SUMS OF MANY PRIMES

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Abstract. Assuming that the Generalized Riemann Hypothesis (GRH) holds, we prove an explicit formula for the number of representations of an integer as a sum of \( k \geq 5 \) primes. Our error terms in such a formula improve by some logarithmic factors an analogous result by Friedlander-Goldston [3].

1. Introduction

Let \( k \geq 2 \) be a fixed integer and set
\[
R_k(n) = \sum_{n_1 + \cdots + n_k = n} \Lambda(n_1) \cdots \Lambda(n_k)
\]
and
\[
\mathcal{S}_k(n) = \sum_{q=1}^{\infty} \frac{\mu(q)^k}{\phi(q)^k} c_q(-n) = \prod_{p|n} \left(1 - \left(-\frac{1}{p-1}\right)^{k-1}\right) \prod_{p\nmid n} \left(1 - \left(-\frac{1}{p-1}\right)^k\right)
\]
where \( c_q \) is the Ramanujan sum defined as
\[
c_q(m) = \sum_{a=1}^{\varphi(q)} e\left(m \frac{a}{q}\right).
\]
Moreover let \( \chi \mod q \) be a Dirichlet character and
\[
c_\chi(m) = \sum_{a=1}^{q} \chi(a) e\left(m \frac{a}{q}\right).
\]
Since we deal with the case \( k = 2 \) in [10], here we assume that \( k \geq 3 \) throughout for simplicity of statement. We just notice that in [10] there is an average of \( R_k(n) \) over \( n \) and the natural hypothesis to make is RH, whereas here and in Friedlander-Goldston [3] there is no such average and the natural hypothesis is GRH. In both cases we are interested into a formula which is “explicit” in the sense that it has the expected main term, a secondary main term depending on the zeros of the \( L \) functions (or just the zeta function when \( k = 2 \)), and an error term of smaller order of magnitude.

We have, for \( k \geq 5 \), the following explicit formula for \( R_k(n) \).

**Theorem.** Let \( k \geq 5 \) be a fixed integer. Assume that the Generalized Riemann Hypothesis (GRH) holds for every Dirichlet series \( L(s, \chi) \), for every \( \chi \mod q \). Then, for every sufficiently large integer \( n \), we have that
\[
R_k(n) = \frac{n^{k-1}}{(k-1)!} \mathcal{S}_k(n) - k \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)^k} \sum_{\chi \mod q} c_\chi(-n) \tau(\chi) \sum_{\rho} \frac{n^{\rho+k-2}}{\rho(\rho+1) \cdots (\rho+k-2)} + O\left(n^{k-7/4} \log^k n\right)
\]
where \( R_k(n), \Theta_k(n), c_\chi(n) \) are respectively defined in \((1)-(2)\) and \((4)\), \( \tau(\chi) \) is the Gauss sum and \( \rho = 1/2 + i\gamma \) runs over the non-trivial zeros of \( L(s, \chi) \), for every \( \chi \mod q \). For \( k \geq 6 \), in the last error term we can replace \( 7/4 \) by \( 2 \). For \( k \geq 7 \), in the last error term we can also replace \( \log k - 1/n \) by \( \log 2n \).

The condition \( k \geq 5 \) essentially arises in two points. The first one is the evaluation of the secondary main term in \((5)\) (see the error term in \((43)\) and the remark at the end of \((3.2)\) while the second one is in the error term estimates (see \((3.3)\)).

This result should be compared with Proposition 1 of Friedlander-Goldston \((3)\). They have a more involved but equivalent form of the secondary main term and worse estimates for the error term. In principle, both here and in \((3)\) one could give a statement with the sum over \( q \) in the “secondary main term” in the right hand side of \((5)\) restricted to \( q \leq n^{1/2}/2 \), and assume only that the GRH holds for the \( L \) functions associated to characters modulo these values of \( q \). For the details, see the remark at the end of \((3.2)\).

The improvement given here is due to the fact that we use the version of the circle method introduced by Hardy and Littlewood in \((6)\) and used also by Linnik in \((11, 12)\), involving series rather than truncated sums: it is essentially equivalent to, but slightly sharper than, the usual approach with truncated sums.

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2. Lemmas

We will use the original Hardy and Littlewood \((6)\) circle method setting, i.e., the weighted exponential sum

\[
\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/N} e(n\alpha), \tag{6}
\]

where \( e(x) = \exp(2\pi i x) \), since it lets us avoid the use of Gallagher’s Lemma (Lemma 1 of \((4)\)) and hence it gives slightly sharper results in this conditional case: see Lemma \((7)\) below.

Let \( 1 \leq Q \leq N \) be a parameter to be chosen later. We will consider the set of the Farey fractions of level \( Q \)

\[
\left\{ \frac{a}{q} : 1 \leq q \leq Q, 0 \leq a \leq q, (a, q) = 1 \right\}.
\]

Let \( a'/q' < a/q < a''/q'' \) be three consecutive Farey fractions,

\[
\mathcal{M}_{q,a} = \left( \frac{a + a'}{q + q'}, \frac{a + a''}{q + q''} \right) \quad \text{if} \quad \frac{a}{q} \neq \frac{1}{1},
\]

and \( \mathcal{M}_{1,1} = (1 - 1/(Q + 1), 1 + 1/(Q + 1)] \) be the Farey arcs centered at \( a/q \). These intervals are disjoint and their union is \((1/(Q + 1), 1 + 1/(Q + 1)]\). Moreover, let

\[
\xi_{q,a} = \left( \frac{-1}{q(q + q')}, \frac{1}{q(q + q'')} \right) \tag{7}
\]

and \( \xi_{1,1} = (-1/(Q + 1), 1/(Q + 1)] \) be the Farey arcs re-centered at the origin. In the following we also use the relation

\[
\left( \frac{-1}{2qQ}, \frac{1}{2qQ} \right) \subseteq \xi_{q,a} \subseteq \left( \frac{-1}{qQ}, \frac{1}{qQ} \right).
\]
Let
\[ z = N^{-1} - 2\pi i \eta \] (8)
for \( \eta \in \xi_{q,a} \), and
\[ V(\eta) = \sum_{m=1}^{\infty} e^{-m/N} e(m\eta) = \sum_{m=1}^{\infty} e^{-mz} = \frac{1}{e^{z} - 1}. \]

**Lemma 1.** If \( z \) satisfies (8) then \( V(\eta) = z^{-1} + \mathcal{O}(1) \).

**Proof.** We recall that the function \( \frac{w}{(e^w - 1)} \) has a power-series expansion with radius of convergence \( 2\pi \) (see for example Apostol [1], page 264). In particular, uniformly for \( |w| \leq 4 < 2\pi \) we have \( \frac{w}{(e^w - 1)} = 1 + \mathcal{O}(|w|) \). Since \( z \) satisfies (8) we have \( |z| \leq 4 \) and the result follows.

Combining Lemma 1 and the inequality
\[ |z|^{-1} \ll \min(N, |\eta|^{-1}), \] (9)
we also have
\[ |V(\eta)| \ll |z|^{-1} + 1 \ll \min(N, |\eta|^{-1}). \] (10)

We will use the approximation
\[ \tilde{S}(\frac{a}{q} + \eta) = \frac{\mu(q)}{\phi(q)} V(\eta) + \tilde{R}(\eta; q, a, V), \] (11)
and
\[ \tilde{R}(\eta; q, a, V) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a)\tau(\overline{\chi}) W(\chi, \eta, V) + \mathcal{O}((\log(qN))^2), \] (12)
where
\[ W(\chi, \eta, V) = \sum_{\ell=1}^{\infty} \Lambda(\ell)\chi(\ell)e^{-\ell/N} e(\ell\eta) - \delta(\chi)V(\eta). \]
\( \delta(\chi) = 1 \) if \( \chi = \chi_0 \mod q \) and 0 otherwise. Recalling (8), by Lemma 1 we can also write
\[ \tilde{S}(\frac{a}{q} + \eta) = \frac{\mu(q)}{\phi(q)} z + \tilde{R}(\eta; q, a, z) + \mathcal{O}\left(\frac{1}{\phi(q)}\right) \]
and
\[ \tilde{R}(\eta; q, a, z) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a)\tau(\overline{\chi}) W(\chi, \eta, z) + \mathcal{O}((\log(qN))^2), \]
where
\[ W(\chi, \eta, z) = \sum_{\ell=1}^{\infty} \Lambda(\ell)\chi(\ell)e^{-\ell/N} e(\ell\eta) - \frac{\delta(\chi)}{z}. \]

Summing up we have
\[ \tilde{S}(\frac{a}{q} + \eta) = \frac{\mu(q)}{\phi(q)} z + \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a)\tau(\overline{\chi}) W(\chi, \eta, z) + \mathcal{O}((\log(qN))^2). \] (13)

Recalling (6), the first ingredient we need is the following explicit formula which slightly sharpens what Linnik [11] (see also eq. (4.1) of [12]) proved.
Lemma 2. If \( \chi \) is a character mod \( q \) and GRH holds for \( L(s, \chi) \) then
\[
W(\chi, \eta, z) = -\sum_{\rho} z^{-\rho} \Gamma(\rho) + E(q, N) 
\]
(14)
where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( L(s, \chi) \) and
\[
E(q, N) \ll \begin{cases} 
1 + \log^2 q & \text{if } \chi \text{ is a primitive character,} \\
1 + \log N \log q + \log^2 q & \text{if } \chi \text{ is not primitive.}
\end{cases}
\]
(15)

Proof. We recall that \( \delta(\chi) = 1 \) if \( \chi = \chi_0 \mod q \) and 0 otherwise. Let
\[
\Sigma(N, \chi, \eta) = \sum_{\ell=1}^{\infty} \Lambda(\ell) \chi(\ell) e^{-\ell/N} e(\ell \eta) = W(\chi, \eta, z) + \frac{\delta(\chi)}{z}.
\]
We notice that if \( \chi \mod q \) is induced by \( \chi_1 \mod q_1 \) then
\[
|\Sigma(N, \chi, \eta) - \Sigma(N, \chi_1, \eta)| \leq \sum_{\ell \geq 1}^{\infty} \Lambda(\ell) e^{-\ell/N} \ll \log q \log N.
\]
We now assume that \( \chi \mod q \) is a primitive character and let \( \alpha = 3/4 \). Following the proof of Lemma 4 in Hardy and Littlewood [6] and §4 in Linnik [11], we have that
\[
W(\chi, \eta, z) = -\sum_{\rho} z^{-\rho} \Gamma(\rho) + C(\chi) - \frac{1}{2\pi i} \int_{(-\alpha)} L'(w, \chi) \Gamma(w) z^{-w} dw,
\]
(16)
where \( C(\chi) \) is a term that depends only on the character \( \chi \). In order to estimate the integral in (16) we need the inequality
\[
\left| \frac{L'}{L} \left( -\frac{3}{4} + it, \chi \right) \right| \ll \log(q(|t| + 2)).
\]
(17)
This follows from equations (1) and (4) of §16 of Davenport [2] since the latter reads
\[
L'(w, \chi) = \sum_{\rho} \frac{1}{w - \rho} + O(\log(q(|t| + 2))),
\]
where the dash means that the sum is restricted to those zeros \( \rho = \beta + i\gamma \) with \( |t - \gamma| < 1 \), while the former implies that the number of such summands is \( \ll \log(q(|t| + 2)) \). Finally, it is obvious that each summand is \( \ll 1 \) on the line of integration \( w = -\alpha + it \).

We notice that \( |z^{-w}| = |z|^\alpha \exp(t \arg(z)) \) where \( |\arg(z)| \leq \frac{1}{2} \pi \). Furthermore the Stirling formula implies that \( \Gamma(w) \ll |t|^{-\alpha-1/2} \exp(-\frac{\pi}{2} |t|) \). Hence
\[
\int_{(-\alpha)} \frac{L'}{L}(w, \chi) \Gamma(w) z^{-w} dw \ll |z|^\alpha \int_0^1 \log(q(t + 2)) dt 
\]
\[
+ |z|^\alpha \int_1^\infty \log(q(t + 2)) t^{-\alpha-1/2} \exp\left((\arg(z) - \frac{\pi}{2}) t\right) dt
\]
\[
\ll |z|^\alpha (1 + \log q) + |z|^\alpha \int_1^\infty \log(q(t + 2)) t^{-\alpha-1/2} dt
\]
\[
\ll |z|^\alpha (1 + \log q).
\]
This is \( \ll 1 + \log(q) \) as stated since \( z \ll 1 \) by [3] and \( \alpha \) is fixed. Finally, we have to deal with the term \( C(\chi) \) in (16). We recall the notation of §19 of Davenport [2]: if \( \chi \) is odd then \( b(\chi) \) denotes \( (L'/L)(0, \chi) \), whereas if \( \chi \) is even it is the constant term in the Laurent expansion of \( (L'/L)(w, \chi) \) around zero. In other words, \( (L'/L)(w, \chi) = w^{-1} + b(\chi) + O(w) \).
If \( \chi \) is odd \( C(\chi) \) is simply \(-\langle L'/L \rangle(0, \chi) = -b(\chi) \) since \( L(0, \chi) \neq 0 \). If \( \chi \) is even then \( L(w, \chi) \) has a simple zero at 0 and therefore \(-\langle L'/L \rangle(w, \chi)\Gamma(w)z^{-w} \) has a double pole at \( w = 0 \) with residue \( C(\chi) = \log(z) - b(\chi) - \Gamma'(1) \). Arguing as on pages 118–119 of Davenport [2], we see that
\[
 b(\chi) = -\sum_{\rho} \left( \frac{1}{\rho} + \frac{1}{2 - \rho} \right) + O(1) = -\sum_{|\gamma| < 1} \frac{1}{\rho} + O(\log q) \ll \log^2 q.
\]
Finally, \( \log(z) \ll 1 \) since \( z \) satisfies \( \xi \).

**Lemma 3.** Let \( N \) be a sufficiently large integer, \( Q \leq N \) and \( z \) be as in \( \xi \) with \( \eta \in \xi_{q,a} \).
We have
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q} \int_{\xi_{q,a}} \left| \frac{\mu(q)}{\phi(q)}z \right|^2 \, d\eta \ll N \log N
\]
and
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q} \int_{\xi_{q,a}} \left| \tilde{S}(\frac{q}{q} + \eta) - \frac{\mu(q)}{\phi(q)}z \right|^2 \, d\eta \ll N \log N.
\]

**Proof.** By Parseval’s theorem and the Prime Number Theorem we have
\[
\int_{-1/2}^{1/2} |\tilde{S}(\alpha)|^2 \, d\alpha = \sum_{m=1}^{\infty} \Lambda^2(m) e^{-2m/N} = \frac{N}{2} \log N + O(N).
\]
Recalling that the equation at the beginning of page 318 of [8] implies
\[
\int_{-1/2}^{1/2} \frac{d\eta}{|z|^2} = \frac{N}{\pi} \arctan \left( \frac{2\pi N}{qQ} \right)
\]
and using Lemma 2 of Goldston [3], we have
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q} \int_{\xi_{q,a}} \left| \frac{\mu(q)}{\phi(q)}z \right|^2 \, d\eta \ll \sum_{q=1}^{Q} \frac{\mu^2(q)}{\phi(q)} \int_{-1/2}^{1/2} \frac{d\eta}{|z|^2} \ll N \log Q.
\]
The Lemma immediately follows using \( Q \leq N \), the relation \(|a - b|^2 = |a|^2 + |b|^2 - 2Re(ab)\)
and the Cauchy-Schwarz inequality. \( \square \)

Let \( x \geq 2 \) be a real number, \( j \geq 0 \), \( q \geq 1 \) be integers and \( \chi \) be a Dirichlet character defined \( \mod q \). We define
\[
\psi_j(x, \chi) := \frac{1}{j!} \sum_{m=1}^{x} (x - m)^j \Lambda(m) \chi(m).
\]  

**Lemma 4.** Let \( x \geq 2 \) be a real number, \( j \geq 0 \), \( q \geq 1 \) be integers and \( \chi \) be a Dirichlet character defined \( \mod q \). Assuming that GRH holds for \( L(s, \chi) \) then
\[
\psi_j(x, \chi) = \delta(\chi) \frac{x^{j+1}}{(j + 1)!} - \sum_{\rho} \frac{x^{\rho+j}}{\rho(\rho + 1) \cdots (\rho + j)} + O_j(x^j E(q,x)) + O'(\log x),
\]
where \( \delta(\chi) = 1 \) if \( \chi \equiv \chi_0 \mod q \) and 0 otherwise, \( \rho \) runs over the non-trivial zeros of \( L(s, \chi) \) and \( E(q,x) \) is defined in \( (15) \). The prime in the last error term means that it is present if and only if \( j = 0 \). For \( j = 0 \) the summation over the zeros at the right hand side should be understood in the symmetric sense.
Proof. For \( j = 0 \) this is a classical result, see e.g. Davenport \[2\], §17 and 19. For \( j \geq 1 \) it follows by a standard Mellin inversion argument using the Cesàro kernel defined at page 142 of Montgomery-Vaughan \[13\]. We sketch here the proof.

If \( q = 1 \), by eq. (5.19) of Montgomery-Vaughan \[13\], we have that
\[
\psi_j(x) = \psi_j(x, \chi_0) = -\frac{1}{2\pi i} \int_{(c)} \frac{\zeta'(w)}{\zeta(w)} \frac{x^{w+j}}{w(w+1) \cdots (w+j)} \, dw,
\]
where \( c > 1 \) is fixed. Moving the line of integration to \( \Re(w) = -3/4 \), we see that the relevant poles are located at the zeros of \( \zeta(w) \) and at \( w = 0, 1 \). They are all simple poles. By the Riemann-von Mangoldt formula, we can choose a large \( T \) such that \( |\Im(\rho) - T| \gg (\log T)^{-1} \) for every non-trivial zero \( \rho \) of \( \zeta(w) \) and hence there is no harm in moving the integration line to \( -3/4 \) since \(|(\zeta'/\zeta)(w)| \ll \log^2 T \) for every \( w = \sigma \pm iT \) with \( \sigma \in [-1, 2] \), and \(|w(w+1) \cdots (w+j)| \gg T^{j+1} \).

By the residue theorem we immediately get
\[
\psi_j(x) = \frac{x^{j+1}}{(j+1)!} - \sum_{\rho} \frac{x^{\rho+j}}{\rho(\rho+1) \cdots (\rho+j)} - \frac{\zeta'(0)}{\zeta(0)} x^j - \frac{1}{2\pi i} \int_{(-3/4)} \frac{\zeta'(w)}{\zeta(w)} \frac{x^{w+j}}{w(w+1) \cdots (w+j)} \, dw. \tag{19}
\]

The vertical integral can be estimated using \( (17) \) in this special case \( (q = 1) \). Its contribution is
\[
\ll x^{j-3/4} \int_{-\infty}^{\infty} \frac{\log(|t|+2)}{(1+|t|)^{j+1}} \, dt \ll x^{j-3/4}. \tag{20}
\]
Combining \( (19)-(20) \) we get the final result in this case \( (j \geq 1, q = 1) \).

Let now \( q \geq 2 \). If \( \chi \) is the principal character \( \mod q \) then
\[
|\psi_j(x) - \psi_j(x, \chi_0)| \ll \frac{1}{j!} \sum_{m \leq x} (x-m)^j \Lambda(m) \ll \frac{1}{j!} x^j \log x \log q \tag{21}
\]
and the result follows using \( (19)-(20) \).

Now assume that \( \chi \mod q \) is not the principal character \( \mod q \). If \( \chi \mod q \) were induced by \( \chi_1 \mod q_1, q_1 \mid q \), then, arguing as in \( (21) \), we would have
\[
|\psi_j(x, \chi) - \psi_j(x, \chi_1)| \ll \frac{1}{j!} x^j \log x \log q. \tag{22}
\]

Now assume that \( \chi \) is a primitive character \( \mod q \). By eq. (5.19) of Montgomery-Vaughan \[13\], we have that
\[
\psi_j(x, \chi) = -\frac{1}{2\pi i} \int_{(c)} \frac{L'(w, \chi)}{L(w, \chi)} \frac{x^{w+j}}{w(w+1) \cdots (w+j)} \, dw,
\]
where \( c > 1 \) is fixed.

We move the line of integration to \( \Re(w) = -3/4 \). The relevant poles are located at the zeros of \( L(w, \chi) \) and at \( w = 0 \). They are all simple poles with the unique exception of \( w = 0 \) which is a double pole for \( (L'/L)(w, \chi) \) when \( \chi \) is even. By the Riemann-von Mangoldt formula, we can choose a large \( T \) such that \( |\Im(\rho) - T| \gg (\log(qT))^{-1} \) for every non-trivial zero \( \rho \) of \( L(s, \chi) \) and hence there is no harm in moving the integration line to \( -3/4 \) since \(|(L'/L)(w, \chi)| \ll \log^2(qT) \) for every \( w = \sigma \pm iT \) with \( \sigma \in [-1, 2] \), and \(|w(w+1) \cdots (w+j)| \gg T^{j+1} \).
By the residue theorem we immediately get

\[
\psi_j(x, \chi) = - \sum_{\rho} \frac{x^{\rho+j}}{\rho(\rho+1) \cdots (\rho+j)} + C(\chi) \frac{x^j}{j!} - \frac{1}{2\pi i} \int_{(-3/4)} L'(w, \chi) \frac{x^{w+j}}{w(w+1) \cdots (w+j)} \, dw,
\]

where \(C(\chi)\) is a term that depends only on the character \(\chi\). The vertical integral can be estimated using (17) and its contribution is

\[
\ll x^{j-3/4} \int_{-\infty}^{\infty} \frac{\log(q(1+|t|))}{(|t|+2)^{j+1}} \, dt \ll_j x^{j-3/4} \log q.
\]

Finally, we have to deal with the term \(C(\chi)\) in (23). If \(\chi\) is odd \(C(\chi)\) is simply \(-(L'/L)(0, \chi) = -b(\chi)\) since \(L(0, \chi) \neq 0\). If \(\chi\) is even then \(L(w, \chi)\) has a simple zero at 0 and therefore \(- (L'/L)(w, \chi)x^{w+j}(w+1) \cdots (w+j)^{-1}\) has a double pole at \(w = 0\) with residue \(x^j C(\chi)/j!\) and \(C(\chi) = - \log x - b(\chi)\). The remaining part of the argument runs as at the bottom of Lemma 2. This, together with (23) - (24), gives the final result for \(j \geq 1\) and a primitive character \(\chi\) mod \(q\). The general result for \(q \geq 2, j \geq 1\), follows using (22). \(\square\)

**Lemma 5.** Let \(k \geq 2\) be an integer, \(N\) be a large integer and \(z\) be as in (8). We have

\[
\int_{-1/2}^{1/2} W(\chi, \eta, V) V(\eta)^{k-1} e(-n\eta) \, d\eta = e^{-n/N} (\psi_{k-2}(n, \chi) - \delta(\chi) \frac{n^{k-1}}{(k-1)!}) + O_k(n^{k-2}),
\]

where \(\psi_{k-2}(n, \chi)\) is defined in (15).

**Proof.** We have

\[
\int_{-1/2}^{1/2} W(\chi, \eta, V) V(\eta)^{k-1} e(-n\eta) \, d\eta
\]

\[
= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} (\Lambda(m_1) \chi(m_1) - \delta(\chi)) e^{-i(\sum_{i=1}^{k} m_i)/N} \int_{-1/2}^{1/2} e^{i \left( \sum_{i=1}^{k} m_i - n \right) \eta} \, d\eta
\]

\[
= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} (\Lambda(m_1) \chi(m_1) - \delta(\chi)) e^{-i(\sum_{i=1}^{k} m_i)/N} \begin{cases} 1 & \text{if } \sum_{i=1}^{k} m_i = n \\ 0 & \text{otherwise} \end{cases}
\]

\[
= e^{-n/N} \sum_{m_1=1}^{n-1} (\Lambda(m_1) \chi(m_1) - \delta(\chi)) \left( \frac{n-1-m_1}{k-2} \right)
\]

\[
= e^{-n/N} \frac{(n-1-m_1)^{k-2}}{(k-2)!} (\Lambda(m_1) \chi(m_1) - \delta(\chi))
\]

\[
+ O_k \left( n^{k-3} \sum_{m_1=1}^{n-1} (\Lambda(m_1) + 1) \right)
\]

\[
= e^{-n/N} \left( \psi_{k-2}(n-1, \chi) - \delta(\chi) \frac{(n-1)^{k-1}}{(k-1)!} \right) + O_k(n^{k-2}),
\]
since the condition $\sum_{i=1}^{k} m_i = n$ implies that the variables are all $< n$. Now
\[
\psi_{k-2}(n, \chi) = \psi_{k-2}(n-1, \chi) + \mathcal{O}_k\left(n^{k-3} \sum_{m=1}^{n-1} \Lambda(m)\right) = \psi_{k-2}(n-1, \chi) + \mathcal{O}_k(n^{k-2})
\]
so that
\[
e^{-n/N} \left(\psi_{k-2}(n-1, \chi) - \delta(\chi) \left(\frac{(n-1)^{k-1}}{(k-1)!}\right)\right) = e^{-n/N} \left(\psi_{k-2}(n, \chi) - \delta(\chi) \frac{n^{k-1}}{(k-1)!}\right) + \mathcal{O}_k(n^{k-2})
\]
and Lemma 5 follows.

The next lemma is a modern version of Lemma 9 of Hardy-Littlewood [6] and should be compared with equation (1.15) of [3].

**Lemma 6.** Assume GRH, $1 \leq q \leq Q$, $Q \leq N$, $\eta \in \xi_{q,a}$ and let $z$ be as in (8). Then
\[
\left|\tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)z}\right| \ll (N(q|\eta|)^{1/2} + (qN)^{1/2}) \log N.
\]

**Proof.** By (13), Lemma 2 and straightforward computations, we have
\[
\left|\tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)z}\right| \ll q^{1/2} \left(\sum_{\chi \mod q} |\sum_{\rho} z^{-\rho} \Gamma(\rho)|\right) + q^{1/2} \log^2(qN). \tag{25}
\]
Since $z^{-\rho} = |z|^{-\rho} \exp(-i\rho \arctan 2\pi N\eta)$, by Stirling’s formula we have
\[
\sum_{\rho} z^{-\rho} \Gamma(\rho) \ll \sum_{\rho} |z|^{-1/2} \exp\left(\gamma \arctan 2\pi N\eta - \frac{\pi}{2} |\gamma|\right).
\]
If $\gamma \eta \leq 0$ or $|\eta| \leq 1/N$ we obtain
\[
\sum_{\rho} z^{-\rho} \Gamma(\rho) \ll N^{1/2} \log(q + 1), \tag{26}
\]
where, in the first case, $\rho$ runs over the zeros with $\gamma \eta \leq 0$.

We can consider only the case $\gamma \eta > 0$ and $|\eta| > 1/N$. So we get
\[
\sum_{\rho} z^{-\rho} \Gamma(\rho) \ll \sum_{\gamma > 0} |z|^{-1/2} \exp\left(-\gamma \arctan\left(\frac{1}{2\pi N\eta}\right)\right) + \sum_{\gamma < 0} |z|^{-1/2} \exp\left(-|\gamma| \arctan\left(\frac{1}{2\pi N\eta}\right)\right).
\]
We investigate only the case $\gamma > 0$ since the other one is similar. We split $\sum_{\gamma > 0}$ according to the cases $\gamma > 1$ and $\gamma \leq 1$ and we denote the first sum as $\sum_1$ and the second one as $\sum_2$. Hence, using (9), we have
\[
\sum_1 \ll |z|^{-1/2} \sum_{m=1}^{\infty} \log(q(m + 1)) \exp\left(-m \arctan\left(\frac{1}{2\pi N\eta}\right)\right) \ll |z|^{-1/2} N \log(q) \ll N|\eta|^{1/2} \log(qN). \tag{27}
\]

Arguing analogously we obtain
\[
\sum_2 \ll |z|^{-1/2} \sum_{0 \leq \gamma \leq 1} \exp\left(-\gamma \arctan\left(\frac{1}{2\pi N\eta}\right)\right) \ll |z|^{-1/2} \log(q + 1) \ll |\eta|^{-1/2} \log(qN). \tag{28}
\]
Lemma 6 now follows inserting (26)-(28) in (25).
Our next lemma concerns the mean-square of the quantity studied in Lemma 1 and it should be considered as a sharper version of equation (7.15) of Friedlander-Goldston [3]. Its proof follows the argument in Theorem 1 of Languasco-Perelli [8]: see also section 5 of [9]. We insert here just the relevant changes.

Lemma 7. Assume GRH, let $z$ be as in [8], $1 \leq q \leq Q$ and $Q < N^{1/2}$. Then

$$
\sum_{a=1}^{q} \int_{-1/4Q}^{1/4Q} \left| \hat{S} \left( \frac{a}{q} + \eta \right) - \frac{\mu(q)}{\phi(q)z} \right|^2 \, d\eta \ll \frac{N}{Q} \log^2 N.
$$

Proof. Assuming GRH, by [13–15] we have

$$
\tilde{S} \left( \frac{a}{q} + \eta \right) - \frac{\mu(q)}{\phi(q)z} = -\frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a) \tau(\chi) \sum_{\rho} z^{-\rho} \Gamma(\rho) + O \left( q^{1/2} \log^2 (qN) \right)
$$

where $\rho = 1/2 + i\gamma$ runs over the non-trivial zeros of $L(s, \chi)$ and $\chi \mod q$ is a Dirichlet character. By the character orthogonality and the previous equation we have

$$
\sum_{a=1}^{q} \int_{-1/4Q}^{1/4Q} \left| \tilde{S} \left( \frac{a}{q} + \eta \right) - \frac{\mu(q)}{\phi(q)z} \right|^2 \, d\eta \ll \frac{q}{\phi(q)} \sum_{\chi \mod q} \int_{-1/4Q}^{1/4Q} \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta + \frac{q \log^4 (qN)}{Q}. \quad (29)
$$

Since $z^{-\rho} = |z|^{-\rho} \exp (-i \rho \arctan 2 \pi N \eta)$, by the Stirling formula we have that

$$
\sum_{\rho} z^{-\rho} \Gamma(\rho) \ll \sum_{\rho} |z|^{-1/2} \exp \left( \gamma \arctan 2 \pi N \eta - \frac{\pi}{2} |\gamma| \right).
$$

If $\gamma \eta \leq 0$ or $|\eta| \leq 1/N$ we get, by (9), that

$$
\sum_{\rho} z^{-\rho} \Gamma(\rho) \ll N^{1/2} \log (q + 1),
$$

where $c_1 > 0$ is an absolute constant and, in the first case, $\rho$ runs over the zeros with $\gamma \eta \leq 0$.

Let $\xi = 1/(qQ)$. From $1 \leq q \leq Q$ and $Q < N^{1/2}$, we have $\xi > 1/N$ and we obtain

$$
\int_{-\xi}^{\xi} \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta \ll \int_{1/N}^{\xi} \left| \sum_{\gamma > 0} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta + \int_{-\xi}^{1/N} \left| \sum_{\gamma < 0} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta + N \xi \log^2 (q + 1). \quad (30)
$$

We will treat only the first integral on the right hand side of (30), the second being completely similar. Clearly

$$
\int_{1/N}^{\xi} \left| \sum_{\gamma > 0} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta = \sum_{k=1}^{K} \int_{\tau}^{2\tau} \left| \sum_{\gamma > 0} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta + O(1) \quad (31)
$$

where $\tau = \tau_k = \xi/2^k$, $1/N \leq \tau \leq \xi/2$ and $K$ is a suitable integer satisfying $K = O(\log N)$.

We can now proceed exactly as at page 312–314 of [8]. We obtain

$$
\int_{2\tau}^{\xi} \left| \sum_{\gamma > 0} z^{-\rho} \Gamma(\rho) \right|^2 \, d\eta \ll \sum_{\gamma_1 > 0} \sum_{\gamma_2 > 0} \frac{1 + \left( \frac{\gamma_1 + \gamma_2}{N\tau} \right)^2}{1 + |\gamma_1 - \gamma_2|^2} \exp \left( -c \left( \frac{\gamma_1 + \gamma_2}{N\tau} \right) \right). \quad (32)
$$
But
\[
\left\{ 1 + \left( \frac{\gamma_1 + \gamma_2}{N\tau} \right)^2 \right\} \exp \left( -c_2 \left( \frac{\gamma_1 + \gamma_2}{N\tau} \right) \right) \ll \exp \left( -\frac{c_2}{2} \frac{\gamma_1}{N\tau} \right),
\]
hence the right hand side of (32) becomes
\[
\ll \sum_{\gamma_1 > 0} \exp \left( -\frac{c_2}{2} \frac{\gamma_1}{N\tau} \right) \sum_{\gamma_2 > 0} \frac{1}{1 + |\gamma_1 - \gamma_2|^2}.
\] (33)

Since the number of zeros \( \rho_2 = 1/2 + i\gamma_2 \) with \( m \leq |\gamma_1 - \gamma_2| \leq m + 1 \) is \( O(\log(q(m + |\gamma_1|))) \), we immediately get
\[
\sum_{\gamma_1 > 0} \exp \left( -\frac{c_2}{2} \frac{\gamma_1}{N\tau} \right) \sum_{\gamma_2 > 0} \frac{1}{1 + |\gamma_1 - \gamma_2|^2} \ll \int_0^\infty (\log^2(qt)) \exp \left( -\frac{c_2}{2} \frac{t}{N\tau} \right) dt.
\]

The function \( (\log^2(qt)) \exp(-\frac{t}{4N\tau}) \) has a maximum attained at \( t_0 \) such that \( t_0 \log(qt_0) = 8N\tau/c_2 \). Hence the right hand side of the previous equation is
\[
\ll \int_0^{1/q} (\log^2(qt)) \exp \left( -\frac{c_2}{2} \frac{t}{N\tau} \right) dt + \int_{1/q}^{t_0} (\log^2(qt)) \exp \left( -\frac{c_2}{2} \frac{t}{N\tau} \right) dt
\]
\[
+ \int_{t_0}^{\infty} (\log^2(qt)) \exp \left( -\frac{c_2}{2} \frac{t}{N\tau} \right) dt
\]
\[
\ll t_0(\log^2(qt_0)) + N\tau(\log^2(qt_0)) \exp \left( -\frac{c_2}{2} \frac{t_0}{N\tau} \right)
\]
\[
\ll N\tau(\log^2(qt_0)) \exp \left( -\frac{c_2}{2} \frac{t_0}{N\tau} \right) \ll N\tau \log^2(qN).
\] (34)

Hence, inserting (34) into (32) - (33), we get
\[
\int_{\tau}^{2\tau} \left| \sum_{\gamma_1 > 0} z^{-\ell_1} \Gamma(\rho) \right|^2 d\eta \ll N\tau \log^2(qN).
\] (35)

Inserting now (34) into (30) - (31) we get
\[
\int_{-1/q}^{1/q} \sum_{\rho} z^{-\ell_1} \Gamma(\rho) \right|^2 d\eta \ll \frac{N}{qQ} \log^2(qN)
\]
and hence, by (29), Lemma 7 follows. \( \square \)

The next lemma will be useful in the computation of the main term in Theorem 1. We insert the proof already contained in Languasco-Perelli [8] for \( k = 2 \) and in Languasco [7] for \( k \geq 3 \).

**Lemma 8.** Let \( k \geq 2 \) be an integer, \( z \) be as in (8) and \( Q \leq N^{1/2}/2 \). Then, uniformly for \( 1 \leq \ell \leq N \), we have
\[
\int_{\xi_{\ell,a}} \frac{e(-\ell \eta)}{z^k} d\eta = e^{-\ell/N} \frac{\ell^{k-1}}{(k-1)!} + O((qQ)^{k-1}).
\]

**Proof.** Let \( T \geq 1/2 \). Using (8) we get
\[
\int_{\xi_{\ell,a}} \frac{e(-\ell \eta)}{z^k} d\eta = \int_{-T}^T \frac{e(-\ell \eta)}{z^k} d\eta + O \left( \int_{-\frac{T}{qQ}}^\infty \frac{d\eta}{|z|^k} + \int_{-T}^{-\frac{T}{qQ}} \frac{d\eta}{|z|^k} \right)
\]
\[
= \int_{-T}^T \frac{e(-\ell \eta)}{z^k} d\eta + O((qQ)^{k-1}).
\] (36)
Using the variable $z = N^{-1} - 2\pi i \eta$ in place of $\eta$, we have
\[
\int_{-T}^{T} \frac{e(-\ell \eta)}{z^k} \, d\eta = \frac{e^{-\ell/N}}{2\pi i} \int_{\frac{1}{N} - 2\pi i T}^{\frac{1}{N} + 2\pi i T} \frac{\exp(\ell z)}{z^k} \, dz.
\] (37)

Let $\Gamma$ denote the left half of the circle $|z - N^{-1}| = 2T$. By the residue theorem we obtain
\[
\frac{e^{-\ell/N}}{2\pi i} \int_{\frac{1}{N} - 2\pi i T}^{\frac{1}{N} + 2\pi i T} \frac{\exp(\ell z)}{z^k} \, dz = e^{-\ell/N} \frac{\ell^{k-1}}{(k-1)!} + \frac{e^{-\ell/N}}{2\pi i} \int_{\Gamma} \frac{\exp(\ell z)}{z^k} \, dz
\]
\[= e^{-\ell/N} \frac{\ell^{k-1}}{(k-1)!} + O\left(\frac{1}{T^{k-1}}\right).
\] (38)

Lemma 8 now follows from (36)-(38) letting $T \rightarrow \infty$. □

Lemma 9 below follows inserting Lemma 7 and (8) in the body of the proof of Lemma 5 of Friedlander-Goldston [3].

**Lemma 9.** Assume GRH and let $z$ be as in (S). Then, for any real $c > 0$, we have
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \frac{1}{\phi(q)^c} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right| \left| \frac{\mu(q)}{\phi(q)} \right|^2 \, d\eta \ll N^2 \log^2 N
\]
and, for $c = 0$, the same result holds replacing $\log^2 N$ with $\log^3 N$.

Let now
\[
S^*(Q) = \max_{q \leq Q} \max_{a,q,a} \max_{\eta \in \xi_{q,a}} |\tilde{R}(\eta; q, a, z)|.
\] (39)

We have

**Lemma 10.** Let $m \geq 2$ and $z$ be as in (S). Then
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right|^m \frac{\mu(q)}{\phi(q)z} \, d\eta \ll (S^*(Q))^{m-2} N \log N.
\]

Assuming GRH we have
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right|^m \left| \frac{\mu(q)}{\phi(q)} \right|^2 \, d\eta \ll (S^*(Q))^{m-2} N^{3/2} \log^2 N,
\]
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right|^m \left| \frac{\mu(q)}{\phi(q)} \right|^2 \, d\eta \ll (S^*(Q))^{m-2} N^2 \log^2 N,
\]
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right|^m \left| \frac{\mu(q)}{\phi(q)} \right|^2 \, d\eta \ll N^{3/2} \log^2 N,
\]
and, for $r \geq 3$,
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right|^m \left| \frac{\mu(q)}{\phi(q)} \right|^r \, d\eta \ll (S^*(Q))^{m-2} N^r \log^2 N,
\]
\[
\sum_{q=1}^{Q} \sum_{a=1}^{q^*} \int_{-1/qQ}^{1/qQ} \left| \tilde{S}\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\phi(q)} \eta \right|^m \left| \frac{\mu(q)}{\phi(q)} \right|^r \, d\eta \ll N^{r-1/2} (\log N)^{3/2}.
\]

The proof of Lemma 10 follows using Lemmas 3, 7, 9 arguing as in Lemma 3 of Friedlander-Goldston [3].
3. Proof of the main result

We consider the usual Farey dissection of level \(Q\) of the unit interval as in (7). By \(\Xi\), we have

\[
e^{-n/N} R_k(n) = \int_0^1 \tilde{S}(\alpha)^k e(-n\alpha) \, d\alpha
\]

\[
= \sum_{q=1}^Q \left( \frac{\mu(q)}{\phi(q)} \right)^k \sum_{a=1}^q e\left(-\frac{n}{q}a\right) \int_{\xi_{q,a}} V(\eta)^k e(-n\eta) \, d\eta
\]

\[
+ k \sum_{q=1}^Q \left( \frac{\mu(q)}{\phi(q)} \right)^{k-1} \sum_{a=1}^q e\left(-\frac{n}{q}a\right) \int_{\xi_{q,a}} \tilde{R}(\eta; q, a, V) V(\eta)^{k-1} e(-n\eta) \, d\eta
\]

\[
+ \sum_{m=2}^k \left( \frac{k}{m} \right) \sum_{q=1}^Q \left( \frac{\mu(q)}{\phi(q)} \right)^{k-m} \sum_{a=1}^q e\left(-\frac{n}{q}a\right) \int_{\xi_{q,a}} \tilde{R}(\eta; q, a, V)^m V(\eta)^{k-m} e(-n\eta) \, d\eta
\]

\[
= M_0(k) + kM_1(k) + \sum_{m=2}^k \left( \frac{k}{m} \right) M_m(k),
\]

say.

3.1. Main term \(M_0(k)\). By Lemma \(\Xi\) we can write

\[
\int_{\xi_{q,a}} V(\eta)^k e(-n\eta) \, d\eta = \int_{\xi_{q,a}} e(-n\eta) \frac{d\eta}{\eta^{k}} + O\left( \int_{1/(qQ)^{2}}^{1/(qQ)^{1}} \frac{d\eta}{\eta^{k-1}} \right)
\]

and, using \(\Xi\), the error term in the previous equation is

\[
\ll \int_{-1/N}^{1/N} N^{k-1} \, d\eta + \int_{1/(qQ)^{1}}^{1/(qQ)^{2}} \frac{d\eta}{|\eta|^{k-1}} \ll \begin{cases} 
\log(N/qQ) & \text{if } k = 2, \\
N^{k-2} & \text{if } k > 2.
\end{cases}
\]

Combining the previous two equations with Lemma \(\Xi\) for \(k \geq 3\) we have

\[
\int_{\xi_{q,a}} V(\eta)^k e(-n\eta) \, d\eta = e^{-\epsilon/N} \frac{n^{k-1}}{(k-1)!} + O_k((qQ)^{k-1} + N^{k-2})
\]

uniformly for \(1 \leq n \leq N\) and \(Q \leq N^{1/2}/2\). Recalling the definition for the Ramanujan sum \(c_q\) in (3) and for the singular series \(\mathcal{S}_k(n)\) in (2), we get

\[
M_0(k) = e^{-n/N} \sum_{q=1}^Q \left( \frac{\mu(q)}{\phi(q)} \right)^k c_q(-n) \frac{n^{k-1}}{(k-1)!} + O\left( \sum_{q=1}^Q \frac{|c_q(-n)\mu(q)|}{\phi(q)^k} ((qQ)^{k-1} + N^{k-2}) \right)
\]

\[
= e^{-n/N} \frac{n^{k-1}}{(k-1)!} \mathcal{S}_k(n) + O\left( \frac{n^{k-1}}{k} \sum_{q>Q} \frac{\mu^2(q)}{\phi(q)^k} |c_q(-n)| \right)
\]

\[
+ O\left( \frac{Q^{k-1}}{k} \sum_{q=1}^Q \frac{\mu^2(q)}{\phi(q)^k} \right) + O\left( N^{k-2} \sum_{q=1}^Q \frac{1}{\phi(q)^{k-1}} \right)
\]

\[
= e^{-n/N} \frac{n^{k-1}}{(k-1)!} \mathcal{S}_k(n) + O\left( Q^k + n^{k-1}Q^{1-k} + N^{k-2} \right),
\]

(41)
since \( k \geq 3 \) and 
\[
\sum_{q>Q} \frac{\mu^2(q)}{\phi(q)^k} |c_q(-n)| \leq \sum_{q>Q} \frac{\mu^2(q)}{\phi(q)^k} \frac{d}{d\eta} \sum_{d|n} \frac{d\mu^2(d)}{\phi(d)^k} \sum_{q'>Q/d} \frac{\mu^2(q')}{\phi(q')^k} \\
\ll Q^{1-k} \sum_{d|n} \frac{d\mu^2(d)}{\phi(d)^k} \ll Q^{1-k} n^\varepsilon,
\]
using also Lemma 2 of Goldston [5]. Then for \( n = N, k \geq 3 \) and \( Q = N^{1/2}/2 \) the error terms are under control, since we have to compare them with the order of magnitude of the secondary main term which is \( \approx N^{k-3/2} \).

3.2. Secondary main term \( M_1(k) \). Equation (40) implies that
\[
M_1(k) = \sum_{q=1}^{Q} \left( \frac{\mu(q)}{\phi(q)} \right)^{k-1} \sum_{a=1}^{q} e\left(-n\frac{a}{q}\right) \int_{\xi_{q,a}} \tilde{R}(\eta; q, a, V) V(\eta)^{k-1} e(-n\eta) \ d\eta.
\]
We set \( \theta_{q,a} = (-1/2, 1/2) \setminus \xi_{q,a} \) so that \( M_1(k) = A - B \), say, where
\[
A := \sum_{q=1}^{Q} \left( \frac{\mu(q)}{\phi(q)} \right)^{k-1} \sum_{a=1}^{q} e\left(-n\frac{a}{q}\right) \int_{1/2}^{1/2} \tilde{R}(\eta; q, a, V) V(\eta)^{k-1} e(-n\eta) \ d\eta,
\]
\[
B := \sum_{q=1}^{Q} \left( \frac{\mu(q)}{\phi(q)} \right)^{k-1} \sum_{a=1}^{q} e\left(-n\frac{a}{q}\right) \int_{\theta_{q,a}} \tilde{R}(\eta; q, a, V) V(\eta)^{k-1} e(-n\eta) \ d\eta.
\]
In order to estimate \( B \), we first remark that
\[
|\tilde{R}(\eta; q, a, V)| \ll |S\left(\frac{a}{q} + \eta\right)| + \frac{\mu^2(q)}{\phi(q)} |V(\eta)| \ll \sum_{n \geq 1} \Lambda(n) e^{-n/N} + \frac{N}{\phi(q)} \ll N
\]
by (10) and the Prime Number Theorem. Hence, since \( \theta_{q,a} \subset (-1/2, -1/(2qQ)) \cup (1/(2qQ), 1/2) \), we obtain
\[
|B| \ll N \sum_{q=1}^{Q} \frac{\mu^2(q)}{\phi^{k-2}(q)} \left( \int_{1/(2qQ)}^{1/2} + \int_{-1/2}^{-1/(2qQ)} \right) |V(\eta)|^{k-1} d\eta \ll_k N Q^{k-2} \sum_{q=1}^{Q} \frac{\mu^2(q)q^{k-2}}{\phi^{k-2}(q)}
\]
\[
\ll_k N Q^{k-1},
\]
by (10) and Lemma 2 of Goldston [5]. We explicitly remark that the usual strategy to estimate \( B \) involves the Cauchy-Schwarz inequality. In this case this would lead to \( |B| \ll_k (N \log N)^{1/2} Q^k \) which is worse than our estimate for \( Q > (N/ \log N)^{1/2} \). In this case the optimal choice of \( Q \) will be \( N^{1/2}/2 \), see (3.3) below, and hence our estimate is slightly sharper. Summing up,
\[
M_1(k) = \sum_{q=1}^{Q} \left( \frac{\mu(q)}{\phi(q)} \right)^{k-1} \sum_{a=1}^{q} e\left(-n\frac{a}{q}\right) \int_{1/2}^{1/2} \tilde{R}(\eta; q, a, V) V(\eta)^{k-1} e(-n\eta) \ d\eta
\]
\[
+ \mathcal{O}_k(N Q^{k-1}).
\]
Inserting the approximation (12), we have
\[
M_1(k) = \sum_{q=1}^{Q} \frac{\mu(q)q^{k-1}}{\phi(q)^k} \sum_{a=1}^{q} e\left(-n\frac{a}{q}\right) \sum_{\chi \bmod q} \chi(a) \tau(\chi) \int_{1/2}^{1/2} W(\chi, \eta, V) V(\eta)^{k-1} e(-n\eta) \ d\eta
\]
where, by Lemma 2 of Goldston [5], the last error term in (42) is
\[ \ll k N^{k-2} \log^2(QN) \sum_{q=1}^{Q} \frac{\mu(q)^{k-1}}{\phi(q)k} \ll k N^{k-2} \log^2(QN) f(Q, k), \]
where
\[ f(Q, k) = \begin{cases} \log Q & \text{if } k = 3, \\ 1 & \text{if } k \geq 4. \end{cases} \]

By (4), we can now write
\[
M_1(k) = \sum_{q=1}^{Q} \frac{\mu(q)^{k-1}}{\phi(q)k} \sum_{\chi \mod q} c_{\chi}(-n) \tau(\chi) \int_{-1/2}^{1/2} W(\chi, \eta, V) (\eta)^{k-1} e(-n\eta) \, d\eta \\
+ O \left( NQ^{k-1} + N^{k-2} \log^2(QN) f(Q, k) \right).
\]

Assuming GRH holds and using Lemma 5 we obtain
\[
M_1(k) = e^{-n/N} \sum_{q=1}^{Q} \frac{\mu(q)^{k-1}}{\phi(q)k} \sum_{\chi \mod q} c_{\chi}(-n) \tau(\chi) \left( \psi_{k-2}(n, \chi) - \delta(\chi) \frac{n^{k-1}}{(k-1)!} \right) \\
+ O \left( NQ^{k-1} + N^{k-2} \log^2(QN) f(Q, k) \right) \\
+ O \left( n^{k-2} \sum_{q=1}^{Q} \frac{\mu(q)^2}{\phi(q)^k} \sum_{\chi \mod q} |c_{\chi}(-n)\tau(\chi)| \right). \tag{42}
\]

Using Lemma 4 with \( j = k - 2 \) and remarking that in this case the error term is \( \log^2(Qn) \) times the last error term of (12), we obtain
\[
M_1(k) = -e^{-n/N} \sum_{q=1}^{Q} \frac{\mu(q)^{k-1}}{\phi(q)k} \sum_{\chi \mod q} c_{\chi}(-n) \tau(\chi) \sum_{\rho} \frac{n^{\rho+k-2}}{\rho(\rho+1) \cdots (\rho+k-2)} \\
+ O \left( NQ^{k-1} + N^{k-2} \log^2(QN) f(Q, k) + n^{k-2} \log^2(Qn) g(Q, k) \right), \tag{43}
\]
where, by Lemma 2 of Goldston [5], the last error term in (12) is
\[
\ll n^{k-2} \sum_{q=1}^{Q} \frac{\mu(q)^2 q^{1/2}}{\phi(q)^{k-2}} \ll k n^{k-2} \begin{cases} Q^{1/2} & \text{if } k = 3, \\ 1 & \text{if } k \geq 4. \end{cases} = n^{k-2} g(Q, k).
\]

We remark that the summation over \( q \) can be extended to all \( q \) by inserting a new error term which is \( O \left( n^{k-2+\Theta} Q^{1-k+\epsilon} \right) \), where \( \Theta = \sup_{\chi \mod q} \{ \beta : L(\beta + i\gamma, \chi) = 0 \} \), see p. 296 of [3]. For \( n = N \) and \( Q = N^{1/2} / 2 \), the previous unconditional estimate becomes admissible for \( k \geq 5 \) and hence we can “just” assume that GRH holds for every \( q \leq N^{1/2} / 2 \) (the order of magnitude of the secondary main term is \( \approx N^{k-3/2} \)). Assuming GRH in its “full strength” the tail of the singular series gives a contribution of \( O \left( n^{k/2+\epsilon} \right) \) which is admissible for \( k \geq 4 \).

Hence we can finally say, for \( n = N, k \geq 5 \) and \( Q = N^{1/2} / 2 \), that the error terms are under control under the assumption of GRH for every \( q \leq N^{1/2} / 2 \).
3.3. The error terms. Essentially, they are estimated as in Languasco [7]. Assuming
the Generalized Riemann Hypothesis, we have, by Lemma 6 and (39), the estimate
\[ S^*(Q) \ll \max_{q \leq Q} \max_{(a,q)=1} \max_{\eta \in \xi_{q,a}} \left[ \log(qN)(N \sqrt{q|\eta|} + \sqrt{qN}) \right] \ll \left( \frac{N}{\sqrt{Q}} + \sqrt{QN} \right) \log(QN). \tag{44} \]

The optimal \( Q \) in (44) is \( Q = N^{1/2}/2 \) and in this case we get
\[ S^*(Q) \ll N^{3/4} \log N. \tag{45} \]

Using this notation and Lemma 10, we can write the following bounds:
i) for \( m = k, m \geq 2 \), we unconditionally get
\[ M_k(k) \ll_k \left( S^*(Q) \right)^{k-2} N \log N; \]
ii) for \( m = k - 1, m \geq 2, k \geq 3 \), assuming GRH, we obtain
\[ M_{k-1}(k) \ll_k \left( S^*(Q) \right)^{k-3} N^{3/2} \log^2 N; \]
iii) for \( m = k - 2, m \geq 2, k \geq 4 \), assuming GRH, we obtain
\[ M_{k-2}(k) \ll_k \left( S^*(Q) \right)^{k-4} N^2 \log^3 N; \]
iv) for \( 2 \leq m \leq k - 3, k \geq 5 \), assuming GRH, we obtain
\[ M_m(k) \ll_k \left( S^*(Q) \right)^{m-2} N^{k-m} \log^2 N. \]

4. Conclusion of the proof

We restrict our analysis to \( k \geq 5 \) since the error terms in 3.2 are under control only
in this case. If \( k = 5 \), the expected main term has size \( N^4 \) and error terms \( O(N^3) \), the
secondary main term has expected size \( N^{7/2} \) and \( M_1(5) \) has an error term \( O(N^{5/2} \log^2 N) \).
Moreover, using (45), we get
\[ M_2(5) \ll N^3 \log^2 N, \quad M_3(5) \ll N^{7/4} \log^4 N, \quad M_4(5) \ll N^3 \log^4 N, \quad M_5(5) \ll N^{13/4} \log^4 N \]
and hence the global error term in this case is \( N^{13/4} \log^4 N \).

If \( k \geq 6 \), the expected main term has size \( N^{k-1} \) and error terms \( O(N^{k-2}) \), the
secondary main term has expected size \( N^{k-3/2} \) and \( M_1(k) \) has an error term \( O(N^{k-2} \log^2 N) \).
Moreover, again by (45), we obtain
\[ M_m(k) \ll_k N^{k-3/2-m/4} \log^m N \quad \text{for} \quad 2 \leq m \leq k - 3, \]
\[ M_{k-2}(k) \ll_k N^{(3/4)k-1} \log^{k-1} N, \quad M_{k-1}(k) \ll_k N^{(3/4)k-3/4} \log^{k-1} N, \]
\[ M_k(k) \ll_k N^{(3/4)k-1/2} \log^{k-1} N. \]

The maximum for \( M_m(k) \) is attained at \( m = 2 \) and is \( \ll_k N^{k-2} \log^2 N \). Hence, for \( k = 6 \)
the global upper bound is \( \ll N^4 \log^5 N \) while for \( k \geq 7 \) it is \( \ll_k N^{k-2} \log^2 N \).

Combining the previous remarks with (44)-(45) and (43), the Theorem follows.
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