Morse-Novikov cohomology of locally conformally Kähler manifolds

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(\(M, J\)) complex manifold, \(\dim_{\mathbb{C}} M \geq 2\), connected.

\((M, J)\) is LCK if it admits a Kähler covering

\[ \Gamma \to (\tilde{M}, J, \Omega) \to (M, J) \]

such that \(\Gamma\) acts by holomorphic homotheties.

Equivalent definition:
\((M, J)\) admits a Hermitian metric \(\omega\) on \(M\) such that

\[ d\omega = \theta \wedge \omega, \quad d\theta = 0 \]

\(\theta\) is called the *Lee form*.
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The weight bundle

- Real line bundle $L_{\mathbb{R}} \longrightarrow M$ associated to the representation

$$\text{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}.$$ 

- The Lee form induces a connection $\nabla = d - \theta$ in $L_{\mathbb{R}}$.

- $\nabla$ is associated to the Weyl covariant derivative determined on $M$ by the LCK metric and the Lee form.

- the Weyl covariant derivative is uniquely defined by the properties $\nabla J = 0$, $\nabla g = \theta \otimes g$; in this context, $\theta$ is called the Higgs field.

- As $d\theta = 0$, then $\nabla^2 = d\theta = 0$, and hence $L_{\mathbb{R}}$ is flat.
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Let $L = L_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$.

- The Weyl connection extends naturally to $L$.
- Its $(0, 1)$-part endows $L$ with a holomorphic structure.
- As $L$ is flat, one can pick a nowhere degenerate section $\lambda$ satisfying
  \[ \nabla(\lambda) = \lambda \otimes (-\theta). \]

Hence, one chooses a Hermitian structure on $L$ such that $|\lambda| = 1$ and considers the associated Chern connection.

- The curvature of the Chern connection on $L$ with respect to the above holomorphic and Hermitian structure is $-2\sqrt{-1}d^c\theta$.
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Vaisman manifolds

- LCK + $\nabla^g \theta = 0$.

- Properties:
  - $\theta^\sharp$ is Killing and real holomorphic ($L_{\theta^\sharp} J = 0$).
  - Conversely (Kamishima, O): A compact LCK manifold admits a LCK metric with parallel Lee form if its Lie group of holomorphic conformalities has a complex one-dimensional Lie subgroup, acting non-isometrically on its Kähler covering.
  - If $F := \{\theta^\sharp, J\theta^\sharp\}$ has compact leaves, then $M/F$ is Kähler orbifold.
  - If $\theta^\sharp$ has compact orbits, then $M/\theta^\sharp$ is Sasakian orbifold.
  - $\|\theta^\sharp\|^2$ is a potential for the Kähler form of the universal cover.
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Structure Theorem

- The monodromy of $L$ is $\mathbb{Z}$.
- Compact Vaisman manifolds are suspensions over $S^1$ with Sasakian fibre:
  - $M$ is a metric cone $N \times \mathbb{R}$.
  - $N$ is Sasaki.
  - $\Gamma$ is $\mathbb{Z}$ generated by $(x, t) \mapsto (\lambda(x), t + q)$ for some $\lambda \in \text{Aut}(N)$, $q \in \mathbb{R}$. 
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- The class of compact LCK manifolds with potential is stable to small deformations.
  - Hence: the Hopf manifold \((\mathbb{C}^n \setminus 0)/\Gamma\), with \(\Gamma\) cyclic group generated by a non-diagonal linear operator, is LCK with potential. This is a generalization of the (non–Vaisman) rank 0 Hopf surface.

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Associated to the operator $d - \theta$. Since $d\theta = 0$, $d_{\theta}^2 = 0$. Denote it $H_{\theta}^\ast(M)$.

- Some call it Lichnerowicz–Poisson (in Poisson and Jacobi geometry).

Clearly $d_{\theta}\omega = 0$. 
$[\omega] \in H_{\theta}^2(M)$ is called the Morse–Novikov class.

- Analogue of the Kähler class.

- The cohomology of the local system $L$ is naturally identified with the cohomology of the Morse–Novikov complex $(\Lambda^\ast(M), d_{\theta})$ (Novikov).
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The cohomology of the local system $L$ is naturally identified with the cohomology of the Morse–Novikov complex $(\Lambda^*(M), d_\theta)$ (Novikov).
Morse–Novikov cohomology of compact Vaisman manifolds is trivial.

- Follows from the Structure theorem.
- Previously proven for locally conformally symplectic manifolds which admit a compatible metric for which the Lee form is parallel (de Leon, Lopez, Marrero, Padron).

More generally: on compact Vaisman manifolds, the Morse–Novikov class of any LCK form vanishes. Precisely:
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More generally: on compact Vaisman manifolds, the Morse–Novikov class of any LCK form vanishes. Precisely:
Theorem 1

Let $M$ be a compact Vaisman manifold, $\dim_{\mathbb{C}} M \geq 3$, $\omega_1$ an LCK-form (not necessarily Vaisman), and $\theta_1$ its Lee form. Then $\theta_1$ is cohomologous with the Lee form of a Vaisman metric, and the Morse–Novikov class of $\omega_1$ vanishes.
Proof of Theorem 1

- Let $\rho$ be the Lee flow corresponding to the Vaisman structure $\omega$.
  - Modulo a deformation, it can be supposed with compact leaves.
  - By averaging over $\rho$, $\theta_1$ and $\omega_1$ can be supposed $\rho$-invariant. The cohomology class does not change.
  - Let $G_0$ be the closure of the group of holomorphic and conformal automorphisms of $M$ generated by $J(\theta^\natural)$: compact and commutative.
  - As above, $\theta_1$ and $\omega_1$ can be supposed $G_0$-invariant.
Let ρ be the Lee flow corresponding to the Vaisman structure ω.

- Modulo a deformation, it can be supposed with compact leaves.
- By averaging over ρ, θ₁ and ω₁ can be supposed ρ-invariant. The cohomology class does not change.
- Let $G_0$ be the closure of the group of holomorphic and conformal automorphisms of $M$ generated by $J(\theta^\ast)$: compact and commutative.
- As above, θ₁ and ω₁ can be supposed $G_0$-invariant.
Proof of Theorem 1

- Let \( \rho \) be the Lee flow corresponding to the Vaisman structure \( \omega \).
  - Modulo a deformation, it can be supposed with compact leaves.
- By averaging over \( \rho \), \( \theta_1 \) and \( \omega_1 \) can be supposed \( \rho \)-invariant. The cohomology class does not change.
- Let \( G_0 \) be the closure of the group of holomorphic and conformal automorphisms of \( M \) generated by \( J(\theta^\natural) \): compact and commutative.
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- Let $G_0$ be the closure of the group of holomorphic and conformal automorphisms of $M$ generated by $J(\theta^\sharp)$: compact and commutative.

- As above, $\theta_1$ and $\omega_1$ can be supposed $G_0$-invariant.
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- By averaging over $\rho$, $\theta_1$ and $\omega_1$ can be supposed $\rho$-invariant. The cohomology class does not change.

- Let $G_0$ be the closure of the group of holomorphic and conformal automorphisms of $M$ generated by $J(\theta^w)$: compact and commutative.

- As above, $\theta_1$ and $\omega_1$ can be supposed $G_0$-invariant.
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- Let \( \rho \) be the Lee flow corresponding to the Vaisman structure \( \omega \).
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Proof of Theorem 1

- Let $\tilde{M}$ be a Kähler covering on which $\tilde{\theta}$ is exact.

- Fact: If $\theta^\sharp$ and $J(\theta^\sharp)$ act conformally and holomorphically and $\theta^\sharp$ cannot be lifted to an isometry of $\tilde{M}$, then $M$ is Vaisman (K–O).

- Hence: suppose $\tilde{\omega}_1$ is $\tilde{\rho}$–invariant.

- Show that $\theta_1$ is basic wrt the foliation $\rho$.

- Hence: $d^c \theta_1 = 0$ (Tsukada), thus:
  
  - $0 = \int_M dd^c \omega_1^{n-1} = \int_M (n-1)^2 \theta_1 \wedge J(\theta_1) \wedge \omega_1^{n-1}$,
  - $\theta_1 \wedge J(\theta_1) \wedge \omega^{n-1} > 0$ unless $\theta_1 = 0$.

- We obtain $\theta_1 = 0$ and $M$ is Kähler.

- But a compact Kähler manifold cannot support a Vaisman structure (different topology).
Proof of Theorem 1

- Let \( \tilde{M} \) be a Kähler covering on which \( \tilde{\theta} \) is exact.
- Fact: If \( \theta^\# \) and \( J(\theta^\#) \) act conformally and holomorphically and \( \theta^\# \) cannot be lifted to an isometry of \( \tilde{M} \), then \( M \) is Vaisman (K–O).
- Hence: suppose \( \tilde{\omega}_1 \) is \( \tilde{\rho} \)-invariant.
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- Let \( \widetilde{M} \) be a Kähler covering on which \( \widetilde{\theta} \) is exact.
- **Fact:** If \( \theta^\# \) and \( J(\theta^\#) \) act conformally and holomorphically and \( \theta^\# \) cannot be lifted to an isometry of \( \widetilde{M} \), then \( M \) is Vaisman (K–O).

**Hence:** suppose \( \tilde{\omega}_1 \) is \( \tilde{\rho} \)-invariant.

**Show that \( \theta_1 \) is basic wrt the foliation \( \rho \).**

**Hence:** \( d^c \theta_1 = 0 \) (Tsukada), thus:

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- We obtain \( \theta_1 = 0 \) and \( M \) is Kähler.
- But a compact Kähler manifold cannot support a Vaisman structure (different topology).
- Main problem with non–Kähler manifolds: do not satisfy the global $\partial \overline{\partial}$-lemma.
- One considers the Bott–Chern complex:
  \[ \cdots \to \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \overline{\partial}} \Lambda^{p,q}(M) \xrightarrow{\partial \overline{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \to \cdots \]
- Its cohomology groups $H_{\partial \overline{\partial}}^{p,q}(M)$ are
  \[
  \ker \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \ker \left( \Lambda^{p,q}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q+1}(M) \right)
  \]
  \[
  \text{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \overline{\partial}} \Lambda^{p,q}(M) \right)
  \]
- For compact manifolds, $H_{\partial \overline{\partial}}^{p,q}(M) \cong R_{\partial \overline{\partial}}^{p,q}(M) \iff$ global $\partial \overline{\partial}$-lemma.
Main problem with non–Kähler manifolds: do not satisfy the global $\partial \bar{\partial}$-lemma.

One considers the Bott–Chern complex:

$$
\cdots \to \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \bar{\partial}} \Lambda^{p,q}(M) \xrightarrow{\partial \bar{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \to \cdots
$$

Its cohomology groups $H_{\partial \bar{\partial}}^{p,q}(M)$ are

$$
\ker \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \ker \left( \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \right)
$$

$$
\text{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \bar{\partial}} \Lambda^{p,q}(M) \right)
$$

For compact manifolds, $H_{\partial \bar{\partial}}^{p,q}(M) = H^{p,q}(M) \iff$ global $\partial \bar{\partial}$-lemma.
Main problem with non–Kähler manifolds: do not satisfy the global $\partial \overline{\partial}$-lemma.

One considers the Bott–Chern complex:

$$\rightarrow \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \overline{\partial}} \Lambda^{p,q}(M) \xrightarrow{\partial \oplus \overline{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \rightarrow$$

Its cohomology groups $H_{\partial \overline{\partial}}^{p,q}(M)$ are

\[
\ker \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \ker \left( \Lambda^{p,q}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q+1}(M) \right)
\]

\[
\operatorname{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \overline{\partial}} \Lambda^{p,q}(M) \right)
\]

For compact manifolds, $H_{\partial \overline{\partial}}^{p,q}(M) \cong H_{\partial \overline{\partial}}^{p,q}(\overline{M})$ $\iff$ global $\partial \overline{\partial}$-lemma.
Main problem with non–Kähler manifolds: do not satisfy the global $\partial \bar{\partial}$-lemma.

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$$

Its cohomology groups $H^{p,q}_{\partial \bar{\partial}}(M)$ are

$$
\ker \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \ker \left( \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \right)
$$

$$
\text{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \bar{\partial}} \Lambda^{p,q}(M) \right)
$$

For compact manifolds, $H^{p,q}_{\partial \bar{\partial}}(M) = H^{p,q}_{\partial \bar{\partial}}(M) \iff$ global $\partial \bar{\partial}$-lemma.
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Its cohomology groups $H^{p,q}_{\partial\bar{\partial}}(M)$ are

$$\text{ker} \left( \Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M) \right) \cap \text{ker} \left( \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \right)$$

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$$

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\text{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial\bar{\partial}} \Lambda^{p,q}(M) \right)
$$

For compact manifolds, $H^{p,q}_{\partial\bar{\partial}}(M) \cong H^{p,q}_{\bar{\partial}\partial}(M) \iff$ global $\partial\bar{\partial}$-lemma.
Main problem with non–Kähler manifolds: do not satisfy the global \( \partial \bar{\partial} \)-lemma.

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\[
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\]

Its cohomology groups \( H^{p,q}_{\partial \bar{\partial}}(M) \) are

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\]

\[
\text{im} \left( \Lambda^{p-1,q-1}(M) \xrightarrow{\partial \bar{\partial}} \Lambda^{p,q}(M) \right)
\]

For compact manifolds, \( H^{p,q}_{\partial \bar{\partial}}(M) \cong H^{p,q}_{\partial}(M) \iff \) global \( \partial \bar{\partial} \)-lemma.
Bott–Chern cohomology of LCK manifolds

- Same complex, but for $d_\theta$:

\[
\Lambda^{p-1,q-1}(M) \xrightarrow{\partial_\theta \bar{\partial}_\theta} \Lambda^{p,q}(M) \xrightarrow{\partial_\theta \oplus \bar{\partial}_\theta} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M)
\]

- Cohomology groups $H^{p,q}_{\partial_\theta \bar{\partial}_\theta}(M) \cong H^{p,q}_{\partial \bar{\partial}}(M, L)$.

- $[\omega] \in H^{1,1}_{\partial \bar{\partial}}(M, L)$ is called Bott–Chern class.
Bott–Chern cohomology of LCK manifolds

- Same complex, but for $d_\theta$:

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$$

Cohomology groups $H^{p,q}_{\partial_\theta \bar{\partial}_\theta}(M) \cong H^{p,q}_{\bar{\partial} \partial}(M, L)$.

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- Cohomology groups $H^{p,q}_{\partial_\theta \overline{\partial}_\theta}(M) \cong H^{p,q}_{\overline{\partial} \overline{\partial}}(M, L)$.

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Bott–Chern cohomology of LCK manifolds

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\[
\begin{align*}
\Lambda^{p-1, q-1}(M) & \xrightarrow{\partial_\theta \bar{\partial}_\theta} \Lambda^{p, q}(M) \\
\Lambda^{p, q}(M) & \xrightarrow{\partial_\theta \oplus \bar{\partial}_\theta} \Lambda^{p+1, q}(M) \oplus \Lambda^{p, q+1}(M)
\end{align*}
\]

- Cohomology groups $H^{p, q}_{\partial_\theta \bar{\partial}_\theta}(M) \cong H^{p, q}_{\partial \bar{\partial}}(M, L)$.

- $[\omega] \in H^{1, 1}_{\partial \bar{\partial}}(M, L)$ is called Bott–Chern class.
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- $[\omega] \in H^{1,1}_{\partial \bar{\partial}}(M, L)$ is called Bott–Chern class.
Meaning of the Bott–Chern class

- \([\omega] = 0 \in H^{1,1}_{\bar{\partial}\partial}(M, L) \Leftrightarrow \tilde{M} \text{ admits an automorphic potential.}
  - \(H^{1,1}_{\bar{\partial}\partial}(M, L) = 0\) is implied by \(H^1(M, L) = 0\) and \(H^2_\theta(M) = 0\) (easier to control).

- Hence: If the Bott–Chern class of an LCK-manifold \(M\) vanishes and the monodromy of \(L\) is \(\mathbb{Z}\), then \(M\) is LCK with potential (will be generalized).

- If \(\omega_1, \omega_2\) are LCK-metrics having the same Lee form \(\theta\), then the following conditions are equivalent:
  - The Bott–Chern classes of \(\omega_1, \omega_2\) are equal.
  - The LCK-structures \(\omega_1\) and \(\omega_2\) are equivalent up to a potential (on a Kähler covering, \(\tilde{\omega}_1 = \tilde{\omega}_2 = \partial\partial\phi\) with automorphic \(\phi\)).
Meaning of the Bott–Chern class

- \([\omega] = 0 \in H^{1,1}_{\partial\bar{\partial}}(M, L) \iff \tilde{M} \text{ admits an automorphic potential.}

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Meaning of the Bott–Chern class

- $[\omega] = 0 \in H^{1,1}_{\partial\bar{\partial}}(M, L) \iff \tilde{M}$ admits an *automorphic* potential.
  
  - $H^{1,1}_{\partial\bar{\partial}}(M, L) = 0$ is implied by $H^1(M, L) = 0$ and $H^2_\theta(M) = 0$ (easier to control).

- Hence: If the Bott–Chern class of an LCK-manifold $M$ vanishes and the monodromy of $L$ is $\mathbb{Z}$, then $M$ is LCK with potential (will be generalized).

- If $\omega_1, \omega_2$ are LCK-metrics having the same Lee form $\theta$, then the following conditions are equivalent:
  
  1. The Bott–Chern classes of $\omega_1, \omega_2$ are equal.
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Meaning of the Bott–Chern class

- \([\omega] = 0 \in H^{1,1}_{\partial\bar{\partial}}(M, L) \iff \tilde{M} \text{ admits an } \textit{automorphic} \text{ potential.}
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- $[\omega] = 0 \in H^{1,1}_{\partial\bar{\partial}}(M, L) \iff \tilde{M}$ admits an automorphic potential.
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[\omega] = 0 \in H^{1,1}(M, L) \iff \tilde{M} \text{ admits an } \textit{automorphic} \text{ potential.}

H^{1,1}(M, L) = 0 \text{ is implied by } H^1(M, L) = 0 \text{ and } H^2_\theta(M) = 0 \text{ (easier to control).}

Hence: If the Bott–Chern class of an LCK-manifold \( M \) vanishes and the monodromy of \( L \) is \( \mathbb{Z} \), then \( M \) is LCK with potential (will be generalized).

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The space of LCK structures

Analogy between Kähler and LCK

- Kähler structures on a complex manifold are determined by:
  - a Kähler class in $H^1(M)$;
  - a choice of a Kähler metric in this Kähler class, obtained by choosing an element in a cone locally modeled on $C^\infty(M)$.

- LCK-structures on a complex manifold with prescribed conformal structure are determined by:
  - a Bott–Chern class in $H^1(\partial\partial(M,L))$;
  - a choice of an LCK-metric with a prescribed Bott–Chern class, obtained by choosing an element in a cone locally modeled on $C^\infty(M)$.
**The space of LCK structures**

**Analogy between Kähler and LCK**

- Kähler structures on a complex manifold are determined by:
  - a Kähler class in $\Omega^{1,1}(M)$;
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The space of LCK structures

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- LCK-structures on a complex manifold with prescribed conformal structure are determined by:
  1. a Bott–Chern class in $H^{1,1}(\partial\partial(M), L)$;
  2. a choice of an LCK-metric with a prescribed Bott–Chern class, obtained by choosing an element in a cone locally modeled on $C^\infty(M)/\ker(\partial\partial)$.
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  1. a Bott–Chern class in $H^{1,1}(\partial M, L)$;
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  2. a choice of an LCK-metric with a prescribed Bott–Chern class, obtained by choosing an element in a cone locally modeled on $\mathcal{C}^\infty(M)/\text{ker}\left(\partial\partial\right)$.
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The space of LCK structures

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The space of LCK structures

Analogy between Kähler and LCK

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  2. a choice of an LCK-metric with a prescribed Bott–Chern class, obtained by choosing an element in a cone locally modeled on $C^\infty(M)/\ker(\partial\partial)$. 
The space of LCK structures

Analogy between Kähler and LCK

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1. a Bott-Chern class in $H^{1,0}(M, L)$;
2. a choice of an LCK-metric with a prescribed Bott-Chern class, obtained by choosing an element in a cone locally modeled on $C^\infty(M)/\ker(\partial)$.
The space of LCK structures

Analogy between Kähler and LCK

- Kähler structures on a complex manifold are determined by:
  1. a Kähler class in $H^{1,1}(M)$;
  2. a choice of a Kähler metric in this Kähler class, obtained by choosing an element in a cone locally modeled on $\mathcal{C}^\infty(M)/\text{const}$.

- LCK-structures on a complex manifold with prescribed conformal structure are determined by:
  1. a Bott–Chern class in $H^{1,1}_{\partial\bar\partial}(M, L)$;
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The space of LCK structures

Analogy between Kähler and LCK

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Theorem 2

Any compact LCK manifold with vanishing Bott–Chern class admits an LCK metric with potential.

Hence, if $\dim_{\mathbb{C}} M \geq 3$, it is embeddable in a Hopf manifold.

Our supposition, connected also with Theorem 1: Let $M$ be a Vaisman manifold, equipped with an additional LCK-form $\omega_1$ (not necessarily Vaisman). Then the Bott–Chern class of $\omega_1$ vanishes; equivalently, $\omega_1$ is an LCK-structure with potential.
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Proof of Theorem 2

- \([\omega] = 0 \in H^{1,1}_{\partial\bar{\partial}}(M, L) \iff \tilde{M} \text{ admits an automorphic potential.}\)
- The weight bundle \(L\) is associated to the monodromy of this covering and the monodromy can be a priori a subgroup of \((\mathbb{R}^+, \cdot) \cong (\mathbb{R}, +)\), which is not necessarily discrete.
- Consider \(L\) as a trivial line bundle with connection \(\nabla_{\text{triv}} - \theta\) and deform \(L\) by adding a small term to \(\theta\) to obtain a bundle \(L'\) with monodromy \(\mathbb{Z}\).
- A local system on \(M\) is defined by a group homomorphism \(H_1(M, \mathbb{Z}) \to \mathbb{R}\). Its monodromy is \(\mathbb{Z}\) if this map is rational. Each real homomorphism from \(H_1(M, \mathbb{Z})\) can be approximated by a rational one.
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- Deforming the monodromy ⇔ deforming $\theta = d \log \varphi$ ⇔ deforming the potential $\varphi$.

- We deform the pair $(L, \varphi)$ to a pair $(L', \varphi')$ in which $\varphi'$ is automorphic function on $\tilde{M}$, with monodromy determined by $L'$.  
  - $\varphi'$ stays plurisubharmonic if $\theta'$ is sufficiently close to $\theta$ in the norm:
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    |\theta - \theta'|_{L^\infty} = \sup_{M} |\theta - \theta'| + \sup_{\tilde{M}} |\nabla \theta - \nabla \theta'|.
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Determine all 1-forms $\theta$ for which there exists a Hermitian two-form $\omega$ having $\theta$ as its Lee form, and all the Morse–Novikov classes which can be realized by an LCK-form.

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Let $M$ be a compact complex manifold, admitting an LCK-metric, and $[\theta] \in H^1(M)$ its Lee class. Determine the set of all classes $[\omega] \in H^1\partial\bar{\partial}_\theta(M)$ such that $[\omega]$ is the Bott–Chern class of an LCK-structure with the Lee class $[\theta]$.

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Is there a global $\partial\bar{\partial}$–lemma?
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