ON THE POSET OF MULTICHAINS

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Abstract. In this article we introduce the poset of \(m\)-multichains of a given bounded poset \(P\). Its elements are the multichains of \(P\) consisting of \(m\) elements, and its partial order is the componentwise partial order of \(P\). We show that this construction preserves quite a few poset-theoretic and topological properties of \(P\) and we give an explicit description of the posets of \(m\)-multichains of a chain and a Boolean lattice, respectively. We conclude this article by giving a formula that computes the number of \(m\)-multichains in the face lattice of the \(n\)-dimensional hypercube.

1. Introduction

The incidence algebra of a locally finite poset \(P = (P, \leq)\) consists of all real-valued functions of two variables, say \(f(x, y)\), where \(x\) and \(y\) range over the elements of \(P\), and where \(f(x, y) = 0\) if \(x \not\leq y\). Among the elements of this algebra are two mutually inverse functions, the \textit{zeta function} and the \textit{Möbius function}, which are given by

\[
\zeta(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\mu(x, y) = \begin{cases} 
1, & \text{if } x = y \\
- \sum_{x \leq z < y} \mu(x, z) & \text{if } x < y, \\
0 & \text{otherwise},
\end{cases}
\]

respectively. Both of these functions are involved in the famous Möbius Inversion Formula [15, Proposition 2], which in a sense generalizes the Inclusion-Exclusion Principle. Moreover, the Möbius function provides a deep link between combinatorics and algebraic topology in that the value of the Möbius function given by the least and the greatest element of \(P\) coincides with the reduced Euler characteristic of the order complex of the proper part of \(P\), i.e. the simplicial complex whose facets are the chains of \(P\) not involving the least and greatest element [16, Proposition 3.8.6]. Much work has been done on the computation of the Möbius function for special posets, and many tools have been developed to aid this process. We refer the reader to [16, Section 3] and [7, 17] for more background.

In a sense, the zeta function can be seen as a starting point for the work presented in this paper. Suppose that \(P\) is a poset with a least element \(\hat{0}\) and a greatest element \(\hat{1}\). Then \(\zeta^m(\hat{0}, \hat{1})\) determines precisely the number of multichains in \(P\) of length \(m - 1\). This gives rise to the definition of the \textit{zeta polynomial} of \(P\) by
setting $Z(\mathcal{P}, m) = \zeta_m(\hat{0}, \hat{1})$. Observe that this polynomial can be evaluated on all integers, since the zeta function is invertible. See [16] for more background and details. Some more enumerative results on the zeta polynomial of certain posets can be found in [10].

We will, however, touch the enumerative aspect of the zeta polynomial only briefly, and will focus more on structural aspects of the set of multichains of length $m$ instead. In particular, we view these multichains inside the $m$-fold direct product of $\mathcal{P}$, which equips them naturally with a partial order, and we will denote this poset of $m$-multichains by $\mathcal{P}^m$. A prominent example of a poset that is defined on the set of multichains of some other poset is the generalized noncrossing partition poset associated with a well-generated complex reflection group $W$ defined by D. Armstrong [1,4], and usually denoted by $\text{NC}_W^m$. The elements of this poset are the multichains in the noncrossing partition lattice associated with $W$, denoted by $\text{NC}_W$, however, the corresponding partial order is not the componentwise one. We remark that the poset of multichains $\text{NC}_W^m$ lives naturally inside the so-called absolute order on the elements of the dual braid monoid of $W$ [2,3]. Other than for motivational reasons, we will not consider the poset of noncrossing partitions in this article. Another subposet of the $m$-fold direct product of $\mathcal{P}$, which is also a subposet of $\mathcal{P}^m$, is the $m$-cover poset of $\mathcal{P}$, and it was studied extensively in [13].

In Section 2 we describe a few structural properties that are preserved under the transition from the poset $\mathcal{P}$ to the poset $\mathcal{P}^m$ of $m$-multichains of $\mathcal{P}$. Our first main result, Theorem 2.4, implies that any lattice variety is closed under forming the poset of $m$-multichains. In Section 3 we prove our second main result, Theorem 3.1, which states that the existence of a well-behaved edge-labeling of $\mathcal{P}$ is preserved as well under this construction. The existence of such an edge-labeling has important topological consequences, for instance it implies that the poset in question is shellable, and hence homotopy equivalent to a wedge of spheres and Cohen-Macaulay. In Section 4, we explicitly describe the poset $\mathcal{P}^m$ in the case where $\mathcal{P}$ is a chain and where $\mathcal{P}$ is a Boolean lattice. In general it is hard to describe the poset of $m$-multichains of an arbitrary poset explicitly in terms of other posets. We did not succeed in describing the structure of the poset of $m$-multichains of the face lattice of the $n$-dimensional hypercube, but we managed to compute its zeta polynomial using the poset perspective. It seems that this result has not appeared in the literature before.

## 2. Structure of Multichain Posets

Let us recall some basic poset and lattice-theoretic terminology. For more background, we refer the reader to [11,12]. Throughout this article we use the abbreviation $[n] = \{1, 2, \ldots, n\}$.

A partially ordered set (poset for short) $\mathcal{P} = (P, \leq)$ is bounded if it has a unique smallest element, denoted by $\hat{0}$, as well as a unique largest element, denoted by $\hat{1}$. A closed interval of $\mathcal{P}$ is a set $[p, q] = \{x \in P \mid p \leq x \leq q\}$. A chain of $\mathcal{P}$ is a subset $C \subseteq P$ that admits a linear order of its elements, i.e. we can write $C = \{p_0 < p_1 < \cdots < p_s\}$ in a unique way. The length of a chain is its cardinality.
Definition 2.1. Let $P = (P, \leq)$ be a bounded poset. For $m > 0$ define the set of $m$-multichains of $P$ by

$$P^{[m]} = \{ (p_1, p_2, \ldots, p_m) \in P^m \mid p_1 \leq p_2 \leq \cdots \leq p_m \}.$$ 

The poset of $m$-multichains of $P$ is then the poset $P^{[m]} = (P^{[m]}, \leq)$, where $\leq$ is to be understood componentwise.

Proposition 2.2. A bounded poset $P$ is graded if and only if $P^{[m]}$ is graded for all $m > 0$.

Proof. By definition $P$ is graded if and only if its rank function $rk$ satisfies $rk(\hat{0}) = 0$ and $rk(p') = rk(p) + 1$ whenever $p < p'$.

For $p = (p_1, p_2, \ldots, p_m) \in P^{[m]}$ define $\mathbf{rk}(p) = \sum_{i=1}^{m} rk(p_i)$. By definition we have $p < p'$ in $P^{[m]}$ if and only if $p' = (p'_1, p'_2, \ldots, p'_m)$ and there exists some $j \in [m]$, such that $p_i = p'_i$ for $i \neq j$, and $p_j < p_{j+1}$, and $p_j < p'_{j'}$. It is now immediate that

$$\mathbf{rk}(p') = \sum_{i=1}^{m} rk(p'_i) = \sum_{i=1}^{m} rk(p_i) + rk(p_j) + 1 = \mathbf{rk}(p) + 1.$$ 

Since the least element $(\hat{0}, \hat{0}, \ldots, \hat{0})$ in $P^{[m]}$ clearly has rank 0, we are done. □

Proposition 2.3. Let $P = (P, \leq_P)$ and $Q = (Q, \leq_Q)$ be bounded posets. For $m > 0$, we have $(P \times Q)^{[m]} \cong P^{[m]} \times Q^{[m]}$.

Proof. Let $((p_1, q_1), (p_2, q_2), \ldots, (p_m, q_m)) \in (P \times Q)^{[m]}$. It follows by definition that $p_1 \leq_P p_2 \leq_P \cdots \leq_P p_m$ and $q_1 \leq_Q q_2 \leq_Q \cdots \leq_Q q_m$, which implies that $((p_1, p_2, \ldots, p_m), (q_1, q_2, \ldots, q_m)) \in P^{[m]} \times Q^{[m]}$. It is clear that this is a bijection, which we will denote by $\zeta$. It is also easy to see that for any $x, x' \in (P \times Q)^{[m]}$ we have $x \leq x'$ if and only if $\zeta(x) \leq \zeta(x')$, which completes the proof. □
all \( p, p' \in P \) we have \( f(p \land_P p') = f(p) \land_Q f(p') \) and \( f(p \lor_P p') = f(p) \lor_Q f(p') \).

(Such a map is a \textit{surjective lattice homomorphism}.)

The next result states that the poset of multichains of a lattice is particularly well-behaved.

\begin{theorem}
Let \( m > 0 \). If \( P \) is a lattice, then \( P^{[m]} \) is a sublattice of the \( m \)-fold direct product \( P^m \).
\end{theorem}

\begin{proof}
It is clear by construction that \( P^{[m]} \) is a subposet of \( P^m \). We need to show that the join and meet operations are preserved. For that we pick \( p = (p_1, p_2, \ldots, p_m) \) and \( p' = (p'_1, p'_2, \ldots, p'_m) \in P^m \). Since \( p_i \leq p_{i+1} \) and \( p'_i \leq p'_{i+1} \) for \( i \in [m-1] \), we have \( p_i \lor p'_i \leq p_{i+1} \lor p'_{i+1} \) and \( p_i \land p'_i \leq p_{i+1} \land p'_{i+1} \). Consequently, the componentwise join and meet of \( p \) and \( p' \), respectively, is contained in \( P^{[m]} \), and must thus be the join and meet of \( p \) and \( p' \) in \( P^{[m]} \), respectively. \( \square \)
\end{proof}

\begin{proposition}
A lattice \( P = (P, \leq_P) \) is a sublattice of a lattice \( Q = (Q, \leq_Q) \) if and only if \( P^{[m]} \) is a sublattice of \( Q^{[m]} \) for every \( m > 0 \).
\end{proposition}

\begin{proof}
Let \( p = (p_1, p_2, \ldots, p_m), p' = (p'_1, p'_2, \ldots, p'_m) \in P^m \). By construction \( p, p' \in Q^{[m]} \), and Theorem 2.4 implies

\[
\begin{align*}
p \land_P p' &= (p_1 \land p'_1, p_2 \land p'_2, \ldots, p_m \land p'_m) \\
&= (p_1 \land_Q p'_1, p_2 \land_Q p'_2, \ldots, p_m \land_Q p'_m) = p \land_Q p',
\end{align*}
\]

and likewise for joins. \( \square \)
\end{proof}

\begin{proposition}
A lattice \( P = (P, \leq_P) \) is a a homomorphic image of a lattice \( Q = (Q, \leq_Q) \) if and only if \( P^{[m]} \) is a homomorphic image of \( Q^{[m]} \) for every \( m > 0 \).
\end{proposition}

\begin{proof}
Let \( f : P \to Q \) be a surjective lattice homomorphism. Define \( \tilde{f} : P^{[m]} \to Q^{[m]} \) componentwise, i.e.

\[
\tilde{f}((p_1, p_2, \ldots, p_m)) = (f(p_1), f(p_2), \ldots, f(p_m)),
\]

for \( (p_1, p_2, \ldots, p_m) \in P^m \). It is immediate that \( \tilde{f} \) is surjective. Moreover, for \( p = (p_1, p_2, \ldots, p_m), p' = (p'_1, p'_2, \ldots, p'_m) \in P^m \), we have \( \tilde{f}(p), \tilde{f}(p') \in Q^m \), and by assumption and Theorem 2.4 follows

\[
\tilde{f}(p \land_P p') = (f(p_1 \land_P p'_1), f(p_2 \land_P p'_2), \ldots, f(p_m \land_P p'_m))
\]

\[
= (f(p_1) \land_Q f(p'_1), f(p_2) \land_Q f(p'_2), \ldots, f(p_m) \land_Q f(p'_m))
\]

\[
= \tilde{f}(p) \land_Q \tilde{f}(p'),
\]

and likewise for joins. \( \square \)
\end{proof}

Recall that a lattice \( P = (P, \leq_P) \) is \textit{distributive} if any three elements \( x, y, z \in P \) satisfy \( x \land (y \lor z) = (x \land y) \lor (x \land z) \). Further \( P \) is \textit{modular} if any three elements \( x, y, z \in P \) satisfy \( (x \land z) \lor (y \land z) = ((x \land z) \lor y) \land z \). We have the following result.
Proposition 2.7. A bounded poset $P$ is a \{ \begin{align*} \text{lattice} & \\
\text{distributive lattice} & \\
\text{modular lattice} & \end{align*} \} if and only if $P^{[m]}$ is a \{ \begin{align*} \text{lattice} & \\
\text{distributive lattice} & \\
\text{modular lattice} & \end{align*} \} for all $m > 0$.

We prove Proposition 2.7 by proving a more general statement. Let $K$ be a class of lattices, and let $\text{Id}(K)$ denote the set of all identities that hold in all members of $K$. Conversely, let $\Sigma$ be a set of identities, and let $\text{Mod}(\Sigma)$ denote the class of all lattices in which all the identities in $\Sigma$ hold. A class $K$ of lattices is a lattice variety if $K = \text{Mod}(\text{Id}(K))$, i.e. if the members of $K$ can be completely described by a set of identities. Consequently the class of distributive and modular lattices, respectively, form a lattice variety. Recall the following result due to G. Birkhoff.

Theorem 2.8 ([5, Theorems 6 and 7]). A class $K$ of lattices is a variety if and only if $K$ is closed under the formation of homomorphic images, sublattices, and direct products.

The following result is a direct consequence of Theorems 2.4 and 2.8.

Corollary 2.9. Let $K$ be a lattice variety, and let $P \in K$. For any $m > 0$, we have $P^{[m]} \in K$.

Now we obtain the proof of Proposition 2.7 almost immediately.

Proof of Proposition 2.7. The class of all lattices forms a lattice variety by virtue of the identity $x = x$, and we have seen before that the class of distributive and modular lattices, respectively, form lattice varieties as well. Hence by Theorem 2.8 any sublattice of the $m$-fold direct product of a \{ \begin{align*} \text{lattice} & \\
\text{distributive lattice} & \\
\text{modular lattice} & \end{align*} \} $P$ is again a \{ \begin{align*} \text{lattice} & \\
\text{distributive lattice} & \\
\text{modular lattice} & \end{align*} \} . In view of Theorem 2.4 this is in particular true for $P^{[m]}$. \qed

The formation of the poset of $m$-multichains of a given lattice $P$ does not only preserve the membership of $P$ in some lattice variety, but also some other lattice properties that do not constitute varieties. A lattice $P = (P, \leq)$ is join-semidistributive if for any three elements $x, y, z \in P$ with $x \vee y = x \vee z$, we have $x \vee y = x \vee (y \wedge z)$. Further $P$ is lower semimodular if for any two elements $p, q \in P$ we have $p, q \preceq p \vee q$, whenever $p \wedge q \preceq p, q$.

Proposition 2.10. A lattice $P$ is join-semidistributive if and only if $P^{[m]}$ is join-semidistributive for all $m > 0$.

Proof. Let $P$ be a join-semidistributive lattice, and pick $x = (x_1, x_2, \ldots, x_m), p = (p_1, p_2, \ldots, p_m), p' = (p'_1, p'_2, \ldots, p'_m) \in P^{[m]}$ such that $x \vee p = x \vee p'$. Theorem 2.4 implies that $x_i \vee p_i = x_i \vee p'_i$ for all $i \in [m]$. Since $P$ is semidistributive we conclude that $x_i \vee p_i = x_i \vee (p_i \wedge p'_i)$ for all $i \in [m]$, and Theorem 2.4 implies $x \vee p = x \vee (p \wedge p')$. This completes the proof. \qed
Proposition 2.11. A lattice $\mathcal{P}$ lower semimodular if and only if $\mathcal{P}^\lfloor m \rfloor$ is lower semimodular for all $m > 0$.

Proof. Let $\mathcal{P}$ be a lower semimodular lattice, and pick $\mathbf{p} = (p_1, p_2, \ldots, p_m), \mathbf{p}' = (p'_1, p'_2, \ldots, p'_m) \in \mathcal{P}^\lfloor m \rfloor$. Suppose further that $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ satisfies $\mathbf{x} = \mathbf{p} \land \mathbf{p}' \prec \mathbf{p}, \mathbf{p}'$. By construction, $\mathbf{p}$ and $\mathbf{p}'$, respectively, differ from $\mathbf{x}$ in exactly one element. Hence we can find $i, j \in [m]$ with $x_k = p_k$ for $k \neq i$, and $x_k = p'_k$ for $k \neq j$. We observe further that necessarily $x_i < p_i$ and $x_j < p'_j$. There are two possibilities:

(i) $i = j$. Hence $x_i < p_i, p'_i$, and since $\mathcal{P}$ is lower semimodular it follows that $p_i, p'_i \prec p_i \lor p'_i$. Consequently, we have $\mathbf{p}, \mathbf{p}' \prec (x_1, x_2, \ldots, x_i-1, p_i \lor p'_i, x_i+1, \ldots, x_m)$.

(ii) $i \neq j$. Without loss of generality we can assume $i < j$, and it is immediate that $\mathbf{p}, \mathbf{p}' \prec \mathbf{x}'$, where $\mathbf{x}' = (x_1, x_2, \ldots, x_i-1, p_i, x_i+1, \ldots, x_j-1, p'_j, x_j+1, \ldots, x_m)$.

This completes the proof. \hfill $\square$

There exist the notions of meet-semidistributivity and upper semimodularity as well, which are defined dually to join-semidistributivity and lower semimodularity, and it is straightforward to show that the analogues of Propositions 2.10 and 2.11 hold for meet-semidistributive and upper semimodular lattices.

3. Topology of Multichain Posets

There is a natural way to associate a topological space with a poset $\mathcal{P}$ via the geometric realization of the order complex of $\mathcal{P}$, i.e. the simplicial complex whose faces are the chains of $\mathcal{P}$. Poset topology is the mathematical discipline that studies topological properties of this simplicial complex from the poset perspective. See for instance [7, 17] for a detailed introduction to this topic. One important concept in poset topology is that of EL-shellability, which was first introduced by A. Björner in [6] for graded posets, and was later generalized to ungraded posets in [8, 9]. We now recall the basic definitions.

Let $\mathcal{E}(\mathcal{P}) = \{(p,q) \mid p < q\}$ denote the set of edges of $\mathcal{P}$. An edge-labeling of $\mathcal{P}$ is a map $\lambda : \mathcal{E}(\mathcal{P}) \rightarrow \Lambda$ for some partially ordered set $\Lambda$. A maximal chain $C$ of $\mathcal{P}$ is rising with respect to $\lambda$ if the sequence of edge labels of this chain, denoted by $\lambda(C)$, is strictly increasing with respect to the partial order of $\Lambda$. A maximal chain $C$ of $\mathcal{P}$ precedes another maximal chain $C'$ of $\mathcal{P}$ if $\lambda(C)$ is lexicographically smaller than $\lambda(C')$. An edge-labeling of a graded poset $\mathcal{P}$ is an EL-labeling if in every closed interval of $\mathcal{P}$ there exists a unique maximal chain that is rising with respect to the labeling, and this chain precedes any other maximal chain in this interval. A graded bounded poset that admits an EL-labeling is EL-shellable.

Theorem 3.1. A bounded poset $\mathcal{P}$ is EL-shellable if and only if $\mathcal{P}^\lfloor m \rfloor$ is EL-shellable for every $m > 0$.

Proof. Suppose that $\mathcal{P}$ is EL-shellable, and let $\lambda$ be an EL-labeling of $\mathcal{P}$ with label set $\Lambda$. Fix $m > 0$. Define the product labeling $\lambda^\lfloor m \rfloor$ of $\mathcal{P}^\lfloor m \rfloor$ by

$$
\lambda^\lfloor m \rfloor((p_1, p_2, \ldots, p_m), (q_1, q_2, \ldots, q_m)) = \begin{cases} 
0, & \text{if } p_i = q_i \\
\lambda(p, q), & \text{if } p_i \neq q_i.
\end{cases}
$$
By definition of \( \mathcal{P}^{|n|} \) it follows immediately that \( \lambda^{|m|} \) labels every edge of \( \mathcal{P}^{|n|} \) by a sequence of the form \((0,0,\ldots,0,a,0,\ldots,0)\) for \( a \in \Lambda \). Suppose (formally) that \( 0 \) is smaller than any label in \( \Lambda \), and order the label set of \( \lambda^{|m|} \) lexicographically.

Now fix \( p = (p_1,p_2,\ldots,p_m) \) and \( q = (q_1,q_2,\ldots,q_m) \) in \( \mathcal{P}^{|m|} \) that satisfy \( p \leq q \). For \( i \in \{1,2,\ldots,m\} \) define elements

\[
    z_i = (p_1,p_2,\ldots,p_{m-i},q_{m-i+1},q_{m-i+2},\ldots,q_m).
\]

Moreover, for \( i \in \{1,2,\ldots,m\} \) let \( C_i \) denote the unique rising chain in the interval \([p_{m-i+1},q_{m-i+1}]\) of \( \mathcal{P} \). By construction, any label along the chain \( C_i \) has the non-zero entry in its \((m-i+1)\)-st position. Thus the concatenation of the chains \( C_1,C_2,\ldots,C_m \) corresponds in a unique fashion to a chain, say \( C \), from \( p \) to \( q \). Moreover, since \( \lambda \) is an EL-labeling of \( \mathcal{P} \), no other chain from \( p \) to \( q \) passing through \( z_1,z_2,\ldots,z_m \) can be rising. Now consider a chain from \( p \) to \( q \) that does not pass through \( z_i \) for some \( i \in \{1,2,\ldots,m\} \). That means that along this chain at least one entry of \( p \) at some position \( j \leq m - i \) is increased before the entry of \( p \) at position \( m - i + 1 \) has reached the element \( q_{m-i+1} \). Hence such a chain cannot be rising. We conclude thus that there exists a unique rising saturated chain from \( p \) to \( q \).

Suppose that there is a chain \( C' \) different from \( C \) that is lexicographically smaller than \( C \). Since \( \lambda \) is an EL-labeling of \( \mathcal{P} \) it follows that \( C' \) cannot pass through all the elements \( z_1,z_2,\ldots,z_m \). Let \( i \) be the first index such that \( C' \) does not pass through \( z_i \). By construction, \( C \) and \( C' \) must coincide from \( p \) to \( z_{i-1} \), because otherwise \( C \) would be lexicographically smaller than \( C' \). Denote by \( C'_i \) the part of \( C' \) that corresponds to the chain (in \( \mathcal{P} \)) from \( p_{m-i+1} \) to \( q_{m-i+1} \). Since \( \lambda \) is an EL-labeling of \( \mathcal{P} \) it follows that \( C'_i \) is lexicographically larger than \( C' \), which contradicts the assumption that \( C' \) is lexicographically smaller than \( C \). It follows that \( C \) is the lexicographically first saturated chain from \( p \) to \( q \). Hence \( \lambda^{|m|} \) is an EL-labeling of \( \mathcal{P}^{|m|} \).

See Figure 1 for an illustration of Theorem 3.1.

4. Applications

In this section, we give explicit descriptions of the poset of multichains of chains and Boolean lattices, and use this perspective to enumerate the \( m \)-multichains of the face lattice of the hypercube.

4.1. Chains. Let \( \mathcal{C}_k \) denote the chain with \( k \) elements, and let \( \mathcal{J}(\mathcal{P}) \) denote the lattice of order ideals of a poset \( \mathcal{P} \).

**Proposition 4.1.** For \( n,m > 0 \), we have \( \mathcal{C}_n^{|m|} \cong \mathcal{J}(\mathcal{C}_{n-1} \times \mathcal{C}_m) \).

**Proof.** In what follows, we identify an order ideal \( A \in \mathcal{J}(\mathcal{C}_{n-1} \times \mathcal{C}_m) \) with the antichain consisting of the maximal elements of \( A \).

Let \( A = \{ (i_1,j_1), (i_2,j_2), \ldots, (i_k,j_k) \} \in \mathcal{J}(\mathcal{C}_{n-1} \times \mathcal{C}_m) \). Since \( A \) is an antichain, it follows that the \( i_k \) are pairwise distinct, and so are the \( j_k \). Without loss of generality, we can assume that \( i_1 < i_2 < \cdots < i_k \), which implies \( j_1 > j_2 > \cdots > j_k \).
Further, let \( x = (a_1, a_2, \ldots, a_m) \in \mathcal{C}_n^m \). By construction, we have \( 1 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq m \). We define

\[
\text{asc}(x) = \left| \{ s \in [m - 1] \mid a_s < a_{s+1} \} \right|.
\]

Finally, let \( \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{else} \end{cases} \) denote the Kronecker symbol.

We first show that \( \mathcal{J}(\mathcal{C}_{n-1} \times \mathcal{C}_m) \) and \( \mathcal{C}_n^m \) are isomorphic as sets, and we explicitly construct a bijection. Let us define a map \( \varphi : \mathcal{J}(\mathcal{C}_{n-1} \times \mathcal{C}_m) \to \mathcal{C}_n^m \), by

\[
\varphi\left( \{ (i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k) \} \right) = \left( \begin{array}{c}
1, 1, \ldots, 1, i_3 + 1, i_1 + 1, \ldots, i_1 + 1, \ldots, i_k + 1, i_{k+1} + 1, \ldots, i_{k+1} \\
\end{array} \right).
\]

If \( A = \emptyset \), then we interpret this as \( \varphi(\emptyset) = (1, 1, \ldots, 1) \). In particular, we have

\[
\text{asc}(\varphi(A)) = k + 1 - \delta_{j_1,m}.
\]

Conversely, let us define a map \( \psi : \mathcal{C}_n^m \to \mathcal{J}(\mathcal{C}_{n-1} \times \mathcal{C}_m) \) by

\[
\psi\left( (a_1, a_2, \ldots, a_m) \right) = \left\{ (a_{j_1} - 1, m), (a_{j_2} - 1, m - j_1), \ldots, (a_{j_{k+1}} - 1, m - j_k) \right\},
\]

where \( \text{asc}\left( (a_1, a_2, \ldots, a_m) \right) = \{j_1, j_2, \ldots, j_k\} \) as well as \( j_{k+1} = m \). If \( a_{j_1} = 1 \), then we identify \((a_{j_1} - 1, m)\) with \( \emptyset \). Hence we have

\[
\left| \psi\left( (a_1, a_2, \ldots, a_m) \right) \right| = k + 1 - \delta_{j_1,1}.
\]
We have
\[ \varphi \circ \varphi \left( \left\{ (i_1, j_1), \ldots, (i_k, j_k) \right\} \right) = \varphi \left( \left\{ 1, \ldots, 1, i_1 + 1, \ldots, i_k + 1 \right\} \right)_m^{1 - j_1}_{j_1 - 1} = \left\{ (a_m - j_1 - 1, m), (a_m - j_2 - 1, j_1), \ldots, (a_m - 1, j_k) \right\}. \]

Let us abbreviate \( A = \left\{ (i_1, j_1), \ldots, (i_k, j_k) \right\} \). If \( j_1 = m \), then \( \varphi(A) \) has \( k \) ascents and starts with some value \( > 1 \), and consequently \( \varphi \circ \varphi(A) \) has \( k \) elements. If \( j_1 < m \), then \( \varphi(A) \) has \( k + 1 \) ascents and starts with \( 1 \), and consequently \( \varphi \circ \varphi(A) \) has \( k \) elements. Further, \( a_m - j_{k+1} - 1 = i_s \) for \( s \in [k] \), where \( j_{k+1} = 0 \). Hence \( \varphi \circ \varphi(A) = A \).

Conversely, we have
\[ \psi \circ \varphi \left( (a_1, a_2, \ldots, a_m) \right) = \varphi \left( \left\{ (a_{j_1} - 1, m), (a_{j_1 + 1} - 1, m - j_1) \right\} \right) = \left( a_{j_1 + 1}, \ldots, a_{j_2}, a_{j_1}, a_{j_2 + 1}, \ldots, a_{j_3}, \ldots, a_{j_k + 1} \right), \]

where \( \text{asc}(a_1, a_2, \ldots, a_m) = \{ j_1, j_2, \ldots, j_k \} \). Let us abbreviate \( x = (a_1, a_2, \ldots, a_m) \). If \( a_{j_1} = 1 \), then \( \varphi(x) \) has \( k \) elements, and consequently \( \psi \circ \varphi(x) \) has \( k \) ascents. If \( a_{j_1} > 1 \), then \( \varphi(x) \) has \( k \) elements, and again it follows that \( \psi \circ \varphi(x) \) has \( k \) ascents. Hence it is immediate that \( \psi \circ \varphi(x) = x \).

Next we show that \( \varphi \) is order-preserving. For \( A \in \mathcal{J} \left( \mathcal{C}_{n-1} \times \mathcal{C}_m \right) \), where
\[ A = \left\{ (i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k) \right\}, \]
we observe that
\[ \varphi(A) = \bigvee_{i=1}^{k} \varphi \left( \left\{ (i_k, j_k) \right\} \right). \]

This implies in particular that \( \varphi \) preserves joins. Now pick \( A, A' \in \mathcal{J} \left( \mathcal{C}_{n-1} \times \mathcal{C}_m \right) \) with \( A \leq A' \). We have
\[ A \leq A' \iff A \cup A' = A' \iff \varphi(A \cup A') = \varphi(A') \implies \varphi(A) \cup \varphi(A') = \varphi(A') \iff \varphi(A) \leq \varphi(A'), \]
which implies that \( \varphi \) is order-preserving.

It remains to show that \( \psi \) is order-preserving. For that, we use induction on \( m \) and \( n \). Observe that if \( m = 1 \), then we claim that \( \mathcal{C}_n \cong \mathcal{J}(\mathcal{C}_{n-1}) \), which is trivially true. If \( n = 1 \), then we claim that \( \mathcal{C}_m^{[m]} \cong \mathcal{J} \left( \mathcal{C}_{n-1} \right) \), which is true since the singleton poset appears on both sides of the isomorphism. Now fix \( m \) and \( n \), and assume that \( \varphi \) is order-preserving for any \( n' < n \) and any \( m' < m \). We pick \( x, x' \in \mathcal{C}_n^{[m]} \) with \( x \leq x' \). Let us write \( x = (a_1, a_2, \ldots, a_m) \) and \( x' = (a'_1, a'_2, \ldots, a'_m) \). If \( a_1 = a'_1 = 1 \), then we can consider \( (a_2, a_3, \ldots, a_m) \) and \( (a'_2, a'_3, \ldots, a'_m) \) as elements of \( \mathcal{C}_n^{[m-1]} \), and the claim follows by induction on \( m \). If \( a_1 > 1 \) and \( a'_1 > 1 \), then we can consider \( (a_1 - 1, a_2 - 1, \ldots, a_m - 1) \) and \( (a'_1 - 1, a'_2 - 1, \ldots, a'_m - 1) \) as elements of \( \mathcal{C}_n^{[m-1]} \), and the claim follows by induction on \( n \). It remains the case where \( a_1 = 1 \) and \( a'_1 > 1 \). Without loss of generality, we can assume that \( a_2 > 1 \), as well as \( a'_1 = 2 \).
For $n$

**Proposition 4.2.**

**Proof.** Let $(A_1, A_2, \ldots, A_m) \in \mathcal{B}_n^m$, where $A_i \subseteq [n]$ for $i \in [m]$. With this multichain we associate the tuple $(j_1, j_2, \ldots, j_m)$ where $j_i = |\{A_k \mid i \in A_k\}| + 1$. It is then immediate that $j_i \in [m+1]$, and that the first set in which $i$ occurs is $A_{m-j_i+2}$, where we put $A_{m+1} = [n]$.

Conversely, for $(j_1, j_2, \ldots, j_m) \in \mathcal{C}_{m+1}$ and $k \in [m]$, we define $A_k = \{i \in [n] \mid j_i \geq m + 2 - k\}$. It is immediate that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$, and hence $(A_1, A_2, \ldots, A_m) \in \mathcal{B}_n^m$.

---

**Figure 2.** An illustration of Proposition 4.1.

and $a'_i = a_i$ for $i > 1$. (Indeed, suppose that $j$ is the first index where $a_j > 1$. Then we have $x \leq (1, a_j, a_{j+1}, \ldots, a_m) < (2, a_j, a_{j+1}, \ldots, a_m) \leq x'$. In particular, we have $x \prec x'$. If $a_2 = 2$, then we can write $\text{asc}(x') = \text{asc}(x) \setminus \{1\}$. Suppose that the second ascent of $x$ is at position $j$, which implies $a_j = 2$. Consequently, we have $\psi(x') \setminus \psi(x) = \{1,m\}$ and $\psi(x) \setminus \psi(x') = \{1,m-1\}$. We see that $(1, m-1) < (1, m)$ in $\mathcal{C}_{m-1} \times \mathcal{C}_m$ which implies that $\psi(x) \leq \psi(x')$.

If $a_2 > 2$, then we have $\text{asc}(x) = \text{asc}(x') = \{j_1, j_2, \ldots, j_k\}$, where $j_1 = 1$. Consequently, we have $\psi(x) \subseteq \psi(x')$, which implies $\psi(x) \leq \psi(x')$, and we are done. □

See Figure 2 for an illustration.

4.2. **Boolean Lattices.** Let $\mathcal{B}_k$ denote the Boolean lattice with $2^k$ elements.

**Proposition 4.2.** For $n, m > 0$, we have $\mathcal{B}_n^m \cong \mathcal{C}_{m+1}^n$.

**Proof.** Let $(A_1, A_2, \ldots, A_m) \in \mathcal{B}_n^m$, where $A_i \subseteq [n]$ for $i \in [m]$. With this multichain we associate the tuple $(j_1, j_2, \ldots, j_m)$ where $j_i = |\{A_k \mid i \in A_k\}| + 1$. It is then immediate that $j_i \in [m+1]$, and that the first set in which $i$ occurs is $A_{m-j_i+2}$, where we put $A_{m+1} = [n]$.

Conversely, for $(j_1, j_2, \ldots, j_m) \in \mathcal{C}_{m+1}^n$ and $k \in [m]$, we define $A_k = \{i \in [n] \mid j_i \geq m + 2 - k\}$. It is immediate that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$, and hence $(A_1, A_2, \ldots, A_m) \in \mathcal{B}_n^m$. 
For an illustration.

We have that as desired. Here, the empty face responds to an empty face of the $n$-cube. It is straightforward to describe the structure of the intervals in $E^m$. The third equality follows from the Binomial Theorem.

Proposition 4.3. For $m, n > 0$, we have $|\mathcal{HC}_n^m| = \sum_{i=0}^m (2i + 1)^n$.

Proof. Let us abbreviate $f(m, n) = \sum_{i=0}^m (2i + 1)^n$.

It is straightforward to describe the structure of the intervals in $\mathcal{HC}_n$. Let $[E, F]$ be such an interval. We have $[E, F] \cong \mathcal{HC}_k$ if $E \neq \emptyset$ and $k$ is the number of $x$'s in $F$ minus the number of $x$'s in $E$, and we have $[E, F] \cong \mathcal{HC}_x$ if $E = \emptyset$, and $k$ is the number of $x$'s in $F$.

To prove the result we use induction on $m$. If $m = 1$, then we can easily check that

$$f(1, n) = |\mathcal{HC}_n| = 1 + \sum_{k=0}^n \binom{n}{k} 2^k = 1 + 3^n,$$

as desired. Here, the empty face $\emptyset$ contributes the 1, and the sum basically counts the faces in which exactly $k$ entries are equal to $x$. The third equality follows from the Binomial Theorem.

Now fix an $(m - 1)$-multichain $(F_1, F_2, \ldots, F_{m-1})$. We want to describe in how many ways this multichain can be extended to an $m$-multichain. If $F_{m-1} = \emptyset$, then it follows that $F_1 = F_2 = \cdots = F_{m-1} = \emptyset$, and there are precisely $f(1, n) = 3^n$.
possible extensions. Clearly, there is a unique such multichain. If \( F_{m-1} \neq \emptyset \), then we denote by \( k \) the number of \( x \)'s in \( F_{m-1} \), and the structure of the intervals in \( \mathcal{H}_n \) implies that there are \( 2^{n-k} \) possible extensions. Clearly, there are \( f(m-1,n) \) such chains. We obtain
\[
f(m,n) = 1 \cdot f(1,n) + \sum_{k=0}^{n} \binom{n}{k} \cdot f(m-1,n) \cdot 2^{n-k}
\]
\[
= 3^n + \sum_{k=0}^{n} \binom{n}{k} \cdot \left( \sum_{i=0}^{m-1} (2i+1)^n \right) \cdot 2^{n-k}
\]
\[
= 3^n + \sum_{i=0}^{m-1} \left( \sum_{k=0}^{n} \binom{n}{k} \cdot (2i+1)^n \cdot 2^{n-k} \right)
\]
\[
= 3^n + \sum_{i=0}^{m-1} (2i+3)^n
\]
\[
= 3^n + \sum_{i=1}^{m} (2i+1)^n
\]
\[
= \sum_{i=0}^{m} (2i+1)^n,
\]
as desired.

Recall from the definition that the evaluation of the zeta polynomial of a bounded poset \( P \) at a positive integer \( m + 1 \) yields the number of \( m \)-multichains of \( P \). Proposition 4.3 now establishes that
\[
Z(\mathcal{H}_n, m) = \sum_{i=0}^{m-1} (2i+1)^n.
\]
It is not ad-hoc clear how to evaluate \( Z(\mathcal{H}_n, m) \) at negative integers, but since \( \mathcal{H}_n \) is the face lattice of the \( n \)-dimensional hypercube, it is in particular an Eulerian poset, and [16, Proposition 3.16.1] implies that \( Z(\mathcal{H}_n, -m) = (-1)^{n+1}Z(\mathcal{H}_n, m) \). In particular, since \( \mu = \zeta^{-1} \), we have
\[
\mu_{\mathcal{H}_n}(\emptyset, (x, x, \ldots, x)) = \zeta^{-1}_{\mathcal{H}_n}(\emptyset, (x, x, \ldots, x)) = Z(\mathcal{H}_n, -1) = (-1)^{n+1}Z(\mathcal{H}_n, 1) = (-1)^{n+1}.
\]

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