RING OF CONDITIONS FOR $\mathbb{C}^n$

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Abstract. The exponential sum (ES) is a linear combination of characters of an additive group $\mathbb{C}^n$. The exponential analytic set (EAS) is a set of common zeroes of a finite tuple of ESs. We consider ES and EAS as analogs of Laurent polynomial and of algebraic variety in complex torus $(\mathbb{C} \setminus 0)^n$. Respectively we construct the ring of conditions for $\mathbb{C}^n$ as an analog of the ring of conditions for $(\mathbb{C} \setminus 0)^n$. The construction of this ring is based on the definition of associated to EAS algebraic subvariety of some multidimensional torus and on the applying tropical algebraic geometry to this subvariety. Just as in the case of a torus, the ring of conditions is generated by hypersurfaces. This preprint is an extended summary of the article proposed to "Izvestiya: Mathematics".

1. ESs, EASs, windings and models

Exponential sum (ES) is a function in $\mathbb{C}^n$ of the form

$$f(z) = \sum_{\lambda \in \Lambda \subset \mathbb{C}^n, c_\lambda \in \mathbb{C}} c_\lambda e^{\langle z, \lambda \rangle},$$

where $\Lambda$ is a finite subset of the dual space $\mathbb{C}^n^*$. The set $\Lambda$ is called the support of ES. The Newton polytope of ES is a convex hull of its support. If $\Lambda \subset \text{Re} \mathbb{C}^n^*$ then we say that $f$ is a quasialgebraic ES. The set of common zeroes of a finite tuple of (quasialgebraic) ESs is called (quasialgebraic) exponential analytic set (EAS). The ring of ESs consists of linear combinations of characters of the additive group $\mathbb{C}^n$, i.e. this ring is similar to the ring of Laurent polynomials, which are linear combinations of characters of a torus $(\mathbb{C} \setminus 0)^n$. Focusing on this similarity, we construct the ring of conditions for the intersection theory for quasialgebraic EASs. The construction is based on the definition of some algebraic variety, associated to EAS, and on the applying tropical algebraic geometry. The intersection theory for arbitrary EASs will be described in the following publications.

Let $G \subset \mathbb{C}^n^*$ be a finitely generated subgroup. Suppose that $G$ contains some basis of a dual space $\mathbb{C}^n^*$. Let $T_G = (\mathbb{C} \setminus 0)^q$ where $q = \text{rank} \ G$. Choosing a basis $\lambda_1, \ldots, \lambda_q$ of $G$ we consider the homomorphism

$$\omega_G: z \mapsto (e^{\langle z, \lambda_1 \rangle}, \ldots, e^{\langle z, \lambda_q \rangle}) \in T_G$$

of the additive group $\mathbb{C}^n^+$ to the torus $T_G$.

Definition 1. The image $\omega_G(\mathbb{C}^n)$ is called the standard winding of the torus $T_G$, and the mapping $\omega_G: \mathbb{C}^n \to T$ itself is called the mapping of standard
winding. If $\omega_G(\text{Im } \mathbb{C}^n)$ is contained in the compact subtorus of the torus $\mathbb{T}_G$ (i.e. if $G \subset \text{Re } \mathbb{C}^{n*}$), then we say that the winding is quasialgebraic.

**Corollary 1.** The torus $\mathbb{T}_G$ is a torus of characters of the group $G$. The invariant definition of the mapping of standard winding is $\omega_G(z)(g) = e^{(z,g)}$.

**Corollary 2.** The standard winding is dense in Zariski topology.

**Corollary 3.** Let $E_G$ be a ring of ESs with supports in $G$. Then the mapping $\omega^*_G: \mathbb{C}[\mathbb{T}_G] \to E_G$ is an isomorphism of the rings.

**Definition 2.** Let $X$ be an EAS with ideal of equations $I \subset E_G$. Keeping the notation $I$ for the corresponding ideal of $\mathbb{C}[\mathbb{T}_G]$ we say that the zero variety $M_G \subset \mathbb{T}_G$ of $I$ is a model of EAS $X$.

**Definition 3.** If the variety $M_G$ is equidimensional, then EAS $X$ is also called equidimensional. Denote $\text{codim } M_G$ by $\text{codim}_a X$ and call algebraic codimension of EAS $X$.

Note that both the equidimensionality and the algebraic codimension of $X$ do not depend on the choice of the group $G$, such that the equations of EAS $X$ are contained in the ring $E_G$. Any EAS is a union of a finite number of equidimensional EASs of different algebraic codimensions. Next, by default, any EAS is assumed to be equidimensional. Equidimensional EAS $X$ of algebraic codimension $n$ is an analogue of a zero-dimensional algebraic variety. The set of points of such EAS is infinite. For example, the set of zeros of the function $e^z - 1$ in $\mathbb{C}^1$ is $2\pi i \mathbb{Z}$. An analogue of the number of points of zero-dimensional algebraic variety is a weak density $d_w(X)$, see the Definition 5.

**Example 1** (см. [Z02, BMZ07, K97]). Let $X$ be an EAS with the equations $f = g = 0$, $f, g \in E_G$. If $f, g$ have no common divisor in the ring $E_G$, then $\text{codim}_a X = 2$, otherwise $\text{codim}_a X = 1$. For example, the algebraic codimension of the point $0 \in \mathbb{C}$, considered as the EAS with the equations $e^z - 1 = e^{\sqrt{2}z} - 1 = 0$, is equal to 2. Therefore, the codimension of EAS, as an analytic set, can be less than $\text{codim}_a X$. Let $(X, z)$ be an irreducible germ of EAS $X$ at $z \in X$. If the codimension of $(X, z)$ less than $\text{codim}_a X$, then the germ is said to be atypical. It is known, that any atypical germ of EAS belongs to some proper affine subspace of $\mathbb{C}^n$. In particular, any atypical component of EAS of algebraic codimension 2 in $\mathbb{C}^2$ is an affine line.

**2. Tropicalization and intersection index of EASs**

Next, we consider quasialgebraic ESs only. We use the notations:

$\mathcal{T}_G$ is a Lie algebra of $\mathbb{T}_G$  
$V = \text{Re } \mathcal{T}_G$ is the space of one-parameter subgroups $\mathbb{T}_G$: $N = \dim V$  
$\mathbb{R}^n = \text{Re } \mathbb{C}^n$  
$s_G: \mathbb{R}^n \to \text{Re } \mathcal{T}_G$ is a restriction of differential $d\omega_G$ to the space $\mathbb{R}^n$. 

Here we define the tropicalization of EAS. Its independence from the choice of the group $G$ is proved in the section \[4\] The tropical notions used below are defined in Section \[8\]

Let $\mathcal{K} \subset \text{Re} \mathcal{T}_G$ be a tropical fan of algebraic variety $M_G \subset \mathcal{T}_G$, $L_G = s_G(\mathbb{R}^n) \subset \text{Re} \mathcal{T}_G$, and let $\mathcal{L}_G$ be a tropical fan consisting of a cone $L_G$ with the Euclidian weight $w(L_G) = 1$; see subsection \[8.1\] Consider the product $\mathcal{L}_G \cdot \mathcal{K}$ of Euclidian tropical varieties. The support of a tropical fan $\mathcal{L}_G \cdot \mathcal{K}$ is contained in $L_G$.

Let $s: U \to V$ be a linear operator. In Subsection \[8.2\] for any tropical variety $M \subset V$ we define the tropical variety $s^*(M)$ in $U$, called the pull back of $M$.

**Definition 4.** Let $M_G$ be a model of EAS $X$ and $\mathcal{K}$ be a tropicalization of algebraic variety $M_G$. The tropical variety $X^{\text{trop}} = s_G^*(\mathcal{L}_G \cdot \mathcal{K}) \subset \mathbb{R}^n$ is called a tropicalization of EAS $X$.

Let $\Delta \subset \mathbb{R}^{n*}$ be a convex polyhedron and let $K_\Delta \subset \mathbb{R}^n$ be a dual cone of a face $\Lambda$. The set of cones $\mathcal{K}_{\Delta,k} = \{K_\Lambda: \Lambda \subset \Delta, \dim \Lambda \geq k\}$ form the $(n-k)$-dimensional fan of cones. For $\dim \Lambda = k$ we put $w(K_\Lambda)$ equal to $k$-dimensional volume $\text{vol}_k(\Lambda)$ of $\Lambda$. Then $\mathcal{K}_{\Delta,k}$ is a Euclidean tropical fan.

**Corollary 4.** Let $X = \{z \in \mathbb{C}^n: f(z) = 0\}$ be a quasialgebraic exponential hypersurface. Then $X^{\text{trop}} = \mathcal{K}_{\Delta,1}$ where $\Delta$ is a Newton polyhedron of ES $f$.

**Proof.** Let $\Gamma$ be a Newton polyhedron of Laurent polynomial $F = \omega_G^*(f)$. Then the tropicalization of algebraic variety $M_G = \{g \in \mathcal{T}_G: F(g) = 0\}$ is a fan $\mathcal{K}_{\Gamma,1} \subset V$ consisting of cones, dual to faces of $\Gamma$. Note that $s_G^*\Gamma = \Delta$, where $s_G^* (\text{Re} \mathcal{T}_G)^* \rightarrow \mathbb{R}^{n*}$ is a linear operator adjoint to $s_G$. Now the statement follows from Theorem \[10\] (3); see also \[4\].

If codim $X = n$, then dim $X^{\text{trop}} = 0$. In this case, denote by $d_w(X)$ the weight of the zero cone in $X^{\text{trop}}$.

**Definition 5.** We call $d_w(X)$ a weak density of EAS $X$.

**Definition 6.** Let $X_1, \ldots, X_k$ be EASs of total algebraic codimension $n$. The weight of a zero cone in a 0-dimensional tropical fan $X_1^{\text{trop}} \cdots X_k^{\text{trop}}$ is denoted by $I(X_1, \ldots, X_k)$ and called the intersection index of EASs $X_1, \ldots, X_k$.

The following statement is a quasialgebraic analogue of BKK theorem; see also \[Kh97, K81\].

**Theorem 1.** Let $X_1, \ldots, X_n$ be zero surfaces of quasialgebraic EASs $f_1, \ldots, f_n$ with Newton polyhedra $\Delta_1, \ldots, \Delta_n$. Then

$$I(X_1, \ldots, X_n) = \frac{n!}{(2\pi)^n} V_n(\Delta_1, \ldots, \Delta_n),$$

where $V_n(\Delta_1, \ldots, \Delta_n)$ is a mixed volume of polyhedra.

**Proof.** Follows from the tropical BKK theorem, see, for example, \[EKK20\] [Theorem 3.1.3].
3. Ring of conditions $\mathcal{E}_G$

Let $\mathcal{V}(E)$ and $\mathcal{V}(V)$ be the rings of Euclidean tropical varieties in $E$ and $V$, $s: E \to V$ be a linear operator. In Subsection 8.2 the mapping of an inverse image $s^*: \mathcal{V}(V) \to \mathcal{V}(E)$ is defined. Below we use the following properties of this mapping:

1. $s^*$ is a homomorphism of $\mathbb{R}$-algebras
2. $s^*$ preserves codimensions of varieties
3. For any linear operator $u: Z \to E$ we have $(s \cdot u)^* = u^* \cdot s^*$
4. Let $s': V^* \to E^*$ be an adjoint to $s$ linear operator and $\Delta \subset V^*$ be a convex polyhedron. Then for any $k \leq \dim V$ we have $s^*K_{\Delta,k} = K_{s^*\Delta,k}$.
5. Let $\rho: \mathbb{T} \to \mathbb{H}$ be a homomorphism of tori and let $M \subset \mathbb{H}$ be an algebraic variety. Denote by $Y_{tr}$ the tropical fan of algebraic variety $Y$. Then $(\rho^{-1}M)_{tr} = s^*M_{tr}$, where $s = d\rho: \text{Re} \mathcal{T} \to \text{Re} \mathcal{H}$.

**Lemma 3.1.** Tropicalizations of EASs in the ring $\mathcal{V}(\mathbb{R}^n)$ forms a semigroup in addition and a semigroup in multiplication.

**Proof.** The set of EAS tropicalizations is a set of tropical fans of the form \{s\_G\mathcal{K}\}, where $\mathcal{K}$ is a tropicalization of some algebraic variety. Therefore, the statement follows from the well-known property of algebraic tropicalizations:

if $\mathcal{P}, \mathcal{Q}$ are tropicalizations of algebraic varieties $P, Q$, then for a general $g \in \mathbb{T}_G$ the tropical varieties $\mathcal{P} + \mathcal{Q}$ and $\mathcal{P} \cdot \mathcal{Q}$ are tropicalizations of respectively $gP \cup Q$ and $gP \cap Q$. □

**Theorem 2.** Integer linear combinations of EAS tropicalizations form a subring $\mathcal{E}_G$ of the ring $\mathcal{V}(\mathbb{R}^n)$. The $\mathbb{Q}$-algebra $\mathcal{E}_G \otimes \mathbb{Q}$ is generated by tropical fans of the form $\mathcal{K}_{\Delta,1}$, where $\Delta \subset \mathbb{R}^{n\ast}$ is a convex polyhedron with vertices at points of the group $G$.

**Proof.** Integer linear combinations of tropicalizations of algebraic varieties form a subring $\mathcal{R}$ of the ring $\mathcal{V}(\text{Re} \mathcal{T}_G)$. In addition, the $\mathbb{Q}$-algebra $\mathcal{R} \otimes \mathbb{Q}$ is generated by tropical fans of the form $\mathcal{K}_{\Lambda,1}$, where $\Lambda \subset (\text{Re} \mathcal{T}_G)^\ast$ is a convex polyhedron with vertices at integer points; see [EKK20]. The linear operator $s\_G': (\text{Re} \mathcal{T}_G)^\ast \to \mathbb{R}^{n\ast}$, adjoint to $s\_G$, translates integer points of $(\text{Re} \mathcal{T}_G)^\ast$ into points of the group $G \subset \mathbb{R}^{n\ast}$. Therefore, the statement follows from the given above properties [1] and [4] of the mapping $s\_G^\ast$. □

Let $G$ be an additive subgroup in a real $N$-dimensional vector space $E$. Now we define a ring of convex polyhedra with vertices at the points of the group $G$. First consider the space $H$ of virtual convex polyhedra with vertices at points of $G$ (recall that a virtual polyhedron is the formal difference of two convex polyhedra). Let $S(H) = \sum_{m \geq 0} S_m(H)$ be a symmetric algebra of $H$. For $S_N(H) \ni s = \Delta_1 \cdot \ldots \cdot \Delta_N$ we set $I(s)$ equal to the mixed volume of $\Delta_1, \ldots, \Delta_N$. We associate with the linear functional $I: S_N(H) \to \mathbb{R}$ the homogeneous ideal $J \subset S(H)$ generated by the following sets of generators: 1) $\ker I$, 2) $\sum_{m \geq N} S_m(H)$ and 3) $\{s \in S_k(H) \mid s \cdot S_{N-k}(H) \subset \ker I, k = 4$.
1, \ldots, N − 1}. Note that the ideal \( J \) does not depend on the choice of Lebesgue measure in the space \( E \).

**Definition 7.** The ring \( \text{Pol}(E; G) = S(H)/J \) is called the ring of convex polyhedra with vertices in \( G \).

**Theorem 3.** The mapping \( \Delta \mapsto \mathcal{K}_{\Delta,1} \) extends to the ring isomorphisms

\( (i) \) \( \text{Pol}(E; E) \to V(E^*) \) for \( G = E \)

\( (ii) \) \( \text{Pol}(E; \mathbb{Z}^N) \otimes \mathbb{Q} \to \mathbb{Q}(E^*) \), where \( \mathbb{Q}(E^*) \) is a ring of rational tropical varieties (see Definition 16)

\( (iii) \) \( \text{Pol}(\mathbb{R}^n; G) \otimes \mathbb{Q} \to \mathcal{E}_G \)

**Proof.** The statements (i), (ii) see in [EKK20]. The statement (iii) follows from Theorem 2. \( \square \)

**Corollary 5.** Let \( X, Y \) be equidimensional \( \text{EASs} \), \( \text{codim}_a X = \text{codim}_a Y = k \). Then the following conditions are equivalent

(1) \( X_{\text{trop}} = Y_{\text{trop}} \)

(2) for any \( \text{EAS} \) \( Z \) of algebraic codimension \( n − k \), \( I(X, Z) = I(Y, Z) \)

**Proof.** It follows from Definition 6 that (1) \( \Rightarrow \) (2). On the other hand, it follows from (2) that

\[ \forall \mathcal{K} \in \mathcal{E}_G : (X_{\text{trop}} − Y_{\text{trop}}) \cdot \mathcal{K} = 0. \]

Now the statement follows from the nondegeneracy of pairing in the ring \( \text{Pol}(\mathbb{R}^n; G) \). This non-degeneracy follows from the non-degeneracy of pairing for tropical varieties (see Theorem 8) and from the standard properties of the mixed volume of polyhedra. \( \square \)

4. **Ring of conditions \( \mathcal{E}^{\text{quasi}} \)**

In this Section we prove that 1) the notion of tropicalization is independent from the choice of the ring \( E_G \), containing the equations \( \text{EAS} \), and 2) any tropical variety in \( \mathbb{R}^n \) is a tropicalization of some \( \text{EAS} \). Thus, the ring \( \mathcal{E}^{\text{quasi}} \), formed by tropicalizations of \( \text{EASs} \), coincides with the ring of tropical varieties in \( \mathbb{R}^n \).

**Proposition 1.** Let \( G \subset H \subset \mathbb{R}^{n*} \), \( \pi_{H,G}: \mathbb{T}_H \to \mathbb{T}_G \) - the character restriction from \( H \) to \( G \). Then the mappings \( s^*_G, s^*_H, (d\pi_{H,G})^*: V(\text{Re} \mathcal{T}_G) \to V(\mathbb{R}^n) \) coincide.

**Proof.** By definition \( s_G = s_H \cdot d\pi_{H,G} \), i.e. the diagram

\[ \begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{s_H} & \text{Re} \mathcal{T}_H \\
\downarrow s_G & & \downarrow \text{Re} \mathcal{T}_G \\
& \text{Re} \mathcal{T}_G & \\
\end{array} \] (4.1)
is commutative. It follows from (3) that the dual diagram

\[
\begin{array}{ccc}
\mathcal{V}(\mathbb{R}^n) & \mathcal{V}(\text{Re} \, \mathcal{T}_H) & \mathcal{V}(\text{Re} \, \mathcal{T}_G) \\
\downarrow s_H & & \downarrow (d\pi_{H,G})^* \\
\downarrow s_G & & \\
\mathcal{V}(\text{Re} \, \mathcal{T}_G) & & \\
\end{array}
\]  

(4.2)

is also commutative. □

**Corollary 6.** The tropicalization \(X_{\text{trop}}\) does not depend on the choice of the group \(G\), containing the equations of EAS \(X\).

**Proof.** Follows from (4.2). □

**Definition 8.** A subring of the ring \(\mathcal{V}(\mathbb{R}^n)\), consisting of EAS tropicalizations, denote by \(\mathcal{E}_{\text{quasi}}\) and call quasialgebraic ring of conditions for an affine space.

**Theorem 4.** \(\mathcal{E}_{\text{quasi}} = \mathcal{V}(\mathbb{R}^n)\).

We start with the proof of the following statement.

**Lemma 4.1.** The ring \(\mathcal{V}(\mathbb{R}^n)\) (regarded as a \(\mathbb{Z}\)-algebra) generated by elements of the form \(s_G^*(K_{\Lambda,1})\), where \(G\) and \(\Lambda\) change respectively in the set of finitely generated subgroups of the space \(\mathbb{R}^{n*}\) and in the set of convex polyhedra with vertices in the character lattice of the torus \(\mathbb{T}_G\).

**Proof.** Let \(\Delta \subset \mathbb{R}^{n*}\) be a convex polyhedron. By construction, the linear operator \(s_G'\colon (\text{Re} \, \mathcal{T}_G)^* \rightarrow \mathbb{R}^{n*}\), adjoint to \(s_G\), takes the points of the character lattice of the torus \(\mathbb{T}_G\) to the points of the group \(G\). Choose the group \(G\), containing the vertices of the polytope \(\Delta\). Let \(\chi_\delta\) be a character of \(\mathbb{T}_G\), such that \(s_G'(\chi_\delta) = \delta\). We denote by \(\Lambda\) the convex hull of the characters \(\chi_\delta\). Then \(\Lambda\) is a convex polyhedron in the space \((\text{Re} \, \mathcal{T}_G)^*\), and \(s_G'(\Lambda) = \Delta\). From here, according to [4] \(\mathcal{K}_{\Delta,1} = s_G^*(\mathcal{K}_{\Lambda,1})\).

Tropical varieties of the form \(\mathcal{K}_{\Delta,1}\) generate the \(\mathbb{R}\)-algebra \(\mathcal{V}(\mathbb{R}^n)\) (see Theorem 2). Now it remains to notice that \(r \cdot \mathcal{K}_{\Delta,1} = \mathcal{K}_{r \cdot \Delta,1}\) for any real \(r > 0\). □

**Proof of Theorem 4.** It follows from Lemma 4.1 because \(\mathcal{K}_{\Delta,1}\) is the tropicalization of an exponential hypersurface with the Newton polyhedron \(\Delta\).

**Corollary 7.** \(\mathbb{Z}\)-algebra \(\mathcal{E}_{\text{quasi}}\) is generated by the tropicalizations of exponential hypersurfaces.

**Proof.** Follows from Lemma 4.1. □
5. **Domains of relatively full measure**

This section defines some of the notions used in the formulation of EASs intersection theorems.

**Definition 9.** Let $Y$ be a subset of a finite-dimensional Euclidean vector space $E$, $B_r \subset E$ be a ball of radius $r$ with center at $0$, and $\sigma_n$ be a volume of the unit ball in $\mathbb{R}^n$. Denote by $N(Y,r)$ the number of isolated points of $Y \cap B_r$. If $\lim_{r \to \infty} \frac{N(Y,r)}{\sigma_n r^n}$ exists, then we denote it by $d_n(Y)$ and call the $n$-density of $Y$.

**Example 2.** If EAS $X \subset \mathbb{C}^1$ is given by the equation $f(z) = 0$, then the $1$-density $d_1(X)$ exists and equals to $\frac{p}{2\pi}$, where $p$ is a semiperimeter of the Newton polygon of $f$. If $f(z) = e^{\alpha z} - c$, then $d_1(X) = \frac{|\alpha|}{2\pi} (\text{the perimeter of the polygon "segment" is considered equal to its doubled length}).$

**Definition 10.** (1) Let $Z \subset E$ be a lattice in the space $E$ with an integer positive multiplicity $m(Z)$, and let $X \subset E$ be a set of points with multiplicities. We call $X$ an $\varepsilon$-perturbation of the shifted lattice $z + Z$, if 1) $X$ belongs to $\varepsilon$-neighborhood $(z + Z)$ of this lattice and 2) in $\varepsilon$-neighborhood of any point $x \in z + Z$ contains exactly $m(Z)$ points.

(2) If the sets $X_1, \ldots, X_m$ are $\varepsilon$-perturbations of shifted lattices $z_j + Z_j$, then the set $\bigcup_{1 \leq j \leq m} X_j$ we call an $\varepsilon$-perturbation of a union of shifted lattices $\bigcup_{1 \leq j \leq m} (z_j + Z_j)$.

**Corollary 8.** Let $X$ be an $\varepsilon$-perturbation of a union of $z_1 + Z_1, \ldots$ and let $\forall j$: rank $Z_j = n$. Then the $n$-density $d_n(X)$ exists and equal $\sum_j d_n(Z_j)$.

Let $\mathcal{I} = \{I\}$ be a finite set of proper subspaces in finite dimensional real vector space $E$ and $B_7 = E \setminus \bigcup_{I \in \mathcal{I}} I$. Denote by $B_{7,1}, B_{7,2}, \ldots$ the connected components of $B_7$. For $0 < R \in \mathbb{R}$ we denote by $B^R \subset E$ the set of points located at a distance $\geq R$ from $\bigcup_{I \in \mathcal{I}} I$.

**Definition 11.** We say that $U \subset E$ is a domain of relatively full measure (RFD), if $\mathcal{I}$ and $R > 0$ exist, such, that $U \supset B^R \supset B_7$. The set of subspaces $\mathcal{I} = \{I\}$ we will call the base of RFD. If an integer lattice is given in the space $E$, and all the subspaces $I \in \mathcal{I}$ are rational (i.e., generated by lattice points), then the base is called rational. We say that RFD with a rational base is a rational RFD.

Let’s list some corollaries of Definition **11**:

**Corollary 9.** (1) The union and intersection of RFDs is also RFD. The rationality property of RFD under unions and intersections of RFDs is preserved.

(2) The property of the domain to be RFD is independent of the choice of metric in the space $E$.

(3) If the subspace $L \subset E$ does not belong to the base of RFD $U$, then $U \cap L$ is RFD in the space $L$. 


The inverse image of RFD under a surjective linear map is RFD.

Domains $B^R_{3,i} = B^R_3 \cap B_{3,i}$ are the connected components of RFD $B^R_3$.

6. INTERSECTIONS OF EASs

Throughout, it is assumed that, if a finite set of EASs and the group $G \subset \mathbb{R}^{n*}$ are involved in the statement, then all these EASs are given by equations from the ring $E_G$. For a fixed $G$ such EASs one-to-one correspond to algebraic subvarieties of the torus $\mathbb{T}_G$ (recall that the variety $M_G$ corresponding to EAS $X$ is called the model $X$). Therefore, the action of the torus on varieties is also defined on EASs. The action of the element $t \in \mathbb{T}_G$ on EAS $X$ is denoted by $t^\ast X$. This action is a "toric shift", those.

Let $X$ be an EAS of algebraic codimension $n$. The following theorem states that there exists a finite set of proper subspaces $I$ of the space $Re \mathbb{T}_G$, such that if $R \gg 0$ and $Re \log t \in B^R_I$ (see Definition [11]) then the toric shift $X^t$ is a small perturbation of a finite union of shifted $n$-dimensional lattices, located in $Im \mathbb{C}^n$.

**Theorem 5.** There is $I$ such that to each of the connected components $B_{3,1}, B_{3,2}, \ldots$ of the domain $B_3$ there corresponds a finite set of $n$-dimensional lattices

$$\{L_{i,j} \subset \mathbb{C}^n: j = 1, \ldots, N_i\}$$

(the lattices $L_{i,1}, L_{i,2}, \ldots$ may sometimes match and equipped with integer positive multiplicities), such that for $R \gg 0$ and $Re \log t \in B^R_{3,i}$ we have:

1. EAS $X^t$ is an $\varepsilon$-perturbation of the union of shifted lattices
   $$z_1(t) + L_{i,1}, \ldots, z_{N_i}(t) + L_{i,N_i},$$
   where functions $z_j(t)$ are continuous and $\varepsilon \to 0$ if $R \to \infty$

2. $n$-density $d_n(X^t)$ independent of the choice $B_{3,i}$.

**Theorem 6.** Let $\text{codim}_aX + \text{codim}_aY = n$. Then there is a finite set of subspaces $I = \{I \subset \mathbb{R}^n\}$, such that the following is true. If $R$ is large enough, then for all $z \in B^R_3 + Im \mathbb{C}^n$ EASs $(z + X)\cap Y$ are equidimensional, their algebraic codimensions are equal to $n$, and weak densities are the same.

Applying tropical intersection index properties leads to the next more familiar in algebra statement of the previous theorem.

**Corollary 10.** There is a quasi-algebraic exponential hypersurface $Z \subset \mathbb{C}^n$, such that for $w \notin Z$, the weak densities of all EASs $(w + X) \cap Y$ are the same.

7. PROOF OF THEOREMS 5, 6

7.1. **Approximation by toric tongues.** The proof of the theorems is based on the use of approximation of algebraic variety by toric tongues. The following notation is used below:
- $V = \text{Re} T_G$, $N = \dim V$
- $K$ is a rational convex polyhedral cone in $V$
- $V_K \subset V$ is a subspace generated by $K$
- $T_K \subset T_G$ is the subtorus generated by the exponentials of the cone $K$
- $M \subset T_G$ is a $k$-dimensional algebraic variety
- $K$ is the tropical fan of the variety $M$
- $K_m \subset K$ is the subfan of cones of dimension $\leq k - m$
- $O_R(K) = \{ \tau \in T_G : \text{Re} \log \tau \notin (\text{supp} K^1)_R \}$.

**Definition 12.** The subset of $T_G$

$$t_{K,\tau} = \tau \exp(K + iV_K) \subset \tau T_K$$

is called a toric tongue. Cone $K$ and $\tau \in T_G$ called respectively the base and the shift of the tongue $t_{K,\tau}$.

Let $S, L$ be subtori of $T_G$, such that $\dim S + \dim L = N$ and $\#(L \cap S) = 1$, and let $U$ be an open domain in some shift of the torus $S$. Consider a domain

$$U_\varepsilon = \{ l \cdot t | l \in L_\varepsilon, t \in U \},$$

where $L_\varepsilon$ is the $\varepsilon$-neighborhood of a unit in $L$ and define the mapping

$$\pi_\varepsilon : U_\varepsilon \to U$$

as $\pi_\varepsilon : l \cdot t \mapsto t$.

**Definition 13.** The subdomain $M \cap U_\varepsilon$ of the variety $M$ we call $\varepsilon$-perturbation of the domain of $U$, if the restriction $\pi_\varepsilon$ to $M \cap U_\varepsilon$ is a finite sheeted unramified covering of $U$.

**Definition 14.** Let $T(M)$ be a finite set of pairwise disjoint $k$-dimensional toric tongues. We call $T(M)$ an approximating set of tongues $M$ with an approximating $k$-dimensional fan $K$, if the following is true

(i) $\dim K = k$ and the set of tongue bases coincide with the set of $k$-dimensional cones $K \in \mathcal{K}$

(ii) for any $\varepsilon$ there is $R$, such that in $O_R(K)$ the variety $M$ coincides with the union $\varepsilon$-perturbations of all domains of the form $O_{R-1}(K) \cap t_{K,\tau}$, where

$$t_{K,\tau} \in T(M).$$

We will call these $\varepsilon$-perturbations the perturbations of toric tongues. The degree of covering of a tongue from Definition 13 is called a weight of tongue.

**Theorem 7.** A finely divided tropical fan of the algebraic variety $M$ is the approximating fan of $M$.

7.2. **Proof Theorem 5.** Let $M \subset T_G$ be a model of EAS $X$, $\text{codim}_a X = n$. Recall that EAS $X^g = \omega_G^{-1}(g^{-1}M)$, where $\omega_G : \mathbb{C}^n \to T_G$ is the mapping of standard winding, called the toric shift EAS $X$. Let $K$ and $T(M) = \{ t_{K,\tau} \}$ be respectively an approximating fan and a set of approximating tongues for $M$. Then $T(g^{-1}(M)) = \{ t_{K,g^{-1}\tau} \}$ is the set of approximating tongues for $g^{-1}M$. The approximating fan for $g^{-1}M$ equal $K$. Recall that $L_G = \text{Re} \log \omega_G(\mathbb{C}^n) \subset V$. We give a sequence of simple statements, leading to the proof of Theorem 5.
(1) If \( \dim K = k \) and the intersection \( L_G \cap V_K \) is transversal, then \( \omega_G^{-1} T_K \) is a \( n \)-dimensional lattice in the space \( \text{Im } \mathbb{C}^n \).

(2) Consider the subset \( D(L_G, \mathcal{K}) \subset V \), consisting of points \( v \), such that the intersection \( (v + L_G) \cap \text{supp } \mathcal{K} \) is nonempty and not transversal. Obviously, \( D(L_G, \mathcal{K}) \) belongs to the union of a finite set \( \mathcal{J} \) of proper subspaces of the space \( V \). Since, by construction, \( \mathcal{K}^1 \subset D(L_G, \mathcal{K}) \), then

\[ \text{if } v \in V \setminus D(L_G, \mathcal{K})_R, \text{ where } R \text{ is big enough then the affine subspace } v + L_G \subset V \text{ is located at a sufficiently large distance from the skeleton } \mathcal{K}^1. \]

(3) Let \( \mathcal{B} \) be a connected component of \( V \setminus D(L_G, \mathcal{K}) \) and \( v \in \mathcal{B} \). Denote by \( \mathcal{K}(\mathcal{B}) \) the set of cones \( K \in \mathcal{K} \), such that \( (v + L_G) \cap K \neq \emptyset \). Then

\[ \text{if } \text{the set is independent of the choice of } v \in \mathcal{B}. \]

\[ \text{iii } \text{the set } \omega_G^{-1} \left( \bigcup_{K \in \mathcal{K}(\mathcal{B})} T_K \right) \text{ is a union of a finite set of shifts of } n \text{-dimensional lattices} \]

\[ Z(\mathcal{B}) = \{ Z_{1,\mathcal{B}}, \ldots, Z_{N_B,\mathcal{B}} \} \] (7.1)

in \( \text{Im } \mathbb{C}^n \); cm. (1).

(4) Let \( T(M; \mathcal{B}) = \{ t_{K,\mathcal{B}} \in T(M) : K \in \mathcal{K}(\mathcal{B}) \} \) and

\[ U_{R,\mathcal{B}} = \{ g \in \mathbb{T}_G : \text{Re log } g \in V \setminus D(L_G, \mathcal{K})_R \}. \]

If \( R \) is large enough, then for \( g \in U_{R,\mathcal{B}} \) the following is true

\[ \text{if } t_{K,\mathcal{B}} \in T(M; \mathcal{B}) \text{, then the intersection } g\omega_G(\mathbb{C}^n) \cap t_{K,\mathcal{B}} \text{ is transversal and consists of a single point, else } g\omega_G(\mathbb{C}^n) \cap t_{K,\mathcal{B}} = \emptyset. \]

\[ \text{ii } (g\omega_G)^{-1} \left( \bigcup_{t_{K,\mathcal{B}} \in T(M; \mathcal{B})} t_{K,\mathcal{B}} \right) = \bigcup_{1 \leq i \leq N_B} (z_i(g) + Z_{i,\mathcal{B}}), \text{ where the functions } z_i : U_{R,\mathcal{B}} \to \mathbb{C}^n \text{ are continuous.} \]

Now applying Theorem [4] we get that EAS \( (g\omega)^{-1}(M) \) is a small perturbation of the union of shifted lattices from [(4), ii]. The first assertion of the theorem [5] is proved. The second statement is that the \( n \)-density of the union of lattices from (7.1) independent of the connected component \( \mathcal{B}. \)

The first statement of the theorem is true for any (including rational) subspace \( L_G \subset \text{Re } \mathbb{T}_G \) (the irrationality property of the space \( L_G \) was not used in the proof). For a rational \( L_G \), the second statement is equivalent to the balance condition for the weights of a fan \( \mathcal{K} \); see Section [8]. Now the second statement follows from the continuous dependence of the density of the union of the lattices from the set \( Z(\mathcal{B}) \) from the mapping of standard winding \( \omega_G \).

7.3. Proof of Theorem [8] Let \( \mathcal{P}, \mathcal{Q} \) be tropicalizations of the models \( P, Q \) of equidimensional EASs \( X, Y \). Recall that (this is proved in tropical geometry) there is RFD \( B_\mathcal{P} \subset V \) with a rational base \( \mathcal{J} \) (see the definition [11]), such that for sufficiently large \( R \) the following is true: if \( \text{Re log } g \in B_\mathcal{J}^R \), then the variety \( P \cap gQ \) is equidimensional, and its tropicalization is equal to \( \mathcal{P} \cdot \mathcal{Q} \). Therefore, if

\[ \text{Re log } \omega_G(z) \in B_\mathcal{J}^R, \] (7.2)
then EAS \((z + X) \cap Y\) is equidimensional and \((X \cdot Y)^\text{trop} = X^\text{trop} \cdot Y^\text{trop}\). Hence, for \(\text{codim}_a X + \text{codim}_a Y = n\) we get that for all such \(z\) the weak density \(d_w((z + X) \cap Y)\) is constant.

Consider a set subspaces \(\mathcal{I} = \{ I \subset \mathbb{R}^n : I = s_{G}^{-1}(J), J \in \mathcal{I} \}\). These subspaces are proper because the standard winding \(\omega_G(\mathbb{C}^n)\) is everywhere dense. Therefore, \(B_I\) is RFD with base \(\mathcal{I}\) and for all \(z \in B_I^R\) the condition (7.2) is satisfied. The theorem is proved.

**Proof of Corollary 10.** From tropical algebraic geometry it’s known that there exists an algebraic hypersurface \(M \subset \mathbb{T}_G\), such that for any \(g /\in M\) the tropicalization of the variety \(gP \cap Q\) is equal to \(P \cdot Q\). Using Theorem 6, we obtain the statement for the exponential hypersurface \(Z = \omega^{-1}_G M \subset \mathbb{C}^n\).

8. **Brief Overview of Tropical Geometry Essentials**

Here we give a brief summary of basic tropical notions and, then, a description of the construction for a pull back \(s^*K\) of tropical fan \(K \subset U\) with respect to the linear operator \(s: V \rightarrow U\), where \(V\) is a vector space with the fixed orientation. The properties of pull back mapping apply to the proof of our main results about EASs; see Section 3. We start with several equivalent definitions of tropical variety.

8.1. **Tropical varieties.**

8.1.1. **Definition of tropical variety.** Let \(K\) be a fan of cones in \(N\)-dimensional vector space \(V\), \(\dim K = k\). For \(K \in K\) we denote by \(V_K \subset V\) the subspace generated by the cone \(K\). The function \(K \mapsto W(K) \in \wedge^q V^*\) on the set of oriented \(p\)-dimensional cones in \(K\) is called a \(p\)-chain of degree \(q\) if \(W(K)\) changes its sign when the orientation of the cone \(K\) changes. As usual, we define a \((p - 1)\)-chain \(dW\), called the boundary of the \(p\)-chain \(W\). A chain \(W\) is said to be closed if \(dW = 0\).

**Definition 15.** A \(k\)-dimensional fan \(K\) with a closed \(k\)-chain of degree \(N - k\) is said to be tropical if \(V_K \subset \ker W(K)\) for any \(k\)-dimensional cone \(K \in K\), i.e. \(W(K)(v_1 \wedge \ldots \wedge v_{N-k}) = 0\) for any \(v_1 \in V_K\). We call \(W(K)\) the weight of the cone \(K\).

Note that \(W(K)\) can be seen as an even volume form in the space \(V/V_K\).

Any partition of the tropical fan \(K\) with the weights inherited from \(K\) is also a tropical fan. Two tropical fans are called equivalent, if they have a common tropical partition. The equivalence class of tropical fans is called the tropical variety. For example, all tropical fans of dimension \(N\) are equivalent.

8.1.2. **Euclidean and rational tropical varieties.** Let a Euclidean metric or an integer lattice be given in the space \(V\). Then we can consider the weight \(W\) as a numerical function \(w\) on the set of cones in the following way. For a \(k\)-dimensional cone \(K \in K\) consider in the quotient space \(V/V_K\) the corresponding quotient metric or quotient lattice. Let \(\Pi \subset V/V_K\) be a unit cube.
of the quotient metric or a fundamental parallelootope of the quotient lattice. We set \( w(K) = W(K)(\xi_1 \land \ldots \land \xi_k) \), where \( \xi_i \) are the sides of \( \Pi \).

**Definition 16.** In the first and second cases we will talk respectively about Euclidean and rational tropical fans and tropical varieties. Further we suppose, that the weights of rational fans are rational.

Note that when choosing a metric in the space \( V \) any tropical variety becomes Euclidean.

From the approximation theorem (see Theorem 7 in Section 7) it follows that to any \( k \)-dimensional algebraic subvariety \( M \) of the torus \( T \) there corresponds a \( k \)-dimensional rational tropical fan in the space \( V = \Re T \). The corresponding tropical variety is called the tropicalization of \( M \).

### 8.1.3. Ring of tropical varieties

Tropical varieties form a commutative graded ring; see [K03]. The following two tropical theorems are conveniently formulated in the language of Euclidean and rational varieties. Denote these algebras by \( E(V) \) and by \( Q(V) \) respectively. Euclidean or rational fan of degree \( N \), those. point \( 0 \) with real or rational weight, we identify respectively with \( \Re \) or \( \Q \). Thus, multiplication operation sets the pairings

\[
P_k : E_k(V) \times E_{N-k}(V) \to \Re, \quad Q_k : Q_k(V) \times Q_{N-k}(V) \to \Q
\]

**Theorem 8.** The pairings \( P_k, \ Q_k \) are non-degenerate.

Recall that to any convex polyhedron \( \Delta \) there corresponds a fan of cones \( K_{\Delta,k} \), consisting of cones, dual to faces of \( \Delta \) of dimension \( \leq k \). We supply cones of codimension \( k \) weights equal areas of dual faces. The fan \( K_{\Delta,k} \) is a Euclidean tropical fan. If the vertices \( \Delta \) are integer, and the face areas are measured using an integer lattice, then \( K_{\Delta,k} \) is a rational tropical fan.

**Theorem 9** (see [K03]). The algebras \( E(V), Q(V) \) are generated by elements degrees 1 of the form \( K_{\Delta,1} \).

### 8.2. Pull backs of tropical varieties

Further we assume that the orientation of the kernel of a linear operator \( s : V \to U \) is fixed. If \( s \) is surjective, then the set \( s^{-1}K = \{ s^{-1}K : K \in \mathcal{K} \} \) form a fan of cones in the space \( V \). Let \( W \) be the weight chain on the fan \( \mathcal{K} \). We agree orientation of any subspace \( E \subset U \) with the orientation of the subspace \( s^{-1}E \subset V \). The map \( s \) gives an isomorphism of the quotient spaces \( U/U_K \) and \( V/V_{s^{-1}K} \). Therefore, the mentioned agreement allows considering \( s^*(W(K)) \) as the weight of the cone \( s^{-1}K \). For the surjective operator \( s \), denote by \( s^*\mathcal{K} \) the fan of cones \( s^{-1}K \) with weights \( s^*(W(K)) \).

**Definition 17.** If \( s \) is surjective then the tropical fan \( s^*\mathcal{K} \) call a pull back of tropical fan \( \mathcal{K} \).

Let \( s \) be injective. We identify its image \( s(V) \) with the subspace \( V \subset U \). Consider \( V \) as a tropical fan in the space \( U \) with a single cone and any nonzero weight \( T(V) \). Then \( \text{supp}(K \cdot V) \subset V \). Let \( L \) be the cone of maximum
dimension of the fan $\mathcal{K} \cdot V$. Then the weight of the cone $L$ is equal to $T(V) \wedge W(L)$, where $W(L)$ is the uniquely defined volume form in the space $V/V_L$. The cones of $\mathcal{K} \cdot V$ form a fan of cones $(\mathcal{K} \cdot V)_s \subset V$. We equip this fan with the weight chain $V$ and consider it as a tropical fan in $V$.

**Definition 18.** If $s$ is injective then we denote the tropical fan $(\mathcal{K} \cdot V)_s \subset V$ by $s^*\mathcal{K}$ and call it the pull back of $\mathcal{K}$.

Define the pull back of tropical fan with respect to any linear operator $s: V \to U$ as follows. Represent $s$ in the form $s = s_{\text{inj}} \cdot s_{\text{surj}}$, where the operator $s_{\text{surj}}: V \to s(V)$ is surjective, and the operator $s_{\text{inj}}: s(V) \to U$ is injective.

**Definition 19.** We put $s^*\mathcal{K} = (s_{\text{surj}}^* \cdot s_{\text{inj}}^*)\mathcal{K}$.

The main result on the pull backs of tropical varieties is as follows.

**Theorem 10.** (1) For any $s: V \to U$ the pull back mapping $\mathcal{K} \mapsto s^*\mathcal{K}$ is a ring isomorphism $s^*: \mathcal{V}(U) \to \mathcal{V}(V)$.

(2) If $s = s_1 \cdot s_2$ then $s^* = s_2^* \cdot s_1^*$.

(3) Let $s': U^* \to V^*$ be an operator adjoint to $s$, and let $\Delta \subset U^*$ be a convex polyhedron. Then $s^*\mathcal{K}_{\Delta,k} = K_{s'\Delta,k}$.

Let $E \subset U$, $E^* = U^*/E^\perp$, where $E^\perp$ is an orthogonal complement of $E$. Denote by $\Delta \subset U^*$ and $\pi\Delta$ respectively a convex polyhedron in $U^*$ and its image under the projection $\pi: U^* \to E^*$. Now choose the Euclidean metric in $U$ and consider the subspace $E$ as a Euclidean tropical fan consisting of a single cone with a weight of 1.

**Proposition 2.** Let $k \leq \dim E$. Then $\mathcal{K}_{\pi\Delta,k} = \iota^*\mathcal{K}_{\Delta,k}$, where $\iota$ is an operator of embedding $E$ into $V$.

**Proof.** By definition [18] we have $\iota^*\mathcal{K}_{\Delta,k} = (\mathcal{K}_{\Delta,k} \cdot E)_{\iota}$. If $k = \dim E = m$, then the statement follows from the tropical theorem BKK (see EKK20 Theorem 3.1.3) for the system of equations $f_1 = \ldots = f_m = g_1 = \ldots = g_{N-m} = 0$, where $f_i$ are tropical polynomials with the common Newton polyhedron $\Delta$, and $g_1 = \ldots = g_{N-m} = 0$ is a system of tropical equations of subspace $E$, consisting of polynomials with the common Newton polyhedron of area 1. For $k < m$, the using of localization of tropical varieties (see [K03]) reduces the statement to the case $k = m$. \(\square\)

Now we turn to the proof of Theorem [10]

First, we prove statement (3). If the operator $s$ is surjective, then the adjoint operator $s'$ is injective and the polyhedron $s'\Delta$ lies in the subspace $s'(U^*) \subset V^*$. In this case, by the definition of the fan $\mathcal{K}_{\Delta,k}$, statement (3) follows from Definition [17]. If the operator $s$ is injective, then the adjoint operator $s'$ is surjective. In this case, according to the definition of [18], statement (3) coincides Proposition [2] for $E = s(V)$. If $s = s_{\text{inj}} \cdot s_{\text{surj}}$, then $s'\Delta = s_{\text{surj}}^*s_{\text{inj}}^*\Delta$. Therefore, according to Definition [19] the required statement is reduced to the previous one. Statement (3) is proved.
We pass to the proof of (2). For tropical fans of the form $K_{\Delta,k}$ statement (2) follows from (3). Really,

$$s_2^*s_1^*K_{\Delta,k} = s_2^*K_{s_1^*\Delta,k} = K_{s_2^*s_1^*\Delta,k} = (s_2s_1^*)^*K_{\Delta,k}.$$  

According to Theorem 9 any tropical variety $K$ of degree $k$ can be represented as a finite sum $\alpha_i \sum_{\Delta} K_{\Delta,k}$. The pull back mapping $s^*: \mathcal{V}(U) \to \mathcal{V}(V)$ is linear. Therefore (2) follows from (3).

To prove (1) we use the ring of convex polyhedra $\text{Pol}(E;E)$; see Definition 7 and Theorem 3.

**Theorem 11.** Any linear operator $s': U^* \to V^*$ can be extended to a ring homomorphism $\text{Pol}(s): \text{Pol}(U^*, U^*) \to \text{Pol}(V^*, V^*)$.

**Proof.** Recall the notation: $J(L)$ is an ideal in symmetric algebra $S(L)$ of a vector space $L$ and $\text{Pol}(L, L) = S(L)/J(L)$ (see Definition 7). The operator $s'$ extends to a ring homomorphism $\text{S}(s): S(U^*) \to S(V^*)$. Thus, it remains to prove that $\text{S}(s)(J_{U^*}) \subset J_{V^*}$.

Known that the mapping $\Delta \mapsto K_{\Delta,1}$ extends to the ring isomorphism $N(L): \text{Pol}(L, L) \to \mathcal{V}(L)$; see [EKK20]. According to Theorem 10 (3), the following diagram is commutative

$$\begin{array}{ccc}
S_k(U^*) & \xrightarrow{S(s)} & S_k(V^*) \\
\downarrow N(U) & & \downarrow N(V) \\
\mathcal{V}(U) & \xrightarrow{s^*} & \mathcal{V}(V)
\end{array}$$  

(8.1)

Let $\mathcal{P} \in J_{U^*}$. Then, according to Theorem 8 $N(U) \mathcal{P} = 0$. The commutativity of the diagram (8.1) implies that $N(V) S(s) \mathcal{P} = 0$. Again, applying Theorem 8, we get that $S(s) \mathcal{P} \in J_{V^*}$.

Theorem 10 (1) follows from Theorem 11.

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