On the Symmetric Space $\sigma$-model Kinematics

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Abstract

The solvable Lie algebra parametrization of the symmetric spaces is discussed. Based on the solvable Lie algebra gauge two equivalent formulations of the symmetric space sigma model are studied. Their correspondence is established by inspecting the normalization conditions and deriving the field transformation laws.

1 Introduction

The symmetric space sigma model [1, 2, 3, 4, 5, 6, 7, 8] has a field content of scalars which parametrize a homogeneous coset manifold $G/K$ which is a Riemannian globally symmetric space for all the $G$-invariant Riemannian structures on it [9]. The theory is invariant under the parametrization preserving global (rigid) $G$-action from the right and the local $K$-action from the left. The scalars transform nonlinearly under these actions. In general the global symmetry group $G$ is a non-compact real form of a semi-simple Lie group and $K$ is its maximal compact subgroup. The formulation of the symmetric space sigma model is based on the Cartan-Maurer form which is induced by the local parametrization of the coset representatives of $G/K$ by the scalar fields of the theory. There are two equivalent formulations of
the symmetric space sigma model, the more conventional one which is also applicable to the cases where the scalar manifold is not a symmetric space is based on the decomposition of the Cartan-Maurer form which reveals the vielbein and the gauge connection identification of the scalar coset manifold $G/K$ \cite{1, 2, 10}. This formulation divides the Cartan-Maurer form into two parts; one is a coset generators valued one-form and the other takes values in the algebra of the maximal compact subgroup $K$. The second formulation of the symmetric space sigma model is based on the introduction of an internal metric \cite{3, 4}. When one uses the solvable Lie algebra parametrization \cite{8, 11} to generate the coset representatives the analysis of the theory has simplifications in both of the formulations \cite{5, 6, 7, 8}. In \cite{8} the Cartan-Maurer forms and the field equations of both of the formulations are studied when the solvable Lie algebra gauge is assumed.

In this work following the construction of the solvable Lie algebra gauge or the solvable Lie algebra parametrization which has an essential role in our analysis we will study the lagrangians of both of the formulations in the solvable Lie algebra gauge to find a correspondence between them. We will inspect the normalization conditions then we will derive the transformation laws between the field contents of both of the formulations by assuming a normalization scheme. We will also mention about the relation between the two formulations when the model is coupled to other fields.

In section two we will introduce the solvable Lie algebra gauge to parameterize the coset manifold $G/K$ of the symmetric space sigma model. By using the solvable Lie algebra parametrization we will derive the lagrangian of the symmetric space sigma model explicitly for both of the above mentioned formulations in section three. We will also compare the normalization conditions and we will obtain the transformation laws between the field contents of the two formulations by choosing a normalization convention.

## 2 Symmetric Spaces and the Solvable Lie Algebra Parametrization

The symmetric space sigma models are based on homogeneous manifolds \cite{12}, the homogeneity is in the sense that there exists a transitive action of a Lie group $G$ on these manifolds. These homogeneous spaces are in the form of a coset manifold $G/K$ where $G$ is in general a non-compact real form of
any other semi-simple Lie group and $K$ is a maximal compact subgroup of $G$. When $G$ is a compact real form its maximal compact subgroup $K$ is $G$ itself thus the coset space $G/G$ is a single point and the corresponding sigma model is an empty set. The numerator group may as well be a split real form (maximally non-compact) of a semi-simple Lie group.

The Lie algebra $k_0$ of the analytical subgroup $K$ is a subalgebra of the semi-simple Lie algebra $g_0$ which is the Lie algebra of $G$. The Lie algebra $k_0$ is a maximal compactly imbedded Lie subalgebra of $g_0$ therefore it is an element of a Cartan decomposition of $g_0$ \[ g_0 = k_0 \oplus p_0, \] (2.1)

which is a vector space direct sum of the Lie subalgebra $k_0$ and a vector subspace $p_0$. Since $G$ is a linear analytical Lie group the corresponding Lie algebra $g_0$ is non-compact for both of the cases when $G$ is a non-compact or a split real form (although we consider it separately we should not forget that it is a limiting non-compact case). The map $u_0 : k + p \longrightarrow k - p$, for all $k \in k_0$ and $p \in p_0$ is the Cartan involution which generates (2.1). The pair $(g_0, u_0)$ is an orthogonal symmetric Lie algebra of the non-compact type \[9\]. Thus we conclude that $(G, K)$ is associated with $(g_0, u_0)$ and it is of non-compact type too. The Cartan decomposition (2.1) is the eigenspace decomposition of $u_0$ where the elements of $k_0$ have $+1$ eigenvalues and the elements of $p_0$ have $-1$ eigenvalues under the involution $u_0$. We also know that $(G, K)$ is a Riemannian symmetric pair therefore the coset space $G/K$ has a unique analytical structure induced by the quotient topology of $G$. The scalar manifold $G/K$ is a Riemannian globally symmetric space for all the $G$-invariant Riemannian structures on $G/K$\[9\]. The crucial consequence of this identification is that the exponential map $Exp : g_0 \longrightarrow G$ induces a diffeomorphism

\[ Exp : p_0 \longrightarrow G/K, \] (2.2)

from the $\mathbb{R}^{\text{dim}p_0}$ manifold $p_0$ onto $G/K$ since it maps the elements of $p_0$ onto the representatives of the left cosets $G/K$ \[9\]. This result will enable us to define a parametrization of the scalar manifold $G/K$ on which the sigma model will be constructed in the next section.

Furthermore one may use the Iwasawa decomposition of $g_0$ which is built

\[ ^1 \text{This is the origin of the name; symmetric space sigma model.} \]
on the Cartan decomposition \[^{2}\text{2}\] and one may make use of the root space decomposition basis of \(g_0\) to parametrize the scalar coset manifold. The Iwasawa decomposition reads
\[
g_0 = k_0 \oplus s_0
\]
\[
= k_0 \oplus h_{p_0} \oplus n_0,
\]
where \(k_0\) is the Lie algebra of \(K\) and the algebra direct sum \(s_0 = h_{p_0} \oplus n_0\) is a solvable Lie subalgebra of \(g_0\) which is isomorphic to the vector space \(p_0\).

In (2.3) \(h_{p_0}\) is generated by \(r\) non-compact Cartan generators \(\{H_i\}\). Also the nilpotent Lie subalgebra \(n_0\) in (2.3) is generated by a subset \(\{E_\beta\}\) of the positive root generators of \(g_0\) where \(\beta \in \Delta^+_\text{nc}\). The roots in \(\Delta^+_\text{nc}\) are the non-compact roots with respect to a Cartan involution \(\theta\) which is composed of the conjugation induced by the Cartan decomposition in (2.3) and the conjugation of \(g_0\) via its complexification \([9]\). The Cartan subalgebra \(h_0\) generates an abelian subgroup in \(G\) which is called the torus. Although we call it torus it is not the ordinary torus topologically in fact it has the topology \((S^1)^m \times \mathbb{R}^n\) for some \(m\) and \(n\) and if it is diagonalizable in \(\mathbb{R}\) (such that \(m = 0\)) then it is called an \(R\)-split torus. These definitions can be generalized for the subalgebras of \(h_0\) as well. The subspace of \(G\) which is generated by \(h_{p_0}\) is the maximal \(R\)-split torus in \(G\) in the sense defined above and its dimension is called the \(R\)-rank which we will denote by \(r\). If \(r\) is maximal such that \(r = l\) where \(l\) is the rank of \(G\) \((l = \dim h_0)\), which also means that \(h_{p_0} = h_0\) then the Lie group \(G\) is said to be in split real form (maximally non-compact). In this case \(h_{p_0} = h_0\) is generated by all the Cartan generators \(\{H_i\}\) and \(\Delta^+_\text{nc} = \Delta^+\) so that the generators \(\{E_\beta\}\) of \(n_0\) correspond to the entire set of positive roots. Thus the solvable Lie subalgebra \(s_0\) coincides with the Borel subalgebra which is generated by the entire Cartan and the positive root generators of \(g_0\) for the split real form case. If on the other hand \(r\) is minimal such that \(r = 0\) then \(G\) is a compact real form. All the other cases in between are called non-compact semi-simple real forms.

For the non-compact real form \(G\) if we consider the Iwasawa decomposition and in (2.2) if we use the basis \(\{H_i, E_\beta\}\) which generates the solvable

\[^2\text{In this respect the Iwasawa decomposition is not a Cartan decomposition but it introduces a solvable Lie algebra of }g_0\text{ which is isomorphic to }p_0\text{ thus which generates another parametrization of }G/K\text{ via (2.2) [9].}\]
Lie subalgebra $s_0$ then we have the parametrization

$$\text{Exp} : \sum \mathbb{R}\{H_i, E_\beta\} \longrightarrow G/K. \quad (2.4)$$

Equation (2.4) is called the solvable Lie algebra parametrization or the solvable Lie algebra gauge of the symmetric space $G/K\ [11]$. On the other hand when $g$ is a split real form (maximally non-compact) if we use the Borel subalgebra basis which is made up of the entire set of the Cartan generators and the positive root generators; then (2.4) is called the Borel parametrization or the Borel gauge of $G/K$.

In summary in this section we have obtained a legitimate parametrization of the symmetric space $G/K$ by using the solvable Lie subalgebra $s_0$ of $g_0$. If we use the notation $\{T_m\}$ for the basis vectors $\{H_j, E_\beta \mid j = 1, \ldots, r ; \beta \in \Delta_{nc}^+\}$ of $s_0$ and if $\{\varphi^m(x)\}$ are $C^\infty$-maps over the $D$-dimensional spacetime then the map

$$\nu(x) = e^{\varphi^m(x)T_m}, \quad (2.5)$$

is an onto $C^\infty$-map from the $D$-dimensional spacetime to the Riemannian globally symmetric space $G/K$. The gauge map, (2.5) which depends on the scalar functions $\{\varphi^m(x)\}$ is the building block in the construction of the symmetric space sigma model.

## 3 Normalization Conditions and the Duality Transformations of the SSSM

In this section we will obtain the field transformations of the two equivalent formulations of the symmetric space sigma model (SSSM) which are based on the solvable Lie algebra parametrization introduced in the last section. We will show that when one assumes a normalization convention relating the matrix representations of the basis that is used in (2.5) of the two separate formulations one can find a correspondence between the sets of field definitions of the two distinct constructions.

In order to construct the symmetric space sigma model we first consider the set of $G$-valued maps $\nu(x)$. They transform onto each other as $\nu \rightarrow k(x)\nu g, \forall g \in G, k(x) \in K$ for some subgroup $K$ of $G$. We will assume that the map $\nu(x)$ corresponds to a parametrization of the coset $G/K$ (for convenience we will consider the left cosets). Thus as mentioned before we
assume that the map $\nu(x)$ is from the $D$-dimensional spacetime into the group $G$ and its range is composed of the representatives of the left cosets of $G/K$. Moreover if $G$ is a non-compact real form of a semi-simple Lie group and $K$ is a maximal compact subgroup of $G$ then $G/K$ becomes a symmetric space and $\nu(x)$ can be taken as the map (2.5) such that $\nu(x) = e^{\phi(x)T_m}$ by using the Cartan and the Iwasawa decompositions as we have mentioned in the last section. In this case the transformation rule $\nu \rightarrow k(x)\nu g$, $\forall g \in G$, $k(x) \in K$ which we assign on $\nu(x)$ preserves the gauge based on the Iwasawa decomposition.

A lagrangian which is invariant under the transformations discussed above can be given as

$$\mathcal{L}_1 = \frac{c}{4} \text{tr}(\star d\mathcal{M}^{-1} \wedge d\mathcal{M}),$$

(3.1)

where the internal metric $\mathcal{M}$ is defined as $\mathcal{M} = \nu^\# \nu$ and $c$ is a constant which may arise when the symmetric space sigma model is coupled to other fields. Here $\#$ is the generalized transpose over the Lie group $G$ such that $(\exp(g))^\# = \exp(g^\#)$. It is induced by the Cartan involution $\theta$ over $g_0$ ($g^\# = -\theta(g) \ \forall g \in g_0$) [9]. As mentioned in [5] it is possible to find a matrix representation of the Lie algebra $g_0$ in which $\#$ coincides with the matrix transpose operator. For this reason one can define an induced $\#$ map over the group $G$ as $(\exp(g))^\# = \exp(g^\#)$. If the subgroup of $G$ generated by the compact generators is an orthogonal group then in the fundamental representation of $g_0$ the generators can be chosen such that $g^\# = g^T$ for $g \in g_0$. Also if the subgroup of $G$ generated by the compact generators is a unitary group then in the fundamental representation $g^\# = g^\dagger$ for $g \in g_0$. In our formulation we will assume that we choose a representation in which $g^\# = g^T$. The lagrangian (3.1) can be expressed in terms of the pullback of the Cartan-Maurer form

$$\mathcal{G} = d\nu \nu^{-1},$$

(3.2)

as follows

$$\mathcal{L}_1 = \frac{c}{4} \text{tr}(\star d\mathcal{M}^{-1} \wedge d\mathcal{M})$$

$$= \frac{c}{4} \text{tr}(- \star d\nu \nu^{-1} \wedge (d\nu \nu^{-1})^\# + \star d\nu^{-1} \nu \wedge \nu^{-1} d\nu$$

$$+ \star d(\nu^\#)^{-1} \nu^\# \wedge (\nu^\#)^{-1} d\nu^\# + (\star d\nu^{-1} \wedge (\nu^{-1})^\# d\nu^\#)^\#)$$
\[ = \frac{c}{4} \text{tr}(-*d\nu^{-1} \wedge (d\nu^{-1})^# - *d\nu^{-1} \wedge d\nu^{-1} + *\nu d\nu^{-1} \wedge d\nu^{-1} - *d\nu^{-1} \wedge (d\nu^{-1})^#) \]

\[ = -\frac{c}{2} \text{tr}(\star \mathcal{G} \wedge \mathcal{G}^\# + \star \mathcal{G} \wedge \mathcal{G}). \quad (3.3) \]

In the above derivation we have made use of the identities

\[ \nu^{-1} d\nu = -d\nu^{-1} \nu, \quad d\nu^{-1} = -\nu d\nu^{-1}. \quad (3.4) \]

Also by bearing in mind the definition of \( \nu \) we have used the properties of the transpose operation such as

\[ (\nu^{-1})^# = (\nu^#)^{-1}, \quad \text{tr}(\nu^#) = \text{tr}(\nu), \quad (\nu_1 \nu_2)^# = \nu_2^# \nu_1^#, \quad (\nu^#)^# = \nu. \quad (3.5) \]

Cyclic permutations are always permissible under the trace operator however one must be careful about the fact that if \( \nu_1 \) and \( \nu_2 \) are matrix valued functions under the representation chosen then

\[ \text{tr}(d\nu_1 \wedge *d\nu_2) = (-1)^{(D-1)} \text{tr}(*d\nu_2 \wedge d\nu_1), \quad (3.6) \]

where \( D \) is the dimension of the spacetime. In the derivation of (3.3) we have also used that \( (d\nu)^# = d(\nu)^# \). Now we will explicitly calculate \( \mathcal{G} \). By using the matrix identity

\[ de^C e^{-C} = dC + \frac{1}{2!}[C, dC] + \frac{1}{3!}[C, [C, dC]] + ...., \quad (3.7) \]

we obtain

\[ \mathcal{G} = de^{\varphi^i T_i} e^{-\varphi^i T_i} \]

\[ = d\varphi^i T_i + \frac{1}{2!}[\varphi^j T_i, d\varphi^j T_j] + \frac{1}{3!}[\varphi^j T_i, [\varphi^j T_j, d\varphi^k T_k]] + .... \]

\[ = d\varphi^i T_i + \frac{1}{2!}\varphi^i d\varphi^j C^k_{ij} T_k + \frac{1}{3!}\varphi^i \varphi^j d\varphi^k C^l_{jk} C^i_{il} T_l + .... \]

\[ = \hat{T} \Delta \hat{d}\varphi. \]

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We have defined
\[ [T_m, T_n] = C^k_{mn} T_k. \] (3.9)
Also the components of the row vector \( \mathbf{T} \) are \( T_i = H_i \) for \( i = 1, ..., r \) and \( T_{\alpha + r} = E_\alpha \) for \( \alpha = 1, ..., \text{dim}n_0 \). \( \mathbf{d} \phi \) is a column vector of the field strengths \( \{ d\phi^m \} \) and the \( \text{dim}n_0 \times \text{dim}n_0 \) matrix \( \Delta \) can be given as
\[
\Delta = \sum_{n=0}^{\infty} \frac{M^n}{(n + 1)!}
\] (3.10)
\[ = (e^M - I)M^{-1}, \]
where \( M^n_m = \varphi^l C^m_{lm} \). If we insert the explicit form of \( G \) calculated in (3.8) in the lagrangian (3.3) we obtain
\[
\mathcal{L}_1 = -\frac{c}{2} \Delta_i^n \ast d\phi^i \wedge \Delta^m_k d\phi^k (tr(T_n T_m^\#) + tr(T_n T_m)). \] (3.11)
If we assume that the scalar maps when coupled to the generators in (2.5) generate ranges in a sufficiently small neighborhood around the identity element of \( g_0 \) then we can introduce another parametrization of the coset elements as a product of two exponential maps
\[
\nu(x) = e^{\frac{1}{2} \phi^i(x) H_i} e^{\chi^\beta(x) E_\beta}, \] (3.12)
where the fields \( \{ \phi^i \} \) are called the dilatons and \( \{ \chi^\beta \} \) are called the axions. The maps in (3.12) and (2.5) can be chosen to be equal to each other if the locality condition of the range is assumed as discussed above [9, 12, 13, 14, 15].

In [7, 8] the Cartan form \( G \) is calculated in terms of the fields \( \{ \phi^i, \chi^\beta \} \) as
\[
G = \frac{1}{2} d\phi^i H_i + \mathbf{E'} \mathbf{\Omega} \mathbf{d}\chi, \] (3.13)
where we have defined the row vector \( \mathbf{(E')}_\alpha = e^{\frac{1}{2} \alpha^i \phi^i} E_\alpha \) and \( \mathbf{d}\chi \) is a column vector of the field strengths of the axions \( \{ d\chi^\beta \} \). Also \( \mathbf{\Omega} \) is a \( \text{dim}n_0 \times \text{dim}n_0 \) matrix
\[
\mathbf{\Omega} = \sum_{n=0}^{\infty} \frac{\omega^n}{(n + 1)!}
\] (3.14)
\[ = (e^\omega - I)\omega^{-1}. \]
The matrix $\omega$ is defined as $\omega^\gamma_{\alpha\beta} = \chi^\alpha K^\gamma_{\alpha\beta}$ where we define the structure constants $K^\gamma_{\alpha\beta}$ and the root vector components $\alpha_i$ as

$$[E_\alpha, E_\beta] = K^\gamma_{\alpha\beta} E_\gamma, \quad [H_i, E_\alpha] = \alpha_i E_\alpha. \quad (3.15)$$

By using (3.13) in (3.3) we can express the lagrangian $L_1$ in terms of the fields $\{\phi^i, \chi^\beta\}$ as

$$L_1 = -\frac{c}{8} \ast d\phi^i \wedge d\phi^j (tr(H_i H_j^\#) + tr(H_i H_j))$$

$$-\frac{c}{4} \ast d\phi^i \wedge e^{\frac{i}{2}\alpha_i \phi^i} F^\alpha (tr(H_i E_\alpha^\#) + tr(E_\alpha H_i^\#) + tr(H_i E_\alpha) + tr(E_\alpha H_i))$$

$$-\frac{c}{2} e^{\frac{i}{2}\alpha_i \phi^i} F^\alpha \wedge e^{\frac{i}{2} \beta \phi^i} F^\beta (tr(E_\alpha E_\beta^\#) + tr(E_\alpha E_\beta)), \quad (3.16)$$

where we have defined

$$F^\alpha = \Omega^\alpha_\beta d\chi^\beta. \quad (3.17)$$

If we compare the lagrangian we have obtained in (3.16) with the one

$$L = -\frac{1}{2} \sum_{i=1}^r \ast d\phi^i \wedge d\phi^i - \frac{1}{2} \sum_{\alpha \in \Delta^+} e^{\alpha \phi^i} \ast F^\alpha \wedge F^\alpha, \quad (3.18)$$

which is given in [5, 7] and whose equations of motion are also derived therein we find that the normalization conditions for the matrix representatives of the basis elements $\{H_i, E_\alpha\}$ must be in the form

$$tr(H_i H_j^\#) + tr(H_i H_j) = \frac{4c}{3} \delta_{ij},$$

$$tr(H_i E_\alpha^\#) + tr(E_\alpha H_i^\#) + tr(H_i E_\alpha) + tr(E_\alpha H_i) = 0,$$

$$tr(E_\alpha E_\beta^\#) + tr(E_\alpha E_\beta) = \frac{1}{c} \delta_{\alpha\beta}. \quad (3.19)$$

As already given in [8] if we compare (3.13) with (3.8) we can find the trans-
formation relations between the field strengths \( \{d\varphi^m\} \) and \( \{d\phi^i, d\chi^\beta\} \) as

\[
\Delta^i_m d\varphi^m = \frac{1}{2} d\phi^i,
\]

\[
\Delta^{\beta + r}_m d\varphi^m = e^{\frac{1}{2} \beta \phi^j} \Omega^{\beta}_\rho d\chi^\rho,
\]

where the indices above are \( i, j = 1, \ldots, r; \beta, \rho = 1, \ldots, \dim n_0; m = 1, \ldots, \dim s_0 = r + \dim n_0 \) and we should remark that we enumerate the roots \( \beta \in \Delta^+_0 \).

As we have discussed in the introduction there exists another formulation of the symmetric space sigma model. By using another set of fields \( \{\varphi'^m\} \) let us consider the coset element

\[
\nu'(x) = e^{\varphi'^m(x) T_m}.
\]

We can define the Cartan form

\[
\mathcal{G}' = \nu'^{-1} d\nu' = P + Q,
\]

where

\[
P = P^m T_m , \quad Q = Q^n K_n,
\]

\( \{K_n\} \) being the generators of \( k_0 \) which is an element of the Cartan decomposition of \( g_0 \) in (2.1). In general in the gauge \( (3.21) \) the coset generators do not have to form up a subalgebra. Thus the Cartan form \( (3.22) \) can have components both in \( P \) and \( Q \) directions since the algebra product of the coset generators may result in the set \( \{K_n\} \). However when we take the coset generators \( \{T_m\} \) to be the generators of the solvable Lie algebra \( s_0 \) of the Iwasawa decomposition \( (2.3) \) we have a simplification. As we will explicitly calculate in the following lines in this case we have

\[
Q = Q^n K_n = 0 \quad \text{and} \quad \mathcal{G}' = P.
\]

This is owing to the fact that the coset generators being the generators of the solvable Lie algebra in this special gauge are not mapped on the generators \( \{K_n\} \) under the algebra product. If we assume a representation in which the generators \( \{K_n\} \) and \( \{T_m\} \) of the Iwasawa decomposition \( (2.3) \) are orthogonal namely

\[
tr(k_0 s_0) = 0,
\]
then the Iwasawa decomposition \((2.3)\) satisfies the requirements of the decomposition needed to build up the equivalent vielbein formulation of the symmetric space sigma model which is invariant under the global \(G\) and the local \(K\) transformations thus an invariant lagrangian can be given as \([1, 2]\)

\[
\mathcal{L}_2 = c' \text{tr}(\ast P \wedge P). \tag{3.26}
\]

We may explicitly calculate this lagrangian by calculating the Cartan form \((3.22)\). By using the matrix identity

\[
e^{-C} d e^C = dC - \frac{1}{2!} [C, dC] + \frac{1}{3!} [C, [C, dC]] - ...., \tag{3.27}
\]

one can calculate \(G'\) as

\[
G' = e^{-\varphi'^T_i d \varphi'^T_i}.
\]

\[
= d \varphi'^T_i - \frac{1}{2!} [\varphi'^T_i, d \varphi'^T_j] + \frac{1}{3!} [\varphi'^T_i, [\varphi'^T_j, d \varphi'^T_k]] - .... \tag{3.28}
\]

\[
= d \varphi'^T_i - \frac{1}{2!} \varphi'^T_i d \varphi'^T_j C^k_{ij} T_k + \frac{1}{3!} \varphi'^T_i \varphi'^T_j d \varphi'^T_k C^l_{jk} C^l_{il} T_i - ....
\]

\[
= \mathbf{T} \mathbf{W} d \varphi'.
\]

We have defined the \(\text{dim}_{s_0} \times \text{dim}_{s_0}\) matrix \(\mathbf{W}\) as

\[
\mathbf{W} = \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n + 1)!}
\]

\[
= (I - e^{-M'}) M'^{-1}, \tag{3.29}
\]

where \(M^n_m = \varphi'^T_i C^m_{in}\). We can now write the lagrangian \((3.26)\) as

\[
\mathcal{L}_2 = c' W^n_i d \varphi'^T_i \wedge W^m_k d \varphi'^T_k \text{tr}(T_n T_m). \tag{3.30}
\]

\(^{3}\)Even if a representation which enables \((3.25)\) does not exist the transformation between the fields which we will derive below justifies the definition of the lagrangian \((3.26)\). Therefore the introduction of \((3.26)\) can legitimately be considered as a redefinition of fields.
As discussed before one can also express the coset map $\nu'$ in terms of the dilatons $\phi'^i$ and the axions $\chi'^\alpha$ as

$$
\nu'(x) = e^{\frac{1}{2}\phi'^i(x)H_i}e^{\chi'^\alpha(x)E_{\beta}}.
$$

(3.31)

In [8] in terms of the fields $\{\phi'^i, \chi'^\alpha\}$ the Cartan form $G'$ is calculated as

$$
G' = \frac{1}{2}d\phi'^iH_i + \vec{E} \Sigma (\vec{U} + d\chi'),
$$

(3.32)

where the components of the row vector $(\vec{E})_\alpha$ are $E_\alpha$ and

$$
U^\alpha = \frac{1}{2}\chi'^\alpha \alpha_i d\phi^i.
$$

(3.33)

Also the $\text{dim}n_0 \times \text{dim}n_0$ matrix $\Sigma$ is

$$
\Sigma = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^m}{(n+1)!},
$$

(3.34)

with $\omega^m_\beta = \chi'^\alpha K^\alpha_{\beta \alpha}$. Now if we compare (3.28) with (3.32) we find the transformation laws between the fields $\{\varphi'^m\}$ and $\{\phi'^i, \chi'^\beta\}$ as

$$
W^i_m d\varphi'^m = \frac{1}{2} d\phi'^i,
$$

(3.35)

$$
W^\alpha+r_m d\varphi'^m = \Sigma^\alpha_\beta(U^\beta + d\chi'^\beta),
$$

where $i = 1, ..., r; \alpha, \beta = 1, ..., \text{dim}n_0; \text{and } m = 1, ..., \text{dim}s_0 = r + \text{dim}n_0$.

Now under a special set of normalization conditions we may find the field transformation laws between the primed and the unprimed fields. To find a straightforward transformation rule after inspecting the two lagrangians in (3.11) and (3.30) we conclude that the normalization conventions in the first (unprimed) formulation and the second (primed) formulation should be chosen such that

$$
(tr(T_nT_m))_2 = -\frac{c}{2c'}(tr(T_nT_m) + tr(T_nT^\#_m))_1,
$$

(3.36)

where the subscripts in $(tr(\ ))_1$ and $(tr(\ ))_2$ correspond to the separate trace conventions for the matrix representatives we choose for the two formulations.
Under the above normalization conventions the comparison of (3.11) with (3.30) now denotes that
\[ L_1 = L_2, \] (3.37)
if we identify
\[ \varphi^m = -\varphi'^m. \] (3.38)
This is obvious since if (3.38) is chosen then
\[ \mathcal{G} = -\mathcal{G}' \quad \text{and} \quad W = \Delta. \] (3.39)
Therefore from (3.20) we have
\[ -\Delta^i_m d\varphi^m = \frac{1}{2} d\phi^i, \] (3.40)
\[ -\Delta^\beta_+^r m d\varphi^m = e^{\frac{1}{2} \beta_j \phi^j} \Omega_\alpha^\beta d\chi_\alpha. \]
Finally comparing this result with (3.35) and using (3.39) leads us to the transformation relations between \( \{\phi^i, \chi^\alpha\} \) and \( \{\phi'^i, \chi'^\alpha\} \) which we wish to find
\[ d\phi^i = -d\phi'^i, \] (3.41)
\[ e^{\frac{1}{2} \beta_j \phi^j} \Omega_\alpha^\beta d\chi_\alpha = -\Sigma^\beta_\rho (U^\rho + d\chi'^\rho). \]
Before concluding as a final remark we will mention about the relevance of the two formulations when the symmetric space sigma model is coupled to other fields \( F^i = dA^i \). The interaction lagrangian in this case can be given as \( [3, 4, 6, 16, 17] \)
\[ L_m = -\frac{c_m}{2} \mathcal{M}^i_k F^k \wedge \ast F^i \] (3.42)
\[ = -\frac{c_m}{2} F \wedge \mathcal{M}' \ast F, \]
where we define
\[ \mathcal{M}' = \nu' \# \nu' \ast \nu'. \] (3.43)
The total lagrangian in the second formulation of the symmetric space sigma model then becomes
\[ L'_{\text{coup}} = c' \text{tr}(\ast P \wedge P) - \frac{c_m}{2} F \wedge \mathcal{M}' \ast F. \] (3.44)
As we have mentioned before if the subgroup of $G$ generated by the compact generators is an orthogonal group then in the fundamental representation of $g_0$ the generators can be chosen such that $g^\# = g^T$ for $g \in g_0$. The orthogonal global symmetry groups obey this property thus if $G = O(m, n)$ then in the fundamental representation we can take $(\nu')^\# = (\nu')^T$. One additional condition can be introduced for the coset elements $\nu'$ and $\nu$. As it can be explicitly seen in [18, 19] and also effectively used in [20, 21, 22] when $G = O(m, n)$ one can choose a basis in which the coset representatives are locally represented by symmetric matrices so that we can assume locally

$$\nu'^T = \nu', \quad \nu^T = \nu.$$  \hspace{1cm} (3.45)

When we choose our representation as above under the transformation law (3.38) we have

$$\mathcal{M} = \eta \mathcal{M}' \eta,$$ \hspace{1cm} (3.46)

where $\eta$ is the indefinite signature metric which takes part in the definition of the orthogonal group $G = O(m, n)$. In general $\eta$ is a symmetric matrix which has $m$ positive and $n$ negative eigenvalues. If $A \in O(m, n)$ then

$$A^T \eta A = \eta.$$ \hspace{1cm} (3.47)

In writing (3.46) we have also used the fact that $\eta^{-1} = \eta$. Now if one assumes the transformation law (3.38) one can use the lagrangian

$$\mathcal{L}_{coup} = \frac{c}{4} tr(*d\mathcal{M}^{-1} \wedge d\mathcal{M}) - \frac{c_m}{2} F'^\wedge \mathcal{M} \wedge F',$$ \hspace{1cm} (3.48)

instead of (3.44) which becomes equal to the former under the prescribed duality transformations. In (3.48) we have also performed the transformation

$$F' = \eta F,$$ \hspace{1cm} (3.49)

on the gauge field strengths assuming the usual dimensional matching between the number of the gauge fields and the dimension of the representation chosen for the coset representatives which is the dimension of the fundamental representation of $O(m, n)$. When one performs the above-mentioned replacement for the lagrangians one should inspect the coexistence of the normalization conventions thus the transformation laws we have derived earlier with the local symmetry conditions (3.45) of the representation chosen.
4 Conclusion

Since our analysis is based on the solvable Lie algebra gauge we have started with a discussion about how one can construct this gauge for the symmetric spaces. Then by assuming the solvable Lie algebra parametrization of the scalar coset, for two equivalent formulations of the symmetric space sigma model we have derived the lagrangians explicitly. We have compared the normalization conditions and obtained the duality transformation laws for the field contents of these separate formulations. Thus in this work we have not only studied the two equivalent formulations of the symmetric space sigma model in detail but we have also obtained the correspondence of their field contents by deriving the transformation laws between them. In addition we have extended the duality transformations to include the matter fields when the symmetric space sigma model has matter couplings.

The majority of the scalar sectors of the supergravity theories with or without matter multiplets are formulated by symmetric space sigma models \[23\]. However the scalar lagrangians are generally expressed in the second formulation mentioned in section three. Therefore one needs to find out the transformation laws between the field contents to express the supersymmetry transformation laws if the first formulation is used. By prescribing a relation between the normalization conditions of the two constructions we have obtained the duality transformation laws. Since the supersymmetry requires certain coefficients for the scalar lagrangians when they are coupled to other fields we have taken general coefficients in our derivation. Our formulation also establishes the transformation rules between two separate parametrizations of the coset elements in either of the constructions.

Although we have assumed a normalization convention for deriving the field transformations we have not questioned the explicit representations which may obey these normalization conditions. This issue can separately be studied in general terms or for specific examples of the supergravity theories. In constructing the representations which obey the normalization convention that we have introduced one has the degrees of freedom which the solvable Lie algebra parametrization provides. This is due to the fact that our analysis is performed in a general and representation free formalism from the algebraic point of view. In this respect one has the freedom of choosing the Cartan decomposition, the root space decomposition, the Cartan subalgebra, the solvable Lie algebra, the basis, also the representation used for the Lie algebra of the global symmetry group \(G\). Another comparison can also be
separately studied which relates the solvable Lie algebra gauge field content and the most general coset parametrization whose Cartan-Maurer form has also a piece in the $K$ generators.

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