Exploring postselection-induced quantum phenomena with the two-time tensor formalism

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Here we present the two-time tensor formalism unifying in a general manner the standard quantum mechanical formalism with no postselection and the time-symmetrized two-state (density) vector formalism, which deals with postselected states. In the proposed approach, a quantum particle’s state, called a two-time tensor, is equivalent to a joined state of two particles propagating in opposite time directions. For a general two-time tensor, we derive outcome probabilities of generalized measurements, as well as mean and weak values of Hermitian observables. We also show how the obtained expressions reduce to known ones in the special cases of no postselection and generalized two-state (density) vectors. Then we develop tomography protocols based on mutually unbiased bases (MUB) and symmetric informationally complete positive operator-valued measure (SIC-POVM), allowing experimental reconstruction of an unknown single qubit two-time tensor. Finally, we employ the developed techniques for experimental tracking of qubit’s time-reversal journey in a quantum teleportation protocol realized with a cloud accessible noisy superconducting quantum processor. The obtained results justify an existence of postselection-induced qubit’s proper time-arrow, which is different from the time-arrow of a classical observer, and demonstrate capabilities of the two-time tensor formalism for exploring quantum phenomena brought forth by a postselection in the presence of noise.

I. INTRODUCTION

The standard quantum formalism is commonly used for calculating a probability distribution of measurement outcomes, given a complete characterization of preparation and evolution of a measured quantum system. This consideration with respect to a preselected initial state contains an implicit time asymmetry related to the concept of ‘collapsing’, or ‘reduction’, of system’s state in the measurement process [1, 2]. Combining the preselection with a postselection, i.e. the consideration of a particular outcome of the measurement, removes this asymmetry and gives rise to the two-state vector formalism (TSVF) [1–3]. Within the TSVF, the quantum particle’s state is described by a pair of vectors $(|\psi\rangle, \langle\phi|)$, where $|\psi\rangle$, determined by the preselection, evolves forward in time, while $\langle\phi|$, determined by the postselection, evolves back from the future to the past.

From a practical point of view, one the most important concepts appearing in the framework of postselection and the TSVF is weak values of observables [4, 5]. Despite of some criticism (see e.g. [6–8]), weak values and related techniques for a weak value amplification [9–12] appear to be useful in the context of quantum metrology [13–22] (for a review, see [23]). Moreover, taking postselection into account also plays an important role in studying complexity theory [24], quantum contextuality [25–29], fundamentals of quantum physics [30–40], design of quantum computing algorithms [41], and quantum communication protocols [42–46].

The postselection with respect to entangled states gives rise to time-reversal phenomena, including an appearance of closed time-like curves (CTC), considered both theoretically [47–56] and experimentally [48, 53]. The basic idea behind these phenomena is that Bell state preparation and Bell state measurement can be considered as a kind of ‘time mirrors’ reflecting quantum state’s propagation in time. Note that this interpretation perfectly agrees with experimental results on delayed entanglement swapping [57, 58].

Originally, the TSVF was formulated with respect to a pair of pure states [2, 3]. An important extension comes with introducing ancillary particle and performing postselection with respect to an entangled state. This creates an entanglement between forward and backward evolving states of a two-state vector, and yields the concept of a generalized two-state vector [3, 59]. Studying statistical ensembles of generalized two-state vectors bring forth a notion of two-state density vectors [60], which can be considered as a manifestation of density matrices in the framework of the TSVF. Another approach of introducing mixedness into the TSVF is presented in Ref. [37], where the case of forward and backward evolving states described with density matrices is considered.

The present work is devoted to a further development of effective description of quantum states in the presence of postselection and pursues the following two main goals. The first goal is closing the gap between the previous approaches for describing mixed, or randomized, quantum states in the presence of the postselection and the stan-
Figure 1. General scheme of a postselection experiment giving rise to a concept of a two-time tensor. First, Alice prepares particles Q and A in some arbitrary state and sends Q to Bob. On his side, Bob applies a unitary operation $U$ to Q and P, then performs an arbitrary measurement on P, and finally returns Q back to Alice. Then Alice makes a joint measurement on A and Q, described by a POVM containing a particular effect $E_{\text{post}}$. If the outcome given by $E_{\text{post}}$ realizes, then Alice tells Bob to keep his measurement result $\mu$, otherwise $\mu$ is discarded.

The second goal is developing practical schemes for tomography of both pre- and postselected states. In the current work, we focus on the case of a single qubit that can be easily generalized to a multiqubit one. Compared to a high level recipe for making a complete set of Kraus operators, sufficient for reconstructing an unknown pre- and postselected state, presented in Ref. [60], here we obtain explicit circuits for running tomography protocols on an arbitrary quantum processor. For this purpose, we borrow two basic approaches for single-qubit tomography: The one based on mutually unbiased bases (MUB) corresponding to measuring of three components of a Bloch vector, and the second based on a symmetric informationally complete POVM (SIC-POVM) allowing reconstruction of unknown state with a measurement of a single type.

To demonstrate capabilities of the two-time tensor formalism and developed tomography techniques, we study experimentally a well-known time-reversal phenomenon appearing in a quantum teleportation protocol [48, 50, 53]. Namely, we track propagation of a qubit state, initially prepared by Alice, (i) forward in time to the moment of a Bell measurement on her qubit and a qubit from a pre-shared Bell pair, then (ii) back in time on Alice’ qubit from the Bell pair to the moment of the Bell pair birth, and then (iii) forward in time on Bob’s particle from the Bell pair. For this purpose we use 7-qubit cloud accessible noisy superconducting quantum processor provided by IBM. Although, experiments on the observation of a postselection-induced time-travel in quantum teleportation were already considered previously [48, 53], to the best of our knowledge, it is the first time where it is experimentally demonstrated, using the developed formalism, how the state, prepared by Alice, propagates back in time on Alice’s physical qubit taken from the pre-shared Bell pair. As it was already mentioned, an important advantage of the developed two-time tensor formalism, compared, e.g., to the time-reversal formalism suggested in Ref. [48], is that the new formalism allows considering decoherence effects. In particular, we observe evidences of irreversible corruption of the Alice’s quantum state during propagation along its proper postselection-induced time-arrow.

The rest of the paper is organized as follows. In Sec. II, we introduce the concept of two-time tensors, provide some illustrative examples, and derive their main mathematical properties. In Sec. III, we discuss description of measurements made on a two-time tensors with a focus on von Neuman measurement of Hermitian observables and measurements of weak values. In Sec. IV, we develop tomography protocols for experimental reconstructing of an unknown single-qubit two-time tensor. In Sec. V, we apply the developed formalism and tomography techniques for an experimental study of time-reversal journey of a qubit’s state in a quantum teleportation protocol. We conclude and provide an outlook in Sec. VI.

II. INTRODUCING TWO-TIME TENSORS

A. General postselection experiment

Let us consider a postselection experiment shown in Fig. 1. The experiment is realized by two parties, named Alice and Bob, that are able to communicate with quantum particles and classical messages. At the start of the experiment, Alice prepares two particles, Q and A, in an arbitrary joint mixed state

$$\rho_{\text{pre}} = \rho_{\text{pre}}^{\text{pre}} |i\rangle_Q \langle i| \otimes |m\rangle_A \langle m|.$$  

Here $|n\rangle_X$ with integer labels $n$ denote computational basis states of particle X and $\rho_{\text{pre}}^{\text{pre}}$ are density matrix elements providing standard conditions $\rho_{\text{pre}}^{\text{pre}} \geq 0$, $Tr\rho_{\text{pre}}^{\text{pre}} = 1$. Note that here and after we apply the Einstein summation convention and omit explicit summation with respect to repeated indices. After its preparation, Q is given to Bob, while A remains with Alice.
On his side, Bob takes an additional particle P, prepared in some mixed state
\[ \rho_{\text{probe}} = U^{\mu}_{kk'} |k\rangle_P \langle k'| \]  
(\( \rho_{\text{probe}} \geq 0, \text{Tr} \rho_{\text{probe}} = 1 \)), and let P and Q evolve according to a unitary evolution operator
\[ U = U_{kj}^{ij} |j\rangle_P \langle k| \otimes |j\rangle_Q \langle i|. \]  
(3)

Then Bob makes a measurement on P described by a POVM \( \{E(\mu')\}_{\mu'} \), satisfying standard conditions \( E(\mu') \geq 0, \sum_{\mu'} E(\mu') = 1 \), where \( 1 \) is the identity matrix. Here we consider outcome labels \( \mu' \) belonging to an arbitrary finite set. Bob keeps the obtained measurement outcome \( \mu \), corresponding to the realized effect \( E(\mu) \), and returns Q back to Alice.

At her side, Alice makes a joint measurement on Q and A, described by another POVM, whose collection of effects includes a particular effect
\[ E^\text{post} = E^\text{post}_{ij}^{mm'} |j\rangle_Q \langle i| \otimes |m\rangle_A \langle m'| \]  
(4)

\( (0 \leq E^\text{post} \leq 1) \). If an outcome of Alice’s measurement corresponds to \( E^\text{post} \), then we say that the postselection passed, it is failed otherwise. Alice transmits the result of the postselection (single bit of information) to Bob, who keeps his measurement outcome \( \mu \) if postselection has passed, or discards \( \mu \) otherwise. We note that the only constraints on particular time moments, when the described operations take place, correspond to the general ordering: Preparations of \( \rho_{\text{pre}} \) and \( \rho_{\text{probe}} \) are in the past light cone of \( U \), while Alice’s and Bob’s measurements are in the future light cone of \( U \) and in the past light cone of the final decision on \( \mu \).

The main quantity of our consideration is conditional probability distribution \( P(\mu) \equiv \text{Pr}[^{\text{postsel.}} \text{pass}] \) of obtaining measurement outcome \( \mu \) by Bob, given the passing postselection on Alice’s side. According to axioms of quantum mechanics, this probability reads
\[ P(\mu) = \frac{\rho_{\text{pre}}^{ikk'} \rho_{\text{probe}}^{i\mu;m\mu';l} U^{i\mu;m\mu'}_{ij} U^{\mu'ji'} E(\mu')^{ii'} \rho_{\text{post}}^{jj';kk'} \rho_{\text{pre}}^{ijk';kk'} U^{ij;j'k'}_{lj} E^\text{post}^{jj';kk'}_{ij}, \]  
(5)
given the probability of the postselection passing
\[ P^\text{post} \equiv \text{Pr}[^{\text{postsel.}} \text{pass}] = \rho_{\text{pre}}^{ikk'} \rho_{\text{probe}}^{i\mu;m\mu';l} U^{i\mu;m\mu'}_{ij} U^{\mu'ji'}_{lj} E^\text{post}^{jj';kk'}_{ij}, \]  
(6)
is greater than zero, and \( P(\mu) = 0 \) otherwise (here and after \( z \) denotes a complex conjugate of \( z \)). We note that Eq. (5) is much easier to follow in the form of a tensor network, shown if Fig. 2(a).

Within Eq. (5) we can separate two mathematical structures, which are related to Alice’s and Bob’s actions correspondingly. The pre- and postselection, which are performed by Alice, are described by a tensor
\[ \eta_{i'i'}^{jj'} := \rho_{\text{pre}}^{i'i';kk'} E^\text{post}^{jj';mm'}_{i'i'} \]  
(7)

while the indirect Bob’s measurement is described by a collection of tensors
\[ K(\mu)^{i'i'}_{jj'} := \rho_{\text{probe}}^{ikk'} U^{k'i';ji'}_{lj} E(\mu)^{ii'}_{l}. \]  
(8)

We then can rewrite Eqs. (5) and (6) in a compact form
\[ P(\mu) = \frac{K(\mu)^{i'i'}_{jj'} \rho_{\text{pre}}^{i\mu;m\mu';l} U^{i\mu;m\mu'}_{ij} U^{\mu'ji'} E(\mu')^{ii'}_{lj} E^\text{post}^{jj';kk'}_{ij}}{K_{\text{post}}^{i'i'}_{jj'}}, \]  
(9)

where \( K := \sum_{\mu} K(\mu) \) and \( \bullet \), in line with Ref. [60], denotes contraction with respect to proper indices [see also Fig. 2(a)].

In what follows, we refer to \( \eta, K(\mu) \), and \( K \) as a two-time tensor, operation outcome \( \mu \) tensor, and operation tensor correspondingly. One can see that \( P(\mu) \) is invariant under multiplying \( \eta \) (as well as both \( K(\mu) \) and \( K \)) by a constant. So we have a freedom of renormalizing \( \eta \) without affecting any of observable quantities. In our work we apply normalization in the form
\[ \eta \mapsto \frac{\eta}{\eta_{ii'}}, \]  
(10)

This normalization condition makes, as we see later, \( \eta \) to be equivalent to a standard ‘preselected’ joint state of two particles. Note that \( \rho_{ii'}^{\text{pre}} \) is proportional to the probability of postselection given that Bob returns Q in the maximally mixed state. So \( \eta_{ii'} = 0 \) only in the trivial case, where the postselection probability \( P^\text{post} \) is 0 regardless of Bob’s actions.

We can also introduce two ‘reduced’ tensors
\[ \eta_{ik}^{\uparrow} = \eta_{kk}, \quad \eta_{ij}^{\downarrow} = \eta_{kk} \]  
(11)

which we call a ‘forward evolving’ and ‘backward evolving’ reduced tensors correspondingly [see Fig. 2(b)]. One can see that the semi-positivity of \( \rho_{\text{pre}} \) and \( E^\text{post} \) implies semi-positivity of \( \eta_{i'}^{\downarrow} \) and \( \eta_{i'}^{\uparrow} \). Provided normalization condition (10), we also have \( \text{Tr} \eta_{i'i'} = \text{Tr} \eta_{ii'} = 1 \).

B. Special cases of two-time tensors

Here we consider some illustrative special cases of \( \eta \) to reveal physical meaning of \( \eta_{i'}^{\downarrow} \) and \( \eta_{i'}^{\uparrow} \). Tensor diagrams for all the considered cases are shown in Fig. 2(c).

First, let us remove the postselection condition by taking \( E^\text{post} = 1 \), which corresponds to a guaranteed passing of the postselection \( P^\text{post} = 1 \). In this case we have
\[ \eta_{i'i'}^{jj'} = \rho_{\text{pre}}^{i'i';kk'} \delta_{jj'} \delta_{kk'}, \]  
(12)
Figure 2. In (a) tensor network representation of Eq. (6) for \( P(\mu) \), as well as definitions of two-time tensor \( \eta \), operation outcome tensor \( K(\mu) \), and operation tensor \( K \) are shown. In (b) construction of reduced forward- and backward evolving reduced tensors \( \eta^f \) and \( \eta^b \) is depicted. In (c) some examples of previously considered special types of two-time tensors are presented.

and after applying normalization condition (10), \( \eta \) factorizes into

\[
\eta = \eta^f \otimes \eta^b, \quad \eta^f = \rho^{\text{pre}}_Q, \quad \eta^b = \rho^{\text{mix}}, \tag{13}
\]

where \( \delta \) denotes a standard Kronecker symbol, \( \rho^{\text{pre}}_Q = \Tr A \rho^{\text{pre}} \) is a reduced preselected state of \( Q \) (here \( \Tr A \) denotes a partial trace with respect to \( A \)), and \( \rho^{\text{mix}} \propto 1 \) is the maximally mixed state. Two-time tensor of the form (13) has a clear physical meaning of the preselected state \( \rho^{\text{pre}}_Q \) evolving forward in time, and a total uncertainty of particle’s future. The probability to obtain \( \mu \) on Bob’s side takes a familiar form

\[
P(\mu) = \frac{\rho^{\text{pre}}_{Q|i;\phi^{\text{pre}}_{\text{probe}} U^{\dagger}_{i;j} U_{i;j}^{\text{pre}} E(\mu)'}{\rho^{\text{pre}}_{Q|i;\phi^{\text{pre}}_{\text{probe}}}} = \Tr \left[ E(\mu) \Tr Q \left( U \rho^{\text{pre}}_Q \otimes \rho^{\text{probe}} U^{\dagger} \right) \right]. \tag{14}
\]

This is exactly the value that one obtains with the use of the standard formalism.

The second illustrative special case is when \( \rho^{\text{pre}}_Q \) is a product state of the pure state \( |\psi^{\text{pre}}\rangle = \sum_i \psi_i^{\text{pre}} |i\rangle \) on \( Q \) and an arbitrary state of \( A \), while postselection is realized only for \( Q \) with respect to the state \( \langle \phi_{\text{post}} | = \sum_i \phi_i^{\text{post}} |i\rangle \) (\( \Tr Q \otimes \Tr A \langle \phi_{\text{post}} | \otimes \Id_A \)). Elements of \( \eta \) then take the form

\[
\eta_{i;\mu}^{j'} = \psi_i^{\text{pre}} \psi_{i'}^{\text{pre}} \phi_{\text{post}}^{j'} \tag{15}
\]

or, simply, \( \eta = \eta^f \otimes \eta^b \) with

\[
\eta^f = |\psi^{\text{pre}}\rangle \langle \psi^{\text{pre}}|, \quad \eta^b = |\phi_{\text{post}}\rangle \langle \phi_{\text{post}}|. \tag{16}
\]

This situation corresponds to a system described by a two-state vector

\[
(|\psi^{\text{pre}}\rangle, |\phi_{\text{post}}\rangle), \tag{17}
\]

where \( |\psi^{\text{pre}}\rangle \) and \( \langle \phi_{\text{post}}| \) evolve forward and backward in time, correspondingly. This is the situation extensively studied in the framework of weak values and weak measurements [1–4].

A similar case of a mixed two-state vector [37] \( \rho^{\text{pre}} \) \( \rho^{\text{post}} \) corresponds to \( \eta = \rho^{\text{pre}} \otimes \rho^{\text{post}} \), and can be realised physically either in ancilla-free way by preparing \( \rho^{\text{pre}} \) on \( Q \) and then postselecting with \( \Tr Q \propto \rho^{\text{post}} \), or by employing several purifying ancillas as it shown in Ref. [37].

We can also consider, so-called, generalized two-state vector [59]

\[
c_i^f (|i\rangle, \langle j|). \tag{18}
\]

In this case \( Q \) and \( A \) are initially prepared in a pure entangled state

\[
|\Psi^{\text{QA}}\rangle = c^{\text{f}}_i |i\rangle_Q \otimes |j\rangle_A \tag{19}
\]

\( \rho^{\text{pre}} = |\Psi^{\text{QA}}\rangle \langle \Psi^{\text{QA}}| \) and postselection corresponds to obtaining the maximally entangled Bell state \( \Tr Q \otimes \Id_A \langle \phi_{\text{post}}| \otimes |\phi_{\text{post}}\rangle \). The elements of \( \eta \) then are given by

\[
\eta_{i;\mu}^{j'} = c_i^{\text{f}} \phi_{i}^{j'} \tag{20}
\]

or one can write

\[
\eta = |\Psi\rangle \langle \Psi|. \tag{21}
\]

Note that in this case forward and backward evolving parts appear to be entangled.

Next we can consider a randomized preparation and postselection scenario, studied in Ref. [60], where Alice randomly chooses to perform a postselection experiment with respect to generalized two-state vector \( c_i^{(r)} (|i\rangle, \langle j|) \) with probability \( p(r) \) \( \sum_r p(r) = 1 \). This scenario corresponds to the two-state tensor in the form of a density vector:

\[
\eta = \sum_r p(r) |\Psi^{(r)}\rangle \langle \Psi^{(r)}|, \tag{22}
\]
with \( |\Psi^{(r)}\rangle = \phi_i^{(r)j} |i\rangle \otimes |j\rangle \). The obtained form of two-tensors is of particular importance, since, as we show further, any two-time tensor \( \eta \) can be represented in the form of the density vector (22).

C. Spectral decomposition and purification of two-time tensors

Here we consider certain mathematical properties of two-time tensors and see how they are equivalent to bipartite states. First, let us introduce multi-indices \( \alpha := (i,j) \), \( \beta := (i',j') \) and consider a two-time tensor \( \eta_{ii'}^{jj'} \) as a matrix \( \eta_{ii'}^{jj'} \). Then one can see that according to the construction of \( \eta \), for an arbitrary vector \( \phi_\gamma \) of an appropriate dimension we have

\[
\bar{\phi}_\alpha \eta_{\alpha \beta} \phi_\beta \geq 0.
\]

Thus, \( \eta_{\alpha \beta} \) is positive semi-definite, and so we can obtain its spectral decomposition

\[
\eta_{\alpha \beta} = \sum_r \lambda^{(r)} \phi_\alpha^{(r)j} \phi_\beta^{(r)} j, \quad \lambda^{(r)} \geq 0, \tag{24}
\]

where \( \phi^{(r)} \) form a set of orthonormal vectors \( \phi_\alpha^{(r)} = \delta^{rr'} \). Provided normalization (10), we also have \( \sum_r \lambda^{(r)} = 1 \). Here we can also see that the only way to obtain \( \sum_r \lambda^{(r)} = 0 \) is to have \( \eta = 0 \).

After splitting multi-indices \( \alpha \) and \( \beta \) back, we come to

\[
\eta_{ii'}^{jj'} = \sum_r \lambda^{(r)} \phi_i^{(r)j} \phi_j^{(r)j'} \tag{25}
\]

that is a two-time density vector [see also Eq. (22)] introduced in Ref. [60]. We note that, similarly to spectral decomposition of mixed states, decomposition (25) completely determines measurement outcome probabilities for a arbitrary measurements on \( \eta \), yet does not specify exactly how \( \eta \) is physically prepared. Like infinite number of possible statistical ensembles can yield the same mixed density matrix, ‘mixed’ \( \eta \) can be realized in infinite number of possible ways.

Finally, we show two-time tensor \( \eta \) also can ‘purified’, i.e. obtained as a partial trace of some ‘pure’ extended tensor \( \tilde{\eta}_{ii'}^{jj'}^{rr'} \):

\[
\eta_{ii'}^{jj'} = \tilde{\eta}_{ii'}^{jj'}^{rr'}, \quad \tilde{\eta}_{ii'}^{jj'}^{rr'} = \Theta_{ii'}^{rr'} \Theta_{ii'}^{rr'}
\]

for some tensor \( \Theta \). This purifying tensor can be taken in the form

\[
\Theta_{ii'}^{rr'} = \sqrt{\lambda^{(r)}} \phi_i^{(r)j} j. \tag{27}
\]

From the physical point of view it can be realized as follows. In the protocol, shown in Fig. 1, Alice splits ancilla \( A \) into two particles: \( A \) and \( R \), and prepare the pure state

\[
|\Theta\rangle_{QAR} := |\Theta_{ii'}^{rr'} |i\rangle_\alpha |j\rangle_A |r\rangle_R . \tag{28}
\]

Here \( R \) stands for a ‘reference’ responsible for the purification. At the final step of the protocol, Alice performs the postselection with respect to \( Q \) and \( A \) in the maximally entangled state proportional to \( |ii\rangle_A |ii\rangle_A \), while keeping \( R \) untouched. The effective state, observed by Bob, is then given by

\[
\eta_{ii'}^{jj'} = \Theta_{ii'}^{rr'} \Theta_{ii'}^{rr'} . \tag{29}
\]

for arbitrary \( \phi_j^{(r)} \). This kind of semi-positivity condition provides non-negative probabilities for measurement outcome \( \mu \). The normalization condition for \( K \) takes the form

\[
K^{ii'}_{kk} = \delta^{ii'} \tag{30}
\]

that actually is the standard normalization condition for Kraus operators. Note that \( K^{kk'} \) is not specified. This asymmetry between upper and lower indices of \( K \) catches an inherent time asymmetry of Bob’s operations: In contrast to Alice’s particle \( Q \), Bob’s particle \( P \) is in a preselected state and no any postselection for \( P \) is considered.

III. MEASURING OBSERVABLES

In Sec. II A, we have considered a general scheme of an indirect measurement on Bob’s side. Here we focus on two particular cases of this measurement: The first one is a projective measurement and the second one is measurement of a weak value.

Before proceed, we note some general properties of operation outcome tensor \( K(\mu) \) and operation tensor \( K \). First, one can see that according to the construction of \( K(\mu) \), we have

\[
|\Theta_{ii'}^{rr'} \Theta_{ii'}^{rr'} |ii\rangle \Theta_{ii'}^{rr'} \Theta_{ii'}^{rr'} = 0
\]

for arbitrary \( \phi_j^{(r)} \). This kind of semi-positivity condition provides non-negative probabilities for measurement outcome \( \mu \). The normalization condition for \( K \) takes the form

\[
K^{ii'}_{kk} = \delta^{ii'} \tag{30}
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that actually is the standard normalization condition for Kraus operators. Note that \( K^{kk'} \) is not specified. This asymmetry between upper and lower indices of \( K \) catches an inherent time asymmetry of Bob’s operations: In contrast to Alice’s particle \( Q \), Bob’s particle \( P \) is in a preselected state and no any postselection for \( P \) is considered.

A. Projective measurements

Consider a finite-dimensional Hermitian observable \( O \) with a spectral decomposition of the form:

\[
O = \sum_s \mu_s \Pi(\mu_s), \tag{31}
\]

where \( \{\mu_s\} \) is a set of distinct real numbers (the size of \( \{\mu_s\} \) does not exceed the dimensionality of \( O \)), and \( \Pi(\mu_s) = \Pi^{(i)}(\mu_s) |i\rangle \langle j| \) are orthogonal projectors forming a complete set \( \sum_s \Pi(\mu_s) = 1 \). The corresponding tensor network diagram is presented in Fig. 3(a). According to the axiomatics of quantum mechanics, measuring \( O \) of a pure state \( |\psi\rangle \) provides a real value \( \mu_i \) with probability \( \Pi(\mu_i) = \langle \psi | \Pi(\mu_i) | \psi \rangle \). At the same time, the original state \( |\psi\rangle \) ‘collapses’ to \( P(\mu_i)^{-1/2} \Pi(\mu_i) |\psi\rangle \).

From the viewpoint of the scheme from Fig. 1, this kind of measurement is equivalent to the scenario with \( \rho_{\text{probes}} = |0\rangle_p \langle 0| \), \( U \) providing transformation

\[
|0\rangle_p \otimes |\psi\rangle_Q \rightarrow \sum_s |s\rangle \otimes \Pi(\mu_s) |\psi\rangle_Q , \tag{32}
\]
and POVM $E$ with effects $E(\mu_s) = \Pi(\mu_s)$.

One can easily check that in this case, the outcome $\mu_i$ tensor and operation tensor are correspondingly given by

$$K(\mu_i) = \Pi(\mu_i) \otimes \overline{\Pi}(\mu_i), \quad K = \sum_s \Pi(\mu_s) \otimes \overline{\Pi}(\mu_s) \quad (33)$$

[see also Fig. 3(b)].

Then for an arbitrary two-time tensor $\eta$, the probability to obtain $\mu_i$ is given by $P(\mu_i) = (K \cdot \eta)^{-1} K(\mu_i) \cdot \eta$, and the mean value of $O$ takes the form

$$\langle O \rangle_{\eta} = \sum_s P(\mu_s) \mu_s = \frac{\tilde{O} \cdot \eta}{K \cdot \eta}, \quad \text{(34)}$$

where we have introduced an auxiliary projective measurement tensor

$$\tilde{O} := \sum_s \mu_s \Pi(\mu_s) \otimes \overline{\Pi}(\mu_s) \quad (35)$$

[see also Fig. 3(c)].

One can see that in the case of no postselection, two-time tensor $\eta = \rho_{\text{Q}}^{\text{pre}} \otimes \rho_{\text{mix}}$, the expression for the mean value reduces to

$$\langle O \rangle_{\eta} = \text{Tr}[O \rho_{\text{Q}}^{\text{pre}}]. \quad \text{(36)}$$

We also note the decomposition (35) can be considered as a generalization of a standard spectral decomposition (31) to the two-time case.

### B. Weak values

We call a weak value of a Hermitian observable $O = \sum_{ij} O_{ij} |i \rangle \langle j|$ with respect to a two-time tensor $\eta$ a quantity

$$O^{\text{weak}}_{\eta} := \frac{\eta_{ik}^{j\prime}}{\eta_{ik'}^{j}}, \quad (\mathbb{1} \otimes O) \cdot \eta = \frac{\langle \text{Id} \rangle}{\langle \mathbb{1} \rangle} \cdot \eta \quad (37)$$

[see also Fig. 3(d)]. To justify this definition, we first note that for $\eta = |\psi^{\text{pre}}\rangle \langle \psi^{\text{pre}}| \otimes |\phi^{\text{post}}\rangle \langle \phi^{\text{post}}|$, which corresponds to a standard two-time vector (17), Eq. (37) reduces to

$$O^{\text{weak}}_{\eta} = \frac{\psi_k^{\text{pre} \cdot \eta} \pi_{\phi^{\text{post}}}^i \phi_k^{\text{post} \cdot \eta}}{\psi_k^{\text{pre} \cdot \eta} \pi_{\phi^{\text{post}}}^j \phi_k^{\text{post} \cdot \eta}} = \frac{\langle \phi^{\text{post}} | O | \psi^{\text{pre}} \rangle}{\langle \phi^{\text{post}} | \psi^{\text{pre}} \rangle}, \quad (38)$$

which is the standard definition of the weak value for the two-time vector $(|\psi^{\text{pre}}\rangle, \langle \phi^{\text{post}}|)$ [3, 4]. Then we show that standard experimental setups devoted to extracting real and imaginary parts of a weak value [23] provide real and imaginary parts of $O^{\text{weak}}_{\eta}$ for general $\eta$ correspondingly.

Let $P$ in our postselection experiment now be a one dimensional particle and let $Q$ and $P$ be its coordinate and momentum operators correspondingly. We consider $P$ initially prepared in a pure state $\rho^{\text{pre}} = |\psi\rangle \langle \psi|$ with a Gaussian wave function

$$\langle q|\psi\rangle = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp \left( -\frac{q^2}{4\sigma^2} \right), \quad \text{(39)}$$

where $|q\rangle$ is an eigen state of $Q$ with an eigen value $q$, and $\sigma^2 > 0$. Let the coupling unitary operation $U$ be realized by applying a measurement Hamiltonian $H = P \otimes O$ during some small time $\epsilon \ll 1$ (here we employ dimensionless units with $\hbar = 1$). Considering orders of $\epsilon$ not higher than the first, we have

$$U = e^{-i\epsilon O \otimes P} \approx 1 - i\epsilon O \otimes P \quad (40)$$

The mean value of coordinate $q$, given passing of the postselection, takes the form

$$\langle Q | \eta \rangle = \frac{\langle \mathbb{1} \otimes \mathbb{1} \rangle \cdot \eta \langle Q | \psi \rangle}{\langle \mathbb{1} \otimes \mathbb{1} - i\epsilon (1 \otimes O \otimes P) | \psi \rangle - \mathcal{O} \otimes \langle \mathbb{1} \otimes | \psi \rangle \rangle}$$

$$= \frac{(1 \otimes O \langle P | \psi \rangle - \mathcal{O} \otimes \langle \mathbb{1} \otimes | \psi \rangle \rangle) \cdot \eta}{\langle \mathbb{1} \otimes \mathbb{1} - i\epsilon (1 \otimes O \otimes P) | \psi \rangle - \mathcal{O} \otimes \langle \mathbb{1} \otimes | \psi \rangle \rangle}, \quad (41)$$

where $\langle \cdot | \psi \rangle \equiv \langle \psi | \cdot \rangle$. Taking into account that $\langle P | \psi \rangle = \langle O | \psi \rangle = 0$, $\langle Q | P \psi \rangle = -\langle P Q | \psi \rangle = i/2$, and $\langle \mathbb{1} \otimes | \eta \rangle = \langle \mathcal{O} \otimes \mathbb{1} \rangle \cdot \eta$, we come to

$$\langle Q | \eta \rangle = \epsilon \text{Re} O^{\text{weak}}_{\eta}. \quad (42)$$

In a similar way, provided $\langle P^2 \rangle_{\psi} = 1/(4\sigma^2)$, measuring (conditional) mean value of momentum gives

$$\langle P | \eta \rangle = \frac{\epsilon}{4\sigma^2} \text{Im} O^{\text{weak}}_{\eta}. \quad (43)$$
We then see that experimentally-accessible real values of \(\langle Q \rangle_\eta\) and \(\langle P \rangle_\eta\), provided definition (37), allow reconstructing complex value of \(O^{\text{weak}}\) for the general two-time tensor \(\eta\).

IV. TOMOGRAPHY OF A SINGLE-QUBIT TWO-TIME TENSOR

Here we consider the problem of an experimental reconstruction of an unknown two-time tensor. Similarly to the case of a standard quantum state tomography, the reconstruction of a two-time tensor \(\eta\) requires its measuring with some sufficient informationally complete collection of outcome tensors sets \(\{K^{(r)}\}_r\), where \(K^{(r)} = \{K^{(r)}(\mu')\}_{\mu'}\) denotes a particular set of outcome tensors. The information completeness condition ensures that observed probabilities

\[
P^{(r)}(\mu) = \frac{K^{(r)}(\mu) \cdot \eta}{K^{(r)} \cdot \eta} \quad (44)
\]

are sufficient for recovering all elements of \(\eta\). An unknown two-time tensor can be recovered, e.g., by maximizing likelihood function

\[
\mathcal{L}(\eta) = \sum_{r,k} \mu^{(r,k)} \log P^{(r)}(\mu^{(r,k)}), \quad (45)
\]

where \(\{\mu^{(r,k)}\}_{r,k}\) denotes a particular measurement outcome obtained in \(K^{(r)}\) measurement. We note this problem can be efficiently solved with gradient descent within a corresponding Riemannian manifold of positive semidefinite matrices [61, 62].

Here we focus on constructing practical schemes for the two-time tensor tomography of a single qubit (the general, yet less detailed, approach for tomography of arbitrary finite-dimensional two-time tensors can be found in Ref. [60]). We consider two basic techniques borrowed from the standard quantum state tomography: The first one based on MUB and employing several operation configurations, and the second one based on SIC-POVM measurements and realized with a single set of operation outcome tensors. We note that the presented schemes can be directly generalized to multiqubit two-time tensors.

Before proceeding further, we highlight some aspects of the two-time tensors tomography that are different compared to the standard quantum state tomography. First, the standard Born rule is replaced with Eq. (44). Second, since we actually deal with two states propagating in opposite time direction, the performed measurement have to be non-destructive. A convenient way to achieve this property is to couple the measured particle to some ancillary particle (like P in Fig. 1), and then perform the final read-out measurement on this ancilla. Third, the measurement employed in the tomography generally affects the postselection probability. It may turn out that that for some measurements and some two-time tensors, the postselection probability can be zero. This fact complicates an analysis of statistical size effects, which we leave for future study.

A. MUB-based approach

In the case of a single qubit, MUB correspond to eigenvectors of standard Pauli operators

\[
s_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad s_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad s_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (46)
\]

The corresponding projectors on eigen spaces read

\[
\Pi^{(r)}(\mu) = \frac{1}{2} (I + \mu \sigma_r), \quad \mu = \pm 1, \quad r = x, y, z \quad (47)
\]

and provide spectral decompositions of the form

\[
\sigma_r = \Pi^{(r)}(1) - \Pi^{(r)}(-1). \quad (48)
\]

As it was already noted, in contrast to standard single qubit quantum state tomography, where an unknown qubit state is measured in three MUB (corresponding to \(x, y, z\) projection on the Bloch sphere), in the case of two-time tensors we actually deal with states of two qubits propagating in opposite time directions. Thereby, we measure forward and backward propagating states in two different MUB. To do so, we take operation outcome tensors of the form

\[
K^{(r_1, r_2)}(\mu) = \Pi^{(r_2)}(\mu_2)\Pi^{(r_1)}(\mu_1) \otimes \Pi^{(r_2)}(\mu_2)\Pi^{(r_1)}(\mu_1) \quad (49)
\]

with \(\mu = (\mu_1, \mu_2)\), where \(\mu_1\) and \(\mu_2\) correspond to measurement results of a forward and backward evolving parts respectively.

If \(r_1 \neq r_2\), Eq. (49) transform to

\[
K^{(r_1, r_2)}(\mu) = \frac{1}{2} \left| \psi^{(r_2)}(\mu_2) \right\rangle \left\langle \psi^{(r_1)}(\mu_1) \right| \otimes \left| \psi^{(r_2)}(\mu_2) \right\rangle \left\langle \psi^{(r_1)}(\mu_1) \right|, \quad (50)
\]

where \(\left| \psi^{(r)}(\mu) \right\rangle\) denotes an eigenvector of \(\sigma_r\) with eigenvalue \(\mu\). Operation tensor in this case is given by

\[
K_{j j'}^{(r_1, r_2)} = \sum_{\mu_1, \mu_2} K^{(r_1, r_2)}(\mu_1, \mu_2)_{j j'} = \frac{1}{2} \delta_{j j'} \delta_{j j'}. \quad (51)
\]

Note that it provides nonzero postselection probability for every \(\eta \neq 0\).

For \(r_1 = r_2\), we have

\[
K^{(r_1, r_2)}(\mu) = \delta_{\mu_1, \mu_2} \left| \psi^{(r_1)}(\mu_1) \right\rangle \left\langle \psi^{(r_1)}(\mu_1) \right| \otimes \left| \psi^{(r_1)}(\mu_1) \right\rangle \left\langle \psi^{(r_1)}(\mu_1) \right|, \quad (52)
\]
i.e. this measurement corresponds to the single projective measurement of $\sigma_r$. The corresponding operation tensor takes the form:

$$K^{(r_1, r_2)} = \sum_{\mu = \pm 1} \Pi^{(r_1)}(\mu) \otimes \Pi^{(r_2)}(\mu).$$  \hfill (53)

Note that for some two-time tensors, the postselection probability for this kind of measurements turns into zero (i.e., $K^{(r_1, r_2)} \bullet \eta = 0$ for $\eta = |00\rangle \langle 00| \otimes |11\rangle \langle 11|$).

The described two-MUB measurement can be realized via quantum circuits shown in Fig. 4(a). Here we employ two ancillary qubits, initialized in $|0\rangle$, in order to realize nondestructive projectile Pauli measurements. The coupling is performed with controlled-NOT (CNOT) gates, surrounded by ‘basis change’ operators $V_r$. Choosing these unitaries in the form

$$V_x := R_y(\pi/2), \quad V_y := R_x(\pi/2) \quad V_z := 1,$$  \hfill (54)

where $R_z(\theta) = e^{-i\sigma_z \theta/2}$ denotes a standard rotation operation, we obtain proper $x$-, $y$-, and $z$-Pauli measurements respectively (computational basis measurement outcomes 0 and 1 have to interpreted as $+1$ as $-1$ correspondingly). Note that due to Eq. (52), in the case of $r_1 = r_2$, only a single ancilla is required.

In total, one needs nine circuits, corresponding to all combinations of $r_1$ and $r_2$, to recover an unknown single-qubit two-time tensor. Next, we will see how to cope with the some problem with a single circuit.

## B. SIC-POVM-based approach

An alternative approach for recovering an unknown state is to make a single informationally complete measurement, e.g. given by a SIC-POVM. In the case of a single qubit, SIC-POVM is defined by four pure states corresponding to vertices of a regular tetrahedron inscribed in the Bloch sphere. For our purpose, we take these states in the following form:

$$|\psi(\mu_1, \mu_2)\rangle = \sigma^{\mu_2}_x \sigma^{\mu_1}_z (\cos \frac{\theta}{2} |0\rangle + e^{-i\pi/4} \sin \frac{\theta}{2} |1\rangle),$$  \hfill (55)

where $\mu_i \in \{0, 1\}$ and $\theta := \arccos(1/\sqrt{3})$ (see also Fig. 5). The corresponding SIC-POVM is given by a set of four operators \(\{\Pi(\mu_1, \mu_2)/2\}_{\mu_1, \mu_2}\) with

$$\Pi(\mu_1, \mu_2) = |\psi(\mu_1, \mu_2)\rangle \langle \psi(\mu_1, \mu_2)|.$$  \hfill (56)

In order to recover both forward and backward evolving states, we perform a doubled SIC-POVM measurement with outcome tensors of the form

$$K(\mu) = \frac{1}{8} |\psi(\mu_1, \mu_2)\rangle \langle \psi(\mu_3, \mu_4)| \otimes |\psi(\mu_1, \mu_2)\rangle \langle \psi(\mu_3, \mu_4)|,$$  \hfill (57)

where $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ and $\mu_i \in \{0, 1\}$. The corresponding operation tensor reads

$$K^{i\beta}_j = \sum_{\mu} K(\mu)^{i\beta}_j = \frac{1}{2} \delta^{i\beta} \delta_{jj'},$$  \hfill (58)

and provides a non-zero postselection probability for every non-trivial two-time tensor $\eta \neq 0$. Note that provided normalization rule (10), $K \bullet \eta$ is the same for every $\eta$, that makes probability (44) to be of the same form as Born rule for standard two-qubit states.

The circuit implementing the considered ‘doubled’ SIC-POVM measurement is shown in Fig. 4(b). To construct this circuit, the scheme for a standard SIC-POVM measurement from Ref. [63] is used. Here, single-qubit unitary $V$ makes the transformation

$$V |0\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\pi/4} \sin \frac{\theta}{2} |0\rangle,$$  \hfill (59)

and can be taken in the form $V = R_x(\pi/4)R_y(\theta)$. To make this measurement, one need at most four ancillary qubits for storing of all elements $\mu$. We note that the number of ancillas can be reduced by using qubits reinitialization. In Fig. 4(b) we show the scheme with two ancillas, reinitializing them back into $|0\rangle$ after reading out the values of $\mu_1$ and $\mu_2$.

## V. TRACKING TIME-REVERSAL STATE PROPAGATION IN A QUANTUM TELEPORTATION PROTOCOL

Here we demonstrate how the developed formalism, as well as developed tomography techniques, allows one to observe a time-reversal propagation of quantum states. Consider a standard single-qubit quantum teleportation protocol [64]. Let Alice be given with some pure qubit state $|\psi\rangle$ on particle $A$. Let also Alice and Bob share maximally entangled state $|\Phi^+\rangle$ on particles $B$ and $C$, s.t. $B$ goes to Alice, and $C$ belongs to Bob. Here and after, standard notation for Bell states is used:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$  \hfill (60)

In order to transmit $|\psi\rangle$ to Bob, Alice measures her particles $A$ and $B$ in Bell basis and transmits Bob the obtained outcome encoded in 2 bits via classical channel. By applying a proper single qubit unitary on $C$, which depends on the message from Alice, Bob obtains $C$ exactly in the state $|\psi\rangle$.

The natural question is how the state $|\psi\rangle$ propagates from $A$ to $C$. Obviously, it can not propagate with the classical message from Alice, which is uncorrelated with $|\psi\rangle$: Four outcomes of Bell measurement appear with the same probability of $1/4$. The only quantum medium that connects Alice and Bob is the entangled pair of $B$ and $C$, so one can suggest that $|\psi\rangle$ propagates from $A$ to $C$ via $B$. 
A hint for this problem can be obtained by considering a fixed outcome of Alice’s measurement. It especially helps to consider the case, where Alice obtains a particular outcome $|\Phi^+\rangle$. In this case, the corresponding Bob’s single qubit transformation equals to identity. It means that Bob already has his particle $C$ in the state $|\psi\rangle$. Since no transformations were applied to $C$, $C$ appears to be in the state $|\psi\rangle$ just after the birth of the Bell pair. Note that this remains to be true even if the state $|\psi\rangle$ is prepared on $A$ after the Bell pair’s birth as it is shown in Fig. 6(a). In this case we effectively have a travel of $|\psi\rangle$ from $A$ back in time to the moment of Bell pair birth via $B$, and then forward in time on $C$.

In order to justify this conclusion, let us consider two-time tensors $\eta_A$, $\eta_B$, and $\eta_C$ of the corresponding particles $A$, $B$, and $C$, given the postselection condition of measuring $|\Phi^+\rangle$ in Alice’s Bell measurement [see Fig. 6(a)]. Notably, we define $\eta_B$ and $\eta_C$ before the preparation of $A$ in $|\psi\rangle$ from the viewpoint of an external macroscopic observer.

One can easily check that the introduced two-time tensors read

$$\eta_A = \eta_C = |\psi\rangle \langle \phi| \otimes \rho_{\text{mix}} \quad \eta_B = \rho_{\text{mix}} \otimes |\psi\rangle \langle \phi| \quad (61)$$

[see also Fig. 6(b)]. The form of these tensors justifies our suggestion about the time travel of $|\psi\rangle$ via the route $A \rightarrow B \rightarrow C$. Note that $|\psi\rangle \langle \phi|$ is in the forward evolving part of $\eta_A$ and $\eta_C$, and is in the backward evolving part of $\eta_B$.

To verify our consideration, we make an experiment on superconducting 7-qubit processor ibm_oslo provided with cloud access by IBM. The architecture of processor is shown in Fig. 6(c). Here the connections between physical qubits, denoted by $q_0$, $q_1$, $q_2$, and $q_3$, correspond to an ability to apply CNOT gates. The circuits realizing tomography of qubits in a quantum teleportation protocol are shown in Fig. 6(d). We use pairs $q_0$, $q_2$ and $q_4$, $q_6$ in order to make SIC-POVM-based tomography according to Fig. 4(b), while $q_1$, $q_3$, and $q_5$ are used for storing states of $A$, $B$, and $C$. Initially the Bell pair of $B$ and $C$ is prepared on $q_3$ and $q_5$ correspondingly. Then in order to make tomography of $B$, we move it from $q_3$ to $q_1$ by applying SWAP gate and then return it back on $q_3$ by another SWAP gate. Next, $A$ is prepared on $q_1$ in the state $|\psi\rangle = [\sqrt{3}/2 \ e^{i\pi/4}/2]^T$ by applying $W = R_x(\pi/4)R_y(\pi/2)R_z(\pi/2)R_y(\pi/4)^T$ to $|0\rangle$. Finally, the Bell measurement on $q_1$ and $q_3$ is performed.

The postselection is made with respect to the both 0 outcome, that corresponds to $|\Phi^+\rangle$ outcome. We apply barriers (denoted by vertical dash lines in Fig. 6(d)) to ensure that the tomography measurements of $\eta_B$ and $\eta_C$ are performed before preparation of $|\psi\rangle$. Also note that in the case of $\eta_A$ and $\eta_C$ tomography, an additional barrier between SWAP gates is put in order to prevent their removing by a transpiler. In total, three circuits corresponding to the tomography of $\eta_A$, $\eta_B$, and $\eta_C$, are run. For each circuit, $N = 20000$ shots (numbers of run) are used.

The tomography results are shown in Fig. 7. Here we show both the full two-time tensors and their reduced forward and backward evolving parts. One can see that obtained tensors are noisy versions of tensors given by
Figure 6. In (a) the scheme of postselective quantum teleportation protocol is depicted. If the postselection is performed with respect to Bell measurement outcome $|Φ^\pm\rangle$ on $A$ and $B$, the teleported state $|ψ\rangle$ can appear on $C$ even before its preparation on $A$. In (b) two-time tensors $η_A$, $η_B$, and $η_C$ are depicted. No possible decoherence effects are taken into account. In (c) the architecture of 7-qubit superconducting processor ibm_oslo is shown. Connections between qubits $q_0, \ldots, q_6$ correspond to an ability to apply CNOT gates. In (d) circuits for the tomography of $A, B, C$ in teleportation experiment, shown in (a), are depicted. Vertical dashed lines shows ‘barriers’, which ensure that each next part of the circuit is realized after the previous one. The path of the state $|ψ\rangle$ through the circuit is highlighted.

Eq. (61). This fact can be explained by an influence of decoherence processes and effects of the finite length statistics. We note that decoherence affects both the quality of gates implementation, especially two-qubit ones, and read-out measurements. It affects a ‘purity’ of the post-selection, and an accuracy of the tomography. We also note that approximately $N/4$ outcomes are used in the tomography due to the postselection condition.

The corresponding fidelities with respect to ideal states, and linear entropies, defined as $S(\rho) = 1 − \text{Tr}(\rho^2)$, are shown in Table I. One can see that the fidelity of the state $|ψ\rangle$ drops down during its travel through the time-reversal path: It starts at 0.89 on $η_A$, then reduces to 0.79 on $η_B$, and finally drops to 0.74 on $η_C$. At the same time, the linear entropy increases from 0.2 through 0.25 to 0.37. This state corruption corresponds to a propagation of $|ψ\rangle$ along its own ‘thermodynamic’ time arrow different from the time arrow of a classical observer. So we see that the two-time tensor formalism allows one to study irreversible processes occurring with quantum states during their travel along non-trivial postselection-induced space-time trajectories.

We note that the observed behavior can be modelled, e.g., by adding depolarizing noise to the Bell state preparation and measurement. Namely, if we suggest that $B$ and $C$ are prepared in the state $f_{\text{prep}} [Φ^+] \langle Φ^+ | + (1 – f_{\text{prep}}) ρ_{\text{mix}} \otimes ρ_{\text{mix}}$ and the postselection effect has the form $f_{\text{ms}} [Φ^+] \langle Φ^+ | + (1 – f_{\text{ms}}) ρ_{\text{mix}} \otimes ρ_{\text{mix}}$, then we obtain

$$η_A = |ψ\rangle \langle ψ | \otimes ρ_{\text{mix}},$$
$$η_B = ρ_{\text{mix}} \otimes [f_{\text{prep}} |ψ\rangle \langle ψ | + (1 – f_{\text{prep}}) ρ_{\text{mix}}],$$
$$η_C = [f_{\text{prep}} f_{\text{ms}} |ψ\rangle \langle ψ | + (1 – f_{\text{prep}} f_{\text{ms}}) ρ_{\text{mix}}] \otimes ρ_{\text{mix}},$$

where noise parameters $f_{\text{prep}}$ and $f_{\text{ms}}$ belong to $[0, 1]$. The form of two-time tensors in Eq. (62) captures the decrease of fidelity and the entropy growth shown in Table I.

Finally, it is worth reminding that even in the presence of this kind of time travel, no logical paradoxes, such as ‘grandfather paradox’, appears [52, 54]. The reason for this is the fundamental impossibility of forcing the desired postselection outcome. In the cases of alternative Bell measurement outcomes $|Ψ^+\rangle$, $|Ψ^-\rangle$, or $|Φ^-\rangle$, the state $|ψ\rangle$ also goes through a time-reversal trajectory, but acquires an additional unitary transformation $u$, given by $σ_x$, $σ_y$ or $σ_z$ correspondingly, in its ‘reflection’ from the Bell measurement (note that this is exactly the transformation which Bob undo after obtaining a message from Alice in the end of the quantum teleportation protocol). Therefore, for the general postselection condi-
tion, we have

\[ \eta_A = |\psi\rangle \langle \psi| \otimes \rho_{\text{mix}}, \]
\[ \eta_B = \rho_{\text{mix}} \otimes u |\psi\rangle \langle \psi| u^\dagger, \]
\[ \eta_C = u |\psi\rangle \langle \psi| u^\dagger \otimes \rho_{\text{mix}}. \] (63)

By removing postselection condition completely, we arrive to

\[ \eta_A = |\psi\rangle \langle \psi| \otimes \rho_{\text{mix}} \quad \eta_B = \eta_C = \rho_{\text{mix}} \otimes \rho_{\text{mix}}, \] (64)

where no any time-reversal phenomenon can be revealed. However, we note that \( \rho_{\text{mix}} \)s have different physical meaning in different parts of two-time tensors. In particular, \( \eta_A \downarrow = \eta_B \downarrow = \eta_C \uparrow = \rho_{\text{mix}} \) correspond to the uncertainty of the Bell measurement, while \( \eta_B \downarrow = \eta_C \uparrow = \rho_{\text{mix}} \) corresponds to the uncertainty about particle’s future.

VI. CONCLUSION AND OUTLOOK

In the present work, we have developed the two-time tensor formalism, which unifies previously proposed time symmetrized two-state (density) vector formalism and the standard ‘no postselection’ formalism in general manner. This goal is achieved by considering a generalized postselection measurement, whose particular postselection outcome is given by an arbitrary POVM effect \( E_{\text{post}} \). By smoothly shifting between the limiting cases of \( E_{\text{post}} \) being a rank-one projector (the case of two-state vectors) and identity operator (the case of no postselection), we cover a large number of possible experimental setups, especially, where decoherence effects can not be neglected.

We have seen that the concept of a two-time tensor \( \eta \) generalizes the concept of a quantum state \( \rho \) in the standard formalism (in no postselection case, \( \eta = \rho \otimes \rho_{\text{mix}} \)), a concept of two-time vector \( (|\psi_{\text{pre}}\rangle \langle \phi_{\text{post}}|) \) (in this case \( \eta = |\psi_{\text{pre}}\rangle \langle \psi_{\text{pre}}| \otimes |\phi_{\text{post}}\rangle \langle \phi_{\text{post}}| \)), as well as other previously considered objects as a generalized two-state vector [59], two-state density vector [60], and mixed two-state vector [37]. We have derived expressions for outcome probabilities of generalized measurements, and also mean and weak values of Hermitian observables. We have also considered practical tomography schemes for reconstructing unknown single-qubit two-time tensors. Namely, we developed two schemes based MUB and SIC-POVM approaches correspondingly. Finally, we have applied the developed formalism and the SIC-POVM-based tomography technique in order to study experimentally a time-reversal quantum state propagation in a quantum teleportation on a noisy cloud-accessible superconducting processor.

The author believes that the presented formalism would be helpful for studying and developing quantum information processing protocols with postselection. Moreover, it rises some fundamental question related to observed postselection-induced phenomena. Can the employed formalism be for studying CTC models other from projective ones [56, 65]? What kind of master equations can describe irreversibility on postselection-induced time arrows? How Markovian and non-Markovian effects are different for standard evolution of a quantum state along the ‘macroscopic’ time-arrow and postselection-induced time arrows? Can the employed two-time tensor formalism be used to study of the inherent time-asymmetry of
the macroscopic world? What kind of effects one can expect with a ‘weak postselection’, where $E_{\text{post}}$ is close but not equal to identity, and so on.

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[1] S. Watanabe, “Symmetry of physical laws, part iii. prediction and retrodiction,” Rev. Mod. Phys. 27, 179 (1955).
[2] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, “Time symmetry in the quantum process of measurement,” Phys. Rev. 134, B1410 (1964).
[3] Y. Aharonov and L. Vaidman, “The two-state vector formalism: an updated review,” Time in quantum mechanics, 399–447 (2008).
[4] Y. Aharonov, D. Z. Albert, and L. Vaidman, “How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100,” Phys. Rev. Lett. 60, 1351 (1988).
[5] Y. Aharonov and L. Vaidman, “Properties of a quantum system during the time interval between two measurements,” Phys. Rev. A 41, 11 (1990).
[6] A. J. Leggett, “Comment on “how the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100”,” Phys. Rev. Lett. 62, 2325 (1989).
[7] A. Peres, “Quantum measurements with postselection,” Phys. Rev. Lett. 62, 2326 (1989).
[8] Y. Aharonov and L. Vaidman, “Aharonov and vaidman reply,” Phys. Rev. Lett. 62, 2327 (1989).
[9] S. Pang, J. Dressel, and T. A. Brun, “Entanglement-assisted weak value amplification,” Phys. Rev. Lett. 113, 030401 (2014).
[10] A. N. Jordan, J. Martínez-Rincón, and J. C. Howell, “Technical advantages for weak-value amplification: when less is more,” Phys. Rev. X 4, 011031 (2014).
[11] J. Harris and J. S. Boyd, R. W. and Lundeen, “Weak value amplification can outperform conventional measurement in the presence of detector saturation,” Phys. Rev. Lett. 118, 070802 (2017).
[12] L. Xu, Z. Liu, A. Datta, G. C. Knee, J. S. Lundeen, Y.-q. Lu, and L. Zhang, “Approaching quantum-limited metrology with imperfect detectors by using weak-value amplification,” Phys. Rev. Lett. 125, 080501 (2020).
[13] O. Hosten and P. Kwiat, “Observation of the spin hall effect of light via weak measurements,” Science 319, 787–790 (2008).
[14] P. B. Dixon, D. J. Starling, A. N. Jordan, and J. C. Howell, “Ultrasonic beam deflection measurement via interferometric weak value amplification,” Phys. Rev. Lett. 102, 173601 (2009).
[15] G. Strübi and C. Bruder, “Measuring ultrasmall time delays of light by joint weak measurements,” Phys. Rev. Lett. 110, 083605 (2013).
[16] X.-Y. Xu, Y. Kedem, K. Sun, L. Vaidman, C.-F. Li, and G.-C. Guo, “Phase estimation with weak measurement using a white light source,” Phys. Rev. Lett. 111, 033604 (2013).
[17] L. Zhou, Y. Turek, C.P. Sun, and F. Nori, “Weak-value amplification of light deflection by a dark atomic ensemble,” Phys. Rev. A 88, 053815 (2013).
[18] G. Jayaswal, G. Mistura, and M. Merano, “Observing angular deviations in light-beam reflection via weak measurements,” Opt. Lett. 39, 6237–6240 (2014).
[19] O. S. Magaña-Loaiza, M. Mirhosseini, B. Rodenburg, and R. W. Boyd, “Amplification of angular rotations using weak measurements,” Phys. Rev. Lett. 112, 200401 (2014).
[20] K. Lyons, J. Dressel, A. N. Jordan, J. C. Howell, and P. G Kwiat, “Power-recycled weak-value-based metrology,” Phys. Rev. Lett. 114, 170801 (2015).
[21] M. Hallají, A. Feizpour, G. Dnochowski, J. Sinclair, and A. M Steinberg, “Weak-value amplification of the non-linear effect of a single photon,” Nat. Phys. 13, 540–544 (2017).
[22] D. R. M. Arvidsson-Shukur, Y. N. Halpern, H. V. Lepage, A. A. Lasek, C. H. W. Barnes, and S. Lloyd, “Quantum advantage in postselected metrology,” Nat. Commun. 11, 1–7 (2020).
[23] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, “Colloquium: Understanding quantum weak values: Basics and applications,” Rev. Mod. Phys. 86, 307 (2014).
[24] S. Aaronson, “Quantum computing, postselection, and probabilistic polynomial-time,” Proc. R. Soc. A: Math. Phys. Eng. Sci. 461, 3473–3482 (2005).
[25] M. S. Leifer and R. W. Spekkens, “Pre-and post-selection paradoxes and contextuality in quantum mechanics,” Phys. Rev. Lett. 95, 200405 (2005).
[26] J. Tollaksen, “Pre-and post-selection, weak values and contextuality,” J. Phys. A: Math. Theor. 40, 9033 (2007).
[27] M. F. Pusey, “Anomalous weak values are proofs of contextuality,” Phys. Rev. Lett. 113, 200401 (2014).
[28] M. F. Pusey and M. S. Leifer, “Logical pre-and post-selection paradoxes are proofs of contextuality,” arXiv preprint arXiv:1506.07850 (2015).
[29] R. Kunjwal, M. Lostaglio, and M. F. Pusey, “Anomalous weak values and contextuality: robustness, tightness, and imaginary parts,” Phys. Rev. A 100, 042116 (2019).
[30] Y. Aharonov, A. Botero, S. Popescu, B. Reznik, and J. Tollaksen, “Revisiting hardy’s paradox: counterfactual statements, real measurements, entanglement and weak values,” Phys. Lett. A 301, 130–138 (2002).
[31] J. Dressel, S. Agarwal, and A. N. Jordan, “Contextual values of observables in quantum measurements,” Phys. Rev. Lett. 104, 240401 (2010).
[32] S. Kocsis, B. Braverman, S. Ravets, M. J. Stevens, R. P. Mirin, L. K. Shalm, and A. M. Steinberg, “Observing the average trajectories of single photons in a two-slit interferometer,” Science 332, 1170–1173 (2011).
[33] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, “Direct measurement of the quantum wavefunction,” Nature 474, 188–191 (2011).
[34] A. Danan, D. Farfurnik, S. Bar-Ad, and L. Vaidman, “Asking photons where they have been,” Phys. Rev. Lett. 111, 240402 (2013).
[35] L. Vaidman, “Past of a quantum particle,” Phys. Rev. A 87, 052104 (2013).
[36] L. Vaidman, “Tracing the past of a quantum particle,” Phys. Rev. A 89, 024102 (2014).
[37] L. Vaidman, A. Ben-Israel, J. Dziewior, L. Knips, M. Weißl, J. Meinecke, C. Schwemmer, R. Ber, and H. Weinfurter, “Weak value beyond conditional expectation value of the pointer readings,” Phys. Rev. A 96, 032114 (2017).
[38] X.-Y. Xu, W.-W. Pan, Qin-Qin Wang, J. Dziewior, L. Knips, Y. Kedem, K. Sun, J.-S. Xu, Y.-J. Han, C.-F. Li, G.-C. Guo, and L. Vaidman, “Measurements of nonlocal variables and demonstration of the failure of the product rule for a pre-and postselected pair of photons,” Phys. Rev. Lett. 122, 100405 (2019).
[39] V. Cimini, I. Gianani, F. Picentini, I. P. Degiovanni, and M. Barbieri, “Anomalous weak values, fisher information, and contextuality in generalized quantum measurements,” Quantum Sci. Technol. 5, 025007 (2020).
[40] E. Rebuffello, F. Picentini, A. Avella, M. A. Souza, M. Gramegna, J. Dziewior, E. Cohen, L. Vaidman, L. P. Degiovanni, and M. Genovese, “Anomalous weak values via a single photon detection,” Light Sci. Appl. 10, 1–6 (2021).
[41] A. W. Harrow, A. Hassidim, and S. Lloyd, “Quantum algorithm for linear systems of equations,” Phys. Rev. Lett. 103, 150502 (2009).
[42] D. R. M. Arvidsson-Shukur and C. H. W. Barnes, “Quantum counterfactual communication without a weak trace,” Phys. Rev. A 94, 062303 (2016).
[43] D. R. M. Arvidsson-Shukur, A. N. O. Gottfries, and C. H. W. Barnes, “Evaluation of counterfactuality in counterfactual communication protocols,” Phys. Rev. A 96, 062316 (2017).
[44] D. R. M. Arvidsson-Shukur and C. H. W. Barnes, “Post-selection and counterfactual communication,” Phys. Rev. A 99, 060102 (2019).
[45] L. Vaidman, “Analysis of counterfactuality of counterfactual communication protocols,” Phys. Rev. A 99, 052127 (2019).
[46] A. Wander, E. Cohen, and L. Vaidman, “Three approaches for analyzing the counterfactuality of counterfactual protocols,” Phys. Rev. A 104, 012610 (2021).
[47] L. Vaidman, “Backward evolving quantum states,” J. Phys. A Math. Theor. 40, 3275 (2007).
[48] M. Laforest, J. Baugh, and R. Laflamme, “Time-reversal formalism applied to maximal bipartite entanglement: Theoretical and experimental exploration,” Phys. Rev. A 73, 032323 (2006).
[49] Y. Aharonov, S. Popescu, J. Tollaksen, and L. Vaidman, “Multiple-time states and multiple-time measurements in quantum mechanics,” Phys. Rev. A 79, 052110 (2009).
[50] B. Cooecke, “Quantum picturalism,” Contemp. Phys. 51, 59–83 (2010).
[51] George Svetlichny, “Time travel: Deutsch vs. teleportation,” Int. J. Theor. Phys. 50, 3903–3914 (2011).
[52] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, and Y. Shikano, “Quantum mechanics of time travel through post-selected teleportation,” Phys. Rev. D 84, 025007 (2011).
[53] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L. A. Rozema, A. Darabi, Y. Soudagar, L. K. Shalm, and A. M. Steinberg, “Closed timelike curves via postselection: theory and experimental test of consistency,” Phys. Rev. Lett. 106, 040403 (2011).
[54] S. M. Korotaeve and E. O. Kiktenko, “Quantum causality in closed timelike curves,” Phys. Scr. 90, 085101 (2015).
[55] O. Oreshkov and N. J. Cerf, “Operational formulation of time reversal in quantum theory,” Nat. Phys. 11, 853–858 (2015).
[56] A. V. Shepelin, A. M. Rostom, V. A. Tomilin, and L. V. Il’ichov, “Multiverse motives by closed time-like curves,” in J. Phys. Conf. Ser., Vol. 2081 (IOP Publishing, 2021) p. 012029.
[57] T. Jennewein, G. Weihs, J.-W. Pan, and A. Zeilinger, “Experimental nonlocality proof of quantum teleportation and entanglement swapping,” Phys. Rev. Lett. 88, 017903 (2001).
[58] E. Megidish, A. Haley, T. Shacham, T. Divr, L. Dovrat, and H. S. Eisenberg, “Entanglement swapping between photons that have never coexisted,” Phys. Rev. Lett. 110, 210403 (2013).
[59] Y. Aharonov and L. Vaidman, “Complete description of a quantum system at a given time,” J. Phys. A Math. 24, 2315 (1991).
[60] R. Silva, Y. Guryanova, N. Brunner, N. Linden, A. J. Short, and S. Popescu, “Pre-and postselected quantum states: Density matrices, tomography, and kraus operators,” Phys. Rev. A 89, 012121 (2014).
[61] I. Luchnikov, A. Ryzhov, S. Filippov, and H. Ouerdane, “Qgopt: Riemannian optimization for quantum technologies,” SciPost Phys. 10, 079 (2021).
[62] I. A. Luchnikov, M. E. Krechetov, and S. N. Filipov, “Riemannian geometry and automatic differentiation for optimization problems of quantum physics and quantum technologies,” New J. Phys. 23, 073006 (2021).
[63] E. O. Kiktenko, A. O. Malyshev, A.S. Mastiukova, V. I. Man’ko, A. K. Fedorov, and D. Chruściński, “Probability representation of quantum dynamics using pseudostochastic maps,” Phys. Rev. A 101, 052320 (2020).
[64] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels,” Phys. Rev. Lett. 70, 1895 (1993).
[65] D. Deutsch, “Quantum mechanics near closed timelike lines,” Phys. Rev. D 44, 3197 (1991).