Concentration bounds for the extremal variogram

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Abstract

In extreme value theory, the extremal variogram is a summary of the tail dependence of a random vector. It is a central ingredient for learning extremal tree structures [Engelke and Volgushev, 2020] and has close connections to the estimation of Hüsler–Reiss models and extremal graphical models [Engelke and Hitz, 2020]. This note presents concentration results for the empirical version of the extremal variogram under general domain of attraction conditions. The results play a role in extending the findings in [Engelke and Volgushev, 2020] to increasing dimensions. The note is also the first building block for penalized estimation of sparse Hüsler–Reiss graphical models [Engelke, 2021].

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1 Introduction and main result

The tail dependence structure of a random vector \( X = (X_1, \ldots, X_d) \) is crucial to the understanding of the extreme values taken by the random variables \( X_j \). It is often defined as the dependence between the components of \( X \) in the regions where at least one of them is large. Multivariate Pareto distributions arise as the only possible limits of the conditional distribution of \( u^{-1} X \parallel X \parallel_\infty > u \) as \( u \rightarrow \infty \), where \( \parallel \cdot \parallel_\infty \) is the maximum norm. As such, they play a central role in tail dependence modelling. Peaks-over-threshold inference methods seek to use all the observations of \( X \) where at least one of the variables exceeds a high (and usually data-dependent) threshold to produce inference on the multivariate Pareto distribution itself.

The family of multivariate Pareto distributions on \( \mathbb{R}^d \) is indexed by the (very large) family of all extreme value copulae on \([0,1]^d\). While a multitude of parametric models for these copulae can be found in the literature, the most popular of which is certainly the Hüsler–Reiss model, most existing inference methods are still limited to moderate dimensions. This issue is particularly present in the context of extreme value theory since the effective sample size is typically only a fraction of the actual number of observations.

The notion of sparsity is useful in constructing parametric models for high-dimensional data that only depend on a relatively low number of non-zero parameters. Graphical modelling uses that notion by imposing conditional independence between many of the pairs of observed variables. Well known results state that given such assumptions, inference on the whole dependence model can be carried out by estimating only the dependence structure of groups of fully conditionally dependent variables. When the model is sufficiently sparse and those groups of variables are moderately sized, the computational gain is massive and the required sample size for a given dimension is greatly reduced, allowing for high-dimensional inference. The task of selecting a graphical model is challenging since the detection of conditional independencies in all generality is especially difficult in high dimension. In the special case of Gaussian data, however, it is well known that the conditional independence relations can be read off as zeros in the inverse covariance matrix. From this simple fact, a number of methods have been derived for Gaussian graphical model selection that are based on estimation of the precision matrix.

The classical notion of conditional independence hardly fits multivariate Pareto distributions, for they are not supported on a product space. By proposing a novel definition of conditional independence tailored for these distributions, Engelke and Hitz (2020) open the door to graphical modelling of extremes. It has further been shown (Engelke and Hitz, 2020; Engelke and Volgushhev, 2020) that in many special cases of interest, including when the underlying multivariate Pareto distribution is indexed by a Hüsler–Reiss distribution, the conditional independence relations are encoded in a collection of \( d \) simple summaries called the variogram matrices.

Extremal graphical model selection methods are arising (Engelke and Volgushhev, 2020) and it appears clear that the efficient estimation of variogram matrices of a high-dimensional multivariate Pareto distribution is essential for their consistent model recovery. In this work, we analyze the behavior of a natural nonparametric estimator which appears in the aforementioned two papers, namely the empirical variogram. Precisely, we prove a concentration inequality that holds simultaneously for the \( d \) empirical variograms under the appropriate domain of attraction condition. The bound is logarithmic in \( d \), providing uniform consistency in exponentially high dimension.

The manuscript is organized as follows. Notational conventions are first given in Section 1.1 followed by
the important definitions leading to the variogram and its empirical counterpart in Section 1.2. Our main result is stated in Section 1.3. Its proof occupies Section 2, with additional details deferred to Section 3.

1.1 Notation

Throughout, \( d \geq 3 \) and \( V := \{1, \ldots, d\} \) indexes the observed variables. The observations are numbered 1 to \( n \), for an arbitrary \( n \geq 3 \). When not specified, we use lower (resp. upper) case bold letters to denote deterministic (resp. random) vectors in \( \mathbb{R}^d \). The entries of a vector \( \mathbf{x} \) are named \( x_1, \ldots, x_d \) and for a subset \( J \subset V \), \( \mathbf{x}_J \) is the subvector indexed by \( J \). The same conventions are used for random vectors \( \mathbf{X} \). Given multivariate observations \( \mathbf{X}_t, t \in \mathbb{N} \), we denote by \( X_{tj} \) the \( j \)-th coordinate of \( \mathbf{X}_t \).

We use \( \| \cdot \|_\infty \) the denote the \( L^\infty \) norm for vectors and the elementwise version thereof for matrices, i.e., \( \| \mathbf{x} \|_\infty = \max_j |x_j| \) and \( \| A \|_\infty = \max_{i,j} |A_{ij}| \).

In Sections 2 and 3 we repeatedly make use of “\( \lesssim \)” to mean “\( \leq \) up to a constant”. Note that said constant may depend on the parameters specified in the statement of Theorem 1 below, but it does not depend explicitly on either of the dimension \( d \), the number of observations \( n \) or the effective sample size \( k \) (to be defined later). In general, if \( a \lesssim b \), then the hidden constant appearing in “\( \lesssim \)” is always independent of every parameter in the upper bound \( b \).

1.2 Definitions

A random vector \( \mathbf{X} \in \mathbb{R}^d \) is said to be in the max-domain of attraction of a random vector \( \mathbf{Z} \) if there exist sequences \( a_n^{(j)} > 0 \) and \( b_n^{(j)} \), \( 1 \leq j \leq d \), such that for independent copies \( \mathbf{X}_1, \mathbf{X}_2, \ldots \) of \( \mathbf{X} \), the vector of normalized maxima

\[
\left( \frac{\max_{1 \leq t \leq n} X_{t1} - b_n^{(1)}}{a_n^{(1)}}, \ldots, \frac{\max_{1 \leq t \leq n} X_{td} - b_n^{(d)}}{a_n^{(d)}} \right)
\]

converges in distribution to \( \mathbf{Z} \). The distribution of \( \mathbf{Z} \) is then one of the so-called multivariate extreme value distributions. If the margins of \( \mathbf{X} \) are unit Pareto, \( a_n^{(j)} \) and \( b_n^{(j)} \) can be chosen without loss of generality as \( n \) and 0, respectively. The margins of \( \mathbf{Z} \) are then unit Fréchet and its distribution function can be written as \( G(z) = \exp\{-L(1/z)\}, \ z \in \mathcal{E} := [0, \infty]^d \setminus \{0\} \). The stable tail dependence function \( L \) satisfies

\[
u(1 - P(\mathbf{X} \leq uz)) \to L(1/z), \quad z \in \mathcal{E},
\]

as \( u \to \infty \). Equivalently,

\[
u P(\mathbf{X} \geq uz) \to R(1/z), \quad z \in \mathcal{E},
\]

for a certain function \( R \) that can be deduced from \( L \). It is positively homogeneous, in that \( R(ax) = aR(x) \), \( a \geq 0 \), it satisfies the bounds \( 0 \leq R(x) \leq \min_j x_j \) and it is concave. Similarly, \( L \) is positively homogeneous, bounded between the maximum and the sum of its arguments, and is convex (de Haan and Ferreira, 2006, Chapter 6).

Multivariate extremes are often defined in terms of the norm of a random vector being large, particularly the supremum norm. The observations of interest are then the ones in \( u\mathcal{L} \), where \( u \) is a large threshold and \( \mathcal{L} := [0, \infty)^d \setminus [0, 1]^d \). Suppose that for the random vector \( \mathbf{X} \in [0, \infty)^d \) with unit Pareto margins, the limit

\[
H(y) := \lim_{u \to \infty} \mathbb{P}(\mathbf{X} \leq uy \mid \mathbf{X} \in u\mathcal{L})
\]

exists for each \( y \in \mathcal{L} \) (in fact, existence for \( y \) s.t. \( \|y\|_\infty = 1 \) implies the existence over \( \mathcal{L} \)). Then \( H \) is a so-called multivariate Pareto distribution function. This limit relation is equivalent to \( \mathbf{X} \) being in the max-domain of attraction of a max-stable random vector \( \mathbf{Z} \) with unit Fréchet margins and stable tail dependence function \( L \). In this case \( H, L \) and \( R \) are equivalent and can be defined from one another. The
random vector $X$ is then said to be in the domain of attraction of a multivariate Pareto random vector $Y$ with distribution function $H$. The distribution of $Y$ is uniquely identified to $R$ and $L$ via

$$
\mathbb{P}(Y \geq y) = \lim_{u \to \infty} \frac{\mathbb{P}(X \geq uy)}{\mathbb{P}(X \in uL)} = \lim_{u \to \infty} u \mathbb{P}(X \geq uy) = \frac{R(1/y)}{L(1)}, \quad y \in \mathcal{L}.
$$

In particular, if $Z$ has a Hüsler–Reiss distribution with parameter matrix $\Gamma$, then $Y$ is said to have a Hüsler–Reiss Pareto distribution with parameter matrix $\Gamma$.

Given a coordinate $m \in V$, we define $Y^{(m)}$ as $Y \mid Y_m > 1$. Informally, whereas $Y$ is the weak limit of $\frac{X}{u} \mid \|X\|_{\infty} > u$ as $u \to \infty$, $Y^{(m)}$ is the weak limit of $\frac{X}{u} \mid X_m > u$. By definition, the joint survival function of $Y^{(m)}$ is given by

$$
\mathbb{P}(Y^{(m)} \geq y) = \lim_{u \to \infty} \frac{\mathbb{P}(X \geq uy, X_m > u)}{\mathbb{P}(X_m > u)} = \lim_{u \to \infty} u \mathbb{P}(X \geq uy, X_m \geq u) = R(1/y^{(m)}), \quad y \in \mathcal{E},
$$

where the $m$-th component of $y^{(m)}$ is equal to $y_m \vee 1$ while its other components are equal to those of $y$. The variogram matrix $\Gamma^{(m)}$ of $Y$ rooted at node $m$ is defined as

$$
\Gamma^{(m)}_{ij} := \text{Var}(\log Y^{(m)}_i - \log Y^{(m)}_j) = e^{(m),2}_i + e^{(m),2}_j - 2e^{(m),1}_i - e^{(m),1}_j, \quad i \neq j, m \in V, \tag{2}
$$

where for $\ell \in \{1, 2\}$, $e^{(m),\ell}_i := \mathbb{E}[(\log Y^{(m)}_i)^\ell]$ and $e^{(m),1}_i := \mathbb{E}[(\log Y^{(m)}_i)(\log Y^{(m)}_j)]$. Note that in the Hüsler–Reiss Pareto case, for each $m$ it can be shown that $\Gamma^{(m)} = \Gamma$, the parameter matrix $\Gamma$.

Consider independent copies $X_1, \ldots, X_n$ of the random vector $X$ and let $k \in \{1, \ldots, n\}$. The empirical variogram $\hat{\Gamma}^{(m)}$ rooted at node $m$ is defined as

$$
\hat{\Gamma}^{(m)}_{ij} := \hat{e}^{(m),2}_i + \hat{e}^{(m),2}_j - 2\hat{e}^{(m),1}_i - \hat{e}^{(m),1}_j, \quad i \neq j, m \in V,
$$

where

$$
\hat{e}^{(m),\ell}_i := \frac{1}{k} \sum_{t=1}^{n} \left\{ \log \left( \frac{k}{nF_i(U_{ti})} \right) \right\}^\ell 1 \left\{ \hat{F}_m(U_{tm}) \leq k/n \right\},
$$

and

$$
\hat{e}^{(m),1}_i := \frac{1}{k} \sum_{t=1}^{n} \log \left( \frac{k}{nF_i(U_{ti})} \right) \log \left( \frac{k}{nF_i(U_{ti})} \right) 1 \left\{ \hat{F}_m(U_{tm}) \leq k/n \right\}
$$

and $U_{ti} = 1/X_{ti}$, $1 \leq t \leq n$, are independent and uniformly distributed. Their empirical distribution function $\hat{F}_i$ is such that $n\hat{F}_i(U_{ti})$ is the rank of $X_{ti}$ among $X_1, \ldots, X_n$ in decreasing order. The upper tail of $X$ corresponds to the lower tail of the copula $F$, the distribution function of $U := 1/X$.

### 1.3 Main result

Equation (1) may be rephrased as

$$
q^{-1} \mathbb{P}(X \geq (qx)^{-1}) \longrightarrow R(x), \quad x \in [0, \infty]^d \setminus \{\infty\}
$$

as $q \to 0$. It implies the corresponding limit relation for every subset of variables $J \subset V$, namely that

$$
q^{-1} \mathbb{P}(X_J \geq (qx_J)^{-1}) \longrightarrow R_J(x_J), \quad x_J \in [0, \infty]^{|J|} \setminus \{\infty\},
$$

where if $x$ is equal to $x_J$ in the components in $J$ and is infinite in $J^c$, $R_J(x_J) = R(x)$. When $J$ has size 1, 2 or 3, we shall write $R_i$, $R_{ij}$ and $R_{ijm}$ for $R_{(i)}$, $R_{(i,j)}$ and $R_{(i,j,m)}$. Our basic distributional assumption is
twofold. First, we require that Equation (11) holds for the subsets \( J \) of size 2 and 3, uniformly on compact sets and with at least a polynomial rate of convergence, therefore controlling how “distant” the thresholded data is from the limiting multivariate Pareto distribution. Second, we make a regularity assumption on the limiting model itself, essentially lower bounding the strength of tail dependence between each pair of variables.

**Assumption 1** (Model assumption). There exist positive constants \( K_1, K_2, \gamma_1 \) and \( \gamma_2 \) such that for all triples of indices \((i,j,m)\) and every \( q \leq 1 \),

\[
\sup_{x,y \in [0,1]} \left| q^{-1}P(X_i \geq (qx)^{-1}, X_j \geq (qy)^{-1}) - R_{ij}(x,y) \right| \leq K_1 q^{\gamma_1}, \tag{5}
\]

\[
\sup_{x,y,z \in [0,1]} \left| q^{-1}P(X_i \geq (qx)^{-1}, X_j \geq (qy)^{-1}, X_m \geq (qz)^{-1}) - R_{ijm}(x,y,z) \right| \leq K_1 q^{\gamma_1} \tag{6}
\]

and

\[
1 - R_{ij}(q^{-1}, 1) \leq K_2 q^{\gamma_2}. \tag{7}
\]

As will be seen later, Equation (7) (or rather its consequence Equation (10)) gives an upper bound on the values \( \Gamma_{ij}^{(m)} \), \( m \in V \) (see Lemma 7). In particular, this assumption enforces asymptotic dependence of every pair of variables, since \( \Gamma_{ij}^{(m)} = \infty \) when \( X_i \) and \( X_j \) are asymptotically independent. Moreover, Equations (5) to (7) jointly imply that the convergence in Equations (5) and (6) is uniform over sets that are growing as \( q \to 0 \), as well as a trivariate version of Equation (7). This is the focus of the next result, which is proved in Section 3.1.

**Proposition 1.** Let Assumption 1 hold. Then for all triples of indices \((i,j,m)\) and every \( q \leq 1 \),

\[
\sup_{(x,y) \in [0,q^{-1}] \times [0,1]} \left| q^{-1}P(X_i \geq (qx)^{-1}, X_j \geq (qy)^{-1}) - R_{ij}(x,y) \right| \leq K q^\xi, \tag{8}
\]

\[
\sup_{(x,y,z) \in [0,q^{-1}]^2 \times [0,1]} \left| q^{-1}P(X_i \geq (qx)^{-1}, X_j \geq (qy)^{-1}, X_m \geq (qz)^{-1}) - R_{ijm}(x,y,z) \right| \leq K q^\xi \tag{9}
\]

and

\[
1 - R_{ijm}(q^{-1},q^{-1},1) \leq 1 - R_{im}(q^{-1},1) + 1 - R_{jm}(q^{-1},1) \leq Kq^\xi, \tag{10}
\]

where \( K := 2(K_1 + K_2) \) and \( \xi := \gamma_1 \gamma_2 / (1 + \gamma_1 + \gamma_2) \).

For simplicity, the proof of Theorem 1 directly assumes Equations (5) to (10). As such, the result as well as several arguments in its proof are stated in terms of the quantities \( K \) and \( \xi \) rather than in terms of the constants appearing in the assumption. Moreover, it can be shown that if Equation (9) holds for some \( K \) and \( \xi \), Assumption 1 is satisfied with \( K_1 = 2K, \ K_2 = K \) and \( \gamma_1 = \gamma_2 = \xi \). That equation itself can therefore be taken as the basic assumption, without much loss of generality.

Finally, while our general tail bound holds without assumptions on \( R \), we are able to obtain a sharper rate of convergence if \( R \) has pairwise densities satisfying a certain bound.

**Assumption 2** (Bounded densities). The measures \( R_{ij} \) have densities \( r_{ij} \) satisfying

\[
r_{ij}(x,y) \leq \frac{K(\beta)}{x^\beta y^{1-\beta}}, \quad (x,y) \in (0, \infty)^2,
\]

for constants \( K(\beta) \) and every \( \beta \in [-\varepsilon,1+\varepsilon] \), for some some \( \varepsilon > 0 \).
Remark. This assumption may appear unnatural at first, but the upper bound arises naturally when studying the bivariate densities of Hüsler–Reiss distributions. Lemma 1 shows that Assumption 2 is satisfied by any non-degenerate Hüsler–Reiss distribution, and that the value ε therein can be chosen arbitrarily large. In addition, it is trivial to check that the assumption holds if the densities r_ij satisfy

\[ r_{ij}(x, 1 - x) \leq K_r(x(1 - x))^\varepsilon, \quad x \in (0, 1), \]

for some positive constants K_r and ε.

We now have the necessary material to state our main result.

Theorem 1. Let Assumption 7 hold and ζ ∈ (0, 1] be arbitrary. There exist positive constants C, c and M only depending on K, ξ and ζ such that for any k ≥ n^c and λ ≤ \( \frac{\sqrt{k}}{(\log n)^4} \),

\[ \mathbb{P}\left( \max_{m \in V} \| \hat{\Gamma}(m) - \Gamma(m) \|_\infty > C\left\{ \left( \frac{k}{n} \right) \xi \left( \log(n/k) \right)^2 + \frac{(\log(n/k))^2(1 + \lambda)}{\sqrt{k}} \right\} \right) \leq M d^2 e^{-c\lambda^2}. \]

If in addition Assumption 2 holds, there exists a positive constant \( \hat{C} \) only depending on K, ξ, ζ, ε and K(β) such that for any k and λ as above,

\[ \mathbb{P}\left( \max_{m \in V} \| \hat{\Gamma}(m) - \Gamma(m) \|_\infty > \hat{C}\left\{ \left( \frac{k}{n} \right) \xi \left( \log(n/k) \right)^2 + \frac{1 + \lambda}{\sqrt{k}} \right\} \right) \leq M d^2 e^{-c\lambda^2}. \]

2 Proof of Theorem 1

Let i ≠ j and m be arbitrary. An expression for the estimation error is given by

\[
\hat{\Gamma}_{ij} - \Gamma_{ij} = (\hat{e}_{ij}(m) - e_{ij}(m))^2 + (\hat{e}_{ij}(m) - e_{ij}(m))^2 - 2(\hat{e}_{ij}(m) - e_{ij}(m)) \\
- 2(\hat{e}_{ij}(m) - e_{ij}(m))^2((\hat{e}_{ij}(m) - e_{ij}(m)) - (\hat{e}_{ij}(m) - e_{ij}(m))) \\
- (\hat{e}_{ij}(m) - e_{ij}(m) - (\hat{e}_{ij}(m) - e_{ij}(m)))^2, \tag{11}
\]

the last two terms stemming from the identity \( y^2 - x^2 = 2x(y - x) + (y - x)^2 \). In order to prove the result, it is sufficient to bound the differences

\[
\hat{e}_{ij}(m) - e_{ij}(m), \quad \hat{e}_{ij}(m) - e_{ij}(m), \quad \hat{e}_{ij}(m) - e_{ij}(m), \quad \hat{e}_{ij}(m) - e_{ij}(m)
\]

for all distinct triples (i, j, m) and \( \eta \in \{1, 2\} \). The terms \( \hat{e}_{ij}(m) - e_{ij}(m) \) are entirely deterministic, since it is known that the observations \( X_{tm} \) that are used for the estimator \( \hat{\Gamma}(m) \) have ranks \( n - k + 1, \ldots, n \) (by continuity, it can be assumed that there are no ties in the data). They are on the order of \( (\log k)^\ell/k \), as is proved in Section 3.2. The rest of the proof thus focuses on the other three differences.

2.1 Preliminaries, additional notation and structure of the proof

Recall that \( \hat{F}_i \) is the empirical distribution function of \((U_t)_{1 \leq t \leq n}\) and denote its left-continuous inverse by \( \hat{F}_i^- \), where \( f^-(t) := \inf\{x : f(x) \geq t\} \). Consider the rescaled tail quantile processes

\[ u_n(\ell)(x) := \frac{n}{k} \hat{F}_i^-(kx/n). \tag{12} \]

Similarly to R and its margins \( R_J, J \subset V \), let

\[ \hat{R}_i(x) := \frac{1}{k} \sum_{t=1}^{n} \mathbb{1}\{U_t \leq \frac{k}{n}x_i, i \in J\}, \quad \hat{R}_J(x) := \hat{R}_J(x_J), \quad x_J \in [0, \infty)^{|J|}, \tag{13} \]
where \( \hat{R}_J := (i_n^{(i)}(x_i))_{i \in J} \).

The function \( R_J \) can be seen as a measure on \([0, \infty)^{|J|}\) and for measurable sets \( A_i \subset \mathbb{R} \), we will write \( R_J(\cup_i A_i) \) to denote \( R_J(\otimes_i A_i) \). If in place of one of the \( A_i \) there is a number \( a_i \), it will be understood that \( A_i = [0, a_i] \). For example, \( R_{ij}(|x, \infty]) = R_{ij}([x, \infty) \times [0, y]) \). We use the same conventions for the functions \( \hat{R}_J \) and \( \hat{R}_J^0 \), as well as \( R_{J,n}, G_{J,n} \) and \( \bar{R}_J \) to be defined later.

From now on, fix a number \( a \in (0, 1) \). It is proved in Lemma 6 that for all distinct triples \((i, j, m)\) and \(\ell \in \{1, 2\}\), we have

\[
\begin{align*}
\hat{e}_i^{(m), \ell} - e_i^{(m), \ell} &= \int_a^1 \frac{(\hat{R}_im(x, 1) - R_{im}(x, 1))}{x} (-2 \log x)^{\ell-1} dx \\
&\quad - \int_1^{n/k} \frac{(\hat{R}_im([x, \infty), 1) - R_{im}([x, \infty), 1))}{x} (-2 \log x)^{\ell-1} dx \\
&\quad + O\left(\frac{k}{n} \xi \log(n/k) + \frac{(\log(n/k) + \log(1/a))^2}{k} + a(\log(n/k) + \log(1/a))\right), \\
\hat{e}_{im}^{(m)} - e_{im}^{(m)} &= \int_a^1 \int_a^1 \frac{\hat{R}_{im}(x, y) - R_{im}(x, y)}{xy} dxdy \\
&\quad - \int_1^{n/k} \frac{\hat{R}_{im}([x, \infty), y) - R_{im}([x, \infty), y)}{xy} dxdy \\
&\quad + O\left(\frac{k}{n} \xi \log(n/k) + \frac{(\log(n/k) + \log(1/a))^2}{k} + a(\log(n/k) + \log(1/a))\right), \\
\hat{e}_{ij}^{(m)} - e_{ij}^{(m)} &= \int_a^1 \int_a^1 \frac{\hat{R}_{ijm}(x, y, 1) - R_{ijm}(x, y, 1)}{xy} dxdy \\
&\quad - \int_1^{n/k} \frac{\hat{R}_{ijm}([x, \infty), y, 1) - R_{ijm}([x, \infty), y, 1)}{xy} dxdy \\
&\quad - \int_1^{n/k} \frac{\hat{R}_{ijm}(x, [y, \infty), 1) - R_{ijm}(x, [y, \infty), 1)}{xy} dxdy \\
&\quad + \int_1^{n/k} \frac{\hat{R}_{ijm}([x, \infty), [y, \infty), 1) - R_{ijm}([x, \infty), [y, \infty), 1)}{xy} dxdy \\
&\quad + O\left(\frac{k}{n} \xi \log(n/k) + \frac{(\log(n/k) + \log(1/a))^2}{k} + a(\log(n/k) + \log(1/a))\right)
\end{align*}
\]

almost surely, where the error terms are not stochastic. We shall separately bound each of the eight integrals above. Denote these integrals, in order of appearance in Equations (14) to (16), by \( I_i^{(m), \ell}, \hat{I}_i^{(m), \ell}, \hat{I}_i^{(m), +}, \hat{I}_i^{(m),} \), \( I_{im}^{(m)+}, I_{im}^{(m)\pm}, I_{ij}^{(m)+}, I_{ij}^{(m)\pm} \) and \( I_{ij}^{(m)++} \).

Recall that \( F \) is the copula of \( U = 1/X \). As an intermediate between \( R_J \) and \( \hat{R}_J \), let \( R_{J,n}(x_J) = \frac{n}{k} F(k x_J/n) = \frac{n}{k} \varphi(x J \geq \frac{n}{k x_J}) \), the pre-asymptotic version of \( R_J \). The processes \( \hat{R}_J - R_J \) can be decomposed into the stochastic error \( \hat{R}_J - R_{J,n} \) and the difference \( R_{J,n} - R_J \) between the tail at finite and infinite levels. Replacing \( \hat{R}_J - R_J \) by \( (\hat{R}_J - R_{J,n}) + (R_{J,n} - R_J) \), each integral \( I_i^{(m), \ell} \) is written as

\[
I_i^{(m), \ell} = I_i^{(m)\ell} + I_i^{(m)\ell+} + I_i^{(m)\ell+} + I_i^{(m)\ell+} + I_i^{(m)++},
\]
where the $A$ terms are stochastic and the $B$ terms represent deterministic bias.

We proceed as follows. In Section 2.2, it is shown that Assumption 1 is sufficient to bound all the bias terms, up to a constant, by $\left(\frac{k}{n}\right)^\xi (\log(n/k) + \log(1/a))^2$. Subsequently, we prove in Section 2.3 concentration results for the stochastic terms which are then leveraged in Section 2.4 to complete the proof.

### 2.2 The bias terms $B$

Equation (8) can be rephrased as

$$\sup_{x \leq n/k, y \leq 1} |R_{ij,n}(x,y) - R_{ij}(x,y)| \leq K \left(\frac{k}{n}\right)^\xi,$$

which directly implies

$$|B^{(m),\ell,-}| \leq \int_a^1 \frac{|R_{im,n}(x,1) - R_{im}(x,1)|(-2 \log x)^{\ell-1}}{x} \, dx \leq K \left(\frac{k}{n}\right)^\xi \int_a^1 (-2 \log x)^{\ell-1} \, dx \lesssim \left(\frac{k}{n}\right)^\xi (\log(1/a))^{\ell},$$

$$|B^{(m),\ell,+}| \leq \int_1^{n/k} \frac{|R_{im,n}(x,\infty,1) - R_{im}(x,\infty,1)|2 \log x)^{\ell-1}}{x} \, dx \leq K \left(\frac{k}{n}\right)^\xi \int_1^{n/k} (2 \log x)^{\ell-1} \, dx \lesssim \left(\frac{k}{n}\right)^\xi (\log(n/k))^{\ell},$$

$$|B^{(m),--}| \leq \int_a^1 \int_a^1 \frac{|R_{im,n}(x,y) - R_{im}(x,y)|}{xy} \, dxdy \leq K \left(\frac{k}{n}\right)^\xi \int_a^1 \int_a^1 \frac{1}{xy} \, dxdy \lesssim \left(\frac{k}{n}\right)^\xi (\log(1/a))^2,$$

$$|B^{(m),+-}| \leq \int_a^1 \int_1^{n/k} \frac{|R_{im,n}(x,\infty,y) - R_{im}(x,\infty,y)|}{xy} \, dxdy \leq K \left(\frac{k}{n}\right)^\xi \int_a^1 \int_1^{n/k} \frac{1}{xy} \, dxdy \lesssim \left(\frac{k}{n}\right)^\xi (\log(n/k))(\log(1/a)).$$

Noting further that Equation (9) implies

$$\sup_{x \leq n/k, y \leq n/k} |R_{ijm,n}(x,y,1) - R_{ijm}(x,y,1)| \leq K \left(\frac{k}{n}\right)^\xi,$$
we easily see that $B_{ij}^{(m),--}$ admits the same bound as $B_{im}^{(m),--}$. Similarly, $B_{ij}^{(m),+-}$, and by symmetry $B_{ij}^{(m),-+}$, admit the same bound as $B_{im}^{(m),+-}$. Finally, since

$$R_{ijm}([x, \infty), [y, \infty), 1) = 1 - R_{jm}(y, 1) - R_{im}(x, 1) + R_{ijm}(x, y, 1)$$

and the same relation holds for $R_{ijm,n}$, Equations (5) and (9) also imply that

$$\sup_{x \leq n/k, y \leq n/k} |R_{ijm,n}([x, \infty), [y, \infty), 1) - R_{ijm}([x, \infty), [y, \infty), 1)| \leq 3K \left( \frac{k}{n} \right)^{\xi}.$$

Deduce that

$$\left| B_{ij}^{(m),+-} \right| \leq \int_{1}^{n/k} \int_{1}^{n/k} \frac{R_{ijm,n}([x, \infty), [y, \infty), 1) - R_{ijm}([x, \infty), [y, \infty), 1)}{xy} dx dy$$

$$\leq 3K \left( \frac{k}{n} \right)^{\xi} \int_{1}^{n/k} \int_{1}^{n/k} \frac{1}{xy} dx dy$$

$$\approx \left( \frac{k}{n} \right)^{\xi} \left( \log(n/k) \right)^2.$$

### 2.3 The stochastic error terms $A$

It remains to bound the stochastic error terms $A$, which entirely depend on the processes $\hat{R}_j - R_{I,n}$. Recall how, for $x \in [0, \infty)^{|J|}$, we define $\tilde{x}_J$ in Equation (13). Consider further the relation $\hat{R}_j(x) = \hat{R}_0^j(\tilde{x}_J)$. We shall rely on the decomposition

$$\hat{R}_j(x) - R_{I,n}(x) = (\hat{R}_0^j(\tilde{x}_J) - R_{I,n}(\tilde{x}_J)) + (R_{I,n}(\tilde{x}_J) - R_{I,n}(x))$$

$$= (G_{I,n}(\tilde{x}_J) - G_{I,n}(x)) + G_{I,n}(x) + (R_{I,n}(\tilde{x}_J) - R_{I,n}(x)),$$

where

$$G_{I,n} := \hat{R}_0^j - R_{I,n}.$$  

Accordingly, each $A$ term is further decomposed into three integrals $A_{1,1}, A_{1,2}$ and $A_{1,3}$. For instance,

$$A_{ij}^{(m),--} = A_{ij,1}^{(m),--} + A_{ij,2}^{(m),--} + A_{ij,3}^{(m),--}$$

$$:= \int_{a}^{1} \int_{a}^{1} G_{ijm,n}(u_n(i)(x), u_n(j)(y), u_n(m)(1)) - G_{ijm,n}(x, y, 1) \frac{dx dy}{xy}$$

$$+ \int_{a}^{1} \int_{a}^{1} G_{ijm,n}(x, y, 1) \frac{dx dy}{xy}$$

$$+ \int_{a}^{1} \int_{a}^{1} R_{ijm,n}(u_n(i)(x), u_n(j)(y), u_n(m)(1)) - R_{ijm,n}(x, y, 1) \frac{dx dy}{xy}.$$  

The first of the three terms in Equation (17) is proportional to a standard empirical process evaluated at a set corresponding to the difference between $x$ and $\tilde{x}_J$. It will be uniformly bounded by using well known concentration inequalities for empirical processes appearing in Kolchinskii (2000) and Massart (2000). The second term, $G_{I,n}$, is now a rescaled sum of $n$ independent and identically distributed (iid) processes which, when integrated as in Equation (19), becomes a sum of iid, bounded random variables. We will be able to control this sum via Bernstein’s inequality. Finally, the third term relates to the difference between $x$ and $\tilde{x}_J$. It can be controlled by weighted approximation results on the uniform quantile processes given in Lemmas 1 and 2. Two bounds are then proved on the corresponding term in Equation (19), the stronger of which only holds under Assumption 2. This gives rise to the two desired results.
2.3.1 Technical preliminaries on uniform quantile processes

Before tackling each term in Equation (19), we prove a few properties of the rescaled quantile functions $u_n^{(i)}$ which will be used throughout. In Lemma 1 we first prove an approximation of $u_n^{(i)}$ by a certain Gaussian process. We then establish various properties of $u_n^{(i)}$ and its approximation in Corollary 1 and Lemma 2.

**Lemma 1.** For any fixed $i \in V$, define the random function $u_n^{(i)}$ as in Equation (12) and let $0 < \nu < 1/2$. There exist random functions $\nu_{n, k}^{(i)}$ defined on a possibly enriched probability space and universal constants $A, B, C \in (0, \infty)$ such that for any $z > 0$,

$$\mathbb{P} \left( \sup_{x \in [0, n/k]} |u_n^{(i)}(x) - \nu_{n, k}^{(i)}(x)| > k^{-1}(A \log n + z) \right) \leq Be^{-Cz}.$$

Moreover, the functions $\nu_{n, k}^{(i)}$, along with constants $\hat{A}, \hat{B}, \hat{C}$ possibly depending on $\nu$, can be chosen such that for any $z > 0$,

$$\mathbb{P} \left( \max \left\{ \sup_{0 \leq x \leq 1} \frac{|\nu_{n, k}^{(i)}(x) - x|}{x^{\nu}}, \sup_{1 \leq x \leq n/k} \frac{|\nu_{n, k}^{(i)}(x) - x|}{x^{1-\nu}} \right\} > k^{-1/2}(\hat{A} + z) \right) \leq \hat{B}e^{-\hat{C}z^2}.$$

**Proof.** By definition, the quantile function $\hat{F}_i^-$ appearing in $u_n^{(i)}(x)$ is the right-continuous function

$$\hat{F}_i^-(x) = U_{n, i, \lfloor nx \rfloor},$$

where $U_{n, i, j}$ is the $j$-th order statistic from the sample $U_{1i}, \ldots, U_{ni}$. We use the convention $U_{n, i, 0} = 0$. Similarly define the left-continuous quantile function

$$\hat{F}_i^+(x) = U_{n, i, \lceil nx \rceil}.$$

Theorem 1 from [Csorgo and Revész, 1978] states that for every $z > 0$,

$$\mathbb{P} \left( \sup_{x \in [0, 1]} |(\hat{F}_i^+(x) - x) - n^{-1/2}B_n(x)| > n^{-1}(A^+ \log n + z) \right) \leq B^+e^{-C^+z},$$

for positive constants $A^+, B^+, C^+$ and a sequence of Brownian bridges $B_n$. We first establish a similar tail bound for the right-continuous quantile function. Using $[y] \geq [y] - 1 = [y - 1]$, note that

$$\hat{F}_i^+(x) \geq \hat{F}_i^-(x) \geq U_{n, i, \lfloor nx - 1 \rfloor} = \hat{F}_i^+(x - 1/n),$$

so for every $x \in [0, 1]$, using the convention $\hat{F}_i^+(x) = \hat{F}_i^+(0)$ if $x < 0$,

$$0 \leq \hat{F}_i^+(x) - \hat{F}_i^-(x) \leq \hat{F}_i^+(x) - \hat{F}_i^+(x - 1/n) \leq 2 \sup_{x \in [0, 1]} |(\hat{F}_i^+(x) - x) - n^{-1/2}B_n(x)| + \left| \left( x + n^{-1/2}B_n(x) \right) - \left( x - \frac{1}{n} + n^{-1/2}B_n(x - 1/n) \right) \right| \leq 2 \sup_{x \in [0, 1]} |(\hat{F}_i^+(x) - x) - n^{-1/2}B_n(x)| + \frac{1}{n} + n^{-1/2}|B_n(x) - B_n(x - 1/n)|.$$

Thus

$$\sup_{x \in [0, 1]} |(\hat{F}_i^+(x) - x) - n^{-1/2}B_n(x)| \leq 3 \sup_{x \in [0, 1]} |(\hat{F}_i^+(x) - x) - n^{-1/2}B_n(x)| + \frac{1}{n} + n^{-1/2}|B_n(x) - B_n(x - 1/n)|.$$
Using the covariance function of the standard Brownian bridge, \( n^{-1/2}(B_n(x) - B_n(x - 1/n)) \) is normally distributed with mean 0 and variance upper bounded by \( 4/n^2 \). Thus, its absolute value is upper bounded by \( \sqrt{2}/n \) with probability greater than \( 1 - e^{-z/4} \). Therefore,

\[
P\left( \sup_{x \in [0,1]} |(F_i^-(x) - x) - n^{-1/2}B_n(x)| > n^{-1}(3A^+ \log n + 3z + 1 + \sqrt{z}) \right) \leq B^+e^{-C^+z} + e^{-z/4}
\]

which implies, for the right choice of \( A, B, C \),

\[
P\left( \sup_{x \in [0,1]} |(F_i^-(x) - x) - n^{-1/2}B_n(x)| > n^{-1}(A \log n + z) \right) \leq Be^{-Cz}. \tag{20}
\]

Now define

\[ w_n^{(i)}(x) := x + n^{-1/2} \frac{n}{k} B_n(kx/n). \]

Then

\[
w_n^{(i)}(x) - w_n^{(i)}(x) = \frac{n}{k} F_i^-(kx/n) - x - n^{-1/2} \frac{n}{k} B_n(kx/n) = \frac{n}{k} \left( F_i^-(kx/n) - \frac{kx}{n} - n^{-1/2} B_n(kx/n) \right)
\]

Thus for \( w_n^{(i)} \) defined above

\[
P\left( \sup_{x \in [0,n/k]} |w_n^{(i)}(x) - w_n^{(i)}(x)| > k^{-1}(A \log n + z) \right)
= \frac{1}{k} \sup_{x \in [0,n/k]} P\left( |\hat{F}_i^-(kx/n) - \frac{kx}{n} - n^{-1/2} B_n(kx/n)| > k^{-1}(A \log n + z) \right)
= \frac{1}{k} \sup_{x \in [0,1]} P\left( |\hat{F}_i^-(x) - x - n^{-1/2} B_n(x)| > n^{-1}(A \log n + z) \right)
\leq Be^{-Cz}
\]

by Equation (20), which proves the first claim.

We now prove the second claim. Let

\[ Z_n(x) := n^{-1/2} \frac{n}{k} B_n(kx/n). \]

Observe that

\[ \{ Z_n(x) \}_{x \in [0,n/k]} \overset{\mathcal{D}}{=} \left\{ n^{-1/2} \frac{n}{k} W_n(kx/n) - n^{-1/2} x W_n(1) \right\}_{x \in [0,n/k]} \overset{\mathcal{D}}{=} \left\{ k^{-1/2} W_n(x) - k^{-1/2} kx/n W_n(n/k) \right\}_{x \in [0,n/k]}, \]

where \( W_n \) are standard Wiener processes on \([0, \infty)\). If the sequences of suprema \( \sup_{0 < x \leq 1} k^{1/2} |Z_n(x)| / x^\nu \) and \( \sup_{1 \leq x \leq n/k} k^{1/2} |Z_n(x)| / x^{1-\nu} \) are uniformly tight, their distributions have finite medians independent of \( n \). Hence by Proposition A.2.1 of van der Vaart and Wellner (1996), there exist constants \( \tilde{A}, \tilde{B}, \tilde{C} \), depending only on those medians, such that

\[
P\left( \max \left\{ \sup_{0 < x \leq 1} \frac{k^{1/2} |Z_n(x)|}{x^\nu}, \sup_{1 \leq x \leq n/k} \frac{k^{1/2} |Z_n(x)|}{x^{1-\nu}} \right\} > \tilde{A} + z \right) \leq \tilde{B}e^{-\tilde{C}z^2}.
\]

and the result follows.

To establish tightness, note that since \( \nu < 1/2 \),

\[
\sup_{0 < x \leq 1} \frac{k^{1/2} |Z_n(x)|}{x^\nu} \leq \sup_{0 < x \leq 1} \frac{|W_n(x)|}{x^\nu} + \frac{k}{n} |W_n(n/k)| = O_p \left( 1 + \sqrt{\frac{k}{n}} \right) = O_p(1)
\]

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and
\[ \sup_{1 \leq x \leq \frac{n}{k}} \frac{k^{1/2}|Z_n(x)|}{x^{1-\nu}} \leq \sup_{1 \leq x < \infty} \frac{|W_n(x)|}{x^{1-\nu}} + \left( \frac{k}{n} \right)^{1-\nu} |W_n(n/k)| = O_P \left( 1 + \left( \frac{k}{n} \right)^{1/2-\nu} \sqrt{\log \log \frac{n}{k}} \right) = O_P(1), \]
where we used the law of the iterated logarithm at 0 and at \( \infty \), respectively, for Wiener processes (see for instance Durrett, 2013, Section 8.11). Note that the probability bounds on the suprema above, and hence the bounds on their medians, are uniform in \( n \) but may depend on \( \nu \), hence the dependence of the constants \( \tilde{A}, \tilde{B}, \tilde{C} \) on this parameter.

Let \( w_n^{(i)} \) be as in the proof of Lemma 1. For \( a \in (0, 1) \) and \( \nu_1, \nu_2 \in [0, 1] \), let
\[ \tilde{\Delta}_n^{(i)}(a, \nu_1, \nu_2) = \max \left\{ \sup_{0 \leq x \leq 1} \frac{|w_n^{(i)}(x) - x|}{x^{\nu_1}}, \sup_{1 \leq x \leq n/k} \frac{|w_n^{(i)}(x) - x|}{x^{1-\nu_2}} \right\} \]
and
\[ \tilde{\Delta}_n^{(i)}(a, \nu) = \tilde{\Delta}_n^{(i)}(a, \nu, \nu). \]
Similarly, let
\[ \hat{\Delta}_n^{(i)}(a, \nu_1, \nu_2) = \max \left\{ \sup_{0 \leq x \leq 1} \frac{|u_n^{(i)}(x) - x|}{x^{\nu_1}}, \sup_{1 \leq x \leq n/k} \frac{|u_n^{(i)}(x) - x|}{x^{1-\nu_2}} \right\} \]
and
\[ \hat{\Delta}_n^{(i)}(a, \nu) = \hat{\Delta}_n^{(i)}(a, \nu, \nu). \]

It is established in Lemma 1 that there are constants \( \tilde{A}, \tilde{B}, \tilde{C} \) only depending on \( \nu \in (0, 1/2) \) such that
\[ \mathbb{P}\left( \tilde{\Delta}_n^{(i)}(0, \nu) > k^{-1/2}(\tilde{A} + \tilde{z}) \right) \leq \tilde{B} e^{-\tilde{C}z^2}. \]

Lemma 1 also allows to obtain a certain bound for the terms \( \hat{\Delta}_n^{(i)} \).

**Corollary 1.** Let \( \tilde{\Delta}_n^{(i)} \) be as above. There exist constants \( \tilde{A}, \tilde{B}, \tilde{C} \) depending only on \( \nu \in (1/2) \) such that for all \( z \geq 0 \),
\[ \mathbb{P}\left( \tilde{\Delta}_n^{(i)}(0, 0, \nu) > \tilde{A} \left( \frac{1}{\sqrt{k}} + \frac{\log n}{k} \right) + \sqrt{\frac{z}{k}} \right) \leq \tilde{B} e^{-\tilde{C}z}. \]

**Proof.** Write
\[ \tilde{\Delta}_n^{(i)}(0, 0, \nu) \leq \tilde{\Delta}_n^{(i)}(0, 0, \nu) + \left| \tilde{\Delta}_n^{(i)}(0, 0, \nu) - \tilde{\Delta}_n^{(i)}(0, 0, \nu) \right| \]
\[ \leq \tilde{\Delta}_n^{(i)}(0, 0, \nu) + \max \left\{ \sup_{0 \leq x \leq 1} \frac{|u_n^{(i)}(x) - w_n^{(i)}(x)|}{x^{1-\nu}}, \sup_{1 \leq x \leq n/k} \frac{|u_n^{(i)}(x) - w_n^{(i)}(x)|}{x^{1-\nu}} \right\} \]
\[ \leq \tilde{\Delta}_n^{(i)}(0, 0, \nu) + \sup_{0 \leq x \leq n/k} |u_n^{(i)}(x) - w_n^{(i)}(x)|, \]
where the second inequality follows from the fact that for two functions \( f \) and \( g \) defined on the same domain,
\[ |\sup_x f(x) - \sup_x g(x)| \leq \sup_x |f(x) - g(x)|. \]

By Lemma 1, the first term above is larger than \( (\tilde{A} + \sqrt{z})/\sqrt{k} \) with probability at most \( \tilde{B} e^{-\tilde{C}z} \) and the second one is larger than \( (A \log n + z)/k \) with probability at most \( B e^{-Cz} \). The result follows by the right choice of \( \tilde{A}, \tilde{B}, \tilde{C} \).

**Lemma 2.** With \( w_n^{(i)} \) as above, there exists a universal positive constant \( c' \) such that for all \( a \in (0, 1) \),
\[ \mathbb{P}\left( \sup_{a \leq x \leq n/k} \frac{|w_n^{(i)}(x) - x|}{x} > 1/2 \right) \leq 6 \exp \left\{ -c' k \left( 1 + \frac{a}{\log \log (1/a)} \right) \right\}. \]
We first consider the terms A.2.1 of van der Vaart and Wellner (1996), that they have sub-Gaussian tails. The same can be said of the

Assume first that \( a \leq e^{-2} \). We are therefore interested in

\[
\sup_{x \in [a,n/k]} \left| Z_n(x) \right| = \frac{1}{\sqrt{k}} \left( \sup_{x \in [a,n/k]} \frac{|W(x)|}{x} + \frac{k}{n} W(n/k) \right)
\]

\[
\leq \frac{1}{\sqrt{k}} \left( \sup_{x \in [a,e^{-2}]} \sqrt{\log \log(1/x)} \frac{|W(x)|}{x} + \sup_{x \in [e^{-2},n/k]} \frac{|W(x)|}{x} + \frac{k}{n} W(n/k) \right)
\]

By the laws of the iterated logarithm at 0 and at infinity, respectively, the above two suprema of Gaussian processes are tight random variables. It follows that they have finite medians and hence, by Proposition A.2.1 of van der Vaart and Wellner (1996), that they have sub-Gaussian tails. The same can be said of the uniformly (in \( n \)) tight random variable \( \frac{1}{n} W(n/k) \). Therefore,

\[
P \left( \sup_{x \in [a,n/k]} \left| Z_n(x) \right| > 1/2 \right) \leq P \left( \sup_{x \in [0,e^{-2}]} \sqrt{\log \log(1/x)} |W(x)| > \frac{\sqrt{k}a}{6\log \log(1/a)} \right)
\]

\[
+ P \left( \sup_{x \in [e^{-2},\infty)} \frac{|W(x)|}{x} > \frac{\sqrt{k}}{6} \right) + P \left( \frac{k}{n} W(n/k) > \frac{\sqrt{k}}{6} \right)
\]

\[
\leq 2 \exp \left\{ -c_1 \frac{ka}{\log \log(1/a)} \right\} + 2e^{-c_2 k} + 2e^{-c_3 k},
\]

for some universal positive constants \( c_1, c_2, c_3 \). The result follows for some \( c' \) depending on those three constants only.

If, instead, \( a > e^{-2} \), then the sum of the last two terms of Equation (21) is a valid bound in itself. The rest of the proof goes through and we obtain the bound \( 4e^{-c'k} \).

In particular, Lemma 2 lower bounds the probability that for all \( x \in [a,n/k] \), \( x/2 \leq u_n^{(i)}(x) \leq 2x \), which will be repeatedly used in Section 2.3.4.

The following sections are respectively dedicated to each of the three terms of the decomposition introduced in Equation (17).

### 2.3.2 Increments of empirical processes

We first consider the terms \( A_{i,1} \). In this section we prove that for any \( \nu \in (0,1/2) \) there exists a constant \( C_1 < \infty \) (which can also depend on the constant \( K \) from Proposition 1) such that for any \( i,j,m \in V \) and \( \varepsilon \leq 1 \),

\[
P \left( \max |A_{i,1}| > C_1 \left( \log(n/k) + \log(1/a) \right)^2 \left\{ \left( \frac{\log(n/k)}{k} \right) \right\}^{1/2} \left( (k/n)^{\varepsilon} + \varepsilon \right)^{1/2}
\]

\[
+ \log(n/k) + \frac{\lambda}{\sqrt{k}} \left( (k/n)^{\varepsilon} + \varepsilon \right)^{1/2} + \frac{\lambda^2}{k} \right\} \right)
\]

\[
\leq d^3 e^{-\lambda^2} + P \left( \max_{i \in V} \Delta_n^{(i)}(a,0,\nu) > \varepsilon \right).
\]
Consider the following decompositions. For all \(x, y \in [a, 1]\), the numerator in the integral \(A_{i,j,1}^{(m),-}\) satisfies

\[
\begin{align*}
\left| G_{ijm,n}(u_n^{(i)}(x), u_n^{(j)}(y), u_n^{(m)}(1)) - G_{ijm,n}(x, y, 1) \right| \\
\leq \left| G_{ijm,n}(u_n^{(i)}(x), u_n^{(j)}(y), 1 \land u_n^{(m)}(1), 1 \lor u_n^{(m)}(1)) \right| + \left| G_{ijm,n}(u_n^{(i)}(x), [y \land u_n^{(j)}(y), y \lor u_n^{(j)}(y)], 1) \right| \\
+ \left| G_{ijm,n}(x \land u_n^{(i)}(x), x \lor u_n^{(i)}(x), y, 1) \right|.
\end{align*}
\]

The numerators in \(A_{i,m,1}^{(m),-}\) and \(A_{i,1}^{(m),\ell,-}\) satisfy a similar bound with only the first two terms, up to a logarithmic factor that is everywhere bounded by \(\log(1/a)\) in the case of \(A_{i,1}^{(m),2,-}\).

For all \(x \in [1, n/k], y \in [a, 1]\), the numerator in the integral \(A_{ij,1}^{(m),+-}\) satisfies

\[
\begin{align*}
\left| G_{ijm,n}(u_n^{(i)}(x), \infty), u_n^{(j)}(y), u_n^{(m)}(1)) - G_{ijm,n}([x, \infty), [y, \infty), 1]) \right| \\
\leq \left| G_{ijm,n}(u_n^{(i)}(x), \infty), u_n^{(j)}(y), [1 \land u_n^{(m)}(1), 1 \lor u_n^{(m)}(1)) \right| \\
+ \left| G_{ijm,n}(u_n^{(i)}(x), \infty), [y \land u_n^{(j)}(y), y \lor u_n^{(j)}(y)], 1)) \right| + \left| G_{ijm,n}(x \land u_n^{(i)}(x), x \lor u_n^{(i)}(x), [y, \infty), 1]) \right|.
\end{align*}
\]

The numerators in \(A_{i,m,1}^{(m),+-}\) and \(A_{i,1}^{(m),\ell,+}\) satisfy a similar bound with only the first two terms, up to a logarithmic factor that is everywhere bounded by \(\log(n/k)\) in the case of \(A_{i,1}^{(m),2,+}\), as well as the numerators in \(A_{ij,1}^{(m),+-}\) by symmetry.

For all \(x, y \in [1, n/k], \), the numerator in the integral \(A_{ij,1}^{(m),++}\) satisfies

\[
\begin{align*}
\left| G_{ijm,n}(u_n^{(i)}(x), \infty), u_n^{(j)}(y), u_n^{(m)}(1)) - G_{ijm,n}([x, \infty), [y, \infty), 1]) \right| \\
\leq \left| G_{ijm,n}(u_n^{(i)}(x), \infty), u_n^{(j)}(y), [1 \land u_n^{(m)}(1), 1 \lor u_n^{(m)}(1)) \right| \\
+ \left| G_{ijm,n}(u_n^{(i)}(x), \infty), [y \land u_n^{(j)}(y), y \lor u_n^{(j)}(y)], 1)) \right| + \left| G_{ijm,n}(x \land u_n^{(i)}(x), x \lor u_n^{(i)}(x), [y, \infty), 1]) \right|.
\end{align*}
\]

Define, for any \(\varepsilon \in (0, 1]\), \(F(\varepsilon) := \cup_{i,j,m} F_{ijm}(\varepsilon)\) where

\[
\begin{align*}
F_{ijm}(\varepsilon) := \bigcup_{i,j,m} \left\{ \frac{n}{k} \mathbb{1}_{i,j,m} \in S \in S_{ijm}^- \cup S_{ijm}^+ \right\},
\end{align*}
\]

and where the classes of sets \(S_{ijm}^-\) and \(S_{ijm}^+\) are defined as

\[
S_{ijm}^- := \left\{ \{w \in [0, \infty)^d : x - \varepsilon \leq w_i \leq x + \varepsilon, a_j \leq w_j \leq b_j, a_m \leq w_m \leq b_m \} : \right. \\
\left. a \leq x \leq 1, a_j, b_j, a_m, b_m \in [0, \infty) \right\},
\]

and

\[
S_{ijm}^+ := \left\{ \{w \in [0, \infty)^d : x - x^{1-\nu} \varepsilon \leq w_i \leq x + x^{1-\nu} \varepsilon, a_j \leq w_j \leq b_j, 0 \leq w_m \leq 1 \} : 
\right. \\
\left. 1 \leq x \leq n/k, a_j, b_j, \in [0, \infty) \right\}.
\]

Recalling the definition of \(\hat{\Delta}_n^{(i)}(a, 0, \nu)\), it follows from the definition of the class \(F(\varepsilon)\) that whenever

\[
\hat{\Delta}_n^{(i)}(a, 0, \nu) = \max_{i \in \mathbb{V}} \max_{a \leq x \leq 1} \left\{ \sup_{1 \leq \varepsilon \leq n/k} \left| u_n^{(i)}(x) - x \right| \right. \\
\left. \sup_{1 \leq \varepsilon \leq n/k} \frac{|u_n^{(i)}(x) - x|}{x^{1-\nu}} \right\} \leq \varepsilon,
\]

the numerator inside any of the integrals \(A_{i,j,1}^-\) can be expressed as a sum of at most three terms of the form \((P_n f_1 - P f_1) + (P_n f_2 - P f_2) + (P_n f_3 - P f_3)\), for functions \(f_1, f_2, f_3 \in F_{ijm}(\varepsilon)\). Here, \(P_n\) is the empirical distribution of the random vectors \(U_1, \ldots, U_n\) whereas \(P\) is their true distribution. In the case of the terms
We now deal with the terms $A_{i,1}^{(m),\pm}$, the sum is multiplied by a logarithmic term. In this case, all the integrals $A_{i,1}$ are upper bounded, in absolute value, by
\[ 3(\log(n/k) + \log(1/a))^2 \sup_{f \in F(\epsilon)} |P_n f - Pf|. \]

What we have established so far is that each such term $A_{i,1}$ satisfies, for any $t > 0$,
\[ \mathbb{P}(|A_{i,1}| \geq t) \leq \mathbb{P}(3(\log(n/k) + \log(1/a))^2 \sup_{f \in F(\epsilon)} |P_n f - Pf| \geq t) + \mathbb{P}\left( \max_{i \in V} \sup_{a \leq x \leq 1} |u_n^{(i)}(x) - x|, \sup_{1 \leq x \leq n/k} \frac{|u_n^{(i)}(x) - x|}{x^{1-\nu}} > \epsilon \right). \]

For any triple $(i, j, m)$, $F_{ijm}$ clearly admits the constant envelope function of the form $n/k$. Moreover it is a VC-subgraph class that satisfies Equation (51) with universal constants $A$ and $V$ (see for instance van der Vaart and Wellner (1996), Theorem 2.6.7). Moreover, the variance of any single function $f$ in $F_{ijm}(\epsilon)$ is bounded by
\[ Pf^2 \leq \left( \frac{n}{k} \right)^2 \left\{ \sup_{a \leq x \leq 1} \mathbb{P}\left( U_i \in \frac{k}{n}[x - \epsilon, x + \epsilon] \right) \vee \sup_{1 \leq x \leq n/k} \mathbb{P}\left( U_i \in \frac{k}{n}[x - x^{1-\nu}\epsilon, x + x^{1-\nu}\epsilon], U_m \leq \frac{k}{n} \right) \right\} \]
\[ \leq \frac{n}{k} \left\{ 2\epsilon \vee \sup_{1 \leq x \leq n/k} R_{im,n}(x) \right\} \]
\[ \leq \frac{n}{k} \left\{ 2\epsilon + \sup_{1 \leq x \leq n/k} R_{im}(x) \right\} \]
\[ \leq \frac{n}{k} \left\{ \epsilon + \left( \frac{k}{n} \right)^{\xi} \right\} \]
where the last two inequalities follow from Assumption 1 and Lemma 8 respectively. By Equation (52) we therefore have
\[ \mathbb{E} \left[ \sup_{f \in F_{ijm}(\epsilon)} |P_n f - Pf| \right] \leq k^{-1/2} \left( \frac{n/k}{\epsilon} + \epsilon \right)^{1/2} \log \left( \left( \frac{n/k}{\epsilon} + \epsilon \right)^{-1/2} \right)^{1/2} \]
\[ + k^{-1} \log \left( \left( \frac{n/k}{\epsilon} + \epsilon \right)^{-1/2} \right) \]
\[ \leq \left( \frac{\log(n/k)}{k} \right)^{1/2} \left( \frac{n/k}{\epsilon} + \epsilon \right)^{1/2} + \frac{\log(n/k)}{k}. \]

It follows from Equation (53) that there exists a constant $c$ such that for each triple $(i, j, m)$ and each $\lambda > 0$,\[ \mathbb{P}\left( \sup_{f \in F_{ijm}(\epsilon)} |P_n f - Pf| \geq c \left( \frac{\log(n/k)}{k} \right)^{1/2} \left( \frac{n/k}{\epsilon} + \epsilon \right)^{1/2} + \frac{\log(n/k)}{k} + \lambda \sqrt{n/k} \right) \leq e^{-\lambda^2}. \]

Combined with the union bound, this completes the proof.

### 2.3.3 Sums of iid processes

We now deal with the terms $A_{i,2}$ involving integrated empirical processes, such as in Equation (19). In this section, we show that there exists a positive constant $c_2$ such that for all $\lambda > 0$,
\[ \mathbb{P}\left( \max_{i \in V} |A_{i,2}| > \left( 1 + \left( \frac{k}{n} \right)^{\xi} \right) (\log(n/k))^2 (\log(n/k) + \log(1/a))^2 \right)^{1/2} \frac{\lambda}{\sqrt{k}} \right) \leq 16d_2 e^{-c_2 \lambda^2/2}. \]
Starting with $A_{ij,2}^{(m),-}$, we have by definition of $G_{ijm,n}$

$$A_{ij,2}^{(m),-} = \int_a^1 \int_a^1 \int_a^1 \frac{G_{ijm,n}(x,y,1)}{xy} dxdy$$

$$= \int_a^1 \int_a^1 \int_a^1 \sum_{t=1}^n \frac{1}{xy} \mathbb{1} \left\{ U_i \leq \frac{k}{n} x, U_j \leq \frac{k}{n} y, U_m \leq \frac{k}{n} \right\} - \frac{1}{xy} \mathbb{P}(U_i \leq \frac{k}{n} x, U_j \leq \frac{k}{n} y, U_m \leq \frac{k}{n}) dxdy$$

$$= \sum_{t=1}^n \left( V_{t,ijm}^{(m),-} - \mathbb{E}[V_{t,ijm}^{(m),-}] \right),$$

where $V_{t,ijm}^{(m),-}, 1 \leq t \leq n$, are independent copies of the random variable

$$V_{ijm}^{(m),-} := \frac{1}{k} \int_a^1 \int_a^1 \int_a^1 \frac{1}{xy} \mathbb{1} \left\{ U_i \leq \frac{k}{n} x, U_j \leq \frac{k}{n} y, U_m \leq \frac{k}{n} \right\} dxdy$$

$$= \frac{1}{k} \log \left( \frac{k}{nU_i} \wedge a^{-1} \right) \log \left( \frac{k}{nU_j} \wedge a^{-1} \right) \mathbb{1} \left\{ U_i \leq \frac{k}{n}, U_j \leq \frac{k}{n}, U_m \leq \frac{k}{n} \right\}.$$

Recall that by assumption $a \leq 1$. We may then write

$$V_{ijm}^{(m),-} = \frac{1}{k} \log(W_i) \log(W_j) \mathbb{1} \{ W_i, W_j > 1 \},$$

with

$$W_i := \left( \frac{k}{nU_i} \wedge a^{-1} \right) \mathbb{1} \{ U_m \leq \frac{k}{n} \}$$

and $W_j$ defined the same way. We easily notice that $0 \leq V_{ijm}^{(m),-} \leq (\log(1/a))^2/k$. Moreover, an application of Lemma [5] (particularly Equation [42]) gives

$$\mathbb{V}ar(V_{ijm}^{(m),-}) \leq \mathbb{E}[(V_{ijm}^{(m),-})^2]$$

$$\leq \frac{4}{k^2} \int_a^1 \int_a^1 \int_a^1 \frac{k}{xy} |R_{ijm,n}(x,y,1)| (\log x) (\log y) \frac{x}{\sqrt{xy}} dxdy$$

$$\leq \frac{4}{kn} \int_0^1 \int_0^1 (\log x)(\log y) \frac{x}{\sqrt{xy}} dxdy$$

$$= \frac{64}{kn},$$

where we used once again that $R_{ijm,n}(x,y,1) \leq x \wedge y \leq \sqrt{xy}$, along with the formula $\int_0^1 \log(x) / \sqrt{x} \, dx = 4$. We may therefore apply Bernstein’s inequality for bounded random variables [van der Vaart and Wellner, 1996, Lemma 2.2.9] with $v = 64/k, M = (\log(1/a))^2/k$, which yields

$$\mathbb{P}\left( \left| A_{ij,2}^{(m),-} \right| > \lambda \right) \leq 2 \exp \left\{ - \frac{k \lambda^2}{2(64 + \lambda(\log(1/a))^2/3)} \right\},$$

Now considering $A_{ij,2}^{(m),+},$ we use the same approach and see that

$$A_{ij,2}^{(m),+} = \sum_{t=1}^n \left( V_{t,ijm}^{(m),+} - \mathbb{E}[V_{t,ijm}^{(m),+}] \right),$$

where

$$V_{ijm}^{(m),+} = -\frac{1}{k} \log(W_i) \log(W_j) \mathbb{1} \{ W_i < 1, W_j > 1 \}.$$
and \( W_i, W_j \) are as before. This time, \( 0 \leq V_{ijm}^{(m),+} \leq (\log(n)) \log(1/a)/k \). An application of Lemma \([5]\) (this time, Equation \([43]\)) gives

\[
\text{Var}(V_{ijm}^{(m),+}) \leq \mathbb{E}
\left(
(\mathbb{E}[V_{ijm}^{(m),+}])^2
\right)
\]

\[
= \frac{4}{k^2} \int_1^{n/k} \int_1^{n/k} R_{ijm,n}([x, \infty), [y, \infty), 1]) (log x)(log y) dxdy
\]

\[
\leq \frac{4}{kn} \int_1^{n/k} \int_1^{n/k} \frac{R_{ijm}([x, \infty), [y, \infty), 1] + 2K(k/n)\xi (log x)(log y) dxdy}{xy},
\]

by Assumption \([1]\) By Proposition \([1]\) \( R_{ijm}([x, \infty), [y, \infty), 1] \leq R_{im}([x, \infty), 1) \wedge R_{jm}(y, 1) \leq K x^{-\xi} \wedge y \leq K x^{-\xi/2} y^{1/2} \). The integral above is thus bounded by

\[
K \int_0^1 \int_1^{\infty} \frac{(\log x)(-\log y)}{x^{1+\xi/2} y^{1/2}} dxdy + 2K \left(\frac{k}{n}\right) \xi \int_1^{\infty} \frac{(\log x)(-\log y)}{xy} dxdy
\]

\[
\leq 16K \left(\frac{k}{n}\right) \xi (\log(n/k))^2 (\log(1/a))^2 \leq C_2 \left(1 + \left(\frac{k}{n}\right)^2 (\log(n/k))^2 (\log(1/a))^2 \right),
\]

for a suitably chosen constant \( C_2 \) depending on \( K \) and \( \xi \) only. Bernstein’s inequality, with

\[
v = \frac{4C_2}{k} \left(1 + \left(\frac{k}{n}\right)^2 (\log(n/k))^2 (\log(1/a))^2 \right)
\]

and \( M = (\log(n/k))^2 (\log(1/a))/k \), therefore implies that for a positive constant \( c_2 \) depending on \( C_2 \) only,

\[
\Pr(|A_{ij,2}^{(m),+} - \lambda| > \lambda) \leq 2 \exp \left\{ -c_2 \frac{k \lambda^2}{1 + (k/n)^2 (\log(n/k))^2 (\log(1/a))^2 + \lambda (\log(n/k))^2 (\log(1/a))} \right\}.
\]

By symmetry, \( A_{ij,2}^{(m),-} \) admits the same bound.

As for \( A_{ij,2}^{(m),-} \), we write it as

\[
A_{ij,2}^{(m),-} = \sum_{t=1}^n (V_{t,ijm}^{(m),-} - \mathbb{E}[V_{t,ijm}^{(m),-}]),
\]

where

\[
V_{ijm}^{(m),-} = \frac{1}{k} \log(W_i) \log(W_j) \mathbb{1}\{W_i, W_j < 1\}
\]

and \( W_i, W_j \) are as before. Again, \( 0 \leq V_{ijm}^{(m),-} \leq (\log(n/k))^2/k \). An application of Lemma \([5]\) (this time, Equation \([44]\)) gives

\[
\text{Var}(V_{ijm}^{(m),-}) \leq \mathbb{E}
\left(
(\mathbb{E}[V_{ijm}^{(m),-}])^2
\right)
\]

\[
= \frac{4}{k^2} \int_1^{n/k} \int_1^{n/k} R_{ijm,n}([x, \infty), [y, \infty), 1]) (log x)(log y) dxdy
\]

\[
\leq \frac{4}{kn} \int_1^{n/k} \int_1^{n/k} \frac{R_{ijm}([x, \infty), [y, \infty), 1] + 3K(k/n)^\xi (log x)(log y) dxdy}{xy},
\]

by Assumption \([1]\) By Proposition \([1]\) \( R_{ijm}([x, \infty), [y, \infty), 1] \leq R_{im}([x, \infty), 1) \wedge R_{jm}([y, \infty), 1) \leq K x^{-\xi} \wedge y^{-\xi} \leq K x^{-\xi/2} y^{-\xi/2} \). The integral above is thus bounded by

\[
K \int_1^{\infty} \int_1^{\infty} \frac{(\log x)(\log y)}{xy^{1+\xi/2}} dxdy + 3K \left(\frac{k}{n}\right) \xi \int_1^{\infty} \int_1^{\infty} \frac{(\log x)(\log y)}{xy} dxdy
\]
\[ \leq \frac{16K}{\xi^4} + 3K \left( \frac{k}{n} \right) \xi (\log(n/k))^4 \leq C_2 \left( 1 + \left( \frac{k}{n} \right) \xi (\log(n/k))^4 \right), \]

after possibly enlarging the constant \( C_2 \). Hence by a similar application of Bernstein’s inequality for bounded random variables as before,

\[
P\left( |A_{ij,2}^{(m),++}| > \lambda \right) \leq 2 \exp \left\{ -c_2 \frac{k\lambda^2}{1 + (k/n)^2 (\log(n/k))^4 + \lambda (\log(n/k))^2} \right\},
\]

after possibly decreasing the (still positive) constant \( c_2 \).

Finally, the terms \( A_{i,2}^{(m),\ell,-}, A_{i,2}^{(m),\ell,+}, A_{m,2}^{(m),--} \) and \( A_{m,2}^{(m),+-} \) can be shown to satisfy similar tail bounds by the same strategy, using Equations (40) and (41) instead of Equations (42) to (44) for \( A_{i,2}^{(m),\ell,-} \) and \( A_{i,2}^{(m),\ell,+} \).

The conclusion of this section is that the positive constant \( c_2 \) can be chosen sufficiently small (only depending on \( K \) and \( \xi \)) such that all the terms \( A_{i,2} \) satisfy

\[
P\left( |A_{i,2}| > \lambda \right) \leq 2 \exp \left\{ -c_2 \frac{k\lambda^2}{1 + (k/n)^2 (\log(n/k))^2 (\log(n/k) + \log(1/a))^2 + \lambda (\log(n/k) + \log(1/a))^2} \right\},
\]

for all \( \lambda > 0 \). The denominator in the exponential above is clearly upper bounded by

\[
2 \max \left\{ 1 + (k/n)^2 (\log(n/k))^2 (\log(n/k) + \log(1/a))^2, \lambda (\log(n/k) + \log(1/a))^2 \right\},
\]

so the whole exponential is upper bounded by

\[
\max \left\{ 2 \exp \left\{ -c_2 \frac{k\lambda^2}{2(1 + (k/n)^2 (\log(n/k))^2 (\log(n/k) + \log(1/a))^2)} \right\}, 2 \exp \left\{ -c_2 \frac{k\lambda}{2(\log(n/k) + \log(1/a))^2} \right\} \right\},
\]

Deduce that at least one of

\[
P\left( |A_{i,2}| > \left( 1 + \left( \frac{k}{n} \right)^\xi (\log(n/k))^2 (\log(n/k) + \log(1/a))^2 \right)^{1/2} \frac{\lambda}{\sqrt{k}} \right) \leq 2e^{-c_2 \lambda^2/2}
\]

or

\[
P\left( |A_{i,2}| > (\log(n/k) + \log(1/a))^2 \frac{\lambda^2}{k} \right) \leq 2e^{-c_2 \lambda^2/2}
\]

holds. Therefore

\[
P\left( |A_{i,2}| > \left( 1 + \left( \frac{k}{n} \right)^\xi (\log(n/k))^2 (\log(n/k) + \log(1/a))^2 \right)^{1/2} \frac{\lambda}{\sqrt{k}} + (\log(n/k) + \log(1/a))^2 \frac{\lambda^2}{k} \right) \leq 2e^{-c_2 \lambda^2/2},
\]

and a union bound allows to conclude.

### 2.3.4 Increments of rescaled copulae

It remains to bound the terms \( A_{i,3} \), corresponding to increments of the measures \( R_{i,n} \) when the rescaled quantile functions \( u_n^{(i)} \) are applied to their arguments. In this section, we prove that under Assumption [1][1] there exists a constant \( C_3 \) such that for any \( \lambda, \tau > 0 \),

\[
P\left( \max |A_{i,3}| > 3\tau (\log(n/k) + \log(1/a))^2 + C_3 (\log(n/k) + \log(1/a))^2 \left( \frac{\tilde{A} + \lambda}{\sqrt{k}} + \left( \frac{k}{n} \right)^\xi \right) \right)
\]

\[
\leq P\left( \max_{i} \tilde{A}_n^{(i)}(a, \nu) > \frac{\tilde{A} + \lambda}{\sqrt{k}} \right) + P\left( \max_{x \in [0,n/k]} |u_n^{(i)}(x) - u_n^{(i)}(x)| > \tau \right)
\]

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We then obtain that for all \( \tau > 0 \) we have for all \( i \)
\[ \tilde{A} \] and such that the slightly larger probability
\[ \mathbb{P}\left( \max \left| A_{i,3} \right| > 3 \tau (\log(n/k) + \log(1/a))^2 + C_3 \left( \frac{A + \lambda}{\sqrt{k}} + \left( \frac{k}{n} \right) \xi (\log(n/k) + \log(1/a))^2 \right) \right) \]

admits the same upper bound.

By Assumption [1] the measure \( R_{J,n} \) in any integrand can be replaced by \( R_J \) at the cost of adding a deterministic error of the order of \( (k/n)^\xi \). After being integrated, such an error is of order at most \((k/n)^\xi (\log(n/k) + \log(1/a))^2 \). We will use this fact on multiple occasions by bounding the increments of \( R_J \) instead of \( R_{J,n} \).

Next we observe that by Lipschitz continuity of \( R_{J,n} \) the quantities \( u_n^{(i)}(x), u_n^{(j)}(y), u_n^{(m)}(1) \) appearing in the arguments of \( R_J \) inside \( A_{i,3} \) can be replaced by \( w_n^{(i)}(x), w_n^{(j)}(y), w_n^{(m)}(1) \) with an error that is controlled by Lemma [1] uniformly over \( x, y, i, j, m \). For example
\[
\max_{i,j,m} \sup_{x,y \in [0,n/k]} \left| R_{ijm,n}(u_n^{(i)}(x), u_n^{(j)}(y), u_n^{(m)}(1)) - R_{ijm,n}(w_n^{(i)}(x), w_n^{(j)}(y), w_n^{(m)}(1)) \right|
\]

\[
\leq 3 \max_i \sup_{x \in [0,n/k]} |u_n^{(i)}(x) - w_n^{(i)}(x)|.
\]

Define
\[
\tilde{A}_{i,3}^{(m)} := \int_a^1 \int_a^1 |R_{ijm,n}(w_n^{(i)}(x), w_n^{(j)}(y), w_n^{(m)}(1)) - R_{ijm,n}(x, y, 1)| dx dy,
\]

and similarly define other terms \( \tilde{A}_{i,3} \) replacing the different \( A_{i,3} \). The difference between those quantities and the original \( A_{i,3} \) terms that they replace can be uniformly controlled as
\[
\max |A_{i,3} - \tilde{A}_{i,3}| \leq 3(\log(n/k) + \log(1/a))^2 \max_i \sup_{x \in [0,n/k]} |u_n^{(i)}(x) - w_n^{(i)}(x)|,
\]

some of those bounds using the fact that
\[
\int \frac{(\log x)^{\ell-1}}{x} dx = \frac{(\log x)^\ell}{\ell} + \text{constant.} \tag{22}
\]

We then obtain that for all \( \tau > 0 \),
\[
\mathbb{P}\left( \max \left| A_{i,3} - \tilde{A}_{i,3} \right| > 3 \tau (\log(n/k) + \log(1/a))^2 \right) \leq \mathbb{P}\left( \max_i \sup_{x \in [0,n/k]} |u_n^{(i)}(x) - w_n^{(i)}(x)| > \tau \right).
\]

Hence it remains to bound the terms \( \tilde{A}_{i,3} \) which are defined in the same way as \( A_{i,3} \) but with \( w_n^{(i)}(x), w_n^{(j)}(y), w_n^{(m)}(1) \) replacing \( u_n^{(i)}(x), u_n^{(j)}(y), u_n^{(m)}(1) \).

Note that whenever
\[
\max_i \sup_{a \leq x \leq n/k} \frac{|w_n^{(i)}(x) - x|}{x} \leq 1/2,
\]
we have for all \( i \) and \( x \in [a,n/k] \) that \( x/2 \leq u_n^{(i)}(x) \leq 2x \). We will assume, for the remainder of the section, that this is realized. Lemma [2] allows to lower bound the probability of that event.

Finally, recall the quantities
\[
\tilde{A}_n^{(i)}(a, \nu) = \max \left\{ \sup_{x \in [a,1]} \frac{|w_n^{(i)}(x) - x|}{x^\nu}, \sup_{x \in [1,n/k]} \frac{|w_n^{(i)}(x) - x|}{x^{1-\nu}} \right\}
\]
from the discussion after Lemma [1]
The general case. We first prove the weaker bound that does not rely on Assumption $A^2$

Firstly, for $x, y \in [a, 1]$ and every triple $(i, j, m)$,

$$R_{ijm,n}(w^{(i)}_n(x), w^{(j)}_n(y), w^{(m)}_n(1)) - R_{ijm,n}(x, y, 1) = R_{ijm,n}(w^{(i)}_n(x), w^{(j)}_n(y), w^{(m)}_n(1)) - R_{ijm,n}(w^{(i)}_n(x), w^{(j)}_n(y), 1) + R_{ijm,n}(w^{(i)}_n(x), w^{(j)}_n(y), 1) - R_{ijm,n}(w^{(i)}_n(x), y, 1) + R_{ijm,n}(w^{(i)}_n(x), y, 1) - R_{ijm,n}(x, y, 1).$$  \tag{23}

Each of the three differences above, by Lipschitz continuity of $R_{ijm}$, is of course bounded by $\max_i \tilde{\Delta}_n^{(i)}(a, 0) \leq \max_i \tilde{\Delta}_n^{(i)}(a, \nu)$, for arbitrary $1/2 > \nu > 0$. Deduce that

$$|\tilde{A}_{ij,3}^{(m),+} | \leq \int_1^1 \int_a^1 |R_{ijm,n}(w^{(i)}_n(x), w^{(j)}_n(y), w^{(m)}_n(1)) - R_{ijm,n}(x, y, 1)| \, dx \, dy \leq (\log(1/a))^2 \max_i \tilde{\Delta}_n^{(i)}(0, \nu).$$

The term $\tilde{A}_{ij,3}^{(m),+}$ is bounded using the same strategy, following an expansion similar to Equation \tag{23} but without the first term. As for the term $\tilde{A}_{ij,3}^{(m),\xi,-}$, it is also bounded after an expansion similar to the third term in Equation \tag{23}, using the indefinite integral \tag{22}.

Secondly, for all $x \in [1, n/k]$, $y \in [a, 1]$ and every triple $(i, j, m)$,

$$R_{ijm,n}([w^{(i)}_n(x), \infty), w^{(j)}_n(y), w^{(m)}_n(1)) - R_{ijm,n}([x, \infty), y, 1) = R_{ijm,n}([w^{(i)}_n(x), \infty), w^{(j)}_n(y), w^{(m)}_n(1)) - R_{ijm,n}([w^{(i)}_n(x), \infty), w^{(j)}_y, 1) + R_{ijm,n}([w^{(i)}_n(x), \infty), w^{(j)}_y, 1) - R_{ijm,n}([w^{(i)}_n(x), \infty), y, 1) + R_{ijm,n}([w^{(i)}_n(x), \infty), y, 1) - R_{ijm,n}([x, \infty), y, 1).$$  \tag{24}

The first two differences are again uniformly bounded by $\max_i \tilde{\Delta}_n^{(i)}(a, \nu)$ by Lipschitz continuity. As for the third difference in Equation \tag{24}, let us replace $R_{ijm,n}$ by $R_{ijm}$ as described at the beginning of this section. We are then left with

$$R_{ijm}([x \wedge w^{(i)}_n(x), \infty), w^{(j)}_n(y), w^{(m)}_n(1)) \leq y \cdot \frac{|w^{(i)}_n(x) - x|}{x \wedge w^{(i)}_n(x)} \leq 2x^{-\nu} \max_i \tilde{\Delta}_n^{(i)}(a, \nu),$$  \tag{25}

by Lemma \tag{8} and the fact that we are on the event $w^{(i)}_n(x) \geq x/2$. Deduce that

$$|\tilde{A}_{ij,3}^{(m),-} | \leq \int_1^{n/k} \int_a^1 |R_{ijm,n}([w^{(i)}_n(x), \infty), w^{(j)}_n(y), w^{(m)}_n(1)) - R_{ijm,n}([x, \infty), y, 1)| \, dx \, dy \leq (\log(n/k))(\log(1/a)) \max_i \tilde{\Delta}_n^{(i)}(a, \nu) + \frac{k}{n} (\log(n/k))(\log(1/a)),$$

where the last term comes from the approximation of $R_{ijm,n}$ by $R_{ijm}$. By symmetry, $A_{ij,3}^{(m),-}$ enjoys the same bound. Moreover, $A_{ij,3}^{(m),+}$ is bounded using the same strategy, following an expansion similar to Equation \tag{24} but without the first term. As for the term $A_{ij,3}^{(m),\xi,+}$, it is also bounded after an expansion similar to the third term in Equation \tag{24}, using Equation \tag{22}.

Thirdly, for all $x, y \in [1, n/k]$ and every triple $(i, j, m)$,

$$R_{ijm,n}([w^{(i)}_n(x), \infty), [w^{(j)}_n(y), \infty), w^{(m)}_n(1)) - R_{ijm,n}([x, \infty), [y, \infty), 1)\]

$$= R_{ijm,n}([w^{(i)}_n(x), \infty), [w^{(j)}_n(y), \infty), w^{(m)}_n(1)) - R_{ijm,n}([w^{(i)}_n(x), \infty), [w^{(j)}_n(y), \infty), 1) + R_{ijm,n}([w^{(i)}_n(x), \infty), [w^{(j)}_n(y), \infty), 1) - R_{ijm,n}([w^{(i)}_n(x), \infty), [y, \infty), 1)$$

$$+ R_{ijm,n}([w^{(i)}_n(x), \infty), [w^{(j)}_n(y), \infty), 1) - R_{ijm,n}([w^{(i)}_n(x), \infty), [y, \infty), 1).$$
The first difference is upper bounded similarly to before by using the Lipschitz continuity of $R_{ijm,n}$. As for the second term, we once again replace $R_{ijm,n}$ by its limit $R_{ijm}$ and obtain
\[
R_{ijm}([w_n^{(i)}(x), \infty), [y \land w_n^{(j)}(y), y \lor w_n^{(j)}(y), 1) \leq R_{jm}([y \land w_n^{(j)}(y), y \lor w_n^{(j)}(y), 1)
\]
\[
\leq \frac{w_n^{(j)}(y) - y}{y \land w_n^{(j)}(y)} \leq 2y^{-\nu} \max_i \Delta_{n}^{(i)}(a, \nu). (27)
\]

The third term of Equation (26) admits the same bound with $x$ replacing $y$. Deduce that
\[
|A_{ijm}^{(m),++}| \leq \int_1^{n/k} \int_1^{n/k} |R_{ijm,n}([w_n^{(i)}(x), \infty), [w_n^{(j)}(y), \infty), w_n^{(m)}(1) - R_{ijm,n}([x, \infty), [y, \infty), 1]|_{dxdy}
\]
\[
\lesssim (\log(n/k))^2 \max_i \Delta_{n}^{(i)}(a, \nu) + \left(\frac{k}{n}\right)^{\xi} (\log(n/k))^2,
\]
the last term again from replacing $R_{ijm,n}$ by $R_{ijm}$.

We have therefore proved that, for any $1/2 > \nu > 0$, each term $A_{ijm}^{(m)}$ is upper bounded by a constant multiple of
\[
(\log(n/k) + \log(1/a))^2 \max_i \Delta_{n}^{(i)}(a, \nu) + \left(\frac{k}{n}\right)^{\xi} (\log(n/k) + \log(1/a))^2.
\]

**Assuming bounded densities.** Let us now suppose that Assumption 2 is satisfied with a certain $\varepsilon \in (0, 4)$. While in the general case above $\nu \in (0, 1/2)$ was arbitrary, let now $\nu = 1/2 - \varepsilon/8$.

The various bounds above on the numerators in the integrals $A_{ijm}^{(m)}$ were for the most part uniform in the integrands $x$ and $y$. By integrating them over a growing domain, a polylogarithmic factor was paid. We shall now derive more subtle bounds that are proportional to functions $f(x, y)$ such that $f(x, y)/xy$ is integrable over the infinite domain, thus allowing us to remove the extra polylogarithmic factors.

Firstly, for $x, y \in [a, 1)$, consider the three terms in Equation (23) and in each one, replace $R_{ijm,n}$ by $R_{ijm}$. By Lemma 9 with $\beta = \nu/2$, the third term is then bounded by
\[
R_{ij}([x \land w_n^{(i)}(x), x \lor w_n^{(i)}(x)], y) \lesssim y^{\nu/2} \frac{|w_n^{(i)}(x) - x|}{x^{\nu/2}} \leq (xy)^{\nu/2} \max_i \Delta_{n}^{(i)}(a, \nu).
\]

The second term admits the same bound up to a factor of $2^{\nu/2}$, since by assumption $w_n^{(i)}(x) \leq 2x$. As for the first term, now using Lemma 9 with $\beta = \nu$, it is upper bounded by both
\[
R_{im}(2x, [1 \land w_n^{(m)}(1), 1 \lor w_n^{(m)}(1)]) \lesssim x^{\nu} \max_i \Delta_{n}^{(i)}(a, \nu)
\]
and
\[
R_{jm}(2y, [1 \land w_n^{(m)}(1), 1 \lor w_n^{(m)}(1)]) \lesssim y^{\nu} \max_i \Delta_{n}^{(i)}(a, \nu),
\]
hence by
\[
(xy)^{\nu/2} \max_i \Delta_{n}^{(i)}(a, \nu)
\]
up to a constant. It then follows that
\[
|A_{ijm}^{(m),--}| \leq \max_i \Delta_{n}^{(i)}(a, \nu) \int_0^1 \int_0^1 (xy)^{\nu/2-1} dxdy + \left(\frac{k}{n}\right)^{\xi} (\log(1/a))^2
\]
\[
\lesssim \max_i \Delta_n^{(i)}(a, \nu) + \left(\frac{k}{n}\right)^\xi (\log(1/a))^2.
\]

The bounds on \(A_{m,3}^{(m),-}\) and \(A_{i,3}^{(m),\ell,+}\) follow from the same argument, noting for the latter that
\[
\int_0^1 \frac{(\log x)^{\ell-1}}{x^{1-\nu/2}} dx < \infty.
\]

Secondly, for \(x \in [1, n/k]\), \(y \in [a, 1]\) and every triple \((i, j, m)\), consider the three terms in Equation (24) and in each one, replace \(R_{ijm,n}\) by \(R_{ijm}\). It was already proved in the general case that by Lemma 8, the third term satisfies (see Equation (25))
\[
R_{ijm}(\lfloor x \wedge w_n^{(i)}(x), x \lor w_n^{(i)}(x) \rfloor, y, 1) \leq R_{ijm}(\lfloor x \wedge w_n^{(i)}(x), x \lor w_n^{(i)}(x) \rfloor, y) \leq 2x^{-\nu} y \max_i \Delta_n^{(i)}(a, \nu).
\]

The second term, by an application of Lemma 10 with \(\beta = -\nu\), is upper bounded by
\[
R_{ijm}(w_n^{(i)}(x), [y \wedge w_n^{(j)}(y), y \lor w_n^{(j)}(y)]) \lesssim w_n^{(i)}(x) y^{-\nu} w_n^{(j)}(y) y^{-\nu} |w_n^{(j)}(y) - y| \lesssim x^{-\nu} y^{\nu+\nu} \max_i \Delta_n^{(i)}(a, \nu).
\]

The first term of Equation (24) is upper bounded by both
\[
R_{jm}(w_n^{(j)}(y), [1 \wedge w_n^{(m)}(1), 1 \lor w_n^{(m)}(1)]) \leq w_n^{(j)}(y) |w_n^{(m)}(1) - 1| \lesssim y \max_i \Delta_n^{(i)}(a, \nu),
\]

by Lemma 9, and
\[
R_{im}(w_n^{(i)}(x), [1 \wedge w_n^{(m)}(1), 1 \lor w_n^{(m)}(1)]) \lesssim w_n^{(i)}(x) y^{-\nu} |w_n^{(m)}(1) - 1| \lesssim x^{-\nu} \max_i \Delta_n^{(i)}(a, \nu),
\]

again by Lemma 10 with \(\beta = \nu\). Hence the first term is in fact bounded by
\[
x^{-\nu/2} y^{1/2} \max_i \Delta_n^{(i)}(a, \nu)
\]
up to a constant. It then follows that
\[
|A_{ijm,3}^{(m),+}| \lesssim \max_i \Delta_n^{(i)}(a, \nu) \int_0^1 \int_0^1 \left( x^{-1-\nu} + x^{-1-\nu} y^{\nu+1} + x^{-1-\nu/2} y^{-1/2} \right) \, dx \, dy
\]
\[
+ \left(\frac{k}{n}\right)^\xi (\log(n/k))(\log(1/a))
\]
\[
\lesssim \max_i \Delta_n^{(i)}(a, \nu) + \left(\frac{k}{n}\right)^\xi (\log(n/k))(\log(1/a)).
\]

The same holds for \(A_{ijm,3}^{(m),-}\) by symmetry. The bounds on \(A_{m,3}^{(m),+}\) and \(A_{i,3}^{(m),\ell,-}\) follow from the same argument, noting for the latter that
\[
\int_1^\infty \frac{(\log x)^{\ell-1}}{x^{1+\zeta}} \, dx < \infty
\]
for any positive \(\zeta\).

Finally, for \(x, y \in [1, n/k]\) and every triple \((i, j, m)\), consider the three terms in Equation (26) and in each one, replace \(R_{ijm,n}\) by \(R_{ijm}\). By Lemma 9 with \(\beta = 1 + \nu\), the third term of Equation (26) satisfies
\[
R_{ijm}(\lfloor x \wedge w_n^{(i)}(x), x \lor w_n^{(i)}(x) \rfloor, y, \infty) \leq R_{im}(\lfloor x \wedge w_n^{(i)}(x), x \lor w_n^{(i)}(x) \rfloor, 1) \lesssim \left| w_n^{(i)}(x) - x \right| \lesssim x^{-\nu-\nu} \max_i \Delta_n^{(i)}(a, \nu)
\]
but at the same time, Lemma 10 with \(\beta = -\nu/2\) yields
\[
R_{ijm}(\lfloor x \wedge w_n^{(i)}(x), x \lor w_n^{(i)}(x) \rfloor, [y, \infty), 1) \leq R_{ijm}(\lfloor x \wedge w_n^{(i)}(x), x \lor w_n^{(i)}(x) \rfloor, [y, \infty))
\]

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We have therefore proved that each term \( A \) as per the statement of the theorem, pick an arbitrary \( \zeta \)
we obtain the following simultaneous upper bound on each integral

\[ R_{ijm}(x) \leq x^{-r + \epsilon} \max_i \Delta_n^{(i)}(a, \nu). \]

The minimum between the bounds above being smaller than their geometric mean, we then have

\[ R_{ijm}(x) \leq x^{-r + \epsilon} \max_i \Delta_n^{(i)}(a, \nu), \]

recalling that \( r = 1/2 - \epsilon/8 \), so that \((-r - \epsilon) + (1 - r + \epsilon/2) = 1 - 2r - \epsilon/2 = -\epsilon/4\). The second term of Equation (26) admits a similar bound since by assumption \( w_n^{(i)}(x) \geq x/2 \). As for the first term, by Lemma 10 with \( \beta = -\epsilon \) it is bounded by

\[ R_{im}(x/2, \infty), [1 \wedge w_n^{(m)}(1), 1 \vee w_n^{(m)}(1)] \leq x^{-\epsilon} |w_n^{(m)}(1) - 1| \leq x^{-\epsilon} \max_i \Delta_n^{(i)}(a, \nu), \]

but also by

\[ R_{jm}(y/2, \infty), [1 \wedge w_n^{(m)}(1), 1 \vee w_n^{(m)}(1)] \leq y^{-\epsilon} |w_n^{(m)}(1) - 1| \leq y^{-\epsilon} \max_i \Delta_n^{(i)}(a, \nu), \]

hence by

\[ (xy)^{-\epsilon/2} \max_i \Delta_n^{(i)}(a, \nu) \]

up to a constant. It then follows that

\[ \max_i \Delta_n^{(i)}(a, \nu) \leq \max_i \Delta_n^{(i)}(a, \nu) \int_1^\infty \int_1^\infty x^{-\epsilon/2} y^{-\epsilon/2} dxdy + \left( \frac{k}{n} \right)^{\xi} (\log(n/k))^2 \]

We have therefore proved that each term \( A \) is upper bounded, up to a constant, by

\[ \max_i \Delta_n^{(i)}(a, \nu) + \left( \frac{k}{n} \right)^{\xi} (\log(n/k) + \log(1/a))^2. \]

\[ \square \]

2.4 Proof of Theorem 1

As per the statement of the theorem, pick an arbitrary \( \zeta \in (0, 1) \) and assume that \( k \geq n^{\xi} \). Since the statement is trivial for \( \lambda < 1 \) (with the right choice of constants), suppose that \( 1 \leq \lambda \leq \sqrt{k}/(\log n)^4 \). Moreover let \( a \) satisfy

\[ \max \left\{ \frac{\lambda^2 \log n}{k}, \left( \frac{k}{n} \right)^{\xi} \right\} \leq a \leq \max \left\{ \frac{\lambda}{\sqrt{k} \log n}, \left( \frac{k}{n} \right)^{\xi} \right\}, \]  

(28)

Note that by our choice of \( \lambda \), the interval above is always non-empty. Introduce the notation \( l_{n,a} := \log(n/k) + \log(1/a) \) and note that by Equation (28), \( l_{n,a} \leq \log(n/k) \). Consider the results of Sections 2.3.2 to 2.3.4 with

\[ \varepsilon := A \left( \frac{1}{\sqrt{k}} + \frac{\log n}{k} \right) + \frac{\lambda}{\sqrt{k}} + \frac{\lambda^2}{k} \leq \frac{\lambda}{\sqrt{k}} \leq \frac{1}{(\log n)^4} \]

and

\[ \tau := \frac{A \lambda}{k}, \]

for \( \hat{A} \) and \( A \) as in Corollary 1 and Lemma 1, respectively. Combining these results with those of Section 2.2, we obtain the following simultaneous upper bound on each integral \( I \) in Equations (14) to (16):

\[ C A_{l_{n,a}} \left\{ \left( \frac{\log(n/k)}{k} \right)^{1/2} \left( \left( \frac{k}{n} \right) + \varepsilon \right)^{1/2} + \frac{\log(n/k)}{k} \right\} + \frac{\lambda}{\sqrt{k}} \left( \left( \frac{k}{n} \right) + \varepsilon \right)^{1/2} + \frac{\lambda^2}{k} \]
\[ + \left( 1 + \left( \frac{k}{n} \right)^\xi (\log(n/k))^2 l_{n,a}^2 \right)^{1/2} \frac{\lambda}{\sqrt{k}} + l_{n,a}^2 \frac{\lambda^2}{k} \]
\[ + 3\tau l_{n,a}^2 + C_3 l_{n,a}^2 \left( \frac{A + \lambda}{\sqrt{k}} + \left( \frac{k}{n} \right)^\xi \right) \]
\[ + O\left( \left( \frac{k}{n} \right)^\xi l_{n,a}^2 \right). \]

Note that \((k/n)^\xi (\log(n/k))^2 l_{n,a}^2 \lesssim (k/n)^\xi (\log(n/k))^4\) can be upper bounded by a constant only depending on \(\xi\). Using this and the fact that \((x + y)^{1/2} \leq x^{1/2} + y^{1/2}\) for \(x, y \geq 0\), we find
\[
\begin{align*}
  l_{n,a}^2 \left( \frac{\log(n/k)}{k} \right)^{1/2} &\left( \left( \frac{k}{n} \right)^\xi + \varepsilon \right)^{1/2} \leq l_{n,a}^2 \left( \frac{k}{n} \right)^{\xi/2} \left( \frac{\log(n/k)}{k} \right)^{1/2} + l_{n,a}^2 \left( \frac{\log(n/k)}{k} \right)^{1/2} \varepsilon^{1/2} \\
  \leq & \frac{1}{\sqrt{k}} + l_{n,a}^2 \left( \frac{\log(n/k)}{k} \right)^{1/2} \lambda^{1/2} k^{-1/4} \leq \frac{\lambda}{\sqrt{k}}
\end{align*}
\]
since \((\log n)^{5/2} \lesssim k^{1/4}\). By similar arguments using that \(\varepsilon \lesssim 1/(\log n)^4\)
\[
l_{n,a}^2 \left( \left( \frac{k}{n} \right)^\xi + \varepsilon \right)^{1/2} \lesssim 1.
\]
Moreover,
\[
\tau l_{n,a}^2 \lesssim \left( \frac{\log(n/k)}{k} \right)^3 + l_{n,a}^2 \frac{\lambda^2}{k}.
\]
In addition, notice that by our choice of \(\lambda\),
\[
\frac{\lambda^2}{k} \leq l_{n,a}^2 \frac{\lambda^2}{k} \lesssim \frac{\lambda(\log n)^{-2} \sqrt{k}}{k} \leq \frac{\lambda}{\sqrt{k}}
\]
and that since \(k \geq n^\xi\),
\[
\frac{(\log n)^3}{k} \lesssim \frac{1}{\sqrt{k}}.
\]
Piecing those results together, Equation (29) can be bounded by
\[
C' \left\{ \left( \frac{k}{n} \right)^\xi (\log(n/k))^2 + \frac{(\log(n/k))^2 (1 + \lambda)}{\sqrt{k}} \right\},
\]
for the right constant \(C'\). If Assumption 2 is made, the same strategy yields the sharper bound
\[
C_1 l_{n,a}^2 \left\{ \left( \frac{\log(n/k)}{k} \right)^{1/2} \left( \left( \frac{k}{n} \right)^\xi + \varepsilon \right)^{1/2} + \frac{\log(n/k)}{k} \right\} + \frac{\lambda}{\sqrt{k}} \left( \left( \frac{k}{n} \right)^\xi + \varepsilon \right)^{1/2} + \frac{\lambda^2}{k} \right\}
\]
\[+ \left( 1 + \left( \frac{k}{n} \right)^\xi (\log(n/k))^2 l_{n,a}^2 \right)^{1/2} \frac{\lambda}{\sqrt{k}} + l_{n,a}^2 \frac{\lambda^2}{k} \]
\[+ 3\tau l_{n,a}^2 + C_3 \left( \frac{A + \lambda}{\sqrt{k}} + \left( \frac{k}{n} \right)^\xi l_{n,a}^2 \right) \]
\[+ O\left( \left( \frac{k}{n} \right)^\xi l_{n,a}^2 \right) \]
\[\leq C' \left\{ \left( \frac{k}{n} \right)^\xi (\log(n/k))^2 + \frac{1 + \lambda}{\sqrt{k}} \right\},
\]
for the right constant \(\bar{C}'\). It is left to control the deterministic error terms in Equations (14) to (16) arising from the truncation of the integrals. Those terms are upper bounded by a constant multiple of
\[
\left( \frac{k}{n} \right)^\xi (\log(n/k)) + l_{n,a}^2 k^{-1} + a l_{n,a} \lesssim \left( \frac{k}{n} \right)^\xi (\log(n/k)) + \frac{1}{\sqrt{k}} + \max \left\{ \frac{\lambda}{\sqrt{k}}, \left( \frac{k}{n} \right)^\xi (\log(n/k)) \right\}
\]
so they are absorbed into the bounds above. Note that this time we have used the upper bound on \( a \) in Equation (28) in order to bound \( a |_{n,a} \).

The probability that each of the two bounds in Equations (30) and (31) holds is at least

\[
1 - d^3 e^{-\lambda^2} - \mathbb{P}\left( \max_{i \in V} \bar{\Delta}_n^{(i)}(a, 0, \nu) > \tilde{A}\left( \frac{1}{\sqrt{k}} + \frac{\log n}{k} \right) + \frac{\lambda^2}{k} \right) - 16d^3 e^{-c \lambda^2 / 2} - \mathbb{P}\left( \max_{i} \bar{\Delta}_n^{(i)}(a, \nu) > \frac{\tilde{A} + \lambda}{\sqrt{k}} \right)
\]

\[= 1 - d^3 e^{-\lambda^2} - \bar{B} e^{-\tilde{C} \lambda^2} - 16d^3 e^{-c \lambda^2 / 2} - \bar{B} e^{-\tilde{C} \lambda^2} - \bar{B} e^{-\tilde{C} \lambda^2} - 6d \exp\left\{ -c' k \left( 1 + \frac{\alpha}{\log \log(1/\alpha)} \right) \right\}
\]

\[
\geq 1 - Md^3 \exp\left\{ - c \min\left\{ \lambda^2, \frac{ka}{\log \log(1/\alpha)} \right\} \right\},
\]

for suitable constants \( M \) and \( c \), where we have used Corollary 1 and Lemmas 1 and 2. By Equation (28), since \( a \geq \lambda^2 (\log n) / k \), we find

\[
\frac{ka}{\log \log(1/\alpha)} \geq \frac{\lambda^2 \log n}{\log k} \geq \lambda^2,
\]

so that the probability above is equal to

\[
1 - Md^3 e^{-c \lambda^2}.
\]

Combining this with Equations (11) and (14) to (16) finally concludes the proof, upon noting that the factor \( e_i^{(m)} - e_j^{(m)} \) appearing in Equation (11) is upper bounded by \( 1 + K/\xi \) (see the proof of Lemma 7) and properly choosing the constants \( C \) and \( \tilde{C} \) in terms of \( C' \) and \( \tilde{C}' \).

\[
\square
\]

### 3 Auxiliary results

Here are collected additional technical results that are necessary to the manuscript, starting with the proof of Proposition 1, followed by lemmas referenced in Section 2.

#### 3.1 Proof of Proposition 1

Recall that \( F \) denotes the copula of \( U = 1 / X \), and let \( F_j \) denote its lower dimensional margins corresponding to the copula of \( U_j \). We start by showing that Equations (5) and (7) imply Equation (8). Let

\[
\psi = \frac{\gamma_1}{1 + \gamma_1 + \gamma_2} \in (0, 1).
\]

and note that both \( -\psi + (1 - \psi) \gamma_1 \) and \( \psi \gamma_2 \) are then equal to \( \xi \). For arbitrary \( x, y \leq q^{-\psi} \), we have

\[
\left| q^{-1} F_{ij}(qx, qy) - R_{ij}(x, y) \right| \leq q^{-1} F_{ij}(q^{-1-\psi} qx, q^{-1-\psi} q^\psi y) - q^{-\psi} R_{ij}(q^\psi x, q^\psi y) \leq K_1 q^{-\psi - (1 - \psi) \gamma_1} = K_1 q^{\xi},
\]

by Equation (5) applied with \( q \) replaced by \( q^{-1-\psi} \). Then, for \( q^{-\psi} \leq x \leq q^{-1} \) and \( 0 < y \leq 1 \), we have both

\[
y \geq R_{ij}(x, y) \geq R_{ij}(q^{-\psi}, y) \geq y R_{ij}(q^{-\psi}, 1) \geq y(1 - K_2 q^{\psi \gamma_2}) \geq y - K_2 q^{\xi}
\]

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and
\[ y = q^{-1}qy \geq q^{-1}F_{ij}(qx, qy) \geq q^{-1}F_{ij}(qq^{-\psi}, qy) \geq R_{ij}(q^{-\psi}, y) - K_1q^\xi \geq y - (K_1 + K_2)q^\xi, \]
by Equation (32) and the preceding lower bound for \( R_{ij}(q^{-\psi}, y) \). Deduce that for all \( x \leq q^{-1}, y \leq 1, \)
\[ \left| q^{-1}F_{ij}(qx, qy) - R_{ij}(x, y) \right| \leq (K_1 + K_2)q^\xi. \tag{33} \]

We now turn to verifying Equation \(10 \). By definition, each function \( R_{ij} \) is the distribution function of a nonnegative measure. It follows that
\[ R_{ijm}(q^{-1}, q^{-1}, 1) = R_{ijm}(q^{-1}, \infty, 1) - (R_{ijm}(q^{-1}, \infty, 1) - R_{ijm}(q^{-1}, q^{-1}, 1)) \]
\[ \geq R_{im}(q^{-1}, 1) - (R_{jm}(\infty, 1) - R_{jm}(q^{-1}, 1)) \]
\[ = R_{im}(q^{-1}, 1) - (1 - R_{jm}(q^{-1}, 1)), \]
so that
\[ 1 - R_{ijm}(q^{-1}, q^{-1}, 1) \leq 1 - R_{im}(q^{-1}, 1) + 1 - R_{jm}(q^{-1}, 1) \leq 2K_2q^{\gamma_2} \tag{34} \]
by Equation (7).

Let us then consider Equation \(9 \). We divide the square \([0, q^{-1}]^2\) of possible values of \((x, y)\) into four quadrants defined by the axes \(x = q^{-\psi}\) and \(y = q^{-\psi}\). First, for all \(x, y, z \leq q^{-\psi}, \)
\[ \left| q^{-1}F_{ijm}(qx, qy, qz) - R_{ijm}(x, y, z) \right| = \left| q^{-1}F_{ijm}(q^{1-\psi}q^\psi x, q^{1-\psi}q^\psi y, q^{1-\psi}q^\psi z) - q^{-\psi}R_{ijm}(q^\psi x, q^\psi y, q^\psi z) \right| \]
\[ = q^{-\psi}\left| q^{1-\psi}F_{ijm}(q^{1-\psi}q^\psi x, q^{1-\psi}q^\psi y, q^{1-\psi}q^\psi z) - R_{ijm}(q^\psi x, q^\psi y, q^\psi z) \right| \]
\[ \leq K_1q^{-\psi + (1-\psi)\gamma_1} = K_1q^\xi, \tag{35} \]
by applying Equation (6) with \(q\) replaced by \(q^{1-\psi}\). Second, for \(q^{-\psi} \leq x, y \leq q^{-1}, z \leq 1, \)
\[ z \geq R_{ijm}(x, y, z) = zR_{ijm}(x/z, y/z, 1) \geq zR_{ijm}(q^{-\psi}, q^{-\psi}, 1) \geq z(1 - 2K_2q^{\gamma_2}) \geq z - 2K_2q^\xi, \tag{36} \]
by the properties of \(R_{ijm}\) and by Equation (34). Similarly, \(q^{-1}F_{ijm}(qx, qy, qz)\) admits similar upper and lower bounds:
\[ z \geq q^{-1}F_{ijm}(qx, qy, qz) \geq q^{-1}F_{ijm}(qq^{-\psi}, qq^{-\psi}, qz) \geq R_{ijm}(q^{-\psi}, q^{-\psi}, z) - K_1q^\xi \]
by Equation (35). Using Equation (36), which holds for every \(x, y \geq q^{-\psi}\), this lower bound is itself lower bounded by
\[ z - (K_1 + 2K_2)q^\xi. \]
Deduce that
\[ \left| q^{-1}F_{ijm}(qx, qy, qz) - R_{ijm}(x, y, z) \right| \leq (K_1 + 2K_2)q^\xi. \tag{37} \]

Third, let \(q^{-\psi} \leq x \leq q^{-1}, y \leq q^{-\psi}, z \leq 1\) (the case where \(q^{-\psi} \leq y \leq q^{-1}\) and \(x \leq q^{-\psi}\) is handled symmetrically). We have
\[ R_{jm}(y, z) \geq R_{ijm}(x, y, z) \]
\[ \geq R_{ijm}(q^{-\psi}, y, z) \]
\[ = R_{jm}(y, z) - (R_{jm}(y, z) - R_{ijm}(q^{-\psi}, y, z)) \]
\[ \geq R_{jm}(y, z) - (z - R_{im}(q^{-\psi}, z)) \]
\[ \geq R_{jm}(y, z) - (1 - R_{im}(q^{-\psi}, 1)) \]
\[ \geq R_{jm}(y, z) - K_2q^{\psi\gamma_2} = R_{jm}(y, z) - K_2q^\xi, \]
where we have used the representation of $R_{ijm}$ as a measure and Equation (7). The term $q^{-1}F_{ijm}(x, y, z)$ enjoys similar upper and lower bounds: by Equation (35) and by the preceding lower bound on $R_{ijm}(q^{-\psi}, y, z)$,

$$q^{-1}F_{ijm}(x, y, z) \geq q^{-1}F_{ijm}(qq^{-\psi}, qy, qz) \geq R_{ijm}(q^{-\psi}, y, z) - K_1 q^\xi \geq R_{jm}(y, z) - (K_1 + K_2) q^\xi,$$

and by Equation (33),

$$q^{-1}F_{ijm}(x, y, z) \leq q^{-1}F_{jm}(qy, qz) \leq R_{jm}(y, z) + (K_1 + K_2) q^\xi.$$ 

Deduce that

$$\left| q^{-1}F_{ijm}(qx, qy, qz) - R_{ijm}(x, y, z) \right| \leq 2(K_1 + K_2) q^\xi.$$ 

We have therefore established that for all $x, y \leq q^{-1}$, $z \leq 1$,

$$\left| q^{-1}F_{ijm}(qx, qy, qz) - R_{ijm}(x, y, z) \right| \leq 2(K_1 + K_2) q^\xi. \tag{38}$$

The result follows by considering Equations (33), (34) and (38) and the definitions of $K$ and $\xi$. □

3.2 The moments $e_{m, \ell}^{(m)}$

Recalling that for any $m$, $Y_{m}^{(m)}$ has a unit Pareto distribution, and thus that $\log Y_{m}^{(m)}$ has a unit exponential distribution, it is evident that $e_{m,1}^{(m)} = 1$ and $e_{m,2}^{(m)} = 2$. As for the empirical versions $\hat{e}_{m,\ell}^{(m)}$ of those moments, they are in fact deterministic, since the terms $\hat{F}_{m}(U_{tm})$ appearing in the sum are exactly the $k$ smallest such terms $\{1/n, \ldots, k/n\}$. Precisely, we have the following result.

**Lemma 3.** As long as $k \geq 3$, we have

$$\left| \hat{e}_{m,1}^{(m)} - 1 \right| \leq \frac{3\log k}{k}, \quad \left| \hat{e}_{m,2}^{(m)} - 2 \right| \leq \frac{8(\log k)^2}{k}.$$

**Proof.** By definition, we have

$$\hat{e}_{m,\ell}^{(m)} = \frac{1}{k} \sum_{j=1}^{k} \{\log(k/j)\}^{\ell} = \begin{cases} \log k - \frac{1}{k} \sum_{j=1}^{k} \log j, & \ell = 1 \\ (\log k)^2 - 2\log k \sum_{j=1}^{k} \log j + \frac{1}{k} \sum_{j=1}^{k} (\log j)^2, & \ell = 2 \end{cases}. \tag{39}$$

Note that

$$\sum_{j=1}^{k} (\log j)^{\ell} = \sum_{j=2}^{k} \int_{j}^{j+1} (\log j)^{\ell} dt \in \left[ \int_{1}^{k} (\log t)^{\ell} dt, \int_{2}^{k+1} (\log t)^{\ell} dt \right].$$

Evaluating those integrals yields

$$k \{\log k - 1\} + 1 \leq \sum_{j=1}^{k} \log j \leq (k + 1) \{\log(k + 1) - 1\} - 2(\log 2 - 1)$$

and

$$k \{(\log k)^2 - 2(\log k + 2)\} - 2 \leq \sum_{j=1}^{k} (\log j)^2 \leq (k + 1) \{(\log(k + 1))^2 - 2\log(k + 1) + 2\} - 2 \{(\log 2)^2 - 2\log 2 + 2\}.$$
Denote by $a_\ell$ and $b_\ell$ the lower and upper bound on $\sum_{j=1}^{k}(\log j)^\ell$ above, $\ell \in \{1, 2\}$. As long as $k \geq 3$, we have by Equation (39) and by simple computations
\[
k|\hat{c}_m^{(m),1} - 1| \leq |a_1 - k \log k + k| \lor |b_1 - k \log k + k| \leq 3 \log k
\]
and
\[
k|\hat{c}_m^{(m),2} - 2| \leq |a_2 - 2(\log k)b_1 + k(\log k)^2 - 2k| \lor |b_2 - 2(\log k)a_1 + k(\log k)^2 - 2k| \leq 8(\log k)^2,
\]
which is the desired result. 

3.3 Verifying the integral representations of different moments

We start by deriving general expressions for the moments of logarithms of random vectors which will lead to proving the representations in Equations (14) to (16). The following result is a multivariate version of the so-called “Darth Vader rule”.

**Lemma 4.** Let $X_1, \ldots, X_d$ be non-negative random variables and $p_1, \ldots, p_d > 0$. Then
\[
\mathbb{E} \left[ \prod_{j=1}^{d} X_j^{p_j} \right] = \int_{[0,\infty)^d} \prod_{j=1}^{d} p_j x_j^{p_j-1} \mathbb{P}(X_1 \geq x_1, \ldots, X_d \geq x_d) dx_1 \ldots dx_d.
\]
Moreover, any number of “$\geq$” can be replaced by “$>$”, as this changes the value of the probability, at most, on a Lebesgue-null set.

**Proof.** Letting $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space containing all the random variables, we have
\[
\mathbb{E} \left[ \prod_{j=1}^{d} X_j^{p_j} \right] = \int_{\Omega} \prod_{j=1}^{d} X_j(\omega)^{p_j} \mathbb{P}(d\omega)
\]
\[
= \int_{\Omega} \left[ \int_{[0,\infty)^d} 1_{\{X_1(\omega) \geq u_1^{1/p_1}, \ldots, X_d(\omega) \geq u_d^{1/p_d}\}} du_1 \ldots du_d \right] \mathbb{P}(d\omega)
\]
\[
= \int_{[0,\infty)^d} \left[ \int_{\Omega} 1_{\{X_1(\omega) \geq u_1^{1/p_1}, \ldots, X_d(\omega) \geq u_d^{1/p_d}\}} \mathbb{P}(d\omega) \right] du_1 \ldots du_d
\]
\[
= \int_{[0,\infty)^d} \mathbb{P}(X_1 \geq u_1^{1/p_1}, \ldots, X_d \geq u_d^{1/p_d}) du_1 \ldots du_d,
\]
where we have used the fact that $X_j(\omega) \geq 0$ for almost every $\omega$ to justify the second equality. The change in the order of integration was allowed by Tonelli’s theorem. Finally, applying the change of variable $x_j = u_j^{1/p_j}$, $du_j/dx_j = p_j x_j^{p_j-1}$ produces the desired result. 

**Lemma 5.** Let $X$ and $Y$ be almost surely positive random variables and let $S$ be the distribution function of $(1/X, 1/Y)$, so that for positive $x, y$, $\mathbb{P}(X \geq x, Y \geq y) = S(1/x, 1/y)$. Then for any $p \in \{1, 2, \ldots\}$,
\[
\mathbb{E}[(\log X)^p 1_{\{X > 1\}}] = p \int_{0}^{1} \frac{S(x, \infty) \log x^{p-1}}{x} \, dx, \quad (40)
\]
\[
\mathbb{E}[-(\log X)^p 1_{\{X < 1\}}] = p \int_{1}^{\infty} \frac{S([x, \infty), \infty) \log x^{p-1}}{x} \, dx, \quad (41)
\]
\[
\mathbb{E}[(\log X)(\log Y)^p 1_{\{X, Y > 1\}}] = p^2 \int_{0}^{1} \int_{0}^{1} \frac{S(x, y)(\log x)(\log y)^{p-1}}{xy} \, dxdy, \quad (42)
\]
\[ \mathbb{E} \left[ - (\log X) (\log Y) \right] \mathbb{I} \{ X < 1, Y > 1 \} = p^2 \int_0^1 \int_1^\infty \frac{S(x, \infty, y)[(\log x)(\log y)]^{p-1}}{xy} \, dxdy, \quad (43) \]
\[ \mathbb{E} \left[ (\log X) (\log Y) \right] \mathbb{I} \{ X, Y \leq 1 \} = p^2 \int_1^\infty \int_1^\infty \frac{S([x, \infty), [y, \infty])[((\log x)(\log y))^{p-1}]}{xy} \, dxdy, \quad (44) \]

where \( S([x, \infty), y) \) and \( S([x, \infty), [y, \infty)) \) are shorthand for \( S(\infty, y) - S(x, y) \) and \( 1 - S(x, \infty) - S(\infty, y) + S(x, y) \), respectively.

**Proof.** First, by Lemma 4 with \( d = 1, p_1 = p \),
\[ \mathbb{E} \left[ (\log X)^p \right] \mathbb{I} \{ X > 1 \} = p \int_0^\infty u^{p-1} \mathbb{P}(\log X \leq u) \, du \]
\[ = p \int_0^\infty u^{p-1} S(e^{-u}, \infty) \, du \]
\[ = p \int_1^\infty S(x, \infty) \left( - \log x \right)^{p-1} \, dx, \]
by the change of variable \( x = e^{-u} \). Similarly,
\[ \mathbb{E} \left[ -(\log X)^p \right] \mathbb{I} \{ X < 1 \} = p \int_0^\infty u^{p-1} \mathbb{P}(\log X \geq -u) \, du \]
\[ = p \int_0^\infty u^{p-1} S([e^u, \infty), \infty) \, du \]
\[ = p \int_1^\infty S([x, \infty), \infty) \left( \log x \right)^{p-1} \, dx, \]
by the change of variable \( x = e^u \). This establishes Equations (40) and (41).

Equations (42) to (44) are proved in a similar fashion by using Lemma 4 with \( d = 2, p_1 = p_2 = p \). First,
\[ \mathbb{E} \left[ ((\log X)(\log Y))^p \right] \mathbb{I} \{ X, Y > 1 \} = p^2 \int_0^\infty \int_0^\infty (uv)^{p-1} \mathbb{P}(\log X \leq u, \log Y \geq v) \, dudv \]
\[ = p^2 \int_0^\infty \int_0^\infty (uv)^{p-1} S(e^{-u}, e^{-v}) \, dudv \]
\[ = p^2 \int_0^1 \int_0^1 S(x, y) \left( (\log x)(\log y) \right)^{p-1} \, dxdy, \]
using the change of variable \( x = e^{-u}, y = e^{-v} \). Second,
\[ \mathbb{E} \left[ ((-\log X)(\log Y))^p \right] \mathbb{I} \{ X < 1, Y > 1 \} = p^2 \int_0^\infty \int_0^\infty (uv)^{p-1} \mathbb{P}(\log X \leq -u, \log Y \geq v) \, dudv \]
\[ = p^2 \int_0^\infty \int_0^\infty (uv)^{p-1} S([e^u, \infty), e^{-v}) \, dudv \]
\[ = p^2 \int_0^1 \int_0^1 S([x, \infty), y) \left( - (\log x)(\log y) \right)^{p-1} \, dxdy, \]
using the change of variable \( x = e^u, y = e^{-v} \). Third,
\[ \mathbb{E} \left[ ((\log X)(\log Y))^p \right] \mathbb{I} \{ X, Y < 1 \} = p^2 \int_0^\infty \int_0^\infty (uv)^{p-1} \mathbb{P}(\log X \leq -u, \log Y \leq -v) \, dudv \]
\[ = p^2 \int_0^\infty \int_0^\infty (uv)^{p-1} S([e^u, \infty), [e^v, \infty)) \, dudv \]
\[ = p^2 \int_0^1 \int_1^1 S([x, \infty), [y, \infty)) \left( (\log x)(\log y) \right)^{p-1} \, dxdy, \]
using the change of variable \( x = e^u, y = e^v \). This establishes Equations (42) to (44). □
Lemma 6. Under Assumption ¹, Equations (14) to (16) hold for any $a \in (0, 1)$.

Proof. Recall that $i,j,m$ are assumed to be distinct indices. It is already proved in (Engelke and Volgushev, 2020, Section S.7) that the moments of interest satisfy

$$e_i^{(m),\ell} = \int_0^1 \frac{R_{im}(x,1)(-2 \log x)^{\ell-1}}{x} dx - \int_1^{\infty} \frac{R_{im}(x,\infty,1)(-2 \log x)^{\ell-1}}{x} dx,$$

$$e_{im}^{(m)} = \int_0^1 \int_0^1 \frac{R_{im}(x,y)}{xy} dx dy - \int_0^1 \int_1^{\infty} \frac{R_{im}(x,\infty,y)}{xy} dx dy,$$

$$e_{ij}^{(m)} = \int_0^1 \int_0^1 \frac{R_{ijm}(x,y,1)}{xy} dx dy - \int_1^1 \int_1^{\infty} \frac{R_{ijm}(x,\infty,y,1)}{xy} dx dy$$

and that their empirical versions satisfy

$$\tilde{e}_i^{(m),\ell} = \int_{1/k}^1 \frac{\tilde{R}_{im}(x,1)(-2 \log x)^{\ell-1}}{x} dx - \int_1^{n/k} \frac{\tilde{R}_{im}(x,\infty,1)(-2 \log x)^{\ell-1}}{x} dx,$$

$$\tilde{e}_{im}^{(m)} = \int_{1/k}^1 \int_{1/k}^1 \frac{\tilde{R}_{im}(x,y)}{xy} dx dy - \int_{1/k}^{n/k} \int_1^{\infty} \frac{\tilde{R}_{im}(x,\infty,y)}{xy} dx dy,$$

$$\tilde{e}_{ij}^{(m)} = \int_{1/k}^1 \int_{1/k}^1 \frac{\tilde{R}_{ijm}(x,y,1)}{xy} dx dy - \int_{1/k}^{n/k} \int_1^{\infty} \frac{\tilde{R}_{ijm}(x,\infty,y,1)}{xy} dx dy,$$

where

$$\tilde{R}_J(x_J) := \frac{1}{k} \sum_{t=1}^n 1 \left\{ \tilde{F}_t(U_{ti}) \leq \frac{k}{n} x_i, i \in J \right\}, \quad x_J := (x_i)_{i \in J} \in [0, \infty)^{|J|}.$$

The integrals in Equations (45) to (47) can be truncated above by using Assumption ¹ which allows to upper bound the tails of the functions $R_J$. In particular, we have

$$\int_{n/k}^{\infty} \frac{R_{im}(x,\infty,1)(2 \log x)^{\ell-1}}{x} dx \lesssim \int_{n/k}^{\infty} \frac{(\log x)^{\ell-1}}{x^{1+1/\ell}} dx \lesssim \left( \frac{k}{n} \right)^{\ell} \log(n/k),$$

$$\int_0^1 \int_{n/k}^{\infty} \frac{R_{im}(x,\infty,y)}{xy} dx dy = \int_0^1 \int_{n/k}^{\infty} \frac{R_{im}(x,y,\infty)}{x} dx dy \lesssim \int_0^1 \int_{n/k}^{\infty} \frac{(x/y)^{-\ell}}{x} dx dy \lesssim \left( \frac{k}{n} \right)^{\ell},$$

and

$$\int_{\{1,\infty\} \setminus [1,n/k]^2} \frac{R_{ijm}(x,\infty,y,\infty,1)}{xy} dx dy$$

$$= \int_{n/k}^{\infty} \int_{n/k}^{\infty} \frac{R_{ijm}(x,\infty,y,\infty,1)}{xy} dx dy + \int_{n/k}^{n/k} \int_1^{\infty} \frac{R_{ijm}(x,\infty,y,\infty,1)}{xy} dx dy$$

$$+ \int_{n/k}^{\infty} \int_1^{n/k} \frac{R_{ijm}(x,\infty,y,\infty,1)}{xy} dx dy$$

$$\leq \int_{n/k}^{\infty} \int_{n/k}^{\infty} \frac{R_{ijm}(x,\infty,y,\infty,1)}{xy} dx dy + \int_{n/k}^{n/k} \int_1^{\infty} \frac{R_{im}(x,\infty,1)}{xy} dx dy + \int_{n/k}^{\infty} \int_1^{n/k} \frac{R_{jm}(y,\infty,1)}{xy} dx dy$$
\[
\int_{n/k}^{\infty} \int_{n/k}^{\infty} \frac{x^{-\xi} \wedge y^{-\xi}}{xy} \, dx \, dy + 2 \int_{1}^{n/k} \int_{n/k}^{\infty} \frac{x^{-\xi}}{xy} \, dx \, dy \\
\lesssim \left( \frac{k}{n} \right)^{\xi} \log(n/k).
\]

Hence we proved that all integral can be truncated above at \( n/k \) while incurring an error of at most \( O((k/n)^{\xi} \log(n/k)) \). Next we show that the integrals can as well be truncated below.

Recall that \( a \in (0, 1) \). Since by their definitions, \( R_J \) and \( \tilde{R}_J \) are both upper bounded by the minimum component of their argument, so is \( |R_J - \tilde{R}_J| \). We then have for \( \ell \in \{1, 2\} \)
\[
\int_{0}^{a} \frac{|\tilde{R}_{im}(x, 1) - R_{im}(x, 1)|}{x} (-2 \log x)^{\ell-1} \, dx \leq \int_{0}^{a} (-2 \log x)^{\ell-1} \, dx \lesssim a(1 + \log(1/a)),
\]
\[
\int_{[0,1]^2 \setminus [a,1]^2} \frac{|\tilde{R}_{im}(x,y) - R_{im}(x,y)|}{xy} \, dx \, dy \leq \int_{0}^{a} \int_{0}^{a} \frac{x \wedge y}{xy} \, dx \, dy + 2 \int_{a}^{1} \int_{a}^{1} \frac{1}{y} \, dx \, dy \lesssim a(1 + \log(1/a)),
\]
\[
\int_{0}^{a} \int_{1}^{n/k} \frac{|\tilde{R}_{im}(x, \infty), y) - R_{im}(x, \infty), y)|}{xy} \, dx \, dy \leq \int_{0}^{a} \int_{1}^{n/k} \frac{1}{x} \, dx \, dy \leq a \log(n/k),
\]
and by symmetry
\[
\int_{1}^{n/k} \int_{0}^{a} \frac{|\tilde{R}_{im}(x, [y, \infty)) - R_{im}(x, [y, \infty))|}{xy} \, dx \, dy
\]
admits the same bound. Finally, the integral
\[
\int_{[0,1]^2 \setminus [a,1]^2} \frac{|\tilde{R}_{ijm}(x,y, 1) - R_{ijm}(x,y, 1)|}{xy} \, dx \, dy
\]
is handled similarly as
\[
\int_{[0,1]^2 \setminus [a,1]^2} \frac{|\tilde{R}_{im}(x,y) - R_{im}(x,y)|}{xy} \, dx \, dy.
\]

We have therefore proved that each of the integrals in Equations 15 to 20 can be truncated below at a point \( a \) and above at \( n/k \), up to a deterministic additive error which satisfies the bound \( \lesssim (k/n)^{\xi} \log(n/k) + a(\log(n/k) + \log(1/a)) \). It follows that with probability 1,
\[
e^{(m),.\ell}_{(i)} - e^{(m),.\ell}_{(i)} = \int_{a}^{1} \frac{\tilde{R}_{im}(x, 1) - R_{im}(x, 1)}{x} (-2 \log x)^{\ell-1} \, dx \\
- \int_{1}^{n/k} \frac{\tilde{R}_{im}(x, \infty), 1) - R_{im}(x, \infty), 1))}{x} (-2 \log x)^{\ell-1} \, dx \\
+ O \left( \left( \frac{k}{n} \right)^{\xi} \log(n/k) + a(\log(n/k) + \log(1/a)) \right),
\]
\[
e^{(m)}_{(i,m)} - e^{(m)}_{(i,m)} = \int_{a}^{1} \int_{a}^{1} \frac{\tilde{R}_{im}(x,y) - R_{im}(x,y)}{xy} \, dx \, dy \\
- \int_{a}^{1} \int_{1}^{n/k} \frac{\tilde{R}_{im}(x, \infty), y) - R_{im}(x, \infty), y)}{xy} \, dx \, dy \\
+ O \left( \left( \frac{k}{n} \right)^{\xi} \log(n/k) + a(\log(n/k) + \log(1/a)) \right),
\]
\[
e^{(m)}_{(i,j)} - e^{(m)}_{(i,j)} = \int_{a}^{1} \int_{a}^{1} \frac{\tilde{R}_{ijm}(x,y, 1) - R_{ijm}(x,y, 1)}{xy} \, dx \, dy
\]

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where the error terms are deterministic. All that remains to obtain the desired result is to replace the functions \( \hat{R}_J \) above by \( \hat{R}_J \). By the result in Appendix C.1 of \cite{Radulovic}, we have

\[
\max_{J:|J|\leq 3} \sup_{x_j \in [0,\infty)^{|J|}} |\hat{R}_J(x_J) - \hat{R}_J(x_J)| \leq \frac{3}{k}
\]

almost surely, so replacing \( \hat{R}_J \) by \( \hat{R}_J \) in the integrals above adds an error that is at most of the order of \((\log(n/k) + \log(1/a))^2/k\). \hfill \Box

**Lemma 7.** Under Assumption \( [7] \), the variograms \( \max_{m \in V} \| \Gamma^{(m)} \|_\infty \) admits an upper bound that depends only on \( K \) and \( \xi \) as in Proposition \( [7] \).

**Proof.** First, as is pointed out in Section \( [3.2] \) for \( \ell \in \{1, 2\} \), \( e_{iJ}^{(m),\ell} = \ell \).

The remaining arguments are based on Equation \( (10) \), which holds by assumption and implies that for all distinct triples \( (i,j,m) \),

\[
R_{ij}([x, \infty), 1) \leq Kx^{-\xi}, \quad R_{ij}([x, \infty), [y, \infty), 1) \leq K(x \wedge y)^{-\xi}, \quad x, y \geq 1.
\]

Equally important is the fact that every function \( R_{ij} \) is upper bounded by its minimum argument. The proof consists of plugging those different bounds in Equations \( (15) \) to \( (17) \) above, which provided expressions for the moments \( e_{iJ}^{(m),\ell}, e_{im}^{(m)} \) and \( e_{ij}^{(m)} \). Repeatedly using the inequality \( a \wedge b \leq (ab)^{1/2} \) for positive \( a, b \), deduce that

\[
|e_{iJ}^{(m),\ell}| \leq \int_0^1 (-2 \log x)^{\ell-1} dx + K \int_1^\infty \frac{(-2 \log x)^{\ell-1}}{x^1+\xi} dx,
\]

\[
|e_{im}^{(m)}| \leq \int_0^1 \int_0^1 (xy)^{-1/2} dxdy + \sqrt{K} \int_0^1 \int_1^\infty x^{-1-\xi/2}y^{-1/2} dxdy,
\]

\[
|e_{ij}^{(m)}| \leq \int_0^1 \int_0^1 (xy)^{-1/2} dxdy + 2\sqrt{K} \int_0^1 \int_1^\infty x^{-1-\xi/2}y^{-1/2} dxdy + K \int_1^\infty \int_1^\infty (xy)^{-1-\xi/2} dxdy.
\]

Simply plugging those bounds in Equation \( (2) \) yields the result. \hfill \Box

### 3.4 Bounds on the measures \( R_{ij} \)

Recall the measure \( R_{ij} \) appearing in Equation \( (11) \), for an arbitrary pair \( i \neq j \). The following bounds necessarily hold.

**Lemma 8.** Let \( 0 < a \leq b \) and \( y > 0 \). Then for every distinct pair \( (i,j) \),

\[
R_{ij}([a,b], y) \leq y \frac{b-a}{a}.
\]
Proof. The idea is that the rectangle \([a,b] \times [0,y]\) is included in the trapezoid \(\{(u,v) \in [0,\infty)^2 : u \leq b, v \leq yu/a\} = S(b)\setminus S(a)\), where
\[
S(x) := \{(u,v) \in [0,\infty)^2 : u \leq x, v \leq yu/a\}.
\]
By homogeneity of \(R\),
\[
R_{ij}(S(b)\setminus S(a)) = (b-a)R_{ij}(S(1)) \leq (b-a)R_{ij}(1,y/a) \leq \frac{b-a}{a^\beta},
\]
since \(R_{ij}\) is always upper bounded by its smallest argument.

The following bound assumes more but is considerably more flexible, as \(\beta\) can be both smaller and larger than 1.

**Lemma 9.** Under Assumption 2 for every \(\beta \in (0,1+\varepsilon]\) there exists \(K(\beta) < \infty\) such that for any \(0 < a \leq b, y > 0\) and every distinct pair \((i,j)\),
\[
R_{ij}([a,b],y) \leq \frac{K(\beta)}{\beta} y^\beta \frac{b-a}{a^\beta}.
\]

**Proof.** The bound in Assumption 2 gives
\[
R_{ij}([a,b],y) = \int_0^y \int_a^b r_{ij}(u,v) du dv \leq K(\beta) \int_0^y v^{\beta-1} dv \int_a^b u^{-\beta} du \leq \frac{K(\beta)}{\beta} y^\beta \int_a^b a^{-\beta} du = \frac{K(\beta)}{\beta} y^\beta \frac{b-a}{a^\beta}.
\]

**Lemma 10.** Under Assumption 2 for every \(\beta \in [-\varepsilon,0)\) there exists \(K(\beta) < \infty\) such that for any \(0 < a \leq b, y > 0\) and every distinct pair \((i,j)\),
\[
R_{ij}([a,b],[y,\infty)) \leq \frac{K(\beta)}{-\beta} y^\beta b^{-\beta}(b-a).
\]

**Proof.** Following the proof of Lemma 9
\[
R_{ij}([a,b],[y,\infty)) \leq K(\beta) \int_y^\infty v^{\beta-1} dv \int_a^b a^{-\beta} du \leq \frac{K(\beta)}{-\beta} y^\beta \int_a^b b^{-\beta} du = \frac{K(\beta)}{-\beta} y^\beta b^{-\beta}(b-a).
\]

### 3.5 Densities of Hüsler–Reiss Pareto distributions

Using the known expression for the stable tail dependence function of the bivariate Hüsler–Reiss distribution, we now calculate the density of the associated functions \(R_{ij}\) and show that it satisfies Assumption 2.

**Lemma 11.** Assume that \(X\) is in the max-domain of attraction of a Hüsler–Reiss distribution with parameter matrix \(\Gamma\) and satisfies Equation (11). Then as long as \(\lambda := \sqrt{\Gamma_{ij}} > 0\), the function \(R_{ij}\) has density
\[
r_{ij}(x,y) = \frac{1}{2\sqrt{2\pi}\lambda\sqrt{xy}} \exp \left\{ -\frac{x^2}{2} - \frac{(\log x - \log y)^2}{8\lambda^2} \right\}, \quad (x,y) \in (0,\infty)^2.
\]
Moreover, for any \(\beta \in \mathbb{R}\), this function enjoys the upper bound
\[
r_{ij}(x,y) \leq \frac{K(\beta)}{x^\beta y^{1-\beta}}, \quad K(\beta) := \frac{\exp\{\lambda^2(2(\beta - 1/2)^2 - 1/2)\}}{2\sqrt{2\pi}\lambda}.
\]
Proof. The pair $(X_i, X_j)$ is in the max-domain of attraction of the bivariate Hüsler–Reiss distribution with parameter $\lambda^2$, so its stable tail dependence function is

$$L(x, y) = x \Phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) + y \Phi \left( \lambda + \frac{\log y - \log x}{2\lambda} \right),$$

and

$$r_{ij}(x, y) := \frac{\partial^2}{\partial x \partial y} R(x, y) = \frac{\partial^2}{\partial x \partial y} (x + y - L(x, y)) = -\frac{\partial^2}{\partial x \partial y} L(x, y).$$

First, we have

$$\frac{\partial}{\partial x} x \Phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) = \Phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) + x \phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) \frac{1}{2\lambda x}$$

$$= \Phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) + \frac{1}{2\lambda} \phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right),$$

so

$$\frac{\partial^2}{\partial x \partial y} x \Phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) = -\frac{1}{2\lambda y} \phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) - \frac{1}{4\lambda^2 y} \phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right)$$

$$= -\frac{1}{4\lambda^2 y} \left( \lambda + \frac{\log y - \log x}{2\lambda} \right) \phi \left( \lambda + \frac{\log x - \log y}{2\lambda} \right),$$

where we used the expression $\phi'(t) = -t \phi(t)$ for the derivative of the standard Gaussian density $\phi$. Now by definition of $\phi$, this is equal to

$$-\frac{1}{4\sqrt{2\pi} \lambda^2 y} \left( \lambda + \frac{\log y - \log x}{2\lambda} \right) \exp \left\{ -\frac{\lambda^2}{2} - \frac{(\log x - \log y)^2}{8\lambda^2} + \frac{\log y - \log x}{2} \right\}$$

$$= -\frac{1}{4\sqrt{2\pi} \lambda^2 x y} \left( \lambda + \frac{\log y - \log x}{2\lambda} \right) \exp \left\{ -\frac{\lambda^2}{2} - \frac{(\log x - \log y)^2}{8\lambda^2} \right\}.$$

Adding this to

$$\frac{\partial^2}{\partial x \partial y} y \Phi \left( \lambda + \frac{\log y - \log x}{2\lambda} \right) = -\frac{1}{4\sqrt{2\pi} \lambda^2 x y} \left( \lambda + \frac{\log x - \log y}{2\lambda} \right) \exp \left\{ -\frac{\lambda^2}{2} - \frac{(\log x - \log y)^2}{8\lambda^2} \right\},$$

obtained by a symmetric argument, yields the desired density. As for the upper bound, note that for any $\beta \in \mathbb{R}$,

$$r_{ij}(x, y) = \frac{1}{2\sqrt{2\pi} \lambda x^\beta \sqrt{y}} \exp \left\{ -\frac{\lambda^2}{2} - \frac{(\log x - \log y)^2}{8\lambda^2} + (\beta - 1/2) \log x \right\}.$$

Writing $u$ and $v$ for $\log x$ and $\log y$, the exponent above is

$$-\frac{\lambda^2}{2} - \frac{(u - v)^2}{8\lambda^2} + (\beta - 1/2) u = -\frac{u^2}{8\lambda^2} + \left(\beta - 1/2\right) u + \frac{v}{4\lambda^2} u - \frac{v^2}{8\lambda^2}$$

which is maximized (in $u$) at $u = v + 4\lambda^2 (\beta - 1/2)$, hence

$$-\frac{\lambda^2}{2} - \frac{(u - v)^2}{8\lambda^2} + (\beta - 1/2) u \leq -\frac{\lambda^2}{2} - 2\lambda^2 (\beta - 1/2)^2 + (\beta - 1/2) v + 4\lambda^2 (\beta - 1/2)^2 = (\beta - 1/2) v + \lambda^2 (2(\beta - 1/2)^2 - 1/2).$$

Conclude that

$$r_{ij}(x, y) \leq \frac{1}{2\sqrt{2\pi} \lambda x^\beta \sqrt{y}^{1-\beta}} \exp \{ \lambda^2 (2(\beta - 1/2)^2 - 1/2) \}. \quad \square$$

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3.6 Technical results from empirical process theory

In this section, we collect two fundamental inequalities from empirical process theory that are used in Section 2.3.2. Denote by $\mathcal{G}$ a class of real-valued functions that satisfies $|f(x)| \leq F(x) \leq U$ for every $f \in \mathcal{G}$ and let $\sigma^2 \geq \sup_{f \in \mathcal{G}} Pf^2$. Additionally, suppose that for some positive $A$, $V$ and for all $\varepsilon > 0$,

$$N(\varepsilon, \mathcal{G}, L_2(\mathbb{P}_n)) \leq \left( \frac{A\|F\|_{L^2(\mathbb{P}_n)}}{\varepsilon} \right)^V$$

(51)

almost surely. In that case, the symmetrization inequality and inequality (2.2) from Koltchinskii (2006) yield

$$\mathbb{E}[\|\mathbb{P}_n - P\|] \leq c_0 \left[ \sigma \left( \frac{V}{n} \log \frac{A\|F\|_{L^2(P)}}{\sigma} \right)^{1/2} + \frac{VU}{n} \log \frac{A\|F\|_{L^2(\mathbb{P}_{n})}}{\sigma} \right]$$

(52)

for a universal constant $c_0 > 0$ provided that $1 \geq \sigma^2 > \text{const} \times n^{-1}$. In fact, the inequality in Koltchinskii (2006) is for $\sigma^2 = \sup_{f \in \mathcal{G}} Pf^2$. However, this is not a problem since we can replace $\mathcal{G}$ by $\mathcal{G}\sigma/(\sup_{f \in \mathcal{G}} Pf^2)^{1/2}$.

The second inequality (a refined version of Talagrand's concentration inequality) states that for any countable class of measurable functions $\mathcal{F}$ with elements mapping into $[-M, M]$,

$$\mathbb{P}\left( \|\mathbb{P}_n - P\|_{\mathcal{F}} \geq 2\mathbb{E}[\|\mathbb{P}_n - P\|_{\mathcal{F}}] + c_1 n^{-1/2} \left( \sup_{f \in \mathcal{F}} Pf^2 \right)^{1/2} \sqrt{t} + n^{-1} c_2 Mt \right) \leq e^{-t},$$

(53)

for all $t > 0$ and some universal constants $c_1, c_2 > 0$. This is a special case of Theorem 3 in Massart (2000) (in the notation of that paper, set $\varepsilon = 1$).

References

Csorgo, M. and P. Revesz (1978). Strong approximations of the quantile process. Annals of Statistics, 882–894.

de Haan, L. and A. Ferreira (2006). Extreme Value Theory. Springer.

Durrett, R. (2013). Probability: Theory and Examples (4.1 ed.).

Engelke, S. (2021). Extremal graphical lasso and high-dimensional extremes. Presented at Extreme Value Analysis 2021, https://media.ed.ac.uk/media/Multivariate%20Extremes%20Sebastian%20Engelke/1_fy5hu26u.

Engelke, S. and A. Hitz (2020). Graphical models for extremes (with discussion). J. R. Stat. Soc. Ser. B Stat. Methodol. 82, 871–932.

Engelke, S. and S. Volgushev (2020). Structure learning for extremal tree models. arXiv preprint arXiv:2012.06179.

Koltchinskii, V. (2006). Local rademacher complexities and oracle inequalities in risk minimization. Annals of Statistics 34(6), 2593–2656.

Massart, P. (2000). About the constants in talagrand’s concentration inequalities for empirical processes. Annals of Probability, 863–884.

Radulović, D., M. Wegkamp, Y. Zhao, et al. (2017). Weak convergence of empirical copula processes indexed by functions. Bernoulli 23(4B), 3346–3384.

van der Vaart, A. W. and J. A. Wellner (1996). Weak Convergence and Empirical Processes With Applications to Statistics. Springer.