A construction of symplectic connections through reduction

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Abstract. We give an elementary construction of symplectic connections through reduction. This provides an elegant description of a class of symmetric spaces and gives examples of symplectic connections with Ricci type curvature, which are not locally symmetric; the existence of such symplectic connections was unknown.

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1. Let \((M, \omega)\) be a smooth 2n-dimensional symplectic manifold; a linear connection \(\nabla\) on \((M, \omega)\) is said to be symplectic if it is torsion free and if \(\omega\) is parallel. If \(n > 1\) the curvature tensor \(R\) of such a connection has two irreducible components under the pointwise action of the linear symplectic group \(Sp(n, \mathbb{R})\) \([1] [2]\). We shall denote them by \(E\) and \(W\):

\[
R = E + W.
\]

The \(E\) component is determined by the Ricci tensor of \(\nabla\); if the \(W\) component vanishes the curvature is said to be of Ricci type.

In \([3]\) the simply connected symmetric symplectic spaces, whose curvature is of Ricci type have been classified algebraically. It was shown that an isomorphism class was determined by the orbit, under the action of the linear symplectic group \(Sp(n, \mathbb{R})\), of an element \(A\) belonging to the Lie algebra \(\mathfrak{sp}(n, \mathbb{R})\) of the linear symplectic group such that

\[
A^2 = \lambda \text{Id}.
\]

where \(\lambda\) is any real number. It was also observed that the only compact symmetric symplectic space with non-zero curvature of Ricci type is the complex projective space \(\mathbb{P}^n(\mathbb{C})\).

In this paper we give a very elementary and geometrical construction of those symmetric symplectic spaces and we provide examples of symplectic connections with Ricci type curvature, which are not locally symmetric. Finally we give a suggestion how to generalize this easy construction.

2. Let \(A \neq 0\) be an element of \(\mathfrak{sp}(n + 1, \mathbb{R})\) and let \(H\) be a homogeneous polynomial of degree 2 on \(\mathbb{R}^{2n+2}\) defined by:

\[
H(x) = \omega(x, Ax), \quad \forall x \in \mathbb{R}^{2n+2}
\]

where \(\omega\) is the standard symplectic structure on \(\mathbb{R}^{2n+2}\). Let \(0 \neq \mu_0 \in \mathbb{R}\) and denote by \(\Sigma_{\mu_0}\) the quadric on \(\mathbb{R}^{2n+2}\):

\[
\Sigma_{\mu_0} = \{ x \in \mathbb{R}^{2n+2} \mid H(x) = \mu_0 \}.
\]
This is a closed embedded submanifold of $\mathbb{R}^{2n+2}$. The tangent space to $\Sigma_{\mu_0}$ at $x$ is $(Ax)^\perp$ (with respect to $\omega$); the restriction of $\omega$ to $\Sigma_{\mu_0}$ admits at the point $x$ a 1-dimensional radical which is spanned by the Hamiltonian vector field

$$X_H(x) = -2Ax. \quad (5)$$

Observe that the condition $\mu_0 \neq 0$, implies that the radial vector $x$ is, at the point $x$, transversal to $\Sigma_{\mu_0}$.

Let $\nabla$ be the standard flat affine connection on $\mathbb{R}^{2n+2}$; it is clearly symplectic relative to $\omega$. Let $Y, Z$ be smooth vector fields on $\Sigma_{\mu_0}$. Define a linear connection $\nabla$ on $\Sigma_{\mu_0}$ by:

$$(\nabla_Y Z)_x = (\nabla\nabla Y Z)_x + \frac{1}{\mu_0}\omega(Z, AY)x. \quad (6)$$

This is indeed a vector belonging to the tangent space $T_x\Sigma_{\mu_0}$:

$$\omega_x(\nabla_Y Z, Ax) = \omega_x(\nabla\nabla Y Z, Ax) + \omega(Z,AY)$$

$$= -\omega(Z,AY) + \omega(Z,AY) = 0.$$ 

This connection is torsion free as:

$$\nabla Y Z - \nabla Z Y = \nabla\nabla Y Z - \nabla\nabla Z Y + \frac{1}{\mu_0}(\omega(Z,AY) - \omega(Y, AZ))x$$

$$= [Y, Z].$$

**1 Lemma.** The Hamiltonian vector field $X_H$, restricted to $\Sigma_{\mu_0}$, is tangent to geodesics of $\nabla$ if and only if

$$A^2 = \lambda \text{Id}.$$ 

**Proof.** Applying the definition:

$$\frac{1}{4} \nabla X_H X_H(x) = A^2 x + \frac{1}{\mu_0} \omega(Ax, A^2 x)x.$$
If $X_H$ is tangent to geodesics of $\nabla$, there exists a function $\nu(x)$ such that

$$A^2x + \frac{1}{\mu_0}\omega(Ax, A^2x)x = \nu(x)Ax.$$ 

But this implies $\nu(x) = 0$. Deriving this relation in any direction $Y$ tangent to $\Sigma_{\mu_0}$:

$$A^2Y + \frac{2}{\mu_0}\omega(AY, A^2x)x + \frac{1}{\mu_0}\omega(Ax, A^2x)Y = 0.$$ 

But using the relation once more:

$$\omega(AY, A^2x) = 0.$$ 

Hence

$$A^2 = \lambda \text{Id}.$$ 

The converse is obvious. ■

2 Lemma. The Hamiltonian vector field $X_H$ generates a one parametric group $\psi_t$ of affine transformations of $(\Sigma_{\mu_0}, \nabla)$.

Proof. Let $Y, Z$ be smooth vector fields along $\Sigma_{\mu_0}$. Then:

$$[X_H, \nabla_Y Z] = [X_H, \tilde{\nabla}_Y Z + \frac{1}{\mu_0}\omega(Z, AY)x]$$

$$= \tilde{\nabla}_{[X_H, Y]} Z + \tilde{\nabla}_Y [X_H, Z] + \frac{1}{\mu_0}(\omega([X_H, Z], AY) + \omega(Z, [X_H, AY]))x$$

as $X_H$ is an affine vector field for $\tilde{\nabla}$.

On the other hand:

$$\nabla_{[X_H, Y]} Z + \nabla_Y [X_H, Z] = \tilde{\nabla}_{[X_H, Y]} Z + \frac{1}{\mu_0}\omega(Z, A[X_H, Y])x$$

$$+ \tilde{\nabla}_Y [X_H, Z] + \frac{1}{\mu_0}\omega([X_H, Z], AY)x.$$ 

Hence the conclusion. ■
3. The action of $\psi_t$ on $\Sigma_{\mu_0}$ is free. We describe, when $A^2 = \lambda \operatorname{Id}, A \neq 0$, the orbit space $M = \Sigma_{\mu_0}/\psi_t$. For all values of $\lambda$, $M$ is a smooth manifold and the canonical projection $\pi : \Sigma_{\mu_0} \to M$ is a smooth submersion.

Case 1. Assume $A^2 = -\operatorname{Id}$. Then:
$$\omega(Ax,Ay) = -\omega(x,A^2y) = \omega(x,y)$$
and thus $A \in \operatorname{Sp}(n+1,\mathbb{R})$. There exists $0 \neq x \in \mathbb{R}^{2n+2}$ such that
$$\omega(x,Ax) = \epsilon_1, \quad \epsilon_1^2 = 1. \quad (7)$$
Then:
$$\omega(Ax,A^2x) = \epsilon_1. \quad (8)$$
Hence, by recurrence, there exists a basis \{e_a; a \leq 2n+2\} of $\mathbb{R}^{2n+2}$ and an integer $p$ ($0 \leq p \leq n+1$) such that
$$\omega(e_{2i-1},e_{2i} = Ae_{2i-1}) = 1, \quad i \leq p \quad (9)$$
$$\omega(e_{2j-1},e_{2j} = Ae_{2j-1}) = -1, \quad p < j \leq n+1. \quad (10)$$

We shall choose $\mu_0 = \pm 1$ and denote $q = n+1-p$. We have the obvious isomorphism
$$\Sigma_{p,q,1} = \Sigma_{q,p,-1}, \quad (11)$$
where $\Sigma_{p,q,\epsilon}$ denotes the quadric
$$\sum_{i=1}^{p} [(x^{2i-1})^2 + (x^{2i})^2] - \sum_{j=p+1}^{n+1} [(x^{2j-1})^2 + (x^{2j})^2] = \epsilon. \quad (12)$$
The following are obvious:
$$\Sigma_{n+1,0,1} = S^{2n+1} \quad (13)$$
$$\Sigma_{p,q,1} = S^{2p-1} \times \mathbb{R}^{2q} \quad n+1 > p \geq 1. \quad (14)$$
The Hamiltonian vector field $X_H$ generates an action of $U(1)$ and one easily checks the

3 Lemma. The reduced manifold $\Sigma_{p,q,1}/U(1)$ is one of the following ones:
$$\Sigma_{n+1,0,1}/U(1) = P_n(\mathbb{C})$$
Remember:

\[ \Sigma_{1,n,1}/U(1) = \mathbb{C}^n. \]

For \( n > p \geq 2 \)

\[ \Sigma_{p,q,1}/U(1) \]

is a rank \( q \) complex vector bundle over \( \mathbb{P}_{p-1}(\mathbb{C}) \).

**Case 2.** Assume \( A^2 = 1 \). Let \( V^\pm = \{ x \in \mathbb{R}^{2n+2} \mid Ax = \pm x \} \). Then, if \( x, y \in V^\pm \), \( \omega(x, y) = \omega(Ax, Ay) = -\omega(x, A^2y) = -\omega(x, y) = 0 \). Hence \( V^+ \) and \( V^- \) are two supplementary lagrangian subspaces. Choose a basis \( \{ e_i; i \leq n+1 \} \) of \( V^+ \) and a basis \( \{ f_j; j \leq n+1 \} \) of \( V^- \) such that

\[ \omega(e_i, f_j) = \delta_{ij}. \] (15)

Then, if \( \mu_0 = -2 \),

\[ \Sigma_{\mu_0} = \left\{ x = \sum_{i=1}^{n+1} x^i e_i + y^i f_i \mid \sum_{i=1}^{n+1} x^i y^i = 1 \right\}. \] (16)

The Hamiltonian vector field

\[ -\frac{1}{2} X_H = \sum_{i=1}^{n+1} x^i e_i - y^i f_i \] (17)

has trajectories:

\[ x^i(t) = x^i(0)e^t \] (18)

\[ y^i(t) = y^i(0)e^{-t}. \] (19)

On the orbit of \((x(0), y(0))\) there is a unique point \((\bar{x}, \bar{y})\) such that \(\sum_{i=1}^{n+1} (\bar{x}^i)^2 = 1\).

Define then the point \( z \in \mathbb{R}^{n+1} \) by

\[ \bar{y}^i = \bar{x}^i + z^i. \] (20)

Then:

\[ \sum_{i=1}^{n+1} \bar{x}^i z^i = 0. \] (21)
Hence we have:

4 Lemma. The reduced manifold $\Sigma_{-2}/\mathbb{R}$ is isomorphic to the tangent bundle $TS^n$.

Case 3. Assume $A^2 = 0 (A \neq 0)$. Let $V = \text{im} A$ and $W = \ker A$. Then $V \subset W$. Furthermore $(\text{im} A)^\perp = \{ y \mid \omega(Ax, y) = 0, \forall x \}$. Hence $\omega(x, Ay) = 0, \forall x$; hence $Ay = 0$ and $(\text{im} A)^\perp \subset \ker A$. Hence by dimension

$$\text{(im} A)^\perp = \ker A.$$  \hspace{1cm} (22)

Let $X$ be a supplementary subspace to $V$ in $W$; then $X$ is symplectic. Let $1 \leq p \leq n + 1$ be the dimension of $V$. Let $\{u_1, \ldots, u_{n+1-p}, v_1, \ldots, v_{n+1-p}\}$ be a basis of $X$ such that:

$$\omega(u_i, v_j) = \delta_{ij}$$  \hspace{1cm} (23)$$
$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0$$  \hspace{1cm} (24)

The subspace $X^\perp$ is symplectic and contains $V$ as a lagrangian subspace. Let $\{e_k; k \leq p\}$ be a basis of a lagrangian subspace $V^*$ of $X^\perp$, supplementary to $V$, such that:

$$\omega(e_k, Ae_\ell) = \epsilon_k \delta_{k\ell},$$  \hspace{1cm} (25)

where $\epsilon_k = 1$ if $k \leq q$, and $\epsilon_k = -1$ if $q < k$. Choose $\mu_0 = \epsilon (\epsilon^2 = 1)$. Then

$$\Sigma_{p,q,\epsilon} = \left\{ x \in \mathbb{R}^{2n+2} \mid \sum_{k=1}^{p} \epsilon_k (x^k)^2 = \epsilon \right\}. $$  \hspace{1cm} (26)

We have the isomorphism $\Sigma_{p,q,\epsilon} = \Sigma_{q,p,-\epsilon}$. If $q = 0$, $\epsilon$ must be chosen to be -1. Clearly,

$$\Sigma_{1,1,1} = 2 \text{ points} \times \mathbb{R}^{2n+1}$$  \hspace{1cm} (27)
$$\Sigma_{p,p,1} = S^{p-1} \times \mathbb{R}^{2n+2-p}, \hspace{0.5cm} p > 1 $$  \hspace{1cm} (28)
$$\Sigma_{p,1,1} = (R^{p-1} \cup R^{p-1}) \times \mathbb{R}^{2n+2-p}, \hspace{0.5cm} p > 1 $$  \hspace{1cm} (29)
$$\Sigma_{p,q,1} = (S^{q-1} \times R^{p-q}) \times \mathbb{R}^{2n+2-p}, \hspace{0.5cm} p > 1, q > 1. $$  \hspace{1cm} (30)
The Hamiltonian vector field reads

\[ X_H = -2 \sum_{k=1}^{p} x^k A e_k. \]  

(31)

It generates an action of \( \mathbb{R} \); if

\[ x = \sum_{k=1}^{p} x^k e_k + y^k A e_k + \sum_{a=1}^{n+1-p} z^a u_a + (z')^a v_a, \]  

(32)

we have:

\[ x^k(t) = x^k(0) \quad z^a(t) = z^a(0) \quad (z')^a(t) = (z')^a(0) \]  

(33)

\[ y^k(t) = y^k(0) - 2x^k(0)t. \]  

(34)

Observe that, if \( \mu_0 = 1 \),

\[ \sum_{k=1}^{p} \epsilon_k x^k y^k(t) = -2t + \sum_{k=1}^{p} \epsilon_k x^k(0)y^k(0). \]  

(35)

Hence there exists a unique point on the orbit such that \( \sum_{k=1}^{p} \epsilon_k x^k y^k = 0 \). Thus:

**5 Lemma.** The reduced manifold \( \Sigma_{p,q,1}/\mathbb{R} \) is one of the following ones:

\[
\Sigma_{1,1,1}/\mathbb{R} = \mathbb{R}^{2n} \cup \mathbb{R}^{2n} \\
\Sigma_{p,1,1}/\mathbb{R} = (\mathbb{R}^{p-1} \cup \mathbb{R}^{p-1}) \times \mathbb{R}^{p-1} \times \mathbb{R}^{2n+2-2p} \quad p > 1 \\
\Sigma_{p,q,1}/\mathbb{R} = T(S^{q-1} \times \mathbb{R}^{p-q}) \times \mathbb{R}^{2n+2-2p} \quad p > 1, q > 1
\]

**Remark.** The condition \( A^2 = \lambda \Id \) is not a necessary condition for the quotient \( \Sigma_{\mu_0}/\psi_t \) to have a natural structure of manifold such that the projection be a smooth submersion. Consider \( A \in \mathfrak{sp}(2, \mathbb{R}) \) having a complex eigenvalue \( \lambda = a + ib \ (ab \neq 0) \). Then \( A \) admits as other eigenvalues \( \bar{\lambda}, -\lambda, -\bar{\lambda} \). Let \( \{ e_\mu \} \) be a basis of the complexified eigenspace corresponding to \( \mu \). Then, we can choose the \( e_\mu \)'s such that:

\[ Ae_\mu = \mu e_\mu, \]
\[ \omega(e_\lambda, e_{\bar{\lambda}}) = \omega(e_\lambda, e_{-\lambda}) = \omega(e_{\bar{\lambda}}, e_{-\lambda}) = \omega(e_{-\lambda}, e_{-\bar{\lambda}}) = 0, \]
\[ \omega(e_\lambda, e_{-\lambda}) = \omega(e_{\bar{\lambda}}, e_{-\bar{\lambda}}) \neq 0. \]

This complex basis is determined up to a transformation
\[ e_\lambda \mapsto \rho e_\lambda, \quad e_{-\lambda} \mapsto \sigma e_{-\lambda}, \quad \rho \sigma \neq 0. \]

Choose the factors in such a way that:
\[ \omega(e_\lambda, e_{-\lambda}) = \omega(e_{\bar{\lambda}}, e_{-\bar{\lambda}}) = 1. \]

Then writing
\[ e_\lambda = e_1 + ie_2 \quad e_{-\lambda} = e_3 + ie_4 \]
we see that:
\[ Ae_1 = ae_1 - be_2 \]
\[ Ae_2 = be_1 + ae_2 \]
\[ Ae_3 = -ae_3 + be_4 \]
\[ Ae_4 = -be_3 - ae_4 \]
\[ \omega(e_1, e_2) = \omega(e_2, e_3) = \omega(e_1, e_4) = \omega(e_3, e_4) = 0 \]
\[ \omega(e_1, e_3) = \frac{1}{2} = -\omega(e_2, e_4). \]

The quadric \( H(x) = \omega(x, Ax) = 1 \) reads:
\[ -x^1(ax^3 + bx^4) + x^2(ax^4 - bx^3) = 1 \]
and is thus diffeomorphic to \( S^1 \times \mathbb{R}^2 \). The orbits of the Hamiltonian vector field are
\[ x^1 + ix^2 = (x^1 + ix^2)(0)\exp((a - ib)t) \]
\[ x^3 + ix^4 = (x^3 + ix^4)(0)\exp(-(a - ib)t). \]

We can rewrite the equation \( H(x) = 1 \) as:
\[ \mathfrak{R}\{[x^1 + ix^2][x^3 + ix^4](-a + ib)] = 1. \]
Hence on each orbit there exists a unique point such that $|x^1 + ix^2| = 1$. Hence the quotient is a cylinder $\mathbb{S}^1 \times \mathbb{R}$.

4. Let $\pi : \Sigma_{\mu_0} \to M = \Sigma_{\mu_0}/\psi_t$ be the canonical projection and let $y = \pi(x)$. Let $H_x$ be the subspace of the tangent space to $\Sigma_{\mu_0}$ at $x$ which is $H_x = [\text{span}(x, Ax)]^\perp$. The differential of $\pi$ at $x$, $\pi_*x$, is a linear isomorphism $H_x \to M_y$. Hence a vector field $Y$ on $M$ admits a unique lift $\bar{Y}$ to $\Sigma_{\mu_0}$, which belongs at each point $x$, to $H_x$.

Notice that

$$\omega_x(\nabla_{\bar{Y}} \bar{Z}, x) = \omega_x(\bar{Y}, \bar{Z}, x) = -\omega_x(\bar{Z}, \bar{Y}).$$

Hence define the reduced connection $\nabla^r$ on $M$ by:

$$\nabla^r_{\bar{Y}} \bar{Z} = \nabla_{\bar{Y}} \bar{Z} + \frac{1}{\mu_0} \omega(\bar{Y}, \bar{Z})Ax.$$  \hfill (37)

6 Lemma. The reduced connection $\nabla^r$ on $M$ is symplectic with respect to the symplectic form $\Omega$:

$$\Omega_y(Y, Z) = \omega_x(\bar{Y}, \bar{Z}).$$ \hfill (38)

Proof. The torsion free condition reads:

$$0 = \nabla^r_{\bar{Y}} \bar{Z} - \nabla^r_{\bar{Z}} \bar{Y} - \nabla \bar{Y}, \bar{Z}$$

$$= \nabla_{\bar{Y}} \bar{Z} + \frac{1}{\mu_0} \omega(\bar{Y}, \bar{Z})Ax - \nabla_{\bar{Z}} \bar{Y} + \frac{1}{\mu_0} \omega(\bar{Z}, \bar{Y})Ax - [\bar{Y}, \bar{Z}].$$

Now:

$$\pi_*[Y, Z] = \pi_*[\bar{Y}, \bar{Z}] = [Y, Z]$$

and

$$\omega([\bar{Y}, \bar{Z}], x) = \omega(\nabla_{\bar{Y}} \bar{Z} - \nabla_{\bar{Z}} \bar{Y}, x)$$

$$= -\omega(\bar{Z}, \bar{Y}) + \omega(\bar{Y}, \bar{Z})$$

i.e.

$$[\bar{Y}, \bar{Z}] = [\bar{Y}, \bar{Z}] + \frac{2}{\mu_0} \omega(\bar{Y}, \bar{Z})Ax.$$
Hence the torsion free condition is satisfied. To prove that $\Omega$ is parallel we note that:

$$Y\Omega(Z,U) = \bar{Y}\omega(\bar{Z},\bar{U}) + \omega(\bar{Z},\bar{Y}\bar{U})$$

$$= \omega(\nabla_{\bar{Y}}\bar{Z},\bar{U}) + \omega(\bar{Z},\nabla_{\bar{Y}}\bar{U})$$

$$= \Omega(\nabla_{\bar{Y}}Z, U) + \Omega(Z, \nabla_{\bar{Y}}U).$$

\[\]

7 Theorem. The curvature tensor $R^r$ of the reduced connection $\nabla^r$ on $(M, \Omega)$ is of Ricci type.

Proof. The lift of the curvature endomorphism of $\nabla^r$ is given by:

$$R^r(Y,Z)T = \nabla^r_{\nabla^r_ZY}T - \nabla^r_{\nabla^r_YZ}T - \nabla^r_{[Y,Z]}T.$$ We have:

$$\nabla^r_{\nabla^r_ZY}T = \nabla^r_{\nabla^r_ZY}T + \frac{1}{\mu_0}\omega(\bar{Y}, \nabla^r_{\nabla^r_ZY}T)Ax$$

$$= \nabla^r(\nabla^r_Z\bar{T} + \frac{1}{\mu_0}\omega(\bar{Z},\bar{T})Ax) + \frac{1}{\mu_0}\omega(\bar{Y}, \nabla^r_{\nabla^r_ZY}T)Ax$$

$$= \nabla^r \nabla^r_T + \frac{1}{\mu_0}Y\omega(Z,T)Ax + \frac{1}{\mu_0}\omega(Z,T)\nabla^r_Ax + \frac{1}{\mu_0}\omega(\bar{Y}, \nabla^r_{\nabla^r_ZY}T)Ax$$

$$= \nabla^r(\nabla^r_Z\bar{T} + \frac{1}{\mu_0}\omega(\bar{T}, A\bar{Z})x) + \frac{1}{\mu_0}\omega(\nabla^r Z\bar{T} + \frac{1}{\mu_0}\omega(\bar{T}, A\bar{Z})x, A\bar{Y})x$$

$$+ \frac{1}{\mu_0}[\omega(\nabla^r Z\bar{T}, \bar{T}) + \omega(\bar{T}, \nabla^r_{\nabla^r_ZY}T)]Ax$$

$$+ \frac{1}{\mu_0}\omega(\bar{Z}, \nabla^r T) [AY + \frac{1}{\mu_0}\omega(Ax, A\bar{Y})x]$$

$$+ \frac{1}{\mu_0}\omega(\bar{Y}, \nabla^r T + \frac{1}{\mu_0}\omega(T, A\bar{Z})x)Ax$$

$$= \nabla^r \nabla^r_{\nabla^r_ZY}T + \frac{1}{\mu_0}[\omega(\nabla^r Z\bar{T}, A\bar{Z}) + \omega(\bar{T}, \nabla^r_{\nabla^r_ZY}T)]Ax$$

$$+ \frac{1}{\mu_0}\omega(\bar{T}, A\bar{Z})\bar{Y} + \frac{1}{\mu_0}\omega(\nabla^r Z\bar{T}, A\bar{Y})x$$

$$+ \frac{1}{\mu_0}[\omega(\nabla^r Z\bar{T}, \bar{T}) + \omega(\bar{T}, \nabla^r_{\nabla^r_ZY}T)]Ax$$

$$+ \frac{1}{\mu_0}\omega(\bar{Z}, \bar{T})\bar{Y} + \frac{1}{\mu_0}\omega(\bar{T}, \nabla^r_{\nabla^r_ZY}T)Ax$$

$$+ \frac{1}{\mu_0}\omega(\bar{Z}, \bar{T})\bar{Y} + \frac{1}{\mu_0}\omega(\bar{T}, \nabla^r_{\nabla^r_ZY}T)Ax.$$
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\[ \bar{\nabla}_Y \bar{\nabla}_Z \bar{T} + \frac{1}{\mu_0} \omega(\bar{T}, A\bar{Z}) \bar{Y} + \frac{1}{\mu_0} \omega(\bar{Z}, \bar{T}) A\bar{Y} \]

\[ + \frac{1}{\mu_0} \left[ \omega(\bar{\nabla}_Y \bar{T}, A\bar{Z}) + \omega(\bar{T}, \bar{\nabla}_Y A\bar{Z}) \right] \]

\[ + \omega(\bar{\nabla}_Z \bar{T}, A\bar{Y}) + \frac{1}{\mu_0} \omega(\bar{Z}, \bar{T}) \omega(Ax, A\bar{Y}) \right] \]

\[ + \frac{1}{\mu_0} \left[ \omega(\bar{\nabla}_Y \bar{T}, T) + \omega(\bar{T}, \bar{\nabla}_Y T) + \omega(\bar{Y}, \bar{\nabla}_Z T) \right] Ax. \]

Also:

\[ \bar{\nabla}_{[Y,Z]} T = \nabla_{[Y,Z]} T + \frac{1}{\mu_0} \omega([Y,Z], T) Ax \]

\[ = \bar{\nabla}_{[Y,Z]} \bar{T} + \frac{1}{\mu_0} \omega(\bar{\nabla} T, A[Y,Z]) x + \frac{1}{\mu_0} \omega([Y,Z], \bar{T}) Ax \]

\[ = \bar{\nabla}_{[Y,Z]} \bar{T} + \frac{2}{\mu_0} \omega(\bar{Y}, \bar{Z}) \bar{\nabla}_A x \bar{T} \]

\[ + \frac{1}{\mu_0} \omega(\bar{T}, A[Y, Z]) + \frac{2}{\mu_0} \omega(\bar{Y}, \bar{Z}) A^2 x \]

\[ + \frac{1}{\mu_0} \omega([\bar{Y}, \bar{Z}] + \frac{2}{\mu_0} \omega(\bar{Y}, \bar{Z}) Ax, \bar{T}) Ax. \]

Hence:

\[ R^r(Y, Z) T = \frac{1}{\mu_0} \omega(\bar{T}, A\bar{Z}) \bar{Y} - \frac{1}{\mu_0} \omega(\bar{T}, A\bar{Y}) \bar{Z} + \frac{1}{\mu_0} \omega(\bar{Z}, \bar{T}) A\bar{Y} \]

\[ - \frac{1}{\mu_0} \omega(\bar{Y}, \bar{T}) A\bar{Z} - \frac{2}{\mu_0} \omega(\bar{Y}, \bar{Z}) A\bar{T} \]

\[ + \frac{1}{\mu_0^2} \left[ \omega(\bar{Z}, \bar{T}) \omega(\bar{Y}, A^2 x) - \omega(\bar{Y}, \bar{T}) \omega(\bar{Z}, A^2 x) \right] \]

\[ - 2\omega(\bar{Y}, \bar{Z}) \omega(\bar{T}, A^2 x) \]

\[ = \frac{1}{\mu_0} \left[ \omega(\bar{Z}, \bar{T})(AY - \frac{1}{\mu_0} \omega(AY, Ax)x) \right] \]

\[ - \omega(\bar{Y}, \bar{T})(AZ - \frac{1}{\mu_0} \omega(AZ, Ax)x) \]

\[ - 2\omega(\bar{Y}, \bar{Z})(AT - \frac{1}{\mu_0} \omega(AT, Ax)x) \]

\[ + \omega(\bar{T}, A\bar{Z}) \bar{Y} - \omega(\bar{T}, A\bar{Y}) \bar{Z}. \]

Notice that

\[ AZ - \frac{1}{\mu_0} \omega(A\bar{Z}, Ax) x \in H_x \]
and thus we can define an operator $A_y$ on $T_y M$ by

$$A_y U = \bar{A} U - \frac{1}{\mu_0} \omega(A \bar{U}, A x) x,$$

where $y = \pi(x)$. Observe that

$$\Omega_y(A_y U, V) = \omega(A \bar{U} - \frac{1}{\mu_0} \omega(A \bar{U}, A x) x, \bar{V})$$

$$= \omega(A \bar{U}, \bar{V}) = - \omega(A \bar{U}, A \bar{V} - \frac{1}{\mu_0} \omega(A \bar{V}, A x) x)$$

$$= - \Omega_y(U, A_y V),$$

i.e. $A_y$ is an element of the symplectic Lie algebra at $y \in M$. We can thus rewrite the curvature formula as:

$$R^{\ast}_y(Y, Z) T = \frac{1}{\mu_0} \left[ \Omega(Z, T) A_y Y - \Omega(Y, T) A_y Z - 2 \Omega(Y, Z) A_y T + \Omega(T, A_y Z) Y - \Omega(T, A_y Y) Z \right].$$

which proves the theorem. ■

8 Theorem. The connection $\nabla^r$ is locally symmetric if and only if $A^2 = \lambda \text{Id}$.

Proof. This is a corollary of Theorem 7. Indeed, from the above one sees that

$$Ric^r(U, V) = \kappa \Omega(U, AV)$$

where $\kappa$ is a constant and $Ric^r$ is the Ricci curvature of the reduced connection.
Hence:
\[(\nabla^r_X Ric')(U, V) = \kappa[X\Omega(U, AV) - \Omega(\nabla^r_X U, AV) - \Omega(U, A\nabla^r_X V)]\]
\[= \kappa[\bar{X}\omega(\bar{U}, AV) - \omega(\nabla^r_X \bar{U}, AV) - \omega(\bar{U}, A\nabla^r_X V)]\]
\[= \kappa[\omega(\nabla^r_X \bar{U}, AV) + \omega(\bar{U}, \nabla^r_X AV) - \omega(\nabla^r_X \bar{U}, AV)\]
\[- \omega(\bar{U}, A(\nabla^r_X \bar{V} + \frac{1}{\mu_0}\omega(\bar{V}, A^2x) + \frac{1}{\mu_0}\omega(\bar{X}, \bar{V})Ax))]\]
\[= \kappa[\omega(\bar{U}, \nabla^r_X (AV + \frac{1}{\mu_0}\omega(\bar{V}, A^2x)x))\]
\[- \omega(\bar{U}, A\nabla^r_X \bar{V} + \frac{1}{\mu_0}\omega(\bar{X}, \bar{V})A^2x)]\]
\[= \frac{\kappa}{\mu_0}[\omega(\bar{V}, A^2x)\omega(\bar{U}, \bar{X}) + \omega(\bar{U}, A^2x)\omega(\bar{V}, \bar{X})].\]

Define the 1-form \(u\) on \(M\) by:
\[u_y(V) = \frac{\kappa}{\mu_0}\omega(\bar{V}, A^2x), \ y = \pi(x).\]

This has a meaning as
\[Ax[\omega(\bar{V}, A^2x)] = 0.\]

The condition for local symmetry is
\[\omega(\bar{V}, A^2x) = 0, \ \forall V\]
and it is identically satisfied if \(A^2 = \lambda \text{ Id}\). The converse is an immediate consequence of Theorem 2 of [3]. ■

9 Corollary. Let \(A \in \mathfrak{sp}(n+1, \mathbb{R})\), \(A^2 \neq \lambda \text{ Id}\) and assume that the projection \(\Sigma \to M\) is a smooth submersion. Then, \(M\) admits a symplectic connection with curvature of Ricci type which is not locally symmetric.

10 Theorem. The reduced spaces \(M\) obtained above, when \(A^2 = \lambda \text{ Id}\), are globally symmetric.

Proof. The argument (identical in all cases \(\lambda > 0, \lambda < 0, \lambda = 0\)) is that the group \(G\) of symplectic transformations of \(\mathbb{R}^{2n+2}\) which commute with \(A\), acts
transitively on the quadric and this action projects onto an action of $G$ on $M$. Furthermore, the isotropy subgroup of a point of $M$ is the group of fixed points of an involutive automorphism of $G$. Finally the action of $G$ on $M$ is symplectic and affine.

5. The extreme simplicity of this construction suggests to generalize it to the situation where one has a family of elements $A_j$ ($j \leq p$) of the symplectic Lie algebra with $[A_i, A_j] = 0$, $\forall i, j$. The submanifold

$$\omega(x, A_jx) = \mu_j \neq 0$$

can be reduced under the action of $\mathbb{R}^p$. In generic situation, the reduced space $M$ is indeed a differentiable manifold and a reduced connection may be defined in a similar way.
References

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