On the Fairness of Rate Allocation in Gaussian Multiple Access Channel and Broadcast Channel

Kenneth W. Shum and Chi Wan Sung
Department of Electronic Engineering
City University of Hong Kong
Email: kshum2006@gmail.com, itcw sung@cityu.edu.hk

Draft, Nov 3, 2006

Abstract

The capacity region of a channel consists of all achievable rate vectors. Picking a particular point in the capacity region is synonymous with rate allocation. The issue of fairness in rate allocation is addressed in this paper. We review several notions of fairness, including max-min fairness, proportional fairness and Nash bargaining solution. Their efficiencies for general multiuser channels are discussed. We apply these ideas to the Gaussian multiple access channel (MAC) and the Gaussian broadcast channel (BC). We show that in the Gaussian MAC, max-min fairness and proportional fairness coincide. For both Gaussian MAC and BC, we devise efficient algorithms that locate the fair point in the capacity region. Some elementary properties of fair rate allocations are proved.

1 Introduction

There are several fairness criteria in rate allocation. The simplest one is to mandate that all users have the same data rate and maximize this common data rate within the capacity region. This allocation is equitable and maximize the data rate of the worst user. However, it often does not effectively utilize system resources.

Another criterion is called max-min fairness. It relaxes equity and allows increasing the rates of some users without lowering the minimum data rate in the system. Under such an allocation policy, nobody can be benefited by worsening anybody who has lower data rate. It can be rephrased as follows. If we take some resources from a wealthy user to a relatively poor user (without reversing the order of wealthiness), the resulting allocation is considered fairer. Such operation is called a Robinhood operation. We say that an allocation is max-min fair if no Robinhood operation is possible without violating feasibility.

The max-min fairness is considered quite stringent. Any decrease of rate of a user with low data rate cannot be compensated by increasing the rate of any user with higher rate, no matter how small the decrease is. Kelly considers logarithmic utility function and proposes proportional fairness for network flow control [1]. Roughly speaking, we say that a rate allocation is proportionally fair if any adjustment will decrease the sum of percentage change over all users. A framework of optimization is thereby introduced using Lagrangian technique. Both max-min and proportional fairness are popular criteria for flow control in unicast and multicast networks [2, 3].

Another classical notion of fairness is the Nash bargaining solution coined by Nash in the 50’s [4]. In fact, proportional fairness is its special case. The Nash bargaining solution is a standard tool in cooperative game theory, and is applied widely in network resource allocation. For example, see [5] for application to orthogonal frequency division multiple-access networks.
Yet there is another fairness criterion based on the theory of majorization. We will give formal definitions of all the above criteria in the next section.

In this paper, the problem of picking a point in the capacity region of Gaussian MAC and BC according to some fairness criteria is considered. Some works have been done on Gaussian MAC. The sum rate at the max-min fair allocation is characterized in [6]. The polymatroid structure of the capacity region of Gaussian MAC is exploited in [7, 8], and algorithms for locating the max-min fair point are found. Implementation issue is addressed in [9].

The organization of this paper is as follows. In Section 2, we give precise definitions of fairness, and review the theory of majorization and Schur-convexity. The results are summarized in Section 3, and the details for the Gaussian MAC and BC are in Section 4 and 5 respectively. The appendix contains some proofs of the theorems in Section 4.

2 Fairness, Majorization and Schur-convexity

The set of all users is denoted by $\Omega = \{1, \ldots, K\}$.

In this section, we review several fairness criteria, and the theory of majorization. We will use the symbol $\mathcal{R}$ to represent capacity region, which is assumed to be a closed and convex set throughout this section.

An allocation is symmetric if every user has the same data rate. Symmetric capacity is the maximal sum rate of all symmetric allocations,

$$C_{\text{sym}}(\mathcal{R}) := \max\{Kr : (r, r, \ldots, r) \in \mathcal{R}\}.$$ 

An allocation is called max-min fair if we cannot increase the rate $r_i$ of user $i$ without decreasing $r_j$ for some $r_j \leq r_i$, while maintaining feasibility. At the max-min fair allocation, no user can increase the data rate without compromising users with lower data rate. Formally speaking, a rate allocation $r^{MM}$ is max-min fair in $\mathcal{R}$ if for any $r \in \mathcal{R}$ such that $r^{MM}_i < r_i$ for some $i$, then we can find $j \in \Omega$ such that $r_j < r^{MM}_j \leq r^{MM}_i$. The sum of rate at the max-min fair allocation for capacity region $\mathcal{R}$ is called the max-min capacity and is denoted by $C_{\text{MM}}(\mathcal{R})$.

Proportional fair (PF) rate allocation $(r^{PF}_i)_{i=1}^K$ is the data rate allocation that maximizes

$$\sum_{i=1}^K \log r_i.$$ 

The proportional fair capacity is the corresponding sum rate,

$$C_{\text{PF}}(\mathcal{R}) := \sum_{i=1}^K r^{PF}_i.$$ 

Since the capacity region $\mathcal{R}$ is closed and convex, and the log function is concave, the maximization is well-defined. Another characterization of the proportional fair allocation $r^{PF}$ is

$$\sum_{i=1}^K \frac{r_i - r^{PF}_i}{r^{PF}_i} \leq 0$$

for all point $r$ in the capacity region $\mathcal{R}$.

In the Nash bargaining solution, there is a notion of disagreement point, which is the default operating point if the users fail to reach any agreement. User $i$ will not accept any data rate lower than $d_i$. The rate
Figure 1: The point $d$ is the disagreement point. The shaded area is the acceptable rate allocation.

allocated to user $i$ in the Nash bargaining solution should be larger than or equal to $d_i$ (Fig. 1). If we are given a disagreement point $d$ in the capacity region, the Nash bargaining solution maximizes

$$\sum_{i=1}^{K} \log(r_i - d_i)$$

over all points in the region

$$\{r \in R : r_i \geq d_i \forall i\}.$$

The Nash bargaining solution satisfies several desirable properties. See [10] for details. It is obviously identical to the proportional fair solution when the origin is chosen as the disagreement point.

For a vector $x = (x_1, \ldots, x_K) \in \mathbb{R}_+^K$, we denote the components in nondecreasing order by $x_1 \leq x_2 \leq \ldots \leq x_K$.

We say that vector $x$ is majorized by vector $y$, written as $x \preceq y$, if for $k = 1, \ldots, K - 1$,

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i,$$

and

$$\sum_{i=1}^{K} x_i = \sum_{i=1}^{K} y_i.$$

In other words, if we sort the components of $x$ and $y$ in nondecreasing order, the cumulative sum of the components of $x$ is larger than the corresponding cumulative sum of $y$.

Majorization induces a partial order that measures dispersion. It is known as the Lorenz order in economics, and is used for comparing income distributions. When $x \preceq y$, we say that the distribution according to $x$ is less spread out, and is thus fairer than that of $y$. A canonical example is that the vector

$$\left( \frac{1}{K}, \ldots, \frac{1}{K} \right)$$

is majorized by any vector in $\mathbb{R}_+^K$ whose components sum to 1.

A function $f : \mathbb{R}_+^K \to \mathbb{R}$ is called Schur convex if

$$f(x) \leq f(y),$$

whenever $x \preceq y$.

If the inequality above is reversed, then we say that the function $f$ is Schur concave. A class of Schur-convex functions is constructed using the following lemma.
Lemma 1. If $\theta : \mathbb{R} \to \mathbb{R}$ is convex (concave), then the function

$$f(x) = \sum_{i=1}^{K} \theta(x_i)$$

is Schur-convex (Schur-concave).

A useful criterion for Schur-convexity is as follows.

**Lemma 2 (Schur’s criterion).** Suppose that $F : \mathbb{R}_+^K \to \mathbb{R}$ is differentiable and symmetric, meaning that $F(x_1, \ldots, x_K) = F(x_{\pi(1)}, \ldots, x_{\pi(K)})$ for any $x$ and permutation $\pi$ of $\{1, \ldots, K\}$. Then

1. $F$ is Schur-convex if $(x_i - x_j) \left( \frac{\partial F}{\partial x_i}(x) - \frac{\partial F}{\partial x_j}(x) \right) \geq 0$ for all $i$ and $j$,

2. $F$ is Schur-concave if $(x_i - x_j) \left( \frac{\partial F}{\partial x_i}(x) - \frac{\partial F}{\partial x_j}(x) \right) \leq 0$ for all $i$ and $j$.

See [11] for more on the theory of majorization and Schur-convexity.

## 3 Capacity-Fairness Tradeoff

Usually there is a tradeoff between sum rate and fairness. The next theorem illustrates such a tradeoff. It shows that the requirement of symmetric fairness is more stringent than proportional fairness in the sense that the symmetric capacity is always less than or equal to the proportional-fair capacity. We use the notation $C_{\text{sum}}$ for the sum capacity, defined as

$$C_{\text{sum}}(R) = \max_{(r_1, \ldots, r_K) \in \mathbb{R}} \sum_{i=1}^{K} r_i.$$

**Theorem 3.** For any convex region $R$,

$$C_{\text{sym}}(R) \leq C_{PF}(R) \leq C_{\text{sum}}(R),$$

$$C_{\text{sym}}(R) \leq C_{MM}(R) \leq C_{\text{sum}}(R).$$

Equality holds in the first inequality in the first line only if the $PF$ allocation is symmetric.

**Proof.** All inequalities are obvious except the the first inequality in the first line, i.e., $C_{\text{sym}}(R) \leq C_{PF}(R)$.

Let $(r_0, \ldots, r_0)$ be the maximal symmetric rate allocation, and $(r_1^{PF}, \ldots, r_K^{PF})$ be the proportional fair rate allocation in region $R$. We have

$$r_0^K \leq \prod_{i=1}^{K} r_i^{PF} \leq \left( \frac{1}{K} \sum_{i=1}^{K} r_i^{PF} \right)^K.$$

The first inequality comes from the defining property of proportional fairness, and the second is the AM-GM inequality. Thus,

$$r_0 K \leq \sum_{i=1}^{K} r_i^{PF}.$$

When equality holds, then we must have $r_1^{PF} = r_2^{PF} = \ldots = r_K^{PF}$. This proves the first inequality.

In general, the max-min capacity may or may not be larger than the proportional fair capacity. However, in Gaussian MAC, both max-min and proportional fair capacity achieve the sum capacity. In fact, the max-min fair and proportional fair rate allocation in Gaussian MAC coincide. It is hence not necessary
Symmetric and Max-min and PF
\[ \eta \in [0, 1] \]
Max. sum rate
\[ \begin{array}{lll}
\text{trivial} & \text{Theorem 8} & \text{trivial} \\
\end{array} \]

Table 1: (Summary of results for Gaussian MAC) Capacities as a function of the power constraint vector \( P \) are Schur concave. For max-min fair or proportional fair, the rates are in increasing order if the power constraints are sorted in increasing order. If the power constraints are more spread out, so does the associated rate allocation.

\[ C_\ast \text{ is Schur concave in } P \]
\[ P_1 \leq \ldots \leq P_K \Rightarrow r_1^\ast \leq \ldots \leq r_K^\ast \]
\[ P \preceq \bar{P} \Rightarrow r^* \preceq \bar{r}^* \]

Table 2: (Summary of results for Gaussian BC) Capacities as a function of the noise vector \( N \) are Schur convex. When the noise powers are sorted in increasing order, then the data rates are in decreasing order, and the power are in increasing order.

to distinguish between max-min and proportional fairness in Gaussian MAC. Meanwhile, in Gaussian BC, max-min capacity is the same as symmetric capacity.

Given a rate allocation \( r \), we define its efficiency, \( \eta \), as the ratio between the sum rate at \( r \) and the maximal sum capacity. Obviously, this is a number between zero and one. For proportional fairness, we have the following lower bound on \( \eta \).

**Theorem 4.** Let \( R \) be a convex region in \( \mathbb{R}_+^K \), the efficiency for proportional fairness, \( \eta_{PF} \), is lower bounded by \( 1/K \).

**Proof.** By the definition of proportional fairness, the region \( R \) is contained in the polyhedron \( P \) defined by
\[
\sum_{i=1}^{K} \frac{r_i}{r_i^{PF}} \leq K,
\]
and \( r_i \geq 0 \) for all \( i \). Therefore, \( C_{sum}(R) \) is not larger than \( C_{sum}(P) \), which is equal to \( K \max_i r_i^{PF} \). Hence,
\[
\eta_{PF} = \frac{C_{PF}(R)}{C_{sum}(R)} \geq \frac{\sum_{i=1}^{K} r_i^{PF}}{K \max_i r_i^{PF}} > \frac{1}{K}.
\]

We will show that in Gaussian MAC, \( \eta_{PF} \) is exactly 1; it achieves the maximal value. In Gaussian BC, the lower bound \( 1/K \) is attained, i.e., \( \inf C_{PF}/C_{sum} = 1/K \) with the infimum taken over all \( K \)-user Gaussian BC.

The major results are summarized in Table 1 and 2. For both Gaussian MAC and BC, we also devise efficient algorithms that compute the fair solutions.
Multiple-Access channels

In a scalar Gaussian MAC with $K$ users, the received signal is

$$Y = \sum_{i=1}^{K} X_i + Z,$$

where $X_i$ is zero-mean Gaussian with variance at most $P_i$ and $Z$ is Gaussian noise with power $N$. Let $C$ denote the Shannon capacity formula

$$C(x) = \frac{1}{2} \log(1 + x).$$

The capacity region of a scalar Gaussian MAC is \[12\]

$$\left\{ R \in \mathbb{R}^K_+ : \sum_{i \in S} R_i \leq C \left( \frac{1}{N} \sum_{i \in S} P_i \right), \text{ for all } S \subseteq \Omega \right\}$$

The faces of the capacity region are hyper-planes in the form $\sum_{i \in S} R_i = c$ for some subset $S$ of $\Omega$ and constant $c$.

In vector Gaussian MAC, the received signal is

$$Y = \sum_{i=1}^{K} X_i s_i + Z.$$

The $s_i$’s are unit-norm column vectors of length $L$, $X_i$ is a Gaussian random variable with zero mean and variance at most $P_i$, and $Z$ is the Gaussian noise vector with zero mean and covariance matrix $N I$, where $I$ is the $L \times L$ identity matrix. The capacity region of a vector Gaussian MAC is \[13\]

$$\left\{ r \in \mathbb{R}^K_+ : \sum_{i \in S} r_i \leq \frac{1}{2} \log \left| I + \frac{1}{N} \sum_{i \in S} P_i s_i s_i^T \right|, \text{ for all } S \subseteq \Omega \right\}.$$
For any function $g$ mapping subsets of $\Omega$ to $\mathbb{R}_+$, let $\mathcal{P}(g)$ denote the polyhedron
\[
\left\{ r \in \mathbb{R}^K_+ : r(S) \leq g(S), \, \forall S \subseteq \Omega \right\}.
\]

If the function $g$ satisfies (i) $g(\emptyset) = 0$, (ii) $g(S) \leq g(T)$ if $S \subseteq T$, and (iii) $g(S) + g(T) \geq g(S \cap T) + g(S \cup T)$ for all subsets $S$ and $T$ of $\Omega$, then $g$ is called a rank function. Property (ii) and (iii) are called the monotonic and submodular property respectively. We say that the polyhedron $\mathcal{P}(g)$ is a polymatroid when $g$ is a rank function.

The capacity regions of scalar and vector Gaussian MAC are polymatroids. For scalar Gaussian MAC, we define the rank function as
\[
g(S) = \frac{1}{2} \log \left(1 + \frac{1}{N} \sum_{i \in S} P_i \right).
\]  

(1)

On the vector case, we define the rank function as
\[
g(S) = \frac{1}{2} \log \left| I + \frac{1}{N} \sum_{i \in S} P_i s_i s_i^T \right|.
\]  

(2)

The collection of points that achieve equality on total data rate $r(\Omega) = g(\Omega)$ is called the dominant face. Given a vector $r$ in $\mathcal{P}(g)$, we say that the set $S$ is a bottleneck of $r$ if $r(S) = g(S)$.

For any function $g : 2^\Omega \to \mathbb{R}_+$, if
\[
g(A) \leq g(B) \text{ implies } g(A \cup C) \leq g(B \cup C)
\]
whenever $A \cap C = \emptyset = B \cap C$ and $|A| = |B|$, then we say that $g$ satisfies the order property. The heuristic meaning is as follow. If the achievable sum rate of group $A$ is less than or equal to that of group $B$, then even if they cooperate with users in group $C$, the sum rate of group $A \cup C$ is still less than that of the group $B \cup C$.

**Example 1** Consider the function $g$ defined as $g(\{a\}) = 1$, $g(\{b\}) = 2$, $g(\{c\}) = g(\{b, c\}) = g(\{a, b\}) = 3$, $g(\{a, c\}) = g(\{a, b, c\}) = 4$. We have $g(\{a\}) < g(\{b\})$ but $g(\{a, c\}) > g(\{b, c\})$. This function does not satisfy the order property.

The scalar Gaussian MAC satisfies the order property, but the vector Gaussian MAC in general does not.

In both the scalar and vector Gaussian MAC, there is a canonical choice of disagreement point for the Nash bargaining solution. Each user can treat the signal of the others as noise and decode independently. For scalar Gaussian MAC, the resulting data rate for user $i$ is
\[
C \left( \frac{P_i}{N + \sum_{j \neq i} P_j} \right).
\]

In the vector case, if user $i$ uses linear MMSE receiver with no joint processing with others, the data rate is
\[
C \left( \frac{1}{N} P_i s_i^T M_i^{-1} s_i \right),
\]
where $M_i$ is the matrix
\[
I + \frac{1}{N} \sum_{j \neq i} P_j s_j^T s_j.
\]
In both cases, they can be expressed in terms of the rank function as
\[ d^*_i = g(\Omega) - g(\Omega \setminus \{i\}). \] (3)

The rate vector \( d^* = (d^*_1, \ldots, d^*_K) \) is called the canonical disagreement point.

The next lemma is a useful consequence of the order property. The proof is straightforward and is omitted.

**Lemma 5.** Let \( g : 2^\Omega \to \mathbb{R}_+ \) be a function that satisfies the order property.

1. If \( g(\{1\}) \leq g(\{2\}) \leq \ldots \leq g(\{K\}) \) are in nondecreasing order, then \( g(\{1, 2, \ldots, i\}) \leq g(S) \) for all \( S \subseteq \Omega \) of size \( i \).
2. For any subset \( A \), the function \( g'(S) = g(S \cup A) - g(A) \) defined for \( S \subseteq \Omega \setminus A \) also satisfies the order property.

### 4.1 Symmetric Rate Allocation

The computation of the symmetric capacity in \( P(g) \) amounts to finding the tightest constraint among \( r(S) \leq g(S) \) for all subsets \( S \). Each component of the symmetric rate cannot exceed \( g(S)/|S| \) for all \( S \subseteq \Omega \). The symmetric rate allocation can be computed by checking \( 2^K - 1 \) constraints.

If \( g \) satisfies the order property, the symmetric capacity can be computed more efficiently using Lemma 5. The computation requires taking the minimum of only \( K \) numbers.

**Theorem 6.** Let \( g : 2^\Omega \to \mathbb{R}_+ \) be a function that satisfies the order property. By relabeling we can assume that \( g(\{i\}) \leq g(\{j\}) \) whenever \( i < j \). The symmetric capacity in the polyhedron \( P(g) \) equals
\[ K \cdot \min \left\{ \frac{1}{k} g(\{1, \ldots, k\}) : k = 1, \ldots, K \right\}. \]

**Corollary 7.** The infimum of fair efficiency of symmetric fairness, taken over all \( K \)-user MAC, is zero.

**Proof.** Suppose that the power of user 1, \( P_1 \), is much less than the others, so that
\[ g(\{1\}) = \min \left\{ \frac{1}{k} g(\{1, \ldots, k\}) : k = 1, \ldots, K \right\}. \]

We see that the symmetric capacity approaches zero when \( P_1 \) approaches zero, while the maximal sum approaches a positive constant. \( \square \)

We next compare two MACs with different power constraints. If the power constraints \( P_1, \ldots, P_K \) become more disperse, then the symmetric capacity will decrease.

**Theorem 8.** Let \( C_{\text{sym}}(P, N) \) be the symmetric capacity of a scalar Gaussian MAC with power constraints \( P \) and noise power \( N \). If \( P \preceq P' \), then \( C_{\text{sym}}(P, N) \geq C_{\text{sym}}(P', N) \), i.e. the symmetric capacity of multiple-access channel with fixed noise power is a Schur-concave function.

**Proof.** Without loss of generality, we assume that the power constraints are sorted in nondecreasing order, \( P_1 \leq P_2 \leq \ldots \leq P_K \), and \( P'_1 \leq P'_2 \leq \ldots \leq P'_K \). Since scalar Gaussian MAC satisfies the order property, the corresponding symmetric rate allocations \( r^{\text{sym}}_i \) and \( s^{\text{sym}}_i \) are given by
\[ r^{\text{sym}}_i = \min \left\{ \frac{1}{k} C \left( \frac{1}{N} \sum_{i=1}^{k} P_i \right) : k = 1, \ldots, K \right\}, \]
and
\[ s^{\text{sym}}_i = \min \left\{ \frac{1}{k} C \left( \frac{1}{N} \sum_{i=1}^{k} P'_i \right) : k = 1, \ldots, K \right\}. \]
for all $i$. As
\[ \sum_{i=1}^{k} P_i \geq \sum_{i=1}^{k} P_i' \]
for all $k$, we have
\[ C\left(\frac{1}{N} \sum_{i=1}^{k} P_i\right) \geq C\left(\frac{1}{N} \sum_{i=1}^{k} P_i'\right). \]
Therefore $r_{sym}^i \geq s_{sym}^i$.  

4.2 Max-min and Proportional Fair Rate Allocation

The following is a useful characterization of max-min fairness [16].

**Lemma 9.** For any function $g : 2^\Omega \to \mathbb{R}_+$, a vector $\mathbf{r}$ is max-min fair in the polyhedron $\mathcal{P}(g)$ if and only if for all $i$, the $i$th component is largest in some bottleneck. In other words, for $i = 1, \ldots, K$, $i$ is contained in a bottleneck $B$ and $r_i \geq r_j$ for all $j \in B$.

The proof of Lemma 9 is contained in the Appendix. The next theorem is the main theorem in this section. The proof of Theorem 10 is in the Appendix (See Prop. 35 and 36.)

**Theorem 10.** Suppose that $g : 2^\Omega \to \mathbb{R}_+$ satisfies the submodular property. The max-min fair point $\mathbf{r}^{MM}$ in $\mathcal{P}(g)$ is on the dominant face, and is majorized by every point on the dominant face.

It is noted that in the theorem we do not assume that the function $g$ is a rank function. The result holds as long as we have the submodular property. The max-min fair allocation is fairer than any other point on the dominant face in the sense of fairness induced by majorization. The max-min fair solution also has the following interpretations.

**Corollary 11.** If $g : 2^\Omega \to \mathbb{R}^K_+$ satisfies the submodular property, the order property and $g(\emptyset) = 0$, then the max-min fair (and hence the proportional fair) solution $\mathbf{r}^*$ maximizes $r_{[1]}, r_{[1]} + r_{[2]}, \ldots$, and $\sum_{j=1}^{K-1} r_{[j]}$ simultaneously over all points on the dominant face.

**Corollary 12.** If $g$ satisfies the submodular property, then the max-min point and the proportional fair point in $\mathcal{P}(g)$ coincide.

**Proof.** Since log is a concave function, $\sum_i \log(r_i)$ is Schur-concave by Lemma 11. Because the max-min point $\mathbf{r}^{MM}$ is majorized by any point $\mathbf{r}$ on the dominant face, we have
\[ \sum_i \log(r_i^{MM}) \geq \sum_i \log(r_i). \]
Hence $\mathbf{r}^{PF} = \mathbf{r}^{MM}$.  

**Corollary 13.** If $g$ satisfies the submodular property, then the max-min (and the proportional fair) point in $\mathcal{P}(g)$ is the point on the dominant face that minimizes the Euclidean norm.

**Proof.** The function $f(x) = x^2$ is convex. The proof is similar to the proof of the last corollary.  

**Corollary 14.** If $g$ satisfies the submodular property, the fairness efficiency of max-min fairness and proportional fairness is equal to 1.

**Theorem 15.** Suppose that $g : 2^\Omega \to \mathbb{R}_+$ satisfies both submodular and order property, and assume $g(\{1\}) \leq g(\{2\}) \leq \cdots \leq g(\{K\})$ after suitable relabeling. Then the components of max-min fair (and hence the proportional fair) solution $\mathbf{r}^*$ in $\mathcal{P}(g)$ are in nondecreasing order, $r_1^* \leq r_2^* \leq \cdots \leq r_K^*$.  

9
Proof. Consider any $i$ in $\Omega$. The index $i$ is contained in a bottleneck $A_i$ of $r^*$ so that $r^*_i = \max\{r^*_j : j \in A_i\}$. If $i - 1 \in A_i$, then $r^*_{i-1} \leq r^*_i$. Otherwise, suppose $i - 1 \not\in A_i$ and let $S$ denote $A_i \setminus \{i\}$. Then

$$r^*(S \cup \{i-1\}) \leq g(S \cup \{i-1\}) \leq g(S \cup \{i\}) = r^*(S \cup \{i\}).$$

We have used the order property in the second inequality. This implies that $r^*_{i-1} \leq r^*_i$. 

A typical class of functions that satisfies the order property is the generalized symmetric functions. A rank function $g$ is said to be generalized symmetric if it has the form

$$g(S) = \phi(Q(S)),$$

where $\phi$ is a monotonic increasing and concave function with $\phi(0) = 0$ and $Q \in \mathbb{R}_+^K$. The rank function in the scalar MAC is an example of generalized symmetric function.

The next theorem compares two Gaussian MACs, with the same total power but different distribution in the power constraints. It shows that if the distribution of power constraints is more spread out, so does the corresponding max-min fair rate allocation. The proof is relegated to the appendix.

**Theorem 16.** Let $g$ be a generalized symmetric rank function defined as $g(S) = \phi(P(S))$, for some vector $P \in \mathbb{R}_+^K$. Let $\hat{P}$ be a vector that majorizes $P$, and $g'$ be the generalized symmetric rank function $g'(S) = \phi(P(S))$. Then the max-min fair capacity associated to $P$ is the same as the max-min fair capacity associated with $\hat{P}$, and he max-min fair point in $P(g)$ is majorized by the max-min fair point in $P(g')$.

**Corollary 17.** Let $r^*(P, n)$ be max-min fair in a Gaussian MAC with power constraints $P_1 \leq \ldots \leq P_K$ and noise power $n$. If $P \preceq P'$, then $r^*(P, n) \preceq r^*(P', n)$.

### 4.3 Algorithm

We present a general recursive algorithm that computes the fair solution in Gaussian MAC. It is a variation of the algorithm in [16, p.527], which computes the max-min fair rate vector in flow control problem. The algorithm to be described below exploits the submodular property, and has shorter running time. The basic idea is contained in the next proposition.

**Proposition 18.** Let $g$ be a function mapping $2^{\Omega}$ to $\mathbb{R}_+$, and let $S_0$ be a subset of $\Omega$ that achieves the minimum

$$\min_{\emptyset \neq S \subseteq \Omega} g(S) / |S|.$$

Let $r^*$ denote the the max-min or proportional fair point (they are the same by Corollary 13) in $P(g)$, we have $r^*_i \geq g(S_0) / |S_0|$ for all $i$, with equality when $i \in S_0$.

**Proof.** For each $i$, $i$ is contained in a bottleneck $B_i$ of $r^*$, so that $r_i \geq r_j$ for all $j \in B_i$. So $r_i$ must be larger than or equal to the average $r^*(B_i) / |B_i|$, and thereby

$$r^*_i \geq \frac{1}{|B_i|} \sum_{j \in B_i} r_j = g(B_i) / |B_i| \geq g(S_0) / |S_0|.$$

Therefore $r^*_i \geq g(S_0) / |S_0|$ for all $i$. Summing over all $i \in S_0$, we obtain

$$r^*(S_0) \geq g(S_0).$$

We must have equality in all the above inequalities. In particular, $r^*_i = g(S_0) / |S_0|$ for all $i \in S_0$. The set $S_0$ is in fact a bottleneck of $r^*$.

This motivates the max-min algorithm.
Max-min algorithm  The algorithm starts by first obtaining the subset $S_0 \in \Omega$ described in Proposition 18, and set the rate of users in $S_0$ to $g(S_0)/|S_0|$. The rate of other users are computed by recursively applying the above computation to $\Omega' := \Omega \setminus S_0$ with

$$g'(S) := g(S \cup S_0) - g(S_0)$$

for $S \subseteq \Omega'$.

Remark: It is noted that we only need the submodular property in proving the correctness of the max-min algorithm. The function $g$ need not satisfy the monotonic property or $g(\emptyset) = 0$. We will use the following lemma in proving the correctness of the algorithm. Note that the lemma holds in general for arbitrary $g$.

Lemma 19. Let $g$ be a function from $2^\Omega$ to $\mathbb{R}_+$, and $S_0 \subseteq \Omega$ be chosen such that

$$g(S_0)/|S_0| = \min_{\emptyset \neq S \subseteq \Omega} g(S)/|S|.$$  

Define the function

$$g'(S) := g(S \cup S_0) - g(S_0)$$

for $S \subseteq \Omega' := \Omega \setminus S_0$.

1. (Non-negativity) $g' \geq 0$, for all $S \subseteq \Omega'$.

2. (Extension of bottleneck) Let $r$ be a vector in $\mathbb{P}(g)$ such that $r(S_0) = g(S_0)$. Let $r'$ be the restriction of the vector $r$ on $\Omega'$. If $B'$ is a bottleneck of $r'$ in $\mathbb{P}(g')$, then $B' \cup S_0$ is a bottleneck of $r$ in $\mathbb{P}(g)$.

3. (Preservation of order property) If $g$ satisfies the order property, so does $g'$.

Proof. 1 By construction, we have

$$g(S \cup S_0) \geq g(S_0)\frac{|S \cup S_0|}{|S_0|}.$$  

Hence $g(S \cup S_0) \geq g(S_0)$.

2 $r'(B')$ equals $g'(B')$, which is $g(B' \cup S_0) - g(S_0)$ by definition. Therefore

$$g(B' \cup S_0) = r'(B') + g(S_0) = r(B') + r(S_0) = r(B' \cup S_0),$$

and $B' \cup A$ is a bottleneck of $r$ in $\mathbb{P}(g)$.

3 Let $S$, $T$ and $U$ be subsets of $\Omega \setminus S_0$ so that $|S| = |T|$ and $S \cap U = \emptyset = T \cap U$. If $g'(S) \leq g'(T)$, then

$$g'(S \cup U) = g(S \cup U \cup A) - g(A) \leq g(T \cup U \cup A) - g(A) = g'(T \cup U).$$

Theorem 20. Suppose $g : 2^\Omega \to \mathbb{R}_+$ satisfies the submodular property. In the polyhedron $\mathbb{P}(g)$, the result obtained by the max-min algorithm is the max-min fair vector.
Proof. Let \( r \) be the vector returned by the max-min algorithm, and let \( S_0 \) be a subset of \( \Omega \) such that 
\[
g(S_0)/|S_0| = \min_{S \subseteq \Omega} g(S)/|S|.
\]
If \( S_0 = \Omega \), then the components in \( r \) are constant and equal \( g(S_0)/|S_0| \). It belongs to \( \mathcal{P}(g) \) because for any \( S \subseteq \Omega \)
\[
r(S) = |S| \frac{g(S_0)}{|S_0|} \leq g(S).
\]
(6)
It is easy to see that \( r \) is max-min fair.

Otherwise, if \( S_0 \subseteq \Omega \), then we have to apply the algorithm recursively. In this case, the vector \( r \) satisfies the following properties: (i) \( r_i = g(S_0)/|S_0| \) for all \( i \in S_0 \) and (ii) the components of \( r \) with index in \( \Omega \setminus S_0 \) yield the max-min solution to the polymatroid \( \mathcal{M}' \) on \( \Omega \setminus S_0 \) with rank function \( g' \) defined as in (5). It is easy to check that \( g' \) also satisfies the submodular property. We first verify that \( r \) is in \( \mathcal{P}(g) \). For any subset \( A \subseteq \Omega \), we can write \( A = A_1 \cup A_2 \) with \( A_1 \subseteq S_0 \) and \( A_2 \cap S_0 = \emptyset \). We decompose \( r(A) \) as
\[
r(A) = r(A_1) + r(A_2) 
\leq |A_1| \frac{g(S_0)}{|S_0|} + g(A_2 \cup S_0) - g(S_0) 
\leq g(A_1) + g(A_2 \cup S_0) - g(S_0)
\]
In the last inequality, we have used the defining property of \( S_0 \), i.e., \( |A_1|g(S_0)/|S_0| \leq g(A_1) \). By the submodularity of \( g \), we have
\[
g(S_0) + g(A_1 \cup A_2) \geq g(A_1) + g(A_2 \cup S_0).
\]
Therefore \( r(A) \leq g(A_1 \cup A_2) \), and thus \( r \) is in \( \mathcal{P}(g) \).

We now show that for each \( i = 1, \ldots, K \), \( i \) is in some bottleneck \( A_i \) such that \( r_i = \max \{ r_j : j \in A_i \} \).
We will apply Lemma \( \text{[9]} \) and conclude that \( r \) is the max-min vector in \( \mathcal{P}(g) \). For \( i \in S_0 \), we can take \( S_0 \) as the required bottleneck \( A_i \). For \( i \notin S_0 \), \( i \) is an element of some bottleneck \( B' \) in the polyhedron \( \mathcal{P}(g') \) such that \( r_i \geq r_j \) for all \( j \in B' \). By part (2) in the previous lemma, \( B' \cup S_0 \) is a bottleneck of \( \mathcal{P}(g') \). By Prop. \( \text{[13]} \) we can show that \( r_i \geq r_j \) for all \( j \in S_0 \). Indeed,
\[
r_i \geq \frac{g'(B')}{|B'|} 
= \frac{1}{|B'|} (g(B' \cup S_0) - g(S_0)) 
\geq \frac{1}{|B'|} \left( g(S_0) \frac{|S_0 \cup B'|}{|S_0|} - g(S_0) \right) 
= \frac{g(S_0)}{|S_0|} = r_j.
\]

Therefore \( i \) is in the bottleneck \( S_0 \cup B' \) and \( r_i \geq r_j \) for all \( j \in B' \cup S_0 \). The vector \( r \) is thereby max-min fair by Lemma \( \text{[9]} \). \( \square \)

**Example 1 (continued)** We compute the max-min fair vector in Example 1. The minimum
\[
\min_{\emptyset \neq S \subseteq \Omega} g(S)/|S|
\]
is achieved when \( S = \{ a \} \). We set \( r^M_1 = g(\{ a \}) = 1 \). Next define
\[
g'(S) := g(\{ a \} \cup S) - g(\{ a \})
\]
for \( S \subseteq \{ b, c \} \). Now \( g'(\{ b \}) = 2 \) and \( g'(\{ c \}) = 3 = g'(\{ b, c \}) \). So in this recursive step, we have the minimum
\[
g'(\{ b, c \})/2 = 3/2.
\]
The resulting max-min fair solution is
\[ r^{\text{MM}} = (1, 3/2, 3/2). \]

In the max-min algorithm, if we compute the minimum of \( g(S)/|S|, \emptyset \neq S \subseteq \Omega \), in a straightforward manner by comparing \( 2^K - 1 \) numbers, the complexity of the algorithm is exponential in the number of users. A more efficient implementation was described in [7] if \( g \) is a rank function. However, when the function \( g \) satisfies the order property, we have a much faster algorithm.

**Proposition 21.** If the function \( g \) satisfies the order property and the submodular property, then the max-min fair point in \( P(g) \) can be computed in \( O(K^2) \) time.

**Proof.** In the max-min algorithm, instead of finding the minimum
\[ \min_{\emptyset \neq S \subseteq \Omega} g(S)/|S| \]
over all subsets of \( \Omega \), we sort \( g(\{1\}), g(\{2\}), \ldots, g(\{K\}) \) in nondecreasing order. This can be done in \( O(K \log(K)) \) time. For notational convenience, we relabel the users so that
\[ g(\{1\}) \leq g(\{2\}) \leq \cdots \leq g(\{K\}). \]

Since function \( g \) satisfies the order property, for any \( k \), the minimum
\[ \min\{g(S)/k : S \subseteq \Omega, |S| = k\} \]
is achieved by \( \{1, 2, \ldots, k\} \) by Lemma 5. Instead of comparing \( g(S)/|S| \) over all subsets \( S \) of \( \Omega \), it is sufficient to examine \( g(\{1, \ldots, k\})/k \), for \( k = 1, \ldots, K \). The minimum can be found in \( O(K) \) time. In the next recursion, the function \( g'(S) = g(S \cup S_0) - g(S_0) \) also satisfies the order property, and
\[ g'(\{k_0 + 1\}) \leq g'(\{k_0 + 2\}) \leq \cdots \leq g'(\{K\}). \]
The recursion can continue without any further sorting. There are at most \( K \) recursive steps and each step takes \( O(K) \) time. The total complexity is therefore \( O(K^2) \). \( \square \)

**Example 2.** Consider a scalar Gaussian MAC with 4 users. Their power constraints are 2, 8, 200, and 300. We let \( P \) be the vector \((2, 8, 200, 300)\). The noise power at the receiver is equal to 1. Let
\[ g_1(S) := 0.5 \log(1 + P(S)) \]
for \( S \subseteq \Omega \). Here we use the natural logarithm function. We want to find the max-min fair rate allocation or the proportional fair rate allocation in \( P(g_1) \). The function \( g_1 \) is a rank function and satisfies the order property.

We first compute the minimum of \( g_1(\{1\}), g_1(\{1, 2\})/2, g_1(\{1, 2, 3\})/3 \) and \( g_1(\{1, 2, 3, 4\})/4 \). The minimum is \( g_1(\{1\}) = 0.5493 \). We set \( r_1^{\text{MM}} = 0.5493 \).

In the next recursive step, set
\[ g_2(S) := g_1(S \cup \{1\}) - g_1(\{1\}) \]
for \( S \subseteq \{2, 3, 4\} \). The minimum of \( g_2(\{2\}), g_2(\{2, 3\})/2 \) and \( g_3(\{2, 3, 4\})/3 \) is \( g_2(\{2\}) \). We set \( r_2^{\text{MM}} = g_2(\{2\}) = 0.6496 \).

Let
\[ g_3(S) := g_2(S \cup \{2\}) - g_2(\{2\}) = g_1(S \cup \{1, 2\}) - g_1(\{1, 2\}) \]
for \( S \subseteq \{3, 4\} \), and compute the minimum of \( g_3(\{3\}) \) and \( g_3(\{3, 4\})/2 \). The minimum is \( g_3(\{3, 4\})/2 = 0.9596 \), and we assign 0.9596 to both \( r_{3MM}^3 \) and \( r_{4MM}^4 \).

The max-min fair rate allocation is thus
\[
{r}_{MM} = (0.5493, 0.6496, 0.9596, 0.9596).
\]

The computation of the Nash bargaining solution in \( P(g) \) with disagreement point \( d \) amounts to finding the max-min fair solution in \( P(g') \), where
\[
g'(S) = g(S) - d(S).
\]

It is noted that if \( g \) is a rank function, the translated \( g' \) in general does not satisfy the monotonic property. However, the max-min algorithm works without assuming the monotonic property. We can apply the max-min algorithm to find the Nash bargaining solution for any disagreement point.

We conclude this section by presenting an algorithm for computing Nash bargaining solution when the rank function is generalized symmetric.

**Lemma 22.** Let \( g \) be a generalized symmetric rank function on \( \Omega \),
\[
g(S) = \phi(P(S)),
\]
for \( S \subseteq \Omega \), and \( P \) is a vector in \( \mathbb{R}_+^K \) such that,
\[
P_1 \leq P_2 \leq \ldots \leq P_K.
\]

Then for any \( k \in \{1, \ldots, K\} \), the minimum
\[
\min \left\{ g(S) + \sum_{i \in S} g(\Omega \setminus \{i\}) : S \subseteq \Omega, |S| = k \right\}
\]
is equal to
\[
g(\{1, 2, \ldots, k\}) + \sum_{i=1}^{k} g(\Omega \setminus \{i\}).
\]

**Proof.** Let \( \tilde{P}_i \) denote \( \sum_{j \neq i} P_j \), where the summation is over all indices except \( i \). The lemma claims that \( \phi(P(\{1, \ldots, k\}) + \sum_{i=1}^{k} \phi(\tilde{P}_i) \) is the minimum. By Lemma 11 it suffices to show that
\[
(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_k, P_1 + \ldots + P_k)
\]
is majorized by
\[
(\tilde{P}_{i_1}, \tilde{P}_{i_2}, \ldots, \tilde{P}_{i_k}, P_{i_1} + \ldots + P_{i_k})
\]
for any choice of \( i_1, \ldots, i_k, 1 \leq i_1 < \ldots < i_k \leq K \).

By subtracting \( \sum_{i=1}^K P_i \) from both vectors, we only need to show that
\[
(P_1, P_2, \ldots, P_k, P_{k+1} + \ldots + P_K) \preceq \left( P_{i_1}, P_{i_1}, \ldots, P_{i_k}, \sum_{j \in B} P_j \right).
\]

where \( B \) denote the set \( \{i_1, \ldots, i_k\} \).

Let \( Q \) be the vector \( \left(P_{i_1}, P_{i_1}, \ldots, P_{i_k}, \sum_{j \in B} P_j \right) \), and \( Q_{[1]}, Q_{[2]}, \ldots, Q_{[K]} \) be the components of \( Q \) in nondecreasing order. It is easy to see that
\[
\sum_{i=1}^{k} P_i \leq Q_{[k]}.
\]

Therefore \( (P_1, P_2, \ldots, P_k, P_{k+1} + \ldots + P_K) \) majorizes \( Q \). This finishes the proof of the lemma. \( \Box \)
Proposition 23. For a generalized symmetric rank function \( g \), the Nash bargaining solution in \( \mathcal{P}(g) \) with the canonical disagreement point can be computed in \( O(K^2) \) time.

Proof. Assume without loss of generality that the power constraints are arranged in nondecreasing order. Let \( d^* \) denote the canonical disagreement point, i.e., for \( i = 1, \ldots, K \), \( d_i^* := g(\Omega) - g(\Omega \setminus \{i\}) \). Let

\[
h(S) := g(S) - d^*(S) = g(S) + \sum_{i \in S} g(\Omega \setminus \{i\}) - |S|g(\Omega).
\]

We relabel the users so that \( h(\{1\}) \leq \ldots \leq h(\{K\}) \). By the previous lemma, the minimum

\[
\min \left\{ h(S) : S \subseteq \Omega, |S| = k \right\}
\]

is \( h(\{1, \ldots, k\}) \). Therefore,

\[
\min_{\emptyset \neq S \subseteq \Omega} h(S)/|S| = \min_{k=1}^{K} \frac{h(\{1, \ldots, k\})}{k}.
\]

The minimum can be obtained efficiently after sorting \( h(\{1\}), \ldots, h(\{K\}) \). Suppose that the minimum is \( h(\{1, \ldots, i_0\})/|i_0| \). We set \( r_{MM}^1 \) to \( h(\{1, \ldots, i_0\})/|i_0| + d_i^* \) for \( i = 1, \ldots, i_0 \).

We next show that the same procedure can be repeated in the next recursive step. Let \( S_0 = \{1, \ldots, i_0\} \), and \( g'(S) = g(S \cup S_0) - g(S_0) \) for \( S \subseteq \Omega' := \{i_0+1, \ldots, K\} \). It is easy to see that \( g' \) is generalized symmetric. Also, we can verify that \( (d_{i_0+1}^*, \ldots, d_K^*) \) is the canonical disagreement point for \( g' \). Indeed,

\[
d_i^* = g(\Omega) - g(\Omega \setminus \{i\})
\]

\[
= g'(\Omega') - g'(\Omega' \setminus \{i\})
\]

for all \( i \not\in S_0 \).

Each recursive step takes \( O(K) \). As there are at most \( K \) steps, the complexity for computing the Nash bargaining solution with the canonical disagreement point is \( O(K^2) \).

Example 2 (cont’d) We compute the Nash bargaining solution in the MAC as in Example 2, with the canonical disagreement point \( d^* \).

\[
d_1^* = g_1(\{1, 2, 3, 4\}) - g_1([2, 3, 4]) = 0.0020
\]

\[
d_2^* = g_1(\{1, 2, 3, 4\}) - g_1([1, 3, 4]) = 0.0079
\]

\[
d_3^* = g_1(\{1, 2, 3, 4\}) - g_1([1, 2, 4]) = 0.2483
\]

\[
d_4^* = g_1(\{1, 2, 3, 4\}) - g_1([1, 2, 3]) = 0.4423
\]

The minimum of

\[
(g_1(\{1, \ldots, k\}) - d_k^*)/k
\]

for \( k = 1, 2, 3, 4 \), is \( g_1(\{1\}) - d_4^* \). Therefore,

\[
r_{MM}^1 = (g_1(\{1\}) - d_4^*) + d_1^* = 0.5493.
\]

For \( S \subseteq \{2, 3, 4\} \), let \( g_2(S) := g_1(S \cup \{1\}) - g_1(\{1\}) \).

\[
g_2(\{2\}) - d_2^* = 0.6418
\]

\[
g_2(\{2, 3\}) - d_2^* - d_3^* = 0.9352
\]

\[
g_2(\{2, 3, 4\}) - d_2^* - d_3^* - d_4^* = 0.6235
\]
The last equation yields the bottleneck. We set

\[ r_2^{MM} = 0.6235 + d_2^* = 0.6314 \]
\[ r_3^{MM} = 0.6235 + d_3^* = 0.8718 \]
\[ r_4^{MM} = 0.6235 + d_4^* = 1.0657 \]

The resulting Nash bargaining solution is

\[ (0.5493, 0.6314, 0.8718, 1.0657). \]

5 Broadcast Channels

In a \( K \)-user Gaussian broadcast channel, the received signal of the \( i \)th user is

\[ Y_i = X + Z_i \]

where \( X \) is a zero-mean Gaussian random variable with variance \( P_T \) and \( Z_i \) is the noise at the \( i \)th receiver, which is modeled as a Gaussian variable with mean zero and variance \( N_i \). We will assume that \( N_1 \leq N_2 \leq \ldots \leq N_K \) throughout this section. Every point \( \mathbf{r} \) on the boundary of the capacity region satisfies

\[ r_i = \frac{1}{2} \log \left( 1 + \frac{\alpha_i P_T}{N_i + \sum_{j=1}^{i-1} \alpha_j P_T} \right) \]

for some \( \alpha_1, \ldots, \alpha_K \) such that

\[ \sum_{j=1}^{K} \alpha_j = 1, \quad \alpha_j \geq 0. \]

5.1 Symmetric Capacity

In order to obtain the symmetric capacity in a BC, we solve a related problem of finding the power distribution so that the users have a common SINR \( \gamma \), i.e.,

\[ \gamma = \frac{p_1}{N_1} = \frac{p_2}{N_2 + p_1} = \cdots = \frac{p_K}{N_K + p_{K-1} + \ldots + p_1}. \]

The noise vector \( \mathbf{N} \) can be expressed in terms of the power vector \( \mathbf{p} \) by a matrix multiplication

\[
\begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_K
\end{bmatrix} =
\begin{bmatrix}
1/\gamma & 0 & \cdots & 0 \\
-1 & 1/\gamma & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 1/\gamma
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
\vdots \\
p_K
\end{bmatrix}.
\]

Let \( G_\gamma \) denote the lower triangular matrix in the above equation. The diagonal elements of \( G_\gamma \) all equal \( 1/\gamma \), and the elements below the diagonal are all \(-1\). The following lemma is obtained by straightforward calculation.

\textbf{Lemma 24.} The inverse of \( G_\gamma \) is a non-negative matrix. The \((i,j)\)-entry of \( G_\gamma^{-1} \) is

\[
[G_\gamma^{-1}]_{ij} =
\begin{cases}
0 & \text{if } i < j, \\
\gamma & \text{if } i = j, \\
\gamma^2(1 + \gamma)^{i-j-1} & \text{otherwise}.
\end{cases}
\]

16
Hence, given the noise powers \( \mathbf{N} = (N_1, \ldots, N_K) \) and the SINR requirement \( \gamma \), we get the corresponding power allocation by multiplying \( G^{-1}_\gamma \) by \( \mathbf{N} \). The following theorem is an immediate consequence. It says that in a Gaussian BC, users with lower noise power use less power.

**Theorem 25.** In a Gaussian BC with noise powers \( N_1 \leq \ldots \leq N_K \), the powers corresponding to the symmetric rate allocation are in increasing order,

\[
p_1^{\text{sym}} \leq p_2^{\text{sym}} \leq \ldots \leq p_K^{\text{sym}}.
\]

**Proof.** Let \( \mathbf{p} \) be the power vector so that all users have SINR \( \gamma \), and let \( h_{ij} \) be the \((i,j)\)-entry of \( G^{-1}_\gamma \). The power of user \( i \) can be obtained by

\[
p_i = h_{i1}N_1 + \ldots + h_{ii}N_i.
\]

Since the first term is nonnegative, we can remove the first term and get

\[
p_i \geq h_{i2}N_2 + h_{i3}N_3 + \ldots + h_{ii}N_i.
\]

It is clear from the previous lemma that \( h_{ij} \) depends only on \( i - j \), hence

\[
p_i \geq h_{(i-1)1}N_1 + h_{(i-1)2}N_2 + \ldots + h_{(i-1)(i-1)}N_{i-1}
\]

Consequently \( p_1 \leq p_2 \leq \ldots \leq p_K \).

We denote the required total power by \( \phi(\mathbf{N}, \gamma) \), which can be computed by

\[
\phi(\mathbf{N}, \gamma) = [1, \ldots, 1] \cdot G^{-1}_\gamma \mathbf{N}
\]

\[
= N_1\theta_1(\gamma) + N_2\theta_2(\gamma) + \ldots + N_K\theta_K(\gamma)
\]

where \( \theta_j(\gamma) \) denote the sum of elements in the \( j \)th column of \( G^{-1}_\gamma \). It is noted that the function \( \theta_j(\gamma) \) is a convex function of \( \gamma \) for all \( j \). Furthermore, we have

\[
\theta_1(\gamma) \geq \theta_2(\gamma) \geq \ldots \geq \theta_K(\gamma). \tag{7}
\]

Given a noise vector \( \mathbf{N} \), the function \( \phi \) is a convex and monotonically increasing function of \( \gamma \).

The next theorem compares the symmetric capacity of two broadcast channels with the same total power constraint.

**Theorem 26.** If \( \mathbf{N} \preceq \mathbf{N'} \), then \( C_{\text{sym}}(\mathbf{N}, P_T) \leq C_{\text{sym}}(\mathbf{N'}, P_T) \). In other words, \( C_{\text{sym}}(\cdot, P_T) \) is a Schur-convex function.

**Proof.** Assume without loss of generality that the components of \( \mathbf{N}_1 \) and \( \mathbf{N}_2 \) are sorted in nondecreasing order. For a given total power constraint and noise vector \( \mathbf{N} \), we can obtain the SINR by solving the equation

\[
\phi(\mathbf{N}, \gamma) = P_T.
\]

The proof is complete if we can show that \( \phi(\mathbf{N}, \gamma) \geq \phi(\mathbf{N'}, \gamma) \) for all \( \gamma \), i.e., \( \phi(\cdot, \gamma) \) is Schur-concave for all \( \gamma \) (Fig. 3). Indeed, for \( i > j \), we have \( N_i \geq N_j \) and

\[
\frac{\partial \phi}{\partial N_i} - \frac{\partial \phi}{\partial N_j} = \theta_i(\gamma) - \theta_j(\gamma) \leq 0.
\]

by (7). Therefore \( \phi(\cdot, \gamma) \) is Schur-concave by Schur’s criterion. \( \square \)
Figure 3: The total power required for symmetric rate allocation as a function of SINR. The noise vector associated to the lower curve majorizes the noise vector associated to the upper one. The dash lines show how to find the SINR when the total power is given.

**Theorem 27.** For fixed noise powers, the symmetric capacity is a convex function of the total power.

**Proof.** For a fixed noise vector $N$, the function $\phi(N, \gamma)$ is a convex function of $\gamma$. Hence the inverse function $\phi^{-1}$ is a concave function. □

**Algorithm**  We have a numerical algorithm computing the symmetric capacity in Gaussian BC by means of the function $\phi(N, \gamma)$. For a given total power $P_T$, we search for the value of $\gamma$ so that $\phi(N, \gamma) = P_T$. This can be done easily as $\phi$ is a monotonic function of $\gamma$. We then compute the data rate from $\gamma$.

**Corollary 28.** The infimum of $\eta_{sym}$, taken over all $K$-user Gaussian BC, is zero.

**Proof.** The sum capacity is attained if we allocate all power to user 1,

$$C_{sum} = \frac{1}{2} \log \left( 1 + \frac{P_T}{N_1} \right).$$

Suppose the noise power of user $K$, $N_K$, is increased, while the others are fixed, the value of the function $\phi(N, \gamma)$ is increased for all $\gamma$. Then the symmetric capacity is decreased, but the sum capacity remains constant. By taking $N_K$ approaching infinity, $\eta_{sym}$ approaches zero. □

### 5.2 Proportional Fair Capacity

The capacity region written in the following form is useful for computing the proportional fair allocation:

$$r_i = \frac{1}{2} \log \left( \frac{N_i + x_i P_T}{N_i + x_{i-1} P_T} \right) \quad \text{for } i = 1, \ldots, K,$$

with

$$0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_{K-1} \leq x_K = 1.$$
The quantity \(x_iP_T\) represents the sum of powers \(P_1 + \ldots + P_i\).

**Theorem 29.** Let \(\Phi_i\) be a concave and monotonically increasing function for \(i = 1, \ldots, K\). Using the notation in (8), the point that maximizes

\[
f(r) := \sum_{i=1}^{K} \Phi_i(r_i)
\]

in the capacity region of a BC with total power \(P_T\) and noise powers \(N_1 \leq \ldots \leq N_K\) satisfies

\[
\frac{\Phi'_i(r_i)}{N_i + x_iP_T} = \frac{\Phi'_{i+1}(r_{i+1})}{N_{i+1} + x_iP_T}
\]

for \(i = 1, \ldots, K - 1\).

In particular a proportional fair allocation satisfies the equation

\[
r_i(N_i + x_iP_T) = r_{i+1}(N_{i+1} + x_iP_T)
\]

for \(i = 1, \ldots, K - 1\).

**Proof.** Suppose that

\[
r = \left(\frac{1}{2} \log \left(\frac{N_i + x_iP_T}{N_i + x_{i-1}P_T}\right)\right)_{i=1, \ldots, K}
\]

maximizes \(f(r)\). This point lies on the boundary of the capacity region. The tangent plane at \(r\) must be orthogonal to the gradient

\[
\nabla f(r) = \left(\Phi'_1(r_1), \Phi'_2(r_2), \ldots, \Phi'_K(r_K)\right)
\]

i.e., it must be orthogonal to \(r' - r\) for all \(r'\) on the tangent plane.

Differentiate (11) with respect to \(x_i\), we obtain for \(i = 1, \ldots, K - 1\),

\[
\frac{1}{2} \left(0, \ldots, 0, \frac{P_T}{N_i + x_iP_T}, -\frac{P_T}{N_{i+1} + x_iP_T}, 0, \ldots, 0\right).
\]

The two fractions in the above vector is in the \(i\)th and \((i + 1)\)st component. This must be orthogonal to the gradient \(\nabla f(r)\). Equating the dot product to zero, we get

\[
\frac{1}{2} \frac{P_T}{N_i + x_iP_T} \Phi'_i(r_i) - \frac{1}{2} \frac{P_T}{N_{i+1} + x_iP_T} \Phi'_{i+1}(r_{i+1}) = 0
\]

For the proportional fair point, we take \(\Phi_i\) to be the log function for all \(i\).

**Corollary 30.** Let \(r^{PF}\) and \(\gamma^{PF}\) be the rate vector and SINR vector corresponding to the proportional fair allocation, with \(N_1 \leq \ldots \leq N_K\). Then the rates and SINR of the users are in decreasing order, i.e.,

\[
r_1^{PF} \geq r_2^{PF} \geq \ldots \geq r_K^{PF}
\]

and

\[
\gamma_1^{PF} \geq \gamma_2^{PF} \geq \ldots \geq \gamma_K^{PF}.
\]

**Proof.** From (10), we have

\[
r_i = r_{i+1} \frac{N_{i+1} + x_iP_T}{N_i + x_iP_T} \geq r_{i+1}.
\]

Since the rate is a monotonically increasing function of the SINR, the inequalities about SINR follows immediately.
Corollary 31. The infimum of $\eta_{PF}$, taken over all $K$-user Gaussian BC, is equal to the lower bound $1/K$.

Proof. Suppose that we fix the noise powers $N_1, \ldots, N_K$ and take $P_T \to 0$. Equation (10) implies that

$$r_i^{PF} N_i \approx r_j^{PF} N_j$$

for all $i$ and $j$. When $P_T$ is small,

$$r_i^{PF} = \frac{1}{2} \log \left( 1 + \frac{p_i^{PF}}{N_i + p_{i-1}^{PF} + \ldots + p_1^{PF}} \right) \approx \frac{1}{2} \frac{p_i^{PF}}{N_i}.$$

Hence $p_i^{PF} \approx p_j^{PF}$ for all $i$ and $j$, and $p_i^{PF} \to P_T/K$ at the proportional fair allocation as $P_T \to 0$. We have the following limits,

$$C_{\text{sum}} \to \frac{1}{2} \frac{P_T}{N_1},$$

and

$$C_{PF} \to \frac{1}{2} \sum_{i=1}^{K} \frac{p_i^{PF}}{N_i} \approx \frac{P_T}{2K} \sum_{i=1}^{K} \frac{1}{N_i}.$$

We obtain

$$\eta_{PF} = \frac{C_{PF}}{C_{\text{sum}}} \to \frac{1}{K} \left( 1 + \frac{N_1}{N_2} + \ldots + \frac{N_1}{N_K} \right).$$

The right hand side can be arbitrarily close to $1/K$ if $N_1 \ll N_i$ for all $i = 2, 3, \ldots, K$.

Theorem 32. In a Gaussian BC with noise power $N_1 \leq \ldots \leq N_K$, the corresponding powers corresponding to the proportional fair allocation are in increasing order,

$$p_1^{PF} < p_2^{PF} < \ldots < p_K^{PF}.$$

Proof. We first prove the theorem in a two-user case. The noise power of user $i$ is $N_i$, ($i = 1, 2$) with $N_1 \leq N_2$. By Theorem 29 the proportional fair rate allocation is

$$r_1 = \frac{1}{2} \log \left( \frac{N_1 + \alpha P_T}{N_1} \right),$$

$$r_2 = \frac{1}{2} \log \left( \frac{N_2 + P_T}{N_2 + \alpha P_T} \right),$$

where $\alpha, 0 \leq \alpha \leq 1$ is chosen such that

$$r_1(N_1 + \alpha P_T) = r_2(N_2 + \alpha P_T).$$

That is, the value of $\alpha$ satisfies

$$\left( N_1 + \alpha P_T \right) \log \left( \frac{N_1 + \alpha P_T}{N_1} \right) = \left( N_2 + \alpha P_T \right) \log \left( \frac{N_2 + P_T}{N_2 + \alpha P_T} \right) \quad (12)$$

The power of user 1 is $\alpha P_T$ and the power of user 2 is $(1 - \alpha) P_T$. We want to show that (12) cannot hold if $\alpha \geq 0.5$. 

20
We will apply the inequality
\[ x \log \left( 1 + \frac{b}{x} \right) < b \]
which holds for \( x > \max \{0, -b\} \). This inequality is an immediate consequence of the inequality
\[ e^x < 1 + x, \forall x \neq 0. \]
Applying the inequality, we get an upper bound
\[ \text{R.H.S. of } (12) = (N_2 + \alpha P_T) \log \left( 1 + \frac{(1 - \alpha)P_T}{N_2 + \alpha P_T} \right) < (1 - \alpha)P_T, \]
and a lower bound
\[ \text{L.H.S. of } (12) = -(N_1 + \alpha P_T) \log \left( 1 + \frac{-\alpha P_T}{N_1 + \alpha P_T} \right) > -(\alpha P_T) = \alpha P_T. \]
If \( \alpha \geq 0.5 \), we can combine the two bounds above,
\[ (N_1 + \alpha P_T) \log \left( \frac{N_1 + \alpha P_T}{N_1} \right) > \alpha P_T \geq (1 - \alpha)P_T \]
\[ > (N_2 + \alpha P_T) \log \left( \frac{N_2 + \alpha P_T}{N_2 + \alpha P_T} \right). \]
Therefore, we must have strict inequality in (12) when \( \alpha \geq 0.5 \). It is noted that the bounds in the above argument is valid for any \( N_1 \) and \( N_2 \).

In a Gaussian BC with \( K \) users, consider the pair of consecutive users \( i \) and \( i + 1 \). At the proportional fair point in the capacity region,
\[ \left( N_i + \sum_{j=1}^{i} P_i \right) \log \left( 1 + \frac{P_i}{N_i + \sum_{j=1}^{i-1} P_j} \right) = \left( N_{i+1} + \sum_{j=1}^{i} P_i \right) \log \left( 1 + \frac{P_{i+1}}{N_{i+1} + \sum_{j=1}^{i} P_j} \right) \]
by Theorem 29. By setting
\[ \tilde{N}_1 := N_i + \sum_{j=1}^{i-1} P_i \]
\[ \tilde{N}_2 := N_{i+1} + \sum_{j=1}^{i-1} P_i \]
\[ \tilde{P}_T := P_i + P_{i+1} \]
\[ \tilde{\alpha} := P_i/(P_i + P_{i+1}), \]
we get
\[ (\tilde{N}_1 + \tilde{\alpha} \tilde{P}_T) \log \left( \frac{\tilde{N}_1 + \tilde{\alpha} \tilde{P}_T}{\tilde{N}_1} \right) = (\tilde{N}_2 + \tilde{\alpha} \tilde{P}_T) \log \left( \frac{\tilde{N}_2 + \tilde{\alpha} \tilde{P}_T}{\tilde{N}_2 + \tilde{\alpha} \tilde{P}_T} \right). \]
We can proceed as in the 2-user case and conclude that \( p_i < p_{i+1} \).

From numerical examples, we have the following conjecture.

**Conjecture 33.** Let \( C_{PF}(N, P_T) \) denote the proportional fair capacity of a Gaussian BC with noise vector \( N \) and total power \( P_T \). If \( N \preceq N' \), then \( C_{PF}(N, P_T) \leq C_{PF}(N', P_T) \).
5.3 Algorithm

We will use the notation for the capacity region in [3], and present an numerical algorithm that maximizes \( \sum_{i=1}^{K} \Phi_i(r_i) \) in the capacity region of a Gaussian BC. It is assumed that \( \Phi_i \) is a strictly monotonically increasing and concave function, so that the inverse of the derivative \( \Phi'_i \) is easy to compute. If \( \Phi_i \) is the logarithm function, then the resulting point is the proportional fair solution. If \( \Phi_i(x) = \log(x - d_i) \), the result is the Nash bargaining solution with disagreement point \( d \).

Theorem [29] says that the equation
\[
\Phi'_{i+1}(r_{i+1}) = \Phi'_i(r_i) \frac{N_i + x_i P_T}{N_i + x_i P_T}
\]
must holds for the optimal solution. We use this equation to express \( x_{i+1} \) in terms of \( x_i \) for \( i = 2, \ldots, K-1 \). It reduces the problem to a one dimensional search. Given any \( x_1 \) we first compute \( r_2 \) by
\[
\begin{align*}
    r_2 &= \Phi'^{-1}_2 \left( \Phi'_1(r_1) \frac{N_2 + x_1 P_T}{N_1 + x_1 P_T} \right) \\
    &= \frac{1}{2} \log \left( \frac{N_2 + x_2 P_T}{N_2 + x_1 P_T} \right)
\end{align*}
\]
and get \( x_2 \) by solving
\[
\begin{align*}
    r_2 &= \frac{1}{2} \log \left( \frac{N_2 + x_2 P_T}{N_2 + x_1 P_T} \right)
\end{align*}
\]
In similar way, we compute \( r_i \) and \( x_i \) for \( i = 3, 4, \ldots, K-1 \). Finally \( r_K \) can be obtained once we know \( x_{K-1} \).

Define the function \( \chi(x_1) \) as
\[
\chi(x_1) := \Phi'_K(r_K)(N_{K-1} + x_{K-1} P_T) - \Phi'_{K-1}(r_{K-1})(N_K + x_{K-1} P_T)
\]
It is a function of \( x_1 \) as the variable \( r_K, r_{K-1}, x_K \) and \( x_{K-1} \) all depend on \( x_1 \). We can now search for the zero of \( \chi(x_1) \) numerically, say \( \chi(x_1^*) = 0 \). From \( x_1^* \), we get \( x_2^* \) by method described above. The vector \( (x_1^*, \ldots, x_K^*) \) will satisfy the condition in Theorem [29] and hence is the optimal solution. Since any zeros of \( \chi \) gives rise to an optimal solution and we know that there the optimal solution is unique, the function \( \chi \) has only one zero.

6 Conclusion

We show how to pick a point in the capacity region of Gaussian MAC and BC according to some fairness criteria. In the Gaussian MAC, there is a strong notion of fairness, namely there is a point on the dominant face that are majorized by all other points on the dominant face, and are both max-min and proportional fair. We can thus call this the fair point in the capacity region. In some particular cases, the fair point can be computed in \( O(K^2) \) time. For the Gaussian BC, the problem of locating the proportional fair solution or Nash bargaining solution reduces to a one-dimensional search. In both channels, fair rate allocation can be compute efficiently.

7 Appendix

Proof of Lemma 4 \( \iff \) For any \( i \in \Omega \), suppose that \( i \) is contained in a bottleneck \( B \), and \( r_i = \max\{r_j : j \in B\} \). If we want to increase \( r_i \), we have to decrease \( r_k \) for some other \( k \in B \). Since \( r_i \) is the largest in \( \{r_j : j \in B\} \), we must have \( r_k \leq r_i \). The vector \( r \) is thus max-min fair.

\( \Rightarrow \) Conversely, suppose that \( r \) is a vector such that \( B_1, \ldots, B_T \) are all the bottlenecks that contain \( i \), and \( r_i \) is not maximal in all such bottlenecks, i.e., \( r_i < \max\{r_j : j \in B_t\} \) for all \( t = 1, \ldots, T \). We can choose \( i_t \) in \( B_t \) such that \( r_{i_t} > r_i \). If we increase \( r_i \) by \( \epsilon \) and decrease each \( r_{i_t} \) by sufficiently small \( \epsilon \), the resulting vector remains in \( \mathcal{P}(g) \). The vector \( r \) is thereby not max-min fair. \( \square \)
Lemma 34. Suppose that a function \( g : 2^\Omega \to \mathbb{R}_+ \) satisfies the submodular property. Union and intersection of two bottlenecks of \( r \) in \( \mathcal{P}(g) \) are also bottlenecks of \( r \).

Proof. Suppose that \( S \) and \( T \) are both bottlenecks of \( r \), i.e., \( r(S) = g(S) \) and \( r(T) = g(T) \).

\[
\begin{align*}
    r(S) + r(T) &= g(S) + g(T) \\
    &\geq g(S \cup T) + g(S \cap T) \\
    &\geq r(S \cup T) + r(S \cap T) \\
    &= r(S) + r(T)
\end{align*}
\]

Therefore, all inequalities above are in fact equalities. In particular, \( r(S \cup T) = g(S \cup T) \) and \( r(S \cap T) = g(S \cap T) \).

The proof of Theorem 10 is divided into the next two propositions.

Proposition 35. Suppose that \( r^{MM} \) be the max-min fair point in \( \mathcal{P}(g) \), where \( g \) satisfies the order property. By relabeling, we can assume without loss of generality that

\[
    0 < i_1 < i_2 < \ldots < i_L = K,
\]

where \( \ell \in \{1, \ldots, L\} \), \( \ell = 1, \ldots, L \), are bottlenecks of \( r^{MM} \). In particular, we have

\[
    r^{MM}(\Omega) = g(\Omega),
\]

i.e., the max-min fair solution lies on the dominant face of \( \mathcal{P}(g) \).

Proof. Let \( B_\ell \) denote the set \( \{1, \ldots, i_\ell\} \) for \( \ell = 1, \ldots, L \).

For any element \( j \in B_1 \), there is a bottleneck \( A_j \) so that \( r_j = \max\{r_i : i \in A_j\} \). If \( j \in B_1 \) and \( k \not\in B_1 \), then \( r_j^{MM} < r_k^{MM} \). Hence \( A_j \) must be a subset of \( B_1 \), for all \( j \in B_1 \). By taking the union of \( A_j \) over all \( j \in B_1 \), we get

\[
    B_1 = \bigcup_{j \in B_1} A_j,
\]

and we can conclude that \( B_1 \) is also a bottleneck of \( r^{MM} \) by Lemma 34.

By similar argument, we can show that \( B_\ell \) is bottleneck of \( r^{MM} \) for all \( \ell \in \{1, \ldots, L\} \).

Proposition 36. For \( \ell = 1, \ldots, L \), let \( B_\ell \) be the set \( \{1, 2, \ldots, i_\ell\} \), with \( 0 < i_1 < i_2 < \ldots < i_L = K \). Let \( w \) be a vector in \( \mathbb{R}_+^K \) such that

\[
    w_1 = \ldots = w_{i_1} < w_{i_1+1} = \ldots = w_{i_2} < \ldots < w_{i_{L-1}+1} = \ldots = w_{i_L}
\]

All vectors in the region

\[
    \mathcal{R} := \{ r \in \mathbb{R}_+^K : r(B_\ell) \leq w(B_\ell), \ \ell = 1, \ldots, L \}
\]

satisfy

\[
    \sum_{j=1}^k r_{|j|} \leq \sum_{j=1}^k w_j
\]

for \( k = 1, \ldots, K \). Consequently, \( w \) is majorized by all points in \( \mathcal{R} \) such that \( r(\Omega) = w(\Omega) \).
Proof. Suppose that \( v \) is a point in \( \mathcal{R} \) that does not majorize \( w \). There is an index \( k, 1 \leq k < K \), such that

\[
\sum_{i=1}^{k} v[i] > \sum_{i=1}^{k} w_i.
\]

We can find an \( \ell \) so that \( i_{\ell} \leq k < i_{\ell+1} \). (Define \( i_0 := 0 \) and \( B_0 := \emptyset \) if necessary.)

Consider the collection \( \mathcal{T} \) of all subsets \( S \subseteq \Omega \) such that \( B_\ell \subseteq S \subseteq B_{\ell+1} \) and \( |S| = k \). The number of such subsets is

\[
a := \binom{i_{\ell+1} - i_\ell}{k - i_\ell}.
\]

Since \( w_i \) is constant for \( i \in B_{\ell+1} \setminus B_\ell \), we have

\[
\sum_{i \in S} w_i = \sum_{i=1}^{k} w_i
\]

for all \( S \in \mathcal{T} \). Hence

\[
\sum_{i \in S} v_i \geq \sum_{i=1}^{k} v[i] > \sum_{i=1}^{k} w_i = \sum_{i \in S} w_i.
\]

We sum the above over all \( S \in \mathcal{T} \),

\[
\sum_{S \in \mathcal{T}} \sum_{i \in S} v_i > \sum_{S \in \mathcal{T}} \sum_{i \in S} w_i. \tag{14}
\]

The left hand side in the above inequality equals

\[
a \sum_{i=1}^{i_\ell} v_i + b \sum_{i=i_\ell+1}^{i_{\ell+1}} v_i,
\]

where

\[
b := a(k - i_\ell) / (i_{\ell+1} - i_\ell).
\]

Similarly, the right hand side of (14) equals

\[
a \sum_{i=1}^{i_\ell} w_i + b \sum_{i=i_\ell+1}^{i_{\ell+1}} w_i.
\]

We rewrite (14) as

\[
(a - b) \sum_{i=1}^{i_\ell} (v_i - w_i) + b \sum_{i=1}^{i_{\ell+1}} (v_i - w_i) > 0,
\]

or equivalently

\[
(a - b)(v(B_\ell) - w(B_\ell)) + b(v(B_{\ell+1}) - w(B_{\ell+1})) > 0.
\]

Since \( a \geq b \), \( w(B_\ell) \geq v(B_\ell) \) and \( w(B_{\ell+1}) \geq v(B_{\ell+1}) \), the left hand side must be less than or equal to zero. We get a contradiction.

\[\square\]

Proof of Theorem 16. Suppose without loss of generality that \( P_1 \leq \ldots \leq P_K \) and \( \tilde{P}_1 \leq \ldots \leq \tilde{P}_K \). Let \( r^* \) and \( \tilde{r}^* \) be the max-min fair point in \( \mathcal{P}(g) \) and \( \mathcal{P}(g') \) respectively. It is clear that \( g(\{1\}) \leq \ldots \leq g(\{K\}) \), \( g'(\{1\}) \leq \ldots \leq g'(\{K\}) \), and both \( g \) and \( g' \) satisfy the order property. By Theorem 15 we obtain \( r^*_1 \leq \ldots \leq r^*_K \) and \( \tilde{r}^*_1 \leq \ldots \leq \tilde{r}^*_K \). We want to show

\[
\sum_{j=1}^{k} r^*_j \geq \sum_{j=1}^{k} \tilde{r}^*_j
\]
for $k = 1, \ldots, K - 1$.

As in the proof of Prop. 35, a bottleneck of the max-min fair vector is of the form $\{1, 2, \ldots, \ell\}$. We can disregard all constraints except those in the form $r(S) \leq g(S)$ for $S = \{1, 2, \ldots, \ell\}$, $\ell = 1, \ldots, K$. Let $k$ be any integer between 1 and $K$. By Prop. 36, the point $r^*$ maximizes $\sum_{j=1}^k r_j^*$ in the region
\[
\begin{align*}
  r_1 + \ldots + r_k &\leq g(\{1, 2, \ldots, k\}) &\text{for } k = 1, \ldots, K, \\
  r_k &\leq r_{k+1} &\text{for } k = 1, \ldots, K - 1, \\
  r_k &\geq 0 &\text{for } k = 1, \ldots, K.
\end{align*}
\]

Expressed in terms of matrix, the constraints become $r^T A \leq b^T$, where
\[
A = \begin{bmatrix}
  1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\
  1 & 1 & 1 & \cdots & 1 & -1 & 1 & \cdots \\
  1 & \cdots & 1 & \cdots & -1 & \cdots \\
  \vdots & \cdots & 1 & \cdots & \vdots & \cdots & \vdots & \cdots \\
  1 & \cdots & \cdots & \cdots & -1 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]
and
\[
b^T = [g(\{1\}), g(\{1, 2\}), \ldots, g(\{1, 2, \ldots, K\}), 0, \ldots, 0].
\]

Let $c$ denote the column vector $[1, \ldots, 1, 0, \ldots, 0]^T$ in which exactly $k$ components are equal to 1. The maximal value in the linear program
\[
\begin{align*}
  \text{max} & \quad \sum_{j=1}^k r_j = c^T \cdot r, \\
  \text{subject to} & \quad r^T A \leq b^T, \\
  & \quad r \geq 0
\end{align*}
\]
is equal to $\sum_{j=1}^k r_j^*$. By duality of linear programming, the minimal value of the dual problem, with $s_j$’s as the dual variables,
\[
\begin{align*}
  \text{min} & \quad \sum_{j=1}^K g(\{1, \ldots, j\}) s_j, \\
  \text{subject to} & \quad A s \geq c, \\
  & \quad s \geq 0
\end{align*}
\]
coinsides with $\sum_{j=1}^k r_j^*$.

Since $P \preceq P'$, we know that $g(\{1, \ldots, j\}) \geq g'(\{1, \ldots, j\})$ for $j = 1, \ldots, K$. If we replace $g(\{1, \ldots, j\})$ by $g'(\{1, \ldots, j\})$, we obtain another linear program,
\[
\begin{align*}
  \text{min} & \quad \sum_{j=1}^K g'(\{1, \ldots, j\}) s_j, \\
  \text{subject to} & \quad A s \geq c, \\
  & \quad s \geq 0.
\end{align*}
\]
The minimal value equals $\sum_{j=1}^k \tilde{r}_j^*$. We are optimizing a smaller objective function over the same feasible region. As a result, we must have $\sum_{j=1}^k r_j \geq \sum_{j=1}^k \tilde{r}_j^*$. \hfill \Box

**Acknowledgements:** We would like to thank Michael Ng for his valuable discussions.
References

[1] F. P. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, 8:33–37, 1997.

[2] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan. Rate control for communication networks: Shadow prices, proportional fairness and stability. *J. Oper. Res. Soc.*, 49(3):237–252, June 1998.

[3] S. Sarkar and L. Tassiulas. Fair allocation of utilities in multirate multicast networks: A framework for unifying diverse fairness objectives. *IEEE Trans. on Automatic Control*, 47(6):931–944, June 2002.

[4] J. F. Nash. The bargaining problem. *Econometrica*, 18:155–162, 1950.

[5] Z. Han, Z. Ji, and K. J. R. Liu. Fair multiuser channel allocation for OFDMA networks using Nash bargaining solutions and coalitions. *IEEE Trans. on Comm.*, 53(8):1366–1376, August 2005.

[6] A. Kapur and M. K. Varanasi. A max-min fair approach to optimize the CDMA capacity region. In *Int. Symp. on Information Theory*, page 435. IEEE, June 2004.

[7] M. A. Maddah-Ali, A. Mobasher, and A. K. Khandani. Using polymatroid structures to provide fairness in multiuser systems. In *Int. Symp. on Inform. Theory*, pages pp.158–162, Seattle, July 2006. IEEE.

[8] K. W. Shum and C. W. Sung. Fair rate allocation in some Gaussian multiple-access channels. In *Int. Symp. on Inform. Theory*, pages pp.163–167, Seattle, July 2006. IEEE.

[9] Yun Shi and Eric Friedman. Algorithms for implementing fair wireless power allocations. In *Canadian Workshop on Information theory*. IEEE, 2005.

[10] M. J. Osborne and A Rubinstein. *Bargaining and Markets*. Academic Press, San Diego, 1990.

[11] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, Cambridge, 2nd edition, 1988.

[12] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley, New York, 1991.

[13] P. Viswanath and V. Anantharam. Optimal sequences and sum capacity of synchronous CDMA systems. *IEEE Trans. Inform. Theory*, 6(45):1984–1991, September 1999.

[14] D. N. C. Tse and S. Hanly. Multiaccess fading channels–part I: polymatroid structure, optimal resource allocation and throughput capacities. *IEEE Trans. Inform. Theory*, 44(7):2769–2815, November 1998.

[15] P. Viswanath, V. Anantharam, and D. Tse. Optimal sequences, power control, and user capacity of synchronous CDMA systems with linear MMSE multiuser receivers. *IEEE Trans. Inform. Theory*, 45(6):1968–1983, September 1999.

[16] D. P. Bertsekas and R. G. Gallager. *Data Networks*. Prentice-Hall, New York, 1987.