Degeneracy of the eigenvalues of hermitian matrices

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Abstract. Degeneracy of the eigenvalues of hermitian matrices is analyzed in terms of algebraic relations between the matrix elements. Only finite-dimensional matrices are considered, representing Hamiltonians of spin systems in particular. The analysis is in terms of linear relations in the space of "diagonal" operators, i.e. a complete set of powers of the Hamiltonian. A relation is found with the theorem formulated by Von Neumann and Wigner that only the variation of three parameters of the Hamiltonian may result in a "crossing" or degeneracy of two eigenvalues.

1. Introduction

In a foregoing paper [1], henceforth referred to as I, conditions were formulated for the crossing of energy levels in the case of a Hamiltonian represented by a hermitian matrix that is a linear function of one real parameter. That paper relied heavily on the epoch-making work of Hund [2, 3, 4] and Von Neumann and Wigner [5]. Consequences of I were worked out in two papers [6, 7] and a summary of this work may be found in Valkering's doctor's thesis [8].

The present work deals with the more complicated problem of determining the degeneracies of the eigenvalues of a general hermitian matrix $H$ of dimension $n$, which generally contains $n^2$ real constants. As a rule it will represent a Hamiltonian. Conditions for these degeneracies will be formulated in terms of algebraic relations between the real constants. An example of such a matrix is found in the representation of a general Hamiltonian of a spin $S$, the spin value being related with $n$ by:

$$n = 2S + 1.$$  

(1)

Solution of the problem of finding the degeneracies for the Hamiltonian $H$ goes in two main steps:

- 1. A definition is given of a complete set of diagonal operators in matrix representation. Such a complete set may be found in the first $n$ powers of the given Hamiltonian $H$:

$$I, H, H^2, H^3, ...H^{n-1}$$  

(2)

in which $I$ represents the unit matrix. Diagonal in this context is supposed to mean commuting with the Hamiltonian $H$ and it is clear that the set (2) may be diagonalized simultaneously. (N.B.: The Hamiltonian may be given in any suitable representation!) Completeness of the set (2) is the regular situation, and degeneracy involves linear dependence of the powers of the Hamiltonian. The number of independent powers is strictly related with the number of degeneracies, in other words: The number of different eigenvalues of $H$ equals the number of independent powers, as was shown in I.
In the next section a suitable set of linear combinations of (2) is given for which degeneracy is easily formulated. The criterion for the linear dependence of the set (2) in this formulation is the vanishing of an algebraic expression in terms of the $n^2$ real parameters.

-2. If the linear dependence of the set (2) is established, one may look in more detail at the set of linear combinations defined in the next section. This set is an orthogonalized one in terms of a suitably defined metric. The last one of this set should be the 0-matrix and consequently it defines, in principle, a set of $n^2$ equations, which, however, are not all independent. It turns out that these equations result in only 3 independent relations for the real parameters of $H$.

-3. These 3 relations correspond to the 3 conditions necessary for the degeneracy of the eigenvalues of a hermitian matrix, formulated by Von Neumann and Wigner [5], already in 1929. These conditions are shortly discussed in the Appendix.

In the last section some simple examples are given of the conditions for ”crossing” for $n = 2$ and 3.

2. A condition for crossing in terms of a complete set of diagonal operators

In ref. [9] a suitable set of orthogonal diagonal operators are defined in terms of linear combinations of the powers given in (2). This was possible after introduction of a positive semi-definite metric defined by the inner product:

$$< FG > = \frac{\text{Tr} e^{-\beta H} F G}{\text{Tr} e^{-\beta H}} = < GF > < F^2 > = \frac{\text{Tr} e^{-\beta H} F^2}{\text{Tr} e^{-\beta H}} \geq 0$$

(3)

in which formulas $F$ and $G$ are two diagonal operators, like the powers of $H$. The expressions (3) may represent ensemble averages with $\beta = 1/kT$. This was the case in the paper cited and here one may take the simple choice $\beta = 0$ ($T = \infty$). Now the orthogonal set takes the form:

\[
\begin{align*}
  f_1 &= I, \\ 
  f_2 &= \begin{bmatrix} I & H \\ < I > & < H > \end{bmatrix}, \\ 
  f_3 &= \begin{bmatrix} I & H & H^2 \\ < I > & < H > & < H^2 > \\ < H > & < H^2 > & < H^3 > \end{bmatrix}, \\ 
  f_n &= \begin{bmatrix} I & H & H^2 & \ldots & H^{n-1} \\ < I > & < H > & < H^2 > & \ldots & < H^{n-1} > \\ < H > & < H^2 > & < H^3 > & \ldots & < H^n > \\ \vdots \\ < H^{n-2} > & < H^{n-1} > & < H^n > & \ldots & < H^{2n-3} > \end{bmatrix}
\end{align*}
\]

(4)

in which expressions $I$ represents the unit matrix of dimension $n$.

One may now formulate the condition for degeneracy taking the following steps:

- In first instance one may conclude, as a consequence of the linear dependence of the set (2), that:

$$f_n = 0$$

(5)

which results in a complicated algebraic relation between the first $n$ powers of the Hamiltonian i.e.: $I, H, ... H^{n-1}$ and the averages of the first $2n - 2$ powers.

- Secondly one may state that:

$$< H^{n-1} f_n > = 0$$

(6)

which may be written in terms of the determinant $D_n$ given by:
This formulation has a straightforward interpretation because:

\[ D_n = \prod_{i<j}(E_i - E_j)^2, \]  

(9)

in which \( E_i \) and \( E_j \) denote eigenvalues of \( H \). In the right member of (9) we may recognize the square of the Vandermonde determinant for the eigenvalues \( E_i \).

The determinant \( D_n \) is a positive semi-definite real polynomial in terms of the real constants defining \( H \), and in general the condition \( D_n = 0 \) will result in more than just one equation, but it is very hard in a practical problem to split \( D_n \) in its positive semi-definite parts, which should all be equal to 0. That there is generally more than one semi-definite part follows from the analysis in the following lines. A consequence of this splitting is the existence of more than one condition for degeneracy.

A more direct way for the determination of a crossing is the analysis of the conditions under which (5) is fulfilled. As was already stated in the Introduction, these conditions generally result in \( n^2 \) equations, which are not all independent. This is in agreement with the theorem of Von Neuman and Wigner [5] that states that for a single degeneracy, i.e. for the identity of 2 eigenvalues there exist 3 conditions for the real parameters of \( H \). This alternative formulation of the problem may best be started with the introduction of the bra – ket or Dirac-notation. A complete set of orthonormal states for our \( n \)-dimensional space will be denoted by:

\[ |i\rangle = 1, 2, ...n. \]  

(10)

The Hamiltonian may be written in terms of this Dirac notation:

\[ H = \sum_{i,j=1}^{n} <i|H|j> |i><j|. \]  

(11)

and the complete set of operators:

\[ |i><j| \]  

(12)

has \( n^2 \) members, corresponding to \( n^2 \) independent real constants in any arbitrary hermitian matrix. The Dirac operators obey the simple algebraic relation:

\[ |i><j| ^\dagger = |j><i| \]  

(13)

in which the symbol ^\dagger (dagger) indicates the hermitian conjugate.

The algebra of the operators implies:

\[ |i><j| h><k| = |i><j|h><k| = \delta_{jk}|i><k|. \]  

(14)
The expression for the powers of $H$ is consistent with the properties of the Dirac operators:

$$
H^2 = \sum_{i,j=1}^{n} \sum_{h,k} <i|H|j><h|H|k> |i><j|h><k|
= \sum_{i,k} <i|H^2|k> |i><k|
$$

(15)

$$
H^n = \sum_{i,k} <i|H^n|k> |i><k|
$$

Now it follows from (4) and (5) that the condition for a crossing takes the form:

$$
\begin{vmatrix}
\delta_{i,j} |i><j| & <i|H|j>|i><j| & <i|H^{n-1}|j>|i><j| & \ldots \\
<i|I>| & <i|H>| & <i|H^{n-1}|> & \ldots \\
<i|H>| & <i|H^2>| & <i|H^n>| & \ldots \\
\ldots \\
<i|H^{n-2}>& <i|H^{n-1}> & <i|H^{2n-3}> & \ldots \\
\end{vmatrix}
= 0
$$

(16)

which results in $n^2$ equations, corresponding to the different operators $|i><j|$. This set of equations correspond to all possible crossings, also those that represent a degeneracy of higher order for which three or more eigenvalues of $H$ coincide. To start with we take a single crossing, which implies that:

$$
D_{n-1} = \begin{vmatrix}
<i|I>| & <i|H>| & <i|H^2>| & \ldots <i|H^{n-2}|>\\
<i|H>| & <i|H^2>| & \ldots <i|H^{n-1}|>\\
<i|H^2>| & <i|H^3>| & \ldots <i|H^n|>\\
\ldots \\
<i|H^{n-2}|> & <i|H^{n-1}|> & <i|H^{2n-4}|> & \ldots \\
\end{vmatrix} \neq 0.
$$

(17)

The determinant $D_{n-1}$ is the minor of the last element of the first row of the determinant (16), apart from sign. The minors of this first row play an important role in the argument of this section. They will be represented by:

$$
m_0, m_1, m_2, \ldots m_{n-1} \quad m_{n-1} = (-)^{n-1}D_{n-1} \neq 0.
$$

(18)

In terms of the minors the set of equations corresponding to the co-efficients of the individual $|i><j|$ can be written:

$$
\delta_{i,j}m_0 + <i|H|j>|m_1 + <i|H^2|j>|m_2 + \ldots <i|H^{n-1}|j>|m_{n-1} = 0.
$$

(19)

Now we consider this set as a set of equations for the $m_l$ ($l = 0, 1, \ldots n-1$) with co-efficients represented by matrix elements of powers of $H$. We know that this set of homogeneous linear
equations has a non-trivial solution, because $m_{n-1} \neq 0$. Consequently the matrix:

$$
\begin{bmatrix}
1 & <1|H|1> & <1|H^2|1> & \cdots & <1|H^{n-2}|1> \\
1 & <2|H|2> & <2|H^2|2> & \cdots & <2|H^{n-2}|2> \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & <n|H|n> & <n|H^2|n> & \cdots & <n|H^{n-2}|n> \\
0 & <1|H|2> & <1|H^2|2> & \cdots & <1|H^{n-2}|2> \\
0 & <1|H|3> & <1|H^2|3> & \cdots & <1|H^{n-2}|3> \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & <n|H|n-1> & <n|H^2|n-1> & \cdots & <n|H^{n-2}|n-1>
\end{bmatrix}
$$

has rank $n-1$.

Along these lines we find conditions for a single crossing in terms of the matrix elements of the powers of $H$. The number of independent conditions turns out to be 3 in the examples we have chosen to test the method. This number is in accordance with the number derived by Von Neumann and Wigner [5]. The single condition in the form (8) represents the condition that a positive semi-definite polynomial in terms of real constants should vanish. This turns out to be only possible by the simultaneous vanishing of 3 positive semi-definite parts of this polynomial.

This will be illustrated by the examples given in the next section. In that section some simple examples for $n = 2$ and $3$ are discussed.

3. Examples

3.1. Spin $S = \frac{1}{2}$ ($n = 2$)

This case is rather simple and suggest triviality but it illustrates in detail the argument of this work.

The Hamiltonian in this case takes the form:

$$H = h_x S_x + h_y S_y + h_z S_z.$$  \hspace{1cm} (21)

All interactions the spin is subjected to may be written in this form, because it is the only hermitian form that could be constructed for this system, if one omits a trivial multiple of the unit operator $I$. The components of the spin vector:

$$\mathbf{S} = \frac{1}{2} \mathbf{\sigma} = \frac{1}{2} (\sigma_x, \sigma_y, \sigma_z), \quad (\hbar = 1)$$  \hspace{1cm} (22)

apart from a factor $\frac{1}{2}$, are the Pauli matrices in the usual representation:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  \hspace{1cm} (23)

One easily finds for this example:

$$<H> = 0 \quad <H^2> = \frac{1}{4} (h_x^2 + h_y^2 + h_z^2)$$

5
The condition for a (single) crossing according to (8) is:

\[ D_2 = \begin{vmatrix} I & 0 \\ 0 & \frac{1}{\lambda}(h_x^2 + h_y^2 + h_z^2) \end{vmatrix} = 0 \]  \hspace{1cm} (24)

which is equivalent to the following set of 3 conditions for the real parameters \( h_x, h_y, h_z \):

\[ h_x = h_y = h_z = 0 \]  \hspace{1cm} (25)

in accordance with the Von Neumann-Wigner criterion.

Alternative way of detecting degeneracies is by imposing the condition (20), i.e.:

\[
\begin{bmatrix}
1 & < 1|H|1 > \\
1 & < 2|H|2 > \\
0 & < 1|H|2 > \\
0 & < 2|H|1 > \\
\end{bmatrix} \hspace{1cm} (26)
\]

should have rank 1. From this condition (25) immediately follows.

N.B.: In the Von Neumann-Wigner argument one has 4 parameters, one corresponding to the unit matrix, which was left out in (21). This parameter drops out in the condition (24).

3.2. Spin \( S = 1 \) (\( n = 3 \))

The most general Hamiltonian for this case has the form:

\[ H = \begin{bmatrix} x & p & r^* \\
p^* & y & q \\
r & q^* & z \end{bmatrix} \]  \hspace{1cm} (27)

and the constants \( x, y, z, p, q, r \) may always be chosen in such a way that this Hamiltonian represents the interaction of a spin 1 with arbitrary external magnetic and electrostatic fields. The diagonal elements \( x, y, z \) are real.

The method of the detection of a degeneracy based on the condition (8) will not be used in this case because of the complicated and lengthy calculations. According to the condition (20) the following matrix has rank 2:

\[
\begin{bmatrix}
1 & < 1|H|1 > & < 1|H^2|1 > \\
1 & < 2|H|2 > & < 2|H^2|2 > \\
1 & < 3|H|3 > & < 3|H^2|3 > \\
0 & < 1|H|2 > & < 1|H^2|2 > \\
0 & < 1|H|3 > & < 1|H^2|3 > \\
0 & < 2|H|1 > & < 2|H^2|1 > \\
0 & < 2|H|3 > & < 2|H^2|3 > \\
0 & < 3|H|1 > & < 3|H^2|1 > \\
0 & < 3|H|2 > & < 3|H^2|2 > \\
\end{bmatrix} \hspace{1cm} (28)
\]

The matrix elements of \( H^2 \) follow in an easy way from (27):

\[
H^2 = \begin{bmatrix}
x^2 + |p|^2 + |r|^2 & (x + y)p + q^*r^* & (x + z)r^* + pq \\
(x + y)p^* + qr & y^2 + |p|^2 + |q|^2 & (y + z)q + p^*r^* \\
(x + z)r^* + pq & (y + z)q^* + pr & z^2 + |q|^2 + |r|^2 \end{bmatrix} \hspace{1cm} (29)
\]

Inserting the matrix elements given in (27) and (29) into (28) the condition for degeneracy is now that the matrix:
has also rank 2. The same condition holds for:

\[
\begin{bmatrix}
1 & x & x^2 + |p|^2 + |r|^2 \\
1 & y & y^2 + |p|^2 + |q|^2 \\
1 & z & z^2 + |q|^2 + |r|^2 \\
0 & p & (x + y)p + q^*r^* \\
0 & r^* & (x + z)r^* + pq \\
0 & p^* & (x + y)p^* + qr \\
0 & q & (y + z)q + p^*r* \\
0 & r & (x + z)r + p^*q^* \\
0 & q^* & (y + z)q^* + pr
\end{bmatrix}
\]

and also for:

\[
\begin{bmatrix}
1 & x & x^2 + |p|^2 + |r|^2 \\
0 & y - x & y^2 - x^2 + |q|^2 - |r|^2 \\
0 & z - x & z^2 - x^2 - |p|^2 + |q|^2 \\
0 & p & (x + y)p + q^*r^* \\
0 & 0 & pqr - p^*q^*r^* \\
0 & q & (y + z)q + p^*r^* \\
0 & 0 & pqr - p^*q^*r^* \\
0 & r & (x + z)r + p^*q^* \\
0 & 0 & pqr - p^*q^*r^*
\end{bmatrix}
\]

Now the rows 5 and 7 of this matrix may be left out and the most convenient form of the condition is that:

\[
\begin{bmatrix}
1 & x & x^2 + |p|^2 + |r|^2 \\
0 & y - x & y^2 - x^2 + |q|^2 - |r|^2 \\
0 & z - x & z^2 - x^2 - |p|^2 + |q|^2 \\
0 & p & (x + y)p + q^*r^* \\
0 & q & (y + z)q + p^*r^* \\
0 & r & (x + z)r + p^*q^* \\
0 & 0 & pqr - p^*q^*r^*
\end{bmatrix}
\]

has rank 2.

One may distinguish the following 2 cases:

**Case I:**

\[pqr - p^*q^*r^* = 0\]
Case II:
\[ pqr - p^*q^*r^* \neq 0 \] (35)
which gives the consequencies:
\[ p = q = r = 0 \quad x = y = z \] (36)
and one immediately sees that this results in a contradiction.
So only Case I remains, for which holds:
\[ C = pqr \text{ is real} \] (37)
and one easily derives by combining the first row of (33) with 2 of the rows 4,5 and 6:
\[
C(x - y) + |p|^2 (|q|^2 - |r|^2) = 0 \\
C(y - z) + |q|^2 (|r|^2 - |p|^2) = 0 \\
C(z - x) + |r|^2 (|p|^2 - |q|^2) = 0
\] (38)
Only 2 of the 3 conditions of (38) are independent, so (37) and (38) result in 3 conditions for crossing, in accordance with the Von Neumann-Wigner Rule. There is, however, a restriction to this conclusion: All 3 parameters \( p, q, r \) should be unequal to 0.
If one of these parameters, e.g. \( p = 0 \), the matrix (33) should be replaced by:
\[
\begin{bmatrix}
1 & x & x^2 + |r|^2 \\
0 & y - x & y^2 - x^2 + |q|^2 - |r|^2 \\
0 & z - x & z^2 - x^2 + |q|^2 \\
0 & 0 & q^*r^* \\
0 & q & (y + z)q \\
0 & r & (x + z)r
\end{bmatrix}
\] (39)
which should again have rank 2. This results in the following 2 solutions, which are similar:
Solution 1:
\[ q = 0, \quad (x - y)(y - z) + |r|^2 = 0, \quad (r \neq 0) \] (40)
Solution 2:
\[ r = 0, \quad (z - x)(x - y) + |q|^2 = 0, \quad (q \neq 0) \] (41)
An analogous solution exists for \((q = r = 0, p \neq 0)\).
For the case that \( p = q = r = 0 \) the matrix for \( H \) given in (27) is diagonal and degeneracies occur for:
\[ x = y \text{ or } y = z \text{ or } z = x \] (42)
which result also follows from (39).
In the underlying example the criterion for degeneracy given in (8) takes the form
\[
D_3 = \begin{bmatrix}
I & < H > & < H^2 > \\
< H > & < H^2 > & < H^3 > \\
< H^2 > & < H^3 > & < H^4 >
\end{bmatrix} = 0.
\] (43)
It should be clear, because of (9), that an overall shift of the eigenvalues does not change the value of $D_3$, so that one may as well impose the extra condition:

$$x + y + z = 0 \text{ or } z = -x - y$$

in the calculation of $\langle H \rangle, ..., \langle H^4 \rangle$. This simplification results for the case $p = 0$, in the following values of the elements of $D_3$:

$$\langle H \rangle = 0 \quad \langle H^2 \rangle = \frac{2}{3}(x^2 + xy + y^2 + |q|^2 + |r|^2)$$
$$\langle H^3 \rangle = -(x + y)xy - (x |q|^2 + y |r|^2)$$
$$\langle H^4 \rangle = \frac{2}{3}(x^2 + xy + y^2 + |q|^2 + |r|^2)^2$$

After some rearrangements one finds with these averages the following expression for the determinant (43):

$$D_3 = \frac{1}{27}[(x - y)(2x + y)(x + 2y) - x(|q|^2 - 2 |r|^2) - y(2 |q|^2 - |r|^2)]^2$$
$$+ \frac{4}{27}(|q|^2 [(x - y)(2x + y) - |q|^2]^2 + |r|^2 [((x - y)(x + 2y) + |r|^2]^2] +$$
$$+ \frac{4}{27} |q|^2 |r|^2 [(7x^2 - 5xy + 7y^2) + 3(|q|^2 + |r|^2)].$$

This expression clearly is a sum of 3 positive semi-definite parts and it vanishes for the 2 solutions (40) and (41).

So far the analysis was relatively simple because of the condition $p = 0$, for which case one could establish the equivalency between the condition $D_3 = 0$ and the von Neumann-Wigner condition. If one starts with a Hamiltonian of the form (27) with all non-diagonal elements $p, q, r \neq 0$ one could easily find a unitary transformation that results in the vanishing of one of these parameters:

$$H'(x', y', z', p', q') = U^\dagger H(x, y, z, p, q, r)U$$

The unitary matrix $U^{(1)}$ is a function of the parameters $(x, y, z, p, q, r)$. In this way the conditions for crossing in terms of the set $(x', y', z', p', q')$ can be expressed in terms of the set $(x, y, z, p, q, r)$

Now the general equivalence of the two methods for determining a crossing is established for the underlying examples. It is shown at the same time that 3 conditions are to be fulfilled for such a crossing, in accordance with the theorem by von Neumann and Wigner [5].

**Appendix. The Von Neumann-Wigner Conditions**

A general $n$-dimensional hermitian matrix $H$ may be written in the form:

$$H = U^\dagger H_{\text{diag}} U$$

in which $H_{\text{diag}}$ is a diagonal representation, which has the eigenvalues of $H$ as its (diagonal) elements:
In (A.1) \( U \) is the unitary matrix that realizes the transformation of \( H \) to diagonal form \( H_{\text{diag}} \). As a start the case of nondegenerate eigenvalues is considered, i.e. all \( E_m (m = 1, 2, \ldots n) \) are different. At first sight one will come to the conclusion that the matrix \( H \) is determined by \( n^2 + n \) real constants, \( n^2 \) corresponding to the matrix \( U \) and \( n \) with \( H_{\text{diag}} \). This is, however, not true because \( U \) may be replaced by a product of 2 matrices, one being \( U \) and the other one \( V \), a unitary matrix that commutes with \( H_{\text{diag}} \) and, consequently, does not change it eigenvalues.

The general form of \( V \) is given by:

\[
V = \begin{bmatrix}
  e^{i\phi_1} & 0 & 0 & 0 & 0 & \ldots & 0 \\
  0 & e^{i\phi_2} & 0 & 0 & 0 & \ldots & 0 \\
  0 & 0 & e^{i\phi_3} & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & e^{i\phi_4} & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & 0 & \ldots & e^{i\phi_n}
\end{bmatrix}
\]  

(A.3)

in which the \( \phi_m (m = 1, 2, \ldots n) \) are arbitrary real constants. Consequently the number of actual "free" constants in \( H \) is:

\[
n^2 + n - n = n^2
\]  

(A.4)

as is well known of an arbitrary hermitian matrix of dimension \( n \).

In the case of degeneracy the diagonal matrix \( H_{\text{diag}} \):

\[
H_{\text{diag}} = \begin{bmatrix}
  E_1 & 0 & 0 & 0 & \ldots & 0 \\
  0 & E_1 & 0 & 0 & 0 & \ldots & 0 \\
  0 & 0 & E_2 & 0 & 0 & \ldots & 0 \\
  0 & 0 & 0 & E_2 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & 0 & \ldots & E_f
\end{bmatrix}
\]  

(A.5)

contains groups of identical eigenvalues. The eigenvalues, which are called \( E_m (m = 1, 2, \ldots f) \) have respective degeneracies \( g_1, g_2, \ldots g_f \) (N.B.: In the example in (A.5) \( g_1 = 2, g_2 = 3, \ldots g_f = 1 \)). The corresponding blocks commute with arbitrary unitary matrices of dimension \( g_1, g_2, \ldots \), which may be denoted by \( V_1, V_2, \ldots V_f \):

\[
V = \begin{bmatrix}
  V_1 & 0 & \ldots & \ldots & 0 \\
  0 & V_2 & \ldots & \ldots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \ldots & \ldots & V_f
\end{bmatrix}
\]  

(A.6)
The total number of real parameters in (53) now is: \( g_1^2 + g_2^2 + \ldots g_f^2 \) and taking into account that the number of parameters in \( H_{\text{diag}} \) is now reduced to \( f \) the number of free parameters in \( H \) is now given by:

\[
\begin{align*}
n^2 + f - (g_1^2 + g_2^2 + \ldots g_f^2). 
\end{align*}
\]

(A.7)

In a general Hamiltonian all \( g_m \) equal 1 and \( f = n \) and the number of free parameters is \( n^2 \). For the case of one degenerate pair of eigenvalues (54) has the value:

\[
\begin{align*}
n^2 + n - 1 - (2^2 + n - 2) = n^2 - 3 
\end{align*}
\]

(A.8)

So the number of free parameters is reduced by 3 representing the number of conditions to be fulfilled for a crossing.

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