Abstract

We present non-parametric identification results for panel models in the presence of a vector of unobserved heterogeneity that is not additively separable in the structural function. We exploit the time-invariance and finite dimension of the heterogeneity to achieve identification of a number of objects of interest with the panel length fixed. Identification does not require that the researcher have access to an instrument that is uncorrelated with the unobserved heterogeneity. Instead the identification strategy relies on an assumption that some lags and leads of observables are independent conditional on the unobserved heterogeneity and some controls. The identification strategy motivates an estimation procedure based on penalized sieve minimum distance estimation in the non-parametric instrumental variables framework. We give conditions under which the estimator is consistent and derive its rate of convergence. We present Monte Carlo evidence of its efficacy in finite samples.

1 Model and Motivation

Consider the following structural model:

\[ Y_{it} = h_t(X_{it}, \eta_i) + \epsilon_{it} \]  \hfill (1)

\( Y_{it} \) is the observation of the dependent variable for a unit \( i \) at time \( t \), \( X_{it} \) is a column vector of regressors, \( \eta_i \) is a column vector that represents time-invariant individual heterogeneity and \( \epsilon_{it} \) is a scalar, unobserved, zero-mean time-varying disturbance. The ‘structural function’ \( h_t \) is not assumed to be of any particular parametric form. Note that the time subscript on the structural function allows for the possibility that the function varies over time. Throughout the discussion it is assumed that the model (1) captures the causal effect of \( X_{it} \) on \( Y_{it} \). In the language of the potential outcomes framework \( h_t(x, \eta_i) + \epsilon_{it} \) is the ‘potential outcome’ at time \( t \) of unit \( i \) for \( X_{it} \) set to the counterfactual level \( x \).
Data consist of \( t = 1, \ldots, T \) observations of \( Y_{it}, X_{it} \) and for each of \( i = 1, \ldots, N \) individuals. The unobserved random vector \( \eta_i \) is not assumed to be independent of nor uncorrelated with some components of \( X_{it} \). This is consistent with the idea that \( \eta_i \) captures underlying characteristics inherent to individual \( i \) which could determine both individual \( i \)'s potential outcomes as well as the observable characteristics of that individual.

The structural function in (1) is not separable in \( \eta_i \). This flexibility is a key feature of the model because it allows for treatment effects (the counterfactual response to a change in \( X_{it} \)) to differ across individuals. The presence of non-separable unobservables presents a substantial challenge for identification. It is well-known that even if exogenous and relevant instruments are available this is not sufficient for the identification of many objects of interest. In this paper we show that the time-invariance of the non-separable heterogeneity can be exploited to achieve both identification and consistent estimation of a range of objects of interest even with a fixed panel length. The identification and estimation strategies we present do not require the presence of instruments that are independent of or even uncorrelated with \( \eta_i \).

The assumption of time-invariance of unobservables has a long history of use in the panel literature. Indeed, the assumption that some unobservables are time-invariant is the foundation of textbook panel methods like fixed effects and first-differencing. These methods exploit time-invariance to provide consistent estimates in the presence of an endogenous scalar additive unobservable. The analysis in this paper can be seen as an extension of this literature that allows for an endogenous and potentially multi-dimensional unobservable that could enter into the structural function in an unrestricted manner. The estimation procedure we propose bears some resemblance to standard methods, in particular the estimators of Arellano & Bond (1991) and Holtz-Eakin et al. (1988) in dynamic panel models. Our procedure is straight-forward to implement, it is no more involved than standard non-parametric instrumental variables (NPIV) estimation as developed in Newey & Powell (2003) Chen & Pouzo (2015), Florens (2011), Horowitz (2011) and others. The method is also computationally light. We derive its asymptotic properties and carry out a Monte Carlo study to evaluate its performance in finite samples.

This paper follows a recent literature that examines identification and estimation in non-parametric settings using panel data with a fixed number of time periods \( T \). Freyberger (2015) studies a related class of non-linear and non-parametric panel models in which the vector of time-invariant unobservables \( \eta_i \) and the scalar error-term \( \epsilon_{it} \) enter the structural function though a scalar-valued term of the form \( \eta_i' \beta_t + \epsilon_{it} \), where \( \beta_t \) is a time-specific vector of coefficients to be estimated and \( \eta_i' \) denotes the transpose of the column vector \( \eta_i \). By contrast, in the model (1) the unobservables \( \eta_i \) enter into the structural function in an unrestricted fashion and the error term \( \epsilon_{it} \) is treated as an additive residual. Freyberger is particularly interested in the estimation of the distribution of \( \eta_i \) and his model can be understood as a factor model with a non-linear or non-parametric link function that depends on observables. Instead, in this paper the goal is to integrate over the heterogeneity conditional on observables, and
one does not need to estimate the distribution of $\eta_i$ even as an intermediate step of the estimation procedure proposed in Section 4. Hence the analysis in this paper is closer to the spirit of classic panel methods rather than to classic factor model methods. Evdokimov (2009) estimates a model that closely resembles (1). However he does not allow the structural function to be time-varying. Evdokimov requires that the heterogeneity $\eta_i$ be scalar and that $\epsilon_{it}$ be fully independent of $\eta_i$ conditional on $X_{it}$. Evdokimov is interested in identification and estimation of the joint distribution of $h_i(X_{it}, \eta_i)$ and $X_{it}$. By contrast, the results in this paper are restricted to identification and estimation of the average of the structural function conditional on observables.

The model (1) can be understood as a non-parametric analogue of the linear correlated random effects model as studied in Chamberlain (1992). This model can be written in the form:

$$Y_{it} = \alpha_i \lambda_t + X_{it} \beta_i + \epsilon_{it}$$

(2)

Where $\alpha_i$ and $\beta_i$ are individual-specific random coefficients that may be correlated with $X_{it}$ and $\lambda_t$ is a time effect that is the same across individuals. Note that the number of individual-specific parameters in the above is equal to the sum of the lengths of $X_{it}$ and $\lambda_t$ which needn’t be scalar. Suppose the lengths of these vectors sum up to some number $K$, then one could capture all the relevant individual heterogeneity in a vector $\eta_i$ of length $K$ (one simply sets $\eta_i = (\alpha_i; \beta_i)$, where ‘;’ denotes vertical concatenation).

One could generalize the model (2) by adding say, squares and cubes of the components of $X_{it}$ as regressors and allowing the coefficients on these regressors to be individual-specific. Of course, as the number of series terms increases the number of individual specific parameters, and hence the length of $\eta_i$, increases to infinity. Unfortunately Chamberlain’s identification and estimation strategy requires that the number individual-specific coefficients be less than the panel length $T$ and so for a given $T$ there is an upper bound on the number of series terms one could include. In fact, consistency of Chamberlain’s method generally requires that $T$ be much larger than the number of parameters (see Graham & Powell (2012)). However, if one is concerned with a finite number of confounders then it may be reasonable to upper bound the dimension of $\eta_i$ even if one is unsure how these confounders enter into the structural function. This interpretation motivates the approach in this paper.

The focus of this paper is on the estimation and identification of the ‘conditional average structural function’ (CSF). Before we formally define this term consider the simpler unconditional ‘average structural function’ (ASF) of $X_{it}$. We define the average structural function of $X_{it}$ as the function $ASF_i(\cdot)$ defined below:

$$ASF_i(x) = E[h_i(x, \eta_i)]$$

Where the expectation integrates over the distribution of the heterogeneity $\eta_i$. In words, $ASF_i(x)$ is the counterfactual average outcome over all agents at
time $t$ were $X_{it}$ exogenously set to level $x$. It is important to note that unless $X_{it}$ is independent of $\eta_i$, then in general $\text{ASF}_t(x_{it}) \neq E[h_t(x_{it}, \eta_i)|X_{it}]$.

In some cases a researcher might only be interested in the effect of a subset of the regressors. Suppose that the researcher is interested in a subvector of $X_{it}$ denoted by $X_{it}^{(1)}$. Denote the other regressors in $X_{it}$ by $X_{it}^{(2)}$, so that $X_{it}$ is partitioned as $X_{it} = (X_{it}^{(1)}; X_{it}^{(2)})$ (recall that ‘;’ denotes vertical concatenation). Then the ‘average structural function of $X_{it}^{(1)}$’ is the function given by:

$$\text{ASF}_t(x^{(1)}) = E_t[h_t(x^{(1)}, X_{it}^{(2)}, \eta_i)]$$

Where the expectation integrates over the joint distribution at time $t$ of the heterogeneity $\eta_i$ and other regressors $X_{it}^{(2)}$ in the population. For notational convenience, we use $h_t(x^{(1)}, X_{it}^{(2)}, \eta_i)$ to mean $h_t((x^{(1)}; X_{it}^{(2)}), \eta_i)$.

Note that the distribution of $X_{it}^{(2)}$ in the population could be different in different periods, the expectation above is taken over the time $t$ distribution of $X_{it}^{(2)}$ in the population and the time subscript on the expectation operator emphasizes this fact.

The average structural function can be understood as a special case of the conditional average structural function. Let $S_{it}$ be some observable characteristics upon which the researcher wishes to condition. $S_{it}$ is generally assumed to be some subset of the observed outcomes and regressors for individual $i$. Let $S$ be a measurable subset of the support of $S_{it}$ with $Pr_t[S_{it} \in S] > 0$. The ‘$t$’ subscript on ‘Pr’ indicates that the probability is with respect to the population distribution of the included random variables at time $t$. The conditional average structural function of $X_{it}$ conditional on $S_{it} \in S$ is a set function defined by:

$$\text{CSF}_t(x, S) = E_t[h_t(x, \eta_i)|S_{it} \in S] \quad (3)$$

Where again the conditional expectation is taken over the distribution of the heterogeneity $\eta_i$ in the population at time $t$. In words, $\text{CSF}_t(x, S)$ is the counterfactual average outcome at time $t$ from an exogenous change of $X_{it}$ to $x$ for the subgroup of agents for whom $S_{it} \in S$.

Again, the researcher may be interested in the effect of an exogenous change to just a subset of the regressors. Suppose the researcher is interested in the effect of a subvector $X_{it}^{(1)}$ of $X_{it}$, with $X_{it}$ partitioned as $X_{it} = (X_{it}^{(1)}; X_{it}^{(2)})$. The CSF of $X_{it}^{(1)}$ conditional on $S_{it} \in S$ is then defined as:

$$\text{CSF}_t(x^{(1)}, S) = E_t[h_t(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S]$$

Where the conditional expectation is taken over the joint distribution of $\eta_i$ and $X_{it}^{(2)}$ in the sub-population for whom $S_{it} \in S$ at time $t$.

As mentioned above, the average structural function is a special case of the CSF. If $S = S$ (where $S$ is the support of $S_{it}$) then $\text{CSF}_t(x^{(1)}, S) = \text{ASF}_t(x^{(1)})$.

If $S_{it}$ is continuously distributed on some subset of its support that includes a point $s$ then $\text{CSF}_t(x^{(1)}, s)$ is not well defined (note that $Pr_t[S_{it} = s] = 0$). However, it will sometimes be convenient to write $\text{CSF}_t(x^{(1)}, S_{it})$. In which
case $CSF_t(x^{(1)}, S_{it})$ is understood to denote a random variable so that for any measurable subset $S$ of $\mathcal{S}$:

$$E_t[CSF_t(x^{(1)}, S_{it})|S_{it} \in S] = CSF_t(x^{(1)}, S)$$

Many other objects of interest can be written in terms of a CSF function and the distribution of the regressors. The average treatment effect can be written in terms of a CSF function, and in the case of a binary treatment so too can the average effect of treatment on the treated. If the structural function is differentiable then conditional average marginal effects are derivatives of the CSF function in its first argument. In Chamberlain’s analysis of the linear random coefficients model, the object of interest is the vector of means of the individual-specific coefficients which is, in the context of his model, equivalent to estimation of the average structural function of $X_{it}$. However, there are certainly interesting objects that cannot be written in terms of the CSF and the distribution of $X_{it}$. In particular, quantile treatment effects and conditional quantile treatment effects cannot be written in this way.

**Application 1: Demand Response to a Proportional Change in Prices**

Let us consider a concrete example. Let $Y_{it}$ be the demand of a household $i$ for some food product at time $t$. In this case $X_{it}$ might be a vector of product prices and observed household characteristics and $\eta_i$ might capture unobserved household characteristics. For instance $\eta_i$ may capture tastes in food, wealth, attitudes towards money, all of which are plausibly constant over the duration of the data. $\epsilon_{it}$ will then capture the influence of other unobserved time-varying factors.

Suppose a researcher wishes to forecast the effect on demand for soda of a proportional sales tax on soda that is expected to raise prices by $100 \times \alpha\%$. In that case the counterfactual question of interest is “what would average demand for soda have been at time $t$ had all consumers faced soda prices that were $100 \times \alpha\%$ higher”. Let us split the regressors into the price of soda $p_{it}$ and all other prices and observed household characteristics $X_{it}^{(2)}$. The object of interest in this case is:

$$E_t[h_t((1 + \alpha)p_{it}, X_{it}^{(2)}, \eta_i)]$$

This object can be written in terms of the CSF of $p_{it}$ with the conditioning variable also $p_{it}$ as follows:

$$E_t[h_t((1 + \alpha)p_{it}, X_{it}^{(2)}, \eta_i)] = E_t[E_t[h_t((1 + \alpha)p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}]]$$

$$= E_t[CSF_t((1 + \alpha)p_{it}, p_{it})]$$

Suppose that $\epsilon_{it}$ is jointly mean independent of $p_{it}$ and $X_{it}^{(2)}$. The true model is then:

$$Y_{it} = h_t(p_{it}, X_{it}^{(2)}, \eta_i) + \epsilon_{it}$$

$$E_t[\epsilon_{it}|p_{it}, X_{it}^{(2)}] = 0$$
Note that $E_t[\epsilon_{it}|p_{it}, X_{it}^{(2)}]$ is a random variable. The notation $E_t[\epsilon_{it}|p_{it}, X_{it}^{(2)}] = 0$ should then be understood to mean that equality holds almost surely, that is $Pr_t[E_t[\epsilon_{it}|p_{it}, X_{it}^{(2)}] = 0] = 1$. This convention is used throughout the paper.

When faced with this model the empirical researcher might be tempted to ignore the heterogeneity and the panel structure and estimate a model of the following form using non-parametric regression methods:

$$Y_{it} = q_t(p_{it}, X_{it}^{(2)}) + \epsilon_{it}$$

(4)

$E_t[\epsilon_{it}|p_{it}, X_{it}^{(2)}] = 0$

Note that the function $q_t(\cdot)$ in the model above is related to the structural function in the true model by:

$q_t(p_{it}, X_{it}^{(2)}) = E_t[h_t(p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}, X_{it}^{(2)}]$

To see this note that the original model implies:

$$Y_{it} = E_t[h_t(p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}, X_{it}^{(2)}] + h_t(p_{it}, X_{it}^{(2)}, \eta_i) - E_t[h_t(p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}, X_{it}^{(2)}] + \epsilon_{it}$$

And note that:

$E_t[h_t(p_{it}, X_{it}^{(2)}, \eta_i) - E_t[h_t(p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}, X_{it}^{(2)}] + \epsilon_{it}|p_{it}, X_{it}^{(2)}] = 0$

Let us calculate the change in demand implied by the true model. If $X_{it}$ increases by $100 \times \alpha$% then the true model (1) implies that:

$$\Delta E_t[Y_{it}] = E_t[E_t[h_t((1 + \alpha)p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}, X_{it}^{(2)}] - E_t[h_t(p_{it}, X_{it}^{(2)}, \eta_i)|p_{it}, X_{it}^{(2)}]] = E_t[CSF_t((1 + \alpha)p_{it}, p_{it}) - CSF_t(p_{it}, p_{it})]$$

But if we ignore the heterogeneity and use (4) to perform the counterfactual analysis we will instead estimate:

$$\Delta E_t[Y_{it}] = E_t[q_t((1 + \alpha)p_{it}, X_{it}^{(2)}) - q_t(p_{it}, X_{it}^{(2)})] = E_t[CSF_t((1 + \alpha)p_{it}, (1 + \alpha)p_{it}) - CSF_t(p_{it}, p_{it})]$$

So unless $CSF_t$ is flat in its second argument (which is true if $\eta_i \perp p_{it}$) the estimation using the model given by (4) will generally give biased and inconsistent estimates of the change in average demand. The size of the bias will be equal to:

$$E_t[CSF_t((1 + \alpha)p_{it}, p_{it}) - CSF_t((1 + \alpha)p_{it}, (1 + \alpha)p_{it})]$$

Note that this problem persists even under the assumption that:

$E_t[\epsilon_{it}|\eta_i, p_{it}, X_{it}^{(2)}] = 0$
And in fact the asymptotic bias from ignoring the heterogeneity will be exactly
the same.

In the demand setting one may be particularly worried that differences in
household wealth, which are often unobserved, are associated with demand and
also with the prices consumers face. Less wealthy consumers may be more
price sensitive than wealthier consumers and are likely to have lower product
demand in general because they face tighter budget constraints. Less wealthy
consumers may live in less affluent neighborhoods and due a greater sensitivity
to the price of petroleum less wealthy consumers may be reluctant to travel
further to cheaper retailers. Therefore retailers in less affluent areas may have
more market power and charge higher prices than retailers located in wealthier
areas for the same goods. If this is the case then greater price sensitivity of
demand and a lower over-all level of demand are associated with higher prices.
Thus a non-parametric regression analysis that ignores the heterogeneity will
overstate the effect on average demand of an across-the-board proportional price
increase.

2 Overview of The Identification and Estimation
Strategy

The purpose of this section is to provide intuition for the formal results pre-
sented in subsequent sections. To preserve parsimony, we focus on the case of a
researcher interested in the conditional average structural function of the whole
vector of regressors \( X_{it} \) (as opposed to some subvector \( X^{(1)}_{it} \)). Furthermore, we
do not consider the case in which \( \epsilon_{it} \) is known to be mean independent of a sub-
set of the regressors \( X_{it} \) at time \( t \). When \( \epsilon_{it} \) is known to be mean independent
of some components of \( X_{it} \) one can relax some of the assumptions mentioned
in this section. In later sections we consider the more general case in which
the researcher is only interested in the conditional average structural function
of a subset of the regressors and the case in which the time-varying shock \( \epsilon_{it} \) is
known to be mean independent of some of the regressors at time \( t \).

Our identification strategy relies on finding a set of observations for an in-
dividual that can be used to proxy for the unobserved heterogeneity. Because
these proxies need not exactly control for \( \eta_i \) they create a measurement error
problem. This necessitates a non-parametric instrumental variables (NPIV) ap-
proach. In theory any NPIV estimator could be used but we focus the Penalized
Sieve Minimum Distance procedure analyzed by [Chen & Pouzo (2012)].

Proxies and Instruments

The method presented in this paper relies on the availability of two observ-
able random vectors \( V_{it} \) and \( Z_{it} \). The first of these, \( V_{it} \) serves as a proxy for
the unobserved heterogeneity \( \eta_i \). The second \( Z_{it} \) serves as a vector of instru-
ments. The high-level conditions for identification provided in Section 3 apply
for general \( V_{it} \) and \( Z_{it} \), however in that same section we also analyze a number
of special cases of the model and in that analysis and throughout most of the
discussion, it is assumed that $V_{it}$ and $Z_{it}$ are each composed of lags and leads
of the regressors.

For example $V_{it}$ could contain all the observations of regressors $X_{is}$ for all
$s > t$ and $Z_{it}$ could contain $X_{is}$ for all $s < t$, that is:

$$V_{it} = (X_{i,t+1}; X_{i,t+2}; \ldots; X_{i,T})$$

$$Z_{it} = (X_{i,1}; X_{i,2}; \ldots; X_{i,t-1})$$

These vectors must have the following properties. First of all the time-
varying shock $\epsilon_{it}$ must be mean-independent of the instruments $Z_{it}$:

$$E_t[\epsilon_{it}|Z_{it}] = 0$$

The condition above is listed as Assumption 1.3 in Section 3. For the choice
of $Z_{it}$ above this condition amounts to ‘predetermination’ of the regressors. The
assumption of predetermination is used extensively in the panel literature (see
e.g. Arellano & Bond (1991)). Note that the mean-independence condition
clearly implies that $Z_{it}$ cannot contain $Y_{it}$ (however, it could contain $Y_{is}$ for
some $s \neq t$).

We also require $Z_{it}$, $X_{it}$, $V_{it}$ and $S_{it}$ satisfy a conditional independence
assumption. Specifically, the vectors $Z_{it}$, $X_{it}$ and $S_{it}$ must be jointly independent
of $V_{it}$ conditional on $\eta_i$, formally:

$$(Z_{it}, X_{it}, S_{it}) \perp V_{it}|\eta_i$$  \hspace{1cm} (5)$$

We must emphasize that this independence is conditional, the proxies $V_{it}$
could be highly dependent on $Z_{it}$, $X_{it}$ and $S_{it}$ when the unobserved individual
specific-characteristics $\eta_i$ are not conditioned upon. Note that if $S_{it}$ is a function
of $X_{it}$ then the condition simplifies to $(Z_{it}, X_{it}) \perp V_{it}|\eta_i$.

In the case in which the researcher is only interested in a subvector of $X_{it}$
denoted $X_{it}^{(1)}$ the condition above can be weakened to:

$$(Z_{it}, X_{it}, S_{it}) \perp V_{it}|\eta_i, X_{it}^{(2)}$$

Where $X_{it}^{(2)}$ is the random vector that contains those components of $X_{it}$ not
included in $X_{it}^{(1)}$. The condition above is Assumption 1.2 stated in the next
section.

Again, note that the independence above is conditional on $\eta_i$ and $X_{it}^{(2)}$. $V_{it}$
could be very strongly associated with $X_{it}^{(1)}$ and $Z_{it}$ but if this association results
from a mutual dependence on some common latent variables $\eta_i$ and observables
$X_{it}^{(2)}$ then the conditional independence is not violated.

\footnote{To see why this is a weaker assumption note that for any random variables $X$, $Y$, $Z$, if $(X, Y) \perp Z$ then $X \perp Z|Y$ and note that $(Z_{it}, X_{it}, S_{it}) \perp V_{it}|\eta_i, X_{it}^{(2)}$ is equivalent to $(Z_{it}, X_{it}^{(1)}, S_{it}) \perp V_{it}|\eta_i, X_{it}^{(2)}$.}
One special case in which $V_{it}$ and $Z_{it}$ can be found that satisfy the conditional independence assumption is the setting in which the time periods are ‘exchangeable’. Exchangability holds for example if the different values of $t$ do not represent time periods at all but say, different cross-sectional observations belonging to a given cluster $i$. In this case the regressors $\{X_{i1}, X_{i2}, ..., X_{iT}\}$ can be thought of as drawn identically and independently from a distribution that depends on a vector of individual-specific and time-invariant latent variables $\xi_i$. Note that the structural function $h_t$ need not depend on all components of $\eta_i$ and so one can simply define $\eta_i$ so that $\xi_i$ is a subvector of $\eta_i$ even if $\xi_i$ may not directly enter the structural function. In fact, there is a more general sense in which $\eta_i$ can be replaced by (rather than appended with) $\xi_i$ that is captured formally in Lemma 2 in the next section. Settings in which exchangability may hold are discussed in Section 3.

Other important data generating processes for the regressors in which one can typically choose $Z_{it}$ and $V_{it}$ that satisfy conditional independence include a stationary moving average process conditional on some latent variables and a Markov process conditional on some latent variables. Both of these cases are analyzed formally in Section 3.

Note that it may not be possible to form $V_{it}$ and $Z_{it}$ with the required properties for all $t$. In this case our results only identify and allow for estimation of the CSF$_t$ for those $t$ such that a $V_{it}$ and $Z_{it}$ with the required properties exist. Of course if we assume that CSF$_t$ does not vary over time then one need only find one such $t$.

Instrumenting for The Unobservables

Suppose that the researcher has available observables $Z_{it}$ and $V_{it}$ that are known to satisfy the conditions above (i.e. Assumptions 1.2 and 1.3 as listed in the next section).

If $\eta_i$ were observed then given $E_t[\epsilon_{it}|Z_{it}] = 0$ the researcher could estimate the structural function $h_t$ by solving the following NPIV conditional moment condition:

$$E_t[Y_{it} - h_t(X_{it}, \eta_i)|Z_{it}] = 0$$

(6)

For $h_t$ to be the unique solution (up to a null set) to the above one needs a completeness assumption to hold. Specifically, if $h_t$ is assumed to be in $L_2(\mathcal{X}, E, F_{X_{it}, \eta_i})$ then the condition above has a unique solution if for any $\delta \in L_2(\mathcal{X}, E, F_{X_{it}, \eta_i})$:

$$E_t[\delta(X_{it}, \eta_i)|Z_{it}] = 0 \iff \delta(X_{it}, \eta_i) = 0$$

Where $\mathcal{X}$ is the support of $X_{it}$, $E$ is the support of $\eta_i$ and $F_{X_{it}, \eta_i}$ is the joint distribution of $X_{it}$ and $\eta_i$. The condition above is given in Assumption 2.c in the next section, and in a weaker form in Assumption 1.4.

In particular the above is an $L_2$-completeness assumption. Completeness can be understood as the non-parametric analogue of the ‘rank’ condition for identification (see Newey & Powell (2003) for discussion of completeness in an
NPIV setting and Andrews (2017) for discussion of $L_2$-completeness). Sufficient conditions for various notions of completeness are given in D'Haultfoeuille (2011), Andrews (2017) and Hu & Shiu (2018).

Of course, one cannot use the moment condition above directly because $\eta_i$ is unobserved. However, under Assumptions 1.1-1.5 and 2.b there exists a $\gamma \in L_2(X \times V, F_{X_{it},V_{it}})$ such that:

$$E_t[\gamma(X_{it},V_{it})|Z_{it}] = E_t[h_t(X_{it},\eta_i)|Z_{it}]$$

Note that a sufficient condition for Assumption 1.5 is another $L_2$-completeness condition. Specifically that for any $\delta \in L_2(Z,F_{Z_{it}})$:

$$E_t[\delta(Z_{it})|X_{it},V_{it}] = 0 \iff \delta(Z_{it}) = 0$$

Substituting into 6, one gets:

$$E_t[Y_{it} - \gamma(X_{it},V_{it})|Z_{it}] = 0 \quad (7)$$

The conditional moment restriction above does not involve $\eta_i$ directly, and so this provides a feasible NPIV moment condition from which one can identify $\gamma$.

One can apply a standard NPIV estimation procedure to estimate $\gamma$ from the conditional moment condition 7. Denote the estimated function by $\hat{\gamma}$.

Under the conditional independence assumption 5 note that:

$$E_t[\gamma(X_{it},V_{it})|X_{it},\eta_i] = E_t[\gamma(X_{it},V_{it})|\eta_i, S_{it}]$$

And so for $F_{X_{it}}$-almost all $x$:

$$E_t[\gamma(x,V_{it})|X_{it},\eta_i] = E_t[\gamma(x,V_{it})|\eta_i, S_{it}]$$

Moreover, the conditional independence assumption 5 also implies that:

$$E_t[\gamma(x,V_{it})|S_{it} \in \mathbb{S}] = E_t[h_t(x,\eta_i)|S_{it} \in \mathbb{S}] = CSF_t(x,\mathbb{S})$$

For $F_{X_{it}}$-almost all $x$ (where $F_{X_{it}}$ is the marginal distribution of $X_{it}$). To see this note that by 5:

$$E_t[\gamma(x,V_{it})|X_{it},\eta_i] = E_t[\gamma(x,V_{it})|\eta_i, S_{it}]$$

And so for $F_{X_{it}}$-almost all $x$:

$$E[h_t(x,\eta_i)|\eta_i, S_{it}] = E_t[\gamma(x,V_{it})|\eta_i, S_{it}]$$

The result then follows by iterated expectations.
Hence identification of $\gamma$ implies identification of the CSF. Let $\hat{\gamma}$ be the NPIV estimator of the solution to 7. Then one can estimate the conditional expectation $E_t[\gamma(x, V_{it}) | S_{it} \in S]$ by taking the sample average of $\hat{\gamma}(x, V_{it})$ for the sub-sample of individuals with $S_{it} \in S$. That is:

$$CSF_t(x, S) \approx \frac{\sum_{i=1}^n \hat{\gamma}(x, V_{it}) 1\{S_{it} \in S\}}{\sum_{i=1}^n 1\{S_{it} \in S\}}$$

NPIV estimation is often referred to as an ‘ill-posed problem’. Finding a solution to 7 generally involves the inversion of a bounded and infinite-dimensional operator, which may require some form of regularization. However, the estimation procedure in this paper may not in fact be ill-posed because the object of interest is a conditional expectation of the solution to the NPIV problem rather than the solution to the NPIV problem itself. In particular under Assumption 1.1 and Assumption 2.c presented in the next section the problem is not ill-posed. This is captured in Lemma 1 stated in the next section. Note that Assumption 2.c requires that the components of $X_{it}^{(1)}$, i.e. the regressors in whose effect the researcher is interested, are included in $Z_{it}$. Hence a necessary condition for Assumptions 1.3 and 2.c to hold is that $E_t[\epsilon_{it} | X_{it}^{(1)}] = 0$.

**Requirements On The Number of Time Periods**

As discussed above, completeness can be understood as a nonparametric analogue of the rank condition in linear IV. Newey & Powell (2003) note that at least in some cases, a necessary condition for completeness is that a corresponding ‘order’ condition hold. In the context of [8] and [9] the order condition states that the dimension of the vector of proxies $V_{it}$ be at least as great as the dimension of $\eta_i$ and that the dimension of instruments $Z_{it}$ be at least as great as that of $(X_{it}; \eta_i)$.

This restriction on the lengths of $Z_{it}$, $V_{it}$, $X_{it}$ and $\eta_i$ is discussed further in the next section. We argue that while this condition may not be strictly necessary for our identification and consistency results, it would be prudent for the empirical researcher to ensure that it holds before applying our estimation procedure.

Suppose $V_{it}$ and $Z_{it}$ are composed of non-overlapping subsets of the lags and leads of $X_{it}$. Then a requirement on the lengths of $V_{it}$ and $Z_{it}$ implies a lower bound on the number of time periods $T$ for which there are available observations for each individual. Suppose for simplicity that $X_{it}$ is a scalar and is included in $Z_{it}$ (i.e. $E_t[\epsilon_{it} | X_{it}] = 0$). Let $\eta_i$ have $d_\eta$ components. Then in order to have enough proxies and instruments one needs at the very least that $T \geq 2d_\eta + 1$. Note that this is the same condition on the number of time periods as assumed in Freyberger (2018).

Again, it is worth noting that this condition may not be necessary in all cases. However, it may provide a useful guide for the empirical practitioner. Note that the particular setting determines what observables can be included as components of $V_{it}$ and $Z_{it}$ without violating the conditional independence

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and instrumental validity conditions. If there are a great many observables that can be included in $V_{it}$ and $Z_{it}$ then $T$ needn’t be as large given the dimension of the unobserved heterogeneity.

3 Formal Identification Results

High Level conditions

Recall that the structural model is given by:

$$Y_{it} = h_t(X_{it}, \eta_i) + \epsilon_{it}$$

A researcher may only be interested in the conditional average structural function of a subset of the regressors. For this reason it will be helpful to further partition the vector of regressors $X_{it}$. The vector is partitioned as $X_{it} = (X_{it}^{(1)}; X_{it}^{(2)})$. The results in this section then pertain to the identification of the conditional average structural function of the sub-vector of regressors $X_{it}^{(1)}$. That is, identification of the function $CSF_t$ that satisfies:

$$CSF_t(x^{(1)}, S) = E_t[h_t(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S]$$

Where $S_{it}$ is some vector of observables upon which the researcher wishes to condition. For example, $S_{it}$ could simply be equal to $X_{it}$. The set $S$ above is some measurable subset of the support of $S_{it}$ such that $Pr_t[S_{it} \in S] > 0$.

In order to state Assumption 2 below it is necessary to distinguish between those regressors (if any) that are known to be mean independent of the time-varying shock $\epsilon_{it}$, and those that are not. The regressors that are mean independent of $\epsilon_{it}$ are referred to as the ‘exogenous regressors’ and are denoted by $W_{it}$. Thus $W_{it}$ is a subvector of the vector of instruments $Z_{it}$ and also of the regressors $X_{it}$. Again we emphasize that there is no assumption that either $W_{it}$ nor any other components of $Z_{it}$ are independent, mean independent nor uncorrelated with the time-invariant heterogeneity $\eta_i$. It is also helpful to refer to those instruments other than those in $W_{it}$. Let $Z_{it}$ refer to the instruments that are not also regressors (i.e. that are not also components of the vector $X_{it}$). Thus $Z_{it}$ is partitioned as $Z_{it} = (W_{it}; \tilde{Z}_{it})$. It will occasionally be useful to denote by $\tilde{X}_{it}$ those components of $X_{it}$ that are not in $W_{it}$ (i.e. that are not used as instruments). Thus the regressors are partitioned as $X_{it} = (W_{it}; \tilde{X}_{it})$.

Theorem 1 below considers identification of the CSF at a given period $t$. In Theorem 1 observations from periods other than $t$ are only referenced insofar as they appear in $V_{it}$, $Z_{it}$ or if they are included in the vector of regressors $X_{it}$ (e.g. in dynamic models). Hence one can think of the observations as tuples of $Y_{it}, X_{it}, V_{it}, Z_{it}$ and $S_{it}$ for the single $t$ for which the CSF is identified and for $i = 1, ..., \infty$.

Let $F$ denote the true joint probability distribution at the period $t$ of the random variables $X_{it}, V_{it}, Z_{it}, S_{it}, \eta_i$ and $\epsilon_{it}$. The notation $F_{X_{it}}$ denotes the marginal distribution of $X_{it}$, and likewise for other combinations of the random
variables. $X$ denotes the support of $X_{it}$, $Z$ the support of $Z_{it}$, $V$ of $V_{it}$, $\mathcal{E}$ of $\eta_{i}$, $S$ of $S_{it}$, $X^{(1)}_{it}$ of $X_{it}^{(1)}$, $X^{(2)}_{it}$ of $X_{it}^{(2)}$ and $W$ of $W_{it}$.

The joint distribution of the observables $Y_{it}, X_{it}, V_{it}, Z_{it}$ is a function of $h_{it}$ and $F$. A functional of $(h_{it}, F)$ is then said to be ‘identified’ if there exists a function from the joint distribution of observables to this functional. Of course, whether or not such a function exists depends on the spaces to which $h_{it}$ and $F$ are a priori restricted.

**Assumption 1.1 (Function Classes)**

The distribution $\tilde{F}_{X_{it}, V_{it}, \eta_{i}, \tilde{Z}_{it}, W_{it}}$ is dominated by the product of its marginals $F_{\tilde{X}_{it}}, F_{V_{it}}, F_{\eta_{i}}, F_{\tilde{Z}_{it}}$, and $F_{W_{it}}$ and has a continuous and square integrable Radon-Nikodym derivative w.r.t the product of its marginals. The structural function $h_{it}$ is continuous in its first argument and $E_{t}[h_{it}(X_{it}, \eta_{i})^{2}]$ exists and is finite.

**Assumption 1.2 (Conditional Independence)**

$$(S_{it}, Z_{it}, X_{it}) \perp V_{it} | X_{it}^{(2)}, \eta_{i}$$

**Assumption 1.3 (Instrumental Validity)**

$$E_{t}[\epsilon_{it}|Z_{it}] = 0$$

**Assumption 1.4 (Uniqueness)**

For any function $\delta \in L_{2}(X \times \mathcal{E}, F_{X_{it}, \eta_{i}})$:

$$E_{t}[\delta(X_{it}, \eta_{i})|Z_{it}] = 0 \implies E_{t}[\delta(x^{(1)}_{it}, X_{it}^{(2)}, \eta_{i})|S_{it} \in S] = 0$$

For $F_{X_{it}^{(1)}}$-almost all $x^{(1)}$.

**Assumption 1.5 (Representation)**

For any $\delta \in L_{2}(Z, F_{Z_{it}})$ : 

$$E_{t}[E_{t}[\delta(Z_{it})|\eta_{i}, X_{it}]|V_{it}, X_{it}] = 0 \iff E_{t}[\delta(Z_{it})|\eta_{i}, X_{it}] = 0$$

**Discussion**

The restrictions that Assumption 1.1 places on the distribution of $F$ are standard in the literature on non-parametric instrumental variables (NPIV) estimation. NPIV estimation is, in effect, an intermediate step in our estimation procedure and the conditions for identification in our model bear some resemblance to those in standard NPIV models. In particular the assumption that $F_{\tilde{X}_{it}, V_{it}, \eta_{i}, W_{it}, \tilde{Z}_{it}}$ is dominated by the product of its marginals and that it has a square-integrable density with respect to the product of the marginals is similar to an assumption in Florens (2011). Florens (2011) use this condition to ensure that the relevant conditional expectations operators are compact, which in turn ensures that they admit a singular value decomposition. For a classic take on this subject (that
is also cited by Florens et al.) see Lancaster (1958). In our case there is an additional complication because we allow for the possibility that the instruments and regressors share some components. The presence of \( W_{it} \) implies that the conditional expectation operators \( A \) and \( B^* \) introduced in Assumption 2 are generally not compact and hence do not admit singular value decompositions. Horowitz (2011) deals with the presence of exogenous regressors by defining a compact operator for each possible value of the exogenous regressors. In order to reduce the notational complexity of Horowitz’s approach we introduce the notion of a ‘pointwise singular system’ which behaves much like a classic singular value decomposition, but which must exist for the operators \( A \) and \( B^* \) under Assumption 1.1. We discuss this in detail along with the properties of the classic singular value decomposition in the appendix.

The assumption that the structural function \( h_t \) is square integrable is also standard (again see Florens (2011)). The continuity of \( h_t \) in its first argument can be relaxed but then \( CSF_t(\cdot, S) \) is only identified up to a \( F_{X^{(1)}_it} \) null set. Continuity of \( h_t \) in its first argument implies \( CSF_t(\cdot, S) \) is continuous which guarantees that if it is identified \( F_{X^{(1)}_it} \)-almost everywhere then it is identified everywhere on the support of \( X^{(1)}_it \).

Assumption 1.2 is less restrictive than it may seem. Independence is only required conditional on the unobserved heterogeneity and perhaps some observables \( X^{(2)}_it \). The tuple \((S_{it}, Z_{it}, X^{(1)}_it)\) and \( V_{it} \) could be very strongly correlated in the population so long as this association is due to factors controlled for in \( \eta_i \) and \( X^{(2)}_it \). This restriction arises naturally in a number of settings. Suppose that \( Z_{it} \) and \( V_{it} \) are each composed of non-overlapping observations of the regressors at periods other than \( t \). Then Assumption 1.2 follows if the regressors are exchangeable (if \( \eta_i \) is defined sufficiently broadly). It also follows for low-order Markov processes that differ between individuals up to a set of factors controlled for in \( \eta_i \) and \( X^{(2)}_it \). The implications of this assumption and the conditions it places on \( \eta_i \) are discussed at length later in this section along with specific examples.

Assumption 1.3 simply states that \( Z_{it} \) is exogenous with regards to the additive time-varying heterogeneity \( \epsilon_{it} \). \( Z_{it} \) is essentially used as an instrument and so this assumption is required just as in a standard instrumental variables framework. Note that this assumption requires mean independence rather than full independence which is sometimes required for identification in non-linear models.

Assumption 1.4 is a weakening of the ‘\( L_2 \)-completeness’ assumption needed for identification in NPIV models. The corresponding \( L_2 \)-completeness assumption (which is imposed in Assumption 2.c) states that for any \( \delta \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \):

\[
E_t[\delta(X_{it}, \eta_i)|Z_{it}] = 0 \iff \delta(X_{it}, \eta_i) = 0 \quad (8)
\]

The above implies that were \( \eta_i \) observed, the instruments \( Z_{it} \) would be relevant and rich enough to identify \( h_t(X_{it}, \eta_i) \).

Assumption 1.5 is a weakening of another \( L_2 \)-completeness condition which
can be stated as follows. For any $\delta \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$:

$$E_t[\delta(X_{it}, \eta_i)|X_{it}, V_{it}] = 0 \iff \delta(X_{it}, \eta_i) = 0$$ (9)

This condition (and similarly Assumption 1.5) can be understood as an assumption that the relationship between $V_{it}$ and $\eta_i$ is sufficiently rich. In particular Assumption 1.5 implies that there is an element $\gamma$ of $L_2$ such that $E_t[\gamma(X_{it}, V_{it})] = 0$ is arbitrarily close to $E_t[h_t(X_{it}, \eta_i)|Z_{it}]$ in the $L_2$ norm. Assumption 2.b, in combination with Assumptions 1.2 and 1.5 guarantees that there is a $\gamma$ in $L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, \eta_i})$ so that the equality holds exactly, that is:

$$E_t[\gamma(X_{it}, V_{it})] = E_t[h_t(X_{it}, \eta_i)|Z_{it}]$$

Note that under Assumption 1.2:

$$E_t[\gamma(X_{it}, V_{it})] = E_t[\gamma(X_{it}, V_{it})|Z_{it}]$$

Again, for sufficient conditions for completeness, bounded completeness and $L_2$-completeness we refer the reader to D’Haultfoeuille (2011), Hu & Shiu (2018) and Andrews (2017).

In addition to the assumptions above one of the following assumptions is required for identification. Assumptions 2.b and 2.c are also useful for establishing the convergence rate of our estimation procedure.

**Assumption 2**

Assumptions 1.1-1.5 hold and any one of the following three conditions holds:

a.

All the regressors are exogenous (i.e. $X_{it} = W_{it}$) and the distribution of $(X_{it}; V_{it}; \eta_i)$ and the distribution of $(Z_{it}; \eta_i)$ are identical. (Note that this implies Assumption 1.5).

Assumption 2.b and 2.c are somewhat technical. We provide a detailed discussion of these assumptions in the appendix including the definition and discussion of the ‘pointwise singular system’.

Define the following two linear operators:

$$A : L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, \eta_i}) \rightarrow L_2(\mathcal{Z}, F_{Z_{it}})$$

$$A[\delta] = [q \in L_2(\mathcal{Z}, F_{Z_{it}}) : q(Z_{it}) = E_t[\delta(X_{it}, V_{it})|Z_{it}]]$$

$$B^* : L_2(\mathcal{Z}, F_{Z_{it}}) \rightarrow L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$$

$$B^*[\delta] = [q \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) : q(X_{it}, \eta_i) = E_t[\delta(Z_{it})|X_{it}, \eta_i]]$$

Let $g_t$ be the element of $L_2(\mathcal{Z}, F_{Z_{it}})$ that satisfies:

$$g_t(Z_{it}) = E_t[Y_{it}|Z_{it}]$$

Note that under Assumptions 1.1 and 1.3:

$$E_t[E_t[Y_{it}|Z_{it}]^2] = E_t[E_t[m(X_{it}, \eta_i)|Z_{it}]^2] < \infty$$
And so $g_t$ is well-defined.

**b.**
Under Assumption 1.1 There must exist a unique ‘pointwise singular system’
\[ \{ v_k, u_k, \mu_k \}_{k=1}^{\infty} \] (defined in the Appendix) for $A$.
Then:
\[
E \left[ \sum_{k=1}^{\infty} \frac{1}{\mu_k(W_{it})} E_t[g_t(Z_{it})u_k(Z_{it})|W_{it}]^2 \right] < \infty
\]

**c.**
The researcher is only interested in the partial function of exogenous regressors. That is, $X_{it}^{(1)}$ is a subvector of the vector of exogenous regressors $W_{it}$.

and
The following stronger version of Assumption 1.4 holds:
For any $\delta \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$:
\[
E_t[\delta(X_{it}, \eta_i)|Z_{it}] = 0 \iff \delta(X_{it}, \eta_i) = 0
\]
and
Assume that the product measure of the probability measures $F_{W_{it}}$ and $F_{X_{it}, \eta_i}|S_{it} \in \mathcal{S}$ is dominated by the probability measure $F_{X_{it}, \eta_i}$. Denote the corresponding Radon-Nikodym derivative (which exists by the Radon-Nikodym theorem) by $f_{S_{it}}$. Assume that $E_t[f_{S_{it}}(X_{it}, \eta_i)^2] < \infty$.

Under Assumption 1.1 There must exist a unique pointwise singular system
\[ \{ v_k, u_k, \mu_k \}_{k=1}^{\infty} \] (defined in the Appendix) for $B^*$.
Then:
\[
E \left[ \sum_{k=1}^{\infty} \frac{1}{\mu_k(W_{it})} E_t[f_{S_{it}}(X_{it}, \eta_i)u_k(X_{it}, \eta_i)|W_{it}]^2 \right] \leq c_3 < \infty
\]
For some scalar $c_3$.

**Discussion**
Assumption 2 can be understood as a regularity condition that is needed to deal with the discontinuity inherent to the inverses of infinite dimensional compact linear operators. Only one of Assumptions 2.a, 2.b and 2.c needs to hold for Theorem 1, however, we assume Assumptions 2.b and 2.c hold (and not 2.a) when presenting the convergence rates for our estimation procedure in the next section.

Assumption 2.b is the only option of the three that places no restrictions on whether or not any components of the regressors $X_{it}$ are exogenous. By contrast Assumption 2.a requires that $E_t[\epsilon_{it}|X_{it}] = 0$ and so all of the regressors can be included as instruments (in short $X_{it} = W_{it}$). Assumption 2.c requires that
\[ E_t[\epsilon_{it}|X_{it}^{(1)}] = 0 \text{ where } X_{it}^{(1)} \text{ is the subvector of regressors in whose partial effect one is interested (in short } \tilde{X}_{it} = X_{it}^{(2)}). \]

In the case where \( Z_{it} \) and \( V_{it} \) consist of leads and lags of the regressors the restriction that \((V_{it}; X_{it}; \eta_i) \) and \((Z_{it}; \eta_i)\) have the same distribution in Assumption 2.a can be thought of as a stationarity assumption.

Both Assumption 2.b and Assumption 2.c place conditions on the ‘pointwise singular systems’ of bounded linear operators and the coefficients of the expansions of functions within these systems. Pointwise singular systems are introduced and discussed in the appendix along with a proof of existence and uniqueness. Conditions on the Fourier coefficients of a function expanded in a singular system are required to ensure solutions to certain classes of linear operator equations using Picard’s theorem. The results for pointwise singular systems are a simple extension of those results. Assumptions of this kind are standard in the NPIV literature, for example Florens (2011) use assumptions of this kind along with Picard’s theorem to guarantee the existence of the NPIV regression function.

In particular, Assumption 2.b guarantees that there exists a \( \gamma \in L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it},V_{it}}) \) so that:

\[ E_t[\gamma(X_{it},V_{it})|Z_{it}] = E_t[Y_{it}|Z_{it}] \]

Assumption 2.c guarantees the existence of a \( \psi_S \in L_2(\mathcal{Z}, F_{Z_{it}}) \) so that:

\[ E_t[\psi_S(Z_{it})|X_{it}, \eta_i] = f_S(X_{it}, \eta_i) \]

Where \( f_S \) is the Radon-Nikodym derivative defined in Assumption 2.c above. The following Lemma is useful for both the proof of identification and establishing the convergence rate of our estimation procedure.

**Lemma 1:**

Suppose Assumption 1.1 and Assumption 2.c hold for a measurable subset \( S \) of the support of \( S_{it} \) with \( P_{Z_{it}}[S_{it} \in S] > 0 \). Then there exists a constant \( c_2 \) so that for any function \( \delta \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it},\eta_i}) \), if \( \tilde{\delta} \) is the corresponding function in \( L_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}}) \) defined by \( \tilde{\delta}(x_{it}^{(1)}) = E_t[\delta(x_{it}^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S] \) then:

\[ E_t[|\tilde{\delta}(X_{it}^{(1)})|] \leq c_2 E_t[E_t[|\delta(X_{it}, \eta_i)|Z_{it}]^2] \]

**Proof:**

See appendix.

Lemma 1 shows that under Assumptions 1.1 and 2.c the problem of estimating the CSF is not ‘ill-posed’. Loosely speaking the result above shows that a small perturbation to \( E_t[Y_{it}|Z_{it}] \) can only lead to a small change in the corresponding conditional average structural function. This is in spite of the fact that a small perturbation to \( E[Y_{it}|Z_{it}] \) may lead to a large change in \( \gamma \) that solves the estimating equation:

\[ E[Y_{it} - \gamma(X_{it},V_{it})|Z_{it}] = 0 \]
This result is key to the convergence rate results presented in Section 4 and is discussed in more detail in that section.

We now present the main result of this section. We show that under the assumptions above the conditional average structural function of $X_{it}^{(1)}$ conditional on $S_{it} \in S$ is identified.

**Theorem 1:**

Suppose Assumptions 1.1, 1.2, 1.3, 1.4, 1.5 and Assumption 2 hold for some time period $t$ and a measurable subset $S$ of the support of $S_{it}$ with $Pr_{it}[S_{it} \in S] > 0$. Then the conditional average structural function (CSF) of $X_{it}^{(1)}$ is identified. That is, there is a mapping from the distribution of observables to a function $CSF_{it}(\cdot;S)$ that satisfies:

$$CSF_{it}(x^{(1)}, S) = E_{it}[h_{it}(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S]$$

For all $x^{(1)}$ in the support of $X_{it}^{(1)}$.

**Proof:**

See Appendix.

Note that the conclusion of Theorem 1 states that $CSF_{it}(x^{(1)}, S)$ is identified for all $x^{(1)}$ in the support of $X_{it}^{(1)}$. Assumption 1.1 requires that the joint distribution of $X_{it}^{(1)}$ and $\eta_i$ be dominated by the product of their marginal distributions. This rules out the case in which $X_{it}^{(1)}$ does not vary in its full support conditional on $\eta_i$. In some practical cases $X_{it}^{(1)}$ may not vary at the individual level but varies at the population level. Assumption 1.1 does not explicitly exclude these cases because it is possible that $X_{it}^{(1)}$ is fixed over time but still varies conditional on $\eta_i$.

As noted above Assumptions 1.4 and 1.5 are weakenings of the completeness assumptions (8) and (9). Newey & Powell (2003) show that in the multivariate Gaussian case a necessary condition for completeness is (in the context of the NPIV model) that the dimension of the vector of instruments be at least as great as the dimension of the vector of endogenous regressors. They conjecture that this condition is also necessary for a broader class of distributions than the Gaussian and note that it is analogous to the ‘order condition’ in standard linear IV. Chen et al. (2014) elaborate on this condition and show that it is connected to the genericity of the completeness property. The sufficient conditions for completeness given in D'Haultfoeuille (2011) all impose this order condition.

The corresponding conditions for (8) and (9) are that i. the vector of proxies $V_{it}$ have more components than $\eta_i$ and ii. that $Z_{it}$ have more components than $(\eta_i; X_{it})$. In the present work we provide only sufficient conditions for identification and consistent estimation and so it is not clear at present whether
these restrictions on the lengths of $Z_{it}$, $V_{it}$ and $\eta_i$ are strictly necessary for identification in our model or consistency of our estimator. Nonetheless, given that the available sufficient conditions for 8 and 9 and hence for Assumptions 1.4 and 1.5 impose the order condition, we suggest that for practical purposes the order condition should be treated as necessary. We refer to this condition as a ‘guidance’ to distinguish it from our other assumptions.

**Guidance**

The dimensions of the vectors $(V_{it}; X_{it})$ and $Z_{it}$ are each at least as great as the dimension of the vector $(\eta_i; X_{it})$.

The high level conditions above, while quite general, do not address how one might go about finding variables $V_{it}$ and $Z_{it}$ with the required properties. In the rest of this section we describe special cases in which collections of lags and leads of the regressors can be used for $V_{it}$ and $Z_{it}$. In these special cases we employ assumptions that are often used to prove consistency of textbook panel methods in linear parametric settings. The use of observations from previous and subsequent periods in place of $V_{it}$ and $Z_{it}$ is redolent of the use of lagged variables as instruments in the Arellano & Bond (1991) estimator for dynamic panel models or in Holtz-Eakin et al. (1988).

The cases below are not intended to be exhaustive but it is hoped that they may provide insight into how one might apply our general identification results to a broad range of settings.

The following lemma will prove useful in the subsequent analysis.

**Lemma 2**

Suppose that $E_t[\epsilon_{it}|Z_{it}] = 0$ and there exists some random variable $\xi_i$ such that $(Z_{it}, X_{it}, S_{it}) \perp \eta_i|\xi_i, X_{it}^{(2)}$. Consider the alternative model:

$$Y_{it} = \tilde{h}_t(X_{it}, \xi_i) + \tilde{\epsilon}_{it}$$

(10)

Where $\tilde{h}_t$ is defined by:

$$\tilde{h}_t(X_{it}, \xi_i) = E_t[h_t(X_{it}, \eta_i)|X_{it}, \xi_i]$$

Then the conditional average structural function associated with (10) is the same as in the original model (1) and $E_t[\tilde{\epsilon}_{it}|Z_{it}] = 0$.

**Proof:**

See Appendix.
Exchangeable Regressors and Strict-Exogeneity

A sequence of random variables is ‘exchangeable’ if permuting the order of the variables does not change their joint distribution. In particular, the regressors $X_{i1}, X_{i2}, ..., X_{iT}$ are exchangeable if, for any function $\pi$ that maps bijectively from $\{1, ..., T\}$ to itself (i.e. any permutation of the time indices) the following is true:

$$(X_{i\pi(1)}, X_{i\pi(2)}, ..., X_{i\pi(T)}) \sim (X_{i1}, X_{i2}, ..., X_{iT})$$

Exchangeable regressors arise naturally in settings where data are ‘clustered’, that is where different ‘individual subscripts’ refer to different sub-populations and different ‘time-subscripts’ do not actually signify different time periods but rather different observations sampled independently and identically from within the sub-population. Even in classic panel settings where the ‘time subscript’ really does indicate the chronology of the observations, exchangability may still hold. For example, if the regressors are serially independent and their distributions are stationary (i.e. conditional on the individual $X_{it}$ is iid over time) then it is easy to show that they are also exchangeable. Note that exchangability of the time-subscripts does not preclude that the distribution of the regressors vary substantially between individuals. Indeed, the regressors for an individual could be perfectly correlated (with probability 1, $X_{it} = X_{is}$ for all $s, t$ and $i$) and this would imply rather than violate exchangability.

‘Strict exogeneity’ here refers to the condition that the additive error be mean independent of the regressors at each $t$. That is:

$$E_t[\epsilon_{it}|X_{i1}, X_{i2}, ..., X_{iT}] = 0$$

The assumption of strict exogeneity is common in empirical work using panel data. It is required for the well-known fixed effects estimator in linear panel models to be consistent with fixed $T$.

Note that strict exogeneity is a condition about the relationship between the time-varying shock $\epsilon_{it}$ and the regressors, and does not directly involve the time-invariant heterogeneity $\eta_i$. In many settings it may be plausible to assume $\epsilon_{it}$ is mean independent of the regressors but not plausible to assume $\eta_i$ is independent of $X_{it}$. For example, consider a model of individual demand for some retail product. The researcher observes prices across a range of stores as well as some features of the individual and individual purchases over a number of months. In this case $X_{it}$ may be composed of prices in those stores at which the individual shops and of individual characteristics. Suppose that the individual characteristics are fixed over time and that the price of a good in a particular store is chosen by the retailer based on costs, and the aggregate demand for the product in that store. Note that individuals with different levels of wealth, different ages and different cultural backgrounds may tend to live in different areas. These characteristics are plausibly fixed over time, thus if they are unobserved then they are captured in $\eta_i$. If these characteristics are related to an individual’s demand for the product, then this suggest the price at a store and
the characteristics of those who shop there are associated. That is, \( X_{it} \) and \( \eta_i \) may become dependent. Since \( h_t \) is time-varying, \( \epsilon_{it} \) only captures the time-varying demand-shock to the individual and not any aggregate shock. Thus, it is plausible that the individual’s demand shock is mean independent of the price in each period.

The following fact about exchangability is key to the proof of Remark 1 below. De Finetti’s theorem states that a sequence of random variables is exchangeable if and only if the random variables are iid conditional on some latent variable. Denoting this latent variable by \( \xi_i \) and its domain by \( \Xi \) for any two disjoint sub-sequences of \{1, ..., \( T \)\} given by \( \{\pi_1', \pi_2', ..., \pi_{\tilde{T}/2}'\} \) and \( \{\pi_1'', \pi_2'', ..., \pi_{\tilde{T}/2}''\} \):

\[
X_{i\pi_1'}, ..., X_{i\pi_{\tilde{T}/2}'} \perp X_{i\pi_1''}, ..., X_{i\pi_{\tilde{T}/2}''} | \xi_i
\] (11)

Let \( \tilde{T} \) be the largest even number weakly less than \( T-1 \) and let \( \{\pi_1', \pi_2', ..., \pi_{\tilde{T}/2}'\} \) and \( \{\pi_1'', \pi_2'', ..., \pi_{\tilde{T}/2}''\} \) be two disjoint sub-sequences of length \( \tilde{T}/2 \) that do not contain \( t \). Define \( V_{it} \) and \( Z_{it} \) as follows:

\[
V_{it} = (X_{i, \pi_1'}; X_{i, \pi_2'}; ...; X_{i, \pi_{\tilde{T}/2}'})
\]
\[
Z_{it} = (X_{i, \pi_1''}; X_{i, \pi_2''}; ...; X_{i, \pi_{\tilde{T}/2}''}; X_{it})
\] (12)

Strict exogeneity then implies \( E_t[\epsilon_{it}|Z_{it}] = 0 \). And furthermore, given (11), it is clear that Lemma 2 applies to the model with the latent variable \( \xi_i \). In short, one can replace \( \eta_i \) in the original model with \( \xi_i \) without changing the conditional average structural function and without violating exogeneity of \( Z_{it} \).

With \( V_{it} \) and \( Z_{it} \) defined as above, exchangenability of the regressors ensures that Assumptions 1.2, 1.5 and 2.a of Theorem 1 hold in the related model. The assumption of strict exogeneity ensures Assumption 1.3 holds. This reasoning is captured in the following remark.

Remark 1

Let \( \tilde{T} \) be the largest even number weakly less than \( T-1 \), suppose the following holds for some \( t \):

1. The regressors \( \{X_{i1}, X_{i2}, ..., X_{iT}\} \) are exchangeable with corresponding latent variable \( \xi_i \).
2. \( \epsilon_{it} \) is strictly exogenous, that is \( E_t[\epsilon_{it}|X_{i1}, X_{i2}, ..., X_{iT}] = 0 \).
3. The conditioning variable \( S_{it} \) is a function of the regressors \( X_{it} \).
4. For any \( \delta \in L_2(\mathcal{X} \times \Xi, F_{X_{it}, \xi_i}) \) then:

\[
E_t[\delta(X_{it}, \xi_i)|X_{i1}, X_{i2}, ..., X_{i, \tilde{T}/2}, X_{i, \tilde{T}/2+1}, X_{it}] = 0 \iff E_t[\delta(x^{(1)}, X_{i, \tilde{T}/2}^{(2)}, \xi_i)|S_{it}] = 0
\]

5. The distribution \( F \) of \( (X_{it}, \xi_i, \eta_i, S_{it}) \) is dominated by the product of its marginals, and has a continuous and square integrable density w.r.t the product of its marginals. The structural function \( h_t \) is continuous in its first argument.
then the conditions of Theorem 1 are satisfied for any measurable subset \( S \) of the support of \( X_{it} \) with \( \Pr_t[X_{it} \in S] > 0 \) and with \( V_{it} \) and \( Z_{it} \) set as in (12). And hence the conditional average structural function of \( X_{it}^{(1)} \) is identified. If \( X_{it} \) has dimension \( d_X \) and \( \xi_i \) has dimension \( d_\xi \) then for the 'Guidance' to hold in this case the number of time periods must satisfy \( T \geq 2 \frac{d_\xi}{d_X} + 2 \) if \( T \) is even and \( T \geq 2 \frac{d_\xi}{d_X} + 1 \) if \( T \) is odd.

**Proof:**

See Appendix.

**Predetermination**

In some cases the assumption of strict exogeneity may be implausible. ‘Predetermination’ is a weaker condition than strict exogeneity that is also assumed to hold in many empirical panel models. Predetermination is here defined as the condition that the additive error be mean independent of contemporaneous and lagged values of the regressors. That is:

\[
E_t[\epsilon_{it}|X_{i1}, X_{i2}, \ldots, X_{it}] = 0
\]

In this case suppose \( T \geq 2t - 1 \) and set \( V_{it} \) and \( Z_{it} \) as follows:

\[
V_{it} = (X_{it+1}; X_{it+2}; \ldots; X_{i,2t-1})
\]

\[
Z_{it} = (X_{i1}; X_{i2}; \ldots; X_{it})
\]

And so under predetermination, with \( Z_{it} \) defined as above:

\[
E_t[\epsilon_{it}|Z_{it}] = 0
\]

Predetermined regressors may be a sensible assumption if the regressors \( X_{it} \) are chosen by some rational agent whose decision may be based on past outcomes but who cannot anticipate future shocks. For example, suppose \( Y_{it} \) is the amount produced of some good and \( X_{it} \) is a vector of inputs like labor and capital. In this case \( \epsilon_{it} \) is the firm-specific productivity shock for firm \( i \) at time \( t \). If past productivity is unexpectedly high then the firm may be able to invest in its capital stock or able to hire additional workers. If the firm faces frictions selling off the capital stock or hiring so that these inputs are persistent then \( \epsilon_{it} \) is associated with future values of the inputs. However if the firm does not anticipate future firm-specific shocks in its factor choices then \( \epsilon_{it} \) may be mean independent of past input levels.

Clearly \( V_{it} \) and \( Z_{it} \) defined above are a special case of (11) and so under exchangability \( Z_{it} \) and \( V_{it} \) defined in this way satisfy the conditional independence assumptions for \( S_{it} \) a function of the regressors. Furthermore, under exchangability it is clear that \( V_{it} \) and \( Z_{it} \) defined above satisfy Assumption 2.a.
With the definitions above Assumption 1.4 is equivalent to the following. For any \( \delta \in L_2(X \times \Xi, F_{X_i, \xi_i}) \):

\[
E_t[\delta(X_{it}, \xi_i)|X_{i1}, X_{i2}, ..., X_{i,t-1}, X_{it}] = 0
\]

If and only if

\[
E_t[\delta(x^{(1)}, X_{it}^{(2)}, \xi_i)|S_{it} \in \mathbb{S}] = 0
\]

For \( F_{X_i^{(1)}} \)-almost all \( x^{(1)} \).

With this in mind one can easily apply the Theorem 1 to this setting.

**Finite Dependence**

The conditional independence in Assumption 1.2 allows for some forms of serial dependence in the regressors. Suppose that the regressors are determined by some dynamic process which differs between individuals only up to some vector of latent variables \( \xi_i \). Suppose also that the serial dependence is ‘finite’ in the sense that \( X_{it} \) is associated with the levels of the regressors up to \( k \) periods in the future, but for periods from \( t + k + 1 \) onward, the association of \( X_{it} \) with the regressors in those periods is zero. Formally:

\[
X_{i1}, X_{i2}, ..., X_{it} \perp X_{it+k+1}, X_{it+k+1}, ..., X_{iT} | \xi_i
\]

We refer to this as a \( k \)-th order ‘finite dependence’ assumption. Processes that satisfy this definition include, for example, \( k \)-th-order moving average processes. That is:

\[
X_{it} = \sum_{s=0}^{k-1} \xi_{is} v_{i,t-s}
\]

Where \( v_{it} \) is an iid random variable and \( \xi_i = (\xi_{i,1}, ..., \xi_{i,k-1}) \) is a vector of individual-specific latent variables.

In this case suppose \( T \geq 2t + k \) and set \( V_{it} \) and \( Z_{it} \) as follows:

\[
V_{it} = (X_{i,t+k+1}; X_{i,t+k+1}; ...; X_{i,2t+k})
\]

\[
Z_{it} = (X_{i1}; X_{i2}; ...; X_{it})
\]

Let \( S_{it} \) be a function of \( X_{it} \). Then one can easily verify that \( V_{it} \) and \( Z_{it} \) above satisfy the conditional independence conditions in Assumption 1.2. If \( X_{it} \) has dimension \( d_X \) and \( \xi_i \) has dimension \( d_\xi \) then for the ‘Guidance’ to hold in this case the number of time periods must satisfy \( T \geq 2 \frac{d_\xi}{d_X} + 2 + k \) if \( T \) is even and \( T \geq 2 \frac{d_\xi}{d_X} + 1 + k \) if \( T \) is odd.
Stationary Markov Process

The ‘finite dependence’ assumption above can be relaxed. Suppose that conditional on some latent variables $\xi_i$ the regressors follow a $k^{th}$-order Markov process. That is, for each $t$:

$$X_{i1}, X_{i2}, ..., X_{it} \perp X_{it+k+1}, ..., X_{iT} | \xi_i, X_{it+1}, ..., X_{it+k}$$

In this case the value of the regressors at period $t$ can be associated with the value of the regressors indefinitely far into the future. However, the value at time $t$ can only ‘directly’ affect the regressors up to $k$ periods in the future. Any correlation between say, $X_{it}$ and $X_{it+k+1}$ is ‘indirect’ in that it can be entirely controlled for using the values of the regressors in the intervening periods.

In this setting it is more difficult to find vectors $Z_{it}$ and $V_{it}$ that satisfy Assumption 1.2. In order to achieve this it is necessary to add an additional set of regressors that contain some leads of the original regressors. In particular define a new set of regressors $\hat{X}_{it}$ by:

$$\hat{X}_{it} = (X_{it}; X_{it+1}; \ldots; X_{it+k})$$

The structural function $h_t(X_{it}, \xi_i)$ does not directly depend on the leads $X_{it+1}, X_{it+2}, \ldots, X_{it+k}$ however there is no requirement that all components of the regressors appear directly in the structural function. Suppose as usual that the researcher is interested in the partial function of a subvector of $X_{it}$, $X_{it}^{(1)}$. Partition $\hat{X}_{it}$ into $(\hat{X}_{it}^{(1)}, \hat{X}_{it}^{(2)})$ where $\hat{X}_{it}^{(1)} = X_{it}^{(1)}$ and $\hat{X}_{it}^{(2)} = (X_{it}^{(2)}; X_{it+1}; \ldots, X_{it+k})$.

The conditional independence assumption for Lemma 2 to hold with the new regressors is:

$$(Z_{it}, \hat{X}_{it}, S_{it}) \perp \eta_i | \xi_i, \hat{X}_{it}^{(2)}$$

If $S_{it}$ is a function of $X_{it}$ and $Z_{it}$ is defined by:

$$Z_{it} = (X_{i1}; X_{i2}; \ldots; X_{it})$$

Then the conditional independence above is equivalent to:

$$X_{i1}, X_{i2}, ..., X_{it} \perp \eta_i | \xi_i, X_{it+1}, ..., X_{it+k}$$

Which has to hold by the $k$-th order Markov assumption. Hence Lemma 2 can be applied and $\eta_i$ replaced by $\xi_i$.

Furthermore if $V_{it}$ is defined by:

$$V_{it} = (X_{i,t+k+1}; X_{i,t+k+1}; \ldots, X_{i,2t+k})$$

Then Assumption 1.2 for the model with $\xi_i$ is equivalent to:

$$X_{i1}, X_{i2}, ..., X_{it} \perp X_{it+k+1}, ..., X_{2t+k} | \xi_i, X_{it+1}, ..., X_{it+k}$$

Which is clearly implied by the latent Markov assumption. With these definition Assumption 1.4 is equivalent to:
\[ E_t[\delta(X_{it}, \ldots, X_{it+k}, \xi_i)|X_{i1}, X_{i2}, \ldots, X_{it-1}, X_{it}] = 0 \]

If and only if:

\[ E_t[\delta(x^{(1)}, X_{it}^{(2)}, X_{it+1}, \ldots, X_{it+k}, \xi_i)|S_{it} \in S] = 0 \]

For \( F_{X^{(1)}} \)-almost all \( x^{(1)} \).

If \( X_{it} \) has dimension \( d_X \) and \( \xi_i \) has dimension \( d_\xi \) then for the 'Guidance' to hold in this case the number of time periods must satisfy \( T \geq 2 \frac{d_\xi}{d_X} + 2 + 3k \) if \( T \) is even and \( T \geq 2 \frac{d_\xi}{d_X} + 1 + 3k \) if \( T \) is odd.

4 Estimation

In this section we describe our estimation method. The procedure is straightforward to implement, requiring only a familiarity with a standard non-parametric regression method like lasso or ridge. Depending on the choice of penalty function the method needn’t involve any numerical optimization.

The procedure is carried out in three stages. The first two stages together correspond to penalized sieve minimum distance (PSMD) estimation in an NPIV model. PSMD estimators and some of their properties are discussed in Chen & Pouzo (2012) and our consistency and convergence rate results rely heavily on their work. Because the estimation procedure is of the “sieve” type, the practitioner must choose an appropriate set of basis functions. In particular a set of functions \( \{\gamma_i\}_{i=1}^\infty \) each of which maps from the product of the supports of \( V_{it} \) and \( X_{it} \) to \( \mathbb{R} \). Conditions on the choice of basis functions that guarantee consistency and a particular rate of convergence are discussed later in this section.

Let \( \hat{P}_n(\cdot) \) be a (possibly data-dependent) penalty function (e.g. an \( L_1 \) penalty (lasso) or \( L_2 \) penalty (ridge)) and let \( \{K(n)\}_{n=1}^\infty \) be a sequence of natural numbers increasing to infinity. Requirements for both \( \hat{P}_n \) and \( \{K(n)\}_{n=1}^\infty \) are discussed later in this section.

In a first stage, for each \( k \in \{1, \ldots, K(n)\} \) the practitioner estimates the conditional means \( E_t[\gamma_k(X_{it}, V_{it})|Z_{it}] \) and \( E_t[Y_{it}|Z_{it}] \). Estimation of each of these functions can be carried out using a standard non-parametric regression method like local-linear regression, polynomial-series regression, Nadaraya-Watson etc.

We will denote the estimated conditional means by:

\[ \hat{E}[^\gamma_k(X_{it}, V_{it})|Z_{it}] \]

And:

\[ \hat{E}[Y_{it}|Z_{it}] \]

In the second step the practitioner estimates a vector of coefficients \( \beta \) by minimizing the objective function given below:

\[
\hat{\beta} \in \arg \min_{\beta \in B_{K(n)}} \sum_{i=1}^n (\hat{E}[Y_{it}|Z_{it}] - \sum_{k=1}^{K(n)} \beta_k \hat{E}[\gamma_k(X_{it}, V_{it})|Z_{it}])^2 + \lambda_n \hat{P}_n(\beta) \quad (13)
\]
Where the scalar $\beta_k$ is the $k$-th component of the vector $\beta$. The space over which the minimum is taken, $B_{K(n)}$, is a closed subset of $\mathbb{R}^{K(n)}$, $\hat{P}_n$ is a continuous and convex penalty function and $\lambda_n$ a scalar. Note that the formulation above does not allow for $\lambda_n$ to be data-dependent but the function $\hat{P}_n(\cdot)$ is possibly data-dependent. However a data-dependent penalty parameter $\hat{\lambda}_n$ and non-random penalty $\bar{P}_n$ can be nested in the framework above by setting $\hat{P}_n = \frac{\lambda_n}{\hat{\lambda}_n} \bar{P}_n$ and likewise for both a data-dependent penalty parameter and function.

In the third step one estimates the CSF of the regressors $X_{it}^{(1)}$ conditional on $S_{it}$ as follows. First, for each value $x^{(1)} \in \mathcal{X}$ of interest the researcher estimates the conditional mean $E_t[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in \mathcal{S}]$ by non-parametric regression. Denote the estimated conditional mean by $\hat{E}_t[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in \mathcal{S}]$. The researcher may then estimate $\text{CSF}_t(x^{(1)}, \mathcal{S})$ as below:

$$\text{CSF}_t(x^{(1)}, \mathcal{S}) = \sum_{k=1}^{K(n)} \hat{\beta}_k \hat{E}_t[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in \mathcal{S}]$$

It is worth noting that this procedure is amenable to sample-splitting techniques. In particular, let the set of individuals $\mathcal{N} = \{1, \ldots, n\}$ be partitioned into $L$ subsets $I_1, I_2, \ldots, I_L$. Suppose that for each $l = 1, \ldots, L$ each of the first stage regressions is carried out using the subset of the population $\mathcal{N} - I_l$ (i.e. the relative complement of $I_l$ in $\mathcal{N}$). Denote the first-stage fitted values from the regression of $Y_{it}$ on $Z_{it}$ using the subset $\mathcal{N} - I_l$ by:

$$\hat{E}_{-I_l}[Y_{it}|Z_{it}]$$

And likewise for the fitted values from the other first-stage regressions. Then one could replace the optimization problem 13 with the following similar optimization problem:

$$\hat{\beta} \in \arg \min_{\beta \in B_{K(n)}} \sum_{l=1}^{L} \sum_{i \in I_l} (\hat{E}_{-I_l}[Y_{it}|Z_{it}]-\sum_{k=1}^{K(n)} \beta_k \hat{E}_{-I_l}[\gamma_k(X_{it}, V_{it})|Z_{it}])^2 + \lambda_n \hat{P}_n(\beta)$$

Sample-splitting in this way has been shown to improve asymptotic efficiency and aid in inference in some semi-parametric settings. See Chernozhukov et al. (2018) for discussion.

**Consistency and Convergence Rate**

To establish the consistency and the convergence rate of the estimator described above, it is necessary to formalize the sense in which the first two stages of the estimation procedure are equivalent to PSMD estimation of an NPIV model.

First consider the functional $m$ given by:

$$m(Z_{it}, h) = E_t[Y_{it} - h(X_{it}, \eta_i)|Z_{it}]$$
And define the functional $\hat{Q}_n$ by:

$$\hat{Q}_n(m) = \frac{1}{n} \sum_{i=1}^{n} \hat{m}(Z_{it}, h)^2 + \lambda_n \tilde{P}_n(h)$$

Where the function $\hat{m}$ is a consistent estimator of the function $m$, $\tilde{P}_n$ is some penalty function and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive scalars. Then define the estimator $\hat{h}$ of the structural function by:

$$\hat{h} = \arg \inf_{h \in \mathcal{H}_{K(n)}} \hat{Q}_n(h)$$

Where $\mathcal{H}_{K(n)}$ is a sieve space. For clarity of analysis we assume that $\hat{m}$ and $\mathcal{H}_{K(n)}$ are chosen so that the infimum above is achieved by a unique $h \in \mathcal{H}_{K(n)}$, however the results in Chen & Pouzo (2012) do not impose this restriction.

Following Chen and Pouzo, let $H$ be the space to which the structural function $h_t$ is a priori confined (here a subset of $L^2(X \times E, F_{X_{it}, \eta_i})$). The estimator $\hat{h}$ described above fits the definition in Chen & Pouzo (2012) of a PSMD estimator of the structural function $h_t$ in the NPIV conditional moment model given by:

$$E_t[Y_{it} - h_t(X_{it}, \eta_i)|Z_{it}] = 0$$

For each natural number $K$, let $\mathcal{H}_K$ be the space defined by:

$$\mathcal{H}_K = \left\{(x, \eta) \mapsto \sum_{k=1}^{K} \beta_k \phi_k(x, \eta) : \beta \in B_K \right\}$$

Where for each $k$, $\phi_k$ is a function in $L^2(X \times E, F_{X_{it}, \eta_i})$ that satisfies:

$$\phi_k(X_{it}, \eta_i) = E_t[\gamma_k(X_{it}, V_{it})|X_{it}, \eta_i]$$

$B_K$ is assumed to be convex and to grow in the following sense. For each $K$ and $L$ define the set $B_K^L$ by $B_K^L = \{ \beta \in B_K : k \geq L \implies \beta_k = 0 \}$ then for each $L$ and each $K$ $B_K^L \subset B_{K+1}^L$ and $\bigcup_{k=L}^{\infty} B_K^k = \mathbb{R}^L$. For example this is satisfied if $B_K = \{ \beta \in \mathbb{R}^K : \|\beta\|_\infty \leq M_K \}$ where $|\cdot|_\infty$ is the maximum of the magnitudes of each coordinate of $\beta$ and $\{M_K\}_{K=1}^{\infty}$ is an increasing sequence of scalars with $M_K \to \infty$. Another sequence of sets $\{B_K\}_{K=1}^{\infty}$ that satisfies these conditions is $B_K = \mathbb{R}^K$.

With $\mathcal{H}_K$ defined as above, under Assumption 1.2, for $h \in \mathcal{H}_{K(n)}$ the functional $m$ can be rewritten as (for the choice of $\beta \in B_{K(n)}$ associated with $h$):

$$m(Z_{it}, h) = E_t[Y_{it}|Z_{it}] - \sum_{k=1}^{K(n)} \beta_k E_t[\gamma_k(X_{it}, V_{it})|Z_{it}]$$

And so a possibly consistent estimator of $m$ is the function $\hat{m}$ given by:
\[ \hat{m}(Z_{it}, h) = \tilde{E}[Y_{it}|Z_{it}] - \sum_{k=1}^{K(n)} \beta_k \tilde{E}[\gamma_k(X_{it}, V_{it})|Z_{it}] \]

Where \( \tilde{E}[Y_{it}|Z_{it}] \) is a consistent estimator of \( E_t[Y_{it}|Z_{it}] \) and \( \tilde{E}[\gamma_k(X_{it}, V_{it})|Z_{it}] \) is a consistent estimator of \( E_t[\gamma_k(X_{it}, V_{it})|Z_{it}] \).

Further suppose that if \( h(X_{it}, \eta_i) = \sum_{k=1}^{K(n)} \beta_k E_t[\gamma_k(X_{it}, V_{it})|X_{it}, \eta_i] \) for some real vector \( \beta \) then the penalty function \( \tilde{P}_n \) satisfies:

\[ \tilde{P}_n(h) = \tilde{P}_n(\beta) \]

For \( \tilde{P}_n \) the same as in [13]. It is easy to see that if \( \cup_{K=1}^{\infty} \mathcal{H}_k \) is dense in \( \mathcal{H} \) and \( \tilde{P}_n \) is continuous and convex then if \( \tilde{P}_n(h) \) satisfies the above it must also be convex on \( \mathcal{H}_{K(n)} \). Then with the sieve space \( \mathcal{H}_{K(n)} \) and the estimated functional \( \hat{m} \) defined as above, one can easily see that the solution \( \hat{h} \) to the problem [14] satisfies:

\[ \hat{h}(X_{it}, \eta_i) = \sum_{k=1}^{K(n)} \beta_k \tilde{E}[\gamma_k(X_{it}, V_{it})|X_{it}, \eta_i] \]

Where \( \beta_k \) is the \( k \)-th component of the vector \( \beta \) that solves [13].

An infeasible estimator of the CSF is given by:

\[ \hat{CSF}_t(x^{(1)}, S) = E_t[\hat{h}(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S] \]

Note that with \( \hat{h} \) defined as above, the infeasible estimator can be written as:

\[ \hat{CSF}_t(x^{(1)}, S) = \sum_{k=1}^{K(n)} \beta_k \tilde{E}[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in S] \]

This allows us to define a feasible estimator by:

\[ \tilde{CSF}_t(x^{(1)}, S) = \sum_{k=1}^{K(n)} \beta_k \tilde{E}[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in S] \]

Where \( \tilde{E}[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in S] \) is a consistent estimate of \( E_t[\gamma_k(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in S] \).

We are interested in the consistency and convergence rate of the feasible estimator. Note that the \( L_1(AX^{(1)}, F_{X_{it}^{(1)}}) \) error of the feasible estimator satisfies the triangle inequality and so:

\[ ||\tilde{CSF}_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} \leq ||\hat{CSF}_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} \]

\[ + ||\hat{CSF}_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} \]
Where $|| \cdot ||_{L_1}$ is the $L_1$-norm with measure given by the true distribution of observables, so for example:

$$||C\hat{S}F_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} = E_t[|C\hat{S}F_t(X^{(1)}_{it}, S) - CSF_t(X^{(1)}_{it}, S)|]$$

Let us define a ‘weak norm’ $|| \cdot ||_{\text{weak}}$ by:

$$||\delta||_{\text{weak}} = E_t [E_t [\delta(X_{it}, \eta_i)|W_{it}, Z_{it}]^2]$$

Technically, the above may not be a ‘norm’ on $L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$ because it may not be point separating. However, one can define a space of equivalence classes of $L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$ functions where two functions $\delta'$ and $\delta''$ are in the same equivalence class if and only if $E_t[\delta'(X_{it}, \eta_i)|Z_{it}] = E_t[\delta''(X_{it}, \eta_i)|Z_{it}]$. Then $|| \cdot ||_{\text{weak}}$ is point separating on this set of equivalence classes.

By Lemma 1, under Assumptions 1.1 and 2.c. there must be a constant $c_S$ so that for any $\delta \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$, letting the function $\delta'$ be defined by $\delta'(x^{(1)}) = E_t[\delta(x^{(1)}, X^{(2)}_{it}, \eta_i)|S_{it} \in S]$,

$$||\delta'||_{L_1} \leq c_S E_t [E_t [\delta(X_{it}, \eta_i)|W_{it}, Z_{it}]^2]$$

And so:

$$||C\hat{S}F_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} \leq ||C\hat{S}F_t(\cdot, S) - C\hat{S}F_t(\cdot, S)||_{L_1} + c_S ||\hat{h} - h_t||_{\text{weak}}$$

And so the $L_1$ error in the feasible estimate of the CSF can be decomposed into the difference between the feasible and infeasible estimate and the error in an NPIV estimate of $h_t$ in the weak norm. The first term is a linear combination of errors from non-parametric regression estimates. The rate of the second term can be handled using the results of Chen and Pouzo for PSMD estimators in NPIV models.

Pleasingly, PSMD estimation in the ‘weak’ norm above is not an ‘ill-posed problem’. To put this in the context of Chen and Pouzo’s analysis of PSMD estimators, the ‘sieve rate of ill-posedness’ that is is central to their rate results is trivially constant and equal to 1.

As discussed in Section 3 Assumption 1.5 is used to guarantee the existence of a set of functions $\{\gamma_k\}_{k=1}^\infty$ so that the corresponding sieve spaces satisfy:

$$\inf_{h \in H_K} ||h - h_t||_{\text{weak}} \to 0$$

The following lemma can be used to upper bound the rate of $\inf_{h \in H_K} ||h - h_t||_{\text{weak}}$ for a particular choice of functions $\{\gamma_k\}_{k=1}^\infty$. 

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Lemma 4

Suppose Assumptions 1.1, 1.5 and 2.b hold. Suppose for each $K$ there is a $K \times K$ matrix of coefficients $\beta(K)$ so that for any function $\delta$ defined on $\mathcal{X} \times \mathcal{V}$ with $E_t[\delta(X_{it}, V_{it})^2] \leq 1$:

$$E_t\left( E_t[\delta(X_{it}, V_{it})|Z_{it}] - \sum_{l=1}^{K} \sum_{k=1}^{K} \beta_{lk}^{(K)} E_t[\delta(X_{it}, V_{it}) \gamma_l(X_{it}, V_{it})] \psi_k(Z_{it}) \right)^2 \leq \rho_K^2$$

For some sequence of functions $\{\gamma_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ such that for each $k$, $E_t[\gamma_k(X_{it}, V_{it})^2] < \infty$ and $E_t[\psi_k(Z_{it})^2] < \infty$ and a sequence of scalars $\{\rho_K\}_{K=1}^{\infty}$ so that $\rho_K \rightarrow 0$.

Suppose as well that $g_t \in L_2(Z, F_{Z_{it}})$ defined by $g_t(Z_{it}) = E_t[h_t(X_{it}, \eta_i)|Z_{it}]$ with $h_t$ the structural function, satisfies the following condition. For each natural number $K$ there is a vector of coefficients $\alpha^{(K)}$ so that:

$$E_t\left( g_t(Z_{it}) - \sum_{k=1}^{K} \alpha_k^{(K)} \psi_k(Z_{it}) \right)^2 \leq \rho_K$$

Then there is a constant $c$ so that for each $K$ there exists a vector of coefficients $\mu^{(K)}$ such that:

$$E_t\left( g_t(Z_{it}) - E_t\left( \sum_{k=1}^{K} \mu_k^{(K)} \gamma_k(X_{it}, V_{it})|Z_{it} \right) \right)^2 \leq c\sqrt{\rho_K}$$

Furthermore, if Assumptions 1.2, and 1.3 hold then letting $\phi_k(X_{it}, \eta_i) = E_t[\gamma_k(X_{it}, V_{it})|X_{it}, \eta_i]$ for each $k$ then:

$$||h_t - \sum_{k=1}^{K} \mu_k^{(K)} \phi_k||_{weak} \leq c\sqrt{\rho_K}$$

Proof:

See Appendix.

The Lemma above shows that approximation results for say, the conditional density $f_{X_{it}, V_{it}|Z_{it}}$ and the conditional mean function $g_t$ can imply a rate for $\inf_{h \in \mathcal{H}_K} ||h - h_t||_{weak}$. In particular, given a set of functions $\{\gamma_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ that satisfy Lemma 4 for some sequence $\{\rho_k\}_{k=1}^{\infty}$ then defining $\mathcal{H}_K$ as in [15] with this set of functions $\{\gamma_k\}_{k=1}^{\infty}$, $\inf_{h \in \mathcal{H}_K} ||h - h_t||_{weak} = O(\sqrt{\rho_K})$.

There is a huge literature on the approximation of smooth functions by finite linear combinations of certain basis functions like wavelets or polynomials. Results from this literature could be used to derive a sequence of scalars $\{\rho_k\}_{k=1}^{\infty}$ and functions $\{\gamma_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ that satisfy Lemma 4 using primitive smoothness conditions on the conditional density $f_{X_{it}, V_{it}|Z_{it}}$ and on the
reduced form function $g_t$ defined by $g_t(Z_{it}) = E_t[Y_{it}|Z_{it}]$.

The following additional assumptions are used to prove consistency and the convergence rate of our estimator.

**Assumption 3.1 (Penalty)**

The following conditions are taken from Assumptions 3.2 (c) and 3.4 from Chen and Pouzo.

i. $\lambda_n > 0$ and $\lambda_n = o(1)$. Suppose that for some continuous functional $P$ and some $\{\Pi_n^i\}_{n=1}^\infty$ a sequence of operators mapping $\mathcal{H}$ to $\mathcal{H}_k(\Pi)$, $\sup_{h \in \mathcal{H}_k(\Pi)} |P(h) - P(h)| = o_p(1)$, $|P(h_t) - P(\Pi_n h_t)| = o(1)$. $P : \mathcal{H} \rightarrow [0, \infty)$ and $P(h_t) < \infty$.

ii. In addition to condition i above, $P$ is lower semi-compact.

iii. In addition to condition i above, $P$ is Gateaux differentiable and there exist constants $C_1$ and $C_2$ so that for any $h', h'' \in \mathcal{H}$:

$$P(h') - P(h'') \leq C_1 ||h' - h''||_{weak}^2 + (C_2 D[P](h'), h'' - h')_{weak}$$

Where $D[P]$ is the Gateaux derivative of $P$. The inner product $\langle \cdot, \cdot \rangle_{weak}$ is defined for any $h', h'' \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_t})$ by:

$$(h', h'')_{weak} = E_t[E_t[h'(X_{it}, \eta_t)]Z_{it}]E_t[h''(X_{it}, \eta_t)]Z_{it}]$$

**Assumption 3.2 (First Stage)**

The following condition corresponds to Assumption 3.3 in Chen and Pouzo.

Suppose that there exists a large positive constant $M$ and a function $P$ that satisfies Assumption 3.1i. that satisfy the following conditions.

For each $k$ define the space $\mathcal{H}_k^n$ by $\mathcal{H}_k^n = \{h \in \mathcal{H}_k : P(h) \leq M\}$. For some $\{\Pi_n^i\}_{n=1}^\infty$ a sequence of operators mapping $\mathcal{H}$ to $\mathcal{H}_k(\Pi)$, $\Pi_k(\Pi_n h_t) \in \mathcal{H}_k^n$ and $P(\Pi_n h_t) \rightarrow 1$. Further, for some sequence of scalars $\{\eta_n\}_{n=1}^\infty$ with $\eta_n \rightarrow 0$

$$\sup_{h \in \mathcal{H}_k^n} \left| \frac{1}{n} \sum_{i=1}^n \hat{m}(Z_{it}, h) - E_t[m(Z_{it}, h)] \right| = O_p(\eta_n)$$

**Assumption 3.3 (Third Stage)**

Let $\varepsilon_{5,n}$ be a random vector-valued function of length $K(n)$ so that the $k$-th component of $\varepsilon_{5,n}(x^{(1)})$ is distributed as:

$$E[\gamma_k((x^{(1)}, X_{it}^{(2)}), V_{it})|S_{it} \in \Sigma] - E[\gamma_k((x^{(1)}, X_{it}^{(2)}), V_{it})|S_{it} \in \Sigma]$$

That is, $\varepsilon_{5,n}$ captures the error from the third stage estimation as a function of $x^{(1)}$.

Let $\Gamma_{5,K(n)}$ be the symmetric $K(n) \times K(n)$ matrix whose $k, l$-th component is:

$$E_t \left[ E_t[\gamma_k(X_{it}, V_{it})|Z_{it}] E_t[\gamma_l(X_{it}, V_{it})|Z_{it}] \right]$$
Assume $\Gamma_{S,K(n)}$ is non-singular. Define the norm $\| \cdot \|_{\Gamma_{S,K(n)}^{-1}}$ on the space of length-$K(n)$ vector-valued functions of $X_{it}^{(1)}$ by:

$$
\| \delta \|_{\Gamma_{S,K(n)}^{-1}} = E_t[\delta(X_{it}^{(1)})^\top \Gamma_{S,K(n)}^{-1} \delta(X_{it}^{(1)})]
$$

Where $\delta^\top$ denotes the transpose of a column vector $\delta$. Assume that for some sequence of scalars $\{\upsilon_{S,n}\}_{n=1}^{\infty}$ with $\upsilon_{S,n} \to 0$ for each $s$, that:

$$
\| \varepsilon_{S,n} \|_{\Gamma_{S,K(n)}^{-1}} = O_p(\upsilon_{S,n})
$$

It is now possible to state Theorem 2, which establishes consistency of our estimator under different combinations of assumptions, and provides a converge rate in terms of the errors in the component non-parametric regressions and the conditions on the approximation error given in Lemma 3.

**Theorem 2:**

Let Assumptions 1.1-1.5, Assumptions 2.b and 2.c. and Assumptions 3.2 and 3.3 hold for sequences of scalars $\{\eta_n\}_{n=1}^{\infty}$ and $\{\upsilon_{S,n}\}_{n=1}^{\infty}$ respectively. Let the conditions for Lemma 3 hold for some sequence of scalars $\{\rho_k\}_{k=1}^{\infty}$ and sequence of functions $\{\gamma_k\}_{k=1}^{\infty}$. Let $K(n) \to \infty$ as $n \to \infty$.

If in addition either

- $K(n)/n \to 0$, $\lambda_n = 0$ and $B_K$ is bounded for each $K$.
- Assumption 3.1 i. and ii. hold and $\max\{\eta_n, \rho_{K(n)}\} = o(\lambda_n)$.
- Assumption 3.1 i. and iii. hold, $\max\{\eta_n, \rho_{K(n)}\} = o(\lambda_n)$ and:

  $$
  \max\{\lambda_n \sup_{h \in \mathcal{H}} |\tilde{P}_n(h) - P(h)|, \lambda_n \|h - \Pi_{K(n)}h_t\|_{weak} \} = O_p(\eta_n)
  $$

Then:

$$
\|CSF_t(\cdot, S) - CSF_t(\cdot, S)\|_{L_1} = O_p(\sqrt{\max\{\eta_n, \rho_{K(n)}, \upsilon_{S,n}^2\}})
$$

Where $\tilde{h}$ is a PSMD estimator defined earlier in this section with the relevant basis functions $\{\gamma_k\}_{k=1}^{\infty}$.

**Proof:**

See Appendix.

The theorem above demonstrates consistency of our estimator of the CSF in the $L_1(X^{(1)}, F_{X_{it}^{(1)}})$ norm for a particular choice of $S$. The convergence rate
of the estimator is in terms of the convergence rates for the component non-parametric regressions and the rate of approximation of \( g_t \) and a linear operator in \( \{\gamma_k\}_{k=1}^\infty \). The rates for the non-parametric regressions can in turn be derived for specific non-parametric regression methods like local linear regression, splines or wavelets using existing results. For example, Lemma C.2 in Chen & Pouzo (2012) provides a result for series least squares that implies their Assumption 3.3 and hence our Assumption 3.2.

5 Monte Carlo

To demonstrate the consistency of our estimation method and the bias of an approach that ignores the unobserved heterogeneity we present the following Monte Carlo exercise. As with all exercises of this nature, the reader should not infer that the error associated with our method will always be as low as in the simulation below. Instead the goal of this section is to provide evidence that the consistency results provided in the previous section are not simply a mathematical trick and that the asymptotic properties say something meaningful about the finite sample performance of the estimator. For the purpose of transparency the code for the Monte Carlo exercise can be downloaded at https://bdeaner.mit.edu/node/6.

The simulation below is modeled loosely after a simple one product demand model. \( p_{it} \) should be understood to represent the price of a good and \( Y_{it} \) the observed demand. We estimate both the average structural function of the price \( p_{it} \) and estimate the counterfactual change in average demand from a uniform rise in the price of 20%. We compare this to the true simulated changes for a sample of 1000 individuals for 8 time periods. We set the price to be strongly auto-correlated.

In particular \( p_{it} \) follows a first-order Markov process. Let \( \xi_{it} \) be iid zero mean Gaussian with variance \( \sigma^2_\xi \). Let \( v_{i1} = \xi_{i1} \) and:

\[
v_{it} = \alpha v_{i,t-1} + \xi_{it}
\]

For a constant \( \alpha \).

The scalar random variable \( p_{it} \) is then given by the formula:

\[
p_{it} = \theta_0 + \theta_1 \frac{1}{2}(\eta_{1i} + \eta_{2i}) + (1 + exp(-v_{it}))^{-1} \left[ \theta_2 + \frac{1}{2} \sum_{k=1}^{2} (\eta_{ki} - \frac{1}{2} (\eta_{1i} + \eta_{2i}))^2 \right]^{\frac{1}{2}}
\]

The two components of the unobserved heterogeneity \( \eta_i \) are each drawn independently from \( U[0,1] \).

Note that for a given individual \( i \), \( p_{it} \) follows a non-linear first-order Markov process that depends non-linearly on \( \eta_i \).

Let the structural function \( h_t \) be given by:

\[
h_t(p_{it}, \eta_i) = \zeta_0 + \zeta_1 \ln(\eta_{1i}^2) - \zeta_2 (1 + \eta_{1i}\eta_{2i})p_{it}^3
\]
\[\epsilon_{it}\] is iid zero-mean Gaussian with variance \(\sigma^2\).

The structural function is also nonlinear in both \(p_{it}\) and \(\eta_i\).

For the following simulation parameter values were as follows \(\alpha = 0.9, \sigma^2 = 2, \sigma^2_1 = 1, \theta_0 = 5, \theta_1 = 2, \theta_2 = 0.2, \phi_0 = 100, \phi_1 = 5, \phi_2 = 0.1\).

These parameters result in a structural function that for a given \(\eta_i\) resembles a demand curve and which varies significantly with \(\eta_i\). The degree of serial correlation was chosen to be high in order to emphasize that our estimation procedure allows for serial correlation of the regressors.

Recall that we simulate data for eight time periods. We let \(t\) (the period for which we estimate the structural function) be equal to 4. We use as instruments the vector of lagged prices \(Z_{it} = (p_{i1}; p_{i2}; p_{i3}; p_{i4}; p_{i5})\) and as proxies the vector of leads of the price \(V_{it} = (p_{i6}; p_{i7}; p_{i8})\). In accordance with the discussion Section 3, to allow for a first-order Markov process in the regressors we control for \(p_{i5}\). In the notation developed in the previous sections \(X_{it} = (X_{it}^{(1)}; X_{it}^{(2)})\) with \(X_{it}^{(1)} = p_{i4}\) and \(X_{it}^{(2)} = p_{i5}\).

We simulate data for 1000 individuals for eight time periods. We then estimate the (unconditional) average structural function and the change in average \(Y_{it}\) from a counterfactual 20% increase in \(X_{it}\).

We evaluate i. the values implied by the true structural function. ii. estimates based on lasso of power series basis functions when \(\eta_i\) is observed. iii. estimates based on lasso of power series basis functions with the heterogeneity ignored and iv. our method where both first stage and second stage regressions are performed using lasso.

All component non-parametric regressions in the simulation are carried out using lasso. With the exception of the second stage of our PSMD procedure we regress on all powers of the variables and interactions of powers of the variables such that the exponents add up to weakly less than 5. Specifically, for procedure ii. we regress by lasso \(Y_{i4}\) on regressors of the form \(\eta_{i1}^j; \eta_{i2}^k p_{i4}^l\) for all triples of integers \(j, k, l\) such that \(j, k, l \geq 0\) and \(j + k + l \leq 5\). For the naive estimator iii. we regress by lasso \(Y_{i4}\) on \(p_{i4}^j\) for \(j = 1, ..., 5\).

For procedure iv., our estimator, we proceed as follows. For each triple of integers \(j, k, l\) so that \(j, k, l \geq 0\) and \(j + k + l \leq 5\) we regress by lasso \(p_{i4}^j p_{i7}^k p_{i8}^l\) on \(p_{i1}^m p_{i2}^n p_{i3}^q p_{i4}^r p_{i5}^s\) for all quintuples of integers \(m, n, q, r, s\) so that \(m, n, q, r, s \geq 0\) and \(m + n + q + r + s \leq 5\). Denote the fitted valued from the first stage regressions by \(\hat{E}[p_{i4}^j p_{i7}^k p_{i8}^l | Z_{i4}]\) for each combination \(j, k, l\) that satisfies the conditions above.

Similarly we regress by lasso \(Y_{it}\) on regressors of the form \(p_{i1}^m p_{i2}^n p_{i3}^q p_{i4}^r p_{i5}^s\) for all quintuples of integers \(m, n, q, r, s\) so that \(m, n, q, r, s \geq 0\) and \(m + n + q + r + s \leq 5\). Denote the fitted values from this regression by \(\hat{E}[Y_{it}| Z_{i4}]\). We then regress by lasso \(\hat{E}[Y_{it}| Z_{i4}]\) on \(\hat{E}[p_{i1}^m p_{i2}^n p_{i3}^q p_{i4}^r p_{i5}^s | Z_{i4}]\) for all quintuples of natural numbers \(j, k, l, r, s\) so that \(j, k, l, r, s \geq 0\), \(j + k + l \leq 5\) and \(r + s \leq 5\).

In all cases apart from the second stage of our PSMD regression the lasso penalty parameter is chosen by two-fold cross-validation. In the second stage of the PSMD estimation procedure cross-validation of the penalty parameter may lead to over-fitting because conditional on the first stage, the dependent variable is deterministic. In order to avoid this problem we first regress by lasso \(Y_{it}\) on...
\( \hat{E}[\eta_i|p_{it}^{\kappa_1}p_{it}^{\kappa_2}|Z_{it}^4] p_{it}^{\kappa_5} \) (where the exponents satisfy the corresponding conditions stated above) and use two-fold cross-validation to choose the penalty parameter. We then use the penalty parameter from this regression in our second stage (i.e. for the same regression but with \( Y_{it} \) replaced by the fitted value \( \hat{E}[Y_{it}|Z_{it}^4] \)).

The figure below shows estimates of the (unconditional) average structural function. The true function is given in blue, ii. in red, iii. in yellow and iv. in purple.

It is clear from the above that approach iii. which ignores the presence of the unobserved heterogeneity results in an estimated APF which is qualitatively different to the true APF. For higher and lower prices this estimated APF appears biased downwards resulting in a strongly concave function that is upward sloping for some low values of the price. By contrast, the APF estimated using our method closely follows the shape of the true APF. The infeasible APF estimate that treats \( \eta_i \) as observed differs almost imperceptibly from the truth.

For further clarity the table below contains the integrated squared errors for the estimated average structural functions plotted above (integrated against the uniform measure between 5.4 and 6.8).

| \eta_i Observed | \eta_i Ignored | Our Method |
|-----------------|----------------|------------|
| 0.24            | 9.62           | 0.49       |

The table below gives the estimated mean change in demand from a 20% increase in \( p_{it} \):

| Predicted Change in Demand |
|-----------------------------|
| True Change | \( \eta_i \) Observed | \( \eta_i \) Ignored | Our Method |
| -21.5       | -25.0            | -55.1        | -27.5       |

While our estimator results in a counterfactual change that is still biased downward by about quarter of the true change, it performs vastly better than
the naive approach which is biased upward in magnitude by close to 150% of the true change.

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Appendix:

For the proofs and discussion below it will be helpful to introduce some new notation and clarify some notation introduced in the main body of the paper.

‘$L_2(Q, \mu)$’ refers to the space of functions that map from $Q$ to $\mathbb{R}$ and that are square integrable with respect to the measure $\mu$. For an element $\delta \in L_2(Q, \mu)$ one can define a random variable $\delta(Q)$ which is the random variable formed by applying an arbitrary element of $\delta$ (which must be $\mu$-measurable) to the random variable $Q$ that is distributed according to $\mu$. ‘$L_2(Q, \mu)$’ is the vector space that is the quotient space of $L_2(Q, \mu)$ with respect to those functions that are $\mu$-almost surely equal to zero. That is, each element $\delta$ of $L_2(Q, \mu)$ is a set of functions that map from the domain of $Q$ to $\mathbb{R}$, so that for any two functions $\delta'$ and $\delta''$ in $\delta$, $\int_Q \delta'^2 d\mu = \int_Q \delta''^2 d\mu$ exists and is finite and $\delta'(Q) = \delta''(Q)$. Note that elements of an $L_2$ space are sets of functions, rather than functions.

Suppose $Q$ and $Y$ are random variables so that $E[Q^2] < \infty$, then one can define a random variable by $E[Q|Y]$ in the usual way and note that it must be the case that $E[E[Q|Y]^2] < \infty$. One can then define an element $\delta$ of $L_2(Y, \mu_Y)$ where $\mathcal{Y}$ is the support of $Y$ and $\mu_Y$ is the marginal distribution of $Y$ by the set of functions $\delta$ in $L_2(Y, \mu_Y)$ such that the corresponding random variable $\tilde{\delta}(Y)$ is almost surely equal to the random variable $E[Q|Y]$.

For short we write that $\delta$ is ‘defined by’:

$$\delta(Y) = E[Q|Y]$$

Note that for an element $\delta \in L_2(Y, \mu_Y)$, for any functions $\delta'$ and $\delta''$ in $\delta$ the random variable $\delta'(Y)$ has the same distribution as the random variable $\delta''(Y)$. For this reason we often write $\delta(Y)$ to mean $\delta'(Y)$ for an arbitrary function $\delta'$ in $\delta$.

‘$|| \cdot ||$’ denotes the $L_2$-norm with respect to the joint distributions of the relevant random variables involved and ‘$(\cdot, \cdot)$’ denotes the $L_2$ inner-product. For instance if functions $\delta$ and $\delta'$ each map from the domains of $X_{it}$ and $V_{it}$ to $\mathbb{R}$ then $||\delta'||$ means $E_t[\delta'(X_{it}, V_{it})^2]$ and $(\delta, \delta')$ means $E_t[\delta(X_{it}, V_{it})\delta'(X_{it}, V_{it})]$. 

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Pointwise Singular Systems

Conditions for the existence of a solution to a linear operator equation usually involve a singular value decomposition of the operator. However, the singular value decomposition generally only exists if the operator is compact. The operators $A$ and $B^*$ defined in Assumption 2 are generally not compact when some regressors are exogenous, and so do not admit a singular value composition. Horowitz (2011) deals with exogenous regressors by considering separate operators for each possible value of the explanatory variables. In order to avoid some of the notational complexity of this approach we introduce what we term ‘pointwise singular systems’.

$A$ and $B^*$ admit a unique pointwise singular system that shares most of the important features of a singular system associated with a classic singular value decomposition and can be used in much the same way. In this subsection we first introduce the classical notion of a singular value decomposition of a compact linear operator. We give conditions for the compactness of a conditional expectations operator. We then define the ‘pointwise singular system’ and prove its existence and uniqueness for a certain class of linear operators that includes both $A$ and $B^*$. We then show that 2.b and 2.c imply the existence of solutions to two particular linear operator equations.

We now present a brief primer on the singular value decompositions of compact linear operators between Hilbert spaces. This part of the discussion follows that found in Kress (2014).

For a compact linear operator $M$ between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the singular system for $M$ is a sequence of tuples $\{v_k, u_k, \mu_k\}_k^{\infty}$. $\{\mu_k\}_k^{\infty}$ is a weakly decreasing sequence of scalars referred to as the singular values of $M$. It must be the case that $\mu_k \to 0$. $\{v_k\}_k^{\infty}$ is a complete orthonormal set for $N(M)^{\perp}$ the orthogonal complement of the null space of $M$. $\{u_k\}_k^{\infty}$ is a complete orthonormal set for $R(M)$ the closure of the range of $M$. For all $k$ $M[v_k] = \mu_k u_k$ and $M^*[u_k] = \mu_k v_k$ (where $M^*$ is the adjoint of $M$ in the inner-product of the Hilbert space $\mathcal{H}_2$). This system exists by e.g. Theorem 15.16 in Kress.

Suppose $M$ is injective. Let $\delta$ be an element of the Hilbert space $\mathcal{H}_2$. Then by Picard’s theorem (e.g. Theorem 15.18 in Kress) there exists an element $\delta'$ in $\mathcal{H}_1$ so that $M[\delta'] = \delta$ if and only if:

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k^2} |(\delta, u_k)_{\mathcal{H}_2}|^2 < \infty$$

Where the bilinear form $(\cdot, \cdot)_{\mathcal{H}_2}$ is the inner product of the Hilbert space $\mathcal{H}_2$. In which case:

$$\delta' = \sum_{k=1}^{\infty} \frac{1}{\mu_k} (\delta, u_k)_{\mathcal{H}_2} v_k$$

And:

$$||\delta'||^2_{\mathcal{H}_1} = \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} |(\delta, u_k)_{\mathcal{H}_2}|^2$$

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Note that $M$ is assumed to be compact. Suppose $\mathcal{H}_1$ is a space $L_2(\Omega_x, \sigma_x)$ and $\mathcal{H}_2$ is a space $L_2(\Omega_y, \sigma_y)$. Suppose $M[\delta]$ is the set of real-valued functions $\delta'$ that satisfy:

$$\delta'(y) = \int_{\Omega_y} K(y, x) \delta(x) d\sigma_x(x)$$

$\sigma_y$-almost surely.

And with $K \in L_2(\Omega_x \times \Omega_y, [\sigma_x \times \sigma_y])$, that is:

$$\int_{\Omega_x \times \Omega_y} K(y, x)^2 d[\sigma_x \times \sigma_y](y, x) < \infty$$

Where $[\sigma_x \times \sigma_y]$ is the product measure. Then it is well known that $M$ is a compact operator.

If $\sigma_x$ and $\sigma_y$ are probability measures (i.e. positive measures with $\sigma_x(\Omega_x) = \sigma_y(\Omega_y) = 1$) then one can understand $M$ to be a conditional expectation operator. Let $X$ and $Y$ be random variables whose marginal distributions correspond to the measures $\sigma_x$ and $\sigma_y$ respectively and whose joint distribution corresponds to a probability measure $\sigma_{x,y}$ defined on the product Borel algebra of $\Omega_x \times \Omega_y$ so that for any event $\Omega'$ in this Borel space:

$$\sigma_{x,y}(\Omega') = \int_{\Omega'} K(y, x) d[\sigma_x \times \sigma_y](y, x)$$

Then $K(y, x)$ is the Radon-Nikodym derivative of $\sigma_{x,y}$ with respect to $[\sigma_x \times \sigma_y]$ and clearly $\sigma_{x,y}$ defined as above is absolutely continuous with $[\sigma_x \times \sigma_y]$.

Note that equivalently $M$ satisfies:

$$M[\delta] = \{ q \in L_2(\Omega_y, \sigma_y) : q(Y) = E[\delta(X)|Y] \}$$

Where the conditional expectation has the usual definition give the probability measure $\sigma_{x,y}$.

Applying this argument backwards one can conclude the following. If $X$ and $Y$ are random variables whose joint probability measure is absolutely continuous with the product of their marginal probability measures, and the Radon-Nikodym derivative of their joint measure with respect to the product of the marginals is square integrable with respect to the product of the marginals, then the operator $M$ as defined above is compact.

However, suppose that there are instead three random variables $X$, $Y$ and $W$. Then the joint distribution of $(X;W)$ and $(Y;W)$ cannot be dominated by the product of the marginal of $(X;W)$ and the marginal of $(Y;W)$. Hence the operator $M$ defined by:

$$M[\delta] = \{ q \in L_2(\Omega_{y,w}, \sigma_{y,w}) : q(Y, W) = E[\delta(X,W)|Y,W] \}$$

Is generally not compact and generally does not admit a singular value decomposition.

However, we now introduce and prove the following proposition:
Proposition 0

Let $M$ be a conditional expectation operator between $L_2(\Omega_{x,w}, \sigma_{x,w})$ and $L_2(\Omega_{y,w}, \sigma_{y,w})$ defined by:

$$M[\delta] = \{ q \in L_2(\Omega_{y,w}, \sigma_{y,w}) : q(Y, W) = E[\delta(X, W)|Y, W] \}$$

Suppose that $\sigma_{x,y,w}$ is dominated by the product of its marginals $[\sigma_x \times \sigma_y \times \sigma_w]$ and that the Radon-Nikodym derivative of $\sigma_{x,y,w}$ with respect to $[\sigma_x \times \sigma_y \times \sigma_w]$ is square integrable with respect to $[\sigma_x \times \sigma_y \times \sigma_w]$.

Then there exists a sequence of tuples $\{v_k, u_k; \mu_k\}_{k=1}^{\infty}$ with the following properties. $\{\mu_k\}_{k=1}^{\infty}$ is a sequence of functions from $\Omega_w$ to $\mathbb{R}$. so that for each $w \in \Omega_w \lim_{k \to \infty} \mu_k(w) = 0$. $\{v_k\}_{k=1}^{\infty}$ is a complete orthonormal set for $N(M)^\perp$ the orthogonal complement of the null space of $M$. $\{u_k\}_{k=1}^{\infty}$ is a complete orthonormal set for $R(M)$ the closure of the range of $M$. For all $k$ $E[v_k(W, X)|W, Y] = \mu_k(W)u_k(W, Y)$ and $E[u_k(W, Y)|W, X] = \mu_k(W)v_k(W, X)$.

Furthermore, if for some function $\delta \in R(M)$:

$$E[\sum_{k=1}^{\infty} E[\frac{\delta(W, Y)u_k(W, Y)}{\mu_k(W)^2}|W]^2] < \infty$$

Then there exists a $\delta' \in \mathcal{L}_2(\Omega_{w,x}, \sigma_{w,x})$ such that:

$$\delta(W, Y) = E[\delta'(W, X)|W, Y]$$

And:

$$E[\delta'(W, X)^2] = \sum_{k=1}^{\infty} E[\frac{\delta(W, Y)u_k(W, Y)}{\mu_k(W)^2}|W]^2]$$

Proof

By assumption the measure $\sigma_{x,y,w}$ is dominated by $[\sigma_x \times \sigma_y \times \sigma_w]$, denote a corresponding Radon-Nikodym derivative by $K$. Note that the Radon-Nikodym derivative is not generally a unique function, but it is important to emphasize that $K$ will represent an arbitrary such function that is fixed throughout the proof.

Now, for every $w \in \Omega_w$ define an operator $M_w$ by:

$$M_w[\delta] = \int_{\Omega_x} \delta(x)K(x, y, w)d\sigma_x(x)$$

Note that if some function $\delta$ and some function $\delta'$ satisfies:

$$\delta'(y, w) = M_w[\delta(w, \cdot)](y)$$

For $\sigma_{y,w}$-almost all $y$ and $w$ then:

$$\delta'(Y, W) = E[\delta(X, W)|Y, W]$$

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And so: \( \delta' \in M[\delta] \)

Now note that by assumption:

\[
\int_{\Omega_{x,y,w}} K(x, y, w)^2 d[\sigma_x \times \sigma_y \times \sigma_w](x, y, w) < \infty
\]

But then for \( \sigma_w \)-almost all \( w \):

\[
\int_{\Omega_{x,y}} K(x, y, w)^2 d[\sigma_x \times \sigma_y](x, y) < \infty
\]

And so for \( \sigma_w \)-almost all \( w \), \( M_w \) is a compact operator.

Let \( \mathcal{W}' \) be the subset of \( \Omega_w \) so that for any \( w \in \mathcal{W}' \) \( M_w \) is compact and recall \( \sigma_w(\mathcal{W}') = 1 \). Then for each \( w \in \mathcal{W}' \) \( M_w \) admits a singular value decomposition \( \{v_k(w, \cdot), u_k(w, \cdot); \mu_k(w)\}_{k=1}^{\infty} \) with the properties discussed earlier in this section. For each \( w \) in the complement of \( \mathcal{W}' \) relative to \( \Omega_w \) let each element of \( \{v_k(w, \cdot), u_k(w, \cdot); \mu_k(w)\}_{k=1}^{\infty} \) be equal to zero.

Now we show that the resulting sequence of tuples \( \{v_k, u_k; \mu_k\}_{k=1}^{\infty} \) has the properties discussed in Proposition 0.

By the definition of a singular system, for \( \sigma_w \)-almost all \( w \):

\[
\int_{\Omega_x} v_k(w, x)K(x, y, w)d\sigma_x(x) = \mu_k(w)u_k(w, y)
\]

Which is equivalent to:

\[
E[v_k(W, X) | Y, W] = \mu_k(W)u_k(W, Y)
\]

Similarly for \( \sigma_w \)-almost all \( w \):

\[
\int_{\Omega_y} u_k(w, y)K(x, y, w)d\sigma_y(y) = \mu_k(w)v_k(w, x)
\]

Which is equivalent to:

\[
E[u_k(W, Y) | X, W] = \mu_k(W)v_k(W, X)
\]

For \( \sigma_w \)-almost all \( w \):

\[
\int_{\Omega_{x,y}} v_k(w, x)^2K(x, y, w)d[\sigma_x \times \sigma_y](x, y) = 1
\]

Which is equivalent to:

\[
E[v_k(W, X)^2] = 1
\]

For \( \sigma_w \)-almost all \( w \):

\[
\int_{\Omega_{x,y}} u_k(w, y)^2K(x, y, w)d[\sigma_x \times \sigma_y](x, y) = 1
\]
Which is equivalent to:
\[ E[u_k(W,Y)^2] = 1 \]
For \( \sigma_w \)-almost all \( w \) and any \( k \neq l \):
\[ \int_{\Omega_{x,y}} v_k(w,x)v_l(w,x)K(x,y,w)d[\sigma_x \times \sigma_y](x,y) = 0 \]
Which is equivalent to, for any \( k \neq l \):
\[ E[v_k(W,X)v_l(W,X)] = 0 \]
For \( \sigma_w \)-almost all \( w \) and any \( k \neq l \):
\[ \int_{\Omega_{x,y}} u_k(w,y)u_l(w,y)K(x,y,w)d[\sigma_x \times \sigma_y](x,y) = 0 \]
Which is equivalent to, for any \( k \neq l \):
\[ E[u_k(W,Y)u_l(W,Y)] = 0 \]
Next note that \( \delta \) is in the null space of \( M \) if and only if \( \delta(w,\cdot) \) is in the null space of \( M_w \) for \( \sigma_w \)-almost all \( w \). Let \( \delta' \) be orthogonal to \( \delta \), that is:
\[ E[\delta(W,X)\delta'(W,X)] = 0 \]
Which is true if and only if for \( \sigma_w \)-almost all \( w \):
\[ \int_{\Omega_{x,y}} \delta(w,x)\delta'(w,x)K(x,y,w)d[\sigma_x \times \sigma_y](x,y) = 0 \]
And so \( \delta'(w,\cdot) \) is in the orthogonal complement of the null space of \( M_w \). Recall that for \( \sigma_w \)-almost all \( w \) \( \{v_k(w,\cdot)\}_{k=1}^{\infty} \) is a complete orthonormal system for \( N(M_w)^\perp \) and so:
\[ \delta'(w,\cdot) = \sum_{k=1}^{\infty} \left( \int_{\Omega_{x,y}} v_k(w,x)\delta'(w,x)K(x,y,w)d[\sigma_x \times \sigma_y](x,y) \right)v_k(w,\cdot) \]
Which holds if and only if for \( \sigma_w \)-almost all \( w \) then:
\[ \delta'(W,X) = \sum_{k=1}^{\infty} E[v_k(W,X)\delta'(W,X)|W]v_k(W,X) \]
So one can conclude that \( \{v_k\}_{k=1}^{\infty} \) is a complete orthonormal set for \( N(M)^\perp \). Since the steps above are equivalences it follows that conversely that if \( \{v_k\}_{k=1}^{\infty} \) is a complete orthonormal set for \( N(M)^\perp \) then for \( \sigma_w \)-almost all \( w \) \( \{v_k(w,\cdot)\}_{k=1}^{\infty} \) is a complete orthonormal system for \( N(M_w)^\perp \).
Next note that clearly \( \delta(w, \cdot) \in \overline{R(M_w)} \) if and only if \( \delta \in \overline{R(M)} \). Note that by the properties of the singular value decomposition for \( \sigma_w \)-almost all \( w \) \( \{u_k(w, \cdot)\}_{k=1}^{\infty} \) is a complete orthonormal set for \( \overline{R(M_w)} \) and so:

\[
\delta'(w, \cdot) = \sum_{k=1}^{\infty} \left( \int_{\Omega_{x,y}} u_k(w, y) \delta'(w, y) K(x, y, w) d[\sigma_x \times \sigma_y](x, y) \right) u_k(w, \cdot)
\]

But the above holds for \( \sigma_w \)-almost all \( w \) if and only if:

\[
\delta'(W, Y) = \sum_{k=1}^{\infty} E[u_k(W, Y) \delta'(W, Y)|W] u_k(W, Y)
\]

It follows that for \( \sigma_w \)-almost all \( w \) \( \{u_k(w, \cdot)\}_{k=1}^{\infty} \) is a complete orthonormal set for \( \overline{R(M_w)} \) if and only if \( \{u_k\}_{k=1}^{\infty} \) is a complete orthonormal set for \( \overline{R(M)} \). The discussion above shows that if, for \( \sigma_w \)-almost all \( w \), \( \{v_k(w, \cdot), u_k(w, \cdot), \mu_k(w)\}_{k=1}^{\infty} \) is a singular system for \( M_w \), then \( \{v_k, u_k, \mu_k\}_{k=1}^{\infty} \) has all the properties of a ‘pointwise orthonormal set’ for \( M \). Conversely, the above shows that if \( \{v_k, u_k, \mu_k\}_{k=1}^{\infty} \) has all the properties of ‘pointwise orthonormal set’ for \( M \) then the discussion above shows that for \( \sigma_w \)-almost all \( w \), \( \{v_k(w, \cdot), u_k(w, \cdot), \mu_k(w)\}_{k=1}^{\infty} \) is a singular system for \( M_w \). Since singular value decompositions are unique it follows that the ‘pointwise singular system’ of \( M \) is also unique.

Finally, note that by Picard’s theorem discussed above, if for some \( w \in W \) and \( \delta \in \mathcal{L}_2(\Omega_{w,y}, \sigma_{w,y}) \), for some \( w \):

\[
\sum_{k=1}^{\infty} \frac{1}{\mu_k(w)^2} \left( \int_{\Omega_{x,y}} \delta(w, y) u_k(w, y) K(x, y, w) d[\sigma_x \times \sigma_y](x, y) \right)^2 < \infty
\]

Then there exists \( \delta_w' \) with:

\[
\int_{\Omega_x} \delta_w'(x) K(x, y, w) d\mu_x(x) = \delta(w, y)
\]

And so if the condition holds for \( \sigma_w \)-almost all \( w \) then letting \( \delta'' \) be a real valued function so that:

\[
\delta''(w, x) = \delta_w'(x)
\]

For \( \sigma_w \) almost all \( w \) then:

\[
E[\delta''(W, X)|Y, W] = \delta(W, Y)
\]

Furthermore note that:

\[
\int_{\Omega_x} \delta_w'(x) K(x, y, w) d[\sigma_x \times \sigma_y](x, y) \leq \sum_{k=1}^{\infty} \frac{1}{\mu_k(w)^2} \left( \int_{\Omega_{x,y}} \delta(w, y) u_k(w, y) K(x, y, w) d[\sigma_x \times \sigma_y](x, y) \right)^2 < \infty
\]
And so:

\[ E[\delta'(W,X)^2] \leq E[\sum_{k=1}^{\infty} E\left[ \frac{\delta(W,Y)u_k(W,Y)}{\mu_k(W)^2}\right][W]^{2}] < \infty \]

But note that conversely, if \[ E[\sum_{k=1}^{\infty} E\left[ \frac{\delta(W,Y)u_k(W,Y)}{\mu_k(W)^2}\right][W]^{2}] < \infty \] then it must be the case that for \[ \sigma_w \]-almost all \[ w \]:

\[ \sum_{k=1}^{\infty} \frac{1}{\mu_k(w)^2} \left( \int_{\Omega_{x,y}} \delta(w,y)v_k(w,y)K(x,y,w)d[\sigma_x \times \sigma_y](x,y) \right)^2 < \infty \]

One can conclude then, that if:

\[ E[\sum_{k=1}^{\infty} E\left[ \frac{\delta(W,Y)u_k(W,Y)}{\mu_k(W)^2}\right][W]^{2}] < \infty \]

There exists a \( \delta' \) so that:

\[ E[\delta'(W,X)|W,Y] = \delta(W,Y) \]

Proposition 0 applies immediately to the operators \( A \) and \( B^* \) defined in the statement of Assumption 2. For \( A \), \( (\tilde{X}_{it}; V_{it}) \) takes the place of \( X \) above, \( \tilde{Z}_{it} \) takes the place of \( Y \) and \( W_{it} \) takes the place of \( W \). For \( B^* \), \( \tilde{Z}_{it} \) takes the place of \( X \) above, \( (\tilde{X}_{it}; \eta_i) \) takes the place of \( Y \) and \( W_{it} \) takes the place of \( W \). In both cases the condition that \( F_{\tilde{X}_{it},V_{it},\eta_i,W_{it},\tilde{Z}_{it}} \) is dominated by the product of its marginals and has a continuous and square integrable Radon-Nikodym derivative w.r.t the product of its marginals from Assumption 1.1 ensures the ‘pointwise singular system’ exists and is unique. It is shown in Propositions 5 that under Assumptions 1.1-1.5 \( g_t \) is in the closure of the range of \( A \). Furthermore, the stronger version of Assumption 1.4 in 2.c is equivalent to the statement that \( B \) (the adjoint of \( B^* \)) is injective, which implies \( B^* \) has dense range and hence \( f_S \) is in the closure of the range of \( B^* \).

Then Proposition above immediately gives the following two results:

Under Assumptions 1.1-1.5 and 2.b there exists a function \( \gamma \in L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it},V_{it}}) \) such that:

\[ E_t[\gamma(X_{it},V_{it})|Z_{it}] = g_t(Z_{it}) \]

Under Assumptions 1.1-1.5 and 2.c there exists a function \( \psi_S \in L_2(\mathcal{Z}, F_{Z_{it}}) \) such that:

\[ E_t[\psi_S(Z_{it})|X_{it}, \eta_i] = f_S(X_{it}, \eta_i) \]

These facts will both be used in the proofs of Lemma 1 and Theorem 1.
Proof of Lemma 1 Theorem 2

Define the following linear operators:

\[ A : \mathcal{L}_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}}) \rightarrow \mathcal{L}_2(\mathcal{Z}, F_{Z_{it}}) \]

\[ A[\delta] = \{ q \in \mathcal{L}_2(\mathcal{Z}, F_{Z_{it}}) : q(Z_{it}) = \mathbb{E}_t[\delta(X_{it}, V_{it})|Z_{it}] \} \]

\[ B : \mathcal{L}_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \rightarrow \mathcal{L}_2(\mathcal{Z}, F_{Z_{it}}) \]

\[ B[\delta] = \{ q \in \mathcal{L}_2(\mathcal{Z}, F_{Z_{it}}) : q(Z_{it}) = \mathbb{E}_t[\delta(X_{it}, \eta_i)|Z_{it}] \} \]

\[ C : \mathcal{L}_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}}) \rightarrow \mathcal{L}_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \]

\[ C[\delta] = \{ q \in \mathcal{L}_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) : q(X_{it}, \eta_i) = \mathbb{E}_t[\delta(X_{it}, V_{it})|X_{it}, \eta_i] \} \]

\[ D_{\delta} : \mathcal{L}_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \rightarrow \mathcal{L}_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}}) \]

\[ D_{\delta}[\delta] = \{ q \in \mathcal{L}_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}}) : q(x_{it}^{(1)}) = \mathbb{E}_t[\delta(x_{it}^{(1)}, X_{it}^{(2)}, \eta)|S_{it} \in \mathbb{S}] \} \]

\[ G_{\delta} : \mathcal{L}_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}}) \rightarrow \mathcal{L}_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}}) \]

\[ G_{\delta}[\delta] = \{ q \in \mathcal{L}_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}}) : q(x_{it}^{(1)}) = \mathbb{E}_t[\delta(x_{it}^{(1)}, X_{it}^{(2)}, \eta)|S_{it} \in \mathbb{S}] \} \]

Note that each of these operators is defined on a space of \( \mathcal{L}_2 \) functions to an \( \mathcal{L}_2 \) space. However, \( \delta \in \mathcal{L}_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}}) \) then \( A[\delta] \) is the image of \( A \) on the set of functions in \( \delta \) (recall \( \mathcal{L}_2 \) is a space whose elements are sets of functions). Note that then \( A[\delta] \) is the same as \( A \) applied to an arbitrary function in \( \delta \). One can easily confirm that the same argument applies for the other operators above.

The above operators are bounded (but not necessarily compact).

It will also be useful to define \( g_t \in \mathcal{L}_2(\mathcal{Z}, F_{Z_{it}}) \) by:

\[ g_t(Z_{it}) = \mathbb{E}_t[Y_{it}|Z_{it}] \]

Although the structural function \( h_t \) is in fact an element of \( \mathcal{L}_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \), \( h_t \) will often be understood to mean the corresponding element of \( \mathcal{L}_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \).

We now state the following six propositions regarding the relationships between the operators and functions defined above under Assumptions 1.1-1.5. The proofs for each of them are provided below.
Proposition 1:
For any function $\delta \in L^2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$:
\[
B \circ C[\delta] = A[\delta]
\]
(16)

Proposition 2:
For any function $\delta \in L^2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$:
\[
D_\mathcal{S} \circ C[\delta] = G_\mathcal{S}[\delta]
\]
(17)

Proposition 3:
\[
B[h_t] = g_t
\]
(18)

Proposition 4:
For any $\delta \in L^2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$ and any $\epsilon > 0$ there is a $\gamma \in L^2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$ such that
\[
||B \circ C[\gamma] - B[\delta]|| \leq \epsilon
\]
(19)

Proposition 5:
For any $\epsilon > 0$ there must exist a $\gamma \in L^2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$ such that:
\[
||A[\gamma] - g_t|| \leq \epsilon
\]
(20)

Proposition 6:
If there is a function from the distribution of observables to $D_\mathcal{S}[h_t]$ then $CSF_i(x^{(1)}, \mathcal{S})$ is identified at $\mathcal{S}$ and all $x^{(1)}$ in the support of $X_{it}^{(1)}$.

Proof of Propositions 1-6

Proof of Proposition 1:
Assumption 1.2 states that $(S_{it}, Z_{it}, X_{it}) \perp V_{it}|X_{it}^{(2)}, \eta_i$ which implies that $Z_{it} \perp V_{it}|X_{it}, \eta_i$, and so for any $\delta \in L^2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$, $E_t[\delta(X_{it}, V_{it})|X_{it}, \eta_i] = E_t[\delta(X_{it}, V_{it})|Z_{it}, X_{it}, \eta_i]$. So by iterated expectations:
\[
E_t[E_t[\delta(X_{it}, V_{it})|X_{it}, \eta_i]|Z_{it}] = E_t[E_t[\delta(X_{it}, V_{it})|Z_{it}, X_{it}, \eta_i]|Z_{it}]
\]
(21)

Which, in terms of the operators defined above, is equivalent to [10].

Proof of Proposition 2:
For some $\delta \in L^2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$ let $\tilde{\delta}$ be some function in $L^2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$ that satisfies $\tilde{\delta}(x^{(1)}, X_{it}^{(2)}, \eta_i) = E_t[\delta(x^{(1)}, X_{it}^{(2)}, V_{it})|X_{it}^{(2)}, \eta_i]$. Assumption 1.2
states that \((S_{it}, Z_{it}, X_{it}) \perp V_{it}|X_{it}^{(2)}, \eta_i\) which implies that \(X_{it}^{(1)} \perp V_{it}|X_{it}^{(2)}, \eta_i\) and so:

\[
\delta(X_{it}^{(1)}, X_{it}^{(2)}, \eta_i) = E_t[\delta(X_{it}, V_{it})|X_{it}^{(1)}, X_{it}^{(2)}, \eta_i]
\]

Furthermore, Assumption 1.3 implies that \(S_{it} \perp V_{it}|X_{it}^{(2)}, \eta_i\), and so:

\[
E_t[\delta(x^{(1)}, X_{it}^{(2)}, V_{it})|X_{it}^{(2)}, \eta_i, S_{it} \in \mathbb{S}] = E_t[\delta(x^{(1)}, X_{it}^{(2)}, V_{it})|X_{it}^{(2)}, \eta_i, S_{it} \in \mathbb{S}]
\]

Let \(\delta'\) be some function in \(L_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}})\) such that \(\delta'(x^{(1)}) = E_t[\delta(x^{(1)}, X_{it}^{(2)}, V_{it})|S_{it} \in \mathbb{S}]\) and \(\delta''\) be some function in \(L_2(\mathcal{X}^{(1)}, F_{X_{it}^{(1)}})\) such that \(\delta''(x^{(1)}) = E_t[\delta(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in \mathbb{S}]\) then by iterated expectations:

\[
\delta'(X_{it}^{(1)}) = \delta''(X_{it}^{(1)})
\]

And in terms of the operators defined above this is equivalent to \([17]\)

**Proof of Proposition 3:**

Note that:

\[
E_t[Y_{it}|Z_{it}] = E_t[h_t(X_{it}, \eta_i)|Z_{it}] + E_t[\epsilon_{it}|Z_{it}]
\]

Where the first equality follows by substitution the model for \(Y_{it}\) and the second follows by assumption 1.3. In terms of the operators this is written as \(g_t = B[h_t]\).

**Proof of Proposition 4:**

Suppose the above is not true and so the closure of the range \(B \circ C\) does not contain the point \(B[h]\). One can apply the hyper-plane separation theorem and so there must exist a function \(\delta\) in \(L_2(Z, F_{Z_{it}})\) and scalar \(c > 0\) so that:

\[
(B \circ C[\gamma] - B[h], \delta) \geq c
\]

For all \(\gamma\) in \(L_2(\mathcal{X} \times \mathcal{Y}, F_{X_{it}^{(2)}}, V_{it})\). But note that:

\[
(B \circ C[\gamma] - B[h], \delta) = (B \circ C[\gamma], \delta) - (B[h], \delta)
\]

\[
= (\gamma, C^* \circ B^*[\delta]) - (h, B^*[\delta])
\]

Where \(C^*\) is the adjoint of \(C\) and \(B^*\) is the adjoint of \(B\) with respect to the inner product \((\cdot, \cdot)\).

But unless \(||C^* \circ B^*[\delta]|| = 0\) and \((h, B^*[\delta]) = 0\) one can always find a \(\gamma\) that sets the above to zero. But assumption 1.5 states that \(||C^* \circ B^*[\delta]|| = 0 \iff ||B^*[\delta]|| = 0\) and so we have a contradiction.

**Proof of Proposition 5:**

Applying \([16]\) and \([18]\) to proposition 4 one gets the above.

**Proof of Proposition 6:**

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Note that by definition of the operator $D_S$, for any function $\delta \in D_S[h_t]$, $\delta(x^{(1)}) = E_t[h_t(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S]$ for some function $h'_t$ in $h_t$ (recall $h_t$ is understood as an element of $L_2$):

$$\delta(x^{(1)}) = E_t[h_t(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S] = CSF_t(x^{(1)}, \delta)$$

For $F_{X_{it}^{(1)}}$-almost all $x^{(1)}$. If $h_t$ is continuous in its first argument then clearly so is $CSF_t$. Note that if two functions are the same almost everywhere and both are continuous then they are the same everywhere. So $CSF_t(\cdot, \delta)$ is equal to the unique function $\delta$ that is continuous and that satisfies the above for $F_{X_{it}^{(1)}}$-almost all $x^{(1)}$. And so if $D_S$ is identified $CSF_t(\cdot, \delta)$ is identified

□

Proof of Lemma 1:

In the discussion of the pointwise singular system above it is shown that Assumptions 1.1-1.5 and Assumption 2.c imply there exists an element $\psi_S \in L_2(\mathcal{Z}, F_{X_{it}}, \eta_i)$ that satisfies:

$$f_S = B^*[\psi_S] \quad (22)$$

And:

$$||\psi_S|| \leq c_S$$

Consider some $b \in L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$, let the function $\delta_S$ satisfy $\delta_S(x^{(1)}) = E_t[b(x^{(1)}, X_{it}^{(2)}, \eta_i)|S_{it} \in S]$, then $\delta_S \in D_S[b]$.

Let $W'$ be a measurable subset of the support of $W_{it}$ with $Pr_t[W_{it} \in W'] \neq 0$. Then by definition of $f_S$:

$$E_t[\delta_S(X_{it}^{(1)})|W_{it} \in W'] = E_t[b(X_{it}, \eta_i)f_S(X_{it}, \eta_i)|W_{it} \in W'] = (b, f_S)_{W'}$$

Where $(\cdot, \cdot)_{W'}$ is an inner product defined on $L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \times L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i})$ by:

$$(\delta', \delta'')_{W'} = E_t[\delta'(X_{it}, \eta_i)\delta''(X_{it}, \eta_i)|W_{it} \in W']$$

And so by 22:

$$(b, f_S)_{W'} = (b, B^*[\psi_S])_{W'}$$

And so:

$$(b, f_S)_{W'} = (B[b], \psi_S)_{W'}$$

And hence by Cauchy-Schwartz:

$$|E_t[\delta_S(X_{it}^{(1)})|W_{it} \in W']| = |(b, f_S)_{W'}| \leq ||B[b]||_{W'} ||\psi_S||_{W'}$$

It follows that for all $W'$:

$$E_t[|E_t[\delta_S(X_{it}^{(1)})|W_{it}]||W_{it} \in W'] \leq ||B[b]||_{W'} ||\psi_S||_{W'}$$

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To see why, suppose the converse were true. Let $\bar{W}$ be the smallest measurable subset of the support of $W_{it}$ such that $Pr_t[E_t[\delta_S(X_{it}^{(1)})|W_{it}] \geq 0|W_{it} \in \bar{W}] = 1$. Let $W$ be the smallest measurable subset of the support of $W_{it}$ such that $Pr_t[E_t[\delta_S(X_{it}^{(1)})|W_{it}] < 0|W_{it} \in W] = 1$. Then either:

$$E_t[|E_t[\delta_S(X_{it}^{(1)})|W_{it}]| |W_{it} \in \bar{W}] > ||B[b]||_W||\psi_S||_W$$

And/or:

$$E_t[|E_t[\delta_S(X_{it}^{(1)})|W_{it}]| |W_{it} \in W] > ||B[b]||_W||\psi_S||_W$$

But note that:

$$E_t[|E_t[\delta_S(X_{it}^{(1)})|W_{it}]| |W_{it} \in \bar{W}] = |E_t[\delta_S(X_{it}^{(1)})|W_{it} \in \bar{W}]|$$

And similarly:

$$E_t[|E_t[\delta_S(X_{it}^{(1)})|W_{it}]| |W_{it} \in W] = |E_t[\delta_S(X_{it}^{(1)})|W_{it} \in W]|$$

And so either way there is a contradiction.

So it has been established that:

$$E_t[|E_t[\delta_S(X_{it}^{(1)})|W_{it}]| |W_{it} \in W'] \leq ||B[b]||_W'||\psi_S||_W'$$

Note that since by Assumption 2.c $X_{it}^{(1)}$ is a subvector or $W_{it}$, $E_t[\delta_S(X_{it}^{(1)})|W_{it}] = \delta_S(X_{it}^{(1)})$. Substituting this fact into the above inequality and setting $W'$ equal to the support of $W_{it}$ one gets:

$$E_t[\delta_S(X_{it}^{(1)})] \leq ||B[b]||_S$$

Where have used that by Assumption 2.c that $||\psi_S|| \leq c_S$. Writing the LHS above in terms of the linear operators defined at the beginning of the appendix:

$$||D_S[b]||_{L_1} \leq ||B[b]||_S$$

$\square$

**Proof of Theorem 1**

**With 2.a**

By assumption 2.c there are no components of $\tilde{X}_{it}$ (all regressors are used as instruments) and so we can suppress arguments that depend on this random variable.

Recall the linear operator $B$ is defined by

$$B[\delta] = E_t[\delta(W_{it}, \eta_i)|Z_{it}]$$
The adjoint of $B$ in the inner-product $(\cdot, \cdot)$, denoted by $B^*$ is defined by:

$$B^*[\delta] = E_t[\delta(Z_{it})|W_it, \eta_i] = E_t[\delta(W_{it}, V_{it})|W_it, \eta_i]$$

Where the second equality holds by Assumption 2.a. In terms of the operators $B^*[\delta] = C[\delta]$.

We now prove that assumption 2.a implies Assumption 1.5. In terms of the operators defined above Assumption 1.5 states that:

$$B \circ B^*[\delta] = 0 \iff B^*[\delta] = 0$$

Suppose $B \circ B^*[\delta] = 0$, then:

$$(B \circ B^*[\delta], \delta) = (B^*[\delta], B^*[\delta]) = ||B^*[\delta]|| = 0$$

And the converse is trivial.

Let $\{v_k, u_k, \mu_k\}_{k=1}^\infty$ be the unique ‘pointwise singular system’ for $B^*$. See above in the appendix for details regarding the pointwise singular system.

One can easily verify from the properties of the pointwise singular system that the system for the composition of $B$ and $B^*$, $B \circ B^*$ can be written in terms of the components of the singular system of $B^*$ as $\{v_k, v_k, \mu_k^2\}_{k=1}^\infty$.

Note that by definition of the singular system:

$$B^*[v_k] = \mu_k(W_{it})u_k(W_{it}, \cdot)$$

[19] shows that $g_t$ is in the closure of the range of $C \circ B^* = B \circ B^*$. Hence one can expand $g_t$ in the pointwise singular system of $B \circ B^*$ as:

$$g_t(Z_{it}) = \sum_{k=1}^\infty E_t[g_t(Z_{it}), v_k(Z_{it})|W_it]v_k(Z_{it})$$

$$= \sum_{k=1}^\infty \mu_k(W_{it})^{-2}E_t[g_t(Z_{it}), v_k(Z_{it})|W_it]B \circ B^*[v_k](Z_{it})$$

Now consider $\gamma_K$ defined by the finite sum:

$$\gamma^{(K)}(X_{it}, V_{it}) = \sum_{\mu_k \geq \mu_K} \mu_k(W_{it})^{-2}E_t[g_t(Z_{it}), v_k(Z_{it})|W_it]v_k(Z_{it})$$

$\gamma_K$ is essentially the spectral cut-off regularized inverse of $B \circ B^*$ applied to $g_t$. Note that by [16] the operator $B \circ B^*$ is equivalent to the operator $A$. And so the pointwise singular systems of $B \circ B^*$ and $A$ are the same. $A$ only depends on the distribution of observables and so $\{\gamma_K\}_{K=1}^\infty$ is a function of the distribution of observables.

Note that:

$$g_t(Z_{it}) - B \circ B^*[\gamma_K](Z_{it}) = \sum_{\mu_K < \mu_k} E_t[g_t(Z_{it}), v_k(Z_{it})|W_it]v_k(Z_{it})$$

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states that \( g_t = B[h_t] \). Using this and the properties of the singular system for \( B^* \) one gets:

\[
B[h_t - B^*[\gamma_K]|(Z_{it}) = \sum_{\mu_K < \mu_k} E_t[h_t(X_{it}, \eta_i), u_k(X_{it}, \eta_i)|W_{it}]B[u_k](Z_{it})
\]

Note that

\[
h_t(X_{it}, \eta_i) = \sum_{k=1}^{\infty} E_t[h_t(X_{it}, \eta_i), u_k(X_{it}, \eta_i)|W_{it}]u_k(X_{it}, \eta_i) + \Pi[h_t](X_{it}, \eta_i)
\]

where \( \Pi \) is the orthogonal projection operator onto the null space of \( B \) and so:

\[
\sum_{k=1}^{\infty} E_t[h_t(X_{it}, \eta_i), u_k(X_{it}, \eta_i)|W_{it}]^2 \leq ||h_t||^2
\]

Which implies that the sum \( \sum_{k=1}^{\infty} E_t[h_t(X_{it}, \eta_i), u_k(X_{it}, \eta_i)|W_{it}]u_k(X_{it}, \eta_i) \) is well defined and finite. Defined \( r_k \) by:

\[
r_k(X_{it}, \eta_i) = \sum_{k=1}^{\infty} E_t[h_t(X_{it}, \eta_i), u_k(X_{it}, \eta_i)|W_{it}]u_k(X_{it}, \eta_i)
\]

Then:

\[
B[h_t - B^*[\gamma_K]] = B[\sum_{\mu_K < \mu_k} r_k]
\] (23)

Recall that by Assumption 1.4:

\[
E_t[\delta(W_{it}, V_{it})|Z_{it}] = 0 \implies E_t[\delta(w^{(1)}, W_{it}^{(2)}, V_{it})|S_{it} \in \mathbb{S}] = 0
\]

For \( F_{W_{it}} \)-almost all \( w^{(1)} \).

Or in terms of the previously defined linear operators:

\[
B[\delta] = 0 \implies D_B[\delta] = 0
\]

And so:

\[
D_B[h_t - B^*[\gamma_K]] = D_B[\sum_{\mu_K < \mu_k} r_k]
\]

Using [17]

\[
D_B[h_t] - G_B[\gamma_K] = D_B[\sum_{\mu_k < \mu_K} r_k]
\]

And so, since \( D_B \) is a bounded operator:

\[
D_B[h_t] = \lim_{K \to \infty} G_B[\gamma_K]
\]

Note that \( G_B \) is a function of the distribution of observables and likewise for \( \gamma_K \) and so \( D_B[h_t] \) is a function of the distribution of observables and so by proposition 6 the conditional average structural function is identified.
With 2.b

It is shown in the discussion of the pointwise singular system above that under Assumptions 1.1-1.5 and Assumption 2.b $g_t$ is in the range of $A$ (as opposed to just the closure of the range). That is, there exists a $\gamma \in L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$ such that:

$$A[\gamma] = g_t$$  \hspace{1cm} (24)

Let $\gamma$ satisfy 24. Note that the set of solutions is a function only of the distribution of observables and is thus identified.

Recall 18, this implies that $\gamma$ satisfies 24 if and only if:

$$A[\gamma] = B[h_i]$$  \hspace{1cm} (25)

From 16:

$$A[\gamma] = B \circ C[\gamma]$$

And so:

$$B[C[\gamma] - h_i] = 0$$

Assumption 1.4 states that:

$$B[C[\gamma] - h_i] \implies D_S[C[\gamma] - h_i] = 0$$

Applying 17 then gives:

$$G_S[\gamma] = D_S \circ B[\gamma] = D_S[h_i]$$

Note that $G_S$ depends only on the distribution of observables, and likewise for $\gamma$, so there is a mapping from the distribution of observables to $D_S[h_i]$ and so by Proposition 6 the conditional average structural function is identified.

With 2.c

By 20 there must be a sequence $\{\gamma_k\}_{k=1}^\infty$ with $\gamma_k \in L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$ for each $k$, such that:

$$\lim_{K \to \infty} ||A[\gamma_k] - g_t|| = 0$$  \hspace{1cm} (26)

Note that 26 depends only on the distribution of observables. Hence there exists a function of the distribution of observables that maps to a sequence $\{\gamma_k\}_{K=1}^\infty$ that satisfies 26.

Let $\{\gamma_k\}_{K=1}^\infty$ satisfy 26. Applying 18 and 16 yields:

$$\lim_{K \to \infty} ||B[C[\gamma_k] - h_i]|| = 0$$  \hspace{1cm} (27)

By Lemma 1:

$$||D_S[C[\gamma_k] - h_i]||_{L_1} \leq c_S ||B[C[\gamma_k] - h_i]||$$

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Using $\{L\}$ the above is equivalent to:
\[
\|G_\delta[\gamma_k] - D_\delta[h_t]\|_{L_1} \leq c_\delta\|B[C[\gamma_k] - h_t]\|
\]
And so by $\{27\}$
\[
D_\delta[h_t] = \lim_{K \to \infty} G_\delta[\gamma_k]
\]
Note that $G_\delta$ is a function of the distribution of observables and likewise for $\{\gamma(K)\}_{K=1}^\infty$ and so $D_\delta[h_t]$ is a function of the distribution of observables and so by Proposition 6 the conditional average structural function is identified.
□

**Proof of Lemma 2:**

The conditional average structural function in the new model $\hat{CSF}_t$ satisfies for each $x^{(1)} \in X^{(1)}$:
\[
\hat{CSF}_t(x^{(1)}, S_{it}) = E_t[\hat{h}_t(x^{(1)}, X^{(2)}_{it}, \xi_{it}) | S_{it}]
\]
Substituting in the definition of $\hat{h}_t$:
\[
\hat{CSF}_t(x^{(1)}, S_{it}) = E_t[E_t[h_t(X_{it}, \eta_i) | X^{(1)}_{it} = x^{(1)}, X^{(2)}_{it}, \xi_{it}] | S_{it}] = \mathbb{E}(x^{(1)}, X^{(2)}_{it}, \xi_{it}) | S_{it}
\]
(28)

Note that $(Z_{it}, X_{it}, S_{it}) \perp \eta_i | \xi_{it}, X^{(2)}_{it}$ implies $(X^{(1)}_{it}, S_{it}) \perp \eta_i | \xi_{it}, X^{(2)}_{it}$ and so:
\[
E_t[h_t(X_{it}, \eta_i)] | X^{(1)}_{it} = x^{(1)}, X^{(2)}_{it}, \xi_{it}] = E_t[h_t(x^{(1)}, X^{(2)}_{it}, \eta_i) | X^{(2)}_{it}, \xi_{it}, S_{it}]
\]
Substituting the above into (28) gives:
\[
\hat{CSF}_t(x^{(1)}_{it}, S_{it}) = E_t[E_t[h_t(x^{(1)}, X^{(2)}_{it}, \eta_i) | X^{(2)}_{it}, \xi_{it}, S_{it}] | S_{it}]
\]
\[
= E_t[h_t(x^{(1)}, X^{(2)}_{it}, \eta_i) | S_{it}]
\]
\[
= CSF_t(x^{(1)}_{it}, S_{it})
\]
Where the second equality follows by iterated expectations and the final equality follows by definition of $CSF_t$.

Next note that:
\[
\hat{\epsilon}_{it} = h_t(X_{it}, \eta_i) - \hat{h}_t(X_{it}, \xi_{it}) + \epsilon_{it}
\]
And so:
\[
E_t[\hat{\epsilon}_{it} | Z_{it}] = E_t[h_t(X_{it}, \eta_i) | Z_{it}] - E_t[\hat{h}_t(X_{it}, \xi_{it}) | Z_{it}] + E_t[\epsilon_{it} | Z_{it}]
\]
By assumption $E_t[\epsilon_{it} | Z_{it}] = 0$, Furthermore, note that $(Z_{it}, X_{it}, S_{it}) \perp \eta_i | \xi_{it}, X^{(2)}_{it}$ implies $Z_{it} \perp \eta_i | \xi_{it}, X_{it}$ and so $E_t[h_t(X_{it}, \eta_i) | X_{it}, \xi_{it}] = E_t[h_t(X_{it}, \eta_i) | X_{it}, \xi_{it}, Z_{it}]$.

So applying the law of iterated expectations:
\[
E_t[\hat{h}_t(X_{it}, \xi_{it}) | Z_{it}] = E_t[h_t(X_{it}, \eta_i) | Z_{it}]
\]
And so:
\[
E_t[\hat{\epsilon}_{it} | Z_{it}] = 0
\]
□
Proof of Remark 1

By De Finetti’s theorem a sequence of random variables is exchangeable if and only if the random variables are iid conditional on some latent variable. Denoting this latent variable by \( \xi_i \) then we then have that for any two distinct sub-sequences of \( \{1, \ldots, T\} \) given by \( \{\pi_1', \pi_2', \ldots, \pi_l'\} \) and \( \{\pi_1'', \pi_2'', \ldots, \pi_k''\} \):

\[ X_{i\pi_1'}, \ldots, X_{i\pi_l'} \perp X_{i\pi_1''}, \ldots, X_{i\pi_k''} | \xi_i \]  \hfill (29)

Note that because the regressors are iid conditional on \( \xi_i \) and \((Z_{it}, S_{it}, X_{it})\)

is composed of leads and lags of the regressors:

\[ (Z_{it}, S_{it}, X_{it}) \perp \eta_i | \xi_i \]

And so by Lemma 1 strict exogeneity also holds in the related model in which \( \eta_i \) is replaced with \( \xi_i \). So assumption 1.3 holds for the related model. Furthermore Lemma 1 states that the conditional average structural functions implied by the original and related models are identical.

Furthermore, because the regressors are iid conditional on \( \xi_i \) and \((Z_{it}, S_{it}, X_{it})\) and \( V_{it} \) are composed of non-overlapping subsets of leads and lags of the regressors:

\[ (Z_{it}, S_{it}, X_{it}) \perp V_{it} | \xi_i, X_{it}^{(2)} \]

And so Assumption 1.2 holds for the related model.

Note that exchangability also clearly implies that assumption 1.6.iii. holds for the related model with \( Z_{it} \) and \( V_{it} \) defined as above.

In the proof of Theorem 1 we show that Assumption 2.a implies Assumption 1.5.

Condition 4. is clearly equivalent to Assumption 1.4.

Condition 5. simply states that Assumption 1.1 holds for this model.

Hence assumptions 1.1-1.5 and Assumption 2.a holds for the related model in which \( \eta_i \) is replaced by \( \xi_i \) and so by Theorem 1 the conditional average structural function is identified in the related model. Since by Lemma 1 the conditional average structural functions of the original and related models are identical, the conditional average structural function of the original model is identified.

\( \square \)

The following lemma is used in the proof of Lemma 4 but may be of independent interest:

Lemma 3:

Let \( A \) be a bounded, linear operator with injective adjoint (and hence dense range) from some Hilbert space \( X \) to Hilbert space \( Y \). Let \( A \) have the property that \(|(I + A^* A)^{-1} A^*| \leq \frac{1}{2\sqrt{\epsilon}} \) (note that \( A \) has to have this property if it is compact see e.g. Kress Theorem 15.23). Let \( f \) be an element of \( Y \) in the range of \( A \).
Suppose that there exists a sequence of scalars \( \{ \rho_k \}_{k=1}^{\infty} \) with \( \rho_k \to 0 \), functions \( \{ f_k \}_{k=1}^{\infty} \) and compact linear operators \( \{ A_k \}_{k=1}^{\infty} \), such that for all \( k \)
\[
||A_k - A|| \leq \rho_k \tag{30}
\]

Where \( || \cdot || \) denotes the operator norm.

And suppose that:
\[
||f_k - f|| \leq \rho_k \tag{31}
\]

Where \( || \cdot || \) denotes the norm of the Hilbert space \( Y \).

For each natural number \( k \), let \( \gamma_k \) be the (unique) solution to the Tikhonov regularized operator equation:
\[
(r_k I + A_k^* A_k) \gamma_k = A_k^* f_k \tag{32}
\]

Then there exists a finite scalar \( c \) such that for all \( k \):
\[
||f - A\gamma_k|| \leq c\sqrt{\rho_k}
\]

**Proof:**

Let \( R_\epsilon \) be the Tikhonov regularized inverse of \( A \) i.e. \( R_\epsilon = (\epsilon I + A^* A)^{-1} A^* \) and let \( R_{\epsilon, k} \) be the Tikhonov regularized inverse of \( A_k \) i.e. \( R_{\epsilon, k} = (\epsilon I + A_k^* A_k)^{-1} A_k^* \) (the existence of the inverse is given for example by Theorem 15.23 in Kress). Denote by \( \gamma_{\epsilon, k} \) the solution to \((\epsilon I + A_k^* A_k)\gamma_{\epsilon, k} = A_k^* f_k \). The triangle inequality tells us that:
\[
||f - A\gamma_{\epsilon, k}|| \tag{33}
\]
\[
= ||A(R_{\epsilon, k} - R_\epsilon)f_k + AR_\epsilon f_k - f||
\]
\[
\leq ||A(R_{\epsilon, k} - R_\epsilon)f_k|| + ||AR_\epsilon f_k - f|| \tag{34}
\]

Now note that \( f \) is in the range of \( A \) and so there exists some \( \gamma \) so that \( A\gamma = f \).

Let us first consider the second term in the above. Note that:
\[
||AR_\epsilon f_k - f|| \leq ||A|| ||R_\epsilon|| ||f_k - f|| + ||AR_\epsilon f - f|| \tag{35}
\]

Recall that:
\[
||R_\epsilon|| \leq \frac{1}{2\sqrt{\epsilon}}
\]

To deal with the second term in \(35 \) suppose that \( f \) is in the range of \( A \) so that there is some \( \gamma \) such that \( f = A\gamma \). Then:
\[ ||AR_\varepsilon f - f|| = (AR_\varepsilon f - f, AR_\varepsilon f - A\gamma) \]
\[ = (A^*AR_\varepsilon f - A^* f, R_\varepsilon f - \gamma) \]
\[ = (\varepsilon R_\varepsilon f, R_\varepsilon f - \gamma) \]
\[ = \varepsilon (R_\varepsilon f, \gamma) - \varepsilon ||R_\varepsilon f|| \]
\[ \leq \varepsilon ||R_\varepsilon f||(||\gamma|| + 1) \]
\[ \leq \frac{1}{2} (||\gamma|| + 1)||f||\sqrt{\varepsilon} \]

Where the first equality follows from \( f = A\gamma \), the third equality from the definition of \( R_\varepsilon \), the first inequality from Cauchy-Schwartz and the final inequality from \( ||R_\varepsilon f|| \leq ||R_\varepsilon|| ||f|| \).

Hence:
\[ ||AR_\varepsilon f_k - f|| \leq \frac{1}{2}\sqrt{\varepsilon} \rho_k ||A|| + \frac{1}{2} (||\gamma|| + 1)||f||\sqrt{\varepsilon} \]

Now we consider the first term in [34]. Using similar reasoning to Kress Theorem 10.1 note that:
\[ (\varepsilon I + A_k^*A_k)(R_{\varepsilon,k} - R_\varepsilon) f_k = (A_k^*-A^*)(f_k+(f-f)+A^*A-A_k^*A_k)(R_\varepsilon f+R_\varepsilon (f_k-f)) \]

Note also that
\[ ||A(\varepsilon I + A_k^*A_k)^{-1}|| \leq ||A_k(\varepsilon I + A_k^*A_k)^{-1}|| + ||A - A_k|| ||(\varepsilon I + A_k^*A_k)^{-1}|| \]
\[ \leq \frac{1}{2}\sqrt{\varepsilon} + \frac{||A - A_k||}{2\varepsilon} \]

Where we have used that \( ||A_k(\varepsilon I + A_k^*A_k)^{-1}|| \leq \frac{1}{2}\sqrt{\varepsilon} \) and \( ||(\varepsilon I + A_k^*A_k)^{-1}|| \leq \frac{1}{2\varepsilon} \) which follows from compactness of \( A_k \).

We thus get:
\[ ||A(R_{\varepsilon,k} - R_\varepsilon) f_k|| \leq ||A(\varepsilon I + A_k^*A_k)^{-1}|| \]
\[ \times \left[ \rho_k(||f|| + \rho_k) + ||A_k^*A_k - A^*A|| \left(||R_\varepsilon f|| + ||R_\varepsilon|| \rho_k\right) \right] \]

Note that:
\[ ||R_\varepsilon f|| \leq ||\gamma|| \]

Also note that:
\[ ||A_k^*A_k - A^*A|| \leq 2\rho_k ||A|| + \rho_k^2 \]
And so we have:

\[
|A(R_{\epsilon,k} - R_{\epsilon}) f_k| \\
\leq (||f|| + \rho_k) \left( \frac{1}{2\sqrt{\epsilon}} + \frac{\rho_k}{2\epsilon} \right) \\
\times \left[ \rho_k + \left( 2\rho_k||A|| + \rho_k^2 \right) \left( ||\gamma|| + \frac{1}{2\sqrt{\epsilon}} \rho_k \right) \right]
\]

Putting this all together:

\[
||f - A\gamma_k|| \\
\leq (||f|| + \rho_k) \left( \frac{1}{2\sqrt{\epsilon}} + \frac{\rho_k}{2\epsilon} \right) \\
\times \left[ \rho_k + \left( 2\rho_k||A|| + \rho_k^2 \right) \left( ||\gamma|| + \frac{1}{2\sqrt{\epsilon}} \rho_k \right) \right] \\
\times \left[ \rho_k + \left( 2\rho_k||A|| + \rho_k^2 \right) \left( ||\gamma|| + \frac{1}{2\sqrt{\epsilon}} \rho_k \right) \right]
\]

\[
||f - A\gamma_k|| = O(\max\{\rho_k, \sqrt{\epsilon}\})
\]

So setting \(\epsilon = \rho_k\)

\[
||f - A\gamma_k|| = O(\sqrt{\rho_k})
\]

And so there exists a constant \(c\) such that:

\[
||f - A\gamma_k|| \leq c\sqrt{\rho_k}
\]

Note that in particular, if \(\epsilon = \rho_k\), \(\rho_k \leq 1\) and \(||A|| \leq 1\) then:

\[
||f - A\gamma_k|| \leq 6(||\gamma|| + 1)(||f|| + 1)\sqrt{\rho_k}
\]

\[
\]

**Proof of Lemma 4**

For each natural number \(K\) let the function \(f^{(K)}\) be defined by:

\[
f^{(K)}(x,v,z) = \sum_{k=1}^{K} \sum_{l=1}^{K} \beta^{(K)}_{kl} \gamma_k(x,v)\psi_l(z)
\]

And let \(g^{(K)}\) be defined by:

\[
g^{(K)}(z) = \sum_{k=1}^{K} O^{(K)}_k \psi_k(z)
\]

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Consider the linear operator equation:

$$
\epsilon r(v, x) + \int_Z f^{(K)}(v, x, z) E_t[f^{(K)}(V_{it}, X_{it}, z)r(V_{it}, X_{it})] F_{Z_{it}}(dz) = E_t[f^{(K)}(v, x, Z_{it})g^{(K)}(Z_{it})]
$$

(36)

Where $F_{Z_{it}}$ is the marginal probability measure of $Z_{it}$.

A function $r^*$ that satisfies the above is:

$$
r^*(x, v) = \sum_{k=1}^{K} \mu_{k}^{(K)} \gamma_k(x, v)
$$

Where the vector of coefficients $\mu^{(K)}$ solves the following linear algebra problem:

$$
\epsilon \mu^{(K)} + (\beta^{(K)}\Gamma^{(K)}) \mu^{(K)} = (\beta^{(K)})\Psi^{(K)}\alpha^{(K)}
$$

Where $\Psi^{(k)}$ is a symmetric $K \times K$ matrix whose $k, l$-th entry is:

$$
E_t[\psi_k(Z_{it})\psi_l(Z_{it})]
$$

And $\Gamma^{(k)}$ is a symmetric $K \times K$ matrix whose $k, l$-th entry is:

$$
E_t[\gamma_k(X_{it}, V_{it})\gamma_l(X_{it}, V_{it})]
$$

The equation (36) is the Tikhonov-regularized approximate operator equation of the form in (32). In particular, let the operator $A_k$ be defined by:

$$
A_k[r](z) = E_t[f^{(K)}(V_{it}, X_{it}, z)r(X_{it}, V_{it})]
$$

And let $A$ be defined by:

$$
A[r](z) = E_t[r(X_{it}, V_{it})|Z_{it} = z]
$$

Note that $A_k$ is a compact linear operator between spaces $L_2(\mathcal{X} \times \mathcal{V}, F_{X_{it}, V_{it}})$ and $L_2(\mathcal{Z}, F_{Z_{it}})$. However, $A$ needn’t be a compact operator between these spaces. Recall that the instruments $Z_{it}$ can be partitioned as $Z_{it} = (W_{it}; \tilde{Z}_{it})$ where $W_{it}$ are those instruments also contained in the vector of regressors $X_{it}$. For each $w \in \mathcal{W}$ the domain of $W_{it}$, under Assumption 1.1 the operator $A_w$ defined below is compact in the norm $\| \cdot \|_w$ defined as in the proof of Theorem 1.

$$
A_w[r](z) = E_t[r(X_{it}, V_{it})|W_{it} = w, \tilde{Z}_{it} = z]
$$

And so by e.g. Kress Theorem 15.23

$$
\|(\epsilon I + A_w^* A_w)^{-1} A_w^*\|_w \leq \frac{1}{2\sqrt{\epsilon}}
$$

Where $\| \cdot \|_w$ is the operator norm relative to the $L_2$ norm $\| \cdot \|_w$. Since the above holds uniformly over $\mathcal{W}$ and since $E_t[\|(\epsilon I + A_w^* A_w)^{-1} A_w^*\|_{W_{it}}] = \|(\epsilon I + A^* A)^{-1} A^*\|$ then:

$$
\|(\epsilon I + A^* A)^{-1} A^*\| \leq \frac{1}{2\sqrt{\epsilon}}
$$

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From the conditions in statement of the lemma:
\[ ||A - A_k|| \leq \rho_k \]

Where \( ||\cdot|| \) is the operator norm.

From the statement of the lemma:
\[ ||g^{(k)} - g_t|| \leq \rho_k \]

Assumption 1.1, Assumption 1.5 and Assumption 1.6 i. together imply \( g_t \) is in the range of \( A \) (see the proof of Theorem 1 for details). So from Lemma 3, since \( r^* \) solves \[ 36 \] which in terms of the operators can be stated as \( \epsilon r^* + A_k^* A_k r^* = A_k^* g^{(k)} \) then there exists a scalar \( c \) such that:
\[ ||g_t - Ar^*|| = E_t\left( g_t(Z_{it}) - \sum_{k=1}^{K} \mu_k^{(K)} E_t[\gamma_k(X_{it},V_{it})|Z_{it}] \right) \leq c\sqrt{\rho K} \]

To prove the final statement in the Lemma note that under Assumption 1.3 \( g_t(Z_{it}) = E_t[h_t(X_{it},\eta_i)|Z_{it}] \) and under Assumption 1.2 \( Z_{it} \perp V_{it} \eta_i, X_{it} \) and so \( E_t[\gamma_k(X_{it},V_{it})|Z_{it}] = E_t[E_t[\gamma_k(X_{it},V_{it})|X_{it},\eta_i]|Z_{it}] \) hence:
\[ E_t\left( g_t(Z_{it}) - E_t\left( \sum_{k=1}^{K} \mu_k^{(K)} \gamma_k(X_{it},V_{it})|Z_{it} \right) \right)^2 = E_t\left[ E\left[ h_t(X_{it},\eta_i) - \sum_{k=1}^{K} \mu_k^{(K)} \epsilon \gamma_k(X_{it},V_{it})|X_{it},\eta_i) \right]^2 \right] \]
\[ = ||h_t - \sum_{k=1}^{K} \mu_k^{(K)} \phi_k||\_{weak} \]

\[ \square \]

Proof of Theorem 2:
Recall that under our Assumptions 1.1 and 2.c
\[ ||C\hat{S}F_t(\cdot,\mathcal{S}) - CSF_t(\cdot,\mathcal{S})||_{L_1} \leq ||C\hat{S}F_t(\cdot,\mathcal{S}) - C\hat{S}F_t(\cdot,\mathcal{S})||_{L_1} + c \rho ||\hat{h} - h_t||\_{weak} \]

Let us first consider the second term on the RHS \( ||\hat{h} - h_t||\_{weak} \).

First note that under our Assumption 1.3
\[ m(Z_{it},h) = E_t[Y_{it} - h(X_{it},\eta_i)|Z_{it}] = E_t[h_t(X_{it},\eta_i) - h(X_{it},\eta_i)|Z_{it}] \]

And so:
\[ E_t[m(Z_{it},h)^2] = ||h - h_t||^2\_{weak} \]

Note that because we are interested in convergence in \( ||\cdot||_\text{weak} \), this norm corresponds to the ‘strong’ norm in Chen and Pouzo (denoted in their paper by
‘||·||’ as well as their weak norm (denoted in their paper by ‘||·||’). Given that 
\[ E_n[m(Z_{it},h)]^2 = ||h - h_n||^2_{weak}\] it is clear that the functional \( E_n[m(Z_{it},h)]^2 \) is lower semi-continuous in the argument \( h \) in the topology induced by the metric 
\[ ||·||_{weak} \] on \( \mathcal{H} \). Furthermore, one can easily verify that \[ ||·||_{weak} \] is convex. Hence by Remark 3.2 in Chen and Pouzo, \( E_n[m(Z_{it},h)]^2 \) is also ‘weak sequentially lower continuous’.

Next note that if \( \bigcup_{k=1}^{\infty} \mathcal{H}_k \) is dense in \( \mathcal{H} \) and \( \hat{P}_n \) is convex, then if \( P \) is continuous it must also be convex. First note that if \( \hat{P}_n \) is convex then \( \hat{P}_n \) is convex on \( \mathcal{H}_k \) for any \( k \). Now suppose \( P \) is not convex on \( \mathcal{H} \), then there is some \( h' \) and \( h'' \) in \( \mathcal{H} \), an \( a \in (0,1) \) and a scalar \( \epsilon > 0 \) such that:

\[ aP(h') + (1-a)P(h'') - P(h''') \geq \epsilon \]

Where \( h''' = ah' + (1-a)h'' \), by convexity of \( \mathcal{H}, h''' \in \mathcal{H} \). By the continuity and density assumptions there must be a \( k \) and an \( \tilde{h}' \) and \( \tilde{h}'' \) in \( \mathcal{H}_k \) such that

\[ |P(h') - P(h')| < \frac{1}{8}\epsilon, \quad |P(h'') - P(h'')| < \frac{1}{8}\epsilon. \]

Let \( \tilde{h}''' = ah' + (1-a)h'' \). By \( \mathcal{H}_k \) and hence \( \mathcal{H}_k \) is convex and so \( \tilde{h}''' \in \mathcal{H}_k \).

\[ |P(h''') - P(h''')| < \frac{1}{8}\epsilon \]

Then by the triangle inequality:

\[ |\left(aP(\tilde{h}') + (1-a)P(\tilde{h}'') - P(\tilde{h}''')\right) - \left(aP(h') + (1-a)P(h'') - P(h''')\right)| \leq \frac{1}{4}\epsilon \]

By Assumption 3.1 above, for \( n \) sufficiently high, \( \text{Prob}\{\sup_{h \in \mathcal{H}_k} |\hat{P}_n(\hat{h}) - P(\hat{h})| \leq \frac{1}{8}\epsilon \} > 0 \) and so for \( \tilde{h}', \tilde{h}'' \in \mathcal{H}_k \):

\[ \text{Prob}\{|a|\hat{P}_n(\tilde{h}') - P(\tilde{h}')| + (1-a)|\hat{P}_n(\tilde{h}'') - P(\tilde{h}'')| + |\hat{P}_n(\tilde{h}'') - P(\tilde{h}'')| \leq \frac{1}{4}\epsilon\} > 0 \]

Hence there must be some realization of \( \hat{P}_n \) such that \( |a|\hat{P}_n(\tilde{h}') - P(\tilde{h}')| + (1-a)|\hat{P}_n(\tilde{h}'') - P(\tilde{h}'')| \leq \frac{1}{4}\epsilon. \) By the triangle inequality and since \( \hat{P}_n \) is convex and so \( |a|\hat{P}_n(\tilde{h}') + (1-a)|\hat{P}_n(\tilde{h}'') - P(\tilde{h}'')| \leq 0. \) Hence:

\[ aP(\hat{h}') + (1-a)P(\hat{h}'') - P(\hat{h}''') \leq \frac{1}{4}\epsilon \] (38)

Combining the inequalities (37) and (38) and the triangle inequality then give:

\[ aP(h') + (1-a)P(h'') - P(h''') \leq \frac{1}{2}\epsilon \]

Which yields a contradiction.

We now show which of the assumptions presented above imply Assumptions 3.1, 3.2, 3.3 (b) and (c) and 4.1 in Chen and Pouzo.
Assumption 3.1 i. This trivially satisfied (the relevant weighting matrix is the identity).

Assumption 3.1 ii. Our Assumptions 1.1 (which guarantees the conditional expectations below exist) and our Assumption 1.3 imply.
\[ E_t[Y_{it} - h_t(X_{it}, \eta_i)|Z_{it}] = 0 \]
The rest of this assumption is trivial since the strong and weak metrics are identical in this case.

Assumption 3.1 iii. It is clear from the constraints on \( B_K \) that for each \( k \) \( \mathcal{H}_k \) is closed and that \( \mathcal{H}_k \subset \mathcal{H}_{k+1} \). That \( \mathcal{H}_k \subset \mathcal{H} \) for each \( k \) follows from the assumption that for each \( k \) \( \gamma_k \) is in \( L_2(\mathcal{X} \times \mathcal{E}, F_{X_{it}, \eta_i}) \) and by Assumption 1.1 the operator that maps \( \gamma \) to the function \( (x, \eta) \mapsto E_t[\gamma(X_{it}, \eta_i)|X_{it} = x, \eta_i = \eta] \) is a bounded linear functional. That there is a sequence of operators \( \{\Pi_k\}_{k=1}^\infty \) that map from \( \mathcal{H} \) to \( \mathcal{H}_k \) such that \( ||h_t - \Pi_k[h_t]||_{weak} = o(1) \) then follows immediately from Lemma 3.

Assumption 3.1 iv. Note that \( E_t[m(Z_{it}, h)^2] = ||h - h_t||_{weak}^2 \) and the weak and strong norms are the same and so Assumption 3.1 iv. follows from 3.1 iii.

Assumption 3.2 (c) (and therefore the weaker assumption 3.2(b) are repeated in Assumption 3.2 above.

Assumption 3.3 is repeated in Assumption 3.1 above.

Assumption 3.4 If \( \mathcal{H} \) is closed, bounded and convex then this assumption clearly follows from Assumption 3.1 iii. above

Assumption 4.1 i. It was shown above if \( P \) is continuous (which is a condition in Assumption 3.1 i.) then convexity of \( \hat{P}_n \) implies convexity of \( P \). And so the space \( \mathcal{H}_{os} \) defined below is convex for any positive scalars \( \epsilon, C_1 \) and \( C_2 \).

\[ \mathcal{H}_{os} = \{ h \in \mathcal{H} : ||h - h_t||_{weak} \leq \epsilon, ||h||_{weak} \leq C_1, P(h) \leq C_2 \} \]

For any \( \epsilon \) and for \( C_1 \) and \( C_2 \) large enough that \( ||h_t||_{weak} \leq C_1 \) and \( P(h_t) \leq C_2 \) then if \( ||h - h_t||_{weak} = o_p(1) \), for sufficiently large \( n \):

\[ \text{Prob}[\hat{h} \notin \mathcal{H}_{os}] < \epsilon \]

Note also that convexity of \( B_{K(n)} \) implies convexity of \( \mathcal{H}_{K(n)} \) and so \( \mathcal{H}_{os} \cap \mathcal{H}_{K(n)} \) is also convex.

First note that in the NPIV case it is trivial to show pathwise differentiability of \( m \), and the pseudo-metric defined by Chen and Pouzo simply corresponds to the metric \( ||\cdot||_{weak} \), the rest of 4.1 i. then follows trivially.

Assumption 4.1 ii. follows trivially from the fact already noted that \( E_t[m(Z_{it}, h)^2] = ||h - h_t||_{weak}^2 \).

So suppose that the conditions for Lemma 3 hold for some sequence \( \{\rho_k\}_{k=1}^\infty \) and set of functions \( \{\gamma_k\}_{k=1}^\infty \) (recall that Lemma 3 also requires that our Assumptions 1.1, 1.5 and 1.6 i. hold). Suppose in addition that Assumption 3.2 holds. Then the PSMD estimator for \( h_t \) described above satisfies Assumptions.
3.1 and 3.3 in Chen and Pouzo where the relevant ‘strong’ norm (’\[\cdot\]\’ in Chen
and Pouzo) is ||\[\cdot\]||\text{weak} defined above. If in addition \(\lambda_n = 0\) then Assumption
3.2a in Chen and Pouzo holds. If \(\lambda_n > 0\) and our Assumption 3.1 (which is the
same as Chen and Pouzo assumption 3.2a) holds then Chen and Pouzo assumption
3.2c holds and hence so too does 3.2b. Furthermore, if \(||\hat{h} - h_t||\text{weak} = o_p(1)\)
then Chen and Pouzo’s Assumption 4.1 holds.

Below we show that Theorems 3.1 and 4.1 of Chen and Pouzo are satisfied
when the conditions for Lemma 3 are satisfied for some sequence \(\{\rho_k\}_{k=1}\), our
Assumption 3.2 is satisfied, \(k(n) \to \infty\), \(k(n)/n \to 0\), \(\lambda_n = 0\) and \(B_K\) is bounded.

First note that, as is shown above, the conditions for Lemma 3, our Assump-
tion 3.2 and \(\lambda_n = 0\) implies Chen and Pouzo’s Assumptions 3.1, 3.2a and 3.3
hold.

Again noting that \(E_t[m(Z_{it}, h)^2] = ||h - h_t||^2\text{weak}\) one gets:

\[
\inf_{h \in \mathcal{H}_K(n)} ||h - h_t||_{\text{weak}} = \inf_{h \in \mathcal{H}_K(n)} ||h - h_t||_{\text{weak}} = \inf_{h \in \mathcal{H}_K(n)} ||h - h_t||^2_{\text{weak}}
\]

For a particular \(\epsilon > 0\), for sufficiently high \(n\), it is clear that \(\Pi_n h \in \mathcal{H}_K(n)\)
and \(||\Pi_n h - h_t||^2_{\text{weak}} \leq \epsilon\), and so for any given \(\epsilon\) there is an \(\bar{n}(\epsilon)\) such that
for \(n \geq \bar{n}(\epsilon)\) \(\inf_{h \in \mathcal{H}_K(n)} ||h - h_t||_{\text{weak}} = \inf_{h \in \mathcal{H}_K(n)} ||h - h_t||^2_{\text{weak}} = \epsilon\). If \(B_K(n)\) is bounded
then \(\mathcal{H}_K(n)\) is bounded for each \(n\), additionally recall that \(E_t[m(Z_{it}, h)^2]\) is
lower semi-continuous. Hence the condition (12) in Chen and Pouzo is trivially
satisfied and so by their Theorem 3.1 one gets:

\[||\hat{h} - h_t||_{\text{weak}} = o_p(1)\]

This then implies that Chen and Pouzo’s Assumption 4.1 holds. Given that
\(\lambda_n = 0\) clearly \(\max\{\eta_n, \lambda_n\} = \eta_n\), and so the condition ii. of Chen and Pouzo’s
Theorem 4.1 is satisfied. Since the strong norm and the weak norm are identical
in our case (and so the sieve measure or ill-posedness is 1) the theorem gives:

\[||\hat{h} - h_t||_{\text{weak}} = O_p(\sqrt{\max\{\eta_n, \rho_{K(n)}\}})\]

Next let us show that Theorems 3.2 and 4.1 of Chen and Pouzo are satisfied
when the conditions for Lemma 3 are satisfied for some sequence \(\{\rho_k\}_{k=1}\), our
Assumptions 2.2, 2.1 i. and 2.1 ii. are satisfied, \(k(n) \to \infty\) and \(\lambda_n > 0\).

As is shown above, the conditions for Lemma 3, and our Assumption 3.2 im-
plies Chen and Pouzo’s Assumptions 3.1, 3.3 hold. Our Assumption 3.1 i. coupled
with the assumption that \(\lambda_n > 0\) is simply Chen and Pouzo’s assumption
3.2c which implies 3.2b. In addition 2.1 ii. states that \(P\) is lower semi-compact.
Now note that condition (13) of Chen and Pouzo is, in our case equivalent to
the condition \(\max\{\eta_n, \rho_{K(n)}\} = o(\lambda_n)\) which is true by assumption. So by Chen
and Pouzo’s Theorem 3.2:

\[||\hat{h} - h_t||_{\text{weak}} = o_p(1)\]
Then Chen and Pouzo’s Assumption 4.1 holds, and since condition ii. of Chen and Pouzo’s Theorem 4.1 is satisfied by assumption, the Theorem again implies:

$$||\hat{h} - h_t||_{weak} = O_p(\sqrt{\max\{\eta_n, \rho_{K(n)}\}})$$

Next let us show that Theorems 3.3 and 4.1 of Chen and Pouzo are satisfied when $H$ is closed and bounded and convex, the conditions for Lemma 3 are satisfied for some sequence $\{\rho_k\}_{k=1}^\infty$, our Assumptions 2.2, 2.1 i. and 2.1 iii. are satisfied, $k(n) \to \infty$ and $\lambda_n > 0$.

As is shown above, the conditions for Lemma 3, and our Assumption 3.2 implies Chen and Pouzo’s Assumptions 3.1, 3.3 hold. Our Assumption 3.1 i. coupled with the assumption that $\lambda_n > 0$ is simply Chen and Pouzo’s assumption 3.2c. In addition 2.1 iii. and $H$ closed, bounded and convex implies Chen and Pouzo’s Assumption 3.4. Now note that condition (14) of Chen and Pouzo is, in our case equivalent to the condition $\max\{\eta_n, \rho_{K(n)}\} = o(\lambda_n)$ which is true by assumption. Furthermore, as noted above $E_t[m(X_t, h)^2]$ is lower sequentially continuous in the topology induced by $||\cdot||_{weak}$ on $H$.

So by Chen and Pouzo’s Theorem 3.3:

$$||\hat{h} - h_t||_{weak} = o_p(1)$$

Then Chen and Pouzo’s Assumption 4.1 holds. Given Chen and Pouzo’s Assumption 3.4 holds, condition iii. of Chen and Pouzo’s Theorem 4.1 is, in this case that:

$$\max\{\lambda_n \sup_{h \in H_{own}} |\hat{P}_n(h) - P(h)|, \lambda_n||\hat{h} - \Pi_{K(n)} h_t||_{weak}\} = O_p(\eta_n)$$

Since this is true by assumption the Theorem again implies:

$$||\hat{h} - h_t||_{weak} = O_p(\sqrt{\max\{\eta_n, \rho_{K(n)}\}})$$

Finally we show that $||C\tilde{S}F_t(\cdot, S) - C\tilde{S}F_t(\cdot, S)||_{L_1} = O_p(v_n)$.

In terms of the notation in our Assumption 3.3 he difference between the feasible and infeasible estimates of the CSF is:

$$C\tilde{S}F_t(x^{(1)}, S) - C\tilde{S}F_t(x^{(1)}, S) = \tilde{\beta}^T \varepsilon_{S,n}(x^{(1)})$$

Note that:

$$\tilde{\beta}^T \varepsilon_{S,n}(x^{(1)}) = \beta^T \Gamma_{S,K(n)} \Gamma_{S,K(n)}^{-1} \varepsilon_{S,n}(x^{(1)}) + (\tilde{\beta} - \beta)^T \Gamma_{S,K(n)} \Gamma_{S,K(n)}^{-1} \varepsilon_{S,n}(x^{(1)})$$

Where $\Gamma_{S,K(n)}$ is defined as in Assumption 3.3.

By Cauchy-Schwartz

$$||\tilde{\beta}^T \Gamma_{S,K(n)} \Gamma_{S,K(n)}^{-1} \varepsilon_{S,n}||_{L_1} \leq ||\varepsilon_{S,n}\|_{\Gamma_{S,K(n)}^{-1}} \sqrt{||\tilde{\beta}^T \Gamma_{S,K(n)} \beta||}$$
And similarly:

\[ ||(\hat{\beta} - \beta)^T \Gamma_{S,K(n)}^{-1} \epsilon_{S,n}||_{L_1} \leq ||\epsilon_{S,n}||_{\Gamma_{S,K(n)}^{-1}} \sqrt{||((\hat{\beta} - \beta)^T \Gamma_{S,K(n)}(\hat{\beta} - \beta)||_{\Gamma_{S,K(n)}^{-1}}} \]

But note that:

\[ |\beta^T \Gamma_{S,K(n)}^T \beta| = ||\Pi_n h||_{weak} \]

And similarly:

\[ |(\hat{\beta} - \beta)^T \Gamma_{S,K(n)}(\hat{\beta} - \beta)| = ||\hat{h} - \Pi_{K(n)} h||_{weak}^2 \]

By the triangle inequality:

\[ ||\Pi_{K(n)} h|| \leq ||h - \Pi_{K(n)} h||_{weak} + ||h||_{weak} \leq \sqrt{p_{K(n)}} + ||h||_{weak} \]

And:

\[ ||\hat{h} - \Pi_{K(n)} h||_{weak} \leq ||\hat{h} - h||_{weak} + ||h - \Pi_{K(n)} h||_{weak} \leq c_3 ||\hat{h} - h||_{weak} + c_3 \sqrt{p_{K(n)}} \]

Combining and using the above gives:

\[ ||\hat{\beta}^T \epsilon_{S,n}||_{L_1} \leq c_3 (||\hat{h} - h||_{weak} + 2\sqrt{p_{K(n)}}||\epsilon_{S,n}||_{\Gamma_{S,K(n)}^{-1}} + ||h||_{weak}||\epsilon_{S,n}||_{\Gamma_{S,K(n)}^{-1}}) \]

Since ||\hat{h} - h||_{weak} = o_p(1) and \( \sqrt{p_{K(n)}} = o(1)\):

\[ ||\hat{\beta}^T \epsilon_{S,n}||_{L_1} = O_p(||\epsilon_{S,n}||_{\Gamma_{S,K(n)}^{-1}}) = O_p(\nu_n) \]

And hence:

\[ ||C\hat{S}F_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} = O_p(\nu_n) \]

Combining with ||\hat{h} - h||_{weak} = O_p(\sqrt{\max\{\eta_n, \rho_{K(n)}\}}) one gets:

\[ ||C\hat{S}F_t(\cdot, S) - CSF_t(\cdot, S)||_{L_1} = O_p(\sqrt{\max\{\eta_n, \rho_{K(n)}, \nu_n^2\}}) \]

□