Normalized Solutions to the mixed fractional Schrödinger equations with potential and general nonlinear term

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Abstract: The purpose of this paper is to establish the existence of solutions with prescribed norm to a class of nonlinear equations involving the mixed fractional Laplacians. This type of equations arises in various fields ranging from biophysics to population dynamics. Due to the importance of these applications, this topic has very recently received an increasing interest. This work extends the results obtained in [18, 24] to the mixed fractional Laplacians. Our method is novel and our results cover all the previous ones.

Keywords: Normalized solutions, Mixed fractional Laplacians, General potentials.

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1 Introduction

This paper concerns the existence of solutions \((u_\lambda, \lambda) \in H^{s_1, s_2}(\mathbb{R}^d) \times \mathbb{R}\) to the following fractional equation

\[
\begin{aligned}
(-\Delta)^{s_1}u(x) + (-\Delta)^{s_2}u(x) + \lambda u(x) + V(x)u(x) &= g(u(x)), \quad x \in \mathbb{R}^d, \\
\int_{\mathbb{R}^d} |u(x)|^2 dx &= a,
\end{aligned}
\tag{1.1}
\]

where \(0 < s_1 < s_2 < 1, 2s_1 < d < \frac{2s_1s_2}{s_2-s_1}, a > 0\). The precise condition on \(g\) and \(V\) will be given later on. The fractional Laplacian is given by

\[
(-\Delta)^{s_i}u(x) = C_{d, s_i} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s_i}} dy, \quad i = 1, 2,
\]

with \(C_{d, s_i} := 2^{2s_i} \pi^{-\frac{d}{2}} s_i \Gamma\left(\frac{d+2s_i}{2} \right) \Gamma\left(1-s_i\right)\), where \(\Gamma\) is the Gamma function, see [35].

Our interest in this problem results from [24]. The authors were interested in finding normalized solutions for a class of mixed fractional equations with simplified potentials. Equation (1.1) arises in the superposition of two stochastic process with a different random walk and a Lévy flight. The associated limit diffusion is described by a sum of two fractional Laplacian with different orders, see [6]. Very recently, Dipierro et al. in [17] tackled another interesting problem related to (1.1). In particular, it turns out that the mixed fractional Laplacians models the population dynamics, some heart anamolies caused by artries issues. Those heart problems can be modeled thanks to the superposition of two to five mixed fractional Laplacians since it is not necessarily the same anomaly in the five artries, see [19]. Equation (1.1) also plays a crucial role in other fields, i.e. chemical reaction design, plasma physics, biophysics, see [1, 16, 41]. The \(L^2\)—norm is a preserved quantity of the solutions of the dynamic version of (1.1). Moreover, the normalized solutions are known to provide stable solutions. This is very attractive for applications and lead many research groups to focus on this area. The study of fractional problems has gained a lot of interest after the pioneering papers of Caffarelli et al. [13, 14, 15]. Inspired with this, several other works have been published in the nonlocal framework, see for instance, [5, 7, 12, 16, 34].

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There are two different ways to deal with equation (1.1) according to the role of $\lambda$:

(i) the frequency $\lambda$ is a fixed and assigned parameter;

(ii) the frequency $\lambda$ is considered as an unknown in the problem.

For case (i), one can apply variational methods looking for critical points of the following functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla^2 u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^d} V(x) u^2 dx - \int_{\mathbb{R}^d} G(u) dx,$$

where $G(s) := \int_0^s g(t) dt$, for $s \in \mathbb{R}$. Also, we can take advantage of some other topological methods such as the fixed point theory, bifurcation or the Lyapunov-Schmidt reduction. A large number of papers are devoted to study of this kind of problems and it is impossible to summarize it here since the related literatures are huge. Alternatively, one can search for solutions to equation (1.1) with the frequency $\lambda$ unknown. In this case, the real parameter $\lambda$ appears as a Lagrange multiplier and $L^2$-norm of solutions are prescribed. They are usually called normalized solutions. Let us introduce and review some important works in this direction that we are going to use in some way in this work.

In [27], Jeanjean studied the existence of solutions with prescribed norm in the framework of semilinear elliptic equations. In particular, he studied equation (1.1) where $s_1 = s_2 = 1$ and $V = 0$. In order to overcome the lack of compactness the author worked in the radial Sobolev space to get some compactness results. Moreover, he considered the nonradial case by using a characterization of the Palais-Smale sequence introduced in [28]. However, he did not work directly with the functional associated to the problem. In his approach, he considered a new modified functional to simplify the obtention of Palais-Smale sequence at a suitable mountain pass level. His ideas were used by many other papers, but they don’t apply to our complex situation. Further details will be pointed later.

We recall that the number $\overline{p} := 2 + \frac{4}{d}$ is called in the literature as $L^2$-critical exponent, which comes from the Gagliardo-Nirenberg inequality , see [9]. It is worth to mention that in [38], Soave studied the existence of normalized solutions for the $L^{\overline{p}}$-subcritical (that is $q \in (2, \overline{p})$) nonlinear Schrödinger equation with combined power nonlinearities of the type

$$g(t) = \mu |t|^{q-2} t + |t|^{p-2} t, \quad \text{with} \quad \mu > 0.$$  

In [29], the authors discussed the existence of ground state normalized solution for the $L^2$–subcritical case while in [30] a multiplicity result is established such that the second solution is not a ground state. We also refer to [2, 39, 42] for the existence of normalized solutions in the $L^2$–supercritical case. In [3], the authors studied the existence of infinitely many normalized solutions.

When $s_1 = s_2 = 1$ : For the potential case, that is $V \neq 0$, Pellacci et al. [36] applied Lyapunov-Schmidt reduction approach to study problem (1.1) for the special case where $g(u) = u^p$. In [26], Ikoma et al. treated the potential case with general nonlinearity. The case of positive potential and vanishing at infinity is considered in a very recent paper [4]. In such a case, the mountain pass structure introduced in [27] is destroyed. By constructing a suitable linking structure, the authors proved the existence of normalized solutions with high Morse index. The case of negative potential is considered in [33] with the particular nonlinearity $g(u) = |u|^p$. Very recently, in [18] Ding et al. treated the case of negative potential and a more general nonlinearity. In [43], Yang et al. studied the existence and multiplicity of normalized solutions to the Schrödinger equations with potentials and non-autonomous nonlinearities. For important contributions to the study of normalized solutions we refer to the works of Hajaiej and Stuart [22, 23].

Contrary to the local case, the situation seems to be in a developing state for the fractional Laplacian, see [31, 40]. It is worth mentioning that there is only one paper devoted to the study of normalized solution for nonlinear equations in which the mixed fractional Laplacians is present. In [24], Hajaiej et al considered the following equation

$$(-\Delta)^{s_1} u(x) + (-\Delta)^{s_2} u(x) + \lambda u(x) = |u|^p u, \quad x \in \mathbb{R}^d$$

where $s_1 < s_2$, $p, d > 0$ and $\lambda \in \mathbb{R}$. The authors discussed the existence and nonexistence of normalized solution for the above problem.
The novelty of our work comes from the presence of a general potential and nonlinearity in Eq. (1.1). This is not an easy situation to deal with for the mixed fractional Laplacians. To the best of our knowledge, this is the first paper proving the existence of normalized solution in this very general context. More precisely, the main difficulties that arise in treating this problem are the following: (i) the lack of compactness due to the unboundedness of the domain; (ii) the absence of the fundamental properties of the mixed fractional Laplacians such as the classification of the Palais Smale sequences; (iii) the presence of the nonradial potential $V$ which forbid the use of the principle of symmetric criticality and perturb the arguments employed for the nonpotential case. To avoid the two first difficulties, we prove a new theorem in which we give a classification of the Palais sequences (see Theorem 3.3). The main key to overcome the third difficulty is to exploit some properties of the functional associated to equation (1.1) with $V = 0$. In particular, we will show that there exists $u \in H^{s_1,s_2}(\mathbb{R}^d)$, such that

$$m_0 = \inf_{v \in \mathcal{P}_{\infty,a}} I(v) = I(u),$$

where $I$ and $\mathcal{P}_{\infty,a}$ are defined in Section 3. There are two ways to prove that $m_0$ is attained; The first idea to establish this result is to work on the radial space to get some compactness properties (see Subsection 3.2). However, the fact that $m_0$ is attained for the radial space is not enough to prove the existence of normalized solution for problem (1.1) with potential, that is, $V \neq 0$. For this reason, we will give a new methods which combine some new technical lemmas with the argument employed by Jeanjean in [27].

For the reader’s convenience, we now state all the conditions on $g$ and $V$:

$(G_1)$ $g : \mathbb{R} \to \mathbb{R}$ is continuous and odd.

$(G_2)$ We assume that there exits $(\alpha, \beta) \in \mathbb{R}^2$ satisfying

$$2 + \frac{4s_2}{d} < \alpha < \beta < \frac{2d}{d - 2s_1},$$

such that

$$\alpha G(s) \leq g(s) s \leq \beta G(s), \quad \forall s \in \mathbb{R}, \quad \text{with} \quad G(s) = \int_0^s g(t) dt.$$  

$(G_3)$ Let $\tilde{G} : \mathbb{R} \to \mathbb{R}$, $\tilde{G}(s) = \frac{g(s)s}{2} - G(s)$. We assume that $\tilde{G}^\prime$ exists and

$$\tilde{G}^\prime(s) s \geq \alpha \tilde{G}^\prime(s), \quad \forall s \in \mathbb{R},$$

where $\alpha$ is given by $(G_2)$.

$(V_1)$ \quad $\lim_{|x| \to +\infty} V(x) = \sup_{x \in \mathbb{R}^d} V(x) = 0$ and there exists some $\sigma_1 \in \left[0, \frac{d(\alpha - 2)}{d(\alpha - 2) - 4}\right]$ such that

$$\int_{\mathbb{R}^d} V(x) u^2 dx \leq \sigma_1 \left( |\nabla_{s_1} u|_2^2 + |\nabla_{s_2} u|_2^2 \right), \quad \forall u \in H^{s_1,s_2}(\mathbb{R}^d).$$

$(V_2)$ $\nabla V(x)$ exists for a.e $x \in \mathbb{R}^d$. Set $W(x) = \frac{1}{2} \langle \nabla V(x), x \rangle$. We assume that $\lim_{|x| \to +\infty} W(x) = 0$, and there exists a positive constant $\sigma_2$ such that

$$0 < \sigma_2 < \min \left\{ \sigma_1 - \frac{(\beta - 2)d}{2\beta}, \frac{d(\alpha - 2)(1 - \sigma_1)}{4} - \sigma_2 \right\}$$

such that

$$\int_{\mathbb{R}^d} W(x) u^2 dx \leq \sigma_2 \left( |\nabla_{s_1} u|_2^2 + |\nabla_{s_2} u|_2^2 \right), \quad \forall u \in H^{s_1,s_2}(\mathbb{R}^d),$$

where $\alpha, \beta$ and $\sigma_1$ are defined in $(G_1)$ and $(V_1)$.

$(V_3)$ $\nabla W(x)$ exists for a.e $x \in \mathbb{R}^d$. Put $Y(x) = (d\alpha/2 - d - 1)W(x) + \langle \nabla W(x), x \rangle$. We suppose that, there exists some $\sigma_3 \in \left[0, s_1^2(d\alpha/2 - d - 2s_2)\right]$ such that

$$\int_{\mathbb{R}^d} Y(x) u^2 dx \leq \sigma_3 \left( |\nabla_{s_1} u|_2^2 + |\nabla_{s_2} u|_2^2 \right), \quad \forall u \in H^{s_1,s_2}(\mathbb{R}^d).$$
As a consequence of the above assumptions, we can deduce the following two remarks, see [27, 42].

**Remark 1.1** (1) It immediately follows from (G1) and (G2) that, for all $t \in \mathbb{R}$ and $s \geq 0$,

\[
\begin{cases}
    s^2 G(t) \leq G(ts) \leq s^a G(t), & s \leq 1, \\
    s^a G(t) \leq G(ts) \leq s^2 G(t), & s \geq 1.
\end{cases}
\]

(2) There exist some $C_1, C_2 > 0$ such that, for all $s \in \mathbb{R}$,

\[
\begin{cases}
    C_1 \min(|s|^\alpha, |s|^\beta) \leq G(s) \leq C_2 \max(|s|^\alpha, |s|^\beta) \leq C_2(|s|^\alpha + |s|^\beta), \\
    \left(\frac{4}{7} - 1\right) G(s) \leq \tilde{G}(s) \leq \left(\frac{4}{7} - 1\right) C_2(|s|^\alpha + |s|^\beta).
\end{cases}
\]

**Remark 1.2** (1) Set $g(t) = |t|^{p-2}t + |t|^{q-2}t$, for $t \in \mathbb{R}$, such that

\[2 + \frac{4s_2}{d} < p < q < \frac{2d}{d - 2s_1}.
\]

A trivial verification shows that (G1) – (G3) are satisfied.

(2) By Sobolev inequality, under some small conditions on $|V|_\infty, |W|_\infty$ and $|Y|_\infty$, one can easily check that (V1) – (V3) are satisfied.

Our main result reads as follows.

**Theorem 1.1** Suppose that assumptions (G1) – (G3) and (V1) – (V3) are satisfied. Then

(1) Problem (1.1) has no nontrivial solution $u \in H^{s_1, s_2}(\mathbb{R}^d)$ provided that $\lambda \leq 0$.

(2) For any $a > 0$, there exists a couple $(\lambda_a, u_a) \in \mathbb{R}^+ \times H^{s_1, s_2}(\mathbb{R}^d)$ solves (1.1).

Next, we consider case where the potential $V(x)$ satisfies the following coercive-type assumptions:

(V1) $V(x) \in C(\mathbb{R}^d, \mathbb{R})$ with $V_0 := \inf_{x \in \mathbb{R}^d} V(x) > 0$.

(V2) For every $M > 0$,

\[
\mu(\{x \in \mathbb{R}^d : V(x) \leq M\}) < \infty,
\]

where $\mu$ denotes the Lebesgue measure in $\mathbb{R}^d$.

Then, we give our second main result.

**Theorem 1.2** Suppose that assumptions (G1) – (G3) and (V1)’ – (V2)’ are satisfied. Then for any $a > 0$, there exists a couple $(\lambda_a, u_a) \in \mathbb{R}^+ \times H^{s_1, s_2}(\mathbb{R}^d)$ solves (1.1).

The paper is organized as follows. In Section 2, we state the main notations and the main results that will be used later. The basic properties of the associated functional to the auxiliary problem with $V = 0$ are then considered in Section 3. In Sections 4 and 5, we will show the proof of Theorems 1.1 and 1.2.

## 2 Preliminary results

In this section, we will focus on the variational structure of equation (1.1). First we introduce the fractional Sobolev space

\[H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : |\nabla_x u|^2 < +\infty\},\]

with

\[|\nabla_x u|^2 = \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d + 2s}} \, dx \, dy.\]

For $0 < s_1 < s_2 < 1$, we set

\[H^{s_1, s_2}(\mathbb{R}^d) = H^{s_1}(\mathbb{R}^d) \cap H^{s_2}(\mathbb{R}^d).\]
One can check that \( H^{s_1,s_2}(\mathbb{R}^d) \) is a Hilbert space with respect to the scalar product
\[
(u,v)_{H^{s_1,s_2}} = \int_{\mathbb{R}^d} u(x)v(x)dx + \frac{1}{2} \sum_{i=1}^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))v(x)}{|x - y|^{d + 2s_i}} dx dy, \quad \forall u,v \in H^{s_1,s_2}(\mathbb{R}^d),
\]
with the induced norm
\[
\|u\|^2 = \|u\|^2_{H^{s_1,s_2}} = |\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2 + |u|^2.
\]
Denote \( 2^*_s = \frac{2d}{d + 2s} \). It is well known that \( H^s(\mathbb{R}^d) \) is continuously embedded in \( L^q(\mathbb{R}^d) \), for any \( q \in [2, 2^*_s] \).
Then
\[
H^{s_1,s_2}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d), \quad \forall q \in [2, 2^*_s].
\]
And we denote \( H^{s_1,s_2}_p(\mathbb{R}^d) \) by
\[
H^{s_1,s_2}_p(\mathbb{R}^d) = \{ u \in H^{s_1,s_2}(\mathbb{R}^d) : u(x) = h(|x|), \quad h : [0, +\infty[ \to \mathbb{R} \}.
\]

**Lemma 2.1** \( H^{s_1,s_2}_p(\mathbb{R}^d) \) is compactly embedding into \( L^p(\mathbb{R}^d) \) for \( p \in (2, 2^*_s) \).

**Proof:** The proof is similar to that in [32].

Recall that from [9], if \( 0 < s < 1, \ d > 2s, \ p < \frac{2d}{d - 2s} \), then the fractional Gagliardo-Nirenberg inequality
\[
|u|^{p+2}_{p+2} \leq B(p,d)|\nabla_s u|^{2p}_{2p} |u|^{2p-2}_{2p}, \quad \forall u \in H^{s_1,s_2}(\mathbb{R}^d),
\]
holds with the optimal constant \( B(d,p,s) > 0 \) given by
\[
B(d,p,s) = \left( \frac{2(p + 2) - dp}{2(p + 2)} \right)^{\frac{2p}{2(p + 2) - dp}} \frac{2s(p + 2)}{2(p + 2)} \|Q_s\|^2_{p+2},
\]
and \( Q_s \) is the ground state of \n\[
(-\Delta)^s u + u = |u|^p u, \quad x \in \mathbb{R}^d,
\]
whose existence and uniqueness have been proved by [21]. For more details on fractional Sobolev spaces, we refer the readers to see [7, 34].

Now, we recall the definition of the Palais Smale sequence.

**Definition 2.1** Let \( I \in C^1(H^{s_1,s_2}(\mathbb{R}^d), \mathbb{R}) \), we say that the sequence \( \{u_n\} \subset H^{s_1,s_2}(\mathbb{R}^d) \) is a \( (PS)_c \)-sequence for \( I \) at level \( c \in \mathbb{R} \) if
\[
I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in} \quad (H^{s_1,s_2}(\mathbb{R}^d))^*.
\]

A solution \( u \) to the problem (1.1) with \( \int_{\mathbb{R}^d} |u(x)|^2 dx = a \) corresponds to a critical point of the following \( C^1 \)-functional
\[
J(u) = \frac{1}{2} |\nabla_{s_1} u|^2 + \frac{1}{2} |\nabla_{s_2} u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} V(x)u^2 dx - \int_{\mathbb{R}^d} G(u) dx,
\]
restricted to the sphere in \( L^2(\mathbb{R}^d) \) given by
\[
S_a := \{ u \in H^{s_1,s_2}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |u(x)|^2 dx = a \}.
\]
Then the parameter \( \lambda \) appears as Lagrange multiplier.

The corresponding functional to (1.1) when \( V = 0 \) is
\[
I(u) = \frac{1}{2} |\nabla_{s_1} u|^2 + \frac{1}{2} |\nabla_{s_2} u|^2 - \int_{\mathbb{R}^d} G(u) dx, \quad \forall u \in H^{s_1,s_2}(\mathbb{R}^d).
\]
It is clear that, under assumptions \((G_1) - (G_3)\), the functional \( I \) is of class \( C^1 \).

In order to follow the same strategy of [18], we need the following definitions to introduce our variational
procedure.
(1) The fiber map,
\[ u(x) \rightarrow (t * u)(x) = t^d u(t x), \]
for \((t, u) \in \mathbb{R}^+ \times H^{s_1, s_2}(\mathbb{R}^d)\), which preserves the \(L^2\)–norm.
(2) The modified functionals associated to equations (1.1) and nonpotential case of (1.1) are respectively

\[ \Psi_{\infty, u}(t) = I(t * u), \quad \forall (t, u) \in \mathbb{R} \times H^{s_1, s_2}(\mathbb{R}^d) \]

and

\[ \Psi_u(t) = J(t * u), \quad \forall (t, u) \in \mathbb{R} \times H^{s_1, s_2}(\mathbb{R}^d) \]

(3) Let

\[ P_{\infty}(u) := s_1 |\nabla s_1 u|^2 + s_2 |\nabla s_2 u|^2 - d \int_{\mathbb{R}^d} \tilde{G}(u) dx, \]

\[ \mathcal{P}_{\infty} := \{ u \in H^{s_1, s_2}(\mathbb{R}^d) : P_{\infty}(u) = 0 \}, \quad \mathcal{P}_{\infty, a} = S_a \cap \mathcal{P}_{\infty}, \]

and

\[ \mathcal{P}_{\infty, a, r} = \mathcal{P}_{\infty, a} \cap H^{s_1, s_2}(\mathbb{R}^d). \]

(4) We introduce the following function:

\[ P(u) = s_1 |\nabla s_1 u|^2 + s_2 |\nabla s_2 u|^2 - \frac{1}{2} \int_{\mathbb{R}^d} (\nabla V(x), x) u^2 dx - d \int_{\mathbb{R}^d} \tilde{G}(u) dx. \]

(5) We set

\[ \mathcal{P} := \{ u \in H^{s_1, s_2}(\mathbb{R}^d) : P(u) = 0 \}, \quad \mathcal{P}_a = S_a \cap \mathcal{P}. \]

3 Properties of the functional \( I \)

In this section, we explore the properties of the functional associated to the auxiliary problem (1.1) without linear potential. More precisely, we will prove that \( m_a := \inf_{u \in \mathcal{P}_{\infty, a}} I(u) \) and \( m_{a, r} := \inf_{u \in \mathcal{P}_{\infty, a, r}} I(u) \) are attained. Here, we would like to mention that since the functional \( I \) is considered on the whole space \( \mathbb{R}^N \), the qualitative properties of \( I \) are more difficult to establish due to the lack of compactness. For this reason, we will consider two cases: the radial and the nonradial cases.

3.1 Technical Lemmas

In this subsection, we prove some technical lemmas which will be useful to prove the main result of Theorem 3.1. We need to use the radial Sobolev space only in the proof of Theorem 3.1. So the rest of Lemmas are considered in \( H^{s_1, s_2}(\mathbb{R}^d) \). We establish our results in the spirit of the papers [2, 42].

First, we observe that

\[ \Psi_{\infty, u}'(t) = s_1 t^{2s_1 - 1} |\nabla s_1 u|^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|^2 - \frac{d}{t^{d+1}} \int_{\mathbb{R}^d} \tilde{G}(t^d u(x)) dx = \frac{1}{t} P_{\infty}(t * u) \]

for all \((t, u) \in \mathbb{R} \times H^{s_1, s_2}(\mathbb{R}^d)\).

**Lemma 3.1** Suppose that \((G_1) - (G_3)\) hold. Then, for any \( a \in S_a \), there exists a unique \( t_a > 0 \) such that \( t_a * u \in \mathcal{P}_{\infty, a} \). Moreover,

\[ I(t_a * u) = \max_{t > 0} I(t * u). \]

Consequently,

\[ m_a := \inf_{u \in S_a} \max_{t > 0} I(t * u) = \inf_{u \in \mathcal{P}_{\infty, a}} I(u). \]
Proof: First, we show that there exists (at least) a $t_u \in \mathbb{R}^+$, such that $t_u * u \in \mathcal{P}_{\infty, a}$. This follows directly from assumptions $(G_1)$ and $(G_2)$. Recall that

$$P_\infty(t * u) = s_1 t^{2s_1} |\nabla_s u|^2_2 + s_2 t^{2s_2} |\nabla_s u|^2_2 - \frac{d}{t^d} \int_{\mathbb{R}^d} \tilde{G}(t^\frac{d}{2} u(x))dx.$$  

In light of Remark 1.1, we can deduce that

$$C_1 t^{2s_1} + C_d t^{2s_2} - C_3 \max\{t^{\frac{\alpha - 2}{2}d}, t^{\frac{\beta - 2}{2}d}\} \leq P_\infty(t * u) \leq C_1 t^{2s_1} + C_d t^{2s_2} - C_3 \min\{t^{\frac{\alpha - 2}{2}d}, t^{\frac{\beta - 2}{2}d}\}, \quad (3.1)$$

where $C_1, C_2, C_3 > 0$ are independent of $t$. Hence, for $t$ large enough and using $(3.1)$, we obtain

$$P_\infty(t * u) \leq C_1 t^{2s_1} + C_d t^{2s_2} - C_3 t^{\frac{\alpha - 2}{2}d},$$

which proves that

$$\lim_{t \to +\infty} P_\infty(t * u) = -\infty. \quad (3.2)$$

Again, for $t$ small enough and using $(3.1)$, we get

$$C_1 t^{2s_1} + C_d t^{2s_2} - C_3 t^{\frac{\alpha - 2}{2}d} \leq P_\infty(t * u) \leq C_1 t^{2s_1} + C_d t^{2s_2} - C_3 t^{\frac{\beta - 2}{2}d},$$

which implies that

$$\lim_{t \to 0^+} P_\infty(t * u) = 0 \quad and \quad P_\infty(t * u) > 0, \quad for \quad t \quad small \quad enough. \quad (3.3)$$

In the above inequalities, we used the fact that $2s_1 < 2s_2 < \frac{\alpha - 2}{2}d < \frac{\beta - 2}{2}d$. Consequently, by combining $(3.2)$ and $(3.3)$, there must exist $t_u > 0$, such that $P_\infty(t_u * u) = 0$. Due to $\Psi_{\infty, a}(t) = \frac{1}{t} P_\infty(t * u)$, we have $t_u$ is a critical point of $\Psi_{\infty, a}$. Therefore, using this fact and $t * u \in S_{s_1}$, we can infer that $t_u * u \in \mathcal{P}_{\infty, a}$.

Next, we show that $t_u$ is unique and $\Psi_{\infty, a}$ reaches its maximum at $t_u$. To reach this goal, we only need to show that for any critical point $t_u$ of $\Psi_{\infty, a}(t)$, we have $\Psi_{\infty, a}'(t_u) |_{t = t_u} < 0$.

Let $t_u$ be a critical point of $\Psi_{\infty, a}$. Then

$$s_1 t_u^{2s_1} |\nabla_s u|^2_2 + s_2 t_u^{2s_2} |\nabla_s u|^2_2 - \frac{d}{t_u^d} \int_{\mathbb{R}^d} \tilde{G}(t_u^\frac{d}{2} u(x))dx = 0. \quad (3.4)$$

Now, by direct computation, we have

$$\Psi_{\infty, a}'(t) = s_1 (2s_1 - 1) t^{2s_1 - 2} |\nabla_s u|^2_2 + s_2 (2s_2 - 1) t^{2s_2 - 2} |\nabla_s u|^2_2 + \frac{d(d + 1)}{t_u^{d+2}} \int_{\mathbb{R}^d} \tilde{G}'(t_u^\frac{d}{2} u(x))dx \cdot u(x)dx. \quad (3.5)$$

Therefore, from $(3.4)$ and $(3.5)$, we infer that

$$\Psi_{\infty, a}''(t)|_{t = t_u} = 2(s_1 - s_2) s_1 t_u^{2s_1 - 2} |\nabla_s u|^2_2 + \frac{(2s_2 + d)d}{t_u^{d+2}} \int_{\mathbb{R}^d} \tilde{G}(t_u^\frac{d}{2} u(x))dx - \frac{d^2}{2 t_u^{d+2}} \int_{\mathbb{R}^d} \tilde{G}'(t_u^\frac{d}{2} u(x)) \cdot t_u^\frac{d}{2} u dx. \quad (3.6)$$

On the other hand, by assumption $(G_3)$, we obtain

$$\frac{(2s_2 + d)d}{t_u^{d+2}} \int_{\mathbb{R}^d} \tilde{G}(t_u^\frac{d}{2} u(x))dx - \frac{d^2}{2 t_u^{d+2}} \int_{\mathbb{R}^d} \tilde{G}'(t_u^\frac{d}{2} u(x)) \cdot t_u^\frac{d}{2} u dx \leq (2s_2 + d - \frac{d\alpha}{2}) \frac{d}{t_u^{d+2}} \int_{\mathbb{R}^d} \tilde{G}(t_u^\frac{d}{2} u(x))dx. \quad (3.7)$$

Consequently, by the fact that $\alpha > 2 + \frac{4s_2}{d}$, $s_1 < s_2$ and using $(3.6)$ and $(3.7)$, we conclude that

$$\Psi_{\infty, a}''(t)|_{t = t_u} < 0.$$ 

This ends the proof. $$\blacksquare$$
Remark 3.1 From the proof of Lemma 3.1, we have that if $P_\infty(u) \leq 0$ ($P_{\infty,r}(u) \leq 0$), then $\exists t_u \in (0,1]$, such that $t_u * u \in P_{\infty,a}$ ($t_u * u \in P_{\infty,a,r}$).

Lemma 3.2 Suppose that $(G_1)-(G_3)$ hold and consider the function $m : \mathbb{R}^+ \to \mathbb{R}$ defined by $m(a) := m_a$, then $m$ is strictly decreasing.

Proof: Let $0 < a_1 < a_2 < \infty$ and $\theta = \frac{A}{s_1} > 1$. For any $u_1 \in S_{a_1}$, set 
\[ u_2(x) = \theta^A u_1(\theta^B x), \]
where $A = \frac{2s_1 - d}{4s_1} < 0$, $B = -\frac{1}{2s_1} < 0$. It is clear that 
\[ 2A - Bd = 1, \quad 2A + B(2s_1 - d) = 0. \]
This implies that 
\[ |\nabla_s(\theta^A u(\theta^B x))|_2 \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\theta^{2A}|u(\theta^B x) - u(\theta^B y)|^2}{|x - y|^{d + 2s}} dx dy = \theta^{2A + B(2s_1 - d)} |\nabla_s u|_2^2, \]
and 
\[ |\theta^A u(\theta^B x)|_2^2 = \theta^{2A - Bd}|u|_2^2. \]
Therefore 
\[ |\nabla_s u_2(x)|_2^2 = |\nabla_s u_1(x)|_2^2, \quad |\nabla_s u_2(x)|_2^2 = \theta^{\frac{d - s_2}{s_1}} |\nabla_s u_1(x)|_2^2, \quad |u_2(x)|_2^2 = a_2. \]
Recall that, from Remark 1.1, we have 
\[ A\beta - Bd = \frac{d}{2s_1} - \frac{d - 2s_1}{4s_1} \beta > 0. \]
Then, by Remark 1.1 and Lemma 3.1, we obtain 
\[
m(a_2) \leq \max_{t > 0} I(t * u_2) = I(t_{u_2} * u_2) = I(t_{u_2} \theta^A u_1(\theta^B t_{u_2} x)) \\
= \frac{1}{2} |\nabla_s (t_{u_2} * u_1)|_2^2 + \frac{\theta^{\frac{s}{s_1}}}{2} |\nabla_s (t_{u_2} * u_1)|_2^2 - \theta^{\frac{d - s_1}{s_1}} \int_{\mathbb{R}^d} G(t^2 \theta^A u_1(t_{u_2} x)) dx \\
\leq \frac{1}{2} |\nabla_s (t_{u_2} * u_1)|_2^2 + \frac{1}{2} \int_{\mathbb{R}^d} \nabla_s (t_{u_2} * u_1)|_2^2 - \theta^{\frac{d - s_1}{s_1}} \int_{\mathbb{R}^d} G(t_{u_2} * u_1(x)) dx \\
\leq \frac{1}{2} |\nabla_s (t_{u_2} * u_1)|_2^2 + \frac{1}{2} \int_{\mathbb{R}^d} \nabla_s (t_{u_2} * u_1)|_2^2 - \int_{\mathbb{R}^d} G(t_{u_2} * u_1(x)) dx \\
= I(t_{u_2} * u_1) \leq \max_{t > 0} I(t * u_1), \]
It follows, since $u_1 \in S_{a_1}$ is arbitrary, that 
\[
m(a_2) \leq \inf_{u \in S_{a_1}} \max_{t > 0} I(t * u) = m(a_1). \]
The equality holds only if there exists $u_{1,n} \in S_{a_1}$ such that $\int_{\mathbb{R}^d} G(t_n * u_{1,n}) dx \to 0$, where $t_n$ satisfies $t_n * u_{2,n} \in P_{a_2}$. Then we have 
\[
\int_{\mathbb{R}^d} G(t_n * u_{2,n}) dx \to 0 \text{ as } n \to \infty. 
\]
Then by $t_n * u_{2,n} \in P_{a_2}$, we have 
\[
|\nabla_s (t_n * u_{2,n})|_2 \to 0, \quad |\nabla_s (t_n * u_{2,n})|_2 \to 0, \text{ as } n \to \infty. 
\]
Therefore, $I(t_n * u_{2,n}) \to 0$. This contradiction the fact that $\inf_{u \in P_{a_2}} I(u) > 0$, which means $m$ is strictly decreasing. 

Lemma 3.3 Suppose that $(G_1)-(G_3)$ hold, then $m$ is continuous on $\mathbb{R}^+$. 

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Proof: Since $m$ is strictly decreasing (see Lemma 3.1), for any fixed $a > 0$, $m(a - h)$ and $m(a + h)$ are monotonic and bounded as $h \to 0^+$, thus they have limits. Moreover, we know that

$$m(a - h) \geq m(a) \geq m(a + h).$$

Thus we have

$$\lim_{h \to 0^+} m(a - h) \geq m(a) \geq \lim_{h \to 0^+} m(a + h).$$

Then we only need to show that $\lim_{h \to 0^+} m(a - h) \leq m(a)$ and $\lim_{h \to 0^+} m(a + h) \geq m(a)$.

Take $u \in P_{\infty, a}$, for $h > 0$, let

$$u_h(x) := \sqrt{1 - \frac{h}{a}} u(x).$$

Evidently, $|u_h|_2^2 = a - h$, and by Lemma 3.1, there exists $t_h > 0$, such that $t_h * u_h \in P_{\infty, a - h}$.

Claim: $t_h$ is bounded in $\mathbb{R}$ and $t_h * u$ is bounded in $H^{s_1, s_2}(\mathbb{R}^d)$ as $h \to 0^+$.

If $t_h < 1$ the result follows directly. So, we can assume that $t_h > 1$. Now, since $t_h * u_h \in P_{\infty, a - h}$, we get

$$s_1 |\nabla s_1 (t_h * u_h)|_2^2 + s_2 |\nabla s_2 (t_h * u_h)|_2^2 = d \int_{\mathbb{R}^d} \tilde{G}(t_h * u_h)dx.$$

Therefore, we have

$$s_2 t_h^{2s_2} |\nabla s_2 u_h|_2^2 + s_2 t_h^{2s_2} |\nabla s_2 u_h|_2^2 = d \int_{\mathbb{R}^d} \tilde{G}(t_h * u_h)dx \geq d \frac{\alpha}{2} - 1)C t_h^{-\frac{\alpha - d}{2}} |u_h|_\alpha^\alpha.$$

It follows, for $h$ small enough, that

$$s_2 t_h^{2s_2} |\nabla s_2 u_h|_2^2 + s_2 t_h^{2s_2} |\nabla s_2 u_h|_2^2 \geq \frac{C}{2} d \frac{\alpha}{2} - 1)C t_h^{-\frac{\alpha - d}{2}} |u_h|_\alpha^\alpha. \quad (3.8)$$

On the other hand, from the fact that $\alpha > 2 + \frac{4s_2}{d}$ (see $(G_2)$), one has

$$\frac{\alpha}{2} - d > 2s_2 > 2s_1. \quad (3.9)$$

Thus, using $(3.8)$ and $(3.9)$, $t_h$ is bounded as $h \to 0^+$, and so, $t_h * u$ is also bounded in $H^{s_1, s_2}(\mathbb{R}^d)$. This proves the claim.

Similar to [42], we obtain

$$\int_{\mathbb{R}^d} (G(t_h * u_h) - G(t_h * u))dx = o(1), \text{ as } h \to 0^+. \quad (3.10)$$

Then, combining $(3.10)$ with the claim, we obtain

$$I(t_h * u) - I(t_h * u_h) = \frac{h}{2a} (|\nabla s_1(t_h * u)|_2^2 + |\nabla s_2(t_h * u)|_2^2) + \int_{\mathbb{R}^d} (G(t_h * u_h) - G(t_h * u))dx = o(1).$$

Thus,

$$m(a - h) \leq I(t_h * u_h) \leq I(t_h * u) + o(1) \leq I(u) + o(1) = m(a) + o(1).$$

This implies

$$\lim_{h \to 0^+} m(a - h) \leq m(a).$$

Now, take $u \in P_{\infty, a}$ and $v_h(x) = \frac{1}{\sqrt{1 + \frac{x}{a}}} u(x)$. By the same argument used above, we can deduce that

$$\lim_{h \to 0^+} m(a + h) \geq m(a).$$

Therefore, $m$ is continuous on $\mathbb{R}^+$ and so we get the proof of our desired result. \qed
Remark 3.2 Similarly, we have $m_r : a \rightarrow m_{a,r}$ is continuous and strictly decreasing.

Lemma 3.4 Let $(G_1)-(G_3)$ be satisfied. Then, the functional $I$ restricted to $\mathcal{P}_{\infty,a}$ is coercive.

Proof: In light of $(G_2)$, we can deduce that for any $u \in \mathcal{P}_{\infty,a}$

$$s_1|\nabla_{s_1} u|^2 + s_2|\nabla_{s_2} u|^2 = d \int_{\mathbb{R}^d} G(u) dx \leq \frac{d}{2} \int_{\mathbb{R}^d} G(u) dx. \quad (3.11)$$

It follows, using again $(G_2)$, that

$$I(u) = \frac{1}{2} |\nabla_{s_1} u|^2 + \frac{1}{2} |\nabla_{s_2} u|^2 - \int_{\mathbb{R}^d} G(u) dx \geq \frac{1}{2s_2} [s_1 |\nabla_{s_1} u|^2 + s_2 |\nabla_{s_2} u|^2] - \int_{\mathbb{R}^d} G(u) dx \geq \frac{d}{4s_2} \int_{\mathbb{R}^d} (g(u) dx - (2 + \frac{4s_2}{d})G(u)) dx \geq \frac{d}{4s_2} \int_{\mathbb{R}^d} G(u) dx \geq \frac{s_1}{2\beta} (\alpha - 2 - \frac{4s_2}{d}) (|\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2),$$

for any $u \in \mathcal{P}_{\infty,a}$. Here, we used the fact $\int_{\mathbb{R}^d} G(u) dx \geq 0$ for any $u \in \mathcal{P}_{\infty,a}$. This end the proof. 

Lemma 3.5 Assume that the conditions of Lemma 3.4 are fulfilled, then

$$m_a > 0.$$

Proof: Let $u \in \mathcal{P}_{\infty,a}$. Then, in view of $(G_2)$, we have

$$s_1|\nabla_{s_1} u|^2 \leq \frac{d}{2} \int_{\mathbb{R}^d} g(u) dx \leq \frac{d}{2} \int_{\mathbb{R}^d} G(u) dx \leq C \int_{\mathbb{R}^d} (|u|^\alpha + |u|^\beta) dx.$$

Here we used Remark 1.1 and the fact that $G(u) \geq 0$. To estimate the right hand side, we apply $(2.1)$ with $p + 2 = \alpha$ and $p + 2 = \beta$, obtaining

$$|\nabla_{s_1} u|^2 \leq C \left( |\nabla_{s_1} u|^2 \frac{d(\alpha-2)}{2s_1} + |\nabla_{s_1} u|^2 \frac{d(\beta-2)}{2s_1} \right). \quad (3.13)$$

Using the same argument, we get

$$|\nabla_{s_2} u|^2 \leq C \left( |\nabla_{s_2} u|^2 \frac{d(\alpha-2)}{2s_2} + |\nabla_{s_2} u|^2 \frac{d(\beta-2)}{2s_2} \right). \quad (3.14)$$

Hence, combining $(3.13)$ and $(3.14)$ and having in mind that $\frac{d(\alpha-2)}{2s_1}$, $\frac{d(\alpha-2)}{2s_2}$, $\frac{d(\beta-2)}{2s_1}$ and $\frac{d(\beta-2)}{2s_2}$ are strictly larger than 2, we infer that

$$|\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2 \geq \delta, \quad (3.15)$$

for some $\delta > 0$. Therefore the result follows by combining $(3.12)$ and $(3.15)$. 

Remark 3.3 Actually, the functional $I$ restricted to $\mathcal{P}_{\infty,a,r}$ is also coercive, therefore, $m_{a,r} > 0$. 

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3.2 The radial case

It should is clear that the lemmas established so far remain unchanged if we work in the subspace $H_1^{s_1,s_2}(\mathbb{R}^d)$.

**Theorem 3.1** Suppose that $(G_1)-(G_3)$ hold, then $m_{a,r}$ is attained, i.e. there exists $u \in H_1^{s_1,s_2}(\mathbb{R}^d)$, such that

$$m_{a,r} = \inf_{v \in \mathcal{P}_{\infty,a,r}} I(v) = I(u),$$

where $\mathcal{P}_{\infty,a} = H_1^{s_1,s_2}(\mathbb{R}^d) \cap \mathcal{P}_{\infty,a}$.

**Proof:** Let $\{u_n\} \subset \mathcal{P}_{\infty,a,r}$ be a minimizing sequence of $m_{a,r}$, i.e.

$$I(u_n) \rightarrow m, P_\infty(u_n) = 0, \ |u_n|_2^2 = a.$$ 

Then, in view of Lemma 3.4, $\{u_n\}$ is bounded in $H_1^{s_1,s_2}(\mathbb{R}^d)$. Subtracting if necessary a subsequence, $u_n \rightharpoonup u$ in $H_1^{s_1,s_2}(\mathbb{R}^d)$. Thus, using $(G_1) - (G_2)$, we can deduce that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} G(u_n) dx = \int_{\mathbb{R}^d} G(u) dx,$$

and

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} g(u_n) u_n dx = \int_{\mathbb{R}^d} g(u) u dx. $$

Therefore

$$I(u) \leq \liminf_{n \rightarrow +\infty} I(u_n) = m_{a,r},$$

$$P_\infty(u) \leq \liminf_{n \rightarrow +\infty} P_\infty(u_n) = 0,$$

and

$$|u|_2^2 \leq \liminf_{n \rightarrow +\infty} |u_n|_2^2 = a.$$ 

Now, from Lemma 3.5 and the fact $P_\infty(u_n) = 0$, we can deduce that $u \neq 0$. Then, let $\mu = |u|_2^2 \in (0,a]$. Observe that from the proof of Lemma 3.1 and the fact that $P_\infty(u) \leq 0$, there is $t_u \in (0,1]$ such that $t_u u \in \mathcal{P}_{\infty,a,r}$. Therefore, having in mind $\frac{1-t_u^2}{2s_2} \leq \frac{1-t_u^2}{2s_1}$, we obtain

$$I(u) - I(t_u u) = \frac{1-t_u^{2s_1}}{2s_1} |\nabla u|^2 + \frac{1-t_u^{2s_2}}{2s_2} |\nabla u|^2 + \frac{t_u'}{d} \int_{\mathbb{R}^d} G(t_u^2 u) dx - \int_{\mathbb{R}^d} G(u) dx (3.16)$$

$$\geq \frac{1-t_u^{2s_2}}{2s_2} [s_1|\nabla u|^2 + s_2|\nabla u|^2] + \frac{t_u'}{d} \int_{\mathbb{R}^d} G(t_u^2 u) dx - \int_{\mathbb{R}^d} G(u) dx$$

$$= \int_{\mathbb{R}^d} g(u) dx - (1 - \frac{1-t_u^{2s_2}}{4s_2} d) \int_{\mathbb{R}^d} G(u) dx + t_u' d \int_{\mathbb{R}^d} G(t_u^2 u) dx.$$ 

By $(G_3)$, we have

$$\frac{g(s)s - 2G(s)}{|s|^{4+\frac{4s}{d}}}$$

is increasing for $s \in \mathbb{R} \setminus \{0\}$,

and so

$$d \left(1 - \frac{1-t_u^{2s_2}}{4s_2} \right) \int_{\mathbb{R}^d} g(u) dx - \left(1 - \frac{1-t_u^{2s_2}}{2s_2} + 1\right) \int_{\mathbb{R}^d} G(u) dx + t_u' d \int_{\mathbb{R}^d} G(t_u^2 u) dx$$

$$= \int_{\mathbb{R}^d} \int_{t_u}^{t_u^2} \frac{|u|^{4+\frac{4s}{d}}}{|t_u^2 u|^{4+\frac{4s}{d}}} \left[ g(u)u - 2G(u) - g(t_u^2 u) + 2G(t_u^2 u) \right] dt dx$$

$$\geq 0.$$
Consequently,
\[ I(u) - I(t_u * u) \geq \frac{1 - t_u^{2s_2}}{2s_2} P_\infty(u), \]
which implies, using Lemma 3.2, that
\[ m_{\mu,r} \geq m_{a,r} = \lim_{n \to \infty} \left( I(u_n) - \frac{1}{2s_2} P_\infty(u_n) \right) \]
\[ \geq \lim_{n \to \infty} \left[ \left( \frac{s_2 - s_1}{2s_2} \right) |\nabla s_1 u_n|^2 + \frac{d}{2s_2} \int \nabla \tilde{G}(u_n) dx - \int \nabla \tilde{G}(u_n) dx \right] \]
\[ \geq \left( \frac{s_2 - s_1}{2s_2} \right) |\nabla s_1 u|^2 + \frac{d}{2s_2} \int \tilde{G}(u) dx - \int \tilde{G}(u) dx \]
\[ = I(u) - \frac{1}{2s_2} P_\infty(u) \geq I(t_u * u) - \frac{t_u^{2s_2}}{2s_2} P_\infty(u) \]
\[ \geq I(t_u * u) \geq m_{\mu,r}. \]
This proves that \( m_{\mu,r} = m_{a,r} \). Hence, by Lemma 3.2, we deduce that \( \mu = a \), which means \( t_u * u \in P_{\infty,a,r} \) and \( I(t_u * u) = m_{a,r} \). Thus \( I \) attains its minimum at \( u \in P_{\infty,a,r} \). This proves our desired result. \( \blacksquare \)

### 3.3 The nonradial case

In this part, we will try to give a version of Theorem 3.1 in the nonradial Sobolev space \( H^{s_1,s_2}(\mathbb{R}^d) \). We divide this subsection into two parts. Due to the lack of compactness, the main purpose of the first part is to classify the \((PS)_c\)-sequences for a suitable functional \( I_\lambda \). In the second part, we prove that \( m_a := \inf_{u \in P_{\infty,a}} I(u) \) is attained.

The main result of this subsection reads as follows.

**Theorem 3.2** Suppose that \((G_1) - (G_3)\) hold, then
\[ m_a := \inf_{u \in P_{\infty,a}} I(u), \]
is attained.

### 3.3.1 Classification of \((PS)_c\)-sequences

Let \( I_\lambda : H^{s_1,s_2}(\mathbb{R}^d) \to \mathbb{R} \) be defined by
\[ I_\lambda(u) = \frac{1}{2} |\nabla s_1 u|^2 + \frac{1}{2} |\nabla s_2 u|^2 + \frac{\lambda}{2} |u|^2 - \int \tilde{G}(u) dx, \quad (3.17) \]
where \( \lambda > 0 \).

The main purpose of this part is, to classify \((PS)_c\)-sequences for the functional \( I_\lambda \). Here, we use some ideas coming from [10, 11].

**Theorem 3.3** Let \( \{u_n\} \subset H^{s_1,s_2}(\mathbb{R}^d) \) be a \((PS)_c\)-sequence. Then, there exists \( k \in \mathbb{N} \), \( k \) functions \( u^1, \ldots, u^k \) in \( H^{s_1,s_2}(\mathbb{R}^d) \) and a subsequence (still denoted \( \{u_n\} \)) such that
1. \( I'_\lambda(u_n) = 0 \), for \( i = 1, \ldots, k \).
2. \( |\nabla s_1 u_n|^2 \to |\nabla s_1 u|^2 + \ldots + |\nabla s_k u|^2 \) and \( |\nabla s_2 u_n|^2 \to |\nabla s_2 u|^2 + \ldots + |\nabla s_2 u|^2 \).
3. \( |u_n|^2 \to |u|^2 + \ldots + |u|^2 \).
4. \( I_\lambda(u_n) \to I_\lambda(u^1) + \ldots + I_\lambda(u^k) \).
5. If \( k = 1 \), then there exists a sequence \( \{y_n\} \subset \mathbb{R}^d \) such that \( u_n(x) - u^1(x - y_n) \to 0 \) strongly in \( H^{s_1,s_2}(\mathbb{R}^d) \).

Theorem 3.3 is also called Representation Lemma in some references. To prove Theorem 3.3, we first give some auxiliary lemmas.
Lemma 3.6 Let $t > 0$ and $2 < q < 2s_2$. If $\{u_n\}$ is a bounded sequence in $H^{s_1, s_2}(\mathbb{R}^d)$ and if
\[
\sup_{y \in \mathbb{R}^d} \int_{B(y,t)} |u_n|^q dx \to 0 \quad \text{as} \quad n \to +\infty,
\]
then $\{u_n\} \to 0$ in $L^r(\mathbb{R}^d)$ for all $r \in (2, 2s_2^*)$.

Proof: It is easy to see that $\{u_n\}$ is bounded in $H^{s_2}(\mathbb{R}^d)$, then the rest of the proof is similar to Lemma 2.1 in [11].

Lemma 3.7 Suppose that $(G_1)$-$($G_3$)$ hold, then $g : H^{s_1, s_2} \to (H^{s_1, s_2})^*$, $u \to g(u)$ is a compact operator.

Proof: The proof follows by applying the same argument used in the proof of Lemma 2.6 in [27].

Lemma 3.8 Let $u \in H^{s_1, s_2}(\mathbb{R}^d) \setminus \{0\}$ be a critical point of $I_\lambda$. Then, there exists a positive constant $M > 0$ independent of $u$ such that
\[
\|u\| \geq M.
\]

Proof: Fix $\lambda > 0$, by the Sobolev embedding theorem, we can put
\[
M'_p = \inf \{ |\nabla s_1 v|^2 + |\nabla s_2 v|^2 + \lambda \int_{\mathbb{R}^d} |v(x)|^2 dx : v \in H^{s_1, s_2}(\mathbb{R}^d), \int_{\mathbb{R}^d} |v(x)|^p dx = 1 \}.
\]
We can see that $M'_p > 0$. Then
\[
|\nabla s_1 u|^2 + |\nabla s_2 u|^2 + \lambda \int_{\mathbb{R}^d} |u(x)|^2 dx \geq M'_p |u|^2.
\]
(3.18)
\[
|\nabla s_1 u|^2 + |\nabla s_2 u|^2 + \lambda \int_{\mathbb{R}^d} |u(x)|^2 dx \geq M'_p |u|^2.
\]
(3.19)
On the other hand, if $u$ is a critical point of $I_\lambda$, we have
\[
|\nabla s_1 u|^2 + |\nabla s_2 u|^2 + \lambda \int_{\mathbb{R}^d} |u(x)|^2 dx = \int_{\mathbb{R}^d} g(u) u dx \leq \beta \int_{\mathbb{R}^d} G(u) dx \leq C (|u|^\alpha + |u|^\beta).
\]
(3.20)
Hence, combining (3.18), (3.19) and (3.20), we obtain
\[
\|u\|^2 \leq C \left( \frac{\|u\|^\alpha}{(M'_p)^{\frac{2\alpha}{2}}} + \frac{\|u\|^\beta}{(M'_p)^{\frac{2\beta}{2}}} \right).
\]
Thus, there exists $M > 0$, such that $\|u\| \geq M$.

Proof of Theorem 3.3:
Claim 1: The $(PS)_c$-sequence $\{u_n\}$ is bounded in $H^{s_1, s_2}(\mathbb{R}^d)$. We have:
\[
c + o(1) \|u_n\| \geq I_\lambda(u_n) - \frac{1}{\alpha} I'_\lambda(u_n, u_n)
\]
\[
= \left( \frac{1}{2} - \frac{1}{\alpha} \right) (|\nabla s_1 u|^2 + |\nabla s_2 u|^2 + \lambda |u|^2) + \int_{\mathbb{R}^d} \left( \frac{1}{\alpha} g(u) u - G(u) \right) dx
\]
\[
\geq C \|u\|^2.
\]
Hence boundedness follows. Thus we may extract a subsequence $\{u_n\}$ such that
\[
u_n \rightharpoonup u^0 \quad \text{in} \quad H^{s_1, s_2}(\mathbb{R}^d),
\]
and
\[
u_n(x) \to u^0(x) \quad \text{a.e. in} \quad \mathbb{R}^d.
\]
By standard arguments, \( u \) is a critical point of \( I_\lambda \). Now let
\[
\Psi^1_n(x) = (u_n - u^0)(x), \quad x \in \mathbb{R}^d.
\]
Then
\[
\Psi^1_n \to 0 \quad \text{in} \quad H^{s_1,s_2}(\mathbb{R}^d).
\]
Furthermore, by applying the Brézis-Lieb lemma [8], we get
\[
\|\Psi^1_n\|^2 = \|u_n\|^2 - \|u^0\|^2 + o(1),
\]
and
\[
\int_{\mathbb{R}^d} G(\Psi^1_n) dx = \int_{\mathbb{R}^d} G(u_n) dx - \int_{\mathbb{R}^d} G(u^0) dx + o(1).
\]
Thus, by taking into account of (3.21), (3.22) and (3.23), we infer that
\[
I_\lambda(\Psi^1_n) = I_\lambda(u_n) - I_\lambda(u^0) + o(1),
\]
and
\[
I'_\lambda(\Psi^1_n) = I'_\lambda(u_n) - I'_\lambda(u^0) + o(1).
\]
Now, using Lemma 3.6 we have, either \( \Psi^1_n \to 0 \) in \( H^{s_1,s_2}(\mathbb{R}^d) \), in that case the proof is over or there exists \( \alpha' > 0 \), such that up to a subsequence
\[
\sup_{y \in \mathbb{R}^d} \int_{B(y,1)} |\Psi^1_n(x)|^2 dx > \alpha' > 0.
\]
Therefore we can find a sequence \( \{y_n\} \subset \mathbb{R}^d \) such that
\[
\int_{B(y_n,1)} |\Psi^1_n(x)|^2 dx \geq \alpha'.
\]
It follows, from the fact that \( \Psi^1_n \to 0 \) in \( H^{s_1,s_2}(\mathbb{R}^d) \) and (3.26), that
\[
|y_n| \to +\infty \quad \text{as} \quad n \to +\infty.
\]
Let us now call \( u^1 \) the weak limit in \( H^{s_1,s_2}(\mathbb{R}^d) \) of the sequence \( \Psi^1_n(\cdot + y_n) \).

**Claim 2:** the weak limit \( u^1 \neq 0 \). Indeed, since \( H^{s_1,s_2}(B(0,1)) \hookrightarrow L^2(B(0,1)) \) compactly embedded, (3.26) concludes the claim. Moreover, by standard argument, \( u^1 \) is a critical point of \( I_\lambda \).

Iterating this procedure, we obtain sequences \( \Psi^1_n = \Psi^{j-1}_n(x + y_n) - u^{j-1} \) \( j \geq 2 \) and sequences of points \( y^i_n \) such that \( |y^i_n| \to +\infty \) as \( n \to +\infty \) and
\[
\Psi^j_n(\cdot + y^i_n) \to u^j \quad \text{in} \quad H^{s_1,s_2}(\mathbb{R}^d),
\]
where each \( u^j \) is a critical point of \( I_\lambda \). Moreover, in view of (3.22) and (3.24), we get
\[
\|\Psi^1_n\|^2 = \|\Psi^{j-1}_n\|^2 - \|u^{j-1}\|^2 + o(1) = \|u_n\|^2 - \|u^0\|^2 - \sum_{i=1}^{j-1} \|u^i\|^2 + o(1),
\]
and
\[
I(\Psi^j_n) = I(\Psi^{j-1}_n) - I(u^{j-1}) + o(1) = I(u_n) - I(u_0) - \sum_{i=1}^{j-1} I(u^i) + o(1).
\]
Thus Lemma 3.8, Claim 1, and (3.27) tell us that the iteration must terminate at some index \( k \geq 0 \).

If \( k = 0 \), put \( u^0_n \equiv u_n \).

If \( k > 0 \), put \( u^k_n \equiv \Psi^k_n(\cdot + y^k_n) \), \( u^i_n \equiv \Psi^i_n(\cdot + y^i_n) - \sum_{j=i+1}^{k} u^j_n(\cdot - y^j_n) \), for \( 0 < i < k \), and
\[
u^0_n \equiv u_n - \sum_{j=1}^{k} u^j_n(\cdot - y^j_n).
\]
Then, the sequences \( \{u^j_n\}, \quad 0 \leq j \leq K \) are the desired sequences.
3.4 Proof of Theorem 3.2

The purpose of this part is to conclude the proof of Theorem 3.2. The main idea is to give a mountain pass characterization of $m_0$, which will be helpful to prove our desired result. Here, we use some ideas coming from [27].

We need the following definition to introduce our variational procedure.

- $\mathcal{H} : H^{s_1,s_2}(\mathbb{R}^d) \times \mathbb{R} \to H^{s_1,s_2}(\mathbb{R}^d)$ with \[
\mathcal{H}(u,t)(x) = e^{\alpha t} u(e^t x).
\]

It is clear that $|\mathcal{H}(u,t)|_2 = |u|_2$, for any $t \in \mathbb{R}$. We start by proving some technical lemmas which will be useful in the sequel.

**Lemma 3.9** Assume that $(G_1) - (G_2)$ hold and let $u \in S(a)$ be arbitrary. Then we have:

1. $|\nabla_{s_1} \mathcal{H}(u,t)|_2 + |\nabla_{s_2} \mathcal{H}(u,t)|_2 \to 0$, and $I(\mathcal{H}(u,t)) \to 0$ as $t \to -\infty$.
2. $|\nabla_{s_1} \mathcal{H}(u,t)|_2 + |\nabla_{s_2} \mathcal{H}(u,t)|_2 \to +\infty$, and $I(\mathcal{H}(u,t)) \to -\infty$ as $t \to +\infty$.

**Proof:** By a straightforward calculation, it follows that

\[
\int_{\mathbb{R}^d} |\mathcal{H}(u,t)(x)|^2 dx = a^2 \text{ and } |\nabla_{s_1} \mathcal{H}(u,t)|_2 + |\nabla_{s_2} \mathcal{H}(u,t)|_2^2 = e^{2ts_1} |\nabla_{s_1} u|_2^2 + e^{2ts_2} |\nabla_{s_2} u|_2^2. \tag{3.29}
\]

Therefore

$|\nabla_{s_1} \mathcal{H}(u,t)|_2 + |\nabla_{s_2} \mathcal{H}(u,t)|_2 \to 0$ as $t \to -\infty$.

Using again (3.29) and Remark 1.1, we find that

\[
|I(\mathcal{H}(u,t))| \leq \frac{1}{2} |\nabla_{s_1} \mathcal{H}(u,t)|_2 + \frac{1}{2} |\nabla_{s_2} \mathcal{H}(u,t)|_2^2 - \int_{\mathbb{R}^d} G(\mathcal{H}(u,t)(x)) dx \leq e^{2ts_1} |\nabla_{s_1} u|_2^2 + e^{2ts_2} |\nabla_{s_2} u|_2^2 + e^{dt}(\frac{a^2}{2}) \int_{\mathbb{R}^d} G(u) dx.
\]

Thus

$I(\mathcal{H}(u,t)) \to 0$ as $t \to -\infty$,

showing (1).

In order to show (2), note that by (3.29),

$|\nabla_{s_1} \mathcal{H}(u,t)|_2^2 + |\nabla_{s_2} \mathcal{H}(u,t)|_2^2 \to +\infty$ as $t \to +\infty$.

On the other hand,

$I(\mathcal{H}(u,t)) \leq e^{2ts_1} |\nabla_{s_1} u|_2^2 + e^{2ts_2} |\nabla_{s_2} u|_2^2 - e^{dt}(\frac{a^2}{2}) \int_{\mathbb{R}^d} G(u) dx$.

Taking into account the fact that $\alpha > 2 + \frac{4s_2}{d}$, it shows that $I(\mathcal{H}(u,t)) \to -\infty$ as $t \to +\infty$. \[\]

**Lemma 3.10** Suppose that the assumptions of Lemma 3.9 are fulfilled. Then, there exists $K(a) > 0$ small enough such that

$0 < \sup_{u \in A} I(u) < \inf_{u \in B} I(u)$

with

$A = \{u \in S(a) : |\nabla_{s_1} u|_2^2 + |\nabla_{s_2} u|_2^2 \leq K(a)\}$ and $B = \{u \in S(a) : |\nabla_{s_1} u|_2^2 + |\nabla_{s_2} u|_2^2 = 2K(a)\}$.
Moreover, exploiting the above inequalities, we deduce that

\[ \int_{\mathbb{R}^d} G(u) \, dx \leq C \left[ \int_{\mathbb{R}^d} |u|^\alpha \, dx + \int_{\mathbb{R}^d} |u|^\beta \, dx \right], \]

where \( C \) is a positive constant. Then, by the Gagliardo-Nirenberg inequality (2.1), we have

\[ \int_{\mathbb{R}^d} G(u) \, dx \leq C \left[ (|\nabla_s u|^2 + |\nabla_s u|^2)^{\frac{d-2}{2}} + (|\nabla_s u|^2 + |\nabla_s u|^2)^{\frac{d-2}{2}} \right]. \]

Since \( G(u) \geq 0 \) for any \( u \in H^{s_1,s_2}(\mathbb{R}^d) \), we can deduce that

\[ I(v) - I(u) = \frac{1}{2} \left[ |\nabla_s v|^2 + |\nabla_s u|^2 \right] - \frac{1}{2} \left[ |\nabla_s v|^2 + |\nabla_s u|^2 \right] - \int_{\mathbb{R}^d} G(v) \, dx + \int_{\mathbb{R}^d} G(u) \, dx \]

\[ \geq \frac{1}{2} \left[ |\nabla_s v|^2 + |\nabla_s u|^2 \right] - \frac{1}{2} \left[ |\nabla_s v|^2 + |\nabla_s u|^2 \right] - \int_{\mathbb{R}^d} G(v) \, dx, \quad \forall u, v \in H^{s_1,s_2}(\mathbb{R}^d). \]

Therefore, if we fix \( u \in A, v \in B \), then \( |\nabla_s u|^2 + |\nabla_s u|^2 \leq K(a) \) and \( |\nabla_s v|^2 + |\nabla_s v|^2 = 2K(a) \),

\[ I(v) - I(u) \geq \frac{K(a)}{2} - C \left[ (K(a))^{\frac{d-2}{2}} + (K(a))^{\frac{d-2}{2}} \right], \]

where \( C = C(\alpha, \beta, d, s_2, a) > 0 \). Due to \( \beta > \alpha > 2 + \frac{4d}{d-2} \), for \( K(a) \) sufficiently small, there exists \( \delta > 0 \), such that

\[ I(v) - I(u) \geq \delta > 0. \]

Moreover, exploiting the above inequalities, we deduce that

\[ \sup_{u \in A} I(u) > 0, \]

where \( K(a) \) small enough.

As a consequence of the above two lemmas we get:

**Lemma 3.11** Let \((G_1) - (G_2)\) be satisfied. Then there exist \( u_1, u_2 \in S(a) \) such that

(1) \( |\nabla_s u_1|^2 + |\nabla_s u_1|^2 \leq K(a) \) (\( K(a) \) is defined in Lemma 3.10).

(2) \( |\nabla_s u_2|^2 + |\nabla_s u_2|^2 > 2K(a) \).

(3) \( I(u_1) > 0 > I(u_2) \).

Moreover setting

\[ \gamma(a) = \inf_{h \in \Gamma(a)} \max_{t \in [0,1]} I(h(t)), \]

with

\[ \Gamma(a) = \{ h \in C([0,1], S(a)), \quad h(0) = u_1, \quad h(1) = u_2 \}. \]

Then

\[ \gamma(a) > \max\{I(u_1), I(u_2)\} > 0. \]

**Proof:** The proof follows directly from Lemmas 3.9 and 3.10.

To get our result we shall draw an additional variational characterization of the mountain pass level \( \gamma(a) \).

**Lemma 3.12** If \( u \in H^{s_1,s_2}(\mathbb{R}^d) \) is a critical point of \( I \), then

\[ P_\infty(u) = 0. \]

**Proof:** Let \( u \) be a critical point of \( I \). Thus, inserting \( u \) and \( x \cdot \nabla u \) respectively in the definition of \( I' \), and then integrating by parts, we get

\[ |\nabla_s u|^2 + |\nabla_s u|^2 + \lambda |u|^2 - \int_{\mathbb{R}^d} g(u) \, dx = 0 \] (3.30)
and
\[
\frac{2s_1 - d}{2} |\nabla s_1 u|_2^2 + \frac{2s_2 - d}{2} |\nabla s_2 u|_2^2 - d \int_{\mathbb{R}^d} G(u) dx = 0. \tag{3.31}
\]

Exploiting (3.30) and (3.31), we infer that
\[
s_1 |\nabla s_1 u|_2^2 + s_2 |\nabla s_2 u|_2^2 - d \int_{\mathbb{R}^d} [g(u)u - G(u)] dx = 0,
\]
which proves \(P_\infty(u) = 0\), thus the proof of the lemma. \(\blacksquare\)

**Lemma 3.13** Assume that \((G_1) - (G_4)\) hold and let \(u \in S(a)\) be arbitrary. Then, there exists a unique \(t_u > 0\) such that \(H(u, t_u) \in P_\infty, a\). Moreover,
\[
I(H(u, t_u)) = \max_{t > 0} I(H(u, t)).
\]

**Proof:** Fix \(u \in S(a)\) and consider the function \(f_u : \mathbb{R} \to \mathbb{R}\) defined by
\[
f_u(t) = I(H(u, t)).
\]
Therefore
\[
f_u'(t) = s_1 |\nabla s_1 H(u, t)|_2^2 + s_2 |\nabla s_2 H(u, t)|_2^2 - d \int_{\mathbb{R}^d} \tilde{G}(H(u, t)(x)) dx.
\]
In light of Lemmas 3.9 and 3.10, there is \(t_0 \in \mathbb{R}\) such that \(f_u'(t_0) = 0\). Thus, by (3.32) we deduce that \(H(u, t_0) \in P_\infty, a\). Now, by direct computation and using (3.32), we get
\[
f_u''(t_0) = 2s_1 |\nabla s_1 H(u, t_0)|_2^2 + 2s_2 |\nabla s_2 H(u, t_0)|_2^2 + d^2 \int_{\mathbb{R}^d} \tilde{G}(H(u, t_0)(x)) dx
\]
\[
- \frac{d^2}{2} \int_{\mathbb{R}^d} \tilde{G}'(H(u, t_0)(x)) H(u, t_0)(x) dx
\]
\[
\leq 2s_2 \left( s_1 |\nabla s_1 H(u, t_0)|_2^2 + s_2 |\nabla s_2 H(u, t_0)|_2^2 \right) + d^2 \int_{\mathbb{R}^d} \tilde{G}(H(u, t_0)(x)) dx
\]
\[
- \frac{d^2}{2} \int_{\mathbb{R}^d} \tilde{G}'(H(u, t_0)(x)) H(u, t_0)(x) dx
\]
\[
= (2s_2d + d^2) \int_{\mathbb{R}^d} \tilde{G}(H(u, t_0)(x)) dx - \frac{d^2}{2} \int_{\mathbb{R}^d} \tilde{G}'(H(u, t_0)(x)) H(u, t_0)(x) dx
\]
\[
\leq (2s_2d + d^2 - \frac{d^2}{2}) \int_{\mathbb{R}^d} \tilde{G}(H(u, t_0)(x)) dx < 0. \tag{3.33}
\]
Hence, from \((G_3)\), \(f_u''(t_0) < 0\) and this proves the unicity of \(t_0\). \(\blacksquare\)

**Lemma 3.14** Assume that \((G_1) - (G_3)\) hold. Then
\[
\gamma(a) = \inf_{u \in P_\infty, a} I(u).
\]

**Proof:** We argue by contradiction. We suppose that there is \(v \in P_\infty, a\) such that
\[
I(v) < \gamma(a). \tag{3.34}
\]
From Lemma 3.9, there exists \(t_0 > 0\) such that \(H(v, -t_0) \in A \) (A defined in lemma 3.10), \( |\nabla s_1 H(v, t_0)|_2^2 + |\nabla s_2 H(v, t_0)|_2^2 \geq 2K(a) \) and \(I(H(v, t_0)) < 0\). Now define the path \(h : [0, 1] \to S(a)\)
\[
h(t) = H(v, (2t - 1)t_0).
\]
Clearly \(h(0) = H(v, -t_0)\) and \(h(1) = H(v, t_0)\). Then, by Lemma 3.13, we know that
\[
\gamma(a) \leq \max_{t \in [0, 1]} I(h(t)) = I(v),
\]
which reach to a contradiction with (3.34).
This concludes the proof. \(\blacksquare\)

Finally, in order to deduce the proof of our main theorem, we shall need the following result.
Lemma 3.15 Assume that \((G_1)-(G_3)\) hold. Let \(k \in \mathbb{N}\) and \(a_1, a_2, \ldots, a_k\) be such that \(a^2 = a_1^2 + \cdots + a_k^2\). Then
\[
\gamma(a) < \gamma(a_1) + \cdots + \gamma(a_k).
\]

Proof: The proof is similar to Lemma 2.11 in [27].

Now, we want to prove that there exists a \((PS)_c\)-sequence for \(I\) restricted to \(S(a)\) at the level \(c = \gamma(a)\). To achieve this goal, we give the following two Lemmas inspired by Lemmas 2.4 and 2.5 from the work of L. Jeanjean [27]. This type of proof has now become classical and we won’t provide the details in the sequel.

Lemma 3.16 Assume that \((G_1)-(G_2)\) hold. Then there exists a sequence \(\{v_n\} \subset S(a)\) such that
1. \(I(v_n) \to \gamma(a)\).
2. \(\{\|v_n\|\}\) and \(\{\int_{\mathbb{R}^d} G(v_n(x))dx\}\) are bounded in \(\mathbb{R}\).
3. \(|(I'(v_n), z)_{(H^{1,2}(\mathbb{R}^d))^*}| \leq \frac{1}{\sqrt{\lambda}}\) for all \(z \in T_{v_n}\), where
\[
T_{v_n} = \left\{ z \in H^{1,2}(\mathbb{R}^d), \int_{\mathbb{R}^d} z v_n dx = 0 \right\}.
\]

Next, we give a characterization of the sequence \(\{v_n\}\) obtained in Lemma 3.16.

Lemma 3.17 Assume that \((G_1)-(G_2)\) hold and let \(\{v_n\} \subset H^{1,2}(\mathbb{R}^d)\) be the sequence obtained in Lemma 3.16. Then, up to a subsequence, we have:
1. \(v_n \rightharpoonup v_a\) in \(H^{1,2}(\mathbb{R}^d)\).
2. \(\int_{\mathbb{R}^d} G(v_n(x))dx \to c\) and \(\int_{\mathbb{R}^d} \tilde{G}(v_n)dx \to d\) with \(c, d > 0\).
3. \(\lambda_n = \frac{1}{\|v_n\|^2} \left(\|\nabla_s v_n\|_2^2 + \|\nabla_s v_n\|_2^2 - \int_{\mathbb{R}^d} g(v_n)v_n dx\right) \to \lambda_a < 0\) in \(\mathbb{R}\).
4. \((\Delta)^s v_n + (-\Delta)^s v_n - \lambda_n v_n - g(v_n) \to 0\) in \((H^{1,2}(\mathbb{R}^d))^*\).

Consequently, we prove that the sequence \(\{v_n\}\) obtained in Lemma 3.16 is a \((PS)\)-sequence of \(I_{-\lambda_a}\) (see (3.17) ) where \(\lambda_a\) is given in Lemma 3.17.

Lemma 3.18 Assume that \((G_1)-(G_3)\) hold. Then, the sequence \(\{v_n\}\) is a \((PS)\)-sequence of \(I_{-\lambda_a}\).

Proof: Using the fact that
\[
\lambda_n = \frac{1}{\|v_n\|^2} \left(\|\nabla_s v_n\|_2^2 + \|\nabla_s v_n\|_2^2 - \int_{\mathbb{R}^d} g(v_n)v_n dx\right) \to \lambda_a
\]
and the boundedness of \(\{v_n\}\) (see Lemma 3.17), we can deduce that
\[
I_{-\lambda_a}(v_n) - \int_{\mathbb{R}^d} \tilde{G}(v_n)dx \to 0.
\]
Thus, again by Lemma 3.17, \(I_{-\lambda_a} \to d > 0\). Moreover, since we have
\[
(\Delta)^s v_n + (-\Delta)^s v_n - \lambda_a v_n - g(v_n) \to 0\text{ in } (H^{1,2}(\mathbb{R}^d))^*,
\]
then \(\{v_n\}\) is a \((PS)\)-sequence of \(I_{-\lambda_a}\).

This completes the proof.

Proof of Theorem 3.2 concluded. By Lemma 3.18 and Theorem 3.3, there exist \(v^1, v^2, \ldots, v^k\) such that
\[
|v_n|^2 \to \sum_{i=1}^k |v^i|^2, \quad |\nabla_s v_n|^2 + |\nabla_s v_n|^2 \to \sum_{i=1}^k |\nabla_s v^i|^2 + |\nabla_s v^i|^2, \quad I_{-\lambda_a}(v_n) \to \sum_{i=1}^k I_{-\lambda_a}(v^i)
\]
which implies that
\[
a^2 = \sum_{i=1}^k |v^i|^2, \quad I(v_n) \to \sum_{i=1}^k I(v^i) \text{ and } \gamma(a) = \sum_{i=1}^k I(v^i).
\]
**Claim.** We shall prove that $k = 1$. Otherwise, we assume that $k \geq 2$. Then, from Theorem 3.3 and Lemma 3.14, we know that

$$
\gamma(a) = \inf_{u \in P_{\infty}} I(u),
$$

and so

$$
\gamma(a) = \sum_{i=1}^{k} I(v^i) \geq \sum_{i=1}^{k} \gamma(a_i).
$$

This reach to a contradiction with Lemma 3.15 and so the claim is proved.

Now the function $v^1$ satisfies

$$
I(v^1) = \gamma(a), \quad |v^1|^2 = a,
$$

and

$$
(-\Delta)^{s_1} v^1 + (-\Delta)^{s_2} v^1 - g(v^1) = \lambda_a v^1.
$$

This ends the proof of Theorem 3.2.

## 4 Proof of Theorem 1.1

Our aim in this section is the proof of Theorem 1.1. This section is divided into two subsections. In the first one, we give some technical lemmas which will be used later. In the second subsection, we conclude the proof of our main result.

### 4.1 Technical Lemmas

In this part, we give some technical lemmas related to problem (1.1) with general nonlinear term since we don’t need to use Theorem 3.3. First, we can remark that:

$$
P(u) = \Psi'(u)(t)|_{t=1}.
$$

Then, we have:

**Proposition 4.1** \( \Psi'(u)(t) = \frac{1}{t} P(t * u) \), for all \( u \in H^{s_1,s_2}(\mathbb{R}^d) \).

**Proof:** First, we claim that

$$
|\nabla_s(t * u)|_2^2 = t^{2s} |\nabla_s u|^2_2, \text{ for all } s \in (0,1):
$$

$$
|\nabla_s(t^x u(tx))|^2_2 = \int_{\mathbb{R}^d} \frac{t^{d} |u(tx) - u(ty)|^2}{|x - y|^{d+2s}} dxdy
$$

$$
= t^{2s} \int_{\mathbb{R}^d} \frac{|u(tx) - u(ty)|^2}{|tx - ty|^{d+2s}} d(tx)d(ty)
$$

$$
= t^{2s} |\nabla_s u|^2_2.
$$

Thus,

$$
\Psi_u(t) = \frac{t^{2s_1}}{2} |\nabla_{s_1} u|^2_2 + \frac{t^{2s_2}}{2} |\nabla_{s_2} u|^2_2 + \frac{1}{2} \int_{\mathbb{R}^d} V\left(\frac{x}{t}\right) u^2 dx - t^{-d} \int_{\mathbb{R}^d} G(t^x u(x))dx.
$$

Therefore,

$$
\Psi'_u(t) = s_1 t^{2s_1 - 1} |\nabla_{s_1} u|^2_2 + s_2 t^{2s_2 - 1} |\nabla_{s_2} u|^2_2 - \frac{1}{2t^d} \int_{\mathbb{R}^d} \langle \nabla V\left(\frac{x}{t}\right), x \rangle u^2 dx - \frac{d}{t^{d+1}} \int_{\mathbb{R}^d} \tilde{G}(t^x u(x))dx
$$

$$
= \frac{1}{t} P(t * u).
$$

This ends the proof.

**Lemma 4.1** If \( u \in H^{s_1,s_2}(\mathbb{R}^d) \) is a solution of equation (1.1), then

$$
P(u) = 0.
$$
**Proof:** Let \( u \) be a solution of (1.1). Thus, multiplying (1.1) by \( u \) and \( x \cdot \nabla u \) respectively, and then integrating by parts, we get

\[
|\nabla s_1 u_1^2| + |\nabla s_2 u_2^2| + \int_{\mathbb{R}^d} V(x)u^2dx + \lambda |u|^2 - \int_{\mathbb{R}^d} g(u)dx = 0 \tag{4.1}
\]

and

\[
\frac{2s_1 - d}{2}|\nabla s_1 u_1^2| + \frac{2s_2 - d}{2}|\nabla s_2 u_2^2| - \frac{d}{2} \int_{\mathbb{R}^d} V(x)u^2dx - \int_{\mathbb{R}^d} \frac{\langle \nabla V(x), x \rangle}{2}u^2dx - \frac{d}{2} \lambda |u|^2 + d \int_{\mathbb{R}^d} G(u)dx = 0. \tag{4.2}
\]

From (4.16) and (4.17), we obtain

\[
s_1|\nabla s_1 u_1^2| + s_2 |\nabla s_2 u_2^2| - \int_{\mathbb{R}^d} \frac{\langle \nabla V(x), x \rangle}{2}u^2dx - \frac{d}{2} \int_{\mathbb{R}^d} \frac{g(u)u}{2} - G(u)dx = 0,
\]

which is exactly \( P(u) = 0 \).

**Proposition 4.2** Let \( u \in S_a \). Then \( t \in \mathbb{R}^+ \) is a critical point for \( \Psi_a(t) \) if and only if \( t \ast u \in \mathcal{P}_a \).

**Proof:** The proof follows directly by using the fact that

\[
\Psi_u'(t) = \frac{1}{t}P[t \ast u], \quad \text{for all } t \in \mathbb{R}^+ \text{ and } u \in S_a.
\]

**Proposition 4.3** For any critical point of \( J|_{\mathcal{P}_a} \), if \( (\Psi_u)''(1) \neq 0 \), then \( \exists \lambda \in \mathbb{R} \) such that

\[
J'(u) + \lambda u = 0.
\]

**Proof:** Let \( u \in H^{s_1,s_2}(\mathbb{R}^d) \) be a critical point of \( J \) constraint on \( \mathcal{P}_a \), then there are \( \lambda, \mu \in \mathbb{R} \) such that

\[
J'(u) + \lambda u + \mu P'(u) = 0.
\]

It is sufficient to prove that \( \mu = 0 \). It is easy to see that \( u \) satisfies the following identity

\[
\frac{\partial}{\partial t} \Phi(t \ast u)|_{t=1} = 0,
\]

where

\[
\Phi(t \ast u) = J(t \ast u) + \frac{\lambda}{2} |t \ast u|^2 + \mu P(t \ast u)
\]

\[
= \Psi_u(t) + \frac{\lambda}{2} |u|^2 + \mu t \Psi_u'(t).
\]

In the last equality we used Proposition 4.1. Then

\[
\frac{\partial}{\partial t} \Phi(t \ast u)|_{t=1} = \Psi_u'(1) + \mu \Psi_u''(1) + \mu \Psi_u''(1)
\]

\[
= (1 + \mu) \Psi_u'(1) + \mu \Psi_u''(1)
\]

\[
= (1 + \mu) P(u) + \mu \Psi_u''(1)
\]

\[
= \mu \Psi_u''(1).
\]

It follows, since \( \Psi_u''(1) \neq 0 \), that \( \mu = 0 \).

**Lemma 4.2** Suppose that \((G_1)-(G_2)\) and \((V_2)\) are satisfied, then for any \( a > 0 \), \( \exists \delta_a > 0 \) such that

\[
\inf \{ t > 0 : \exists u \in S_a, |\nabla s_1 u_1^2| + |\nabla s_2 u_2^2| = 1, \text{ such that } t \ast u \in \mathcal{P}_a \} \geq \delta_a.
\]

Consequently,

\[
\inf_{u \in \mathcal{P}_a} (|\nabla s_1 u_1^2| + |\nabla s_2 u_2^2|) \geq \delta_a^2 > 0.
\]
Proof: We recall that

$$\Psi'_u(t) = s_1 t^{2s_1 - 1} |\nabla s_1 u|_2^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|_2^2 - \frac{1}{t} \int_{R^d} W\left(\frac{x}{t}\right) u^2 dx - \frac{d}{t^{d+1}} \int_{R^d} \tilde{G}(t\frac{\nabla u(x)}{t}) dx. \quad (4.3)$$

In light of condition (V$_2$), we infer that

$$s_1 t^{2s_1 - 1} |\nabla s_1 u|_2^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|_2^2 - \frac{1}{t} \int_{R^d} W\left(\frac{x}{t}\right) u^2 dx = s_1 t^{2s_1 - 1} |\nabla s_1 u|_2^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|_2^2 - \frac{1}{t} \int_{R^d} W(x)(t * u)^2 dx \quad (4.4)$$

$$\geq s_1 t^{2s_1 - 1} |\nabla s_1 u|_2^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|_2^2 - \frac{\alpha^2}{4} \left( |\nabla s_1 (t * u)|_2^2 + |\nabla s_2 (t * u)|_2^2 \right)$$

$$\geq s_1 t^{2s_1 - 1} |\nabla s_1 u|_2^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|_2^2 - \sigma_2 [t^{2s_1 - 1} |\nabla s_1 (u)|_2^2 + t^{2s_2 - 1} |\nabla s_2 (u)|_2^2]$$

$$\geq (s_1 - \sigma_2)[t^{2s_1 - 1} |\nabla s_1 (u)|_2^2 + t^{2s_2 - 1} |\nabla s_2 (u)|_2^2].$$

Let $u \in S_\alpha$ with $|\nabla s_1 u|_2^2 + |\nabla s_2 u|_2^2 = 1$ and $t > 0$ such that $t * u \in P_\alpha$. Then, invoking Proposition 4.2 and exploiting (4.3) and (4.4), we deduce that

$$(s_1 - \sigma_2) \min(t^{2s_1 - 1}, t^{2s_2 - 1}) \leq s_1 t^{2s_1 - 1} |\nabla s_1 u|_2^2 + s_2 t^{2s_2 - 1} |\nabla s_2 u|_2^2 - \frac{1}{t} \int_{R^d} W\left(\frac{x}{t}\right) u^2 dx$$

$$= \frac{d}{t^{d+1}} \int_{R^d} \tilde{G}(t\frac{\nabla u(x)}{t}) dx.$$ 

That is

$$(s_1 - \sigma_2) \leq \frac{d \int_{R^d} g(t^{\frac{\nabla u}{t}} u)dx}{2t^{\frac{d+1}{2}} + 1} - \frac{d \int_{R^d} G(t^{\frac{\nabla u}{t}} u)dx}{t^{d+1} \min(t^{2s_1 - 1}, t^{2s_2 - 1})} \quad (4.5)$$

Now, to conclude our proof, we distinguish two cases:

**Case 1: If $t > 1$.** Then, from (4.5), we get

$$(s_1 - \sigma_2) \leq \frac{d \int_{R^d} g(t^{\frac{\nabla u}{t}} u)dx}{2t^{\frac{d+2s_1}{2}} + 1} - \frac{d \int_{R^d} G(t^{\frac{\nabla u}{t}} u)dx}{t^{d+2s_1}} \quad (4.6)$$

This implies, using condition (G$_2$), that

$$(s_1 - \sigma_2) \leq \frac{d \int_{R^d} g(t^{\frac{\nabla u}{t}} u)dx}{2t^{\frac{d+2s_1}{2}} + 1} - \frac{d \int_{R^d} g(t^{\frac{\nabla u}{t}} u)dx}{\beta t^{\frac{d+2s_1}{2}} + 1} = dt^{-d-2s_1}\left(\frac{1}{2} - \frac{1}{\beta}\right) \int_{R^d} g(t^{\frac{\nabla u}{t}} u)\frac{\nabla u}{t} dx. \quad (4.7)$$

Again by (G$_2$) and the fact that $|\nabla s_1 u|_2^2 + |\nabla s_2 u|_2^2 = 1$ and $|u|_2^2 = \alpha$, there is $C > 0$ such that

$$\int_{R^d} g(t^{\frac{\nabla u}{t}} u)\frac{\nabla u}{t} dx \leq C[t^{\frac{d\alpha}{2}} + t^{4\alpha}]. \quad (4.8)$$

Consequently, by combining (4.7) and (4.8), we infer that

$$(s_1 - \sigma_2) \leq Cd\left(\frac{1}{2} - \frac{1}{\beta}\right)[t^{\frac{d\alpha}{2}} - d^{-2s_1} + t^{\frac{d\alpha}{2}} - d^{-2s_1}].$$

So by, $2 + \frac{d\alpha}{2} < \alpha < \beta$, we obtain the lower bound of $\delta_\alpha$.

**Case 2: If $t < 1$.** By the same argument used in case 1, we obtain

$$(s_1 - \sigma_2) \leq Cd\left(\frac{1}{2} - \frac{1}{\beta}\right)[t^{\frac{d\alpha}{2}} - d^{-2s_2} + t^{\frac{d\alpha}{2}} - d^{-2s_2}].$$

So by, $2 + \frac{d\alpha}{2} < \alpha < \beta$, we obtain the lower bound of $\delta_\alpha$.

This completes the proof.  

**Lemma 4.3** Assume that (G$_1$) – (G$_3$) and (V$_3$) hold. Then for any $u \in P_\alpha$, we have $(\Psi_u)'(1) < 0$. And it is a natural constraint of $J|_{S_\alpha}$. 

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\textbf{Proof:} For any $u \in \mathcal{P}_a$, we have
\[ s_1|\nabla_s u|^2 + s_2|\nabla_d u|^2 - \int_{\mathbb{R}^d} W(x)u^2dx = d \int_{\mathbb{R}^d} \tilde{G}(u)dx. \] (4.9)

On the other hand, a direct computation shows that
\[ \Psi'_u((2s_1 - 1)s_1|\nabla_s u|^2 + (2s_2 - 1)s_2|\nabla_d u|^2 + \int_{\mathbb{R}^d} W(x)u^2dx + \int_{\mathbb{R}^d} (\nabla W(x), x)u^2dx \] (4.10)

It follows, using (G3), (V3) and (4.10), that
\[ \Psi'_u((2s_1 - 1)s_1|\nabla_s u|^2 + (2s_2 - 1)s_2|\nabla_d u|^2 + \int_{\mathbb{R}^d} W(x)u^2dx + \int_{\mathbb{R}^d} (\nabla W(x), x)u^2dx \] (4.11)

Thus, $\mathcal{P}^+_a = \mathcal{P}^-_a = \emptyset$. Furthermore, by Proposition 4.2, one can see that it is a natural constraint of $J|_{S_a}$. This ends the proof. \[ \square \]

\textbf{Corollary 4.1} Suppose that (G1) – (G3) and (V1) – (V3) hold. Then:
(i) For any $u \in H^{s_1, s_2}(\mathbb{R}^d) \setminus \{0\}$, there exists an unique $t_u > 0$ such that $t_u * u \in \mathcal{P}$. Moreover,
\[ \Psi_u(t_u) = J(t_u * u) = \max_{t > 0} J(t * u). \]
(ii) We have
\[ C_a := \inf_{u \in \mathcal{P}_a} J(u) = \inf_{u \in S_a, t > 0} \max J(t * u) > 0. \]

\textbf{Proof:} (i) Let $a := |u|^2$, then $|\nabla_s u|^2, |\nabla_d u|^2 > 0$. From (4.4) and assumptions (G1) – (G2), for $t$ small enough, we find that
\[ \Psi'_u(t) = s_1t^{2s_1-1}|\nabla_s u|^2 + s_2t^{2s_2-1}|\nabla_d u|^2 - \frac{1}{t} \int_{\mathbb{R}^d} W(x)u^2dx \]
\[ - \frac{d}{t^{d+1}} \int_{\mathbb{R}^d} \tilde{G}(t^{-\frac{d}{2}}u(x))dx \]
\[ \geq (s_1 - \sigma_2)|t^{2s_1-1}|\nabla_s u|^2 + t^{2s_2-1}|\nabla_d u|^2 \]
\[ - \frac{\beta}{2} - 1 \frac{d}{t^{d+1}} \int_{\mathbb{R}^d} G(t^{-\frac{d}{2}}u(x))dx \]
\[ \geq (s_1 - \sigma_2)|t^{2s_1-1}|\nabla_s u|^2 + t^{2s_2-1}|\nabla_s u|^2 \]
\[ - \frac{\beta}{2} - 1 \frac{d}{t^{-\frac{d}{2}}+1} \int_{\mathbb{R}^d} G(u(x))dx > 0. \]
Here, we used the fact that $\alpha > 2 + \frac{4s_2}{d}$. Hence, there exists some $t_0 > 0$ such that $\Psi_u$ increases in $(0,t_0)$. On the other hand, for $t$ large enough, we have

\[
\Psi_u(t) = \frac{t^{2s_1}}{2} |\nabla_u u|^2 + \frac{t^{2s_2}}{2} |\nabla_{s_2} u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} V(\frac{x}{t})u^2 dx - t^{-d} \int_{\mathbb{R}^d} G(t^d u(x)) dx
\]

\[
\leq \frac{t^{2s_1}}{2} |\nabla_u u|^2 + \frac{t^{2s_2}}{2} |\nabla_{s_2} u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} V(\frac{x}{t})u^2 dx - t^{-d} \int_{\mathbb{R}^d} G(u(x)) dx,
\]

which implies that $\lim_{t \to +\infty} \Psi_u(t) = -\infty$. So there exists some $t_1 > t_0$ such that

\[
\Psi_u(t_1) = \max_{t > 0} \Psi_u(t).
\]

Hence, $\Psi'_u(t_1) = 0$. It follows, from Proposition 4.2, that $t_1 \star u \in \mathcal{P}$.

In what follows we show that $t_1$ is the unique maximum of $\Psi_u$. Otherwise, suppose that there exists another $t_2 > 0$ such that $t_2 \star u \in \mathcal{P}$. Then, by Lemma 4.3, we have that $t_1$ and $t_2$ are two strict local maxima of $\Psi_u$. Without loss of generality, we assume that $t_1 < t_2$. Then there exists some $t_3 \in (t_1,t_2)$ such that

\[
\Psi_u(t_3) = \min_{t \in [t_1,t_2]} \Psi_u(t).
\]

So, $\Psi'_u(t_3) = 0$ and $\Psi'_u(t_3) \geq 0$. This reaches to a contradiction with Lemma 4.3.

(ii) From assertion (i), it follows directly that

\[
C_a := \inf_{u \in \mathcal{P}_x} J(u) = \inf_{u \in S_a} \max_{t > 0} J(t \star u) > 0.
\]

It remains to show that $C_a > 0$. Let $u \in \mathcal{T}_a$. In light of assumptions $(G_1)-(G_3)$ and $(V_2)$, we have

\[
d^{\alpha - 2} \int_{\mathbb{R}^d} G(u) dx \leq d \int_{\mathbb{R}^d} \bar{G}(u) dx = s_1 |\nabla_{s_1} u|^2 + s_2 |\nabla_{s_2} u|^2 - \int_{\mathbb{R}^d} W(u) u^2 dx \tag{4.13}
\]

\[
\leq (s_1 + s_2)|\nabla_{s_1} u|^2 + (s_2 + \sigma_2)|\nabla_{s_2} u|^2.
\]

Then, using (4.13), we infer that

\[
J(u) = \frac{1}{2} |\nabla_{s_1} u|^2 + \frac{1}{2} |\nabla_{s_2} u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} V(x)u^2 dx - \int_{\mathbb{R}^d} G(u) dx \tag{4.14}
\]

\[
\geq \frac{1 - \sigma_1}{2} (|\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2) - \int_{\mathbb{R}^d} G(u) dx
\]

\[
\geq \frac{1 - \sigma_1}{2} - \frac{2(s_1 + s_2)}{d(\alpha - 2)} |\nabla_{s_1} u|^2 + \frac{1 - \sigma_1}{2} - \frac{2(s_2 + \sigma_2)}{d(\alpha - 2)} |\nabla_{s_2} u|^2
\]

\[
\geq \frac{1 - \sigma_1}{2} - \frac{2(s_1 + s_2)}{d(\alpha - 2)} (|\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2).
\]

On the other hand, from conditions $(V_1)$ and $(V_2)$, we can deduce that

\[
\frac{1 - \sigma_1}{2} - \frac{2(s_1 + s_2)}{d(\alpha - 2)} > 0 \quad \text{and} \quad \frac{1 - \sigma_1}{2} - \frac{2(s_2 + \sigma_2)}{d(\alpha - 2)} > 0. \tag{4.15}
\]

It follows, from (4.14), (4.15) and Lemma 4.2, that

\[
C_a \geq \frac{1 - \sigma_1}{2} - \frac{2(s_1 + s_2)}{d(\alpha - 2)} > 0.
\]

This completes the proof.

\[\blacksquare\]

**Corollary 4.2** Suppose that the assumptions of Corollary 4.1 are fulfilled. Then, $J|_{\mathcal{P}_x}$ is coercive, i.e.,

\[
\lim_{u \in \mathcal{P}_x, \ |\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2 \to \infty} J(u) = +\infty.
\]
Proof: By the same argument used in the proof of Corollary 4.1, we have

\[ J(u) \geq \left( \frac{1 - \sigma_1}{2} - \frac{2(s_1 + \sigma_2)}{d(\alpha - 2)} \right) (|\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2) \rightarrow +\infty, \]

as \( u \in P_a \) and \( |\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2 \rightarrow \infty. \)

This ends the proof. ■

4.2 Proof of Theorem 1.1 concluded

Now, we are ready to conclude the proof of Theorem 1.1. We start by proving point (1), that is, the nonexistence of normalized solution.

Proposition 4.4 Suppose that assumptions \((G_1) - (G_3)\) and \((V_1) - (V_3)\) are satisfied. Then, problem (1.1) has no nontrivial solution \( u \in H^{s_1,s_2}(\mathbb{R}^d) \) for \( \lambda \leq 0. \)

Proof: Let \( u \) be a solution of (1.1). Thus, multiplying (1.1) by \( u \) and \( x \cdot \nabla u \) respectively, and then integrating by parts, we get

\[ |\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2 + \int_{\mathbb{R}^d} V(x)|u|^2 dx + \lambda|u|^2 - \int_{\mathbb{R}^d} g(u)u dx = 0 \quad (4.16) \]

and

\[ \frac{2s_1 - d}{2}|\nabla_{s_1} u|^2 + \frac{2s_2 - d}{2}|\nabla_{s_2} u|^2 = \frac{d}{2} \int_{\mathbb{R}^d} V(x)|u|^2 dx + \int_{\mathbb{R}^d} W(x)|u|^2 dx - \frac{d\lambda}{2}|u|^2 + \frac{d}{2} \int_{\mathbb{R}^d} G(u) dx. \quad (4.17) \]

From (4.17), we obtain

\[ \frac{2s_1 - d}{2}|\nabla_{s_1} u|^2 + \frac{2s_2 - d}{2}|\nabla_{s_2} u|^2 = \frac{d}{2} \int_{\mathbb{R}^d} V(x)|u|^2 dx + \int_{\mathbb{R}^d} W(x)|u|^2 dx - \frac{d\lambda}{2}|u|^2 + \frac{d}{2} \int_{\mathbb{R}^d} G(u) dx \leq \sigma_2|\nabla_{s_1} u|^2 + \sigma_2|\nabla_{s_2} u|^2 + \frac{d}{2} \int_{\mathbb{R}^d} V(x)|u|^2 dx + \frac{d\lambda}{2}|u|^2 - d \int_{\mathbb{R}^d} G(u) dx. \]

Thus,

\[ \frac{d\lambda}{2}|u|^2 \geq d \int_{\mathbb{R}^d} G(u) dx - \left( \sigma_2 + \frac{d - 2s_1}{2} \right)|\nabla_{s_1} u|^2 - \left( \sigma_2 + \frac{d - 2s_1}{2} \right)|\nabla_{s_2} u|^2 + \frac{d}{2} \int_{\mathbb{R}^d} V(x)|u|^2 dx. \]

By (4.16), we obtain

\[ \frac{d\lambda}{2}|u|^2 \geq d \int_{\mathbb{R}^d} G(u) dx + (s_2 - s_1)|\nabla_{s_1} u|^2 - (s_1 - \sigma_2) \int_{\mathbb{R}^d} V(x)|u|^2 dx \]

\[ + \left( \sigma_2 + \frac{d - 2s_1}{2} \right)|u|^2 - \left( \sigma_2 + \frac{d - 2s_1}{2} \right) \int_{\mathbb{R}^d} g(u) u dx, \]

Then by \( \int_{\mathbb{R}^d} V(x)|u|^2 dx \leq 0, \) we have

\[ (s_1 - \sigma_2)|u|^2 \geq d \int_{\mathbb{R}^d} G(u) dx - \left( \sigma_2 + \frac{d - 2s_1}{2} \right) \int_{\mathbb{R}^d} g(u) u dx \geq \tau \int_{\mathbb{R}^d} G(u) dx, \]

where \( \tau = d - (\sigma_2 + \frac{d - 2s_1}{2}) \beta. \) By \( \sigma_2 < s_1 - \frac{(\beta - 2d)}{2d}, \) we have \( \tau > 0. \) Therefore, if \( \lambda \leq 0, \) we have \( \int_{\mathbb{R}^d} G(u) dx = 0, \) then \( u \) is trivial. ■

Remark 4.1 Although there are only trivial solution \( u \in H^{s_1,s_2} \) of (1.1) provided \( \lambda \leq 0. \) There may be some nontrivial solutions in \( W^{s_1,p} \cap W^{s_2,q}, \) for some \( p, q \in (2, \frac{2d}{d-2s_1}). \) For example, \([20]\) showed the existence result for nonlinear Helmholtz equation by dual variational methods.

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Let \( \{a \} \subset P \) be a minimal sequence of \( C \), i.e. \( J(a) = C_a > 0 \).

Claim 1. We have
\[
C_a < m_a.
\]
Indeed, invoking Theorem 4.1, we can assume that \( m_a \) is attained by \( v_a \in S_a \). Therefore, in view of \((V_1)\), one can see that there exists \( C > 0 \) and a domain \( \Omega \subset \mathbb{R}^d \), such that,
\[
- \int_{\Omega} V \left( \frac{x}{t_v} \right) |v_a|^2(x) \, dx \geq C|\Omega| > 0,
\]
where \( t_v \) is given in Corollary 4.1. Thus, again by Corollary 4.1, we infer that
\[
C_a \leq \max_{t > 0} J(t \ast v_a) = J(t_v \ast v_a) = I(t_v \ast v_a) + \frac{1}{2} \int_{\mathbb{R}^d} V \left( \frac{x}{t_v} \right) |v_a|^2(x) \, dx
\]
\[
< I(t_v \ast v_a) + \frac{1}{2} \int_{\Omega} V(x)|v_a|^2(x) \, dx < I(t_v \ast v_a) - C|\Omega|
\]
\[
\leq \max_{t > 0} I(t \ast v_a) = I(v_a) = m_a,
\]
which implies that \( C_a < m_a \).

From Corollary 4.2, we know that \( \{u_n\} \) is bounded in \( H^{s_1,s_2}(\mathbb{R}^d) \), and so, \( u_n \rightharpoonup u \) in \( H^{s_1,s_2}(\mathbb{R}^d) \).

Claim 2. The weak limit \( u \) is nontrivial, that is, \( u \neq 0 \).

If we assume that \( u = 0 \), we would have by Brézis-Lieb lemma and assumption \((V_1)\), that
\[
I(u_n) = C_a + o(1), \quad \Psi'_{\infty,u_n}(1) = o(1).
\]
Then by the uniqueness, there exists \( t_n = 1 + o(1) \), such that
\[
t_n \ast u_n \in P_{\infty,a}.
\]
Thus, we have
\[
m_a \leq I(t_n \ast u_n) = I(u_n) + o(1) = C_a + o(1),
\]
which reach to a contradiction with claim 1. Hence, \( u \neq 0 \).

From the fact that \( \{u_n\} \) is bounded in \( H^{s_1,s_2}(\mathbb{R}^d) \), and \( I(u_n) \) is bounded in \( \mathbb{R} \), we can deduce that \( \lambda_n := -\frac{\langle J(u_n) \rangle}{\langle u_n \rangle} \) is also bounded in \( \mathbb{R} \). Then, we can assume that for some subsequence \( \lambda_n \rightarrow \lambda_a \in \mathbb{R} \).

Claim 3. We must have \( \lambda_a > 0 \).

Since \( \{u_n\} \subset P_a \), we have \( P(u_n) = 0 \), and so,
\[
s_1|\nabla_{s_1} u_n|^2 + s_2|\nabla_{s_2} u_n|^2 - \int_{\mathbb{R}^d} \tilde{W}(x)|u_n|^2 \, dx - d \int_{\mathbb{R}^d} \left( \frac{1}{2} g(u_n) u_n - G(u_n) \right) \, dx = 0 \tag{4.18}
\]
On the other hand, in light of \((V_2)\), one has
\[
\frac{\beta - 2}{2} d \int_{\mathbb{R}^d} G(u_n) \, dx \geq d \int_{\mathbb{R}^d} \left( \frac{1}{2} g(u_n) u_n - G(u_n) \right) \, dx = d \int_{\mathbb{R}^d} \tilde{G}(u_n) \, dx \tag{4.19}
\]
\[
= s_1|\nabla_{s_1} u_n|^2 + s_2|\nabla_{s_2} u_n|^2 - \int_{\mathbb{R}^d} W(x)|u_n|^2 \, dx \geq (s_1 - s_2)|\nabla_{s_1} u_n|^2 + (s_2 - s_2)|\nabla_{s_2} u_n|^2.
\]
Consequently, by combining (4.18) and (4.19), we obtain

$$\lambda_n|u_n|^2 - (J'(u_n), u_n) = \langle \nabla_{s_1} u_n^2 - \nabla_{s_2} u_n^2 \rangle - \int_{\mathbb{R}^d} V(x)u_n^2 dx + \int_{\mathbb{R}^d} g(u_n)u_n dx \tag{4.20}$$

$$\geq -|\nabla_{s_1} u_n^2| - \int_{\mathbb{R}^d} g(u_n)u_n dx$$

$$= \frac{s_2 - s_1}{s_1}\langle \nabla_{s_2} u_n^2 \rangle - \frac{1}{s_1} \int_{\mathbb{R}^d} W(x)|u_n|^2 dx - \frac{d}{s_1} \int_{\mathbb{R}^d} \tilde{G}(u_n) dx + \int_{\mathbb{R}^d} g(u_n)u_n dx$$

$$\geq \frac{s_2 - s_1}{s_1}\langle \nabla_{s_2} u_n^2 \rangle + \left( \frac{d}{s_1} - \frac{d - 2s_1}{2s_1} \right) \int_{\mathbb{R}^d} G(u_n) dx - \frac{s_2}{s_1} (|\nabla_{s_1} u_n|^2 + |\nabla_{s_2} u_n|^2)$$

$$\geq \left( \frac{2\beta(s_1 - s_2)}{(\beta - 2)d} - 1 \right)|\nabla_{s_1} u_n|^2 + \left( \frac{2\beta(s_2 - s_2)}{(\beta - 2)d} - 1 \right)|\nabla_{s_2} u_n|^2.$$

Recall that from assumption (V₂), we have

$$\sigma_2 < s_1 - \frac{(\beta - 2)d}{2\beta} < s_2 - \frac{(\beta - 2)d}{2\beta}$$

It follows, from this fact and (4.20), that

$$\lambda_n|u_n|^2 \geq C(|\nabla_{s_1} u_n|^2 + |\nabla_{s_2} u_n|^2),$$

for some $C > 0$. Thus, in light of Lemma 4.2 and claim 2, there exists $\delta > 0$ such that $\lambda_n \alpha > \delta$ for all $n \in \mathbb{N}$. This implies that, up to a subsequence, we may assume $\lambda_n \to \lambda > 0$. This proves claim 3.

Consequently, $u$ solves the following equation

$$(-\Delta)^s u(x) + (\Delta)^{s_2} u(x) + \lambda_n u(x) + V(x)u(x) = g(u(x)).$$

**Claim 4.** We have $J(u) > 0$.

By (V₂), (G₂), (G₃) and Lemma 4.1, one has

$$(s_1 + \sigma_2)|\nabla_{s_1} u|^2 + (s_2 + \sigma_2)|\nabla_{s_2} u|^2 \geq s_1|\nabla_{s_1} u|^2 + s_2|\nabla_{s_2} u|^2 - \int_{\mathbb{R}^d} W(x)|u|^2 dx \tag{4.21}$$

$$= d \int_{\mathbb{R}^d} \tilde{G}(u) dx$$

$$\geq \frac{d(\alpha - 2)}{2} \int_{\mathbb{R}^d} G(u) dx.$$

Again, from (V₁) and (V₂), we can remark that

$$\frac{1 - \sigma_1}{2} > \frac{2(s_2 + \sigma_2)}{d(\alpha - 2)} > \frac{2(s_1 + \sigma_2)}{d(\alpha - 2)}.$$ \tag{4.22}

Then, in view of (4.21), (4.22) and (V₁), we obtain

$$J(u) = \frac{1}{2} |\nabla_{s_1} u|^2 + \frac{1}{2} |\nabla_{s_2} u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} V(x)u^2 dx - \int_{\mathbb{R}^d} G(u) dx$$

$$\geq \frac{1 - \sigma_1}{2} (|\nabla_{s_1} u|^2 + |\nabla_{s_2} u|^2) - \int_{\mathbb{R}^d} G(u) dx$$

$$\geq \left( \frac{1 - \sigma_1}{2} - \frac{2(s_1 + \sigma_2)}{d(\alpha - 2)} \right) |\nabla_{s_1} u|^2 + \left( \frac{1 - \sigma_1}{2} - \frac{2(s_2 + \sigma_2)}{d(\alpha - 2)} \right) |\nabla_{s_2} u|^2.$$

This proves, from claim 2 and Lemma 4.2 that $J(u) > 0$.

**Claim 5.** We show that $|u|^2 = \alpha$.

Assume that $b := |u|^2 \neq \alpha$. Put $c := u - b$, due to the norm is weakly lower semi-continuous, we have $c \in (0, \alpha)$. Set $\phi_n := u_n - u$. Then, by (V₁), we get

$$\int_{\mathbb{R}^d} V(x)|\phi_n|^2 dx \to 0, \text{ as } n \to \infty.$$
Moreover, by using the Brézis-Lieb lemma, we have $|φ_n|^2 = c + o(1)$, $I(φ_n) + J(u) = C_α + o(1)$, and $Ψ_{∞, φ_n}(1) = o(1)$. If $\lim \inf (|∇_s φ_n|^2 + |∇_s^2 φ_n|^2) = 0$, then by
\[
s_1 |∇_s φ_n|^2 + s_2 |∇_s^2 φ_n|^2 = d \int_\mathbb{R}^d \tilde{G}(φ_n)dx + o(1),
\]
we have $\lim \inf \int_\mathbb{R}^d \tilde{G}(φ_n)dx = 0$. Similar to (4.20), by using the Brézis-Lieb lemma, we have
\[
λ_n |u_n|^2 - λ_n |u|^2 = -|∇_s φ_n|^2 - |∇_s^2 φ_n|^2 + \int_\mathbb{R}^d g(φ_n)φ_n dx + o(1).
\]
Thus, we have $λ_n c = 0$, which is impossible. Therefore, we have
\[
\lim \inf (|∇_s φ_n|^2 + |∇_s^2 φ_n|^2) > 0.
\]
By the uniqueness, there exists $t_n = 1 + o(1)$ such that $t_n * φ_n ∈ \mathcal{P}_{∞, |φ_n|^2}$. Then, by Lemmas 3.2 and 3.3, we have
\[
m_{|φ_n|^2} ≤ I(t_n * φ_n) = I(φ_n) + o(1) = C_α - J(u) + o(1).
\]
Therefore,
\[
m_α ≤ m_c ≤ \lim \inf_{n \to ∞} m_{|φ_n|^2} ≤ C_α - J(u) ≤ m_α - J(u).
\]
Hence, using claim 4, we reach to a contradiction.

Claim 5 leads to the fact that $u_n → u$ in $L^2(\mathbb{R}^d)$, and $u ∈ \mathcal{P}_α$. By the fractional Gagliardo-Nirenberg inequality, see [25], we have
\[
|u_n - u|_p ≤ B_1^{\frac{d}{2}}|∇_s u_n - ∇_s u|_{\frac{d}{2}} ≤ |u_n - u|_2^{\frac{d}{2}} - λ_n |u|^2,
\]
where $A = \frac{(p - 2d)}{2s_1}$, $1 ≤ p < \frac{2d}{2s_1}$, and $B$ is the best constant give by
\[
B = 2^{2s_1} \pi^{-s_1} \Gamma((d - 2s_1)/2) \Gamma((d + 2s_1)/2) / \Gamma(d/2)^{2s_1/d}.
\]
Since $2 + \frac{4s_1}{d} < α < β < \frac{2d}{2s_1}$, we have $\int_\mathbb{R}^d G(u_n)dx → \int_\mathbb{R}^d G(u)dx$. Thus,
\[
C_α ≤ J(u) ≤ \lim \inf_{n \to ∞} J(u_n) = C_α,
\]
which implies $J(u) = C_α$. Moreover, $|∇_s u_n|^2 → |∇_s u|^2$, $|∇_s^2 u_n|^2 → |∇_s^2 u|^2$, as $n → ∞$. This shows that $u_n → u$ in $H^{s_1, s_2}(\mathbb{R}^d)$, and $u ∈ \mathcal{P}_α$ attains $C_α$.

Then by Proposition 4.3, there exists $λ ∈ \mathbb{R}^+$ such that $(λ, u)$ solves
\[
(-Δ)^{s_1} u(x) + (-Δ)^{s_2} u(x) + λu(x) + V(x)u(x) = g(u(x)),
\]
with $\int_{\mathbb{R}^d} |u(x)|^2 dx = a$.

This ends the proof.

5 Proof of Theorem 1.2

We show the proof of Theorem 1.2 in this section. First, we introduce some notations here. Recall that
\[
J(u) = \frac{1}{2} |∇_s u|^2 + \frac{1}{2} |∇_s^2 u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} V(x)u^2 dx - \int_{\mathbb{R}^d} G(u)dx.
\]
This functional is defined on the following Hilbert space
\[
H^{s_1, s_2}_V(\mathbb{R}^d) := \{ u ∈ H^{s_1, s_2}(\mathbb{R}^d) : \int_{\mathbb{R}^d} V(x)|u|^2 dx < ∞ \},
\]
with the norm
\[
\|u\|^2 := \int_{\mathbb{R}^d} (|∇_s u|^2 + |∇_s^2 u|^2 + V(x)|u|^2) dx.
\]
which is induced by the following inner product
\[
\langle u, v \rangle_V := \frac{1}{2} \sum_{i=1}^{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))v(x) - u(y)v(x)}{|x - y|^{2s}} \, dxdy + \int_{\mathbb{R}^d} V(x)u(x)v(x) \, dx,
\]
Thus we have
\[
J(u) = \frac{1}{2} \|u\|_V - \int_{\mathbb{R}^d} G(u) \, dx, \quad J'(u)v = \langle u, v \rangle_V - \int_{\mathbb{R}^d} g(u) \, dx.
\]

**Lemma 5.1** Assume \((V')_i, (V'_j)\) are satisfied, then \(H^{s_i, s_j}_F (\mathbb{R}^d)\) is continuously embedded in \(L^p(\mathbb{R}^d)\), for any \(p \in [2, 2^*_s]\). Moreover, \(H^{s_i, s_j}_F (\mathbb{R}^d)\) is compactly embedded in \(L^p(\mathbb{R}^d)\), for any \(p \in [2, 2^*_s]\).

Equivalently, we consider the following constrained minimizing problem:
\[
E_a = \inf \{ J(u) : u \in S_a \}.
\]
Let \(\{u_n\} \subset S_a\) be a minimizing sequence of \(J\) with respect to \(E_a\), i.e. \(J(u_n) \to E_a\), and \(|u_n|^2 = a\).

**Claim 1.** \(\{u_n\}\) is bounded in \(H^{s_i, s_j}_F (\mathbb{R}^d)\).

Thus, going to a subsequence if necessary, \(u_n \to u\) in \(L^p(\mathbb{R}^d), 2 \leq p < 2^*_s\). By Lemma 5.1, we can extract a subsequence such that \(u_n \to u\) in \(L^p(\mathbb{R}^d), 2 \leq p < 2^*_s\).

**Claim 2.** \(\lim_{n \to +\infty} \int_{\mathbb{R}^d} G(u_n) \, dx = \int_{\mathbb{R}^d} G(u) \, dx, \) and \(\lim_{n \to +\infty} \int_{\mathbb{R}^d} g(u_n) u_n \, dx = \int_{\mathbb{R}^d} g(u) u \, dx\).

Therefore, we have
\[
J(u) \leq \liminf_{n \to +\infty} J(u_n) = E_a, \quad |u|^2 = \lim_{n \to +\infty} |u_n|^2 = a.
\]
Thus, \(u \in S_a\), and \(J(u) = E_a\), which means \(E_a\) is attained. Let
\[
\lambda_n = \frac{1}{a} \langle J'(u_n), u_n \rangle = \frac{1}{a} \left( \int_{\mathbb{R}^d} g(u_n) u_n \, dx - \|u_n\|_V \right).
\]

Similar to the proof of Theorem 1.1, we have \(\{\lambda_n\}\) is bounded and for some subsequence \(\lambda_n \to \lambda_a > 0\).
Take \(\lambda = \lambda_a\), we have \((\lambda, u)\) solves
\[
(-\Delta)^{s_i} u(x) + (-\Delta)^{s_j} u(x) + \lambda u(x) + V(x)u(x) = g(u(x)),
\]
with \(\int_{\mathbb{R}^d} |u(x)|^2 \, dx = a\).
This ends the proof.

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