ON THE INTEGRABILITY OF STATIONARY AND
RESTRICTED FLOWS OF THE KDV HIERARCHY

G. TONDO

Abstract. A bi–Hamiltonian formulation for stationary flows of
the KdV hierarchy is derived in an extended phase space. A map
between stationary flows and restricted flows is constructed: in a
case it connects an integrable Hénon–Heiles system and the Gar-
nier system. Moreover a new integrability scheme for Hamiltonian
systems is proposed, holding in the standard phase space.

Date: Dipartimento di Scienze Matematiche, Università degli Studi di Trieste,
Piaz.le Europa 1, I34127 Trieste, Italy.
1991 Mathematics Subject Classification. Primary 58F07; Secondary 35Q58.
Work partially supported by the GNFM of the Italian CNR and by the project
"Metodi Geometrici e probabilistici in Fisica Matematica" of the Italian MURST.
1. Introduction

In the last years there has been an increasing interest for the construction of finite–dimensional dynamical systems from soliton equations, through the so–called methods of stationary flows and restricted flows (see [1], [2] and references therein). The discovery of suitable sets of coordinates has allowed one to write the reduced systems as physically interesting Hamiltonian systems. In the case of the KdV hierarchy, the $q$–representation for stationary flows has given rise to the Hénon–Heiles system [3, 4], the square eigenfunctions representation for restricted flows has furnished the Neumann and the Garnier systems [5, 6]. However the relation between dynamical systems which are obtained through different reduction techniques from the same soliton hierarchy is not clear; moreover a systematic way to find the second Hamiltonian formulation for stationary flows of any order, without the use of a Miura map, is still lacking.

The aim of this paper is to give a contribution in these directions. In particular:

i) A bi–Hamiltonian formulation for stationary flows of the KdV hierarchy in a suitably extended phase space is derived in a systematic way. As an example, the bi–Hamiltonian structure of Hénon–Heiles–type systems is explicitly shown.

ii) A map between stationary and restricted flows of the KdV hierarchy is obtained, based on the generating function of the Gelfand–Dickey (GD) polynomials. As an application, a map between an integrable Hénon–Heiles system and the Garnier system with two degrees of freedom is constructed.

iii) An integrability criterion is proposed, which can be applied to both stationary and restricted flows. Though weaker than the bi–Hamiltonian formulation, it does not require the extension of the phase spaces.

The paper is organized as follows. In Sect. 2 we construct the stationary flows associated to the the KdV hierarchy through the kernel of the Poisson pencil. Using the generating function of GD polynomials as in [7], we give a bi–Lagrangian and a bi–Hamiltonian formulation of the Lax–Novikov stationary equations of any order; as an application, we exhibit a generalized Hénon–Heiles system.

In Sects. 3–4 we formulate the method of restricted flows in terms of the Poisson pencil instead of the spectral problem as in [5, 6]. This formulation allows us to explicitly construct a map between restricted
ON THE INTEGRABILITY OF STATIONARY AND RESTRICTED FLOWS

and stationary flows, by means of an appropriate extension of the corresponding phase spaces. The previous map is specialized to the Hénon–Heiles and the Garnier systems.

In Sect. 5 we show that the entire bi–Hamiltonian hierarchy of the Hénon–Heiles and the Garnier systems cannot be reduced from the extended to the standard phase space. For this reason, we propose an integrability criterion holding for a generic finite–dimensional Hamiltonian system. It generalizes the criterion introduced in [8] for the particular case of the Hénon–Heiles system. Though weaker than the bi–Hamiltonian scheme, it assures Liouville–integrability of a Hamiltonian system [9] in its standard phase space, i.e. without the introduction of supplementary coordinates. This criterion is applied to the generalized Hénon–Heiles system and to the Garnier system with two degrees of freedom.

Now we give some preliminaries, mainly to fix notations and terminology. Let $M$ be a $n$–dimensional manifold. At any point $u \in M$, the tangent and cotangent spaces are denoted by $T_u M$ and $T^*_u M$, the pairing between the two spaces by $\langle \cdot , \cdot \rangle : T^*_u M \times T_u M \to \mathbb{R}$. For each smooth function $f \in C^\infty (M)$, $df$ denotes the differential of $f$. $M$ is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{ , \} : C^\infty (M) \times C^\infty (M) \to C^\infty (M)$, possibly a degenerate one; the associated Poisson tensor $P$ is defined by $\{ f , g \}(u) := \langle df(u) , P_u dg(u) \rangle$. So, at each point $u$, $P_u$ is a linear map $P_u : T^*_u M \to T_u M$, skew–symmetric and with vanishing Schouten bracket [10]. A function $h \in C^\infty (M)$ with a non trivial differential $df \in \text{Ker} P$ is called a Casimir of $P$: $P_u df(u) = 0$. A map $\Phi : M \to M$ is a Poisson morphism if $\{ f , g \} \circ \Phi = \{ f \circ \Phi , g \circ \Phi \}$, for each $f , g \in C^\infty (M)$; $\Phi$ leaves invariant the Poisson tensor $P$: $P_{\Phi(u)} = \Phi_* P_u \Phi^*$, where $\Phi_*$ and $\Phi^*$ denote, respectively, the tangent and the cotangent maps associated to $\Phi$. In particular, if the Poisson bracket is non degenerate, i.e. if $P$ is invertible, and the Poisson morphism is a diffeomorphism, $\Phi$ defines a symplectic (canonical) transformation. $M$ is said to be a bi–Hamiltonian manifold if it is endowed with two Poisson tensors $P_0$ and $P_1$ such that the associated pencil $P^\lambda := P_1 - \lambda P_0$ be itself a Poisson tensor for any $\lambda \in \mathbb{C}$ [11, 12].

2. STATIONARY FLOWS AND HÉNON–HEILES SYSTEMS

2.1. KdV hierarchy and Gelfand–Dickey polynomials. Let $M$ be a bi–Hamiltonian manifold: if the associated Poisson pencil $P^\lambda := P_1 - \lambda P_0$ admits as a Casimir a formal Laurent series $h(\lambda)$
\[ h(\lambda) := \sum_{j \geq 0} h_j \lambda^{-j}, \quad (2.1) \]

then \( h_0 \) is a Casimir of \( P_0 \) and the coefficients \( h_j \ (j \geq 1) \) are the Hamiltonian functions of a hierarchy of bi–Hamiltonian vector fields \( X_j \):

\[ X_j = P_1 dh_j = P_0 dh_{j+1} \quad (j \geq 0) . \quad (2.2) \]

At any point \( u \in M \), the bi–Hamiltonian flows are given by \( du/dt_j = X_j(u) \), \( t_j \) being the evolution parameter of the \( j \)th flow. The vector fields \((2.2)\) are Hamiltonian also with respect to the Poisson pencil \( P^\lambda \).

In fact the recursion relation \((2.2)\) can be written as

\[ X_j = P^\lambda dh^{(j)}(\lambda), \quad h^{(j)}(\lambda) := (\lambda^j h(\lambda))_+ , \quad (2.3) \]

where the index + means the projection of a Laurent series onto the purely polynomial part.

Let \( M \) be the algebra of polynomials in \( u, u_x, u_{xx}, \ldots \) \((u = u(x)\) is a \( C^\infty \) function of \( x \) and the subscript \( x \) means the derivative with respect to \( x \)), and let \( P_0 \) and \( P_1 \) be the two Poisson tensors of the KdV hierarchy \([11]\)

\[ P_0 := \frac{d}{dx}, \quad P_1 := \frac{d^3}{dx^3} + 4u \frac{d}{dx} + 2u_x . \quad (2.4) \]

The gradients of the Casimirs of the associated Poisson pencil \( P^\lambda \) can be obtained searching for the 1–forms \( v(\lambda) := \sum_{j \geq 0} v_j \lambda^{-j} \) which are solutions of the following equation

\[ B^\lambda(v(\lambda), v(\lambda)) = a(\lambda) , \quad (2.5) \]

where \( a(\lambda) = \sum_{j \geq -1} a_j \lambda^{-j} \), \( a_j \) are constant parameters and \( B^\lambda \) is the bilinear function

\[ B^\lambda(w_1, w_2) := w_{1xx}w_2 + w_1w_{2xx} - w_{1x}w_{2x} + (4u - \lambda)w_1w_2 . \quad (2.6) \]

In fact \( B^\lambda \) is related to the Poisson pencil through the relation

\[ \frac{d}{dx} B^\lambda(w_1, w_2) = w_1 P^\lambda w_2 + w_2 P^\lambda w_1 \quad (\forall w_1, w_2) . \quad (2.7) \]

Eq. \((2.5)\) can be solved developing the left hand side as a Laurent series.
\[ B^\lambda(v(\lambda), v(\lambda)) = \sum_{k \geq -1} B_k \lambda^{-k} \quad (2.8) \]

so that, for each \( a(\lambda) \), it furnishes the coefficients of the solution \( v(\lambda) \) (unique up to a sign). The solution corresponding to \( \bar{a}(\lambda) = -\lambda \) is the so-called basis solution \( \bar{v}(\lambda) \); its first coefficients are:

\[ \bar{v}_0 = 1, \quad \bar{v}_1 = 2 u, \quad \bar{v}_2 = 2(u_{xx} + 3u^2), \quad \bar{v}_3 = 2(u^{(4)} + 5u_x^2 + 10u_{xx}u + 10u^3) \quad (2.9) \]

and so on, namely the gradients of the first KdV Hamiltonians. In the following we shall consider also the 1–form \( v(\lambda) = c(\lambda)\bar{v}(\lambda) \), which is solution of (2.5) for

\[ a(\lambda) = -\lambda c^2(\lambda), \quad c(\lambda) = 1 + \sum_{j \geq 1} c_j \lambda^{-j} \quad (2.10) \]

where the coefficient \( c_j \) are free parameters. In this case the first 1–forms of the hierarchy are \( v_0 = 1, v_1 = \bar{v}_1 + c_1, v_2 = \bar{v}_2 + c_1 \bar{v}_1 + c_2 \) and so on.

The coefficient \( B_k \) in (2.8) can be expressed through the GD polynomials. For each Laurent series \( v(\lambda) \) let us consider the functions \( B^{(k)}(\lambda) := B^\lambda \left( v(\lambda), v^{(k)}(\lambda) \right) \), where \( v^{(k)}(\lambda) := \left( \lambda^k v(\lambda) \right)_+ \); these functions have the form

\[ B^{(k)}(\lambda) = \lambda^{k+1} v_0^2 + \sum_{j=1}^{k-1} \lambda^{k-j} (p_{0j} - v_0 v_{j+1}) + \sum_{j \geq 0} \lambda^{-j} p_{jk} \quad (j, k \in \mathbb{N}_0) \quad (2.11) \]

It can be shown that

\[ B_{-1} = -v_0^2, \quad B_k = p_{0k} - v_0 v_{k+1} \quad (k \in \mathbb{N}_0) \quad (2.12) \]

furthermore, if \( v(\lambda) \) is a solution of Eq. (2.5), the coefficients \( p_{jk} \) in (2.11) are polynomials in \( u \) and its \( x \)–derivatives. They will be referred to as Gelfand–Dickey (GD) polynomials and the function \( B^\lambda \) as their generating function.

The fundamental property of the GD polynomials, stemming from (2.11), (2.5), (2.7) and (2.3), is the following relation with the gradients \( v_j = dh_j \) and the bi–Hamiltonian vector fields \( X_k \) :
\[ \frac{d}{dx} p_{jk} = v_j X_k. \] (2.13)

We report some GD polynomials to be used in the following \((v_0 = 1)\):

\[
\begin{align*}
p_{00} &= 4u - v_1, \\
p_{01} &= 8uv_1 - v_1^2 - v_2 + 2v_1x, \\
p_{02} &= 4uv_1^2 + 8uv_2 - 2v_1v_2 - v_3 - v_1^2 + 2v_1v_1x + 2v_2x, \\
p_{12} &= 8uv_1v_2 - v_2^2 + 4uv_3 - v_1v_3 - v_4 + 2v_1xv_2x + 2v_2v_1xx + 2v_1v_2xx + (2.14), \\
p_{kk} &= 2v_kxx v_k - v_k^2 + 4uv_k^2.
\end{align*}
\]

The GD polynomials corresponding to the basis solution \(\bar{v}(\lambda)\) are the polynomials defined in [1, Prop. 12.1.12].

2.2. The method of stationary flows. The method of stationary flows [13, 14, 15] was developed in order to reduce the flows of the KdV hierarchy onto the set \(M_n\) of fixed points of the \(n\)th flow \(X_n\) of the hierarchy:

\[ M_n := \{ u \mid X_n(u, u_x, \ldots, u^{(2n+1)}) = 0 \}. \] (2.15)

As \(M_n\) is odd-dimensional it cannot be a symplectic manifold; nevertheless we will show that it is a bi–Hamiltonian manifold: it will be referred to as extended phase space. Moreover, \(M_n\) is naturally foliated, on account of (2.2) and (2.4), by a one–parameter family of \(2n\)–dimensional submanifolds \(S_n\) given by

\[ S_n := \{ u \mid v_{n+1}(u, u_x, \ldots, u^{(2n)}) = c \} \] (2.16)

(c being a constant parameter), which are invariant manifolds with respect to each vector field of the KdV hierarchy, due to the invariance of the 1–forms \(v_k\). So \(M_n\) can be naturally parametrized by \(v_1, \ldots, v_{n+1}\) and by their \(x\)–derivatives \(v_{1x}, \ldots, v_{nx}\). We shall use these coordinates in the following.

Here we perform two different stationary reductions of the KdV flows by improving the procedure introduced in [7]. On one side, we choose as a reduction submanifold \(S_n^{(0)}\) just the leaf \(S_n\) of the foliation (2.16) corresponding to \(c = 0\); it is a level set of the GD polynomial \(p_{bn}\), due to (2.5), (2.8) and (2.12). On account of Eq. (2.13), also the GD polynomials \(p_{jn}\), restricted to \(M_n\), are invariant with respect to each flow of the hierarchy; thus we can choose as a second reduction submanifold \(S_n^{(1)}\) a level set of \(p_{nn}\). The one–parameter family of the
level sets of \( p_{nn} \) forms a foliation of the manifold \( M_n \) different from the previous one. Finally we construct the bi–Hamiltonian structure in the ground manifold \( M_n \).

From the computational point of view, one proceeds as follows.

**i)** Due to (2.3) and (2.5), the manifold \( M_n \) is defined by the solutions \( u \) of the equation

\[
B^\lambda \left( v(\lambda), v^{(n)}(\lambda) \right) = \lambda^n a(\lambda) ,
\]

(2.17)

where \( v(\lambda) = \sum_{j=0}^{n} v_j \lambda^{-j} \), \( a(\lambda) = \sum_{j=-1}^{2n} a_j \lambda^{-j} \). In particular if \( a(\lambda) = -\lambda c^2(\lambda) \), as in (2.10), \( M_n \) is given by

\[
M_n = \left\{ u | \bar{X}_n + \sum_{j=1}^{n} c_j \bar{X}_{n-j} = 0 \right\} ,
\]

(2.18)

i.e. by the solutions of the Lax–Novikov equations \( [13] \). Taking into account Eq. (2.11) and choosing \( a_{-1} = -1 \), equating in Eq. (2.17) the coefficients of \( \lambda^{n+1} \) we get \( v_0^2 = 1 \); from now on we put \( v_0 = 1 \). Moreover equating the coefficients of the other powers of \( \lambda \) we get the following system:

\[
p_{0k} - v_{k+1} = a_k \quad (k = 0, \ldots, n-1) , \quad p_{jn} = a_{n+j} \quad (j = 0, \ldots, n) .
\]

(2.19)

**ii)** In order to obtain the first Poisson tensor \( P_0 \), we eliminate from Eqs. (2.19) \( u = v_1/2 + a_0/4 \) using the first equation \( (k = 0) \) and we extract the system of \( n \) second order ODE’s in the \( v_j \) \( (j = 1, \ldots, n) \):

\[
p_{0k} - v_{k+1} = a_k \quad (k = 1, \ldots, n-1) ; \quad p_{0n} = a_n ,
\]

(2.20)

which will be referred to as \( P_0 \)–system. The remaining equations (2.19) will furnish a set of \( n \) independent integrals of motion. In order to obtain a second Poisson structure, we consider the following system (\( P_1 \)–system)

\[
p_{0k} - v_{k+1} = a_k \quad (k = 1, \ldots, n-1) ; \quad p_{nn} = a_{2n} ,
\]

(2.21)

with \( u \) as above.
The system (2.20) can be written in Lagrangian form. To this purpose, we use the so-called Newton or $r$–representation introduced in [16]. Namely, we choose as new coordinates in $S_n$ the first $n$ coefficients $r_j$ of the formal series $r(\lambda) := \sqrt{v(\lambda)}$,

$$r_k = \Delta_{-k} \left( \sqrt{v(\lambda)} \right) \quad (k = 1, \ldots, n) ,$$  \hspace{1cm} (2.22)

where $\Delta_k$ means the coefficient of $\lambda^k$ in a Laurent series. Taking into account Eq.(2.17), and observing that $2r_n + 1 = -\sum_{j=1}^{n} r_j r_{n+1-j}$, Eqs. (2.20) are equivalent to

$$\left( \lambda^n (r_{xx} + (r_1 + a_0 - \lambda/4)r - a/4 r^3) \right)_+ = 0 .$$  \hspace{1cm} (2.23)

This system is Lagrangian, with Lagrangian function

$$L_n^{(0)} = \Delta_{-(n+1)} \left( \mathcal{L}(\lambda; \nabla(\lambda)) \right) ,$$  \hspace{1cm} (2.24)

where $\mathcal{L}(\lambda; \Xi(\lambda))$ is given, for each Laurent series $w(\lambda)$, by

$$\mathcal{L}(\lambda; \Xi(\lambda)) := \frac{\infty}{\infty} \left( \Xi_{\frac{\lambda}{\lambda}}(\lambda) \right) - \frac{\infty}{\infty} (\Xi_{\infty} + \frac{1 - \lambda}{\Delta}) \Xi(\lambda) - \frac{\Xi(\lambda)}{\forall \Xi(\lambda)} .$$  \hspace{1cm} (2.25)

The Lagrangian gradients $\frac{\delta}{\delta r_k} := \frac{\partial}{\partial r_k} - \frac{d}{dx} \frac{\partial}{\partial r_{xx}}$ of $L_n^{(0)}$ are

$$\frac{\delta L_n^{(0)}}{\delta r_k} = \Delta_{k-1} \left( \lambda^n ( -r_{xx} - (r_1 + a_0 - \lambda/4)r + a/4 r^3) \right)_+ \quad (k = 1, \ldots, n) .$$  \hspace{1cm} (2.26)

We remark that it is also possible to put also the $P_1$–system (2.21) in Lagrangian form. To this purpose, we take as coordinates in $S_n$ $q_k = r_k$ ($k = 1, \ldots, n-1$) and $q_n = \sqrt{-v_n}$. By this choice the system (2.21) is equivalent to

$$\frac{1}{2} q_n^2 + \left( \lambda^{n-1} (q_{xx} + (q_1 + a_0 - \lambda/4)q - a/4 q^3) \right)_+ = 0 ,$$

$$q_{nxx} + (q_1 + a_0/4)q_n - a_{2n}/4 q_n^3 = 0 ,$$  \hspace{1cm} (2.27)

where $\left( \lambda^{n-1} q(\lambda) \right)_+ := \left( \lambda^{n-1} \sqrt{v(\lambda)} \right)_+$. This is a Lagrangian system with Lagrangian
\[ L_n^{(1)} = \Delta_{-n} \left( \mathcal{L}(\lambda) \Pi(\lambda) \right) + \frac{\infty}{\varepsilon} \Pi_{\varepsilon} - \frac{\infty}{\varepsilon} (\Pi_{\infty} + \frac{1}{\Delta}) \Pi_{\varepsilon} - \frac{1}{\Pi_{\varepsilon}}. \]  

Indeed it can be verified that the Lagrangian gradients of \( L_n^{(1)} \) are 

\[ \frac{\delta L_n^{(1)}}{\delta q_1} = \Delta_0 \left( \lambda^{n-1} - q_{xx} - (q_1 + \frac{a_0 - \lambda}{4} q + \frac{a}{4q^3}) \right) - \frac{1}{2} q_n^2 \]

\[ \frac{\delta L_n^{(1)}}{\delta q_k} = \Delta_{k-1} \left( \lambda^{n-1} - q_{xx} - (q_1 + \frac{a_0 - \lambda}{4} q + \frac{a}{4q^3}) \right) + (k = 2, \ldots, n-1) \]  

\[ \frac{\delta L_n^{(1)}}{\delta q_n} = -q_{nxx} - (q_1 + \frac{a_0}{4}) q_n + \frac{a_2}{4q_n^3} \]  

The two previous Lagrangian systems can be put in canonical Hamiltonian form. For the \( P_0 \)-system the canonical momenta are \( s_{n+1-k} = r_{kx} \) \( (k = 1, \ldots, n) \) and the Hamiltonian function

\[ H_n^{(0)} = \Delta_{-(n+1)} \left( \mathcal{H}(\lambda; \nabla(\lambda), f(\lambda)) \right), \]

where \( s(\lambda) = \sum_{j=1}^{n} s_j \lambda^{-j} \) and \( \mathcal{H}(\lambda; \exists(\lambda), \forall(\lambda)) \) is given by

\[ \mathcal{H}(\lambda; \exists(\lambda), \forall(\lambda)) = \frac{\infty}{\varepsilon} \nabla(\lambda) + \frac{\infty}{\varepsilon} \left( \exists_{\infty} + \frac{-1}{\Delta} \right) \nabla(\lambda) + \frac{1}{\nabla_{\varepsilon}(\lambda)} \]

For the \( P_1 \)-system the canonical momenta are \( p_n = q_{nx}, p_{n-k} = q_{kx} \) \( (k = 1, \ldots, n-1) \), and the Hamiltonian function is

\[ H_n^{(1)} = \Delta_{-n} \left( \mathcal{H}(\lambda; \Pi(\lambda), \sqrt{\lambda}) \right) + \frac{\infty}{\varepsilon} \sqrt{\lambda} + \frac{\infty}{\varepsilon} (\Pi_{\infty} + \frac{1}{\Delta}) \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} \]

with \( p(\lambda) = \sum_{j=1}^{n} p_j \lambda^{-j} \).

The two Hamiltonian functions depend, respectively, on the two sets of coordinates and momenta \( (r_k, s_k), (q_k, p_k) \) and on the two sets of free parameters \( (a_0, \ldots, a_{n-1}, a_n) \) and \( (a_0, \ldots, a_{n-1}, a_{2n}) \).

iv) Now let us consider the manifold \( M_n \) \((2.13)\), which can be parametrized either by \( (r_k, s_k, a_n) \), or by \( (q_k, p_k, a_{2n}) \), with \( a_n \) and \( a_{2n} \) as additional dynamical variables in \( M_n \). On this manifold one can extend trivially the canonical Poisson structures, the Hamiltonians and the vector fields associated with each one of the two systems as in \([17]\). In particular the vector fields can be extended in such a way that they are tangent...
to one of the foliations $S^{(0)}_n$ and $S^{(1)}_{a_0}$. Taking into account, on one side, the relation between the two sets of coordinates through the original variables $(v_k, v_{kx})$, on the other side the relation between the two integrals of motion $a_n$ and $a_{2n}$ through the GD polynomials $p_{0n}$ and $p_{nn}$, a map $\Phi : M_n \to M_n, (r_k, s_k, a_n) \mapsto (q_k, p_k, a_{2n})$ can be systematically constructed. It relates the Hamiltonians and the vector fields of one system with the corresponding ones of the other system. Since this map is not a Poisson morphism, the extended canonical Poisson structures associated with one chart is mapped into a Poisson structure different from the extended canonical structure associated with the other chart. If this second Poisson tensor is compatible with the extended canonical one, a bi–Hamiltonian formulation of the two systems is obtained.

In conclusion we can state the following:

**Proposition 2.1.** The $P_0$–system (2.20) and the $P_1$–system (2.21), written respectively in the coordinates $r_k$ and $q_k$ are natural Lagrangian systems. The corresponding canonical Hamiltonian systems

$$
\begin{align*}
  r_{kx} &= \frac{\partial H^{(0)}_n}{\partial s_k}, & s_{kx} &= -\frac{\partial H^{(0)}_n}{\partial r_k}, \\
  q_{kx} &= \frac{\partial H^{(1)}_n}{\partial p_k}, & p_{kx} &= -\frac{\partial H^{(1)}_n}{\partial q_k},
\end{align*}
$$(2.33)

have $n$ integrals of motion given by

$$
K_j \equiv -\frac{1}{8}p_{jn|Y} = a_{n+j} \quad (j = 1, \ldots, n), \quad H_j \equiv -\frac{1}{8}p_{jn|X} = a_{n+j} \quad (j = 0, \ldots, n - 1).
$$

(2.35)

Moreover, the map $\Phi : M_n \to M_n$ in the extended phase space generates a second Poisson structure.

**Remark 2.1.** The symbols $|Y$ and $|X$ in (2.33) mean that, in the GD polynomials $p_{jk}$, the coordinates $(v_k, v_{kx})$ must be replaced by the canonical coordinates $(r_k, s_k)$ and $(q_k, p_k)$ respectively and that the first order $x$–derivatives of momenta must be eliminated by means of the Hamiltonian dynamical equations (2.33), (2.34).

In the next Subsection we shall give some applications of the results stated in this proposition.

2.3. **The bi–Hamiltonian structure of a Hénon–Heiles system.**

We consider a generalized Hénon–Heiles system with two degrees of freedom. Its Hamiltonian is
where \( q_1, q_2, p_1, p_2 \) are the canonical coordinates and momenta and \( a_0, a_1, a_4 \) are free constant parameters. This Hamiltonian encompasses the two cases \( a_0 = a_4 = 0 \) and \( a_0 = a_1 = 0 \) introduced in [18]. Moreover \( H_0 \) is related with the Hamiltonian

\[
H_H = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} (Aq_1' + Bq_2') + \frac{1}{2} q_1'q_2' + a_4q_2'^2,
\]

through the map

\[
q_1 = q_1' + \frac{A}{2} - 2B, \quad q_2 = q_2', \quad a_0 = -2A + 12B, \quad a_1 = -A^2 + 16AB - 48B^2.
\]

The function \( H_H \) is the Hamiltonian of a classical integrable Hénon–Heiles system [19] with the additional term \( a_4/8q_2'^2 \).

The function (2.36) is the Hamiltonian of the the vector field obtained reducing \( X_0(u) = u_x \) to the stationary manifold \( M_2 \) given by the fixed points of the flow \( X_2 + c_1X_1 + c_2X_0 \)

\[
M_2 = \left\{ u|u^{(5)} + 10u_{xxx}u + 20u_{xx}u_x + 30u_xu^2 + c_1(u_{xxx} + 6u_xu) + 2c_2u_x = 0 \right\},
\]

where \( c_1 = -a_0/2, c_2 = -a_1/2 + a_0^2/4 \).

It can be obtained specializing to the case \( n = 2 \) the Hamiltonian (2.32) of the \( P_1 \)-system. In this case \( H_2^{(1)} = H_0 \) and the canonical coordinates and momenta are, respectively, \( q_1 = v_1/2, q_2 = \sqrt{-v_2}, p_1 = q_1x, p_2 = q_2x \). The integrals of motion obtained by the reduction of the GD polynomials are

\[
\begin{align*}
H_0 &= -\frac{1}{8}p_{02x}q_1 - \frac{1}{4}q_1'^2, \\
H_1 &= -\frac{1}{8}p_{12x} = p_2^2q_1 - p_1p_2q_2 - \frac{1}{2}q_1'^2q_2' - \frac{1}{8}q_2'^2 - \frac{a_4q_1}{4q_2'^2} - \frac{a_0}{4}q_1q_2^2 + \frac{a_1}{8q_2'^2}.
\end{align*}
\]

The corresponding Hamiltonian vector fields will be denoted by \( X_{j+1} := E_dH_j \) \((j = 0, 1, 2)\); \( E \) being the canonical \((4 \times 4)\) Poisson matrix. The Hénon–Heiles vector field \( X_1 \) is:
\[ X_1 = [p_1, p_2, -3q_1^2 - \frac{1}{2}q_2^2 - a_0q_1 + \frac{a_1}{4}, -q_1q_2 + \frac{a}{4q_2^2} - \frac{a_0}{4}q_2]^T. \] 

(2.41)

The second Hamiltonian formulation can be obtained specializing to the case \( n = 2 \) the Hamiltonian (2.30) of the \( P_0 \)-system:

\[ H_2^{(0)} = s_1s_2 - \frac{5}{8}r_1^4 + \frac{5}{2}r_1^2r_2 - \frac{1}{2}r_2^2 - \frac{a_0}{2}r_1^3 + \frac{3}{8}a_1r_1^2 + a_0r_1r_2 - \frac{a_2}{4}r_1 - \frac{a_1}{4}r_2, \] 

(2.42)

where the canonical coordinates (2.22) and momenta are, respectively, \( r_1 = v_1/2, r_2 = v_2/2 - v_1^2/4, s_1 = r_{2x}, s_2 = r_{1x} \). The integrals of motion obtained by the reduction of the GD polynomials are

\[ K_0 = -\frac{1}{8}p_{02\gamma} = -\frac{a_2}{8}, \quad K_1 = -\frac{1}{8}p_{12\gamma} = H_2^{(0)}, \]

\[ K_2 = -\frac{1}{8}p_{22\gamma} = -s_2^2r_2 + s_1s_2r_1 + \frac{1}{2}s_1^2 - \frac{1}{2}r_1^2 + 2r_1r_2 - \frac{3}{8}a_0r_1^4 + \frac{a_1}{4}r_1^3 - \frac{a_0}{2}r_1^2r_2 + \frac{a_1}{2}r_1r_2 + \frac{a_2}{2}r_2^2 - \frac{a_2}{8}r_2^2 - \frac{a_1}{4}r_2, \]

(2.43)

and the corresponding Hamiltonian vector fields will be denoted by \( Y_j := E dK_j \).

Now we construct the bi–Hamiltonian structure of the Hénon–Heiles system. Let \( M_2 \) be the 5-dimensional extended phase space parametrized by \( (r_1, r_2, s_1, s_2; a_2) \) or \( (q_1, q_2, p_1, p_2; a_4) \). It is convenient to make use of block notations. So, for example, we denote with \( (r, s; a) \) the 5-tuple \( (r_1, r_2, s_1, s_2; a_2) \), with \( \tilde{X} = [\tilde{X}^r, \tilde{X}^s, \tilde{X}^a]^T \) the generic vector field and with \( d\tilde{K} = [\frac{\partial \tilde{K}}{\partial r}, \frac{\partial \tilde{K}}{\partial s}; \frac{\partial \tilde{K}}{\partial a}]^T \) the generic gradient of a function \( \tilde{K} \) (the superscript \( T \) means transposition). In this notation a vector field \( \tilde{X} = \tilde{P} d\tilde{K} \) with Hamiltonian function \( \tilde{K} \) with respect to a Poisson tensor \( \tilde{P} \) will be written

\[
\begin{bmatrix}
\tilde{X}^r \\
\tilde{X}^s \\
\tilde{X}^a \\
\end{bmatrix} =
\begin{bmatrix}
P^{rr} & P^{rs} & P^{ra} \\
P^{sr} & P^{ss} & P^{sa} \\
P^{ar} & P^{as} & P^{aa} \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \tilde{K}}{\partial r} \\
\frac{\partial \tilde{K}}{\partial s} \\
\frac{\partial \tilde{K}}{\partial a} \\
\end{bmatrix}, \tag{2.44}
\]

where \( P^{rs} = -(P^{sr})^T \) etc. . . . From the definition of \( r_1, r_2 \) and \( q_1, q_2 \) in terms of \( v_1 \) and \( v_2 \), and from (2.40) and (2.43) one obtains the following map \( \Phi : M_2 \to M_2, (r, s; a_2) \mapsto (q, p; a_4) \).
ON THE INTEGRABILITY OF STATIONARY AND RESTRICTED FLOWS

\[ q_1 = r_1 , \quad q_2 = (-2r_2 - r_1^2)^{1/2} \]
\[ p_1 = s_2 , \quad p_2 = -\frac{s_1 + r_1s_2}{(-2r_2 - r_1^2)^{1/2}} , \quad a_4 = -8K_2 \quad (2.45) \]

with \( K_2 \) given by Eq.(2.43). In these two charts let us consider the extended Hamiltonians \( \tilde{H}_j \) and \( \tilde{K}_j \), the vector fields \( \tilde{X}_j \) (\( \tilde{X}_j^r = X_j^r \), \( \tilde{X}_j^s = X_j^s \), \( \tilde{X}_j^a = 0 \)) and \( \tilde{Y}_j \) (\( \tilde{Y}_j^r = Y_j^r \), \( \tilde{Y}_j^s = Y_j^s \), \( \tilde{Y}_j^a = 0 \)), the extension of the canonical Poisson structure, \( \tilde{E} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). The following proposition holds

**Proposition 2.2.** The action of the map \( \Phi : M_2 \to M_2 \) defined by (2.45) on the Hamiltonians \( \tilde{H}_j \), the vector fields \( \tilde{Y}_j \) and the Poisson tensor \( \tilde{P}_0 := \tilde{E} \) is given by \( \Phi^*(\tilde{H}_j) = \tilde{K}_j \), \( \Phi^*(\tilde{Y}_j) = \tilde{X}_j \) and by

\[ \tilde{P}_0 := \Phi^*\tilde{P}_0\Phi^* = \begin{pmatrix} 0 & A & -8\tilde{X}_2^q \\ -A^T & B & -8\tilde{X}_2^p \\ 8(\tilde{X}_2^q)^T & 8(\tilde{X}_2^p)^T & 0 \end{pmatrix} , \quad (2.46) \]

where \( A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -q_2 \\ -q_1 & 0 \end{pmatrix} \), \( B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -p_2 \\ p_1 & 0 \end{pmatrix} \).

Thus we have recovered in the extended phase space \( M_2 \) a second Poisson tensor \( \tilde{P}_0 \). We can check that \( \tilde{P}_0 \) is compatible with \( \tilde{P}_1 = \tilde{E} \). Furthermore \( \tilde{P}_0 \) and \( \tilde{P}_1 \) give rise to the following bi–Hamiltonian hierarchy

\[ \tilde{X}_{j+1} := \tilde{P}_1 d\tilde{H}_j = \tilde{P}_0 d\tilde{H}_{j+1} \quad (j = 0, 1) , \quad (2.47) \]

the Hamiltonians \( \tilde{H}_0 \) and \( \tilde{H}_2 \) being Casimirs of \( \tilde{P}_0 \) and \( \tilde{P}_1 \) respectively.

3. **Restricted flows and Garnier systems**

The method of restricted flows was introduced in [20] as a non linearization of the KdV spectral problem and was generalized in [3, 2]. We formulate this method putting the emphasis on the role of the GD polynomials and of their generating function; this formulation allows us to construct a map between stationary and restricted flows in the next section. In view of the applications, we begin by applying the method to the KdV hierarchy, recovering the Garnier system.

Let us consider the following system
\[ p_{00} - v_1 = a_0, \quad P_0 \left( v_1 - \sum_{j=1}^{n} \beta_j \right) = 0, \quad P^\lambda_k \beta_k = 0 \quad (k = 1, \ldots, n) \quad (3.1) \]

where: \( \lambda_1, \ldots, \lambda_n \) are distinct fixed parameters, \( P^\lambda_k := P_1 - \lambda_k P_0 \) (\( P_0 \) and \( P_1 \) being the two KdV Poisson tensors \( \text{[2.4]} \)). This is a system of \((n + 2)\) equations in \( u, v_1, \beta_1, \ldots, \beta_n \). The second equation will be referred to as the \( P_0 \)–\textit{restriction} of the first KdV flow \( X_0 = P_0 v_1 = v_1 \), and the last \( n \) equations define the kernel of \( n \) Poisson tensors extracted from the Poisson pencil. On account of \( \text{[2.14]}, \text{[2.4]} \) and \( \text{[2.7]} \) this system is equivalent to the following one

\[ u = \frac{v_1}{2} + \frac{a_0}{4}, \quad v_1 = \sum_{j=1}^{n} \beta_j + c, \quad B^\lambda_k (\beta_k, \beta_k) = f_k, \quad (3.2) \]

where \( c \) and \( f_k \) are free parameters and \( B^\lambda \) is just the generating function \( \text{[2.6]} \) of the GD polynomials.

Using the first two equations to eliminate \( u \) and \( v_1 \) from the last \( n \) equations, one gets a system of \( n \) ODE’s of second order for \( \beta_1, \ldots, \beta_n \):

\[ 2 \beta_{kxx} \beta_k - \beta_{kx}^2 + 2 \beta_k^2 \left( \sum_{j=1}^{n} \beta_j + d \right) - \lambda_k \beta_k^2 = f_k \quad (k = 1, \ldots, n), \quad (3.3) \]

where \( d := c + a_0/2 \). Introducing the so–called eigenfunction variables \( \psi_j = \sqrt{\beta_j} \) and the momenta \( \chi_j = \psi_{jx} \), Eqs. \( \text{[3.3]} \) can be written in canonical Hamiltonian form

\[ \psi_{jx} = \frac{\partial K_G}{\partial \chi_j}, \quad \chi_{jx} = -\frac{\partial K_G}{\partial \psi_j} \quad (j = 1, \ldots, n). \quad (3.4) \]

with Hamiltonian

\[ K_G = \frac{1}{2} \sum_{j=1}^{n} \chi_j^2 + \frac{1}{8} \left[ \left( \sum_{k=1}^{n} \psi_j^2 \right)^2 - \sum_{j=1}^{n} (\lambda_j - 2d) \psi_j^2 + \sum_{j=1}^{n} f_j \psi_j^2 \right] \]. \quad (3.5) \]

The corresponding Hamiltonian vector field \( Y_G = \mathcal{E} [K_G] \) is

\[ Y_G = [\chi|, -\infty (\psi_\infty^\epsilon + \psi_\infty^\delta) \psi | + \frac{\infty}{\Delta} (\lambda | - \infty) \psi | + \frac{\{l|}{\Delta \psi_l^\epsilon} ]^T (l = \infty, \ldots, \}) \]. \quad (3.6) \]
\( \mathcal{E} \) being the \((2n \times 2n)\) canonical Poisson matrix. Eqs. \((3.4)\) are just the equations of the Garnier system with \(n\) degrees of freedom \([2]\). A set of integrals of motion is

\[
I_j = \chi_j^2 + \frac{\psi_j^2}{4} \left( 2d - \lambda_j + \sum_{k=1}^{n} \frac{\psi_k^2}{\psi_j^2} + \frac{f_j}{\psi_j^2} \sum_{k=1}^{n} \frac{1}{4\lambda_{jk}} \left( \frac{f_k \psi_k^2}{\psi_j^2} + \frac{f_j \psi_k^2}{\psi_k^2} + (\psi_j \chi_k - \psi_k \chi_j)^2 \right) \right),
\]

with \(\sum_{j=1}^{n} I_j = 2K_G\). These integrals were obtained in \([23]\) by means of a Lax representation; we shall recover them in the next section by the use of the generating function of the GD polynomials.

Let us consider the \((2n+1)\) extended phase space \(\mathcal{M}_{\xi}\) with coordinates \((\psi_k, \chi_k, d)\) and the extended Hamiltonian \(\tilde{K}_G\), the vector field \(\tilde{\mathcal{Y}}_G = \tilde{E} d\tilde{K}_G\) with \(\tilde{\mathcal{Y}}_G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\). In this space the Garnier system has a second Hamiltonian structure given by

\[
\tilde{P}_1 := \begin{bmatrix} 
0 & \Lambda - \psi \otimes \psi & 4 \tilde{Y}_G^\psi \\
-(\Lambda - \psi \otimes \psi)^T & \chi \otimes \psi - \psi \otimes \chi & 4 \tilde{Y}_G^\chi \\
-4(\tilde{Y}_G^\psi)^T & -4(\tilde{Y}_G^\chi)^T & 0 
\end{bmatrix},
\]

where \(\otimes\) denotes the tensor product, \(\psi = [\psi_1, \ldots, \psi_n]^T\), \(\chi = [\chi_1, \ldots, \chi_n]^T\), \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\). This structure is an extension of the one constructed in \([3]\) for \(f_k = 0, (k = 1, \ldots, n)\). In view of the applications we specialize the above structure to the case \(n = 2\), in the five–dimensional extended phase space \(\mathcal{M}_{\xi}\) with coordinates \((\psi_1, \psi_2, \chi_1, \chi_2, d)\). The following proposition holds:

**Proposition 3.1.** The Garnier vector field \(\tilde{\mathcal{Y}}_1 = \tilde{\mathcal{Y}}_G\) belongs to the following bi–Hamiltonian hierarchy

\[
\tilde{\mathcal{Y}}_{j+1} = \tilde{P}_1 d\tilde{G}_j = \tilde{P}_0 d\tilde{G}_{j+1} \quad (j = 0, 1),
\]

where the Hamiltonians \(\tilde{G}_j\) are given by

\[
\tilde{G}_0 = \frac{d}{4}, \quad \tilde{G}_1 = -(\lambda_1 + \lambda_2)\frac{d}{4} + \frac{1}{2}(\tilde{I}_1 + \tilde{I}_2) \\
\tilde{G}_2 = \lambda_1 \lambda_2 \frac{d}{4} - \frac{1}{2}(\lambda_1 + \lambda_2)(\tilde{I}_1 + \tilde{I}_2) + \frac{1}{2}(\lambda_1 \tilde{I}_1 + \lambda_2 \tilde{I}_2),
\]

\(\tilde{G}_0\) and \(\tilde{G}_2\) being Casimirs of \(\tilde{P}_0\) and \(\tilde{P}_1\) respectively, and \(\tilde{I}_1, \tilde{I}_2\) being the extensions to \(\mathcal{M}_{\xi}\) of the integrals of motion \((3.7)\).
As in the case of the Hénon–Heiles system, a bi–Hamiltonian structure for the Garnier system seems to naturally exist only in its extended phase space. Nevertheless in Subsect. 5.3 a realization of the integrability structure introduced in Prop. 5.1 will be constructed in the original four–dimensional phase space.

4. A MAP BETWEEN STATIONARY AND RESTRICTED FLOWS

Now we shall construct a map between the \( n \)th stationary flow and the previous restricted flow of the KdV hierarchy. To this end we extend the corresponding phase spaces, regarding some free parameters in the Hamiltonian functions as additional dynamical variables.

4.1. The general case. As for the \( P_1 \)–formulation of the stationary flow (2.34) we extend its phase space to a \((3n + 1)\)–dimensional space, \( \tilde{M}_n \), with coordinates \((q_k, p_k; a_0, \ldots, a_{n-1}, a_{2n})\); analogously we consider the \( P_0 \)–formulation of the first restricted flow (3.4) in the extended space \( \tilde{M}_n \) with coordinates \((\psi_k, \chi_k; f_1, \ldots, f_n, d)\).

Let us consider the solutions \( q_k \) of the dynamical equations (2.34); then \( v^{(n)}(\lambda) \) given by

\[
v^{(n)}(\lambda) = \lambda \left(q^2(\lambda)\right)^{(n-1)} - q_n^2, \tag{4.1}
\]

with \( q(\lambda) = 1 + \sum_{j=1}^{n} q_j \lambda^{-j} \), satisfies (2.17) and consequently the following equation

\[
B^\lambda \left(v^{(n)}(\lambda), v^{(n)}(\lambda)\right) = \lambda^{2n} a(\lambda), \tag{4.2}
\]

where, as above, we put \( u = v_1/2 + a_0/4 \). So, for each \( n \)–tuple of distinct complex parameters \( \lambda_j \), any solution \( v^{(n)}(\lambda) \) of Eq. (4.2) fulfills the system

\[
B^\lambda_k \left(v^{(n)}(\lambda_k), v^{(n)}(\lambda_k)\right) = \lambda_k^{2n} a(\lambda_k) \quad (k = 1, \ldots, n), \tag{4.3}
\]

where \( v^{(n)}(\lambda_k) := v^{(n)}(\lambda)|_{\lambda = \lambda_k} \). In order to have a solution satisfying also the second equation (3.4), the Lagrange interpolation formula can be used [21, 22]. It allows us to represent the polynomial \( v^{(n)}(\lambda) \) by

\[
v^{(n)}(\lambda) = p(\lambda) + \sum_{j=1}^{n} \frac{p(\lambda)}{\lambda - \lambda_j} \beta_j, \tag{4.4}
\]

where \( p(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j) \), and
\beta_k = \frac{u^{(n)}(\lambda_k)}{p'(\lambda_k)} \quad (k = 1, \ldots, n) , \quad (4.5)

( p'(\lambda) \text{ means the derivative of } p(\lambda) \text{ with respect to } \lambda).\text{ Obviously the } n \text{ functions } \beta_k \text{ are solutions of the following system}

\begin{align*}
2\beta_{kxx} - \beta_{kx}^2 + 2\beta_k^2(\sum_{j=1}^n \beta_j + \frac{a_0}{2} - \sum_{j=1}^n \lambda_j) - \lambda_k \beta_k^2 = \frac{\lambda_k^{2n} a(\lambda_k)}{(p'(\lambda_k))^2} \quad (k = 1, \ldots, n) ;
\end{align*}

(4.6)

furthermore \beta_k satisfy the so–called Bargmann constraint

\begin{align*}
\sum_{j=1}^n (\beta_j - \lambda_j) = v_1 ,
\end{align*}

(4.7)

as one can verify by means of (4.4). Comparing (4.6) with (3.3), we can state the following

**Proposition 4.1.** Let \( \Psi : \tilde{M}_n \to \tilde{M}_n, (q, p; a_0, \ldots, a_{2n-1}, a_{2n}) \mapsto (\psi, \chi; f_1, \ldots, f_n, d) \) be the map:

\begin{align*}
\psi_k &= \left( \sum_{j=0}^{n-1} \sum_{l=0}^j q_{j-l} \lambda_k^{n-j} - q_n^2 \right)^{1/2} , \\
\chi_k &= \frac{\sum_{j=0}^{n-1} \sum_{l=0}^j q_{j-l} \lambda_k^{n-j} - q_n^2}{\left( p'(\lambda_k) \left( \sum_{j=0}^{n-1} \sum_{l=0}^j q_{j-l} \lambda_k^{n-j} - q_n^2 \right) \right)^{1/2}} , \\
f_k &= \frac{1}{(p'(\lambda_k))^2} \left( a_{2n} - 8 \sum_{j=0}^n H_{n-j} \lambda_k^j + \sum_{j=n+1}^{2n+1} a_{2n-j} \lambda_k^j \right) , \\
& \quad d = \frac{a_0}{2} - \sum_{j=1}^n \lambda_j
\end{align*}

(4.8)

\( (k = 1, \ldots, n) \), where \( H_j \) are the Hamiltonian functions (2.35). If \( (q_k, p_k) \) are solutions of the stationary flows (2.34), then \( (\psi_k, \chi_k) \) are solutions of the Garnier system (3.4) for \( f_k \) and \( d \) given by (4.8).

**Remark 4.1.** The function \( B^\lambda \) is also a generating function of integrals of motion for the Garnier system. Indeed evaluating the function \( B^\lambda \) by means of (4.4) and eliminating the first \( x \)–derivatives of \( \chi_k \) by means of the Hamilton equations (3.4), one gets

\begin{align*}
4 \sum_{j=1}^n \frac{I_j}{\lambda - \lambda_j} + \sum_{j=1}^n \frac{f_j}{(\lambda - \lambda_j)^2} + 2d - \lambda = \frac{\lambda^{2n} \hat{a}(\lambda)}{(p(\lambda))^2} 
\end{align*}

(4.9)

where \( I_j \) are the functions (3.7). Taking in this equation the residues at \( \lambda = \lambda_j \) it follows that the functions \( I_j \) are integrals of motion along the flow (3.4). \( \square \)
4.2. The map between the Hénon–Heiles and the Garnier system. Now we specialize the map of Prop. 4.1 to the Hénon–Heiles and the Garnier systems with two degrees of freedom: we obtain the surprising result that the Hénon–Heiles vector field is mapped into the Garnier vector field. Let us consider the seven–dimensional phase space of the Hénon–Heiles system $\tilde{\mathcal{M}}_2$ with coordinates $(q, p; a_0, a_1, a_4)$. Similarly, for the Garnier systems let us select the parameters $f_1, f_2, d$ and enlarge the phase space to a seven–dimensional phase space $\mathcal{M}_2$, with coordinates $(\psi, \chi; f_1, f_2, d)$. It is easy to prove the following

**Proposition 4.2.** Let $\Psi : \tilde{\mathcal{M}}_2 \rightarrow \mathcal{M}_2, (q, p; a_0, a_1, a_4) \mapsto (\psi, \chi; f_1, f_2, d)$ be defined by

$$
\psi_1 = \lambda_1^{−1/2}(\lambda_1^2 + 2\lambda_1q_1 - q_2^2)^{1/2} , \quad \psi_2 = \lambda_1^{−1/2}(−\lambda_1^2 − 2\lambda_2q_1 + q_2^2)^{1/2} ,
$$

$$
\chi_1 = \frac{(\lambda_1p_1 − q_2p_2)}{(\lambda_1^2 + 2\lambda_1q_1 - q_2^2)^{1/2}} , \quad \chi_2 = \frac{(\lambda_2p_1 − q_2p_2)}{(\lambda_1^2 + 2\lambda_2q_1 - q_2^2)^{1/2}} ,
$$

$$
f_1 = \lambda_1^{−2}(-\lambda_1^5 + a_0\lambda_4^4 + a_1\lambda_2^3 - 8H_0\lambda_1^2 - 8H_1\lambda_1 + a_4) ,
$$

$$
f_2 = \lambda_1^{−2}(-\lambda_2^5 + a_0\lambda_4^4 + a_1\lambda_2^3 - 8H_0\lambda_2^2 - 8H_1\lambda_2 + a_4) , \quad d = \frac{a_0}{2} − (\lambda_1 + \lambda_2) ,
$$

where $\lambda_1 = \lambda_1 − \lambda_2$. The tangent map $\Psi_*$ maps the extended Hénon–Heiles vector fields $\tilde{X}_1, \tilde{X}_2$ (2.47) into the extended Garnier vector fields $\tilde{Y}_1, \tilde{Y}_2$ (3.3):

$$
\Psi_*(\tilde{X}_1) = \tilde{Y}_\infty , \quad \Psi_*(\tilde{X}_2) = \tilde{Y}_\varepsilon .
$$

Moreover the pull–back of the Garnier integrals of motion $G_\infty$ and $G_\varepsilon$ are integrals of motion for the Hénon–Heiles system

$$
\Psi^*(G_\infty) = -\frac{1}{8}(\lambda_1^2 + \lambda_2^2) + \frac{a_0}{8}(\lambda_1 + \lambda_2) + \frac{a_1}{8} ,
$$

$$
\Psi^*(G_\varepsilon) = \lambda_1^{−2} \left(2\lambda_1\lambda_2H_0 + (\lambda_1 + \lambda_2)H_1 + 2H_2 \right) + \frac{\lambda_1^{−2}\lambda_1\lambda_2}{4} \left((\lambda_1^3 + \lambda_2^3) - \frac{a_0}{2}(\lambda_1^2 + \lambda_2^2) - \frac{a_1}{2}(\lambda_1 + \lambda_2) \right) ,
$$

The action of the map $\Psi$ on the Poisson tensor $\tilde{E}$ of the Hénon–Heiles system, furnishes a new Poisson tensor for the Garnier system compatible with $\tilde{E}$. Moreover the action of $\Psi$ on the Poisson tensor $\tilde{P}_0$ is given by
\[
\Psi^* \tilde{P}_0 \Psi_* = \lambda_{12}^{-2} \begin{bmatrix}
0 & \mathcal{A} & 0 \\
-\mathcal{A}^T & \mathcal{B} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where

\[
\mathcal{A} = \frac{1}{\psi_1^2 \psi_2^2} \begin{bmatrix}
\psi_2^2 (\psi_1^2 + \psi_2^2 + \lambda_1 - \lambda_2) & -\psi_1 \psi_2 (\psi_1^2 + \psi_2^2) \\
-\psi_1 \psi_2 (\psi_1^2 + \psi_2^2) & \psi_1^2 (\psi_1^2 + \psi_2^2 + \lambda_2 - \lambda_1)
\end{bmatrix},
\]

\[
\mathcal{B} = \frac{\psi_1^2 + \psi_2^2}{\psi_1^2 \psi_2^2} \begin{bmatrix}
0 & (\chi_2 \psi_1 - \chi_1 \psi_2) \\
-(\chi_2 \psi_1 - \chi_1 \psi_2) & 0
\end{bmatrix}.
\]

So the map \(\Psi\) is not a Poisson morphism. However, according to Eqs. (4.11), the orbits of the Hénon–Heiles system are mapped into the orbits of the Garnier system.

5. **A NEW INTEGRABILITY STRUCTURE**

### 5.1. The reduced structures of the Hénon–Heiles and the Garnier systems.

In order to have a bi–Hamiltonian hierarchy also in the original phase space for the Hénon–Heiles and the Garnier systems, one can try to apply the reduction techniques known from the literature [10, 24]. In particular, two methods can be followed: a restriction to the standard phase space or a projection onto it. However, in both cases, these attempts fail.

As for the Hénon–Heiles system, if the restriction submanifold is chosen to be a leaf \(S_{a_4}^{(1)}\) of the second natural foliation in \(M_2\), the Hamiltonians \(\tilde{H}_j\), the vector fields \(\tilde{X}_j\) and the Poisson structure \(\tilde{P}_1\) can be trivially restricted respectively to \(H_j\), \(X_j\) and \(E\); but it turns out that \(\tilde{P}_0\) cannot be restricted. So two integrable Hamiltonian vector fields are obtained in \(S_{a_4}^{(1)}\) but not a bi–Hamiltonian hierarchy.

If \(\Pi : M_2 \to S_2, (q_1, q_2, p_1, p_2; a_{2n}) \mapsto (q_1, q_2, p_1, p_2)\) is the projection map, the Hamiltonians \(\tilde{H}_j\) and the vector fields \(\tilde{X}_j\) cannot be projected onto \(S_2\), because they depend on the fiber coordinate. Instead, the Poisson tensors \(\tilde{P}_0\) and \(\tilde{P}_1\) are projected onto:

\[
P_H := \Pi_* \tilde{P}_0 \Pi^* = \begin{bmatrix} 0 & \mathcal{A} \\ -\mathcal{A}^T & \mathcal{B} \end{bmatrix}, \quad \Pi_* \tilde{P}_1 \Pi^* = E,
\]

with \(A, B\) as in Prop. 2.2. Because these operators are compatible and invertible, one obtains the following Nijenhuis tensor [25].
\[ N_H := P_H E^{-1} = \begin{bmatrix} A & 0 \\ B & A^T \end{bmatrix} \] (5.2)

and consequently the hierarchy of Poisson tensors \( P_k := N_H^k P_H, k \in \mathbb{Z} \). However these tensors are not invariant along the flow of the Hénon–Heiles vector field \( X_1 \). In other words \( X_1 \) is neither a symmetry of \( P_0 \) nor of \( P_1 \), so that these tensors cannot generate a bi–Hamiltonian hierarchy starting from \( X_1 \).

As in the case of the Hénon–Heiles system, one cannot reduce the bi–Hamiltonian structure of the Garnier system with \( n \) degrees of freedom onto the restricted phase space. If \( \Pi : \mathcal{M} \rightarrow \mathcal{S}^{(\infty)}, (\psi_\parallel, \chi_\parallel; \parallel) \mapsto (\psi_\parallel, \chi_\parallel) \) is the projection map, the Poisson tensor \( \tilde{P}_0 \) and \( \tilde{P}_1 \) are projected onto two compatible tensors

\[ \Pi_* \tilde{P}_0 \Pi^* = \mathcal{E}, \quad \mathcal{P}_G := \Pi_* \tilde{P}_\infty \Pi^* = \begin{bmatrix} 0 & \Lambda - \psi \otimes \psi \\ - (\Lambda - \psi \otimes \psi)^T & \chi \otimes \psi - \psi \otimes \chi \end{bmatrix}. \] (5.3)

They give rise to the Nijenhuis tensor \( N_G := \mathcal{P}_G \mathcal{E}^{-\infty} \) together with the hierarchy of Poisson tensor fields \( \mathcal{P}_\parallel := N_G^\parallel \mathcal{E}, \parallel \in \mathbb{Z} \). However these tensor fields are not invariant along the flow of the Garnier vector field \( \mathcal{Y}_G \), so they do not generate a bi–Hamiltonian hierarchy starting from \( \mathcal{Y}_G \).

5.2. A new integrability criterion. In the previous subsection we have put into evidence some problems arising in the geometrical reduction of a bi–Hamiltonian structure from an extended phase space onto the original one. As an alternative construction, here we introduce a new integrability scheme, weaker than the bi–Hamiltonian one, but living in the standard phase space. We shall define this new structure for a generic Hamiltonian system with \( n \) degrees of freedom; for \( n = 2 \) it coincides with the one introduced in [8] for the Hénon–Heiles system with the Hamiltonian \( (2.37) \) and \( a_4 = 0 \). As new examples of this integrability structure, the case of the Garnier system with two degrees of freedom will be discussed here whereas multidimensional extensions of the Hénon–Heiles system will be presented elsewhere.

**Proposition 5.1.** Let \( M \) be a \( 2n \)–dimensional Poisson manifold equipped with a Poisson tensor \( Q_0 \), and \( Z_0 \) a Hamiltonian vector field with Hamiltonian \( h_0: Z_0 = Q_0 \, dh_0 \). Let there exist a tensor \( \mathcal{N} : TM \rightarrow TM \) and a skew–symmetric tensor \( Q_1 : T^* M \rightarrow TM \) such that

\[ Q_1 = \mathcal{N} Q_0. \] (5.4)
Denote by $Z_i$ and $\alpha_i$ the vector fields and the 1–forms obtained, respectively, by the iterated action of the tensor $N$ on $Z_0$ and its adjoint $N^* : T^* M \to T^* M$ on $\alpha_0 := dh_0$

$$Z_i := N^i Z_0, \quad \alpha_i := N^{*i} \alpha_0 \quad (i = 1, \ldots, n - 1). \quad (5.5)$$

Let there exist $n - 1$ independent functions $h_i$ $(i = 1, \ldots, n - 1)$ and $(n^2 + n - 2)/2$ functions $\mu_{ij}$ $(i = 1, \ldots, n - 1; 0 \leq j \leq i)$ with $\mu_{00} = 1$, $\mu_{ii} \neq 0$ $(i = 1, \ldots, n - 1)$, such that the 1–forms $\alpha_i$ can be written as

$$\alpha_i = \sum_{j=0}^{i} \mu_{ij} dh_j \quad (i = 1, \ldots, n - 1). \quad (5.6)$$

Under the previous assumptions the following results hold:

i) the vector fields $Z_i$ satisfy the recursion relations

$$Z_{i+1} = Q_0 \alpha_{i+1} = Q_1 \alpha_i \quad (i = 0, \ldots, n - 2). \quad (5.7)$$

ii) the functions $h_i$ are in involution with respect to the Poisson bracket defined by $Q_0$ and they are constants of motion for the fields $Z_k$

$$\{h_i, h_j\} Q_0 = 0, \quad \mathcal{L}_{Z_{i}}(\langle \rangle) = \text{I}, \quad (5.8)$$

where $\mathcal{L}_{Z_{i}}$ denotes the Lie derivative with respect to the vector field $Z_k$.

iii) the Hamiltonian system corresponding to the vector field $Z_0$ is Liouville–integrable. In addition if $Q_1$ is a Poisson tensor field, then also $Z_1$ is an integrable Hamiltonian vector field and the functions $h_i$ are in involution also with respect to the Poisson bracket defined by $Q_1$.

Proof.
i) From Eq. (5.4) and the skew–symmetry of $Q_0$ and $Q_1$ it follows that $Q_0 N^* = N Q_0$ and $Q_1 N^* = N Q_1$. Then

$$Z_1 - Q_0 \alpha_1 = Z_1 - Q_0 N^* \alpha_0 = Z_1 - N Q_0 \alpha_0 = 0 \quad (5.9)$$

and the first relation (5.7) is proved by induction since it is

$$Z_{i+1} - Q_0 \alpha_{i+1} = N Z_i - Q_0 N^* \alpha_i = N (Z_i - Q_0 \alpha_i) \quad . \quad (5.10)$$

The second relation (5.7) follows from
\[ Z_{i+1} - Q_1 \alpha_i = \mathcal{N} Z_i - Q_1 \alpha_i = \mathcal{N}(Z_i - Q_0 \alpha_i) \quad (5.11) \]

**ii)** By (5.6), the gradients \( dh_k \) can be expressed for any \( k \) in terms of \( dh_0 \)

\[ dh_k = (\sum_{i=0}^{k} \nu_{ki}N^*i)dh_0 \quad (5.12) \]

where \( \nu_{ki} \) are the elements of the matrix \( a^{-1} \), \( a \) being the lower triangular matrix defined by \( a_{ij} = \mu_{ij} \) (\( i \geq j \)), \( a_{ij} = 0 \) (\( i < j \)), \( (i,j = 0, \ldots, n - 1) \). Thus

\[ \{h_i, h_j\}_Q^0 := \langle dh_i, Q_0 dh_j \rangle = \sum_{i=0}^{j} \sum_{a=0}^{j} \nu_{ia}N^*a \langle dh_0, Q_0 N^*b dh_0 \rangle \]

\[ = \sum_{i=0}^{j} \sum_{a=0}^{j} \nu_{ia}N^*a \langle dh_0, N^*a+b Q_0 dh_0 \rangle \quad (5.13) \]

and the first relation (5.8) follows from the skew–symmetry of the tensor \( \mathcal{N}^m Q_0 \) for any \( m \). Furthermore

\[ L_{Z_\parallel}(\xi) = \langle dh_i, Q_0 \alpha_{k-1} \rangle \]

\[ = \langle dh_i, Q_0 \sum_{j=0}^{k} \mu_{kj} dh_j \rangle \]

\[ = \sum_{j=0}^{k} \mu_{kj} \{h_i, h_j\}_Q^0 \]

\[ = 0 \quad (5.14) \]

**iii)** Since \( Z_0 \) is a Hamiltonian vector field, it is Liouville–integrable on account of the previous result. Moreover, since it is

\[ \{h_i, h_j\}_Q^1 := \langle dh_i, Q_1 dh_j \rangle \]

\[ = \sum_{i=0}^{j} \sum_{a=0}^{j} \nu_{ia}N^*a \langle dh_0, Q_1 N^*b dh_0 \rangle \]

\[ = \sum_{i=0}^{j} \sum_{a=0}^{j} \nu_{ia}N^*a \langle dh_0, N^*a+b Q_1 dh_0 \rangle \]

\[ = 0. \quad (5.15) \]
it follows that if $Q_1$ is also a Poisson tensor, $\{\cdot,\cdot\}_{Q_1}$ is a Poisson bracket, $Z_1$ is a Hamiltonian vector field and then it is Liouville-integrable.

**Remark 5.1.** The recursion scheme and the integrability of the vector field $Z_0$ do not require that the skew-symmetric tensor $Q_1$ be a Poisson tensor; so $M$ is a Poisson manifold, not a bi-Hamiltonian one. □

In view of the applications of the next subsection, it may be worthwhile to remark that the results of Prop. 5.1 hold true if the role of $Q_0$ and $Q_1$ are interchanged; to be more precise, one can prove (just as for Prop. 5.1)

**Proposition 5.2.** The integrability scheme of Prop. 5.1 is still valid if $Q_0$ is skew-symmetric, $Q_1$ is a Poisson tensor and the role of $Z_0$ is now played by $Z_1 = Q_1 dh_0$. The involution relations (5.8) become $\{h_i, h_j\}_{Q_1} = 0$.

### 5.3. The integrability structure of the Hénon–Heiles and the Garnier systems.

In Subsect. 5.1 we have recovered by projection onto the quotient manifold $S_2$ the Nijenhuis tensor (5.2) and a hierarchy of compatible Poisson tensors; however, it is not possible to associate to these tensors and to the Hénon–Heiles vector field $X_1$ (2.41) a bi-Hamiltonian hierarchy of vector fields. Nevertheless it is possible to use these elements to construct an example of the integrability structure introduced in Prop. 5.2. To this purpose, let us make the following choices:

i) $Q_1 = E$, the vector field $Z_1 := X_1$ (2.41) with Hamiltonian $h_0 := H_0$ (2.36);  
ii) the tensor field $\mathcal{N} := N_H$ (5.2) and $Q_0 := P_{-2} = N_H^{-2}P_H$, with $P_H$ as in (5.1);  
iii) the function $h_1 := H_1$ (2.40) and the functions $\mu_{ij}$ as $\mu_{10} = 0, \mu_{11} = 1/q_2^2$;

then it is immediate to check that the conditions of Prop. 5.2 are satisfied. Moreover the vector field $Z_0 := Q_0 dh_0 = P_{-2} dH_0$ is a new integrable vector field:

\[
Z_0 = \begin{bmatrix}
-2p_1q_1 - p_2q_2 \\
-p_1q_2 - 2p_1^2 + 6q_1^2 + 2q_1q_2 - \frac{a_4}{4q_2^2} - \frac{a_1}{2}q_1 - 2a_0q_1^2 + \frac{a_0}{4}q_2^2 \\
p_1p_2 + 2p_2^2 + 3q_1^2q_2 - \frac{a_4}{4}q_2 + a_0q_1q_2
\end{bmatrix}, \quad (5.16)
\]

This integrability structure is related, through the map (2.38), to the one introduced in [8] for the Hamiltonian (2.37) with $a_4 = 0$. 

For the Garnier system with two degrees of freedom one can construct an example of the integrability structures of Prop. 5.1. Indeed if one uses the elements of Subsect. 5.1 and makes the following choices:

i) \( Q^0 := E \), \( h^0 := \tilde{G}_1 (3.10) \), \( Z^0 := Y G (3.6) \);

ii) \( \mathcal{N} := N^{-1} = E \mathcal{P}^{-\infty}_g \), with \( \mathcal{P}_g \) as in (5.3), \( Q^\infty := \mathcal{P}^{-\infty} = N^{-\infty} E \);

iii) the functions \( h_1 := \tilde{G}_2 (3.10) \), \( \mu_{10} = 0 \), \( \mu_{11} = -\frac{\lambda_1^2 \lambda_2^2}{\lambda_2^2 \psi_2^2 + \lambda_1 \psi_2^2 - \lambda_1 \lambda_2} \);

then the conditions of Prop. 5.1 are satisfied. Moreover the vector field \( Z_1 := \mu_{11} Y \) is a new integrable vector field (\( Y \) is the restriction to the submanifold of \( \mathcal{M}_\varepsilon \), \( d = \text{cost} \), of the vector field \( \bar{Y}_2 (3.3) \)).

At last, we compute how the map between the standard phase spaces of the Hénon–Heiles and of the Garnier systems, induced by the map (4.10), acts on the recursion operators of the previous integrability structures.

**Proposition 5.3.** Let us consider the map \( \Psi : (q_1, q_2, p_1, p_2) \mapsto (\psi_1, \psi_2, \chi_1, \chi_2) \)

\[
\psi_1 = \lambda_{12}^{-1/2} \left( \lambda_1^2 + 2 \lambda_1 q_1 - q_2^2 \right)^{1/2}, \quad \psi_2 = \left( \lambda_{12} \right)^{-1/2} \left( -\lambda_2^2 - 2 \lambda_2 q_1 + q_2^2 \right)^{1/2}, \\
\chi_1 = \frac{\left( \lambda_1 p_1 - q_2 p_2 \right)}{\left( \lambda_{12} \left( \lambda_1^2 + 2 \lambda_1 q_1 - q_2^2 \right) \right)^{1/2}}, \quad \chi_2 = \frac{\left( \lambda_2 p_1 - q_2 p_2 \right)}{\left( \lambda_{12} \left( -\lambda_2^2 - 2 \lambda_2 q_1 + q_2^2 \right) \right)^{1/2}}.
\]

(5.17)

The map \( \Psi \) relates the recursion operators of the Hénon–Heiles and of the Garnier systems: \( \Psi_\ast N_H = N_g^{-\infty} \Psi_\ast \).

6. **Concluding remarks**

In this paper we have derived a bi–Hamiltonian formulation for stationary flows, and for the first restricted flows of the KdV hierarchy. Our approach amounts to searching the kernel of, respectively, the Poisson pencil and \( n \)-Poisson structures extracted from the Poisson pencil of the KdV hierarchy. In this approach the generating function of the GD polynomials plays a relevant role. Moreover it allows us to construct a map between stationary flows and restricted flows; in the case of the fifth-order stationary KdV equation, this map relates solutions of the Hénon–Heiles system with solutions of the Garnier system. However, to obtain these results one must extend the phase space of the reduced flows by means of some free parameters naturally contained in the corresponding Hamiltonian functions. This difficulty can be overcome, at least if one analyzes complete integrability of a Hamiltonian system without requiring an explicit knowledge of a bi–Hamiltonian structure. To this purpose, we have introduced a new integrability
scheme in the standard phase space, which implies Liouville integrability of the reduced Hamiltonian systems. For brevity we have applied this scheme only to the Hénon–Heiles and the Garnier systems with two degrees of freedom. Other examples such as Hénon–Heiles type systems with three and four degrees of freedom, constructed by means of the reduction method of Sect. 2, will be discussed elsewhere.

Acknowledgments. I wish to thank F. Magri, who pointed out the role of the GD polynomials in the bi–Hamiltonian formulation of the KdV hierarchy and C. Morosi for many valuable discussions and suggestions.
References

[1] L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific, Singapore 1991.

[2] M. Antonowicz, S. Rauch–Wojciechowski, *How to construct finite dimensional bi–Hamiltonian systems from soliton equations: Jacobi integrable potentials*, J. Math. Phys. 33 (1992), 2115–2125.

[3] A.P. Fordy, *The Hénon–Heiles system revisited*, Physica D 52 (1991), 204–210.

[4] M. Antonowicz, S. Rauch–Wojciechowski, *Bi–Hamiltonian Formulation of the Hénon–Heiles System and its Multidimensional Extensions*, Phys. Lett. A 163 (1992), 167–172.

[5] C. W. Cao, *Non Linearization of Eigenvalue Problem*, Non Linear Physics, (Gu C. et al., eds.), Springer–Verlag, Berlin 1990, pp. 66–78.

[6] M. Antonowicz, S. Rauch–Wojciechowski, *Constrained flows of integrable PDEs and bi–Hamiltonian structure of the Garnier system*, Phys. Lett. A 147 (1990), 455–462.

[7] G. Tondo, *A connection between the Hénon–Heiles system and the Garnier system*, Theor. Math. Phys. 33 (1994), 796–802 and Teoret. Matem. Fiz. 99 (1994), 552–559.

[8] R. Caboz, V. Ravoson, L. Gavrilov, *Bi–Hamiltonian structure of an integrable Hénon–Heiles system*, J. Phys. A 24 (1991), L523–L525.

[9] V.I. Arnold, *Mathematical Methods in Classical Mechanics*, Springer–Verlag, New York 1989.

[10] P. Libermann, C. M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht 1987.

[11] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. 19 (1978), 1156–1162.

[12] P. Casati, F. Magri, M. Pedroni, *Bihamiltonian Manifolds and τ–function*, Contemporary Mathematics 132, ( M. J. Gotay et al., eds.), American Mathematical Society, Providence 1992, pp. 213–234.

[13] P. Lax, *Periodic Solutions of the KdV Equation*, Comm. Pure Appl. Math. XXVIII (1975), 141–188.

[14] S. P. Novikov, *Integrable Systems*, Lect. Not. Series 60 , Cambridge University Press 1981, pp. 1–12.

[15] O.I. Bogoyavlenskii, S.P. Novikov, *The relationship between Hamiltonian formalism of stationary and non stationary problems*, Funct. Anal. Appl. 176 (1976), 8–11.

[16] S. Rauch–Wojciechowski, *Newton representation for stationary flows of the KdV hierarchy*, Phys. Lett. A 170 (1992), 91–94.

[17] M. Antonowicz, A.P. Fordy, S. Wojciechowski, *Integrable stationary flows, Miura maps and Bi–Hamiltonian Structures*, Phys. Lett. A 124 (1987), 143–150.

[18] M. Blaszak, S. Rauch–Wojciechowski, *A Hénon–Heiles system and related integrable Newton equations*, J. Math. Phys. 35 (1994), 1693–1709.

[19] M. Tabor, *Chaos and Dynamical systems*, Wiley, New York 1989.

[20] J. Moser, *Various aspects of integrable Hamiltonian systems*, in Dynamical Systems, (CIME 1978), Progress in Mathematics 8, Birkhauser, Basle 1980, pp. 233–289.
[21] S. I. Alber, *On Stationary Problems for Equations of Korteweg-de Vries Type*, Comm. Pure Appl. Math. **XXXIV** (1981), 259–272.

[22] C. W. Cao, *Parametric representation of the finite-band solution of the Heisenberg equation*, Phys. Lett. A **184** (1994), 333–338.

[23] S. Wojciechowski, *Integrability of One Particle in a Perturbed Central Quartic Potential*, Phys. Scripta, **31** (1985), 433–438.

[24] J. E. Marsden, T. Ratiu, *Reduction of Poisson Manifolds*, Lett. Math. Phys. **11** (1986), 161–169.

[25] F. Magri, *A geometrical approach to the non linear solvable equations*, in Lect. Not. Phys. **120**, M. Boiti, F. Pempinelli, G. Soliani eds., Springer–Verlag, Berlin 1980, 233–263.

G. Tondo, Dipartimento di Scienze Matematiche, Università degli Studi di Trieste, Piazz.le Europa 1, I34127 Trieste, Italy.

E-mail address: TONDO@UNIV.TRIESTE.IT