REPRESENTATION OF MAXITIVE MEASURES: AN OVERVIEW

PAUL PONCET

ABSTRACT. We review and extend several Radon–Nikodym like theorems proven in the literature for the idempotent integral. Idempotent integration was originally introduced by Shilkret as an analogue of the Lebesgue integration where \( \sigma \)-additive measures are replaced by \( \sigma \)-maxitive (\( \max \)-additive) measures. This integral is a widespread tool used in mathematical areas such as optimization, idempotent analysis, large deviation theory, fuzzy set theory, or extreme value theory, where the existence of Radon–Nikodym derivatives turns out to be crucial.

1. INTRODUCTION

Maxitive measures were introduced by Shilkret \[94\] as an analogue of classical finitely additive measures or charges with the supremum operation, denoted \( \oplus \), in place of the addition \( + \). A maxitive measure on a \( \sigma \)-algebra \( \mathcal{B} \) is then a map \( \nu : \mathcal{B} \to \mathbb{R}_+ \) such that \( \nu(\emptyset) = 0 \) and

\[
\nu(B_1 \cup B_2) = \nu(B_1) \oplus \nu(B_2),
\]

for all \( B_1, B_2 \in \mathcal{B} \). It is \( \sigma \)-maxitive if it commutes with countable unions of elements of \( \mathcal{B} \).

In this paper we are interested in representing maxitive measures \( \nu \) under the form

\[
\nu(B) = \int_B f \odot d\tau,
\]

where \( \int_B f \odot d\tau \) denotes (in Gerritse’s notation \[39\]) the idempotent \( \odot \)-integral of the measurable map \( f \) with respect to the maxitive measure \( \tau \). Here \( \odot \) is a pseudo-multiplication, i.e. an associative binary relation satisfying a series of natural properties. If \( \odot \) is the usual multiplication (resp. the minimum \( \wedge \)), then the idempotent \( \odot \)-integral specializes to the Shilkret integral \[94\] (resp. the Sugeno integral \[96\]).

Date: May 12, 2014.

2010 Mathematics Subject Classification. Primary 28B15; Secondary 03E72, 49J52.

Key words and phrases. idempotent integration, Shilkret integral, Sugeno integral, essential supremum, Radon–Nikodym theorem, maxitive measures, \( \sigma \)-principal measures, localizable measures, countable chain condition, optimal measures, possibility theory.
Since the early work of Shilkret [94], idempotent integration has been re-discovered under various forms and studied by several authors with motivations from dimension theory and fractal geometry, optimization, capacities and large deviations of random processes, fuzzy sets and possibility theory, idempotent analysis and max-plus (tropical) algebra.

Because of these numerous fields of application, the wording around maxitive measures is not unique, thus deserves to be reviewed. The term of idempotent integration that we use was coined by Maslov and derived from the mathematical area of idempotent analysis originally developed by Kolokoltsov and Maslov (see [48], [49]).

Many authors have focused on the search for Radon–Nikodym like theorems with respect to the idempotent integral, since the existence of Radon–Nikodym derivatives is often crucial in applications. Sugeno and Murofushi [97] actually showed that, if \( \nu \) and \( \tau \) are \( \sigma \)-maxitive measures on a \( \sigma \)-algebra \( \mathcal{B} \), with \( \tau \) \( \sigma \)-\( \otimes \)-finite and \( \sigma \)-principal, then \( \nu \) is \( \otimes \)-absolutely continuous with respect to \( \tau \) if and only if there exists some \( \mathcal{B} \)-measurable map \( c : E \to \mathbb{R}_+ \) such that \( \nu(B) = \int_B c \otimes d\tau \) for all \( B \in \mathcal{B} \).

This result looks like the classical Radon–Nikodym theorem, except that one needs an unusual condition on the dominating measure \( \tau \), namely \( \sigma \)-principal. This condition roughly says that every \( \sigma \)-ideal of \( \mathcal{B} \) has a greatest element “modulo negligible sets”. Although \( \sigma \)-finite \( \sigma \)-additive measures are always \( \sigma \)-principal, this is not true for \( \sigma \)-finite \( \sigma \)-maxitive measures. The conditions of \( \sigma \)-principality and \( \sigma \)-\( \otimes \)-finiteness have proved to be essential, since I showed in [82] a converse statement to the Sugeno–Murofushi theorem.

After the article [97], many results of Radon–Nikodym flavour for maxitive measures have been published. This is the case of Agbeko [2], de Cooman [23], Akian [5], Barron et al. [12], Puhalskii [84], and Drewnowski [31]. By linking several properties of maxitive measures together (see Table 1), we shall see why some of these results are already encompassed in the Sugeno–Murofushi theorem. In addition, we shall prove a new Radon–Nikodym type theorem in the case where the \( \sigma \)-maxitive measures \( \nu \) and \( \tau \) are associated (meaning, roughly speaking, that they are “strongly dominated” by a common \( \sigma \)-maxitive measure).

The paper is organized as follows. Section 2 introduces the notion of \( \sigma \)-maxitive measure and recalls some key theorems and examples. Maxitive measures that can be represented as essential suprema are studied in Section 3; we also discuss Barron et al.’s theorem whose proof draws a link between maxitive measures and classical additive measures. Section 4 develops the idempotent integral and its properties. In Section 5 we review existing Radon–Nikodym theorems for the idempotent integral and prove a variant that generalizes results due to de Cooman and Puhalskii; we also
| of bounded variation | $\implies$ | finite  |
|---------------------|------------|---------|
|                     | $\iff$    |         |
| exhaustive          | $\iff$    | $\sigma$-finite |
| optimal             | $\iff$    | semi-finite   |
| essential           | $\iff$    |         |
| $\sigma$-principal  | $\iff$    | autocontinuous       |
| CCC                 | $\iff$    |         |
| localizable         | $\iff$    |         |

**Table 1.** Many properties of $\sigma$-maxitive measures defined on a $\sigma$-algebra are considered in this paper; we shall prove many links between these properties, that we have represented here as a summary. In blue, the conditions of $\sigma$-finiteness and $\sigma$-principality taken together are equivalent to the Radon–Nikodym property, as recalled by Theorem 5.8. Note that for $\sigma$-additive measures, $\sigma$-finiteness implies $\sigma$-principality, while this is not the case for $\sigma$-maxitive measures.

make the connection with Section 3. Section 6 focuses on the important particular case of optimal measures, i.e. maxitive fuzzy measures. Section 7 proposes new foundations for possibility theory, relying on the concept of $\sigma$-principal maxitive measures developed in Section 5.

### 2. Preliminaries on maxitive measures

2.1. **Notations.** Let $E$ be a nonempty set. A *prepaving* on $E$ is a collection of subsets of $E$ containing the empty set and closed under finite unions. An *ideal* of a prepaving $\mathcal{E}$ is a nonempty subset $\mathcal{I}$ of $\mathcal{E}$ that is closed under finite unions and such that $A \subset G \in \mathcal{I}$ and $A \in \mathcal{E}$ imply $A \in \mathcal{I}$. A collection of subsets of $E$ containing $E$, the empty set, and closed under finite intersections and countable unions is a *semi-$\sigma$-algebra*. In a semi-$\sigma$-algebra, a *$\sigma$-ideal* is an ideal that is closed under countable unions. A semi-$\sigma$-algebra (resp. a topology) closed under the formation of complements is...
a $\sigma$-algebra (resp. a $\tau$-algebra). When explicitly referring to a $\sigma$-algebra, we shall preferentially call it $\mathcal{B}$ instead of $\mathcal{E}$.

Assume in all the sequel that $\mathcal{E}$ is a preposing on $E$. A set function on $\mathcal{E}$ is a map $\mu : \mathcal{E} \to \mathbb{R}_+$ equal to zero at the empty set. A set function $\mu$ is

- **monotone** if $\mu(G) \leq \mu(G')$ for all $G, G' \in \mathcal{E}$ such that $G \subset G'$,
- **null-additive** if $\mu(G \cup N) = \mu(G)$ for all $G, N \in \mathcal{E}$ with $\mu(N) = 0$,
- **finite** if $\mu(G) < \infty$ for every $G \in \mathcal{E}$,
- **$\sigma$-finite** if $\mu(G_n) < \infty$ for all $n$, where $(G_n)$ is a countable family of elements of $\mathcal{E}$ covering $E$,
- **continuous from below** if $\mu(G) = \lim_{n} \mu(G_n)$, for all $G_1 \subset G_2 \subset \ldots \in \mathcal{E}$ such that $G = \bigcup_n G_n \in \mathcal{E}$.

We shall need the following notion of negligibility. If $\mu$ is a null-additive monotone set function on $\mathcal{E}$, a subset $N$ of $E$ is $\mu$-negligible if it is contained in some $G \in \mathcal{E}$ such that $\mu(G) = 0$. A property $P(x)$ ($x \in E$) is satisfied $\mu$-almost everywhere (or $\mu$-a.e. for short) if there exists some negligible subset $N$ of $E$ such that $P(x)$ is true, for all $x \in E \setminus N$.

### 2.2. Definition of maxitive measures.**

In this section, $\mathcal{E}$ will denote a preposing on some nonempty set $E$.

A maxitive (resp. completely maxitive) measure on $\mathcal{E}$ is a set function $\nu$ on $\mathcal{E}$ such that, for every finite (resp. arbitrary) family $\{G_j\}_{j \in J}$ of elements of $\mathcal{E}$ with $\bigcup_{j \in J} G_j \in \mathcal{E}$,

\[
\nu\left(\bigcup_{j \in J} G_j\right) = \bigoplus_{j \in J} \nu(G_j).
\]

A $\sigma$-maxitive measure is a continuous from below maxitive measure. One should note that a $\sigma$-maxitive measure does not necessarily commute with intersections of nonincreasing sequences, unlike $\sigma$-additive measures.

**Remark 2.1.** The term “maxitive” qualifying a set function that satisfies Equation (1) was coined by Shilkret [94], and has been widely used, especially in the fields of probability theory and fuzzy theory. However, one can find many other terms in the literature for maxitive or $\sigma$-maxitive measures, say: $f$-additive or fuzzy additive measures [96, 68, 101], contactability measures [100], measures of type $\vee$ [18], idempotent measures [61, 5], max-measures [97], stable measures [34], cost measures [4, 16], semi-additive measures [38], possibility measures [63], generalized possibility measures [33], performance measures [29], sup-decomposable measures [64], set-additive measures [9, 58, 59]. As for completely maxitive measures, one finds: $\text{sup}$-measures [71, 73], idempotent measures when $\mathcal{E} = 2^E$ or $\tau$-maxitive measures for general $\mathcal{E}$ [84], (generalized) possibility measures [92, 103, 93, 23, 101], supremum-preserving measures [52].
Some differences may appear in the definitions, essentially depending on the choice of the range of the measure and on the structure of the space \((E, \mathcal{E})\). See also the historical notes in [84, Appendix B].

The definition of the term “possibility measure” remains unclear, and mainly oscillates between “normed \(\sigma\)-maxitive measure” and “normed completely maxitive measure”. We shall propose in Section 7 a different definition, aiming at founding an operational possibility theory.

Note that every maxitive measure is null-additive and monotone. Actually a much stronger property than monotonicity holds, namely the alternating property. For a map \(f : \mathcal{E} \to \mathbb{R}\) we classically define \(\Delta_{G_1} \ldots \Delta_{G_n} f(G)\) after Choquet [20] by iterating the formula \(\Delta_{G_1} f(G) = f(G \cup G_1) - f(G)\) (with the convention that \(-\infty + \infty = \infty - \infty = 0\)). Then \(f\) is alternating of infinite order (or alternating for short) if

\[
(-1)^{n+1} \Delta_{G_1} \ldots \Delta_{G_n} f(G) \geq 0,
\]

for all \(n \in \mathbb{N} \setminus \{0\}\), \(G, G_1, \ldots, G_n \in \mathcal{E}\), where \(\mathbb{N}\) denotes the set of non-negative integers. Nguyen et al. [70] gave a combinatorial proof of the fact that every finite maxitive measure is alternating (see also Harding et al. [43, Theorem 6.2]). This is actually true for every (finite or not) maxitive measure, as the following proposition states.

**Proposition 2.2.** Every maxitive measure on \(\mathcal{E}\) is alternating.

**Proof.** Recall the convention \(\infty - \infty = 0\). We write \(s \wedge t\) for the infimum of \(\{s, t\}\). Let \(G_1, \ldots, G_n \in \mathcal{E}\), and define \(\nu_0(G) = -\nu(G)\), \(\nu_n(G) = (-1)^{n+1} \Delta_{G_n} \ldots \Delta_{G_1} \nu(G)\). A proof by induction shows that the property “\(\nu_0(G \cup G') = \nu_0(G) \wedge \nu_0(G')\) and \(\nu_n(G) = 0 \oplus (\nu_{n-1}(G) - \nu_{n-1}(G_n)) \geq 0\), for all \(G, G' \in \mathcal{E}\)” holds for all \(n \in \mathbb{N} \setminus \{0\}\). □

2.3. **Elementary and advanced examples.** Here we collect some examples given in the literature, especially on metric spaces where maxitive measures appear naturally. Some examples are also linked with extreme value theory, which is the branch of probability theory that aims at the modelling of rare events.

**Example 2.3** (Essential supremum). Let \(\mu\) be a null-additive monotone set function, and let \(f : E \to \mathbb{R}_+\) be a map. If one sets

\[
\nu(G) = \inf \{t > 0 : G \in \mathcal{A}_t\}
\]

with \(\mathcal{A}_t := \{G \in \mathcal{E} : G \cap \{f > t\} \) is \(\mu\)-negligible\}, then \(\nu\) is a maxitive measure, called the \(\mu\)-essential supremum of \(f\), and we write

\[
\nu(G) = \bigoplus_{x \in G} \mu f(x).
\]
In this case, $f$ is a relative density of $\nu$ (with respect to $\mu$). Sufficient conditions for the existence of a relative density, when $\nu$ and $\mu$ are given, are discussed in Section 3.

**Example 2.4** (Cardinal density of a maxitive measure). In the previous example, one can take for $\mu$ the maxitive measure $\delta_\#$ defined by $\delta_\#(G) = 1$ if $G$ is nonempty, $\delta_\#(G) = 0$ otherwise. Then the essential supremum in Equation (2) reduces to an “exact” supremum, i.e.

$$\nu(G) = \bigoplus_{x \in G} f(x) = \bigoplus_{x \in G} f(x).$$

In this special case we say that $f$ is a cardinal density of $\nu$. Note also that a maxitive measure with a cardinal density is necessarily completely maxitive. Conversely, complete maxitivity happens to be a sufficient condition for guaranteeing the existence of a cardinal density. This question was treated in detail by the author in [79] and [81].

**Examples 2.5** (Measures of non-compactness). Let $E$ be a Banach space. Following Appell [9], a measure of non-compactness (or monc for short) on $E$ is a maxitive measure $\nu$ on the collection of bounded subsets of $E$, satisfying the following axioms, for all bounded subsets $B$ of $E$:

- $\nu(B + K) = \nu(B)$, for all compact subsets $K$ in $E$,
- $\nu(tB) = t\nu(B)$, for all $t > 0$,
- $\nu(\text{co}(B)) = \nu(B)$, where $\text{co}$ denotes the closed convex hull.

The definition may differ from one author to the other, see e.g. Mallet-Paret and Nussbaum [58, 59] for a quite different list of axioms. Note that if $E = \mathbb{R}^d$, then $\nu(B) = 0$ for all bounded subsets $B$. As Appell recalled, three important examples of moncs appear in the literature, namely the ball monc (or Hausdorff monc)

$$\alpha(B) = \inf\{t > 0 : \text{there are finitely many balls of radius } t \text{ covering } B\};$$

the set monc (or Kuratowski monc)

$$\beta(B) = \inf\{t > 0 : \text{there are finitely many subsets of diameter at most } t \text{ covering } B\};$$

and the lattice monc (or Istrătescu monc)

$$\gamma(B) = \sup\{t > 0 : \text{there is a sequence } (x_n)_n \text{ in } B \text{ with } \|x_m - x_n\| \geq t \text{ for } m \neq n\},$$

and we have the classical relations $\alpha \leq \gamma \leq \beta \leq 2\alpha$. Since moncs vanish on compact subsets, hence on singletons, they are a source of examples of maxitive measures with no cardinal density.
Examples 2.6 (Dimensions).

- If $E$ is a topological space, the topological dimension is a maxitive measure on the collection of its closed subsets (see e.g. Nagata [69, Theorem VII-1]). If $E$ is normal, the topological dimension is even $\sigma$-maxitive [69, Theorem VII-2].
- If $E$ is a metric space, the Hausdorff dimension and the packing-dimension are $\sigma$-maxitive measures on $2^E$, and the upper box dimension is a maxitive measure on $2^E$ (see e.g. Falconer [34]).
- If $E$ is the Cantor set $\{0, 1\}^\mathbb{N}$, the constructive Hausdorff dimension and the constructive packing-dimension are completely maxitive measures on $2^E$, see Lutz [56, 57].
- If $E$ is the set of positive integers, the zeta dimension is a maxitive measure on $2^E$, see Doty et al. [30].

Example 2.7 (Random closed sets). Let $(\Omega, \mathcal{A}, P)$ be a probability space and $E$ be a locally-compact, separable, Hausdorff topological space. We denote by $\mathcal{F}$ the collection of closed subsets of $E$, and by $\mathcal{K}$ the collection of compact subsets. A random closed set is a measurable map $C : \Omega \to \mathcal{F}$. For measurability a $\sigma$-algebra on $\mathcal{F}$ is needed. The usual $\sigma$-algebra considered is the Borel $\sigma$-algebra generated by the Vietoris (or hit-and-miss) topology on $\mathcal{F}$. Choquet’s fundamental theorem is that the distribution of a random closed set $C$ is characterized by its Choquet capacity $T : \mathcal{K} \to [0, 1]$ defined by $T(K) = P[C \cap K \neq \emptyset]$. Moreover, $T$ is an alternating set function that is continuous from above on $\mathcal{K}$ (see the definition in Section 6), and every $[0, 1]$-valued alternating, continuous from above set function on $\mathcal{K}$ is the Choquet capacity of some random closed set.

Recall that every maxitive measure is alternating (see Proposition 2.2). For a given upper-semicontinuous map $c : E \to [0, 1]$, the following construction explicitly gives a random closed set whose Choquet capacity has cardinal density $c$ [70]. Let $U$ be a uniformly distributed random variable on $[0, 1]$. Then $C = \{x \in E : c(x) \geq U\}$ is a random closed set on $E$, and its Choquet capacity $T$ is maxitive and satisfies $T(K) = \bigoplus_{x \in K} c(x)$, for all $K \in \mathcal{K}$.

One may observe that this random closed set is such that

$$C(\omega) \subset C(\omega') \text{ or } C(\omega') \subset C(\omega),$$

for all $\omega, \omega' \in \Omega$. More generally, Miranda et al. called consonant (of type C2) a random closed set $C$ satisfying the above relation for all $\omega, \omega' \in \Omega_0$, for some event $\Omega_0$ of probability 1. These authors showed that a random closed set is consonant if and only if its Choquet capacity is maxitive [65, Corollary 5.4].
Elements of random set theory may be found in the reference book by Matheron [62], or in the monograph by Molchanov [66].

**Example 2.8** (Random sup-measures). Let \((\Omega, \mathcal{A})\) and \((E, \mathcal{B})\) be measurable spaces, \(P\) be a probability measure on \(\mathcal{A}\), and \(m\) be a finite \(\sigma\)-additive measure on \(\mathcal{B}\). Consider a Poisson point process \((X_k, T_k)_{k \geq 1}\) on \(\mathbb{R}_+ \times E\) with intensity \(\beta x^{-\beta-1}dx \times m(dt)\), where \(\beta > 0\). Then the random process defined on \(\mathcal{B}\) by

\[
M(B) = \bigoplus_{k \geq 1} X_k \cdot 1_B(T_k)
\]

is, \(\omega\) by \(\omega\), a completely maxitive measure. Moreover, this is a \(\beta\)-Fréchet random sup-measure with control measure \(m\) in the sense of Stoev and Taqqu [95, Definition 2.1], for it is a map \(M : \Omega \times \mathcal{B} \to \mathbb{R}_+\) satisfying the following axioms:

- for all pairwise disjoint collections \((B_j)_{j \in \mathbb{N}}\) of elements of \(\mathcal{B}\), the random variables \(M(B_j), j \in \mathbb{N}\), are independent, and, almost surely,
  \[
  M(\bigcup_{j \in \mathbb{N}} B_j) = \bigoplus_{j \in \mathbb{N}} M(B_j);
  \]

- for all \(B \in \mathcal{B}\) the random variable \(M(B)\) has a Fréchet distribution with shape parameter \(1/\beta\), in such a way that, for all \(x > 0\),
  \[
  P[M(B) \leq x] = \exp(-m(B)x^{-\beta}).
  \]

The Poisson process \((X_k, T_k)_{k \geq 1}\) was introduced by de Haan [27] as a tool for representing continuous-time max-stable processes. These processes play an important role in extreme value theory. See also Norberg [71] and Resnick and Roy [87] for elements on random sup-measures.

**Example 2.9** (The home range). Let \((X_n)_{n \geq 1}\) be a sequence of independent, identically distributed \(\mathbb{R}^2\)-valued random variables, and assume that the common distribution has compact support. We write this sequence in polar coordinates \((R_n, \Theta_n)_{n \geq 1}\). Define the map \(h\) on Borel subsets \(B\) of \([0, 2\pi]\) by:

\[
h(B) = \sup\{r \in \mathbb{R}_+ : P[R_1 > r, \Theta_1 \in B] > 0\}.
\]

Then, according to de Haan and Resnick [28, Proposition 2.1], \(h\) is a completely maxitive measure, and \(h\) may be thought of as the boundary of the natural habitat of some animal, called the home range in ecology. The sequence \((X_n)_{n \geq 1}\) is then seen as the successive sightings of the animal. De Haan and Resnick aimed at finding consistent estimates of the boundary \(h\).

The following paragraph contradicts an assertion made by van de Vel [98, Exercise II-3.19.1].
Example 2.10 (Carathéodory number of a convexity space). A collection \( \mathcal{C} \) of subsets of a set \( X \) that contains \( \emptyset \) and \( X \) is a convexity on \( X \) if it is closed under arbitrary intersections and closed under directed unions. The pair \( (X, \mathcal{C}) \) is called a convexity space, and elements of \( \mathcal{C} \) are called convex subsets of \( X \). If \( A \subset X \), the convex hull \( \text{co}(A) \) of \( A \) is the intersection of all convex subsets containing \( A \). Advanced abstract convexity theory is developed in the monograph by van de Vel [98]. The Carathéodory number \( c(A) \) of some \( A \subset X \) is the least integer \( n \) such that, for each subset \( B \) of \( A \) and \( x \in \text{co}(B) \cap A \), there exists some finite subset \( F \) of \( B \) with cardinality \( \leq n \) such that \( x \in \text{co}(F) \). In [98, Exercise II-3.19.1], van de Vel asserted that the map \( A \mapsto c(A) \) is a maxitive (integer-valued) measure on \( \mathcal{E} \), where \( \mathcal{E} \) is the prepanel made up of finite unions of convex subsets of \( X \). However, a simple counterexample is built as follows. Let \( X \) be the three-element semilattice \( \{x_1, x_2, x_3\} \) with \( x_2 = x_1 \land x_3 \), endowed with the convexity made up of all subsets of \( X \) but \( \{x_1, x_3\} \). Let \( A_i = \{x_i\} \) for \( i = 1, 2, 3 \). Then \( c(A_i) = 1 \) for \( i = 1, 2, 3 \), hence \( \max_{i=1,2,3} c(A_i) = 1 \). However, \( c(\bigcup_{i=1,2,3} A_i) = c(X) = 2 \), for if \( B := \{x_1, x_3\} \), one has \( x_2 \in \text{co}(B) \cap X = X \), while every nonempty subset \( F \) of \( B \) with cardinality \( \leq 1 \) is either \( \{x_1\} \) or \( \{x_3\} \), hence does not contain \( x_2 \).

Example 2.11 (Interpretation of maxitive measures). Finkelstein et al. [36] suggested to use maxitive measures as a model for a physicist’s reasoning and beliefs about probable, possible, and impossible events. Kreinovich et al. [53] advocated the use of maxitive measures for modelling rarity of events, for maxitive measures are limits of probability measures in a large deviation sense (for a justification see e.g. the work by O’Brien and Vervaat [74], Gerrits [39], O’Brien [72], Akian [5], Puhalskii [83, 84]). This interpretation is in accordance with Bouleau’s criticism of extreme value theory [17]. This author noted that some events, although possible, are so rare (Bouleau gave the example of the extinction of Neanderthal Man) that they cannot be appropriately understood by classical probability theory (and in particular by extreme value theory). Since probability theory relies on the frequentist paradigm, the question of the probability of such events would make no sense. For further discussion on the intuitive and the formalized distinction between probable and possible events, see also El Rayes and Morsi [33, Paragraph 2] and Nguyen et al. [70].

3. MAXITIVE MEASURES AS ESSENTIAL SUPREMA

3.1. Introduction. In this section, we shall be interested in representing a maxitive measure \( \nu \) defined on a \( \sigma \)-algebra \( \mathcal{B} \) as an essential supremum
with respect to some null-additive monotone set function $\mu$, i.e. as

\begin{equation}
\nu(B) = \bigoplus_{x \in B} \mu f(x),
\end{equation}

for all $B \in \mathcal{B}$, as introduced in Example 2.3. Note that, for such a $\mu$, the set function $\tau := \delta_\mu$, defined by $\tau(B) = 1$ if $\mu(B) > 0$, $\tau(B) = 0$ otherwise, is a maxitive measure, and Equation (4) is satisfied if and only if

\begin{equation}
\nu(B) = \bigoplus_{x \in B} \tau f(x),
\end{equation}

for all $B \in \mathcal{B}$. Thus, we can restrict our attention to essential suprema with respect to some maxitive measure $\tau$, without loss of generality.

**Definition 3.1.** Let $\nu$ and $\tau$ be null-additive monotone set functions on a $\sigma$-algebra $\mathcal{B}$ on $E$. Then $\nu$ is absolutely continuous with respect to $\tau$ (or $\tau$ dominates $\nu$), in symbols $\nu \ll \tau$, if for all $B \in \mathcal{B}$, $\tau(B) = 0$ implies $\nu(B) = 0$. We shall say that $\nu$ is strongly absolutely continuous with respect to $\tau$ (or $\tau$ strongly dominates $\nu$), in symbols $\nu \lll \tau$, if $\nu$ admits a $\mathcal{B}$-measurable relative density with respect to $\tau$, i.e. if there exists a $\mathcal{B}$-measurable map $f : E \to \mathbb{R}_+$ such that Equation (4) holds for all $B \in \mathcal{B}$.

Absolute continuity, although necessary in Equation (4), seems a priori too poor a condition for ensuring the existence of a (relative) density, i.e. $\nu \ll \tau$ does not imply $\nu \lll \tau$ in general. For instance, every maxitive measure $\nu$ satisfies $\nu \ll \delta_\#$, while $\nu$ does not necessarily have a cardinal density (see for instance Example 2.5 on measures of non-compactness). We shall understand in Section 5 that absolute continuity is actually a necessary and sufficient condition for the existence of a density whenever the dominating measure is $\sigma$-principal (and the measure $\delta_\#$ is not $\sigma$-principal in general).

The next proposition ensures that, under the absolute continuity condition, a relative density exists whenever a cardinal density already exists. Given a $\sigma$-algebra $\mathcal{B}$ on $E$, we say that a maxitive measure $\nu$ on $\mathcal{B}$ is autocontinuous if $\nu \lll \nu$.

**Proposition 3.2.** Let $\nu$ be a maxitive measure on $\mathcal{B}$ with a $\mathcal{B}$-measurable cardinal density $c$. Then for every maxitive measure $\tau$ on $\mathcal{B}$, we have $\nu \ll \tau$ if and only if $\nu \lll \tau$. In particular, $\nu$ is autocontinuous.

**Proof.** Let $B \in \mathcal{B}$, and let $x \in B$, $t \in \mathbb{R}_+$ such that $\tau(N) = 0$ with $N \supset B \cap \{c > t\}$. If $c(x) > t$, then $x \in N$. Since $\tau(N) = 0$, we have $\nu(N) = 0$, so that $c(x) = 0$, a contradiction. Thus $c(x) \leq t$, and we get

\[ \nu(B) = \bigoplus_{x \in B} c(x) \leq \bigoplus_{x \in B} c(x). \]

Now we show the converse inequality. If $\nu(B)$ is infinite, this is evident. If not, let $a > \nu(B) = \bigoplus_{x \in B} c(x)$. Then $B \cap \{c > a\} = \emptyset$ is negligible with respect to $\tau$, hence $a \geq \bigoplus_{x \in B} c(x)$, and the result is proved. \qed
3.2. Existence of a relative density. The following theorem on existence and “uniqueness” of relative densities is due to Barron et al. [12, Theorem 3.5]. We add the following component: we define a maxitive measure $\tau$ on a $\sigma$-algebra $\mathcal{B}$ to be essential if there exists a $\sigma$-finite, $\sigma$-additive measure $m$ such that $\tau(B) > 0$ if and only if $m(B) > 0$, for all $B \in \mathcal{B}$.

**Theorem 3.3 (Barron–Cardaliaguet–Jensen).** Let $\nu, \tau$ be $\sigma$-maxitive measures on $\mathcal{B}$. Assume that $\tau$ is essential. Then $\nu \ll \tau$ if and only if $\nu \ll \tau$. In this situation, the relative density of $\nu$ with respect to $\tau$ is unique $\tau$-almost everywhere.

**Sketch of the proof.** Since $\tau$ is essential we can replace, without loss of generality, $\tau$ by some $\sigma$-finite, $\sigma$-additive measure $m$ in the statement of Theorem 3.3. We first assume that both $m$ and $\nu$ are finite. The ingenious proof given by Barron et al. relies on the following idea: to $\nu$ they associate the map $m_\nu$ defined on $\mathcal{B}$ by

$$m_\nu(B) = \inf \left\{ \sum_{j \geq 1} \nu(B_j)m(B_j) : \bigcup_{j \geq 1} B_j = B, B_k \in \mathcal{B}, \forall k \geq 1 \right\}.$$  

This formula is certainly inspired by the Carathéodory extension procedure in classical measure theory, see e.g. [8, Definition 10.21]. As intuition suggests, $m_\nu$ turns out to be a $\sigma$-additive measure, absolutely continuous with respect to $m$. Thanks to the classical Radon–Nikodym theorem there is some $\mathbb{R}_+$-valued map $c \in L^1(m)$ such that

$$m_\nu(B) = \int_B c \, dm,$$

for all $B \in \mathcal{B}$. The definition of $m_\nu$ gives $c \in L^\infty(m)$, and one can prove that $\nu(\cdot) = \bigoplus_{x \in \mathcal{B}} c(x)$ using the following “reconstruction” formula for $\nu$:

$$\nu(B) = \sup \left\{ \frac{m_\nu(B')}{m(B')} : B' \subset B, B' \in \mathcal{B}, m(B') > 0 \right\},$$

for all $B \in \mathcal{B}$.

Now take some (not necessarily finite) $\nu$, and let $\nu_1 : B \mapsto \arctan \nu(B)$. Then $\nu_1$ is a finite $\sigma$-maxitive measure, absolutely continuous with respect to $\tau$, hence one can write $\nu_1(B) = \bigoplus_{x \in B} c_1(x)$. Since $\nu_1(E) \leq \pi/2$, we can choose $c_1$ to be ($\mathcal{B}$-measurable and) such that $0 \leq c_1 \leq \pi/2$. It is now an easy task to show that, for all $B \in \mathcal{B}$, $\nu(B) = \bigoplus_{x \in B} c(x)$, where $c(x) = \tan(c_1(x))$.

The case where $m$ is $\sigma$-finite is easily deduced. □

**Corollary 3.4.** Let $\nu$ be an essential $\sigma$-maxitive measure on $\mathcal{B}$. Then $\nu$ is autocontinuous. Moreover, if the empty set is the only $\nu$-negligible subset, then $\nu$ has a cardinal density.
Barron et al.’s theorem is interesting because of its proof, which points out a correspondence between $\sigma$-maxitive and $\sigma$-additive measures. However, a part of the mystery persists, for it relies on the classical Radon–Nikodym theorem: the construction of the density remains hidden.

Note that Acerbi et al. [1, Theorem 3.2] used Theorem 3.3 for resolving some non-linear minimization problems. They considered a $\sigma$-finite, $\sigma$-additive measure $m$ on $(E, \mathcal{B})$, and derived sufficient conditions for a functional $F : L^\infty(m; E, \mathbb{R}^n) \times \mathcal{B} \to \mathbb{R}$ to be of the form

$$F(u, B) = \bigoplus_{x \in B} m f(x, u(x)),$$

for some measurable map $f : E \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ such that $f(x, \cdot)$ is lower-semicontinuous on $\mathbb{R}^n$, $m$-almost everywhere. This study was carried on by Cardaliaguet and Prinari [19], with the search for representations of the form

$$F(u, B) = \bigoplus_{x \in B} m f(x, u(x), Du(x)),$$

where $u$ runs over the set of Lipschitz continuous maps on $E$.

Theorem 3.3 was rediscovered by Drewnowski [31, Theorem 1], with a notably different proof. He applied this result to the representation of Köthe function $M$-spaces as $L^\infty$-spaces. Actually, we shall see in Section 5 that Theorem 3.3 is a direct consequence of a more general result, proved years earlier by Sugeno and Murofushi [97], which expresses it as a Radon–Nikodym like theorem with respect to the Shilkret integral (see Theorem 5.5).

3.3. **Maxitive measures of bounded variation.** Considering Theorem 3.3, a natural interest is to derive sufficient conditions for a maxitive measure to be essential. A null-additive set function on $\mathcal{B}$ satisfies the *countable chain condition* (or is $\text{CCC}$) if each family of non-negligible pairwise disjoint elements of $\mathcal{B}$ is countable. (A $\text{CCC}$ set function is sometimes called $\sigma$-*decomposable*, but this terminology should be avoided, because of possible confusion with the notion of decomposability used e.g. by Weber [102].) It is not difficult to show that every essential maxitive measure is $\text{CCC}$. The converse statement was the object of Mesiar’s hypothesis, proposed in [63]. Murofushi [67] showed that this hypothesis as such is wrong, by providing a counterexample; see also Poncet [78]. We now give the following sufficient condition for a maxitive measure to be essential. A null-additive set function $\mu$ on $\mathcal{B}$ is *of bounded variation* if $|\mu| := \sup_\pi \sum_{B \in \pi} \mu(B) < \infty$, where the supremum is taken over the set of finite $\mathcal{B}$-partitions $\pi$ of $E$.

**Proposition 3.5.** *Every $\sigma$-maxitive measure of bounded variation on $\mathcal{B}$ is finite and essential.*
Proof. Let \( \nu \) be a \( \sigma \)-maxitive measure of bounded variation on \( \mathcal{B} \) and \( m \) be the map defined on \( \mathcal{B} \) by

\[
m(B) = \sup_{\pi} \sum_{B' \in \pi} \nu(B \cap B'),
\]

where the supremum is taken over the set of finite \( \mathcal{B} \)-partitions \( \pi \) of \( E \). Then \( m \), called the disjoint variation of \( \nu \), is the least \( \sigma \)-additive measure greater than \( \nu \) (see e.g. Pap [75, Theorem 3.2]). Since \( \nu \) is of bounded variation, \( m \) is finite, and \( \nu(B) > 0 \) if and only if \( m(B) > 0 \), so that \( \nu \) is essential (and finite, as one can check easily).

\[ \square \]

4. THE IDEMPOTENT INTEGRAL

4.1. Introduction. Until today, the Lebesgue integral has given rise to many extensions. The first of them dates back to Vitali [99], who proposed to replace \( \sigma \)-additive measures by some more general set functions (see the historical note of Marinacci [60]). In [20] Choquet built on the same idea to create the tool now called the Choquet integral, which has found numerous applications, as in fuzzy set theory, game theory, statistics, or mathematical economics. After Choquet, many authors have examined the properties of integrals where the operations \((+, \times)\) used for both the Lebesgue and the Choquet integrals are swapped for some more general pair \((\dot{+}, \dot{\times})\) of associative binary relations on \( \mathbb{R}_+ \) or \( \mathbb{R}_+^* \). In the case where \((+, \times)\) is the pair \((\max, \min)\), one gets the Sugeno integral or fuzzy integral discovered by Sugeno [96]. In the general case, one talks about the pan-integral or seminormed fuzzy integral, see e.g. Weber [102], Sugeno and Murofushi [97], Wang and Klir [101], Pap [75, 77]. Interestingly, under reasonable continuity assumptions, one can explicitly describe these general additions \((\dot{+}, \dot{\times})\) (sometimes called triangular conorms or pseudo-additions) as “mixtures” between classical addition and the maximum operation (see e.g. Sugeno and Murofushi [97], Benvenuti and Mesiar [15]). Note however that the structure of general multiplications \( \dot{\times} \) remains unknown [15].

Beyond the simple replacement of arithmetical operations, another direction of generalization is to integrate \( L \)-valued functions (giving rise to \( L \)-valued integrals) rather than real-valued functions, where \( L \) has an appropriate semiring or semimodule structure. In this process, measures can either remain real-valued if \( L \) is a (semi)module (as in the Bochner integral which is a well-known extension of the Lebesgue integral, where \( L \) is a Banach space), or can also be \( L \)-valued if \( L \) is a semiring. Maslov [61] developed an integration theory for measures with values in an ordered semiring. Other authors considered the case where \( L \) is a complete lattice, see e.g. Greco [41], Liu and Zhang [55], de Cooman et al. [26], Kramosil [50].
In the line of Maslov, Akian [5] focused on defining an integral for dioid-valued functions, and showed how crucial the assumption of continuity of the underlying partially ordered set can be (see the monograph by Gierz et al. [40] for background on continuous lattices and domain theory; see also [81]). Jonasson [47] had a similar approach, but managed to mix the powerful tool of continuous poset theory with a general ordered-semiring structure for \( L \). See also Heckmann and Huth [44] for the role of continuous posets in integration theory. For extensions of the Riemann integral driven by the idea of approximation and still using arguments from continuous poset theory, see Edalat [32], Howroyd [45], Lawson and Lu [54], and references therein.

A review of integration theory in mathematics should include a number of prolific developments (e.g. the Birkhoff integral, the Pettis integral, or the stochastic Itô integral among many others). Needless to say this is far beyond the scope of this work; the reader may refer to the book [76] for a broad overview of measure and integration theory. In this paper, we shall limit our attention to the case where \( \oplus \) is the maximum operation \( \max = \oplus \) and \( \times \) is a pseudo-multiplication (i.e. a binary relation \( \odot \) satisfying the properties given in Paragraph 4.2).

This section is devoted to the construction of the idempotent integral. This corresponds to an evolution of the integral introduced by Shilkret [94], who made the earliest attempt of this nature, as far as we know.

4.2. Pseudo-multiplications and their properties. In the remaining part of this paper, we consider a binary relation \( \odot \) defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \) with the following properties:

- associativity;
- continuity on \((0, \infty) \times [0, \infty]\);
- continuity of the map \( s \mapsto s \odot t \) on \((0, \infty]\), for all \( t \);
- monotonicity in both components;
- existence of a left identity element \( 1_{\odot} \), i.e. \( 1_{\odot} \odot t = t \) for all \( t \);
- absence of zero divisors, i.e. \( s \odot t = 0 \Rightarrow 0 \in \{s, t\} \), for all \( s, t \);
- \( 0 \) is an annihilator, i.e. \( 0 \odot t = t \odot 0 = 0 \), for all \( t \).

We call such a \( \odot \) a pseudo-multiplication. Pseudo-multiplications and more generally pseudo-arithmetic operations have been studied e.g. by Benvenuti and Mesiar [15]. Note that the axioms above are stronger than in [97], where associativity was not assumed. For more on pseudo-multiplications see also [82].

We consider the map \( O : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) defined by \( O(t) = \inf s > 0 s \odot t \). An element \( t \) of \( \mathbb{R}_+ \) is \( \odot \)-finite if \( O(t) = 0 \) (and \( t \) is \( \odot \)-infinite otherwise). We conventionally write \( t \ll_{\odot} \infty \) for a \( \odot \)-finite element \( t \). If \( O(1_{\odot}) = 0 \),
we say that the pseudo-multiplication $\odot$ is non-degenerate. This amounts to say that the set of $\odot$-finite elements differs from $\{0\}$.

4.3. **Definition and elementary properties.** Throughout this section, $\mathcal{B}$ is a $\sigma$-algebra on $E$. A map $f : E \to \mathbb{R}_+$ is $\mathcal{B}$-measurable if $\{f > t\} := \{x \in E : f(x) > t\} \in \mathcal{B}$, for all $t \in \mathbb{R}_+$.

**Definition 4.1.** Let $\nu$ be a maxitive measure on $\mathcal{B}$, and let $f : E \to \mathbb{R}_+$ be a $\mathcal{B}$-measurable map. The idempotent $\odot$-integral of $f$ with respect to $\nu$ is defined by

$$\nu(f) = \int_{E}^{\infty} f \odot d\nu = \bigoplus_{t \in \mathbb{R}_+} t \odot \nu(f > t).$$

The occurrence of $\infty$ in the notation $\int_{E}^{\infty}$ is not an integration bound, see [80, Theorem I-5.7] for a justification.

According to Gerritse [39, Proposition 3], the following identity holds:

$$\int_{E}^{\infty} f \odot d\nu = \bigoplus_{B \in \mathcal{B}} \left( f^\wedge(B) \odot \nu(B) \right),$$

where $f^\wedge(A)$ stands for $\inf_{x \in A} f(x)$. Also, notice that the supremum in Equation (5) may be reduced to a countable supremum, for

$$\int_{E}^{\infty} f \odot d\nu = \bigoplus_{t \in \mathbb{R}_+} t \odot \nu\left( \bigcup_{r \in \mathbb{Q}_+, r \geq t} \{ f > r \} \right) = \bigoplus_{t \in \mathbb{R}_+} t \odot \bigoplus_{r \in \mathbb{Q}_+, r \geq t} \nu(f > r) = \bigoplus_{r \in \mathbb{Q}_+, r \geq t} r \odot \nu(f > r),$$

so that Equation (5) is now given in a countable form.

**Proposition 4.2.** Let $\nu$ be a $\sigma$-maxitive measure on $\mathcal{B}$. Then, for all $\mathcal{B}$-measurable maps $f, g : E \to \mathbb{R}_+$, and all $r \in \mathbb{R}_+$, $B \in \mathcal{B}$, the following properties hold:

- $\nu(1_B) = \nu(B)$,
- homogeneity: $\nu(r \odot f) = r \odot \nu(f)$,
- $\sigma$-maxitivity: $\nu(\bigoplus_n f_n) = \bigoplus_n \nu(f_n)$, for every sequence of $\mathcal{B}$-measurable maps $f_n : E \to \mathbb{R}_+$,
- $B \mapsto \int_{E}^{\infty} f \odot d\nu$ is a $\sigma$-maxitive measure on $\mathcal{B}$.

**Proof.** See Sugeno and Murofushi [97, Proposition 6.1].

In order to study the idempotent integral more deeply, it would be natural to fix a measurable space $(E, \mathcal{B})$ endowed with a $\sigma$-maxitive measure $\nu$, and, by analogy with the additive case, to look at the spaces $L^p(\nu)$, $p > 0$. These are Banach spaces, as noticed by Shilkret [94] in the case where $\odot$ is the usual multiplication, and it is easily seen that the monotone
and dominated convergence theorems, the Chebyshev and Hölder inequalities, etc. are satisfied (see [84, Lemmata 1.4.5 and 1.4.7] and [84, Theorem 1.4.19]). However, these spaces are less interesting to study than their classical counterpart, since $L^p(\nu) = L^1(\nu^{1/p})$, so that all of them can be viewed as $L^1$ spaces. In particular, $L^2(\nu)$ is not a Hilbert space. Nonetheless, these spaces can be considered as generalizations of the spaces $L^\infty(m)$ (with $m$ a $\sigma$-additive measure), since $L^\infty(m) = L^1(\delta_m)$.

Further properties of the Shilkret integral with respect to an optimal measure (see Definition 6.1) were studied by Agbeko [3] and applied to characterizations of boundedness and uniform boundedness of measurable functions. We also refer the reader to Puhalskii [84] and to de Cooman [23], who both gave a pretty exhaustive treatment of the Shilkret integral. We note however that their approach is essentially limited to completely maxitive measures defined on $\tau$-$\sigma$-algebras (also called ample fields, i.e. $\sigma$-algebras closed under arbitrary intersections, see Janssen et al. [46]), but this framework has the disadvantage of breaking the parallel with classical measure theory. We shall come back to this debate in Section 7.

4.4. Examples. We pursue the study of two examples introduced above, namely the essential supremum and the Fréchet random sup-measures. We also generalize the latter with the concept of regularly-varying random sup-measure.

Example 4.3 (Example 2.3 continued). Let $\mu$ be a null-additive monotone set function and let $f : E \to \mathbb{R}_+$ be some $\mathcal{B}$-measurable map. Then the $\mu$-essential supremum of $f$ is the maxitive measure $B \mapsto \bigoplus_{x \in B} f(x)$; it can be seen as an $\odot$-idempotent integral, i.e.

$$
\bigoplus_{x \in B} f(x) = \int_B f \odot d\delta_\mu,
$$

where $\delta_\mu$ is the maxitive measure defined by $\delta_\mu(B) = 1$ if $\mu(B) > 0$, $\delta_\mu(B) = 0$ otherwise. Moreover, integration with respect to the $\mu$-essential supremum (call it $\tau$) gives

$$
\int_E g \odot d\tau = \bigoplus_{x \in E} g(x) \odot f(x) = \int_E g \odot f \odot d\delta_\mu.
$$

Example 4.4 (Example 2.8 continued). Let $(\Omega, \mathcal{A})$ and $(E, \mathcal{B})$ be measurable spaces, $P$ be a probability measure on $\mathcal{A}$, and $m$ be a finite $\sigma$-additive measure on $\mathcal{B}$. Let $M$ be a $\beta$-Fréchet random sup-measure with control measure $m$. For all measurable maps $f : E \to \mathbb{R}_+$, we can consider the
Shilkret integral $M(f)$ defined as usual by

$$\int_E^\infty f \cdot dM = \bigoplus_{t \in \mathbb{R}^+} t \cdot M(f > t).$$

This coincides with the extremal integral of Stoev and Taqqu [95] (note that these authors did not seem to know about Shilkret’s or Maslov’s works). It can be seen as a kind of stochastic integral with a deterministic integrand, very similar to the well-known $\alpha$-stable (or sum-stable) integral (see Samorodnitsky and Taqqu [90]). Note that $M(f)$ is indeed a random variable, for the supremum over $\mathbb{R}^+$ can be replaced by a countable supremum (see Paragraph 4.3). Moreover, if $f \in L^\beta(m)$, then $M(f)$ follows a Fréchet distribution with

$$P[M(f) \leq x] = \exp(-\|f\|_\beta^\beta x^{-\beta}),$$

where $\|f\|_\beta$ denotes the Lebesgue $\beta$-norm of $f$ with respect to $m$, i.e.

$$\|f\|_\beta = (\int f^\beta dm)^{1/\beta}.$$  

This implies that, whenever $\|f\|_\beta < \infty$, $B \mapsto \int_B^\infty f \cdot dM$ is itself a $\beta$-Fréchet random sup-measure with control measure $B \mapsto \int_B f^\beta dm$. See [95] for additional properties. In the particular case where

$$M(B) = \bigoplus_{k \geq 1} X_k \cdot 1_B(T_k),$$

for some Poisson point process $(X_k, T_k)_{k \geq 1}$ on $\mathbb{R}^+ \times E$ with intensity measure $\beta x^{-\beta-1}dx \times m(dt)$, we have

$$\int_E^\infty f \cdot dM = \bigoplus_{k \geq 1} X_k \cdot f(T_k).$$

De Haan [27] introduced this latter integral process and showed that, if $(X_t)_{t \in \mathbb{R}}$ is a continuous-time simple max-stable process, then there exists a Poisson process with the above properties, and a collection $(f_t)_{t \in \mathbb{R}}$ of nonnegative $L^1$ maps such that

$$(X_t)_{t \in \mathbb{R}} \overset{d}{=} (\int_E^\infty f_t \cdot dM),$$

where $d$ means equality in finite-dimensional distributions [27, Theorem 3].

**Example 4.5** (Regularly-varying sup-measures). A variant on the previous example can be done as follows. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(E, \mathcal{B})$ be a measurable space, and $m$ be a finite $\sigma$-additive measure on $\mathcal{B}$. We define a $\beta$-regularly-varying random sup-measure with control measure $m$ to be a map $M: \Omega \times \mathcal{B} \rightarrow \mathbb{R}^+$ satisfying the following conditions:
for all pairwise disjoint collections \((B_j)_{j \in \mathbb{N}}\) of elements of \(\mathcal{B}\), the random variables \(M(B_j), j \in \mathbb{N}\), are independent, and, almost surely,

\[
M(\bigcup_{j \in \mathbb{N}} B_j) = \bigoplus_{j \in \mathbb{N}} M(B_j);
\]

for all \(B \in \mathcal{B}\) the random variable \(M(B)\) is regularly-varying of index \(\beta\), i.e. there exists a function \(L\), slowly-varying at \(\infty\), such that, when \(x \to \infty\),

\[
P[M(B) > x] \sim m(B) x^{-\beta} L(x).
\]

Recall that \(L : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}\) is slowly-varying at \(\infty\) if, for all \(a > 0\),

\[
\lim_{x \to \infty} L(ax)/L(x) = 1.
\]

See e.g. Resnick [86] for more on regularly- and slowly-varying functions. For all measurable maps \(f : E \to \mathbb{R}_+\), the random variable \(M(f)\) defined as the Shilkret integral of \(f\) with respect to \(M\)

\[
P[M(f) > x] \sim \|f\|_{\beta} x^{-\beta} L(x),
\]

when \(x \to \infty\), for all \(f \in L^\beta(m)\). Let us prove this assertion. First, consider the case where \(f\) is a nonnegative (measurable) simple map, i.e. a map of the form \(f = \sum_{j=1}^k t_j 1_{B_j}\), where \(B_1, \ldots, B_k \in \mathcal{B}\) are pairwise disjoint and \(t_j > 0\) for \(j = 1, \ldots, k\). One can write \(f = \bigoplus_{j=1}^k t_j 1_{B_j}\). Thus, \(M(f) = \bigoplus_{j=1}^k t_j M(B_j)\), almost surely, so that

\[
P[M(f) > x] \sim -\log P[M(f) \leq x] = \sum_{j=1}^k -\log P[M(B_j) \leq x/t_j],
\]

since the random variables \(M(B_1), \ldots, M(B_k)\) are independent. We get

\[
P[M(f) > x] \sim \sum_{j=1}^k P[M(B_j) > x/t_j](1 + o(1))
\]

\[
= \sum_{j=1}^k m(B_j) t_j^\beta x^{-\beta} L(x/t_j)(1 + o(1))
\]

\[
= \sum_{j=1}^k m(B_j) t_j^\beta x^{-\beta} L(x)(1 + o(1)),
\]

since \(L\) is slowly-varying. This shows that \(P[M(f) > x] \sim \|f\|_{\beta} x^{-\beta} L(x)\).

In the general case where \(f\) is a nonnegative map in \(L^\beta(m)\), let \((\varphi_n)\) be a nondecreasing sequence of nonnegative simple maps that converges pointwise to \(f\). Then \(\|\varphi_n\|_{\beta} \to \|f\|_{\beta}\) when \(n \to \infty\). As a consequence,

\[
P[M(\varphi_n) > x] \sim_{x \to \infty} \|\varphi_n\|_{\beta} x^{-\beta} L(x) \to_n \|f\|_{\beta} x^{-\beta} L(x),
\]

But we also have \(P[M(\varphi_n) > x] \to_n P[M(f) > x]\), and the result follows.
5. The Radon–Nikodym Theorem

5.1. Introduction. A widespread proof of the Radon–Nikodym theorem for \( \sigma \)-additive measures, due to von Neumann, uses the representation of bounded linear forms on a Hilbert space (see e.g. Rudin [89]). But for \( \sigma \)-maxitive measures the space \( L^2 \), as already noticed, actually reduces to an \( L^1 \) space, for \( L^2(\nu) = L^1(\nu^{1/2}) \) for every \( \sigma \)-maxitive measure \( \nu \). That is why such an approach is not possible\(^1\), and we have to find another way for proving a Radon–Nikodym theorem for \( \sigma \)-maxitive measures. Sugeno, in relation to the Sugeno integral, was confronted with the same problem in his thesis, and gave sufficient conditions for the existence of a Radon–Nikodym derivative [96] at the cost of a topological structure on \( E \). This first result was refined by Candeloro and Pucci [18, Theorem 3.7] and Sugeno and Murofushi [97, Corollary 8.3].

In this section, we give a general definition of the density of a maxitive measure with respect to the Shilkret integral. Then we recall the main theorem stating the existence of such a density [97, Corollary 8.4]. Here, \( \mathcal{B} \) still denotes a \( \sigma \)-algebra.

The literature is not unanimous in the meaning of the term “density” applied to maxitive measures. For Akian [5], a density is any map \( c \) such that \( \nu(\cdot) = \bigoplus_{x \in \cdot} c(x) \), i.e. what we called cardinal density. For Barron et al. [12] and Drewnowski [31], a density corresponds to our concept of relative density (see Section 3). The following definition encompasses both points of view. Let \( \nu \) and \( \tau \) be maxitive measures on \( \mathcal{B} \). Then \( \nu \) has a density with respect to \( \tau \) if there exists some \( \mathcal{B} \)-measurable map (called density) \( c : E \to \mathbb{R}_+ \) such that

\[
\nu(B) = \int_B c \circ d\tau,
\]

for all \( B \in \mathcal{B} \).

**Definition 5.1.** Let \( \nu, \tau \) be monotone set functions on \( \mathcal{B} \). Then \( \nu \) is \( \bigcirc \)-absolutely continuous with respect to \( \tau \) (or \( \tau \) \( \bigcirc \)-dominates \( \nu \)), in symbols \( \nu \ll_{\bigcirc} \tau \), if for all \( B \in \mathcal{B} \), \( \nu(B) \leq \infty \circ \tau(B) \).

**Remark 5.2.** In [82], I have given a slightly different definition of \( \bigcirc \)-absolute continuity, which was that \( \nu \) is \( \bigcirc \)-absolutely continuous with respect to \( \tau \) if for all \( B \in \mathcal{B} \) such that \( \tau(B) \) be \( \bigcirc \)-finite, \( \nu(B) \leq \infty \circ \tau(B) \).

It is easily seen that the two definitions coincide when either \( \nu \) is semi-\( \bigcirc \)-finite, or \( \tau \) is \( \sigma \)-\( \bigcirc \)-finite and \( \nu \) is \( \sigma \)-maxitive (see the definitions of semi-\( \bigcirc \)-finiteness and \( \sigma \)-\( \bigcirc \)-finiteness below). For that reason, all the results of [82]\(^1\)
that involve the latter definition of $\odot$-absolute continuity are still valid with
the former definition.

In the case where $\odot$ is the usual multiplication $\times$ (resp. the infimum $\wedge$),
then $\ll_\odot$ coincides with the usual relation $\ll$ (resp. with $\leq$). If $\nu$ has a
density with respect to $\tau$, then $\nu$ is $\odot$-absolutely continuous with respect to
$\tau$, according to Definition 3.1. Taking $\tau = \delta_\#_\mu$ in Equation (6), one gets
$\nu(B) = \bigoplus_{x \in B} c(x)$, i.e. one recovers the notion of cardinal density introduced in Example 2.4. If $\mu$ is a null-additive monotone set function, then Equation (6) with $\tau = \delta_\#_\mu$ rewrites as
$\nu(B) = \bigoplus_{x \in B} c(x)$, which fits with the case of essential suprema and relative densities introduced in Example 2.3.

5.2. Uniqueness and finiteness of the density. Let $(E, \mathcal{B})$ be a measurable space. A set function $\nu : \mathcal{B} \to \mathbb{R}_+$ is semi-$\odot$-finite if, for all $B \in \mathcal{B}$,
$\nu(B) = \bigoplus_{A \subset B} \nu(A)$, where the supremum is taken over \{\text{all } A \in \mathcal{B} : A \subset B, \nu(A) \ll_\odot \infty\}.

Proposition 5.3. Let $\nu, \tau$ be $\sigma$-maxitive measures on $\mathcal{B}$. Assume that $\nu$ is semi-$\odot$-finite and admits a $\mathcal{B}$-measurable density $c$ with respect to $\tau$. Then $\nu$ admits a $\odot$-finite-valued $\mathcal{B}$-measurable density with respect to $\tau$.

Proof. See [82, Proposition 3.2].

Paralleling the classical case, we have the following result on “uniqueness” of the density.

Proposition 5.4. Let $\nu, \tau$ be $\sigma$-maxitive measures on $\mathcal{B}$. If $\nu$ admits a $\mathcal{B}$-measurable density $c$ with respect to $\tau$, then this density is unique, $\tau$-almost everywhere.

Proof. The assertion can be proved along the same lines as the case of the Lebesgue integral, see e.g. Rudin [89, Theorem 1.39(b)].

5.3. Principality and existence of a density. Let $(E, \mathcal{B})$ be a measurable space. Sugeno and Murofushi [97, Corollary 8.4] proved a Radon–Nikodym theorem for the Shilkret integral when the dominating measure is $\sigma$-$\odot$-finite and $\sigma$-principal.

A null-additive monotone set function $\tau$ on $\mathcal{B}$ is $\odot$-finite if $\tau(E) \ll_\odot \infty$, and $\sigma$-$\odot$-finite if there exists some countable family $\{B_n\}_{n \in \mathbb{N}}$ of elements of $\mathcal{B}$ covering $E$ such that $\tau(B_n) \ll_\odot \infty$ for all $n$. It is $\sigma$-principal if, for every $\sigma$-ideal $\mathcal{I}$ of $\mathcal{B}$, there exists some $L \in \mathcal{I}$ such that $S \setminus L$ is $\tau$-negligible, for all $S \in \mathcal{I}$. See [82, Proposition 4.1] for a justification of this terminology.
**Theorem 5.5** (Sugeno–Murofushi). Let $\nu, \tau$ be $\sigma$-maxitive measures on $\mathcal{B}$. Assume that $\tau$ is $\sigma$-$\mathcal{C}$-finite and $\sigma$-principal. Then $\nu \ll \tau$ if and only if there exists some $\mathcal{B}$-measurable map $c : E \to \mathbb{R}_+$ such that

$$
\nu(B) = \int_B c \circ d\tau,
$$

for all $B \in \mathcal{B}$. If these conditions are satisfied, then $c$ is unique $\tau$-almost everywhere. Moreover, if $\nu$ is semi-$\circ$-finite, one can choose a map $c$ taking only $\circ$-finite values.

**Proof.** See [97, Theorem 8.2] for the original proof. See also [80, Chapter III] for another proof of this theorem that makes use of order-theoretical arguments, in the case where $\circ$ is the usual multiplication. \qed

If $\circ$ is the usual multiplication, the hypothesis of $\sigma$-$\mathcal{C}$-finiteness of $\tau$ cannot be removed: consider for instance a finite set $E$, and let $\nu = \delta_{\#}$ and $\tau = \infty \cdot \delta_{\#}$ be $\sigma$-maxitive measures defined on the power set of $E$. Then $\tau$ is $\sigma$-principal and $\nu$ is absolutely continuous with respect to $\tau$, but $\nu$ never has a density with respect to $\tau$.

**Theorem 5.5** encompasses **Theorem 3.3**, for if $\tau$ is an essential $\sigma$-maxitive measure, then $\delta_\tau$ is ($\sigma$-finite and) $\sigma$-principal (use Theorem A.1). We can thus state the following corollary.

**Corollary 5.6** (Generalization of Barron–Cardaliaguet–Jensen). Let $\nu, \tau$ be $\sigma$-maxitive measures on $\mathcal{B}$. Assume that $\tau$ is $\sigma$-principal. Then $\nu \ll \tau$ if and only if $\nu \ll \ll \tau$. In this situation, the relative density of $\nu$ with respect to $\tau$ is unique $\tau$-almost everywhere.

We have another simple consequence, which generalizes Corollary 3.4.

**Corollary 5.7.** Let $\nu$ be a $\sigma$-principal $\sigma$-maxitive measure on $\mathcal{B}$. Then $\nu$ is autocontinuous. Moreover, if the empty set is the only $\nu$-negligible subset, then $\nu$ is completely maxitive (and has a cardinal density).

**Proof.** Simply take $\tau = \delta_\nu$ in the previous theorem. \qed

At this stage we think it useful to recall the characterization of those $\sigma$-maxitive measures $\tau$ with the Radon–Nikodym property, i.e. such that all $\sigma$-maxitive measures $\circ$-dominated by $\tau$ have a measurable density with respect to $\tau$.

**Theorem 5.8.** Given a non-degenerate pseudo-multiplication $\circ$, a $\sigma$-maxitive measure $\tau$ on $\mathcal{B}$ satisfies the Radon–Nikodym property with respect to the idempotent $\circ$-integral if and only if $\tau$ is $\sigma$-$\mathcal{C}$-finite and $\sigma$-principal.

**Proof.** See [82]. \qed
Corollary 5.9. Let $\tau$ be a $\sigma$-maxitive measure on $\mathcal{B}$. Then $\tau$ satisfies the Radon–Nikodym property with respect to the Shilkret integral if and only if $\tau$ is $\sigma$-finite and $\sigma$-principal.

Corollary 5.10. Let $\tau$ be a $\sigma$-maxitive measure on $\mathcal{B}$. Then $\tau$ satisfies the Radon–Nikodym property with respect to the Sugeno integral if and only if $\tau$ is $\sigma$-principal.

Two $\sigma$-maxitive measures $\nu$ and $\tau$ on $\mathcal{B}$ are associated if there exists a third $\sigma$-maxitive measure $\mu$ on $\mathcal{B}$ such that $\nu \ll \mu$ and $\tau \ll \mu$. A reformulation of Corollary 5.6 is that, if $\tau$ is $\sigma$-principal and $\nu \ll \tau$, then $\nu$ and $\tau$ are associated. With this notion of associated maxitive measures we can give a variant of the Radon–Nikodym type theorem, which is a generalization of Puhalskii [84, Theorem 1.6.34] and de Cooman [23, Theorem 7.2].

Theorem 5.11 (Idempotent Radon–Nikodym theorem, variant). Let $\circ$ be a pseudo-multiplication that makes $\mathbb{R}_+$ into an exact residual semigroup (see Section B in the appendix). Let $\nu$, $\tau$ be $\sigma$-maxitive measures on $\mathcal{B}$, and assume that $\nu$ and $\tau$ are associated. Then $\nu \ll \circ \tau$ if and only if there exists some $\mathcal{B}$-measurable map $c : E \to \mathbb{R}_+$ such that

$$\nu(B) = \int_B c \circ d\tau,$$

for all $B \in \mathcal{B}$. If these conditions are satisfied, then $c$ is unique $\tau$-almost everywhere. Moreover, if $\nu$ is semi-$\circ$-finite, one can choose a map $c$ taking only $\circ$-finite values.

Proof. We assume that $\nu$ and $\tau$ are associated and such that $\nu \ll \circ \tau$. By definition, there is a $\sigma$-maxitive measure $\mu$ on $\mathcal{B}$ such that $\nu \ll \mu$ and $\tau \ll \mu$. So there are $\mathcal{B}$-measurable maps $c_1, c_2 : E \to \mathbb{R}_+$ such that $\nu(B) = \bigoplus_{x \in B} c_1(x)$ and $\tau(B) = \bigoplus_{x \in B} c_2(x)$, for all $B \in \mathcal{B}$.

We use the notations of Section B in the appendix. Let $A$ be the subset

$$A = \{x \in E : c_1(x) \ll_\circ c_2(x)\}.$$

We show that $A$ is $\mu$-negligible. We have

$$A = \{x \in E : c_1(x) > \infty \circ c_2(x)\}$$

$$= \bigcup_{q \in \mathbb{Q}_+} \{x \in E : c_1(x) > q \text{ and } q \geq \infty \circ c_2(x)\}$$

$$= \bigcup_{q \in \mathbb{Q}_+} B_q \cap \{c_1 > q\},$$

where $B_q$ is the subset $\{x \in E : \infty \circ c_2(x) \leq q\}$. Notice that $B_q$ is $\mathcal{B}$-measurable since

$$B_q = \bigcap_{r \in \mathbb{Q}_+} \{x \in E : r \circ c_2(x) \leq q\},$$

22
and hence \( A \) is \( \mathcal{B} \)-measurable too. To prove that \( A \) is \( \mu \)-negligible first note that

\[
\bigoplus_{x \in B_q} \infty \circ \tau(B_q) = \bigoplus_{x \in B_q} \infty \circ c_2(x) \leq q,
\]

for all \( q \in \mathbb{Q}_+ \). Since \( \nu \ll \infty \circ \tau \) this implies \( \nu(B_q) \leq q \) for all \( q \in \mathbb{Q}_+ \). Since \( \nu(B_q) \) is the \( \mu \)-essential supremum of \( c_1 \) on \( B_q \), i.e.

\[
\nu(B_q) = \inf\{ t > 0 : \mu(B_q \cap \{ c_1 > t \}) = 0 \},
\]

this shows that \( \mu(B_q \cap \{ c_1 > q \}) = 0 \). Consequently,

\[
\mu(A) = \bigcup_{q \in \mathbb{Q}_+} \mu(B_q \cap \{ c_1 > t \}) = 0.
\]

By definition of \( A \), we have \( c_1(x) \ll c_2(x) \) for all \( x \in E \setminus A \), so we can define the map \( c : E \to \mathbb{R}_+ \) by \( c(x) = 0 \) if \( x \in A \) and \( c(x) = (c_1(x)/c_2(x)) \circ \) if \( x \in E \setminus A \) (see again Section B for the notations). The map \( c \) is \( \mathcal{B} \)-measurable because

\[
\{ x \in E : c(x) \leq t \} = A \cup \{ x \in E \setminus A : (c_1(x)/c_2(x)) \circ \leq t \}
\]

\[
= A \cup \{ x \in E \setminus A : c_1(x) \leq t \circ c_2(x) \},
\]

for all \( t \in \mathbb{R}_+ \). By assumption \((\mathbb{R}_+, \circ)\) is exact, so \( c_1(x) = c(x) \circ c_2(x) \) for all \( x \in E \setminus A \). As a consequence,

\[
\nu(B) = \int_{\mathcal{B}} c_1(x) \circ d\delta_\mu
\]

\[
= \int_{\mathcal{B} \cap (E \setminus A)} c_1(x) \circ d\delta_\mu
\]

\[
= \int_{\mathcal{B} \cap (E \setminus A)} c(x) \circ c_2(x) \circ d\delta_\mu
\]

\[
= \int_{\mathcal{B} \cap (E \setminus A)} c(x) \circ d\tau,
\]

for all \( B \in \mathcal{B} \), and the result is proved. \( \square \)

6. Optimality of maxitive measures

6.1. Definition of optimal measures. In this section we focus on the special case of optimal measures. Let \((E, \mathcal{B})\) be a measurable space. A set function \( \nu \) on \( \mathcal{B} \) is continuous from above if \( \nu(B) = \lim_n \nu(B_n) \), for all \( B_1 \supset B_2 \supset \ldots \in \mathcal{B} \) such that \( B = \bigcap_n B_n \) (we do not impose the condition \( \nu(B_{n_0}) < \infty \) for some \( n_0 \)). A monotone null-additive set function that is both continuous from above and from below is a fuzzy measure. Continuity from above is automatically satisfied for finite \( \sigma \)-additive measures, but this
is untrue for (finite) $\sigma$-maxitive measures (see Puri and Ralescu [85] for a counterexample, see also Wang and Klir [101, Example 3.13]), so special care is needed. The following definition is given by Agbeko [2].

**Definition 6.1.** An *optimal measure* is a maxitive fuzzy measure.

Surprisingly, it suffices for a maxitive measure to be continuous from above in order to satisfy continuity from below:

**Proposition 6.2** (Murofushi–Sugeno–Agbeko). A set function $\nu$ on $\mathcal{B}$ is an optimal measure if and only if it is a continuous from above maxitive measure. In this case, for all sequences $(B_n)$ of elements of $\mathcal{B}$,

$$\nu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \max_{n \in \mathbb{N}} \nu(B_n),$$

where the max operator signifies that the supremum is reached.

**Proof.** Murofushi and Sugeno [68] and after them Agbeko [2, Lemma 1.4] and Kramosil [51] showed that every continuous from above maxitive measure $\nu$ satisfies the identity of the proposition; the first part of the proposition is then an easy consequence.

The property of continuity from above in Definition 6.1 is thus a strong condition. It becomes even more obvious with the following result. It was proved by Agbeko [2, Theorem 1.2] using Zorn’s lemma, and Fazekas [35, Theorem 9] supplied an elementary proof. To formulate it, recall first that a $\nu$-atom (called indecomposable $\nu$-atom by Agbeko) is an element $H$ of $\mathcal{B}$ such that $\nu(H) > 0$, and for each $B \in \mathcal{B}$ either $\nu(H \setminus B) = 0$, or $\nu(H \cap B) = 0$.

**Theorem 6.3** (Agbeko–Fazekas). Let $\nu$ be an optimal measure on $\mathcal{B}$. Then there exists an at most countable collection $(H_n)_{n \in \mathbb{N}}$ of pairwise disjoint $\nu$-atoms $H_n \in \mathcal{B}$ such that

$$\nu(B) = \max_{n \in \mathbb{N}} \nu(B \cap H_n),$$

for all $B \in \mathcal{B}$, where the max operator signifies that the supremum is reached. In particular, $\nu$ takes an at most countable number of values.

A consequence of this theorem is that every optimal measure takes an at most countable number of values.

An optimal measure $\nu$ satisfies the *exhaustivity* property, according to the terminology used by Pap [75], i.e. $\nu(B_n) \to 0$ when $n \to \infty$ for all pairwise disjoint $B_1, B_2, \ldots \in \mathcal{B}$. In fact, exhaustivity is exactly what a $\sigma$-maxitive measure needs to be optimal:

**Proposition 6.4.** A $\sigma$-maxitive measure is optimal if and only if it is exhaustive.
Proof. The easy proof is left to the reader. □

Optimal measures were also studied (under various names) by Riečanová [88], Murofushi and Sugeno [68], Arslanov and Ismail [10]. In particular, the last-mentioned authors proved that the cardinality of some nonempty set $E$ is non-measurable if and only if all optimal measures on $2^E$ have a cardinal density [10, Theorem 19]. In [81] we studied $L$-valued optimal measures defined on the Borel algebra of a topological space, where $L$ is a partially ordered-set.

In Section 5 we introduced semi-$\odot$-finiteness for maxitive measures. For optimal measures, this merely reduces to $\odot$-finiteness.

**Proposition 6.5.** An optimal measure is semi-$\odot$-finite if and only if it is $\odot$-finite.

*Proof.* Let $\nu$ be a semi-$\odot$-finite optimal measure on $\mathcal{B}$. If $\nu(E) = 0$, the result is clear. Otherwise, let $0 < s < \nu(E)$. In view of Fazekas [35, Remark 5], the set $\{\nu(B) : B \in \mathcal{B}, \nu(B) > s\}$ is finite, thus

$$\bigoplus_{B \subseteq E, \nu(B) \ll \odot \infty} \nu(B) = \nu(B_0)$$

for some $B_0 \in \mathcal{B}$ such that $\nu(B_0) \ll \odot \infty$. By semi-$\odot$-finiteness, $\nu(E) = \nu(B_0) \ll \odot \infty$, so $\nu$ is $\odot$-finite. □

6.2. **Densities of optimal measures.** In this paragraph, we use previous results on the existence of densities for $\sigma$-maxitive measures, and apply them to optimal measures.

Agbeko proved Theorem 5.5 independently of Sugeno and Murofushi [97] in the particular case where $\tau$ is a normed optimal measure and $\nu$ is a finite optimal measure on $\mathcal{B}$ [2, Theorem 2.4]. This is indeed a particular case thanks to [68, Lemma 2.1], which states that every optimal measure is CCC, hence $\sigma$-principal under Zorn’s lemma. Below we show without Zorn’s lemma that every optimal measure is $\sigma$-principal (hence CCC by [82, Proposition 4.1]). We actually show the stronger result that every optimal measure is essential -although not of bounded variation in general, as asserted by the next proposition.

**Proposition 6.6.** For every optimal measure $\nu$ we have $|\nu| = \sum_n n\nu(H_n)$, where $(H_n)_{n \in \mathcal{N}}$ is a collection satisfying the conditions of Theorem 6.3. In particular, $\nu$ is of bounded variation if and only if $\sum_n \nu(H_n) < \infty$.

---

2A cardinal $|E|$ is measurable if there exists a two-valued probability measure on $2^E$ making all singletons negligible. The existence of measurable cardinals remains an open question.
Proof. Let \( \nu \) be an optimal measure on a \( \sigma \)-algebra \( \mathcal{B} \), and let \((H_n)_{n \in N}\) be a collection satisfying the conditions of the Agbeko-Fazekas Theorem (Theorem 6.3).

Recall that \( |\nu| \) is defined as \( |\nu| = \sup_{\pi} \sum_{B \in \pi} \nu(B) \), where the supremum is taken over the set of finite \( \mathcal{B} \)-partitions \( \pi \) of \( E \). Let \( \pi_n \) denote the finite \( \mathcal{B} \)-partition \( \{H_1, \ldots, H_n, E \setminus \bigcup_{k=1}^{n} H_k\} \). Then \( \sum_{k=1}^{n} \nu(H_k) \leq \sum_{k=1}^{n} \nu(H_k) + \nu(\bigcap_{k=1}^{n} E \setminus H_k) \leq |\nu| \), so that \( \sum_{k=1}^{\infty} \nu(H_k) \leq |\nu| \).

Conversely, let \( \{B_1, \ldots, B_n\} \) be a finite \( \mathcal{B} \)-partition of \( E \). We can suppose without loss of generality that \( \nu(B_k) > 0 \) for all \( 1 \leq k \leq n \). By the Agbeko–Fazekas Theorem, for every \( k = 1, \ldots, n \) there exists some \( n_k \) such that \( 0 < \nu(B_k) = \nu(B_k \cap H_{n_k}) \leq \nu(H_{n_k}) \). Moreover, \( k \neq k' \) implies \( n_k \neq n_{k'} \), because if \( H := H_{n_k} = H_{n_{k'}} \) and \( k \neq k' \), then \( B_k \cap B_{k'} = \emptyset \), so \( \nu(H) = \nu(H \setminus (B_k \cup B_{k'})) = \nu(H \setminus B_k) + \nu(H \setminus B_{k'}) = 0 \), a contradiction. Consequently, \( \sum_{k=1}^{n} \nu(B_k) \leq \sum_{k=1}^{n} \nu(H_{n_k}) \leq \sum_{k=1}^{\infty} \nu(H_k) \), so that \( |\nu| \leq \sum_{k=1}^{\infty} \nu(H_k) \).

\[ \square \]

Proposition 6.7. Every optimal measure is essential (hence \( \sigma \)-principal, hence CCC and autocontinuous).

Proof. Let \( \nu \) be an optimal measure on a \( \sigma \)-algebra \( \mathcal{B} \), and let \((H_n)_{n \in N}\) be a collection satisfying the conditions of the Agbeko-Fazekas Theorem (Theorem 6.3). We can suppose, without loss of generality, that \( \nu \) is finite. We define \( m \) on \( \mathcal{B} \) by

\[ m(B) = \sum_{n} \nu(B \cap H_n). \]

Then one can show that \( m \) is a \( \sigma \)-finite, \( \sigma \)-additive measure on \( \mathcal{B} \) such that \( m(B) > 0 \) if and only if \( \nu(B) > 0 \). What makes \( m \) additive is that \( \nu((B \cup B') \cap H_n) = \nu(B \cap H_n) + \nu(B' \cap H_n) \) whenever \( B \cap B' = \emptyset \). This is because, if \( B \cap B' = \emptyset \), then \( \nu(B \cap H_n) > 0 \) implies \( \nu(B' \cap H_n) = 0 \), since \( \nu(H_n) = \nu(H_n \setminus (B \cup B')) = \nu(H_n \setminus B) + \nu(H_n \setminus B') = \nu(H_n \setminus B') > 0 \).

As a consequence, we derive the Radon–Nikodym like theorem for optimal measures due to Agbeko.

Corollary 6.8 (Agbeko). Let \( \nu, \tau \) be \( \sigma \)-maxitive measures on \( \mathcal{B} \). Assume that \( \tau \) is \( \odot \)-finite and optimal. Then \( \nu \ll \odot \tau \) if and only if there exists some \( \mathcal{B} \)-measurable map \( c : E \rightarrow \mathbb{R}_+ \) such that

\[ \nu(B) = \int_B c \odot d\tau, \]

for all \( B \in \mathcal{B} \). If these conditions are satisfied, then \( c \) is unique \( \tau \)-almost everywhere.

Proof. Combine Theorem 5.5 and Proposition 6.7, or use Agbeko [2, Theorem 2.4] for the original statement. \[ \square \]
Problem 6.9. Characterize those $\sigma$-maxitive measures $\tau$ that satisfy the *optimal* Radon–Nikodym property, i.e. such that all optimal measures that are $\ominus$-absolutely continuous with respect to $\tau$, have a measurable density with respect to $\tau$.

7. FOUNDATIONS OF POSSIBILITY THEORY

7.1. Towards an appropriate definition of possibility measures. Possibility theory is an analogue of probability theory, where probability measures are replaced by their maxitive counterpart. It has been developed over the last few years by several authors including Bellalouna [14], Akian et al. [6, 7], Akian [4], Del Moral and Doisy [29], de Cooman [22, 23, 24, 25], Puhalskii [84], Barron et al. [13], Fleming [37] among others. See also Bac- celli et al. [11]. Analogies with probability theory, especially stressed by de Cooman [22] and Akian et al. [7], arise in the definitional aspects (such as the notion of independent events, or the concept of *maxingale* which replaces that of martingale [84, 13]) as well as in important results such as the law of large numbers or the central limit theorem. Nonetheless, possibility theory has its own specificities, for instance the surprising fact that convergence in “possibility” implies almost sure convergence$^3$ (see [4, Proposition 28] and [84, Theorem 1.3.5]).

In a stochastic context, the Radon–Nikodym property is highly desirable if one wants to dispose of conditional laws. In the $\sigma$-additive case this property is achieved by the classical Radon–Nikodym theorem$^4$, but in the $\sigma$-maxitive case this property may fail in absence of the $\sigma$-principality condition. To overcome this drawback, most of the publications require the possibility measure under study $\Pi$ to have a cardinal density, i.e. to be of the form

\[ \Pi[A] = \bigoplus_{\omega \in \mathcal{A}} c(\omega). \]

This condition was imposed by Akian et al. [6, 7], Akian [4], Del Moral and Doisy [29], de Cooman [22, 23, 24, 25], Puhalskii [84], Fleming [37]. Hypothesis (8) then facilitates the definition of conditioning, for $\Pi [X | Y]$ can be defined by the data of its cardinal density $c_{X|Y}$ given by:

\[ c_{X|Y}(x|y) = \frac{c_{(X,Y)}(x,y)}{c_Y(y)}, \]

if $c_Y(y) > 0$, and $c_{X|Y}(x|y) = 0$ otherwise, where $c_X$ and $c_Y$ are the respective (maximal) cardinal densities of $\Pi_X := \Pi \circ X^{-1}$ and $\Pi_Y$, and $c_{(X,Y)}$.

---

$^3$Recall that probabilists are familiar with the converse implication.

$^4$Notice that every probability measure is $\sigma$-principal, see Theorem A.1 in the Appendix.
that of the random variable \((X, Y) : \Omega \times \Omega \to \mathbb{R}_+\). In [26] and [84], another restrictive hypothesis was adopted, for their authors only considered completely maxitive measures defined on \(\tau\)-algebras. A \(\tau\)-algebra \(\mathcal{A}\) on \(\Omega\) being atomic, every \(\omega \in \Omega\) is contained in a smallest event, denoted by \([\omega]_{\mathcal{A}}\). This particularity enables one to give an explicit formula of conditional laws, \(\omega\) by \(\omega\).

The assumption of complete maxitivity and the use of \(\tau\)-algebras instead of \(\sigma\)-algebras, if easier to handle, are certainly not satisfactory, especially if one wants to parallel probability theory. A more general framework is possible, and we suggest to adopt the following definition of a possibility measure.

**Definition 7.1.** Let \((\Omega, \mathcal{A})\) be a measurable space. A possibility measure (or a possibility for short) on \((\Omega, \mathcal{A})\) is a \(\sigma\)-principal \(\sigma\)-maxitive measure \(\Pi\) on \(\mathcal{A}\) such that \(\Pi[\Omega] = 1\). Then \((\Omega, \mathcal{A}, \Pi)\) is called a possibility space.

7.2. **Conditional law with respect to a possibility measure.** A conjunction of factors tends to confirm that this is the right definition. Firstly, properties of \(\Pi\) are transferred to the “laws” of random variables. If \((E, \mathcal{B})\) is a measurable space and \(X : \Omega \to E\) is a random variable, its (possibility) law \(\Pi_X\) on \(\mathcal{B}\) is the set function defined by \(\Pi_X(B) = \Pi[X \in B] := \Pi[X^{-1}(B)]\), and this is a possibility measure. Moreover, if \(\Pi\) is optimal (resp. completely maxitive), then \(\Pi_X\) is optimal (resp. completely maxitive).

Secondly, the \(\sigma\)-principality property ensures that the Radon–Nikodym property is satisfied for the Shilkret integral \(\Sigma[X] := \int X \cdot d\Pi\) of some random variable \(X : \Omega \to \mathbb{R}_+\). Thus, following the classical approach of Halmos and Savage [42], conditioning can be defined as follows. Let \(X : \Omega \to \mathbb{R}_+\) be a random variable and \(\mathcal{D}\) be a sub-\(\sigma\)-algebra of \(\mathcal{A}\). The \(\sigma\)-maxitive measure defined on \(\mathcal{D}\) by \(A \mapsto \Sigma[X.1_A] = \int_A X \cdot d\Pi\) is absolutely continuous with respect to the possibility \(\Pi|_{\mathcal{D}}\). Thus, there exists some \(\mathcal{D}\)-measurable random variable from \(\Omega\) into \(\mathbb{R}_+\), written \(\Sigma[X|\mathcal{D}]\), such that \(\Sigma[X.1_A] = \Sigma[\Sigma[X|\mathcal{D}].1_A]\) for all \(A \in \mathcal{D}\).

Barron et al. [13] considered the special case \(\Pi := \delta_P\), where \(P\) is a probability measure. Then \(\Pi\) is essential, hence \(\sigma\)-principal, so it is a possibility measure, and the Shilkret integral \(\Sigma[X]\) of a random variable \(X\) coincides with the \(P\)-essential supremum of \(X\), i.e. \(\Sigma[X] = \bigoplus_{\omega \in \Omega} X(\omega)\). Also, whenever \(\Sigma[X] < \infty\), one has \(\Sigma[X|\mathcal{D}] = \lim_{p \to \infty} E[X^p|\mathcal{D}]^{1/p}\), \(P\)-almost surely (where \(E[X]\) denotes the usual expected value of \(X\) with respect to the probability measure \(P\)), see [13] Proposition 2.12. Barron et al. derived a number of properties that still work in our more general context, as asserted by the next result (whose proof is left to the reader).
Proposition 7.2. Let $X : \Omega \to \mathbb{R}_+$ be a random variable and $\mathcal{D}$ be a sub-$\sigma$-algebra of $\mathcal{B}$. Then the following assertions hold:

- $Y$ is $\Pi$-almost surely equal to $\Sigma[X|\mathcal{D}]$ if and only if $\Sigma[XZ] = \Sigma[YZ]$ for all $\mathcal{D}$-measurable random variables $Z$, 
- $X \leq \Sigma[X|\mathcal{D}]$, $\Pi$-almost surely, 
- if $Y : \Omega \to \mathbb{R}_+$ is a $\mathcal{D}$-measurable random variable such that $X \leq Y$, $\Pi$-almost surely, then $\Sigma[X|\mathcal{D}] \leq Y$, $\Pi$-almost surely, 
- $X \mapsto \Sigma[X|\mathcal{D}]$ is a $\oplus$-linear form, 
- $\Sigma[\Sigma[X|\mathcal{D}]] = \Sigma[X]$, 
- if $X$ is $\mathcal{D}$-measurable then $\Sigma[X|\mathcal{D}] = X$, $\Pi$-almost surely,

where “$\Pi$-almost surely” stands for “$\Pi$-almost everywhere”.

Remark 7.3. Considering the second and third properties, $\Sigma[X|\mathcal{D}]$ can be interpreted as a projection (in an order-theoretical sense) of $X$ on the set of $\mathcal{D}$-measurable random variables.

From these properties, Barron et al. deduced an ergodic theorem for maxima and, with the concept of maxingales, developed a theory of optimal stopping in $L^\infty$.

Our new perspective on possibility measures should encourage us to recast possibility theory. The next step would be to confirm that convergence theorems given in [4] and [84] remain unchanged.

8. Conclusion and Perspectives

In this paper, we have emphasized the link between essential suprema representations and Radon–Nikodym like theorems for the idempotent integral. We have shown that the Radon–Nikodym type theorem proved by Sugeno and Murofushi encompasses similar results including those of Agbeko, Barron et al., Drewnowski. We have proved a variant of this theorem that generalizes results due to de Cooman, Puhalskii. We have also recalled a converse statement to the Sugeno–Murofushi theorem, i.e. the characterization of those $\sigma$-maxitive measures satisfying the Radon–Nikodym property as being $\sigma$-$\ominus$-finite $\sigma$-principal.

Acknowledgements. I am grateful to Colas Bardavid who carefully read a preliminary version of the manuscript and made very accurate suggestions. I also thank Marianne Akian who made useful remarks and provided a counterexample to [98, Exercise II-3.19.1] inserted as Example 2.10, and Prof. Jimmie D. Lawson for his advice and comments.
REFERENCES

[1] Emilio Acerbi, Giuseppe Buttazzo, and Francesca Prinari. The class of functionals which can be represented by a supremum. J. Convex Anal., 9(1):225–236, 2002.

[2] Nutefe Kwami Agbeko. On the structure of optimal measures and some of its applications. Publ. Math. Debrecen, 46(1-2):79–87, 1995.

[3] Nutefe Kwami Agbeko. How to characterize some properties of measurable functions. Math. Notes (Miskolc), 1(2):87–98, 2000.

[4] Marianne Akian. Theory of cost measures: convergence of decision variables. Rapport de recherche 2611, INRIA, France, 1995.

[5] Marianne Akian. Densities of idempotent measures and large deviations. Trans. Amer. Math. Soc., 351(11):4515–4543, 1999.

[6] Marianne Akian, Jean-Pierre Quadrat, and Michel Viot. Bellman processes. In Proceedings of the 11th International Conference on Analysis and Optimization of Systems held at Sophia Antipolis, June 15–17, 1994, volume 199 of Lecture Notes in Control and Information Sciences, pages 302–311, Berlin, 1994. Springer-Verlag. Edited by Guy Cohen and Jean-Pierre Quadrat.

[7] Marianne Akian, Jean-Pierre Quadrat, and Michel Viot. Duality between probability and optimization. In Idempotency (Bristol, 1994), volume 11 of Publ. Newton Inst., pages 331–353, Cambridge, 1998. Cambridge University Press.

[8] Charalambos D. Aliprantis and Kim C. Border. Infinite dimensional analysis. Springer, Berlin, third edition, 2006. A hitchhiker’s guide.

[9] Jürgen Appell. Lipschitz constants and measures of noncompactness of some pathological maps arising in nonlinear fixed point and eigenvalue theory. In Proceedings of the Conference on Function Spaces, Differential Operators and Nonlinear Analysis held at Praha, May 27 - June 1, 2004, pages 19–27. Math. Inst. Acad. Sci. of Czech Republic, 2005. Edited by P. Drábek, J. Rákosník.

[10] M. Z. Arslanov and E. E. Ismail. On the existence of a possibility distribution function. Fuzzy Sets and Systems, 148(2):279–290, 2004.

[11] François L. Baccelli, Guy Cohen, Geert Jan Olsder, and Jean-Pierre Quadrat. Synchronization and linearity. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Ltd., Chichester, 1992. An algebra for discrete event systems.

[12] Emmanuel N. Barron, Pierre Cardaliaguet, and Robert R. Jensen. Radon–Nikodym theorem in $L^\infty$. Appl. Math. Optim., 42(2):103–126, 2000.

[13] Emmanuel N. Barron, Pierre Cardaliaguet, and Robert R. Jensen. Conditional essential suprema with applications. Appl. Math. Optim., 48(3):229–253, 2003.

[14] Faouzi Bellalouha. Un point de vue linéaire sur la programmation dynamique. Détectque de ruptures dans le cadre des problèmes de fiabilité. PhD thesis, Université Paris-IX Dauphine, France, 1992.

[15] Pietro Benvenuti and Radko Mesiar. Pseudo-arithmetical operations as a basis for the general measure and integration theory. Inform. Sci., 160(1-4):1–11, 2004.

[16] Pierre Bernhard. Max-plus algebra and mathematical fear in dynamic optimization. Set-Valued Anal., 8(1-2):71–84, 2000. Set-valued analysis in control theory.

[17] Nicolas Bouleau. Splendeurs et misères des lois de valeurs extrêmes. Risques, 4:85–92, 1991.

[18] Domenico Candeloro and Sabrina Pucci. Radon–Nikodym derivatives and conditioning in fuzzy measure theory. Stochastica, 11(2-3):107–120, 1987.

[19] Pierre Cardaliaguet and Francesca Prinari. Supremal representation of $L^\infty$ functionals. Appl. Math. Optim., 52(2):129–141, 2005.
[20] Gustave Choquet. Theory of capacities. *Ann. Inst. Fourier, Grenoble*, 5:131–295, 1953–1954.

[21] Guy Cohen, Stéphane Gaubert, and Jean-Pierre Quadrat. Duality and separation theorems in idempotent semimodules. *Linear Algebra Appl.*, 379:395–422, 2004. Tenth Conference of the International Linear Algebra Society.

[22] Gert de Cooman. The formal analogy between possibility and probability theory. In *Foundations and applications of possibility theory, Proceedings of the International Workshop (FAPT ’95) held in Ghent, December 13–15, 1995*, volume 8 of *Advances in Fuzzy Systems—Applications and Theory*, pages 71–87, River Edge, NJ, 1995. World Scientific Publishing Co. Inc. Edited by Gert de Cooman, Da Ruan and Etienne E. Kerre.

[23] Gert de Cooman. Possibility theory. I. The measure- and integral-theoretic groundwork. *Internat. J. Gen. Systems*, 25(4):291–323, 1997.

[24] Gert de Cooman. Possibility theory. II. Conditional possibility. *Internat. J. Gen. Systems*, 25(4):325–351, 1997.

[25] Gert de Cooman. Possibility theory. III. Possibilistic independence. *Internat. J. Gen. Systems*, 25(4):353–371, 1997.

[26] Gert de Cooman, Guangquan Zhang, and Etienne E. Kerre. Possibility measures and possibility integrals defined on a complete lattice. *Fuzzy Sets and Systems*, 120(3):459–467, 2001.

[27] Laurens de Haan. A spectral representation for max-stable processes. *Ann. Probab.*, 12(4):1194–1204, 1984.

[28] Laurens de Haan and Sidney I. Resnick. Estimating the home range. *J. Appl. Probab.*, 31(3):700–720, 1994.

[29] Pierre Del Moral and Michel Doisy. Maslov idempotent probability calculus. I. *Teor. Veroyatnost. i Primenen.*, 43(4):735–751, 1998.

[30] Dave Doty, Xiaoyang Gu, Jack H. Lutz, Elvira Mayordomo, and Philippe Moser. Zeta-dimension. In *Proceedings of the Thirtieth International Symposium on Mathematical Foundations of Computer Science (Gdansk, Poland, August 29 - September 2, 2005)*, pages 283–294. Springer-Verlag, 2005.

[31] Lech Drewnowski. A representation theorem for maxitive measures. *Indag. Math. (N.S.)*, 20(1):43–47, 2009.

[32] Abbas Edalat. Domain theory and integration. *Theoret. Comput. Sci.*, 151(1):163–193, 1995. Topology and completion in semantics (Chartres, 1993).

[33] Ahmed Bahaa El-Rayes and Nehad N. Morsi. Generalized possibility measures. *Inform. Sci.*, 79(3-4):201–222, 1994.

[34] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.

[35] István Fazekas. A note on “optimal measures”. *Publ. Math. Debrecen*, 51(3-4):273–277, 1997.

[36] Andrei M. Finkelstein, Olga Kosheleva, Tanja Magoc, Erik Madrid, Scott A. Starks, and Julio Urenda. To properly reflect physicists reasoning about randomness, we also need a maxitive (possibility) measure. *J. Uncertain Systems*, 1(2):84–108, 2007.

[37] Wendell H. Fleming. Max-plus stochastic processes. *Appl. Math. Optim.*, 49(2):159–181, 2004.

[38] Boris D. Gel’dman. Topological properties of the set of fixed points of multivalued mappings. *Mat. Sb.*, 188(12):33–56, 1997.
[39] Bart Gerritse. Varadhan’s theorem for capacities. *Comment. Math. Univ. Carolin.*, 37(4):667–690, 1996.

[40] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael W. Mislove, and Dana S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[41] Gabriele H. Greco. Fuzzy integrals and fuzzy measures with their values in complete lattices. *J. Math. Anal. Appl.*, 126(2):594–603, 1987.

[42] Paul R. Halmos and Leonard J. Savage. Application of the Radon–Nikodym theorem to the theory of sufficient statistics. *Ann. Math. Statistics*, 20:225–241, 1949.

[43] John Harding, Massimo Marinacci, Nhu T. Nguyen, and Tonghui Wang. Local Radon–Nikodym derivatives of set functions. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 5(3):379–394, 1997.

[44] Reinhold Heckmann and Michael Huth. Quantitative semantics, topology, and possibility measures. *Topology Appl.*, 89(1-2):151–178, 1998. Domain theory.

[45] John D. Howroyd. A domain-theoretic approach to integration in Hausdorff spaces. *LMS J. Comput. Math.*, 3:229–273 (electronic), 2000.

[46] Hugo J. Janssen, Gert de Cooman, and Etienne E. Kerre. Ample fields as a basis for possibilistic processes. *Fuzzy Sets and Systems*, 120(3):445–458, 2001.

[47] Johan Jonasson. On positive random objects. *J. Theoret. Probab.*, 11(1):81–125, 1998.

[48] Vassili N. Kolokoltsov and Victor P. Maslov. Idempotent analysis as a tool of control theory and optimal synthesis. I. *Funktsional. Anal. i Prilozhen.*, 23(1):1–14, 1989.

[49] Vassili N. Kolokoltsov and Victor P. Maslov. Idempotent analysis as a tool of control theory and optimal synthesis. II. *Funktsional. Anal. i Prilozhen.*, 23(4):53–62, 1989.

[50] Ivan Kramosil. Generalizations and extensions of lattice-valued possibilistic measures, part I. Technical Report 952, Institute of Computer Science, Academy of Sciences of the Czech Republic, 2005.

[51] Ivan Kramosil. Generalizations and extensions of lattice-valued possibilistic measures, part II. Technical Report 985, Institute of Computer Science, Academy of Sciences of the Czech Republic, 2006.

[52] Volker Krätschmer. When fuzzy measures are upper envelopes of probability measures. *Fuzzy Sets and Systems*, 138(3):455–468, 2003.

[53] Vladik Kreinovich and Luc Longpré. Kolmogorov complexity leads to a representation theorem for idempotent probabilities (σ-maxitive measures). *ACM SIGACT News*, 36(3):107–112, 2005.

[54] Jimmie D. Lawson and Bin Lu. Riemann and Edalat integration on domains. *Theoret. Comput. Sci.*, 305(1-3):259–275, 2003. Topology in computer science (Schloß Dagstuhl, 2000).

[55] Xue Cheng Liu and Guangquan Zhang. Lattice-valued fuzzy measure and lattice-valued fuzzy integral. *Fuzzy Sets and Systems*, 62(3):319–332, 1994.

[56] Jack H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.

[57] Jack H. Lutz. Effective fractal dimensions. *MLQ Math. Log. Q.*, 51(1):62–72, 2005.

[58] John Mallet-Paret and Roger D. Nussbaum. Inequivalent measures of noncompactness. *Annali di Matematica Pura ed Applicata*, 48, 2010.
[59] John Mallet-Paret and Roger D. Nussbaum. Inequivalent measures of noncompactness and the radius of the essential spectrum. *Proc. Amer. Math. Soc.*, 139(3):917–930, 2011.

[60] Massimo Marinacci. Vitali’s early contribution to non-additive integration. *Riv. Mat. Sci. Econom. Social.*, 20(2):153–158, 1997.

[61] Victor P. Maslov. *Méthodes opératorielles*. Éditions Mir, Moscow, 1987. Translated from the Russian by Djilali Embarek.

[62] Georges Matheron. *Random sets and integral geometry*. John Wiley & Sons, New York-London-Sydney, 1975. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

[63] Radko Mesiar. Possibility measures, integration and fuzzy possibility measures. *Fuzzy Sets and Systems*, 92(2):191–196, 1997.

[64] Radko Mesiar and Endre Pap. Idempotent integral as limit of g-integrals. *Fuzzy Sets and Systems*, 102(3):385–392, 1999. Fuzzy measures and integrals.

[65] Enrique Miranda, Inés Couso, and Pedro Gil. A random set characterization of possibility measures. *Inform. Sci.*, 168(1–4):51–75, 2004.

[66] Ilya S. Molchanov. *Theory of random sets*. Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2005.

[67] Toshiaki Murofushi. Two-valued possibility measures induced by $\sigma$-finite $\sigma$-additive measures. *Fuzzy Sets and Systems*, 126(2):265–268, 2002.

[68] Toshiaki Murofushi and Michio Sugeno. Continuous-from-above possibility measures and $f$-additive fuzzy measures on separable metric spaces: characterization and regularity. *Fuzzy Sets and Systems*, 54(3):351–354, 1993.

[69] Jun-iti Nagata. *Modern dimension theory*, volume 2 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, revised edition, 1983.

[70] Hung T. Nguyen and Bernadette Bouchon-Meunier. Random sets and large deviations principle as a foundation for possibility measures. *Soft Comput.*, 8:61–70, 2003.

[71] Tommy Norberg. Random capacities and their distributions. *Probab. Theory Relat. Fields*, 73(2):281–297, 1986.

[72] George L. O’Brien. Sequences of capacities, with connections to large-deviation theory. *J. Theoret. Probab.*, 9(1):19–35, 1996.

[73] George L. O’Brien, Paul J. J. F. Torfs, and Wim Vervaat. Stationary self-similar extremal processes. *Probab. Theory Related Fields*, 87(1):97–119, 1990.

[74] George L. O’Brien and Wim Vervaat. Capacities, large deviations and loglog laws. In *Stable processes and related topics (Ithaca, NY, 1990)*, volume 25 of *Progr. Probab.*, pages 43–83, Boston, MA, 1991. Birkhäuser Boston.

[75] Endre Pap. *Null-additive set functions*, volume 337 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995.

[76] Endre Pap, editor. *Handbook of measure theory. Vol. I, II*. North-Holland, Amsterdam, 2002.

[77] Endre Pap. Pseudo-additive measures and their applications. In *Handbook of measure theory. Vol. II*, pages 1403–1468. North-Holland, Amsterdam, 2002.

[78] Paul Poncet. A note on two-valued possibility ($\sigma$-maxitive) measures and Mesiar’s hypothesis. *Fuzzy Sets and Systems*, 158(16):1843–1845, 2007.

[79] Paul Poncet. A decomposition theorem for maxitive measures. *Linear Algebra Appl.*, 435(7):1672–1680, 2011.
[80] Paul Poncet. Infinite-dimensional idempotent analysis: the role of continuous posets. PhD thesis, École Polytechnique, Palaiseau, France, 2011.
[81] Paul Poncet. How regular can maxitive measures be? Topology Appl., 160(4):606–619, 2013.
[82] Paul Poncet. The idempotent Radon–Nikodym theorem has a converse statement. Inform. Sci., 271:115–124, 2014.
[83] Anatolii A. Puhalskii. Large deviations of semimartingales via convergence of the predictable characteristics. Stochastics Stochastics Rep., 49(1-2):27–85, 1994.
[84] Anatolii A. Puhalskii. Large deviations and idempotent probability, volume 119 of Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2001.
[85] Madan L. Puri and Dan A. Ralescu. A possibility measure is not a fuzzy measure. Fuzzy Sets and Systems, 7(3):311–313, 1982.
[86] Sidney I. Resnick. Extreme values, regular variation, and point processes, volume 4 of Applied Probability. A Series of the Applied Probability Trust. Springer-Verlag, New York, 1987.
[87] Sidney I. Resnick and Rishin Roy. Random usc functions, max-stable processes and continuous choice. Ann. Appl. Probab., 1(2):267–292, 1991.
[88] Zdena Riečanová. Regularity of semigroup-valued set functions. Math. Slovaca, 34(2):165–170, 1984.
[89] Walter Rudin. Real and complex analysis. McGraw-Hill Book Co., New York, 3 edition, 1987.
[90] Gennady Samorodnitsky and Murad S. Taqqu. Stable non-Gaussian random processes. Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance.
[91] Irving E. Segal. Equivalences of measure spaces. Amer. J. Math., 73:275–313, 1951.
[92] Glenn Shafer. A mathematical theory of evidence. Princeton University Press, Princeton, N.J., 1976.
[93] Glenn Shafer. Belief functions and possibility measures. In Analysis of fuzzy information, Vol. I, pages 51–84. CRC, Boca Raton, FL, 1987.
[94] Niel Shilkret. Maxitive measure and integration. Nederl. Akad. Wetensch. Proc. Ser. A 74 = Indag. Math., 33:109–116, 1971.
[95] Stilian A. Stoev and Murad S. Taqqu. Extremal stochastic integrals: a parallel between max-stable processes and α-stable processes. Extremes, 8(4):237–266, 2005.
[96] Michio Sugeno. Theory of fuzzy integrals and its applications. PhD thesis, Tokyo Institute of Technology, Japan, 1974.
[97] Michio Sugeno and Toshiaki Murofushi. Pseudo-additive measures and integrals. J. Math. Anal. Appl., 122(1):197–222, 1987.
[98] Marcel L. J. van de Vel. Theory of convex structures, volume 50 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 1993.
[99] Giuseppe Vitali. On the definition of integral of functions of one variable. Riv. Mat. Sci. Econom. Social., 20(2):159–168, 1997. Translated from the Italian by Massimo Marinacci.
[100] P.-Z. Wang. Fuzzy contactability and fuzzy variables. Fuzzy Sets and Systems, 8(1):81–92, 1982.
[101] Zhen Yuan Wang and George J. Klir. Fuzzy measure theory. Plenum Press, New York, 1992.
APPENDIX A. SOME PROPERTIES OF $\sigma$-ADDITIVE MEASURES

The notions of $\sigma$-principal or CCC measures were originally introduced for the study of $\sigma$-additive measures. Recall that a $\sigma$-additive measure $m$ defined on a $\sigma$-algebra $\mathcal{B}$ is CCC (resp. $\sigma$-principal) if the $\sigma$-maxitive measure $\delta_m$ is. Also, following Segal [91], $m$ is localizable if, for all $\sigma$-ideals $\mathcal{I}$ of $\mathcal{B}$, there exists some $L \in \mathcal{B}$ such that

1. $m(S \setminus L) = 0$, for all $S \in \mathcal{I}$;
2. if there is some $B \in \mathcal{B}$ such that $m(S \setminus B) = 0$ for all $S \in \mathcal{I}$, then $m(L \setminus B) = 0$.

The next theorem establishes a link between these notions for $\sigma$-additive measures. It enlightens the fact that being finite is a very strong condition for a $\sigma$-additive measure (while it is of little consequence for a $\sigma$-maxitive measure).

**Theorem A.1.** Let $(E, \mathcal{B})$ is a measurable space and $m$ be a $\sigma$-additive measure on $\mathcal{B}$. Consider the following assertions:

1. $m$ is finite,
2. $m$ is $\sigma$-finite,
3. $m$ is $\sigma$-principal,
4. $m$ is CCC,
5. $m$ is localizable.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. Moreover, $(4) \Rightarrow (3)$ under Zorn’s lemma.

**Sketch of the Proof.** Assume that $m$ is finite, and let us show that $m$ is $\sigma$-principal. Let $\mathcal{I}$ be a $\sigma$-ideal of $\mathcal{B}$. Let $a = \sup\{m(S) : S \in \mathcal{I}\}$. We can find some sequence $S_n \in \mathcal{I}$ such that $m(S_n) \uparrow a$. Defining $L := \bigcup_n S_n \in \mathcal{I}$, we have $m(L) = a$. If there exists some $S \in \mathcal{I}$ such that $m(S \setminus L) > 0$, then $m(S \cup L) > a$ (since $m$ is finite), which contradicts $S \cup L \in \mathcal{I}$. Thus, $m(S \setminus L) = 0$, for all $S \in \mathcal{I}$, which gives $\sigma$-principality of $m$. The other implications in Theorem A.1 can be proved along the same lines as for $\sigma$-maxitive measures. □

APPENDIX B. RESIDUAL SEMIGROUPS

An ordered semigroup is a semigroup $(S, \odot)$ equipped with a partial order $\leq$ compatible with the structure of semigroup, i.e. such that $r \leq s$ and $r' \leq s'$ imply $r \odot r' \leq s \odot s'$.
If \((S, \odot)\) is an ordered semigroup and \(r, s \in S\), we say that \(r\) is absolutely continuous with respect to \(s\), written \(r \ll_{\odot} s\), if there exists some \(t \in S\) such that \(r \leq t \odot s\). We say that \(S\) (or \(\odot\)) is residual if for all \(r, s \in S\) with \(r \ll_{\odot} s\), there is an element of \(S\) denoted by \((r/s)_{\odot}\) such that \(r \leq t \odot s \iff (r/s)_{\odot} \leq t\), for all \(t \in S\). Note that in this situation we have \(r \leq (r/s)_{\odot} \odot s\). A residual semigroup \((S, \odot)\) is exact if \(r = (r/s)_{\odot} \odot s\) for all \(r, s \in S\) with \(r \ll_{\odot} s\).

**Examples B.1.** In \(\mathbb{R}_+\) here is what we have for different choices of semigroup binary operations (recall that \(\oplus\) denotes the maximum and \(\land\) the minimum):

- \(r \ll_{\times} s \iff (r = s = 0 \text{ or } s \neq 0)\), in which case \((r/s)_{\times} \times s = r\). So \((\mathbb{R}_+, \times)\) is an exact residual semigroup.
- \(r \ll_{+} s\) always holds, and \((r/s)_{+} = 0 \oplus (r - s)\). So \((\mathbb{R}_+, +)\) is a non-exact residual semigroup.
- \(r \ll_{\oplus} s\) always holds, and \((r/s)_{\oplus} = 0\) if \(r \leq s\), \((r/s)_{\oplus} = r\) otherwise. So \((\mathbb{R}_+, \oplus)\) is a non-exact residual semigroup.
- \(r \ll_{\land} s \iff r \leq s\), in which case \((r/s)_{\land} = r\), so \((\mathbb{R}_+, \land)\) is an exact residual semigroup.

**Proposition B.2.** Let \((S, \odot)\) be an ordered semigroup. If \(S\) is residual, then for all nonempty subsets \(T\) of \(S\) with infimum and all \(s \in S\), \(\{t \odot s : t \in T\}\) has an infimum and

\[
\bigwedge_{t \in T} (t \odot s) = (\bigwedge T) \odot s.
\]

Conversely, if every non-empty subset of \(S\) has an infimum and Equation (9) is satisfied for all nonempty subsets \(T\) of \(S\) with infimum and all \(s \in S\), then \(S\) is residual.

**Proof.** First assume that \(S\) is residual. Let \(T\) be a nonempty subset of \(S\) with infimum, and let \(s \in S\). Then \((\bigwedge T) \odot s\) is a lower-bound of the set \(A = \{t \odot s : t \in T\}\). Now let \(\ell\) be a lower-bound of \(A\). Since \(T\) is nonempty we have \(\ell \ll_{\odot} s\). Moreover, \(\ell \leq t \odot s\) for all \(t \in T\), so that \((\ell/s)_{\odot} \leq t\) for all \(t \in T\). This shows that \((\ell/s)_{\odot} \leq \bigwedge T\), i.e. that \(\ell \leq (\bigwedge T) \odot s\). So \((\bigwedge T) \odot s\) is the greatest lower bound of \(A\), i.e. its infimum, and we have proved Equation (9).

Conversely, assume that every non-empty subset of \(S\) has an infimum and that Equation (9) is satisfied, and let \(r, s \in S\) such that \(r \ll_{\odot} s\). Define \((r/s)_{\odot} = \bigwedge T\), where \(T\) is the nonempty set \(\{t \in S : r \leq t \odot s\}\). Thanks to Equation (9), the equivalence \(r \leq t \odot s \iff (r/s)_{\odot} \leq t\), for all \(t \in S\), is now obvious. \(\square\)
CMAP, École Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France, and INRIA, Saclay–Île-de-France

E-mail address: poncet@cmap.polytechnique.fr