ON THE SOLVABILITY OF SINGULAR BOUNDARY VALUE PROBLEMS ON THE REAL LINE IN THE CRITICAL GROWTH CASE

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Abstract. Combining fixed point techniques with the method of lower-upper solutions we prove the existence of at least one weak solution for the following boundary value problem

\[
\begin{cases}
(\Phi(a(t,x(t))) x'(t))' = f(t,x(t),x'(t)) & \text{in } \mathbb{R}, \\
x(-\infty) = \nu_1, & x(+\infty) = \nu_2
\end{cases}
\]

where \(\nu_1,\nu_2 \in \mathbb{R}\), \(\Phi : \mathbb{R} \to \mathbb{R}\) is a strictly increasing homeomorphism extending the classical \(p\)-Laplacian, \(a\) is a nonnegative continuous function on \(\mathbb{R} \times \mathbb{R}\) which can vanish on a set having zero Lebesgue measure and \(f\) is a Carathéodory function on \(\mathbb{R} \times \mathbb{R}^2\).

1. Introduction. This paper is concerned with the existence of at least one weak solution for the following boundary value problem

\[
\begin{cases}
(\Phi(a(t,x(t))) x'(t))' = f(t,x(t),x'(t)) & \text{in } \mathbb{R}, \\
x(-\infty) = \nu_1, & x(+\infty) = \nu_2
\end{cases}
\]

(BVP)

where \(\nu_1,\nu_2 \in \mathbb{R}\), the function \(a \in C(\mathbb{R} \times \mathbb{R}, [0, +\infty))\) and may vanish on a set with zero Lebesgue measure and \(f : \mathbb{R}^3 \to \mathbb{R}\) is a Carathéodory function, that is

(i) for any \((x,y) \in \mathbb{R}^2, t \mapsto f(t,x,y)\) is a measurable function on \(\mathbb{R}\),

(ii) \((x,y) \mapsto f(t,x,y)\) is a continuous function for a.e. \(t \in \mathbb{R}\).

Here \(\Phi : \mathbb{R} \to \mathbb{R}\) is the so-called \(\Phi\)-Laplacian operator already studied, e.g., in [11, 12, 13, 18] (see also the survey [10] and the references therein), which generalizes the classical \(r\)-Laplacian operator \(\Phi(y) := y|y|^{r-2}\), with \(r > 1\). More precisely, we assume that \(\Phi\) is a strictly increasing homeomorphism such that \(\Phi(0) = 0\) and

\[
\lim_{z \to 0^+} \frac{\Phi(z)}{z^p} > 0 \quad \text{for some positive } p. \quad (1)
\]

It is worth emphasizing that the first author in [5] established the existence of at least one weak solution to the following problem

\[
\begin{cases}
(\Phi(a(t,x(t))) x'(t))' = f(t,x(t),x'(t)) & \text{in } I = [0, \infty), \\
x(0) = \nu_1, & x(+\infty) = \nu_2
\end{cases}
\]

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under the same assumptions on \( \Phi, a \) and \( f \). One of the main differences between our main result (see Theorem 2.3) and Theorem 1 in [5] concerns the relative behavior of \( f(t, x, \cdot) \) and \( \Phi(\cdot) \) as \( y \to 0 \), and of \( f(\cdot, x, y) \) as \( t \to \infty \). To be more precise, one of the key assumptions in [5, Theorem 1] is the existence of a constant \( \theta > 1 \) with the following property: for every \( L > 0 \) one can find \( K_L \in W^{1,1}_{loc}(I) \), such that

\[
\int_c^\infty \frac{1}{a_*(t)} K_L(t)^{-\frac{\rho}{\rho - 1}} \, dt < \infty \quad (\text{for a suitable } c > 0) \tag{2}
\]

where \( \rho \) is as in (1) and \( a_* \) is a continuous function satisfying \( a_*(t) \geq a(t, x) \) for every \( (t, x) \in I \times \mathbb{R} \) and \( 1/a_* \in L^p_{loc}(I) \) for some \( p > 1 \) (actually, \( a_* \) is defined essentially as in (7)). As it is clear from some concrete examples (see Section 5 in [5]), assumption (2) excludes the possibility that \( f \) has the critical rate of decay \(-1\) as \( t \to \infty \), that is

\[
f(t, \cdot, \cdot) \approx \frac{1}{t} \quad \text{as } t \to \infty. \tag{3}
\]

The main aim of the present paper is to obtain an existence result which covers the case (3) and which takes care of the fact that problem (BVP) is formulated on the whole of \( \mathbb{R} \).

Roughly speaking, in order to deal with the critical case (3), one needs to reformulate assumption (2) for \( \theta = 1 \). As it will be clear from the proof of Proposition 2, the appropriate assumption in such a case is of exponential type, that is

\[
\int_{|t| \geq T_0} \frac{1}{a_*(t)} e^{-\frac{K_L(t)}{\rho}} < +\infty \quad (\text{for a suitable } T_0 > 0).
\]

Another interesting phenomenon, which is peculiar of the critical case (3), is that the solvability of (BVP) is also influenced by the relative behavior of \( f \) and \( \Phi \) with respect to \( x \).

One of the main reason to face with problem (BVP) comes from the amount of applications of the \( \Phi \)-Laplacian operators in e.g. non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces; see for instance [9, 19].

In view of this fact several papers have been devoted to \( \Phi \)-Laplacian type equations where

\[
\Phi : (-a, a) \to (-b, b)
\]

is a strictly increasing homeomorphism for some \( 0 < a, b \leq \infty \). When \( a = \infty \) and \( b < \infty \), the map \( \Phi \) is usually called non-surjective \( \Phi \)-Laplacian, and the main prototype is the mean curvature operator

\[
\Phi(s) = \frac{s}{\sqrt{1 + s^2}} \quad \text{for } s \in \mathbb{R};
\]

we refer the interested reader to [1, 3] and references therein. When \( a < \infty \), the \( \Phi \)-Laplacian is said to be singular, and in this case the main prototype is the relativistic operator

\[
\Phi(s) = \frac{s}{\sqrt{1 - s^2}} \quad \text{for } s \in (-1, 1);
\]

see [2, 4, 14, 20]. In all the aforementioned papers the ODEs considered are non-singular in the following sense: they take the form (for suitable \( \Phi \) and \( f \))

\[
\left( \Phi(a(t, x(t)) x'(t)) \right)' = f(t, x(t), x'(t)) \quad \text{a.e. on } I \subseteq \mathbb{R} \tag{4}
\]

where the function \( a \) is assumed to be continuous and strictly positive. This is one of the main differences with our setting: indeed we allow the function \( a \) to vanish
on a subset of $\mathbb{R} \times \mathbb{R}$ having zero Lebesgue measure. If this is the case, we shall say that the ODE (4) is singular.

Concerning the existence of heteroclinic solutions for (BVP) (which is an interesting problem of relevance in physics/dynamic of populations) we recall the papers [16, 17] in which the authors deal with the following BVP (see also [8] for the case $a \equiv 1$, $\nu_1 = 0$ and $\nu_2 = 1$)
\[
\begin{cases}
\smash{a(x(t)) (\Phi(x'(t)))' = f(t, x(t), x'(t)) \quad \text{a.e. on } \mathbb{R},} \\
\smash{x(-\infty) = \nu_1, \quad x(\infty) = \nu_2,}
\end{cases}
\]
and the more recent papers [21, 22] where the author considered the following BVP
\[
\begin{cases}
\smash{a(t, x(t)) (\Phi(x'(t)))' = f(t, x(t), x'(t)) \quad \text{a.e. on } \mathbb{R},} \\
\smash{x(-\infty) = \nu_1, \quad x(\infty) = \nu_2.}
\end{cases}
\]
We stress the fact that also in these papers the authors assumed $a > 0$ (that is the ODE is non-singular in the previous sense).

The literature concerning BVP for singular ODEs of the form (4) (both on compact and non-compact intervals) seems to be rather incomplete. To the best of our knowledge, we mention the paper [15] in which the authors obtained the solvability of (4) assuming that $a(t, x) = k(t)$ and $I = [0, T]$ for some $T > 0$; more recently, this paper has been generalized in [7] to the case where $a$ depends on $(t, x)$ (and again $I = [0, T]$).

As regards the case $I = (0, \infty)$ or $I = \mathbb{R}$, we highlight the paper [6] in which the authors establish the existence of at least one weak solution of
\[
\begin{cases}
\smash{(\Phi(k(t)) x'(t)))' = f(t, x(t), x'(t)) \quad \text{in } [0, \infty),} \\
\smash{x(0) = \nu_1, \quad x(+\infty) = \nu_2,}
\end{cases}
\]
where the function $k$ can vanish on a set with zero Lebesgue measure. As already said, this existence result has been generalized in [5] to BVPs of the form
\[
\begin{cases}
\smash{(\Phi(a(t, x(t)) x'(t)))' = f(t, x(t), x'(t)) \quad \text{in } [0, \infty),} \\
\smash{x(0) = \nu_1, \quad x(+\infty) = \nu_2,}
\end{cases}
\]
where $a(t, x)$ is allowed to vanish on a set with zero Lebesgue measure and $f(t, \cdot, \cdot) \equiv |t|^{\gamma}$ as $t \to \infty$, for suitable $\gamma > -1$.

The main aim of this paper is to give a further contribute in this direction: in fact we prove the solvability of (BVP) assuming that $f$ has the critical rate of decay $-1$ as $|t| \to \infty$, that is
\[
f(t, \cdot, \cdot) \approx \frac{1}{|t|} \quad \text{as } |t| \to \infty.
\]
Together with this assumption, we require that $f$ satisfies a suitable form of the so-called Nagumo-Wintner growth condition (see, precisely, assumption $(H_2)$); we stress that such condition (in some stronger forms) has been profitably exploited in [16, 17, 21, 22]. In our context, the Nagumo-Wintner growth condition allows us to obtain a priori estimates of the first derivative of any solution of (BVP) on any compact interval of $\mathbb{R}$. These estimates play a fundamental role in the proof of Theorem 2.3: in fact, first we use a fixed point technique and the method of the lower/upper solutions (see assumption $(H_1)$) to prove the existence of a solution $u_n$ to
\[
(\Phi(a(t, x(t)) x'(t)))' = f(t, x(t), x'(t)) \quad \text{a.e. } t \in I_n = [-n, n]
\]
with $n \in \mathbb{N}$ sufficiently large; then by means of the a priori estimates provided by the Nagumo-Wintner growth condition we are able to prove that the sequence $u_n$
converges (in a suitable sense) to a solution of (BVP). We point out that, since we are assuming (5) we need to require a balance between the behavior of $f(t, x, \cdot)$ with respect to $\Phi$ as $y \to 0$ (see assumption (H3)).

Despite assumptions (H$_1$)-(H$_3$) seem rather technical, in Section 5 we shall prove that these assumptions are fulfilled by any BVP of the following form

$$\begin{cases}
\left( \Phi(a(t, x(t)) x'(t)) \right)' = f_1(t, x(t)) f_2(x'(t)) & \text{a.e. on } \mathbb{R}, \\
x(-\infty) = \nu_1, \ x(\infty) = \nu_2,
\end{cases}$$

where $f_1$ and $f_2$ satisfies suitable growth conditions and either

1. $a(t, x) = k_1(t) k_2(x)$ with $k_1 \geq 0$, and $k_2 > 0$ on $(\nu_1, \nu_2)$, or
2. $a(t, x) = k_1(t) + k_2(x)$ with $k_1, k_2 \geq 0$.

As a concrete example, Theorem 2.3 applies to the following BVPs:

$$\begin{cases}
\Phi \left( e^{-x(t)^2} \min \{ \sqrt{|t|}, 1/t^2 \} x'(t) \right)' = -\frac{m t}{t^2 + 1} |x'(t)|^{\theta} & \text{a.e. on } \mathbb{R}, \\
x(-\infty) = 0, \ x(\infty) = 1,
\end{cases}$$

where $m \in (1, \infty)$ is sufficiently large, $\theta \in (0, 1)$ is arbitrarily chosen and

$$\Phi(z) = z + \sin(z).$$

Finally, we explicitly note that Theorem 2.3 comprehends also the case when $a(t, x) = a(t)$, that is $a$ is independent of $x$. As a consequence, the result in this paper complete the study started in [6, 7]. To the best of our knowledge it remains open the problem when

$$f(t, \cdot, \cdot) \approx |t|^{\gamma} \quad \text{as } t \to \infty, \text{ for suitable } \gamma < -1.$$

2. Assumptions and main theorem. We begin by giving a couple of preliminary definitions which we will use along the paper.

**Definition 2.1.** We say that $x \in C(\mathbb{R}, \mathbb{R})$ is a solution to the problem (BVP) if

1. $x \in W^{1,1}_{loc}(\mathbb{R})$ and $t \mapsto \Phi(a(t, x(t)) x'(t)) \in W^{1,1}_{loc}(\mathbb{R}),$
2. $(\Phi(a(t, x(t)) x'(t)))' = f(t, x(t), x'(t))$ for a.e. $t \in \mathbb{R},$
3. $\lim_{t \to -\infty} x(t) = \nu_1$ and $\lim_{t \to +\infty} x(t) = \nu_2.$

**Definition 2.2.** We say that $\alpha \in C(\mathbb{R}, \mathbb{R})$ is a lower (upper) solution of

$$(\Phi(a(t, x(t)) x'(t)))' = f(t, x(t), x'(t))$$

if it satisfies

1. $\alpha \in W^{1,1}_{loc}(\mathbb{R})$ and $t \mapsto \Phi(a(t, \alpha(t)), \alpha'(t)) \in W^{1,1}_{loc}(\mathbb{R}),$
2. $(\Phi(a(t, \alpha(t)), \alpha'(t)))' \geq (\leq) f(t, \alpha(t), \alpha'(t))$ for a.e. $t \in \mathbb{R}.$

We point out that if $x \in W^{1,p}(\mathbb{R})$ is such that $t \mapsto \Phi(a(t, x(t)) x'(t)) \in W^{1,1}(\mathbb{R}),$ then being $\Phi$ a homeomorphism there exists a unique $A_x \in C(\mathbb{R}, \mathbb{R})$ such that

$$A_x(t) = a(t, x(t)) x'(t) \quad \text{for a.e. } t \in \mathbb{R}.$$  

At this point we can state our main result.

**Theorem 2.3.** Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing homeomorphism such that $\Phi(0) = 0$ and satisfying (1); moreover, let $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function. We assume that
(H1) there exists a pair of lower and upper solutions \( \alpha, \beta \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) to (ODE) such that

(i) \( \alpha(t) \leq \beta(t) \) for every \( t \in \mathbb{R} \),

(ii) \( \alpha \) is increasing in \((-\infty, -T_0)\) and \( \beta \) is decreasing in \((T_0, +\infty)\), for some \( T_0 > 0 \),

(iii) \( \lim_{t \to -\infty} \alpha(t) < +\infty \) and \( \lim_{t \to +\infty} \beta(t) < +\infty \);

(iv) introducing the function

\[
a_*(t) := \min_{x \in [\alpha(t), \beta(t)]} a(t, x) > 0,
\]

we have \( 1/a_* \in L^p_{\text{loc}}(\mathbb{R}) \) for some \( p > 1 \).

(H2) there exist a constant \( H > 0 \), a non-negative function \( \mu \in L^q([-T_0, T_0]) \) (for some \( q > 1 \)), a non-negative function \( \ell \in L^1([-T_0, T_0]) \) and a measurable function \( \psi : (0, +\infty) \to (0, +\infty) \) such that

(i) \( 1/\psi \in L^1_{\text{loc}}(0, +\infty) \) and \( \int_1^{+\infty} \frac{d\tau}{\psi(\tau)} = \infty \),

(ii) \( |f(t, x, y)| \leq \psi(\Phi(a(t, x(t))y)) \cdot \left( \ell(t) + \mu(t)|y|^{\frac{q-1}{q}} \right) \), for almost every \( t \in [-T_0, T_0] \), every \( x \in [\alpha(t), \beta(t)] \), and every \( y \in \mathbb{R} \) with \( |y| \geq H \);

(H3) for every \( L > 0 \) there exists a nonnegative function \( \eta_L \in L^1(\mathbb{R}) \) and a continuous function \( K_L \in W^{1,1}_{\text{loc}}([0, +\infty)) \) null on \([0, T_0]\) and strictly increasing on \([T_0, +\infty)\) such that

(i) \( \int_{|t| \geq 0} \frac{1}{a_*(t)} e^{-\frac{K_L(|t|)}{p}} < +\infty \),

(ii) setting, for every \( t \in \mathbb{R} \),

\[
N_L(t) = \Phi^{-1} \left( \Phi(L)e^{-\frac{K_L(|t|)}{p}} \right)
\]

then, for a.e. \( t \in (-\infty, -T_0] \cup [T_0, +\infty) \), every \( x \in [\alpha(t), \beta(t)] \) and every \( y \in \mathbb{R} \) verifying the bound \( |y| \leq N_L(t)/a_*(t, x) \), we have

\[
|f(t, x, y)| \geq K_L'(|t|) |\Phi(a(t, x) y)|;
\]

(iii) setting, for almost every \( t \in \mathbb{R} \),

\[
\gamma_L(t) := \frac{N_L(t)}{a_*(t)} \quad \text{and} \quad \dot{\gamma}_L(t) := \gamma_L(t) + |\alpha'(t)| + |\beta'(t)|,
\]

then, for almost every \( t \in \mathbb{R} \), every \( x \in [\alpha(t), \beta(t)] \) and every \( y \in \mathbb{R} \) verifying the bound \( |y| \leq \dot{\gamma}_L(t) \) we have

\[-\eta_L(t) \leq f(t, x, y) \leq \eta_L(t);\]

(iv) for a.e. \( t \in (-\infty, -T_0] \cup [T_0, +\infty) \), every \( x \in [\alpha(t), \beta(t)] \) and every \( y \in \mathbb{R} \) verifying the bound \( |y| \leq \dot{\gamma}_L(t) \), we have

\[t \cdot f(t, x, y) \leq 0.\]

Then there exists at least one weak solution \( x \in W^{1,p}_{\text{loc}}(\mathbb{R}) \) to the problem

\[
\begin{cases}
\left( \Phi(a(t, x(t)) x'(t)) \right)' = f(t, x(t), x'(t)) & \text{in } \mathbb{R} \\
\alpha(t) \leq x(t) \leq \beta(t) & \text{in } \mathbb{R} \\
x(-\infty) = \lim_{t \to -\infty} \alpha(t), \quad x(+\infty) = \lim_{t \to +\infty} \beta(t). \end{cases}
\]
Remark 1. Let us point out that $\mathcal{N}_L(t)$ is continuous, and using the fact that $1/a_* \in L_{loc}^p(\mathbb{R})$ we have that $\gamma_L \in L_{loc}^p(\mathbb{R})$. Moreover, since $K_L(t)$ is strictly increasing in $[T_0, +\infty)$ and $\Phi$ is a strictly increasing homeomorphism, we deduce that $\mathcal{N}_L(t)$ is strictly decreasing for any $t \in (-\infty, -T_0]$ and for any $t \in [T_0, +\infty)$. In particular, gathering the definition of $\mathcal{N}_L(t)$ and the monotonicity of $\Phi$, we can infer that $\mathcal{N}_L(t) > L$ for any $t \leq -T_0$, $\mathcal{N}_L(t) = L$ if $t \in [-T_0, T_0]$ and $\mathcal{N}_L(t) < L$ if $t \geq T_0$. Especially, by (1) it follows that

$$\limsup_{\xi \to 0^+} \frac{\Phi^{-1}(\xi)}{\xi^\frac{1}{p}} < +\infty,$$

and combining the above considerations together with $(H_3)$-(i) and the fact that $1/a_* \in L_{loc}^p(\mathbb{R})$ we obtain $\mathcal{N}_L/a_* = \gamma_L \in L^1(\mathbb{R})$. Finally, since $\alpha, \beta \in W_{loc}^{1,1}(\mathbb{R})$, we also have that $\hat{\gamma}_L = \gamma_L + |\alpha'| + |\beta'| \in L_{loc}^1(\mathbb{R})$.

Remark 2. Since, by definition of $a_*$, we have

$$\frac{\mathcal{N}_L(t)}{a(t, x)} \leq \gamma_L(t) \leq \hat{\gamma}_L(t), \quad \text{for a.e. } t \in \mathbb{R} \text{ and every } x \in [\alpha(t), \beta(t)],$$

on account of $(H_3)$-(iv) we can re-write $(H_3)$-(ii) as follows:

\begin{itemize}
  \item[(*)] $f(t, x, y) \leq -K_L(t) |\Phi(a(t, x) y)|,$
  \end{itemize}

for a.e. $t \geq T_0$, any $x \in [\alpha(t), \beta(t)]$ and any $y \in \mathbb{R}$ s.t. $|y| \leq \mathcal{N}_L(t)/a(t, x);$ 

\begin{itemize}
  \item[(*)] $f(t, x, y) \geq K_L(t) |\Phi(a(t, x) y)|,$
  \end{itemize}

for a.e. $t \leq -T_0$, any $x \in [\alpha(t), \beta(t)]$ and any $y \in \mathbb{R}$ s.t. $|y| \leq \mathcal{N}_L(t)/a(t, x)$.

3. Solvability on compact sets. In this section we establish the existence of at least one weak solution to the following auxiliary boundary value problem on the compact interval $I_n = [-n, n]$

$$\begin{cases}
(\Phi(a(t, x(t)) x'(t)))' = f(t, x(t), x'(t)) & \text{a.e. } t \in I_n \\
x(-n) = \alpha(-n), \quad x(n) = \beta(n),
\end{cases}$$

(11)

where $n \in \mathbb{N}$ is such that $n > T_0$. Let $J = [-T_0, T_0]$, and define

$$M := \max_{t \in J} \beta(t) - \min_{t \in J} \alpha(t) \geq 0$$

(12)

and

$$a_0 := \max\{a(t, x) : t \in J \text{ and } \alpha(t) \leq x \leq \beta(t)\}.$$ 

(13)

Fix $N \in \mathbb{R}_+$ such that

$$N > \max \left\{ H, \frac{M}{2T_0} \right\} a_0,$$

(14)

and let us observe that since $\Phi$ is a strictly increasing homeomorphism on $\mathbb{R}$ satisfying $\Phi(0) = 0$, it follows that $\Phi(N) = \Phi(-N) < 0$. Take $L \geq N$ such that

$$\min \left\{ \int_{\Phi(N)}^{\Phi(L)} \frac{d\tau}{\psi(\tau)}, \quad \int_{\Phi(-N)}^{-\Phi(-L)} \frac{d\tau}{\psi(\tau)} \right\} > \|\ell\|_{L^1(J)} + \|\mu\|_{L^1(J)} M^{\frac{p-1}{p}}.$$ 

(15)

Let us introduce the truncating operators $T : W_{loc}^{1,p}(I_n) \to W_{loc}^{1,1}(I_n)$ defined as

$$T(x)(t) := \begin{cases}
\alpha(t) & \text{if } x(t) < \alpha(t), \\
x(t) & \text{if } \alpha(t) \leq x(t) \leq \beta(t), \\
\beta(t) & \text{if } x(t) > \beta(t),
\end{cases}$$

(16)
Moreover, we introduce the truncating function $f^* : I_n \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f^*(t, x, y) := \begin{cases} f(t, \alpha(t), \alpha'(t)) + \arctan(x - \alpha(t)) & \text{if } x < \alpha(t), \\ f(t, x, y) & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t, \beta(t), \beta'(t)) + \arctan(x - \beta(t)) & \text{if } x > \beta(t). \end{cases}$$

We then consider the following truncated problem

$$\begin{cases} (\Phi(A(x)(t)x'(t)))' = f^*(t, x(t), D(T(x')(t))) & \text{a.e. } t \in I_n \\ x(-n) = \alpha(-n), \quad x(n) = \beta(n). \end{cases} \tag{16}$$

The forthcoming result guarantees the existence of a solution to (16) and it is based on the following abstract result [7, Theorem 2.1].

**Theorem 3.1.** Let $A : W^{1, p}(I_n) \subseteq C(I_n, \mathbb{R}) \to C(I_n, \mathbb{R})$ and $F : W^{1, p}(I_n) \to L^1(I_n)$ be general operators satisfying the following properties:

(H1) $A$ is continuous with respect to the uniform topology of $C(I_n, \mathbb{R})$; moreover, there exist two functions $h_1, h_2 \in C(I_n, \mathbb{R})$ such that

(H1)$_1$ $h_1, h_2 \geq 0$ on $I$ and $\frac{1}{h_1}, \frac{1}{h_2} \in L^p(I_n)$;

(H1)$_2$ $h_1(t) \leq A(x)(t) \leq h_2(t)$ for every $x \in W^{1, p}(I_n)$ and every $t \in I_n$;

(H2) $F$ is continuous (with respect to the usual norms) and there exists a nonnegative function $\Theta \in L^1(I_n)$ such that

$$|F(x)(t)| \leq \Theta(t) \quad \text{for every } x \in W^{1, p}(I_n) \text{ and a.e. } t \in I_n.$$

Then, for every $\nu_1, \nu_2 \in \mathbb{R}$ there exists a solution $x \in W^{1, p}(I_n)$ of the problem

$$\begin{cases} (\Phi(A(x)(t)x'(t)))' = F(x)(t) & \text{a.e. on } I_n, \\ x(0) = \nu_1, x(n) = \nu_2. \end{cases}$$

More precisely, there exists a function $x \in W^{1, p}(I_n)$ such that

(a) $\Phi \circ (A(x) \cdot x') \in W^{1, 1}(I_n)$ and $(\Phi \circ (A(x) \cdot x'))' = F$ in $L^1(I_n)$;

(b) $x(0) = \nu_1$ and $x(n) = \nu_2$.

**Theorem 3.2.** Assume that the assumptions (H1)-(H3) hold true. Then, there exists at least one weak solution $u_n \in W^{1, p}_{loc}(I_n)$ to (16).

**Proof.** Let us define the operators $A : W^{1, p}(I_n) \to C(I_n, \mathbb{R})$ and $F : W^{1, p}(I_n) \to L^1(I_n)$ as follows

$$A(x)(t) := a(t, T(x)(t)) \quad \text{and} \quad F(x)(t) := f^*(t, x(t), D(T(x')(t))).$$

Then, setting $\nu_1 := \alpha(-n)$ and $\nu_2 := \beta(n)$, problem (16) becomes

$$\begin{cases} (\Phi(A(x)(t)x'(t)))' = F(x)(t) & \text{a.e. } t \in I_n, \\ x(-n) = \nu_1, x(n) = \nu_2. \end{cases} \tag{17}$$

Put $\alpha_n := \min_{t \in I_n} \alpha(t)$ and $\beta_n := \max_{t \in I_n} \beta(t)$. Exploiting the definition of $T(x)$ and using assumption (H1) we get

$$\alpha_n \leq \alpha(t) \leq T(x)(t) \leq \beta(t) \leq \beta_n \quad \text{for any } t \in I_n. \tag{18}$$

Now, since $T$ is continuous as an operator on $C(I_n, \mathbb{R})$, the uniform continuity of $a$ on $I_n \times [\alpha_n, \beta_n]$ implies that $A$ is continuous with respect to the uniform topology of $C(I_n, \mathbb{R}).$
Moreover, if \( x \in W^{1,p}(I_n) \), then \( T(x)(t) \in [\alpha(t), \beta(t)] \) for any \( t \in I_n \), and from (7) and (18) we can infer that for any \( t \in I_n \) it holds
\[
0 < a_+(t) = \min_{\zeta \in [\alpha(t), \beta(t)]} a(t, \zeta) \leq A(x)(t) \leq \max_{\zeta \in [\alpha_n, \beta_n]} a(t, \zeta) =: a_-(t).
\]

Arguing as in Theorem 3.1 in [15] (and by exploiting the fact that \( D(\alpha') = \alpha' \) and \( D(\beta') = \beta' \)), we can prove that \( F \) is continuous; furthermore,
\[
|F(x)(t)| \leq \Theta(t) := \eta_L(t) + \frac{\pi}{2} \text{ for every } x \in W^{1,p}(I_n) \text{ and for a.e. } t \in I_n.
\]

Since by assumption \((H_3)-(iii)\) we have \( \Theta \in L^1(I_n) \), we are in the position to apply Theorem 3.1 to deduce the existence of a solution to (16).

**Lemma 3.3.** Under the assumptions \((H_1)-(H_3)\), let \( u_n \in W^{1,p}(I_n) \) be any solution to (16). Then
\[
\alpha(t) \leq u_n(t) \leq \beta(t) \quad \text{for every } t \in I_n.
\]

**Proof.** First we show that \( \alpha(t) \leq u_n(t) \) for every \( t \in I_n \). Assume by contradiction that there exists \( \ell \in I_n \) such that \( \alpha(\ell) > u_n(\ell) \).

Let \( z(\ell) := u_n(\ell) - \alpha(\ell) \). Since \( u_n \) solves (16), by assumption \((H_1)\) we can see that \( z(-n) = 0, z(n) \geq 0 \) and \( z(\ell) < 0 \). Thus, by the continuity of \( z \), there exists \( \vartheta \in I_n \) such that
\[
z(\vartheta) = \min_{t \in I_n} z(t) < 0.
\]

Therefore, we can find \( t_1 \in [-n, \vartheta) \) and \( t_2 \in (\vartheta, n] \) such that \( z(t_1) = z(t_2) = 0 \) and
\[
z(t) < 0 \quad \text{for any } t \in (t_1, t_2).
\]

From the definition of \( T(x) \) we deduce that
\[
T(u_n)(t) = \alpha(t) \quad \text{for any } t \in (t_1, t_2).
\]

Now, recalling that \( u_n \) is a weak solution to (16), \( \alpha \) is a lower solution to (ODE), by (20) and the definition of \( f^* \), we can infer that
\[
(\Phi(a(t, T(u_n)(t)) u_n'(t)))' = f^*(t, u_n(t), D(T(u_n))^\prime(t))
= f(t, \alpha(t), \alpha'(t)) + \arctan z(t)
< f(t, \alpha(t), \alpha'(t)) \leq (\Phi(a(t, \alpha(t)) \alpha'(t)))' \quad \text{for any } t \in (t_1, t_2).
\]

Now we set
\[
Z_1 = \{ t \in (t_1, \vartheta) : z'(t) < 0 \} \quad \text{and} \quad Z_2 = \{ t \in (\vartheta, t_2) : z'(t) > 0 \}.
\]

Firstly, we point out that \( Z_1 \) and \( Z_2 \) have positive Lebesgue measure, furthermore we can find \( t_1^* \in Z_1 \) and \( t_2^* \in Z_2 \) such that \( a(t_i^*, x(t_i^*)) > 0 \) for \( i = 1, 2 \). In particular, from (21) it follows that
\[
T(u_n)(t_i^*) = \alpha(t_i^*) \quad \text{for } i = 1, 2.
\]

Thus, integrating (22) on \([t_i^*, \vartheta)\) we obtain
\[
\Phi(a(\vartheta, T(u_n)(\vartheta)) u_n'(\vartheta)) - \Phi(a(t_i^*, T(u_n)(t_i^*)) u_n'(t_i^*))
\leq \Phi(a(\vartheta, \alpha(t_i^*)) \alpha'(\vartheta)) - \Phi(a(t_i^*, \alpha(t_i^*)) \alpha'(t_i^*)),
\]
which is equivalent to
\[
\Phi(a(\vartheta, T(u_n)(\vartheta))) u'_{n}(\vartheta)) - \Phi(a(\vartheta, \alpha(\vartheta)) \alpha'(\vartheta)) \\
\leq \Phi(a(t_1^*, T(u_n)(t_1^*)) u'_{n}(t_1^*)) - \Phi(a(t_1^*, \alpha(t_1^*)) \alpha'(t_1^*)).
\]
Combining this last inequality with (23), the fact that \(\Phi\) is strictly increasing, \(a(t_1^*, u_n(t_1^*)) > 0\) and \(z'(t_1^*) < 0\), we get
\[
\Phi(a(\vartheta, T(u_n)(\vartheta))) u'_{n}(\vartheta)) - \Phi(a(\vartheta, \alpha(\vartheta)) \alpha'(\vartheta)) < 0. \tag{24}
\]
On the other hand, integrating (22) on \([\vartheta, t_2^*]\) we get
\[
\Phi(a(\vartheta, T(u_n)(\vartheta))) u'_{n}(\vartheta)) - \Phi(a(\vartheta, \alpha(\vartheta)) \alpha'(\vartheta)) \\
\geq \Phi(a(t_2^*, T(u_n)(t_2^*)) u'_{n}(t_2^*)) - \Phi(a(t_2^*, \alpha(t_2^*)) \alpha'(t_2^*)),
\]
and using (23), \(z'(t_2^*) > 0\) and the strict monotonicity of \(\Phi\) we have
\[
\Phi(a(\vartheta, T(u_n)(\vartheta))) u'_{n}(\vartheta)) - \Phi(a(\vartheta, \alpha(\vartheta)) \alpha'(\vartheta)) > 0,
\]
which contradicts (24). Thus we conclude that \(\alpha(t) \leq u_n(t)\) for every \(t \in I_n\).

Similarly we can prove that \(u_n(t) \leq \beta(t)\) for every \(t \in I_n\). \qed

In the statement of Proposition 1 below, we use the definition of \(A_n\) in (6); we also recall that \(J = [-T_0, T_0]\) (where \(T_0\) is defined in Theorem 2.3, see assumption \((H_1)-(ii))\), \(I_n = [-n, n]\) (with \(n \in \mathbb{N}\) satisfying \(n \geq T_0\)) and \(L \geq N\) is a constant such that (15) holds.

**Proposition 1.** Under the assumptions \((H_1)-(H_3)\), let \(u_n \in W^{1,p}(I_n)\) be any solution to (16). Then \(|A_{u_n}(t)| < L\) for every \(t \in J\).

**Proof.** Assume by contradiction that there exists \(\tilde{t} \in J\) such that \(|A_{u_n}(\tilde{t})| \geq L\); then either \(A_{u_n}(\tilde{t}) \geq L\) or \(A_{u_n}(\tilde{t}) \leq -L\). Assume that
\[A_{u_n}(\tilde{t}) \geq L.\]

Let us note that \(\min_{t \in J} |A_{u_n}(t)| \leq N \leq L\). Indeed, assuming by contradiction \(A_{u_n}(t) > N\) for a.e. \(t \in J\), then integrating over \([-T_0, T_0]\) we obtain
\[
2NT_0 = \int_{-T_0}^{T_0} N dt < \int_{-T_0}^{T_0} A_{u_n}(t) dt = \int_{-T_0}^{T_0} a(t, u_n(t))u'_n(t) dt
\]
(by (19) and (13))
\[
\leq a_0 \int_{-T_0}^{T_0} |u'_n(t)| dt
\]
(since \(u'_n(t) = A_{u_n}(t)/a(t, u_n(t)) > N/a(t, u_n(t)) > 0\) for a.e. \(t \in J\))
\[
= a_0 \int_{-T_0}^{T_0} u'_n(t) dt = a_0 (u_n(T_0) - u_n(-T_0))
\]
(again by (19))
\[
\leq a_0 (\beta(T_0) - \alpha(-T_0))
\]
(using (12) and (14))
\[
\leq a_0 M < 2NT_0,
\]
which is a contradiction. Similarly, if we assume by contradiction that $A_{u_n}(t) < -N$ for a.e. $t \in J$, then using (7), (19), (13) and (14) we get a contradiction.

Therefore we can find $-T_0 \leq t_1 < t_2 \leq T_0$ such that

$$A_{u_n}(t_1) = N, \quad A_{u_n}(t_2) = L$$

and

$$N < A_{u_n}(t) < L \quad \text{for any } t \in (t_1, t_2).$$

(25)

From (25), by gathering together the definition of $A_{u_n}$ in (6), the definition of $a_0$ in (13) (see also (19)) and the definition of $a_\ast$ in (7), we infer that

$$\frac{N}{a_0} < u'_n(t) < \frac{L}{a_\ast(t)} \quad \text{for a.e. } t \in (t_1, t_2);$$

on the other hand, by (14) and the fact that $N_L \equiv L$ on $[-T_0, T_0]$ (see Remark 1), we obtain

$$0 < H < u'_n(t) < \gamma_L(t) \leq \gamma_L(t), \quad \text{for a.e. } t \in (t_1, t_2).$$

In particular, it follows that $D(u'_n) \equiv \gamma'_L$ (and $u'_n > 0$) almost everywhere on $(t_1, t_2)$. Then, being $T(u_n) \equiv u_n$, using assumption $(H_2)$ and reminding that $u_n$ solves (16), we get

\[
\left| (\Phi(a(t, u_n(t)) u'_n(t)))' \right| = \left| (\Phi(a(t, u_n(t)) u'_n(t)))' \right| \\
= |f^*(t, u_n(t), D(T(u_n))'(t))| = |f(t, u_n(t), u'_n(t))| \\
\leq \psi(|\Phi(a(t, u_n(t)) u'_n(t))|) \cdot \left(\ell(t) + \mu(t)|u'_n(t)|^{\frac{\gamma - 1}{\gamma}}\right).
\]

On the other hand, since $\Phi(a(t, u_n(t)) u'_n(t)) = \Phi(0) = 0$ (as $\Phi$ is increasing, $\Phi(0) = 0$ and the function $a$ is non-negative) and since $u'_n > 0$ a.e. on $(t_1, t_2)$, we get

\[
\left| (\Phi(a(t, u_n(t)) u'_n(t)))' \right| \leq \psi(\Phi(a(t, u_n(t)) u'_n(t))) \cdot \left(\ell(t) + \mu(t)|u'_n(t)|^{\frac{\gamma - 1}{\gamma}}\right).
\]

(26)

Now, using a change of variable, (26), assumption $(H_2)$, Hölder inequality and (12) we obtain

\[
\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(t)} \, d\tau = \int_{\Phi(A_{u_n}(t_2))}^{\Phi(A_{u_n}(t_1))} \frac{1}{\psi(t)} \, d\tau \\
= \int_{t_1}^{t_2} \frac{(\Phi(a(t, u_n(t)) u'_n(t)))'}{\psi(\Phi(a(t, u_n(t)) u'_n(t)))} \, d\tau \\
\leq \int_{t_1}^{t_2} \left(\ell(t) + \mu(t)|u'_n(t)|^{\frac{\gamma - 1}{\gamma}}\right) \, d\tau \\
\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^\gamma(J)} \left(\int_{t_1}^{t_2} |u'_n(t)|^{\frac{\gamma}{\gamma - 1}} \, d\tau\right)^{\frac{\gamma - 1}{\gamma}} \\
= \|\ell\|_{L^1(J)} + \|\mu\|_{L^\gamma(J)} (u_n(t_1) - u_n(t_2))^{\frac{\gamma - 1}{\gamma}} \\
\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^\gamma(J)} (\beta(t_1) - \alpha(t_2))^{\frac{\gamma - 1}{\gamma}} \\
\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^\gamma(J)} M^{\frac{\gamma - 1}{\gamma}},
\]

which is in contrast with (15). Similarly, one can prove that also the case $A_{u_n}(\tilde{t}) \leq -L$ leads to a contradiction. \qed
Proposition 2. Assume that the hypotheses (H1)-(H3) hold true. If \( u_n \in W^{1,p}(I_n) \) is any solution to (16), then

(i) \( A_{u_n} \) is increasing on \([-n,-T_0]\) and decreasing on \([T_0,n]\);

(ii) \( A_{u_n} \geq 0 \) in \([-n,-T_0] \cup [T_0,n]\);

(iii) if there exists \( \bar{t}_1 \in [-n,-T_0] \) such that \( A_{u_n}(\bar{t}_1) = 0 \) then \( A_{u_n}(t) = 0 \) for any \( t \in [-n,\bar{t}_1] \); similarly, if there exists \( t_2 \in [T_0,n] \) such that \( A_{u_n}(t_2) = 0 \), then \( A_{u_n}(t) = 0 \) for any \( t \in [\bar{t}_2,n] \);

(iv) \( |A_{u_n}(t)| \leq N_L(t) \) for every \( t \in I_n \);

(v) \( |u_n'(t)| \leq \gamma_L(t) \) for a.e. \( t \in I_n \).

Proof. Firstly, we point out that since \( u_n \) is a solution to (16), from Lemma 3.3 we have that \( u_n(t) \in [\alpha(t), \beta(t)] \) for any \( t \in I_n \). Then we deduce that

\[
(*) \quad \mathcal{T}(u_n)(t) = u_n(t);
\]

\[ (* ) \quad |\mathcal{D}(u_n')(t)| \leq \hat{\gamma}_L(t) = \gamma_L(t) + |\alpha'(t)| + |\beta'(t)|; \]

\[ (* ) \quad f^*(t,u_n(t),\mathcal{D}(u_n')(t)) = f(t,u_n(t),\mathcal{D}(u_n')(t)). \]

(i) In view of assumption (H3)-(iv), for every \( t \in [-n,-T_0] \) we get

\[ (\Phi(a(t,u_n(t)),u_n'(t)))' = f(t,u_n(t),\mathcal{D}(u_n')(t)) \geq 0, \]

that is \( (\Phi(A_{u_n}(t)))' \geq 0 \) for any \( t \in [-n,-T_0] \). Since \( \Phi \) is increasing, we deduce the thesis. Similarly, we can prove that \( (\Phi(A_{u_n}(t)))' \leq 0 \) every \( t \in [-n,-T_0] \). From the monotonicity of \( \Phi \) it follows the desired result.

(ii) Assume by contradiction that there exists \( t_1 \in [-n,-T_0] \) such that \( \mathcal{A}_{u_n}(t_1) < 0 \). From (i) for any \( t \in [-n,t_1] \) we have

\[ \mathcal{A}_{u_n}(t) \leq \mathcal{A}_{u_n}(t_1) < 0; \]

as a consequence, by taking into account the definition of \( \mathcal{A}_x \) in (6), we deduce that

\[ u_n'(t) = \frac{\mathcal{A}_{u_n}(t)}{a(t,u_n(t))} < 0 \quad \text{a.e. } t \in [-n,t_1]. \] (28)

Now, since \( u_n \) solves (16), using (28), (19) and assumption (H1)-(ii), we have

\[ \alpha(-n) = u_n(-n) = u_n(t_1) - \int_{-n}^{t_1} u_n'(\tau) d\tau > u_n(t_1) \geq \alpha(t_1) \geq \alpha(-n), \]

which gives a contradiction.

Similarly, arguing by contradiction we can find \( t_2 \in [T_0,n] \) such that \( \mathcal{A}_{u_n}(t_2) < 0 \). From (i) for any \( t \in [t_2,n] \) we have

\[ \mathcal{A}_{u_n}(t) \leq \mathcal{A}_{u_n}(t_2) < 0; \]

as a consequence, we obtain

\[ u_n'(t) = \frac{\mathcal{A}_{u_n}(t)}{a(t,u_n(t))} < 0 \quad \text{a.e. } t \in [t_2,n]. \] (29)
By proceeding exactly as above, from (29) we obtain
\[ \beta(n) = u_n(n) = u_n(t_2) + \int_{t_2}^{n} u_n'(\tau) \, d\tau < u_n(t_2) \leq \beta(t_2) \leq \beta(n), \]
which is a contradiction.

(iii) The proof of this statement easily follows by combining (i)-(ii).

(iv) Combining Proposition 1 and Remark 1 we have that \(|A_{u_n}(t)| < L = N_L(t)\) for any \(t \in J\). In the light of (ii) it remains to prove that
\[ A_{u_n}(t) \leq N_L(t) \quad \forall t \in I_n \setminus J. \] (30)

Let
\[ \hat{t} := \inf \{ t < -T_0 : A_{u_n}(\tau) < N_L(\tau) \quad \forall \tau \in [t, -T_0] \}. \]

Assume by contradiction that \(\hat{t} > -n\). Firstly, let us note that \(A_{u_n}(t) > 0\) for all \(t \in [\hat{t}, -T_0]\). Indeed, from the definition of \(\hat{t}\) it follows that \(A_{u_n}(t) < N_L(t)\) for any \(t \in [\hat{t}, -T_0]\). Moreover, by (iii) if there exists \(\tilde{t} \in [\hat{t}, -T_0]\) such that \(A_{u_n}(\tilde{t}) = 0\), then \(A_{u_n}(t) = 0\) for any \(t \in [-n, -\tilde{t}]\), and recalling that \(N_L(t) > 0\) we have that \(A_{u_n}(t) < N_L(t)\) for any \(t \in [-n, -T_0]\) in contrast with \(\hat{t} > -n\).

Now, since \(0 < A_{u_n}(t) \leq N_L(t)\) for any \(t \in [\hat{t}, -T_0]\), by (6) and (7) we get
\[ 0 < u_n'(t) \leq \frac{N_L(t)}{K(t, u_n(t))} \leq \frac{N_L(t)}{N_n(t)} = \gamma_L(t) \leq \gamma_L(t) \quad \text{a.e. } t \in [\hat{t}, -T_0]. \] (31)

Since \(u_n\) solves (16), exploiting (31), the definition of \(D\) and assumption \((H_3)-(ii)\) we get
\[ (\Phi(A_{u_n}(t)))' = f(t, u_n(t), u_n'(t)) \geq K_L(|t|)\Phi(A_{u_n}(t)) \quad \text{for } t \in [\hat{t}, -T_0] \]
where we have also used assumption \((H_3)-(iv)\). Recalling that \(A_{u_n}(t) > 0\) for any \(t \in [\hat{t}, -T_0]\), \(\Phi\) is strictly increasing and \(\Phi(0) = 0\) we can infer that
\[ \frac{(\Phi(A_{u_n}(t)))'}{\Phi(A_{u_n}(t))} \geq K_L(|t|) \quad \text{for } t \in [\hat{t}, -T_0]. \]

Integrating both sides of this last inequality on \([\hat{t}, -T_0]\), and taking into account that \(K_L(t) \equiv 0\) for any \(t \in [-T_0, 0]\) we get
\[ \log \Phi(A_{u_n}(-T_0)) - \log \Phi(A_{u_n}(\hat{t})) \geq K_L(|\hat{t}|). \] (32)

Now, let us note that by Proposition 1 we get \(A_{u_n}(-T_0) < L\), thus \(\Phi(A_{u_n}(-T_0)) < \Phi(L)\) (as \(\Phi\) is increasing); as a consequence, by combining this last inequality with (32), we have
\[ \Phi(A_{u_n}(\hat{t})) < \Phi(L)e^{-K_L(|\hat{t}|)} \quad \text{for every } t \in [\hat{t}, -T_0]. \]

From this, we obtain
\[ A_{u_n}(\hat{t}) < \Phi^{-1}\left(\Phi(L)e^{-K_L(|\hat{t}|)}\right) = N_L(\hat{t}), \]
and this contradicts the definition of \(\hat{t}\). Thus we conclude that \(\hat{t} = -n\). In a very similar way one can show that \(|A_{u_n}| \leq N_L\) on \([T_0, n]\), and the proof is complete. \(\square\)
4. Solvability on the real line. Now, using a limit argument, we give the proof of Theorem 2.3. In what follows, we inherit all the notation introduced so far; in particular, we use the definition of $A_{\pi}$ in (6), the definitions of $N_{L}$ and $\gamma_{L}$ in (8)-(9) (with $L \geq N$ as in (15)), and $I_{n} = [−n, n]$. 

Proof of Theorem 2.3. Let $u_{n} \in W^{1,p}(I_{n})$ be a solution to (16) given by Theorem 3.2. Taking into account the definitions of $T$, $D$ and $f^{*}$, and using Proposition 2-(v), we can infer

$$(\Phi(a(t, u_{n}(t)) u^{\prime}_{n}(t)))' = f^{*}(t, u_{n}(t), D(u^{\prime}_{n}(t))) = f(t, u_{n}(t), u^{\prime}_{n}(t)).$$

Thus $u_{n}$ is a solution to (11). Let us define $\{x_{n}\}_{n \in \mathbb{N}} \subset W^{1,p}(\mathbb{R})$ as follows

$$x_{n}(t) = \begin{cases} a(-n) & \text{if } t < -n, \\ u_{n}(t) & \text{if } t \in I_{n}, \\ \beta(n) & \text{if } t > n. \end{cases}$$

(33)

For every $n > T_{0}$, let

$$y_{n}(t) := x^{\prime}_{n}(t) = \begin{cases} u^{\prime}_{n}(t) & \text{if } t \in I_{n}, \\ 0 & \text{if } |t| > n, \end{cases}$$

and

$$z_{n}(t) := \begin{cases} (\Phi(A_{n}(t)))' & \text{if } t \in I_{n}, \\ 0 & \text{otherwise}. \end{cases}$$

From Remark 1 and Proposition 2-(v) we get

$$|y_{n}(t)| = |x^{\prime}_{n}(t)| \leq \gamma_{L} \in L^{1}(\mathbb{R}).$$

(34)

Taking into account that $u_{n}$ solves (11), verifies (19) and using assumption $(H_{3})$-(iii) we get

$$|z_{n}(t)| = |(\Phi(A_{n}(t)))'| = |f(t, u_{n}(t), u^{\prime}_{n}(t))| \leq \eta_{L}(t) \in L^{1}(\mathbb{R}).$$

Thus, by applying the Dunford-Pettis Theorem we deduce the existence of two functions $g, h$ in $L^{1}(\mathbb{R})$ such that, up to a sub-sequence, $y_{n} \to g$ and $z_{n} \to h$ in $L^{1}(\mathbb{R})$. Therefore, for every measurable subset $A \subset \mathbb{R}$ we have

$$\int_{A} y_{n}(t) \, d\tau \to \int_{A} g(\tau) \, d\tau \quad \text{and} \quad \int_{A} z_{n}(t) \, d\tau \to \int_{A} h(\tau) \, d\tau \quad \text{as } n \to \infty.$$

Now, from Lemma 3.3 and Proposition 1 we deduce that the sequences $\{u_{n}(0)\}_{n \in \mathbb{N}}$ and $\{A_{u_{n}}(0)\}_{n \in \mathbb{N}}$ are bounded, so we can assume that $u_{n}(0) \to u_{0}$ and $A_{u_{n}}(0) \to y_{0}$ for some $u_{0}, y_{0} \in \mathbb{R}$. Hence

$$x_{n}(t) = x_{n}(0) + \int_{0}^{t} x^{\prime}_{n}(\tau) \, d\tau \to u_{0} + \int_{0}^{t} g(\tau) \, d\tau, \quad \text{as } n \to +\infty.$$

Let us define

$$x(t) := u_{0} + \int_{0}^{t} g(\tau) \, d\tau.$$ 

(35)

Then we have that $x \in C(\mathbb{R})$, $x(0) = u_{0}$ and $x^{\prime}(t) = g(t)$ a.e. $t \in \mathbb{R}$. Moreover, from the definition of $x_{n}(t)$ and the fact that $\alpha, \beta \in C(\mathbb{R}, \mathbb{R})$, it follows that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in \mathbb{R}$. Now, if $t \in I_{n}$, then $x_{n}(t) = u_{n}(t)$ and $y_{n}(t) = u^{\prime}_{n}(t)$, Thus

$$\Phi(A_{u_{n}}(t)) = \Phi(A_{u_{n}}(0)) + \int_{0}^{t} z_{n}(\tau) \, d\tau.$$

(36)
Taking into account that \( a \in C([\mathbb{R} \times \mathbb{R}, (0, +\infty)) \), from (36) we get
\[
y_n(t) = x'_n(t) \to \frac{1}{a(t, x(t))} \Phi^{-1} \left( y_0 + \int_0^t h(\tau) d\tau \right) =: \frac{1}{a(t, x(t))} S(t) \quad \text{a.e.} t \in \mathbb{R}.
\] (37)

Recalling (34), we can apply the Dominated Convergence Theorem to prove that
\[
x'_n(t) \to \frac{1}{a(t, x(t))} S(t) \quad \text{in } L^1(\mathbb{R}).
\] (38)

Since \( x'_n = y_n \to g \) in \( L^1(\mathbb{R}) \), we can infer that
\[
g(t) = \frac{1}{a(t, x(t))} S(t) \quad \text{a.e. } t \in \mathbb{R}.
\] (39)

From (38), (39) and \( x'(t) = g(t) \) a.e. \( t \in \mathbb{R} \) we have
\[
x'_n \to g = x' \quad \text{in } L^1(\mathbb{R}).
\]

Let us observe that by (37) and (39) we have \( x'_n(t) \to g(t) = x'(t) \) a.e. \( t \in \mathbb{R} \) as \( n \to +\infty \). Moreover, taking into account that \( u_n \) is a solution to (11) on \( I_n \) and \( x'_n = u'_n \) a.e. \( t \in I_n \), it is possible to find a set \( F \subset \mathbb{R} \), independent on \( n \) with vanishing Lebesgue measure, such that, for every \( n > T_0 \) and every \( t \in I_n \setminus F \) we have
\[
z_n(t) = (\Phi(a(t, u_n(t))u'_n(t)))' = f(t, u_n(t), u'_n(t)) = f(t, x_n(t), x'_n(t)).
\]

Since \( x_n \to x \) and \( f \) is a Carathéodory function, we get
\[
\lim_{n \to +\infty} z_n(t) = f(t, x(t), x'(t)) \quad \text{for every } t \in \mathbb{R} \setminus F.
\]

On the other hand, applying the dominated convergence theorem we obtain
\[
z_n(t) \to f(t, x(t), x'(t)) \quad \text{in } L^1(\mathbb{R}).
\]

As a consequence, since \( z_n \to h \) in \( L^1(\mathbb{R}) \) we obtain
\[
(\Phi(a(t, x(t))x'(t)))' = h(t) = f(t, x(t), x'(t)) \quad \text{a.e. } t \in \mathbb{R},
\]

and this proved that \( x \) is a solution to the (ODE). Finally, we observe that
\[
\sup_{t \in \mathbb{R}} |x_n(t) - x(t)| \leq |x_n(0) - x(0)| + \|x'_n - x'\|_{L^1(\mathbb{R})}
\]

(see (33) and (35))
\[
= |u_n(0) - u_0| + \|x'_n - x'\|_{L^1(\mathbb{R})} \quad \text{for every } n \in \mathbb{N};
\]

as a consequence, since \( x'_n \to x' \) in \( L^1(\mathbb{R}) \) and \( u_n(0) \to u_0 \) as \( n \to \infty \), we conclude that \( x_n \to x \) uniformly on \( \mathbb{R} \). In particular,
\[
\lim_{t \to -\infty} x(t) = \lim_{n \to +\infty} \left( \lim_{t \to -\infty} x_n(t) \right) = \lim_{n \to +\infty} \alpha(-n) = \lim_{t \to -\infty} \alpha(t),
\]
\[
\lim_{t \to +\infty} x(t) = \lim_{n \to +\infty} \left( \lim_{t \to +\infty} x_n(t) \right) = \lim_{n \to +\infty} \beta(n) = \lim_{t \to +\infty} \beta(t).
\]
5. A class of examples. Let \( \nu_1, \nu_2 \in \mathbb{R} \) be such that \( \nu_1 < \nu_2 \). We consider the following BVP:

\[
\begin{align*}
\{ & (\Phi(a(t, x(t)))x'(t))' = f_1(t, x(t))f_2(x'(t)) \quad \text{a.e. on } \mathbb{R}, \\
& x(-\infty) = \nu_1, \quad x(\infty) = \nu_2,
\end{align*}
\]

(40)

where the functions \( a, \Phi, f_1 \) and \( f_2 \) fulfill the assumptions listed below:

(I) \( a : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a non-negative continuous function on \( \mathbb{R} \times \mathbb{R} \) which is also bounded on \( \mathbb{R} \times [\nu_1, \nu_2] \). Furthermore, we suppose that it is possible to find a real \( p > 1 \) and a real \( \sigma > 0 \) such that, setting

\[
h(t) := \min_{x \in [\nu_1, \nu_2]} a(t, x) \geq 0,
\]

we have \( 1/h \in L^p_{loc}(\mathbb{R}) \) and

\[
\int_{|t| \geq 1} \frac{1}{|t|^\sigma h(t)^p} \, dt < \infty.
\]

(II) \( \Phi : \mathbb{R} \to \mathbb{R} \) is an odd, strictly increasing homeomorphism from \( \mathbb{R} \) onto itself; moreover, there exists \( \rho > 0 \) such that

\[
\lim_{z \to 0^+} \frac{\Phi(z)}{z^\rho} > 0.
\]

(III) \( f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function enjoying the following properties:

(III)_1 there exists a function \( \lambda \in L^1_{loc}(\mathbb{R}) \) such that

\[
|f_1(t, x)| \leq \lambda(t) \quad \text{for a.e. } t \in \mathbb{R} \text{ and every } x \in [\nu_1, \nu_2];
\]

(42)

(III)_2 it is possible to find a real number \( T_0 > 0 \), two real constants \( c_1, c_2 > 0 \) and a real number \( \delta \geq -1 \) such that

\[
c_1 |t|^{-1} \leq |f_1(t, x)| \leq c_2 |t|^\delta,
\]

(43)

for a.e. \( |t| \geq T_0 \) and every \( x \in [\nu_1, \nu_2] \);

(III)_3 \[ f_1(t, x) \leq 0 \] for every \( |t| \geq T_0 \) and every \( x \in [\nu_1, \nu_2] \).

(IV) \( f_2 \in C(\mathbb{R}, \mathbb{R}) \) and it enjoys the following properties:

(IV)_1 \( f_2 > 0 \) on \( \mathbb{R} \setminus \{0\} \) and \( f_2(0) = 0 \);

(IV)_2 there exists a real number \( y^* > 0 \), two real constants \( c'_1, c'_2 > 0 \) and a real number \( \gamma \leq 1 \) such that

\[
c'_1 |\Phi(y)| \leq f_2(y) \leq c'_2 |\Phi(y)|^{\gamma},
\]

(44)

for every \( y \in \mathbb{R} \) with \( |y| < y^* \);

(IV)_3 there exist a real \( H > 0 \) and a real constant \( c'_3 > 0 \) such that, if \( y \in \mathbb{R} \) and \( |y| \geq H \), the following estimate holds true:

\[
f_2(y) \leq c'_3 |y|^{1-1/q} \quad \text{for some } 1 < q \leq \infty;
\]

(45)

(IV)_4 \( b \) is homogeneous of degree \( d > 0 \) on \( \mathbb{R} \), that is,

\[
f_2(sy) = s^d f_2(y) \quad \text{for every } s > 0 \text{ and every } y \in \mathbb{R}.
\]

Finally, introducing the constant (see also assumption (I))

\[
M_a := \sup_{\mathbb{R} \times [\nu_1, \nu_2]} a(t, x) < \infty,
\]

(46)

we suppose that the following relations hold:

\[
\frac{c_1 c'_1}{(M_a)^d} \geq \sigma \rho \quad \text{and} \quad \frac{c_1 c'_1}{(M_a)^d} \geq \sigma + \delta.
\]

(47)
Our aim is to prove that, in the present setting, all the hypotheses of Theorem 2.3 are satisfied; as a consequence, there exists a solution \( x \in W^{1,p}_{\text{loc}}(\mathbb{R}) \) of (40).

We explicitly point out that in the particular case when (43) holds with \( \delta = -1 \) we cover the critical case

\[
f(t, \cdot, \cdot) = f_1(t, \cdot)f_2(\cdot) \approx \frac{1}{|t|} \quad \text{as } |t| \to \infty.
\]

**Remark 3.** Before proceeding we highlight, for a future reference, a few consequences of the above assumptions (I)-to-(IV) we shall use in the sequel.

(a) For every \( v \in (-\infty, p] \) one has

\[
\int_{\{|t| \geq 1\}} \frac{1}{|t|^\sigma h(t)^v} \, dt < \infty.
\]

In fact, since \( a(t, x) \geq h(t) \geq 0 \) for any \( (t, x) \in \mathbb{R} \times [\nu_1, \nu_2] \) and \( a \) is bounded on the same set, we obviously have that \( h \) is bounded on \( \mathbb{R} \); hence, the map \( t \mapsto |t|^{-\sigma}/h^p \) being integrable on \( \{|t| \geq 1\} \), for every \( v \in (-\infty, p] \) we have

\[
\int_{\{|t| \geq 1\}} \frac{1}{|t|^\sigma h(t)^v} \, dt \leq \left( \sup_{\mathbb{R}} h \right)^{p-v} \int_{\{|t| \geq 1\}} \frac{1}{|t|^\sigma h(t)^p} \, dt < \infty.
\]

(b) For every \( \zeta > 0 \) one has

\[
\max_{|y| \leq \zeta} |\Phi(y)| = \Phi(\zeta).
\]

In fact, if \( y \in \mathbb{R} \) is such that \( |y| \leq \zeta \), from the fact that \( \Phi \) is an odd increasing function on \( \mathbb{R} \) (see assumption (II)) we infer that

\[
-\Phi(\zeta) = \Phi(-\zeta) \leq \Phi(y) \leq \Phi(\zeta), \quad \text{whence } |\Phi(y)| \leq \Phi(\zeta).
\]

(c) By combining assumption (III)$_3$ with estimate (43) we easily infer that

\[
t \cdot f_1(t, x) < 0 \quad \text{for every } t \in \mathbb{R} \setminus [-T_0, T_0] \text{ and every } x \in [\nu_1, \nu_2].
\]

Indeed, if \( x \in [\nu_1, \nu_2] \) and \( |t| \geq T_0 > 0 \) we have

\[
t \cdot f_1(t, x) = -|t| \cdot f_1(t, x) \leq -c_1 |t|^{-1} < 0.
\]

(d) By combining the growth condition (45) in assumption (IV)$_3$ with the \( \delta \)-homogeneity of \( f_2 \) in assumption (IV)$_4$ it is readily seen that

\[
0 < d \leq 1 - \frac{1}{q} \leq 1 < p.
\]

In fact, if \( H > 0 \) is as in assumption (IV)$_3$ and if \( y > H \) is arbitrarily fixed, by (45) and the \( \delta \)-homogeneity of \( f_2 \) one has

\[
s^d f_2(y) = f_2(sy) \leq c_3 s^{1-1/q} y^{1-1/q}, \quad \text{for every } s \geq 1,
\]

but this is possible only if \( d \leq 1 - 1/q \).

We now prove that all the hypotheses of Theorem 2.3 are satisfied. First of all, on account of assumptions (II), we have that \( \Phi \) satisfies (1) (with a suitable \( \rho > 0 \)); furthermore, since \( f_1 \) is a Carathéodory function on \( \mathbb{R} \times \mathbb{R} \) and \( f_2 \) is continuous on \( \mathbb{R} \) (as it follows from assumptions (III) and (IV)), the function

\[
f : \mathbb{R}^3 \to \mathbb{R}, \quad f(t, x, y) := f_1(t, x) f_2(y),
\]

is a Carathéodory function in \( \mathbb{R}^3 \).
Hypothesis ($H_1$). We now prove that hypotheses ($H_1$) in the statement of Theorem 2.3 is satisfied. To this end we observe that, since $f_2(0) = 0$ (see (IV)1), the (constant) functions 
\[
\alpha(t) = \nu_1 \quad \text{and} \quad \beta(t) = \nu_2 \quad (t \in \mathbb{R})
\]  
are, respectively, a lower and an upper solution of the ODE 
\[
\left(\Phi(a(t, x(t)) x'(t))\right)' = f_1(t, x(t)) f_2(x'(t)) = f(t, x(t), x'(t)).
\]  
Moreover, it is straightforward to check that
\begin{itemize}
  \item $\alpha \leq \beta$ on $\mathbb{R}$ (since $\nu_1 < \nu_2$);
  \item $\alpha$ is increasing on $(-\infty, -T_0]$ and $\beta$ is increasing on $[T_0, \infty)$ (where $T_0 > 0$ is the same number appearing in assumption (III));
  \item $\alpha \to \nu_1 = \alpha_0 \in \mathbb{R}$ as $t \to -\infty$ while $\beta \to \nu_2 = \beta_0 \in \mathbb{R}$ as $t \to \infty$.
\end{itemize}
Finally, since we have (by (7) and (48)) 
\[
a_*(t) = \min_{x \in [\alpha(t), \beta(t)]} a(t, x) = \min_{x \in [\nu_1, \nu_2]} a(t, x) = h(t) \quad \text{for every } t \in \mathbb{R},
\]  
from assumption (I) we infer that $1/a_* = 1/h \in L^p_{\text{loc}}(\mathbb{R})$ (for a suitable $p > 1$).

Hypothesis ($H_2$). In this paragraph we prove that also hypothesis ($H_2$) is satisfied. In fact, if $H > 0$ is as in assumption (IV)3, by exploiting estimate (45) and taking into account assumption (III)$_1$ we obtain 
\[
|f(t, x, y)| = |f_1(t, x)| \cdot f_2(y) \leq c_3 \lambda(t) |y|^{1-1/q},
\]  
for a.e. $t \in \mathbb{R}$, every $x \in [\nu_1, \nu_2]$ and every $y \in \mathbb{R}$ with $|y| \geq H$. As a consequence, hypothesis ($H_2$) is fulfilled with the choice 
\[
\psi(s) \equiv 1, \quad l(t) \equiv 0, \quad \mu(t) = c_3 \lambda(t),
\]  
and $T_0 > 0$ as in assumption (III)$_3$. We explicitly point out that, since the function $\lambda$ is in $L^\infty_{\text{loc}}(\mathbb{R})$ (see (III)$_1$), we have $\mu \in L^p([-T_0, T_0])$.

Hypothesis ($H_3$). In this last paragraph of the section we prove that, in our setting, hypothesis ($H_3$) in the statement of Theorem 2.3 is satisfied.

To begin with, if $T_0 > 0$ is as in assumption (III), we consider the function 
\[
K_0 : [0, \infty) \to \mathbb{R}, \quad K_0(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq T_0; \\ \int_{T_0}^t \gamma(s) \, ds, & \text{if } t > T_0, \end{cases}
\]  
where $\gamma : \mathbb{R} \to \mathbb{R}$ is defined as follows: 
\[
\gamma(t) := \min \left\{ \min_{[\nu_1, \nu_2]} |f_1(t, \cdot)|, \min_{[\nu_1, \nu_2]} |f_1(-t, \cdot)| \right\}.
\]  
Since, by assumption (III), the map $x \mapsto f_1(t, x)$ is continuous on $\mathbb{R}$ for a.e. $t \in \mathbb{R}$, the function $K_0$ is well-defined; moreover, as 
\[
|f_1(t, x)| \leq \lambda(t) \text{ for a.e. } (t, x) \in \mathbb{R} \times [\nu_1, \nu_2],
\]  
and $\lambda$ is in $L^\infty_{\text{loc}}(\mathbb{R})$, it is readily seen that $K_0 \in W^{1,1}_{\text{loc}}([0, \infty))$. Finally, $K_0$ is strictly increasing on $[T_0, \infty)$ (see Remark 3-(c)) and, by (43), we have 
\[
K_0(t) \geq c_1 \int_{T_0}^t \frac{1}{s} \, ds = c_1 \log \left( \frac{t}{T_0} \right), \quad \text{for every } t \geq T_0.
\]  
(51)
Let now $L > 0$ be arbitrarily fixed and let

- $m_{f_2}(L) := \min_{y^* \leq |y| \leq L} f_2(y) \quad \text{(where } y^* \text{ is as in assumption (IV)$_2$});$
- $M_{\Phi}(L) := \max_{y^* \leq |y| \leq L} |\Phi(z)|.$

If $c_1' > 0$ is as in (44), we set

$$c(L) := \min \left\{ \frac{c_1'}{(M_a)^d}, \frac{m_{f_2}(L)}{M_{\Phi}(L)} (M_a)^d \right\} > 0, \quad (52)$$

and we consider the function $H_L : \mathbb{R} \to \mathbb{R}$ defined by

$$H_L(t) := \Phi^{-1} \left( \Phi(L) \exp \left(-c(L) K_0(|t|) \right) \right).$$

Since, by (51), $K_0(t) \to \infty$ as $t \to \infty$, we have that $H_L(t) \to 0$ as $t \to \pm \infty$; as a consequence, it is possible to find a real $\overline{t}_L > T_0$ such that

$$H_L(t) \leq y^*, \quad \text{for every } t \in \mathbb{R} \text{ with } |t| \geq \overline{t}_L. \quad (53)$$

We then claim that the function $K_L$ defined by

$$K_L(t) := \begin{cases} c(L) K_0(t), & \text{if } 0 \leq t \leq \overline{t}_L, \\ c(L) K_0(\overline{t}_L) + \frac{c_1'}{(M_a)^d} \int_{\overline{t}_L}^t \gamma(s) \, ds, & \text{if } t > \overline{t}_L, \end{cases}$$

satisfies all the properties in hypothesis (B4). Indeed, on the one hand we have

(*) $K_L \in W^{1,1}_{\text{loc}}([0, \infty))$, since the same is true of $K_0$ (see also (42));

(*) $K_L \equiv K_0 \equiv 0$ on $[0, T_0]$ and $K_L$ is strictly increasing on $[T_0, \infty)$, since the same is true of $K_0$ and $\overline{t}_L > T_0$ (see also Remark 3-(c) and (52)).

Furthermore, since $K_L \geq c(L) K_0$ on $[0, \infty)$ (note that $c(L) \leq c_1'/(M_a)^d$, see the above (52)), by the very definition of $\mathcal{N}_L$ we have

$$\mathcal{N}_L(t) \equiv \Phi^{-1} \left( \Phi(L) \exp \left(-K_L(|t|) \right) \right) \leq \Phi^{-1} \left( \Phi(L) \exp \left(-c(L) K_0(|t|) \right) \right) = H_L(t).$$

By combining this last inequality with (53) we then get

$$\mathcal{N}_L(t) \leq y^*, \quad \text{for every } t \in \mathbb{R} \text{ with } |t| \geq \overline{t}_L. \quad (54)$$

As a consequence, for almost every $t \in \mathbb{R}$ with $|t| > \overline{t}_L$, every $x \in [\nu_1, \nu_2]$ and every $y \in \mathbb{R}$ with $|a(t, x) y| \leq \mathcal{N}_L(t) \leq y^*$ we obtain

$$|f(t, x, y)| = |f_1(t, x)| \cdot f_2(y) = \frac{|f_1(t, x)|}{a(t, x)^d} f_2(a(t, x) y) \quad \text{(by (IV)$_4$)}$$

$$\geq \frac{c_1'}{(M_a)^d} \frac{|f_1(t, x)| \cdot |\Phi(a(t, x) y)|}{a(t, x)^d} \quad \text{(by (44) and (46))}$$

$$\geq K_L'(|t|) \cdot |\Phi(a(t, x) y)| \quad \text{(by the very definition of } K_L).$$
On the other hand, since \( N_L \leq L \) on \( \mathbb{R} \), for almost every \( t \in \mathbb{R} \) with \( |t| \in [T_0, \tilde{t}_L] \), every \( x \in [\nu_1, \nu_2] \) and every \( y \in \mathbb{R} \) with \( |a(t, x)y| \leq N_L(t) \) we have

\[
|f(t, x, y)| = |f_1(t, x) f_2(y)| \geq \frac{1}{(M_a)^d} |f_1(t, x)| f_2(a(t, x)y)
\]

\[
\geq \begin{cases} 
\frac{c_1'}{(M_a)^d} |f_1(t, x)| \Phi(a(t, x)y), & \text{if } |a(t, x)y| \leq y^* \quad (\text{by (44)}), \\
\left( \frac{m_{f_2}(L)}{\Phi(L) (M_a)^d} \right) |f_1(t, x)| \Phi(a(t, x)y), & \text{if } y^* \leq |a(t, x)y| \leq N_L(t).
\end{cases}
\]

\[
\geq K'_L(|t|) |\Phi(a(t, x)y)| \quad (\text{by the very definition of } K_L).
\]

Summing up, we have proved that

\[
|f(t, x, y)| \geq K'_L(|t|) |\Phi(a(t, x)y)|,
\]

for almost every \( t \in \mathbb{R} \) with \( |t| \geq T_0 \), every \( x \in [\nu_1, \nu_2] \) and every \( y \in \mathbb{R} \) with \( |y| \leq N_L(t)/a(t, x) \); hence, \((H_3)-(i)\) holds.

We now turn to prove that \( K_L \) satisfies assumption \((H_3)-(i)\). To this end we first notice that, since \( K_L \) is continuous on \([0, \infty)\) and \( 1/a_* = 1/h \) is in \( L^p_{loc}(\mathbb{R}) \) (see assumption (I) and (50)), we obviously have

\[
\int_{T_0 \leq |t| \leq \tilde{t}_L} \frac{1}{a_* (t)} \exp \left( - \frac{K_L(|t|)}{\rho} \right) dt < \infty. \tag{55}
\]

On the other hand, by the very definition of \( K_L \) and by using (43) one has

\[
\int_{|t| > \tilde{t}_L} \frac{1}{a_* (t)} \exp \left( - \frac{K_L(|t|)}{\rho} \right) dt
\]

\[
= \exp \left( - \frac{c(L) K_0(\tilde{t}_L)}{\rho} \right) \int_{|t| > \tilde{t}_L} \frac{1}{a_* (t)} \exp \left( - \frac{c_1'}{\rho (M_a)^d} \int_{\tilde{t}_L}^{|t|} \gamma(s) ds \right) dt
\]

\[
\leq \exp \left( - \frac{c(L) K_0(\tilde{t}_L)}{\rho} \right) \int_{|t| > \tilde{t}_L} \frac{1}{a_* (t)} \left( \frac{\tilde{t}_L}{|t|} \right)^{\frac{c_1 c_1'}{\rho (M_a)^d}} dt
\]

\[
= c \int_{|t| > \tilde{t}_L} \frac{1}{a_* (t)} \left( \frac{1}{|t|} \right)^{\frac{c_1 c_1'}{\rho (M_a)^d}} dt,
\]

where \( c > 0 \) is a suitable constant. From this, since the map

\[
t \mapsto |t|^{-\sigma} / a_* (t) = |t|^{-\sigma} / h(t)
\]

is integrable on \( \{|t| \geq 1\} \) (see Remark 3-(a)) and since, by (47), we have

\[
\frac{c_1 c_1'}{\rho (M_a)^d} \geq \sigma,
\]
we conclude that
\[
\int_{|t|>\tau_L} \frac{1}{a_+(t)} \left( \frac{1}{|t|} \right)^{\frac{c_1 c_2}{\rho (\alpha \beta')}} dt < \infty, \quad \text{whence}
\]
\[
\int_{|t|>\tau_L} \frac{1}{a_+(t)} \exp \left( -\frac{K_L(|t|)}{\rho} \right) dt < \infty.
\] (56)

By combining (55) with (56) we finally obtain the validity of (H3)-(i).

To conclude the demonstration of the validity of hypothesis (H3) we are left to prove (H3)-(iii) and (H3)-(iv) (with a suitable \( \eta_L \in L^1(\mathbb{R}) \)). As regards (H3)-(iii), by combining assumptions (III)_3 and (IV)_1 we readily see that
\[
|t| \cdot f(t, x, y) = t \cdot f_1(t, x) f_2(y) \leq 0
\]
for \(|t| \geq T_0\), every \( x \in [\nu_1, \nu_2] \) and every \( y \in \mathbb{R} \).

We then turn to prove the existence of a non-negative function \( \eta_L \in L^1(\mathbb{R}) \) satisfying (note that, since \( \alpha, \beta \) are constant, we have \( \alpha' = \beta' = 0 \) on \( \mathbb{R} \))
\[
|f(t, x, y)| \leq \eta(t) \quad \text{for a.e.} \ t \in \mathbb{R}, \text{every} \ x \in [\nu_1, \nu_2] \text{and} \ y \leq \mathcal{N}_L(t)/a_+.
\]

To this end we observe that, on account of (54), for a.e. \( t \in \mathbb{R} \), every \( x \in [\nu_1, \nu_2] \) and every \( y \in \mathbb{R} \) such that \( a_+(t) y \leq \mathcal{N}_L(t) \) we have the following estimate:
\[
|f(t, x, y)| = |f_1(t, x)| f_2(y) = \frac{|f_1(t, x)|}{a_+(t)} f_2(a_+(t) y) \quad \text{(by (50))}
\]
\[
\leq \begin{cases} 
\frac{c_2 c_2'}{a_+(t)^d} |t|^{\delta} \Phi(\mathcal{N}_L(t))^\gamma, & \text{if} \ |t| > \tau_L \quad \text{(by (43), (44) and Rem. 3-(b))}, \\
\left( \max_{[0, L]} f_2 \right) \cdot \frac{\lambda(t)}{a_+(t)^d}, & \text{if} \ |t| \leq \tau_L \quad \text{(by (42))},
\end{cases}
\]
\[
=: \eta(t).
\]

We now show that \( \eta_L \) is in \( L^1(\mathbb{R}) \). On the one hand, since \( 0 < d \leq p \) (see Remark 3-(c)) and \( \lambda \in L^\infty(\mathbb{R}) \), we have (also remind that \( 1/a_+ = 1/h \in L^p_{\text{loc}}(\mathbb{R}) \))
\[
\int_{\{|t| \leq \tau_L\}} \eta_L(t) dt = \left( \max_{[0, L]} f_2 \right) \int_{\{|t| \leq \tau_L\}} \frac{\lambda(t)}{a_+(t)^d} dt
\]
\[
\leq \left( \max_{[0, L]} f_2 \right) \cdot \|\lambda\|_{L^\infty([-\tau_L, \tau_L])} \int_{\{|t| \leq \tau_L\}} \frac{1}{a_+(t)^d} dt < \infty.
\] (57)

On the other hand, by crucially exploiting estimate (43) (and taking into account the very definition of \( \mathcal{N}_L \), see (8)) we obtain
\[
\int_{\{|t| > \tau_L\}} \eta_L(t) dt = c_2 c_2' \int_{\{|t| > \tau_L\}} \frac{|t|^{\delta}}{a_+(t)^d} \Phi(\mathcal{N}_L(t))^\gamma
\]
\[
= c_2 c_2' \Phi(L)^\gamma \int_{\{|t| > \tau_L\}} \frac{|t|^{\delta}}{a_+(t)^d} \exp \left( -\gamma K_L(|t|) \right) dt
\]
\[
= c_1 \int_{\{|t| > \tau_L\}} \frac{|t|^{\delta}}{a_+(t)^d} \exp \left( -\frac{\gamma c_1}{(M_\alpha)^d} \int_{\tau_L}^{|t|} \gamma(s) ds \right) dt.
\]
\[
\leq c_1 \int_{\{|t|>\tau_L\}} \frac{|t|^\delta}{a_*(t)^d} \left( \frac{t_L}{|t|} \right)^{\frac{\gamma c_1 c'_1}{(Ma)^d} - \delta} \, dt
= \varpi \int_{\{|t|>\tau_L\}} \frac{1}{a_*(t)} \left( \frac{1}{|t|} \right)^{\frac{\gamma c_1 c'_1}{(Ma)^d} - \delta} \, dt,
\]
for suitable real constants \(c_1, \varpi > 0\). From this, since
\[
t \mapsto |t|^{-\sigma}/a_*(t)^d = |t|^{-\sigma}/h(t)^d \in L^1((1, \infty)),
\]
(as \(d \leq p\), see Remark 3) and since, by (47), we have
\[
\gamma \cdot \frac{c_1 c'_1}{(Ma)^d} - \delta \geq \sigma,
\]
it is readily seen that
\[
\int_{\{|t|>\tau_L\}} \frac{1}{a_*(t)} \left( \frac{1}{|t|} \right)^{\frac{\gamma c_1 c'_1}{(Ma)^d} - \delta} \, dt < \infty, \quad \text{whence}
\int_{\{|t|>\tau_L\}} \eta_L(t) \, dt < \infty.
\]
By combining (57) with (59) we then conclude that
\[
\int_{\mathbb{R}} \eta_L(t) \, dt < \infty,
\]
and this completes the demonstration of the validity of \((H_3)\).

Gathering together all the facts established so far, we are entitled to apply our
Theorem 2.3 to the boundary value problem (40): in fact, since \textit{all the hypotheses}
\((H_1)\)-to-\((H_3)\) are satisfied and, by (48),
\[
\lim_{t \to -\infty} \alpha(t) = \nu_1 \quad \text{and} \quad \lim_{t \to \infty} \beta(t) = \nu_2,
\]
it is possible to find \textit{(at least) one solution} \(x \in W^{1, p}_{\text{loc}}(\mathbb{R})\) (where \(p > 1\) is as in
assumption (I)) of the BVP (40), further satisfying
\[
\nu_1 \leq x(t) \leq \nu_2 \quad \text{for every} \ t \in \mathbb{R}.
\]

\textbf{Remark 4.} It is worth noting that, in the particular case when also the homeo-
morphism \(\Phi\) in assumption (II) is homogeneous of a certain degree \(g \in (0, p]\), our
growth assumption \((IV)_3\) can be replaced with the following one:

\textbf{(IV)}\(\prime_3\) there exist a real \(H > 0\) and a real constant \(c'_3 > 0\) such that, if \(y \in \mathbb{R}\) and
\(|y| \geq H\), the following estimate holds:
\[
f_2(y) \leq c'_3 |\Phi(y)|^\alpha \quad \text{for some} \ \alpha \leq 1.
\]
In fact, if (60) is satisfied, we have
\[
|f_1(t, x)||f_2(y)| \leq c'_3 |f_1(t, x)||\Phi(y)|^\alpha = \frac{|f_1(t, x)|}{a(t, x)^\alpha} |\Phi(a(t, x)y)|^\alpha
\]
(by (41) and assumption \((III)_1\))
\[
\leq c'_3 \frac{\lambda(t)}{h(t)^\alpha} |\Phi(a(t, x)y)|^\alpha,
\]
for a.e. \( t \in \mathbb{R} \), every \( x \in [\nu_1, \nu_2] \) and every \( y \in \mathbb{R} \) with \( |y| \geq H \). As a consequence, hypothesis \((H_2)\) in Theorem 2.3 is again fulfilled with the choice

\[
\psi(s) = s^\alpha, \quad l(t) = c_3 \frac{\lambda(t)}{h(t)^{\alpha g}}, \quad \mu(t) \equiv 0.
\]

Notice that, since \( \alpha \leq 1 \), the function \( \psi \) satisfies \((H_2)-(i)\); furthermore, since \( \lambda \) is in \( L^\infty_{loc}(\mathbb{R}) \), since the function \( 1/h \) is in \( L^p_{loc}(\mathbb{R}) \) and since \( \alpha g \leq g \leq p \), we have

\[
l \in L^1([-T_0, T_0]).
\]

It is also worth highlighting that, by combining the growth condition (60) with \( d \)-homogeneity of \( f_2 \) in assumption (IV)_4 and the \( g \)-homogeneity of \( \Phi \), we get

\[
d \leq \alpha g \leq p.
\]

This bound is used in the proof of the validity of hypothesis \((H_3)\) (see, precisely, (58)).

We conclude this section by presenting a couple of concrete examples of BVPs of the form (40) and satisfying assumptions (I)-to-(IV) introduced above.

**Example 1.** Let us consider the following boundary value problem:

\[
\begin{aligned}
\Phi \left( e^{-x(t)^2} \min\{\sqrt{|t|}, 1/t^2\} x'(t) \right)' &= -\frac{m t}{t^2 + 1} |x'(t)|^\theta \quad \text{a.e. on } \mathbb{R}, \\
x(-\infty) &= 0, \quad x(\infty) = 1,
\end{aligned}
\]

where \( m \in (1, \infty) \) will be fixed later on, \( \theta \in (0, 1) \) is arbitrarily chosen and

\[
\Phi(z) = z + \sin(z).
\]

Obviously, the above problem takes the form (40) with

- \((*)\) \( a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad a(t, x) = e^{-x^2} \min\{\sqrt{|t|}, 1/t^2\}; \)
- \((*)\) \( \Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(z) = z + \sin(z); \)
- \((*)\) \( f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f_1(t, x) := -\frac{m t}{t^2 + 1}; \)
- \((*)\) \( f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(y) := |y|^\theta. \)

We claim that the functions \( a, \Phi, f_1 \) and \( f_2 \) satisfy all the assumptions (I)-(IV) introduced in this section, with suitable constants fulfilling (47).

(I) Clearly, \( a \) is non-negative and continuous on \( \mathbb{R} \times \mathbb{R} \); moreover, it is bounded on \( \mathbb{R} \times [0, 1] \) (note that \( a(t, x) \leq 1 \) for every \((t, x) \in \mathbb{R} \times [0, 1] \)) and

\[
h(t) := \min_{x \in [0,1]} a(t, x) = \frac{1}{e} \min\{\sqrt{|t|}, 1/t^2\}, \quad \text{for every } t \in \mathbb{R}.
\]

It is then very easy to recognize that \( 1/h \in L^p_{loc}(\mathbb{R}) \) for every fixed \( p \in (1, 2) \) and that, for any such a \( p \), we also have

\[
\int_{\{|t| \geq 1\}} \frac{1}{|t|^\sigma h^p(t)} dt < \infty, \quad \text{for every } \sigma > 1 + 2p.
\]

As a consequence, \( a \) fulfills assumption (I).
We now claim that, once one have fixed $p$, we conclude that $\Phi$ fulfills assumption (II) with the choice $\rho = 1$.

(III) First of all, $f_1$ is a Carathéodory function on $\mathbb{R} \times \mathbb{R}$ (as it is continuous on the same set); moreover, for every $t \geq 0$ and every $x \in [0,1]$ we have

$$[f_1(t,x)] = \frac{m|t|}{t^2 + 1} =: \lambda(t) \in L^\infty_{loc}(\mathbb{R}),$$

and thus $f_1$ fulfills assumption $(\text{III})_1$. As for assumption $(\text{III})_2$ we observe that, since we obviously have

$$\lim_{|t| \to \infty} |f_1(t,x)| = \lim_{|t| \to \infty} \frac{m|t|^2}{t^2 + 1} = m,$$

for every fixed $\varepsilon \in (0,1/2)$ there exists $T_0 = T_0(\varepsilon) > 0$ such that

$$(m - \varepsilon)|t|^{-1} \leq |f_1(t,x)| \leq (m + \varepsilon)|t|^{-1},$$

for every $t \in \mathbb{R}$ with $|t| \geq T_0(\varepsilon)$ and every $x \in [0,1]$. As a consequence, estimate (43) holds true with the choice (note that $m - \varepsilon > 1/2 > 0$)

$$\delta = -1, \quad c_1(\varepsilon) = m - \varepsilon, \quad c_2(\varepsilon) = m + \varepsilon \quad \text{and} \quad T_0 = T_0(\varepsilon). \quad (62)$$

Finally, since $t \cdot f_1(t,x) < 0$ for every $t \neq 0$ and every $x \in [0,1]$, we conclude that $f_1$ also fulfills assumption $(\text{III})_3$ (again with $T_0 = T_0(\varepsilon)$).

(IV) Obviously, $f_2(y) = |y|^\vartheta$ is continuous on the whole of $\mathbb{R}$ (as $\vartheta > 0$) and it fulfills assumption $(\text{IV})_1$; furthermore, since we clearly have

$$\lim_{y \to 0} \frac{f_2(y)}{|\Phi(y)|^\vartheta} = \lim_{y \to 0} \frac{|y|^\vartheta}{|y + \sin(y)|^\vartheta} = \frac{1}{2\vartheta},$$

and since, by assumption, $\vartheta \in (0,1)$, for every fixed $\varepsilon \in (0,1/2^\vartheta)$ it is possible to find a real $y^* = y^*(\varepsilon) > 0$ such that

$$\left(\frac{1}{2\vartheta} - \varepsilon\right)|\Phi(y)| \leq f_2(y) \leq \left(\frac{1}{2\vartheta} + \varepsilon\right)|\Phi(y)|^\vartheta,$$

for every $y \in \mathbb{R}$ with $|y| < y^*(\varepsilon)$. As a consequence, estimate (44) in assumption $(\text{IV})_2$ is satisfied with the choice (note that $1/2^\vartheta - \varepsilon > 0$)

$$\gamma = \vartheta \in (0,1), \quad c_1'(\varepsilon) = \frac{1}{2\vartheta} - \varepsilon, \quad \text{and} \quad c_2'(\varepsilon) = \frac{1}{2\vartheta} + \varepsilon. \quad (63)$$

We now observe that, since $f_2(y) = |y|^\vartheta$, our growth condition (45) in assumption $(\text{IV})_3$ is trivially satisfied with the choice

$$H = 1 \quad \text{and} \quad q = \frac{1}{1 - \vartheta} \in (1, \infty).$$

Finally, we note that $f_2$ is homogeneous of degree $d = \vartheta < 1$.

We now claim that, once one have fixed $p \in (1,2)$, it is possible to choose $m > 1$ and $\varepsilon \in (0,1/2^\vartheta)$ in such a way that the relations in (47) hold true.

To prove the claim we first notice that, by definition, one has

$$M_n = \sup_{\mathbb{R} \times [0,1]} \left(e^{-x^2} \min\{|t|, 1/t^2\}\right) = 1;$$
moreover, since we obviously have
\[ \lim_{\epsilon \to 0^+} (m - \epsilon) \cdot \left( \frac{1}{2^p} - \epsilon \right) = \frac{m}{2^p}, \]
it is possible to choose \( m > 1 \) and \( \epsilon \in (0, 1/2^p) \) such that
\[ (m - \epsilon) \cdot \left( \frac{1}{2^p} - \epsilon \right) > \max \left\{ 1 + 2p, \frac{2p}{\rho} \right\}. \]
As a consequence, by (62), (63) and the fact that \( \rho = 1 \) we have
\[ \frac{c_1(\epsilon) c'_1(\epsilon)}{(M_\rho)^\delta} = c_1(\epsilon) c'_1(\epsilon) = (m - \epsilon) \cdot \left( \frac{1}{2^p} - \epsilon \right) > 1 + 2p = (1 + 2p) \rho. \]
Furthermore, again by (62) and (63) we also have
\[ \gamma \cdot \frac{c_1(\epsilon) c'_1(\epsilon)}{(M_\rho)^\delta} = \vartheta \cdot c_1(\epsilon) \cdot c_2(\epsilon) = \vartheta \cdot (m - \epsilon) \cdot \left( \frac{1}{2^p} - \epsilon \right) > 2p = (1 + 2p) + \delta. \]
From this, since \( \sigma \) can be chosen arbitrarily close to \( 1 + 2p \), we conclude that both the relations in (47) are satisfied. We are then entitled to apply our Theorem 2.3, which ensures the existence of a solution \( x \in W^{1,p}_{loc}(\mathbb{R}) \) of (61).

**Example 2.** Let \( \overline{p} \in (1, \infty) \) be arbitrarily fixed and let \( m \in (1, \infty) \). Moreover, let \( r, \vartheta \in \mathbb{R} \) be two real numbers satisfying the following relations:
\[ 1 < r < \overline{p} + 1 \quad \text{and} \quad 0 < \vartheta < r - 1. \]
Finally, let \( \Phi_r(z) = |z|^{r-2} z \) the usual \( r \)-Laplace operator on \( \mathbb{R} \) and let \( \nu_1, \nu_2 \in \mathbb{R} \) be such that \( \nu_1 < \nu_2 \). We then consider the following BVP on \( \mathbb{R} \):
\[ \begin{cases} (\Phi_r(a(t, x) x'(t)))' = k(t) \ |x'(t)|^{\vartheta}, \quad \text{a.e. on } \mathbb{R}, \\ x(-\infty) = \nu_1, \ x(\infty) = \nu_2, \end{cases} \tag{64} \]
where the functions \( a : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( k : \mathbb{R} \to \mathbb{R} \) are defined as follows:
\[ a(t, x) = \frac{\sin(t)|t|^{1/p} + \sin^2(t)}{2} \quad \text{and} \quad k(t) = -\text{sgn}(t) \cdot \frac{m + |t\cos(t)|}{1 + |t|}. \]
Obviously, the above problem takes the form (40) with
\[ (*) \Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(z) = \Phi_r(z) = |z|^{r-2} z; \]
\[ (*) f_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f_1(t, x) := k(t); \]
\[ (*) f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(y) := |y|^\vartheta. \]
We claim that, also in this case, the functions \( a, \Phi_r, f_1 \) and \( f_2 \) satisfy all the assumptions (I)-to-(IV) with suitable constants fulfilling (47).

(1) Clearly, \( a \) is non-negative and continuous on \( \mathbb{R} \times \mathbb{R} \); moreover, it is bounded on \( \mathbb{R} \times [\nu_1, \nu_2] \) (note that \( a(t, x) \leq 1 \) for every \( (t, x) \in \mathbb{R} \times [\nu_1, \nu_2] \)) and
\[ h(t) := \min_{x \in [\nu_1, \nu_2]} a(t, x) = \frac{1}{2} |\sin(t)|^{1/p}, \quad \text{for every } t \in \mathbb{R}. \]
Since \(1/h \in L^p_{\text{loc}}(\mathbb{R})\) for every \(p \in (1, \overline{p})\) (as is very easy to see), we choose
\[
p \in \left( \max\{1, r - 1\}, \overline{p} \right) .
\]

Furthermore, since we also have (for any \(\sigma > 1\))
\[
\int_{\{|t| \geq 2\pi\}} \frac{1}{|t|^\sigma} h(t)^{p/\overline{p}}\,dt = 4 \int_{2\pi}^\infty \frac{1}{t^\sigma} |\sin(t)|^{p/\overline{p}}\,dt
\]
\[
= 4 \sum_{n=1}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \frac{1}{t^\sigma} |\sin(t)|^{p/\overline{p}}\,dt
\]
\[
= 4 \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{1}{(t + 2n\pi)^\sigma} |\sin(t)|^{p/\overline{p}}\,dt
\]
\[
\leq 4 \left( \int_0^{2\pi} \frac{1}{|\sin(t)|^{p/\overline{p}}}\,dt \right) \sum_{n=1}^{\infty} \frac{1}{(2n\pi)^\sigma} < \infty,
\]
we conclude that \(a\) fulfills assumption (I).

(II) It is straightforward to recognize that \(\Phi\) is a strictly increasing homeomorphism from \(\mathbb{R}\) into itself and that \(\Phi_r\) is odd; moreover, since we have
\[
\lim_{z \to 0^+} \frac{\Phi_r(z)}{z^{r-1}} = 1,
\]
we conclude that \(\Phi\) fulfills assumption (II) with the choice \(\rho = r - 1\).

(III) First of all, \(f_1\) is a Carathéodory function on \(\mathbb{R} \times \mathbb{R}\) (as \(k\) is continuous out of \(\{0\}\)); moreover, for every \(t \in \mathbb{R}\) and every \(x \in [\nu_1, \nu_2]\) we have
\[
|f_1(t, x)| \leq \frac{m + |t \cos(t)|}{1 + |t|} =: \lambda(t) \in L^\infty_{\text{loc}}(\mathbb{R}),
\]
and thus \(f_1\) fulfills assumption (III)_1. As for assumption (III)_2 we observe that, for every \(t \in \mathbb{R}\) with \(|t| \geq 1\) and every \(x \in [\nu_1, \nu_2]\), we have
\[
\frac{m}{2} |t|^{-1} \leq \frac{m}{1 + |t|} \leq |f_1(t, x)| \leq \frac{m + |t \cos(t)|}{1 + |t|} \leq (m + 1) |t|;
\]
as a consequence, (43) holds true with the choice
\[
\delta = 1, \quad c_1 = \frac{m}{2}, \quad c_2 = m + 1 \quad \text{and} \quad T_0 = 1.
\]

Finally, since \(t \cdot f_1(t, x) \leq 0\) for every \(t \in \mathbb{R} \setminus \{0\}\) and every \(x \in \mathbb{R}\), we conclude that \(f_1\) also fulfills assumption (III)_3 (again with \(T_0 = 1\)).

(IV) Obviously, \(f_2(y) = |y|^\vartheta\) is continuous on the whole of \(\mathbb{R}\) (since \(\vartheta > 0\)) and it fulfills assumption (IV)_1; moreover, since we have
\[
|\Phi(y)| = |y|^\vartheta - 1 \quad \text{for every} \quad y \in \mathbb{R},
\]
and, by assumption, \(\vartheta/(r - 1) < 1\), it is straightforward to recognize that condition (44) in assumption (IV)_2 is satisfied with the choice
\[
\gamma := \frac{\vartheta}{r - 1} \quad \text{and} \quad c_1 = c_2 = 1.
\]

We now turn our attention to assumption (IV)_3. Actually, since the homeomorphism \(\Phi_r\) is homogeneous of degree \(g = r - 1 < p\) (see (65)), we prove that \(f_2\) satisfies condition (IV)_3 in Remark 4.
To this end it suffices to observe that, since \( \vartheta \leq r - 1 \), we have
\[
f_2(y) = |y|^\vartheta \leq |y|^{r-1} \quad \text{for every } y \in \mathbb{R} \text{ with } |y| \geq 1;
\]
thus, condition (60) is satisfied with the choice
\[
H = 1 \quad \text{and} \quad \alpha = 1.
\]

Finally, we note that \( f_2 \) is homogeneous of degree \( d = \vartheta \).

We finally claim that, if we choose \( m > 1 \) in such a way that
\[
m > (2r - 2) \cdot \max \{1, 2/\vartheta\},
\]
then all the relations in (47) are fulfilled.

To prove this claim we first notice that, by definition, one has
\[
M_a = \sup_{\mathbb{R} \times [\nu_1, \nu_2]} \left( \frac{|\sin(t)|^{1/\varphi} + \sin^2(x)}{2} \right) \leq 1;
\]
as a consequence, by (66), (67) and the fact that \( \rho = r - 1 \) we have
\[
\frac{c_1 c'_1}{(M_a)^d} \geq c_1 c'_1 = \frac{m}{2} > r - 1 = \rho.
\]
Furthermore, again by exploiting (66) and (67) we get
\[
\gamma \cdot \frac{c_1 c'_1}{(M_a)^d} \geq \gamma \cdot (c_1 c'_1) = \frac{\vartheta}{r - 1} \cdot \frac{m}{2} > 2 = 1 + \delta.
\]
From this, since \( \sigma \) can be chosen arbitrarily close to 1, we conclude that both the relations in (47) are satisfied. We are then entitled to apply our Theorem 2.3, which ensures the existence of a solution \( x \in W^{1,p}_{loc}(\mathbb{R}) \) of (61).

We end this section by explicitly notice that the above examples do not fall in the class of examples considered in [5]. Indeed, in the cited paper condition (43) in assumption (III) is replaced by the following one: there exists \( c_1, c_2 > 0 \) and \( \delta_1, \delta_2 \in \mathbb{R} \) with \( \delta_2 \geq \delta_1 > -1 \) such that
\[
c_1 t^{\delta_1} \leq |f_1(t, x)| \leq c_2 t^{\delta_2}, \quad \text{for a.e. } t \geq T_0 \text{ and } x \in [\nu_1, \nu_2].
\]

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