Stability of new exact solutions of the nonlinear Schrödinger equation in a Pöschl–Teller external potential

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Abstract

We discuss the stability properties of the solutions of the general nonlinear Schrödinger equation in 1+1 dimensions in an external potential derivable from a parity-time (PT) symmetric superpotential W(x) that we considered earlier in Kevrekidis et al (2015 Phys. Rev. E 92 042901). In particular we consider the nonlinear partial differential equation \( \{ i \partial_t + \partial_x^2 - V(x) + g|\psi(x,t)|^{2\kappa} \} \psi(x,t) = 0, \) for arbitrary nonlinearity parameter \( \kappa, \) where \( g = \pm 1 \) and \( V \) is the well known Pöschl–Teller potential which we allow to be repulsive as well as attractive. Using energy landscape methods, linear stability analysis as well as a time dependent variational approximation, we derive consistent analytic results for
the domains of instability of these new exact solutions as a function of the strength of the external potential and $\kappa$. For the repulsive potential we show that there is a translational instability which can be understood in terms of the energy landscape as a function of a stretching parameter and a translation parameter being a saddle near the exact solution. In this case, numerical simulations show that if we start with the exact solution, the initial wave function breaks into two pieces traveling in opposite directions. If we explore the slightly perturbed solution situations, a 1% change in initial conditions can change significantly the details of how the wave function breaks into two separate pieces. For the attractive potential, changing the initial conditions by 1% modifies the domain of stability only slightly. For the case of the attractive potential and negative $g$ perturbed solutions merely oscillate with the oscillation frequencies predicted by the variational approximation.

Keywords: \( \mathcal{PT} \)-symmetric superpotential, variational approximation, translational instability, Derrick’s theorem, collective coordinates

(Some figures may appear in colour only in the online journal)

1. Introduction

The study of open systems with balanced loss and gain, typically defined by Parity-Time (\( \mathcal{PT} \)) symmetry, has elicited significant attention from physics, nonlinear science and mathematics communities during the past decade. This is in part due to their emerging applications in many physical contexts and in part due to their intriguing mathematical structure. The initial investigation of such systems [2–5] arose in the context of whether non-Hermitian quantum systems could lead to entirely real eigenvalues. Keeping in perspective the formal similarity of the Schrödinger equation with Maxwell’s equations in the paraxial approximation, many experimentalists realized that such \( \mathcal{PT} \)-invariant systems can indeed be fabricated using optical means [6–15]. Motivated by this success, in the ensuing years, \( \mathcal{PT} \)-invariant phenomena were also observed in electronic circuits [16, 17], mechanical constructs [18], whispering-gallery microcavities [19], among many other physical systems.

In a parallel development, the concept of supersymmetry (SUSY) prevalent in high-energy physics was also experimentally studied in optics [20, 21]. The underlying notion is that for a given potential we can obtain a SUSY partner potential such that both potentials possess identical spectrum (with possibly one eigenvalue different) [22, 23]. A simultaneous presence of \( \mathcal{PT} \)-symmetry and SUSY can lead to unexpected phenomena and is likely to be very useful in achieving transparent and one-way reflectionless complex optical potentials [24–28]. Previously [1] we studied the interplay between nonlinearity, \( \mathcal{PT} \)-symmetry and supersymmetry as well as the rich consequences of this interplay. There we obtained exact solutions of the general nonlinear Schrödinger equation (NLSE) in 1 + 1 dimensions in the presence of a \( \mathcal{PT} \)-symmetric complex potential [22, 29]. In a recent paper [30] we studied the stability properties of the solutions of NLSE in the real partner potential of the problem studied in [1] which was a Pöschl–Teller potential [31, 32].

Here our objective is to discuss the stability properties of two related exact solutions which exist when we change the sign of the nonlinear coupling $g$ to being negative keeping the potential attractive, or keep the sign of the nonlinear term unchanged but consider a repulsive potential. In the latter case we will find that the solutions are translationally unstable, whereas in the former case the solutions are stable to small perturbations.
1.1. Different solutions to the NLSE in an external Pöschl–Teller potential

By allowing the nonlinearity coupling \( g = \pm 1 \) and the sign of the potential \( \lambda = \pm 1 \) we have found different classes of exact solutions when the NLSE is in the presence of a Pöschl–Teller potential centered at \( x = 0 \). Schrödinger’s equation for these cases is given by:

\[
\{ i \partial_t + \partial_x^2 + g |\psi(x,t)|^{2\kappa} - V(x) \} \psi(x,t) = 0 ,
\]

where

\[
V(x) = -\lambda \tilde{b}^2 \sech^2(x) , \quad \tilde{b}^2 = b^2 - 1/4 ,
\]

with \( \tilde{b}^2 > 0 \) and \( \kappa > 0 \). Since \( g \) can be scaled out of the equation by letting \( \psi(x,t) \to g^{-1/(2\kappa)} \psi(x,t) \), we can restrict ourselves to \( g = \pm 1 \) in what follows. The signs here are chosen such that the nonlinear term is attractive for \( g = +1 \) and repulsive for \( g = -1 \) and the external Pöschl–Teller potential is attractive for \( \lambda = +1 \) and repulsive for \( \lambda = -1 \). This potential is a special case of potentials obtainable from the complex \( \mathcal{PT} \)-symmetric SUSY superpotential

\[
W(x) = (m - 1/2) \tanh x - ib \sech x ,
\]

with \( m = 1 \), which gives rise to \( \mathcal{PT} \)-symmetric partner potentials \( V_{\pm} = W^2 \pm W' \). Our real \( V(x) \) corresponds to \( V_+ \). There are several cases of equation (1.1) which have exact solutions. These are

(I) Attractive nonlinear term and attractive potential: \( g = +1, \lambda = +1 \). In this case, the exact solution is given by

\[
\psi_0(x,t) = A_0(b,\gamma) \sech^\gamma(x) e^{i\gamma t} ,
\]

\[
A_0^{2/\gamma}(b,\gamma) = \gamma(\gamma + 1) - \tilde{b}^2 ,
\]

where \( \gamma = 1/\kappa \). In this case,

\[
\tilde{b}^2 \equiv \gamma(\gamma + 1) \geq \tilde{b}^2 \geq 0 .
\]

We studied this case in a previous paper \([30]\), where we found that all solitary waves for \( \kappa < 2 \) and \( 0 < \tilde{b}^2 < \tilde{b}_0^2 \) are stable, as for the case of solitary waves in the NLSE (\( \tilde{b}^2 = 0 \)). However, we also found a new region above \( \kappa = 2 \) where these solutions are stable.

(II) Attractive nonlinear term and repulsive potential: \( g = +1, \lambda = -1 \). For this case, the exact solution is given by

\[
\psi_0(x,t) = A_0(b,\gamma) \sech^\gamma(x) e^{i\gamma t} ,
\]

\[
A_0^{2/\gamma}(b,\gamma) = \gamma(\gamma + 1) + \tilde{b}^2 .
\]

In this case, we only require \( \tilde{b}_0^2 \geq 0 \). This solution goes over to a particular moving solitary wave solution of the NLSE when \( \tilde{b} \to 0 \). Since the solutions of the NLSE are stable to deformations of the width for all \( \kappa < 2 \) we expect (and we will find) that in that regime there will be a critical value of \( \tilde{b} \) above which the solution will be unstable to width deformations. We expect and we find that again the solutions are always unstable for \( \kappa > 2 \). What we will also find is that for all values of \( \kappa \) these solutions are unstable to a slight translation, even if induced by numerical noise.
(III) Repulsive nonlinear term and attractive potential: $g = -1$, $\lambda = +1$. For this case, the exact solution is given by

$$\psi_0(x, t) = A_0(\tilde{b}, \gamma) \operatorname{sech}^{\gamma}(x) e^{i\gamma t},$$

$$(1.7a)$$

$$A_0^{2\gamma}(\tilde{b}, \gamma) = \tilde{b}^2 - \gamma(\gamma + 1).$$

$$(1.7b)$$

In this case, we require $\tilde{b}^2 \geq \tilde{b}_0^2$. For this choice of $g$ there are no bright solitary wave solutions in the absence of the potential. For the special case of $\kappa = 1$ there are both gray and black dark soliton solutions (see Chabchoub, et al [33]). We will find that these new bright solutions in the presence of a potential for $g = -1$ are linearly stable.

In all these cases, we find that the quantity

$$g A_0^{2\gamma}(\tilde{b}, \gamma) = \gamma(\gamma + 1) - \lambda \tilde{b}^2.$$

$$(1.8)$$

is independent of $g$ and depends only on the sign of $\lambda$. The normalization, or ‘mass’ of these exact wave functions is given by

$$M_0(\tilde{b}, \gamma) = \int dx |\psi_0(x, t)|^2 = A_0^2(\tilde{b}, \gamma) c_1[\gamma].$$

$$(1.9)$$

where

$$c_1[\gamma] = \int dz \operatorname{sech}^{2\gamma}(z) = \sqrt{\pi} \frac{\Gamma[\gamma]}{\Gamma[\gamma + 1/2]}.$$ 

$$(1.10)$$

This paper is structured as follows: in section 2 we discuss Hamilton’s principle of least action and the time-dependent variational approximation. In section 3, we use Derrick’s theorem to study the stability of these solutions to width instabilities. In section 4 we discuss the energy landscape when we include translations of the origin of the solution. In section 5 we perform a linear stability analysis. In section 6, we introduce a four-parameter trial wave function to study the dynamics of the model, and in section 7 we provide results of the direct numerical solutions of the nonlinear Schrödinger equation in the Pöschl–Teller external potential. Our main conclusions are summarized in section 8.

2. Time-dependent variational principle

The time-dependent version of the variational approximation can be traced to an obscure appendix in the 1930 Russian edition of the ‘Principles of Wave Mechanics’, by Dirac[11]. In this version of the variational approximation, the wave function is taken to be a function of a number of time-dependent parameters. Variation of the action, as defined by Dirac, leads to a classical set of Hamiltonian equations of motion for the parameters. These classical equations are then solved as a function of time to provide an approximation to the evolution of the wave function.

The action which leads to equation (5.1) is given by

$$\Gamma[\psi, \psi^*] = \int dt L[\psi, \psi^*]$$

$$(2.1)$$

[11] Dirac, appendix to the Russian edition of The Principles of Wave Mechanics, as cited by Frenkel [34]. Pattanayak and Schieve [35] point out that the reference often quoted, Dirac [36], does not contain this equation.
where
\[
L[\psi, \psi^*] = \frac{i}{2} \int dx \left[ \psi^* \left( \partial_t \psi - (\partial_x \psi^*) \psi \right) - H[\psi, \psi^*] \right],
\]
and
\[
H[\psi, \psi^*] = \int dx \left[ |\partial_x \psi|^2 - \frac{g}{\kappa + 1} |\psi|^{2\kappa + 2} + V(x) |\psi|^2 \right].
\]
The NLSE and its complex conjugate follow from minimizing the action via,
\[
\frac{\delta \Gamma}{\delta \psi^*} = \frac{\delta \Gamma}{\delta \psi} = 0.
\]
\[\text{(2.3)}\]

2.1 Symplectic formulation

In this section it will be useful to introduce a symplectic formulation of Lagrange’s equations for the variational parameters. We consider a variational wave function of the form:
\[
\tilde{\psi}[x, Q(t)], \quad Q(t) = \{Q^1(t), Q^2(t), \ldots, Q^{2n}(t)\}.
\]
Introducing the notation \(\partial_i \equiv \partial / \partial Q^i\), the Lagrangian (2.2a) is given by
\[
L[Q, \dot{Q}] = \sum_i \pi_i(Q) \dot{Q}^i - H[Q],
\]
where
\[
\pi_i(Q) = \frac{i}{2} \int dx \left\{ \tilde{\psi}^* \left[ \partial_i \tilde{\psi} - [\partial_i \tilde{\psi}^*] \tilde{\psi} \right] \right\},
\]
and \(H(Q)\) is given by
\[
H(Q) = \int dx \left[ |\partial_x \tilde{\psi}|^2 - \frac{g}{\kappa + 1} |\tilde{\psi}|^{2\kappa + 2} + V(x) |\tilde{\psi}|^2 \right].
\]
The Euler–Lagrange equations now become
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}^i} \right) - \frac{\partial L}{\partial Q^i} = 0.
\]
From (2.4) this gives
\[
\sum_j f_{ij}(Q) \dot{Q}^j = v_i(Q),
\]
where we have set \(v_i(Q) \equiv \partial_i H(Q)\), and where
\[
f_{ij}(Q) = \partial_i \pi_j(Q) - \partial_j \pi_i(Q)
\]
is an antisymmetric \(2n \times 2n\) symplectic matrix. If \(\det f(Q) \neq 0\), we can define an inverse as the contra-variant matrix with upper indices,
\[
\sum_j f^{ij}(Q) f_{jk}(Q) = \delta^i_k,
\]
in which case the equations of motion (2.8) can be put in the form:
\[
\dot{Q}^i = \sum_j f^{ij}(Q) v_j(Q).
\]
Conservation of energy is expressed as
\[
\frac{dH(Q)}{dt} = \sum_i \dot{Q}_i v_i(Q) = \sum_{ij} f^{ij}(Q) v_j(Q) v_i(Q) = 0 ,
\]
(2.12)
since \( f^{ij}(Q) \) is an antisymmetric tensor. Poisson brackets are defined using this antisymmetric tensor. If \( A(Q) \) and \( B(Q) \) are functions of \( Q \), Poisson brackets are defined by
\[
\{ A(Q), B(Q) \} = \sum_{ij} \left( \frac{\partial A(Q)}{\partial Q_j} \right) f^{ij}(Q) \left( \frac{\partial B(Q)}{\partial Q_i} \right) .
\]
(2.13)
In particular,
\[
\{ Q^i, Q^j \} = f^{ij}(Q) .
\]
(2.14)
This definition satisfies Jacobi’s identity. That is, what we have shown here is that the 2n quantities \( Q^i \) are classical symplectic variables.

3. Derrick’s theorem

Derrick’s theorem \cite{37} states that for a Hamiltonian dynamical system, an exact solution of the equation of motion is unstable if under a scale transformation, \( x \mapsto \beta x \) with fixed normalization, the energy of the system is lowered. The stretched wave function for Derrick’s theorem is given by
\[
\psi_{\beta}(x, t) = A(\beta x, \beta, \gamma) \text{sech}^\gamma(\beta x) ,
\]
(3.1)
with the normalization fixed by the requirement,
\[
M[\beta, \beta, \gamma] = \int dx |\psi_{\beta}(x, t)|^2 = A^2(\beta, \beta, \gamma) c_1[\gamma]/\beta = M_0[\beta, \gamma] = A_0^2(\beta, \gamma) c_1[\gamma] .
\]
(3.2)
So \( A^2(\beta, \beta, \gamma) = \beta A_0^2(\beta, \gamma) \). Evaluation of the Hamiltonian \((2.2b)\) with Derrick’s wave function gives:
\[
H(\beta, \gamma) = H_1(\beta, \gamma) + H_2(\beta, \gamma) + H_3(\beta, \gamma) ,
\]
(3.3)
where
\[
H_1(\beta, \gamma) = \int dx |\partial_x \psi_{\beta}|^2 = A_0^2 \beta^2 \gamma^2 \int dz \text{sech}^{2\gamma+2}(z) \sinh^2(z) = \frac{A_0^2 \beta^2 \gamma}{2} c_1[\gamma + 1] ,
\]
(3.4a)
\[
H_2(\beta, \gamma) = -\frac{g}{k+1} \int dx |\psi_{\beta}|^{2\gamma + 2} = -g A_0^{2/\gamma} A_0^2 \frac{\beta^1/\gamma}{\gamma + 1} \int dz \text{sech}^{2\gamma+2}(z) ,
\]
(3.4b)
\[
= -A_0^2 \frac{\gamma}{\gamma + 1} \left[ \frac{\gamma}{\gamma + 1} - \lambda \beta^2 \right] c_1[\gamma + 1] ,
\]
\[
H_3(\beta, \gamma) = \int dx V(x) |\psi_{\beta}|^2 = -\lambda \beta^2 A_0^2 \gamma \int dx \text{sech}^2(x) \text{sech}^{2\gamma}(\beta x) ,
\]
(3.4c)
\[
= -\lambda \beta^2 A_0^2 g_1[\beta, \gamma] .
\]
where
\[ g_1(\beta, \gamma) = \int dz \text{sech}^2(\gamma z) \text{sech}^2(z/\beta). \] (3.5)

So then \( H(\beta, \gamma)/M_0(\beta, \gamma) \) is given by
\[
h(\tilde{b}, \beta, \gamma) \equiv \frac{H(\beta, \gamma)}{A_0(\beta, \gamma) c_1[\gamma + 1]} = \frac{1}{2} \beta^2 \gamma + \frac{\gamma \beta^{1/\gamma}}{\gamma + 1} \left[ \gamma(\gamma + 1) - 1 \right] \rho \tilde{b}^2 - \lambda \tilde{b}^2 \frac{g_1(\beta, \gamma)}{c_1[\gamma + 1]}, \]
(3.6)
and is independent of \( g. \) \( h(\tilde{b}, \beta, \gamma) \) is stationary when \( \beta = 1. \) We have
\[
\frac{\partial h(\tilde{b}, \beta, \gamma)}{\partial \beta} = \gamma \left[ \beta - \beta^{1/\gamma - 1} \right] + \lambda \tilde{b}^2 \left[ \frac{\beta^{1/\gamma - 1}}{\gamma + 1} + \frac{1}{c_1[\gamma + 1]} \frac{\partial g_1(\beta, \gamma)}{\partial \beta} \right]. \]
(3.7)
From equation (A.4) in appendix, we find
\[
\frac{\partial g_1(\beta, \gamma)}{\partial \beta} \bigg|_{\beta = 1} = \frac{c_1[\gamma + 1]}{\gamma + 1},
\]
so that at \( \beta = 1, \)
\[
\left. \frac{\partial h(\tilde{b}, \beta, \gamma)}{\partial \beta} \right|_{\beta = 1} = 0,
\]
(3.9)
for all values of \( \gamma \) and \( \tilde{b}^2. \) The sign of the second derivative of \( h(\tilde{b}, \beta, \gamma) \) with respect to \( \beta \) at \( \beta = 1 \) determines whether the solution is unstable to small changes in the width. If \( \frac{\partial^2 h(\tilde{b}, \beta, \gamma)}{\partial \beta^2} \) evaluated at \( \beta = 1 \) is negative the solution is unstable. We find
\[
\frac{\partial^2 h(\tilde{b}, \beta, \gamma)}{\partial \beta^2} = \gamma + (\gamma - 1) \beta^{1/\gamma - 2} - \lambda \tilde{b}^2 \left[ \frac{\gamma - 1}{\gamma(\gamma + 1)} \beta^{1/\gamma - 2} + \frac{1}{c_1[\gamma + 1]} \frac{\partial^2 g_1(\beta, \gamma)}{\partial \beta^2} \right].
\]
(3.10)
From appendix, we have
\[
\left. \frac{\partial^2 g_1(\beta, \gamma)}{\partial \beta^2} \right|_{\beta = 1} = 4 c_2[\gamma + 1] - 6 c_2[\gamma + 2] - \frac{2 c_1[\gamma + 1]}{\gamma + 1},
\]
where
\[
c_2[\gamma] = \int dz z^2 \text{sech}^2(z) = 2^{\gamma - 1} \gamma F_3[\gamma, \gamma, \gamma, 2 \gamma; 1 + \gamma, 1 + \gamma, 1 + \gamma; -1]/\gamma^3.
\]
(3.11)
Inserting this into (3.10) and evaluating it at \( \beta = 1 \) gives:
\[
\left. \frac{\partial^2 h(\tilde{b}, \beta, \gamma)}{\partial \beta^2} \right|_{\beta = 1} = 2 \gamma - 1 + \lambda \tilde{b}^2 \left[ \frac{1 - 2 \gamma + 1 \gamma}{c_1[\gamma]} \frac{2 c_2[\gamma + 1] - 3 c_2[\gamma + 2]}{c_1[\gamma]} \right],
\]
(3.12)
which is independent of \( g. \) The critical value of \( \tilde{b}^2, \) when (3.12) vanishes, is
\[
\tilde{b}_c^2 = -\lambda \frac{\gamma(\gamma - 1)}{1 - (2 \gamma + 1) \frac{2 c_2[\gamma + 1] - 3 c_2[\gamma + 2]}{c_1[\gamma]}}.
\]
(3.13)
which is independent of $g$. One can easily check that
\[ 1 - (2\gamma + 1) \frac{2c_2[\gamma + 1] - 3c_2[\gamma + 2]}{c_1[\gamma]} > 0, \] (3.14)
for all $\gamma$. In appendix, we give an alternative form for equation (3.13), which is in agreement with [30]. In figure 1, we plot $\tilde{b}_{\text{crit}}^2$ for the three cases. Referring to the figure, according to Derrick’s theorem:

(I) For case I with $g = +1$, $\lambda = +1$, and $0 \leq \tilde{b}^2 \leq \tilde{b}_\gamma^2$, we see from equation (3.12) that $\partial^2 h(\tilde{b}, \beta, \gamma)/\partial \beta^2 \geq 0$ for all $\kappa < 2$, so Derrick’s theorem predicts that solutions are width stable for $\kappa < 2$. For $\kappa > 2$, there is another region for $\tilde{b}_{\text{crit}}^2 < \tilde{b}^2 < \tilde{b}_\gamma^2$ where width stable solutions are also possible.

(II) For case II with $g = +1$, $\lambda = -1$, and $0 \leq \tilde{b}^2 \leq \tilde{b}_{\text{crit}}^2$, width stable solutions are possible for $\kappa < 2$, but we will find that this region is unstable to translation perturbations.

(III) For case III with $g = -1$, $\lambda = +1$, and $\tilde{b}_\gamma^2 < \tilde{b}^2 < \tilde{b}_{\text{crit}}^2$ width stable solutions are possible for all $\kappa$.

Derrick’s theorem is a version of the time-independent variational method and only provides information about the stability of the system under a change in $\beta$, the width of the wave function. Thus Derrick’s theorem only gives a sufficient condition for an instability to occur. To see if there are translational instabilities as well as width instabilities we will consider what happens to the energy when we make a small translation of the position of the solution.

4. Translational stability landscape

Using Derrick’s theorem we explored whether the solution was a maximum or minimum of the energy landscape as a function of the stretching parameter $\beta$. Here we would like to do a similar analysis to study whether the energy increases or decreases as we let $x \rightarrow x + a$ where $a$ is a small translation. Again we will posit that there is a translational instability if the energy $H[a]$ decreases as $a$ departs from zero. Let us again consider the NLSE plus real potential defined in equations (1.1) and (1.2). We want to see how the energy of the system changes under the translation $x \rightarrow x + a$ with the normalization fixed by the requirement that the mass
$M$ is preserved. Clearly choosing $\psi_a = \psi(x + a)$ preserves the mass for the wave function of the exact solution whose $x$ dependence is displayed by letting

$$\psi(x) = A(b, \gamma) \text{sech}^2(x/e^{\gamma/2}).$$

(4.1)

It is also clear that both $H_1$ and $H_2$ remain unchanged under $x \to x + a$. Only $H_3$ is not translationally invariant. We define

$$H_3(a, \gamma) = \int dx \, V(x)|\psi_a|^2 - \lambda \tilde{b}^2 A_0^2 \int dy \text{sech}^2(y - a) \text{sech}^{2\gamma}(y).$$

(4.2)

We need to ensure that as a function of $a$ the energy is stationary. Since both $H_1$ and $H_2$ are independent of $a$ we only need to consider:

$$\frac{\partial H_3(a, \lambda)}{\partial a} \bigg|_{a=0}$$

(4.3)

at $\beta = 1$ and then calculate the second derivative at $\beta = 1$. Clearly

$$\frac{\partial H_3}{\partial a} = \frac{\partial H_3}{\partial a} = -2 \lambda \tilde{b}^2 A_0^2 \int dy \text{sech}^2(y - a) \tanh(y - a) \text{sech}^{2\gamma}(y),$$

(4.4)

which is indeed zero when evaluated at $a = 0$. The second derivative evaluated at $a = 0$ is

$$\frac{\partial^2 H_3}{\partial a^2} \bigg|_{a=0} = 2 \lambda \tilde{b}^2 A_0^2 \sqrt{\pi} \gamma \Gamma[\gamma + 1] \Gamma[\gamma + 5/2].$$

(4.5)

Thus we indeed find that the solitary wave has translational instability if $\lambda < 0$, i.e. for the repulsive potential, while it is stable in case $\lambda > 0$, i.e. attractive potential. Note that the answer does not depend on the sign of $g$. We show in figure 2 the three-dimensional landscape for $H(\beta, a)$ for a stretching and displacement shift, $x \to \beta x + a$. One can see stability for an attractive potential (case I) but a saddle point for a repulsive potential (case II).

### 5. Linear stability analysis

The traditional way to study stability under small perturbations is to perform a linear stability analysis which we now present. The results obtained agree with the simpler analysis using Derrick’s theorem and looking at the effects of translation on the energy landscape. Starting with the NLSE equation in an external potential:

$$\{ i \partial_t + \partial_x^2 + g \mid \psi(x, t) \}^{2\kappa} - V(x) \} \psi(x, t) = 0,$$

(5.1)

we write down solitary wave solutions as $\psi(x, t) = \phi_\omega(x) e^{-i\omega t}$, with $\phi_\omega(x) \in \mathbb{R}$. Here $\phi_\omega(x)$ satisfies

$$\{ \omega + \partial_x^2 + g \mid \phi_\omega(x) \}^{2\kappa} - V(x) \} \phi_\omega(x) = 0.$$

(5.2)

For $\omega = -\gamma^2$, one has the explicit expression:

$$\phi_{-\gamma}(x) = A_0(b, \gamma) \text{sech}(x).$$

(5.3)

We consider perturbations in the form $\psi(x, t) = (\phi_\omega(x) + r(x, t)) e^{-i\omega t}$ with $r(x, t) \in \mathbb{C}$ and linearize equation (5.1) with respect to $r(x, t)$. The linearized equation is of the form

$$\{ \omega + i\partial_t + \partial_x^2 + V(x) + g \mid \phi_\omega(x) \}^{2\kappa} r(x, t) + 2\kappa g \mid \phi_\omega(x) \}^{2\kappa} \text{Re} \{ r(x, t) \} = 0.$$

(5.4)
Because of the term $\text{Re}\{r(x,t)\}$, equation (5.4) is not a $C$-linear operator. For computational convenience we separate the real and imaginary parts of $r(x,t)$ and define

$$R(x,t) = \begin{pmatrix} p(x,t) \\ q(x,t) \end{pmatrix} = \begin{pmatrix} \text{Re}\{r(x,t)\} \\ \text{Im}\{r(x,t)\} \end{pmatrix}.$$

The equation (5.4) can then be written as

$$\frac{\partial}{\partial t} R = JL R$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

with self-adjoint operators

$$L_-(\omega) = -\partial_x^2 + V(x) - \omega - g |\phi_\omega(x)|^{2\kappa},$$

$$L_+(\omega) = L_-(\omega) - 2\kappa g |\phi_\omega(x)|^{2\kappa}.$$ (5.8a)

From (5.2) and its derivative with respect to $\omega$, we find

$$L_-(\omega) \phi_\omega(x) = 0, \quad L_+(\omega) \partial_\omega \phi_\omega(x) = \phi_\omega(x).$$ (5.9)

To explore the linear stability we consider eigenvalues for the operator $JL$. Since

$$(JL)^2 = \begin{pmatrix} -L_-(\omega) L_+(\omega) & 0 \\ 0 & -L_+(\omega) L_-(\omega) \end{pmatrix}$$

we can consider eigenvalues of the operator $(-L_-L_+)$ instead. If $(-L_-L_+)$ has positive eigenvalues, so does $JL$, and the solitary wave solutions are linearly unstable; if $L_-L_+$ has zero or negative eigenvalues only, $JL$ only has purely imaginary eigenvalues and the solitary wave solutions are spectrally stable.

For case (II) $g = 1, \lambda = -1$ we have $\omega = -\gamma^2$. The amplitude $\phi_\omega(x)$ is a positive function in $L^2$ and $L_-(\omega) \phi_\omega = 0$. By proposition 2.8 in [38] $L_-(\omega)$ is nonnegative and the kernel $\ker(L_-(\omega))$ is span $\{\phi_\omega\}$. Since $L_+(\omega) \partial_\omega \phi_\omega = \phi_\omega$, $\partial_\omega \phi_\omega$ is an eigenfunction of $L_-L_+$, corresponding to zero eigenvalue. We will show that the smallest eigenvalue of $L_-L_+$ is negative. According to [39] the smallest eigenvalue of $L_-L_+$ is given by,
\[ \min \sigma_d(L_-L_+) = \min \left\{ \frac{\langle u, L_+u \rangle}{\langle u, L_-u \rangle}, u \in \ker(L_-) \right\}. \quad (5.11) \]

\( L_- \) is positive definite in \( \ker(L_-)^\perp \), so the sign of smallest eigenvalue is decided by that of \( \langle u, L_+u \rangle \). Since \( \phi_\omega \) is an even function, \( \phi'_\omega \) is an odd function in \( \ker(L_-)^\perp \). We know that
\[ (L_- \phi_\omega(x))' = L_+ \phi'_\omega(x) + V'(x) \phi_\omega(x) = 0, \quad (5.12) \]
which implies that \( L_+ \phi'_\omega(x) = -V'(x) \phi_\omega(x) \). Hence we have
\[ \langle \phi'_\omega(x), L_+ \phi'_\omega(x) \rangle = \langle \phi'_\omega(x), -V'(x) \phi_\omega(x) \rangle, \]
\[ = \frac{1}{2} \langle \phi_\omega(x), V''(x) \phi_\omega(x) \rangle \approx -0.455 < 0. \quad (5.13) \]

It follows that \( \langle u, L_+u \rangle \) can be negative and thus \( -(L_-L_+) \) has at least one positive eigenvalue. We conclude that the solitary wave solutions are linearly unstable.

For case III \( g = -1, \lambda = +1 \) we know that \( L_- \) is nonnegative as in case II. Since \(-2\kappa g |\phi_\omega(x)|^2 \) is positive, \( L_+ \) is positive as well. It implies that \(-L_-L_+ \) has only negative eigenvalues and the solitary wave solutions are spectrally stable.

6. Four parameter time-dependent trial wave function

A simple alternative to the linear stability analysis presented above is to parametrize the wave function for the displaced solitary wave using collective coordinates and apply the variational principle outlined in section 2 to obtain the dynamics of these collective coordinates. Studying the behavior of the position and width collective coordinates immediately allows one to see if these coordinates oscillate under small perturbations or grow or collapse. As a bonus this method also gives the frequencies of small oscillations, or the initial growth of instabilities as well as an approximate description of the perturbed solitary wave. An equivalent approach (the ‘generalized traveling wave method’) has been used recently by Quintero, Mertens, and Bishop [40] to study stability of the NLSE with a class of external potentials. One of the purposes of this paper is to see how well and under what conditions such an approach works.

In this section, we consider a four-parameter trial wave function of the form:
\[ \tilde{\psi}(x,t) = A(t) \text{sech}^2 \left[ \beta(t) y(x,t) \right] e^{i \tilde{\phi}(x,t)}, \quad (6.1) \]
where
\[ \tilde{\phi}(x,t) = -\theta(t) + p(t) y(x,t) + \Lambda(t) y^2(x,t). \quad (6.2) \]
Here we have put \( y(x,t) = x - q(t) \). The parameter \( \Lambda \) is related to the canonical conjugate variable to the average value of \( y^2 \). It arises naturally in the Hartree–Fock approximation to the dynamics of the Schrödinger equation [41].

It will be useful to define a reciprocal width parameter \( G(t) = 1/\beta(t) \) and use this parameter as a generalized coordinate. Conservation of probability gives the ‘mass’ equation,
\[ M = \int dx \rho(x,t) = G(t) A^2(t) c_1[\gamma], \quad (6.3) \]
where \( c_1[\gamma] \) is given in equation (1.10). In order to maintain probability conservation, we want to keep \( M \) constant. That is, we put
so \( A(t) \) and \( G(t) \) are not independent variables. The phase \( \theta(t) \) does not enter into the Hamiltonian, and we will ignore it in what follows. The trial wave function we will assume is:

\[
\tilde{\psi}(x, t) = \sqrt{\frac{M}{Gc_1[\gamma]}} \text{sech}^\gamma \left( \frac{y}{G} \right) e^{i[p_y + \Lambda y^2]}, \tag{6.5}
\]

The four variational parameters are labeled by \( Q_i(t) = \{q(t), p(t), G(t), \Lambda(t)\} \).

Taking the appropriate derivatives we find that

\[
L_0 = \frac{1}{2} \int dx \left[ \tilde{\psi}^* \tilde{\psi}_t - \tilde{\psi}_t^* \tilde{\psi} \right] = \pi_i(Q) \dot{Q}_i, \tag{6.7}
\]

where

\[
\pi_q = p M, \quad \pi_p = 0, \quad \pi_G = -MG^2c_2[\gamma]/c_1[\gamma], \quad \pi_\Lambda = -2MGc_2[\gamma]/c_1[\gamma], \tag{6.8}
\]

with \( c_2[\gamma] \) given in equation (3.11). The only non-zero derivatives of the \( \pi_i \) are

\[
\partial_p \pi_q = M, \quad \partial_G \pi_\Lambda = -2MGc_2[\gamma]/c_1[\gamma], \tag{6.9}
\]

so the symplectic tensor is

\[
f_{ij}(Q) = M \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -C \\
0 & 0 & C & 0
\end{pmatrix}, \quad C = 2Gc_2[\gamma]/c_1[\gamma], \tag{6.10}
\]

and the inverse

\[
f^{ij}(Q) = \frac{1}{MC} \begin{pmatrix}
0 & C & 0 & 0 \\
-C & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}. \tag{6.11}
\]

We also find that for the four-parameter trial wave function

\[
H = H_1 + H_2 + H_3, \tag{6.12}
\]

where

\[
H_1 = \int dx |\tilde{\psi}_t|^2 = M p^2 + \frac{M}{G^2} \frac{\gamma c_1[\gamma + 1]}{c_1[\gamma]} + 4MG^2\Lambda^2 \frac{c_2[\gamma]}{c_1[\gamma]}; \tag{6.13a}
\]

\[
H_2 = -\frac{g}{\kappa + 1} \int dx |\tilde{\psi}|^{2\kappa+2} = -\frac{gM\gamma}{\gamma + 1} \left( \frac{M}{Gc_1[\gamma]} \right)^{1/\gamma} \frac{c_1[\gamma + 1]}{c_1[\gamma]}, \tag{6.13b}
\]

\[
H_3 = \int dx V(x) |\tilde{\psi}|^2 = -\lambda \bar{b}^2 M \frac{f_j[G, q, \gamma]}{c_1[\gamma]}, \tag{6.13c}
\]
with

\[ f_1[G, q, \gamma] = \int dz \text{sech}^{2\gamma}(z)\text{sech}^2(Gz + q). \]  

(6.14)

The Hamiltonian then becomes

\[ \frac{H(Q)}{M} = p^2 + \frac{\gamma G^2}{2} \frac{c_1[\gamma + 1]}{c_1[\gamma]} + 4G^2\Lambda^2 \frac{c_2[\gamma]}{c_1[\gamma]} - \frac{g \gamma}{\gamma + 1} \left( \frac{M}{G c_1[\gamma]} \right)^{1/\gamma} \frac{c_1[\gamma + 1]}{c_1[\gamma]} - \lambda \bar{b}^2 \frac{f_1[G, q, \gamma]}{c_1[\gamma]}. \]  

(6.15)

The equations of motion are

\[ \dot{Q}^i = \sum_j f^{ij}(Q) v_j(Q). \]  

(6.16)

where

\[ v_i = \partial_i H(Q). \]

In terms of the functions \( f_2 \) and \( f_3 \) defined by

\[ f_2[G, q, \gamma] = -\frac{1}{2} \partial_q f_1[G, q, \gamma] = \int dz \text{sech}^{2\gamma}(z)\text{sech}^2(Gz + q) \tanh(Gz + q), \]  

(6.17a)

and recalling that as a result of Mass conservation we have from (6.4),

\[ g \left( \frac{M}{G(t) c_1[\gamma]} \right)^{1/\gamma} = g A_0^{2\gamma}(t), \]  

(6.18)

and since \( G(0) = 1 \) at \( t = 0 \), we can use equation (1.7) to find:

\[ g \left( \frac{M}{c_1[\gamma]} \right)^{1/\gamma} = g A_0^{2\gamma} = \gamma(\gamma + 1) - \lambda \bar{b}^2. \]  

(6.19)

Therefore the collective coordinate equations reduce to:

\[ \dot{q} = 2p, \]  

(6.20a)

\[ \dot{p} = -2\lambda \bar{b}^2 \frac{f_2[G, q, \gamma]}{c_1[\gamma]}, \]  

(6.20b)

\[ \dot{G} = 4G\Lambda, \]  

(6.20c)

\[ \dot{\Lambda} = -4\Lambda^2 + \frac{1}{G^2} \frac{\gamma^2}{2\gamma + 1} \frac{c_1[\gamma]}{c_2[\gamma]} - \frac{\gamma(\gamma + 1) - \lambda \bar{b}^2}{(2\gamma + 1)(\gamma + 1)} \frac{1}{G^{2+1/\gamma}} \frac{c_1[\gamma]}{c_2[\gamma]} \frac{\lambda \bar{b}^2 f_3[G, q, \gamma]}{G c_2[\gamma]}, \]  

(6.20d)

and are independent of \( g \). We need to remember that for case III when \( g = -1 \) and \( \lambda = +1 \) there is no solution unless \( \bar{b}^2 > \gamma(\gamma + 1) \).

6.1. Small oscillations

We require that the variational wave function starts out so that it agrees with the exact solutions at \( t = 0 \),
\[\tilde{\psi}(x,0) = \psi_0(x,0),\quad (6.21)\]

with \(A_0\) fixed by (6.19). This means that we want to choose
\[q_0 = 0, \quad p_0 = 0, \quad G_0 = 1, \quad \Lambda_0 = 0.\quad (6.22)\]

Setting
\[q(t) = q_0 + \delta q(t), \quad p(t) = p_0 + \delta p(t), \quad G(t) = G_0 + \delta G(t), \quad \Lambda(t) = \Lambda_0 + \delta \Lambda(t), \quad (6.23)\]
to first order, we have
\[f_1[G,q,\gamma] = c_1[\gamma + 1] + g_1[\gamma] \delta G, \quad (6.24a)\]
\[f_2[G,q,\gamma] = g_2[\gamma] \delta q, \quad (6.24b)\]
\[f_3[G,q,\gamma] = \frac{c_1[\gamma + 1]}{2[\gamma + 1]} + g_3[\gamma] \delta G, \quad (6.24c)\]

where
\[g_1[\gamma] = -2 \int dz \sech^{2\gamma+2}(z) \tanh(z) = -\frac{2\gamma}{(2\gamma + 1)(\gamma + 1)} c_1[\gamma]. \quad (6.25a)\]
\[g_2[\gamma] = \int dz \sech^{2\gamma+2}(z) [1 - 3 \tanh^2(z)] = \frac{4\gamma^2}{(2\gamma + 1)(2\gamma + 3)} c_1[\gamma], \quad (6.25b)\]
\[g_3[\gamma] = \int dz z^2 \sech^{2\gamma+2}(z) [1 - 3 \tanh^2(z)] = -2 c_2[\gamma + 1] + 3 c_2[\gamma + 2]. \quad (6.25c)\]

Substituting equations (6.23) into equations (6.20) gives
\[\delta q = 2 \delta p, \quad (6.26a)\]
\[\delta \dot{p} = -\lambda \tilde{b}^2 \frac{8\gamma(\gamma + 1)}{(2\gamma + 1)(2\gamma + 3)} \delta q, \quad (6.26b)\]
\[\delta \dot{\Lambda} = 4 \delta \Lambda, \quad (6.26c)\]
\[\delta \dot{\lambda} = \left\{-2\gamma \frac{c_1[\gamma + 1]}{c_2[\gamma]} + \frac{2\gamma + 1}{2\gamma(\gamma + 1)} [\gamma(\gamma + 1) - \lambda \tilde{b}^2] \frac{c_1[\gamma + 1]}{c_2[\gamma]} + \lambda \tilde{b}^2 \frac{f_1[1,0,\gamma]}{c_2[\gamma]} - \lambda \tilde{b}^2 \frac{g_3[\gamma]}{c_2[\gamma]} \right\} \delta G. \quad (6.26d)\]

Thus, to first order, the \((q,p)\) and \((G,\Lambda)\) modes uncouple and reduce to equations of the form:
\[\delta \dot{q} + \omega_q^2 \delta q = 0, \quad \delta \dot{G} + \omega_G^2 \delta G = 0, \quad (6.27)\]

where the \((q,p)\) mode frequency is given by
\[\omega_q^2 = 4\lambda \tilde{b}^2 \frac{g_2[\gamma]}{c_1[\gamma]} = \lambda \tilde{b}^2 \frac{16\gamma^2}{(2\gamma + 1)(2\gamma + 3)}, \quad (6.28)\]
independent of \( g \). For \( \lambda = +1 \), translational motion is \textit{stable} for all \( \gamma \), and \textit{unstable} for \( \lambda = -1 \). This is easily explained by the fact that as long as the solitary wave is near \( q = 0 \), for \( \lambda = +1 \) it sees an attractive force that brings it back to the origin. For the opposite sign, \( \lambda = -1 \), it sees a repulsive force that moves it from the origin, assuming it maintains its shape. We illustrate this behavior in section 7.2 by a numerical solution of Schrödinger’s equation using a split-operator method.

The \((G, \Lambda)\) mode frequency is given by

\[
\omega_G^2 = A(\gamma) + \lambda \hat{b}^2B(\gamma) \ .
\]  

(6.29)

where

\[
A(\gamma) = \frac{4\gamma(2\gamma - 1)}{2\gamma + 1} \frac{c_1(\gamma)}{c_2[\gamma]} ,
\]

\[
B(\gamma) = 4 \left\{ \frac{1}{2\gamma + 1} \frac{c_1(\gamma)}{c_2[\gamma]} - \frac{2c_2[\gamma + 1] - 3c_2[\gamma + 2]}{c_2[\gamma]} \right\} ,
\]  

(6.30)

and the critical value of \( \hat{b}^2 \) is then given by

\[
\hat{b}_{\text{crit}}^2 = -\lambda \frac{A(\gamma)}{B(\gamma)} = -\lambda \frac{\gamma(2\gamma - 1)}{1 - (2\gamma + 1)} \frac{2c_2[\gamma + 1] - 3c_2[\gamma + 2]}{c_2[\gamma]} ,
\]  

(6.31)

which is the same result for \( \hat{b}_{\text{crit}}^2 \) that we found in equation (3.13) using Derrick’s theorem. Here in addition, we find a value for the dynamical evolution of \( \Lambda \) determined from the small oscillation equations is in good agreement with what is found from Derrick’s theorem. Here it sees an attractive force that brings it back to the origin. For the opposite sign, \( \lambda = -1 \), we illustrate this behavior in section 7.2 by a numerical solution of Schrödinger’s equation using a split-operator method.

6.2. Dynamics of the collective coordinates

In this section we plot representative time evolutions of the collective coordinates of the four-parameter variational calculation given in equations (6.20) for the three cases. The parameters \((q, p, G, \Lambda)\) can be related to the expectation values of \((x, p_{\text{op}}, x^2, x_{\text{op}}^2)\) where \( p_{\text{op}} = -i\partial / \partial x \). Here \((x)/M = q(t)\), etc. One can show that using the equations obeyed by these four collective coordinates, that the evolution equations for these expectation values are \textit{exactly} satisfied. We expect, and find that these time evolutions qualitatively agree with the numerically calculated values of these collective coordinates especially when the actual form of the numerically determined wave function is preserved. In case III, although we predict the translational instability quite well, we did not anticipate that the form of the wave function would bifurcate. Of course \textit{a posteriori} one could assume a two humped variational wave function with more parameters to actually capture better the time evolution of the initial exact solution. What we find, is that when we are in the oscillatory regime of either \( q(t) \) or \( G(t) \), the oscillation period determined from the small oscillation equations is in good agreement with what is found from the dynamical evolution of \( q(t) \) and \( G(t) \) from their evolution equations.

6.2.1. \( g = +1 \) and \( \lambda = +1 \) (Case I). This is the case that we studied in our previous paper [30]. However, in that paper we did not consider oscillations in the spatial direction, nor did we numerically solve the NLSE to compare with our analytic results. For this case in the \( G \) unstable region the behavior in \( q \) is oscillatory in our approximation. We will choose \( \kappa = 5/2 \) and two values of \( \hat{b}^2 \), namely \( \hat{b}^2 = 1/10 \) which is in the unstable regime and \( \hat{b}^2 = 1/5 \) which is in the stable regime to display the two types of behavior for the parameters \( q(t) \) and \( G(t) \) as a function of time. For the blowup case, the period for \( q(t) \) is oscillatory with a period of
\[ T = 32.5, \] which agrees with the numerical results in figure 3(a). However the solitary wave blows up, \( G(t) \to 0 \), as shown in figure 3(b).

For \( \tilde{b}^2 = \frac{1}{5} \) one is in the oscillatory regime for \( G(t) \) and \( q(t) \) and we get the results shown in figures 4. For this case the small oscillation equation predicts that the period of \( q(t) \) is \( T = 23 \), and the period of \( G(t) \) is \( T = 37 \).

6.2.2. \( g = +1 \) and \( \lambda = -1 \) (Case II). The case where we have an exact solution for a repulsive potential leads to the most unexpected behavior, as we will show later in our numerical simulations. The first interesting thing is that although Derrick’s theorem shows that the answer is stable to changing the width when \( \kappa < \frac{3}{2} \) and \( \tilde{b}^2 \) is below \( \tilde{b}_c^2 \) in figure 1(b), we find that if we shift the position by a small amount, because of the repulsive potential, the solitary wave is pushed out of the region of the potential and then oscillates about a potential free solution in this approximation. This result is suggested by the four-component variational calculation, and confirmed by a numerical calculation shown in figure 9(b). Choosing \( \kappa = 1 \) and \( \tilde{b}^2 = 1 \), which is in the regime which is stable to width changes, the solutions of the dynamic equations (6.18) give the results shown in figure 5. Whereas choosing \( \kappa = 1.85 \), which is unstable to width changes gives the results shown in figure 6.

6.2.3. \( g = -1 \) and \( \lambda = +1 \) (Case III). For this case, all allowed solutions \( \tilde{b}^2 > (\kappa + 1)/\kappa^2 \) should be stable to small changes in both \( G \) and \( q(t) \). This oscillatory behavior for the case \( \kappa = 3, \tilde{b}^2 = 1/2, q[0] = 0.01 \) is shown in figures 7.
7. Numerical study of stability

7.1. Domains of stability

In order to study the stability of the soliton solutions, the actual numerical simulation of the soliton evolution has been performed. For that purpose, we have numerically solved equation (5.1) with the initial conditions described in section 1.1 using a Crank–Nicolson scheme [42].

The complex soliton in the spatial domain was represented on a regular grid with mesh size $\Delta x = 5 \times 10^{-3}$, and free boundary conditions were imposed.

In order to study the stability regimes, we calculate the normalized correlation between the initial intensity profile (at $t = 0$), and the intensity profile for $t > 0$, i.e.

$$
corr(0, t) = \frac{\int_{-\infty}^{\infty} dx |\psi(x, 0)|^4 |\psi(x, t)|^2}{\int_{-\infty}^{\infty} dx |\psi(x, 0)|^4}. \tag{7.1}
$$

Notice that $0 \leq corr(0, t) = C_t < \infty$ and its value $C_t$ can be interpreted as follows:

$$
corr(0, t) = \begin{cases} 
C_t = 1 & \text{stable regime,} \\
C_t > 1 & \text{blow–up regime,} \\
0 \leq C_t < 1 & \text{translational instability,}
\end{cases}
$$

for any $t > 0$. Theoretically, it means that the evolution of the soliton solutions should be checked up to $t \to \infty$. However, in practice, the inherent numerical noise of the simulations randomly perturbs the soliton shape during evolution, so finite evolution times are enough to determine the stability of soliton solutions. In this regard, we have found that the evolution of solitons up to $5 \times 10^2$ time units with step size $\Delta t = 3 \times 10^{-2}$ is enough to study their stability. First let us return to the problem we studied earlier (Case I) where $\lambda = g = 1$. For that case three methods predicted the same width stability region, namely Derrick’s theorem, setting $\omega_0^2 = 0$ and the Vakhitov–Kolokolov criterion. In this case, if we consider the domain of numerical stability using the exact initial conditions, we agree with the results of Derrick’s theorem.

Next look at the case when our variational method predicts stability, namely for the attractive potential with $g = -1$. The stability should occur as long as there is a solution namely $\tilde{b}^2 > (\kappa + 1)/\kappa^2$ as shown in figure 1(a). The numerical solutions lead to the same conclusions as shown in figure 8(a). Figures 8(a) and (b) show the distribution of the stability.
In figure 8(b) we show the regions of translational and width instabilities for case II.

7.2. Effect of translational instability

This is illustrated for parameters in region II in figure 9 by numerical solutions of Schrödinger’s equation using a split-operator method. In figure 9(a), the initial conditions are such that the solution ‘rolls’ off the top of the potential in both directions and bifurcates, reminiscent of the quantum roll problem discussed in connection with the inflationary universe [41]. Although classically if we are at a maximum or saddle point we expect a particle to go to the right or to the left, since the underlying theory is essentially quantum mechanical, some of the mass density can go to the right or to the left. If we start at the top of the saddle, and the repulsive potential is strong ($\tilde{\beta}^2 \sim 1$) then the solution bifurcates. In figure 9(b), we give a slight kick away from the origin by setting $q(0) = 0.01$, which leads to translational motion away from the origin. In this region of parameter space, the results are strongly dependent on the initial conditions. Another problem where a saddle node instability causes a bifurcation of solutions in the NLSE occurs when one looks at the stationary states of the NLSE in a symmetric double well potential. This interesting case was discussed in the paper of Andrea Sacchetti [43].
Figure 8. Distribution of the stability regions for (a) case III and (b) case II. (a) Stability region (red dots) for an attractive potential for case III with $\lambda = +1$ and $g = -1$. In the empty region below the solid line no soliton solution exists. (b) Distribution of width instability regime (blue filled squares), and translational instability regime (yellow filled diamonds) for case II with $\lambda = -1$ and $g = 1$.

Figure 9. Density $\rho(x,t)$ calculated by a numerical solution of Schrödinger’s equation at intervals of $\Delta t = 2$ for region II with $\kappa = 0.8$ and $b^2 = 1$, $G(0) = 1$, showing in (a) a bifurcation when $q(0) = 0$ of the initial wave function into two waves going in opposite directions and in (b) initial condition instability when $q(0) = 0.01$. (a) $\rho(x,t)$ at intervals of $\Delta t = 2$ for $q(0) = 0$. (b) $\rho(x,t)$ at intervals of $\Delta t = 2$ for $q(0) = 0.01$.

Figure 10. The circles are numerical calculations of $\tilde{b}^2_{\text{crit}}$ for $G(0) = 1.01$, $q(0) = 0$. The green curve fit to these points lies above the curve from Derrick’s theorem (red line). The upper curve (blue line) is $\tilde{b}^2_\gamma$. 
7.3. Effects of initial conditions

A small perturbation of the initial conditions can also lead to instabilities. These effects smear out the instability regions predicted by Derrick’s theorem as shown in figure 1. As an illustration of such effects, we show in figure 10 the modification of region I for $\kappa > 2$ as a result of a 1% change in $G(0)$ (using $G(0) = 1.01$) instead of the exact solution value of $G(0) = 1.00$. The unstable regime found by a numerical solution of Schrödinger’s equation with this initial condition is somewhat bigger than that found from Derrick’s theorem (see figure 10). Similar effects are found for other initial conditions.

8. Conclusions

In this paper we studied the stability of two new exact classes of solutions of the NLSE in a real Pöschl–Teller potential which is the SUSY partner of a complex $\mathcal{PT}$-symmetric potential studied previously [1]. Since the exact equations are derivable from an action principle, if we approximate the wave function by a set of collective coordinates we obtain a symplectic formulation of the dynamics of the collective coordinates with a conserved Hamiltonian. This reduced system is amenable to several approaches to studying the stability problem such as Derrick’s theorem, looking at the energy landscape as a function of translations, as well as a time dependent variational approach. Using these methods we have mapped the domain of stability of these exact solutions and found good agreement with what we obtain from numerical simulations. For the case of the solution with an attractive potential but $g = -1$, the solutions are stable to both small width and small position deformations for all $\kappa$ as long as $\tilde{b}^2 > \gamma (\gamma + 1)$. In that situation, the small oscillation equations of the variational approach give good agreement with numerical simulations. For the more interesting case of $g = 1$ and a repulsive potential, there is a translational instability. This can be seen by looking at the energy landscape or by looking at the small oscillation equations of our variational approximation. The stability results obtained from looking at the energy landscape of the solution as a function of both width stretching and translations are in agreement with the results of a more rigorous linear stability analysis. For the case of the repulsive potential which has a translational instability we find by performing numerical simulations quite interesting results. If one starts with the exact initial solution, the solution breaks into two equal amplitude pulses moving in opposite directions. By perturbing the solution in one direction, the majority of the wave goes in that direction, but still some of the wave goes in the opposite direction.

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Appendix. Useful integrals and definitions

\[ c_1[\gamma] = \int dz \operatorname{sech}^2(z) = \frac{\sqrt{\pi} \Gamma[\gamma]}{\Gamma[\gamma + 1/2]} . \]  
(A.1)

A useful result is

\[ c_1[\gamma + 1] = \frac{2\gamma}{2\gamma + 1} c_1[\gamma] , \quad \frac{c_1[\gamma + 1]}{c_1[\gamma]} = \frac{2\gamma}{2\gamma + 1} . \]  
(A.2)

In section 3, we defined an integral,

\[ g_1[\beta, \gamma] = \int dz \operatorname{sech}^2(\gamma) \operatorname{sech}^2(z/\beta) . \]  
(A.3)

The first derivative of \( g_1[\beta, \gamma] \) with respect to \( \beta \) evaluated at \( \beta = 1 \) is given by

\[ \left. \frac{\partial g_1[\beta, \gamma]}{\partial \beta} \right|_{\beta=1} = 2 \int dz z \operatorname{sech}^{2\gamma+2}(z) \tanh(z) , \]

\[ = \frac{1}{\gamma + 1} \int dz \operatorname{sech}^{2\gamma+2}(z) = c_1[\gamma + 1] \]  
(A.4)

where we have integrated by parts. The second derivative, evaluated at \( \beta = 1 \) is

\[
\left. \frac{\partial^2 g_1[\beta, \gamma]}{\partial \beta^2} \right|_{\beta=1} = 2 \int dz \left[ -z^2 \operatorname{sech}^{2\gamma+2}(z) - 2z \operatorname{sech}^{2\gamma+2}(z) \tanh(z) 
+ 3z^2 \operatorname{sech}^{2\gamma+2}(z) \tan^2(z) \right].
\]  
(A.5)

Using hyperbolic function identities and integration by parts we find

\[
\left. \frac{\partial^2 g_1[\beta, \gamma]}{\partial \beta^2} \right|_{\beta=1} = 4 c_2[\gamma + 1] - 6 c_2[\gamma + 2] - \frac{2 c_1[\gamma + 1]}{\gamma + 1} .
\]  
(A.6)

We use this result in section 3. A second form of the critical curve can be found using an identity for \( c_2[\gamma + 1] \), which we derive here. First note that

\[ c_2[\gamma + 1] = \int dz z^2 \operatorname{sech}^{2\gamma+2}(z) , \]

\[ = \int dz z^2 \operatorname{sech}^{2\gamma}(z) \left[ 1 - \tanh^2(z) \right] = c_2[\gamma] - I[\gamma] , \]  
(A.7)

where

\[ I[\gamma] = \int dz z^2 \sinh^2(z) \operatorname{sech}^{2\gamma+2}(z) . \]  
(A.8)
Using the identity,
\[
\frac{\partial^2}{\partial \lambda^2} \text{sech}^2(y(\lambda z)) = 2 \gamma (2\gamma + 1) z^2 \sinh^2(\lambda z) \text{sech}^{2\gamma + 2}(\lambda z)
\]
\[
- 2 \gamma z^2 \text{sech}^{2\gamma}(z)
\] (A.9)
integrating it over \(z\), and evaluating at \(\lambda \to 1\) gives:
\[
2 c_1[\gamma] = 2 \gamma (2\gamma + 1) I[\gamma] - 2 \gamma c_2[\gamma].
\] (A.10)
Substituting in \(I[\gamma]\) from equation (A.7) gives:
\[
c_2[\gamma + 1] = \left(\frac{2\gamma}{2\gamma + 1}\right) c_2[\gamma] - \left(\frac{1}{\gamma(2\gamma + 1)}\right) c_1[\gamma].
\] (A.11)
Using this result, and after a bit of algebra, it is easy to show that the critical curve equation (3.13) can be written as
\[
\tilde{b}_{\text{crit}}^2 = -\lambda \frac{\gamma(2\gamma - 1)}{(\gamma - 1)(2\gamma - 1)} + \frac{4\gamma^2 c_2[\gamma]}{5\gamma + 3 c_1[\gamma]}.
\] (A.12)
This form is identical to equation (4.28) in [30].

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