Singularity for Solutions of Linearized KdV Equations

by Keiichi Kato 1 Masaki Kawamoto 2 Koichiro Nanbu

1 Department of mathematics, Faculty of science, Tokyo university of science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan.
Email: kato@rs.tus.ac.jp
2 Department of Engineering for Production, Graduate School of Science and Engineering, Ehime University, 3 Bunkyo-cho Matsuyama, Ehime, 790-8577. Japan.
Email: kawamoto.masaki.zs@ehime-u.ac.jp

Abstract. We investigate the time propagation of singularity of a solution to linearized KdV equation by using the characterization of wave front sets with using to the wave packet transform (short time Fourier transform).

Keywords: Singularity of solution; KdV equation; Wave packet transform; Wave front sets

1 Introduction

In this paper, we investigate singularities of solutions to a linearized KdV equation written as the form

\[
\begin{aligned}
\partial_t u(t, x) + \partial_x^3 u(t, x) + \partial_x (a(t, x)u(t, x)) &= 0, \\
u(0, x) = u_0(x) \in L^2(\mathbb{R}),
\end{aligned}
\]

where \( \partial_t = \partial/\partial t, \partial_x = \partial/\partial x, (t, x) \in \mathbb{R} \times \mathbb{R}, a(t, x) \) be a smooth given function which decays fast at infinity and \( u(t, x) \) be a real valued unknown function.

The motivation to consider the above problem is the following: Consider the KdV equation

\[
f_t(t, x) + af(t, x)f_x(t, x) + \gamma f_{xxx}(t, x) = 0,
\]

where \( (t, x) \in \mathbb{R} \times \mathbb{R} \) and \( a, \gamma \) be positive parameters. For \( x_0 \in \mathbb{R} \) and \( abcd\gamma \neq 0 \) with ratio \( ac = 12b^2\gamma = 3d \), the following function which is called solution

\[
f(t, x) = c \cosh^{-2}(b(x - dt - x_0))
\]

solves the equation (2) with the initial condition \( f(0, x) = c \cosh(b(x - x_0)) \). Let \( \varepsilon > 0 \) be a sufficiently small constant and consider the nonlinear equation

\[
v_t + avv_x + \gamma v_{xxx} = \varepsilon F(v).
\]

If we consider a solution \( v(t, x) \) to (3) of the form \( v(t, x) = f(t, x) + \varepsilon w(t, x) \), then we have

\[
\varepsilon (w_t + a(wf_x + fw_x) + \gamma w_{xxx}) = \varepsilon F(f) - \varepsilon^2 ww_x - \varepsilon^2 \int_0^1 F'(f + \varepsilon \theta w) d\theta w.
\]
If we neglect higher order term with respect to $\varepsilon$ and smooth and rapidly decaying function $F(f)$, we have the first equation of (1) with $a = 1$, $\gamma = 1$, $b = 1$ and $a(t, x) = f(t, x)$. Hence to investigate the behavior of a solution to the linearized problem (1) of the KdV equation would be a first step to know the singularities of the solution $s$ to the KdV equation.

Throughout this paper, we assume the following assumptions on coefficients $a(t, x)$;

**Assumption 1.1.** Let $a \in C^1(R; C^\infty(R))$ is a real-valued function and suppose that there exist a real number $\rho > 0$ such that for all $l_1 \in \{0, 1\}$ and $l_2 \in N \cup \{0\}$ we have

$$\left| \partial_t^l \partial_x^2 a(t, x) \right| \leq C_{l_1, l_2} (1 + |x|)^{-\rho - l_1 - l_2}$$

for some $C_{l_1, l_2} > 0$.

**Remark 1.2.** Clearly the soliton $f$ in (3) satisfies Assumption 1.1.

Under this assumption, we have a unique solution of (1).

**Theorem 1.3.** Under the assumption 1.1 there exists a unique solution $u = u(t, x)$ of (1), which is included in $C(R; L^2(R))$ and satisfies that for any $T \in R$,

$$\sup_{t \in [0, T]} \| u(t, \cdot) \|_2 \leq C_T \| u_0 \|_2.$$  \hspace{1cm} (6)

Moreover if $u_0 \in H^3(R)$, then $u \in C^1(R; L^2(R)) \cap C(R; H^3(R))$ and satisfies that for any $T \in R$,

$$\sup_{t \in [0, T]} \| u(t, \cdot) \|_{H^3(R)} \leq C_T \| u_0 \|_{H^3(R)}.$$  \hspace{1cm} (7)

The aim of this paper is to consider the propagation of the singularities for solution $u(t, x)$. In order to consider such issue, it is important to analyze the "wave front set", which is a set of the singularities of the solution $u(t, \cdot)$ and is defined as follows;

**Definition 1.4** (Wave front sets). For $f \in \mathcal{S}'(R)$, the wave front set $WF(f)$ of $f$ is a subset of $R \times R \setminus \{0\}$ which is determined as follows: We say $(x_0, \xi_0) \notin WF(f)$ if there exists a function $\chi \in C^\infty_0(R)$ with $\chi(x_0) \neq 0$ and a conic neighborhood $\Gamma$ of $\xi_0$ such that for all $N \in N$ there exists a positive constant $C_N$ so that

$$\left| \hat{\chi} f(\xi) \right| \leq C_N (1 + |\xi|)^{-N}$$

holds for all $\xi \in \Gamma$, where $\hat{\cdot}$ denotes the Fourier transform. Otherwise we say $(x_0, \xi_0) \in WF(f)$.

In order to analyze the wave front sets of $u(t, \cdot)$, we employ the approach with using Wave packet transform, which was firstly considered by Folland [9] and was developed by Okaji [18], Kato-Kobayashi-Ito [12], Pilipovic-Prangoski [20] and so on. Here the wave packet transform is defined as follows;

**Definition 1.5** (Wave packet transform). Let $\varphi \in \mathcal{S}(R) \setminus \{0\}$ and $f \in \mathcal{S}'(R)$. The wave packet transform $W_\varphi f(x, \xi)$ with window function $\varphi$ is defined as

$$W_\varphi f(x, \xi) := \int_R \varphi(y - x) f(y) e^{-iy \xi} dy.$$  \hspace{1cm} (7)
As is seen in §3, the wave packet transform reduces (1) to the first order partial differential equations on \( R \times R^2 \) with error terms and such equations are closely related to Hamilton equation. By using the method of characteristics, we can obtain an integral equation which has the solutions to Hamilton equation associated to (1). We can characterize the wave front sets of the solutions just by the asymptotic behavior of solution with respect to the parameter \( \lambda \) which is equivalent to |\( \xi | \) to the integral equation with the aide of characterization of wave front set via wave packet transform (Theorem 3.22 of [9], Theorem 2.2 of [19] and Theorem 1.1 of [12]). This is the merit of using the wave packet transform in the study of characterization of wave front set. As the result, we obtain the following characterization of wave front set is obtained;

**Theorem 1.6.** Let \( u \) be a solution to (1) in \( C(R; L^2(R)) \). Then the following two statements are equivalent:

(i): \((x_0, \xi_0) \notin WF(u(t_0, \cdot))\).

(ii): For any \( N \in N, b \geq 1 \) and \( \varphi_0 \in \mathcal{S}(R) \setminus \{0\} \), there exist a neighborhood \( K \) of \( x_0 \), a neighborhood \( \Gamma \) of \( \xi_0 \) and a constant \( C_{N,b,\varphi_0} > 0 \) such that for all \( x \in K, b^{-1} \leq |\xi| \leq b \) for \( \xi \in \Gamma \) and \( \lambda \geq 1 \),

\[
|W_{\varphi(-t_0)}u_0(x(0; \lambda), \lambda \xi)| \leq C_{N,b,\varphi} \lambda^{-N}
\]

holds, where \( x(t; \lambda) \) is a solution to

\[
\dot{x}(t) = -3\lambda^2 \xi^2 + a(t, x(t)), \quad x(t_0) = x,
\]

and \( \varphi_{0,\lambda}(x) = \lambda^{d/2} \varphi_0(\lambda^d x) \) with \( \min(\rho, 1/4) < d < 2 \min(\rho, 1/4) \), i.e.,

\[
W_{\varphi(-t_0)}u_0(x(0; \lambda), \lambda \xi) = \int_R \varphi_0(-t_0, y - x(0; \lambda), \lambda \xi) u_0(y) e^{-iy \cdot \lambda \xi} dy
= \int_R e^{i\xi_0(\partial_x^2 - 3i\xi \partial_y^2)} \varphi_{0,\lambda}(y - x(0; \lambda)) u_0(y) e^{-iy \cdot \lambda \xi} dy. \tag{8}
\]

In the case that \( a(t, x) \equiv 0 \), the operator \( x - 3t \partial_x^3 \) commutes with \( \partial_t + \partial_x^3 \), hence \((x - 3t \partial_x^3) u(t, x)\) solves the equation if \( u(t, x) \) solves the equation. This shows that \( u(t, \cdot) \) is smooth for \( t \neq 0 \) if the initial data \( u(0, x) = u_0(x) \) decays rapidly. Theorem 1.6 is a refinement of this phenomena. Our theorem determines the condition of initial data in which each point \((x_0, \xi_0) \in T^* R^3 \setminus 0 \) is not in the wave front set of the solution for given time \( t_0 \). \( x(0, \lambda) = x + 3\lambda^2 \xi^2 t_0 \) in (8) corresponds to the operator \( x - 3t \partial_x^3 \). If \( \lambda \) tends to \( \infty \), \( x(0, \lambda) = x + 3\lambda^2 \xi^2 t_0 \) tends to \( +\infty \), so the condition (ii) is the condition of the initial data \( u_0(x) \) at \( +\infty \).

This phenomena is called **smoothing effect** and which acts very important role in analyzing the solution to nonlinear equations. This kind of characterization was studied in several equations, in particular, nonlinear equations, see e.g., Beals [1], Biswas et al [3], Farah [8], Kato [15], Levandosky et al [16], Pilipovic [21], Rauch [22] and Rauch-Reed [23, 24]. Moreover as the Mathematically and Physically interests, there are lot of works
of characterization of wave front set of solutions of linear equations, see e.g., Cordoba-
Fefferman [4], Ito-Kato-Kobayashi [12], Johansson [10], [18] and [20]. In particular, in
[20], the new characterization of wave front set is introduced.

As for the KdV equations, there are lot of studies associated to singularity. However,
many papers deal with the (generalized) KdV equation and there are no results for not
only perturbed KdV equation [4] but also linearized KdV equation. In [3], de Bouard,
Hayashi and Kato studies Gevrey regularizing effect for the KdV equation. In [13], one
of the authors and Ogawa study analyticity of solutions to the KdV equation with the initial
data as Dirac’s delta. In the papers Mann [17], Biswas-Konar [2] and references their in,
one can find some recent studies for perturbed KdV equations, however there are no works
associated with propagation of singularities as far as we know. Hence, as the first step
of characterizing the wave front set of solution of (4), we consider the simplified linear
model. This attempt is new and not only mathematically but also physically important.

2 Preliminaries and proof of Theorem 1.3

In this section, we introduce some notations and give the proof of Theorem 1.3. Throughout
this paper, ||·|| denotes the norm on $L^2(\mathbb{R})$, and suppose that $x$ and $\xi$ are always
included in the compact neighborhood of $x_0$ and conic neighborhood of $\xi_0$, respectively.
For fixed $x_0$ and $t_0$, assume that $\lambda$ is always enough great compared to $|x|$ and $|t_0|$. If
we write $C$, $C > 0$ is a constant and which never depends on any parameters under
considering.

2.1 Proof of Theorem 1.3

Now we prove the existence and uniqueness of solution to (1). In the proof, we employ
the result of Enss-Veselić [7], that is

**Lemma 2.1.** Let $H_0(t)$ be a family of selfadjoint operators on $L^2(\mathbb{R})$. Suppose that, for
fixed $s \in \mathbb{R}$ and for every $t \in \mathbb{R}$, the operator

$$
\frac{H_0(t) - H_0(s)}{t - s} (H_0(s) + i)^{-1}
$$

can be extended to bounded operators, then there uniquely exists a family of unitary
operators $U_0(t, s)$ such that

$$
i \frac{\partial}{\partial t} U_0(t, s) = H_0(t) U_0(t, s), \quad i \frac{\partial}{\partial s} U_0(t, s) = -U_0(t, s) H_0(s),$$

$$U_0(t, s) = U_0(t, 0) U_0(s, 0)^*, \quad U_0(s, s) = \text{Id}_{L^2(\mathbb{R}^n)}$$

and

$$U_0(t, s) \mathcal{D}(H_0(s)) \subset \mathcal{D}(H_0(s))$$

hold.
Here we say $U_0(t, s)$ is the propagator for $H_0(t)$. Now we apply this lemma to our model. First, we reduce (1) to a form

$$\iota \partial_t u(t, x) = -i \partial^3 u(t, x) + i \left( (\partial_x (a(t, x)u(t, x)) + a(t, x)(\partial_x u(t, x))) / 2 + i(\partial_x a)(t, x)u(t, x) \right).$$

Let $s = 0$. Employing the substitution $p = -i \nabla$, it follows

$$\iota \partial_t u(t, x) = (-p^3 - (p a(t) + a(t)p)/2 + i(\partial_x a)(t))u(t, x),$$

where $a(t)$ is the multiplication operator of $a(t, x)$. Hence we define

$$H_0(t) = -p^3 - (p a(t) + a(t)p)/2, \quad H(t) = H_0(t) + V(t) := H_0(t) + i(\partial_x a)(t).$$

and first we prove the unique existence of unitary propagator for $H_0(t)$ and after we prove that $u(t, x)$ uniquely exists by using Duhamel’s formula.

It can be easily seen that $\mathcal{D}(H_0(0)) = \mathcal{D}(p^3)$ holds since $a(t, x)$ is bounded, i.e., for all $\phi \in L^2(\mathbb{R}),$

$$\sum_{k=0}^3 \|p^k(H_0(0) + i)^{-1}\phi\| \leq C\|\phi\|.$$

Then for $\phi \in L^2(\mathbb{R})$

$$\left\| \frac{(H_0(t) - H_0(0))}{t - 0} (H_0(0) + i)^{-1}\phi \right\|$$

$$= \left\| \frac{p(a(t, x) - a(0, x)) + a(t, x) - a(0, x)}{2t} \right\| (H_0(0) + i)^{-1}\phi \right\|$$

$$\leq \left\| \frac{(\partial_x a)(t, x) - (\partial_x a)(0, x)}{2t} \right\| (H_0(0) + i)^{-1}\phi \right\| + \left\| \frac{a(t, x) - a(0, x)}{t} \right\| p(H_0(0) + i)^{-1}\phi \right\|$$

$$\leq \left( C_{1,0} + \frac{C_{1,1}}{2} \right) \|(p + 1)(H_0(0) + i)^{-1}\phi\| \leq C\|\phi\|.$$

holds. By this inequality and the result of (7), we get the unique existence of $U_0(t, 0)$. Now we show the unique existence of $U(t, 0)$. By Duhamel’s formula we have

$$u(t, x) = U_0(t, 0) u_0 + \int_0^t U_0(t, s)(\partial_x a(s))u(s, x)ds.$$

Since $(\partial_x a(t))$ is bounded, we can use the standard argument in proving the existence of propagator, see e.g., Kato [14], IX §2, and that proves the unique existence of the propagator $U(t, 0)$ so that $u(t, x) = U(t, 0)u_0$. Moreover since $\mathcal{D}(H_0(0)) = \mathcal{D}(p^3)$ and $U_0(t, s)\mathcal{D}(H_0(s)) \subset \mathcal{D}(H_0(s))$ hold, for $u(t) = u(t, \cdot)$, we have

$$\|((H_0(0) + i)u(t))\| \leq \|(H_0(0) + i)U_0(t, 0) u_0\| + \int_0^t \left\{ \|((H_0(0) + i)U_0(t, s)(H_0(s) + i)^{-1}\|_{\mathcal{B}} \times \|((H_0(s) + i)(\partial_x a(s))(H_0(0) + i)^{-1})\|_{\mathcal{B}} \|(H_0(0) + i)u(s)\| \right\} ds,$$

where $\| \cdot \|_{\mathcal{B}}$ denotes the operator norm from $L^2(\mathbb{R})$ to itself. By the norm equivalence $c\|u\|_{H^3(\mathbb{R})} \leq \|(H_0(0) + i)u\| \leq C\|u\|_{H^3(\mathbb{R})}$, we have,

$$\|u(t)\|_{H^3(\mathbb{R})} \leq C\|u_0\|_{H^3(\mathbb{R})} + C \int_0^t \|u(s)\|_{H^3(\mathbb{R})} ds,$$

where the bound $\|((H_0(s) + i)(\partial_x a(s))(H_0(0) + i)^{-1}\|_{\mathcal{B}} \leq C$ follows from $a(t, \cdot) \in C^\infty(\mathbb{R})$ and (3). By using this inequality, the proof of Theorem 1.3 completes.
3 Transformed equation via wave packet transform

In this section, we transform (1) through wave packet transform and construct the solution of the transformed equation by using the solution of Hamilton-Jacobi equation.

Let us define \( \varphi(t, x, \xi) = e^{-it(\partial^2_x + 3i\xi \partial^2_y)} \varphi_0(x) \), \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \setminus \{0\} \). By (7),

\[
W_{\varphi(t)} \partial_t u(t, x, \xi) + W_{\varphi(t)} \partial^3_x u(t, x, \xi) + W_{\varphi(t)} \partial_x (au)(t, x, \xi) = 0
\]

holds. Clearly

\[
W_{\varphi(t)} \partial_t u(t, x, \xi) = \partial_t W_{\varphi(t)} u(t, x, \xi) - W_{\partial_t \varphi(t)} u(t, x, \xi)
\]

holds by the definition of wave packet transform (7). Furthermore, integration by parts and the straightforward calculation shows

\[
W_{\varphi(t)} \partial^3_x u(t, x, \xi) = \int_{\mathbb{R}} \varphi(t, y-x, \xi) \partial^3_y u(t, y) e^{-iy\xi} dy
\]

By the equation

\[
W_{\partial \varphi(t)} u(t, x, \xi) + W_{\partial^2 \varphi(t)} u(t, x, \xi) + W_{3i\xi \partial^2 \varphi(t)} u(t, x, \xi) = 0,
\]

and equations (9), (10) and (11), we get

\[
\partial_t W_{\varphi(t)} u(t, x, \xi) = W_{\varphi(t)} \partial_t u(t, x, \xi) + W_{\partial_t \varphi(t)} u(t, x, \xi)
\]

To deal with the term \( W_{\varphi(t)} \partial_x (au)(t, x, \xi) \), we use the Taylor expansion;

\[
a(t, y) = a(t, x + (y-x)) = a(t, x) + \sum_{k=1}^{L-1} \frac{\partial^k_x a(t, x)}{k!}(y-x)^k + r_L(t, x, y)(y-x)^L,
\]

\[
r_L(t, x, y) := \frac{1}{(L-1)!} \int_0^1 \partial^L_x a(t, x + \theta(y-x))(1-\theta)^{L-1} d\theta.
\]

Then by the similar calculation as in (11),

\[
W_{\varphi(t)} (\partial_x (au))(t, x, \xi)
\]

\[
= - \int_{\mathbb{R}} \partial_y \varphi(t, y-x, \xi) (au)(t, y) e^{-iy\xi} dy + i\xi \int_{\mathbb{R}} \varphi(t, y-x, \xi) (au)(t, y) e^{-iy\xi} dy
\]

\[
= - \int_{\mathbb{R}} \partial_y \varphi(t, y-x, \xi) (a(t, x) + \tilde{r}(t, x, y)) u(t, x, \xi) dy
\]

\[
+ i\xi \int_{\mathbb{R}} \varphi(t, y-x, \xi) (au)(t, y) e^{-iy\xi} dy.
\]
Hence we finally obtain equations
\[
\begin{cases}
(\partial_t + (\mathbf{3}x^2 + a(t, x)) \partial_x - i\xi^3 + i\xi a(t, x)) W_{\varphi(t)} u(t, x, \xi) = R_u(t, x, \xi), \\
W_{\varphi(0)} u(0, x, \xi) = W_{\varphi_0} u_0(x, \xi),
\end{cases}
\]
(12)
where
\[
R_u(t, x, \xi) = \sum_{k=1}^{L-1} \frac{\partial_x^k a(t, x)}{k!} (-W_{x^3 \partial_x \varphi(t)} u(t, x, \xi) + i\xi W_{x^2 \varphi(t)} u(t, x, \xi)) + R_L u(t, x, \xi)
\]
with
\[
R_L u(t, x, \xi) = - \int \partial_y \varphi(e, y - x, \xi) r_L(t, x, y)(y - x)^L u(t, y) e^{-i\xi y} dy \\
+ i\xi \int \varphi(t, y - x, \xi) r_L(t, x, y)(y - x)^L u(t, y) e^{-i\xi y} dy.
\]
Now we solve (12) by using the method of characteristic curve. Fix \( t_0 > 0 \) and let \( x(t) \) and \( \xi(t) (\equiv \xi) \) be solutions to
\[
\begin{cases}
\dot{x}(t) = -3\xi^2 + a(t, x(t)), \\
\dot{\xi}(t) = 0,
\end{cases}
\]
(13)
Then by the similar calculations in §5 of [1,2], we get
\[
W_{\varphi(t)} u(t, x(t), \xi) := e^{i \int_0^t (\xi^3 - \xi a(\tau, x(\tau))) d\tau} W_{\varphi_0} u_0(x(0), \xi) \\
+ \int_0^t e^{i \int_0^\tau (\xi^3 - \xi a(\tau, x(\tau))) d\tau} R_u(s, x(s), \xi) ds
\]
will be a solution to (12). Since the solution \( W_{\varphi(t)} u(t, x(t), \xi) \) does not depend on the choice of \( \varphi \), we substitute \( \varphi_\lambda(t - t_0, x, \xi) \) instead of \( \varphi(t, x, \xi) \), where we remark that
\[
\varphi_\lambda(t - t_0, x, \xi) = e^{-(t - t_0)(\partial_x^2 - 3i\xi \partial_x^2)} \varphi_{0, \lambda}, \\
\varphi_{0, \lambda}(x) = \lambda^{d/2} \varphi_0(\lambda^d x)
\]
with \( 0 < d \leq 1/2 \), which is fixed later. Then we have
\[
W_{\varphi_\lambda(t-t_0)} u(t, x(t), \xi) = e^{i \int_0^t (\xi^3 - \xi a(\tau, x(\tau))) d\tau} W_{\varphi_\lambda(-t_0)} u_0(x(0), \xi) \\
+ \int_0^t e^{i \int_0^\tau (\xi^3 - \xi a(\tau, x(\tau))) d\tau} R_u(s, x(s), \xi) ds.
\]
Furthermore, by introducing \( x(t; \lambda) \) as solution to
\[
\dot{x}(t) = -3\lambda^2 \xi^2 + a(t, x(t)), \\
x(t_0) = x,
\]
we finally get
\[
W_{\varphi_\lambda(t-t_0)} u(t, x(t; \lambda), \lambda \xi) = e^{i \int_0^t (\lambda^3 \xi^3 - \lambda \xi a(\tau, x(\tau; \lambda))) d\tau} W_{\varphi_\lambda(-t_0)} u_0(x(0; \lambda), \lambda \xi) \\
+ \int_0^t e^{i \int_0^\tau (\lambda^3 \xi^3 - \lambda \xi a(\tau, x(\tau; \lambda))) d\tau} R_u(s, x(s; \lambda), \lambda \xi) ds.
\]
(14)
By using this solution, we prove Theorem [1,6]
4 Proof of Theorem 1.6

In this section, we prove Theorem 1.6. The approach is based on the argument of [11]. The proof is obtained by showing both relations (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). The second relation can be proven quite simple by using the approach of proving (ii) $\Rightarrow$ (i) and hence we first prove that (ii) $\Rightarrow$ (i) and after we prove (i) $\Rightarrow$ (ii).

Before the proof of main theorem, we introduce the following important lemma;

**Lemma 4.1.** Let $0 < \theta < 2$. For some $\lambda_0 \geq 1$, the solution $x(s; \lambda)$ of (13) satisfies
\[ |x(s; \lambda)| \geq \frac{3}{2b^2} \lambda^2 |s - t_0| \]
for all $x \in K$, $\xi \in \Gamma$ with $b^{-1} \leq |\xi| \leq b$, $\lambda \geq \lambda_0$ and $s \geq 0$ with $|t_0 - s| \geq \lambda^{-\theta}$.

**Proof.** The proof is the almost same as in the proof of (11) in [11], hence we only give the sketch of the proof. We use Picard’s iteration method. Let us define
\[
\begin{cases}
x^{(N+1)}(s) := x - 3(s - t_0)\lambda^2 \xi^2 - \int_{t_0}^{s} (s - s_1) \nabla_x V(s_1, x^{(N)}(s_1)) ds_1, \\
x^{(0)}(s) := x - 3(s - t_0)\lambda^2 \xi^2
\end{cases}
\]
and use the induction scheme; Since $\theta < 2$, by taking $\lambda > 1$ enough large, it can be seen that
\[ |x^{(0)}(s)| \geq 3|s - t_0|\lambda^2 \xi^2 - |x| \geq 3b^{-2}|s - t_0|\lambda^2 - C \geq 3b^{-2}|s - t_0|\lambda^2/2 \]
holds, where we use $K$ is compact. Hence suppose that for some $M \in \mathbb{N} \cup \{0\}$,
\[ |x^{(M)}(s)| \geq 3b^{-2}|s - t_0|\lambda^2/2 \]
holds. Then we prove that
\[ |x^{(M+1)}(s)| \geq 3b^{-2}|s - t_0|\lambda^2/2 \]
also holds. Indeed
\[ |x^{(M+1)}(s)| \geq 5b^{-2}|s - t_0|\lambda^2/2 - \left| \int_{t_0}^{s} (s - s_1) \nabla_x V(s_1, x^{(N)}(s_1)) ds_1 \right| \]
holds and by using Assumption [11] for a positive constant $c > 0$, we have
\[ |x^{(M+1)}(s)| \geq 5b^{-2}|s - t_0|\lambda^2/2 - c \int_{t_0}^{s} |s - s_1| \left( 1 + c\lambda^2 b^{-2}(s_1 - t_0) \right)^{-\rho - 1} ds_1 \]
\[ \geq 5b^{-2}|s - t_0|\lambda^2/2 - C \]
\[ \geq 3b^{-2}|s - t_0|\lambda^2/2, \]
which is the desired result. \qed
4.1 Proof for (ii) ⇒ (i)

It suffices to prove the following assertion \( P(N, \varphi_0) \) for all \( N \in \mathbb{N} \) and \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \backslash \{0\} \) by induction.

**P(N, \varphi_0):** For \( b \geq 1 \), there exist a neighborhood \( K \) of \( x_0 \), a neighborhood \( \Gamma \) of \( \xi_0 \) and \( C_{N,b,\varphi_0} \) such that for all \( x \in K, \xi \in \Gamma \) with \( b^{-1} \leq |\xi| \leq b, \lambda \geq 1 \) and \( 0 \leq t \leq t_0 \),

\[
|W_{\varphi_\lambda(t-t_0)}u(t, x(t; \lambda), \lambda\xi)| \leq C_{N, b, \varphi_0} \lambda^{-N}
\]

holds.

In fact, if the assertion \( P(N, \varphi_0) \) holds for \( N \in \mathbb{N} \) and \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \backslash \{0\} \),

\[
|W_{\varphi_\lambda, \lambda}u(t_0, x, \lambda\xi)| \leq C_{N, b, \varphi_0} \lambda^{-N}
\]

(15)
holds for all \( x \in K, \xi \in \Gamma \) with \( b^{-1} \leq |\xi| \leq b, \lambda \geq 1 \) by taking \( t = t_0 \), since \( \varphi_\lambda(0) = \varphi_{0,\lambda} \)
and \( x(t_0; \lambda) = x \). Then Theorem 1.1 of [12] which is a refinement of Theorem 2.2 of [19] and Theorem 3.22 of [9], shows that (15) is equivalent to \( (x_0, \xi_0) \notin \text{WF}(u(t_0, \cdot)) \), i.e., we have (i) in Theorem 1.6. Hence we prove \( P(MM_0, \varphi_0) \) by the induction with respect to \( N \) for some small constant \( M_0 \). We note that we can assume that \( 0 < \rho < 1/4 \) without loss of generality;

(I): Proof of \( P(0, \varphi_0) \).

Since \( u(t, x) \) is in \( C(\mathbb{R}; L^2(\mathbb{R})) \), we have by Schwarz’s inequality, \( L^2 \) conservativity for \( e^{-t(\partial_x^2 - 3\xi^2/2)} \) and \( (6) \)

\[
|W_{\varphi_\lambda(t-t_0)}u(t, x(t; \lambda), \lambda\xi)| \leq C\|\varphi_\lambda(t-t_0)\|_{L^2}\|u(t, \cdot)\|_{L^2} \leq \|\varphi_0\|_{L^2}\|u_0\|_{L^2}.
\]

(16)

(II): Assuming that for \( M \in \mathbb{N}, P(MM_0, \varphi_0) \) is valid for all \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \backslash \{0\} \), we show that \( P((M + 1)M_0, \varphi_0) \) holds for all \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \backslash \{0\} \) for some small constant \( 0 < M_0 < 1/2 \). Since the condition (ii) in Theorem 1.6 says that the modulus of the first term of the right hand side of (14) is less than or equal to \( C\lambda^{-(M+1)M_0} \), it is enough to prove that

\[
\left| \int_0^t Ru(s, x(s; \lambda), \lambda\xi)ds \right| \leq C_{N, b, \varphi_0} \lambda^{-(M+1)M_0}
\]

holds. In the followings, for simplicity, we may use notations \( x_\lambda(t) = x(t; \lambda), U(t, \xi) = e^{-t(\partial_x^2 - 3\xi^2/2)} \) and \( (Uf)(t, x) = U(t, \xi)f(x) \). We put

\[
Ru(t, x, \xi) = \Gamma_1(t) + R_Lu(t, x, \xi),
\]

where

\[
\Gamma_1(t) = \sum_{k=1}^{L-1} \frac{\partial^k a(t, x)}{k!} (-W_{x^k\partial_x^k}(t, x, \xi) + i\xi W_{x^k\partial_x^k}(t, x, \xi)).
\]

(II-I) Estimation for \( \Gamma_1(t) \). By the simple calculation, we can see

\[
(x^k)\varphi_\lambda(t, x; \lambda\xi) = U(t, \lambda\xi)(x + 3t\partial_x^2 - 6it\lambda\xi\partial_x)^k\varphi_{0,\lambda}(x)
= U(t, \lambda\xi) \sum_{\alpha+\beta+\gamma=k} C_{\alpha, \beta, \gamma} \lambda^\gamma t^{\beta+\gamma}(x^\alpha\partial_x^{2\beta+\gamma})\varphi_{0,\lambda}(x)
= \sum_{\alpha+\beta+\gamma=k} C_{\alpha, \beta, \gamma} \lambda^{-d\alpha + 2d\beta + (1+d)\gamma} \xi^{\gamma} t^{\beta+\gamma} U(t, \lambda, \xi)(x^\alpha\partial_x^{2\beta+\gamma}\varphi_0)(x).
\]
By the assumption of induction, we have
\[
\left| \int_0^t \Gamma_1(s) ds \right| \\
\leq C \int_0^t \sum_{k=1}^{L-1} \sum_{\alpha+\beta+\gamma=k} |\partial_2^k a(s, x_\lambda(s))| \lambda^{-d\alpha+2d\beta+(1+d)\gamma} |\xi| |s - t_0|^{\beta+\gamma} \\
\times (\lambda^d |W_{U(x_\theta,\varphi_0)}(s-t_0)| u(s, x_\lambda(s), \lambda \xi)| \lambda | |W_{U(x_\theta,\varphi_0)}(s-t_0)| u(s, x_\lambda(s), \lambda \xi)|) \ ds \\
\leq C \int_0^t \sum_{k=1}^{L-1} \sum_{\alpha+\beta+\gamma=k} (1 + |x_\lambda(s)|)^{-\rho-k} \lambda^{-d\alpha+2d\beta+(1+d)\gamma+1} |s - t_0|^{\beta+\gamma} \lambda^{-M_0} ds \\
\leq \Gamma_2 + \Gamma_3
\]
with, for $0 < \theta < 2$,
\[
\Gamma_2 := C \int_{|s-t_0|\geq \lambda^\theta} \sum_{k=1}^{L-1} \sum_{\alpha+\beta+\gamma=k} (1 + |x_\lambda(s)|)^{-\rho-k} \lambda^{-d\alpha+2d\beta+(1+d)\gamma+1} |s - t_0|^{\beta+\gamma} \lambda^{-M_0} ds
\]
and
\[
\Gamma_3 := C \int_{|s-t_0|\leq \lambda^\theta} \sum_{k=1}^{L-1} \sum_{\alpha+\beta+\gamma=k} (1 + |x_\lambda(s)|)^{-\rho-k} \lambda^{-d\alpha+2d\beta+(1+d)\gamma+1} |s - t_0|^{\beta+\gamma} \lambda^{-M_0} ds.
\]
By using Lemma 4.1 we have
\[
\Gamma_2 \leq C \int_{|s-t_0|\geq \lambda^\theta} \left\{ \sum_{k=1}^{L-1} \sum_{\alpha+\beta+\gamma=k} (1 + 3b^{-2} \lambda^2 |s - t_0|/2)^{-\rho-k} \times \lambda^{-d\alpha+2d\beta+(1+d)\gamma+1} |s - t_0|^{\beta+\gamma} \lambda^{-M_0} \right\} ds.
\]
Here by employing the inequality
\[
k \geq \min(k, \beta + \gamma + 1) - 2d \min(k - \beta - \gamma, 1),
\]
we have the estimate
\[
\Gamma_2 \leq \sum_{k=1}^{L-1} \sum_{\alpha+\beta+\gamma=k} \lambda^{-d\alpha+2d\beta+(1+d)\gamma+1-2 \min(k, \beta + \gamma + 1) - 2\rho+2d \min(k - \beta - \gamma, 1) - M_0} \\
\times \int_0^t |s - t_0|^{\beta+\gamma-\min(k, \beta + \gamma + 1) - \rho+d \min(k - \beta - \gamma, 1)} ds \\
\leq C \lambda^{-d-2\rho-M_0},
\]
since $-d\alpha+2d\beta+(1+d)\gamma-2 \min(k, \beta + \gamma + 1) + 2d \min(k - \beta - \gamma, 1)$ takes the maximum $d - 1$ when $k = 1$, $\alpha = \beta = 0$ and $\gamma = 1$. If we take $\rho < d < 2\rho$ and $0 < M_0 \leq 2\rho - d$, we have
\[
\Gamma_2 \leq C \lambda^{-(M+1)M_0}.
\]
Since \( \int_{|s-t_0| \leq \lambda^{-6}} ds \leq C \lambda^{-\theta} \) holds, by taking \( \theta \) so that \( 1 + d < \theta < 2 \), we get
\[
\Gamma_3 \leq C \lambda^{-\theta-d\alpha+2d\beta+(1+d)\gamma+1-\theta(\beta+\gamma)}MM_0 \leq C \lambda^{-(M+1)M_0}
\]
for \( 0 < M_0 < d \). Consequently we get
\[
\left| \int_0^t \Gamma_1(s) ds \right| \leq C \lambda^{-(M+1)M_0}.
\]

(II-II) Estimation for \( R_L u(t,x(t;\lambda),\lambda \xi); \)
Let us define \( \psi_1 \in C_0^\infty(\mathbb{R}) \) and \( \psi_2 \in C^\infty(\mathbb{R}) \) as follows
\[
\psi_1(x) = \begin{cases} 
1 & |x| \leq 1, \\
0 & |x| \geq 2,
\end{cases} \quad \psi_2(x) = \begin{cases} 
0 & |x| \leq 1, \\
1 & |x| \geq 2,
\end{cases} \quad \psi_1 + \psi_2 = 1.
\]

Devide \( R_L u = R_{L,1} u + R_{L,2} u \) with
\[
R_{L,j} u(s,x(s;\lambda),\lambda \xi) = \int (\partial_s \varphi(s,t) + i\lambda \xi \varphi(s,t)) (s,y-x(s;\lambda),\lambda \xi) r_L(s,x(s;\lambda),y)(y-x(s;\lambda))^L 
\times \psi_j \left( \frac{\lambda^{1/4}(y-x(s;\lambda))}{1 + \lambda^2|s-t_0|} \right) u(y) e^{-i\lambda \xi} dy ds.
\]

First, we estimate the term associated to \( R_{L,1} \).
\cdot The case that \( |s-t_0| \leq \lambda^{-15/8} \).
On the support of \( \psi_1 \) and \( |s-t_0| \leq \lambda^{-15/8} \), it holds that
\[
|y-x(s;\lambda)| \leq 2\lambda^{-1/4}(1 + \lambda^2|s-t_0|) \leq C \lambda^{-1/8},
\]
and which deduces
\[
\int_{|s-t_0| \leq \lambda^{-15/8}} |R_{L,1} u(s,x(s;\lambda),\lambda \xi)| ds 
\leq C \int_{|s-t_0| \leq \lambda^{-15/8}} \int \left( \lambda^d ||\partial_s \varphi||_\lambda(s,y-x(s;\lambda),\lambda \xi)| + \lambda ||\varphi(s,y-x(s;\lambda),\lambda \xi)|| \right) 
\times |r_L(s,x(s;\lambda),y)||y-x|^L |u(y)| dy ds 
\leq C \lambda^{-L/8+1-15/8} (||\partial_s \varphi||_\lambda + ||\varphi||_\lambda) ||u||.
\]

\cdot The case of \( |s-t_0| \geq \lambda^{-15/8} \).
On the supp \( \psi_1 \) \( (\lambda^{1/4}(y-x(s;\lambda))/(1 + \lambda^2|s-t_0|)) \), it holds that
\[
|y-x(s;\lambda)| \leq 2\lambda^{-1/4}(1 + \lambda^2|s-t_0|).
\]
Together with Lemma 4.1, we get for \( \lambda \geq \lambda_0 \)
\[
|x(s;\lambda) + \theta(y-x(s;\lambda))| \geq |x(s;\lambda)| - |y-x(s;\lambda)| \geq C \lambda^2|s-t_0|.
\]
Consequently,
\[
\left| \partial_s^L u(s, x(s); \lambda) + \theta(y - x(s; \lambda)) \right| \leq C \left( 1 + |x(s; \lambda) + \theta(y - x(s; \lambda))|^{-\rho} \right)
\]
holds. Then we have
\[
\int_{|s-t_0| \geq \lambda^{-\theta}} |R_{L,1} u(s, x_\lambda(s), \lambda \xi)| \, ds 
\]
\[
\leq C \int_{|s-t_0| \geq \lambda^{-\theta}} \int \left( \lambda^d \left| (\partial_x \varphi)(s, y - x_\lambda(s), \lambda \xi) \right| + \lambda \left| \varphi(s, y - x_\lambda(s), \lambda \xi) \right| \right) 
\times \left( (1 + \lambda^2 |s-t_0|)^{-\rho} \lambda^{-L/4} (1 + \lambda^2 |s-t_0|)^L |u(y)| dy ds \right) 
\]
\[
\leq C \lambda^{-L/4 + 1} \int_0^t (1 + \lambda^2 |s-t_0|)^{-\rho} ds \times (\| (\partial_x \varphi) \| + \| \varphi \|) \| u \|
\]
\[
\leq C \lambda^{-L/4 - 1}.
\]
By choosing \( L \) large, we have
\[
\int_0^t |R_{L,1} u(s, x_\lambda(s), \lambda \xi)| \, ds \leq C \lambda^{-(M+1)M_0}.
\]
Next, we estimate the term associated to \( R_{L,2} \). For \( m \in \mathbb{N} \),
\[
(1 + |x|^2)^m \varphi_\lambda(t, x, \lambda \xi) = e^{-t(\partial_x^2 - 3i\lambda \xi \partial_x)} \{ 1 + \left( x + 3it \partial_t^2 - 6it \lambda \xi \partial_x \right) \}^m \varphi_0(\xi)(x)
\]
\[
= e^{-t(\partial_x^2 - 3i\lambda \xi \partial_x)} \sum_{\alpha + \beta + \gamma \leq 2m} C_{\alpha, \beta, \gamma} \lambda^{\gamma} t^{\beta+\gamma} \left( x^\alpha \partial_x^{2\beta+\gamma} \right) \varphi_0(\xi)(x)
\]
\[
= e^{-t(\partial_x^2 - 3i\lambda \xi \partial_x)} \sum_{\alpha + \beta + \gamma \leq 2m} C_{\alpha, \beta, \gamma} \lambda^{-da + 2d\beta + (1+d)\gamma} \xi^{\beta+\gamma} \left( x^\alpha \partial_x^{2\beta+\gamma} \varphi_0 \right)(x).
\]
By using this, we can obtain
\[
\int_0^t |R_{L,2} u(s, x_\lambda(s), \lambda \xi)| \, ds 
\]
\[
\leq \int_0^t \int (1 + |y - x_\lambda(s)|^2)^{-m} \left( 1 + |y - x_\lambda(s)|^2 \right)^m (|\partial_x (\varphi_\lambda)| + |i \lambda \xi \varphi_\lambda(s-t_0, y - x_\lambda(s), \lambda \xi)|) \times |r_L(s, x_\lambda(s), y)| |y - x_\lambda(s)|^L |u(y)| dy ds 
\]
\[
\leq C \int_0^t \int (1 + |y - x_\lambda(s)|^2)^{-m} 
\times \left( \lambda^d \sum_{\alpha + \beta + \gamma = 2m} \lambda^{-da + 2d\beta + (1+d)\gamma} |s-t_0|^{\beta+\gamma} |U(s-t_0) \left( x^\alpha \partial_x^{2\beta+\gamma} \varphi_0 \right)_\lambda(y-x)| 
\right.
\]
\[
\left. + \lambda \sum_{\alpha + \beta + \gamma = 2m} \lambda^{-da + 2d\beta + (1+d)\gamma} |s-t_0|^{\beta+\gamma} |U(s-t_0) \left( x^\alpha \partial_x^{2\beta+\gamma} \varphi_0 \right)_\lambda(y-x)| \right) 
\times |y - x_\lambda(s)|^L |u(y)| dy ds 
\]
\[
\leq C \int_0^t \int (1 + |y - x_\lambda(s)|^2)^{-m + L/2} \lambda^{2(1+d)m+1} (1 + |s - t_0|^{2m}) 
\times |U(s-t_0) \left( \partial_x^{2m} \varphi_0 \right)_\lambda(y-x)| |u(y)| dy ds.
\]

If we transform the equation (12) with the initial condition

\[ W \]

where

\[ i \]

shows that

\[ \psi_2 \]

we have

\[ \int_0^t |R_{L,2}u(s, x_\lambda(s), \lambda \xi)| \, ds \]

\[ \leq C \int_0^t \int (\lambda^{-1/4} \lambda^2 |s - t_0|)^{-2m + L} \lambda^{2(1 + d)m + 1} (1 + |s - t_0|^{2m}) \]

\[ \times |U(s - t_0) (\partial_x^2 \varphi_0) (y - x) | \, u(s, y) | dy \, ds \]

\[ \leq C \lambda^{-(3/2 - 2d)m + 1 + (7/4)L} \int_0^t (1 + |s - t_0|^L) \| U(s - t_0) (\partial_x^2 \varphi_0) (y - x) \| \| u(s, y) \| \, ds \]

\[ \leq C \lambda^{-(3/2 - 2d)m + 1 + (7/4)L} \int_0^t (1 + |s - t_0|^L) ds \| (\partial_x^2 \varphi_0) (y - x) \| \| u_0(\cdot) \| \]

\[ \leq C \lambda^{-(1/2)m + 1 + (7/4)L} \leq C \lambda^{-(M + 1)M_0}, \]

if we take \( m \) sufficiently large compared to \( L \). Hence we get \( P((M + 1)M_0, \varphi_0) \) for any \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \backslash \{0\} \). This completes the proof. \( \square \)

4.2 Proof for \((i) \Rightarrow (ii)\)

If we transform the equation (12) with the initial condition \( W_{\varphi(x_0)}u(t_0, x, \xi) \) for \( t = t_0 \), we have

\[ W_{\varphi(t_0)}u(t, x_\lambda(t), \lambda \xi) = e^{i \int_{t_0}^t (\varepsilon^3 - \xi a(\tau, x(t)) \) dr \} W_{\varphi_0, \lambda} u(t_0, x_\lambda(t_0), \lambda \xi) \]

\[ + \int_{t_0}^t e^{i \int_{t_0}^r (\varepsilon^3 - \xi a(\tau, x_\lambda(\tau)) \) dr \} Ru(s, x_\lambda(s), \lambda \xi) \, ds, \]

where \( x_\lambda(t) \) is a solution of \( \dot{x}(t) = -3\lambda^2 \xi^2 + a(t, x(t)) \) with \( x(t_0) = x \). Hence we have

\[ |W_{\varphi(t_0)}u_0(x_\lambda(t), \lambda \xi)| \leq |W_{\varphi_0, \lambda} u(t_0, x_\lambda(t), \lambda \xi)| + \int_{t_0}^t |Ru(s, x_\lambda(s), \lambda \xi)| \, ds. \]

We note that the condition \((i)\) is equivalent to \( |W_{\varphi_0, \lambda} u(t_0, x, \lambda \xi)| \leq C \lambda^{-N} \). Hence by the same argument as in the proof for \((ii) \Rightarrow (i)\), we can show that \( P(N, \varphi_0) \) is valid for all \( N \in \mathbb{N} \) and \( \varphi_0 \in \mathcal{S}(\mathbb{R}) \backslash \{0\} \) by induction. If we take \( t = t_0 \), we have \((ii)\) of Theorem 1.6 \( \square \)

References

[1] Beals, M.: Self-spreading and strength of singularities for solutions to semilinear wave equations, Ann., of Math., 118, 187–214 (1983).

[2] Biswas, A., Konar, S.: Soliton perturbation theory for the compound KdV equation, Int. J. Theo. Phys., 46, 237–243 (2007).

[3] Biswas, A., Ebadi, G., Johnson, S., Strong, J. A., Wang, G. W., Xu, T. Z.: Singular solitons, shock waves, and other solutions to potential KdV equation, Nonlinear Dyn., 76, 1059–1068 (2014).
[4] Craig, W., Kappeler, T., Strauss, W.: Gain of regularity for equations of KdV type, Ann. Inst. H. Poincaré 9, 147–186 (1992).

[5] Cordoba, A., Fefferman, C.: Wave packets and Fourier integral operators, Com. P. D. E., 3, 979–1005 (1978).

[6] de Bouard, A., Hayashi, N., Kato, K.: Gevrey regularizing effect for the (generalized) Korteweg-de Vries equation and nonlinear Schrödinger equations, Annals de l’Institut Henri Poincaré: Analyse non linéaire, 12, 673-725 (1995).

[7] Enss, V., Veselić, K.: Bound states and propagating states for time-dependent Hamiltonians, Ann., Inst., Henri Poincaré, 39, 159–191 (1983).

[8] Farah, L., G.: Global rough solutions to the critical generalized KdV equations, J. Diff. Eqn., 249, 1968–1985 (2010).

[9] Folland, G. B.: Harmonic analysis in phase space, Ann. of Math. Studies, 122, Princeton Univ. Press, Princeton, NL, (1989).

[10] Johansson, K.: Propagation of singularities for pseudo-differential operators and generalized Schrödinger propagators, Licentiate Thesis in Linnaeus Univ., (2010).

[11] Kato, K., Ito, S.: Singularity for solutions to time dependent Schrödinger equations with sub-quadratic potential, SUT J. of Math., 50, 383–398 (2014).

[12] Kato, K., Ito, S., Kobayashi, M.: Remarks on characterization of wave front set by wave packet transform, Osaka Math. J., 54, 209–228 (2017).

[13] Kato, K., Ogawa, T.: Analyticity and smoothing effect for the Korteweg-de Vries equation with a single point singularity, Math. Annalen 316, 577–608 (2000).

[14] Kato, T.: Perturbation Theory for linear operators, 2nd edition, Springer-Verlag, (1976).

[15] Kato, T.: On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Adv. Math., (Suppl. Studies: Studies in Appl. Math.) 8, 93–128 (1983).

[16] Levandosky, J., Sepúlveda, M., Villagrán, O. V.: Gain of regularity for the KP-I equation, J. Diff. Eqn., 245, 762–808 (2008).

[17] Mann, E.: The perturbed Korteweg-de Vries equation considered anew, J. Math. Phys., 38, 3772 (1997).

[18] Okaji, T.: Propagation of wave packets and its application, P.D.E. and Spectral Theory., 239–243 (2001).

[19] Okaji, T.: A note on the wave packet transforms, Tsukuba J. Math., 25 383–397 (2001).

[20] Pilipovic, S., Prangoski, B.: On the characterizations of wave front sets via the short-time Fourier transform, arXiv:1801.05999 (2018).
[21] Pokhozhaev, S. I.: On the singular solutions of the Korteweg de Vries equation, Math. Notes, 88, 741–747 (2010).

[22] Rauch, J.: Singularity of solutions to semilinear wave equations, J. Math. Pures et appl., 58, 299–308 (1979).

[23] Rauch, J., Reed, M.: Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimensions, Duke Math. J., 49, 397–475 (1982).

[24] Rauch, J., Reed, M.: Propagation of singularities for semilinear hyperbolic equation in one space variable, Ann. of Math., 111, 531–552 (1980).