Flux-Approximation Limits of Solutions to the Brio System with Two Independent Parameters

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Abstract By the flux-approximation method, we study limits of Riemann solutions to the Brio system with two independent parameters. The Riemann problem of the perturbed system is solved analytically, and four kinds of solutions are obtained constructively. It is shown that, as the two-parameter flux perturbation vanishes, any two-shock-wave and two-rarefaction-wave solutions of the perturbed Brio system converge to the delta-shock and vacuum solutions of the transport equations, respectively. In addition, we specially pay attention to the Riemann problem of a perturbed simplified system of conservation laws derived from the perturbed Brio system by neglecting some quadratic term. As one of the parameters of the perturbed Brio system goes to zero, the solution of which consisting of two shock waves tends to a delta-shock solution to this simplified system. By contrast, the solution containing two rarefaction waves converges to a contact discontinuity and a rarefaction wave of the simplified system. What is more, the formation mechanisms of delta shock waves under flux approximation with both two parameters and only one parameter are clarified. Some numerical simulations presenting the formation processes of delta shock waves and vacuum states are also presented to confirm the theory analysis.

Keywords Brio system · Transport equations · Riemann problem · Delta shock wave · Vacuum · Flux approximation · Numerical simulations

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1 Introduction

As to our knowledge, in the past over two decades, the delta shock wave has been systematically studied by a large number of scholars. For example, see the results in [10, 12, 20–24, 27, 28] and the references cited therein. Particularly, in the related researches of delta shock waves, one of the interesting topics is to explore the formation of delta shock waves and vacuum states in solutions, which correspond to the phenomena of concentration and cavitation, respectively. At this moment, an effective approach is to use the so called vanishing pressure limit method, which was early proposed by Chen and Liu [6, 7] to study the formation of delta shock waves and vacuums for the Euler equations of isentropic and nonisentropic gas dynamics, respectively. See also Li [13] for the isothermal Euler equations with zero temperature.

In view that the vanishing pressure limit method is only focused on pressure perturbation, for more general of physical consideration, Yang and Liu recently introduced the flux-approximation method [25] to study the limit behavior of solutions to the isentropic Euler equations with flux perturbation. The flux-approximation method, generally speaking, is a natural generalization of the vanishing pressure limit method. The main idea of it is to introduce some small perturbed parameters in the flux function of the system, then the limits of solutions to the perturbed system can be studied by taking the perturbed parameters go to zero. Physically, a reasonable perturbation can be used to control some dynamical behaviors of fluids [25], so it is worth studying the flux perturbation problem which plays an important role in the theory, application, and numerical computation [5, 19]. This method has been successfully applied to study the formation of delta shock waves, say, Yang and Liu [26] for the nonisentropic fluid flows, Yang and Zhang [29, 31] for the relativistic Euler equations, Sun [17] for the transport equations, etc.

Motivated by the works mentioned above, we in this paper introduce the following system of conservation laws

\[
\begin{align*}
    u_t + \left( \frac{1}{2}u^2 + \frac{1}{2}\epsilon_1 v^2 \right)_x &= 0, \\
    v_t + (uv - \epsilon_2 v)_x &= 0,
\end{align*}
\]

(1.1)

where \(u(x, t), v(x, t)\) are the unknowns, and \(\epsilon_1, \epsilon_2 > 0\) are two independent parameters. Here, we are pleased to mention that, even though the parameters \(\epsilon_1\) and \(\epsilon_2\) are considered very small and govern the strength of flux, they do not vanish in general. We just propose to include these small parameters to explore the limit behaviors of solutions to the system (1.1).

System (1.1) is called the perturbed Brio system, since it can be viewed as a perturbed model coming from the Brio system [3]

\[
\begin{align*}
    u_t + \left( \frac{u^2 + v^2}{2} \right)_x &= 0, \\
    v_t + (uv - v)_x &= 0,
\end{align*}
\]

(1.2)

which arises as a simplified model in ideal magnetohydrodynamics (MHD) and corresponds to coupling the fluid dynamic equations with Maxwell’s equations of electrodynamics. The Brio system (1.2) is strictly hyperbolic and genuinely nonlinear at \(\{(u, v) : u \in R, v > 0\}\) and \(\{(u, v) : u \in R, v < 0\}\), but not on the whole of \(R^2\). In [9], Hayes et al. found that the genuine nonlinearity was lost when \(v\) changes sign. For solutions crossing the line \(v = 0\), delta-shock solution might have to be used, and non-uniqueness of Riemann solutions was anticipated.
Letting $\epsilon_1, \epsilon_2 \to 0$, the system (1.1) transforms formally into the following transport equations

\[
\begin{align*}
    u_t + \left(\frac{1}{2} u^2\right)_x &= 0, \\
    v_t + (uv)_x &= 0.
\end{align*}
\]

Here we should mention that the small parameters $\epsilon_1$ and $\epsilon_2$ are artificial, the choice of them in (1.1) makes the system (1.3) into a strictly hyperbolic system such that all the calculations are performed analytically. The system (1.3) has a physical context and describes some important physical phenomena, in which $u$ is velocity and $v$ the density. It can be used to model the motion of free particles which stick under collision [4], and to describe the formation of large-scale structures of the universe [18, 30]. See also [1, 8] for more related applications. It is non-strictly hyperbolic, with a linearly degenerate characteristic field, and has been studied in the numerous papers such as [2, 11, 15, 21]. Interestingly, the delta shock wave and vacuum state do appear in the Riemann solutions.

However, if we only take $\epsilon_1 \to 0$, then (1.1) becomes the following single-parameter-perturbation model

\[
\begin{align*}
    u_t + \left(\frac{1}{2} u^2\right)_x &= 0, \\
    v_t + (uv - \epsilon_2 v)_x &= 0,
\end{align*}
\]

which was used to approximate the transport equation by Shen et al. [16]. They proved that, as $\epsilon_2 \to 0$, the Riemann solutions of (1.4) converge to those of the transport equations (1.3). Specifically, as $\epsilon_2 = 1$, system (1.4) is called the simplified Brio system, which was derived in [9] by neglecting the $v^2$ term in the flux function of the first equation of Brio system (1.2). What is more, it is found that the delta shock wave occurs in the solutions of (1.4).

The main purpose of this paper is to discuss the formation of delta shock waves and vacuum states in the vanishing flux-approximation limit of solutions to the perturbed Brio model (1.1). It remains to be seen whether or not the limits $\epsilon_1, \epsilon_2 \to 0$ and $\epsilon_1 \to 0$ of solutions to the Riemann problem for the perturbed Brio system (1.1) are identical with those for the transport equations (1.3) and the single-parameter-perturbation model (1.4), respectively. In what follows, we outlook the context of each section of this paper.

In Sect. 2, we solve delta shock waves and vacuum states for transport equations (1.3).

Section 3 deals with the Riemann problem for the single-parameter-perturbation system (1.4) with initial data

\[
(u, v)(0, x) = \begin{cases} 
(u_-, v_-), & x < 0, \\
(u_+, v_+), & x > 0,
\end{cases}
\]

where $u_\pm$ and $v_\pm$ are arbitrary constants. Two kinds of Riemann solutions consisting of rarefaction waves, shock waves and contact discontinuities are constructed when $u_- - u_+ < 2\epsilon_2$. While when $u_- - u_+ > 2\epsilon_2$, the delta shock wave appears in solutions. Nevertheless, the velocity of delta shock wave in system (1.4) is quite different from that of (1.3), since it will no longer equal to the value of $u$ on the discontinuity line, which implies that the flux perturbation works in the transport equations (1.3).

In Sect. 4, we investigate the Riemann problem of the perturbed Brio system (1.1). For convenience, we only consider the case when $v > 0$, that is, the system (1.1) is strictly hyperbolic and genuinely nonlinear, which also means that the perturbation adopted here
transforms the non-strictly hyperbolic system (1.3) into the strictly hyperbolic one. By analyzing in phase plane, we construct four kinds of Riemann solutions with the classical waves involving rarefaction waves and shock waves.

Section 5 studies the flux-approximation limits of solutions to (1.1), (1.5) as both parameters go to zero. Concretely, it is rigorously proved that, as $\epsilon_1, \epsilon_2 \to 0$, any two-shock Riemann solution to the perturbed Brio system (1.1) tends to a delta-shock solution to the transport equations (1.3), and the intermediate state between the two shock waves tends to a weighted $\delta$-measure that forms a delta shock wave. Meanwhile, it is also shown that, any two-rarefaction-wave Riemann solution to the perturbed Brio system (1.1) converges to a two-contact-discontinuity solution to the transport equations (1.3), whose non-vacuum intermediate state in between tends to a vacuum state. These results present that the delta-shock and vacuum solutions of (1.3) can be obtained by a flux-approximation limit of solutions to the perturbed Brio system (1.1).

In Sect. 6, we discuss the limit behaviors of Riemann solutions to the perturbed Brio system (1.1) as $\epsilon_1 \to 0$. It is shown that the Riemann solutions of (1.1) converge to the corresponding Riemann solutions of the single-parameter-perturbation model (1.4). Especially, any two-shock-wave Riemann solution of (1.1) tends to a delta-shock solution to the perturbed model (1.4).

Following the above analysis, from the point of hyperbolic conservation laws, the above two different convergence processes show the two kinds of occurrence mechanism on the formation of delta shock wave. When $\epsilon_1$ and $\epsilon_2$ decrease simultaneously, the strict hyperbolicity of the limiting system fails (see Sect. 5), which leads to the formation of delta shock wave. This point is the same as that in [16, 17]. While when only one parameter $\epsilon_1$ decreases, the strict hyperbolicity of the limiting system is preserved (see Sect. 6), but the delta shock wave still occurs, which is different from that in [16, 17]. In addition, the above results also indicate the fact that different flux approximations have their respective effects on the formation of delta shock waves.

Finally, in Sect. 7, by employing the Nessyahu-Tadmor scheme [14], we present some numerical results to examine the formation process of delta shock waves and vacuum states as $\epsilon_1$ and $\epsilon_2$ decrease simultaneously, or only $\epsilon_1$ decreases, which completely confirm the theoretical analysis.

2 Preliminaries

In this section, let us recall the Riemann solutions to the transport equations (1.3). More details can be found in [15, 21].

The system (1.3) has duplicate eigenvalues $\lambda = u$ with corresponding right eigenvectors $r = (1, 0)^T$. Since $\nabla \lambda \cdot \vec{r} = 0$, so (1.3) is full linear degenerate and elementary waves involve only contact discontinuities. The left state $(u_-, v_-)$ and right state $(u_+, v_+)$ can be connected by classical elementary waves (contact discontinuity and vacuum) or a delta shock wave. Depending on the choice of initial data, there are two possible wave patterns for solutions of Riemann problem (1.3) and (1.5).

When $u_- < u_+$, the Riemann solution consists of two contact discontinuities $J$ with a vacuum in between, which can be shown as

$$
(u, v)(t, x) = \begin{cases} 
(u_-, v_-), & -\infty < x < u_-t, \\
(\frac{x}{t}, 0), & u_-t \leq x \leq u_+t, \\
(u_+, v_+), & u_+t < x < +\infty.
\end{cases}
$$ (2.1)
When $u_- > u_+$, the Riemann solution can not be constructed by using the classical waves, and the delta shock wave appears. To define this kind of solution, the following weighted delta function supported on a curve should be introduced.

A two-dimensional weighted delta function $w(s)\delta_S$ supported on a smooth curve $S$ parameterized as $t = t(s), x = x(s) (a \leq s \leq b)$ can be defined by

$$\left\langle w(t(s))\delta_S, \varphi(t, x) \right\rangle = \int_a^b w(t(s))\varphi(t(s), x(s))\,ds$$ (2.2)

for all test functions $\varphi \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))$.

Based on this definition, a delta-shock solution of (1.3) can be represented in the following form

$$u(t, x) = u_0(x, t), \quad v(t, x) = v_0(t, x) + w(t)\delta_S,$$ (2.3)

where $S = \{(t, \sigma t) : 0 \leq t < \infty\}$, and

$$u_0(t, x) = u_- + \lbrack u \rbrack H(x - \sigma t), \quad v_0(t, x) = v_- + \lbrack v \rbrack H(x - \sigma t),$$

$$w(t) = \sigma \lbrack v \rbrack - \lbrack uv \rbrack t,$$

in which $\lbrack G \rbrack = G_+ - G_-$ expresses the jump of the quality $G$ across the curve $S$, $\sigma$ is the velocity of the delta shock wave, and $H(x)$ the Heaviside function.

As mentioned in [21] that the solution $(u, v)$ constructed above satisfies that

$$\left\langle u, \varphi_t \right\rangle + \left\langle \frac{1}{2}u^2, \varphi_x \right\rangle = 0,$$

$$\left\langle v, \varphi_t \right\rangle + \langle uv, \varphi_x \rangle = 0$$ (2.4)

for all test functions $\varphi \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))$, where

$$\left\langle v, \varphi \right\rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} v_0\varphi\,dx\,dt + \left\langle w\delta_S, \varphi \right\rangle,$$

$$\left\langle uv, \varphi \right\rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} u_0v_0\varphi\,dx\,dt + \left\langle \sigma w\delta_S, \varphi \right\rangle.$$

Then, a unique solution of (1.3) and (1.5) containing a weighted $\delta$-measure is given as

$$(u, v)(t, x) = \begin{cases} (u_-, v_-), & x < x(t), \\ (\sigma, w(t)\delta(x - \sigma t)), & x = x(t), \\ (u_+, v_+), & x > x(t), \end{cases}$$ (2.5)

in which $x(t), \sigma$ and $w(t)$ satisfy the generalized Rankine-Hugoniot relation

$$\begin{cases} \frac{dx}{dt} = \sigma, \\ \frac{dw(t)}{dt} = [v]\sigma - [uv], \\ [u]\sigma = \left[ \frac{u^2}{2} \right]. \end{cases}$$ (2.6)
In addition, the entropy condition
\[ u_+ < \sigma < u_- \]
should be supplemented in order to guarantee the uniqueness.

By solving the generalized Rankine-Hugoniot relation (2.6) with the initial data \( w(0) = 0, x(0) = 0 \), one can obtain that
\[
\sigma = \frac{1}{2}(u_- + u_+), \quad w(t) = \frac{1}{2}(v_- + v_+)(u_- - u_+). 
\] (2.7)
Thus, the delta-shock solution defined by (2.5) with (2.7) is obtained.

3 Riemann Solutions of System (1.4)

In this section, we shall solve the Riemann problem (1.4) and (1.5).

System (1.4) has two eigenvalues
\[
\lambda_{1}^{\epsilon_2} = u - \epsilon_2, \quad \lambda_{2}^{\epsilon_2} = u
\]
with the corresponding right eigenvectors
\[
\vec{r}_1^{\epsilon_2} = (0, 1)^T, \quad \vec{r}_2^{\epsilon_2} = (\epsilon_2, v)^T.
\]
Therefore, the system is strictly hyperbolic for \( \epsilon_2 > 0 \). In addition, it is easily to check that \( \nabla \lambda_{1}^{\epsilon_2} \cdot \vec{r}_1^{\epsilon_2} = 0 \) and \( \nabla \lambda_{2}^{\epsilon_2} \cdot \vec{r}_2^{\epsilon_2} = \epsilon_2 \), which mean that \( \lambda_{1}^{\epsilon_2} \) is always linearly degenerate and \( \lambda_{2}^{\epsilon_2} \) is genuinely nonlinear for \( \epsilon_2 > 0 \).

Seeking the self-similar solution
\[
(u, v)(t, x) = (u, v)(\xi), \quad \xi = x/t.
\]
Equivalently, the Riemann problem (1.4) and (1.5) is reduced to
\[
\begin{cases}
-\xi u_\xi + \left(\frac{1}{2}u^2\right)_\xi = 0, \\
-\xi v_\xi + (uv - \epsilon_2 v)_\xi = 0, \\
(u, v)(0, \pm \infty) = (u_\pm, v_\pm),
\end{cases}
\] (3.1)
which provides either the constant state solution or the rarefaction wave
\[
R(u_-, v_-) : \begin{cases}
\xi = \lambda_{2}^{\epsilon_2} = u, \\
\epsilon_2 \ln v - u = \epsilon_2 \ln v_- - u_-, \\
u > u_-.
\end{cases}
\] (3.2)
For a bounded discontinuity at \( \xi = w \), the following Rankine-Hugoniot relation
\[
\begin{cases}
w[u] = \left[\frac{1}{2}u^2\right], \\
w[v] = [uv - \epsilon_2 v]
\end{cases}
\] (3.3)
holds. When \([u] \neq 0\), we get the shock wave

\[
S(u_-, v_-) : \begin{cases} 
  w_1 = \frac{u_- + u}{2}, \\
  v = \frac{u - u_- + 2\epsilon_2}{u_- - u + 2\epsilon_2}, \\
  v_- = \frac{u_- - u + 2\epsilon_2}{u_- - u + 2\epsilon_2}, \\
  u < u_- < u + 2\epsilon_2.
\end{cases} \tag{3.4}
\]

While when \([u] = 0\), it corresponds to a contact discontinuity

\[
J : \quad w^{\epsilon_2} = u_- - \epsilon_2 = u_+ - \epsilon_2. \tag{3.5}
\]

However, when \(u_- - u_+ > 2\epsilon_2\), the solution can not be constructed by classical waves. At this moment, the delta-shock solution containing Dirac delta function in the state variable \(v\) will be considered. Under the definition (2.2), we give the definition of the delta-shock solution to (1.4) and (1.5).

**Definition 3.1** A pair \((u, v)\) is called a delta shock wave type solution of (1.4) in the sense of distributions if there exist a smooth curve \(S\) and a function \(w^{\epsilon_2}(t)\) such that \(u\) and \(v\) are represented in the following form

\[
u = \bar{u}(t, x), \quad v = \bar{v}(t, x) + w^{\epsilon_2}(t)\delta_S, \tag{3.6}
\]

where \(\bar{u}, \bar{v} \in L^\infty(R \times (0, +\infty); R)\), \(w^{\epsilon_2}(t) \in C^1(S)\), \(u|_S = u^{\epsilon_2}_S = \sigma^{\epsilon_2} + \epsilon_2\), \(\sigma^{\epsilon_2}\) is the tangential derivative of curve \(S\), and they satisfy

\[
\langle u, \varphi_t \rangle + \langle \frac{1}{2}u^2, \varphi_x \rangle = 0, \\
\langle v, \varphi_t \rangle + \langle uv - \epsilon_2v, \varphi_x \rangle = 0 \tag{3.7}
\]

for all test functions \(\varphi \in C^\infty_{0}\left((-\infty, +\infty) \times [0, +\infty)\right)\), where

\[
\langle v, \varphi \rangle = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \bar{v}\varphi dx dt + \langle w^{\epsilon_2}(t)\delta_S, \varphi \rangle, \\
\langle uv, \varphi \rangle = \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \bar{u}\bar{v}\varphi dx dt + \langle w^{\epsilon_2}(t)u^{\epsilon_2}_S \delta_S, \varphi \rangle.
\]

Using this definition, we can define the delta-shock solution of (1.4) with the discontinuity \(x = x(t)\) of the form

\[
(u, v)(t, x) = \begin{cases} 
  (u_-, v_-), & x < x(t), \\
  (u^{\epsilon_2}_S, w^{\epsilon_2}(t)\delta(x - x(t))), & x = x(t), \\
  (u_+, v_+), & x > x(t),
\end{cases} \tag{3.8}
\]

where \((u_-, v_-)\) and \((u_+, v_+)\) are piecewise smooth solutions of (1.4), \(\delta(\cdot)\) is the standard Dirac measure, \(x(t) \in C^1\), \(w^{\epsilon_2}(t)\) is strength of delta shock wave and \(u^{\epsilon_2}_S\) is the correspon-
The solution \((u, v)(t, x)\) defined in (3.8) satisfies (3.7) in the sense of distributions if it satisfies the relation

\[
\begin{align*}
\frac{dx}{dt} &= \sigma^\varepsilon_2, \\
\frac{d\varepsilon^2(t)}{dt} &= [v]\sigma^\varepsilon_2 - [u v - \varepsilon_2 v], \quad (3.9) \\
[u]\sigma^\varepsilon_2 &= \left[\frac{u^2}{2}\right]
\end{align*}
\]

and

\[
[u]_{x=x(t)} = u^\sigma_\delta = \sigma^\varepsilon_2 + \varepsilon_2. \quad (3.10)
\]

In fact, for any test function \(\varphi \in C^\infty_0((-\infty, +\infty) \times [0, +\infty))\), if (3.9) and (3.10) hold, then by Green’s formulation and integrating by parts, it yields that

\[
\langle v, \varphi_t \rangle + \langle u v - \varepsilon_2 v, \varphi_x \rangle
\]

\[
= \int_0^{+\infty} \int_{-\infty}^{x(t)} (v_- \varphi_t + v_- (u_- - \varepsilon_2) \varphi_x) dx dt
\]

\[
+ \int_0^{+\infty} \int_{x(t)}^{+\infty} (v_+ \varphi_t + v_+ (u_+ - \varepsilon_2) \varphi_x) dx dt
\]

\[
+ \int_0^{+\infty} w\varepsilon^2(t) (\varphi_t + (u^\sigma_\delta - \varepsilon_2) \varphi_x) dt
\]

\[
= -\int_0^{+\infty} -(v_- (u_- - \varepsilon_2) \varphi) dt + v_- \varphi dx + \int_0^{+\infty} -(v_+ (u_+ - \varepsilon_2) \varphi) dt + v_+ \varphi dx
\]

\[
+ \int_0^{+\infty} w\varepsilon^2(t) (\varphi_t + (u^\sigma_\delta - \varepsilon_2) \varphi_x) dt
\]

\[
= \int_0^{+\infty} (v_- (u_- - \varepsilon_2) - v_+ (u_+ - \varepsilon_2)) \varphi dt
\]

\[
+ \int_0^{+\infty} (v_+ - v_-) \sigma\varepsilon^2 \varphi dt - \int_0^{+\infty} \frac{d\varepsilon^2(t)}{dt} \varphi dt
\]

\[
= \int_0^{+\infty} \left( \sigma\varepsilon^2 [v] - [u v - \varepsilon_2 v] - \frac{d\varepsilon^2(t)}{dt} \right) \varphi dt
\]

\[
= 0.
\]

That is, the second equation of (3.7) holds. The rest one can be checked in a similar way.

Equations (3.9) and (3.10) are called the generalized Rankine-Hugoniot relation of delta shock waves. Furthermore, the entropy condition

\[
u_- - \varepsilon_2 > \sigma^\varepsilon_2 > u_+ + \varepsilon_2 \quad (3.11)
\]

should be supplemented to guarantee the uniqueness.
By solving the ordinary differential equations (3.9) and (3.10) with initial data \( t = 0 \): \( x(0) = 0 \), \( \psi^L(0) = 0 \), we get that
\[
\sigma^L = u^L - \sqrt{1 + 4\epsilon_1^2 v^2}
\]
\[
\psi^L(t) = \frac{1}{2} (v^L(u_- - u_+ + 2\epsilon_2) - v^L(u_+ - u_- + 2\epsilon_2))t.
\]

Given a constant state \((u_-, v_-)\), these wave curves divide the half-phase plane into three regions
\[
I = \{(u, v) | u_- < u < +\infty\}; \quad II = \{(u, v) | u_- - 2\epsilon_2 < u < u_-\}; \quad III = \{(u, v) | -\infty < u < u_- - 2\epsilon_2\}.
\]

According to the state \((u_+, v_+)\) in the different regions of the half-phase plane, the solution is \( J + R \) when \((u_+, v_+) \in I(u_-, v_-)\), \( J + S \) when \((u_+, v_+) \in II(u_-, v_-)\), and delta shock wave when \((u_+, v_+) \in III(u_-, v_-)\) (see Fig. 1), where the symbol “+” means “followed by”.

4 Riemann Solutions of the Perturbed Brio System (1.1)

Now, we construct the Riemann solutions to (1.1) and (1.5). The eigenvalues of (1.1) are
\[
\lambda_{1,2}^{\epsilon_1\epsilon_2} = u - \frac{1}{2} \epsilon_2 \pm \frac{\sqrt{\epsilon_2^2 + 4\epsilon_1^2 v^2}}{2},
\]
with \( \lambda_1^{\epsilon_1\epsilon_2} < \lambda_2^{\epsilon_1\epsilon_2} \), so the system (1.1) is strictly hyperbolic for \( \epsilon_1, \epsilon_2 > 0 \). The corresponding right eigenvectors are given by
\[
\vec{r}_1^{\epsilon_1\epsilon_2} = \left( \frac{1}{2} \epsilon_2 - \frac{\sqrt{\epsilon_2^2 + 4\epsilon_1^2 v^2}}{2}, v \right)^T, \quad \vec{r}_2^{\epsilon_1\epsilon_2} = \left( \frac{1}{2} \epsilon_2 + \frac{\sqrt{\epsilon_2^2 + 4\epsilon_1^2 v^2}}{2}, v \right)^T.
\]

Checking genuine nonlinearity for the perturbed Brio system (1.1), we find that
\[
\nabla \lambda_i^{\epsilon_1\epsilon_2} \cdot \vec{r}_i^{\epsilon_1\epsilon_2} = \epsilon_1 v \left( 2 \pm \frac{\epsilon_2}{\sqrt{\epsilon_2^2 + 4\epsilon_1^2 v^2}} \right), \quad i = 1, 2.
\]
Therefore, both two characteristic fields are genuinely nonlinear when \( v > 0 \) and \( \epsilon_1, \epsilon_2 > 0 \).

As before, we look for the self-similar solution, then (1.1) becomes

\[
\begin{align*}
-\xi u_\xi + \left( \frac{1}{2} u^2 + \frac{1}{2} \epsilon_1 v^2 \right)_\xi &= 0, \\
-\xi v_\xi + (uv - \epsilon_2 v)_\xi &= 0
\end{align*}
\]  

(4.2)

with the boundary condition

\[
(u, v)(0, \pm \infty) = (u_\pm, v_\pm).
\]  

(4.3)

For the smooth solution, we write (4.2) in matrix form as

\[
\begin{pmatrix}
-\xi + u \\
v
\end{pmatrix}
\begin{pmatrix}
\epsilon_1 v \\
-\xi + u - \epsilon_2
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}_\xi = 0.
\]

Besides the constant state

\[
(u, v)(\xi) = \text{constant},
\]

it provides the backward rarefaction wave

\[
R_1(u_-, v_-) : \\
\begin{align*}
\xi &= \lambda_{\epsilon_1 \epsilon_2}^{\epsilon_1} = u - \frac{1}{2} \epsilon_2 - \frac{\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2}, \\
v du &= \left( \frac{1}{2} \epsilon_2 - \frac{\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2} \right) dv,
\end{align*}
\]  

(4.4)

or the forward rarefaction wave

\[
R_2(u_-, v_-) : \\
\begin{align*}
\xi &= \lambda_{\epsilon_1 \epsilon_2}^{\epsilon_2} = u - \frac{1}{2} \epsilon_2 + \frac{\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2}, \\
v du &= \left( \frac{1}{2} \epsilon_2 + \frac{\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2} \right) dv.
\end{align*}
\]  

(4.5)

From (4.4) and (4.5), we calculate that

\[
\frac{d\lambda_{\epsilon_1 \epsilon_2}}{dv}^{\epsilon_1} = \frac{\epsilon_2 \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} - \epsilon_2^2 - 8 \epsilon_1 v^2}{2v \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}} < 0
\]  

(4.6)

and

\[
\frac{d\lambda_{\epsilon_2 \epsilon_2}}{dv}^{\epsilon_2} = \frac{\epsilon_2 \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} + \epsilon_2^2 + 8 \epsilon_1 v^2}{2v \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}} > 0,
\]  

(4.7)

which mean that the velocity of backward (forward) rarefaction wave \( \lambda_{\epsilon_1 \epsilon_2}^{\epsilon_1} \) (\( \lambda_{\epsilon_2 \epsilon_2}^{\epsilon_2} \)) is monotonically decreasing (increasing) about \( v \).

Those states which can be connected to \((u_-, v_-)\) by \( R_i \) must lie in a direction in which \( \lambda_{\epsilon_1 \epsilon_2}^{\epsilon_1} \) is monotonically increasing, so the left state \((u_-, v_-)\) and right state \((u, v)\) can be connected by \( R_1 \) with \( v < v_- \) and \( R_2 \) with \( v > v_- \).
Integrating the second equations of (4.4) and (4.5) respectively, one can get that

$$R_1(u_-, v_-): \begin{cases} 
\xi = \lambda_1^{\epsilon_1 \epsilon_2} = u - \frac{1}{2} \epsilon_2 - \frac{\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2}, \\
\frac{1}{2} \left(-\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} + \epsilon_2 \ln(\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} + \epsilon_2)\right) = C_1,
\end{cases} \quad (4.8)$$

and

$$R_2(u_-, v_-): \begin{cases} 
\xi = \lambda_2^{\epsilon_1 \epsilon_2} = u - \frac{1}{2} \epsilon_2 + \frac{\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2}, \\
\frac{1}{2} \left(\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} + \epsilon_2 \ln(\sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} - \epsilon_2)\right) = C_2,
\end{cases} \quad (4.9)$$

where

$$C_i = u_- - \frac{1}{2}\left((-1)^i \sqrt{\epsilon_2^2 + 4 \epsilon_1 v_-^2} + \epsilon_2 \ln(\sqrt{\epsilon_2^2 + 4 \epsilon_1 v_-^2} + (-1)^{i+1} \epsilon_2)\right), \ i = 1, 2. \quad (4.10)$$

In order to depict the geometric properties of rarefaction wave curves, we show the following lemma.

**Lemma 4.1** For the back and forward rarefaction waves based on the given left state $(u_-, v_-)$, we have

$$R_1(u_-, v_-): \quad \frac{du}{dv} < 0, \quad \frac{d^2 u}{dv^2} < 0, \quad \lim_{v \to 0^+} u = u_{**},$$

$$R_2(u_-, v_-): \quad \frac{du}{dv} > 0, \quad \frac{d^2 u}{dv^2} < 0, \quad \lim_{v \to +\infty} u = +\infty,$$

where $u_{**} = \frac{1}{2}(\epsilon_2 + \epsilon_2 \ln 2 \epsilon_2) + C_1$.

**Proof** From the second equation of (4.4), we can obtain that

$$\frac{du}{dv} = \frac{\epsilon_2 - \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2v} < 0 \quad (4.11)$$

and

$$\frac{d^2 u}{dv^2} = \frac{-2 \epsilon_2 \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2} + 2 \epsilon_2^2}{4v^2 \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}} < 0, \quad (4.12)$$

which implies that the backward rarefaction wave curve $R_1$ is monotonically decreasing and concave. Similarly, the second equation of (4.5) yields that

$$\frac{du}{dv} = \frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 4 \epsilon_1 v^2}}{2v} > 0 \quad (4.13)$$
and

\[
\frac{d^2 u}{dv^2} = \frac{-2\epsilon_2 \sqrt{\epsilon_2^2 + 4\epsilon_1 v^2} - 2\epsilon_2^2}{4v^2 \sqrt{\epsilon_2^2 + 4\epsilon_1 v^2}} < 0. \quad (4.14)
\]

Thus, the forward rarefaction wave curve \( R_2 \) is monotonically increasing and concave.

The asymptotic properties of \( R_1 \) and \( R_2 \) can be easily obtained from (4.8) and (4.9). The proof is completed. \( \square \)

For a bounded discontinuity at \( \xi = \sigma \epsilon_1 \epsilon_2 \), the Rankine-Hugoniot condition for system (1.1) can be written as

\[
\begin{align*}
\sigma \epsilon_1 \epsilon_2 [u] &= \left[ \frac{1}{2} u^2 + \frac{1}{2} \epsilon_1 v^2 \right], \\
\sigma \epsilon_1 \epsilon_2 [v] &= \left[ uv - \epsilon_2 v \right]. \quad (4.15)
\end{align*}
\]

Eliminating \( \sigma \epsilon_1 \epsilon_2 \) in (4.15), we have

\[
\frac{u_+ - u_-}{v_+ - v_-} = \frac{\epsilon_2 \pm \sqrt{\epsilon_2^2 + 4 \epsilon_1 (v_+ + v_-)^2}}{v_+ + v_-}. \quad (4.17)
\]

In view of the classical Lax entropy conditions, the propagation speed \( \sigma_1 \epsilon_1 \epsilon_2 \) should satisfy

\[
\sigma_1 \epsilon_1 \epsilon_2 < \lambda_1(u_-, v_-), \quad \lambda_1(u_+, v_+) < \sigma_1 \epsilon_1 \epsilon_2 < \lambda_2(u_+, v_+) \quad (4.18)
\]

for the backward shock wave, and the propagation speed \( \sigma_2 \epsilon_1 \epsilon_2 \) should satisfy

\[
\lambda_1(u_-, v_-) < \sigma_2 \epsilon_1 \epsilon_2 < \lambda_2(u_-, v_-), \quad \lambda_2(u_+, v_+) < \sigma_2 \epsilon_1 \epsilon_2 \quad (4.19)
\]

for the forward shock wave.

Furthermore, from the second equation of (4.15), we have

\[
\sigma_1 \epsilon_1 \epsilon_2 = \frac{u_- + \frac{v_+ (u_- - u_+)}{v_- - v_+} - \epsilon_2}{v_- - v_+} = \frac{v_- (u_- - u_+)}{v_- - v_+} - \epsilon_2. \quad (4.20)
\]

Together with (4.18)-(4.20), one has

\[
\frac{-2\epsilon_1 v_+^3}{\epsilon_2 + \sqrt{\epsilon_2^2 + 4 \epsilon_1 v_+^2}} < \frac{v_- v_+ (u_- - u_+)}{v_- - v_+} < \frac{-2\epsilon_1 v_+^3}{\epsilon_2 + \sqrt{\epsilon_2^2 + 4 \epsilon_1 v_+^2}} \quad (4.21)
\]

and

\[
\frac{-2\epsilon_1 v_+^3}{\epsilon_2 - \sqrt{\epsilon_2^2 + 4 \epsilon_1 v_+^2}} < \frac{v_- v_+ (u_- - u_+)}{v_- - v_+} < \frac{-2\epsilon_1 v_+^3}{\epsilon_2 - \sqrt{\epsilon_2^2 + 4 \epsilon_1 v_+^2}}. \quad (4.22)
\]
(4.21) means that \( v_+ > v_- \), \( u_- > u_+ \), and the minus sign is taken in (4.17) for backward shock wave. In contrast, (4.22) means that \( v_+ < v_- \), \( u_- > u_+ \), and the plus sign is chosen in (4.17) for forward shock wave.

Let us fix \((u_-, v_-)\) in the \((u, v)\) phase plane, the backward shock wave curve is given by

\[
S_1(u_-, v_-) : \begin{cases} 
\sigma_1^{\epsilon_2} = u_- + \frac{v(u - u_-)}{v - v_-} - \epsilon_2, \\
\frac{u - u_-}{v - v_-} = \frac{\epsilon_2 - \sqrt{\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2}}{v + v_-}, \\
u < u_-, \quad v > v_-
\end{cases}
\tag{4.23}
\]

and the forward shock wave curve can be expressed as

\[
S_2(u_-, v_-) : \begin{cases} 
\sigma_2^{\epsilon_2} = u + \frac{v_- (u - u_-)}{v - v_-} - \epsilon_2, \\
\frac{u - u_-}{v - v_-} = \frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2}}{v + v_-}, \\
u < u_-, \quad v < v_-
\end{cases}
\tag{4.24}
\]

The shock wave curves possess the following geometric properties.

**Lemma 4.2** For the back and forward shock waves based on the given left state \((u_-, v_-)\), we have

\[
S_1(u_-, v_-) : \frac{du}{dv} < 0, \quad \frac{d^2u}{dv^2} < 0, \quad \lim_{v \to -\infty} u = -\infty,
\]

\[
S_2(u_-, v_-) : \frac{du}{dv} > 0, \quad \frac{d^2u}{dv^2} < 0, \quad \lim_{v \to 0^+} u = u_*,
\]

where \( u_* = u_- - (\epsilon_2 + \sqrt{\epsilon_2^2 + 2 \epsilon_1 v_-^2}) \).

**Proof** By a simple calculation, it follows from the second equation of (4.23) that

\[
\frac{d^2u}{dv^2} = \frac{4v_- (v + v_-)(\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2)(\epsilon_2^2 - \epsilon_2 \sqrt{\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2})}{(v + v_-)^4(\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2)^2} + \frac{-4\epsilon_1 \epsilon_2 (v + v_-)^3 (v - v_-)}{(v + v_-)^4(\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2)^2} < 0.
\]

Therefore, the backward shock wave curve \( S_1 \) is monotonically decreasing and concave.

Similarly, from the second equation of (4.24), we can get

\[
\frac{d^2u}{dv^2} = \frac{-\epsilon_2 - \epsilon_2 \sqrt{\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2}}{(v + v_-)^2 \sqrt{\epsilon_2^2 + 2 \epsilon_1 (v + v_-)^2}} > 0,
\]
We at first discuss the limit behavior of the Riemann solutions as $\epsilon \to 0$.

**5.1 Formation of Delta Shock Wave for the System (1.1)**

In this section, we investigate the formation process of delta shock wave and vacuum in the Riemann solutions for (1.4) and (1.5) depend on the position of the right state $(u_+, v_+)$. The solution is $R_1 + R_2$ when $(u_+, v_+) \in R_1 R_2(u_-, v_-)$, $S_1 + R_2$ when $(u_+, v_+) \in S_1 R_2(u_-, v_-)$, $R_1 + S_2$ when $(u_+, v_+) \in R_1 S_2(u_-, v_-)$, and $S_1 + S_2$ when $(u_+, v_+) \in S_1 S_2(u_-, v_-)$ (see Fig. 2). In next section, we only consider the limit process for the two cases $(u_+, v_+) \in S_1 S_2(u_-, v_-)$ and $(u_+, v_+) \in R_1 R_2(u_-, v_-)$, because the other two regions $S_1 R_2(u_-, v_-)$ and $R_1 S_2(u_-, v_-)$ have empty interiors when $\epsilon_1, \epsilon_2 \to 0$.

That is, the forward shock curve $S_2$ is monotonically increasing and concave. In addition, the asymptotic properties of $S_1$ and $S_2$ are obviously checked from (4.23) and (4.24). We finish the proof.

Through the above analysis, for any given left state $(u_-, v_-)$, we know that the elementary wave curves divide the half-upper $(u, v)$ phase plane into four regions. The structures of Riemann solutions for (1.4) and (1.5) depend on the position of the right state $(u_+, v_+)$. The solution is $R_1 + R_2$ when $(u_+, v_+) \in R_1 R_2(u_-, v_-)$, $S_1 + R_2$ when $(u_+, v_+) \in S_1 R_2(u_-, v_-)$, $R_1 + S_2$ when $(u_+, v_+) \in R_1 S_2(u_-, v_-)$, and $S_1 + S_2$ when $(u_+, v_+) \in S_1 S_2(u_-, v_-)$ (see Fig. 2). In next section, we only consider the limit process for the two cases $(u_+, v_+) \in S_1 S_2(u_-, v_-)$ and $(u_+, v_+) \in R_1 R_2(u_-, v_-)$, because the other two regions $S_1 R_2(u_-, v_-)$ and $R_1 S_2(u_-, v_-)$ have empty interiors when $\epsilon_1, \epsilon_2 \to 0$.

5 Formation of Delta Shock Wave and Vacuum when $\epsilon_1, \epsilon_2 \to 0$

In this section, we investigate the formation process of delta shock wave and vacuum in the Riemann solutions to (1.1) and (1.5) as $\epsilon_1, \epsilon_2 \to 0$. Our discussions will be divided into the following two cases.

5.1 Formation of Delta Shock Wave for the System (1.1)

We at first discuss the limit behavior of the Riemann solutions as $\epsilon_1, \epsilon_2 \to 0$.

When $(u_+, v_+) \in S_1 S_2(u_-, v_-)$, the solution of Riemann problem (1.1) and (1.5) consists of $S_1$ and $S_2$. Denoted the intermediate state by $(u_1^{\epsilon_1 \epsilon_2}, v_1^{\epsilon_1 \epsilon_2})$, then we have

$$
S_1(u_-, v_-): \begin{align*}
\sigma_1^{\epsilon_1 \epsilon_2} &= u_- + \frac{v_1^{\epsilon_1 \epsilon_2} - u_-}{u_1^{\epsilon_1 \epsilon_2} - v_-} - \epsilon_2, \\
u_1^{\epsilon_1 \epsilon_2} &= u_- + \sqrt{\frac{\epsilon_2 - 4\epsilon_1 (v_1^{\epsilon_1 \epsilon_2} + v_-)^2}{v_1^{\epsilon_1 \epsilon_2} + v_-}}, \\
u_1^{\epsilon_1 \epsilon_2} &< u_-, \quad v_1^{\epsilon_1 \epsilon_2} > v_-
\end{align*}
$$

(5.1)
and

\[
S_2(u_-, v_-): \begin{cases}
\sigma_2^{\epsilon_1 \epsilon_2} = u_+ + \frac{u_+ - u_-}{v_+ - v_-} \epsilon_2, \\
u_+ = u_+^{\epsilon_1 \epsilon_2} + (v_+ - v_-^{\epsilon_1 \epsilon_2}) \\
u_+^{\epsilon_1 \epsilon_2} > u_+, \ v_-^{\epsilon_1 \epsilon_2} > v_+.
\end{cases}
\tag{5.2}
\]

We need to establish the following lemmas.

\textbf{Lemma 5.1} \( \lim_{\epsilon_1, \epsilon_2 \to 0} v_+^{\epsilon_1 \epsilon_2} = +\infty. \)

\textbf{Proof} The second equations of (5.1) and (5.2) yield that

\[
u_- u_+ = (v_-^{\epsilon_1 \epsilon_2} - v_+) \epsilon_2 + \sqrt{\epsilon_2^2 + 4\epsilon_1(v_-^{\epsilon_1 \epsilon_2} + v_-)^2} \frac{v_-^{\epsilon_1 \epsilon_2} + v_+}{v_-^{\epsilon_1 \epsilon_2} + v_-},
\tag{5.3}
\]

If \( \lim_{\epsilon_1, \epsilon_2 \to 0} v_-^{\epsilon_1 \epsilon_2} = M_1 \in (\max(v_-, v_+) + \infty), \) then by passing to the limit of (5.3) as \( \epsilon_1, \epsilon_2 \to 0, \) we directly get \( u_- - u_+ = 0, \) which is a contradiction. Thus, one has \( \lim_{\epsilon_1, \epsilon_2 \to 0} v_-^{\epsilon_1 \epsilon_2} = +\infty. \) This concludes the proof.

\textbf{Lemma 5.1} shows that the intermediate state \( v_-^{\epsilon_1 \epsilon_2} \) between the two shock waves becomes singular when \( \epsilon_1 \) and \( \epsilon_2 \) drop to 0.

\textbf{Lemma 5.2} \( \lim_{\epsilon_1, \epsilon_2 \to 0} 2\sqrt{\epsilon_1} v_-^{\epsilon_1 \epsilon_2} = \frac{u_- - u_+}{2}. \)

\textbf{Proof} From Lemma 5.1 and (5.3), we have

\[
u_- u_+ = \lim_{\epsilon_1, \epsilon_2 \to 0} \left( \sqrt{4\epsilon_1(v_-^{\epsilon_1 \epsilon_2} + v_-)^2} + \sqrt{4\epsilon_1(v_-^{\epsilon_1 \epsilon_2} + v_+)^2} \right) \]

\[
= \lim_{\epsilon_1, \epsilon_2 \to 0} 2\sqrt{\epsilon_1} (2v_-^{\epsilon_1 \epsilon_2} + v_+ + v_-).
\tag{5.4}
\]

Then, the desired result can be immediately obtained. The proof is completed.

\textbf{Lemma 5.3} \( \lim_{\epsilon_1, \epsilon_2 \to 0} u_+^{\epsilon_1 \epsilon_2} = \lim_{\epsilon_1, \epsilon_2 \to 0} \sigma_1^{\epsilon_1 \epsilon_2} = \lim_{\epsilon_1, \epsilon_2 \to 0} \sigma_2^{\epsilon_1 \epsilon_2} = \frac{u_- + u_+}{2}. \)

\textbf{Proof} Combining (5.1) and (5.2) with Lemma 5.2, we obtain that

\[
u_- u_+ = \lim_{\epsilon_1, \epsilon_2 \to 0} \left( u_- - \sqrt{\epsilon_2^2 + 4\epsilon_1(v_-^{\epsilon_1 \epsilon_2} + v_-)^2} \right) = \frac{u_- + u_+}{2}
\]

and

\[
u_- u_+ = \lim_{\epsilon_1, \epsilon_2 \to 0} \left( u_- + u_+ - \frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 4\epsilon_1(v_-^{\epsilon_1 \epsilon_2} + v_-)^2}}{v_-^{\epsilon_1 \epsilon_2} + v_-} - \epsilon_2 \right) = \frac{u_- + u_+}{2}. \]
Similarly, we can prove that \( \lim_{\epsilon_2, \epsilon_3 \to 0} \sigma_2^{\epsilon_1 \epsilon_2} = \frac{u_- + u_+}{2} \). Thus, this lemma is right.

Lemma 5.3 means that, when \( \epsilon_1, \epsilon_2 \to 0 \), \( S_1 \) and \( S_2 \) coincide, and the velocities \( \sigma_1^{\epsilon_1 \epsilon_2} \) and \( \sigma_2^{\epsilon_1 \epsilon_2} \) approach the quantity \( \sigma \) given in (2.7), which is just the propagating speed of the delta shock wave of the transport equations.

**Lemma 5.4** \( \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\epsilon_1 \epsilon_2}^{\epsilon_2} \psi_{\epsilon_1}^{\epsilon_2} d\xi = \sigma [v] - [uv] = \frac{1}{2} (v_- + v_+) (u_- - u_+) \).

**Proof** \( S_1 \) and \( S_2 \) satisfy the following Rankine-Hugoniot relation

\[
\begin{cases}
\sigma_1^{\epsilon_1 \epsilon_2} (v_- - \psi_{\epsilon_1}^{\epsilon_2}) = u_- v_- - \epsilon_2 v_- - \psi_{\epsilon_1}^{\epsilon_2} v_- + \epsilon_2 v_+ \\
\sigma_2^{\epsilon_1 \epsilon_2} (v_+ - v_-) = u_+ v_+ - \epsilon_2 v_+ - \psi_{\epsilon_1}^{\epsilon_2} v_- + \epsilon_2 v_- 
\end{cases}
\]  

Adding them together, one has

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\epsilon_1 \epsilon_2}^{\epsilon_2} \psi_{\epsilon_1}^{\epsilon_2} d\xi = \lim_{\epsilon_1, \epsilon_2 \to 0} \left( \sigma_2^{\epsilon_1 \epsilon_2} v_+ - \sigma_1^{\epsilon_1 \epsilon_2} v_- + u_- v_- - u_+ v_+ + \epsilon_2 v_+ - \epsilon_2 v_-ight) = \sigma [v] - [uv] = \frac{1}{2} (v_- + v_+) (u_- - u_+).
\]

This result is consistent with the strength of delta shock wave given by (2.7) in Sect. 2. Thus, we complete the proof of this lemma.

Now, we present the limit of solutions to (1.1) and (1.5) as two-parameter flux perturbation vanishes. The following theorem is necessary because it gives a very nice depiction of the limit.

**Theorem 5.1** Let \( u_- > u_+ \). For any fixed \( \epsilon_1, \epsilon_2 > 0 \), assume that \((u^{\epsilon_1 \epsilon_2}, v^{\epsilon_1 \epsilon_2})\) is a two-shock Riemann solution of the system (1.1) and (1.5) constructed in Sect. 4. Then, as \( \epsilon_1, \epsilon_2 \to 0 \), the limit of solution \((u^{\epsilon_1 \epsilon_2}, v^{\epsilon_1 \epsilon_2})\) is a delta shock wave of (1.3) and (1.5) connecting two constant states \((u_\pm, v_\pm)\), which is expressed by (2.5) with (2.7).

**Proof** (1) Firstly, set \( \xi = x/t \). For any \( \epsilon_1, \epsilon_2 > 0 \), the two-shock Riemann solution can be expressed as

\[
(u^{\epsilon_1 \epsilon_2}, v^{\epsilon_1 \epsilon_2})(\xi) = \begin{cases} (u_-, v_-), & \xi < \sigma_1^{\epsilon_1 \epsilon_2}, \\
(u_+^{\epsilon_1 \epsilon_2}, v_+^{\epsilon_1 \epsilon_2}), & \sigma_1^{\epsilon_1 \epsilon_2} < \xi < \sigma_2^{\epsilon_1 \epsilon_2}, \\
(u_+, v_+), & \xi > \sigma_2^{\epsilon_1 \epsilon_2}, \end{cases}
\]  

which satisfies the weak formulations

\[
\int_{-\infty}^{+\infty} (\xi u^{\epsilon_1 \epsilon_2} - \frac{1}{2} (u^{\epsilon_1 \epsilon_2})^2 - \frac{1}{2} \epsilon_1 (v^{\epsilon_1 \epsilon_2})^2) \psi d\xi + \int_{-\infty}^{+\infty} u^{\epsilon_1 \epsilon_2} \psi d\xi = 0, \quad (5.7)
\]

and

\[
\int_{-\infty}^{+\infty} (\xi v^{\epsilon_1 \epsilon_2} - (u^{\epsilon_1 \epsilon_2} v^{\epsilon_1 \epsilon_2} - \epsilon_2 v^{\epsilon_1 \epsilon_2})) \psi d\xi + \int_{-\infty}^{+\infty} v^{\epsilon_1 \epsilon_2} \psi d\xi = 0 \quad (5.8)
\]
for any function $\psi \in C_0^1(\mathbb{R})$.

(2) Secondly, we consider the limits of $u^{e_1e_2}$ and $u^{e_1e_2}$ depending on $\xi$. Dividing the integral interval $(-\infty, +\infty)$ into $(-\infty, \sigma_1^{e_1e_2})$, $(\sigma_1^{e_1e_2}, \sigma_2^{e_1e_2})$ and $(\sigma_2^{e_1e_2}, +\infty)$, for the first integral on the left side of (5.8), we have

$$\int_{-\infty}^{+\infty} (u^{e_1e_2} v^{e_1e_2} - \epsilon_2 u^{e_1e_2} - \xi v^{e_1e_2}) \psi' d\xi$$

By integrating by parts and using Lemmas 5.3 and 5.4, one can calculate that

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \left( \int_{-\infty}^{\sigma_1^{e_1e_2}} + \int_{\sigma_1^{e_1e_2}}^{+\infty} \right) (u^{e_1e_2} v^{e_1e_2} - \epsilon_2 u^{e_1e_2} - \xi v^{e_1e_2}) \psi' d\xi$$

$$= \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{-\infty}^{\sigma_1^{e_1e_2}} (u_- v_+ - \epsilon_2 u_- - \xi v_+) \psi' d\xi + \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\sigma_1^{e_1e_2}}^{+\infty} (u_+ v_+ - \epsilon_2 v_+ - \xi v_+) \psi' d\xi$$

$$= (\sigma [v] - [uv]) \psi'(\sigma) + \int_{-\infty}^{+\infty} \psi H_v(\xi - \sigma) d\xi,$$

where

$$H_v(x) = \begin{cases} v_-, & x < 0, \\ v_+, & x > 0. \end{cases}$$

Meanwhile, we have

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\sigma_1^{e_1e_2}}^{\sigma_2^{e_1e_2}} (u^{e_1e_2} v^{e_1e_2} - \epsilon_2 u^{e_1e_2} - \xi v^{e_1e_2}) \psi' d\xi$$

$$= \lim_{\epsilon_1, \epsilon_2 \to 0} \left( u^*_e v^*_e (\psi(\sigma_2^{e_1e_2}) - \psi(\sigma_1^{e_1e_2})) - \epsilon_2 u^*_e v^*_e \psi(\sigma_2^{e_1e_2}) + \epsilon_2 u^*_e v^*_e \psi(\sigma_1^{e_1e_2}) \right) - \sigma_2^{e_1e_2} v^*_e \psi(\sigma_2^{e_1e_2}) + \sigma_1^{e_1e_2} v^*_e \psi(\sigma_1^{e_1e_2}) + \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\sigma_1^{e_1e_2}}^{\sigma_2^{e_1e_2}} v^*_e \psi' d\xi.$$

Considering Lemmas 5.3 and 5.4, we can obtain

$$\lim_{\epsilon_1, \epsilon_2 \to 0} u^*_e v^*_e (\psi(\sigma_2^{e_1e_2}) - \psi(\sigma_1^{e_1e_2}))$$

$$= \lim_{\epsilon_1, \epsilon_2 \to 0} u^*_e v^*_e (\sigma_2^{e_1e_2} - \sigma_1^{e_1e_2}) \frac{\psi(\sigma_2^{e_1e_2}) - \psi(\sigma_1^{e_1e_2})}{\sigma_2^{e_1e_2} - \sigma_1^{e_1e_2}}$$

$$= (\sigma [v] - [uv]) \psi'(\sigma).$$

Similarly, we also get

$$\lim_{\epsilon_1, \epsilon_2 \to 0} (\epsilon_2 v^*_e \psi(\sigma_1^{e_1e_2}) - \epsilon_2 v^*_e \psi(\sigma_2^{e_1e_2}))$$

$$= \lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_2 v^*_e (\sigma_1^{e_1e_2} - \sigma_2^{e_1e_2}) \frac{\psi(\sigma_1^{e_1e_2}) - \psi(\sigma_2^{e_1e_2})}{\sigma_1^{e_1e_2} - \sigma_2^{e_1e_2}}.$$
\[= 0 \]

and

\[
\begin{aligned}
\lim_{\epsilon_1, \epsilon_2 \to 0} & \left( \sigma_1^{\epsilon_1 \epsilon_2} v_*^{\epsilon_1 \epsilon_2} \psi(\sigma_1^{\epsilon_1 \epsilon_2}) - \sigma_2^{\epsilon_1 \epsilon_2} v_*^{\epsilon_1 \epsilon_2} \psi(\sigma_2^{\epsilon_1 \epsilon_2}) \right) \\
= & \lim_{\epsilon_1, \epsilon_2 \to 0} v_*^{\epsilon_1 \epsilon_2} (\sigma_1^{\epsilon_1 \epsilon_2} - \sigma_2^{\epsilon_1 \epsilon_2}) \frac{\psi(\sigma_1^{\epsilon_1 \epsilon_2}) - \psi(\sigma_2^{\epsilon_1 \epsilon_2})}{\sigma_1^{\epsilon_1 \epsilon_2} - \sigma_2^{\epsilon_1 \epsilon_2}} \\
= & -(\sigma [v] - [uv]) (\sigma' \psi(\sigma) + \psi(\sigma)).
\end{aligned}
\]

Moreover, the Lemma 5.4 yields that

\[
\begin{aligned}
\lim_{\epsilon_1, \epsilon_2 \to 0} & \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} v_*^{\epsilon_1 \epsilon_2} \psi d\xi \\
= & \lim_{\epsilon_1, \epsilon_2 \to 0} (\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}) v_*^{\epsilon_1 \epsilon_2} \cdot \lim_{\epsilon_1, \epsilon_2 \to 0} \frac{1}{\sigma_2^{\epsilon_1 \epsilon_2} - \sigma_1^{\epsilon_1 \epsilon_2}} \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} \psi(\xi) d\xi \\
= & (\sigma [v] - [uv]) \psi(\sigma).
\end{aligned}
\]

Returning to (5.10), we immediately get

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_{\sigma_1^{\epsilon_1 \epsilon_2}}^{\sigma_2^{\epsilon_1 \epsilon_2}} (u^{\epsilon_1 \epsilon_2} v^{\epsilon_1 \epsilon_2} - \epsilon_2 v^{\epsilon_1 \epsilon_2} - \xi v^{\epsilon_1 \epsilon_2}) \psi' d\xi = 0.
\tag{5.11}
\]

Therefore

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_{-\infty}^{+\infty} (v^{\epsilon_1 \epsilon_2} - H_v(\xi - \sigma)) \psi d\xi = (\sigma [v] - [uv]) \psi(\sigma).
\tag{5.12}
\]

Now, let us focus on (5.7). Noticing the fact that \(\epsilon_1 (v_*^{\epsilon_1 \epsilon_2})^2\) is bounded and using the same way as above, one can obtain that

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_{-\infty}^{+\infty} (u^{\epsilon_1 \epsilon_2} - H_u(\xi - \sigma)) \psi d\xi = (\sigma [u] - [u^2]) \psi(\sigma) \\
= \left(\frac{u_- + u_+}{2} [u] - \left[\frac{1}{2} u^2\right]\right) \psi(\sigma) = 0,
\tag{5.13}
\]

where

\[
H_u(x) = \begin{cases} 
  u_-, & x < 0, \\
  u_+, & x > 0.
\end{cases}
\]
Finally, we study the limits of \( u^\epsilon_1 \epsilon_2 (x, t) \) and \( v^\epsilon_1 \epsilon_2 (x, t) \) depending on \( t \). For any test function \( \phi(x, t) \in C^0_\infty (R \times R^+) \), by (5.13), we have

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} u^\epsilon_1 \epsilon_2 (x/t) \phi(x, t) dx dt = \lim_{\epsilon_1, \epsilon_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} u^\epsilon_1 \epsilon_2 (\xi) \phi (\xi t, t) d\xi dt = \lim_{\epsilon_1, \epsilon_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} H_u(\xi - \sigma) \phi(x, t) dx dt = \int_0^{+\infty} \int_{-\infty}^{+\infty} H_u(\xi - \sigma) \phi(x, t) dx dt,
\]

which means that

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} (u^\epsilon_1 \epsilon_2 (x/t) - H_u(\xi - \sigma)) \phi(x, t) dx dt = 0. \tag{5.15}
\]

In a similar way, we have

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} v^\epsilon_1 \epsilon_2 (x/t) \phi(x, t) dx dt = \int_0^{+\infty} \int_{-\infty}^{+\infty} H_v(\xi - \sigma) \phi(x, t) dx dt + \int_0^{+\infty} (\sigma [v] - [uv]) t \phi(\sigma t, t) dt, \tag{5.16}
\]

that is

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} (v^\epsilon_1 \epsilon_2 (x/t) - H_v(\xi - \sigma)) \phi(x, t) dx dt = \int_0^{+\infty} (\sigma [v] - [uv]) t \phi(\sigma t, t) dt. \tag{5.17}
\]

According to the definition (2.2), the right side of (5.17) can be rewritten as

\[
\int_0^{+\infty} (\sigma [v] - [uv]) t \phi(\sigma t, t) dt = \left\{ w_1(\cdot) \delta_S, \varphi(\cdot, \cdot) \right\},
\]

where

\[
w_1(t) = t (\sigma [v] - [uv]).
\]

The proof of Theorem 5.1 is completed.

\[\square\]

5.2 Formation of Vacuum State for the System (1.1)

Now we investigate the formation of vacuum state for the system (1.1) when \( (u_+, v_+) \in \mathcal{R}_1 \mathcal{R}_2(u_-, v_-) \). For this case, the solution of Riemann problem (1.1) and (1.5) consists of \( \mathcal{R}_1 \)
and $R_2$. Denoted the intermediate state by $(u_1^{ε}, v_1^{ε})$, then we have

$$
R_1(u_-, v_-) : \begin{cases}
\xi = λ_1^{ε} = u - \frac{1}{2} ε_2 - \sqrt{\frac{e_{1}^{2} + 4e_1 v^2}{2}}, \\
u_1^{ε} = \frac{1}{2} \left( - \frac{e_{1}^{2} + 4e_1 (v_1^{ε})^2}{2} + ε_2 \ln(\sqrt{e_{1}^{2} + 4e_1 (v_1^{ε})^2} + ε_2) \right) = C_1, \\
u_1^{ε} > u_-, \quad v_1^{ε} < v_- 
\end{cases}
$$

(5.18)

and

$$
R_2(u_-, v_-) : \begin{cases}
\xi = λ_2^{ε} = u - \frac{1}{2} ε_2 + \sqrt{\frac{e_{1}^{2} + 4e_1 v^2}{2}}, \\
u_2^{ε} = \frac{1}{2} \left( \sqrt{e_{1}^{2} + 4e_1 (v_2^{ε})^2} + ε_2 \ln(\sqrt{e_{1}^{2} + 4e_1 (v_2^{ε})^2} + ε_2) \right) = C_2, \\
u_2^{ε} < u_+, \quad v_2^{ε} < v_+ 
\end{cases}
$$

(5.19)

where $C_i$ is expressed as (4.10).

It is easy to find that

$$
u_+ - u_- = \frac{1}{2} \left( - 2 \sqrt{e_{1}^{2} + 4e_1 (v_1^{ε})^2} + \sqrt{e_{1}^{2} + 4e_1 v_-^2} + \sqrt{e_{1}^{2} + 4e_1 v_+^2} \right) + ε_2 \ln(\sqrt{e_{1}^{2} + 4e_1 (v_1^{ε})^2} + ε_2) + ε_2 \ln(\sqrt{e_{1}^{2} + 4e_1 (v_2^{ε})^2} + ε_2) - ε_2 \ln(\sqrt{e_{1}^{2} + 4e_1 v_-^2} - ε_2) \right).$$

(5.20)

If $\lim_{ε_1, ε_2 \to 0} v_1^{ε} = K \in (0, \min(v_-, v_+))$, then we immediately get $u_+ = u_-$ from (5.20), which contradicts with $u_+ > u_-$. So we can conclude that $\lim_{ε_1, ε_2 \to 0} v_1^{ε} = 0$, that is, the vacuum state will appear. In addition, from (5.18) and (5.19), we have $\lim_{ε_1, ε_2 \to 0} λ_1^{ε} = \lim_{ε_1, ε_2 \to 0} λ_2^{ε} = u$. At this moment, two rarefaction waves become two contact discontinuities $ξ = x/t = u_±$. Therefore, we have the following theorem.

**Theorem 5.2** Let $u_+ > u_-$. For any fixed $ε_1, ε_2 > 0$, assume that $(u_1^{ε}, v_1^{ε})$ is a two-rarefaction wave solution of the system (1.1) and (1.5). Then, as $ε_1, ε_2 \to 0$, the two rarefaction waves become two contact discontinuities connecting the constant states $(u_±, v_±)$ and the vacuum $(v = 0)$, which form a vacuum solution of (1.3) and (1.5).

**6 Limits of Riemann Solutions to (1.1) when $ε_1 \to 0$**

This section discusses the limit behaviors of Riemann solutions to (1.1) and (1.5) as $ε_1 \to 0$. Because our focus is the delta shock wave, so we first discuss the situation $(u_+, v_+) \in III(u_-, v_-)$, that is, $u_+ < u_- - 2ε_2$.

**Lemma 6.1** When $(u_+, v_+) \in III(u_-, v_-)$, there exists a positive parameter $ε_0$ such that $(u_+, v_+) \in S_1 S_2(u_-, v_-)$ when $0 < ε_1 < ε_0$. 

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Proof When \((u_+, v_+) \in III(u_-, v_-)\), we have \(u_- > u_+ + 2\epsilon_2\). All states \((u, v)\) connected with \((u_-, v_-)\) by a backward shock wave \(S_1\) or a forward shock wave \(S_2\) satisfy

\[
\begin{align*}
\frac{u - u_-}{v - v_-} &= \frac{\epsilon_2 - \sqrt{\epsilon_2^2 + 4\epsilon_1(v + v_-)^2}}{v + v_-}, \quad u < u_-, \; v > v_-,
\frac{u - u_-}{v - v_-} &= \frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 4\epsilon_1(v + v_-)^2}}{v + v_-}, \quad u < u_-, \; v < v_-.
\end{align*}
\]  

(6.1)

If \(v_- = v_+\), then we may take \(\epsilon_0\) as any real positive number. Otherwise, by taking

\[
\left(\frac{u_+ - u_-}{v_+ - v_-} (v_- + v_+) - \epsilon_2\right)^2 = \left(\sqrt{\epsilon_2^2 + 4\epsilon_1(v_+ + v_-)^2}\right)^2,
\]

one can solve that

\[
\epsilon_0 = \frac{\frac{u_+ - u_-}{v_+ - v_-} (v_- + v_+) - \epsilon_2)^2 - \epsilon_2^2}{4(v_- + v_+)^2}.
\]

(6.2)

Considering that \(u_+ < u_- - 2\epsilon_2\), so this lemma is true.

When \(0 < \epsilon_1 < \epsilon_0\), besides two constant states \((u_\pm, v_\pm)\), the Riemann solution consists of a backward shock wave \(S_1\) and a forward shock wave \(S_2\) with the intermediate state \((u_\epsilon^{\epsilon_1, \epsilon_2}, v_\epsilon^{\epsilon_1, \epsilon_2})\), which is determined by

\[
S_1(u_-, v_-) : \begin{cases}
\sigma_1^{\epsilon_1, \epsilon_2} = u_- + \frac{\epsilon_1^{\epsilon_1, \epsilon_2} (u_\epsilon^{\epsilon_1, \epsilon_2} - u_-)}{v_\epsilon^{\epsilon_1, \epsilon_2} - v_-} - \epsilon_2, \\
u_\epsilon^{\epsilon_1, \epsilon_2} = u_- + (v_\epsilon^{\epsilon_1, \epsilon_2} - v_-) \frac{\epsilon_2 - \sqrt{\epsilon_2^2 + 4\epsilon_1(v_\epsilon^{\epsilon_1, \epsilon_2} + v_-)^2}}{v_\epsilon^{\epsilon_1, \epsilon_2} + v_-}, \\
u_\epsilon^{\epsilon_1, \epsilon_2} < u_-, \; v_\epsilon^{\epsilon_1, \epsilon_2} > v_-.
\end{cases}
\]

(6.3)

and

\[
S_2(u_-, v_-) : \begin{cases}
\sigma_2^{\epsilon_1, \epsilon_2} = u_+ + \frac{\epsilon_1^{\epsilon_1, \epsilon_2} (u_\epsilon^{\epsilon_1, \epsilon_2} - u_+)}{v_\epsilon^{\epsilon_1, \epsilon_2} - v_+} - \epsilon_2, \\
u_\epsilon^{\epsilon_1, \epsilon_2} = u_+ + (v_\epsilon^{\epsilon_1, \epsilon_2} - v_+) \frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 4\epsilon_1(v_\epsilon^{\epsilon_1, \epsilon_2} + v_+)^2}}{v_\epsilon^{\epsilon_1, \epsilon_2} + v_+}, \\
u_\epsilon^{\epsilon_1, \epsilon_2} > u_+, \; v_\epsilon^{\epsilon_1, \epsilon_2} > v_+.
\end{cases}
\]

(6.4)

Lemma 6.2 \(\lim_{\epsilon \to 0, \epsilon_1^{\epsilon_1, \epsilon_2} = +\infty}\).

Proof From the second equations of (6.3) and (6.4), one can obtain

\[
\begin{align*}
(u_- - u_+) &= (v_\epsilon^{\epsilon_1, \epsilon_2} - v_+) \frac{\frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 4\epsilon_1(v_\epsilon^{\epsilon_1, \epsilon_2} + v_+)^2}}{v_\epsilon^{\epsilon_1, \epsilon_2} + v_+}}{v_\epsilon^{\epsilon_1, \epsilon_2} + v_-}
+ (v_- - v_\epsilon^{\epsilon_1, \epsilon_2}) \frac{\epsilon_2 - \sqrt{\epsilon_2^2 + 4\epsilon_1(v_\epsilon^{\epsilon_1, \epsilon_2} + v_-)^2}}{v_\epsilon^{\epsilon_1, \epsilon_2} + v_-}.
\end{align*}
\]
If \( \lim_{\epsilon_1 \to 0} v^{\epsilon_1, \epsilon_2} = M_2 \in (\max(v_-, v_+), +\infty) \), then taking \( \epsilon_1 \to 0 \) yields that

\[
u_+ - u_+ = 2\epsilon_2 \frac{M_2 - v_+}{M_2 + v_+} < 2\epsilon_2,
\]

(6.5)

which contradicts with \( u_- > u_+ + 2\epsilon_2 \). Thus, \( \lim_{\epsilon_1 \to 0} v^{\epsilon_1, \epsilon_2} = +\infty \). The proof is completed. \( \square \)

Using the same way as used in Sect. 5, we can easily obtain the following lemmas.

**Lemma 6.3** \( \lim_{\epsilon_1 \to 0} \sqrt{\epsilon_2^2 + 4\epsilon_1(v^{\epsilon_1, \epsilon_2}_+ + v_-)^2} = \lim_{\epsilon_1 \to 0} \sqrt{\epsilon_2^2 + 4\epsilon_1(v^{\epsilon_1, \epsilon_2}_+ + v_+)^2} = \frac{u_- - u_+}{2} \).

**Lemma 6.4** \( \lim_{\epsilon_1 \to 0} u^{\epsilon_1, \epsilon_2}_+ = \frac{u_- + u_+}{2} + \epsilon_2, \lim_{\epsilon_1 \to 0} \sigma^{\epsilon_1, \epsilon_2}_1 = \lim_{\epsilon_1 \to 0} \sigma^{\epsilon_1, \epsilon_2}_2 = \frac{u_- + u_+}{2} \).

**Lemma 6.5** \( \lim_{\epsilon_1 \to 0} \int_{v^{\epsilon_1, \epsilon_2}_+}^{v^{\epsilon_1, \epsilon_2}_-} v^{\epsilon_1, \epsilon_2}_+ d\xi = \frac{1}{2}(v_- (u_- - u_+ + 2\epsilon_2) - v_- (u_+ - u_- + 2\epsilon_2)) \).

These lemmas show that, when \( \epsilon_1 \to 0 \), the velocities of shock waves \( S_1 \) and \( S_2 \) approach to \( \sigma^{\epsilon_2}_1 \), which means that \( S_1 \) and \( S_2 \) coincide. Besides, the intermediate state \( v^{\epsilon_1, \epsilon_2}_+ \) becomes singular which determines the delta-shock solution of (1.4) and (1.5). Similar to the proof in Theorem 5.1, we can draw the conclusion as follows.

**Theorem 6.1** Let \( (u_+, v_+) \in \text{III}(u_-, v_-) \). For any fixed \( \epsilon_1, \epsilon_2 > 0 \), assume that \( (u^{\epsilon_1, \epsilon_2}, v^{\epsilon_1, \epsilon_2}) \) is a two-shock Riemann solution of (1.1) and (1.5) constructed in Sect. 4. Then, as \( \epsilon_1 \to 0 \), the limit of solution \( (u^{\epsilon_1, \epsilon_2}, v^{\epsilon_1, \epsilon_2}) \) is a delta shock wave of (1.4) and (1.5) connecting \( (u_\pm, v_\pm) \), which is expressed by (3.8) with (3.10) and (3.12).

Next, we discuss the limit behavior of the Riemann solution to (1.1) and (1.5) in the case \( (u_+, v_+) \in \text{I}(u_-, v_-) \) as \( \epsilon_1 \to 0 \), that is \( u_- < u_+ \).

**Lemma 6.6** When \( (u_+, v_+) \in \text{I}(u_-, v_-) \), there exists a positive parameter \( k_0 \) such that \( (u_+, v_+) \in R_1 R_2(u_-, v_-) \) when \( 0 < \epsilon_1 < k_0 \).

**Proof** In this case, if \( v_+ < v_- \), there exists a \( k_1 > 0 \), such that \( (u_+, v_+) \in R_{1k_1} \). In fact, from \( (u_+, v_+) \in R_{1k_1} \), we have

\[
u_+ - u_- = \int_{v_-}^{v_+} \frac{\epsilon_2 - \sqrt{\epsilon_2^2 + 4\epsilon_1 v^2}}{2v} dv, \quad v_+ < v_-.
\]

(6.6)

Using the mean value theorem, we take

\[
\kappa_1 = \frac{(2v_0 u_+ - u_-)^2 - \epsilon_2^2}{4v_0^2}, \quad v_+ < v_0 < v_-.
\]

(6.7)

While if \( v_- < v_+ \), there exists a \( k_2 > 0 \), such that \( (u_+, v_+) \in R_{2k_2} \). In fact, from \( (u_+, v_+) \in R_{2k_2} \), we get

\[
u_+ - u_- = \int_{v_-}^{v_+} \frac{\epsilon_2 + \sqrt{\epsilon_2^2 + 4\epsilon_1 v^2}}{2v} dv, \quad v_- < v_+.
\]

(6.8)
By the mean value theorem, we take
\[ k_2 = \frac{(2\bar{v}_0 \frac{u_+ - u_-}{v_+ - v_-} - \epsilon_2)^2 - \epsilon_2^2}{4\bar{v}_0^2}, \quad v_- < \bar{v}_0 < v_+. \]  
(6.9)
Taking \( k_0 = \min\{k_1, k_2\} \), we get the result. \( \square \)

When \( 0 < \epsilon_1 < k_0 \), the Riemann solution to (1.1) and (1.5) just consists of two rarefaction waves. Letting \( \epsilon_1 \to 0 \) in (4.4) and (4.5), \( R_1 \) and \( R_2 \) become the contact discontinuity \( J \) and rarefaction wave \( R \), which can be expressed as

\[ J: \begin{cases} \xi = \lambda_1^2 = u - \epsilon_2, \\ du = 0, \\ u_{\epsilon_1}^2 = u_- \end{cases} \]  
(6.10)
and

\[ R(u_-, v_-): \begin{cases} \xi = \lambda_2^2 = u, \\ d(u - \epsilon_2) = \epsilon_2 dv, \\ u_{\epsilon_1}^2 < u_+ \end{cases} \]  
(6.11)
where \((u_{\epsilon_1}^2, v_{\epsilon_1}^2)\) is the intermediate state of \( J \) and \( R \). After a simple calculation, we can get
\[ (u_{\epsilon_1}^2, v_{\epsilon_1}^2) = (u_-, v_+ \exp(\frac{u_- - u_+}{\epsilon_2})). \]  
(6.12)
Then, we have the following theorem.

**Theorem 6.2** Let \((u_+, v_+) \in I(u_-, v_-)\). For any fixed \( \epsilon_1, \epsilon_2 > 0 \), assume that \((u_{\epsilon_1}^2, v_{\epsilon_1}^2)\) is a two-rarefaction wave Riemann solution of the system (1.1) and (1.5) constructed in Sect. 4. Then as \( \epsilon_1 \to 0 \), the limit of solution \((u_{\epsilon_1}^2, v_{\epsilon_1}^2)\) is a contact discontinuity \( J \) and a rarefaction wave \( R \) of (1.4) and (1.5) connecting \((u_{\pm}, v_{\pm})\).

We have proven that when \( u_- > u_+ + 2\epsilon_2 \) or \( u_- < u_+ \), the solutions to the Riemann problem (1.1) and (1.5) are just the solutions to the Riemann problem for (1.4) with the same initial data as \( \epsilon_1 \to 0 \). The same conclusions are true for the rest cases, and we omit the discussions.

7 Numerical Simulations

In order to confirm the theoretical analysis for the formation of delta shock wave and vacuum mentioned in Sects. 5 and 6, we present some representative numerical results. Many more numerical tests have been performed to make sure that what are presented are not numerical artifacts. To discretize the system, we employ the Nessyahu-Tadmor scheme [14] with \( 300 \times 300 \) cells and \( CFL = 0.475 \). In what follows, we exhibit the numerical results by two cases.

**Case 1**. Formation of delta shock wave and vacuum as \( \epsilon_1, \epsilon_2 \to 0 \)

We first show the formation of delta shock wave. At this moment, the initial data should satisfy \( u_- > u_+ \) and \((u_+, v_+) \in S_1 S_2(u_-, v_-)\). We simulate the solution of the Riemann problem (1.1) and take initial data as follows
According to the different choices of \( \epsilon_1 \) and \( \epsilon_2 \), we exhibit the results of numerical simulations in Figs. 3, 4, 5.

From these numerical results, one can clearly observe that when \( \epsilon_1 \) and \( \epsilon_2 \) decrease, the location of \( S_1 \) and \( S_2 \) is getting closer and closer, the intermediate state \( v_{\epsilon_1, \epsilon_2}^* \) increases sharply, which yields a weighted \( \delta \)-measure, while the velocity \( u \) is a step function. Eventually, as \( \epsilon_1, \epsilon_2 \to 0 \), \( S_1 \) and \( S_2 \) coincide to form a delta shock wave of (1.3).

When \( u_- < u_+ \) and \( (u_+, v_+) \in R_1 R_2(u_-, v_-) \), the formation of vacuum can be observed clearly. We choose the following initial data

\[
(u, v)(0, x) = \begin{cases} 
(2, \ 4), & x < 0, \\
(-6, \ 1), & x > 0.
\end{cases}
\]  

(7.1)

The numerical results are presented by Figs. 6, 7, 8 on the basis of the different choices of parameters \( \epsilon_1 \) and \( \epsilon_2 \).
Fig. 5  Numerical results of $u$ (left) and $v$ (right) for $\epsilon_1 = 0.0002$ and $\epsilon_2 = 0.0003$ at $t = 0.1$

Fig. 6  Numerical results of $u$ (left) and $v$ (right) for $\epsilon_1 = \epsilon_2 = 1$ at $t = 0.2$

The above numerical results clearly illustrate that, when $\epsilon_1$ and $\epsilon_2$ decrease, the boundaries of $R_1$ and $R_2$ become closer and closer, and the intermediate state $v^{\epsilon_1,\epsilon_2}$ also decreases, while the velocity $u$ approximates to a linear function. As a result, $R_1$ and $R_2$ tend to two contact discontinuities with an intermediate vacuum state.

**Case 2.** Formation of delta shock wave as $\epsilon_1 \to 0$.

In this case, we take $\epsilon_2 = 0.2$ and select initial data to be

$$(u, v)(0, x) = \begin{cases} (2, 2), & x < 0, \\ (-2, 1), & x > 0. \end{cases} \quad (7.3)$$

The numerical results are given by Figs. 9, 10, 11 under the different choices of $\epsilon_1$.

Figures 9-11 clearly show that, for a fixed $\epsilon_2$, as $\epsilon_1$ decrease, the location of two shock waves become closer and closer, and the intermediate state $v^{\epsilon_1,\epsilon_2}$ increases dramatically, while the velocity tends to be a step function. Finally, as $\epsilon_1 \to 0$, $S_1$ and $S_2$ coincide to form a delta shock wave of (1.4), while the velocity is a step function.

To sum up, all of the above numerical results coincide with the theoretical analysis in Sects. 5 and 6.
Fig. 7  Numerical results of \( u \) (left) and \( v \) (right) for \( \epsilon_1 = 0.6 \) and \( \epsilon_2 = 0.2 \) at \( t = 0.2 \).

Fig. 8  Numerical results of \( u \) (left) and \( v \) (right) for \( \epsilon_1 = 0.0005 \) and \( \epsilon_2 = 0.0001 \) at \( t = 0.2 \).

Fig. 9  Numerical results of \( u \) (left) and \( v \) (right) for \( \epsilon_1 = 0.4 \) at \( t = 0.2 \).
Fig. 10 Numerical results of $u$ (left) and $v$ (right) for $\epsilon_1 = 0.04$ at $t = 0.2$

Fig. 11 Numerical results of $u$ (left) and $v$ (right) for $\epsilon_1 = 0.0004$ at $t = 0.2$

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