Depletion Forces in Athermally Sheared Mixtures of Disks and Rods

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We carry out numerical simulations to study uniformly sheared two-dimensional suspensions consisting of a mixture of athermal, frictionless, circular disks and elongated rods. We find that shear-induced fluctuations lead to depletion forces that cause rods to group in parallel oriented clusters. As the fraction of rods increases, this clustering increases, leading to an increase in the average rate of rotation of the rods, and a decrease in the magnitude of their nematic ordering.

Entropic excluded volume forces are known to play a key role in systems of elongated, aspherical particles. For hard rods in thermal equilibrium, Onsager\textsuperscript{1} explained the isotropic to nematic phase transition by such effects: as the particle packing increases, aligned particles have a smaller excluded volume, which reduces rotational entropy but causes an even greater increase in translational entropy, causing the system to transition to an orientationally ordered phase. A similar effect, known as the depletion force, was proposed by Oosawa and Asakura\textsuperscript{2} to describe the effective attraction between large particles in a colloid of smaller particles\textsuperscript{3}. Depletion forces are observed not only in thermally equilibrated systems, but also in athermal but vibrated dry granular systems, in particular mixtures of spheres and rods\textsuperscript{1,4,5}. Depletion forces are usually argued to be the basis for the \textit{“Brazil nut effect”}\textsuperscript{6–8} in which, upon shaking, large particles rise to the top of a size-polydisperse mixture of athermal hard particles.

Here we ask whether depletion forces can arise in strictly athermal granular systems undergoing a uniform linear deformation. When local fluctuations in the granular system arise solely from such global linear deformations, with no additional vibrations or mechanical agitation, can these fluctuations still result in the entropic effects that give rise to depletion forces?

Some of our previous work gives reason for doubt. For size-bidisperse but shape-monodisperse systems of circular disks and of elongated rods, where the ratio of big to small particle lengths is a modest 1.4, we have found the following. Isotropic compression of athermal rods, unlike thermally equilibrated rods, gives no nematic ordering as the packing increases\textsuperscript{1}. Bidisperse circular disks\textsuperscript{10} and bidisperse rods\textsuperscript{11} show no size segregation in steady-state simple shear; shearing tends to mix different particle sizes when starting from initial configurations that are more ordered. However, here we will give evidence that depletion forces do arise when mixtures of elongated rods and circular disks are subjected to uniform, steady-state, simple shearing.

We consider a two-dimensional (2D) athermal system of $N$ total particles, of which a fraction $f$ are size-monodisperse rods and the remaining $1-f$ are size-bidisperse circular disks. We take equal numbers of big and small disks with diameter ratio $D_b/D_s = 1.4$. For rods we use elongated 2D spherocylinders, composed of a rectangle of axis length $L$ capped by semi-circular endcaps of diameter $D_b$ (see inset to Fig. 1(a)). The asphericity of the spherocylinders is $\alpha = L/D_b = 4$, giving a tip-to-tip length of $L + D_b = 5D_b$. We use $N = 2048$ and consider systems with $N_{\text{rod}} = 1, 64, 128, 256, \text{and } 512$ rods, corresponding to fractions $f = 0.00049, 0.03125, 0.0625, 0.125, \text{and } 0.25$. A more geometric measure of the density of the rods is the ratio of the packing fraction of the rods $\phi_{\text{rod}}$ to the total packing fraction $\phi$ of all particles. Our cases correspond to $\phi_{\text{rod}}/\phi = 0.00392, 0.206, 0.350, 0.536, \text{and } 0.729$.

For the elastic contact interaction between particles we use a one sided harmonic potential as detailed in Ref.\textsuperscript{2}. A spherocylinder $i$ which is contact with spherocylinder $j$ feels a force $F_{ij}^{\text{el}} = (k_e/d_{ij})(1-r_{ij}/d_{ij})\hat{n}_{ij}$. Here $r_{ij}$ is the shortest distance between the axes of the two spherocylinders and $d_{ij} \equiv (D_i + D_j)/2$. Two particles are in contact whenever $r_{ij} < d_{ij}$; $\hat{n}_{ij}$ is the unit vector pointing normally inwards to spherocylinder $i$ at the point of contact with $j$, and $k_e$ is the stiffness of the repulsion. For contacts between a spherocylinder and a disk, or between two disks, we regard the disk as a spherocylinder with $L = 0$ and apply the same expression.

We model energy dissipation as a viscous drag between particles and a background host medium. If $\dot{\mathbf{r}}_i$ is the center of mass velocity of particle $i$, and $\dot{\theta}_i$ its angular velocity about the center of mass, then the local velocity at position $\mathbf{r}$ on the particle is $\mathbf{v}_i(\mathbf{r}) = \dot{\mathbf{r}}_i + \dot{\theta}_i \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_i)$. The dissipative force density is then $\mathbf{f}^{\text{dis}}(\mathbf{r}) = -k_d [\mathbf{v}_i(\mathbf{r}) - \mathbf{v}_{\text{host}}(\mathbf{r})]$, where $\mathbf{v}_{\text{host}}(\mathbf{r})$ is the local velocity of the host medium. Integrating over the area of the particle then gives the total dissipative force, $\mathbf{F}^{\text{dis}} = \int d^2 r \mathbf{f}^{\text{dis}}(\mathbf{r})$. We are interested in the case of linear deformations, for which $\mathbf{v}_{\text{host}}(\mathbf{r}) = \dot{\mathbf{r}} \cdot \hat{\mathbf{r}}$, where $\dot{\mathbf{r}}$ is a constant strain rate tensor.

The elastic and dissipative forces give rise to elastic and dissipative torques. The elastic torque is $\mathbf{\tau}^{\text{el}}_{ij} = \mathbf{s}_{ij} \times \mathbf{F}^{\text{el}}_{ij}$, where $\mathbf{s}_{ij}$ is the moment arm from the center of mass of $i$ to the point of contact with $j$. The dissipative torque is $\mathbf{\tau}^{\text{dis}}_{ij} = \int d^2 r (\mathbf{r} - \mathbf{r}_i) \times \mathbf{f}^{\text{dis}}(\mathbf{r})$. We then use overdamped equations of motion, $\mathbf{F}^{\text{dis}}_i + \sum_j \mathbf{F}^{\text{el}}_{ij} = 0$ and $\mathbf{\tau}^{\text{dis}}_i + \sum_j \mathbf{\tau}^{\text{el}}_{ij} = 0$, where the sum is over all $j$ in contact with $i$, to determine the translational and orienta-
tional trajectories, \( r_i(t) \) and \( \theta_i(t) \). We take as unit of length \( D = 1 \), unit of energy \( k_B = 1 \), and unit of time \( t_0 = D^2 k_B A_i / k_B = 1 \), where \( A_i \) is the area of particle \( i \). For simplicity we choose the viscous drag \( k_d \) to vary with particle size so that \( k_d A_i = 1 \) is the same for all particles. We integrate using the Heun method with step size \( \Delta t / t_0 = 0.02 \). See \([1,3,5]\) for further details.

For simple shearing at strain rate \( \dot{\gamma} \), we have \( \mathbf{v}_{\text{host}}(r) = \dot{\gamma} y \mathbf{\hat{x}} \). We start shearing from an initial configuration in which particles are placed at random positions and rods have random orientation, however care is taken so no two rods have axes that intersect, as that would correspond to the unphysical situation of one rod penetrating through another. We then shear at the fixed rate \( \dot{\gamma} = 10^{-5} \), using Lees-Edwards boundary conditions to impose the shear strain \([15]\).

To characterize the behavior of our system we compute the stress tensor \( \mathbf{p} [2,5] \), and the resulting pressure, \( p = [p_{xx} + p_{yy}] / 2 \), and shear stress, \( \sigma_{xy} = -p_{xy} \). We also measure the magnitude \( S_2 \) and orientation \( \theta_2 \) of the nematic order parameter \([4,5]\). As a measure of the parallel clustering of rods we define \( Z_{\text{side}} / Z_{\text{rod}} \), where \( Z_{\text{rod}} \) is the average number of contacts a rod has with any other particle, and \( Z_{\text{side}} \) is the average number of side-to-side contacts that a given rod has with other rods. A side-to-side contact is when two rods make contact along their respective flat sides.

In Fig. 1(a) we plot the pressure \( p \) vs the net shear strain \( \gamma = \dot{\gamma} t \), as the system is sheared at a packing \( \phi = 0.60 \), well below jamming. We show results for systems with \( N_{\text{rod}} = 64, 128, 256, \) and \( 512 \) rods. Because we start in a random initial configuration with many unphysically large particle overlaps, \( p \) is initially large. As we begin to shear, the system quickly relaxes these overlaps to small values, pushing the particles away from each other. The configurations obtained just after this initial quench are ones in which particles are evenly distributed throughout the system, so as to avoid large overlaps, but otherwise without any spatial correlations. As the system is further sheared, the pressure continues to relax, but now more slowly. Over a strain of \( \gamma \approx 10 \) the particles evolve into configurations representative of the sheared steady-state, after which the pressure stays constant, aside from small fluctuations.

In Fig. 1(b) we plot the corresponding value of the clustering parameter \( Z_{\text{side}} / Z_{\text{rod}} \) vs \( \gamma \). Not surprisingly, we see that \( Z_{\text{side}} / Z_{\text{rod}} \) increases as \( N_{\text{rod}} \) increases; the higher the density of rods, the greater the probability for there to be side-to-side contacts between them. More interesting, however, is the dependence of \( Z_{\text{side}} / Z_{\text{rod}} \) on the shear strain \( \gamma \) for fixed \( N_{\text{rod}} \). As \( \gamma \) increases, \( Z_{\text{side}} / Z_{\text{rod}} \) first takes a sharp drop, from the value of the random initial configuration to a small value characteristic of the configuration in which the initial large overlaps have relaxed, particles are more evenly spread throughout the system, but no correlations have yet been introduced by the shearing. Then, as the shearing continues, \( Z_{\text{side}} / Z_{\text{rod}} \) increases significantly, saturating to a constant value in the steady-state after a strain of roughly \( \gamma \approx 10 \), the same strain needed to relax the pressure to steady-state. The strong correlation between the behavior of \( p \) and \( Z_{\text{side}} / Z_{\text{rod}} \) is simple to understand. The clustering of rods with side-to-side contacts allows a more efficient packing of the system and thus a decrease in the system pressure.

Finally in Fig. 1(c) we plot the magnitude of the nematic order parameter \( S_2 \) vs \( \gamma \). It is well known that elong-
gated particles in an athermal shear flow show nematic orientational ordering [3]. We see that, similar to the behavior of $p$, $S_2$ rises rapidly from the value $S_2 \approx 0$ of the random initial state, starts to plateau, but only reaches its steady-state value after the strain $\gamma \approx 10$.

In Figs. 1(d), 1(e), and 1(f) we similarly show $p$, $Z_{\text{side}}/Z_{\text{rod}}$, and $S_2$ vs $\gamma$ for the larger packing $\phi = 0.85$ near jamming; a system of only size-bidisperse disks has $\phi_j^{(0)} = 0.8433$ [26], while a system of only size-bidisperse spherocylinders of $\alpha = 4$ has $\phi_j^{(4)} \approx 0.906$ [2]. We see behavior qualitatively the same as at the lower packing $\phi = 0.60$. The only significant differences are that the steady-state value of $p$ now depends strongly the number of rods $N_{\text{rod}}$, and the case $N_{\text{rod}} = 64$ decays to the steady-state more rapidly.

In Figs. 2(a) and 2(b) we show snapshots of typical configurations in the sheared steady-state of $N_{\text{rod}} = 64$ rods at packings $\phi = 0.60$ and $0.85$ respectively. The side-to-side clustering of the rods is evident to the eye. We also see examples where two parallel rods are separated by a single row of disks, as was previously observed in experiments on vibrated mixtures of rods and spheres [5].

Next we examine the effect that adding rods to the disks has on the rheology of the system. In Figs. 3(a) and (b) we plot the steady-state pressure $p$ and shear viscosity $\eta = \sigma_{xy}/\dot{\gamma}$ vs packing $\phi$, for a fixed strain rate $\dot{\gamma} = 10^{-5}$. We show results for $N_{\text{rod}} = 1$, 64, 128, 256, and 512. For comparison, we also show results for a system composed entirely of $N = 2048$ spherocylinders; in this case we take a size-bidisperse distribution to avoid spatial ordering. As $\phi$ increases, the dependence on $N_{\text{rod}}$ noticeably increases. This is primarily due to the dependence of the jamming $\phi_j$ on the density of rods; we expect that $\phi_j$ must vary from $\phi_j^{(0)} = 0.8433$ at a vanishingly low density of rods, to $\phi_j^{(4)} \approx 0.906$ as the system becomes entirely rods. Thus, at a fixed large packing $\phi \gtrsim \phi_j^{(0)}$, we see that the shear viscosity $\eta$ decreases as more rods are added to the system. In Fig. 3(c) we show the macroscopic friction $\mu = \sigma_{xy}/\dot{\gamma}$ vs $\phi$. In contrast to $\eta$, for fixed $\phi \gtrsim \phi_j^{(0)}$ we find that $\mu$ generally increases as $N_{\text{rod}}$ increases. In experiments, one often creates packings under the condition of constant pressure rather than constant volume. In Fig. 3(d) we therefore show the shear viscosity $\eta$ vs $N_{\text{rod}}$ at three different fixed values of constant pressure $p$ that put the system close to jamming (see horizontal dotted lines in Fig. 3(a)). In each case $\eta$ decreases slightly as $N_{\text{rod}}$ increases.

Finally we examine the rotational motion and orientational ordering of the rods. In a system of only rods, shearing serves both to orient the rods as well as to cause them to rotate [3, 17, 26]. So it is interesting to see how such behavior is modified when the rods are immersed in a background of disks. In Figs. 4(a) and 4(b) we plot the average angular velocity scaled by the strain rate $-\langle \dot{\theta}_i \rangle /\dot{\gamma}$ and the magnitude of the ensemble averaged nematic order parameter $S_2$ (see [3] for the nematic orientation $\theta_2$) vs the total packing fraction $\phi$, for the different values

![FIG. 2. Snapshots of configurations in the sheared steady-state of $N_{\text{rod}} = 64$ spherocylinders in a sea of 1984 size bidisperse disks at packing (a) $\phi = 0.60$ and (b) $\phi = 0.85$; redish hues are used for spherocylinders, while bluish hues are for circular disks. Systems are sheared at the rate $\dot{\gamma} = 10^{-5}$. Animations of these configurations are available in our Supplemental Material [5].](image)

![FIG. 3. For $N_{\text{rod}} = 1$, 64, 128, 256, 512, and 2048 spherocylindrical rods in a system with $N = 2048$ total particles: steady-state values of (a) pressure $p$, (b) shear viscosity $\eta = \sigma_{xy}/\dot{\gamma}$, and (c) macroscopic friction $\mu = \sigma_{xy}/\dot{\gamma}$ vs packing $\phi$; (d) $\eta$ vs $N_{\text{rod}}$ at the three different values of constant pressure $p$ indicated by the horizontal dotted lines in (a). For the case, $N_{\text{rod}} = 2048$, where all particles are rods, we use a size-bidisperse distribution of rods. The vertical dashed lines indicate the jamming transition of size-bidisperse disks, $\phi_j^{(4)} = 0.8433$. Error bars are smaller than the symbol size.](image)
We believe that the dependence of \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\) and \(S_2\) on the number of rods \(N_{\text{rod}}\) is closely related to the depletion forces that cause the rods to form parallel oriented clusters. For a rod of length \(\ell = (1 + \alpha)D_0\) in a dense packing to rotate, it is necessary to have a local packing fluctuation on the length scale \(\ell\), so that sufficient free volume opens up to allow the rod to rotate. Rods that are in parallel side-to-side contact have more local free volume than rods in isolation; that is the origin of the depletion force. The sliding of one rod over another is a relatively low energy fluctuation that facilitates packing fluctuations on the length scale \(\ell\), and so facilitates rod rotation. In contrast, a rod in isolation from other rods is surrounded by disks; the motion of any one disk creates a packing fluctuation on the length scale \(D\), and it would thus take a correlated motion of several disks to create sufficient free volume to allow the rod to rotate. Such correlated spatial motion is rare, and consequently we find that for a system with only a single isolated rod, the rod ceases to rotate on the strain scale \(\gamma \approx 100\) of our simulations. But as the fraction of rods \(N_{\text{rod}}/N\) increases, the clustering of rods as measured by \(Z_{\text{side}}/Z_{\text{rod}}\) increases (see Fig. 1), and hence the rate of rotation, \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\), increases. The increasing rate of rotation then leads to a decrease in the magnitude of the nematic ordering \(S_2\) [3].

In a recent experimental work [28] it was observed that the addition of elongated rod shaped particles to a quasi-2D granular system of glass beads increased the rate of discharge of the beads in hopper flow. As the number of rods initially increased, the rate of discharge increased. It was argued that the mechanism for this increasing discharge rate is the rotation of the rods near the surface layer that causes a secondary flow of the glass beads and a significant increase in the thickness of the flowing layer. While our simulations are spatially uniform and have no surface layer, our observation that increasing the fraction of rods increases the clustering of rods, which then results in a decrease in the shear viscosity at constant pressure (see Fig. 3(d)) as well as an increase in the average rate of rod rotation, may play some role in this effect.

We have also considered the behavior of mixtures of rods and disks under isotropic compression, as well as pure shear, in which the system is compressed in one direction while expanded at equal rate in the orthogonal direction. We find no evidence for depletion forces in isotropic compression, however we do find evidence in pure shear, although the magnitude of the effect seems smaller than for simple shear. Our results for these cases can be found in our Supplemental Material [5].

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FIG. 4. (a) Average angular velocity \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\) and (b) magnitude of the nematic order parameter \(S_2\) vs packing \(\phi\), for mixtures of size-bidisperse circular disks and size-monodisperse elongated rods (i.e., spherocylinders of \(\alpha = 4\)). The system has \(N = 2048\) total particles, and \(N_{\text{rod}} = 1, 64, 128, 256, 512\) rods. For comparison, results are also shown for a system of only \(N = 2048\) size-bidisperse spherocylinders. The strain rate is \(\dot{\gamma} = 10^{-5}\). Vertical dashed lines indicate the jamming transitions of a system of only size-bidisperse disks, \(\phi_j^{(0)} = 0.8433\), and of only size-bidisperse \(\alpha = 4\) spherocylinders, \(\phi_j^{(4)} \approx 0.906\). Of \(N_{\text{rod}} = 1\) to 512. In computing these quantities, we average only over the \(N_{\text{rod}} = fN\) rods, since the circular disks experience no collisional elastic torques and so rotate uniformly and do not order. For comparison, we show the same quantities for a system of only \(N = 2048\) size-bidisperse, \(\alpha = 4\), spherocylinders.

We see in Fig. 4 that the behavior of the mixture of rods and disks is qualitatively similar to that of only rods [3]. The angular velocity \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\) is non-monotonic, decreasing to a minimum and then increasing as \(\phi\) increases. The magnitude of the nematic order parameter \(S_2\) is similarly non-monotonic, increasing to a maximum and then decreasing as \(\phi\) increases. For the entire range of \(\phi\) we see that as \(N_{\text{rod}}\) decreases, \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\) decreases, while \(S_2\) increases; the fewer the number of rods, the more slowly they rotate and the more orientationally ordered they are. For the case of only a single rod, \(N_{\text{rod}} = 1\), we see that \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\approx 0\) within the estimated errors and \(S_2\) is close to unity. This indicates that, for the range of \(\phi\) shown, the angular motion of an isolated rod consists only of small angular deflections about a fixed direction. An isolated rod in a sea of sheared disks ceases to rotate, except at very low packings.

We believe that the dependence of \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\) and \(S_2\) on the number of rods \(N_{\text{rod}}\) is closely related to the depletion forces that cause the rods to form parallel oriented clusters. For a rod of length \(\ell = (1 + \alpha)D_0\) in a dense packing to rotate, it is necessary to have a local packing fluctuation on the length scale \(\ell\), so that sufficient free volume opens up to allow the rod to rotate. Rods that are in parallel side-to-side contact have more local free volume than rods in isolation; that is the origin of the depletion force. The sliding of one rod over another is a relatively low energy fluctuation that facilitates packing fluctuations on the length scale \(\ell\), and so facilitates rod rotation. In contrast, a rod in isolation from other rods is surrounded by disks; the motion of any one disk creates a packing fluctuation on the length scale \(D\), and it would thus take a correlated motion of several disks to create sufficient free volume to allow the rod to rotate. Such correlated spatial motion is rare, and consequently we find that for a system with only a single isolated rod, the rod ceases to rotate on the strain scale \(\gamma \approx 100\) of our simulations. But as the fraction of rods \(N_{\text{rod}}/N\) increases, the clustering of rods as measured by \(Z_{\text{side}}/Z_{\text{rod}}\) increases (see Fig. 1), and hence the rate of rotation, \(-\langle \dot{\theta}_i \rangle / \dot{\gamma}\), increases. The increasing rate of rotation then leads to a decrease in the magnitude of the nematic ordering \(S_2\) [3].

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Supplemental Material

**EQUATIONS OF MOTION**

The balance of elastic and dissipative forces and torques gives rise to the following equations of motion for the translational and rotational degrees of freedom,

\[
\dot{r}_i = v_{\text{host}}(r_i) + \frac{F_{\text{el}}^i}{k_d A_i}, \quad (\text{SM-1})
\]

\[
\dot{\theta}_i = -\dot{\gamma} f(\theta_i) + \frac{\tau_{\text{el}}^i}{k_d A_i I_i}, \quad (\text{SM-2})
\]

For simple shearing at the rate \(\dot{\gamma}\), with flow in the \(\hat{x}\) direction, the host velocity is \(v_{\text{host}}(r) = \dot{\gamma} y \hat{x}\). The rotational drive is given by \(f(\theta) = 1 - (\Delta I_i / I_i) \cos 2\theta_i / 2\). Here \(A_i\) is the area of the particle, \(I_i\) is the sum of the two eigenvalues of the particles’ moment of inertia tensor, while \(\Delta I_i\) is the absolute value of their difference.

For pure shearing, with compression along the \(\hat{y}\) direction and expansion along the \(\hat{x}\) direction, both at rate \(\dot{\gamma}\), the host velocity is \(v_{\text{host}}(r) = \dot{\gamma} |x \hat{x} - y \hat{y}| / 2\). The rotational drive is given by \(f(\theta) = (\Delta I_i / I_i) \sin 2\theta_i / 2\).

For isotropic compression at rate \(\dot{\gamma}\) the host velocity is \(v_{\text{host}}(r) = -\dot{\gamma} r\). In this case there is no rotational drive and so \(f(\theta) = 0\). Details of the derivation of these equations of motion can be found in Refs. [1-3].

**OBSERVABLES**

The stress tensor \(p\) is comprised of two pieces, one due to the elastic forces and one due to the dissipative forces. The elastic part is

\[
p^\text{el} = -\frac{1}{L_x L_y \sum_{i=1}^N} \Sigma^\text{el}, \quad \Sigma^\text{el}_i = \sum_j s_{ij} \otimes F^\text{el}_j, \quad (\text{SM-3})
\]

where \(s_{ij}\) is the moment arm from the center of mass of particle \(i\) to the point of contact with particle \(j\), and the sum is over all particles \(j\) in contact with \(i\). The dissipative part is

\[
p^\text{dis} = -\frac{1}{L_x L_y \sum_{i=1}^N} \Sigma^\text{dis}, \quad \Sigma^\text{dis}_i = \int d^2r (r - r_i) \otimes f^\text{dis}_i(r), \quad (\text{SM-4})
\]

where \(f^\text{dis}_i(r)\) is the dissipative force density and the integral is over the area of the particle. Details may be found in Refs. [2,3]. For most of our parameters, except at fairly low \(\phi\), we find that the dissipative contribution \(p^\text{dis}\) is negligible compared to the elastic contribution \(p^\text{el}\).

In two dimensions, the magnitude \(S_m\) and orientation \(\theta_m\) of the \(m\)-fold orientational order parameter can be written as

\[
S_m = \left[ \frac{1}{N'} \sum_i \cos m \theta_i \right]^2 + \left[ \frac{1}{N'} \sum_i \sin m \theta_i \right]^2, \quad (\text{SM-5})
\]

\[
\tan m \theta_m = \left[ \frac{1}{N'} \sum_i \sin m \theta_i \right] / \left[ \frac{1}{N'} \sum_i \cos m \theta_i \right], \quad (\text{SM-6})
\]

For the instantaneous values of \(S_m\) and \(\theta_m\) in a given configuration, the above sums are over all the \(N'\) non-circular particles in that configuration. For the ensemble average of \(S_m\) and \(\theta_m\), the terms \([\ldots]\) in the above should be taken as averages over all configurations in the ensemble. In the present work we are interested in the nematic orientational order, \(m = 2\).

**SIMPLE SHEAR**

Here we provide some additional results for simple shearing of the mixture of rods and disks. In Fig. 3a we plot the steady-state value of the orientational angle \(\theta_2\) of the nematic order parameter vs the packing \(\phi\), for mixtures of rods and disks with \(N_{\text{rod}} = 1, 64, 128, 256\) and \(512\) rods in a sea of bidisperse disks with \(N = 2048\) total particles. We also show \(\theta_2\) for a system of only 2048 bidisperse rods. It is interesting that, in the dense region near jamming, as \(N_{\text{rod}}\) decreases the orientation angle \(\theta_2\) increases, indicating a closer alignment of the rod with the direction of minimal stress, \(\theta = 45^\circ\). In Fig. 3b we show the steady-state value for the rod clustering parameter \(Z_{\text{side}} / Z_{\text{rod}}\) vs \(\phi\), for systems with \(N_{\text{rod}} = 64, 128, 256\) and \(512\) rods. It is interesting that \(Z_{\text{side}} / Z_{\text{rod}}\), for fixed \(N_{\text{rod}}\), varies relatively little over the entire range of \(\phi\). As noted in the main text, \(Z_{\text{side}} / Z_{\text{rod}}\) increases as \(N_{\text{rod}}\) increases. For both quantities the system is sheared at the strain rate \(\dot{\gamma} = 10^{-5}\).

**PURE SHEAR**

We now consider the behavior of the mixture of rods and disks under pure shearing. Under pure shearing, the orientation of rod shaped particles will relax to the direction of minimal stress, unlike the continuous rotation of particles that occurs under simple shearing. Thus, under pure shearing with compression along \(\hat{y}\), the orientation of the nematic order parameter quickly relaxes to \(\theta_2 = 0\).
Unlike simple shearing, where the Lees-Edwards boundary conditions allow us to shear to arbitrarily large total strains $\gamma$, in pure shear one compresses in one direction (here the $\hat{y}$ direction) so the system will shrink to too narrow a height if one strains to too large $\gamma$. It is thus not always possible to shear long enough to reach the steady-state with finite system sizes [3]. We consider here mixtures with $N_{rod} = 128$ and 512 rods and $N = 2048$ total particles. To allow for a larger total strain, we start with a system of aspect ratio $L_y/L_x = 12$, and shear until we reach $L_x/L_y = 12$. This allows us to reach a maximum total strain of $\gamma_{\text{max}} = 2\ln 12 \approx 4.97$. Our results are for a shear rate $\dot{\gamma} = 10^{-5}$ and are averaged over four independent runs starting from four different random initial configurations.

In Figs. SM-2(a) and SM-2(d) we plot the pressure $p$ vs the net shear strain $\dot{\gamma} = \gamma t$ at the packings $\phi = 0.60$, well below jamming, and at $\phi = 0.85$, slightly above jamming. Except for the few smallest $\gamma$ points, the data points here (and similarly for the other panels of Fig. SM-2) represent an average of the instantaneous values over a strain window of $\Delta \gamma = 0.2$. Since we can only shear to the relatively small $\gamma_{\text{max}} \approx 5$, we see that our systems have not quite reached the steady state; the pressure $p$ continues to change gradually, rather than plateauing to a constant, at the largest $\dot{\gamma}_{\text{max}}$. In Figs. SM-2(b) and SM-2(e) we plot the rod clustering parameter $Z_{\text{side}}/Z_{\text{rod}}$ vs $\gamma$ for $\phi = 0.60$ and 0.85, respectively. Although we have not quite reached the steady state, as the system is strained we see that $Z_{\text{side}}/Z_{\text{rod}}$ clearly increases from the small value obtained immediately after the quench from the random initial configuration, thus indicating the presence of depletion forces. As for simple shearing (see Fig. 1 of the main text) we see that $Z_{\text{side}}/Z_{\text{rod}}$ increases as $N_{\text{rod}}$ increases, though for the smaller $\phi = 0.60$ the values of $Z_{\text{side}}/Z_{\text{rod}}$ seem smaller than those found for simple shearing. Finally, in Figs. SM-2(c) and SM-2(f) we plot the magnitude of the nematic order parameter $S_2$ vs $\gamma$ for $\phi = 0.60$ and 0.85, respectively.

In Fig. SM-3(a) and SM-3(b) we plot the clustering parameter $Z_{\text{side}}/Z_{\text{rod}}$ vs $\gamma$ for $N_{rod} = 128$ and 512, respectively. Here we show results for a range of different

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**FIG. SM-1.** For mixtures of $N_{rod}$ size-monodisperse spherocylindrical rods of $\alpha = 4$, embedded in a sea of size-bidisperse circular disks with $N = 2048$ total particles, sheared at a strain rate of $\dot{\gamma} = 10^{-5}$: (a) Orientation angle $\theta$ vs packing $\phi$ for $N_{rod} = 1, 64, 128, 256$ and 512 rods. Also shown are results for a system of only 2048 size-bidisperse rods (no disks). (b) Rod clustering parameter $Z_{\text{side}}/Z_{\text{rod}}$ vs $\phi$ for $N_{rod} = 64, 128, 256$ and 512 rods. The vertical dashed lines locate the jamming packing of bidisperse disks, $\phi_j^{(0)} = 0.8433$, and bidisperse spherocylinders of $\alpha = 4$, $\phi_j^{(4)} \approx 0.906$.

**FIG. SM-2.** For systems of $N_{rod} = 128$ and 512 spherocylindrical rods of $\alpha = 4$ in a sea of size-bidisperse circular disks with $N = 2048$ total particles, undergoing pure shearing at a strain rate $\dot{\gamma} = 10^{-5}$: (a) pressure $p$, (b) fraction of contacts on a spherocylinder that are side-to-side with another spherocylinder, $Z_{\text{side}}/Z_{\text{rod}}$, and (c) magnitude of the nematic order parameter $S_2$ vs net strain $\gamma = \dot{\gamma}t$, for packing $\phi = 0.60$. Similarly, (d) pressure $p$, (e) fraction $Z_{\text{side}}/Z_{\text{rod}}$, and (f) $S_2$ vs $\gamma$ for $\phi = 0.85$. Each data point, except for the few smallest, represents an average of the instantaneous values over a strain window of $\Delta \gamma = 0.2$, so as to reduce fluctuations.
packings $\phi$. Comparing the large $\gamma$ values of $Z_{\text{side}}/Z_{\text{rod}}$ seen here in the steady state values found in simple shear, shown in Fig. SM-1(b), it seems that $Z_{\text{side}}/Z_{\text{rod}}$ varies more with the packing $\phi$ in pure shear as compared to simple shear.

Finally, although we have not quite reached the steady state, the plots of $S_2$ in Figs. SM-2(c) and SM-2(f) suggest that $S_2$ at the largest $\gamma$ is not far from its steady state value. For a rough estimate of that steady-state value we therefore compute as follows. We first compute the ensemble average of $S_2$ for each individual run, averaging only over configurations in the strain window $4 < \gamma < 5$, at the end of the run. We then average the resulting values of $S_2$ over the four different independent runs (for $N_{\text{rod}} = 1$ we use eight independent runs), and estimate the statistical error from the variance of those values. The resulting $S_2$ is plotted vs $\phi$ in Fig. SM-3. We show results for $N_{\text{rod}} = 1$, 128, and 512. As was found for simple shear (see Fig. 4(b) of the main text), we find that $S_2$ decreases as $N_{\text{rod}}$ increases. We thus conclude that depletion forces are present in a pure sheared system, though at some packings they may be smaller than we have found in simple shearing. Animations of pure shearing with different $N_{\text{rod}}$ are available as additional Supplemental Material.[6]

**ISOTROPIC COMPRESSION**

We now consider the behavior of the mixture of rods and disks under isotropic compression. Our results here are for a compression rate of $\dot{\gamma} = 10^{-6}$ and represent an average over 8 independent runs starting from different random initial configurations. At each compression step of strain increment $\Delta \gamma = \dot{\gamma} \Delta t$, the packing fraction increases by $\Delta \phi/\phi = 2 \Delta \gamma$. We will therefore plot our results vs $\phi$ rather than $\gamma$. We start our compression runs from a random initial configuration at the dilute packing $\phi_{\text{init}} = 0.25$.

In Fig. SM-3(a) we plot the pressure $p$ vs $\phi$ for systems with the different values of $N_{\text{rod}}$. We therefore present our results vs $\phi$ rather than $\gamma$. We start our compression runs from a random initial configuration at the dilute packing $\phi_{\text{init}} = 0.25$.

In Fig. SM-4 we plot the pressure $p$ vs $\phi$ for systems with the different values of $N_{\text{rod}}$. The vertical dashed lines indicate the jamming packings of systems of only bidisperse circular disks, $\phi_J^{(0)} = 0.8417$ [4], and only bidisperse $\alpha = 4$ spherocylinders $\Pi$, $\phi_J^{(4)} \approx 0.866$. Note, these values of $\phi_J$ for compression-driven jamming are lower than those for simple shear-driven jamming; this is particularly so for the case of spherocylinders, due to the nematic ordering that occurs for spherocylinders under shear [3] but not under compression [4]. We see that, unlike the behavior of $p$ in simple shear, as shown in Fig. 3(a) of the main text, there is relatively little dependence of $p$ on $N_{\text{rod}}$. The small dependence that exists shows $p$ to increase as $N_{\text{rod}}$ increases below $\phi_J^{(0)}$, but $p$ to decrease as $N_{\text{rod}}$ increases above $\phi_J^{(0)}$.

The weak dependence of $p$ on $N_{\text{rod}}$ we believe is due to the absence of orientational ordering of the compressed rods, as we have shown previously to be the case for a system of only size-bidisperse rods [1]. It is the ordering of the rods under shear that allows the system to pack more efficiently and to relax the pressure; this process is absent in compression. We argue for the absence of orientational ordering of the rods as follows. If the rods had completely random orientations, a finite number of rods would still possess some small finite nematic ordering as a statistical fluctuation. However we would expect the magnitude of that nematic ordering to scale with the number of rods as $S_2 \sim 1/\sqrt{N_{\text{rod}}}$, and so vanish in the infinite system limit. In Fig. SM-5(b) we therefore plot $\sqrt{N_{\text{rod}}} S_2$ vs $\phi$, for systems with $N_{\text{rod}} = 64$, 128, 256, and 512 rods. Error bars are determined from the variance of values found in the 8 independent compression runs. We see that $\sqrt{N_{\text{rod}}} S_2$ is, within the estimated errors, inde-
out inducing any correlations that might be created by compression. The values of \(Z_{\text{side}}/Z_{\text{rod}}\) so obtained are shown as the solid black circles in Fig. SM-5(c). We see that these values roughly approximate (indeed they are slightly larger than) the values obtained by compression of the initial dilute configuration. We thus conclude that the increasing \(Z_{\text{side}}/Z_{\text{rod}}\) found for \(N_{\text{rod}} = 512\) is simply an effect of the increasing density of particles. We conclude that no depletion forces develop from athermal isotropic compression of mixtures of rods and disks. Finally, in Fig. SM-5(d) we show a snapshot of a configuration of \(N_{\text{rod}} = 256\) rods, compressed above jamming to the packing \(\phi = 0.92\). Visual inspection is consistent with our result that there is no nematic ordering of the rods, and little tendency for them to group into parallel clusters. Animations of compressions with different \(N_{\text{rod}}\) are available as additional Supplemental Material [5].

FIG. SM-5. For systems of \(N_{\text{rod}} = 64, 128, 256, \text{and} 512\) spherocylindrical rods of \(\alpha = 4\) in a sea of size-bidisperse circular disks with \(N = 2048\) total particles, sheared at a strain rate of \(\dot{\gamma} = 10^{-6}\). (a) pressure \(p\), (b) magnitude of the nematic order parameter \(S_2\) scaled by \(\sqrt{N_{\text{rod}}}\), and (c) rod clustering parameter \(Z_{\text{side}}/Z_{\text{rod}}\) vs packing \(\phi\). In (a) the vertical dashed lines indicate the jamming packings of systems of only bidisperse circular disks, \(\phi_J^{(0)} = 0.8417\), and only bidisperse \(\alpha = 4\) spherocylinders, \(\phi_J^{(4)} \approx 0.866\). In (c) solid black circles represent values obtained from relaxing a random configuration with \(N_{\text{rod}} = 512\) at each \(\phi\) without compression. For clarity, in each panel symbols are shown only on a subset of data points. In (a) error bars are typical smaller than the data point symbol; in (b) and (c) representative error bars are shown on a subset of the data points. (d) Snapshot of a configuration of \(N_{\text{rod}} = 256\) rods at the densest packing \(\phi = 0.92\).

Finally we consider the rod clustering parameter \(Z_{\text{side}}/Z_{\text{rod}}\), which is plotted vs \(\phi\) in Fig. SM-5(c). For \(N_{\text{rod}} \leq 256\) we see that \(Z_{\text{side}}/Z_{\text{rod}}\) barely changes as the system is compressed and \(\phi\) increases. For \(N_{\text{rod}} = 512\), however, we see a steady increase in \(Z_{\text{side}}/Z_{\text{rod}}\) with increasing \(\phi\), although the values of \(Z_{\text{side}}/Z_{\text{rod}}\) found remain small compared to those found in shearing. To determine if this increase in \(Z_{\text{side}}/Z_{\text{rod}}\) is due to the development of depletion forces as the system is compressed, or whether it is just an effect of the increasing density of particles, we do the following. At each value of \(\phi = 0.05, 0.06, \ldots, 0.90\) we create a random initial configuration with \(N_{\text{rod}} = 512\) rods in the same manner that we do for simple shearing. We then relax the energy of that configuration by simulating the equations of motion Eq. [SM-1], only setting \(v_{\text{host}} = 0\) so there is no compression. This relaxation reduces the unphysically large particle overlaps of the initial random configuration, spreading the particles more evenly throughout the system, but with-

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