THE SYMPLECTIC LEAVES FOR THE ELLIPTIC POISSON BRACKET ON PROJECTIVE SPACE DEFINED BY FEIGIN-ODESSKII AND POLISHCHUK

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ABSTRACT. This paper determines the symplectic leaves for a remarkable Poisson structure on $\mathbb{CP}^{n-1}$ discovered by Feigin and Odesskii, and, independently, by Polishchuk. It is determined by a holomorphic line bundle of degree $n \geq 3$ on a compact Riemann surface of genus one.

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2020 Mathematics Subject Classification. 53D17, 14H52.
Key words and phrases. elliptic Poisson bracket, symplectic leaves, elliptic curve, elliptic algebras.
1. Introduction

We always work over the field of complex numbers, \( \mathbb{C} \).

Always, \( E = (E, 0, +) \) denotes an elliptic curve over \( \mathbb{C} \).

An elliptic normal curve \( E \subseteq \mathbb{P}^{n-1} \) is a degree-\( n \) elliptic curve that is not contained in any hyperplane. We always assume \( n \geq 3 \).

1.1. “Elliptic” Poisson brackets on projective spaces. In 1998, Feigin and Odesskii [FO98] and, independently, Polishchuk [Pol98], discovered a remarkable family of Poisson structures on projective spaces: in short, an elliptic normal curve \( E \subseteq \mathbb{P}^{n-1} \) determines a Poisson bracket, \( \Pi_E \), on \( \mathbb{P}^{n-1} \).

Given an elliptic normal curve \( E \subseteq \mathbb{P}^{n-1} \), Feigin-Odesskii defined \( \Pi_E \) via an explicit, though mysterious, formula involving theta functions. In sharp contrast, Polishchuk defined \( \Pi_E \) in abstract terms without explicit formulas. Twenty years later Hua and Polishchuk showed these two Poisson structures are the same [HP18, Thm. 5.2].

Given an elliptic normal curve \( E \subseteq \mathbb{P}^{n-1} \), \( n \geq 3 \), we write \( \Pi_E \) for the associated Poisson bracket on \( \mathbb{P}^{n-1} \). It is often called the Feigin-Odesskii, or Feigin-Odesskii-Sklyanin, bracket, and is often denoted by \( g_{n,1}(E) \). Polishchuk’s name should also be attached to \( \Pi_E \). As a consequence, \( \mathbb{P}^{n-1} \) can be identified with \( \mathbb{P}H^0(E, \mathcal{L})^* \) and hence, via Serre duality, with the set of isomorphism classes of non-split extensions\(^3\) of \( \mathcal{L} \) by \( O_E \); i.e., \( \mathbb{P}^{n-1} \) can be identified with

\[
\mathbb{P}_\mathcal{L} := \mathbb{P} \text{Ext}^1(\mathcal{L}, O_E),
\]

and \( \Pi_E \) can be defined in terms of the geometry associated to such extensions.

We always identify \( E \) with its image under the composition \( E \to \mathbb{P}H^0(E, \mathcal{L})^* \to \mathbb{P} \text{Ext}^1(\mathcal{L}, O_E) = \mathbb{P}_\mathcal{L} \).

As for any Poisson bracket on a complex manifold, there is a unique stratification of \( \mathbb{P}_\mathcal{L} \) such that the restriction of \( \Pi_E \) to each stratum is non-degenerate; i.e., each stratum is a symplectic manifold with respect to the restriction of \( \Pi_E \). These strata are called the symplectic leaves for \( \Pi_E \). Since \( \Pi_E \) was first discovered several groups of mathematicians have wanted a description of the symplectic leaves in terms of the geometry related to \( E \) as a subvariety of \( \mathbb{P}_\mathcal{L} \).

This paper solves the problem for all \( n \geq 3 \) (the solution for \( n = 3, 4 \) is folklore).

1.2. Homological leaves. An extension of \( \mathcal{L} \) by \( O_E \) is a rank-two locally free \( O_E \)-module whose determinant is \( \mathcal{L} \). Given an extension \( \xi \in \mathbb{P}_\mathcal{L} \), represented by the non-split exact sequence \( 0 \to O_E \to \mathcal{E} \to \mathcal{L} \to 0 \), we call \( \mathcal{E} \) the middle term of \( \xi \) and denote it by \( m(\mathcal{E}) \). Thus

\[
m : \mathbb{P}_\mathcal{L} \to \text{Bun}(2, \mathcal{L}) := \text{the moduli space of rank-two bundles on } E \text{ such that } \det \mathcal{E} \cong \mathcal{L},
\]

For each \( \mathcal{E} \in \text{Bun}(2, \mathcal{L}) \) we define

\[
L(\mathcal{E}) := \{ \xi \in \mathbb{P}_\mathcal{L} | m(\xi) \cong \mathcal{E} \} = m^{-1}(\mathcal{E}).
\]

We call \( L(\mathcal{E}) \) a homological leaf. Clearly, \( \mathbb{P}_\mathcal{L} \) is the disjoint union of the \( L(\mathcal{E}) \)'s.

Even if one ignores the “symplectic” origin of the problem, the classification of homological leaves is a natural problem in a wide range of settings. A solution to such a problem consists of

(1) a “list” of those \( \mathcal{E} \)'s for which \( L(\mathcal{E}) \neq \emptyset \);
(2) a geometric description of each \( L(\mathcal{E}) \);
(3) a determination of whether \( L(\mathcal{E}) \) is a quasi-affine, quasi-projective, affine, or projective variety;
(4) answers to questions like: what is the dimension of \( L(\mathcal{E}) \)? when is one leaf contained in the closure of another? And so on.

\(^3\)See Appendix A for the definition of isomorphic extensions. If we fix \( \mathcal{A} \) and \( \mathcal{C} \) such that \( \text{End}(\mathcal{A}) \cong \text{End}(\mathcal{C}) \cong \mathbb{C} \), then \( \mathbb{P} \text{Ext}^1(\mathcal{C}, \mathcal{A}) \) is in natural bijection with the set of isomorphism classes of extensions that are isomorphic to a non-split extension of the form \( 0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \) (Proposition A.1). These definitions make sense in a larger context: if \( X \) is a projective scheme over an algebraically closed field \( k \), one can fix \( \mathcal{A}, \mathcal{C} \in \text{coh}(X) \) such that \( \text{End}(\mathcal{A}) \cong \text{End}(\mathcal{C}) \cong k \) and address the question of determining the homological leaves in \( \mathbb{P} \text{Ext}^1(\mathcal{C}, \mathcal{A}) \). Another interesting case would be to replace \( \text{coh}(X) \) by the category of left modules over a finite dimensional \( k \)-algebra.
We answer these and other questions about the $L(\mathcal{E})$’s.

In their 1998 paper, Feigin and Odesskii claimed without proof that the symplectic leaves for $\Pi_E$ are precisely the homological leaves [FO98, Thm. 1, p.66]. Here we prove that the $L(\mathcal{E})$’s are indeed the symplectic leaves (Theorem 5.9). Before proving this one must show that every $L(\mathcal{E})$ is smooth of even dimension (Theorem 5.6).

Before stating our main results in §1.4, we need the notation in §1.3.

1.3. **Higher secant varieties to $E$.** Let

$$ E^{[d]} := \text{the $d^{th}$ symmetric power of $E$}. $$

We write $((x_1, \ldots, x_d))$ for the image in $E^{[d]}$ of $(x_1, \ldots, x_d) \in E^d$. An effective divisor of degree $d$ can be thought of in three ways: as a divisor, as a closed subscheme of $E$ having length $d$, and as a point in $E^{[d]}$. The morphism

$$ \sigma : E^{[d]} \to E, \quad \sigma((x_1, \ldots, x_d)) := x_1 + \cdots + x_d, $$

presents $E^{[d]}$ as a $\mathbb{P}^{d-1}$-bundle over $E$.\footnote{The symbol $\sigma$ reminds us of the symbol $\Sigma$ and the word “sum”.} We also define $\sigma(N) := \sigma(D)$ if $N \cong \mathcal{O}_E(D)$.

All intersections in this paper are scheme-theoretic intersections.

Given an effective divisor $D$ on $E$ we write $\overline{D}$ for its linear span; i.e.,

$$ \overline{D} := \text{the smallest linear subspace $L \subseteq \mathbb{P}_E$ such that $L \cap E$ contains $D$}. $$

We call $\overline{D}$ a secant plane or a $d$-secant if its dimension is $d$.\footnote{The following facts are well-known. If $\deg D \geq n + 1$, then $\overline{D} = \mathbb{P}_E$. If $\deg D = n$ and $\mathcal{O}_E(D) \not\cong \mathcal{L}$, then $\overline{D} = \mathbb{P}_E$. If $\deg D = n$ and $\mathcal{O}_E(D) \cong \mathcal{L}$, then $\dim \overline{D} = n - 2$ and $\overline{D} \cap E = D$. If $\deg D = n - 1$, then $\dim \overline{D} = n - 2$ and $\deg(\overline{D} \cap E) = n$. If $\deg D \leq n - 2$, then $\dim \overline{D} = \deg D - 1$ and $\overline{D} \cap E = D$.}

The $d^{th}$ secant variety to $E$ is

$$ \text{Sec}_d(E) := \bigcup_{D \in E^{[d]}} \overline{D}. $$

(1.2) It is well-known that Sec$_d(E)$ is a closed irreducible subvariety of $\mathbb{P}_E$ of dimension $\min\{2d - 1, n - 1\}$; see [Lan84, Lem. 1 and the Theorem on p. 266] or [Fis, p. 11] or [EH16, Prop. 10.11].

For each $x \in E$, we define

$$ E^{[d]}_x := \sigma^{-1}(x) = \{(x_1, \ldots, x_d) \mid x_1 + \cdots + x_d = x\} $$

and the partial secant variety or secant slice

$$ \text{Sec}_{d,x}(E) := \bigcup_{D \in \sigma^{-1}(x)} \overline{D}. $$

(1.3) This is a closed subvariety of Sec$_d(E)$ of dimension $\min\{2d - 2, n - 1\}$ (Proposition 2.12). Divisors $D$ and $D'$ in $E^{[d]}$ are linearly equivalent if and only if $\sigma(D) = \sigma(D')$ so, if $D \in E^{[d]}_x$, then

$$ \text{Sec}_{d,x}(E) = \bigcup_{D' \sim D} \overline{D}' .$$

(1.4) 

1.4. **Main results.**

1.4.1. **The list of those $\mathcal{E}$’s for which $L(\mathcal{E}) \neq \emptyset$.** Given an integer $d \in [1, \frac{n}{2}]$, a point $x \in E$, and a divisor $D \in E^{[d]}_x$, the isomorphism class of $\mathcal{E}_{d,x} := \mathcal{O}_E(D) \oplus \mathcal{L}(-D)$ does not depend on the choice of $D$.

Suppose $n$ is even. We define $\Omega := \{x \in E \mid 2x = \sigma(\mathcal{L})\}$. For each $\omega \in \Omega$ and divisor $D \in E^{[d]}_{\omega[n/2]}$, the isomorphism class of $\mathcal{L}_\omega := \mathcal{O}_E(D)$ does not depend on the choice of $D$. Let $\mathcal{E}_\omega$ be the unique-up-to-isomorphism non-split extension of $\mathcal{L}_\omega$ by $\mathcal{L}_\omega$ (see §2.3.1). Note that $\mathcal{E}_{-\omega} \cong \mathcal{L}_\omega \oplus \mathcal{L}_\omega$.

When $n$ is odd, we write $\mathcal{E}_o$ for the unique-up-to-isomorphism indecomposable locally free $\mathcal{O}_E$-module of rank two and determinant $\mathcal{L}$.
Theorem 1.1 (Proposition 2.3, Theorem 2.4, and Corollary 3.16). Let $E \in \text{Bun}(2, \mathcal{L})$. Then $L(E) \neq \emptyset$ if and only if either

1. $E$ is indecomposable or
2. $E \cong N_1 \oplus N_2$ where $N_1$ and $N_2$ are invertible $\mathcal{O}_E$-modules of positive degree.

Thus, if $n$ is even then $L(E) \neq \emptyset$ if and only if

$$E \in \{ E_{d,x} \mid d \in [1, \frac{n}{2}] \text{ and } x \in E \} \cup \{ E_\omega \mid \omega \in \Omega \},$$

and if $n$ is odd then $L(E) \neq \emptyset$ if and only if

$$E \in \{ E_{d,x} \mid d \in [1, \frac{n}{2}] \text{ and } x \in E \} \cup \{ E_\omega \}.$$ 

In the rest of §1.4 we assume $E$ is such that $L(E) \neq \emptyset$; i.e., $E$ is either $E_{d,x}$, $E_\omega$, or $E_\omega$.

1.4.2. Geometric description of $L(E)$.

Theorem 1.2 (Theorems 4.5 to 4.7).

1. If $n$ is odd, then
   
   (a) $L(E_{d,x}) = \text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E)$;
   
   (b) $L(E_\omega) = \mathbb{P}_L - \text{Sec}_{\frac{n-1}{2}}(E)$, which is a dense open subset of $\mathbb{P}_L$.

2. If $n$ is even, then
   
   (a) $L(E_{d,x}) = \text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E)$ if $d < \frac{n}{2}$, or $d = \frac{n}{2}$ and $x \notin \Omega$;
   
   (b) $L(E_\omega) \cup L(E_{\frac{n}{2},\omega}) = \text{Sec}_{\frac{n}{2},\omega}(E) - \text{Sec}_{\frac{n-1}{2}}(E)$ if $\omega \in \Omega$;
   
   (c) $L(E_{\frac{n}{2},\omega})$ consists of those points in $\text{Sec}_{\frac{n}{2},\omega}(E) - \text{Sec}_{\frac{n-1}{2}}(E)$ that lie on at least two, and hence infinitely many, distinct $\frac{n}{2}$-secant planes, for each $\omega \in \Omega$;

   (d) $L(E_\omega)$ is a dense open subset of $\text{Sec}_{\frac{n}{2},\omega}(E)$ and consists of those points in $\text{Sec}_{\frac{n}{2},\omega}(E) - \text{Sec}_{\frac{n-1}{2}}(E)$ that lie on a unique $\frac{n}{2}$-secant plane, for each $\omega \in \Omega$.

Theorem 1.3 (Theorems 4.5 to 4.7).

1. If $n$ is odd, then
   
   (a) $\dim L(E_{d,x}) = 2d - 2$;
   
   (b) $\dim L(E_\omega) = n - 1$.

2. If $n$ is even, then
   
   (a) $\dim L(E_{d,x}) = 2d - 2$ if $d < \frac{n}{2}$, or $d = \frac{n}{2}$ and $x \notin \Omega$;
   
   (b) $\dim L(E_{\frac{n}{2},\omega}) = n - 4$ if $\omega \in \Omega$;
   
   (c) $\dim L(E_\omega) = n - 2$ if $\omega \in \Omega$.

In other words, if $n$ is odd, the homological leaves are

1. the individual points $x \in E$, $\{x\} = \text{Sec}_{1,x}(E) = L(E_{1,x})$, and
2. $\text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E)$ for each $x \in E$ and $d \in [2, \frac{n}{2}]$, which has dimension $2d - 2$, and
3. $\mathbb{P}_L - \text{Sec}_{\frac{n-1}{2}}(E)$.

On the other hand, if $n$ is even, the homological leaves are

1. the individual points $x \in E$, $\{x\} = \text{Sec}_{1,x}(E) = L(E_{1,x})$, and
2. $\text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E)$ for each $x \in E$ and $d \in [2, \frac{n}{2}]$, which has dimension $2d - 2$, and
3. $\text{Sec}_{\frac{n}{2},x}(E) - \text{Sec}_{\frac{n-1}{2}}(E)$ for each $x \in E - \Omega$, which has dimension $n - 2$, and
4. the four $L(E_{\frac{n}{2},\omega})$’s of dimension $n - 4$ described in Theorem 1.2(2)(c), and
5. the four $L(E_\omega)$’s of dimension $n - 2$ described in Theorem 1.2(2)(d).

Every point on $E \subseteq \mathbb{P}_L$ is a homological leaf, and therefore a symplectic leaf by [HP19, §5.2]. These are the only 0-dimensional leaves when $n \neq 4$. When $n = 4$ there are four additional 0-dimensional leaves, namely the $L(E_{\frac{n}{2},\omega})$’s for $\omega \in \Omega$.

There are inclusions

$$\text{Sec}_{d,x}(E) \subseteq \text{Sec}_d(E) \subseteq \text{Sec}_{d+1,y}(E) \subseteq \text{Sec}_{d+1}(E)$$
for all $x, y \in E$. From this one can determine when one leaf is contained in the closure of another. If $1 \leq d < \frac{n}{2} - 1$, then, reading from left to right, the dimensions of the varieties in (1-5) are $2d - 2$, $2d - 1$, $2d$, and $2d + 1$, respectively; Proposition 2.12 shows that $\dim \text{Sec}_{d,2}(E) = 2d - 2$; in §1.3 there is a reference for the known equality $\dim \text{Sec}_{d}(E) = \min\{2d - 1, n - 1\}$.

**Proposition 1.4 (Proposition 5.15).**

1. If $n$ is odd, then $L(E_o)$ is affine.
2. If $n$ is even, then $L(E_{x,x})$ is affine for all $x \in E$, and the four $L(E_w)$'s are quasi-affine but not affine.

1.4.3. **Smoothness of $L(E)$ and the symplectic leaves.** Given a Poisson manifold $(X, \Pi)$, its symplectic leaves are, by definition, those submanifolds $Y \subseteq X$ such that $T_xY = \text{the image of } \Pi_x : T_{x}^*X \rightarrow T_xX$ for all $x \in Y$ [CFM21, Prop. 1.8, p.6; (2.10), p.29; Thm. 4.1, p.63]. The following result is needed before showing that the $L(E)$'s are the symplectic leaves.

**Theorem 1.5 (Theorem 5.6).** Every $L(E)$ is smooth of even dimension.

Of course, this is a consequence of the fact that the $L(E)$'s are the symplectic leaves but before proving that we need to know that the $L(E)$'s are smooth quasi-projective varieties.

**Theorem 1.6 (Theorem 5.9).** The symplectic leaves for $(\mathbb{P}_L, \Pi)$ are the homological leaves $L(E)$.

The proof of this theorem makes essential use of a result of Hua and Polishchuk [HP20, Prop. 2.3].

1.5. **Some special cases.**

1. By definition, $\text{Sec}_0(E) = \emptyset$, $\text{Sec}_1(E) = E$, $\text{Sec}_2(E)$ is the union of all the secant lines.
2. If $d \geq \frac{n}{2}$, then $\text{Sec}_d(E) = \mathbb{P}_L$.
3. Each $x \in E$ is a 0-dimensional leaf, $\{x\} = L(E_{1,x})$ where $E_{1,x} \cong O_E(x) \oplus L(-x)$.
4. When $n = 3$, the 0-dimensional leaves are the individual points of $E$ and $\mathbb{P}_L - E$ is the unique 2-dimensional leaf.
5. When $n = 4$, $\{\text{Sec}_{2,x}(E) \mid x \in E\} = \{\text{quadrics containing } E\}$; the union of these quadrics is $\mathbb{P}_L = \mathbb{P}^3$ and, if the embedding of $E$ in $\mathbb{P}^3$ is chosen such that $O(1)|_E \cong O_E(4 \cdot (0))$, then $\text{Sec}_{2,x}(E) = \text{Sec}_{2,-x}(E)$ and is singular if and only if $x \in \Omega$. The symplectic leaves in this case are the individual points of $E$, the vertices of the four singular quadrics that contain $E$, the complement of $E \cup \{\text{the vertex}\}$ in each singular quadric, and the complement of $E$ in each smooth quadric that contains $E$.
6. When $n \geq 5$, the only 0-dimensional leaves are the points of $E$.
7. When $n = 5$, the secant variety $\text{Sec}_2(E)$ is a quintic hypersurface. Its complement, $\mathbb{P}^4 - \text{Sec}_2(E)$, is the unique 4-dimensional leaf; it is $L(E_o)$ where $E_o$ is the unique-up-to-isomorphism rank-two indecomposable locally free $O_E$-module whose determinant is isomorphic to $L$. The 3-dimensional variety $\text{Sec}_2(E) - E$ is the disjoint union of the (2-dimensional) “partial” secant varieties $\text{Sec}_{2,x}(E) - E$, indexed by $x \in E$; $\text{Sec}_{2,x}(E)$ is the union of the secant lines $\overline{y,x-y}$, $y \in E$.

1.6. **Methods.** A key step in our analysis is showing that $L(E)$ is a geometric quotient of a certain subvariety $X(E) \subseteq \text{Hom}(O_E, E)$. In terms of of vector bundles, $X(E)$ consists of the homomorphisms $f : O_E \rightarrow E$ that correspond to embeddings of the trivial bundle $E \times \mathbb{C}$ in the rank-two vector bundle $V(E)$ that corresponds to $E$. In algebraic terms, $X(E)$ consists of the homomorphisms $f : O_E \rightarrow E$ whose cokernel is invertible, and hence isomorphic to $L$. The left action of $\text{Aut}(E)$ on $\text{Hom}(O_E, E)$ leaves $X(E)$ stable so one may consider the geometric quotient $X(E)/\text{Aut}(E)$. This exists and is isomorphic to $L(E)$ via the morphism $X(E) \rightarrow \mathbb{P}_L$ that sends $f$ to the isomorphism class of the non-split extension $0 \rightarrow O_E \rightarrow E \rightarrow \text{coker}(f) \rightarrow 0$. It is easy to see that $\text{Aut}(E)$ acts freely on $X(E)$ so $\dim L(E) = n - \dim \text{Aut}(E)$.

Several other properties of $L(E)$ follow from the fact that $L(E) \cong X(E)/\text{Aut}(E)$. 
The key to relating the leaves to the secant and partial secant varieties is Proposition 4.1, which was inspired by an erroneous remark in [Ber92] (see Remark 4.3). Proposition 4.1 says that a point $\xi \in \mathbb{P}_L$ belongs to $\overline{D}$ if and only if there is a non-zero map $m(\xi) \to \mathcal{O}_E(D)$. In fact, if $D$ has minimal degree such that $\xi \in \overline{D}$, then there is an epimorphism $m(\xi) \to \mathcal{O}_E(D)$ (Theorem 4.2).

1.7. Definitions of $\Pi_E$. In 1989, Feigin and Odesskii defined a family of graded $\mathbb{C}$-algebras $Q_{n,k}(E, \eta)$ that depend on a pair of relatively prime integers $n > k \geq 1$, a point $\eta \in \mathbb{C}$, and on $E$ [FO89, OF89]. Only the algebras $Q_{n,1}(E, \eta)$ are relevant to this paper.

Fix a point $\tau \in \mathbb{H}$, the upper half-plane, and the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, such that $\mathbb{C}/\Lambda \cong E$. The algebra $Q_{n,k}(E, \eta)$ is the free algebra $\mathbb{C}\langle x_0, \ldots, x_{n-1} \rangle$ modulo the $n^2$ homogeneous quadratic relations\(^4\)

\begin{equation}
\sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r}x_{i+r} = 0
\end{equation}

where the indices $i$ and $j$ belong to $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\theta_0, \ldots, \theta_{n-1}$ are certain theta functions of order $n$, also indexed by $\mathbb{Z}_n$, that are quasi-periodic with respect to $\Lambda$. The quasi-periodicity of the $\theta_\alpha$’s implies that $Q_{n,k}(E, \eta)$ depends only on the image of $\eta$ in $E$, so we often treat $\eta$ as a point on $E$.\(^5\)

Most of the $Q_{n,1}(E, \eta)$ are not commutative, but $Q_{n,1}(E, 0)$ is isomorphic to the polynomial ring $\mathbb{C}\langle x_0, \ldots, x_{n-1} \rangle$ on $n$ variables. In fact, the $Q_{n,1}(E, \eta)$’s form a flat family of deformations of that polynomial ring, i.e., the Hilbert series of $Q_{n,1}(E, \eta)$ is the same as that of $\mathbb{C}\langle x_0, \ldots, x_{n-1} \rangle$.\(^6\)

Since $Q_{n,1}(E, 0)$ is a polynomial ring on $n$ variables, the formula

\begin{equation}
\{x_i, x_j\} := \lim_{\eta \to 0} \frac{[x_i, x_j]}{\eta}
\end{equation}

extends to a Poisson bracket on the polynomial ring $Q_{n,k}(E, 0)$. Since $[x_i, x_j]$ is homogeneous of degree two, so is $\{x_i, x_j\}$. It follows that the Poisson bracket on $Q_{n,1}(E, 0)$ induces a Poisson structure on $\text{Proj} Q_{n,1}(E, 0) \cong \mathbb{P}^{n-1}$. In [HP19, §5.2], Hua and Polishchuk give explicit formulas for $\{x_i, x_j\}$ and $\{t_i, t_j\}$, again involving theta functions, where $t_i = \frac{\tau}{x_i}$. We do not need these so we omit them.

In [HP20, Lem. 2.1], Hua and Polishchuk show that the value, $\Pi_\xi$, of $\Pi_E$ at a point $\xi \in \mathbb{P}_L$ can be given by a triple Massey product. In [Pol22, Thm. A(1)], Polishchuk uses this triple Massey product to give a simple explicit formula for $\Pi_E$ that only involves the defining equation(s) for the higher secant variety $\text{Sec}_d(E)$ that is of codimension one in $\mathbb{P}_L$ when $n$ is odd, and of codimension two when $n$ is even (see Proposition 2.13).

1.8. The singular locus of $\text{Sec}_d(E)$. Although we do not need it, we record the fact that if $d \leq \frac{n}{2}$, then

\begin{equation}
\text{Sing}(\text{Sec}_d(E)) = \text{Sec}_{d-1}(E),
\end{equation}

where $\text{Sing}(-)$ denotes the singular locus. At first, the history of this result confused us. The fact that $\text{Sec}_d(E) - \text{Sec}_{d-1}(E)$ is smooth follows from a result proved in 1992 by Bertram [Ber92, Corollary on p. 440].\(^7\) Bertram’s result was also proved in [GvBH04, Prop. 8.15] but that paper doesn’t cite [Ber92]. The first paragraph in [Fis10] says that [GvBH04] proved the equality in (1-8) even though they only proved that $\text{Sing}(\text{Sec}_d(E)) \subseteq \text{Sec}_{d-1}(E)$. The paper [Fis10] “replaces” the unpublished paper [Fis]; the first paragraph in [Fis10], which is essentially the same as that in [Fis], states (1-8); just after [Fis, Thm. 1.4] it is said that (1-8) is a consequence of [Fis, Thm. 1.4] but, paraphrasing Fisher, the proof is omitted because it is closely related to that in [GvBH04, Prop. 8.15]. Fortunately, (the very simple

\(^4\)The original definition uses $x_{(j-r)k+(i+r)}$ instead of $x_{j-r}x_{i+r}$; see [CKS21a, §3.1.1] for an explanation.

\(^5\)Some extra care is needed when the denominator in (1-6) vanishes. This is addressed in [CKS21a]. The $\theta_\alpha$’s are defined in [CKS21a, Prop. 2.6].

\(^6\)To be honest, we don’t know if this is true as $\eta$ ranges over all of $\mathbb{C}$, but it is true for all $\eta$ in some analytic neighborhood of 0 $\in \mathbb{C}$. This does not affect the results in this paper because $Q_{n,1}(E, \eta)$ plays no role in this paper beyond its use by Feigin-Odesskii to define $\Pi_E$; for precise statements regarding the family of $Q_{n,1}(E, \eta)$’s see [CKS20].

\(^7\)That corollary concerns smooth projective curves of arbitrary genus.
argument at) [Cop04, p. 18] shows that if \( X \subseteq \mathbb{P}^{n-1} \) is any smooth irreducible projective variety such that \( \text{Sec}_d(X) \neq \mathbb{P}^{n-1} \) and \( X \) is not contained in any hyperplane, then
\[
\text{(1-9)} \quad \text{Sec}_{d-1}(X) \subseteq \text{Sing}(\text{Sec}_d(X)).
\]
The equality (1-8) follows from this and Bertram’s result.

We will later use the fact that the argument at [Cop04, p. 18] shows more than (1-9): it shows that the dimension of the tangent space to \( \text{Sec}_d(X) \) at a point \( x \in \text{Sec}_{d-1}(X) \) is \( n - 1 \).

1.9. Acknowledgements. A.C. was partially supported by NSF grant DMS-2001186. R.K. was supported by JSPS KAKENHI Grant Numbers JP16H06337, JP17K14164, JP20K14288, and JP21H04994, Leading Initiative for Excellent Young Researchers, MEXT, Japan, and Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JP-MXP0619217849. S.P.S. thanks JSPS (Japan Society for the Promotion of Science) for financial support, and Osaka University and Osaka City University for their hospitality in August 2019 and March 2020 when some of this work was done.

We thank Tom Fisher, Mihai Fulger, Sándor Kovács, Sasha Polishchuk, and Brent Pym for useful conversations.

2. Notation and preliminaries

2.1. The notation \( n, E, L, H, \text{ and } \Omega \). For the rest of the paper, we fix an integer \( n \geq 3 \), a degree-\( n \) elliptic normal curve \( E \subseteq \mathbb{P}^{n-1} \) over \( \mathbb{C} \), and define \( L := \mathcal{O}_{\mathbb{P}^{n-1}}(1)|_E \), which is an invertible \( \mathcal{O}_E \)-module of degree \( n \). Always, we make the identifications
\[
\mathbb{P}^{n-1} = \mathbb{P}H^0(E, L)^* = \mathbb{P}\text{Ext}^1(L, \mathcal{O}_E) =: \mathbb{P}L.
\]
Each \( \xi \in \mathbb{P}L = \mathbb{P}\text{Ext}^1(L, \mathcal{O}_E) \) is represented by a non-split exact sequence\(^8\)
\[
0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E} \longrightarrow L \longrightarrow 0.
\]
The middle term, \( \mathcal{E} \), of \( \xi \) is denoted by \( m(\xi) \). It is an element in
\[
\text{Bun}(2, L) := \{ \text{isomorphism classes of rank-two locally free } \mathcal{O}_E \text{-modules } \mathcal{E} \text{ such that } \text{det } \mathcal{E} \cong L \}.
\]
For each \( \mathcal{E} \in \text{Bun}(2, L) \), we define
\[
L(\mathcal{E}) := m^{-1}(\mathcal{E}) = \{ \xi \in \mathbb{P}\text{Ext}^1(L, \mathcal{O}_E) \mid \text{the middle term of } \xi \text{ is isomorphic to } \mathcal{E} \}.
\]
Symmetric powers \( E^{[d]} \), secant varieties \( \text{Sec}_d(E) \), their subvarieties \( E^{[d]} \subseteq E^{[d]} \) and \( \text{Sec}_{d,x}(E) \subseteq \text{Sec}_d(E) \), and related notations, such as the linear span \( \overline{D} \) of an effective divisor \( D \) and the summation map \( \sigma : E^{[d]} \to E \), are defined in §1.3. We also fix an effective divisor \( H \) such that \( \mathcal{O}_E(H) \cong L \), and define
\[
\Omega := \{ \omega \in E \mid 2\omega = \sigma(H) \}.
\]
Clearly, \( \Omega \) is a coset of the 2-torsion subgroup \( E[2] = \{ x \in E \mid 2x = 0 \} \), and does not depend on the choice of \( H \).

2.2. The assumption on \( \mathcal{E} \). We will say that \( \mathcal{E} \) satisfies the assumptions in §2.2 if
\[
\text{(1) } \mathcal{E} \text{ is a rank-two locally free } \mathcal{O}_E \text{-module,}
\]
\[
\text{(2) } \text{det } \mathcal{E} \cong L, \text{ i.e., } \mathcal{E} \in \text{Bun}(2, L), \text{ and}
\]
\[
\text{(3) } \text{no invertible } \mathcal{O}_E \text{-module of degree } \leq 0 \text{ is a direct summand of } \mathcal{E}.
\]
Theorem 2.4 shows (1), (2), (3) hold if \( \mathcal{E} \in \text{Bun}(2, L) \) is such that \( L(\mathcal{E}) \neq \emptyset \); this is the case of interest.

2.3. The bundles \( \mathcal{E}_{d,x}, \mathcal{E}_o, \text{ and } \mathcal{E}_\omega \text{ in } \text{Bun}(2, L) \).

\(^8\)See Appendix A for the basics on extensions and their isomorphism classes.
2.3.1. Indecomposable bundles in \( \text{Bun}(2, \mathcal{L}) \). The degree of a locally free \( \mathcal{O}_E \)-module is defined to be the degree of its determinant.

Suppose \( n \) is even. If \( \omega \in \Omega \), we define
\[
\mathcal{L}_\omega := \mathcal{O}_E((\omega + \frac{n-2}{2}) \cdot (0)),
\]
\[
\mathcal{E}_\omega := \text{the unique non-split self-extension of } \mathcal{L}_\omega.
\]
We have \( \mathcal{L}_\omega \cong \mathcal{L} \) because \( \mathcal{O}_E(2(\omega) + (n-2) \cdot (0)) \cong \mathcal{O}_E((2\omega) + (n-1) \cdot (0)) \). Up to isomorphism the four \( \mathcal{L}_\omega \)'s are the only \( \mathcal{N} \)'s such that \( \mathcal{N} \cong \mathcal{L} \). By Atiyah's classification of indecomposable bundles on an elliptic curve, the four \( \mathcal{E}_\omega \)'s are pairwise non-isomorphic and, up to isomorphism, are the only indecomposables in \( \text{Bun}(2, \mathcal{L}) \) (see [Ati57, Thm. 7] and [Har77, Cor. V.2.16]).

If \( n \) is odd and \( \mathcal{E} \in \text{Bun}(2, \mathcal{L}) \), then the rank and degree of \( \mathcal{E} \) are relatively prime so, by Atiyah's classification, [Ati57, Cor. (i) to Thm. 7], there is a unique indecomposable \( \mathcal{E} \) in \( \text{Bun}(2, \mathcal{L}) \); this also follows from [Ati57, Thm. 10] with his \((r, d)\) equal to \((2, n)\). Up to isomorphism, that \( \mathcal{E} \) is
\[
\mathcal{E}_o := \text{the unique non-split extension of } \mathcal{O}_E((\omega + \frac{n-1}{2} \cdot (0)) \text{ by } \mathcal{O}_E((\omega + \frac{n-3}{2} \cdot (0)))
\]
where \( \omega \) is any element in \( \Omega \); the isomorphism class of \( \mathcal{E}_o \) does not depend on the choice of \( \omega \) because \( 2(\omega) + (n-2) \cdot (0) \sim (2\omega) + (n-1) \cdot (0) = (\sigma(H)) + (n-1) \cdot (0) \sim H \).

2.3.2. The decomposable \( \mathcal{E} \)'s in \( \text{Bun}(2, \mathcal{L}) \) for which \( L(\mathcal{E}) \neq \emptyset \). If \( d \in [1, \frac{n}{2}] \cap \mathbb{Z} \) and \( x \in E \), we define
\[
\mathcal{E}_{d,x} := \mathcal{O}_E(D) \oplus \mathcal{L}(-D)
\]
where \( D \in E_x^{[d]} \). The isomorphism class of \( \mathcal{O}_E(D) \), and hence that of \( \mathcal{E}_{d,x} \), does not depend on the choice of \( D \). If \( n \) is even and \( \omega \in \Omega \), then \( \mathcal{E}_{d,x,\omega} \cong \mathcal{L}_\omega \oplus \mathcal{L}_\omega \).

The next four results are elementary, but we record them for completeness.

**Lemma 2.1.** Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_E \)-module, and let \( \xi \in \text{Ext}^1(\mathcal{L}, \mathcal{O}_E) \) be the extension
\[
0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0.
\]
It follows that

(1) \( \mathcal{E} \) is locally free of rank two;

(2) \( \text{det } \mathcal{E} \cong \mathcal{L} \);

(3) if \( \mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2 \), where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are invertible \( \mathcal{O}_E \)-modules, then

(a) \( \mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L} \), and

(b) if \( \text{deg } \mathcal{L} \geq 1 \) and \( \xi \neq 0 \), then both \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) have positive degree.

**Proof.** (1) Trivial.

(2) If \( 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0 \) is an exact sequence of locally free \( \mathcal{O}_E \)-modules, then \( \text{det } \mathcal{F} \cong (\text{det } \mathcal{F}_1) \otimes (\text{det } \mathcal{F}_2) \) (see [Har77, Exer. II.5.16 (d)]). Note also that, if \( \mathcal{N} \) is an invertible \( \mathcal{O}_E \)-module, then \( \text{det } \mathcal{N} = \mathcal{N} \).

(3a) This is a special case of (2) because the determinant of a line bundle is itself.

(3b) Suppose \( \text{deg } \mathcal{L} \geq 1 \) and the sequence does not split.

For \( j \in \{1, 2\} \), let \( \pi_j : \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{L}_j \) be the projection.

Suppose \( H^0(\mathcal{E}, \mathcal{L}_1) = 0 \). Then \( \pi_1 f = 0 \), so \( f(\mathcal{O}_E) \subseteq \ker(\pi_1) = \mathcal{L}_2 \). Since \( \text{deg } \mathcal{L}_1 \leq 0 \), \( \text{deg } \mathcal{L}_2 \geq \text{deg } \mathcal{L} \geq 1 \). Thus \( f(\mathcal{O}_E) \) is a proper submodule of \( \mathcal{L}_2 \). Since \( \operatorname{coker}(f) \) contains a copy of \( \mathcal{L}_2 / f(\mathcal{O}_E) \), it is not torsion-free and therefore not isomorphic to \( \mathcal{L} \). This is a contradiction so we conclude that \( H^0(\mathcal{E}, \mathcal{L}_1) \neq 0 \).

Hence \( \text{deg } \mathcal{L}_1 \geq 0 \). We need to show that \( \text{deg } \mathcal{L}_1 > 0 \). Since \( H^0(\mathcal{E}, \mathcal{L}_1) \neq 0 \) we only need to rule out the possibility that \( \mathcal{L}_1 \) is isomorphic to \( \mathcal{O}_E \). Suppose to the contrary that \( \mathcal{L}_1 \cong \mathcal{O}_E \). Then \( \text{deg } \mathcal{L}_2 = \text{deg } \mathcal{L} \geq 1 \). If \( f(\mathcal{O}_E) \subseteq \mathcal{L}_2 \), then \( \mathcal{L}_2 / f(\mathcal{O}_E) \), and hence \( \operatorname{coker}(f) \), is not torsion-free so, as in the previous paragraph, \( \operatorname{coker}(f) \) would not be isomorphic to \( \mathcal{L} \). Hence \( f(\mathcal{O}_E) \nsubseteq \mathcal{L}_2 \). Thus \( \pi_1 f(\mathcal{O}_E) \neq 0 \), whence \( \pi_1 f \) is an isomorphism \( \mathcal{O}_E \rightarrow \mathcal{L}_1 \). Let \( g : \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{O}_E \) be the map \( ((\pi_1 f)^{-1}, 0) \). The image
of \(g \circ f\) is a non-zero map \(\mathcal{O}_E \to \mathcal{O}_E\) so some scalar multiple of \(g\) splits the sequence, contradicting the hypothesis that the sequence does not split.

We therefore conclude that \(\deg \mathcal{L}_1 > 0\). Similarly, \(\deg \mathcal{L}_2 > 0\).

**Lemma 2.2.** Fix invertible \(\mathcal{O}_E\)-modules \(N_1\) and \(N_2\), both having positive degree. If \(\deg(N_1 \otimes N_2) \geq 3\), then there is a non-split extension of the form

\[
0 \to \mathcal{O}_E \to N_1 \oplus N_2 \to N_1 \otimes N_2 \to 0.
\]

**Proof.** Since \(\deg N_1 + \deg N_2 \geq 3\), there are effective divisors \(D_1\) and \(D_2\) such that \(N_i \cong \mathcal{O}_E(D_i)\) and \(D_1 \cap D_2 = \emptyset\). Let \(\mathcal{I}_j\) denote the ideal vanishing on \(D_j\).

Since \(D_1 \cap D_2 = \emptyset\), \(\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{O}_E\) and \(\mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_1 \mathcal{I}_2 \cong \mathcal{O}_E(-D_1 - D_2)\).

The kernel of the map \(\mathcal{I}_1 \oplus \mathcal{I}_2 \to \mathcal{I}_1 + \mathcal{I}_2 = \mathcal{O}_E\), \((a, b) \mapsto a - b\), is \(\mathcal{I}_1 \cap \mathcal{I}_2\). Hence there is an exact sequence \(0 \to \mathcal{I}_1 \mathcal{I}_2 \to \mathcal{I}_1 \oplus \mathcal{I}_2 \to \mathcal{O}_E \to 0\), which does not split because \(D_1\) and \(D_2\) are effective. If we now apply the functor \(\mathcal{O}_E(D_1 + D_2) \otimes -\), which is isomorphic to \(N_1 \otimes N_2 \otimes -\), to the sequence we obtain a non-split sequence of the form \((2.1)\). \(\Box\)

**Proposition 2.3.** If \(\mathcal{E} \in \text{Bun}(2, \mathcal{L})\) is decomposable, then the following are equivalent:

1. \(\mathcal{L}(\mathcal{E}) \neq \emptyset\).
2. \(\mathcal{E} \cong \mathcal{N}_1 \oplus \mathcal{N}_2\) for some invertible \(\mathcal{O}_E\)-modules \(\mathcal{N}_1\) and \(\mathcal{N}_2\), both of positive degree.
3. \(\mathcal{E} \cong \mathcal{E}_{d,x}\) for some \(d \in [1, \frac{n}{2}]\) and \(x \in E\).

**Proof.** Lemmas 2.1 and 2.2 show \((1) \Leftrightarrow (3)\). Since \((3) \Rightarrow (2)\) is obvious, it remains to show \((2) \Rightarrow (3)\). Since \(n = \deg \mathcal{E} = \deg \mathcal{N}_1 + \deg \mathcal{N}_2\), we can assume that \(d := \deg \mathcal{N}_1 \in [0, \frac{n}{2}]\). If \(\sigma(\mathcal{N}_1) = x\), then \(\mathcal{N}_1 \cong \mathcal{O}_E(D)\) for some \(D \in E^{[d]}\). Since \(\mathcal{N}_1 \otimes \mathcal{N}_2 \cong \mathcal{L}\), we conclude that \(\mathcal{N}_2 \cong \mathcal{L}(-D)\) and \(\mathcal{E} \cong \mathcal{E}_{d,x}\). \(\Box\)

**2.3.3. The \(\mathcal{E}\)'s in \(\text{Bun}(2, \mathcal{L})\) for which \(\mathcal{L}(\mathcal{E}) \neq \emptyset\).** We summarize the content of §§2.3.1 and 2.3.2.

**Theorem 2.4.** Let \(\mathcal{E} \in \text{Bun}(2, \mathcal{L})\). If \(\mathcal{L}(\mathcal{E}) \neq \emptyset\), then \(\mathcal{E}\) satisfies the assumptions in §2.2.

More particularly:

1. If \(\mathcal{E}\) is indecomposable and \(n\) is even, then \(\mathcal{E} \cong \mathcal{E}_\omega\) for some \(\omega \in \Omega\).
2. If \(\mathcal{E}\) is indecomposable and \(n\) is odd, then \(\mathcal{E} \cong \mathcal{E}_o\).
3. If \(\mathcal{E}\) is decomposable, then \(\mathcal{E} \cong \mathcal{E}_{d,x}\) for some \(d \in [1, \frac{n}{2}]\) and \(x \in E\). In particular, \(\mathcal{E}\) has no invertible direct summand of degree \(\leq 0\).

**Proof.** In §2.3.1 we made note of the fact that \((1)\) and \((2)\) hold (without assuming that \(\mathcal{L}(\mathcal{E}) \neq \emptyset\)).

(3) Suppose \(\mathcal{E}\) is decomposable. By Proposition 2.3, \(\mathcal{E} \cong \mathcal{E}_{d,x}\) for some \(d \in [1, \frac{n}{2}]\) and \(x \in E\). Since the Krull-Schmidt theorem holds for locally free \(\mathcal{O}_E\)-modules of finite rank ([Ati56, Thm. 3]), \(\mathcal{E}\) does not have an invertible \(\mathcal{O}_E\)-module of degree \(\leq 0\) as a direct summand.

We will prove the converse of Theorem 2.4 in Corollary 3.16; i.e., that \(\mathcal{L}(\mathcal{E}) \neq \emptyset\) for the \(\mathcal{E}\)'s listed in \((1)\), \((2)\), and \((3)\). Since Proposition 2.3 already shows that \(\mathcal{L}(\mathcal{E}_{d,x}) \neq \emptyset\), it remains to show that \(\mathcal{L}(\mathcal{E}) \neq \emptyset\) when \(\mathcal{E}\) is indecomposable. Thus, after Corollary 3.16, we will know that \(\mathcal{E}_{d,x}, \mathcal{E}_\omega, \mathcal{E}_o\) is the complete list of \(\mathcal{E}\)'s such that \(\mathcal{L}(\mathcal{E}) \neq \emptyset\).

**2.4. The linear span of effective divisors on \(E\).** Since \(E\) is a smooth curve, every non-zero ideal in \(\mathcal{O}_E\) is a product of maximal ideals in a unique way. It follows that the map

\[
\{\text{length-}d\ \text{subschemes of } E\} \longrightarrow E^{[d]}
\]

that sends \(\text{Spec}(\mathcal{O}_E/m_{x_1}^{r_1} \cdots m_{x_k}^{r_k})\) to the divisor \(\sum_{i=1}^t r_i(x_i)\) is injective; it is obviously surjective, hence bijective. Accordingly, we do not distinguish between closed subschemes of \(E\) having length \(d\), effective divisors of degree \(d\), and points on \(E^{[d]}\).

Let \(H\) be as in §2.1, and set \(z := \sigma(H)\). Scheme-theoretic intersections of \(E\) with the hyperplanes in \(\mathbb{P}_\mathcal{L}\) are linearly equivalent to one another so all of them belong to \(E_x^{[n]}\); every \(D \in E_x^{[n]}\) arises in this way, and the morphism \((\mathbb{P}_\mathcal{L})^\vee \to E_x^{[n]}\), \(L \mapsto E \cap L\), is an isomorphism whose inverse is the map \(D \mapsto \overline{D}\).

Linear subspaces of \(\mathbb{P}_\mathcal{L}\) of dimension \(d\) are called \(d\)-planes, or planes if we do not specify \(d\).
Proposition 2.5. If \( d \leq n - 2 \) and \( D \in E^{[d]} \), then \( \overline{D} \) is the unique \((d-1)\)-plane \( L \) such that \( E \cap L = D \).\(^9\)

The case \( d = n - 1 \) is a little different.

Proposition 2.6. If \( D \in E^{[n-1]} \), then \( \overline{D} \) is the unique hyperplane \( L \) such that \( D \subseteq E \cap L \). It follows that \( E \cap \overline{D} = D + (x) \) where \( x = \sigma(H) - \sigma(D) \).

If \( D_1 \) and \( D_2 \) are effective divisors we define
\[
\gcd(D_1, D_2) := D_1 \cap D_2, \\
\lcm(D_1, D_2) := D_1 + D_2 - D_1 \cap D_2.
\]

If \( L_1 \) and \( L_2 \) are linear subspaces of \( \mathbb{P}_E \) we define
\[
\langle L_1, L_2 \rangle := \text{the smallest linear subspace that contains } L_1 \cup L_2.
\]

Lemma 2.7 (Fisher). [Fis10, Lem. 2.6] [Fis18, Lem. 9.4] If \( D, D_1, \) and \( D_2 \) are effective divisors on \( E \), then

1. \( \dim \overline{D} = \begin{cases} 
\deg D - 1 & \text{if } \deg D < n, \\
n - 2 & \text{if } \overline{D} \sim H, \\
n - 1 & \text{otherwise.}
\end{cases} \)
2. \( \langle \overline{D_1}, \overline{D_2} \rangle = \lcm(D_1, D_2). \)
3. \( \overline{D_1} \cap \overline{D_2} = \gcd(D_1, D_2) \) if \( \deg(\lcm(D_1, D_2)) \leq n \) and \( \lcm(D_1, D_2) \not\sim H \).

Proposition 2.8. Let \( D_1 \) and \( D_2 \) be effective divisors on \( E \subseteq \mathbb{P}^{n-1} \).

1. If \( \deg D_1 + \deg D_2 < n \), then \( D_1 \cap D_2 = \emptyset \) if and only if \( \overline{D_1} \cap \overline{D_2} = \emptyset \).
2. If \( \deg D_1 + \deg D_2 = n \) and \( \lcm(D_1, D_2) \not\sim H \), then \( D_1 \cap D_2 = \emptyset \) if and only if \( \overline{D_1} \cap \overline{D_2} = \emptyset \).

Proof. 1. Certainly \( D_1 \cap D_2 = \emptyset \) if \( \overline{D_1} \cap \overline{D_2} \neq \emptyset \).

Suppose the reverse implication is false. Then there are effective divisors \( D_1 \) and \( D_2 \) such that \( D_1 \cap D_2 = \emptyset \) but \( \overline{D_1} \cap \overline{D_2} \neq \emptyset \). Since \( D_1 \cap D_2 = \emptyset \), \( \lcm(D_1, D_2) = D_1 + D_2 \). Hence \( \deg(\lcm(D_1, D_2)) < n \) so, by Lemma 2.7(3), \( \overline{D_1} \cap \overline{D_2} = \gcd(D_1, D_2) = \emptyset = \emptyset \). This is a contradiction so we conclude that the proposition holds.

2. Assume \( \deg D_1 + \deg D_2 = n \) and \( \lcm(D_1, D_2) \not\sim H \).

Certainly \( \overline{D_1} \cap \overline{D_2} = \emptyset \) implies \( D_1 \cap D_2 = \emptyset \). Conversely, if \( D_1 \cap D_2 = \emptyset \), then \( \lcm(D_1, D_2) = D_1 + D_2 \not\sim H \) so, by Lemma 2.7(3), \( \overline{D_1} \cap \overline{D_2} = \gcd(D_1, D_2) = \emptyset = \emptyset \).

Remark 2.9. The hypothesis in Proposition 2.8(1) that \( \deg D_1 + \deg D_2 < n \) can’t be improved: if \( n = 4 \) and \( L_1 \) and \( L_2 \) are distinct lines on a singular quadric that contains \( E \), then there are degree-two divisors \( D_1 \) and \( D_2 \) such that \( L_1 = \overline{D_1} \) and \( L_2 = \overline{D_2} \), and \( D_1 \cap D_2 = \emptyset \neq \overline{D_1} \cap \overline{D_2} \). More precisely, if \( n = 4 \) and \( \omega \in \Omega \), then \( \Sec_{2,\omega}(E) \) is a rank 3 quadric containing \( E \) and a point in \( \Sec_{2,\omega}(E) - E \) lies on either a unique secant line or on infinitely many; those that lie on infinitely many secant lines are the vertices of the quadric cones \( \Sec_{2,\omega}(E) \).

2.5. The varieties \( \Sec_{d,x}(E) - \Sec_{d-1}(E) \). The next result says that every point in \( \Sec_{d,x}(E) - \Sec_{d-1}(E) \) belongs to \( \overline{D} \) for a unique \( D \in E^{[d]}_x \) when \( 1 \leq d < \frac{n}{2} \).

Proposition 2.10. If either \( d \in [1, \frac{n}{2}] \) or \( d = \frac{n}{2} \) and \( x \notin \Omega \), then

\[
(2-2) \quad \Sec_{d,x}(E) - \Sec_{d-1}(E) = \text{the disjoint union} \bigcup_{D \in E^{[d]}_x} (\overline{D} - \Sec_{d-1}(E)).
\]

\(^9\)A proof of a slightly weaker version of this result, which can easily be adapted to prove the proposition, can be found in [Hul86, Lem. IV.1.1, p.32]. See, also, [GvBH04, Lem. 13.2].
Proposition 2.13. Since \( p < \frac{n}{2} \), \( \deg D_1 + \deg D_2 < n \) whence \( p \in \overline{D_1} \cap \overline{D_2} = \gcd(D_1, D_2) \); hence \( D_1 \cap D_2 \neq \emptyset \). In particular, \( \deg(D_1 \cap D_2) \leq d - 1 \), so \( p \in \sec_{d-1}(E) \). This contradicts the choice of \( p \) so we conclude there can be no such \( p \). The equality in (2-2) therefore holds.

Now assume that \( d = \frac{n}{2} \) and \( x \notin \Omega \). Suppose \( D_1, D_2 \in E_{x}^{[n/2]} \) and

\[
p \in (\overline{D_1} - \sec_{\frac{n}{2}-1}(E)) \cap (\overline{D_2} - \sec_{\frac{n}{2}-1}(E)).
\]

Since \( x \notin \Omega \), \( D_1 + D_2 \not\sim H \). Therefore \( p \in \overline{D_1} \cap \overline{D_2} = \gcd(D_1, D_2) \) where the equality follows from Lemma 2.7(3). Since \( p \notin \sec_{\frac{n}{2}-1}(E) \),\( \deg(\gcd(D_1, D_2)) = \frac{n}{2} \) from which it follows that \( D_1 = \gcd(D_1, D_2) = D_2 \). Hence the union in (2-2) is disjoint as claimed. 

Corollary 2.11. If \( d \in [1, \frac{n}{2}) \), then

\[
(2-3) \quad \sec_{d}(E) - \sec_{d-1}(E) = \text{the disjoint union} \bigcup_{x \in E} (\sec_{d,x}(E) - \sec_{d-1}(E)).
\]

Proof. The left- and right-hand sides of (2-3) are equal so we only need to check that the union on the right is disjoint. Suppose \( p \in (\sec_{d,x}(E) - \sec_{d-1}(E)) \cap (\sec_{d,y}(E) - \sec_{d-1}(E)) \). Then \( p \in \overline{D_1} \cap \overline{D_2} \) for some \( D_1 \in E_{x}^{[d]} \) and some \( D_2 \in E_{y}^{[d]} \). Since \( \deg D_1 + \deg D_2 < n \), \( \overline{D_1} \cap \overline{D_2} = \gcd(D_1, D_2) \). But \( p \notin \sec_{d-1}(E) \) so \( \deg \gcd(D_1, D_2) \geq d \); hence \( \deg \gcd(D_1, D_2) = d \) and we conclude that \( \gcd(D_1, D_2) = D_1 = D_2 \), whence \( x = \sigma(D_1) = \sigma(D_2) = y \). The union is therefore disjoint.

Proposition 2.12. If \( d < \frac{n}{2} \), then

\[
(2-4) \quad \dim \sec_{d,x}(E) = 2d - 2.
\]

Proof. Let \( Y \subseteq E_{x}^{[d]} \times \mathbb{P}^{n-1} \) be the incidence variety

\[
Y := \{(D,p) \in E_{x}^{[d]} \times \mathbb{P}^{n-1} \mid p \in \overline{D}\}.
\]

Let \( \pi_1 : Y \to E_{x}^{[d]} \) and \( \pi_2 : Y \to \mathbb{P}^{n-1} \) be the projections. Clearly, \( \pi_2(Y) = \sec_{d,x}(E) \). Since \( \pi_1^{-1}(D) = \{D\} \times \overline{D} \), every fiber of \( \pi_1 \) is isomorphic to \( \mathbb{P}^{d-1} \). Since \( \dim(E_{x}^{[d]}) = d - 1 \), \( \dim Y = 2d - 2 \).

To show that \( \dim \sec_{d,x}(E) = 2d - 2 \) it suffices to show that \( \pi_2 \) is injective on a non-empty Zariski-open subset of \( Y \). Following the argument after Lemma 4.2 in the unpublished manuscript [Fis], we will show that \( \pi_2 \) is injective on the non-empty open subset

\[
U := \{(D,p) \in E_{x}^{[d]} \times \mathbb{P}^{n-1} \mid p \in \overline{D} - \sec_{d-1}(E)\},
\]

but this follows from Proposition 2.10. 

When \( n \) is even, (2-4) also holds for \( d = \frac{n}{2} \); that is the content of Proposition 2.13(2) below. Indeed, Proposition 2.13 says that when \( n \) is even, the varieties \( \sec_{\frac{n}{2},x}(E) \), \( x \in E \), form a pencil of hypersurfaces of degree \( \frac{n}{2} \) that contain \( \sec_{d-1}(E) \). This is the appropriate generalization to higher dimensions of the familiar fact that the quartic elliptic normal curve \( E \subseteq \mathbb{P}^{3} \) is contained in a pencil of quadrics.

Proposition 2.13 is due to Room [Roo38, 9.22.1, 9.26.1]. The notation and terminology in Room’s book is that of 1938; it will challenge the modern reader. Fortunately there are “modern” proofs of the results we need: see Fisher’s unpublished paper [Fis] and his published paper [Fis10]; in particular, [Fis, Lem. 7.1, Prop. 7.2] and [Fis10, Lemmas 2.7 and 2.9].
2.5.1. Notation and terminology. [Fis10]. A divisor pair \((D_1, D_2)\) is a pair of effective divisors on \(E\) such that \(D_1 + D_2 \sim H\). Given a divisor pair \((D_1, D_2)\), let \(\{e_i\}\) and \(\{f_j\}\) be bases for \(H^0(E, \mathcal{O}_E(D_1))\) and \(H^0(E, \mathcal{O}_E(D_2))\), respectively, and define

\[
\Phi(D_1, D_2) := \text{the matrix whose } i j^{\text{th}} \text{ entry is } \mu(e_i \otimes f_j)
\]

where

\[
\mu : H^0(E, \mathcal{O}_E(D_1)) \otimes H^0(E, \mathcal{O}_E(D_2)) \longrightarrow H^0(E, \mathcal{L})
\]

is the multiplication map. Elements in \(H^0(E, \mathcal{L})\) are linear forms on \(\mathbb{P}H^0(E, \mathcal{L})^* = \mathbb{P}_L\), so the entries in \(\Phi(D_1, D_2)\) are linear forms on \(\mathbb{P}_L\).

Divisor pairs \((D_1, D_2)\) and \((D'_1, D'_2)\) are equivalent if either \(D_1 \sim D'_1\) or \(D_1 \sim D'_2\).

**Proposition 2.13** (Room). Let \((D_1, D_2)\) be a divisor pair and let \(x := \sigma(D_1)\); note that \(x + x_2 = \sigma(H)\).

1. If \(d := \deg(D_1) \leq \deg(D_2)\), then

\[
\text{Sec}_{D_1}^d(E) = \bigcup_{D \sim D_1} D' = \{ p \in \mathbb{P}^{n-1} \mid \text{rank} \Phi(D_1, D_2)_p < d \},
\]

where \(\Phi(D_1, D_2)_p\) denotes the evaluation of the matrix \(\Phi(D_1, D_2)\) at the point \(p\).

2. If \(n\) is even and \(\deg(D_1) = \deg(D_2) = \frac{n}{2}\), then

\[
\text{Sec}_{D_1}^{\frac{n}{2}}(E) = \{ p \in \mathbb{P}^{n-1} \mid \text{det} \Phi(D_1, D_2)_p = 0 \}.
\]

In particular, \(\text{Sec}_{D_1}^{\frac{n}{2}}(E)\) has dimension \(n - 2 = 2\frac{n}{2} - 2\).

3. If \(n\) is even and \((D_1, D_2)\) and \((D'_1, D'_2)\) are inequivalent divisor pairs such that all \(D_i\) and \(D'_i\) have degree \(\frac{n}{2}\), then

\[
\text{Sec}_{D_1}^{\frac{n}{2}-1}(E) = \{ p \in \mathbb{P}^{n-1} \mid \text{det} \Phi(D_1, D_2)_p = \text{det} \Phi(D'_1, D'_2)_p = 0 \}.
\]

**Proof.** See [Fis10, Lem. 2.7] or [Fis, Lem. 7.1] for (1). As in the proof of [Fis, Prop. 7.2(i)], (2) is a special case of (1). See [Fis10, Lem. 2.9] or [Fis, Prop. 7.2(ii)] for (3). \(\square\)

Part (2) of Proposition 2.13 allows us to improve Proposition 2.12.

**Corollary 2.14.** \(\dim \text{Sec}_{D_1}^{d}(E) = 2d - 2\) for all \(d \in [1, \frac{n}{2}]\).

**Corollary 2.15** (Room). If \(n\) is even, then \(\text{Sec}_{D_1}^{\frac{n}{2}-1}(E)\) has codimension 2 in \(\mathbb{P}^{n-1}\) and is the intersection of two hypersurfaces of degree \(\frac{n}{2}\). The varieties \(\text{Sec}_{D_1}^{\frac{n}{2}}(E), x \in E\), form a pencil, parametrized by the projective line \(E/\sim\) where \(x \sim \sigma(H) - x\), of degree-\(\frac{n}{2}\) hypersurfaces that contain \(\text{Sec}_{D_1}^{\frac{n}{2}-1}(E)\).

**Proposition 2.16.** If \(n\) is even, then \(\text{Sec}_{D_1}^{\frac{n}{2}}(E) = \text{Sec}_{D_1}^{\frac{n}{2}}(E)\) if and only if either \(x = y\) or \(x + y = \sigma(H)\).

**Proof.** \((\Leftarrow)\) This is the content of Proposition 2.13(2).

\((\Rightarrow)\) Suppose the claim is false; i.e., \(\text{Sec}_{D_1}^{\frac{n}{2}}(E) = \text{Sec}_{D_1}^{\frac{n}{2}}(E)\) and \(x \neq y\) and \(x + y \neq \sigma(H)\).

Let \(p \in \text{Sec}_{D_1}^{\frac{n}{2}}(E) - \text{Sec}_{D_1}^{\frac{n}{2}-1}(E)\). (Such \(p\) exists since \(\text{codim} \text{Sec}_{D_1}^{\frac{n}{2}}(E) = 1\) while \(\text{codim} \text{Sec}_{D_1}^{\frac{n}{2}-1}(E) = 2\).)

Let \(D \in E^{[n/2]}\) and \(D' \in E^{[n/2]}\) be such that \(p \in D \cap D'\).

If \(\text{lcm}(D, D') \sim H\), then \(\deg(\text{lcm}(D, D')) = n\), whence \(\text{lcm}(D, D') = D + D'\). But then \(D + D' \sim H\) so \(x + y = \sigma(D + D') = \sigma(H)\) which contradicts the assumption that \(x + y \neq \sigma(H)\). We conclude that \(\text{lcm}(D, D') \not\sim H\). However, \(\deg(\text{lcm}(D, D')) \leq \deg(D + D') = n\) so, by Lemma 2.7(3), \(D \cap D' = \text{gcd}(D, D')\). Thus \(p \in \text{gcd}(D, D')\). But \(p \notin \text{Sec}_{D_1}^{\frac{n}{2}-1}(E)\) so \(\deg(\text{gcd}(D, D')) = \frac{n}{2}\). Hence \(D = \text{gcd}(D, D') = D'\), which implies that \(x = \sigma(D) = \sigma(D') = y\); this contradicts the assumption that \(x \neq y\) so we conclude that the proposition is true. \(\square\)

---

\(^{10}\)Of course, this “value” is only defined up to a non-zero scalar multiple but that does not affect its rank.
2.6. **Locally free** $\mathcal{O}_E$-modules and vector bundles on $E$. Sometimes we consider vector bundles, and their associated projective-space bundles, on a variety $X$ as schemes over $X$ via the familiar correspondence of [Har77, Exer. II.5.18] and [Har77, Exer. II.7.10]. In this paper we use the notation

$$(2-5) \quad (\text{locally free sheaf } \mathcal{E}) \rightsquigarrow \begin{cases} \mathbb{V}\mathcal{E} \text{ or } \mathbb{V}(\mathcal{E}) := \text{the vector bundle } \text{Spec}(\text{Sym}(\mathcal{E}^\vee)) \\ \mathbb{P}\mathcal{E} \text{ or } \mathbb{P}(\mathcal{E}) := \text{the projective bundle } \text{Proj}(\text{Sym}(\mathcal{E}^\vee)) \end{cases},$$

where $\mathcal{E}^\vee$ is the dual sheaf, the Spec and Proj constructions are as in [Har77, Exer. II.5.17] and [Har77, Construction before Ex. II.7.8.7] respectively, and Sym denotes the symmetric algebra of the sheaf [Har77, Exer. II.5.16]. One can recover $\mathcal{E}$ as the sheaf of sections of $\mathbb{V}\mathcal{E}$ [Har77, Exer. II.5.18 (c)]. We conflate $\mathcal{E}$ and $\mathbb{V}\mathcal{E}$ and refer to $\mathcal{E}$ itself as a bundle; this is not uncommon: e.g., see [EH16, Chapter 0].

Although the correspondence in (2-5) is often used: for example

- in [Ful98, §B.5.5] the projective bundle of $\mathcal{E}$ is what (2-5) refers to as $\mathbb{P}\mathcal{E}$, and is denoted there by $P(\mathcal{E})$ for $E := \mathbb{V}\mathcal{E}$;
- similarly, the projectivization of $\mathcal{E}$ in [EH16, §9.1] is our $\mathbb{P}\mathcal{E}$. As explained there, the points of $\mathbb{P}\mathcal{E}$ over $x \in X$ are precisely the lines in the vector space $\mathcal{E}_x$ (fiber of $\mathcal{E}$ at $x$); it is not always observed: for example, [Ful98, Note following §B.5.5] observes that $P(\mathcal{E})$ (our $\mathbb{P}\mathcal{E}$) would be the $P(\mathcal{E}^\vee)$ in [Gro61, §8.4] and, similarly, the $\mathbb{P}(\mathcal{E})$ of [Har77, Definition preceding Prop. II.7.11] omits the dualization in (2-5). The same goes for the $P(E)$ of [Laz04, Appendix A]; if anything, the convention opposite ours (i.e., (2-5) without dualization) might well be more common.

This difference in conventions can cause some confusion (it did for some of us): it explains, for instance, the apparent discrepancy in defining Chern classes in various sources: compare the simple summation of [EH16, Definition 5.10] with the alternating sum of [Har77, §A.3, Defn.]. The two convention switches ($\mathcal{E}$ versus $\mathcal{E}^\vee$ and the sign difference) cancel out to accord the same significance to the Chern classes $c_k(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ as introduced in either reference.

3. **Trivial subbundles of** $\mathcal{E} \in \text{Bun}(2, \mathcal{L})$

3.1. The $\text{Aut}(\mathcal{E})$-action on the quasi-affine variety $X(\mathcal{E}) := \text{Hom}(\mathcal{O}_E, \mathcal{E}) - (Z_\mathcal{E} \cup \{0\})$. As the next lemma suggests, the following definition plays a central role in all that follows; also see Remark 3.4.

**Definition 3.1.** Let $\mathcal{E}$ be a rank-two locally free $\mathcal{O}_E$-module. Define

$$(3-1) \quad Z_\mathcal{E} := \{\text{non-zero maps } f : \mathcal{O}_E \to \mathcal{E} \mid \text{coker}(f) \text{ is not invertible}\}$$

$$= \{\text{non-zero maps } f : \mathcal{O}_E \to \mathcal{E} \mid \text{coker}(f) \text{ is not torsion-free}\}$$

$$\subseteq \text{Hom}_E(\mathcal{O}_E, \mathcal{E}) - \{0\}$$

and

$$X(\mathcal{E}) := \text{Hom}_E(\mathcal{O}_E, \mathcal{E}) - (Z_\mathcal{E} \cup \{0\}).$$

In other words, $X(\mathcal{E})$ consists of the embeddings of the trivial bundle $E \times \mathbb{C}$ as a subbundle of $\mathbb{V}(\mathcal{E})$.

**Lemma 3.2.** If $\mathcal{E}$ is a rank-two locally free $\mathcal{O}_E$-module, then

$$X(\mathcal{E}) = \{\text{non-zero maps } f : \mathcal{O}_E \to \mathcal{E} \mid \text{coker}(f) \cong \text{det } \mathcal{E}\}.$$ 

In other words, if $f \in X(\mathcal{E})$, then there is an exact sequence

$$\xi : \quad 0 \longrightarrow \mathcal{O}_E \xrightarrow{f} \mathcal{E} \longrightarrow \text{det } \mathcal{E} \longrightarrow 0,$$

which is unique up to non-zero scalar multiple in $\text{Ext}^1(\text{det } \mathcal{E}, \mathcal{O}_E)$.

**Proof.** Suppose $f \in X(\mathcal{E})$. Since $\mathcal{O}_E$ is of rank one, $f$ has to be a monomorphism. So we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_E \xrightarrow{f} \mathcal{E} \longrightarrow \text{coker}(f) \longrightarrow 0$$
of locally free $\mathcal{O}_E$-modules, which implies that $\det \mathcal{E} \cong \det \mathcal{O}_E \otimes \det(\text{coker}(f)) \cong \text{coker}(f)$. The uniqueness follows from Proposition A.1. □

The left action of $\text{End}(\mathcal{E})$ on $\text{Hom}(\mathcal{O}_E, \mathcal{E})$ via composition makes $\text{Hom}(\mathcal{O}_E, \mathcal{E})$ into a left $\text{End}(\mathcal{E})$-module. That action restricts to give a left action of the automorphism group $\text{Aut}(\mathcal{E})$ on $\text{Hom}(\mathcal{O}_E, \mathcal{E})$ and hence on $X(\mathcal{E})$.

**Lemma 3.3.** Let $\mathcal{E}$ be an arbitrary $\mathcal{O}_E$-module, let $f_1, f_2 : \mathcal{O}_E \to \mathcal{E}$ be arbitrary monomorphisms, and let $g_i$ be a cokernel of $f_i$. The extensions

$$
\xi_i : \begin{array}{c}
0 \\ \downarrow \mu \\ \downarrow \nu \\ \downarrow \tau \\
0 \\
\end{array}
\begin{array}{c}
\mathcal{O}_E \\
\mathcal{E} \\
\text{coker}(f_i) \\
0 \\
\end{array}
\begin{array}{c}
f_1 \\
g_1 \\
f_2 \\
g_2 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
$$

are isomorphic if and only if $f_2 \in \text{Aut}(\mathcal{E}) f_1$.

**Proof.** ($\Rightarrow$) Suppose $\xi_1 \cong \xi_2$. There is a commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \mu \\
\downarrow \nu \\
\downarrow \tau \\
0 \\
\end{array}
\begin{array}{c}
\mathcal{O}_E \\
\mathcal{E} \\
\text{coker}(f_1) \\
0 \\
\end{array}
\begin{array}{c}
f_1 \\
g_1 \\
0 \\
0 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
$$

in which $\mu, \nu,$ and $\tau$ are isomorphisms. Since $\text{End}(\mathcal{O}_E) = \mathbb{C}$, $\mu$ is a scalar multiple of the identity. We can therefore treat $\mu$ as a scalar and rewrite the equality $\nu f_1 = f_2 \mu$ as $f_2 = (\mu^{-1} \nu) f_1$. Hence $f_2 \in \text{Aut}(\mathcal{E}) f_1$.

($\Leftarrow$) If $f_2 = \nu f_1$ for some $\nu \in \text{Aut}(\mathcal{E})$, then there is a commutative diagram as above with $\mu$ being the identity map $\mathcal{O}_E \to \mathcal{O}_E$, and $\tau$ being some isomorphism, so $\xi_1 \cong \xi_2$. □

**Remark 3.4.** Let $\mathcal{E}$ be a rank-two locally free $\mathcal{O}_E$-module such that $\deg \mathcal{E} \geq 1$. It follows from Lemma 3.3 that the map $f \mapsto \xi$ in Lemma 3.2 descends to a surjective set map

$$\Psi_\mathcal{E} : X(\mathcal{E}) \longrightarrow \mathcal{L}(\mathcal{E})$$

whose fibers are the $\text{Aut}(\mathcal{E})$-orbits in $X(\mathcal{E})$. Proposition 3.14 shows that $Z_\mathcal{E} \cup \{0\}$ is a hypersurface in $\text{Hom}(\mathcal{O}_E, \mathcal{E})$, whence $X(\mathcal{E})$ is quasi-affine. Theorem 5.6 shows that $\Psi_\mathcal{E}$ is the geometric quotient of $X(\mathcal{E})$ modulo the action of the algebraic group $\text{Aut}(\mathcal{E})$. Lemma 3.5 shows that the action of $\text{Aut}(\mathcal{E})$ on $X(\mathcal{E})$ is free in the set-theoretic sense; i.e., the only group element fixing a point is the identity. ◊

**Lemma 3.5.** Suppose $\mathcal{E}$ satisfies the assumptions in §2.2.

1. If $f \in X(\mathcal{E})$, then the left $\text{End}(\mathcal{E})$-module generated by $f$ is free.

2. $\text{Aut}(\mathcal{E})$ acts freely on $X(\mathcal{E})$.

**Proof.** (1) Suppose to the contrary that the cokernel of $0 \neq f \in \text{Hom}_E(\mathcal{O}_E, \mathcal{E})$ is torsion-free and $\gamma f = 0$ for some non-zero $\gamma \in \text{End}(\mathcal{E})$. The hypotheses on $\mathcal{E}$ imply $\text{coker}(f) \cong \mathcal{L}$. Let $g: \mathcal{E} \to \mathcal{L}$ be a cokernel of $f$. By the universal property of the cokernel there is a unique $h: \mathcal{L} \to \mathcal{E}$ such that $\gamma = hg$. Since $\gamma \neq 0$, $h \neq 0$. Since $\deg \mathcal{L} > 0$, the image of $h$ is not contained in the image of $f$. Hence $gh$ is a non-zero endomorphism of $\mathcal{L}$ which is invertible. It follows that the sequence splits.

(2) This is an immediate consequence of (1). □

Suppose $\mathcal{E}$ satisfies the assumption in §2.2. Our next task, which we complete in Corollary 3.16, is to show there is a non-split extension of the form $0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$; i.e., $\mathcal{L}(\mathcal{E}) \neq \varnothing$. Lemma 2.2 already showed that every $\mathcal{E}_{d,x}$ is the middle term of a non-split $\xi \in \text{Ext}^1(\mathcal{L}, \mathcal{O}_E)$. Thus, by Lemma 3.2, it remains to show that if $\mathcal{E} \cong \mathcal{E}_n$ or $\mathcal{E} \cong \mathcal{E}_\omega$ ($\omega \in \Omega$), depending on the parity of $n$, then there is a non-zero map $f: \mathcal{O}_E \to \mathcal{E}$ whose cokernel is torsion-free; i.e., $X(\mathcal{E}) \neq \varnothing$. 
Lemma 3.6. Let \( \mathcal{E} \) be a rank-two locally free \( \mathcal{O}_E \)-module. Under the adjunction \( (- \otimes \mathcal{E}) \dashv (- \otimes \mathcal{E}^\vee) \) the epimorphism \( \mathcal{E} \otimes \mathcal{E} \to \wedge^2 \mathcal{E} = \det \mathcal{E} \) corresponds to an \( \text{Aut}(\mathcal{E}) \)-equivariant isomorphism
\[
\nu : \mathcal{E} \to (\det \mathcal{E}) \otimes \mathcal{E}^\vee.
\]

Proof. By definition, \( \det \mathcal{E} = \wedge^2 \mathcal{E} \). The adjunction isomorphism \( \text{Hom}(\mathcal{E} \otimes \mathcal{E}, \wedge^2 \mathcal{E}) \cong \text{Hom}(\mathcal{E}, \wedge^2 \mathcal{E} \otimes \mathcal{E}^\vee) \) sends the epimorphism \( \mathcal{E} \otimes \mathcal{E} \to \wedge^2 \mathcal{E} \) to a homomorphism \( \nu : \mathcal{E} \to \wedge^2 \mathcal{E} \otimes \mathcal{E}^\vee \). The fact that the map \( \mathcal{E} \otimes \mathcal{E} \to \wedge^2 \mathcal{E} \) gives a non-degenerate pairing on stalks implies that \( \nu \) is an isomorphism on stalks, and hence an isomorphism. The fact that \( \nu \) is \( \text{Aut}(\mathcal{E}) \)-equivariant is a general fact about rigid symmetric monoidal abelian categories \( (\mathcal{C}, \otimes, 1) \); see Remark B.4.

Let \( \nu \) be as in Lemma 3.6 and let
\[
\nu_* : \text{Hom}(\mathcal{O}_E, \mathcal{E}) \to \text{Hom}(\mathcal{O}_E, (\det \mathcal{E}) \otimes \mathcal{E}^\vee)
\]
be the map \( \nu_*(a) = \nu a \); clearly, \( \nu_* \) is \( \text{Aut}(\mathcal{E}) \)-equivariant because \( \nu \) is. Let
\[
\theta : \text{Hom}(\mathcal{O}_E, (\det \mathcal{E}) \otimes \mathcal{E}^\vee) \to \text{Hom}(\mathcal{E}, \det \mathcal{E})
\]
be the isomorphism associated to the the adjunction \( (- \otimes \mathcal{E}) \dashv (- \otimes \mathcal{E}^\vee) \); it is also \( \text{Aut}(\mathcal{E}) \)-equivariant by Proposition B.3.

Lemma 3.7. Let \( \mathcal{E} \) be a rank-two locally free \( \mathcal{O}_E \)-module. The \( \text{Aut}(\mathcal{E}) \)-equivariant isomorphism
\[
(3-3) \quad \theta \circ \nu_* : \text{Hom}(\mathcal{O}_E, \mathcal{E}) \to \text{Hom}(\mathcal{E}, \det \mathcal{E})
\]
has the following properties:

1. If \( s \in \text{Hom}(\mathcal{O}_E, \mathcal{E}) \), then \( (\theta \nu_*)(s) \circ s = 0 \).
2. If \( s_1, s_2 \in \text{Hom}(\mathcal{O}_E, \mathcal{E}) \), then \( (\theta \nu_*)(s_1) \circ s_2 = -(\theta \nu_*)(s_2) \circ s_1 \).

Proof. Since both statements can be verified locally, it suffices to prove the corresponding claims for a rank-two free module \( F := \mathcal{E}_x \) over the stalk \( R := \mathcal{O}_{E,x} \) at each point \( x \in E \). Fix a basis \( (e_1, e_2) \) for \( F \) and let \( (e_1^\vee, e_2^\vee) \) be its dual basis.

The isomorphism \( \nu \) induces the \( R \)-module isomorphism
\[
F \to (\det F) \otimes F^\vee, \quad v \mapsto \sum_i (v \wedge e_i) \otimes e_i^\vee,
\]
and \( \theta \) induces the \( R \)-module isomorphism
\[
(\det F) \otimes F^\vee \cong \text{Hom}_R(R, (\det F) \otimes F^\vee) \to \text{Hom}_R(F, \det F), \quad \omega \otimes g \mapsto (u \mapsto g(u)\omega),
\]
so \( \theta \circ \nu_* \) induces the \( R \)-module isomorphism
\[
(3-4) \quad F = \text{Hom}_R(R, F) \to \text{Hom}_R(F, \det F), \quad v \mapsto (u \mapsto \sum_i e_i^\vee(u)(v \wedge e_i) = v \wedge u).
\]

1. If the isomorphism (3-4) sends \( s \in \text{Hom}_R(R, F) \) to \( \varphi \in \text{Hom}_R(F, \det F) \), then \( \varphi(u) = s(1) \wedge u \) for all \( u \in F \), so \( (\varphi \circ s)(1) = s(1) \wedge s(1) = 0 \). Therefore \( \varphi \circ s = 0 \).
2. If the isomorphism (3-4) sends \( s_1 \) to \( \varphi_1 \) and \( s_2 \) to \( \varphi_2 \), then
\[
(\varphi_1 \circ s_2)(1) = s_1(1) \wedge s_2(1) = -s_2(1) \wedge s_1(1) = -(\varphi_2 \circ s_1)(1).
\]

Therefore \( \varphi_1 \circ s_2 = -\varphi_2 \circ s_1 \). \( \square \)
3.2.1. **Notation for epimorphisms.** Given quasi-coherent $O_E$-modules $F$ and $G$, we define

\[(3-5) \quad \text{Epi}(F, G) := \{ \text{epimorphisms } \pi : F \to G \} \subseteq \text{Hom}(F, G).\]

**Lemma 3.8.** Let $E$ be a rank-two locally free $O_E$-module. The composition of the linear isomorphisms in the top row of the following diagram restrict to an Aut$(E)$-equivariant bijection between the sets in the bottom row:

\[
\begin{array}{ccc}
\text{Hom}(O_E, E) & \xrightarrow{\nu} & \text{Hom}(O_E, (\det E) \otimes E^\vee) \\
X(E) & \xrightarrow{\pi} & \text{Epi}(E, \det E).
\end{array}
\]

If $f \in X(E)$ is sent to $\pi \in \text{Epi}(E, \det E)$, then the sequence

\[
0 \longrightarrow O_E \xrightarrow{f} E \xrightarrow{\pi} \det E \longrightarrow 0.
\]

is exact.

**Proof.** Let $f : O_E \to E$ be a non-zero homomorphism and define $\pi := \theta \nu_*(f) = \theta(\nu f)$. We will show that $\text{coker}(f)$ is torsion-free if and only if $\pi$ is epic.

Since $\pi f = 0$ by Lemma 3.7(1), the morphism $\pi : E \to \det E$ is the composition of the cokernel morphism $E \to \text{coker}(f)$ and a non-zero morphism $g : \text{coker}(f) \to \det E$. If $\text{coker}(f)$ is torsion-free, then $\text{coker}(f) \cong \det E$ by Lemma 3.2, so $g$ is an isomorphism, and hence $\pi$ is an epimorphism.

Conversely, if $\pi : E \to \det E$ is epic, then the sequence

\[
0 \longrightarrow \ker(\pi) \longrightarrow E \xrightarrow{\pi} \det E \longrightarrow 0
\]

is exact so $\ker(\pi) \cong O_E$. Since $\pi f = 0$ a similar argument shows that $f$ is the composition of an isomorphism $O_E \to \ker(\pi)$ and the kernel morphism $\ker(\pi) \to E$. Hence $\text{coker}(f) \cong \det E$, which is torsion-free.

**Remark 3.9.** The bijection $f \leftrightarrow \pi = \theta(\nu f)$ in Lemma 3.8 puts the two middle arrows in exact sequences of the form $0 \to O_E \to E \to \det E \to 0$ in bijective correspondence with each other. We will freely switch perspective between the two points of view, parametrizing the non-split extensions of $\det E$ by $O_E$ either by torsion-free-cokernel monomorphisms $O_E \to E$ or by epimorphisms $E \to \det E$.

**Lemma 3.10.** If $0 \longrightarrow O_E \xrightarrow{f} E \xrightarrow{\pi} L \longrightarrow 0$ is exact, then $\text{Hom}(E, L)f = \pi \text{Hom}(O_E, E)$.

**Proof.** By Lemma 3.8 and the universality of cokernels, we can assume that $L = \det E$, and that $f$ and $\pi$ correspond to each other via $\theta \circ \nu_*$ in (3-3); i.e., $\pi = (\theta \nu_*)(f)$.

To prove “$\subseteq$”, let $\pi' \in \text{Hom}(E, L)$; then $\pi' = (\theta \nu_*)(f')$ for a unique $f' \in \text{Hom}(O_E, E)$; by Lemma 3.7(2), $\pi' f = (\theta \nu_*)(f') \circ f = -(\theta \nu_*)(f) \circ f' \in \pi \text{Hom}(O_E, E)$.

To prove “$\supseteq$”, let $f' \in \text{Hom}(O_E, E)$; then $\pi f' = (\theta \nu_*)(f) \circ f' = -(\theta \nu_*)(f') \circ f \in \text{Hom}(E, L)f$. □

3.3. The subvarieties $Z_x \subseteq Z_E$ and the properties of $\mathbb{P}Z_E \subseteq \mathbb{P} \text{Hom}(O_E, E)$. In this section we show that if $E$ satisfies the assumptions in §2.2, then $\mathbb{P}Z_E$ is a hypersurface in $\mathbb{P} \text{Hom}(O_E, E)$. (At this stage we don’t even know that $Z_E \cup \{0\}$ is Zariski-closed in $\text{Hom}(O_E, E) - \{0\}$.) It will follow that $L(E)$ is non-empty for all $E$ in §2.2.

For each $x \in E$, let

\[(3-6) \quad Z_x := \{ \text{non-zero maps } f : O_E \to E \mid \text{coker}(f) \text{ contains a copy of } O_x \} \subseteq \text{Hom}_E(O_E, E) - \{0\}.
\]

Clearly,

\[(3-7) \quad Z_E = \bigcup_{x \in E} Z_x.
\]

**Lemma 3.11.** Let $E \in \text{Bun}(2, L)$. Fix a point $x \in E$. 

---

\[\text{Diagram and equations omitted.} \]

---

\[\text{End of text.} \]
(1) If $\mathcal{E} = \mathcal{N}_1 \oplus \mathcal{N}_2$ for some invertible $\mathcal{O}_E$-modules $\mathcal{N}_1, \mathcal{N}_2$ of degree $\geq 1$, then

$$h^0(\mathcal{E}(-x)) = \begin{cases} h^0(\mathcal{E}) - 2 & \text{if } \mathcal{N}_1 \not\cong \mathcal{O}_E(x) \text{ and } \mathcal{N}_2 \not\cong \mathcal{O}_E(x), \\ h^0(\mathcal{E}) - 1 & \text{if } \mathcal{N}_1 \cong \mathcal{O}_E(x) \text{ or } \mathcal{N}_2 \cong \mathcal{O}_E(x). \end{cases}$$

(2) If $\mathcal{E}$ is indecomposable, then $h^0(\mathcal{E}(-x)) = h^0(\mathcal{E}) - 2 = \deg \mathcal{E} - 2$.

More succinctly, if $\mathcal{E} \in \text{Bun}(2, \mathcal{L})$ and $x \in E$, then

$$h^0(\mathcal{E}(-x)) = \begin{cases} h^0(\mathcal{E}) - 1 & \text{if } \mathcal{O}_E(x) \text{ is a direct summand of } \mathcal{E}, \\ h^0(\mathcal{E}) - 2 & \text{otherwise}. \end{cases}$$

Proof. (1) If both $\mathcal{N}_1$ and $\mathcal{N}_2$ have degree $\geq 2$, then both $\mathcal{N}_1(-x)$ and $\mathcal{N}_2(-x)$ have degree $\geq 1$ so $h^0(\mathcal{E}(-x)) = \deg \mathcal{N}_1(-x) + \deg \mathcal{N}_2(-x) = \deg \mathcal{N}_1 + \deg \mathcal{N}_2 - 2 = h^0(\mathcal{E}) - 2$.

If $\deg \mathcal{N}_1 = 1$ and $\mathcal{N}_1 \not\cong \mathcal{O}_E(x)$, then $h^0(\mathcal{N}_1) = 0$ and, since $\deg \mathcal{L} \geq 3$, $h^0(\mathcal{N}_2(-x)) = h^0(\mathcal{N}_2) - 1$, whence $h^0(\mathcal{E}(-x)) = h^0(\mathcal{E}) - 2$.

If $\mathcal{N}_1 \cong \mathcal{O}_E(x)$, then $h^0(\mathcal{N}_1(-x)) = h^0(\mathcal{N}_1)$ and $h^0(\mathcal{N}_2(-x)) = h^0(\mathcal{N}_2) - 1$, whence $h^0(\mathcal{E}(-x)) = h^0(\mathcal{E}) - 1$.

(2) Suppose $\mathcal{E}$ is indecomposable. Every indecomposable locally free $\mathcal{O}_E$-module is semistable (a proof can be found in [Tu93, Appendix A]) so $h^0(\mathcal{E}) = \deg \mathcal{E}$ by [Tu93, Lem. 17]. Since $\deg \mathcal{E} = \deg \mathcal{L} \geq 3$, $\mathcal{E}(-x)$ is also semistable of positive degree so $h^0(\mathcal{E}(-x)) = \deg \mathcal{E} = \deg \mathcal{E} - 2 = h^0(\mathcal{E}) - 2$. \hfill $\Box$

Lemma 3.12. Let $\mathcal{E} \in \text{Bun}(2, \mathcal{L})$. Fix points $x, y \in E$.

(1) If $\mathcal{E} = \mathcal{N}_1 \oplus \mathcal{N}_2$ for some invertible $\mathcal{O}_E$-modules $\mathcal{N}_1, \mathcal{N}_2$ of degree $\geq 2$, then

$$h^0(\mathcal{E}(-x - y)) = \begin{cases} h^0(\mathcal{E}) - 4 & \text{if neither } \mathcal{N}_1 \text{ nor } \mathcal{N}_2 \text{ is isomorphic to } \mathcal{O}_E(x + y), \\ h^0(\mathcal{E}) - 3 & \text{if exactly one of } \mathcal{N}_1, \mathcal{N}_2 \text{ is isomorphic to } \mathcal{O}_E(x + y), \\ h^0(\mathcal{E}) - 2 & \text{if both } \mathcal{N}_1 \text{ and } \mathcal{N}_2 \text{ are isomorphic to } \mathcal{O}_E(x + y). \end{cases}$$

The third case occurs if and only if $n = 4$ and $\mathcal{E} \cong \mathcal{O}_E(\omega + 0)^{\oplus 2}$ for some $\omega \in \Omega$ and $x + y = \omega$.

(2) If $\mathcal{E}$ is indecomposable, then

$$h^0(\mathcal{E}(-x - y)) = \begin{cases} h^0(\mathcal{E}) - 3 = 0 & \text{if } n = 3, \\ h^0(\mathcal{E}) - 3 = 1 & \text{if } n = 4, \mathcal{E} \cong \mathcal{E}_\omega \ (\omega \in \Omega), \text{ and } x + y = \omega, \\ h^0(\mathcal{E}) - 4 & \text{otherwise}. \end{cases}$$

Proof. (1) This can be proved in a similar way to Lemma 3.11(1). The only point that might need additional explanation is the last sentence: if $\mathcal{E} \cong \mathcal{O}_E(x+y)^{\oplus 2}$, then $\mathcal{L} \cong \det(\mathcal{E}) \cong \mathcal{O}_E(x+y)^{\oplus 2}$ so $n = \deg(\mathcal{L}) = 4$, and $x + y + x + y \sim \mathcal{L}$ so $x + y \in \Omega$; in particular, $x + y = \omega$ for some $\omega \in \Omega$, so $\mathcal{O}_E(x + y) \cong \mathcal{O}_E((\omega) + (0)) = \mathcal{L}_\omega$ ($\mathcal{L}_\omega$ is defined in §2.3.1).

(2) If $n = 3$, then $\mathcal{E}(-x - y)$ is semistable of negative degree, so $h^0(\mathcal{E}(-x - y)) = 0$.

Suppose $n = 4$. As in §2.3.1, $\mathcal{E} \cong \mathcal{E}_\omega$ for some $\omega \in \Omega$, and $\mathcal{E}_\omega$ is a non-split self-extension of $\mathcal{L}_\omega = \mathcal{O}_{\mathcal{E}}((\omega) + (0))$. If $x + y \neq \omega$, then $h^0(\mathcal{L}_\omega(-x - y)) = 0$ so $h^0(\mathcal{E}(-x - y)) = 0$. If $x + y = \omega$, then $\mathcal{E}(-x - y)$ is a non-split self-extension of $\mathcal{O}_E$, so applying Hom$(\mathcal{O}_E, -)$ to that extension produces an exact sequence

$$0 \longrightarrow \text{Hom}(\mathcal{O}_E, \mathcal{O}_E) \longrightarrow \text{Hom}(\mathcal{O}_E, \mathcal{E}(-x - y)) \longrightarrow \text{Hom}(\mathcal{O}_E, \mathcal{O}_E) \longrightarrow \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E)$$

in which the right-most morphism is non-zero. Hence $H^0(\mathcal{E}(-x - y)) = \text{Hom}(\mathcal{O}_E, \mathcal{E}(-x - y))$ has dimension $1 = h^0(\mathcal{E}) - 3$.

If $n \geq 5$ this can be proved in a similar way to Lemma 3.11(2). \hfill $\Box$

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\textsuperscript{11}When $n = 4$, $\mathcal{O}_E(\omega + 0)$ is the sheaf $\mathcal{L}_\omega$ defined in §2.3.1. Hence the third case occurs if and only if $n = 4$ and $\mathcal{E} \cong \mathcal{L}_\omega \oplus \mathcal{L}_\omega$. \hfill $\Box$
Remark on notation. We will use the symbol “\( \mathbb{P} \)” to denote the projectivizations of spaces, varieties, etc. In all cases, it will be clear from the context which action of \( \mathbb{C}^\times \) we are quotienting out. For example, \( \mathbb{P}Z_x \) is the quotient of \( Z_x \) by the action that is the restriction of the natural action of \( \mathbb{C}^\times \) on \( \text{Hom}(\mathcal{O}_E, \mathcal{E}) - \{0\} \). Similarly, \( \mathbb{P}Z_\mathcal{E} = Z_\mathcal{E}/\mathbb{C}^\times \).

Proposition 3.13. Suppose \( \mathcal{E} \) satisfies the assumptions in §2.2. If \( x \in E \), then
\[
\mathbb{P}Z_x \subseteq \mathbb{P}\text{Hom}(\mathcal{O}_E, \mathcal{E})
\]
is a linear subspace of \( \mathbb{P}\text{Hom}(\mathcal{O}_E, \mathcal{E}) \) and
\[
\text{codim} \mathbb{P}Z_x = \begin{cases} 1 & \text{if } \mathcal{O}_E(x) \text{ is a direct summand of } \mathcal{E}, \\ 2 & \text{otherwise}. \end{cases}
\]

Proof. The cokernel of a morphism \( f : \mathcal{O}_E \to \mathcal{E} \) contains a copy of the skyscraper sheaf \( \mathcal{O}_x \) if and only if \( f \) extends to the central term in the extension
\[
0 \to \mathcal{O}_E \to \mathfrak{m}_x^{-1} \to \mathcal{O}_x \to 0.
\]
Hence \( Z_x \cup \{0\} \) is the image of the natural map \( \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{E}) \to \text{Hom}(\mathcal{O}_E, \mathcal{E}) \). This map is injective because \( \text{Hom}(\mathcal{O}_x, \mathcal{E}) = 0 \). Thus, \( \dim Z_x = \dim \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{E}) = h^0(\mathcal{E}(-x)) \). The result now follows from Lemma 3.11.

Piecing together the various projective spaces \( \mathbb{P}Z_x \) of Proposition 3.13, we have the following result.

Proposition 3.14. If \( \mathcal{E} \) satisfies the assumptions in §2.2, then \( \mathbb{P}Z_\mathcal{E} \) is Zariski-closed in \( \mathbb{P}\text{Hom}(\mathcal{O}_E, \mathcal{E}) \):
\[
\mathbb{P}Z_\mathcal{E} = \begin{cases} \text{a union of two hyperplanes if } \mathcal{O}_E(x) \text{ is a direct summand of } \mathcal{E} \text{ for some } x \in E, \\ \text{an irreducible hypersurface otherwise}. \end{cases}
\]

Hence \( X(\mathcal{E}) = \text{Hom}(\mathcal{O}_E, \mathcal{E}) - (\text{a hypersurface}) \).

Proof. Since \( \mathcal{E} \) is fixed, we will write \( Z \) in place of \( Z_\mathcal{E} \).

(1) Suppose \( \mathcal{O}_E(x) \) is a direct summand of \( \mathcal{E} \). Then \( \mathcal{E} \cong \mathcal{O}_E(x) \oplus \mathcal{L}(-x) \). We fix such an isomorphism, and regard \( \mathcal{O}_E(x) \) and \( \mathcal{L}(-x) \) as submodules of \( \mathcal{E} \), and regard \( \text{Hom}(\mathcal{O}_E, \mathcal{O}_E(x)) \) and \( \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)) \) as subspaces of \( \text{Hom}(\mathcal{O}_E, \mathcal{E}) \). We first show that
\[
\bigcup_{y \in E - \{x\}} (Z_y \cup \{0\}) \subseteq \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)) \subseteq Z \cup \{0\}.
\]

Let \( y \in E - \{x\} \). As in the proof of Proposition 3.13, \( Z_y \cup \{0\} \) is the image of the natural injection \( \text{Hom}(\mathfrak{m}_y^{-1}, \mathcal{E}) \to \text{Hom}(\mathcal{O}_E, \mathcal{E}) \). Since \( \text{Hom}(\mathfrak{m}_y^{-1}, \mathcal{O}_E(x)) \cong \text{Hom}(\mathcal{O}_E(y), \mathcal{O}_E(x)) = 0 \), that image is equal to the image of \( \text{Hom}(\mathfrak{m}_y^{-1}, \mathcal{L}(-x)) \to \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)) \). Hence the first inclusion in (3-10) holds.

Let \( 0 \neq f \in \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)) \). Since \( \deg \mathcal{L}(-x) \geq 2 \), the cokernel of \( f \) is not torsion-free, so, as an element of \( \text{Hom}(\mathcal{O}_E, \mathcal{E}) \), \( f \) belongs to \( Z \). Hence the second inclusion in (3-10) holds.

On the other hand, since \( Z_x \cup \{0\} \) is the image of the natural injection \( \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{E}) \to \text{Hom}(\mathcal{O}_E, \mathcal{E}) \), it is equal to the direct sum of
\[
V_1 := \text{the image of } \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{O}_E(x)) \longrightarrow \text{Hom}(\mathcal{O}_E, \mathcal{O}_E(x)), \\
V_2 := \text{the image of } \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{L}(-x)) \longrightarrow \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)).
\]

Since \( \mathfrak{m}_x^{-1} \cong \mathcal{O}_E(x) \), \( \dim V_1 = 1 \). Since \( \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{L}(-x)) \cong \text{Hom}(\mathcal{O}_E, \mathcal{L}(-2x)) \), \( \dim V_2 = n - 2 \). It follows from this, and (3-10), that
\[
Z \cup \{0\} = \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)) \cup (V_1 \oplus V_2).
\]

Both \( \text{Hom}(\mathcal{O}_E, \mathcal{L}(-x)) \) and \( V_1 \oplus V_2 \) have codimension one in \( \text{Hom}(\mathcal{O}_E, \mathcal{E}) \). Since these subspaces are different, \( \mathbb{P}Z \) is the union of two hyperplanes.

(2) Suppose no \( \mathcal{O}_E(x) \) is a direct summand of \( \mathcal{E} \).
The natural map $H^0(E, \mathcal{E}) \otimes \mathcal{O}_E \to \mathcal{E}$ is epic: if $\mathcal{E}$ is decomposable, then it is a direct sum of two invertible sheaves of degree $\geq 2$ (because no $\mathcal{O}_E(x)$ is a direct summand of $\mathcal{E}$) so is generated by their global sections; if $\mathcal{E}$ is indecomposable it is also generated by its global sections (by [CKS21b, Lem. 4.8(4)], for example) because it is then semistable and, by hypothesis, its slope is $\frac{n}{2} > 1$. Thus, there is an exact sequence

(3-11) \[ 0 \to Z \to H^0(E, \mathcal{E}) \otimes \mathcal{O}_E \to \mathcal{E} \to 0. \]

Since $Z$ is a submodule of a locally free $\mathcal{O}_E$-module it is also locally free (of rank $n - 2$ because the right-most map in (3-11) is epic). Let $x \in E$. Since $\text{Tor}_1(\mathcal{O}_E/\mathfrak{m}_x, \mathcal{E}) = 0$, applying the functor $\mathcal{O}_E/\mathfrak{m}_x \otimes -$ to (3-11) produces an exact sequence

(3-12) \[ 0 \to Z/\mathfrak{m}_x Z \to H^0(E, \mathcal{E}) \to \mathcal{E}/\mathfrak{m}_x \mathcal{E} \to 0 \]

of vector spaces. The kernel of the map $H^0(E, \mathcal{E}) \to \mathcal{E}/\mathfrak{m}_x \mathcal{E}$ is the image of the map $H^0(E, \mathcal{E}(-x)) \cong \text{Hom}(\mathfrak{m}_x^{-1}, \mathcal{E}) \to H^0(E, \mathcal{E})$ that appeared in the proof of Proposition 3.13, so the fiber over $x$ of the vector bundle associated to $Z$ is $Z_x \cup \{0\}$.

The map $Z \to H^0(E, \mathcal{E}) \otimes \mathcal{O}_E$ gives rise to a morphism $\phi : \mathbb{P}(Z) \to E \times \mathbb{P}H^0(E, \mathcal{E})$ between the corresponding projective-space bundles. The composition of $\phi$ with the projection to $\mathbb{P}H^0(E, \mathcal{E})$ is a morphism $\mathbb{P}(Z) \to \mathbb{P}H^0(E, \mathcal{E})$ whose image is the union of all the $\mathbb{P}Z_x$'s; i.e., its image is $\mathbb{P}Z \subseteq \mathbb{P}H^0(E, \mathcal{E})$. Since $\phi$ is a projective morphism, $\mathbb{P}Z$ is a closed subvariety of $\mathbb{P}H^0(E, \mathcal{E})$, and is irreducible because $\mathbb{P}(Z)$ is irreducible. Since rank $Z = n - 2$, dim $\mathbb{P}(Z) = n - 2$.

Finally, the morphism $\mathbb{P}(Z) \to \mathbb{P}H^0(E, \mathcal{E})$ has finite fibers, because the cokernel of a morphism $\mathcal{O}_E \to \mathcal{E}$ can only contain finitely many $\mathcal{O}_x$'s, and is therefore finite because it is a projective morphism ([Har77, Exer. III.11.2]). Hence dim $\mathbb{P}Z =$ dim $\mathbb{P}(Z) = n - 2$; i.e., $\mathbb{P}Z$ is a hypersurface, as claimed. \(\square\)

Remark 3.15. The argument in part (2) of the proof of Proposition 3.14 is a surreptitious application of Grauert’s Theorem [Har77, Cor. III.12.9]. To see this, let $\pi_1, \pi_2 : E \times E \to E$ be the projections to the first and second factors, respectively, and let $\Delta \subseteq E^2$ denote the diagonal.

From the exact sequence $0 \to \mathcal{O}_{E^2}(-\Delta) \to \mathcal{O}_{E^2} \to \mathcal{O}_\Delta \to 0$ we obtain exact sequences

(3-13) \[ 0 \to \mathcal{O}_{E^2}(-\Delta) \otimes \pi_1^* \mathcal{E} \to \pi_1^* \mathcal{E} \to \mathcal{O}_\Delta \otimes \pi_1^* \mathcal{E} \to 0 \]

and

(3-14) \[ 0 \to \pi_{2*}(\mathcal{O}_{E^2}(-\Delta) \otimes \pi_1^* \mathcal{E}) \to \pi_{2*}\pi_1^* \mathcal{E} \to \pi_{2*}(\mathcal{O}_\Delta \otimes \pi_1^* \mathcal{E}) \cong \mathcal{E} \]

where the isomorphism at the right follows from the fact that the restriction of $\pi_2$ to $\Delta$ is an isomorphism onto $E$. We also note that $\pi_{2*}\pi_1^* \mathcal{E} \cong H^0(E, \mathcal{E}) \otimes \mathcal{O}_E$ and that the right-most map in (3-14) is the natural map $H^0(E, \mathcal{E}) \otimes \mathcal{O}_E \to \mathcal{E}$, which is an epimorphism as shown in the proof of Proposition 3.14. Hence the kernel, $\pi_{2*}(\mathcal{O}_{E^2}(-\Delta) \otimes \pi_1^* \mathcal{E})$, is isomorphic to $Z$ in (3-11). Each term in (3-14) is the sheaf of sections of a bundle on $E$, and the bundle associated to the middle term $\pi_{2*}\pi_1^* \mathcal{E}$ is the trivial bundle $E \times H^0(E, \mathcal{E})$.

The module $\mathcal{E}(-x)$ is the $\pi_x$-fiber over $x \in E$ of the bundle corresponding to $\mathcal{O}_{E^2}(-\Delta) \otimes \pi_1^* \mathcal{E}$. Since all $H^0(E, \mathcal{E}(-x))$'s have the same dimension, $n - 2$ (by Lemma 3.11 because $\mathcal{O}_E(x)$ is not a direct summand of $\mathcal{E}$), [Har77, Cor. III.12.9] tells us that

$$H^0(E, \mathcal{E}(-x)) \cong \text{the fiber over } x \text{ of the bundle corresponding to } \pi_{2*}(\mathcal{O}_{E^2}(-\Delta) \otimes \pi_1^* \mathcal{E}).$$

Similarly, since $\pi_{2*}\pi_1^* \mathcal{E} \cong H^0(E, \mathcal{E}) \otimes \mathcal{O}_E$,

$$H^0(E, \mathcal{E}) \cong \text{the fiber over } x \text{ of the bundle corresponding to } \pi_{2*}\pi_1^* \mathcal{E}.$$
The map $\pi_{2*}(O_{E^2}(-\Delta) \otimes \pi_1^*\mathcal{E}) \to \pi_{2*}\pi_1^*\mathcal{E}$ is by construction a fiber-wise linear embedding, and hence induces a map

$$\mathbb{P}(\pi_{2*}(O_{E^2}(-\Delta) \otimes \pi_1^*\mathcal{E})) \to \mathbb{P}(\pi_{2*}\pi_1^*\mathcal{E}) \cong E \times \mathbb{P}H^0(E, \mathcal{E})$$

of projective-space bundles over $E$. Projecting onto the second component of the last term gives a map

$$\mathbb{P}(\pi_{2*}(O_{E^2}(-\Delta) \otimes \pi_1^*\mathcal{E})) \to \mathbb{P}\text{Hom}(O_E, \mathcal{E})$$

of projective varieties which maps the fiber $\mathbb{P}H^0(E, \mathcal{E}(-x))$ above $x$ isomorphically to the subspace $\mathbb{P}Z_x \subseteq \mathbb{P}\text{Hom}(O_E, \mathcal{E})$. Because $Z$ is the union of the $Z_x$'s, it follows that $\mathbb{P}Z$ is the image of the morphism (3-15) of projective varieties, and hence Zariski-closed and irreducible. \hfill \Box

**Corollary 3.16.** If $\mathcal{E}$ satisfies the assumptions in §2.2, then $L(\mathcal{E}) \neq \emptyset$; i.e., there is a non-split extension of the form $0 \to O_E \to \mathcal{E} \to \mathcal{L} \to 0$.

**Proof.** By Proposition 3.14, $\mathbb{P}H^0(E, \mathcal{E}) - \mathbb{P}Z_E \neq \emptyset$, so there is a non-zero map $f : O_E \to \mathcal{E}$ having torsion-free cokernel. Now apply Lemma 3.2. \hfill \Box

Together with Proposition 2.3 and Theorem 2.4, we have now completed the proof of Theorem 1.1; $L(\mathcal{E}) \neq \emptyset$ if and only if either $\mathcal{E} \cong \mathcal{E}_{d, x}$ for some $d \in [1, \frac{n}{2}]$ and some $x \in E$ or $\mathcal{E}$ is indecomposable, in which case it is isomorphic to $\mathcal{E}_\omega$ when $n$ is odd and is one of the four $\mathcal{E}_\omega$'s, $\omega \in \Omega$, when $n$ is even.

**Corollary 3.17.** If $\mathcal{E}$ satisfies the assumptions in §2.2, then $\mathbb{P}X(\mathcal{E})$ is an affine variety.

**Proof.** By Proposition 3.14, $\mathbb{P}Z_E \subseteq \mathbb{P}H^0(E, \mathcal{E})$ is the zero locus of a homogeneous polynomial so its open complement is affine. \hfill \Box

The bundles $\mathcal{L}_\omega$ for $\omega \in \Omega = \{z \in E \mid 2z = \sigma(H)\}$ were defined in §2.3.1. When $n = 4$, $\mathcal{L}_\omega = O_E(\omega + (0))$.

**Proposition 3.18.** If $\mathcal{E}$ satisfies the assumptions in §2.2, then

- (1) $\deg \mathbb{P}Z_E = 2$ if $\mathcal{E} \cong \mathcal{E}_{1, x}$ for some $x \in E$, i.e., if some $O_E(x)$ is a direct summand of $\mathcal{E}$,
- (2) $\deg \mathbb{P}Z_E = \frac{n}{2} = 2$ if $n = 4$ and $\mathcal{E} \cong \mathcal{L}_\omega \oplus \mathcal{L}_\omega$ for some $\omega \in \Omega$, and
- (3) $\deg \mathbb{P}Z_E = n$ otherwise.

**Proof.** We will write $Z$ in place of $Z_E$.

(1) If $\mathcal{E} \cong \mathcal{E}_{1, x}$, then $\mathbb{P}Z$ is a union of two hyperplanes by Proposition 3.14, so $\deg \mathbb{P}Z = 2$.

(2,3) We now assume that no $O_E(x)$ is a direct summand of $\mathcal{E}$. Hence, by (3-8), $h^0(E, \mathcal{E}(-x)) = h^0(E, \mathcal{E}) - 2$ for all $x \in E$.

By definition,

$$\mathbb{P}Z = \bigcup_{x \in E} \mathbb{P}Z_x = \bigcup_{x \in E} \mathbb{P}\text{Hom}(O_E, \mathcal{E}(-x)) \subseteq \mathbb{P}\text{Hom}(O_E, \mathcal{E}).$$

Since $\mathbb{P}Z_x$ is a codimension-two linear subspace of $\mathbb{P}\text{Hom}(O_E, \mathcal{E})$, there is a morphism

$$E \to \mathbb{G}(n - 3, n - 1), \quad x \mapsto \mathbb{P}Z_x,$$

where $\mathbb{G}(n - 3, n - 1)$ is the Grassmannian of $(n - 3)$-planes in $\mathbb{P}\text{Hom}(O_E, \mathcal{E})$: as noted in Remark 3.15, the inclusions $H^0(E, \mathcal{E}(-x)) \subseteq H^0(E, \mathcal{E})$ glue to an embedding of a rank-$(n - 2)$ vector bundle over $E$ into the trivial vector bundle $H^0(E, \mathcal{E}) \times E$, and the fact that this gives a morphism into the Grassmannian then follows from the latter’s universal property [EH16, Thm. 3.4].

---

13That the complement is affine is a special case of a more general fact that we will use later: if $D$ is an ample effective divisor on a projective scheme $X$, then the complement $X - D$ is affine. To see this, first note that if $Y$ is a closed subvariety of $X$, then $X - Y = X - Y_{red}$, so there is no loss of generality in replacing $D$ by $mD$ where $m$ is a positive integer. Thus, we can assume that $D$ is very ample. After identifying $X$ with its image under the morphism $X \to \mathbb{P}^r := \mathbb{P}H^0(X, O_X(D))^*$ associated to the complete linear system $|D|$, there is a hyperplane $H \subseteq \mathbb{P}^r$ such that $X - D = X - (X \cap H) = X \cap (\mathbb{P}^r - H)$; hence $X - D$ is a closed subscheme of the affine scheme $\mathbb{P}^r - H$ and is therefore an affine scheme also.
We are now in the setting of [EH16, §10.2], and wish to apply [EH16, Prop. 10.4]. To do this we must determine the number $d$ in [EH16, Prop. 10.4]. By definition, $d$ is the unique number having the following property: a general point in $\mathbb{P}Z$ belongs to $\mathbb{P}Z_x$ for exactly $d$ different $x$’s; i.e., $d$ is the unique positive integer such that there is a non-empty open set $U \subseteq \mathbb{P}Z$ with the property that every point in $U$ belongs to $\mathbb{P}Z_x$ for exactly $d$ different $x$’s. We will show that $d = 2$ in the case of (2) and that $d = 1$ in the case of (3).

If $x \neq y$, then a section of $\mathcal{E}$ vanishes at both $x$ and $y$ if and only if it is a section of $\mathcal{E}(-x - y)$ so, as subspaces of $H^0(\mathcal{E})$,
\[
H^0(\mathcal{E}(-x)) \cap H^0(\mathcal{E}(-y)) = H^0(\mathcal{E}(-x - y))
\]
whence
\[
\mathbb{P}Z_x \cap \mathbb{P}Z_y = \mathbb{P}H^0(\mathcal{E}(-x - y)).
\]
Hence
\[(3-16) \quad \{p \in \mathbb{P}Z \mid p \text{ belongs to } \mathbb{P}Z_x \text{ for at least two different } x \text{'s} \} = \bigcup_{D \in E^{[2]} - \Delta} \mathbb{P}H^0(\mathcal{E}(-D))
\]
where $\Delta = \{(x, x) \mid x \in E\} \subseteq E^{[2]}$.

Proof of (2). Suppose that $n = 4$ and $\mathcal{E} \cong \mathcal{L}_\omega \oplus \mathcal{L}_\omega$ for some $\omega \in \Omega$.

Claim: If $p$ in $\mathbb{P}Z_x$, then $p$ belongs to $\mathbb{P}Z_y$ if and only if $y \in \{x, \omega - x\}$. Proof: Suppose $x \neq y$. It follows from Lemma 3.11(1) that $h^0(\mathcal{E}(-x)) = h^0(\mathcal{E}(-y)) = 2$, and follows from Lemma 3.12(1) that
\[h^0(\mathcal{E}(-x - y)) = \begin{cases} 2 & \text{if } x + y = \omega \\ 0 & \text{if } x + y \neq \omega. \end{cases}\]
Hence $\mathbb{P}Z_x \cap \mathbb{P}y = \emptyset$ if $x + y \neq \omega$, and $\mathbb{P}Z_x = \mathbb{P}Z_y$ if $x + y = \omega$. ◇

Hence $d = 2$.$^{14}$

Proof of (3). We are still assuming that no $\mathcal{O}_E(x)$ is a direct summand of $\mathcal{E}$. And, since we are not in case (2), if $n = 4$, then $\mathcal{E}$ is not isomorphic to $\mathcal{L}_\omega \oplus \mathcal{L}_\omega$ for any $\omega \in \Omega$. Hence, by Lemma 3.11, $h^0(\mathcal{E}(-x)) = h^0(\mathcal{E}) - 2$ for all $x \in E$.

Let $D \in E^{[2]}$ be a degree-two divisor. If $n = 3$, then $h^0(\mathcal{E}(-D)) = 0$ by Lemma 3.12 so $\mathbb{P}Z_x \cap \mathbb{P}Z_y = \emptyset$ for all $(x, y) \in E^{[2]} - \Delta$, whence $d = 1$. If $n \geq 4$, then we are not in the third case of Lemma 3.12(1) so
\[h^0(\mathcal{E}(-D)) = \begin{cases} h^0(\mathcal{E}) - 4 & \text{if } \mathcal{O}_E(D) \text{ is not a direct summand of } \mathcal{E} \\ h^0(\mathcal{E}) - 3 & \text{if } \mathcal{O}_E(D) \text{ is a multiplicity-one direct summand of } \mathcal{E}. \end{cases}\]
The second case occurs for only finitely many $D$ up to linear equivalence. Hence the subset, $U$ say, of $E^{[2]} - \Delta$ for which the first case occurs is a non-empty and open subset of $E^{[2]}$. Hence
\[\bigcup_{D \in U} \mathbb{P}H^0(\mathcal{E}(-D))\]
is a non-empty open subset of $\mathbb{P}Z$ and, arguing as in [EH16, discussion preceding Prop. 10.4] (where their $m$ denotes the dimension of their $B$), we see that this open subset has dimension $2 + (n - 4) - 1 = n - 3$. However, $\dim \mathbb{P}Z = n - 2$ so a general point in $\mathbb{P}Z$ does not belong to that open subset. In other words, a general point in $\mathbb{P}Z$ belongs to exactly one $\mathbb{P}Z_x$, whence $d = 1$.

This concludes the proof that $d = 1$ in the case of (3).

We can now translate [EH16, Prop. 10.4] to the present setting:

- Let $\mathcal{F}$ be the vector bundle on $E$ whose $x$-fiber is $H^0(\mathcal{E}(-x))$ (which made an appearance in the proof of Proposition 3.14, where we argued that it is indeed a bundle).

---

$^{14}$The proof shows that $\{p \in \mathbb{P}Z \mid p \text{ belongs to } \mathbb{P}Z_x \text{ for a unique } x \in E\} = \bigcup\{x \mid 2x = \omega\} \mathbb{P}Z_x$. The set of such $x$’s has four elements because it is an $E^{[2]}$-coset.
• Taking into account the relation between Chern and Segre classes in the Chow ring of $E$ ([EH16, §10.1, especially Defn. 10.1 and Prop. 10.3]) and the fact that in the present situation the number $m$ in [EH16, Prop. 10.4] is 1, that result says that

$$\text{the degree of } \mathbb{P}Z \text{ as a subvariety of } \mathbb{P}H^0(E, \mathcal{E}) = -\frac{\deg \mathcal{F}}{d}$$

(3-17)

(where, as usual, the degree of $\mathcal{F}$ is the degree of its top exterior power: e.g., [Har77, proof of Prop. V.2.8], [Dal07, Definition 2.14], etc.).

Consider, now, the exact sequence

$$0 \to \mathcal{E}(-x) \to \mathcal{E} \to \mathcal{E}_x \to 0$$

(3-18)

where $\mathcal{E}_x = \mathcal{E}/m_x \mathcal{E}$. The pullback of $\mathcal{E}_x$ to Spec($\mathbb{C}$) = $\{x\}$ via the morphism $x \to E$ is a 2-dimensional vector space. In the cases under consideration ($\mathcal{E}$ of degree $\geq 3$, either indecomposable or isomorphic to $\mathcal{O}_E(D) \oplus \mathcal{O}_E(D')$ with both $D$ and $D'$ having degree $\geq 2$), $H^1(E, \mathcal{E}(-x)) = 0$ so there is an exact sequence

$$0 \to H^0(E, \mathcal{E}(-x)) \to H^0(E, \mathcal{E}) \to H^0(E, \mathcal{E}_x) \to 0$$

(3-19) canonically in $x \in E$, and hence an exact sequence

$$0 \to \mathcal{F} \to H^0(E, \mathcal{E}) \otimes \mathcal{O}_E \to \mathcal{F}' \to 0$$

where $\mathcal{F}'$ is the bundle whose $x$-fiber is $H^0(E, \mathcal{E}_x)$. But $\mathcal{F}'$ is easily seen to be precisely $\mathcal{E}$: $\mathcal{E}_x$ is the $x$-fiber of $\Delta_* \mathcal{E}$ through the second projection $\pi_2 : E^2 \to E$, where

$$\Delta : E \to E \times E$$

is the diagonal. Since the dimension of $H^0(E, \mathcal{E}_x)$ is the same for all $x \in E$, Grauert’s theorem, [Har77, Cor. III.12.9], ensures that $\mathcal{F}'$ is indeed a bundle, and specifically

$$\pi_2^* \Delta_* \mathcal{E} \cong \mathcal{E} \quad \text{because} \quad \pi_2 \circ \Delta = \text{id}.$$ 

It follows that $\deg \mathcal{F}' = \deg E = n$ whence $\deg \mathcal{F} = -\deg \mathcal{F}' = -n$ by (3-19) and additivity of degree [Har77, Exer. II.6.12]. Given (3-17), this finishes the proof. \hfill \Box

Remark 3.19. The latter, more elaborate, part of the proof of Proposition 3.18 also applies to the simple case $\mathcal{E} \cong \mathcal{O}_E \oplus \mathcal{L}$: there too, the computation leading up to the equality $\deg \mathcal{F} = -\deg \mathcal{F}' = -n$ still works. The difference now is that in this degenerate situation we have $\mathbb{P}Z = \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{L}) \subset \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{E})$ so the generic element of that hyperplane in $\mathbb{P}H^0(E, \mathcal{E})$ will lie on $n$ subspaces $\mathbb{P}Z_x$: a section of $\mathcal{L}$ vanishes at $n$ points of $E$, including multiplicities. This means that the $d$ factor in [EH16, Prop. 10.4], which was 1 throughout the bulk of the proof of Proposition 3.18, would in this case be $n$ instead. As claimed, then,

$$\text{degree of } \mathbb{P}Z \subset \mathbb{P}(\text{Hom}(\mathcal{O}_E, \mathcal{E})) = -\frac{\deg \mathcal{F}}{d} = -\frac{-n}{n} = 1.$$ 

As for the case $\mathcal{E} \cong \mathcal{O}_E(x) \oplus \mathcal{L}(-x)$, it is a mixture of sorts: here $\mathbb{P}Z$ is a union of two hyperplanes, and the generic point of one of them lies on a unique $\mathbb{P}Z_x$, while the generic point of the other lies on $(n - 1)$ such planes. \hfill \diamondsuit

3.4. Serre duality, the notation $\xi^\perp$, and $T_i \mathbb{P}_L$ and $T^*_i \mathbb{P}_L$. The tangent and cotangent spaces at a point $\xi \in \mathbb{P}_L$ are described in [HP20, §2.1, p.3]. Since our conventions and notation are not quite the same as those in [HP20] we provide proofs in Lemmas 3.22 and 3.23.
3.4.1. **Serre duality and related notation.** Fix an isomorphism $t : \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) \to \mathbb{C}$ of vector spaces and define the map
\begin{equation}
\text{Ext}^1(\mathcal{L}, \mathcal{O}_E) \times \text{Hom}(\mathcal{O}_E, \mathcal{L}) \to \mathbb{C}, \quad (\eta, s) \mapsto t(\eta \cdot s),
\end{equation}
where $\eta \cdot s \in \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E)$ is the pullback of $\eta$ along the homomorphism $s : \mathcal{O}_E \to \mathcal{L}$.\(^{15}\) The fact that the map in (3-20) is non-degenerate is the essence of Serre duality in this situation. We use the pairing in (3-20) to make the identifications
\begin{align}
\text{Ext}^1(\mathcal{L}, \mathcal{O}_E)^* &= \text{Hom}(\mathcal{O}_E, \mathcal{L}), \quad t(- \cdot s) = s, \\
\text{Ext}^1(\mathcal{L}, \mathcal{O}_E) &= \text{Hom}(\mathcal{O}_E, \mathcal{L})^*, \quad \eta = t(\eta \cdot -).
\end{align}
If $\eta \in \text{Ext}^1(\mathcal{L}, \mathcal{O}_E)$, we define
\begin{equation}
\eta^\perp := \{ s \in \text{Hom}(\mathcal{O}_E, \mathcal{L}) \mid \eta \cdot s = 0 \};
\end{equation}
although the identifications in (3-21) and (3-22) depend on the choice of $t$, $\eta^\perp$ does not. Given a point $\xi = \mathbb{C} \eta \in \mathbb{P}_E$ we define
\begin{equation}
\xi^\perp := \eta^\perp.
\end{equation}
The next lemma gives a description of $\xi^\perp$ that will be used repeatedly in what follows.

The symbol $\perp$ is also used in the following way: given a subspace $W \subseteq \text{Hom}(\mathcal{O}_E, \mathcal{L})$, we define
\begin{equation}
W^\perp := \{ \lambda \in \text{Hom}(\mathcal{O}_E, \mathcal{L})^* \mid \lambda(s) = 0 \text{ for all } s \in W \}.
\end{equation}
When $\text{codim} W = 1$ we often consider $W^\perp$ as a point in $\mathbb{P}_E \text{Hom}(\mathcal{O}_E, \mathcal{L})^* = \mathbb{P} \text{Ext}^1(\mathcal{L}, \mathcal{O}_E) = \mathbb{P}_E$; because $W^\perp \in \mathbb{P}_E$ we can use (3-24) to define $(W^\perp)^\perp$; we have $(W^\perp)^\perp = W$. Thus the definitions in (3-24) and (3-25) are compatible with each other.

Given a homomorphism $\pi : \mathcal{E} \to \mathcal{L}$, we define the map
\begin{equation}
\pi_* : \text{Hom}(\mathcal{O}_E, \mathcal{E}) \to \text{Hom}(\mathcal{O}_E, \mathcal{L}), \quad \pi_*(b) := \pi b.
\end{equation}

**Lemma 3.20.** Let $\xi \in \text{Ext}^1(\mathcal{L}, \mathcal{O}_E)$ be a non-split extension $0 \to \mathcal{O}_E \to \mathcal{E} \xrightarrow{\pi} \mathcal{L} \to 0$.

1. The image of the map $\pi_*$ in (3-26) is a codimension-one subspace of $\text{Hom}(\mathcal{O}_E, \mathcal{L})$, namely
   \[ \text{im}(\pi_*) = \{ s \in \text{Hom}(\mathcal{O}_E, \mathcal{L}) \mid \xi \cdot s = 0 \} = \xi^\perp. \]

2. Under the Serre duality identifications in (3-21), $\xi^\perp = \text{im}(\pi_*)$ and $\mathbb{C}\xi = \text{im}(\pi_*)^\perp$.

**Proof.** (1) This follows from the long exact sequence obtained by applying $\text{Hom}(\mathcal{O}_E, -)$ to $\xi$.

(2) The natural pairing $\text{Ext}^1(\mathcal{L}, \mathcal{O}_E) \times \text{Hom}(\mathcal{O}_E, \mathcal{L}) \to \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E)$ that gives rise to the Serre duality identifications is the map $(\xi, s) \mapsto \xi \cdot s$ where $\xi \cdot s$ is the top row in the pullback
\begin{equation}
\begin{array}{ccccccc}
\xi \cdot s : & 0 & \to & \mathcal{O}_E & \to & \mathcal{E} \times_{\mathcal{L}} \mathcal{O}_E & \xrightarrow{\pi^*} & \mathcal{O}_E & \to & 0 \\
\xi : & 0 & \to & \mathcal{O}_E & \to & \mathcal{E} & \xrightarrow{\pi} & \mathcal{L} & \to & 0.
\end{array}
\end{equation}
Since $\xi \cdot s = 0$ if and only if the top row splits, $\xi \cdot s = 0$ if and only if there is a map $g : \mathcal{O}_E \to \mathcal{E}$ such that $\pi g = s$; i.e., if and only if $s \in \text{im}(\pi_*)$. Hence $\xi^\perp = \text{im}(\pi_*)$. The equality $\xi = \text{im}(\pi_*)^\perp$ now follows from the remarks about $(W^\perp)^\perp$ at the end of §3.4.1. \(\square\)

**Remark 3.21.** It follows from this lemma that if $\varphi \in \text{Hom}(\mathcal{E}, \mathcal{L})$, then $\varphi f \in \xi^\perp$ because Lemma 3.10 showed that $\varphi f \in \text{im}(\pi_*)$. \(\square\)

\(^{15}\)See the proof of Lemma 3.20(2) for more about this.
3.4.2. Tangent and cotangent spaces. We write

\[ G := \mathcal{G}(n - 1, \text{Hom}(O_E, \mathcal{L})) \]

for the Grassmannian of codimension-one subspaces of Hom\((O_E, \mathcal{L})\). We will consider the isomorphism \(F : \mathbb{P}_L \to G\) given by

\[ \mathbb{P}_L = \mathcal{G}(1, \text{Ext}^1(\mathcal{L}, O_E)) = \mathcal{G}(1, \text{Hom}(O_E, \mathcal{L})^*) \sim G \]

\[ \mathbb{C} \xi = \text{im}(\pi_*) \quad \text{and} \quad \text{im}(\pi_*) = \xi^\perp, \]

i.e., \(F(\xi) = \xi^\perp\), and its differential \(dF_\xi : T_\xi \mathbb{P}_L \to T_{\text{im}(\pi_*)}G\).

**Lemma 3.22.** Let \(\xi \in \mathbb{P}_L\) be the isomorphism class of the non-split extension

\[ 0 \to O_E \xrightarrow{f} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \to 0. \]

The tangent spaces to \(\mathbb{P}_L\) at \(\xi\) and to \(G\) at \(\xi^\perp\) are

\[ T_\xi \mathbb{P}_L = \text{Hom} \left( \mathbb{C} \xi, \text{Ext}^1(\mathcal{L}, O_E)/\mathbb{C} \xi \right) \quad \text{and} \quad (3-29) \]

\[ T_\xi G = \text{Hom} \left( \xi^\perp, \text{Hom}(O_E, \mathcal{L})/\xi^\perp \right) \quad \text{and} \quad (3-30) \]

\[ \cong \{ \alpha : \text{Hom}(O_E, \mathcal{E}) \to \text{Hom}(O_E, \mathcal{L})/\xi^\perp | \alpha(f) = 0 \}. \]

**Proof.** The equalities in (3-29) and (3-30) follow from the proof of [EH16, Thm. 3.5, p. 96] (see, for example, the sentence just after its proof).

To prove (3-31), consider the exact sequence

\[ 0 \to \text{Hom}(O_E, O_E) \xrightarrow{f_*} \text{Hom}(O_E, \mathcal{E}) \xrightarrow{\pi_*} \text{Hom}(O_E, \mathcal{L}). \]

where \(f_*(a) = fa\) and \(\pi_*(b) = \pi b\). Since

\[ \text{im}(\pi_*) \cong \frac{\text{Hom}(O_E, \mathcal{E})}{\ker(\pi_*)} = \frac{\text{Hom}(O_E, \mathcal{E})}{\text{im}(f_*)} = \frac{\text{Hom}(O_E, \mathcal{E})}{\mathbb{C} f}, \]

we have

\[ \text{Hom} \left( \text{im}(\pi_*), \text{Hom}(O_E, \mathcal{L})/\text{im}(\pi_*) \right) \cong \text{Hom} \left( \frac{\text{Hom}(O_E, \mathcal{E})}{\mathbb{C} f}, \frac{\text{Hom}(O_E, \mathcal{L})}{\text{im}(\pi_*)} \right), \]

thus giving the isomorphism in (3-31).

**Lemma 3.23.** Let \(\xi\) be a non-split exact sequence \(0 \to O_E \xrightarrow{f} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \to 0\).

1. The cotangent space to \(\mathbb{P}_L\) at \(\xi\) is \(T^*_\xi \mathbb{P}_L = \xi^\perp \subseteq \text{Hom}(O_E, \mathcal{L}).\)
2. The cotangent space to \(G\) at \(\xi^\perp = \text{im}(\pi_*)\) is

\[ T^*_\xi G = \text{Hom} \left( \text{Hom}(O_E, \mathcal{L})/\xi^\perp, \xi^\perp \right) \]

\[ \cong \{ \alpha : \text{Hom}(O_E, \mathcal{L}) \to \xi^\perp | \alpha(\xi^\perp) = 0 \}. \]

3. Let \(\delta : \text{Hom}(O_E, \mathcal{L}) \to \text{Ext}^1(O_E, O_E)\) be the connecting homomorphism, and fix \(s \in (t\delta)^{-1}(1)\). The map \(T^*_\xi G \to T^*_\xi \mathbb{P}_L, \alpha \mapsto \alpha(s)\), that sends \(\alpha\) in (3-32) to \(\alpha(s)\) is an isomorphism.

**Proof.** (1) The cotangent space \(T^*_x M\) at a point \(x\) on a manifold \(M\) is naturally dual to \(T_x M\). Hence

\[ T^*_\xi \mathbb{P}_L = \text{Hom}(\mathbb{C} \xi, \text{Ext}^1(\mathcal{L}, O_E)/\mathbb{C} \xi)^* \]

\[ = \text{Hom}(\text{Ext}^1(\mathcal{L}, O_E)/\mathbb{C} \xi, \mathbb{C} \xi) \]

\[ = \text{Hom}(\text{Hom}(O_E, \mathcal{L})^*/\mathbb{C} \xi, \mathbb{C} \xi). \]

The last of these spaces is the kernel of the map \(\text{Hom}(\text{Hom}(O_E, \mathcal{L})^*, \mathbb{C} \xi) \to \text{Hom}(\mathbb{C} \xi, \mathbb{C} \xi), \rho \mapsto \rho|_{\mathbb{C} \xi}, \) which is \(\xi^\perp\).
Lemma 3.24. Suppose $E$ satisfies the assumptions in §2.2. The map
\begin{equation}
\psi_E : X(E) \rightarrow \mathbb{P}_L
\end{equation}
that sends $f \in X(E)$, or the corresponding $\pi \in \text{Epi}(E, L)$, to the isomorphism class of the extension
\begin{equation}
\xi := \begin{array}{ccc}
0 & \xrightarrow{f} & O_E \\
& \xrightarrow{\pi} & E \\
& & \xrightarrow{L} 0
\end{array}
\end{equation}
is a morphism whose image is $L(E)$. 

\textbf{Proof.} As in §3.4, we set $G := G(n - 1, \text{Hom}(O_E, L))$.

Let $f \in X(E)$, let $\pi \in \text{Epi}(E, L)$ be the corresponding epimorphism in Lemma 3.8, and let $\xi = \psi_E(f)$ be the corresponding extension in (3-34). We give $\text{Epi}(E, L)$ its usual algebraic structure as a Zariski-open subset of $\text{Hom}(E, L)$. The linear bijection $X(E) \rightarrow \text{Epi}(E, L)$ in Lemma 3.8 is an isomorphism of quasi-affine varieties so it suffices to show that the map $\text{Epi}(E, L) \rightarrow \mathbb{P}_L$, $\pi \mapsto \xi$, is a morphism. Since the map $\xi \mapsto \text{im}(\pi_*) = \xi^\perp$ is an isomorphism $\mathbb{P}_L \rightarrow G$, it suffices to show that the map $\text{Epi}(E, L) \rightarrow G$, $\pi \mapsto \text{im}(\pi_*)$, is a morphism. That is what we will do.

The map that sends $\pi : E \rightarrow L$ to the linear map $\pi_*$ in (3-26) is itself a linear map, hence a morphism. Restricting that map to \textit{epimorphisms} produces linear maps $\text{Hom}(O_E, E) \rightarrow \text{Hom}(O_E, L)$ of maximal rank, i.e., of rank $n - 1 = \deg L - 1 = \deg E - 1$. It is a standard fact that under these conditions the map $\text{Epi}(E, L) \rightarrow G$, $\pi \mapsto \text{im}(\pi_*)$, is a morphism: see, for example, [Sal99, Prop. 13.4] or [CKS21a, Prop. 3.17(1)]. \hfill \Box

Remark 3.25. We will often use the projectivized version $\mathbb{P}X(E) \rightarrow \mathbb{P}_L$ of Lemma 3.20.

We will eventually see that the homological leaves $L(E)$ are locally closed subsets of $\mathbb{P}_L$. For now, the next lemma suffices. Recall that a set is \textit{constructible} if it is a disjoint union of locally closed subsets (see [Bor69, §AG.1.3] or [Har77, Exer. II.3.18], for example).

Lemma 3.26. If $E$ satisfies the assumptions in §2.2, then $L(E)$ is an irreducible constructible subset of $\mathbb{P}_L$ and
\begin{equation}
\dim L(E) = n - \dim \text{Aut}(E).
\end{equation}

\textbf{Proof.} Let $\psi : X(E) \rightarrow L(E)$ be the corestriction of the morphism $\psi_E$ in Lemma 3.24. Since $L(E)$ is the image of $\psi$, it is constructible by Chevalley’s Theorem [Har77, Exer. II.3.19]. But $L(E)$ is dense in $L(E)$, so $\psi$ is dominant. Since the domain of $\psi$ is irreducible (by Proposition 3.14) its image is also irreducible; hence $L(E)$ is irreducible.

All non-empty fibers of $\psi$ are isomorphic as schemes to the algebraic group $\text{Aut}(E)$ (Lemmas 3.3 and 3.5) so have the same dimension, namely $\dim \text{Aut}(E)$ (which is given in Lemma 3.28). However, by [Har77, Exer. II.3.22(b),(c)], there is a dense open subset of $L(E)$ (hence also one of $L(E)$) over which the fibers all have irreducible components of dimension exactly
\begin{equation}
e_{\psi} := \dim(\text{domain of } \psi) - \dim(\text{codomain of } \psi).
\end{equation}
It follows that $e_{\psi} = \dim \text{Aut}(E)$, which is the desired conclusion rephrased. \hfill \Box
Remark 3.27. The statement about \( \dim L(\mathcal{E}) \) in Lemma 3.26 is compatible with [HP20, Prop. 2.3] which says that the rank of the Poisson bracket \( \Pi \) on \( \mathbb{P}_L \) at a point \( \xi \in \mathbb{P}_L \) equals \( n - \dim \text{End}(\mathcal{E}) \) where \( \mathcal{E} \) is the middle term of the extension \( \xi \). In short, rank \( \Pi_\xi = n - \dim \text{End}(\mathcal{E}) \).

The next result, Lemma 3.28, computes the dimension of \( \text{Aut}(\mathcal{E}) \); that together with Lemma 3.26 allows us to determine \( \dim L(\mathcal{E}) \) and Theorem 1.3 follows at once.

We follow customary practice (e.g., [Hum75, §7.1]) by writing
\[
\begin{align*}
\mathbb{G}_a & \text{ for the additive } 1\text{-dimensional algebraic group, which is } (\mathbb{C}, +) \text{ in this paper, and } \\
\mathbb{G}_m & \text{ for the multiplicative group } (\mathbb{C}^*, \cdot).
\end{align*}
\]

**Lemma 3.28.** If \( \mathcal{E} \) satisfies the assumptions in §2.2, then \( \text{Aut}(\mathcal{E}) \) is isomorphic to
\[
\begin{align*}
(1) & \quad \mathbb{G}_m \text{ if } \mathcal{E} \text{ is indecomposable and } n \text{ is odd;} \\
(2) & \quad \mathbb{G}_a \times \mathbb{G}_m \text{ if } \mathcal{E} \text{ is indecomposable and } n \text{ is even;} \\
(3) & \quad \left( \mathbb{G}_a^{\text{deg}N_1 - \text{deg}N_2} \times \mathbb{G}_m \right) \times \mathbb{G}_m \text{ if } \mathcal{E} \cong N_1 \oplus N_2 \text{ and } N_1 \not\cong N_2, \text{ with the first copy of } \mathbb{G}_m \text{ acting on } \\
& \quad \mathbb{G}_a \text{ by scaling;} \\
(4) & \quad \text{GL}(2) \text{ if } \mathcal{E} \cong N' \oplus N.
\end{align*}
\]

In particular, the dimension of \( \text{Aut}(\mathcal{E}) \) in the four cases is, respectively,
\[
1, \quad 2, \quad 2 + |\text{deg}N_1 - \text{deg}N_2| \quad \text{and} \quad 4.
\]

**Proof.** In the first case, \( \mathcal{E} \cong A' \otimes L \) for some invertible \( \mathcal{O}_E \)-module \( L \) so \( \text{End}(\mathcal{E}) \cong \text{End}(A'_1) = \mathbb{C} \). In the second case, \( \mathcal{E} \cong A \otimes L \) for some invertible \( \mathcal{O}_E \)-module so \( \text{End}(\mathcal{E}) \cong \text{End}(A) \cong \mathbb{C}[x]/(x^2) \). The last two cases follow from the fact that
\[
\text{End}(N_1 \oplus N_2) \cong \begin{pmatrix} \mathbb{C} & \text{Hom}(N_2,N_1) \\ \text{Hom}(N_1,N_2) & \mathbb{C} \end{pmatrix}.
\]
The proof is complete. \( \square \)

We define
\[
(3-36) \quad \mathbb{P} \text{ Aut}(\mathcal{E}) := \frac{\text{Aut}(\mathcal{E})}{\text{the central copy of } \mathbb{G}_m}.
\]

**4. The relation between the \( L(\mathcal{E})'s \) and the partial secant varieties \( \text{Sec}_{d,x}(\mathcal{E}) \)**

4.1. A criterion for \( \xi \in \mathbb{P}_L \) to belong to \( \text{Sec}_{d,x}(\mathcal{E}) \). The next result was prompted by the remarks just before Theorem 1 in [Ber92] (but see Remark 4.3). We could not find a proof of the correct version of Bertram’s remark so we include one here for the convenience of the reader.

**Proposition 4.1.** Let \( D_0 \) be an effective divisor of degree \( d \) and let \( z = \sigma(D_0) \). If either \( 1 \leq d < n \) or \( D_0 \sim H \), then a non-split extension
\[
(4-1) \quad \xymatrix{ \xi : & 0 & \mathcal{O}_E & \mathcal{E} \ar[r]^\pi & L & 0 }
\]
belongs to \( \text{Sec}_{d,x}(\mathcal{E}) \) if and only if there is a non-zero morphism \( \mathcal{E} \to \mathcal{O}_E(D_0) \).

If \( \xi \in \text{Sec}_{d,x}(\mathcal{E}) \) and \( d \) is minimal such that \( \xi \in \text{Sec}_d(\mathcal{E}) \), then there is an epimorphism \( \mathcal{E} \to \mathcal{O}_E(D_0) \).

**Proof.** Since \( \text{Sec}_{d,z}(\mathcal{E}) \) is the union of the secants \( \overline{D} \) as \( D \) varies over all effective divisors linearly equivalent to \( D_0 \), we must prove the equivalence of the statements
\[
(\text{a}) \quad \xi \in \overline{D} \text{ for some effective } D \sim D_0; \\
(\text{b}) \quad \text{there is a non-zero morphism } \mathcal{E} \to \mathcal{O}_E(D_0).
\]

Certainly, (a) and (b) are equivalent when \( D_0 \sim H \) so from now on we assume \( d < n \).

When \( D \) is an effective divisor of degree \( < n \) we write \( \mathcal{I}_D \) for the ideal in \( \mathcal{O}_E \) vanishing on \( D \) and write \( i : \mathcal{I}_D L \to L \) for the natural inclusion.
The condition $\xi \in \overline{D}$ says that $\xi$ belongs to every hyperplane in $\mathbb{P}H^0(E, \mathcal{L})^* \cong \mathbb{P} \text{Ext}^1_1(\mathcal{L}, \mathcal{O})$ that contains $D$ (counting multiplicities). Hyperplanes in $\mathbb{P}H^0(E, \mathcal{L})^*$ correspond to non-zero sections $s \in H^0(E, \mathcal{L})$, and containing $D$ simply means that the corresponding section vanishes on $D$. Thus

$$\overline{D} = \mathbb{P}\left(\text{the kernel of the map } H^0(E, \mathcal{L})^* \to H^0(E, \mathcal{I}_D\mathcal{L})^*\right),$$

where the map between the two projective spaces is induced by $i$. The condition $\xi \in \overline{D}$, then, is equivalent to saying that $\xi$, regarded as a functional on $H^0(E, \mathcal{L})$, vanishes on $H^0(E, \mathcal{I}_D\mathcal{L})$. Serre duality provides a commutative diagram

$$\begin{array}{c}
H^0(E, \mathcal{L})^* \\
\downarrow \downarrow \\
\text{Ext}^1_1(\mathcal{L}, \mathcal{O}_E) \\
\downarrow \downarrow \\
\text{Ext}^1_1(\mathcal{I}_D\mathcal{L}, \mathcal{O}_E),
\end{array}$$

in which the map $\iota$ sends an extension $\xi$ to the extension $\xi \cdot i$ which is the top row of the pullback diagram

$$\begin{array}{c}
0 \\
\downarrow \downarrow \\
\mathcal{O}_E \\
\downarrow \downarrow \\
\pi^{-1}(\mathcal{I}_D\mathcal{L}) \\
\downarrow \downarrow \\
\mathcal{I}_D\mathcal{L} \\
\downarrow \downarrow \\
\mathcal{E} \\
\downarrow \downarrow \\
\mathcal{L} \\
\downarrow \downarrow \\
0.
\end{array}$$

It follows that $\xi \in \overline{D}$ if and only if $\xi \cdot i = 0$ in $\text{Ext}^1_1(\mathcal{I}_D\mathcal{L}, \mathcal{O}_E)$, i.e., if and only if the top row of (4-3) splits. Hence $\xi \in \overline{D}$ if and only if there exists a morphism $g : \mathcal{I}_D\mathcal{L} \to \mathcal{E}$ such that $\pi g = i$. Since $\mathcal{I}_D\mathcal{L} \cong \mathcal{L}(-D)$ has degree $n - d > 0$, there is no non-zero morphism from $\mathcal{I}_D\mathcal{L}$ to $\mathcal{O}_E$, so every non-zero morphism $g : \mathcal{I}_D \to \mathcal{E}$ satisfies $\pi g \neq 0$, and hence, replacing $g$ by a suitable multiple of itself, $\pi g = i$. Thus $\xi \in \overline{D}$ if and only if there is a non-zero morphism $g : \mathcal{I}_D\mathcal{L} \to \mathcal{E}$ (which is necessarily monic).

We now consider the two implications in turn, always assuming $d < n$.

(a) $\Rightarrow$ (b). Suppose that there is a non-zero morphism $\mathcal{E} \to \mathcal{O}_E(D_0)$, and let $\mathcal{N}$ be its image. By Theorem 2.4, $\mathcal{E}$ is either indecomposable (hence semistable of positive slope) or a direct sum of two line bundles of positive degree. This implies that there is no non-zero morphism from $\mathcal{E}$ to a line bundle of degree $\geq 0$. So $\mathcal{N}$ has positive degree, and thus it is isomorphic to $\mathcal{O}_E(D)$ for some effective divisor $D$. Since there is a monomorphism $\mathcal{O}_E(D) \to \mathcal{O}_E(D_0)$, there is an effective divisor $D_1$ such that $D \leq D_1 \sim D_0$. Since $\det \mathcal{E} \cong \mathcal{L}$, the kernel of the epimorphism $\mathcal{E} \to \mathcal{O}_E(D)$ must be isomorphic to $\mathcal{L}(-D)$. We thus obtain a monomorphism $\mathcal{I}_D\mathcal{L} \to \mathcal{E}$. This implies that $\xi \in \overline{D}$ as discussed above, whence $\xi \in \overline{D_1}$.

(b) $\Rightarrow$ (a). Suppose that $\xi \in \overline{D}$ for some effective $D \sim D_0$. Since there is always a monomorphism $\mathcal{O}_E(D') \to \mathcal{O}_E(D_0)$ for any divisor $D'$ of degree $< d = \deg D_0$, we can assume that $D_0$ is of minimal degree satisfying (a) (equivalently, $d$ is minimal such that $\xi \in \text{Sec}_d(\mathcal{E})$ as in the last sentence in the statement); thus $D$ is an effective divisor of minimal degree such that $\xi \in \overline{D}$. We prove that there is an epimorphism $\mathcal{E} \to \mathcal{O}_E(D)$.

Since $\xi \in \overline{D}$, the top row in (4-3) splits; i.e., $\pi^{-1}(\mathcal{I}_D\mathcal{L}) \cong \mathcal{O}_E \oplus \mathcal{I}_D\mathcal{L}$. Thus, we have a monomorphism $\mathcal{O}_E \oplus \mathcal{I}_D\mathcal{L} \to \mathcal{E}$ whose cokernel is isomorphic to $\mathcal{L}/\mathcal{I}_D\mathcal{L} \cong \mathcal{O}_D$. Let $\mathcal{N}$ denote the image of $\mathcal{I}_D\mathcal{L}$ in $\mathcal{E}$. The assumed degree minimality of $D$ then implies that $\mathcal{E}/\mathcal{N}$ is a torsion-free extension of $\mathcal{O}_D$ by $\mathcal{O}_E$, and hence isomorphic to $\mathcal{O}_E(D)$; indeed, if $\mathcal{E}/\mathcal{N}$ were not torsion-free, then there would be a some skyscraper subsheaf $\mathcal{O}_p \subseteq \mathcal{O}_D$ and a submodule $\mathcal{E} \supseteq \mathcal{M} \supseteq \mathcal{N}$ such that $\mathcal{M}/\mathcal{N} \cong \mathcal{O}_p$. Since $\mathcal{E}$ is torsion-free, $\mathcal{M}$ must be isomorphic to $\mathcal{N}(p) \cong \mathcal{I}_D(-p)\mathcal{L}$. Note that $D - (p)$ is an effective divisor since $\mathcal{O}_p \subseteq \mathcal{O}_D$. Thus we have a monomorphism $\mathcal{I}_D(-p)\mathcal{L} \to \mathcal{E}$, and hence $\xi \in \overline{D - (p)}$ as discussed above. This contradicts the minimality of $\deg D$. □
Theorem 4.2. Suppose $\mathcal{E}$ satisfies the assumptions in §2.2. Let $\xi \in \mathbb{P}_L$ and write $\mathcal{E} := m(\xi)$. If
\begin{align*}
d_1 & := \min \{ d \mid \xi \in \text{Sec}_d(E) \}, \\
d_2 & := \min \{ d \mid \xi \in \overline{D} \text{ for some } D \in E^{[d]} \}, \\
d_3 & := \min \{ d \mid \text{there is a non-zero morphism } \mathcal{E} \rightarrow \mathcal{O}_E(D) \text{ for some } D \in E^{[d]} \}, \\
d_4 & := \min \{ d \mid \text{there is an epimorphism } \mathcal{E} \rightarrow \mathcal{O}_E(D) \text{ for some } D \in E^{[d]} \},
\end{align*}
then $d_1 = d_2 = d_3 = d_4$. If we write $d$ for this common integer and let $D \in E^{[d]}$, then there is an epimorphism $\mathcal{E} \rightarrow \mathcal{O}_E(D)$ if and only if $\xi \in \overline{D} \subseteq \text{Sec}_{d,x}(E)$.

Proof. The equality $d_1 = d_2$ is a consequence of the fact that $\text{Sec}_d(E)$ is the union of the $\overline{D}$’s as $D$ varies over $E^{[d]}$. The rest is a consequence of Proposition 4.1; indeed, $d_1 = d_2 = d_3$ is follows from the “if and only if” statement, and they are also equal to $d_4$ by the last sentence of Proposition 4.1. \hfill \Box

Remark 4.3. In contrast to Proposition 4.1, the discussion at [Ber92, p.430, preceding Thm. 1] suggests that $\xi \in \overline{D}$ if and only if there is an epimorphism $\mathcal{E} \rightarrow \mathcal{O}_E(D)$. This cannot be: the condition $\xi \in \overline{D}$ depends on $D$ as a subscheme of $E$, but the existence of an epimorphism $\mathcal{E} \rightarrow \mathcal{O}_E(D)$ depends only on the linear equivalence class of $D$.

A subtle point in the proof of Proposition 4.1 is the necessity, in the proof of the implication
\[ \xi \in \overline{D} \implies \text{there is an epimorphism } \mathcal{E} \rightarrow \mathcal{O}_E(D), \]
to consider a $D$ of minimal degree such that $\xi \in \overline{D}$. Example 4.4 shows this is not a moot point: it is possible for the epimorphism $\pi$ of (4-1) to split over some $L(-D) \leq L$ (which in the different notation adopted below will be $\mathcal{O}_E(D) \leq L$) but with non-zero torsion in the corresponding quotient $\mathcal{E}/L(-D)$.

Example 4.4. Choose an effective divisor $D$ and distinct points $x, y \in E$ such that $H = D + x + y$. The exact sequence $0 \rightarrow \mathcal{O}_E(D) \rightarrow L \rightarrow \mathcal{O}_x \oplus \mathcal{O}_y \rightarrow 0$ corresponds to an element
\[ \alpha \in \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E(D)) \cong \text{Ext}^1(\mathcal{O}_x, \mathcal{O}_E(D)) \oplus \text{Ext}^1(\mathcal{O}_y, \mathcal{O}_E(D)) \]
whose two components are both non-zero.

Now construct an extension $\mathcal{E}$ (a rank-2 bundle) of $\mathcal{O}_x \oplus \mathcal{O}_y$ by $\mathcal{O}_E \oplus \mathcal{O}_E(D)$ such that its class
\[ \beta \in \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E \oplus \mathcal{O}_E(D)) \cong \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E) \oplus \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E(D)) \]
satisfies the following conditions: its right-hand component, $\beta_r \in \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E(D))$, is $\alpha$ and its left-hand component, $\beta_l \in \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E)$, is non-zero and is in $\text{Ext}^1(\mathcal{O}_y, \mathcal{O}_E) \subseteq \text{Ext}^1(\mathcal{O}_x \oplus \mathcal{O}_y, \mathcal{O}_E)$.

Let $\mathcal{N}$ denote the image of the morphism $\mathcal{O}_E(D) \rightarrow L$.

By construction, there is an extension $0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{E} \xrightarrow{\pi} L \rightarrow 0$ which is non-split because $\beta_l \neq 0$. However, the subsequence $0 \rightarrow \mathcal{O}_E \rightarrow \pi^{-1}(\mathcal{N}) \rightarrow \mathcal{N} \rightarrow 0$ splits via a, necessarily monic, morphism $g : \mathcal{N} \cong \mathcal{O}_E(D) \rightarrow \pi^{-1}(\mathcal{N}) \subseteq \mathcal{E}$; the quotient $\mathcal{E}/g(\mathcal{N})$ is not torsion-free because the condition on $\beta_l$ implies that $\mathcal{E}/g(\mathcal{N}) \cong \mathcal{O}_x \oplus \mathcal{O}_E(y)$. \hfill \Box

4.2. Description of the leaves $L(\mathcal{E})$ in terms of the spaces $\text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E)$.

Theorem 4.5. Suppose either $d \in [1, \frac{n}{2}]$ and $x \in E$ or $d = \frac{n}{2}$ and $x \in E - \Omega$. Then
\begin{equation}
L(\mathcal{E}_{d,x}) = \text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E).
\end{equation}
In particular, $L(\mathcal{E}_{d,x})$ is locally closed in $\mathbb{P}_L$ and $\dim L(\mathcal{E}_{d,x}) = 2d - 2$.

Proof. By definition, $\mathcal{E}_{d,x} \cong \mathcal{O}_E(D) \oplus L(-D)$ where $D$ is an arbitrary point in $E_{d,x}^{[d]}$. Since $\text{Hom}(\mathcal{E}_{d,x}, \mathcal{O}_E(D)) \neq 0$ but $\text{Hom}(\mathcal{E}_{d,x}, \mathcal{O}_E(D')) = 0$ for all divisors $D'$ of degree $< d$, (4-4) follows from Proposition 4.1. The local closure of $L(\mathcal{E}_{d,x})$ follows from (4-4), and its dimension is given by Proposition 2.12. \hfill \Box

If $n$ is odd and $\mathcal{E}$ is indecomposable (i.e., if $\mathcal{E} \cong \mathcal{E}_o$), a similar argument proves the next result.
Theorem 4.6. Suppose $\mathcal{E}$ satisfies the assumptions in §2.2. If $n$ is odd, then
\[ L(\mathcal{E}_o) = \mathbb{P}_L - \text{Sec}_{\omega-1}(E). \]

In particular, $L(\mathcal{E}_o)$ is a Zariski-dense open subset of $\mathbb{P}_L$.

Proof. This follows from Proposition 4.1 and the observation that an odd-degree $\mathcal{E}$ satisfying the assumptions in §2.2 is indecomposable precisely when it admits no non-zero morphisms to any line bundles of degree $\leq \frac{n-1}{2}$.

The next result disposes of the still-missing even-$n$ cases.

Theorem 4.7. Suppose $n$ is even and fix $\omega \in \Omega$. As in §2.3.1, let $\mathcal{L}_\omega$ be the unique-up-to-isomorphism invertible $\mathcal{O}_E$-module of degree $\frac{n}{2}$ such that $\sigma(\mathcal{L}_\omega) = \omega$, and let $\mathcal{E}_\omega$ be the unique-up-to-isomorphism non-split extension of $\mathcal{L}_\omega$ by $\mathcal{L}_\omega$.

1. $L(\mathcal{E}_\omega) \cup L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega) = \text{Sec}_{\frac{n}{2}}(E) - \text{Sec}_{\frac{n}{2}-1}(E)$.
2. $L(\mathcal{E}_\omega)$ is open dense in $\text{Sec}_{\frac{n}{2}}(E)$, has dimension $n-2$, and consists of those points in $\text{Sec}_{\frac{n}{2}}(E) - \text{Sec}_{\frac{n}{2}-1}(E)$ that lie on a unique $\frac{n}{2}$-secant.
3. $L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega)$ has dimension $n-4$ and consists of those points in $\text{Sec}_{\frac{n}{2}}(E) - \text{Sec}_{\frac{n}{2}-1}(E)$ that lie on infinitely many $\frac{n}{2}$-secants.
4. A point in $\text{Sec}_{\frac{n}{2}}(E) - \text{Sec}_{\frac{n}{2}-1}(E)$ lies on infinitely many $\frac{n}{2}$-secants if and only if it lies on at least two distinct $\frac{n}{2}$-secants.

Proof. Let $\xi \in \text{Sec}_{\frac{n}{2}}(E) - \text{Sec}_{\frac{n}{2}-1}(E)$. Suppose $0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$ represents $\xi$, and that $\mathcal{N}$ is an invertible $\mathcal{O}_E$-module of minimal degree that is a quotient of $\mathcal{E}$.

1. If there is a non-zero map $\mathcal{E} \to \mathcal{L}_\omega$ but to no other line bundles of degree $\leq \frac{n}{2}$, then $\mathcal{E}$ is isomorphic to either $\mathcal{E}_\omega$ or $\mathcal{L}_\omega \oplus \mathcal{L}_\omega$. It follows from this and Proposition 4.1 that
\[ \text{Sec}_{\frac{n}{2}}(E) - \text{Sec}_{\frac{n}{2}-1}(E) = L(\mathcal{E}_\omega) \cup L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega). \]

However, the proof of Proposition 4.1 shows more.

Suppose $\mathcal{E} \cong \mathcal{E}_\omega$. The exact sequence $0 \to \mathcal{L}_\omega \to \mathcal{E}_\omega \to \mathcal{L}_\omega \to 0$ yields an exact sequence $0 \to \text{Hom}(\mathcal{L}_\omega, \mathcal{N}) \to \text{Hom}(\mathcal{E}_\omega, \mathcal{N}) \to \text{Hom}(\mathcal{L}_\omega, \mathcal{N})$ which implies that $\text{Hom}(\mathcal{L}_\omega, \mathcal{N}) \neq 0$, whence $\text{deg} \mathcal{N} \geq \frac{n}{2}$. But $\mathcal{L}_\omega$ is a quotient of $\mathcal{E}_\omega$ so $\frac{n}{2}$ is the smallest degree of an invertible $\mathcal{O}_E$-module that is a quotient of $\mathcal{E}$. Since $\dim \text{Hom}(\mathcal{E}_\omega, \mathcal{L}_\omega) = 1$, the proof of Proposition 4.1 shows there is a unique effective divisor $D$ such that $\xi \in \overline{D}$.

Suppose $\mathcal{E} \cong \mathcal{L}_\omega \oplus \mathcal{L}_\omega$. As in the previous paragraph, we conclude that $\text{deg} \mathcal{N} \geq \frac{n}{2}$ and, since $\mathcal{L}_\omega$ is a quotient of $\mathcal{L}_\omega \oplus \mathcal{L}_\omega$, $\frac{n}{2}$ is the smallest degree of an invertible $\mathcal{O}_E$-module that is a quotient of $\mathcal{E}$. Since $\dim \text{Hom}(\mathcal{E}, \mathcal{L}_\omega) = 2$ in this case, there are at least two effective $D$’s of degree $\frac{n}{2}$ such that $\xi \in \overline{D}$. In fact, because $\mathbb{P} \text{Hom}(\mathcal{E}, \mathcal{L}_\omega) \cong \mathbb{P}^1$, there are infinitely many $D$’s of degree $\frac{n}{2}$ such that $\xi \in \overline{D}$.

This proves (1) and the concrete descriptions of the two leaves in (2) and (3). It also proves (4).

2. The dimension of $\text{Sec}_{\frac{n}{2}}(E)$ is $n - 2$ by Proposition 2.12.

The density claim in (2) follows from openness and irreducibility, so this leaves only the openness claim of (2) unaddressed. One argument (that might be of independent use) runs as follows.

Applying the functor $\text{Hom}(-, \mathcal{L}_\omega)$ to any extension $0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{L} \to 0$ gives rise, via the long exact cohomology sequence attached to it, to a morphism
\[ H^0(E, \mathcal{L}_\omega) \to \text{Ext}^1(\mathcal{L}, \mathcal{L}_\omega) \cong \text{Ext}^1(\mathcal{L}_\omega, \mathcal{O}_E) \cong H^0(E, \mathcal{L}_\omega)^* \]
with cokernel $\text{Ext}^1(\mathcal{E}, \mathcal{L}_\omega)$ (the first isomorphism exists because $\mathcal{L} \cong \mathcal{L}_\omega \otimes \mathcal{L}_\omega$).

Globalizing this over all $\xi \in \mathbb{P}_L = \mathbb{P} \text{Ext}^1(\mathcal{L}, \mathcal{O})$, we obtain a morphism $\Psi$ from the tautological bundle $\mathcal{O}_{\mathbb{P}_L}(-1)$, whose fiber over the point $x \in \mathbb{P}_L$ is the line $x$, to the trivial bundle
\[ \text{Hom}(H^0(E, \mathcal{L}_\omega), H^0(E, \mathcal{L}_\omega)^*) \otimes \mathcal{O}_{\mathbb{P}_L} \cong \text{End}(H^0(E, c\mathcal{L}_\omega)) \otimes \mathcal{O}_{\mathbb{P}_L}, \]
which on open affine patches of $\mathbb{P}_L$ can simply be regarded as a section of that trivial bundle. Restricting that section to 

$$L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega) \sqcup L(\mathcal{E}_\omega) = \text{Sec}_\mathcal{L}_\omega - \text{Sec}_\mathcal{L}_{-1},$$

we see that

- $L(\mathcal{E}_\omega)$ is the locus where said section has maximal rank $n - 1$, since the cokernel $\text{Hom}(\mathcal{E}, \mathcal{L}_\omega)$ of the morphism (4-5) in question is 1-dimensional;
- while $L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega)$ is the locus where the rank is $n - 2$ (or at any rate, less than maximal).

Since the condition rank $\leq n - 2$ is given by the vanishing of various minors, $L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega)$ is a Zariski-closed subset of $L(\mathcal{E}_\omega) \sqcup L(\mathcal{L}_\omega \oplus \mathcal{L}_\omega)$. □

Proposition 2.10 showed that every point in $\text{Sec}_{d,x}(E) - \text{Sec}_{d-1}(E)$ belongs to $D$ for a unique $D \in E^{[d]}_x$ when $d \in [1, \frac{n}{2}]$ or when $d = \frac{n}{2}$ and $x \notin \Omega$. Theorem 4.7(2) allows us to improve this by determining what happens when $d = \frac{n}{2}$ and $x \in \Omega$. We record the combination of those two results in the next proposition.

**Proposition 4.8.** Suppose $n$ is even and that $\mathcal{E}$ satisfies the assumptions in §2.2. If $\omega \in \Omega$, then every point in $L(\mathcal{E}_\omega) - \text{Sec}_{2-1}(\mathcal{E})$ belongs to $D$ for a unique $D \in E^{[n/2]}_\omega$.

5. **$L(\mathcal{E})$ is a symplectic leaf**

In this section we show that every $L(\mathcal{E})$ is smooth, then that the $L(\mathcal{E})$’s are the symplectic leaves for $(\mathbb{P}_L, \Pi)$. A first step towards this is to show that the map $X(\mathcal{E}) \to L(\mathcal{E})$ is a geometric quotient with respect to the action of $\text{Aut}(\mathcal{E})$. We show that this quotient map has other good properties.

5.1. **The differential of $\Psi_\mathcal{E}$.** We fix $\mathcal{E}$ that satisfies the assumptions in §2.2, and a homomorphism $f \in X(\mathcal{E})$. Let $\pi \in \text{Epi}(\mathcal{E}, \mathcal{L})$ be the epimorphism corresponding to $f$, and let $\xi$ be the isomorphism class of the non-split extension

$$\xi: \quad 0 \longrightarrow \mathcal{O}_E \xrightarrow{f} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0. \quad (5-1)$$

We will consider the differentials of the vertical maps in the commutative diagram

$$\begin{array}{ccc}
\xi & : & 0 \longrightarrow \mathcal{O}_E \xrightarrow{f} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0, \\
\Psi & : & \mathcal{E} \xrightarrow{\tau} \text{Epi}(\mathcal{E}, \mathcal{L}), \\
\mathbb{P}_L & \xrightarrow{\Phi} & \mathcal{G}, \\
\end{array} \quad (5-2)$$

where $\mathcal{G} = \mathbb{G}(n - 1, \text{Hom}(\mathcal{O}_E, \mathcal{L}))$, $\Psi := \Psi_\mathcal{E}$ (defined in Remark 3.4), $\tau$ is the restriction of the linear isomorphism $\theta_{\nu_*} \in \text{Hom}(\mathcal{O}_E, \mathcal{L}) \to \text{Hom}(\mathcal{E}, \mathcal{L})$ in Lemma 3.8, and $\Phi$ is the map $\Phi(\pi) := \text{im}(\pi) = \xi^\perp$. By definition, $\Psi(f) = \xi$ so the image of $\Psi$ is $L(\mathcal{E})$.

For brevity, we will often write $X$ rather than $X(\mathcal{E})$.

The tangent spaces at points in the spaces in (5-2) are

$$T_fX = \text{Hom}(\mathcal{O}_E, \mathcal{E})$$
$$T_{\xi} \mathbb{P}_L = \text{Hom}(\mathbb{C}\xi, \text{Ext}^1(\mathcal{L}, \mathcal{O}_E)/\mathbb{C}\xi),$$
$$T_{\pi} \mathcal{G} = \text{Hom}(\xi^\perp, \text{Hom}(\mathcal{O}_E, \mathcal{L})/\xi^\perp),$$

and

$$T_{\pi} \text{Epi}(\mathcal{E}, \mathcal{L}) = \text{Hom}(\mathcal{E}, \mathcal{L}),$$

the first of these because $X(\mathcal{E})$ is a dense open subset of $\text{Hom}(\mathcal{O}_E, \mathcal{E})$, and the last of these because $\text{Epi}(\mathcal{E}, \mathcal{L})$ is a non-empty Zariski-open subset of $\text{Hom}(\mathcal{E}, \mathcal{L})$. 

5.1.1. Remarks about tangent spaces to points on Grassmannians. Our next result, Lemma 5.1, gives an explicit description of the differential $d\Phi_x$. Its proof is expressed in the language of infinitesimal deformations. See [Har77, Exers. II.2.8, II.8.6, II.8.7, III.4.10] for some general background, and [Cop04, pp.18-20] for an application of these techniques very close in spirit to what we need. We describe the general idea in some detail for the convenience of the reader.

We write 
\[ \mathbb{C}[\varepsilon] := \mathbb{C}[x]/(x^2) \]
for the complex ring of dual numbers; so $\varepsilon$ is a formal variable whose square is zero.

As a preliminary, we recall two equivalent pictures of the tangent space to the Grassmannian $G(k, V)$ of $k$-planes in a vector space $V$. With that in mind, fix a $k$-dimensional subspace $V' \subseteq V$ and consider it as a point of $G(k, V)$.

On the one hand, [EH16, Thm. 3.5] gives an identification
\[ T_{V'}G(k, V) = \text{Hom}(V', V/V') \]
that is canonical in the sense that it glues over all $V'$ to give an isomorphism of bundles.

On the other hand, by [Har77, Exer. II.2.8],
\[ T_{V'}G(k, V) = \{ \Psi : \text{Spec} \mathbb{C}[\varepsilon] \to G(k, V) \mid \Psi(\text{closed point}) = V' \} \]
By the universal property of the Grassmannians (as in [EH00, Exer. VI-18] and [EH16, discussion immediately preceding §3.2.4]), this amounts to
\[ T_{V'}G(k, V) \cong \text{rank-}k \mathbb{C}[\varepsilon]\text{-summands } V'_\varepsilon \subseteq V[\varepsilon] := V \otimes \mathbb{C}[\varepsilon] \text{ for which} \]
the map $V'_\varepsilon/\varepsilon V'_\varepsilon \to V = V[\varepsilon]/\varepsilon V[\varepsilon]$ is an isomorphism onto $V'$.

A bijection between (5-3) and (5-4) can be effected as follows. First, let $\iota : V' \to V$ be the inclusion, $\alpha : V \to V/V'$ the natural map, and fix a linear map $\beta : V/V' \to V$ such that $\alpha \beta = \text{id}_{V/V'}$.

(a) If we are given $\theta \in \text{Hom}(V', V/V')$ as in (5-3), we define
\[ V'_\varepsilon := \{ x + \varepsilon \beta \theta(x) \mid x \in V' \} + \varepsilon V' = \varepsilon V' + \text{image}(\iota + \varepsilon \beta \theta). \]

Roughly speaking, the construction $\theta \mapsto V'_\varepsilon$ sends $\theta$ to an isomorphic copy of its graph: first, $V'_\varepsilon$ always contains $\varepsilon V' \subseteq V[\varepsilon]$; second, there is an isomorphism;
\[ \mu : V \times (V/V') \longrightarrow V[\varepsilon]/\varepsilon V', \quad \mu(w, z) := w + \varepsilon(\beta(z) + \varepsilon V'); \]
third, $V'_\varepsilon/\varepsilon V'_\varepsilon = \mu(\Gamma)$ where $\Gamma := \{(x, \theta(x)) \in V \times V/V' \mid x \in V'\}$ is the graph of $\theta$.

(b) Conversely, given $V'_\varepsilon$ as in (5-4), the condition that the map $V'_\varepsilon/\varepsilon V'_\varepsilon \to V$ is an isomorphism onto $V'$ implies that $V'_\varepsilon \cap \varepsilon V = \varepsilon V'_\varepsilon$; hence for each $x \in V'$ there is an element $y \in V$, unique modulo $V'$, such that $x + \varepsilon y \in V'_\varepsilon$; we may therefore define $\theta \in \text{Hom}(V', V/V') = T_{V'}G(k, V)$ by the formula
\[ \theta(x) := y + V' \]
to recover the equality in (5-3).

(c) The procedures in (a) and (b) are mutual inverses.

Suppose we are given $\theta_1 \in \text{Hom}(V', V/V')$. Define $V'_\varepsilon := \varepsilon V' + \text{image}(\iota + \varepsilon \beta \theta_1)$ as in (5-5). If $x \in V'$, then $x + \varepsilon \beta \theta_1(x) = (\iota + \varepsilon \beta \theta_1)(x) \in V'_\varepsilon$ so, in (b), we can take $y = \beta \theta_1(x)$; the map $\theta_2$ defined by (5-6) is therefore $\theta_2(x) = \beta \theta_1(x) + V' = \alpha \beta \theta_1(x) = \theta_1(x)$; i.e., $\theta_2 = \theta_1$.

Suppose we are given $V'_{\varepsilon,1}$ satisfying (5-4). Define $\theta$ by (5-6), then define $V'_{\varepsilon,2}$ as in (a); i.e., $V'_\varepsilon := \varepsilon V' + \text{image}(\iota + \varepsilon \beta \theta)$. Let $x \in V'$ and let $y \in V$ be such that $x + \varepsilon y \in V'_{\varepsilon,1}$, since $\theta(x) = y + V' = \alpha(y)$,
\[ (\iota + \varepsilon \beta \theta)(x) = x + \varepsilon \beta \alpha(y) = x + \varepsilon y + \varepsilon (\beta \alpha(y) - y) \in V'_{\varepsilon,1} + \varepsilon V' = V'_{\varepsilon,1} \]
because $\beta \alpha(y) - y \in \text{ker}(\alpha) = V'$. Hence $\text{im}(\iota + \varepsilon \beta \theta) \subseteq V'_{\varepsilon,1}$ and $V'_{\varepsilon,2} \subseteq V'_{\varepsilon,1}$. But $\dim V'_{\varepsilon,2} = \dim V'_{\varepsilon,1}$ so $V'_{\varepsilon,2} = V'_{\varepsilon,1}$. \(\diamond\)
Let $\xi$ be the non-split extension $0 \to O_E \xrightarrow{f} E \xrightarrow{\pi} L \to 0$ above. Since $\xi^\perp = \ker(\pi_*)$, there is an exact sequence

$$0 \to \text{Hom}(O_E, O_E) \xrightarrow{f_*} \text{Hom}(O_E, E) \xrightarrow{\pi_*} \xi^\perp \to 0$$

where $f_*(a) = fa$.

**Lemma 5.1.** Let $\xi$ be as above and let $\varphi \in \text{Hom}(E, \xi) = T^\pi \text{Epi}(E, \xi)$. The differential

$$d\Phi_\pi : T^\pi \text{Epi}(E, \xi) = \text{Hom}(E, \xi) \rightarrow T^{\xi^\perp} \mathbb{C} = \text{Hom}(\xi^\perp, \text{Hom}(O_E, E)/\xi^\perp)$$

has two interpretations:

1. as a map $\xi^\perp \to \text{Hom}(O_E, \xi)/\xi^\perp$,

$$d\Phi_\pi(\varphi)(\pi s) = \varphi s + \xi^\perp$$

for all $\pi s \in \xi^\perp$; equivalently, if we first fix a linear map $\mu : \xi^\perp \to \text{Hom}(O_E, E)$ such that

$$\pi_* \mu = \text{id}_{\xi^\perp},$$

then

$$d\Phi_\pi(\varphi)(x) = \varphi \circ \mu(x) + \xi^\perp$$

for all $x \in \xi^\perp$;

2. as a map $\text{Hom}(O_E, E) \to \text{Hom}(O_E, L)/\xi^\perp$,

$$d\Phi_\pi(\varphi)(s) = \varphi s + \xi^\perp$$

i.e., if $\varphi \in \text{Hom}(E, \xi)$, then $d\Phi_\pi(\varphi)$ is the composition

$$\text{Hom}(O_E, E) \xrightarrow{\varphi_*} \text{Hom}(O_E, L) \xrightarrow{\gamma} \text{Hom}(O_E, L)/\xi^\perp$$

where $\gamma(v) = v + \xi^\perp$ and $\varphi_*(a) = \varphi a$.

**Proof.** Fix $\varphi \in \text{Hom}(E, \xi)$. When writing the equality $T^\pi \text{Epi}(E, \xi) = \text{Hom}(E, \xi)$ we are identifying $\varphi$, which belongs to $\text{Hom}(E, \xi)$, with the homomorphism $\pi + \varepsilon \varphi : E \otimes \mathbb{C}[\varepsilon] \to L \otimes \mathbb{C}[\varepsilon]$ over the ring $\mathbb{C}[\varepsilon]$ of dual numbers. The $\varepsilon$-thickening of the map $\Phi$ in (5-2) is the map $\tilde{\Phi}$ given by the formula

$$\tilde{\Phi}(\pi + \varepsilon \varphi) := \text{im}(\pi + \varepsilon \varphi)_*$$

$$= \{(\pi + \varepsilon \varphi) \circ (s + \varepsilon s') = \pi s + \varepsilon (\pi s' + \varphi s) \mid s, s' \in \text{Hom}(O_E, E)\}$$

The image $\text{im}(\pi + \varepsilon \varphi)_*$ is a $\mathbb{C}[\varepsilon]$-point of $\mathbb{C}$. The map $\tilde{\Phi}$ specializes back to $\Phi$.

(1) The discussion preceding Lemma 5.1 shows that, regarded as a map $\xi^\perp \to \text{Hom}(O_E, E)/\xi^\perp$, the tangent vector $d\Phi_\pi(\varphi)$ is the map $\pi s \mapsto y + \xi^\perp$ where $y \in \text{Hom}(O_E, L)$ is the unique element modulo $\xi^\perp$ such that

$$\pi s + \varepsilon y \in \text{im}(\pi + \varepsilon \varphi)_*.$$ 

Since $\pi s + \varepsilon \varphi s \in \text{im}(\pi + \varepsilon \varphi)_*$, we may take $y = \varphi s$; thus $d\Phi_\pi(\varphi)(\pi s) = \varphi s + \xi^\perp$. This gives the first description of $d\Phi_\pi(\varphi)$ in (1).

To verify the second description of $d\Phi_\pi(\varphi)$ in (1), use the fact that every $x$ in $\xi^\perp$ equals $\pi s$ for some $s \in \text{Hom}(O_E, E)$, then observe that $\pi_*(s) = \pi s = \pi_*(\mu s))$ whence $s - \mu(x) = s - \mu(\pi s) \in \text{ker}(\pi_*) = \mathbb{C} f$; then, since $\varphi f \in \xi^\perp$, by Lemma 3.10, $\varphi s - \varphi \circ \mu(x) \in \xi^\perp$, i.e., $\varphi s + \xi^\perp = \varphi \circ \mu(x) + \xi^\perp$.17

(2) As an element belonging to the set in (5-8) the tangent vector $d\Phi_\pi(\varphi)$ is the composition

$$\text{Hom}(O_E, E) \to \text{Hom}(O_E, E)/\mathbb{C} f \xrightarrow{\sim} \xi^\perp \to \text{Hom}(O_E, L)/\xi^\perp, \quad s \mapsto s + \mathbb{C} f \mapsto \pi s \mapsto \varphi s + \xi^\perp;$$

---

17It is implicit in this paragraph that $\varphi \circ \mu(x) + \xi^\perp$ does not depend on the choice of $\mu$. One can also see this directly by the following argument: if $\mu_1, \mu_2 : \xi^\perp \to \text{Hom}(O_E, E)$ are such that $\pi_* \mu_1 = \pi_* \mu_2 = \text{id}_{\xi^\perp}$, then $(\mu_1 - \mu_2)(x) \in \text{ker}(\pi_*) = \mathbb{C} f$ for all $x \in \xi^\perp$; hence $\varphi \circ (\mu_1 - \mu_2)(x) \in \mathbb{C} f \subseteq \text{im}(\pi_*) \subseteq \xi^\perp$; thus $\varphi \circ \mu_1(x) + \xi^\perp = \varphi \circ \mu_2(x) + \xi^\perp$. 
Proposition 5.2. If $\xi$ is the non-split extension extension $0 \rightarrow \mathcal{O}_E \xrightarrow{f} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \rightarrow 0$, then

$$\{\varphi \in \text{Hom}(\mathcal{E}, \mathcal{L}) \mid \xi \cdot \varphi = 0 \} = \pi \text{End}(\mathcal{E}) = \{\varphi \in \text{Hom}(\mathcal{E}, \mathcal{L}) \mid d\Phi_{\pi}(\varphi) = 0\}.$$ 

Equivalently, $\xi \cdot \varphi = 0 \iff d\Phi_{\pi}(\varphi) = 0 \iff \varphi \in \pi \text{End}(\mathcal{E})$.

Proof. Applying the functor $\text{Hom}(\mathcal{E}, -)$ to $\xi$ yields the exact sequence

$$\cdots \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}) \xrightarrow{\pi_*} \text{Hom}(\mathcal{E}, \mathcal{L}) \xrightarrow{\delta} \text{Ext}^1(\mathcal{E}, \mathcal{O}_E) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{E}) \rightarrow \cdots \enspace .$$

Since $\delta(\varphi) = \xi \cdot \varphi$, the left-most term in (5-10) is the kernel of $\delta$. The middle term in (5-10) is the image of the map $\text{Hom}(\mathcal{E}, \mathcal{L}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{L})$, $g \mapsto \pi g$. Those terms are equal because $\ker(\delta) = \{\xi \cdot \varphi = 0\}$.

Suppose $d\Phi_{\pi}(\varphi) = 0$. Then $\varphi \in \xi^\perp = \text{im}(\pi_*)$ for all $s \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$; i.e., $\varphi s = \pi s'$ for some $s' \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$. But $\xi \cdot \pi = 0$ by the first equality in (5-10), so $(\xi \cdot \varphi) s = 0$ for all $s \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$.

In other words, $(\xi \cdot \varphi, s)$ is in the kernel of the pairing

$$\text{Ext}^1(\mathcal{E}, \mathcal{O}_E) \times \text{Hom}(\mathcal{O}_E, \mathcal{E}) \rightarrow \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) \cong \mathbb{C}$$

for all $s \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$; but this pairing is non-degenerate so $\xi \cdot \varphi = 0$. Hence $\varphi \in \pi \text{End}(\mathcal{E})$.

Suppose $\varphi \in \pi \text{End}(\mathcal{E})$. Then $\varphi \text{Hom}(\mathcal{O}_E, \mathcal{E}) \subseteq \pi \text{Hom}(\mathcal{O}_E, \mathcal{E})$. Thus, given $s \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$ there is an $s' \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$ such that $\varphi s = \pi s'$ in $\xi^\perp$. Hence $\varphi s + \xi = 0$; i.e., $d\Phi_{\pi}(\varphi) = 0$.

For the next result we first note that $\text{Epi}(\mathcal{E}, \mathcal{L})$ is stable under the action of $\text{Aut}(\mathcal{E})$ on $\text{Hom}(\mathcal{E}, \mathcal{L})$; we recall from Lemma 3.8 that $\text{Aut}(\mathcal{E})$ is acting on $\text{Hom}(\mathcal{E}, \mathcal{L})$ by acting on both $\mathcal{E}$ and $\mathcal{L} = \text{det} \mathcal{E}$.

Corollary 5.3. The differential $d\Phi$ vanishes precisely on the tangent spaces to the $\text{Aut}(\mathcal{E})$-orbits in $\text{Epi}(\mathcal{E}, \mathcal{L})$; i.e., if $\pi \in \text{Epi}(\mathcal{E}, \mathcal{L})$, then the kernel of $d\Phi_{\pi} : T_{\pi}\text{Epi}(\mathcal{E}, \mathcal{L}) \rightarrow T_{\pi}(G, \pi)$ where $G = \text{Aut}(\mathcal{E})$.

Proof. Let $a : G \rightarrow G, \pi$ be the map $g \mapsto g, \pi$. Since $a$ is an isomorphism, by Lemma 3.5, and $T_1G$ naturally identifies with $\text{End}(\mathcal{E})$, the differential $da_1 : T_1G \rightarrow T_\pi(G, \pi)$ at the identity is an isomorphism $\text{End}(\mathcal{E}) \rightarrow T_\pi(G, \pi) = \pi \text{End}(\mathcal{E}) \subseteq \text{Hom}(\mathcal{E}, \mathcal{L}) = T_{\pi}\text{Epi}(\mathcal{E}, \mathcal{L})$. By Proposition 5.2, $\ker(d\Phi_{\pi}) = \pi \text{End}(\mathcal{E}) = T_{\pi}(G, \pi)$.

Corollary 5.4. Suppose $\mathcal{E}$ satisfies the assumptions in §2.2. The differential $d\Phi$ has constant rank.

Proof. The kernels of $d\Phi$ are the tangent spaces to the orbits. But those orbits are all isomorphic to $\text{Aut}(\mathcal{E})$ by Lemma 3.5, so they, and their tangent spaces, have the same dimension.

5.2. $L(\mathcal{E})$ is the geometric quotient $X(\mathcal{E})/\text{Aut}(\mathcal{E})$ and is smooth. The following definition is needed for the statement of the next theorem.

Definition 5.5. A geometric quotient for an action $\alpha : G \times X \rightarrow X$ of a linear algebraic group $G$ on a variety $X$ is a pair $(Y, q)$ consisting of a scheme $Y$ and a morphism $q : X \rightarrow Y$ such that

1. $q$ is open and surjective,
2. $(q_*\mathcal{O}_X)^G = \mathcal{O}_Y$, and
3. the geometric fibers of $q$ are the $G$-orbits for the action.

As noted in [GP93, discussion following Defn. 1.1], this is equivalent to [MFK94, Defn. 0.6] in reasonable cases (finite-type schemes over fields, etc; this need not concern us, as we will always meet the requirements). Also, as Borel remarks in [Bor69, §6.16], what we are calling a geometric quotient is the same as what he simply calls a “quotient” (which is defined in [Bor69, §6.3]).

Theorem 5.6. Suppose $\mathcal{E}$ satisfies the assumptions in §2.2.  

\textsuperscript{18}Since the map displayed on the previous line factors through $\text{Hom}(\mathcal{O}_E, \mathcal{O}_E)/\mathbb{C}f$ it sends $f$ to $0$, i.e., $\varphi f \in \xi^\perp$, because $\pi f = 0$. It follows that $\varphi f = \pi s'$ for some $s' \in \text{Hom}(\mathcal{O}_E, \mathcal{E})$. 

The homological leaf $L(\mathcal{E})$ is smooth.

2. The corestriction $X(\mathcal{E}) \to L(\mathcal{E})$ of the map $X(\mathcal{E}) \to \mathbb{P}_\mathcal{E}$ in Lemma 3.20 realizes $L(\mathcal{E})$ as the geometric quotient

\[
L(\mathcal{E}) \cong \frac{X(\mathcal{E})}{\operatorname{Aut}(\mathcal{E})}.
\]

Proof. (1) The differential of $\Psi : X(\mathcal{E}) \to \mathbb{P}_\mathcal{E}$ has constant rank, by Corollary 5.4, so it is a submersion in the sense of [Ser06, p.86, Definition] by [Ser06, Theorem]. In particular, its image $L(\mathcal{E}) \subset \mathbb{P}_\mathcal{E}$ is a smooth closed subvariety locally in the complex analytic (rather than Zariski) topology.

The conclusion follows from the fact that smoothness at a point $x \in X$ of a complex algebraic variety is equivalent to smoothness in the complex-analytic sense. Indeed,

- The two smoothness properties can be phrased as the requirements that the local rings $\mathcal{O}_x$ (of germs of regular functions) and $\mathcal{O}_{\text{hol},x}$ (of germs of holomorphic functions) be regular as local rings [Ser56, §1.4];
- As observed in loc.cit., $\mathcal{O}_{\text{hol},x}$ is noetherian;
- So that $\mathcal{O}_x$ and $\mathcal{O}_{\text{hol},x}$ are local noetherian rings of equal dimension $d$, say [Ser56, §2.6, Corollaire 2];
- Whence by definition ([AM69, Thm. 11.22]), regularity means that the quotient $\mathfrak{m}/\mathfrak{m}^2$ is $d$-dimensional over the ground field, where $\mathfrak{m}$ denotes, in each of the two cases, the maximal ideal of the ring in question.
- The conclusion then follows from the fact that for a noetherian local ring the quotient $\mathfrak{m}/\mathfrak{m}^2$ is unaltered upon passing to the $\mathfrak{m}$-adic completion [AM69, Prop. 10.15];
- And the canonical inclusion $\mathcal{O}_x \leq \mathcal{O}_{\text{hol},x}$ induces an isomorphism of adic completions [Ser56, §2.6, Prop. 3].

(2) This will follow from [Bor69, Prop. 6.6] which says the following: let $G$ act on $X$ and assume that the irreducible components of $X$ are open; if $Y$ is normal and $q : X \to Y$ is a separable [Bor69, §AG.8.2] orbit map (in the sense of [Bor69, §6.3], i.e., it is surjective and its fibers are the $G$-orbits), then $(Y,q)$ is the geometric quotient\[^{19}\] of $X$ by $G$.

All the hypotheses of [Bor69, Prop. 6.6] hold for the morphism $X(\mathcal{E}) \to L(\mathcal{E})$: first, it is surjective and its fibers are precisely the $\operatorname{Aut}(\mathcal{E})$-orbits (as mentioned numerous times via Lemma 3.3) so it is an orbit map; second, its domain, $X(\mathcal{E})$, is irreducible; third, $L(\mathcal{E})$ is normal because it is smooth; fourth, $X(\mathcal{E}) \to L(\mathcal{E})$ is separable because the base field is $\mathbb{C}$.

Remark 5.7. Together, Theorem 5.6(1) and Theorem 4.5 show that the varieties $\text{Sec}_{d,x}(E) - \text{Sec}_{d,1}(E)$ are smooth when $d < \frac{n}{2}$ or $d = \frac{n}{2}$ and $x \not\in \Omega$. Compare this to [GvBH04, Prop. 8.15], which shows that $\text{Sec}_{d}(E) - \text{Sec}_{d,1}(E)$, which is the disjoint union of all $\text{Sec}_{d,x}(E) - \text{Sec}_{d,1}(E), x \in E$, by Proposition 2.10, is smooth when $d < \frac{n}{2}$.

Corollary 5.8. Under the hypotheses of Theorem 5.6 the quotient morphism $X(\mathcal{E}) \to L(\mathcal{E})$ is smooth and faithfully flat.

Proof. (This is a consequence of Theorem 5.6 and its proof.)

The two varieties are smooth (the codomain by Theorem 5.6 (1)) and in the course of the proof we showed that the differential of the morphism induces surjections of tangent spaces. That the morphism is smooth then follows from the criterion of [Gro67, Théorème 17.11.1].

Smoothness then in turn implies flatness [Gro67, Théorème 17.5.1], whence faithful flatness, the latter being nothing but flatness plus surjectivity [Gro60, Chapitre 0, §6.7.8].

5.3. The $L(\mathcal{E})$’s are the symplectic leaves. We first recall some standard material.

\[^{19}\]See the sentence just before the statement of this theorem.
If we view the Poisson structure $\Pi$ on $\mathbb{P}_E$ as a tensor providing linear maps $\Pi_\xi : T^*_\xi \mathbb{P}_E \rightarrow T^*_\xi \mathbb{P}_E$ for all $\xi \in \mathbb{P}_E$, then the symplectic leaves are, by definition, those submanifolds $S \subseteq \mathbb{P}_E$ such that

$$T^*_\xi S = \text{the image of } \Pi_\xi$$

for all $\xi \in S$ [CFM21, Prop. 1.8, p.6; (2.10), p.29; Thm. 4.1, p.63].

Consider a Lie group $G$ acting smoothly on a smooth manifold $X$. If the action of $G$ on $X$ is free and proper, then $X/G$ is a smooth manifold and the quotient $\psi : X \rightarrow X/G$ is a smooth submersion, by [Lee13, Thm. 7.10], and $G.x$ is an embedded submanifold of $X$ [Lee13, Prop. 21.7]. It now follows from [Lee13, Prop. 5.8] that

$$\ker\left( d\psi_x : T_x X \rightarrow T_{\psi(x)}(X/G) \right) = T_x(G.x).$$

Since $\dim(G.x) = \dim X - \dim(G.x)$ it follows that $d\psi_x$ is surjective and hence, for all $x \in X$, there are natural isomorphisms

$$T_{\psi(x)}(X/G) \cong T_x X / T_x(G.x).$$

**Theorem 5.9.** The symplectic leaves for $(\mathbb{P}_E, \Pi)$ are precisely the homological leaves $L(E)$.

**Proof.** Let $G = \text{Aut}(E)$. By Corollary 5.3,

$$\ker(d\Phi_\pi : T_\pi \text{Epi}(E, \mathcal{L}) \rightarrow T^*_\pi \mathbb{G}) = T_\pi(G, \pi)$$

so, by the paragraph before the statement of this theorem and Lemma 5.1,

$$T_{\Phi(\pi)}\left( \begin{array}{c} \text{Epi}(E, \mathcal{L}) \\ \text{Aut}(\mathcal{E}) \end{array} \right) \cong \frac{T_\pi \text{Epi}(E, \mathcal{L})}{T_\pi(G, \pi)} = \text{the image of } d\Phi_\pi$$

$$= \{d\Phi_\pi(\varphi) = \gamma \circ \varphi_\ast : \text{Hom}(O_E, \mathcal{E}) \rightarrow \text{Hom}(O_E, \mathcal{L})/\xi^\perp | \varphi \in \text{Hom}(E, \mathcal{E})\}. $$

Hence, by the remarks at the start of §5.3, to prove that the $L(E)$'s are the symplectic leaves we must show, after appropriate identifications, that

the image of $d\Phi_\pi = \text{the image of } \Pi_\xi$.

**Claim:** The image of $d\Phi_\pi$ equals $\{\xi, \varphi | \varphi \in \text{Hom}(E, \mathcal{L})\} \subseteq \text{Ext}^1(E, O_E)$.

**Proof:** Fix a non-zero $\varphi \in \text{Hom}(E, \mathcal{L})$. Recall that $\varphi_\ast : \text{Hom}(O_E, E) \rightarrow \text{Hom}(O_E, \mathcal{L})$ is the map $s \mapsto \varphi s$. The kernel of the map $d\Phi_\pi(\varphi) = \gamma \circ \varphi_\ast : \text{Hom}(O_E, \mathcal{E}) \rightarrow \text{Hom}(O_E, \mathcal{L})/\xi^\perp$, given by $s \mapsto \varphi s + \xi^\perp = \varphi_\ast(s) + \xi^\perp$, is $\varphi_\ast^{-1}(\xi^\perp)$. The kernel of the map $\xi \cdot \varphi : \text{Hom}(O_E, \mathcal{E}) \rightarrow \text{Ext}^1(O_E, O_E)$, given by $s \mapsto \xi \cdot \varphi s = \xi \cdot \varphi_\ast(s)$, is also $\varphi^{-1}(\xi^\perp)$. Since $\xi^\perp$ has codimension one in $\text{Hom}(O_E, \mathcal{L})$, $\varphi^{-1}(\xi^\perp)$ has codimension one in $\text{Hom}(O_E, \mathcal{E})$. There is therefore an isomorphism $\alpha : \text{Hom}(O_E, \mathcal{L})/\xi^\perp \rightarrow \text{Ext}^1(O_E, O_E)$ such that $\varphi \mapsto \alpha \circ d\Phi_\pi(\varphi) = \xi \cdot \varphi_\ast$.

**Claim:** The map $\text{im}(\Pi_\xi) \rightarrow \text{im}(d\Phi_\pi)$, $\eta \mapsto \eta \cdot \pi$, is an isomorphism.

**Proof:** We begin with the following observation: it follows from the exact sequence

$$0 \rightarrow \text{Hom}(L, L) \rightarrow \text{Hom}(E, L) \rightarrow \text{Hom}(O_E, L) \rightarrow \text{Ext}^1(L, L) \rightarrow 0$$

that $\xi^\perp = \{\varphi f | \varphi \in \text{Hom}(E, \mathcal{L})\}$.

The proof of [HP20, Prop. 2.3] says that $\Pi_\xi$, which is their $\Pi_\psi$, fits into a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(E, \mathcal{L}) & \rightarrow & \xi^\perp \rightarrow \text{Hom}(O_E, \mathcal{L}) \\
\delta \downarrow & & \downarrow \Pi_\xi \\
\text{Ext}^1(E, O_E) & \rightarrow & \text{Ext}^1(L, O_E) / \xi
\end{array}$$

(5.13)
Definition 5.10. An action \( \triangleright : G \times X \to X \) of an algebraic group on an algebraic variety is principal if
\[
\text{can} = \text{can}_\triangleright : G \times X \xrightarrow{(\cdot, \pi_2)} X \times X
\]
is a closed immersion, where \( \pi_2 \) denotes the projection \( G \times X \to X \).

We might occasionally refer to principal actions as algebrao-geometrically-free or \( \text{ag-free} \) for short.

There is a distinction: [MFK94, Example 0.4] is an otherwise fairly well-behaved set-theoretically free action that is not principal in the sense of Definition 5.10.

Proposition 5.11. Suppose \( \mathcal{E} \) satisfies the assumptions in \$2.2. \) The actions of Aut(\( \mathcal{E} \)) on \( X = X(\mathcal{E}) \) and \( \mathbb{P} \text{Aut}(\mathcal{E}) \) on \( \mathbb{P}X \) are principal in the sense of Definition 5.10.

Proof. We focus on the first claim regarding Aut(\( \mathcal{E} \)); the second is entirely analogous. Let \( G = \text{Aut}(\mathcal{E}) \), and identify \( X = X(\mathcal{E}) \) with \( \text{Epi}(\mathcal{E}, \mathcal{L}) \) as in Lemma 3.8.

The image of (5-14) is the graph of the action \( \triangleright \), i.e., the image is the relation
\[
\Gamma := \{(g x, x) \mid x \in X, g \in G\} \subseteq X \times X.
\]
Note first that \( \Gamma \subseteq X \times X \) is closed because it is the fiber product
\[
X \times_{L(\mathcal{E})} X \subseteq X \times X,
\]
where the left-hand side denotes the pullback along the two quotient maps \( X(\mathcal{E}) \to L(\mathcal{E}) \). The map \( G \times X \to \Gamma \) induced by (5-14) is bijective because it is free in the set-theoretic sense. Its inverse, \( \Gamma \to G \times X \), is \( (g, x) \mapsto (g x, x) \) where \( g \in G \) is the unique element such that \( g x = y \).

Claim: (5-14) induces embeddings on tangent spaces. In other words, the differential \( d\text{can}_{(g, x)} \) is one-to-one at every point \( (g, x) \in G \times X \).

Proof: We only need to consider the points \((1, x) \in G \times X \) (injectivity at other points follows from a translation argument because (5-14) is \( G \)-equivariant). Having fixed \((1, x)\), with \( x \) regarded as an epimorphism \( \mathcal{E} \to \mathcal{L} \) via Lemma 3.8, the domain of \( d\text{can}_{(1, x)} \) is the tangent space \( T_{(1, x)}(G \times X) \cong T_1 G \times T_x X \cong \text{End}(\mathcal{E}) \times \text{Hom}(\mathcal{E}, \mathcal{L}) \) and \( d\text{can}_{(1, x)} : \text{End}(\mathcal{E}) \times \text{Hom}(\mathcal{E}, \mathcal{L}) \to \text{Hom}(\mathcal{E}, \mathcal{L})^2 \cong T_{(x, x)}(X \times X) \) is the map \((\alpha, \varphi) \mapsto (x \circ \alpha, \varphi, \varphi)\), which is injective by Lemma 3.5.

It follows from the claim that (5-14) is a continuous bijection onto a closed subspace of \( X \times X \), and is an immersion in the complex-analytic sense of [Ser06, Part II, §III.10]. It follows from [Ser06, p.84, Theorem] that, complex-analytically, \( \text{can} \) can be identified locally with a linear embedding of complex vector spaces. It is in particular a closed immersion in the complex-analytic sense, whence also an algebraic closed immersion [GR02, Prop. XII.3.2].

\[\text{can} = \text{can}_\triangleright : G \times X \xrightarrow{(\cdot, \pi_2)} X \times X\]

is one-to-one at every point \((g, x) \in G \times X\).

\[\text{can} = \text{can}_\triangleright : G \times X \xrightarrow{(\cdot, \pi_2)} X \times X\]

is one-to-one at every point \((g, x) \in G \times X\).\]

The image of the map \( \text{Hom}(\mathcal{O}_E, \mathcal{O}_E) \to \text{Ext}^1(\mathcal{L}, \mathcal{O}_E) \) in the exact sequence
\[
0 \to \text{Hom}(\mathcal{O}_E, \mathcal{O}_E) \to \text{Ext}^1(\mathcal{L}, \mathcal{O}_E) \xrightarrow{\pi} \text{Ext}^1(\mathcal{E}, \mathcal{O}_E) \to \text{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) \to 0
\]
is \( \mathbb{C} \xi \) so \( \{\eta \in \text{Ext}^1(\mathcal{L}, \mathcal{O}_E) \mid \eta \cdot \pi = 0\} = \mathbb{C} \xi \).
Remark 5.12. The proof of Proposition 5.11 invoked a “GAGA principle”, so named for [Ser56], whereby analytic-geometric results imply algebraic-geometric results, as a matter of convenience. An alternative argument proceeds as follows.

Write $Y := L(\mathcal{E})$. Because $X \to Y$ is smooth, by Corollary 5.8, of relative dimension $d := \dim \mathbb{P} \mathrm{Aut}(\mathcal{E})$ [Har77, §III.10, Definition], the same is true of the two projections $X \times_Y X \to X$ [Har77, Prop. III.10.1(b)]. In particular, $X \times_Y X$ is smooth of dimension $d + \dim X = d + n - 1$. It follows from this and the injectivity of the differential proved while proving Proposition 5.11 that in fact the (abusively labeled) corestriction can $g : G \times X \to X \times_Y X$ in fact induces isomorphisms of tangent spaces, and is thus étale [Har77, Exer. III.10.3]: smooth of relative dimension 0. It is also radicial [Gro67, Définition 3.5.4 and §3.5.5] by injectivity: that injectivity can be phrased as an elementary sentence in the sense of [Sei58, p.687], so transports over to arbitrary algebraically closed extensions of $\mathbb{C}$ by the Lefschetz principle stated on [Sei58, p.687, bottom].

It follows from [Gro67, Théorème 17.9.1] that $\mathcal{G} : G \times X \to X \times_Y X$ is an open immersion and, because it is also a bijection onto the closed subscheme $X \times_Y X \subseteq X \times X$, the conclusion follows. \hfill \Box

For a further variation on the theme that the morphism $X(\mathcal{E}) \to L(\mathcal{E})$ is as well-behaved as a geometric quotient could possibly be, recall the following notion (e.g., [GR02, §4.7], [Ser58, §2.2], etc.).

Definition 5.13. A geometric quotient $\pi : X \to X/G$ for an action of a linear algebraic group is locally trivial if there is an open cover

$$X/G = \bigcup V_i$$

such that for each $i$ there is a $G$-equivariant isomorphism $\pi^{-1}(V_i) \cong G \times V_i$, where $G$ acts on $G \times V_i$ via $g : (h, v) = (gh, v)$. Thus, if we identify $\pi^{-1}(V_i)$ with $G \times V_i$, then $\pi$ is the projection onto the second coordinate.

Theorem 5.14. Let $X = X(\mathcal{E})$. The geometric quotient $\mathbb{P} X \to L(\mathcal{E})$ is locally trivial.

Proof. In this proof we will use the notation $G := \mathbb{P} \mathrm{Aut}(\mathcal{E})$ and $Y := L(\mathcal{E}) = X/G$.

It follows from Proposition 5.11 and [MK94, Prop. 0.9, p.16] that the corestriction $X \times G \to X \times_Y X \subseteq X \times X$ of (5-14) is an isomorphism. In particular, since the quotient morphism $X \to Y$ is smooth and surjective (Corollary 5.8), we have local triviality in the smooth (rather than the Zariski) topology [The18, Tag 021Y]: substitute a smooth cover for the Zariski cover of Definition 5.13. This means that the claim also holds in the étale topology of [The18, Tag 021Y] by [The18, Tag 055V]. To see that the claim holds as stated, in the coarser Zariski topology, we consider two separate cases.

Suppose $\mathcal{E} \not\cong \mathcal{N} \oplus \mathcal{N}$ for any $\mathcal{N}$. In this case, $G$ is solvable and connected by Lemma 3.28. It follows that étale-locally trivial quotients by $G$ are Zariski-locally trivial by [Ser58, Prop. 14, p.1-25] and the definition of special at the start of [Ser58, §4.1].

Suppose $\mathcal{E} \cong \mathcal{N} \oplus \mathcal{N}$. By Lemma 3.28, $G \cong \mathrm{PGL}(2)$ in this case. However, as noted in [Ser58, §4.4, p.1-026], $\mathrm{PGL}(n)$ is not special if $n \geq 2$ so we can’t conclude that the quotient $X \to Y = X/\mathrm{PGL}(2)$ is Zariski-local trivial simply because is étale-locally trivial. However, by [Ser58, Prop. 18], $X \to Y = X/\mathrm{PGL}(2)$ will be locally trivial if, as a principal fiber space with group $\mathrm{PGL}(2)$ [Ser58, §2.2], it is obtained by structure-group extension [Ser58, §3.3] along $\mathrm{GL}(n) \to \mathrm{PGL}(n)$. But this is the case here because $Y \cong X(\mathcal{E})/\mathrm{Aut}(\mathcal{E}) = X(\mathcal{E})/\mathrm{GL}(2)$. This concludes the proof. \hfill \Box

Proposition 5.15. Suppose $\mathcal{E}$ satisfies the assumptions in §2.2. The homological leaf $L(\mathcal{E})$ is

1. affine if either
   1. $n$ is odd and $\mathcal{E}$ is indecomposable, or
   2. $n$ is even and $\mathcal{E}$ is a direct sum of two line bundles of degree $\frac{n}{2}$.
2. quasi-affine but not affine if $n$ is even and $\mathcal{E}$ is indecomposable.

Proof. By Theorem 5.6(2), $L(\mathcal{E})$ is a geometric quotient of $X = X(\mathcal{E})$ for the action of $\mathrm{Aut}(\mathcal{E})$. Hence $L(\mathcal{E})$ is also a geometric quotient $\mathbb{P}X/\mathbb{P} \mathrm{Aut}(\mathcal{E})$ (where the projective version $G := \mathbb{P} \mathrm{Aut}(\mathcal{E})$ of the automorphism group in (3-36)), and the scheme $\mathbb{P} X$ being acted upon is affine by Corollary 3.17.

By Lemma 3.28, the group $G$ is
trivial when $E$ is indecomposable of odd degree;
- the additive group $G_a$ when $E$ is indecomposable of even degree;
- the semidirect product $G_a^{d_2-d_1} \rtimes G_m = G_a^{-2d_1} \rtimes G_m$ when $E \cong N_1 \oplus N_2$ where $d_1 = \deg N_1 \leq \deg N_2 = d_2$ and $N_1 \not\cong N_2$;
- $\operatorname{PGL}(2)$ when $E \cong N \oplus N'$ for a line bundle $N$.

(1) In (1a) and (1b), $G$ is reductive in the sense of [MFK94, Defm. 1.4], as seen by examining the above list. It is well-known that the geometric quotient is therefore affine; see, e.g., [MFK94, Thm. 1.1 and Amplification 1.3].

(2) Now $G$ is $G_a$, so in particular unipotent [Hum75, §17.5]. Because the variety $\mathbb{P}(\operatorname{Hom}(O_E, E) - Z \cup \{0\})$ on which $G$ acts is affine, the geometric quotient is quasi-affine by [FM76, Prop. 3].

It remains to prove the non-affineness claim when $E \cong E_\omega$. By definition, $E_\omega$ is the middle term of a non-split extension $0 \to L_\omega \to E_\omega \to L_\omega \to 0$. There is an associated exact sequence $0 \to H^0(E, L_\omega) \to H^0(E, E_\omega) \to H^0(E, L_\omega) \to 0$. We choose a splitting $H^0(E, E_\omega) = H^0(E, L_\omega) \oplus H^0(E, L_\omega)$; it is not canonical, but can be chosen once and for all for the rest of the proof.

The non-zero maps $f : O_E \to E_\omega$ that are not in $Z$ are precisely those whose components in each copy of $H^0(E, L_\omega)$ are non-zero and the whose zero loci do not intersect (i.e., their zero loci are effective divisors of degree $\frac{\alpha}{2}$ that do not meet).

Because this is also the description of $X (L_\omega \oplus L_\omega) = \operatorname{Hom}(O_E, L_\omega \oplus L_\omega) - Z \cup \{0\}$, $\mathbb{P} \operatorname{Aut}(L_\omega \oplus L_\omega)$, which is isomorphic to $\operatorname{PGL}(2)$, also acts on the space $\mathbb{P} X (E_\omega)$ with the action restricting to that of

$$G = \mathbb{P} \operatorname{Aut}(E_\omega) \cong G_a.$$ 

Since $\pi : X \to X/G$ is open and surjective, it is also a quotient map in the usual, topological, sense [Mun00, §22]: a subset of $X/G$ is open if and only if its preimage is open in $X$. This follows for instance from the fact that if $\pi^{-1}(U)$ is open then so is its saturation [Mun00, §22]

$$\bigcup_{g \in G} gU$$

under the action by $G$, so that $U$ itself must be open by the openness of $\pi$. In particular, the image of any (closed!) $\operatorname{PGL}(2)$-orbit in $X$ will be a copy of $\operatorname{PGL}(2)/G$ embedded as a closed subvariety of $X/G$.

It remains to show that $F := \operatorname{PGL}(2)/G = \operatorname{PGL}(2)/G_a$ is not affine. To see this, note that $F$ is the quotient of $\operatorname{SL}(2)/G = \operatorname{SL}(2)/G_a \cong \mathbb{A}^2 - \{(0,0)\}$ by the order-2 central subgroup $\mu_2 \subset \operatorname{SL}(2)$, so there is a finite [Mum70, §12, Thm. 1] morphism $\mathbb{A}^2 - \{(0,0)\} \to F$. This morphism is affine so $F$ cannot be affine because $\mathbb{A}^2 - \{(0,0)\}$ is not [Har77, Exer. III.4.3].

This finishes the proof.

Example 5.16. Suppose $n = 2d \geq 4$. Assume $E = N_1 \oplus N_2$ where $\deg N_i = d$ and $N_1 \not\cong N_2$. By Proposition 5.15(b), $L(E)$ is affine. It is, however, illuminating to give a direct proof of this, not least because it illustrates how the familiar $n = 4$ case extends to larger even $n$. When $n = 4$, $E \subset \mathbb{P} E \cong \mathbb{P}^3$ is contained in a pencil of quadrics, four of which are singular and, if $Q$ is one of the smooth quadrics, then $Q - E$ is an affine variety (and a symplectic leaf); on the other hand if $Q$ is singular, then it is a cone whose vertex is a symplectic leaf (the vertices are the leaves in Theorem 4.7(3)), and $Q - (E \cup$ the vertex) is a symplectic leaf that is quasi-affine but not affine (Proposition 5.15(2)); the leaves of the form $Q - (E \cup$ the vertex) are those in Theorem 4.7(2). More about the geometry of $E \subset \mathbb{P}^3$ can be found in [Hui86].

Clearly, $\operatorname{Aut}(N_1 \oplus N_2) \cong \mathbb{C}^\times \times \mathbb{C}^\times$.

Let $D_1$ and $D_2$ be effective divisors such that $N_1 \cong O_E(D_1)$ and $N_2 \cong O_E(D_2)$.

Let $f = (f_1, f_2) : O_E \to E = N_1 \oplus N_2$ be a non-zero map and let $(f_i) \in \operatorname{Div}(E)$ denote the divisor of zeros of $f_i \in \operatorname{Hom}(O_E, N_i) = H^0(E, N_i)$. It is routine to show that $\operatorname{coker}(f)$ is torsion-free if and only if $f_1 \neq 0$ and $f_2 \neq 0$ and $(f_1) \cap (f_2) = \emptyset$. It follows from the first two of these three conditions that

$$Z_E \cup \{0\} \supseteq \left( \operatorname{Hom}(O_E, N_1) \times \{0\} \right) \cup \left( \{0\} \times \operatorname{Hom}(O_E, N_2) \right),$$
whence

\[ X(\mathcal{E}) = \text{Hom}(\mathcal{O}_E, \mathcal{E}) - \mathcal{Z}_E \cup \{0\} \subseteq \left( \text{Hom}(\mathcal{O}_E, \mathcal{N}_1) - \{0\} \right) \times \left( \text{Hom}(\mathcal{O}_E, \mathcal{N}_2) - \{0\} \right) \subseteq \text{Hom}(\mathcal{O}_E, \mathcal{E}). \]

Since each copy of \( C^n \) in Aut(\( \mathcal{E} \)) acts on the appropriate \( \text{Hom}(\mathcal{O}_E, \mathcal{N}_i) \) by scaling,

\[ L(\mathcal{E}) \cong X(\mathcal{E}) / \text{Aut}(\mathcal{E}) \subseteq \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{N}_1) \times \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{N}_2) \cong \mathbb{P}^d \times \mathbb{P}^{d-1}. \]

(When \( n = 4 \), the right-hand side is a smooth quadric in \( \mathbb{P}^d \cong \mathbb{P}^3 \) that contains \( E \).)

It now follows from the condition that \((f_1) \cap (f_2) = \emptyset\) that the image of \( Z \cup \{0\} \) under the map

\[ \text{Hom}(\mathcal{O}_E, \mathcal{E}) - \text{the right-hand side of (5-15)} \longrightarrow \mathbb{P}^d \times \mathbb{P}^{d-1} \]

is the effective divisor \( D \in \text{Div}(\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}) \) consisting of those pairs

\[ (s_1, s_2) \in \mathbb{P}H^0(\mathcal{E}, \mathcal{N}_1) \times \mathbb{P}H^0(\mathcal{E}, \mathcal{N}_2) \]

whose zero loci, which are effective degree-\( d \) divisors, which are equivalent to \( D_1 \) and \( D_2 \) respectively, do not intersect. The divisor of zeros of a generic \( s_1 \in \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{N}_1) \) consists of \( d \) distinct points on \( E \). Avoiding these in the zero locus of \( s_2 \in \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{N}_2) \) means avoiding \( d \) hyperplanes in \( \mathbb{P} \text{Hom}(\mathcal{O}_E, \mathcal{N}_2) \). Hence the intersection of \( D \) with the generic \( \mathbb{P}^{d-1} \) fiber of the second projection

\[ \mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \longrightarrow \mathbb{P}^{d-1} \]

consists of \( d \) hyperplanes. Interchanging the roles of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), the same goes for the other projection. By [Har77, Exer. III.12.6], the Picard group \( \text{Pic}(\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}) \) is \( \mathbb{Z} \oplus \mathbb{Z} \); the class of \( D \) in it is \( (d, d) \), so \( D \) is ample; its complement is therefore affine. \( \diamond \)

**Remark 5.17.** Proposition 5.15(2) is a natural example of a \( \mathbb{G}_a \)-action on an affine scheme whose geometric quotient is quasi-affine but not affine; [GP93, Ex. 2.5] is another case in point, as is the quotient

\[ \text{SL}(2) \rightarrow \text{SL}(2)/(\text{upper triangular unipotent matrices}) \cong \mathbb{A}^2 - \{(0, 0)\} \]

that makes an oblique appearance in the above proof of non-_affineness.

Recall, however, that (under mild conditions we need not be concerned with here) quotients of unipotent-group actions on quasi-affine schemes are again quasi-affine [FM76, Prop. 3 and Thm. 4]. This is manifestly not so for \( \mathbb{G}_m \):

\[ (\text{quasi-affine}) \quad \mathbb{A}^n - \{0\} \longrightarrow (\mathbb{A}^n - \{0\}) / \mathbb{G}_m \cong \mathbb{P}^{n-1} \quad \text{(non-quasi-affine)}. \]

See [Fau83, Thm. 3], though, for a positive result: if a linear algebraic group does not map onto \( \mathbb{G}_m \), then quotients of quasi-affine varieties by its actions are quasi-affine. \( \diamond \)

**Appendix A. Isomorphism classes of extensions**

Let \( k \) be a commutative ring with identity, and let \( \mathcal{A} \) be a \( k \)-linear abelian category. Given objects \( A \) and \( C \) in \( \mathcal{A} \), we define the category \( \mathcal{A}(C, A) \) as follows: The objects in \( \mathcal{A}(C, A) \) are the exact sequences

\[ (A-1) \]

\[ 0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0 \]

in which \( A' \cong A \) and \( C' \cong C \). We call \((A-1)\) an extension of \( C' \) by \( A' \).\(^{21}\) A morphism \( \xi_1 \rightarrow \xi_2 \) between exact sequences

\[ (A-2) \]

\[ \xi_i : 0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0, \quad (i = 1, 2), \]

is a triple \((\mu, \nu, \tau)\) such that the diagram

\[ (A-3) \]

\[ \begin{array}{ccc}
0 & \longrightarrow & A_1 \\
& \downarrow{\mu} & \searrow{\nu} \\
0 & \longrightarrow & A_2
\end{array} \]

\[ \begin{array}{ccc}
& & 0 \\
& \downarrow{\tau} & \\
& & 0
\end{array} \]

\[ \begin{array}{ccc}
& \longrightarrow & B_1 \\
& \longrightarrow & B_2
\end{array} \]

\[ \begin{array}{ccc}
& \longrightarrow & C_1 \\
& \longrightarrow & C_2
\end{array} \]

\[ \longrightarrow \]

\[ 0 \rightarrow A'_1 \rightarrow B'_1 \rightarrow C'_1 \rightarrow 0 \]

\[ \longrightarrow \]

\[ 0 \rightarrow A'_2 \rightarrow B'_2 \rightarrow C'_2 \rightarrow 0 \]

\[ \longrightarrow \]

\(^{21}\)The terminology in the Stacks Project differs: extensions that we call equivalent, they call isomorphic (Tag 010I).
commutes.\footnote{MacLane has a section on the category of short exact sequences in his book *Homology*, CH. XII \S\,6.} If \((\mu, \nu, \tau)\) are such that this diagram commutes, then \((\mu, \nu, \tau)\) is an isomorphism if and only if \(\mu, \nu\) and \(\tau\) are isomorphisms; we then write \(\xi_1 \cong \xi_2\).

If \(f : A \to B\) is monic and \(g_i : B \to C_i, i = 1, 2,\) are cokernels for \(f,\) then the sequences

\[
0 \to A \xrightarrow{f} B \xrightarrow{g} C_i \to 0 \quad (i = 1, 2)
\]

are isomorphic.

\(A\) is a \(k\)-linear category: the addition \((\mu, \nu, \tau) + (\mu', \nu', \tau') := (\mu + \mu', \nu + \nu', \tau + \tau')\) gives \(\text{Hom}_k(\xi_1, \xi_2)\) the structure of an abelian group; it is a \(k\)-module with scalar multiplication given by \(\lambda(\mu, \nu, \tau) := (\lambda \mu, \lambda \nu, \lambda \tau)\) for \(\lambda \in k\).

Assume \(k\) is a field. Proposition A.1 shows that in the situation of interest in this paper, where \(A\) and \(C\) are sheaves of sections of stable vector bundles on \(E\), the points in the projective space \(\mathbb{P} \text{Ext}^1_A(C, A)\) of 1-dimensional subspaces of \(\text{Ext}^1_A(C, A)\) are the isomorphism classes of exact sequences \(0 \to A' \to B' \to C' \to 0\) in which \(A' \cong A\) and \(C' \cong C\).

Let \(\alpha : A' \to A\) and \(\gamma : C' \to C\) be isomorphisms. The map that sends an extension

\[
(A-4) \quad \xi' : \quad 0 \longrightarrow A' \xrightarrow{f} B' \xrightarrow{g} C' \longrightarrow 0
\]

to the extension

\[
(A-5) \quad \xi : \quad 0 \longrightarrow A \xrightarrow{f \alpha^{-1}} B' \xrightarrow{g \gamma} C \longrightarrow 0
\]

induces a \(k\)-linear isomorphism

\[\phi_{\alpha, \gamma} : \text{Ext}^1_A(C', A') \to \text{Ext}^1_A(C, A).\]

Let \(\Phi_{\alpha, \gamma}\) be the unique morphism such that the diagram

\[
\begin{array}{ccc}
\text{Ext}^1_A(C', A') - \{0\} & \xrightarrow{\phi_{\alpha, \gamma}} & \text{Ext}^1_A(C, A) - \{0\} \\
\downarrow & & \downarrow \\
\mathbb{P} \text{Ext}^1_A(C', A') & \xrightarrow{\Phi_{\alpha, \gamma}} & \mathbb{P} \text{Ext}^1_A(C, A)
\end{array}
\]

commutes. Although \(\phi_{\alpha, \gamma}\) depends on the choice of \(\alpha\) and \(\gamma,\) Proposition A.1 shows that in the situation of interest in this paper the isomorphism \(\Phi_{\alpha, \gamma}\) is the same for all choices of \(\alpha\) and \(\gamma\) and, as a consequence, \(\mathbb{P} \text{Ext}^1_A(C, A)\) is in natural bijection with isomorphism classes of non-split exact sequences \(0 \to A' \to B' \to C' \to 0\) in which \(A' \cong A\) and \(C' \cong C\).

**Proposition A.1.** Let \(k\) be a field, and \(A\) and \(C\) objects in a \(k\)-linear abelian category \(A.\) If \(\text{End}(A) = \text{End}(C) = k,\) then the map \(\Phi_{\alpha, \gamma}\) in (A-6) does not depend on the choice of \(\alpha\) or \(\gamma,\) and the map

\[
\begin{array}{c}
\left\{ \begin{array}{l}
\text{non-split exact sequences} \\
0 \to A' \to B' \to C' \to 0
\end{array} \right\} \bigg/ \cong \\
\xrightarrow{\cong} \\
\mathbb{P} \text{Ext}^1_A(C, A)
\end{array}
\]

that sends (A-4) to (A-5) is a well-defined bijection that does not depend on the choice of \(\alpha\) or \(\gamma.\)

**Proof.** Consider two non-split isomorphic exact sequences

\[
(A-8) \quad \xi_i : \quad 0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0 \quad (i = 1, 2)
\]

in which \(A_i \cong A\) and \(C_i \cong C.\) There is a commutative diagram

\[
\begin{array}{c}
0 \longrightarrow A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \longrightarrow 0 \\
\mu \downarrow \quad \quad \quad \nu \downarrow \quad \quad \quad \tau \\
0 \longrightarrow A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C_2 \longrightarrow 0
\end{array}
\]
in which $\mu$, $\nu$, and $\tau$ are isomorphisms.

Fix isomorphisms $\alpha_i : A_i \rightarrow A$ and $\gamma_i : C_i \rightarrow C$. For brevity write $\phi_i = \phi_{\alpha_i, \gamma_i}$ and $\Phi_i = \Phi_{\alpha_i, \gamma_i}$. Then $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$ are the exact sequences

\[(A-9)\quad 0 \rightarrow A \xrightarrow{f_{1\alpha_1}} B_1 \xrightarrow{\gamma_1} C \rightarrow 0 \quad (i = 1, 2).\]

The maps $\lambda := \alpha_2\mu\alpha_1^{-1}$ and $\theta := \gamma_2\tau\gamma_1^{-1}$ are isomorphisms $A \rightarrow A$ and $C \rightarrow C$, respectively.

Since the diagram

\[
\begin{array}{ccc}
\phi_1(\xi_1) & : & 0 \rightarrow A \xrightarrow{f_{1\alpha_1}} B_1 \xrightarrow{\gamma_1} C \rightarrow 0 \\
& & \downarrow \lambda \quad \downarrow \nu \quad \downarrow \theta \\
\phi_2(\xi_2) & : & 0 \rightarrow A \xrightarrow{f_{2\alpha_2}} B_2 \xrightarrow{\gamma_2} C \rightarrow 0 
\end{array}
\]

commutes, $(\lambda, \nu, \theta)$ is an isomorphism $\phi_1(\xi_1) \rightarrow \phi_2(\xi_2)$.

We will now show that the equivalence classes of $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$, which are points in $\text{Ext}^1_A(C, A)$, are non-zero scalar multiples of each other.

Because $\text{End}_A(A) = \text{End}_A(C) = k$, $\lambda$ and $\theta$ are scalar multiples of the identity maps $\text{id}_A$ and $\text{id}_C$, so, with a small abuse of notation, we can view $\lambda$ and $\theta$ as scalars and deduce that the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A \xrightarrow{f_{1\alpha_1}} B_1 \xrightarrow{\theta\gamma_1} C \rightarrow 0 \\
& & \downarrow \lambda \quad \downarrow \nu \\
0 & \rightarrow & A \xrightarrow{f_{2\alpha_2}} B_2 \xrightarrow{\gamma_2} C \rightarrow 0 
\end{array}
\]

commutes. But the rows of this diagram are scalar multiples of $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$ so the equivalence classes of $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$ become equal in $\mathbb{P}\text{Ext}^1_A(C, A)$, i.e., $\Phi_1(\xi_1) = \Phi_2(\xi_2)$. This completes the proof that the map in (A-7) is well defined.

If we consider the case $A_1 = A_2$ and $C_1 = C_2$, the argument above also shows that the morphisms $\Phi_1, \Phi_2 : \mathbb{P}\text{Ext}^1_A(C_1, A_1) \rightarrow \mathbb{P}\text{Ext}^1_A(C, A)$ are the same.

The map in (A-7) is certainly surjective. It remains to show it is injective.

Let $\xi_1$ and $\xi_2$ be non-split exact sequences as in (A-8). As before, fix isomorphisms $\alpha_i : A_i \rightarrow A$ and $\gamma_i : C_i \rightarrow C$, and write $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$ for the exact sequences in (A-9). Suppose that $\Phi_1(\xi_1) = \Phi_2(\xi_2)$. Then the equivalence classes containing $\phi_1(\xi_1)$ and $\phi_2(\xi_2)$ are scalar multiples of each other. Hence there is a non-zero scalar, $\delta \in k$, such that $\delta\phi_1(\xi_1)$ and $\phi_2(\xi_2)$ are equivalent, i.e., there is an isomorphism $\nu : B_1 \rightarrow B_2$ such that the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A \xrightarrow{f_{1\alpha_1}} B_1 \xrightarrow{\delta\gamma_1} C \rightarrow 0 \\
& & \downarrow \nu \quad \downarrow \lambda \quad \downarrow \nu \\
0 & \rightarrow & A \xrightarrow{f_{2\alpha_2}} B_2 \xrightarrow{\gamma_2} C \rightarrow 0 
\end{array}
\]

commutes. It follows that the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \rightarrow & 0 \\
& & \downarrow \alpha_2^{-1}\alpha_1 & \downarrow \nu & \downarrow \delta\gamma_1^{-1}\gamma_1 & & & & \\
0 & \rightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \rightarrow & 0 
\end{array}
\]

commutes. Thus, $\xi_1 \cong \xi_2$. The map in (A-7) is therefore injective. \hfill \Box
Proposition B.1. Equivalence of extensions. Two extensions

\[ \xi_1 : 0 \to A \overset{f_1}{\longrightarrow} B_1 \overset{g_1}{\longrightarrow} C \to 0 \quad (i = 1, 2) \]

are equivalent, denoted \( \xi_1 \equiv \xi_2 \), if there is a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow f_1 \\
A \\
\downarrow f_2 \\
0
\end{array}
\end{array}
\begin{array}{c}
0 \\
\downarrow \cong \\
B_1 \\
\downarrow g_2 \\
B_2 \\
\downarrow \cong \\
C \\
\downarrow 0
\end{array}
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow g_1 \\
C \\
\downarrow 0
\end{array}
\end{array}
\]

By definition, \( \text{Ext}^1_A(C,A) \) is the set of equivalence classes of extensions of \( C \) by \( A \). It is a \( k \)-module.

If \( p \) is an odd prime and \( \pi : \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}p \) the map \( \pi(x) = x + \mathbb{Z}p \), then the sequences \( 0 \to \mathbb{Z} \overset{\pi}{\to} \mathbb{Z} \overset{\pi}{\to} \mathbb{Z}/\mathbb{Z}p \to 0 \) and \( 0 \to \mathbb{Z} \overset{\pi}{\to} \mathbb{Z} \overset{\pi}{\to} \mathbb{Z}/\mathbb{Z}p \to 0 \) are isomorphic but not equivalent.

If \( X \) and \( Y \) are objects in \( A \) there is a “composition” map

\[
\text{Hom}(X, A) \times \text{Ext}^1_A(Y, X) \times \text{Hom}(C, Y) \to \text{Ext}^1_A(C, A), \quad (\alpha, \xi, \beta) \mapsto \alpha \cdot \xi \cdot \beta.
\]

A group action on objects in rigid monoidal categories

Let \( (C, \otimes, 1) \) be a rigid monoidal category (see [EGNO15] for the terminology on monoidal categories).

In this section, the letter \( G \) always denotes a group.

A \( G \)-action on an object \( x \in C \) is a group homomorphism \( \rho_x : G \to \text{Aut}(x) \). If \( G \) acts on \( x \) and \( y \), then it acts on \( x \otimes y \) by \( \rho_{x \otimes y} = \rho_x \otimes \rho_y \). It also acts on the set \( \text{Hom}_C(x, y) \) by

\[ (\alpha \triangleright f) := \rho_y(\alpha) \circ f \circ \rho_x(\alpha)^{-1} \quad (\alpha \in G, f \in \text{Hom}_C(x, y)). \]

If we are given a \( G \)-action on only one of \( x \) and \( y \) we can impose the trivial action of \( G \) on the other one and so obtain an action of \( G \) on \( \text{Hom}_C(x, y) \).

A morphism \( f \in \text{Hom}_C(x, y) \) is said to be \( G \)-equivariant if \( \alpha \triangleright f = f \) for all \( \alpha \in G \).

Proposition B.1. Suppose \( G \) acts on \( x, y, z, w \in C \). If a map \( \varphi : \text{Hom}_C(x, y) \to \text{Hom}_C(z, w) \) is \( G \)-equivariant, then \( \varphi \) preserves \( G \)-equivariance of morphisms.

Proof. Suppose \( f \in \text{Hom}_C(x, y) \) is \( G \)-equivariant. If \( \alpha \in G \), then \( \alpha \triangleright f = f \) so \( \alpha \triangleright \varphi(f) = \varphi(\alpha \triangleright f) = f. \n\]

Proposition B.2. Suppose \( G \) acts on \( x, x', y \in C \). If \( f : x \to x' \) is a \( G \)-equivariant morphism, then the maps

1. \( f_* : \text{Hom}_C(y, x) \to \text{Hom}_C(y, x'), h \mapsto f \circ h, \)

2. \( f^* : \text{Hom}_C(x', y) \to \text{Hom}_C(x, y), h \mapsto h \circ f, \)

are \( G \)-equivariant.

Consequently, the maps \( f_* \) and \( f^* \) preserve \( G \)-equivariance of morphisms.

Proof. Let \( \alpha \in G \) and \( h \in \text{Hom}_C(y, x) \). Since \( f \) is \( G \)-equivariant, \( f \circ \rho_x(\alpha) = \rho_{x'}(\alpha) \circ f \), so

\[
\begin{align*}
f_*(\alpha \triangleright h) &= f_* (\rho_x(\alpha) \circ h \circ \rho_y(\alpha)^{-1}) = f \circ \rho_x(\alpha) \circ h \circ \rho_y(\alpha)^{-1} = \rho_{x'}(\alpha) \circ f \circ h \circ \rho_y(\alpha)^{-1} \\
&= \rho_{x'}(\alpha) \circ f_*(h) \circ \rho_y(\alpha)^{-1} = \alpha \triangleright f_*(h).
\end{align*}
\]

The second statement is proved dually. The last statement follows from Proposition B.1. \( \square \)

Given an object \( x \in C \), let \( x^* \) denote its left dual (when \( C \) is symmetric, which it will be after the next result, the left dual is isomorphic to the right dual, but we will consistently use the left dual).

If \( G \) acts on \( x \), then it also acts on \( x^* \); for each \( \alpha \in G \), the automorphism \( \rho_x(\alpha) : x \to x \) induces the automorphism \( \rho_x(\alpha)^* : x^* \to x^* \), and \( \rho_x^*(\alpha) := (\rho_x(\alpha)^*)^{-1} \) defines the action on \( x^* \).
Proposition B.3. Suppose $G$ acts on $x, y, z \in \mathcal{C}$. The bijection
\begin{equation}
\text{Hom}_\mathcal{C}(y \otimes x, z) \to \text{Hom}_\mathcal{C}(y, z \otimes x^*)
\end{equation}
given by the adjunction $(- \otimes x) \dashv (- \otimes x^*)$ is $G$-equivariant, so it preserves $G$-equivariance of morphisms.

Proof. Let $f \in \text{Hom}_\mathcal{C}(y \otimes x, z)$. There is a commutative diagram
\begin{equation}
\begin{array}{ccc}
y & \xrightarrow{id \otimes \text{coev}} & y \otimes x \otimes x^* \\
\rho_y(\alpha) & \downarrow & \rho_y(\alpha) \otimes \rho_x(\alpha) \otimes \rho_x(\alpha)^{-1} \\
y & \xrightarrow{id \otimes \text{coev}} & y \otimes x \otimes x^* \\
\end{array}
\end{equation}
\begin{equation}
\begin{array}{ccc}
f & \xrightarrow{id} & z \otimes x^* \\
\rho_y(\alpha) \otimes \rho_x(\alpha)^{-1} & \downarrow & \rho_y(\alpha) \otimes \rho_x(\alpha)^{-1} \\
f & \xrightarrow{id} & z \otimes x^* \\
\end{array}
\end{equation}
(in which we have omitted the isomorphism $(y \otimes x) \otimes x^* \to y \otimes (x \otimes x^*)$); the left-hand square is $y \otimes (-)$ applied to the diagram
\begin{equation}
\begin{array}{ccc}
1 & \xrightarrow{\text{coev}} & x \otimes x^* \\
\text{id} & \downarrow & \rho_x(\alpha) \otimes \rho_x(\alpha)^{-1} \\
1 & \xrightarrow{\text{coev}} & x \otimes x^* \\
\end{array}
\end{equation}
whose commutativity follows from the definition of the left dual $\rho_x(\alpha)^*$. The first and the second rows of (B-2) are the images of $f$ and $\alpha \circ f$, respectively, under the map (B-1), so the commutativity of (B-2) implies that (B-1) is $G$-equivariant. \qed

For the rest of this subsection we assume that $\mathcal{C}$ is also symmetric with braiding $c_{x,y} : x \otimes y \to y \otimes x$, and also abelian.

We define the second exterior power $\wedge^2 x$ of $x \in \mathcal{C}$ to be the image of the morphism
\begin{equation}
x \otimes x \xrightarrow{1 - c_{x,x}} x \otimes x.
\end{equation}
If $G$ acts on $x$, then $\rho_{x \otimes x}(\alpha) = \rho_x(\alpha) \otimes \rho_x(\alpha)$ induces an automorphism $\rho_{\wedge^2 x}(\alpha)$ of $\wedge^2 x$. Thus $G$ also acts on $\wedge^2 x$, and the quotient morphism $x \otimes x \to \wedge^2 x$ is $G$-equivariant.

Remark B.4. The equivariance of the maps $\nu$ in Lemma 3.6 and $\theta \circ \nu_*$ in Lemma 3.8 can now be deduced as follows: the group $\text{Aut}(x)$ acts on $x$ and the quotient morphism $x \otimes x \to \wedge^2 x$ is $\text{Aut}(x)$-equivariant, so the morphism $\nu : x \to \wedge^2 x \otimes x^*$ obtained by the adjunction $(- \otimes x) \dashv (- \otimes x^*)$ is $\text{Aut}(x)$-equivariant by Proposition B.3. Therefore the map $\text{Hom}_\mathcal{C}(1, x) \to \text{Hom}_\mathcal{C}(x, \wedge^2 x)$ that is the composition
\begin{equation}
\text{Hom}_\mathcal{C}(1, x) \xrightarrow{\nu_*} \text{Hom}_\mathcal{C}(1, \wedge^2 x \otimes x^*) \xrightarrow{\text{adj}} \text{Hom}_\mathcal{C}(x, \wedge^2 x),
\end{equation}
where the latter map is the adjunction, is $\text{Aut}(x)$-equivariant by Propositions B.2 and B.3, where 1 is equipped with the trivial action of $\text{Aut}(x)$.

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