Abstract: In this paper, we investigate the Koszul behavior of finitely semi-graded algebras by the distributivity of some associated lattice of ideals. The Hilbert series, the Poincaré series, and the Yoneda algebra are defined for this class of algebras. Moreover, the point modules and the point functor are introduced for finitely semi-graded rings. Finitely semi-graded algebras and rings include many important examples of non-$N$-graded algebras coming from mathematical physics that play a very important role in mirror symmetry problems, and for these concrete examples, the Koszulity will be established, as well as the explicit computation of its Hilbert and Poincaré series.

Keywords: graded algebras; distributive lattices; Koszul algebras; Hilbert and Poincaré series; Yoneda algebra; point modules; point functor; Zariski topology; skew PBW extensions

MSC: Primary: 16W70; Secondary: 16S37, 16S38, 16S36, 16W50

1. Introduction

Finitely graded algebras over fields cover many important classes of non-commutative rings and algebras coming from mathematical physics; examples of these algebras are the multi-parameter quantum affine $n$-space, the Jordan plane, the Manin algebra $M_q(2)$, and the multiplicative analogue of the Weyl algebra, among many others. There has recently been interest in developing the non-commutative projective algebraic geometry for finitely graded algebras (see, for example, [1–6]). However, for non-$N$-graded algebras, only a few works in this direction have been realized ([7,8]). Some examples of non-$N$-graded algebras generated in degree one are the dispin algebra $\mathcal{U}(osp(1,2))$, the Woronowicz algebra $\mathcal{W}_q(s(2,K))$, the quantum algebra $\mathcal{U}'(so(3,K))$, the quantum symplectic space $O_q(s(p(2n)))$, and some algebras of operators, among others. Two of the most important algebraic properties studied in non-commutative algebraic geometry for graded algebras are the Koszulity and the functor parametrization of its point modules. Koszul graded algebras were defined by Priddy in [9] and have many equivalent characterizations involving the Hilbert series, the Poincaré series, and some associated lattices of vector spaces. In this paper, we were interested in investigating the Koszul behavior for algebras over fields not being necessarily $N$-graded, and also in defining the set of point modules for finitely semi-graded rings. Finitely semi-graded algebras extend finitely graded algebras over fields generated in degree one and conform a particular subclass of finitely semi-graded rings defined in [8]. In addition, for finitely semi-graded algebras, we defined its Hilbert series, the Poincaré series, the Yoneda algebra, and we investigated some associated lattices of vector spaces similarly, as this is done in the classical graded case.

For finitely semi-graded algebras, we studied the uniqueness of the Hilbert series (Corollary 2); for this, we used a beautiful paper by Bell and Zhang ([10]), where this property was established for connected graded algebras finitely generated in degree 1. The uniqueness of the Poincaré series of a
given finitely semi-graded algebra was proved assuming that its Yoneda algebra is finitely generated in degree one and the base field has a free homogeneous resolution (Corollary 4). We see that a finitely
semi-graded algebra has a natural induced $\mathbb{N}$-filtration, so we show that the Hilbert series of the
algebra coincides with the Hilbert series of its associated graded algebra. We associated to a finitely
semi-graded algebra a lattice of vector spaces defined with the ideal of relations of its presentation,
and from a result that gives conditions over the distributiveness of this lattice (Theorem 6), we
defined the semi-graded Koszul algebras, extending this way the well-known notion of graded Koszul
algebras. On the other hand, we also studied in the present paper the set of point modules for finitely
semi-graded rings, and a standard Zariski topology is defined for them as well as the point functor.
One important part of the present paper consists in giving many examples of finitely semi-graded
algebras and rings as well as examples of semi-graded Koszul algebras. Most of the examples that
we present arise in mathematical physics and can be interpreted as skew PBW extensions.
This class of non-commutative rings of polynomial type were introduced in [11], and they are a good
global way of describing rings and algebras not being necessarily $\mathbb{N}$-graded. Thus, the general results
that we proved for finitely semi-graded algebras are in particular applied to skew PBW extensions;
in Corollary 3, we explicitly computed the Hilbert series of skew PBW extensions that are finitely
semi-graded algebras over fields, covering this way many examples of quantum algebras. Finally, in
Theorem 7 and Example 6, we present examples of non-$\mathbb{N}$-graded algebras that have Koszul behavior,
I.e., they are semi-graded Koszul.

The paper is organized as follows: In Section 1, we review the basic facts on semi-graded rings and
skew PBW extensions that we need for the rest of the work. In Section 2, we introduce the semi-graded
algebras and we present many examples of them. The list of examples include not only skew PBW
extensions that are algebras over fields, but also other non-graded algebras that cannot be described
as skew extensions. Section 3 is dedicated to constructing and proving the uniqueness of the Hilbert
series, the Poincaré series, and the Yoneda algebra of a finitely semi-graded algebra. In Section 4,
we study the Koszul behavior of finitely semi-graded algebras, and we show that some non-$\mathbb{N}$-graded
algebras coming from quantum physics are semi-graded Koszul. In Section 5, we introduce and study
the collection of point modules for finitely semi-graded rings. A standard Zariski topology is defined
for them as well as the point functor. In a forthcoming paper, we will compute the set of point modules
for many concrete examples of skew PBW extensions.

If not otherwise noted, all modules are left modules, and $K$ will be an arbitrary field. In order to
appreciate better the results of the paper, we first recall the definition of finitely graded algebras over
fields and its Hilbert series (see [5]). Let $A$ be a $K$-algebra, $A$ is finitely graded if: (a) $A$ is $\mathbb{N}$-graded, i.e.,
$A$ has a graduation $A = \bigoplus_{n \geq 0} A_n$, $A_n A_m \subseteq A_{n+m}$ for every $n, m \geq 0$; (b) $A$ is connected, i.e., $A_0 = K$;
(c) $A$ is finitely generated as $K$-algebra. Thus, $A$ is locally finite, i.e., $\dim_K A_n < \infty$ for every $n \geq 0$, and
hence, the Hilbert series of $A$ is defined by:
$$h_A(t) := \sum_{n=0}^{\infty} (\dim_K A_n) t^n.$$ 

1.1. Semi-Graded Rings and Modules

In this starting subsection, we recall the definition and some basic facts about semi-graded rings
and modules; more details and the proofs omitted here can be found in [8].

Definition 1. Let $B$ be a ring. We say that $B$ is semi-graded (SG) if there exists a collection $\{B_n\}_{n \geq 0}$ of
subgroups $B_n$ of the additive group $B^+$ such that the following conditions hold:

(i) \( B = \bigoplus_{n \geq 0} B_n. \)

(ii) \( \text{For every } m, n \geq 0, B_m B_n \subseteq B_0 \oplus \cdots \oplus B_{m+n}. \)

(iii) \( 1 \in B_0. \)

The collection $\{B_n\}_{n \geq 0}$ is called a semi-gradation of $B$, and we say that the elements of $B_n$ are homogeneous
of degree $n$. Let $B$ and $C$ be semi-graded rings and let $f : B \to C$ be a ring homomorphism; we say that $f$
is homogeneous if $f(B_n) \subseteq C_n$ for every $n \geq 0$. 

Definition 2. Let \( B \) be an SG ring and let \( M \) be a \( B \)-module. We say that \( M \) is a \( \mathbb{Z} \)-semi-graded, or simply semi-graded, if there exists a collection \( \{ M_n \}_{n \in \mathbb{Z}} \) of subgroups \( M_n \) of the additive group \( M^+ \) such that the following conditions hold:

(i) \( M = \bigoplus_{n \in \mathbb{Z}} M_n \).
(ii) For every \( m \geq 0 \) and \( n \in \mathbb{Z} \), \( B^m M_n \subseteq \bigoplus_{k \leq m+n} M_k \).

The collection \( \{ M_n \}_{n \in \mathbb{Z}} \) is called a semi-graduation of \( M \), and we say that the elements of \( M_n \) are homogeneous of degree \( n \). We say that \( M \) is positively semi-graded, also called \( \mathbb{N} \)-semi-graded, if \( M_n = 0 \) for every \( n < 0 \).

Let \( B \) be a semi-graded ring and \( M \) be a semi-graded \( B \)-module, let \( N \) be a submodule of \( M \); we say that \( N \) is a semi-graded submodule of \( M \) if \( N = \bigoplus_{n \in \mathbb{Z}} N_n \).

We present next an important class of semi-graded rings that includes finitely graded algebras.

Definition 3. Let \( B \) be an SG ring and \( M \) be a semi-graded module over \( B \). Let \( N \) be a submodule of \( M \); we say that \( N \) is a semi-graded submodule of \( M \) if \( N = \bigoplus_{n \in \mathbb{Z}} N_n \).

We get the following elementary but key facts.

Proposition 1. Let \( B = \bigoplus_{n \geq 0} B_n \) be an SG ring. Then:

(i) \( B_0 \) is a subring of \( B \). Moreover, for any \( n \geq 0 \), \( B_0 \oplus \cdots \oplus B_n \) is a \( B_0 \)-\( B_0 \)-bimodule, as well as \( B \).
(ii) \( B \) has a standard \( \mathbb{N} \)-filtration given by:

\[
F_n(B) := B_0 \oplus \cdots \oplus B_n.
\]

(iii) The associated graded ring \( Gr(B) \) satisfies:

\[
Gr(B)_n \cong B_n, \text{ for every } n \geq 0 \text{ (isomorphism of abelian groups)}.\]

(iv) Let \( M = \bigoplus_{n \in \mathbb{Z}} M_n \) be a semi-graded \( B \)-module and \( N \) a submodule of \( M \). The following conditions are equivalent:

(a) \( N \) is semi-graded.
(b) For every \( z \in N \), the homogeneous components of \( z \) are in \( N \).
(c) \( M/N \) is semi-graded with semi-graduation given by:

\[
(M/N)_n := (M_n + N)/N, n \in \mathbb{Z}.
\]

Remark 1. (i) According to (ii)–(b) in the previous proposition, if \( N \) is a semi-graded submodule of \( M \), then \( N \) can be generated by homogeneous elements; however, if \( N \) is a submodule of \( M \) generated by homogeneous elements, then we cannot assert that \( N \) is semi-graded.

(ii) Let \( B \) be an SG ring, as we saw in (ii) of the previous proposition, then \( B \) is \( \mathbb{N} \)-filtered. Conversely, if we assume that \( B \) is a \( \mathbb{N} \)-filtered ring with filtration \( \{ F_n(B) \}_{n \geq 0} \) such that for any \( n \geq 0 \), \( F_n(B)/F_{n-1}(B) \) is
Hilbert series based on a recent paper by Bell and Zhang \[10\].

In Section 3, we introduce the semi-graded algebras over fields, and for them, we discuss the uniqueness of the generalized Hilbert series.

**Remark 2.**

(i) Note that if \( K \) is a field and \( B \) is a finitely graded \( K \)-algebra, then the generalized Hilbert series \( \text{Gh}_B(t) \) coincides with the usual Hilbert series, i.e., \( \text{Gh}_B(t) = h_B(t) \).

(ii) Observe that if an FSG ring \( B = \bigoplus_{n \geq 0} B_n \) has another semi-graduation \( B = \bigoplus_{n \geq 0} B'_n \), then its generalized Hilbert series depends on the semi-graduation, in particular on \( B_0 \). For example, consider the usual real polynomial ring in two variables \( B := \mathbb{R}[x,y] \), then \( \text{Gh}_B(t) = \frac{1}{(1-t)^2} \), but if we view this ring as \( B = (\mathbb{R}[x])[y] \) then \( \text{Gh}_B(t) = \mathbb{R}[x] \), its generalized Hilbert series is \( \frac{1}{1-t} \). However, in Section 3, we introduce the semi-graded algebras over fields, and for them, we discuss the uniqueness of the Hilbert series based on a recent paper by Bell and Zhang \[10\].

1.2. Skew PBW Extensions

As was pointed out above, finitely graded algebras over fields are examples of FSG rings. In order to present many other examples of FSG rings not being necessarily graded algebras, we recall in this subsection the notion of skew PBW extension defined first in \[11\].

**Definition 6 (\[11\]).** Let \( R \) and \( A \) be rings. We say that \( A \) is a skew PBW extension of \( R \) (also called a \( \sigma - PBW \) extension of \( R \)) if the following conditions hold:

(i) \( R \subseteq A \).

(ii) There exist finitely many elements \( x_1, \ldots, x_n \in A \) such \( A \) is a left \( R \)-free module with basis:

\[
\text{Mon}(A) := \{ x^\alpha = x_1^{a_1} \cdots x_n^{a_n} \mid \alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n \}, \text{ with } \mathbb{N} := \{0, 1, 2, \ldots \}.
\]

The set \( \text{Mon}(A) \) is called the set of standard monomials of \( A \).

(iii) For every \( 1 \leq i \leq n \) and \( r \in R - \{0\} \), there exists \( c_i \in R - \{0\} \) such that:

\[
x_ir - c_ix_i \in R.
\]

(iv) For every \( 1 \leq i, j \leq n \), there exists \( c_{ij} \in R - \{0\} \) such that:

\[
x_jx_i - c_{ij}x_ix_j \in R + Rx_1 + \cdots + Rx_n.
\]

Under these conditions, we write \( A := \sigma(R)(x_1, \ldots, x_n) \).

**Example 1.** Many important algebras and rings coming from mathematical physics are particular examples of skew PBW extensions: the habitual ring of polynomials in several variables, Weyl algebras, enveloping algebras of finite dimensional Lie algebras, algebra of \( q \)-differential operators, many important types of Ore algebras, algebras of diffusion type, additive and multiplicative analogues of the Weyl algebra, dispin algebra \( \mathcal{U}(\mathfrak{osp}(1,2)) \), quantum algebra \( \mathcal{U}'(\mathfrak{so}(3,K)) \), Woronowicz algebra \( \mathcal{W}_q(\mathfrak{sl}(2,K)) \), Manin algebra \( \mathcal{O}_q(M_2(K)) \), coordinate
algebra of the quantum group $SL_q(2)$, $q$-Heisenberg algebra $H_n(q)$, Hayashi algebra $W_0(f)$, differential operators on a quantum space $D_q(S_q)$. Witten’s deformation of $U(sl_2(K))$, multi-parameter Weyl algebra $A_{2,1}^T(K)$, quantum symplectic space $O_q(sp(K^2))$, some quadratic algebras in 3 variables, some 3-dimensional skew polynomial algebras, particular types of Sklyanin algebras, homogenized enveloping algebra $A(G)$, and Sridharan enveloping algebra of 3-dimensional Lie algebra $G$, among many others. For a precise definition of any of these rings and algebras, see [6,12–15].

Associated to a skew PBW extension $A = σ(R)⟨x_1, \ldots, x_n⟩$, there are $n$ injective endomorphisms $σ_1, \ldots, σ_n$ of $R$ and $σ_i$-derivations, as the following proposition shows.

**Proposition 2** ([11]). Let $A$ be a skew PBW extension of $R$. Then, for every $1 ≤ i ≤ n$, there exists an injective ring endomorphism $σ_i : R → R$ and a $σ_i$-derivation $δ_i : R → R$ such that:

$$x_ir = σ_i(r)x_i + δ_i(r),$$

for each $r ∈ R$.

A particular case of skew PBW extension is when all derivations $δ_i$ are zero. Another interesting case is when all $σ_i$ are bijective and the constants $c_{ij}$ are invertible. We recall the following definition.

**Definition 7** ([6,11,14,15]). Let $A$ be a skew PBW extension.

(a) $A$ is quasi-commutative if the conditions (iii) and (iv) in Definition 6 are replaced by:

(iii′) For every $1 ≤ i ≤ n$ and $r ∈ R \setminus \{0\}$, there exists $c_{ij} ∈ R \setminus \{0\}$ such that:

$$x_ir = c_{ij}x_i.$$  \hspace{1cm} (4)

(iv′) For every $1 ≤ i, j ≤ n$, there exists $c_{ij} ∈ R \setminus \{0\}$ such that:

$$x_jx_i = c_{ij}x_ix_j.$$  \hspace{1cm} (5)

(b) $A$ is bijective if $σ_i$ is bijective for every $1 ≤ i ≤ n$, and $c_{ij}$ is invertible for any $1 ≤ i, j ≤ n$.

(c) $A$ is constant if the condition (ii) in Definition 6 is replaced by: For every $1 ≤ i ≤ n$ and $r ∈ R$,

$$x_ir = rx_i.$$  \hspace{1cm} (6)

(d) $A$ is pre-commutative if the condition (iv) in Definition 6 is replaced by: For any $1 ≤ i, j ≤ n$ there exists $c_{ij} ∈ R \setminus \{0\}$ such that:

$$x_jx_i - c_{ij}x_ix_j ∈ Rx_1 + \cdots + Rx_n.$$  \hspace{1cm} (7)

(e) $A$ is called semi-commutative if $A$ is quasi-commutative and constant.

**Remark 3.** Later below, we need the following classification given in [6,14,15] of skew PBW extensions of Example 1. The extensions are classified as constant (C), bijective (B), pre-commutative (P), quasi-commutative (QC), and semi-commutative (SC); in Tables 1–3, the symbols $\ast$ and $\checkmark$ denote negation and affirmation, respectively:
Table 1. Classification of Skew PBW Extensions.

| Skew PBW Extension                                           | C | B | P | QC | SC |
|-------------------------------------------------------------|---|---|---|----|----|
| Classical polynomial ring                                   | ✓ | ✓ | ✓ | ✓ | ✓ |
| Ore extensions of bijective type                            | * | ✓ | ✓ | * | * |
| Weyl algebra                                                | ✓ | ✓ | ✓ | * | * |
| Particular Sklyanin algebra                                 | ✓ | ✓ | ✓ | ✓ | ✓ |
| Universal enveloping algebra of a Lie algebra               | ✓ | ✓ | ✓ | * | * |
| Homogenized enveloping algebra $A(G)$                       | ✓ | ✓ | ✓ | * | * |
| Tensor product                                              | ✓ | ✓ | ✓ | * | * |
| Crossed product                                             | ✓ | ✓ | * | * | * |
| Algebra of $q$-differential operators                        | * | ✓ | ✓ | * | * |
| Algebra of shift operators                                  | ✓ | ✓ | ✓ |   | * |
| Mixed algebra                                               | ✓ | * | * | * | * |
| Algebra of discrete linear systems                          | ✓ | ✓ | ✓ | * | * |
| Linear partial differential operators                        | * | ✓ | ✓ | * | * |
| Linear partial shift operators                               | * | ✓ | ✓ | * | * |
| Algebra of linear partial difference operators               | * | ✓ | ✓ | * | * |
| Algebra of linear partial q-dilation operators               | * | ✓ | ✓ | * | * |
| Algebra of linear partial q-differential operators           | * | ✓ | ✓ | * | * |
| Algebras of diffusion type                                   | ✓ | ✓ | * | * | * |
| Additive analogue of the Weyl algebra                        | ✓ | ✓ | * | * | * |
| Multiplicative analogue of the Weyl algebra                  | ✓ | ✓ | * | * | * |
| Quantum algebra $U(sl(3, K))$                                | ✓ | ✓ | ✓ |   |   |
| Dispens algebra                                              | ✓ | ✓ | * | * | * |
| Woronowicz algebra                                          | ✓ | ✓ | * | * | * |
| Complex algebra                                              | ✓ | * | * | * | * |
| Algebra $A$                                                  | ✓ | * | * | * | * |
| Manin algebra                                               | ✓ | * | * | * | * |
| $q$-Heisenberg algebra                                       | ✓ | * | * | * | * |
| Quantum enveloping algebra of $sl(2, K)$                    | ✓ | * | * | * | * |
| Hayashi's algebra                                           | ✓ | * | * | * | * |
| The algebra of differential operators on a quantum space $S_q$ | ✓ | * | * | * | * |
| Witt's deformation of $U(sl(2, K))$                         | ✓ | * | * | * | * |
| Quantum Weyl algebra of Maltsinioti                         | * | ✓ | * | * | * |
| Quantum Weyl algebra                                         | * | ✓ | * | * | * |
| Multi-parameter quantized Weyl algebra                       | ✓ | ✓ | * | * | * |
| Quantum symplectic space                                     | * | ✓ | * | * | * |
| Quadratic algebras in 3 variables                           | ✓ | * | * | * | * |

Table 2. Classification of 3-Dimensional Skew Polynomial Algebras.

| Cardinal | 3-Dimensional Skew Polynomial Algebras | C | B | P | QC | SC |
|----------|----------------------------------------|---|---|---|----|----|
| $|\{a, \beta, \gamma\}| = 3$ | $yz - axy = 0, \ zx - \beta xx = 0, \ xy - \gamma yx = 0$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| $|\{a, \beta, \gamma\}| = 2, \beta \neq a = \gamma = 1$ | $yz - zy = z, \ zx - \beta xx = y, \ xy - yx = x$ | ✓ | ✓ | ✓ | * | * |
| $|\{a, \beta, \gamma\}| = 2, \beta \neq a \neq \gamma = 1$ | $yz - zy = 0, \ zx - \beta xx = b, \ xy - yx = 0$ | ✓ | ✓ | * | * | * |
| $\alpha = \beta = 1$ | $yz - zy = a, \ zx - \beta xx = 0, \ xy - yx = 0$ | ✓ | ✓ | ✓ |   |   |
| $\alpha = \beta = \gamma = 1$ | $yz - zy = x, \ zx - xx = y, \ xy - yx = z$ | ✓ | ✓ | ✓ |   |   |
| $\alpha = \beta = \gamma = 1$ | $yz - zy = 0, \ zx - xx = 0, \ xy - yx = 0$ | ✓ | ✓ | * | * | * |
| $\alpha = \beta = \gamma = 1$ | $yz - zy = y, \ zx - xx = x, \ xy - yx = 0$ | ✓ | ✓ | ✓ | * | * |
| $\alpha = \beta = \gamma = 1$ | $yz - zy = az, \ zx - xx = x, \ xy - yx = 0$ | ✓ | ✓ | ✓ | * | * |
Let $A$ be a skew PBW extension of the ring $R$. Then, $A$ is a skew PBW extension if and only if $A$ is a quasi-commutative skew PBW extension of $R$. Any skew PBW extension is an FSG ring.

Definition 8. Let $A = \sigma(R) \langle x_1, \ldots, x_n \rangle$ be a skew PBW extension. $R$, $n$, $\sigma_k$, $\delta_k$, $c_{ij}$, $d_{ij}$, $a^{(k)}_{ij}$, with $1 \leq i < j \leq n$, defined as before, are called the parameters of $A$.

Some notation will be useful in what follows.

Definition 9. Let $A$ be a skew PBW extension of $R$.

(i) For $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $|a| := a_1 + \cdots + a_n$.
(ii) For $X = x^a \in \text{Mon}(A)$, $\exp(X) := a$ and $\deg(X) := |a|$.
(iii) Let $0 \neq f \in A$, and $t(f)$ is the finite set of terms that conform $f$, i.e., if $f = c_1X_1 + \cdots + c_lX_l$, with $X_i \in \text{Mon}(A)$ and $c_i \in R - \{0\}$, then $t(f) := \{c_1X_1, \ldots, c_lX_l\}$.
(iv) Let $f$ be as in (iii), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^l$.

Skew PBW extensions have been investigated enough, with many homological properties of them having been studied, as well as their Gröbner theory ([8,11–13,16–24]). We conclude this introductory section with some known results about skew PBW extensions and semi-graded rings that we use in the present paper.

Theorem 1 ([12]). Let $A$ be an arbitrary skew PBW extension of the ring $R$. Then, $A$ is a filtered ring with filtration given by:

$$F_m := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A | \deg(f) \leq m\}, & \text{if } m \geq 1, \end{cases}$$

and the graded ring $\text{Gr}(A)$ is a quasi-commutative skew PBW extension of $R$. If the parameters that define $A$ are as in Definition 8, then the parameters that define $\text{Gr}(A)$ are $R$, $n$, $\sigma_k$, $c_{ij}$, with $1 \leq i < j \leq n$, $1 \leq k \leq n$. Moreover, if $A$ is bijective, then $\text{Gr}(A)$ is bijective.

Proposition 3 ([8]). (i) Any $\mathbb{N}$-graded ring is SG.
(ii) Let $K$ be a field. Any finitely graded $K$-algebra is an FSG ring.
(iii) Any skew PBW extension is an FSG ring.

For skew PBW extensions, the generalized Hilbert series has been computed explicitly.
Theorem 2 ([8]). Let \( A = \sigma(R)(x_1, \ldots, x_n) \) be an arbitrary skew PBW extension. Then:

\[
Gh_A(t) = \frac{1}{(1-t)^n}.
\] (9)

Remark 4. (i) Note that the class of SG rings properly includes the class of \( \mathbb{N} \)-graded rings: In fact, the enveloping algebra of any finite-dimensional Lie algebra proves this statement. This example proves also that the class of FSG rings properly includes the class of finitely graded algebras.

(ii) The class of FSG rings properly includes the class of skew PBW extensions: For this, consider the Artin–Schelter regular algebra of global dimension 3 defined by the following relations:

\[
\begin{align*}
yx &= xy + z^2, \\
yz &= yz + x^2, \\
zx &= zx + y^2.
\end{align*}
\]

Observe that this algebra is a particular case of a Sklyanin algebra, which in general is defined by the following relations:

\[
\begin{align*}
axy + bxy + cz^2 &= 0, \\
azy + byz + cx^2 &= 0, \\
axz + bzx + cy^2 &= 0, \\
a, b, c \in K.
\end{align*}
\]

2. Finitely Semi-Graded Algebras

In the present section, we define the finitely semi-graded algebras. All of the examples that we study, in particular, the semi-graded Koszul algebras that we introduce later, are additionally finitely presented. Let us recall first this notion. Let \( B \) be a finitely generated \( K \)-algebra, so there exist finitely many elements \( g_1, \ldots, g_n \in B \) that generate \( B \) as \( K \)-algebra, and we have the \( K \)-algebra homomorphism \( f : K\{x_1, \ldots, x_n\} \to B \), with \( f(x_i) := g_i, 1 \leq i \leq n; \) let \( I := \ker(f) \), then we get a presentation of \( B \):

\[
B \cong K\{x_1, \ldots, x_n\}/I.
\] (10)

Recall that \( B \) is said to be finitely presented if \( I \) is finitely generated.

2.1. Definition

In the previous section, we defined the finitely semi-graded rings, and we observed that they generalize finitely graded algebras over fields and skew PBW extensions. In this section, we concentrate in some particular class of this type of rings which satisfy some other extra natural conditions.

Definition 10. Let \( B \) be a \( K \)-algebra. We say that \( B \) is finitely semi-graded (FSG) if the following conditions hold:

(i) \( B \) is an FSG ring with semi-graduation \( B = \bigoplus_{p \geq 0} B_p \).

(ii) For every \( p, q \geq 1 \), \( B_p B_q \subseteq B_{1} \oplus \cdots \oplus B_{p+q} \).

(iii) \( B \) is connected, i.e., \( B_0 = K \).

(iv) \( B \) is generated in degree 1.

Remark 5. Let \( B \) be an FSG \( K \)-algebra;

(i) Since \( B \) is locally finite and \( B \) is finitely generated in degree 1, then any \( K \)-basis of \( B_1 \) generates \( B \) as \( K \)-algebra.

(ii) The canonical projection \( \varepsilon : B \to K \) is a homomorphism of \( K \)-algebras, called the augmentation map, with \( \ker(\varepsilon) = \bigoplus_{n \geq 1} B_n \). Therefore, the class of FSG algebras is contained in the class of augmented algebras, i.e., algebras with augmentation (see [25]); however, as we see, a semi-gradation is a nice tool for defining some invariants useful for the study of the algebra. \( B_{\geq 1} := \bigoplus_{n \geq 1} B_n \) is called the augmentation ideal. Thus, \( K \) becomes a \( B \)-bimodule with products given by \( b \cdot \lambda := b_0 \lambda, \lambda \cdot b := \lambda b_0 \), with \( b \in B, \lambda \in K, \) and \( b_0 \) is the homogeneous component of \( b \) of degree zero.

(iii) It is well known that \( B \) is finitely graded if and only if the ideal \( I \) in (10) is homogeneous ([5]). In general, finitely semi-graded algebras do not need to be finitely presented. Any finitely graded algebra
generated in degree 1 is FSG, but \( B := K\{x, y\} / (xy - x) \) with semi-graduation \( B_n := K<yx^nx^{-k}|0 \leq k \leq n> \), \( n \geq 0 \), is an FSG algebra, and it is not finitely graded generated in degree 1. Thus, the class of FSG algebras includes properly all finitely graded algebras generated in degree 1.

(iv) Any FSG algebra is \( \mathbb{N} \)-filtered (see Proposition 1), but note that the Weyl algebra \( A_{1}(K) = K\{t, x\} / (tx - xt - 1) \) is \( \mathbb{N} \)-filtered but not FSG, i.e., the class of FSG algebras do not coincide with the class of \( \mathbb{N} \)-filtered algebras.

**Proposition 4.** Let \( B \) be an FSG algebra over \( K \). Then \( B_{\geq 1} \) is the unique two-sided maximal ideal of \( B \) semi-graded as left ideal.

**Proof.** From Remark 5, we have that \( B_{\geq 1} \) is a two-sided maximal ideal of \( B \), and of course, semi-graded as left ideal. Let \( I \) be another two-sided maximal ideal of \( B \) semi-graded as left ideal; since \( I \) is proper, \( I \cap B_0 = I \cap K = 0 \); let \( x \in I \), then \( x = x_0 + x_1 + \cdots + x_n \), with \( x_i \in B_i, 1 \leq i \leq n \), but since \( I \) is semi-graded, \( x_i \in I \) for every \( i \), so \( x_0 = 0 \), and hence, \( x \in B_{\geq 1} \). Thus, \( I \subseteq B_{\geq 1} \) and \( I = B_{\geq 1} \). \( \square \)

### 2.2. Examples of FSG Algebras

In this subsection, we present a wide list of FSG algebras, many of them within the class of skew PBW extensions. For the explicit set of generators and relations for these algebras, see [6,12–15].

**Example 2** (Skew PBW extensions that are FSG algebras). Note that a skew PBW extension of the field \( K \) is an FSG algebra if and only if it is constant and pre-commutative. Thus, we have:

(i) By the classification presented in the tables of Remark 3, the following skew PBW extensions of the field \( K \) are FSG algebras: The classical polynomial algebra; the particular Sklyanin algebra; the universal enveloping algebra of a Lie algebra; the quantum algebra \( U'(so(3, K)) \); the dispin algebra; the Woronowicz algebra; the \( q \)-Heisenberg algebra; nine types 3-dimensional skew polynomial algebras; and six types of Sridharan enveloping algebra of 3-dimensional Lie algebras.

(ii) Many skew PBW extensions in the first table of Remark 3 are marked as non-constant; however, reconsidering the ring of coefficients, some of them can be also viewed as skew PBW extensions of the base field \( K \); this way, they are FSG algebras over \( K \): the algebra of shift operators; the algebra of discrete linear systems; the multiplicative analogue of the Weyl algebra; the algebra of linear partial shift operators; and the algebra of linear partial \( q \)-dilation operators.

(iii) In the class of skew quantum polynomials (see [12]), the multi-parameter quantum affine \( n \)-space is another example of a skew PBW extension of the field \( K \) that is an FSG (actually finitely graded) algebra. In particular, this is the case for the quantum plane.

(iv) The following skew PBW extensions of the field \( K \) are FSG but not finitely graded: the universal enveloping algebra of a Lie algebra; the quantum algebra \( U'(so(3, K)) \); the dispin algebra; the Woronowicz algebra; the \( q \)-Heisenberg algebra; eight of the nine types 3-dimensional skew polynomial algebras; and five of the six types of Sridharan enveloping algebra of 3-dimensional Lie algebras.

**Example 3** (FSG algebras that are not skew PBW extensions of \( K \)). The following algebras are FSG but not skew PBW extensions of the base field \( K \) (however, in every example below, the algebra is a skew PBW extension of some other subring):

(i) The Jordan plane \( A \) is the \( K \)-algebra generated by \( x, y \) with relation \( xy = xy + x^2 \), so \( A = K\{x, y\} / (yx - xy - x^2) \). \( A \) is not a skew PBW extension of \( K \), but of course, it is an FSG algebra over \( K \). Actually, it is a finitely graded algebra over \( K \) (observe that \( A \) can be viewed as a skew PBW extension of \( K[x] \), i.e., \( A = \sigma(K[x])(y) \)).

(ii) The \( K \)-algebra in Example 1.18 of [5] is not a skew PBW extension of \( K \):

\[
A = K\{x, y, z\} / (z^2 - xy - yx, zx - xz, zy - yz).
\]

However, \( A \) is an FSG algebra; actually, it is a finitely graded algebra over \( K \) (note that \( A \) can be viewed as a skew PBW extension of \( K[z] \): \( A = \sigma(K[z])(x, y) \)).
(iii) The following examples are similar to the previous ones: the homogenized enveloping algebra $\mathcal{A}(G)$; algebras of diffusion type; the Manin algebra, or more generally, the algebra $O_q(M_n(K))$ of quantum matrices; the complex algebra $V_q(sl_2(C))$; the algebra $U_{\mathfrak{p}}$: Witten’s deformation of $U(sl(2, K))$; the quantum symplectic space $O_q(\mathfrak{sp}(2n))$; and some quadratic algebras in 3 variables.

Example 4 (FSG algebras that are not skew PBW extensions). The following FSG algebras are not skew PBW extensions:

(i) Consider the Sklyanin algebra with $c \neq 0$ (see Remark 4), then $S$ is not a skew PBW extension, but clearly, it is an FSG algebra over $K$.

(ii) The finitely graded $K$-algebra in Example 1.17 of [5]:

$$B = K\{x, y\} / (yx^2 - x^2y, y^2x - xy^2).$$

(iii) Any monomial quadratic algebra:

$$B = K\{x_1, \ldots, x_n\} / \langle x_i x_j, (i, j) \in S \rangle,$$

with $S$ any finite set of pairs of indices ([26]).

(iv) $B = K\{w, x, y, u\} / \langle xy, xu - xu - uw \rangle$ ([27]).

(v) $B = K\{x, y\} / \langle x^2y, y^2x \rangle$ ([27]).

(vi) $B = K\{x, y\} / \langle x^2 - xy, yx, y^3 \rangle$ ([28]).

(vii) $B = K\{w, x, y, z\} / \langle z^2y^2, y^3x^2, x^2w, zy^3x \rangle$ ([28]).

(viii) $B = K\{x, y, z\} / \langle x^4, yx^3, x^3y \rangle$ ([28]).

(ix) $B = K\{x, y, z\} / \langle xz - zx, yz - zy, x^3z, y^4 + xz^3 \rangle$ ([28]).

(x) $B = K\{x, y, z, w, g\} / \langle y^2z, x^2y, y^2w^2, y^3z, zg - gx, yg - gy, wg - gw, zy - yg \rangle$ ([28]).

(xi) $B = K\{x, y\} / \langle x^2y - yx^2, yx^2 - y^2x \rangle$ ([28]).

(xii) $B = K\{x, y\} / \langle xy, xy^2, y^3 \rangle$ ([28]).

3. Some Invariants Associated to FSG Algebras

Now we study some invariants associated to finitely semi-graded algebras: the Hilbert series, the Yoneda algebra, and the Poincaré series. The topics that we consider here for FSG algebras extend to some well-known results on finitely graded algebras.

3.1. The Hilbert Series

In Definition 5, we presented the notion of generalized Hilbert series of an FSG ring. We shall prove next that if $B$ is an FSG algebra over a field $K$, then $\text{Gh}_B(t)$ is well-defined, i.e., it does not depend on the semi-graduation (compare with Remark 2). This theorem was proved recently by Bell and Zhang in [10] for connected graded algebras finitely generated in degree 1; we applied the Bell–Zhang result to our semi-graded algebras.

Theorem 3 ([10]). Let $A$ and $B$ be connected graded algebras finitely generated in degree 1. Then, $A \cong B$ as $K$-algebras if and only if $A \cong B$ as graded algebras.

Corollary 1 ([10]). Let $A$ be a connected graded algebra finitely generated in degree 1. If $A$ has two gradations $A = \bigoplus_{n \geq 0} A_n = \bigoplus_{n \geq 0} B_n$, then there exists an algebra automorphism $\phi: A \to A$ such that $\phi(A_n) = B_n$ for every $n \geq 0$. In particular, $\dim_K A_n = \dim_K B_n$ for every $n \geq 0$, and the Hilbert series of $A$ is well-defined. Moreover, if $\text{Aut}(A) = \text{Aut}_{gr}(A)$, then $A_n = B_n$ for every $n \geq 0$.

We shall prove that the generalized Hilbert series of FSG algebras is well-defined.

Proposition 5. If $B$ is an FSG algebra, then $Gr(B)$ is a connected graded algebra finitely generated in degree 1.

Proof. This is a direct consequence of part (iii) of Proposition 1.  □
Theorem 4. Let B and C be FSG algebras over the field K. If \( \phi : B \to C \) is a homogeneous isomorphism of K-algebras, then \( \text{Gr}(B) \cong \text{Gr}(C) \) as graded algebras.

Proof. From the previous proposition, we know that \( \text{Gr}(B) \) and \( \text{Gr}(C) \) are connected graded algebras finitely generated in degree 1; according to Theorem 3, we only have to show that \( \text{Gr}(B) \) and \( \text{Gr}(C) \) are isomorphic as K-algebras. For every \( n \geq 0 \), we have the homomorphism of K-vector spaces \( \varphi_n : \text{Gr}(B)_n \to \text{Gr}(C)_n \), \( b_n \mapsto c_n \) with \( \phi(b_n) := c_n \) (observe that \( \text{Gr}(B)_n \cong B_n \) and \( \text{Gr}(C)_n \cong C_n \) as K-vector spaces); from this, we obtain a homomorphism of K-vector spaces \( \varphi : \text{Gr}(B) \to \text{Gr}(C) \) such that \( \varphi \circ \mu_n = \varphi_n \), for every \( n \geq 0 \), where \( \mu_n : \text{Gr}(B)_n \to \text{Gr}(B) \) is the canonical injection. Considering \( \varphi := \phi^{-1} \), we get a homomorphism of K-vector spaces \( \varphi : \text{Gr}(C) \to \text{Gr}(B) \) such that \( \varphi \circ \nu_n = \varphi_n \), for every \( n \geq 0 \), where \( \nu_n : \text{Gr}(C)_n \to \text{Gr}(C) \) is the canonical injection. However, observe that \( \varphi \circ \varphi = i_{\text{Gr}(C)}, \varphi \circ \varphi = i_{\text{Gr}(B)} \). In fact, \( \varphi \varphi(b_n) = \varphi \varphi(\mu_n(b_n)) = \varphi \varphi_n(b_n) = \varphi(c_n) = \varphi\nu_n(c_n) = \varphi_n(c_n) \equiv \varphi^{-1}(c_n) = b_n \). In a similar way, we can prove the first identity. It is obvious that \( \varphi \) is multiplicative. \( \square \)

Corollary 2. Let B be an FSG algebra. If B has two semi-graduations \( A = \bigoplus_{n \geq 0} B_n = \bigoplus_{n \geq 0} C_n \), then \( \dim_K B_n = \dim_K C_n \) for every \( n \geq 0 \), and the generalized Hilbert series of B is well-defined. Moreover, \( \text{Gh}_B(t) = h_{\text{Gr}(B)}(t) \).

Proof. We consider the identical isomorphism \( i_B : B \to B \): By Theorem 4, there exists an isomorphism of graded algebras \( \phi : \text{Gr}_1(B) \to \text{Gr}_2(B) \), where \( \text{Gr}_1(B) \) is the graded algebra associated to the semi-gradation \( \{B_n\}_{n \geq 0} \) and \( \text{Gr}_2(B) \) is the graded algebra associated to \( \{C_n\}_{n \geq 0} \); from the proof of Corollary 1, we know that \( \dim_K(\text{Gr}_1(B)_n) = \dim_K(\text{Gr}_2(B)_n) \) for every \( n \geq 0 \), but from the part (iii) of Proposition 1, \( \text{Gr}_1(B)_n \cong B_n \) and \( \text{Gr}_2(B)_n \cong C_n \); moreover, these isomorphisms are K-linear, so \( \dim_K B_n = \dim_K C_n \) for every \( n \geq 0 \). \( \square \)

Corollary 3. Each of the algebras presented in Examples 2–4 have generalized Hilbert series well-defined. In addition, let \( A = \sigma(K)(x_1, \ldots, x_n) \) be a skew PBW extension of the field K; if A is an FSG algebra, then the generalized Hilbert series is well-defined and given by:

\[
\text{Gh}_A(t) = \frac{1}{(1-t)^n}.
\]

Proof. Direct consequence of the previous corollary and Theorem 2. \( \square \)

Example 5. In this example, we show that the condition (iv) in Definition 10 is necessary in order for the generalized Hilbert series of FSG algebras to be well-defined. Let \( \mathcal{L} \) be the 3-dimensional (Heisenberg) Lie algebra that has a K-basis \( \{x, y, z\} \) with Lie bracket:

\[
[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0.
\]

The universal enveloping algebra \( \mathcal{U}(\mathcal{L}) \) is connected graded with \( \deg x = \deg y = 1, \deg z = 2 \). With this grading, the homogeneous component of degree 1 of \( \mathcal{U}(\mathcal{L}) \) is \( Kx +Ky \). Thus, \( \mathcal{U}(\mathcal{L}) \) is not generated in degree 1, i.e., with this grading, \( \mathcal{U}(\mathcal{L}) \) can not be viewed as an FSG algebra. In this case, the generalized Hilbert series is:

\[
\frac{1}{(1-t)^2(1-t^2)}.
\]

On the other hand, \( \mathcal{U}(\mathcal{L}) \) is FSG by setting \( \deg x = \deg y = \deg z = 1 \). According to Corollary 3, in this case, the generalized Hilbert series is:

\[
\frac{1}{(1-t)^3}.
\]
3.2. The Yoneda Algebra

The collection $\text{SGR} - B$ of semi-graded modules over $B$ is an abelian category, where the morphisms are the homogeneous $B$-homomorphisms; $K$ is an object of this category with the trivial semi-graduation given by $K_0 := K$ and $K_n := 0$ for $n \neq 0$. We can associate to $B$ the Yoneda algebra defined by:

$$E(B) := \bigoplus_{i \geq 0} \text{Ext}^i_B(K,K); \quad (11)$$

recall that in any abelian category, the $\text{Ext}^i_B(K,K)$ groups can be computed either by projective resolutions of $K$ or by extensions of $K$. Here, we take into account both equivalent interpretations; the first one is used in the proof of Theorem 5. For the second interpretation (see [29]), the groups $\text{Ext}^i_B(K,K)$ are defined by equivalence classes of exact sequences of finite length with semi-graded $B$-modules and homogeneous $B$-homomorphisms from $K$ to $K$:

$$\xi : 0 \rightarrow K \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow 0;$$

the addition in $\text{Ext}^i_B(K,K)$ is the Baer sum (see [29], Section 3.4):

$$\chi \oplus [\xi] : 0 \rightarrow K \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow 0,$$

$$[\xi] \oplus [\chi] : 0 \rightarrow K \rightarrow Y \rightarrow X_{i-1} \oplus X_{j-1} \rightarrow \cdots \rightarrow X_2 \oplus X_2 \rightarrow Y_1 \rightarrow K \rightarrow 0,$$

where $Y_1$ is the pullback of homomorphisms $X_1 \rightarrow K$ and $X_2' \rightarrow K$, and $Y_1$ is the pushout of $K \rightarrow X_i$ and $K \rightarrow X_j'$. The zero element of $\text{Ext}^i_B(K,K)$ is the class of any split sequence $\xi$.

The product in $E(B)$ is given by concatenation of sequences:

$$\text{Ext}^i_B(K,K) \times \text{Ext}^j_B(K,K) \rightarrow \text{Ext}^{i+j}_B(K,K)$$

$$([\chi], [\xi]) \mapsto [\chi][\xi] := [\chi \xi],$$

where:

$$\xi : 0 \rightarrow K \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow 0,$$

$$\chi : 0 \rightarrow K \rightarrow X_1' \rightarrow \cdots \rightarrow X_n' \rightarrow K \rightarrow 0,$$

$$\chi \xi : 0 \rightarrow K \rightarrow X_1' \rightarrow \cdots \rightarrow X_n' \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow 0.$$

Note that the unit of $E(B)$ is the equivalence class of $0 \rightarrow K \xrightarrow{i} K \rightarrow 0$.

Thus, $E(B) = \bigoplus_{i \geq 0} E^i(B)$ is a connected $\mathbb{N}$-graded algebra, where $E^i(B) := \text{Ext}^i_B(K,K)$ is a $K$-vector space. Observe that Definition (11) extends the usual notion of Yoneda algebra of graded algebras.

3.3. The Poincaré Series

Another invariant that we want to consider is the Poincaré series; let $B$ be an FSG algebra; as we observed above, $E(B)$ is connected and graded; if $E(B)$ is finitely generated, then $E(B)$ is locally finite, and hence, the Poincaré series of $B$ is defined as the Hilbert series of $E(B)$, i.e.:

$$P_B(t) := \sum_{n=0}^{\infty} (\dim_K \text{Ext}^n_B(K,K)) t^n. \quad (12)$$

By Corollary 1, $P_B(t)$ is well-defined if $E(B)$ is generated in degree 1. In our next theorem, we show that in this case, $P_B(t)$ can also be defined by the $\text{Tor}$ vector spaces (compare with [30,31]).
Theorem 5. Let $B$ be an FSG algebra such that $K$ has a $B$-free homogeneous resolution and $E(B)$ is finitely generated. Then, for every $n \geq 0$:

$$\text{Tor}_n^B(K, K) \cong \text{Ext}_n^B(K, K).$$

Proof. Consider the $B$-free homogeneous resolution of $K$:

$$\cdots \to B(X_0) \xrightarrow{\beta_0} B(X_{n-1}) \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_2} B(X_1) \xrightarrow{\beta_1} B(X_0) \xrightarrow{\beta_0} K \to 0,$$

the $\text{Ext}_n^B(K, K)$ and the $\text{Tor}_n^B(K, K)$ spaces can be computed applying $\text{Hom}_B(-, K)$ and $K \otimes_B -$, respectively:

$$0 \to \text{Hom}_B(K, K) \xrightarrow{\beta_0^*} \cdots \xrightarrow{\beta_{n-1}^*} \text{Hom}_B(B(X_{n-1}), K) \xrightarrow{\beta_n^*} \text{Hom}_B(B(X_n), K) \to \cdots$$

$$\cdots \to K \otimes_B B(X_n) \xrightarrow{i_0 \otimes \beta_0} K \otimes_B B(X_{n-1}) \xrightarrow{i_0 \otimes \beta_{n-1}} \cdots \xrightarrow{i_0 \otimes \beta_1} K \otimes_B B(X_0) \xrightarrow{i_0 \otimes \beta_0} K \to 0;$$

it is easy to show that $K \cong \text{Hom}_B(K, K)$ as $B$-bimodules ($k \mapsto a_k$, with $a_k(x) := xk$, for $x, k \in K$), whence we have:

$$\text{Hom}_B(B(X_n), K) \cong \text{Hom}_B(B(X_n), \text{Hom}_B(K, K)) \cong \text{Hom}_B(K \otimes_B B(X_n), K),$$

but $B \geq 1(K \otimes_B B(X_1)) = 0$, then from the previous isomorphism and considering that $\text{Tor}_n^B(K, K)^* = \text{Hom}_K(\text{Tor}_n^B(K, K), K)$, we get:

$$\text{Tor}_n^B(K, K) \cong \text{Ext}_n^B(K, K),$$

but since $E(B)$ is finitely generated, then $\dim_K \text{Ext}_n^B(K, K) < \infty$, and from this:

$$\text{Tor}_n^B(K, K) \cong \text{Tor}_n^B(K, K)^* \cong \text{Ext}_n^B(K, K).$$

\[\square\]

Corollary 4. Let $B$ be an FSG algebra such that $K$ has a $B$-free homogeneous resolution and $E(B)$ is finitely generated in degree 1, then $P_B(t)$ is well-defined, and it is also given by:

$$P_B(t) = \sum_{n=0}^{\infty} (\dim_K \text{Tor}_n^B(K, K)) t^n.$$

(13)

Proof. This follows from (12) and the previous theorem. \[\square\]

4. Koszulity

Koszul algebras were defined by Priddy in [9]. Later in 2001, Berger in [32] introduced a generalization of Koszul algebras which are called generalized Koszul algebras or $N$-Koszul algebras. The 2-Koszul algebras of Berger are the Koszul algebras of Priddy (for the definition of Koszul algebras adopted in this paper, see Remark 6). $N$-Koszul algebras are finitely graded, where all generators of the ideal $I$ of relations are homogeneous and have the same degree $N \geq 2$. In 2008, Cassidy and Shelton ([28]) generalized the $N$-Koszul algebras, introducing the $K_2$ algebras; these type of algebras accept that the generators of $I$ have different degrees, but again, all generators are homogeneous since the $K_2$ algebras are graded. Later, Phan in [25] extended this notion to $K_m$ algebras for any $m \geq 1$.

In this section, we study the semi-graded version of Koszulity, and for this purpose, we follow the lattice interpretation of this notion (see [26,30–33]).
4.1. Semi-Graded Koszul Algebras

Recall that a lattice is a collection $L$ endowed with two idempotent commutative and associative binary operations $\land, \lor : L \times L \rightarrow L$ satisfying the following absorption identities: $a \land (a \lor b) = a$, $(a \land b) \lor b = b$. A sublattice of a lattice $L$ is a non-empty subset of $L$ closed under $\land$ and $\lor$. A lattice is called distributive if it satisfies the following distributivity identity: $a \land (b \lor c) = (a \land b) \lor (a \land c)$. If $X \subseteq L$, the sublattice generated by $X$, denoted $[X]$, consists of all elements of $L$ that can be obtained from the elements of $X$ by the operations $\land$ and $\lor$. We say that $X$ is distributive if $[X]$ is a distributive lattice. The (direct) product of the family of lattices $\{L_\omega\}_{\omega \in \Omega}$ is defined as follows:

$$\prod_{\Omega} L_\omega := (\prod_{\Omega} L_\omega, \land, \lor),$$

which is the cartesian product with $\land$ and $\lor$ operating component-wise. A semidirect product of the family $\{L_\omega\}_{\omega \in \Omega}$ is a sublattice $L$ of $\prod_{\Omega} L_\omega$ such that for every $\omega_0 \in \Omega$, the composition:

$$L \to \prod_{\Omega} L_\omega \to L_{\omega_0}$$

is surjective.

**Proposition 6** ([30]). If $L$ is a semidirect product of the family $\{L_\omega\}_{\omega \in \Omega}$, then $L$ is distributive if and only if for all $\omega \in \Omega$, $L_\omega$ is distributive.

Let $K$ be a field and $V$ be a $K$-vector space; the set $L(V)$ of all its linear subspaces is a lattice with respect to the operations of sum and intersection.

**Proposition 7** ([26]). Let $V$ be a vector space and $X_1, \ldots, X_n \subseteq V$ be a finite collection of subspaces of $V$. The following conditions are equivalent:

(i) The collection $X_1, \ldots, X_n$ is distributive.

(ii) There exists a basis $B := \{\omega_i\}_{i \in C}$ of $V$ such that each of the subspaces $X_i$ is the linear span of a set of vectors $\omega_i$.

(iii) There exists a basis $B$ of $V$ such that $B \cap X_i$ is a basis of $X_i$ for every $1 \leq i \leq n$.

With the previous elementary facts about lattices, we have the following notions associated to any FSG algebra presented as in (10) (compare with [30]).

**Definition 11.** Let $B = K\{x_1, \ldots, x_n\}/I$ be an FSG algebra. The lattice associated to $B$ is the sublattice $L(B)$ of subspaces of the free algebra $F := K\{x_1, \ldots, x_n\}$ generated by $\{F_{\geq 1}F_{\geq 1}^{3} | s, g, h \geq 0\}$. For any integer $j \geq 2$, the $j$-th lattice associated to $B$ is defined by:

$$L_j(B) := \{F_j, F_jF_h | s, h \geq 0, g \geq 2, s + g + h = j\} \subset \text{subspaces of } F_j \cap, +,$$

where $F_jF_h$ is the subspace of $F_j$ consisting of finite sums of elements of the form $abc$, with $a \in F_s, b \in I_g, c \in F_h$, and:

$$I_g := \{a_g \in F_g | a_g \text{ is the } g\text{-th component of some element in } I\}.$$}

For any two-sided ideal $H$ of $F$, the $K$-subspace $H_g$ is defined similarly. From now on, in this section, we denote $F := K\{x_1, \ldots, x_n\}$.

**Theorem 6.** Let $B = K\{x_1, \ldots, x_n\}/I$ be an FSG algebra with $I = \langle b_1, \ldots, b_m \rangle$ such that $b_i \in F_{\geq 1}$ for $1 \leq i \leq m$. Then, $L(B)$ is a semidirect product of the family of lattices:

$$\{L_j(B) \cup \{0, F_j\}_{j \geq 2}\} \cup \{\{0, K\}, \{0, F_1\}\}.$$
In particular, $L(B)$ is distributive if and only if for all $j \geq 2$, $L_j(B)$ is distributive.

**Proof.** The proof of Lemma 2.4 in [30] can be easily adapted.

Step 1. For any $j \geq 2$ and any $X \subseteq L_j(B)$, we have $0 \subseteq X \subseteq F_j$. Thus, $L_j(B) \cup \{0, F_j\}$ is, in fact, a lattice.

Step 2. If $s \geq 0$, $g \geq 1$, $h \geq 0$ and $j \geq 2 + s + h$, then:

$$\left(F^s_{\geq 1}F^h_{\geq 1}\right)_{\cdot} = F_s \left(F^h_{\cdot}\right)_{\cdot}.$$

We only have to prove that $(F^s_{\geq 1}F^h_{\geq 1})_{\cdot} \subseteq F_s \left(F^h_{\cdot}\right)_{\cdot}$ since the other containment is trivial. Recall that one element of $(F^s_{\geq 1}F^h_{\geq 1})_{\cdot}$ is the $j$-th component of some element of $F^s_{\geq 1}F^h_{\geq 1}$, let $z_j \in (F^s_{\geq 1}F^h_{\geq 1})_{\cdot}$, then there exists $y \in F^s_{\geq 1}F^h_{\geq 1}$ such that $z_j$ is the $j$-th component of $y$; the element $y$ is a finite sum of elements of the form $a_{bc}$, with $a \in F^s_{\geq 1} = F_{s+1}$, $b \in F^h_{\geq 1} = F_{h+1}$, so the $j$-th component of $y$ is a sum of the $j$-th components of elements of the form $a_{bc}$, with $k \geq s$, $b \in F^h_{\cdot}$, $j \geq h$, but since $F_k = F_k F_{-s}$ for $k \geq s$ and $F_1 = F_{-h} F_1$ for $t \geq h$, then the $j$-th component of $a_{bc}$ is the $j$-th component of $a_{bc} = a_{bc1} a_{bc2}$, i.e., it is an element of $F_s \left(F^h_{\cdot}\right)_{\cdot}$.

Step 3. For $g \geq 1$ and $j \geq 2$:

$$(F^g_{\cdot})_{\cdot} = \sum K_{k_0} I_{k_1} I_{k_2} \cdots I_{k_{g-1}} I_{k_g},$$

where the sum is taken over all relevant $k_0, \ldots, k_g, l_1, \ldots, l_g$ such that $\sum m k_m + \sum n l_n = j$. Indeed, if $p \in F^g_{\cdot}$, then $p$ is a finite sum of elements of the form $a^{(k)} p_1 a^{(1)} p_2 \cdots a^{(g-1)} p_g a^{(g)}$, with $a^{(k)} \in F$, $p_i \in \{b_1, \ldots, b_m\}$, $0 \leq r \leq s, 1 \leq i \leq g$.

Step 4. For any $g \geq 2$ and any $2g + 1$ non-negative integers $k_0, \ldots, k_g, l_1, \ldots, l_g$, we have:

$$F_{k_0} I_{k_1} I_{k_2} \cdots I_{k_{g-1}} I_{k_g} = \bigcap_{a=1}^g F_{k_0 + l_1 + \cdots + k_{a-1}} I_{k_a} F_{k_{a+1} + \cdots + k_g}.$$

In fact, let $g = a_0 p_1 a_1 \cdots p_g a_g \in F_{k_0} I_{k_1} I_{k_2} \cdots I_{k_{g-1}} I_{k_g}$, with $a_r \in F_{k_r}, p_i \in I_{k_i}$, $0 \leq r \leq g, 1 \leq i \leq g$, then $g \in F_{k_0 + l_1 + \cdots + k_{a-1}} I_{k_a} F_{k_{a+1} + \cdots + k_g}$ for every $1 \leq a \leq g$; the converse follows from the fact that for any $a \in F \setminus \{0\}$ homogeneous with $a = bc = de$, then $b, c, d, e$ are homogeneous; in addition, if $b \in F, d \in F_1$ with $t \geq s$, then there is $f$ such that $a = b f e, d = b f$ and $c = f e$.

Step 5. For any $s \geq 0, g \geq 1, h \geq 0$ and $j < s + h$, we have $(F^s_{\geq 1}F^h_{\geq 1})_j = 0$ since $b_i \in F_{g+1}$ for $1 \leq i \leq m$; likewise, for $j < g$, $(F^g_{\cdot})_j = 0$.

From these steps, $L(B)$ is a sublattice of the product of the given family, i.e.:

$$L(B) \hookrightarrow \{0, K\} \times \{0, F_1\} \times \left( \prod_{j \geq 2} L_j(B) \cup \{0, F_j\} \right).$$

Finally, fix $j \geq 2$, then $L(B) \rightarrow L_j(B) \cup \{0, F_j\}$ is a lattice surjective map since: (a) $(F^g_{\cdot})_j = 0$ if $j < g$; (b) $(F^s_{\geq 1})_j = F_j$ if $j \geq s$; (c) if $s, h \geq 0, g \geq 2$ and $s + g + h = j$, then $F_j I_{s} F_{h} = (F^s_{\geq 1}F^h_{\geq 1})_{\cdot}$. The cases $j = 0, 1$ can be proved by the same method. Thus, $L(B)$ is a semidirect product of the given family.

**Definition 12.** Let $B = \mathbb{K}\{x_1, \ldots, x_n\} / I$ be an FSG algebra. We say that $B$ is semi-graded Koszul, denoted $SK$, if $B$ satisfies the following conditions:

(i) $B$ is finitely presented with $I = \langle b_1, \ldots, b_m \rangle$ and $b_i \in F_{g+1}$ for $1 \leq i \leq m$.

(ii) $L(B)$ is distributive.

**Remark 6.** (i) In the present paper, we adopt the following definition of Koszul algebras (see [26, 30–33]). Let $B$ be a $K$-algebra; it is said that $B$ is Koszul if $B$ satisfies the following conditions: (a) $B$ is $\mathbb{N}$-graded, connected, finitely generated in degree one; (b) $B$ is quadratic, i.e., the ideal $I$ in (10) is finitely generated by homogeneous elements of degree 2; (c) $L(B)$ is distributive.
Theorem 7. If $A$ is a skew PBW extension of a field $K$ with presentation $A = K\{x_1, \ldots, x_n\} / I$, where:

$$I = \langle x_i x_j - c_{ij} x_j x_i - a_{ij}^{(k_{ij})} x_{k_{ij}} | c_{ij}, a_{ij}^{(k_{ij})} \in K, c_{ij} \neq 0, 1 \leq i < j \leq n \rangle,$$

then $A$ is Koszul.

Proof. Note that $A$ is an FSG algebra. Let $F := K\{x_1, \ldots, x_n\}$, $N := \{x_1, \ldots, x_n\}$, and $J := \{k_{ij} \in \{1, \ldots, n\} | a_{ij}^{(k_{ij})} \neq 0, 1 \leq i < j \leq n \}$. We are going to show that $L_m(A)$ is distributive lattice for $m \geq 2$.

If $|J| = n$, we define:

$$B_m := \left( \bigcup_{r=1}^{m} D_r^{(m)} \right),$$

where:

$$D_r^{(m)} := \{ a_1 \cdots a_{r-1} x_i a_{r+1} \cdots a_m | a_t \in N_t, t = 1, \ldots, r-1, r+1, \ldots, n; 1 \leq i \leq n \};$$

$B_m$ is a basis of $F_m$. Now, consider $F_s I_g F_h \leq F_m$ with $s, h \geq 0, g \geq 2$ and $s + g + h = m$. Since $F_s I_g F_h$ is generated by $D_s^{(m+1)}, \ldots, D_s^{(m)}$, then $F_s I_g F_h \cap B_m = \bigcup_{r=s+1}^{m} D_r^{(m)}$, which is a basis of $F_s I_g F_h$.

If $|J| = n - 2$, define:

$$B_m := \left( \bigcup_{r=1}^{m} D_r^{(m)} \right) \cup \{ x_i^m \},$$

where $l \notin J$, and:

$$D_r^{(m)} := \{ a_1 \cdots a_{r-1} x_i a_{r+1} \cdots a_m | a_t \in N_t, t = 1, \ldots, r-1, r+1, \ldots, n; i \in J \};$$

again, $B_m$ is a basis of $F_m$. As before, consider $F_s I_g F_h \leq F_m$ with $s, h \geq 0, g \geq 2$ and $s + g + h = m$; since $F_s I_g F_h$ is generated by $D_s^{(m+1)}, \ldots, D_s^{(m)}$, then $F_s I_g F_h \cap B_m = \bigcup_{r=s+1}^{m} D_r^{(m)}$, which is the basis of $F_s I_g F_h$.

If $|J| \leq n - 2$, we define:

$$B_m := \left( \bigcup_{r=1}^{m-1} B_r^{(m)} \right) \cup \left( \bigcup_{r=1}^{m-1} C_r^{(m)} \right) \cup \left( \bigcup_{r=1}^{m} D_r^{(m)} \right) \cup E,$$
where:

\[ B_r^{(m)} := \{ a_1 \cdots a_{r-1} x_i a_{r+2} \cdots a_m | a_i \in \mathbb{N}; t = 1, 2, \ldots, r-1, r+2, \ldots, m; i, j \notin \{ j \}; i < j \}, \]

\[ C_r^{(m)} := \{ a_1 \cdots a_{r-1} (x_j x_i - c_j x_i) a_{r+2} \cdots a_m | a_i \in \mathbb{N}; t = 1, 2, \ldots, r-1, r+2, \ldots, m; i, j \notin \{ j \}; i < j \}, \]

\[ D_r^{(m)} := \{ a_1 \cdots a_{r-1} x_i a_{r+1} \cdots a_m | a_i \in \mathbb{N}, t = 1, \ldots, r, 1, \ldots, n; l \in \}, \]

\[ E = \{ x_i^m | i \notin \{ j \} \}. \]

\( B_m \) is a basis of \( A_m \); consider \( F_s I_g F_h \leq F_m \) with \( s, h \geq 0, g \geq 2 \) and \( s + g + h = m \); since \( F_s I_g F_h \) is generated by \( C_s^{(m)} \), \( D_s^{(m)} \), \( D_s^{(m)} \), then \( F_s I_g F_h \cap B_m = \bigcup_{r=s+1}^{s+g} C_r^{(m)} \cup \bigcup_{r=s+1}^{s+g} D_r \), which is the basis of \( F_s I_g F_h \).

**Example 6.** (i) The following algebras satisfy the conditions of the previous theorem, and hence, they are SK (but not Koszul): the dispin algebra \( U(\mathfrak{osp}(1, 2)) \); the \( q \)-Heisenberg algebra; the quantum algebra \( U_q(\mathfrak{sl}(3, \mathbb{K})) \); the Woronowicz algebra \( W_i(\mathfrak{sl}(2, \mathbb{K})) \); the algebra \( S_h \) of shift operators; the algebra \( D \) for multidimensional discrete linear systems; and the algebra of linear partial shift operators.

(ii) The following algebras do not satisfy the conditions of the previous theorem, but by direct computation, we proved that the lattice \( L(B) \) is distributive, so they are SK (but not Koszul): The algebra \( V_q(\mathfrak{sl}(3, \mathbb{C})) \); the Witten’s deformation of \( U(\mathfrak{sl}(2, \mathbb{K})) \); and the quantum symplectic space \( O_q(\mathfrak{sp}(2^n)) \).

**Example 7.** Consider the algebra \( A = K\{x, y\}/(x^2 - xy, yx, y^3) \) (see ([28])), which is not a skew PBW extension but is an FSG algebra. This algebra satisfies that \( L(A) \) is a subdirect product of the family of lattices:

\[ \{ L_i(A) \cup \{ 0, A_i \} \}_{i \geq 2} \cup \{ \{ 0, K \} \cup \{ 0, A_1 \} \}, \]

but \( L_3(A) \) is not distributive. In fact, note that the lattice \( L_3(A) \) is generated by \( A_1 I_2, I_2 A_1, I_3 \), and:

1. \( A_1 I_2 \) is \( K \)-generated by \( D = \{ x^3 - xy x, x^2 y - yx, y x y, yx y \} \), and \( D \) is \( K \)-linearly independent; therefore, \( \dim_K(A_1 I_2) = 4 \).
2. \( I_2 A_1 \) is \( K \)-generated by \( C = \{ x^3 - x^2 y, y x^2 - yx, yx x, y x^2 \} \), which is \( K \)-linearly independent; therefore, \( \dim_K(I_2 A_1) = 4 \).

Now, let us suppose \( B = \{ a_1, a_2, \ldots, a_8 \} \) is a \( K \)-basis of \( A_3 \) such that \( X := B \cap A_1 I_2 \) is the basis of \( A_1 I_2 \) and \( Y := B \cap I_2 A_1 \) is the basis of \( I_2 A_1 \).

Without loss of generality, suppose that \( X = \{ a_1, \ldots, a_4 \} \), then \( yx y = \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 \) and \( yx y = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 + \beta_4 a_4 \) with \( \lambda_i, \beta_i \in K \) for \( 1 \leq i \leq 4 \), \( \lambda_1 \neq \beta_1 \)(maybe organizing), \( \lambda_1 \neq 0 \) and \( \lambda_j \neq \beta_j \), for some \( j = 2, 3, 4 \), otherwise, if \( \lambda_j = \beta_j \), for \( j = 2, 3, 4 \), then \( yx y - \frac{\lambda_j}{\beta_j} y x^2 = 0 \), which is impossible. Thus:

\[ y x^2 - y x y = (\lambda_1 - \beta_1) a_1 + (\lambda_2 - \beta_2) a_2 + (\lambda_3 - \beta_3) a_3 + (\lambda_4 - \beta_4) a_4, \]

with at least \( a_1, a_j \in X \cap Y \), and consequently:

\[ a_1 = a_1(x^3 - xy x) + a_2(x^2 y - y x^2) + a_3(y x^2) + a_4(y x y), \]

\[ = \gamma_1(x^2 - x^2 y) + \gamma_2(y x^2 - y x y) + \gamma_3(x y x) + \gamma_4(y^2 x), \]

\[ a_j = \eta_1(x^3 - xy x) + \eta_2(x^2 y - y x^2) + \eta_3(y x^2) + \eta_4(y x y), \]

\[ = \mu_1(x^2 - x^2 y) + \mu_2(y x^2 - y x y) + \mu_3(x y x) + \mu_4(y^2 x), \]

with \( \lambda_i, \gamma_i, \eta_i, \mu_i \in K \) for \( 1 \leq i \leq 4 \). Therefore, there exist two different \( K \)-combinations non-trivial of \( C \cup D \) equal to 0, and hence, the base \( B \) does not exist. Thus, \( A \) is an FSG algebra but is not SK.
### 4.2. Poincaré Series of Skew PBW Extensions

Now we compute the Poincaré series of some skew PBW extensions of $K$.

**Theorem 8.** Let $A = \sigma(K)[x_1, \ldots, x_n]$ be a skew PBW extension of the field $K$ that is a Koszul algebra, then the Poincaré series of $A$ is well-defined and given by $P_A(t) = (1 + t)^n$.

**Proof.** Since $A$ is Koszul, then $h_A(t)P_A(-t) = 1$ and $E(A)$ is Koszul, whence $E(A)$ is finitely generated in degree 1 (see [31,32], or [26]); therefore, the theorem follows from Corollaries 3 and 4.

**Example 8.** From Remark 6 and Theorem 8, we present in Table 4 the Poincaré series of some skew PBW extensions of the base field $K$:

| SK Algebra | $P_A(t)$ |
|------------|----------|
| Classical polynomial algebra $K[x_1, \ldots, x_n]$ | $(1 + t)^n$ |
| Some Sridharan enveloping algebras of 3-dimensional Lie algebras | $(1 + t)^3$ |
| Particular Sklyanin algebra | $(1 + t)^3$ |
| L. Partial $q$-dilation operators $K[t_1, \ldots, t_n][H_1^{(q)}, \ldots, H_m^{(q)}]$ | $(1 + t)^{n+m}$ |
| Multiplicative analogue of the Weyl algebra $O_n(\lambda_{ji})$ | $(1 + t)^n$ |
| Some 3-dimensional skew polynomial algebras | $(1 + t)^3$ |
| Multi-parameter quantum affine n-space | $(1 + t)^n$ |

### 5. Point Modules and the Point Functor for FSG Rings

We conclude the paper with another important topic studied in non-commutative algebraic geometry for graded algebras, the point modules and its parametrization by the point functor. Thus, in this last section, we introduce and study the collection of point modules for FSG rings. A standard Zariski topology is defined for them as well as the point functor.

**Definition 13.** Let $B = \bigoplus_{n \geq 0} B_n$ be an FSG ring that is generated in degree 1.

(i) A point module for $B$ is a finitely $\mathbb{N}$-semi-graded $B$-module $M = \bigoplus_{n \in \mathbb{N}} M_n$ such that $M$ is cyclic, generated in degree 0, i.e., there exists an element $m_0 \in M_0$ such that $M = Bm_0$, and $\dim B_m(M_n) = 1$ for all $n \geq 0$.

(ii) Two point modules $M$ and $M'$ for $B$ are isomorphic if there exists a homogeneous $B$-isomorphism between them.

(iii) $P(B)$ is the collection of isomorphism classes of point modules for $B$.

The following result is the first step in the construction of the geometric structure for $P(B)$.

**Theorem 9.** Let $B = \bigoplus_{n \geq 0} B_n$ be an FSG ring generated in degree 1. Then, $P(B)$ has a Zariski topology generated by finite unions of sets $V(J)$ defined by:

$$V(J) := \{ M \in P(B) \mid \text{Ann}(M) \supseteq J \},$$

with $J$ a semi-graded left ideal of $B$.

**Proof.** Taking $J = B$, we get from Definition 13 that $V(B) = \emptyset$; for $J = 0$, we have $V(0) = P(B)$. Let $\{J_i\}_{i \in \mathcal{C}}$ be a family of semi-graded as left ideals of $B$, then from (iv) of Proposition 1, $\bigcap_{i \in \mathcal{C}} J_i$ is a semi-graded as left ideal and we have:

$$\bigcap_{i \in \mathcal{C}} V(J_i) = V(\bigcap_{i \in \mathcal{C}} J_i).$$

\[\square\]
Definition 14. Let $B = \bigoplus_{n \geq 0} B_n$ be an FSG ring generated in degree 1 such that $B_0$ is commutative and $B$ is a $B_0$-algebra. Let $S$ be a commutative $B_0$-algebra. An $S$-point module for $B$ is an $\mathbb{N}$-semi-graded $S \otimes B_0$ $B$-module $M$ which is cyclic, generated in degree 0, $M_n$ is a locally free $S$-module with $\text{rank}_S(M_n) = 1$ for all $n \geq 0$, and $M_0 = S$. $P(B; S)$ denotes the set of $S$-point modules for $B$.

Remark 7. (i) Note that $S \otimes B_0$ $B$ is an FSG ring generated in degree 1 and with $S$ in degree 0:

$$S \otimes B_0 = S \otimes B_0 (\sum_{n \geq 0} \otimes B_n) = \sum_{n \geq 0} (S \otimes B_0 B_n),$$

so $(S \otimes B_0 B_n)_{n+m} = \sum_{n \geq 0} (S \otimes B_0 B_n)_{n+m}$.

(i) Taking $S = B_0$, we get that $P(B) \subseteq P(B; B_0)$. If $B_0 = K$ is a field, then clearly $P(B) = P(B; K)$.

Theorem 10. Let $B = \bigoplus_{n \geq 0} B_n$ be an FSG ring generated in degree 1 such that $B_0$ is commutative and $B$ is a $B_0$-algebra. Let $B_0$ be the category of commutative $B_0$-algebras and let $\text{Set}$ be the category of sets. Then, $P$, defined by:

$$B_0 \xrightarrow{P} \text{Set}$$

$$S \mapsto P(B; S) \mapsto P(B; T),$$

given by $M \mapsto T \otimes S M$, is a covariant functor called the point functor for $B$.

Proof. (i) Firstly note that if $M$ is an $S$-point module, then $T \otimes S M$ is a $T$-point module:

(a) $M$ is an $S$-module because of homomorphism $S \mapsto T \otimes B_0 B$, $s \mapsto s \otimes 1$, in addition, by the hypothesis, every homogeneous component $M_n$ is an $S$-module; $M$ is a left $B$-module because of homomorphism $B \mapsto T \otimes B_0 B$, $b \mapsto 1 \otimes b$; $T \otimes S M$ is a $T \otimes B_0 B$-module with a product given by

$$(t \otimes b) \cdot (t' \otimes m) := t'^t \otimes b \cdot m = t'^t \otimes (1 \otimes b) \cdot m;$$

in a similar way, $T \otimes S M$ is a $T \otimes S (S \otimes B_0 B)$-module with product $t \otimes (s \otimes b) \cdot (t' \otimes m) := t'^t \otimes (s \otimes b) \cdot m$.

(b) Since $M$ is $S \otimes B_0 B$-cyclic, generated in degree zero, there exists $m_0 \in M_0$ such that $M = (S \otimes B_0 B) \cdot m_0$, whence:

$$T \otimes S M = T \otimes S [(S \otimes B_0 B) \cdot m_0] = [T \otimes S (S \otimes B_0 B)] \cdot (1 \otimes m_0) = ((T \otimes S) S \otimes B_0 B) \cdot (1 \otimes m_0) = (T \otimes B_0 B) \cdot (1 \otimes m_0),$$

i.e., $T \otimes S M$ is $T \otimes B_0 B$-cyclic with generator $1 \otimes m_0$.

(c) $T \otimes S M$ is $\mathbb{N}$-semi-graded with respect to $T \otimes B_0 B$ with semi-gradation:

$$(T \otimes S M)_n := T \otimes S M_n, n \in \mathbb{N}.$$

In fact, $T \otimes S M = T \otimes S (\sum_{n \in \mathbb{N}} \otimes M_n) = \sum_{n \in \mathbb{N}} (T \otimes S M_n); (T \otimes B_0 B_m)(T \otimes S M_n) \subseteq T \otimes S B_m \cdot M_n$, but $B_m \cdot M_n = (1 \otimes B_0 M_n) \subseteq M_0 \oplus \cdots \oplus M_{m+n}$ since $M$ is $S \otimes B_0 B$-semi-graded. Thus, $(T \otimes B_0 B_m)(T \otimes S M_n) \subseteq (T \otimes S M_0) \oplus \cdots \oplus (T \otimes S M_{m+n})$.

(d) $1 \otimes m_0 \in T \otimes S M_0 \in (T \otimes S M)_0$.

(e) $T \otimes S M_0 = T \otimes S S = T$, i.e., $(T \otimes S M)_0 = T$.

(f) It is clear that $(T \otimes S M)_n = T \otimes B_0 M_n$ is a $T$-module; let $L$ be a prime ideal of $T$ and $Q := f^{-1}(L)$, where $f : S \rightarrow T$ is the given homomorphism, then it is easy to check that $S_Q \rightarrow T_L, u \mapsto \frac{f(u)}{f(1)}$ is a ring homomorphism, and from this, we get:

$$(T \otimes S M_n)_L \cong T_L \otimes (T \otimes S M_n) \cong (T_L \otimes T) \otimes S M_n \cong T_L \otimes S M_n \cong (T_L \otimes S_Q S_Q) \otimes S M_n \cong T_L \otimes S_Q (S_Q \otimes S M_n) \cong T_L \otimes S_Q S_Q \cong T_L.$$
This proves that $T \otimes_S M_R$ is locally free of rank 1.

(ii) $P$ is a covariant functor: It is clear that $P(i_R) = i_{P(R)}$; if $R \xrightarrow{f} S \xrightarrow{\phi} T$ are morphisms in $B_0$, then $P(\phi \circ f) = P(\phi)P(f)$. In fact, $T \otimes_S (S \otimes_R M) \cong T \otimes_R M$.  

Next, we recall some basic facts about schemes (see [34]). A scheme is a local ringed space $(X, \mathcal{F})$ for which every point $x \in X$ has a neighborhood $U_x$ such that the induced local ringed space $(U_x, \mathcal{F}|_{U_x})$ is isomorphic as local ringed space to $(\text{Spec}(R_x), \mathcal{O}_x)$, where $R_x$ is some commutative ring. Let $B_0$ be a commutative ring; recall that a $B_0$-scheme is a scheme $(X, \mathcal{F})$ such that $\mathcal{F}(U)$ is a $B_0$-algebra for every open $U \subseteq X$. For example, if $R$ is a commutative $B_0$-algebra, then the affine scheme $(\text{Spec}(R), \mathcal{O})$ is a $B_0$-scheme, with $\mathcal{O}$ defined by $\mathcal{O}(U) := \lim_{U_f \subseteq \mathcal{U} \subseteq U} R_f$, where $U_f$ ranges over all basic open sets contained in the open $U$ and $R_f$ is the localization of $R$ with respect to $f \in R$. The category of $B_0$-schemes is a subcategory of the category of schemes, and in turn, this last one is a subcategory of the category of local ringed spaces. A morphism between $B_0$-schemes is a morphism of the corresponding local ringed spaces such that the ring homomorphisms are $B_0$-algebra homomorphisms. Given two $B_0$-schemes $(X, \mathcal{F}), (Y, \mathcal{G})$, the set of morphisms from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ will be denoted by $\text{Hom}_{B_0\text{-schemes}}(X, Y)$. Fixing a $B_0$-scheme $(X, \mathcal{F})$, which we denote simply by $X$, we have the representable functor $h_X := \text{Hom}_{B_0\text{-schemes}}(-, X)$ defined in the following way, where $A_{ff}$ is the category of the affine schemes:

\[
B_0 \xrightarrow{h_X} \text{Set} \quad \text{where } \tilde{\phi} \in \text{Hom}_{K\text{-schemes}}(\text{Spec}(S), \text{Spec}(R)) \text{ is the image of } \phi \text{ under the } \text{Spec} \text{ functor:}
\]

\[
R \xrightarrow{\phi} \text{Spec}(R) \quad \text{for } \phi \in \text{Hom}_{K\text{-schemes}}(\text{Spec}(S), \text{Spec}(R)) \text{ and } \\
R \xrightarrow{\phi} \text{Spec}(R) \xrightarrow{\phi} \text{Spec}(S) \xrightarrow{\phi} \text{Spec}(R)
\]

**Definition 15.** Let $B = \bigoplus_{n \geq 0} B_n$ be an FSG ring generated in degree 1 such that $B_0$ is commutative and $B$ is a $B_0$-algebra. We say that a $B_0$-scheme $X$ parametrizes the point modules of $B$ if the point functor $P$ is naturally isomorphic to $h_X$.

**Theorem 11.** Let $B = \bigoplus_{n \geq 0} B_n$ be an FSG ring generated in degree 1 such that $B_0 = K$ is a field and $B$ is a $K$-algebra. Let $X$ be a $K$-scheme that parametrizes $P(B)$. Then, there exists a bijective correspondence between the closed points of $X$ and $P(B)$.

**Proof.** According to Remark 7, $P(B; K) = P(B)$; moreover, since $\text{Spec}(K) = \{0\}$, then every morphism of $\text{Hom}_{K\text{-schemes}}(\text{Spec}(K), X)$ determines one closed point of $X$, and viceversa. Thus, we have the bijective correspondence:

\[
\text{Hom}_{K\text{-schemes}}(\text{Spec}(K), X) \leftrightarrow \text{closed points of } X.
\]

Now, since $X$ parametrizes $P(B)$, the point functor $P$ is naturally isomorphic to $h_X$, so we have a bijective function between $\text{Hom}_{K\text{-schemes}}(\text{Spec}(K), X)$ and $P(B)$. Therefore, we get a bijective function between $P(B)$ and the closed points of $X$.  

**Remark 8.** Using the parametrization of the point modules for the quantum affine $n$-space, in a forthcoming paper, we will compute the set of point modules for many examples of skew PBW extensions.
Author Contributions: The mainly contribution of the second author was in the Section 4 about Koszulity. The other sections are due to the first author.

Funding: Universidad Nacional de Colombia, HERMES project 40482.

Acknowledgments: The authors are grateful to James Jim Zhang for valuable corrections, comments, and suggestions.

Conflicts of Interest: The author declares no conflict of interest.

References
1. Ginzburg, V. Calabi–Yau algebras. *arXiv* 2006, arXiv:math.AG/0612139v3.
2. Kanazawa, A. Non-commutative projective Calabi-Yau schemes. *J. Pure Appl. Algebra* 2015, 219, 2771–2780. [CrossRef]
3. Liu, L.-Y.; Wu, Q.-S. Twisted Calabi-Yau property of right coideal subalgebras of quantized enveloping algebras. *J. Algebra* 2014, 399, 1073–1085. [CrossRef]
4. Liu, L.-Y.; Wu, Q.-S.; Wang, S.-Q. Twisted Calabi–Yau property of Ore extensions. *J. Noncommut. Geom.* 2014, 8, 587–609. [CrossRef]
5. Rogalski, D. Noncommutative projective geometry. In *Noncommutative Algebraic Geometry*; Vol. 64 of Mathematical Sciences Research Institute Publications; Cambridge University Press: New York, NY, USA, 2016; pp. 13–70.
6. Suárez, H. Koszulity for graded skew PBW extensions. *Commun. Algebra* 2017, 45, 4569–4580. [CrossRef]
7. Gaddis, J.D. PBW Deformations of Artin-Schelter Regular Algebras and Their Homogenizations. Ph.D. Thesis, The University of Wisconsin-Milwaukee, Milwaukee, WI, USA, 2013.
8. Lezama, O.; Latorre, E. Non-commutative algebraic geometry of semi-graded rings. *Int. J. Algebra Comput.* 2017, 27, 361–389. [CrossRef]
9. Priddy, S. Koszul Resolutions. *Trans. Am. Math. Soc.* 1970, 152, 39–60. [CrossRef]
10. Bell, J.; Zhang, J.J. An isomorphism lemma for graded rings. *Proc. Am. Math. Soc.* 2017, 145, 989–994. [CrossRef]
11. Gallego, C.; Lezama, O. Gröbner bases for ideals of skew PBW extensions. *Commun. Algebra* 2011, 39, 50–75. [CrossRef]
12. Lezama, O.; Reyes, M. Some homological properties of skew PBW extensions. *Commun. Algebra* 2014, 42, 1200–1230. [CrossRef]
13. Reyes, M.A. Ring and Module Theoretic Properties of σ-PBW Extensions. Ph.D. Thesis, Universidad Nacional de Colombia, Bogotá, Colombia, 2013.
14. Suárez, H. N-Koszul Algebras, Calabi-Yau Algebras and Skew PBW Extensions. Ph.D. Thesis, Universidad Nacional de Colombia, Bogotá, Colombia, 2017.
15. Suárez, H.; Reyes, M.A. Koszulity for skew PBW extensions over fields. *IP J. Algebra Number Theory Appl.* 2017, 39, 181–203. [CrossRef]
16. Acosta, J.; Lezama, O. Universal property of skew PBW extensions. *Algebra Discrete Math.* 2015, 20, 1–12.
17. Acosta, J.P.; Lezama, O.; Reyes, M.A. Prime ideals of skew PBW extensions. *Rev. Unión Mat. Argent.* 2015, 56, 39–55.
18. Acosta, J.P.; Chaparro, C.; Lezama, O.; Ojeda, I.; Venegas, C. Ore and Goldie theorems for skew PBW extensions. *Asian-Eur. J. Math.* 2003, 6, 1350061.
19. Artamonov, V. Derivations of Skew PBW-Extensions. *Commun. Math. Stat.* 2015, 3, 449–457. [CrossRef]
20. Gallego, C.; Lezama, O. Projective modules and Gröbner bases for skew PBW extensions. *Dissertationes Math.* 2017, 521, 1–50.
21. Lezama, O.; Venegas, H. Some homological properties of skew PBW extensions arising in non-commutative algebraic geometry. *Discuss. Math. Gen. Algebra Appl.* 2017, 37, 45–57. [CrossRef]
22. Reyes, A.; Suárez, H. A notion of compatibility for Armendariz and Baer properties over skew PBW extensions. *Rev. Unión Mat. Argent.* 2018, 59, 157–178. [CrossRef]
23. Venegas, C. Automorphisms for skew PBW extensions and skew quantum polynomial rings. *Commun. Algebra* 2015, 42, 1877–1897. [CrossRef]
24. Reyes, A.; Suárez, H. Sigma-PBW extensions of skew Armendariz rings. *Adv. Appl. Clifford Algebr.* 2017, 27, 3197–3224. [CrossRef]
25. Phan, C. Koszul and Generalized Koszul Properties for Noncommutative Graded Algebras. Ph.D. Thesis, University of Oregon, Eugene, OR, USA, 2009.

26. Polishchuk, A.; Positselski, C. Quadratic Algebras; American Mathematical Society: Providence, RI, USA, 2005; Volume 37.

27. Phan, C. The Yoneda algebra of a graded Ore extension. arXiv 2010, arXiv:1002.2318v1.

28. Cassidy, T.; Shelton, B. Generalizing the notion of Koszul algebra. Math. Z. 2008, 260, 93–114. [CrossRef]

29. Weibel, C. An Introduction to Homological Algebra; Cambridge University Press: Cambridge, UK, 1997.

30. Backelin, J. A Distributiveness Property of Augmented Algebras and Some Related Homological Results. Ph.D. Thesis, Stockholm University, Stockholm, Sweden, 1981.

31. Fröberg, R. Koszul Algebras. In Advances in Commutative Ring Theory, Proceedings of the 3rd International Conference on Lecture Notes in Pure Applied Mathematics, Fez, Morocco, 6 March 1999; Marcel Dekker: New York, NY, USA, 1999; Volume 205, pp. 337–350.

32. Berger, R. Koszulity for nonquadratic algebras. J. Algebra 2001, 239, 705–734. [CrossRef]

33. Backelin, J.; Fröberg, R. Koszul algebras, Veronese subrings and rings with linear resolutions. Rev. Roumaine Math. Pures Appl. 1985, 30, 85–97.

34. Shafarevich, I.R. Basic Algebraic Geometry; Springer: Berlin/Heidelberg, Germany, 2013; Volumes 1–2.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).