General Concentric Black Rings

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Supersymmetric black ring solutions of five dimensional supergravity coupled to an arbitrary number of vector multiplets are constructed. The solutions are asymptotically flat and describe configurations of concentric black rings which have regular horizons with topology $S^1 \times S^2$ and no closed time-like curves at the horizons.

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I. INTRODUCTION

An interesting development in black hole studies is the discovery of black rings in five-dimensions. These are asymptotically flat black hole solutions with an event horizon of topology $S^1 \times S^2$, rather than the more familiar $S^3$. They were first discovered in pure Einstein gravity [1] and were further generalised in [2, 3, 4]. More recently, supersymmetric black rings have been found in D=5 minimal supergravity [5] (see [6] for earlier related work) and these have been generalised to give supersymmetric multi-concentric black ring solutions of the same theory [7]. Furthermore, the construction in [7] also allows for the possibility of a rotating black hole to sit at the common centre of the rings.

The black ring solutions all have non-vanishing rotation which supports them from collapsing. A single supersymmetric black ring has two angular momenta which are necessarily different, and hence the single black ring is distinguished by its asymptotic charges from the supersymmetric rotating black hole solution of [8, 9], (see also [10]). An interesting feature of the concentric ring solutions [7] is that they can carry the same conserved charges as the rotating black hole and with an entropy that can be smaller than, equal to or greater than that of the black hole. This shows that the conserved charges of the multi-ring configurations in themselves are not sufficient to distinguish the microstates that presumably account for the entropy of the rings after embedding them in string theory.

In this paper we will further generalize the results of [7] and construct supersymmetric black ring solutions of D=5 minimal supergravity coupled to an arbitrary number of vector multiplets. We will also generalize the results of [7] by constructing multi-concentric black ring solutions of this more general class of supergravity theories. Recall that these theories can arise, for example, as part of the low-energy effective action of M-theory reduced on a Calabi-Yau 3-fold [11]. The scalar field in each vector multiplet then corresponds to a coordinate on the Kähler moduli space of the Calabi-Yau (excluding the overall volume, which appears in a hypermultiplet). Note that general static and rotating black hole solutions of this theory have been found in [12] and [13], respectively.

An important feature of black hole solutions of these theories is that the near horizon geometry is essentially independent of the asymptotic values of the scalar fields at infinity (this was first discovered in a four-dimensional context in [14] and further explored in [15, 16]. A discussion of the five-dimensional case is also given in [17, 18, 19].) In the case that the supergravity theory is obtained from a compactification of a Calabi-Yau manifold, the asymptotic values of the scalars specify particular moduli of the Calabi-Yau. The near horizon limit of the black hole geometry is therefore independent of these moduli.

In order to have a statistical interpretation, it is expected that the black hole entropy should not depend on adiabatic changes of the environment and hence not on the scalar moduli [20]. Thus, one motivation for the present work is to see if a similar phenomenon happens for general supersymmetric black rings. For a single black ring carrying multiple charges, we shall see that it indeed does: the area of the horizon of the ring depends only via the difference of the two conserved angular momentum. We also discuss the generalisation of this statement to concentric rings.

The construction of the solutions will follow the strategy of [7]. We use the classification of the most general supersymmetric solutions of D=5 gauged supergravity coupled to an arbitrary number of vector multiplets [21], generalising that of [22, 23]. Although the classification developed in [21] is for the gauged theory, we can easily extract the appropriate results for the ungauged theory. In particular we are only interested in the “time-like case”, where the vector that can be constructed as a
bi-linear from the Killing spinor is not everywhere null. In a neighbourhood where this vector is time-like, we have a canonically defined hyper-Kähler metric. Here, generalising a similar analysis of \cite{22}, we study the case when this base is a Gibbons-Hawking space \cite{24}, when the analysis simplifies considerably. With this technology in hand, and the results of \cite{21}, our construction of the general concentric rings is straightforward.

The plan of the rest of the paper is as follows. We first summarise the general results of the classification of supersymmetric solutions in section 2, and then analyse the special case of a Gibbons-Hawking base in section 3. In section 4 we present our new multi-concentric black ring solutions, which once again can have an optional rotating black hole sitting at the common centre. In section 5 we present some further details for the special case of the three-charge “STU” model. This model can be obtained from the dimensional reduction of \( D=11 \) supergravity on a six-torus. It is therefore trivial to uplift our solutions to obtain solutions of \( D=11 \) supergravity and, after reduction and T-duality, solutions of type IIB supergravity reduced on a five-torus. Section 6 briefly concludes.

II. SUPERSYMMETRIC SOLUTIONS OF \( \mathcal{N}=1 \) SUPERGRAVITY

A. \( \mathcal{N}=1 \) supergravity

The action of \( \mathcal{N}=1 \ D=5 \) ungauged supergravity coupled to \( n-1 \) abelian vector multiplets is given by \cite{25}

\[
S = \frac{1}{16\pi G} \int \left( \frac{1}{2} R - Q_{IJJ} F^I \wedge \ast F^J - Q_{IJJ} dX^I \wedge \ast dX^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right)
\]

where we use a positive signature metric and the fermions have been set to zero. \( I,J,K \) take values \( 1, \ldots, n \) and \( C_{IJK} \) are constants that are symmetric on \( IJK \). The \( X^I \) are scalars which are constrained via

\[
\frac{1}{6} C_{IJK} X^I X^J X^K = 1.
\]

We may regard the \( X^I \) as being functions of \( n-1 \) unconstrained scalars \( \phi^0 \). It is convenient to define

\[
X_I \equiv \frac{1}{6} C_{IJK} X^J X^K
\]

so that the condition \( \mathcal{2} \) becomes

\[
X_I X^I = 1.
\]

In addition, the coupling \( Q_{IJJ} \) depends on the scalars via

\[
Q_{IJJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K.
\]

In the special case that the scalars \( X^I \) take values in a symmetric space we have the important identity

\[
C_{IJK} C_{J\ell M} C_{MPQ} \delta^{JJ'} \delta^{KK'} = \frac{4}{3} \delta_{I(L} C_{MPQ)}.
\]

In this case we have the relation:

\[
X_I = \frac{9}{2} C^{IJK} X_J X_K
\]

where \( C^{IJK} \equiv \delta^{II'} \delta^{JJ'} \delta^{KK'} C_{I'J'K'}. \) The constraints \( \mathcal{4} \) are also sufficient to ensure that the matrix \( Q_{IJ} \) is invertible with an inverse \( Q^{IJ} \) given by

\[
Q^{IJ} = 2 X^I X^J - 6 C^{IJK} X^K.
\]

Note that the equations of motion and supersymmetry transformations for this theory (with opposite signature) can be found, for example, in \cite{21}.

By definition, a bosonic supersymmetric background admits a Killing spinor \( \epsilon^a \). From this Killing spinor we can construct tensors from spinor bi-linears, which can be used to classify the general supersymmetric solutions of this theory. Building on the work \cite{22,23} this has been carried out for the more general gauged theory in \cite{21}. The relevant equations for the ungauged theory, of interest here, can be obtained from those in \cite{21} by simply setting the gauge parameter \( \chi \) to vanish (with some care).

There are two types of supersymmetric geometries, specified by whether the vector \( V \) that can be constructed from the Killing spinor is null everywhere, the “null case”, or not, the “time-like case”. Here, we will only be interested in the latter case. We also note that in general \( V \) is a Killing vector field that generates a symmetry of the full solution.

B. The time-like case

In the time-like case we work in a neighbourhood \( U \) where \( V \) is a time-like Killing vector field. Introduce coordinates \((t, x^m)\) such that \( V = \partial / \partial t \). The conditions for the existence of time-like Killing spinors can then be summarised as follows. The metric can be written locally as

\[
ds^2 = -f^2(\partial t + \omega)^2 + f^{-1} ds^2(M_4)
\]

where \( M_4 \) is an arbitrary four-dimensional hyper-Kähler manifold, and \( f \) and \( \omega \) are a scalar and a one-form on \( M_4 \), respectively. We define

\[
e^0 = f(\partial t + \omega),
\]

and choose the orientation of \( M_4 \) so that \( e^0 \wedge \eta_4 \) is positively oriented in five dimensions, where \( \eta_4 \) is the volume form of \( M_4 \). We can split the two-form \( d\omega \) into self-dual and anti-self-dual parts on \( M_4 \) and it is convenient to define:

\[
f d\omega \equiv G^+ + G^-.
\]
The gauge field can then be written
\[ F^I = d(X^I e^0) + \Theta^I, \]
where \( \Theta^I \) is a self-dual two-form on \( M_4 \) satisfying
\[ X_I \Theta^I = -\frac{2}{3} G^+. \]  
(13)
These conditions are sufficient to ensure the existence of a Killing spinor preserving 4 of the 8 supersymmetries.

We now consider the consequence of imposing the Bianchi identities \( dF^I = 0 \) and the Maxwell equations. The Bianchi identities give
\[ d\Theta^I = 0, \]
so the \( \Theta^I \) are harmonic self-dual two-forms on the base. The Maxwell equations reduce to
\[ \nabla^2_{HK} (f^{-1} X_I) = \frac{1}{6} C_{IJK}(\Theta^J \cdot \Theta^K), \]
(15)
where \( \nabla^2_{HK} \) denotes the Laplacian on the hyper-Kähler base \( M_4 \); and contracting \[ \Box \] with \( X^I \) we obtain
\[ \nabla^2_{HK} f^{-1} = -\frac{1}{3} Q_{IJ} [(\Theta^I \cdot \Theta^J) + 2 f^{-1}(dX^I \cdot dX^J)] + \frac{2}{3} (G^+ \cdot G^+), \]
(16)
where we have used the convention that for \( p \)-forms \( \alpha, \beta \) on \( B \), we set
\[ (\alpha \cdot \beta) = \frac{1}{p!} \alpha_{m_1 \ldots m_p} \beta^{m_1 \ldots m_p}. \]  
(17)
The integrability conditions for the existence of a Killing spinor guarantee that the Einstein equation and scalar equations of motion are satisfied as a consequence of the above equations.

**III. GIBBONS-HAWKING SOLUTIONS**

An interesting set of solutions arises when we consider solutions for which the base manifold admits a tri-holomorphic Killing vector i.e. a Killing vector which preserves the hyper-Kähler structure. It has been shown \[ \text{[20]} \text{[24]} \] that such manifolds are Gibbons-Hawking spaces \[ \text{[20]} \text{[24]}. \] We also make the important assumption, which leads to much simplification, that the Killing vector generates a symmetry of the full solution (including all scalars and gauge fields). The analysis of this section generalises section 3.7 of \[ \text{[22]} \].

Locally, we can choose co-ordinates \( x^5, x^i \) for \( i = 1, 2, 3 \) on \( M_4 \) with the tri-holomorphic Killing-vector given by \( \partial_5 \). The Gibbons-Hawking base metric can then be written as
\[ ds_4^2 = H^{-1}(dx^5 + \chi)^2 + H \delta_{ij} dx^i dx^j \]  
(18)
where \( \chi = \chi dx^i \), and \( H, \chi \) are independent of \( x^5 \). In addition
\[ \nabla \times \chi = \nabla H \]  
(19)
so in particular \( H \) is harmonic on \( \mathbb{R}^3 \). In this section \( \nabla \) will be the gradient and \( \nabla^2 \) will be the Laplacian on \( \mathbb{R}^3 \).

We will find it convenient to introduce the vierbein
\[ e^5 = H^{-\frac{1}{2}}(dx^5 + \chi), \quad e^i = H^{\frac{1}{2}} dx^i \]  
(20)
and positive orientation on the base is defined with respect to \( e^5 \land e^1 \land e^2 \land e^3 \).

To proceed we introduce one-forms
\[ \Lambda^I \equiv (\Lambda^I_j) dx^j \]
(21)
so that we can write
\[ \Theta^I = -\frac{1}{2} \left( (dx^5 + \chi) \land (\Lambda^I_j) dx^j \right) - \frac{1}{4} H \epsilon_{ijk} (\Lambda^I_k) dx^i \land dx^j \]
(22)
Closure of \( \Theta^I \), \[ \Box \], implies that \( d\Lambda^I = 0 \), so that locally \( \Lambda^I = dW^I \) for some functions \( W^I \). In addition, \( d\Theta^I = 0 \) also implies
\[ \nabla^2(HW^I) = 0 \]
(23)
Hence
\[ W^I = H^{-1} K^I \]
(24)
where \( K^I \) are harmonic functions on \( \mathbb{R}^3 \).

Next, we consider the gauge equations \[ \Box \]. It is straightforward to see that this is equivalent to
\[ \nabla^2(f^{-1} X_I) = \frac{1}{24} \nabla^2(H^{-1} C_{I PQ} K^P K^Q) \]
(25)
which we solve by taking
\[ f^{-1} X_I = \frac{1}{24} H^{-1} C_{I PQ} K^P K^Q + L_I \]
(26)
where \( L_I \) are some more harmonic functions on \( \mathbb{R}^3 \).

Lastly, we shall solve for \( \omega \). It is convenient to set
\[ \omega = \omega_5 (dx^5 + \chi) + \hat{\omega} \]
(27)
where
\[ \hat{\omega} = \hat{\omega}_i dx^i. \]
(28)
Recall that
\[ (d\omega)^+ = -\frac{3}{2} f^{-1} X_I \Theta^I \]
(29)
where \( ^+ \) denotes the self-dual projection on the Gibbons-Hawking base. On using the expression for \( f^{-1} X_I \) given in \[ \Box \] we obtain
\[ \nabla \times \hat{\omega} = H \nabla \omega_5 - \omega_5 \nabla H + \frac{3}{2} \left( \frac{1}{24} C_{I PQ} K^P K^Q + H L_I \right) \nabla(H^{-1} K^I) \]
(30)
From the integrability condition of this equation we find the constraint

\[ \nabla^2 \omega_5 = \nabla^2 \left( -\frac{1}{48} H^{-2} C_{I PQ} K^I K^P K^Q \right) - \frac{3}{4} H^{-1} L_I K^I \]  

(31)

which we solve by taking

\[ \omega_5 = -\frac{1}{48} H^{-2} C_{I PQ} K^I K^P K^Q - \frac{3}{4} H^{-1} L_I K^I + M(32) \]

where \( M \) is another harmonic function on \( \mathbb{R}^3 \). Substituting [27] back into (30) we see that \( \hat{\omega} \) must satisfy

\[ \nabla \times \hat{\omega} = H \nabla M - M \nabla H + \frac{3}{4} (L_I \nabla K^I - K^I \nabla L_I) \]  

(33)

This expression fixes \( \hat{\omega} \) up to a gradient which can be removed locally by making a shift in \( f \).

The general solution with Gibbons-Hawking base is specified by \( 2n+2 \) harmonic functions \( H, K^I, L_I \) and \( M \) on \( \mathbb{R}^3 \). \( H \) determines the Gibbons-Hawking base, and \( \omega \) is given by [27, 28] and [33]. The scalars \( X_I \) and \( f \) are determined from [26]. For example, we can multiply [26] by \( X^I \) to get an expression for \( f \) and then substitute this back into [20] to solve for \( X^I \). Finally, the gauge field is determined from [14].

Until now we have not used the condition [4]. If we assume it we obtain some important simplifications. In particular, the identity [4] together with [26] implies that

\[ f^{-3} = \frac{1}{2304} H^{-3} (C_{MNQ} K^M K^N K^Q)^2 + \frac{1}{32} H^{-2} (C_{MNQ} K^M K^N K^Q) K^I L_I + \frac{9}{16} H^{-1} C^{IJM} C_{I PQ} K^P K^Q L_J L_M + \frac{9}{2} C^{IJM} L_I L_J L_M \]  

(34)

Observe also that in the full 5-dimensional metric \( g_{55} = f^2 [f^{-3} H^{-1} - (\omega_5)^2] \) and using [34] we obtain

\[ f^{-3} H^{-1} - (\omega_5)^2 = \frac{1}{24} H^{-2} C_{MNQ} K^M K^N K^Q M + \frac{9}{16} H^{-2} C^{IJM} C_{I PQ} K^P K^Q L_J L_M + \frac{3}{2} H^{-1} M L_I K^I - \frac{9}{16} H^{-2} (K^I L_I)^2 + \frac{9}{2} H^{-1} C^{IJM} L_I L_J L_M - M^2 \]  

(35)

which contains no products of more than four of the harmonic functions \( K^I, L_I \) in each term.

Henceforth we will take the hyper-Kähler base to be \( \mathbb{R}^4 \) equipped with metric

\[ ds^2(\mathbb{R}^4) = H \left( dx^i dx^j \right) + H^{-1} (dy^i + \chi_i dx^i)^2 \]

\[ = H (dy^2 + r^2 \left[ d\theta^2 + \sin^2(\theta) d\phi^2 \right]) + H^{-1} (dy + \cos \theta d\phi)^2 \]  

(36)

with \( H = 1/|x| \equiv 1/r \) and we observe that \( \chi_i dx^i = \cos \theta d\phi \) satisfies \( \nabla \times \chi = \nabla H \). The range of the angular coordinates are \( 0 < \theta < \pi, 0 < \phi < 2\pi \) and \( 0 < \psi < 4\pi \).

Before proceeding to examine some black ring solutions, it is useful to compare our conventions with the Gibbons-Hawking solutions of the minimal theory. These are given in terms of four harmonic functions \( H, K, L \) and \( M \), and were presented in [22]. In particular, we note that for the minimal solution, \( C_{111} = \frac{2}{\sqrt{3}} \) and \( X^1 = \sqrt{\frac{3}{2}} \). Moreover, we have \( \Theta^4 = -\frac{2}{\sqrt{3}} G^+ \). Hence, after a straightforward computation we obtain

\[ K^1 = -2\sqrt{\frac{3}{2}} K, \quad L_1 = \frac{1}{\sqrt{3}} L \]  

(37)

IV. BLACK RING SOLUTIONS

Our ansatz for the multi-black ring solutions is given by

\[ K^I = \sum_{i=1}^N q^I_i h_i \]

\[ L_I = \lambda_I + \frac{1}{24} \sum_{i=1}^N (Q_{II} - C_{IJK} q^J_i q^K_i) h_i \]

\[ M = \frac{3}{4} \sum_{i=1}^N \lambda_I q^I_i - \frac{3}{4} \sum_{i=1}^N \lambda_I q^I_i |x_i| h_i \]  

(38)

where \( h_i \) are harmonic functions in \( \mathbb{R}^3 \) centred at \( x_i \), \( h_i = 1/|x - x_i| \), and \( Q_{II} \), \( q^I_i \) and \( \lambda_I \) are constants.

To see that this includes the multi black ring solutions of the minimal theory found in [7], we simply make the identifications

\[ q^I_i = \sqrt{3} q_i, \quad Q_{II} = 2\sqrt{3} Q_i, \quad \lambda_1 = \frac{1}{\sqrt{3}} \]  

(39)

Note that \( i \) labels the \( N \) rings. We also note that for the special case when \( N = 1 \) and \( x_1 = 0 \), i.e. all of the harmonic functions in \( \mathbb{R}^3 \) are centred at the origin, that we obtain the general rotating black hole solution of [18].

For simplicity, we will now continue our analysis of these solutions in the special case that \( X^I \) take values in a symmetric space. In this case we have the identity [4] and we can use the expression for \( f \) given in [34].

To ensure asymptotic flatness, we shall require that \( f \to 1 \) as \( r \to \infty \), and hence we must have

\[ \frac{9}{2} C^{IJM} \lambda_I \lambda_J \lambda_M = 1 \]  

(40)

We observe that

\[ X_I \to \lambda_I, \quad X^I \to \lambda^I = \frac{9}{2} C^{IJK} \lambda_J \lambda_K \quad \text{as } r \to \infty . \]  

(41)

The sub-leading corrections are easy to calculate and we find, for example,

\[ X_I = \lambda_I + \frac{1}{24} [\mu_I - \lambda_I (\lambda^J \mu_J)] \frac{1}{r} + \ldots \]  

(42)
where
\[ \mu_I \equiv \sum_{i=1}^{N} (Q_{Ii} - C_{IJK} q^J i q^K i) + C_{IJK} \sum_{i,j=1}^{N} q^j i q^K j \] (43)
and we see that the solutions carry scalar charge. It is also straightforward to calculate the electric charges carried by the solution. We find that
\[ \frac{1}{2 \pi^2} \int_{S^3} Q_{IJ} * F^J = \frac{1}{2} \mu_I . \] (44)

For the ADM mass we find
\[ M_{ADM} = \frac{\pi}{8G} \lambda^I \mu_I \] (45)

Noting that if we contract (43) with \( \lambda^I \) we obtain \( (1/2) \lambda^I \mu_I \), we see that the ADM mass is consistent with the BPS bound. Explicit expressions for the angular momentum will be given below for the special case that all of the harmonic functions have centres lying on the z-axis.

A. Near-Horizon Analysis

We now analyse what happens as \( x \to x_i \) for some fixed \( i \). Our analysis essentially follows that of [7]. We first make a rotation so that \( x_i \) is at \( (0,0,-R_i^2/4) \) and set up new spherical polar coordinates \( (\epsilon_i, \theta_i, \phi_i) \) in \( \mathbb{R}^3 \) centred on \( x_i \) and then consider an expansion in \( \epsilon_i \). After doing this, and solving (43), we find a coordinate singularity at \( \epsilon_i = 0 \). To see this, it is useful to note the following expansions:
\[ f = \frac{16}{R_i^2 \nu_i^2} \epsilon_i^2 + O(\epsilon_i^3) \]
\[ H^{-1} = \frac{\nu_i^2}{4} \epsilon_i^{-2} + \kappa^1(\epsilon_i)^{-1} + O((\epsilon_i)^0) \]
\[ f^2 \omega_5 = -\frac{2}{\nu_i} \epsilon_i + \kappa^2(\epsilon_i)^2 + O((\epsilon_i)^3) \]
\[ f^2 (f^{-3} H^{-1} - (\omega_5)^2) = \frac{1}{4} (\ell_i)^2 + \kappa^3 \epsilon_i + O((\epsilon_i)^3) \] (46)

where \( \kappa^1, \kappa^2, \kappa^3 \) are constants whose value is not important in the context of this discussion, and we have set
\[ \nu_i = \left( \frac{1}{6} C_{IPQ} q^I q^P q^Q i \right)^{1/3} \] (47)
and
\[ \ell_i = \nu_i^{-2} \left( \frac{1}{16} C^{IJM} C_{IPQ} q^I q^P (Q_{JI} - C_{JST} q^S i q^T i) \times (Q_{MJ} - C_{MUV} q^U i q^V i) \right. \]
\[ \left. - \frac{1}{16} [q^I i (Q_{Ii} - C_{IJK} q^K i)]^2 - 3 \nu_i^3 (R_i)^2 \lambda_i q^I i \right)^{1/3} \] (48)

where we have assumed that \( \nu_i > 0 \) and \( \ell_i^2 > 0 \).

Motivated by a similar analysis in [7] we then introduce new coordinates
\[ dt = dv + \left( \frac{b_2}{\ell_i^2} + \frac{b_1}{\epsilon_i} \right) d\epsilon_i \]
\[ d\psi = d\phi_i + 2(d\psi' + \epsilon_i d\epsilon_i) \]
\[ \phi_i = \phi_i' \] (49)

for constants \( b_j \) and \( c_j \). In order to eliminate a \( 1/\epsilon_i \) divergence in \( g_{\psi \psi} \) and a \( 1/\ell_i \) divergence in \( g_{\psi \epsilon} \), we take
\[ b_2 = \frac{\ell_i \nu_i^2}{8} \]
\[ c_1 = -\frac{\nu_i}{2 \ell_i} \] (50)

A \( 1/\epsilon_i \) divergence in \( g_{\psi \epsilon} \) can be eliminated by a suitable choice for \( b_1 \), whose explicit expression is not illuminating. The metric can now be written
\[ ds^2 = -\frac{256}{R_i^4 \nu_i^4} \epsilon_i^4 dv^2 - \frac{4}{\ell_i} d\epsilon_i + \frac{32}{R_i^4 \nu_i^4} \sin^2 \theta_i \epsilon_i^3 d\psi' d\phi_i' \]
\[ + \frac{8}{\nu_i} \epsilon_i d\psi d\psi' + \ell_i^2 d\psi'^2 + \frac{\nu_i^2}{4} \left[ d\theta_i^2 + \sin^2 \theta_i d\phi_i'^2 \right] \]
\[ + 2 g_{\psi \psi'} d\epsilon_i d\psi' + 2 g_{\psi \theta} d\psi d\theta_i + 2 g_{\psi \phi} d\psi d\phi_i + \ldots \] (51)

where \( g_{\psi \psi'} \) and \( g_{\psi \epsilon} \) are \( O(\epsilon_i^4) \); \( g_{\psi \theta} \) and \( g_{\psi \phi} \) are \( O(\epsilon_i^2) \); and \( g_{\psi \phi'} \) is \( O(\epsilon_i^3) \) whose explicit forms are unimportant for our considerations here, and the ellipsis denotes terms involving sub-leading (integer) powers of \( \epsilon_i \) in all of the metric components explicitly indicated.

The determinant of this metric is analytic in \( \epsilon_i \). At \( \epsilon_i = 0 \) it vanishes if and only if \( \sin^2 \theta_i = 0 \), which just corresponds to coordinate singularities. It follows that the inverse metric is also analytic in \( \epsilon_i \) and hence the above coordinates define an analytic extension of our solution through the surface \( \epsilon_i = 0 \).

The supersymmetric Killing vector field \( V = \theta_i \) is null at \( \epsilon_i = 0 \). Furthermore \( V_{\psi} dx^\mu = -(2/\ell_i) d\epsilon_i \) at \( \epsilon_i = 0 \), so \( V \) is normal to the surface \( \epsilon_i = 0 \). Hence \( \epsilon_i = 0 \) is a null hypersurface and a Killing horizon of \( V \), i.e., the black ring has an event horizon which is the union of the Killing Horizons for each \( \epsilon_i = 0 \). Furthermore, by expanding out the determinant of the metric obtained by restricting to the surface on which \( v, \theta_i \) and \( \epsilon_i \) are constant, it is straightforward to show that there are no closed time-like curves (CTCs) at the horizon.

In the near horizon limit defined by scaling \( v \to v/\delta, \epsilon_i \to \delta \epsilon_i \) and then taking the limit \( \delta \to 0 \), we find that the metric is locally the product of \( AdS_3 \) with radius \( \nu_i \) and a two-sphere of radius \( \frac{\ell_i}{2} \).

We can read off the geometry of a spatial cross-section of the horizon:
\[ ds^2_{horizon} = \ell_i^2 d\psi'^2 + \frac{\nu_i^2}{4} \left[ d\theta_i^2 + \sin^2 \theta_i d\phi_i'^2 \right] . \] (52)
We see that the horizon has geometry $S^1 \times S^2$, where the $S^1$ and round $S^2$ have radii $\ell_i$ and $R$, respectively. This is precisely the geometry of the event horizon for a single supersymmetric black ring. The area of this specific horizon is given by

$$A = 2\pi^2 \ell_i \nu_i^2 .$$

(53)

Note that since $\ell_i$ depends on $\lambda_i$, the black ring horizon area depends on the values taken by the scalars at asymptotic infinity. We will discuss how this is related to the angular momentum in the next sub-section.

We also note that near the pole

$$X_I = \frac{1}{6} \nu_i^{-2} C_{I ST} q^S q^T i + O(\epsilon_i)$$

$$X^I = \nu_i^{-1} q^i + O(\epsilon_i)$$

(54)

so the scalars are regular near the horizon. Moreover, although the $\Theta^I$ are not regular at the horizon, it is straightforward to show, using the expansions given in (46), that the gauge field strengths $F^I$ are also regular at the horizon.

The $i$th ring has dipole charges defined by

$$D^I_i = \frac{1}{16\pi G} \int_{S^2} F^I = \frac{q^I_i}{8G}$$

(55)

where $S^2$ encloses the $i$th black ring only once, and can be taken to be the $S^2$ at the $i$th horizon.

The $S^1$ direction of the rings all lie on an orbit of the tri-holomorphic Killing vector $\partial_\theta$ and hence describe concentric rings (see [7] for further comments). We also note that if we set $x_i = 0$ for one value of $i$ we obtain a general rotating black hole with topology $S^3$ of the kind found in [12], sitting at the centre of the rings.

B. Poles on the $z$-axis.

We now consider the solutions with all poles located along the $z$-axis, where we can analyse the solutions in more detail. Consider the general solution [28] with $x_i = (0, 0, -k_i R_i^2/4)$ and $k_i = \pm 1$. Thus

$$h_i = (r^2 + \frac{k_i R_i^2}{2} \cos \theta + \frac{R_i^4}{16})^{-1/2} .$$

(56)

We can solve [28] with $\hat{\omega}$ only having a non-zero $\phi$ component, $\hat{\omega}_\phi$, that is a function of $r$ and $\theta$ only. In particular, in addition to $\partial_\theta$ and $\partial_\phi$, these solutions have an extra $U(1)$ symmetry generated by $\partial_\theta$.

To solve [28] we write $\hat{\omega} = \hat{\omega}^L + \hat{\omega}^Q$ where $\hat{\omega}^L$ is linear in the charges $q_i$ and independent of $Q$ and $\hat{\omega}^Q$ contains the dependence on $Q$. We find

$$\hat{\omega}^L = -\sum_{i=1}^N \frac{3q_i \lambda_i}{4} [1 - (r + \frac{R_i^2}{4})h_i](\cos \theta + k_i)d\phi .$$

(57)

and

$$\hat{\omega}^Q = -\frac{1}{256} \sum_{i<j} \frac{1}{(k_i R_i^2 - k_j R_j^2)}$$

$$\times [q^i_i (Q_{ij} - C_{IJK} q^j_J q^K_k)$$

$$- q^j_j (Q_{ii} - C_{IJK} q^i_I q^K_K)]$$

$$\times h_i h_j [16 \frac{1}{h_i^2} + 16 \frac{1}{h_j^2} - \frac{32}{h_i h_j} - (k_i R_i^2 - k_j R_j^2)^2] d\phi$$

(58)

By considering the asymptotic form of the solution we find that the angular momentum is given by

$$J_1 = \frac{\pi}{48G} \sum_{i,j,m=1}^N C_{IPQ} q^I_i q^P_j q^Q_m$$

$$+ 3 \sum_{i,j=1}^N q^i_i (Q_{ij} - C_{IJK} q^j_J q^K_k)$$

$$- \frac{3\pi}{8G} \sum_{i=1}^N j_i (k_i - 1)$$

(59)

$$J_2 = J_1 + \frac{3\pi}{4G} \sum_{i=1}^N j_i k_i .$$

where we have defined

$$j_i = q^I_i \lambda_i R_i^2$$

(60)

If we have a single black ring with $N = 1$ we now see that the moduli dependence of the area of the event horizon appearing in the expression for $\ell_i$ in (43) can be re-expressed in terms of the conserved charge $J_2 - J_1$, consistent with [20]. More generally, for the multi-ring solutions, with poles on the $z$-axis, the moduli dependence in $\ell_i$ can be expressed in terms of $j_i$, which, from [50], have the natural interpretation as fixing the contribution to $J_2 - J_1$ coming from the $i$th ring. It would be interesting to check that this also holds for poles not all on the $z$-axis. The black hole entropy also depends on $Q_i$ which are quantised electric charges when the model comes from, for example, the reduction of D=11 supergravity on a Calabi-Yau three-fold (e.g. [12]). It also depends on the $q^i$, which we saw above are dipole charges, which are expected to be quantised, similarly.

We would also like to check whether there are any Dirac-Misner strings that might require making periodic identifications of the time coordinate. From the reasoning given for the computation of black ring solutions in [6], we demand that $\hat{\omega} = 0$ at $\theta = 0, \pi$. The expression for $\hat{\omega}^L$ in (57) satisfies these conditions. In order for the same to hold for $\hat{\omega}^Q$ we require that

$$q^i_i (Q_{ij} - C_{IJK} q^j_J q^K_k) = q^j_j (Q_{ii} - C_{IJK} q^i_I q^K_K)$$

(61)

for $i \neq j$. A condition which is sufficient (though not generally necessary) for this to hold is

$$Q_{ii} - C_{IJK} q^i_I q^K_K = \Lambda q_{ii}$$

(62)
for all $I$, $i$ where $\Lambda$ is constant and $q_{II} \equiv \delta_{IJ}q^{J}$. However, in general, this condition is excessively constraining on the parameters of the solution. If we do impose this condition we have $L_I = \lambda_I + \frac{\Lambda}{2\ell}I_{IJ}K^{J}$.

V. THREE-CHARGE SOLUTIONS

In the special case of the 3-charge STU-model with $C_{123} = 1$ we obtain some useful simplifications. In particular we find

$$f^{-3} = \frac{1}{64}(12L_1 + H^{-1}K^2K^3)(12L_2 + H^{-1}K_1K^2)$$

$$\times (12L_3 + H^{-1}K^2)$$

and so we find

$$X_1 = \frac{1}{3}(12L_1 + H^{-1}K^2K^3)^{2/3}$$

$$X_2 = \frac{1}{3}(12L_2 + H^{-1}K^2K^3)^{1/3}(12L_3 + H^{-1}K^1K^2)^{1/3}$$

$$X_3 = \frac{1}{3}(12L_3 + H^{-1}K^2K^3)^{1/3}(12L_2 + H^{-1}K^1K^2)^{1/3}$$

Furthermore:

$$\nu_i = (q_1i^2q_3i^3)^{\frac{1}{2}}$$

and

$$\ell_i = (q_1i^2q_3i^3)^{-\frac{1}{2}}[\frac{1}{16}(q_1i^2)(Q_{1i} - 2q_2q_3i^3)]^2$$

$$- \frac{1}{16}(q_2i^2)(Q_{2i} - 2q_1q_3i^3)^2$$

$$- \frac{1}{16}(q_3i^2)(Q_{3i} - 2q_1q_2i^3)^2$$

$$+ \frac{1}{8}q_1q_2q_3i^3(Q_{1i} - 2q_2q_3i^3)(Q_{2i} - 2q_1q_3i^3)$$

$$+ \frac{1}{8}q_1q_2q_3i^3(Q_{1i} - 2q_2q_3i^3)(Q_{3i} - 2q_1q_2i^3)$$

$$+ \frac{1}{8}q_2^3q_3i^3(Q_{2i} - 2q_1q_3i^3)(Q_{3i} - 2q_1q_2i^3)$$

$$- 3q_1q_2q_3i^3R_i^2(\lambda_1q_1i + \lambda_2q_2i + \lambda_3q_3i)^{\frac{1}{2}}$$

If we impose the constraint (62) which removes the string singularities, this simplifies to

$$\ell_i = (q_1i^2q_3i^3)^{-\frac{1}{2}}[-3R_i^2q_1i^2q_3i^3\lambda_1q_1i$$

$$\frac{\Lambda^2}{16}(q_1^i + q_2^i + q_3^i)(q_1^i + q_2^i - q_3^i)$$

$$\times (q_1^i - q_2^i + q_3^i)(-q_1^i + q_2^i + q_3^i)^{\frac{1}{2}}$$

VI. CONCLUSION

In this paper we have constructed new supersymmetric solutions corresponding to concentric black rings carrying multiple charges. We have shown that the rings have regular horizons, and have computed the conserved charges associated with these rings. We have shown that there are no closed time-like curves at the horizon. Although we think it is unlikely that there are any CTCs elsewhere, this remains to be proven; in particular, it would be interesting to be able to examine the global causal structure of the most general solutions in which the poles of the harmonic functions are not all co-linear. The main obstacle to this would appear to be the complicated nature of the solution to (43).

The circumference of the $i$th black ring horizon in our solutions is given by $2\pi\ell_i$, where $\ell_i$ is defined in (48). It is interesting that in the special case of a single three charge ring, it has been argued in [27] that the geometry with $\ell_i = 0$ corresponds to a regular three-charge super-tube [28, 29] without horizon. It would be interesting to prove that such geometries are indeed regular and free from CTCs, since our solutions would then also include superpositions of concentric multi-charge regular super-tubes with black rings and a black hole at the common centre.

Moreover, one could also investigate whether even more general black ring solutions could be found for which the hyper-Kähler base is no longer flat. Finally, it is an interesting open question as to whether or not there are supersymmetric black ring solutions in gauged supergravity theories.

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Note added: After this work was completed an interesting paper appeared [27] which has some overlap with section 5 of this paper. In particular, in the three-charge case, a solution describing a single black ring with a possible black hole at the centre is explicitly constructed in [27]. However, the solutions we present for this model are
more general in that we can have an arbitrary number of concentric rings and we also allow for the possibility of arbitrary asymptotic values of the scalar fields.

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