ON RELATIVISTIC PERTURBATIONS OF SECOND AND HIGHER ORDER

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ABSTRACT

We present the results of a study of the gauge dependence of spacetime perturbations. In particular, we consider gauge invariance in general, we give a generating formula for gauge transformations to an arbitrary order $n$, and explicit transformation rules at second order.

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1. Introduction

Second order treatments have been recently proposed, both in cosmology and compact object theory, as a way of obtaining more accurate results to be compared with present and future observations. Also, second order perturbations provide a reliable measure of the accuracy of the linearized theory. However, as it is well known, relativistic perturbations are, in general, gauge dependent. Here we illustrate some results we have recently derived concerning this issue and the one of gauge invariance. We shall omit proofs, for reasons of space.

2. Knight diffeomorphisms

Let \( M \) be a \( m \)-dimensional differentiable manifold, and let \( \xi \) be a vector field on \( M \), generating a flow \( \phi : \mathbb{R} \times M \to M \), where \( \phi(0, p) = p, \forall p \in M \). For any given \( \lambda \in \mathbb{R} \), we shall write, as usual, \( \phi_\lambda(p) := \phi(\lambda, p), \forall p \in M \). If \( T \) is a tensor field on \( M \), the pull–back \( \phi^*_\lambda T \) defines a new field \( \phi^*_\lambda T \) on \( M \), which is thus a function of \( \lambda \). Then it is known that \( \phi^*_\lambda T \) admits the following expansion around \( \lambda = 0 \):

\[
\phi^*_\lambda T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} L^k \xi T = e^{\lambda L_\xi} T ,
\]

where \( L_\xi \) denotes the Lie derivative along \( \xi \). It is worth pointing out that the proof of (1) uses the group property \( \phi_{\lambda+\sigma} = \phi_\lambda \circ \phi_\sigma \).

Let us now suppose that there are two vector fields \( \xi(1) \) and \( \xi(2) \) on \( M \), generating the flows \( \phi^{(1)} \) and \( \phi^{(2)} \). We can combine \( \phi^{(1)} \) and \( \phi^{(2)} \) to define a new one-parameter family of diffeomorphisms \( \Psi : \mathbb{R} \times M \to M \), whose action is given by \( \Psi_\lambda := \phi^{(2)}_{\lambda^2/2} \circ \phi^{(1)}_\lambda \). Thus, \( \Psi_\lambda \) displaces a point of \( M \) a parameter interval \( \lambda \) along the integral curve of \( \xi(1) \), and then an interval \( \lambda^2/2 \) along the integral curve of \( \xi(2) \). With a chess-inspired terminology, we shall call it a knight diffeomorphism, or simply a knight. This concept can be immediately generalized to the case in which \( n \) vector fields \( \xi(1), \ldots, \xi(n) \) are defined on \( M \), corresponding to the flows \( \phi^{(1)}, \ldots, \phi^{(n)} \). Then we define a one-parameter family \( \Psi : \mathbb{R} \times M \to M \) of knights of rank \( n \) by

\[
\Psi_\lambda := \phi^{(n)}_{\lambda^n/n!} \circ \cdots \circ \phi^{(2)}_{\lambda^2/2} \circ \phi^{(1)}_\lambda ,
\]

and the vector fields \( \xi(1), \ldots, \xi(n) \) will be called the generators of \( \Psi \). Of course, \( \Psi_\sigma \circ \Psi_\lambda \neq \Psi_{\sigma+\lambda} \); consequently, (1) cannot be applied if we want to expand in \( \lambda \) the pull-back \( \Psi^*_\lambda T \) of a tensor field \( T \) defined on \( M \). However, the result is easily extended, because the pull-back \( \Psi^*_\lambda T \) of a tensor field \( T \) by a one-parameter family of knights \( \Psi \) with generators \( \xi(1), \ldots, \xi(k), \ldots \) can be expanded around \( \lambda = 0 \) as follows:

\[
\Psi^*_\lambda T = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \cdots \sum_{l_k=0}^{+\infty} \sum_{l_{k+1}=0}^{+\infty} \cdots \lambda^{l_1+2l_2+\cdots+kl_k+\cdots} \frac{\lambda^{l_1+2l_2+\cdots+kl_k+\cdots} T}{2^{l_2} \cdots (k!)^{l_k} l_1! l_2! \cdots l_k! \cdots (\xi(1) \xi(2) \cdots \xi(k))} \cdot \ldots \cdot \ldots \cdot \ldots T .
\]

\(^*\)In order not to burden the discussion unnecessarily, we suppose that \( \phi \) is a one-parameter group of diffeomorphisms, defining global transformations of \( M \).
The proof of this simply requires the repeated application of (4). The explicit form of (3) up to the second order in \( \lambda \) is

\[
\Psi^* T = T + \lambda \mathcal{L}_{\xi(1)} T + \frac{\lambda^2}{2} \left( \mathcal{L}_{\xi(1)}^2 + \mathcal{L}_{\xi(2)} \right) T + \cdots.
\]

Equations (3) and (4) apply to a one-parameter family of knights of arbitrarily high rank, and can be specialized to the particular case of rank \( n \) simply by setting \( \xi(k) \equiv 0, \forall k > n \). Applying (3) to one of the coordinate functions on \( M \), \( x^\mu \), we have, since \( \Psi^* x^\mu(p) = x^\mu(\Psi(p)) \), the extension to second order in \( \lambda \) of the action of an “infinitesimal point transformation”:

\[
\tilde{x}^\mu := x^\mu(\Psi(p)) = x^\mu(p) + \lambda \xi^\mu(1) + \frac{\lambda^2}{2} \left( \xi^\mu_{(1,\nu)} \xi^\nu(1) + \xi^\mu_{(2)} \right) + \cdots.
\]

Knights are rather special, and (3) may seem of limited applicability. This is, however, not the case, as shown by the following

**Theorem:** Let \( \Psi : \mathbb{R} \times M \to M \) be a one-parameter family of diffeomorphisms. Then \( \exists \phi^{(1)}, \ldots, \phi^{(k)}, \ldots, \) one-parameter groups of diffeomorphisms of \( M \), such that

\[
\Psi = \cdots \circ \phi^{(k)}_{\lambda^k/k!} \circ \cdots \circ \phi^{(2)}_{\lambda^2/2} \circ \phi^{(1)}_{\lambda}.
\]

The meaning of this Theorem is that any one-parameter family of diffeomorphisms can always be regarded as a one-parameter family of knights — of infinite rank, in general — and can be approximated by a family of knights of suitable rank. 

### 3. Gauge transformations

Consider now a family of spacetime models \( \{(M, g_\lambda, \tau_\lambda)\} \), where the metric \( g_\lambda \) and the matter fields (here collectively referred to as \( \tau_\lambda \)) satisfy the field equation \( \mathcal{E}[g_\lambda, \tau_\lambda] = 0 \), and \( \lambda \in \mathbb{R} \). We assume that \( g_\lambda \) and \( \tau_\lambda \) depend smoothly on the dimensionless parameter \( \lambda \), so that \( \lambda \) itself is a measure of the amount by which a specific \( (M, g_\lambda, \tau_\lambda) \) differs from the background solution \( (M, g_0, \tau_0) \), which is supposed to be known. This situation is most naturally described by introducing an \((m + 1)\)-dimensional manifold \( \mathcal{N} \), foliated by submanifolds diffeomorphic to \( M \), so that \( \mathcal{N} = M \times \mathbb{R} \). We shall label each copy of \( M \) by the corresponding value of the parameter \( \lambda \). Now, if a tensor field \( T_\lambda \) is given on each \( M_\lambda \), a tensor field \( T \) is automatically defined on \( \mathcal{N} \).

In order to define the perturbation in \( T \), we must find a way to compare \( T_\lambda \) with \( T_0 \). This requires a prescription for identifying points of \( M_\lambda \) with those of \( M_0 \), which is given by a diffeomorphism \( \varphi_\lambda : \mathcal{N} \to \mathcal{N} \) such that \( \varphi_\lambda|_{M_0} : M_0 \to M_\lambda \). Clearly, \( \varphi_\lambda \) can be regarded as the member of a flow \( \varphi \) on \( \mathcal{N} \), corresponding to the value of \( \lambda \) of \( \mathcal{N} \).

\[\text{†} \text{ We have supposed so far that maps and fields are analytic, but it is possible to give versions of (3), (4), and (6), that hold only for } C^n \text{ objects. The main change then is the substitution of Taylor series like the one in (4) by a finite sum of } n - 1 \text{ terms plus a remainder.}\]
the group parameter. Therefore, we could equally well give the vector field $X$ that generates $\varphi$, and we shall refer both to the point identification map $\varphi$ and to $X$ as a *gauge choice*. The perturbation can now be defined simply as

$$\Delta T_\lambda := \varphi^*_\lambda T|_{M_0} - T_0.$$  

(7)

The first term on the right hand side of (7) can be Taylor-expanded to get\(^1\)

$$\Delta T_\lambda = \sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \delta^k T, \quad \delta^k T := \left[ \frac{d^k \varphi^*_\lambda T}{d\lambda^k} \right]_{\lambda=0,M_0} = \mathcal{L}^k_X T|_0 .$$

(8)

Equation (8) defines then the $k$-th order perturbation of $T$. Notice that $\Delta T_\lambda$ and $\delta^k T$ are defined on $M_0$; this formalizes the statement one commonly finds in the literature, that “perturbations are fields living in the background.” In the particular case when $T$ is the metric or the matter fields, $\delta^k g$ and $\delta^k \tau$ obey *linear* equations, obtained by differentiating $\mathcal{E}[g_\lambda, \tau_\lambda] = 0$ with respect to $\lambda$. This gives an iterating procedure to calculate $g_\lambda$ and $\tau_\lambda$ when the field equation is too difficult to solve exactly.

Let us now suppose that *two* gauges $X$ and $Y$ are defined, associated with $\varphi$ and $\psi$ on $\mathcal{N}$, that connect any two leaves of the foliation. Thus $X$ and $Y$ are everywhere transverse to the $M_\lambda$, and points lying on the same integral curve of either of the two are to be regarded as *the same point* within the respective gauge, i.e., $\varphi$ and $\psi$ are both point identification maps. Both can be used to pull back a generic tensor field $T$, and to construct therefore two other tensor fields $\varphi^*_\lambda T$ and $\psi^*_\lambda T$, for any given value of $\lambda$. In particular, on $M_0$ we now have three tensor fields, i.e., $T_0$, and

$$T^X_\lambda := \varphi^*_\lambda T|_0 , \quad T^Y_\lambda := \psi^*_\lambda T|_0 .$$

(9)

Since $X$ and $Y$ represent gauge choices for mapping a perturbed manifold $M_\lambda$ onto the unperturbed one $M_0$, $T^X_\lambda$ and $T^Y_\lambda$ are the representations, in $M_0$, of the perturbed tensor according to the two gauges. We can write, using (7), (8) and (9),

$$T^X_\lambda = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \delta^k T^X = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \mathcal{L}^k_X T|_0 , \quad T^Y_\lambda = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \delta^k T^Y = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \mathcal{L}^k_Y T|_0 .$$

(10)

If $T^X_\lambda = T^Y_\lambda$, for any pair of gauges $X$ and $Y$, we say that $T$ is *totally gauge-invariant*. This is a strong condition, because then (10) imply that $\delta^k T^X = \delta^k T^Y$, for all $X$ and $Y$ and $\forall k$. But in practice, one is interested in perturbations to order $n$; it is thus convenient to weaken the definition, saying that $T$ is *gauge-invariant to order $n$* iff $\delta^k T^X = \delta^k T^Y$ for any $X$ and $Y$, and $\forall k \leq n$. One can then prove the following

**Proposition 1:** A tensor field $T$ is gauge-invariant to order $n \geq 1$ iff $\mathcal{L}_\xi \delta^k T = 0$, for any vector field $\xi$ on $\mathcal{M}$ and $\forall k < n$.

As a consequence, $T$ is gauge-invariant to order $n$ iff $T_0$ and all its perturbations of order lower than $n$ are, in any gauge, a combination of Kronecker deltas with

\(^1\)For the sake of simplicity, we denote the restriction to $M_0$ of a tensor field defined over $\mathcal{N}$ simply by the suffix 0.
constant coefficients. Further, it then follows that $T$ is totally gauge-invariant iff it is a combination of Kronecker deltas with coefficients depending only on $\lambda$.

If a tensor $T$ is not gauge-invariant, it is important to know how its representation on $\mathcal{M}_0$ changes under a gauge transformation. To this purpose, it is useful to define, for each value of $\lambda \in \mathbb{R}$, the diffeomorphism $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ given by $\Phi_\lambda := \varphi_\lambda \circ \psi_\lambda$. We must stress that $\Phi : \mathbb{R} \times \mathcal{M}_0 \to \mathcal{M}_0$ so defined, is not a flow on $\mathcal{M}_0$. In fact, $\Phi_{-\lambda} \neq \Phi_\lambda^{-1}$, and $\Phi_{\lambda+\sigma} \neq \Phi_\sigma \circ \Phi_\lambda$, essentially because $X$ and $Y$, in general, do not commute. However, the Theorem above guarantees that, to order $n$ in $\lambda$, the one-parameter family of diffeomorphisms $\Phi$ can always be approximated by a one-parameter family of knights of rank $n$. It is very easy to see that the tensor fields $T^X_\lambda$ and $T^Y_\lambda$ defined by the gauges $\phi$ and $\psi$ are connected by the linear map $\Phi^*_\lambda$:

$$T^Y_\lambda = \psi^*_\lambda T^X_\lambda |_0 = (\psi^*_\lambda \phi^*_\lambda \varphi^*_\lambda T^X_\lambda |_0) = \Phi^*_\lambda (\varphi^*_\lambda T^X_\lambda |_0).$$

Thus, the Theorem allows us to use (3) as a generating formula for a gauge transformation to an arbitrary order $n$. To second order, we have explicitly

$$T^Y_\lambda = T^X_\lambda + \lambda \mathcal{L}_{\xi(1)} T^X_\lambda + \frac{\lambda^2}{2} \left( \mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(1)} \right) T^X_\lambda + \ldots,$$

where $\xi(1)$ and $\xi(2)$ are now the first two generators of $\Phi_\lambda$, or of the gauge transformation, if one prefers. We can now relate the perturbations in the two gauges. To the lowest orders, this is easy to do explicitly, just substituting (10) into (12):

**Proposition 2:** Given a tensor field $T$, the relations between its first and second order perturbations in two different gauges are:

$$\delta T^Y - \delta T^X = \mathcal{L}_{\xi(1)} T_0;$$

$$\delta^2 T^Y - \delta^2 T^X = \left( \mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(1)} \right) T_0 + 2 \mathcal{L}_{\xi(1)} \delta T^X.$$

This result is consistent with Proposition 1, of course. Equation (13) implies that $T_\lambda$ is gauge-invariant to the first order iff $\mathcal{L}_{\xi} T_0 = 0$, for any vector field $\xi$ on $\mathcal{M}$. In particular, one must have $\mathcal{L}_{\xi(2)} T_0 = 0$, and therefore Eq. (14) leads to $\mathcal{L}_{\xi_2} \delta T = 0$. Similar conditions hold at higher orders. It is also possible to find the explicit expressions, in terms of $X$ and $Y$, for the generators $\xi(k)$ of a gauge transformation. In fact, it is easy to prove that the first two generators of the one-parameter family of diffeomorphisms $\Phi$ are $\xi(1) = Y - X$, and $\xi(2) = [X,Y]$.

4. Conclusions

In this contribution we have briefly illustrated how a gauge transformation in the theory of spacetime perturbations is a member of a family — not a group — of diffeomorphisms. The family can be approximated by a flow only when attention is restricted to linear perturbations. When $n$-th order perturbations are considered, gauge transformations can instead be approximated by $n$-th rank knights diffeomorphisms. In fact, in introducing these objects, we have proved that a generic family
of diffeomorphisms can always be regarded as a knight, in general of $\infty$-th rank. We have also considered gauge invariance, and given a generating formula for gauge transformations of arbitrary order. The formalism presented here has been applied in the context of cosmology to obtain the explicit transformations between the synchronous and the Poisson (generalized longitudinal) gauges.

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1. K. Tomita, Prog. Theor. Phys. 37, 831 (1967); S. Matarrese, O. Pantano, & D. Saez, Phys. Rev. Lett. 72, 320 (1994); MNRAS 271, 513 (1994).
2. D. S. Salopek, J. M. Stewart, & K. M. Croudace, MNRAS 271, 1005 (1994); H. Russ, M. Morita, M. Kasai, & G. Börner, Phys. Rev. D 53, 6881 (1996).
3. R. J. Gleiser, C. O. Nicasio, R. H. Price, & J. Pullin, Class. Quantum Grav. 13, L117 (1996); preprint gr-qc/9609022.
4. R. K. Sachs, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).
5. J. M. Stewart & M. Walker, Proc. R. Soc. London A 341, 49 (1974).
6. M. Bruni, S. Matarrese, S. Mollerach, & S. Sonego, preprint IC/96/174, SISSA–136/96/A, gr-qc/9609040, submitted for publication.
7. J. M. Bardeen, Phys. Rev. D 22, 1882 (1980); H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. 78, 1 (1984); J. M. Stewart, Class. Quantum Grav. 7, 1169 (1990); V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. 215, 203 (1992); R. Durrer, Fund. Cosmic Phys. 15, 209 (1994).
8. G. F. R. Ellis, & M. Bruni, Phys. Rev. D 40, 6, 1804–1818, (1989).
9. J. A. Schouten, *Ricci-Calculus* (Springer, Berlin, 1954), p. 108.
10. S. Sonego & M. Bruni, (1996) submitted for publication.