INVERSE PROBLEMS FOR RANDOM WALKS ON TREES:
NETWORK TOMOGRAPHY

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Abstract. Let $G$ be a finite tree with root $r$ and associate to the internal
vertices of $G$ a collection of transition probabilities for a simple nondegenerate
Markov chain. Embed $G$ into a graph $G'$ constructed by gluing finite linear
chains of length at least 2 to the terminal vertices of $G$. Then $G'$ admits
distinguished boundary layers and the transition probabilities associated to
the internal vertices of $G$ can be augmented to define a simple nondegenerate
Markov chain $X$ on the vertices of $G'$. We show that the transition probabilities
of $X$ can be recovered from the joint distribution of first hitting time and first
hitting place of $X$ started at the root $r$ for the distinguished boundary layers
of $G'$.

1. Introduction

Computing networks consist of hardware devices (hosts, routers, end terminals,
etc), together with a collection of connections between pairs of such devices, along
which packets of information are passed. This rudimentary structure is readily
modeled by a graph $\Lambda = (V, E)$ where the vertex set $V$ represents hardware devices
and the edge set $E \subset V \times V$ represents direct connections between devices. If, in
addition to the underlying graph structure, parameters are associated to vertices
and edges, it is possible to produce a more accurate model of a given computing
environment. When the parameters defining the model (including the underlying
graph) are dynamic, network performance can be expected to vary, and network
control and/or predictability become issues of serious consequence. For real world
applications, the first step in addressing such issues involves the accurate monitoring
of the parameters which define the network.

For networks modeled as above, it is often the case that direct monitoring of
system parameters is impossible and one must rely on inference methods to produce
reliable estimates for parameter values (cf [CCLY] for a recent survey). The
search for good estimates provides a rich source of challenging inverse problems
(for related work see [BDF], [LY], [RCW], [TYBW] and references therein). In this
paper we formulate and solve one such problem.

The problems in which we are interested involve a fixed network topology in
which one designated device can send packets to a collection of endusers and monitor
packet arrival at enduser positions, but cannot directly observe any behavior for
devices between packet origin and packet collection. We are interested in using
packet arrival times to determine network parameters associated to devices which
are not directly observable. Thus, we are interested in a type of network tomography
problem. To concisely state our results, we begin by formalizing the discussion.
Let $\Lambda = (V, E)$ be a finite tree with root vertex $r$, terminal vertices $V_{\text{term}}(\Lambda) \subset V$, and internal vertices $V \setminus V_{\text{term}}(\Lambda)$ (see section 2 for background and notation). At each terminal vertex of $\Lambda$ glue on a finite linear tree with vertex set of size at least 2 (the size of the linear chain is determined by the geometry of $\Lambda$ and the terminal vertex in question: see section 2). Call the resulting graph, denoted $\Lambda'$, an augmentation of $\Lambda$ and note that $\Lambda$ is naturally embedded in $\Lambda'$.

The proof of Theorem 1.1 (see also Theorem 2.4) relies on a close examination of the structure of the path space associated to the process $X$ and recursion. The argument generalizes that given for chains in [DGM1]; it depends on the tree structure of $\Lambda$.

While we have chosen to present our results in the context of computing, it is clear that Theorem 1.1 should have applications in a variety of applied environments.
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(our interests were originally directed towards inverse problems for one-dimensional diffusions). Indeed, there is a variety of related literature (for applications in medical imaging, see the survey [A]; for an application involving neuroscience, see [BC]). A number of such applications have been discussed by Grünbaum and his collaborators (cf [Gr1], [Gr2], [GM], [P] and references therein). Our results involve different techniques and focus on detailed “time of flight” information, distinguishing it from the work cited above.

2. Background and Notation

Let $\Lambda = (V, E)$ be a finite rooted tree. Thus, $\Lambda$ is a connected graph without cycles, with finite vertex set $V$ and edge set $E \subset V \times V$, and a distinguished element $r \in V$. We will say that a vertex $v$ is terminal if there is exactly one vertex $u$ such that $uv \in E$. We will write $V_{\text{term}}(\Lambda)$ for the set of terminal vertices. We refer to vertices which are not terminal as internal vertices.

By a path in $\Lambda$ we will mean an ordered tuple of vertices, $(v_1, v_2, \ldots, v_n)$ where $v_i v_{i+1} \in E$ for all $i < n$. Given a path $\gamma = (v_1, v_2, \ldots, v_n)$ we will say that $\gamma$ connects $v_1$ to $v_n$. We write $P_{uv}$ for the collection of paths connecting $u$ and $v$.

Associated to every path $\gamma$ is a length: the length of $\gamma$, denoted by $l(\gamma)$, is the number of edges defined by $\gamma$ (i.e. if $\gamma = (v_1, v_2, \ldots, v_n)$, then $l(\gamma) = n - 1$). There is a natural notion of distance between vertices of $\Lambda$:

$$\text{dist}(u, v) = \min_{\gamma \in P_{uv}} \{l(\gamma)\}.$$ 

The length function and the root give rise to a norm:

$$|v| = \text{dist}(v, r).$$

The norm gives rise to a partition of the vertices of $\Lambda$ by shells:

$$V_k(\Lambda) = \{v \in V(\Lambda) : |v| = k\}.$$ 

Because $\Lambda$ is a tree, if $v \in V_{k+1}$, there is a unique vertex $v^* \in V_k$ such that $v^*v$ is an edge.

A simple but important example is given by a discrete interval: given integers $-k \leq 0 < l$, the $(-k, l)$-segment is the rooted tree with $k+l+1$ vertices obtained by taking as vertices the integers $\{-k, -k+1, \ldots, l\}$, and edges given by $i(i+1)$, $-k \leq i < l$, and root vertex 0 (when $k = 0$ we will refer to the corresponding segment as the $l$-segment). For this example, the shells of $\Lambda$ contain either one or two vertices.

If $\Lambda$ is a finite tree with root $r$, we can associate to $\Lambda$ an outer radius and an inner radius:

$$R_{\text{out}}(\Lambda) = \max_{v \in V(\Lambda)} \{|v|\}$$
$$R_{\text{in}}(\Lambda) = \min_{v \in V_{\text{term}}(\Lambda)} \{|v|\}.$$ 

We will say that $\Lambda$ is spherical if $R_{\text{out}}(\Lambda) = R_{\text{in}}(\Lambda)$. Note that for spherical rooted trees we have

$$V_{R_{\text{out}}(\Lambda)} = V_{\text{term}}(\Lambda).$$

Definition 2.1. Suppose that $\Lambda$ is a rooted tree, $v \in V_{\text{term}}(\Lambda)$, and $l \in \mathbb{N}$. The $l$-augmentation of $\Lambda$ at $v$ is the finite rooted tree obtained by gluing a copy of the $l$ segment to $\Lambda$ by identifying the root of the $l$-segment with the terminal vertex $v$ and taking the root of the resulting tree to be the root of $\Lambda$. We say that the rooted tree $\Lambda'$ is an augmentation of $\Lambda$ if $\Lambda'$ is obtained from $\Lambda$ by a series of augmentations at
boundary vertices. We say that \( \Lambda' \) is an \( m \)-complete augmentation of \( \Lambda \) if \( \Lambda' \) is an augmentation of \( \Lambda \) which can be obtained by first performing an \( m \)-augmentation of \( \Lambda \) at each terminal vertex of \( \Lambda \).

Thus, if we \( l \)-augment the \( l \)-interval at \( 0 \in V_{\text{term}}([0, l]) \) we obtain the \((-l, l)\)-interval. Similarly, given \( n \) copies of the \( l \)-interval, we can glue them together at \( 0 \) and take as root the gluing point to obtain a spherical rooted tree of radius \( l \) which we will refer to as the \((l, n)\)-star.

Given a rooted tree \( \Lambda \) we can always construct an augmentation of \( \Lambda \) which is spherical:

**Definition 2.2.** Suppose that \( \Lambda \) is a rooted tree and that \( l \) is any natural number. For each \( v \in V_{\text{term}}(\Lambda) \), perform an \((R_{\text{out}}(\Lambda) - |v| + l)\)-augmentation of \( \Lambda \) at \( v \). The resulting augmentation is called the \( l \)-spherical augmentation of \( \Lambda \).

Thus, if \( \Lambda \) is a finite rooted tree and \( \Lambda' \) is the \( l \)-spherical augmentation of \( \Lambda \), then \( \Lambda' \) is spherical and

\[
R_{\text{in}}(\Lambda') = R_{\text{out}}(\Lambda') = R_{\text{out}}(\Lambda) + l.
\]

Given a finite rooted tree \( \Lambda \) and an \( l \)-spherical augmentation \( \Lambda' \), suppose that \( X \) is a nondegenerate simple Markov chain on \( \Lambda' \). We are interested in determining the structure of \( X \) on \( \Lambda \) via first passage probabilities at the boundary of \( \Lambda' \). More precisely,

**Definition 2.3.** Let \( \Lambda \) be a finite tree with root \( r \) and \( \Lambda' \) an \( l \)-spherical augmentation of \( \Lambda \). Let \( X \) be a nondegenerate simple Markov chain on \( \Lambda' \). We say that \( X \) is determined by \( l \)-spherical first hitting times if the transition probabilities for vertices in \( \Lambda \) are completely determined by the triple \((W, P_{\text{out}}^r, P_{\text{in}}^r)\) where

- \( W \) is the collection of transition probabilities for vertices \( V(\Lambda') \setminus V(\Lambda) \),
- \( P_{\text{out}}^r \) is the joint distribution of hitting time and hitting place of \( X \) started at the root for vertices in the shell \( V_{R_{\text{out}}(\Lambda')} \), and
- \( P_{\text{in}}^r \) is the joint distribution of hitting time and hitting place of \( X \) started at the root for vertices in the shell \( V_{R_{\text{out}}(\Lambda') - 1} \).

We can now concisely state our main theorem:

**Theorem 2.4.** Let \( \Lambda \) be a finite rooted tree and suppose that \( X \) is a nondegenerate simple Markov chain on the \( 2 \)-spherical augmentation of \( \Lambda \). Then \( X \) is determined by \( 2 \)-spherical first hitting times.
Before proceeding to the proof of Theorem 2.4, we note that the main result of [DGM1] establishes an important special case:

**Theorem 2.5.** [DGM1] Let $k$ and $l$ be natural numbers and suppose that $X$ is a simple nondegenerate Markov chain on the $2$-spherical augmentation of the $(-k, l)$-interval. Then $X$ is determined by $2$-spherical first hitting times.

Thus, as noted in [DGM1], there are simple counterexamples to the most straightforward generalizations of Theorem 2.4 (in particular, it is, in general, impossible to determine transition probabilities with a single pair of symmetrically placed detectors, i.e. with a single boundary layer).

The proof of Theorem 2.4 involves an extension of the ideas developed for the proof of Theorem 2.5. As an important illustrative example of how the argument proceeds, we fix a natural number $m$ and study the special case of the $(1, m)$-star.

Enumerate the vertices of $(1, m)$-star $\Lambda$ as $\{v_i\}_{i=0}^{m}$ with $v_0 = 0$.

Using the norm to partition the vertices by shells (cf (2.1)), we write

- $V_2 = \{v_{m+j}\}_{j=1}^{m}$ where there is an edge between $v_j$ and $v_{m+j}$
- $V_3 = \{v_{2m+j}\}_{j=1}^{m}$ where there is an edge between $v_{m+j}$ and $v_{2m+j}$.

Let $\tau_{\text{in}}$ be the first hitting time of $V_2$

$$\tau_{\text{in}} = \inf\{n \geq 0 : X_n \in V_2\}$$

and similarly, let $\tau_{\text{out}}$ be the first hitting time of $V_3$. We will write the joint distribution of first hitting time and first hitting place as in Definition 2.3:

$$P_{\text{in}}^0(\tau_{\text{in}} = k, X_{\tau_{\text{in}}} = v_{m+j})$$

$$P_{\text{out}}^0(\tau_{\text{out}} = k, X_{\tau_{\text{out}}} = v_{2m+j}).$$

We write $t_{j,l}$ for the probability of transitioning from vertex $v_j$ to vertex $v_l$ in one time step. Then

$$P_{\text{in}}^0(2, j) = t_{0,j}t_{j,m+j}$$
$$P_{\text{out}}^0(3, j) = t_{0,j}t_{j,m+j}t_{m+j,2m+j}. \quad (2.5)$$

Similarly,

$$P_{\text{in}}^0(4, j) = P_{\text{in}}^0(2, j) \sum_{i=1}^{m} t_{0,i}t_{i,0} \quad (2.6)$$
$$P_{\text{out}}^0(5, j) = P_{\text{out}}^0(3, j) \sum_{i=1}^{m} t_{0,i}t_{i,0} + t_{m+j,3}t_{j,m+j}. \quad (2.7)$$

We conclude

$$t_{j,m+j} = \frac{1}{t_{m+j,j}} \left[ P_{\text{out}}^0(5, j) - P_{\text{out}}^0(3, j) \right] - P_{\text{in}}^0(4, j) - P_{\text{in}}^0(2, j) \quad (2.8)$$

from which it follows that the transition probabilities $t_{j,m+j}$ are determined for all $j$. From (2.4), it follows that the transition probabilities $t_{0,j}$ are determined for all $j$. Since the walk is by assumption simple, the transition probabilities $t_{j,0}$ are determined for all $j$. We conclude:

**Lemma 2.6.** For every $m$, every nondegenerate simple Markov chain on the $(1, m)$-star is determined by $2$-spherical first hitting times.
3. Proof of Theorem 2.4

To establish the general case, we begin with an observation: Every rooted tree naturally embeds in its spherical augmentations. Thus,

**Lemma 3.1.** Theorem 2.4 is true for general rooted trees if and only if it is true for all spherical rooted trees.

Let $\Lambda$ be a spherical rooted tree. To prove Theorem 2.4 we give a careful analysis of the structure of paths beginning at the root and having certain prescribed hitting properties for the outer shells of $\Lambda'$ where $\Lambda'$ is a (general) spherical augmentation of $\Lambda$. We first demonstrate that we can recursively determine transition probabilities associated to terminal vertices of $\Lambda'$. Using this result and the methods employed to obtain it, we establish a recursion algorithm for determining all unknown transition probabilities.

To this end, let $R > 0$ be the radius of the spherical rooted tree $\Lambda$. Let $m > 0$ and let $\Lambda'$ be the $(m + 1)$-spherical augmentation of $\Lambda$. We refer to elements of $V_{R+m+1}$ (respectively, $V_{R+m}$) as outer boundary vertices (respectively, inner boundary vertices). Fix a vertex $v \in V_{R+m+1}$ and define (for the remainder of the paper)

$$ T = (R + m + 1) + 2m. \tag{3.1} $$

We consider paths beginning at the root and having $T$ as the first hitting time for the vertex $v \in V_{R+m+1}$. More precisely, we define

$$ \Gamma_v = \{ \gamma : \gamma(0) = r, \gamma(T) = v, \gamma(j) \neq v \text{ for all } j < T \}. \tag{3.2} $$

Elements of $\Gamma_v$ do not visit most boundary vertices of $\Lambda'$:

**Lemma 3.2.** Let $v \in V_{R+m+1}$ and suppose $v^* \in V_{R+m}$ is the unique vertex such that $v^*v$ is an edge. For $\Gamma_v$ as in (3.2), if $\gamma \in \Gamma_v$, then $\gamma(j) \notin V_{R+m} \setminus \{v^*\}$ for all $j \leq T$.

**Proof.** Let $\gamma$ be a curve in $\Gamma_v$ and suppose $|\gamma(j)| = R + m, \gamma(j) \neq v^*$. Then $j > R + m - 1$. Denote by $\tau$ the first time that $\gamma$ visits $v$. Then

$$ \tau \geq j + (m + 1) + (m + 2) > R + m - 1 + 3 + 2m. $$

We conclude that $\tau > T$, which completes the proof of the lemma.

We partition $\Gamma_v$ by first hitting times of $v^*$:

**Lemma 3.3.** Let $v \in V_{R+m+1}$ be an element of the outer boundary layer of $\Lambda'$ and let $\Gamma_v$ be as in (3.2). Suppose that $v^*$ is the unique vertex for which $v^*v$ is an edge and define

$$ \Gamma_{v,i} = \{ \gamma \in \Gamma_v : \gamma(T - (2i - 1)) = v^*, |\gamma(j)| < R + m \text{ for all } j < T - (2i - 1) \}. \tag{3.3} $$

Then

1. $\Gamma_{v,i} \cap \Gamma_{v,j} = \emptyset$ if $i \neq j$
2. $\bigcup_{i=1}^{m+1} \Gamma_{v,i} = \Gamma_v$. 


Lemma 3.5. We have: of an initial segment which has nice first hitting properties, followed by an end segment which never visits the tree $\Lambda$.

Lemma 3.4. We make this precise: Denoting starting positions by a superscript, we write

\[ \chi_{v,l} \]

P as required. By choice of $\gamma

Proof The first statement is obvious. By definition, for $\gamma \in \Gamma_{v,l}$, the first hitting time of $v^*$ is $T - (2l - 1)$. From this and Lemma 3.4, we conclude that (2) and (3) hold.

Truncation provides for a decomposition of paths in $\Gamma_{v,l}$: Each such path consists of an initial segment which has nice first hitting properties, followed by an end segment which never visits the tree $\Lambda$. We make this precise: Denoting starting positions by a superscript, we write

\[ (3.4) \quad \mathbb{P}^\gamma(t, v) = \text{Probability}(\{\text{starting at } r, X \text{ first hits } v \text{ at time } t\}). \]

We have:

Lemma 3.5. Let $\Gamma_{v,l}$ be as defined in lemma 3.3. For $1 \leq l \leq m + 1$,

\[ (3.5) \quad \mathbb{P}^\gamma(\Gamma_{v,l}) = \mathbb{P}^\gamma_{in}(T - (2l - 1), v^*) \chi_{v,l} \]

where $\chi_{v,l}$ is an expression which involves only the transition probabilities for vertices $z$ with $R + 1 \leq |z| < R + m + 1$.

Proof By Lemma 3.4, each $\gamma \in \Gamma_{v,l}$ can be decomposed as a path $\mathcal{C}_l(\gamma)$ starting at the root $r$ with first hitting time of $\{v^*\}$ occurring at time $T - (2l - 1)$, followed by a path of length $(2l - 1)$ which begins at $v^*$ and ends when it makes its first visit to $v$. We will write

\[ \Gamma_{v,l} = \{\gamma: \gamma(0) = v^*, \gamma(2l - 1) = v, \gamma(j) < R + m + 1 \text{ for all } j < 2l - 1\}. \]

(3.6)

By choice of $l$, if $\gamma \in \Gamma_{v,l}$, $|\gamma(j)| > R$ for all $0 \leq j \leq 2l - 1$. Thus, if $\gamma \in \Gamma_{v,l}$ we can compute $\mathbb{P}^\gamma(\{\gamma\})$ in terms of the transition probabilities associated to vertices $z$ with $R < |z| \leq R + m$. Summing over all elements $\Gamma_{v,l}$ gives an expression

\[ \chi_{v,l} = \mathbb{P}^\gamma(\Gamma_{v,l}) \]

which involves only the transition probabilities for vertices $z$ with $R < |z| \leq R + m$.

Finally, using Lemma 3.4, we compute $\mathbb{P}^\gamma(\Gamma_{v,l})$

\[ (3.6) \quad \mathbb{P}^\gamma(\Gamma_{v,l}) = \mathbb{P}^\gamma_{in}(T - (2l - 1), v^*) \chi_{v,l} \]

as required.

The next result establishes that the transition probabilities at the terminal vertices of $\Lambda$ are $m$-spherically determined by first hitting times. It is also the first step in an inductive proof of Theorem 2.4.
Lemma 3.6. Let \( m > 1 \). Let \( \Lambda \) be a spherical tree with root \( r \) and radius \( R \), \( \Lambda' \) the \((m+1)\)-spherical augmentation of \( \Lambda \). Let \( u \in V_R \), and suppose \( w \) is the unique vertex of \( \Lambda' \) for which \( uw \) is an edge of \( \Lambda' \). Then the transition probability \( t_{uw} \) is determined by the triple \((W, \mathbb{P}_{out}^{\gamma}, \mathbb{P}_{in}^{\gamma})\), where \( W \) is the set of transition probabilities for vertices \( V(\Lambda') \setminus V(\Lambda) \), \( \mathbb{P}_{out}^{\gamma} \) is the joint distribution of first hitting time and first hitting place for \( V_{R+m+1} \), and \( \mathbb{P}_{in}^{\gamma} \) is the joint distribution of first hitting time and first hitting place for \( V_{R+m} \).

Proof. Let \( v \) be a terminal vertex of \( \Lambda' \) a distance \( m+1 \) from \( u \). Let \( T \) be as in (3.1) and let \( v^* \) be the unique vertex such that \( v^*v \) is an edge of \( \Lambda' \). From Lemma 3.3 and Lemma 3.5 we have

\[
\mathbb{P}^r(T, v) = \mathbb{P}^r(\Gamma)
\]

\[
= \sum_{l=1}^{m} \mathbb{P}_{in}^{\gamma}(T - (2l - 1), v^*)\chi_{v,l} + \mathbb{P}^r(\Gamma_{v,m+1}).
\]

We let \( \gamma_* \) be the element of \( \Gamma_{v,m+1} \) which “changes direction exactly twice.” From (3.1) and (3.3), \( \gamma_* \) is the path which starts at \( r \), moves out radially \( R + m \) units, moves in radially \( m \) units and moves out radially \( m + 1 \) units. Since we know all transitions associated to \( \gamma_* \) we can explicitly compute the probability that \( \gamma_* \) occurs:

\[
\mathbb{P}^r(\{\gamma_*\}) = \chi_* t_{uw}
\]

where

\[
\chi_* = \mathbb{P}_{out}^{\gamma}(R + m + 1, v)\rho_{uv}
\]

and \( \rho_{uv} \) involves only transition probabilities along the path of length \( m + 1 \) from \( u \) to \( v \) (if we write the unique such path as \( u_0 u_1 \ldots u_{m+1} \) with \( u_0 = u \) and \( u_{m+1} = v \), then \( \rho_{uv} = \prod_{i=0}^{m+1} t_{u_i+1,u_i} \)).

Recall, an element of \( \Gamma_{v,m+1} \) starts at position \( r \), first hits position \( v \) at time \( R + m + 1 + 2m \), and first hits position \( v^* \) at time \( R + m \). Thus, if \( \gamma \in \Gamma_{v,m+1} \setminus \{\gamma_*\} \), then, as in Lemma 3.5, we may view \( \gamma \) as a truncation followed by a path which never visits \( u \). Thus, as in Lemma 3.5, we can write

\[
\mathbb{P}^r(\Gamma_{v,m+1} \setminus \{\gamma_*\}) = \mathbb{P}_{in}^{\gamma}(R + m, v^*)\chi_{v,m+1}
\]

where \( \chi_{v,m+1} \) depends only on transition probabilities for vertices \( z \) such that \( R < |z| \leq R + m \). Using (3.7), (3.8), and (3.10) we can solve for \( t_{uw} \):

\[
t_{uw} = \frac{1}{\chi_*} \left[ \frac{\mathbb{P}_{out}^{\gamma}(T, v)}{\sum_{l=1}^{m+1} \mathbb{P}_{in}^{\gamma}(T - (2l - 1), v^*)\chi_{v,l}} \right].
\]

This completes the proof of the lemma.

The next result provides for the inductive step in the proof of Theorem 2.4

Lemma 3.7. Let \( \Lambda \) be a rooted tree, \( \Lambda' \) the 2-spherical augmentation of \( \Lambda \). Suppose that \( X \) is a simple nondegenerate Markov chain on \( \Lambda' \), and that the transition probabilities for the vertices \( V(\Lambda') \setminus (\bigcup_{j=0}^{k} V_j) \) are known. Let \( u \in V_k \) and \( w \in V_{k+1} \) be such that \( uw \) is an edge. Then the transition probability \( t_{uw} \) is determined by the transition probabilities at the vertices \( V(\Lambda') \setminus (\bigcup_{j=0}^{k} V_j) \) and the joint distributions of first hitting time and place, \( \mathbb{P}_{out}^{\gamma} \) and \( \mathbb{P}_{in}^{\gamma} \).
Proof Let $V_{\text{term}}(w, \Lambda')$ be the terminal vertices of $\Lambda'$ which can be connected to $w$ by a path of length $R + 2 - (k + 1)$. Let

$$V^*_{\text{term}}(w, \Lambda') = \{ v^* \in V_{R+1} : \text{there exists } v \in V_{\text{term}}(w, \Lambda') \text{ with } v^*v \text{ an edge} \}.$$ 

Set

$$T_w = R + 1 + (R + 1 - k) + (R + 2 - k) = 3R + 4 - 2k$$

and let

$$\Gamma_{uw} = \{ \gamma : \gamma(0) = r, \gamma(T_w) \in V_{\text{term}}(w, \Lambda'), \gamma(j) \notin V_{\text{term}}(w, \Lambda') \text{ for all } j < T_w \}.$$ 

As in Lemma 3.2, if $\gamma \in \Gamma_{uw}$, then $\gamma(j) \notin V_{R+1} \setminus V^*_{\text{term}}(w, \Lambda')$ for all $j < T_w$. For $1 \leq l \leq k + 1$, set

$$\Gamma_{uw,l} = \{ \gamma \in \Gamma_{uw} : \gamma(T_w - (2l - 1)) \in V^*_{\text{term}}(w, \Lambda'), |\gamma(j)| < R + 1 \text{ for all } j < T_w - (2l - 1) \}.$$

Then, as in Lemma 3.3, the sets $\Gamma_{uw,l}$ partition $\Gamma_{uw}$. Moreover, as in Lemma 3.4 paths behave nicely under truncation in that for all $l$ with $1 \leq l \leq k + 1$, paths in $\Gamma_{uw,l}$ have truncations which start at the root $r$ and first hit $V_{R+1} \setminus V^*_{\text{term}}(w, \Lambda')$ at time $T_w - (2l - 1)$. As in the case $k = R$, if $1 \leq l < k + 1$, a path $\gamma \in \Gamma_{uw,l}$ can be decomposed as a path $C_\ell(\gamma)$ which starts at the root $r$ and first hits $V_{R+1} \setminus V^*_{\text{term}}(w, \Lambda')$ at time $T_w - (2l - 1)$, followed by a path that never visits the $k$ shell $V_k(\Lambda)$. As in Lemma 3.6, we conclude

$$P_r(\Gamma_{uw,l}) = \sum_{v^* \in V^*_{\text{term}}(w, \Lambda')} P^r_{\text{in}}(T_w - (2l - 1), v^*) \chi_{v^*,l}$$

where $\chi_{v^*,l}$ is an expression which involves only the transition probabilities for vertices $z$ with $k + 1 \leq |z| < R + 2$. As in Lemma 3.6, the paths $\Gamma_{uw,k+1}$ contain a distinguished subset of elements: those with an initial segment which moves to a radial distance of $R + 1$ in time $R + 1$, followed by a segment that moves in a radial distance of $k$ units in $k$ time units, followed by a segment which moves a radial distance of $k + 1$ units (see 3.13). If we denote this subset by $\Gamma^*_{uw}$, then, as in Lemma 3.6, we have

$$P_r(\Gamma^*_{uw}) = t_{uw} \left[ \sum_{v \in V_{\text{term}}(w, \Lambda')} P^r_{\text{out}}(R + 2, v) \rho_{uw} \right]$$

where $\rho_{uw}$ involves only transition probabilities along the path from $w$ to $v$ (and these transition probabilities are by assumption known). Finally, if $\gamma \in \Gamma_{uw,k+1} \setminus \Gamma^*_{uw}$, then $\gamma(R + 1) \in V^*_{\text{term}}(w, \Lambda')$ and $\gamma(R + 1 + 2k) \in V_{\text{term}}(w, \Lambda')$ which implies that $\gamma$ does not visit the $k$ shell $V_k(\Lambda)$ once it has left it. We conclude

$$P_r(\Gamma_{uw,k+1} \setminus \Gamma^*_{uw}) = \sum_{v^* \in V^*_{\text{term}}(w, \Lambda')} P^r_{\text{in}}(R + 1, v^*) \chi_{v^*}$$

where $\chi_{v^*}$ involves only transition probabilities for vertices $z$ satisfying $k < |z| \leq R + 1$. Using 3.15, 3.16, and 3.17, we can, as in Lemma 3.6, solve for $t_{uw}$. This concludes the proof of the lemma.
The proof is recursive; an induction on distance to the inner boundary of the 2-spherical augmentation. The formal argument is as follows:

By Lemma 2.6 it suffice to consider the case of spherical trees Λ of radius $R > 0$ is arbitrary. Let $\Lambda'$ be the 2-spherical augmentation of Λ and for $u \in V(\Lambda)$, let $d = d(u)$ be the distance of $u$ from the inner boundary, $V_{R+1}(\Lambda')$, of $\Lambda'$. If $d = 1$, then by Lemma 3.6, $t_{uw}$ is determined by $P_{\text{in}}$ and $P_{\text{out}}$. If the result holds when $d = k - 1$, by Lemma 3.7 it is true when $d = k$. This finishes the proof.

From the proofs of Lemma 3.6 and Lemma 3.7, we note that, given a finite rooted tree, Λ, embedded in its 2-spherical augmentation $\Lambda'$ and a simple nondegenerate Markov chain, we only require a finite number of values of the joint distribution of exit time and place to determine the transition probabilities for a simple nondegenerate Markov chain on the embedded tree. More precisely, we have:

**Corollary 3.8.** Let Λ be a rooted tree with outer radius $R_{\text{out}}(\Lambda)$. Let $\Lambda'$ be the 2-spherical augmentation of Λ and suppose that $X$ is a simple nondegenerate Markov chain of $\Lambda'$. Then there is an algorithmic procedure for explicitly determining the transition probabilities of $X$ on Λ from the 2-spherical hitting times. The algorithm depends on data from the joint distribution of exit time and place for time $t \leq 3R+4$.

**Proof** From the proof of Lemma 3.7 to determine transition probabilities for elements of the $k$ shell $V_k(\Lambda)$ we need to sample times up to $3R+4 - 2k$ (cf. (3.13)). The corollary follows immediately.

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