Representations of the braid group $B_n$ and the highest weight modules of $U(\mathfrak{sl}_{n-1})$ and $U_q(\mathfrak{sl}_{n-1})$

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Abstract

In [1] we have constructed a $\left[\frac{n+1}{2}\right] + 1$ parameters family of irreducible representations of the Braid group $B_3$ in arbitrary dimension $n \in \mathbb{N}$, using a $q$–deformation of the Pascal triangle. This construction extends in particular results by S.P. Humphries (2000), who constructed representations of the braid group $B_3$ in arbitrary dimension using the classical Pascal triangle. E. Ferrand (2000) obtained an equivalent representation of $B_3$ by considering two special operators in the space $\mathbb{C}^n[X]$. Slightly more general representations were given by I. Tuba and H. Wenzl (2001). They involve $\left[\frac{n+1}{2}\right]$ parameters (and also use the classical Pascal’s triangle). The latter authors also gave the complete classification of all simple representations of $B_3$ for dimension $n \leq 5$. Our construction generalize all mentioned results and throws a new light on some of them. We also study the irreducibility and equivalence of the constructed representations.

In the present article we show that all representations constructed in [1] may be obtained by taking exponent of the highest weight modules of $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$. We generalize these connections between the representation of the braid group $B_n$ and the highest weight modules of the $U_q(\mathfrak{sl}_{n-1})$ for arbitrary $n$ using the well-known reduced Burau representations.

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1 Introduction. Braid group representations

Our aim is to describe the dual $\hat{B}_n$ of the braid group $B_n$. It is natural to compare the representation theory of the symmetric group $S_n$ and of the braid group $B_n$. We know almost everything about representation theory of the symmetric group $S_n$. We know the description of the dual $\hat{S}_n$ in terms of Young diagrams. We know even the Plancherel measure on $\hat{S}_n$. The Young graph explains how to decompose the restriction $\pi|_{S_{n-1}}$ of the representation $\pi \in \hat{S}_n$, etc.

The braid groups $B_n$ are defined by the generators $\sigma_i$, $1 \leq i \leq n - 1$ and by the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$. The dual $\hat{B}_n$ of the group $B_n$ is known only for the commutative case when $n = 2$. In this case $B_2 \cong \mathbb{Z}$ hence $\hat{B}_2 \cong S^1$. The representation theory for the braid groups $B_n$ is much more complicated than for $S_n$. The reason is the following. In the case of the group $S_n$ we have the essential (quadratic) relation $\sigma_i^2 = 1$, hence $Sp(\pi(\sigma_i)) \subseteq \{-1, 1\}$. In the case of the group $B_n$ we do not have these conditions. Since $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ we have $Sp(\pi(\sigma_i)) = Sp(\pi(\sigma_{i+1}))$, but the spectra $Sp(\pi(\sigma_i))$ may be almost arbitrary.

The Hecke algebra $H_n(q)$ see f.e.[15] appears as the factor algebra of the group algebra of the group $B_n$ subject to the following quadratic relation $\sigma_i^2 = (q-1)\sigma_i + q$, $1 \leq i \leq n - 1$, hence $Sp(\pi(\sigma_i)) \subseteq \{-1, q\}$ and $H_n(q) \cong \mathbb{C}[S_n]$. This is a reason why the representation theory of Hecke algebras is well developed.

The next step is to impose the polynomial condition $p_k(\sigma_i) = 0$ on the generators $\sigma_i$ where $k$ is the order of the polynomial $p_k(x)$. For $k = 3$ the corresponding algebra is called Birman–Murakami–Wenzl type algebra or simple BMW algebra see [26, 32] (see also [27]) and so on.

The situation becomes much more complicated if no additional conditions on the spectra are imposed. We shall study this general case for $B_3$.

In [29] I.Tuba and H.Wenzl gave the complete classification of all simple representations of $B_3$ for dimension $\leq 5$.

In [12] E.Formanek et al. gave the complete classification of all simple representations of $B_n$ for dimension $\leq n$.

We generalize the results I.Tuba and H.Wenzl for $B_3$, give new representations of $B_n$ for large dimension and establish connection between the representations of $B_n$ and the highest weight modules of the quantum group.
$U_q(sl_{n-1})$.

More precisely, in the work [1] with S.Albeverio we have constructed a $\left[\frac{n+1}{2}\right] + 1$ parameter family of irreducible representations of the braid group $B_3$ in arbitrary dimension $n \in \mathbb{N}$, using a $q-$deformation of the Pascal triangle. This construction extends in particular results by S.P. Humphries [14], I. Tuba and H. Wenzl [29], and E. Ferrand [11]. The irreducibility and the equivalence of the constructed representations is studied. For example the representations corresponding to different $q$ and $n$ are nonequivalent.

In this article we show that there is a striking connection between these representations of $B_3$ and a highest weight modules of the quantum group $U_q(sl_2)$, a one-parameter deformation of the universal enveloping algebra $U(sl_2)$ of the Lie algebra $sl_2$. The starting point for all these considerations is some homomorphism $\rho_3$ of the braid group $B_3$ into $SL(2,\mathbb{Z})$:

$$\rho_3 : B_3 \mapsto sl_2 \mapsto SL(2,\mathbb{Z})$$

$$\sigma_1 \mapsto (\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}) \mapsto (\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}), \ \sigma_2 \mapsto (\begin{array}{cc} 0 & 0 \\ -1 & 0 \end{array}) \mapsto (\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}) .$$

The constructed representations may be treated as the $q-$symmetric power of this fundamental representation or as an appropriate $q-$exponential of the highest weight modules of $U_q(sl_2)$.

We generalize these connections between the representation of the braid group $B_n$ and the highest weight modules of the quantum group $U_q(sl_{n-1})$ for arbitrary $n$ using the well-known reduced Burau representation $b^{(t)}_n$ see c.f. [15]. We note that in particular $\rho_3 = b^{(-1)}_3$.

Let $g$ be the Lie algebra defined by a Cartan matrix $A$ and let $B$ be the corresponding braid group. Denote by $U(g)$ the quantized enveloping algebra of $g$ over the field $\mathbb{C}(v)$, and let $V$ be the integrable $U(g)-$module. In [24] G. Lusztig defined a natural action of $B$ on $V$ which permutes the weight space of $V$ according to the action of the Weyl group on the weights. This rather general but different approach allows us also to construct the irreducible representations of the braid group $B$ (see [22]).
0. Definition of the Artin braid group $B_n$

$$B_n = \langle (\sigma_i)_{i=1}^{n-1}, \ | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ \sigma_i\sigma_j = \sigma_i\sigma_j, \ | \ i-j \ | \geq 2 \rangle.$$ 

$B_n = \pi_1(X)$ is the fundamental group $\pi_1$ of the configuration space $X = \{\mathbb{C}^n \setminus \Delta\}/S_n$ where $\Delta = \{(z_1, \ldots, z_n) \ | \ x_i = z_j \text{ for some } i \neq j\}$ and the group $S_n$ act freely on $\mathbb{C}^n \setminus \Delta$ by permuting coordinates.

A **Braid** on $n$ strings is a collection of curves in $\mathbb{R}^3$ joining $n$ points in a horizontal plane to the $n$ points directly below them on another horizontal plane. Operation: concatenation.

Knot theory : Alexander, Jones, HOMFLYPT, Kauffman polynomials.
Respectively: Temperley-Lieb, Hecke, BMW algebras.

Geometry, physics etc.

Relation with the symmetric group $S_n : \sigma_i^2 = 1$

$$\sigma_i^2 = 1 \Rightarrow Sp(\rho(\sigma_i)) \subseteq \{-1, 1\}$$

$$Rep(S_n) \quad Rep(B_n)？$$

$\hat{S}_n = \{\text{Young diagrams}\}, \quad$ Plancherel measure on $\hat{S}_n$.

The Young graph explains how to decompose the restriction $\rho |_{S_{n-1}}$ of the representation $\rho \in \hat{S}_n$, etc.

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \Rightarrow Sp(\rho(\sigma_i)) = Sp(\rho(\sigma_{i+1})).$$

The Hecke algebra is defined by

$$H_n(q) = \langle (\sigma_i)_{i=1}^{n-1} \ | \ ...\sigma_i^2 = (q-1)\sigma_i + q, \ p_2(\sigma_i) = 0 \rangle,$$

hence $Sp(\rho(\sigma_i)) \subseteq \{-1, q\}$ and $H_n(q) \cong \mathbb{C}[S_n].$
1. **Definition** \( B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \).

2. **Homomorphism** \( \rho : B_3 \mapsto \text{SL}(2, \mathbb{Z}) \),
   \[ \sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \sigma_2 = (\sigma_1^{-1})^2. \]

3. \( B_3/\mathbb{Z}(B_3) \simeq \text{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}_2 \ast \mathbb{Z}_3. \)

4. P. Humphries result, Pascal’s triangle
   \[ \sigma_1 \mapsto \sigma_1(1,n), \quad \sigma_2 \mapsto \sigma_2(1,n). \]

5. Ferrand result \( \Phi_n, \Psi_n \in \text{End} \mathbb{C}^n[X] \).

6. Tubo-Wenzl example
   \[ \sigma_1 \mapsto \sigma_1(1,n) \Lambda_n, \quad \sigma_2 \mapsto \Lambda_n^\sharp \sigma_2(1,n), \quad \Lambda_n \Lambda_n^\sharp = cI. \]

7. Tubo-Wenzl classifications of \( B_3 \) - mod, \( \dim V \leq 5 \).

8. Generalizations
   \[ \sigma_1 \mapsto \sigma_1^\Lambda(q,n) := \sigma_1(q,n) D_n(q)^\sharp \Lambda_n, \]
   \[ \sigma_2 \mapsto \sigma_2^\Lambda(q,n) := \Lambda_n^\sharp D_n(q) \sigma_2(q,n), \]
   where \( \sigma_2(q,n) = (\sigma_1^{-1}(q^{-1},n))^\sharp, \quad \Lambda_n = \text{diag}(\lambda_r)_{r=0}^n, \quad \Lambda_n \Lambda_n^\sharp = cI, \]
   \[ D_n(q) = \text{diag}(q_r)_{r=0}^n, \quad q_r = q^{r-1 \frac{r-1}{2}}, \quad r, n \in \mathbb{N}. \]

9. The connection between \( \text{Rep}(B_3) \) and \( U_q(\mathfrak{sl}_2)\text{-mod}. \)

10. The Burau representation \( \rho_n : B_n \mapsto \text{GL}_n(\mathbb{Z}[t, t^{-1}]) \).

11. Lawrence-Kramer representations

12. Generalization of 8 and 9 for \( B_n \).

13. Formanek classifications of \( B_n \) - mod, for \( \dim V \leq n \).
1. \( B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \).
2. \( \rho : B_3 \mapsto \text{SL}(2, \mathbb{Z}), \)

\[
\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.
\]

3. \( B_3/Z(B_3) \cong \text{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3. \)

Hint: the Pascal triangle, \( \sigma_1 \mapsto \sigma_2? (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})^{-1} = (\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}). \)

\[
\sigma_1(1, 2) := \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^{-1}(1, 2)^\sharp := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.
\]

Notations the central symmetry:

\[
A^\sharp := (A^t)^{\ast}, \quad A^\sharp = (a^\sharp_{ij}), \quad a^\sharp_{ij} = a_{n-i,n-j},
\]

\[
\sigma_1 \mapsto \sigma_1(1, 2), \quad \sigma_2 \mapsto \sigma_2(1, 2) := \sigma_1^{-1}(1, 2)^\sharp.
\]

4. P. Humphries, [14] representations of \( B_3 \) in \( \mathbb{C}^{n+1} \)

\[
\sigma_1 \mapsto \sigma_1(1, n), \quad \sigma_2 \mapsto \sigma_2(1, n) := \sigma_1^{-1}(1, n)^\sharp.
\]  

5. Ferrand result, [11]. \( \Phi_n, \Psi_n \in \text{End} \mathbb{C}^n[X] : \Phi_n \Psi_n \Phi_n = \Psi_n \Phi_n \Psi_n. \)

\[
(\Phi_n p)(X) := p(X + 1), \quad (\Psi_n p)(X) := (1 - X)^n p \left( X/(1 - X) \right).
\]

6. Tubo-Wenzl example [29]: representations \( \sigma^h(1, n) \) of \( B_3 \) in \( \mathbb{C}^{n+1} \)

\[
\sigma_1 \mapsto \sigma_1(1, n) \Lambda_n, \quad \sigma_2 \mapsto \Lambda_n^\sharp \sigma_2(1, n),
\]  

conditions on the complex diagonal matrix \( \Lambda_n = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_n) \) are the following:

\[
\Lambda_n \Lambda_n^\sharp = cI, \quad c \in \mathbb{C}.
\]
7. Tubo - Wenzl classifications of $B_3 - \text{mod}$, $\dim V \leq 5$.

See [29]. Let $V$ be a simple $B_3$ module of dimension $n = 2, 3$. Then there exist a basis for $V$ for which $\sigma_1$ and $\sigma_2$ act as follows $(\lambda = (\lambda_k)_k)$ for $n = 2$ and $n = 3$

$\sigma_1^\lambda := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \sigma_2^\lambda := \begin{pmatrix} -\lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$

$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 \lambda_2 \lambda_3 - \lambda_4 \lambda_2 & \lambda_2 \\ 0 & \lambda_2 \lambda_3 - \lambda_4 \lambda_2 \end{pmatrix}, \quad \sigma_2 \mapsto \sigma_2^\lambda = \begin{pmatrix} -\lambda_2 & 0 \\ \lambda_2 \lambda_3 - \lambda_4 & 0 \\ 0 & \lambda_2 \lambda_3 - \lambda_4 \lambda_2 \end{pmatrix}. \quad (5)$

Let us set $D = \sqrt{\lambda_2 \lambda_3 / \lambda_1 \lambda_4}$. All simple modules for $n = 4$ are the following:

$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \frac{\lambda_1 (1+D^{-1}+D^{-2}) \lambda_2 (1+D^{-1}+D^{-2}) \lambda_3 \lambda_4}{D} \\ 0 & \lambda_2 \\ 0 & 0 & (1+D^{-1}) \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad (6)$

$\sigma_2 \mapsto \sigma_2^\lambda = \begin{pmatrix} \frac{\lambda_4}{D} & 0 & 0 \\ -\lambda_3 & \frac{\lambda_3}{D} & 0 \\ -D \lambda_2 & -(D+1) \lambda_2 & \lambda_2 \\ -(D^2+D+1) \lambda_1 & -(D^2+D+1) \lambda_1 & \lambda_1 \end{pmatrix}. \quad (7)$

Let us set $\gamma = (\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5)^{1/5}$. All simple modules for $n = 5$ are the following:

$\sigma_1 \mapsto \sigma_1^\lambda = \begin{pmatrix} \lambda_1 (1+\frac{\gamma^2}{\lambda_2 \lambda_4}) (\lambda_2 + \frac{\gamma^3}{\lambda_3 \lambda_4}) (\lambda_3 + \frac{\gamma^4}{\lambda_4 \lambda_5}) (1+\frac{\lambda_3 \lambda_4}{\lambda_2 \lambda_4}) (1+\frac{\lambda_3 \lambda_4}{\lambda_2 \lambda_4}) (1+\frac{\lambda_3 \lambda_4}{\lambda_2 \lambda_4}) (1+\frac{\lambda_3 \lambda_4}{\lambda_2 \lambda_4}) (1+\frac{\lambda_3 \lambda_4}{\lambda_2 \lambda_4}) (1+\frac{\lambda_3 \lambda_4}{\lambda_2 \lambda_4}) \\ 0 & \lambda_2 \\ \frac{\gamma^2}{\lambda_3 \lambda_4} + \lambda_3 + \gamma \\ \frac{\gamma^3}{\lambda_1 \lambda_4} + \lambda_3 + \gamma \\ \frac{\gamma^4}{\lambda_1 \lambda_5} + \lambda_3 + \gamma \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}. \quad (8)$

The formula for $\sigma_2^\lambda$ was not given in [29].
8. Equivalence of Tuba-Wenzl’s representations in the case \( \dim \leq 5 \) and our representations.

General formulas for \( 1 \leq n \leq 4 \) gives us (we set \( q_r = \frac{(r-1)r}{2} \)):

\[
\begin{align*}
\sigma_1 & \mapsto \sigma_1^\Lambda := \sigma_1(q, n) \Lambda_n, \quad \sigma_2 \mapsto \sigma_2^\Lambda := \Lambda_n^\sharp \sigma_2(q, n), \\
\Lambda_n \Lambda_n^\sharp &= \lambda_0 \lambda_n \Lambda_n(q), \quad \Lambda_n(q) = q_n^{-1} D_n(q) D_n^\sharp(q), \quad D_n(q) = \text{diag}(q_r)_{r=0}^n, \\
\lambda_r \lambda_{n-r} &= \lambda_0 \lambda_n q^{-(n-r)r}, \quad 0 \leq r \leq n. 
\end{align*}
\] (9)

Let \( n = 1 \) we have

\[
\sigma_1^\Lambda = \begin{pmatrix} 1 & 1 \end{pmatrix} \Lambda_1, \quad \sigma_2^\Lambda = \Lambda_1^\sharp \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}.
\]

Let \( n = 2 \), conditions (9) gives us \( \Lambda_2 = \text{diag}(\lambda_r)_{r=0}^3 \)

\[
\text{diag}(\lambda_0 \lambda_2, \lambda_2^2, \lambda_0 \lambda_2) = \lambda_0 \lambda_2 \text{diag}(1, q^{-1}, 1), \quad \text{so} \quad q^{-1} = \lambda_2^2 / \lambda_0 \lambda_2.
\]

\[
\sigma_1^\Lambda(q, 2) = \begin{pmatrix} 1 & 1 & q^1 \\ 0 & 1 & 0 \end{pmatrix} \Lambda_2, \quad \sigma_2^\Lambda(q, 2) = \Lambda_2^\sharp \begin{pmatrix} 1 & 0 & 0 \\ -1 & q^{-1} - (1+q^{-1}) & 1 \end{pmatrix}.
\]

For \( n = 3 \) conditions (9) gives us \( q^{-2} = \lambda_1 \lambda_2 / \lambda_0 \lambda_3 \) for \( r = 1 \).

\[
\sigma_1(q, 3) = \begin{pmatrix} 1 & 1+q+q^2 & 1+q+q^2 \\ 0 & 1 & 0 \\ 0 & 1+q & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},
\]

\[
\sigma_2(q, 3) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & q^{-1} - (1+q^{-1}) & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

For \( n = 4 \) conditions (9) gives us \( q^{-3} = \lambda_1 \lambda_3 / \lambda_0 \lambda_4 \) for \( r = 1 \) and \( q^{-4} = \lambda_2^2 / \lambda_0 \lambda_4 \) for \( r = 2 \).

\[
\sigma_1(q) = \begin{pmatrix} 1 & (1+q)(1+q^2) & (1+q^2)(1+q+q^2) & (1+q)(1+q^2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+q+q^2 & 1+q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},
\]

\[
\sigma_2(q, 4) = (\sigma_1^{-1}(q^{-1}, 4))^\sharp.
\]
\[ \sigma_1 \mapsto \sigma_1(1, n)\Lambda_n, \quad \sigma_2 \mapsto \Lambda_n^z \sigma_2(1, n), \quad (2) \]
\[ \Lambda_n = \text{diag}(\lambda_r)_{r=0}^n, \quad \Lambda \Lambda^z = cI, \quad c \in \mathbb{C}, \quad (3) \]

8. Generalization of (2) for \( q \neq 1 \), with the condition (3)
\[ \sigma_1 \mapsto \sigma_1^A(q, n) := \sigma_1(q, n)D_n^z(q)\Lambda_n, \quad \sigma_2 \mapsto \sigma_2^A(q, n) := \Lambda_n^z D_n(q)\sigma_2(q, n), \quad (10) \]
\[ \sigma_2(q, n) := \sigma_1^{-1}(q^{-1}, n)^2, \quad D_n(q) = \text{diag}(q_r)_{r=0}^n, \quad q_r = \frac{q^{r+1}-1}{q-1}, \quad (11) \]
where \( q \)— binomial coefficients or Gaussian polynomials are defined as follows
\[ \binom{n}{k}_q := \frac{(n)_q}{(k)_q(n-k)_q}, \quad [n]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad (12) \]

Corresponding to two forms of \( q \)–natural numbers, defined by
\[ (n)_q := \frac{q^n - 1}{q - 1}, \quad [n]_q := \frac{q^n - q^{-1}}{q - q^{-1}}, \quad (13) \]

Theorem 1 [1] The formulas (10) \( \sigma_1 \mapsto \sigma_1^A(q, n), \quad \sigma_2 \mapsto \sigma_2^A(q, n) \) give the representation of \( B_3 \).

Theorem 2 [1] The representation \( \sigma^A(q, n) \) defined by (10) generalize the Tubo-Wenzl representations for arbitrary \( n \in \mathbb{N} \).

Definition. We say that the representation is subspace irreducible or irreducible (resp. operator irreducible) when there no nontrivial invariant close subspaces for all operators of the representation (resp. there no nontrivial bounded operators commuting with all operators of the representation).

Let us define for \( n, r, q, \lambda \) such that \( n \in \mathbb{N}, \ 0 \leq r \leq n, \ \lambda \in \mathbb{C}^{n+1}, \ q \in \mathbb{C} \) the following operators
\[ F_{r,n}(q, \lambda) = \text{exp}(q) \left( \sum_{k=0}^{n-1} (k+1)_q E_{kk+1} \right) - q_{n-r} \lambda_r (D_n(q)\Lambda_n^z)^{-1}, \quad (14) \]
where \( \text{exp}(X) = \sum_{m=0}^{\infty} X^m / (m)!_q \). For the matrix \( C \in \text{Mat}(n+1, \mathbb{C}) \) we denote by
\[ M_{i_1i_2...i_r}(C), \ (\text{resp. } A_{j_1j_2...j_r}(C)), \quad 0 \leq i_1 < ... < i_r \leq n, \ 0 \leq j_1 < ... < j_r \leq n \]
its minors (resp. the cofactors) with \( i_1, i_2, ..., i_r \) rows and \( j_1, j_2, ..., j_r \) columns.
Theorem 3 [1] The representation of the group $B_3$ defined by (10) have the following properties:
1) for $q = 1$, $\Lambda_n = 1$, it is subspace irreducible in arbitrary dimension $n \in \mathbb{N}$;
2) for $q \neq 1$, $\Lambda_n = \text{diag}(\lambda_k)_{k=0}^n \neq 1$ it is operator irreducible if and only if for any $0 \leq r \leq \left[\frac{n}{2}\right]$ there exists $0 \leq i_0 < i_1 < \ldots < i_r \leq n$ such that

$$M_{r+1r+2..n}^{i_0\ldots i_r}(F_{r,n}(q, \lambda)) \neq 0;$$

(15)

3) for $q \neq 1$, $\Lambda_n = 1$ it is subspace irreducible if and only if $(n)_q \neq 0$.

The representation has $\left[\frac{n+1}{2}\right] + 1$ free parameters.

9. The connection between $\text{Rep}(B_3)$ and $U_q(\mathfrak{sl}_2)$-mod.

The algebra $U(\mathfrak{sl}_2)$ is the associative algebra generated by three generators $X, Y, H$ with the relations (7).

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,$$

(16)

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ in } \mathfrak{sl}_2.$$

$U_q(\mathfrak{sl}_2)$ is the algebra generated by four variables $E, F, K, K^{-1}$ with the relations

$$KK^{-1} = K^{-1}K = 1,$$

(17)

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$

(18)

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} = \frac{q^H - q^{-H}}{q - q^{-1}}.$$  

(19)

Comultiplication $\Delta$, counit $\varepsilon$ and antipod $S$ are as follows:

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K,$$

$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF,$$

$$\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.$$

All finite-dimensional $U-$module $V$ being the highest weight module of highest weight $\lambda$ are of the following form (see Kassel, [17, Theorem V.4.4.])

$$\rho(n)(X) = \begin{pmatrix} 0 & n & 0 & \ldots & 0 \\ 0 & 0 & n-1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \rho(n)(Y) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix},$$

10
\[ \rho(n)(H) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -n+2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}. \]

where \( \lambda = \dim(V) - 1 \in \mathbb{N} \).

All finite-dimensional \( U_q \)-module \( V \) being the highest weight module of highest weight \( \lambda \) are of the following form (see Kassel, [17, Theorem VI.3.5.])

\[ \rho_{\varepsilon,n}(E) = \varepsilon \begin{pmatrix} n \ 0 \\ 0 \ 0 \ \vdots \ 0 \\ 0 \ 0 \ \vdots \ 1 \end{pmatrix}, \]

\[ \rho_{\varepsilon,n}(F) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q^n \end{pmatrix}, \]

\[ \rho_{\varepsilon,n}(K) = \varepsilon \begin{pmatrix} q^n & 0 & \cdots & 0 & 0 \\ 0 & q^n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q-n+2 & 0 \end{pmatrix}, \]

where \( \varepsilon = \pm 1, \lambda = \varepsilon q^n \) and \( n \in \mathbb{N} \).

The main observation is the following:

\[ \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & (2)_q \ 0 \\ 0 & 1 \ \vdots \ 0 \\ 0 & 0 \ \vdots \ 1 \end{pmatrix} = \exp(q) \begin{pmatrix} 0 & (2)_q & 0 \\ 0 & 0 & (1)_q \\ 0 & 0 & 0 \end{pmatrix}, \]

where

\[ \begin{pmatrix} 0 & (2)_{q^2} & 0 \\ 0 & 0 & (1)_{q^2} \end{pmatrix} = \begin{pmatrix} 0 & [2]_q & 0 \\ 0 & 0 & [1]_q \end{pmatrix} \begin{pmatrix} q^2 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp(q^2)X := \sum_{m=0}^{\infty} \frac{1}{(m)!q^m} X^m. \]

**Theorem 4** For \( q = 1 \) holds

\[ \sigma_1(1, n) = \exp(\rho(n)(X)), \quad \sigma_2(1, n) = \exp(\rho(n)(-Y)). \] \hspace{1cm} (20)

**Theorem 5** For \( q \neq 1 \) we have

\[ \sigma_1(q^2, n)D_n(q^2) = \exp(q^2) \left( q^{n/2} \rho_{1,n}(EK^{1/2}) \right) D_n(q^2), \] \hspace{1cm} (21)

\[ D_n(q^2)\sigma_2(q^2, n) = \exp(q^2) \left( -q^{n/2} \rho_{1,n}(FK^{-1/2}) \right) D_n(q^2). \] \hspace{1cm} (22)

**Proof.** The two forms of \( q \)-natural numbers are connected as follows (see Kassel, [17])

\[ [n] = q^{-(n-1)}(n)_{q^2}, \quad [n]! = q^{-(n-1)n/2}(n)!_{q^2} \] \hspace{1cm} (23)
\[
\begin{pmatrix}
0 & (n) & 0 & \cdots & 0 \\
0 & 0 & (n-1) & \cdots & 0 \\
0 & 0 & 0 & \cdots & (1) \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
0 & [n] & 0 & \cdots & 0 \\
0 & 0 & [n-1] & \cdots & 0 \\
0 & 0 & 0 & \cdots & [1] \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} \operatorname{diag}(q^n, q^{n-1}, \ldots, 1)
\]
\[= q^{n/2} \rho_{1,n}(EK^{1/2}), \text{ and}
\]
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & (1) & 0 & \cdots & 0 \\
0 & (2) & \cdots & 0 & 0 \\
0 & \cdots & (n) & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0 \\
2 & 0 & \cdots & 0 & 0 \\
0 & \cdots & [n] & 0
\end{pmatrix} \operatorname{diag}(1, q, \ldots, q^{n-1}, q^n)
\]
\[= q^{n/2} \rho_{1,n}(FK^{-1/2}), \text{ since}
\]
\[
\operatorname{diag}(1, q, \ldots, q^{n-1}, q^n) = q^{n/2} \rho_{1,n}(K^{1/2})
\]
and
\[
\operatorname{diag}(q^n, q^{n-1}, \ldots, 1) = q^{n/2} \rho_{1,n}(K^{-1/2}).
\]

At last we conclude that
\[
\begin{pmatrix}
0 & (n) & 0 & \cdots & 0 \\
0 & 0 & (n-1) & \cdots & 0 \\
0 & 0 & 0 & \cdots & (1) \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
= q^{n/2} \rho_{1,n}(EK^{1/2}),
\]
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & (1) & 0 & \cdots & 0 \\
0 & (2) & \cdots & 0 & 0 \\
0 & \cdots & (n) & 0
\end{pmatrix}
= q^{n/2} \rho_{1,n}(FK^{-1/2}).
\]

Further we observe that
\[
X \otimes I + I \otimes X \mid_{S^2(C^2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I + I \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mid_{S^2(C^2)} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}
\]
\[
\Delta \rho(1)(X) \mid_{S^2(C^2)} = \rho(2)(X),
\]
\[
(I + X) \otimes (I + X) = \exp(\Delta(X)), \quad \sigma_1(1, 1) \otimes \sigma_1(1, 1) \mid_{S^2(C^2)} = \sigma(1, 2).
\]

**Lemma 6** We have for \( q \neq 1 \)
\[
\rho_{1,n} = \Delta^{n-1} \rho_{1,1} \mid_{S^n(q)(C^2)}, \tag{24}
\]
where \( S^n(q)(C^2) \) is \( q \)-symmetric tensor power of \( C^2 \).
Proof. For $n = 1$ we have the following operators

$$\rho_{1,1}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho_{1,1}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho_{1,1}(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = q^H.$$

For $n = 2$ we get

$$\rho_{1,2}(E) = \begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_{1,2}(F) = \begin{pmatrix} 0 & 0 & 0 \\ [1] & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}, \quad \rho_{1,2}(K) = \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}.$$

We have $\Delta(\rho_{1,1}(E)) =$

$$\rho_{1,1}(E) \otimes \rho_{1,1}(K) + 1 \otimes \rho_{1,1}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}.$$

Further $\Delta(\rho_{1,1}(F)) =$

$$\rho_{1,1}(F) \otimes 1 + \rho_{1,1}(K^{-1}) \otimes \rho_{1,1}(F) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} q^{-1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \ end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

and

$$\Delta(\rho_{1,1}(K)) = \rho_{1,1}(K) \otimes \rho_{1,1}(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}.$$

In the $q$–symmetric basis of the submodule $S^2(q)(\mathbb{C}^2)$ of the module $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$e^{s,q}_{00} = e_0 \otimes e_0, \quad e^{s,q}_{01} = q^{-1} e_0 \otimes e_1 + e_1 \otimes e_0, \quad e^{s,q}_{11} = e_1 \otimes e_1$$

the operator $\Delta(\rho_{1,1}(E))$ has the following form:

$$\Delta(\rho_{1,1}(E)) \mid_{S^2(q)(\mathbb{C}^2)} = \begin{pmatrix} 0 & [2] \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The basis in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is generated by vectors $e_{kn}$, $0 \leq k, n \leq 1$ where $e_{kn} = e_k \otimes e_n$. Operator $\Delta(\rho_{1,1}(E))$ acts as follows $e_{00} \mapsto 0$, $e_{01} \mapsto e_{00}$, $e_{10} \mapsto qe_{00}$, $e_{11} \mapsto q^{-1}e_{01} + e_{10}$, hence $e^{s,q}_{00} \mapsto 0$,

$$e^{s,q}_{01} = q^{-1}e_{01} + e_{10} \mapsto (q + q^{-1})e_{00} = [2]e^{s,q}_{00}, \quad e^{s,q}_{11} \mapsto q^{-1}e_{01} + e_{10} = e^{s,q}_{01}.$$
Similarly we get

$$\Delta(\rho_{1,1}(F))|_{S^2,q(C)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \Delta(\rho_{1,1}(K))|_{S^2,q(C)} = \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}. $$

hence (24) holds for \(n = 2\). For \(n > 2\) the proof is similar.

10. The Burau representation \(\rho : B_n \hookrightarrow GL_n(\mathbb{Z}[t, t^{-1}])\) is defined for a non-zero complex number \(t\) by

$$\sigma_i \mapsto \beta_i = I_{i-1} \oplus (1-t \ 1) \oplus I_{n-i-1}$$

where \(1-t\) is the \((i, i)\) entry. Representation \(\rho\) splits into 1-dimesional and \(n-1\)-dimensional irreducible representations, known as reduced Burau representation \(\overline{\rho} : B_n \hookrightarrow GL_{n-1}(\mathbb{Z}[t, t^{-1}])\)

$$\sigma_1 \mapsto b_1 = \begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix} \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto b_{n-1} = I_{n-3} \oplus \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix},$$

$$\sigma_i \mapsto b_i = I_{i-2} \oplus \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \oplus I_{n-i-2}, \quad 2 \leq i \leq n-2.$$

**Problem.** Whether the reduced Burau representation \(\overline{\rho} : B_n \hookrightarrow GL_{n-1}(\mathbb{Z}[t, t^{-1}])\) is faithful?

YES for \(n = 3\) (Birman [8]). NO for \(n \geq 9\) Moody [25] Long and Paton [23], Bigelow [6] improved further for \(n \geq 5\).

**Open problem:** Whether the reduced Burau representation of \(B_4 \hookrightarrow GL_3(\mathbb{Z}[t, t^{-1}])\)

$$b_1 = \begin{pmatrix} -t & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}$$

is faithful

11. Lawrence-Kramer representations, [20]

$$\lambda : B_n \hookrightarrow GL_m(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]), \quad m = n(n-1)/2.$$ The basis in the space \(\mathbb{C}^{n(n-1)/2}\) is \(x_{ik}, \ 1 \leq i < k \leq n.\)
Faithfulness for all \( n \), Bigelow [7], Kramer [21] \( \Rightarrow \) \( B_n \) is a linear group for all \( n \).

\[
\sigma_k x_{k,k+1} = t q^2 x_{k,k+1}
\]

\[
\sigma_k x_{ik} = (1 - q)x_{ik} + qx_{i,k+1}
\]

for \( i < k \)

\[
\sigma_k x_{i,k+1} = x_{ik} + t q^{k-i+1}(q - 1)x_{k,k+1}
\]

for \( i < k \)

\[
\sigma_k x_{kj} = t q(q - 1)x_{k,k+1} + qx_{k+1,j}
\]

for \( k + 1 < j \)

\[
\sigma_k x_{k+1,j} = x_{kj} + (1 - q)x_{k+1,j}
\]

for \( k + 1 < j \)

\[
\sigma_k x_{ij} = x_{ij} + t q^{k-i}(q - 1)^2 x_{k,k+1}
\]

for \( i < k < k \)

1. 12. Generalization of 8 and 9 for \( B_n \). For \( n = 4 \) and \( t = -1 \) we have \( \overline{\rho}_4 : B_4 \mapsto \text{SL}(3, \mathbb{Z}) \)

\[
b_1 = \left( \begin{array}{ccc}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \quad b_2 = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right), \quad b_3 = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right).
\]

\[
b_1 = \exp(-F_1), \quad b_2 = \exp(E_1 - F_2), \quad b_3 = \exp(E_2).
\]

We can show that the symmetric powers \( b_i \otimes b_i \mid S \) are the following

\[
b_1 \otimes b_1 \mid S = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right), \quad b_2 \otimes b_2 \mid S = \left( \begin{array}{cccc}
1 & 2 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right),
\]

\[
b_3 \otimes b_3 \mid S = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right).
\]

We have for \( n = 5 \) and \( t = -1 \) \( b^{(5)} : B_5 \mapsto \text{SL}(4, \mathbb{Z}) \)

\[
b_1 = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right), \quad b_2 = \left( \begin{array}{cccc}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right), \quad b_3 = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right), \quad b_4 = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right).
\]

Let \( \overline{\rho} : B_n \mapsto \text{SL}_{n-1}(\mathbb{Z}) \) be the reduced Burrau representation for \( t = -1 \).

The quantum group \( U_q(\mathfrak{sl}_{n-1}) \) is the algebra generated by \( 4(n-1) \) variables \( E_i, F_i, K_i, K_i^{-1} \) with relations as (17)–(19). Let

\[
\rho_m : U_q(\mathfrak{sl}_{n-1}) \mapsto \text{End}(\mathbb{C}^m)
\]
be the highest weight $U_q(\mathfrak{sl}_{n-1})$–module. Then

$$\sigma_1 \mapsto \exp(-\rho_m(F_1)), \quad \sigma_k \mapsto \exp(\rho_m(E_{k-1} - F_k)), \quad \sigma_n \mapsto \exp(\rho_m(E_{n-1})).$$

gives the representation of $B_n$ for $q = 1$ (see (20)).

For $q \neq 1$ we can obtain formulas similar to (21)–(22).

13. Formanek classifications of $B_n \mod \ dim V \leq n$.

In [12] E.Formanek et al. gave the complete classification of all simple representations of $B_n$ for dimension $\leq n$.

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