ON A QUESTION OF EREMENKO
CONCERNING ESCAPING COMPONENTS
OF ENTIRE FUNCTIONS

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Abstract. Let $f$ be an entire function with a bounded set of singular values, and suppose furthermore that the postsingular set of $f$ is bounded. We show that every component of the escaping set $I(f)$ is unbounded. This provides a partial answer to a question of Eremenko.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function, and consider the escaping set

$$I(f) := \{ z \in \mathbb{C} : f^n(z) \to \infty \}$$

In 1989, Eremenko [E] showed that $I(f)$ is nonempty, and that every component of $I(f)$ is unbounded. He also states that the following seem plausible:

(a) Every component of $I(f)$ is unbounded.
(b) Every point of $I(f)$ can be connected to $\infty$ by a curve in $I(f)$.

An interesting class of functions to consider for these questions is the Eremenko-Lyubich class

$$\mathcal{B} := \{ f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire : } \text{sing}(f^{-1}) \text{ is bounded} \}.$$  
(Recall that $\text{sing}(f^{-1})$ consists of all critical and asymptotic values of $f$.) Recently [R3S], it was shown that [E] fails, even in this class. In fact, there exists $f \in \mathcal{B}$ for which $J(f) \supset I(f)$ contains no nontrivial curve at all. On the other hand [R], the dynamics “near $\infty$” remains the same throughout any given parameter space in $\mathcal{B}$ (in the sense of Eremenko and Lyubich; see [EL, Section 3] or [R, Section 2]). Taken together, these results show that there exist parameter spaces in class $\mathcal{B}$ in which [E] fails for every map.

In this note, we show that this is not the case for property (a). Recall that the postsingular set is defined as

$$\mathcal{P}(f) := \bigcup_{j \geq 0} f^j(\text{sing}(f^{-1})).$$

We shall prove the following result (which applies, in particular, to hyperbolic and postsingularly finite maps).
1.1. **Theorem** (Escaping components of postsingularly bounded maps).

Suppose that \( f \in \mathcal{B} \) and \( P(f) \) is bounded. Then every component of \( I(f) \) is unbounded.

**Remark.** Note that, if \( g \in \mathcal{B} \) is arbitrary, then the function \( f := g(z)/K \) will have a bounded postsingular set for sufficiently large \( K \). Hence every parameter space in class \( \mathcal{B} \) contains some functions to which our theorem applies.

There have previously been many cases in which property (a) has been established by in fact proving the stronger property (b), beginning with the case of a hyperbolic exponential map, which was completely described by Devaney in the early 1980s (see e.g. [DK]); the most recent result of this type \([RS, \text{Theorem } 1.2]\) does so for any (not necessarily hyperbolic) function \( f \in \mathcal{B} \) of finite order. (Compare the latter reference for a more comprehensive discussion of previous contributions.) It is interesting to note that each of these results also established the following property, which is stronger than (a):

(c) For every \( z \in I(f) \), there is an unbounded, connected set \( A \ni z \) such that \( f^n|_A \to \infty \) uniformly.

The methods of \([RS]\) can be used to show the existence of hyperbolic entire functions \( f \in \mathcal{B} \) for which this property fails, so our proof will need to establish (a) without any uniform properties.

There is an interesting case of functions outside class \( \mathcal{B} \) where this has previously been done. Rippon and Stallard \([RS]\) prove that, for any transcendental entire function \( f \), the set \( A(f) \subset I(f) \) introduced by Bergweiler and Hinkkanen \([BH]\) has only unbounded components (and this set does, in fact, satisfy the analog of (c)). In the case of an entire function which has a multiply-connected wandering domain, they also show that \( A(f) \) is connected; since \( A(f) \) is dense in \( I(f) \cap J(f) \), this easily implies that \( I(f) \) is also connected. This argument does not require (c); in fact, it seems plausible that this property will fail for functions of this kind.

It is now well-known that Rippon and Stallard’s construction of unbounded connected subsets of \( I(f) \) can be adapted to give more precise information for functions of class \( \mathcal{B} \). We will use a result of this type (Proposition 3.2) as a starting point of our proof; however, for the functions we are considering, \( I(f) \) is usually disconnected. Thus, we need to find another way to connect an arbitrary escaping point to an unbounded escaping component.

We shall achieve this by using three additional ingredients: the combinatorial coding of escaping points in class \( \mathcal{B} \) by *external addresses*, hyperbolic expansion properties of our map \( f \), and (perhaps most importantly) separation properties of the plane.

**Background and Notation.** We refer the reader to \([M]\) for background on holomorphic dynamics and an introduction to plane hyperbolic geometry. We denote the complex plane by \( \mathbb{C} \); all closures will be taken in \( \mathbb{C} \) unless stated otherwise. We denote the right half-plane by \( \mathbb{H} := \{ \text{Re } z > 0 \} \); more generally, we write \( \mathbb{H}_R := \{ \text{Re } z > R \} \). We also denote the unit disk by \( \mathbb{D} \). If \( U \subset \mathbb{C} \) is a domain and omits at least two points, we will denote hyperbolic distance in \( U \) by \( \text{dist}_U \), while Euclidean distance is denoted, as usual, by \( \text{dist} \).
2. Preliminaries

Let us define a tract to be any Jordan domain in which Re $z$ is unbounded from above and which is disjoint from its $2\pi i \mathbb{Z}$-translates. As in [R], we denote by $\mathcal{B}_{\log}$ the set of all functions $F : \mathcal{V} \to \mathbb{H}$, where

(a) $H$ is a $2\pi i$-periodic Jordan domain which contains a right half-plane.
(b) $\mathcal{V}$ is $2\pi i$-periodic and Re $z$ is bounded from below, but not from above, in $\mathcal{V}$.
(c) Each component $T$ of $\mathcal{V}$ is a tract. (We call these components the tracts of $F$).
(d) For each such $T$, $F_T := F|_T$ is a conformal isomorphism between $T$ and $\mathbb{H}$ with $F_T(\infty) = \infty$.
(e) The components of $\mathcal{V}$ accumulate only at $\infty$; i.e., if $z_n \in \mathcal{V}$ is a sequence of points all belonging to different components of $\mathcal{V}$, then $z_n \to \infty$.

Recall that any entire function $f \in \mathcal{B}$ has a logarithmic transform $F \in \mathcal{B}_{\log}$, with $\exp \circ F = f \circ \exp$. Thus the study of the dynamical behavior of $f$ near $\infty$ reduces to that of $F$.

Note that any $F \in \mathcal{B}_{\log}$ extends continuously to $\overline{\mathcal{V}}$ by Carathéodory’s theorem. We define the escaping set of $F$ as

$$I(F) := \{ z \in \overline{\mathcal{V}} : F^n(z) \in \mathcal{V} \text{ for all } n \text{ and } \text{Re } F^n(z) \to \infty \}. $$

It is not difficult to see that $\overline{I(F)}$ does not separate the plane and has no interior. Indeed, this is a simple application of Koebe’s distortion theorem, analogous to the proof of [EL, Theorem 1].

A function $F \in \mathcal{B}_{\log}$ is called normalized if $H = \mathbb{H}$ and $|F'(z)| \geq 2$ for all $z \in \mathcal{V}$. (By [EL] Lemma 1, any map $F \in \mathcal{B}_{\log}$ can be normalized using a suitable restriction and conjugacy by a translation.) Let $z_0 \in I(F)$, and let $T_j$ be the tract of $F$ with $F^j(z_0) \in T_j$. Then the sequence $s = T_0T_1T_2 \ldots$ is called the external address of $z_0$. The shift map on external addresses is denoted by $\sigma(s) = T_1T_2 \ldots$.

3. Proof of the Theorem

Let us begin by noting a simple topological fact.

3.1. Lemma (Separation of Tracts).

Let $T$ be a tract, let $R > 0$ and suppose that $z$ belongs to the unbounded component $U$ of $\{ z \in T : \text{Re } z > R \}$. Then the unbounded component $\tilde{U}$ of

$$T \setminus \{ z + it : t \in (-2\pi, 2\pi) \}.$$

is contained in $U$.

Proof. We will prove the contrapositive. That is, suppose that $z \in T$ with Re $z > R$, and that there is a curve $\gamma$ in $\tilde{U}$ which connects $\infty$ to a point $w$ with Re $w = R$. We may assume that all points of $\gamma$ except the endpoint $w$ have real parts larger than $R$; then $\gamma$ belongs to $U$. We need to show that $z \notin U$. 
For some integer $j$, the translates $\gamma_1 = \gamma + 2\pi ij$ and $\gamma_2 = \gamma + 2\pi i(j + 1)$ surround $z$ in $\mathbb{H}_R$ (compare Figure 1(a)). Clearly these translates must intersect the line segment $[z - 2\pi i, z + 2\pi i]$. Since $\gamma \subset U$, it follows that $j \not\in \{0, -1\}$. Hence $\gamma_1$ and $\gamma_2$ separate $z$ from $\gamma$, and hence from $U$, in $\mathbb{H}_R$. ■

Using this fact, we can provide an adaption of the aforementioned result of Rippon and Stallard [RS] to class $\mathcal{B}$. (See also [R3S] Theorem 3.3.)

3.2. Proposition (Unbounded sets of escaping points).
Let $F \in \mathcal{B}_\log$ be normalized, and let $z_0 \in I(F)$ have external address $s = T_0 T_1 T_2 \ldots$.
Set $D_j := \mathbb{D}_{2\pi}(F^j(z_0))$. Then there exists a sequence $(B_j)_{j \geq j_0}$ of unbounded subsets of $I(F)$ such that

(i) $B_j \subset T_j \setminus D_j$;
(ii) $B_j \cup \{\infty\}$ is compact and connected;
(iii) $B_j \cap \partial D_j \neq \emptyset$;
(iv) $F(B_j) \subset B_{j+1}$.

Proof. Let us set $z_j := F^j(z_0)$. Since $\Re z_j \to \infty$, we may assume without loss of generality that all $z_j$ belong to the unbounded connected component of $T_j \cap \mathbb{H}$.

Let $A_j$ be the unbounded connected component of $\overline{T_j \setminus D_k}$. It follows from the previous lemma that $A_j \subset \mathbb{H}$, and furthermore $\inf_{z \in A_j} \Re z \to \infty$.

We now define inductively $A_0^j := A_j$ and let $A_j^{k+1}$ denote the unbounded component of $F_{T_j}^{-1}(A_{j+1}^k) \setminus D_j$. (In other words, $A_j^k$ is obtained by pulling back the set $A_{j+k}$ to $T_j$, cutting off at the disks $(D_i)$ in every step.)
Then each $A^k_j$ is a closed, unbounded, connected set, and $A^{k+1}_j \subset A^k_j$. Furthermore, by the expanding property of $F$, we have $\text{dist}(A^k_j, z_j) \leq 2\pi$.

We now set

$$B_j := \bigcap_k A^k_j,$$

and claim that these sets meet all our conditions. Indeed, $B_j \cup \{\infty\}$ is compact and connected as the nested intersection of compact and connected sets. Since each $A^k_j$ intersects $\partial D_j$, so does $B_j$. By construction, we have $B_j \subset T_j \setminus D_j$, but in fact we also have $B_j \subset T_j$ since $F(B_j) \subset B_{j+1} \subset \mathbb{H}$.

The main problem in applying the previous proposition to prove our main result is that we do not know a priori that the sets $B_j$ are connected. Hence it is conceivable that, for every component $C$ of $B_0$, there is some $j$ such that the component of $B_j$ containing $F_j(C)$ is far away from $z_j$. However, we will now show that the $B_j$ actually are connected, so that this does not happen. The main tool required to do this is the following topological fact.

3.3. Lemma.

Let $T$ be a tract, and let $C_0, C_1 \subset T$ be unbounded connected sets. Then there exists $k \in \{0, 1\}$ such that every point of $C_k$ has distance at most $2\pi$ from $C_{1-k}$.

Proof. Suppose that there is $z_0 \in C_0$ such that $\text{dist}(z_0, C_1) > 2\pi$ (otherwise, there is nothing to prove). Since $T$ is disjoint from its own $2\pi i \mathbb{Z}$-translates, we see that $C_1$ is disjoint from the set

$$A := [z_0 - 2\pi i, z_0 + 2\pi i] \cup C_0 + 2\pi i \cup C_0 - 2\pi i.$$

Since $T$ is a Jordan domain which contains $C_0$ but is disjoint from $C_0 + 2\pi i$ and $C_0 - 2\pi i$, there is a component $U$ of $\mathbb{C} \setminus A$ which contains all sufficiently large points of $T$. In particular, we will have $C_1 \subset U$. Furthermore, by construction of $U$, for every point $z \in U$ there is $t \in (-2\pi, 2\pi)$ with $z + ti \in C_0$. This completes the proof.

3.4. Corollary (Uniqueness of unbounded components).

Let $F \in \mathcal{B}_{\log}$ be normalized, and suppose that $C_0, C_1 \subset I(F)$ are connected and unbounded. Suppose furthermore that all points in $C_0 \cup C_1$ have the same external address. Then $C_0 \cup C_1$ is connected.

Proof. Let $z_k \in C_k$ be two arbitrary points. By the previous lemma, for every $j \geq 0$ there is some $k \in \{0, 1\}$ such that $\text{dist}(F^j(z_k), F^j(C_{1-k})) \leq 2\pi$. In particular, there is some $k$ for which this happens for infinitely many $j$. Since $|F'| \geq 2$, it follows that

$$\text{dist}(z_k, C_{1-k}) \leq 2^{1-j}\pi$$

infinitely often; hence $\text{dist}(z_k, C_{1-k}) = 0$. So $C_0 \cup C_1$ is connected, as required.

In particular, this implies that the sets $B_j$ from Proposition 3.2 are connected, as desired. Collecting these results together, we obtain:
3.5. Theorem (Unbounded escaping components). Let \( z_0 \in I(F) \) have external address \( \underline{a} = T_0 T_1 \ldots \). Then the set
\[
I_\underline{a}(F) := \{ z \in I(F) : z \text{ has address } \underline{a} \}
\]
has a unique unbounded component \( C_\underline{a} \). Furthermore,
\[
\text{dist}(F^j(z_0), C_{\sigma^j(\underline{a})}) \leq 2\pi
\]
for all sufficiently large \( j \). (In fact, it suffices to take \( j \) so large that \( F^i(z_0) \) belongs to the unbounded component of \( T_i \cap \mathbb{H} \) for all \( i \geq j \).)

Proof of Theorem 3.5. Let us assume without loss of generality that \( 0 \in \mathcal{P}(f) \). By conjugating \( f \) with \( z \mapsto Kz \) for sufficiently large \( K \), we may furthermore assume that \( \mathcal{P}(f) \subset \mathbb{D} \). We set \( W := \{ |z| > 1 \} \) and \( \tilde{V} := f^{-1}(W) \). Then every component of \( \tilde{V} \) is an unbounded Jordan domain; these components are called the tracts of \( f \). We also set \( U := \mathbb{C} \setminus \mathcal{P}(f) \); then \( W \subset U \) by definition, and also \( f^{-1}(U) \subset U \).

Now let \( z_0 \in I(f) \). Note that finitely many iterates of \( z_0 \) might not belong to a tract of \( f \); however, we will nonetheless be able to assign an external address to \( z_0 \). Let us begin by associating to \( z_0 \) a sequence \( \tilde{T}_1, \tilde{T}_2, \ldots \) of tracts of \( f \). For sufficiently large \( k \) (say \( k \geq k_0 \)), the point \( f^k(z_0) \) will belong to the unbounded component of \( \tilde{T} \cap W \) for some tract \( \tilde{T} \). (Note that only finitely many tracts can intersect \( \partial W \), and recall that \( |f^n(z_0)| \to \infty \).) We will denote this tract by \( \tilde{T}_k \). To define \( \tilde{T}_j \) for \( j < k \) is a little bit trickier. To do so, take \( k \geq k_0 \) and let \( \tilde{V}_k \) be the component of \( f^{-k}(\tilde{T}_k) \) containing \( z_0 \). Then \( \tilde{V}_k \) is a Jordan domain, and (since \( \tilde{T}_k \) is disjoint from the postsingular set) \( f^k : \tilde{V}_k \to \tilde{T}_k \) is a conformal isomorphism.

Furthermore, for \( j < k \), all sufficiently large points of \( f^j(\tilde{V}_k) \) will be contained in a unique tract \( \tilde{T}_j \). This completes our definition of the sequence \( (\tilde{T}_j) \). (Note that the definition is independent of \( k \), and that it agrees with our original definition for \( j \geq k_0 \).)

However, a sequence of tracts of \( f \) is not quite sufficient to identify a suitable escaping component. The easiest way to adapt this is to instead consider a logarithmic transform \( F \in \mathcal{B}_{\log} \). More precisely, let \( \mathcal{V} := \exp^{-1}(\tilde{V}) \). Then there is a function \( F : \mathcal{V} \to \mathbb{H} \) with \( \exp \circ F = f \circ \exp \). If \( K \) was chosen sufficiently large in the beginning of the proof, then \( F \) will be normalized; i.e. \( |F'| \geq 2 \). (See [EL, Section 2] or [R, Section 2] for more details on this logarithmic change of variable.)

We now define an external address \( \underline{a} = T_1 T_2 \ldots \) of \( F \) with \( \exp(T_j) = \tilde{T}_j \) as follows. Let \( T_1 \) be an arbitrary component of \( \exp^{-1}(\tilde{T}_1) \), and let \( T_k \) be the unique component of \( \exp^{-1}(\tilde{V}_k) \) which has an unbounded intersection with \( T_1 \). Then \( T_j \) is defined to be the unique tract of \( F \) with \( F^j(\zeta) \in T_j \) for all sufficiently large \( \zeta \in V_k \) (\( k \geq j \)).

To complete the proof, let \( C_\underline{a} \) be the unbounded component of \( I(F) \) having address \( \underline{a} \), and let \( C \) be the component of \( I(f) \) containing \( \exp(C_\underline{a}) \). For \( k \geq k_0 \), let \( \zeta_k \) be the unique point in \( T_k \) with \( \exp(\zeta_k) = f^k(z_0) \). We have \( \text{dist}(\zeta_k, C_{\sigma^k(\underline{a})}) \leq 2\pi \) by Theorem 3.5. Since \( \text{Re} z_{k+1} \to +\infty \), this means that the hyperbolic distance \( \text{dist}_H(\zeta_k, C_{\sigma^k(\underline{a})}) \) tends to zero. The map
\[
\exp \circ F_{\tilde{T}_k}^{-1} : \mathbb{H} \to \tilde{T}_k
\]
is a conformal isomorphism, and it follows that
\[ \text{dist}_{T_k}(f^k(z_0), f^k(C)) \rightarrow 0. \]

Finally, because \( f^k : \widetilde{V}_k \rightarrow \widetilde{T}_k \) is a conformal isomorphism, we have
\[ \text{dist}_U(z_0, C) \leq \text{dist}_{T_k}(z_0, C) = \text{dist}_{\widetilde{T}_k}(f^k(z_0), f^k(C)) \rightarrow 0. \]
Thus \( z_0 \in C \), as required. \( \square \)

Finally, let us note the following interesting corollary of Theorem 3.5 for disjoint-type maps.

3.6. Corollary (Escaping components of disjoint-type maps).

Suppose that \( F \in \mathcal{B}_{\log} \) is of disjoint type (that is, \( \overline{V} \subset \mathbb{H} \)). Then the set \( I_\mathbb{A}(F) \) is connected and unbounded for every external address \( \mathbb{A} \).

Proof. This follows by a similar (but simpler) hyperbolic contraction argument to that in the previous proof. Indeed, in this setting we have \( F(I_\mathbb{A}) = I_\sigma(\mathbb{A}) \) (rather than the inclusion which usually holds). So if \( z_0 \in I_\mathbb{A} \), then \( \text{dist}(F^n(z_0), F^n(I_\mathbb{A})) \leq 2\pi \), and \( F \) is again a (even strict) hyperbolic expansion. The claim follows. \( \square \)

Remark. [R, Theorem 1.3] states that every hyperbolic map \( f \in \mathcal{B} \) is actually conjugate, on its escaping set, to a map of disjoint type. Hence the previous result provides a slightly alternative proof of Theorem [1.4] in the hyperbolic case.

Acknowledgments

I would like to thank Walter Bergweiler, Helena Mihaljevic-Brandt, Jörn Peter, Phil Rippon and Hendrik Schubert for interesting discussions about this work.

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