Steady-state distributions and non-steady dynamics in non-equilibrium systems

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We search for steady states in a class of fluctuating and driven physical systems that exhibit sustained currents. We find that the physical concept of a steady state, well known for systems at equilibrium, must be generalised to describe such systems. In these, the generalisation of a steady state is associated with a stationary probability density of micro-states and a deterministic dynamical system whose trajectories the system follows on average. These trajectories are a manifestation of non-stationary macroscopic currents observed in these systems. We determine precise conditions for the steady state to exist as well as the requirements for it to be stable. We illustrate this with some examples.

I. INTRODUCTION

The study and classification of non-equilibrium systems remains one of the major open problems in statistical physics [1–7]. A large class of non-equilibrium systems are driven systems whose behaviour is characterised by the presence of sustained non-zero currents. Unlike undriven systems whose dynamics is the relaxation towards equilibrium where all currents are zero, such systems show complex dynamics: oscillations, dynamic order-disorder transitions, pattern formation and phase separation [8–23]. They are also thought to be the framework for new theories of “active or living matter” [24–35] required to describe biological systems [36–50].

The Gibbs-Boltzmann distribution of equilibrium statistical mechanics describes the probability density of states $\rho$ of a macroscopic system at equilibrium with a fixed temperature, $T$, as a function of the total energy $\mathcal{H}$ of the system in that state, $\rho \propto e^{-\mathcal{H}/T}$ and forms the starting point for studies of many interacting particle systems at equilibrium [51]. The average (statistical) properties of equilibrium systems can thus all be expressed as integrals (or sums) over this distribution and because of the average deterministic dynamical system are in general not stationary but varying in time. Whilst some of our results are rigorous, our interest here is in concrete physical realisations of these steady states in experimentally feasible systems. We study several examples of driven fluctuating systems characterised by these generalised steady states: a driven oscillator, a model of a chemical reacting system and a thin film of active nematic in a disordered flowing state sometimes referred to as ‘active turbulence’ [15, 58].

II. NON-EQUILIBRIUM SYSTEMS

Non-equilibrium systems are defined dynamically, i.e. by a set of dynamic rules encoding their evolution. We thus consider systems with $N > 1$ “microscopic” degrees of freedom $\vec{x}(t) = (x_1(t), \ldots, x_N(t))$ generically undergoing dynamics (a sum of deterministic and fluctuating parts) given by the Langevin equation,

$$\frac{d}{dt} \vec{x} = -\nabla \mathcal{H}(\vec{x}) + \vec{w}(\vec{x}) + \xi(t)$$

(1)
where $\mathcal{H}(\vec{x})$ is a scalar function of $\vec{x}$, $\bf D$ is a mobility matrix and the gradient operator, $\nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) \equiv (\nabla_1, \ldots, \nabla_N)$. The fluctuations, $\vec{\xi} = (\xi_1, \ldots, \xi_N)$ are white with zero mean and autocorrelation function:

$$\langle \xi_i(t) \rangle = 0 \quad \langle \xi_i(t) \xi_j(t') \rangle = 2\theta D_{ij} \delta(t - t') \quad \theta > 0 \quad . \quad (2)$$

To be concrete we consider diagonal mobility matrices \cite{59} of the form $D_{ij} = D_{ii} \delta_{ij}$ where $D_i > 0$ are independent of $\vec{x}$ and $\delta_{ij}$ is the Kronecker delta \cite{60}. We can w.l.g. rewrite $\vec{w}(\vec{x}) = \bf D : \vec{v}(\vec{x})$ where the differentiable vector-valued function $\vec{v}(\vec{x}) = (v_1(\vec{x}), \ldots, v_N(\vec{x}))$ cannot be written as the derivative of a scalar function. This implies the microscopic breaking of “detailed balance” \cite{53}. Equations like this emerge in many models for slow dynamics of driven physical systems \cite{51, 61}. The Langevin equation is equivalent to a Fokker-Planck equation \cite{62} for the probability density, $P(\vec{x}, t)$:

$$\partial_t P = \sum_{i=1}^{N} \nabla_i D_i \left( \theta \nabla_i P + P(\nabla_i \mathcal{H} - v_i) \right) \quad . \quad (3)$$

We assume $P$ is well behaved, i.e. $P, \nabla P \rightarrow 0$ as $|\vec{x}| \rightarrow \infty$. We find a steady state probability density, by requiring RHS of eqn. (3) to vanish. However even more important is determining if it is stable, i.e. that the system moves towards it and remains there. This is our goal. We now state our main result more formally.

**Definition:** It is useful to define a function $h(\vec{x})$ as follows: the system has a steady state probability density $P_{ss} = \rho(\vec{x}) = \frac{1}{Z} e^{-h(\vec{x})}$ if a function $h(\vec{x})$ can be found that satisfies

$$\sum_{i=1}^{N} D_i L_i(h) = 0 \quad (4)$$

where $L_i(h) = \theta (\nabla_i \mathcal{H})^2 + \nabla_i \mathcal{H} \cdot \nabla_i h + \nabla_i \mathcal{H} \cdot (v_i - \nabla_i \mathcal{H}) - \theta \nabla_i^2 h - \nabla_i v_i$ and the normalisation $Z = \int d^N x e^{-h(\vec{x})}$ is chosen so that $\int d^N x \rho(\vec{x}) = 1$.

**Theorem 1a:** For functions $h(\vec{x})$ which satisfy eqn. (4), $\rho$ remains constant on the trajectories: $\vec{x}(t) = \vec{X}(t)$,

$$\frac{dX_i}{dt} = V_i(\vec{X}) \quad , \quad V_i = D_i (v_i - \nabla_i \mathcal{H} + \theta \nabla_i h) \quad (5)$$

The set $\{ \rho, \vec{V} \}$ characterise a generalized steady state.

A useful decomposition of the equation (4) are solutions, $h(\vec{x})$ which satisfy both the following conditions:

$$\sum_{i=1}^{N} \nabla_i V_i = -C \quad , \quad \sum_{i=1}^{N} V_i \nabla_i h = C \quad , \quad (6)$$

and we base our subsequent analysis on this observation.

**Theorem 1b:** If $C(\vec{x}) \geq 0$, $\forall \vec{x}$ then the generalized steady state is stable and the system will always evolve towards it for any arbitrary initial condition. If not, i.e. if $C < 0$ for some values of $\vec{x}$ then the situation is inconclusive, the steady state may be generically unstable or the stability of the steady state may depend on initial conditions and the values of parameters.

To obtain more information about how quickly the system relaxes to the generalised steady state, it is helpful to decompose the set of stable scenarios into two classes.

**Theorem 2a:** If $\mathcal{C} = 0$, then the system relaxes exponentially fast to the steady state if all the eigenvalues of the Hessian matrix $\nabla_i \nabla_i h$ are all positive.

$$\nabla \nabla h > 0 \quad . \quad (7)$$

Note that “equilibrium” systems with $v_i(\vec{x}) = 0$ have $h = \mathcal{H}(\vec{x})/\theta$ (the Boltzmann distribution at temperature $\theta$), $V_i = 0$ and hence $\mathcal{C} = 0$.

**Theorem 2b:** If $\mathcal{C} > 0$ and $\lim_{|\vec{x}| \rightarrow 0} \mathcal{C} > 0$ then the system relaxes exponentially fast to the stationary state, irrespective of the form of $h(\vec{x})$. Such systems I denote as super stable.

We note that while systems satisfying these conditions will always relax exponentially fast, it is also quite possible that systems which do not satisfy them might also relax exponentially fast under certain conditions, and that the bounds we have obtained can be improved to include a wider class of systems. Furthermore, it is important to note that when $C = 0$ and $\nabla \nabla \mathcal{C} \neq 0$, then the system is still stable, just that it could possibly relax very slowly to the generalised steady state. We also point out that these results are valid for arbitrarily large noise amplitude, $\theta$.

By obtaining $V_i(h) \neq 0$, we have explicitly calculated the macroscopic current. When the amplitude of the noise, $\theta = 0$, the typical trajectories are those of the deterministic equation. As other trajectories do not keep the probability density constant, typical trajectories act as attractors. Finally, we note that in one dimension, $N = 1$, the only possible steady state dynamical system is one with $\rho V$ constant.

We emphasize that the statement that $h(\vec{x})$ which satisfies eqn. (4) determines a probability density that is stationary is by itself not particularly helpful or new \cite{63}. This is because in practice the nonlinear steady state equation will yield several (in general, approximate) solutions for $h$, and it will not be obvious even if any of them is stable, i.e. an attractor for the dynamics. The Fokker-Planck equation can be written as

$$\partial_t P + \sum_{i=1}^{N} \nabla_i J_i = 0 \quad \text{where } J_i \text{ is a probability current; the stationary condition, eqn. (4) is simply the statement that the steady-state is associated with a divergenceless current, } \sum_{i=1}^{N} \nabla_i J_i = 0 \quad [63]. \text{ This gives rise to a complicated non-linear partial differential equation for } h \text{ which}
may have no, or more than one, solution. Furthermore since it is determined by a nonlinear PDE, one must have a method to calculate it (even approximately). Hence identifying conditions that make \( \rho \propto e^{-h} \) stable and outlining a systematic way to obtain \( h \) is the main result of the theorem above and the subject of this article. We will be particularly interested in situations where \( h = \mathcal{H}/\theta + \epsilon \) with \( \epsilon < \mathcal{H}/\theta \); this is the case for many examples of active matter [22].

We now outline a proof below.

Proof: That \( h \) determines a stationary probability density and remains constant on typical trajectories follows trivially by substitution. To show that it determines a stable (in the probabilistic sense) probability density, we show that if we start with an arbitrary density \( P(\vec{x}, t) = \rho(\vec{x}) \pi(\vec{x}, t) \), with the conditions above on \( h, \pi \to h \) and \( d\vec{x}/dt \to \vec{V} \) exponentially fast. Substituting this density into eqn. (3) we get a modified backward Kolmogorov equation for \( \pi \):

\[
\partial_t \pi = \mathcal{L} \pi , \quad \mathcal{L} = \sum_i D_i \{ \theta \nabla_i^2 + W_i \nabla_i \} \tag{8}
\]

where \( W_i = (\nabla_i \mathcal{H} - v_i - 2\theta \nabla_i h) \). Moving along trajectories \( \vec{X} = \vec{V}_i \pi \), \( \pi \) evolves according to the comoving time derivative

\[
\frac{d\pi}{dt} = \partial_t \pi + \sum_i V_i \nabla_i \pi = \mathcal{L}' \pi \tag{9}
\]

where \( \mathcal{L}' = \sum_i \theta D_i \{ \nabla_i^2 - \nabla_i h \nabla_i \} \).

We sum over all trajectories by integrating over all possible deviations from the typical trajectories. Defining the inner product \( \langle f, g \rangle_{\rho} = \int \mathcal{N} y(\vec{x}) f(\vec{x}) g(\vec{x}) \), where \( \vec{x} = \vec{X} + \vec{y} \), and the norm \( \|A\|_{\rho}^2 = \langle A, A \rangle_{\rho} \), then for any \( C^2 \) function \( f(\vec{x}) \) using integration by parts, it is easy to show that

\[
\langle \mathcal{L}' f, f \rangle_{\rho} = -\theta \sum_i D_i \| \nabla_i f \|_{\rho}^2 \tag{10}
\]

To show that motion along these trajectories is stable to noise we can look for the dynamics of the deviation of the probability density from the steady state, \( \|P - \rho\|_{\rho} = \left[ \int \mathcal{N} y(\pi - 1)^2 \right]^{1/2} = \|\pi - 1\|_{\rho} \) given by

\[
\frac{d}{dt} \|\pi - 1\|_{\rho}^2 = 2 \left( \frac{d\pi}{dt}, \pi - 1 \right)_{\rho} - \int \mathcal{N} \mathcal{C}(\vec{x}) \rho(\pi - 1)^2\rho
\]

\[ = 2 \langle \mathcal{C}'(\pi - 1), \pi - 1 \rangle_{\rho} - \langle \mathcal{C}, (\pi - 1)^2 \rangle_{\rho}
\]

\[ = -2 \sum_i \theta D_i \| \nabla_i (\pi - 1) \|_{\rho}^2 - \langle \mathcal{C}, (\pi - 1)^2 \rangle_{\rho} \]

Hence, if \( \mathcal{C}(\vec{x}) = \vec{V} \cdot \nabla h \geq 0 \),

\[
\frac{d}{dt} \|\pi - 1\|_{\rho}^2 \leq 0 \tag{11}
\]

where \( V_i = D_i (v_i - \nabla_i \mathcal{H} + \theta \nabla_i h) \) [64]. This proves that \( \|\pi - 1\|_{\rho} \) always decreases with time if \( C \geq 0 \). However one would also like to know how quickly the system relaxes to the steady-state. In what follows we set all the \( D_i = 1 \) to simplify formulas.

First we consider the case \( C = 0 \). For this we use Bakry-Émery inequality [54–57]. The inequality is obtained in this setting by taking the comoving time derivative of \( \|\nabla_i (\pi - 1)\|_{\rho}^2 \), integrating by parts:

\[
\frac{d}{dt} \|\nabla_i (\pi - 1)\|_{\rho}^2 \leq -\sum_j \frac{\theta}{2} \int \rho \nabla_i \pi \nabla_j \pi \nabla_j h \]

\[ \leq -\theta \lambda_0 \|\nabla_i (\pi - 1)\|_{\rho}^2 \tag{12}
\]

where \( \lambda_0 > 0 \) is the smallest eigenvalue of the Hessian matrix \( \nabla^2 \pi \). Hence once \( \nabla^2 \pi \) relaxes exponentially fast to zero on a timescale of order \( (\theta \lambda_0)^{-1} \).

Next we consider the case \( C > 0 \) and \( \lim_{\mathcal{A} \to 0} \mathcal{C}(\vec{x}) = \mathcal{C}_0 > 0 \). Here,

\[
\frac{d}{dt} \|\pi - 1\|_{\rho}^2 \leq -\mathcal{C}_0 \|\pi - 1\|_{\rho}^2 \tag{13}
\]

and \( \|\pi - 1\|_{\rho}^2 \) relaxes exponentially fast to zero on a timescale of order \( (\mathcal{C}_0)^{-1} \), irrespective of the form of \( h \) as long as \( \mathcal{C}, \mathcal{C}_0 > 0 \). Clearly if there are several (possibly approximate) values for \( h \), this provides a way to rank them.

The proof above relied on being in a finite-dimensional vector space \( \vec{x} \in \mathbb{R}^N \), hence these results can be generalized to regularized stochastic field \( \mathbf{f}(\vec{r}, t) \) dynamics where \( \vec{r} \in \mathbb{R}^d \) in the following sense. We define an expansion (e.g. Fourier) in a set of orthonormal basis functions, \( \mathbf{f}(\vec{r}, t) = \sum_{\mathbf{q}} f_{\mathbf{q}}(t) \Psi_{\mathbf{q}}(\vec{r}) \), \( \int \Psi_{\mathbf{q}}^*(\vec{r}) \Psi_{\mathbf{q'}}(\vec{r}) = \delta_{\mathbf{q}, \mathbf{q'}} \) for Fourier series, \( \Psi_{\mathbf{q}} = e^{i \mathbf{q} \cdot \vec{r}} \), \( \frac{1}{a} \mathbf{q} \in \mathbb{Z}^d \) which can be regularized by restricting the number of modes to a finite number, \( \mathbf{f}_{\mathcal{A}}(\vec{r}, t) = \sum_{\mathbf{q} = \mathbf{q}_{\min}}^{\mathbf{q}_{\max}} f_{\mathbf{q}}(t) \Psi_{\mathbf{q}}(\vec{r}) \). In the Fourier expansion, \( |\mathbf{q}_{\min}| \sim \pi/L \) and \( |\mathbf{q}_{\max}| \sim \pi/a \) where \( a \) is a short-distance lengthscale. The restricted modes \( \{ f_{\mathbf{q}} \} \) are a finite vector space with \( N \gg 1 \) which satisfy the theorem above.

Now we illustrate the theorem with some examples for which we calculate macroscopic currents. It turns out that many examples of driven active systems have \( \mathcal{C} = 0 \).

III. EXAMPLES

A. The noisy Hopf oscillator

The study of the effects of fluctuations on the normal form of an oscillator that can go through a Hopf bifurcation provides a relatively simple non-trivial two dimensional system where one can study the implications of
our main result. The degrees of freedom $\vec{x}(t) = (x_1, x_2)$ have equation of motion
\[
\frac{d\vec{x}}{dt} = A\vec{x} - B|\vec{x}|^2\vec{x} + \Omega \cdot \vec{x} + \xi(t) ,
\]
where $\Omega = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$, and $A, B > 0, \Omega$ are constants.

The noise $\xi = (\xi_1, \xi_2)$ has zero mean and mean square fluctuations
\[
\langle \xi_i(t)\xi_j(t') \rangle = 2\theta\delta_{ij}\delta(t-t') .
\]
This is of the form
\[
\frac{d\vec{x}}{dt} = -\nabla H + \vec{v}(\vec{x}) + \vec{\xi}(t) ,
\]
where $H = -\frac{A}{2}|\vec{x}|^2 + \frac{B}{4}|\vec{x}|^4$ and $\vec{v} = \Omega \cdot \vec{x}$ cannot be written as the gradient of a scalar function. For varying noise amplitude, $\theta = 0$ the system is deterministic and for $A > 0$ undergoes oscillations with frequency $\Omega$ and amplitude $\sqrt{A/B}$.

We look for solutions of the form $h = \frac{H}{\theta} + \epsilon$, where
\[
\epsilon = \frac{1}{2} \vec{x} \cdot \mathbf{M} \cdot \vec{x} + \frac{g}{4}|\vec{x}|^4 + \ldots , \quad \mathbf{M} = \begin{pmatrix} m_1 & m_2 \\ m_4 & m_3 \end{pmatrix} ,
\]
which is reasonable if $|\vec{x}|$ is not too large. We allow the most general quadratic form and assume the form of the stabilising quartic term is unchanged but that its coupling constant may change. This can be substituted into eqn. (4) to obtain a power series which can be zeroed to term by term starting with the lowest powers to obtain simultaneous nonlinear equations for the coefficients, $m_1, g$ [65]. We find one solution, $m_1 = m_2 + m_4 = m_3 = g = 0$, which when we expand around the critical point of the corresponding expression for $h(\vec{x})$, has two positive eigenvalues for the matrix $\nabla\nabla, \nabla_j$. We thus expect the system to relax exponentially fast to this distribution, i.e. $\rho \propto e^{-H/\theta}$, so we have shown that for this system, the steady state distribution has the same form as the equilibrium one as long as the typical value of $|\vec{x}| \sim \sqrt{A/B}$ is small enough [66]. The preferred trajectories are thus given by $V_i = v_i$ and correspond to oscillations with frequency $\Omega$. The effect of fluctuations is to generate a cloud of points around the deterministic limit cycle.

\section*{B. The noisy Brusselator:} A more complicated example is provided by the effect of fluctuations on the dynamics of the Brusselator. The Brusselator is a simple two dimensional dynamical system that shows oscillatory behaviour [67]. We now use the results above to study the effects of fluctuations on this system. We consider equations for species $x, y$
\[
\frac{dx}{dt} = \mu x + x^2 y - \lambda x + x + \xi_1(t) , \quad \frac{dy}{dt} = \lambda x - x^2 y + \xi_2(t)
\]
with $\langle \xi_1 \rangle = 0$ and $\langle \xi_1(t)\xi_2(t') \rangle = 2\theta\delta_{ij}\delta(t-t')$. In the absence of fluctuations, $\theta = 0$ the system has a fixed point at $x^* = \mu, y^* = \lambda/\mu$ which becomes unstable to oscillations for $\lambda > 1 + \mu^2$. We now systematically construct an expression for the steady-state density $\rho = \frac{1}{2}e^{-h}$ when $\theta \neq 0$. In this system $H = 0$ and $\vec{v} = (\mu + x^2 y - (\lambda + 1)x, \lambda x - x^2 y)$. As long as $x, y$ are not too large, we may look for a power series expansion for $h(x,y)$ and keep terms up to a particular order, e.g. 4th order : $h = a_1 x + \frac{1}{2}a_2 x^2 + \frac{1}{3}a_3 x^3 + \frac{1}{4}a_4 x^4 + b_0 y + b_1 xy + \frac{1}{2}b_2 x^2 y + \frac{1}{3}b_3 x^3 y + \frac{1}{4}b_4 x^4 y + \frac{1}{2}c_1 xy^2 + \frac{1}{4}c_2 x^2 y^2 + \frac{1}{4}d_3 y^3 + \frac{1}{3}d_4 y^4 + \ldots$. This can be substituted into eqn. (4) to obtain a power series which can be zeroed to term by term starting with the lowest powers to obtain simultaneous nonlinear equations for the coefficients, $a_i, b_i, c_i, d_i, e_i$ [65]. Once we have the steady state we can obtain the typical trajectories. These are given by $\frac{d\vec{X}}{dt} = \vec{V} = (\mu + x^2 y - (\lambda + 1)x + \partial_t h, \lambda x - x^2 y + \partial_t h)$. The solution is most illustrative if we consider particular parameters. In Fig. 1 (a,b) we plot a single trajectory from a solution of the stochastic differential equation (SDE) for the noisy Brusselator, eqn. (18) with $\mu = 1, \lambda = 3$ for each of the two values of $\theta = \frac{1}{2}(0.2)^2, \theta = \frac{1}{2}(0.1)^2$ above plus a trajectory of the deterministic Brusselator ($\theta = 0$) and the typical trajectory of the noisy system.

We see that the effect of the noise is to shift the typical trajectory from the deterministic limit cycle to a new limit cycle as well as of course generating a cloud of state points around the new limit cycle.

\section*{C. Fluctuating d=2 active nematic:} Active matter consists of interacting self-driven particles that individually consume energy and collectively generate motion and mechanical stresses in the bulk [22, 68–71]. Due to the orientable nature of their constituents, active suspensions can exhibit liquid crystalline order and have been modeled as active liquid crystals (LCs) [22, 69, 71]. An astonishing property of active LCs is their ability to spontaneously flow in the absence of any mechanical forcing [58, 72–78]. We study a 2d active nematic film in the $Re = 0$ limit. The degrees of freedom of the system are a local nematic order parameter $Q(r,t)$, traceless symmetric $2 \times 2$ matrix and local fluid velocity $v(r,t)$, a 2d vector. The equations of motion are those of nematodynamics augmented to include activity [79, 80],
\[
0 = \eta \nabla^2 v_i + \delta_{ij} \left( \nabla_k \sigma_{kj} + \xi^Q_{ij}(r,t) \right) \quad (19)
\]
\[
(\partial_t + v \cdot \nabla) Q_{ij} = \Omega_{ij}^r + \Omega_{ij}^\sigma + \xi^Q_{ij}(r,t) \quad (20)
\]
where \( \delta^T_{ij} = \delta_{ij} - \partial_i \partial_j / \nabla^2 \), \( \nabla^2 = \sum_i \partial_i^2 \),

\[
\Omega_i^j = \frac{1}{\gamma} H_{ij} \ , \quad \Omega_{ij}^\gamma = \lambda \|Q\| u_{ij} - (\omega_{ik} Q_{kj} - Q_{ik} \omega_{kj}) \ ,
\]

with \( u_{ij} = \frac{1}{2} (\partial_i v_j + \partial_j v_i) \), \( \omega_{ij} = \frac{1}{2} (\partial_i v_j - \partial_j v_i) \), and

\[
H_{ij}(r, t) = -\delta F / \delta Q_{ij}(r, t) \ , \quad F = \int_r \left[ -\frac{A}{2} \|Q\|^2 + \frac{B}{4} \|Q\|^4 + \frac{K}{2} (\partial_i Q_{jk})^2 \right] ,
\]

\[
\sigma_{ij} = -\lambda \|Q\| H_{ij} + Q_{ik} H_{kj} - H_{ik} Q_{kj} + \alpha Q_{ij} \ ,
\]

where \( \|Q\|^2 = \text{Tr} Q^2 \).

The fluctuations have zero mean \( \langle \xi^Q \rangle = 0 \). \( \xi^Q \) is traceless symmetric with \( \xi^Q_{11} = \xi^Q_{22} = \xi^Q, \)

\[
\langle \xi^Q_i(r, t) \xi^Q_k(r', t') \rangle = \frac{2 \theta}{\gamma} \delta_{ik} \delta(r - r') \delta(t - t')
\]

and

\[
\langle \xi^Q_i(r, t) \xi^Q_k(r', t') \rangle = -4 \theta \eta (\delta_{ik} \nabla^2 + \partial_i \partial_k) \delta(r - r') \delta(t - t') .
\]

We set \( \gamma = 1, \theta = 1 \). The active contribution to the stress \( \sigma^a_{ij} = \alpha Q_{ij} \) is what makes this system non-equilibrium in the manner described above and we denote \( \alpha = 0 \) as passive and \( \alpha \neq 0 \) as active [81]. This system shows a generic instability of the nematic ordered state to a disordered flowing state [15, 22, 69, 73]. Characterising this state quantitavely in the presence of fluctuations remains an open question which we address below.

There is only one independent component of the strain rate tensor, e.g. vorticity, \( \omega(r, t) = \omega_{12} \). The linearity of the Stokes eqn. (19) means that \( \omega \) is slaved to \( Q \), so that the only truly independent fields are \( Q_{11}, Q_{22}, \) and \( Q_{22} \). We consider the system in a square box of area \( A = L^2 \), \( r = (x, y), \{0 \leq x, y \leq L\} \) with \( Q_1 = Q_1 = S_0 \) and \( Q_2 = Q_2 = 0 \) on the boundary \( \partial A \) where \( S_0 = \sqrt{2A/B} \). We consider deviations of \( Q_i = Q_i + \delta Q_i \) about the value on the boundary and \( \omega \) around a stationary fluid. We have boundary conditions, \( \delta Q_1 = 0, \hat{n} \cdot \nabla \delta Q_2 = 0 \) on \( \partial A \). Taking Fourier transforms, \( Q_i(r, t) = \int_r \Psi_q(r) \delta Q_1(r, t), \quad \bar{Q}_2(q, t) = \int_r \Phi_q(r) \delta Q_2(r, t) \) with similar expressions for \( \xi^Q_i(r, t), \xi^Q_i(r, t), \xi^Q_i(r, t) \), where \( \Psi_q(r) = N \sin(q_1 x) \sin(q_2 y), \quad \Phi_q(r) = N \cos(q_1 x) \cos(q_2 y) \) with \( N \) chosen so that \( \frac{1}{4} \pi \int_r \Psi_q^2 = 1 \) and \( q = (q_1, q_2) = \frac{\pi}{L} (m, n) \) with \( m, n \in \mathbb{Z}^+ \) [65].

Linear stability analysis shows that the uniform nematic state is unstable once, \( \exists q \) s.t. \( K q^2 + \frac{\alpha' S_0 \cos 2 \theta \eta q^2}{\Delta_q} < 0 \) where \( N_q = (1 + \lambda \cos 2 \theta \eta), \Delta_q = \eta \left( \frac{1 + \lambda q^2}{q^2} + \frac{\lambda^2 q^2}{2} + \frac{2 \lambda q^2}{q^2} \cos 2 \theta \eta \right), \cos 2 \theta \eta = (q_2^2 - q_1^2) / q^2 \), \( \alpha' = \alpha(1 + \lambda S_0) \), \( q^2 = |q|^2 \) [58, 73]. In what follows we restrict ourselves to \( q < q_{\text{max}} \approx \frac{\pi}{L} \) with \( a \) a microscopic length, keeping the number of modes finite. We restrict our analysis to \( \lambda < 1 \). For \( \alpha > 0 \), the instability is driven by modes, \( q_2 < q_1 \) while for \( \alpha < 0 \), it is driven by modes \( q_2 > q_1 \). At very small \( |\alpha| \) only the lowest \( q \) modes are unstable. To illustrate our approach, we have studied parameter ranges where \( |\alpha| \) small enough so that only a very small number of modes are linearly unstable: for \( \alpha = \alpha_+ > 0 \), such that modes \( q_+ = \frac{\pi}{L}(1, 0), 2q_+ = \frac{\pi}{L}(2, 0), q_+ = \frac{\pi}{L}(2, 1), 2q_+ = \frac{\pi}{L}(0, 2), q_+ = \frac{\pi}{L}(1, 2) \) are unstable while for \( \alpha = \alpha_- < 0 \), modes \( q_- = \frac{\pi}{L}(0, 1), 2q_- = \frac{\pi}{L}(0, 2), q_+ = \frac{\pi}{L}(1, 2) \) are unstable. We treat each case, \( \alpha_+ \) separately.
FIG. 2. Director configurations of a deterministic typical trajectory of active nematic \( \sqrt{\frac{\lambda}{2\eta}} = 1, \sqrt{\frac{\lambda'}{2\eta'}} = \frac{1}{2}, \lambda = 0 \). (a) at \( t = 0 \); (b) at \( t = T/4 \) and (c) at \( t = T/2 \) where \( T = 2\pi/\nu \).

\( \alpha \). Case \( \alpha = \alpha_+ \): Analysis of the steady-state distribution and typical trajectories leads to the following observations. All modes, \( \tilde{Q}(q) \) apart from the unstable modes fluctuate about zero (they are equilibrium-like). The unstable modes have amplitudes: 
\[ Q_1(q_+ = 0, Q_2(q_+) \equiv Q_{10}, Q_1(2q_+) = 0, Q_2(2q_+) \equiv Q_{20}, Q_1(q_1) \equiv S_{21}, Q_2(q_1) \equiv Q_{21}, S_{21} = (S_{21}, Q_{21}) \]
that on long timescales fluctuate about the deterministic trajectories with \( Q_{10} = 0; Q_{20}^2 = \frac{C'}{3\eta} S_0, Q_{20} = 0; Q_{20}^2 = \frac{C'}{\eta} S_0 \), and

\[ \frac{d}{dt}S_{21} = -D(q_1) \cdot \alpha(q_1) \cdot S_{21}; |S_{21}|^2 \simeq \frac{C}{3B} S_0, \quad (21) \]

where \( C = \alpha' \left( 1 - \frac{\lambda}{2} \right) \ll A, B \) and keeping only leading order terms in \( \alpha \). The mean amount of nematic order is renormalized by activity to \( \bar{Q}_1 = S = S_0 + \delta S_0 \simeq \sqrt{\frac{A}{2\eta}} - \frac{C}{3\eta} \). \( \alpha, D \) are matrices. They are

\[ \alpha(q_1) = \alpha' \left( \begin{array}{cc} -a & 0 \\ a & 0 \end{array} \right), \quad D(q_1) = \left( \begin{array}{cc} D_{11} & D_{12} \\ D_{21} & D_{22} \end{array} \right) \]

where

\[ D_{11} = (1 + \frac{\lambda^2 S^2}{2\eta} \sin^2 \theta_q), \quad D_{22} = (1 + \frac{2S^2}{\eta} N_q^2), \]
\[ D_{12} = D_{21} = -\frac{\lambda S^2}{\eta} \sin \theta_q N_q, a = -\frac{S_0 \sin \theta_0 q}{2\Delta_0}, \quad \sin \theta_0 q = 2q_1 q_2 / q^2. \]

\( \alpha \) is antisymmetric, hence any term proportional to \( \alpha \) cannot be written as the derivative of a scalar function and makes the system non-equilibrium as defined above. \( D \) is a mobility matrix [65]. This leads to oscillatory behaviour of mode \( q_1 \) with frequency \( \nu = |\alpha'| \sqrt{\det(D(q_1))} \). Hence we can construct the evolution of the average dynamics of the nematic director as illustrated in Figure 2. When \( A \simeq 0 \), anomalous fluctuations near critical points mean that these results must be augmented by RG analysis. As such points are rare, one expects to find few experiments in their vicinity [15].

\( b. \) Case \( \alpha = \alpha_- \): Here the unstable modes have amplitudes: 
\[ Q_1(q_-) = 0, Q_2(q_-) \equiv Q_{01}, Q_1(2q_-) = 0, Q_2(2q_-) \equiv Q_{02}, Q_1(q_1) \equiv S_{12}, Q_2(q_1) \equiv Q_{12}, S_{12} = (S_{12}, Q_{12}) \]
As above these modes fluctuate about deterministic trajectories with \( Q_{01}, Q_{02}, S_{12} \) following the same equation as \( Q_{10}, Q_{20}, S_{21} \) respectively with \( C \) replaced by \( C' \).

IV. DISCUSSION

Thermodynamic equilibrium is characterised by the macroscopic quantities of a system being stationary in time. Equilibrium statistical mechanics provides a link to microscopic degrees of freedom via a steady-state probability distribution of microstates (that maximises entropy). The utility of equilibrium statistical mechanics rests on the ability to express macroscopic quantities in terms of sums over microstates weighted by this probability distribution. In general, these sums are very difficult to evaluate, however their very existence justifies numerous approximations that can be made which allow many of these quantities to be calculated to a controllable accuracy using a variety of analytical and numerical techniques [51].

In this article we show how to place a number of classical non-equilibrium systems on a similar footing. We find that in order to do this one must add a new dynamical aspect to the concept of the steady state. These non-equilibrium steady states are intrinsically dynamic in the sense that they are steady only at the level of the probability density of microstates. The probability density of states can only remain steady if the systems moves through the microstates in a particular determin-
istic manner. The generalised steady states are thus characterised by two related quantities, a probability density and a dynamical system. Unsurprisingly, one is not in general able to show that generalised steady states exist for every non-equilibrium system, however we are able to show when they can be found, under what conditions such generalised steady states are stable. Furthermore, we show that if the steady state distribution satisfies specific properties, then the system relaxes exponentially fast to that generalised steady state on a timescale that we can calculate. This has been done by reformulating and extending some results from the mathematics of stochastic systems [54–57]. The presentation has been kept non-technical and physically intuitive. We hope this will lead to wide application in studies of realistic driven fluctuating physical systems, thus allowing the implications of experimental measurements on active matter systems to be more precisely quantified [18–20]. In practice, one will be restricted to situations where one can only obtain approximate expressions for steady-state distributions and dynamical systems that characterise the steady states. The nonlinearity of the problem means that there will in general be more than one candidate steady state. Hence one can use this relaxation timescale as a criterion to rank steady states.

Once in a generalised steady state, macroscopic quantities can like equilibrium systems be calculated as sums over microstates weighted by the steady state distribution. However unlike equilibrium systems in which they do not change in time, these macroscopic properties evolve in time according to the typical dynamical system. A trivial example of a typical dynamical system is one that is constant in time giving equilibrium-like behaviour. The simplest non-trivial example of a typical dynamical system which cannot be mapped to an equivalent equilibrium system is one which shows cyclic motion. We illustrate our approach with a few examples of this type. In particular we obtain a new way to characterise the behaviour of active nematics beyond the generic instability of active liquid crystals. We find that beyond the instability, some of the soft goldstone modes of the nematic, instead of fluctuating about zero become excited by activity and their amplitudes develop oscillatory behaviour. However the period of oscillation of each mode (which we explicitly calculate) is different. This leads to a highly dynamic but deterministic disordered structure of director orientations and fluid velocity on average which we can explicitly describe and predict.

In a generalised steady state, the average long time behaviour of this class of non-equilibrium systems can thus be quantified by (1) picking an ensemble of initial conditions randomly from the steady state distribution $\rho$, (2) following each realisation’s evolution along the typical trajectory which goes through its initial point; (3) finally one can average over typical trajectories to obtain temporal correlations.

We have studied dynamics in the overdamped limit in which momentum degrees of freedom are assumed to have relaxed to their steady-state values, however our analysis can also be extended to timescales for which momentum degrees of freedom are still relevant [60]. We conclude by noting that while we have found conditions in the form of strict bounds on the derivatives of the steady state distribution, it is possible (and indeed expected) that with more sophisticated analysis, one can find more accurate bounds which will make these conditions valid for an even wider class of steady states [60].

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The oscillator has \( H = -\frac{A}{2}|\vec{x}|^2 + \frac{B}{4}(|\vec{x}|^2)^2 \) and \( \vec{v} = (v_1, v_2) = (\Omega x_2, -\Omega x_1) \) and the condition that determines the stationary distribution, \( \rho(\vec{x}) = \frac{1}{Z} \exp \left[ -\frac{H(\vec{x})}{\theta} - \epsilon(\vec{x}) \right] \) is

\[
\sum_{i=1}^{2} L_i(\epsilon) = 0 \tag{A1}
\]

where \( L_i(\epsilon) = \theta(\nabla_i \epsilon)^2 + v_i \nabla_i \epsilon + \nabla_i \epsilon \nabla_i H - \theta \nabla_i^2 \epsilon \). We look for a power series expansion for \( \epsilon(\vec{x}) \) and keep terms up to a particular order, e.g.

\[
\epsilon = \frac{1}{2} \vec{x} \cdot M \cdot \vec{x} + \frac{1}{4} g|\vec{x}|^4 \ldots , \quad M = \begin{pmatrix} m_1 & m_2 \\ m_4 & m_3 \end{pmatrix} .
\]

This can be substituted into eqn. (4) to obtain a power series which can be set to zero term by term starting with the lowest powers and stopping at the highest powers kept in the expansion for \( \epsilon \) to obtain simultaneous nonlinear equations for the coefficients, \( m_i, g \). We obtain the equations

\[
\begin{align*}
g &= 0 , \quad \theta(m_1 + m_3) = 0 , \quad \theta(m_2 + m_4) = 0 , \quad \theta(\frac{1}{4} (m_2 + m_4)^2 + m_1^2) = 0 , \quad \theta(\frac{1}{4} (m_2 + m_4)^2 + m_3^2) = 0 , \quad \theta(m_2 + m_4) (m_1 + m_3) = 0 ,
\end{align*}
\]

giving us 5 equations for 5 unknowns. Solving the equations give the solution described in the main text. The solution has \( \mathcal{C}(\vec{x}) = 0 \).

Appendix B: Brusselator

The Brusselator has \( H = 0 \) and \( \vec{v} = (v_x, v_y) = (\mu + x^2 y - (\lambda + 1)x, \lambda x - x^2 y) \) and hence the condition that determines the stationary distribution, \( \rho(x, y) = \frac{1}{Z} e^{-h(x, y)} \) is

\[
\sum_{i=1}^{2} L_i(h) = 0 \tag{B1}
\]

where \( L_i(h) = \theta(\nabla_i h)^2 + \nabla_i^2 H + \nabla_i h (v_i - \nabla_i H) - \theta \nabla_i^2 h - \nabla_i v_i \). We look for a power series expansion for \( h(x, y) \) and keep terms up to a particular order, e.g.

\[
h = a_1 x + \frac{1}{2} a_2 x^2 + \frac{1}{3} a_3 x^3 + \frac{1}{4} a_4 x^4 + y \left( b_0 + b_1 x + \frac{1}{2} b_2 x^2 + \frac{1}{3} b_3 x^3 \right) + \frac{y^2}{2} \left( c_0 + c_1 x + \frac{1}{2} c_2 x^2 \right) + \frac{y^3}{3} \left( d_0 + d_1 x + \frac{1}{4} c_0 y^4 \right) + \ldots
\]
This can be substituted into eqn. (4) to obtain a power series which can be set to zero term by term starting with the lowest powers to obtain simultaneous nonlinear equations for the coefficients, \( a_i, b_i, c_i, d_i, e_i \). We obtain the equations

\[
\begin{align*}
-1 - \lambda - a_1 \mu - a_1^2 \theta + a_2 \theta - b_2 \theta + c_0 &= 0 \\
0 + a_1 \lambda - b_0 \lambda - a_2 \mu - a_1 a_2 \theta + 2a_1 \lambda + 2b_0 b_1 \lambda + c_0 &= 0 \\
b_1 \mu - 2a_1 b_1 \theta + b_2 \theta - 2b_0 c_0 \theta + 2d_0 &= 0 \\
-1 + a_2 + a_2 \lambda - b_0 \lambda - a_3 + a_2^2 \theta - 2a_1 a_3 \theta + 3a_4 \theta - b_2 \theta + b_0 b_2 \theta + (c_2 \theta)/2 &= 0 \\
-c_1 \mu - 2b_1^2 \theta - 2c_3 \theta - 2a_1 c_1 \theta + c_0 \theta + 4b_0 d_0 \theta + 6c_0 &= 0 \\
2 + b_1 + b_1 \lambda - c_0 \lambda - b_2 \mu - 2a_2 b_1 \theta - 2a_1 b_2 \theta + 2b_3 \theta - 2b_1 c_0 \theta - 2b_0 c_1 \theta + 2d_1 &= 0 \\
-a_1 + b_0 + b_2 + b_2 \lambda - c_1 \lambda - b_3 \mu - 2a_3 b_1 \theta - 2a_2 b_2 \theta - 2a_1 b_3 \theta - 2b_0 c_0 \theta - 2b_1 c_1 \theta - b_0 c_2 \theta &= 0 \\
c_1 + c_1 \lambda - 2d_0 \lambda - c_1 \lambda - 4b_1 b_2 \theta - 2a_3 c_1 \theta - 3a_0 \theta - 2a_1 c_1 \theta - 4b_0 d_0 \theta - 4b_2 d_1 &= 0 \\
a_3 + 3a_3 \lambda - (b_2 \lambda)/2 - a_4 \mu - 2a_2 a_3 \theta - b_1 b_2 \theta - (2b_0 b_3 \theta)/3 &= 0 \\
-(d_1 \mu)/3 - b_3 c_1 \theta - 2c_0 d_0 \theta - (2a_1 d_1 \theta)/3 - 2b_0 c_0 \theta &= 0 \\
a_4 + a_4 \lambda - (b_3 \lambda)/3 - a_2^2 \theta - 2a_2 a_4 \theta - (b_2 \theta)/4 - (2b_0 b_2 \theta)/3 &= 0 \\
-(c_1^2 \theta)/4 - d_0^2 \theta - (2b_1 d_1 \theta)/3 - 2c_0 c_0 \theta &= 0 \\
-a_2 + b_1 + b_3 + b_3 \lambda - (c_2 \lambda)/2 - 2a_2 b_1 \theta - 2a_2 b_3 \theta - 2a_3 b_2 \theta - (2b_0 c_3 \theta)/3 - b_2 b_1 \theta + b_0 c_2 \theta &= 0 \\
-2b_1 + 2c_0 + c_2 + 2c_0 - 2d_0 \lambda - 2b_2 \theta - 2b_2 \theta - 2c_0 c_2 \theta - 2c_0 c_2 \theta - 2b_2 \theta - 4b_1 \theta &= 0 \\
(B2) & \quad (B3) & \quad (B4) & \quad (B5) & \quad (B6) & \quad (B7) & \quad (B8) & \quad (B9) & \quad (B10) & \quad (B11) & \quad (B12) & \quad (B13) & \quad (B14) & \quad (B15)
\end{align*}
\]

These equations may be solved for the specific parameters considered in the main text. These nonlinear equations yield a large number of possible solutions for the coupling constants, \( \{a_i, b_i, c_i, d_i, e_i\} \), all having \( \mathcal{C}(x, y) \neq 0 \). For parameters \( \mu = 1, \lambda = 3, \theta = \frac{1}{2}(0.1)^2 \), there are 90 possible solutions for sets of constants, \( \{a_i, b_i, c_i, d_i, e_i\} \), and for parameters \( \mu = 1, \lambda = 3, \theta = \frac{1}{2}(0.2)^2 \), there are 96 possible sets of constants. For \( \theta \) small, we find one family of solutions that have \( \mathcal{C} > 0 \) in the region close to the attractor (a limit cycle), i.e. they correspond to a stable non-equilibrium steady state, however as \( \theta \) increases that is no longer the case. For example, when \( \mu = 1, \lambda = 3, \theta = \frac{1}{2}(0.2)^2 \), we obtain

\[
\begin{align*}
a_1 &= -5.99184, a_2 = 0.673657, a_3 = 0.153112, a_4 = -0.0245148, b_0 = -8.10039, b_1 = 0.54483, b_2 = 0.407974, b_3 = -0.0979923, c_0 = 1.25294, c_1 = -0.15968, c_2 = -0.344532, d_0 = -0.0606681, d_1 = 0.011561, e_0 = -0.00567803, \\
\text{for } \mu = 1, \lambda = 3, \theta = \frac{1}{2}(0.1)^2, \text{ we find } a_1 = -4.25216, a_2 = 0.309472, a_3 = 0.137682, a_4 = -0.00944351, b_0 = -5.76712, b_1 = 0.038488, b_2 = 0.372361, b_3 = -0.08008, c_1 = 0.598897, c_2 = 0.000139246, c_3 = -0.282185, d_0 = 0.0450479, d_1 = -0.0011471, e_0 = -0.00166964.
\end{align*}
\]

Appendix C: \( d=2 \) active nematic

The equations of motion for the fields \( \mathbf{v}(\mathbf{r}, t), \mathbf{Q}(\mathbf{r}, t) \), the fluid velocity and the traceless symmetric nematic order parameter respectively, are those of incompressible viscous nematodynamics augmented to include activity [79]. The fields are explicitly

\[
\mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} Q_1(\mathbf{r}, t) & Q_2(\mathbf{r}, t) \\ Q_2(\mathbf{r}, t) & -Q_1(\mathbf{r}, t) \end{pmatrix} = \mathbf{v} = (v_1(\mathbf{r}, t), v_2(\mathbf{r}, t))
\]

Using the velocity field we can define the symmetric and asymmetric parts of the strain rate tensor: \( u_{ij} = \frac{1}{2} (\partial_i v_j + \partial_j v_i), \omega_{ij} = \frac{1}{2} (\partial_i v_j - \partial_j v_i) \) respectively. The dynamics can be reduced to coupled equations for \( Q_1(\mathbf{r}, t) = Q_{11}, Q_2(\mathbf{r}, t) = Q_{12}, u_1(\mathbf{r}, t) = u_{11}, u_2(\mathbf{r}, t) = u_{12} \) and \( \mathbf{w}(\mathbf{r}, t) = \omega_{12} \). We consider the system in a square box of area \( \mathcal{A} = L^2, \mathbf{r} = (x, y), \{0 \leq x, y \leq L\} \) with \( Q_1 = Q_1 = S_0 \) and \( Q_2 = Q_2 = 0 \) on the boundary \( \partial \mathcal{A} \) where \( S_0 = \sqrt{2A/B} \). We consider deviations of \( Q_i = \bar{Q}_i + \delta Q_i \) around the value on the boundary and \( u_1, u_2, \omega \) around a stationary fluid. We have boundary conditions, \( \delta Q_1 = 0, \mathbf{n} \cdot \nabla \delta Q_2 = 0, u_1 = 0, \mathbf{n} \cdot \nabla u_2 = 0, \mathbf{n} \cdot \nabla \omega = 0 \) on \( \partial \mathcal{A} \) which are also respected by the fluctuations (\( \mathbf{n} \) is local normal to \( \partial \mathcal{A} \)). Both \( u_1, u_2 \) can be expressed in terms of \( \omega : u_2 = (\partial_2^2 - \partial_1^2) \nabla^{-2} \omega, u_1 = 2 \partial_1 \partial_2 \nabla^{-2} \omega \) so there is only one independent component of the strain rate tensor. The
where $\tilde{\xi}_1(t)$ is antisymmetric and hence the term proportional to $\alpha$ cannot be written as the derivative of a scalar function and makes the system non-equilibrium in the manner defined in the main text.

We now consider the case when $\alpha = \alpha_+ > 0$, such that modes with $q_+ = \frac{\pi}{L}(1, 0), 2q_+ = \frac{\pi}{L}(2, 0), q_t = \frac{\pi}{L}(2, 1)$ are linearly unstable. The unstable modes have amplitudes: $Q_1(q_+) = 0, Q_2(q_+) = Q_2(q_+), Q_1(2q_+) = 0, Q_2(2q_+)$ cannot be written as the derivative of a scalar function and makes the system non-equilibrium in the manner defined in the main text.

We get equations for the modes

$$\frac{d}{dt} \hat{Q}(q) = \mathbf{D}(q) \cdot \left[ - \frac{\partial \mathcal{H}}{\partial \hat{Q}(q)} - \alpha(q) \cdot \hat{Q}(q) \right] + \mathbf{W}(q, t),$$

with noise $\mathbf{W} = (W_1, W_2)$, and $\mathcal{H}(\hat{Q}, \alpha)$ is an effective non-equilibrium "Hamiltonian". The matrix $\alpha$ is antisymmetric and hence the term proportional to $\alpha$ cannot be written as the derivative of a scalar function and makes the system non-equilibrium in the manner defined in the main text.
where the sum $\sum_{q'}$ is over all the stable modes and

$$k(q) = \begin{pmatrix} Kq^2 + 4BS_0^2 & 0 \\ 0 & Kq^2 \end{pmatrix} + \alpha'(q) \begin{pmatrix} k_{11}(q) & k_{12}(q) \\ k_{21}(q) & k_{22}(q) \end{pmatrix},$$  \hspace{1cm} (C7)$$

with $k_{11} = \frac{S_0 \lambda \sin^2 \theta}{2\Delta q}$, $k_{22} = \frac{S_0 \cos 2\theta \Delta q}{\Delta q}$, $k_{12} = k_{21} = -\frac{S_0 \sin 2\theta \Delta q}{2\Delta q}$. It is noteworthy that for the stable modes both eigenvalues of $k(q)$ are positive while for the unstable modes, one or more is negative. Hence the “non-equilibrium effective Hamiltonian” can be written as $H = \frac{1}{2}\sum_q H_q$ where $H_q = -\frac{1}{2}A_q|Q(q)|^2 + \frac{1}{2}B_q|Q(q)|^4$ where $A_q > 0$ for the unstable modes and $A_q < 0$ for the stable modes, $(B_q > 0$ for all modes). This means that any pair of unstable modes $Q(q)$ can be described as effective Hopf oscillators.

The “Hamiltonian”, $H$ is minimised for a manifold of mode amplitudes given by

$$Q_{10}^2 = f_{10} C \frac{2B}{2B} S_0, \quad Q_{20}^2 = f_{20} C \frac{2B}{2B} S_0, \quad |S_{21}|^2 = f_{21} C \frac{2B}{2B} S_0, \quad \tilde{Q}_1 = S_0 - \frac{1}{2} \left( f_{21} + f_{10} + f_{20} \right) C \frac{2B}{2B}, \quad \tilde{Q}(q) = 0. \hspace{1cm} (C8)$$

where we have taken $C \ll A, B$ and ignore terms of $O(C^2)$, and taken the limit of large system size, $L \gg 1$. The values of the scaling factors, $f_{ij}$ are functions of all the parameters in a manner dependent on the closure approximations taken in finding the minimum. In the main text we have taken all $f_{ij} = 2/3$. Hence we consider the dynamics in the vicinity of this minimum and obtain an expression for $h$. Our method for obtaining an approximate expression for $h = H + \epsilon$ is as follows. We write down a Taylor series expansion expansion for $\epsilon(q, \alpha)$ keeping terms up to linear order in $\alpha$ and quadratic order in the fields, $Q(q)$. Truncating the expansion at low order is a reasonable approximation as $Q$ is not too large. Clearly such a method can systematically improved by including higher order terms. We thus look for an expression for $\epsilon$ of the form

$$\epsilon = \sum_{q} \frac{1}{q_{max}} Q(q) \cdot M(q, \alpha) \cdot Q(q) + \cdots, \hspace{1cm} (C9)$$

which satisfies the condition for the stationary probability density $\rho(Q) = \frac{1}{Z} \exp (-h|Q|)$ :

$$\sum_{q} \text{Tr} (D(q) \cdot L(q)) = 0 \hspace{1cm} (C10)$$

where

$$L(q) = \theta \left( \frac{\partial h}{\partial Q(q)} \right)^2 + \frac{\partial^2 h}{\partial Q(q)^2} + \frac{\partial h}{\partial Q(q)} \left( \alpha \cdot \tilde{Q} - \frac{\partial h}{\partial Q(q)} \right) - \theta \frac{\partial^2 h}{\partial Q(q)^2} - \alpha(q). \hspace{1cm} (C11)$$

To satisfy the stationarity condition, $D \cdot M$ must also be traceless and symmetric. A matrix, $M$ can always be found that keeps the minimum of $H$ unchanged so all modes apart from the linearly unstable ones fluctuate about zero. The solution for $h$ corresponding to this matrix $M$ has $C = 0$.

Therefore all modes except $q_+, 2q_+, q_1$ fluctuate about zero and make no contribution to the typical trajectories of the system. Hence we obtain the deterministic equations in the main text for the typical trajectories for these modes. A similar analysis can be performed for the $\alpha_-$ case.