Learning Time Varying Risk Preferences from Investment Portfolios using Inverse Optimization with Applications on Mutual Funds

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The fundamental principle in Modern Portfolio Theory (MPT) is based on the quantification of the portfolio’s risk related to performance. Although MPT has made huge impacts on the investment world and prompted the success and prevalence of passive investing, it still has shortcomings in real-world applications. One of the main challenges is that the level of risk an investor can endure, known as risk-preference, is a subjective choice that is tightly related to psychology and behavioral science in decision making. This paper presents a novel approach of measuring risk preference from existing portfolios using inverse optimization on the mean-variance portfolio allocation framework. Our approach allows the learner to continuously estimate real-time risk preferences using concurrent observed portfolios and market price data. We demonstrate our methods on real market data that consists of 20 years of asset pricing and 10 years of mutual fund portfolio holdings. Moreover, the quantified risk preference parameters are validated with two well-known risk measurements currently applied in the field. The proposed methods could lead to practical and fruitful innovations in automated/personalized portfolio management, such as Robo-advising, to augment financial advisors’ decision intelligence in a long-term investment horizon.

1. Introduction
Risk preference (risk tolerance or risk aversion) is a fundamental concept modeling individual preference for certainty under uncertainty. In portfolio allocation, one primary goal has been to reconcile empirical information about securities prices with theoretical models of asset pricing under conditions of inter-temporal uncertainty (Cohn 1975). The notion of risk preference has been an essential assumption underlying almost all such models (Cohn 1975). However, measurement of risk preference has been treated in separate paths. In finance domain, quantification of risk can be roughly summarized as ratio comparisons between wealth and risky assets under market volatility.

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However, the underlying biological, behavioral, and social factors behind risk appetite are commonly studied in many other disciplines, i.e., social science (Payne et al. 2017, Guiso and Paiella 2008), behavior science (Brennan and Lo 2011), mathematics (von Neumann and Morgenstern 1947), psychology (Sokol-Hessner et al. 2009, Mcgraw et al. 2010), and genetics (Linner 2019). For more than half of a century, many measures of risk preference have been developed in various fields, including curvature measures of utility functions (Arrow 1971, Pratt 1964), human subject experiments and surveys (Rabin and Thaler 2001, Holt and Laury 2002), portfolio choice for financial investors (Guiso and Paiella 2008), labor-supply behavior (Chetty 2006), deductible choices in insurance contracts (Cohen and Einav 2007, Szpiro 1986), contestant behavior on game shows (Post et al. 2008), option prices (Aït-Sahalia and Lo 2000) and auction behavior (Lu and Perrigne 2008). Nowadays, investor-consumers’ risk preferences are mainly investigated through one or combination of three ways. The first one is assessing actual behavior. Two examples are inferring households’ risk attitudes using regression analysis on historical financial data (Schooley and Worden 1996) and inferring investors’ risk preferences from their trading decision using reinforcement learning (Alsabah et al. 2020, Wang et al. 2020). The second one is assessing responses to hypothetical scenarios about investment choices (see Barsky et al. (1997) and (Hey 1999)). In practice, online questionnaires are widely adopted by Robo-advising firms to evaluate investors’ risk profiles (Alsabah et al. 2020). The third one is subjective questions (see Hanna et al. (1998) for a survey of these different techniques).

Despite its profound importance in economics, there remain some limitations with respect to learning the risk preference using classic approaches. First, many existing methods assume inherent small deviations of the risk preference, which is “valid only for potential losses that are relatively unthreatening to the individuals’ wealth” (Thomas 2016). Second, most practices in place are often insufficient to deal with scenarios where investment decisions are managed by machine learning processes (Robo-advising), and risk preferences are expected as input parameters that can generate new decisions directly (auto-rebalancing). Currently, Robo-advisors first communicate and categorize clients’ risk preferences based on human interpretation, and later map them to the nearest values of a finite set of representative risk preference levels (Capponi et al. 2019). Those limitations make existing theories and approaches challenging to deal with prominent situations in which risk preference changes dramatically in an inter-temporal dimension, such as savings, investment, consumption problems, dynamic labor supply decisions, and health decisions (O’Donoghue and Somerville 2018). Real-world risk preference is clearly not as straightforward as many theories have assumed, and perhaps individuals even do not exhibit risk appetite consistently in their behaviors across domains (O’Donoghue and Somerville 2018). Namely, risk preference is
time varying in reality. Nowadays, due to the technological advances, we already have overwhelming behavioral data across all domains, providing a myriad of additional sources to help us decipher the perplexity of risk preference from different angles. Although these new approaches might not ultimately prove to be the best models for studying risk preference, they can be used in conjunction with traditional methods with additional, more nuanced implications that are borne out by data (O’Donoghue and Somerville 2018).

To tackle aforementioned limitations, we present a novel inverse optimization approach to measure risk preference directly from market signals and portfolios. In a volatile market, the same portfolio created by buy and hold investors could reflect very different risk levels when market conditions change, regardless most of the time, investor’s subjective risk preference remains unchanged. Our approach’s primary motivation is to complement traditional approaches through continuous monitoring and evaluation of embedded portfolio risks and to facilitate the automated decision process of portfolio adjustment when necessary to ensure that real portfolio risk is aligned with investor’s true risk preference. Our method is developed based on two fundamental methodologies: convex optimization based Modern Portfolio Theory (MPT) and learning decision-making scheme through inverse optimization. We assume investors are rational and their portfolio decisions are near-optimal. Consequently, their decisions are affected by the risk preference factors through portfolio allocation model. We propose an inverse optimization framework to infer the risk preference factor that must have been in place for those decisions to have been made. Furthermore, we assume risk preference stays constant at the point of decision, but varies across multiple decisions over time, and can be inferred from joint observations of time-series market price and asset allocations.

Our inverse optimization approach represents an integral component of a general view of using machine learning to learn individual investor’s risk preference, which embodies data and models from three aspects. The first aspect is leveraging demographic features, such as education, financial status, gender, age, to learn risk preference. Such type of information is collected once, and its impact on risk preference can be learned by applying statistical models on a large amount of client data. The second aspect reflects influences on risk preference from lifetime events, such as mortgage payment, disaster loss, accidental medical conditions and expense. These significant events typically have huge impacts on investment goals or contribution plans, leading to subsequent changes in risk preferences. The third aspect is extracting insights from consumer financial behavior, which involves understanding how investor-consumers make financial decisions, and how these decisions are reflected in the interactions of financial products. The data that captures these behaviors is probably the best source to investigate the underlying risk preferences behind decisions. In an automated financial advice system, risk evaluation coming from the first and second aspects are often sporadic, whereas the estimation from the third aspect is real-time insights based on continuous
monitoring and measurement. Therefore, a successful investment management system shall always compare and validate results from all three components and ensure their consistency throughout the entire investment horizon. Unfortunately, the third aspect was considered intractable until the recent advances of big data and machine learning offer new approaches to solve those challenging problems. Our approach offers a tractable way to monitor and measure investors’ risk preferences by observing their portfolios, since portfolios are considered as outcomes of an investor’s investment decisions.

We demonstrate our approach using 20 years of market data and 10 years of mutual fund quarterly holding data. The reasons of using mutual fund portfolios are threefold. First, mutual fund holdings are freely accessible public data and it is easier to discuss and interpret results using public data than clients’ private data. Second, mutual funds are usually constructed by tracking industry capitalization indices, or managed by fund managers. Thus, they can be considered as optimal portfolios constructed through rational decision making. Moreover, risk measurements of mutual funds are common knowledge reported by several well-known metrics and estimations with our methods can be directly validated by these metrics. Third, mutual funds usually are diversified on large numbers of assets and they fit the underlying MPT used in our model. Moreover, high-dimensional portfolios are challenging to optimize in the inverse optimization problem and they really test the efficiency of our approach. Numerical experiments show that our approach can directly tackle learning tasks involving hundreds of assets. For tasks involving more than one thousand assets, we propose two different approaches, sector aggregation and factor projection, to transform the problems into lower dimensional space.

**Our contributions** We summarize the major contributions of our paper as follows:

- To the best of our knowledge, we propose the first inverse optimization approach for learning time varying risk preference parameters of the mean-variance portfolio allocation model, based on a set of observed mutual fund portfolios and underlying asset price data. The flexibility of our approach also enables us to move beyond mean-variance and adopt more general risk metrics.

- The proposed method provides an effective solution in Robo-advising where risk-return trade-off needs to be dynamically updated based on the risk profile communicated by the client. Risk preference values learned from our approach can be used directly as input parameters, or as references of market risk preference, in Robo-advising portfolio construction.

- Our inverse optimization approach is able to handle learning task that consists of hundreds of assets in portfolio, and efficiently learn from long sequences of time-series data composed by thousands of observations. For portfolios composed by more than one thousand assets, we propose Sector-based and Factor-base aggregation to improve the computational efficiency. In particular, to our knowledge, it is the first time factor analysis is introduced in inverse optimization approach on portfolio allocation.
• We collect and process 10 years of mutual fund portfolio holding and 20 years of market price data to demonstrate the proposed algorithms. Our data collection and engineering process is scalable to gather all available mutual fund historical holdings. To the best of our knowledge, historical fund portfolio holding data has been very difficult to find, and we aim to share this data to the public to facilitate related researches.

2. Background

The fundamental mean-variance portfolio optimization model developed by Markowitz (1952) and its variants assume that investors estimate the risk of the portfolio according to the variability of the expected return. Moreover, Markowitz (1952) assumes that investors make decisions solely based on the preferences of two objectives: the expected return and the risk. The trade-off between the two objectives is typically denoted by a positive coefficient and referred to as risk tolerance (or risk aversion). Later, Black and Litterman (1992) extends the framework in Markowitz (1952) by blending investors’ private expectations, known as Black-Litterman (BL) model. A Bayesian statistical interpretation of BL model is proposed in He and Litterman (2002) and an inverse optimization perspective is derived in Bertsimas et al. (2012). Most of these mean-variance based approaches assume an investor’s risk preference is known. For example, in Bertsimas et al. (2012), the risk-award trade-off \( \delta \) is denoted by the ratio between expected profit and variance. As they mention: Even though there are various proposals in the literature, there is no consensus on how to fit \( \delta \). (Bertsimas et al. 2012) Later, in their experiments, \( \delta \) is exogenously set to 1.25 based on the suggestion of He and Litterman (2002). Suggested empirical risk aversion values also vary from expert to expert. Ang (2014) suggests a range from 1 to 10 for retail investors and believes it is rare to have risk aversion greater than 10 (Ang 2014). Fabozzi et al. (2007), however, states that risk aversion value should be somewhere between 2 and 4.

In expected utility theory, risk aversion relies on the choice of (usually nonlinear) a von Neumann–Morgenstern utility function \( u(c) \), where \( c \) represents value change in wealth. Absolute risk aversion \(-u''(c)/u'(c)\) and relative risk aversion \(-u''(c)c/u'(c)\) (Arrow 1971, Pratt 1964) are used to measure how much utility an investor gains (or loses) as the increase (or decrease) of wealth, and how risk aversions are compared across different individuals. Unfortunately, selection of such utility functions to fit representative investors is very challenging because the exact form and parameters of utility functions are generally unknown, and their selections depend on the objectives and preferences of the investor (Warren 2018). This is essentially a subjective process, and the literature has reached no consensus over which utility function provides the best description of individual behavior (Starmer 2000).

In CAPM (Capital Asset Pricing Model) (Trevnor 1961, Sharpe 1964, Lintner 1969, Mossin 1966), the relationship between risk and expected return is modeled through \( \beta \), an indicator
measures relative volatility of the target security/portfolio comparing to market. Market benchmark portfolio has a *beta* of 1.0, and larger *beta* value usually means the security/portfolio can potentially outperform the average market return in a larger margin, and thus allow investors to gauge whether the cost (price) is consistent with such a likely return.

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