively by both chemical potentials \( \mu \) and \( \tilde{\mu} \).

In the next subsection we apply the above results for deriving the electric and heat conductance and compute the associated Lorenz number.

4.3. Lorenz Number and Wiedemann–Franz Law

Using Equation (53) one gets for the electric conductance in the gate \( G_i \)

\[ E(G_i) = e^2 \frac{\partial}{\partial \mu_i} (j_e) = (1 - 1) \frac{e^2}{2\pi \xi_i^2} \] (64)

where the value \( e \) of the electric charge has been restored. On the other hand, since

\[ \beta_i = \frac{1}{\xi_i \kappa} \] (65)

\( \kappa \), being the Boltzmann constant, one obtains from (61, 62)

\[ H(G_i) = \frac{\partial}{\partial T_i} \langle q_x(G_i) \rangle = -\beta_i^2 k \frac{\partial}{\partial \mu} \langle q_x(G_i) \rangle \]

\[ = (-1) \frac{\pi \nu^2 \kappa}{6 \beta_i} = (-1) \frac{\pi \nu^2 k\kappa}{6 \beta_i} \] (66)

which is linear in the temperature \( T_i \) as observed in ref. [47].

In terms of Equations (64) and (66) the Lorenz number\textsuperscript{48} in the gate \( G_i \) is

\[ L(G_i) = \frac{\beta_i k \kappa}{E(G_i)} = L_0 \nu^2 \xi_+^2 \] (67)

where

\[ L_0 = \frac{\pi^2}{3} \left( \frac{k}{e} \right)^2 \] (68)

is the free electron value. As expected

\[ L(G_i) = L(G_2) \equiv L \] (69)

and using Equations (33) and (34) one finally gets

\[ L = L_0 \left( \kappa + g_+ \right)^{1/2} \left( \kappa + g_- \right)^{1/2} \] (70)

displaying explicitly the dependence on the statistical parameter \( \kappa \) and the coupling constants \( g_{\pm} \). We observe in this respect that

\[ L \bigg|_{\xi_{\pm}=0} = \frac{\pi^2}{3} \left( \frac{k}{e} \right)^2 \kappa \equiv L^{ae}(\kappa) \] (71)
represents the free anyon Lorenz number with statistical parameter $\kappa$. As expected, for canonical fermions one has $L_{\text{stat}}^0(1) = L_0$.

Combining Equations (70) and (71) one gets

$$R \equiv \frac{L}{L_{\text{stat}}^0(\kappa)} = \frac{(\kappa + g_1)(\kappa + g_2)^{1/2}}{\kappa^2}$$

(72)

which codifies a temperature-independent deviation from the Wiedemann–Franz law\cite{49} generated by the interaction. The vanishing of the Lorenz number $L$ at the boundary $\kappa = -g_2$ of the domain (35) is a physical consequence of the vanishing of the velocity $\langle \tilde{\nu} \rangle$ in these points.

4.4. Mean Entropy Production and Energy Transmutation

We focus in this subsection on the entropy production, which represents the key quantity quantifying the departure from equilibrium. In order to simplify the notation, we adopt here the modified chemical potentials

$$v_i = \frac{\mu_i}{\zeta}, \quad \tilde{v}_i = \frac{\mu_i}{\zeta}$$

(73)

Plugging Equations (61) and (62) in the general expression (21) one finds for the mean value of the entropy production

$$\langle \dot{S} \rangle = \frac{\pi v^2(\beta_1 + \beta_2)(\beta_1 - \beta_2)^2}{12\beta_1^2 \beta_2^2}
+ \frac{\beta_1}{4\pi} B_1(v_i, \tilde{v}_i) + \frac{\beta_2}{4\pi} B_2(v_i, \tilde{v}_i)$$

(74)

where

$$B_1(v_i, \tilde{v}_i) = (v_i - \tilde{v}_i - \tilde{v}_1 + \tilde{v}_2)^2 + 4\nu_1 \tilde{v}_1,$$

$$B_2(v_i, \tilde{v}_i) = (v_i - \tilde{v}_i + \tilde{v}_1 - \tilde{v}_2)^2 - 4\nu_2 \tilde{v}_2$$

(75)

In order to implement the second law of thermodynamics we require that

$$\langle \dot{S} \rangle \geq 0, \quad \forall \beta_i \geq 0$$

(76)

which imposes some restriction on the chemical potentials $\{v_i, \tilde{v}_i\}$. In fact performing the repeated limits

$$\lim_{\beta_1 \to 0} \lim_{\beta_2 \to 0} \frac{1}{\beta_1} \langle \dot{S} \rangle = B_1(v_i, \tilde{v}_i)$$

(77)

$$\lim_{\beta_1 \to 0} \lim_{\beta_2 \to 0} \frac{1}{\beta_2} \langle \dot{S} \rangle = B_2(v_i, \tilde{v}_i)$$

(78)

we deduce that

$$B_i(v_i, \tilde{v}_i) \geq 0, \quad i = 1, 2$$

(79)

are necessary conditions for the bound (76). From the explicit form (74) of $\langle \dot{S} \rangle$ we infer that Equation (79) is sufficient as well.

The above considerations lead to the following conclusions:

1. The non-negativity (76) of the mean entropy production imposes non-trivial conditions (79) on the chemical potentials in the GGE heat reservoirs.

2. In the Gibbs limit in which one of the pairs $(\tilde{v}_1, \tilde{v}_2)$ or $(v_1, v_2)$ vanishes, one has that

$$B_i(v_i, 0) = (v_i - \tilde{v}_2)^2 \geq 0, \quad B_i(0, \tilde{v}_i) = (\tilde{v}_1 - \tilde{v}_2)^2 \geq 0$$

(80)

being identically satisfied. Therefore in that limit the entropy production is non-negative for any value of the chemical potentials. 3. The conditions (79) imply that

$$\langle \dot{Q} \rangle = \frac{1}{4\pi} B_1(v_i, \tilde{v}_i) + \frac{1}{4\pi} B_2(v_i, \tilde{v}_i) \geq 0$$

(81)

where $Q$ is the observable defined by Equation (15). Therefore in the physical regime (80) of non-negative mean entropy production our non-equilibrium anyon TL liquid converts chemical to heat energy without dissipation. Let us illustrate this aspect assuming without loss of generality that

$$\beta_2 \geq \beta_1 \geq 0 \Rightarrow r \equiv \frac{\beta_1}{\beta_2} \in [0, 1]$$

(82)

Accordingly the hot and cold reservoirs in Figure 1 are respectively $R_1$ and $R_2$ because $T_1 > T_2$. In this setting the heat flows in the gates $G_i$, where the leads $L_i$ are oriented as in Figure 1, are shown in Figure 2. We see that the heat current through the cold gate $G_2$ is always negative, indicating that the corresponding heat flow enters the cold reservoir. Concerning the heat current flowing in the hot gate $G_1$, there is a critical value $r_0$ (for the parameters chosen in Figure 2 one has $r_0 \sim 0.18$) for which $q_i(G_i)$ inverts his direction: for $0 \leq r < r_0$ and $r_0 < r \leq 1$ it leaves and enters the hot reservoir, respectively. Therefore, despite of the fact that the energy and particle currents have the same direction and intensity (see Equations (10) and (11)) in the gates $G_i$, this is not the case for the heat current because of the explicit dependence of $q_i(G_i)$ on the chemical potentials.
5. Anyon Quantum Noise

This section focusses on the quantum fluctuations described by the connected two-point electric current correlation function

\[
(j_x(t_1, x_1) j_x(t_2, x_2))^\text{con} = (j_x(t_1, x_1) j_x(t_2, x_2)) - \langle j_x(t_1, x_1) \rangle \langle j_x(t_2, x_2) \rangle
\]

(83)

Our main goal here is to investigate the dependence of these fluctuations on the statistical parameter \( \kappa \), which opens the possibility to study experimentally the nature of the anyon TL excitations by measuring the noise.

Using the definitions (31) and (42) one finds

\[
j_x(t_1, x_1) j_x(t_2, x_2))^\text{con} = -\frac{\nu^2}{4e^2} \left\{ \left[ \beta_1 \sinh \left( \frac{\pi}{\beta_1} (vt_{12} - x_{12}) - i\epsilon \right) \right]^2 + \left[ \beta_2 \sinh \left( \frac{\pi}{\beta_2} (vt_{12} + x_{12}) - i\epsilon \right) \right]^2 \right\}
\]

(84)

which shows that the second moment of the probability distribution generated by the current \( j_x \) depends on the temperatures but does not involve the chemical potentials. The noise power at frequency \( \omega \) is obtained \([50]\) by performing the Fourier transform

\[
P(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \langle j_x(t, x) j_x(0, x) \rangle^\text{con}
\]

(85)

In what follows we use the temperatures \( T_i \) defined by Equation (65) and set

\[
T_1 = T + \frac{\delta}{2}, \quad T_2 = T - \frac{\delta}{2}
\]

(86)

The integral in Equation (85) can be performed explicitly and one finds in the limit \( \epsilon \to 0^+ \)

\[
P(\omega) = \frac{\omega}{4\pi^2} \left[ 2 + \coth \left( \frac{\omega}{2T_1 - \nu \delta} \right) + \coth \left( \frac{\omega}{2T_1 + \nu \delta} \right) \right]
\]

(87)

which is the subject of the analysis below.

First of all, the zero-frequency limit in Equation (87) gives

\[
P_0 = \lim_{\omega \to 0} P(\omega) = \frac{\nu}{\pi^2} T \frac{(\kappa + g_\perp) T}{\pi \kappa^2} T
\]

(88)

which shows the typical linear in the temperature Johnson–Nyquist behavior. The interesting feature is the dependence on the statistical parameter \( \kappa \in D \) (see Equation (35)). There are two characteristic regimes depending on the sign of coupling constant \( g_\perp \). For \( g_\perp < 0 \) the admissible values of \( \kappa \) are \( \kappa \geq -g_\perp \) and the coefficient in Equation (88) has a maximum at \( \kappa = -2g_\perp \). In this case the behavior of \( P_0 \) is shown in Figure 3 for three different values of the temperature. For \( g_\perp \geq 0 \) the allowed values for \( \kappa \) are \( \kappa > 0 \) and the noise \( P_\omega \) is monotonically decreasing as shown in Figure 4. In both cases \( P_0 \) decays as \( 1/\kappa \) for large \( \kappa \).

Let us explore finally the frequency dependence of \( P(\omega) \). For this purpose we expand Equation (87) around \( \delta = 0 \). One has

\[
P(\omega) = Q(\omega) \left[ 1 + R_2(\omega) \delta^2 + R_4(\omega) \delta^4 + \cdots \right]
\]

(89)

where

\[
Q(\omega) = \frac{\omega}{2\pi \xi_{\perp}^2} \left[ 1 + \coth \left( \frac{\omega}{2T_1} \right) \right]
\]

(90)

\[
R_2(\omega) = \frac{\omega}{16T_1^2 \nu^2} \frac{\omega \coth \left( \frac{\omega}{2T_1} \right) - 2T_1}{\omega \coth \left( \frac{\omega}{2T_1} \right) + 1} \sinh^2 \left( \frac{\omega}{2T_1} \right)
\]

(91)

and a similar but longer expression for \( R_4 \), whose explicit form is not reported for conciseness. The \( \kappa \)-dependence of the coefficients \( R_1 \) is carried by the velocity \( \nu \) given by Equation (34). Figures 5 and 6 illustrate the impact of \( \kappa \) on the frequency behavior. The frequencies where \( R_2 \) reaches his maximum and \( R_4 \) his minimum and maximum manifestly depend on \( \kappa \). Therefore the frequency behavior of the noise is sensitive to the specific value of the statistical parameter of the anyon excitations which are propagated.

The noise generated by the helical current \( \tilde{j}_x \) can be investigated along the above lines as well. As a consequence of the axial-vector duality, in this case the power \( P(\omega) \) is simply obtained from Equation (87) by the substitution \( \xi_{\perp} \leftrightarrow \xi_{\parallel} \).

In conclusion, both the zero and finite frequency current quantum fluctuations carry the imprint of the anyon statistics and
offer therefore relevant experimental tools for detecting the statistical parameter $\kappa$. A similar conclusion has been reached\textsuperscript{[20]} also in the case of non-ballistic anyon transport, which takes place in the presence of impurities.

6. Discussion

We performed a systematic study of the dependence of the anyon particle and heat transport on the statistical parameter $\kappa$ of the TL anyon liquid in contact with two GGE heat reservoirs. Each of them depends on two chemical potentials, which implement the axial-vector duality of the model. The system is a specific case of non-equilibrium CFT with central charge $c = 1$ and provides an instructive example for testing some general ideas in this context.

In this setting we derived in explicit form the mean value of the entropy production $\langle S \rangle$, generated by the spontaneous breaking of time reversal, and established the conditions on the chemical potentials implementing the physical requirement $\langle S \rangle \geq 0$. We observed that the mean values of the electric and helical currents depend on the chemical potentials but not on the temperatures. Precisely the opposite behaviour is characterising instead the associated quantum fluctuations, which is consistent with the general CFT predictions.\textsuperscript{[45,46]} We have shown in addition that the quadratic fluctuations of the anyon electric current in the zero frequency limit obey the Johnson-Nyquist law with $\kappa$-dependent pre-factor. The noise at finite frequencies carries specific $\kappa$-dependence as well, providing attractive experimental applications.

The framework, developed in this paper, can be applied in different contexts and generalised in various directions. An attractive subject in the context of CFT current algebras\textsuperscript{[51]} is the detailed study of the non-equilibrium representation of the axial-vector current algebra generated in the TL model. Moreover, it would be interesting to extend the results of this paper to other types of anyon quantum liquids, which have been considered in the literature.\textsuperscript{[52–56]} Another challenging issue is to explore along the lines of\textsuperscript{[57,58]} the behaviour of the higher moments of the probability distribution generated by the entropy production operator. One will obtain in this way a complete picture of the departure from equilibrium, induced by the contact with the two GGE reservoirs. We will come back to these issues elsewhere.

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Conflict of Interest

The authors declare no conflict of interest.

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anyon quantum transport, entropy production, Lorenz number, noise power

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