On the minimax principle for Coulomb–Dirac operators

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Abstract

Let \( q \) and \( v \) be symmetric sesquilinear forms such that \( v \) is a form perturbation of \( q \). Then we can associate a unique self-adjoint operator \( B \) to \( q + v \). Assuming that \( B \) has a gap \((a, b) \subset \mathbb{R}\) in the essential spectrum, we prove a minimax principle for the eigenvalues of \( B \) in \((a, b)\) using a suitable orthogonal decomposition of the domain of \( q \).

This allows us to justify two minimax characterisations of eigenvalues in the gap of three–dimensional Dirac operators with electrostatic potentials having strong Coulomb singularities.

1 Introduction and main results

1.1 General discussion

Since the early days of quantum mechanics the Dirac operators with potentials having a Coulomb singularity are used to describe relativistic electrons in atomic fields. We say that a measurable Hermitian \( 4 \times 4 \)–matrix function \( V \) on \( \mathbb{R}^3 \) belongs to the class \( \mathcal{P}_\nu \), if for some \( \tilde{\nu} \in [0, \nu) \) the inequalities

\[
0 \geq V(x) \geq -\frac{\tilde{\nu}}{|x|}1_{C^4}\quad \text{hold for almost every } x \in \mathbb{R}^3.
\] (1.1)

If \( V \in \mathcal{P}_1 \) and (1.1) is satisfied with \( \tilde{\nu} = \nu \), we say that \( V \in \overline{\mathcal{P}}_\nu \).

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Let $H_0$ be the free Dirac operator (see Appendix). If $V \in \mathcal{P}_1$, one can define a physically meaningful self-adjoint operator $H$ formally corresponding to $H_0 + V$, see Subsection 1.3 below. For the essential spectra we have (see [9])

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = (-\infty, -1] \cup [1, \infty).$$

The eigenvalues of $H$ in $(-1, 1)$ are of particular interest; for example, the lowest eigenvalue $\lambda_1$ in this gap is interpreted as the ground state energy of the electron.

In the rest of this subsection we assume that $V$ is an electric potential, i.e., is proportional to $\frac{1}{C^4}$.

Talman [11] and Datta and Devaiah [2] proposed a formal minimax characterisation of $\lambda_1$:

$$\lambda_1 = \min_{x \in \text{Ran}T_+} \max_{y \in \text{Ran}T_-} \frac{\langle x + y, (H_0 + V)(x + y) \rangle}{\|x + y\|^2}.$$

Here $T_\pm$ are the projectors on the upper and lower two components of 4-spinors, i.e.,

$$T_+ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} := \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \quad T_- \begin{pmatrix} \varphi \\ \psi \end{pmatrix} := \begin{pmatrix} 0 \\ \psi \end{pmatrix}, \quad \text{for } \varphi, \psi \in L^2(\mathbb{R}^3, \mathbb{C}^4). \quad (1.2)$$

Esteban and Séré [5] replaced $T_\pm$ by the spectral projectors of the unperturbed Dirac operator $H_0$

$$P_+ := P_{H_0}([-1, \infty)), \quad P_- := P_{H_0}((\infty, -1]) \quad (1.3)$$

and announced that for $V \in \mathcal{P}_{1/2}$ the $k$th eigenvalue in the gap (counted from below with multiplicity) coincides with the minimax level

$$\lambda_k = \inf_{\text{subspace of } P_+ \mathcal{H}^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \times (\mathbb{C} \oplus P_- \mathcal{H}^{1/2}(\mathbb{R}^3, \mathbb{C}^4)) \setminus \{0\}} \sup_{x \in \mathbb{C} \oplus P_- \mathcal{H}^{1/2}(\mathbb{R}^3, \mathbb{C}^4)} \frac{h_0[x] + v[x]}{\|x\|^2}, \quad (1.4)$$

where $h_0$ and $v$ are the quadratic forms of $H_0$ and $V$, respectively.

A general result on the variational characterisation of the eigenvalues of operators with gaps in the essential spectrum was proved by Griesemer and Siedentop [7]. As a corollary they found that the variational characterisation of the lowest eigenvalue by Talman, Datta, and Devaiah is correct for $-21 < V \leq 0$ provided $V(x) \to 0$ as $|x| \to \infty$. Griesemer, Lewis, and
Siedentop \cite{6} proved that the approach of \cite{5} holds for $V \in P_\gamma$ where $\gamma \approx 0.3$ is the real solution of $2\gamma^3 - 3\gamma^2 + 4\gamma = 1$. Dolbeault, Esteban and Séré \cite{4} extended the result of \cite{5} to a class of $V$ which, under an extra assumption of slow decay at infinity, contains $P_{2/(2+\pi/2)}$. In \cite{3}, the same authors have claimed the validity of both Esteban–Séré and Talman–Datta–Devaiah minimax principles for $V \in P_1$. However, they replaced $P_\pm H_0^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ in (1.4) by $P_\pm C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ (accordingly, $T_\pm C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$), and their argument relies on the statement that $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ is an operator core for $H$, which is only true for $V \in P_{\sqrt{3}/2}$, see Theorem 2.1.6 of \cite{1}.

Trying to overcome this difficulty we have returned to the minimax principle (1.4). The corresponding abstract formulation, which is the main result of our paper, naturally applies to self–adjoint operators obtained as form perturbations of symmetric sesquilinear forms. Moreover, we only deal with the domain of the unperturbed quadratic form. In the case of Dirac operators we prove the minimax characterisation of eigenvalues (1.4) for all $V \in P_1$ and a version of the Talman–Datta–Devaiah minimax principle for $V \in P_{2/(2+\pi/2)}$. Our proofs are based on the ones of \cite{3} and \cite{7}, but we consistently work with forms instead of operators.

The main abstract result of the article is explicitly formulated in Subsection 1.2, and the applications to Dirac operators can be found in Subsection 1.3. In Section 2 we give the definition of form perturbations, which is the key element in the construction of the operators we study. The rest of the paper contains proofs. In the appendix the (very standard) definition of the free Dirac operator is given for convenience.

Throughout the text for any sesquilinear form $f : \Omega \times \Omega \to \mathbb{C}$ (linear in the second argument) we say that $f$ is defined on $D[f] := \Omega$. The corresponding quadratic form is defined on $\Omega$ by $f[x] := f[x, x]$. If we start from a quadratic form $f$ on $D[f]$, then the corresponding sesquilinear form is naturally defined on $D[f]$ by

$$f[x, y] = \frac{1}{4}(f[x + y] - f[x - y] - if[x + iy] + if[x - iy]).$$

For a linear operator $A$ its domain is denoted by $D(A)$.

### 1.2 The abstract minimax principle

In order to treat the Dirac operators with strong Coulombic singularities, Nenciu \cite{9} has introduced the concept of form perturbations of self–adjoint
operators, which generalises the pseudo–Friedrichs extension of Kato ([8], VI.3.4). We will slightly modify this definition and introduce form perturbations of symmetric sesquilinear forms in Section [2]. The concept of form perturbation is needed for the following theorem:

**Theorem 1.** Let a symmetric sesquilinear form \( v \) be a form perturbation of a symmetric sesquilinear form \( q \). Then there exists a unique self–adjoint operator \( B \) satisfying the conditions

\[
\begin{aligned}
& (j) \quad D(B) \subset D[q]; \\
& (jj) \quad \langle Bx, y \rangle = q[x, y] + v[x, y] \quad \text{for all } x \in D(B), \ y \in D[q].
\end{aligned}
\]

Moreover,

\[
D(B) = \left\{ x \in D[q] : \sup_{y \in D[q]\{0\}} \frac{|q[x, y] + v[x, y]|}{\|y\|} < \infty \right\}. \tag{1.7}
\]

The proof of Theorem 1 is identical to the one of Theorem 2.1 of [9].

Our main result is a minimax principle for the eigenvalues of \( B \) in the gaps of its essential spectrum \( \sigma_{\text{ess}}(B) \):

**Theorem 2.** Let a symmetric sesquilinear form \( v \) be a form perturbation of a symmetric sesquilinear form \( q \). Let \( \mathfrak{H}_\pm \) be orthogonal subspaces of \( \mathfrak{H} \) such that \( \mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_- \) and \( \Lambda_+, \ \Lambda_- \) the projectors onto \( \mathfrak{H}_+ \) and \( \mathfrak{H}_- \), respectively. We assume that

\[
\begin{aligned}
& (i) \quad \mathfrak{D}_\pm := \Lambda_\pm D[q] \subset D[q]; \\
& (ii) \quad a := \sup_{x \in \mathfrak{D}_- \{0\}} \frac{q[x] + v[x]}{\|x\|^2} < \infty; \\
& (iii) \quad \lambda_1 > a, \quad \text{where} \\
& \quad \lambda_k := \inf_{\mathfrak{V} \text{ subspace of } \mathfrak{D}_+} \sup_{x \in (\mathfrak{H} \oplus \mathfrak{D}_-) \{0\}} \frac{q[x] + v[x]}{\|x\|^2}. \tag{1.10}
\end{aligned}
\]

Let \( B \) be the self–adjoint operator defined in Theorem 1 and

\[
b := \inf \left( \sigma_{\text{ess}}(B) \cap (a, \infty) \right) \in [a, \infty].
\]

For \( k \in \mathbb{N} \), we denote by \( \mu_k \) the \( k \)-th eigenvalue of \( B \) in the interval \( (a, b) \) in non-decreasing order, counted with multiplicity, if such eigenvalue exists. If there is no \( k \)-th eigenvalue, we let \( \mu_k := b \).

Then

\[
\lambda_k = \mu_k \quad \text{for all } \ k \in \mathbb{N}. \tag{1.12}
\]
The proof of Theorem 2 can be found in Section 3.

1.3 Application to Dirac operators with Coulomb singularities

In this subsection we elaborate and improve upon the results of [3] and [7] using Theorem 2. In the following $h_0$ is the quadratic form of the free Dirac operator $H_0$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with $D[h_0] = H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, see Appendix for more details. Let $V \in \mathcal{P}_1$, see (1.1), and $v$ be the sesquilinear form of $V$.

It is shown in [9] that $v$ is a form perturbation of $h_0$ for $V \in \mathcal{P}_1$. Applying Theorem 1 we define a unique self–adjoint operator $H$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ that satisfies

$$D(H) \subset H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$$

and

$$\langle Hx, y \rangle = h_0[x, y] + v[x, y] \quad \text{for all } x \in D(H) \text{ and } y \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4).$$

This construction of $H$ is by Nenciu [9] and coincides with the self–adjoint extensions constructed by Schmincke [10] and Wüst [15].

We start with a minimax principle choosing $\Lambda$ to be the spectral projectors $P_\pm$ defined in (1.3).

**Theorem 3.** Let $h_0$, $v$ and $H$ be as defined above. Then the $k$th eigenvalue $\mu_k$ of $H$ in $(-1, 1)$, counted from below with multiplicity, is given by

$$\mu_k = \inf_{\mathcal{D}_+ \text{ subspace of } \mathcal{D}_+} \sup_{\dim \mathcal{D} = k} \frac{h_0[x] + v[x]}{\|x\|^2},$$

where $\mathcal{D}_+: = P_+ H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$.

Another possible choice of $\Lambda_\pm$ are $T_\pm$, see (1.2). In this case we will have to further restrict the maximal admissible strength of the Coulomb singularity:

**Theorem 4.** Let $h_0$, $v$ and $H$ be as defined above. Assume furthermore that $V \in \mathcal{P}_2((2/\pi + \pi/2)$. Then the $k$th eigenvalue $\mu_k$ of $H$ in $(-1, 1)$, counted from below with multiplicity, is given by

$$\mu_k = \inf_{\mathcal{T}_+ \text{ subspace of } \mathcal{T}_+} \sup_{\dim \mathcal{T} = k} \frac{h_0[x] + v[x]}{\|x\|^2},$$

where $\mathcal{T}_+: = T_+ H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$.

The proofs of Theorems 3 and 4 can be found in Section 4.
2 Form perturbations

In this section we define the concept of form perturbations for symmetric sesquilinear forms.

Let $q$ be a symmetric sesquilinear form on a dense domain $D[q]$ in a complex Hilbert space $\mathcal{H}$. We assume that two orthogonal projections $P_\pm$ with $P_+ + P_- = 1_{\mathcal{H}}$ satisfy

1. $P_\pm D[q] \subset D[q]$;
2. $q[x_+] > 0$ for all $x_+ \in P_+ D[q] \setminus \{0\}$;
3. $q[x_-] \leq 0$ for all $x_- \in P_- D[q]$;
4. $q[x_+, x_-] = 0$ for all $x_+ \in P_+ D[q]$ and $x_- \in P_- D[q]$.

For $\alpha > 0$ we define the inner product in $D[q]$ by

$$\langle x, y \rangle_\alpha := q[P_+ x, P_+ y] - q[P_- x, P_- y] + \alpha \langle x, y \rangle$$  \hspace{1cm} (2.1)

and assume that

5. $\mathfrak{Q}_\alpha := (D[q], \langle \cdot, \cdot \rangle_\alpha)$ is a Hilbert space (i.e., is complete).

Note that

$$\| \cdot \|_\alpha^2 \leq \| \cdot \|_\alpha^2 \leq \frac{\tilde{\alpha}}{\alpha} \| \cdot \|_\alpha^2 \quad \text{for} \quad \tilde{\alpha} > \alpha > 0,$$  \hspace{1cm} (2.2)

so the topology of $\mathfrak{Q}_\alpha$ is independent of $\alpha > 0$. We introduce

$$U := 1 \oplus (-1) \quad \text{on} \quad P_+ \mathcal{H} \oplus P_- \mathcal{H}.$$

6. Let $v$ be a symmetric sesquilinear form in $\mathcal{H}$ with $D[v] \supseteq D[q]$.

7. We assume that $v$ is bounded on $\mathfrak{Q}_\alpha$, i.e. there exists a constant $C_\alpha > 0$ such that

$$|v[x, y]| \leq C_\alpha \|x\|_\alpha \|y\|_\alpha \quad \text{for all} \quad x, y \in D[q].$$  \hspace{1cm} (2.3)

Then $v$ defines on $\mathfrak{Q}_\alpha$ a bounded self–adjoint operator $V_\alpha$ by

$$\langle V_\alpha x, y \rangle_\alpha = v[x, y] \quad \text{for all} \quad x, y \in D[q].$$  \hspace{1cm} (2.4)

Note that by (2.2) (7) holds (or not) for all $\alpha > 0$ at the same time.
(8) At last, we assume that for $\alpha$ big enough the operator $U + V_\alpha$ has a bounded inverse in $\mathcal{Q}_\alpha$.

**Definition 5.** If the assumptions (1)–(8) are satisfied, we say that $v$ is a form perturbation of $q$.

**Lemma 6.** If $q$ is a sesquilinear form of a self–adjoint operator $Q$, then the assumptions (1)–(4) are satisfied if and only if

$$P_+ = P_Q^+ := P_Q((0, \infty)), \quad P_- = P_Q^- := P_Q((\infty, 0]),$$

where $P_Q(\Omega)$ is the spectral projector of $Q$ corresponding to a Borel set $\Omega \subset \mathbb{R}$.

**Proof.** Since $q[x_+, x_-] = 0$ for all $x_\pm \in P_\pm D[q]$, $P_+ Q P_- = P_- Q P_+ = 0$ holds. Hence $[P_+, Q] = 0$ and, therefore, $[P_+, P_Q^\pm] = 0$. We thus have

$$0 \leq q[P_\pm P_Q^\mp x] = q[P_Q^\mp P_\pm x] \leq 0 \quad \text{for all} \quad x \in D[q].$$

This implies

$$P_\pm P_Q^\mp x = P_Q^\mp P_\pm x = 0 \quad \text{for all} \quad x \in D[q].$$

Thus for every $x \in D[q]$

$$P_Q^\pm x = P_Q^\pm (P_+ x + P_- x) = (1 - P_Q^\mp) P_\pm x = P_\pm x.$$ \hfill \Box

**Remark 7.** If $q$ is a sesquilinear form of a self–adjoint operator $Q$, and $v$ is a form perturbation of $q$, then by Lemma 6 $v$ is a form perturbation of $Q$ in the sense of Nenciu [9].

### 3 Proof of the abstract minimax principle

The inequality $\lambda_k \leq \mu_k$ for all $k \in \mathbb{N}$ follows from the proof of Theorem 1 of [7]. It remains to prove that $\lambda_k \geq \mu_k$. We follow the ideas of the proof of Theorem 1.1 of [3], but consistently work with forms instead of operators.

We first introduce a sesquilinear form

$$s := q + v \quad \text{on} \quad D[q] \quad (3.1)$$

and

$$s_- : \mathcal{D}_- \times \mathcal{D}_- \to \mathbb{C}, \quad s_-[x_-, y_-] := -s[x_-, y_-]. \quad (3.2)$$
Furthermore, for $u > a$ let
\[ m_u : \mathcal{D} \to [0, \infty), \quad m_u[y_] := s_- [y_] + u \| y_- \|^2. \] (3.3)

By (1.9), $m_u^{1/2}$ are equivalent norms in $\mathcal{D}_-$. We denote the completion of $\mathcal{D}_-$ in $\mathcal{H}$ with respect to $m_{a+1}$ by $\overline{\mathcal{D}}_-$, and the unique continuous extensions of $m_u$ to $\overline{\mathcal{D}}_-$ by $\overline{m_u}$. Since $s_-$ is continuous with respect to $m_{a+1}$, we can uniquely extend it to
\[ \overline{s_-} : \overline{\mathcal{D}}_- \times \overline{\mathcal{D}}_- \to \mathbb{C}. \] (3.4)

For $x_+ \in \mathcal{D}_+$ and $u > a$ let
\[ \varphi_{u,x_+} : \mathcal{D}_- \to \mathbb{R}, \quad \varphi_{u,x_+}(y_-) := s[x_+ + y_-] - u \| x_+ + y_- \|^2. \] (3.5)

Then for $u > a$ and $x_+ \in \mathcal{D}_+$ we have
\[ \sup_{y_- \in \mathcal{D}_-} \varphi_{u,x_+}(y_-) = \sup_{y_- \in \mathcal{D}_-} \left( s[x_+] - u \| x_+ \|^2 + 2 \Re [s[x_+, y_-] - m_u[y_-]] \right). \] (3.6)

Since the norms $m_u^{1/2}$ are equivalent to each other,
\[ \sup_{y_- \in \mathcal{D}_-} \varphi_{u,x_+}(y_-) < \infty \text{ for } u > a \text{ if and only if } x_+ \in \mathcal{S}, \]
where
\[ \mathcal{S} := \left\{ x_+ \in \mathcal{D}_+ : \sup_{y_- \in \mathcal{D}_- \setminus \{0\}} \frac{|s[x_+, y_-]|}{m_{a+1/2}[y_-]} < \infty \right\} \subset \mathcal{D}_+. \] (3.7)

For $x_+ \in \mathcal{S}$ and $u > a$, $s[x_+, \cdot]$ extends to a linear bounded functional $s_{x_+}$ in the Hilbert space $(\overline{\mathcal{D}}_-, \overline{m_u})$. Hence by the Riesz’s theorem there exist a unique linear operator
\[ L_u : \mathcal{S} \to \overline{\mathcal{D}}_- \text{ such that } s_{x_+}(y_-) = \overline{m_u}[L_u x_+, y_-] \text{ for all } y_- \in \overline{\mathcal{D}}_. \] (3.8)

Let $\overline{\varphi_{u,x_+}}$ be the unique continuous extension of $\varphi_{u,x_+}$ to $\overline{\mathcal{D}}_-$ for $x_+ \in \mathcal{S}$. By (3.6) we have
\[ \sup_{y_- \in \mathcal{D}_-} \overline{\varphi_{u,x_+}}(y_-) = s[x_+] - u \| x_+ \|^2 + m_u[L_u x_+] - \inf_{y_- \in \mathcal{D}_-} \overline{m_u}[L_u x_+ - y_-]. \] (3.9)

This obviously implies that $L_u x_+$ is the unique maximiser in (3.9).
Lemma 8.

$$\lambda_k = \inf_{\mathcal{V} \text{ subspace of } \mathcal{S}, \dim \mathcal{V} = k} \sup_{x \in (\mathcal{V} \oplus \mathcal{D} - \{0\})} \frac{s[x]}{\|x\|^2}. \quad (3.10)$$

Proof. If for $x_+ \in \mathcal{D} \setminus \{0\}$ there exists $u \in (a, \infty)$ such that

$$\sup_{x_- \in \mathcal{D} - \{0\}} \frac{s[x_+ + x_-]}{\|x_+ + x_-\|^2} < u,$$

then by (3.2) and (3.3)

$$0 > \sup_{y_- \in \mathcal{D} - \{0\}} \frac{s[x_+ + y_-] - u\|x_+ + y_-\|^2}{\|x_+ + y_-\|^2} \geq \frac{1}{\|x_+\|^2} \sup_{y_- \in \mathcal{D} - \{0\}} \varphi_{u,x_+}(y_-)$$
holds. But then $x_+ \in \mathcal{S}$ and we can reformulate (1.11) as (3.10).

For $u > a$ we define

$$g_u : \mathcal{S} \to \mathbb{R}, \quad g_u[x_+] := \overline{\varphi_{u,x_+}}(L_u x_+); \quad (3.11)$$
$$n_u : \mathcal{S} \to \mathbb{R}, \quad n_u[x_+] := \|x_+\|^2 + \|L_u x_+\|^2. \quad (3.12)$$

Lemma 9. Assume that (1.8) and (1.9) are satisfied. If $a < u < u'$, then

$$\| \cdot \| \leq n_u^{1/2} \leq n_u^{1/2} \leq \frac{u' - a}{u - a} n_u^{1/2}; \quad (3.13)$$

$$(u' - u)n_u \leq g_u - g_{u'} \leq (u' - u)n_u. \quad (3.14)$$

Moreover, for any $u > a$:

$$\lambda_1 > u \iff g_u[x_+] > 0 \text{ for all } x_+ \in \mathcal{S} \setminus \{0\}; \quad (3.15)$$
$$\lambda_1 \geq u \iff g_u[x_+] \geq 0 \text{ for all } x_+ \in \mathcal{S}. \quad (3.16)$$

As a consequence, (1.10) is equivalent to

$$(iii') \text{ For some } u > a, \ g_u[x_+] \geq 0 \text{ for all } x_+ \in \mathcal{S}. \quad (3.17)$$

Proof. We define (recall (3.4))

$$B^\#: \overline{\mathcal{D} -} \to (\overline{\mathcal{D} -})^*, \quad (B^\# x_-)(y_-) := \overline{\varphi_{-x_-}}[y_-]. \quad (3.18)$$
and introduce the embedding operator

\[ J : \mathfrak{H} \to \mathfrak{H}^* \quad \text{where} \quad (Jx)(y) := \langle x, y \rangle. \]  

(3.19)

We first prove that the operator \( B_-^# + uJ : \overline{\mathfrak{D}_-} \to (\overline{\mathfrak{D}_-})^* \) is invertible for all \( u > a \).  

(3.20)

The injectivity follows from (1.9). Now for any \( f \in (\overline{\mathfrak{D}_-})^* \) there is \( c_f > 0 \) such that \( |f(y)| \leq c_f m_u^{-1/2}(y) \) for all \( y \in \overline{\mathfrak{D}_-} \). Hence by the Riesz representation theorem there exists \( x_u \in \overline{\mathfrak{D}_-} \) such that

\[ f(y) = m_u[x_u, y] = \overline{s_-}[x_u, y] + u\langle x_u, y \rangle \]

for all \( y \in \overline{\mathfrak{D}_-} \). This implies \( f(y) = ((B_-^# + uJ)x_u)(y) \) for all \( y \in \overline{\mathfrak{D}_-} \) which means that \( B_-^# + uJ : \overline{\mathfrak{D}_-} \to (\overline{\mathfrak{D}_-})^* \) is surjective.

We know that \( \overline{s_-} \) is a densely defined, closed and bounded below sesquilinear form in \( \mathfrak{H}_- \). By the Friedrichs theorem (see e.g. [14], Theorem 5.37) there exists a self-adjoint operator \( B_- \) such that

\[ D(B_-) := \{ x \in \overline{\mathfrak{D}_-} : \text{there exists } \tilde{x} \in \mathfrak{H}_- \text{ such that } \langle \tilde{x}, y \rangle = \overline{s_-}[x, y] \text{ for all } y \in \overline{\mathfrak{D}_-} \}, \]

\[ B_- x := \tilde{x} \quad \text{for } x \in D(B_-). \]  

(3.21)

(3.22)

By (3.18) we obtain

\[ D(B_-) = \{ x \in \overline{\mathfrak{D}_-} : \quad B_-^# x \in \mathfrak{H}_-^* \subset (\overline{\mathfrak{D}_-})^* \}, \]

\[ J(B_- x) = B_- x \quad \text{for all } x \in D(B_-). \]

(3.23)

As in Lemma 2.1 of [3], using the spectral decomposition of \( B_- \) we obtain for \( u' > u > a \)

\[ \|(B_- + u')^{-1}y\| \leq \|(B_- + u)^{-1}y\| \leq \frac{u' - a}{u - a} \|(B_- + u')^{-1}y\| \quad \text{for all } y \in \mathfrak{H}_-. \]

By the density of \( \mathfrak{H}_-^* \) in \( (\overline{\mathfrak{D}_-})^* \) we get

\[ \|(B_-^# + u'J)^{-1}y\| \leq \|(B_-^# + uJ)^{-1}y\| \leq \frac{u' - a}{u - a} \|(B_-^# + u'J)^{-1}y\| \]  

(3.23)
for all \( y \in \overline{D}^- \). Let us introduce (recall (3.8))
\[
S_+^\# : \mathcal{S} \to \overline{D}^-, \quad (S_+^\#x_+)(y) := s_{x_+}(y). 
\]
(3.24)

By (3.3), (3.18) and (3.20) we observe that
\[
L_u x_+ = (B_+^\# + uJ)^{-1}S_+^\#x_+.
\]
Substituting this into (3.23), we obtain
\[
\|L_u x_+\| \leq \|L_u' x_+\| \leq \frac{u' - a}{u - a} \|L_u' x_+\| \quad \text{for all } x_+ \in \mathcal{S}.
\]
This together with (3.12) implies (3.13). The remaining statements follow in the same way as in Lemma 2.1 of [3], where the role of \( F_+ \) is now played by \( \mathcal{S} \) and we use (3.10) instead of (1.11).

Let the Hilbert space \((\mathcal{X}, \overline{\mathcal{X}})\) be the completion of \((\mathcal{S}, \mathcal{N}_u)\). Note that by Lemma 9 \( \mathcal{X} \) is contained in \( \mathcal{H}_+ \) and does not depend on \( u > a \).

By (3.16), \( g_u[x_+] \geq 0 \) for all \( x_+ \in \mathcal{S} \) if \( a < u \leq \lambda_1 \). On the other hand, for \( u \geq \lambda_1 \), by (3.14) and (3.13) we obtain
\[
g_u \geq g_{\lambda_1} + (\lambda_1 - u)n_{\lambda_1} \geq (\lambda_1 - u)\left(\frac{u - a}{\lambda_1 - a}\right)^2 n_u. 
\]
Hence for any \( u > a \)
\[
g_u \geq -c_u n_u, \quad c_u := \max \left\{ 0, (u - \lambda_1)\left(\frac{u - a}{\lambda_1 - a}\right)^2 \right\}.
\]
Now we define
\[
h_u : \mathcal{S} \longrightarrow \mathbb{R}, \quad h_u[x_+] := g_u[x_+] + (c_u + 1)n_u[x_+]. 
\]
(3.25)

We claim that \( h_u^{1/2} \) and \( h_u'^{1/2} \) are equivalent norms for \( u' > u > a \). By (3.14) and (3.13)
\[
g_u \leq g_u' + (u' - u)n_u \leq g_u' + (u' - u)\left(\frac{u' - a}{u - a}\right)^2 n_u',
\]
which implies that \( h_u \leq g_u' + (1 + c_u + u' - u)(u' - a)^2(u - a)^{-2}n_{u'} \) and that there exists a constant \( c_2(u, u') \) such that \( h_u \leq c_2(u, u')h_{u'} \). By Lemma 9 we
get that \( g_u \geq g_{u'} + (u' - u)n_{u'} \) and hence \( h_u \geq g_{u'} + (u' - u + 1 + c_{u'})n_{u'} \) which means that there is a constant \( c_1(u, u') > 0 \) such that \( h_u \geq c_1(u, u')h_{u'} \). Hence the norms are equivalent.

For \( u > a \) let the Hilbert space \( \mathfrak{G}_u = (\mathfrak{G}, \overline{h}_u) \) be the completion of \( (\mathfrak{G}, h_u) \). Note that \( \mathfrak{G} \subset \mathfrak{X} \) does not depend on \( u \).

The extension of \( g_u \) to \( \mathfrak{G} \) is denoted by \( \overline{g}_u \). It is a closed, semi-bounded quadratic form with the domain \( \mathfrak{G} \). By the Friedrichs theorem there is a unique self–adjoint operator \( T_u : D(T_u) \subset \mathfrak{X} \to \mathfrak{X} \) with the form domain \( \mathfrak{G} \), such that \( \overline{g}_u[x +] = \overline{n}_u[x +, T_u x +] \) for all \( x + \in D(T_u) \) and \( \mathfrak{G} \) is a form-core for \( T_u \).

The following lemma is a simple consequence of Courant minimax principle.

**Lemma 10.** Let \( T \) be a self–adjoint, bounded below operator in a Hilbert space \( \mathfrak{X} \) with the domain \( D(T) \) and \( t \) the corresponding sesquilinear form with the domain \( D[t] \). We define

\[
 l_k(T) := \inf_{\mathfrak{V} \text{ subspace of } D[t]} \sup_{x \in \mathfrak{V} \setminus \{0\}} \frac{t[x]}{\|x\|_X^2},
\]

\[
 w_k(T) := \text{card}\{k' \geq 1, l_{k'}(T) = l_k(T)\},
\]

and

\[
 T^\#: D[t] \to D[t]^*, \quad (T^z)(v) := t[z, v] \quad \text{for all } v, z \in D[t].
\]

If \( l_k(T) < \inf \sigma_{\text{ess}}(T) \) then \( l_k(T) \) is an eigenvalue of \( T \) with multiplicity \( w_k(T) \).

As a consequence, if \( C \subset D[t] \) is a form-core for \( T \), then there is a sequence \((3_n)_{n \in \mathbb{N}}\) of subspaces of \( C \) with \( \dim 3_n = w_k(T) \) and (recall (3.19))

\[
 \sup_{z \in 3_n \atop \|z\|_X = 1} \left\| (T^z - l_k(T))z \right\|_{D[t]^*} \to 0 \quad \text{for } n \to \infty.
\]

Applying Lemma 10 we obtain

\[
 l_k(T_u) = \inf_{\mathfrak{V} \text{ subspace of } \mathfrak{G}} \sup_{x + \in \mathfrak{V} \setminus \{0\}} \frac{\overline{g}_u[x +]}{\overline{n}_u[x +]},
\]

\[
 w_k(T_u) = \text{card}\{k' \geq 1, l_{k'}(T_u) = l_k(T_u)\}.
\]
If \( l_k(T_u) < \inf \sigma_{\text{ess}}(T_u) \) then \( l_k(T_u) \) is an eigenvalue of \( T_u \) with multiplicity \( w_k(T_u) \). As in Lemma 10 we define for \( u > a \)

\[
T_u^\#: \mathfrak{G}_u \to \mathfrak{G}_u^*, \quad (T_u^\# z)(w) := \left[ T_u z, w \right] \quad \text{for all } w, z \in \mathfrak{G}. \tag{3.30}
\]

Starting from (3.10) and following the proof of Lemma 2.2 of [3] we obtain Lemma 11.

**Lemma 11.** Let (1.8), (1.9) and (1.10) be satisfied. Then for any \( k \geq 1 \), \( \lambda_k \) is the unique solution in \((a, \infty)\) of the non-linear equation

\[
l_k(T_\lambda) = 0. \tag{3.31}
\]

We thus have \( \lambda_k = \lambda_{k'} \) if and only if \( l_{k'}(T_{\lambda_k}) = 0 \). Let

\[
w_k := \text{card}\{k' \geq 1 : \lambda_k = \lambda_{k'}\}. \tag{3.32}
\]

Then Lemma 10 implies the existence of a sequence of subspaces \((\mathfrak{Z}_n)_{n \in \mathbb{N}}\) of \( \mathfrak{G} \) with \( \dim \mathfrak{Z}_n = w_k \) for all \( n \geq 1 \) such that

\[
\sup_{x_+ \in \mathfrak{Z}_n \atop \lambda_k[x_+] = 1} \| T_{\lambda_k} x_+ \| \mathfrak{G}_{\lambda_k}^* \to 0 \quad \text{for } n \to \infty. \tag{3.33}
\]

According to (1.9), (3.2) and (3.3) \((u - a)\|y_-\|^2 \leq m_u[y_-] \) holds for all \( y_- \in \mathcal{D}_- \). Hence by (1.6)

\[
|s[x, y_-]| \leq (u - a)^{-1/2} \| Bx \| m_u^{1/2}[y_-] \quad \text{for all } y_- \in \mathcal{D}_-, \; x \in D(B).
\]

We thus get for \( x \in D(B), \; y_- \in \mathcal{D}_- \)

\[
|s[\Lambda_+ x, y_-]| \leq |s[x, y_-]| + |s[\Lambda_- x, y_-]| \\
\leq (u - a)^{-1/2} \| Bx \| m_u^{1/2}[y_-] + m_u[\Lambda_- x, y_-] + |u(\Lambda_- x, y_-)| \\
\leq \left( \frac{\| Bx \|}{\sqrt{u - a}} + m_u^{1/2}[\Lambda_- x] + \frac{|u|}{\sqrt{u - a}} \| \Lambda_- x \| \right) m_u^{1/2}[y_-].
\]

Hence

\[
\Lambda_+ D(B) \subset \mathfrak{G}. \tag{3.34}
\]

Let \( x_+ \in \mathfrak{G} \) and \( y_+ \in \Lambda_+ D(B) \). Then by (3.30)

\[
(T_{\lambda_k}^\# x_+)(y_+) = g_{\lambda_k}[x_+, y_+]. \tag{3.35}
\]
By (3.11), (3.35) and (3.38) we have
\[ g_{\lambda_k}[x_+] = s[x_+] - \lambda_k\|x_+\|^2 + 2s_{x_+}(L_{\lambda_k}x_+) - \overline{s-}[L_{\lambda_k}x_+] - \lambda_k\|L_{\lambda_k}x_+\|^2 \]
(3.36)
for all \( x_+ \in \mathcal{G} \).

For \( u > a \) we define (recall (3.35))
\[ E_u : \mathcal{G} \oplus \mathcal{D} \to \overline{\mathcal{D}}_-, \quad E_u x := L_u \Lambda_+ x - \Lambda_- x. \]
(3.37)

By (3.35) and (3.36) we obtain for all \( x = x_+ \oplus x_- \in \mathcal{G} \oplus \mathcal{D}_- \) and \( y \in D(B) \) with \( y_\pm := \Lambda_\pm y \):
\[
(T_{\lambda_k}^\# x_+)(y_+) = s[x_+, y_+] - \lambda_k\langle x_+, y_+ \rangle + 2s_{x_+}(L_{\lambda_k}y_+) - \overline{s-}[L_{\lambda_k}x_+, L_{\lambda_k}y_+]
- \lambda_k(L_{\lambda_k}x_+, L_{\lambda_k}y_+)
= s[x, y] - s[x_-, y] - s[x_+, y_-] + 2s_{x_+}(L_{\lambda_k}y_+) - \overline{s-}[L_{\lambda_k}x_+, L_{\lambda_k}y_+]
- \lambda_k(x_+ + L_{\lambda_k}x_+, y_+ + L_{\lambda_k}y_+)
\]
(3.38)

For all \( x_- \in \mathcal{D}_- \) and \( y \in D(B) \) such that \( \Lambda_\pm y = y_\pm \) we get
\[ s_{y_+}(x_-) = s[y_+, x_-] = s[y, x_-] - s[y_-, x_-] = \langle B y, x_- \rangle + \overline{s-}[y_-, x_-]. \]

Thus for all \( x_- \in \overline{\mathcal{D}}_- \) and \( y \in D(B) \) such that \( \Lambda_\pm y = y_\pm \)
\[ s_{y_+}(x_-) = \langle B y, x_- \rangle + \overline{s-}[y_-, x_-] \]
(3.39)
holds. Now by (3.38)
\[ s_{x_+}(L_{\lambda_k}y_+) = \overline{s_{y_+}}[L_{\lambda_k}x_+, L_{\lambda_k}y_+] = s_{y_+}(L_{\lambda_k}x_+). \]

This together with (3.39) implies
\[
2s_{x_+}(L_{\lambda_k}y_+) - s[x_+, y_-] = s_{y_+}(L_{\lambda_k}x_+) + s_{x_+}(L_{\lambda_k}y_+) - s_{x_+}(y_-)
= s_{y_+}(L_{\lambda_k}x_+) + s_{x_+}(L_{\lambda_k}y_+ - y_-) = \langle y_-, B x \rangle + \overline{s-}[y_-, x_-] + s_{x_+}(E_{\lambda_k} y),
\]
(3.40)

Inserting (3.40) into (3.38) and using (3.37), we obtain
\[
(T_{\lambda_k}^\# x_+)(y_+) = \langle x + E_{\lambda_k} x, (B - \lambda_k)y \rangle
+ s_{x_+}(E_{\lambda_k} y) - \overline{s-}(L_{\lambda_k}x_+, E_{\lambda_k} y) - \lambda_k(L_{\lambda_k}x_+, E_{\lambda_k} y).
\]
By (3.8) and (3.3) all the terms in the last line cancel. We thus get that for \( x_+ \in \mathcal{G}, y_+ \in \Lambda_+ D(B) \) the relation
\[
(T_{\lambda_k}^# x_+)(y_+) = (x_+ + L_{\lambda_k} x_+, (B - \lambda_k) y)
\]
holds for any \( y \in D(B) \) with \( \Lambda_+ y = y_+ \).

We now estimate \( h_{\lambda_k} \). Let \( y \in D(B) \). By (3.34) and (3.25) we get
\[
h_{\lambda_k} [\Lambda_+ y] = (c_{\lambda_k} + 1) n_{\lambda_k} [\Lambda_+ y] + g_{\lambda_k} [\Lambda_+ y].
\]

Now by (3.30) and (3.41)
\[
|g_{\lambda_k} [\Lambda_+ y]| \leq \langle y + E_{\lambda_k} y, (B - \lambda_k) y \rangle \leq \|y + E_{\lambda_k} y\| \| (B - \lambda_k) y \|
\]
which by (3.37), (3.2) and (1.9) implies
\[
\| (B - \lambda_k) y \| \| E_{\lambda_k} y \| \geq \langle (B - \lambda_k) y, E_{\lambda_k} y \rangle \geq (\lambda_k - a) \| E_{\lambda_k} y \|^2.
\]

Substituting (3.44) into (3.43), we obtain
\[
|g_{\lambda_k} [\Lambda_+ y]| \leq (1 + |\lambda_k|) \left( 1 + \frac{1 + |\lambda_k|}{\lambda_k - a} \right) \|y\|^2_{D(B)}.
\]

By (3.12), (3.37) and (3.44),
\[
n_{\lambda_k} [\Lambda_+ y] = \| \Lambda_+ y + L_{\lambda_k} \Lambda_+ y \|^2 = \| y + E_{\lambda_k} y \|^2 \leq \left( 1 + \frac{1 + |\lambda_k|}{\lambda_k - a} \right)^2 \|y\|^2_{D(B)}.
\]

Substituting (3.45) and (3.46) into (3.42) we find a constant \( c(\lambda_k, a) > 0 \) such that
\[
h_{\lambda_k}^{1/2} [\Lambda_+ y] \leq c(\lambda_k, a) \|y\|_{D(B)} \quad \text{for all } y \in D(B).
\]
By (3.41) and (3.47) we get for all $x_+ \in \mathcal{G}, y_+ \in (\Lambda_+ D(B)) \setminus \{0\}$ and $y \in D(B)$ such that $\Lambda_+ y = y_+$:

$$\frac{|(T_{\lambda_k}^# x_+)(y_+)|}{h_{\lambda_k}^{1/2} (y_+)} \geq \frac{|(x_+ + L_{\lambda_k} x_+, (B - \lambda_k) y)|}{c(\lambda_k, a) \|y\|_{D(B)}}.$$  \hspace{1cm} (3.48)

According to (3.34), for $x_+ \in \mathcal{G}_{\lambda_k}$

$$\|T_{\lambda_k}^# x_+\|_{\mathcal{H}_{\lambda_k}} \geq \sup_{y_+ \in (\Lambda_+ D(B)) \setminus \{0\}} \frac{|(T_{\lambda_k}^# x_+)(y_+)|}{h_{\lambda_k}^{1/2} [y_+]}.$$  \hspace{1cm} From this and (3.33) it follows that

$$\sup_{x_+ \in \mathcal{G}_n} \sup_{y_+ \in (\Lambda_+ D(B)) \setminus \{0\}} \frac{|(T_{\lambda_k}^# x_+)(y_+)|}{h_{\lambda_k}^{1/2} [y_+]} \rightarrow 0 \text{ for } n \rightarrow \infty.$$  \hspace{1cm} (3.49)

Hence we get by (3.48)

$$\sup_{x_+ \in \mathcal{G}_n} \sup_{y_+ \in (\Lambda_+ D(B)) \setminus \{0\}} \frac{|(x_+ + L_{\lambda_k} x_+, (B - \lambda_k) y)|}{\|y\|_{D(B)}} \rightarrow 0 \text{ for } n \rightarrow \infty.$$  \hspace{1cm} (3.49)

We now prove that either $\lambda_k \in \sigma_{\text{ess}}(B) \cap (a, \infty)$ or $\lambda_k$ is an eigenvalue of $B$ in $(a, \infty)$ with multiplicity greater than or equal to $w_k$. First we define

$$\tilde{3}_n := (1 + L_{\lambda_k}) 3_n.$$  \hspace{1cm} (3.50)

We know that $\dim \tilde{3}_n = w_k$ and so

$$\dim \tilde{3}_n = w_k.$$  \hspace{1cm} (3.51)

Relations (3.49) and (3.51) imply the existence of sequences $(\tilde{x}_n^{(l)})_{n \in \mathbb{N}} \subset \mathcal{H}$, $l \in \{1, 2, \ldots, w_k\}$ such that $\{\tilde{x}_n^{(1)}, \ldots, \tilde{x}_n^{(w_k)}\}$ is orthonormal in $\mathcal{H}$ for all $n$ and

$$\lim_{n \rightarrow \infty} \sup_{y \in D(B) \setminus \{0\}} \frac{|\langle \tilde{x}_n^{(l)}, (B - \lambda_k) y \rangle|}{\|y\|_{D(B)}} \rightarrow 0 \text{ for all } l \in \{1, 2, \ldots, w_k\}.$$
Since $D(B)$ is dense in $\mathcal{H}$ with respect to $\| \cdot \|$, without loss of generality $(\tilde{x}_n^{(l)}) \subset D(B)$ for all $l \in \{1, 2, \ldots, w_k\}$. Since $\|y\|_{D(B)} = \|(B + i)y\|$ and $\mathcal{H} = (B + i)D(B)$, we conclude that

$$\lim_{n \to \infty} \sup_{y \in \mathcal{H} \setminus \{0\}} \|((B - i)^{-1}(B - \lambda_k)\tilde{x}_n^{(l)}, y)\| \to 0 \quad \text{for all} \quad l \in \{1, 2, \ldots, w_k\},$$

which means that $\lim_{n \to \infty} (B - i)^{-1}(B - \lambda_k)\tilde{x}_n^{(l)} = 0$ for all $l \in \{1, 2, \ldots, w_k\}$. If $\lambda_k \notin \sigma(B)$ then $(B - \lambda_k)^{-1}(B - i)$ is a bounded operator and thus for $l \in \{1, 2, \ldots, w_k\}$ we get

$$1 = \|\tilde{x}_n^{(l)}\| \leq \|(B - \lambda_k)^{-1}(B - i)\| \|(B - i)^{-1}(B - \lambda_k)\tilde{x}_n^{(l)}\| \to 0 \quad \text{for} \quad n \to \infty,$$

which is a contradiction. Hence either $\lambda_k \in \sigma_{\text{ess}}(B) \cap (a, \infty)$ or $\lambda_k \in (a, \infty)$ is an eigenvalue of $B$ with multiplicity greater than or equal to $w_k$. This implies that $\lambda_k \geq \mu_k$ for all $k \in \{1, w_1\}$. By induction we conclude that $\lambda_k \geq \mu_k$ for all $k \geq 1$.

### 4 Applications to Dirac operators with singular potentials: proofs

#### 4.1 Proof of Theorem 3

We want to apply Theorem 2 with $q := h_0$. The assumption (i) obviously holds; the assumptions (ii) with $a = -1$ follows from the non–positivity of $V$. It remains to prove (iii).

By monotonicity and (1.1) it is clearly enough to deal with the case

$$V(x) = V_{\tilde{\nu}, 0}(x) := -\frac{\tilde{\nu}}{|x|} \mathbb{1}_{\mathbb{C}^4}.$$

For this we consider $V_{\tilde{\nu}, 0}$ as an element of a family of potentials

$$V_{\nu, \varepsilon}(x) := -\frac{\nu}{|x| + \varepsilon} \mathbb{1}_{\mathbb{C}^4}, \quad \nu \in [0, \tilde{\nu}], \quad \varepsilon \in [0, \infty).$$

In the First Step of the proof of Theorem 4.2 in [3] it is proved that for $\varepsilon > 0$ the first minimax value $\lambda_1(V_{\nu, \varepsilon})$ of $H_0 + V_{\nu, \varepsilon}$ satisfies

$$\lambda_1(V_{\nu, \varepsilon}) \geq 0 \quad \text{for all} \quad \nu \in [0, \tilde{\nu}] \quad \text{and} \quad \varepsilon > 0. \quad (4.1)$$
For $\nu \in [0, \bar{\nu}]$ and $\varepsilon \in [0, \infty)$ we define (cf. (3.1))

$$s_{\nu, \varepsilon} : h_0 + v_{\nu, \varepsilon} \quad \text{on} \quad D[s_{\nu, \varepsilon}] := H^{1/2}(\mathbb{R}^3, \mathbb{C}^4),$$

(4.2)

where $v_{\nu, \varepsilon}$ is the sesquilinear form of $V_{\nu, \varepsilon}$, and

$$g_{\nu, \varepsilon} : \mathcal{D}_+ \to \mathbb{R} \cup \{\infty\}, \quad g_{\nu, \varepsilon}[x_+] := \sup_{x_- \in \mathcal{D}_-} s_{\nu, \varepsilon}[x_+ + x_-].$$

(4.3)

Introducing

$$m_{\nu, \varepsilon} : \mathcal{D}_- \to [0, \infty), \quad m_{\nu, \varepsilon}[x_-] := -s_{\nu, \varepsilon}[x_-]$$

(4.4)

we observe that $\mathcal{D}_-$ is closed with respect to the norm $m_{\nu, \varepsilon}^{1/2}$, which is equivalent to the $H^{1/2}$–norm on $\mathcal{D}_-$. As in the proof of Theorem 2 there exists a linear operator $L_{\nu, \varepsilon} : \mathcal{S} \to \mathcal{D}_-$ such that

$$g_{\nu, \varepsilon}[x_+] = s_{\nu, \varepsilon}[x_+ + L_{\nu, \varepsilon}x_+].$$

(4.5)

By the equivalence of $m_{\nu, \varepsilon}^{1/2}$ and $H^{1/2}$–norm on $\mathcal{D}_-$ we observe that $\mathcal{S} = \mathcal{D}_+$. Letting $x_+^* := L_{\nu, \varepsilon}x_+$ and using that $x_+^*$ is a maximizer of $s_{\nu, \varepsilon}[x_+ + \cdot]$, we obtain

$$0 = \frac{d}{d\alpha} \bigg|_{\alpha=0} s_{\nu, \varepsilon}[x_+ + x_-^* + \alpha y_-] \text{ for all } y_- \in \mathcal{D}_-. $$

Let us now assume that

$$x_+ \in \mathcal{C}_+ := P_+ C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \subset H^1(\mathbb{R}^3; \mathbb{C}^4).$$

(4.6)

Then

$$\langle P_- V_{\nu, \varepsilon} x_+, y_- \rangle = -h_0[x_-^*, y_-] - v_{\nu, \varepsilon}[x_-^*, y_-] = m_{\nu, \varepsilon}[x_-^*, y_-] \text{ for all } y_- \in \mathcal{D}_-. $$

(4.7)

We observe that $c_{\nu, \varepsilon} := -h_0 - v_{\nu, \varepsilon}$ defined on $\mathcal{D}_-$ is a densely defined, closed, symmetric and bounded below sesquilinear form in $\mathcal{S}_- := P_- L^2(\mathbb{R}^3, \mathbb{C}^4)$. By Friedrichs theorem there is a unique self–adjoint operator $C_{\nu, \varepsilon}$ in $\mathcal{S}_-$ corresponding to $c_{\nu, \varepsilon}$. Moreover, for all $\nu \in [0, \bar{\nu}]$ and $\varepsilon \in [0, \infty)$ we have

$$C_{\nu, \varepsilon} \geq 1_{\mathcal{S}_-}$$

(4.8)
and

\[ D(C_{\nu,0}) \subset D(C_{\nu,\epsilon}). \]  \hspace{1cm} (4.9)

Relation (4.7) implies that \( x^* \in D(C_{\nu,\epsilon}) \) and

\[ x^* = L_{\nu,\epsilon}x_+ = C_{\nu,\epsilon}^{-1}P_{-}V_{\nu,\epsilon}x_+, \]  \hspace{.5cm} \text{for all } x_+ \in \mathfrak{C}_+.

Now we claim that

\[ \|C_{\nu,0}^{-1}P_{-}V_{\nu,0}x_+ - C_{\nu,\epsilon}^{-1}P_{-}V_{\nu,\epsilon}x_+\| \to 0 \]  \hspace{.5cm} \text{for } \epsilon \searrow 0. \]  \hspace{1cm} (4.10)

With the help of the triangle inequality and the resolvent identity (which we can apply by (4.9)) we can estimate

\[ \|C_{\nu,0}^{-1}P_{-}V_{\nu,0}x_+ - C_{\nu,\epsilon}^{-1}P_{-}V_{\nu,\epsilon}x_+\| \leq \|C_{\nu,\epsilon}^{-1}(C_{\nu,\epsilon} - C_{\nu,0})C_{\nu,0}^{-1}P_{-}V_{\nu,0}x_+\| + \|C_{\nu,\epsilon}^{-1}P_{-}(V_{\nu,0} - V_{\nu,\epsilon})x_+\|. \]  \hspace{1cm} (4.11)

The last term tends to zero as \( \epsilon \searrow 0 \) by (4.8) and dominated convergence (Note that \( V_{\nu,0}x_+ \in L^2(\mathbb{R}^3, \mathbb{C}^4) \) by the Hardy inequality and (4.6)).

Let \( y_- := C_{\nu,0}^{-1}P_{-}V_{\nu,0}x_+ \). Since \( y_- \in D(C_{\nu,0}) \) and \( C_{\nu,0} \geq -V_{\nu,0} \geq 0 \) we get \( y_- \in D(V_{\nu,0}) \). Hence, again by dominated convergence,

\[ \|(C_{\nu,\epsilon} - C_{\nu,0})y_-\| = \|(V_{\nu,0} - V_{\nu,\epsilon})y_-\| \to 0 \]  \hspace{.5cm} \text{for } \epsilon \searrow 0.

The claim (4.10) is thus proven.

By (4.7) we have

\[ g_{\nu,\epsilon}[x_+] = s_{\nu,\epsilon}[x_+] + C_{\nu,\epsilon}^{-1}P_{-}V_{\nu,\epsilon}x_+ \]
\[ = s_{\nu,\epsilon}[x_+] + s_{\nu,\epsilon}[C_{\nu,\epsilon}^{-1}P_{-}V_{\nu,\epsilon}x_+, x_+] \]  \hspace{1cm} (4.12)
\[ = s_{\nu,\epsilon}[x_+] + \langle C_{\nu,\epsilon}^{-1}P_{-}V_{\nu,\epsilon}x_+, V_{\nu,\epsilon}x_+ \rangle.

By (4.10), (4.12) and dominated convergence we get

\[ g_{\nu,0}[x_+] = \lim_{\epsilon \nearrow 0} g_{\nu,\epsilon}[x_+], \]  \hspace{.5cm} \text{for all } x_+ \in \mathfrak{C}_+ \text{ and } \nu \in [0, \bar{\nu}]. \]  \hspace{1cm} (4.13)

Let \( x_+ \in \mathfrak{D}_+, \nu \in [0, \bar{\nu}] \) and \( \epsilon \in [0, \infty) \) be arbitrary. By (4.5) we obtain

\[ g_{\nu,\epsilon}[x_+] = s_{\nu,\epsilon}[x_+] + s_{\nu,\epsilon}[L_{\nu,\epsilon}x_+, x_+] + 2\mathfrak{R}s_{\nu,\epsilon}[x_+, L_{\nu,\epsilon}x_+]. \]  \hspace{1cm} (4.14)
Setting \( y_- := x_-^* = L_{\nu, \varepsilon} x_+ \) in (4.14) we can rewrite the last term in (4.14):

\[
2 \Re s_{\nu, \varepsilon}[x_+, L_{\nu, \varepsilon} x_+] = 2 v_{\nu, \varepsilon}[x_+, L_{\nu, \varepsilon} x_+] = 2 m_{\nu, \varepsilon}[L_{\nu, \varepsilon} x_+].
\]

(4.15)

Combining (4.15), (4.14) and (4.4) we arrive at

\[
g_{\nu, \varepsilon}[x_+] = s_{\nu, \varepsilon}[x_+] + m_{\nu, \varepsilon}[L_{\nu, \varepsilon} x_+].
\]

(4.16)

Now by (4.15), the Kato inequality and (4.4)

\[
m_{\nu, \varepsilon}[L_{\nu, \varepsilon} x_+] = v_{\nu, \varepsilon}[x_+, L_{\nu, \varepsilon} x_+] \leq \left( - v_{\nu, \varepsilon}[x_+] \right)^{1/2} - v_{\nu, \varepsilon}[L_{\nu, \varepsilon} x_+]^{1/2}
\]

\[
\leq \sqrt{\frac{\pi}{2}} \| x_+ \|_{H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)} (m_{\nu, \varepsilon}[L_{\nu, \varepsilon} x_+])^{1/2},
\]

i.e.

\[
m_{\nu, \varepsilon}[L_{\nu, \varepsilon} x_+] \leq \frac{\pi}{2} \| x_+ \|_{H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)}^{2}.
\]

(4.17)

This shows that the right hand side of (4.16) is continuous in \( x_- \) in the \( \mathcal{H}^{1/2} \)-norm. Thus by density the non–negativity of \( g_{\nu, \varepsilon} \) on \( \mathcal{D}_+ \) is equivalent to its non–negativity on \( \mathcal{C}_+ \) for all \( \varepsilon \in [0, \infty) \) and \( \nu \in [0, \tilde{\nu}] \). For \( \varepsilon > 0 \) and \( \nu \in [0, \tilde{\nu}], \ (3.17) \) and (4.1) imply \( g_{\nu, \varepsilon}[x_+] \geq 0 \) for all \( x_+ \in \mathcal{C}_+ \). According to (4.13), \( g_{\nu, \varepsilon}[x_+] \geq 0 \) also holds for \( \varepsilon = 0 \) for all \( x_+ \in \mathcal{C}_+ \), and thus for all \( x_+ \in \mathcal{D}_+ \). Another application of (3.17) finally yields \( \lambda_1 \geq 0 \).

### 4.2 Proof of Theorem 4

The statement follows from Theorem 2 with \( q := h_0 \). The assumption (i) obviously holds; the assumption (ii) with \( a = -1 \) follows from the non–positivity of \( V \). To establish (iii) we observe that for any 2–spinor \( \varphi \in \mathcal{H}^{1/2}(\mathbb{R}^3, \mathbb{C}^2) \) the 4–spinor

\[
\begin{pmatrix}
\varphi \\
\mathcal{F}^{-1} \sigma \cdot \mathbf{p} \mathcal{F} \varphi \\
\frac{\sqrt{p^2 + 1} - 1}{\sqrt{p^2 + 1} + 1} \mathcal{F} \varphi
\end{pmatrix}
\]

(where \( \mathcal{F} \) is the Fourier transform) belongs to \( P_{H_0}([1, \infty)) \mathcal{H}^{1/2}(\mathbb{R}^3, \mathbb{C}^4) \), which follows from e.g. Subsection 1.4.2 of [12]. Hence (iii) is an easy consequence of Theorem 1 of [13].
Appendix: free Dirac operator

In $L^2(\mathbb{R}^3, \mathbb{C}^4)$ the free Dirac operator

$$H_0 = -i\alpha \cdot \nabla + \beta$$

is self-adjoint on the domain $D(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ are defined as

$$\beta := \begin{pmatrix} 1_C & 0 \\ 0 & 1_C \end{pmatrix}, \quad \alpha_k := \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3;$$

where $\sigma_k$ are the Pauli matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

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1See e.g. [12], Chapter I
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