**On Non-uniqueness of Continuous Entropy Solutions to the Isentropic Compressible Euler Equations**

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**Abstract**

We consider the Cauchy problem for the isentropic compressible Euler equations in a three-dimensional periodic domain under general pressure laws. For any smooth initial density away from the vacuum, we construct infinitely many entropy solutions with no presence of shock. In particular, the constructed density is smooth and the momentum is $\alpha$-Hölder continuous for $\alpha < 1/7$. Also, we provide a continuous entropy solution satisfying the entropy inequality strictly.

**1. Introduction**

We consider the Euler equations for an isentropic compressible fluid on the spatially periodic domain $[0, T] \times \mathbb{T}^3$, $\mathbb{T}^3 = [-\pi, \pi]^3$ and $T \in (0, \infty)$ with initial conditions

$$
\begin{align*}
\partial_t \varrho + \text{div}(\varrho u) &= 0 \\
\partial_t (\varrho u) + \text{div} (\varrho u \otimes u) + \nabla[p(\varrho)] &= 0 \\
(\varrho, u)|_{t=0} &= (\varrho_0, u_0).
\end{align*}
$$

The two unknowns are the mass density $\varrho : [0, T] \times \mathbb{T}^3 \rightarrow [0, \infty)$ of the fluid (or gas) and its velocity $u : [0, T] \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$. In this paper, we consider mass densities bounded below by a positive constant (i.e. that is no formation of vacuum). The equations express the conservation laws of mass and linear momentum, respectively, and are called the continuity equation and the momentum equation. The pressure $p$ is given as a function of the density, $p = p(\varrho)$. A typical example of the pressure is given by the polytropic pressure law $p(\varrho) = \kappa \varrho^\gamma$ for $\kappa > 0$ and $\gamma > 1$, but various pressure laws appear in the study of real gases and complex fluids, see [14,39]. In particular, the equations (1.1) are a hyperbolic system when $p'(\varrho) > 0$.

For general hyperbolic system, the unique existence of a smooth solution for short time is well-known [32,34]. As we see from the one-dimensional example, the
Burgers equations, however, a smooth solution develops discontinuity in finite time. In an attempt to continue the solution after the singularity occurs, weak solutions in $L^\infty_{t,x}$ have been considered, but they are known to be non-unique. This leads the Lax entropy inequalities as a selection principle and they play a successful role in the Burgers equations. Similarly, the isentropic Euler equations (1.1) have many smooth solutions with finite-time blow-up. To single out physically relevant solutions among bounded weak solutions, the entropy inequality is imposed as

$$\partial_t \left( \frac{\varrho |u|^2}{2} + \varrho e(\varrho) \right) + \text{div} \left( \left( \frac{\varrho |u|^2}{2} + \varrho e(\varrho) + p(\varrho) \right) u \right) \leq 0. \tag{1.2}$$

Here, $e : \mathbb{R} \to \mathbb{R}$ denotes the specific internal energy, which is related to pressure $p(\varrho)$ through $\varrho^2 e'(\varrho) = p(\varrho)$. We often use $P(\varrho) := \varrho e(\varrho)$ instead, called pressure potential, which satisfies $\varrho P'(\varrho) = P(\varrho) + p(\varrho)$. Since the equality demonstrates the energy conservation law, (1.2) is also called the energy inequality. Indeed, $\varrho |u|^2/2 + \varrho e(\varrho)$ represents the total energy density, the sum of the kinetic energy density and the internal energy density. We call weak solutions $(\varrho, u)$ in $L^\infty_{t,x}$ which solve (1.1) and satisfy (1.2) in distribution sense as entropy solutions. See [15, 33] for further discussion on general hyperbolic systems.

Entropy solutions have a weak-strong uniqueness principle when $p'(\varrho) > 0$; if there exists a short-time classical solution (strong solution) to (1.1) then any entropy solutions (weak solutions) with the same initial data must coincide with it, [15, 25, 40]. However, it turns out that the entropy inequality (1.2) cannot serve as a selection principle. After conjectured by Elling [23], De Lellis and Székelyhidi [18] established the first non-uniqueness result, finding a bounded initial data which has infinitely many entropy solutions on $(0, \infty) \times \mathbb{R}^d$, $d \geq 2$. This was extended in the works of Chiodaroli [10] allowing regular initial density $\varrho_0$. In the special case of $d = 2$ and $p(\varrho) = \varrho^2$, the non-uniqueness results were obtained for some Lipschitz initial data in [11] and even with smooth initial data in [12]. Indeed, the constructed solution coincides with the classical solution for finite time but collapsed into a perturbed Riemann state. Also, in the case of $d = 2$ and under the polytropic pressure law, Markfelder and Klingenberg [36] showed the non-uniqueness that encompasses a large variety of shocks and rarefaction waves. For a more comprehensive survey of these results, we refer the reader to [35]. Recently, in the case of $p(\varrho) = \varrho^\gamma$, $1 < \gamma < 1 + \frac{2}{d}$, $d = 2, 3$, Chen et al. [9] obtained a dense subset of the energy space such that any initial data in the set generates infinitely many weak solutions on $(0, \infty) \times \mathbb{T}^d$ with no increment of the average of the total energy on the torus.

All the previous non-uniqueness results for entropy solutions produce merely bounded solutions. The paper, on the other hand, provides infinite many solutions even without the presence of discontinuities. The constructed solutions furthermore satisfy the energy equation. The theorem is stated for (1.1) formulated in terms of

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1 For discussions on different-type of admissible conditions such as mixing entropy or maximal entropy production, see [7, 8] for example.
the density and the linear momentum \( m = \varrho u \),

\[
\begin{aligned}
&\partial_t \varrho + \text{div} \ m = 0 \\
&\partial_t m + \text{div} \left( \frac{m \otimes m}{\varrho} \right) + \nabla p(\varrho) = 0 \\
&\varrho(0, \cdot) = \varrho_0(\cdot), \quad m(0, \cdot) = \varrho_0(\cdot) u_0(\cdot),
\end{aligned}
\tag{1.3}
\]

and the corresponding energy inequality,

\[
\partial_t \left( \frac{|m|^2}{2\varrho} + \varrho e(\varrho) \right) + \text{div} \left( \frac{m}{\varrho} \left( \frac{|m|^2}{2\varrho} + \varrho e(\varrho) + p(\varrho) \right) \right) \leq 0. \tag{1.4}
\]

**Theorem 1.1.** For any \( 0 \leq \beta < 1/7 \), initial density \( \varrho_0 \in C^\infty(\mathbb{T}^3) \) with \( \varrho_0 \geq \varepsilon_0 \) for some positive constant \( \varepsilon_0 \), and pressure \( p \in C^\infty([\varepsilon_0, \infty)) \), there exists an initial data \( m_0 \in C^\beta(\mathbb{T}^3) \) such that we can find infinitely many distinct entropy solutions, \( \varrho \in C^\infty([0, T] \times \mathbb{T}^3) \) and \( m \in C^\beta([0, T] \times \mathbb{T}^3) \), to the isentropic compressible Euler equations (1.3) satisfying the energy equation

\[
\partial_t \left( \frac{|m|^2}{2\varrho} + \varrho e(\varrho) \right) + \text{div} \left( \frac{m}{\varrho} \left( \frac{|m|^2}{2\varrho} + \varrho e(\varrho) + p(\varrho) \right) \right) = 0 \tag{1.5}
\]

in a distribution sense.

With a suitable modification of our arguments, one can also produce an analogous example of infinitely many entropy solutions with the same initial data that produce entropy.

Another aspect of the entropy inequality (1.4) is the energy balance law. Any smooth solutions to (1.3) conserve the energy locally for any time, satisfying (1.5). The conservation is still valid when a weak solution \((\varrho, m)\) satisfies \( \varrho \in L^3(0, T; B^\infty_{3, \infty}(\mathbb{T}^3)) \cap L^\infty([0, T] \times \mathbb{T}^3) \), \( \varrho \geq \varepsilon_0 > 0 \), \( m \in L^3(0, T; B^\infty_{3, \infty}(\mathbb{T}^3)) \) for \( \alpha > 1/3 \) and \( p \in C^2(\varepsilon_0, \|\varrho\|_{L^\infty}) \), see [24,26] (we also refer to [22] for the conservation in the Euler system). The result is sharp because shock solutions possess spatial regularity \( B^{1/3}_{3, \infty} \) (see [24]), yet even in the absence of shocks, an entropy solution can dissipate the total energy.

**Theorem 1.2.** For any \( 0 \leq \beta < 1/7 \), \( \varrho_0 \in C^\infty(\mathbb{T}^3) \) with \( \varrho_0 \geq \varepsilon_0 \) for some positive constant \( \varepsilon_0 \), and \( p \in C^\infty([\varepsilon_0, \infty)) \), there exists an initial data \( m_0 \in C^\beta(\mathbb{T}^3) \) such that there is an entropy solution \((\varrho, m) \in C^\infty([0, T] \times \mathbb{T}^3) \times C^\beta([0, T] \times \mathbb{T}^3) \) to the isentropic compressible Euler equations (1.3) such that \((\varrho, m)\) satisfies the entropy inequality (1.4) strictly in distribution sense.

**Remark 1.3.** The constructed solution presents a distinct mechanism from shock solutions. While the entropy production measure (the left hand side of (1.4)) of a shock is supported on a neighborhood of the shock front, that of the constructed solution remains strictly negative on whole domain \([0, T] \times \mathbb{T}^3 \).
Remark 1.4. The constructed solution satisfies the total energy/entropy dissipation
\[
\int_{T^3} \frac{|m(t, x)|^2}{2\rho(t, x)} + \varrho(t, x)e(\varrho(t, x)) \, dx < \int_{T^3} \frac{|m_0|^2}{2\rho_0} + \varrho_0e(\varrho_0) \, dx, \quad \forall t \in (0, T)
\] (1.6)
where \((\varrho_0, m_0)\) denotes the initial data of the solution \((\varrho, m)\).

The proof is relying on the convex integration scheme starting from De Lellis and Székelyhidi [19,20], used with great success in proving the longstanding open Onsager Conjecture [38] for incompressible Euler equations. Indeed, the full conjecture is established by Isett [29] (see also [6]), after a series of developments [1–5,16,20,21,27,30]. In the effort of a rigorous mathematical validation of the classical Kolmogorov’s theory of turbulence, a stronger version of Onsager conjecture has been introduced, namely for \(\alpha < 1/3\) the existence of \(\alpha\)-Hölder continuous weak solutions of the incompressible Euler equations satisfying the local energy inequality strictly,
\[
\partial_t \frac{|u|^2}{2} + \text{div} \left( u \left( \frac{|u|^2}{2} + p \right) \right) < 0.
\] (1.7)
Building upon [18,28], De Lellis and the second author in [17] obtain the result up to the threshold 1/7, which is the same as the threshold exponent 1/7 in Theorem 1.2. Indeed, we adapt the convex integration used in [17] to the compressible case, using the structural similarity between the entropy inequality (1.2) and the local energy inequality for the incompressible Euler equations.

2. Outline of the Proof

We construct entropy solutions approximations with sequences of dissipative Euler-Reynolds flows.

Definition 2.1. For a given \(\varrho \in C^\infty([T_1, T_2] \times \mathbb{T}^3)\) with \(\varrho \geq \epsilon_0\) for some positive constant \(\epsilon_0\), a tuple of smooth tensors \((m, c, R, \varphi)\) is a dissipative Euler-Reynolds flow with global energy loss \(E = E(t, x)\) if it solves the system
\[
\begin{align*}
\partial_t \varrho + \text{div} \, m &= 0 \\
\partial_t m + \text{div} \left( \frac{m \otimes m}{\varrho} \right) + \nabla p(\varrho) &= \text{div}(\varrho(R - c \text{Id})) \\
\partial_t \left( \frac{|m|^2}{2\varrho} + P(\varrho) \right) + \text{div} \left( \frac{m}{\varrho} \left( \frac{|m|^2}{2\varrho} + \varrho P'(\varrho) \right) \right) &= \varrho \left( \partial_t + \frac{m}{\varrho} \cdot \nabla \right) \frac{1}{2} \text{tr}(R) + \text{div}((R - c \text{Id})m) + \text{div}(\varrho \varphi) + \partial_t E
\end{align*}
\] (2.1)
in distribution sense, where \((\text{div} S)_i = \partial_j S_{ij}\) and \text{Id} is the identity matrix. To be consistent with the term dissipative, we assume that \(\partial_t E \leq 0\).
Compared with a dissipative Euler-Reynolds flow for the incompressible Euler equations introduced in [17, 28], the counterpart for the compressible Euler equations has the constant \( c \). When the fluid is incompressible (\( \rho = \text{const.} \)), the terms involved with \( c \) vanish, while in the compressible case, the constant \( c \) plays an important role in cancelling the Reynolds stress from the previous step and estimating the new unresolved current error in the convex integration scheme.

2.1. Induction scheme

At each \( q \)th step, we construct a dissipative Euler-Reynold flow \((m_q, c_q, R_q, \varphi_q)\) with some fixed global energy loss \( E = E(t) \), where \( c_q \) is prescribed and \((c_q, R_q, \varphi_q)\) converges to 0 in \( \mathbb{R} \times C^0([0, T] \times \mathbb{T}^3) \times C^0([0, T] \times \mathbb{T}^3) \) as \( q \) goes to infinity. Then, we see that limit solution will solve (1.3), (1.4) in distribution sense.

More precisely, for \( q \in \mathbb{N} \cup \{0\} \) we introduce the frequency \( \lambda_q \) and the amplitude \( \delta_{\frac{1}{2}q} \) of the momentum \( m_q \), which have the form

\[
\lambda_q = [\lambda_0^{(b^q)}], \quad \delta_q = \lambda_q^{-2\alpha},
\]

where \( \alpha \) is a positive parameter smaller than 1 and \( b, \lambda_0 \) are real parameters larger than 1 (however, while \( b \) will be typically chosen close to 1, \( \lambda_0 \) will be typically chosen very large). In particular, \( \lambda_q \delta_{\frac{1}{2}q} \) is a monotone increasing sequence. We also set \( c_q \) as

\[
c_q = \sum_{j=q+1}^{\infty} \delta_j.
\]

At \((q + 1)\)th step, we then find a correction \( n_{q+1} := m_{q+1} - m_q \) to make the error \((R_q, \varphi_q)\) get smaller (in \( C^0 \) space) as \( q \) goes to infinity.

In the induction hypothesis, we will assume several estimates on the tuple \((m_q, R_q, \varphi_q)\). For technical reasons, the domains of definition of the tuples is changing at each step and it is given by \([-\tau_{q-1}, T + \tau_{q-1}] \times \mathbb{T}^3\), where \( \tau_{-1} = (\lambda_0 \delta_{\frac{1}{2}0})^{-1} \) and for \( q \geq 0 \) the parameter \( \tau_q \) is defined by

\[
\tau_q = \left( C_\rho M \lambda_q^{\frac{1}{2}} \lambda_q^{\frac{1}{2}} \delta_{\frac{1}{2}q} \delta_{\frac{1}{2}q+1} \right)^{-1}
\]

for some constant \( C_\rho \) depending only on \( \rho \) and a constant \( M = M(\rho, p) \) depending only on \( \rho \) and \( p \); they will be specified later in (3.9) and Proposition 2.3, respectively. Note the important fact that \( \tau_q \) is decreasing in \( q \) for the choice of sufficiently large \( \lambda_0 \). In order to shorten our formulas, it is convenient to introduce the following notation:

- Given an interval \( \mathcal{I} = [a, b] \), \( |\mathcal{I}| \) means its length \((b - a)\), and \( \mathcal{I} + \sigma \) denotes the concentric enlarged interval \((a - \sigma, b + \sigma)\).
- Given a function \( f \) on \([0, T] \times \mathbb{T}^3\), \( \text{supp}_t(f) \) denotes its temporal support,

\[
\text{supp}_t(f) := \{ t : \exists x \text{ with } f(t, x) \neq 0 \}.
\]
functions\( (2.2) \)

for the same energy loss \( E \) which satisfies\( (2.2) \)

\[
\| \bar{F}_q \|_{N} := \| F_q \|_{C^{0}(\{0, T\} + \tau_q^{-1}; C^{N}(T^{3}))}.
\]

When \( F \) is a function of time or space only, we abuse notation and write \( \| F \|_0 = \| F \|_{C^{0}(0, T)} \) for a time function \( F \) or \( \| F \|_0 = \| F \|_{C^{0}(T^{3})} \) for a space function \( F \).

We are now ready to detail the inductive estimates

\[
\| m_q \|_0 \leq M - \delta_q^{\frac{1}{2}}, \quad \| m_q \|_{N} \leq M \lambda_q^{N} \delta_q^{\frac{1}{2}}, \quad N = 1, 2 \tag{2.2}
\]

and

\[
\| R_q \|_{N} \leq \lambda_q^{N-3y} \delta_q^{\frac{1}{2}}, \quad \| D_{t,q} R_q \|_{N-1} \leq \lambda_q^{N-3y} \delta_q^{\frac{1}{2}} \delta_q^{\frac{1}{2}}, \quad N = 0, 1, 2 \tag{2.3}
\]

\[
\| \varphi_q \|_{N} \leq \lambda_q^{N-3y} \delta_q^{\frac{3}{2}}, \quad \| D_{t,q} \varphi_q \|_{N-1} \leq \lambda_q^{N-3y} \delta_q^{\frac{3}{2}} \delta_q^{\frac{3}{2}}, \quad N = 0, 1, 2 \tag{2.4}
\]

where \( D_{t,q} = \partial_t + \frac{m_q}{\bar{q}} \cdot \nabla \) and \( \gamma = (b-1)^2 \) and constant \( M = M(q, p) \geq 1 \) will be determined in Section 2.2.

**Remark 2.2.** Note that in a writing like (2.3) and (2.4), for \( N = 0 \) we are not claiming any negative Sobolev estimate on \( D_{t,q} F \): the reader should just consider the advective derivative estimate to be an empty statement when \( N = 0 \). The reason for this convention is just to make the notation easier, as we do not have to state in a separate line the estimate for \( \| F \|_0 \) in many future statements.

Notice that (2.1) is invariant under addition to \( E \) of a constant and we adopt the normalization condition \( E(0) = 0 \). We will therefore assume that

\[
E(0) = 0, \quad E' \leq 0. \tag{2.5}
\]

Under this setting, the core inductive proposition is given as follows:

**Proposition 2.3.** (Inductive proposition) Let \( \alpha \in (0, 1/7) \) and let \( q \in C^{\infty}([-\tau_{-1}, T + \tau_{-1}] \times T^{3}) \) be a function with \( q \geq \varepsilon_0 \) for some positive constant \( \varepsilon_0 > 0 \) and \( \frac{d}{dt} \int_{T^{3}} q(t, x) dx = 0 \) for all \( t \). There exists a constant \( M = M(q) > 1 \), functions \( \tilde{b}(\alpha) > 1 \) and \( \Lambda_0 = \Lambda_0(\alpha, b, M, q, p) > 0 \) such that the following property holds. Let \( b \in (1, \tilde{b}(\alpha)) \) and \( \lambda_0 \geq \Lambda_0 \) and let \( c_q = \sum_{j=q+1}^{\infty} \delta_j \) for any \( q \in \mathbb{N} \).

Assume that a tuple of tensors \((m_q, c_q, R_q, \varphi_q)\) is a dissipative Euler- Reynolds flow defined on the time interval \([0, T] + \tau_{q-1}\) satisfying (2.2)-(2.4) for an energy loss \( E \) satisfying (2.5). Then, we can find a corrected dissipative Euler- Reynolds flow \((m_{q+1}, c_{q+1}, R_{q+1}, \varphi_{q+1})\) with prescribed \( c_{q+1} \) on the time interval \([0, T] + \tau_{q} \) for the same energy loss \( E \) which satisfies (2.2)-(2.4) for \( q + 1 \) and

\[
\| m_{q+1} - m_q \|_{C^{0}(\{0, T\} \times T^{3})} + \frac{1}{\lambda_q} \| m_{q+1} - m_q \|_{C^{0}(\{0, T\}; C^{1}(T^{3}))} \leq M \delta_q^{\frac{1}{2}}. \tag{2.6}
\]

While the latter proposition would be enough to prove Theorem 1.2, we will indeed need a technical refinement in order to show Theorem 1.1.
Remark 2.4. The assumption \( \frac{d}{dt} \int_{\mathbb{T}^3} \varrho(t, x) \, dx = 0 \) for all \( t \) is a compatibility condition of the continuity equation.

Remark 2.5. The constant \( M \) depends on an upper bound of \( \| \varrho \|_{C(\{0, T\} + \tau_{-1}; C^2(\mathbb{T}^3))} \) and \( \varepsilon_0 \) more precisely. Also, the dependence of \( \Lambda_0 \) on \( \varrho \) and \( p \) means the dependence on \( \varepsilon_0 \) and upper bounds of \( \| \varrho \|_{C(\{0, T\} + \tau_{-1}; C^{n_0+1}(\mathbb{T}^3))} \), \( \| \partial_t \varrho \|_{C(\{0, T\} + \tau_{-1}; C^{n_0+1}(\mathbb{T}^3))} \) and \( \| p \|_{C^{n_0+1}(\mathbb{T}^3)} \), where \( n_0 = \lceil \frac{2b(2b+\alpha)}{(b-1)(1-\alpha)} \rceil \). The norm on \( p \) can be weakened, but we don’t pursue that direction in this paper.

Proposition 2.6. (Bifurcating inductive proposition). Let \( \varrho \in C^\infty \left( (-\tau_{-1}, T + \tau_{-1}] \times \mathbb{T}^3 \right) \) be a function with \( \varrho \geq \varepsilon_0 \) for some positive constant \( \varepsilon_0 \) and \( \frac{d}{dt} \int_{\mathbb{T}^3} \varrho(t, x) \, dx = 0 \) for all \( t \). Let a constant \( M \), the functions \( \tilde{\varrho} \) and \( \Lambda_0 \), the parameters \( \alpha, b, \lambda_0 \) and the tuple \((m_q, c_q, R_q, \varphi_q)\) be as in the statement of Proposition 2.3. For any time interval \( I \subset (0, T) \) with \( |I| \geq 3\tau_q \) we can produce a first tuple \((m_{q+1}, c_{q+1}, R_{q+1}, \varphi_{q+1})\) and a second one \((\tilde{m}_{q+1}, c_{q+1}, \tilde{R}_{q+1}, \tilde{\varphi}_{q+1})\) which share the same initial data, satisfy the same conclusions of Proposition 2.3 and additionally

\[
\|m_{q+1} - \tilde{m}_{q+1}\|_{C^0(\{0, T\}; L^2(\mathbb{T}^3))} \geq \varepsilon_0 \delta_{q+1}^\frac{1}{2}, \quad \text{supp}_t(m_{q+1} - \tilde{m}_{q+1}) \subset I. \quad (2.7)
\]

Furthermore, if we are given two tuples \((m_q, c_q, R_q, \varphi_q)\) and \((\tilde{m}_q, c_q, \tilde{R}_q, \tilde{\varphi}_q)\) satisfying (2.2)–(2.4) and

\[
\text{supp}_t(m_q - \tilde{m}_q, R_q - \tilde{R}_q, \varphi_q - \tilde{\varphi}_q) \subset \mathcal{J}
\]

for some interval \( \mathcal{J} \subset (0, T) \), we can exhibit corrected counterparts \((m_{q+1}, c_{q+1}, R_{q+1}, \varphi_{q+1})\) and \((\tilde{m}_{q+1}, c_{q+1}, \tilde{R}_{q+1}, \tilde{\varphi}_{q+1})\) again satisfying the same conclusions of Proposition 2.3 together with the following control on the support of their difference:

\[
\text{supp}_t(m_{q+1} - \tilde{m}_{q+1}, R_{q+1} - \tilde{R}_{q+1}, \varphi_{q+1} - \tilde{\varphi}_{q+1}) \subset \mathcal{J} + (\lambda_q \delta_q^\frac{1}{2})^{-1}. \quad (2.8)
\]

2.2. Construction of a Starting Tuple with Stationary Density

For the simplicity, we choose stationary density \( \varrho(t, \cdot) = \varrho_0 \) for all \( t \in \mathbb{R} \). In order to construct a starting momentum, unlike the incompressible case, we need to cancel out the prescribed pressure. To this end, we use Mikado flows as building-blocks, which are stationary solutions of the incompressible Euler equation first introduced in [16]. In order to define them, consider a function \( \vartheta \) on \( \mathbb{R}^2 \) and let \( \tilde{U}(x) = e_3 \vartheta(x_1, x_2) \), where \( e_3 = (0, 0, 1) \). Then, we apply a stretching factor \( s > 0 \), a general rotation \( O \), and a translation by a vector \( \tilde{x} \) to define

\[
U(x) = sO\tilde{U}(O^{-1}(x - \tilde{x})).
\]

Observe that the periodization of this function solves the stationary incompressible Euler equations on \( \mathbb{T}^3 \) when \( Oe_3 \) belongs to \( a\mathbb{Q}^3 \) for some \( a > 0 \). From now on, with a slight abuse of our terminology, a Mikado flow always refers to such
periodization. Moreover the vector \( f = sOe_3 \) will be, without loss of generality assumed to belong to \( \mathbb{Z}^3 \) and will be called the direction of the Mikado flow, while \( \bar{x} \) will be called its shift. For each \( f \) we will specify an appropriate choice of \( \vartheta \), which will be smooth and compactly supported in a disk \( B(0, r_0) \) for some small \( r_0 > 0 \). Also, \( \vartheta \) will not depend on the shift \( \bar{x} \) and we will denote by \( U_f \) the corresponding Mikado flows when \( \bar{x} = 0 \). The Mikado flows then can be written as \( U_f = f \psi_f \) for some smooth \( \psi_f \in C_c^\infty(\mathbb{R}^3) \) with \( f \cdot \nabla \psi_f = 0 \). We recall the following elementary lemma, used since the pioneering work [16].

**Lemma 2.7.** Let \( \mathcal{F} \) be a set of vectors in \( \mathbb{Z}^3 \) with finite cardinality. For each \( f \in \mathcal{F} \), let \( \bar{x}(f) \in \mathbb{R}^3 \), \( \gamma_f \in \mathbb{R} \) and \( \lambda \in \mathbb{N} \). If the supports of the maps \( U_f(\cdot - \bar{x}(f)) \) are pairwise disjoint, then

\[
\sum_{f \in \mathcal{F}} \gamma_f U_f(\lambda(x - \bar{x}(f)))
\]

is a stationary solution of the incompressible Euler equations on \( \mathbb{T}^3 \).

Note that the supports of the functions \( U_f(\cdot - \bar{x}) \) and \( \psi_f(\cdot - \bar{x}) \) are contained in a \( r_0 \)-neighborhood of

\[
I_f + \bar{x} := \left\{ x \in \mathbb{T}^3 : (x - \sigma f - \bar{x}) \in 2\pi \mathbb{Z}^3 \text{ for some } \sigma \in \mathbb{R} \right\}.
\]

(2.9)

If \( r_0 \) is sufficiently small, depending on \( f \), the latter is a “thin tube” winding around the torus a finite number of time.

Another ingredient for defining the starting errors \( R_0 \) and \( \varphi_0 \) is the inverse divergence operator introduced in [20].

**Definition 2.8.** *(Inverse divergence operator).* For any \( f \in C^\infty(\mathbb{T}^3; \mathbb{R}^3) \), the inverse divergence operator is defined by

\[
(R f)_{ij} = \mathcal{R}_{ijk} f_k = -\frac{1}{2} \Delta^{-2} \partial_{ijk} f_k + \frac{1}{2} \Delta^{-1} \partial_k f_k \delta_{ij} - \Delta^{-1} \partial_i f_j - \Delta^{-1} \partial_j f_i.
\]

**Remark 2.9.** The image of the divergence free operator \( R f(x) \) is designed to be a trace-free symmetric matrix at each point \( x \) and to solve

\[
div(R f) = f - \langle f \rangle.
\]

To define \( \varphi_0 \), we abuse the notation and define an inverse divergence operator on \( C^\infty(\mathbb{T}^3; \mathbb{R}) \),

\[
(R g)_{i} = \Delta^{-1} \partial_i g.
\]

Indeed, that maps a smooth scalar function to a vector-valued function, and solves \( \div R g = g - \langle g \rangle \).
We are now ready to find a starting tuple. Set $\rho(t, x) = \varrho_0(x)$ for all $t \in \mathbb{R}$, so that $\partial_t \rho = 0$ and $\text{div} m_0 = 0$. By the choice of $\varrho$, one can fix $M, \alpha, b$, and $\lambda_0$ by Propositions 2.3 and 2.6. We also set the preponderant part $\bar{m}_0$ of $m_0$ as

$$\bar{m}_0 = \varrho^{\frac{1}{2}} (C_{\varrho, p} - p(\varrho) - c_0 \varrho)^{\frac{1}{2}} \sum_{i=1}^{3} \psi_i(\bar{x} x) e_i,$$

where $C_{\varrho, p}$ is a positive constant defined by $C_{\varrho, p} = 2(\| p(\varrho) \|_0 + c_0 \| \varrho \|_0)$, and $e_i, i = 1, 2, 3$, are standard unit vectors whose $i$th component is 1. Also, $\psi_i = \psi_{e_i}(-\bar{x}_i)$ is associated functions to each Mikado direction $e_i$, which is compactly supported and satisfies $e_i \cdot \nabla \psi_i = 0$ and

$$\int_{T^3} \psi_i dx = \int_{T^3} \psi_i^3 dx = 0, \quad \int_{T^3} \psi_i^2 dx = 1.$$

Suitable shifts $\bar{x}_i$ are chosen to have pairwise disjoint supp($\psi_i$). We remark that $\partial_t \bar{m}_0 = 0$ because $\varrho$ is time-independent. Since we have

$$\frac{\bar{m}_0 \otimes \bar{m}_0}{\varrho} = (C_{\varrho, p} - p(\varrho) - c_0 \varrho) \text{Id} + (C_{\varrho, p} - p(\varrho) - c_0 \varrho) \sum_{i=1}^{3} \psi_i^2(\bar{x} x) - 1) e_i \otimes e_i,$$

its divergence satisfies

$$\text{div} \left( \frac{\bar{m}_0 \otimes \bar{m}_0}{\varrho} \right) + \nabla (p(\varrho) + c_0 \varrho) = \sum_{i=1}^{3} \partial_i (p(\varrho) + c_0 \varrho)(\psi_i^2(\bar{x} x) - 1) e_i.$$

(2.10)

The last equation follows from $\text{div}((\psi_i^2(\bar{x} x) - 1) e_i \otimes e_i) = 0$ because of $e_i \cdot \nabla \psi_i = 0$. Since $\psi_i$ is a smooth periodic function on $T^3$ with zero-mean, one can represent it as its Fourier series $\psi_i = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \tilde{b}_{i,k} e^{i k \cdot x}$. Note that the divergence-free condition of $\psi_i e_i$ implies $\tilde{b}_{i,k}(e_i \cdot k) = 0$ for all non-zero $k \in \mathbb{Z}^3$. Using this condition, one can write $\bar{m}_0$ as

$$\bar{m}_0 = \sum_{i=1}^{3} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \varrho^{\frac{1}{2}} (C_{\varrho, p} - p(\varrho) - c_0 \varrho)^{\frac{1}{2}} \text{curl} \left( \frac{i \tilde{b}_{i,k} e^{i k \cdot x}}{i \lambda |k|^2} e^{i x \bar{x} k} \right).$$

Then, adding a correction

$$\nu = \sum_{i=1}^{3} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \nabla (\varrho^{\frac{1}{2}} (C_{\varrho, p} - p(\varrho) - c_0 \varrho)^{\frac{1}{2}}) \times \left( \frac{i \tilde{b}_{i,k} e^{i k \cdot x}}{i \lambda |k|^2} e^{i x \bar{x} k} \right),$$

we get a divergence-free, mean-zero, approximate momentum $m_0 = \bar{m}_0 + \nu$.

Now, we choose a Reynold stress error $R_0$. To make it solve the relaxed momentum equation in (2.1),...
\[
\text{div}(\varrho R_0) = \text{div} \left( \frac{m_0 \otimes m_0}{\varrho} \right) + \nabla (p(\varrho) + c_0 \varrho)
\]
\[
= -\sum_{i=1}^{3} \partial_i (p(\varrho) + c_0 \varrho)(\psi_i^2(\vec{x}, t) - 1)e_i + \text{div} \left( \frac{1}{q} (\bar{m}_0 \otimes v + v \otimes \bar{m}_0 + v \otimes v) \right),
\]

we set
\[
\varrho R_0 := \mathcal{R} \left( -\sum_{i=1}^{3} \partial_i (p(\varrho) + c_0 \varrho)(\psi_i^2(\vec{x}, t) - 1)e_i \right) + \bar{m}_0 \otimes v + v \otimes \bar{m}_0 + v \otimes v
\]
\[
=\frac{2}{3} E(t) \text{Id}
\]
\[
=: \varrho \bar{R}_0 - \frac{2}{3} E(t) \text{Id}.
\]

We remark that the argument of the inverse divergence operator is mean-zero (see (2.11)), so that \( \text{div}(\varrho R_0) \) is the argument itself. Also, \( E \) is a global energy loss depending only on time, which will be specified later. Note that \( \bar{R}_0 \) is independent of time, so that \( \partial_t (\varrho \kappa_0) = -E' \), where \( \kappa_0 = \text{tr}(R_0)/2 \).

Lastly, we choose an unresolved current error \( \varphi_0 \). Since both \( m_0 \) and \( \varrho \) are time independent, it is enough to find a solution \( \varphi_0 \) to

\[
\text{div}(\varrho \varphi_0) = \text{div} \left( \frac{m_0 \otimes m_0}{\varrho} \left( \frac{|m_0|^2}{2 \varrho} + \varrho P'(\varrho) \right) \right) - \partial_t (\varrho \kappa_0) - \text{div}(\kappa_0 m_0) - \text{div}(R_0 m_0) - E',
\]

where we used \( \varrho \left( \partial_t + \frac{m_0}{\varrho} \cdot \nabla \right) \kappa_0 = \partial_t (\varrho \kappa_0) + \text{div}(\kappa_0 m_0) \) and \( \text{div}(c_0 m_0) = 0 \).

Since we have \( \partial_t (\varrho \kappa_0) = -E' \), we set
\[
\varrho \varphi_0 := \mathcal{R} \left( \sum_{i=1}^{3} \partial_i \left( \frac{(C_{\varrho, p} - p(\varrho) - c_0 \varrho)^{\frac{3}{2}}}{2 \varrho} \right) \psi_i^3(\vec{x}, t)e_i \right) + \left( \frac{|m_0|^2 m_0}{2 \varrho} - \frac{|m_0| m_0^2}{2 \varrho} \right)
\]
\[
\quad + \mathcal{R}(m_0 \cdot \nabla P'(\varrho)) - \kappa_0 m_0 - R_0 m_0.
\]

Here we used \( e_i \cdot \nabla \psi_i^3 = 0 \), and hence,
\[
\text{div} \left( \frac{\bar{m}_0}{\varrho} \left( \frac{|m_0|^2}{2 \varrho} \right) \right) = \text{div} \left( \frac{(C_{\varrho, p} - p(\varrho) - c_0 \varrho)^{\frac{3}{2}}}{2 \varrho} \sum_{i=1}^{3} \psi_i^3(\vec{x}, t)e_i \right)
\]
\[
= \sum_{i=1}^{3} \partial_i \left( \frac{(C_{\varrho, p} - p(\varrho) - c_0 \varrho)^{\frac{3}{2}}}{2 \varrho} \right) \psi_i^3(\vec{x}, t)e_i.
\]

Then, \( (m_0, c_0, R_0, \varphi_0) \) solves (2.1), and can be estimated as follows:
\[ \|m_0\|_0 \leq \|\bar{m}_0\|_0 + \|v\|_0 \leq \frac{1}{4} M(\varrho, p) \left(1 + \lambda_1^{-1}\right) \leq M - \delta_0^{\frac{1}{2}} \]
\[ \|m_0\|_N \leq \|\bar{m}_0\|_N + \|v\|_N \lesssim_{\varrho, p} \bar{\lambda}^N \quad \forall N = 1, 2 \]

for some \( M(\varrho, p) \geq 2 \), and
\[ \|R_0\|_N \lesssim_{\varrho, p} \bar{\lambda}^{N-1} + \|E\|_0, \quad \|\varphi_0\|_N \lesssim_{\varrho, p} \bar{\lambda}^{N-1} + \|E\|_0 \bar{\lambda}^N \]
\[ \|D_{t,q} R_0\|_{N-1} \lesssim \|\bar{\partial}_t R_0\|_{N-1} + \left\| \frac{m_0}{\varrho} \cdot \nabla R_0 \right\|_{N-1} \lesssim_{\varrho, p} \|E'\|_0 + \bar{\lambda}^{N-1}(1 + \|E\|_0) \]
\[ \|D_{t,q} \varphi_0\|_{N-1} \lesssim \|\bar{\partial}_t \varphi_0\|_{N-1} + \left\| \frac{m_0}{\varrho} \cdot \nabla \varphi_0 \right\|_{N-1} \lesssim_{\varrho, p} \|E'\|_0 \bar{\lambda}^{N-1} + \|E\|_0 \bar{\lambda}^{N-1} + \bar{\lambda}^{N-1} \]

for any \( N = 0, 1, 2 \) (the cases \( \|\cdot\|_{-1} \) are empty statements).\(^2\) Let \( C(\varrho, p) \geq 1 \) be the maximum of all implicit constants in the above inequalities. For \( \bar{b}(\alpha) \) sufficiently close to 1 and sufficiently large \( \lambda_0 \geq \Lambda_0(\alpha, b, M(\varrho), \varrho, p) \), we can always find a positive integer \( \bar{\lambda} \) to satisfy
\[ 2C(\varrho, p)\lambda_0^{3\gamma} \delta_1^{\frac{3}{2}} \leq \bar{\lambda} \leq (2C(\varrho, p))^{-1}\lambda_0^{1/2} \delta_0^{1/2}. \quad (2.12) \]

For the choice of such \( \bar{\lambda} \) and \( E = 0 \), the constructed initial approximate solution \((m_0, R_0, \varphi_0)\) satisfies \((2.2)\)–\((2.4)\), and hence serves as a starting tuple for Theorem 1.1. On the other hand, one can find a non-trivial \( E \) satisfying \((2.5)\) (moreover, \( E' < 0 \)) and
\[ 4C(\varrho, p)\|E\|_0 \leq \lambda_0^{-3\gamma} \delta_1^{\frac{3}{2}}, \quad \text{and} \quad 4C(\varrho, p)\|E'\|_0 \leq \lambda_0^{-3\gamma} \delta_1^{\frac{3}{2}} \delta_0^{\frac{3}{2}}. \quad (2.13) \]

For example,
\[ \bar{E}(t) = \frac{\lambda_0^{-3\gamma} \delta_1^{\frac{3}{2}}}{8C(\varrho, p)} (1 - \exp(-\lambda_0 \delta_0^{\frac{1}{2}} t)). \quad (2.14) \]

For such choice of \( \bar{\lambda} \) and \( \bar{E} \), the constructed initial approximate solution \((m_0, R_0, \varphi_0)\) again satisfies \((2.2)\)–\((2.4)\), which works as a starting tuple for Theorem 1.2.

\(^2\) Here and in the rest of that pages, given two quantities \( A_q \) and \( B_q \) depending on the induction parameter \( q \) we will use the notation \( A \lesssim B \) meaning that \( A \leq CB \) for some constant \( C \) which is independent of \( q \). We also use the notation \( A \lesssim_{\varrho, p} B \) to mean \( A \leq C(\varrho, p)B \) for some constant that depends on \( \varrho \) and \( p \) and is independent of \( q \). More precisely, the constant will depend on \( \varepsilon_0 \) and the \( C^N \) norms of \( \varrho \) and \( p \) for \( N \leq 2n_0 \) where \( n_0 \) is the constant defined in Corollary 7.2. In some situations we will need to be more specific and then we will explicitly the dependence of \( C \) on the various parameters involved in our arguments.
2.3. Construction of a Starting Tuple with Time-Dependent Density

The construct density for Theorems 1.1 and 1.2 can be time-dependent. In this subsection, we provide a new starting tuple \((\tilde{m}_0, \tilde{R}_0, \tilde{\varrho}_0)\) with a time-dependent density \(\varrho\) defined by perturbing the stationary density \(\varrho_0\). We first fix the numbers \(\alpha\) and \(b\) and define the functions \(M\) and \(\Lambda_0\) by Propositions 2.3 and 2.6, which gives a fixed number \(n_0\) (see Remark 2.5). We then write a time-dependent density as

\[ \varrho(t, x) := \varrho_0(x) + \varepsilon \hat{\varrho}(t, x), \quad (2.15) \]

where \(\hat{\varrho}(t, x)\) can be any smooth function such that \(\int_{\mathbb{T}^3} \partial_t \hat{\varrho} dx = 0\) on \([-1, T + 1]\) and

\[ \|\hat{\varrho}\|_{C^0([-1, T + 1]; C^N(\mathbb{T}^3))} \leq \|\varrho_0\|_{C^N(\mathbb{T}^3)}, \quad \|\partial_t \hat{\varrho}\|_{C^0([-1, T + 1]; C^N(\mathbb{T}^3))} \leq \|\varrho_0\|_{C^N(\mathbb{T}^3)} \quad \forall N \in [0, n_0 + 1]. \]

We consider \(\varepsilon \in (0, 1/2)\) to have \(|\varrho| \geq \varepsilon_0/2\). Then, \(M(\varrho)\) and \(\Lambda_0(\alpha, b, M(\varrho), \varrho, p)\) are independent of \(\varepsilon\) and \(\hat{\varrho}\) (see Remark 2.5)).

To define \((\tilde{m}_0, \tilde{R}_0, \tilde{\varrho}_0)\), we let \((m_0, R_0, \varrho_0)\) be as in the previous Section 2.2; if needed, we adjust \(\lambda_0\) to satisfy \(\lambda_0 \geq \Lambda_0(\alpha, b, M(\varrho), \varrho, p)\) and then \(\tilde{\lambda}\) to satisfy (2.12) with the replacement of \(C(\varrho, p)\) and \(\lambda_0\). We set \(\tilde{m}_0\) first as

\[ \tilde{m}_0(t, x) := m_0(x) + \hat{m}_0(t, x) \]

where \(\hat{m}_0 := -\mathcal{R} \partial_t \varrho = -\varepsilon \mathcal{R} \partial_t \hat{\varrho}\).

Since the average of \(\partial_t \hat{\varrho}\) on \(\mathbb{T}^3\) is zero and \(\text{div} m_0 = 0\), it solves \(\partial_t \varrho + \text{div} \tilde{m}_0 = 0\). Also, for sufficiently small \(\varepsilon\), we have

\[ \|\tilde{m}_0\|_0 \leq \|m_0\|_0 + \|\hat{m}_0\|_0 \leq \frac{1}{4} M(\varrho_0, p) \left(1 + 1/\lambda_0\right) + \varepsilon \|\partial_t \varrho\|_0 \leq M(\varrho, p) - \delta_0^\frac{1}{2} \]

\[ \|\tilde{m}_0\|_N \leq \|m_0\|_N + \|\hat{m}_0\|_N \lesssim_{\varrho, p} \lambda_0^N + \varepsilon \|\partial_t \varrho\|_N \lesssim \lambda_0^N, \quad \forall N = 1, 2 \]

for some \(M(\varrho, p) \geq 2\). Then, we set the initial Reynolds stress as

\[ \varrho \tilde{R}_0 = \varrho_0 R_0 + c_0(\varrho - \varrho_0) \text{Id} \]

\[ + \mathcal{R} \left[ \partial_t \tilde{m}_0 + \text{div} \left( \frac{\tilde{m}_0 \otimes \tilde{m}_0}{\varrho} - \frac{m_0 \otimes m_0}{\varrho_0} \right) + \nabla (p(\varrho) - p(\varrho_0)) \right] + \zeta \text{Id}. \]

Since we will choose \(\zeta = \zeta(t)\), \((\tilde{m}_0, c_0, \tilde{R}_0)\) solves the relaxed momentum equation in (2.1) with \(\varrho\) defined in (2.15). The purpose of \(\zeta\), on the other hand, is as follows. The relaxed energy equation in (2.1) now can be seen as an equation for \(\tilde{\varrho}_0\). Since \(\varrho(\partial_t + (\tilde{m}_0/\varrho) \cdot \nabla) \tilde{\varrho}_0 = \partial_t (\tilde{\varrho} \tilde{\varrho}_0) + \text{div} (\tilde{m}_0 \tilde{\varrho}_0),\) where \(\tilde{\varrho}_0 = \text{tr} \tilde{R}_0/2,\) we need the compatibility condition

\[ \int_{\mathbb{T}^3} \partial_t \left( \frac{\|	ilde{m}_0\|^2}{2\varrho} + P(\varrho) \right) - \partial_t (\tilde{\varrho} \tilde{\varrho}_0) - \partial_t E \; dx = 0. \quad (2.16) \]
Recalling that (2.16) holds for \((\varrho_0, m_0, R_0)\), we choose \(\zeta\) as

\[
\int_{\mathbb{T}^3} \left( \frac{|\tilde{m}_0|^2}{2\varrho_0} - \frac{|m_0|^2}{2\varrho_0} + P(\varrho) - P(\varrho_0) - \frac{3}{2} c_0(\varrho - \varrho_0) \right) dx =: \frac{3}{2} \zeta(t)
\]

to have (2.16) for \((\varrho, \tilde{m}_0, \tilde{R}_0)\). Then, we choose \(\tilde{\varphi}_0\) as

\[
\varrho \tilde{\varphi}_0 := \varrho_0 \varphi_0 + \mathcal{R} \partial_t \left( \frac{|\tilde{m}_0|^2}{2\varrho_0} - \frac{|m_0|^2}{2\varrho_0} + (P(\varrho) - P(\varrho_0)) - (\varrho \tilde{\kappa}_0 - \varrho_0 \kappa_0) \right) \nolinebreak + \left( \frac{\tilde{m}_0}{\varrho} \right) \left( \frac{|\tilde{m}_0|^2}{2\varrho_0} + \varrho P'(\varrho) \right) - \frac{m_0}{\varrho_0} \left( \frac{|m_0|^2}{2\varrho_0} + \varrho_0 P'(\varrho_0) \right) \nolinebreak - (\tilde{\kappa}_0 \tilde{m}_0 - \kappa_0 m_0) - (\tilde{R}_0 \tilde{m}_0 - R_0 m_0) + c_0(\tilde{m}_0 - m_0),
\]

so that \((\tilde{m}_0, c_0, \tilde{R}_0, \tilde{\varphi}_0)\) satisfies the relaxed energy equation and hence solves (2.1). Notice that for every \(\delta > 0\), we can find sufficiently small \(\epsilon = \epsilon(\delta) \in (0, 1/2)\) so that \(|\zeta|, |\partial_t \zeta|, |\partial^2_t \zeta| \leq \delta\). Therefore, the new starting tuple \((\tilde{m}_0, c_0, \tilde{R}_0, \tilde{\varphi}_0)\) also satisfies the inductive estimates (2.2)–(2.4) for sufficiently small \(\epsilon\); in the estimate (2.3) we use \(\partial_t \tilde{m}_0 = \partial_t \tilde{m}_0\).

### 2.4. Proof of Theorem 1.2

Fix \(\beta < \frac{1}{4}\) and choose \(\alpha \in (\beta, \frac{1}{2})\). We let the density \(\varrho(t, x) = \varrho_0(x)\) be stationary for all \(t \in \mathbb{R}\), and set \(M, \bar{b}(\alpha), \) and \(\Lambda_0\) as in Proposition 2.3. For the non-trivial global energy loss \(E\) defined by (2.14), which is a function of time, the constructed starting tuple \((m_0, R_0, \varphi_0)\) in Section 2.2 solves (2.1) for \(c = c_0\) and satisfies (2.2)–(2.4), by adjusting \(\bar{b}(\alpha)\) and \(\Lambda_0\) if necessary.

Now, we apply Proposition 2.3 iteratively to produce a sequence of approximate solutions \((m_q, R_q, \varphi_q)\), which solves (2.1) with the energy loss \(E\) and \(c = c_q\), and satisfies (2.2)–(2.4). Since the sequence \(\{m_q\}\) also satisfies (2.6), it is Cauchy in \(C^0([0, T]; C^\beta(\mathbb{T}^3))\). Indeed, for any \(q < q'\), we have the estimates

\[
\|m_{q'} - m_q\|_{C^0([0, T]; C^\beta(\mathbb{T}^3))} \lesssim \sum_{l=1}^{q'-q} \|m_{q+1} - m_{q+1-l-1}\|_{C^0([0, T]; C^\beta(\mathbb{T}^3))} \nolinebreak \lesssim \sum_{l=1}^{q'-q} \|m_{q+1} - m_{q+1-l-1}\|_{1-\beta} \|m_{q+1} - m_{q+1-l-1}\|_{1} \nolinebreak \lesssim \sum_{l=1}^{q'-q} \lambda^\beta_{q+1} \delta^{\frac{1}{2}}_{q+1} = \sum_{l=1}^{q'-q} \lambda^\beta_{q+1} - \alpha \rightarrow 0
\]

as \(q\) goes to infinity because of \(\beta - \alpha < 0\). Therefore, we obtain its limit \(m\) in \(C^0([0, T]; C^\beta(\mathbb{T}^3))\). Since \((c_q, R_q, \varphi_q)\) converges to 0 in \(C^0([0, T] \times \mathbb{T}^3)\), the limit \(m\) solves the compressible equation (1.3) with stationary density \(\varrho\) and satisfies

\[
\partial_t \left( \frac{|m|^2}{2\varrho} + \varrho e(\varrho) \right) + \text{div} \left( \frac{m}{\varrho} \left( \frac{|m|^2}{2\varrho} + \varrho e(\varrho) + p(\varrho) \right) \right) = E'
\]
in the distributional sense. Here, we used the identity \( \varrho \left( \partial_t + \frac{m}{\varrho} \cdot \nabla \right) \kappa = \partial_t \kappa + \text{div}(\kappa m) \), \( \kappa = \frac{1}{2} \text{tr}(R) \). Since \( E' < 0 \) for all \( t \in [0, T] \), the constructed solution satisfies the entropy inequality (1.4) strictly.

In particular, testing with a time-dependent function \( \chi \in C_c^\infty((0, T)) \), we conclude that
\[
- \int_0^\infty \chi'(t) \int_{\mathbb{T}^3} \frac{|m|^2}{2\varrho} (t, x) \, dx \, dt = \int_0^\infty E'(t) \chi(t) \, dt.
\]
Given that \( E \) is \( C^1 \), the latter implies that the total kinetic energy is a \( C^1 \) function and it in facts coincides with \( E(t)/(2\pi)^3 \) up to a constant addition, namely
\[
\int_{\mathbb{T}^3} \frac{|m|^2}{2\varrho} (T, x) \, dx - \int_{\mathbb{T}^3} \frac{|m|^2}{2\varrho} (0, x) \, dx = E(T, x) - E(0, x) < 0.
\]
Therefore, the constructed solution has total kinetic energy dissipation (1.6).

Lastly, estimating
\[
\| \partial_t m_q \|_0 \lesssim_\varrho \| m_q \|_0 \| m_q \|_1 + \| \nabla p(\varrho) \|_0 + \| R_q \|_1 + \| \nabla \varrho \|_0 \lesssim_{0, \varrho} \lambda_q \delta_q^{\frac{1}{2}},
\]
where \( \| \cdot \|_N = \| \cdot \|_{C^0([0, T]; C^N(\mathbb{T}^3))} \), we have
\[
\| m_q \|_{C^1([0, T] \times \mathbb{T}^3)} \lesssim_{0, \varrho} \lambda_q \delta_q^{\frac{1}{2}}.
\]
Then, the time regularity of the limit momentum \( m \) can be concluded by interpolation argument. Hence \( m \in C^\alpha([0, T] \times \mathbb{T}^3) \).

2.5. Proof of Theorem 1.1

In this argument we assume \( T \geq 20 \) for convenience. Let \( \varrho(t, \cdot) = \varrho_0 \) for all \( t \in \mathbb{R} \). Fix \( \beta < \frac{1}{2} \) and \( \alpha \in (\beta, \frac{1}{2}) \). Then, choose \( b \) and \( \lambda_0 \) in the range suggested in Proposition 2.3. Also, choose the initial approximate solution \( (m_0, c_0, R_0, \varphi_0) \) as in Section 2.2 with \( E \equiv 0 \). As before, we see that it solves (2.1) on \( [0, T] + \tau_{-1} \) and satisfies (2.2)–(2.4).

Now, we apply Proposition 2.3 iteratively to produce a sequence of approximate solutions \( (m_q, c_q, R_q, \varphi_q) \), which solves (2.1) with the energy loss 0, and satisfies (2.2)–(2.4). Since the sequence \( \{m_q\} \) also satisfies (2.6), it is Cauchy in \( C^0([0, T]; C^\beta(\mathbb{T}^3)) \), and estimating the equation as before, we conclude that it is also Cauchy in \( C^\beta([0, T] \times \mathbb{T}^3) \).

On the other hand, fix \( \tilde{q} \in \mathbb{N} \cup \{0\} \) satisfying \( b^\tilde{q} \geq \tilde{q} \). At the \( \tilde{q} \)th step using Proposition 2.6 we can produce two distinct tuples, one which we keep denoting as above and the other which we denote by \( (\tilde{m}_q, c_q, \tilde{R}_q, \tilde{\varphi}_q) \) and satisfies (2.7), namely
\[
\| \tilde{m}_q - m_q \|_{C^0([0, T]; L^2(\mathbb{T}^3))} \geq \varepsilon_0 \delta_q^{\frac{1}{2}}, \quad \text{supp}_t (m_q - \tilde{m}_q) \subset \mathcal{I},
\]
with $\mathcal{I} = (10, 10 + 3\bar{q} - 1)$. Applying now Proposition 2.3 iteratively, we can build a new sequence $(\tilde{m}_q, c_q, \tilde{R}_q, \tilde{\varphi}_q)$ of approximate solutions which satisfy (2.2)–(2.6) and (2.8), inductively. Arguing as above, this second sequence converges to a solution $(\tilde{\varrho}_0, \tilde{m})$ to the compressible Euler equation (1.3). Indeed, $\tilde{m} \in C^\beta([0, T] \times \mathbb{T}^3)$. We remark that for any $q \geq \bar{q}$,

$$\text{supp}_t(m_q - \tilde{m}_q) \subset \mathcal{I} + \sum_{q = \bar{q}}^\infty (\lambda_q \delta_q^{\frac{1}{2}})^{-1} \subset [9, T],$$

(by adjusting $\lambda_0$ to be even larger than chosen above, if necessary), and hence $\tilde{m}_q$ shares initial data with $m_q$ for all $q$. As a result, two solutions $\tilde{m}_q$ and $m_q$ have the same initial data. However, the new solution $\tilde{m}$ differs from $m$ because

$$\|m - \tilde{m}\|_{C^0([0, T]; L^2(\mathbb{T}^3))} \geq \|m_q - \tilde{m}_q\|_{C^0([0, T]; L^2(\mathbb{T}^3))} - \sum_{q = \bar{q}}^\infty |m_{q+1} - m_q - (\tilde{m}_{q+1} - \tilde{m}_q)|_{C^0([0, T]; L^2(\mathbb{T}^3))}$$

$$\geq \|m_q - \tilde{m}_q\|_{C^0([0, T]; L^2(\mathbb{T}^3))} - (2\pi)^{\frac{1}{2}} \sum_{q = \bar{q}}^\infty (\|m_{q+1} - m_q\|_0 + \|\tilde{m}_{q+1} - \tilde{m}_q\|_0)$$

$$\geq \varepsilon_0 \delta_{\bar{q}}^{\frac{1}{2}} - 2(2\pi)^{\frac{1}{2}} M \sum_{q = \bar{q}}^\infty \delta_{q+1}^{\frac{1}{2}} > 0.$$

The last inequality follows from adjusting $\lambda_0$ to a larger one if necessary. By changing the choice of time interval $\mathcal{I}$ and the choice of $\bar{q}$, we can easily generate infinitely many solutions. Since we choose $E = 0$, the infinitely many solutions satisfy the entropy equality.

3. Construction of the Momentum Correction

In this section we detail the choice of the correction $n := m_{q+1} - m_q$. As in the literature which started from the paper [20], the perturbation $n$ is, in first approximation, obtained from a family of highly oscillatory stationary solution of the incompressible Euler equation, which are modulated by the errors $R_q$ and $\varphi_q$ and transported along the coarse grain flow of the background vector field $m_q/\varrho$. For stationary Euler flows, we use Mikado flows, introduced in Section 2.2,

$$\sum_{f \in \mathcal{F}} \gamma_f U_f(\lambda(x - x(f))).$$

In the construction of perturbation, $f$ will vary in a finite fixed set of directions $\mathcal{F}$ (which in fact will have cardinality 270) and for each $f$ we will specify an appropriate choice of $\varrho$. In a first approximation we wish to define our perturbation $m_{q+1} - m_q$ as

$$\sum_{f \in \mathcal{F}} \gamma_f (R_q(t, x), \varphi_q(t, x), \varrho(t, x)) U_f(\lambda(x - x(f))).$$
where the coefficients $\gamma_f$ are appropriately chosen smooth functions (later on called “weights”), $\lambda$ is a very large parameter and the $x(f)$ are appropriately chosen shifts to ensure the disjoint support condition of Lemma 2.7. As already pointed out such Ansatz must be corrected and we need to modify the perturbation so that it is approximately advected by the velocity $m_q/\varrho$. Note that on large time-scales the flow of the velocity $m_q/\varrho$ does not satisfy good estimates, while it satisfies good estimates on a sufficiently small scale $\tau_q$. Following [3,17,27], this issue is solved by introducing a partition of unity in time and restarting the flow at a discretized set of times, roughly spaced according to the parameter $\tau_q$. Like the case in [17], one has to face the delicate problem of keeping the supports of the various Mikado flows disjoint. This is done by discretizing the construction in space too, taking advantage of the fact that for sufficiently small space and time scales, the supports of the transported Mikado flows remain roughly straight thin tubes: the argument requires then a subtle combinatorial choice of the “shifts”. As in [17], the introduction of the space-time discretization deteriorates the estimates and accounts for the Hölder threshold $1/7$.

3.1. Mikado directions

To determine a set of suitable directions $f$, we recall two geometric lemmas [17,20,37]. In the first we denote by $\mathbb{S}$ the subset of $\mathbb{R}^{3\times3}$ of all symmetric matrices, set $|K|_\infty := |(k_{lm})|_\infty = \max_{l,m} |k_{lm}|$ for $K \in \mathbb{R}^{3\times3}$.

**Lemma 3.1.** (Geometric Lemma I). Let $\mathcal{F} = \{f_i\}_{i=1}^{6}$ be a set of vectors in $\mathbb{Z}^3$ and $C$ a positive constant such that

$$\sum_{i=1}^{6} f_i \otimes f_i = C \text{Id}, \quad \text{and} \quad \{f_i \otimes f_i\}_{i=1}^{6} \text{forms a basis of } \mathbb{S}. \quad (3.1)$$

Then, there exists a positive constant $N_0 = N_0(\mathcal{F})$ such that for any $N \leq N_0$, we can find functions $\{\Gamma_{f_i}\}_{i=1}^{6} \subset C^\infty(S_N; (0, \infty))$, with domain $S_N := \{\text{Id} - K : K \text{ is symmetric}, |K|_\infty \leq N\}$, satisfying

$$\text{Id} - K = \sum_{i=1}^{6} \Gamma_{f_i}^2 (\text{Id} - K)(f_i \otimes f_i), \quad \forall (\text{Id} - K) \in S_N.$$

**Lemma 3.2.** (Geometric Lemma II). Suppose that

$\{f_1, f_2, f_3\} \subset \mathbb{Z}^3 \setminus \{0\}$ is an orthogonal frame and $f_4 = -(f_1 + f_2 + f_3)$. \quad (3.2)

Then, for any $N_0 > 0$, there are affine functions $\{\Gamma_{f_k}\}_{1 \leq k \leq 4} \subset C^\infty(V_{N_0}; [N_0, \infty))$ with domain $V_{N_0} := \{m \in \mathbb{R}^3 : |m| \leq N_0\}$ such that

$$m = \sum_{k=1}^{4} \Gamma_{f_k}(m)f_k \quad \forall m \in V_{N_0}.\quad (3.3)$$
Based on these lemmas, we choose 27 pairwise disjoint families \( \mathcal{F}^j \) indexed by 
\( j \in \mathbb{Z}_3^3 \), where each \( \mathcal{F}^j \) consists further of two (disjoint) subfamilies \( \mathcal{F}^j, R \cup \mathcal{F}^j, \varphi \) with cardinalities \( |\mathcal{F}^j, R| = 6 \) and \( |\mathcal{F}^j, \varphi| = 4 \), chosen so that \( \mathcal{F}^j, R \) and \( \mathcal{F}^j, \varphi \) satisfy (3.1) and (3.2), respectively. For example, for \( j = (0, 0, 0) \) we can choose
\[
\mathcal{F}^j, R = \{(1, \pm 1, 0), (1, 0, \pm 1), (0, 1, \pm 1)\},
\]
\[
\mathcal{F}^j, \varphi = \{(1, 2, 0), (-2, 1, 0), (0, 0, 1), (1, -3, -1)\}
\]
and then we can apply 26 suitable rotations (and rescalings). Next, the function \( \vartheta \) will be chosen for each \( f \) in two different ways, depending on whether \( f \in \mathcal{F}^j, R \) or \( f \in \mathcal{F}^j, \varphi \). Introducing the shorthand notation \( \langle u \rangle = \int_{\mathbb{R}^3} u(x) \, dx \), we impose the moment conditions
\[
\langle \psi_f \rangle = \langle \psi^3_f \rangle = 0, \quad \langle \psi^2_f \rangle = 1 \quad \forall f \in \mathcal{F}^j, R,
\]
\[
\langle \psi_f \rangle = 0, \quad \langle \psi^3_f \rangle = 1 \quad \forall f \in \mathcal{F}^j, \varphi.
\]
(3.3)
The main point is that the Mikado directed along \( f \in \mathcal{F}^j, R \) will be used to “cancel the error \( R_q \)”, while the ones directed along \( f \in \mathcal{F}^j, \varphi \) will be used to “cancel the error \( \varphi_q \)” and the different moment conditions will play a major role. In both cases we assume also that
\[
\text{supp}(\psi_f) \subset B \left( l_f, \frac{\eta}{10} \right) := \{ x \in \mathbb{R}^3 : |x - y| < \frac{\eta}{10} \text{ for some } y \in l_f \}, \quad (3.4)
\]
where \( l_f \) is as defined in (2.9) and \( \eta \) is a geometric constant which will be specified later, cf. Proposition 3.4.

3.2. Regularization and Drift

We start with smoothing the tuple \((m_q, R_q, \varphi_q)\). To this end, we first introduce the parameters \( \ell \) and \( \ell_t \), defined by
\[
\ell = \frac{1}{\lambda_q^{2/3}} \left( \frac{\delta_q + 1}{\delta_q} \right)^{3/8}, \quad \ell_t = \frac{1}{\lambda_q - 3 \gamma_q} \left( \frac{1}{\lambda_q + 1} \delta_q \delta_q^{-1} \delta_q^{1/3} \right).
\]
The space regularization of \( m_q \) is defined by applying a “low-pass filter” which roughly speaking eliminates all the waves larger than \( \ell^{-1} \). In order to do so we first introduce some suitable notation. First of all, for a function \( f \) in the Schwartz space \( S(\mathbb{R}^3) \), the Fourier transform of \( f \) and its inverse on \( \mathbb{R}^3 \) are denoted by
\[
\hat{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} \, dx, \quad \check{f}(x) = \int_{\mathbb{R}^3} f(\xi) e^{ix \cdot \xi} \, d\xi.
\]
As usual, we understand the Fourier transform on more general functions as extended by duality to \( S'(\mathbb{R}^3) \). Since practically all the objects considered in this note are functions, vectors and tensors defined on \( I \times T^3 \) for some time domain \( I \subset \mathbb{R} \), regarding them as spatially periodic functions on \( I \times \mathbb{R}^3 \), we will consider their Fourier transform as time-dependent elements of \( S'(\mathbb{R}^3) \). We then follow the
standard convention on Littlewood-Paley operators. We let $\phi(\xi)$ be a radial smooth function supported in $B(0, 2)$ which is identically 1 on $B(0, 1)$. For any number $j \in \mathbb{Z}$ and distribution $f$ in $\mathbb{R}^3$, we set

$$P_{\leq 2^j} f(\xi) := \phi\left(\frac{\xi}{2^j}\right) \hat{f}(\xi), \quad P_{> 2^j} f(\xi) := \left(1 - \phi\left(\frac{\xi}{2^j}\right)\right) \hat{f}(\xi),$$

and, for $j \in \mathbb{Z}$,

$$P_{2^j} f(\xi) := \left(\phi\left(\frac{\xi}{2^j}\right) - \phi\left(\frac{\xi}{2^{j-1}}\right)\right) \hat{f}(\xi).$$

For a positive real number $S$, we finally let $P_{\leq S}$ equal the operator $P_{\leq 2^j}$ for the largest $J$ such that $2^J \leq S$. We are thus ready to introduce the coarse scale velocity $m_\ell$ defined by

$$m_\ell = P_{\leq \ell - 1} m_q,$$ (3.5)

Note that, regarding $m_q$ as a spatially periodic function on $I \times \mathbb{T}^3$, $P_{\leq \ell - 1} m$ can be written as the space convolution of $m$ with the kernel $2^{-3J} \hat{\phi}(2^J \cdot)$, which belongs to $\mathcal{S}(\mathbb{R}^3)$. In particular $m_\ell$ is also spatially periodic and will be in fact regarded as a function on $I \times \mathbb{T}^3$. Similar remarks apply to several other situations in the rest of this note.

The regularization of the errors $R_q$ and $\varphi_q$ is more laborious and follows the intuition that, while we need to regularize them in time and space, we want such regularization to give good estimates on their advective derivatives along $\frac{m_\ell}{\epsilon}$, for which we introduce the ad hoc notation

$$D_{t, \ell} := \partial_t + \frac{m_\ell}{\epsilon} \cdot \nabla.$$

First of all we let $\Phi(\tau, x; t)$ be the forward flow map with the drift velocity $\frac{m_\ell}{\epsilon}$ defined on some time interval $[a, b]$ starting at the initial time $t \in [a, b]$:

$$\begin{cases} 
\partial_\tau \Phi(\tau, x; t) = \frac{m_\ell}{\epsilon} (\Phi(\tau, x; t)) \\
\Phi(t, x; t) = x.
\end{cases}$$ (3.6)

Remark 3.3. Strictly speaking the map above is defined on $[a, b] \times \mathbb{R}^3$. Note however that the periodicity of $\frac{m_\ell}{\epsilon}$ implies that $\Phi$ induces a well-defined map from $[a, b] \times \mathbb{T}^3$ into $\mathbb{T}^3$. From now on we will implicitly identify both maps.

We then take a standard mollifier $\rho$ on $\mathbb{R}$, namely a nonnegative smooth bump function satisfying $\|\rho\|_{L^1(\mathbb{R})} = 1$ and supp $\rho \subset (-1, 1)$. As usual we set $\rho_\delta(s) = \delta^{-1} \rho(\delta^{-1}s)$ for any $\delta > 0$. We can thus introduce the mollification along the trajectory

$$(\rho_\delta * \Phi \ F)(t, x) = \int_{\mathbb{R}} F(t + s, \Phi(t + s, x; t)) \rho_\delta(s) \, ds.$$
(Note that if $F$ and $m_\ell$ are defined on some time interval $[a, b]$, then $\rho_\delta \ast F$ is defined on $[a, b] - \delta$.) This mollification can be found in [27] and is designed to satisfy

$$
D_{t, \ell}(\rho_\delta \ast F)(t, x) = \int (D_{t, \ell}F)(t + s, \Phi(t + s, x; t))\rho_\delta(s) \, ds = -\int F(t + s, \Phi(t + s, x; t))\rho_\delta'(s) \, ds.
$$

The regularized errors are then given by

$$
R_\ell = \rho_{\ell, t} \ast F P_{\ell - 1} R_q, \quad \varphi_\ell = \rho_{\ell, t} \ast F P_{\ell - 1} \varphi_q.
$$

These errors can be defined on $[0, T] + 2\tau_q$ by the choice of sufficiently large $\lambda_0$. We will need later quite detailed estimates on the difference between the original tuple and the regularized one and on higher derivatives of the latter. Such estimates are in fact collected in Section 5.

### 3.3. Partition of Unity and Shifts

We first introduce nonnegative smooth functions $\{\chi_v\}_{v \in \mathbb{Z}^3}$ and $\{\theta_u\}_{u \in \mathbb{Z}}$ whose sixth powers give suitable partitions of unity in space $\mathbb{R}^3$ and in time $\mathbb{R}$, respectively:

$$
\sum_{v \in \mathbb{Z}^3} \chi_v^6(x) = 1, \quad \sum_{u \in \mathbb{Z}} \theta_u^6(t) = 1.
$$

Here, $\chi_v(x) = \chi_0(x - 2\pi v)$ where $\chi_0$ is a non-negative smooth function supported in $Q(0, 9/8\pi)$ satisfying $\chi_0 = 1$ on $Q(0, 7/8\pi)$, where from now on $Q(x, r)$ will denote the cube $\{y : |y - x|_\infty < r\}$ (with $|z|_\infty := \max(|z_i|)$). Similarly, $\theta_u(t) = \theta_0(t - u)$ where $\theta_0 \in C_c^\infty(\mathbb{R})$ satisfies $\theta_0 = 1$ on $[1/8, 7/8]$ and $\theta_0 = 0$ on $(-1/8, 9/8)^c$. Then, we divide the integer lattice $\mathbb{Z}^3$ into 27 equivalent families $[j]$ with $j \in \mathbb{Z}_3^3$ via the usual equivalence relation

$$
v = (v_1, v_2, v_3) \sim \tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \iff v_i \equiv \tilde{v}_i \mod 3 \quad \text{for all} \quad i = 1, 2, 3.
$$

We use these classes to define the set of indices

$$
\mathcal{I} := \{(u, v, f) : (u, v) \in \mathbb{Z} \times \mathbb{Z}^3 \quad \text{and} \quad f \in \mathcal{F}^v\}.
$$

For each $I$ we denote by $f_I$ the third component of the index and we further subdivide $\mathcal{I}$ into $\mathcal{I}_R \cup \mathcal{I}_\psi$ depending on whether $f_I \in \mathcal{F}^v[R]$ or $f_I \in \mathcal{F}^v[\psi]$. Next we introduce the parameters $\tau = \tau_q$ and $\mu = \mu_q$ with $\tau_q^{-1} > 0$ and $\mu_q^{-1} \in \mathbb{N} \setminus \{0\}$, which are explicitly given by

$$
\mu_q^{-1} = 3[\lambda_q^{\frac{1}{2}} \lambda_{q + 1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \delta_{q + 1}^{\frac{1}{2}}], \quad \tau_q^{-1} = 40\pi C_1 M q^{-1} \cdot \lambda_q^{\frac{1}{2}} \lambda_{q + 1}^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \delta_{q + 1}^{\frac{1}{2}},
$$

(3.9)
where $\hat{C}_q$ is chosen as a constant depending on $q$ such that $\|\nabla (m_q/\rho)\|_0 \leq \hat{C}_q M \lambda q \delta_q^2$. (For the motivation of $\tau_q$ and $\eta$, see (3.15) and Proposition 3.4). We define

$$\theta_I(t) = \begin{cases} \theta_3^u(t^{-1}), & I \in \mathcal{I}_R \\ \theta_2^u(t^{-1}), & I \in \mathcal{I}_\varphi, \end{cases} \quad \chi_I(x) = \begin{cases} \chi_3^v(\mu^{-1}x), & I \in \mathcal{I}_R \\ \chi_2^v(\mu^{-1}x), & I \in \mathcal{I}_\varphi. \end{cases}$$

Next, for each $I$ let $U_{f_I}$ be the corresponding Mikado flow. Moreover, given $I = (u, v, f)$, denote by $t_u$ the time $t_u = u\tau$ and let $\xi_I = \xi_u$ be the solution of the following PDE (which we understand as a map on $\mathbb{R} \times \mathbb{T}^3$ taking values in $\mathbb{T}^3$, cf. Remark 3.3):

$$\begin{cases} \partial_t \xi_u + \left( \frac{m_\ell}{\rho} \cdot \nabla \right) \xi_u = 0 \\ \xi_u(t_u, x) = x \end{cases} \quad (3.10)$$

In the rest of the paper $\nabla \xi_I$ will denote the Jacobi Matrix of the partial derivatives of the components of the vector map $\xi_I$ and we will use the shorthand notations $\nabla \xi_I^T$, $\nabla \xi_I^{-1}$ and $\nabla \xi_I^{-T}$ for, respectively, its transpose, inverse and transport of the inverse. Moreover, for any vector $f \in \mathbb{R}^3$ and any matrix $A \in \mathbb{R}^{3 \times s}$ the notation $\nabla \xi_I f$ and $\nabla \xi_I A$ (resp. $\nabla \xi_I^{-1} f$, etc.) will be used for the usual matrix product, regarding $f$ as a column vector (that is a $\mathbb{R}^{3 \times 1}$-matrix).

For each $I = (u, v, f)$ we will also choose a shift

$$z_I = z_{u,v} + \bar{x}_f \in \mathbb{R}^3$$

and, setting $\lambda = \lambda_{q+1}$, we are finally able to introduce the main part of our perturbation, which is achieved using the following “master function”

$$\hat{N}(R, \varphi, \rho, t, x) := \sum_{I \in \mathcal{I}} \theta_I(t) \chi_I(\xi_I(t, x)) \gamma_I(R, \varphi, \rho, t, x) \nabla \xi_I^{-1}(t, x) U_{f_I}(\lambda(\xi_I(t, x) - z_I)),$$

where the $\gamma_I$’s are smooth scalar functions (the “weights”) whose choice will be specified in the next section. In order to simplify our notation we will use $U_I$ for $U_{f_I}(\cdot - z_I)$, $\psi_I$ for $\psi_{f_I}(\cdot - z_I)$ and $\tilde{f}_I$ for $\nabla \xi_I^{-1} f_I$. We therefore have the writing

$$\hat{N} := \sum_{I \in \mathcal{I}} \theta_I \chi_I(\xi_I) \gamma_I \tilde{f}_I \psi_I(\lambda \xi_I). \quad (3.12)$$

Note that, since we want $\hat{N}$ to be a periodic function of $x$, we will impose that

$$z_{u,v} = z_{u,v'} \quad \text{if } \mu(v - v') \in 2\pi \mathbb{Z}^3. \quad (3.13)$$

Finally, the preponderant part of the correction $m_{q+1} - m_q$ will take the form

$$n_o(t, x) := \hat{N}(R_{\ell}(t, x), \varphi_{\ell}(t, x), \rho(t, x), t, x), \quad (3.14)$$
which is well-defined on $[0, T] + 2\tau_q$. (Indeed, it is possible to have $[0, T] + 3\tau_q \subset [0, T] + \tau_{q-1}$ by the choice of sufficiently large $\lambda_0$). In the rest of Section 3, without mentioning, our analysis is done in the time interval $[0, T] + 2\tau_q$. Given the complexity of several formulas and future computations, it is convenient to break down the functions $\hat{N}$ and $n_o$ in more elementary pieces. To this end we introduce the scalar maps

$$n_I(t, x) := \theta_I(t) \chi_I(\xi_I(t, x))\psi_I(\lambda(\xi_I(t, x))),$$

using which we can write

$$\hat{N}(R, \varphi, \varrho, t, x) = \sum_{I \in \mathcal{I}} \gamma_I(R, \varphi, \varrho, t, x)\nabla \xi_I(t, x)^{-1} f_I n_I(t, x)$$

$$= \sum_{I \in \mathcal{I}} \gamma_I(R, \varphi, \varrho, t, x) \tilde{f}_I(t, x)n_I(t, x)$$

and

$$n_o = \sum_{I \in \mathcal{I}} \gamma_I \nabla \xi_I^{-1} f_I n_I = \sum_{I \in \mathcal{I}} \gamma_I \tilde{f}_I n_I.$$

The crucial point in our construction is the following proposition, obtained in [17, Proposition 3.5]

**Proposition 3.4.** There is a constant $\eta = \eta(\mathcal{F})$ in (3.4) such that it allows a choice of the shifts $z_I = z_{u, v} + \bar{x}_f$ which ensure that for each $(\mu_q, \tau_q, \lambda_{q+1})$, the conditions $\text{supp}(n_I) \cap \text{supp}(n_J) = \emptyset$ hold for every $I \neq J$ and that (3.13) holds for every $u, v$ and $v'$.

Note that the difference between Proposition 3.4 and Proposition 3.5 in [17] is that in [17] the drift velocity is divergence-free while $m/\varrho$ is not. However, the proof is not relying on the divergence-free condition. Also, we remark that the proof of Proposition 3.4 requires the choice of parameters satisfying the relations

$$\mu_q^{-1} \ll \lambda_{q+1} \in \mathbb{N}, \quad \tau_q \left\| \nabla (m_q/\varrho) \right\|_0 \leq \frac{1}{10}, \quad \mu_q \tau_q \left\| \nabla (m_q/\varrho) \right\|_0 \leq \frac{\eta}{10\pi \lambda_{q+1}},$$

where $\eta$ is a positive constant determined by $\mathcal{F} = \bigcup_{j \in \mathbb{Z}^3} \mathcal{F}^j$, which has finite cardinality.

### 3.4. Choice of the Weights

We next detail the choice of the functions $\gamma_I$, subdividing it into two cases.
3.4.1. Energy Weights  

The weights $\gamma_I$ for $I \in \mathcal{I}_\varphi$ will be chosen so that the low frequency part of $\frac{1}{2}|n_o|^2 n_o$ makes a cancellation with the mollified unresolved current $\varrho^3 \varphi_\ell$. Because of Proposition 3.4, we have

$$|n_o|^2 n_o = \sum_{I \in \mathcal{I}} \frac{3}{2} \chi_I^3(\xi_I) \gamma_I^3 \psi_I^3(\lambda q + 1) \xi_I^{-1} f_I^2 \xi_I^{-1} f_I = \sum_{I \in \mathcal{I}} \frac{3}{2} \chi_I^3(\xi_I) \gamma_I^3 \psi_I^3(\lambda q + 1) \xi_I^{-1} f_I^2 \xi_I^{-1} f_I.$$

In order to find the desired $\gamma_I$, we introduce the notation $\mathcal{I}_{u,v,\varphi}$ for $\{I \in \mathcal{I}_\varphi : I = (u, v, f)\}$ and we observe that, by (3.3), it suffices to achieve

$$(|n_o|^2 n_o)_L = \sum_{u,v} \theta^6_u \left( \frac{t}{\tau_q} \right) \chi^6_v \left( \frac{\xi}{\mu_q} \right) \sum_{I \in \mathcal{I}_{u,v,\varphi}} \gamma_I^3 |\tilde{f}_I|^2 \tilde{f}_I. \quad (3.16)$$

Next we look for our coefficients in the following form:

$$\gamma_I = \frac{\lambda_q^{-\varphi} \delta_q^2}{|\tilde{f}_I|^2 \tilde{f}_I}.$$

there $\Gamma_I$ will be specified in a moment.

Recall that $\xi_I$ is a solution to (3.10) and satisfies $\nabla \xi_I \big|_{t=\tau_u} = \text{Id}$ and

$$\nabla \xi_I^{-1} \big|_{(t, x)} = \nabla \Phi_u (t, \xi_I (t, x)),$$

where $\Phi_u$ is the “forward flow” $\Phi(t, x; t_u)$ introduced in (3.6) and thus solves

$$\left\{ \begin{array}{l}
\partial_t \Phi_u (t, x) = \frac{m_q}{q} (t, \Phi_u (t, x)) \\
\Phi_u (t_u, x) = x.
\end{array} \right. \quad (3.17)$$

This implies that

$$\|\nabla \xi_I\|_{C^0(\mathcal{I}_u \times \mathbb{R}^3)} \leq \exp \left( 2 \tau_q \|\nabla (m \ell / \varphi)\|_0 \right) \leq \exp (2C_\varphi M \tau_q \lambda_q \delta_q^2), \quad (3.18)$$

$$(\|\text{Id} - \nabla \xi_I^{-1}\|_{C^0(\mathcal{I}_u \times \mathbb{R}^3)} = \|\text{Id} - \nabla \Phi_u\|_{C^0(\mathcal{I}_u \times \mathbb{R}^3)} \leq 2 \tau_q \|\nabla \Phi_u\|_{C^0(\mathcal{I}_u \times \mathbb{R}^3)} \|\nabla (m \ell / \varphi)\|_0 \leq 2C_\varphi M \tau_q \lambda_q \delta_q^2 \exp (2C_\varphi M \tau_q \lambda_q \delta_q^2) \quad (3.19)$$

for the time interval $\mathcal{I}_u := [t_u - \frac{1}{2} \tau_q, t_u + \frac{3}{2} \tau_q] \cap [0, T] + 2 \tau_q$. Therefore, for sufficiently large $\lambda_0$, we have

$$|\tilde{f}_I| = |\nabla \xi_I^{-1} f_I| \geq \frac{3}{4} \quad \|2\lambda_q^3 \delta_q^{-3} (\nabla \xi_I) \varphi_\ell\|_{C^0} \leq 3C_\varphi$$
on the support of \( \theta_f \) for some positive constant \( C_1(\varphi) \). Since \( \{f_I : I \in \mathcal{I}_{u,v,\varphi}\} = \mathcal{F}^{[v],\varphi} \) satisfies (3.2), we can apply Lemma 3.2 with \( N_0 = 3C_1(\varphi) \) to solve
\[
-2Q^3 \varphi_\ell = \sum_{I \in \mathcal{I}_{u,v,\varphi}} \gamma_I^3 |\tilde{f}_I|^2 \tilde{f}_I \iff -2Q^3 \varphi_\ell = \sum_{I \in \mathcal{I}_{u,v,\varphi}} \Gamma_I^3 f_I
\]
on each support of \( \theta_f \) (observe that we have crucially used that \( \xi_I = \xi_u \) is independent of \( f_I \) for \( I \in \mathcal{I}_{u,v,\varphi} \)). We are thus in the position to apply Lemma 3.2 to the set \( \mathcal{F}^{[v],\varphi} = \{f_I : I \in \mathcal{I}_{u,v,\varphi}\} \) and we let \( \Gamma f_I, I \in \mathcal{I}_{u,v,\varphi} \) be the corresponding functions. As a result, we can set
\[
\Gamma f_I(t, x) = \Gamma_{f_I}^{1/3} \left(-2Q^3 \varphi_\ell \right) \Gamma f_I(t, x).
\]
Note that the smoothness of the selected functions \( \Gamma_{f_I} \) depends only on \( C_1 \) and that in fact Lemma 3.2 is just applied 27 times, taking into consideration that \( |v| \in \mathbb{Z}_3^2 \).

For later use we record here the important “cancellation property” that the choice of our weights achieves:
\[
\frac{1}{2Q^2} (|n_o|^2 n_o)_L = -Q \varphi_\ell.
\]

### 3.4.2. Reynolds Weights

Similarly to the previous section we decompose \( n_o \otimes n_o \) into the low and high frequency parts,
\[
n_o \otimes n_o = \sum_{I} \theta_I^2 \chi_I^2 (\xi_I) \gamma_I^2 \psi_I^2 (\lambda_{q+1} \xi_I) \tilde{f}_I \otimes I
\]
\[
= \sum_{I} \theta_I^2 \chi_I^2 (\xi_I) \gamma_I^2 (\psi_I^2) \tilde{f}_I \otimes I + \sum_{I} \theta_I^2 \chi_I^2 (\xi_I) \gamma_I^2 (\lambda_{q+1} \xi_I) - (\psi_I^2) \tilde{f}_I \otimes I.
\]
Since the weights for \( I \in \mathcal{I}_\varphi \) have already been established, for each fixed \( (u, v) \) we denote by \( I(u, v) \) the sets of indices \( (u', v') \) such that \( \max(|u-u'|_\infty, |v-v'|_\infty) \leq 1 \) (where \( |w|_\infty := \max(|w_1|, |w_2|, |w_3|) \) for any \( w \in \mathbb{R}^3 \)) and rewrite
\[
(n_o \otimes n_o)_L = \sum_{u,v} \theta_u^6 \left( \frac{t}{\tau_q} \right) \chi_v^6 \left( \frac{\xi_u}{\mu_q} \right) \sum_{I \in \mathcal{I}_{u,v,R}} \gamma_I^2 \tilde{f}_I \otimes I
\]
\[
+ \sum_{J \in \mathcal{I}_\varphi} \theta_J^2 \chi_J^2 (\xi_J) \gamma_J^2 (\psi_J^2) \tilde{f}_J \otimes I
\]
\[
= \sum_{u,v} \theta_u^6 \left( \frac{t}{\tau_q} \right) \chi_v^6 \left( \frac{\xi_u}{\mu_q} \right) \left[ \sum_{I \in \mathcal{I}_{u,v,R}} \gamma_I^2 \tilde{f}_I \otimes I + \sum_{J \in \mathcal{I}_{u',v',\varphi}} \theta_J^2 \chi_J^2 (\xi_J) \gamma_J^2 (\psi_J^2) \tilde{f}_J \otimes J \right].
\]
To make \( n_o \otimes n_o \) cancel out \( q^2(\delta_{q+1} \text{Id} - R_\ell) \), we recall that \( \tilde{f}_I \otimes \tilde{f}_I = \nabla \xi^{-1}_I (f_I \otimes f_I) \nabla \xi^{-T}_I \) and set
\[
\sum_{I \in \mathcal{F}_{u,v}} \gamma^2_I f_I \otimes f_I = \nabla \xi_I^2(\delta_{q+1} \text{Id} - R_\ell) - \sum_{(u',v') \in I(u,v)} \sum_{J \in \mathcal{F}_{u',v',\varphi}} \theta^2_J \gamma^2_J(\xi_J) (\psi^2_J f_J \otimes f_J) \nabla \xi^-_I.
\]

We now define \( M_I \) as
\[
M_I = \delta_{q+1}[\nabla \xi_I \nabla \xi^-_I - \text{Id}] - \sum_{(u',v') \in I(u,v)} \sum_{J \in \mathcal{F}_{u',v',\varphi}} \theta^2_J \gamma^2_J(\xi_J) (\psi^2_J f_J \otimes f_J) \nabla \xi^-_I
\]
and \( \gamma_I = \delta_{q+1}^2 \Gamma_I \) (for \( I \in \mathcal{F}_{u,v,R} \)) and impose
\[
\sum_{I \in \mathcal{F}_{u,v,R}} \Gamma^2_I f_I \otimes f_I = \text{Id} + \delta_{q+1}^{-1} M_I
\]

In order to show that such a choice is possible observe that we can make \( \| \delta_{q+1}^{-1} M_I \|_{C^0(\text{supp}(\theta_I) \times \mathbb{R}^3)} \) sufficiently small, provided that \( \lambda_0 \) is sufficiently large, because of (3.18), (3.19), \( \| \delta_{q+1}^{-1} R_\ell \|_0 \lesssim \lambda_0^{-3\gamma} \) and, \( \| \delta_{q+1}^{-1} \gamma_J^2 \|_0 \lesssim \lambda_0^{-2\gamma} \) when \( J \in \mathcal{F}_{u',v',\varphi} \). We can thus apply Lemma 3.1 to \( \{f_I : I \in \mathcal{F}_{u,v,R}\} = \mathcal{F}^{[v],R} \) and, denoting by \( \Gamma_f \) the corresponding functions, we just need to set
\[
\Gamma_I = \Gamma f_I (\text{Id} + \delta_{q+1}^{-1} M_I).
\]

Observe once again that this means applying Lemma 3.1 just 27 times, given that there are 27 different families \( \mathcal{F}^{[v],R} \). We finally record the desired “cancellation property” that the choice of the weights achieves
\[
(n_o \otimes n_o)_L = \sum_{u,v} \theta^6_u \left( \frac{t}{\mu_q} \right) \chi^6_u \left( \frac{\xi_u}{\mu_q} \right) \left[ (n_o \otimes n_o)_L \right] = q^2(\delta_{q+1} \text{Id} - R_\ell) = q^2(\delta_{q+1} \text{Id} - R_\ell).
\]

### 3.5. Fourier Expansion in Fast Variables and Corrector \( n_c \)

In the rest of this article, we use a representation of \( n_o, n_o \otimes n_o, \) and \( \frac{1}{2} |n_o|^2 n_o \) based on the Fourier series of \( \psi_I, \psi^2_I \) and \( \psi^3_I \). Indeed, since \( \psi_I \) is a smooth function on \( \mathbb{T}^3 \) with zero-mean, we have
\[
\psi_I(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \hat{b}_{I,k} e^{ik \cdot x}, \quad \psi^2_I(x) = \hat{c}_{I,0} + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \hat{\tilde{c}}_{I,k} e^{ik \cdot x},
\]
\[
\psi^3_I(x) = \hat{d}_{I,0} + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \hat{\tilde{d}}_{I,k} e^{ik \cdot x}.
\]
In particular,
\[ \hat{c}_{I,0} = \langle \psi_I^2 \rangle, \quad \hat{d}_{I,0} = \langle \psi_I^3 \rangle. \]

Since \( \psi_I \) is in \( C^\infty(\mathbb{T}^3) \), we have
\[
\sum_{k \in \mathbb{Z}^3} |k|^{n_0+2} |b_{I,k}| + \sum_{k \in \mathbb{Z}^3} |k|^{n_0+2} |c_{I,k}| + \sum_{k \in \mathbb{Z}^3} |k|^{n_0+2} |d_{I,k}| \lesssim 1, \quad \sum_{k \in \mathbb{Z}^3} |\hat{c}_{I,k}|^2 \lesssim 1.
\] (3.26)

for \( n_0 = \lceil \frac{2b(2+a)}{b-1} \rceil - 1 \). Also, it follows from \( f_I \cdot \nabla \psi_I = f_I \cdot \nabla \psi_I^2 = f_I \cdot \nabla \psi_I^3 = 0 \) that
\[
\hat{b}_{I,k}(f_I \cdot k) = \hat{c}_{I,k}(f_I \cdot k) = \hat{d}_{I,k}(f_I \cdot k) = 0.
\] (3.27)

Next, as a consequence of (3.21), (3.24), and (3.25), we have
\[
n_o = \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} \delta_{q+1} b_{u,k} e^{i\lambda_{q+1} k \cdot \xi_I} \] (3.28)
\[
n_o \otimes n_o = \hat{\varphi}^2 (\delta_{q+1} \text{Id} - R_\ell) + \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} c_{u,k} e^{i\lambda_{q+1} k \cdot \xi_I} \] (3.29)
\[
\frac{1}{2} |n_o|^2 n_o = -\dot{\varphi}^2 \varphi_\ell + \frac{1}{2} \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} d_{u,k} e^{i\lambda_{q+1} k \cdot \xi_I} \] (3.30)
\[
\frac{1}{2} |n_o|^2 = -\hat{\varphi}^2 \kappa_\ell + \frac{3}{2} \dot{\varphi}^2 \delta_{q+1} + \frac{1}{2} \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} \text{tr}(c_{u,k}) e^{i\lambda_{q+1} k \cdot \xi_I},
\] (3.31)

where \( \kappa_\ell = \frac{1}{2} \text{tr}(R_\ell) \) and the relevant coefficients are defined as follows:
\[
b_{u,k} = \sum_{I: u_I = u} \theta_I \chi_I(\xi_I) \delta_{q+1} b_{I,k} \tilde{f}_I =: \sum_{I: u_I = u} B_{I,k} \tilde{f}_I,
\]
\[
c_{u,k} = \sum_{I: u_I = u} \theta^2_I \chi^2_I(\xi_I) \delta_{q+1} c_{I,k} \tilde{f}_I \otimes \tilde{f}_I,
\] (3.32)
\[
d_{u,k} = \sum_{I: u_I = u} \theta^3_I \chi^3_I(\xi_I) \delta_{q+1} d_{I,k} |\tilde{f}_I|^2 \tilde{f}_I.
\]

Observe that, by the choice of \( \theta_I \), if \(|u - u'| > 1\), then
\[
\text{supp}_{t,x}(b_{u,k}) \cap \text{supp}_{t,x}(b_{u',k'}) = \text{supp}_{t,x}(c_{u,k}) \cap \text{supp}_{t,x}(c_{u',k'}) = \text{supp}_{t,x}(d_{u,k}) \cap \text{supp}_{t,x}(d_{u',k'}) = \emptyset
\]
for any \( k, k' \in \mathbb{Z}^3 \setminus \{0\} \).

We next prescribe an additional correction \( n_c \) to make \( n = n_o + n_c \) divergence-free. Since we have (3.27) and the identity \( \nabla \times (\nabla \nabla^T U(\xi_I)) = \text{cof}(\nabla \nabla^T U)(\nabla \times \)
Using this, the preponderant part $n_o$ of the momentum correction can be written as
\[
 n_o = \sum_{u \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\delta_{q+1}^2}{\lambda_{q+1}} B_{I,k} \nabla \xi_I^{-1} f_I e^{i\lambda_{q+1}k \cdot \xi_I}.
\]

Note that $\det(\nabla \xi_I)$ is away from 0 on the support of $B_{I,k}$. Therefore, we define
\[
 n_c = \frac{\delta_{q+1}^2}{\lambda_{q+1}} \sum_{u \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \nabla \left( \frac{B_{I,k}}{\det(\nabla \xi_I)} \right) \times \left( \nabla \xi_I^{-1} \frac{i k \times f_I}{|k|^2} \right) e^{i\lambda_{q+1}k \cdot \xi_I}.
\]
where
\[
 e_{u,k} = \mu_q \sum_{I:u_I=u} \nabla (\det(\nabla \xi_I^{-1}) \partial_I \chi_I(\xi_I)) \delta_{q+1}^2 \lambda_{q+1}^{-1} b_{I,k} \times \left( \nabla \xi_I^{-1} \frac{i k \times f_I}{|k|^2} \right).
\]

In this way, the final momentum correction $m_{q+1} - m_q =: n = n_o + n_c$ can be written as
\[
 n = \nabla \times \left( \frac{\delta_{q+1}^2}{\lambda_{q+1}} \sum_{I,k} \det(\nabla \xi_I^{-1}) B_{I,k} \nabla \xi_I^{-1} \frac{i k \times f_I}{|k|^2} e^{i\lambda_{q+1}k \cdot \xi_I} \right),
\]
and hence it is divergence-free and mean-zero. For later use, we remark that if $|u - u'| > 1$, $\text{supp}_{I,x}(e_{u,k}) \cap \text{supp}_{I,x}(e_{u',k'}) = \emptyset$ holds for any $k, k' \in \mathbb{Z}^3 \setminus \{0\}$. Also, by its definition, the correction $n$ has the representation
\[
 n = \sum_{u \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\delta_{q+1}^2}{\lambda_{q+1}} (b_{u,k} + (\lambda_{q+1} \mu_q)^{-1} e_{u,k}) e^{i\lambda_{q+1}k \cdot \xi_I}.
\]
4. Definition of the New Errors

4.1. New Reynolds Stress

With the correction $n$ of the momentum defined as in the previous section, we reorganize the Euler-Reynolds system and the relaxed energy equation as the equations for the new Reynolds stress $R_{q+1}$ and for the unresolved current $\varphi_{q+1}$, respectively.

We first define $R_{q+1}$. Using the momentum equation

$$\partial_t m_q + \nabla \cdot \left( \frac{m_q \otimes m_q}{\varrho} \right) + \nabla p(\varrho) = \text{div}(\varrho (R_q - c_q \text{Id}))$$

at $q$th step with $c_q = \sum_{j=q+1}^{\infty} \delta_j$, we can write the equation for $R_{q+1}$ as

$$\text{div}(\varrho R_{q+1}) = \varrho D_{t,\ell} \frac{n}{\varrho} - \text{div}(m_q - m_\ell) \frac{n}{\varrho} + \frac{\frac{n \otimes n}{\varrho} + \varrho R_\ell - \delta_{q+1} \varrho \text{Id}}{\varrho} =: \nabla \cdot (\varrho R_T)$$

$$+ (n \cdot \nabla) m_\ell + \frac{\varrho}{\varrho} \left( \frac{m_q - m_\ell}{\varrho} \otimes n + \frac{n \otimes (m_q - m_\ell)}{\varrho} + \varrho (R_q - R_\ell) \right),$$

where $D_{t,\ell} = \left( \partial_t + \frac{m_\ell}{\varrho} \cdot \nabla \right)$, and decompose $R_\ell$ further as

$$\nabla \cdot (\varrho R_\ell) = \text{div} \left( \frac{n_o \otimes n_o}{\varrho} + \varrho R_\ell - \delta_{q+1} \varrho \text{Id} \right) =: \nabla \cdot (\varrho R_{O1})$$

$$+ \text{div} \left( \frac{n_o \otimes n_c}{\varrho} + \frac{n_c \otimes n_o}{\varrho} + \frac{n_c \otimes n_c}{\varrho} \right), =: \nabla \cdot (\varrho R_{O2}).$$

Then, define $R_{q+1}$ as

$$R_{q+1} = R_T + R_N + R_{O1} + R_{O2} + R_M + \frac{2}{3} \frac{\zeta(t)}{\varrho} \text{Id}. \quad (4.1)$$

Here the last term does not affect $\text{div}(\varrho R_{q+1})$ because $\zeta$ is a function of time, which will be specified in Section 4.2. Our choice of $R_{O2}$ and $R_M$ are

$$\varrho R_{O2} = \frac{n_o \otimes n_c}{\varrho} + \frac{n_c \otimes n_o}{\varrho} + \frac{n_c \otimes n_c}{\varrho}. \quad (4.2)$$

and

$$\varrho R_M = \varrho (R_q - R_\ell) + \frac{m_q - m_\ell}{\varrho} \otimes n + \frac{n \otimes (m_q - m_\ell)}{\varrho} =: \varrho R_{M1} + \varrho R_{M2}. \quad (4.3)$$
which are the only two Reynolds stress errors which might have nonzero trace. For the other errors, we solve the divergence equation by using the inverse divergence operator \( R \) in Definition 2.8 to get trace-free errors, namely we set

\[
\begin{align*}
\varrho R_{O1} &= R \left( \text{div} \left( \frac{n_o \otimes n_o}{\varrho} + \varrho R_{\ell} - \delta_{q+1} \varrho \text{Id} \right) \right) \\
\varrho R_N &= R \left( (n \cdot \nabla) \frac{m_\ell}{\varrho} \right) \\
\varrho R_T &= R \left( \varrho D_{t, \ell} \frac{n}{\varrho} - \text{div}(m_q - m_\ell) \frac{n}{\varrho} \right).
\end{align*}
\]

Here, we used that

\[
\varrho D_{t, \ell} \frac{n}{\varrho} - \text{div}(m_q - m_\ell) \frac{n}{\varrho} = \partial_t n + \text{div} \left( \frac{n \otimes m_\ell}{e} \right)
\]

and hence, they have zero average. As a result, we have

\[
\text{tr}(\varrho R_{q+1}) = \text{tr}(\varrho R_{O2} + \varrho R_M) + 2\zeta,
\]

which gives

\[
\kappa_{q+1} := \frac{1}{2} \text{tr} R_{q+1} = \frac{1}{2} \text{tr} (R_{O2} + R_M) + \frac{\zeta}{\varrho} \tag{4.4}
\]

4.2. New Current

Applying the frequency cut-off \( P_{\leq \ell-1} \) to the Euler-Reynolds system, we have

\[
\partial_t m_\ell + \nabla \cdot \left( \frac{m_\ell \otimes m_\ell}{\varrho} \right) + \nabla p_{\ell}(\varrho) = \nabla \cdot P_{\leq \ell-1}(\varrho R_q - \varrho c_q \text{Id}) + Q(m_q, m_q),
\]

where \( p_{\ell}(\varrho) = P_{\leq \ell-1} p(\varrho) \) and \( Q(m_q, m_q) = Q_{\ell, \varrho}(m_q, m_q) \) is defined as

\[
Q(m_q, m_q) := \nabla \cdot \left( \frac{m_\ell \otimes m_\ell}{\varrho} - P_{\leq \ell-1} \left( \frac{m_q \otimes m_q}{\varrho} \right) \right). \tag{4.5}
\]

Also, we recall that the tuple \((m_q, c_q, R_q, \varphi_q)\) solves

\[
\partial_t \left( \frac{|m_q|^2}{2\varrho} + P(\varrho) \right) + \text{div} \left( \frac{m_q}{\varrho} \left( \frac{|m_q|^2}{2\varrho} + \varrho P'(\varrho) \right) \right) = \varrho \left( \partial_t + \frac{m_q}{\varrho} \cdot \nabla \right) \kappa_q + \text{div}((R_q - c_q \text{Id}) m_q) + \text{div}(\varrho \varphi_q) + \partial_t E
\]

with \( \kappa_q := \frac{1}{2} \text{tr}(R_q) \). Using these equations, we can write the equation for \( \varphi_{q+1} \) as

\[
\begin{align*}
\varrho D_{t, \ell, q+1} \kappa_{q+1} + \text{div}(\varrho \varphi_{q+1}) &= \varrho D_{t, \ell, q} \left( \frac{|n|^2}{2\varrho^2} + \kappa_q + \frac{(m_q - m_\ell) \cdot n}{\varrho^2} \right) + \nabla \cdot \left( \frac{|n|^2 n}{2\varrho^2} + \varrho \varphi_\ell \right) \\
&= \varrho D_{t, q+1} \kappa_{q+1} - (\zeta_1 + \zeta_2 + \zeta_3) \nabla (\varrho \varphi_T) - \nabla (\varrho \varphi_O)
\end{align*}
\]
where $D_{t,q} = \left( \partial_t + \frac{m_q}{\varrho} \cdot \nabla \right)$. The functions $\zeta_1$, $\zeta_2$, $\zeta_3$, and $\zeta_4$ will be defined to invert the divergence. Then, we set

$$
\varrho \varphi_O := \mathcal{R} \left( \nabla \cdot \left( \frac{|n_o|^2 n_o}{2\varrho^2} + \varrho \varphi_\ell \right) \right) + \frac{|n|^2 - |n_o|^2 n_o}{2\varrho^2} \quad =: \varrho \varphi_O^1
$$

$$
\varrho \varphi_R := -R_{q+1} n
$$

$$
\varrho \varphi_{M1} := \frac{|m_q - m_\ell|^2 n}{2\varrho} + \varrho (\varphi_q - \varphi_\ell)
$$

$$
\varrho \varphi_{M2} := \left( \frac{n \otimes n}{\varrho} + \varrho R_q - \varrho R_{q+1} - \delta_{q+1} \varrho \text{Id} \right) \frac{m_q - m_\ell}{\varrho}.
$$

Next, recall that the definition of $\kappa_q$, $R_{O_2}$, $R_M$ and $\kappa_{q+1} = \frac{1}{2} \text{tr}(R_{O_2} + R_M) + \frac{\zeta}{\varrho}$ to get

$$
\frac{|n|^2}{2\varrho^2} + \kappa_q + \frac{(m_q - m_\ell) \cdot n}{\varrho^2}
\quad = \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} \text{Id} + R_\ell \right) + \frac{3}{2} \delta_{q+1} + \frac{1}{2} \text{tr}(R_M + R_{O_2}) \quad \text{(4.6)}
\quad = \frac{3}{2} \delta_{q+1} + \kappa_{q+1} - \frac{\zeta}{\varrho} + \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} \text{Id} + R_\ell \right).
$$

Set $\zeta = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4$, where $\xi_i$ will be determined below. (4.6) then gives the equation for $\varphi_T$; since we have $\varrho D_{t,q} (\xi/\varrho) = \xi' + \text{div}(m_q \xi/\varrho)$,

$$
\nabla \cdot (\varrho \varphi_T) + \xi' = \text{div} \left( -\kappa_{q+1} n + \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} \text{Id} + R_\ell \right) (m_q - m_\ell) - \frac{m_q \zeta}{\varrho} \right)
\quad + \frac{\varrho}{2} D_{t,\ell} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} \text{Id} + R_\ell \right) - \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} \text{Id} + R_\ell \right) \text{div}(m_q - m_\ell)
$$
Now, we define $\zeta_0$ to make the divergence equation solvable and set $\varphi_T = \varphi_{T1} + \varphi_{T2}$ as

$$Q\varphi_{T1} = -\kappa_{q+1} n + \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{Q^2} - \delta_{q+1} \text{Id} + R_\ell \right) (m_q - m_\ell) - \frac{m_q \zeta}{Q} \quad (4.7)$$

$$\zeta_0(t) = \int_0^t \left\{ \frac{1}{2} D_{t,\ell} \text{tr} \left( \frac{n_o \otimes n_o}{Q^2} - \delta_{q+1} \text{Id} + R_\ell \right) \right\}(s) \, ds$$

$$\varphi_{T2} = R \left( \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{Q^2} - \delta_{q+1} \text{Id} + R_\ell \right) \right)$$

Here, we remark that, by the definition of $R$, we have

$$R(g(t, \cdot)) := R(g(t, \cdot) - h(t)) \quad (4.9)$$

for every smooth periodic time-dependent vector field $g$ and for every $h$ which depends only on time.

In a similar way, we let

$$\zeta_1(t) := \int_0^t \left\{ \frac{n}{Q} \cdot (\text{div} \, P_{\leq \ell-1}(Q(R_q - c_q \text{Id})) + Q(m_q, m_\ell)) \right\}(s) \, ds$$

$$\zeta_2(t) := \int_0^t \left\{ \text{div}(m_q - m_\ell) \frac{n \cdot m_\ell}{Q^2} \right\}(s) \, ds$$

$$\zeta_3(t) := \int_0^t \left\{ \left( \frac{n \otimes n}{Q} - \delta_{q+1} \text{Id} + \varrho R_q - \left( \varrho R_{q+1} - \frac{2}{3} \varrho \zeta \text{Id} \right) \right) : \nabla \frac{m_\ell}{Q} \right\}(s) \, ds$$

$$+ \int_0^t \left\{ \left( \frac{(m_q - m_\ell) \otimes n}{Q} + \frac{n \otimes (m_q - m_\ell)}{Q} \right) : \nabla \frac{m_\ell}{Q} \right\}(s) \, ds$$

$$\zeta_4(t) := \int_0^t \left\{ \frac{n}{Q} \cdot \nabla (p(\varrho) - p_\ell(\varrho)) \right\}(s) \, ds \quad (4.10)$$

and

$$Q\varphi_{H1} := R \left( \frac{n}{Q} \cdot (\text{div} \, P_{\leq \ell-1}(Q(R_q - c_q \text{Id})) + Q(m_q, m_\ell)) \right)$$

$$Q\varphi_{M3} := R \left( \text{div}(m_q - m_\ell) \frac{n \cdot m_\ell}{Q^2} \right)$$

$$Q\varphi_{M4} := R \left( \nabla \cdot (p(\varrho) - p_\ell(\varrho)) \right)$$

$$Q\varphi_{H2} := R \left( \left( \frac{n \otimes n}{Q} - \delta_{q+1} \varrho \text{Id} + \varrho R_q - \left( \varrho R_{q+1} - \frac{2}{3} \varrho \zeta \text{Id} \right) \right) : \nabla \frac{m_\ell}{Q} \right)$$

$$+ R \left( \left( \frac{(m_q - m_\ell) \otimes n}{Q} + \frac{n \otimes (m_q - m_\ell)}{Q} \right) : \nabla \frac{m_\ell}{Q} \right) - \frac{2m_\ell \zeta}{3Q} .$$

Here, $\zeta_3$ is well-defined because $\varrho R_{q+1} - (2\zeta)/3 \text{Id}$ is independent of $\zeta$ (see (4.1)).
5. Preliminary Estimates

We now start detailing the estimates which will lead to the proof of the inductive propositions. In this section, we set $\| \cdot \|_N = \| \cdot \|_{C^0([0,T]+\tau;C^N(\mathbb{T}^3))}$.

5.1. Regularization

First of all we address a series of a-priori estimates on the regularized tuple and on their differences with the original one. By its construction, we can easily see that

$$\|m_\ell\|_N \lesssim N \ell^{1-N} \lambda_q \delta_q^{\frac{1}{2}}, \quad \forall N \geq 1,$$

$$\|D_{\ell,\ell}^s R_\ell\|_0 \lesssim s \ell^{-s} \lambda_q^{-3} \delta_q^{\frac{3}{2}}, \quad \|D_{\ell,\ell}^s q_\ell\|_0 \lesssim s \ell^{-s} \lambda_q^{-3} \delta_q^{\frac{3}{2}}, \quad \forall s \geq 0.$$

Also, there exists $\bar{b}(\alpha) > 1$ such that for any $b \in (1, \bar{b}(\alpha))$ we can find $\Lambda_0 = \Lambda_0(\alpha, b, M, \varrho)$ with the following property: if $\lambda_0 \geq \Lambda_0$, then $|\nabla^{N+1}\phi(t + \tau, x; t)| \lesssim M, \varrho \ell^{-N}$ holds for $N \geq 0$ and $\tau \in [-\ell, \ell]$. This implies

$$\ell^{2} \| D_{\ell,\ell}^s R_\ell\|_{s,N,M,\varrho} \ell^{-N} \lambda_q^{-3} \delta_q^{\frac{3}{2}},$$

$$\ell^{2} \| D_{\ell,\ell}^s q_\ell\|_{s,N,M,\varrho} \ell^{-N} \lambda_q^{-3} \delta_q^{\frac{3}{2}}.$$
Also, we have
\[(Q_{\ell_t} * \Phi F - F)(t, x) = \int_{\mathbb{R}} (F(t + s, \Phi(t + s, x; t)) - F(t, x))Q_{\ell_t}(s) \, ds\]
\[= \int_{\mathbb{R}} \int_0^s D_{t, \ell} F(t + \tau, \Phi(t + \tau, x; t)) \, d\tau Q_{\ell_t}(s) \, ds,\]
from which we conclude \(\| F - Q_{\ell_t} * \Phi F \|_{C^0([a, b] \times \mathbb{T}^3)} \lesssim \ell_t \| D_{t, \ell} F \|_{C^0([a, b] + \ell_t \times \mathbb{T}^3)}\) because of \(\text{supp}(Q_{\ell_t}) \subset (-\ell_t, \ell_t)\). In addition, we have the following decomposition,
\[F - Q_{\ell_t} * \Phi P_{\leq \ell-1} F = (F - P_{\leq \ell-1} F) + (P_{\leq \ell-1} F - Q_{\ell_t} * \Phi P_{\leq \ell-1} F), \quad (5.7)\]
\[D_{t, \ell} P_{\leq \ell-1} F = P_{\leq \ell-1} D_{t, \ell} F + \left[ \frac{m_{\ell}}{Q} \cdot \nabla, P_{\leq \ell-1} \right] F, \quad (5.8)\]
where as usual \([A, B]\) denotes the commutator \(AB - BA\) of the two operators \(A\) and \(B\). Note that \(D_{t, \ell} F\) can be further decomposed as \(D_{t, \ell} F = D_{t, q} F + \frac{(m_q - m_{\ell})}{Q} \cdot \nabla F\). Then, using \((2.3), (2.4), (3.8),\) and \((A.4),\) we obtain
\[\| R_q - R_{\ell_t} \|_0 \lesssim \| P_{\geq \ell-1} R_q \|_0 + \ell_t \| D_{t, \ell} P_{\leq \ell-1} R_q \|_{C(\mathcal{I}^q \times \mathbb{T}^3)}\]
\[\lesssim \ell^2 \| R_q \|_2 + \ell_t \| D_{t, \ell} R_q \|_{C(\mathcal{I}^q \times \mathbb{T}^3)} + \ell \| \nabla \frac{m_{\ell}}{Q} \|_{C(\mathcal{I}^q \times \mathbb{T}^3)} \| \nabla R_q \|_{C(\mathcal{I}^q \times \mathbb{T}^3)}\]
\[\lesssim ((\ell \lambda_q)^2 + \ell_t \lambda_q \delta_q^{\frac{1}{2}}) \lambda_q^{-3\gamma} \delta_{q+1} \lesssim \frac{1}{2} \frac{1}{2} \frac{3}{2} \lambda_q \lambda_q \delta_{q+1} \delta_{q+1},\]
and
\[\| \varphi_q - \varphi_{\ell_t} \|_0 \lesssim \| P_{\geq \ell-1} \varphi_q \|_0 + \ell_t \| D_{t, \ell} P_{\leq \ell-1} \varphi_q \|_{C(\mathcal{I}^q \times \mathbb{T}^3)}\]
\[\lesssim \ell^2 \| \varphi_q \|_2 + \ell_t \| D_{t, \ell} \varphi_q \|_{C(\mathcal{I}^q \times \mathbb{T}^3)} + \ell \| \nabla \frac{m_{\ell}}{Q} \|_{C(\mathcal{I}^q \times \mathbb{T}^3)} \| \nabla \varphi_q \|_{C(\mathcal{I}^q \times \mathbb{T}^3)}\]
\[\lesssim ((\ell \lambda_q)^2 + \ell_t \lambda_q \delta_q^{\frac{1}{2}}) \lambda_q^{-3\gamma} \delta_{q+1}^{\frac{3}{2}} \lesssim \frac{1}{2} \frac{1}{2} \frac{3}{2} \lambda_q \lambda_q \delta_{q+1} \delta_{q+1},\]
where \(\mathcal{I}^q = [0, T] + \tau_{q-1}\). Furthermore, we have for \(N = 1, 2\)
\[\| R_q - R_{\ell_t} \|_N \lesssim \| R_q \|_N + \| R_{\ell_t} \|_N \lesssim \lambda_q^N \lambda_q^{-3\gamma} \delta_{q+1} \lesssim \lambda_q^{N+1} \lambda_q^{-\frac{3}{2}} \delta_{q+1}^\frac{3}{2} \delta_{q+1}^{\frac{3}{2}}\]
\[\| \varphi_q - \varphi_{\ell_t} \|_N \lesssim + \| \varphi_q \|_N + \| \varphi_{\ell_t} \|_N \lesssim \lambda_q^N \lambda_q^{-3\gamma} \delta_{q+1}^{\frac{3}{2}} \lesssim \lambda_q^{N+1} \lambda_q^{-\frac{3}{2}} \delta_{q+1}^\frac{3}{2} \delta_{q+1}^{\frac{3}{2}}.\]
Now, we consider the advective derivatives. We remark that for \(F_{\ell_t} = P_{\leq \ell-1} F\), we can write
\[D_{t, \ell} (F - F_{\ell_t}) = D_{t, \ell} P_{\geq \ell-1} F = P_{\geq \ell-1} D_{t, \ell} F + \left[ \frac{m_{\ell}}{Q} \cdot \nabla, P_{\geq \ell-1} \right] F.\]
Then, we apply this to \(F = m\) and \(F = p(\rho)\) and use \((5.6)\) and \((A.5)\) to obtain
\[\| D_{t, \ell} (m_q - m_{\ell}) \|_{N-1} \lesssim \| P_{\geq \ell-1} D_{t, \ell} m_q \|_{N-1} + \| \frac{m_{\ell}}{Q} \cdot \nabla, P_{\geq \ell-1} \| \| m_q \|_{N-1}\]
\[\lesssim \| P_{\geq \ell-1} D_{t, q} m_q \|_{N-1} + \ell \| (m_q - m_{\ell}) / \rho \cdot \nabla m_q \|_{N-1}\]
\[\lesssim \ell^{2-N} (\lambda_q \delta_q^{\frac{1}{2}})^2. \quad (5.9)\]
Here, we obtain the estimate for $\| P_{> \ell^{-1}} D_t q m_q \|_{N-1}$ from the relaxed momentum equation:

$$\| P_{> \ell^{-1}} D_t q m_q \|_{N-1} \leq \| P_{> \ell^{-1}} p(\varrho) \|_N + \| P_{> \ell^{-1}} (\varrho R_q) \|_N + \| P_{> \ell^{-1}} (\text{div}(m_q/\varrho) m_q) \|_N - 1$$

$$\lesssim \ell^2 (\| p(\varrho) \|_{N+2} + \| \varrho \|_{N+2}) + \ell^{2-N} \| \varrho R_q \|_2 + \ell \| m_q ( - \partial_t \varrho/\varrho + m_q \cdot \nabla \varrho^{-1} ) \|_N$$

$$\lesssim \ell^{2-N} (\varrho_q \delta_q^2)^2.$$

In a similar way, we have

$$\| D_t,\ell P_{> \ell^{-1}} R_q \|_{N-1} \lesssim \| D_t,\ell p \|_{N-1} + \| (m_q/\varrho) \cdot \nabla, \ P_{> \ell^{-1}} R_q \|_{N-1} \lesssim \lambda_{q+1}^{N-1} \lambda_q^{1/2} \lambda_q^{-3} \delta_q^{3/2}$$

$$\| D_t,\ell P_{> \ell^{-1}} \varrho \|_{N-1} \lesssim \| (m_q/\varrho) \cdot \nabla, \ P_{> \ell^{-1}} \varrho \|_{N-1} \lesssim \lambda_{q+1}^{N-1} \lambda_q^{1/2} \lambda_q^{-3} \delta_q^{3/2}.$$

Simply applying the triangle inequality, it can be easily shown that

$$\| D_t,\ell (P_{\leq \ell^{-1}} R_q - \varrho, * \Phi, P_{\leq \ell^{-1}} R_q) \|_{N-1} \leq 2 \| D_t,\ell P_{\leq \ell^{-1}} R_q \|_{N-1} \lesssim \lambda_{q+1}^{N-1} \lambda_q^{1/2} \lambda_q^{-3} \delta_q^{3}$$

(5.10)

$$\| D_t,\ell (P_{\leq \ell^{-1}} \varrho - \varrho, * \Phi, P_{\leq \ell^{-1}} \varrho) \|_{N-1} \leq 2 \| D_t,\ell P_{\leq \ell^{-1}} \varrho \|_{N-1} \lesssim \lambda_{q+1}^{N-1} \lambda_q^{1/2} \lambda_q^{-3} \delta_q^{3}.$$

(5.11)

Combining (5.7), (5.10), and (5.11), it follows that

$$\| D_t,\ell (R_q - R_\ell) \|_{N-1} \lesssim \lambda_{q+1}^{N} \delta_q^{1/2} \cdot \lambda_q^{1/2} \lambda_q^{-1/2} \delta_q^{5/2} \delta_q^{3}$$

$$\| D_t,\ell (\varrho - \varrho_\ell) \|_{N-1} \lesssim \lambda_{q+1}^{N} \delta_q^{1/2} \cdot \lambda_q^{1/2} \lambda_q^{-1/2} \delta_q^{5/2} \delta_q^{3}.$$

5.2. Quadratic Commutator

We next deal with a quadratic commutator estimate, which is a version of the estimate in [13] leading to the proof of the positive part of the Onsager conjecture for the incompressible Euler equations. In the compressible case, the situation is more complicated due to the presence of the density $\varrho$. Indeed, it leads to additional commutator terms which need to be estimated. Another difference from the incompressible case found in [17] is that we can do the estimate for a fixed, finite number of derivatives but cannot estimate all derivatives. This is because $\varrho$ is just a smooth function and thus only a fixed number of its derivatives can be controlled.
Lemma 5.2. For any integer $N \in [0, \overline{N}]$, $\overline{N} \in \mathbb{N}$ independent of $q$, $Q_e(m_q, m_q)$ defined as in (4.5) satisfies
\[
\|Q(m_q, m_q)\|_N \lesssim \ell^{1-N} (\lambda_q \delta^2_q)^2, \quad \|D_t,\ell Q(m_q, m_q)\|_N \lesssim \ell^{-N} \delta_q (\lambda_q \delta^2_q)^2.
\]

Here, we allow the implicit constants to be depending on $M, \varrho, p$ and $\overline{N}$.

Proof. For the convenience, we drop the index $q$ from $Q_e$ and $q$ from $m_q$. We write $Q(m, m)$ as follows
\[
Q(m, m) = \nabla \cdot \left( \frac{m_\ell \otimes m_\ell}{\varrho} - \frac{P_{\leq \ell^{-1}} (m \otimes m)}{\varrho} \right) + \nabla \cdot \left( \frac{P_{\leq \ell^{-1}} (m \otimes m)}{\varrho} - \frac{P_{\leq \ell^{-1}} (m \otimes m)}{\varrho} \right)
\]
\[
= Q_1 + Q_2
\]
Writing $Q_1$ as
\[
Q_1 = \varrho^{-1} \nabla \cdot \left( m_\ell \otimes m_\ell - P_{\leq \ell^{-1}} (m \otimes m) \right) + \left( m_\ell \otimes m_\ell - P_{\leq \ell^{-1}} (m \otimes m) \right) : \nabla \varrho^{-1}
\]
and using (A.2) on setting $f = g = m$, we have
\[
\|Q_1\|_N \leq \|Q_{11}\|_N + \|Q_{12}\|_N \lesssim \|m_\ell \otimes m_\ell - P_{\leq \ell^{-1}} (m \otimes m)\|_{N+1} \lesssim \ell^{1-N} \|m\|_1^2 \lesssim \ell^{1-N} (\lambda_q \delta^2_q)^2.
\]
On the other hand, $Q_2$ can be written as
\[
Q_2 = \frac{P_{\leq \ell^{-1}} \text{div} (m \otimes m)}{\varrho} - \frac{P_{\leq \ell^{-1}} (\varrho^{-1} \text{div} (m \otimes m))}{\varrho} + P_{\leq \ell^{-1}} (m \otimes m) : \nabla \varrho^{-1} - P_{\leq \ell^{-1}} (m \otimes m : \nabla \varrho^{-1}).
\]
Here, $\nabla \varrho^{-1}$ means that $\nabla (1/\varrho)$ rather than $(\nabla \varrho)^{-1}$. Since both the first two and last two terms are of the form $P_{\leq \ell^{-1}} (f)g - P_{\leq \ell^{-1}} (fg)$, we can apply (A.3) to get that
\[
\|Q_2\|_N \lesssim \ell^{1-N} \lambda_q \delta^2_q + \ell^{1-N} \lesssim \ell^{1-N} (\lambda_q \delta^2_q)^2.
\]
Now, we consider the advective derivative. We first show $\|D_t,\ell (Q Q_{11})\|_N \lesssim \ell^{-N} (\lambda_q \delta^2_q)^2$. Then, the desired estimate $\|D_t,\ell Q_{11}\|_N \lesssim \ell^{-N} (\lambda_q \delta^2_q)^2$ easily follows because
\[
\|D_t,\ell Q_{11}\|_N \lesssim \|D_t,\ell (Q Q_{11})\|_N + \sum_{N_1 + N_2 = N} \|D_t,\ell Q\|_{N_1} \|Q_{11}\|_{N_2}.
\]
We first note that using the equation, obtained by taking mollification \( P_{\leq \ell^{-1}} \) to the relaxed momentum equation,
\[
D_{t, \ell} m_\ell = - \text{div}(m_\ell q) m_\ell - \nabla p_\ell(q) - P_{\leq \ell^{-1}} \text{ div}(\varrho R_q - c_q \varrho \text{ Id}) + Q(m, m)
= (m_\ell P_{\leq \ell^{-1}} \partial_t q) q - m_\ell (m_\ell \cdot \nabla) q^{-1} - \nabla p_\ell(q)
+ P_{\leq \ell^{-1}} \text{ div}(\varrho R_q - c_q \varrho \text{ Id}) + Q(m, m),
\]
we get
\[
\| D_{t, \ell} m_\ell \|_0 \lesssim_{\varrho, \varrho} \| m_\ell \|_0 + \| m_\ell \|_0^2 + 1 + \| R_q \|_1 + \| Q(m, m) \|_0 \lesssim_{\varrho, \varrho} \lambda_q \delta_q
\] (5.13)
and for \( N \geq 1, \)
\[
\| D_{t, \ell} m_\ell \|_N \lesssim \ell^{-N} \| m_{\ell} P_{\leq \ell^{-1}} \partial_t q \|_1 \cdot \| q^{-1} \|_N + \| m_{\ell} \otimes m_{\ell} \|_1 \| \nabla q^{-1} \|_N + \| p_\ell(q) \|_2 + \| q \|_2) \]
\[
+ \ell^{-N} \| R_q \|_2 + \| Q(m, m) \|_N
\lesssim \ell^{-N} (\lambda_q \delta_q^2)^2.
\] (5.14)
As a consequence, we have
\[
\| D_{t, \ell} \nabla m_\ell \|_N \leq \| \nabla D_{t, \ell} m_\ell \|_N + \| (\nabla (m_\ell q) \cdot \nabla) m_\ell \|_N \lesssim \ell^{-N} (\lambda_q \delta_q^2)^2.
\] (5.15)
Also, similar to (5.9), one can obtain
\[
\| D_{t, \ell} m_\ell \|_0 \lesssim_{\varrho, \varrho} \lambda_q \delta_q.
\] (5.16)
Since \( \varrho Q_{11} \) can be decomposed into
\[
\varrho Q_{11} = (m_\ell - m) \cdot \nabla m_\ell + [m \cdot \nabla, P_{\leq \ell^{-1}}] m + m_\ell \text{ div } m_\ell - P_{\leq \ell^{-1}} (m \text{ div } m)
\] (5.17)
their advective derivative can be estimated as follows; using the density equation,
\[
\| D_{t, \ell} (m_\ell \text{ div } m_\ell - P_{\leq \ell^{-1}} (m \text{ div } m)) \|_N = \| D_{t, \ell} (m_\ell \partial_t q) \|_N
\leq \| (D_{t, \ell} m_\ell) \partial_t q + m_\ell (D_{t, \ell} \partial_t q) \|_N + \| m_\ell P_{\leq \ell^{-1}} (m \partial_t q) \|_N
\lesssim \| D_{t, \ell} m_\ell \|_N + \sum_{N_1 = N_2 = N} \| m_\ell \|_{N_1} \| D_{t, \ell} \partial_t q \|_{N_2} + \ell^{-N} \| D_{t, \ell} (m \partial_t q) \|_0
\lesssim \ell^{-1} (\lambda_q \delta_q^2)^2 + \ell^{-N} \delta_q^2 + \ell^{-N} \lambda_q \delta_q + \ell^{-1} (\lambda_q \delta_q^2)^2 + \ell^{-1} (\lambda_q \delta_q^2)^2 \lesssim \ell^{-N} (\lambda_q \delta_q^2)^2.
\]
Here, we denoted \( P_{\leq \ell^{-1}} \partial_t q \) by \( (\partial_t q) \ell \) and used (5.14), (2.2), (5.16), (A.4), and (A.5).

To estimate the first two terms in (5.17), we recall that
\[
\hat{P}_{\leq \ell^{-1}} f(\xi) = \hat{P}_{\leq 2^j} f(\xi) = \phi \left( \frac{\xi}{2^j} \right) \hat{f}(\xi)
\] for some radial function \( \phi \in S \), where \( J \in \mathbb{N} \cup \{0\} \) is the maximum number satisfying \( 2^J \leq \ell^{-1} \). For the convenience,
we set \( \tilde{\phi}_t(x) = 2^J \phi(2^J x) \). Then, by Poison summation formula, \( P_{\leq \ell^{-1}} f(x) = \int_{\mathbb{R}^3} f(x - y) \tilde{\phi}_t(y)\,dy \) holds. Using this, the advective derivative of the commutator term can be written as follows,

\[
D_{t,\ell}[m \cdot \nabla, P_{\leq \ell^{-1}}]m = (\partial_t + \frac{m_\ell}{Q}(x) \cdot \nabla) \int ((m(x) - m(x - y)) \cdot \nabla)m(x - y) \tilde{\phi}_t(y)\,dy
\]

\[
= \int ((D_{t,\ell} m(x) - D_{t,\ell} m(x - y)) \cdot \nabla)m(x - y) \tilde{\phi}_t(y)\,dy
\]

\[
- \int (\frac{m_\ell}{Q}(x) - \frac{m_\ell}{Q}(x - y)) a \nabla_a m_b (x - y) \nabla b m(x - y) \tilde{\phi}_t(y)\,dy
\]

\[
+ \int ((m(x) - m(x - y)) \cdot D_{t,\ell} \nabla)m(x - y) \tilde{\phi}_t(y)\,dy
\]

\[
+ \int (m(x) - m(x - y)) a (\frac{m_\ell}{Q}(x) - \frac{m_\ell}{Q}(x - y)) b (\partial_{ab} m)(x - y) \tilde{\phi}_t(y)\,dy.
\]

Based on the decomposition, we use (2.2), (5.3), and \( \|y^n \tilde{\phi}_t\|_{L^1(\mathbb{R}^3)} \lesssim \ell^n, n \geq 0 \), to get

\[
\|D_{t,\ell}(\partial Q_{11})\|_0 \\
\lesssim \|D_{t,\ell}(m - m_{\ell})\|_0 \|\nabla m\|_0 + \|m - m_{\ell}\|_0 \|D_{t,\ell} \nabla m_{\ell}\|_0 + \ell \|\nabla D_{t,\ell} m\|_0 \|\nabla m\|_0 + \ell \|m\|_1^3
\]

\[
+ \ell \|\nabla m\|_0 \|D_{t,\ell} \nabla m\|_0 + \ell^2 \|m\|_1 \|\nabla^2 m\|_0 + \delta_q^\frac{1}{2} (\lambda_q \delta_q^\frac{1}{2})^2 \lesssim \delta_q^\frac{1}{2} (\lambda_q \delta_q^\frac{1}{2})^2
\]

where we used (5.14). In the case of \( N \geq 1 \), we write

\[
\|D_{t,\ell}(\partial Q_{11})\|_N \\
\lesssim \|(\partial_t + m_{\ell} P_{\leq \ell^{-1}}^{-1} \cdot \nabla)(\partial Q_{11})\|_N + \|(m_{\ell} P_{\leq \ell^{-1}}^{-1} \cdot \nabla) (\partial Q_{11})\|_0
\]

\[
\lesssim \ell^{-N} \|(\partial_t + m_{\ell} P_{\leq \ell^{-1}}^{-1} \cdot \nabla) (\partial Q_{11})\|_0
\]

\[
+ \sum_{N_1 + N_2 + N_3 = N} \|m_{\ell}\|_{N_1} \|P_{\leq \ell^{-2}} Q_{11}^{-1}\|_{N_2} \|\nabla (\partial Q_{11})\|_{N_3}
\]

\[
\lesssim \ell^{-N} (\|D_{t,\ell}(\partial Q_{11})\|_0 + \|(m_{\ell} P_{\leq \ell^{-1}}^{-1} \cdot \nabla) Q_{11}\|_0) + \ell^2 \|\lambda_q \delta_q^\frac{1}{2}\|^2
\]

\[
\lesssim \ell^{-N} (\|D_{t,\ell}(\partial Q_{11})\|_0 + \delta_q^\frac{1}{2} (\lambda_q \delta_q^\frac{1}{2})^2) + \ell^{1-N} (\lambda_q \delta_q^\frac{1}{2})^3
\]

\[
\lesssim \ell^{-N} \delta_q^\frac{1}{2} (\lambda_q \delta_q^\frac{1}{2})^2.
\]

The second inequality follows from Bernstein’s inequality.

In order to estimate the advective derivative of \( Q_{12} \), we first estimate
Observe that we can estimate the advective derivative of
\[ Q = \frac{D_t \ell (m_\ell \otimes m_\ell - P_{\leq \ell - 1} (m \otimes m))}{N} \]
follows from (A.5) for the latter two terms in the estimate. The third inequality
\[ \ell^2 N |D_{t,\ell}m| |m_\ell| + \sum_{N_1 + N_2 = N} \ell_1 |\nabla (m_\ell P_{\leq \ell - 1} q^{-1})|_{N_1} |m_\ell| |m_\ell|_{N_2} \]
\[ + \ell^1 |\nabla (m_\ell P_{\leq \ell - 1} q^{-1})|_{N_1} |m_\ell| |m_\ell|_{N_2} \]
\[ \lesssim \ell^2 N \lambda_2 \delta_q^2 \frac{1}{\lambda_q \delta_q^2} + \ell^1 \lambda q \delta_q^2 + \ell^1 \lambda q \delta_q^2 + \ell^1 \lambda q \delta_q^2 \]
\[ \lesssim \ell^1 \lambda q \delta_q^2 \]

In the second inequality, we have used (A.2) for the first term, (A.4) for the second and third term and (A.5) for the latter two terms in the estimate. The third inequality follows from \( |m_\ell P_{\leq \ell - 1} q^{-1}|_{N_1} \lesssim q_1 \). Then, it follows that
\[ |D_{t,\ell} Q_{12}| \lesssim |\nabla q^{-1} : D_{t,\ell} (m_\ell \otimes m_\ell - P_{\leq \ell - 1} (m \otimes m))|_{N} \]
\[ + |(m_\ell \otimes m_\ell - P_{\leq \ell - 1} (m \otimes m)) : D_{t,\ell} \nabla q^{-1}|_{N} \lesssim \ell^1 \delta_q^2 (\lambda q \delta_q^2)^2 . \]

Finally, it remains to estimate the term \( Q_2 \). We write this as
\[ Q_2 = \frac{P_{\leq \ell - 1} \text{div}(m \otimes m)}{q} - P_{\leq \ell - 1} \left( \frac{\text{div}(m \otimes m)}{q} \right) + (P_{\leq \ell - 1} (m \otimes m)) \nabla q^{-1} \]
\[ - P_{\leq \ell - 1} (m \otimes m \nabla q^{-1}) \]
Since both the first and last two terms are of the form \( P_{\leq \ell - 1} f g - P_{\leq \ell - 1} (f g) \), we can apply (A.3) to get that
\[ |Q_2|_{N} \lesssim \ell^1 \lambda q \delta_q^2 + \ell^1 \lambda q \delta_q^2 \]
Observe that we can estimate the advective derivative of \( P_{\leq \ell - 1} f g - P_{\leq \ell - 1} (f g) \) as follows
\[ |D_{t,\ell} (f g - (f g)\ell)|_{N} = |D_{t,\ell} (f g) - D_{t,\ell} (f g)\ell)|_{N} \]
\[ = |(D_{t,\ell} f) g + f \ell D_{t,\ell} g - (D_{t,\ell} f g)\ell - [m_\ell / q \cdot \nabla, P_{\leq \ell - 1}] (f g)|_{N} \]
\[ \lesssim |(D_{t,\ell} f) g - ((D_{t,\ell} f) g)\ell|_{N} + \| f \ell D_{t,\ell} g - (f D_{t,\ell} g)\ell\|_{N} \]
\[ + \| (m_\ell / q) \cdot \nabla, P_{\leq \ell} \|_{N} \sum_{N_1 + N_2 = N} \| f \ell D_{t,\ell} g - (f D_{t,\ell} g)\ell\|_{N_2} \]
\[ + \| f \ell (D_{t,\ell} g)\ell - (f D_{t,\ell} g)\ell\|_{N} + \| (m_\ell / q) \cdot \nabla, P_{\leq \ell - 1} \|_{N} \cdot \]
We estimate this by applying (A.3) to the first norm, (A.2) to the third. For the last term, we first note that since $Q$ is smooth, we can bound a finite number of its derivatives and so we can bound $\|m_t/Q\|_N \lesssim_N \ell^{-N} \delta_q^{\frac{1}{2}}$ for $N \in [1, \overline{N}]$; then we use Lemma A.6. Therefore, we get the estimate for advective derivative of $Q_2$ as

$$\|D_{t, \ell} Q_2\|_N \lesssim \ell^{1-N} \|D_{t, \ell} \text{div}(m \otimes m)\|_0 \|Q^{-1}\|_{\max(1, N)} + \sum_{N_1+N_2=N} \epsilon^{-N_1} \|\text{div}(m \otimes m)\|_0 \|D_{t, \ell} \text{div}(m \otimes m)\|_0$$

Finally we address the estimates on the backward flow $\xi_1$.

**Lemma 5.3.** For every $b > 1$ there exists $\Lambda_0 = \Lambda_0(b, \varrho)$ such that for $\lambda_0 \geq \Lambda_0$ the backward flow map $\xi_1$ satisfies, for any $N \geq 0$ and $s = 0, 1, 2$, the following estimates on the time interval $\mathcal{I}_u = [t_u - \frac{1}{\tau_q} \tau_q, t_u + \frac{3}{2} \tau_q] \cap [0, T] + 2 \tau_q$:

$$\|\text{Id} - \nabla \xi_1\|_{C^0(\mathcal{I}_u \times \mathbb{R}^3)} \lesssim \frac{1}{5}$$ (5.18)

$$\|D_{t, \ell}^s \nabla \xi_1\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim_{\varrho, p, N, M} \ell^{-N}(\lambda_q \delta_q^{\frac{1}{2}})^s$$ (5.19)

$$\|D_{t, \ell}^s (\nabla \xi_1)^{-1}\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim_{\varrho, p, N, M} \ell^{-N}(\lambda_q \delta_q^{\frac{1}{2}})^s$$ (5.20)

Note that the implicit constants in the inequalities are independent of the index $I = (m, n, f)$. In particular,

$$\|\nabla \xi_1\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} + \|(\nabla \xi_1)^{-1}\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim_{\varrho, N} \ell^{-N}.$$ (5.21)

The implicit constant in this inequality is also independent of $M$.
Proof. First, we can find $\lambda_0(b, \varrho)$ such that for any $\lambda \geq \lambda_0(b)$, $\tau_q \| \nabla \frac{m_{\ell}}{\varrho} \|_0 \leq \frac{1}{10}$ holds. Then, (5.18) easily follows from (3.19). Also,

$$\| \nabla \xi_I \|_{C^0([0,T]; C^N(\mathbb{R}^3))} \lesssim N 1 + \tau_q \| \nabla \frac{m_{\ell}}{\varrho} \|_N \lesssim \epsilon_N 1 + \epsilon^{-N} \lesssim \epsilon_N \epsilon^{-N}$$

(5.22)

from which follows $\| (\nabla \xi_I)^{-1} \|_{C^0([0,T]; C^N(\mathbb{R}^3))} \lesssim N \epsilon^{-N}$. Since we have

$$D_{t, \ell} \nabla \xi_I = -(\nabla \xi_I)(\nabla \frac{m_{\ell}}{\varrho}), \quad D_{t, \ell}^2 \nabla \xi_I = (\nabla \xi_I)(\nabla \frac{m_{\ell}}{\varrho})^2 - (\nabla \xi_I) D_{t, \ell} \nabla \frac{m_{\ell}}{\varrho},$$

using (5.15) and (5.22), $\| D_{t, \ell}^s \nabla \xi_I \|_{C^0([0,T]; C^N(\mathbb{R}^3))} \lesssim \epsilon, \epsilon, M \epsilon^{-N}(\lambda q \delta^q_1)^s$ easily follows. Lastly, we have

$$D_{t, \ell}(\nabla \xi_I)^{-1} = \nabla \frac{m_{\ell}}{\varrho}(\nabla \xi_I)^{-1}, \quad D_{t, \ell}^2(\nabla \xi_I)^{-1} = D_{t, \ell} \nabla \frac{m_{\ell}}{\varrho}(\nabla \xi_I)^{-1} + \left( \nabla \frac{m_{\ell}}{\varrho} \right)^2 (\nabla \xi_I)^{-1}.$$

Therefore, (5.20) can be obtained similarly. $\square$

6. Estimates in the Momentum Correction

The main point of this section is to get the estimates on the momentum correction. In this section, we set $\| \cdot \|_N = \| \cdot \|_{C^0([0,T] + \tau_q; C^N(\mathbb{T}^3))}$.

The following proposition provides the estimates for the perturbation $n$:

**Proposition 6.1.** For $N = 0, 1, 2$ and $s = 0, 1, 2$, the following estimates hold for $n_o, n_c$, and $n = n_o + n_c$:

$$\tau_q^s \| D_{t, \ell}^s n_o \|_N \lesssim \epsilon, \epsilon, M \lambda_{q+1}^N \delta_{q+1}^s$$

(6.1)

$$\tau_q^s \| D_{t, \ell}^s n_c \|_N \lesssim \epsilon, \epsilon, M \lambda_{q+1}^N \delta_{q+1}^s$$

(6.2)

$$\tau_q^s \| D_{t, \ell}^s n \|_N \lesssim \epsilon, \epsilon, M \lambda_{q+1}^N \delta_{q+1}^s$$

(6.3)

there the implicit constants are independent of $s, N$, and $q$. Moreover,

$$\| n \|_N \lesssim \epsilon, \lambda_{q+1}^N \delta_{q+1}^s$$

(6.4)

where the implicit constant is additionally independent of $p$ and $M$.

The latter estimates are in fact a simple consequence of estimates on the functions $b_{u,k}, c_{u,k}, d_{u,k}$ and $e_{u,k}$ defined in (3.32) and (3.34)
Lemma 6.2. For any $N \geq 0$ and $s = 0, 1, 2$, the coefficients $b_{u,k}$, $c_{u,k}$, $d_{u,k}$, and $e_{u,k}$ defined by (3.32) and (3.34) satisfy the following,

$$
\tau_q^s \| D_{t,\ell}^s b_{u,k} \|_N \lesssim \mu_q^{-N} \max_I |\hat{b}_{I,k}| \quad (6.5)
$$

$$
\tau_q^s \| D_{t,\ell}^s c_{u,k} \|_N \lesssim \mu_q^{-N} \max_I |\hat{c}_{I,k}| \quad (6.6)
$$

$$
\tau_q^s \| D_{t,\ell}^s d_{u,k} \|_N \lesssim \mu_q^{-N} \max_I |\hat{d}_{I,k}| \quad (6.7)
$$

$$
\tau_q^s \| D_{t,\ell}^s e_{u,k} \|_N \lesssim \mu_q^{-N} \max_I |\hat{b}_{I,k}|. \quad (6.8)
$$

where the implicit constants in the inequalities (6.5)–(6.8) depend on $q$, $p$, $N$, and $M$. Moreover, for $N = 0, 1, 2$,

$$
\|b_{u,k}\|_N + \|e_{u,k}\|_N \lesssim \mu_q^{-N} \max_I |\hat{b}_{I,k}|. \quad (6.9)
$$

where the implicit constant is independent of $M$ and $N$.

Remark 6.3. Observe that, by the definition of the respective coefficients, the moduli $|\hat{b}_{I,k}|$, $|\hat{c}_{I,k}|$ and $|\hat{d}_{I,k}|$ just depend on the third component of the index $I = (u, v, f)$, since they involve the functions $\psi_f$, but not the “shifts” $z_{u,v}$. In particular, the set of their possible values is a finite number, independent of $q$ and just depending on the collection of the family of functions $\psi_f$ and on the frequency $k$.

Proof. First of all, it is easy to see that for any $s = 0, 1, 2$ and $N \geq 0$,

$$
\| D_{t,\ell}^s \partial I \|_{C^0(\mathbb{R})} = \| \partial_{t,\ell}^s \partial I \|_{C^0(\mathbb{R})} \lesssim \tau_q^{-s}, \quad (6.10)
$$

$$
\| \chi_I(\xi) \|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim \mu_q^{-N}, \quad D_{t,\ell}^s [\chi_I(\xi)] = 0,
$$

where $\mathcal{I}_u = [t_u - \frac{1}{2} \tau_q, t_u + \frac{3}{2} \tau_q]$. Indeed, the estimate of $\chi_I(\xi)$ follows from (5.19), Lemma A.1, and $\ell^{-1} \leq \mu_q^{-1}$. We remark that the implicit constants are independent of $I$.

Recall that when $f \in \mathcal{I}_\varphi$,

$$
\gamma_I = \frac{\lambda_q^{-\gamma} \delta_{q+1}^2 \Gamma_I}{|f_I|^3} = \frac{\lambda_q^{-\gamma} \delta_{q+1}^2 \Gamma_I}{|\nabla \xi_{f}^{-1} f_I|^3}
$$

for

$$
\Gamma_I(x) = \Gamma_f^3 (-2 \lambda_q^{3\gamma} \delta_{q+1}^{-3} (\nabla \xi_I) e_3 \varphi_f)
$$

where $\Gamma_f$’s are the functions given by Lemma 3.2. First it is easy to see that (5.20) implies

$$
\| D_{t,\ell}^s ([\nabla \xi_I]^{-1} f_I) \|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim q, p, N, M (\lambda_q \delta_{q+1}^{\frac{1}{4}})^s \ell^{-N}. \quad (6.11)
$$

Also, using (5.19) and (5.2),
\[ \|D_t^s (2\lambda_0^3 \delta_{q+1}^2 (\nabla \xi_{I}) \varphi_{\ell})\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim_0 \rho, p, N, M (\ell_t^{-s} + (\lambda \delta_q^2 \frac{1}{\ell})^s) \ell_t^{-N} \lesssim_{\tau} \ell_t^{-s} \ell^{-N}. \] (6.12)

Next, for any smooth functions \( \Gamma = \Gamma(x) \) and \( g = g(t, x) \) we have

\[ \|D_{t, \ell}^s \Gamma(g)\|_{C^0_x} \lesssim \sum_{N_1 + N_2 = N} \|D_{t, \ell} g\|_{C^N_x} \| (\nabla \Gamma)(g)\|_{C^N_x}, \]

\[ \|D_{t, \ell}^2 \Gamma(g)\|_{C^0_x} \lesssim \sum_{N_1 + N_2 = N} \|D_{t, \ell}^2 g\|_{C^N_x} \| (\nabla \Gamma)(g)\|_{C^N_x} + \|D_{t, \ell} g \otimes D_{t, \ell} g\|_{C^N_x} \| (\nabla^2 \Gamma)(g)\|_{C^N_x}, \] (6.13)

and therefore we obtain, by Lemma A.1, that

\[ \|D_{t, \ell}^s (\nabla \xi_{I})^{-1} f_{I}\|_{N} \lesssim_{\rho, p, N, M} (\lambda \delta_q^2 \frac{1}{\ell})^s \ell_t^{-N} \]

\[ \|D_{t, \ell}^s (\Gamma f_{I})^{-1} \frac{1}{2} (2\lambda_0^3 \delta_{q+1} (\nabla \xi_{I}) \varphi_{\ell})\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim_0 \rho, p, N, M \tau_t^{-s} \ell^{-N}. \]

Here, we used (6.11) and (6.12) which we can apply thanks to the fact that \( \| (\nabla \xi_{I})^{-1} f_{I}\|_{N} \geq \frac{3}{4} \) and \( \Gamma f_{I} \geq 3 \) (according to our choice of \( N_0 \) in applying Lemma 3.2). Also the implicit constant in the second inequality can be chosen to be independent of \( I \) because of the finite cardinality of the functions \( f_{I} \). On the other hand, in the case of \( s = 0 \) and \( N = 0, 1, 2 \), because of (2.4) and (5.21), the implicit constants in both inequalities can be chosen to be independent of \( N \) and \( M \). Therefore, it follows that when \( I \in \mathcal{J}_\varphi \),

\[ \|D_{t, \ell}^s \delta_{q+1} \gamma_{I}\|_{N} \lesssim_{\rho, p, N, M} \tau_t^{-s} \ell^{-N}. \] (6.14)

In particular, for \( N = 0, 1, 2 \),

\[ \|\delta_{q+1} \gamma_{I}\|_{N} \lesssim_{\rho} \ell^{-N}, \]

where the implicit constant depends on an upper bound of \( \|\varphi\|_{C([0, T] + \tau_t^{-1}; C^2(\mathbb{R}))} \) and \( \rho_0 \) more precisely.

On the other hand, when \( I \in \mathcal{J}_R \), recall that \( \delta_{q+1} \Omega^{-1} \gamma_{I} = \Gamma_I = \Gamma f_{I} (\text{Id} + \delta_{q+1} \mathcal{M}_I) \) for a finite collection of smooth functions \( f_{I} \) chosen through Lemma 3.1. First \( \mathcal{M}_I \) can be estimated as

\[ \|\mathcal{M}_I\|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \lesssim_{N} \delta_{q+1} \| (\nabla \xi_{I}) (\nabla \xi_{I})^\top - \text{Id} \|_{C^0(\mathcal{I}_u; C^N(\mathbb{R}^3))} \]

\[ + \sum_{N_1 + N_2 + N_3 = N} \|\nabla \xi_{I}\|_{C^0(\mathcal{I}_u; C^N_{x_1})} \| R_{\ell} \|_{N_2} \| \nabla \xi_{I}\|_{C^0(\mathcal{I}_u; C^N_{x_3})} + \|\rho^{-2}\|_{C^0(\mathcal{I}_u; C^N)} \]

\[ \sum_{1} \sum_{2} \|\nabla \xi_{I}\|_{C^0(\mathcal{I}_u; C^N_{x_1})} \| \chi_{I}^2 (\xi_{I})\|_{C^0(\mathcal{I}_u; C^N_{x_2})} \| \chi_{I}^2 (\xi_{I})\|_{C^0(\mathcal{I}_u; C^N_{x_3})} \|\nabla \xi_{I}\|_{C^0(\mathcal{I}_u; C^N_{x_4})} \]

\[ \lesssim_{\rho, N, M} \delta_{q+1} \mu_q^{-N}, \]
where the double summations in the third line is taken over \( N_1 + N_2 + N_3 + N_4 = N \) and \( J = (u', v', f') \) satisfying \( f \in \mathcal{F}(u, v, f) := \mathcal{F}(v'). \phi \) and the last line follows from (5.18), (5.19), (5.1), (6.10), and (6.14). In the case of \( N = 0, 1, 2 \), we note that the implicit constants can be chosen to be independent of \( N \) and \( M \). Similarly, we have

\[
\| D_{t, \ell}^s \mathcal{M}_I \|_N \lesssim_{\varepsilon, \rho, N, M} \| \varepsilon \| N, N \delta q_{q+1} - s \mu q_{q+1} - N.
\]

Then, (6.13) and Lemma A.1 imply that when \( f_1 \in \mathcal{F}_{I, R} = \mathcal{F}_{(u, v, f), R} := \mathcal{F}(u, v, f) \), for \( s = 0, 1, 2 \) and \( N \geq 0 \),

\[
\| D_{t, \ell}^s \delta q_{q+1}^{\frac{1}{2}} \gamma \|_N = \| D_{t, \ell}^s (\Gamma f_1 (\text{Id} + \delta q_{q+1}^{\frac{1}{2}} \mathcal{M}_I)) \|_N \lesssim_{\varepsilon, \rho, N, M} \delta q_{q+1} - s \mu q_{q+1} - N. \tag{6.15}
\]

In particular, for \( s = 0, N = 0, 1, 2 \), the implicit constant can be chosen to be independent of \( M \) and \( N \) but to depend only on \( \varepsilon_0 \) and an upper bound of \( \| \varepsilon \|_{C([0, T] + \tau; C^2 (\mathbb{T}^3))} \):

\[
\| \delta q_{q+1}^{\frac{1}{2}} \gamma \|_N \lesssim_{\varepsilon} \mu q_{q+1} - N.
\]

Finally, recall the definition of \( b_{u,k}, c_{u,k}, d_{u,k}, \) and \( e_{u,k} \). Then, the estimates (6.5)–(6.9) follows from (6.10), (6.14), (6.15), and (5.21).

\[\Box\]

**Proof of Proposition 6.1.** Using (6.9), (5.21), and (3.26), we easily have \( \lambda q_{q+1} - N \| n_0 \| N \lesssim_{\varepsilon, N} \delta q_{q+1}^{\frac{1}{2}} \) and \( \lambda q_{q+1} - N \| n_e \| N \lesssim_{\varepsilon, N} (\lambda q_{q+1} + \mu q_{q+1})^{-\frac{1}{2}} \delta q_{q+1}^{\frac{1}{2}} \), recalling Remark 6.3. On the other hand, we observe that \( D_{t, \ell} \epsilon \lambda q_{q+1} + \mu q_{q+1}^{\frac{1}{2}} = 0 \) because of \( D_{t, \ell} \epsilon \xi = 0 \). Hence the remaining inequalities in (6.1) and (6.2) are obtained in a similar fashion. Finally, (6.3) follows from (6.1) and (6.2). Note that all estimates used in the proof have implicit constants independent of \( q \). Moreover, the finite cardinalities of the range of \( N \) and \( s \) make it possible to choose the implicit constants in (6.1), (6.2), and (6.3) independent of \( N \) and \( s \) too. Furthermore, when \( s = 0 \), we can also make the implicit constants independent of \( M \).

\[\Box\]

### 7. A Microlocal Lemma

We will need in the sequel a suitable extension of [31, Lemma 4.1] whose proof can be found in [17, Lemma 8.1]. We will use the notation

\[
\mathcal{F}[f](k) = \int_{\mathbb{T}^3} f(x) e^{-ik \cdot x} \, dx, \quad f(x) = \sum_{k \in \mathbb{Z}^3} \mathcal{F}[f](k) e^{ik \cdot x}
\]

for the Fourier series of periodic functions.

**Lemma 7.1.** (Microlocal Lemma). Let \( T \) be a Fourier multiplier defined on \( C^\infty(\mathbb{T}^3) \) by

\[
\mathcal{F}[Th](k) = m(k) \mathcal{F}[h](k), \quad \forall k \in \mathbb{Z}^3
\]
for some $m$ which has an extension in $\mathcal{S}(\mathbb{R}^3)$ (which for convenience we keep denoting by $m$). Then, for any $n_0 \in \mathbb{N}$, $\lambda > 0$, and any scalar functions $a$ and $\xi$ in $C^\infty(\mathbb{T}^3)$, $T(\ae^{i\lambda \xi})$ can be decomposed as

$$T(\ae^{i\lambda \xi}) = \left[ am(\lambda \nabla \xi) + \sum_{k=1}^{2n_0} C_k^\lambda(\xi, a) : (\nabla^k m)(\lambda \nabla \xi) + \varepsilon_{n_0}(\xi, a) \right] \ae^{i\lambda \xi}$$

for some tensor-valued coefficient $C_k^\lambda(\xi, a)$ and a remainder $\varepsilon_{n_0}(\xi, a)$ which is specified in the formula

$$\varepsilon_{n_0}(\xi, a)(x) = \sum_{n_1+n_2=n_0+1} \frac{(-1)^{n_1} c_{n_1,n_2}}{n_0!} \cdot \int_0^1 \int_{\mathbb{R}^3} \tilde{m}(y) e^{-i\lambda \nabla \xi(x)} ((y \cdot \nabla)^{n_1} a)(x - ry) e^{i\lambda Z[\xi](r)} \beta_{n_2}[\xi](r)(1-r)^{n_0} \, dy \, dr,$$

where $c_{n_1,n_2}$ is a constant depending on $n_1$ and $n_2$, and the function $\beta_n[\xi]$ is

$$\beta_n[\xi](r) = B_n(i\lambda Z'(r), i\lambda Z''(r), \ldots, i\lambda Z^{(n)}(r)),$$

$$Z[\xi](r) = Z[\xi]_{x,y}(r) = r \int_0^1 (1-s)(y \cdot \nabla)^2 \xi(x - rsy) \, ds,$$

with $B_n$ denoting the $n$th complete exponential Bell polynomial;

$$B_n(x_1, \ldots, x_n) = \sum_{k=1}^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \quad (7.2)$$

where

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{j_1!j_2! \cdots j_{n-k+1}!} \left( \frac{x_1}{1!} \right)^{j_1} \left( \frac{x_2}{2!} \right)^{j_2} \cdots \left( \frac{x_{n-k+1}}{(n-k+1)!} \right)^{j_{n-k+1}},$$

and the summation is taken over $\{j_k\} \subset \mathbb{N} \cup \{0\}$ satisfying

$$j_1 + j_2 + \cdots + j_{n-k+1} = k, \quad j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1)j_{n-k+1} = n. \quad (7.3)$$

We now collect an important consequence on the anti-divergence operator $\mathcal{R}$.

**Corollary 7.2.** Let $N = 0, 1, 2$ and $F = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} a_{u,k} e^{i\lambda_q + k \xi u}$. Assume that a function $a_{u,k}$ fulfills the following requirements.

(i) The support of $a_{u,k}$ satisfies $\text{supp}(a_{u,k}) \subset (t_u - \frac{1}{2} \tau_q, t_u + \frac{3}{2} \tau_q) \times \mathbb{R}^3$. In particular, for $u$ and $u'$ neither same nor adjacent, we have

$$\text{supp}(a_{u,k}) \cap \text{supp}(a_{u',k'}) = \emptyset, \quad \forall k, k' \in \mathbb{Z}^3 \setminus \{0\}. \quad (7.4)$$

For any $0 \leq j \leq n_0 + 1$ and $(u, k) \in \mathbb{Z} \times \mathbb{Z}^2$,
\[
\|a_{u,k}\|_j + (\lambda_{q+1} \delta_{q+1}^{\frac{1}{2}})^{-1} \|D_{t,\ell}a_{u,k}\|_j \lesssim_j \mu_{q}^{-j} |\hat{a}_k|, \quad \sum_k |k|^{n_0+2} |\hat{a}_k| \leq a_F,
\]
(7.5)
for some $a_F > 0$, where $n_0 = \lceil \frac{2b(2+\alpha)}{(b-1)(1-\alpha)} \rceil$ and $\| \cdot \|_j = \| \cdot \|_{C(I;C^j(\mathbb{T}^3))}$ on some time interval $I \subset \mathbb{R}$.

Then, for any $b \in (1, 3)$, we can find $\Lambda_0 = \Lambda_0(b, \varrho, p)$ such that for any $\lambda_0 \geq \Lambda_0$, $\mathcal{R}F$ satisfies the following inequalities:
\[
\|\mathcal{R}F\|_N \lesssim \lambda_{q+1}^{N-1} a_F, \quad \|D_{t,q+1}\mathcal{R}F\|_{N-1} \lesssim \lambda_{q+1}^{N-1} \delta_{q+1}^{\frac{1}{2}} a_F
\]
upon setting $D_{t,q+1} = \partial_t + \frac{m_{q+2}}{\varrho} \cdot \nabla$.

The proof of Corollary 7.2 follows from the same argument in the proof of [17, Corollary 8.2] with slight revision because the the backward flow map $\xi_u$ has velocity $m_\ell/\varrho$ instead of $\nu_\ell$ in [17]. The flow map $\xi_u$ and $m_\ell/\varrho$, on the other hand, satisfy the same estimates as for the original flow map and its velocity leading to in [17] (see Lemma 5.3, (2.2), (5.3)), while the velocity $m_\ell/\varrho$ no longer has frequency localization to $\leq \ell^{-1}$. As a result, the same argument works except for one part in the material derivative estimate which relies on frequency localization. Also, we remark that through careful examination of the proof in [17, Corollary 8.2] one could see that (7.5) only for $0 \leq j \leq n_0 + 1$ is needed. For the completeness, we sketch the proof and point out the needed revision below.

**Sketch of the proof.** The proof is relying on the decomposition
\[
F = \mathcal{P}_{\gtrsim \lambda_{q+1}} \left( \sum_{u,k} a_{u,k}e^{i\lambda_{q+1}k \xi_u} \right) - \sum_{u,k} \varepsilon_{n_0}^{\lambda_{q+1}}(k \cdot \xi_u, a_{u,k})e^{i\lambda_{q+1}k \xi_u},
\]
(7.6)
where $\mathcal{P}_{\gtrsim \lambda_{q+1}}$ is defined by
\[
\mathcal{P}_{\gtrsim \lambda_{q+1}} = \sum_{2^j \geq \frac{1}{2} \lambda_{q+1}} P_{2^j}
\]
and
\[
\varepsilon_{n_0}^{\lambda_{q+1}}(k \cdot \xi_u, a_{u,k}) = \sum_{2^j \geq \frac{1}{2} \lambda_{q+1}} \varepsilon_{n_0,j}(k \cdot \xi_u, a_{u,k}).
\]
The remainder $\varepsilon_{n_0,j}(\xi, a)$ is obtained by applying Lemma 7.1 to $P_{2^j}$ and $n_0 = \lceil \frac{2b(2+\alpha)}{(b-1)(1-\alpha)} \rceil$. In particular, the remainder part of $F$ has frequency localization
\[
\mathcal{P}_{\gtrsim \lambda_{q+1}} F := F - \mathcal{P}_{\gtrsim \lambda_{q+1}} F = - \sum_{k,u} \varepsilon_{n_0}^{\lambda_{q+1}}(k \cdot \xi_u, a_{u,k})e^{i\lambda_{q+1}k \xi_u}
\]
(7.7)
and satisfies
\[
\| \sum_{u,k} \varepsilon \partial_{k} (k \cdot \xi_{u}, a_{u,k}) \|_{0} \lesssim_{n_{0}} (\lambda_{q+1} \mu)_{-1}^{(n_{0}+1)} a_{F} \lesssim \lambda_{q+1}^{-2} a_{F},
\]
\[
(7.8)
\]
\[
\| \sum_{u,k} D_{t} \varepsilon \partial_{k} (k \cdot \xi_{u}, a_{u,k}) \|_{0} \lesssim_{n_{0}} \lambda_{q+1} \delta_{q+1}^{2} (\lambda_{q+1} \mu)_{-1}^{(n_{0}+1)} a_{F} \lesssim \lambda_{q+1} \delta_{q+1}^{2} \lambda_{q+1}^{-2} a_{F}.
\]
\[
(7.9)
\]
Using this, one can easily obtain \( \| R F \|_{N} \lesssim \lambda_{q+1}^{-1} a_{F} \). To estimate the material derivative of \( R F \), we use the following decomposition,
\[
D_{t} \partial_{q+1} R F = R D_{t} \partial_{q+1} F + \left[ m_{q} \frac{\rho}{\rho} \cdot \nabla, R \partial_{q+1} F \right] + \left[ m_{q} \partial_{q+1} \nabla, R \partial_{q+1} F \right].
\]
The first and the last terms on the right hand side can be estimated as in [17, Corollary 8.2]. To estimate the second term, we further decompose it into
\[
\left[ m_{q} \partial_{q+1} \nabla, R \partial_{q+1} F \right] + \left[ m_{q} \partial_{q+1} \nabla, R \partial_{q+1} F \right].
\]
Since \( m_{q} \partial_{q+1} \nabla = P_{\xi_{q+1}} (m_{q} \partial_{q+1} \nabla) \), we can estimate it as in [17, Corollary 8.2] (see also the proof of Lemma A.7.) Therefore, it suffices to estimate the remaining term;
\[
\left[ m_{q} \partial_{q+1} \nabla, R \partial_{q+1} F \right] + \left[ m_{q} \partial_{q+1} \nabla, R \partial_{q+1} F \right],
\]
where we used \( \| m_{q} \partial_{q+1} \nabla \|_{N_{1}} \lesssim \xi^{2} \| m_{q} \|_{0} \nabla^{-1} \|_{N_{1}+2} \) and the choice of \( b < 3 \) and sufficiently large \( \Lambda_{0} \).

8. Estimates on the Reynolds Stress

In this section, we obtain the relevant estimates for the new Reynolds stress and its new advective derivative \( D_{t} \partial_{q+1} R_{q+1} = \partial_{t} R_{q+1} + \frac{m_{q+1} \cdot \nabla R_{q+1}, \sum_{u,k} D_{t} \varepsilon \partial_{k} (k \cdot \xi_{u}, a_{u,k})} \), summarized in the following proposition. For technical reasons it is however preferable to estimate rather \( R_{q+1} - \xi \partial \xi \), as indeed the estimates on the function \( \xi(t) \) are akin to those for the new current, which will be detailed in the next section. For the remaining sections, we set \( \| \cdot \|_{N} = \| \cdot \|_{C^{0}_{0}(0,T)+t_{*}:[T_{*}]} \).

Proposition 8.1. There exists \( b(\alpha) > 1 \) with the following property. For any \( 1 < b < b(\alpha) \) we can find \( \Lambda_{0} = \Lambda_{0}(\alpha, b, M, \rho, p) \) such that the following estimates...
hold for every $\lambda_0 \geq \Lambda_0$:

$$
\| R_{q+1} - \frac{2}{3} \zeta / \varrho \text{Id} \|_N 
\lesssim C_{\varrho, p, M} \lambda_{q+1}^N \cdot \lambda_q^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{3}{2}} \lesssim \frac{1}{2} \lambda_{q+1}^{N-3y} \delta_{q+2}.
$$

(8.1)

for all $N = 0, 1, 2$ ( $\| \cdot \|_1$ is an empty statement), where $C_{\varrho, p, M}$ depends only upon $\varrho$, $p$ and the $M > 1$ of Propositions 2.3 and 2.6.

Taking into account (4.1), we will just estimate the separate terms $R_T$, $R_N$, $R_{O1}$, $R_{O2}$ and $R_M$. For the errors $R_{O2}$ and $R_M$, we use a direct estimate, while the other errors, including the inverse divergence operator, are estimated by Corollary 7.2. In the following subsections, we fix $n_0 = \lceil \frac{2b(2+a)}{(b-1)(1-a)} \rceil$ so that $\lambda_{q+1}^2 (\lambda_{q+1} + \mu_\varrho)^{-n_0+1} \lesssim \delta_{q+1}^{\frac{1}{2}}$ for any $q$ and allow the dependence on $M$ of the implicit constants in $\lesssim$. Also, we remark that

$$
\frac{1}{\lambda_{q+1}^2 \tau_q} + \frac{\delta_{q+1}^{\frac{1}{2}}}{\lambda_{q+1} \mu_\varrho} \lesssim M \lambda_{q+1}^{\frac{1}{2}} \lambda_q^{\frac{1}{2}} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{3}{2}}.
$$

(8.2)

We note that it is enough to estimate $\varrho R - \frac{2}{3} \zeta$ and its advective derivative for the various Reynolds error terms. For the convenience, we restrict the range of $N$ as in (8.1) in this section, without mentioning it further.

8.1. Transport Stress Error

Recall that

$$
\varrho R_T = R \left( \varrho D_{t, \ell} \frac{n}{\varrho} - \text{div}(m_q - m_\ell) \frac{n}{\varrho} \right)
$$

Since $\varrho D_{t, \ell} (n/\varrho) - \text{div}(m_q - m_\ell)(n/\varrho) = \partial_t n + \text{div} ((n \otimes m_\ell)/\varrho)$, we see that it has zero mean and thus we can apply the inverse divergence operator. Now as $D_{t, \ell} \xi_I = 0$, we have

$$
\varrho D_{t, \ell} (n/\varrho) = \varrho D_{t, \ell} \left( \frac{1}{\varrho} \sum_{u \in \mathbb{Z}^3, k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{u \in \mathbb{Z}^3, k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1}^{\frac{1}{2}} (b_{u, k} + (\lambda_{q+1} + \mu_\varrho)^{-1} e_{u, k}) e^{i \lambda_{q+1} k \cdot \xi_I} \right)
$$

$$
= \sum_u \sum_k \delta_{q+1}^{\frac{1}{2}} Q D_{t, \ell} \left( \frac{b_{u, k}}{\varrho} + (\lambda_{q+1} + \mu_\varrho)^{-1} \frac{e_{u, k}}{\varrho} \right) e^{i \lambda_{q+1} k \cdot \xi_I}.
$$

Since $b_{u, k}$ and $e_{u, k}$ satisfy $\text{supp}(b_{u, k})$, $\text{supp}(e_{u, k}) \subset (t_u - \frac{1}{2} \tau_q, t_u + \frac{3}{2} \tau_q) \times \mathbb{R}^3$ and
for any $\tilde{N} \geq 0$ by (6.5) and (6.8). We now have

$$\div (m_q - m_\ell) = \sum_{u \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{\delta^2 q + 1} \div (m_q - m_\ell) (b_{u,k} + (\lambda_{q+1} + \mu_q)^{-1} e_{u,k}) e^{i\lambda_{q+1} k \cdot \xi}$$

Since $\div (m_q - m_\ell) = -\partial_t \varrho + (\partial_t \varrho)_{\ell}$, we can estimate it similarly as to

$$\|e^{-1} \div (m_q - m_\ell) (b_{u,k} + (\lambda_{q+1} + \mu_q)^{-1} e_{u,k}) \|_{\tilde{N}}$$

We can thus apply Corollary 7.2 to get

$$\| R_T \|_{N} \lesssim \lambda_{q+1}^{\frac{\delta^2 q + 1}{\lambda_{q+1}^{\frac{\delta^2 q + 1}}}} \| D_{t,q+1} R_T \|_{N-1} \lesssim \lambda_{q+1}^{\frac{\delta^2 q + 1}{\lambda_{q+1}^{\frac{\delta^2 q + 1}}}}.$$

8.2. Nash Stress Error

Recall $\varrho \varphi N = \mathcal{R} \left( (n \cdot \nabla) \frac{m_\ell}{Q} \right)$ and observe that

$$(n \cdot \nabla) \frac{m_\ell}{Q} = \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta^2 q + 1 ((b_{u,k} + (\lambda_{q+1} + \mu_q)^{-1} e_{u,k}) \cdot \nabla) \frac{m_\ell}{Q} e^{i\lambda_{q+1} k \cdot \xi}.$$

Since $b_{u,k}$ and $e_{u,k}$ satisfy $\text{supp}(b_{u,k}), \text{supp}(e_{u,k}) \subset (t_u - \frac{1}{2} \tau_q, t_u + \frac{3}{2} \tau_q) \times \mathbb{R}^3$ and

$$\| (b_{u,k} + (\lambda_{q+1} + \mu_q)^{-1} e_{u,k}) \cdot \nabla \|_{\tilde{N}} \lesssim \lambda_{q+1}^{\frac{\delta^2 q + 1}{\lambda_{q+1}^{\frac{\delta^2 q + 1}}}} \| \nabla \|_{\tilde{N}} \lesssim \lambda_{q+1}^{\frac{\delta^2 q + 1}{\lambda_{q+1}^{\frac{\delta^2 q + 1}}}} |b_{I,k}| \lambda_{q+1}^{\frac{1}{2}} \delta_q$$

for any $\tilde{N} \geq 0$ by (2.2), (5.15), (6.5), and (6.8), we apply Corollary 7.2 and obtain

$$\| R_N \|_{N} \lesssim \lambda_{q+1}^{\frac{\delta^2 q + 1}{\lambda_{q+1}^{\frac{\delta^2 q + 1}}}} \| D_{t,q+1} R_N \|_{N-1} \lesssim \lambda_{q+1}^{\frac{\delta^2 q + 1}{\lambda_{q+1}^{\frac{\delta^2 q + 1}}}}.$$
8.3. Oscillation Stress Error

Recall that $RO = RO_1 + RO_2$ where

$$
\psi RO_1 = \mathcal{R} \left( \text{div} \left( \frac{n_o \otimes n_o}{\psi} + \psi R_{\ell} - \delta_{q+1} \psi \text{Id} \right) \right)
$$

$$
\psi RO_2 = \frac{n_o \otimes n_c}{\psi} + \frac{n_c \otimes n_o}{\psi} + \frac{n_c \otimes n_c}{\psi}.
$$

We compute

$$
\text{div} \left( \frac{n_o \otimes n_o}{\psi} + \psi R_{\ell} - \delta_{q+1} \psi \text{Id} \right) = \text{div} \left( \sum_{u \in \mathbb{Z}, \, k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} \frac{c_{u,k}}{\psi} e^{i \lambda_{q+1} k \cdot \xi_i} \right)
$$

$$
= \sum_{u,k} \delta_{q+1} \text{div} \left( \frac{c_{u,k}}{\psi} \right) e^{i \lambda_{q+1} k \cdot \xi_i},
$$

because of $\hat{c}_{I,k}(f_I \cdot k) = 0$. Also, since we have

$$
D_{t,\ell} \text{ div} \frac{c_{u,k}}{\psi} = \text{div} \left( D_{t,\ell} \frac{c_{u,k}}{\psi} \right) - \left( \nabla \frac{m_{\ell}}{\psi} \right)_{ij} \left( \nabla \frac{c_{u,k}}{\psi} \right)_{ji},
$$

it follows from (6.6) that

$$
\| \text{div} \frac{c_{u,k}}{\psi} \|_{N} + (\lambda_{q+1} \delta_{q+1} \frac{1}{\mu_q} - 1) \| D_{t,\ell} \text{ div} \frac{c_{u,k}}{\psi} \|_{N} \lesssim N_1 (N, M) \frac{\mu_q - N |\hat{c}_{I,k}|}{\mu_q}
$$

for any $\tilde{N} \geq 0$.

Finally using $\text{supp}(c_{u,k}) \subset (t_u - \frac{1}{2} \tau_q, t_u + \frac{3}{2} \tau_q) \times \mathbb{R}^3$, we apply Corollary 7.2 to get

$$
\| D_{t,q+1} RO_1 \|_{N-1} \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}}{\lambda_{q+1} \mu_q} (8.5)
$$

On the other hand, we use (6.1), (6.2), (6.3), and (5.3) to estimate $RO_2$ as follows,

$$
\| RO_2 \|_{N} \lesssim \sum_{N_0 + N_1 + N_2 = N} \| e^{-1} \|_{N_0} \| n_o \|_{N_1} \| n_c \|_{N_2}
$$

$$
+ \sum_{N_0 + N_1 + N_2 = N} \| e^{-1} \|_{N_0} \| n_c \|_{N_1} \| n_c \|_{N_2} \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}}{\lambda_{q+1} \mu_q},
$$

$$
\| D_{t,q+1} RO_2 \|_{N-1} \leq \| D_{t,\ell} RO_2 \|_{N-1} + \left\| \frac{n + m_q - m_\ell}{\psi} \cdot \nabla RO_2 \right\|_{N-1}
$$

$$
\lesssim \| e^{-1} \|_{N} \sum_{N_1 + N_2 = N-1} (\| D_{t,\ell} n_o \|_{N_1} \| n_c \|_{N_2} + \| n_o \|_{N_1} \| D_{t,\ell} n_c \|_{N_2} + \| D_{t,\ell} n_c \|_{N_1} \| n_c \|_{N_2})
$$

$$
+ \| e^{-1} \|_{N} \sum_{N_1 + N_2 = N-1} (\| n \|_{N_1} + \| m_q - m_\ell \|_{N_1}) \| RO_2 \|_{N_2+1} \lesssim \lambda_{q+1}^N \frac{1}{\delta_{q+1}} \cdot \delta_{q+1} \frac{1}{\lambda_{q+1} \mu_q}.
$$

Therefore, we have

$$
\| RO \|_{N} \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}}{\lambda_{q+1} \mu_q}, \quad \| D_{t,q+1} RO \|_{N-1} \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}}{\lambda_{q+1} \mu_q}.
$$

(8.6)
8.4. Mediation Stress Error

Recall that

$$\varrho R_M = \varrho (R_q - R_\ell) + \frac{(m_q - m_\ell) \otimes n}{\varrho} + \frac{n \otimes (m_q - m_\ell)}{\varrho}.$$  

Using (5.4), (5.3), and (6.3), we have

$$\| R_M \|_N \lesssim \| \varrho \|_N \| R_q - R_\ell \|_N + \sum_{N_0 + N_1 + N_2 = N} \| \varrho^{-2} \|_{N_0} \| m_q - m_\ell \|_{N_1} \| n \|_{N_2} \lesssim \lambda_{q+1}^N \cdot (\lambda_q \gamma_q^{-1} \delta_q^\frac{1}{2} \delta_{q+1}^\frac{3}{2} + (\ell \lambda_q) \gamma_q^{-\frac{1}{2}} \delta_q^\frac{1}{2} \delta_{q+1}^\frac{3}{2}) \lesssim \lambda_{q+1}^N \cdot \lambda_q \gamma_q^{-\frac{1}{2}} \delta_q^\frac{1}{2} \delta_{q+1}^\frac{3}{2}.$$  

To estimate $D_{t,q+1} R_M$, we use the decomposition $D_{t,q+1} R_M = D_{t,\ell} R_M + \left(\frac{m_q - m_\ell + n}{\varrho}\right) \cdot \nabla R_M$ additionally to obtain

$$\| D_{t,q+1} (\varrho R_M) \|_{N-1} \lesssim \| D_{t,\ell} (\varrho (R_q - R_\ell)) \|_{N-1} + \| D_{t,\ell} \left(\frac{(m_q - m_\ell) \otimes n}{\varrho}\right) \|_{N-1} \lesssim \lambda_{q+1}^N \cdot \lambda_q \gamma_q^{-\frac{1}{2}} \delta_q^\frac{1}{2} \delta_{q+1}^\frac{3}{2}.$$  

To summarize, we obtain

$$\| R_M \|_N \lesssim \lambda_{q+1}^N \lambda_q \gamma_q^{-\frac{1}{2}} \delta_q^\frac{1}{2} \delta_{q+1}^\frac{3}{2}, \quad \| D_{t,q+1} R_M \|_{N-1} \lesssim \lambda_{q+1}^N \lambda_q \gamma_q^{-\frac{1}{2}} \delta_q^\frac{1}{2} \delta_{q+1}^\frac{3}{2}.$$  

Finally, Proposition 8.1 follows from (8.3)–(8.7) and (8.2).

9. Estimates for the New Current

In this section, we obtain the last needed estimates, on the new unresolved current $\varphi_{q+1}$ and on the remaining part of the Reynolds stress $\frac{2}{3} \zeta / \varrho \text{Id}$, which we summarize in the following proposition.

**Proposition 9.1.** There exists $\tilde{b}(\alpha) > 1$ with the following property. For any $1 < b < \tilde{b}(\alpha)$ there is $\Lambda_0 = \Lambda_0(\alpha, b, M, \varrho, p)$ such that the following estimates hold for $\lambda_0 \geq \Lambda_0$:

$$\| \varphi_{q+1} \|_N \leq \lambda_{q+1}^{N-3\gamma} \delta_{q+2}^\frac{3}{2}, \quad \forall N = 0, 1, 2,$$

$$\| D_{t,q+1} \varphi_{q+1} \|_{N-1} \leq \lambda_{q+1}^{N-3\gamma} \delta_{q+2}^\frac{3}{2}, \quad \forall N = 1, 2,$$

$$\| \zeta \|_0 + \| \zeta' \|_0 \leq \frac{\varepsilon_0^2}{20 M} \lambda_{q+1}^{-3\gamma} \delta_{q+2}^\frac{3}{2}$$

for $M$ defined as in (2.2).
Without mentioning, we assume that $N$ is in the range above and allow the dependence on $M$ of the implicit constants in $\lesssim$ in this section. For convenience, we single out the following fact, which will be repeatedly used: note that there exists $\bar{b}(\alpha) > 1$ such that for any $1 < b < \bar{b}(\alpha)$ and a constant $\bar{C}_{M, q, p}$ depending only on $q$, $p$, and $M$, we can find $\Lambda_0 = \Lambda_0(\alpha, b, M, q, p)$ which gives

$$
\bar{C}_{M, q, p} \left[ \frac{\delta_{q+1}}{\lambda_{q+1} \tau_q} + \frac{\delta_{q+1}^2}{\lambda_{q+1} \mu_q} + \frac{\lambda_q^\frac{1}{2} \delta_{q+1}^\frac{5}{4}}{\lambda_{q+1}^\frac{1}{2} \delta_{q+1}^\frac{5}{4}} \right] \leq \lambda_{q+1}^{-3y} \delta_{q+1}^\frac{3}{2},
$$

for any $\lambda_0 \geq \Lambda_0$. This is possible because $\alpha < \frac{1}{7}$.

### 9.1. High Frequency Current Error

We start by observing that $\varphi_{H1}$ is

$$
\varphi_{H1} = \mathcal{R} \left( \frac{n}{Q} \cdot (\text{div} P_{\leq \ell^{-1}}(\varphi(R_q - c_q \text{Id})) + Q(m_q, m_q)) \right)
$$

(9.3)

by (4.9). We thus can apply Corollary 7.2 to

$$
\frac{n}{Q} \cdot (\text{div} P_{\leq \ell^{-1}}(\varphi(R_q - c_q \text{Id})) + Q(m_q, m_q))
$$

$$
= \sum_{u \cdot k} \frac{1}{Q} (\text{div} P_{\leq \ell^{-1}}(\varphi(R_q - c_q \text{Id})) + Q(m_q, m_q)) \delta_{q+1}^\frac{1}{2} (b_{u,k})
$$

$$
+ (\lambda_{q+1} \mu_q)^{-1} e_{u,k} e^{i\lambda_{q+1} k \cdot \xi_l}.
$$

Indeed, using (2.3), (5.12), (6.5), (6.8), we obtain

$$
\|\varphi_{H1}\|_N \lesssim \lambda_{q+1}^{-1} \delta_{q+1}^\frac{1}{2} (\lambda_q^{-3y} \delta_{q+1} + (\ell \lambda_q) \lambda_q \delta_q) \lesssim \lambda_{q+1}^{N} \lambda_q^{-1} \delta_{q+1}^\frac{3}{2}.
$$

Furthermore, (2.2), (2.3), (5.3), (A.4), and (A.5) imply

$$
\|D_{t, \ell} \text{div} P_{\leq \ell^{-1}}(\varphi R_q)\|_{N-1}
$$

$$
\leq \|\text{div} P_{\leq \ell^{-1}}(D_{t, \ell} \varphi R_q)\|_{N-1} + \|\text{div}[m_{\ell} P_{\leq \ell^{-1}} Q^{-1} \cdot \nabla, P_{\leq \ell^{-1}}] \varphi R_q\|_{N-1}
$$

$$
+ \|\text{div}[m_{\ell} P_{\leq \ell^{-1}} Q^{-1} \cdot \nabla, P_{\leq \ell^{-1}}] \varphi R_q\|_{N-1} + \|\nabla(m_{\ell} / \varphi)\|_{k_i \delta_k P_{\leq \ell^{-1}}(\varphi R_q)_{ij}}\|_{N-1}
$$

$$
\lesssim \lambda_{q+1}^{N} (\|D_{t, \ell} \varphi R_q\|_1 + \|[m_{\ell} P_{\leq \ell^{-1}} Q^{-1} \cdot \nabla, P_{\leq \ell^{-1}}] \varphi R_q\|_N)
$$

$$
+ \|[m_{\ell} P_{\leq \ell^{-1}} Q^{-1} \cdot \nabla, P_{\leq \ell^{-1}}] \varphi R_q\|_{N} + \sum_{N_1 + N_2 = N-1} \|m_{\ell}\|_{N_1 + \ell^{-1} = N_2-1} \|\varphi R_q\|_0
$$

$$
\lesssim \lambda_{q+1}^\frac{1}{2} \delta_{q+1}^\frac{1}{2} \lambda_q^{-3y} \delta_{q+1}.
$$
In the last inequality, we used
\[ \|m_\ell P_{>\ell} Q^{-1}\|_{N'} \lesssim \|m_\ell\|_0 \|P_{>\ell} Q^{-1}\|_{N'} + \sum_{N_1+N_2=N'} \|m_\ell\|_{N_1} \|P_{>\ell} Q^{-1}\|_{N_2} \]
\[ \lesssim 1 + \sum_{N_1+N_2=N'} \delta_q^\frac{1}{2} \|\nabla^N Q^{-1}\|_{N_2} \lesssim 1 \]
for \( N' \in [1, n_0 + 1] \). In a similar way, we also get
\[ \|D_t,\ell \text{ div } P_{\leq \ell-1} (Q c_q \text{ Id})\|_{N-1} \lesssim \lambda_{q+1}^N \delta_q^\frac{1}{2} \lambda_q^{1-3\gamma} \delta_q+1. \]
Then, using (5.12), it follows that
\[ \|D_{t, q+1} \varphi H_1\|_{N-1} \lesssim \lambda_{q+1}^N \delta_q^\frac{1}{2} + \frac{\lambda_{q+1}^{1-3\gamma}}{\lambda_q} \delta_q+1. \]

In order to deal with \( \varphi H_2 \), we use the definition of \( R_{q+1} \) to get
\[ n \otimes n - \delta_{q+1} Q \text{ Id} + Q R_q - \varrho R_{q+1} + \frac{2}{3} \zeta \text{ Id} \]
\[ = \left( n \otimes n - \delta_{q+1} Q \text{ Id} + Q R_\ell \right) - \varrho R_{O1} - \varrho R_T - \varrho R_N - \varrho R_{M2}. \]
Thus, we can write
\[ \varphi H_2 = \mathcal{R} \left( \left( n \otimes n - \delta_{q+1} Q \text{ Id} + Q R_q - \varrho R_{q+1} + \frac{2}{3} \zeta \text{ Id} \right) : \nabla \frac{m_\ell}{\varrho} \right) \]
\[ + \mathcal{R} \left( \left( (m_q - m_\ell) \otimes n + n \otimes (m_q - m_\ell) \right) : \nabla \frac{m_\ell}{\varrho} \right) - \frac{2m_\ell \zeta}{3\varrho} \]
\[ = \mathcal{R} \left( \left( n \otimes n - \delta_{q+1} Q \text{ Id} + Q R_\ell \right) - \varrho R_{O1} - \varrho R_T - \varrho R_N \right) : \nabla \frac{m_\ell}{\varrho} \right) - \frac{2m_\ell \zeta}{3\varrho}. \]

To estimate the term with \( \zeta \) first, assuming (9.2), we have
\[ \| \left( m_\ell \zeta \right)/\varrho^2 \|_0 \lesssim \| \zeta \|_N \| m_\ell/\varrho^2 \|_N \lesssim \frac{1}{10} \lambda_{q+1}^{N-3\gamma} \delta_q^\frac{3}{2} \]
for \( N = 0, 1, 2 \) and
\[ \| D_{t, q+1} \left( (m_\ell \zeta)/\varrho^2 \right) \|_{N-1} \]
\[ \lesssim \left( \| D_{t, q+1} (m_\ell \zeta)/\varrho^2 \|_{N-1} + \| m_\ell D_{t, q+1} \varrho^{-2} \|_{N-1} \| \zeta \|_0 + \| \zeta' \|_0 \| m_\ell/\varrho^2 \|_{N-1} \right) \]
\[ \lesssim \frac{1}{10} \lambda_{q+1}^{N-3\gamma} \delta_q^\frac{1}{2} \lambda_q^{\frac{3}{2}} \]
for \( N = 1, 2 \). In the last inequality, we used
\[ \| D_{t, q+1} m_\ell \|_{N-1} \lesssim \lambda_{q+1}^{N-1} \delta_q^\frac{1}{2} \lambda_q \delta_q^\frac{1}{2} + \lambda_{q+1}^N + \varrho^{-N} (\varrho \delta_q)^3 \delta_q, \]
(9.6)
obtained from (5.13), (5.14), (5.6), and (6.4) and
\[ \| D_{t,q+1} e \|_0 \lesssim \| \partial_t e \|_0 + \| m_{q+1}/\rho \| \| \nabla e \|_0 \lesssim 1 \]
\[ \| D_{t,q+1} e \|_N \lesssim \| \partial_t e \|_N + \sum_{N_1+N_2=N} \| m_{q+1}/\rho \|_{N_1} \| \nabla e \|_{N_2} \lesssim \lambda_{q+1}^{N} \delta_{q+1}^{\frac{1}{2}} \quad \forall N = (9.7) \]

Apply, on the other hand, Corollary 7.2 with (3.29), (6.6), and (2.2), we have
\[
\left\| \mathcal{R} \left( \left( \frac{n_0 \otimes n_0}{\rho} - \delta_{q+1} \mathbb{I} + \rho R_{\ell} \right) : \nabla \frac{m_{\ell}}{\rho} \right) \right\|_N \lesssim \lambda_{q+1}^{N} \lambda_{q+1} \delta_{q+1}^{\frac{1}{2}} \frac{\delta_{q+1}}{\lambda_{q+1}},
\]
\[
\left\| D_{t,q+1} \mathcal{R} \left( \left( \frac{n_0 \otimes n_0}{\rho} - \delta_{q+1} \mathbb{I} + \rho R_{\ell} \right) : \nabla \frac{m_{\ell}}{\rho} \right) \right\|_{N-1} \lesssim \lambda_{q+1}^{N} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \delta_{q+1}^{\frac{1}{2}} \frac{\delta_{q+1}}{\lambda_{q+1}}.
\]

To estimate the remaining term, by (7.6) and (7.7), recall that the Reynolds stress errors \( R_\Delta \), which represents either \( R_{O1}, R_T \), or \( R_N \), can be written as \( R_\Delta = \mathcal{R} G_\Delta \) satisfying
\[
\| G_\Delta \|_N \lesssim \lambda_{q+1}^{N} \left( \frac{\delta_{q+1}}{\mu_q} + \frac{\delta_{q+1}^{\frac{1}{2}}}{\tau_q} \right), \quad \| D_{t,\ell} G_\Delta \|_{N-1} \lesssim \lambda_{q+1}^{N} \delta_{q+1}^{\frac{1}{2}} \left( \frac{\delta_{q+1}}{\mu_q} + \frac{\delta_{q+1}^{\frac{1}{2}}}{\tau_q} \right) \quad (9.8)
\]

Furthermore, such \( G_\Delta \) has the form \( \sum_{u,k} \xi_{\Delta}^{u,k} e^{i \lambda_{q+1}^{k} \xi_{\ell}} \) and has a decomposition
\[
G_\Delta = \mathcal{P}_{\geq \lambda_{q+1}} G_\Delta + \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta, \quad (9.9)
\]
as in (7.6) and (7.7), where \( \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta \) satisfies
\[
\| \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta \|_0 \lesssim \lambda_{q+1}^{-2} \delta_{q+1}^{\frac{1}{2}} \left( \frac{\delta_{q+1}}{\mu_q} + \frac{\delta_{q+1}^{\frac{1}{2}}}{\tau_q} \right),
\]
\[
\| D_{t,\ell} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta \|_0 \lesssim \lambda_{q+1}^{-1} \delta_{q+1} \left( \frac{\delta_{q+1}}{\mu_q} + \frac{\delta_{q+1}^{\frac{1}{2}}}{\tau_q} \right). \quad (9.10)
\]

Indeed, they follow from (7.8) and (7.9). Now, we write
\[
\nabla \frac{m_{\ell}}{\rho} = \nabla \left( m_{\ell} \mathcal{P}_{\geq \lambda_{q+1}} \frac{1}{\rho} \right) + \nabla \left( m_{\ell} \mathcal{P}_{\leq \lambda_{q+1}} \frac{1}{\rho} \right)
\]
\[
= : \nabla (m_{\ell}/\rho)_1 + \nabla (m_{\ell}/\rho)_2
\]

Now \( (m_{\ell}/\rho)_1 \) has the frequency localized to \( \leq \frac{1}{2} \lambda_{q+1} \) since \( m_{\ell} \) had frequency localized to \( \ell^{-1} \) and \( \ell^{-1} \leq \frac{1}{2} \lambda_{q+1} \) for sufficiently large \( \lambda_0 \). Thus, \( \mathcal{R} \mathcal{P}_{\geq \lambda_{q+1}} G_\Delta : \nabla (m_{\ell}/\rho)_1 \) has the frequency localized to \( \geq \lambda_{q+1} \) and
\[
\| \mathcal{R} \left( \mathcal{R} \mathcal{P}_{\geq \lambda_{q+1}} G_\Delta : \nabla (m_{\ell}/\rho)_1 \right) \|_N \lesssim \frac{1}{\lambda_{q+1}} \| \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m_{\ell}/\rho)_1 \|_N
\]
\[
\lesssim \frac{1}{\lambda_{q+1}^{\frac{1}{2}}} \sum_{N_1+N_2=N} \| G_\Delta \|_{N_1} \| \nabla \frac{m_{\ell}}{\rho} \|_{N_2} \quad (9.11)
\]
On the other hand, \( \mathcal{R} P_{\leq \lambda_{q+1}} G_\Delta : \nabla (m_\ell / q)_1 \) has the frequency localized to \( \lesssim \lambda_{q+1}^{-1} \), so that

\[
\| \mathcal{R} \left( \mathcal{R} P_{\leq \lambda_{q+1}} G_\Delta : \nabla (m_\ell / q)_1 \right) \|_N \lesssim \lambda_{q+1}^N \| \mathcal{R} \left( \mathcal{R} P_{\leq \lambda_{q+1}} G_\Delta : \nabla (m_\ell / q)_1 \right) \|_0 \\
\lesssim \lambda_{q+1}^N \| P_{\leq \lambda_{q+1}} G_\Delta \|_0 \| \nabla \frac{m_\ell}{q} \|_0.
\]

(9.12)

Observe that since \( q \) is smooth in space-time and bounded below by a positive constant \( \varepsilon_0 \), we have by Bernstein’s inequality that

\[
\| P_{\leq \lambda} q^{-1} \|_{N'} + \| \partial_t P_{\leq \lambda} q^{-1} \|_{N'} \lesssim \lambda^{-2} \left( \| \nabla^2 q^{-1} \|_{N'} + \| \nabla^2 \partial_t q^{-1} \|_{N'} \right) \lesssim \lambda^{-2}
\]

(9.13)

for any \( \lambda \geq 1 \) and for any \( N' = 0, 1, 2, 3 \). So we see \( \| (m_\ell / q)_2 \|_{N'} \lesssim \frac{1}{\lambda_{q+1}^2} \| m_\ell \|_{N'} \) for \( N' = 0, 1, 2, 3 \). It then follows that

\[
\| \mathcal{R} (RG_\Delta : \nabla (m_\ell / q)_2) \|_N \lesssim \sum_{N_1 + N_2 = N} \| G_\Delta \|_{N_1} \| \nabla (m_\ell / q)_2 \|_{N_2} \\
\lesssim \frac{1}{\lambda_{q+1}^2} \sum_{N_1 + N_2 = N} \| G_\Delta \|_{N_1} \| m_\ell \|_{N_2 + 1}.
\]

(9.14)

Therefore, using (9.8) and (9.10), we obtain

\[
\left\| \mathcal{R} (R_{O_1} : \nabla \frac{m_\ell}{q}) \right\|_N + \left\| \mathcal{R} (R_T : \nabla \frac{m_\ell}{q}) \right\|_N + \left\| \mathcal{R} (R_N : \nabla \frac{m_\ell}{q}) \right\|_N \\
\lesssim \lambda_{q+1}^N \left( \frac{\delta_{q+1}}{\lambda_{q+1} \tau_q} + \frac{\delta_{q+1}}{\lambda_{q+1} \mu_q} \right) \frac{\lambda_q \delta_q^2}{\lambda_{q+1}}.
\]

To estimate their advective derivatives, consider the decomposition

\[
D_{t,q+1} \mathcal{R} \left( R_\Delta : \nabla \frac{m_\ell}{q} \right) \\
= D_{t,\ell} \mathcal{R} \left( R_\Delta : \nabla \frac{m_\ell}{q} \right) + \left( \frac{n + (m_q - m_\ell)}{q} \cdot \nabla \right) \mathcal{R} \left( R_\Delta : \nabla \frac{m_\ell}{q} \right).
\]

We can easily see that

\[
\left\| \frac{n + (m_q - m_\ell)}{q} \cdot \nabla \mathcal{R} \left( R_\Delta : \frac{m_\ell}{q} \right) \right\|_{N-1} \\
\lesssim \lambda_{q+1}^N \delta_{q+1} \left( \frac{\delta_{q+1}^2}{\lambda_{q+1} \tau_q} + \frac{\delta_{q+1}}{\lambda_{q+1} \mu_q} \right) \frac{\lambda_q \delta_q^2}{\lambda_{q+1}}.
\]
As for the first term, we use again the decompositions \( R_\Delta = R P_{\gtrsim \lambda_{q+1}} G_\Delta + R P_{\lesssim \lambda_{q+1}} G_\Delta \) and \( \nabla \frac{m_\ell}{\varrho} = \nabla (m_\ell/\varrho)_1 + \nabla (m_\ell/\varrho)_2 \) and consider
\[
\| D_{t,\ell} R (R : \nabla (m_\ell/\varrho)_1) \|_{N-1} \leq \| D_{t,\ell} R (R P_{\gtrsim \lambda_{q+1}} G_\Delta : \nabla (m_\ell/\varrho)_1) \|_{N-1} + \| D_{t,\ell} R (R P_{\lesssim \lambda_{q+1}} G_\Delta : \nabla (m_\ell/\varrho)_1) \|_{N-1}.
\]
In order to estimate the first term on the right hand side, consider the decomposition
\[
D_{t,\ell} R P_{\gtrsim \lambda_{q+1}} H = R P_{\gtrsim \lambda_{q+1}} D_{t,\ell} H
\]
for any smooth vector-valued function \( H \) and Littlewood-Paley operator \( P_{\gtrsim \lambda_{q+1}} \) projecting to the frequency \( \gtrsim \lambda_{q+1} \). Since Lemmas A.4 is still valid when \( \ell^{-1} \) is replaced by \( C \lambda_{q+1} \), additionally using (9.13), we have
\[
\| (m_\ell/\varrho)_1 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \|_{N-1} \lesssim \lambda_{q+1}^{-\frac{1}{2}} \| \nabla (m_\ell/\varrho)_1 \|_0 \| \nabla H \|_0 \leq \lambda_{q+1}^{-\frac{1}{2}} m_q \| \nabla H \|_0
\]
\[
\| (m_\ell/\varrho)_2 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \|_{N-1} \lesssim \lambda_{q+1}^{-\frac{1}{2}} \| \nabla (m_\ell/\varrho)_2 \|_1 \| \nabla H \|_0 \leq \lambda_{q+1}^{-\frac{3}{2}} m_q \| \nabla H \|_0
\]
Also, by Lemma A.7 we obtain
\[
\left\| \frac{m_\ell}{\varrho} \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \right\|_{N-1} \lesssim \sum_{N_1+N_2=N-1} \| \nabla (m_\ell P_{\lesssim \lambda_{q+1}}) \|_{N_1} \| H \|_{N_2},
\]
and using \( \| R f \|_0 \lesssim \| f \|_0 \) and \( \| P_{\lesssim \lambda_{q+1}} \|_{N'} \lesssim \epsilon^3 \| \nabla^3 \varrho^{-1} \|_{N'} \) we get
\[
\left\| \frac{m_\ell}{\varrho} \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \right\|_{N-1} \lesssim \sum_{N_1+N_2+N_3=N-1} \| m_\ell \|_{N_1} \| P_{\lesssim \lambda_{q+1}} \|_{N_2} \| \nabla P_{\gtrsim \lambda_{q+1}} H \|_{N_3} \lesssim \epsilon^3 \sum_{N_1+N_2+N_3=N-1} \| m_\ell \|_{N_1} \| \nabla P_{\gtrsim \lambda_{q+1}} H \|_{N_3}.
\]
Since \( P_{\gtrsim \lambda_{q+1}} D_{t,\ell} H \) and \((m_\ell/\varrho)_1 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \) have frequencies localized to \( \gtrsim \lambda_{q+1} \), it follows that
\[
\| D_{t,\ell} R P_{\gtrsim \lambda_{q+1}} H \|_{N-1} \lesssim R P_{\gtrsim \lambda_{q+1}} D_{t,\ell} H \|_{N-1} + \frac{1}{\lambda_{q+1}} \| (m_\ell/\varrho)_1 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \| H \|_{N-1}
\]
\[
+ \| R (m_\ell/\varrho)_2 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \| H \|_{N-1} + \frac{1}{\lambda_{q+1}} \| (m_\ell/\varrho)_1 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \| H \|_{N-1}
\]
\[
\lesssim \frac{1}{\lambda_{q+1}} \| D_{t,\ell} H \|_{N-1} + \frac{1}{\lambda_{q+1}} \| (m_\ell/\varrho)_1 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \| H \|_{N-1}
\]
\[
+ \| R (m_\ell/\varrho)_2 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \| H \|_{N-1} + \| (m_\ell/\varrho)_1 \cdot \nabla, P_{\gtrsim \lambda_{q+1}} \| H \|_{N-1}
\]
\[
\lesssim \frac{1}{\lambda_{q+1}} \| D_{t,\ell} H \|_{N-1} + \lambda_{q+1}^{-\frac{3}{2}} m_q \| \nabla H \|_0 + \sum_{N_1+N_2=N-1} \| \nabla \frac{m_\ell}{\varrho} \|_{N_1} \| H \|_{N_2}.
\]
Now, we apply it to $H = R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} : \nabla (m\ell / \varrho)_1$. For such $H$, we have $H = P_{\geq \frac{1}{3} \lambda_{q+1}} H$ for sufficiently large $\lambda_0$, so that

$$\| D_{t, \ell} R (R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} : \nabla (m\ell / \varrho)_1) \|_{N-1} \leq \lambda_{q+1}^{N-2} \| D_{t, \ell} (R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} : \nabla (m\ell / \varrho)_1) \|_{N-1}$$

Indeed, the second inequality can be obtained by applying (9.16) again to $H = G_{\Delta}$,

$$\| D_{t, \ell} R (R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} : \nabla (m\ell / \varrho)_1) \|_{N-1} \leq \sum_{N_1 + N_2 = N-1} \| D_{t, \ell} R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} \|_{N_1} \|
abla (m\ell / \varrho)_1 \|_{N_2} + \| R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} \|_{N_1} \| D_{t, \ell} \nabla (m\ell / \varrho) \|_{N_2}$$

where we used

$$\| D_{t, \ell} \nabla (m\ell / \varrho) \|_{N_2} \leq \| \varrho^{-1} D_{t, \ell} \nabla m\ell \|_{N_2} + \| D_{t, \ell} m\ell \otimes \nabla \varrho^{-1} \|_{N_2}$$

which follows from (5.14) and (5.15). Also, we get

$$\| R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} : \nabla (m\ell / \varrho)_1 \|_{N-1} \leq \sum_{N_1 + N_2 = N-1} \| R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} \|_{N_1} \| \nabla (m\ell / \varrho) \|_{N_2}$$

As for the remaining term $\| D_{t, \ell} R (R \mathcal{P}_{\lambda_{q+1}} G_{\Delta} : \nabla (m\ell / \varrho)_1) \|_{N-1}$, we set $D_{t, \ell}^L := \partial_t + (m\ell / \varrho)_1 \cdot \nabla$ and estimate
We observe that by setting $D_t \triangleq \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1$ we get
\[
\|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0 \lesssim \|\mathcal{R} D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0
+ \|\nabla (m \ell / \varrho)_1 \cdot \nabla (m \ell / \varrho)_1\|_0
\lesssim \|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0
+ \|\mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0
\lesssim \lambda_{q+1}^{-1}\frac{1}{\lambda_{q+1}} \left( \frac{\delta_{q+1}^2}{\lambda_{q+1} \tau_q} + \frac{\delta_{q+1}^2}{\lambda_{q+1} \mu_q} \right) \frac{\lambda_q \delta_q^2}{\lambda_{q+1}}.
\]

Here, we used $\|\mathcal{R} g\|_0 \lesssim \|g\|_0$. Now observe that the frequency of $D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1$ is localized to $\lesssim \lambda_{q+1}$, so that

\[
\|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_{N-1} \lesssim \lambda_{q+1}^N \|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0.
\]

Thus we see
\[
\|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_{N-1} \\
\lesssim \lambda_{q+1}^N \|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0
+ \|\mathcal{R} (m \ell / \varrho)_2 \cdot \nabla (m \ell / \varrho)_1\|_0
\lesssim \lambda_{q+1}^N \|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_0
+ \lambda_{q+1}^{-2} \|\mathcal{R} (m \ell / \varrho)_2 \cdot \nabla (m \ell / \varrho)_1\|_0
\lesssim \lambda_{q+1}^N \frac{1}{\lambda_{q+1}} \left( \frac{\delta_{q+1}^2}{\lambda_{q+1} \tau_q} + \frac{\delta_{q+1}^2}{\lambda_{q+1} \mu_q} \right) \frac{\lambda_q \delta_q^2}{\lambda_{q+1}}.
\]

As a result, combining with (9.17), we obtain
\[
\|D_t \mathcal{R} \mathcal{P}_{\leq \lambda_{q+1}} G_\Delta : \nabla (m \ell / \varrho)_1\|_{N-1} \lesssim \lambda_{q+1}^N \frac{1}{\lambda_{q+1}} \left( \frac{\delta_{q+1}^2}{\lambda_{q+1} \tau_q} + \frac{\delta_{q+1}^2}{\lambda_{q+1} \mu_q} \right) \frac{\lambda_q \delta_q^2}{\lambda_{q+1}}.
\]

Now we need to estimate $D_t \mathcal{R} (R_\Delta : \nabla (m \ell / \varrho)_2)$. Consider the decomposition
\[
D_t \mathcal{R} H = \mathcal{R} D_t H + \left[ \frac{m \ell}{\varrho} \cdot \nabla, \mathcal{R} \right] H,
\]
We observe that by setting $H = R_\Delta : \nabla (m \ell / \varrho)_2$, we get
Thus we get, using (9.13),
\[
\| D_{t, \ell} R \|_{N-1} \lesssim \sum_{N_1 + N_2 = N-1} \| D_{t, \ell} R \|_{N_1} \| \nabla (m \ell / \varrho) \|_{N_2} + \| R \|_{N_1} \| D_{t, \ell} \nabla (m \ell / \varrho) \|_{N_2}
\]
\[
\lesssim \lambda_N^{N} \delta_{q+1} \frac{1}{\lambda_{q+1}^2} \lambda_q \delta_q^2 .
\]
Here, we used the estimate \( \| D_{t, \ell} \nabla (m \ell / \varrho) \|_{N_2} \lesssim \delta_q^2 + \lambda_q \delta_q^2 \), obtained similar to (9.18) but additionally using (9.13). The remaining term can be estimated as
\[
\| \left[ \frac{m \ell}{\varrho} \cdot \nabla, R \right] (R_\Delta : \nabla (m \ell / \varrho)) \|_{N-1}
\]
\[
\lesssim \sum_{N_1 + N_2 + N_3 = N-1} \frac{\| \frac{m \ell}{\varrho} \|_{N_1}}{\| R_\Delta \|_{N_2}} \| \nabla (m \ell / \varrho) \|_{N_3} + \frac{\| \frac{m \ell}{\varrho} \|_{N_1}}{\| R_\Delta \|_{N_2}} \| \nabla (m \ell / \varrho) \|_{N_3+1}
\]
\[
\lesssim \lambda_N^{N} \delta_{q+1} \frac{1}{\lambda_{q+1}^2} \delta_q^2 .
\]
We therefore get
\[
\| D_{t, \ell+1} R (R_\Delta : \nabla \frac{m \ell}{\varrho}) \|_{N-1} \lesssim \lambda_N^{N} \delta_{q+1} \frac{1}{\lambda_{q+1}} \left( \frac{\delta_{q+1}}{\lambda_{q+1}^2} + \frac{\delta_{q+1}}{\lambda_{q+1}^2} \right) \frac{\lambda_q \delta_q^2}{\lambda_{q+1}} ,
\]
and the estimates for \( \varphi H_2 \) follow,
\[
\| \varphi H_2 \|_{N-1} \lesssim \lambda_N^{N} \delta_{q+1} \delta_q^2 , \quad \| D_{t, \ell+1} \varphi H_2 \|_{N-1} \lesssim \lambda_N^{N} \delta_{q+1} \delta_q^2 .
\]
To summarize, we get
\[
\| \varphi H \|_{N} \lesssim \frac{1}{2} \lambda_{q+1}^{N-3} \delta_{q+2}^3 , \quad \| D_{t, \ell+1} \varphi H \|_{N-1} \lesssim \frac{1}{2} \lambda_{q+1}^{N-3} \delta_{q+1}^3 .
\]

9.2. Estimates on \( \zeta \)

In this section, we prove
\[
\| \lambda \|_{0} \lesssim \frac{\delta_0^3}{40 (1 + T + \tau_0)} \lambda_{q+1}^{3} \delta_q^3 ,
\]
which implies (9.2) by integration in time.
9.2.1. Estimates on $\zeta_1$ and $\zeta_3$  

By (9.4), $\zeta_3$ can be written as

$$
\zeta_3' = \int_{T^3} \left( \frac{n_o \otimes n_o}{q} - \delta_{q+1} \text{Id} + \varrho R_{\ell} \right) : \nabla \frac{m_{\ell}}{q} \, dx - \int_{T^3} \varrho (RO_1 + RT + RN) : \nabla \frac{m_{\ell}}{q} \, dx.
$$

where $\varrho$ has zero-mean. The magnitude of the first term can be estimated by

$$
\langle \cdot \rangle \leq \frac{\varepsilon_0^2}{100M(1 + T + \tau_0)} \lambda_{q+1}^{-3} \delta_{q+2}^3
$$

First of all, we have $\| \varrho R_\Delta \| \leq \varepsilon_0^2 / (200M(1 + T + \tau_0)) \lambda_{q+1}^{-3} \delta_{q+2}^3$, taking advantage of Lemma A.2 and the representations (3.28) and (3.29).

To estimate $\tilde{\zeta}_{32}'$, we argue as we did in the previous section to estimate $\| \varphi H_2 \|$.

More precisely, using the decomposition (9.9), we have

$$
\left\{ \varrho R_\Delta : \nabla \frac{m_{\ell}}{q} \right\} = \left\{ \mathcal{R} \mathcal{P} \lesssim_{\lambda q+1} G_\Delta : \nabla \frac{m_{\ell}}{q} \right\} + \left\{ \mathcal{R} \mathcal{P} \gtrsim_{\lambda q+1} G_\Delta : \nabla \left( m_{\ell} P \lesssim_{\ell-1} q^{-1} \right) \right\}
$$

where $\varrho R_\Delta$ represents either $\varrho RO_1$, $\varrho RT$, or $\varrho RN$ and can be written as $\mathcal{R} \mathcal{G} \Delta$. Since the argument of $\langle \cdot \rangle$ in the second term has frequency localized to $\gtrsim \lambda_{q+1}$, it has zero-mean. The magnitude of the first term can be estimated by $\varepsilon_0^2 / (400M(1 + T + \tau_0)) \lambda_{q+1}^{-3} \delta_{q+2}^3$ as in (9.12) and (9.14). The last term can be estimated using

$$
\| \mathcal{R} \mathcal{P} \gtrsim_{\lambda q+1} G_\Delta : \nabla \left( m_{\ell} P \lesssim_{\ell-1} q^{-1} \right) \| \leq \mathcal{R} \mathcal{P} \gtrsim_{\lambda q+1} G_\Delta \| \nabla \left( m_{\ell} P \lesssim_{\ell-1} q^{-1} \right) \|
$$

where the second inequality follows from (9.13) and the last one from (9.8). Combining the estimates, we get

$$
\| \zeta_1' \| + \| \zeta_3' \| \leq \frac{\varepsilon_0^2}{100M(1 + T + \tau_0)} \lambda_{q+1}^{-3} \delta_{q+2}^3
$$

9.2.2. Estimates on $\zeta_0$, $\zeta_2$, and $\zeta_4$  

Decompose $\zeta_0'$ into $\zeta_{01}'$ and $\zeta_{02}'$ where

$$
(2\pi)^3 \zeta_{01}' = \int_{T^3} \varrho D_{t,\ell} \text{tr} \left( \frac{n_o \otimes n_o}{q^2} - \delta_{q+1} \text{Id} + R_{\ell} \right) \, dx
$$

$$
(2\pi)^3 \zeta_{02}' = -\int_{T^3} \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{q^2} - \delta_{q+1} \text{Id} + R_{\ell} \right) \, dx
$$

Since the integrands can be written as

$$
(2\pi)^3 \zeta_{01}' = \int \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\delta_{q+1}}{2} \varrho \text{tr} (D_{t,\ell} c_{u,k}) e^{i\lambda_{q+1} k \cdot \xi_{t}} \, dx
$$

$$
(2\pi)^3 \zeta_{02}' = -\int \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\delta_{q+1}}{2} \text{tr} (c_{u,k} \text{div} (m_q - m_\ell)) e^{i\lambda_{q+1} k \cdot \xi_{t}} \, dx
$$

$$
= -\int \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\delta_{q+1}}{2} \text{tr} (c_{u,k} (-\partial_t Q + (\partial_t Q)_\ell)) e^{i\lambda_{q+1} k \cdot \xi_{t}} \, dx,
$$

where $\lambda_q$, $\delta_q$, and $\lambda_{q+1}$ are non-negative integers.
it thus suffices to use Lemma A.2 to estimate

$$\|\xi_0\|_0 \leq \frac{\varepsilon_0^2}{200M(1 + T + \tau_0)} \lambda^{\frac{3}{2}} q^{\frac{3}{2}} + 2.$$  

In a similar way, we write $$\xi_2'$$ and $$\xi_4'$$ as

$$\begin{align*}
(2\pi)^3 \xi_2' &= \int \text{div}(m_q - m_\ell) \frac{n \cdot m_\ell}{Q^2} \, dx \\
&= \int \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_1^q \text{div}(m_q - m_\ell) \frac{m_\ell}{Q^2} \cdot b_{u,k} e^{i\lambda_q + 1 k \xi_I} \, dx \\
&= \int \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_1^q (-\partial_t Q + (\partial_t Q) \ell) \frac{m_\ell}{Q^2} \cdot b_{u,k} e^{i\lambda_q + 1 k \xi_I} \, dx
\end{align*}$$

and

$$\begin{align*}
(2\pi)^3 \xi_4' &= \int \frac{n}{Q} \cdot \nabla (p(\varrho) - p_\ell(\varrho)) \, dx \\
&= \int \sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_1^q \frac{1}{Q} \nabla (p(\varrho) - p_\ell(\varrho)) \cdot b_{u,k} e^{i\lambda_q + 1 k \xi_I} \, dx.
\end{align*}$$

Therefore, as before, we apply Lemma A.2 to get

$$\|\xi_2\|_0 + \|\xi_4\|_0 \leq \frac{\varepsilon_0^2}{200M(1 + T + \tau_0)} \lambda^{\frac{3}{2}} q^{\frac{3}{2}} + 2.$$  

Thus, we get the desired estimate (9.19).

### 9.3. Transport Current Error

We use the definition of $$\varphi_T$$ and recall its splitting into $$\varphi_{T1} + \varphi_{T2}$$. Since we have $$\|e^{i\lambda_q + 1 k \xi_I}\|_N \lesssim \lambda^N |k|^2$$ for any $$k \in \mathbb{Z}^3 \setminus \{0\}$$ and $$N \leq 2$$, $$D_{t,\ell} e^{i\lambda_q + 1 k \xi_I} = 0$$, and almost disjoint support of $$c_{u,k}, (3.26), (3.29)$$ and (6.6) imply

$$\begin{align*}
&\left\| \frac{n_o \otimes n_o}{Q^2} - \delta_{q+1} \text{Id} + R_\ell \right\|_N \lesssim \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{u \in \mathbb{Z}} \delta_{q+1} \frac{c_{u,k}}{Q^2} e^{i\lambda_q + 1 k \xi_I} \right\|_N \lesssim \lambda^N q + \frac{1}{2} \cdot \delta_{q+1}, \\
\end{align*}$$

(9.20)

$$\begin{align*}
&\left\| D_{t,q+1} \left( \frac{n_o \otimes n_o}{Q^2} - \delta_{q+1} \text{Id} + R_\ell \right) \right\|_{N-1} \lesssim \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{u \in \mathbb{Z}} \delta_{q+1} \left(D_{t,\ell} \frac{c_{u,k}}{Q^2}\right) e^{i\lambda_q + 1 k \xi_I} \right\|_{N-1} \\
&+ \left\| (n + (m_q - m_\ell) \cdot \nabla) \left( \frac{n_o \otimes n_o}{Q^2} - \delta_{q+1} \text{Id} + R_\ell \right) \right\|_{N-1} \lesssim \lambda^N q + \frac{1}{2} \cdot \delta_{q+1}.
\end{align*}$$

(9.21)
Recall that
\[ Q\varphi T_1 = -\kappa_{q+1} n + \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} I_d + R_\ell \right) (m_q - m_\ell) - \frac{m_q \zeta}{\varrho} \]

We then can use (8.1), (6.3), (9.20), (9.21), (5.3), (9.2), and \( \kappa_{q+1} = \frac{1}{2} \text{tr}(R_{q+1}) \) to estimate

\[
\|Q\varphi T_1\|_N \lesssim \sum_{N_1+N_2=N} \|R_{q+1} - \frac{2}{3} \frac{\xi}{\varrho} I_d + \frac{2}{3} \frac{\xi}{\varrho} I_d \|_{N_1} \|n\|_{N_2} \\
+ \sum_{N_1+N_2=N} \left\| \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} I_d + R_\ell \right\|_{N_1} \|m_q - m_\ell\|_{N_2} + \|m_q / \varrho\|_N \|\varrho\|_0 \\
\lesssim \lambda_{q+1}^N \left( \frac{1}{\lambda_q} \frac{1}{\lambda_{q+1}} \delta_q + (\frac{1}{\lambda_q} \frac{1}{\lambda_{q+1}} \delta_q)^2 \right) \delta_{q+1} \lesssim \lambda_{q+1}^N \lambda_q \lambda_{q+1} \delta_q \delta_{q+1},
\]

and, using (9.6) and (9.7) as well,

\[
\|D_{t,q+1} \varphi T_1\|_{N-1} \lesssim \sum_{N_1+N_2=N-1} \|D_{t,q+1}(R_{q+1} - \frac{2}{3} \frac{\xi}{\varrho} I_d + \frac{2}{3} \frac{\xi}{\varrho} I_d)\|_{N_1} \|n\|_{N_2} \\
+ \|R_{q+1} - \frac{2}{3} \frac{\xi}{\varrho} I_d + \frac{2}{3} \frac{\xi}{\varrho} I_d\|_{N_1} \|D_{t,q+1}n\|_{N_2} \\
+ \sum_{N_1+N_2=N-1} \left\| D_{t,q+1} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} I_d + R_\ell \right) \right\|_{N_1} \|m_q - m_\ell\|_{N_2} \\
+ \sum_{N_1+N_2=N-1} \left\| \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} I_d + R_\ell \right\|_{N_1} \|D_{t,q+1}(m_q - m_\ell)\|_{N_2} \\
\lesssim \lambda_{q+1}^N \delta_q \frac{1}{\lambda_q} \delta_q \lambda_{q+1} \delta_q \delta_{q+1},
\]

As for \( \varphi T_2 \), observe that

\[
Q\varphi T_2 = R \left( \frac{\varrho}{2} D_{t,\ell} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} I_d + R_\ell \right) \right) \\
- R \left( \frac{1}{2} \text{tr} \left( \frac{n_o \otimes n_o}{\varrho^2} - \delta_{q+1} I_d + R_\ell \right) \right) \left( \text{div} m_q - \text{div} m_\ell \right),
\]

from (4.8) and since \( \text{div} m_q = -\partial_t q \). By (3.29),

\[
Q\varphi T_2 = \frac{1}{2} R \left( \sum_{m} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} \text{tr}(QD_{t,\ell}c_{m,k}) e^{i \lambda_{q+1} k \cdot \xi \ell} \right) \\
+ \frac{1}{2} R \left( \sum_{m} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1} \text{tr}((\partial_t q - (\partial_t \varrho) \ell)c_{m,k}) e^{i \lambda_{q+1} k \cdot \xi \ell} \right)
\]
and estimate it using Corollary 7.2 with (6.6) as follows
\[ \| \varphi T_2 \|_N \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}}{\lambda_{q+1}^1 \tau_q}, \quad \| D_{t,q+1} \varphi T_2 \|_{N-1} \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}}{\lambda_{q+1}^1 \tau_q}. \]
To summarize, we have
\[ \| \varphi T \|_N \lesssim \frac{1}{5} \lambda_{q+1}^{N-3} \delta_{q+1}^3 \quad \| D_{t,q+1} \varphi T \|_{N-1} \lesssim \frac{1}{5} \lambda_{q+1}^{N-3} \delta_{q+1}^3 \]
for sufficiently small \( b - 1 > 0 \) and large \( \lambda_0 \).

### 9.4. Oscillation Current Error

Recall that \( \varphi_{\Omega 1} = \mathcal{R} \left( \nabla \cdot \left( \frac{|n_o|^2 n_o}{2 \rho^2} + \varphi_{\ell} \right) \right) \). We remark that (3.30) gives
\[ \text{div} \left( \frac{|n_o|^2 n_o}{2 \rho^2} + \varphi_{\ell} \right) = \text{div} \left( \sum_{u \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_{q+1}^3 \frac{d_{u,k}}{\rho^2} e^{i \lambda_{q+1} k \cdot \xi} \right) \]
\[ = \sum_{u,k} \delta_{q+1}^3 \text{div} \left( \frac{d_{u,k}}{\rho^2} e^{i \lambda_{q+1} k \cdot \xi} \right), \]
because of \( \hat{d}_{l,k}(\hat{f}_l \cdot k) = 0 \). Also, we have
\[ \| D_{t, \ell} \text{div} \left( \frac{d_{u,k}}{\rho^2} \right) \|_{\tilde{N}} \lesssim \| D_{t, \ell} \frac{d_{u,k}}{\rho^2} \|_{\tilde{N}+1} + \| \nabla (m_{\ell}/\varrho) : \nabla \frac{d_{u,k}}{\rho^2} \|_{\tilde{N}} \]
\[ \lesssim M \frac{N}{\lambda_{q+1}} \frac{\lambda_{q+1}^N}{\mu q}, \quad \forall \tilde{N} \in [0, n_0+1]. \]
Therefore, using \( \text{supp}(d_{u,k}) \subset (t_u - \frac{1}{4} \tau_q, t_u + \frac{3}{4} \tau_q) \times \mathbb{R}^3 \), it follows from Corollary 7.2 with (6.7) that
\[ \| \varphi_{\Omega 1} \|_N \lesssim \lambda_{q+1}^N \frac{\delta_{q+1}^3}{\lambda_{q+1}^1 \mu q}, \quad \| D_{t,q+1} (\varphi_{\Omega 1}) \|_{N-1} \lesssim \lambda_{q+1}^N \frac{1}{\lambda_{q+1}^1 \mu q} \]
Next recall that \( \varphi_{\Omega 2} = \frac{|n_o|^2 n_0 n_o}{2 \rho^2} \). Then, (6.1)–(6.3) imply
\[ \| \varphi_{\Omega 2} \|_N \lesssim \left( \frac{|n_o \cdot n_c n_o|}{\rho^2} \right), \quad \| D_{t,q+1} (\varphi_{\Omega 2}) \|_{N-1} \lesssim \left( \frac{|n_o|^2 n_c}{2 \rho^2} \right), \quad \| D_{t,q+1} (\varphi_{\Omega 2}) \|_{N-1} \lesssim \lambda_{q+1}^N \frac{1}{\lambda_{q+1}^1 \mu q}. \]
Therefore, combining the estimates, we get
\[ \| \varphi O \|_N \lesssim \frac{1}{5} \lambda_{q+1}^{N-3} \delta_{q+1}^3 \quad \| D_{t,q+1} \varphi O \|_{N-1} \lesssim \frac{1}{5} \lambda_{q+1}^{N-3} \delta_{q+1}^3 \]
for sufficiently small \( b - 1 > 0 \) and large \( \lambda_0 \).
9.5. Reynolds Current Error

Recall that \( \varphi \varphi_R = (R_{q+1} - \frac{3}{2} \zeta / \varphi \text{Id})n + \frac{2}{3}(\zeta / \varphi)n \). Similar to the estimate for \( \kappa_{q+1}n \) in \( \varphi T_1 \), we have

\[
\| \varphi_R \|_N \lesssim \lambda_{q+1}^{N-3\gamma} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{5}{2}}, \\
\| D_{t,q+1} \varphi_R \|_{N-1} \lesssim \lambda_{q+1}^{N-3\gamma} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{5}{2}}
\]

for sufficiently small \( b - 1 > 0 \) and large \( \lambda_0 \).

9.6. Mediation Current Error

Recall that \( \varphi \varphi_M = \varphi \varphi_{M1} + \varphi \varphi_{M2} + \varphi \varphi_{M3} + \varphi \varphi_{M4} \). We estimate each term separately. Recall now

\[
\varphi \varphi_{M1} = \frac{|m_q - m_\ell|^2}{2\varphi} n + \varphi(\varphi_q - \varphi_\ell), \\
\varphi \varphi_{M2} = \left( \frac{n \otimes n}{\varphi} + \varphi R_q - \varphi R_{q+1} - \delta_{q+1} \varphi \text{Id} \right) \frac{m_q - m_\ell}{\varphi}, \\
\varphi \varphi_{M3} = \mathcal{R} \left( \text{div}(m_q - m_\ell) \frac{n \cdot m_\ell}{\varphi^2} \right), \\
\varphi \varphi_{M4} := \mathcal{R} \left( \frac{n}{\varphi} \cdot \nabla (p(\varphi) - p_\ell(\varphi)) \right)
\]

For \( \varphi \varphi_{M1} \), we use (5.3), (5.5), and (6.3) to get

\[
\| \varphi \varphi_{M1} \|_N \lesssim \frac{1}{10} \lambda_{q+1}^{N-3\gamma} \delta_{q+1}^{\frac{3}{2}}, \\
\| D_{t,q+1} \varphi \varphi_{M1} \|_{N-1} \lesssim \frac{1}{10} \lambda_{q+1}^{N-3\gamma} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{5}{2}}.
\]

For \( \varphi \varphi_{M2} \), we use (9.4), and we estimate \( \varphi \varphi_{M2} \) in a similar way as \( \varphi T_1 \) and \( \varphi H_2 \),

\[
\left\| \left( \frac{n \otimes n}{\varphi} + \varphi R_q - \varphi R_{q+1} - \delta_{q+1} \varphi \text{Id} \right) \frac{m_q - m_\ell}{\varphi} \right\|_N \\
\leq \lambda_{q+1}^{N} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{5}{2}}, \\
\left\| D_{t,q+1} \left[ \left( \frac{n \otimes n}{\varphi} + \varphi R_q - \varphi R_{q+1} - \delta_{q+1} \varphi \text{Id} \right) \frac{m_q - m_\ell}{\varphi} \right] \right\|_{N-1} \\
\leq M \lambda_{q+1}^{N} \delta_{q+1}^{\frac{3}{2}} \delta_{q+1}^{\frac{5}{2}} \delta_{q+1}^{\frac{5}{2}}
\]

For \( \varphi \varphi_{M3} \) and \( \varphi \varphi_{M4} \), observe that the argument of \( \mathcal{R} \) have the respective forms

\[
\sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1}^{\frac{1}{2}} \text{div}(m_q - m_\ell) \frac{m_\ell}{\varphi} \cdot b_{u,k} e^{i\lambda_{q+1}k \cdot \xi_l}, \\
\sum_u \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \delta_{q+1}^{\frac{1}{2}} \nabla (p(\varphi) - p_\ell(\varphi)) \frac{1}{\varphi} \cdot b_{u,k} e^{i\lambda_{q+1}k \cdot \xi_l}
\]
Therefore, using \( \text{supp}(b_{u,k}) \subset (t_u - \frac{1}{2}\tau_q, t_u + \frac{3}{2}\tau_q) \times \mathbb{R}^3 \), the desired estimate follows from Corollary 7.2:

\[
\| \varphi_M \|_N \leq \frac{1}{5} \lambda^N_{q+1} \cdot 3^3 \frac{1}{\delta_q} \delta_{q+2}^2, \\
\| D_t, q+1 \varphi_{\Lambda} \|_{N-1} \leq \frac{1}{5} \lambda^N_{q+1} \cdot 3^3 \frac{1}{\delta_q} \delta_{q+1}^2 \delta_{q+2}.
\]

### 10. Proofs of the Key Inductive Propositions

#### 10.1. Proof of Proposition 2.3

For a given dissipative Euler-Reynolds flow \((m_q, c_q, R_q, \varphi_q)\) on the time interval \([0, T] + \tau_q-1\), we recall the construction of the corrected one: \(m_{q+1} = m_q + n_{q+1}\), where \(n_{q+1}\) is defined by (3.35). Furthermore, we find a new Reynolds stress \(R_{q+1}\) and an unresolved flux current \(\varphi_{q+1}\) which solve (2.1) together with \(m_{q+1}\), and satisfy (8.1), (9.1) and (9.2) for sufficiently small \(b - 1 > 0\) and large \(\lambda_0\). In other words, \((m_{q+1}, c_{q+1}, R_{q+1}, \varphi_{q+1})\) is a dissipative Euler-Reynolds flow for the energy loss \(E\) and the error \((R_{q+1}, \varphi_{q+1})\) satisfies (2.3)–(2.4) for \(q + 1\) as desired. Now, denote the absolute implicit constant in the estimate (6.4) for \(n\) by \(M_0\) and define \(M = 2M_0\). Then, one can easily see that

\[
\|m_{q+1} - m_q\|_0 + \frac{1}{\lambda_{q+1}} \|m_{q+1} - m_q\|_1
\]

\[
= \|n_{q+1}\|_0 + \frac{1}{\lambda_{q+1}} \|n_{q+1}\|_1 \leq 2M_0 \delta_{q+1}^{\frac{1}{2}} = M \delta_{q+1}^{\frac{1}{2}}.
\]

Also, using (2.2) and (6.3), we have that

\[
\|m_{q+1}\|_0 \leq \|m_q\|_0 + \|n_{q+1}\|_0 \leq M - \delta_{q+1}^{\frac{1}{2}} + M_0 \delta_{q+1}^{\frac{1}{2}} \leq M - \delta_{q+1}^{\frac{1}{2}},
\]

\[
\|m_{q+1}\|_N \leq \|m_q\|_N + \|n_{q+1}\|_N \leq M \lambda^N_{q} \delta_{q+1}^{\frac{1}{2}} + \frac{1}{2} M \lambda^N_{q+1} \delta_{q+1}^{\frac{1}{2}} \leq M \lambda^N_{q+1} \delta_{q+1}^{\frac{1}{2}},
\]

for \(N = 1, 2\), provided that \(\lambda_0\) is sufficiently large. Therefore, we construct a desired corrected flow \((m_{q+1}, c_{q+1}, R_{q+1}, \varphi_{q+1})\).

#### 10.2. Proof of Proposition 2.6

Consider a given time interval \(I \subset (0, T)\) with \(|I| \geq 3\tau_q\). Then, we can always find \(u_0\) such that \(\text{supp}(\theta_{u_0}(\tau_q^{-1} \cdot)) \subset I\). Now, if \(I = (u_0, v, f)\) belongs to \(\mathcal{I}_0\), we replace \(\gamma_I\) in \(n_{q+1,0}\) by \(\tilde{\gamma}_I = -\gamma_I\). In other words, we replace \(\Gamma_I\) by \(\tilde{\Gamma}_I = -\Gamma_I\). Otherwise, we keep the same \(\gamma_I\). Note that \(\tilde{\Gamma}_I\) still solves (3.23) and hence \(\tilde{\gamma}_I\) satisfies (3.22). Since the replacement does not change the estimates for \(\Gamma_I\) used in the proof of Lemma 6.2, the corresponding coefficients \(\tilde{b}_{u,k}, \tilde{c}_{u,k}, \tilde{d}_{u,k}, \text{and } \tilde{e}_{u,k}\) satisfy (6.5)–(6.8), and \(\tilde{n}_a, \tilde{n}_c,\) and \(\tilde{n}_{q+1}\), generated by them, also fulfill (6.1)–(6.3). As a result, the corrected dissipative Euler-Reynolds flow \((\tilde{m}_{q+1}, c_{q+1}, \tilde{R}_{q+1}, \varphi_{q+1})\)
satisfies (2.2)–(2.4) for \( q + 1 \) and (2.6) as desired. On the other hand, by the construction, the correction \( \tilde{n}_{q+1} \) differs from \( n_{q+1} \) on the support of \( \theta_{u_0}(\tau_q^{-1} \cdot) \). Therefore, we can easily see

\[
\text{supp}_t(m_{q+1} - \tilde{m}_{q+1}) = \text{supp}_t(n_{q+1} - \tilde{n}_{q+1}) \subset I.
\]

Furthermore, by (3.22) and (3.23), we have

\[
\sum_{I \in \mathcal{S}_{u,v,R}} \gamma_I^2 |(\nabla \xi_I)^{-1} f_I|^2 = \text{tr} \left( \left( \nabla \xi_I \right)^{-1} \sum_{f \in \mathcal{F}_{u,v,R}} \gamma_I^2 f_I \otimes f_I [(\nabla \xi_I)^{-1}]^\top \right)
= \text{tr}(\varphi^2(\delta_{q+1} \text{Id} - R_{\ell} - \tilde{M}_I)),
\]

where \( \tilde{M}_I := \sum_{(u', v') \in I(u, v)} \theta^2 f (\xi_{I'}) \sum_{f' \in \mathcal{F}_{u', v', \phi}} \gamma_{I'}' \left( \int_{\mathbb{T}^3} \phi_{I'} dx \right) (\nabla \xi_{I'})^{-1} f' \otimes (\nabla \xi_{I'})^{-1} f'. \) In particular,

\[
\| \tilde{M}_I \|_0 \lesssim \lambda^{2\gamma} \delta_{q+1}
\]

(see the proof in Section 3.4.2). In this proof, \( \| \cdot \|_N \) denotes \( \| \cdot \|_{C([0, T]; C^N(\mathbb{T}^3))} \). Then, it follows that

\[
|n_o - \tilde{n}_o|^2 = \sum_{I \in \mathcal{S}_{R:u_0}} 4\theta^2 f (t) \gamma^2 |(\nabla \xi_I)^{-1} f_I|^2 (1 + (\psi_I^2(\lambda_{q+1} - \xi_I) - 1))
\]

\[
= \sum_{I \in \mathcal{S}_{R:u_0}} 4\theta^2 f (t) \gamma^2 |(\nabla \xi_I)^{-1} f_I|^2 (\varphi^2(3\delta_{q+1} - \text{tr}(R_{\ell})) - \text{tr}(\tilde{M}_I))
+ \sum_{k \in \mathbb{Z}^3 \setminus \{0\} : I \in \mathcal{S}_{R:u_0}} \sum_{I \in \mathcal{S}_{R:u_0}} 4\theta^2 f (t) \gamma^2 |(\nabla \xi_I)^{-1} f_I|^2 \tilde{c}_{I,k} e^{i\lambda_{q+1} k \cdot \xi_I}
\]

\[
= 4\varphi^2(\tau_q^{-1} I)(\varphi^2(3\delta_{q+1} - \text{tr}(R_{\ell}) - \text{tr}(\tilde{M}_I)) + \sum_{k \in \mathbb{Z}^3 \setminus \{0\} : I \in \mathcal{S}_{R:u_0}} \sum_{I \in \mathcal{S}_{R:u_0}} 4\theta^2 f (t) \gamma^2 |(\nabla \xi_I)^{-1} f_I|^2.
\]

where

\[
\text{tr}(\tilde{c}_{u_0,k}^R) = \sum_{I \in \mathcal{S}_{R:u_0}} \theta^2 f (t) \gamma^2 |(\nabla \xi_I)^{-1} \delta_{q+1} \gamma f \tilde{c}_{I,k} |(\nabla \xi_I)^{-1} f_I|^2.
\]

Since we can obtain \( \| \text{tr}(\tilde{c}_{u_0,k}^R) \|_N \lesssim \mu_{q^N} |\tilde{c}_{I,k}| \) for \( N = 0, 1, 2 \) in the same way used to get the estimate (6.6) for \( c_{u_0,k} \), we conclude

\[
\| n_o - \tilde{n}_o \|^2_{C^0([0, T]; L^2(\mathbb{T}^3))}
\]

\[
\geq 12\delta_{q+1} \varphi_{u_0}^2 \|e\|^2_{C^0([u_0, u_0 + \frac{1}{8} \tau_q; L^2(\mathbb{T}^3))} - 4(2\pi)^3(\| \varphi^2 R_{\ell} \|_0 + \| \text{tr}(\tilde{M}_I) \|_0)
\]

\[
- \sup_{t \in [0, T]} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} 4 \left| \delta_{q+1} \int \text{tr}(\tilde{c}_{u_0,k}^R) e^{i\lambda_{q+1} k \cdot \xi_I} dx \right|
\]

\[
\geq 12\delta_{q+1} \varphi_{u_0}^2 \|e\|^2_{C^0([u_0, u_0 + \frac{1}{8} \tau_q; L^2(\mathbb{T}^3))} - c_o \delta_{q+1} \lambda_{q-3}^2 + \lambda_{q-2}^2 + (\lambda_{q+1} \mu_q)^2)
\]

\[
\geq 4\delta_{q+1} \varepsilon_0
\]
for sufficiently large \( \lambda_0 \). Indeed, in the second inequality, we used Lemma A.2 to get

\[
\sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left| \int \text{tr}(Z_{u_0,k}) e^{\lambda_{q+1} k \cdot \xi} \, dx \right| \lesssim \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\| \text{tr}(Z_{u_0,k}) \|_2 + \| \text{tr}(\tilde{Z}_{u_0,k}) \|_2 \| \nabla \xi \| C^0((0,T); \mathbb{R}^3)}{\lambda_{q+1}^2 |k|^2}
\]

\[
\lesssim (\lambda_{q+1} \mu_q)^{-2} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|\hat{e}_{L,k}|}{|k|^2} \lesssim (\lambda_{q+1} \mu_q)^{-2} \left( \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|\hat{e}_{L,k}|^2}{|k|^4} \right) \left( \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|k|^4} \right)^{1/2}.
\]

Therefore, we obtain

\[
\| m_{q+1} - \tilde{m}_{q+1} \|_{C^0((0,T); L^2(\mathbb{T}^3))} = \| n_{q+1} - \tilde{n}_{q+1} \|_{C^0((0,T); L^2(\mathbb{T}^3))}
\]

\[
\geq \| n_o - \tilde{n}_o \|_{C^0((0,T); L^2(\mathbb{T}^3))} - (2\pi)^{3/2} (\| n_c \|_0 + \| \tilde{n}_c \|_0)
\]

\[
\geq 2\delta_{q+1}^2 \epsilon_0 - \frac{(2\pi)^{3/2} 2M_0 \delta_{q+1}^2 \lambda_{q+1} \mu_q}{\lambda_{q+1} \mu_q} \delta_{q+1} \geq \frac{1}{2} \delta_{q+1} \epsilon_0
\]

for sufficiently large \( \lambda_0 \).

Lastly, we suppose that a dissipative Euler-Reynolds flow \((\tilde{m}_q, c_q, \tilde{R}_q, \tilde{\phi}_q)\) satisfies (2.2)–(2.4) and

\[
\text{supp}_t (m_q - \tilde{m}_q, R_q - \tilde{R}_q, \varphi_q - \tilde{\varphi}_q) \subset \mathcal{J}
\]

for some time interval \( \mathcal{J} \). Proceed to construct the regularized flow, \( \tilde{R}_\ell \) and \( \tilde{\varphi}_\ell \) as we did for \( R_\ell \) and \( \varphi_\ell \) and note that they differ only in \( \mathcal{J} + \ell t \subset \mathcal{J} + (\lambda_{q} \delta_{q}^2)^{-1} \). Consequently, \( n_{q+1} \) differ from \( \tilde{n}_{q+1} \) only in \( \mathcal{J} + (\lambda_{q} \delta_{q}^2)^{-1} \) and hence the corrected dissipative Euler-Reynolds flows \((m_{q+1}, c_{q+1}, R_{q+1}, \varphi_{q+1})\) and \((\tilde{m}_{q+1}, c_{q+1}, \tilde{R}_{q+1}, \tilde{\varphi}_{q+1})\) satisfy

\[
\text{supp}_t (m_{q+1} - \tilde{m}_{q+1}, R_{q+1} - \tilde{R}_{q+1}, \varphi_{q+1} - \tilde{\varphi}_{q+1}) \subset \mathcal{J} + (\lambda_{q} \delta_{q}^2)^{-1}.
\]

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Appendix A. Some Technical Lemmas

The proof of the following two lemmas can be found in [6, Appendix]:

**Lemma A.1.** (Hölder norm of compositions). Suppose $F : \Omega \to \mathbb{R}$ and $\Psi : \mathbb{R}^n \to \Omega$ are smooth functions for some $\Omega \subset \mathbb{R}^m$. Then, for each $N \in \mathbb{N}$, we have

$$\| \nabla^N (F \circ \Psi) \|_0 \lesssim \| \nabla F \|_0 \| \nabla \Psi \|_{N-1} + \| \nabla F \|_{N-1} \| \Psi \|_N \| \nabla \Psi \|_0,$$

where the implicit constant in the inequalities depends only on $n$, $m$, and $N$.

**Lemma A.2.** Let $N \geq 1$. Suppose that $a \in C^\infty([0, T] \times \mathbb{T}^3)$ and $\xi \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ satisfies

$$\frac{1}{C} \leq |\nabla \xi| \leq C$$

for some constant $C > 1$. Then, we have

$$\left| \int_{\mathbb{T}^3} a(x) e^{ik \cdot \xi} \, dx \right| \lesssim \frac{\| a \|_N + \| a \|_0 \| \nabla \xi \|_N}{|k|^N},$$

where the implicit constant in the inequality is depending on $C$ and $N$, but independent of $k$.

The following lemmas contain various commutator estimates, used in the proof.

**Lemma A.3.** Let $f$ and $g$ be in $C^\infty([0, T] \times \mathbb{T}^3)$ and set $f_\ell = P_{\leq \ell-1} f$, $g_\ell = P_{\leq \ell-1} g$ and $(fg)_\ell = P_{\leq \ell-1} (fg)$. Then, for each $N \geq 0$, the following holds,

$$\| f_\ell g_\ell - (fg)_\ell \|_N \lesssim_N \ell^{2-N} \| f \|_1 \| g \|_1.$$

**Proof.** Since the expression that we need to estimate is localized in frequency, by Bernstein’s inequality it suffices to prove the case $N = 0$. Recall now the function $\phi$ used to define the Littlewood-Paley operators and the number $J$, which is the maximal natural number such that $2^J \leq \ell^{-1}$. Denoting by $\tilde{\phi}$ the inverse Fourier transform and by $\tilde{\phi}_\ell$ the function $\tilde{\phi}_\ell(x) = 2^3 \tilde{\phi}(2^J x)$, a simple computation (see for instance [13]) gives

$$(f_\ell g_\ell - (fg)_\ell)(x)$$

$$= \int (f(x) - f(x - y))(g(x) - g(x - y))\tilde{\phi}_\ell(y) \, dy - (f - f_\ell)(g - g_\ell)(x)$$

and the claim easily follows. \qed
Lemma A.4. Let $f$ and $g$ be in $C^\infty([0, T] \times \mathbb{T}^3)$ and set $f_\ell = P_{\leq \ell^{-1}}f$ and $(fg)_\ell = P_{\leq \ell^{-1}}(fg)$. Then, for each $N \geq 0$, the following holds,

\begin{align}
\|g, P_{\leq \ell^{-1}}f\|_0 \lesssim \ell \|f\|_0 \|\nabla g\|_0 \\
\|g, P_{\leq \ell^{-1}}f\|_N \lesssim_N \ell^{1-N} \|f\|_0 \|g\|_{\max\{1, N\}}.
\end{align}

(A.3)

In particular, for any smooth function $v$, $F \in C^\infty(\mathbb{T}^3)$ and for each $N \geq 0$, we have

\begin{align}
\|v \cdot \nabla, P_{\leq \ell^{-1}}f\|_0 &= \|v \cdot \nabla, P_{> \ell^{-1}}f\|_0 \lesssim \ell \|\nabla F\|_0 \|\nabla v\|_0 & (A.4) \\
\|v \cdot \nabla, P_{\leq \ell^{-1}}f\|_N &= \|v \cdot \nabla, P_{> \ell^{-1}}f\|_N \lesssim_N \ell^{1-N} \|\nabla F\|_0 \|v\|_{\max\{1, N\}}.
\end{align}

(A.5)

Remark A.5. When $v$ has the frequency localized to $\ell^{-1}$, using the Bernstein’s inequality, (A.5) can be improved to $\|v \cdot \nabla, P_{\leq \ell^{-1}}f\|_N \lesssim_N \ell^{1-N} \|\nabla v\|_0 \|\nabla F\|_0$.

Proof. We first write

$$f_\ell g - (fg)_\ell = \int f(y)(g(x) - g(y))\tilde{\phi}_\ell(x - y) \, dy.$$ 

Then, (A.3) is obtained as $\|f_\ell g - (fg)_\ell\|_0 \lesssim \ell \|f\|_0 \|\nabla g\|_0$ and

$$|\nabla^N (f_\ell g - (fg)_\ell)| \leq \int |f(y)||g(x) - g(y)||\nabla^N \tilde{\phi}_\ell(x - y)| \, dy \\
+ \sum_{N_1=1}^N c_{N_1, N} \int |f(y)||\nabla^{N_1} g(x)||\nabla^{N-N_1} \tilde{\phi}_\ell(x - y)| \, dy \\
\lesssim \ell^{1-N} \|f\|_0(\|\nabla g\|_0 + \|g\|_N)$$

for some constants $c_{N_1, N}$. Since we have

$$[v \cdot \nabla, P_{> \ell^{-1}}f](x) = v \cdot \nabla (P_{> \ell^{-1}}f - F) + (v \cdot \nabla F) - P_{> \ell^{-1}}(v \cdot \nabla F) = -v \cdot \nabla P_{\leq \ell^{-1}}F + P_{\leq \ell^{-1}}(v \cdot \nabla F) = -[v \cdot \nabla, P_{\leq \ell^{-1}}]F,$$

we apply (A.3) to $g = v_i$ and $f = \partial_t F$, then (A.4) and (A.5) follow. □

Lemma A.6. For a fixed $N \in \mathbb{N}$, if $v$ and $g$ satisfy

$$\|v\|_N \lesssim_N \ell^{-N} v_F, \quad \|g\|_N \lesssim_N g_F$$

for all integer $N \in [1, N]$ and for some positive constants $v_F$ and $g_F$, then we have

$$\|[v \cdot \nabla, P_{\leq \ell^{-1}}](fg) - ([v \cdot \nabla, P_{\leq \ell^{-1}}]f)g\|_N \lesssim \ell^{1-N} \|\nabla f\|_0 v_F g_F + \ell^{-N} \|f\|_0 v_F g_F.$$

(A.6)

for any integer $N \in [0, N]$. 

Proof. We first write

\[
[v \cdot \nabla, P_{\leq t-1}](fg) - ([v \cdot \nabla, P_{\leq t-1}]f)g = \int_{\mathbb{R}^3} (v(x) - v(y)) \cdot \nabla(fg)(y) \tilde{\phi}_e(x - y) \, dy
\]

\[
- \int_{\mathbb{R}^3} (v(x) - v(y)) \cdot \nabla f(y) g(x) \tilde{\phi}_e(x - y) \, dy
\]

\[
= \int_{\mathbb{R}^3} (v(x) - v(y)) \cdot \nabla f(y)(g(y) - g(x)) \tilde{\phi}_e(x - y) \, dy
\]

\[
+ \int_{\mathbb{R}^3} (v(x) - v(y)) \cdot \nabla g(y) f(y) \tilde{\phi}_e(x - y) \, dy.
\]

Then, (A.6) follows from

\[
\|\nabla^N_x (v(x) - v(y))\|_0 \lesssim \varepsilon^{-N} v_F, \quad |g(x) - g(y)| \lesssim |x - y|g_F, \quad \int |y||\tilde{\phi}_e(y)|dy \lesssim \varepsilon.
\]

\[\square\]

Lemma A.7. For vector-valued functions \(H\) and \(v\) in \(C^\infty([0, T] \times \mathbb{T}^3)\), it holds that

\[
\|\left[ P_{\leq t-1} v \cdot \nabla, \mathcal{R} \right] P_{\geq \lambda q+1} H \|_{N-1} \lesssim \sum_{N_1 + N_2 = N - 1} \epsilon \|\nabla v\|_{N_1} \|H\|_{N_2}
\]

for \(N = 1, 2\), where \(\mathcal{R} = \Delta^{-1}\nabla\).

Proof. For convenience, we write \(v_\ell := P_{\leq t-1} v\) and \(H_j = P_{2^j} H\) for for \(2^j \gtrsim \lambda q + 1\). We first use the Fourier expansion and the Taylor’s theorem to get

\[
-\left[ v_\ell \cdot \nabla, \mathcal{R} \right] H_j
\]

\[
= \sum_{k, \eta \in \mathbb{Z}^3} (\mathcal{F}[\mathcal{R}](k) - \mathcal{F}[\mathcal{R}](\eta)) i\eta \cdot \mathcal{F}[v_\ell](k - \eta) \mathcal{F}[H_j](\eta) e^{i k \cdot x}
\]

\[
= \sum_{k, \eta \in \mathbb{Z}^3} \sum_{l=1}^{l_0} \frac{1}{l!} [(k - \eta) \cdot \nabla]^{l} [\mathcal{F}[\mathcal{R}](\eta) i\eta \cdot \mathcal{F}[v_\ell](k - \eta) \mathcal{F}[H_j](\eta) e^{i k \cdot x}
\]

\[
+ \frac{1}{l_0!} \sum_{k, \eta \in \mathbb{Z}^3} \int_0^1 [(k - \eta) \cdot \nabla]^{l_0+1} \mathcal{F}[\mathcal{R}](\eta + \sigma(k - \eta))(1 - \sigma)^{l_0} d\sigma i\eta
\]

\[
\cdot \mathcal{F}[v_\ell](k - \eta) \mathcal{F}[H_j](\eta) e^{i k \cdot x}.
\]

(A.7)

where \(l_0 > 2\), independent of \(q\), is chosen to satisfy \((\varepsilon^{-1} \lambda_{q+1}^{-1})^{l_0} \lambda_{q+1}^3 \lesssim 1\). The first term can be written as \(\sum_{l=1}^{l_0} \frac{1}{l!} \nabla^l v_\ell : \nabla \mathcal{R}_l H_j\) where the operator \(\mathcal{R}_l\) has a Fourier multiplier defined by \(\mathcal{F}[\mathcal{R}_l g](\eta) = (-i)^l \nabla^l \mathcal{F}[\mathcal{R}](\eta) \mathcal{F}[g](\eta)\).
Using this decomposition, we now estimate
\[
\left\| \sum_{l=1}^{l_0} \frac{1}{l!} \nabla^l \eta \cdot \nabla R_l P_{\geq \lambda_{q+1}} H \right\|_{N-1}
\]
\[
\lesssim \sum_{N_1+N_2=N-1} \sum_{l=1}^{l_0} \frac{1}{l!} \left\| \nabla^l \eta \right\|_{N_1} \left\| \nabla R_l P_{\geq \lambda_{q+1}} H \right\|_{N_2}
\]
\[
\lesssim \sum_{N_1+N_2=N-1} \sum_{l=1}^{l_0} \frac{\ell^{1-l}}{l!} \left\| \nabla v \right\|_{N_1} \sum_{l \geq \lambda_{q+1}} \left\| K_{l,j} \right\|_{L^1} \left\| H \right\|_{N_2}
\]
\[
\lesssim \sum_{N_1+N_2=N-1} \ell \left\| \nabla v \right\|_{N_1} \left\| H \right\|_{N_2},
\]
where \( K_{l,j} \) is the kernel of the operator \( \nabla R_l P_{2j} \) and the last estimate follows from
\[
K_{l,j} = 2^{j-(-l+3)} K_{l,0}(2^j), \quad \sum_{2^j \geq \lambda_{q+1}} \left\| K_{l,j} \right\|_{L^1(\mathbb{R}^3)} \lesssim \lambda_{q+1}^{-l} \left\| K_{l,0} \right\|_{L^1(\mathbb{R}^3)}.
\]

The remaining term can be estimated as follows: since \(|k - \eta| \leq \frac{1}{2} |\eta|\) and hence \(|k| \lesssim |\eta|\), we get
\[
\left\| \nabla^{N-1} \sum_{2^j \geq \lambda_{q+1}} (A.7) \right\|_{N-1} \lesssim \sum_{2^j \geq \lambda_{q+1}} \sum_{k, \eta \in \mathbb{R}^3} \left| k \right|^{-N-1} \left| k - \eta \right|^{l_0+1} \left| \eta \right|^{l_0+1} \left\| \nabla \eta \right\|_{N-1} \left\| \nabla v \right\|_{N-1}
\]
\[
\lesssim \sum_{2^j \geq \lambda_{q+1}} \sum_{k, \eta \in \mathbb{R}^3} \left| k - \eta \right|^{l_0} \left| \eta \right|^{l_0+1} \left\| \nabla \eta \right\|_{N-1} \left\| \nabla v \right\|_{N-1} \left\| \nabla H \right\|_{L^2(\mathbb{T}^3)}
\]
\[
\lesssim \sum_{2^j \geq \lambda_{q+1}} \sum_{k \leq 2^j} (\ell^{-1} 2^{-j})^{l_0} 2^{-j} \left\| \nabla v \right\|_{L^2(\mathbb{T}^3)} \left\| \nabla H \right\|_{L^2(\mathbb{T}^3)}
\]
\[
\lesssim \sum_{2^j \geq \lambda_{q+1}} (\ell^{-1} \lambda_{q+1}^{-1})^{l_0} 2^j \left\| \nabla v \right\|_{L^2(\mathbb{T}^3)} \left\| \nabla H \right\|_{N-1} \lesssim \lambda_{q+1}^{-1} \left\| \nabla v \right\|_{N-1} \left\| \nabla H \right\|_{N-1}.
\]

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