A faithful description of the state of a complex dynamical network would require, in principle, the measurement of all its $d$ variables, an unfeasible task for systems with practical limited access and composed of many nodes with high dimensional dynamics. However, even if the network dynamics is observable from a reduced set of measured variables, how to reliably identifying such a minimum set of variables providing full observability remains an unsolved problem. From the Jacobian matrix of the governing equations of nonlinear systems, we construct a pruned fluence graph in which the nodes are the state variables and the links represent only the linear dynamical interdependences encoded in the Jacobian matrix after ignoring nonlinear relationships. From this graph, we identify the largest connected sub-graphs where there is a path from every node to every other node and there are not outgoing links. In each one of those sub-graphs, at least one node must be measured to correctly monitor the state of the system in a $d$-dimensional reconstructed space. Our procedure is here validated by investigating large-dimensional reaction networks for which the determinant of the observability matrix can be rigorously computed.

When dealing with large complex systems, observability becomes a key concept that addresses the ability of examining the system dynamics from a reduced set of measurements collected in a finite time. Indeed, to properly understand the functioning of many biological or technological networks, it is fundamental to be able to retrieve the complex behavior emerging from the local interactions of dynamical units when just a limited amount of information is available.

The idea of observability was first introduced by Kalman for linear systems [1] which was further extended to nonlinear systems by several other researchers, e.g. [2]. That now classical way of investigating observability provides a yes-or-no answer, that is, the system is either fully observable or not through a given set of measurements. In order to bypass this binary classification of observability, Friedland proposed the use of a conditioning number between 0 (non-observable) and 1 (fully observable) to quantify the observability of linear systems [3]. Later on, Aguirre showed that observability depends on the chosen coordinate set to describe the system dynamics [4]. This work led to the introduction of the observability coefficients to characterize the observability of many low-dimensional chaotic systems [3][5][6].

Recently, an attempt to apply those coefficients to small dynamical networks was reported [7] but, it was pointed out that such an assessment is out of scope for large dynamical systems due to the impossibility of calculating the determinant of the corresponding observability matrix [10]. One way to tackle this drawback is by introducing symbolic observability coefficients [10][11] that allow treating larger dimensional systems although the number of variable combinations to investigate increases exponentially with the system dimension.

In order to avoid the use of a brute-force search for a minimum sensor set, observability is addressed in [12] by means of graph-theoretic methods. Mainly based on performing a linearization of the system (all dynamical interdependences between variables are considered constant as in linear systems), such a technique reveals that sparse networks with heterogeneous degree distributions are less observable while the observability of denser and homogeneous networks relies on just a few nodes [12]. However, these latter results may not hold for nonlinear systems as the presence of nonlinearities are one of the main causes of observability loss [7, 10]. Some variants of this graphical approach were developed in [13][14] by considering the effect of connection types in the resulting topologies and in the change of the number of the necessary sensors. However, none of them actually takes into account the nonlinear nature of the dynamical interdependence between the state variables. Moreover, it was recently shown that Liu and coworkers’ graphical approach may not provide the right reduced set of variables to measure (see the supplement material in Ref. [15] and Ref. [16]).

Our goal is therefore to address the observability of a complex system to identify a minimum set of variables providing access to the rest of state variables, following an analogous graph-theoretic approach as in Liu and coworkers (and inspired in structured system theory [17][18]), but properly handling the effect of nonlinear dynamical interdependences among variables in the system’s observability. We show the correctness of this approach by using benchmark reaction networks coming from biology or physics and by comparing the obtained results with rigorous algebraic computations of the determinant of the observability matrix. Our results contradict the conclusions drawn in [12] evidencing important discrepancies mainly resulting from treating linear and
nonlinear interdependences on an equal footing [15].

Let us start by considering a dynamical system whose variables \( x_i, \ i = 1, 2, \ldots, d \) evolve according to

\[
\dot{x}_i = f_i(x),
\]

where \( x \in \mathbb{R}^d \) is the state vector, and \( f_i \) is the \( i \)th component of the vector field \( f \). The dynamical system \( \dot{x} = f(x) \) is said to be state observable at time \( t \) if the initial state \( x(0) \) can be uniquely determined from the knowledge of a variable \( s = h(x) \in \mathbb{R}^m \), with \( m < d \), measured in the interval \([0; t]\) [18]. In practice, the observability of \( \dot{x} = f(x) \) through \( s \) is assessed by computing the rank of the observability matrix

\[
\mathcal{O}_s(x) = \begin{bmatrix}
\frac{\partial h(x)}{\partial x} \\
\frac{\partial \mathcal{L}_f h(x)}{\partial x} \\
\vdots \\
\frac{\partial \mathcal{L}_f^{d-1} h(x)}{\partial x}
\end{bmatrix},
\]

where \( d \equiv \frac{\partial}{\partial x} \) and \( \mathcal{L}_f h(x) \) is the Lie derivative of \( h \) along the vector field \( f \). This is thus the Jacobian matrix of the Lie derivatives of \( s \) [2]. The system \( \dot{x} = f(x) \) is said to be state observable if and only if the observability matrix has full rank, that is, \( \text{rank}(\mathcal{O}_s) = d \) [26]. Notice that, the full observability of a system is determined by the space spanned not only by the measured variables but also by their appropriate Lie derivatives [15].

A systematic check of all the possible combinations turns out to be a daunting task for large \( d \). Therefore, it becomes crucial to furnish methods to unveil a tractable set of variables providing full observability of a system. A first attempt was reported in Liu et al. [12] using a graphical representation of the functional relationship among the system variables. We follow such an approach by choosing as the network representation of the system \( \dot{x} = f(x) \) its corresponding “fluence graph” where a directed link \( x_j \rightarrow x_i \) is drawn whenever \( x_j \) appears in the differential equation of \( x_i \), that is, if the element \( J_{ij} \) of the Jacobian matrix of the Eq. \( \dot{x} = f(x) \) is non-zero [27].

An illustrative example is provided in Fig. 1(a) for the Rössler system \( (x, y, z) \) and \( f = (-y - z, x + ay, b + z(x - c)) \). In this procedure, a link from \( x_j \) to \( x_i \) (with \( i \neq j \)) is present whenever \( J_{ij} \neq 0 \) independently on the linear (solid lines) or nonlinear (thick dashed lines) nature of the functional dependence. At this point is where we deviate from Liu and coworkers approach as it ignores the fact that a lack of observability most often originates from the nonlinear relationship between variables [8]. In order to correct this shortcoming, we propose to distinguish linear from nonlinear couplings [10, 11] by pruning from the fluence graph all nonlinear links and keeping only those associated with the constant elements in the Jacobian matrix of the system. We call this reduced fluence graph, the pruned fluence graph (Fig. 1(b)) that we take as the minimum graph containing the information flow that will allow us to select the minimum set of sensors to ensure observability of the whole system while working in a \( d \)-dimensional reconstructed space.

![Fluence graphs](image-url)  
**FIG. 1:** Fluence graphs of the Rössler system. (a) Full fluence graph where an edge is plotted between variables \( x_i \) and \( x_j \) whenever \( J_{ij} \neq 0 \). The thick dashed line indicates a nonlinear term in the Jacobian matrix. (b) The same as (a) but edges nonlinearly relating two variables are removed from the full fluence graph. A dashed circle surrounds a root strongly connected component (SCC). In both graphs, edges \( x_i \rightarrow x_i \) are omitted since they do not contribute to the determination of the SCC.

![Pruned fluence graphs](image-url)  
**FIG. 2:** Pruned fluence graphs of (a) the 5D rational system for the circadian oscillations in the Drosophila period protein, (b) the 9D Rayleigh-Bénard convection, and (c) the 13D reaction network for the replication of fission yeast. Numbers \( i \) label the variable \( x_i \) of the models, continuous lines from \( i \) to \( j \) represent that variable \( x_j \) is linearly influencing variable \( x_i \) and variables surrounded by a dashed circle are part of an SCC without outcoming edges.

In the following, the graph analysis described in [12] to isolate the minimum set of sensors still holds. Namely, a node in the pruned fluence graph is a sensor if it belongs to a root strongly connected component (SCC) of the graph (a subgraph in which there is a directed path from each node to every node in the subgraph) and with no outcoming links, that is, such an SCC is either an isolated subgraph or a root (a sink) of information flowing from any other subgraphs in the network [28]. By measuring at least one of the nodes in each subgraph classified as
a root SCC in the pruned fluence graph, we make the conjecture that such a selection is not only minimal but also provides a good observability.

In order to validate our hypothesis, we applied the above procedure to several nonlinear dynamical systems widely known in the physical and biological scientific community. For each of them we confirm that a candidate set of variables to be measured actually provides full observability by checking that the determinant \( \det \mathcal{O}_s \) of the analytical observability matrix \( \mathcal{O}_s \) as defined in Eq. (2), is always nonzero \( [6, 20] \). Note that for dimensions larger than 4, the determinant cannot always be computed due to its complexity (Maple software fails to compute some observability matrices for a 5D rational system \( [10] \)). To deal with this difficulty, a symbolic formalism was introduced in \([10, 11]\) that allows to quantify the observability of a given measure by means of a symbolic observability coefficient \( \eta = 1 \), if the observability is full, \( \eta > 0.75 \) if good, and poor otherwise \([21]\). Briefly, it is based on a symbolic Jacobian matrix \( J_{ij} \) whose elements can be either 1, 1 and \( \bar{1} \) which encode, respectively, constant, nonlinear and rational terms (whose denominators contain variables \( x_j \)) of the Jacobian matrix \( J_{ij} = \partial f_i / \partial x_j \) \((4)\).

It turns out that the symbolic observability coefficients are inversely proportional to the complexity of the determinant of the observability matrix \( \eta \). Therefore, in general, in those cases in which the sensor set is providing full observability, the determinant \( \det \mathcal{O}_s \) can thus be analytically computed. We used this property as a validation of our hypothesis, and check whether a non-vanishing determinant is obtained for a given set of variables potentially providing full observability.

**The Rössler system.** Let us now consider the Rössler system whose Jacobian matrix

\[
J_{\text{Ros}} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & -c \end{bmatrix}
\]

(3)

has two non-constant terms and whose symbolic form accounting only for the linear dynamical interdependencies is reduced to

\[
\tilde{J}_{\text{Ros}}^{\text{lin}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

(4)

The two nonlinear terms, \( J_{31} \) and \( J_{33} \) are, therefore, not considered for constructing the pruned fluence graph shown in Fig. (1b). On the other hand, when all terms in the Jacobian matrix are considered equivalent independently of their linear or nonlinear nature as in \([12]\), the resulting graph is the one shown in Fig. (1a) except that the thick dashed line will be drawn as a thin line. For this latter case, the decomposition in SCC singles out a root SCC composed by the three variables, suggesting that any of the three variables can be measured to achieve observability. However, it is known that measuring variable \( z \) alone provides poor observability \((\eta_{z, z, z} = 0.44)\) of the Rössler dynamics \([5, 10]\). The picture changes completely if only the linear dependencies are considered as depicted in the pruned fluence graph (Fig. (1b)). In this case, this graph contains a single root SCC composed only of variables \( x \) and \( y \) for which \( \eta_{x, z, z} = 0.84 \) and \( \eta_{y, y, y} = 1 \), respectively. Variable \( z \) is therefore no longer part of the recommended set of measurements when taking into account the pruned fluence graph, in full agreement with the symbolic observability coefficients \([13]\).

Moreover, and not surprisingly confirming these results, when looking at the determinants of the corresponding analytical observability matrices we find that \( \det \mathcal{O}_{yyy} = 1 \), whose constant value means full observability, \( \det \mathcal{O}_{xxy} = x - (a + c) \), meaning that observability is good as long as the determinant is not vanishing and it depends on variable \( x \) (order 1), and \( \det \mathcal{O}_{zxx} = z^2 \), indicating that observability is very poor as it depends on the square of \( z \) (order 2).

A 5D model for the circadian oscillations of the *Drosophila.* Let us now consider a 5D rational model for the circadian oscillations in the *Drosophila* period protein \([22]\). Dynamical equations and the corresponding symbolic Jacobian matrix reduced to just the constant elements are reported in the Supplemental Material for all the systems considered in the subsequent part of this letter. The pruned fluence graph (Fig. (2a)) presents three root SCC, thus suggesting that just measuring variables \( x_2 \) and \( x_3 \) and either \( x_4 \) or \( x_5 \), is sufficient to fully and efficiently account for the dynamics of the whole system. This prediction about the appropriate set of variables to monitor is confirmed by the symbolic observability coefficients \( \eta \) of the reconstructed spaces \{\( x_2^2 x_3 x_5 \)\} and \{\( x_2^2 x_3 x_4^2 \)\} \([23]\) whose values are both equal to 1, and by the constant analytical determinants \( \det \mathcal{O}_{x_2 x_3 x_5} = -k_5 k_2 \) and \( \det \mathcal{O}_{x_2^2 x_3 x_4^2} = k_1 k_2 \), where \( k_1, k_2, \) and \( k_5 \) are parameters of the model \([13]\).

A 9D Rayleigh-Bénard convection model. Let us now consider a 9D system describing the Rayleigh-Bénard convection in a square platform \([23]\). Its pruned fluence graph shown in Fig. (3) exhibits six root SCC, suggesting that a good observability might be obtained by measuring variables \( x_2, x_4, x_5, \) and \( x_6 \), and either one between \( x_3 \) or \( x_8 \), and another between \( x_1 \) or \( x_7 \). Therefore, at least 6 variables are needed in this 9D reaction network to effectively ensure full observability. This selection of variables is in agreement with the fact that most of the combinations whose observability coefficient is equal to 1, do not involve variables \( x_1, x_3, x_7, x_8, \) and \( x_9 \) \([14]\). For instance, \( \det \mathcal{O}_{x_2^2 x_3 x_5 x_4 x_6} = -b_2 / \sigma^3 / 2 \) where \( b_2 \) and \( \sigma \) are parameters of the model, confirming that a full observability can be indeed obtained with this reduced set of measured variables.

A 13D model for the DNA replication in fission yeast. A more challenging dynamical system, that is also analysed in \([12]\), is the model for cell cycle control in fission yeast governing the concentrations of the state variables \([24]\). The corresponding pruned fluence graph
TABLE I: Minimum set of variables that are needed to measure according to \(i\) Liu and coworkers’ inference graph, and \(ii\) the proposed pruned fluence graph. The cardinality of both minimum sensor sets \(m_1\) and \(m_2\) is reported in the column next to each case. Last column indicates the required number of variables to measure for getting full observability according to exact analytical calculations [13].

\[
\begin{array}{cccc}
\text{Model} & \text{Inference graph} & m_1 & \text{Pruned fluence graph} \\
3D & x \lor y \lor z & 1 & x \lor y \\
5D & x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5 & 1 & x_2 \land x_3 \land (x_4 \lor x_5) & 3 & 3 \\
9D & x_6 \land (x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5 \lor x_7 \lor x_8 \lor x_9) & 2 & (x_1 \lor x_2) \land x_3 \land x_4 \land x_5 \land x_6 \land (x_3 \lor x_8) & 6 & 6 \\
13D & (x_1 \lor x_2 \lor x_3 \lor x_4 \lor x_5 \lor x_8 \lor x_9 \lor x_{10}) & 5 & (x_1 \lor x_9) \land x_3 \land x_5 \land x_6 \land x_7 \land x_{11} \land x_{12} & 9 & 10 \\
\end{array}
\]

(Fig. 2) presents 9 root SCCs, meaning that, at least 9 variables — one from each SCC — must be measured. The detection of the SCCs of the pruned fluence graph tells us immediately that variables \(x_4, x_9\) and \(x_{10}\) can be discarded from the minimum sensor set as well as that either variable \(x_1\) or \(x_8\), can be excluded but not simultaneously. Indeed, the combinations providing good observability (\(\eta = 0.93\)) with 9 variables measured do not involve the sets \(\{x_4, x_8, x_9, x_{10}\}\) or \(\{x_1, x_4, x_9, x_{10}\}\). For instance, the reconstructed space spanned by \(\{x_1 x_2 x_3 x_4 x_5 x_7 x_{11} x_{12} x_{13}\}\) yields a \(\eta = 0.93\) and \(\det \mathcal{O} = (k_7 + k_8)(k_9 + k_4)(k_{11} + k_{12})\) which never vanishes. These results are in full agreement with those reported in [12] where the authors build a fluence graph treating all dynamical interdependences as linear. Just for illustration, their analysis gives rise to the existence of two SCCs: \(\{x_1, x_3\}\) and another one with the rest of the 12 variables which is a root SCC. Therefore, their conclusion is that by just monitoring any variable in the root SCC, the system is observable as verified by the Sedoglavic’s algorithm [25], an algorithm that certifies the local (not global) observation in a probabilistic way.

A more thorough comparison of the two approaches is given in Table I where the minimum sensor set is reported in each case for the four nonlinear dynamical systems considered in this Letter. As observed in the last column of the table, where the minimum number \(m\) of variables needed to get full observability according to exact analytical calculations, Liu and coworker’s approach tends to underestimate the number of variables.

Complex networks are large dimensional systems for which it is not possible to measure all the variables required for a full description of any of their states.

Considering that nonlinear links are generally responsible for the lack of local observability, we proposed constructing a pruned fluence graph considering only linear links – corresponding to the constant non zero terms of the Jacobian matrix of the network. We showed that identifying the root SCCs of the pruned fluence graph allowed to correctly identify the reduced set of measurements providing a good observability of the network dynamics. This technique was validated with the use of symbolic observability coefficients and the analytical determinants of observability matrices. We thus presented an easy-to-implement technique for selecting the variables to be measured for reconstructing a \(d\)-dimensional space of a reaction network. The extension to networks of dynamical systems is straightforward as long as the Jacobian matrix describes the node dynamics and their connectivity.

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[26] The observability matrix $O_s$ corresponds in fact to the Jacobian matrix of the change of coordinates $\Phi_s : x \to X$ where $X \in \mathbb{R}^d$ is the reconstructed state vector from the $m$ measured variables and their adequately chosen $d - m$ Lie derivatives.
[27] In [12] they flip the direction of each edge using the “inference diagram” by drawing a directed link $x_i \to x_j$ if $x_j$ appears in $x_i$’s differential equation.
[28] Notice that, since in [12] they flip the edge direction of the edges with respect to our choice for the graph representation of the dynamical system, their definition of root SCC is reversed here.
[29] The notation $\{ x_i^j \}$ is equivalent to the vector $\{ x_i, \dot{x}_i, \ddot{x}_i, \ldots \}$ up to the $j$th time derivative.