ADDITIVE LIE (\(\xi\)-LIE) DERIVATIONS AND GENERALIZED LIE (\(\xi\)-LIE) DERIVATIONS ON PRIME ALGEBRAS

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Abstract. The additive (generalized) \(\xi\)-Lie derivations on prime algebras are characterized. It is shown, under some suitable assumption, that an additive map \(L\) is an additive (generalized) Lie derivation if and only if it is the sum of an additive (generalized) derivation and an additive map from the algebra into its center vanishing all commutators; is an additive (generalized) \(\xi\)-Lie derivation with \(\xi \neq 1\) if and only if it is an additive (generalized) derivation satisfying \(L(\xi A) = \xi L(A)\) for all \(A\). These results are then used to characterize additive (generalized) \(\xi\)-Lie derivations on several operator algebras such as Banach space standard operator algebras and von Neumann algebras.

1. Introduction

Let \(A\) be an associative ring (or an algebra over a field \(F\)). Then \(A\) is a Lie ring (Lie algebra) under the Lie product \([A, B] = AB - BA\). Recall that an additive (linear) map \(\delta\) from \(A\) into itself is called an additive (linear) derivation if \(\delta(AB) = \delta(A)B + A\delta(B)\) for all \(A, B \in A\). More generally, an additive (linear) map \(L\) from \(A\) into itself is called an additive (linear) Lie derivation if \(L([A, B]) = [L(A), B] + [A, L(B)]\) for all \(A, B \in A\). The questions of characterizing Lie derivations and revealing the relationship between Lie derivations and derivations have received many mathematicians’ attention recently (for example, see [1, 4, 8, 10, 13]).

Note that an important relation associated with the Lie product is the commutativity. Two elements \(A, B \in A\) are commutative if \(AB = BA\), that is, their Lie product is zero. More generally, if \(\xi \in F\) is a scalar and if \(AB = \xi BA\), we say that \(A\) commutes with \(B\) up to the factor \(\xi\). The conception of commutativity up to a factor for pairs of operators is also important and has been studied in the context of operator algebras and quantum groups (ref. 3, 9). Motivated by this, we introduced an binary operation \([A, B]_\xi = AB - \xi BA\), called the \(\xi\)-Lie product of \(A\) and \(B\), and a conception of (generalized) \(\xi\)-Lie derivations in [12]. Recall that an additive (linear) map \(L : A \to A\) is called a \(\xi\)-Lie derivation if \(L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi\) for all \(A, B \in A\); an additive (linear) map \(\delta : A \to A\) is called an additive (linear) generalized \(\xi\)-Lie derivation if there exists an additive (linear) \(\xi\)-Lie derivations.

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derivation \( L \) from \( \mathcal{A} \) into itself such that \( \delta([A, B]_ξ) = \delta(A)B - ξ\delta(B)A + AL(B) - ξBL(A) \) for all \( A, B \in \mathcal{A} \), and \( L \) is called the relating \( ξ \)-Lie derivation of \( \delta \). These conceptions unify several important conceptions such as (generalized) derivations, (generalized) Jordan derivations and (generalized) Lie derivations (see [6, 7]). It is clear that a (generalized) \( ξ \)-Lie derivation is a (generalized) derivation if \( ξ = 0 \); is a (generalized) Lie derivation if \( ξ = 1 \); is a (generalized) Jordan derivation if \( ξ = -1 \). Moreover, a characterization of (generalized) \( ξ \)-Lie derivations on triangular algebras for all possible \( ξ \) is given in [12]. Note that triangular algebras are not prime.

The purpose of the present paper is to discuss the questions of characterizing the Lie (\( ξ \)-Lie) derivations and generalized Lie (\( ξ \)-Lie) derivations, and revealing the relationship between such additive maps to derivations (generalized derivations) on prime algebras. As every (generalized) \( ξ \)-Lie derivation is a (generalized) derivation if \( ξ = 0 \), we need only consider the case that \( ξ \neq 0 \).

Let us recall some notions and notations. Throughout this paper, \( \mathcal{A} \) denotes a prime algebra over a field \( \mathbb{F} \) (i.e. \( \mathcal{A}AB = 0 \) implies \( A = 0 \) or \( B = 0 \) for any \( A, B \in \mathcal{A} \)) with the center \( \mathcal{Z}(\mathcal{A}) \) and maximal right ring of quotients \( \mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A}) \). The center \( \mathcal{C} \) of \( \mathcal{Q} \) is a field which is called the extended centroid of \( \mathcal{A} \). The central closure \( \mathcal{AC} \) of \( \mathcal{A} \) is the \( \mathcal{C} \)-subalgebra of \( \mathcal{Q} \) generated by \( \mathcal{A} \). An element \( A \in \mathcal{A} \) is algebraic over \( \mathcal{Z}(\mathcal{A}) \), if there exists a polynomial \( p \in \mathcal{P}(\mathcal{Z}(\mathcal{A})) \) such that \( p(A) = 0 \), that is, there exist \( Z_0, Z_1, \cdots, Z_n \in \mathcal{Z}(\mathcal{A}) \) such that \( Z_n \neq 0 \) and \( p(A) = Z_0 + Z_1A + \cdots + Z_nA^n = 0 \). In this case \( n = \deg(p) \) is called the degree of \( p \), and \( \min\{\deg(p) : p(A) = 0\} \) is called the degree of algebraicity of \( A \) over \( \mathcal{Z}(\mathcal{A}) \), denoted by \( \deg(A) \). If \( A \) is not algebraic over \( \mathcal{Z}(\mathcal{A}) \), then we write \( \deg(A) = \infty \). The degree of algebraicity of \( \mathcal{A} \) is defined as \( \deg(\mathcal{A}) = \sup\{\deg(A) : A \in \mathcal{A}\} \) (Ref. [2] for details).

This paper is organized as follows. Let \( \mathcal{A} \) be a prime algebra over a field \( \mathbb{F} \). Assume that \( ξ \in \mathbb{F} \) is a nonzero scalar and \( L : \mathcal{A} \rightarrow \mathcal{A} \) is an additive map. It is known that, if \( \deg(A) \geq 3 \), then \( L \) is an additive Lie derivation if and only if it is the sum of an additive derivation and an additive map into its centroid vanishing each commutator [1]; when \( \mathbb{F} \) is of characteristic not 2, then \( L \) is a Jordan derivation if and only if \( L \) is an additive derivation [5]. In Section 2, we show that, when \( \mathbb{F} \) is of characteristic not 2 and \( \mathcal{A} \) is unital containing a nontrivial idempotent \( P \), then \( L \) is a \( ξ \)-Lie derivation with \( ξ \neq ±1 \) if and only if \( L \) is an additive derivation satisfying \( L(ξA) = ξL(A) \) for all \( A \in \mathcal{A} \) (Theorem 2.1). This result then is used to give a characterization of additive \( ξ \)-Lie derivations on factor von Neumann algebras (Theorem 2.2). For Banach space standard operator algebras, a little more can be said. Let \( \mathcal{A} \) be a standard operator algebra in \( \mathcal{B}(X) \), i.e., \( \mathcal{A} \) contains all finite rank operators (note that, we do not require that \( \mathcal{A} \) contains the unit \( I \) and is closed under norm topology), where \( \mathcal{B}(X) \) is the Banach algebra of all bounded linear operators acting on \( X \). Let \( L : \mathcal{A} \rightarrow \mathcal{B}(X) \) be an additive map. We obtain that, if \( \text{dim } X \geq 3 \), then \( L \) is an additive Lie derivation if
and only if $L$ is the sum of an additive derivation on $A$ and an additive map from $A$ into $\mathbb{F}I$ annihilating each commutator; $L$ is an additive $\xi$-Lie derivation with $\xi \neq 1$ if and only if $L$ is an additive derivation satisfying $L(\xi A) = \xi L(A)$ (Theorem 2.3).

Section 3 is devoted to characterizing the generalized $\xi$-Lie derivations. Assume that $A$ is unital and $\delta : A \to A$ is an additive map. We show that, if $\deg A \geq 3$, then $\delta$ is a generalized Lie derivation if and only if $\delta$ is the sum of an additive generalized derivation on $A$ and an additive map from $A$ into its center annihilating all commutators; if $\mathbb{F}$ is of characteristic not 2, then $\delta$ is a generalized Jordan derivation if and only if $\delta$ is an additive generalized derivation; if $\mathbb{F}$ is of characteristic not 2 and $A$ contains a nontrivial idempotent $P$, then $\delta$ is a generalized $\xi$-Lie derivation with $\xi \neq \pm 1$ if and only if $\delta$ is an additive generalized derivation satisfying $\delta(\xi A) = \xi \delta(A)$ for all $A \in A$ (Theorem 3.1). As an application, a characterization of additive generalized $\xi$-Lie derivations on factor von Neumman algebras and Banach space standard operator algebras is obtained (Theorem 3.2 and Theorem 3.3).

2. Additive Lie and $\xi$-Lie derivations

In this section, we consider the question of characterizing the additive Lie and $\xi$-Lie derivations on prime algebras. It is obvious that if an additive map $L$ on an algebra $A$ is the sum of an additive derivation and an additive map from $A$ into its center vanishing the commutators, then $L$ is a Lie derivation. Also, it is clear that, for $\xi \neq 1$, every additive derivation $L$ satisfying $L(\xi A) = \xi L(A)$ is a $\xi$-Lie derivation. Our main purpose in this section is to show that the inverses of these facts are true under some weak assumptions.

The following is the main result.

**Theorem 2.1.** Let $A$ be a prime algebra over a field $\mathbb{F}$. Assume that $\xi \in \mathbb{F}$ is a nonzero scalar and $L : A \to A$ is an additive $\xi$-Lie derivation.

1. If $\xi = 1$, that is, if $L$ is a Lie derivation, and if $\deg A \geq 3$, then $L(A) = \tau(A) + h(A)$ for all $A \in A$, where $\tau : A \to AC$ (the central closure of $A$) is an additive derivation and $h : A \to C$ (the extended centroid of $A$) is an additive map vanishing each commutator.

2. If $\xi = -1$, that is, if $L$ is a Jordan derivation, and if $\mathbb{F}$ is of characteristic not 2, then $L$ is an additive derivation.

3. If $\xi \neq \pm 1$, $\mathbb{F}$ is of characteristic not 2, $A$ is unital and contains a nontrivial idempotent $P$, then $L$ is an additive derivation and satisfies $L(\xi A) = \xi L(A)$ for all $A \in A$.

**Proof.** By [1], the statement (1) is true; by [5], the statement (2) is true.

We’ll prove the statement (3) by checking several claims. In the sequel, we always assume that $L : A \to A$ is an additive $\xi$-Lie derivation with $\xi \neq \pm 1$.

Let $A_{11} = PAP$, $A_{12} = PA(I - P)$, $A_{21} = (I - P)AP$ and $A_{22} = (I - P)A(I - P)$. It is clear that $A = A_{11} + A_{12} + A_{21} + A_{22}$. 
Claim 1. \( L(P) = PL(P) + (I - P)L(P)P \) and \( L(I - P) = -PL(P)(I - P) - (I - P)L(P)P + (I - P)L(I - P)(I - P) \).

Since
\[
0 = L([P, I - P]_\xi) = [L(P), I - P]_\xi + [P, L(I - P)]_\xi = L(P)(I - P) - \xi(I - P)L(P) + PL(I - P) - \xi L(I - P)P,
\]
multiplying by \( I - P \) from both sides in Eq.(2.1), we get \( (I - P)L(P)(I - P) - \xi (I - P)L(P)(I - P) = 0 \), that is, \( (1 - \xi)(I - P)L(P)(I - P) = 0 \). Note that \( \xi \neq 1 \). It follows that \( (I - P)L(P)(I - P) = 0 \). Hence \( L(P) = PL(P)P + PL(P)(I - P) + (I - P)L(P)P = PL(P) + (I - P)L(P)P \).

By Eq.(2.1), we have
\[
0 = PL(P)(I - P) - \xi(I - P)L(P)P + PL(I - P) - (1 - \xi)PL(I - P)P - \xi(I - P)L(I - P)P - \xi((I - P)L(P)P + (I - P)L(I - P)P).
\]
Since \( \xi \neq 0, 1 \), we get \( PL(I - P)P = 0, PL(I - P)(I - P) = -PL(P)(I - P) \) and \( (I - P)L(I - P)P = -(I - P)L(P)P \). So
\[
L(I - P) = PL(I - P)(I - P) + (I - P)L(I - P)P + (I - P)L(I - P)(I - P) = -PL(P)(I - P) - (I - P)L(P)P + (I - P)L(I - P)(I - P).
\]

Claim 2. \( L(I) = 0 \).

Define a map \( L' : A \to A \) by
\[
L'(A) = L(A) - [A, PL(P)(I - P) - (I - P)L(P)P] \quad \text{for all} \quad A \in A.
\]

By Claim 1, it is easy to check that \( L' \) is also an additive \( \xi \)-Lie derivation and satisfies that
\[
L'(P) = PL(P)P \in A_{11} \quad \text{and} \quad L'(I - P) = (I - P)L(I - P)(I - P) \in A_{22}. \tag{2.2}
\]

For any \( A_{12} \in A_{12} \), by Eq.(2.2), we have \( A_{12}L'(P) = 0 \) and \( L'(I - P)A_{12} = 0 \). Since
\[
L'(A_{12}) = L'([P, A_{12}]_\xi) = L'(P)A_{12} - \xi A_{12}L'(P) + PL'(A_{12}) - \xi L'(A_{12})P, \tag{2.3}
\]
multiplying by \( (I - P) \) from the right side in Eq.(2.3), we get
\[
L'(A_{12})(I - P) = L'(P)A_{12}(I - P) + PL'(A_{12})(I - P) = L'(P)A_{12} + PL'(A_{12})(I - P).
\]

Multiplying by \( P \) from the left side in the above equation, we have \( L'(P)A_{12} = 0 \). Hence we have proved that
\[
A_{12}L'(P) = L'(P)A_{12} = 0. \tag{2.4}
\]

Similarly, by using of the relation \( L'(I - P) \in A_{22} \), one can show that
\[
L'(I - P)A_{12} = A_{12}L'(I - P) = 0. \tag{2.5}
\]
Combining Eq.(2.4) with (2.5), we obtain \[ L'(I)A_{12} = A_{12}L'(I) = 0. \] Since \( \mathcal{A} \) is prime, it follows that
\[
L'(I)P = 0 \quad \text{and} \quad (I - P)L'(I) = 0. \tag{2.6}
\]

Now for any \( A \in \mathcal{A}, \) since \( \xi \neq 1 \) and \( 0 = L'([A, I]_\xi) - L'([I, A]_\xi) = (1 - \xi)[A, L'(I)] \), we get \( L'(I) \in Z(\mathcal{A}). \) Thus, by Eq.(2.6), we have \( L'(I) = 0, \) and so \( L(I) = 0. \) Complete the proof of the claim.

**Claim 3.** For any \( A \in \mathcal{A}, \) we have \( L(\xi A) = \xi L(A) \) and \( L \) is an additive derivation.

For any \( A \in \mathcal{A}, \) by the definition of \( L, \) we have
\[
L((1 - \xi)A) = L([I, A]_\xi) = L(I)A - \xi AL(I) + L(A) - \xi L(A),
\]
that is,
\[
-L(\xi A) = L(I)A - \xi AL(I) - \xi L(A).
\]

This and Claim 2 yield to
\[
L(\xi A) = \xi L(A). \tag{2.7}
\]

Now take any \( A, B \in \mathcal{A}. \) Note that \( (1 - \xi)[A, B]_{-1} = [A, B]_\xi + [B, A]_\xi \) and \( \xi \neq 1. \) Then, by Eq.(2.7), we have
\[
L((1 - \xi)[A, B]_{-1}) = L([A, B]_\xi) + L([A, B]_\xi)
\]
\[
= L(A)B - \xi BL(A) + AL(B) - \xi L(B)A + L(B)A - \xi AL(B) + BL(A) - \xi L(A)B
\]
\[
= (1 - \xi)(L(A)B + AL(B) + L(B)A + BL(A),
\]
that is,
\[
L(AB + BA) = L(A)B + AL(B) + L(B)A + BL(A).
\]

Hence \( L \) is an additive Jordan derivation from \( \mathcal{A} \) into itself. By statement (2), \( L \) is an additive derivation, completing the proof of the theorem. \( \square \)

As an application of Theorem 2.1 to the factor von Neumann algebras case, we have

**Theorem 2.2.** Let \( \mathcal{M} \) be a factor von Neumann algebra and \( \xi \in \mathbb{C} \) a nonzero scalar. Assume that \( L : \mathcal{M} \to \mathcal{M} \) is an additive \( \xi \)-Lie derivation.

1. If \( \xi = 1 \) and \( \deg \mathcal{M} \geq 3, \) then there exist an additive derivation \( \tau \) on \( \mathcal{M} \) and an additive functional \( h : \mathcal{M} \to \mathbb{C} \) vanishing on each commutator such that \( L(A) = \tau(A) + h(A)I \) for all \( A \in \mathcal{M}. \)

2. If \( \xi \neq 1, \) then \( L \) is an additive derivation and satisfies that \( L(\xi A) = \xi L(A) \) for all \( A \in \mathcal{M}. \)

Recall that a subalgebra \( \mathcal{A} \subseteq \mathcal{B}(X) \) is called a standard operator algebra if it contains all finite rank operators of \( \mathcal{B}(X). \) Note that \( \mathcal{A} \) may not contain the unit operator \( I \) and Theorem 2.1 can not be applied. For the standard operator algebra \( \mathcal{A}, \) we have the following result.
Multiplying \( I \) in Eq.(2.9), we get \((I A F) \equiv \{A \}_{P} \not\subseteq F\) since \(\{A \}_{P} \not\subseteq A\). Thus if \(P\) is idempotent, \(A \) is in fact inner, that is, there exists an operator \(f: A \rightarrow F\) such that \(f(A) = TA - AT\) for all \(A \in A\); if \(X\) is finite dimensional, then \(\forall L: A \rightarrow B(X)\) is an additive derivation and satisfies \(L(\xi A) = \xi L(A)\) for all \(A \in A\).

We remark that, if \(X\) is infinite dimensional, then, by [11], every additive derivation \(\tau\) on \(A\) is in fact inner, that is, there exists an operator \(T \in B(X)\) such that \(\tau(A) = TA - AT\) for all \(A \in A\); if \(X\) is finite dimensional, then every additive derivation \(\tau\) on \(M_n(F)\) has the form \(\tau(A) = TA - AT + (f(\alpha_{ij}))_{n \times n}\) for all \(A = (\alpha_{ij})_{n \times n} \in M_n(F)\), where \(T \in M_n(F)\) and \(f: F \rightarrow F\) is an additive derivation.

**Proof of Theorem 2.3.** By Theorem 2.1(1), the statements (1) and (2) are true.

We'll complete the proof of the statement (3) by checking several claims. Fix a nontrivial idempotent \(P \in A\). In the sequel, as a notational convenience, we denote \(A_{11} = PAP\), \(A_{12} = \{PA - PAP: A \in A\}\), \(A_{21} = \{AP - PAP: A \in A\}\) and \(A_{22} = \{A - AP - PA + PAP: A \in A\}\). Thus \(A = A_{11} \uparrow A_{12} \uparrow A_{21} \uparrow A_{22}\). Similarly, write \(B(X) = B_{11} \uparrow B_{12} \uparrow B_{21} \uparrow B_{22}\). Assume that \(\xi \neq 1\) and \(L: A \rightarrow B(X)\) is an additive \(\xi\)-Lie derivation.

**Claim 1.** \(PL(P)P = (I - P)L(P)(I - P) = 0\).

For any \(A_{22} \in A_{22}\), by the definition of \(L\), we have

\[
0 = L([P, A_{22}]_{\xi}) = L(P)A_{22} - \xi A_{22}L(P) + PL(A_{22}) - \xi L(A_{22})P. \tag{2.8}
\]

Multiplying \(I - P\) from the both sides of Eq.(2.8), we get

\[
(I - P)L(P)A_{22} = \xi A_{22}L(P)(I - P) \quad \text{for all } A_{22} \in A_{22}. \tag{2.9}
\]

Since \(\mathcal{F}(X) \subseteq A\) is dense in \(B(X)\) under the strong operator topology, there exists a net \(\{A_{\alpha}\} \subset \mathcal{F}(X)\) such that \(\text{SOT-lim}_{\alpha} A_{\alpha} = I\). Note that \(A_{\alpha} - PA_{\alpha} - A_{\alpha}P + PA_{\alpha}P \in A_{22}\) and \(A_{\alpha} - PA_{\alpha} - A_{\alpha}P + PA_{\alpha}P \rightarrow I - E\) strongly. Replacing \(A_{22}\) by \(A_{\alpha} - PA_{\alpha} - A_{\alpha}P + PA_{\alpha}P\) in Eq.(2.9), we get \((I - P)L(P)(I - P) = 0\) since \(\xi \neq 1\).

For any \(A_{12} \in A_{12}\), we have

\[
L(A_{12}) = L(PA_{12} - \xi A_{12}P) = L(P)A_{12} - \xi A_{12}L(P) + PL(A_{12}) - \xi L(A_{12})P.
\]

Multiplying \(I - P\) from the right side of the above equation, we get

\[
L(A_{12})(I - P) = L(P)A_{12}(I - P) - \xi A_{12}L(P)(I - P) + PL(A_{12})(I - P),
\]
that is,
\[(I - P)L(A_{12})(I - P) = L(P)A_{12} - \xi A_{12}(I - P)L(P)(I - P)
= PL(P)A_{12} + (I - P)L(P)A_{12}.
\]
This implies that \(PL(P)A_{12} = 0\). Since \(A\) is prime, it follows that \(PL(P)P = 0\), completing the proof of the claim.

Now, define a map \(L' : A \to B(X)\) by
\[L'(A) = L(A) - [A, PL(P)(I - P) - (I - P)L(P)P] \quad \text{for all } A \in A.
\]
By Claim 1, it is easy to check that \(L'\) is also an additive \(\xi\)-Lie derivation and satisfies that \(L'(P) = 0\).

The following we’ll prove that \(L'\) is an additive derivation, and so \(L\) is an additive derivation, as desired.

**Claim 2.** \(L'(A_{ii}) \subseteq B_{ii}, i = 1, 2.\)

For any \(A_{22} \in A_{22}\), we have
\[0 = L'([P, A_{22}]\xi) = L'(P)A_{22} - \xi A_{22}L'(P) + PL'(A_{22}) - \xi L'(A_{22})P
= PL'(A_{22}) - \xi L'(A_{22})P.
\]
That is,
\[PL'(A_{22})P + PL'(A_{22})(I - P) - \xi PL'(A_{22})P - \xi(I - P)L'(A_{22})P = 0.
\]
Note that \(\xi \neq 1\). It follows that \(PL'(A_{22})P = PL'(A_{22})(I - P) = (I - P)L'(A_{22})P = 0\), and so \(L'(A_{22}) \in B_{22}\).

Taking any \(A_{11} \in A_{11}\) and \(A_{22} \in A_{22}\), we have
\[0 = L'([A_{11}, A_{22}]\xi) = L'(A_{11})A_{22} - \xi A_{22}L'(A_{11}) + A_{11}L'(A_{22}) - \xi L'(A_{22})A_{11}
= L'(A_{11})A_{22} - \xi A_{22}L'(A_{11})
= PL'(A_{11})(I - P)A_{22} + (I - P)L'(A_{11})(I - P)A_{22}
- \xi A_{22}(I - P)L'(A_{11})P - \xi A_{22}(I - P)L'(A_{11})(I - P).
\]
This implies that
\[PL'(A_{11})(I - P)A_{22} = 0, \quad \xi A_{22}(I - P)L'(A_{11})P = 0 \quad (2.10)
\]
and
\[(I - P)L'(A_{11})(I - P)A_{22} = \xi A_{22}(I - P)L'(A_{11})(I - P). \quad (2.11)
\]
Since \(F(X) \subseteq A\) is dense in \(B(X)\) under the strong operator topology, there exists a net \(\{A_\alpha\} \in F(X)\) such that SOT-lim_\alpha A_\alpha = I. Note that \(A_\alpha - PA_\alpha - A_\alpha P + PA_\alpha P \in A_{22}\) and \(A_\alpha - PA_\alpha - A_\alpha P + PA_\alpha P \to I - E\) strongly. Replacing \(A_{22}\) by \(A_\alpha - PA_\alpha - A_\alpha P + PA_\alpha P\) in Eqs.\((2.10)\)-\((2.11)\), we get \(PL'(A_{11})(I - P) = (I - P)L'(A_{11})P = (I - P)L'(A_{11})(I - P) = 0\) since \(\xi \neq 0, 1\). Hence \(L'(A_{11}) \in B_{11}\), completing the proof of the claim.

**Claim 3.** \(L'(A_{ij}) \subseteq B_{ij}, 1 \leq i \neq j \leq 2.\)
For any $A_{12} \in \mathcal{A}_{12}$, noting that $L'(P) = 0$, we have

$$L'(A_{12}) = L'(PA_{12} - \xi A_{12}P)$$

$$= L'(P)A_{12} - \xi A_{12}L'(P) + PL'(A_{12}) - \xi L'(A_{12})P$$

$$= PL'(A_{12}) - \xi L'(A_{12})P.$$  \hfill (2.12)

Multiplying $P$ from both sides of the above equation, we get $\xi PL'(A_{12})P = 0$, which implies that $PL'(A_{12})P = 0$. Similarly, multiplying $I - P$ from the left side of Eq.(2.12) leads to

$$(I - P)L'(A_{12}) = -\xi (I - P)L'(A_{12})P.$$ \hfill (2.13)

Multiplying $I - P$ from the right side of Eq.(2.13), we get $(I - P)L'(A_{12})(I - P) = 0$. Multiplying $P$ from the right side of Eq.(2.13), we get $(1 + \xi)(I - P)L'(A_{12})P = 0$, which implies that $(I - P)L'(A_{12})P = 0$ since $\xi \neq -1$.

Similarly, for any $A_{21} \in \mathcal{A}_{21}$, by using of the equation $L'(A_{21}) = L'([A_{21}, P]_{\xi})$, one can check that $PL'(A_{21})P = 0$, $(I - P)L'(A_{21})(I - P) = 0$ and $PL'(A_{21})(I - P) = 0$.

Thus we obtain $L'(A_{ij}) \in \mathcal{B}_{ij}$ with $i \neq j$.

**Claim 4.** $L'$ has the following properties:

(a) $L'(A_{ii}B_{ij}) = L'(A_{ii})B_{ij} + A_{ii}L'(B_{ij})$ holds for all $A_{ii} \in \mathcal{A}_{ii}$ and $B_{ij} \in \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$.

(b) $L'(A_{ij}B_{jj}) = L'(A_{ij})B_{jj} + A_{ij}L'(B_{jj})$ holds for all $A_{ij} \in \mathcal{A}_{ij}$ and $B_{jj} \in \mathcal{A}_{jj}$, $1 \leq i \neq j \leq 2$.

(c) $L'(A_{ij}B_{ji}) = L'(A_{ij})B_{ji} + A_{ij}L'(B_{ji})$ holds for all $A_{ij} \in \mathcal{A}_{ij}$ and $B_{ji} \in \mathcal{A}_{ji}$, $1 \leq i \neq j \leq 2$.

(d) $L'(A_{ii}B_{ii}) = L'(A_{ii})B_{ii} + A_{ii}L'(B_{ii})$ holds for all $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$.

For any $A_{ii} \in \mathcal{A}_{ii}$ and $B_{ij} \in \mathcal{A}_{ij}$, it follows from Claims 2-3 that

$$L'(A_{ii}B_{ij}) = L'([A_{ii}, B_{ij}]_{\xi})$$

$$= L'(A_{ii})B_{ij} - \xi B_{ij}L'(A_{ii}) + A_{ii}L'(B_{ij}) - \xi L'(B_{ij})A_{ii}$$

$$= L'(A_{ii})B_{ij} + A_{ii}L'(B_{ij}),$$

and so (a) holds true.

Similarly, (b) is true for all $A_{ij} \in \mathcal{A}_{ij}$ and $B_{jj} \in \mathcal{A}_{jj}$.

For any $A_{ij} \in \mathcal{A}_{ij}$ and $B_{ji} \in \mathcal{A}_{ji}$, by Claim 3 and the additivity of $L'$, we get

$$L'(A_{ij}B_{ji}) - L'(\xi B_{ji}A_{ij}) = L'([A_{ij}, B_{ji}]_{\xi})$$

$$= L'(A_{ij})B_{ji} - \xi B_{ji}L'(A_{ij}) + A_{ij}L'(B_{ji}) - \xi L'(B_{ji})A_{ij}$$

$$= (L'(A_{ij})B_{ji} + A_{ij}L'(B_{ji})) - \xi (B_{ji}L'(A_{ij}) + L'(B_{ji})A_{ij}).$$

Note that $L'(A_{ij}B_{ji}), L'(A_{ij})B_{ji} + A_{ij}L'(B_{ji}) \in \mathcal{B}_{ii}$ and $L'(\xi B_{ji}A_{ij}), B_{ji}L'(A_{ij}) + L'(B_{ji})A_{ij} \in \mathcal{B}_{jj}$. It follows that $L'(A_{ij}B_{ji}) = L'(A_{ij})B_{ji} + A_{ij}L'(B_{ji})$, and so (c) is true.
For any $A_{ii}, B_{ii} \in A_{ii}$ and any $C_{ij} \in A_{ij}$, by (a), we have

\[
L'(A_{ii}B_{ii}C_{ij}) = L'(A_{ii})B_{ii}C_{ij} + A_{ii}L'(B_{ii})C_{ij}
\]

and

\[
L'(A_{ii}B_{ii}C_{ij}) = L'(A_{ii}B_{ii})C_{ij} + A_{ii}B_{ii}L'(C_{ij}).
\]

Comparing the above two equations gives

\[
(L'(A_{ii}B_{ii}) - L'(A_{ii})B_{ii} - A_{ii}L'(B_{ii}))C_{ij} = 0
\]

for all $C_{ij} \in A_{ij}$. Since $\mathcal{A}$ is prime, it follows that $L'(A_{ii}B_{ii}) - L'(A_{ii})B_{ii} - A_{ii}L'(B_{ii}) = 0$, that is, (d) holds true.

**Claim 5.** $L'$ is an additive derivation, and therefore, $L$ is an additive derivation and satisfies $L(\xi A) = \xi L(A)$ for all $A \in \mathcal{A}$.

For any $A, B \in \mathcal{A}$, write $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$. By Claim 4 and the additivity of $L'$, it is easily checked that $L'(AB) = L'(A)B + AL'(B)$, that is, $L'$ is an inner derivation on $\mathcal{A}$. Note that the map $A \mapsto [A, PL(P)(I - P) - (I - P)L(P)P]$ is an additive derivation of $\mathcal{A}$. So $L$ is also an additive derivation.

Finally, for any $A, B \in \mathcal{A}$, we have

\[
[L(A), B]_{\xi} + [A, L(B)]_{\xi} = L([A, B]_{\xi}) = L(AB) - L(\xi BA)
\]

\[
= L(A)B + AL(B) - L(B)(\xi A) - BL(\xi A),
\]

which implies that

\[
BL(\xi A) = \xi BL(A)
\]  \hspace{1cm} (2.14)

holds for all $A, B \in \mathcal{A}$. Taking a net $\{B_{\alpha}\}$ in $\mathcal{A}$ such that $B_{\alpha} \to I$ strongly, and replacing $B$ by $B_{\alpha}$ in Eq.(2.14), we obtain $L(\xi A) = \xi L(A)$. This completes the proof of the statement (3) in Theorem 2.3.

3. ADDITIVE GENERALIZED LIE AND $\xi$-LIE DERIVATIONS

In this section, we discuss the question of characterizing the additive generalized Lie derivations and generalized $\xi$-Lie derivations. It is obvious that, for $\xi \neq 1$, every additive generalized derivation $\delta$ satisfying $\delta(\xi A) = \xi \delta(A)$ is an additive generalized $\xi$-Lie derivation; and the sum of an additive generalized derivation and an additive map into the center vanishing all commutators is an additive generalized Lie derivation. We show that the inverses of above facts are true for most prime algebras.

The following is the main result in this section.

**Theorem 3.1.** Let $\mathcal{A}$ be a unital prime algebra over a field $F$ and $\xi \in F$ with $\xi \neq 0$. Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is an additive generalized $\xi$-Lie derivation with $L : \mathcal{A} \to \mathcal{A}$ the relating $\xi$-Lie derivation.
(1) If $\xi = 1$, that is, if $\delta$ is a generalized Lie derivation, and if $\text{deg} A \geq 3$, then $\delta(A) = \delta'(A) + h(A)$ for all $A \in \mathcal{A}$, where $\delta' : \mathcal{A} \to \mathcal{A}C$ is an additive generalized derivation and $h : \mathcal{A} \to C$ is an additive map vanishing each commutator.

(2) If $\xi = -1$, that is, if $\delta$ is a generalized Jordan derivation, and if $\mathbb{F}$ is of characteristic not 2, then $\delta$ is an additive generalized derivation.

(3) If $\xi \neq \pm 1$, $\mathbb{F}$ is of characteristic not 2, and if $\mathcal{A}$ contains a nontrivial idempotent, then $\delta$ is an additive generalized derivation and $\delta(\xi A) = \xi \delta(A)$ for all $A \in \mathcal{A}$.

**Proof.** Since $\delta : \mathcal{A} \to \mathcal{A}$ is an additive generalized $\xi$-Lie derivation with $L : \mathcal{A} \to \mathcal{A}$ the relating $\xi$-Lie derivation, we have

$$
\delta([A, B]_\xi) = \delta(A)B - \xi \delta(B)A + AL(B) - \xi BL(A)
$$

for all $A, B \in \mathcal{A}$. Taking $B = I$ in the above equation, we get

$$
\delta(-\xi A) = -\xi L(A) - \xi \delta(I)A + AL(I) - \xi L(I),
$$

that is,

$$
\delta(-\xi A) = -\xi L(A) - \xi \delta(I)A + AL(I) \quad \text{for all } A \in \mathcal{A}. \tag{3.1}
$$

If $\xi = 1$, then Eq.(3.1) becomes $\delta(A) = L(A) + \delta(I)A + AL(I)$ for all $A \in \mathcal{A}$. By [II], $L$ has the form of $L(A) = \tau(A) + h(A)$, where $\tau$ is an additive derivation of $\mathcal{A}$ and $h : \mathcal{A} \to C$ is an additive map satisfying $h([A, B]) = 0$ for all $A$ and $B$. Define $\delta' : \mathcal{A} \to \mathcal{A}$ by $\delta'(A) = \tau(A) + \delta(I)A + AL(I)$ for all $A \in \mathcal{A}$. Thus we get $\delta(A) = \delta'(A) + h(A)$. It is easily seen that $\delta'$ is an additive generalized derivation. Hence the statement (1) of Theorem 3.1 holds true.

If $\xi \neq 1$, then, substituting $A$ by $-\xi^{-1}A$ in Eq.(3.1), one gets

$$
\delta(A) = -\xi L(-\xi^{-1}A) + \delta(I)A - \xi^{-1}AL(I) \quad \text{for all } A \in \mathcal{A}. \tag{3.2}
$$

for all $A \in \mathcal{A}$. Since $L$ is an additive $\xi$-Lie derivation, by Theorem 2.1(2) and (3), we see that $L$ is an additive derivation satisfying $L(\xi A) = \xi L(A)$ for all $A$. It follows from Eq.(3.2) that $\delta(A) = L(A) + \delta(I)A - \xi^{-1}AL(I)$, which is a generalized derivation. Furthermore, $\delta(\xi A) = L(\xi A) + \delta(I)\xi A - \xi^{-1}\xi AL(I) = \xi L(A) + \delta(I)\xi A - \xi^{-1}\xi AL(I) = \xi \delta(A)$. Hence, the statement (2) of Theorem 3.1 is true. \qed

For the von Neumann algebra case, we have

**Theorem 3.2.** Let $\mathcal{M}$ be a factor von Neumann algebra and $\xi \in \mathbb{C}$ a nonzero scalar. Assume that $\delta : \mathcal{M} \to \mathcal{M}$ be an additive generalized $\xi$-Lie derivation.

(1) If $\xi = 1$ and $\text{deg} \mathcal{M} \geq 3$, then there exist an additive generalized derivation $\tau$ on $\mathcal{M}$ and an additive functional $h : \mathcal{M} \to \mathbb{C}$ vanishing on each commutator such that $\delta(A) = \tau(A) + h(A)I$ for all $A \in \mathcal{M}$.

(2) If $\xi \neq 1$, then $\delta$ is an additive generalized derivation and $\delta(\xi A) = \xi \delta(A)$ for all $A \in \mathcal{M}$.

For Banach space standard operator algebras, we have
Theorem 3.3. Let $X$ be a Banach space over the real or complex field $\mathbb{F}$ and $\mathcal{A}$ a standard operator subalgebra of $\mathcal{B}(X)$ containing the identity $I$. Assume that $\xi \in \mathbb{F}$ with $\xi \neq 0$ and $\delta: \mathcal{A} \to \mathcal{B}(X)$ is an additive generalized $\xi$-Lie derivation.

(1) If $\xi = 1$ and $\dim X \geq 3$, then $\delta(A) = \tau(A) + h(A)I$ for all $A \in \mathcal{A}$, where $\tau: \mathcal{A} \to \mathcal{B}(X)$ is an additive generalized derivation and $h: \mathcal{A} \to \mathbb{F}$ is an additive map vanishing all commutators.

(2) If $\xi \neq 1$, then $\delta$ is an additive generalized derivation and satisfies $\delta(\xi A) = \xi \delta(A)$ for all $A \in \mathcal{A}$.

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