POSITIVE KERNELS AND QUANTIZATION

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Abstract
In the paper we investigate a method of quantization based on the concept of positive definite kernel on a principal $G$-bundle with compact structural group $G$. For $G = U(1)$ our approach leads to Kostant–Souriau geometric quantization as well as to coherent state method of quantization. So, the theory proposed here can be treated as a generalization of both mentioned quantizations to the case of general compact group.

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1 Introduction

The geometric quantization initiated in \[\text{Kos, Kir, Sou}\] provides an effective machinery for quantization of Hamiltonian systems. In fact the main ingredient of Kostant-Souriau theory is a principal \(U(1)\)-bundle \(\pi : P \rightarrow M\) over a symplectic manifold \((M, \omega)\) with connection form \(\vartheta\) which satisfies the consistency condition \(\pi^*\omega = icurvature(\vartheta)\) with the symplectic form \(\omega\).

The other essential element of this theory is a complex distribution \(P \subset T^C M\) which is maximal and isotropic with respect to \(\omega\), called the polarization. The crucial step for quantization of a physical classical observable \(f \in \mathcal{C}^\infty(M)\) is the proper choice of polarization \(P\). Then one constructs for \(f\) a selfadjoint operator \(\hat{F}\) (quantum observable) which acts in the Hilbert space consisting of such sections of the corresponding prequantum bundle which are annihilated by \(P\), e.g. see \[\text{Sn}\].

The coherent state method of quantization is based on the concept of coherent state map, i.e. a symplectic map \(K\) of the phase space \((M, \omega)\) into the quantum phase space of pure states \((\mathbb{C}P(H), \omega_{FS})\) which is the complex projective Hilbert space with Fubini–Study form \(\omega_{FS}\) as the symplectic form. The Kähler form \(\omega_{FS}\) is the curvature form of the connection form \(\vartheta_{FS}\) defined canonically on the tautological principal \(U(1)\)-bundle \(\pi : P_{FS} \rightarrow \mathbb{C}P(H)\) by the metric and complex structure of \(H\).

Moreover the scalar product in Hilbert space \(\mathcal{H}\) also defines the positive definite kernel \(K_{FS} : P_{FS} \times P_{FS} \rightarrow \mathbb{C}\), which after normalization has a physical interpretation as the transition amplitude between two pure states. The canonical prequantum \(U(1)\)-bundle \((\pi : P_{FS} \rightarrow \mathbb{C}P(H), \vartheta_{FS}, \omega_{FS})\) as well as \((\pi : P_{FS} \rightarrow \mathbb{C}P(H), K_{FS})\) are the universal objects in the category of all prequantum \(U(1)\)-bundles and in the category of principal bundles with fixed positive definite kernels \((\pi : P \rightarrow M, K)\), respectively. Between these categories there is functorial dependence, see \[\text{O92}\], i.e. any prequantum bundle \((\pi : P \rightarrow M, \vartheta, \omega)\) is obtained from \((\pi : P \rightarrow M, K)\) for some properly chosen kernel \(K\). In \[\text{O92}\] also it is shown that one can quantize those Hamiltonian flows on \((M, \omega)\) which preserve the kernel \(K\).

Motivated by the fundamental role of positive definite kernels in the
geometry of the prequantum bundles as well as their physical interpretation as transition amplitudes we investigate here method of quantization entirely based on the notion of this type of kernels in the case of general compact structural group $G$ (for $G = U(1)$ see [O92]). In fact we quantize the one-parameter groups $\{\tau_t\}_{t \in \mathbb{R}}$ of automorphisms of the principal $G$-bundles with fixed positive definite kernels $(\pi : P \rightarrow M, K)$. The projection $\sigma_t(\pi(p)) := \pi(\tau_t(p)), p \in P$, of the flow $\{\tau_t\}_{t \in \mathbb{R}}$, on $M$ is described by the equation (3.26) which is a version of the Hamiltonian equation for the flow $\{\sigma_t\}_{t \in \mathbb{R}}$. This equation connect the vector field $X$ tangent to $\{\sigma_t\}_{t \in \mathbb{R}}$ (Hamiltonian vector field) with generating function $F : P \rightarrow \mathcal{B}(V)$ which is $G$-equivariant and operator valued (Hamiltonian). See (3.26) and Proposition 3.3.

This paper is organized as follows. In Section 2 we give a short outline of the theory of positive definite kernels. Especially we investigate these kernels on the principal G-bundles describing their relationship with the notion of connection.

In Section 3 and Section 4 we show that for the pair $(F, X)$, satisfying (3.26), one can construct the $G$-version of Kostant–Souriau operator $Q_{(F, X)}$, see (3.49) and (3.53) or (3.54) in non-singular case. The differential operator $Q_{(F, X)}$ can be extended to a self-adjoint operator in Hilbert space completely defined by the positive definite kernel $K : P \times P \rightarrow \mathcal{B}(V)$ (see Theorem 4.1 and Proposition 4.1).

Finally, we present two simple examples illustrating the method of quantization proposed here. One can find other examples important for physical applications in [O-´S, H-O, H-O-T].

2 Positive definite kernels on principal bundles

We begin this section with a short presentation of the theory of operator-valued positive definite kernels. A more exhaustive treatment can be found for example in Chapter I of [N].

Let us take a set $P$ and complex Hilbert spaces $V$ and $\mathcal{H}$. By $\mathcal{B}(V, \mathcal{H})$ we denote the Banach space of bounded linear operators from $V$ into $\mathcal{H}$. By $\mathcal{B}(V)$ we denote $\mathcal{B}(V, V)$. For adjoint of $A \in \mathcal{B}(V, \mathcal{H})$ we will write $A^* \in \mathcal{B}(\mathcal{H}, V)$. 

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Now we will show that there exist functorial correspondences between three categories whose objects are the following:

(i) $\mathcal{B}(V)$-valued positive definite kernels, i.e. maps $K : P \times P \to \mathcal{B}(V)$ such that for any finite sequences $p_1, \ldots, p_J \in P$ and $v_1, \ldots, v_J \in V$ one has

$$\sum_{i,j=1}^{J} \langle v_i, K(p_i, p_j)v_j \rangle \geq 0,$$  

(2.1)

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $V$. Everywhere in the paper we assume that scalar products are anti-linear in the first argument and linear in the second argument.

The positivity condition (2.1) implies that $K$ is Hermitian, i.e. for each $q, p \in P$ one has

$$K(q, p) = K(p, q)^*.$$  

(2.2)

(ii) maps $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ satisfying the condition

$$\{\mathfrak{K}(p)v : p \in P \text{ and } v \in V\}^\perp = \{0\}.$$  

(2.3)

(iii) Hilbert spaces $\mathcal{K} \subset V^P$ realized by $V$-valued functions $f : P \to V$ such that the evaluation functionals

$$E_pf := f(p)$$  

(2.4)

are continuous maps of Hilbert spaces $E_p : \mathcal{K} \to V$ for every $p \in P$.

These functorial correspondences for the case $G = U(1)$ and $\dim_{\mathcal{C}} V = 1$ are proved in [O92]. The proofs of these correspondences for the case of general group $G$ and general Hilbert space $V$ can be given in a similar way. Here we restrict our considerations to the main steps of these proofs.

Equivalence between (ii) and (iii) is given as follows. For $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ we define monomorphism of vector spaces $J : \mathcal{H} \to V^P$ by

$$J(\psi)(p) := \mathfrak{K}(p)^*\psi,$$  

(2.5)

where $\psi \in \mathcal{H}$. Using this monomorphism we obtain Hilbert space structure on $\mathcal{K} := J(\mathcal{H})$. The continuity of the evaluation functionals follows from the inequality

$$\|E_pJ(\psi)\| = \|\mathfrak{K}(p)^*\psi\| \leq \|\mathfrak{K}(p)^*\| \cdot \|\psi\| = \|\mathfrak{K}(p)^*\| \cdot \|J(\psi)\|.$$  

(2.6)
Taking Hilbert space $\mathcal{K} \subset V^P$ such as in (iii) we put by definition $\mathcal{H} := \mathcal{K}$ and define $\mathcal{K}(p) : V \rightarrow \mathcal{H}$ by

$$\mathcal{K}(p) := E_p^*.$$ (2.7)

In order to check (2.3) note that

$$\langle \mathcal{K}(p)v \mid f \rangle = \langle E_p^*v \mid f \rangle = \langle v, E_pf \rangle = \langle v, f(p) \rangle$$ (2.8)

where $\langle \cdot \mid \cdot \rangle$ is the scalar product in $\mathcal{H}$. Thus if $\langle \mathcal{K}(p)v \mid f \rangle = 0$ for every $v \in V$ and $p \in P$, then (2.3) implies $f(p) = 0$ for all $p \in P$.

Since one has

$$0 \leq \left\| \sum_{i=1}^{J} \mathcal{K}(p_i)v_i \right\|^2 = \sum_{i,j=1}^{J} \langle v_i, \mathcal{K}(p_i)^* \mathcal{K}(p_j)v_j \rangle,$$ (2.9)

the passage from (ii) to (i) is given by

$$K(q, p) := \mathcal{K}(q)^* \mathcal{K}(p).$$ (2.10)

In order to show the implication (i) $\Rightarrow$ (iii) let us take vector subspace $\mathcal{K}_0 \subset V^P$ consisting of functions

$$f(p) := \sum_{i=1}^{I} K(p, p_i)v_i,$$ (2.11)

defined for finite sequences $p_1, \ldots, p_I \in P$ and $v_1, \ldots, v_I \in V$. Due to positive definiteness of the kernel $K : P \times P \rightarrow \mathcal{B}(V)$ we can define a scalar product between $g(\cdot) = \sum_{j=1}^{J} K(\cdot, q_j)w_j \in \mathcal{K}_0$ and $f \in \mathcal{K}_0$ by the formula

$$\langle g \mid f \rangle := \sum_{i=1}^{I} \sum_{j=1}^{J} \langle K(p_i, q_j)w_j, v_i \rangle.$$ (2.12)

Substituting $g(\cdot) = K(\cdot, v) \in \mathcal{K}_0$ into (2.12) we obtain the reproducing property

$$\langle v, f(p) \rangle = \langle v, \sum_{i=1}^{I} K(p, p_i)v_i \rangle = \sum_{i=1}^{I} \langle K(p_i, p)v, v_i \rangle = \langle K(\cdot, v) \mid f \rangle.$$ (2.13)
From (2.13) one has the inequality
\[ \| f(p) \| \leq \sqrt{\| K(p,p) \|} \| f \|, \]
which proves that (2.12) is a positive definite scalar product. Inequality (2.14) implies that if \( \{ f_n \} \) is a fundamental sequence in \( K_0 \), then \( \{ f_n(p) \} \) is a fundamental sequence in \( V \). Thus one can realize equivalence classes of fundamental sequences \( \{ \{ f_n \} \} \in K_0 \) by the function
\[ f(p) := \lim_{n \to \infty} f_n(p) \] (2.15)
from \( V^P \). In consequence we embed \( \iota : K_0 \hookrightarrow V^P \) as the abstract complement \( K_0 \) of \( K_0 \) into \( V^P \). Summing up we define Hilbert space \( K \subset V^P \) from (iii) as \( K := \iota(K_0) \). The continuity of the evaluation functionals \( E_p, p \in P \), for the Hilbert space \( K \) follows from (2.14). This completes the proof of the implication (i) \( \Rightarrow \) (iii).

Let us note that maps \( \mathfrak{R}_1 : P \longrightarrow \mathcal{B}(V, \mathcal{H}_1) \) and \( \mathfrak{R}_2 : P \longrightarrow \mathcal{B}(V, \mathcal{H}_2) \) factorize the same kernel, i.e.
\[ K(p,q) = \mathfrak{R}_1^*(p) \mathfrak{R}_1(q) = \mathfrak{R}_2^*(p) \mathfrak{R}_2(q) \] (2.16)
if and only if there exists a Hilbert space isomorphism \( U_{21} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2 \) such that \( \mathfrak{R}_2(p) = U_{21} \mathfrak{R}_1(p) \) for any \( p \in P \). Let us define \( U_{21} \) by
\[ U_{21} \sum_{i=1}^l \mathfrak{R}_1(p_i) v_i := \sum_{i=1}^l \mathfrak{R}_2(p_i) v_i, \]
(2.17)
where \( v_i \in V \) and \( p_i \in P \).

We also note that if the property (2.16) is fulfilled then the Hilbert spaces \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) defined in (2.5) coincide, i.e., \( J_1(\mathcal{H}_1) = J_2(\mathcal{H}_2) = \mathcal{K} \).

Subsequently we will be interested in the case when all objects defined above are smooth. So, further on we will assume that \( P \) is a smooth manifold.

**Proposition 2.1** Let \( P \) be a smooth \( n \)-dimensional manifold and \( V \) a finite-dimensional complex Hilbert space. Then the following properties are equivalent:

(a) the positive definite kernel \( K : P \times P \to \mathcal{B}(V) \) is a smooth map.
(b) the map $\hat{\mathcal{R}} : P \to \mathcal{B}(V, \mathcal{H})$ is smooth.

(c) the Hilbert space $\mathcal{K} \subset V^P$ defined in (iii) consists of smooth functions, i.e. $\mathcal{K} \subset C^\infty(P, V)$.

Proof:

(a) $\Rightarrow$ (b). Let $x = (x^1, \ldots, x^n)$ be local coordinates of $p \in P$, $y = (y^1, \ldots, y^n)$ be local coordinates of $q \in P$ and let $(e_1, \ldots, e_n)$ be the canonical basis in $\mathbb{R}^n$.

Firstly, we prove existence of the partial derivatives. To this end observe that for $v \in V$ and $t_1, t_2 \in \mathbb{R}$ due to (2.10) one has

\[
\left\| \frac{1}{t_1} (\hat{\mathcal{R}}(x + t_1 e_1) - \hat{\mathcal{R}}(x)) - \frac{1}{t_2} (\hat{\mathcal{R}}(x + t_2 e_1) - \hat{\mathcal{R}}(x)) \right\|^2 v = \tag{2.18} \]
\[
= \frac{1}{t_1^2} \langle v, [K(x + t_1 e_1, x + t_1 e_1) - K(x, x + t_1 e_1, x) - K(x, x + t_1 e_1) + K(x, x)|v] \rangle + \]
\[
+ \frac{1}{t_2^2} \langle v, [K(x + t_2 e_1, x + t_2 e_1) - K(x, x + t_2 e_1, x) - K(x, x + t_2 e_1) + K(x, x)|v] \rangle + \]
\[
+ \frac{1}{t_1 t_2} \langle v, [-K(x + t_1 e_1, x + t_2 e_1) + K(x + t_1 e_1, x) + K(x, x + t_2 e_1) - K(x, x) + K(x + t_2 e_1, x + t_1 e_1) + K(x + t_2 e_1, x) + K(x, x + t_1 e_1) - K(x, x)|v]. \rangle.
\]

Since kernel $K$ is a smooth map, we have

\[
K(x + t e_i, x) = K(x, x) + t \frac{\partial K}{\partial y^i}(y, x)|_{y=x} + \frac{1}{2} t^2 \frac{\partial^2 K}{\partial (y^i)^2}(y, x)|_{y=x} + r_1(x; t), \tag{2.19}
\]

\[
K(x, x + t e_i) = K(x, x) + t \frac{\partial K}{\partial x^i}(y, x)|_{y=x} + \frac{1}{2} t^2 \frac{\partial^2 K}{\partial (x^i)^2}(y, x)|_{y=x} + r_2(x; t), \tag{2.20}
\]

\[
K(x + t_1 e_1, x + t_2 e_i) = K(x, x) + t_1 \frac{\partial K}{\partial y^i}(y, x)|_{y=x} + t_2 \frac{\partial K}{\partial x^i}(y, x)|_{y=x} + \]
\[
+ \frac{1}{2} t_1^2 \frac{\partial^2 K}{\partial (y^i)^2}(y, x)|_{y=x} + \frac{1}{2} t_2^2 \frac{\partial^2 K}{\partial (x^i)^2}(y, x)|_{y=x} + \frac{1}{2} t_1 t_2 \frac{\partial^2 K}{\partial y^i \partial x^j}(y, x)|_{y=x} + r_{12}(x; t_1, t_2), \tag{2.21}
\]

where

\[
\lim_{t \to 0} \frac{r_1(x; t)}{t^2} = \lim_{t \to 0} \frac{r_1(x; t)}{t^2} = \lim_{t_1, t_2 \to 0} \frac{r_{12}(x; t_1, t_2)}{t_1^2 + t_2^2} = 0. \tag{2.22}
\]
Therefore from (2.10) we obtain

\[
\left\| \frac{1}{t_1} (\tilde{\mathbf{K}}(x + t_1 e_i) - \tilde{\mathbf{K}}(x)) - \frac{1}{t_2} (\tilde{\mathbf{K}}(x + t_2 e_i) - \tilde{\mathbf{K}}(x)) \right\|^2 = (2.23)
\]

\[
= \left\langle v, \left[ \begin{array}{c} r_{12}(x; t_1, t_1) - r_1(x; t_1) - r_2(x; t_1) + r_{12}(x; t_2, t_2) - r_1(x; t_2) + \\
- \frac{r_2(x; t_2)}{t_2} - \frac{r_{12}(x; t_1, t_2)}{t_1 t_2} + \frac{r_1(x; t_1)}{t_1 t_2} + \frac{r_2(x; t_2)}{t_1 t_2} + \\
- \frac{r_{12}(x; t_2, t_1)}{t_1 t_2} + \frac{r_1(x; t_2)}{t_1 t_2} + \frac{r_2(x; x)}{t_1 t_2} \end{array} \right] v \right\rangle \xrightarrow{t_1, t_2 \to 0} 0.
\]

Since Hilbert space $\mathcal{V}$ is finite-dimensional the above proves existence of the partial derivatives

\[
\frac{\partial \tilde{\mathbf{K}}}{\partial x^i}(p) := \lim_{t \to 0} \frac{1}{t} (\tilde{\mathbf{K}}(x + t e_i) - \tilde{\mathbf{K}}(x)).
\] (2.24)

In order to verify continuity of $\frac{\partial \tilde{\mathbf{K}}}{\partial x^i} : P \to \mathcal{B}(\mathcal{V}, \mathcal{H})$ note that

\[
\left\| \left[ \frac{\partial \tilde{\mathbf{K}}}{\partial x^i} (x + \Delta x) - \frac{\partial \tilde{\mathbf{K}}}{\partial x^i} (x) \right] v \right\|^2 = (2.25)
\]

\[
= \lim_{t_1, t_2 \to 0} \left\langle v, \left[ \begin{array}{c} \frac{1}{t_1} (\tilde{\mathbf{K}}(x + \Delta x + t e_i) - \tilde{\mathbf{K}}(x + \Delta x)) - \frac{1}{t_2} (\tilde{\mathbf{K}}(x + t e_i) - \tilde{\mathbf{K}}(x)) \end{array} \right] v \right\rangle =
\]

\[
= \left\langle v, \left[ \frac{\partial^2 K}{\partial y^i \partial x^i} (x + \Delta x, x + \Delta x) + \frac{\partial^2 K}{\partial y^j \partial x^i} (x, x) - \frac{\partial^2 K}{\partial y^j \partial x^i} (x + \Delta x, x) + \\
- \frac{\partial^2 K}{\partial y^j \partial x^i} (x, x + \Delta x) \right] v \right\rangle \xrightarrow{\Delta x \to 0} 0.
\]

Continuity of partial derivatives of $\tilde{\mathbf{K}} : P \to \mathcal{B}(\mathcal{V}, \mathcal{H})$ implies existence of its Fréchet derivative. Existence of higher order Fréchet derivatives of $\tilde{\mathbf{K}} : P \to \mathcal{B}(\mathcal{V}, \mathcal{H})$ can be verified in a similar way.

Implication (b) $\Rightarrow$ (a) follows from (2.10).

Let us now prove that (b) $\Rightarrow$ (c). Smoothness of $\tilde{\mathbf{K}} : P \to \mathcal{B}(\mathcal{V}, \mathcal{H})$ implies smoothness of $\tilde{\mathbf{K}}^* : P \to \mathcal{B}(\mathcal{H}, \mathcal{V})$. Thus for any $\psi \in \mathcal{H}$ the function $f : P \to \mathcal{V}$ defined by $f(p) := \tilde{\mathbf{K}}^*(p)\psi$ depends smoothly on $p \in P$. 

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Finally we verify implication (c) ⇒ (a). For any \( q \in P \) and \( v \in V \) one has \( J(\mathfrak{K}(q)v) \in \mathcal{K} \subset C^\infty(P,V) \). Thus from (2.5) and (2.10) it follows that \( K(p,q)v = \mathfrak{K}^*(p)\mathfrak{K}(q)v \) is smooth with respect to the variable \( q \). Since \( K(q,p)^* = K(p,q) \) we obtain the smooth dependence of the kernel \( K \) on both arguments.

\[ \square \]

One of the most interesting situations arises when \( P \) is a complex analytic manifold and the scalar product in Hilbert space \( \mathcal{K} \subset \mathcal{V} \) consisting of holomorphic functions, is defined by the integral taken with respect to some measure \( \mu \) on \( P \). Therefore Hilbert space \( \mathcal{K} \) as well as kernel \( K \) depends on the choice of \( \mu \). This case and the dependence of \( K \) on \( \mu \) in particular was studied in [PW]. Many other interesting facts and statements concerning the meaning of reproducing kernel for the quantization can be found in [B-G, G, O88, H-O, O-´S].

From now on we will assume that \( P \) is a principal bundle

\[
\begin{array}{ccc}
G & \longrightarrow & P \\
\downarrow & & \downarrow \pi \\
M & \longrightarrow & M
\end{array}
\]

(2.26)

over the smooth manifold \( M \) with some Lie group \( G \) as the structural group. Additionally we will assume that one has a faithful unitary representation of \( G \)

\[
T : G \longrightarrow \text{Aut}(V)
\]

(2.27)

in Hilbert space \( V \) and we will suppose that positive definite kernel \( K : P \times P \to \mathcal{B}(V) \) has equivariance property

\[
K(p,qg) = K(p,q)T(g)
\]

(2.28)

where \( p, q \in P \) and \( g \in G \). This property is equivalent to each of the following two:

\[ \mathfrak{K}(pg) = \mathfrak{K}(p)T(g) \]

(2.29)

and

\[ f(pg) = T(g^{-1})f(p) \]

(2.30)

for \( f \in \mathcal{K} \), where the map \( \mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H}) \) and Hilbert space \( \mathcal{K} \) are defined in (ii) and (iii), respectively.
Using the action of $G$ on $P \times V$ defined by
\[ P \times V \ni (p, v) \mapsto (pg, T(g^{-1})v) \in P \times V \]
(2.31)
one obtains $T$-associated vector bundle
\[ V \xrightarrow{\tilde{\pi}} \mathbb{V} \]
(2.32)
over $M$ with the quotient manifold $\mathbb{V} := (P \times V)/G$ as its total space. The equivariance properties (2.28), (2.29) and (2.30) allow us to transpose the geometric objects defined in (i), (ii) and (iii) on the vector bundle (2.32). Note that the fiber $\mathbb{V}_m = \tilde{\pi}^{-1}(m)$ consists of equivalence classes $[(p, v)] \in (P \times V)/G$ for which $\pi(p) = m$.

Given $\pi(p) = m$, $\pi(q) = n$, we define by
\[ K_T(m, n)([(p, v)], [(q, w)]) := \langle v, K(p, q)w \rangle, \]
(2.33)
the section
\[ K_T : M \times M \rightarrow pr_1^*\mathbb{V} \otimes pr_2^*\mathbb{V}^* \]
(2.34)
of the bundle $pr_1^*\mathbb{V} \otimes pr_2^*\mathbb{V}^* \rightarrow M \times M$ which is the tensor product of the pullbacks $pr_1^*\mathbb{V} \rightarrow M \times M$ and $pr_2^*\mathbb{V}^* \rightarrow M \times M$ of (2.32), where $pr_i : M \times M \rightarrow M$ is the projection on the $i$-th component of $M \times M$.

Let us define the map $\tilde{\mathcal{R}} : \mathbb{V} \rightarrow \mathcal{H}$ by
\[ \tilde{\mathcal{R}}([(p, v)]) := \mathcal{R}(p)v. \]
(2.35)
Note that the kernel (2.33) and the map (2.35) are interrelated by the equality
\[ K_T(m, n) = \iota_m \circ \iota_n \]
(2.36)
or equivalently
\[ K_T(m, n)([(p, v)], [(q, w)]) = \langle \tilde{\mathcal{R}}([(p, v)]), \tilde{\mathcal{R}}([(q, w)]) \rangle, \]
(2.37)
where $\iota_m := \tilde{\mathcal{R}}_{|\mathbb{V}_m} : \mathbb{V}_m \rightarrow \mathcal{H}$.

We also can realize Hilbert space $\mathcal{H}$ by smooth sections of vector bundles $\tilde{\pi}^* : \mathbb{V}^* \rightarrow M$ or $\tilde{\pi} : \mathbb{V} \rightarrow M$. The first realization is given by the anti-linear
monomorphism of vector spaces $I^*: \mathcal{H} \to C^\infty(M,\mathbb{V}^*)$ defined for $\psi \in \mathcal{H}$ in the following way

$$I^*(\psi)(\pi(p))([[p,v]]) := \langle \psi | \mathcal{R}(p)v \rangle = \langle J(\psi)(p), v \rangle. \quad (2.38)$$

The second realization is as follows

$$I(\psi)(\pi(p)) := [(p, \mathcal{R}(p)^*\psi)] = [(p, J(\psi)(p))]. \quad (2.39)$$

The map $I: \mathcal{H} \to C^\infty(M,\mathbb{V}^*)$ is also a linear monomorphism of vector spaces.

By $C^\infty(M,\mathbb{V}^*)$ and $C^\infty(M,\mathbb{V})$ we denote vector spaces of smooth sections of the bundles $\tilde{\pi}^*: \mathbb{V}^* \to M$ and $\tilde{\pi}: \mathbb{V} \to M$, respectively.

The restriction $K_{T|\Delta}$ of the kernel $K_T$ to the diagonal $\Delta := \{(m,n) \in M \times M : m = n\}$ determines a semi-positive definite Hermitian structure $H_K := K_{T|\Delta}$ on the bundle $\tilde{\pi}: \mathbb{V} \to M$. For subsequent considerations we will need to deal with the positive definite Hermitian structure. So, further on we will assume that linear operator $K(p,p)$ is invertible for every $p \in P$. The last condition is equivalent to the condition $\ker \mathcal{R}(p) = \{0\}$, for $p \in P$.

Recall that we assumed $\dim \mathbb{V} =: N < \infty$.

Apart from the Hermitian structure $H_K$ the positive Hermitian kernel $K$ defines on $P$ a $\mathcal{B}(\mathbb{V})$-valued differential one-form

$$\vartheta(p) := (\mathcal{R}(p)^*\mathcal{R}(p))^{-1}\mathcal{R}(p)^*d\mathcal{R}(p) = K(p,p)^{-1}d_qK(p,q)|_{q=p}, \quad (2.40)$$

where $d_q$ denotes the exterior derivative with respect to the variable $q$.

Let $g_\mathcal{X}(t) := \exp(t\mathcal{X})$ denote a one-parameter group generated by an element $\mathcal{X} \in \mathfrak{g} = T_eG$ of Lie algebra of $G$. By $\xi_\mathcal{X} \in C^\infty(P,TP)$ we denote vector field tangent to the vertical flow $p \mapsto pg_\mathcal{X}(t)$. From (2.40) and (2.29) it follows that

$$(\xi_\mathcal{X},\vartheta)(p) = (\mathcal{R}(p)^*\mathcal{R}(p))^{-1}\mathcal{R}(p)^*(\xi_\mathcal{X} \cdot d\mathcal{R})(p) = \frac{d}{dt}T(g_\mathcal{X}(t))|_{t=0} = DT(e)(\mathcal{X})$$

(2.41)

and

$$\vartheta(pg) = T(g^{-1})\vartheta(p)T(g), \quad (2.42)$$

where $DT(e): \mathfrak{g} \to \mathcal{B}(\mathbb{V})$ is the derivative of the representation (2.27) taken at the unit element $e \in G$.

Taking (2.41) and (2.42) into account we conclude from

$$\langle v, K(p,p)\vartheta(p)w \rangle + \langle \vartheta(p)v, K(p,p)w \rangle = d\langle v, K(p,p)w \rangle \quad (2.43)$$
that $\vartheta \in C^\infty(P, T^*P \otimes \mathcal{B}(V))$ is the one-form of a metric connection $\nabla_K$ consistent with the Hermitian structure $H_K$.

Now let us consider the Grassmannian $Gr(N, \mathcal{H})$ of $N$-dimensional Hilbert subspaces of $\mathcal{H}$ and the tautological vector bundle

\[ V \rightarrow \mathbb{E} \]

\[ \begin{array}{c}
\downarrow^{pr_1} \\
Gr(N, \mathcal{H})
\end{array} \quad (2.44)\]

where total space of (2.44) is defined by

\[ \mathbb{E} := \{(z, \psi) \in Gr(N, \mathcal{H}) \times \mathcal{H} : \psi \in z \}. \quad (2.45)\]

The scalar product in $\mathcal{H}$ defines an Hermitian structure $H_\mathbb{E}$ on the vector bundle $pr_2 : \mathbb{E} \rightarrow Gr(N, \mathcal{H})$ in the canonical way. There is also a unique connection on this bundle

\[ \nabla_\mathbb{E} : C^\infty(Gr(N, \mathcal{H}), \mathbb{E}) \rightarrow C^\infty(Gr(N, \mathcal{H}), \mathbb{E} \otimes T^*(Gr(N, \mathcal{H}))), \quad (2.46)\]

consistent with $H_\mathbb{E}$ and complex analytic structures of $Gr(N, \mathcal{H})$. See [K-N], Volume 1, Chapter 2, for the definition of such connections.

Since $\ker \tilde{\mathcal{R}}_{\mathbb{V}_m} = \{0\}$ one has a map $\mathcal{K} : M \rightarrow Gr(N, \mathcal{H})$ defined by

\[ \mathcal{K}(m) := \tilde{\mathcal{R}}(\mathbb{V}_m). \quad (2.47)\]

Thus one has also the following vector bundle morphism

\[ \begin{array}{ccc}
V & \xrightarrow{\tilde{\mathcal{R}}} & \mathbb{E} \\
\downarrow^{\tilde{\pi}} & & \downarrow^{pr_1} \\
M & \xrightarrow{\mathcal{K}} & Gr(N, \mathcal{H})
\end{array} \quad (2.48)\]

where

\[ \tilde{\mathcal{R}}([(p, v)]) := (\tilde{\mathcal{R}}([(p, v)]), \mathcal{K}(m)). \quad (2.49)\]

Taking into account the definitions of $H_K$ and $\nabla_K$ we find that they are pull-backs $H_K = \mathcal{K}^*H_\mathbb{E}$ and $\nabla_K = \mathcal{K}^*\nabla_\mathbb{E}$ of $H_\mathbb{E}$ and $\nabla_\mathbb{E}$, respectively. So, (2.48)
gives a vector bundle morphism which preserves Hermitian and connection structures.

It is important to mention that in geometric models of physical systems one considers a bundle \( \tilde{\pi} : V \to M \) as the space of states, where the fibers \( \tilde{\pi}^{-1}(m), m \in M \) describe internal degrees of freedom and the base manifold \( M \) is responsible for the external degrees of freedom of the system. Within such an interpretation a positive definite kernel \( K(p, q) \) after normalization can be considered as a transition amplitude

\[
a((p, v), (q, w)) := \frac{\langle v, K(p, q)w \rangle}{\langle v, K(p, p)v \rangle^{1/2} \langle w, K(q, q)w \rangle^{1/2}} \quad (2.50)
\]

between the states \( \tilde{\mathcal{R}}((p, v)), \tilde{\mathcal{R}}((q, w)) \in \mathcal{H} \). Let us mention here that from physical point of view transition amplitude \((2.50)\) is the most fundamental object which is usually obtained in an experimental way (see [F-L-S] Chapter 3).

Finally let us make some comments:

i) The maps \( \tilde{\mathcal{R}} : P \to \mathcal{B}(V, \mathcal{H}), \tilde{\mathcal{R}} : V \to \mathcal{H} \) and kernel \( K_T \) define each other explicitly.

ii) The scalar product \( \langle \cdot | \cdot \rangle \) of \( \mathcal{H} \) defines positive definite kernel \( K_E : \text{Gr}(N, \mathcal{H}) \times \text{Gr}(N, \mathcal{H}) \to \text{pr}_1^* \mathbb{E}^* \otimes \text{pr}_2^* \mathbb{E}^* \) on the tautological bundle \( \pi : \mathbb{E} \to \text{Gr}(N, \mathcal{H}) \) by

\[
K_E((z_1, \psi_1), (z_2, \psi_2)) := \langle \psi_1 | \psi_2 \rangle. \quad (2.51)
\]

iii) The pullback of the kernel \( K_E \) on the vector bundle \( \tilde{\pi} : V \to M \) by the map \( \tilde{\mathcal{R}} : V \to \mathcal{H} \) gives the kernel \( K_T \) defined in \((2.33)\). Therefore one can consider \((\mathbb{E}, K_E)\) as the universal object in the category of vector bundles \((V, K_T)\) with the fixed positive definite kernel \( K_T \).

Similarly, the prequantum bundle \((\mathbb{E}, \nabla_E, H_E)\), where \( \nabla_E \) and \( H_E \) are the connection and the Hermitian structure on \( \pi : \mathbb{E} \to \text{Gr}(N, \mathcal{H}) \) defined by kernel \( K_E \), is the universal object in the category of prequantum bundles \((V, \nabla_K, H_K)\) defined by \((V, K_T)\). The relationship between \((V, K_T)\) and \((V, \nabla_K, H_K)\) has functorial character.

In subsequent sections we will call \( \tilde{\mathcal{R}} : V \to \mathcal{H} \) as well as \( \tilde{\mathcal{R}} : P \to \mathcal{B}(V, \mathcal{H}) \) the coherent state map. See [O92] and [O88] for a physical motivation of this terminology.
3 One-parameter groups of automorphisms and prequantization

In this section we introduce various Lie algebras, see Proposition 3.1, Proposition 3.2 and Proposition 3.3, the elements of which generate the one-parameter group of automorphisms of the principal bundle \( \pi : P \to M \). We will show the special importance of the Lie algebras appearing in the short exact sequence (3.31). This is important because this sequence generalizes the exact sequence of Lie algebras

\[
0 \to \mathbb{R} \to C^\infty(M, \mathbb{R}) \to H_0(M, \omega) \to 0,
\]

for the symplectic manifold \( M \) with symplectic form \( \omega \), where \( H_0(M, \omega) \) is the Lie algebra of Hamiltonian vector fields. We also introduce here equation (3.8) which is a natural generalization of Hamiltonian equation for the case of general gauge group \( G \).

Now we extend Kostant-Souriau prequantization procedure on the case of general principal \( G \)-bundle \( \pi : P \to M \) with a fixed \( DT(e)(g) \)-valued connection form \( \vartheta \in C^\infty_G(T^*P \otimes DT(e)(g)) \).

Let \( \xi \in C^\infty(P, TP) \) be the vector field tangent to the flow of automorphisms \( \tau : (\mathbb{R}, +) \to \text{Aut}(P, \vartheta) \) of the principal bundle (2.26)

\[
\tau_t(pg) = \tau_t(p)g,
\]

where \( g \in G \) and \( p \in P \), which preserve the connection form \( \vartheta \)

\[
\tau_t^* \vartheta = \vartheta.
\]

Then one has

\[
\xi(pg) = DR_g(p)\xi(p),
\]

and

\[
\mathcal{L}_\xi \vartheta = 0,
\]

where \( R_g(p) := pg \), while \( DR_g(p) \) is the derivative of \( R_g \) at \( p \) and \( \mathcal{L}_\xi \) is Lie derivative with respect to \( \xi \).

The space of vector fields satisfying (3.3) will be denoted by \( C^\infty_G(P, TP) \). By \( \mathcal{E}_G^0 \subset C^\infty_G(P, TP) \) we denote the subspace consisting of those \( \xi \in C^\infty_G(P, TP) \) which satisfy (3.4).
Recall that the covariant differential $D\varphi$ of $DT(e)(\mathfrak{g})$-valued pseudotensorial r-form $\varphi$ on $P$ is defined as 

$$D\varphi(\xi_1, \ldots, \xi_{r+1}) := d\varphi(pr_{\text{hor}}\xi_1, \ldots, pr_{\text{hor}}\xi_{r+1}),$$

where $pr_{\text{hor}}(p) : T_pP \to T^h_pP$ is projection associated with the decomposition $T_pP = T^h_pP \oplus T^v_pP$ of the tangent space $T_pP$ in the horizontal and vertical parts taken with respect to the connection form \((2.40)\). In particular, for connection 1-form $\vartheta$ and $DT(e)(\mathfrak{g})$-valued pseudotensorial 0-form, i.e. $DT(e)(\mathfrak{g})$-valued function such that

$$F(pg) = T(g^{-1})F(p)T(g),$$

one has

$$D\vartheta = d\vartheta + \frac{1}{2}[\vartheta, \vartheta],$$

$$DF = dF + [\vartheta, F],$$

where \((3.6)\) is the structure equation for the curvature form $\Omega := D\vartheta$, (see for example \[K-N\]). In the subsequent we will use the notation taken from \[K-N\], Volume 1 Chapter 2.

By $C^\infty_G(P, DT(e)(\mathfrak{g}))$ we denote the space of $DT(e)(\mathfrak{g})$-valued functions satisfying condition \((3.5)\).

Now let us consider the vector space $\mathcal{P}_G$ which by definition consists of pairs $(F, \xi) \in C^\infty_G(P, DT(e)(\mathfrak{g})) \times C^\infty_G(P, TP)$ such that

$$\xi \lrcorner \Omega = DF.$$  \hspace{1cm} (3.8)

**Proposition 3.1** For $(F, \xi), (G, \eta) \in \mathcal{P}_G$ we have

$$\mathcal{L}_{[\xi, \eta]}\vartheta = D\{F, G\} + \vartheta([\xi, \eta]),$$

where

$$\{F, G\} := 2\Omega(\xi, \eta) + DG(\xi) - DF(\eta) + [F, G].$$

**Proof:**

Due to the identity

$$\mathcal{L}_\xi\vartheta = \xi \lrcorner \Omega + D(\vartheta(\xi)),$$

the condition \((3.8)\) is equivalent to

$$\mathcal{L}_\xi\vartheta = D(F + \vartheta(\xi)).$$

\hspace{1cm} (3.12)
Proposition 3.2

The space \( \mathcal{P}_G \) with the bracket

\[
[(F, \xi), (G, \eta)] := \{F, G\}, [\xi, \eta]
\]

is a Lie algebra.

Proof:

For arbitrary \((F, \xi), (G, \eta), (H, \lambda) \in \mathcal{P}_G\) by direct calculations we obtain

\[
\{F, \{G, H\}\} = D((DH)(\eta) - (DG)(\xi) + [G, H] + 2\Omega(\xi, \eta))(\lambda) + [F, (DH)(\eta) - (DG)(\xi) + [G, H] + 2\Omega(\xi, \eta)], (3.15)
\]

Adding the cyclic permutations of both sides of \((3.15)\) we find that

\[
\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. (3.16)
\]
Thus we conclude that bracket (3.14) satisfies Jacobi identity and hence \((\mathcal{P}_G, [\cdot, \cdot])\) is a Lie algebra.

\[\square\]

Let \(\mathcal{E}_G\) be the Lie algebra of vector fields \(\xi \in C^\infty_G(P, TP)\) for which there exist \(F \in C^\infty_G(P, DT(e)(\mathfrak{g}))\) such that \((F, \xi) \in \mathcal{P}_G\). Denote by \(N_G\) the set of \(F \in C^\infty_G(P, DT(e)(\mathfrak{g}))\) for which \(DF = 0\). One has the following exact sequence of Lie algebras

\[0 \to N_G \xrightarrow{\iota_1} \mathcal{P}_G \xrightarrow{pr_2} \mathcal{E}_G \to 0, \tag{3.17}\]

where

\[\iota_1(F) := (F, 0), \quad pr_2(F, \xi) := \xi. \tag{3.18}\]

From now on we will assume that \(M\) is a connected manifold and denote by \(P(p)\) the set of elements of \(P\) which one can join with \(p\) by curves which are horizontal with respect to the connection \(\vartheta\). By \(G(p)\) we denote the subgroup \(G(p) \subset G\) consisting of those \(g \in G\) for which \(pg \in P(p)\), i.e. \(G(p)\) is the holonomy group based at \(p\). Let us recall (e.g. see [K-N]) that for connected base manifold \(M\) all holonomy groups \(G(p)\) and their Lie algebras \(\mathfrak{g}(p)\) are conjugate in \(G\) and \(\mathfrak{g}\), respectively. Recall also that Lie algebra \(DT(e)(\mathfrak{g}(p))\) is generated by \(\Omega_p(X(p'), Y(p'))\), where \(p' \in P(p)\) and \(X(p'), Y(p') \in T_{p'}P\).

Taking this into account we conclude from condition (3.14) that \((F, \xi) \in \mathcal{P}_G\) function \(F\) takes values \(F(p')\) in \(\mathfrak{g}(p)\) if \(p' \in P(p)\). In the special case when \(F \in N_G\), i.e. when \(DF = 0\), function \(F\) is constant on \(P(p)\) and \((F, \xi) \in DT(e)(\mathfrak{g}(p)) \cap DT(e)(\mathfrak{g}'(p))\), where \(\mathfrak{g}'(p)\) is the centralizer of Lie subalgebra \(\mathfrak{g}(p)\) in \(\mathfrak{g}\).

For the sake of completeness let us note that

\[\mathcal{L}_\xi \Omega = [\Omega, F + \vartheta(\xi)] = D^2(F + \vartheta(\xi)) = D\mathcal{L}_\xi \vartheta, \tag{3.19}\]

for \((F, \xi) \in \mathcal{P}_G\).

It follows from (3.11) that \(\mathcal{E}_G^0 \subset \mathcal{E}_G\). Thus we can consider the subspace \(\mathcal{P}_G^0 \subset \mathcal{P}_G\) of such elements \((F, \xi) \in \mathcal{P}_G\) that \(\xi \in \mathcal{E}_G^0\) and \(F = F_0 - \vartheta(\xi)\), where \(DF_0 = 0\). For \(\xi, \eta \in \mathcal{E}_G^0\) we have

\[\vartheta([\xi, \eta]) = 2\Omega(\xi, \eta) - [\vartheta(\xi), \vartheta(\eta)]. \tag{3.20}\]

Thus for \((F, \xi), (G, \eta) \in \mathcal{P}_G^0\) we obtain

\[[(F, \xi), (G, \eta)] = [(F_0, 0) + (-\vartheta(\xi), \xi), (G_0, 0) + (-\vartheta(\eta), \eta)] =
\[\]

\[= [(-\vartheta(\xi), \xi), (-\vartheta(\eta), \eta)] = ([\vartheta(\xi), \vartheta(\eta)], [\xi, \eta]) = (-\vartheta([\xi, \eta]), [\xi, \eta]) \tag{3.21}\]
From (3.9) and (3.21) we see that \( \mathcal{P}_G^0 \) is a Lie subalgebra of \( \mathcal{P}_G \) which contains the ideal \( \mathcal{N}_G \).

Summing up we accumulate the above facts in the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & N_G & \xrightarrow{\iota_1} & \mathcal{P}_G & \xrightarrow{pr_2} & \mathcal{E}_G & \rightarrow & 0,
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & N_G & \xrightarrow{\iota_1} & \mathcal{P}_G^0 & \xrightarrow{pr_2} & \mathcal{E}_G^0 & \rightarrow & 0,
\end{array}
\]

where horizontal arrows form the exact sequences of Lie algebras and vertical arrows are Lie algebra monomorphisms.

In order to describe Lie algebra \( \mathcal{P}_G^0 \) we define a linear monomorphism \( \Phi : \mathcal{E}_G^0 \rightarrow \mathcal{P}_G^0 \) by

\[
\Phi(\xi) := (-\vartheta(\xi), \xi).
\]

(3.23)

It follows from (3.21) that \( \Phi \) is a monomorphism of Lie algebras. Due to (3.12) one has the decomposition

\[
\mathcal{P}_G^0 = \iota_1(N_G) \oplus \Phi(\mathcal{E}_G^0)
\]

(3.24)

of \( \mathcal{P}_G^0 \) into the direct sum of Lie subalgebra \( \Phi(\mathcal{E}_G^0) \) and ideal \( \iota_1(N_G) \) of central elements of \( \mathcal{P}_G^0 \). Hence we conclude that the lower exact sequence of Lie algebras in (3.22) is trivial.

Now let us define the following Lie subalgebra

\[
\mathcal{H}_G^0 := D\pi(\mathcal{E}_G^0),
\]

(3.25)

of \( C^\infty(M,TM) \), where \( D\pi : TP \rightarrow TM \) is the tangent map of the bundle map \( \pi : P \rightarrow M \). We also define \( \mathcal{F}_G^0 \subset C^\infty_G(P, DT(e)(\mathfrak{g})) \times \mathcal{H}_G^0 \) as the vector subspace consisting of elements \( (F,X) \in C^\infty_G(P, DT(e)(\mathfrak{g})) \times \mathcal{H}_G^0 \) which satisfy the condition

\[
X^* \lrcorner \Omega = DF,
\]

(3.26)

where \( X^* \) is the horizontal lift of \( X \) with respect to \( \vartheta \). From condition (3.26) and identity (3.11) it follows that

\[
\xi := X^* - F^* \in \mathcal{E}_G^0,
\]

(3.27)

where \( F^* \) is a vertical field defined by the function \( F \in C^\infty_G(P, DT(e)(\mathfrak{g})) \) in the following way

\[
(F^* f)(p) = \frac{d}{dt} f(p \exp(tF'(p)))|_{t=0},
\]

(3.28)
where \( f \in C^\infty(P) \) and function \( F' : P \to \mathfrak{g} \) is such that \( DT(e)(F'(p)) = F(p) \). Note that (3.27) gives the decomposition of \( \xi = \xi^h + \xi^v \) on the horizontal \( \xi^h = X^* \) and vertical \( \xi^v = -F^* \) components.

On the other side decomposing \( \xi = \xi^h + \xi^v \in \mathcal{E}_G^0 \) on the horizontal and vertical parts we define

\[
(F, X) = (-\partial(\xi^v), D\pi(\xi^h)) \in \mathcal{F}_G^0.
\] (3.29)

Summing up the above facts we formulate the following

**Proposition 3.3** The relations (3.27) and (3.29) define a Lie algebras isomorphism between \((\mathcal{E}_G^0, [\cdot, \cdot])\) and \((\mathcal{F}_G^0, \{\cdot, \cdot\})\), where the Lie bracket of \((F, X), (G, Y) \in \mathcal{F}_G^0\) is defined by

\[
\{ (F, X), (G, Y) \} := (-2\Omega(X^*, Y^*) + [F, G], [X, Y]).
\] (3.30)

One has the following exact sequence of Lie algebras

\[
0 \rightarrow N_G \overset{i_1}{\rightarrow} \mathcal{F}_G^0 \overset{pr_2}{\rightarrow} \mathcal{H}_G^0 \rightarrow 0,
\] (3.31)

where \( i_1(F) := (F, 0) \) and \( pr_2(F, X) := X \).

The integration of the horizontal part \( \xi^h = X^* \) of \( \xi \in \mathcal{E}_G^0 \) gives the flow \( \{\tau^h_t\}_{t \in \mathbb{R}} \) being the horizontal lift of the flow

\[
\sigma : (\mathbb{R}, +) \rightarrow \text{Diff}(M)
\] (3.32)
defined by the projection of \( \{\tau_t\}_{t \in \mathbb{R}} \) on the base \( M \) of the principal bundle \( P \). The vector field \( X \in \mathcal{H}_G^0 \) is the velocity vector field of \( \{\sigma_t\}_{t \in \mathbb{R}} \).

Since \( \{\tau_t\}_{t \in \mathbb{R}} \) and \( \{\tau^h_t\}_{t \in \mathbb{R}} \) are the liftings of \( \{\sigma_t\}_{t \in \mathbb{R}} \) there exists a \( G \)-valued cocycle on \( P \), namely a map \( c : \mathbb{R} \times P \rightarrow G \) such that

\[
c(t + s, p) = c(t, \tau^h_s(p))c(s, p) = c(s, p)c(t, \tau_s(p)), \tag{3.33}
\]

\[
c(t, pg) = g^{-1}c(t, p)g, \tag{3.34}
\]

which intertwines both flows

\[
\tau_t(p) = \tau^h_t(p)c(t, p). \tag{3.35}
\]

Applying the representation \( T : G \rightarrow \text{Aut}(V) \) (see (2.27)) to (3.33) and subsequently differentiating (3.33) with respect to the parameter \( t \) at \( t = 0 \) we obtain differential equation

\[
\frac{d}{ds}T(c(s, p)) = T(c(s, p)) \frac{d}{dt}T(c(t, \tau_s(p)))_{t=0} \tag{3.36}
\]
with initial condition \( T(c(0,p)) = \mathbb{1} \).

In order to solve (3.36) note that from definition (2.40) one has

\[
\vartheta_p(\xi(p)) = K(p,p) - 1 \lim_{\Delta t \to 0} \frac{1}{\Delta t} [K(p,\tau_{\Delta t}(p)c(\Delta t, p)) - K(p,p)] = \quad (3.37)
\]

\[
= K(p,p)^{-1} \lim_{\Delta t \to 0} \frac{1}{\Delta t} [K(p,\tau_{\Delta t}^h(p)c(\Delta t, p)) - K(p,p)] =
\]

\[
= K(p,p)^{-1} \lim_{\Delta t \to 0} \frac{1}{\Delta t} [K(p,\tau_{\Delta t}^h(p)c(\Delta t, p)) - K(p,p)] +
\]

\[
\left. + \right. K(p,p)^{-1} \lim_{\Delta t \to 0} \frac{1}{\Delta t} [K(p,pc(\Delta t, p)) - K(p,p)] =
\]

\[
= \vartheta_p(\xi^h(p)) + K(p,p)^{-1} K(p,p) \lim_{\Delta t \to 0} \frac{1}{\Delta t} [T(c(\Delta t, p)) - \mathbb{1}] = \frac{d}{dt} T(c(t,p))|_{t=0}.
\]

By virtue of

\[
\mathcal{L}_\xi(\xi_L\vartheta) = [\xi,\xi_L\vartheta] + \xi_L(\mathcal{L}_\xi\vartheta) = 0, \quad (3.38)
\]

we obtain

\[
\frac{d}{dt} T(c(t,\tau_s(p)))|_{t=0} = \vartheta(\xi)(\tau_s(p)) = \vartheta(\xi)(p). \quad (3.39)
\]

Now, solving equation

\[
\frac{d}{ds} T(c(s,p)) = T(c(s,p))\vartheta(\xi)(p) \quad (3.40)
\]

with \( T(c(0,p)) = \mathbb{1} \) we get

\[
T(c(t,p)) = \exp(t\vartheta(\xi)(p)) = \exp(-tF(p)). \quad (3.41)
\]

Taking (3.41) into account and making use of the fact that \( \{\sigma_t\}_{t \in \mathbb{R}} \) is defined by \( X \in \mathcal{H}_G^0 \) we conclude from (3.35) that \( \{\tau_t\}_{t \in \mathbb{R}} \) is determined in a unique way by \( (F,X) \in \mathcal{F}_G^0 \).

In the case when \( G = U(1), \dim V = 1 \) and the curvature \( \Omega \) is a non-singular 2-form equation (3.26) reduces to the Hamilton equation with \( F \in C^\infty_G(P) \cong C^\infty(M) \) as a Hamiltonian (total energy function). So, it is natural to consider (3.26) as a generalization of the Hamilton equations to the case of general gauge group \( G \). If \( \Omega \) is non-singular one can consider the vector field \( X_F \in \mathcal{H}_G^0 \) defined by \( F \in \text{pr}_1(\mathcal{F}_G^0) \) as the Hamiltonian field. By definition
the space \( pr_1(F^0_G) \) consists of \( F \in C^\infty_G(P, DT(e)(g)) \) such that \( (F, X) \in F^0_G \).

Note that \( pr_1(F^0_G) \subsetneq C^\infty_G(P, DT(e)(g)) \) in general case. However, if \( G = U(1) \) one has equality \( pr_1(F^0_G) = C^\infty_G(P) \cong C^\infty(M) \).

Now we generalize the Kostant–Souriau prequantization procedure to the general gauge group case. To this end let us consider the space \( C^\infty_G(P, V) \) of \( V \)-valued functions \( f : P \to V \) equivariant with respect to the gauge group \( G \)

\[
f(pg) = T(g^{-1})f(p),
\]

where \( p \in P \) and \( g \in G \).

The flow \( \{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, \vartheta) \) defines a one-parameter group \( \Sigma_t : C^\infty_G(P, V) \to C^\infty_G(P, V) \) of automorphisms of the vector space \( C^\infty_G(P, V) \)

\[
(\Sigma_t f)(p) := f(\tau_{-t}(p)).
\]

Defining the flow

\[
\tilde{\tau}_t[(p, v)] := [(\tau_t(p), v)]
\]

on the vector bundle \( \tilde{\pi} : V \to M \) one obtains the one-parameter group \( \tilde{\Sigma}_t : C^\infty(M, V) \to C^\infty(M, V) \) acting on the sections \( \psi \in C^\infty(M, V) \) in the following way

\[
(\tilde{\Sigma}_t \psi)(\pi(p)) := \tilde{\tau}_t \psi(\sigma_{-t} \circ \pi(p)) = \tilde{\tau}_t \psi(\pi(\tau_{-t}(p))) = \tilde{\tau}_t \psi(\pi(\tau_{-t}^h(p))).
\]

The isomorphism \( R : C^\infty_G(P, V) \simto C^\infty(M, V) \) defined by

\[
(R f)(\pi(p)) := [(p, f(p))]
\]

intertwines the flows (3.43) and (3.45)

\[
R \circ \Sigma_t = \tilde{\Sigma}_t \circ R
\]

and moreover

\[
R \circ J = I,
\]

where \( I : H \longrightarrow C^\infty(M, V) \) is defined in (2.39).

If \( \xi \) is the velocity vector field of the flow \( \{\tau_t\}_{t \in \mathbb{R}} \), then one can consider \( -L_\xi \) as the generating operator for \( \{\Sigma_t\}_{t \in \mathbb{R}} \). On the other hand the generator \( Q_{(F, X)} \) of the flow \( \{\tilde{\Sigma}_t\}_{t \in \mathbb{R}} \) has the form

\[
Q_{(F, X)} := -(\nabla_X + F),
\]
where \((F, X) \in \mathcal{F}_G^0\). The operator \(\nabla_X\) is the covariant derivative with respect to vector field \(X \in C^\infty(M, TM)\) and the second ingredient \(\tilde{F}\) of the right-hand-side of (3.49) is an endomorphism of \(C^\infty(M, V)\) defined by the function \(F \in pr_1(\mathcal{F}_G^0)\) in the following way:

\[
\tilde{F}([(p, v)]) := [(p, F(p)v)].
\] (3.50)

From (3.47) we find that

\[
\mathcal{R} \circ \mathcal{L}_\xi = (\nabla_X + \tilde{F}) \circ \mathcal{R}.
\] (3.51)

Since \([\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}\) we see from (3.50) that the linear monomorphism

\[
Q : \mathcal{F}_G^0 \longrightarrow \text{End}(C^\infty(M, V))
\] (3.52)

satisfies the prequantization property

\[
[Q_{(F,X)}, Q_{(G,Y)}] = Q_{\{ (F,X), (G,Y) \}},
\] (3.53)

where the bracket \([\cdot, \cdot]\) on the left-hand-side of (3.53) is the commutator of the 1-st order differential operators and the Lie bracket \(\{\cdot, \cdot\}\) is defined in (3.30). In the non-degenerate case, i.e. when \((F, X)\) is defined by \(F\) (see (3.26)) the property (3.53) reduces to

\[
[Q_F, Q_G] = Q_{\{F,G\}},
\] (3.54)

where \(Q_F := Q_{(F,X_F)}\) and the bracket \(\{F,G\}\) is defined by

\[
\{F, G\} := -2\Omega(X_F^* Y_G^*) + [F, G].
\] (3.55)

In the case \(G = U(1)\) the operator (3.49) is the Kostant–Souriau prequantization operator. So, one can consider the construction presented above as a natural generalization of the Kostant–Souriau prequantization procedure.

### 4 Quantization

It is well known that in order to quantize a function \(F \in C^\infty_{U(1)}(P, i\mathbb{R}) \cong C^\infty(M, \mathbb{R})\) (a classical physical quantity) in the Kostant–Souriau geometric quantization one needs to choose a proper polarization \(\mathcal{P} \subset T^c M\) on the symplectic manifold \((M, \Omega)\). Further, using \(\mathcal{P}\) one realizes Hilbert space \(\mathcal{H}\).
by sections of the vector bundle $\tilde{\pi} : V \to M$ in such a way that differential operator $Q_F$, defined in (3.49), preserves $\mathcal{H}$ and admits a self-adjoint extension in it, (for details see e.g. [Sni]).

In the method of quantization which will be discussed here we avoid the notion of polarization. For the construction of the Hilbert space $\mathcal{H}$ we will instead use the $\mathcal{B}(V)$-valued positive definite kernel discussed in Section 2. In fact Hilbert space $K \cong \mathcal{H}$ was defined in Section 2 as one of the triple of equivalent objects used for the description of $\mathcal{B}(V)$-valued positive definite kernels.

In Section 3 we described the flows $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, \vartheta)$ of automorphisms of the principal bundle $\pi : P \to M$ with the fixed $DT(e)(g)$-valued connection form $\vartheta$, see (2.40). Here we restrict ourselves to those flows $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, K) \subset \text{Aut}(P, \vartheta)$ which preserve $\mathcal{B}(V)$-valued positive definite kernel $K$, i.e. such ones that

$$K(\tau_t(p), \tau_t(q)) = K(p, q),$$

for any $p, q \in P$ and $t \in \mathbb{R}$.

The following statement is valid

**Theorem 4.1** The flow $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P)$ satisfies invariance condition (4.1) if and only if there exists a unitary flow $U_t : \mathcal{H} \to \mathcal{H}$ such that

$$\mathfrak{K}(\tau_t(p)) = U_t \mathfrak{K}(p),$$

where $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ is the map satisfying the condition (2.3) of the definition (ii) and is related to the kernel $K(p, q)$ by (2.10).

**Proof:**

Provided the map $\mathfrak{K} : P \to \mathcal{B}(V, \mathcal{H})$ has property (4.2) we obtain (4.1) from the equality (2.11).

Let us take $f, g \in K_0$, where elements of the vector subspace $K_0$ are defined in (2.11). We define the flow $\{U_t\}_{t \in \mathbb{R}}$ on $K_0$ by

$$\begin{align*}
(U_t f)(p) := f(\tau_{-t}(p)) &= \sum_{i=1}^f K(\tau_{-t}(p), p_i) v_i.
\end{align*}$$

The invariance condition (4.1) and (2.11) implies the equality

$$\langle U_t f | U_t g \rangle = \langle f | g \rangle$$

The following statement is valid

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$$\mathfrak{K}(\tau_t(p)) = U_t \mathfrak{K}(p),$$

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$$\begin{align*}
(U_t f)(p) := f(\tau_{-t}(p)) &= \sum_{i=1}^f K(\tau_{-t}(p), p_i) v_i.
\end{align*}$$

The invariance condition (4.1) and (2.11) implies the equality

$$\langle U_t f | U_t g \rangle = \langle f | g \rangle$$

for any $p, q \in P$ and $t \in \mathbb{R}$.
for the scalar product $\langle \cdot | \cdot \rangle$ defined in (2.12). So, we can consider $\{U_t\}_{t \in \mathbb{R}}$ as a unitary flow on the Hilbert space $\mathcal{H} := \mathcal{K}_0$. Let $E_p : \mathcal{H} \to V$ be the evaluation functional at $p \in P$. Let us define the map $\hat{\mathcal{R}} : P \to \mathcal{B}(V, \mathcal{H})$ by $\hat{\mathcal{R}}(p) := E_p^*$. Then for any $v \in V$ and $f \in \mathcal{K}_0$ one has

$$\langle \hat{\mathcal{R}}(\tau_t(p))v|f \rangle = \langle v, E_{\tau_t(p)}^*f \rangle = \langle v, \sum_{i=1}^t K(\tau_t(p), p_i)v_i \rangle = \langle v, E_p U_{-t}f \rangle = \langle \hat{\mathcal{R}}(p)v|U_{-t}f \rangle = \langle U_t \hat{\mathcal{R}}(p)v|f \rangle.$$  

(4.6)

Thus we obtain (4.2).

□

The above theorem implies

**Proposition 4.1** For any flow $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, K)$ one has

$$\mathcal{H} \xrightarrow{\tilde{\mathcal{R}}} \mathcal{K} \xrightarrow{\tilde{\tau}_t} \mathcal{H} \xrightarrow{U_t} \mathcal{H},$$  

(4.7)

i.e. $U_t \circ \tilde{\mathcal{R}} = \tilde{\mathcal{R}} \circ \tilde{\tau}_t$ for $t \in \mathbb{R}$, where $\tilde{\mathcal{R}}$ is the coherent state map defined in (2.35) and the flow $\tilde{\tau}_t : \mathbb{V} \to \mathbb{V}$ is given in (3.44). Moreover, if $K$ is a smooth positive definite kernel, then $\mathcal{H}_0 := \text{span}\{\hat{\mathcal{R}}(\mathbb{V})\} \subset \mathcal{H}$ is an essential domain of the generator $\hat{F}$ of the flow $U_t =: e^{i\hat{F}t}$.

**Proof:**

The equivariance property (4.7) follows from the definition (2.35) and the property (4.2).

From (4.2) we have

$$e^{i\hat{F}t} \hat{\mathcal{R}}([(p, v)]) = \hat{\mathcal{R}}(\tilde{\tau}_t([(p, v)])) = \hat{\mathcal{R}}(\tilde{\tau}_t([(p, v)])) = \hat{\mathcal{R}}(\tilde{\tau}_t([(p, v)])),$$

(4.8)

Since the coherent state map $\tilde{\mathcal{R}}$ is smooth (see Proposition 2.1) and $\{\tilde{\tau}_t\}_{t \in \mathbb{R}}$ is a smooth flow we find that the right-hand-side of (4.8) is differentiable with respect to $t$. Then by Stone theorem $\hat{\mathcal{R}}([(p, v)]) \in \mathcal{D}(\hat{F})$, i.e. $\mathcal{H}_0 \subset \mathcal{D}(\hat{F})$. Moreover, the vector subspace $\mathcal{H}_0 \subset \mathcal{H}$ is invariant with respect to the action
Thus, since \( H_0 \) is dense in \( \mathcal{H} \), it turns to be (see [R-S], Volume 1, Theorem VIII.11) an essential domain of \( \hat{F} \), i.e.

\[ \overline{\hat{F}|_{H_0}} = \hat{F}, \]

(4.9)

where the bar in (4.9) denotes the closure of the symmetric operator \( \hat{F}|_{H_0} \).

Note that within the representation of Hilbert space \( \mathcal{H} \) by the \( V \)-valued functions on \( P \), see (2.5), one has

\[ U_t = J^{-1} \circ \Sigma_t \circ J. \]

(4.10)

Thus for generating operator \( \hat{F} \) we obtain

\[ \hat{F} = iJ^{-1} \circ \mathcal{L}_\xi \circ J. \]

(4.11)

Taking realization of \( \mathcal{H} \) by sections of the bundle \( \tilde{\pi} : V \rightarrow M \), see (2.39) and (3.49), one has

\[ U_t = I^{-1} \circ \hat{\Sigma}_t \circ I \]

(4.12)

and

\[ \hat{F} = iI^{-1} \circ (\nabla_X + \tilde{F}) \circ I. \]

(4.13)

Let us recall in this context that \( J(H_0) = \mathcal{K}_0 \). Note, also that the above two representations of \( \{U_t\}_{t \in \mathbb{R}} \) are intertwined by the operator \( \mathcal{R} \) defined in (3.46).

Since \( H_0 \) is an essential domain of \( \hat{F} \) we have the commutation relation

\[ \hat{F} e^{i\hat{F}_t} = e^{i\hat{F}_t} \hat{F}, \]

(4.14)

valid on the elements of \( \mathcal{H}_0 \). Now, let us define the vector subspace \( \mathcal{U}_1 := \mathcal{H}_0 + \hat{F}(\mathcal{H}_0) \subset \mathcal{H} \). From (4.14) we obtain, in particular, that \( e^{i\hat{F}_t} \mathcal{U}_1 \subset \mathcal{U}_1 \). Due to (4.11) one obtains that for a smooth vector field \( \xi \) and smooth coherent state map \( \tilde{\mathcal{K}} \), the left-hand-side of \( \hat{F} e^{i\hat{F}_t} \tilde{\mathcal{K}}([(p, v)]) = e^{i\hat{F}_t} \hat{F} \tilde{\mathcal{K}}([(p, v)]) \) is differentiable with respect to \( t \). Then by Stone theorem \( \hat{F} \tilde{\mathcal{K}}([(p, v)]) \in \mathcal{D}(\hat{F}) \) and \( \tilde{\mathcal{K}}([(p, v)]) \in \mathcal{D}(\hat{F}^2) \). Thus we have \( \mathcal{U}_1 \subset \mathcal{D}(\hat{F}) \) and \( \mathcal{H}_0 \subset \mathcal{D}(\hat{F}^2) \).

In such a way, step by step, we prove that \( \mathcal{U}_l \subset \mathcal{D}(\hat{F}) \), where \( \mathcal{U}_l \) is defined by

\[ \mathcal{U}_l := \mathcal{U}_{l-1} + \hat{F}(\mathcal{U}_{l-1}), \quad \mathcal{U}_0 := \mathcal{H}_0, \]

(4.15)

for \( l = 1, 2, \ldots \). By \( \mathcal{U}_\infty \) we denote the vector space spanned by all \( \mathcal{U}_l \), \( l \in \mathbb{N} \cup \{0\} \). Summing up the above considerations we formulate the following
Proposition 4.2  One has the filtration
\[ U_0 \subset U_1 \subset \ldots \subset U_\infty \subset \mathcal{D}(\hat{F}) \]  (4.16)
of the domain \( \mathcal{D}(\hat{F}) \) of the operator \( \hat{F} \) onto its essential domains. This filtration is preserved
\[ e^{i\hat{F}t}U_l \subset U_l, \]  (4.17)
by the flow \( \{e^{i\hat{F}t}\}_{t \in \mathbb{R}} \). Moreover
\[ \hat{F}U_l \subset U_{l+1} \]  (4.18)
and
\[ U_\infty \subset \mathcal{D}(\hat{F}^l), \]  (4.19)
for \( l \in \mathbb{N} \cup \{0\} \).

Next proposition shows how to reconstruct the classical Hamiltonian \( F \) from the quantum Hamiltonian \( \hat{F} \).

Proposition 4.3  The generating function \( F : P \longrightarrow DT(e)(g) \) is obtained as the coherent states mean values function of \( \hat{F} \), i.e.
\[ F(p) = i(\mathcal{R}(p)^*\mathcal{R}(p))^{-1}\mathcal{R}^*(p)\hat{F}\mathcal{R}(p). \]  (4.20)

Proof:
Using \( U_t = e^{it\hat{F}} \), from (2.40), (3.27), and (4.2) we have
\[ \mathcal{R}(p)^*\hat{F}\mathcal{R}(p) = -i\frac{d}{dt}[\mathcal{R}^*(p)U_t\mathcal{R}(p)]|_{t=0} = -i\mathcal{R}^*(p)\frac{d}{dt}\mathcal{R}(\tau_t(p))|_{t=0} = \]  (4.21)
\[ = -i\mathcal{R}^*(p)(\xi\mathcal{R})(p) = -i\mathcal{R}^*(p)\mathcal{R}(p)\vartheta(\xi)(p) \]  (4.22)
\[ = -i\mathcal{R}^*(p)\mathcal{R}(p)\vartheta(F^*)(p) = -i\mathcal{R}^*(p)\mathcal{R}(p)F(p) \]  (4.23)
\[ \square \]
5 The coordinate description and examples

In this section we will investigate the quantization procedure which was proposed in Section 4 in terms of a concrete trivialization of the principal bundle \( \pi : P \to M \). Because of its importance for physical application we will discuss the holomorphic case in details. Finally we will present two examples, where in the second example we obtain a holomorphic (anti–holomorphic) realization for any self–adjoint operator with simple spectrum.

Now, for the further investigation of the quantum flow generator \( \hat{F} \) given in \((4.13)\) we will describe its representation in a trivialization

\[
\begin{align*}
\alpha : \Omega \to P, & \quad \pi \circ \alpha = \text{id}_\Omega, \\
\end{align*}
\]

of \( \pi : P \to M \), where \( \bigcup_{\alpha \in A} \Omega_\alpha = M \) is a covering of \( M \) by the open subsets.

We note that on \( \pi^{-1}(\Omega_\alpha) \) one has

\[
\Omega(p) = T(h^{-1}) \left( d\vartheta_\alpha(m) + \frac{1}{2}[\vartheta_\alpha(m), \vartheta_\alpha(m)] \right) T(h),
\]

\[
\mathbf{D}F(p) = T(h^{-1})(dF_\alpha(m) + [\vartheta_\alpha(m), F_\alpha(m)]) T(h),
\]

for \( p = \alpha(m)h \), where

\[
\vartheta_\alpha := \alpha^*\vartheta \quad \text{and} \quad F_\alpha := F \circ \alpha.
\]

The positive definite kernel \( K : P \times P \to \mathcal{B}(V) \) in the trivialization \((5.1)\) is described by

\[
\mathfrak{R}_\alpha(m) := \mathfrak{R} \circ \alpha(m),
\]

\[
K_{\mathfrak{R}\beta}(m, n) := \mathfrak{R}_\alpha(m) \mathfrak{R}_\beta(n),
\]

for \( m \in \Omega_\alpha \) and \( n \in \Omega_\beta \). Using \((5.4)\) from \((2.40)\) we obtain

\[
\vartheta_\alpha(m) = (\mathfrak{R}_\alpha(m)^* \mathfrak{R}_\alpha(m))^{-1} \mathfrak{R}_\alpha(m)^* d\mathfrak{R}_\alpha(m).
\]

Let us take the flow \( \{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P, K) \) and define local cocycle \(- \varepsilon, \varepsilon[\times \Omega_\alpha \exists (t, m) \mapsto g_\alpha(t, m) \in G \)

\[
\tau_t(s_\alpha(m)) = s_\alpha(\sigma_t(m))g_\alpha(t, m).
\]

For \( p = s_\alpha(m)h \) we have

\[
\mathfrak{R}(\tau_t(p)) = \mathfrak{R}_\alpha(\sigma_t(m)) T(g_\alpha(t, m)) T(h).
\]
From (5.9) and (4.2) one gets
\[
i^\alpha \tilde{F}_K(m)v = (X_{\tilde{F}_K})(m)v + \tilde{R}_\alpha(m)\phi_\alpha(m)v, \tag{5.10}
\]
where \( v \in V \) and
\[
\phi_\alpha(m) := \frac{d}{dt}T(g_\alpha(t,m))|_{t=0}. \tag{5.11}
\]

**Proposition 5.1** If we assume that \( \xi = X^* - F^* \in \mathcal{E}_G^0 \) is the velocity vector field for \( \{\tau_t\}_{t \in \mathbb{R}} \), then the map \( \phi_\alpha : \Omega_\alpha \to \mathcal{B}(V) \) is given by
\[
- \phi_\alpha = F_\alpha + \vartheta_\alpha(X) \tag{5.12}
\]

**Proof:**
From (3.35), (5.8) and (5.2) we find that
\[
\tau_t^h(s_\alpha(m),v) = s_\alpha(\sigma_t(m))\kappa_\alpha(t,m) \tag{5.13}
\]
Comparing (5.14) and (5.15) one obtains
\[
T(g_\alpha(t,m)) = T(\kappa_\alpha(t,m)c(t,s_\alpha(m))). \tag{5.16}
\]
Differentiating (5.16) at \( t = 0 \) and taking into account
\[
\vartheta(s_\alpha(m),h) = T(h^{-1})\vartheta_\alpha(m)T(h) + T(h^{-1})dT(h), \tag{5.17}
\]
where \( h \in G \), gives (5.12). \( \square \)

Using (5.2) and (5.3) we find that \( \xi = X^* - F^* \in \mathcal{E}_G^0 \), i.e., \( \mathcal{L}_\xi \vartheta = 0 \), if and only if
\[
\mathcal{L}_X \vartheta_\alpha \equiv X \left[ d\vartheta_\alpha + d(\vartheta_\alpha(X)) \right] = d\phi_\alpha + [\vartheta_\alpha, \phi_\alpha]. \tag{5.18}
\]
The selfadjointess of $\hat{F}$ implies the following relation

$$K_\beta(n)^*(X_{K_\alpha}(m)) + (X_{K_\beta}(n))^{*}K_\alpha(m) + \phi_\alpha(m) + \phi_\beta(n)^*K_\beta(n)^*K_\alpha(m) \equiv 0$$

(5.19)

between the kernel map $K_\alpha : \Omega_\alpha \to \mathcal{B}(V, \mathcal{H})$ and $(F, X) \in \mathcal{F}_G^0$. The transition cocycle $g_{\alpha\beta} : \Omega_\alpha \cap \Omega_\beta \to G$ defined by

$$s_\beta(m) = s_\alpha(m)g_{\alpha\beta}(m),$$

(5.20)

for $m \in \Omega_\alpha \cap \Omega_\beta$ leads to the corresponding gauge transformation of the formulae (5.10) which is given by

$$K_\beta(m) = K_\alpha(m)T(g_{\alpha\beta}(m))$$

(5.21)

and

$$\phi_\beta(m) = T(g_{\beta\alpha}(m))\phi_\alpha(m) + \frac{d}{dt}T(g_{\beta\alpha}(\sigma_t(m)))|_{t=0}T(g_{\alpha\beta}(m)).$$

(5.22)

For the sake of completeness of our exposition in a fixed gauge let us find the expression for the action of Kostant–Souriau operator $Q(F, X) = iI \circ \hat{F} \circ I^{-1}$ on the part of its essential domain spanned by sections of the form $I(\psi)$, where $\psi = K_\beta(n)v$ for $n \in \Omega_\beta, v \in V$. In the $s_\alpha$-gauge section $I(\psi) \in C^\infty(M, V)$ and $Q(F, X)I(\psi)$ are given by

$$I(\psi)(m) = [(s_\alpha(m), K_\alpha^*(m)K_\beta(n)v)]$$

(5.23)

and by

$$Q(F, X)(\psi)(m) = iI(\hat{F}\psi)(m) = [(s_\alpha(m), iK_\alpha^*(m)\hat{F}K_\beta(n)v)]$$

(5.24)

respectively, $m \in \Omega_\alpha$. Hence, using the relation (5.19) we obtain the coordinate expression on $Q(F, X)$ in terms of the kernel $K_{\alpha\beta}(m, n)$:

$$Q(F, X)(K_{\alpha\beta}(\cdot, n))(m)v = -(XK_{\alpha\beta}(\cdot, n))(m)v - \phi_\alpha(m)^*K_{\alpha\beta}(m, n)v.$$  

(5.25)

Recall here that the operator-valued maps $\phi_\alpha : \Omega_\alpha \to \mathcal{B}(V)$ are related to the generating function $F$ by (5.2).

From the viewpoint of physical applications, see e.g. [H-O], one of the most interesting cases appears when $\pi : P \to M$ is a complex analytic principal $GL(N, \mathbb{C})$-bundle. Consequently we will assume that the
coherent state map $\mathcal{R} : P \longrightarrow \mathcal{B}(V, \mathcal{H})$ is a complex analytic map which satisfies the condition $\mathcal{R}(\pi) = \mathcal{H}$ for $T = \text{id}$ and $g \in \text{GL}(N, \mathbb{C}) \cong \text{GL}(V, \mathbb{C})$.

Taking a complex analytic trivialization $s^{\text{hol}}_\alpha : \Omega_\alpha \longrightarrow P$ we find that $\mathcal{R}^{\text{hol}} = \mathcal{R} \circ s^{\text{hol}}_\alpha : \Omega_\alpha \longrightarrow \mathcal{B}(V, \mathcal{H})$ is holomorphic map and so does the transition map $h_{\alpha \beta} : \Omega_\alpha \cap \Omega_\beta \longrightarrow \text{GL}(N, \mathbb{C})$, where $s^{\text{hol}}_\alpha(m) = s^{\text{hol}}_\beta(m) h_{\alpha \beta}(m)$. Thus the kernel $K^{\text{hol}}_{\alpha \beta} = (\mathcal{R}^{\text{hol}})^\ast \mathcal{R}^{\text{hol}} : \Omega_\alpha \times \Omega_\beta \longrightarrow \text{GL}(N, \mathbb{C})$ is a map anti-holomorphic in the first argument and holomorphic in the second one. Using $K^{\text{hol}}_{\alpha \beta}$ we define another trivialization $s^{\text{hol}}_\alpha(m) := s^{\text{hol}}_\alpha(m) K^{\text{hol}}_{\alpha \alpha}(m, m)^{-\frac{1}{2}}$ (5.26)

with a transition cocycle

$$g_{\alpha \beta}(m) := K^{\text{hol}}_{\alpha \alpha}(m, m)^{\frac{1}{2}} h_{\alpha \beta}(m) K^{\text{hol}}_{\beta \beta}(m, m)^{-\frac{1}{2}}.$$ (5.27)

It follows from $s^{\text{hol}}_\beta(m) = s^{\text{hol}}_\alpha(m) h_{\alpha \beta}(m)$ that $g_{\alpha \beta} : \Omega_\alpha \cup \Omega_\beta \longrightarrow U(N) \subset \text{GL}(N, \mathbb{C})$. So, we can reduce the holomorphic coherent state map $\mathcal{R} : P \longrightarrow \mathcal{B}(V, \mathcal{H})$ to the principal $U(N)$-bundle $\pi : P \longrightarrow M$ which is a subbundle of $\pi : P \longrightarrow M$ defined by the trivialization (5.26). Therefore, we can apply the method of quantization investigated in this section to the holomorphic case.

In this connection we note that holomorphic flow $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(P)$ preserves the kernel $K^{\text{hol}}$ defined by $K^{\text{hol}}_{\alpha \beta}$ if and only if

$$K^{\text{hol}}_{\alpha \beta}(\sigma_t(m), \sigma_t(n)) = h_{\alpha}(t, m)^{\ast} K^{\text{hol}}_{\alpha \beta}(m, n) h_{\beta}(t, n),$$ (5.28)

where the holomorphic cocycle $h_{\alpha}(t, m)$ is defined by

$$\tau_t(s^{\text{hol}}_\alpha(m)) = s^{\text{hol}}_\alpha(\sigma_t(m)) h_{\alpha}(t, m)$$ (5.29)

The cocycles $g_{\alpha}(t, m)$ and $h_{\alpha}(t, m)$ corresponding to the sections $s_\alpha : \Omega_\alpha \longrightarrow P$ and $s^{\text{hol}}_\alpha : \Omega_\alpha \longrightarrow P$, respectively, are related by

$$g_{\alpha}(t, m) = K^{\text{hol}}_{\alpha \alpha}(\sigma_t(m), \sigma_t(m))^{\frac{1}{2}} h_{\alpha}(t, m) K^{\text{hol}}_{\alpha \alpha}(m, m)^{-\frac{1}{2}},$$ (5.30)

where $g_{\alpha}(t, m)$ is defined in (5.8).

Note here that

$$\mathcal{R}_\alpha(m) = s^{\text{hol}}_\alpha(m) K^{\text{hol}}_{\alpha \alpha}(m, m)^{-\frac{1}{2}}.$$ (5.31)
Using (5.30) we find that

\[
\phi_\alpha(m) = K^{\text{hol}}_{\alpha\bar{\alpha}}(m, m) \frac{d}{dt} \phi_\alpha(m) K^{\text{hol}}_{\bar{\alpha}\alpha}(m, m)^{-\frac{1}{2}} - K^{\text{hol}}_{\alpha\bar{\alpha}}(m, m)^{\frac{1}{2}} \frac{d}{dt} K^{\text{hol}}_{\alpha\bar{\alpha}}(\sigma_t(m), \sigma_t(m))^{-\frac{1}{2}} |_{t=0},
\]

(5.32)

where \( \phi^{\text{hol}}_\alpha : \Omega_\alpha \rightarrow GL(N, \mathbb{C}) \) defined by

\[
\phi^{\text{hol}}_\alpha(m) := \frac{d}{dt} h_\alpha(t, m)|_{t=0}
\]

(5.33)
is a holomorphic map.

One also has

\[
F^{\alpha}(m) = K^{\text{hol}}_{\alpha\bar{\alpha}}(m, m) \frac{d}{dt} F^{\text{hol}}_{\alpha}(m) K^{\text{hol}}_{\bar{\alpha}\alpha}(m, m)^{-\frac{1}{2}}
\]

(5.34)

where \( F^{\alpha} = F \circ s_\alpha \) and \( F^{\text{hol}}_{\alpha} = F \circ s^{\text{hol}}_{\alpha} \) and

\[
\vartheta^{\alpha}(m) = K^{\text{hol}}_{\alpha\bar{\alpha}}(m, m) \frac{d}{dt} \vartheta^{\text{hol}}_{\alpha}(m) K^{\text{hol}}_{\bar{\alpha}\alpha}(m, m)^{-\frac{1}{2}} + K^{\text{hol}}_{\alpha\bar{\alpha}}(m, m)^{\frac{1}{2}} \frac{d}{dt} K^{\text{hol}}_{\alpha\bar{\alpha}}(m, m)^{-\frac{1}{2}}.
\]

(5.35)

From (5.31) and (5.32) we obtain the holomorphic representation

\[
i \hat{F} \hat{\kappa}^{\text{hol}}_{\alpha}(m)v = (X^{(1,0)} \hat{\kappa}^{\text{hol}}_{\alpha})(m)v + \hat{\kappa}^{\text{hol}}_{\alpha}(m) \phi^{\text{hol}}_\alpha(m)v
\]

(5.36)
of the generating operator \( \hat{F} \). Vector field \( X^{(1,0)} \) appearing in (5.36) is \((1,0)\)-component of the vector field \( X = X^{(1,0)} + X^{(0,1)} \) tangent to the flow \( \{ \tau_t \}_{t \in \mathbb{R}} \). Note that \( X^{(0,1)} = \bar{X}^{(1,0)} \) and vectors \( \hat{\kappa}^{\text{hol}}_{\alpha}(m)v \in \mathcal{H} \), such that \( m \in \Omega_\alpha \), \( v \in V \) and \( \alpha \in A \), span an essential domain of \( \hat{F} \).

Using (5.36) we obtain anti-holomorphic representation of Kostant-Souriau operator

\[
(Q_{(F,X)} K^{\text{hol}}_{\alpha\beta}(\cdot, n))(m)v = -(X^{(0,1)} K^{\text{hol}}_{\alpha\beta}(\cdot, n))(m)v - \phi^{\text{hol}}_{\alpha}(m)^* K^{\text{hol}}_{\alpha\beta}(m, n)v.
\]

(5.37)

Hence we see, that in the holomorphic case the essential domain of \( Q_{(F,X)} \) consists of those anti-holomorphic sections of \( \pi : V \rightarrow M \) which are locally spanned by \( K^{\text{hol}}_{\alpha\beta}(\cdot, n)v \).

Finally we present two simple examples illustrating our method of quantization.
Example 1 We consider the trivial principal $GL(2, \mathbb{C})$-bundle $P = M \times GL(2, \mathbb{C})$ where $M = \mathbb{D} \times \mathbb{D}$ is the product of two unit discs $\mathbb{D} \subset \mathbb{C}$. Let $z = \left( \frac{z_1}{z_2} \right)$, $w = \left( \frac{w_1}{w_2} \right)$ be elements of $\mathbb{D} \times \mathbb{D}$. We introduce the positive definite kernel $K^{\text{hol}} : P \times P \longrightarrow \mathcal{B}(\mathbb{C}^2)$ which in the standard trivialization $s(z) := (z, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ has the form

$$K^{\text{hol}}_{\mathbb{D} \times \mathbb{D}}(\bar{z}, w) := \left( 1 + \frac{1 - \bar{z}_1 w_1}{\bar{z}_1} \right)^{\frac{1}{2}} \left( 1 + \frac{w_1}{1 - \bar{z}_2 w_2} + \bar{z}_1 w_1 \right)$$  \hspace{1cm} (5.38)

and the holomorphic flow $\sigma_t : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D} \times \mathbb{D}$, $t \in \mathbb{R}$, defined by

$$\sigma_t \left( \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) := \left( e^{it} z_1 e^{-it} z_2 \right).$$  \hspace{1cm} (5.39)

Kernel (5.38) satisfies the relationship

$$K^{\text{hol}}_{\mathbb{D} \times \mathbb{D}}(\overline{\sigma_t(z)}, \sigma_t(w)) = h^+(t) K^{\text{hol}}_{\mathbb{D} \times \mathbb{D}}(\bar{z}, w) h(t),$$  \hspace{1cm} (5.40)

where $h(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, $t \in \mathbb{R}$. Thus the flow $\tau_t : P \longrightarrow P$ defined by $\tau_t(s(z)) = s(\sigma_t(z))h(t)$ preserves the positive definite kernel

$$K^{\text{hol}}((\bar{z}, g^+), (w, h)) := g^+ K^{\text{hol}}_{\mathbb{D} \times \mathbb{D}}(\bar{z}, w),$$  \hspace{1cm} (5.41)

where $(z, g^+), (w, h) \in (\mathbb{D} \times \mathbb{D}) \times GL(2, \mathbb{C})$.

The vector field

$$X = i \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right)$$  \hspace{1cm} (5.42)

tangent to the one-parameter group (5.39) and function $F : \mathbb{D} \times \mathbb{D} \times U(2) \longrightarrow \text{Mat}_{2 \times 2}(\mathbb{C})$ defined by

$$F(z, g) = g^{-1} K^{\text{hol}}(\bar{z}, z)^{\frac{1}{2}} F^{\text{hol}}(z) K^{\text{hol}}(\bar{z}, z)^{-\frac{1}{2}} g,$$  \hspace{1cm} (5.43)

where $F^{\text{hol}}(m)$ is given by

$$F^{\text{hol}}(m) = \left( \begin{array}{cc} \frac{i(4 - 3|z_1|^2 - 2|z_2|^2)^2 + |z_1|^2 + |z_2|^2}{(1 - |z_2|^2)(1 - |z_2|^2|z_2|^2 - |z_2|^4)} & \frac{i z_2 (1 - |z_1|^2)(2 - |z_2|^2 - 2|z_2|^2)}{(1 - |z_2|^2)(1 - |z_2|^2|z_2|^2 - |z_2|^4)} \\ \frac{i z_1 (1 - |z_1|^2)(2 - |z_2|^2 - 2|z_2|^2)}{(1 - |z_1|^2)(1 - |z_1|^2|z_1|^2 - |z_1|^4)} & \frac{i(1 - |z_1|^2)(-4 + 2|z_1|^2 + 4|z_2|^2 - 2|z_2|^4 - 4|z_2|^2|z_2|^2)}{(1 - |z_2|^2)(4 - |z_1|^2 - 2|z_1|^4) - 4|z_2|^2|z_2|^4 \right).$$  \hspace{1cm} (5.44)
satisfy equation (3.26). Applying the formulae (5.37) in this case we obtain Kostant–Souriau operator

\[
(Q_{(F,X)} \psi)(\bar{z}_1, \bar{z}_2) = i \left( \bar{z}_1 \frac{\partial \psi_1(\bar{z}_1, \bar{z}_2)}{\partial \bar{z}_1} + \psi_1(\bar{z}_1, \bar{z}_2) \right)
\]

(5.45)

where \( \psi \in D(Q_{(F,X)}) \) is given by

\[
\psi(\bar{z}_1, \bar{z}_2) = \left( \psi_1(\bar{z}_1, \bar{z}_2) \psi_2(\bar{z}_1, \bar{z}_2) \right) = \left( \sum_{k=1}^{K} v_{1k} \frac{b_{1k}}{1-\bar{z}_1 b_{1k}} + c \sum_{k=1}^{K} v_{2k} \frac{b_{2k}}{1-\bar{z}_2 b_{2k}} + c \bar{z}_1 \right)
\]

(5.46)

and constant \( c, v_{1k}, w_{1k}, v_{2k}, w_{2k} \in \mathbb{C} \) satisfy the condition

\[
c = \sum_{k=1}^{K} v_{1k} + w_{1k} v_{2k}.
\]

(5.47)

The unitary flow generated by \( Q_{(F,X)} \) is the quantization of the flow (5.39).

**Example 2**  Let \( \hat{F} \) be a self-adjoint operator with simple spectrum acting in Hilbert space \( \mathcal{H} \). Fix standard Hilbert space isomorphism \( U : \mathcal{H} \rightarrow L^2(\mathbb{R}, d\sigma) \), where measure \( d\sigma \) is determined by spectral measure \( dE \) of \( \hat{F} \) and a certain choice of normalized cyclic vector \(|0\rangle \in \mathcal{H}\) for \( \hat{F} \):

\[
d\sigma(\omega) := \langle 0 | dE(\omega) | 0 \rangle,
\]

(5.48)

\( \omega \in \mathcal{R} \). The homogeneous polynomials \( \{\omega^n\}_{n=0}^{\infty} \) form a subset linearly dense in \( L^2(\mathbb{R}, d\sigma) \). After Gram-Schmidt orthonormalization procedure they give an orthonormal polynomial basis \( \{P_n\}_{n=0}^{\infty} \), deg\( P_n = n \), in \( L^2(\mathbb{R}, d\sigma) \). Acting by \( P_n(\hat{F}) \) on \(|0\rangle \) one obtains the orthonormal basis

\[
|n\rangle := P_n(\hat{F}) |0\rangle
\]

(5.49)

in Hilbert space \( \mathcal{H} \).

We now assume the condition

\[
\limsup_{n \to \infty} \sqrt[n]{|\mu_n|} < +\infty
\]

(5.50)

on the absolute moments

\[
|\mu_n| := \int_{\mathbb{R}} |\omega|^n d\sigma(\omega) = \frac{1}{P_0^2} \langle 0 | \hat{F}^n | 0 \rangle
\]

(5.51)
of the operator $\hat{F}$. It follows from (5.50) that there exists the maximal open strip $\Sigma \subset \mathbb{C}$ in complex plane $\mathbb{C}$, which is invariant under the translations

$$\sigma_t(z) := z + t$$

$t \in \mathbb{R}$ and such that the characteristic functions

$$\chi(s) = \int_{\mathbb{R}} e^{-i\omega s} d\sigma(\omega),$$

$s \in \mathbb{R}$, of the measure $d\sigma$ possesses holomorphic extension $\chi_\Sigma$ to $\Sigma$, see [H-O-T].

Hence, one can define on the principal $U(1)$-bundle $P := \Sigma \times U(1)$ the positive definite kernel:

$$K[[z, g], (v, h)] := \bar{g}K_\Sigma(\bar{z}, v)h$$

(5.54)

where

$$K_\Sigma(\bar{z}, v) := \chi_\Sigma(\bar{z} - v)$$

(5.55)

and $(z, g), (v, h) \in \Sigma \times U(1)$. The map $\mathfrak{R}_\Sigma : \Sigma \rightarrow \mathcal{H} \cong \mathcal{B}(\mathbb{C}, \mathcal{H})$ defined by

$$\mathfrak{R}_\Sigma(\tau) := \sum_{n=0}^{\infty} \chi_n(z)|n\rangle$$

(5.56)

where

$$\chi_n(z) := \int e^{-iz\omega} P_n(\omega)d\sigma(\omega),$$

(5.57)

for $z \in \Sigma$, gives factorization

$$K_\Sigma(\bar{z}, v) = \mathfrak{R}_\Sigma(z)^* \mathfrak{R}_\Sigma(v)$$

(5.58)

of the kernel (5.55). From (5.56) and (5.57) it follows that

$$e^{-i\hat{F}} \mathfrak{R}_\Sigma(z) = \mathfrak{R}_\Sigma(z + t).$$

(5.59)

Thus we find that coherent states $\mathfrak{R}_\Sigma(z), z \in \Sigma$, span an essential domain $\mathcal{D}(\hat{F})$ of $\hat{F}$ and

$$\hat{F}\mathfrak{R}_\Sigma(z) = i\frac{d}{dz}\mathfrak{R}_\Sigma(z).$$

(5.60)
According to (5.2) the curvature form $\Omega_\Sigma$ of the connection form $\vartheta_\Sigma$ defined by the kernel (5.56) is given by

$$\Omega_\Sigma = i\partial \bar{\partial}(\log \circ K_\Sigma)(\bar{z}, z) = i(\log \circ \chi_\Sigma)'(\bar{z} - z)d\bar{z} \wedge dz. \quad (5.61)$$

For the vector field $X = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$ tangent to the translation flow (5.52) one has

$$X_L \Omega_\Sigma = dF \quad (5.62)$$

where

$$F = (\log \circ \chi_\Sigma)'(\bar{z} - z). \quad (5.63)$$

We summarize the above facts in the proposition

**Proposition 5.2** Operator $\hat{F}$ can be obtained by quantization of the classical Hamiltonian $F$, given in (5.63), which generates Hamiltonian flow (5.52) on symplectic manifold $(\Sigma, \Omega_F)$.

Representing $\hat{F}$ in the Hilbert space $I(\mathcal{H})$ spanned by the anti-holomorphic function $K_\Sigma(\cdot, v)$, $v \in \Sigma$, we find for $F$ its Kostant–Souriau operator

$$Q_F = I \circ \hat{F} \circ I^{-1} = i \frac{d}{d\bar{z}}. \quad (5.64)$$

In conclusion let us note that our procedure of quantization, applied to the case considered in this example, leads to realization (5.64) of the operator $\hat{F}$ in function Hilbert space $I(\mathcal{H})$ which is an alternative to its spectral representation in $L^2(\mathbb{R}, d\sigma)$.

**References**

[B-G] D. Beltiţă, J. E. Galé, “Universal objects in categories of reproducing kernels”, Revista Matematica Iberoamericana 27 no. 1 (2011) 123-179.

[F-L-S] R. Feynman, R. Leighton, M. Sands, “The Feynman lectures on physics. Volume III”, Addison-Wesley Publishing 1965.

[G] K. Gawędzki, “Fourier-like kernels in geometric quantization”, Dissertationes Mathematicae CXXVIII (1976).

[H-O] M. Horowski, A. Odzijewicz, “Geometry of the Kepler System in Coherent States Approach”, Ann. Inst. Henri Poincaré 59 1 (1993) 69-89.
[H-O-T] M. Horowski, A. Odzijewicz, A. Tereszkiewicz, “Some integrable systems in nonlinear quantum optics”, J. Math. Phys. 44 (2003) 480-506.

[Kir] A. A. Kirillov, “Unitary representations of nilpotent Lie groups”, Usp. Mat. Nauk 17 (1962) 53104 (in Russian).

[Kos] B. Kostant, “Quantization and Unitary Representations”, Lect. Notes in Math., Vol. 170, Berlin, Heidelberg, New York: Springer 1970, 87-208.

[K-N] S. Kobayashi, K. Nomizu, “Foundations of differential geometry”, Interscience Publishers New York London 1963.

[N] K.-H. Neeb, “Holomorphy and convexity in Lie theory”, Walter de Gruyter, Berlin New York 2000.

[O88] A. Odzijewicz, “On reproducing kernels and quantization of states”, Commun. Math. Phys. 114 (1988) 577-597.

[O92] A. Odzijewicz, “Coherent states and geometric quantization”, Commun. Math. Phys. 150 (1992) 385-413.

[O-´S] A. Odzijewicz, M. ´Swi¸ etochowski, “Coherent states map for MIC-Kepler system”, J. Math. Phys. 38 10 (1997).

[PW] Z. Pasternak–Winiarski, “On the dependence of the reproducing kernel on the weight of integration”, J. Funct. Anal. 94 (1990) 110-134.

[R-S] M. Reed, B. Simon, “Methods of Modern Mathematical Physics”, Academic Press New York London 1972.

[´Sni] J. Śniatycki, “Geometric Quantization and Quantum Mechanics”, Applied Mathematical Science, Vol. 30, Springer Verlag, New York, 1980.

[Sou] J. M. Souriau, “Structure des systèmes dynamiques”, Paris: Dunod, 1970.