Fast OT for Latent Domain Adaptation

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Abstract—In this paper, we address the problem of unsupervised Domain Adaptation. The need for such an adaptation arises when the distribution of the target data differs from that which is used to train a model and the ground truth information of the target data is unknown. We propose an algorithm that uses optimal transport theory with a verifiably efficient and implementable solution to learn the best latent feature representation. This is achieved by minimizing the cost of transporting the samples from the target domain to the distribution of the source domain.

Index Terms—Optimal Transport, Unsupervised Domain Adaptation

I. INTRODUCTION

Adapting a classifier trained on a source domain to recognize instances from a new target domain is an important problem of increasing research interest [1], [2], [3]. Difficulties often arise in practice, as is the case when the data is different from that which is used to train a model. Specifically, consider an inference problem where a model is learned using a certain set of data and then applied to a new domain. This is often the case in practice, as is the case when the data is different from what is used to train a model. Consider the problem of increasing research interest [4], [5], [6]. The need for such an adaptation arises when the distribution of the target data differs from that which is used to develop the model and the ground truth information of the target data is unknown. We propose an algorithm that uses optimal transport theory with a verifiably efficient and implementable solution to learn the best latent feature representation. This is achieved by minimizing the cost of transporting the samples from the target domain to the distribution of the source domain.

A. Generative modeling

The Generative Adversarial Network was first introduced by Goodfellow et al. [7] in 2014. In this framework, a generative model is pitted against an adversary: the discriminator. The generator aims to decease the discriminator by synthesizing realistic samples from some underlying distribution. The discriminator on the other hand, attempts to discriminate between a real data sample and that from the generator. Both models are approximated by neural networks. When trained alternatively, the generator learns to produce random samples from the data distribution which are very close to the real data samples. Following this, Conditional Generative Adversarial Networks (CGANs) were proposed in [8]. These networks were trained to generate realistic samples from a class conditional distribution, by replacing the random noise input to the generator by some useful information. As a result, the generator now aims to generate realistic data samples, when given the conditional information. CGANs have been used to generate random faces when given facial attributes [9] as well as to produce relevant images given text descriptions [10].

Many works have recently attempted to use GANs for performing domain adaptation. In [11], the authors use the generator to learn the features for classification and the discriminator to differentiate between the source and target domain features produced by the generator. Figure 1 depicts the block diagram for this approach. In [12], a cyclic GAN was used to perform image translation between unpaired images.
In [17] a cyclic GAN was implemented to adapt semantic segmentation of street images from GTA5 to CityScapes data.

### B. Optimal Transport

Optimal Transport [11] is a pointwise comparative analytical tool that provides a distance measure between two probability distributions. The distance measure is based on a cost \( c(\cdot, \cdot) \) which is imputed to transporting a source distribution to a target distribution. Formally, given two densities \( \mu_s \) and \( \mu_t \) on two measurable spaces \( X_s \) and \( X_t \), the Kantorovich-Monge relaxation/formulation of the optimal transport problem entails finding a transport plan which is a probabilistic coupling \( \gamma^* \) defined over \( X_s \times X_t \) such that,

\[
\arg\min_{\gamma \in \Gamma} \int_{X_s \times X_t} c(x_s, x_t)d\gamma(x_s, x_t), \tag{1}
\]

where \( c : X_s \times X_t \to [0, +\infty] \) and \( c(x, y) \) denotes the cost of transporting a unit of mass from \( x \) to \( y \). \( \gamma^*(x, y) \) denotes the coupling that provides the minimum \( \mathbb{E}_{x,y}[c(x,y)] \).

In most practical applications, one has access only to the samples of the distribution where discrete measures \( \mu_s = \sum_{i=1}^{N_s} p_{s_i} \delta_{x_{s_i}} \) and \( \mu_t = \sum_{i=1}^{N_t} p_{t_i} \delta_{x_{t_i}} \), where \( \delta_{x_{s_i}}, p_{s_i} \) and \( \delta_{x_{t_i}}, p_{t_i} \) denote the Dirac function and the probability mass at \( x_{s_i} \in X_s \) and \( x_{t_i} \in X_t \), respectively. The optimal transport plan under the discrete case is the solution to a linear programming problem which is defined as follows,

\[
\gamma^* = \arg\min_{\gamma \in \Gamma} <C, \gamma>; \tag{2}
\]

where \( C \geq 0 \) is the cost matrix with \( C_{ij} = \| (x_{s_i} - x_{t_j}) \|_2^2 \) and

\[
\Gamma = \{ \gamma \in \mathbb{R}_+^{N_s \times N_t} | \gamma 1_{N_s} = \mu_s, \gamma^T 1_{N_t} = \mu_t \}. \tag{3}
\]

is the set of probabilistic coupling matrices and \( 1 \) is a vector of ones of appropriate dimension.

### III. Problem Formulation

Consider data from a source domain, \( X_s = \{x_s\}_i=1,...,N \) with a corresponding set of labels \( Y_s = \{y_s\}_i=1,...,N \), where \( N \) is the total number of samples in the dataset. Let \( g_s : X_s \to L_s \) be a function that transforms the data into a latent feature space, \( L_s = g_s(X_s) \). Following this, a classifier function \( f(.) \) is used to assign a labels to the data samples, \( Y_s = f(L_s) = f(g_s(X_s)) \). If the classifier is well trained, \( Y_s \approx Y_t \).

Now, consider target domain data \( X_t \) for which the ground truth labels are unavailable. One may consider using the classifier trained on \( X_s \) to classify the data \( X_t \) if similar classes as in the source domain are of interest. Such a procedure would yield optimal performance if and only if the distributions of \( X_s \) and \( X_t \) are the same. This usually fails to be the case in practical applications, and hence resulting in sub optimal classification performance.

In order to mitigate this problem, Domain Adaptation (DA) is required. Note that our goal here is to take on the classification problem where labels for the target distribution are completely unknown, and hence to learn the function \( g_t : X_t \to L_t \) such that \( Y_t = f(g_t(X_t)) \) leads to optimal classification performance in the absence of any information about the target domain.

### IV. Proposed Approach

As noted in the previous section, the inference model must be optimal for the source domain. In order to ensure this, we propose to learn the functions \( g_s(.) \) and \( f(.) \) so that they minimize the cross entropy loss between the ground truth labels, \( Y_s \) and those predicted by the model, \( Y_s = f(g_s(X_s)) \),

\[
\min_{f,g_s} C_{Loss}(Y_s, f(g_s(X_s))), \tag{4}
\]

where, \( C_{Loss}(Y_s, f(g_s(X_s))) = \sum_{i=1}^{N} -y_{s_i} \log f(g_s(x_{s_i})) \).

#### A. Learning the Optimal Latent Space

We first aim to learn the optimal latent spaces \( L_s \) and \( L_t \) such that the same classifier be used for both the source and target by minimizing the cost of transporting the samples in the latent space of source domain to that of the target domain. This leads to learning of latent spaces \( L_s = g_s(X_s) \) and \( L_t = g_t(X_t) \) with minimum discrepancy. The function that must be optimized is given as,

\[
\min_{f,g_s,g_t} C_{Loss}(Y_s, Y_t) + \lambda_1 T_{Loss}(L_s, L_t), \tag{5}
\]

where, \( T_{Loss}(L_s, L_t) = \sum_{i,j} \gamma^*_ij C_{ij}. \gamma^* \) is the optimal transport mapping for going from \( L_t \) to \( L_s \), and \( C_{ij} \) is the corresponding cost. The determination of \( \gamma^* \) is further discussed in Section IV-B. \( \lambda_1 \) in Equation [5] is a hyperparameter which controls the importance of the second term with respect to the first one.

To ensure an optimal adaptation of the source domain classifier to that of the target domain, we proceed to minimize the cost for transporting between \( L_s \) and \( L_t \), while safeguarding the invariance of the predictive power of the source domain classifier \( f(.) \) when applied to target domain, i.e. \( P(Y_s|L_s) \approx P(Y_t|L_t) \). To best integrate this constraint, we opt to include the classification cost in the loss to be optimized as a nonlinear transformation of the latent representation.
of the input data. A probabilistic interpretation of such an augmentation of the divergence/discrepancy loss, as the energy function of a Boltzmann distribution. In sum, the overall transportation loss may be written as,

$$\min_{f,g} C_{\text{Loss}}(Y_s, \hat{Y}_s) + \lambda_1 T_{\text{Loss}}([L_s; f(L_s)]; [L_t; f(L_t)])$$

Hence, $C_{ij}$ is now the cost of transporting the vector $[L_{ij}; f(L_{ij})]$ to $[L_{ij}; f(L_{ij})]$. While the source domain data and their associated labels regularize the problem and reduce the search space, it is also practically important to regularize the target domain whose labels are unknown. To that end, we constrain the entropy of the predicted target labels, thereby not only tying the target latent representation but also its associated labels, thus reducing the search space again. This entropy criterion in some sense encourages the model to make more confident decisions on the target space, resulting in the overall transport loss as,

$$\min_{f,g} C_{\text{Loss}}(Y_s, \hat{Y}_s) + \lambda_1 T_{\text{Loss}}([L_s; f(L_s)]; [L_t; f(L_t)]) + \lambda_2 H(\sigma(f(g_t(X_t)))),$$  \hspace{1cm} (7)$$

where, $\sigma$ is the softmax function, $H(z) = \sum_i -z_i \log z_i$, and $\lambda_1$ and $\lambda_2$ control the contributions of the last two terms.

**B. Discrete Optimal Transport Problem: An Efficient Resolution**

The solution of the proposed model evolves along two directions: the first solves for the optimal transport map $\gamma^*$ while keeping the functions $g_s(\cdot), g_t(\cdot)$, and $f(\cdot)$ constant, and the next one solves for the functions $g_s(\cdot), g_t(\cdot)$, and $f(\cdot)$ as in Equation (7).

In order to compute $\gamma^*$, we proceed to solve the following optimization problem,

$$\gamma^* = \arg \min_{\gamma \in \mathbb{R}^{N^2}} \sum_{i,j} \gamma_{ij} C_{ij}$$

subject to: $\gamma^T 1 = \mu_s$; $\gamma 1 = \mu_t$ \hspace{1cm} (8)

Assuming $N_s = N_t = N$, there are $N^2$ unknowns and $2N$ constraints. This at best leads to a computational complexity of $O(N^3 \log N)$, to address this difficulty, accounting for the fact that the transport plan $\gamma$ is sparse (at most $2N - 1$ non-zero elements) and its support known, would reduce the search space. The dual of Equation (8) which is the dual discrete form of the Kantorovic formulation can be written as,

$$\max_{\phi, \psi} \sum_{i=1}^{N_s} \phi_i \mu_{s_i} + \sum_{j=1}^{N_t} \psi_j \mu_{t_j}$$

subject to: $\phi_i + \psi_j \leq C_{ij} \forall i,j$, \hspace{1cm} (9)$$

where the support is $(i, j)$ for which $\phi_i + \psi_j = C_{ij}$. We now have $2N$ unknowns, but there are $N^2$ constraints. This leads to the same difficulty as in the primal formulation. But, what the dual formulation does allow is the possibility of a Stochastic Gradient Descent (SGD) based approach to perform the optimization.

**Proposition 1.** The dual problem is of the form,

$$\min_{x} \sum_{i} m_i(x);$$

s.t $x \in \cap_{k=1}^K S_k$ \hspace{1cm} (10)$$

Proof. Set $x = \psi_j$ for some $j \in \{1, \ldots, N\}$ and let $m_i(x) = -[\phi_i \mu_{s_i} + \psi_j \mu_{t_j}]$. If $S_k$ is the half-space defined by $\phi_i + \psi_j \leq C_{ij}$, we get the dual of the Kantorovic formulation described in Equation (9). \hfill \square

SGD relies on approximation of the gradient $\sum_{i=1}^{N} \nabla m_i(x)$. This is carried out by randomly selecting $i \in \{1, \ldots, N\}$ and applying $x^{t+1} = \text{proj}_S(x^t - \lambda \nabla m_i(x))$. This estimated gradient has a high variance across samples. In order to stabilize the updates it is important to store the observed gradients in a cumulative manner to improve the estimate of overall gradient. This is done by using a Stochastic Variance Reduction (SVR) methods \cite{18}, $x^{t+1} = \text{proj}_S(x^t - \lambda r(\nabla m_i(x)))$, where $r$ refines the gradient estimate. Applying this to Equation (9), the following updates must be performed,

$$\hat{\phi}_i^{t+1} = \hat{\phi}_i^t - \lambda r_{\hat{\phi}_i} (\mu_{s_i});$$

$$\hat{\psi}_j^{t+1} = \hat{\psi}_j^t - \lambda r_{\hat{\psi}_j} (\mu_{t_j}).$$ \hspace{1cm} (11, 12)$$

The solution is then found by projection onto the half space $\hat{\phi}_i + \hat{\psi}_j \leq C_{ij}$. The support for $\gamma$ is now $(i, j)$ for which $\hat{\phi}_i + \hat{\psi}_j = C_{ij}$. The optimization for Equation (8) is now over
(i, j) ∈ A, where |A| ≤ N_1 + N_2 − 1, hence significantly reducing the computational complexity.

V. EXPERIMENTS AND RESULTS

A. Domain Adaptation for Computer Vision

In order to substantiate the described approach we evaluate it on various public datasets that have commonly been used in the literature to demonstrate Domain Adaptation. In each case a CNN is used to realize the functions \( g_s(\cdot), g_t(\cdot), \) and \( f(\cdot) \).

The first dataset utilized includes handwritten digits from MNIST, USPS, and SVHN that are to be recognized, with all of the 10 classes. MNIST is used as the source domain and USPS and SVHN are considered the target domain. Figure 3 provides an example of the samples in the dataset. The performance on this dataset is summarized in Tables I and II. As can be observed, the proposed approach boosts the performance in comparison to the state of art in Domain Adaptation. It is also demonstrated that ensuring the minimization of transport cost between the predictions on source and target labels is critical towards achieving a successful adaptation to the target domain. Figure 4 depicts the classification loss for the source (MNIST) and target (USPS) during training. Note that the ground truth labels for the target domain are not used in the training process, but are only used to calculate the classification loss in order to visualize the training progress.

The second dataset that has been evaluated is a slightly more challenging one. This involves an object classification task and consist of images from Amazon (31 classes), DSLR (31 classes), Webcam (31 classes), and Caltech10 (10 classes). The Amazon and DSLR data have higher resolution images while the Webcam and Caltech10 has a lower resolution. Figure 6 shows the examples from this dataset. Table II shows the performance of each of the domains when a separate classifier is trained on each of them, with all the ground truth labels assumed to be available. Table II demonstrates the performance when domain adaptation was used assuming the ground truth labels are only available for the source. In each case, the OT-inspired approach demonstrates a superior adaptation performance.
Fig. 5: An example comparing OT with a CGAN for Shape Morphing

Fig. 6: Samples from Amazon, DSLR, Webcam, and Caltech10

| Dataset       | Accuracy |
|---------------|----------|
| Amazon        | 64.2 %   |
| DSLR          | 96.1 %   |
| Webcam        | 98.6 %   |
| Caltech10     | 82.7 %   |

TABLE III: Performance on Amazon, DSLR, Webcam, Caltech10 when all labels are available

B. Shape Morphing

In addition to the computer vision applications, we also evaluate the optimal transport approach detailed in Section IV-B toward shape morphing, and compare it with a Conditional Generative Adversarial Network (CGAN). Shape morphing is the task of converting a source shape defined by points \( x_s \in \mathcal{X}_s \) into a target shape \( x_t \in \mathcal{X}_t \). If we consider \( N_1 = N_2 = 1000 \), a standard method minimizing Equation 8 requires memory of 8 MB while the approach detailed in Section IV-B requires 36 KB. We consider Optimal Transport between curves on \( \mathbb{R}^2 \) by treating them as distributions. Figure 7 shows the case with a circular curve as the source and a square as the target along with the computed transport maps. In Figure 5 we show the case when there are two circles in the target and the source is a single circle. The result of such a transform is consequently compared with a CGAN and it can be observed that the CGAN fails in such a case most likely due to the discontinuity in the input. The failure of the GAN to perform as expected is due to its inherent stability issues which are discussed and addressed in detail in [19].

VI. CONCLUSION

In this paper we proposed a Domain Adaptation approach based on Optimal Transport theory with a new efficient OT algorithm with demonstrably greater computational and effective performance, ensuring that a classifier trained on some source domain can still perform at or better than state of the art classification on a target domain that has a different data distribution. We also show that it is important to consider the distribution of the model predictions when learning the transport map. Finally we compare the performance of this OT Domain Adaptation, with the Adversarial Domain Adaptation and show that we can outperform them using this approach. Furthermore we also demonstrate the strength of OT when it comes to shape morphing in comparison to a CGAN.

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