Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group of type $E_7$

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Abstract

Let $G$ be a connected reductive algebraic group defined over an algebraically closed field $k$. Following Serre, a closed subgroup $H$ of $G$ is called $G$-completely reducible if whenever $H$ is contained in a parabolic subgroup $P$ of $G$, $H$ is contained in some Levi subgroup $L$ of $P$. The aim of this paper is to present a method to find triples $(G, M, H)$ with the following three properties. Property 1: $G$ is a simple algebraic group defined over $k$ of characteristic 2. Property 2: $H$ and $M$ are closed reductive subgroups of $G$ such that $H < M < G$, and $(G, M)$ is a reductive pair. Property 3: $H$ is $G$-completely reducible, but not $M$-completely reducible. We exhibit our method by presenting a new example of such a triple in $G = E_7$. Then we consider a rationality problem and a problem concerning conjugacy classes as important applications of our construction.

Keywords: algebraic groups, separable subgroups, complete reducibility

1 Introduction

Let $G$ be a connected reductive algebraic group defined over an algebraically closed field $k$ of characteristic $p$. In [Ser98, Lec. 1], J.P. Serre defined that a closed subgroup $H$ of $G$ is $G$-completely reducible ($G$-cr for short) if whenever $H$ is contained in a parabolic subgroup $P$ of $G$, $H$ is contained in a Levi subgroup $L$ of $P$. This is a faithful generalization of the notion of semisimplicity in representation theory since if $G = GL_n(k)$, a subgroup $H$ of $G$ is $G$-cr if and only if $H$ acts completely reducibly on $k^n$ [Ser98, Lec. 1]. It is known that if a closed subgroup $H$ of $G$ is $G$-cr, then $H$ is reductive [Ser98 Property 4]. Moreover, if $p = 0$, the converse holds [BMR05 Lem. 2.6]. Therefore the notion of $G$-complete reducibility is not interesting if $p = 0$. In this paper, we assume that $p > 0$.

Completely reducible subgroups of connected reductive algebraic groups have been much studied. See [Ser98, Ser], [LS03, LT04, Ser97, Ser97]. In investigations of the subgroup structure of connected reductive algebraic groups, a study of completely reducible subgroups was the core of research, see [Dyn57] and [Dyn00] for $G$ classical, and see [LS96] and [LT99] for $G$ exceptional. Recently, studies of complete reducibility via Geometric Invariant Theory (GIT for short) have been fruitful. See [BMR10, BMR05, BMRT, BMR10, LMS09, Mar03e, Mar03a]. In this paper, we see another application of GIT to study complete reducibility (Proposition 5.5).

Here is the main problem we consider. Let $H$ and $M$ be closed reductive subgroups of $G$ such that $H \leq M \leq G$. It is natural to ask whether $H$ being $M$-cr implies that $H$ is $G$-cr and
vice versa. It is not difficult to find a counterexample for the forward direction. For example, take $H = M = \text{PGL}_2(k)$ and $G = \text{SL}_3(k)$ where $p = 2$ and $H$ sits inside $G$ via the adjoint representation. Another such example is [BMR05, Ex. 3.45]. However, it is hard to get a counterexample for the reverse direction, and it necessarily involves a small $p$. In [BMRT10, Sec. 7], Bate et al. presented the only known counterexample for the reverse direction where $p = 2$, $H \cong S_3$, $M \cong A_1 \times A_1$, and $G = G_2$, which we call “the $G_2$ example”. The aim of this paper is to prove the following.

**Theorem 1.1.** Let $G$ be a simple algebraic group of type $E_7$ defined over $k$ of characteristic $p = 2$. Then there exists a connected reductive subgroup $M$ of type $A_7$ of $G$ and a reductive subgroup $H \cong D_{14}$ (the dihedral group of order 14) of $M$ such that $(G, M)$ is a reductive pair and $H$ is $G$-cr but not $M$-cr.

We call our example “the $E_7$ example”. Our $E_7$ example is different from the $G_2$ example in the following sense. In the $E_7$ example, the $G$-cr and non-$M$-cr subgroup $H$ sits in a rank 6 Levi subgroup, as opposed to a rank 1 Levi subgroup in the $G_2$ example. In Section 4, we show that it is natural to look at a subgroup $H$ of $G$ sitting in a Levi subgroup of rank greater than 1 to find a new such triple.

With our method, in this paper we find the $E_7$ example, and we rediscover the $G_2$ example. Moreover, with the same method we have found four more new such triples, one in $E_7$ and three in $E_6$. Currently, we are trying to find all such triples using the same method as in this paper where $H$ is generated by some reflections corresponding to a subgroup of the Weyl group of $G$ with the assistance of the computer algebra system Magma. The result will appear in the forthcoming paper [Ch].

Our discussion is motivated by [BMRT10]. We recall a few relevant definitions and results here. We denote the Lie algebra of $G$ by $\text{Lie} G = g$. From now on, by a subgroup of $G$, we always mean a closed subgroup of $G$.

**Definition 1.2.** Let $H$ be a subgroup of $G$ acting on $G$ by inner automorphisms. Let $H$ act on $g$ by the corresponding adjoint action. Then $H$ is called separable if $\text{Lie} C_G(H) = c_g(H)$.

Recall that we always have $\text{Lie} C_G(H) \subseteq c_g(H)$. In [BMRT10], Bate et al. investigated the relationship between $G$-complete reducibility and separability, and showed the following [BMRT10, Thm. 1.2 and Thm. 1.4].

**Proposition 1.3.** Suppose that $p$ is very good for $G$. Then any subgroup of $G$ is separable in $G$.

**Proposition 1.4.** Suppose that $(G, M)$ is a reductive pair. Let $H$ be a subgroup of $M$ such that $H$ is a separable subgroup of $G$. If $H$ is $G$-cr, then it is also $M$-cr.

Recall that a pair of reductive groups $G$ and $M$ is called a reductive pair if $\text{Lie} M$ is an $M$-module direct summand of $g$. This is automatically satisfied if $p = 0$. Propositions [1.3] and [1.4] tell that the subgroup $H$ in Theorem 1.1 must be non-separable, which is possible for small $p$ only.

Now, we explain our method. First, we introduce a notion of separable action, which is a slight generalization of the notion of a separable subgroup. This notion of separable action is essential to our method and it enables us to explain $G_2$ example more clearly.

**Definition 1.5.** Let $H$ and $N$ be subgroups of $G$ where $H$ acts on $N$ by group automorphisms. The action of $H$ is called separable in $N$ if $\text{Lie} C_N(H) = c_{\text{Lie} N}(H)$. Note that the condition means that the fixed points of $H$ acting on $N$, taken with their natural scheme structure, are smooth.
Here is a brief sketch of our method. **Note that we need** \( p \) **to be** 2.

1. Pick a parabolic subgroup \( P \) of \( G \) with a Levi subgroup \( L \) of \( P \). Find a subgroup \( K \) of \( L \) such that \( K \) acts non-separably on the unipotent radical \( R_u(P) \) of \( P \).

2. Conjugate \( K \) by a suitable element \( v \) of \( G \), and set \( H = vKv^{-1} \). Then there is a natural way to choose a subgroup \( M \) of \( G \) (Remark 3.6, Remark 5.4). Show that \( H \) is not \( M \)-cr using a recent result from GIT (Proposition 2.4). Note that \( K \) is \( M \)-cr in our case.

3. Prove that \( H \) is \( G \)-cr.

**Remark 1.6.** It can be shown using [Spr98 Thm. 13.4.2] that such a \( K \) in Step 1 is a non-separable subgroup of \( G \).

First of all, for Step 1, \( p \) cannot be very good for \( G \) by Proposition 1.3 and 1.4. It is known that 2 and 3 are bad for \( E_7 \) and \( G_2 \). In the following sections, we explain the reason why we choose \( p = 2 \), not \( p = 3 \). In the \( G_2 \) example, Bate et al. [BMRT10 Sec. 7] assumed \( p = 2 \) and followed Step 1, but did not explain the importance of the non-separable action of \( K \) on \( R_u(P) \).

We go through the \( G_2 \) example in Section 3 to explain our method in a simpler example than the \( E_7 \) example, and we explain why it works. In Section 5, we look at the \( E_7 \) example where \( K \) in the \( E_7 \) is generated by elements corresponding to certain reflections in the Weyl group of \( G \). Because of this particular form of \( K \), we are able to turn a problem of non-separability into a purely combinatorial problem involving the root system of \( G \).

As for Step 2, we explain the reason of our choice of \( v \) and \( M \) explicitly, which was not done in [BMRT10 Sec. 7]. Our use of Proposition 2.4, which was not used in [BMRT10 Sec. 7], gives an alternative way to prove that \( H \) is not \( M \)-cr in the \( G_2 \) example, and simplifies the calculation considerably in the \( E_7 \) example. It also gives a conceptual understanding of the relationship between a non-separable action and complete reducibility.

Finally, Step 3 is easy in both the \( G_2 \) and the \( E_7 \) example.

Our \( E_7 \) example is not only interesting in its own right, but also has many important consequences and applications. For example, in Section 6, we consider a rationality problem as an application of the \( E_7 \) example. We need a definition first to explain our result there.

**Definition 1.7.** Let \( k_0 \) be a subfield of an algebraically closed field \( k \). Let \( H_0 \) be a \( k_0 \)-defined closed subgroup of a \( k_0 \)-defined reductive algebraic group \( G_0 \). Then \( H_0 \) is called \( G_0 \)-cr over \( k_0 \) if whenever \( H_0 \) is contained in a \( k_0 \)-defined parabolic subgroup \( P_0 \) of \( G_0 \), it is contained in some \( k_0 \)-defined Levi subgroup of \( P_0 \).

Note that if \( k \) is algebraically closed then \( G \)-cr over \( k \) means \( G \)-cr in the usual sense. Here is the main result of Section 6.

**Theorem 1.8.** Let \( k_0 \) be a nonperfect field of characteristic \( p = 2 \), and let \( G_0 \) be a split simple algebraic group defined over \( k_0 \) of type \( E_7 \). Let \( k \) be the algebraic closure of \( k_0 \). Then there exists a \( k_0 \)-defined subgroup \( H_0 \) of \( G_0 \) such that \( H_0 \) is \( G_0 \)-cr over \( k \), but not \( G \)-cr over \( k \).

As another important application of the \( E_7 \) example, we consider a problem concerning conjugacy classes. Given \( n \in \mathbb{N} \), we let \( G \) act on \( G^n \) by simultaneous conjugation:

\[
g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).
\]

In [Slo97], Slodowy proved the following fundamental result applying Richardson’s tangent space argument, [Ric67 Sec. 3], [Ric82 Lem. 3.1].
Proposition 1.9. Let $M$ be a reductive subgroup of a reductive group $G$. Let $n \in \mathbb{N}$, let $(m_1, \ldots, m_n) \in M^n$ and let $H$ be the subgroup of $M$ generated by $m_1, \ldots, m_n$. Suppose that $(G, M)$ is a reductive pair and that $H$ is separable in $G$. Then the intersection $G \cdot (m_1, \ldots, m_n) \cap M^n$ is a finite union of $M$-conjugacy classes.

Proposition 1.9 has many consequences. See [BMR05], [Slo97], and [Vin96, Sec. 3] for example. In [BMR T10, Ex. 7.15], Bate et al. found a counterexample in $G = G_2$ showing that Proposition 1.9 fails without the separability hypothesis. In Section 7, we present a new counterexample to Proposition 1.9. Here is the main result of Section 7.

Theorem 1.10. Let $G$ be a simple algebraic group of type $E_7$ defined over $k$ of characteristic $p = 2$. Let $M$ be the connected reductive subsystem subgroup of type $A_7$. Then there exists $n \in \mathbb{N}$ and a tuple $m \in M^n$ such that $G \cdot m \cap M^n$ is an infinite union of $M$-conjugacy classes. Note that $(G, M)$ is a reductive pair in this case.

Now, we give the outline of our paper. In Section 2, we fix our notation which basically follows [Bor91], [Hum91], and [Spr98]. Also, we recall some preliminary results, in particular Proposition 2.4 from GIT. After that, we review the $G_2$ example in Section 3 to illustrate our method. In Section 4, we prove Theorem 4.1 which shows that the $G_2$ example is the only example where the same form of $K$ in a rank 1 Levi subgroup of $G$ acts non-separably on $R_u(P)$, which is necessary for our Step 1 to go through. Since the choice of $K$ in the $G_2$ example is “canonical” in the sense which we explain in the first paragraph of Section 4, we are naturally led to look at $K$ sitting in a higher rank Levi subgroup. Then, in Section 5, we prove our main result, Theorem 5.1. Section 5 is the heart of this paper. Then, in Section 6 and Section 7, we consider important applications of our $E_7$ example. In Section 6, we consider a rationality problem, and prove Theorem 1.8. Finally, in Section 7, we discuss a problem concerning conjugacy classes, and prove Theorem 1.10.

2 Preliminaries

2.1 Notation

Throughout the paper, we denote by $k$ an algebraically closed field of positive characteristic $p$. We denote the multiplicative group of $k$ by $k^*$. We use a capital roman letter, $G$, $H$, $K$, etc., to represent an algebraic group, and the corresponding lowercase gothic letter, $g$, $h$, $k$, etc., to represent its Lie algebra. We sometimes use another notation for Lie algebras: $\text{Lie } G$, $\text{Lie } H$, and $\text{Lie } K$ are the Lie algebras of $G$, $H$, and $K$ respectively.

We denote the identity component of $G$ by $G^\circ$. We write $[G, G]$ for the derived group of $G$. The unipotent radical of $G$ is denoted by $R_u(G)$. An algebraic group $G$ is reductive if $R_u(G) = \{1\}$. In particular, $G$ is simple as an algebraic group if $G$ is connected and all proper normal subgroups of $G$ are finite.

In this paper, when a subgroup $H$ of $G$ acts on $G$, $H$ always acts on $G$ by inner automorphisms. The adjoint representation of $G$ is denoted by $\text{Ad}_g$ or just $\text{Ad}$ if no confusion arises. We write $C_G(H)$ and $c_H$ for the global and the infinitesimal centralizers of $H$ in $G$ and $g$ respectively.

We write $X(G)$ and $Y(G)$ for the set of characters and cocharacters of $G$ respectively.
2.2 Complete reducibility and GIT

Let $G$ be a connected reductive algebraic group. We recall Richardson’s formalism [Ric88 Sec. 2.1-2.3] for the characterization of a parabolic subgroup $P$ of $G$, a Levi subgroup $L$ of $P$, and the unipotent radical $R_u(P)$ of $P$ in terms of a cocharacter of $G$ and state a result from GIT (Proposition 2.4).

**Definition 2.1.** Let $X$ be an affine variety. Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say that $\lim_{a \to 0} \phi(a)$ exists if there exists a morphism $\hat{\phi} : k \to X$ (necessarily unique) whose restriction to $k^*$ is $\phi$. If this limit exists, we set $\lim_{a \to 0} \phi(a) = \hat{\phi}(0)$.

**Definition 2.2.** Let $\lambda$ be a cocharacter of $G$. Define

$$P_\lambda := \{ g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists} \},$$

$$L_\lambda := \{ g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} = g \},$$

$$R_u(P_\lambda) := \{ g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} = 1 \}.$$

Then, $P_\lambda$ is a parabolic subgroup of $G$, $L_\lambda$ is a Levi subgroup of $P_\lambda$, and $R_u(P_\lambda)$ is a unipotent radical of $P_\lambda$ [Ric88 Sec. 2.1-2.3]. By [Spr98 Prop. 8.4.5], any parabolic subgroup $P$ of $G$, any Levi subgroup $L$ of $P$, and any unipotent radical $R_u(P)$ of $P$ can be expressed in this form. It is well known that $L_\lambda = C_G(\lambda(k^*))$.

Let $M$ be a reductive subgroup of $G$. Then, there is a natural inclusion $Y(M) \subseteq Y(G)$ of cocharacter groups. Let $\lambda \in Y(M)$. We write $P_\lambda(G)$ or just $P_\lambda$ for the parabolic subgroup of $G$ corresponding to $\lambda$, and $P_\lambda(M)$ for the parabolic subgroup of $M$ corresponding to $\lambda$. It is obvious that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

**Definition 2.3.** Let $\lambda \in Y(G)$. Define a map $c_\lambda : P_\lambda \to L_\lambda$ by

$$c_\lambda(g) := \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1}.$$

Note that the map $c_\lambda$ is the usual canonical projection from $P_\lambda$ to $L_\lambda \cong P_\lambda/R_u(P_\lambda)$.

We state a result from GIT (see [BMR05 Lem. 2.17 and Thm. 3.1] and [BMRT Thm. 3.3]).

**Proposition 2.4.** Let $H$ be a subgroup of $G$. Let $\lambda$ be a cocharacter of $G$ with $H \subseteq P_\lambda$. If $H$ is $G$-cr, there exists $v \in R_u(P_\lambda)$ such that $c_\lambda(h) = vhv^{-1}$ for every $h \in H$.

2.3 Root subgroups and root subspaces

Let $G$ be a connected reductive algebraic group. Fix a maximal torus $T$ of $G$. Let $\Psi(G, T)$ denote the set of roots of $G$ with respect to $T$. We sometimes write $\Psi(G)$ for $\Psi(G, T)$. Fix a Borel subgroup $B$ containing $T$. Then $\Psi(B, T) = \Psi^+(G)$ is the set of positive roots of $G$ defined by $B$. Let $\Sigma(G, B) = \Sigma$ denote the set of simple roots of $G$ defined by $B$. Let $\zeta \in \Psi(G)$. We write $U_\zeta$ for the corresponding root subgroup of $G$ and $u_\zeta$ for the Lie algebra of $U_\zeta$. We define $G_\zeta := \langle U_\zeta, U_{-\zeta} \rangle$.

Let $H$ be a subgroup of $G$ normalized by some maximal torus $T$ of $G$. Consider the adjoint representation of $T$ on $\mathfrak{h}$. The root spaces of $\mathfrak{h}$ with respect to $T$ are also root spaces of $\mathfrak{g}$ with respect to $T$, and the set of roots of $H$ relative to $T$, $\Psi(H, T) = \Psi(H) = \{ \zeta \in \Psi(G) \mid g_\zeta \subseteq \mathfrak{h} \}$, is a subset of $\Psi(G)$. 

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Let $\zeta, \xi \in \Psi(G)$. Let $\xi^\vee$ be the coroot corresponding to $\xi$. Then $\zeta \circ \xi^\vee : k^* \to k^*$ is a homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. We define $\langle \zeta, \xi^\vee \rangle := n$. Let $s_\xi$ denote the reflection corresponding to $\xi$ in the Weyl group of $G$. Each $s_\xi$ acts on the set of roots $\Psi(G)$ by the following formula [Spr98, Lem. 7.1.8].

$$s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee \rangle_\xi.$$  

By [Car72, Prop. 6.4.2 and Lem. 7.2.1], we can choose homomorphisms $\epsilon_\zeta : k \to U_\zeta$ so that

$$n_\xi \epsilon_\xi(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \xi}(\pm a),$$

where $n_\xi = \epsilon_\xi(1) \epsilon_{-\xi}(-1) \epsilon_\xi(1)$.

We define $e_\xi := \epsilon_\xi'(0)$. Then we have

$$\text{Ad}(n_\xi)e_\xi = \pm e_{s_\xi \cdot \xi}.$$  

Now, we list four lemmas which we need in our calculations in the following sections. The first one is elementary [Spr98, Prop. 8.2.1].

**Lemma 2.5.** Let $P$ be a parabolic subgroup of $G$. Any element $u$ in $R_u(P)$ can be expressed uniquely as

$$u = \prod_{i \in \Psi(R_u(P))} \epsilon_i(a_i),$$

for some $a_i \in k$, where the product is taken with respect to a fixed ordering of $\Psi(R_u(P))$.

The next two lemmas [Hum91, Lem. 32.5 and Lem. 33.3] are important in our calculation of $C_{R_u(P)}(K)$.

**Lemma 2.6.** Let $\xi, \zeta \in \Psi(G)$. If no positive integral linear combination of $\xi$ and $\zeta$ is a root of $G$, then

$$\epsilon_\xi(a) \epsilon_\zeta(b) = \epsilon_\zeta(b) \epsilon_\xi(a).$$

**Lemma 2.7.** Let $\Psi$ be the root system of type $A_2$ spanned by roots $\xi$ and $\zeta$. Then

$$\epsilon_\xi(a) \epsilon_\zeta(b) = \epsilon_\zeta(b) \epsilon_\xi(a) \epsilon_{\xi+\zeta}(\pm ab).$$

The last result is useful when we calculate $c_{\text{Lie}(R_u(P))}(K)$.

**Lemma 2.8.** Suppose that $p = 2$. Let $W$ be a subgroup of $G$ generated by all the $n_\xi$ where $\xi \in \Psi(G)$. Let $K$ be a subgroup of $W$. Let $\{O_i \mid i = 1 \cdots m\}$ be the set of orbits of the action of $K$ on $\Psi(R_u(P))$. Then,

$$c_{\text{Lie}(R_u(P))}(K) = \left\{ \sum_{i=1}^m a_i \sum_{\zeta \in O_i} e_\zeta \mid a_i \in k \right\}.$$  

**Proof.** When $p = 2$, (2.2) yields

$$\text{Ad}(n_\xi)e_\zeta = \pm e_{s_\xi \cdot \zeta}.$$  

Then an easy calculation gives the desired result.

**Remark 2.9.** $p = 2$ is essential to get Lemma 2.8. In particular, if $p = 3$, Lemma 2.8 fails.
3 The $G_2$ example

Assumption 3.1. For the rest of the paper, we assume $p = 2$.

We recall the $G_2$ example [BMRT10 Sec. 7] to exhibit our method to find a triple $(G, M, H)$ with the desired property. Our approach to the $G_2$ example is different from [BMRT10 Sec. 7] in the following sense.

1. We show the importance of a non-separable action of $K$ on $R_u(P)$.
2. We explain how to choose $v$ to get $H = vKv^{-1}$.
3. We use Proposition 2.4; see Remark 3.8.

3.1 Step 1

Let $G$ be a simple algebraic group of type $G_2$ defined over $k$ of characteristic 2. Fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ of $G$ containing $T$. Then we have $\Psi^+(G) = \{\alpha, \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$ where $\alpha$ is short and $\beta$ is long [Hum91 Sec. 33.5]. We call the roots whose coefficient of $\beta$ is 2 weight-2 roots, and the roots whose coefficient of $\beta$ is 1 weight-1 roots respectively. Define

$$L_\alpha := \langle T, G_\alpha \rangle, \quad P_\alpha := \langle L_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$ 

Then $P_\alpha$ is a parabolic subgroup of $G$, and $L_\alpha$ is a Levi subgroup of $P_\alpha$. We have

$$\Psi(R_u(P_\alpha)) = \{\beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$ 

Pick $\alpha^\vee(t) \in \alpha^\vee(k^*)$ of order 3, and define

$$K := \langle n_\alpha, \alpha^\vee(t) \rangle.$$ 

From [BMRT10 Sec. 7], we obtain the orbits of $\langle n_\alpha \rangle$ on $\Psi(R_u(P_\alpha))$.

$$O_1 = \{\beta, 3\alpha + \beta\}, \quad O_2 = \{\alpha + \beta, 2\alpha + \beta\}, \quad O_3 = \{3\alpha + 2\beta\}. \quad (3.1)$$

By [BMRT10 Sec. 7]

$$\alpha^\vee(t) \text{ acts trivially on } U_\beta, U_{3\alpha+\beta}, U_{3\alpha+2\beta}, U_\beta, U_{3\alpha+\beta}, U_{3\alpha+2\beta}.$$ 

$$\alpha^\vee(t) \text{ acts non-trivially on } U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{\alpha+\beta}, U_{2\alpha+\beta}. \quad (3.2)$$

Since $K$ is generated by $n_\alpha$ and $\alpha^\vee(t)$, a slight variant of Proposition 2.2 with (3.1) and (3.2) yields

**Proposition 3.2.** $\mathfrak{c}_{\text{Lie}(R_u(P_\alpha))}(K) = \{a(e_\beta + e_{3\alpha+\beta}) + b(e_{3\alpha+2\beta}) \mid a, b \in k\}.$

The next result is crucial for our argument.

**Proposition 3.3.** $C_{R_u(P_\alpha)}(K) = U_{3\alpha+2\beta}.$
Proof. Let \( u \in C_{R_\alpha(P_\alpha)}(K) \). By Lemma 2.5, \( u \) can be expressed uniquely as

\[
u = \epsilon_\beta(a_\beta) \epsilon_{3\alpha+\beta}(a_{3\alpha+\beta}) \epsilon_{\alpha+\beta}(a_{\alpha+\beta}) \epsilon_{2\alpha+\beta}(a_{2\alpha+\beta}) \epsilon_{3\alpha+2\beta}(a_{3\alpha+2\beta})
\]

for some \( a_\beta, a_{3\alpha+\beta}, a_{\alpha+\beta}, a_{2\alpha+\beta}, a_{3\alpha+2\beta} \in k \).

Since \( \alpha^\vee(t) \) must act trivially on \( U_{\alpha+\beta} \) and \( U_{2\alpha+\beta} \), (3.2) yields

\[a_{\alpha+\beta} = a_{2\alpha+\beta} = 0.\]

From [BMRT10, Sec. 7], we obtain

\[n_\alpha u^{-1}_\alpha = \epsilon_\beta(a_{3\alpha+\beta}) \epsilon_{3\alpha+\beta}(a_\beta) \epsilon_{3\alpha+2\beta}(a_\beta a_{3\alpha+\beta} + a_{3\alpha+2\beta}).\]

Since \( u \) is centralized by \( n_\alpha \), we have

\[a_\beta = a_{3\alpha+\beta}, a_\beta a_{3\alpha+\beta} = 0.\]

Thus we have

\[a_\beta = a_{3\alpha+\beta} = 0.\]

Proposition 3.4. \( K \) acts non-separably on \( R_\alpha(P_\alpha) \).

Proof. This follows from Proposition 3.2 and 3.3. \( \square \)

3.2 Step 2

Let

\[C := \{ \epsilon_\beta(a) \epsilon_{3\alpha+\beta}(a) \mid a \in k \}.\]

Remark 3.5. The 1-dimensional curve \( C \) was chosen so that \( T_1(C) \) is tangent to \( \epsilon_{\text{Lie}(R_\alpha(P_\alpha))}(K) \) but not tangent to \( \text{Lie} C_{R_\alpha(P_\alpha)}(K) \).

Pick any \( a \in k^* \). Let \( v(a) \in C \). We have

\[v(a)n_\alpha v^{-1}_\alpha = n_\alpha \epsilon_{3\alpha+2\beta}(a^2),\]

\[v(a)\alpha^\vee(t)v(a^{-1}) = \alpha^\vee(t).\]

Now set

\[H = v(a)Kv(a^{-1}) = \langle n_\alpha \epsilon_{3\alpha+2\beta}(a^2), \alpha^\vee(t) \rangle,\]

\[M = \langle L_\alpha, G_{3\alpha+2\beta} \rangle.\]

Remark 3.6. \( M \) was chosen so that \( M \) is generated by a Levi subgroup \( L_\alpha \) containing \( H \) and all root subgroups of even \( \beta \)-weight where \( \beta \) is the simple root not contained in \( \Psi(L_\alpha) \). It is easy to see that \( (G, M) \) is a reductive pair since \( \Psi(M, T) \) is a closed subsystem of \( \Psi(G, T) \), [BMRT10, Lem. 3.9].

Then we have

\[H \subset M, H \not\subset L_\alpha.\]

Proposition 3.7. \( H \) is not \( M \)-cr.
Proof. First, let
\[ \lambda = \alpha^\vee + 2\beta^\vee. \]
Since we have \( \langle \alpha, \lambda \rangle = 0 \) and \( \langle \beta, \lambda \rangle = 1 \), it is easy to see that
\[ L_\alpha = L_\lambda, \ P_\alpha = P_\lambda. \]
Let \( c_\lambda : P_\lambda \to L_\lambda \) be the homomorphism from Definition 2.3. Suppose that \( H \) is \( M \)-cr. By Proposition 2.4, it is enough to show that there is an element \( h \in H \) which is not \( R_\alpha(P_\lambda(M)) \)-conjugate to \( c_\lambda(h) \). Set
\[ h := v(a)n_\alpha v(a)^{-1} = n_\alpha e^{3\alpha + 2\beta}(a^2). \]
Then we have
\[
c_\lambda(h) = \lim_{x \to 0} \left( \lambda(x)v(a)n_\alpha v(a)^{-1}\lambda(x)^{-1} \right)
= \lim_{x \to 0} \left( \lambda(x)n_\alpha e^{3\alpha + 2\beta}(a^2)\lambda(x)^{-1} \right)
= n_\alpha.
\]
Now suppose that \( h \) is \( R_\alpha(P_\lambda(M)) \)-conjugate to \( c_\lambda(h) = n_\alpha \). Then there exists \( m \in R_\alpha(P_\lambda(M)) \) such that
\[ mv(a)n_\alpha v(a)^{-1}m^{-1} = n_\alpha. \]
Since \( m \in U_{3\alpha + 2\beta} \) centralizes \( n_\alpha \) and \( v(a) \), this implies
\[ v(a)n_\alpha v(a)^{-1} = n_\alpha. \]
This contradicts \( 3.3 \).

Remark 3.8. In [BMRT10, Sec. 7, Prop. 7.17], Bate et al. used [BMR05, Lem. 2.17 and Thm. 3.1] to turn a problem on \( M \)-complete reducibility into a problem involving \( M \)-conjugacy. We have used Proposition 2.4 to turn the same problem into a problem involving \( R_\alpha(P \cap M) \)-conjugacy, which is easier.

3.3 Step 3

Proposition 3.9. \( H \) is \( G \)-cr.

Proof. See [BMRT10, Lem. 7.10(a)].

4 The rank 1 result

First, we point out that the form of \( K = \langle n_\alpha, \alpha^\vee(t) \rangle \) in the \( G_2 \) example is “canonical” in the following sense; in the proof of Proposition 3.3 it was necessary for \( K \) to contain some \( n_i \) for \( i \in \Sigma \) which acts on \( u \in C_{R_u(P)}(K) \) by swapping the order of non-commuting pair of \( \epsilon_j(a_j) \) odd times. But \( n_i \) by itself does not generate a \( G \)-cr subgroup of \( G \), so Bate et al. added some extra element from \( T \) to obtain a \( G \)-cr subgroup \( K \) of \( G \). The next theorem shows that the \( G_2 \) example is the only case where a subgroup \( K \) of this form acts non-separably on \( R_\alpha(P) \) where \( P \) is a rank 1 parabolic subgroup of a simple algebraic group \( G \). Thus, we are naturally led to look at \( K \) sitting in a higher rank Levi subgroup in the following section.
**Theorem 4.1.** Let $G$ be a simple algebraic group of any type except type $G_2$ defined over an algebraically closed field $k$ of characteristic 2. Fix a maximal torus $T$. Pick any root $\zeta$ of $G$, and choose $\zeta'(t)$ of odd order $n \geq 3$ in $T$. Let $K = \langle n_\zeta, \zeta'(t) \rangle \cong D_{2n}$ (the dihedral group of order $2n$). Then $K$ acts separably on $R_u(P_\zeta)$. Also, the same is true if $G$ is of type $G_2$ and $\zeta$ is a long root of $G$.

We use the next lemma to prove Theorem 4.1.

**Lemma 4.2.** Let $G$ be a simple algebraic group of any type except type $G_2$. Fix a maximal torus $T$. Pick any root $\zeta$ and coroot $\xi'$ of $G$. Then the absolute value of $\langle \zeta, \xi' \rangle$ is always less than 3. Also, the statement holds if $G$ is of type $G_2$ and $\xi$ is a long root of $G$.

**Proof.** This is a standard result, see [Hum72, Sec. 9.4].

**Proof of Theorem 4.1.** Let $G$ and $K$ as in the hypothesis. Suppose that $K$ acts non-separably on $R_u(P_\zeta)$. Then there exists $x \in \mathfrak{c}(\text{Lie}(R_u(P_\zeta)))(K) \setminus \text{Lie}(C_{R_u(P_\zeta)})(K)$. We can write $x = \sum_{i \in I} a_i e_i$ for some subset $I$ of $\Psi(P_\zeta)$ and for some $a_i \in k^*$. We have
\[
\text{Ad}(\zeta'(t)) x = \sum_{i \in I} a_i \text{Ad}(\zeta'(t)) (e_i) = \sum_{i \in I} a_i (t^{i, \xi'} e_i).
\]
Since $\zeta'(t)$ centralizes $x$ and the order of $t$ is 3 or greater, $\langle i, \xi' \rangle$ is zero for each $i \in I$ by Lemma 4.2. Hence $\zeta'(t)$ and $n_\zeta$ centralize $U_i$ for each $i \in I$. Therefore $U_i \subseteq C_{R_u(P_\zeta)}(K)$, and it follows that $x \in \text{Lie}(C_{R_u(P_\zeta)})(K)$. This is a contradiction.

**Remark 4.3.** In the $G_2$ case, we have $\langle 3\alpha + \beta, \alpha' \rangle = 3$, and the assertion of Theorem 4.1 is false.

5 The $E_7$ example

5.1 Step 1

Let $G$ be a simple algebraic group of type $E_7$ defined over $k$ of characteristic 2. Fix a maximal torus $T$ of $G$. Fix a Borel subgroup $B$ of $G$ containing $T$. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ be the set of simple roots of $G$. Figure 1 defines how each simple root of $G$ corresponds to each node in the Dynkin diagram of $E_7$.

![Figure 1: Dynkin diagram of $E_7$](image)

From [EdV69 Appendix, Table B], we have the coefficients of all positive roots of $G$. We label all positive roots of $G$ in Table 1 in the Appendix. Our ordering of roots is different from [EdV69 Appendix, Table B], which is convenient later on.
The set of positive roots is 
\[ \Psi^+(G) = \{1, 2, \cdots, 63\}. \]

Note that \( \{1, \cdots, 35\} \) and \( \{36, \cdots, 42\} \) are precisely the roots of \( G \) such that the coefficient of \( \sigma \) is 1 and 2 respectively. We call the roots of the first type \textit{weight-1 roots}, and the second type \textit{weight-2 roots}. Define
\[ L_\alpha \beta \gamma \delta \epsilon \eta := \langle T, G_{43}, \cdots, G_{63} \rangle, \]
\[ P_\alpha \beta \gamma \delta \epsilon \eta := \langle L_\alpha \beta \gamma \delta \epsilon \eta, U_1, \cdots, U_{42} \rangle. \]

Then \( P_\alpha \beta \gamma \delta \epsilon \eta \) is a parabolic subgroup of \( G \), and \( L_\alpha \beta \gamma \delta \epsilon \eta \) is a Levi subgroup of \( P_\alpha \beta \gamma \delta \epsilon \eta \). We have
\[ \Psi(R_u(P_\alpha \beta \gamma \delta \epsilon \eta)) = \{1, \cdots, 42\}. \]

Define
\[ q_1 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta, \]
\[ q_2 := n_\epsilon n_\beta n_\gamma n_\alpha n_\eta n_\delta n_\beta, \]
\[ K := \langle q_1, q_2 \rangle. \]

Let \( \zeta_1, \zeta_2 \) be simple roots of \( G \). From the Cartan matrix of \( E_7 \) [Hum72, Sec. 11.4] we have
\[ \langle \zeta_1, \zeta_2 \rangle = \begin{cases} 2, & \text{if } \zeta_1 = \zeta_2, \\ -1, & \text{if } \zeta_1 \text{ is adjacent to } \zeta_2 \text{ in the Dynkin diagram}, \\ 0, & \text{otherwise}. \end{cases} \]

From this, it is not difficult to calculate \( \langle \xi, \zeta_\vee \rangle \) for all \( \xi \in \Psi(R_u(P_\alpha \beta \gamma \delta \epsilon \eta)) \) and for all \( \zeta \in \Sigma \). These calculations show how \( n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta \) act on \( \Psi(R_u(P_\alpha \beta \gamma \delta \epsilon \eta)) \). Let \( \pi : \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta \rangle \to \text{Sym}(\Psi(R_u(P_\alpha \beta \gamma \delta \epsilon \eta))) \cong S_{42} \) be the corresponding homomorphism. Then we have
\[ \pi(n_\alpha) = (2 17)(3 22)(5 33)(6 35)(10 20)(13 27)(21 32)(24 34)(25 26)(30 31)(36 37), \]
\[ \pi(n_\beta) = (1 10)(3 11)(6 5 23)(7 24)(12 26)(13 28)(15 17)(19 30)(21 29)(37 38), \]
\[ \pi(n_\gamma) = (1 16)(3 34)(5 31)(6 25)(9 15)(14 28)(18 29)(22 24)(26 35)(30 33)(38 39), \]
\[ \pi(n_\delta) = (1 19)(3 21)(8 9)(10 30)(11 29)(12 28)(13 26)(20 31)(22 32)(25 27)(39 40), \]
\[ \pi(n_\epsilon) = (1 15)(2 20)(3 6)(4 11)(7 12)(9 16)(10 17)(22 35)(24 36)(34 31)(38 40), \]
\[ \pi(n_\eta) = (1 29)(3 30)(5 24)(7 23)(10 21)(11 19)(16 18)(20 32)(22 31)(33 34)(41 42). \]

From this, we obtain
\[ \pi(q_1) = (1 2)(3 6)(4 7)(9 10)(11 12)(13 14)(15 20)(16 17)(18 21)(19 23)(22 25)(24 26) \]
\[ (27 28)(29 32)(31 33)(34 35)(36 38)(37 39)(40 41), \]
\[ \pi(q_2) = (1 6 7 5 4 3 2)(8 10 12 14 13 11 9)(15 16 21 23 26 27 22)(17 20 25 28 24 19 18) \]
\[ (29 30 32 33 35 34 31)(36 38 39 41 42 40 37). \]

It is easy to see that \( K \cong D_{14} \). The orbits of \( K \) in \( \Psi(R_u(P_\alpha \beta \gamma \delta \epsilon \eta)) \) are
\[ O_1 = \{1, \cdots, 7\}, O_8 = \{8, \cdots, 14\}, O_{15} = \{15, \cdots, 28\}, O_{29} = \{29, \cdots, 35\}, \]
\[ O_{36} = \{36, \cdots, 42\}. \]

Thus Lemma 2.8 yields
Proposition 5.1. 
\[ \xi_{\text{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\theta
}))}(K) = \left\{ a \left( \sum_{\lambda \in O_1} e_\lambda \right) + b \left( \sum_{\lambda \in O_8} e_\lambda \right) + c \left( \sum_{\lambda \in O_{15}} e_\lambda \right) + d \left( \sum_{\lambda \in O_{29}} e_\lambda \right) + m \left( \sum_{\lambda \in O_{36}} e_\lambda \right) \right\} \]
\[ a, b, c, d, m \in k. \]

The following is the most important technical result in this paper.

Proposition 5.2. Let \( u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\theta
})}(K) \). Then \( u \) must have the form,
\[ u = \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a + b + c) \prod_{i=36}^{42} \epsilon_i(a) \]
for some \( a, b, c, a_i \in k \).

Proof. By Lemma 2.5, \( u \) can be expressed uniquely as
\[ u = \prod_{i=1}^{42} \epsilon_i(b_i), \] for some \( b_i \in k \).

By (2.1), we have
\[ n_\xi \epsilon_\zeta(a) n_\zeta^{-1} = \epsilon_{\xi^{-1}\zeta}(a) \] for any \( a \in k \) and \( \xi, \zeta \in \Psi(G) \).

Thus we have
\[ q_1 u q_1^{-1} = q_1 \left( \prod_{i=1}^{42} \epsilon_i(b_i) \right) q_1^{-1} \]
\[ = \left( \prod_{i=1}^{7} \epsilon_{q_1^{-1}i}(b_i) \right) \left( \prod_{i=8}^{14} \epsilon_{q_1^{-1}i}(b_i) \right) \left( \prod_{i=15}^{28} \epsilon_{q_1^{-1}i}(b_i) \right) \left( \prod_{i=29}^{35} \epsilon_{q_1^{-1}i}(b_i) \right) \left( \prod_{i=36}^{42} \epsilon_{q_1^{-1}i}(b_i) \right). \]
(5.1)

We reorder the terms \( \epsilon_{q_1^{-1}i}(b_i) \) in (5.1) into the natural order. Note that given \( i, j \in \{1, \cdots, 42\} \), either \( U_i \) and \( U_j \) commute by Lemma 2.6 or \( \{i, j, i+j\} \) forms an \( A_2 \) subsystem. (We use + for the sum of roots as vectors, not for the sum of labels). In the latter case, if we swap the order of \( \epsilon_i(m) \) and \( \epsilon_j(n) \), then we get a “correction term” \( \epsilon_{i+j}(mn) \) by Lemma 2.7. We list all pairs of weight-1 roots \( \{i, j\} \) corresponding to the weight-1 non-commuting root subgroups \( \{U_i, U_j\} \) of \( R_u(P_{\alpha\beta\gamma\delta\epsilon\theta
}) \) with the value of \( i+j \) in Table 2 in the Appendix. Abusing the language, we say that \( \{i, j\} \) is a non-commuting pair of roots.

We apply the following (⋆) to reorder the terms in the first factor of (5.1), which is \( \prod_{i=1}^{7} \epsilon_{q_1^{-1}i}(b_i) \). The terms in the other factors can be reordered in a similar way.

(⋆) Move the \( \epsilon_1 \) term to the left, and if a weight-2 term occurs, this can be moved to the right since weight-2 terms commute with any other term by Lemma 2.6. Then move the \( \epsilon_2 \) term to the left until it appears immediately after \( \epsilon_1 \) term. Continue with this process until all terms corresponding to weight-1 roots are rearranged into the natural order. Then rearrange weight-2 terms in the natural order.
Thus we have
\[
\prod_{i=1}^{7} \epsilon_{q_i}(b_i) = \epsilon_2(b_1)\epsilon_1(b_2)\epsilon_6(b_3)\epsilon_7(b_4)\epsilon_5(b_5)\epsilon_3(b_6)\epsilon_4(b_7)
\]
\[
= \epsilon_1(b_2)\epsilon_2(b_1)\epsilon_3(b_6)\epsilon_4(b_7)\epsilon_5(b_5)\epsilon_6(b_3)\epsilon_7(b_4) \left( \prod_{i=36}^{41} \epsilon_i(c_i) \right) \epsilon_{42}(b_4b_7)
\]
for some \(c_i \in k\).  \hspace{1cm} (5.2)

Note that we can express the \(c_i\) terms in terms of the \(b_j\), but we do not do this because it is not necessary for our purpose. Likewise for the \(d_i, f_i, g_i, h_i\) terms in (5.3), (5.4), (5.5), and (5.6) below. Similarly, we have
\[
\prod_{i=8}^{14} \epsilon_{q_i}(b_i) = \epsilon_8(b_8)\epsilon_{10}(b_9)\epsilon_9(b_{10})\epsilon_{12}(b_{11})\epsilon_{11}(b_{12})\epsilon_{14}(b_{13})\epsilon_{13}(b_{14})
\]
\[
= \epsilon_8(b_8)\epsilon_9(b_{10})\epsilon_{10}(b_9)\epsilon_{11}(b_{12})\epsilon_{12}(b_{11})\epsilon_{13}(b_{14})\epsilon_{14}(b_{13}) \left( \prod_{i=36}^{41} \epsilon_i(d_i) \right) \epsilon_{42}(b_{11}b_{12})
\]
for some \(d_i \in k\).  \hspace{1cm} (5.3)

\[
\prod_{i=15}^{28} \epsilon_{q_i}(b_i) = \epsilon_{20}(b_{15})\epsilon_{17}(b_{16})\epsilon_{16}(b_{17})\epsilon_{21}(b_{18})\epsilon_{23}(b_{19})\epsilon_{15}(b_{20})\epsilon_{18}(b_{21})\epsilon_{25}(b_{22})\epsilon_{19}(b_{23})\epsilon_{26}(b_{24})
\]
\[
\quad \epsilon_{22}(b_{25})\epsilon_{24}(b_{26})\epsilon_{28}(b_{27})\epsilon_{27}(b_{28})
\]
\[
= \epsilon_{15}(b_{20})\epsilon_{16}(b_{17})\epsilon_{17}(b_{16})\epsilon_{18}(b_{21})\epsilon_{19}(b_{23})\epsilon_{20}(b_{15})\epsilon_{21}(b_{18})\epsilon_{22}(b_{25})\epsilon_{23}(b_{19})\epsilon_{24}(b_{26})
\]
\[
\quad \epsilon_{25}(b_{22})\epsilon_{26}(b_{24})\epsilon_{27}(b_{28})\epsilon_{28}(b_{27}) \left( \prod_{i=36}^{41} \epsilon_i(f_i) \right) \epsilon_{42}(b_{22}b_{25}) \text{ for some } f_i \in k.  \hspace{1cm} (5.4)
\]

\[
\prod_{i=29}^{35} \epsilon_{q_i}(b_i) = \epsilon_{32}(b_{29})\epsilon_{30}(b_{30})\epsilon_{33}(b_{31})\epsilon_{29}(b_{32})\epsilon_{31}(b_{33})\epsilon_{35}(b_{34})\epsilon_{34}(b_{35})
\]
\[
= \epsilon_{29}(b_{32})\epsilon_{30}(b_{30})\epsilon_{31}(b_{33})\epsilon_{32}(b_{29})\epsilon_{33}(b_{31})\epsilon_{34}(b_{35})\epsilon_{35}(b_{34}) \left( \prod_{i=36}^{41} \epsilon_i(g_i) \right) \epsilon_{42}(b_{34}b_{35}) \text{ for some } g_i \in k.  \hspace{1cm} (5.5)
\]

We also have
\[
\prod_{i=36}^{42} \epsilon_{q_i}(b_i) = \left( \prod_{i=36}^{41} \epsilon_i(h_i) \right) \epsilon_{42}(b_{42}) \text{ for some } h_i \in k.  \hspace{1cm} (5.6)
\]

Combining (5.1), (5.2), (5.3), (5.4), (5.5), and (5.6), we obtain
\[
q_{1uq_1^{-1}} = \epsilon_1(b_2)\epsilon_2(b_1)\epsilon_3(b_6)\epsilon_4(b_7)\epsilon_5(b_5)\epsilon_6(b_3)\epsilon_7(b_4)\epsilon_8(b_8)\epsilon_9(b_{10})\epsilon_{10}(b_9)\epsilon_{11}(b_{12})\epsilon_{12}(b_{11})\epsilon_{13}(b_{14})
\]
\[
\epsilon_{14}(b_{13})\epsilon_{15}(b_{20})\epsilon_{16}(b_{17})\epsilon_{17}(b_{16})\epsilon_{18}(b_{21})\epsilon_{19}(b_{23})\epsilon_{20}(b_{15})\epsilon_{21}(b_{18})\epsilon_{22}(b_{25})\epsilon_{23}(b_{19})\epsilon_{24}(b_{26})
\]
\[
\epsilon_{25}(b_{22})\epsilon_{26}(b_{24})\epsilon_{27}(b_{28})\epsilon_{28}(b_{27})\epsilon_{29}(b_{32})\epsilon_{30}(b_{30})\epsilon_{31}(b_{33})\epsilon_{32}(b_{29})\epsilon_{33}(b_{31})\epsilon_{34}(b_{35})\epsilon_{35}(b_{34})
\]
\[
\left( \prod_{i=36}^{41} \epsilon_i(c_i + d_i + f_i + g_i + h_i) \right) \epsilon_{42}(b_4b_7 + b_{11}b_{12} + b_{22}b_{25} + b_{34}b_{35} + b_{42}).  \hspace{1cm} (5.7)
\]
Since $q_1$ centralizes $u$, we have
\[ b_1 = \cdots = b_7, \quad b_8 = \cdots = b_{14}, \quad b_{15} = \cdots = b_{28}, \quad b_{29} = \cdots = b_{35}. \]
Set
\[ b_1 = a, \quad b_8 = b, \quad b_{15} = c, \quad b_{29} = d, \quad c_i + d_i + f_i + g_i + h_i = a_i \text{ for } i \in \{36, \cdots, 41\}. \]
Then (5.7) simplifies to
\[ q_1 u q_1^{-1} = \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(d) \left( \prod_{i=36}^{41} \epsilon_i(a_i) \right) \epsilon_{42}(a^2 + b^2 + c^2 + d^2 + b_{42}). \]
Since $q_1$ centralizes $u$, comparing the arguments of the $\epsilon_{42}$ term on both sides, we must have
\[ b_{42} = a^2 + b^2 + c^2 + d^2 + b_{42}, \]
which is equivalent to
\[ a + b + c + d = 0. \]
Then we obtain the desired result.

Proposition 5.3. $K$ acts non-separably on $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$.

Proof. In view of Proposition 5.1, it suffices to show that $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \not\in \text{Lie } C_{R_u(P_{\lambda})}(K)$. Suppose the contrary. Since by [Spr98, Cor. 14.2.7] $C_{R_u(P_{\lambda})}(K)$ is isomorphic as a variety to $k^n$ for some $n \in \mathbb{N}$, there exists a morphism of varieties $v : k \to C_{R_u(P_{\lambda})}(K)$ such that $v(0) = 1$ and $v'(0) = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$. By Lemma 2.5, $v(a)$ can be expressed uniquely as
\[ v(a) = \prod_{i=1}^{42} \epsilon_i(f_i(a)), \quad (5.8) \]
where $f_i \in k[X]$.

Differentiating (5.8), and evaluating at $a = 0$, we obtain
\[ v'(0) = \sum_{i \in \{1, \cdots, 42\}} (f_i)'(0)e_i. \]
Since $v'(0) = \sum_{i \in O_1} e_i$, we have
\[ (f_i)'(0) = \begin{cases} 1 & \text{if } i \in O_1, \\ 0 & \text{otherwise}. \end{cases} \]
Then we have
\[ f_i(a) = \begin{cases} a + g_i(a) & \text{if } i \in O_1, \\ g_i(a) & \text{otherwise}, \end{cases} \]
where $g_i \in k[X]$ has no constant or linear term.

Then from Proposition 5.2, we obtain
\[ (a + g_1(a)) + g_8(a) + g_{15}(a) = g_{29}(a). \]
This is a contradiction. \qed
5.2 Step 2

Let

\[ C_1 := \left\{ \prod_{i=1}^{7} \epsilon_i(a) \mid a \in k \right\}. \]

By Lemma 2.8 and Proposition 5.2, \( T_1(C_1) \) is tangent to \( \epsilon_{\text{Lie}(R_u(P_{\alpha,\beta,\gamma,\eta}))}(K) \) but not tangent to \( \text{Lie} C_{R_u(P_{\alpha,\beta,\gamma,\eta})}(K) \). Pick any \( a \in k^{*} \). Let \( v(a) \in C_1 \). We have

\[
\begin{align*}
v(a)q_1v(a)^{-1} &= q_1\epsilon_40(a^2)\epsilon_41(a^2)\epsilon_42(a^2), \\
v(a)q_2v(a)^{-1} &= q_2\epsilon_{36}(a^2)\epsilon_{39}(a^2).
\end{align*}
\]

Set

\[
H := v(a)Kv(a)^{-1} = \langle q_1\epsilon_40(a^2)\epsilon_41(a^2)\epsilon_42(a^2), q_2\epsilon_{36}(a^2)\epsilon_{39}(a^2) \rangle.
\]

\[
M := \langle L_{\alpha,\beta,\gamma,\delta,\epsilon,\eta}, G_{36}, \ldots, G_{42} \rangle.
\]

Remark 5.4. In this case \( \sigma \) is the unique simple root not contained in \( \Psi(L_{\alpha,\beta,\gamma,\delta,\epsilon,\eta}) \). \( M \) was chosen so that \( M \) is generated by a Levi subgroup \( L_{\alpha,\beta,\gamma,\delta,\epsilon,\eta} \) containing \( K \) and all root subgroups of even \( \sigma \)-weight.

We have

\[ H \subset M, H \not\subset L_{\alpha,\beta,\gamma,\delta,\epsilon,\eta}. \]

Note that we have

\[ \Psi(M) = \{ \pm36, \ldots, \pm63 \}. \]

Since \( M \) is generated by all root subgroups for roots of even \( \sigma \)-weight, it is easy to see that \( \Psi(M) \) is a closed subsystem of \( \Psi(G) \), thus \( M \) is reductive by [BMRT10 Lem. 3.9]. It is easy to check that \( M \) is of type \( A_7 \).

**Proposition 5.5.** \( H \) is not \( M \)-cr.

**Proof.** Let

\[ \lambda = 3\alpha^\vee + 6\beta^\vee + 9\gamma^\vee + 12\delta^\vee + 8\epsilon^\vee + 4\eta^\vee + 7\sigma^\vee. \]

We have

\[
\begin{align*}
\langle \alpha, \lambda \rangle &= 0, \langle \beta, \lambda \rangle = 0, \langle \gamma, \lambda \rangle = 0, \langle \delta, \lambda \rangle = 0, \\
\langle \epsilon, \lambda \rangle &= 0, \langle \eta, \lambda \rangle = 0, \langle \sigma, \lambda \rangle = 2.
\end{align*}
\]

So we have

\[ L_{\alpha,\beta,\gamma,\delta,\epsilon,\eta} = L_{\lambda}, \quad P_{\alpha,\beta,\gamma,\delta,\epsilon,\eta} = P_{\lambda}. \]

It is easy to see that \( L_{\lambda} \) is of type \( A_7 \), so \([L_{\lambda}, L_{\lambda}]\) is isomorphic to either \( SL_7 \) or \( PGL_7 \). We rule out the latter. Pick \( x \in k^{*} \) such that \( x \neq 1, x^2 = 1 \). Then \( \lambda(x) \neq 1 \) since \( \sigma(\lambda(x)) = x^2 \neq 1 \).

Also, we have \( \lambda(x) \in Z([L_{\lambda}, L_{\lambda}]) \). Therefore \([L_{\lambda}, L_{\lambda}] \cong SL_7 \). It is easy to check that the map \( k^{*} \times [L_{\lambda}, L_{\lambda}] \to L_{\lambda} \) is separable, so we have \( L_{\lambda} \cong GL_7 \).

Let \( c_\lambda : P_{\lambda} \to L_{\lambda} \) be the homomorphism as in Definition 2.3. In order to prove that \( H \) is not \( M \)-cr, by Theorem 2.4 it suffices to find a tuple \((h_1, h_2) \in H^2\) which is not \( R_u(P_{\lambda}(M))\)-conjugate to \( c_\lambda ((h_1, h_2)) \). Set

\[ h_1 := v(a)q_1v(a)^{-1}, \quad h_2 := v(a)q_2v(a)^{-1}. \]

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By (5.9) we have
\[ c_\lambda ((h_1, h_2)) = \lim_{x \to \infty} (\lambda(x)q_1\epsilon_40(a^2)\epsilon_41(a^2)\epsilon_42(a^2)\lambda(x)^{-1}) \]
\[ = (q_1, q_2). \]

Now suppose that \((h_1, h_2)\) is \(R_u(P_\lambda(M))\)-conjugate to \(c_\lambda ((h_1, h_2))\). Then there exists \(m \in R_u(P_\lambda(M))\) such that
\[ mv(a)q_1v(a)^{-1}m^{-1} = q_1. \]
\[ mv(a)q_2v(a)^{-1}m^{-1} = q_2. \]

Thus we have
\[ mv(a) \in C_{R_u(P_\lambda)}(K). \]

Note that we have
\[ \Psi(R_u(P_\lambda(M))) = \{36, \ldots, 42\}. \]

So, by Lemma 2.5, \(m\) can be expressed uniquely as
\[ m := \prod_{i=36}^{42} \epsilon_i(a_i), \text{ for some } a_i \in k. \]

Then we have
\[ mv(a) = \epsilon_1(a)\epsilon_2(a)\epsilon_3(a)\epsilon_4(a)\epsilon_5(a)\epsilon_6(a)\epsilon_7(a) \left( \prod_{i=36}^{42} \epsilon_i(a_i) \right) \in C_{R_u(P_\lambda)}(K). \]

This contradicts Proposition 5.7.

**Remark 5.6.** Instead of using \(C_1\) to define \(v(a)\), we can take \(C_8 := \{\prod_{i=8}^{14} \epsilon_i(a) \mid a \in k\}\), \(C_{15} := \{\prod_{i=15}^{28} \epsilon_i(a) \mid a \in k\}\), or \(C_{29} := \{\prod_{i=29}^{35} \epsilon_i(a) \mid a \in k\}\). In each case, a similar argument goes through and gives rise to a different example with the desired property.

### 5.3 Step 3

**Proposition 5.7.** \(H\) is \(G\)-cr.

**Proof.** First note that \(H\) is conjugate to \(K\), so \(H\) is \(G\)-cr if and only if \(K\) is \(G\)-cr. Then, by [BMR05 Lem. 2.12, Cor. 3.22], it suffices to show that \(K\) is \([L_\lambda, L_\lambda]\)-cr. We can identify \(K\) with the image of the corresponding subgroup of \(S_7\) under the permutation representation \(\pi_1 : S_7 \to SL_7(k)\). It is easy to see that \(K \cong D_{14}\). A quick calculation shows that this representation of \(D_{14}\) is a direct sum of a trivial 1-dimensional and 3 irreducible 2-dimensional subrepresentations. Therefore \(K\) is \([L_\lambda, L_\lambda]\)-cr.

### 6 A rationality problem

We consider a rationality question. In particular, we prove Theorem 1.8. The key here is again the existence of a 1-dimensional curve \(C_1\) which is tangent to \(\epsilon_{1,\text{Lie}(R_u(P_\lambda))}(K)\) but not tangent to \(\text{Lie}C_{R_u(P_\lambda)}(K)\). The same phenomenon was seen in [BMR10 Ex. 7.22] (but the cause of that was not mentioned explicitly) where Bate et al. presented an example with the same property in \(G_0\) of type \(G_2\).
Proof of Theorem 1.8. Let $k_0$, $k$, and $G_0$ be as in the hypothesis. We choose a $k_0$-defined $k_0$-split maximal torus $T_0$ so that for each $\zeta \in \Psi(G_0)$ the corresponding root $\zeta$, coroot $\zeta'$, and homomorphism $\epsilon_\zeta$ are defined over $k_0$. Since $k_0$ is not perfect, there exists $\tilde{a} \in k \setminus k_0$ such that $\tilde{a}^2 \in k_0$. Use the notation $q_1, q_2, K, P_\lambda, L_\lambda$ of Section 5. Let

$$H_0 = (v(\tilde{a})q_1v(\tilde{a})^{-1}, v(\tilde{a})q_2v(\tilde{a})^{-1}).$$

$$= (q_1\epsilon_{40}(\tilde{a}^2)\epsilon_{41}(\tilde{a}^2)\epsilon_{42}(\tilde{a}^2), q_2\epsilon_{36}(\tilde{a}^2)\epsilon_{39}(\tilde{a}^2)).$$

Now it is obvious that $H_0$ is $k_0$-defined. We already know that $H_0$ is $G_0$-cr by Proposition 5.7. Since $G_0$ and $T_0$ are $k_0$-split, $P_\lambda$ and $L_\lambda$ are defined over $k_0$ by [Bor91] V.20.4, V.20.5. Suppose that there exists a Levi subgroup $L'_0$ of $P_\lambda$ defined over $k_0$ such that $L'_0$ contains $H_0$. Then there exists $w_0 \in R_u(P_\lambda)(k_0)$ such that $L'_0 = w_0L_\lambda w_0^{-1}$ by [Bor91] V.20.5. Then $w_0^{-1}H_0w_0 \subseteq L_\lambda$ and $v(\tilde{a})^{-1}H_0v(\tilde{a}) \not\subseteq L_\lambda$. So we have $c_\lambda(w_0^{-1}h_0w_0) = w_0^{-1}h_0w_0$ and $c_\lambda(v(\tilde{a})^{-1}h_0v(\tilde{a})) = v(\tilde{a})^{-1}h_0v(\tilde{a})$ for any $h_0 \in H_0$. We also have $c_\lambda(w_0) = c_\lambda(v(\tilde{a})) = 1$ since $w_0, v(\tilde{a}) \in R_u(P_\lambda)(k)$. Therefore we obtain

$$w_0^{-1}h_0w_0 = c_\lambda(w_0^{-1}h_0w_0) = c_\lambda(h_0) = c_\lambda(v(\tilde{a})^{-1}h_0v(\tilde{a})) = v(\tilde{a})^{-1}h_0v(\tilde{a})$$

for any $h_0 \in H_0$. So we have

$$w_0 = v(\tilde{a})z,$$ where $z \in C_{R_u(P_\lambda)}(K)(k)$.

By Proposition 5.2, $z$ must be in the following form:

$$z := \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i) \text{ for some } a, b, c, a_i \in k.$$

Then we have

$$w_0 = \left( \prod_{i=1}^{7} \epsilon_i(\tilde{a}) \right) \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i)$$

$$= \prod_{i=1}^{7} \epsilon_i(\tilde{a}+a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(b_i) \text{ for some } b_i \in k.$$

Since $w_0$ is a $k_0$-point, $b$, $c$, and $a+b+c$ all belong to $k_0$, so $a \in k_0$. But $a + \tilde{a}$ belongs to $k_0$ as well, so $\tilde{a} \in k_0$. This is a contradiction.

Remark 6.1. As in Section 5, we can take $v(a)$ from $C_8$, $C_{15}$, or $C_{29}$. In each case, a similar argument goes through, and gives rise to a different example.

Remark 6.2. [BMR93] Ex. 5.11] shows that there is a $k_0$-defined subgroup of $G_0$ of type $A_n$ which is not $G_0$-cr over $k$ even though it is $G_0$-cr over $k_0$. Note that this example works for any $p > 0$.

7 A problem of conjugacy classes

As another important application of the $E_7$ example, we consider a problem concerning conjugacy classes. We present a new counterexample to Proposition 1.9 with the hypothesis of separability removed. Here, the key is again the existence of a 1-dimensional curve $C_1$ as in [BMRT10] Ex. 7.15]. Use the notation $G, q_1, q_2, K, \epsilon_\lambda, L_\lambda$ of Section 5.
From Table 2 it is easy to see that
\[ Z(R_u(P)) = \langle U_{36}, U_{37}, U_{38}, U_{39}, U_{40}, U_{41}, U_{42} \rangle. \]

Let
\[ K_0 := (K, Z(R_u(P))). \]

It is standard that there exists a finite subset \( F = \{ z_1, z_2, \cdots, z_{\tilde{n}} \} \) of \( Z(R_u(P)) \) such that
\[ C_{P_\lambda}(K, F) = C_{P_\lambda}(K_0). \]

Let
\[ m := (q_1, q_2, z_1, \cdots, z_{\tilde{n}}). \]

Let \( n := \tilde{n} + 2 \). For \( \tilde{a} \in k^* \), define
\[ m(\tilde{a}) := v(\tilde{a}) \cdot m \in P_\lambda(M)^n. \]

**Lemma 7.1.** \( C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0) \).

**Proof.** It is obvious that \( C_{R_u(P_\lambda)}(K_0) \subseteq C_{P_\lambda}(K_0) \). We prove the converse. Let \( lu \in C_{P_\lambda}(K_0) \) where \( l \in L_\lambda \) and \( u \in R_u(P_\lambda) \). Then \( lu \) centralizes \( Z(R_u(P_\lambda)) \), so \( l \) centralizes \( Z(R_u(P_\lambda)) \) since \( u \) does. It suffices to show that \( l = 1 \). Let \( l = t\tilde{l} \) where \( t \in Z(L_\lambda)^* = \lambda(k^*) \) and \( \tilde{l} \in [L_\lambda, L_\lambda] \).

We have \( (i, \lambda) = 4 \) for any \( i \in \{36, \cdots, 42\} \).

Now \( Z(R_u(P_\lambda)) \) has the structure of a vector space over \( k \) in the obvious way, and the action of \( \lambda(k^*) \) on \( Z(R_u(P_\lambda)) \) is linear. So, for any \( z \in Z(R_u(P_\lambda)) \) there exists \( \alpha \in k^* \) such that \( t \cdot z = \alpha z \). Then we have \( \tilde{l} \cdot z = \alpha^{-1}z \). Now define
\[ A := \{ \tilde{l} \in [L_\lambda, L_\lambda] \mid \tilde{l} \text{ acts on } Z(R_u(P_\lambda)) \text{ by multiplication by a scalar} \} \]

Then it is easy to see that \( A \subseteq [L_\lambda, L_\lambda] \). Since \( [L_\lambda, L_\lambda] \cong SL_7 \) and \( L_\lambda \cong GL_7 \), we have \( A = Z([L_\lambda, L_\lambda]) \). Therefore we obtain \( \tilde{l} \in A = Z([L_\lambda, L_\lambda]) \subseteq \lambda(k^*) \). So we have \( l = c\tilde{l} \in \lambda(k^*) \).

Then we obtain \( \tilde{l} \in C_{\lambda(k^*)}([Z(R_u(P_\lambda))]) \). By (7.1) this implies \( l = 1 \).

**Lemma 7.2.** \( G \cdot m \cap P_\lambda(M)^n \) is an infinite union of \( P_\lambda(M) \)-conjugacy classes.

**Proof.** By Lemma 7.1 we have
\[ C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0) \subseteq C_{R_u(P_\lambda)}(K). \]

Then we obtain
\[ C_{P_\lambda}(v(b)K_0v(b)^{-1}) = v(b)C_{P_\lambda}(K_0)v(b)^{-1} \subseteq v(b)C_{R_u(P_\lambda)}(K)v(b)^{-1}. \]

Suppose that \( m(\tilde{a}) \) is \( P_\lambda(M) \)-conjugate to \( m(\tilde{b}) \). Then there exists \( m \in P_\lambda(M) \) such that \( m \cdot m(\tilde{a}) = m(\tilde{b}) \).

By (7.2), we have
\[ mv(\tilde{a})v(\tilde{b})^{-1} \in C_{P_\lambda}(v(b)K_0v(b)^{-1}) \subseteq v(b)C_{R_u(P_\lambda)}(K)v(b)^{-1}. \]

Then by Proposition 5.2 we have
\[ v(b)^{-1}mv(\tilde{a}) = \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a + b + c) \prod_{i=36}^{42} \epsilon_i(a_i), \text{ for some } a, b, c, a_i \in k. \]
This yields
\[ m = \prod_{i=1}^{7} \epsilon_i(a + \tilde{a} + \tilde{b}) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a + b + c) \prod_{i=36}^{42} \epsilon_i(b_i), \]
for some \( a, b, c, b_i \in k \).

But \( m \in P_\lambda(M) \), so we have
\[ a + \tilde{a} + \tilde{b} = 0, \quad b = 0, \quad c = 0, \quad a + b + c = 0. \]

Hence we have
\[ \tilde{a} = \tilde{b}. \]

Thus we have shown that if \( \tilde{a} \neq \tilde{b} \), then \( m(\tilde{a}) \) is not \( P_\lambda(M) \)-conjugate to \( m(\tilde{b}) \). So, in particular, \( G \cdot m \cap P_\lambda(M)^n \) is an infinite union of \( P_\lambda(M) \)-conjugacy classes. \( \square \)

We need the next result \[Lon13\] Lem. 4.4. We include the proof to make this paper self-contained.

**Lemma 7.3.** \( G \cdot m \cap P_\lambda(M)^n \) is a finite union of \( M \)-conjugacy classes if and only if it is a finite union of \( P_\lambda(M) \)-conjugacy classes.

**Proof.** Suppose that \( m_1 \) and \( m_2 \) are in the same \( M \)-conjugacy class of \( G \cdot m \cap P_\lambda(M)^n \). Then there exists \( m \in M \) such that \( m \cdot m_1 = m_2 \). Let \( Q = m^{-1}P_\lambda(M)m \). Then we have \( m_1 \in (P_\lambda(M) \cap Q)^n \). Now let \( S \) be a maximal torus of \( M \) contained in \( P_\lambda(M) \cap Q \). Since \( S \) and \( m^{-1}Sm \) are maximal tori of \( Q \), they must be \( Q \)-conjugate. So there exists \( q \in Q \) such that
\[ qSm^{-1} = m^{-1}Sm. \] (7.3)

Since \( Q = m^{-1}P_\lambda(M)m \), there exists \( p \in P_\lambda(M) \) such that \( q = m^{-1}pm \). Then from (7.3), we obtain
\[ pmSm^{-1}p^{-1} = S. \]

This implies
\[ m^{-1}p^{-1} \in N_M(S). \]

Fix a finite set \( N \subseteq N_M(S) \) of coset representatives for the Weyl group \( W = N_M(S)/S \). Then we have
\[ m^{-1}p^{-1} = ns \text{ for some } n \in N, s \in S. \]

Then we obtain
\[ m_1 = m^{-1} \cdot m_2 = (nsp) \cdot m_2 \in (nP_\lambda(M)) \cdot m_2. \]

Since \( N \) is a finite set, this shows that a \( M \)-conjugacy class in \( G \cdot m \cap P_\lambda(M)^n \) is a finite union of \( P_\lambda(M) \)-conjugacy classes. The converse is obvious. \( \square \)

**Proof of Theorem 1.10.** By Lemma 7.2 and Lemma 7.3 we conclude that \( G \cdot m \cap P_\lambda(M)^n \) is an infinite union of \( M \)-conjugacy classes. Now it is evident that \( G \cdot m \cap M^n \) is an infinite union of \( M \)-conjugacy classes. \( \square \)

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### Table 1: The set of positive roots of $G = E_7$

| # | 1 | 1 | 1 | 1 | 0 | 2 | 1 | 1 | 1 | 0 | 0 | 3 | 0 | 1 | 1 | 2 | 1 | 1 | 4 | 0 | 0 | 0 | 1 | 2 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 1 | 2 | 2 | 1 | 0 | 6 | 0 | 1 | 1 | 2 | 2 | 1 | 7 | 1 | 2 | 2 | 2 | 1 | 1 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 1 | 0 | 0 | 10 | 0 | 1 | 1 | 1 | 1 | 0 | 11 | 0 | 0 | 0 | 1 | 2 | 1 | 1 | 12 | 1 | 2 | 2 | 2 | 1 | 1 |
| 13 | 1 | 1 | 2 | 3 | 2 | 1 | 14 | 1 | 2 | 3 | 3 | 2 | 1 | 15 | 0 | 0 | 1 | 1 | 0 | 0 | 16 | 0 | 0 | 0 | 1 | 1 | 0 |
| 17 | 0 | 1 | 1 | 1 | 0 | 0 | 18 | 0 | 0 | 0 | 1 | 1 | 1 | 19 | 0 | 0 | 1 | 2 | 1 | 0 | 20 | 1 | 1 | 1 | 1 | 1 | 0 |
| 21 | 0 | 1 | 1 | 1 | 1 | 1 | 22 | 1 | 1 | 1 | 2 | 1 | 1 | 23 | 1 | 2 | 2 | 2 | 1 | 0 | 24 | 1 | 1 | 1 | 2 | 2 | 1 |
| 25 | 0 | 1 | 2 | 2 | 2 | 1 | 26 | 0 | 1 | 1 | 2 | 2 | 1 | 1 | 27 | 0 | 1 | 2 | 3 | 2 | 1 | 28 | 0 | 1 | 2 | 2 | 3 | 2 |
| 29 | 0 | 0 | 1 | 1 | 1 | 1 | 30 | 0 | 1 | 1 | 2 | 1 | 0 | 31 | 1 | 1 | 1 | 2 | 1 | 0 | 32 | 1 | 1 | 1 | 1 | 1 | 1 |
| 33 | 0 | 1 | 2 | 2 | 1 | 0 | 34 | 0 | 1 | 2 | 2 | 1 | 1 | 35 | 1 | 1 | 1 | 2 | 2 | 1 | 36 | 0 | 1 | 2 | 3 | 2 | 1 |
| 37 | 1 | 1 | 2 | 3 | 2 | 1 | 38 | 1 | 2 | 2 | 3 | 2 | 1 | 39 | 1 | 2 | 3 | 3 | 2 | 1 | 40 | 1 | 2 | 3 | 4 | 2 | 1 |
| 38 | 2 | 1 | 1 | 2 | 3 | 2 | 1 | 39 | 1 | 2 | 2 | 3 | 2 | 1 | 40 | 1 | 2 | 3 | 3 | 2 | 1 |
| 41 | 1 | 2 | 3 | 4 | 3 | 1 | 42 | 1 | 2 | 3 | 4 | 3 | 2 | 43 | 1 | 0 | 0 | 0 | 0 | 0 | 44 | 0 | 1 | 0 | 0 | 0 | 0 |
| 45 | 0 | 0 | 1 | 0 | 0 | 0 | 46 | 0 | 0 | 0 | 1 | 0 | 0 | 47 | 0 | 0 | 0 | 0 | 1 | 0 | 48 | 0 | 0 | 0 | 0 | 0 | 1 |
| 49 | 0 | 0 | 1 | 1 | 0 | 0 | 50 | 0 | 1 | 1 | 1 | 0 | 0 | 51 | 0 | 0 | 1 | 1 | 0 | 0 | 52 | 0 | 0 | 0 | 1 | 1 | 0 |
| 53 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 54 | 0 | 1 | 1 | 1 | 0 | 0 | 55 | 0 | 1 | 1 | 1 | 0 | 0 | 56 | 0 | 0 | 1 | 1 | 1 | 0 |
| 57 | 0 | 0 | 0 | 1 | 1 | 1 | 58 | 0 | 1 | 1 | 1 | 1 | 0 | 59 | 0 | 1 | 1 | 1 | 1 | 0 | 60 | 0 | 0 | 1 | 1 | 1 | 1 |
| 61 | 1 | 1 | 1 | 1 | 1 | 1 | 62 | 0 | 1 | 1 | 1 | 1 | 1 | 63 | 1 | 1 | 1 | 1 | 1 | 1 |

The above table represents the set of positive roots of the Lie algebra $E_7$. Each row corresponds to a root vector, with the entries indicating the coefficients of each basis element. Roots are typically denoted as $\alpha_i$, where $i$ is the index of the basis element. In the table, for simplicity, we use 1s to indicate a non-zero coefficient, and 0s for zero coefficients. The table includes both simple and non-simple roots, with the latter often represented as linear combinations of simple roots.
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Table 2: The triples of roots where the first two entries in each triple form a non-commuting pair of weight-1 roots, and the last entry in each triple is the weight-2 root for the corresponding correction term.

References

[BMR05] M. Bate, B. Martin, and G. Röhrle, *A geometric approach to complete reducibility*, Inventiones Mathematicae **161** (2005), 177–218.

[BMR08] ______, *Complete reducibility and commuting subgroups*, J. Reine Angew. Math. **621** (2008), 213–235.

[BMRT] M. Bate, B. Martin, G. Röhrle, and R. Tange, *Closed orbits and uniform S-instability in geometric invariant theory*, Trans. Amer. Math. Soc., to appear.

[BMRT10] ______, *Complete reducibility and separability*, Trans. Amer. Math. Soc. **362** (2010), no. 8, 4283–4311.

[Bor91] A. Borel, *Linear Algebraic Groups*, second enlarged ed., Springer, Graduate Texts in Mathematics, 1991.

[Car72] R. Carter, *Simple Groups of Lie Type*, John Wiley & Sons, 1972.

[Dyn57] E. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Translations **6** (1957), 111–244.

[Dyn00] ______, *Selected papers of E.B. Dynkin with commentary*, AMS, 2000, [Dyn57] is reproduced on pp.175-308 with corrections on pp.309-312.

[FdV69] H. Freudenthal and H. de Vries, *Linear Algebraic Groups*, Academic Press, New York and London, 1969.
[Hum72] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer, Graduate Texts in Mathematics, 1972.

[Hum91] ______, Linear Algebraic Groups, Springer, Graduate Texts in Mathematics, 1991.

[LMS05] M. Liebeck, B. Martin, and A. Shalev, On conjugacy classes of maximal subgroups of finite simple groups, and a related zeta function, Duke Math. J. 128 (2005), no. 3, 541–557.

[Lon13] D. Lond, On reductive subgroups of algebraic groups and a question of Kulshammer, PhD thesis, University of Canterbury, New Zealand, 2013.

[LS96] M. Liebeck and G. Seitz, Reductive subgroups of exceptional algebraic groups, Mem. Amer. Math. Soc. 580 (1996).

[LS03] ______, Variations on a theme of Steinberg. Special issue celebrating the 80th birthday of Robert Steinberg, J. Algebra 260 (2003), no. 1, 261–297.

[LT99] R. Lawther and D. Testerman, A1 subgroups of exceptional algebraic groups, Mem. Amer. Math. Soc. 141 (1999).

[LT04] M. Liebeck and D. Testerman, Irreducible subgroups of algebraic groups, Q.J. Math 55 (2004), 47–55.

[Mar03a] B. Martin, A normal subgroup of a strongly reductive subgroup is strongly reductive, J. Algebra 265 (2003), no. 2, 669–674.

[Mar03b] ______, Reductive subgroups of reductive groups in nonzero characteristic, J. Algebra 262 (2003), no. 1, 265–286.

[Ric67] R. Richardson, Conjugacy classes in Lie algebras and algebraic groups, Ann. of Math. 86 (1967), 1–15.

[Ric82] ______, On orbits of algebraic groups and Lie groups, Bull. Austral. Math. Soc 25 (1982), no. 1, 1–28.

[Ric88] ______, Conjugacy classes of ntuples in Lie algebras and algebraic groups, Duke Math. J. 57 (1988), 1–35.

[Sei97] G. Seitz, Abstract homomorphisms of algebraic groups, J. London. Math. Soc. 56 (1997), no. 1, 104–124.

[Ser] J.P. Serre, Complète réductibilité, Séminaire Bourbaki, 56ème année, 2003-2004, no. 932.

[Ser97] ______, Semisimplicity and tensor products of group representations: converse theorems (With an appendix by Walter Feit), J. Algebra 194 (1997), no. 2, 496–520.

[Ser98] ______, The notion of complete reducibility in group theory, Moursund Lectures, Part II, University of Oregon (1998), arXiv:math/0305257v1.

[Slo97] P. Slodowy, Two notes on a finiteness problem in the representation theory of finite groups, Austral. Math. Soc. Lect. Ser. 9, Algebraic groups and Lie groups, 331-348, Cambridge Univ. Press, Cambridge, 1997.
[Spr98] T. Springer, *Linear Algebraic Groups*, second ed., Birkhäuser, Progress in Mathematics, 1998.

[Uch] T. Uchiyama, *Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group of type $E_6$, $E_7$, and $E_8$: a classification*, preprint.

[Vin96] E. Vinberg, *On invariants of a set of matrices*, J. Lie Theory 6 (1996), 249–269.