Existence of the equilibrium in choice

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Abstract In this paper, we prove the existence of the equilibrium in choice for games in choice form. These games have recently been introduced by A. Stefanescu, M. Ferrara and M. V. Stefanescu. Our results link the recent research to the older approaches, regarding games in normal form or qualitative games.

Key words game in choice form, equilibrium in choice, selection theorem, fixed point theorem

1 Introduction

It has been widely agreed upon the fact that Nash’s concept of equilibrium reflects the possibility of challenging the choices of the unilateral acts of the players involved in noncooperative games. Having the aim of gaining the most they can do, the players are in the situation of making choices in a process described, mathematically, by the notion of "game" (this was proposed by Nash, and it was initially called "game in normal form"). The original definitions of Nash [4,5] have been extended, but the derived notions of qualitative game or abstract economy and their corresponding concepts of equilibrium reflect the initial meaning of flexible elections in any given context to permit the players not to deviate, once they agreed on the best solution for them.

The new extension of a game, called "the game in choice form", is due to Stefanescu, Ferrara and Stefanescu [9]. The game in choice form is a family of the sets of individual strategies and choice profiles. The authors

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also defined the concept of equilibrium in choice for this type of game. Their interpretations evolved along with the problem of noncooperative solutions of the voting operators. Firstly, Stefanescu and Ferrara proposed the concept of Nash equilibrium in choice in [8] and they renamed it in [9], founding conditions under which the equilibrium in choice exists. The new adopted definitions are coherent with the old underlying formalism. For instance, when the utility functions represent the players’ options, a choice profile can be seen as the family of players’ graphs best reply mappings. In this case, the set of equilibria in choice coincides with the set of Nash equilibria. The generality of the new concepts raise the interest of the formalist theorist of games, which explicitly can show their significance. The authors themselves developed these ideas in their work. They referred to the fact that the players’ preferences need not be explicitly represented, at the same time considering the possibility of recuperating the known solutions as particular cases. The second problem raised is the one of the nonexistence of a best reply. This interest is obviously at the core of our original research. In this paper, we are looking for new conditions, in order to obtain the existence of the equilibrium in choice. Our assumptions are different from the ones proposed when the new theory was framed. They concern the properties of the sets of choice profiles. We are now exploring a method of proof based on continuous selection and fixed point theorems for correspondences defined by using the upper sections of the sets which form the game.

The rest of the paper is organised as follows: In Section 2, some notation, terminological convention, basic definition and results about correspondences and games in choice form are given. Section 3 contains existence results for equilibrium in choice.

2 Preliminaries and notation

Throughout this paper, we shall use the following notations and definitions:

Let $A$ be a subset of a vector space $X$, $2^A$ denotes the family of all subsets of $A$ and $coA$ denotes the convex hull of $A$. If $A$ is a subset of a topological space $X$, $clA$ denotes the closure of $A$ in $X$. If $T,T' : X \to 2^Y$ are correspondences, then $coT$ and $T \cap T' : X \to 2^Y$ are correspondences defined by $(coT)(x) := coT(x)$ and $(T \cap T')(x) := T(x) \cap T'(x)$, for each $x \in X$, respectively.

Given a correspondence $T : X \to 2^Y$, for each $x \in X$, the set $T(x)$ is called the upper section of $T$ at $x$. For each $y \in Y$, the set $T^{-1}(y) := \{ x \in X : y \in T(x) \}$ is called the lower section of $T$ at $y$. The correspondence $T^{-1} : Y \to 2^X$, defined by $T^{-1}(y) = \{ x \in X : y \in T(x) \}$ for $y \in Y$, is called the (lower) inverse of $T$.

Let $X, Y$ be topological spaces. The correspondence $T : X \to 2^Y$ is called lower semicontinuous if for each $x \in X$ and for each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x$ in $X$ so that $T(y) \cap V \neq \emptyset$ for each $y \in U$.
The following lemma will be useful to our study of existence of equilibrium in choice.

**Lemma 1** (see Yuan [15]). Let $X$ and $Y$ be two topological spaces and let $W$ be an open (resp. closed) subset of $X$. Suppose $T_1 : X \to 2^Y$, $T_2 : X \to 2^Y$ are upper semicontinuous (resp. lower semicontinuous) correspondences such that $T_2(x) \subset T_1(x)$ for all $x \in W$. Then the correspondence $T : X \to 2^Y$ defined by

$$T(x) = \begin{cases} T_1(x), & \text{if } x \notin W, \\ T_2(x), & \text{if } x \in W \end{cases}$$

is also upper semicontinuous (resp. lower semicontinuous).

Further, we present the main models of noncooperative games we will deal with in this paper. The corresponding notions of equilibrium are also recalled.

Let $(X_i)_{i \in N}$ be the family of the individual sets of strategies and let $X = \prod_{i \in N} X_i$.

The *normal form* of an $n$-person game is $(X, r_i)_{i \in N}$, where, for each $i \in N$, $X_i$ is a nonempty set (the set of individual strategies of player $i$) and $r_i$ is the preference relation on $X$ of player $i$. The individual preferences $r_i$ are often represented by utility functions, i.e. for each $i \in N$ there exists a real valued function $u_i : X \to \mathbb{R}$ (called the utility function of $i$), such that: $x_i y \iff u_i(x) \geq u_i(y), \forall x, y \in X$. Then the normal form of $n$-person game is $(X_i, u_i)_{i \in N}$.

We denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $X_{-i} = \prod_{i \neq j} X_i$ and $(x_{-i}, X_i) = \{(x_{-i}, x_i) : x_i \in X_i\}$.

The Nash equilibrium for the game $(X_i, u_i)_{i \in N}$ is a point $x^* \in X$ which satisfies for each $i \in N : u_i(x^*) \geq u_i(x^*_{-i}, x_i)$ for each $x_i \in X_i$.

For each $i \in N$, the player’s $i$’s best reply mapping is the correspondence $B_i : X_{-i} \to 2^{X_i}$, defined by $B_i(x_{-i}) = \{x_i \in X_i : u_i(x_{-i}, x_i) \geq u_i(x_{-i}, y_i) \text{ for each } y_i \in X_i\}$. Then, $x^* \in X$ is a Nash equilibrium if only if $x^* \in \bigcap_{i \in N} \text{Gr}(B_i)$.

The element $x^* \in X$ is called weak Nash equilibrium (Stefanescu, Ferrara, Stefanescu [9]):

$$u_i(x^*) \geq u_i(x^*_{-i}, x_i) \text{ for each } x_i \in X_i \text{ and } i \in N \text{ such that } B_i(x^*_{-i}) \neq \emptyset.$$ 

A qualitative game $\Gamma = (X_i, P_i)_{i \in N}$ is defined as a family of $n$ ordered triplets $(X_i, P_i)$, where for each $i \in N$: $P_i : X \to 2^{X_i}$ is a preference correspondence. An equilibrium for $\Gamma$ is a point $x^* \in X$ which satisfies for each $i \in N : P_i(x^*) = \emptyset$. A weak equilibrium (Stefanescu, Ferrara, Stefanescu [9]) of $\Gamma = (X_i, P_i)_{i \in N}$ is a point $x^* \in X$ which satisfies $P_i(x^*) = \emptyset$ for each $i \in N$ such that $\{x_i \in X_i : P_i(x^*_{-i}, x_i) = \emptyset\} \neq \emptyset$.

A choice profile (Stefanescu, Ferrara, Stefanescu [9]) is any collection $C := (C_i)_{i \in N}$ of nonempty subsets of $X$. A game in choice form is a double family $((X_i)_{i \in N}, (C_i)_{i \in N})$, where $C = (C_i)_{i \in N}$ is a choice profile.

We denote $C(x_{-i})$ the upper section through $x_{-i}$ of the set $C_i$ i.e., $C(x_{-i}) = \{y_i \in X_i : (x_{-i}, y_i) \in C_i\}$. 

We will make the following assumption:

(A) assume that for each \( x \in X \), there exists \( i \in N \) such that \( C_i(x_{-i}) \neq \emptyset \).

The game strategy \( x^* \in X \) is an \textit{equilibrium in choice} (denoted EC) (A. Stefanescu, M. Ferrara, V. Stefanescu [9]) if \( \forall i \in N, (x^*_i, X_i) \cap C_i \neq \emptyset \Rightarrow x^* \in C_i \), equivalently, \( x^*_i \in C_i(x_{-i}) \), for every \( i \in N \) for which \( C_i(x_{-i}) \neq \emptyset \). The strategy \( x^* \in X \) is a \textit{strong equilibrium in choice} (denoted SEC) if \( x^*_i \in \bigcap_{i \in N} C_i \).

3 Equilibrium results

This section provides a summary of different theorems concerning the existence of equilibria in choice for games in choice form. In order to underline the novelty and the importance of our work, we also must discuss here the additional benefit of obtaining corollaries which state, under new conditions, the existence of the weak Nash equilibria for games in normal form, or the existence of the weak equilibria of qualitative games. To prove our point, we will use continuous selection theorems or fixed point theorems for the correspondences that we will form, considering upper sections of the sets defining the game of the choice form. The advantage we have by doing this, deserves a great prominence in the assessment of the new assumptions which characterize the new statements. These results differ very much from the ones obtained by Stefanescu, Ferrara and Stefanescu in [9] and they link the recent research to the older approaches, regarding games in normal form or qualitative games. In order to suggest priorities to the reader, we keep the relationship between the main theorems and their consequences on particular games. For a better understanding of the paper, we recall all properties of the correspondences which will be used and the tools of proofs. Our study gives a new perspective of unifying of different approaches and results on the equilibrium concepts and the existence of noncooperative theory of games. Finally, we note that we obtain the existence of the strong equilibrium in choice for all situations considered in this section, if we suppose, in addition, that \( C_i(x_{-i}) \neq \emptyset \) for each \( x_{-i} \in X_{-i} \).

Let \( X, Y \) be topological spaces. We recall that the correspondence \( T : X \to 2^Y \) has the \textit{local intersection property} if \( x \in X \) with \( T(x) \neq \emptyset \) implies the existence of an open neighborhood \( V(x) \) of \( x \) such that \( \cap_{z \in V(x)} T(z) \neq \emptyset \).

To prove Theorem 1, we need the following lemma.

\textbf{Lemma 2} (Wu, Shen, \[13\]). Let \( X \) be a nonempty paracompact subset of a Hausdorff topological space \( E \) and \( Y \) be a nonempty subset of a Hausdorff topological vector space. Let \( S, T : X \to 2^Y \) be correspondence which verify:

\( a \) for each \( x \in X \), \( S(x) \neq \emptyset \) and \( \text{co} S(x) \subset T(x) \);
\( b \) \( S \) has the local intersection property.

Then, \( T \) has a continuous selection.
Our main result cites conditions which ensure the existence of equilibria in choice for a game in choice form in the lack of convexity of the upper sections of the sets $C_i$. The framework for our general next theorem consists of Hausdorff topological vector spaces. The proof is based on an argument that implies the above lemma.

**Theorem 1** Let $((X_i)_{i \in N}, (C_i)_{i \in N})$ be a game in choice form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;

b) there exists a nonempty subset $D_i$ of $C_i$ such that $W_i = \{x_i \in X_i : D_i(x_i) \neq \emptyset\}$ is closed and $D_i(x_i) \neq \emptyset$ if only if $C_i(x_i) \neq \emptyset$;

c) if $D_i(x_i) \neq \emptyset$, there exists an open neighborhood $V(x_i)$ of $x_i$ such that $\cap_{z_i \in V(x_i)} D_i(z_i) \neq \emptyset$;

d) $D_i(x_i)$ is convex or empty for each $x_i \in X_i$.

Then, the game admits equilibria in choice.

**Proof** For each $i \in N$, let us define the correspondences $S_i, T_i : X_i \to 2^{X_i}$, by

$$S_i(x_i) = \begin{cases} \co(\bigcup_{y_i \in D_i(y_i) \neq \emptyset} D_i(y_i)) & \text{if } x_i \notin W_i; \\ D_i(x_i) & \text{if } x_i \in W_i, \end{cases}$$

$$T_i(x_i) = \begin{cases} \co(\bigcup_{y_i \in C_i(y_i) \neq \emptyset} C_i(y_i)) & \text{if } x_i \notin W_i; \\ C_i(x_i) & \text{if } x_i \in W_i. \end{cases}$$

We call $T_i$ the correspondence of the upper sections of the sets $C_i$.

The correspondence $S_i$ has nonempty and convex values and $S_i(x_i) \subset T_i(x_i)$ for each $x_i \in X_i$.

Assumption c) implies that $S_i|_{(X_i \cap W_i)}$ has the local intersection property. If $x_i \in W_i$, then, $S_i(x_i) = \co(\bigcup_{y_i \in D_i(y_i) \neq \emptyset} D_i(x_i)) \neq \emptyset$. According to b), there exists an open neighborhood $V_i(x_i)$ of $x_i$ such that $D_i(z_i) \neq \emptyset$ for each $z_i \in V_i(x_i)$. Then, $S_i(z_i) = \co(\bigcup_{y_i \in D_i(y_i) \neq \emptyset} D_i(y_i))$ for each $z_i \in V_i(x_i)$ and, $\bigcap_{z_i \in V_i(x_i)} S_i(z_i) \neq \emptyset$. It follows that $S_i : X_i \to 2^{X_i}$ has the local intersection property.

The Wu-Shen Lemma implies that $T_i$ has a continuous selection $f_i : X_i \to X_i$.

Let $f : X \to X$ be defined by $f(x) = \prod_{i \in N} f_i(x_i)$ for each $x \in X$. The function $f$ is continuous, and, according to the Brouwer fixed point Theorem, there exists $x^* \in X$ such that $f(x^*) = x^*$. Hence, $x^* \in \prod_{i \in N} T_i(x^*_i)$ and obviously, $x^*_i \in T_i(x^*_i)$ for each $i \in N$. Suppose that $(x^*_i, X_i) \cap C_i \neq \emptyset$, for some $i \in N$. Then, $C_i(x^*_i) \neq \emptyset$ and $x^*_i \in C_i(x^*_i)$, which implies $x^* = (x^*_i, x^*_i) \in C_i$.

As a corollary, we obtain sufficient conditions for a game in normal form to admit weak Nash equilibria. The main assumption is new in literature and it refers to the existence of a best reply for each player $i$ to the common strategy of the other players, which lies in open intervals of the product
space $X_i$. The meaning is that each player $i$ can remain stable in the choice of his own best strategy in the situation that the decisions of the opponents can vary slightly in any manner profitable to themselves.

**Corollary 1** Let $((X_i)_{i \in N}, (u_i)_{i \in N})$ be a game in normal form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;
b) the set $\{x \in X : u_i(x) \geq u_i(x_{-i}, y_i) \}$ for each $y_i \in X_i$ is nonempty;
c) $W_i = \{x_{-i} \in X_{-i} : B_i(x_{-i}) \neq \emptyset \}$ is closed;
d) if $B_i(x_{-i}) \neq \emptyset$, there exists an open neighborhood $V(x_{-i})$ of $x_{-i}$ so that $\bigcap_{z_{-i} \in V(x_{-i})} B_i(z_{-i}) \neq \emptyset$;
e) $B_i(x_{-i})$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits weak Nash equilibria.

The following corollary concerns the existence of the weak equilibria for qualitative games.

**Corollary 2** Let $((X_i)_{i \in N}, (P_i)_{i \in N})$ be a qualitative game. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;
b) the set $\{x \in X : P_i(x) = \emptyset \}$ is nonempty and $W_i = \{x_{-i} \in X_{-i} : \exists x_i \in X_i$ such that $P_i(x_{-i}, x_i) = \emptyset \}$ is closed;
c) for each $x_{-i} \in X_{-i}$, if there exists $x_i \in X_i$ such that $P_i(x) = \emptyset$, then there exist an open neighborhood $V_i(x_{-i})$ of $x_{-i}$ and $z_i \in X_i$ such that $P_i(z_{-i}, z_i) = \emptyset$ for each $z_{-i} \in V_i(x_{-i})$;
d) $\{x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset \}$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits weak equilibria.

A new approach to the existence of equilibria in choice implies the concept of weakly convex graph for a correspondence, proposed by X. Ding and Y He in [2]. Firstly, we recall the definition.

**Definition 1** (see [2]). Let $X$ be a nonempty convex subset of a topological vector space $E$ and $Y$ be a nonempty subset of $E$. The correspondence $T : X \rightarrow 2^Y$ is said to have weakly convex graph (in short, it is a WCG correspondence) if for each $n \in \mathbb{N}$ and for each finite set $\{x_1, x_2, ..., x_n \} \subset X$, there exists $y_i \in T(x_i)$, $(i = 1, 2, ..., n)$, such that

$$\text{co}(\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}) \subset \text{Gr}(T).$$

We note that either the graph $\text{Gr}(T)$ is convex, or $\bigcap \{T(x) : x \in X\} \neq \emptyset$, then $T$ has a weakly convex graph.

It is obvious that a WCG correspondence may have nonempty values and may not be convex-valued.
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Example 1 \( T : [0, 2] \to [0, 2], \ T(x) = \begin{cases} \frac{1}{3} & \text{if } x = 1, \\ \frac{4}{3} - \frac{x}{2} & \text{if } x \in [0, 2] \setminus \{1\} \end{cases} \) is a WCG correspondence (since \( \cap \{ T(x) : x \in [0, 2] \} = \{0\} \neq \emptyset \)), but \( T(1) \) is not convex and \( \text{Gr}(T) \) is not convex either.

The theorem below is a continuous selection theorem for correspondences having weakly convex graph.

**Lemma 3** (Patricle, [6]). Let \( Y \) be a nonempty subset of a topological vector space \( E \) and \( K \) be a \((n - 1)\)-dimensional simplex in a topological vector space \( F \). Let \( T : K \to 2^Y \) be a WCG correspondence. Then, \( T \) has a continuous selection on \( K \).

The following lemma guarantees the existence of a fixed point for a product of lower semi-continuous correspondences. It will be useful for proving our second result on equilibria in choice.

**Lemma 4** (Wu, [12]). Let \( I \) be an index set. For each \( i \in I \), let \( X_i \) be a nonempty convex subset of a Hausdorff locally convex topological vector space \( E_i \), \( D_i \) a nonempty compact metrizable subset of \( X_i \) and \( S_i, T_i : X \to 2^{D_i} \) two correspondences with the following conditions:

1. for each \( x \in X \), \( \text{clco} S_i(x) \subset T_i(x) \) and \( S_i(x) \neq \emptyset \).
2. \( S_i \) is lower semi-continuous.

Then, there exists \( x^* = \prod_{i \in I} x_i^* \in D = \prod_{i \in I} D_i \) such that \( x_i^* \in T_i(x^*) \) for each \( i \in I \).

The assumption that each correspondence of the upper sections of the sets \( C_i \) has a selection which is an WCG correspondence also assures the existence of the equilibria in choice. The following theorem presents precisely this.

**Theorem 2** Let \( ((X_i)_{i \in N}, (C_i)_{i \in N}) \) be a game in choice form. Assume that, for each \( i \in N \), the following conditions are fulfilled:

a) \( X_i \) is a nonempty convex compact metrizable set in a locally convex space \( E_i \) and \( C_i \) is nonempty;

b) for each \( x_i \in X_i \), \( W_i = \{ x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset \} \) is a \((n - 1)\)-dimensional simplex in \( X \);

c) there exists a WCG correspondence \( S_i : W_i \to 2^{X_i} \) such that \( S_i(x_{-i}) \subset C_i(x_{-i}) \) for each \( x_{-i} \in W_i \);

d) \( C_i(x_{-i}) \) is convex or empty for each \( x_{-i} \in X_{-i} \).

Then, the game admits equilibria in choice.

Proof. Let be \( i \in I \). From the assumption (c) and the selection Lemma 3, it follows that there exists a continuous function \( f_i : W_i \to X_i \) so that for each \( x_{-i} \in W_i \), \( f_i(x_{-i}) \in S_i(x_{-i}) \subset C_i(x_{-i}) \).

Let us define the correspondence \( T_i : X_{-i} \to 2^{X_i} \), by
Let $B$ we can apply this theorem.

According to Wu’s fixed-point Theorem, applied for the correspondences $S_i = T_i$ and $T_i : X \to 2^{X_i}$, there exists $x^* \in X$ such that for each $i \in I$, $x_i^* \in T_i(x^*)$. Suppose that $(x_{-i}^*, X_i) \cap C_i \neq \emptyset$, for some $i \in N$. Then, $C_i(x_{-i}^*) \neq \emptyset$ and $x_i^* \in C_i(x_{-i}^*)$, which implies $x^* = (x_{-i}^*, x_i^*) \in C_i$.

**Remark 1** We can obtain two corollaries to the above theorem, if we replace assumption c) with:

(c') there exists a correspondence $S_i : W_i \to 2^{X_i}$ such that $S_i$ has a convex graph and $S_i(x_{-i}) \subset C_i(x_{-i})$ for each $x_{-i} \in W_i$;

or

(c") there exists a correspondence $S_i : W_i \to 2^{X_i}$ with closed values such that $S_i$ has the property that for any finite set $\{x_{-i}^1, x_{-i}^2, \ldots, x_{-i}^m\} \subset X$, $\bigcap_{j=1}^m S_i(x_{-i}^j) \neq \emptyset$ and $S_i(x_{-i}) \subset C_i(x_{-i})$ for each $x_{-i} \in W_i$.

**Proof.** In the first case, since a correspondence with a convex graph is a WCG one, it follows that $S_i$ verifies Assumption c) from Theorem 2, then we can apply this theorem.

In the second case, $X$ is a compact space and for each $i \in I$ the closed sets $S_i(x_{-i})$, $x_{-i} \in X_{-i}$ have the finite intersection property, then $\bigcap \{S_i(x_{-i}) : x_{-i} \in X_{-i}\} \neq \emptyset$. It follows that $S_i$ is a WCG correspondence and the conclusion comes from Theorem 2.

As in the first case, we obtain the following corollaries concerning the existence of the weak Nash equilibria for games in normal form and, respectively, of the weak equilibria for qualitative games.

**Corollary 3** Let $((X_i)_{i \in N}, (u_i)_{i \in N})$ be a game in normal form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;

b) the set $\{x \in X : u_i(x) \geq u_i(x_{-i}, y_i) \text{ for each } y_i \in X_i\}$ is nonempty;

c) for each $x_i \in X_i$, $W_i = \{x_{-i} \in X_{-i} : B_i(x_{-i}) \neq \emptyset\}$ is a $(n_i - 1)$-dimensional simplex in $X$;

d) there exists a WCG correspondence $S_i : W_i \to 2^{X_i}$ such that $S_i(x_{-i}) \subset B_i(x_{-i})$ for each $x_{-i} \in W_i$;

e) $B_i(x_{-i})$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits weak Nash equilibria.

**Corollary 4** Let $((X_i)_{i \in N}, (P_i)_{i \in N})$ be a qualitative game. Assume that, for each $i \in N$, the following conditions are fulfilled:
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a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;
b) the set $\{x \in X : P_i(x) = \emptyset\}$ is nonempty;
c) for each $x_i \in X_i$, $W_i = \{x_{-i} \in X_{-i} : \exists x_i \in X_i \text{ such that } P_i(x_{-i}, x_i) = \emptyset\}$ is a $(n_i - 1)$-dimensional simplex in $X$;
d) there exists a WCG correspondence $S_i : W_i \to 2^{X_i}$ such that $S_i(x_{-i}) \subset \{x_i \in X_i \text{ such that } P_i(x_{-i}, x_i) = \emptyset\}$ for each $x_{-i} \in W_i$;
e) $\{x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset\}$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits equilibria in choice.

Now, we present a continuous selection lemma on Banach spaces which was proposed by Yuan [15].

Lemma 5 (see [15]). Let $X$ be a paracompact space, $Y$ be a Banach space and $T : X \to 2^Y$ be a lower semicontinuous correspondences with closed convex values. Let $S : X \to 2^Y$ be a correspondence whose graph is open in $X \times Y$ such that $T(x) \cap S(x) \neq \emptyset$ for each $x \in X$. Then, there exists a continuous function $f : X \to Y$ such that $f(x) \in \text{co}(T(x) \cap S(x))$ for each $x \in X$.

The previous lemma leads us to the enunciation of Theorem 3, which gives new conditions under which the equilibria in choice exist.

Theorem 3 Let $((X_i)_{i \in N}, (C_i)_{i \in N})$ be a game in choice form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Banach space $E_i$;
b) $C_i$ is nonempty and open;
c) the set $W_i = \{x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset\}$ is nonempty and open;
d) there exists a lower semicontinuous correspondence $S_i : W_i \to 2^{X_i}$ with closed convex values such that $C_i(x_{-i}) \cap S_i(x_{-i}) \neq \emptyset$ for each $x_{-i} \in X_{-i}$;
e) $C_i(x_{-i})$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits equilibria in choice.

Proof Let be $i \in I$. From the assumption d) and the above lemma, it follows that there exists a continuous function $f_i : W_i \to X_i$ such that $f_i(x_{-i}) \in C_i(x_{-i}) \cap S_i(x_{-i})$ for each $x_{-i} \in W_i$.

Let us define the correspondence $T_i : X_{-i} \to 2^{X_i}$, by

$$
T_i(x_{-i}) := \begin{cases} 
\{f_i(x_{-i})\} & \text{if } x_{-i} \in W_i; \\
\text{co}(\bigcup_{y_{-i} \in C_i(y_{-i}) \neq \emptyset} C_i(y_{-i})) & \text{if } x_{-i} \notin W_i.
\end{cases}
$$

We appeal to Lemma 1 to conclude that $T_i$ is upper semicontinuous on $X$.

According to Kakutani’s fixed-point Theorem, there exists $x^* \in X$ such that for each $i \in I$, $x^*_i \in T_i(x^*_{-i})$. Suppose that $(x^*_i, X_i) \cap C_i \neq \emptyset$, for some $i \in N$. Then, $C_i(x^*_i) \neq \emptyset$ and $x^*_i = f_i(x^*_i) \in C_i(x^*_i)$, which implies $x^* = (x^*_i, x^*_i) \in C_i$.
As corollaries of Theorem 3, we obtain the following results which assume the lower semicontinuity of the involved correspondences. We note that these results are different from the old ones obtained in literature so far.

**Corollary 5** Let \( ((X_i),(u_i)) \) be a game in normal form. Assume that, for each \( i \in \mathbb{N} \), the following conditions are fulfilled:

a) \( X_i \) is a nonempty, convex and compact set in a Banach space \( E_i \);

b) the set \( \{x \in X : u_i(x) \geq u_i(x_i, y_i)\text{ for each } y_i \in X_i\} \) is nonempty and open;

c) \( W_i = \{x_{-i} \in X_{-i} : B_i(x_{-i}) \neq \emptyset\} \) is nonempty and open;

d) there exists a lower semicontinuous correspondence \( S_i : X_{-i} \to 2^{X_i} \), with closed convex values such that \( S_i(x_{-i}) \cap B_i(x_{-i}) \neq \emptyset \) for each \( x_{-i} \in X_{-i} \);

e) \( B_i(x_{-i}) \) is convex or empty for each \( x_{-i} \in X_{-i} \).

Then, the game admits weak Nash equilibria.

**Corollary 6** Let \( ((X_i),(P_i)) \) be a qualitative game. Assume that, for each \( i \in \mathbb{N} \), the following conditions are fulfilled:

a) \( X_i \) is a nonempty, convex and compact set in a Banach space \( E_i \);

b) the set \( \{x \in X : P_i(x) = \emptyset\} \) is nonempty and open;

c) \( W_i = \{x_{-i} \in X_{-i} : \exists x_i \in X_i \text{ such that } P_i(x_{-i}, x_i) = \emptyset\} \) is nonempty and open;

d) there exists a lower semicontinuous correspondence \( S_i : W_i \to 2^{X_i} \), with closed convex values such that \( S_i(x_{-i}) \cap \{x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset\} \neq \emptyset \) for each \( x_{-i} \in X_{-i} \);

e) \( \{x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset\} \) is convex or empty for each \( x_{-i} \in X_{-i} \).

Then, the game admits weak equilibria.

Further, we will prove the existence of the equilibrium in choice under new conditions. The proof we will provide explicitly relies on lemmas concerning the fixed points for the correspondences we will construct based on the upper sections of the sets \( (C_i)_{i \in \mathbb{N}} \).

First, we recall the following definition.

If \( X \) is a nonempty set and \( Y \) is a topological space, the correspondence \( T : X \to 2^Y \) is said to be transfer open-valued if for any \( (x, y) \in X \times Y \) with \( y \in T(x) \), there exists an \( x' \in X \) such that \( y \in \text{int}T(x') \).

Further, we present the following useful statement about the transfer open-valued correspondences (Proposition 1 in [3]).

**Lemma 6** Let \( Y \) be a nonempty set, \( X \) be a topological space and \( S : X \to 2^Y \) be a correspondence. The following assertions are equivalent:

a) \( S^{-1} : Y \to 2^X \) is transfer open-valued and \( S \) has nonempty values;

b) \( X = \bigcup_{y \in Y} \text{int}S^{-1}(y) \).

In this context, Ansari and Yao proved in [1] a fixed point result.
Existence of the equilibrium in choice

**Lemma 7** (Ansari, Yao [7]). Let $X$ be a compact convex subset of a Hausdorff topological vector space. Let $S : X \rightarrow 2^X$ be a correspondence with nonempty convex values. If $X = \cup \{ \text{int}_X S^{-1}(y) : y \in Y \}$ (or, $S^{-1} : Y \rightarrow 2^X$ is transfer open-valued), then, $S$ has fixed points.

We will apply the previous lemma in order to prove the existence of the equilibria in choice for games in choice form.

**Theorem 4** Let $((X_i)_{i \in N}, (C_i)_{i \in N})$ be a game in choice form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$ and $C_i$ is nonempty;

b) $X = \bigcup_{y \in X} \{ \text{int}_X \bigcap_{i \in N} (C_i(y_i)) \}$, where $W_i = \{ x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset \}$;

c) $C_i(x_{-i})$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits equilibria in choice.

**Proof** For each $i \in N$, let us define the correspondence $T_i : X_{-i} \rightarrow 2^{X_i}$, by

$$T_i(x_{-i}) = \begin{cases} \text{co}(\bigcup_{y_{-i}, C_i(y_{-i}) \neq \emptyset} C_i(y_{-i})) & \text{if } x_{-i} \notin W_i, \\ C_i(x_{-i}) & \text{if } x_{-i} \in W_i. \end{cases}$$

The correspondence $S_i$ has nonempty and convex values.

Let $T : X \rightarrow 2^X$ be defined by $T(x) = \prod_{i \in N} T_i(x_{-i})$ for each $x \in X$.

The correspondence $S$ also has nonempty and convex values.

If $y \in X$, then $T^{-1}(y) = \bigcap_{i \in N} \{ x \in X : y_i \in T_i(x_{-i}) \} = \bigcap_{i \in N} (C_i(y_i)) = \bigcup_{y \in X} \text{int}_X T^{-1}(y)$, according to assumption b).

We can apply the Ansari and Yao Lemma and we obtain that there exists $x^* \in X$ such that $x^* \in T(x^*)$. Obviously, $x^*_i \in T_i(x^*_{-i})$ for each $i \in N$. Suppose that $(x^*_i, x^*) \cap C_i \neq \emptyset$, for some $i \in N$. Then, $C_i(x^*_i) \neq \emptyset$ and $x^*_i \in C_i(x^*_i)$, which implies $x^* = (x^*_i, x^*_i) \in C_i$.

**Remark 2** According to Lemma 7, we can replace condition b) in Theorem 4 with

b') if $x_i \in C_i(x_{-i})$, then, there exists $z_i \in X_i$ such that $y_{-i} \in \text{int}_X \{ x_{-i} \in X_{-i} : z_i \in C_i(x_{-i}) \}$ and the set $W_i = \{ x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset \}$ is closed.

A new result involving the equilibria in choice will naturally follow directly from Theorem 4.

**Theorem 5** Let $((X_i)_{i \in N}, (C_i)_{i \in N})$ be a game in choice form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$ and $C_i$ is nonempty;

b) for each $x_i \in X_i$, $\{ x_{-i} \in X_{-i} : x_i \in C_i(x_{-i}) \} \cup C_i W_i$ is open, where $W_i = \{ x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset \}$;
c) \( C_i(x_{-i}) \) is convex or empty for each \( x_{-i} \in X_{-i} \).

Then, the game admits equilibria in choice.

Another proof of the above theorem appeals to Yannelis and Prabhakar’ continuous selection lemma \([14]\) applied for the correspondences \( T_i : X_{-i} \to 2^{X_i} \) defined by

\[
T_i(x_{-i}) = \begin{cases} 
\text{co}(\bigcup_{y_{-i} \in C_i(y_{-i}) \neq \emptyset} C_i(y_{-i})) & \text{if } x_{-i} \notin W_i; \\
C_i(x_{-i}) & \text{if } x_{-i} \in W_i.
\end{cases}
\]

for each \( i \in N \).

We present below the lemma.

**Lemma 8** (Yannelis and Prabhakar, \([14]\)). Let \( X \) be a paracompact Hausdorff topological space and \( Y \) be a Hausdorff topological vector space. Let \( T : X \to 2^Y \) be a correspondence with nonempty convex values and for each \( y \in Y \), \( T^{-1}(y) \) is open in \( X \). Then, \( T \) has a continuous selection that is, there exists a continuous function \( f : X \to Y \) so that \( f(x) \in T(x) \) for each \( x \in X \).

For the proof, we note that for each \( i \in N \), the correspondence \( T_i \) has nonempty and convex values and if \( x_i \in X_i \), then \( T_i^{-1}(x_i) = C_i W_i \cup \{ x_{-i} \in X_{-i} : x_{-i} \in C_i(x_{-i}) \} \) is an open set, according to assumption \( b \).

We can apply the Yannelis and Prabhakar Lemma and we obtain that there exists \( f_i : X_{-i} \to X_i \), a continuous selection of \( T_i \). Let \( f : X \to X \) be defined by \( f(x) = \prod_{i \in N} f_i(x_{-i}) \) for each \( x \in X \). The function \( f \) is continuous, and, according to the Brouwer fixed point Theorem, there exists \( x^* \in X \) such that \( f(x^*) = x^* \). Hence, \( x^* \in \prod_{i \in N} T_i(x^*_{-i}) \) and obviously, \( x^*_i \in T_i(x^*_{-i}) \) for each \( i \in N \). Suppose that \( (x^*_{-i}, X_i) \cap C_i \neq \emptyset \) for some \( i \in N \). Then, \( C_i(x^*_{-i}) \) is open, and the set \( W_i = \{ x_{-i} \in X_{-i} : x_{-i} \in C_i(x_{-i}) \} \) is closed, then the above Theorem holds.

**Remark 3** If for each \( x_i \in X_i \), \( \{ x_{-i} \in X_{-i} : x_{-i} \in C_i(x_{-i}) \} \) is open, and the set \( W_i = \{ x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset \} \) is closed, then the above Theorem holds.

Corollary 7 is mainly obtained by verifying an assumption concerning the union of all lower sections of the best reply correspondences for a game in normal form.

**Corollary 7** Let \( ((X_i)_{i \in N}, (u_i)_{i \in N}) \) be a game in normal form. Assume that, for each \( i \in N \), the following conditions are fulfilled:

- a) \( X_i \) is a nonempty, convex and compact set in a Hausdorff topological vector space \( E_i \);
- b) the set \( \{ x \in X : u_i(x) \geq u_i(x_{-i}, y_i) \text{ for each } y_i \in X_i \} \) is nonempty;
- c) \( X = \bigcup_{y_i \in X} \{ \text{int}_{X_i} \bigcap_{i \in N} (C_i W_i \cup B_i^{-1}(y_i)) \} \), where \( W_i = \{ x_{-i} \in X_{-i} : B_i(x_{-i}) \neq \emptyset \} \); 
- d) \( B_i(x_{-i}) \) is convex or empty for each \( x_{-i} \in X_{-i} \).

Then, the game admits weak Nash equilibria.
Remark 4 In Corollary 7, condition c) can also be replaced with

c') the best reply correspondence $B_i : X_{-i} \rightarrow 2^{X_i}$ is transfer open valued and the set $W_i = \{ x_{-i} \in X_{-i} : B_i(x_{-i}) \neq \emptyset \}$ is closed.

A new statement can be deduced explicitly from Corollary 7.

**Corollary 8** Let $((X_i)_{i\in N}, (u_i)_{i\in N})$ be a game in normal form. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;

b) the set $\{ x \in X : u_i(x) \geq u_i(x_{-i}, y_i) \}$ for each $y_i \in X_i$ is nonempty;

c) $B_i^{-1}(x_i) \cup \{ x_{-i} \in X_{-i} : B_i(x_{-i}) = \emptyset \}$ is open for each $x_i \in X_i$;

d) $B_i(x_{-i})$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits weak Nash equilibria.

The following results refer to the existence of weak equilibria for the qualitative games. They are consequences of Theorem 5.

**Corollary 9** Let $((X_i)_{i\in N}, (P_i)_{i\in N})$ be a qualitative game. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;

b) the set $\{ x \in X : P_i(x) = \emptyset \}$ is nonempty;

c) $X = \bigcup_{i \in N} \{ \text{int} X \cap (\bigcup_{i \in N} (W_i \cup \{ x_{-i} \in X_{-i} : P_i(x_{-i}) = \emptyset \})) \}$, where $W_i = \{ x_{-i} \in X_{-i} : \exists x_i \in X_i \text{ such that } P_i(x_{-i}, x_i) = \emptyset \}$;

d) $\{ x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset \}$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits weak equilibria.

Remark 5 In Corollary 9, condition c) can also be replaced with

c') if $P_i(y_{-i}, x_i) = \emptyset$, then, there exists $z_i \in X_i$ such that $y_{-i} \in \text{int} Y_{-i} \{ x_{-i} \in X_{-i} : P_i(x_{-i}, z_i) = \emptyset \}$ and the set $W_i = \{ x_{-i} \in X_{-i} : \exists x_i \in X_i \text{ such that } P_i(x_{-i}, x_i) = \emptyset \}$ is open.

**Corollary 10** Let $((X_i)_{i\in N}, (P_i)_{i\in N})$ be a qualitative game. Assume that, for each $i \in N$, the following conditions are fulfilled:

a) $X_i$ is a nonempty, convex and compact set in a Hausdorff topological vector space $E_i$;

b) the set $\{ x \in X : P_i(x) = \emptyset \}$ is nonempty;

c) $\{ x_{-i} \in X_{-i} : \exists x_i \in X_i \text{ such that } P_i(x_{-i}, x_i) = \emptyset \} \cup \{ x_{-i} \in X_{-i} : P_i(x_{-i}, x_i) \neq \emptyset \}$ is open for each $x_i \in X_i$;

d) $\{ x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset \}$ is convex or empty for each $x_{-i} \in X_{-i}$.

Then, the game admits weak equilibria.

A very great importance in the fixed point theory has Tarafdar’s fixed point Theorem, which we present below.
Lemma 9 (Tarafdar, [10]). Let \( \{X_i\}_{i \in I} \) be a family of nonempty compact convex sets, each in a topological vector space \( E_i \), where \( I \) is an index set. Let \( X = \prod_{i \in I} X_i \). For each \( i \in I \), let \( S_i : X \to 2^{X_i} \) be a correspondence such that

- a) for each \( x \in X \), \( S_i(x) \) is a nonempty, convex subset of \( X_i \);
- b) for each \( x_i \in X_i \), \( S_i^{-1}(x_i) \) contains a relatively open subset \( O_{x_i} \) of \( X \) such that
  \[
  \bigcup_{x_i \in X_i} O_{x_i} = X \quad (O_{x_i} \text{ may be empty for some } x_i).
  \]
  Then, there exists a point \( x \in X \) such that \( x \in S(x) = \prod_{i \in I} S_i(x_i) \), that is, \( x_i \in S_i(x) \) for each \( i \in I \), where \( x_i \) is the projection of \( x \) onto \( X_i \) for each \( i \in I \).

By using Lemma 9, we establish Theorem 6, which is a slightly different version of Theorem 4.

Theorem 6 Let \( (X_i)_{i \in I}, (C_i)_{i \in I} \) be a game in choice form. Assume that, for each \( i \in I \), the following conditions are fulfilled:

- a) \( X_i \) is a nonempty, convex and compact set in a Hausdorff topological vector space \( E_i \) and \( C_i \) is nonempty;
- b) for each \( x_i \in X_i \), \( \{x_{-i} \in X_{-i} : x_i \in C_i(x_{-i})\} \) contains a relatively open subset \( O_{x_i} \) of \( X_{-i} \), such that
  \[
  \bigcup_{x_i \in X_i} O_{x_i} = X_{-i} \quad (O_{x_i} \text{ may be empty for some } x_i);
  \]
- c) \( C_i(x_{-i}) \) is convex or empty for each \( x_{-i} \in X_{-i} \).

Then, the game admits equilibria in choice.

Proof For each \( i \in I \), let us define the correspondence \( T_i : X_{-i} \to 2^{X_i} \), by

\[
T_i(x_{-i}) = \begin{cases} \text{co}\bigcup_{y \in C_i(x_{-i}) \neq \emptyset} C_i(y_{-i}) & \text{if } x_{-i} \notin W_i; \\ C_i(x_{-i}) & \text{if } x_{-i} \in W_i, \end{cases}
\]
where \( W_i = \{x_{-i} \in X_{-i} : C_i(x_{-i}) \neq \emptyset\} \).

The correspondence \( T_i \) has nonempty and convex values.

If \( x_i \in X_i \), then \( T_i^{-1}(x_i) = \bigcap_{x_i \in X_i} T_i(x_{-i}) \). According to assumption b), for each \( x_i \in X_i \), \( T_i^{-1}(x_i) \) contains a relatively open subset \( O_{x_i} \) of \( X \) such that

\[
\bigcup_{x_i \in X_i} O_{x_i} = X_{-i} \quad (O_{x_i} \text{ may be empty for some } x_i).
\]

We can apply the previous lemma and we obtain that there exists \( x^* \in X \) such that \( x_i^* \in S_i(x^*) = T_i(x_{-i}^*) \) for each \( i \in I \). Suppose that \( x_{-i}^* \in X_{-i} \) and \( C_i \neq \emptyset \), for some \( i \in I \). Then, \( C_i(x_{-i}^*) \neq \emptyset \) and \( x_i^* \in C_i(x_{-i}^*) \), which implies \( x^* = (x_{-i}^*, x_i^*) \in C_i \).

Now, we get the following corollaries from the previous result.
Corollary 11 Let \((X_i)_{i \in N}, (u_i)_{i \in N}\) be a game in normal form. Assume that, for each \(i \in N\), the following conditions are fulfilled:

a) \(X_i\) is a nonempty, convex and compact set in a Hausdorff topological vector space \(E_i\);
b) the set \(\{x \in X : u_i(x) \geq u_i(x_{-i}, y_i) \text{ for each } y_i \in X_i\}\) is nonempty;
c) for each \(x_i \in X_i\), \(B_i^{-1}(x_i)\) contains a relatively open subset \(O_{x_i}\) of \(X\) such that

\[
\bigcup_{x_i \in X_i} O_{x_i} = X_{-i} \text{ (may be empty for some } x_i) ;
\]
d) \(B_i(x_{-i}) = \{x_i \in X_i : u_i(x) \geq u_i(x_{-i}, y_i) \text{ for each } y_i \in X_i\}\) is convex or empty for each \(x_{-i} \in X_{-i}\).

Then, the game admits weak Nash equilibria.

Corollary 12 Let \((X_i)_{i \in N}, (P_i)_{i \in N}\) be a qualitative game. Assume that, for each \(i \in N\), the following conditions are fulfilled:

a) \(X_i\) is a nonempty, convex and compact set in a Hausdorff topological vector space \(E_i\);
b) the set \(\{x \in X : P_i(x) = \emptyset\}\) is nonempty;
c) for each \(x_i \in X_i\), \(\{x_{-i} \in X_{-i} : P_i(x_{-i}, x_i) = \emptyset\}\) contains a relatively open subset \(O_{x_i}\) of \(X\) such that

\[
\bigcup_{x_i \in X_i} O_{x_i} = X_{-i} \text{ (may be empty for some } x_i) ;
\]
d) \(\{x_i \in X_i : P_i(x_{-i}, x_i) = \emptyset\}\) is convex or empty for each \(x_{-i} \in X_{-i}\).

Then, the game admits weak equilibria.

Remark 6 In a particular case, we can weaken condition b) of Theorem 6 by condition b'):

b') for each \(x_i \in X_i\), \(\{x_{-i} \in X_{-i} : x_i \in C_i(x_{-i})\} = O_{x_i}\) is an open subset of \(X\) such that

\[
\bigcup_{x_i \in X_i} O_{x_i} = X_{-i} \text{ (may be empty for some } x_i) .
\]
According to Lemma 7, this condition is equivalent with the fact that the correspondence \(T_i^{-1} : X_i \rightarrow 2^{X_{-i}}\) is transfer open-valued and \(T_i\) has nonempty values, where \(T_i : X_i \rightarrow 2^{X_i}\) is defined by \(T_i(x_{-i}) = C_i(x_{-i})\) for each \(x_{-i} \in X_{-i}\).

In this case, we obtain the following theorem concerning the existence of the strong equilibrium in choice.

Theorem 7 Let \((X_i)_{i \in N}, (C_i)_{i \in N}\) be a game in choice form. Assume that, for each \(i \in N\), the following conditions are fulfilled:

a) \(X_i\) is a nonempty, convex and compact set in a Hausdorff topological vector space \(E_i\) and \(C_i\) is nonempty;
b) for each \(x_i \in X_i\), \(\{x_{-i} \in X_{-i} : x_i \in C_i(x_{-i})\} = O_{x_i}\) is an open subset of \(X\) such that

\[
\bigcup_{x_i \in X_i} O_{x_i} = X_{-i} \text{ (may be empty for some } x_i) ;
\]
c) \(C_i(x_{-i})\) is convex or empty for each \(x_{-i} \in X_{-i}\).

Then, the game admits strong equilibria in choice.
4 Concluding remarks

Our study is a new perspective unifying different approaches and results on the equilibrium concepts and the existence of noncooperative theory of games. We have proposed to the reader a synthesis of theorems and consequences which state, under new conditions, the existence of the equilibrium for games in choice form, in normal form and also for qualitative games. Our approach differs essentially from the one of Stefanescu, Ferrara and Stefanescu, who proposed the new concept of game in choice form and the corresponding equilibrium in choice (2012). A further research may consist of the integration of all research instruments and perspectives. The advantage of using these new ideas is that they are more systematic and can cover more general situations. This paper reflects the integrity of this kind of thinking and can reopen the problem of the equilibrium existence under new conditions.

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