PENTAGON INTEGRALS FOR HEAVY QUARK PHYSICS

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Abstract

In this paper we present the calculation of a scalar pentagon integral with two consecutive massive external legs having an equal mass propagator embedded between them. We also deal with the two situations where the farest external leg is either massive or not. The relevance of the calculation comes from its application in many perturbative QCD calculations as well as in QCD corrections for weak precesses.

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I. INTRODUCTION

A series of rare elementary processes involving more than two particles in the final state are going to be measured with increasing precision. The multiplicity of the final state makes it difficult to extract predictions by the standard gauge theories even if simplifications arise when either participants are all massless or only some of the external particles are massive. However more accurate rate measurements of processes with heavy quark hadrons in the final state will soon be available as is the case of the CHORUS experiment where direct evidence for the associate charm production in charged current neutrino nucleon scattering has been shown [1]. In the one loop calculation of such processes we encounter pentagon integrals with a massive line as sketched in figure 1 where massive particles are bold, massless ones thin and dashed ones can be either massive or not.

![Pentagon with one massive line](image)

FIG. 1. Pentagon with one massive line

In general the inclusion of masses makes things more involved, although the calculation simplifies when either external masses are equal to each other or they are equal to the internal masses or both eventualities occur as it is often the case in normal gauge theories. Recently a lot of progress has been made in the technics for perturbative calculations with different approaches. A non-comprehensive list is given in [2] and reference therein and in [3], [4], [5], [6] and [7]. In particular adopting the dimensional regularization approach for Feynman parametrized integrand the authors of ref. [3] derived simplifications and
recursion formulas by the implementation of algebraic technic. Using these methods the problem of the evaluation of a one loop $n$ points scalar integral is translated to the evaluation of a combination of $n - 1$ points scalar integrals and the original $n$ points integral in $D = 6 - 2\varepsilon$ dimensions; moreover the original $n$ points one loop integral can be represented as the solution of a partial differential equation system. In the present paper we use this approach to perform the calculation of the pentagon integral represented in figure 1. Other massive pentagon integrals have been recently evaluated in next to leading order calculations of processes in which an Higgs particle can be generated at hadron colliders. In particular two independent groups report the NLO corrections for the process in which an Higgs particle is generated together with a $t\bar{t}$ pair, [8] and [9]. Another NLO calculation involving massive pentagon integrals is given in [10] in which the final state considered consists of an Higgs particle plus two jets. The general methods employed here do not concern with the specific processes and the results must be analytically continued to describe a specific process. Finally only the most simple tensor integral is given while we postpone other cases to a dedicated paper [11]. The paper is organized as follow: in section II relevant formulas from ref. [3] are collected, in section III they are applied to the scalar massive pentagon represented in figure 1 transforming it in a combination of four points integrals; section IV is devoted to four point integrals evaluation and in section V more simple tensor integral (vector) is given with the conclusions. The initial condition for the differential equations originated in the four points evaluation are calculated in the appendix.

II. BASIC FORMULAS

The starting point is the integral in $D = 4 - 2\varepsilon$ dimensions

$$I_n = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}l}{(2\pi)^{4-2\varepsilon}} \frac{1}{(l^2 - M_1^2)((l - p_1)^2 - M_2^2)\cdots((l - p_{n-1})^2 - M_n^2)}$$

(1)

with the momenta $k_i$ taken to be outgoing, $k_i^2 = m_i^2$ and

$$p_i \equiv \sum_{j=1}^{i} k_j$$
Applying Feynman parametrization, Wick rotating and integrating over loop momentum this integral can be cast in the form

\[ I_n = i (-1)^n (4\pi)^{\varepsilon-2} \mu^{2\varepsilon} I_n \]

(3)

\[ I_n = \Gamma(n - 2 + \varepsilon) \int_0^1 d^n a_i \delta(1 - \sum_i a_i) \frac{1}{D(a_i)^{n-2+\varepsilon}} \]

(4)

with

\[ D(a_i) = \sum_{i,j=1}^n S_{ij} a_i a_j \]

(5)

and the matrix \( S \) given by

\[ S = \frac{1}{2} (M_i^2 + M_j^2 - p_{ij}^2) \]

(6)

with \( p_{ii} = 0 \) and

\[ p_{ij} \equiv k_i + k_{i+1} + \ldots + k_{j-1} \]

(7)

for \( i < j \). We will not repeat the derivations obtained in ref. [3] but, to introduce notation and to be self-consistent, in the rest of this section we just collect relevant formulas that will be used in section III and IV. Performing a projective transformation [12] with parameters \( \alpha_i \) in such a way that the denominator in Eq.(4) has no \( \alpha_i \) dependence the definition of a new matrix follows (indices are not summed)

\[ \rho_{ij} = S_{ij} \alpha_i \alpha_j. \]

(8)

Using the following definitions

\[ \hat{I}_n = \left( \prod_{j=1}^n \alpha_j \right)^{-1} I_n \]

\[ N_n = 2^{n-1} \det(\rho) \]

\[ \eta = N_n \rho^{-1} \]

\[ \gamma_i = \sum_{j=1}^n \eta_{ij} \alpha_j \]

\[ \hat{\Delta}_n = \sum_{i,j=1}^n \eta_{ij} \alpha_i \alpha_j \]

(9)
the authors of ref. [3] find

\[ \hat{I}_n = \frac{1}{2N_n} \left[ \sum_{i=1}^{n} \gamma_i \hat{I}^{(i)}_{n-1} + (n - 5 + 2\varepsilon) \hat{\Delta}_n \hat{I}^{D=6-2\varepsilon}_n \right] \] (10)

\[ \frac{1}{n-4+2\varepsilon} \frac{\partial \hat{I}_n}{\partial \alpha_i} = \frac{1}{2N_n} \left[ \sum_{j=1}^{n} \eta_{ij} \hat{I}^{(j)}_{n-1} + (n - 5 + 2\varepsilon) \gamma_i \hat{I}^{D=6-2\varepsilon}_n \right] \] (11)

where \( \hat{I}^{(i)}_{n-1} \) stands for the \( n-1 \) integral with the denominator obtained from an \( \hat{I}_n \) integral eliminating the propagator between legs \( i - 1 \) and \( i \); once Feynman parameter has been introduced in the usual way for \( \hat{I}_n \) the denominator in \( \hat{I}^{(i)}_{n-1} \) is obtained putting \( a_i = 0 \). By the observation that \( \hat{I}_4 \) and \( \hat{I}_5 \) are finite in 6 dimensions, performing one-loop calculation one can limit to evaluate

\[ \hat{I}_5 = \frac{1}{2N_5} \sum_{i=1}^{5} \gamma_i \hat{I}^{(i)}_4 + \mathcal{O}(\varepsilon) \] (12)

\[ \frac{\partial \hat{I}_4}{\partial \alpha_i} = \frac{\varepsilon}{N_4} \sum_{j=1}^{4} \eta_{ij} \hat{I}^{(j)}_3 + \mathcal{O}(\varepsilon) \] (13)

taking only the divergent part from the \( \hat{I}^{(j)}_3 \) integrals in Eq.(13).

III. PENTAGON WITH ONE MASSIVE LINE

To write down the integral in figure 1 we set \( k_1^2 = k_4^2 = M_1^2 = M_4^2 = M_5^2 = 0 \), \( k_2^2 = k_3^2 = M_3^2 = m^2 \) and \( k_5^2 = q^2 \) giving

\[ I_5 = \mu^{2\varepsilon} \int \frac{d^{1-2\varepsilon}l}{(2\pi)^{1-2\varepsilon} l_1^2 (l- p_1)^2 ((l- p_2)^2 - m^2)(l- p_3)^2(l- p_4)^2} \] (14)

and

\[ I_5 = \Gamma(3 + \varepsilon) \int_0^1 d^5 a_i \frac{1}{D(a_i)} \frac{1}{(a_i)^{3+\varepsilon}} \] (15)

with \( D \) given in Eq.(5) and the matrix \( S \) given by
with $s_{i,i+1} = (k_i + k_{i+1})^2$. We define $\bar{s}_{i,i+1} = s_{i,i+1} - m^2$ and in the following we will assume $\bar{s}_{12}, \bar{s}_{23}, \bar{s}_{34}, \bar{s}_{45}, \bar{s}_{51}, q^2 < 0$. Performing the projective transformation with

$$
\begin{pmatrix}
0 & 0 & m^2 - s_{12} & -s_{45} & -q^2 \\
0 & 0 & 0 & -s_{23} & -s_{51} \\
m^2 - s_{12} & 0 & 2m^2 & 0 & m^2 - s_{34} \\
-s_{45} & -s_{23} & 0 & 0 & 0 \\
-q^2 & -s_{51} & m^2 - s_{34} & 0 & 0
\end{pmatrix}
$$

(16)

we get for the $\rho_{ij}$ matrix in Eq.(8)

$$
\begin{pmatrix}
0 & 0 & 1 & 1 & \lambda \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 2M^2 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
\lambda & 1 & 1 & 0 & 0
\end{pmatrix}
$$

(18)

with

$$
\lambda = \frac{q^2 s_{23}}{s_{45} s_{51}}
$$

(19)

$$
M^2 = -\frac{m^2 s_{45} s_{51}}{s_{12} s_{23} s_{34}}
$$

(20)

If $k_5^2 = q^2 = 0$ we only have to take $\lambda = 0$ in Eq.(18). The coefficient relevant for the evaluation of the pentagon by Eq.(12) are given in the table 1, keeping apart the case
\[ q^2 = 0. \] Due to the presence of masses we have not cyclic relations between the coefficients but only the relations

\[
\begin{align*}
\gamma_4 &= \gamma_2 |_{\alpha_4 \leftrightarrow \alpha_2, \alpha_5 \leftrightarrow \alpha_1} \\
\gamma_5 &= \gamma_1 |_{\alpha_4 \leftrightarrow \alpha_2, \alpha_5 \leftrightarrow \alpha_1}
\end{align*}
\]  

(21)

Table 1. Coefficients to be used in Eq.(12)

| \( \text{par} \) | \( q^2 = \text{any} \) | \( q^2 = 0 \) |
|---------------|----------------|---------|
| \( N_5 \)    | \( 1 + M^2 - (1 + 2M^2) \lambda + M^2 \lambda^2 \) | \( 1 + M^2 \) |
| \( \gamma_1 \) | \( \alpha_1 - \alpha_2 + \alpha_3 - \lambda \alpha_3 + \alpha_4 + 2M^2 \alpha_4 \) | \( \alpha_1 - \alpha_2 + \alpha_3 + (1 + 2M^2) (\alpha_4 - \alpha_5) \) |
|               | \( -2M^2 \lambda \alpha_4 - \alpha_5 - 2M^2 \alpha_5 + 2M^2 \lambda \alpha_5 \) |                     |
| \( \gamma_2 \) | \( -\alpha_1 + \alpha_2 - 3 + \lambda \alpha_3 + \alpha_4 - 2\lambda \alpha_4 - 2M^2 \lambda \alpha_4 \) | \( -\alpha_1 + \alpha_2 - 3 + \alpha_4 + (1 + 2M^2) \alpha_5 \) |
|               | \( + 2M^2 \lambda^2 \alpha_4 + \alpha_5 + 2M^2 \alpha_5 - 2M^2 \lambda \alpha_5 \) |                     |
| \( \gamma_3 \) | \( \alpha_1 - \lambda \alpha_1 - \alpha_2 + \lambda \alpha_2 + \alpha_3 - 2 \lambda \alpha_3 \) | \( \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 \) |
|               | \( + \lambda^2 \alpha_3 - \alpha_4 + \lambda \alpha_4 + \alpha_5 - \lambda \alpha_5 \) |                     |
| \( \gamma_4 \) | \( \alpha_1 + 2M^2 \alpha_1 - 2M^2 \lambda \alpha_1 + \alpha_2 - 2 \lambda \alpha_2 \) | \( (1 + 2M^2) (\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5) \) |
|               | \( -2M^2 \lambda \alpha_2 + 2M^2 \lambda^2 \alpha_2 - \alpha_3 + \lambda \alpha_3 + \alpha_4 - \alpha_5 \) |                     |
| \( \gamma_5 \) | \( -\alpha_1 - 2M^2 \alpha_1 + 2M^2 \lambda \alpha_1 + \alpha_2 + 2M^2 \alpha_2 \) | \( (1 + 2M^2) (\alpha_2 - \alpha_1) + \alpha_3 - \alpha_4 + \alpha_5 \) |
|               | \( -2M^2 \lambda \alpha_2 + \alpha_3 - \lambda \alpha_3 - \alpha_4 + \alpha_5 \) |                     |

In terms of new kinematical variables \( \alpha_i, \lambda \) and \( M \) the denominator in the \( \hat{I}_5 \) integral represented in Eq.(5) is given by

\[
\frac{M^2 \alpha_3^2}{\alpha_3^2} + \frac{\alpha_1 \alpha_3}{\alpha_1 \alpha_3} + \frac{\alpha_1 \alpha_4}{\alpha_1 \alpha_4} + \frac{\alpha_2 \alpha_4}{\alpha_2 \alpha_4} + \frac{\lambda \alpha_1 \alpha_5}{\alpha_1 \alpha_5} + \frac{\alpha_2 \alpha_5}{\alpha_2 \alpha_5} + \frac{\alpha_3 \alpha_5}{\alpha_3 \alpha_5}
\]  

(22)

and the four points denominators in the \( \hat{I}_4^{(i)} \) integrals in Eq.(12) can be obtained putting \( a_i \) to zero in the expression above. It is easy to verify the relations
\[
\hat{I}_4^{(4)}(\alpha_1, \alpha_2, \alpha_3, \alpha_5) = \hat{I}_4^{(2)}(\alpha_1, \alpha_3, \alpha_4, \alpha_5)\bigg|_{\alpha_4 \rightarrow \alpha_2, \alpha_5 \leftrightarrow \alpha_1} \\
\hat{I}_4^{(5)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \hat{I}_4^{(1)}(\alpha_2, \alpha_3, \alpha_4, \alpha_5)\bigg|_{\alpha_5 \rightarrow \alpha_1, \alpha_4 \leftrightarrow \alpha_2}.
\]

(23) In the next section we proceed to the evaluation of \(\hat{I}_4^{(1)}\), \(\hat{I}_4^{(2)}\) and \(\hat{I}_4^{(3)}\) using the set of partial differential Eqs.(13).

**IV. FOUR POINTS FUNCTIONS**

Here we evaluate the integrals \(\hat{I}_4^{(1)}\) and \(\hat{I}_4^{(2)}\), corresponding to massive boxes with an internal massive line, in the variables defined in Eqs.(17, 19, and 20) and translate the integrals \(\hat{I}_4^{(3)}\) that are well known and correspond to massive boxes with massless internal lines.

**A. The integral \(\hat{I}_4^{(1)}\)**

After putting \(a_1 = 0\) in Eq.(22) we have for the denominator in \(\hat{I}_4^{(1)}\)

\[
\frac{M^2 a_3^2}{\alpha_3^2} + \frac{a_2 a_4}{\alpha_2 \alpha_4} + \frac{a_2 a_5}{\alpha_2 \alpha_5} + \frac{a_3 a_5}{\alpha_3 \alpha_5}.
\]

(24) Before solving the integral we perform the following kinematic transformation:

\[
\alpha_2 = M c_2 \\
\alpha_3 = M c_3 \\
\alpha_4 = c_4/M \\
\alpha_5 = c_5/M.
\]

(25) In terms of the new variables we get for the denominator:

\[
\frac{a_3^2}{c_3^2} + \frac{a_2 a_4}{c_2 c_4} + \frac{a_2 a_5}{c_2 c_5} + \frac{a_3 a_5}{c_3 c_5}.
\]

(26) and
\[ N_4 = 1/2 \]
\[ \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & -2 & 2 \\ 0 & 1 & 2 & -2 \end{pmatrix} \]  

(27)

The only divergent three points functions extracted by the expression above are \( \hat{I}_3^{(1)} \) and \( \hat{I}_3^{(2)} \) obtained putting \( a_2 = 0 \) and \( a_3 = 0 \) respectively; these correspond to two two-mass triangles, while the other two obtained putting \( a_4 = 0 \) and \( a_5 = 0 \) respectively are three-mass triangles checked to be finite. At the \( \mathcal{O}(\varepsilon^{-1}) \) we have

\[ \hat{I}_3^{(1)} = \Gamma(1 + \varepsilon) \left( \frac{1}{2 \varepsilon^2 c_4} + \frac{\log(c_5)}{\varepsilon c_4} \right) + \mathcal{O}(\varepsilon^0) \]

(28)

\[ \hat{I}_3^{(2)} = \frac{\Gamma(1 + \varepsilon)}{\varepsilon (c_4 - c_5)} \log \left( \frac{c_5}{c_4} \right) + \mathcal{O}(\varepsilon^0). \]

(29)

The system of partial differential equations in Eq.(13) is then given by

\[ \frac{\partial \hat{I}_4^{(1)}}{\partial c_2} = 0 \]

\[ \frac{\partial \hat{I}_4^{(1)}}{\partial c_3} = 0 \]

\[ \frac{\partial \hat{I}_4^{(1)}}{\partial c_4} = \frac{\Gamma(1 + \varepsilon)}{c_4 - c_5} \left[ \frac{1}{\varepsilon} \left( 1 - \frac{c_5}{c_4} \right) + 2 \log(c_4) - 2 \frac{c_5}{c_4} \log(c_5) \right] \]

\[ \frac{\partial \hat{I}_4^{(1)}}{\partial c_5} = \frac{2 \Gamma(1 + \varepsilon)}{c_4 - c_5} \log \left( \frac{c_5}{c_4} \right) \]

(30)

with the solution

\[ \hat{I}_4^{(1)} = \Gamma(1 + \varepsilon) \left[ \frac{1}{\varepsilon} \log(c_4) + \left( \log \left( 1 - \frac{c_4}{c_5} \right) - \log(c_4 - c_5) \right)^2 + 2 \log(c_4) \log(c_5 - c_4) + 2 \text{Li}_2 \left( \frac{c_4}{c_5} \right) - 2 \log(c_5) \log(c_4 - c_5) + k_1 \right]. \]

(31)

The integration constant \( k_1 \) is evaluated in the appendix and its value is

\[ k_1 = \frac{1}{2 \varepsilon^2} + 5 \zeta(2) \]

(32)

where \( \text{Li}_2 \) is the dilogarithm function and \( \zeta(2) = \pi^2/6 \). After some manipulation we have
\[
\Gamma(1 + \epsilon) \left( \frac{c_4^{2\epsilon}}{2\epsilon^2} - \log^2 \left( \frac{c_4}{c_5} \right) - 2 \text{Li}_2 \left( 1 - \frac{c_4}{c_5} \right) + \frac{\pi^2}{6} \right) \tag{33}
\]

Reintroducing the original variables inverting Eq.(25) we get for \( \hat{I}_4^{(1)} \):

\[
\hat{I}_4^{(1)} = \Gamma(1 + \epsilon) \left( \frac{(M \alpha_4)^{2\epsilon}}{2\epsilon^2} - \log^2 \left( \frac{\alpha_4}{\alpha_5} \right) - 2 \text{Li}_2 \left( 1 - \frac{\alpha_4}{\alpha_5} \right) + \frac{\pi^2}{6} \right) \tag{34}
\]

Being \( \hat{I}_4^{(1)} \) independent from \( \lambda \) its value does not change in the limit \( q^2 \to 0 \).

**B. The integral \( \hat{I}_4^{(2)} \)**

Here and in the following subsection we proceed performing the same steps as in the derivation of \( \hat{I}_4^{(1)} \). The limit \( q^2 \to 0 \) now gives a different situation; in fact in this limit there will be three divergent three-point integrals extracted by \( \hat{I}_4^{(2)} \) so as explained in [3] the limit procedure is not smooth and the two case have to be taken separately.

1. \( \hat{I}_4^{(2)} \), \( q^2 \neq 0 \)

In this case \( \hat{I}_4^{(2)} \) is a three external mass box but, differently from \( \hat{I}_4^{(1)} \), it has all external masses different from each other and so it needs evaluation. After putting \( a_2 = 0 \) in Eq.(22) we have for the denominator in \( \hat{I}_4^{(2)} \)

\[
\frac{M^2 a_3^2}{\alpha_3^2} + \frac{a_1 a_3}{\alpha_1 \alpha_3} + \frac{a_1 a_4}{\alpha_1 \alpha_4} + \frac{\lambda a_1 a_5}{\alpha_1 \alpha_5} + \frac{a_3 a_5}{\alpha_3 \alpha_5} . \tag{35}
\]

Rescaling the variables with

\[
\alpha_1 = \frac{c_1}{M} \\
\alpha_3 = M c_3 \\
\alpha_4 = M c_4 \\
\alpha_5 = \frac{c_5}{M} \\
\lambda = \delta / M^2
\]

we get
\[
\frac{a_3^2}{c_3^2} + \frac{a_1 a_3}{c_1 c_3} + \frac{a_1 a_4}{c_1 c_4} + \frac{\delta a_1 a_5}{c_1 c_5} + \frac{a_3 a_5}{c_3 c_5}.
\]  
(37)

giving \(N_4 = 1/2\) and 

\[
\eta = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -\delta & 1 \\
1 & -\delta & 2\delta(1-\delta) & 2\delta - 1 \\
0 & 1 & 2\delta - 1 & -2
\end{pmatrix}.
\]
(38)

The only divergent three points functions extracted by the expression above are \(\hat{I}_3^{(1)}\) and \(\hat{I}_3^{(2)}\) obtained putting \(a_1 = 0\) and \(a_3 = 0\) respectively; these correspond to two two-mass triangle, while the other two obtained putting \(a_4 = 0\) and \(a_5 = 0\) respectively are three-mass triangle checked to be finite. At the \(O(\varepsilon^{-1})\) we have

\[
\hat{I}_3^{(1)} = \Gamma(1 + \varepsilon) \left( \frac{1}{2\varepsilon^2 c_4} + \frac{\log(c_5)}{\varepsilon c_4} \right) + O(\varepsilon^0)
\]
(39)

\[
\hat{I}_3^{(2)} = \frac{\Gamma(1 + \varepsilon)}{\varepsilon(c_5 - \delta c_4)} \log \left( \frac{\delta c_4}{c_5} \right) + O(\varepsilon^0).
\]
(40)

The system in Eq.(13) is then given by

\[
\frac{\partial \hat{I}_4^{(2)}}{\partial c_1} = 0
\]

\[
\frac{\partial \hat{I}_4^{(2)}}{\partial c_3} = 0
\]

\[
\frac{\partial \hat{I}_4^{(2)}}{\partial c_4} = \frac{\Gamma(1 + \varepsilon)}{c_5 - \delta c_4} \left[ \frac{1}{\varepsilon} \left( \frac{c_5}{c_4} - \delta \right) + 2 \frac{c_5}{c_4} \log(c_5) - 2\delta \log(\delta c_4) \right]
\]

\[
\frac{\partial \hat{I}_4^{(2)}}{\partial c_5} = \frac{2\Gamma(1 + \varepsilon)}{c_5 - \delta c_4} \log \left( \frac{\delta c_4}{c_5} \right)
\]
(42)

with the solution

\[
\hat{I}_4^{(2)} = \Gamma(1 + \varepsilon) \left[ \frac{1}{\varepsilon} \log(c_4) + 2 \log(c_4) \log(c_5) - 2 \log(c_5 - \delta c_4) \log \left( 1 - \frac{\delta c_4}{c_5} \right) \\
+ 2 \text{Li}_2 \left( \frac{\delta c_4}{c_5} \right) + \log^2(c_5 - \delta c_4) - 2 \log(c_5) \log(c_5 - \delta c_4) \\
+ \log^2 \left( 1 - \frac{\delta c_4}{c_5} \right) + 2 \log(\delta) \log(c_5 - \delta c_4) + 2 \log(c_4) \log \left( 1 - \frac{\delta c_4}{c_5} \right) + k_2 \right] .
\]
(43)
The integration constant $k_2$ is evaluated in the appendix and its value is

$$k_2 = \frac{1}{2 \varepsilon^2} + \frac{1}{\varepsilon} \log(\delta) - \frac{1}{2} \log^2(\delta) - \text{Li}_2(1 - \delta) - 2 \zeta(2).$$

After some manipulation we have

$$\hat{I}_4^{(2)} = \Gamma(1 + \varepsilon) \left[ \left( \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} + 1 \right) \frac{(c_4 \delta)^2}{2} - \frac{\log^2(\delta)}{2} - \log^2 \left( \frac{c_4 \delta}{c_5} \right) - \text{Li}_2(1 - \delta) - 2 \text{Li}_2 \left( 1 - \frac{c_4 \delta}{c_5} \right) \right]$$

Reintroducing the original variables inverting Eq.(36) we get for $\hat{I}_4^{(2)}$:

$$\hat{I}_4^{(2)} = \Gamma(1 + \varepsilon) \left[ \left( \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} + 1 \right) \frac{(M \lambda \alpha_4)^2}{2} - \frac{1}{2} \log^2(M^2 \lambda) - \log^2 \left( \frac{\lambda \alpha_4}{\alpha_5} \right) \right.$$

$$\left. - \text{Li}_2(1 - M^2 \lambda) - 2 \text{Li}_2 \left( 1 - \frac{\lambda \alpha_4}{\alpha_5} \right) \right]$$

2. $\hat{I}_4^{(2)}$, $q^2 = 0$

In this case $\hat{I}_4^{(2)}$ is a two external mass box. Putting $\lambda = 0$ the denominator in Eq.(35) became

$$\frac{M^2 a_3^2}{\alpha_3^2} + \frac{a_1 a_3}{\alpha_1 \alpha_3} + \frac{a_1 a_4}{\alpha_1 \alpha_4} + \frac{a_3 a_5}{\alpha_3 \alpha_5}$$

Rescaling the variables as in Eq.(36) we get

$$\frac{a_3^2}{c_3^2} + \frac{a_1 a_3}{c_1 c_3} + \frac{a_1 a_4}{c_1 c_4} + \frac{a_3 a_5}{c_3 c_5}$$

The divergent three point functions $\hat{I}_3^{(1)}$, $\hat{I}_3^{(2)}$ and $\hat{I}_3^{(3)}$ are obtained putting $a_1$, $a_2$ and $a_3 = 0$ respectively.

$$\hat{I}_3^{(1)} = \Gamma(1 + \varepsilon) \left( \frac{1}{\varepsilon^2 c_4} + \frac{1}{\varepsilon c_4} \log(c_5) \right) + \mathcal{O}(\varepsilon^0)$$

$$\hat{I}_3^{(2)} = \Gamma(1 + \varepsilon) \left( \frac{1}{\varepsilon^2 c_5} + \frac{1}{\varepsilon c_5} \log(c_1 c_4) \right) + \mathcal{O}(\varepsilon^0)$$

$$\hat{I}_3^{(3)} = \Gamma(1 + \varepsilon) \frac{1}{\varepsilon (c_1 - c_5)} \log \left( \frac{c_5}{c_1} \right) + \mathcal{O}(\varepsilon^0)$$
while \( \hat{I}^{(4)} \) is a three-mass triangle checked to be finite. The partial differential equation system is given by

\[
\frac{\partial \hat{I}^{(2)}_4}{\partial c_1} = 2 \Gamma(1 + \varepsilon) \log \left( \frac{c_5}{c_1} \right) \\
\frac{\partial \hat{I}^{(2)}_4}{\partial c_3} = 0 \\
\frac{\partial \hat{I}^{(2)}_4}{\partial c_4} = \frac{\Gamma(1 + \varepsilon)}{c_4} \left( \frac{1}{\varepsilon} + 2 \log(c_5) \right) \\
\frac{\partial \hat{I}^{(2)}_4}{\partial c_5} = \frac{2 \Gamma(1 + \varepsilon)}{c_5} \left( \frac{1}{\varepsilon} + \frac{c_1}{c_1 - c_5} \log(c_1) - \frac{c_5}{c_1 - c_5} \log(c_5) + \log(c_4) \right)
\]

with the solution

\[
\hat{I}^{(2)}_4 = \Gamma(1 + \varepsilon) \left[ \frac{1}{\varepsilon} (\log(c_4) + 2 \log(c_5)) - \left( \log \left( \frac{1 - \frac{c_1}{c_5}}{c_1 - c_5} \right) - \log(c_1 - c_5) \right)^2 - 2 \log(c_1) \log \left( \frac{1 - \frac{c_1}{c_5}}{c_1 - c_5} \right) + 2 \log(c_4c_1 - c_4c_5) \log(c_5) - 2 \text{Li}_2 \left( \frac{c_1}{c_5} \right) + k_3 \right]
\]

The integration constant \( k_3 \) is evaluated in the appendix and its value is

\[
k_3 = \frac{3}{2\varepsilon^2} - 8\zeta(2)
\]

After some manipulation we find

\[
\hat{I}^{(2)}_4 = \Gamma(1 + \varepsilon) \left[ \frac{(c_4)^{2\varepsilon} - 2 \varepsilon}{2\varepsilon^2} + \frac{(c_5)^{2\varepsilon}}{\varepsilon^2} - \log \left( \frac{c_4}{c_5} \right)^2 + 2 \text{Li}_2 \left( 1 - \frac{c_1}{c_5} \right) - 4 \zeta(2) \right]
\]

Reintroducing the original variables we have

\[
\hat{I}^{(2)}_4 = \Gamma(1 + \varepsilon) \left[ \left( \frac{\alpha_4}{M} \right)^{2\varepsilon} \frac{1}{2\varepsilon^2} + \frac{(M \alpha_5)^{2\varepsilon}}{\varepsilon^2} - \log \left( \frac{\alpha_4}{M^2 \alpha_5} \right)^2 + 2 \text{Li}_2 \left( 1 - \frac{\alpha_1}{\alpha_5} \right) - 4 \zeta(2) \right]
\]

C. The integral \( \hat{I}^{(3)}_4 \)

Putting \( a_3 = 0 \) in Eq.(22) we eliminate the massive propagator and obtain the easy (opposite) two mass box [13] or the one external massive box if we take respectively \( q^2 \neq 0 \)
or $q^2 = 0$. These integrals are well-known and are reported also in [3]. Here we just put these integrals in the kinematics specified in section 3.

1. $\hat{I}_4^{(3)}$, $q^2 \neq 0$

After putting $a_3 = 0$ in Eq.(22) the denominator is given by

$$\frac{a_1 a_4}{\alpha_1 \alpha_4} + \frac{a_2 a_4}{\alpha_2 \alpha_4} + \frac{\lambda a_1 a_5}{\alpha_1 \alpha_5} + \frac{a_2 a_5}{\alpha_2 \alpha_5}$$

(58)

Using Eq.(4.44) from the third paper in ref. [3] the integral in the kinematics of section 3 reads

$$\hat{I}_4^{(3)} = \frac{2 \Gamma(1 + \varepsilon)}{1 - \lambda} \left[ \frac{1}{\varepsilon^2} \left( (\alpha_1 \alpha_4) - (\alpha_2 \alpha_4) + \varepsilon - \left( \frac{\alpha_1 \alpha_5}{\lambda} \right) \right) - \frac{1}{2} \log^2 \left( \frac{\alpha_2 \alpha_5}{\alpha_1 \alpha_4} \right) 
- \text{Li}_2 \left( 1 - \frac{\alpha_1}{\alpha_2} \right) - \text{Li}_2 \left( 1 - \frac{\alpha_5}{\alpha_4} \right) + \text{Li}_2 (1 - \lambda) - \text{Li}_2 \left( 1 - \frac{\alpha_2 \lambda}{\alpha_1} \right) - \text{Li}_2 \left( 1 - \frac{\alpha_4 \lambda}{\alpha_5} \right) \right]$$

(59)

2. $\hat{I}_4^{(3)}$, $q^2 = 0$

Putting $\lambda = 0$ in Eq.(58) the denominator of this integrals is given by

$$\frac{a_1 a_4}{\alpha_1 \alpha_4} + \frac{a_2 a_4}{\alpha_2 \alpha_4} + \frac{a_2 a_5}{\alpha_2 \alpha_5}$$

(60)

Using Eqs.(4.27, 4.40) from the third paper in ref. [3] the integral in the kinematics of section 3 reads

$$\hat{I}_4^{(3)} = \Gamma(1 + \varepsilon) \left[ \frac{2}{\varepsilon^2} \left( (\alpha_1 \alpha_4) - (\alpha_2 \alpha_4) + \varepsilon \right) - \log^2 \left( \frac{\alpha_1 \alpha_4}{\alpha_2 \alpha_5} \right) 
- 2 \text{Li}_2 \left( 1 - \frac{\alpha_1}{\alpha_2} \right) - 2 \text{Li}_2 \left( 1 - \frac{\alpha_5}{\alpha_4} \right) - 4 \zeta(2) \right]$$

(61)

V. CONCLUSIONS

An expression for the scalar pentagon integral shown in figure 1 can be built via Eqs.(12), (23), the four points integrals evaluated in the last section and the coefficients in table 1. The expressions for $\hat{I}_5$ are very long and are not reported.
More familiar kinematics is realized by replacing the variables $\alpha_i$, $\lambda$ and $M^2$ with their definitions in terms of $s_{ij}$, $q^2$ and $m^2$. Tensor integrals will be considered in a separate paper [11], however the simplest one of them, the vector integral, is related to the scalar integrals with one Feynman parameter in the numerator by the following relation

$$I^D_n[l^\mu] \rightarrow I^D_n[\mathcal{P}^\mu]$$

in which the arrow means integration over loop momentum $l$, the integrand numerator is in the square brackets and

$$\mathcal{P}^\mu = \sum_{i=1}^{n-1} a_{i+1} p_i^\mu$$

with $p_i$ given in Eqs.(2). The integrals $\hat{I}_5[a_j]$ can be evaluated by [3,14]

$$\hat{I}_5[a_j] = \frac{1}{2N_5} \sum_{i=1}^5 \eta_{ji} \hat{I}_4^{(i)} + \mathcal{O}(\varepsilon)$$

where $\eta$ defined in Eq.(9) is deduced by $\rho$ given in Eq.(18)

$$\eta = \begin{pmatrix}
1 & -1 & 1 - \lambda & 1 - 2M^2 (-1 + \lambda) & -1 + 2M^2 (-1 + \lambda) \\
-1 & 1 & -1 + \lambda & 1 + 2 (-1 + M^2 (-1 + \lambda)) \lambda & 1 - 2M^2 (-1 + \lambda) \\
1 - \lambda & -1 + \lambda & (-1 + \lambda)^2 & -1 + \lambda & 1 - \lambda \\
1 - 2M^2 (-1 + \lambda) & 1 + 2 (-1 + M^2 (-1 + \lambda)) \lambda & -1 + \lambda & 1 & -1 \\
-1 + 2M^2 (-1 + \lambda) & 1 - 2M^2 (-1 + \lambda) & 1 - \lambda & -1 & 1
\end{pmatrix}$$

Higher tensor integrals can be evaluated considering that they are linked to scalar integrals with more powers of Feynman parameters in the numerator [3]. Such a decomposition can also be organized in a way that drastically reduces numerical instabilities generated by the presence of inverse powers of Gram determinants [14]. Besides the deep inelastic case mentioned in the introduction, the results obtained in the present paper with $q^2 \neq 0$ can be useful in the evaluation at one loop of the decay amplitude of a real $W$ boson or a virtual photon in a heavy quark-antiquark pair and two light quarks. Let us consider the case in which all massless particles and $k_5$ are gluons, then the pentagon studied with $q^2 = 0$ can be identified with one of the four pentagon in the perturbative evaluation of the one-loop associated production of heavy quark in the gluon-gluon-fusion with a...
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**APPENDIX A: FOUR POINTS INITIAL CONDITIONS**

In this appendix we report the calculation of the integration constants for the four points integral of section four systematically neglecting $O(\varepsilon)$ terms. Instead of reporting all length passages, we give the steps that can be followed by programs of function manipulation like the used Mathematica.

1. Integration constant for $\hat{I}_4^{(1)}$ integral

   The point chosen to evaluate $\hat{I}_4^{(1)}$ is $c_2 = c_3 = (c_4/2) = c_5 = 1$ where the expression in Eq.(31) gives
   \[ \hat{I}_4^{(1)} = \Gamma(1 + \varepsilon) \left( \frac{1}{\varepsilon} \log(2) - 3 \zeta(2) + k_1 \right). \]  

   The expression for the integral at the point selected deduced using Eq.(4) and the first of Eqs.(9) is
   \[ \hat{I}_4^{(1)} = \frac{\Gamma(2 + \varepsilon)}{2} \int_0^1 d^4a_i \frac{\delta(1 - \sum_i a_i)}{(a_3^2 + \frac{1}{2} a_2 a_4 + a_2 a_5 + a_3 a_5)^{2+\varepsilon}}. \]  

   The factor 2 is given by $\Pi_{c_i}$. Renaming $a_2$ with $x$, $a_3$ with $y$ and $a_4$ with $z$, and performing the transformation $x \rightarrow 1 - x$, $y \rightarrow x - y$ and $z \rightarrow z$ we arrive at the expression
\[ \hat{I}_4^{(1)} = \frac{\Gamma(2 + \varepsilon)}{2} \int_0^1 dx \int_0^x dy \int_0^y dz \frac{1}{(x^2 - 2 x y - \frac{1}{2} x z + y z + y - \frac{1}{2} \varepsilon)^{2+\varepsilon}}. \tag{A3} \]

Putting apart the Gamma function for the moment, the \( z \) integration gives
\[ -\frac{(x^2 + y - 2 x y)^{-1-\varepsilon}}{(1 + \varepsilon) (1 + x - 2 y)} + \frac{2^{1+\varepsilon} (2 x^2 + y - 5 x y + 2 y^2)^{-1-\varepsilon}}{(1 + \varepsilon) (1 + x - 2 y)}. \tag{A4} \]

The two integrals can be evaluated by shifting both in \( y \)
\[ y \rightarrow y + \frac{x}{2} \tag{A5} \]

simplifying the \( x \) integral
\[ -\frac{2^{1+\varepsilon} (x + 2 y - 4 x y)^{-1-\varepsilon}}{(1 + \varepsilon) (1 - 2 y)} + \frac{2^{2+2\varepsilon} (x + 2 y - 6 x y + 4 y^2)^{-1-\varepsilon}}{(1 + \varepsilon) (1 - 2 y)} \tag{A6} \]

and inverting the integration order [12]. After some manipulation and expanding some Hypergeometric and Generalized Hypergeometric functions the result is
\[ \hat{I}_4^{(1)} = \Gamma(2 + \varepsilon) \left( \frac{1}{2 \varepsilon^2} - \frac{1}{2 \varepsilon} + \frac{1}{\varepsilon} \log(2) + \frac{1}{2} + 2 \zeta(2) - \log(2) \right). \tag{A7} \]

Finally, making the substitution \( \Gamma(2 + \varepsilon) = (1 + \varepsilon) \Gamma(1 + \varepsilon) \) in Eq.(A7) and taking into account Eq.(A1) we find \( k_1 \) in Eq.(32).

2. Integration constant for \( \hat{I}_4^{(2)} \) integral with \( q^2 \neq 0 \)

To evaluate the integration constant we evaluate the integral in the point:
\[ c_1 = \frac{1}{\delta - 1} \]
\[ c_2 = \frac{1}{\delta - 1} \]
\[ c_3 = \frac{1}{1 - \delta} \]
\[ c_4 = -1 \tag{A8} \]
\[ c_5 = \frac{1}{\delta - 1} \tag{A9} \]

with
The integrations are trivial but the expression is very long. The $c_i$ chosen cannot be simultaneously positive so we checked the result in the point $(\delta/2) = c_1 = c_3 = c_4 = c_5 = 1$ where the expression in Eq.(43) gives

$$\tilde{I}_4^{(2)} = \Gamma(2 + \varepsilon) \left( \frac{1}{2 \varepsilon^2} - \frac{1}{2 \varepsilon} + \frac{1}{\varepsilon} \log(2) + \frac{1}{2} + \frac{3}{2} \zeta(2) - \log(2) - \frac{1}{2} \log^2(2) \right)$$

(A11)

The expression for the integral at the point selected is

$$\tilde{I}_4^{(2)} = \Gamma(2 + \varepsilon) \int_0^1 d^4a_i \frac{\delta (1 - \sum_i a_i)}{(a_3^2 + a_1a_3 + a_4a_1 + 2a_1a_5 + a_5a_3)^{2+\varepsilon}}$$

(A12)

Renaming $a_1$ with $x$, $a_3$ with $y$ and $a_4$ with $z$, and performing the transformation $x \rightarrow 1-x$, $y \rightarrow x - y$ and $z \rightarrow z$ we arrive at the expression

$$\tilde{I}_4^{(2)} = \Gamma(2 + \varepsilon) \int_x^1 dx \int_y^x dy \int_0^y dz \frac{1}{(-2xy + yz + x + y - z)^{2+\varepsilon}}$$

(A13)

Putting apart the Gamma function for the moment, the $z$ integration gives

$$-\frac{(x + y - 2xy)^{-1-\varepsilon}}{(1+\varepsilon)(1-y)} + \frac{(x - 2xy + y^2)^{-1-\varepsilon}}{(1+\varepsilon)(1-y)}$$

(A14)

Performing the $x$ integration before and adding and subtracting terms we find Eq.(A11).

3. Integration constant for $\tilde{I}_4^{(2)}$ integral with $q^2 = 0$

The point chosen to evaluate $\tilde{I}_4^{(2)}$ is given by $2c_1 = (c_3/2) = (c_4/2) = c_5 = 1$ in which the expression in Eq.(54) gives

$$\tilde{I}_4^{(2)} = \Gamma(1 + \varepsilon) \left( \frac{1}{\varepsilon} \log(2) - \log^2(2) + 5 \zeta(2) + k_3. \right)$$

(A15)

The expression for the integral at the point selected is

$$\tilde{I}_4^{(2)} = \frac{\Gamma(2 + \varepsilon)}{2} \int_0^1 d^4a_i \frac{\delta (1 - \sum_i a_i)}{(\frac{1}{4} a_3^2 + a_1a_3 + a_4a_1 + \frac{1}{2} a_5a_3)^{2+\varepsilon}}.$$ 

(A16)

Renaming $a_3$ with $x$, $a_4$ with $y$ and $a_5$ with $z$, and performing the transformation $x \rightarrow 1-x$, $y \rightarrow x - y$ and $z \rightarrow z$ we arrive at the expression
\[ \hat{I}_4^{(2)} = \frac{\Gamma(2 + \varepsilon)}{2} \int_0^1 dx \int_0^x dy \int_0^y dz \frac{1}{\left( \frac{1}{4}x^2 - y^2 - \frac{1}{2}xz + yz - \frac{1}{2}x + y - \frac{1}{2}z + \frac{1}{4} \right)^{2+2\varepsilon}}. \] (A17)

Putting apart the Gamma function for the moment, the \( z \) integration gives

\[ \frac{2^{2+2\varepsilon} (1-x)^{-1-\varepsilon} (1-x+2y)^{-1-\varepsilon}}{(1+\varepsilon) (1+x-2y)} - \frac{2^{2+2\varepsilon} (1-2x+x^2+4y-4y^2)^{-1-\varepsilon}}{(1+\varepsilon) (1+x-2y)}. \] (A18)

Performing the shift

\[ y \rightarrow y + \frac{x}{2} \] (A19)

gives

\[ \frac{2^{2+2\varepsilon} (1-x)^{-1-\varepsilon} (1+2y)^{-1-\varepsilon}}{(1+\varepsilon) (1-2y)} - \frac{2^{2+2\varepsilon} (1+4y-4xy-4y^2)^{-1-\varepsilon}}{(1+\varepsilon) (1-2y)}, \] (A20)

finally performing the \( x \) integration before and adding and subtracting terms before expanding in \( \varepsilon \) the result is

\[ \hat{I}_4^{(2)} = \Gamma(2 + \varepsilon) \left( \frac{3}{2\varepsilon^2} - \frac{3}{2 \varepsilon} + \frac{1}{\varepsilon} \log(2) + \frac{3}{2} - 3 \zeta(2) - \log(2) - \log^2(2) \right). \] (A21)

Substituting \( \Gamma(2 + \varepsilon) = (1 + \varepsilon) \Gamma(1 + \varepsilon) \) in Eq.(A21) and taking into account Eq.(A15) we find \( k_3 \) in Eq.(55).
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