Matrix bispectrality and noncommutative algebras: beyond the prolate spheroidals

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Dedicated to Vaughan Jones, a truly inspirational friend, who left us some great memories.

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1 Preamble

The bispectral problem discussed in this paper has its origin in Duistermaat and Grünbaum (1986). It is motivated, as mentioned there, by an effort to understand and extend a remarkable phenomenon in Fourier analysis on the real line: the operator of time-and-band limiting is an integral operator admitting a second order differential operator with simple spectrum in its commutator. This property, which gives a good numerical way to compute the eigenfunctions of the integral operator, was put to good use in a series of papers by D. Slepian, H. Landau and H. Pollak at Bells Labs back in the 1960’s, see Slepian and Pollak (1961); Landau and Pollak (1962, 1961, 1962); Slepian (1964, 1978, 1983, 1976) and is of interest in one other contribution in this issue, see Connes and Consani (2021); Connes and Marcolli (2008); Connes and Moscovici (2021). We are thankful to Luc Haine, of Louvain-la-Neuve, Belgium who alerted one of us (AG) on Nov 2021 about a talk by Alain Connes in the series Mathematical Picture Language. This talk delivered on Dec 7th. 2021 can be seen in Youtube and covers some of the contents of his work with various collaborators on the zeta function and its relations to the prolate spheroidal functions. For readers interested in this fascinating connection there is no better way of learning about this than watching the lecture. See Connes (2021).

If one tries to extend this property beyond the Fourier case by adding a potential $V(x)$ to the operator $-\left(\frac{d}{dx}\right)^2$ and replacing an expansion in exponentials by an expansion in term of the eigenfunctions $\psi(x, z)$ of $-\left(\frac{d}{dx}\right)^2 + V(x)$ it appears plausible that a certain property of these eigenfunctions $\psi(x, z)$ could play a useful role. This property is now known as the bispectral property which will be formulated below. Its solution in the scalar case was the purpose of Duistermaat and Grünbaum (1986).

The version of the bispectral problem that we discuss in this paper is a noncommutative one, obtained by allowing all objects in the original formulation to be matrix valued. The details are given later.

Now for a bit of history: one of us (AG) gave a couple of talks, one at Vanderbilt in Nov 2013 and one in IMPA, Rio de Janeiro in March 2014 . The first talk was at the invitation of Vaughan Jones and started by saying that the topic was most likely of no interest to him. At some point in the talk Vaughan said “all of this is about bi-modules and subfactors”. The second talk had both Vaughan and another one of us (JZ) in the audience, there was some more mention of these
topics and then in Grünbaum (2014), written in April 2014, a reference is made to a future joint paper with Vaughan "On the bimodule structure of the bispectral problem, in preparation". The occasion of this second talk was a visit that Vaughan (and for part of it with Wendy) did to Argentina, Uruguay and Brazil. In Buenos Aires he delivered one lecture, in Rio he delivered a series of talks but in Montevideo besides giving a lecture he was received in a private audience by President Jose Mujica, described as "the world humblest head of state" by Wikipedia. Vaughan whose command of the spanish language was quite high lamented that he had troubles understanding President Mujica.

The examples in this short paper Grünbaum (2014) were discussed with Vaughan who showed some level of interest but we never managed to get him fully on board. Of course we assumed that we always had time to get him onto this project. The short paper Grünbaum (2014) makes three separate conjectures going with three different examples. These conjectures were proved to be correct in the thesis of the third of us (BVC) under the supervision of (JZ).

And now that sadly we have no chance of benefiting from the insights that Vaughan would have brought in, we present the problem and some results to a wider audience in the hope that someone may surmise what it was that Vaughan had in mind. Getting someone involved in this effort would be a nice way of honoring his memory.

2 Commuting integral and differential operators

The bispectral problem, introduced in the next section is motivated by the following very concrete problem in signal communication: a signal of support in the interval \([-T, T]\) is transmitted over a channel that has bandwidth \([-W, W]\) i.e., all frequencies in the signal beyond absolute value of \(W\) cannot be sent over. A mathematical formulation is as follows: an arbitrary signal in \(L^2(\mathbb{R})\) is chopped to the interval \([-T, T]\) and then its Fourier transform is chopped to the interval \([-W, W]\). Denoting for simplicity these two chopping operations by \(T\) and \(W\) we are dealing with the operator

\[
E = WFT
\]

where \(F\) stands for the Fourier transform. The spectral analysis of this operator, i.e., a look at its singular functions and singular values requires the consideration of the operator \(E^*E\). It is easy to see that \(E^*E\) is an integral operator acting in \(L^2(-T,T)\) whose kernel is given by

\[
\frac{\sin W(t-s)}{t-s}
\]

and this bounded operator acts on a function in the space \(L^2(-T,T)\) by

\[
(Kf)(s) = \int_{-T}^{T} \frac{\sin W(t-s)}{t-s} f(t)dt
\]

for \(f \in L^2(-T,T)\) and \(s \in (-T, T)\). This \(K\) commutes with the operator

\[
(Lf)(x) = \left(-\frac{d}{dx}\right)(T^2 - x^2) \left(\frac{df}{dx}\right) + W^2 x^2 f(x)
\]

defined on \(C^2\) functions.

One can show that this densely defined operator has a unique selfadjoint extension in \(L^2(-T,T)\) with eigenfunctions and eigenvalues that depend on the parameter \(W\).

Its eigenfunctions are known as the prolate spheroidal wave functions, since this is one of the differential operators resulting in separating variables when solving for the eigenfunctions of the Laplacian on a prolate spheroid.

What other naturally appearing integral operators allow for commuting differential operators? Two other examples are the Bessel and Airy kernels, as in the work of Tracy and Widom, Tracy and Widom (1994a,b) in the context of Random Matrix Theory. For the Bessel case see also Slepian (1964); Grünbaum et al. (1982). There are other examples, but the search is nowhere close to finished. The bispectral property, to be formulated below was put forward in Duistermaat and Grünbaum (1986) as an important ingredient in the search for more examples of this commuting property.

3 The bispectral problem

The problem was posed and solved in Duistermaat and Grünbaum (1986). It is as follows:
Find all nontrivial instances where a function $\psi(x, z)$ satisfies

$$L \left( x, \frac{d}{dx} \right) \psi(x, z) \equiv \left( -\left( \frac{d}{dx} \right)^2 + V(x) \right) \psi(x, z) = z\psi(x, z)$$

as well as

$$B \left( z, \frac{d}{dz} \right) \psi(x, z) \equiv \left( \sum_{i=0}^{M} b_i(z) \left( \frac{d}{dz} \right)^i \right) \psi(x, z) = \Theta(x) \psi(x, z).$$

All the functions $V(x), b_i(k), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here $M$ is arbitrary (finite). The operator $L$ could be of higher order, but in Duistermaat and Grünbaum (1986), attention is restricted to order two.

The complete solution is given as follows:

**Theorem** If $M = 2$, then $V(x)$ is (except for translation) either $\frac{c}{x^2}$ or $ax$, i.e. we have a Bessel or an Airy case. If $M > 2$, there are two families of solutions

a) $L$ is obtained from $L_0 = -\left( \frac{d}{dx} \right)^2$ by a finite number of Darboux transformations ($L = AA^* \rightarrow \tilde{L} = A^* A$). In this case $V$ is a rational solution of the Korteweg-deVries hierarchy of equations. Here $A$ is a first order differential operator.

b) $L$ is obtained from $L_0 = -\left( \frac{d}{dx} \right)^2 + \frac{1}{4x^2}$ after a finite number of (rational) Darboux transformations.

In all cases we have a solution of the ad-conditions, a complicated system of nonlinear differential equations. These conditions are necessary and sufficient.

Notice that the solutions organize themselves into nice manifolds.

The simplest example of case a) follows from $L_0 = -\left( \frac{d}{dx} \right)^2$ by two Darboux transformations, one gets the operator

$$L_2 = -\left( \frac{d}{dx} \right)^2 + \frac{6(x^4 + 12t_3x)}{(x^3 - t_3)^2}.$$ 

In this case $\Theta(x) = x^4 - 4t_3x$ and the differential operator in the spectral parameter is

$$B_2 \left( z, \frac{d}{dz} \right) = \left( -\left( \frac{d}{dz} \right)^2 + \frac{6}{z^2} \right)^2 + 4it_3 \left( \frac{d}{dz} \right).$$

The potential in the operator $L_2 = -\left( \frac{d}{dx} \right)^2 + V(x, t)$ above satisfies the KdV equation.

It was later observed by Magri and Zubelli, see Zubelli and Magrì (1991) that in case b) we are dealing with rational solutions of the Virasoro equations (i.e. master symmetries of KdV). The bigger picture became more apparent in the work Zubelli and Valerio Silva (2000) where it is shown that the generic rational potentials that decay at infinity and remain rational by all the flows of the master-symmetry KdV hierarchy are bispectral potentials for the Schrödinger operator.

In case a) the space of common solutions has dimension one and in case b) it has dimension two. One refers to these as the rank one and rank two situations.

Observe that the “trivial cases” when $M = 2$ are self-dual in the sense that since the eigenfunctions $\psi(x, z)$ are functions either of the product $xz$ or of the sum $x + z$, one gets $B$ by replacing $z$ for $x$ in $L$. The bispectral involution introduced in Wilson (1993) shows how this can be adapted in the “higher order cases”.

3
4 The non-commutative version of the bispectral problem

A first noncommutative (or matrix) version of the bispectral problem was considered in J. Zubelli Ph.D. thesis at Berkeley, see also Zubelli (1990, 1992, Zubelli 1992), in the situation where both the physical space and spectral operators act on the same line of the eigenfunction and the eigenvalues are both scalar valued. Later on, several other versions were considered. See Sakhnovich and Zubelli (2001), Kasman (2015), Bovallian and Liberati (2005), Grünbaum and Iliev (2003), Grünbaum (2014), and references therein. The noncommutative version of the bispectral problem displayed interesting connections with soliton equations as well. Indeed, in Zubelli (1992) it was shown that a large class of rational solutions to the AKNS hierarchy [Ablowitz et al. (1974)] led to matrix differential operators that displayed the bispectral property. Among the important equations in Mathematical Physics that are covered by the AKNS hierarchy one finds the modified KdV and the nonlinear Schrödinger equation. The matrix differential operator that appeared in this case was in turn related to Dirac operators. The connection between bispectrality and another important topic in Mathematical Physics, namely Huygens’ principle in the strict sense [Berest and Veselov (1994)] turned out to appear also in the context of Matrix Bispectrality. Indeed, in Chalub and Zubelli (2006, 2005, 2003), it was shown that rational solutions to the AKNS hierarchy led to matrix differential operators which satisfy Huygens’ principle in the strict sense. In other words, the fundamental solutions of the perturbed Dirac equation in a suitably high space-time dimension had its support precisely on the surface of the light cone and not in its interior. Another interesting connection between Matrix Bispectrality to soliton equations of Mathematical Physics was explored in Sakhnovich and Zubelli (2001).

In this present paper we take the bi-module structure of the problem into account and let the operators act on different sides as well as allow both eigenvalues to be matrix valued.

We consider triplets \((L, \psi, B)\) satisfying the equations

\[
L\psi(x, z) = \psi(x, z)F(z) \quad (\psi B)(x, z) = \theta(x)\psi(x, z)
\]

with \(L = L(x, \frac{\partial}{\partial x}), B = B(z, \frac{\partial}{\partial z})\) linear matrix differential operators, i.e., \(L\psi = \sum_{i=0}^{l} a_i(x) \cdot \left(\frac{\partial}{\partial x}\right)^i \psi, \psi B = \sum_{j=0}^{m} (\frac{\partial}{\partial z})^j \psi \cdot b_j(z)\). The functions \(a_i : U \subset \mathbb{C} \to M_N(\mathbb{C}), b_j : V \subset \mathbb{C} \to M_N(\mathbb{C}), F : V \subset \mathbb{C} \to M_N(\mathbb{C}), \theta : U \subset \mathbb{C} \to M_N(\mathbb{C})\) and the nontrivial common eigenfunction \(\psi : U \times V \subset \mathbb{C}^2 \to M_N(\mathbb{C})\) are in principle compatible sized meromorphic matrix valued functions defined in suitable open subsets \(U, V \subset \mathbb{C}\).

A triplet \((L, \psi, B)\) satisfying (1) is called a bispectral triplet.

The study of the structure of the algebra of possible \(\theta(x)\) going with a fixed bispectral \(\psi(x, z)\) was first raised in Castro and Grünbaum (2006) and analyzed in Tiraô (2011), Grünbaum and Tiraô (2007). See also Casper (2018) and Zurrián (2017).

We consider now the examples and conjectures given in Grünbaum (2014) as well as their validation and further description given in Vasquez and Zubelli (2021).

For the benefit of the reader we give a few definitions before giving some explicit results in the next section.

**Definition 1.** Let \(\mathbb{K}\) be a field, \(C\) be a \(\mathbb{K}\)-algebra, \(A\) a subring of \(C\) and \(S \subset C\). We define

\[
A \cdot < S > := \text{span}_{\mathbb{K}} \left\{ \prod_{j=1}^{n} s_j \mid s_1, \ldots, s_n \in S \cup A, n \in \mathbb{N} \right\},
\]

where the noncommutative product is understood from left to right, i.e., \(\prod_{j=1}^{n+1} s_j := (\prod_{j=1}^{n} s_j) s_{n+1}\), for \(n = 0, 1, 2, \ldots\). For completion, \(\prod_{j=1}^{0} s_j := 1\).

The set \(A \cdot < S >\) is called the subalgebra generated by \(S\) over \(A\) and we call an element \(f \in A \cdot < S >\) a noncommutative polynomial with coefficients in \(A\) and set of variables \(S\).

**Definition 2.** Let \(C\) be a noncommutative ring and \(A\) a subring of \(C\). We say that an element \(\alpha \in C\) is integral over \(A\) if there exists a noncommutative polynomial \(f\) with coefficients in \(A\) such that \(f(\alpha) = 0\). Furthermore, we say that \(\beta \in C\) is integral over \(\alpha \in C\) if \(\beta\) is integral over \(A \cdot < \alpha >\). Finally, \(\alpha\) and \(\beta\) are associated integral if \(\alpha\) is integral over \(\beta\) and \(\beta\) is integral over \(\alpha\).

In order to characterize the algebraic structure of bispectrality in the present noncommutative context, we start with the following definitions.
**Definition 3.** Let $\mathbb{K}$ be a field, we denote by $\mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle$ the free algebra generated by the letters $x_\lambda$, $\lambda \in \Lambda$ i.e.,

$$\mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle = \bigoplus_{F \subseteq \Lambda, \text{finite}} \mathbb{K} \cdot x_\lambda.$$ 

**Definition 4.** Let $A$ be a $\mathbb{K}$-algebra. A presentation for an algebra $A$ is a triple $(\mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle, f, I)$ such that $I \subset A$ is an ideal and $f : \mathbb{K}\langle x_\lambda \mid \lambda \in \Lambda \rangle/I \to A$ is an isomorphism. Furthermore, we say that $A$ is finitely generated if there exists a presentation with $\Lambda$ finite and finitely presented if there exists a presentation with $\Lambda$ finite and the ideal $I$ is generated by finitely many elements.

## 5 Three examples

Take for $\Psi(x, z)$ the matrix valued function

$$\Psi(x, z) = e^{xz} \begin{pmatrix} z - 1/x & 1/x^2 \\ 0 & z - 1/x \end{pmatrix}$$

and consider all instances of matrix-valued polynomials $\theta(x)$ and differential operators $B$ (with matrix coefficients $b_i(z)$) such that

$$\Psi B = \sum_{i=1}^m \left( \frac{d}{dz} \right)^i \Psi b_i = \theta(x) \Psi(x, z).$$

In this case one has

$$L \Psi = -z^2 \Psi$$

with

$$L = -\left( \frac{d}{dx} \right)^2 + 2 \begin{pmatrix} 1/x^2 & -2/x^3 \\ 0 & 1/x^2 \end{pmatrix}.$$ 

In other words for this specific differential operator in the variable $x$ we are asking for all bispectral "partners" of $L$.

One finds that one such pair $(B, \theta)$ is given by

$$B = \left( \frac{d}{dz} \right)^3 - 3 \left( \frac{d}{dz} \right)^2 \cdot \frac{1}{x} + 3 \left( \frac{d}{dz} \right) \cdot \frac{1}{x^2} + 3 \begin{pmatrix} 0 & 1/z^2 \\ 0 & 0 \end{pmatrix}$$

and $\theta(x)$ the scalar-valued polynomial

$$\theta(x) = x^3.$$ 

The set of all possible $\theta(x)$ is given by the following subalgebra $A$. The complete statement is given by

**Theorem 1.** Let $\Gamma$ be the sub-algebra of $M_2(\mathbb{C})[x]$ of the form

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{01} & r_{02} \end{pmatrix} + \begin{pmatrix} r_{11} & r_{12} \\ r_{01} & r_{02} \end{pmatrix} x + \begin{pmatrix} r_{11} & r_{12} \\ r_{01} & r_{02} \end{pmatrix} x^2 + \begin{pmatrix} r_{11} & r_{12} \\ r_{01} & r_{02} \end{pmatrix} x^3 + x^4 p(x),$$

where $p \in M_2(\mathbb{C})[x]$ and all the variables $r_{01}, r_{11}, r_{12}, r_{02}, r_{12}, r_{22}, r_{11}, r_{31}, r_{32}, r_{22} \in \mathbb{C}$. Then $\Gamma = A$. Moreover, for each $\theta$ we have an explicit expression for the operator $B$.

Furthermore, we have the presentation $A = \mathbb{C} \cdot \langle \alpha_0, \alpha_1 \mid I = 0 \rangle$ with the ideal $I$ given by

$$I := \langle \alpha_0^2, \alpha_1^3 + \alpha_0 \alpha_1, 3 \alpha_1 \alpha_0 \alpha_1 + \alpha_0^2 + \alpha_1^2 \rangle.$$ 

This is an example of an algebra with an integral element over a nilpotent one.

For the next example take for $\Psi(x, z)$ the matrix-valued function

$$\psi(x, z) = \left[ \frac{d}{dx} - \begin{pmatrix} 1/x & -1/x^2 & 1/x^3 \\ 0 & 1/x & -1/x^2 \\ 0 & 0 & 1/x \end{pmatrix} \right] e^{xz} I = e^{xz} \begin{pmatrix} z - 1/x & 1/x^2 & -1/x^3 \\ 0 & z - 1/x & 1/x^2 \\ 0 & 0 & z - 1/x \end{pmatrix}.$$
Here one can see that
\[ L\psi = -z^2 \psi \]
with
\[ L = -\left( \frac{d}{dx} \right)^2 + 2 \begin{pmatrix} 1/x^2 & -2/x^3 & 3/x^4 \\ 0 & 1/x^2 & -2/x^3 \\ 0 & 0 & 1/x^2 \end{pmatrix}. \]

The results in this case about the set of all possible \( \theta(x) \) are given below.

**Theorem 2.** Let \( \Gamma \) the sub-algebra of \( M_3(\mathbb{C})[x] \) of the form

\[
\begin{pmatrix}
  r_{11}^0 & r_{12}^0 & r_{13}^0 \\
  0 & r_{22}^0 & r_{23}^0 \\
  0 & 0 & r_{33}^0
\end{pmatrix}
+ \begin{pmatrix}
  r_{11}^1 & r_{12}^1 & r_{13}^1 \\
  0 & r_{22}^1 & r_{23}^1 \\
  0 & 0 & r_{33}^1
\end{pmatrix} x
+ \begin{pmatrix}
  r_{11}^2 & r_{12}^2 & r_{13}^2 \\
  0 & r_{22}^2 & r_{23}^2 \\
  0 & 0 & r_{33}^2
\end{pmatrix} x^2
+ \begin{pmatrix}
  r_{11}^3 & r_{12}^3 & r_{13}^3 \\
  0 & r_{22}^3 & r_{23}^3 \\
  0 & 0 & r_{33}^3
\end{pmatrix} x^3
+ \begin{pmatrix}
  r_{11}^4 & r_{12}^4 & r_{13}^4 \\
  0 & r_{22}^4 & r_{23}^4 \\
  0 & 0 & r_{33}^4
\end{pmatrix} x^4
+ \begin{pmatrix}
  r_{11}^5 & r_{12}^5 & r_{13}^5 \\
  0 & r_{22}^5 & r_{23}^5 \\
  0 & 0 & r_{33}^5
\end{pmatrix} x^5 + x^6 p(x),
\]

where \( p \in M_3(\mathbb{C})[x] \) and all the variables \( r_{11}^0, r_{12}^0, \ldots, r_{33}^5 \in \mathbb{C} \) are arbitrary.

Then, \( \Gamma = \mathbb{A} \) and for each \( \theta \) we have an explicit expression for the operator \( B \).

Furthermore, we have the presentation \( \mathbb{A} = \mathbb{C} \cdot \langle \alpha_2, \alpha_3 \mid I = 0 \rangle \) with
\[ I = \langle \alpha_2^3, \alpha_2^3 - \alpha_3, (\alpha_3\alpha_2)^2 \alpha_3 - 4\alpha_3\alpha_2^2 \alpha_3 \rangle. \]

This is an example of an algebra with nilpotent and idempotent associated elements.

As the last example we consider a case when both "eigenvalues" \( F \) and \( \theta \) are matrix valued. Let
\[
\psi(x, z) = \frac{e^{xz}}{(x - 2)zx} \left( \frac{z^3 - z^2 + z - 1}{x^2 z^2} \right) \]
and
\[
L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \left( \frac{d}{dz} \right)^2 + \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \left( \frac{d}{dz} \right) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
then \( L\psi = \psi F \) with
\[ F(z) = \begin{pmatrix} 0 & 0 \\ 0 & z^2 \end{pmatrix}. \]

It is easy to check that \( \psi B = \theta \psi \) for
\[
B = \begin{pmatrix} 0^3 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} d^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2(z-1) & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -z^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
and
\[
\theta(x) = \begin{pmatrix} x & 0 \\ x^2(x - 2) & x^2 \end{pmatrix}.
\]

In this case we characterize the algebra \( \mathbb{A} \) of all polynomial \( F \) such that there exist \( L = L(x, \frac{d}{dx}) \) with \( L\psi = \psi F \) as follows.
Theorem 3. Let $\Gamma$ be the sub-algebra of $M_2(\mathbb{C})[z]$ of the form

$$\left( \begin{array}{cc} a & 0 \\ b & a \end{array} \right) + \left( \begin{array}{cc} c & e \\ d & -c \end{array} \right) z + \left( \begin{array}{cc} a-b-c & c+a-b \\ d & e \end{array} \right) \frac{z^2}{2} + z^3 p(z),$$

where $p \in M_2(\mathbb{C})[z]$ and all the variables $a, b, c, d, e$ are arbitrary. Then $\Gamma = \mathbb{A}$.

Furthermore, we have the presentation $\mathbb{A} = \mathbb{C} \cdot \langle \theta_1, \theta_3, \theta_4, \theta_5 \mid I = 0 \rangle$ with

$$I = \langle \theta_1^2 - \theta_1, \theta_3, \theta_4, \theta_5 \rangle + \theta_1 \theta_3 + 2 \theta_4 - \theta_5 \theta_1 - \theta_3 \theta_2 - \theta_1 \theta_5 - \theta_3 \theta_4 - \theta_1 \theta_5$$

$$\theta_3 \theta_1 - \theta_1 - \theta_4 - 1 \theta_1 \theta_3 + \theta_5 \theta_4 - \frac{1}{2} \theta_3 \theta_1 + \frac{1}{2} \theta_4 \theta_3 + \theta_5 \theta_1 - 1 \theta_3 \theta_4 + \frac{1}{2} \theta_5 \theta_4 - 1 \theta_3 \theta_5$$

$$\theta_1 \theta_3 - \theta_3 + \theta_4 + \theta_5 - \frac{3}{2} \theta_4 \theta_1 + \frac{3}{2} \theta_4 \theta_3 - 2 \theta_3 \theta_1 - \frac{3}{2} \theta_5 \theta_4 + \frac{3}{2} \theta_5 \theta_3$$

$$\theta_5 \theta_1 - \theta_4 \theta_3 - \theta_5 \theta_1 - \theta_3 \theta_1 + \frac{1}{2} \theta_2 \theta_1 + \theta_5 \theta_1 - \theta_3 \theta_2 + \theta_5 \theta_4 - \frac{1}{2} \theta_5 \theta_4 + \frac{1}{2} \theta_5 \theta_4 + \frac{1}{2} \theta_5 \theta_4 + \frac{1}{2} \theta_5 \theta_4 + \frac{1}{2} \theta_5 \theta_4$$

This is an example of an algebra with two integer elements over one nilpotent and one idempotent. This is linked to the spin Calogero systems whose relation with bispectrality can be found in [Bergvelt et al. 2009; Kasman 2015]. Theorems 1, 2 and 3 give positive answers to the Conjectures 1, 2 and 3 of Grünbaum (2014) about three bispectral algebra. Moreover, these algebras are Noetherian and finitely generated because they are contained in the $N \times N$ matrix polynomial ring $M_N(\mathbb{K}[z])$ (See [Vasquez and Zubelli 2021]).

We close this section by remarking the important role played by Darboux transformations, which goes back to Duistemaat and Grünbaum (1986) and Zubelli and Maier (1991). Indeed, the three examples in this section are instances of rational Darboux transformations from the scalar matrix exponential functions. All such Darboux transformations were shown to be bispectrals in Theorems 1.1 and 1.2 of [Geiger et al. 2017].

6 A few extensions of the problems discussed above

Since one of the goals of this paper is to serve as an “invitation” to look at this problem extended to a wider audience, we give a road map with some selected references. Indeed, the bispectral problem has many different incarnations, and in our opinion we are still far from having a unified theory.

In the scalar case there are some early papers that should be mentioned are Kasman and Rothstein (1997), Wilson (1993, 1998), Bakalov et al. (1996, 1997, 1998a, b), Grünbaum and Yakimova (2002), Geiger et al. (2017).

One natural issue concerns the numerical aspects involving the prolate spheroidal functions. In this case, the reader may want to consult the book [Osipov et al. 2013].

Another direction concerns the purely discrete (actually finite) version of time-band-limiting. In an effort to better understand the commuting property in question, one of us looked at the case when the real line is replaced by the $N$ roots of unity. See [Grünbaum 1981] as well as [Perlzdorf 1984, Perline 1987]. Expression (11) in [Grünbaum 1981] contains a small typo: the $r(2)$ in the denominator should be replaced by $r(1)$.

Moving on to the discrete-continuous version of the bispectral problem, we have that for the scalar case, involving orthogonal polynomials satisfying differential equations the problem had already been considered by S. Bochner (and even earlier). A very good introduction to this is given in [Haine 2001] and its references. See also [Grünbaum (2001); Grünbaum and Haine (1999); Grünbaum et al. (1997); Grünbaum et al. (1993)].

For the matrix valued case there are two sources of early examples, one resulting from the theory of matrix valued spherical functions, see [Grünbaum et al. (2001, 2002, 2003); Grünbaum et al. (2004); Grünbaum et al. (2005)], and another one see [Durán and Grünbaum (2004, 2007); Durán and de la Iglesia (2008)]. See also [Grünbaum and Nuyens (2003) as well as [Boyallian and Liberati (2008); Miranian (2005); Reach (1988)].

Solutions of the bispectral problem can be used to obtain integral operators which reflect some ordinary differential operator in the sense of [for instance Casper et al. 2013]. This fact generalizes the commuting property in the scalar case. It would be interesting to see whether this could be extended to the matrix case.

In the (noncommutative) matrix case, by considering operators in the physical and spectral variables acting from opposite directions, we maintain the Ad-conditions that played a substantial role in [Duistemaat and Grünbaum (1986)].
In this case, this leads to the embedding of the bispectral algebras of eigenvalues into the matrix polynomial algebra $M_N(C[x])$. See [Vasquez and Zubelli (2021)].

Another natural direction would be to look for a characterization for algebras relating to the spin-Calogero system for matrices of arbitrary size of matrix $N$. The examples $N=1$ and $N=2$ were generalized to arbitrary matrix size $N$ and was characterized as a subalgebra of $M_N(C[x])$ using a family of maps $\{P_k\}_{k=0}^\infty$ satisfying some nice properties such as translation and a product rule similar to the Leibniz rule. See [Vasquez and Zubelli (2021)].

Finally, let us go back to the original problem. We recall that both in the scalar and in the matrix case the motivation behind the bispectral problem was a desire to understand "what is behind" the remarkable commutativity property between the operator of time-and-band limiting in the Fourier, Bessel and Airy cases. It was hard to suspect that this problem would have connections with many of the recent developments in integrable systems. There has been some progress in connecting the bispectral problem with the commutativity property mentioned above, and here again-at least in the scalar case-there are some connections with Integrable systems, see for instance [Grünbaum (2021); Casper and Yakimov (2020); Casper et al. (2019); Casper et al. (2021a,b)]. For connections between the bispectral problem and the commuting property see [Castro and Grünbaum (2017); Grünbaum et al. (2017, 2018)].

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