Conformal couplings of Galileons to other degrees of freedom

Gianmassimo Tasinato

Institute of Cosmology & Gravitation, University of Portsmouth,
Dennis Sciama Building, Portsmouth, PO1 3FX, United Kingdom

May 22, 2014

Abstract

We discuss a formulation of Galileon actions in terms of matrix determinants in four dimensions. This approach allows one to straightforwardly determine derivative couplings between Galileons and scalar or vector degrees of freedom that lead to equations of motion with at most two space-time derivatives. We use this method to easily build generalizations of Galileon set-ups preserving conformal symmetry, finding explicit examples of couplings between Galileons and additional degrees of freedom that preserve the Galileon conformal invariance. We discuss various physical applications of our method and of our results.

1 Introduction

Explaining the current acceleration of our universe is an open theoretical problem. Observational data are consistent with cosmological acceleration driven by a cosmological constant, whose size is however much smaller than what is expected by an effective field theory approach: see [1] for a review. A possibility is that the cosmological constant is set to zero by some yet unknown mechanism, while cosmological acceleration is driven by the dynamics of additional degrees of freedom, or by a modification of gravity at sufficiently large distances, as reviewed in [2]. It is however hard to find consistent theories that lead to cosmological acceleration, and that at the same time are able to avoid solar system constraints on deviations from general relativity.

Recently, a new scenario has been pushed forward, based on a scalar Lagrangian with derivative self-interactions respecting a Galilean symmetry [3]. Although the Galileon Lagrangian contains higher order derivative self-interactions, it is designed in such a way that the corresponding equations of motion have at most two space-time derivatives, hence avoiding Ostrogradsky instabilities. These field equations have been shown in [3] to admit de Sitter solutions at small scales, that can describe present day acceleration even in the absence of scalar potential. Although the scalar field is massless, it does not necessarily lead to dangerous long range fifth forces. Derivative self-interactions are able to screen its effects in proximity of spherically symmetric sources, realizing the so-called Vainshtein screening mechanism [4]. Galileon theories received much attention over the past few years, for their phenomenological consequences. But there have also been interesting theoretical developments suggesting possible directions for consistent infrared completions of these theories. Galileon interactions have been explicitly shown to arise in appropriate limits of higher dimensional brane probe constructions [5, 6], they appear in compactifications of higher dimensional Lovelock theories [7], and in the decoupling limit of a four dimensional theory of massive gravity [8, 9]. Typically, such constructions add new degrees of freedom, (scalar, vector and/or tensor) to the short-distance action that respects the Galileon symmetry.

The original work [3] proposes an intriguing four dimensional infrared completion of the single field Galileon scalar Lagrangian, based on a conformally invariant scalar action. At short distances, and for small values of the field, the conformal symmetry reduces to the Galilean symmetry, hence the conformally invariant action approaches the Galilean one in these limits. The 4d conformal group SO(4,2) admits the SO(4,1) de Sitter group as maximal subgroup: a non-trivial scalar profile can spontaneously break the conformal group to the de Sitter group [10].
Hence, after symmetry breaking, the resulting effective action describes scalar fluctuations around the de Sitter configuration, and at short distances is well approximated by a Galilean invariant action. Such an approach suggests a possible explanation for the observed cosmological acceleration in terms of symmetries: our FRW universe is asymptotically approaching a maximally symmetric de Sitter configuration, resulting from the spontaneous breaking of the conformal symmetry of an IR complete Galileon set-up.

As discussed already in [3], this proposal is difficult to implement when taking into full account the gravitational backreaction of the conformally invariant scalar Lagrangian. The Weyl rescaling symmetry that has to be imposed on the gravitational action coupled to the conformal system allows one to ‘gauge away’ the Galileon scalar, and work with a purely gravitational action that does not admit theoretically interesting de Sitter solutions. One might wonder whether this negative conclusion might be avoided by considering more general conformal Galileon systems, that couple the Galileon to other degrees of freedom (scalars and vectors) in such a way to maintain the conformal invariance. When coupling to gravity, there is the possibility that while one of the scalars is gauged away, the action for the remaining degrees of freedom still respects the structure of a conformal Galileon set-up and is nevertheless able to generate interesting de Sitter configurations.

Besides this motivation, the question of finding conformal couplings of the Galileon to other fields is interesting on its own, since it can reveal new unforeseen symmetries or structures in the more general Galileon action, that can suggest new ideas to address the problem of explaining current cosmological acceleration. Moreover, conformal Galileon actions are known to be more amenable of supersymmetrization [11] with respect to standard Galileon actions [12, 13]. Finding these conformal couplings is the scope of this work. We implement a novel method based on a four dimensional analysis of the Galileon action. We make use of a very compact, determinantal form of the Galileon action, that renders straightforward to extend it to set-ups that couple the Galileon to scalar and vector degrees of freedom, in a way that maintain Galilean as well as gauge symmetries. Hence, we show how conformally invariant versions of the resulting actions can be easily determined, and express them again in a compact form that renders manifest the symmetries of the system. See also [14, 15, 16] for approaches to Galileons using determinants.

In our conclusions, we analyze various possible applications of our results. We discuss how they can lead to new actions with derivative couplings among several degrees of freedom (scalar and vector) that avoid Ostrogradsky instabilities, and that can be relevant for finding new consistent de Sitter solutions or new realizations of screening mechanisms. We speculate how our approach to express Galileon invariant actions in a determinantal form can make more explicit new symmetries and structures of the set-ups under consideration. We also discuss problems left open in this work, as for example the issue of consistently including gravity in these generalized conformal Galileon set-ups.

2 A convenient way to express Galileon Lagrangians

We use a convenient way to express Galileon Lagrangians in terms of matrix determinants. Besides leading to compact expressions, this method renders more explicit the symmetries of the set-ups under consideration. We work in four dimensions, although our approach can be straightforwardly applied to other dimensions. We start with some technical formulae that we will need in what follows. Given a squared $4 \times 4$ real matrix $M_{\mu \nu}$, its determinant is

$$
\det M_{\mu \nu} = -\frac{1}{4!} \epsilon_{\alpha_1 \ldots \alpha_4} \epsilon^{\beta_1 \ldots \beta_4} M_{\beta_1}^{\alpha_1} \ldots M_{\beta_4}^{\alpha_4}.
$$

The contractions between Levi-Civita symbols can be managed using the identity

$$
\epsilon_{\alpha_1 \ldots \alpha_j, \alpha_{j+1} \ldots \alpha_4} \epsilon^{\alpha_1 \ldots \alpha_j, \beta_{j+1} \ldots \beta_4} = -(4-j)! j! \delta_{\alpha_{j+1} \ldots \alpha_4}^{[\beta_{j+1} \ldots \beta_4]},
$$

where $[\beta_{j+1} \ldots \beta_4]$ denotes anti-symmetrization.

The determinant of $(\delta_{\mu \nu} + M_{\mu \nu})$ can be also expressed in terms of traces as follows

$$
\det (\delta_{\mu \nu} + M_{\mu \nu}) = 1 + \text{tr} M + \frac{1}{2} \left[ (\text{tr} M)^2 - \text{tr} (M^2) \right] + \frac{1}{6} \left[ (\text{tr} M)^3 - 3 \text{tr} M \text{tr} (M^2) + 2 \text{tr} (M^3) \right] + \frac{1}{24} \left[ (\text{tr} M)^4 - 6 \text{tr}(M^2)(\text{tr} M)^2 + 3 (\text{tr} (M^2))^2 + 8 \text{tr} M \text{tr}(M^3) - 6 \text{tr} (M^4) \right].
$$


2.1 The Galileon invariant Lagrangian

In order to describe the standard single field Galileon set-up in a compact way, the basic building block we need is the following scalar Lagrangian defined in Minkowski space, \( g_{\mu\nu} = \eta_{\mu\nu} \)

\[
S_{gal} = \int d^4x \det (\delta_{\mu}^{\nu} + \frac{1}{\Lambda} \Pi_{\mu}^{\nu} - \partial_{\mu} \pi \partial^{\nu} \pi). \tag{2.4}
\]

This Lagrangian describes the dynamics of a single scalar field \( \pi(x) \), and \( \Pi_{\mu\nu} \equiv \partial_{\mu} \partial_{\nu} \pi \). Indexes are raised and lowered with the Minkowski metric. A parameter \( \Lambda \) of dimension of mass has been introduced for dimensional reasons. The same Lagrangian can also be rewritten more elegantly as

\[
S_{gal} = \int d^4x \det (\delta_{\mu}^{\nu} - \frac{1}{\Lambda^2} e^{\Lambda x} \partial_{\mu} \partial^{\nu} e^{-\Lambda x}). \tag{2.5}
\]

Remarkably, this action contains all the Galileon interactions, as we are going to discuss now.

From now on, for simplicity, we choose units for which \( \Lambda = 1 \). Using formula (2.1), the determinant contained in (2.4) can be expanded as a sum of various contributions. The contributions that contain only \( \Pi_{\mu\nu} \) tensors are total derivatives. Indeed, one observes that

\[
\Pi_{\alpha_1\beta_1} \cdots \Pi_{\alpha_k\beta_k} \epsilon^{\alpha_1 \ldots \alpha_k \beta_k + \ldots \beta_n} \epsilon^{\beta_1 \ldots \beta_k \beta_{k+1} \ldots \beta_n} = \partial_{\alpha_1} \left( \Pi_{\beta_1 \pi} \Pi_{\alpha_2 \beta_2} \cdots \Pi_{\alpha_k \beta_k} \epsilon^{\alpha_1 \ldots \alpha_k \beta_k + \ldots \beta_n} \epsilon^{\beta_1 \ldots \beta_k \beta_{k+1} \ldots \beta_n} \right) - \partial_{\beta_1} \partial_{\alpha_1} \left( \Pi_{\alpha_2 \beta_2} \cdots \Pi_{\alpha_k \beta_k} \epsilon^{\alpha_1 \ldots \alpha_k \beta_k + \ldots \beta_n} \epsilon^{\beta_1 \ldots \beta_k \beta_{k+1} \ldots \beta_n} \right) \tag{2.6}
\]

but the second line vanishes due to the antisymmetric Levi-Civita symbols.

The contributions that contain powers of the tensor \( \partial_{\mu} \pi \partial_{\nu} \pi \) higher or equal to two vanish. Indeed, one can write such contributions as

\[
(\partial_{\alpha_1} \pi \partial_{\beta_1} \pi) (\partial_{\alpha_2} \pi \partial_{\beta_2} \pi) K^{(3)}_{\alpha_3 \beta_3} \cdots K^{(n)}_{\alpha_n \beta_n} \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon^{\beta_1 \beta_2 \ldots \beta_n} \tag{2.7}
\]

where the \( K^{(i)}_{\alpha_i \beta_i} \) can be \( \delta_{\alpha_i \beta_i} \), \( \Pi_{\alpha_i \beta_i} \), or \( \partial_{\alpha_i} \pi \partial_{\beta_i} \pi \). For our argument, it is sufficient to know that the \( K^{(i)}_{\alpha_i \beta_i} \) are symmetric tensors in the indexes \( \alpha_i, \beta_i \). Then we can write

\[
(\partial_{\alpha_1} \pi \partial_{\beta_1} \pi) (\partial_{\alpha_2} \pi \partial_{\beta_2} \pi) K^{(3)}_{\alpha_3 \beta_3} \cdots K^{(n)}_{\alpha_n \beta_n} \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon^{\beta_1 \beta_2 \ldots \beta_n} = (\partial_{\alpha_2} \pi \partial_{\beta_1} \pi) (\partial_{\alpha_1} \pi \partial_{\beta_2} \pi) K^{(3)}_{\alpha_3 \beta_3} \cdots K^{(n)}_{\alpha_n \beta_n} \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon^{\beta_1 \beta_2 \ldots \beta_n} = - (\partial_{\alpha_2} \pi \partial_{\beta_1} \pi) (\partial_{\alpha_1} \pi \partial_{\beta_2} \pi) K^{(3)}_{\alpha_3 \beta_3} \cdots K^{(n)}_{\alpha_n \beta_n} \epsilon^{\alpha_1 \alpha_2 \ldots \alpha_n} \epsilon^{\beta_1 \beta_2 \ldots \beta_n} \tag{2.8}
\]

Being equal to its opposite, this combination vanishes.

Hence, when using formula (2.1), the only non-trivial field dependent contributions from the expansion of the matrix determinant in eq. (2.4) are the ones that contain one power of \( \partial_{\mu} \pi \partial_{\nu} \pi \), and powers of \( \Pi_{\mu\nu} \) from zero to three. These contributions lead to equations of motion with at most two space-time derivatives, due to the properties of the antisymmetric Levi-Civita symbol. Moreover, the action (2.4) preserves the Galileon invariance \( \pi(x) \rightarrow \pi(x) + a + b_{\mu} x^{\mu} \). By performing such transformation, the determinant becomes

\[
\det \left[ \delta_{\mu}^{\nu} + \Pi_{\mu}^{\nu} - (\partial_{\mu} \pi + b_{\mu})(\partial_{\nu} \pi + b_{\nu}) \right].
\]

When evaluating it, one finds that terms that depend on powers of \( (\partial_{\mu} \pi + b_{\mu})(\partial_{\nu} \pi + b_{\nu}) \) higher or equal to two vanish, for the same arguments developed in eq. (2.8). Then, when considering the terms proportional to \( (\partial_{\mu} \pi + b_{\mu})(\partial_{\nu} \pi + b_{\nu}) \), one finds that all contributions depending on \( b_{\mu} \) are total derivatives: this proves that (2.4) is invariant under Galileon symmetry.
Not surprisingly, expanding the determinant in (2.4) in terms of traces using formula (2.3), one finds a scalar Lagrangian corresponding exactly to the Galileon Lagrangian discussed in [3], where the derivative interactions of different orders combine in Galileon invariant combinations (dubbed \( \mathcal{L}_2 \ldots \mathcal{L}_5 \) in [3]) with fixed overall coefficients:

\[
S_{gal} = \int d^4x \left[ 1 + 2 \mathcal{L}_2 + 3 \mathcal{L}_3 + 2 \mathcal{L}_4 + \frac{5}{6} \mathcal{L}_5 \right].
\] (2.9)

In order to find a general combination of Galileon Lagrangians, in which each of the previous contributions has arbitrary overall coefficient, it is sufficient to sum four copies of actions (2.4) (or (2.5)) in the following way

\[
\sum_{i=1}^{4} \int d^4x \det (\delta^{\nu\rho} + c_i \varepsilon^{\nu\rho \mu} \partial_{\mu} e^{-\pi})
\] (2.10)

with arbitrary parameters \( c_i \). Expanding the determinants and summing the different contributions, one can fix the relative coefficients among the Galileon invariant combinations \( \mathcal{L}_i \) by appropriately choosing the \( c_i \) and using Newton identities (similar considerations have already been done in [17]). Notice that a cosmological constant term appears, that can be tuned to zero by adding a bare cosmological constant to the initial action.

To end this subsection, it is worth pointing out that the determinantial Lagrangian contained in (2.4) is not the only one that leads to the Galileon invariant combinations of [3]. By considering a quadratic shift on the scalar,

\[
\pi = \frac{c_0}{2} x^2 + \hat{\pi},
\] (2.11)

for some dimensionful parameter \( c_0 \), one obtains an effective action for the field \( \hat{\pi} \) of the form

\[
S = \int d^4x \det \left[ (1 + c_0) \delta^{\nu\rho} + \hat{\Pi}^{\nu\rho} - \partial_{\rho} \hat{\pi} \partial^{\nu} \hat{\pi} - c_0 \partial_{\rho} \hat{\pi} x^{\nu} - c_0 x^{\mu} \partial_{\mu} \hat{\pi} - c_0^2 x^{\mu} x^{\nu} \right].
\] (2.12)

Expanding the determinant using formula (2.1), one can show that the various contributions can be re-expressed as a sum of Galileon invariant combinations \( \mathcal{L}_i \), with coefficients depending on \( c_0 \). Hence, the Lagrangian for fluctuations around a quadratic scalar profile as (2.11) still obeys the Galileon symmetry. This fact has indeed been used in [3] to characterize the Lagrangian for fluctuations around self-accelerating de Sitter configurations, for which a scalar profile of the form (2.11) solves the background equations of motion.

### 2.2 The conformal Galileon

The Galileon action (2.4) is not the unique one that leads to equations of motion with at most two time derivatives. Simple extensions exist with this property, that can allow one to study systems with larger symmetries, and that possibly reduce to the Galileon symmetric case in appropriate limits.

Consider for example the following action in Minkowski space

\[
S = \int d^4x \det (\varepsilon^{\gamma_1 \pi} \delta^{\nu\rho} + c_2 \varepsilon^{\gamma_2 \pi} \Pi^{\nu\rho} + c_3 \varepsilon^{\gamma_3 \pi} \partial_{\nu} \pi \partial^{\rho} \pi + c_4 \varepsilon^{\gamma_4 \pi} \delta^{\nu\rho} \partial_{\nu} \pi \partial^{\rho} \pi)
\] (2.13)

for arbitrary parameters \( c_i \) and \( \gamma_i \). Expanding the determinant inside the integral by means of eq. (2.1) we get a sum of different contributions. In this case, contributions depending only on the tensors \( \Pi_{\mu\nu} \) (weighted by appropriate powers of \( e^\pi \)) are not in general total derivatives. On the other hand, for the same arguments developed in eq. (2.8), terms that depend on powers of \( \partial_{\nu} \pi \partial^{\rho} \pi \) higher or equal to two vanish. It is straightforward to check that the last term weighted by \( c_4 \) can be added maintaining the property that the resulting equations of motion contain at most two-space time derivatives; this is due to the properties of the antisymmetric Levi-Civita symbol.

In general the resulting system does not preserve exact Galilean invariance, but it might do so in appropriate limits.

In this section, we will show that a special case of the previous action (2.13) is invariant under conformal transformations, that reduce to Galileon transformations in a short-distance, small \( \pi \) limit. The conformal group is characterized by the following symmetry laws:

4
- Dilations $D$ controlled by a parameter $\lambda$
  \[\pi(x) \rightarrow \pi(\lambda x) + \ln \lambda\]  
  (2.14)

- Infinitesimal special conformal transformations $K_\mu$ controlled by a vector $c_\mu$
  \[\pi(x) \rightarrow \pi(x + (x^2) c - 2(c \cdot x)) - 2 c_\mu x^\mu\]  
  (2.15)

- Translations $P_\mu$ controlled by a vector $a_\mu$
  \[\pi(x) \rightarrow \pi(x + a)\]  
  (2.16)

- Boosts $M_{\mu\nu}$ controlled by a tensor $\Lambda_{\mu\nu}$
  \[\pi(x) \rightarrow \pi(\Lambda \cdot x)\]  
  (2.17)

The symmetry group that characterizes de Sitter space in four dimension, $SO(4,1)$, is a subgroup of the above conformal group $SO(3,2)$. Besides the boosts $M_{\mu\nu}$, the generators $S_\mu$ of de Sitter algebra are a combination of translations and infinitesimal special conformal transformations: $S_\mu = P_\mu - \frac{1}{2} H^2 K_\mu$.

The infinitesimal special conformal transformation requires that $\pi(x) \rightarrow \pi(x') - 2c_\mu x^\mu$ with

\[x'^\mu = x^\mu + (x^2) c^\mu - 2(c \cdot x) x^\mu\]  
(2.18)

for an infinitesimal vector $c^\mu$. This implies that at linear order in $c^\mu$

\[dx'^\mu = [(1 - 2c_\mu x^\rho) \delta^\mu_\nu + 2(e^\mu x_\nu - x^\mu c_\nu)] dx^\nu.\]  
(2.19)

Calling

\[A^\mu_\nu \equiv (e^\mu x_\nu - x^\mu c_\nu)\]  
(2.20)

the antisymmetric combination appearing in eq. (2.19), we find that at leading order in $c^\mu$ the derivative of $\pi$ transforms as

\[\partial_\mu \pi(x) \rightarrow e^{-2c_\mu} \partial'_\mu \pi(x') + 2 \partial'_\mu \pi(x') A^\mu_\rho - 2 c_\mu.\]  
(2.21)

Hence ($\partial'$ denotes derivatives along the coordinates $x'_\mu$)

\[
\begin{align*}
\partial_\mu \pi(x) \partial_\nu \pi(x) &\rightarrow e^{-4c_\mu} \partial'_\mu \pi(x') \partial'_\nu \pi(x') - 2 (\partial'_\mu \pi c_\nu + \partial'_\nu \pi c_\mu) + 2 (\partial_\mu \pi \partial_\lambda \pi A^\lambda_\nu + \partial_\nu \pi \partial_\lambda \pi A^\lambda_\mu), \\
\partial_\mu \partial_\nu \pi(x) &\rightarrow e^{-4c_\mu} \partial'_\mu \partial'_\nu \pi(x') - 2 (\partial'_\mu \pi c_\nu + \partial'_\nu \pi c_\mu) + 2 \eta_{\mu\nu} e^\rho \partial'_\rho \pi + 2 (\partial_\mu \partial_\lambda \pi A^\lambda_\nu + \partial_\nu \partial_\lambda \pi A^\lambda_\mu). 
\end{align*}
\]  
(2.22)

We will be interested in the combination

\[C_\mu^\nu \equiv e^{-2\pi} \left[\partial_\mu \partial_\nu \pi - \partial_\mu \pi \partial_\nu \pi + \frac{1}{2} \delta_\mu^\nu \partial_\rho \pi \partial_\rho \pi\right]\]  
(2.24)

Under conformal transformations, it becomes

\[C'_\mu^\nu(x') = C_\mu^\nu(x) + e^{-2\pi} \left[(\partial_\mu \partial_\lambda \pi - \partial_\lambda \pi \partial_\lambda \pi) A^\lambda_\nu + A^\mu_\rho (\partial_\rho \partial_\nu \pi - \partial_\nu \pi \partial_\rho \pi) \right]\]  
(2.25)

\[\equiv C_\mu^\nu(x) + B^\lambda_\mu A^\lambda_\nu + A^\mu_\rho B^\rho_\nu\]  
(2.26)

in the second line, the matrix $A$ defined in eq. (2.20) is antisymmetric, while the matrixes $B$ and $C$ are symmetric. Consider now the determinant

\[\det \left[\delta_\mu^\nu + C_\mu^\nu\right].\]  
(2.27)

Using eq. (2.3), we can expand the determinant in terms of traces of $C$ and its powers, schematically as $\text{tr}[C^n]$. Under an infinitesimal special conformal transformation, at linear order in $c_\mu$ each one of these traces transforms
schematically to \( \text{tr}[C^n] \rightarrow \text{tr}[C^n] = \text{tr}[C^n]+n \text{tr}[C^{n-1} B A]+n \text{tr}[C^{n-1} A B] \). But the last two contributions vanish, due to the antisymmetry of \( A \). Hence, each trace involving \( C \) is invariant under infinitesimal special conformal transformations, \( \text{tr}[C^n] = \text{tr}[C^n] \), and the complete determinant (2.27) as well. It is also straightforward to prove that the same quantity is also invariant under dilations.

This implies that the action

\[
S_{\text{conf}} = \int d^4x \, e^{4\pi} \det \left[ \delta_{\mu}^{\nu} + c_0 e^{-2\pi} \left( \partial_{\mu} \partial^{\nu} \pi - \partial_{\nu} \pi \partial_{\nu} \pi + \frac{1}{2} \delta_{\mu}^{\nu} \partial_\rho \pi \partial_\rho \pi \right) \right]
\]

(2.28)
is invariant under conformal transformations, for arbitrary constant \( c_0 \). Indeed, the determinant as we have seen is conformally invariant. The overall factor of \( e^{4\pi} \) has been included to compensate for the transformation properties of the measure \( d^4x \).

This action corresponds to the so-called conformally invariant Galileon system. Expanding the determinant using formula (2.3) and integrating by parts, one finds a sum of combinations of derivative interactions involving \( \pi \) (with fixed coefficients) that are weighted by different powers of \( e^\pi \), each of them separately conformally invariant. They reproduce the conformal Galileon Lagrangian described in [3], obtained combining curvature invariants of a conformally flat metric \( g_{\mu\nu} \equiv e^{2\pi} \eta_{\mu\nu} \).

An exception is the conformally invariant contribution weighted by zero powers of \( e^\pi \) in the expansion of (2.28), that turns out to be a total derivative: already in [3] such a term had to be put by hand, or obtained as a suitable limit of a \( d \neq 4 \) combination. Similar considerations can be done here.

In order to find the most general conformal Lagrangian with arbitrary coefficients in front of the different conformally invariant contributions, it is enough to proceed exactly as in the case of standard Galileon (see the discussion around eq. (2.10)): sum four copies of the action (2.28), with different arbitrary coefficients \( c_0 \), and then properly use Newton identities to fix the preferred coefficient in front of each conformally invariant combination.

As reviewed in the introduction, the main motivation of [3] for studying a conformally invariant Galileon set-up is to provide a consistent IR completion of Galileon actions in four dimensions. In this view, motivated by the results of [10], the fact that our universe is apparently approaching a de Sitter configuration can be interpreted as a process of spontaneous symmetry breaking. Namely, the spontaneous breakdown of the SO(4,1) conformal symmetry to an SO(4,1) de Sitter symmetry by means of a non-trivial profile for the scalar \( \pi \) (acting as dilaton). An example of such profile is

\[
e^{\pi(x)} = \frac{1}{1 + \frac{1}{3} H^2 x^2}
\]

(2.29)
with \( H \) corresponding to the de Sitter scale. This scalar profile preserves the subgroup of de Sitter symmetries listed between eqs. (2.17) and (2.18). Hence by general grounds one expects that it solves the equations of motion for the conformal symmetric Lagrangian, by appropriately choosing the free parameters that characterizes it. The Lagrangian for fluctuations around this solution respects the de Sitter symmetry, and in a short distance/small field limit acquires a Galileon symmetry. On the other hand, it provides a potentially consistent IR completion of the Galileon action valid also at large distances. There are issues with this proposal when coupling the action to gravity, as explained in [3] and reviewed in the introduction, that can motivate the question of determining conformal couplings of the Galileon to additional degrees of freedom, as scalars and vectors. We will address this question in what follows.

3 The bi-Galileon and its conformal version

An interesting feature of our method is that it allows one to generalize the previous action and determine couplings between Galileons and other fields, respecting Galileon or conformal symmetries. In this section, we will discuss couplings between Galileon and scalar fields, focussing on the case of the bi-Galileon [18, 19].

Consider two scalar fields \( \pi \) and \( \sigma \): we require that both of them respect Galileon symmetries

\[
\pi \rightarrow \pi + c^{(\pi)} + b^{(\pi)}_{\mu} x^\mu \quad \quad \sigma \rightarrow \sigma + c^{(\sigma)} + b^{(\sigma)}_{\mu} x^\mu.
\]

(3.1)
For the moment, we choose arbitrary parameters \( b^{(i)}_\mu, c^{(i)} \). The general structure of an action that couples these scalar fields in a way that preserves the symmetry (3.1) is the following

\[
S_{\text{bi-gal}} = \int d^4 x \left( \det \left[ \delta_\mu^\nu + e^\pi \partial_\mu e^{-\pi} + \partial_\mu \partial^\nu \pi \right] + \det \left[ \delta_\mu^\nu + e^\sigma \partial_\mu \partial^\nu e^{-\sigma} + \partial_\mu \partial^\nu \pi \right] \right). \tag{3.2}
\]

Using the arguments we developed in the single Galileon case, it is straightforward to check that this action respects the Galileon invariance (3.1). Expanding the determinants, one obtains all the interactions that characterize bi-Galileon theories of [18]. There appear various Galileon-invariant combinations with fixed coefficients: to determine a general action with arbitrary coefficients in front of those combinations one proceeds as explained around eq (2.10), summing various copies of (3.2) with suitable coefficients weighting the second and third term inside each of the two determinants. Multi-Galileon theories can be straightforwardly expressed in the same manner, by adding more terms in the action depending on the additional fields, with the same structure as the ones above.

It is not difficult to find a conformal version of the bi-Galileon theory, symmetric under the conformal symmetry discussed in section 2.2, that applies to both scalars \( \pi \) and \( \sigma \). It is sufficient to build analogues of the combination \( C_\mu^\nu \) of eq. (2.24) for each scalar \( \pi, \sigma \), and linearly combine them. An example is the following action

\[
S_{\text{conf}} = \int d^4 x e^{4\pi} \det \left[ \delta_\mu^\nu + a_1 e^{-2\pi} \left( \partial_\mu \partial^\nu \pi - \partial_\mu \pi \partial^\nu \pi + \frac{1}{2} \delta_\mu^\nu \partial^\rho \pi \partial_\rho \pi \right) \\
+a_2 e^{-2\pi} \left( \partial_\mu \partial^\nu \sigma - \partial_\mu \sigma \partial^\nu \sigma + \frac{1}{2} \delta_\mu^\nu \partial^\rho \sigma \partial_\rho \sigma \right) \right] \tag{3.3}
\]

for arbitrary coefficients \( a_1, a_2 \). One proves that this action is conformally invariant repeating exactly the same steps we already made for the single scalar case, before eq (2.28). Indeed, the determinant is by construction conformally invariant. The overall coefficient \( e^{4\pi} \) in front of it has been placed to take care of the transformation of the measure. For simplicity, for this purpose we have chosen to put an overall factor depending only on \( \pi \), but a different choice depending also on \( \sigma \) can also be made. Expanding the determinant of (3.3), one finds a sum of several conformally invariant combinations, each of them weighted by different powers of \( e^\pi \), \( e^\sigma \). The advantage of (3.1) is that it condenses in a compact formula all such combinations. Notice that, in the small field and short distance limit, one recovers a Galileon symmetric set-up in which both fields \( \pi, \sigma \) are invariant under a Galileon symmetry with the same parameters appearing in eq (3.1), \( b^{(\pi)}_\mu = b^{(\sigma)}_\mu, c^{(\pi)} = c^{(\sigma)} \).

As for the case of single Galileon field, the conformal symmetry can be spontaneously broken by a non-trivial profile for one (or both) the scalars \( \pi, \sigma \). If the profile respects de Sitter symmetry, a subgroup of the full conformal symmetry, one expects to have it as particular solution of the conformal Lagrangian: fluctuations around such profile will respect de Sitter symmetry and, in the small-scale limit, will be governed by a bi-Galileon invariant theory expanded around a de Sitter configuration. It would be interesting to understand whether the resulting multi-scalar system can be consistently coupled to dynamical gravity preserving the conformal Galileon scalar symmetry, and at the same time admitting non-trivial and interesting de Sitter solutions. This topic is left for future work.

4 Conformal couplings of Galileons to vectors

Another sector to which Galileons can couple are vector degrees of freedom: also this case can also be discussed by means of an elegant determinantal formulation. Galileons coupling to vectors are theoretically well motivated, since they arise in a suitable decoupling limit of massive gravity [20, 21, 22]. In order to couple Galileons to vectors, some further formula for conveniently handling determinants will be needed. Consider an abelian vector field \( A_\mu \) with field strength \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \) invariant under an abelian \( U(1) \) symmetry \( A_\mu \rightarrow A_\mu + \partial_\mu \xi \). The field strength is an antisymmetric tensor in the indexes \( (\mu, \nu) \). We will be interested in matrices \( M_{\mu}^{\nu} \) that combine symmetric and antisymmetric parts, of the form

\[
M_{\mu}^{\nu} = S_{\mu}^{\nu} + F_{\mu}^{\nu} \tag{4.1}
\]
where $S_{\mu \nu}$ an arbitrary symmetric matrix. Writing schematically \((\delta^{\nu}_{\mu} + M_{\mu}^{\nu}) = (1 + M)\), we can write
\[
\det (1 + M) = \det (1 + S) - \frac{1}{2} \text{tr} F^2 + \frac{1}{8} \left[ (\text{tr} F^2)^2 - 2 \text{tr} F^4 \right] + \text{tr} (S F^2) - \frac{1}{2} \text{tr} F^2 \text{tr} S + \text{tr} S \text{tr} (S F^2) - \text{tr} (S^2 F^2) - \frac{1}{4} \text{tr} F^2 \left( (\text{tr} S)^2 - \text{tr} S^2 \right) - \frac{1}{2} \text{tr} (S F S F). \tag{4.2}
\]

By means of this result, it is straightforward to couple scalars $\pi$ to vectors in a way that preserve Galilean symmetries. To build a set-up that preserves both the scalar Galilean symmetry $\pi \rightarrow \pi + b_{\mu} x^{\mu} + c$ and vector abelian symmetry, we choose $S_{\mu}^{\nu} = \Pi_{\mu}^{\nu}$, and write the action as
\[
S_{\text{vec}} = \int d^4 x \det (\delta^{\nu}_{\mu} + \Pi_{\mu}^{\nu} + F_{\mu}^{\nu}). \tag{4.3}
\]

Expanding the determinant with the help of eq \((4.2)\) (using the symmetric matrix $\Pi_{\mu}^{\nu}$ in place of $S_{\mu}^{\nu}$), one finds all the vector Galileon combinations determined in \([19, 23]\) with fixed coefficients. These combinations couple the vector to the scalar preserving Galileon and gauge symmetries, and lead to equations of motion with at most two space-time derivatives. Combining various copies of the action \((4.3)\) with different choices of the coefficients in front of each of them, one can reproduce the Galileon and gauge symmetric vector-scalar combinations of \([23]\), with arbitrary coefficients in front of each of them.

We can go beyond the Galileon invariant case, and look for an action coupling scalars to vectors by means of derivative interactions breaking the Galileon symmetry, and that nevertheless maintain equations of motion with at most second space-time derivatives. An example of such an action is
\[
S = \int d^4 x \det (e^{7\pi} \delta^{\nu}_{\mu} + c_2 e^{7\pi} \Pi_{\mu}^{\nu} + c_3 e^{7\pi} \partial_{\mu} \partial^{\nu} \pi + c_4 e^{7\pi} \delta^{\nu}_{\mu} \partial_{\rho} \partial^{\nu} \pi + c_5 e^{7\pi} F_{\mu \nu}) \tag{4.4}
\]
for arbitrary parameters $c_i, \gamma_i$. By means of the tools explained in the previous sections, this action can be shown to lead to equations of motion with at most two space-time derivatives, although it breaks the Galileon symmetry (while maintaining the abelian gauge symmetry on the vectors).

A particularly interesting special case of \((4.4)\) is an action that preserves conformal symmetry. Under infinitesimal special conformal transformations, the field strength $F_{\mu \nu}$ transforms as (see e.g. \([24]\])
\[
e^{-2\pi} F_{\mu \nu} \rightarrow e^{-2\pi} F_{\mu \nu} + 2 e^{-2\pi} (A^{\lambda}_{\mu} F_{\lambda \nu} + A^{\lambda}_{\nu} F_{\lambda \mu}) \tag{4.5}
\]
with the antisymmetric matrix $A_{\mu}^{\nu}$ defined in eq. \((2.20)\). Using this fact, an action that preserves conformal invariance results
\[
S_{\text{conf}} = \int d^4 x e^{4\pi} \det \left[ \delta^{\nu}_{\mu} + c_1 e^{-2\pi} \left( \partial_{\mu} \partial^{\nu} \pi - \partial_{\mu} \partial^{\nu} \pi + \frac{1}{2} \delta^{\nu}_{\mu} \partial^{\rho} \partial^{\rho} \pi \right) + c_2 e^{-2\pi} F_{\mu \nu} \right] \tag{4.6}
\]
for arbitrary coefficients $c_1, c_2$. By means of equations \((2.3)\) and \((4.2)\) the determinant inside eq \((4.6)\) can be expanded in terms of traces. As for the case of the scalar Galileon, using the arguments developed in the previous sections it is straightforward to prove that each trace is invariant under infinitesimal special conformal transformations and dilations, hence the entire determinant is conformally invariant. The conformally invariant action can be expanded in a sum of several combinations weighted by different powers of $e^\pi$, each of them invariant under conformal and abelian symmetries: eq \((4.6)\) condenses in a compact expression these combinations. The previous conformal vector-scalar Lagrangian can also be straightforwardly extended to a multi-scalar version, for example adding inside the determinant a scalar degree of freedom $\sigma$ with a combination as the $C_{\mu}^{\nu}$ of eq. \((2.24)\) that preserves conformal symmetry by itself.

Hence, also the conformal version of Galileon couplings to vectors can also be elegantly described in a determinant form. It would be interesting to study whether the resulting system can improve on the instabilities of de Sitter configurations driven by vector fields in a Galileon set up, analyzed in \([23]\).
5 Outlook

In this work we discussed an elegant formulation of Galileon actions in terms of matrix determinants in four dimensions. This formulation allows one to straightforwardly determine derivative couplings between Galileons and scalar and vector degrees of freedom that lead to equations of motion with at most two space-time derivatives. We have shown how to use this method to build generalizations of Galileon set-ups preserving conformal symmetry, finding the first examples of couplings between Galileons and additional degrees of freedom able to maintain the Galileon conformal invariance.

Having a novel point of view to determine consistent derivative couplings of scalars to other degrees of freedom is interesting, because it allows one to generalize known results in new directions. Let us discuss some examples of future investigations.

Recently it has been shown that Galileon systems satisfy a duality relation \[25\] (see also \[26\]) associated with a field dependent coordinate transformation \(x^\mu \rightarrow x^\mu + \partial^\mu \pi\). Using this fact, it is straightforward to prove one of the results of section 2.1, namely that the action

\[S = \int d^4x \det (\delta_{\mu}^\nu + \partial_{\mu} \partial^\nu \pi)\]

is trivial since it leads to non-dynamical equations of motion for the scalar \(\pi\). This fact can be shown by expanding the determinant and showing that the various terms assemble in total derivatives, as explained in section 2.1. Or more simply, observing that the determinant inside the previous integral is nothing but the Jacobean of the coordinate transformation above, hence it is not surprise that this action corresponds to a non-dynamical theory. One can extend the analysis studying the effect of the previous coordinate transformation for dynamical cases using our determinantal approach, studying generalizations of the Galileon duality to the case of new couplings between Galileons and other degrees of freedom. We will develop this investigation elsewhere.

An important consequence of derivative self-couplings of scalar fields is the realization of screening mechanisms as the Vainsthein effect, in which non-linearities of the scalar action induce strong interactions in proximity of spherically symmetric sources that are able to hide the effects of light scalar fields. Our determinantal formulation of actions with derivative couplings render the analysis of spherically symmetric set-ups particularly straightforward, and allow one to generalize known results in various ways, that we will present in a separate work.

One main motivation for studying the conformally invariant actions considered in our work is the proposal pushed forward in \[3\], that we reviewed in the Introduction. Namely, the possibility to find infrared completions of Galileon systems that admit de Sitter configurations able to explain the current acceleration of our universe. In the single scalar Galileon case, as argued already in \[3\], such proposal has problems once the dynamics of gravity coupling to the conformal action is taken into account. The reason is that Weyl symmetry is able to gauge away the scalar, leaving with a system that does not admit interesting de Sitter configurations. The hope is to improve the situation adding new degrees of freedom coupling to the Galileon, that respect the Galilean conformal symmetry. When coupling to gravity, there is the possibility that while one of the scalars is gauged away, the action for the remaining degrees of freedom still respects the structure of a conformal Galileon set-up and is able to generate interesting de Sitter configurations. Conformal couplings of gravity to Galileons are not however easy to include in the elegant determinantal formulation adopted in this work. Articles as \[27\] offer useful ideas on this respect, that we leave for developments in a future work.

Acknowledgments

It is a pleasure to thank Kazuya Koyama, Gustavo Niz and Ivonne Zavala for comments on the draft. GT is supported by an STFC Advanced Fellowship ST/H005498/1.

References

[1] C. P. Burgess, “The Cosmological Constant Problem: Why it’s hard to get Dark Energy from Micro-physics,” arXiv:1309.4133 [hep-th].
[2] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, “Modified Gravity and Cosmology,” Phys. Rept. 513 (2012) 1 [arXiv:1106.2476 [astro-ph.CO]].

[3] A. Nicolis, R. Rattazzi and E. Trincherini, “The Galileon as a local modification of gravity,” Phys. Rev. D 79 (2009) 064036 [arXiv:0811.2197 [hep-th]].

[4] A. I. Vainshtein, “To the problem of nonvanishing gravitation mass,” Phys. Lett. B 39 (1972) 393.

[5] C. de Rham and A. J. Tolley, “DBI and the Galileon reunited,” JCAP 1005 (2010) 015 [arXiv:1003.5917 [hep-th]].

[6] K. Hinterbichler, M. Trodden and D. Wesley, “Multi-field Galileons and higher co-dimension branes,” Phys. Rev. D 82 (2010) 124018 [arXiv:1008.1305 [hep-th]].

[7] K. Van Acoleyen and J. Van Doorsselaere, “Galileons from Lovelock actions,” Phys. Rev. D 83 (2011) 084025 [arXiv:1102.0487 [gr-qc]].

[8] C. de Rham and G. Gabadadze, “Generalization of the Fierz-Pauli Action,” Phys. Rev. D 82 (2010) 044020 [arXiv:1007.0443 [hep-th]].

[9] C. de Rham, G. Gabadadze and A. J. Tolley, “Resummation of Massive Gravity,” Phys. Rev. Lett. 106 (2011) 231101 [arXiv:1011.1232 [hep-th]].

[10] S. Fubini, “A New Approach to Conformal Invariant Field Theories,” Nuovo Cim. A 34 (1976) 521.

[11] J. Khoury, J.-L. Lehners and B. A. Ovrut, “Supersymmetric Galileons,” Phys. Rev. D 84 (2011) 043521 [arXiv:1103.0003 [hep-th]].

[12] M. Koehn, J.-L. Lehners and B. Ovrut, “Supersymmetric Galileons Have Ghosts,” Phys. Rev. D 88 (2013) 023528 [arXiv:1302.0840 [hep-th]].

[13] F. Farakos, C. Germani and A. Kehagias, “Ghost-free Supersymmetric Galileons,” arXiv:1306.2961 [hep-th].

[14] D. Fairlie, “Comments on Galileons,” J. Phys. A 44 (2011) 305201 [arXiv:1102.1594 [hep-th]].

[15] T. L. Curtright and D. B. Fairlie, “A Galileon Primer,” arXiv:1212.6972 [hep-th].

[16] C. Deffayet, S. Deser and G. Esposito-Farese, “Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors,” Phys. Rev. D 80 (2009) 064015 [arXiv:0906.1967 [gr-qc]].

[17] K. Koyama, G. Niz and G. Tasinato, “Strong interactions and exact solutions in non-linear massive gravity,” Phys. Rev. D 84 (2011) 064033 [arXiv:1104.2143 [hep-th]].

[18] A. Padilla, P. M. Saffin and S.-Y. Zhou, “Bi-Galileon theory I: Motivation and formulation,” JHEP 1012 (2010) 031 [arXiv:1007.5424 [hep-th]].

[19] C. Deffayet, S. Deser and G. Esposito-Farese, “Arbitrary p-form Galileons,” Phys. Rev. D 82 (2010) 061501 [arXiv:1007.5278 [gr-qc]].

[20] K. Koyama, G. Niz and G. Tasinato, “The Self-Accelerating Universe with Vectors in Massive Gravity,” JHEP 1112 (2011) 065 [arXiv:1110.2618 [hep-th]].

[21] G. Gabadadze, K. Hinterbichler, D. Pirtskhalava and Y. Shang, “On the Potential for General Relativity and its Geometry,” arXiv:1307.2245 [hep-th].

[22] N. A. Ondo and A. J. Tolley, “Complete Decoupling Limit of Ghost-free Massive Gravity,” arXiv:1307.4769 [hep-th].
[23] G. Tasinato, K. Koyama and N. Khosravi, “The role of vector fields in modified gravity scenarios,” arXiv:1307.0077 [hep-th]; G. Tasinato, K. Koyama and G. Niz, “Exact Solutions in Massive Gravity,” Class. Quant. Grav. 30 (2013) 184002 [arXiv:1304.0601 [hep-th]]; G. Tasinato, K. Koyama and G. Niz, “Vector instabilities and self-acceleration in the decoupling limit of massive gravity,” Phys. Rev. D 87 (2013) 064029 [arXiv:1210.3627 [hep-th]].

[24] R. Jackiw and S.-Y. Pi, “Tutorial on Scale and Conformal Symmetries in Diverse Dimensions,” J. Phys. A 44 (2011) 223001 [arXiv:1101.4886 [math-ph]].

[25] C. de Rham, M. Fasiello and A. J. Tolley, “Galileon Duality,” arXiv:1308.2702 [hep-th].

[26] P. Creminelli, M. Serone and E. Trincherini, “Non-linear Representations of the Conformal Group and Mapping of Galileons,” arXiv:1306.2946 [hep-th].

[27] S. Deser and G. W. Gibbons, “Born-Infeld-Einstein actions?,” Class. Quant. Grav. 15 (1998) L35 [hep-th/9803049].