The emergent gravity’s entropy Ansatz from an augmented variational principle

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Abstract. In the last decade, physical and geometrical investigations about the relationship between horizon thermodynamics and gravitational dynamics suggest that gravity could be an emergent phenomenon. Among the others, Padmanabhan’s theory of “emergent gravity” focuses on the concept of spacetime as an effective macroscopic description of a more fundamental microscopic theory (“atoms of spacetime”) at the Planck scales, thus aiming to reconcile the large scale description of gravity and the small one of quantum physics. However, some mathematical aspects of this approach are still not clear as, for example, the derivation of Einstein equations from a suitable entropy functional (and not from an action as in standard General Relativity). Thus, in this work, a direct and non-trivial link between Padmanabhan’s entropy used in emergent gravity and standard General Relativity action is established. To do that, Augmented Variational Principles and Kerr-Schild metrics will be used. We shall also discuss how this link accounts for the details of the variation of Padmanabhan’s action based on gravitational entropy. It will also clarify the role of the background metric used in Padmanabhan’s functional and its non-dynamical role.

1. Introduction

Quantum gravity is one of the most intriguing and puzzling field in physics. The need to find such a theory becomes obvious when one studies black holes in which the action of gravity is relevant at least down to Planck scales. Moreover, the possibility to associate quantities as temperature and entropy to black holes horizon (the three dimensional surface which surrounds black holes in spacetime) is a clear manifestation of the existence of some quantum microscopic origin of gravity and spacetime itself [1, 2, 3, 4].

Although there are well-developed proposals for a theory of quantum gravity, as string theory or loop quantum gravity, a completely satisfactory and accepted theory is yet to come. All this fundamental theories are affected by the presence of intricate mathematical difficulties which are rather complicated to solve. Thus, in the last years, big efforts have been done in order to develop alternative approaches to get information about the quantum nature of gravity. One of these has been developed by Padmanabhan and it is a thermodynamical approach called “emergent gravity” [5, 6, 7].

Since in standard thermodynamics quantities as temperature and entropy arise from the existence of some internal degrees of freedom of the system, according to Padmanabhan, the...
possibility to attribute some entropy to spacetime is an indication of some coarse-grained structure of spacetime at Planck scales. Given a certain region of spacetime, it is possible to define an entropy functional in terms of suitably parametrized microscopic degrees of freedom. In particular, the variation of this functional leads to Einstein equations. However, this procedure is not completely clear from a mathematical (as well as physical) point of view. In fact, we know that the standard procedure to find some field equation starting from a functional is to define a Lagrangian, or a suitable action. By minimizing the latter with respect to some dynamical field, one obtains the equations of motion of the system. This is true for every general covariant theory as in the case of General Relativity, where the dynamical field is the metric. However, an essential part of the definition of covariant theories is that any field appearing in the functional has to be treated as a dynamical field and eventually determined by field equations. That is not the case for Padmanabhan’s functional, where the two fields appearing in the functional, the metric and some lightlike vector parametrizing the microscopic degrees of freedom cited above, are treated, respectively, as a fixed and a dynamical field (see [5]).

The aim of this paper is both to clarify this procedure by giving an explicit mathematical definition of the entropy ansatz by finding an action with the same structure of Padmanabhan’s entropy ansatz and to find a way to mimic the dynamic of the metric with the same mathematical objects used by Padmanabhan. We use a mathematical tool called Augmented Variational Principle [8] applied for a special class of metrics called Kerr-Schild metrics [9, 10].

The paper is organized as follow: in Sec. 2 we will give a brief review on the emergent gravity approach. In Sec. 3 we will introduce the Augmented Variational Principle and in Sec. 4 we will show how one can recover Padmanabhan’s entropy functional in that context. In order to do that, Kerr-Schild metrics will be used. Finally, in Sec. 5 we will summarize our results, leaving some open questions about this work to be studied in future.

2. Emergent Gravity

Emergent gravity is a classical approach to quantum gravity with the goal to understand which are the fundamental geometric quantities one should use to develop a complete theory of quantum gravity. The name may be unfortunate since it does not suggest that one derives a mesoscopic description from an underlying microscopic description, but it has become common in that specific case, so we adopt it. The main idea of this approach is that horizons are totally observer dependent, so one can perceive them and study their thermodynamical properties, even if there is no black hole (see [5]). In fact, we know that, for a Schwarzschild black hole, the surface \( r = 2Gm \) defines the horizon, \( \mathcal{H} \). One can compute the so-called surface gravity \( \kappa \) on \( \mathcal{H} \) which corresponds to the gravitational acceleration at the horizon. Therefore, in view of equivalence principle, starting from a flat spacetime, an observer sitting at spatial infinity can be accelerated along one direction, e.g. the \( x \)-direction, with an acceleration \( \kappa \) and, as a consequence, he will perceive a horizon. This kind of observers are called the Rindler observers and they will attribute to this horizon every physical properties as the observers in presence of a real horizon of a real black hole region. This suggests to modify the Hamilton principle to obtain Einstein equations from quantities defining the horizons, namely the (lightlike) normal unit vector \( n^a \) to the horizon, and related to thermodynamics.

The starting point is to define an entropy functional, the Padmanabhan’s ansatz for the total entropy of the system as a function of the lightlike vectors \( n^a \) is [5]:

\[
S[n^a] = -\frac{1}{8\pi G} \int_V d^4x \sqrt{-g} \left[ 2P_{cd} \nabla_c n^a \nabla_d n^b - 8\pi G (T_{ab} + \lambda(x) g_{ab}) n^a n^b \right] \tag{1}
\]

where \( G \) is the Newton constant, \( g \) is the determinant of the (Rindler) metric tensor and the lightlike vectors \( n^a \) are normal to the Rindler horizon, \( \mathcal{H} \), and they parametrize the spacetime internal degrees of freedom. The tensor \( P_{cd} \) has the same symmetries of the Riemann tensor, it
depends on the theory one assumes to describe gravitational dynamics and it satisfies $\nabla_c P^{cd} = 0$. The Lagrange multiplier $\lambda(x)$ is put in the functional to take into account that the vectors $n^a$ are lightlike. Finally, $T_{ab}$ is the stress-energy tensor describing the matter contribution to the total action and it is conserved, viz. $\nabla_a T^{ab} = 0$.

Conversely to what happens in General Relativity, this entropy functional depends on the lightlike vectors $n^a$, which are the dynamical field one has to study in order to describe the dynamics of spacetime. The metric acts as a fixed background. The marvellous fact is that the minimization of the entropy with respect to the vectors $n^a$ leads to "Einstein-like" field equations:

$$G_{ab} = 8\pi G \left( T_{ab} + \frac{\Lambda}{8\pi G} g_{ab} \right),$$

where $\Lambda$ is a cosmological constant arising as an integration constant.

In what follows, we will clearly show how one can recover this result in General Relativity by shedding light on the role of the metric (and that of any other dynamical field) in the variational principle used by Padmanabhan.

### 3. Augmented Variational Principle

The Augmented Variational Principle (AVP) [8] has been introduced with the aim to generalize previous works on general covariant theories [11, 12], to solve in a natural way the problem of the so-called "Katz anomalous factor" [13] (the latter related to the definition of the mass of a black hole without adding by hand any term in the Lagrangian, as in [11]). Moreover it tries to generalize the concept of black hole entropy without using Killing vectors or Killing horizons, as for example in [11].

By using the AVP it is also possible to obtain the field equations and the relative conserved quantities for generic gauge covariant theories, i.e. the amount of conserved quantity which has to be expended to drive a physical system from one configuration of the system, $\bar{g}_{ab}$, to another one, $g_{ab}$.

In general we need two different metrics, $g_{ab}$ and $\bar{g}_{ab}$, the latter considered as a reference metric. We want to study the relation between the dynamics of these two different configurations of the system. In any general covariant theory, this can be achieved by writing the Lagrangian and then by minimizing it with respect to suitable dynamical fields. Specifically, in General Relativity the Lagrangian is [8]

$$L = \sqrt{-g} R - \sqrt{-\bar{g}} R + \nabla_c \left( \sqrt{-g} g^{ab} w^c_{ab} \right),$$

where $\sqrt{-g}$ ($\sqrt{-\bar{g}}$) is the square root of the determinant of the (reference) metric and the third term is a surface term, it is built up by the contraction between the metric tensor and the difference of the traceless connections of the two metrics [8]. The pure divergence is fundamental in the Lagrangian since, for example, it contributes to the definition of the Noether charges with an extra term which corrects the value of the black hole energy, correcting the anomalous factor cited above.

As usual, the dynamic of the system arises from the variation of the Lagrangian with respect to the metric tensor, thus leading to

$$\delta L = \sqrt{-\bar{g}} G_{ab} \delta \bar{g}^{ab} - \sqrt{g} G_{ab} \delta g^{ab} + \nabla_c \left[ \sqrt{-g} \delta \left( g^{ab} w^c_{ab} \right) \right],$$

where $G_{ab}$ is the Einstein tensor. In order to have a well posed variational principle, the metric and the tensor $w^c_{ab}$ have to satisfy the following conditions:

$$g_{ab} - \bar{g}_{ab} |_{\partial V} = 0, \quad w |_{\partial V} = 0,$$

where $V$ is a $d$–dimensional part of the manifold and $\partial V$ its boundary.
4. Augmented Variational Principle in Kerr-Schild metrics

Given two metrics $g_{ab}$ and $\bar{g}_{ab}$ and a covector $\zeta_a$, a Kerr-Schild metric [10, 14] has the following form

$$ g_{ab} = \bar{g}_{ab} + \zeta_a \zeta_b, $$

where $\bar{g}_{ab}$ can be set as a reference metric. The vectors $\zeta^a$ are light-like vectors defined everywhere in spacetime and they are the same for both $g_{ab}$ and $\bar{g}_{ab}$, i.e. $\zeta_a = \zeta_a$.

One nice feature of Kerr-Schild metrics is that they linearize Einstein’s gravity [15] and, for example, they include charged and rotating black holes and stars, de Sitter and anti-de Sitter space-times [16, 17]. Recently, Kerr-Schild metrics have been used in some approach to quantum gravity [18, 19].

The Kerr-Schild metrics represent the natural candidate to satisfy the conditions imposed by AVP method, since they consist in two different metrics, $g_{ab}$ and $\bar{g}_{ab}$, in which one can be set a reference field. In particular, from eq. (6), as the difference between the two metrics depends on the null vectors $\zeta^a$, this allows us to relate the dynamics of $g_{ab} - \bar{g}_{ab}$ to the one of the null vectors and to establish a direct and clear link between Padmanabhan’s ansatz in eq. (1) and standard General Relativity action. Therefore, by inserting the Kerr-Schild ansatz (6) into the augmented Lagrangian given in eq.(3) we have [20]

$$ R = \bar{R} + \nabla a \left( 2\zeta^a \nabla b \zeta^b \right) - 2P^d g_{ab} \nabla c \zeta^a \nabla d \zeta^b, $$

where $P^d_{ab} = \frac{1}{2} \left( \delta^d_a \delta^c_b - \delta^d_b \delta^c_a \right)$ for General Relativity and $\nabla$ is the covariant derivative with respect to the total metric $g_{ab}$. Without loss of generality, we can set $\bar{g}_{ab}$ as the Minkowski metric, then $\bar{R} = 0$ and the augmented gravitational action becomes $^1$

$$ A_{grav}^* = \int_V d^4x \left[ \left( -2\sqrt{-g} P^d_{ab} \nabla c \zeta^a \nabla d \zeta^b \right) + \nabla c \left( \sqrt{-g} 2P^d_{ab} \zeta^a \nabla d \zeta^b \right) \right] $$

where $V$ is a certain volume of spacetime. For the matter action we consider the contribution of a perfect fluid described by an energy-momentum tensor $T_{ab}$. The integral curves of $\zeta^a$ must be null geodesics of $g_{ab}$, thus $\zeta^a \nabla a \zeta^b = 0$ (which in turns satisfies the null energy condition). Then the matter action can be written as

$$ A_{mat}^* = \int_V d^4x \sqrt{-g} T_{ab} \zeta^a \zeta^b, $$

with the matter Lagrangian density satisfying the null-energy condition and it is conserved.

One can easily realize that by using Kerr-Schild metrics, the variation of the action with respect to the metric $g_{ab}$ now becomes a variation with respect to the null vectors $\zeta^a$ [20]. In fact, the variation of the total Kerr-Schild metric is a transformation of the type

$$ \delta g^{ab} = g^{ab} - \bar{g}^{ab} = \zeta^a \zeta^b. $$

Moreover, the condition $\zeta_a = \bar{\zeta}_a$ over the null vectors $\zeta^a$ leads to

$$ \delta \zeta^a = \bar{\zeta}^a - \zeta^a = 0, $$

thus we can add a Lagrange multiplier $\lambda(x)$ to the action to take into account the condition $\zeta_a \delta \zeta^a = 0$. Finally, the variation of the total action $A^*_{tot} = A^*_{grav} + A^*_{mat}$ with respect to the null vectors $\zeta$ leads to the following field equations

$$ G_{ab} = 8\pi G \left( T_{ab} + \frac{\Lambda}{8\pi G} g_{ab} \right). $$

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$^1$ This choice is due to simplify the variation of the total action. However, one can consider a Ricci scalar built up on a generic background metric and then minimize the action with respect to the null vectors $\zeta^a$ by keeping the background as fixed. In this way, the result in eq. (7) is still valid.
where $\Lambda$ is an integration constant taking part of its value from the Ricci scalar $R$ and part from the Lagrange multiplier $\lambda(x)$ [20]. It plays the role of a cosmological constant, whose value is completely determined by the geometrical features of the system. We notice that the equations of motion in eq. (12) do not explicitly depend on $\zeta^a$, so the relation between the latter and the null vectors $n^a$ used by Padmanabhan is irrelevant (even if it can be found, see [20]). The minimization of the augmented action leads to some “Einstein-like” field equations as expected from standard General Relativity. Notably, the cosmological constant $\Lambda$ appears as an integration constant (as in Padmanabhan’s derivation, see [5]), instead of being the result of the presence of some term added by hand in the action.

5. Conclusions and open questions
In this work we have established a nontrivial relation between Padmanabhan’s entropy action and the AVP. Since the entropy functional in eq.(1) has the same expression of the augmented action shown in eq.(8), now it becomes clear how the Einstein field equations can arise from the variation of the entropy. As a consequence of our results, the AVP could give a better and clearer mathematical foundation to the physical intuition of emergent gravity. Remarkably, if one takes the on-shell solution ($R = 0$) and then evaluates the surface term in eq. (8), one finds $A^*_{grav} = \frac{1}{4} A_H$, where $H$ is the boundary of the region over we are integrating [20]. This is something already known in General Relativity, for example for the entropy of a black hole [5, 11], and it confirms the goodness of our results.

However, there is some point which needs some further comment. Our results depend on the choice of the Kerr-Schild metric and, as far as we know, we do not know if this is the most general solution of Einstein equations. However, as noted before, the validity of the Kerr-Schild ansatz is wide and it covers a lot of physical solutions, so we are encouraged to deepen this topic. Moreover, we want to stress the fact that a natural way to incorporate the presence of a cosmological constant in the Hilbert-Einstein action is then found. This is not obvious and one should investigate about the value of this constant in a cosmological context by comparing it with the one already obtained by the $\Lambda$CDM model or by Padmanabhan himself [21]. Finally, recently Padmanabhan and collaborators have obtained a result similar to the one shown in eq.(8) for the gravitational action but in a more complicated framework, by considering quantum corrections to General Relativity [22, 23, 24, 25]. Is it possible to incorporate these corrections in the AVP Lagrangian in order to extend this approach at least at a semi-classical level? All these points will be the subjects of future works in this field.

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