COISOTROPIC INTERSECTIONS

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Abstract. In this paper we make the first steps towards developing a theory of intersections of coisotropic submanifolds, similar to that for Lagrangian submanifolds.

For coisotropic submanifolds satisfying a certain stability requirement we establish persistence of coisotropic intersections under Hamiltonian diffeomorphisms, akin to the Lagrangian intersection property. To be more specific, we prove that the displacement energy of a stable coisotropic submanifold is positive, provided that the ambient symplectic manifold meets some natural conditions. We also show that a displaceable, stable, coisotropic submanifold has non-zero Liouville class. This result further underlines the analogy between displacement properties of Lagrangian and coisotropic submanifolds.

1. Introduction

In this paper we make the first steps towards developing a theory of coisotropic intersections similar to that for Lagrangian submanifolds. The main objective of the paper is to establish persistence of coisotropic intersections under Hamiltonian diffeomorphisms for a certain class of coisotropic submanifolds, the so-called stable coisotropic submanifolds. We also show that displaceable, stable coisotropic submanifolds have non-zero Liouville class, generalizing the results of Bolle, [Bo1, Bo2], for submanifolds of \( \mathbb{R}^{2n} \).

1.1. Coisotropic intersections. The Lagrangian intersection property or persistence of Lagrangian intersections is unquestionably one of the most fundamental results in symplectic topology. This result asserts that a Lagrangian submanifold necessarily intersects its image under a Hamiltonian diffeomorphism that is in some sense close to the identity, e.g., has sufficiently small energy. Depending on the notion of closeness and on the requirements imposed on the manifolds, various forms of the Lagrangian intersection property have been proved in [Ch1, Ch2, Fl1, Fl2, Fl3, Fl5, Gr, LS1, Oh1, Oh2, Oh3, We2, We3], to mention just some of the pertinent references.

Recall that a submanifold \( M \) of a symplectic manifold \( (W^{2n}, \omega) \) is said to be coisotropic if for every \( p \in M \) the symplectic orthogonal \( (T_p^o M)^\omega \) to the tangent space \( T_p M \) is contained in \( T_p M \). For instance, Lagrangian submanifolds are coisotropic, as are hypersurfaces in \( W \). Furthermore, \( \dim M \geq n \), when \( M \) is coisotropic. The examples discussed below indicate that coisotropic submanifolds enjoy the same kind of Hamiltonian rigidity as Lagrangian submanifolds and lead to the following

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**Question 1.1** (Coisotropic intersections). Can a coisotropic submanifold be displaced by a Hamiltonian diffeomorphism arbitrarily close to the identity in a suitable sense? For instance, interpreting closeness in the sense of Hofer’s metric (see, e.g., [HZ3, Po3]), we can ask if there is a lower bound, depending only on the submanifold, on the energy of a Hamiltonian diffeomorphism displacing the submanifold.

This question can be further restricted in a number of ways: by imposing additional assumptions on the codimension of $M$ or on the dynamics of the characteristic foliation or via other types of symplectic-topological requirements on $M$ and the ambient manifold, or through requiring the Hamiltonian diffeomorphism to be close to the identity in a particularly strong way. For instance, when $M$ is Lagrangian, the question reduces to the Lagrangian intersection property. In terms of codimension, the other extreme case is that of codimension zero: $M = W$. In this case, the coisotropic intersection property obviously holds. When $M$ is a hypersurface, the answer to the question is affirmative due to non-degeneracy of Hofer’s metric, [LMc1], and the fact that a displaceable connected hypersurface necessarily bounds. The extreme case in terms of closeness of the diffeomorphism to the identity is that of infinitesimal displacement:

**Example 1.2** (Infinitesimal intersections). Let $M$ be a coisotropic submanifold and let $H$ be a Hamiltonian near $M$. On the infinitesimal level, the intersections of $M$ with its image under the Hamiltonian flow of $H$ correspond to the points where the Hamiltonian vector field $X_H$ of $H$ is tangent to $M$. These are precisely the leaf-wise critical points of $H$ along the characteristic foliation $F$ on $M$. This observation suggests that the “amount” of coisotropic intersections is governed by the foliated Morse theory of $F$.

When $M$ is Lagrangian, this observation readily implies persistence of intersections for any Hamiltonian diffeomorphism generated by a $C^2$-small Hamiltonian, by the Weinstein symplectic neighborhood theorem, [We1, We4].

Here we answer Question 1.1 affirmatively only for coisotropic manifolds satisfying a certain additional stability requirement (see Definition 2.1), introduced by Bolle in [Bo2] and similar to stability of hypersurfaces (cf. [EKP, HZ3]). Namely, for such a submanifold we establish a lower bound on the Hofer norm of a displacing Hamiltonian diffeomorphism (see Theorem 2.7) and hence prove that the submanifold has positive displacement energy. The stability condition, discussed in detail in Section 2.1, is quite restrictive: see, in particular, Example 2.2. (We also impose some natural and not-too-restrictive assumptions on the ambient manifold.) Nevertheless, these results combined with the examples given in this section appear to provide enough evidence to conjecture that the coisotropic intersection property holds in general.

Next let us examine the case where the characteristic foliation is a fibration. We do this by passing to the graph of the foliation and interpreting it as a Lagrangian submanifold. This will also lead us to a refinement of Question 1.1.

**Example 1.3** (Leaf-wise coisotropic intersections). Let $M$ be a coisotropic submanifold of $(W,\omega)$ and let $\Gamma \subset M \times M \subset W \times W$ be the graph of the characteristic foliation $F$ on $M$. In other words, $\Gamma$ is formed by pairs $(x, y) \in M \times M$ with $x$ and $y$ lying on the same leaf of $F$. Then $\Gamma$ is a one-to-one immersed Lagrangian submanifold of $W \times W$, where the latter is equipped with the symplectic form $\omega \oplus -\omega$. (In general, $\Gamma$ is not a true submanifold, e.g., $\Gamma$ can be dense in $M \times M$.)
Consider a Hamiltonian diffeomorphism \( \varphi \) of \( W \). Then \( \tilde{\varphi} = (id, \varphi) \) is a Hamiltonian diffeomorphism of \( W \times W \) and the intersection points \( \tilde{\varphi}(\Gamma) \cap \Gamma \) are in one-to-one correspondence with \( x \in M \) such that \( x \) and \( \varphi(x) \in M \) lie on the same leaf. Hence, the leaf-wise intersection property for \( M \), i.e., the existence of a leaf \( F \) of \( M \) with \( \varphi(F) \cap F \neq \emptyset \). For instance, assume that \( F \) is a fibration. Then \( \Gamma \) is a true smooth Lagrangian submanifold of \( W \times W \) and we conclude that in this case leaf-wise intersections do exist, provided that \( \varphi \) is not far from \( id \) in a suitable sense; see, e.g., [Ch2, Fl1, Fl2, Fl3, Fl5, Gr, LS1, Oh1, Oh3, We2].

Taking this example as a motivation let us call \( x \in M \) a leaf-wise intersection of \( M \) and \( \varphi(M) \) if \( x \in F \cap \varphi(F) \) for some leaf \( F \) of the coisotropic foliation. Note that in the infinitesimal setting of Example 1.2, leaf-wise intersections correspond to the critical points of \( H \) on \( M \).

**Question 1.4 (Leaf-wise coisotropic intersections).** Do leaf-wise intersections exist (perhaps, under some additional conditions on the coisotropic submanifold) whenever the Hamiltonian diffeomorphism is sufficiently close to the identity, e.g., in the sense of Hofer’s metric?

When \( M \) is Lagrangian, this question is of course equivalent to Question 1.1, for the characteristic foliation in this case has only one leaf, the entire manifold \( M \). In the other extreme case \( M = W \), where \( W \) is closed, Question 1.4 is equivalent to the Arnold conjecture – the existence of fixed points of Hamiltonian diffeomorphisms – and hence the answer to the question is affirmative in this case. (See, e.g., [Fl1, Fl2, Fl3, Fl5, FO, Gr, HZ3, LT1, McSa, Sa] and references therein.)

Question 1.4, arising also from some problems in Hamiltonian dynamics, was originally raised by Moser in 1978 in [Mo]. In [Ba, Mo], persistence of leaf-wise intersections was proved for closed coisotropic submanifolds and Hamiltonian diffeomorphisms which are \( C^1 \)-close to the identity. Moser’s theorem was later extended by Hofer, [Ho], to hypersurfaces of restricted contact type in \( \mathbb{R}^{2n} \) and Hamiltonian diffeomorphisms with energy smaller than a certain symplectic capacity of the region bounded by the hypersurface. (See also [EH, Li].) Recently, Dragnev, [Dr], generalized this result to arbitrary closed coisotropic submanifolds in \( \mathbb{R}^{2n} \) that have contact type, but need not be of restricted contact type (see Definition 2.1). In this case, the energy of the diffeomorphism is required to be smaller than the homological capacity of the submanifold itself. Note that this capacity is positive for any coisotropic submanifold which is displaceable and stable as is shown below in Remark 6.7.

Here we do not consider the problem of extending Dragnev’s theorem to other ambient manifolds; this question will be addressed elsewhere. Instead, we prove a simple preliminary result in this direction and generalize the results of [EH, Ho] to subcritical Stein manifolds. Namely, we show that leaf-wise intersections of hypersurfaces in such manifolds persist for Hamiltonian diffeomorphisms with energy smaller than the homological capacity of the region bounded by the hypersurface. (See Theorem 2.9.) As stated, this result does not hold when the contact type condition is dropped; see Example 7.2.

Finally note that Moser’s theorem and the results on Lagrangian intersections and the Arnold conjecture discussed above suggest the existence, under suitable additional assumptions, of more than one leaf-wise intersection.
Remark 1.5 (Totally non-coisotropic displacement, [Gü]). A theorem of Polterovich and of Laudenbach and Sikorav, [LS2, Po2], asserts that Hamiltonian persistence of intersections is an exclusive feature of Lagrangian submanifolds among submanifolds of middle dimension. Namely, a middle-dimensional submanifold \( N \) admits an infinitesimal Hamiltonian displacement if and only if \( N \) is not Lagrangian and its normal bundle has a non-vanishing section. This clear-cut dichotomy does not carry over to lower codimensions, but the general picture is somewhat similar. To be more specific, it has been shown by Gürel, [Gü], that a totally non-coisotropic submanifold admits an infinitesimal Hamiltonian displacement, provided that its normal bundle has a non-vanishing section. Now, in contrast with the Lagrangian case, it is not sufficient to assume (under the same normal bundle condition) that \( N \) is simply not coisotropic, for \( N \) may contain a Lagrangian submanifold.

1.2. Coisotropic Liouville class. The Lagrangian intersection property is intimately connected to the fact that a Lagrangian submanifold that is displaceable must have non-zero Liouville class. Moreover, under suitable hypotheses, the displacement energy can be bounded from below via the size of the Liouville class. This connection can also be extended to coisotropic manifolds. Thus assume, for the sake of simplicity, that the ambient symplectic manifold \((W^{2n},\omega)\) is symplectically aspherical and exact: \(\omega = d\lambda\). Then the restriction \(\lambda|_{\mathcal{F}}\) of \(\lambda\) to the characteristic foliation \(\mathcal{F}\) of \(M\) is leaf-wise closed, and the cohomology class \([\lambda|_{\mathcal{F}}]\in H^1_{dR}(\mathcal{F})\) in the foliated de Rham cohomology is defined. (Recall that \(H^*_{dR}(\mathcal{F})\) is the cohomology of the complex of smooth differential forms along the leaves of \(\mathcal{F}\); see, e.g., [MoSc] and references therein for a discussion of foliated de Rham cohomology.) Note that \([\lambda|_{\mathcal{F}}]\) depends on the choice of \(\lambda\). When \(M\) is Lagrangian, this is the ordinary Liouville class of \(M\). In general, we will refer to it as the \textit{coisotropic Liouville class}. By analogy with Lagrangian manifolds, we ask

\textbf{Question 1.6 (Coisotropic Liouville class).} Is \([\lambda|_{\mathcal{F}}]\neq 0\), provided that \(M\) is displaceable and closed?

When \(M\) is Lagrangian, this answer is affirmative by [Ch1, Ch2, Gr, Oh3, Po1]. If \(M\) is a hypersurface, the answer is also affirmative. (Indeed, a displaceable hypersurface bounds a region and then \(\int_M \lambda \wedge \omega^{n-1}\) is equal to the symplectic volume of the region. On the other hand, this integral would be zero if \(\lambda\) were leaf-wise exact.) Furthermore, again by analogy with the Lagrangian case, one can ask whether the displacement energy of \(M\) can be bounded from below via the “size” of \([\lambda|_{\mathcal{F}}]\). A sufficiently good lower bound would imply an affirmative answer to Question 1.1 by a version of the figure-eight trick.

\textbf{Example 1.7.} In the setting of Example 1.3, assume that \(\omega = d\lambda\). Then the manifold \(W \times W\) is exact and the Liouville class \([\lambda_{\Gamma}]\) of the graph \(\Gamma\) is defined, when \(\Gamma\) is interpreted as an immersed manifold. Denote by \(\pi_1\) and \(\pi_2\) the natural projections of \(\Gamma\) to the first and the second factor in \(W \times W\). Then \(\lambda_{\Gamma} = \pi_1^*\lambda - \pi_2^*\lambda\). Furthermore, it is not hard to show that \([\lambda|_{\mathcal{F}}]\neq 0\) if \([\lambda_{\Gamma}]\neq 0\). Assume now that \(\mathcal{F}\) is a fibration and \(M\) is displaceable. Then \(\Gamma\) is a genuine, displaceable Lagrangian submanifold of \(W \times W\). Under natural additional assumptions on \(W\), we infer that \([\lambda_{\Gamma}]\neq 0\) (see, e.g., [Ch1, Ch2, Gr, Oh3, Po1]), and hence \([\lambda|_{\mathcal{F}}]\neq 0\). Note also that, as a consequence, the first cohomology of the fiber of \(\mathcal{F}\) is non-trivial.
We prove that, for a stable coisotropic submanifold, there exists a loop $\eta$ which is tangent to $\mathcal{F}$, contractible in $W$ and bounds non-zero symplectic area (Theorem 2.7(ii)). This theorem requires natural minor assumptions on the behavior of $\omega$ at infinity in $W$, but holds even when $W$ is not exact. As a consequence of the theorem, we obtain an affirmative answer to Question 1.6 for stable submanifolds. However, this approach does not lead to an answer to this question in general, for such a loop $\eta$ need not exist when the stability condition is dropped even if the Liouville class is non-zero. (This follows from counterexamples to the Hamiltonian Seifert conjecture; see, e.g., [Ci, Gi1, Gi2, GG1, Ke1] and references therein.)

Furthermore, if $M$ has restricted contact type, the displacement energy of $M$ is greater than or equal to the symplectic area bounded by $\eta$. (See Definition 2.1 and Theorem 2.7(iii).)

For $W = \mathbb{R}^{2n}$, these results were proved by Bolle, [Bo1, Bo2], using the finite-dimensional reduction methods of [CZ]. Our proof draws heavily on Bolle’s ideas and is in fact just a translation of his argument to the Floer theoretic setting.\footnote{The author is grateful to Claude Viterbo for calling his attention to Bolle’s papers, [Bo1, Bo2], which played a crucial role in this work.}

1.3. Dense existence of non-contractible loops. The stability condition and the proof of Theorem 2.7 suggest that the loop $\eta$, tangent to a leaf of the characteristic foliation but not contractible in it, can be viewed as a generalization of a closed characteristic on a hypersurface; cf. [Bo1, Bo2]. Then assertions (ii) and (iii) of Theorem 2.7 are interpreted as generalizations of the existence of closed characteristics on stable hypersurfaces in $\mathbb{R}^{2n}$, established in [HZ2, HZ3, St]. (See also [Gi3] for further references.)

Continuing this analogy, consider a map $\tilde{\mathcal{K}}: W \to \mathbb{R}^k$ whose components are proper, Poisson–commuting Hamiltonians. Then $M_a = \tilde{\mathcal{K}}^{-1}(a)$ is a coisotropic submanifold in $W$ whenever $a \in \mathbb{R}^k$ is a regular value of $\tilde{\mathcal{K}}$. Assume, for the sake of simplicity, that all levels $M_a$ are displaceable. Then we prove that for a subset $A \subset \mathbb{R}^k$ dense in the set of regular values of $\tilde{\mathcal{K}}$, a level $M_a$, with $a \in A$, carries a loop tangent to the characteristic foliation, contractible in $W$ but not in its leaf, and bounding a positive symplectic area. This result (Theorem 2.11) can be regarded as a generalization of the dense existence theorem, [FH, HZ3]; see also [Gi3]. Here, as above, we need to impose some natural additional conditions on $W$, but the coisotropic submanifolds $M_a$ need not be stable. This theorem leads to the question whether a version of the almost existence theorem, [HZ2, HZ3, St], for such loops holds for commuting Hamiltonians.

1.4. Organization of the paper. The main results of the paper and the necessary definitions are stated and further discussed in detail in Section 2.

The goal of Section 3 is purely technical: here we set conventions and recall relevant results concerning filtered Floer homology, homotopy maps, and action selectors.

In Sections 4 and 5 we establish an auxiliary result on which the proofs of our main theorems hinge. This result, which may be of independent interest (cf. [En, KL, LMc2, McSl, Oh4]), asserts the existence of a Floer connecting trajectory descending from a one-periodic orbit to the maximum of a Hamiltonian and having energy bounded from above by the displacement energy of the support of the Hamiltonian, provided that the maximum is large enough and the Hamiltonian
is “slow” near its maximum. For non-degenerate Hamiltonians this result (Proposition 4.1) is proved in Section 4. In Section 5 we deal with the degenerate case (Proposition 5.1) and also discuss the space of finite energy Floer trajectories.

Section 6 is devoted to the proofs of the main results of the paper. Here we establish Theorem 2.7 giving affirmative answers to Questions 1.1 and 1.6 for stable, coisotropic submanifolds. In this section we also prove a version of the dense existence theorem for commuting Hamiltonians (Theorem 2.11).

Finally, in Section 7 we prove persistence of leaf-wise intersections for hypersurfaces of restricted contact type in subcritical Stein manifolds (Theorem 2.9).

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2. Displacement of stable coisotropic submanifolds

2.1. Contact type and stable coisotropic submanifolds. Let $(\mathbb{W}^{2n}, \omega)$ be a symplectic manifold and let $M \subset \mathbb{W}$ be a closed coisotropic submanifold of codimension $k$. Set $\omega_0 = \omega|_M$. Then, as is well known, the distribution $\ker \omega_0$ has dimension $k$ and is integrable. Denote by $\mathcal{F}$ the characteristic foliation on $M$, i.e., the $k$-dimensional foliation whose leaves are tangent to the distribution $\ker \omega_0$.

Definition 2.1. The coisotropic submanifold $M$ is said to be stable if there exist one-forms $\alpha_1, \ldots, \alpha_k$ on $M$ such that $\ker d\alpha_i \supset \ker \omega_0$ for all $i = 1, \ldots, k$ and

$$\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \omega_{n-k}^0 \neq 0 \quad (2.1)$$

anywhere on $M$. We say that $M$ has contact type if the forms $\alpha_i$ can be taken to be primitives of $\omega_0$. Furthermore, $M$ has restricted contact type if the forms $\alpha_i$ extend to global primitives of $\omega$ on $W$.

Stable and contact type coisotropic submanifolds were introduced by Bolle in [Bo1, Bo2]. The nature of the requirements of Definition 2.1, which are very restrictive, is illustrated by the following examples.

Example 2.2.

(i) A contact type coisotropic submanifold is automatically stable. Furthermore, a coisotropic submanifold which is $C^1$-close to a coisotropic submanifold of contact type also has contact type. (Apparently the latter is not true for stable coisotropic submanifolds.)

(ii) A hypersurface has contact type as a coisotropic submanifold if and only if it has contact type in the standard sense. A hypersurface is stable as a coisotropic submanifold if and only if it is stable as a hypersurface, i.e., there exists a vector field $Z$ transverse to $M$ and such that $\ker (\varphi^*_t \omega|_M) = \ker (\omega|_M)$ for small $|t|$, where $\varphi_t$ is the flow of $Z$; cf. [HZ3, p. 122] and [EKP]. We will generalize this observation to coisotropic submanifolds of codimension $k \geq 1$ in Proposition 2.6.

(iii) The product of stable submanifolds is also stable. More precisely, let $M_1 \subset W_1$ and $M_2 \subset W_2$ be stable. Then $M_1 \times M_2 \subset W_1 \times W_2$ is stable. For instance, the product of contact type hypersurfaces is a stable coisotropic submanifold. The product $M \times S^1 \subset W \times T^*S^1$ has (restricted) contact type, provided that $M$ has (restricted) contact type. However, unless $M_1$
Indeed, $\alpha$ turns means that $a_T$ is independent in $V$ be the vector space formed by linear combinations $a_T$ are constants. The forms $\alpha$ are closed along $\mathcal{F}$ and the natural map from $V$ to the foliated de Rham cohomology $H^1_{dR}(\mathcal{F})$ along $\mathcal{F}$, sending $\alpha$ to its cohomology class, is a monomorphism. (Here, as above, $M$ is closed.) In particular, $\dim H^1_{dR}(\mathcal{F}) \geq k$. Indeed, $\alpha|_x = df|_x$ would imply that $\alpha_x|_{\mathcal{T}x, \gamma} = 0$ at a critical point $x$ of $f$, which in turn means that $a_1 = \cdots = a_k = 0$ since the forms $\alpha_1, \ldots, \alpha_k$ are, by (2.1), linearly independent in $T^*_x \mathcal{F}$, cf. [Bo2, Remark 3].

When $M$ has contact type, consider the vector space $V_0 \subset V$ formed by $\alpha = a_1 \alpha_1 + \cdots + a_k \alpha_k$ with $a_1 + \cdots + a_k = 0$. The forms $\alpha$ are closed on $M$ and the natural map $V_0 \to H^1(M; \mathbb{R})$ is a monomorphism. (As a consequence, $\dim H^1(M; \mathbb{R}) \geq k - 1$.) The proof of this observation due to Bolle, [Bo2], is similar to the argument for stable manifolds above, 2.5.

Let, as above, $M$ be a closed, stable, coisotropic submanifold. Consider the product $M \times \mathbb{R}^k$. Let $(p_1, \ldots, p_k)$ be coordinates on $\mathbb{R}^k$ and let us use the same symbols $\omega_0$ and $\alpha_i$ for differential forms on $M$ and for their pull-backs to $M \times \mathbb{R}^k$. Then the form

$$\omega = \omega_0 + \sum_{i=1}^k d(p_i \alpha_i),$$

is symplectic near $M = M \times \{0\}$ in $M \times \mathbb{R}^k$. The normal bundle to $M$ in $W$ is trivial, for it is isomorphic to $T^* \mathcal{F}$, and thus can be identified with $M \times \mathbb{R}^k$. Then, as is immediately clear (see [Bo1, Bo2]) from the Weinstein symplectic neighborhood theorem (see, e.g., [We4]), the local normal form of $\omega$ near $M$ is given by (2.2). From now on, we identify a neighborhood of $M$ in $W$ with a neighborhood of $M$ in $T^* \mathcal{F} = M \times \mathbb{R}^k$ equipped with the symplectic form (2.2).

**Proposition 2.5.** Let $M$ be a stable coisotropic submanifold.

(i) The leaf-wise metric $(\alpha_1)^2 + \cdots + (\alpha_k)^2$ on $\mathcal{F}$ is leaf-wise flat.

(ii) The Hamiltonian flow of $\rho = (p_1^2 + \cdots + p_k^2)/2$ is the leaf-wise geodesic flow of this metric.

This proposition is essentially proved in [Bo2]. Here we just briefly outline the argument for the sake of completeness.
Proof. To prove the first assertion, note that by (2.1) the forms $\alpha_i$ are linearly independent and leaf-wise closed. Thus, locally on every leaf, their primitives form a coordinate system in which the metric is isometric to the standard metric on $\mathbb{R}^k$. Let $Y_1, \ldots, Y_k$ be the coordinate vector fields for this coordinate system, i.e., $\alpha_i(Y_j) = \delta_{ij}$. To prove the second assertion, we now simply observe from (2.2) that the Hamiltonian vector field $X_\rho$ of $\rho$ is given by

$$X_\rho = \sum_{i=1}^k p_i Y_i, \quad (2.3)$$

which is the geodesic spray of the metric. □

The leaf-wise metric $\rho$ has some other relevant properties that do not hold for leaf-wise flat metrics in general. For instance, the length spectrum of $\rho$ is nowhere dense; see Lemma 6.6.

The next proposition, also quite elementary, clarifies the nature of the stability condition and generalizes Example 2.2(ii).

**Proposition 2.6.** A coisotropic submanifold $M$ is stable if and only if there exists a tubular neighborhood $M \times U$ of $M = M \times \{0\}$ in $W$, where $U \subset \mathbb{R}^k$ is a neighborhood of the origin, such that the submanifolds $M_p = M \times \{p\}$ are coisotropic for $p \in U$ and $\ker \omega_p = \ker \omega_0$, where $\omega_p = \omega|_{M_p}$.

Proof. The fact that stability implies the existence of such a neighborhood is a consequence of the normal form (2.2). Conversely, set $\alpha_i = \left. \left( i_{\partial/\partial p_i} \omega \right) \right|_M$. Then, (2.1) follows immediately from the fact that $\partial/\partial p_1, \ldots, \partial/\partial p_k$ are linearly independent. Furthermore,

$$d\alpha_i = \left. \left( L_{\partial/\partial p_i} \omega \right) \right|_M = \left. \frac{d}{dt} \omega_p(t) \right|_{t=0},$$

where $p(t) = (0, \ldots, 0, t, 0, \ldots, 0)$ with $t$ in the $i$th slot, and hence $\ker d\alpha_i \supset \ker \omega_0$ as required. □

### 2.2. Hamiltonian displacement of coisotropic submanifolds.

To state our results on Hamiltonian displacement of stable coisotropic submanifolds, we need to impose some natural conditions on the ambient symplectic manifold $(W, \omega)$. Namely, in what follows $W$ is always assumed to be symplectically aspherical, i.e., $\omega|_{\pi_2(W)} = 0 = c_1|_{\pi_2(W)}$.

Furthermore, we require $W$ to be closed or geometrically bounded and wide. The condition that $W$ is geometrically bounded means that $W$ admits a complete metric which is compatible with $\omega$ in a rather weak sense and has injectivity radius bounded away from zero and sectional curvature bounded from above; see [AL] for the precise definition. A symplectic manifold is said to be wide if it admits a proper Hamiltonian bounded from below whose Hamiltonian flow has no non-trivial contractible periodic orbits of period less than or equal to one; see [Gü]. Among wide manifolds are manifolds convex at infinity (e.g., cotangent bundles to closed manifolds and $\mathbb{R}^{2n}$), twisted cotangent bundles, and non-compact covering spaces of closed manifolds. In fact, the author is not aware of any example of a geometrically bounded, open manifold that is not wide.

The essence of these requirements is that the standard machinery of Floer homology is applicable to symplectically aspherical, geometrically bounded manifolds (see, e.g., [CGK, GG2] and Section 3 below). Furthermore, one of the main tools
utilized in this paper is the technique of action selectors. This technique, developed for closed and convex at infinity symplectically aspherical manifolds in [FS, Sc2], has recently been extended to geometrically bounded, wide manifolds by Gürel, [Gü].

Recall also that the energy of a compactly supported, time-dependent Hamiltonian $H: [0, 1] \times W \to \mathbb{R}$ is defined as
\[
\| H \| = \int_0^1 (\max H_t - \min H_t) \, dt,
\]
where $H_t = H(t, \cdot)$. The Hamiltonian diffeomorphism $\varphi_H$, i.e., the time-one Hamiltonian flow of $H$, is said to displace $M$ if $\varphi_H(M) \cap M = \emptyset$. When such a map $\varphi_H$ exists, we call $M$ displaceable. For instance, every compact subset of $\mathbb{R}^{2n}$ is displaceable.

Now we are in a position to state the main result of the paper.

**Theorem 2.7.** Let $W$ be symplectically aspherical, and closed or wide and geometrically bounded. Let $M$ be a closed, stable, coisotropic submanifold of $W$.

(i) Then there exists a constant $\Delta > 0$ such that $\| H \| > \Delta$ for any compactly supported Hamiltonian $H$ with $\varphi_H$ displacing $M$.

(ii) When $M$ is displaceable, there exists a loop $\eta$ tangent to $\mathcal{F}$, contractible in $W$, and bounding a positive symplectic area $A(\eta)$.

(iii) Moreover, if $M$ is displaceable and has restricted contact type, there exists a loop $\eta$ tangent to $\mathcal{F}$, contractible in $W$, and such that $0 < A(\eta) \leq \| H \|$.

**Remark 2.8.** Note that the loop $\eta$ from assertions (ii) and (iii) is necessarily not contractible in the leaf containing it. Moreover, $\eta$ is not contractible in the class of loops tangent to $\mathcal{F}$. This follows immediately from the observation that the area spectrum of $M$ has zero measure. (By definition, the area spectrum of $M$ is the set formed by symplectic areas bounded by loops in $M$ that are tangent to $\mathcal{F}$ and contractible in $W$.) The same holds for the curve $\gamma$ from Theorem 2.11 below.

Theorem 2.7 will be proved in Section 6. For stable coisotropic submanifolds in $\mathbb{R}^{2n}$ assertions (ii) and (iii) were established by Bolle, [Bo1, Bo2].

When $M$ is Lagrangian, and hence necessarily a torus, the first assertion is a particular case of the Lagrangian intersection property discussed in Section 1, see, e.g., [Ch1, Ch2, Fl1, Fl2, Fl3, Fl5, Gr, LS1, Oh1, Oh2, Oh3, We2, We3] for similar and more general results. When $M$ is a hypersurface, assertion (i) is trivial. However, in this case, we prove a sharper theorem concerning leaf-wise intersections and complementing the results of [Ba, Dr, EH, Ho, Li, Mo], cf. Example 1.3 and Question 1.4.

**Theorem 2.9.** Let $M$ be a connected closed hypersurface of restricted contact type, bounding a domain $U$ in a subcritical Stein manifold $W$. Then for a compactly supported Hamiltonian diffeomorphism $\varphi_H: W \to W$ with $\| H \| < c_{\mathrm{hom}}(U)$, there exists a leaf $F$ of the characteristic foliation on $M$ such that $\varphi_H(F) \cap F \neq \emptyset$.

We refer the reader to Section 3.3.4 for the definition of the homological capacity $c_{\mathrm{hom}}$. Theorem 2.9 will be established in Section 7. For $W = \mathbb{R}^{2n}$, this result was proved in [Ho].

As stated, with the upper bound on $\| H \|$, Theorem 2.9 does not hold for hypersurfaces that do not have contact type. In Example 7.2, we construct a Hamiltonian flow $\varphi^t$ on $\mathbb{R}^{2n}$ and a sequence of coisotropic submanifolds $M_i$, $C^0$-converging to
$S^{2n-1}$, such that $M_i$ and $\varphi^{t_i}(M_i)$ have no leaf-wise intersections for some sequence of times $t_i \to 0+$.

As is pointed out in Section 1, assertion (ii) of Theorem 2.7 fails when the requirement that $M$ is stable is dropped. Indeed, for hypersurfaces in $\mathbb{R}^{2n}$, (ii) implies the existence of a closed characteristic, while counterexamples to the Hamiltonian Seifert conjecture show that in general closed characteristics need not exist; see, e.g., [Ci, Gi1, Gi2, GG1, Ke1] and references therein. Recall also that the condition that $M$ is displaceable is essential in (ii): the Liouville class of the zero section of a cotangent bundle is zero.

Assertion (ii) gives an affirmative answer to Question 1.6 for stable coisotropic manifolds. Namely, assume that $\omega$ is exact, i.e., $\omega = d\lambda$. Then the restriction $\lambda|_F$ is closed along $F$, and hence the foliated Liouville class $[\lambda|_F] \in H^1_{\text{dR}}(\mathcal{F})$ is defined.

**Corollary 2.10.** Assume in addition to the hypotheses of Theorem 2.7 that $\omega$ is exact and $M$ is displaceable. Then $[\lambda|_F] \neq 0$.

### 2.3. Commuting Hamiltonians.

Let, as in Section 1.3 and Example 2.3, $M$ be a regular level $\vec{K}^{-1}(0)$ of the map $\vec{K} = (K_1, \ldots, K_k): W \to \mathbb{R}^k$ whose components are Poisson–commuting Hamiltonians. (Now, in contrast with Example 2.3, we do not assume that $\vec{K}$ gives rise to a torus action on $M$.) As above, $W$ is required to be symplectically aspherical, and closed or wide and geometrically bounded. In the latter case, we also require the map $\vec{K}$ to be proper (on its image) near $M$ to insure that the coisotropic manifolds $M_a = \vec{K}^{-1}(a)$ are compact and close to $M$ when $a \in \mathbb{R}^k$ is near the origin. Then, as in Section 1.3, consider loops on $M_a$ which are tangent to the characteristic foliation, contractible in $W$, but not contractible in the leaf of the foliation. Such a loop can be thought of as an analogue of a closed characteristic on a hypersurface; see Section 1.3. Hence, the existence of such a loop can be interpreted as a generalization of the dense existence theorem of Hofer and Zehnder and of Struwe, [HZ1, HZ3, St], to the moment map $\vec{K}$. In Section 6 we prove

**Theorem 2.11.** Assume that $M$ is displaceable. Then, for a dense set of regular values $a \in \mathbb{R}^k$ near the origin, the level set $M_a$ carries a closed curve $\gamma$, which is contractible in $W$, tangent to the characteristic foliation $\mathcal{F}_a$ on $M_a$, and bounds positive symplectic area.

### 3. Filtered Floer homology

In this section we recall the definition of filtered Floer homology for geometrically bounded symplectically aspherical manifolds, set conventions and notation used in this definition, and revisit the construction of action selectors for the manifolds in question. Most of the results mentioned here are either well known or established elsewhere or can be proved by adapting standard arguments. For this reason, the proofs are omitted or just very briefly outlined; however, in each case detailed references are provided although not necessarily to the original proofs. We refer the reader to Floer’s papers [Fl1, Fl2, Fl3, Fl4, Fl5], to [BPS, CGK, FH, FHS, Oh2, SZ], or to [HZ3, McSa, Sa] for introductory accounts of the construction of Floer homology in this setting.

#### 3.1. Preliminaries: notation and conventions.

Let $(W^{2n}, \omega)$ be a symplectically aspherical manifold. Denote by $\Delta W$ the space of smooth contractible loops
\begin{align*}
\gamma & : S^1 \to W \text{ and consider a time-dependent Hamiltonian } H : S^1 \times W \to \mathbb{R}, \text{ where } S^1 = \mathbb{R}/\mathbb{Z}. \text{ Setting } H_t = H(t, \cdot) \text{ for } t \in S^1, \text{ we define the action functional } \\
A_H(\gamma) &= A(\gamma) + \int_{S^1} H_t(\gamma(t)) \, dt,
\end{align*}
where \( A(\gamma) \) is the negative symplectic area bounded by \( \gamma \), i.e.,
\begin{equation}
A(\gamma) = -\int_z \omega,
\end{equation}
where \( z : D^2 \to W \) is such that \( z|_{S^1} = \gamma \).

The least action principle asserts that the critical points of \( A_H \) are exactly contractible one-periodic orbits of the time-dependent Hamiltonian flow \( \varphi^t_H \) of \( H \), where the Hamiltonian vector field \( X_H \) is defined by \( i_{X_H} \omega = -dH \). We denote the collection of such orbits by \( \mathcal{P}_H \) and let \( \mathcal{P}_{H(a,b)} \subset \mathcal{P}_H \) stand for the collection of orbits with action in the interval \((a, b)\). The action spectrum \( \mathcal{S}(H) \) of \( H \) is the set of critical values of \( A_H \). In other words, \( \mathcal{S}(H) = \{ A_H(\gamma) \mid \gamma \in \mathcal{P}_H \} \). This is a zero measure set; see, e.g., \([HZ3, Sc2]\).

In what follows we will always assume that \( H \) is compactly supported and set \( \supp H = \bigcup_{t \in S^1} \supp H_t \). In this case, \( \mathcal{S}(H) \) is compact and hence nowhere dense.

Let \( J_t \) be a time-dependent almost complex structure on \( W \). A Floer anti-gradient trajectory \( u \) is a map \( u : \mathbb{R} \times S^1 \to W \) satisfying the equation
\begin{equation}
\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = -\nabla H_t(u), \tag{3.1}
\end{equation}
Here the gradient is taken with respect to the time-dependent Riemannian metric \( \omega(\cdot, J_t(\cdot)) \). This metric gives rise to an \((L^2-)\) Riemannian metric on \( \Lambda W \) and (3.1) can formally be interpreted as the equation \( \partial u/\partial s = -\nabla_{L^2} A_H(u(s, \cdot)) \). In other words, \( u \) is a trajectory of the \( L^2 \)-anti-gradient flow of \( A_H \) on \( \Lambda W \). In what follows, we denote the curve \( u(s, \cdot) \in \Lambda W \) by \( u(s) \).

The energy of \( u \) is defined as
\begin{equation}
E(u) = \int_{-\infty}^{\infty} \left\| \frac{\partial u}{\partial s} \right\|^2_{L^2(S^1)} \, ds = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial t} - J \nabla H(u) \right\|^2 \, dt \, ds. \tag{3.2}
\end{equation}
We say that \( u \) is asymptotic to \( x^\pm \in \mathcal{P}_H \) as \( s \to \pm \infty \) or connecting \( x^- \) and \( x^+ \) if \( \lim_{s \to \pm \infty} u(s) = x^\pm \) in \( \Lambda W \). More generally, \( u \) is said to be partially asymptotic to \( x^\pm \in \mathcal{P}_H \) at \( \pm \infty \) if \( u(s^\pm_k) \to x^\pm \) for some sequences \( s^\pm_k \to \pm \infty \). In this case
\[ A_H(x^-) - A_H(x^+) = E(u). \]
We denote the space of Floer trajectories connecting \( x^- \) and \( x^+ \), with the topology of uniform \( C^\infty \)-convergence on compact sets, by \( \mathcal{M}_H(x^-, x^+, J) \) or simply by \( \mathcal{M}_H(x^-, x^+) \) when the role of \( J \) is not essential, even though \( \mathcal{M}_H(x^-, x^+, J) \) depends on \( J \). This space carries a natural \( \mathbb{R} \)-action \((\tau \cdot u)(t, s) = u(t, s + \tau) \) and we denote by \( \mathcal{M}_H(x^-, x^+, J)/\mathbb{R} \) the quotient \( \mathcal{M}_H(x^-, x^+, J)/\mathbb{R} \).

Recall that \( \gamma \in \mathcal{P}_H \) is said to be non-degenerate if \( d\varphi_H : T_{\gamma(0)}W \to T_{\gamma(0)}W \) does not have one as an eigenvalue. In this case, the so-called Conley-Zehnder index \( \mu_{CZ}(\gamma) \in \mathbb{Z} \) is defined; see, e.g., \([Sa, SZ]\). Here we normalize \( \mu_{CZ} \) so that \( \mu_{CZ}(\gamma) = n \) when \( \gamma \) is a non-degenerate maximum of an autonomous Hamiltonian with a small Hessian. Assume that all periodic orbits with actions in the interval
by applying the argument from [FH, FHS, SZ], including \( x^\pm \), are non-degenerate. Then, for a generic \( J \), suitable transversality conditions are satisfied and \( \mathcal{M}_H(x^-, x^+, J) \) is a smooth manifold of dimension \( \mu_{cz}(x^+) - \mu_{cz}(x^-) \); see, e.g., [FH, SZ] and references therein.

3.2. **Filtered Floer homology and homotopy.** The objective of this section is two-fold. In its first part we briefly outline the construction of filtered Floer homology following closely [FH]; see also [BPS, CGK, Sc2]. (Note that in the case of open geometrically bounded manifolds the necessary compactness property of the moduli spaces of connecting trajectories is guaranteed by Sikorav’s version of the Gromov compactness theorem; see [AL].) In the second part, we discuss properties of monotonicity maps. Here, we depart slightly from the setting of [BPS, CGK, FH], for we need to consider also non-monotone homotopies and to account for possible non-compactness of \( W \). For this reason, some of the proofs, still quite standard, are outlined below.

Throughout the discussion of the filtered Floer homology \( HF^{(a,b)}(W) \), we assume, when \( W \) is open, that all the intervals are in the positive range of actions, i.e., \( a > 0 \) for any interval \((a, b)\). This condition can be relaxed in some instances and replaced by the requirement that \((a, b)\) does not contain zero. (The latter is clearly necessary with the definitions we adopt here, for \( H \) is assumed to be compactly supported and thus \( H \) always has trivial degenerate periodic orbits if \( W \) is open.)

### 3.2.1. Filtered Floer homology: definitions

Let \( H \) be a compactly supported Hamiltonian on \( W \). Assume that all contractible one-periodic orbits of \( H \) are non-degenerate if \( W \) is closed or that all such orbits with positive action are non-degenerate when \( W \) is open. This is a generic condition. Consider an interval \((a, b)\), with \( a > 0 \) when \( W \) is open, such that \( a \) and \( b \) are outside \( S(H) \). Then the collection \( \mathcal{P}^{(a,b)}_H \) is finite. Assume furthermore that \( J \) is regular, i.e., the necessary transversality conditions are satisfied for moduli spaces of Floer trajectories connecting orbits from \( \mathcal{P}^{(a,b)}_H \). This is again a generic property as can be readily seen by applying the argument from [FH, FHS, SZ].

Let \( CF_k^{(a,b)}(H) \) be the vector space over \( \mathbb{Z}_2 \) generated by \( x \in \mathcal{P}^{(a,b)}_H \) with \( \mu_{cz}(x) = k \). Define

\[
\partial : CF_k^{(a,b)}(H) \to CF_{k-1}^{(a,b)}(H)
\]

by

\[
\partial x = \sum_y \#(\mathcal{M}_H(x, y, J)) \cdot y.
\]

Here the summation extends over all \( y \in \mathcal{P}^{(a,b)}_H \) with \( \mu_{cz}(y) = \mu_{cz}(x) - 1 \) and \( \#(\mathcal{M}_H(x, y, J)) \) is the number of points, modulo 2, in \( \mathcal{M}_H(x, y, J) \). (Recall that in this case \( \mathcal{M}_H(x, y, J) \) is a finite set by the compactness theorem.) Then, as is well known, \( \partial^2 = 0 \). The resulting complex \( CF^{(a,b)}(H) \) is the filtered Floer complex for \((a, b)\). Its homology \( HF^{(a,b)}(H) \) is called the filtered Floer homology. This is essentially the standard definition of Floer homology with critical points outside \((a, b)\) being ignored. Then \( HF(H) := HF^{(-\infty, \infty)}(H) \) is the ordinary Floer homology when \( W \) is compact. (As is well-known, \( HF_*(H) = H_{*+n}(W; \mathbb{Z}_2) \).) In general, \( HF^{(a,b)}(H) \) depends on the Hamiltonian \( H \), but not on \( J \); see Section 3.2.2.

Let \( a < b < c \). Assume that all of the above assumptions are satisfied for all three intervals \((a, c)\) and \((a, b)\) and \((b, c)\). Then clearly \( CF^{(a,b)}(H) \) is a subcomplex
of \( \text{CF}^{(a, c)}(H) \), and \( \text{CF}^{(b, c)}(H) \) is naturally isomorphic to the quotient complex \( \text{CF}^{(a, c)}(H)/\text{CF}^{(a, b)}(H) \). As a result, we have the long exact sequence

\[
\ldots \rightarrow \text{HF}^{(a, b)}(H) \rightarrow \text{HF}^{(a, c)}(H) \rightarrow \text{HF}^{(b, c)}(H) \rightarrow \ldots .
\]

(3.3)

We will refer to the first map in this sequence as the inclusion map and to the second one as the quotient map and to the whole sequence as the \((a, b, c)\) exact sequence for \( H \).

In the construction of the action selector for open manifolds given in Section 3.3.1, it will be convenient to work with filtered Floer homology for the interval \((0, b)\) even though 0 is necessarily a critical value of the action functional. This homology is defined as

\[
\text{HF}^{(0, b)}(H) = \lim_{\epsilon \to 0^+} \text{HF}^{(\epsilon, b)}(H),
\]

where the inverse limit is taken with respect to the quotient maps and \( \epsilon \to 0^+ \) in the complement of \( S(H) \). It is clear that this definition is equivalent to the original one when \( W \) is closed and 0 is not in \( S(H) \).

3.2.2. Homotopy. Let us now examine the dependence of \( \text{HF}^{(a, b)}(H) \) on \( H \). Consider a homotopy \( H^s \) of Hamiltonians from \( H^0 \) to \( H^1 \). By definition, this is a family of Hamiltonians parametrized by \( s \in \mathbb{R} \), and such that \( H^s \equiv H^0 \) when \( s \) is large negative and \( H^s \equiv H^1 \) when \( s \) is large positive. Furthermore, let \( J^s \) be a family of \( t \)-dependent almost complex structures such that again \( J^s \equiv J^0 \) when \( s \ll 0 \) and \( J^s \equiv J^1 \) when \( s \gg 0 \). (In what follows, both the family of Hamiltonians \( H^s \) and the pair of families \( (H^s, J^s) \) will be referred to as a homotopy, depending on the context, when no confusion can arise.) For \( x \in P^{(a_0, b_0)} \) and \( y \in P^{(a_1, b_1)} \) denote by \( \mathcal{M}_{H^s}(x, y, J^s) \) the space of solutions of (3.1) with \( H = H^s \) and \( J = J^s \).

Next we need to address the regularity issue. When \( W \) is closed, regularity of a homotopy \( (H^s, J^s) \) is understood in the standard sense, i.e., that the standard transversality requirements are met by the homotopy \( (H^s, J^s) \); see [FH, FHS, SZ]. (This is a generic property, [FH, FHS, SZ].) If \( W \) is open, these conditions are never satisfied, for Hamiltonians are compactly supported. In this case, we say that \( (H^s, J^s) \) is regular as long as the transversality requirements are met along all homotopy trajectories connecting periodic orbits with positive action. (This is also a generic property; the argument of [FH, FHS, SZ] applies to this case.)

When the transversality conditions are satisfied, \( \mathcal{M}_{H^s}(x, y, J^s) \) is a smooth manifold of dimension \( \mu_{cz}(x) - \mu_{cz}(y) \). Moreover, \( \mathcal{M}_{H^s}(x, y, J^s) \) is a finite set when \( \mu_{cz}(x) = \mu_{cz}(y) \). Define the homotopy map

\[
\Psi_{H^0 H^1}: \text{CF}^{(a_0, b_0)}(H^0) \rightarrow \text{CF}^{(a_1, b_1)}(H^1)
\]

by

\[
\Psi_{H^0 H^1}(x) = \sum_y \#(\mathcal{M}_{H^s}(x, y, J^s)) \cdot y.
\]

Here the summation is over all orbits \( y \in P^{(a_1, b_1)} \) with \( \mu_{cz}(y) = \mu_{cz}(x) \) and \( \#(\mathcal{M}_{H^s}(x, y, J^s)) \) is the number of points, modulo 2, in this moduli space.

The map \( \Psi_{H^0 H^1} \) depends on the entire homotopy \( (H^s, J^s) \) and in general is not a map of complexes. However, \( \Psi_{H^0 H^1} \) becomes a homomorphism of complexes when \( (a_0, b_0) = (a_1, b_1) \) and the homotopy is monotone decreasing, i.e., \( \partial H^s \leq 0 \) pointwise. Moreover, the induced map in homology is then independent of the homotopy – as long as the homotopies are decreasing – and commutes with the maps from
the exact sequence (3.3). (The reader is referred to, e.g., [BPS, CGK, FH, Sa, SZ, Sc2, Vi4], for the proofs of these facts for both open and closed manifolds.) There are other instances when the same is true. This is the case, for instance, when the location of the intervals \((a_0, b_0)\) and \((a_1, b_1)\) is compatible with the growth of the Hamiltonians in the homotopy. We now analyze this particular case in more detail, for it is essential for the proof of Proposition 4.1 below.

**Definition 3.1.** A homotopy \(H^s\) is said to be \(C\)-bounded, \(C \in \mathbb{R}\), if

\[
\int_{-\infty}^{\infty} \int_{S^1} \max_{W} \partial_t H_t^s \, dt \, ds \leq C.
\]

(3.5)

It is clear that a \(C\)-bounded homotopy is also \(C'\)-bounded for any \(C' \geq C\). In what follows we will always assume that \(C \geq 0\).

**Example 3.2.** Let us give some examples of \(C\)-bounded homotopies.

(i) A monotone decreasing homotopy is 0-bounded.

(ii) Every homotopy is \(C\)-bounded if \(C\) is large enough.

(iii) Let \(H^s\) be a linear homotopy from \(H^0\) to \(H^1\), i.e.,

\[
H^s = (1 - f(s))H^0 + f(s)H^1,
\]

where \(f: \mathbb{R} \to [0, 1]\) is a monotone increasing compactly supported function equal to zero near \(-\infty\) and equal to one near \(\infty\). Then \(H^s\) is \(C\)-bounded for any \(C \geq \int_{S^1} \max_W (H^1 - H^0) \, dt\).

The following three observations (H0)–(H2) show that the standard properties of homotopy maps in ordinary Floer homology extend to the maps induced by \(C\)-bounded homotopies in filtered Floer homology. The proof of (H0)–(H2) will further clarify the essence of Definition 3.1. Note also that in (H0)–(H2) we will assume that the end points of the intervals in question are outside the action spectra of the Hamiltonians and, as above, \(a > 0\) when \(W\) is open. Furthermore, we will require all Hamiltonians to be non-degenerate (in the positive action range for open manifolds) and the homotopies to be regular. However, this latter requirement is not essential, as we will show in Section 3.2.3, and is not met in Examples 3.3 and 3.4.

(H0) Let \(H^s\) be a \(C\)-bounded homotopy. Then

\[
\Psi_{H^0, H^1}^*: \text{CF}^{(a, b)}(H^0) \to \text{CF}^{(a+C, b+C)}(H^1)
\]

is a homomorphism of complexes for any interval \((a, b)\). Hence, \(\Psi_{H^0, H^1}^*\) induces a map in Floer homology, which we, abusing notation, also denote by \(\Psi_{H^0, H^1}^*\). This map sends the \((a, b, c)\) exact sequence for \(H^0\) to the \((a + C, b + C, c + C)\) exact sequence for \(H^1\), i.e., on the level of homology \(\Psi_{H^0, H^1}^*\) commutes with all maps in the long exact sequence (3.3).

Here, and in (H1) and (H2) below, we suppress the dependence of this map on the homotopy \(J^s\) of complex structures. In fact, as we will see in Section 3.2.3, the induced map on the level of homology is independent of \(J^s\).

**Example 3.3.** For any \(C \geq 0\) the identity homotopy from \(H\) to \(H\) induces a map which is the composition

\[
\text{HF}^{(a, b)}(H) \to \text{HF}^{(a, b+C)}(H) \to \text{HF}^{(a+C, b+C)}(H)
\]
of the inclusion and quotient maps from the exact sequences (3.3) for the intervals \( a < b < b + C \) and \( a < a + C < b + C \), respectively. We will refer to this map as the inclusion-quotient map.

In view of Example 3.2(ii), note also that \( \text{CF}^{(a+C,b+C)}(H) = 0 \) whenever \( a+C > \max S(H) \), and hence the assertion \( (H0) \) becomes trivial if, for a fixed homotopy, \( C \) is taken to be large enough. For instance, the inclusion-quotient map is automatically zero if \( a + C > b \).

\( (H1) \) For a homotopy of \( C \)-bounded homotopies \( H^{a,\lambda} \), i.e., a family of \( C \)-bounded homotopies parametrized by \( \lambda \in [0,1] \), the induced map

\[
\Psi_{H^0,H^1}: \text{HF}((a,b),(H^0)) \rightarrow \text{HF}((a+C,b+C),(H^1))
\]

is independent of the choice of homotopy in the family \( H^{a,\lambda} \).

Consider now a \( C \)-bounded homotopy \( H^s \) from \( H^0 \) to \( H^1 \) and a \( C' \)-bounded homotopy \( G^s \) from \( G^0 = H^1 \) to \( G^1 \). Define the gluing or composition of these homotopies \( H^s \#_RG^s \), where \( R > 0 \) is large enough, as \( H^s \#_RG^s = H^s + R \) if \( s \leq 0 \) and \( H^s \#_RG^s = G^{s-R} \) if \( s \geq 0 \). (In other words, in \( H^s \#_RG^s \) we first, for \( s \leq 0 \), perform the homotopy obtained from \( H^s \) by shifting it sufficiently far to the left in \( s \) and then, for \( s \geq 0 \), the homotopy resulting from \( G^s \) by shifting it to the right.) It is clear that \( H^s \#_RG^s \) is \((C+C')\)-bounded.

\( (H2) \) When \( R > 0 \) is large enough, the map

\[
\Psi_{H^0,G^1}: \text{HF}((a,b),(H^0)) \rightarrow \text{HF}((a+C',b+C+C'),(G^1))
\]

induced by the homotopy \( H^s \#_RG^s \) is equal to the composition \( \Psi_{G^0,G^1} \circ \Psi_{H^0,H^1} \).

\textbf{Example 3.4.} Let \( H^s \) be an increasing linear homotopy from \( H^0 \) to \( H^1 \geq H^0 \) and let \( G^s \) be an arbitrary decreasing homotopy from \( G^0 = H^1 \) to \( G^1 = H^0 \). By Example 3.2, \( H^s \) is \( C \)-bounded with \( C = \int_{S^1} \max_W (H^1 - H^0) \, dt \) and \( G^s \) is \( 0 \)-bounded. Hence, the composition of the homotopies is also \( C \)-bounded and homotopic to the identity homotopy in the class of \( C \)-bounded homotopies. By \( (H2) \) and \( (H3) \), this composition induces the same map as the identity homotopy, i.e., the inclusion-quotient map from Example 3.3.

\textbf{Outline of the proof of \( (H0)-(H2) \).} Consider first a solution \( u \) of the homotopy Floer equation (3.1) for a homotopy \( H^s \), which is (partially) asymptotic to \( y \) at \( \infty \) and \( x \) at \( -\infty \). Then, as a direct calculation shows (see, e.g., [Sc2]), we have

\[
A_{H^s}(u(s^+)) - A_{H^s}(u(s^-)) \leq \int_{s^-}^{s^+} \int_{S^1} (\partial_u H^s_t)(u(s)) \, dt \, ds - \int_{s^-}^{s^+} \int_{S^1} ||\partial_u u||^2 \, dt \, ds.
\]

Hence,

\[
A_{H^1}(y) - A_{H^0}(x) \leq \int_{-\infty}^{\infty} \int_{S^1} \max_W \partial_u H^s_t \, dt \, ds - E(u),
\]

where \( E(u) \) is still defined by (3.2). In particular, when \( H^s \) is \( C \)-bounded, we infer that

\[
A_{H^1}(y) - A_{H^0}(x) \leq C,
\]

by Definition 3.1, and

\[
E(u) \leq C + (A_{H^0}(x) - A_{H^1}(y)).
\]
by our assumptions, all periodic orbits of $H^0$ and $H^1$ are non-degenerate and the homotopy is regular, and (H0) follows by the standard gluing and compactness argument; see, e.g., [Fl15, FH, SZ, Sa, Sc1]. The only additional point to check is that gluing and compactification of the moduli spaces in question do not involve periodic orbits outside the range of actions. Assume, for instance, that a homotopy connecting trajectory $u$ from $x \in \mathcal{P}_{H^0}^{(a, b)}$ to $z \in \mathcal{P}_{H^1}^{(a+C, b+C)}$ with $\mu_{cz}(z) = \mu_{cz}(x)+1$ is obtained by gluing an $H^0$-downward trajectory from $x$ to $x' \in \mathcal{P}_{H^0}^{(a, b)}$ with a homotopy trajectory from $x'$ to $z$. Then, another component of the boundary of $\mathcal{M}_{H^1}(x, z, J^s)$ is a broken trajectory connecting first $x$ to some $y \in \mathcal{P}_{H^1}$ by a homotopy trajectory and then $y$ to $z$ by an $H^1$-downward trajectory. We need to check that $y \in \mathcal{P}_{H^1}^{(a+C, b+C)}$. It is clear that $A_{H^1}(y) \geq A_{H^1}(z) > a+C$. Furthermore, by (3.7), we have $A_{H^1}(y) \leq A_{H^1}(x') + C < b+C$. The rest of the proof proceeds in a similar fashion.

When $W$ is open, extra care is needed because the Hamiltonians are compactly supported and hence there are degenerate critical points. However, these points do not enter the calculation. Indeed, let, for instance, $x$ and $z$ be as above. Note that the energy of trajectories from $\mathcal{M} := \mathcal{M}_{H^1}(x, z, J^s)$ is uniformly bounded due to (3.8). By the compactness theorem, the closure $\bar{\mathcal{M}}$ of $\mathcal{M}$ is compact in $C^\infty(\mathbb{R} \times S^1, W)$ equipped with the topology of uniform $C^\infty$-convergence on compact sets. Then a trajectory $v \in \bar{\mathcal{M}}$ also has bounded energy and thus is partially asymptotic to some orbits $x' \in \mathcal{P}_{H^0}$ and $y' \in \mathcal{P}_{H^1}$. (This can be verified by adapting, for instance, the argument from the proof of Proposition 10 on page 235 of [HZ3].) Moreover, $A_{H^0}(x') \leq A_{H^0}(x)$ and $A_{H^1}(y') \geq A_{H^1}(z)$; cf. [Sc1, p. 66].

To prove, for example, the second of these inequalities, pick $k \to \infty$ such that $v(s_k) \to y'$ and all $s_k$ are large enough so that $H^s(s_k) = H^1$. Then $A_{H^1}(u(s_k)) \geq A_{H^1}(z)$ for all $u \in \mathcal{M}$. For every $k$ we have a sequence $u_n \in \mathcal{M}$ with $u_n(s_k) \to v(s_k)$. Hence, $A_{H^1}(v(s_k)) \geq A_{H^1}(z)$ and $A_{H^1}(y') = \lim A_{H^1}(v(s_k)) \geq A_{H^1}(z)$.) It follows immediately that $y'$ is a non-degenerate orbit, since $A_{H^1}(y') \geq A_{H^1}(z) > a+C > 0$. Moreover, $A_{H^0}(x') > 0$, for otherwise we would have $a+C < A_{H^1}(y') \leq C$ by (3.7). Thus $x'$ is also a non-degenerate orbit with positive action. Since $H^s$ is regular, the standard description of $\bar{\mathcal{M}}$ via broken trajectories applies, and the proof is finished as for compact manifolds.

Properties (H1) and (H2) are established by a similar reasoning invoking the standard gluing and compactification argument, checking that the orbits are in the required ranges of action, and, when $W$ is open, verifying as above that the compactifications of the relevant moduli spaces do not involve orbits with non-positive action.

\begin{remark}
\textit{Remark 3.5.} Assertions (H0)--(H2) still hold when the interval $(a+C, b+C)$ is replaced by any interval $(a^1, b^1)$ with end points outside $S(H^1)$ and such that $a^1 \geq a+C$ and $b^1 \geq b+C$. This generalization, however, adds little new information, for then the homotopy map $HF^{(a,b)}(H^0) \to HF^{(a',b')}(H^1)$ is the composition of the homotopy $HF^{(a,b)}(H^0) \to HF^{(a+C,b+C)}(H^1)$ with the inclusion-quotient map $HF^{(a+C,b+C)}(H^0) \to HF^{(a',b')}(H^1)$.
\end{remark}

\subsection{Invariance of filtered Floer homology}
Properties (H0)--(H2) of the homotopy maps have a number of standard consequences (see, e.g., [BPS, CGK, FH, Sa, SZ,
are \[ \theta \] the case: the isomorphism is not induced by the homotopy of the Hamiltonians. Moreover, this map is actually independent of the homotopy of the Hamiltonians as long as the latter is \( C \)-bounded. Indeed, any two \( C \)-bounded homotopies can be connected by a linear family of \( C \)-bounded homotopies. Furthermore, since regular homotopies are dense, a \( C \)-bounded homotopy can be approximated by a regular \( C' \)-bounded homotopy with \( C' \) arbitrarily close \( C \). Then, the assumption that the end-points of the intervals are outside the action spectra guarantees that the approximating homotopy still induces a map for the action ranges \( (a, b) \) and \( (a + C, b + C) \), provided that \( C' \) is close enough to \( C \). Thus \( \Psi_{H^0 H^1} \) is in fact a canonical map depending only on the Hamiltonians \( H^0 \) and \( H^1 \), the interval \( (a, b) \) with \( a > 0 \) when \( W \) is open, and the constant \( C \geq 0 \). This justifies our neglect of the homotopy regularity condition in Examples 3.3 and 3.4. Note, however, that this canonical map is defined only when a \( C \)-bounded homotopy between \( H^0 \) and \( H^1 \) exists which is not always the case. (For instance, such a homotopy fails to exist with \( C = 0 \) whenever \( H^1 > H^0 \) point-wise.)

The next result, also quite standard, is the continuity property for filtered homology; see [BPS, CGK, FH, Sc1, Vi4].

(H3) Let \( (a^s, b^s) \) be a family (smooth in \( s \)) of non-empty intervals such that \( a^s \) and \( b^s \) are outside \( S(H^s) \) for some homotopy \( H^s \) and such that \( (a^s, b^s) \) is independent of \( s \) when \( s \) is near \( \pm \infty \) and, abusing notation, equal to \( (a^0, b^0) \) and \( (a^1, b^1) \), respectively. Then there exists an isomorphism of homology

\[
\text{HF}^{(a^0, b^0)}(H^0) \xrightarrow{\cong} \text{HF}^{(a^1, b^1)}(H^1). \tag{3.9}
\]

We will outline the proof of (H3) below, after the next assertion (H4) is stated. When the interval is fixed and the homotopy is monotone decreasing, the isomorphism (3.9) is in fact \( \Psi_{H^0 H^1} \). We emphasize however that in general this is not the case: the isomorphism is not induced by the homotopy \( H^s \), for the homotopy need not map \( \text{CF}^{(a^0, b^0)}(H^0) \) to \( \text{CF}^{(a^1, b^1)}(H^1) \) even for a fixed interval of actions. Leaving aside the general question in what sense this isomorphism is canonical and natural with respect to the long exact sequence (3.3), we focus on the particular case of a fixed interval.

(H4) Assume that \( a < b < c \) are outside \( S(H^s) \) for all \( s \). Then the isomorphisms (3.9) from (H3) for all three intervals can be chosen to map the \( (a, b, c) \) exact sequence (3.3) for \( H^0 \) to the \( (a, b, c) \) exact sequence for \( H^1 \).

Outline of the proof of (H3) and (H4). Let us recall the construction of the isomorphism (3.9) for a fixed interval, say, \( (a, b) \). (Here we follow the argument from [BPS] essentially word-for-word.) First, we break up \( H^s \), up to a homotopy of homotopies, into a composition of a finite sequence of homotopies \( K_i^s \) each of which is “small”. By this we mean that for all \( i \) both \( K_i^s \) and the inverse homotopy \( K_i^{-s} \) are \( \epsilon \)-bounded, where \( \epsilon > 0 \) is so small that the intervals \( (a, a + 2\epsilon) \) and \( (b, b + 2\epsilon) \) do not meet \( S(K_i^s) \) for all \( s \). Now it is sufficient to prove (H3) and (H4) for one homotopy \( K^s = K_i^s \). In this case, the isomorphism (3.9) is defined as

\[
\text{HF}^{(a, b)}(K^0) \to \text{HF}^{(a+\epsilon, b+\epsilon)}(K^1) \xrightarrow{\cong} \text{HF}^{(a, b)}(K^1), \tag{3.10}
\]
where the second arrow is the inverse of the quotient-inclusion map. (The latter map is obviously an isomorphism due to the requirement imposed on $\epsilon$.) To see

that (3.10) is an isomorphism, observe that its inverse is given by

$$HF(a, b)(K^1) \xrightarrow{\cong} HF(a + \epsilon, b + \epsilon)(K^1) \rightarrow HF(a + 2\epsilon, b + 2\epsilon)(K^0) \xrightarrow{\cong} HF(a, b)(K^0),$$

where the middle arrow is the map induced by the inverse homotopy $K^{-s}$. This

concludes the proof of (H3) for a fixed interval. The general case is established in a similar fashion. Finally, from the second part of (H0), it is clear that the maps

(3.10) defined for all three intervals send the $(a, b, c)$ exact sequence for $K^0$ to that for $K^1$. This proves (H4). □

Another standard consequence of (H0)–(H2), along the lines of (H3) and (H4), is

that the filtered Floer homology can be defined “by continuity” for any Hamiltonian $H$, not necessarily non-degenerate, and any $a < b$ outside $S(H)$. (See, e.g., [Vi4].) Moreover, the long exact sequence and (H0)–(H4) also hold in this case. Here, as above, we are assuming that $a > 0$ when $W$ is open.

3.3. The action selector. In this section we recall the definition and the properties of the action selector that are used in the proof of Proposition 4.1. The constructions differ somewhat depending on whether the manifold is open and wide or closed. We start by dealing with the case of wide manifolds, closely following [Gü], for this case appears to be more relevant to the displacement questions and, from the author’s perspective, more transparent. Then, we very briefly review the construction in the case where $W$ is closed. Here we follow [Sc2] with some minor alterations. Note also that for $\mathbb{R}^{2n}$ and cotangent bundles actions selectors were constructed in [HZ3] and [Vi2], respectively, and the approach of [Sc2] has been extended to manifolds convex at infinity in [FS].

3.3.1. The action selector for wide manifolds. Assuming that $W^{2n}$ is geometrically bounded and wide (see Section 2.2 or [Gü]), let us recall from [Gü] the definition of the action selector $\sigma(K)$ for a compactly supported, non-negative Hamiltonian $K: S^1 \times W \to \mathbb{R}$. It is easy to see that since $W$ is wide, there exists a smooth compactly supported function $F: W \to [0, \infty)$ without non-trivial contractible periodic orbits with period $T \leq 1$ and such that $F \geq K$ point-wise. Without loss of generality, we may assume that $\text{supp } F$ is a smooth connected manifold with boundary and that the restriction of $F$ to the interior of its support is a Morse function with finitely many critical points.

Under these assumptions, we have

$$HF_{*}^{0, \infty}(F) \cong HM_{*+n}^{0, \infty}(F) \cong H_{*+n}(\supp F, \partial \supp F; \mathbb{Z}_2)$$

and, in particular,

$$HF_{*}^{0, \infty}(F) \cong \mathbb{Z}_2.$$ (Recall that the filtered Floer homology for $(0, \infty)$ is defined by (3.4).) Denote the generator of this group – the fundamental class – by $\left[\text{max } F\right]$. A monotone decreasing homotopy from $F$ to $K$ induces a map

$$\Psi_{F,K}: HF^{(0, \infty)}(F) \rightarrow HF^{(0, \infty)}(K),$$

independent of the choice of homotopy. (This follows from the results of the previous section and (3.4).) Set

$$[\text{max } K] = \Psi_{F,K}(\left[\text{max } F\right]) \in HF_{*}^{0, \infty}(K).$$
It is not hard to show that $[\max K]$ is well defined, i.e., independent of the choice of $F$. Then, for $K \geq 0$, we set
\[ \sigma(K) = \inf\{ a > 0 \mid j_K^a([\max K]) = 0 \} \in S(K), \]
where
\[ j_K^a : HF^{(0, \infty)}(K) \to HF^{(a, \infty)}(K) \]
is the quotient map.

Finally note that this definition makes sense even when $K$ is not assumed to be non-negative. However, it is not clear whether the resulting action selector has the required properties (S0)–(S4) stated below. If $W$ is convex at infinity, the actions selector described above coincides, for non-negative Hamiltonians, with the one of [FS].

3.3.2. The action selector for closed manifolds. Assume that $W^{2n}$ is closed. Then for any Hamiltonian $K$, we have the canonical identification
\[ HF_*(K) \cong H_{*+n}(W; \mathbb{Z}_2). \]
Denoting by $[\max K]$ the image of the fundamental class under this isomorphism, set, following [Sc2],
\[ \sigma(K) = \inf\{ a \in \mathbb{R} \mid j_K^a([\max K]) = 0 \} \in S(K), \]
where
\[ j_K^a : HF(K) \to HF^{(a, \infty)}(K) \]
is the quotient map. Observe that when $K \geq 0$ we can assume that $a \geq 0$ by (S1) below, as in the definition of the action selector for open manifolds.

Note that here, in contrast with [Sc2], we consider non-normalized Hamiltonians, i.e., we are not assuming that $\int_W K_t \omega^n = 0$. As a consequence, neither $S(K)$ nor $\sigma(K)$ is uniquely determined by $\phi_K$.

Alternatively, one can define the class $[\max K]$ as follows. Let $F$ be a Morse function which is $C^2$-close to a constant. Then, canonically,
\[ HF_*(F) \cong HM_{*+n}(F) \cong H_{*+n}(W; \mathbb{Z}_2). \]
Let $\max F \in CM_{2n}(F)$ be the “fundamental class”, i.e., the sum of all local maxima. Then $[\max F]$ is the generator of $HF_*(F) \cong \mathbb{Z}_2$. Now, for a fixed $K$, we take $F \geq K$ and define $[\max K]$ to be the image of $[\max F]$ under the monotone decreasing homotopy map
\[ \Psi_{F,K} : HF(F) \to HF(K). \]

3.3.3. Properties of the action selector. The action selector $\sigma$, defined as above, has the following properties for non-negative Hamiltonians, regardless of whether $W$ is wide or closed:

(S0) $\sigma$ is monotone, i.e., $\sigma(K) \leq \sigma(H)$, whenever $0 \leq K \leq H$ point-wise;
(S1) $0 \leq \sigma(K) \leq E^+(K)$ for any $K \geq 0$, where
\[ E^+(K) = \int_{S^1} \max K_t \ dt; \]
(S2) $\sigma(K) > 0$, provided that $K \geq 0$ is not identically zero;
(S3) $\sigma(K)$ is continuous in $K$ in the $C^0$-topology;
(S4) $\sigma(K) \leq E^+(H)$, whenever $\phi_H$ displaces supp $K$ and $H \geq 0$. 


We refer the reader to [Gü] for the proofs of (S1)–(S3) when \( W \) is wide and to [Sc2] when \( W \) is closed. As stated, (S4) is established in [FGS] for closed manifolds and in [Gü] for wide manifolds in the stronger form \( \sigma(K) \leq \sigma(H) \), following the earlier versions of this upper bound from [FS, Gi3, HZ3, Oh3, Sc2, Vi2].

Note also that if all contractible one-periodic orbits of \( K \) are non-degenerate, setting \( A_K(c) = \sup \lambda_i A_{K_i}(c_i) \) for \( c = \sum c_i \in CF^{[0,\infty)}(K) \), we have

\[
\sigma(K) = \inf_{|c| = \max K} A_K(c).
\]

As a consequence, there exists a cycle \( c \) (possibly non-unique) with \( A_K(c) = \sigma(K) \). We call such a cycle a carrier of the action selector.

### 3.3.4. Homological capacity

Let \( U \neq W \) be an open set in \( W \), where \( W \) is either closed or geometrically bounded and wide. Recall that the homological capacity \( c_{\text{hom}}(U) \in [0,\infty] \) of \( U \) is defined as

\[
c_{\text{hom}}(U) = \sup_{\text{supp } K \subset U} \sigma(K).
\]

For a compact subset \( Z \) of \( W \) define the displacement energy \( e(Z) \) as \( \inf \|H\| \), where the infimum is taken over all compactly supported Hamiltonians \( H \) with \( \varphi_H(Z) \cap Z = \emptyset \), and set \( e(U) = \sup_{Z \subset U} e(Z) \). Then \( c_{\text{hom}}(U) \leq e(U) \). This well-known fact (see, e.g., [FS, Gi, HZ3, Sc2, Vi2]) follows, for instance, from (S4).

For a closed set \( M \subset U \) (e.g., a closed submanifold), we set

\[
c_{\text{hom}}(M) = \inf_{M \subset U} c_{\text{hom}}(U).
\]

Note that \( c_{\text{hom}}(M) = 0 \) whenever \( M \) is infinitesimally displaceable. On the other hand, drawing on the results of [Vi4], one can expect that a closed Lagrangian submanifold necessarily has positive homological capacity. In Remark 6.7, we show that this is also true for stable coisotropic submanifolds. It seems to be unknown whether the homological capacity of every closed hypersurface in \( \mathbb{R}^{2n} \) is positive.

### 4. Displacement and connecting trajectories

Let, as above, \( W \) be symplectically aspherical and either closed or geometrically bounded and wide. The key to the proof of Theorem 2.7 is the following result.

**Proposition 4.1.** Let \( U \) be an open subset of \( W \) displaced by a Hamiltonian diffeomorphism \( \varphi_H \). Furthermore, let \( K \geq 0 \) be a Hamiltonian supported in \( U \) and such that \( \max K \) is large enough (e.g., \( \max K > \|H\| \) and \( \max K > \max S(H) \), provided that \( H \geq 0 \)). Assume furthermore that

- all one-periodic orbits from \( P^{[0,\infty)}_K \) are non-degenerate;
- for all \( t \in [0,1] \) the functions \( K_t \) attain their maximum at the same point \( p \in U \), which is thus a non-degenerate maximum of \( K_t \) for all \( t \);
- \( K \) is autonomous (i.e., independent of \( t \)) near \( p \);
- the eigenvalues of the Hessian \( d^2 K_t \) of \( K_t \) at \( p \) are close to zero, and thus \( K \) has no one-periodic orbits other than \( p \) in a neighborhood of \( p \).

Then the flow of \( K \) has a contractible one-periodic orbit \( \gamma \) in \( U \) which is connected to \( p \) by a Floer anti-gradient trajectory and such that

\[
0 < A_K(\gamma) - \max K \leq \|H\|.
\]
Remark 4.2. Here we assume that $W$ is equipped with an almost complex structure, possibly time-dependent, which is compatible with $\omega$ and such that the regularity requirements are satisfied. Hence the Floer complex of $K$ is defined.

Remark 4.3. As stated, with the upper bound in (4.1), the proposition is apparently new. (This upper bound is absolutely crucial for the proof of Theorem 2.7 and without it the proposition is quite straightforward.) However, it should be noted that in the context of Hofer’s geometry a number of somewhat similar results have been established, often under much less restrictive conditions on the ambient manifold; see [En, KL, LMc2, McSl, Oh4]. One can expect that along the lines of some of these results the difference $A_K(\gamma) - \max K$ can hypothetically be bounded from above in terms of the Hofer norm of the time-one flow $\varphi_K$.

Outline of the proof. Without loss of generality, we may assume that $E^+(H) = ||H||$. Since $\sigma(K) \leq ||H|| < K(p) = A_K(p)$, the maximum $p$ is not homologically essential, i.e., there exists a carrier $y$ of the action selector, not containing $p$, and the “maximum cycle” $z \in CF^0_n(K)$ containing $p$ such that $z - y = \partial x$ for a chain $x \in CF_{n+1}^0(K)$. As a consequence, there is a Floer connecting trajectory $u$ from some orbit $\gamma$ in $x$ to $p$. We can choose $x$ so that the class $[x']$ induced by $x$ in $HF_{n+1}^{0,\infty}(K)$ is non-zero as long as $K(p) < c < A_K(\gamma)$ and hence for $K(p) < a < b < A_K(\gamma)$ the quotient map $HF^{0,\infty}(K) \to HF^{b,\infty}(K)$ sending $[x']$ to $[x']$ is also non-zero.

Note that, since $\varphi_H$ displaces $\text{supp} K$, we have $HF^{(b,\infty)}(K \# H) = HF^{(b,\infty)}(H) = 0$ once $b > \max S(H)$. Finally, arguing by contradiction, assume that the gap between $K(p)$ and $A_K(\gamma)$ is greater than $E^+(H)$, i.e., $E(u) = A_K(\gamma) - K(p) > E^+(H)$. Now we utilize the fact that the homology of $K$ cannot be completely destroyed by a relatively small perturbation $H$. This implies that $HF^{(b,\infty)}(K \# H) \neq 0$ as long as $[b - E^+(H), b] \subset (\max K, A_K(\gamma))$. Hence, when $\max K$ is large enough, the gap cannot exceed $E^+(H)$ and $E(u) = A_K(\gamma) - K(p) > E^+(H)$.

Proof of Proposition 4.1. First note that without loss of generality we may assume that $H$ meets the following requirements:

- $H$ is periodic in time and $H_0 = H_1 \equiv 0$;
- $H$ is compactly supported and $\min_W H_t = 0$ for all $t \in S^1$.

The first requirement can be satisfied by reparametrizing $H$, i.e., replacing $H$ by a new Hamiltonian of the form $\lambda(t)H(\lambda(t), x)$ where $\lambda: [0, 1] \to [0, 1]$ is a monotone increasing function identically equal to zero for $t$ near zero and to one for $t$ near one. To have the second requirement met, we replace $H$ by a Hamiltonian of the form $f_t \cdot (H_t - \min_W H_t)$, where $f_t$ is a non-negative cut-off function equal to one on $\varphi_H^t(U)$. Note that neither of these alterations changes $||H||$.

Both of the requirements are purely technical rather than essential. The first one is needed to ensure that the composition $K \# H$ is one-periodic in time. The second one enables us to treat wide and closed manifolds in the same way, with only superficial discrepancies. In what follows we will assume that $H$ meets these conditions. Note that since $\min_W H_t = 0$, we have $H \geq 0$ and $||H|| = E^+(H)$.

The proof of the theorem proceeds slightly differently depending on whether $W$ is wide or closed. Below, we first consider the case of wide manifolds and then indicate modifications needed when $W$ is closed.
Wide manifolds. Let $F \geq K$ be an autonomous Hamiltonian as in the definition of the action selector. Without loss of generality, we may assume that $F$ has a unique local maximum which is also located at $p$ and that $F(p) = K(p)$. Then under a monotone homotopy from $F$ to $K$ the Floer cycle $p = \max F \in \text{CF}_{n}^{(0, \infty)}(F)$ is mapped to a Floer cycle $z = z_1 + \cdots + z_m$ of $K$ with $z_1 = p$. We emphasize that the point $p$ does occur in $z$. The reason is that the trivial Floer connecting trajectory from $p$ for $F$ to $p$ for $K$ is the only connecting trajectory between these two trivial orbits by the standard energy argument. (Here we use the assumption that $F(p) = K(p)$.) Furthermore, by utilizing the condition that the eigenvalues of $d^2 K_t$ are close to zero, it is easy to arrange the homotopy so that this connecting trajectory is non-degenerate, cf. [FHS].

Note also that since the homotopy is decreasing, we have

$$F(p) = K(p) = A_K(p) \geq A_K(z_i) \text{ for all } i = 2, \ldots, m,$$

and hence

$$A_K(z) = K(p) = \max K.$$

Clearly, $[z] = [\max K]$, but $z$ cannot be the carrier of the action selector for $K$, for

$$A_K(p) = \max K > E^+(H) \geq \sigma(K),$$

where the last inequality follows from (S4). (In other words, $p$ is not homologically essential.) Let $y \in \text{CF}_{n}^{(0, \infty)}(K)$ be a carrier of the action selector $\sigma(K)$. Then $[y] = [\max K] = [z]$ and $z - y = \partial x$ for some $x \in \text{CF}_{n+1}^{(0, \infty)}(K)$. Furthermore, it is clear from this chain of inequalities that $p$ does not enter the cycle $y$ and, as a consequence, $x$ contains a periodic orbit $\gamma$ connected with $p$ by a Floer anti-gradient trajectory. Then

$$0 < A_K(\gamma) = \max K.$$

However, to ensure that (4.1) holds in its entirety, we need to impose an additional requirement on the chain $x$ and the orbit $\gamma$ which are in general not unique.

To this end, for a given $x$ denote by $x_1$ an orbit, occurring in $x$, which is connected to $p$ and has the smallest possible action $A_K(x_1)$ among all such orbits in $x$. Then we chose $x$ with $A_K(x_1)$ attaining the smallest value for all $x$ with $z - y = \partial x$. We will show that for any $x$ and $\gamma = x_1$, meeting this action minimization condition,

$$A_K(x_1) \leq \max K + E^+(H),$$

(4.2)

which implies (4.1).

First let us consider the standard quotient map

$$j: \text{HF}_{n+1}^{(a, \infty)}(K) \to \text{HF}_{n+1}^{(b, \infty)}(K).$$

We claim that, provided that $x$ and $x_1$ are action minimizing as above,

$$j \neq 0, \text{ when max } K < a < b < A_K(x_1).$$

(4.3)

(In particular, both of the groups are non-zero.) To see this, denote by $x^a$ and $x^b$ the images of the chain $x$ in $\text{CF}_{n+1}^{(a, \infty)}(K)$ and $\text{CF}_{n+1}^{(b, \infty)}(K)$, respectively. By definition, $j(x^a) = x^b$ on the level of complexes. Furthermore, both $x^a$ and $x^b$ are closed, since $\partial x = z - y \in \text{CF}_{n}^{(0, a)}(K)$. Hence, it suffices to show that $[x^b] \neq 0$ in $\text{HF}_{n+1}^{(b, \infty)}(K)$.

Assume the contrary: there exists $w \in \text{CF}_{n+2}^{(b, \infty)}(K)$ with $\partial w = x^b$. In other words, there exists a chain in $w' \in \text{CF}_{n+2}^{(0, \infty)}(K)$ such that $x' = x - \partial w' \in \text{CF}_{n+2}^{(0, b)}(K)$.
Then clearly $\partial x' = z - y$ and no orbit entering $x'$ has action in the interval $(b, \infty)$, where $b < A_K(x_1)$, which is impossible due to our choice of $x$. This contradiction completes the proof of (4.3).

Proceeding with the proof of (4.2), we again make use of the fact that $U$ is displaced by $\varphi_H$. Denote by $K\#H$ the Hamiltonian
\[
(K\#H)_t = K_t + H_t \circ (\varphi'_K)^{-1}
\]
generating the time-dependent flow $\varphi'_K \circ \varphi'_H$. Since $\varphi_H$ displaces supp $K$, the one-periodic orbits of $K\#H$ are exactly the one-periodic orbits of $H$ and moreover $S(K\#H) = S(H)$, as is well known (see, e.g., [HZ3]). Thus, by the continuity property (H3) of Floer homology,
\[
HF^{(b', b)}(K\#H) = HF^{(b', b)}(H)
\]
for any $b < b'$ which are not in $S(H)$. In particular,
\[
HF^{(b, \infty)}(K\#H) \cong HF^{(b, \infty)}(H) = 0, \text{ whenever } b > \max S(H). \quad (4.4)
\]

Consider a linear monotone increasing homotopy from $K$ to $K\#H$. (Recall that $H \geq 0$.) By Example 3.4, such a homotopy induces a map
\[
\Psi_{K, K\#H} : HF^{(a, \infty)}(K) \to HF^{(b, \infty)}(K\#H), \text{ where } b = a + E^+(H),
\]
whenever $a \notin S(K)$ and $b \notin S(K\#H)$. The composition of this map with the map
\[
\Psi_{K\#H, K} : HF^{(b, \infty)}(K\#H) \to HF^{(b, \infty)}(K)
\]
induced by a monotone decreasing homotopy from $K\#H$ to $K$ is the quotient map $j$. (See Example 3.4.)

To finish the proof in the case where $W$ is wide, assume that (4.2) fails: $E^+(H) < A_K(x_1) - \max K$. Pick $a > \max K$ such that $b = a + E^+(H) < A_K(x_1)$, and $a$ and $b$ are not in $S(K)$ and also $b \notin S(K\#H)$. Then $j = \Psi_{K\#H, K} \circ \Psi_{K, K\#H}$ factors through the group $HF^{(b, \infty)}(K\#H)$. Since $b > a > \max K > \max S(H)$, this group is zero by (4.4). Hence, $j = 0$, which contradicts (4.3).

Closed manifolds. Only two points of the above argument require modifications when $W$ is closed.

The first of these is the definition of the cycle $z$. Now we take as $F \geq K$ a function which is $C^2$-close to a constant and has a unique local maximum, equal to $K(p)$, that is also located at $p$. Then as above $z$ is the image of the Floer cycle $p = \max_F \in CF^{(0, \infty)}(F)$ under a monotone homotopy from $F$ to $K$. It is clear that $p$ occurs in $z$, i.e., $z = p + z_2 + \cdots + z_m$ for the same reason as for open manifolds.

The second point of the proof that is not obvious when $W$ is closed, for we work with non-normalized Hamiltonians, is the equality of action spectra $S(K\#H) = S(H)$. To see that this is the case, let us first recall how the action spectrum depends on the Hamiltonian. Let $G^s$ be a homotopy of Hamiltonians such that the time-one flows $\varphi_{G^s}$ are independent of $s$, i.e., the Hamiltonians $G^s$ determine the same element in the universal covering of the group Ham($W$). Then
\[
S(G^1) = S(G^0) + \int_{S^1} \int_W (G_1^1 - G_0^0) \omega^1 \, dt. \quad (4.5)
\]
This can be established by arguing as in the proof of [Sc2, Lemma 3.3].

\footnote{The author is grateful to Felix Schlenk for his help in clarifying the behavior of action spectra under homotopy.}
Following, e.g., [HZ3] or [FGS], consider a homotopy from $K = K^0$ to $K^1$ through reparametrizations $K^1_t = \lambda(t)K_{\lambda(t)}(t)$ of $K$ such that $K^1_t \equiv 0$ when $t \in [0, 1/2]$. Let $H^*$ be a similar homotopy beginning with $H^0 = H$ and ending with $H^1$, and such that $H^1_t \equiv 0$ for $t \in [1/2, 1]$. Clearly, these homotopies do not change the action spectra. Let $G^* = K^*\#H^*$. It is easy to see that $S(K^1\#H^1) = S(H^1) = S(H)$. (The first equality follows from the fact that $K^1\#H^1$ and $H^1$ have literally the same one-periodic orbits, with the same parametrizations, and the two Hamiltonians are equal along these orbits.) Furthermore, a direct calculation shows that

$$
\int_{S^1} \int_W G^*_t \omega^n \, dt = \int_{S^1} \int_W (K^1_t + H^1_t) \omega^n \, dt \\
= \int_{S^1} \int_W (K_t + H_t) \omega^n \, dt \\
= \int_{S^1} \int_W G^0_t \omega^n \, dt.
$$

Thus, by (4.5), $S(K^1\#H^1) = S(K\#H)$. As a consequence, $S(H) = S(K\#H)$.

The rest of the proof proceeds exactly as in the case of open manifolds.

5. Connecting trajectories for degenerate displaceable Hamiltonians

In this section we extend Proposition 4.1 to Hamiltonians whose one-periodic orbits are degenerate. This result will be used in the proof of Theorem 2.7.

5.1. The space of finite energy anti-gradient trajectories. Let, as above, $W$ be a geometrically bounded, symplectically aspherical manifold and let $K$ be a compactly supported Hamiltonian on $W$. Denote by $\mathcal{B} = \mathcal{B}(K)$ the space of contractible Floer anti-gradient trajectories with finite energy, i.e., the space of solutions of (3.1) such that $E(u) < \infty$ and $u(s)$ is contractible. We equip this space with the weak $C^\infty$-topology, i.e., the topology of uniform $C^\infty$-convergence on compact sets. Note that $\mathcal{P}_K \subset \mathcal{B}$.

Recall that by the compactness theorem any sequence of $u_i \in \mathcal{B}$ such that $u_i(0, 0)$ is bounded contains a converging subsequence. In fact, $\mathcal{B}$ is a locally compact, separable, metrizable space, and the evaluation map

$$
ev: \mathcal{B} \to W, \text{ defined as } \ev(u) = u(0, 0),$$

is continuous and proper; see, e.g., [HZ3] and references therein. Moreover, this map is a homeomorphism on a complement of a compact subset of $\mathcal{B}$.

For $u \in \mathcal{B}$, set $A_K(u)$ to be the action of $K$ on the closed curve $u(0)$. Then $A_K: \mathcal{B} \to \mathbb{R}$ is a continuous function, which is identically zero outside a compact set.

The anti-gradient flow $\Phi$ of $A_K$ on $\mathcal{B}$ is defined as the shift in the $\tau$-direction:

$$\Phi^\tau(u)(s) = u(s + \tau) \text{ for all } \tau \in \mathbb{R}.$$  

Obviously, the fixed points of this flow, i.e., finite energy trajectories independent of $s$, are exactly the one-periodic orbits of $K$. The function $A_K$ is decreasing along the orbits of $\Phi^\tau$, i.e., $A_K(\Phi^\tau(u)) \leq A_K(u)$ for any $\tau \geq 0$. Moreover, $A_K$ is strictly decreasing along non-trivial orbits: $A_K(\Phi^\tau(u)) < A_K(u)$ for any $\tau > 0$, when $u$ is not a fixed point of the flow.

Denote by $\Gamma(u) = \{\Phi^\tau(u) \mid \tau \in \mathbb{R}\}$ the orbit of $\Phi$ through $u$. By definition, the limit set $\omega^+(u)$ is formed by the limits of all converging sequences $\Phi^\tau(u)$ with
$\tau_i \to \infty$. The limit set $\omega^{-}(u)$ is defined similarly, but with $\tau_i \to -\infty$. The following properties of the closure $\bar{\Gamma}(u) = \Gamma(u) \cup \omega^{+}(u) \cup \omega^{-}(u)$ and of the limit sets $\omega^{\pm}(u)$ are well known:

- the sets $\bar{\Gamma}(u)$ and $\omega^{\pm}(u)$ are compact, connected, and invariant under the flow $\Phi$;
- the action functional $A_K$ is constant on $\omega^{\pm}(u)$, and $A_K|_{\omega^{+}(u)} \equiv \min_{\bar{\Gamma}(u)} A_K$ and $A_K|_{\omega^{-}(u)} \equiv \max_{\bar{\Gamma}(u)} A_K$;
- both of the limit sets $\omega^{\pm}(u)$ are non-empty and entirely comprised of the fixed points of $\Phi$.

It is clear that $u$ is partially asymptotic to $x^{\pm}$ as $s \to \pm \infty$, i.e., $u(s^{\pm}) \to x^{\pm}$ in $C^\infty(S^1, W)$ for some sequences $s_i^{\pm} \to \pm \infty$, if and only if $x^{\pm} \in \omega^{\pm}(u)$. (Here we treat $x^{\pm}$ simultaneously as periodic orbits (elements of $P_K$) and as finite energy trajectories (elements of $B$).) If $x^{\pm}$ are non-degenerate, we necessarily have $\omega^{\pm}(u) = \{x^{\pm}\}$. Otherwise, the limit sets can be quite large and $x^{\pm}$ are not unique.

### 5.2. Existence of connecting trajectories for degenerate Hamiltonians

In this section we prove an analogue of Proposition 4.1 for degenerate Hamiltonians. Let, as in Section 4, $W$ be a symplectically aspherical manifold, which is either closed or geometrically bounded and wide.

**Proposition 5.1.** Let $U$ be an open subset of $W$ displaced by a Hamiltonian diffeomorphism $\varphi_H$. Furthermore, let $K \geq 0$ be a Hamiltonian supported in $U$ and such that for all $t \in [0, 1]$ the functions $K_t$ attain their maxima at the same connected set $M \subset U$ and are autonomous near $M$. Assume also that $K$ is $C^2$-close to a constant on a sufficiently small neighborhood of $M$. Then, provided that $\max K$ is large enough, there exists $u \in B(K)$ partially asymptotic to a point of $M$ and to a contractible one-periodic orbit $\gamma$ such that (4.1) holds:

$$0 < A_K(\gamma) - \max K \leq \|H\|.$$  

**Remark 5.2.** In the context of this proposition, $W$ is equipped with an arbitrary almost complex structure $J$ compatible with $\omega$ in the sense of geometrically bounded manifolds and independent of time at infinity. Note also that (4.1) guarantees that $A(\gamma) > 0$ and, in particular, $\gamma$ is a non-trivial one-periodic orbit when $K$ is autonomous. Furthermore, $A_K(\gamma) - \max K = E(u)$.

**Remark 5.3.** Proposition 5.1 will be applied in the setting where $M$ is a Morse–Bott non-degenerate critical set of $K$. Then it might be possible to conclude that $u$ is genuinely asymptotic to a point on $M$ as $s \to \infty$; cf. [Poz]. However, the remaining one-periodic orbits of $K$ are still very degenerate and in this case $u$ need not be truly asymptotic to any orbit at $-\infty$. (This phenomenon, occurring already for the anti-gradient flow of a function on a smooth manifold, is overlooked in [Bo2]. However, the argument of [Bo2] still goes through for partial asymptotics.)

Note also that some degree of control over periodic orbits of $K$ near $M$ is essential. Here, it is achieved through requiring $K$ to be $C^2$-close to a constant near $M$ and thus to have no one-periodic orbits near $M$ with action greater than $\max K$. Without a restriction on the behavior of $K$ near $M$ the proposition probably fails.

The idea of the proof is, of course, to approximate $K$ by a sequence $K_t \to K$ of non-degenerate Hamiltonians satisfying the hypotheses of Proposition 4.1 and to define the orbit $u$ as the limit of a sequence of trajectories $u^0_i$ for $K_i$ such as in
Proposition 4.1. The nuance is that an arbitrary sequence \( u_l^0 \) need not have an orbit with the required properties as its limit point. (For instance, \( u_l^0 \) can converge to a point on \( M \) or to a trajectory which is not partially asymptotic to a point on \( M \).) However, the trajectories \( u_l^0 \) are not unique and can be replaced by \( u_l = \Phi_{\tau_l}(u_l^0) \) for any sequence of shifts \( \tau_l \). We show that \( u_l \) converges to the required \( u \) for some sequence \( \tau_l \), using an elementary, point–set topological argument.

**Proof.** Pick a sequence of non-degenerate Hamiltonians \( K_l \) which are supported in \( U \) and \( C^\infty \)-converge to \( K \) and satisfy the hypotheses of Proposition 4.1. Without loss of generality, we may assume that all \( K_l \) attain their maxima at the same point \( p \in M \) and that \( \max K_l = K_l(p) = \max K \).

Furthermore, we can approximate \( J \) by almost complex structures \( J_l \) (possibly time-dependent) which are compatible with \( \omega \) and equal to \( J \) at infinity and such that the pairs \((K_l, J_l)\) are regular. Let \( u_l^0 \) be an anti-gradient trajectory for \( K_l \) whose existence is established in Proposition 4.1: \( u_l^0 \) connects a contractible one-periodic orbit \( \gamma_l^0 \) of \( K_l \) to \( p \) and (4.1) holds. By passing if necessary to a subsequence, we may assume that the sequence \( \gamma_l^0 \) converges to a one-periodic orbit \( \gamma^0 \) of \( K \).

Recall that \( K \) is assumed to be autonomous and \( C^2 \)-close to a constant near \( M \). It follows that every one-periodic orbit of \( K \), with action in the interval \([\max K, \infty)\), that meets a sufficiently small neighborhood \( V \) of \( M \) must be a point of \( M \). Furthermore, \( K_l \) are also \( C^2 \)-close to a constant on \( V \). As a consequence, every one-periodic orbit of \( K_l \) entering \( V \) is trivial and we may require, in addition, that the actions of \( K_l \) on these orbits (with the exception of \( p \)) are strictly smaller than \( \max K_l \). (It is easy to show that the approximations \( K_l \) with this property do exist.) Since \( K_l(\gamma_l^0) > \max K_l \) by (4.1), we conclude that \( \gamma_l^0 \) does not enter \( V \). Hence, \( \gamma^0 \) also lies entirely outside \( V \).

Recall that by the compactness theorem any sequence \( u_l \in \mathcal{B}(K_l) \) contains a converging subsequence, provided that the sequence \( \text{ev}(u_l) \) is bounded in \( W \). (Here convergence is again understood as \( C^\infty \)-convergence uniform on compact subsets of \( \mathbb{R} \times S^1 \), i.e., in the weak \( C^\infty \)-topology on the space \( \mathcal{C} \) of smooth maps \( \mathbb{R} \times S^1 \to W \).)

Set \( \Gamma_l = \Gamma(u_l^0) \) and denote by \( \Sigma \subset \mathcal{C} \) the set of all limit points of sequences \( u_l \in \Gamma_l \) (or, equivalently, \( u_l \in \Gamma_l \)). This set has the following properties:

- (a) the set \( \Sigma \) contains \( p \) and \( \gamma_0^0 \);
- (b) the sets \( \Gamma_l \) converge to \( \Sigma \) in the Hausdorff topology: for every neighborhood \( \mathcal{U} \) of \( \Sigma \) in \( \mathcal{C} \) we have \( \Gamma_l \cap \mathcal{U} \) when \( l \) is large enough;
- (c) the set \( \Sigma \) is connected, compact, and invariant under the flow \( \Phi \);
- (d) the flow \( \Phi \) on \( \Sigma \) is non-trivial;
- (e) the action functional \( A_K \) is not constant on \( \Sigma \), and \( \min_{\Sigma} A_K = A_K(p) = \max K \) and \( \max_{\Sigma} A_K \leq \max K + \| H \| \).

**Proof of (a)–(e).** The first assertion (a) is obvious. To prove (b), we just note that otherwise we would have a sequence \( u_{l_i} \in \Gamma_{l_i} \cap \mathcal{U} \) for some \( l_i \to \infty \). By compactness, \( u_{l_i} \) must have a limit point outside \( \mathcal{U} \) and hence not in \( \Sigma \), which is impossible. In (c), only the fact that \( \Sigma \) is connected requires a proof. This readily follows from (b) and from the fact that all \( \Gamma_l \) contain the point \( p \). Assertion (e) is obvious except for the statement that \( A_K \) is non-constant, which is a consequence of (d). To prove (d), observe that \( \text{ev}(\Sigma) \) contains both \( p \in M \subset V \) and \( \gamma^0(0) \not\in V \). By (c), \( \text{ev}(\Sigma) \) is connected and therefore there exists \( u \in \Sigma \) such that \( u(0,0) = \text{ev}(u) \in V \setminus M \).
Since no one-periodic orbit of $K$ with action in $[\max K, \infty)$, other than the points of $M$, enter $V$, we conclude that $u(0)$ is not a contractible one-periodic orbit of $K$ and, in particular, the entire anti-gradient trajectory $u$ cannot be a contractible one-periodic orbit of $K$. Thus $u$ is not a fixed point of the flow $\Phi$. □

Let $\Sigma_{\min}$ be the set of $u \in \Sigma$ at which $A_K|_{\Sigma}$ attains its minimum $\max K$. Note that $\Sigma_{\min}$ is entirely comprised of fixed points of $\Phi$ or, equivalently, of periodic orbits of $K$. Regarding $M$ as a subset of $B(K)$, set $C = M \cap \Sigma_{\min} = M \cap \Sigma$. Then $C$ is a compact, proper subset of $\Sigma$. Indeed, compactness of $C$ is obvious. By (e), $\Sigma_{\min} \neq \Sigma$, and hence $C \neq \Sigma$. On the other hand, $C \neq \emptyset$, for $p \in C$. Next, we claim that

(f) the set $C$ is a union of connected components of $\Sigma_{\min}$, i.e., $\Sigma_{\min} \setminus C$ is closed.

Proof of (f). Indeed, assume the contrary. Then, there exists a sequence of periodic orbits $u_l \in \Sigma_{\min} \setminus C$ converging to a point of $C$. As a consequence, $\text{ev}(u_l)$ converges to a point of $M$. Then $u_l$ must be a trivial periodic orbit when $l$ is large enough. (For $u_l$ enters $V$.) In addition, $A_K(u_l) = \max K$ and thus $u_l$ is a point of $M$. This is impossible since $u_l \in \Sigma_{\min} \setminus C$. □

Set $\mathcal{N}_\epsilon = \{u \in \Sigma \mid A_K(u) < \max K + \epsilon\}$. Fix a connected component $C_0$ of $C$ and let $\mathcal{U}_\epsilon$ be the connected component of $\mathcal{N}_\epsilon$ that contains $C_0$. Clearly, the sets $\mathcal{U}_\epsilon$ with $\epsilon > 0$ are open, nested, and invariant under the positive flow $\Phi^+$. Our next goal is to prove

(g) the open sets $\mathcal{U}_\epsilon$ with $\epsilon > 0$ form a fundamental system of neighborhoods of $C_0$, i.e., for any open set $U \supset C_0$, we have $\mathcal{U}_\epsilon \subset U$, when $\epsilon > 0$ is sufficiently small.

Proof of (g). Let us first show that the sets $\mathcal{N}_\epsilon$ form a fundamental system of neighborhoods of $\Sigma_{\min}$ for $\epsilon > 0$. Assume the contrary. Then for some open set $\mathcal{W} \supset \Sigma_{\min}$, there exists a sequence $u_l \in \mathcal{N}_\epsilon \setminus \mathcal{W}$ with $\epsilon_l \to 0+$. Passing if necessary to a subsequence, we have $u = \lim u_l \in \Sigma \setminus \mathcal{W}$ by the compactness theorem, and hence $u \not\in \Sigma_{\min}$. This is impossible, for $A_K(u) = \lim A_K(u_l) = \min A_K|_{\Sigma}$ and thus $u \in \Sigma_{\min}$.

Let now $\mathcal{U}$ be a neighborhood of $C_0$. We need to show that $\mathcal{U}_\epsilon \subset \mathcal{U}$, when $\epsilon > 0$ is small. Assume that this is not the case: the sets $\mathcal{U}_\epsilon$ are not entirely contained in $\mathcal{U}$ for some sequence $\epsilon_i \to 0+$. (Since the family $\mathcal{U}_\epsilon$ is nested, this is true for all $\mathcal{U}_\epsilon$, but we prefer to work with a sequence.) Therefore, the intersection $\bigcap \mathcal{U}_\epsilon \subset \Sigma_{\min}$ is connected since $\mathcal{U}_\epsilon$, are connected, contains $C_0$, and a point of $\Sigma_{\min} \setminus C$. This is impossible due to (f). □

Now we are in a position to finish the proof of Proposition 5.1. Utilizing (g), pick $\epsilon > 0$ so small that $\text{ev}(\mathcal{U}_\epsilon) \subset V$. Then every fixed point of $\Phi$ in $\mathcal{U}_\epsilon$ must belong to $C_0$. (Otherwise, there would be a one-periodic orbit with action in $[\max K, \infty)$, other than a point of $M$, entering $V$.) Thus $\omega^+(u) \in C_0$ for any $u \in \mathcal{U}_\epsilon$. Observe now that $\mathcal{U}_\epsilon \setminus C_0 \neq \emptyset$, since $\Sigma$ is connected and $C_0$ is closed. Let $u \in \mathcal{U}_\epsilon \setminus C_0$. Then $u$ is a non-trivial, anti-gradient trajectory partially asymptotic to a point in $M$ at $\infty$. Therefore,

$$\max K = \min_{\Sigma} A_K < A_K(u).$$
As a consequence, for any $\gamma \in \omega^-(u)$ we have

$$0 < A_K(\gamma) - \max K \leq \|H\|,$$

where the second inequality follows from (e). This completes the proof of (4.1) and thus the proof of the proposition. \hfill \Box

Proposition 5.1 has a counter-part asserting the existence of homotopy connecting trajectories “transferring” action selectors; cf. [CGK, Ke2]. This is a much more standard result and we treat it in lesser detail.

**Proposition 5.4.** Let $U$ be an open subset of $W$ displaced by a Hamiltonian diffeomorphism $\varphi_H$. Furthermore, let $K \geq 0$ be a Hamiltonian supported in $U$ and let $f \geq 0$ be a $C^2$-small autonomous Hamiltonian such that $f \leq K$. Consider a monotone decreasing homotopy from $K$ to $f$. Then there exists a homotopy trajectory $u$ partially asymptotic to a point $p \in \mathcal{P}(f)$ at $\infty$ and to $\gamma \in \mathcal{P}(K)$ at $-\infty$ and such that

- $f$ attains its maximum at $p$,
- $A_K(\gamma) \leq \sigma(K)$ and $E(u) \leq A_K(\gamma) - f(p) \leq \|H\|$.

**Remark 5.5.** When the Hamiltonians $K$ and $f$ are non-degenerate and the homotopy is regular, the inequality $A_K(\gamma) \leq \sigma(K)$ can be replaced by equality $A_K(\gamma) = \sigma(K)$.

**Outline of the proof.** First note that similarly to the proof of Proposition 4.1 we may assume without loss of generality that $H$ is compactly supported and $\min H_t = 0$. In particular, $H \geq 0$ and $\|H\| = E^+(H)$. Furthermore, the lower bound $E(u) \leq A_K(\gamma) - f(p)$ follows immediately from (3.8).

When $K$ and $f$ are non-degenerate and the homotopy is regular the assertion is well known; see, e.g., the proof of Proposition 4.1 or [FS, Gi3, Gü, Ke2, KL, Oh4, Sc2, Vi2] to mention just a few sources where similar results have been proved. Moreover, in this case we have $A_K(\gamma) = \sigma(K)$. Indeed, in the notation of the proof of Proposition 4.1, $\gamma$ is an orbit in the chain $y$, a carrier of the action selector, such that $A_K(\gamma) = \sigma(K)$.

To deal with the general case, we argue as in the proof of Proposition 5.1 and approximate $f$ and $K$ by non-degenerate Hamiltonians $f_t \to f$ and $K_t \to K$ meeting the requirements of Proposition 5.4. We also approximate the homotopy $K^s$ from $K$ to $f$ by regular monotone decreasing homotopies $K^s_t$ from $K_t$ to $f_t$. Furthermore, we may assume that $\max f_t = \max f$. Applying the non-degenerate case of the proposition to $K_t$ and $f_t$, we obtain a critical point $p_t$, an orbit $\gamma_t$, and a homotopy connecting trajectory $u_t$.

Since $E(u_t) \leq \|H\|$, the compactness theorem implies that the sequence $u_t$ contains a converging subsequence. Passing to this subsequence and taking the limit, we obtain a homotopy connecting trajectory $u$ from $K$ to $f$ such that $E(u) \leq \|H\|$. Then, by compactness again, $u$ is partially asymptotic to $p \in \mathcal{P}_f$ at $\infty$ and to $\gamma \in \mathcal{P}_K$ at $-\infty$. Note that although we can assume that $\gamma_t$ converges to an orbit of $K$ due to the Arzela–Ascoli theorem, we cannot claim that $\gamma_t \to \gamma$ and hence cannot conclude that $\sigma(K) = A_K(\gamma)$. However,

$$f(p) = A_f(p) \geq \max f = f_t(p_t) \quad (5.1)$$

and

$$A_K(\gamma) \leq \lim_{t \to \infty} A_{K_t}(\gamma_t) = \sigma(K). \quad (5.2)$$
To prove (5.2), we argue as in the proof of (H0)–(H2) in Section 3.2.2. Namely,

\[ A_K(\gamma) = \sup_s A_K_s(u(s)) = \sup_{s,l} A_{K_l}(u_l(s)) \]

\[ \leq \lim_{l \to \infty} \sup_s A_{K_l}(u_l(s)) = \lim_{l \to \infty} A_{K_l}(\gamma_l) = \lim_{l \to \infty} \sigma(K_l) = \sigma(K), \]

where the last equality follows from (S3), continuity of the action selector. The first inequality, (5.1), is established in a similar fashion and immediately implies that \( f(p) = \max f \). Finally, from (5.1) and (5.2), we infer that

\[ A_K(\gamma) - f(p) \leq \lim_{l \to \infty} (A_{K_l}(\gamma_l) - f_l(p_l)) \leq \|H\|, \]

which concludes the proof. \( \square \)

**Remark 5.6.** Propositions 5.1 and 5.4 (and their non-degenerate counterparts) represent two different Floer homological (broadly understood) approaches to proving the existence of (infinitely many) periodic orbits of Hamiltonians, either in the autonomous case (the Weinstein conjecture and the almost existence theorem) or for time-dependent Hamiltonians (the Conley conjecture). One approach comprises a class of methods that lead to “low-lying” orbits with action smaller than the displacement energy as in Proposition 5.4. These methods are utilized in, for instance, [CGK, FS, FHW, FGS, Gi3, G"u, HZ3, Vi2, Vi4]. The second class of methods detects orbits lying above the action selector value as in Propositions 4.1 and 5.1. In this class are the variational methods of [HZ2], the Floer homological results of [Ke2, GG2] and apparently some of the results utilizing Hofer's geometry, e.g., [Oh4, Sch]. It goes without saying that many methods do not fit into this crude classification. These include, for instance, the equivariant (Floer) homological methods (see, e.g., [HZ1, Vi1, Vi3]) and holomorphic curve methods, e.g., [HV, Lu1, Lu2, LT2].

### 6. Displacement energy for stable manifolds

Let, as in Section 2, \( M \) be a stable, closed, coisotropic submanifold of \( W \). Recall from Section 2.1 that a neighborhood of \( M \) in \( W \) is identified with a neighborhood of \( M \times R^k \) with the symplectic form (2.2). Using this identification, we denote by \( U_r \), with \( r > 0 \) sufficiently small, the neighborhood of \( M \) in \( W \) corresponding to \( M \times D_r^k \), where \( D_r^k \) is the ball of radius \( r \). Recall also that by definition \( \rho = (p_1^2 + \cdots + p_k^2)/2 \), where \( (p_1, \ldots, p_k) \) are the coordinates on \( R^k \). (Thus \( U_r = \{ \rho < r^2/2 \} \).)

Let \( K \) be a smooth function on \([0, r]\) such that

- \( K \) is monotone decreasing and \( K \equiv 0 \) near \( r \);
- all odd-order derivatives of \( K \) at 0 are zero, and \( K''(0) < 0 \) is close to zero.

Abusing notation, we also denote by \( K \) the function on \( W \) equal to \( K(|p|) \) on \( U_r \), where \( |p| = \sqrt{2\rho} \), and extended to be identically zero outside \( U_r \). Then the Hamiltonian \( K \) satisfies the hypotheses of Proposition 5.1. By Proposition 2.5, the Hamiltonian flow of \( K \) on \( U_r \) is a reparametrization of the leaf-wise geodesic flow on \( M \). Outside \( U_r \), the flow is the identity map.

**Theorem 6.1.** Assume that a Hamiltonian diffeomorphism \( \varphi_H \) displaces \( M \) and that \( r > 0 \) is sufficiently small so that, in particular, \( \varphi_H \) also displaces \( U_r \). Then
there exists a constant $\Delta > 0$, independent of $r$ and $K$, such that $K$ has a contractible periodic orbit $\gamma$ with

$$\Delta \leq A_K(\gamma) - \max K \leq \|H\|,$$

provided that $\max K$ is large enough.

Proof. The proof of the theorem relies on Proposition 5.1 and the following two lemmas, which are essentially contained in [Bo1, Bo2] and which will also be used in the proof of Theorem 2.7.

For a closed curve $\eta: S^1 \to M$ lying in a leaf of the foliation $F$, set

$$\delta(\eta) = \sum_{i=1}^k \left| \int_{\eta} \alpha_i \right|.$$

**Lemma 6.2.** There exists a constant $\delta_M > 0$ such that

$$\delta_M \leq \delta(\eta)$$

for all non-trivial closed geodesics $\eta$ of the leaf-wise metric $\alpha_1^2 + \cdots + \alpha_k^2$ (see Proposition 2.5).

Proof. Note that since, by Proposition 2.5, the metric is leaf-wise flat we have $\alpha_i(\dot{\eta}(t)) = \text{const}$ for every leaf-wise geodesic $\eta$. Hence

$$\left| \int_{\eta} \alpha_i \right| = \int_{S^1} |\alpha_i(\dot{\eta}(t))| \, dt.$$

As a consequence,

$$\delta(\eta) \geq \text{length}(\eta) \tag{6.1}$$

which follows immediately from the fact that $\sum_i |\alpha_i(v)| \geq 1$ for every unit vector $v$ tangent to $F$.

Assume that the required constant $\delta_M > 0$ does not exist, i.e., $\delta(\eta_j) \to 0$ for some sequence of closed non-trivial leaf-wise geodesics $\eta_j$. Thus, we also have $\text{length}(\eta_j) \to 0$. Passing if necessary to a subsequence, we conclude that all $\eta_j$ are contained in a small neighborhood in $M$. (Indeed extending the metric from $F$ to $M$, we can view $\eta_j$ as a sequence of closed curves whose length converges to zero. Now it is clear that the sequence $\eta_j$ contains a subsequence lying in a neighborhood of arbitrarily small radius.) Without loss of generality we may assume that this neighborhood is foliated. Then, when $j$ is large enough, $\eta_j$ is contained in a small ball in the leaf $F \supset \eta_j(S^1)$. In particular $\eta_j$ is contractible in $F$. This is impossible since the metric is leaf-wise flat.

**Remark 6.3.** When $M$ has restricted contact type, Lemma 6.2 implies that the “area spectrum” of $M$ is separated from zero by $\delta_M/k$, for $\delta(\eta) = k|A(\eta)|$.

Fix a neighborhood $V = U_R$, for some $R > 0$. In what follows, we will always assume that $0 < r < R/2$ and that $U_r$ is displaced by $\varphi_H$.

**Lemma 6.4.** There exists a constant $c_V > 0$, depending on $V$ but not on $K$ or $r > 0$, such that for any Floer anti-gradient trajectory $u$ for $K$ partially asymptotic to a non-trivial one-periodic orbit $\gamma$ at $-\infty$ and to a point of $M$ at $\infty$, we have

$$c_V \delta(\pi(\gamma)) \leq A_K(\gamma) - \max K.$$

**Remark 6.5.** Note that the right hand side of this inequality is the energy $E(u)$. 

Proof. Let $f$ be a non-negative, smooth, decreasing function on $[0,R]$ identically equal to 1 on $[0,R/2]$ and vanishing near $R$. Abusing notation, we also denote by $f$ the function on $W$ equal to $f(|p|)$ on $V = U_R$, where $|p| = \sqrt{2p}$ as above, and extended to be identically zero outside $V$.

Following [Bo2], set $\beta_i = f\pi^* \alpha_i$. This is a smooth one-form on $W$. A feature of the form $\beta_i$, important in what follows, is that

$$i_{X_K} d\beta_i = 0. \quad (6.2)$$

To prove this, we first note that $(6.2)$ trivially holds outside $U_r$. On the other hand, $f \equiv 1$ on $U_r$ since $r < R/2$. Thus on $U_r$ we have $\beta_i = \pi^* \alpha_i$ and, point-wise,

$$i_{X_K} d\beta_i = i_{X_K} \pi^* d\alpha_i = K'(\rho) i_{\pi^* X_\rho} d\alpha_i = 0.$$

The last equality follows from the fact that $\pi_* X_\rho$ is tangent to $\mathcal{F}$ by Proposition 2.5 and that $T\mathcal{F} \subset \ker d\alpha_i$ since $M$ is stable.

Set $c_i = \| d\beta_i \|_{C^0} > 0$ so that

$$|d\beta_i(X,Y)| \leq c_i \| X \| \| Y \|,$$

for any two tangent vectors $X$ and $Y$. Here, on the right hand side, the norm is taken with respect an arbitrary metric compatible with $\omega$ as in Section 2.2. (We emphasize that, since the argument relies only on the results of Section 5, which hold for general Hamiltonians and metrics, the metric need not meet any regularity requirements for Floer anti-gradient trajectories; cf. Remark 5.2.) It is clear that $c_i$ is independent of $K$.

We claim that

$$A_K(\gamma) - \max K = E(u) \geq c_i^{-1} \left| \int_{\pi(\gamma)} \alpha_i \right|. \quad (6.3)$$

The assertion of the lemma immediately follows from $(6.3)$ by adding up these inequalities for $i = 1, \ldots, k$ and setting $c_V = \min c_i^{-1}/k$.

To prove $(6.3)$, fix $s_j^\pm \to \pm \infty$ such that $u(s_j^+)$ converges to a point of $M$ and $u(s_j^-)$ converges to $\gamma$ in $C^\infty(S^1, W)$. Then utilizing the definition of $c_i$ and $(6.2)$ in the last step, we have

$$E(u) = \int_{\mathbb{R} \times S^1} \left\| \frac{\partial u}{\partial s} \right\| \left| \frac{\partial u}{\partial t} - X_K \right| ds \ dt \geq c_i^{-1} \int_{\mathbb{R} \times S^1} \left| d\beta_i \left( \frac{\partial u}{\partial s} \frac{\partial u}{\partial t} - X_K \right) \right| ds \ dt \geq c_i^{-1} \lim_{j \to \infty} \left| \int_{s_j^-}^{s_j^+} \int_{S^1} d\beta_i \left( \frac{\partial u}{\partial s} \frac{\partial u}{\partial t} - X_K \right) dt \ ds \right| \geq c_i^{-1} \lim_{j \to \infty} \left| \int_{s_j^-}^{s_j^+} \int_{S^1} d\beta_i \left( \frac{\partial u}{\partial s} \frac{\partial u}{\partial t} \right) dt \ ds \right|.$$

Note that at this stage we still do not know if the limit in question exists. However, by applying Stokes’ formula, we see that

$$\left| \int_{s_j^-}^{s_j^+} \int_{S^1} d\beta_i \left( \frac{\partial u}{\partial s} \frac{\partial u}{\partial t} \right) dt \ ds \right| = \left| \int_{u(s_j^+)} \beta_i - \int_{u(s_j^-)} \beta_i \right| \to \left| \int \beta_i \right|.$$
as \( j \to \infty \). Furthermore, recall that \( \gamma \) is contained in \( U_r \) and \( f|_{U_r} \equiv 1 \). Thus,

\[
\left| \int_{\gamma} \beta_i \right| = \left| \int_{\gamma} \pi^* \alpha_i \right| = \left| \int_{\pi(\gamma)} \alpha_i \right|,
\]

which completes the proof of (6.3) and of the lemma.

To finish the proof of the theorem, we apply Proposition 5.1 to the Hamiltonian \( K \). Next, applying Lemma 6.4 to \( u \) and \( \gamma \) whose existence is guaranteed by this proposition, we have

\[
\|H\| \geq A_K(\pi(\gamma)) - \max K \geq c_V\delta(\pi(\gamma)) \geq c_V\delta_M =: \Delta > 0,
\]

where the last inequality follows from Lemma 6.2 and the fact that \( \pi(\gamma) \) is, up to parametrization and orientation, a closed non-trivial geodesic of the leaf-wise geodesic flow on \( \mathcal{F} \).

Proof of Theorem 2.7. Let \( K \) be as above and let \( \max K \) be large enough. Assertion (i) follows immediately from Theorem 6.1. To prove (ii), we argue as in [Bo2]. Pick \( \gamma \) as in the proof of Theorem 6.1. This is a non-trivial periodic orbit and, since \( K \) attains its maximum along \( M \), we see from (6.4) that

\[
A(\gamma) \geq A_K(\gamma) - \max K \geq c_V\delta(\pi(\gamma)).
\]

Then, utilizing the normal form (2.2) and setting \( \eta = \pi(\gamma) \), we have

\[
A(\eta) = A(\gamma) - \int_{\gamma} \sum p_i \alpha_i \\
\geq A(\gamma) - r \sum \left| \int_{\eta} \alpha_i \right| \\
\geq c_V\delta(\eta) - r\delta(\eta) \\
= (c_V - r)\delta(\eta) \\
\geq (c_V - r)\delta_M > 0,
\]

provided that \( r > 0 \) is small enough. This proves (ii) for \( \eta = \pi(\gamma) \). Moreover, we have also established the a priori lower bound

\[
A(\eta) \geq (c_V - r)\delta_M.
\]

It remains to prove (iii). To this end, we first need to establish a general property of the metric \( \rho \), which holds for arbitrary stable coisotropic submanifolds. Recall that the length spectrum of a (leaf-wise) metric on a foliated manifold \( (M, \mathcal{F}) \) is the collection of lengths of all closed leaf-wise geodesics in \( M \). In contrast with the length spectrum of a metric on \( M \), the length spectrum of a leaf-wise metric need not in general be nowhere dense, even if the metric is leaf-wise flat. However, as our next observation shows, this is the case for the metric \( \rho \).

Lemma 6.6. Assume that \( M \) is stable. The length spectrum of \( \rho \) is closed, has zero measure, and is, therefore, nowhere dense.

Proof. The assertion that the length spectrum is closed holds obviously for any metric. To show that the spectrum has zero measure, consider a level \( \{\rho = \text{const}\} \subset M \times \mathbb{R}^k \) and recall that by Proposition 2.5 closed geodesics of \( \rho \) are projections to \( M \) of closed characteristics on this level. Furthermore, again by Proposition 2.5, the length of a geodesic is equal to the integral of \( \lambda = \sum p_i \alpha_i \) over the corresponding
closed characteristics. (Strictly speaking, the equality holds only up to a factor depending only on the level, and we chose \( c \) so that this factor is equal to one.) The characteristic foliation on the level is tangent to the distribution \( \text{ker} \, d\lambda \) as is easy to see from (2.3). It follows that the integrals of \( \lambda \) over closed characteristics form a zero measure set. (The distribution \( \text{ker} \, d\lambda \) need not have constant rank, but the standard argument still applies.)

Let now \( \gamma \) be as above and \( \eta = \pi(\gamma) \). Our next goal is to obtain an upper bound on \( A(\gamma) \) and \( A(\eta) \). To this end, we need to impose an extra requirement on \( K \) guaranteeing that \( K(\gamma) \) is close to \( \max K \).

Namely, fix \( r_- \) and \( r_+ \) such that \( 0 < r_- < r_+ < r \) and pick a sufficiently small constant \( \epsilon > 0 \). We can chose \( K \) so that on \( U_r \), we have

1. \( \max K - K < \epsilon \) on \([0, r_-]\) and \( K < \epsilon \) on \([r_+, r]\);
2. on \([r_-, r_+]\) the Hamiltonian \( K \), thought of as function of \(|p|\), has constant slope lying outside the length spectrum of the metric \( \rho \);
3. \( K \) has a sufficiently large variation over the constant slope range, i.e., \( K(r_-) - K(r_+) > C \cdot \|H\| \), where \( C \) is a constant, to be specified later, depending only on \( V \).

In particular, these conditions ensure that the only non-trivial periodic orbits of \( K \) occur within the shells \((0, r_-)\) and \((r_+, r)\).

Furthermore, we claim that \( \gamma \) lies in the shell \((0, r_-)\), and hence

\[
\max K - K(\gamma) < \epsilon. \tag{6.7}
\]

Indeed, first observe that since \( M \) has restricted contact type, we have

\[
A(\eta) = \int_\eta \alpha_i \quad \text{for } i = 1, \ldots, k.
\]

Thus, by (6.3),

\[
|A(\eta)| \leq c_i \|H\| \quad \text{for } i = 1, \ldots, k
\]

and, as a consequence,

\[
|A(\gamma)| = \left| A(\eta) + \int_\gamma \sum_i p_i \alpha_i \right|
\leq |A(\eta)| + r \sum_i \left| \int_\eta \alpha_i \right|
= (1 + kr)|A(\eta)|
\leq C \cdot \|H\|,
\]

where, for instance, \( C = (1 + kr)c_1 \). On the other hand, if \( \gamma \) were in the shell \((r_+, r)\), we would have

\[
\max K - K(\gamma) > K(r_-) - K(r_+) > C \cdot \|H\|,
\]

and hence,

\[
A(\gamma) \geq \max K - K(\gamma) > C \cdot \|H\|.
\]

Thus \( \gamma \) is indeed in the shell \((0, r_-)\) and (6.7) holds.

Next note that

\[
A(\eta) = (A_K(\gamma) - \max K) + (\max K - K(\gamma)) - \int_\gamma \sum p_i \alpha_i.
\]
Here the first term in bounded from above by $\|H\|$ due to Proposition 5.1, the second term is bounded from above by $\epsilon$ due to (6.7), and the last term is bounded from above by $r\delta(\eta) \leq rc_V^{-1}\|H\|$ according to (6.4). As a consequence,

$$ (c_V - r)rM \leq A(\eta) \leq \|H\| + \epsilon + rc_V^{-1}\|H\|, $$

(6.8)

where the first inequality is (6.6). Note that we also have an a priori upper bound on the length of $\eta$,

$$ \text{length}(\eta) \leq \delta(\eta) \leq c_V^{-1}\|H\|, $$

resulting from (6.1) and (6.4).

Consider a sequence of Hamiltonians $K_i$ as above with $r_i \to 0$ and $\epsilon_i \to 0$. Then by the Arzela–Ascoli theorem the geodesics $\eta_i$ converge to a geodesic $\eta$ and, passing to the limit in (6.8), we see that

$$ 0 < c_V\delta_M \leq A(\eta) \leq \|H\|, $$

which concludes the proof. □

Remark 6.7. The first assertion of Theorem 2.7 can also be established by making use of Proposition 5.4 instead of Proposition 5.1, although in a somewhat less direct way. Let $K$ and $r$ be as in Theorem 6.1 with $\max K > \|H\|$ and let $\epsilon > 0$ be so small that $f = \epsilon K$ is $C^2$-small. Denote by $u$ and $\gamma$ the homotopy connecting trajectory and the periodic orbit from Proposition 5.4 for a linear homotopy from $K$ to $f$. Since $\max K > \|H\| > \sigma(K)$, we conclude that the orbit $\gamma$ is non-trivial. It is not hard to see that the proof of Lemma 6.4 goes through for $u$. (Note that the assumptions that $f = \epsilon K$ and that the homotopy is linear are essential to make sure that (6.2) holds for every Hamiltonian $K^s$ in the homotopy.) Hence, as in the proof of Theorem 6.1, we have

$$ \|H\| \geq E(u) \geq c_V\delta(\pi(\gamma)) \geq c_V\delta_M, $$

which proves assertion (i). Even though this approach does not lead to assertion (ii) in its full generality, it does imply that $\mathcal{F}$ has a leaf-wise non-trivial closed geodesic, contractible in $W$. This is an analogue of the existence result for closed characteristics on closed, stable or contact type hypersurfaces, [HZ3, Vi1]; see also the survey [Gi3] for a discussion of more recent results.

Finally note that passing to the limit as $f \to 0+$, we infer from Proposition 5.4 that $\sigma(K) \geq c_V\delta_M$. As a consequence, $c_{\text{hom}}(U) \geq c_V\delta_M$ for any neighborhood $U$ of $M$. Thus $c_{\text{hom}}(M) \geq c_V\delta_M > 0$, whenever $M$ is stable and displaceable; cf. [Dr].

Proof of Theorem 2.11. It suffices to show that the levels $M_\alpha$ carrying an orbit $\gamma$ with the required properties exist arbitrarily close to $M$. Set $K = f(K_1, \ldots, K_k)$, where $f: \mathbb{R}^k \to \mathbb{R}$ is a bump-function supported in a small neighborhood of the origin in $\mathbb{R}^k$ and such that $\max f$ is large enough. Since the support of $f$ is small, we may assume that $\supp K$ is displaceable and all $a \in \supp f$ are regular values of $K$. By Proposition 5.1, the flow of $K$ has a contractible in $W$ one-periodic orbit $\gamma$ with $A(\gamma) > 0$; see, e.g., Remark 5.2. Furthermore, the Hamiltonian $K$ Poisson–commutes with all $K_i$, and thus $\gamma$ is tangent to a regular level $M_\alpha$. □

7. Leaf-wise intersections for hypersurfaces

The goal of this section is to prove the leaf-wise intersection property (Theorem 2.9) for hypersurfaces of restricted contact type in subcritical Stein manifolds. In fact the theorem holds for a slightly broader class of ambient manifolds than
subcritical Stein. Namely, let \( W \) be an exact symplectically aspherical manifold convex in the sense of [FS] and let as above \( M \) be a closed hypersurface of restricted contact type in \( W \) bounding a domain \( U \). Assume that \( H \) is a compactly supported Hamiltonian on \( W \) such that

\[
\|H\| < c_{\text{hom}}(U)
\]

and

\[
c_{\text{hom}}\left(\bar{U} \cup \text{supp}(H)\right) < \infty.
\]

Then Theorem 2.9 (in a slightly generalized form) asserts that \( \varphi(F) \cap F \neq \emptyset \) for some leaf \( F \) of the characteristic foliation of \( M \) and \( \varphi = \varphi_H \).

It is worth mentioning that the requirement (7.1) is quite restrictive. (Note that (7.1) implies that \( c_{\text{hom}}(U) < \infty \).) The reason that (7.1) holds if \( W \) is a subcritical Stein manifold is that every compact set in \( W \) is displaceable; see, e.g., [BC]. However, (7.1) fails, as can be seen from the results of [Vi4], when \( U \) is a tubular neighborhood of the zero section in a cotangent bundle. Essentially the only case where (7.1) can be verified is where \( \bar{U} \cup \text{supp}(H) \) is displaceable in \( W \).

Hereafter, we assume that \( W \) and \( M \) are as above. In this case, the action selector \( \sigma \) from Section 3.3.1 can also be defined as in [FS] for all compactly supported Hamiltonians (not necessarily positive) and has, in addition to (S0)–(S4), a number of other properties. For instance, \( \sigma(H) = 0 \), whenever \( H \leq 0 \), and \( \sigma \) is sub-additive, i.e., \( \sigma(H \# K) \leq \sigma(H) + \sigma(K) \); see [FS]. Furthermore, \( \sigma(K) \) depends only on the time-one flow \( \varphi_K \) and we will also use the notation \( \sigma(\varphi_K) \).

**Proof of Theorem 2.9.** First note that as in the proof of Proposition 4.1 we may require that \( \min H_t = 0 \) for all \( t \) and hence \( \|H\| = E^+(H) \) and \( H \geq 0 \). (This can be achieved, keeping (7.1), by replacing \( H \) by the Hamiltonian \( f \cdot (H - \min H) \), where \( f \) is a cut-off function identically equal to one near \( \bar{U} \cup \text{supp}(H) \).) Furthermore, without loss of generality we may also assume that the fixed point set \( \text{Fix}(\varphi) \) does not meet \( M \), for otherwise the assertion is obvious. Then \( \varphi \) has no fixed points near \( M \).

Fix a global primitive \( \lambda \) of \( \omega \), restricting to a contact form on \( M \). Let \( U_\varepsilon = M \times [-\varepsilon,0] \) be a narrow shell inside \( U \), containing no points of \( \text{Fix}(\varphi) \). We choose the projection \( U_\varepsilon \to [-\varepsilon,0] \) so that the Hamiltonian flow of this projection (thought of as a function on \( U_\varepsilon \)) is exactly equal to the Reeb flow of \( \lambda|_{M_\varepsilon} \), where \( M_\varepsilon = M \times \{\tau\} \), for all \( \tau \in [-\varepsilon,0] \). The projection \( U_\varepsilon \to M \) is chosen so that the characteristic foliation on \( M_\tau \) projects to the characteristic foliation on \( M \). (It is well known that such a shell exists, when \( M \) has contact type.)

Let \( K \geq 0 \) be a non-negative function which is equal to zero on \( W \setminus U \) and to \( \max K > 0 \) on \( U \setminus U_\varepsilon \), and which is a monotone decreasing function of \( \tau \) on \( U_\varepsilon \). Set \( \psi_s = \varphi_s \varphi_K \) for \( s \in [0,1] \). Then we have the following disjoint union decomposition

\[
\text{Fix}(\psi_s) = \text{Fix}(\varphi) \cup Z_s, \text{ where } Z_s = \{x \in U_\varepsilon | \varphi^{-1}(x) = \varphi_{sK}(x)\};
\]

as is easy to see recalling that \( \text{Fix}(\varphi) \cap U_\varepsilon = \emptyset \).

Since \( H \geq 0 \), the Hamiltonian generating the map \( \varphi^{-1} \) is non-positive and hence \( \sigma(\varphi^{-1}) = 0 \). Then, by conjugation invariance and sub-additivity of the action selector \( \sigma \) (see [FS] and also [Se2, Vi2]), we have

\[
\sigma(\varphi_{sK}) = \sigma(\varphi_s \varphi_K \varphi^{-1}) \leq \sigma(\varphi_s \varphi_K) + \sigma(\varphi^{-1}) = \sigma(\varphi_s \varphi_K) = \sigma(\psi_s).
\]
From the monotonicity property, (S0), of \( \sigma \) we infer that \( c_{\text{hom}}(U) = \sup \sigma(\varphi_K) \), where the supremum is taken over all \( K \) as above. Pick \( K \) such that \( \sigma(\varphi_K) \) is close to \( c_{\text{hom}}(U) \):

\[
E^+(H) < \sigma(\varphi_K) \leq c_{\text{hom}}(U).
\]

(7.3)

As \( s \) varies through the interval \([0, 1]\), the action selector \( \sigma(\psi_s) \) changes from

\[
\sigma(\psi_0) = \sigma(\varphi) \leq E^+(H)
\]

to

\[
\sigma(\psi_1) \geq \sigma(\varphi_K) > E^+(H),
\]

where the first inequality follows form (7.2) and the second one from (7.3). By continuity of the action selector, we see that \( \sigma(\psi_s) \) cannot be independent of \( s \).

Furthermore, \( \psi_0 = \varphi \), and thus \( \text{Fix}(\psi_0) = \text{Fix}(\varphi) \) and \( Z_0 = \emptyset \). As a consequence, the part \( Z_s \) of \( \text{Fix}(\psi_s) \) must be non-empty for some \( s_0 \in (0, 1] \) and, moreover, \( \sigma(\psi_{s_0}) \) is the action value of \( \psi_{s_0} \) on \( x \in Z_{s_0} \).

As in the proof of Proposition 4.1, we may reparametrize the Hamiltonians \( H \) and \( K \) (making \( K \) now time-dependent), without altering the time-one maps, the action spectra, the Hofer norms, and the action selectors so that \( K \) is independent of \( t \) in \([1/2, 1]\), and \( H_t \equiv 0 \) when \( t \in [0, 1/2] \). From now on, we assume that \( H \) and \( K \) have this property. We will also denote \( \varphi^t_H \) by \( \varphi^t \).

Consider the orbit \( \gamma(t) \) through \( x = \gamma(0) \) of the time-dependent flow \( \psi^t_{s_0} = \varphi^t \varphi^t_{s_0 K} \). Let \( G = H#(s_0 K) \) be the Hamiltonian generating this flow. Due to the above reparametrizations of \( H \) and \( K \), the orbit \( \gamma \) is comprised of two parts: \( \gamma_1(t) = \varphi^t_{s_0 K}(x) \) ending at \( y = \varphi_{s_0 K}(x) \) and \( \gamma_2(t) = \varphi^t(y) \) ending at \( x \). Note that \( x \) and \( y \) lie on the same Reeb orbit (i.e., a leaf of characteristic foliation) on some level \( M_t \) and \( \varphi(y) = x \). Furthermore,

\[
\sigma(\psi_{s_0}) = A_G(\gamma) = -\int_{\gamma_1} \lambda + \int_0^{1/2} s_0 K_t(\gamma_1(t)) \ dt + A_H(\gamma_2), \tag{7.4}
\]

where

\[
A_H(\gamma_2) = -\int_{\gamma_2} \lambda + \int_{1/2}^1 H_t(\gamma_2(t)) \ dt.
\]

The term \( T = -\int_{\gamma_1} \lambda \) is the time required for the Reeb flow on \( M_t \) to move \( x \) to \( y \) and our next goal is to establish an upper bound on \( T \) independent of \( K \).

Consider the function \( f(z) \) equal to the action of \( H \) on the orbit \( \varphi^t(z), \ t \in [0, 1] \), defined using the primitive \( \lambda \). (For instance, \( f(y) = A_H(\gamma_2) \).) This function is independent of \( K \). Clearly \( f \) is a compactly supported function and \( C = -\min f \) is also independent of \( K \) and \( A_H(\gamma_2) \geq -C \). Furthermore, the middle term in (7.4) is non-negative, for \( K \geq 0 \). As a consequence,

\[
\sigma(\psi_{s_0}) \geq T - C.
\]

Finally note that the Hamiltonian \( G \) generating \( \psi_{s_0} \) is supported in \( U \cup \text{supp}(H) \). Thus \( \sigma(\psi_{s_0}) \leq c_{\text{hom}}(U \cup \text{supp}(H)) \). Therefore,

\[
T \leq c_{\text{hom}}(U \cup \text{supp}(H)) + C \leq c(U \cup \text{supp}(H)) + C < \infty,
\]

where the upper bounds on the right hand side are clearly independent of \( K \).

To finish the proof, consider a sequence of Hamiltonians \( K_i \) such as \( K \), non-constant on a more and more narrow range of \( \tau \) in \([-\epsilon, 0]\) eventually converging to zero. For each \( K_i \) we have a pair of points \( x_i \) and \( y_i \) lying on the same Reeb
orbit on some $M_i$, with $t_i \to 0$ and such that $\varphi(y_i) = x_i$. Furthermore, the Reeb flow requires time $T_i \leq c_{\text{hom}}(U \cup \text{supp}(H)) + C$ to move $x_i$ to $y_i$. Applying the Arzela–Ascoli theorem and passing if necessary to a subsequence, we obtain points $x = \lim x_i$ and $y = \lim y_i$ on $M$ lying on the same Reeb orbit and such that $\varphi(y) = x$. This completes the proof of the theorem.

**Remark 7.1.** The proof of Theorem 2.9 also yields the upper bound $c_{\text{hom}}(\bar{U} \cup \text{supp}(H)) + C$ for the “Reeb distance” from $x$ to $y$.

In the next example we show that Theorem 2.9, as stated with the upper bound on $\|H\|$, does not extend to hypersurfaces in $\mathbb{R}^{2n}$ that do not have contact type. To be more precise, we construct a Hamiltonian flow $\varphi^t$ on $\mathbb{R}^{2n}$ and a sequence of hypersurfaces $M_i$, $C^0$-converging to $S^{2n-1}$, such that $M_i$ and $\varphi^t(M_i)$ have no leaf-wise intersections for some sequence of times $t_i \to 0+$.

**Example 7.2.** Let $S^{2n-1}$ be the unit sphere in $\mathbb{R}^{2n}$ and let $\varphi^t$ be the Hamiltonian flow of $H = f \cdot p_1$, where $(p_1, q_1, \ldots, p_n, q_n)$ are the standard coordinates on $\mathbb{R}^{2n}$ and $f$ is a cut-off function equal to one near $S^{2n-1}$. For $t > 0$ small, the only leaf-wise intersections of $S^{2n-1}$ and $\varphi^t(S^{2n-1})$ are two points $x_1$ and $x_2$ on the unit circle $S$ in the $(p_1, q_1)$-plane. (The points $\varphi^t(x_1)$ and $\varphi^t(x_2)$ are the intersections of $S$ and the transported circle $S + (t, 0)$.) Let us now insert two symplectic plugs into $S^{2n-1}$ centered at points on $S$ between $x_1$ and $\varphi^t(x_1)$ and between $x_2$ and $\varphi^t(x_2)$; see, e.g., [Ci, Gi1, Gi2, GG1, Ke1]. We choose the plugs so small and center them in such a way that they are displaced by $\varphi^t$. As a result, we obtain a new hypersurface $M$ that is $C^0$-close to $S^{2n-1}$, differs from $S^{2n-1}$ only within the plugs, and such that the leaf $S$ is broken into two leafs: one containing $x_1$ and $x_2$ and the other one containing $\varphi^t(x_1)$ and $\varphi^t(x_2)$. We claim that $M$ and $\varphi^t(M)$ have no leaf-wise intersections. Indeed, $x_1$ and $x_2$ are no longer leaf-wise intersections for $M$ and $\varphi^t(M)$, and since the plugs are displaced and due to the plug-symmetry conditions, no new leaf-wise intersections are created.

Applying this construction to a sequence $t_i \to 0+$, we obtain a sequence of perturbations $M_i$ of $S^{2n-1}$, $C^0$-converging to $S^{2n-1}$, and such that $M_i$ and $\varphi^t(M_i)$ have no leaf-wise intersections. Note that $\varphi^t M_i \to id$ while $c_{\text{hom}}(U_i) \to c_{\text{hom}}(U) = \pi$, where $U_i$ is the domain bounded by $M_i$ and $U$ is the unit ball. It is also clear that $\|H_i\| \to 0$, where $H_i = t_i H$ is a Hamiltonian generating $\varphi^t$.

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