TWO TOPICS IN CHIRAL EFFECTIVE LAGRANGIANS

Hidenaga YAMAGISHI
4 Chome 11-16-502, Shimomeguro, Meguro, Tokyo, 153, Japan

Ismail ZAHED
Department of Physics, SUNY, Stony Brook, New York 11794, USA.
e-mail: zahed@nuclear.physics.sunysb.edu

In the absence of nucleons, we use partial wave unitarity, to show that the chiral expansion parameter must be close to $p^2/4\pi f^2_\pi$ rather than $p^2/16\pi^2 f^2_\pi$ as previously suggested, where $p$ is a typical pion momentum and $f_\pi$ the pion decay constant. When nucleons are included, we apply the Tani-Foldy-Wouthuysen (TFW) transformation to the pion-nucleon effective Lagrangian to obtain an expansion in powers of $1/m_N$ (inverse nucleon mass). The results are presented up to order $O(1/m_N^3)$, corresponding to $O(p^4)$ in the momentum. In this case partial wave-unitarity is also lost in about the same range of momenta.

1 Introduction

Effective Lagrangians have been an important tool in our understanding of hadronic processes. If we consider processes without baryons or other heavy particles for simplicity, then the loop expansion of the effective Lagrangian is equivalent to an expansion in the momentum $p$. Georgi and Manohar have suggested that the quantitative expansion parameter in this case is $p^2/16\pi^2 f^2_\pi \sim (p/1.2\text{GeV})^2$, where the factor of $16\pi^2$ arises from a one-loop Feynman diagram.

For $p = m_\pi$, this gives 1.4% accuracy. However, it is seldom in chiral physics to achieve an accuracy of this order. The standard tree level result for $\pi\pi \rightarrow \pi\pi$ gives scattering lengths which are off by 40% and 25%, depending on the isospin and angular momentum. In $\gamma \gamma \rightarrow \pi^0\pi^0$ where the Born term is absent, the one-loop result is off by 25-30%.

In this note, we show that the true expansion parameter must be closer to $p^2/4\pi f^2_\pi \sim (p/0.33\text{GeV})^2$ through a consideration of partial wave unitarity for $\pi\pi$ scattering. For $\pi N$ scattering, we first show that a pion-nucleon Lagrangian can be organized in terms of the Tani-Foldy-Wouthuysen transformation, and then observe that partial wave unitarity in the S31 channel yields a bound that is close to the $\pi\pi$ one.
2 Partial Wave Unitarity

In the chiral limit, the invariant amplitude $T^I(s,t)$ with isospin $I$ can be decomposed into partial waves as

$$T^I(s,t) = 32\pi \sum_l (2l + 1) P_l(\cos\theta) \eta^I_l e^{i\delta^I_l} \sin\delta^I_l$$

(1)

where $s$ and $t$ are Mandelstam variables, $P_l$ Legendre polynomials, $\theta$ the scattering angle in the center of mass frame, $\eta^I_l$ the partial-wave inelasticties, and $\delta^I_l$ the partial-wave phase shifts. Projecting out the S-wave gives the bound

$$|T^I_0| \leq 32\pi$$

(2)

On the other hand the tree result for massless pions is

$$T^0(s,t) = T^0_0(s) = 8k^2 f_\pi^2$$

(3)

where $k$ is the pion momentum in the center of mass frame. It follows that the unitarity bound for massless pions is

$$k^2 \leq 4\pi f_\pi^2$$

(4)

and accordingly, the chiral loop expansion parameter must be closer to $p^2 / 4\pi f_\pi^2$.

For massive pions, Eqs. (2) and (3) are modified to

$$|T^I_0(s)| \leq 32\pi \left(1 + \frac{m^2_\pi}{k^2}\right)^{\frac{3}{2}}$$

(5)

$$T^0(s,t) = T^0_0(s) = \frac{(8k^2 + 7m^2_\pi)}{f_\pi^2}$$

(6)

Since Eq. (5) is monotonically decreasing and Eq. (6) monotonically increasing, we obtain the bound

$$k^2 \leq 5.2m^2_\pi \sim (0.32 \text{ GeV})^2$$

(7)

which is numerically close to Eq. (4).

It follows that the loop expansion of conventional chiral perturbation theory must break down before 300 MeV as far as $\pi\pi$ scattering is concerned, in agreement with a similar observation in...
3 Tani-Foldy-Wouthuysen Transformation

For purely pionic processes such as previously discussed, the loop expansion of the effective Lagrangian is equivalent to an expansion in the momentum $p$. This is no longer true when nucleons are put in Gasser et al.\textsuperscript{9}. To deal with this situation, heavy baryon chiral perturbation theory was proposed\textsuperscript{10}, and used as a $1/m_N$ expansion\textsuperscript{11}. In all of this work, the projected fields\textsuperscript{12}

$$N^\pm_\xi(x) = e^{imv\cdot x} \frac{1}{2}(1 \pm \gamma_5) \psi(x)$$ \hspace{1cm} (8)

with $v^2 = 1$, were employed.

In the nonrelativistic reduction of the Dirac equation, it is perhaps more convenient to work with the TFW transformation\textsuperscript{13}, rather than Eq. (8). In this note, we apply the TFW transformation to the pion-nucleon effective Lagrangian and obtain terms up to $O(p^4)$.

We take the model relativistic pion-nucleon Lagrangian as\textsuperscript{1}

$$\mathcal{L} = + \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{4} f_\pi^2 m_\pi^2 \text{Tr}(U + U^\dagger)$$

$$+ \psi^\dagger (i\partial_0 - \mathcal{H}) \psi$$

$$\mathcal{H} = + \vec{\alpha} \cdot (\vec{p} + i \vec{\Gamma}) - i \Gamma_0$$

$$- ig_\sigma \sigma^i \Delta_i - ig_A \gamma_5 \Delta_0 + \beta m_N$$

$$+ \frac{m_\pi^2}{2\Lambda} (\beta(U + U^\dagger - 2) + \beta \gamma_5(U^\dagger - U))$$ \hspace{1cm} (9)

where $U = \xi^2$ is the chiral field and

$$\Gamma_\mu = \frac{1}{2} [\xi^\dagger, \partial_\mu \xi] \quad \Delta_\mu = \frac{1}{2} \xi^\dagger (\partial_\mu U) \xi$$ \hspace{1cm} (10)

$\Gamma_\mu$ and $\Delta_\mu$ count as $O(p)$ and $m_\pi^2/\Lambda$ as $O(p^2)$. The Lagrangian is standard except for the term proportional to $m_\pi^2/\Lambda$, which is the pion-nucleon sigma term at tree level. In QCD the quark mass term generates both the pion mass and the sigma term, so the two should go together\textsuperscript{14}.

The idea of TFW is to perform a series of unitary transformations so that the upper components of $\psi$ are decoupled from the lower components to a given order in $1/m_N$. This makes the evaluation of the fermion determinant

$$\det(i\partial_0 - \mathcal{H}) = \int [d\psi][d\psi^\dagger] \exp(i \int d^4 x \psi^\dagger (i\partial_0 - \mathcal{H}) \psi)$$ \hspace{1cm} (11)

\textsuperscript{a}The general case can be found in Steele et al.\textsuperscript{14}, for which the present arguments also apply.
straightforward. Since we are interested in terms of $O(p^4)$ in $H$, we must work to $O(1/m_N^3)$.

Operators which do not connect the upper and lower components will be called even; operators which connect upper components only with lower components will be called odd. Algebraically, an even operator $\mathcal{E}$ obeys $E\beta = \beta E$, and an odd operator $O$ obeys $O\beta = -\beta O$. We may write

$$H = \beta m_N + \mathcal{E} + O$$

$$\mathcal{E} = -i\Gamma_0 - ig_A\sigma^i\Delta_i + \frac{m^2}{2\Lambda} \beta(U + U^\dagger - 2)$$

$$O = \vec{\alpha} \cdot (\vec{p} + i\vec{\Gamma}) - ig_A\gamma_5\Delta_0 + \frac{m^2}{2\Lambda} \beta\gamma_5(U^\dagger - U)$$

We now apply the unitary transformation $\psi = e^{-iS}\psi'$, where $S$ is taken as $O(1/m_N)$. Expanding the exponential to the desired order

$$\psi^\dagger(i\partial_0 - H)\psi = \psi'^\dagger(i\partial_0 - H')\psi'$$

$$H' = H + i[S, H] - \frac{1}{2}[S, [S, H]]$$

$$-\frac{i}{6}[S, [S, [S, H]]] + \frac{1}{24}[S, [S, [S, \beta m_N]]]$$

$$-\dot{S} - \frac{i}{2}[S, \dot{S}] + \frac{1}{6}[S, [S, \dot{S}]]$$

where the dot denotes the time derivative. To cancel the odd term to $O(m^0_N)$, we choose $S = -i\beta O/2m_N$, which is consistent with our initial assumption.

Substitution into Eq. (13) gives

$$H' = \beta m_N + \mathcal{E}' + O'$$

$$\mathcal{E}' = \mathcal{E} + \frac{1}{2m_N} \beta O^2 - \frac{1}{8m_N^2} [O, [O, \mathcal{E}]]$$

$$-\frac{1}{8m_N^2} \beta O^4 - \frac{i}{8m_N^2} [O, \dot{O}]$$

$$O' = \frac{1}{2m_N} \beta [O, \mathcal{E}] - \frac{1}{3m_N^2} O^3 - \frac{1}{48m_N^3} [\beta O, [O, [O, \mathcal{E}]]]$$

$$+ \frac{i}{2m_N} \beta \dot{O} - \frac{i}{48m_N^3} [\beta O, [O, \dot{O}]]$$

The odd term is now $O(1/m_N)$. Applying a second unitary transformation $\psi'' = \exp(-\beta O'/2m_N)\psi''$

$$H'' = \beta m_N + \mathcal{E}'' + O''$$
\[ E'' = E' - \frac{1}{4m_N} \beta([\mathcal{O}, \mathcal{E}] + i\dot{\mathcal{O}})^2 \]
\[ \mathcal{O}'' = \frac{1}{2m_N} \beta[\mathcal{O}', \mathcal{E}'] + \frac{i}{2m_N} \beta \dot{\mathcal{O}}' \]  
(15)

The odd term is now \( \mathcal{O}(1/m_N^2) \). Applying a third unitary transformation

\[ \psi'' = \exp(-\beta \mathcal{O}''/2m_N) \psi'' \]
\[ \mathcal{H}''' = \beta m_N + \mathcal{E}'' + \mathcal{O}'' \]
\[ \mathcal{O}''' = \frac{1}{2m_N} \beta[\mathcal{O}''', \mathcal{E}'] + \frac{i}{2m_N} \beta \dot{\mathcal{O}}''' \]  
(16)

The odd term is now \( \mathcal{O}(1/m_N^3) \). Applying a fourth unitary transformation

\[ \psi''' = \exp(-\beta \mathcal{O}'''/2m_N) \psi''' \]
\[ \mathcal{H}'''' = \beta m_N + \mathcal{E}'' + \mathcal{O}(\frac{1}{m_N^3}) \]  
(17)

so we have an even Hamiltonian to the desired order.

We may note that the TFW transformations preserve charge conjugation symmetry, so the Hamiltonian Eq. (17) can be used for the nonrelativistic \( NN \) system, in contrast with previous work.

There is one worry, namely that the functional Jacobian of the transformations. However, one can plausibly argue they are 1. In the case of the chiral anomaly, the natural basis for expanding \( \psi \) was the eigenfunctions of the massless Dirac operator, which anticommuted with the generator of axial transformations \( \gamma_5 \). In our case the natural basis for expanding \( \psi \) should be \( \beta \) diagonal, since it anticommutes with the generators \( \beta \mathcal{O}, \beta \mathcal{O}', \beta \mathcal{O}'' \), and \( \beta \mathcal{O}''' \). However, \( \beta \) has no zero modes, so the anomaly should vanish.

Counterterms necessary to absorb loop divergences may be derived in the standard manner, by using the BPHZ scheme for instance with on-shell subtractions. In this way, the nucleon mass appearing in the expansion at tree level is the renormalized mass. Incidentally, one observes that the relativistic one-loop calculations yield terms of order \( p^2m_N/m_N^0 \). How these terms are generated after nonrelativistic reduction deserves further investigation. Furthermore, the \( \mathcal{O}(m_N^0) \) term from Eq. (17) gives

\[ \mathcal{H}_0 = -i \Gamma_0 - i g_{\sigma i} \Delta_i + \frac{m_N^2}{2 \Lambda} \beta (U + U^\dagger - 2) \]  
(18)

as the improved version of the static model. Does it account for the \( \Delta \)?

In terms of the present construction, the \( \pi N \) scattering amplitude can be constructed and partial wave unitarity tested. Explicit calculations using
the tree results show that the S31 wave gives a bound on the pion momentum to be $k \leq 0.28$ GeV. This is close to the bound established above using $\pi\pi$ scattering.

4 Conclusions

In $\pi\pi$ and $\pi N$ scattering, if we are to reach 300 MeV and beyond, there are three possible courses of action. Use generalized chiral perturbation theory. However, this tends to decrease predictive power. Try unitarization. However, this breaks crossing symmetry. The one we favor is the master formula approach developed recently. In this approach, the $\pi\pi$ scattering amplitude is reduced to a sum of Green’s functions and form factors, some of which are measurable. By making educated guesses about the unknown pieces, one may test chiral symmetry even at $\rho$ energies.

In a way, in $\pi\pi$ scattering say, one should not be worried if lowest order predictions of chiral symmetry are off by 20 % rather than 1.4 % for $p \sim 140$ MeV. However, one should be aware that the predictions can be significantly off already for $p \sim 300$ MeV.

Acknowledgements

The results in this work are dedicated to Mannque Rho for his sixtieth birthday. Mannque has inspired us throughout our careers, and we take this opportunity to thank him for his friendship and support. This work was supported in part by the US DOE grant DE-FG-88ER40388.

References

1. A.V. Manohar and H. Georgi, *Nucl. Phys.* B234 (1984) 189.
2. J.L. Petersen, *Phys. Rep.* C2 (1971) 155.
3. J. Bijnens and F. Cornet, *Nucl. Phys.* B296 (1988) 557; J.F. Donoghue, B.R. Holstein, and Y.C. Lin, *Phys. Rev.* D37 (1988) 2423.
4. J. Gasser and H. Leutwyler, *Ann. Phys.* 158 (1984) 142.
5. J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model*, Camb. Univ. Press. 1992.
6. J. Stern, H. Sazdjian and N.H. Fuchs, *Phys. Rev.* D47 (1993) 3814.
7. T.N. Truong, *Phys. Rev. Lett.* 22 (1988) 2526; D. Morgand and M.R. Pennington, *Phys. Lett.* B272 (1991) 134.
8. H. Yamagishi and I. Zahed, *Phys. Rev.* D53 (1996) 2288; *Ann. Phys.* 247 (1996) 292.
9. J. Gasser, M.E. Sainio and A. Svarc, *Nucl. Phys.* B307 (1988) 779.
10. E. Jenkins and A. Manohar, *Phys. Lett.* B255 (1991) 558.
11. V. Bernard, N. Kaiser, J. Kambor and U. Meissner, *Nucl. Phys.* B338 (1992) 315; G. Ecker and M. Mojmír, “Low-energy expansion of the Pion-Nucleon Lagrangian”, e-print: hep-ph/9508204.
12. H. Georgi, *Phys. Lett.* B240 (1990) 447.
13. S. Tani, Soryushiron Kenkyu, 1 (1949) 15 (in Japanese); *Prog. Theor. Phys.* 6 (1951) 267; L.L. Foldy and S.A. Wouthuysen, *Phys. Rev.* 78 (1950) 29.
14. J. Steele, H. Yamagishi and I. Zahed, hep-ph : 9707399 ; *Phys. Rev.* D 1998 in Press.
15. K. Fujikawa, *Phys. Rev. Lett.* 42 (1979) 1195; S.N. Vergeles as quoted in A.A. Migdal, *Phys. Lett.* 81B (1979) 38.