A GENERALIZATION OF KING’S EQUATION VIA NONCOMMUTATIVE GEOMETRY

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Abstract. We introduce a framework in noncommutative geometry consisting of a *-algebra, a bimodule endowed with a derivation ("1-forms") and a Hermitian structure (a “noncommutative Kähler form”), and a cyclic 1-cochain whose coboundary is determined by the previous structures. This data leads to moment map equations on the space of connections on arbitrary finitely-generated projective Hermitian module. As particular cases, we obtain a large class of equations in algebra (King’s equations for representations of quivers, including ADHM equations), in classical gauge theory (Hermitian Yang-Mills equations, Hitchin equations, Bogomolny and Nahm equations, etc.), as well as in noncommutative gauge theory by Connes, Douglas and Schwarz. We also discuss Nekrasov’s beautiful proposal for re-interpreting noncommutative instantons on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ as an infinite-dimensional solution of King’s equation

$$\sum_{i=1}^{n} [T_i^\dagger, T_i] = \hbar \cdot n \cdot \text{id}_{\mathcal{H}}$$

where $\mathcal{H}$ is a Hilbert space completion of a finitely-generated $\mathbb{C}[T_1, \ldots, T_n]$-module (e.g. an ideal of finite codimension).

1. Introduction

There is a remarkable similarity between self-dual Yang-Mills equations and equations introduced by King in [12] for representations of quivers. The underlying reason is that both equations are obtained from appropriate moment maps. We introduce in this paper a common generalization based on noncommutative geometry. In this setup the moment map equation is governed by a cyclic 1-cochain. Examples of a generalized King’s equation include ADHM equations, noncommutative instantons, vortex equations (in particular Hitchin and Vafa-Witten equations), as well as Bogomolny and Nahm equations for the gauge group $U(k)$. Furthermore, we discuss Nekrasov’s suggestion to reinterpret noncommutative instantons as infinite-dimensional versions of King’s equation, also related to Quantum minimal surfaces considered recently in [3].

2. Some motivations and backgrounds

2.1. Mumford stability and harmonic representatives: examples. One of major recurrent themes in Kähler geometry is an equivalence between the algebro-geometric property of a polystability, and the existence of a kind of harmonic metric. Let us start with several motivating examples.

2.1.1. Kempf-Ness Theorem. Let $G$ be an algebraic reductive group over $\mathbb{C}$ acting linearly on a finite dimensional vector space $V$ over $\mathbb{C}$.

Definition 2.1. A non-zero orbit $G \cdot v \subset V - \{0\}$ is called semistable iff its closure does not contain 0.

It is easy to see that the union of all semistable orbits forms an open $G$-invariant subset of $V$ (possibly empty).

Definition 2.2. A semistable orbit is called polystable iff it is closed (equivalently, closed in the semistable locus).

Let us choose a maximal compact subgroup $K \subset G$ and a Hermitian norm $\|\cdot\|$ on $V$ invariant under the $K$-action. By definition, on a semistable orbit $G \cdot v$ the function $\log(\text{norm})$ is bounded below.
Theorem 2.3. (Kempf-Ness [11]) A semistable orbit $G \cdot v$ is polystable iff the restriction of the function $\log(\text{norm})$ to this orbit achieves a minimum. Moreover, in this case the locus of minima is a unique orbit of $K$.

The set of polystable orbits coincides with the set of $\mathbb{C}$-points of the reduced scheme $\mathcal{M} := \text{Spec}(A) - \{0\}$, where $A = \mathbb{C}[V]^G$ is the algebra of invariants.

The function,
\[
H : G \cdot v \mapsto \min_{g \in G} \log(\|g \cdot v\|) \in \mathbb{R}
\]
is a plurisubharmonic continuous function on $\mathcal{M}$. Moreover, on the smooth locus of $\mathcal{M}$, the function $H$ is the potential of a Kähler metric $\omega_M = i \partial \bar{\partial} H$.

Example 2.4. Fix integers $r, n \geq 1$. If $G = GL(r, \mathbb{C})$ (with the maximal compact subgroup $K = U(r)$) and the representation $V$ is the direct sum of $n$ copies of the adjoint representation of $G$, then the local minima of the function $\log(\text{norm})$ on non-zero orbits are non-zero collections $(T_1, \ldots, T_n)$ of $n$ operators in $\mathbb{C}^r$ satisfying
\[
\sum_{i=1}^n [T_i^\dagger, T_i] = 0
\]
where $T_i^\dagger$ is the Hermitian conjugate to $T_i$. The polystable orbits, together with the zero orbit, are exactly the conjugacy classes of $r$-dimensional semisimple representations of the free algebra $\mathbb{C}(T_1, \ldots, T_n)$.

2.1.2. King’s Theorem. A quiver is a finite oriented graph. Here is the formal definition:

Definition 2.5. A quiver $Q = (Q_0, Q_1, s, t)$ is a tuple consisting of finite sets $Q_0$, $Q_1$ (whose elements are called vertices and arrows of $Q$ respectively), and two maps $s : Q_0 \to Q_1$, $t : Q_1 \to Q_0$, (called the source and the target maps).

Definition 2.6. A representation $\mathcal{E}$ of a quiver $Q$ over a field $k$ is given by a collection of $k$-vector spaces $\mathcal{E}_v$ for each vertex $v \in Q_0$, and a collection of morphisms $T_a : \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)}$ for each arrow $a \in Q_1$.

The representations of a given quiver form an abelian category.

Definition 2.7. Let us fix a collection of numbers $\eta = (\eta_v \in \mathbb{R})_{v \in Q_0}$ associated with the vertices of $Q$. Let $\mathcal{E}$ be a non-zero finite dimensional representation of a quiver $Q$ such that,
\[
\sum_{v \in Q_0} \eta_v \cdot \dim \mathcal{E}_v = 0 \in \mathbb{R}.
\]

Then, $\mathcal{E}$ is called semistable with slope $\eta$ (or, equivalently $\eta$-semistable) iff for any subrepresentation $\mathcal{E}' \subset \mathcal{E}$ such that $\mathcal{E}' \neq 0, \mathcal{E}$, one has $\sum_{v \in Q_0} \eta_v \cdot \dim \mathcal{E}'_v \leq 0$. A $\eta$-semistable representation is called $\eta$-stable iff in the previous condition one has strict inequality $\sum_{v \in Q_0} \eta_v \cdot \dim \mathcal{E}'_v < 0$. A $\eta$-semistable representation is called polystable iff it is a direct sum of $\eta$-stable ones.

For any given $\eta$, the semistable representations with slope $\eta$, together with the zero representation, form an artinian abelian category. The simple objects in this category are exactly the $\eta$-stable representations, whereas the non-zero semisimple objects are exactly the $\eta$-polystable representations.

Theorem 2.8. (A. D. King [12]) In the case $k = \mathbb{C}$, a representation is $\eta$-polystable iff there exists a collection of Hermitian norms $(\|\cdot\|_v)_{v \in Q_0}$ on vector spaces $(\mathcal{E}_v)_{v \in Q_0}$ such that on the orthogonal direct sum $E := \bigoplus_v \mathcal{E}_v$ one has the following equality:
\[
\sum_{a \in Q_1} [T_a^\dagger, T_a] = \sum_{v \in Q_0} \eta_v \cdot \text{Pr}_{\mathcal{E}_v}
\]
taking place in the algebra of operators in $E$, where $\text{Pr}_{\mathcal{E}_v}$ is the orthogonal projection to the direct summand $\mathcal{E}_v$. 

\[
(\|\cdot\|_v)_{v \in Q_0}
\]

\[
\sum_{v \in Q_0} \eta_v \cdot \dim \mathcal{E}_v = 0 \in \mathbb{R}.
\]

\[
(\|\cdot\|_v)_{v \in Q_0}
\]
Notice that (2.4) is equivalent to a collection of individual constraints for each vertex \( v \in Q_0 \):

\[
\forall v \in Q_0 : \quad \sum_{a \in Q_1} Pr_{\mathcal{E}_v} \cdot [T^1_a, T_a] \cdot Pr_{\mathcal{E}_v} = \eta_0 \cdot Pr_{\mathcal{E}_v} \in Pr_{\mathcal{E}_v} : \text{End}(E) \cdot Pr_{\mathcal{E}_v} \simeq \text{End}(\mathcal{E}_v).
\]

Similarly to the Kempf-Ness theorem, the set of isomorphism classes of \( \eta \)-polystable representation of \( Q \) with a given dimension vector, \[ \dim(\mathcal{E}) := (\dim(\mathcal{E}_v)_{v \in Q_0}) \in \mathbb{Z}_{\geq 0}^n, \] is the set of \( \mathbb{C} \)-points of a reduced separated scheme over \( \mathbb{C} \). Moreover, its open dense subset of smooth points is endowed with a natural Kähler metric.

2.1.3. Donaldson-Uhlenbeck-Yau (DUY) Theorem. Let \( X/\mathbb{C} \) be a smooth connected Kähler manifold of complex dimension \( n > 0 \), and \( \nu \in H^2(X; \mathbb{R}) \cap H^{1,1}(X) \) be a Kähler class. We assume that

\[
(2.7) \quad \langle [X], \nu^n \rangle = 1.
\]

**Definition 2.9.** For \( \lambda \in \mathbb{R} \), a holomorphic vector bundle \( \mathcal{E} \) on \( X \) is called \( \lambda \)-stable if

\[
(2.8) \quad \langle [X], c_1(\mathcal{E}) \cdot \nu^{n-1} \rangle = \lambda \cdot \text{rank}(\mathcal{E})
\]

and for any torsion-free coherent subsheaf \( 0 \neq \mathcal{E}' \subset \mathcal{E} \) such that \( \text{rank}(\mathcal{E}') < \text{rank}(\mathcal{E}) \) one has

\[
(2.9) \quad \langle [X], c_1(\mathcal{E}) \cdot \nu^{n-1} \rangle < \lambda \cdot \text{rank}(\mathcal{E}')
\]

Equivalently, in (2.9) one can replace torsion-free subsheaves by subbundles of \( \mathcal{E} \) restricted to the complements \( X - Z \) to closed analytic subsets \( Z \subset X \) of complex codimension at least 2. A \( \lambda \)-polystable bundle is defined as a finite sum of \( \lambda \)-stable ones.

**Theorem 2.10.** ([8], [16]) For a choice of a Kähler \((1,1)\)-form \( \omega^{1,1} \) on \( X \) with \( [\omega^{1,1}] = \nu \), we have the following: a vector bundle \( \mathcal{E} \) is \( \lambda \)-polystable if and only if it admits a Hermitian metric \( h_\mathcal{E} \) such that the curvature form \( F = F_{h_\mathcal{E}} \) of the canonical connection associated with \( h_\mathcal{E} \) satisfies the Hermitian Yang-Mills equation (HYM in short):

\[
(2.10) \quad \frac{1}{2\pi} F \cdot (\omega^{1,1})^{n-1} = \lambda \cdot \text{id}_{\mathcal{E}} \cdot (\omega^{1,1})^n \in \Gamma(\mathcal{E}^* \otimes \mathcal{E} \otimes \Omega_X^{n,n}).
\]

The DUY theorem is a famous example of Kobayashi-Hitchin type correspondences in differential geometry.

Later this result was generalized in [6] by S. Bando and Y.-T. Siu to so-called reflexive sheaves

\[
(2.11) \quad \mathcal{E} \in \text{Coh}(X), \quad \mathcal{E} = \mathcal{E}^{**} \quad \text{where} \ \mathcal{E}^* := \mathcal{H}om(\mathcal{E}, \mathcal{O}_X),
\]

which can be alternatively viewed as vector bundles defined outside of closed analytic subsets of complex codimension at least 2.

2.2. Geometry of moment maps. Let \( (M, \omega_M) \) be a symplectic manifold. Let a connected compact Lie group \( K \) with Lie algebra \( \mathfrak{k} \) acts smoothly on \( M \) and preserves the symplectic form \( \omega_M \). Then we get a homomorphism of Lie algebras

\[
(2.12) \quad u \in \mathfrak{k} \mapsto X_u \in \Gamma(M, T_M), \quad X_{[u_1, u_2]} = [X_{u_1}, X_{u_2}], \quad \mathcal{L}_{X_u} \omega_M = 0.
\]

The condition \( \mathcal{L}_{X_u} \omega_M = 0 \) implies that the 1-form \( i_{X_u} \omega_M \) is closed, as follows from the Cartan formula \( \mathcal{L}_{X_u} = d \circ i_{X_u} + i_{X_u} \circ d \) and the closedness of \( \omega_M \).

The symplectic action as above is called Hamiltonian if a homomorphism is chosen

\[
(2.13) \quad \mathfrak{R} \to (C^\infty(M), \{\cdot, \cdot\}), \quad u \in \mathfrak{R} \mapsto H_u
\]

\[1\]This is a simplifying assumption which holds in the context of our paper.
to the Lie algebra of functions on $M$ endowed with the standard Poisson bracket $\{\cdot, \cdot\}$, lifting the homomorphism $u \mapsto X_u$ from $\mathfrak{k}$ to the Lie algebra of symplectic vector fields on $M$. Explicitly, it means that

\begin{equation}
(2.14) \quad dH_u = i_{X_u} \omega_M \quad \forall u \in \mathfrak{k},
\end{equation}

\begin{equation}
(2.15) \quad H_{[u_1, u_2]} = \{H_{u_1}, H_{u_2}\} := \omega_M(i_{X_{u_1}}, i_{X_{u_2}}) \quad \forall u_1, u_2 \in \mathfrak{k}.
\end{equation}

The collection of Hamiltonians $(H_u)_{u \in \mathfrak{k}}$ gives a moment map

\begin{equation}
(2.16) \quad \mu : M \to \mathfrak{k}^*, \quad x \mapsto (u \mapsto H_u(x) \in \mathbb{R}).
\end{equation}

This is a $K$-equivariant map. We define the symplectic quotient of $(M, \omega_M)$ for a given Hamiltonian action to be the quotient of the space $\mu^{-1}(0) \subset M$ by the action of $K$. This quotient is a locally compact singular space in general, but it is symplectic on its open dense subset of smooth points. Moreover, if $M$ is endowed with a complex structure such that $\omega_M$ is the imaginary part of a Kähler (1,1)-form and $K$ acts by Kähler isometries, then the quotient space $\mu^{-1}(0)/K$ is a reduced complex-analytic space with a Kähler metric on its smooth locus.

The constraint $\mu(x) = 0$ on a point $x \in M$ is called the moment map equation.

Remark 2.11. For a given symplectic $K$-action, the obstruction to the existence of a Hamiltonian lift is a class in $H^2(\mathfrak{k}, \mathbb{R})$. If the obstruction vanishes, then the set of all various lifts to a Hamiltonian action is a torsor over the group of abelian characters $Hom_{\text{Lie}}(\mathfrak{k}, \mathbb{R}) = H^1(\mathfrak{k}, \mathbb{R})$.

Example 2.12. (King’s equations as moment map equations)

Let $Q$ be a finite quiver. Fix a finite-dimensional Hermitian vector space $E_v$ for each vertex $v \in Q_0$. Then the compact Lie group

\begin{equation}
(2.17) \quad K := \prod_{v \in Q_0} U(E_v)
\end{equation}

acts on the finite-dimensional complex vector space

\begin{equation}
(2.18) \quad M := \prod_{a \in Q_1} \text{Hom}(E_{s(a)}, E_{t(a)})
\end{equation}

parameterizing representations of $Q$ in $(E_v)_{v \in Q_0}$. We endow $M$ with the constant (i.e., translationally invariant) Kähler metric associated with the Hermitian norm on $M$ given by

\begin{equation}
(2.19) \quad \| (T_a)_{a \in Q_1} \|^2 := \sum_{a \in Q_1} \text{Trace}(T_a^* T_a).
\end{equation}

The moment map in this example is given (in terms of Hamiltonians) by the formula, where $u = (u_v)_{v \in Q_0} \in \mathfrak{k}$,

\begin{equation}
(2.20) \quad H_u((T_a)_{a \in Q_1}) := \sqrt{-1} \cdot \text{Trace} \left( \sum_{v \in Q_0} u_v \cdot \left( \sum_{a \in Q_1} [T_a^*, T_a] - \sum_{v \in Q_0} \eta_v \cdot P r e_v \right) \right).
\end{equation}

We see that the vanishing of the moment map is equivalent to King’s equation (2.4).

Example 2.13. (Hermitian Yang-Mills equations as moment map equations)

Let $\mathcal{E} \to X$ be a complex vector bundle over a Kähler manifold $(X, \omega_X^{1,1})$, endowed with a Hermitian metric. We define the “compact” group $K$ to be the group of unitary automorphisms of $\mathcal{E}$. The infinite-dimensional manifold $M$ on which $K$ acts will be the affine space of unitary automorphisms $\mathcal{E}$ (not necessarily integrable). The space of $\partial$-connections has the tangent space (at each point) equal to $\Gamma(X, \text{End} \mathcal{E} \otimes \Omega_X^{1,1})$, and it is endowed with the Hermitian structure given by

\begin{equation}
(2.21) \quad (\alpha, \beta) := \sqrt{-1} \int_X \text{Trace}(\alpha \wedge \beta) \wedge (\omega_X^{1,1})^{\dim_X X - 1}.
\end{equation}
We define a constant (i.e. translationally invariant) Kähler metric $\omega^{1,1}_M$ on the affine space of connections by the form (2.21) on each tangent space. The action of group $K$ is by Kähler isometries, hence symplectic. Moreover, this action has a canonical Hamiltonian lift, with the moment map given by

$$(2.22) \quad H_u(\nabla^{0,1}) := \int_X \text{Trace}(u \cdot \frac{1}{2\pi\sqrt{-1}} F_{\nabla^{0,1}} \cdot (\omega^{1,1}_X)_{\text{dimc} X-1} - \lambda u \cdot (\omega^{1,1}_X)_{\text{dimc} X}).$$

Again, we see that the vanishing of the moment map is equivalent to the HYM equation.

### 2.3. Further examples of harmonic representatives.

#### 2.3.1. ADHM construction.

In physics (gauge theory) one is interested in solutions of HYM equations (2.10) in the case of a non-compact space $X = \mathbb{R}^4 = \mathbb{C}^2$ endowed with the standard flat metric. The solution with finite energy $\int \| F \|^2 < \infty$ are called instantons. A classical result [4] identifies instantons for the gauge group $U(k)$ and total charge $N \in \mathbb{Z}_{\geq 0}$ (the second Chern class $c_2$), with a conjugacy classes (under the natural action of $U(k) \times U(N)$) of solutions of the system of ADHM equations

$$(2.23) \quad [\alpha, \beta] + ba = 0, \quad [\alpha^\dagger, \alpha] + [\beta^\dagger, \beta] + b^\dagger b - a a^\dagger = 0$$

where

$$(2.24) \quad \alpha, \beta \in \text{End}(\mathbb{C}^N), \quad a \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^N), \quad b \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^k)$$

satisfying the following non-degneracy condition:

$$(2.25) \quad \text{the stabilizer of } (a, b, \alpha, \beta) \text{ in } \text{GL}(k) \text{ is trivial.}$$

*Framed* instantons are defined as solutions of ADHM equations satisfying the nondegeneracy condition (2.25), modulo the (free) action of the group $U(N)$ only. In terms of algebraic geometry, framed instantons on $\mathbb{R}^4$ correspond to polystable holomorphic vector bundles $\mathcal{E}$ on $\mathbb{C}P^2 \supset \mathbb{C}^2 \simeq \mathbb{R}^4$ with the Chern classes

$$(2.26) \quad \text{rank } \mathcal{E} = k, \quad c_1(\mathcal{E}) = 0, \quad \langle [\mathbb{C}P^2], c_2(\mathcal{E}) \rangle = N$$

and with the trivialization of the restriction of $\mathcal{E}$ to the projective line at infinity $\mathbb{C}P^1_\infty := \mathbb{C}P^2 - \mathbb{C}^2$. The residual action of $U(k) \subset \text{GL}(k, \mathbb{C})$ is via changing the trivialization isomorphism

$$(2.27) \quad \mathcal{E}_{\mathbb{C}P^1_\infty} \simeq \mathbb{C}^k \otimes \mathcal{O}_{\mathbb{C}P^1_\infty}.$$

One can view instantons on $\mathbb{R}^4 = \mathbb{C}^2$ as solutions of HYM on $\mathbb{C}P^2$ for a singular Kähler metric (which is the flat metric on $\mathbb{C}^2$), with singularities at $\mathbb{C}P^1_\infty \subset \mathbb{C}P^2$.

(Framed) ADHM equations can be re-interpreted as King’s equation for the following quiver $Q^{(k)}$. The set of vertices is two-element set $\{1, 2\}$. Quiver $Q^{(k)}$ has two arrows $\alpha, \beta$ connecting vertex 1 with itself, $k$ arrows $a_1, \ldots, a_k$ connecting 2 with 1, and $k$ arrows $b_1, \ldots, b_k$ connecting 1 with 2.

![Diagram of quiver Q(k)](image-url)
A solution of ADHM equations gives a representation $\mathcal{F}$ of $Q^{(k)}$ in the Hermitian spaces $\mathcal{F}_1 = \mathbb{C}^N$, $\mathcal{F}_2 = \mathbb{C}^1$ (endowed with the standard Hermitian norm), satisfying the constraints

\begin{equation}
[\alpha, \beta] + \sum_{i=1}^{k} b_i a_i = 0,
\end{equation}

\begin{equation}
[\alpha^\dagger, \alpha] + [\beta^\dagger, \beta] + \sum_{i=1}^{k} b_i^\dagger b_i - \sum_{i=1}^{k} a_i a_i^\dagger = 0.
\end{equation}

Equation (2.28) can be viewed as a relation in the path algebra of $Q^{(k)}$, and equation (2.29) can be viewed as King’s equation at vertex 1, cf. (2.5). Notice that the King’s equation at vertex 2 is automatically satisfied by the following reason: we have an obvious trace identity

\begin{equation}
\text{Trace} \left( [\alpha^\dagger, \alpha] + [\beta^\dagger, \beta] + \sum_{i=1}^{k} b_i^\dagger b_i + \sum_{i=1}^{k} a_i a_i^\dagger \right) = 0.
\end{equation}

Therefore, equation (2.29) implies that the l.h.s. of the King’s equation at the vertex 2 has also trace 0, but it is an endomorphism of the 1-dimensional space $\mathcal{F}_2 = \mathbb{C}^1$, hence it is equal to 0 as an operator.

2.3.2. Instantons on noncommutative $\mathbb{R}^4$ and deformed ADHM construction. About 20 years ago, motivated by ideas from string theory, following pioneering work [1], N. Nekrasov and A. Schwarz in [15] proposed a generalization of ADHM construction and HYM equations to the case of noncommutative flat space $\mathbb{R}^4_\theta$. The latter is understood as certain completion of quantum algebra $A_\theta$ generated by coordinates $x_1, x_2, x_3, x_4$ satisfying commutation relations

\begin{equation}
[x_1, x_j] = \sqrt{-1} \cdot \theta_{ij}
\end{equation}

where $\theta = (\theta_{ij})_{1 \leq i, j \leq 4}$ is a real non-degenerate skew-symmetric $4 \times 4$ matrix. A bundle over the noncommutative space, corresponding to $A_\theta$, is understood as a finitely-generated projective $A_\theta$-module. The space of framed instantons on noncommutative $\mathbb{R}^4_\theta$ is in one-to-one correspondence with the set of solutions of the deformed ADHM equations

\begin{equation}
[\alpha, \beta] + ba = 0, \quad [\alpha^\dagger, \alpha] + [\beta^\dagger, \beta] + b^\dagger b - aa^\dagger = \eta \cdot \text{id}_{\mathbb{C}^N}, \quad \eta \neq 0
\end{equation}

without any non-degeneracy condition like (2.25). The deformed ADHM equations can be (again) interpreted as King’s equations for the same quiver $Q^{(k)}$ but with the deformed moment map (parameters $\eta$ as in (2.3)).

Each instanton on noncommutative space $\mathbb{R}^4_\theta$ gives a torsion-free module $E$ over $\mathbb{C}[z_1, z_2]$ where $z_1, z_2$ are two complex coordinates on $\mathbb{C}^2 \simeq \mathbb{R}^4$, which is extended to a coherent sheaf on $\mathbb{C}P^2$ trivialized as a bundle at $\mathbb{C}P^1$. In contrast with the commutative case, $E$ is not necessarily locally-free (i.e. not a vector bundle globally). For example, $E$ could be an ideal of finite codimension in $\mathbb{C}[z_1, z_2]$, giving a large class of examples of instantons of rank $k = 1$ on $\mathbb{R}^4_\theta$ which does not have any analog in the commutative limit $\theta \to 0$. Notice that such torsion-free coherent sheaves are not reflexive (see (2.11), hence are are excluded in the classical (commutative) Kobayashi-Hitchin correspondence.

2.3.3. Nekrasov’s proposal: an infinite-dimensional King’s equation. Soon after [15] it was observed in works by K. Furuuchi [9] and by N. Nekrasov [14] that the equations for an instanton on $\mathbb{R}^4_\theta$ for $\theta \neq 0$ are in a sense equivalent to a structure of pre-Hilbert space on $\mathbb{C}[z_1, z_2]$-module $E$ satisfying certain constraint which is an infinite-dimensional generalization of King’s equation, which differs drastically from ADHM equations. This equivalence is not translationally invariant, in a sense it depends on a specific coherent state for algebra $A_\theta$ which is “centered” at point $0 \in \mathbb{R}^4$.

Many years ago one of us (M.K) was told by N. Nekrasov that the correspondence between solutions of HYM equations on flat noncommutative spaces and solutions of the infinite-dimensional King’s equation should exist in any complex dimension $n$ of the flat space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, beyond the hyperkähler case $n = 2$ where we have ADHM construction at our disposal.
In what follows we will describe informally the infinite-dimensional King’s equation from Nekrasov’s proposal. In the last section of the paper 6.5 we will sketch a derivation of the infinite-dimensional King’s equation from HYM equations on flat noncommutative spaces $\mathbb{R}^2_n$ for arbitrary $n$.

Let $E = E_{\text{global}}$ be a finitely generated torsion-free $\mathbb{C}[z_1, z_2, \ldots, z_n]$-module, corresponding to an algebraic coherent sheaf $\mathcal{E}$ on $CP^n$ which is a vector bundle outside of a finite set of points in $C^n \subset CP^n$, together with the trivialized restriction to $CP^{n-1} := CP^n - C^n$.

The infinite-dimensional King’s-like equation (which we suggest to call Nekrasov equation) is the equation on a positive Hermitian inner product $h = h_{\text{global}}$ on $E_{\text{global}}$. Let us denote by $\mathcal{H} = \mathcal{H}_h$ the completion of the vector space $E_{\text{global}}$ with respect to $h$. The action of generators $z_i \in \mathbb{C}[z_1, z_2, \cdots, z_n]$ give rise to commuting unbounded operators $Z_i$ on $\mathcal{H}$. The proposed equation is,

$$\sum_{i=1}^n [Z_i^\dagger, Z_i] = \hbar \cdot n \cdot \text{id}_\mathcal{H} \tag{2.33}$$

where the ”Planck’s constant” $\hbar > 0$ is only a real parameter, and Hermitian conjugates $Z_i^\dagger$ are taken with respect to $h$.

We cannot help but ask the reader to notice the remarkable similarity between King’s equation (2.2) (for the quiver with one vertex and $n$ loops) and Nekrasov equation (2.33).

This is not yet a precise mathematical formulation because one should specify the ”behaviour at infinity”. Presumably, it is given by the condition

$$\forall 1 \leq i, j \leq n: \quad [Z_i^\dagger, Z_j] = \hbar \delta_{ij} \cdot \text{id}_\mathcal{H} + \text{trace class operator} \tag{2.34}$$

Also, Nekrasov argued that for torsion-free algebraic coherent sheaves on $C^n$ of higher ($k > 1$) rank, the solutions of noncommutative HYM should approximate the solutions of the usual HYM equation in the limit $h \to 0$, at least at the open locus in $C^n$ where the sheaf is a bundle. First, the space of positive Hermitian products on $E_{\text{global}}$ is an approximation to the space of Hermitian metrics on a holomorphic vector bundle over $X$. Indeed, e.g. for $\mathcal{E} = \mathcal{O}_{C^n}^{\otimes k}$ the Hermitian product on $E_{\text{global}} = \mathcal{O}_{C^n}^{\otimes k} \otimes \mathbb{C}[z_1, z_2, \cdots, z_n]$ is given (roughly) by a positive self-adjoint element in

$$E_{\text{global}} \otimes \overline{E}_{\text{global}} = \mathbb{C}[z_1, z_2, \cdots, z_n] \otimes \mathbb{C}[\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_n] \otimes (\mathbb{C}^k \otimes \overline{\mathbb{C}}^k) \cong C^\infty(\mathbb{R}^{2n}) \otimes \mathbb{R} \text{Mat}(k \times k, \mathbb{C}),$$

and then should give a metric in the trivial bundle of rank $k$ on $C^n$.

Following two (informal) conjectures are due to Nekrasov.

**Conjecture 1.** Equation (2.33) has a unique solution with a given appropriate boundary condition at infinity.

**Conjecture 2.** In the limit $h \to 0$ solutions of the equation (2.33) approaches to the solutions of the equation (2.10) with parameter $\lambda = 0$.

It seems that one can generalize all this to arbitrary coherent sheaves on $C^n$, not necessarily torsion-free. Presumably, the sheaf should be pure of certain dimension $m \leq n$ (meaning that the dimension of support of the sheaf is $m$), and the sheaf has no non-zero subsheaves with at most $(m - 1)$-dimensional support. Moreover, the trivialization at infinity (in the case $m = n$) should be replaced by an extension to $CP^{n-1}_\infty$ together with a metric on it satisfying HYM equation. The corresponding Nekrasov equation is

$$\sum_{i=1}^n [Z_i^\dagger, Z_i] = \hbar \cdot m \cdot \text{id}_\mathcal{H} \tag{2.36}$$

As an example we mention King’s equation for finite-dimensional representations of $\mathbb{C}[z_1, \ldots, z_n]$ (the case $m = 0$, the equation is literally the same as (2.2)), and the case $m = 1$ for curves in affine spaces studied partially before (see [3] and references therein).
3. Algebraic formalism: synopsis

Let us fix some notations. For an associative unital algebra $A$ over $\mathbb{C}$, we denote by $\overline{A}$ the complex-conjugate algebra:

$$f \cdot \overline{g} = \overline{f \cdot g}, \quad \overline{f + g} = \overline{f} + \overline{g}, \quad \overline{\lambda f} = \overline{f}, \quad \forall f, g \in A, \forall \lambda \in \mathbb{C},$$

and by $A^{op}$ the opposite algebra

$$f^{op} + g^{op} = (f + g)^{op}, \quad f^{op} \cdot g^{op} = (g \cdot f)^{op}, \quad (\lambda f)^{op} = \lambda \cdot f^{op} \quad \forall f, g \in A, \forall \lambda \in \mathbb{C}.$$ 

There are canonical isomorphisms

$$(3.3) \quad (A_1 \otimes A_2)^{op} \simeq A_2^{op} \otimes A_1^{op}, \quad \overline{A_1 \otimes A_2} \simeq \overline{A_1} \otimes \overline{A_2}, \quad A^{op} \simeq A^{op}, \quad \overline{A} \simeq (A^{op})^{op} \simeq A.$$ 

If $E$ is a left module over $A$ then $E^{op}$ is a left module over $\overline{A}$. Similarly, a right module over $A$ is the same as a left module over $A^{op}$. We have a duality between finitely-generated projective left module $E$ over $A$ and finitely-generated projective right modules

$$(3.4) \quad E \leftrightarrow E^{\vee} := \text{Hom}_{A^{mod}}(E, A) \in \text{mod} - A, \quad E = \text{Hom}_{\text{mod} - A}(E^{\vee}, A).$$

A $*$-algebra is an associative unital algebra $A$ over $\mathbb{C}$ endowed with an anti-linear involution $f \mapsto f^*$ satisfying

$$(3.5) \quad (f^*)^* = f, \quad f^* + g^* = (f + g)^*, \quad f^* \cdot g^* = (g \cdot f)^*, \quad (\lambda f)^* = \overline{\lambda} \cdot f^* \quad \forall f, g \in A, \forall \lambda \in \mathbb{C}.$$ 

For any $*$-algebra $A$ we have a canonical isomorphism $\overline{A} \simeq A^{op}$, $f \mapsto (f^{op})$. An element $f \in C$ is called Hermitian if $f = f^*$, and non-negative iff it can be written as a finite sum of the form $\sum f_i^* f_i$.

In particular, for a $*$-algebra $A$ and a bimodule $B$ over $A$ (i.e. a module over $A \otimes A^{op}$, we can write $B \in A - \text{mod} - A$), the complex-conjugate $\overline{B}$ (which is a module over $\overline{A} \otimes A^{op}$) is naturally again a bimodule over $A$ via the chain of canonical isomorphisms of algebras

$$(3.6) \quad \overline{A \otimes A^{op}} \simeq A \otimes A^{op} \simeq A^{op} \otimes A \simeq A \otimes A^{op}.$$ 

The setup (in which later we will define the moment map equations) is the following: we are given

(A1) an associative unital $*$-algebra $A$ over $\mathbb{C}$,

(A2) a bimodule $\Omega^1$ over $A$,

(A3) a derivation $d : A \to \Omega^1$, i.e. a $\mathbb{C}$-linear map $d$ satisfying the Leibniz rule

$$(3.7) \quad d(f \cdot g) = f \cdot d(g) + d(f) \cdot g, \quad \forall f, g \in A,$$

(A4) a bilinear form $\omega : \Omega^1 \otimes_{\mathbb{C}} \Omega^1 \to \mathbb{C}$ (a “noncommutative Kähler form”) satisfying the properties

$$(3.8) \quad \omega(\alpha, \overline{\beta}) = \overline{\omega(\beta, \alpha)}, \quad \omega(f \cdot \alpha \cdot g, \beta) = \omega(\alpha, f^* \cdot \beta \cdot g^*), \quad \omega(\alpha, \overline{\beta}) > 0 \quad \forall \alpha \neq 0,$$

(A5) a linear functional $\eta : A \to \mathbb{C}$ satisfying

$$(3.9) \quad \eta(f^*) = -\overline{\eta(f)}, \quad \eta(\{f, g\}) = \frac{-1}{2\sqrt{-1}} \left( \omega(df, d(g^*)) - \omega(dg, d(f^*)) \right).$$ 

This setup will be applied to

(M1) a finitely-generated projective $A$-module $E$,

(M2) a connection on $E$ which is defined as a $\mathbb{C}$-linear map $\nabla : E \to \Omega^1 \otimes_A E$ satisfying

$$(3.10) \quad \nabla(f \cdot \phi) = df \otimes \phi + f \cdot \nabla(\phi), \quad \forall f \in A, \phi \in E,$$
(M3) a Hermitian form on $E$ which is defined to be a bilinear map $H : E \otimes_C E \to \mathcal{A}$ satisfying
\begin{equation}
H(f\phi_1, g\phi_2) = f \cdot H(\phi_1, \overline{\phi_2}) \cdot g^*
\end{equation}
and such that the induced morphism of right modules over $\mathcal{A}$
\begin{equation}
E \to E^\vee = \text{Hom}_{\mathcal{A}-\text{mod}}(E, \mathcal{A}), \quad \phi_2 \mapsto (\phi_1 \mapsto H(\phi_1, \overline{\phi_2}))
\end{equation}
is an isomorphism and is positive-definite, in the sense $H(\phi, \overline{\phi}) \geq 0$ for all $\phi \in E$.

We will explain in the next section (see Proposition 2) that the action of the gauge group of unitary automorphisms of $E$ on the space of connections on $E$ can be lifted using (3.9) to a Hamiltonian action. In particular, we will get the notion of a harmonic representative.

**Definition 3.1.** For a finitely-generated projective $\mathcal{A}$-module $\mathcal{E}$ endowed with connection $\nabla$, a Hermitian form $H$ is called harmonic iff it satisfies the moment map equation (5.70) defined later in section 5.3.

**Remark 3.2.** Our setup differs from the one proposed in [10]. It would be interesting to compare two formalisms.

4. **Explanations in two basic examples**

We will illustrate our axiomatics in the case of a quiver, or a compact $C^\infty$-manifold $X$.

**(A1)+(M1):** The algebra $\mathcal{A}$ is either a finite sum $\mathbb{C}^Q_0$ of copies of $\mathbb{C}$ (quiver case), or the algebra $C^\infty_C(X) := C^\infty(X) \otimes_R \mathbb{C}$ of smooth $\mathbb{C}$-valued functions on a manifold $X$, with the involution $*$ given by the complex conjugation. In these examples $\mathcal{A}$ happen to be commutative, although this property does not play any role in the general formalism. In the noncommutative gauge theory the algebra $\mathcal{A}$ is the algebra of functions on a noncommutative deformation of $\mathbb{R}^4$.

In general, a finitely-generated projective $\mathcal{A}$-module $E$ is a left $\mathcal{A}$-module which is isomorphic to $\mathcal{A}^n \cdot P$ where $P \in \text{Mat}(n \times n, \mathcal{A})$ is a projector, $P^2 = P$.

Such a module is the same data as a collection of finite-dimensional complex vector spaces $(\mathcal{E}_v)_{v \in Q_0}$ where $E := \oplus_v \mathcal{E}_v$ (quiver case), or the same data as a finite-dimensional complex vector bundle $\mathcal{E}$ over $X$ where $E = \Gamma(X, \mathcal{E})$ (manifold case).

**A2:** The bimodule $\Omega^1$ in the quiver case is the complex vector space $\mathbb{C}^{Q_1}$ spanned by the set of arrows $Q_1$ of the quiver, with the structure of a bimodule over $\mathcal{A}$ given by
\begin{equation}
a = \pi_{s(a)} \cdot a \cdot \pi_{t(a)},
\end{equation}
where $\pi_v \in \mathcal{A} = \mathbb{C}^{Q_0}$ denotes the projector (the base vector) corresponding to arbitrary $v \in Q_0$.

In the case of a manifold, the bimodule $\Omega^1$ is the space of complex-valued 1-forms on $X$ with both the left and the right action given by the point-wise multiplication. More generally, one can consider pairs $(X, \mathcal{F})$ where $\mathcal{F} \subset T_X \otimes_R \mathbb{C}$ is a complex vector subbundle of the complexified tangent bundle $T_X$ to $X$ such that
\begin{equation}
\mathcal{F} + \overline{\mathcal{F}} = T_X \otimes_R \mathbb{C}.
\end{equation}
We define in this case the bimodule $\Omega^1$ as the space of sections of the dual bundle $\Gamma(X, \mathcal{F}^*)$, which is the quotient of the space $\Gamma(X, T_X^* \otimes \mathbb{C})$ of complex-valued 1-forms on $X$.

The condition (4.2) is satisfied e.g. when $X$ is endowed with a complex structure and $\mathcal{F} = T_X^{0,1}$. More generally, the case when (4.2) is satisfied and $\mathcal{F}$ is formally integrable (which means that $\Gamma(X, \mathcal{F}) \subset \Gamma(X, T_X \otimes \mathbb{C})$ is closed under the Lie bracket), corresponds to a foliation on $X$ with a transversal holomorphic structure. The foliation is given by the real distribution $\mathcal{F} \cap T_X$. In this case the sheaf of functions on $X$ are killed by all the complex-valued vector fields which are local sections of $\mathcal{F}$, is the same as the sheaf of functions which are locally constant along the foliation and holomorphic on the complex quotient.

In what follows, we will call the case $\Omega^1 = \Gamma(X, T_X^{0,1} \otimes \mathbb{C})$ the totally real case, and the case $\Omega^1 = \Gamma(X, (T_X^{0,1})^*)$ when $X$ is endowed with a complex structure, the totally complex case.
(A3)+(M2): the derivation \( d \) is equal to zero in the quiver case, and to the de Rham differential in the manifold case when \( \Omega^1 = \Gamma(X, T_X^* \otimes_{\mathbb{R}} \mathbb{C}) \). More generally, in the case of a complex distribution \( \mathcal{F} \) as above, the differential \( d \) is the composition of the de Rham differential \( \mathcal{A} = C^\infty_C(X) \to \Gamma(X, T_X^* \otimes_{\mathbb{R}} \mathbb{C}) \) and of the projection \( \Gamma(X, T_X^* \otimes_{\mathbb{R}} \mathbb{C}) \to \Gamma(X, \mathcal{F}^*) \).

In the quiver case, a connection on a finitely-generated projective module \( E = (\mathcal{E}_x)_{x \in Q_0} \) is the same as an action of arrows
\[
T_a : \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)} \quad \forall a \in Q_1
\]
which extend to an action of the path algebra of the quiver.

In the manifold case, a connection is the usual connection on a complex vector bundle, or a connection along distribution \( \mathcal{F} \). In the totally complex case when \( \mathcal{F} = T_X^{0,1} \), the connection in algebraic sense is the same as \( \mathcal{E} \)-connection on \( \mathcal{E} \).

In the general algebraic setup, the differential \( d : \mathcal{A} \to \Omega^1 \) gives rise to a structure of a bimodule on \( B := \mathcal{A} \oplus \Omega^1 \) given by
\[
f \cdot (h, \alpha) \cdot g := (f \cdot h \cdot g, f \cdot \alpha \cdot g + df \cdot h \cdot g), \quad \forall f, h, g \in \mathcal{A}, \alpha \in \Omega^1
\]
endowed with an epimorphism \( \pi_B \) onto the diagonal bimodule \( \mathcal{A}_{\text{diag}} \) given by \( (h, \alpha) \mapsto h \), and a splitting \( h \mapsto (h, 0) \) which is a monomorphism \( i_B \) of right modules over \( \mathcal{A} \). Conversely, any \( \mathcal{A} \)-bimodule \( B \) together with morphisms\(^2\)
\[
\pi_B \in \text{Hom}_{\mathcal{A} \text{-mod} - \mathcal{A}}(B, \mathcal{A}_{\text{diag}}), \quad i_B \in \text{Hom}_{\mod - \mathcal{A}}(\mathcal{A}_{\text{diag}}, B)
\]
such that \( \pi_B \circ i_B = \text{id}_B \) is the same data as a bimodule \( \Omega^1 := \text{ker}(\pi_B) \) together with a derivation \( d : \mathcal{A} \to \Omega^1 \) satisfying the analogous condition (3.10). The notion of a connection satisfying the analogous condition (3.10) can be rephrased as a homomorphism of left \( \mathcal{A} \)-modules
\[
\nabla : E \to B \otimes_{\mathcal{A}} E, \quad \nabla \in \text{Hom}_{\mathcal{A} \text{-mod}}(E, B \otimes_{\mathcal{A}} E)
\]
satisfying the constraint
\[
(\pi_B \circ i_B) \circ \nabla : E \to \mathcal{A}_{\text{diag}} \otimes_{\mathcal{A}} E \simeq E \quad \text{is equal to } \text{id}_E.
\]
Explicitly, the correspondence is given by
\[
\nabla \rightsquigarrow \text{morphism } \nabla : \phi \mapsto (\phi, \nabla(\phi)) \in E \oplus (\Omega^1 \otimes_{\mathcal{A}} E) = B \otimes_{\mathcal{A}} E.
\]
Assume that \( B \) is a finitely-generated projective when considered as a right module over \( \mathcal{A} \) (equivalently, one can replace \( B \) by \( \Omega^1 \) because \( \Omega^1 \oplus \mathcal{A}_{\text{diag}} \simeq B \) in \( \mod - \mathcal{A} \)). Then \( B \) can be represented as the dual to a finitely-generated projective left \( \mathcal{A} \)-module which we denote by \( \text{Diff}_{\leq 1} : \)
\[
\text{Diff}_{\leq 1} \simeq \text{Hom}_{\mathcal{A} \text{-mod} - \mathcal{A}}(B, \mathcal{A}), \quad B \simeq \text{Hom}_{\mod - \mathcal{A}}(\text{Diff}_{\leq 1}, \mathcal{A})
\]
In the manifold case and \( \Omega^1 = \Gamma(X, T_X^* \otimes_{\mathbb{R}} \mathbb{C}) \) the space \( \text{Diff}_{\leq 1} \) can be naturally identified with the space of differential operators of order \( \leq 1 \), hence the notation.

The left \( \mathcal{A} \)-action on \( \Omega^1 \) gives a right action on \( \text{Diff}_{\leq 1} \), therefore we have \( \text{Diff}_{\leq 1} \in \mathcal{A} - \mod - \mathcal{A} \). The epimorphism \( \pi_B \) gives (by duality) a monomorphism of bimodules \( \pi_B^* : \mathcal{A}_{\text{diag}} \to \text{Diff}_{\leq 1} \). We define the algebra \( \text{Diff} \) of “noncommutative differential operators” as the quotient of the tensor algebra
\[
T_{\mathcal{A}}(\text{Diff}_{\leq 1}) := \mathcal{A} \oplus \text{Diff}_{\leq 1} \oplus (\text{Diff}_{\leq 1} \otimes_{\mathcal{A}} \text{Diff}_{\leq 1}) \oplus \ldots
\]
by the two-sided ideal generated by the subspace
\[
\{ f - \pi_B^*(f) \mid f \in \mathcal{A} \} \subset \mathcal{A} \oplus \text{Diff}_{\leq 1} \subset T_{\mathcal{A}}(\text{Diff}_{\leq 1}).
\]
The algebra \( \text{Diff} \) is filtered (with the component \( \text{Diff}_{\leq n} \subset \text{Diff} \) defined as the image of the subspace \( \text{Diff}_{\leq 1} \subset T_{\mathcal{A}}(\text{Diff}_{\leq 1}) \)), and endowed with a homomorphism \( \mathcal{A} \to \text{Diff} \). It follows from definitions
\[\text{In the formulation of the notion of a connection below, the homomorphism } i_B \text{ plays no role. It can be completely omitted. What we really need is just a bimodule } B \text{ and a morphism } \pi_B : B \to \mathcal{A}_{\text{diag}} \text{ of bimodules.}\]
that finitely-generated $\mathcal{A}$-modules with connections can be identified with $\text{Diff}$-modules which are finitely-generated projective as $\mathcal{A}$-modules. The algebra $\text{Diff}$ is the usual path algebra in the quiver case, and a “free analog” of the algebra of differential operators in the manifold case. In the totally real case $
abla^1 = \Gamma(X, T_X \otimes \mathbb{R})$, in the local coordinates $(x_1, \ldots, x_k)$ on $X$, an element of $\text{Diff}$ can be written as a finite sum

$$
(4.12) \sum_{l \leq N, i_1, \ldots, i_l \in \{1, \ldots, k\}} f_{i_1, \ldots, i_l} \cdot \partial_{i_1} \cdot \cdots \cdot \partial_{i_l} \quad \text{for some} \ N < \infty, \ f_{i_1, \ldots, i_l} \in C^\infty_{\mathbb{C}}(X),
$$

where $\partial_i$ are free noncommutative variables obeying the exchange relation with the elements of $C$:

$$
(4.13) \partial_i \cdot f - f \cdot \partial_i = \frac{\partial f}{\partial x_i} \in \mathcal{A} = C^\infty_{\mathbb{C}}(X).
$$

In the totally complex case one replaces free variables ($\partial_i = \partial_x$) by the antiholomorphic derivatives ($\partial_x$) by the antiholomorphic derivatives. If we are interested e.g. in flat connections (or bundles with a holomorphic structure in the complex case), we should impose certain additional relations in $\text{Diff}$ (e.g. the commutativity relation $\partial_i \cdot \partial_j = \partial_j \cdot \partial_i$). The corresponding quotient algebra is either the usual algebra of (complex-valued) differential operators in the totally real case, or its subalgebra of differential operators in $\bar{\partial}$-direction in the totally complex case.

**A4:** In the quiver case, a choice of $\omega$ is equivalent to a choice of a collection of Hermitian norms on vector spaces

$$
(4.14) \Omega^1_{v_1, v_2} := \pi_{v_1} \cdot \Omega^1 \cdot \pi_{v_2} = \mathbb{C} \{ a \in \mathbb{Q} | s(a) = v_1, t(a) = v_2 \}
$$

for all pairs $(v_1, v_2)$ of vertices of $Q$. For example, one can declare the generating set

$$
\{ a \in \mathbb{Q} | s(a) = v_1, t(a) = v_2 \}
$$

to be an orthonormal basis of $\Omega^1_{v_1, v_2}$.

In the manifold case, the choice of $\omega$ is equivalent to a choice of a Hermitian form on the vector bundle $\mathcal{F} \subset T_X \otimes \mathbb{R}$. In the totally real (resp. totally complex) cases, a particular choice of such a form is given by a Riemannian metric (resp. a Kähler metric) on $X$.

**A5:** Let us denote by $d^1$ the derivation $\mathcal{A} \to \overline{\Omega^1}$ given by

$$
(4.15) d^1(f) := -\overline{\partial f}.
$$

Two derivations $d, d^1$ with values in $\mathcal{A}$-bimodules $\Omega^1, \overline{\Omega^1}$ and a linear map $\omega : \Omega^1 \otimes \overline{\Omega^1} \to \mathbb{C}$ satisfying

$$
(4.16) \omega((f \cdot \alpha \cdot g) \otimes \alpha') = \omega(\alpha \otimes (g \cdot \alpha' \cdot f)), \quad \alpha \in \Omega^1, \alpha' \in \overline{\Omega^1}, \ f, g \in \mathcal{A}
$$

give rise to a skew-symmetric functional on $\mathcal{A}$

$$
(4.17) \Psi(f \otimes g) := \omega(df \otimes d^1 g) - \omega(dg \otimes d^1 f)
$$

satisfying an additional reality constraint

$$
(4.18) \overline{\Psi(f \otimes g)} = -\Psi(f^* \otimes g^*).
$$

**Lemma 4.1.** The functional $\Psi$ satisfies the identity

$$
(4.19) \Psi(f_0 f_1 \otimes f_2) + \Psi(f_1 f_2 \otimes f_0) + \Psi(f_2 f_1 \otimes f_0) = 0.
$$

**Proof:** A direct calculation using (3.7) and (4.16) gives

$$
(4.20) \Psi(f_0 f_1 \otimes f_2) + \Psi(f_1 f_2 \otimes f_0) + \Psi(f_2 f_1 \otimes f_0) = \Psi(f_0 f_1 \otimes f_2) + \cdots = \omega(df_0 f_1 \otimes d^1 f_2) + \omega(df_0 f_1 \otimes d^1 f_2) - \omega(df_2 \otimes d^1 f_0 f_1) - \omega(df_2 \otimes f_0 d^1 f_1) + \cdots = \omega(df_0 \otimes f_1 d^1 f_2) + \omega(df_0 \otimes f_1 d^1 f_2) - \omega(df_2 \otimes d^1 f_0 f_1) - \omega(df_2 \otimes d^1 f_0 f_1) + \cdots = 0
$$

where triple dots in each line denote terms obtain by cyclic permutation of indices $0 \to 1 \to 2 \to 0$. ■
So, we see that \( \Psi \) is a 2-cocycle in the cyclic cochain complex of \( \mathcal{A} \). Recall that the latter is defined by
\[
C_{\text{cycl}}^n(\mathcal{A}) := \{ \psi : \mathcal{A}^{\otimes n} \to \mathbb{C} \mid \psi(f_2 \otimes \cdots \otimes f_n \otimes f_1) = (-1)^{n-1} \psi(f_1 \otimes \cdots \otimes f_n) \}
\]
with the differential
\[
d\psi(f_0 \otimes \cdots \otimes f_n) = \sum_{i \in \mathbb{Z}/(n+1)\mathbb{Z}} (-1)^m \psi(f_i f_{i+1} \otimes f_{i+2} \otimes \cdots \otimes f_{i-1}).
\]

The existence of \( \eta \) satisfying the constraint (4.9) means that the 2-cocycle \( \Psi \) is a coboundary. The obstruction lies in \( H^2_{\text{cycl}}(\mathcal{A}) \).

In the quiver case for \( \mathcal{A} = \mathbb{C} Q_0 \), there is no obstructions as \( H^2_{\text{cycl}}(\mathbb{C} Q_0) = 0 \). In the manifold case, the 2-nd continuous cyclic cohomology of \( \mathcal{A} = C_c^\infty(X) \) coincides with the continuous dual to \( \Omega^1(X)/d\Omega^0(X) \). Assume for simplicity that \( X \) is oriented. In this case, a dense subset of the continuous dual, as above, consists of closed forms on \( X \) of degree equal to \( \dim(X) - 1 \). Any closed form
\[
\beta \in \Gamma(X, \wedge^{\dim X - 1} T^* X \otimes_R \mathbb{C}), \quad d\beta = 0
\]
gives a cyclic 2-cochain by the formula
\[
f_1 \otimes f_2 \mapsto \int_X f_1 df_2 \wedge \beta.
\]

In our example of a complex distribution \( \mathcal{F} \subset TX \otimes \mathbb{C} \) and a Hermitian form on \( \mathcal{F} \), the corresponding obstruction class in \( H^2_{\text{cycl}}(\mathcal{A}) \) is represented by the differential of certain form \( \delta \) of degree \( \dim_R X - 2 \). The vanishing of the obstruction means that \( \delta \) is closed. This is a necessary and sufficient condition for the existence of a solution \( \eta \) for the constraint (3.9). In the case of HYM equations on complex Kähler manifolds the form \( \delta \) is equal to \( (\omega_X)^{\dim_X X - 1} \), where \( \omega_X \) is the Kähler form on \( X \).

**Remark 4.2.** We already observed that (for a given data \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \)) the obstruction to the existence of functional \( \eta \) is a class in \( H^2_{\text{cycl}}(\mathcal{A}) \) satisfying the reality constraint (4.18). If the obstruction vanishes, the set of choices of possible functionals \( \eta \) is a torsor over the real subspace of \( H^1_{\text{cycl}}(\mathcal{A}) = \text{Hom}(\mathcal{A}/[A,A], \mathbb{C}) \) given by the fixed points of the anti-linear involution
\[
\eta \mapsto \eta^*, \quad \eta^*(f) := -\overline{\eta(f^*)}.
\]

Notice the similarity with the analogous question for the liftings of a symplectic action to a Hamiltonian one, cf. Remark 2.11.

**(M3):** In the quiver case, a Hermitian \( \mathcal{A} \)-valued form on \( \mathcal{A} \)-module \( E \) is equivalent to the collection of Hermitian forms on the individual complex vector spaces \( E_v \) for all vertices \( v \in Q_0 \).

In the manifold case (independently on the choice of complex distribution \( \mathcal{F} \)), a Hermitian \( \mathcal{A} \)-valued form on an \( \mathcal{A} \)-module is equivalent to a Hermitian norm on the corresponding complex vector bundle \( E \).

In general, when a projector \( P \in \text{Mat}(n \times n, \mathcal{A}) \), \( P^2 = P \) is self-adjoint:
\[
P = (p_{ij})_{1 \leq i,j \leq n} \in \text{Mat}(n \times n, \mathcal{A}), \quad p_{ij}^* = p_{ji} \forall i,j, \quad P^2 = P,
\]
then the submodule \( E := \mathcal{A}^n \cdot P \) carries an \( \mathcal{A} \)-valued Hermitian form given by the restriction to \( E \subset \mathcal{A}^n \) of the standard form on \( \mathcal{A}^n \):
\[
H_{\text{standard}}((f_1, \ldots, f_n), (g_1, \ldots, g_n)) := \sum_i f_i g_i^*.
\]

**Remark 4.3.** The framework of [1] (and then of [15]) fits (partially) into our setup. In order to define the notion of a connection, authors of [1] use a collection \( (\partial_i)_{i=1,\ldots,n} \) of derivations of an algebra \( \mathcal{A} \) closed under the Lie bracket. In our formalism the corresponding bimodule is \( \Omega^1 := A_{\text{diag}}^{\otimes n} \) endowed with the derivation
\[
d(f) := (\partial_1 f, \ldots, \partial_n f) \in \Omega^1.
\]
5. Formula for the Hamiltonian action.

5.1. The case of a trivial bundle.
Let us assume that \( E \cong A^n = \mathbb{C}^n \otimes A \) is a free finitely generated left module over \( A \), endowed with the canonical Hermitian \( A \)-valued form (see (4.27)).

The set \( M \) of connections on \( E \) can be identified in the usual way with the space of matrices of 1-forms:

\[
(5.1) \quad A = (A_{ij})_{1 \leq i, j \leq n} \in Mat(n \times n, \Omega^1) \quad \mapsto \quad \nabla_A : E \to \Omega^1 \otimes_A E, \quad \nabla_A(\phi) = d\phi + \phi \cdot A.
\]

The Lie algebra of the “compact gauge group” is defined as

\[
(5.2) \quad \mathfrak{k} := \{(u_{ij})_{1 \leq i, j \leq n} \in Mat(n \times n, A) \mid u_{ij} = -u_{ji} \quad \forall i, j\}.
\]

It acts on the (infinite-dimensional) complex affine space \( M \) of connections by the infinitesimal affine transformations

\[
(5.3) \quad (d + \cdot A) \mapsto (1 - \epsilon u) \circ (d + \cdot A) \circ (1 - \epsilon u)^{-1} = d + \cdot A + \epsilon(du + [u, A])
\]

where \( \epsilon \) is a formal variable satisfying \( \epsilon^2 = 0 \), and notation \( \cdot A \) stays for the operator of right multiplication by \( A \), and similarly for other symbols. In other words, the value of the vector field \( X_u \) corresponding to \( u \in \mathfrak{k} \) on \( M \) at the point \( A \) is

\[
(5.4) \quad X_u|_A = du + [u, A].
\]

An \( A \)-valued Hermitian form \( H_0 \) on \( E \) together with a “noncommutative Kähler metric” \( \omega \) produces a usual \( \mathbb{C} \)-valued Hermitian form on the complex vector space \( Mat(n \times n, \mathbb{C}) \otimes \Omega^1 \) given by

\[
(5.5) \quad \omega_0(A^{(1)}, A^{(2)}) := \sum_{ij} \omega(A^{(1)}_{ij}, A^{(2)}_{ij}).
\]

This form is strictly positive on non-zero vectors by (3.8), and the infinitesimal action of \( \mathfrak{k} \) via \( A \mapsto A + \epsilon[u, A] \) preserves \( \omega_0 \). Therefore, the infinitesimal action of \( \mathfrak{k} \) on affine space \( M \) of connections endowed with the “constant” Kähler metric corresponding to \( \omega_0 \) is by Kähler isometries, because the vector field \( X_u \) is the sum of the infinitesimal generator of the linear action \( A \mapsto A + \epsilon[u, A] \) (which is an isometry), and of the shift by a constant vector \( A \mapsto A + \epsilon \cdot du \) (which is also an isometry).

In what follows, we will use an identity which follows directly from (3.8) and the definition (5.5)

\[
(5.6) \quad \omega_0([A^{(1)}, u], [A^{(2)}]) = \omega_0(A^{(1)}, [u, A^{(2)}]) \quad \forall A^{(1)}, A^{(2)} \in M, \quad \forall u \in \mathfrak{k}.
\]

The constant (i.e. invariant under shifts) symplectic form \( \omega^{\text{sym}}_M \) on \( M \) corresponding to the Kähler metric \( \omega_0 \) is given by the real skew-symmetric form on the tangent space

\[
(5.7) \quad \omega^{\text{sym}}_M(A^{(1)}, A^{(2)}) := \text{Im} \, \omega_0(A^{(1)}, A^{(2)}) = \frac{1}{2\sqrt{-1}} \left( \omega_0(A^{(1)}, \bar{A}^{(2)}) - \omega_0(A^{(2)}, \bar{A}^{(1)}) \right).
\]

For a given \( u \in \mathfrak{k} \), the corresponding vector field \( X_u \) is an infinitesimal Kähler isometry, hence it preserves the symplectic form \( \omega^{\text{sym}}_M \). We claim that this symplectic action of \( \mathfrak{k} \) can be lifted to a Hamiltonian action. Let us denote for \( u \in \mathfrak{k} \) by \( H_u \) the following real-valued function on \( M \):

\[
(5.8) \quad H_u(A) := \eta(\text{Trace}(u)) - \omega^{\text{sym}}_M(A, du) + \frac{1}{2}\omega^{\text{sym}}_M(A, [A, u]).
\]

Proposition 1. The assignment \( u \mapsto H_u \) is a Lie algebra homomorphism lifting the action \( u \mapsto X_u \).
Proof: First, it is immediate to see that the vector field $X_u$ corresponds to the Hamiltonian $H_u$:

\begin{equation}
\label{eq:5.10}
\iota_{X_u} \omega_M^{sym} = dH_u.
\end{equation}

It suffices (see (2.15)) to prove that

\begin{equation}
\label{eq:5.11}
\omega_M^{sym}(X_{u_1}, X_{u_2}) = H_{[u_1, u_2]} \quad \forall u_1, u_2 \in \mathfrak{g}.
\end{equation}

In other words, we have to check that for any $A \in M$

\begin{equation}
\label{eq:5.12}
\omega_M^{sym}(du_1 + [u_1, A], du_2 + [u_2, A]) = \eta(\text{Trace}([u_1, u_2])) - \omega_M^{sym}(A, d[u_1, u_2]) + \frac{1}{2} \omega_M^{sym}(A, [u_1, [u_1, u_2]]).
\end{equation}

Indeed, we have

\begin{equation}
\label{eq:5.13}
\omega_M^{sym}(du_1 + [u_1, A], du_2 + [u_2, A]) = \text{Im} \omega_0\left(du_1 + [u_1, A], \overline{du_2 + [u_2, A]}\right)
= \text{Im} \omega_0(du_1, \overline{du_2}) + \text{Im} \omega_0([u_1, A], \overline{du_2}) + \text{Im} \omega_0\left(du_1, [u_2, A]\right) + \text{Im} \omega_0\left([u_1, A], [u_2, A]\right).
\end{equation}

Then we use

\begin{equation}
\label{eq:5.14}
\text{Im} \omega_0([u_1, A], \overline{du_2}) + \text{Im} \omega_0\left(du_1, \overline{du_2, A}\right) = -\text{Im} \omega_0([A, u_1], \overline{du_2}) + \text{Im} \omega_0\left([A, u_2], \overline{du_1}\right) = -\omega_M^{sym}(A, d[u_1, u_2]),
\end{equation}

and, utilizing the antisymmetry of $\omega_M^{sym}$,

\begin{equation}
\label{eq:5.15}
\text{Im} \omega_0\left([u_1, A], [u_2, A]\right) = \frac{1}{2} \left(\text{Im} \omega_0\left([A, u_1], [A, u_2]\right) - \text{Im} \omega_0\left([A, u_2], [A, u_1]\right)\right) = \frac{1}{2} \text{Im} \omega_0\left(A, [A, [u_1, u_2]]\right) = \frac{1}{2} \omega_M^{sym}(A, [A, [u_1, u_2]]).
\end{equation}

This calculation finishes the proof of (5.11).
Later we will need a formula for $H_u(A)$ written in a slightly different form:

$$H_u(A) = \eta(\text{Trace}(u)) - \omega^{symp}(A, du) + \frac{1}{2} \omega^{symp}(A, [A, u]) = \eta(\text{Trace}(u)) + \frac{1}{2 \sqrt{-1}} \left[ -\omega_0(A, du) + \omega_0(du, A) + \frac{1}{2} \omega_0(A, [A, u]) - \frac{1}{2} \omega_0([A, u], A) \right].$$

5.2. General bundle.

Let $P$ be a self-adjoint (see (4.26)) projector in $Mat(n \times n, A)$. Then the free module $E = A^n$ splits into the orthogonal sum of two submodules (here we denote $id_{A^n}$ as 1 for brevity)

$$E \simeq E_1 \oplus E_2, \quad E_1 := E \cdot P, \quad E_2 := E \cdot (1 - P).$$

We will consider the action of the gauge group of unitary automorphisms of $E_1$ on the space $M_1$ of connections on $E_1$. First, consider the Lie subalgebra $\mathfrak{t}_{1+2}$ of $\mathfrak{t}$ consisting of infinitesimal unitary symmetries preserving the direct sum decomposition (5.17)

$$\mathfrak{t}_{1+2} := \{ u \in \mathfrak{t} \mid u = PuP + (1 - P)u(1 - P) \}.$$

It is clear that $\mathfrak{t}_{1+2}$ is the direct sum of two subalgebras

$$\mathfrak{t}_1 := \{ u \in \mathfrak{t} \mid u = PuP \}, \quad \mathfrak{t}_2 := \{ u \in \mathfrak{t} \mid u = (1 - P)u(1 - P) \}$$

and $\mathfrak{t}_1$ is the Lie algebra of infinitesimal unitary symmetries of $E_1$.

Next, consider the space of connections on $E$ preserving the direct sum decomposition (5.17):

$$\mathfrak{M}_{1+2} := \{ A \in Mat(n \times n, A) \mid d + A = P \cdot (d + A) \cdot P + (1 - P) \cdot (d + A) \cdot (1 - P) \}.$$

It is an affine subspace of the affine space $M$ of connections on $E$, and it is isomorphic to the product of the space $M_1$ of connections in $E_1$ and the space $M_2$ of connections in $E_2$.

There is a distinguished point $A_{can} \in M_{1+2}$ given by

$$A_{can} = P \cdot dP + (1 - P) \cdot d(1 - P) = (2P - 1) \cdot dP$$

which gives points $A_{can,1} \in M_1$, $A_{can,2} \in M_2$ after the identification $M_{1+2} \simeq M_1 \times M_2$. Then we identify $M_1$ with an affine subspace $M_{(1)} \subset M_{1+2}$ consisting of connections whose restriction to $E_2$ is $A_{can,2}$. Explicitly, we have

$$M_{(1)} = \{ A \in M \mid A = A_{can} + \delta_A, \quad \delta_A = P\delta_A P \}.$$

The Lie subalgebra $\mathfrak{t}_1 \subset \mathfrak{t}$ preserves the submanifold $M_{(1)} \subset M$. In particular, for any $u \in \mathfrak{t}_1$ the value of the vector field $X_u$ restricted to $M_{(1)}$ is given (see (5.4)) at the point $A_{can} + \delta_A$ by

$$X_u|_{A_{can} + \delta_A} = du + [u, A_{can} + \delta_A].$$

Using (5.11), this formula implies for any $u_1, u_2 \in \mathfrak{t}_1$

$$\omega^{symp}(X_{u_1}, X_{u_2}) = \omega^{symp}(du_1 + [u_1, A], du_2 + [u_2, A]) = \eta(\text{Trace}(u_1, u_2)) - \omega^{symp}(A, du_1, du_2 + [u_2, A]) = \frac{1}{2} \omega^{symp}(A, [A, u_1, u_2]),$$

where $A := A_{can} + \delta_A$. We conclude
Proposition 2. The assignment

\[ H_{(1),u}(A_{\text{can}} + \delta A) := \eta(\text{Trace}(u)) - \omega^\text{symp}(A_{\text{can}} + \delta A, du) + \frac{1}{2}\omega^\text{symp}(A_{\text{can}} + \delta A, [A_{\text{can}} + \delta A, u]) \]

gives a Hamiltonian action of \( k \) on \( M(1) \simeq M_1 \) lifting the symplectic action by gauge transformations. ■

5.3. Universal formula for the moment map.

In this section we propose a formula for the moment map written in an “invariant” way, which does not refer explicitly to the representation of finitely-generated projective \( \mathcal{A} \)-module \( E \) as an image of a self-adjoint projector \( P \in \text{Mat}(n \times n, \mathcal{A}) \) for some \( n < \infty \).

In order to be able to write the formula, we will need to introduce some notations and constructions.

5.3.1. More about Hermitian modules. In this section \( \mathcal{A} \) denotes an arbitrary \(*\)-algebra. Recall (see (3.12)) that a Hermitian structure \( H \) on a finitely-generated \( \mathcal{A} \)-module \( E \) gives rise to an isomorphism of \( \mathcal{A} \text{-}\)modules

\[ \text{iso}_H : E \simeq E^\vee \]

(here we consider the \( \overline{\mathcal{A}} \)-module \( \overline{E} \) as an \( \mathcal{A}^{op} \)-module via the canonical isomorphism of algebras \( \overline{\mathcal{A}} \simeq \mathcal{A}^{op} \)).

With any endomorphism \( u : E \rightarrow E, \ u \in \text{Hom}_{\mathcal{A}\text{-}mod}(E,E) \) we can associate

1. the complex-conjugate morphism \( \overline{u} : E \rightarrow E, \ u \in \text{Hom}_{\mathcal{A}\text{-}mod}(E,E) \),

2. the adjoint morphism, by applying the contravariant functor \( \text{Hom}_{\mathcal{A}\text{-}mod}(-, \mathcal{A}) \)

\[ u^t : E^\vee \rightarrow E^\vee, \ u^t \in \text{Hom}_{\mathcal{A}\text{-}mod}(E^\vee,E^\vee). \]

The Lie algebra of infinitesimal unitary symmetries of \((E,H)\) is defined (generalizing (5.2),(5.19)) as

\[ \mathfrak{k} := \{ u \in \text{Hom}_{\mathcal{A}\text{-}mod}(E,E) | u^t = -u \}, \ u^t := \overline{u^t} = \overline{u^\vee}. \]

Recall (see (4.4),(4.5) and (4.6),(4.7)) that a connection \( \nabla \) on \( \mathcal{A} \)-module \( E \) we can recast as a homomorphism of \( \mathcal{A} \)-modules

\[ \overline{\nabla} \in \text{Hom}_{\mathcal{A}\text{-}mod}(E, B \otimes \mathcal{A} E) \]

such that

\[ (\pi_B \otimes \text{id}_E) \circ \overline{\nabla} = \text{id}_E. \]

The complex conjugation gives

\[ \overline{\nabla} \in \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}, \overline{B} \otimes \overline{\mathcal{A}} E), \ (\pi_{\overline{B}} \otimes \text{id}_{\overline{E}}) \circ \overline{\nabla} = \text{id}_{\overline{E}}. \]

Applying the isomorphism \( \text{iso}_{\overline{\mathcal{A}}} \) from (5.26) we obtain another morphism

\[ \overline{\nabla} \in \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}^\vee, \overline{E}^\vee \otimes \overline{\mathcal{A}} \overline{B}), \ (\text{id}_{\overline{E}^\vee} \otimes \pi_{\overline{B}}) \circ \overline{\nabla} = \text{id}_{\overline{E}^\vee}, \]

where we treat \( \overline{B} \) as a \( \mathcal{A} \otimes \mathcal{A}^{op} \)-module via the canonical isomorphism of algebras \( \mathcal{A} \otimes \mathcal{A}^{op} \simeq \overline{\mathcal{A}} \otimes \overline{\mathcal{A}}^{op} \).

Finally, using the following chain of isomorphisms

\[ \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}^\vee, \overline{E}^\vee \otimes \overline{\mathcal{A}} \overline{B}) \simeq \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}^\vee, \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}, \overline{B})) \simeq \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}, \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}^\vee, \overline{B})) \simeq \text{Hom}_{\overline{\mathcal{A}}\text{-}mod}(\overline{E}, \overline{B} \otimes \overline{\mathcal{A}} E) \]
we obtain the Hermitian-conjugate connection (formulated in terms of a morphism of \(\mathcal{A}\)-modules)

\[
(5.33) \quad \nabla^\dagger \in \text{Hom}_{\mathcal{A}-\text{mod}}(E, B \otimes_{\mathcal{A}} E), \quad (\pi_{\overline{\mathcal{B}}} \otimes \text{id}_E) \circ \nabla^\dagger = \text{id}_E.
\]

Here in (5.32) we use the fact that for any finitely-generated projective \(\mathcal{A}\)-module \(E\) and an arbitrary \(\mathcal{A}\)-module \(F\), the canonical map

\[
(5.34) \quad E^\vee \otimes_{\mathcal{A}} F \to \text{Hom}_{\mathcal{A}-\text{mod}}(E, F),
\]

\[
E^\vee \otimes_{\mathcal{A}} F = \text{Hom}_{\mathcal{A}-\text{mod}}(E, \mathcal{A}) \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{A}-\text{mod}}(\mathcal{A}, F) \overset{\text{composition}}{\longrightarrow} \text{Hom}_{\mathcal{A}-\text{mod}}(E, F)
\]

is an isomorphism.

Alternatively, let us use the Hermitian-conjugate derivation \(d^\dagger\) with values in \(\overline{\Omega^1}\) (see (4.15))

\[
(5.35) \quad d^\dagger : \mathcal{A} \to \overline{\Omega^1}, \quad d^\dagger(f) := -d(f^\dagger), \quad d^\dagger(f \cdot g) = f \cdot d^\dagger(g) + d^\dagger(f) \cdot g \quad \text{for free}.
\]

The bimodule \(\overline{B}\) is identified with

\[
(5.36) \quad \mathcal{A} \oplus \overline{\Omega^1}, \quad \text{with the bimodule structure } f \cdot (h, \omega) \cdot g := (f \cdot h \cdot g, f \cdot \alpha \cdot g + d^\dagger f \cdot h \cdot g) \quad \text{(as in (4.4))}
\]

by the map

\[
(5.37) \quad (h, \omega) \in \overline{B} \mapsto (h^\dagger, \overline{\omega} - d\overline{\omega}) \in \mathcal{A} \oplus \overline{\Omega^1}.
\]

For the trivial \(\mathcal{A}\)-module \(E = \mathcal{A}^n\) with the canonical Hermitian \(\mathcal{A}\)-valued form (4.27), for any connection \(A\) given by a \((n \times n)\) matrix

\[
(5.38) \quad A = (A_{ij})_{1 \leq i, j \leq n} \in \text{Mat}(n \times n, \Omega^1)
\]

the Hermitian conjugate connection is given by

\[
(5.39) \quad A^\dagger = ((A^\dagger)_{ij})_{1 \leq i, j \leq n} \in \text{Mat}(n \times n, \overline{\Omega^1}), \quad (A^\dagger)_{ij} := \overline{A_{ji}} \quad \forall i, j.
\]

5.3.2. Bimodules and traces. In this section \(\mathcal{A}\) denotes an arbitrary associative algebra over \(\mathbb{C}\) (not necessarily a \(*\)-algebra). With every \(\mathcal{A}\)-bimodule \(G\) we associate a vector space \(#(G)\) by the formula

\[
(5.40) \quad #(G) := G/\{\text{linear span of } \alpha \cdot g - g \cdot \alpha \mid \alpha \in \mathcal{A}, g \in G\} \cong G \otimes_{\mathcal{A} \otimes_{\mathcal{A}^\text{op}} \mathcal{A}} \mathcal{A}_{\text{diag}}.
\]

It follows form the definition that for any finite sequence of bimodules \(G_1, \ldots, G_n\) one has a chain of canonical isomorphisms

\[
(5.41) \quad #(G_1 \otimes_{\mathcal{A}} G_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} G_n) \simeq #(G_2 \otimes_{\mathcal{A}} G_3 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} G_1) \simeq #(G_n \otimes_{\mathcal{A}} G_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} G_{n-1}).
\]

For any finitely-generated \(\mathcal{A}\)-module \(E\), any \(\mathcal{A}\)-bimodule \(G\) and any morphism of \(\mathcal{A}\)-modules

\[
(5.42) \quad \Phi : E \to G \otimes_{\mathcal{A}} E
\]

we define its trace along \(E\) (denoted by \(\text{Trace}_E(\Phi)\)) with values in \(#(G)\), via the chain of isomorphisms

\[
(5.43) \quad \Phi \in \text{Hom}_{\mathcal{A}-\text{mod}}(E, G \otimes_{\mathcal{A}} E) \overset{(5.34)}{=} E^\vee \otimes_{\mathcal{A}} G \otimes_{\mathcal{A}} E \simeq G \otimes_{\mathcal{A} \otimes_{\mathcal{A}^\text{op}} \mathcal{A}} \mathcal{A}_{\text{diag}} \simeq #(G) \ni \text{Trace}_E(\Phi),
\]

and a map

\[
(5.44) \quad G \otimes_{\mathcal{A} \otimes_{\mathcal{A}^\text{op}} \mathcal{A}} \mathcal{A}_{\text{diag}} \overset{\text{id}_G \otimes_{\mathcal{A} \otimes_{\mathcal{A}^\text{op}} \mathcal{A}} \text{id}_E}{\longrightarrow} G \otimes_{\mathcal{A} \otimes_{\mathcal{A}^\text{op}} \mathcal{A}} \mathcal{A}_{\text{diag}} = #(G) \ni \text{Trace}_E(\Phi),
\]

where

\[
(5.45) \quad \delta_E : E \otimes_{\mathcal{A}} E^\vee \to \mathcal{A}_{\text{diag}}, \quad \delta_E(\alpha \otimes \alpha^\vee) := \alpha^\vee(\alpha) \in \mathcal{A}, \quad \forall \alpha \in E, \forall \alpha^\vee \in \text{Hom}_{\mathcal{A}-\text{mod}}(E, \mathcal{A}) = E^\vee
\]

is the canonical morphism of \(\mathcal{A}\)-bimodules.
Remark 5.1. The constraints for the left and right action form on the noncommutative Kähler form $\omega$ (see (3.8)) can be interpreted as follows: $\omega$ is equal to the composition of a linear functional
\begin{equation}
\omega' : #(\Omega^1 \otimes_A \Omega^2) \to \mathbb{C}
\end{equation}
and of the canonical surjection
\begin{equation}
\Omega^1 \otimes_C \Omega^2 \to #(\Omega^1 \otimes_A \Omega^2).
\end{equation}

5.3.3. Linear functional on the triple tensor product. In this section we work in the setup (A1)−(A5). Recall that we have automatically two derivations $d, d^\dagger$ (see (5.35)), hence we can define a doubled bimodule by
\begin{equation}
\mathcal{B} := \mathcal{A} \oplus \Omega^1 \oplus \Omega^2, \quad f \cdot (h, \alpha, \beta) \cdot g := (f \cdot h \cdot g, f \cdot \alpha \cdot g + df \cdot h \cdot g, f \cdot \beta \cdot g + d^\dagger f \cdot h \cdot g).
\end{equation}
Define a linear map
\begin{equation}
\Xi : \mathcal{B} \otimes_C \mathcal{B} \otimes_C \mathcal{B} \to \mathbb{C}
\end{equation}
by the following formulas (the missing terms map to zero):
\begin{align}
&f_1 \otimes f_2 \otimes f_3 \mapsto \frac{1}{3} \left( -2\sqrt{-1} \eta(f_1 f_2 f_3) - \omega(df_1, df_1 f_2 f_3) + \omega(df_2, f_3, df_3 f_1 f_2 f_3) \right) + (1 \to 2 \to 3), \\
&\alpha_1 \otimes f_2 \otimes f_3 \mapsto +\omega(\alpha_1, d f_2 f_3), \quad \text{and the same r.h.s. for } f_3 \otimes \alpha_1 \otimes f_2, \ f_2 \otimes f_3 \otimes \alpha_1, \\
&\nu_1 \otimes f_2 \otimes f_3 \mapsto -\omega(df_2, f_3, \nu_1), \quad \text{and the same r.h.s. for } f_3 \otimes \nu_1 \otimes f_2, \ f_2 \otimes f_3 \otimes \nu_1, \\
&\alpha_1 \otimes \nu_2 \otimes f_3 \mapsto -\omega(\alpha_1, \nu_2, f_3) \quad \text{and the same r.h.s. for } f_3 \otimes \alpha_1 \otimes \nu_2, \ \nu_2 \otimes f_3 \otimes \alpha_1, \\
&\nu_1 \otimes \alpha_2 \otimes f_3 \mapsto +\omega(\alpha_2, f_3, \nu_1) \quad \text{and the same r.h.s. for } f_3 \otimes \nu_1 \otimes \alpha_2, \ \alpha_2 \otimes f_3 \otimes \nu_1.
\end{align}

Proposition 3. The map $\Xi$ descends to a map
\begin{equation}
\Xi' : #(\mathcal{B} \otimes_A \mathcal{B} \otimes_A \mathcal{B}) \to \mathbb{C}.
\end{equation}

Proof: It follows from the definition that the map $\Xi$ is $\mathbb{Z}/3\mathbb{Z}$-invariant, where $\mathbb{Z}/3\mathbb{Z}$ acts by cyclic permutations of factors in $\mathcal{B}^\otimes_3$. This symmetry reduces the number of possible checks to the following list:
\begin{align}
-f_1 \cdot g \otimes f_2 \otimes f_3 + f_1 \otimes g \cdot f_2 \otimes f_3 + f_1 \otimes dg \cdot f_2 \otimes f_3 + f_1 \otimes d^3 g \cdot f_2 \otimes f_3 \mapsto 0, \\
-f_1 \cdot g \otimes f_2 \otimes \alpha_3 + f_1 \otimes g \cdot f_2 \otimes \alpha_3 + f_1 \otimes d^3 g \cdot f \otimes \alpha_3 \mapsto 0, \\
-f_1 \cdot g \otimes f_2 \otimes \nu_3 + f_1 \otimes g \cdot f_2 \otimes \nu_3 + f_1 \otimes dg \cdot f \otimes \nu_3 \mapsto 0, \\
-f_1 \cdot g \otimes \alpha_2 \otimes f_3 + f_1 \otimes g \cdot \alpha_2 \otimes f_3 \mapsto 0, \\
-f_1 \cdot g \otimes \nu_2 \otimes f_3 + f_1 \otimes g \cdot \nu_2 \otimes f_3 \mapsto 0, \\
-f_1 \cdot g \otimes \alpha_2 \otimes f_3 + \alpha_1 \otimes g \cdot f_2 \otimes f_3 + \alpha_1 \otimes d^3 g \cdot f_2 \otimes f_3 \mapsto 0, \\
-\nu_1 \cdot g \otimes f_2 \otimes f_3 + \nu_1 \otimes g \cdot f_2 \otimes f_3 + \nu_1 \otimes dg \cdot f_2 \otimes f_3 \mapsto 0.
\end{align}

\footnote{Notice, that the cyclic group action descends to #(B \otimes_A B \otimes_A B) by (5.41). The functional $\Xi'$ is cyclically invariant as well.
All the checks are straightforward corollaries of the Leibniz rule, of the fact that \( \omega \) descends to a functional \( \omega' : \#(\Omega^1 \otimes_A \Omega_A) \to \mathbb{C} \) (see (5.47)), and of the relation (3.9). Here is the most non-trivial check (5.56).

\[
\begin{align*}
- f_1 \cdot g & \otimes f_2 \otimes f_3 + f_1 \cdot g \cdot f_2 \otimes f_3 + f_1 \otimes dg \cdot f_2 \otimes f_3 + f_1 \otimes d^i g \cdot f_2 \otimes f_3 \\
& \quad \quad + \frac{1}{3} \left( \omega(f_1 \cdot dg, d^i f_2 \cdot f_3) + \omega(df_3, f_1 \cdot d^i f_2) - \omega(df_3, f_1 \cdot f_3) - \omega(df_1, d^i g \cdot f_2 \cdot f_3) - \\
& \quad \quad - \omega(f_1 \cdot dg, f_2, d^i f_3) - \omega(df_2, f_3, f_1 \cdot d^i g) + \omega(df_2, f_3, d^i f_1) + \omega(df_3, f_1, d^i g \cdot f_2) \right) + \\
& \quad \quad + \omega(df_1, f_2, d^i f_3) - \omega(df_1, f_1, d^i g) = \\
& \quad \quad = \frac{1}{3} \left( \omega(dg, d^i (f_2 f_3 f_4)) \right) + \omega(d(f_2 f_3 f_4), d^i g) \\
& \quad \quad + \omega(df_2, f_3, f_1, d^i g) + \omega(df_3, f_1, d^i g) + \omega(f_2 f_3, f_1, d^i g) = 0.
\end{align*}
\]

The rest is a routine calculation. \( \blacksquare \)

5.3.4. **Formula for the moment map in terms of \( \Xi' \).** Our goal (in the setup \( \text{(A1)-(A5)} \)) is to associate with any Hermitian module \((E, \mathcal{H})\) endowed with a connection \( \nabla \), a \( \mathbb{R} \)-linear functional on the Lie algebra \( \mathfrak{t} \) defined as in (5.27). In other words, we want to define a number

\[
H_u(\mathcal{H}, \nabla) \in \mathbb{R}
\]

depending \( \mathbb{R} \)-linearly on \( u \in \mathfrak{t} \), extending the formulas (5.8),(5.25).

We can form the following a chain of morphisms of \( \mathcal{A} \)-modules:

\[
\begin{align*}
E & \xrightarrow{i_E} E \xrightarrow{\nabla} \mathcal{B} \mathcal{A} E \xrightarrow{id_{\mathcal{B} \mathcal{A} E}} \mathcal{B} \mathcal{A} E \\
& \quad \quad \quad \xrightarrow{\mathrm{id}_{\mathcal{B} \mathcal{A} E}} \mathcal{B} \mathcal{A} E
\end{align*}
\]

where \( \nabla : E \to \mathcal{B} \mathcal{A} E \) is the morphism of \( \mathcal{A} \)-modules associated with the connection \( \nabla \otimes \nabla^1 : E \to (\Omega^1 \otimes \Omega_A) \otimes_A E \).

The composition in (5.65) is a morphism of \( \mathcal{A} \)-modules

\[
C_3 : E \to \mathcal{B} \mathcal{A} E.
\]

Applying the trace along \( E \) to the morphism \( C_3 \) we obtain an element

\[
\text{Trace}_E(C_3) \in \#(\mathcal{B} \mathcal{A} \mathcal{B} \mathcal{A} \mathcal{B} \mathcal{A} E).
\]

**Definition 5.2.** The moment map (see (5.64)) is given by

\[
H_u(\mathcal{H}, \nabla) := \frac{\sqrt{-1}}{2} \cdot \Xi'(\text{Trace}_E(C_3))
\]

where \( \Xi' \) is well-defined by Proposition 3. The equation on the Hermitian form

\[
H_u(\mathcal{H}, \nabla) = 0 \quad \forall u \in \mathfrak{t}
\]

we call the **universal moment map equation.**

**Proposition 4.** In the case when Hermitian finitely-generated projective \( \mathcal{A} \)-module \( E \) is isomorphic to the image of a self-adjoint projector \( P \in \text{Mat}(n \times n, \mathcal{A}) \) for some \( n < \infty \) endowed with the induced Hermitian \( \mathcal{A} \)-valued pairing, the definition of the moment map via (5.69) and as in (5.25) agree.
Proof: In order to alleviate the notations we will perform the check in the simplest case when $E$ is the free module of rank 1 endowed with the standard Hermitian form. Hence, $u$ is a $(1 \times 1)$-matrix, which is just an element of $\mathcal{A}$ satisfying $u^* = -u$. Similarly, the connection is an element $\alpha \in \Omega^1$.

The morphism $\nabla: E \to \mathbb{B} \otimes \mathcal{A}$ is given (on the base element $1 \in E = \mathcal{A}$) by
\begin{equation}
\nabla: 1 \mapsto (1, \alpha, \overline{\alpha}) \otimes 1.
\end{equation}

The chain (5.65) applied to the element $1 \in \mathcal{A} = E$ is given by
\begin{equation}
1 \mapsto u \mapsto (u, u \cdot \alpha, u \cdot \overline{\alpha}) \otimes (1, \alpha, \overline{\alpha}) \mapsto (u, u \cdot \alpha, u \cdot \overline{\alpha}) \otimes (1, \alpha, \overline{\alpha}) \otimes 1.
\end{equation}

Hence, we have to calculate
\begin{equation}
\Xi(\Xi' \text{Trace}_E(C_3)) = -2\sqrt{-1} \eta(u) + \omega(\alpha, d^1u) - \omega(du, \overline{\alpha}) + \omega(\alpha \cdot u, \overline{\alpha}) - \omega(u \cdot \alpha, \overline{\alpha}).
\end{equation}

Using the fact $d^1 = d 1 = 0$.

6. Examples

6.1. Quiver type. The case of a quiver was essentially described above. The algebra $\mathcal{A}$ is $\mathbb{C}Q_0$, the bimodule $\Omega^1$ is $\mathbb{C}Q_1$, the derivation $d$ is 0. The choice of functional $\eta$ corresponds to the choice of a real cyclic 1-cocycle of $\mathcal{A}$. The resulting moment map equation is thus the general King’s equation.

As particular examples relevant for gauge theory we would mention ADHM equations (2.23), deformed ADHM equations (2.32), and the 0-dimensional reduction of HYM: $[z_1, z_2] = 0, [\overline{z}_1, z_1] + [\overline{z}_2, z_2] = 0$. 

6.2. Manifold type. For a real Riemannian or for a complex Kähler manifold \( X \) we set \( \mathcal{A} := C^\infty(X) \otimes_{\mathbb{R}} \mathbb{C} \), the bimodule \( \Omega^1 \) is either \( \Gamma(X, T_X^\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \) or \( \Gamma(X, (T^{0,1})^*) \). We get HYM equations in the complex case, and a real version in the totally real case. In the case of flat connection over a Riemannian manifold we obtain the well-known equation for the harmonic metric on a non-unitary local system.

In the mixed real/complex case one gets a generalization which coincides with Bogomolny equations when \( \dim_X = 3 \) and the complex distribution \( \mathcal{F} \) is in local coordinates \((x_1, x_2, x_3)\) generated by
\[
\mathbb{C} \cdot \partial_{x_1} + \mathbb{C} \cdot (\partial_{x_2} + i \partial_{x_3}) .
\]

6.3. Mixed manifold/quiver case.

6.3.1. Twisted quiver bundles (following [2]). Suppose that we are given a Kähler manifold \( X \) with a Kähler form \( \omega_X^{1,1} \), a finite quiver \( Q \), and a collection of holomorphic vector bundles \( M_a \) over \( X \) for each arrow \( a \in Q_1 \), endowed with Hermitian metrics \( H_a \). Then we have the following algebra \( \mathcal{A} := \mathbb{C}^{Q_0} \otimes \mathbb{C}^{Q_1}(X) \). The bimodule \( \Omega^1 \) defined as
\[
\Omega^1 := \left( \bigoplus_{v \in Q^0} \pi_v \cdot \Omega^0(X) \cdot \pi_v \right) \oplus \left( \bigoplus_{a \in Q_1} \pi_{s(a)} \cdot \Gamma(X, C^\infty_X, \mathcal{O}_X \otimes \mathcal{M}^*_a) \cdot \pi_{t(a)} \right) ,
\]
and the derivation \( d : \mathcal{A} \to \Omega^1 \) is a \( \mathcal{F} \)-operator taking values in the first summand of \( (6.2) \).

An example of a module with a connection is an \( M \)-twisted \( Q \)-bundle, which is by definition (see [2]) a collection of holomorphic vector bundles \( (\mathcal{E}_v)_{v \in Q_0} \) together with a collection of holomorphic morphisms
\[
\forall a \in Q_1 : \quad \phi_a : M_a \otimes \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)} .
\]

For such a module an \( \mathcal{A} \)-valued Hermitian form is a collection of Hermitian metrics \( (h_v)_{v \in Q_0} \) on the individual bundles \( \mathcal{E}_v \). Let \( \rho \) and \( \sigma \) be collections of real numbers \( \rho_v \) and \( \sigma_v > 0 \). The harmonicity equation on \( (h_v)_{v \in Q_0} \) (e.g. the moment map equation) is called twisted quiver \( (\rho, \sigma) \)-vortex equation, and it is:
\[
\forall v \in Q_0 : \quad \sigma_v \sqrt{-1} \Lambda F_{H_v} + \sum_{a \in s^{-1}(v)} \phi_a \circ \phi_a^{H_a} - \sum_{a \in t^{-1}(v)} \phi_a^{H_a} \circ \phi_a = \rho_v \text{id}_{\mathcal{E}_v} ,
\]
where \( \Lambda \) is the contraction with the bivector field \( (\omega_X^{1,1})^{-1} \), and \( F_H = (F_{H_v}) \) is the curvature corresponding to the metric \( H = H_v \), \( \forall v \in Q_0 \). Here the compositions on the l.h.s. are defined as
\[
\phi_a \circ \phi_a^{H_a} : \mathcal{E}_{s(a)} \to M_a \otimes \mathcal{R}_{t(a)} \to \mathcal{E}_{s(a)} , \quad \phi_a^{H_a} \circ \phi_a : \mathcal{E}_{t(a)} \to M_a \otimes \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)} .
\]

A special case of the above vortex equation is when \( Q \) is one vertex \( v \) with one loop \( a \), and map \( M_a \otimes \mathcal{E}_v \to \mathcal{E}_v \) gives a map from \( M := M_a \) to commuting endomorphisms of \( \mathcal{E} := \mathcal{E}_v \). Such an object can be interpreted as a coherent sheaf on the total space of the dual bundle \( M^* \) with \( \dim_X \) -dimensional support which is proper and finite over \( X \), and such that the direct image to \( X \) is a vector bundle. In the case \( M = T_X \) this is equivalent to the Hitchin equation. When \( \dim_X = 2 = M = \wedge^2 T_X^\mathbb{C} \) we get Vafa-Witten equation, and when \( n = \dim_X > 2 \) and \( M = \wedge^n T_X^\mathbb{C} \) we get a generalization of Vafa-Witten equations considered by one of us (G.B.) in an unpublished manuscript. In all these examples the total space of \( M^* \) is a non-compact Calabi-Yau space in the algebroid-geometric sense, i.e. it is endowed with a non-vanishing holomorphic volume form.

Remark 6.1. For any quiver \( Q \) and a collection of bundles \( M_a \) labeled by the arrows of \( Q \) one can construct a new quiver \( Q' \) with the same set of vertices \( Q'_0 = Q_0 \) and with exactly one edge \( a'_{ij} \) for every ordered pair \((i, j)\) of vertices. The new bundles \( M'_{ij} \) can be defined as the direct sums
\[
M'_{ij} := \oplus_{a \in Q_1 : s(a) = i, t(a) = j} M_a .
\]

There is an obvious equivalence between the \( M \)-twisted \( Q \)-bundles and the \( M' \)-twisted \( Q' \)-bundles, and the corresponding harmonic metrics. Nevertheless, for bookkeeping purposes, it is more convenient to work with the original description.
6.3.2. **Nahm’s equation.** The algebra $C$ is $C^\infty(X)$ where $X$ is a 1-dimensional manifold. The bimodule is supported on the diagonal and is $\Gamma(X, T_X^* \otimes \mathbb{R}) \oplus C^\infty(X)$, looks like the tensor product of 1-forms on $X$ and the quiver with one vertex and one loop. The equation for harmonic representatives is exactly Nahm equation for the group $U(k)$: $\dot{A}_i = \epsilon_{ijk}[A_j, A_k]$ where $A_i = -A_i^\dagger \in Mat(k \times k, \mathbb{C})$ are functions of time.

6.4. **Noncommutative instantons.** Ignoring the problem related to the noncompactness of the noncommutative space $\mathbb{R}^2_n$, the corresponding framework is the following. The algebra $A$ is certain $C^\infty$-version of the algebra generated by generators $z_1, \ldots, z_n$ and their Hermitian conjugates $z_1^*, \ldots, z_n^*$ satisfying relations\footnote{One can further generalize these relations and get holomorphic noncommutative spaces, via replacing (6.7) by $[z_i, z_j] = c_{ij}$ and $[z_i^*, z_j^*] = -\Sigma_{ij}$ where $(c_{ij})_{1 \leq i, j \leq n}$ is any skew-symmetric complex $n \times n$ matrix.} (6.7)

$$[z_i, z_j] = 0, \quad [z_i^*, z_j^*] = 0,$$

(6.8)

$$[z_i^*, z_j] = \hbar \delta_{ij}.$$

The algebra $A$ is endowed with commuting derivations $\partial_1, \ldots, \partial_n$ and $\overline{\partial}_1, \ldots, \overline{\partial}_n$ given by (6.9)

$$\partial_i(z_j) = \overline{\partial}_i(z_j^*) = \delta_{ij},$$

(6.10)

$$\overline{\partial}_i(z_j^*) = \partial_i(z_j) = 0.$$

A noncommutative HYM instanton is a finitely-generated projective $A$-module $E$ endowed with a $A$-valued Hermitian form (see (3.11))

$$H : E \otimes \mathbb{C} E \to A, \quad H(f \phi_1, g \phi_2) = f \cdot H(\phi_1, \overline{\phi}_2) \cdot g^*$$

endowed with $\mathbb{C}$-linear endomorphisms $\nabla_1, \ldots, \nabla_n$ and $\overline{\nabla}_1, \ldots, \overline{\nabla}_n$ satisfying relations (6.12)

$$[\nabla_i, \nabla_j] = [\nabla_i, \nabla_j^*] = 0,$$

(6.13)

$$[\nabla_i, z_j] = [\nabla_i, z_j^*] = \delta_{ij},$$

(6.14)

$$[\overline{\nabla}_i, \overline{\nabla}_j] = [\overline{\nabla}_i, \overline{\nabla}_j^*] = 0,$$

(6.15)

$$\sum_{i=1}^n [\overline{\nabla}_i, \nabla_i] = 0,$$

and (6.16)

$$H(\overline{\nabla}_i(\phi_1), \overline{\phi}_2) + H(\phi_1, \overline{\nabla}_i(\overline{\phi}_2)) = \overline{\partial}_i(H(\phi_1, \overline{\phi}_2)).$$

6.5. **From noncommutative HYM to infinite-dimensional King’s equation.** The algebra $A$ has a positive functional (state) $\int_\rho : A \to \mathbb{C}$ (depending on arbitrary constant $\rho > 0$) satisfying

$$(6.17) \quad \int_\rho aa^* \geq 0 \quad \forall a \in A$$

and given by

$$\int_\rho \prod_i z_i^{-k_i} \prod_i (z_i^*)^{l_i} = \prod_{i=1}^n \delta_{k_i, l_i} \rho^{k_i l_i}.$$

One can check using (6.18) that one has $\forall a \in A$, $\forall i \in \{1, \ldots, n\}$:

$$\int_\rho \overline{\nabla}_i(a) = \frac{1}{\rho + \hbar} \int_\rho a \cdot z_i,$$

$$\int_\rho z_i \cdot a = \frac{\rho}{\rho + \hbar} \int_\rho a \cdot z_i.$$
Let us introduce a non-negative $\mathbb{C}$-valued pre-Hermitian pairing on $E$ by

$$\langle \phi_1, \phi_2 \rangle := \int_{\rho} H(\phi_1, \overline{\phi_2}).$$

We conclude from (6.16) and (6.19) that

$$\langle \nabla_i \phi_1, \phi_2 \rangle = \int_{\rho} H(\nabla_i(\phi_1), \overline{\phi_2}) = \int_{\rho} \nabla_i(H(\phi_1, \overline{\phi_2})) - \int_{\rho} H(\phi_1, \nabla_i(\overline{\phi_2})) = \frac{1}{\rho + \hbar} \int_{\rho} H(\phi_1, \overline{\phi_2}) \cdot z_i - \langle \phi_1, \nabla_i(\phi_2) \rangle = \langle \phi_1, \frac{1}{\rho + \hbar} z_i^* \cdot \phi_2 - \nabla_i(\phi_2) \rangle.$$

Also, it follows from (6.20) that

$$\langle z_i \cdot \phi_1, \phi_2 \rangle = \int_{\rho} H(z_i \cdot \phi_1, \phi_2) = \int_{\rho} z_i \cdot H(\phi_1, \phi_2) = \frac{\rho}{\rho + \hbar} \int_{\rho} H(\phi_1, \phi_2) \cdot z_i = \langle \phi_1, \frac{\rho}{\rho + \hbar} z_i^* \cdot \phi_2 \rangle.$$

Let us introduce operators in the Hilbert space $\mathcal{H}$ which is the completion of $E$ with respect to $\langle \cdot, \cdot \rangle$:

$$Z_i = z_i - \rho \nabla_i.$$

The equations (6.22) and (6.23) imply that

$$Z_i^\dagger = \rho \nabla_i.$$

Finally, using (6.13) and (6.15) we conclude that

$$\sum_{i=1}^{n} [Z_i^\dagger, Z_i] = \rho \cdot n \cdot \text{id}_\mathcal{H}.$$

Consider the subspace $\mathcal{H}_0 \subset \mathcal{H}$ which the common kernel of operators $\nabla_i$, $i = 1, \ldots, n$. This subspace is preserved by the operators $z_i$, hence it is preserved by the operators $Z_i$. We claim (the argument in not totally rigorous) that $\mathcal{H}_0 \subset \mathcal{H}$ is also preserved by the adjoint operators $Z_i^\dagger$. Indeed, it is the case when $E$ is the trivial bundle of rank one (in this case $\mathcal{H}_0$ is a completion of $\mathbb{C}[z_1, \ldots, z_n]$). In general, let us consider the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad \mathcal{H}_1 := \mathcal{H}_0^\perp.$$

In this splitting we have for any $i = 1, \ldots, n$:

$$Z_i = \begin{pmatrix} Z_{i00}^0 & Z_{i01}^0 \\ 0 & Z_{i11}^1 \end{pmatrix}, \quad Z_i^\dagger = \begin{pmatrix} (Z_{i00}^0)^\dagger & 0 \\ (Z_{i01}^1)^\dagger & (Z_{i11}^1)^\dagger \end{pmatrix}.$$

We conclude that

$$\sum_i [Z_i^00]^\dagger, Z_i^{00}] + \sum_i (Z_i^{01})^\dagger Z_i^{01} = \rho \cdot n \cdot \text{id}_{\mathcal{H}_0}.$$

For each $i$ the operator $[Z_i^{00}, Z_i^{00}] - \rho \cdot \text{id}_{\mathcal{H}_0}$ is of trace class, hence its trace is equal to zero (reasoning: the trace does not change by small deformations). Together with (6.29) this vanishing of traces implies that

$$\sum_i \text{Trace}((Z_i^{01})^\dagger Z_i^{01}) = 0,$$

and therefore all operators $Z_i^{01}$ vanish. Hence, the equation (6.26) holds on $\mathcal{H}_0$ as well. This concludes the argument.
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