A STOCHASTIC FORWARD-BACKWARD SPLITTING METHOD FOR SOLVING MONOTONE INCLUSIONS IN HILBERT SPACES

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Abstract. We prove novel convergence results for a stochastic forward-backward splitting algorithm for solving monotone inclusions given by the sum of a maximal monotone operator and a single-valued maximal monotone cocoercive operator. We derive convergence rates in expectation in the strongly monotone case, as well as almost sure convergence results under weaker assumptions.

1. Introduction. Maximal monotone operators have been extensively studied since [37], largely because they have wide applications in pure and applied sciences, and because they provide a convenient framework for a unified treatment of equilibrium problems, variational inequalities, and convex optimization, see [5, 54] and references therein. Let \( \mathcal{H} \) be a real Hilbert space and let \( T: \mathcal{H} \to 2^{\mathcal{H}} \) be a set valued maximal monotone operator [5]. In this context, a key problem is to find an element \( \overline{w} \in \mathcal{H} \) such that \( 0 \in T(\overline{w}) \). One notable example of maximal monotone operator is the subdifferential of a proper, lower semicontinuous, convex function \( G \), and in this case to solve the inclusion \( 0 \in T(\overline{w}) \) is equivalent to minimizing \( G \).

In this paper, we consider the case where \( T \) is the sum of two maximal monotone operators denoted by \( A \) and \( B \), with \( B \) single valued and cocoercive. The problem is then to find \( \overline{w} \in \mathcal{H} \) such that

\[
0 \in A\overline{w} + B\overline{w},
\]  

under the assumption that such a \( \overline{w} \) exists. The problem in (1.1) includes not only variational inequalities, which can be recovered choosing \( A \) equal to the normal cone of a nonempty closed and convex set, but also composite minimization problems, where \( A \) is the subdifferential of a proper lower semicontinuous convex functions and \( B \) is the gradient of a smooth convex function.

There is a vast literature on algorithmic schemes for solving (1.1), that separate the contribution of \( A \) and \( B \). One well-known method is the forward-backward splitting algorithm [5, 14]:

\[
(\forall n \in \mathbb{N}) \quad w_{n+1} = J_{\gamma_n A} \circ (I - \gamma_n B)w_n,
\]  

where \( \gamma_n \in [0, +\infty[ \) and \( J_{\gamma_n A} = (I + \gamma_n A)^{-1} \) is the resolvent operator of \( A \). Since the seminal works [34, 43], forward-backward splitting methods have been considerably developed to be more flexible, to achieve better convergence properties, and to allow for numerical errors [5, 17, 42, 6, 51, 16].

In the important situation where only stochastic estimates of the operators \( A \) or \( B \) are available, convergence of the sequence in (1.2) has not been proved in the general case. To the best of our knowledge, convergence properties of algorithm (1.2) and its variants have been studied only for the special cases of minimization and variational inequalities (see Section 2 for a review of related work). This paper is a first step in studying stochastic forward-backward splitting methods for monotone inclusions, assuming to have access only to a stochastic oracle representation of \( B \), while we assume exact knowledge of \( A \) and of its resolvent.

We consider an infinite dimensional setting, and focus on almost sure convergence and convergence in expectation of the iterates. Moreover, we do not assume \( B \) to be an expectation, and the stochastic estimates of \( B \) do not necessarily come from independent samples, as it is usually assumed in stochastic approximation,
see e.g. [40]. Our convergence results on the sequence of the iterates do not require averaging, and allow the choice of step-sizes of the form \( n^{-\theta} \) with \( \theta \in [0, 1] \). When restricting to sparsity based optimization, avoiding averaging is relevant to preserve sparsity of the solutions [32]. A ubiquitous assumption in the stochastic approximation literature is boundedness (of some kind) of the stochastic estimates. A contribution of our approach is a relaxation of this requirement: we use a relative boundedness criterion, imposing the variance of the estimates to be upper bounded by the operator \( B \) itself.

The analysis in the paper is divided in three parts. In the first part, we establish almost sure convergence of the iterates. In the second part, we study convergence in expectation in the strongly monotone case. We provide a non-asymptotic analysis of stochastic forward-backward splitting algorithm, where the bounds depend explicitly on the parameters of the problem. The \( O(1/n) \) convergence rate is achieved. Finally, we study two special cases: variational inequalities and optimization. For the case of variational inequalities, we obtain an additional convergence result without imposing stronger monotonicity properties on \( B \), which requires averaging of the iterates. The present paper extends a short conference version [47] restricted to the minimization case.

The paper is organized as follows. We first review related work in Section 2. Section 3 collects some preliminaries and Section 4 contains the main results of the paper. Section 5 focuses on variational inequalities and minimization problems.

2. Related work. We are not aware of any paper studying stochastic forward-backward splitting methods for monotone inclusions. Therefore in this section we discuss related algorithms which have been developed for the two special cases of convex optimization and variational inequalities, that will be studied in Section 5. The field of stochastic approximation theory began with the seminal papers of Robbins and Monro [45] and Kiefer and Wolfowitz [26]. After those papers, stochastic approximation algorithms were widely used in stochastic optimization, see e.g. [8, 21, 23, 29] and references therein. An improvement of the original stochastic approximation method, is proposed by [41] and [44]. Their method relies on the averaging of the trajectories and allow for larger step-sizes.

In the special case of composite minimization problems, namely \( A = \partial g \), for some proper, lower semi-continuous, and convex \( G: \mathcal{H} \to \mathbb{R} \cup \{+\infty\} \), and \( B = \nabla f \) for some smooth and convex function \( F: \mathcal{H} \to \mathbb{R} \), stochastic implementations of forward-backward splitting algorithms, and more generally of first order methods, received much attention and have been recently studied in several papers [18, 31, 32, 19]. In particular, [31] proposes an accelerated method and derives a rate of convergence for the objective function values which is optimal both with respect to the smooth component and the non-smooth term. Similar accelerated proximal gradient algorithms have been also studied in the machine learning community, see [30, 52], and also [10, 48, 49, 55]. When restricted to the composite optimization case, our stochastic forward-backward splitting algorithm is related to the FOBOS algorithm presented in [20]. With respect to [20], we allow for an additional relaxation step, which in practice can speed up convergence. It is worth noting that FOBOS algorithm can be applied also when \( F \) is not differentiable. In the optimization setting, the study of almost sure convergence has a long history, see e.g. [46, 28, 8, 13] and references therein. Recent results on almost sure convergence of projected stochastic gradient algorithm can be found in [7, 38], under rather technical assumptions. Our results are generalizations to monotone inclusions of the analysis of the stochastic projected subgradient algorithm in [4]. Note that the latter is also one of the few papers dealing with the infinite dimensional setting. Our results about convergence rates in expectation are a generalization to monotone inclusions of [3, Section 3] (see also the very recent preprint [1]).

Going beyond optimization problems, there is a line of research studying stochastic algorithms for variational inequalities on finite dimensional spaces. The sample average approximation has been studied e.g. in [50, 12] (see also references therein), and a mirror proximal stochastic approximation algorithm to solve variational inequalities corresponding to a maximal monotone operator has been proposed in [25]. A stochastic iterative proximal method has been proposed in [27], and almost sure convergence properties of a stochastic forward-backward splitting algorithm for solving strongly monotone variational inequalities has been studied.
in [24]. Our results are a generalization of this kind of analysis to the monotone inclusion case.

**Notation.** Throughout, \((\Omega, \mathcal{A}, \mathbb{P})\) is a probability space, \(\mathbb{N}^* = \mathbb{N}\setminus\{0\}, \mathcal{H}\) is a real separable Hilbert space, and \(2^\mathcal{H}\) is its power set. We denote by \(\langle \cdot | \cdot \rangle\) and \(\| \cdot \|\) the scalar product and the associated norm of \(\mathcal{H}\). The symbols \(\rightarrow\) and \(\Rightarrow\) denote weak and strong convergence, respectively. We denote by \(\ell_1^+(\mathbb{N})\) the set of summable sequences in \([0, +\infty[.\) The class of all lower semicontinuous convex functions \(G: \mathcal{H} \rightarrow [-\infty, +\infty]\) such that \(\text{dom} G = \{x \in \mathcal{H} \mid G(x) < +\infty\} \neq \emptyset\) is denoted by \(\Gamma_0(\mathcal{H})\). We denote by \(\sigma(X)\) the \(\sigma\)-field generated by a random variable \(X: \Omega \rightarrow \mathcal{H}\), where \(\mathcal{H}\) is endowed with the Borel \(\sigma\)-algebra. The expectation of a random variable \(X\) is denoted by \(\mathbb{E}[X]\). The conditional expectation of \(X\) given \(Y\) is denoted by \(\mathbb{E}[X|Y]\). The shorthand notation ‘a.s’ stands for ‘almost sure’ or ‘almost surely’.

3. **Preliminaries.** Let \(A: \mathcal{H} \rightarrow 2^\mathcal{H}\) be a set-valued operator. The domain and the graph of \(A\) are respectively defined by \(\text{dom} A = \{w \in \mathcal{H} \mid Aw \neq \emptyset\}\) and \(\text{gra} A = \{(w, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Aw\}\). We denote by \(\text{zer} A = \{w \in \mathcal{H} \mid 0 \in Aw\}\) the set of zeros of \(A\). The inverse of \(A\) is \(A^{-1}: \mathcal{H} \rightarrow 2^\mathcal{H}: u \mapsto \{w \in \mathcal{H} \mid u \in Aw\}\).

The following notion is central in the paper.

**Definition 3.1.** Let \(A: \mathcal{H} \rightarrow 2^\mathcal{H}\) be a set-valued operator. \(A\) is monotone if
\[
\forall (w, u) \in \text{gra} A \, \forall (y, v) \in \text{gra} A \quad \langle w - y | u - v \rangle \geq 0,
\]
and maximally monotone if it is monotone and there exists no monotone operator \(B: \mathcal{H} \rightarrow 2^\mathcal{H}\) such that \(\text{gra} B\) properly contains \(\text{gra} A\).

Let \(A\) be a monotone operator. We say that \(A\) is uniformly monotone at \(y \in \text{dom} A\) if there exists an increasing function \(\phi: [0, +\infty[ \rightarrow [0, +\infty]\) vanishing only at \(0\) such that
\[
\forall (w, u) \in \text{gra} A \, \forall v \in Ay \quad \langle w - y | u - v \rangle \geq \phi(\|w - y\|).
\]
In the case when \(\phi = \mu |\cdot|^2\), for some \(\mu \in ]0, +\infty[\), we say that \(A\) is \(\mu\)-strongly monotone at \(y\). If \(A - \mu I\) is monotone, for some \(\mu \in ]0, +\infty[\), we say that \(A\) is \(\mu\)-strongly monotone. We say that \(A\) is strictly monotone at \(y \in \mathcal{H}\) if
\[
\forall (w, u) \in \text{gra} A \, \forall v \in Ay \quad w \neq y \Rightarrow \langle w - y | u - v \rangle > 0.
\]
Let \(\beta \in ]0, +\infty[\). A single-valued operator \(B: \mathcal{H} \rightarrow \mathcal{H}\) is \(\beta\)-cocoercive if
\[
\forall w, y \in \mathcal{H} \quad \langle w - y | Bw - By \rangle \geq \beta \|Bw - By\|^2.
\]

The resolvent of any maximally monotone operator \(A\) is
\[
J_A = (I + A)^{-1}.
\]
We recall that \(J_A\) is well defined and single valued [37], and can therefore be identified with an operator \(J_A: \mathcal{H} \rightarrow \mathcal{H}\). When \(A = \partial G\) for some \(G \in \Gamma_0(\mathcal{H})\), then \(J_A\) coincides with the proximity operator of \(G\) [39], which is defined as
\[
\text{prox}_G: \mathcal{H} \rightarrow \mathcal{H}: w \mapsto \arg\min_{v \in \mathcal{H}} G(v) + \frac{1}{2}\|w - v\|^2.
\]
Finally, let \(\beta \in \mathbb{R}\). We define the family of functions
\[
\varphi_\beta: ]0, +\infty[ \rightarrow \mathbb{R}: t \mapsto \begin{cases} \beta^{-1}(t^{\beta} - 1) & \text{if } \beta \neq 0; \\ \log t & \text{if } \beta = 0. \end{cases}
\]

The following notions and results will be required in the following sections.

**Definition 3.2.** [22] Let \(S\) be a non-empty subset of \(\mathcal{H}\). Then,
(i) A sequence \((w_n)_{n \in \mathbb{N}^*}\) in \(\mathcal{H}\) is quasi-Fejér monotone with respect to \(S\) if there exists \((\varepsilon_n)_{n \in \mathbb{N}^*}\) such that
\[
(\forall w \in S)(\forall n \in \mathbb{N}^*)\quad \|w_{n+1} - w\|^2 \leq \|w_n - w\|^2 + \varepsilon_n.
\] (3.7)

(ii) A sequence of random vectors \((w_n)_{n \in \mathbb{N}^*}\) in \(\mathcal{H}\) is stochastic quasi-Fejér monotone with respect to \(S\) if
\[
(\forall w \in S)(\forall n \in \mathbb{N}^*)\quad \mathbb{E}[\|w_{n+1} - w\|^2 | \sigma(w_1, \ldots, w_n)] \leq \|w_n - w\|^2 + \varepsilon_n \quad \text{a.s.}
\] (3.8)

Proposition 3.3. [4, Lemma 2.3] Let \(S\) be a non-empty closed subset of \(\mathcal{H}\), and let \((w_n)_{n \in \mathbb{N}^*}\) be a stochastic quasi-Fejér monotone sequence with respect to \(S\). Then the following hold.

(i) Let \(w \in S\). Then, \((\mathbb{E}[\|w_n - w\|^2])_{n \in \mathbb{N}^*}\) converges to some \(z_w \in \mathbb{R}\) and \((\|w_n - w\|^2)_{n \in \mathbb{N}^*}\) converges a.s. to an integrable random vector \(z_w\).

(ii) \((w_n)_{n \in \mathbb{N}^*}\) is bounded a.s.

(iii) The set of weak subsequential limits of \((w_n)_{n \in \mathbb{N}^*}\) is non-empty a.s.

We note that a version of Proposition 3.3 in the finite dimensional setting can be found in [22]. The following lemma establishes a convergence rate for numerical sequences satisfying a given recursive inequality; see [3] and [47] for a proof.

Lemma 3.4. Let \(\alpha\) be in \([0, 1]\), and let \(c\) and \(\tau\) be in \([0, +\infty[\), let \((\eta_n)_{n \in \mathbb{N}^*}\) be the sequence defined by \((\forall n \in \mathbb{N}^*)\) \(\eta_n = cn^{-\alpha}\). Let \((s_n)_{n \in \mathbb{N}^*}\) be such that
\[
(\forall n \in \mathbb{N}^*)\quad 0 \leq s_{n+1} \leq (1 - \eta_n)s_n + \tau_n^2.
\] (3.9)

Let \(n_0\) be the smallest integer such that \((\forall n \geq n_0 > 1)\) \(\eta_n \leq 1\) and set \(t = 1 - 2^{1-\alpha} \geq 0\). Then, for every \(n \geq 2n_0\),
\[
s_{n+1} \leq \begin{cases} \tau c \varphi_{1-2\alpha}(n) + s_{n_0} \exp \left(\frac{c t (n + 1)}{1 - \alpha}\right) \exp \left(\frac{-ct(n + 1)\varphi_{1-2\alpha}}{1 - \alpha}\right) + \frac{\tau c^2}{(n - 2)\alpha} & \text{if } \alpha \in [0, 1[ \\
\left(s_{n_0} + \frac{n_0}{n + 1}\right)^c + \frac{\tau c^2}{(n + 1)c} \left(1 + \frac{1}{n_0}\right)^c \varphi_{c-1}(n) & \text{if } \alpha = 1,
\end{cases}
\] (3.10)

where \(\varphi_{1-2\alpha}\) and \(\varphi_{c-1}\) are defined as in (3.7).

Lemma 3.5. [5, Proposition 23.7] Let \(A: \mathcal{H} \to 2^{\mathcal{H}}\) be maximally monotone. Then, the resolvent of \(A\) is firmly-expansive, i.e.,
\[
(\forall w \in \mathcal{H})(\forall u \in \mathcal{H})\quad \|J_A w - J_A u\|^2 \leq \|w - u\|^2 - \|(w - J_A w) - (u - J_A u)\|^2.
\] (3.11)

Lemma 3.6 (Baillon-Haddad Theorem). Let \(L \in \Gamma_0(\mathcal{H})\) be a convex differentiable function with \(\beta^{-1}\) Lipschitzian gradient, for some \(\beta \in [0, +\infty[\). Then, \(\nabla L\) is \(\beta\)-cocoercive, i.e,
\[
(\forall u \in \mathcal{H})(\forall w \in \mathcal{H})\quad \langle w - u | \nabla L(w) - \nabla L(u) \rangle \geq \beta \|\nabla L(w) - \nabla L(u)\|^2.
\] (3.12)

4. Main results. The following is the main problem studied in this paper.

Problem 4.1. Let \(A: \mathcal{H} \to 2^{\mathcal{H}}\) be a maximally monotone operator, let \(B: \mathcal{H} \to \mathcal{H}\) be a \(\beta\)-cocoercive operator, for some \(\beta \in [0, +\infty[\). Assume that \(\text{zer}(A + B) \neq \emptyset\). The goal is to find \(\overline{w} \in \mathcal{H}\) such that
\[
0 \in A\overline{w} + B\overline{w}.
\] (4.1)

Problems that can be decomposed into sums of monotone operators have been the subject of intensive study in the last years, see e.g. [5, 14] and references therein.
4.1. Algorithm. We propose the following stochastic forward-backward splitting algorithm for solving Problem 4.1. The key difference with respect to the classical setting is that we assume to have access only to a stochastic estimate of $B$.

**Algorithm 4.2.** Let $(\gamma_n)_{n \in \mathbb{N}^*}$ be a sequence in $]0, +\infty[$, $(\lambda_n)_{n \in \mathbb{N}^*}$ be a sequence in $[0, 1]$, and let $(\mathcal{B}_n)_{n \in \mathbb{N}^*}$ be a $\mathcal{H}$-valued random process. Fix $w_1 \in \mathcal{H}$ and set

$$\begin{align*}
(\forall n \in \mathbb{N}^*) \quad z_n &= w_n - \gamma_n \mathcal{B}_n \\
y_n &= J_{\gamma_n A^*} z_n \\
w_{n+1} &= (1 - \lambda_n) w_n + \lambda_n y_n.
\end{align*}$$

(4.2)

**Condition 4.3.** We will consider the following conditions for the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}^*}$, $\mathcal{F}_n = \sigma(w_1, \ldots, w_n)$.

(A1) For every $n \in \mathbb{N}^*$, $\mathbb{E}[\mathcal{B}_n | \mathcal{F}_n] = Bw_n$.

(A2) For every $n \in \mathbb{N}^*$, there exists $\alpha_n \in ]0, +\infty[$ such that $\mathbb{E}[\|\mathcal{B}_n - Bw_n\|^2 | \mathcal{F}_n] \leq \sigma^2 (1 + \alpha_n \|Bw_n\|^2)$.

(A3) There exists $\varepsilon \in ]0, +\infty[$ such that for every $n \in \mathbb{N}^*$ it holds $1 - \gamma_n (1 + 2 \sigma^2 \alpha_n) \geq \varepsilon.$

(A4) Let $\overline{w}$ be a solution of Problem 4.1. Then the following hold.

$$\sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n = +\infty \quad \text{and} \quad \sum_{n \in \mathbb{N}^*} \chi_n^2 < +\infty, \quad \text{where} \quad \chi_n^2 = \lambda_n \gamma_n^2 (1 + 2 \alpha_n \|B\overline{w}\|^2). \quad (4.3)$$

When $(\forall n \in \mathbb{N}^*) \mathcal{B}_n = Bw_n$, our algorithm reduces to the well known forward–backward splitting in [15, Section 6]. However, under Condition 4.3, weak convergence of $(w_n)_{n \in \mathbb{N}^*}$ is not guaranteed since $\lambda_n \gamma_n^2 \to 0$, while to apply the classic theory we need $\inf \gamma_n$ to be strictly greater than 0. Assuming additionally that $(\forall n \in \mathbb{N}^*) \lambda_n = 1$, ergodic convergence of $(w_n)_{n \in \mathbb{N}^*}$ has been proved in [43].

In the case $A = \partial G$ and $B = \nabla L$, for some $G$ and $L \in \Gamma_0(\mathcal{H})$ such that $L$ is differentiable with $\beta^{-1}$-Lipschitz continuous gradient, Algorithm 4.2 reduces to the stochastic proximal forward-backward splitting in [47] which is a variant of the algorithm in [20]. Also, very recently, a related algorithm has been studied in [1].

**Example 4.4.** When, for every $w \in \mathcal{H}$, $Bw$ can be written as an expectation, namely

$$Bw = \int_G b(w, y) P(dy), \quad (4.4)$$

where $P$ is a probability measure on the measurable space $(\mathcal{G}, \mathcal{B})$, and $b: \mathcal{H} \times \mathcal{G} \to \mathcal{H}$ is a measurable function such that $\int_G \|b(w, y)\| P(dy) < +\infty$, there are several ways to find a stochastic estimate of $Bw$. In particular, if a sample $\{y_1, \ldots, y_n\}$ of i.i.d. points is available, then one can take $b(w, y_i)$. If moreover $B$ is a gradient operator, we are in the classical setting of stochastic optimization [40].

**Remark 4.5.** Condition (A2) can be seen as a relative error criterion, and has been considered in [4] for the case of constrained minimization problems on infinite dimensional spaces. This is a more general condition than the one usually assumed in the context of stochastic optimization, where $\alpha_n = 0$. If $A = 0$, then $B\overline{w} = 0$ for every solution of Problem 4.1. In this case, $\chi_n = \lambda_n \gamma_n^2$ in (4.3) and condition (A4) becomes $\sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n = +\infty$ and $\sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n^2 < +\infty$. The latter are the usual conditions required on the stepsize in the study of stochastic gradient descent algorithms (see e.g. [9]). These conditions guarantee a sufficient, but not too fast, decrease of the stepsize length.

4.2. Almost sure convergence. We now state our first result for the general setting, collecting some basic properties satisfied by the sequences appearing in Algorithm 4.2.

**Proposition 4.6.** Suppose that (A1), (A2), (A3) and (A4) are satisfied. Let $(w_n)_{n \in \mathbb{N}^*}$ be the sequence generated by Algorithm 4.2 and $\overline{w}$ a solution of Problem 4.1. Then the following hold.
(i) The sequence \((E[∥w_n - \omega∥^2])_{n∈N^*}\) converges to a finite value.
(ii) \(\sum_{n∈N^*} \lambda_n E[∥w_n - y_n - \gamma_n(\mathfrak{B}_n - B\omega)∥^2] < +∞\) and \(\sum_{n∈N^*} \lambda_n E[∥w_n - y_n∥^2] < +∞\).
(iii) \(\sum_{n∈N^*} \lambda_n \gamma_n E[⟨w_n - \omega | B\omega_n - B\omega⟩] < +∞\). Consequently,
\[
\lim_{n→∞} E[(w_n - \omega | B\omega_n - B\omega)] = 0 \quad \text{and} \quad \lim_{n→∞} E[∥B\omega_n - B\omega∥^2] = 0. \quad (4.5)
\]

**Proof.** Let \(\omega\) be a solution of Problem 4.1. We have
\[
(∀n ∈ N^*) \quad \omega = J_{γ_n A}(w_n - γ_n B\omega) \quad (4.6)
\]
Let us set
\[
(∀n ∈ N^*) \quad y_n = J_{γ_n A}(w_n - γ_n \mathfrak{B}_n) \quad \text{and} \quad u_n = w_n - y_n - γ_n(\mathfrak{B}_n - B\omega). \quad (4.7)
\]
Then it follows from the convexity of \(∥·∥^2\) that
\[
(∀n ∈ N^*) \quad ∥u_{n+1} - \omega∥^2 = ∥(1 - \lambda_n)(w_n - \omega) + \lambda_n(y_n - \omega)∥^2 \leq (1 - \lambda_n)∥w_n - \omega∥^2 + \lambda_n∥y_n - \omega∥^2. \quad (4.8)
\]
Since \(J_{γ_n A}\) is firmly non-expansive by Lemma 3.5, we have
\[
(∀n ∈ N^*) \quad ∥y_n - \omega∥^2 \leq ∥(w_n - \omega) - γ_n(\mathfrak{B}_n - B\omega)∥^2 - ∥u_n∥^2 = ∥w_n - \omega∥^2 - 2γ_n ⟨w_n - \omega | \mathfrak{B}_n - B\omega⟩ + γ_n^2 ∥\mathfrak{B}_n - B\omega∥^2 - ∥u_n∥^2, \quad (4.9)
\]
Hence, using assumption (A1), by noting that \(E[⟨w_n - \omega | \mathfrak{B}_n - B\omega⟩] < +∞\) and \(E[∥\mathfrak{B}_n - B\omega∥^2] < +∞\), we obtain
\[
(∀n ∈ N^*) \quad E[⟨w_n - \omega | \mathfrak{B}_n - B\omega⟩] = E[E[⟨w_n - \omega | \mathfrak{B}_n - B\omega⟩ | F_n]] = E[E[⟨w_n - \omega | \mathfrak{B}_n - B\omega⟩F_n]] = E[⟨w_n - \omega | B\omega_n - B\omega⟩]. \quad (4.10)
\]
Moreover, using assumption (A2), we have
\[
E[∥\mathfrak{B}_n - B\omega∥^2] \leq 2E[∥B\omega_n - B\omega∥^2] + 2E[∥\mathfrak{B}_n - B\omega_n∥^2] \leq 2E[∥B\omega_n - B\omega∥^2] + 2σ^2(1 + 2α_n E[∥B\omega_n∥^2]) \leq (2 + 4σ^2α_n)E[∥B\omega_n - B\omega∥^2] + 2σ^2(1 + 2α_n)∥B\omega∥^2 \leq \frac{2 + 4σ^2α_n}{β} E[ ⟨w_n - \omega | B\omega_n - B\omega⟩ ] + 2σ^2(1 + 2α_n)∥B\omega∥^2. \quad (4.11)
\]
where the last inequality follows from the cocoercivity of \(B\). Recalling the definition of \(ε\), from (4.8), (4.9), (4.10), and (4.11) we get that
\[
E[∥w_{n+1} - \omega∥^2] \leq (1 - λ_n)E[∥w_n - \omega∥^2] + λ_n E[∥y_n - \omega∥^2] \leq E[∥w_n - \omega∥^2] - 2γ_n λ_n \frac{1 - γ_n(1 + 2σ^2α_n)}{β} E[⟨w_n - \omega | B\omega_n - B\omega⟩] + 2σ^2χ_n - λ_n E[∥u_n∥^2] \quad (4.12)
\]
\[
\leq E[∥w_n - \omega∥^2] - 2εγ_n λ_n E[⟨w_n - \omega | B\omega_n - B\omega⟩] + 2σ^2χ_n^2 - λ_n E[∥u_n∥^2]. \quad (4.13)
\]
(i): Since the sequence \((χ_n)_{n∈N^*}\) is summable by assumption (A4), we derive from (4.12) that \(E[∥w_{n+1} - \omega∥^2])_{n∈N^*}\) converges to a finite value.

(ii)(iii): It follows from (4.12) that
\[
\sum_{n∈N^*} λ_n E[∥u_n∥^2] < +∞, \quad \text{and} \quad \sum_{n∈N^*} γ_n λ_n E[⟨w_n - \omega | B\omega_n - B\omega⟩] < +∞. \quad (4.14)
\]
Since \( \sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n = +\infty \), we obtain,
\[
\lim_{n \to \infty} \mathbb{E}[(w_n - \overline{w} \mid Bw_n - B\overline{w})] = 0
\]  (4.15)
which implies, by cocoercivity, \( \lim_{n \to \infty} \mathbb{E}[[Bw_n - B\overline{w}]^2] = 0. \)

Since \( B \) is cocoercive, it is Lipschitzian. Therefore, by (i), there exists \( M \in \] 0, +\infty [ \) such that
\[
(\forall n \in \mathbb{N}^*) \quad \mathbb{E}[(w_n - \overline{w} \mid Bw_n - B\overline{w})] \leq \beta^{-1} \mathbb{E}[[w_n - \overline{w}]^2] \leq M. \]  (4.16)
Hence, we derive from (4.3) and (4.11) that
\[
\sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n^2 \mathbb{E}[[\mathbb{B}_n - B\overline{w}]^2] < +\infty. \]  (4.17)
Now, using (4.14) and (4.17), we obtain
\[
\sum_{n \in \mathbb{N}^*} \lambda_n \mathbb{E}[[w_n - y_n]^2] \leq 2 \sum_{n \in \mathbb{N}^*} \lambda_n \mathbb{E}[[u_n]^2] + 2 \sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n^2 \mathbb{E}[[\mathbb{B}_n - B\overline{w}]^2] < +\infty. \]  (4.18)
Therefore, (ii) and (iii) are proved. \( \square \)

In the next theorem, we focus on almost sure convergence of Algorithm 4.2. This kind of convergence of the iterates is the one traditionally studied in the stochastic optimization literature. However, most papers focus on the finite dimensional setting, and require boundedness of the variance of the stochastic estimate of the gradients or subgradients (namely, \( \alpha_n = 0 \) in assumption (A2)). One exception is \([4]\), dealing with a stochastic projected subgradient algorithm on a Hilbert space. Our proof is based on quasi-martingale techniques (see e.g. \([36]\)). Depending on the monotonicity properties of the operator \( B \), we get two different convergence properties.

**Theorem 4.7.** Suppose that (A1), (A2), (A3) and (A4) are satisfied. Let \((w_n)_{n \in \mathbb{N}^*}\) be the sequence generated by Algorithm 4.2 and let \( \overline{w} \) be a solution of Problem 4.1. Then

(i) \((||w_n - \overline{w}||^2)_{n \in \mathbb{N}^*}\) converges almost surely to some integrable random variable \( \zeta_{\overline{w}} \).

(ii) If \( B \) is uniformly monotone at \( \overline{w} \), then \( w_n \to \overline{w} \) a.s.

(iii) If \( B \) is strictly monotone at \( \overline{w} \) and weakly continuous, then there exists a subsequence \((w_{n_k})_{n \in \mathbb{N}^*}\) such that \( w_{n_k} \to \overline{w} \) a.s.

**Proof.** Let \( \overline{w} \) be a solution of problem 4.1. Reasoning as in the proof of Proposition 4.6, we have
\[
(\forall n \in \mathbb{N}^*) \quad ||y_n - \overline{w}||^2 \leq ||w_n - \overline{w}||^2 - 2\gamma_n (w_n - \overline{w} \mid \mathbb{B}_n - B\overline{w}) + \gamma_n^2 ||\mathbb{B}_n - B\overline{w}||^2 - ||u_n||^2, \]  (4.19)
where \( u_n \) is defined as in (4.7). We next estimate the conditional expectation with respect to \( \mathcal{F}_n \) of each term in the right hand side of (4.19). Since \( w_n \) is \( \mathcal{F}_n \)-measurable, we have
\[
\mathbb{E}[(||w_n - \overline{w}||^2 \mid \mathcal{F}_n)] = ||w_n - \overline{w}||^2. \]  (4.20)
and using condition (A1),
\[
(\forall n \in \mathbb{N}^*) \quad \mathbb{E}[(w_n - \overline{w} \mid \mathbb{B}_n - B\overline{w} \mid \mathcal{F}_n)] = (w_n - \overline{w} \mid \mathbb{E}[\mathbb{B}_n - B\overline{w} \mid \mathcal{F}_n]) = (w_n - \overline{w} \mid Bw_n - B\overline{w}). \]  (4.21)
We next use condition (A2) and the fact that \( Bw_n \) is \( \mathcal{F}_n \)-measurable since \( x_n \) is \( \mathcal{F}_n \)-measurable and \( B \) is continuous, to derive
\[
\mathbb{E}[(||\mathbb{B}_n - B\overline{w}||^2 \mid \mathcal{F}_n)] \leq 2\mathbb{E}[(||Bw_n - B\overline{w}||^2 \mid \mathcal{F}_n)] + 2\mathbb{E}[(||\mathbb{B}_n - Bw_n||^2 \mid \mathcal{F}_n)] \\
\leq 2||Bw_n - B\overline{w}||^2 + 2\sigma^2(1 + \alpha_n||Bw_n||^2) \\
\leq 2||Bw_n - B\overline{w}||^2 + 2\sigma^2(1 + 2\alpha_n||Bw_n - B\overline{w}||^2 + 2\alpha_n||B\overline{w}||^2) \\
\leq \frac{(2 + 4\sigma^2\alpha_n)}{\beta} (w_n - \overline{w} \mid Bw_n - B\overline{w}) + 2\sigma^2(1 + 2\alpha_n||B\overline{w}||^2), \]  (4.22)
where the last inequality follows from the cocoercivity of $B$. Now, note that as in (4.8), we have
\[
(\forall n \in \mathbb{N}^*) \quad \|w_{n+1} - w\|^2 \leq (1 - \lambda_n)\|w_n - w\|^2 + \lambda_n\|y_n - w\|^2.
\] (4.23)

Taking the conditional expectation and invoking (4.19), (4.21), (4.22), we obtain,
\[
\mathbb{E}[\|w_{n+1} - w\|^2 | \mathcal{F}_n] \leq (1 - \lambda_n)\|w_n - w\|^2 + \lambda_n\mathbb{E}[\|y_n - w\|^2 | \mathcal{F}_n]
\leq \|w_n - w\|^2 - 2\gamma_n\lambda_n \left(1 - \frac{\gamma_n(1 + 2\sigma^2\alpha_n)}{\beta}\right) \langle Bw_n - B\bar{w} | w_n - w\rangle
+ 2\sigma^2\lambda_n^2 - \lambda_n\mathbb{E}[\|u_n\|^2 | \mathcal{F}_n]
\leq \|w_n - w\|^2 - 2\epsilon\gamma_n\lambda_n \langle Bw_n - B\bar{w} | w_n - w\rangle + 2\sigma^2\lambda_n^2 - \lambda_n\mathbb{E}[\|u_n\|^2 | \mathcal{F}_n].
\] (4.24)

Hence $(w_n)_{n \in \mathbb{N}^*}$ is a random quasi-Fejér sequence with respect to the nonempty closed convex set $\text{zer}(A + B)$.

(i): It follows from Proposition 3.3(i) that $(\|w_n - \bar{w}\|^2)_{n \in \mathbb{N}^*}$ converges a.s to some integrable random variable $\zeta$.

(ii) Since $B$ is uniformly monotone at $\bar{w}$, then there exists an increasing function $\phi: [0, +\infty] \to [0, +\infty]$ vanishing only at 0 such that
\[
\langle Bw_n - B\bar{w} | w_n - \bar{w}\rangle \geq \phi(\|w_n - \bar{w}\|).
\] (4.25)

and thus $\bar{w}$ is the unique solution of Problem 4.1. Therefore, we derive from Proposition 4.6 (iii) that
\[
\sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n \mathbb{E}[\phi(\|w_n - \bar{w}\|)] < \infty,
\] (4.26)

and hence
\[
\sum_{n \in \mathbb{N}^*} \lambda_n \gamma_n \phi(\|w_n - \bar{w}\|) < \infty \quad \text{a.s.}
\] (4.27)

Since $(\lambda_n \gamma_n)_{n \in \mathbb{N}^*}$ is not summable by (A4), we have $\lim \phi(\|w_n - \bar{w}\|) = 0$ a.s. Consequently, there exists a subsequence $(k_n)_{n \in \mathbb{N}^*}$ such that $\phi(\|w_{k_n} - \bar{w}\|) \to 0$ a.s, which implies that $\|w_{k_n} - \bar{w}\| \to 0$ a.s. In view of (i), we get $w_{k_n} \to \bar{w}$ a.s.

(iii): By Proposition 4.6(i), $\mathbb{E}[\|w_n - \bar{w}\|^2]$ converges, hence it is uniformly bounded, and $\lim \mathbb{E}[\|Bw_n - B\bar{w}\|^2] = 0$, and hence there exists a subsequence $(k_n)_{n \in \mathbb{N}^*}$ such that
\[
\lim_{n \to \infty} \mathbb{E}[\|Bw_{k_n} - B\bar{w}\|^2] = 0.
\] (4.28)

Therefore, there exists a subsequence $(p_n)_{n \in \mathbb{N}^*}$ of $(k_n)_{n \in \mathbb{N}^*}$ such that
\[
\|Bw_{p_n} - B\bar{w}\|^2 \to 0 \quad \text{almost surely.}
\] (4.29)

Let $\tau$ be a weak cluster point of $(w_{p_n})_{n \in \mathbb{N}^*}$, then there exists a subsequence $(w_{q_p})_{n \in \mathbb{N}^*}$ such that for almost all $\omega$, $w_{q_p}(\omega) \to \tau(\omega)$. Since $B$ is weakly continuous, for almost all $\omega$, $Bw_{q_p}(\omega) \to B\tau(\omega)$. Therefore, for almost all $\omega$, by (4.29), $B\bar{w} = B\tau(\omega)$, and hence $\langle B\tau(\omega) - B\bar{w} | \tau(\omega) - \bar{w}\rangle = 0$. Since $B$ is strictly monotone at $\bar{w}$, we obtain, $\bar{w} = \tau(\omega)$. This shows that $w_{q_p} \to \bar{w}$. Defining $(t_n)_{n \in \mathbb{N}^*}$ by setting $t_n = q_p$, the statement follows.

Remark 4.8. Under the same assumptions as in Theorem 4.7, suppose in addition that $B$ is strictly monotone at $\bar{w}$. We can prove almost sure weak convergence to a solution of a subsequence $(w_{t_n})_{n \in \mathbb{N}^*}$ of $(w_n)_{n \in \mathbb{N}^*}$ in the following two cases:

(i) $\mathcal{H}$ is finite dimensional, indeed in this case the weak and strong topology coincide, and therefore $B$ is weakly continuous.

(ii) $B$ is bounded and linear, since in this case $B$ is weakly continuous. For instance, this covers the case of regularized quadratic minimization on Hilbert spaces.
4.3. Convergence rate. In this section we focus on convergence in expectation. We provide some results for the case when either $A$ or $B$ is strongly monotone. We are able to derive an explicit convergence rate for $\mathbb{E}[\|w_n - \overline{w}\|^2]$ similarly to what has been done for smooth minimization in the finite dimensional case [3, Theorem 1]. In the following theorem we will consider the following assumption.

**Assumption 4.9.** Let $\overline{w}$ be a solution of Problem 4.1. Furthermore, suppose that $A$ is $\nu$-strongly monotone and $B$ is $\mu$-strongly monotone at $\overline{w}$, for some $(\nu, \mu) \in [0, +\infty]^2$ such that $\nu + \mu > 0$.

**Theorem 4.10.** Assume that conditions (A1), (A2), (A3), and (4.9) are satisfied. Let $(w_n)_{n \in \mathbb{N}^*}$ be the sequence generated by Algorithm 4.2 with $\inf_{n \in \mathbb{N}^*} \lambda_n \geq \frac{\lambda}{\tau c} > 0$, $\sup_{n \in \mathbb{N}} \nu_n \leq \bar{\alpha} < +\infty$ and $\gamma_n = c_1 n^{-\theta}$ for some $\theta \in [0, 1]$ and for some $c_1 \in [0, +\infty]$. Set $t = 1 - 2^{\theta - 1} \geq 0$, $c = 2 c_1 \Delta (\nu + \mu \varepsilon)/(1 + \nu)^2$, $\tau = 2 \sigma^2 c_1^2 (1 + t\|B\overline{w}\|)/c^2$ and let $n_0$ be the smallest integer such that $(\forall n \geq n_0 > 1)\max\{c_1\} n^{-\theta} \leq 1$. Then, by setting

$$s_n = \mathbb{E}[\|w_n - \overline{w}\|^2],$$

we have, for every $n \geq 2n_0$,

$$s_{n+1} \leq \left\{ \begin{array}{ll}
\tau^2 c \varphi_1 - \theta (n) + s_n \exp \left( \left( \frac{\alpha_0 n}{1 - \theta} \right) \right) \exp \left( -\frac{ct(n + 1)^{1-\theta}}{1 - \theta} \right) + \tau^2 c \varphi_1 (n) & \text{if } \theta \in [0, 1], \\
\frac{n_0}{n + 1} c + \frac{\tau c^2}{(n + 1)^c} (1 + 1/n_0)^c \varphi_1 (n) & \text{if } \theta = 1.
\end{array} \right. \quad (4.30)$$

Therefore, if $\theta = 1$ and $c_1$ is chosen such that $c > 1$, then

$$\mathbb{E}[\|w_n - \overline{w}\|^2] = O(n^{-1}). \quad (4.31)$$

**Proof.** Since $\mu + \nu > 0$, then $A + B$ is strongly monotone at $\overline{w}$. Hence, problem (4.1) has a unique solution, i.e., $\text{zer}(A + B) = \{\overline{w}\}$. Moreover, since $\gamma_n A$ is $\gamma_n \nu$-strongly monotone, by [5, Proposition 23.11] $J_{\gamma_n A}$ is $(1 + \gamma_n \nu)$-cocoercive, and then

$$\langle y_n - \overline{w}, J_{\gamma_n A}(w_n - \gamma_n \mathfrak{B}_n) - J_{\gamma_n A}(\overline{w} - \gamma_n \mathfrak{B}_n) \rangle^2 \leq \frac{1}{(1 + \gamma_n \nu)^2} \| (w_n - \overline{w}) - \gamma_n (\mathfrak{B}_n - B\overline{w}) \|^2. \quad (4.32)$$

Next, proceeding as in the proof of Proposition 4.6 and recalling (4.10)-(4.11), we obtain

$$\mathbb{E}[\|y_n - \overline{w}\|^2] \leq \frac{1}{(1 + \gamma_n \nu)^2} \left( \mathbb{E}[\|w_n - \overline{w}\|^2] - 2 \gamma_n \left( 1 - \frac{1 + 2 \sigma^2 \alpha_n}{\beta} \right) \right).$$

$$\cdot \mathbb{E}(\|w_n - \overline{w}\|) \right) + \frac{2 \gamma_n^2 \sigma^2 (1 + \alpha_n \|B\overline{w}\|)}{n \overline{w}^2}. \quad (4.33)$$

Since $B$ is strongly convex of parameter $\mu$,

$$\langle Bw_n - B\overline{w} | w_n - \overline{w} \rangle \geq \mu \|w_n - \overline{w}\|^2. \quad (4.34)$$

Therefore, from (4.33), using the $\mu$-strong convexity of $B$ at $\overline{w}$ and recalling the definition of $\varepsilon$, we get

$$\mathbb{E}[\|y_n - \overline{w}\|^2] \leq \frac{1}{(1 + \gamma_n \nu)^2} \left( (1 - 2 \gamma_n \mu \varepsilon) \mathbb{E}[\|w_n - \overline{w}\|^2] + 2 \gamma_n^2 \chi_n^2 \right). \quad (4.35)$$

Hence, by definition of $w_{n+1}$,

$$\mathbb{E}[\|w_{n+1} - \overline{w}\|^2] \leq \left( 1 - \frac{\lambda_n \gamma_n (2 \nu + \gamma_n \nu^2 + 2 \mu \varepsilon)}{(1 + \gamma_n \nu)^2} \right) \mathbb{E}[\|w_n - \overline{w}\|^2] + \frac{2 \sigma^2 \chi_n^2}{(1 + \gamma_n \nu)^2}. \quad (4.36)$$
Now, let $\gamma_n = c_1 n^{-\theta}$ and fix $n \geq n_0$. Since $\gamma_n \leq \gamma_n \leq c_1 n_0^{-\theta} \leq 1$, we have
\[
\frac{\lambda_n \gamma_n (2\nu + \gamma_n \nu^2 + 2\mu)}{(1 + \gamma_n \nu)^2} \geq \frac{2\lambda (\nu + \mu \varepsilon)}{(1 + \nu)^2} \gamma_n = cn^{-\theta},
\] (4.37)
where we set $c = c_1 \lambda (\nu + \mu \varepsilon)/(1 + \nu)^2$. On the other hand,
\[
\frac{2\sigma \gamma_n^2}{(1 + \gamma_n \nu)^2} \leq 2\sigma^2 (1 + \|B\|) c_1 n^{-2\theta}. \tag{4.38}
\]
Then, putting together (4.36), (4.37), and (4.38), we get
\[
\mathbb{E}[\|w_{n+1} - \widehat{w}\|^2] \leq (1 - \eta_n)\mathbb{E}[\|w_n - \widehat{w}\|^2] + \eta_n^2,
\] (4.39)
with $\tau = 2\sigma^2 c_1^2 (1 + \|B\|)/c^2$ and $\eta_n = cn^{-\theta}$. Then, (4.30) follows from Lemma 3.4.

**Remark 4.11.** Note that we focused on convergence of the iterates. Indeed, differently from the case of optimization, where convergence of function values can be studied, in the general case of monotone inclusions such a quantity cannot be defined. In some cases, for instance for variational inequalities, a merit function can be introduced [2]. This merit function has been used to quantify the inaccuracy of an approximation of the solution in [25].

5. **Special cases.** In this section, we study two special instances of Problem 4.1, namely variational inequalities and minimization problems. Moreover, for variational inequalities, we prove a new result, showing that a suitably defined merit function goes to zero when it is evaluated on the iterates of stochastic forward-backward algorithm.

5.1. **Variational Inequalities.** In this section we focus on a special case of Problem 4.1, assuming that $A$ is the subdifferential of $G \in \Gamma_0(H)$.

**Problem 5.1.** Let $B : H \rightarrow H$ be a $\beta$-cocoercive operator, for some $\beta \in [0, \infty[$, let $G$ be a function in $\Gamma_0(H)$. The problem is to solve the following variational inequality [33, 34, 5]
\[
\text{find } \widehat{w} \in H \text{ such that } \langle \forall w \in H \rangle \| \widehat{w} - w \| B\widehat{w} + G(\widehat{w}) \leq G(w), \tag{5.1}
\]
under the assumption that (5.1) has at least one solution. In the setting of Problem 5.1, let $(\mathfrak{B}_n)_{n \in \mathbb{N}^\ast}$ be a random process taking values in $H$, $(\forall n \in \mathbb{N}^\ast)$ $\gamma_n = c_1 n^{-\theta}$ for some $\theta \in [0, 1]$ and for some $c_1 \in [0, \infty[$. Let $(\lambda_n)_{n \in \mathbb{N}^\ast}$ be a sequence in $[\underline{\lambda}, 1]$ for some $\underline{\lambda} \in [0, \infty[$. Set
\[
(\forall n \in \mathbb{N}^\ast) \quad \begin{align*}
  z_n &= w_n - \gamma_n \mathfrak{B}_n \\
  y_n &= \text{prox}_{\tau_n G} z_n \\
  w_{n+1} &= (1 - \lambda_n)w_n + \lambda_n y_n.
\end{align*} \tag{5.2}
\]
The following assumption will be used in the next corollary.

**Assumption 5.2.** Let $\widehat{w}$ be a solution of (5.1). Suppose that $G$ is $\nu$-strongly convex and $B$ is $\mu$-strongly monotone at $\widehat{w}$, for some $(\nu, \mu) \in [0, \infty[^2$ such that $\nu + \mu > 0$.

Note that, while on $B$ we can assume a local strong monotonicity property, on the function $G$ we need a global assumption.

**Corollary 5.3.** Assume that conditions (A1), (A2), (A3) and (5.2) are satisfied, with $\sup_{n \in \mathbb{N}^\ast} \alpha_n \leq \bar{\alpha} < \infty$. Let $(u_n)_{n \in \mathbb{N}^\ast}$ be the sequence defined by (5.2). Set $t = 1 - 2^{\theta-1} \geq 0$, $c = 2c_1 \lambda (\nu + \mu \varepsilon)/(1 + \nu)^2$, etc.
\[ \tau = 2\sigma^2 c_1^2 (1 + \mathbb{E}[B\mathbb{E}[|]])/c^2 \] and let \( n_0 \) be the smallest integer such that \((\forall n \geq n_0 > 1)\) \(\max\{c, c_1\} n^{-\theta} \leq 1\). Then \((4.30)\) holds. In particular, if \( \theta = 1 \) and \( c_1 \) is chosen such that \( c > 1 \), then
\[
\mathbb{E}[||w_n - \overline{w}||^2] = O(n^{-1}).
\]

Proof. Set \( A = \partial G \). Then Problem 5.1 reduces to a particular case of Problem 4.1. Hence, the results follow from Theorem 4.10. \(\Box\)

**Corollary 5.4.** In the same setting as in Corollary 5.3, suppose that \( (A4) \) is also satisfied. Then

(i) If \( B \) is uniformly monotone at \( \overline{w} \), then \( w_n \to \overline{w} \) a.s.

(ii) If \( B \) is strictly monotone at \( \overline{w} \) and weakly continuous, then there exists a subsequence \((w_{n_k})_{n \in \mathbb{N}}\)

\( \text{such that} \ w_{n_k} \to \overline{w} \) a.s.

(iii) If \( B \) is strictly monotone, and either \( \mathcal{H} \) is finite dimensional space or \( B \) is a bounded and linear, then there exists a subsequence \((w_{n_k})_{n \in \mathbb{N}}\)

\( \text{such that} \ w_{n_k} \to \overline{w} \) a.s.

Proof. The results follow from Theorem 4.7. \(\Box\)

When \( G \) is the indicator function of a non-empty, closed, convex subset \( C \) of \( \mathcal{H} \), Problem 5.1 reduces to the problem of solving a classic variational inequality \([35, 33]\), namely to find \( \overline{w} \) such that
\[
(\forall w \in C) \quad \langle B\overline{w} - w, w - \overline{w} \rangle \leq 0.
\]

Proximal algorithms are often used to solve this problem, see \([5, \text{Chapter 25}]\) and references therein. When \( B \) is accessible only through a stochastic oracle, the available methods are limited. In \([12]\) a smoothing sample average approximation method is analyzed when \( B \) can be written as an expectation. An iterative Tikhonov regularization method, based on an iterative projected scheme is studied in \([27]\). Recently, a stochastic mirror-prox algorithm has been proposed in \([25]\) for the case when \( C \) is a nonempty, compact, convex subset of \( \mathbb{R}^d \), and \( B \) is only Lipschitz continuous. We also remark that in \([24]\) a almost sure convergence of a forward-backward splitting algorithm with respect to a noneuclidean metric is studied in a finite dimensional space, when \( B \) is given as an expectation.

Note that, by \([11, \text{Lemma 1}]\), since cocoercivity of \( B \) implies Lipschitz continuity, \( \overline{w} \) is a solution of (5.3) if and only if
\[
(\forall w \in C) \quad \langle Bw - \overline{w}, w - \overline{w} \rangle \leq 0.
\]

As it has been done in \([25]\), it is therefore natural to quantify the inaccuracy of a candidate solution \( u \in \mathcal{H} \) by
\[
V(u) = \sup_{w \in C} \langle Bw - u, w - u \rangle.
\]

In particular, note that \((\forall u \in \mathcal{H})\), \( V(u) \geq 0 \) and \( V(u) = 0 \) if and only \( u \) is a solution of (5.4). We will consider convergence properties of the following iteration, which differs from the on in ?? only by the averaging step.

**Algorithm 5.5.** Let \( C \) be a nonempty bounded closed convex subset of \( \mathcal{H} \). Let \((\gamma_t)_{t \in \mathbb{N}^*}\) be a sequence in \([0, +\infty[\), \((\lambda_t)_{t \in \mathbb{N}^*}\) be a sequence in \([0, 1] \), and let \((\mathcal{B}_t)_{t \in \mathbb{N}^*}\) be a \( \mathcal{H} \)-valued random process. Fix \( w_1 \in \mathcal{H} \) and set
\[
(\forall n \in \mathbb{N}^*)
\begin{align*}
  z_t &= w_t - \gamma_t \mathcal{B}_t, \\
  y_t &= P_C z_t, \\
  w_{t+1} &= (1 - \lambda_t) w_t + \lambda_t y_t, \\
  \overline{w}_n &= \left( \sum_{t=1}^n \gamma_t \lambda_t w_t \right) / \sum_{t=1}^n (\gamma_t \lambda_t).
\end{align*}
\]
The next theorem gives an estimate of the function $V$ when evaluated on the expectation of $\bar{w}_n$. Note that we do not impose any additional monotonicity property on $B$.

**Theorem 5.6. (Ergodic convergence)** In the setting of problem (5.1), assume that $G = \iota_C$ for some nonempty bounded closed convex set $C$ in $\mathcal{H}$. Let $(\bar{w}_n)_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 5.5 and assume conditions (A1) and (A2) hold. Set

$$
\theta_0 = \sup_{u \in C} ||w_1 - u||^2 \quad \text{and} \quad \theta_{1,n} = \sum_{t=1}^{n} (\lambda_t \gamma_t^2 (1 + \sigma^2 \alpha_t) E[||Bw_t||^2] + \sigma^2 \lambda_t \gamma_t^2),
$$

then

$$
V(E[\bar{w}_n]) \leq (\theta_0 + \theta_{1,n}) \left( \sum_{t=1}^{n} \lambda_t \gamma_t \right)^{-1}.
$$

(5.7)

Moreover, if the conditions (A3) and (A4) are satisfied, then $\lim_{n \to +\infty} V(E[\bar{w}_n]) = 0$. In particular, if $(\forall t \in \mathbb{N}^*) \lambda_t = 1$ and $\gamma_t = t^{-\theta}$ for some $\theta \in [1/2, 1]$, we get

$$
V(E[\bar{w}_n]) = O(n^{1-\theta}).
$$

(5.8)

**Proof.** Since $C$ is a non-empty closed convex set, $P_C$ is non-expansive, and for every $u \in C$, $u = P_C u$. Hence, from the convexity of $|| \cdot ||^2$

$$
(\forall t \in \mathbb{N}^*)(\forall u \in C) \quad ||w_{t+1} - u||^2 \leq (1 - \lambda_t)||w_t - u||^2 + \lambda_t ||y_t - u||^2
$$

(5.9)

(5.10)

We derive from conditions (A1) that

$$
E[\langle w_t - u | \mathcal{F}_t \rangle | \mathcal{F}_t] = \langle w_t - u | Bw_t \rangle
$$

(5.11)

and from (A3) that

$$
E[||B_t||^2 | \mathcal{F}_t] \leq 2E[||B_t - Bw_t||^2 | \mathcal{F}_t] + 2E[||Bw_t||^2 | \mathcal{F}_t]
$$

$$
\leq 2||Bw_t||^2 + 2\sigma^2 (1 + \alpha_t)||Bw_t||^2.
$$

(5.12)

Therefore, by (5.10) and monotonicity of $B$, for every $t \in \mathbb{N}^*$ and every $u \in C$, $\langle w_t - u | Bu - Bw_t \rangle \leq 0$, we obtain

$$
2\lambda_t \gamma_t \langle w_t - u | Bu \rangle \leq ||w_t - u||^2 - E[||w_{t+1} - u||^2 | \mathcal{F}_t]
$$

$$
+ 2\lambda_t \gamma_t^2 (1 + \sigma^2 \alpha_t)||Bw_t||^2 + 2\sigma^2 \lambda_t \gamma_t^2,
$$

(5.13)

which implies that

$$
2E[\langle \bar{w}_n - u | Bu \rangle] \leq \left( \sum_{t=1}^{n} \lambda_t \gamma_t \right)^{-1} \sum_{t=1}^{n} \left( E[||w_t - u||^2] - E[||w_{t+1} - u||^2] \right.
$$

$$
+ 2\lambda_t \gamma_t^2 (1 + \sigma^2 \alpha_t) E[||Bw_t||^2] + 2\sigma^2 \lambda_t \gamma_t^2
$$

$$
\leq \left( \sum_{t=1}^{n} \lambda_t \gamma_t \right)^{-1} \left( E[||w_1 - u||^2] + \sum_{t=1}^{n} \left( 2\lambda_t \gamma_t^2 (1 + \sigma^2 \alpha_t) E[||Bw_t||^2] + 2\sigma^2 \lambda_t \gamma_t^2 \right) \right).
$$

(5.14)
Therefore,
\[
\sup_{u \in C} \mathbb{E}[\langle \overline{w}_n - u, Bu \rangle] \leq (\theta_0 + \theta_{1,n}) \left( \sum_{t=1}^{n} \lambda_t \gamma_t \right)^{-1},
\]
which proves (5.8). Finally, since $C$ is bounded, $\theta_0 < +\infty$. Now, assume that (A3) and (A4) are satisfied. Then, (A4) implies that $\sum_{t=1}^{+\infty} \lambda_t \gamma_t = +\infty$, therefore it is enough to prove that $\theta_0 + \theta_{1,n}$ is uniformly bounded.

To this aim, since we derive from (A4) that $\sum_{t=1}^{+\infty} \lambda_t \gamma_t^2 < +\infty$ and $\sum_{t=1}^{+\infty} \lambda_t \gamma_t^2 \alpha_t < +\infty$, we are left to prove that $(\mathbb{E}[\|Bu_t\|^2])_{t \in \mathbb{N}^*}$ is bounded. This directly follows from Proposition 4.6(i). The last assertion is a direct consequence of (5.8) when evaluated for $(\forall t \in \mathbb{N}^*) \gamma_t = t^{-\theta}$ and $\lambda_t = 1$. \(\square\)

**Remark 5.7.** As we mentioned before, when $\dim \mathcal{H}$ is finite, $C$ is a non-empty convex, compact subset of $\mathcal{H}$, and $B$ is bounded, an alternative method to solve Problem 5.1 can be found in [25], where $\alpha_n = 0$ and the assumption of cocoercivity of $B$ is replaced by the weaker Lipschitz continuity assumption. Note that, with respect to forward-backward, the mirror-prox algorithm proposed in [25], requires two projections per iteration, rather than one. With such procedure, in [25] it is proved that $\mathbb{E}[V(\overline{w}_n)]$ goes to zero. Note that in general $\mathbb{E}[V(\overline{w}_n)] \leq \mathbb{E}[V(\overline{w}_n)]$.

**5.2. Minimization problems.** In this section, we show that the results in [47] can be obtained as a corollary of the results in the previous sections specialized to minimization problems.

**Problem 5.8.** Let $\beta \in [0, +\infty[, \ G \in \Gamma_0(\mathcal{H})$, and let $L: \mathcal{H} \to \mathbb{R}$ be a convex differentiable function, with a $\beta^{-1}$-Lipschitz continuous gradient. The problem is to
\[
\min_{w \in \mathcal{H}} \Phi(w) = L(w) + G(w),
\]
under the assumption that the set of solution to (5.16) is non-empty.

In the setting of Problem 5.8, let $(\forall n \in \mathbb{N}^*) \gamma_n = c_1 n^{-\theta}$ for some $\theta \in [0, 1]$ and for some $c_1 \in [0, +\infty[$. Let $(\lambda_n)_{n \in \mathbb{N}^*}$ be a sequence in $[\underline{\lambda}, 1]$ for some $\underline{\lambda} \in [0, +\infty[$. Define
\[
(\forall n \in \mathbb{N}^*) \quad \begin{align*}
   z_n &= w_n - \gamma_n \mathfrak{B}_n \\
   y_n &= \text{prox}_{\gamma_n G} z_n \\
   w_{n+1} &= (1 - \lambda_n) w_n + \lambda_n y_n,
\end{align*}
\]
where $(\mathfrak{B}_n)_{n \in \mathbb{N}^*}$ is a random process taking values in $\mathcal{H}$.

The following results from [47] are special cases of Corollary 5.3 and Corollary 5.4 when $B$ is the gradient of the smooth component $L$.

**Corollary 5.9.** Assume that conditions (A1), (A2) and (A3) are satisfied with $B = \nabla L$ and $\sup_{n \in \mathbb{N}^*} \alpha_n \leq \overline{\alpha} < +\infty$. Let $\overline{w}$ be a solution of Problem 5.8. Suppose that $G$ is $\nu$-strongly convex and $L$ is $\mu$-strongly convex at $\overline{w}$ for some $(\nu, \mu) \in [0, +\infty]^2$ such that $\nu + \mu > 0$. Set $t = 1 - 2^{\theta - 1} \geq 0$, $c = 2c_1 \Delta(\nu + \mu) / (1 + \nu)^2$, $\tau = 2\sigma^2 c_1^2 (1 + \nu \|\nabla L(\overline{w})\|) / c^2$ and let $n_0$ be the smallest integer such that $(\forall n \geq n_0 > 1) \max\{c, c_1\} n^{-\theta} \leq 1$. Then (4.30) holds. In particular, if $\theta = 1$ and $c_1$ is chosen such that $c > 1$, then
\[
\mathbb{E}[\|w_n - \overline{w}\|^2] = O(n^{-1}).
\]

**Proof.** Set $B = \nabla L$ and $A = \partial G$. The results follow from Corollary 5.3, Lemma 3.6 and the fact that $(\forall n \in \mathbb{N}^*) J_{\gamma_n A} = \text{prox}_{\gamma_n G}$. \(\square\)

**Corollary 5.10.** Under the same assumptions as in Corollary 5.9, suppose that (A4) is also satisfied. Then
(i) If \( L \) is uniformly convex at \( \overline{w} \), then \( w_n \to \overline{w} \) a.s.

(ii) If \( L \) is strictly convex at \( \overline{w} \) and \( \nabla L \) is weakly continuous, then there exists a subsequence \( (w_{n_k})_{k \in \mathbb{N}} \) such that \( w_{n_k} \to \overline{w} \) a.s.

(iii) If \( L \) is strictly convex at \( \overline{w} \), and either \( \mathcal{H} \) is finite dimensional space or \( L \) is a bounded and linear, then there exists a subsequence \( (w_{n_k})_{k \in \mathbb{N}} \) such that \( w_{n_k} \to \overline{w} \) a.s.

Proof. Set \( B = \nabla L \) and \( A = \partial G \). The results follow from Corollary 5.4, Lemma 3.6 and the fact that \((\forall n \in \mathbb{N}^+)\ J_{\gamma n A} = \text{prox}_{\gamma_n G} \).

In the case when \( G \) is the indicator function of a non empty closed convex set and \((\forall n \in \mathbb{N}^+)\ \lambda_n = 1\), a similar result on the rate of convergence of \( \mathbb{E}[\|w_n - \overline{w}\|] \) has been obtained in \cite{53}, under similar assumptions to (A1), . . . , (A4) for the case where \( L \) is strongly convex, under the additional assumption of boundedness of the conditional expectations of \((\|\mathfrak{B}_n - \nabla L(w_n)\|^2)_{n \in \mathbb{N}^+}\).

Corollary 5.9 is the extension to the nonsmooth case of \cite[Theorem 1]{3}, in particular, when \( G = 0 \), we obtain the same convergence rate. Note however that the assumptions on the stochastic approximations of the gradient of the smooth part are slightly different. In particular, we replace the boundedness condition at the solution and the Lipschitz continuity assumption on \( \mathfrak{B}_n \) with assumption (A2). We briefly contrast our results with those in \cite{20}. This latter paper considers convergence of the average of the iterates with respect to the function values and assume \( \mathcal{H} \) to be finite dimensional. Also uniform boundedness of the iterations and the subdifferentials are required. Our convergence results consider convergence of the iterates (with no averaging) and hold in an infinite dimensional setting, without boundedness assumptions. The non asymptotic rate \( O(n^{-1}) \) which we obtain for the iterates improves the \( O((\log n)/n) \) rate derived from \cite[Corollary 10]{20} for the average of the iterates. However, it should be noted that convergence of the objective values is studied in \cite{20} also for the non strongly convex case. Assumption (A2) has been considered in the context of stochastic gradient descent in \cite{9}. Note that under such a condition, the variance of the stochastic approximation is allowed to grow with \( \|\nabla L(w_n)\| \) and therefore can be unbounded.

As we mentioned in the introduction, the study of almost sure convergence has a long history. However, most paper consider only the finite dimensional setting. An analysis of a stochastic projected subgradient algorithm in an infinite dimensional Hilbert space can be found in \cite{4}. Corollary 5.10 can be seen as an extension of \cite[Theorem 3.1]{4}, where the case where \( G \) is an indicator function is considered.

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