Insights into the inner structures of the fully charmed tetraquark state $X(6900)$

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Abstract

The recently discovered fully-charmed tetraquark candidate $X(6900)$ is analyzed within the frameworks of effective-range expansion, compositeness relation and width saturation, and a three-coupled channel dynamical study. By taking into account constraints from heavy-quark spin symmetry, the coupled-channel amplitude including the $J/\psi J/\psi$, $\chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c1}$ is constructed to fit the experimental di-$J/\psi$ event distributions around the energy region near 6.9 GeV. The three different theoretical approaches lead to similar conclusions that the two-meson components do not play dominant roles in the $X(6900)$. Our determinations of the resonance poles from the refined couple-channel study are found to be consistent with the experimental analyses. We give predictions to the line shapes of the $\chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c1}$ channels, which could provide useful guides for the future experimental measurements.

1 Introduction

The first fully-heavy-flavor tetraquark meson candidate $X(6900)$ is recently observed in the di-$J/\psi$ spectra by the LHCb Collaboration [1]. This intriguing observation has sparked fruitful theoretical discussions [2]. Among the many interpretations, diquark-antidiquark cluster mechanism in the valence-quark picture is currently the most popular theoretical models to explain the narrow state observed around 6.9 GeV [2]. Although different theoretical methods, including the quark models, QCD sum rules and effective Lagrangians, could account for the right mass of the tetraquark candidate $X(6900)$ without big difficulties, its $J^{PC}$ quantum numbers and decay patterns are still under vivid debate. Our present study aims at pushing forward the clarification of the nature of the $X(6900)$, its possible decay patterns and the extraction of its resonance pole position from the experimental event distributions by employing a sophisticated coupled-channel framework based on general principles of $S$-matrix theory.

In a series of recent works [3–10], we have developed a theoretical framework that is specially useful to bring insights on the inner structures of the resonance states near some underlying two-hadron thresholds. It is based on the effective range expansion (ERE) and the compositeness relation [3][10][16], and has been widely and successfully used to study many possible exotic

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hadrons, such as the charmed baryon \( \Lambda_c(2545) \) \([1]\), the hidden bottom tetraquark candidates \( Z_b(10610)/Z_b(10650) \) \([5]\), the narrow state \( X(3872) \) \([6]\), the pentaquark candidates \( P_c(4312), P_c(4440), P_c(4457) \) \([8]\) and the hidden charm mesons \( Z_c(3900), X(4020), \chi_{c1}(4140), \psi(4260), \psi(4660) \) \([7]\). This formalism is specially powerful to describe the elastic single-channel scattering. Just with the inputs of the mass and width of the resonance, we can estimate the scattering length, effective range and the compositeness coefficient \([3]\), that is the probability to find the two-hadron component inside the resonance. In this work we will first tentatively apply this formalism in the elastic scattering processes of \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) to explore the newly observed \( X(6900) \).

Clearly a realistic study needs to include the \( J/\psi J/\psi \) channel, in which the \( X(6900) \) is observed by the experiments \([1]\). Similar situations also happen for the pentaquark candidates \( P_c(4312), P_c(4440), P_c(4457) \), which require to simultaneously include at least the \( J/\psi p \) and \( \Sigma_c D^{(*)} \) channels \([8]\). Generally speaking when the mass of the resonance is below or rather close to the underlying two-hadron threshold, other lighter hadronic degrees of freedom (d.o.f) usually need to be introduced to account for the decay width of the observed resonances. Then coupled-channel formalism should be instead used in this situation. A simple but efficient approach based on the simultaneous requirements of the decay widths and the compositeness coefficients is developed in Refs. \([5,8,9]\). The most important merit of this approach is that we can predict the partial compositeness coefficients (namely the probabilities to find the different components inside the considered resonance) and the partial decay widths of the resonance just with the minimal inputs, i.e. the total compositeness of the considered channels, apart from the mass and width of the resonance from the experimental determinations. Within the aforementioned formalism we will perform a three-coupled channel \( (J/\psi J/\psi, \chi_{c0}\chi_{c0}, \chi_{c1}\chi_{c1}) \) study for the \( X(6900) \). We point out that the quantum numbers \( J^{PC} = 0^{++} \) are assumed for the \( X(6900) \) throughout, which is also one of the possibilities for the ground states in the quark model picture \([2]\).

The total compositeness coefficient entering in the above approach should be provided beforehand and usually it is taken as free parameters in practice. In order to reach more definite conclusion on the nature of the \( X(6900) \), we further construct the scattering amplitudes involving the \( J/\psi J/\psi, \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) channels by imposing the heavy quark symmetry to reduce the number of free parameters. We use a general coupled-channel near-threshold (of the \( \chi_{c}\chi_{c} \) states) parameterization driven by the presence of a Castillejo-Dalitz-Dyson (CDD) pole \([17]\), which is indeed more general than the ERE \([6]\), without assuming any specific dynamical model. The resulting coupled-channel scattering amplitudes are used to fit the experimental di-\( J/\psi \) event distributions, by taking into account the re-scattering due to the two-hadron systems out of a \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) S-wave source. The resonance pole positions, their couplings to the \( J/\psi J/\psi, \chi_{c0}\chi_{c0}, \chi_{c1}\chi_{c1} \) channels and the compositeness coefficients can then be obtained. We will also give further predictions for the event distributions of the \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \).

This paper is organized as follows. First the tentative study of the elastic scattering of \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) based on the ERE is explored to address the emergence of the \( X(6900) \) at its experimental pole position. Then, we combine the decay widths and compositeness relations to perform a three-coupled-channel study by assuming several different values of the total compositeness coefficients. Next, the fits to the experimental \( J/\psi J/\psi \) event distributions with the \( J/\psi J/\psi, \chi_{c0}\chi_{c0}, \) and \( \chi_{c1}\chi_{c1} \) coupled-channel scattering amplitudes are carried out. As a result, the pole positions of the \( X(6900) \), their coupling strengths, width and compositeness coefficients are determined, and compared with those in the previous approaches. We also give the predictions to the line shapes of the \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \). Finally we give a short summary.
and conclusions.

2 Effective range expansion for the elastic scattering

The standard ERE formalism for the elastic $S$-wave two-body scattering is given by

$$T(E) = \frac{1}{-\frac{a}{2} + r k^2 - i k},$$

where $k$ represents the three-momentum in the center of mass (CM) frame, and $a$ and $r$ in this case correspond to the scattering length and the effective range, respectively. The CM three-momentum $k$ in the non-relativistic limit is related to the CM energy $E$ via

$$k = \sqrt{2\mu_m(E - m_{\text{th}})},$$

where the reduced mass $\mu_m$ and the threshold $m_{\text{th}}$ are given by

$$\mu_m = \frac{m_1 m_2}{m_1 + m_2}, \quad m_{\text{th}} = m_1 + m_2,$$

with $m_1$ and $m_2$ the masses of the two scattering particles. It is easy to verify that the ERE amplitude in Eq. (1) fulfills the unitarity relation

$$\text{Im} T(E)^{-1} = -k, \quad (E > m_{\text{th}}).$$

Regularly speaking, the ERE formula in Eq. (1) works well in the energy region near the two-particle threshold. Two exceptional objects may hinder the application energy region of the ERE formalism: the left-hand cuts and the preexisting bare poles [6–18] (also named as the CDD poles in literature [17]). Since the exchanges of the color-singlet light hadrons between the two-charmonium states are highly suppressed, the contributions from the left-hand cuts can be safely neglected in the two-charmonium scattering. The situation for the CDD pole is more subtle. If the CDD pole is distant from the two-particle threshold, one can apply the ERE formalism in Eq. (1) without any problem. While in the special circumstance when the CDD pole is close to the threshold, the ERE formula in Eq. (1) becomes inaccurate for the description of the near-threshold dynamics [4]. As a consequence, one should explicitly introduce the CDD pole terms into the scattering amplitudes. However, it is impracticable to predict whether the CDD poles near the thresholds exist or not. Alternatively, Refs. [4,5] provide an indirect but practicable way to discern the validity of the ERE formula. It is obtained in Refs. [4,5] that when the mass of the CDD pole $M_{\text{CDD}}$ approaches to the threshold $m_{\text{th}}$ the resulting $a$ and $r$ will be linearly and quadratically inversely proportional to $M_{\text{CDD}} - m_{\text{th}}$ respectively, i.e.

$$a \propto M_{\text{CDD}} - m_{\text{th}}, \quad r \propto \frac{1}{(M_{\text{CDD}} - m_{\text{th}})^2}.$$  

Since other sources could also contribute to the scattering length [1] it may not be so reliable to infer about the CDD pole information from the value of $a$. In contrast, comparing with the standard strong interaction scale around 1 fm, a large value of the magnitude for the effective

\footnote{The quantum mechanical example of a square-well is analyzed in Refs. [5,21], where one can observe explicitly that the size of $|a|$ could be very different to the radius $R$ of the square-well potential by small changes in its depth.}
range \( r \), would strongly hint the existence of the near-threshold CDD pole. In another word, the large effective range \( r \) clearly provides an intuitive and practicable criteria for the existence of the CDD pole near threshold, which also indicates that the ERE formalism in Eq. (1) probably is invalid to describe the dynamics around the threshold. On the contrary, if the resulting magnitude of the effective range \( r \) is around 1 fm, it is unlike that one would need to introduce necessarily a CDD pole around the threshold energy region, and the ERE formula in Eq. (1) would be sufficient to describe the underlying physics.

Through the analytical continuation one can extrapolate the amplitudes into the second Riemann sheet (RS), where the resonance poles lie. The scattering amplitude in the second RS, \( T_{II}(E) \), takes the form

\[
T_{II}(E) = \frac{1}{-\frac{1}{a} + \frac{1}{2} r k^2 + i k}.
\]  

(6)

The imaginary part of the three-momentum \( k \) should be taken to be positive, i.e. \( \text{Im} k > 0 \), in Eqs. (1) and (6). Alternatively, one can still use Eq. (1) for \( T(E) \), but for calculating \( k = \sqrt{2\mu m(E - m_{th})} \) in the first RS the argument of the radicand is taken between \([0, 2\pi)\), while in the second RS it is between \([2\pi, 4\pi)\). The resonance pole \( E_R = M_R - i\Gamma_R/2 \), with \( M_R \) the resonance mass and \( \Gamma_R \) the width, corresponds to the solution of \( T_{II}(E_R)^{-1} = 0 \), that is

\[
-\frac{1}{a} + \frac{1}{2} r k^2_R + i k_R = 0.
\]  

(7)

\( k_R \) stands for the three momentum at the pole \( E_R \), i.e. \( k_R = \sqrt{2\mu m(E_R - m_{th})} \). For simplicity in the writing, we introduce \( k_r \) and \( k_i \) to denote the real and imaginary parts of \( k_R \), respectively,

\[
k_r = \text{Re} k_R, \quad k_i = \text{Im} k_R,
\]  

(8)

where \( k_i > 0 \) is taken, consistently with the convention for \( k \) in Eqs. (1) and (6). It is straightforward to solve Eq. (7) to obtain \( a \) and \( r \) in terms of \( k_r \) and \( k_i \). The solutions of \( a \) and \( r \) have been worked out in Ref. [4],

\[
a = -\frac{2k_i}{|k_R|^2}, \quad r = -\frac{1}{k_i}.
\]  

(9)

By combining Eqs. (9) and (6), the Laurent expansion of the S-wave scattering amplitude in the second RS reads [5]

\[
T_{II}(k) = \frac{-k_i/k_r}{k - k_R} + \ldots,
\]  

(10)

where one can easily identify \(-k_i/k_r\) as the residue of the partial-wave amplitude at the pole position in the variable \( k \). In our previous study [3, 5], the compositeness coefficient \( X \), corresponding to the weight of the two-particle component inside the resonance, is shown to be equal to this residue,

\[
X = -\frac{k_i}{k_r}.
\]  

(11)

It is also proved in Ref. [5] that \( X \) in the previous equation is bounded within the range \([0, 1]\), and hence it meets the requirement for the probabilistic interpretation, when the mass of the resonance pole lies above the considered threshold [3].
According to Eqs. (9) and (11), once the mass and width of the resonance are known, the scattering length, effective range and the compositeness coefficient can be correspondingly predicted within the assumption of single-channel scattering. We give the numerical results in Table 1 for the \( \chi_{c0}\chi_{c0} \) uncoupled scattering, where the two different sets of the masses and the widths of the \( X(6900) \) from the Models I and II of LHCb [1] are separately analyzed for the \( \chi_{c0}\chi_{c0} \) scattering. The values of the mass and width of the \( X(6900) \) resonance from Ref. [1] are

\[
\text{Model I: } M = 6905 \pm 11 \pm 7 \text{ MeV}, \quad \Gamma = 80 \pm 19 \pm 33 \text{ MeV},
\]

\[
\text{Model II: } M = 6886 \pm 11 \pm 11 \text{ MeV}, \quad \Gamma = 168 \pm 33 \pm 69 \text{ MeV},
\]

where the distinction is based according to the treatment of the non-resonant background. For both sets, the masses of the \( X(6900) \) are below the \( \chi_{c1}\chi_{c1} \) threshold and hence we can not interpret the \( X \) defined in Eq. (11) as the probability [3]. The scattering length and effective range resulting in the elastic \( \chi_{c1}\chi_{c1} \) channel are found to be

\[
a = -0.59 \pm 0.04, \quad r = -0.31 \pm 0.02, \quad \text{(case I)},
\]

\[
a = -0.51 \pm 0.05, \quad r = -0.28 \pm 0.02, \quad \text{(case II)},
\]

which are given in units of fm.

The small value of the \( X \) obtained for \( \chi_{c0}\chi_{c0} \) scattering indicates that including this channel alone is not sufficient. This is a contradiction with respect to our onset assumption on the dominance of this channel in order to justify the single-channel treatment. Therefore, this result clearly indicates that a coupled-channel analysis is required for the \( X(6900) \). Similar conclusions can be also made for the case II, although the value of the \( X \) is larger in this case, being compatible with the previous one at the level of one standard deviation. The mild values for the effective range \( r \) imply that the scenario with a near-threshold CDD pole is disfavored in the uncoupled case. It should be pointed out that the total width of the \( X(6900) \) state is implicitly assumed to be saturated by the \( \chi_{c0}\chi_{c0} \) or \( \chi_{c1}\chi_{c1} \) in this simplified framework based on assuming the dominance of only one channel within the elastic ERE. This assumption could be unrealistic, since the partial decay width to the \( J/\psi J/\psi \) channel is likely non-negligible. Therefore a more realistic study requires the information of the partial decay width to \( \chi_{c0}\chi_{c0} \), which will be found out in the next section within a coupled-channel analysis, after performing the fits to the experimental event distributions.

| Resonance | Mass (MeV) | Width (MeV) | Threshold (MeV) | \( a \) (fm) | \( r \) (fm) | \( X \) |
|-----------|-----------|-------------|----------------|-------------|-------------|-------|
| \( X(6900)\)-I | 6905 ± 13 | 80 ± 38 | \( \chi_{c0}\chi_{c0} \) (6829.4) | −0.18 ± 0.07 | −1.52 ± 0.69 | 0.25 ± 0.11 |
| \( X(6900)\)-II | 6886 ± 16 | 168 ± 77 | \( \chi_{c0}\chi_{c0} \) (6829.4) | −0.32 ± 0.06 | −0.72 ± 0.26 | 0.53 ± 0.16 |

Table 1: Scattering lengths, effective ranges and the compositeness coefficients from the ERE study. The two different sets of the masses and widths are taken from the LHCb determinations [1]. For the discussions on the compositeness coefficients \( X \), see the text for details.
3 Coupled-channel study

From the previous discussion, it is noticed that the coupled-channel formalism is needed to describe the \(X(6900)\). Under the assumption of the \(J^{PC} = 0^{++}\), we consider the \(J/\psi J/\psi\) (labeled as 1), \(\chi_{c0}\chi_{c0}\) (labeled as 2) and \(\chi_{c1}\chi_{c1}\) (labeled as 3) three couple channels. The contribution from the \(\eta_c\eta_c\) channel, which threshold is rather distant from the interested energy region around 6.9 GeV, is effectively reabsorbed in the \(J/\psi J/\psi\) channel. In the infinite quark mass limit, the spin symmetry of the heavy quarks become exact, which in turn predicts special patterns for the heavy-hadron spectra. For the heavy-quarkonium system with charm quarks, the heavy-quark symmetry predicts that \(J/\psi\) and \(\eta_c\) will form a spin doublet, and \(\chi_{c0}, \chi_{c1}, \chi_{c2}\) and \(h_c\) form another spin multiplet [22]. These predictions are in reasonable agreement with the experimental measurements [23]. In order to reduce the number of free parameters, we will impose the heavy-quark symmetry to the couplings between the charmonia and the \(X(6900)\) state. To be more specific, the coupling of the \(J/\psi\) pair with \(X(6900)\) will be denoted as \(g_a\). The couplings of the \(\chi_{c0}\chi_{c1}\) and the \(X(6900)\), denoted as \(g_b\), will be imposed to be universal, apart from the Clebsch-Gordan factors. To properly account for the threshold effects, the masses of the charmonia will be taken by their physical values.

To fix the two couplings \(g_a\) and \(g_b\), we solve the two equations of the decay width and the compositeness relation. The partial compositeness coefficient \(X_j\), i.e. the fraction of the two-particle state of the \(j\)th channel in the resonance, is given by [3]

\[
X_j = |g_j|^2 \left| \frac{dG_j(s_R)}{ds} \right|, \quad (14)
\]

where \(g_j\) denotes the coupling strength between the two particles in \(j\)th channel and the resonance. \(G_j(s)\) represents the two-point one-loop function for the \(j\)th channel and its explicit expression from the dimensional regularization by replacing the divergent term with a subtraction constant takes the form

\[
G_j(s) = -\frac{1}{16\pi^2} \left[ a(\mu^2) + \log \frac{m_2^2}{\mu^2} - x_+ \log \frac{x_+ - 1}{x_+} - x_- \log \frac{x_- - 1}{x_-} \right],
\]

\[
x_\pm = \frac{s + m_1^2 - m_2^2 \pm g_j(s)}{2s}. \quad (15)
\]

In this equation one has to fix \(m_1\) and \(m_2\) to the masses of the two particles in this channel. The CM three-momentum in the channel \(j\) is \(q_j(s)\), given by the standard kinematical expression

\[
q_j(s) = \sqrt{s - (m_{j1} + m_{j2})^2} \left| s - (m_{j1} - m_{j2})^2 \right|, \quad (16)
\]

The function \(G_j(s)\) is independent of the regularization scale \(\mu\), due to the mutual cancellation of the \(\mu\) dependence between the first two terms in Eq. [15], and to be more specific we set \(\mu = 770\) MeV throughout the present work. Notice that the derivative of the \(G_j(s)\) function is independent of the subtraction constant term \(a(\mu)\). It is noted that there are several proposals in literature about the extension of the Weinberg’s compositeness relation [11] from the bound state to the resonance situation [3,12,16]. We refer to Ref. [7] for further details about the comparisons of different approaches.

When the mass of the resonance clearly lies above the threshold of the \(j\)th channel, its partial decay width will be taken by the standard formula [23]

\[
\Gamma_j = |g_j|^2 \frac{q_j(M_R^2)}{8\pi M_R^2}, \quad (17)
\]
being \( q_j(M_R^2) \) the relativistic CM three momentum at the resonance mass. In the situation when the mass of the resonance lies close to or even below the \( j \)th threshold, the above formula is not applicable any more and we introduce the Lorentzian distribution to account for the finite-width effect \[5,9\]

\[
\Gamma_j = |g_j|^2 \int_{m_{th}}^{M_R+2\Gamma_R} dw \frac{q_j(w^2)}{16\pi^2 w^2 (M_R - w)^2 + \Gamma_R^2/4},
\]

(18)

which naturally recovers the standard decay-width formula (17) in the narrow-width limit for \( M_R > m_{th} \).

When assuming the \( J^{PC} = 0^{++} \) for the \( X(6900) \), the three channels \( J/\psi J/\psi \) (1), \( \chi_{c0}\chi_{c0} \) (2) and \( \chi_{c1}\chi_{c1} \) (3), are all in the \( S \) wave. The equations to fix \( g_a \) and \( g_b \) then read

\[
X = X_1 + X_2 + X_3 = |g_a|^2 \left| \frac{dG_1^I(s_R)}{ds} \right| + |g_b|^2 \left| \frac{dG_2^I(s_R)}{ds} \right| + \left| g_b \right|^2 \left| \frac{dG_3(s_R)}{ds} \right|,
\]

(19)

and

\[
\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 = |g_a|^2 \frac{q_1(M_R^2)}{8\pi M_R^2} + |g_b|^2 \int_{m_{th}}^{M_R+2\Gamma_R} dw \frac{q_2(w^2)}{16\pi^2 w^2 (M_R - w)^2 + \Gamma_R^2/4} \Gamma_R
\]

\[+ \left| g_b \right|^2 \int_{m_{th},2}^{M_R+2\Gamma_R} dw \frac{q_3(w^2)}{16\pi^2 w^2 (M_R - w)^2 + \Gamma_R^2/4},
\]

(20)

where the 1/3 factors in \( X_3 \) and \( \Gamma_3 \) correspond to the Clebsch-Gordan coefficient from the angular-momentum superposition. The two equations (19) and (20) can uniquely determine the coupling strengths \( |g_a| \) and \( |g_b| \) as a function of the total compositeness \( X \). In terms of these couplings we can then calculate the partial decay widths and compositeness coefficients for the \( J/\psi J/\psi \), \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) channels. The results are summarized in Table 2 where \( X \) in Eq. (19) should be taken as an external input (which we also fix below by implementing a coupled-channel dynamical study). It is found that there exists a maximum value of \( X \), which is 0.4 for the case I and 0.9 for the case II, in order for Eqs. (19) and (20) to have solutions. We have tried several different values for the \( X \). Larger values of \( X \) lead to a smaller magnitude of \( |g_a| \) and a larger one for \( |g_b| \). As a result, the partial width of the \( J/\psi J/\psi \) channel becomes smaller and the width of \( \chi_{c0}\chi_{c0} \) tends to be larger. Since the threshold of the \( \chi_{c1}\chi_{c1} \) is clearly higher than the resonance mass, its partial decay width is always tiny. With increasing values of \( X \), the partial compositeness coefficient of the \( \chi_{c0}\chi_{c0} \) also tends to increase, while the compositeness values for \( J/\psi J/\psi \) and \( \chi_{c1}\chi_{c1} \) always remain small.\(^2\)

In order to have a more definite conclusion in the above approach, it is necessary to pin down the value of the total compositeness coefficient \( X \), which is however not practicably obtainable. Another way to proceed is to further constrain the coupling strengths \( g_a \) and \( g_b \), which will in turn give more definite values for the \( X \). In the following, we perform fits to the \( J/\psi J/\psi \) invariant-mass distributions from the LHCb II, in order to obtain more definite values for the couplings and the resonance pole position.

The coupled-channel scattering amplitudes take the form \[24\]

\[
T(s) = [1 - \mathcal{V}(s) \cdot G(s)]^{-1} \cdot \mathcal{V}(s),
\]

(21)

\(^2\)Despite the threshold for \( \chi_{c1}\chi_{c1} \) is clearly larger than \( M_R \) we have still calculated the compositeness for this channel because its resulting values are clearly meaningful. This is because they are driven by \( |g_b|^2/3 \), with \( g_b \) the same as for \( \chi_{c0}\chi_{c0} \), times the modulus squared of the derivative of the \( \chi_{c1}\chi_{c1} \) \( G_3(s) \) function. The latter is also necessarily smaller than for \( \chi_{c0}\chi_{c0} \) because its threshold is further away from the resonance pole position.
admitting solutions are 0.4 for case I and 0.9 for case II, respectively. They are related by

\[ \sqrt{s T} = \frac{b_{12}}{b_{23}} \left( s - M_{\text{CDD}}^2 \right) \cdot \frac{b_{13}}{b_{33}} \left( s - M_{\psi}^2 \right) \left( s - M_{\psi}^2 \right), \]

where the \( b_{ij} \) parameters are dimensionless. There is a normalization difference between the unitarized phenomenological amplitude \( \mathcal{T} \) in Eq. (21) and the ERE amplitude \( T \) in Eq. (11). They are related by \( \mathcal{T} = 8\pi \sqrt{sT} \). To reduce the free parameters, we impose the heavy-quark symmetry to constrain the parameters in the perturbative amplitudes

\[ b_{13} = \frac{b_{12}}{\sqrt{3}}, \quad b_{23} = \frac{b_{22}}{\sqrt{3}}; \quad b_{33} = \frac{b_{22}}{3}. \]

The Clebsch-Gordan coefficients of the angular momenta for \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) have been taken into account. We point out that general parameterizations of the perturbative amplitudes \( \mathcal{V}(s) \) have been exploited by introducing polynomial terms, and it turns out that the successful fits generally prefer the form of \( s - M_{\text{CDD}}^2 \) in the \( \mathcal{V}(s) \) for the \( \chi_{c0}\chi_{c0} \) and \( \chi_{c1}\chi_{c1} \) scattering amplitudes, which corresponds to a single CDD pole to be associated with the unique resonance around. While for the transition amplitudes of the \( J/\psi J/\psi \rightarrow \chi_{c0}\chi_{c0}, \chi_{c1}\chi_{c1} \) we find that it is enough to take the constant terms to obtain reasonable fits, as already shown in Eq. (22). This fact indicates a smooth direct interaction kernel involving the \( J/\psi J/\psi \) channel with the other channels, as expected for a far-threshold channel only relevant in providing the total decay width of the resonance.

The formula to describe the experimental \( J/\psi J/\psi \) event distributions reads

\[ \frac{dN(s)}{d\sqrt{s}} = \left| B_1(s) \right|^2 \frac{q_{J/\psi J/\psi}(s)}{M_{J/\psi}^2}, \]

where the production amplitudes are given by the general parameterization [25]

\[ B(s) = \left[ 1 - \mathcal{V}(s) \cdot G(s) \right]^{-1} \cdot \mathcal{P}. \]

In this equation, \( \mathcal{P} \) is the vector of production vertexes, which is taken as the constant array

\[ \mathcal{P} = \left( \begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array} \right). \]
subsequently modulated by the final-state interactions driven by the $\chi_c0\chi_c0$ and $\chi_c1\chi_c1$ strong re-scattering [26][27]. The production parameters $d_j$ are dimensionless, due to the introduction of the $M^2_{J/\psi}$ in Eq. (24).

Again we impose the heavy quark symmetry to further constrain the vertexes, so that $d_3 = d_2/\sqrt{3}$. In order to be consistent with the assumptions in Eq. (22) that the $J/\psi J/\psi$ channel is weakly coupled to the $X(6900)$ state, we set the production vertex $d_1$ to zero and only allow $d_2$ to vary in the fits. It is pointed out that when releasing $d_1$ the fits indeed prefer very large ratios of $d_2/d_1$, but typically give extremely large uncertainties, showing strong correlations between the two parameters. Therefore our treatment to fix $d_1 = 0$ is not only motivated by the scattering amplitude (22), but also helps to stabilize the various fits.

For the subtraction constant, its natural value can be estimated by matching the functions $G_j(s)$ calculated in dimensional regularization and with a three-momentum cut-off $q_{\text{max}}$ at threshold, as explained in Refs. [25,28,29]. This leads to

$$a = -2 \log \left( 1 + \sqrt{1 + \frac{m^2}{q_{\text{max}}}} \right) + \cdots \approx -3.0,$$

by taking $q_{\text{max}} = 1.0$ GeV and $m = m_{\chi_c0}$. We will take a universal value for the subtraction constants in the three channels (since masses are rather similar) and fix it to the one given in Eq. (27). It is further verified that other natural values ranging from $-3$ to $-2$ lead to quite similar results, as explicitly shown later.

Regarding to the parameter $M_{\text{CDD}}$, we have scanned its values around the range of 6.9 GeV and there is a clear minimum for the resulting $\chi^2$. To obtain the stable fits, we will fix the $M_{\text{CDD}}$ at the minimum values.

We focus on the experimental data in the energy region around 6.9 GeV [1], which amount to 12 data points, as shown in Fig. 1. We take the background contributions from the experimental analyses [1], called there models I and II, to distinguish the two different types of fits that we then perform and which are denoted Fits I and II, respectively. In the fits the free parameters within our approach are finally $b_{12}, b_{22}$ in $V(s)$ (22) and $d_2$ in $P$ (26), with $M_{\text{CDD}}$ determined as explained.

|       | $\chi^2$/d.o.f | $a(\mu)$ | $M_{\text{CDD}}$ | $b_{22}$ | $b_{12}$ | $d_2$ |
|-------|----------------|----------|------------------|----------|----------|-------|
| Fit-I | 1.6/(12 – 3)   | -3.0*    | 6910*            | 10817$^{+8378}_{-2096}$ | 151$^{+153}_{-99}$ | 2213$^{+2106}_{-316}$ |
| Fit-II| 4.9/(12 – 3)   | -3.0*    | 6885*            | 21073$^{+15141}_{-7359}$ | 484$^{+239}_{-112}$ | 3645$^{+1325}_{-714}$ |

Table 3: The fits Fit-I and -II obtained with background contributions taken from the models I and II of Ref. [1], respectively. The entries marked with asterisks are fixed during the fits. Other types of fits with different values of $a(\mu)$, and discussions about the variances of the fixed parameters, are given in the text for further details. $M_{\text{CDD}}$ is given in units of MeV, and the parameters $b_{22}, b_{12}$ and $d_2$ are dimensionless.

We give the outputs of the central fits in Table 3 labeled as Fit-I and Fit-II. For each fit performed $M_{\text{CDD}}$ is fixed to some value and, as indicated above, after this scanning there is a clear minimum in the $\chi^2$ for the values $M_{\text{CDD}} = 6910$ and 6885 MeV in Fit-I and Fit-II,
respectively. With the fitted parameters, we then calculate the resonance pole positions, their residues and the compositeness coefficients. The resonance poles lie in the complex energy plane of an unphysical RS, which can be accessed via the analytical extrapolation of the $G_j(s)$ functions. The expression given in Eq. (15) represents $G_j(s)$ in the first or physical RS, and its corresponding formula on the unphysical RS reads

$$G_j(s)_{\Pi} = G_j(s) - \frac{iG_j(s)}{4\pi\sqrt{s}} .$$

(28)

The imaginary part of $G_j(s)_{\Pi}$ is opposite with the one of $G_j(s)$ above the threshold. Different unphysical RS’s of the coupled-channel scattering amplitudes in Eq. (21) can be accessed by properly taking $G_j(s)$ or $G_j(s)_{\Pi}$ for different channels. The second RS can be labeled as $(-,+,+)$, where the plus(minus) sign in the jth entry indicates taking $G_j(s)(G_j(s)_{\Pi})$ in the jth channel. In this convention, the first, third, fourth and fifth sheets are labeled as $(+,+,+)$, $(-,-,+)$, $(-,-,+)$ and $(-,-,-)$, respectively. The most relevant resonance poles are found to lie in the third RS (which connects continuously with the physical RS between the $\chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c1}$ thresholds). The matrix elements of the scattering matrix in the unphysical RS around the resonance pole region can be written as

$$T_{kj}(s) = \frac{G_{kj}^s}{s - M_{pole}^2} + \cdots ,$$

(29)

where $G_{kj}^{s}$ are the couplings of the resonance to the corresponding channels and can then be obtained by working out the residue of the partial-wave amplitudes at the resonance pole $M_{pole}$. The omitted terms in Eq. (29) are the regular parts in the $s - M_{pole}^2$ Laurent expansion. The pole positions and the resonance couplings $|\gamma_{i=1,2,3}|$ to the different channels are summarized in Table 4. The partial compositeness coefficients $X_{i=1,2,3}$ can be calculated via Eq. (14), and the results are also given in Table 4. The total compositeness $X$ is simply given by the sum of $X_{i=1,2,3}$. The masses and widths of the resonances from Fit-I and Fit-II are well compatible with the experimental determinations [1], and given in Eq. (12).

Let us stress the small value obtained for the total compositeness, with $X < 0.2$ for the two fits. This fact clearly indicates the dominance of a bare component for the $X(6900)$ and the small weight of the two-hadronic components in its nature. This conclusion has been reached without assuming any specific dynamical model, but just relying on a general $S$-matrix parameterization, Eqs. (21) and (22). The smallness of $X$ is a reflection of the value of $M_{CDD}$ lying so closed to the resonance mass [6,19]. This is the basic point stressed in the Morgan’s counting-pole criterion on the nature of a resonance [18,19], because it drives to the proliferation of similar pole positions in different RS’s [19]. We have that this is the case here too, and poles are found in different RS’s associated to the inelastic channel $\chi_{c1}\chi_{c1}$ with little variation in their positions, as required by this criterion.

Since now the total compositeness coefficient $X$ is known from the fits, it is interesting to redo the analyses by combining Eqs. (19) and (20). For the case I when fixing $X = 0.17$, the solutions of Eqs. (19) and (20) read

$$|g_a| = 6.2 \text{ GeV} , \quad |g_b| = 9.5 \text{ GeV} , \quad \Gamma_1 = 49.7 \text{ MeV} , \quad \Gamma_2 = 30.1 \text{ MeV} , \quad \Gamma_3 = 0.2 \text{ MeV} , \quad X_1 = 0.018 , \quad X_2 = 0.126 , \quad X_3 = 0.026 ,$$

(30)

which agree well with the Fit-I results from the sophisticated coupled-channel study in Table 4.
and $J/\psi J/\psi$ decay widths in terms of the central values of Table 1, as they should, and indeed they are pretty compatible with the results for parameters turn out to be

By taking these realistic partial decay widths, the central values for the single-channel ERE study the total width of the $X(6900)$ is assumed to be saturated by the $\chi_{c0}\chi_{c0}$ channel, which does not seem to be consistent with the partial decay width predicted by the resonance dynamics.

For the case II when fixing $X = 0.13$, the solutions of Eqs. (19) and (20) are

\[
|g_a| = 11.1 \text{ GeV}, \quad |g_b| = 6.7 \text{ GeV}, \quad \Gamma_1 = 154.7 \text{ MeV}, \quad \Gamma_2 = 12.8 \text{ MeV}, \quad \Gamma_3 = 0.5 \text{ MeV},
\]
\[
X_1 = 0.06, \quad X_2 = 0.06, \quad X_3 = 0.01,
\]
which are also in good accord with the Fit-II results in Table 4. When obtaining the values in Eqs. (30) and (31), we take the masses and widths of the $X(6900)$ from the experimental analyses in Ref. [1]. These equations also provide the partial decay widths of the $X(6900)$ to the different channels. It is worthy pointing out that when using the masses and widths of the resonances from our fits in Table 4 the agreements of the residues and partial compositeness coefficients turn out to become excellent. Therefore here we provide a solid demonstration that the coupled-channel methods by utilizing the compositeness relations and the decay widths, namely Eqs. (19) and (20), indeed offer a very convenient and reliable approach to study the resonance dynamics.

By taking a first glance at the results in Table 4 from the single-channel ERE study, it seems that the compositeness coefficients predicted by the elastic ERE are larger and in contradiction with the results in Table 4 from the coupled-channel fits. However, it should be stressed that in the elastic ERE study the total width of the $X(6900)$ is assumed to be saturated by the $\chi_{c0}\chi_{c0}$ channel, which does not seem to be consistent with the partial decay width predicted by the coupling/residue in Table 4. To be more specific, we calculate next the partial decay widths to $\chi_{c0}\chi_{c0}$, $\Gamma_2$, by removing to the central values of the total widths in Table 4 the easily calculable $J/\psi J/\psi$ decay widths in terms of the central values of $|\gamma_1|$, cf. Eq. (17) with $|g_1|$ substituted by $|\gamma_1|$. The relative uncertainty for the resulting $\Gamma_2$ is estimated as twice the relative error for $|\gamma_2|$ (because it depends quadratically in the coupling). We then obtain the values

\[
\Gamma_2 = 40^{+11}_{-20} \text{ MeV (Fit – I)}, \quad \Gamma_2 = 26^{+11}_{-14} \text{ MeV (Fit – II)}.
\]

By taking these realistic partial decay widths, the central values for the single-channel ERE parameters turn out to be

\[
a = -0.10 \text{ fm}, \quad r = -3.0 \text{ fm}, \quad X = 0.13 \text{ (Fit – I)},
\]
\[
\text{and}
\]
\[
a = -0.09 \text{ fm}, \quad r = -4.1 \text{ fm}, \quad X = 0.10 \text{ (Fit – II)}.
\]

It is clear that the compositeness coefficients from the ERE are now much smaller than in Table 1 as they should, and indeed they are pretty compatible with the results for $X_2$ in Table 4 from the fits.

| Mass (MeV) | Width/2 (MeV) | $|\gamma_1|$ (GeV) | $|\gamma_2|$ (GeV) | $|\gamma_3|$ (GeV) | $X_1$ | $X_2$ | $X_3$ | $X = \sum_{i=1}^{3} X_i$ |
|------------|--------------|-------------------|------------------|------------------|------|-------|-------|-----------------|
| Fit-I      | 6907$^{+5}_{-3}$ | 33$^{+14}_{-10}$ | 4.6$^{+2.5}_{-2.8}$ | 9.7$^{+1.4}_{-2.6}$ | 5.6$^{+0.8}_{-1.5}$ | 0.01$^{+0.01}_{-0.01}$ | 0.13$^{+0.04}_{-0.06}$ | 0.03$^{+0.01}_{-0.01}$ | 0.17$^{+0.04}_{-0.07}$ |
| Fit-II     | 6892$^{+2}_{-2}$ | 80$^{+24}_{-17}$ | 10.3$^{+1.8}_{-1.4}$ | 6.9$^{+1.4}_{-1.9}$ | 4.0$^{+0.8}_{-1.1}$ | 0.05$^{+0.02}_{-0.01}$ | 0.06$^{+0.03}_{-0.03}$ | 0.01$^{+0.01}_{-0.01}$ | 0.13$^{+0.03}_{-0.01}$ |

Table 4: Resonance poles of the $X(6900)$ on the third Riemann sheets. $|\gamma_{i=1,2,3}|$ represent the resonance couplings to the $J/\psi J/\psi, \chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c1}$ channels respectively. $X_{i=1,2,3}$ denote the partial compositeness coefficients, i.e. the probabilities to find the $J/\psi J/\psi, \chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c1}$ components in the $X(6900)$ state, respectively.
Other fits by taking different values of the subtraction constants \( a(\mu) \) are also obtained. In order to make a clear comparison of the fits in Table 3, the values of \( M_{CDD} \) of the Fit-I and Fit-II will be fixed at the same values as shown in the former table. For the Fit-IA by fixing \( a(\mu) = -2.5 \), the fitted parameters read: \( b_{22} = 5234.6 \), \( b_{12} = 155.0 \) and \( d_2 = 1229.7 \), with the \( \chi^2 = 3.5 \). The resulting resonance pole is \( 6925 - i \times 41 \) MeV and the total compositeness coefficient is \( X = 0.11 \). For the Fit-IB by fixing \( a(\mu) = -3.5 \), the fit gives \( b_{22} = 45382.3 \), \( b_{12} = 538.0 \) and \( d_2 = 3353.1 \), with the \( \chi^2 = 3.5 \). The resulting resonance pole is \( 6923 - i \times 42 \) MeV and the total compositeness is \( X = 0.04 \). For the Fit-IIA, taking now \( a(\mu) = -2.5 \), it leads to \( b_{22} = 3442.7 \), \( b_{12} = 200.0 \) and \( d_2 = 1549.6 \), with the \( \chi^2 = 7.7 \). The resulting pole position is \( 6909 - i \times 99 \) MeV and the total compositeness is \( X = 0.23 \). For the Fit-IIB with \( a(\mu) = -3.5 \) fixed, the results are \( b_{22} = 46782.2 \), \( b_{12} = 871.3 \) and \( d_2 = 5734.4 \), with the \( \chi^2 = 10.1 \). The resulting resonance pole is \( 6879.5 - i \times 104 \) MeV and \( X = 0.09 \). Here we only show the most relevant resonance poles found on the third RS. It is clear that in all the cases the total compositeness coefficients \( X \) remain clearly small, the pole positions moderately vary and the \( \chi^2 \) gets worse by around a factor 2.

![Figure 1: Event distributions of the J/ψJ/ψ. The data are taken from Ref. [1]. The blue and red dotted lines represent the background contributions extracted from the model I and II from Ref. [1], respectively. The shaded areas correspond to the error bands at the one standard deviation by using the parameters shown in Table 3.](image-url)

The resulting fit plots are illustrated in Fig. 1 where the error bands at the one standard deviation for the Fit-I and Fit-II, are provided. The blue and red dotted lines in Fig. 1 represent the background contributions of model I and II, respectively, extracted from Ref. [1]. The resulting curves obtained in the scenarios of the Fit-IA/B and Fit-IIA/B are not explicitly shown, since they look quite similar as those from the Fit-I and Fit-II, respectively. In this way we verify that different fits by taking different values for the subtraction constants \( a(\mu) \) within the same type of background contributions lead to rather similar results for the di-J/ψ event.
Figure 2: The left and right panels show our predictions for the distributions of $\chi_{c0}\chi_{c0}$ and $\chi_{c1}\chi_{c1}$, respectively.

Figure 3: Our predictions for the scattering amplitudes. The top left and right panels show the amplitudes of $J/\psi J/\psi \to J/\psi J/\psi$ and $J/\psi J/\psi \to \chi_{c0}\chi_{c0}$. The panels in the bottom left and right give the predictions for the amplitudes of $J/\psi J/\psi \to \chi_{c1}\chi_{c1}$ and $\chi_{c0}\chi_{c0} \to \chi_{c0}\chi_{c0}$, respectively. Since the heavy-quark symmetry is imposed, the shapes of the amplitudes involving $\chi_{c1}$ look rather similar to those with $\chi_{c0}$ and we do not explicitly show other amplitudes.
distributions, while the two different types of fits by using different models for the background, as provided by the LHCb analyses [1], give obvious different results. Therefore, after taking a more sophisticated coupled-channel analysis, our results confirm the findings of Ref. [1]. The resonance pole positions from Fit-I and Fit-II are well compatible with the masses and widths determined in the model I and II from Ref. [1], respectively. As a novelty, our coupled-channel study provides new information on the couplings of the $\chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$, which in turn allows us to calculate the partial compositeness coefficients. Furthermore, the coupled-channel analyses also enable us to predict the line shapes of the distributions of the $\chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$, as shown in Fig. 2 which could provide useful guidelines for the experimental study in the next step. Future measurements on the distributions of the $\chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$ will be definitely helpful to discriminate different scenarios proposed here and to reach a more definite conclusion for the properties of the $X(6900)$ state, since the $\chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$ event distributions look very different for Fit-I and Fit-II, particularly for the former channel. The predictions for the scattering amplitudes from different fits are given in Fig. 3, where the shaded areas correspond to the error bands at the one-sigma level from the Fit-I and Fit-II in Table 3. Strong cusp effects around the $\chi_c^0\chi_c^0$ threshold are clearly seen in all the scattering amplitudes. In the Fit-I case, the resonant enhancements obviously show up in the amplitudes of $J/\psi J/\psi \rightarrow J/\psi J/\psi, \chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$, while in the transition amplitudes $\chi_{0,1}\chi_{0,1} \rightarrow \chi_{0,1}\chi_{0,1}$ the resonance manifests as a dip. For the Fit-II case, except the $J/\psi J/\psi \rightarrow J/\psi J/\psi$, the resonance barely shows any structure in the scattering amplitudes, which seems consistent with the rather large widths from the Fit-II. The global fits by including the di-$J/\psi$ distribution data in the lower energy region could be crucial to discriminate the two scenarios from Fit-I and Fit-II.

4 Conclusions

In this work we focus on the narrow peak around 6.9 GeV observed in the $J/\psi J/\psi$ event distributions from the LHCb measurements [1], which is the first discovered fully-heavy-flavored tetraquark candidate. Three different theoretical approaches, including the effective range expansion, the combination of the compositeness relation and the saturation of the decay width, and the unitarized phenomenological amplitudes, are used to investigate the $X(6900)$ state. It is remarkable that different theoretical methods lead to similar conclusions for the $X(6900)$: the $J/\psi J/\psi, \chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$ components do not play dominant roles in the $X(6900)$. The most important component should be a bare or elementary one, e.g. as a compact four-charm-quark state or other microscopic degrees of freedom. We have also provided the resulting pole positions, couplings to the different channels and partial as well as total compositeness coefficients.

Our sophisticated coupled-channel study confirms that two types of different resonance poles can be obtained for the $X(6900)$, by taking the two estimates of the background contributions from the LHCb [1]. Different line shapes for the $\chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$ distributions are predicted. Future experimental measurements on the $\chi_c^0\chi_c^0$ and $\chi_c^1\chi_c^1$ are shown to be definitely helpful to further pin down the properties of the $X(6900)$.

Acknowledgements

We thank Feng-Kun Guo for useful discussions. This work is funded in part by the Natural Science Foundation of China under Grant Nos. 11975090 and 11575052, the Natural Science
Foundation of Hebei Province under Contract No. A2015205205, and the MINECO (Spain) and FEDER (EU) grants FPA2016-77313-P and MICINN (Spain) PID2019-106080GB-C22.

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