Higher order analysis of the geometry of singularities using the Taylorlet transform

Thomas Fink

Abstract

We consider an extension of the continuous shearlet transform which additionally uses higher order shears. This extension, called the Taylorlet transform, allows for a detection of the position, the orientation, the curvature, and other higher order geometric information of singularities. Employing the novel vanishing moment conditions of higher order, \( \int_\mathbb{R} g(\pm t^k)t^mdt = 0 \) for \( k, m \in \mathbb{N} \), \( k \geq 1 \), on the analyzing function \( g \in \mathcal{S}(\mathbb{R}) \), we can show that the Taylorlet transform exhibits different decay rates for decreasing scales depending on the choice of the higher order shearing variables. This enables a faster detection of the geometric information of singularities in terms of the decay rate with respect to the dilation parameter. Furthermore, we present a construction that yields analyzing functions which fulfill vanishing moment conditions of different orders simultaneously.

Keywords Shearlets · Edge classification · Curvature

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1 Introduction

Edges are crucial features in image processing as they contain a considerable part of an image’s visual information. Hence, their detection and analysis play a major role in computer vision and are vital in various fields of application including object recognition, image enhancement, and feature extraction. One of the most important characteristics of an edge is its curvature. It is a distinctive size for contours and, hence, especially valuable for shape recognition [10, 11]. Furthermore, the edge curvature plays a vital role in the human visual perception—especially corners and edge points with high curvature are crucial for the eye’s recognition of shapes [1, 2].
A plethora of different methods exists for the task of edge detection. Multi-scale approaches play a special role as they offer a good noise robustness and are motivated from a continuous setting where the term of an edge finds a more general mathematical analogue in the singular support. The detection of this feature, i.e., the distinction between regular and singular points by the decay rate of continuous multi-scale methods such as the continuous wavelet transform, has been thoroughly discussed in the literature, e.g., for the continuous wavelet transform in [9]. In addition to the identification of singularities, the continuous curvelet and shearlet transform also allow for a detection of directional information, i.e., a resolution of the wavefront set [3, 6]. When the latter is additionally endowed with second order shears

$$S_s : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + s_1 x_1 + s_2 x_2^2 \\ x_2 \end{pmatrix} ,$$

the resulting bendlet transform is capable of extracting the curvature of an edge [8].

In this paper, we introduce the Taylorlet transform, which utilizes higher order shears like the bendlet transform and allows for an extraction of position, orientation, curvature, and higher order geometric information of edges. It extends the bendlet transform by conditions that ensure a high decay rate for a faster detection of the desired features in terms of the decay rate with respect to the dilation parameter. The approach is based on a modeling of a singular support as graph of a singularity function \( q \in C^\infty(\mathbb{R}) \), i.e., \( \text{sing supp}(f) = \{ x \in \mathbb{R}^2 : x_1 = q(x_2) \} \). The geometrical data accessible by the Taylorlet transform consists of the Taylor coefficients of \( q \). In this perspective, the 2D continuous wavelet and shearlet transform essentially identify the 0th rsp. the 0th and 1st Taylor coefficients of the singularity function.

For these detection properties, vanishing moment conditions for the respective analyzing function are essential. They are responsible for the ability of the continuous wavelet transform to detect singularities of high regularity [9, Thm 3] and ensure the decay rate of the continuous shearlet transform for decreasing scales [5, Thm 3.1]. We will show in this paper that analogously an analyzing Taylorlet \( \tau = g \otimes h \), where \( g, h \in \mathcal{S}(\mathbb{R}) \) fulfilling vanishing moment conditions of order \( n \in \mathbb{N} \), that is

$$\int_{\mathbb{R}} g(\pm x_1^k) x_1^m dx_1 = 0 \in \mathbb{R} \text{ for all } k \in \{1, \ldots, n\}, \ m \in \{0, \ldots, kr - 1\}$$

for \( r \in \mathbb{N} \) is of similar importance for the extraction of higher order geometric information.

The paper is organized as follows. In Section 2, the Taylorlet transform and all basic definitions are introduced, before we describe the construction of analyzing Taylorlets of arbitrary order in Section 3. In Section 4, the main result is stated and explained in detail and afterwards proved in Section 5. Section 6 shows some numerical examples of the Taylorlet transform. Finally, in Section 7, we discuss open problems and possible extensions of the Taylorlet transform.
2 Basic definitions and notation

The goal of the Taylorlet transform is a precise analytical description of the singular support of the analyzed function $f$. To this end, we assume that we can represent $\text{sing supp}(f)$ locally as the graph of a singularity function $q \in C^\infty(\mathbb{R})$ and describe $\text{sing supp}(f)$ by the Taylor coefficients of $q$. These coefficients can be found by observing the decay rate of the Taylorlet transform. In this way, the continuous shearlet transform essentially delivers a local linear approximation to the singular support which can be regarded as a first-order Taylor polynomial of the singularity function $q$. Hence, we will use it as a starting point for the construction of the Taylorlet transform. To this end, we need an extension of the classical shear: we will use a modification of the higher order shearing operators introduced in [8].

**Definition 1** For $n \in \mathbb{N}$ and $s = (s_0, \ldots, s_n)^T \in \mathbb{R}^{n+1}$ the $n$th order shearing operator is defined as

$$S_s^{(n)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad S_s^{(n)}(x) := \left( x_1 + \sum_{\ell=0}^{n} \frac{s_{\ell}}{\ell!} x_2^\ell \right).$$

In contrast to [8], here, the higher order shearing operator also includes a simple translation along the $x_1$-axis in the form of $s_0$. This is included to emphasize the Taylor coefficient perspective on the singular support of the analyzed function.

Furthermore, for $a, \alpha > 0$ we use an $\alpha$-scaling matrix [8]

$$A_a^{(\alpha)} := \begin{pmatrix} a & 0 \\ 0 & a^\alpha \end{pmatrix}.$$ 

**Definition 2** (Iterated integrals) Let $k \in \mathbb{N}$, $u \in \mathbb{R}$ and $\phi \in L^1(\mathbb{R})$. We define

$$I_t^+ \phi(u) := \int_{-\infty}^{u} \phi(v) \, dv, \quad I_t^- \phi(u) := \int_{u}^{\infty} \phi(v) \, dv \quad \text{and} \quad I_t^{k+1} \phi = I_t^\pm \circ I_t^k \phi.$$

A central property of analyzing functions of continuous multi-scale transforms is the vanishing moment condition which plays a crucial role for wavelets in order to detect singularities of a certain smoothness [9]. For shearlets, there exist analogous results [5]. Pursuing a similar goal, the following definition of analyzing Taylorlets incorporates some special vanishing moment properties.

**Definition 3** (Vanishing moments of higher order, analyzing Taylorlet, restrictiveness) We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has $r$ vanishing moments of order $n$ if

$$\int_{\mathbb{R}} f(\pm t^k) t^m \, dt = 0$$

for all $m \in \{0, \ldots, kr - 1\}$ and for all $k \in \{1, \ldots, n\}$. 

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Let $S(\mathbb{R}^d)$ denote the Schwartz space on $\mathbb{R}^d$ and let $g, h \in S(\mathbb{R})$ such that $g$ has $r$ vanishing moments of order $n$. We call the function

$$\tau = g \otimes h$$

an analyzing Taylorlet of order $n$ with $r$ vanishing moments.

We say $\tau$ is restrictive, if additionally

(i) $I^j_+ g(0) \neq 0$ for all $j \in \{0, \ldots, r\}$ and

(ii) $\int_{\mathbb{R}} h(t) dt \neq 0$.

The concept of the restrictiveness is a generalization of the non-vanishing moment conditions employed on certain shearlets in [7, section 2.2] for the purpose of edge classification. Furthermore, in Theorem 1, we show that for arbitrary $n \in \mathbb{N}$, the set of restrictive analyzing Taylorlets of order $n$ is not empty.

Moreover, in order to show the effects of the higher order vanishing moments, we will employ techniques of the Fourier transform.

**Definition 4** (Fourier transform) Let $f \in L^1(\mathbb{R}^n)$. Then, the Fourier transform of $f$ is defined as

$$\hat{f}(\omega) = \int_{\mathbb{R}^n} f(x) e^{-i\omega \cdot x} dx.$$ 

We define the Taylorlet transform as follows.

**Definition 5** (Taylorlet transform) Let $r, n \in \mathbb{N}$ and let $\tau \in S(\mathbb{R}^2)$ be an analyzing Taylorlet of order $n$ with $r$ vanishing moments. Moreover, let $\alpha > 0$, $t \in \mathbb{R}$, $a > 0$ and $s \in \mathbb{R}^{n+1}$. We define

$$\tau^{(n, \alpha)}_{a,s,t}(x) := \tau \left( A^{(\alpha)}_s S^{(n)}(x) \left( \frac{x_1}{x_2 - t} \right) \right)$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Whenever the values of $\alpha$ and $n$ are clear, we will omit these indices and write $\tau_{a,s,t}$ instead.

The Taylorlet transform w.r.t. $\tau$ of a tempered distribution $f \in S'(\mathbb{R}^2)$ is defined as

$$\mathcal{T}^{(n, \alpha)}_\tau f(a, s, t) = \left\{ f, \tau^{(n, \alpha)}_{a,s,t} \right\}.$$

In order to properly state the mapping properties of the Taylorlet transform, we now introduce the most important function spaces and their respective topologies.

**Definition 6** (Function spaces and topologies) Let $X$ be a topological vector space and let $X'$ be its dual space. A set $B \subset X$ is called bounded, if

$$\sup_{x \in B} |\langle x', x \rangle| < \infty$$

for all $x' \in X'$. 

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Let $\mathcal{B}$ denote the set of all bounded subsets of $X$. The strong dual topology of $X'$ is generated by the following family of semi-norms:
\[
\| \cdot \|_B : X' \to [0, \infty), \quad \| x' \|_B = \sup_{x \in B} |\langle x', x \rangle|, \quad \text{where } B \in \mathcal{B}.
\]
The Schwartz space is defined as
\[
\mathcal{S}(\mathbb{R}^d) = \left\{ \phi \in C^\infty(\mathbb{R}^d) \left| \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_\beta x \phi(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^d \right. \right\}.
\]
Its topology is generated by the semi-norms
\[
\| \cdot \|_{\alpha, \beta} : \mathcal{S}(\mathbb{R}^d) \to [0, \infty), \quad \| \phi \|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_\beta x \phi(x)|, \quad \text{where } \alpha, \beta \in \mathbb{N}^d.
\]
The space $S'(\mathbb{R}^d)$ of tempered distributions is the dual space of $\mathcal{S}(\mathbb{R}^d)$ and will be endowed with the strong dual topology.

The space $C^\infty(\mathbb{R}^d)$ of smooth functions is endowed with the topology generated by the semi-norms
\[
\| \cdot \|_{N,K} : C^\infty(\mathbb{R}^d) \to [0, \infty), \quad \| \phi \|_{N,K} = \sup_{|\beta| \leq N} \sup_{x \in K} |\partial_\beta x \phi(x)|,
\]
where $N \in \mathbb{N}$ and $K \subset \mathbb{R}^d$ is compact.

In the following proposition, we will show some basic properties of the Taylorlet transform.

**Proposition 1** (Properties of the Taylorlet transform) Let $\tau \in \mathcal{S}(\mathbb{R}^2)$ be an analyzing Taylorlet of order $n \in \mathbb{N}$. Let $\alpha > 0$ and let $V = \mathbb{R}_+ \times \mathbb{R}^{n+1} \times \mathbb{R}$ denote the parameter space of the Taylorlet transform. Then, the following statements hold:

1. For all $f \in S'(\mathbb{R}^2)$, the map $\mathcal{T}_\tau f : V \to \mathbb{C}$ is well-defined and linear.
2. For all $f \in S'(\mathbb{R}^2)$, we have $\mathcal{T}_\tau f \in C^\infty(V)$.
3. Let $S'(\mathbb{R}^2)$ be endowed with the strong dual topology. Then, the Taylorlet transform $\mathcal{T}_\tau : S'(\mathbb{R}^2) \to C^\infty(V)$ is a continuous linear operator.
4. Let $f \in L^2(\mathbb{R}^2)$ and let $\tau$ have at least one vanishing moment. If $\mathcal{T}_\tau f \equiv 0$, then $f \equiv 0$.

**Proof** 1. Since $f \in S'(\mathbb{R}^2)$ and $\tau_v \in \mathcal{S}(\mathbb{R}^2)$ for all $v \in V$, the Taylorlet transform
\[
\mathcal{T}_\tau f(v) = \langle f, \tau_v \rangle
\]
is well-defined. It is linear due to its definition as linear functional.

2. In order to prove that for all $f \in S'(\mathbb{R}^d)$ the Taylorlet transform of $f$ is smooth, we first show that the parameter derivatives of the Taylorlet are Schwartz functions, that is
\[
\partial_\beta v \tau_v \in \mathcal{S}(\mathbb{R}^2) \quad \text{for all } \beta \in \mathbb{N}^{n+3}.
\]
A simple computation yields the derivatives of $\tau_v$ for all $x \in \mathbb{R}^2$:

$$
\partial_a \tau_v (x) = -\frac{x_1 - \sum_{k=0}^{n} \frac{\delta_k A^k}{a^2 k^2}}{a^2} \cdot \partial_{x_1} \tau_v (x) - \frac{x_2 - t}{a^{\alpha + 1}} \cdot \partial_{x_2} \tau_v (x),
$$

$$
\partial_{x_k} \tau_v (x) = -\frac{x_k^2}{a \cdot k!} \cdot \partial_{x_1} \tau_v (x),
$$

$$
\partial_t \tau_v (x) = -\frac{1}{a^{\alpha}} \cdot \partial_{x_2} \tau_v (x).
$$

Since the derivatives with respect to $v$ are sums of products of polynomials in $x$ and derivatives of $\tau_v$ with respect to $x$, any derivative with respect to $v$ is a Schwartz function.

For $f \in S'(\mathbb{R}^2)$, any first-order partial derivative of the Taylorlet transform of $f$ can be represented as the limit of a difference quotient, that is

$$
\partial_{v_\ell} T_{\tau} f (v) = \lim_{h \to 0} \frac{T_{\tau} f (v + he_\ell) - T_{\tau} f (v)}{h}
$$

for $\ell \in \{1, \ldots, n + 3\}$, where $e_\ell$ denotes the $\ell$th unit vector. We obtain

$$
\partial_{v_\ell} T_{\tau} f (v) = \lim_{h \to 0} \frac{T_{\tau} f (v + he_\ell) - T_{\tau} f (v)}{h}

= \lim_{h \to 0} \frac{1}{h} \left[ \langle f, \tau_v + he_\ell \rangle - \langle f, \tau_v \rangle \right]

= \lim_{h \to 0} \left( f, \frac{\tau_v + he_\ell - \tau_v}{h} \right).
$$

We now prove that for all $f \in S'(\mathbb{R}^2)$, we have

$$
\lim_{h \to 0} \langle f, \Delta_{v_\ell, h} \tau_v \rangle = \langle f, \partial_{v_\ell} \tau_v \rangle.
$$

This coincides with the weak $S(\mathbb{R}^2)$-convergence

$$
\Delta_{v_\ell, h} \tau_v \rightharpoonup \partial_{v_\ell} \tau_v \quad \text{for} \ h \to 0.
$$

We will show the strong $S(\mathbb{R}^2)$-convergence of this function sequence, that is, the convergence in the semi-norms $|| \cdot ||_{\alpha, \beta}$ for $\alpha, \beta \in \mathbb{N}^2$, which implies the weak convergence. For this, we utilize Taylor’s theorem which yields for all $h > 0$ and $x \in \mathbb{R}^2$:

$$
\Delta_{v_\ell, h} \tau_v (x) - \partial_{v_\ell} \tau_v (x) = \frac{1}{h} \int_{v_\ell}^{v_\ell + h} \partial_{v_\ell}^2 \tau_v (x) (v_\ell - \tilde{v}_\ell) d\tilde{v}_\ell,
$$

(1)
where $\tilde{v} = (v_1, \ldots, v_{\ell-1}, \tilde{v}_\ell, v_{\ell+1}, \ldots, v_{n+3})$. We thus obtain for each $\alpha, \beta \in \mathbb{N}^2$

$$\|\Delta_{\nu, h} \tau_v - \partial_{\nu} \tau_v\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^2} \left| x^\alpha \partial^\beta \left( \Delta_{\nu, h} \tau_v(x) - \partial_{\nu} \tau_v(x) \right) \right|$$

$$= \sup_{x \in \mathbb{R}^2} \left| x^\alpha \partial^\beta \frac{1}{h} \int_{v_\ell}^{v_\ell+h} \partial^2_{v_\ell} \tau_v(x) (v_\ell - \tilde{v}_\ell) d\tilde{v}_\ell \right|$$

$$\leq \sup_{x \in \mathbb{R}^2} \left| \int_{v_\ell}^{v_\ell+h} x^\alpha \partial^\beta \partial^2_{v_\ell} \tau_v(x) d\tilde{v}_\ell \right|$$

$$\leq h \cdot \sup_{x \in \mathbb{R}^2} \sup_{\tilde{v}_\ell \in [v_\ell, v_\ell+h]} \left| x^\alpha \partial^\beta \partial^2_{v_\ell} \tau_v(x) \right|.$$ 

Since $\partial^\gamma \tau_v \in S(\mathbb{R}^2)$ for all $\gamma \in \mathbb{N}^{n+3}$, we have

$$\sup_{x \in \mathbb{R}^2} \sup_{\tilde{v}_\ell \in [v_\ell, v_\ell+h]} \left| x^\alpha \partial^\beta \partial^2_{v_\ell} \tau_v(x) \right| < \infty$$

for all $h \in K$, where $K \subset [0, \infty)$ bounded. Hence, we obtain for all $\alpha, \beta \in \mathbb{N}^2$:

$$\lim_{h \to 0} \|\Delta_{\nu, h} \tau_v - \partial_{\nu} \tau_v\|_{\alpha, \beta} = 0.$$ 

Consequently, $\partial^\gamma \tau_v f(v) = \langle f, \partial^\gamma \tau_v \rangle$ for all $\ell \in \{1, \ldots, n+3\}$ and so the partial derivatives exist. This argument can be iterated to show that any derivative of the Taylorlet transform $\tau f$ exists for all $f \in S'(\mathbb{R}^2)$ and thus $\tau f$ is smooth.

3. Since the topology of $C^\infty(V)$ is generated by the semi-norms $\| \cdot \|_{N, K}$, it suffices to show that for all $N \in \mathbb{N}$, $K \subset V$ compact and $\varepsilon > 0$, there exists an environment $U \subset S'(\mathbb{R}^2)$ of 0 such that

$$\sup_{|\beta| \leq N} \sup_{v \in K} |\partial^\beta \tau_v f(v)| < \varepsilon$$

for all $f \in U$. We will now show that the sets

$$T_{\beta, K} = \{\partial^\beta \tau_v : v \in K\}$$

are bounded for all $\beta \in \mathbb{N}^{n+3}$. For this, let $f \in S'(\mathbb{R}^2)$. Then,

$$\sup_{\phi \in T_{\beta, K}} |\langle f, \phi \rangle| = \sup_{v \in K} |\langle f, \partial^\beta \tau_v \rangle| < \infty,$$

since $\partial^\beta \tau_v \in S(\mathbb{R}^2)$ and $K \subset V$ is compact. Consequently, the sets $T_{\beta, K}$ are bounded and so is $T_{N, K} = \bigcap_{|\beta| \leq N} T_{\beta, K}$ for $N \in \mathbb{N}$. According to the definition of the strong dual topology, the set $T_{N, K}$ induces a semi-norm in the strong topology of $S'(\mathbb{R}^2)$ by

$$\| \cdot \|_{T_{N, K}} : S'(\mathbb{R}^2) \to \mathbb{C}, \quad \| f \|_{T_{N, K}} = \sup_{\phi \in T_{N, K}} |\langle f, \phi \rangle|.$$ 

Thus, the set

$$U_{N, K, \varepsilon} = \{ f \in S'(\mathbb{R}^2) : \| f \|_{T_{N, K}} < \varepsilon \}$$
is open in the strong dual topology of $S'(\mathbb{R}^2)$ and for all $f \in U_{N,K,\epsilon}$ we have
\[
\sup_{\beta \leq N} \sup_{v \in K} |\partial^\beta \tau \tau f(v)| = \sup_{|\beta| \leq N} \sup_{v \in K} |\langle f, \partial^\beta \tau \tau v \rangle| \\
= \sup_{\phi \in T_{N,K}} |\langle f, \phi \rangle| = \| f \|_{T_{N,K}} < \epsilon.
\]

4. If $\tau \tau f \equiv 0$, we especially have
\[
\tau \tau f(a,s,t) = 0 \quad \text{for all } a > 0, s, t \in \mathbb{R} \text{ and for } s_2 = \ldots = s_n = 0.
\]
In this situation, the Taylorlet transform reduces to a shearlet transform utilizing an $\alpha$-scaling matrix. As shown in [4], a shearlet transform of this type offers a reconstruction formula for $f \in L^2(\mathbb{R}^2)$. As $\tau$ has at least one vanishing moment of order 1, it is an admissible shearlet and yields the following reconstruction formula for $s_2 = \ldots = s_n = 0$:
\[
f(x) = \frac{1}{C_{\tau}} \int_0^\infty \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \tau \tau f(a,s,t) \tau \tau \tau \tau (x) \frac{da}{a^{n/2}} \, dt \, ds_0 \, ds_1 \quad \text{f. a. a. } x \in \mathbb{R}^2.
\]
Thus, we obtain $f \equiv 0$.

We want to distinguish between the right choice of shearing parameters $s = (s_0, \ldots, s_n)$ which are those corresponding to the Taylor coefficients of $q$, and the incorrect ones by comparing the respective decay rates of the Taylorlet transform. We will show that, if the choice is incorrect, the Taylorlet transform decays fast because of the vanishing moments of higher order. The restrictiveness on the other hand makes sure that the Taylorlet transform decays slowly at a correct choice of shearing parameters.

3 Construction of a Taylorlet

In the setting of the continuous shearlet transform the vanishing moment property w.r.t. the $x_1$–direction is inherently given by the definition of a classical shearlet $\psi$ [6]
\[
\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1) \cdot \hat{\psi}_2 \left( \frac{\xi_2}{\xi_1} \right),
\]
since $0 \notin \text{supp}(\hat{\psi}_1)$. In the Taylorlet setting, the function $g$ essentially takes over the role of $\psi_1$. Unfortunately, vanishing moments of $g$ alone are not sufficient for the construction of an analyzing Taylorlet, since we additionally need vanishing moments of $g(\pm t^k)$ for all $k \leq n$. This complicates the situation as we cannot rely on a construction via the Fourier transform.

We will first present a construction procedure and images of each construction step starting from the exemplary function $\phi(t) = e^{-t^2}$. Later, we prove that the resulting function is a restrictive Taylorlet of order $n$ with $r$ vanishing moments. The remarks in the steps of the general setup can be understood as a guideline for the aforementioned proof. The respective properties will only be shown later.
General setup

I. We start with an even function \( \phi \in S(\mathbb{R}) \) fulfilling
\[
\phi^{(k)}(0) \neq 0 \iff k \mod 2 = 0.
\]
This condition is necessary for the Taylorlet to be a Schwartz function. For instance, we can choose \( \phi(t) = e^{-t^2} \).

II. Let \( v_n \) be the least common multiple of the numbers 1, \ldots, \( n \). We define
\[
\phi_n(t) := \phi(t^v_n) \quad \text{for all } t \in \mathbb{R}.
\]
This function is still in \( S(\mathbb{R}) \) and fulfills
\[
\phi_n^{(k)}(0) \neq 0 \iff k \mod 2v_n = 0.
\]

III. We define, for each \( r \in \mathbb{N} \),
\[
\phi_{n,r} := \frac{1}{(2rv_n)!} \cdot \phi_n^{(2rv_n)}.
\]
This function has \( 2rv_n \) vanishing moments since each derivative generates one further vanishing moment. Furthermore, the function is also in \( S(\mathbb{R}) \) and fulfills
\[
\phi_{n,r}^{(k)}(0) \neq 0 \iff k \mod 2v_n = 0.
\]

IV. We define, for each \( r \in \mathbb{N} \)
\[
\tilde{\phi}_{n,r}(t) = \phi_{n,r} \left( |t|^{1/v_n} \right) \quad \text{for all } t \in \mathbb{R}.
\]
The concatenation with the function $| \cdot |^{1/n}$ ensures that $\tilde{\phi}_{n,r}$ has vanishing moments of order $n$. The function $\tilde{\phi}_{n,r}$ fulfills

$$\tilde{\phi}_{n,r}^{(k)}(0) \neq 0 \iff k \mod 2 = 0.$$ 

So, it is smooth despite the singularity of $| \cdot |^{1/\nu}$ and belongs to $S(\mathbb{R})$, as well.

V. For all $t \in \mathbb{R}$, we define

$$g(t) = (1 + t)\tilde{\phi}_{n,r}(t).$$

This step guarantees that the necessary properties of $g$ for the restrictiveness are fulfilled. Furthermore, $g \in S(\mathbb{R})$.

VI. We choose a function $h \in S(\mathbb{R})$ such that $\int_{\mathbb{R}} h(t)dt \neq 0$ and define the Taylorlet $\tau := g \otimes h$. Since $g, h \in S(\mathbb{R})$, we have $\tau \in S(\mathbb{R}^2)$ (Fig. 1).

In the following theorem we will prove some properties of the function $\tau$ generated by steps I–VI above.

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Fig. 1 Plots of the Taylorlet example, starting from the function $\phi(t) := e^{-t^2}$, $g(t) := (1 + t) \cdot \tilde{\phi}_{2,2}(t)$, $h(t) := e^{-t^2}$, given in (20). Increasing curvature from left to right with $s_2 = 0$ (left), $s_2 = 1$ (center), $s_2 = 2$ (right).
Theorem 1 Let \( r, n \in \mathbb{N} \). The function \( \tau \) described in the general setup exhibits the following properties:

i) \( \tau \in S(\mathbb{R}^2) \).

ii) \( \tau \) is an analyzing Taylorlet of order \( n \) with \( 2r - 1 \) vanishing moments.

iii) \( \tau \) is restrictive.

Proof i) Since \( \tau = g \otimes h \) and as \( h \in S(\mathbb{R}) \) due to the general setup, we only need to prove that \( g \in S(\mathbb{R}) \). To this end, we show that the Schwartz properties are consecutively passed on to the next function through every step of the general setup.

First, we observe that with the change of the argument \( \phi_n := \phi(\cdot)^{vn} \) in step II the Schwartz properties of \( \phi \) remain. Furthermore, we obtain with the condition from step I that

\[
\phi_n^{(k)}(0) \neq 0 \iff k \mod 2v_n = 0.
\]

This property is invariant under the action of step III, i.e.,

\[
0 \neq \phi_{n,r}^{(k)}(0) = \phi_n^{(2rv_n+k)}(0) \iff k \mod 2v_n = 0
\]

for all \( r \in \mathbb{N} \). After step IV, the function \( \tilde{\phi}_{n,r} := \phi_{n,r} \left( |\cdot|^{1/v_n} \right) \) is clearly smooth on every set not containing the origin. In order to show the smoothness of \( \tilde{\phi}_{n,r} \) in the origin, we use Taylor’s theorem to approximate \( \phi_{n,r} \) by a Taylor polynomial. We thus obtain that

\[
\phi_{n,r}(t) = \sum_{k=0}^{K} \frac{\phi_n^{(2kv_n)}(0)}{(2kv_n)!} \cdot t^{2kv_n} + o\left(t^{2Kv_n}\right) \quad \text{for } t \to 0.
\]

Hence, we can approximate \( \tilde{\phi}_{n,r} \) by a sequence of polynomials, as well.

\[
\tilde{\phi}_{n,r}(t) = \phi_{n,r} \left( |t|^{1/v_n} \right) = \sum_{k=0}^{K} \frac{\phi_n^{(2kv_n)}(0)}{(2kv_n)!} \cdot t^{2kv_n} + o\left(t^{2Kv_n}\right) \quad \text{for } t \to 0.
\]

Consequently, \( \tilde{\phi}_{n,r} \) is smooth and inherits the Schwartzian decay property of \( \phi_{n,r} \). Hence, \( \tilde{\phi}_{n,r} \in S(\mathbb{R}) \). In the last step, we get that

\[
g(t) := (1 + t) \cdot \tilde{\phi}_{n,r}(t)
\]

is a Schwartz function.

ii) We will prove this statement in three steps. First, we will show that \( \phi_{n,r} \) has \( 2rv_n \) vanishing moments. In a second step, we will prove that \( \tilde{\phi}_{n,r} \) has \( 2r \) vanishing moments of order \( n \), and in the last part, we show that \( g \) has \( 2r - 1 \) vanishing moments of order \( n \).

STEP 1

As shown in the proof of i), \( \phi_n, \phi_{n,r} \in S(\mathbb{R}) \). Hence, their Fourier transforms exist and we obtain

\[
\tilde{\phi}_{n,r}(\omega) = \left( \phi_n^{(2rv_n)} \right)^\wedge (\omega) = \frac{1}{(2rv_n)!} \cdot (-1)^{rv_n} \omega^{2rv_n} \phi_n(\omega).
\]
Consequently, \( \tilde{\phi}_{n,r} \) has a root of order at least \( 2r v_n \) in the origin and hence \( \phi_{n,r} \) has at least \( 2r v_n \) vanishing moments.

**STEP 2**

We now prove that \( \tilde{\phi}_{n,r} := \phi_{n,r} \left( |\cdot|^{1/v_n} \right) \) has \( 2r \) vanishing moments of order \( n \). Now let \( k \in \{1, \ldots, n\} \). Then,

\[
\int_{\mathbb{R}} \tilde{\phi}_{n,r}(\pm t^k) t^m dt = \int_{\mathbb{R}} \phi_{n,r}(|t|^{k/v_n}) t^m dt
\]

\[
= \int_{\mathbb{R}} \phi_{n,r}(|u|) u^{mv_n/k} \frac{v_n}{k} u^{v_n/k-1} du
\]

\[
= \frac{v_n}{k} \int_{\mathbb{R}} \phi_{n,r}(u) u^{(m+1)v_n/k-1} du.
\]

Since \( v_n \) is the least common multiple, \( v_n/k \in \mathbb{N} \) and thus the integral vanishes for all \( m < 2k \cdot r \). That is, \( \tilde{\phi}_{n,r} \) has \( 2r \) vanishing moments of order \( n \).

**STEP 3**

Now, we will show that \( g \) has \( 2r - 1 \) vanishing moments of order \( n \). For all \( k \in \{1, \ldots, n\} \) and for all \( m \in \mathbb{N} \), we have

\[
\int_{\mathbb{R}} g \left( \pm t^k \right) t^m dt = \int_{\mathbb{R}} \left( 1 \pm t^k \right) \tilde{\phi}_{n,r}(\pm t^k) t^m dt
\]

\[
= \int_{\mathbb{R}} \tilde{\phi}_{n,r}(\pm t^k) t^m dt + \int_{\mathbb{R}} \tilde{\phi}_{n,r}(\pm t^k) t^{k+m} dt.
\]

Due to the result of **STEP 2**, this expression vanishes if \( m + k < 2k \cdot r \). Hence, \( g \) has \( 2r - 1 \) vanishing moments of order \( n \) and \( \tau \) is an analyzing Taylorlet of order \( n \) with \( 2r - 1 \) vanishing moments.

**iii)** In order to prove that \( \tau = g \otimes h \) is restrictive, it is sufficient to show that

\[
I_j^i g(0) \neq 0 \quad \text{for all } j \in \{0, \ldots, 2r - 1\}
\]

since \( \int_{\mathbb{R}} h(t) dt \neq 0 \) is already given in step VI of the general setup. This property will be shown in two steps. First, we will prove the sufficiency of

\[
\int_{0}^{\infty} \tilde{\phi}_{n,r}(t) t^{2m+1} dt \neq 0
\]

for all \( m \in \{0, \ldots, r - 1\} \). Afterwards, we will reduce this property to the already proven property that

\[
\phi_{n,r}^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \mod 2v_n = 0.
\]
STEP 1

Due to a well-known formula for iterated integrals, we obtain

\[ (j - 1)! \cdot I_j^i g(0) = \int_0^\infty g(t) t^{j-1} dt \]
\[ = \int_0^\infty \tilde{\phi}_{n,r}(t) (1 + t) t^{j-1} dt \]
\[ = \int_0^\infty \tilde{\phi}_{n,r}(t) t^{j-1} dt + \int_0^\infty \tilde{\phi}_{n,r}(t) t^j dt. \]

(2)

Since \( \tilde{\phi}_{n,r} \) is an even function with 2r vanishing moments, we obtain for \( j < r \) that

\[ \int_0^\infty \tilde{\phi}_{n,r}(t) t^{2j} dt = \frac{1}{2} \int_{\mathbb{R}} \tilde{\phi}_{n,r}(t) t^{2j} dt = 0. \]

(3)

Hence, we can conclude for the iterated integral of \( g \) that

\[ (j - 1)! \cdot I_j^i g(0) = \begin{cases} 
\int_0^\infty \tilde{\phi}_{n,r}(t) t^{j-1} dt, & \text{if } j \mod 2 = 0, \\
\int_0^\infty \tilde{\phi}_{n,r}(t) t^j dt, & \text{if } j \mod 2 = 1.
\end{cases} \]

Since \( g \) has \( 2r - 1 \) vanishing moments,

\[ 0 = \int_{\mathbb{R}} g(t) t^j dt = \int_{-\infty}^0 g(t) t^j dt + \int_0^\infty g(t) t^j dt = I_{-}^{j+1} g(0) + I_{+}^{j+1} g(0) \]

for all \( j \in \{0, \ldots, 2r - 1\} \). Hence, the statements \( I_{+}^{j} g(0) \neq 0 \) and \( I_{-}^{j} g(0) \neq 0 \) are equivalent. With the equations (2) and (3), we then obtain that \( I_{+}^{j} g(0) \neq 0 \) for all \( j \in \{0, \ldots, 2r - 1\} \) is equivalent to

\[ \int_0^\infty \tilde{\phi}_{n,r}(t) t^{2j+1} dt \neq 0 \quad \text{for all } j \in \{0, \ldots, r - 1\}. \]

STEP 2

\[ \int_0^\infty \tilde{\phi}_{n,r}(t) t^{2m+1} dt = \int_0^\infty \phi_{n,r} \left( \frac{1}{v_n} \right) t^{2m+1} dt \]
\[ = v_n \cdot \int_0^\infty \phi_{n,r}(u) u^{v_n(2m+1)} \cdot u^{v_n-1} du \]
\[ = v_n \cdot \int_0^\infty \phi_{n,r}(u) u^{2v_n(m+1)-1} du \]
\[ = v_n \cdot \int_0^\infty \phi_{n,r}^{(2v_n)}(u) u^{2v_n(m+1)-1} du \]

part. int.
\[ = [2v_n(m + 1) - 1]! \cdot v_n \cdot \int_0^\infty \phi_n^{(2v_n[r-m-1]+1)}(u) du \]
\[ = -[2v_n(m + 1) - 1]! \cdot v_n \cdot \phi_n^{(2v_n[r-m-1])}(0). \]
The last expression does not vanish for any \( m \in \{0, \ldots, r - 1\} \) because \( \phi_n^k(0) \neq 0 \iff k \mod 2v_n = 0 \). Hence,

\[
\int_0^\infty \tilde{\phi}_{n,r}(t)t^{2m+1}dt \neq 0 \quad \text{for all} \ m \in \{0, \ldots, r - 1\}.
\]

Due to STEP 1, we can conclude that \( \tilde{I}_+g(0) \neq 0 \) for all \( j \in \{0, \ldots, r - 1\} \). \( \square \)

**Remark 1** The sequence \( v_n \) of the least common multiples of the numbers \( 1, \ldots, n \) is innately connected to the second Chebyshev function which plays a crucial role for the prime number theorem. The second Chebyshev function is defined as

\[
\psi(x) = \sum_{p \in \mathbb{P}, k \in \mathbb{N}} \log p.
\]

It is related to the sequence \( v_n \) by the equation \( v_n = e^{\psi(n)} \). The prime number theorem can be proven via a connection to \( \psi \) and its asymptotic behavior

\[
\lim_{x \to \infty} \frac{\psi(x)}{x} = 1.
\]

This limit also provides an asymptotic estimate for \( v_n \):

\[
\lim_{n \to \infty} \frac{\log v_n}{n} = 1.
\]

### 4 Main result

We first introduce the class of feasible functions which is used in the main result.

**Definition 7** (Feasible function, singularity function) Let \( \delta \) denote the Dirac distribution. Furthermore, let \( j \in \mathbb{N}, q \in C^\infty(\mathbb{R}) \) and let

\[
f(x) := \tilde{I}_+^j\delta(x_1 - q(x_2)).
\]

Then \( f \) is called a \( j \)-feasible function with singularity function \( q \).

The variable \( j \) describes the smoothness of \( f \). In terms of smoothness, we obtain for \( j \geq 1 \), that \( f \in C^{j-1}(\mathbb{R}^2) \). For instance, by choosing \( q(x_2) = x_2^2 \) for all \( x_2 \in \mathbb{R} \) and \( j \geq 1 \), we obtain the function

\[
f(x) = \frac{(x_1 - x_2)^{j-1}}{(j-1)!} \cdot H\left(\pm(x_1 - x_2^2)\right),
\]

where \( H: \mathbb{R} \to \mathbb{R}, t \mapsto 1_{\mathbb{R}^+}(t) \) is the Heaviside step function (Fig. 2).

In order to classify the shearing variables w.r.t. the local properties of the singularity function \( q \), we introduce the concept of the highest approximation order.

**Definition 8** (Highest approximation order) Let \( j, n \in \mathbb{N} \) and let \( f \) be a \( j \)-feasible function with singularity function \( q \). Furthermore, let \( \alpha > 0, t \in \mathbb{R}, \alpha > 0 \) and \( k \in \{0, \ldots, n - 1\} \). If \( s_\ell = q^{(\ell)}(t) \) for all \( \ell \in \{0, \ldots, k\} \) and \( s_{k+1} \neq q^{(k+1)}(t) \), we
say that $k$ is the highest approximation order of the shearing variable $s = (s_0, \ldots, s_n)$ for $f$ in $t$.

The following theorem states the main result of this article and treats the classification of the Taylorlet transform’s decay w.r.t. the highest approximation order.

**Theorem 2** Let $r, n \in \mathbb{N}$ and let $\tau$ be an analyzing Taylorlet of order $n$ with $r$ vanishing moments. Let furthermore $j < r$, $t \in \mathbb{R}$ and let $f$ be a $j$-feasible function.

1. Let $\alpha > 0$. If $s_0 \neq q(t)$, the Taylorlet transform has a decay of
   \[
   \mathcal{T}^{(n, \alpha)} f(a, s, t) = O(a^N) \quad \text{for } a \to 0
   \]
   for all $N > 0$.

2. Let $\alpha < \frac{1}{n}$ and let $k \in \{0, \ldots, n-1\}$ be the highest approximation order of $s$ for $f$ in $t$. Then, the Taylorlet transform has the decay property
   \[
   \mathcal{T}^{(n, \alpha)} f(a, s, t) = O\left( a^{j-1+(r-j)[1-(k+1)\alpha]} \right) \quad \text{for } a \to 0.
   \]

3. Let $\alpha > \frac{1}{n+1}$ and let $\tau$ be restrictive. If $n$ is the highest approximation order of $s$ for $f$ in $t$, then the Taylorlet transform has the decay property
   \[
   \mathcal{T}^{(n, \alpha)} f(a, s, t) \sim a^{j-1} \quad \text{for } a \to 0.
   \]

The strategy for the proof of this theorem consists of multiple reductions to simpler cases. In the proof of Theorem 2, we show that it is sufficient to consider 0-feasible functions, i.e., functions of the form $f(x) = \delta(x_1 - q(x_2))$ for $x \in \mathbb{R}^2$. In Lemma
we then prove that all cases of the Taylorlet transform can be reduced to a linear combination of integrals of the form
\[
\int_{\mathbb{R}} \partial^m \tau \left( \frac{z^k}{t} \right) t^{km+\ell} dt,
\]
(4)
where $\ell, m \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. In order to obtain the decay rate of the Taylorlet transform, we have to determine the behavior of the integrals (4) for $z \to \pm \infty$. In Lemma 2, we show that we can ensure a fast decay of the integrals (4) for $z \to \pm \infty$ by imposing vanishing moment conditions of higher order to the analyzing Taylorlet $\tau \in S(\mathbb{R}^2)$. Thus, the vanishing moments of higher order are the key feature to the fast decay.

**Remark 2** At this point, we want to highlight the role of the vanishing moments of higher order for the classification result. Suppose, we wish to analyze the example function
\[
f(x) = \delta \left( x_1 - \frac{c}{2} x_2^2 \right)
\]
and intend to find out the curvature of its singular support in the origin with the help of the Taylorlet transform. We furthermore assume that the analyzing Taylorlet $\tau = g \otimes h$ is only of order 1, and that $g$ has infinitely many vanishing moments of order 1, but no vanishing moments of order 2, i.e.,
\[
\int_{\mathbb{R}} g(x_1) x_1^m dx_1 = 0 \quad \text{for all } m \in \mathbb{N} \quad \text{and} \quad \int_{\mathbb{R}} g(\pm x_1^2) dx_1 \neq 0.
\]

Case 1: $s_0 \neq 0$ or $s_1 \neq 0$:
Then, the theory of the shearlet transform delivers for $\alpha \in (0, 1)$ that
\[
\mathcal{T}^{(2, \alpha)} f(a, s, 0) = \int_{\mathbb{R}^2} \tau \left( \left[ x_1 - \frac{s_2}{2} x_2^2 - s_1 x_2 - s_0 \right] / a \right) \delta \left( x_1 - \frac{c}{2} x_2^2 \right) dx = \mathcal{O}(a^N),
\]
for $a \to 0$ for all $N \in \mathbb{N}$. Yet, we would also like to have that
\[
\mathcal{T}^{(2, \alpha)} f(a, s, 0) = \mathcal{O}(a^N), \quad \text{for } a \to 0
\]
for some large $N \in \mathbb{N}$, if $s_0 = 0, s_1 = 0$ and $s_2 \neq c$.

Case 2: $s_0 = 0, s_1 = 0$ and $s_2 \neq c$:
If $g$ does not have vanishing moments of second order, we obtain for $\alpha \in \left(0, \frac{1}{2}\right)$ that
\[
\mathcal{T}^{(2, \alpha)} f(a, s, 0) = \int_{\mathbb{R}^2} \tau \left( \left[ x_1 - \frac{s_2}{2} x_2^2 \right] / x_2 / a^\alpha \right) \delta \left( x_1 - \frac{c}{2} x_2^2 \right) dx
\]
\[
= \int_{\mathbb{R}} \tau \left( \frac{c-s_2}{2} \cdot x_2 / a^\alpha \right) dx_2 \quad \text{(substituting } x_2 = a^\alpha u) \]
\[
= a^\alpha \int_{\mathbb{R}} g \left( a^{2\alpha-1} \cdot \frac{c-s_2}{2} \cdot u^2 \right) h(u) du.
\]

By defining the function
\[
g_{\pm, 2} : \mathbb{R} \to \mathbb{R}, u \mapsto g(\pm u^2)
\]
and applying Plancherel’s theorem to the last integral, we obtain
\[
\mathcal{T}^{(2,\alpha)} f(a, s, 0) = \frac{\sqrt{a}}{\sqrt{2|c-s_2|} \pi} \cdot \int_{\mathbb{R}} \hat{g}_{\text{sgn}(c-s_2), 2} (\sqrt{2|c-s_2|} - \frac{1}{2} a^{\frac{1}{2} - \alpha} \omega) \hat{h}(\omega) d\omega.
\]
Due to \(g, h \in S(\mathbb{R})\), \(\alpha < \frac{1}{2}\) and the dominated convergence theorem, we get
\[
\lim_{a \to 0} a^{-\frac{1}{2}} \mathcal{T}^{(2,\alpha)} f(a, s, 0) = \frac{1}{\sqrt{2|c-s_2|} \pi} \cdot \int_{\mathbb{R}} \hat{g}_{\text{sgn}(c-s_2), 2}(0) \hat{h}(\omega) d\omega
= \int_{\mathbb{R}} g(\text{sgn}(c-s_2) u^2) du.
\]
Hence, for \(h(0) \neq 0\), e.g., for \(h(u) = e^{-u^2}\) from the example in the general setup, we obtain that
\[
\mathcal{T}^{(2,\alpha)} f(a, s, 0) \sim \sqrt{a} \quad \text{for } a \to 0. \quad (5)
\]
Case 3: \(s_0 = 0, s_1 = 0\) and \(s_2 = c\):
We obtain
\[
\mathcal{T}^{(2,\alpha)} f(a, s, 0) = \int_{\mathbb{R}} \tau \left( \frac{0}{x_2/a^\alpha} \right) dx_2 = a^{\alpha} g(0) \cdot \int_{\mathbb{R}} h(u) du.
\]
Thus, for \(g(0) \neq 0\) and \(\int_{\mathbb{R}} h(u) du \neq 0\), we have
\[
\mathcal{T}^{(2,\alpha)} f(a, s, 0) \sim a^{\alpha} \quad \text{for } a \to 0. \quad (6)
\]
Due to statement 3 of Theorem 2, we need \(\alpha > \frac{1}{4}\) for the detection of the curvature. Hence, the ratio of the decay rates for \(c = s_2\) (6) and \(c \neq s_2\) (5) is \(a^{\frac{1}{2} - \alpha}\) and hence at best \(a^{\frac{1}{6}}\). Detecting this difference can become difficult in numerical practice without vanishing moments of higher order. As identifying this difference in the decay rates is necessary for the detection of the edge curvature, the task of determining the local edge curvature thus might get numerically unstable.

Remark 3 It is reasonable to emphasize the significance of the restrictiveness for the Taylorlet transform. This property ensures that a Taylorlet transform of order \(n\) decays slowly if the highest approximation order is \(n\). If a Taylorlet lacks the restrictiveness, we can construct an example function whose Taylorlet transform is equal to zero for all \(a > 0\) if the highest approximation order is \(n\).

Let \(\tau = g \otimes h\) be a Taylorlet of order \(n\) with \(r\) vanishing moments such that there is a \(k \in \mathbb{N}\) with
\[
\int_0^\infty g(t)t^k dt = 0.
\]
Furthermore, let \(\alpha \in \left(\frac{1}{n+1}, \frac{1}{n}\right)\) and
\[
f(x) := x_1^k \mathbb{1}_{\mathbb{R}_+}(x_1).
\]
Then, we obtain for the Taylorlet transform of \( f \) that
\[
\mathcal{T}^{(n,\alpha)}_{\tau} f(a, 0, 0) = \int_{\mathbb{R}^2} f(x) \cdot \tau \left( \frac{x_1}{a}, \frac{x_2}{d^\alpha} \right) dx
\]
\[
= a^{1+\alpha} \cdot \int_{\mathbb{R}} (ay_1)^k \mathbb{I}_{\mathbb{R}^+}(y_1) \cdot g(y_1) dy_1 \cdot \int_{\mathbb{R}} h(y_2) dy_2
\]
\[
= a^{k+1+\alpha} \cdot \int_{0}^{\infty} y_1^k \cdot g(y_1) dy_1 \cdot \int_{\mathbb{R}} h(y_2) dy_2 = 0.
\]

Remark 4 Furthermore, we want to stress the importance of the choice of \( \alpha \) for the Taylorlet transform.

As the general setup involves the least common multiple \( v_n \) of the numbers 1, \ldots, \( n \), it is possible that the order of the Taylorlet is higher than originally intended. For instance, consider an analyzing Taylorlet \( \tau \) of order 5. When built according to the general setup, we have \( v_5 = v_6 = 60 \). To this end, Theorem 1 states that \( \tau \) is also an analyzing Taylorlet of order 6.

The problems that arise from a wrong choice of \( \alpha \) become clear when we consider a case where \( \alpha < \frac{1}{6} \), \( f \) is a \( j \)-feasible function and \( \tau \) is the analyzing Taylorlet of order 5 (and 6) described above. If the highest approximation order of \( s \in \mathbb{R}^6 \) is 5, we can treat the Taylorlet transform \( \mathcal{T}^{(5,\alpha)} f(a, s, t) \) like \( \mathcal{T}^{(6,\alpha)} f(a, \sigma, t) \), where \( \sigma = (s_0, \ldots, s_5, 0) \). We are allowed to do so, because for all \( k \in \mathbb{N} \) we can write every shearing operator \( S^{(k)}_s \) of order \( k \) as shearing operator \( S^{(k+1)}_{s'} \) where \( s' = (s_0, \ldots, s_k, 0) \). Let \( t \in \mathbb{R} \). If the highest approximation order of \( \sigma \) for \( f \) in \( t \) is 5, all conditions of case 2 of Theorem 2 are met and so the Taylorlet transform has a decay of \( \mathcal{O} \left( a(r-j)(1-6\alpha)^{-1} \right) \) for \( a \rightarrow 0 \). This can be significantly faster than the decay of \( \sim a^{j-1} \) for \( a \rightarrow 0 \) which occurs for the choice of \( \alpha > \frac{1}{6} \).

5 Proof of the main result

In order to prove Theorem 2, we need the following auxiliary results.

Lemma 1 Let \( f \in C(\mathbb{R}) \) such that for all \( n \in \mathbb{N} \) there exists a constant \( c_n \in \mathbb{R}_+ \) with
\[
\sup_{t \in \mathbb{R}} |t^n \cdot f(t)| = c_n < \infty.
\]
Then,
\[
\int_{\mathbb{R}\setminus[-a^\beta, a^\beta]} f(t/a) dt = \mathcal{O}(a^N) \quad \text{for } a \rightarrow 0
\]
for all \( \beta < 1 \) and \( N \in \mathbb{N} \).

Proof By applying the decay condition, we obtain for \( n > 1 \)
\[
\left| \int_{\mathbb{R}\setminus[-a^\beta, a^\beta]} f(t/a) dt \right| \leq 2c_n \int_{a^\beta}^\infty (a/t)^n dt = \frac{2c_n}{n-1} a^{(1-\beta)n+\beta}.
\]
Since we can choose \( n \in \mathbb{N} \) arbitrarily large and since \( \beta < 1 \), we get the desired result. \( \square \)

The next lemma provides a relation between the vanishing moments of order \( n \) and the decay rate of integrals over the graph of a monomial. This is of great importance as the Taylorlet transform of a feasible function can be represented as a sum over integrals of this type.

**Lemma 2** \( \text{Let } r, n \in \mathbb{N} \text{ and let } \tau \text{ be an analyzing Taylorlet of order } n \text{ with } r \text{ vanishing moments. Then for all } \ell, m \in \mathbb{N} \text{ and for all } k \in \{1, \ldots, n\}, \text{we have} \)

\[
\int_{\mathbb{R}} \partial_1^m \tau \left( \frac{z \cdot t^k}{t} \right) t^{km + \ell} \, dt = \mathcal{O} \left( |z|^{-(m+r)} \right) \text{ for } z \to \pm \infty. \tag{7}
\]

**Proof** The idea is to represent the integral in (7) as a Fourier transform, to utilize the separation approach \( \tau = g \otimes h \), and to show the decay result via the Fourier transforms of \( g \) and \( h \). We define the function

\[
\hat{\tau}_k(z, \omega) := \int_{\mathbb{R}} \tau \left( \frac{z \cdot t^k}{t} \right) e^{-it\omega} \, dt.
\]

Then, we can rewrite the left side of (7) into

\[
\int_{\mathbb{R}} \partial_1^m \tau \left( \frac{z \cdot t^k}{t} \right) t^{km + \ell} \, dt = i\ell \partial_\omega \partial_m \hat{\tau}_k(z, 0). \tag{8}
\]

For \( k \in \mathbb{N} \), we introduce the function

\[
g_{\pm,k} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto g(\pm t^k).
\]

As the vanishing moment property can be rewritten as

\[
\int_{\mathbb{R}} g_{\pm,k}(t)t^m \, dt = 0 \quad \text{for all } m \in \{0, \ldots, kr - 1\}, \ k \in \{1, \ldots, n\},
\]

we can conclude that

\[
g_{\pm,k}^{(v)}(0) = 0 \quad \text{for all } v \in \{0, \ldots, kr - 1\}, \ k \in \{1, \ldots, n\}.
\]

Consequently, we get the decay rate

\[
g_{\pm,k}^{(v)}(\omega) = \mathcal{O}(\omega^{kr-1-v}) \quad \text{for } \omega \to 0. \tag{9}
\]

By utilizing the convolution theorem, we now obtain

\[
\hat{\tau}_k(z, \omega) = \frac{1}{2\pi} \left( \frac{1}{|z|^{1/k}} \left( g_{\text{sgn}(z),k} \right)^\wedge \left( \frac{\cdot}{|z|^{1/k}} \right) * \hat{h} \right)(\omega)
\]

and hence

\[
\partial_\omega \hat{\tau}_k(z, 0) = \frac{1}{2\pi |z|^{1/k}} \int_{\mathbb{R}} \left( g_{\text{sgn}(z),k} \right)^\wedge \left( -\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) \, d\omega.
\]

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We will now check the decay rate of \( \partial_\omega^\ell \partial_z^m \tilde{r}_k(z, 0) \). For this, we observe that by applying the product rule we obtain
\[
\partial_\omega^\ell \partial_z^m \tilde{r}_k(z, 0) = \frac{1}{2\pi} \sum_{\nu=0}^m c_\nu |z|^{-[m+(\nu+1)/k]} \int_{\mathbb{R}} \omega^\nu \left[ \left( g_{\text{sgn}(z), k} \right)^\wedge \right]^\wedge(v) \left( -\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) d\omega
\]
with \( c_\nu \in \mathbb{R} \) for all \( \nu \in \{0, \ldots, m\} \). By applying (9) and \( h \in S(\mathbb{R}) \), we estimate the terms in this equation as
\[
\left| |z|^{-[m+(\nu+1)/k]} \int_{\mathbb{R}} \omega^\nu \left[ \left( g_{\text{sgn}(z), k} \right)^\wedge \right]^\wedge(v) \left( -\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) d\omega \right| \leq 2 \left| |z|^{-[m+(\nu+1)/k]} \int_0^\infty \omega^\nu \cdot (\omega |z|^{-1/k})^{kr-1-v} \cdot |\hat{h}^{(\ell)}(\omega)| d\omega \right| = 2 \left| |z|^{-(m+r)} \int_0^\infty \omega^{kr-1} |\hat{h}^{(\ell)}(\omega)| d\omega \right| = c \cdot |z|^{-(m+r)}
\]
for some constant \( c > 0 \).

The next lemma’s statement is essentially the same as in Theorem 2, but we restrict the choice of analyzed functions to 0-feasible functions.

**Lemma 3** Let \( r, n \in \mathbb{N} \) and let \( \tau \) be an analyzing Taylorlet of order \( n \) with \( r \) vanishing moments. Let furthermore \( t \in \mathbb{R} \) and let \( f \) be a 0-feasible function.

1. Let \( \alpha > 0 \). If \( s_0 \neq q(t) \), the Taylorlet transform has a decay of
\[
\mathcal{T}^{(n, \alpha)} f(a, s, t) = O(a^{-N}) \quad \text{for } a \to 0
\]
for all \( N > 0 \).

2. Let \( \alpha < \frac{1}{n} \) and let \( k \in \{0, \ldots, n-1\} \) be the highest approximation order of \( s \) for \( f \) in \( t \). Then, the Taylorlet transform has the decay property
\[
\mathcal{T}^{(n, \alpha)} f(a, s, t) = O \left( a^{r[1-(k+1)\alpha]-1} \right) \quad \text{for } a \to 0.
\]

3. Let \( \alpha > \frac{1}{n+1} \) and let \( \tau \) be restrictive. If \( n \) is the highest approximation order of \( s \) for \( f \) in \( t \), then the Taylorlet transform has the decay property
\[
\mathcal{T}^{(n, \alpha)} f(a, s, t) \sim a^{-1} \quad \text{for } a \to 0.
\]

**Proof** We restrict ourselves to the case \( t = 0 \) as all other cases are equivalent to treating a shifted version of \( f \). Furthermore, we note that \( f \in S'(\mathbb{R}^2) \). Hence, the Taylorlet transform \( \mathcal{T}^{(n, \alpha)} f(a, s, 0) = \langle \tau_{a,s,0}, f \rangle \) is well-defined.

1. The idea is to exploit the special form of \( f \) in order to simplify its Taylorlet transform and to use the Schwartz class decay condition of \( \tau \) in order to estimate the integral.
The structure of $f$ leads to the following form of the Taylorlet transform:

$$
\mathcal{T}^{(n, \alpha)} f(a, s, 0) = \int_{\mathbb{R}^2} \delta(x_1 - q(x_2)) \tau_{a,s,0}(x) dx
$$

$$
= \int_{\mathbb{R}} \tau \left( \left[ q(x_2) - \sum_{\ell=0}^{n} \frac{s}{\ell!} \cdot x_2^{\ell} \right] / a \right) dx_2
$$

$$
= \int_{\mathbb{R}} g(\tilde{q}(x_2)/a) h(x_2/a^\alpha) dx_2,
$$

where $\tilde{q}(x_2) = q(x_2) - \sum_{k=0}^{n} \frac{s_k}{k!} \cdot x_2^k$. Since $g, h \in \mathcal{S}(\mathbb{R})$, the integrand in the last line fulfills the necessary decay condition of Lemma 1. By applying this lemma, we can conclude that

$$
\left| \int_{\mathbb{R}\setminus[-a^\beta,a^\beta]} g(\tilde{q}(x_2)/a) h(x_2/a^\alpha) dx_2 \right| \leq \|g\|_{L^\infty} \cdot \int_{\mathbb{R}\setminus[-a^\beta,a^\beta]} h(x_2/a^\alpha) dx_2 \overset{\text{Lemma 1}}{=} O(a^N) \quad \text{for } a \to 0
$$

for all $N \in \mathbb{N}$ if $\beta < \alpha$. Hence,

$$
\mathcal{T}^{(n, \alpha)} f(a, s, 0) = \int_{-a^\beta}^{a^\beta} g(\tilde{q}(x_2)/a) h(x_2/a^\alpha) dx_2 + O(a^N) \quad \text{for } a \to 0
$$

for all $N > 0$. Due to the conditions of this lemma, $\tilde{q} \in C^\infty(\mathbb{R})$ and $\tilde{q}(0) \neq 0$. Hence, there exists an $\varepsilon > 0$ such that $d := \min_{x_2 \in [-\varepsilon, \varepsilon]} |\tilde{q}(x_2)| > 0$. By employing the boundedness of $h$ and the Schwartz decay condition that for all $M \in \mathbb{N}$ there exists $c_M > 0$ such that $\sup_{x_2 \in \mathbb{R}} |x_2^M \cdot g(x_2)| = c_M < \infty$, we get

$$
|\mathcal{T}^{(n, \alpha)} f(a, s, 0)| \leq \|h\|_{L^\infty} c_M d^{-M} a^{M+\beta}.
$$

Since we can choose $M$ to be arbitrarily large, the result follows immediately.

2. The general idea of this proof is to represent the Taylorlet transform as a sum of integrals of the form (7) and to apply Lemma 2 in order to obtain the desired decay rate. To this end, we will divide the proof into four steps.

**STEP 1**

In the first step, we will show that the Taylorlet transform $\mathcal{T}^{(n, \alpha)} f(a, s, 0)$ is an integral over a curve and we will prove that only a small neighborhood of the origin is relevant for the decay of the Taylorlet transform for $a \to 0$.

First, we rewrite (10):

$$
\mathcal{T}^{(n, \alpha)} f(a, s, 0) = \int_{\mathbb{R}} \tau \left( \left[ q(x_2) - \sum_{\ell=0}^{n} \frac{s}{\ell!} \cdot x_2^{\ell} \right] / a \right) dx_2
$$

$$
= \int_{\mathbb{R}} \tau \left( \tilde{q}(x_2) \cdot \frac{x_2^{k+1}}{a} \right) dx_2,
$$

where $\tilde{q}(x_2) = q(x_2) - \sum_{k=0}^{n} \frac{s_k}{k!} \cdot x_2^k$. Since $g, h \in \mathcal{S}(\mathbb{R})$, the integrand in the last line fulfills the necessary decay condition of Lemma 1. By applying this lemma, we can conclude that

$$
\left| \int_{\mathbb{R}\setminus[-a^\beta,a^\beta]} g(\tilde{q}(x_2)/a) h(x_2/a^\alpha) dx_2 \right| \leq \|g\|_{L^\infty} \cdot \int_{\mathbb{R}\setminus[-a^\beta,a^\beta]} h(x_2/a^\alpha) dx_2 \overset{\text{Lemma 1}}{=} O(a^N) \quad \text{for } a \to 0
$$

for all $N \in \mathbb{N}$ if $\beta < \alpha$. Hence,

$$
\mathcal{T}^{(n, \alpha)} f(a, s, 0) = \int_{-a^\beta}^{a^\beta} g(\tilde{q}(x_2)/a) h(x_2/a^\alpha) dx_2 + O(a^N) \quad \text{for } a \to 0
$$

for all $N > 0$. Due to the conditions of this lemma, $\tilde{q} \in C^\infty(\mathbb{R})$ and $\tilde{q}(0) \neq 0$. Hence, there exists an $\varepsilon > 0$ such that $d := \min_{x_2 \in [-\varepsilon, \varepsilon]} |\tilde{q}(x_2)| > 0$. By employing the boundedness of $h$ and the Schwartz decay condition that for all $M \in \mathbb{N}$ there exists $c_M > 0$ such that $\sup_{x_2 \in \mathbb{R}} |x_2^M \cdot g(x_2)| = c_M < \infty$, we get

$$
|\mathcal{T}^{(n, \alpha)} f(a, s, 0)| \leq \|h\|_{L^\infty} c_M d^{-M} a^{M+\beta}.
$$

Since we can choose $M$ to be arbitrarily large, the result follows immediately.
where \( \tilde{q}(x_2) = \left\{ \begin{array}{ll} x_2^{-(k+1)} & [q(x_2) - \sum_{\ell=0}^{n} \frac{s_\ell}{\ell!} \cdot x_2^\ell] \text{ for } x_2 \neq 0, \\ \frac{1}{(k+1)!} \cdot (q^{(k+1)}(0) - s_{k+1}) \end{array} \right. \)

for \( x_2 = 0. \)

Since \( k \) is the highest approximation order of \( s \) for \( f \) in \( t = 0 \), we have \( s_{k+1} \neq q^{(k+1)}(0) \). Hence, \( \tilde{q}(0) \neq 0. \) Furthermore, we have \( \tilde{q} \in C^\infty(\mathbb{R}) \) due to the conditions of this Lemma. In order to show that just a small neighborhood of the origin is responsible for the decay of the Taylorlet transform for \( a \to 0 \), we observe that the integrand in (11) fulfills the decay condition of Lemma 1. By applying this lemma, we obtain for \( \beta \in \left( 0, \frac{1}{k+1} \right) \) and an arbitrary \( N \in \mathbb{N} \) that

\[
|\mathcal{T}^{(n,a)} f(a, s, 0)| = \left| \int_{-a^\beta}^{a^\beta} \tau \left( \tilde{q}(x_2) \cdot x_2^{k+1}/a \right) dx_2 \right| + O(a^N) \quad \text{for } a \to 0. \tag{12}
\]

**STEP 2**

If we replaced the term \( \tilde{q}(x_2) \) in the argument of the integrand by some constant \( c \neq 0 \), the integral would be a truncated version of the desired form (7). Hence, we could apply Lemma 2 to obtain an estimate for the decay of the Taylorlet transform. In order to get closer to this form, we will approximate the integrand by a Taylor polynomial in this step.

Now, we expand the integrand of (12) into a Taylor series with respect to the first component in a neighborhood of the point \( \tilde{q}(0) \cdot x_2^{k+1}/a. \)

\[
|\mathcal{T}^{(n,a)} f(a, s, 0)| = \left| \int_{-a^\beta}^{a^\beta} \tau \left( \tilde{q}(x_2) \cdot x_2^{k+1}/a \right) dx_2 \right| + O(a^N)
\]

\[
\leq \sum_{m=0}^{M} \int_{-a^\beta}^{a^\beta} \partial^m \tau \left( \tilde{q}(0) \cdot x_2^{k+1}/a \right) \cdot \left[ x_2^{k+1} (\tilde{q}(x_2) - \tilde{q}(0)) \right]^m a^m \cdot m! dx_2
\]

\[
+ c^{M+1} \int_{-a^\beta}^{a^\beta} \left( |x_2|^{k+1} / a \right)^{M+1} \cdot |x_2|^{M+1} dx_2 + O(a^N). \tag{13}
\]

For the last estimate we used that due to the smoothness of \( \tilde{q} \), there exists a \( c > 0 \) such that \( |\tilde{q}(x_2) - \tilde{q}(0)| \leq c|x_2| \) for all \( x_2 \in [-a^\beta, a^\beta] \). We now prove that it is possible to choose \( M \in \mathbb{N} \) such that the rest term in (13) behaves like \( O(a^N) \) for \( a \to 0 \) for an arbitrary, but fixed \( N \in \mathbb{N} \). We have

\[
\int_{-a^\beta}^{a^\beta} \left( |x_2|^{k+1} / a \right)^{M+1} \cdot |x_2|^{M+1} dx_2 \sim a^{(M+1)(\beta(k+2)-1)+\beta} \quad \text{for } a \to 0. \tag{14}
\]

By restricting the choice of \( \beta \in \left( 0, \frac{1}{k+1} \right) \) to \( \beta \in \left( \frac{1}{k+2}, \frac{1}{k+1} \right) \), we obtain the desired decay rate of \( O(a^N) \) for

\[
M = \left\lfloor \frac{N - \beta}{\beta(k + 2) - 1} \right\rfloor - 1.
\]
STEP 3

In the third step, we will expand $\tilde{q}$ in a Taylor series about the origin to obtain a representation of the Taylorlet transform as a sum of truncated versions of integrals of the form (7).

We now expand the term $\tilde{q}(x_2) - \tilde{q}(0)$ about the point $x_2 = 0$.

$$
\mathcal{T}^{(n, \alpha)} f(a, s, 0) = \sum_{m=0}^{M} \int_{-a^\alpha}^{a^\alpha} \partial_1^m \tau \left( \tilde{q}(0) \cdot \frac{x_2^{k+1}}{a} \right) \left( \frac{x_2^{k+1}}{a} \right)^m \frac{\tilde{q}(x_2) - \tilde{q}(0)}{m!} dx_2 + \mathcal{O}(a^N)
$$

where $\rho(x_2)$ is the remainder term of the Taylor series expansion with the property $\rho(x_2) = \mathcal{O}(x_2^{Lm+1})$ for $x_2 \to 0$. Now, we estimate the summands for each $m \in \{0, \ldots, M\}$.

$$
\int_{-a^\alpha}^{a^\alpha} \partial_1^m \tau \left( \tilde{q}(0) \cdot \frac{x_2^{k+1}}{a} \right) \left( \frac{x_2^{k+1}}{a} \right)^m \left[ \rho(x_2) + \sum_{\ell=1}^{Lm} \tilde{q}^{(\ell)}(0) \cdot \frac{x_2^\ell}{\ell!} \right] dx_2
$$

$$
= \sum_{v=0}^{m} \int_{-a^\alpha}^{a^\alpha} \partial_1^m \tau \left( \tilde{q}(0) \cdot \frac{x_2^{k+1}}{a} \right) \frac{x_2^{(k+1)m}}{a^m} \left[ \rho(x_2) \right]^v \left[ \sum_{\ell=1}^{Lm} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot \frac{x_2^\ell}{\ell!} \right] dx_2.
$$

Since $\tau \in \mathcal{S}(\mathbb{R}^2)$, $\rho(x_2) = \mathcal{O}(x_2^{Lm+1})$ for $x_2 \to 0$ and $\sum_{\ell=1}^{Lm} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell = \mathcal{O}(x_2)$ for $x_2 \to 0$, for every $v \in \{1, \ldots, m\}$, there exist $c_v, a_0 > 0$ such that for all $a < a_0$

$$
\left| \int_{-a^\alpha}^{a^\alpha} \partial_1^m \tau \left( \tilde{q}(0) \cdot \frac{x_2^{k+1}}{a} \right) \frac{x_2^{(k+1)m}}{a^m} \left[ \rho(x_2) \right]^v \left[ \sum_{\ell=1}^{Lm} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot \frac{x_2^\ell}{\ell!} \right] dx_2 \right|
$$

$$
\leq c_v \cdot a^{-m} \left| \int_{-a^\alpha}^{a^\alpha} |x_2|^{(k+1)m} |x_2|^{(Lm+1)v} \cdot |x_2|^{m-v} dx_2 \right|
$$

$$
= \mathcal{O}(a^{[(k+2)\beta-1]m + \beta(L_m+2)}) \quad \text{for} \quad a \to 0.
$$

We now compare the exponent $[(k + 2)\beta - 1]m + \beta(L_m + 2)$ of the decay rate in (16) to the exponent $(M + 1)(\beta(k + 2) - 1) + \beta$ of the decay rate in (14), where the latter decay rate is equal to $\mathcal{O}(a^N)$. By considering that $\beta < \frac{1}{k+1}$, we see that the...
choice $L_m = M - m$ is sufficient to obtain a decay rate of $O(a^N)$ in (16). Hence, for all $m \in \{0, \ldots, M\}$, we have

$$f_{-a^\beta}^a \partial_1^m \tau \left( \tilde{g}(0) \cdot \frac{x_2^{k+1}}{x_2/a^\alpha} \right) \left( \frac{x_2^{k+1}}{a} \right)^m \cdot \left[ \rho(x_2) + \sum_{\ell=1}^{L_m} \frac{\tilde{g}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell \right]^m dx_2$$

$$= f_{-a^\beta}^a \partial_1^m \tau \left( \tilde{g}(0) \cdot \frac{x_2^{k+1}}{x_2/a^\alpha} \right) \left( \frac{x_2^{k+1}}{a} \right)^m \cdot \left[ \sum_{\ell=1}^{L_m} \frac{\tilde{g}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell \right]^m dx_2 + O(a^N)$$

for $a \to 0$. By inserting this result into (15), we get

$$T(n, \alpha) f(a, s, 0) = \sum_{m=0}^M a^{-m} \sum_{\ell=m}^{(M-m)m} c_{\ell, m} f_{-a^\beta}^a \partial_1^m \tau \left( \frac{\tilde{g}(0) \cdot x_2^{k+1}}{x_2/a^\alpha} \right) \cdot x_2^{(k+1)m+\ell} dx_2 + O(a^N)$$

for $a \to 0$ for appropriate constants $c_{\ell, m} \in \mathbb{R}$. By comparing the summand for $m = 0$ in the equation (15) with the summand for $m = 0$ in the upper equality, we obtain that

$$c_{0, 0} = 1.$$  

(17)

This constant will become important in the proof of statement 3 of this lemma.

**STEP 4**

In this final step, we extend the integration limits to $\pm \infty$ and apply Lemma 2 to estimate the decay of the Taylorlet transform.

Applying Lemma 1 again, we can change back the integration limits to $\pm \infty$ by only adding another $O(a^N)$-term. Furthermore, we substitute $x_2 = a^\alpha v$ and obtain

$$T(n, \alpha) f(a, s, 0) = a^{-1} \sum_{m=0}^M a^{-m} \sum_{\ell=m}^{(M-m)m} c_{\ell, m} f_{-a^\beta}^a \partial_1^m \tau \left( \frac{\tilde{g}(0) \cdot a^{(k+1)\alpha-1} v^{k+1}}{v} \right) \cdot (a^\alpha v)^{(k+1)m+\ell} dv + O(a^N).$$

(18)

Finally, we brought the Taylorlet transform into a shape that is fit for an application of Lemma 2. Since $\tilde{g}(0) \neq 0$ and $\lim_{a \to 0} a^{(k+1)\alpha-1} = \infty$, Lemma 2 delivers

$$|T(n, \alpha) f(a, s, 0)| \leq O \left( a^{(1-(k+1)\alpha)r-1} \right) \text{ for } a \to 0.$$  

**3.** For this case, we use the same argumentations as in case 2 to obtain (18) with the choices of $k = n$ and

$$\tilde{g}(x_2) = \begin{cases} \frac{x_2^{n+1}}{(n+1)!} \cdot \left[ q(x_2) - \sum_{\ell=0}^n \frac{q^{(\ell)}(0)}{\ell!} \cdot x_2^\ell \right], & \text{for } x_2 \neq 0, \\ \frac{x_2^{n+1}}{(n+1)!} \cdot q^{(n+1)}(0), & \text{for } x_2 = 0. \end{cases}$$
In spite of the similarities, there is a major difference in the situations, namely that
\[
\lim_{a \to 0} a^{(n+1)\alpha - 1} = 0.
\]
Hence, we obtain for the integrals in (18) that
\[
\lim_{a \to 0} \int_{\mathbb{R}} \partial^m_1 \tau \left( \tilde{q}(0) a^{(n+1)\alpha - 1} u^{n+1} u \right) u^{(n+1)m + \ell} du = \int_{\mathbb{R}} g^{(m)}(0) \tilde{q}(0) a^{(n+1)\alpha - 1} u^{n+1} h(u) u^{(n+1)m + \ell} du = g^{(m)}(0) \int_{\mathbb{R}} h(u) u^{(n+1)m + \ell} du.
\]
(19)

We now focus on the powers of \(a\) appearing in the summands of (18). For the indices \(\ell\) and \(m\) of the double sum’s summands in (18), we obtain that
\[
S_{\ell,m}(a) := c_{\ell,m} \cdot a^{-s} \int_{\mathbb{R}} \partial^m_1 \tau \left( \tilde{q}(0) a^{(k+1)\alpha - 1} v^{k+1} \right) \cdot (a^s v)^{(k+1)m + \ell} dv = O \left( a^{[s(n+1)\alpha - 1]m + \ell \alpha - 1} \right) \text{ for } a \to 0
\]
for all \(m \in \{0, \ldots, M\}\) and \(\ell \in \{m, \ldots, (M - m)m\}\). Due to the restrictiveness, \(g(0) \neq 0\) and \(\int_{\mathbb{R}} h(u) du \neq 0\). Hence, together with (19) and \(c_{0,0} = 1\) due to (17), we obtain that
\[
S_{0,0}(a) \sim a^{-1} \cdot c_{0,0} \cdot g(0) \cdot \int_{\mathbb{R}} h(u) du \sim a^{-1} \text{ for } a \to 0.
\]
Since \((n + 1)\alpha - 1 > 0\), \(S_{0,0}\) is the slowest decaying summand. Thus,
\[
\mathcal{T}^{(n,\alpha)} f(a, s, 0) \sim a^{-1} \text{ for } a \to 0.
\]
\(\square\)

With this lemma, we are now able to prove Theorem 2.

Proof of Theorem 2 The proof strategy is to reduce the case \(f(x) = I^j_+ \delta(x_1 - q(x_2))\) to the case \(f(x) = \delta(x_1 - q(x_2))\) of Lemma 3 by partial integration and to show that the resulting iterated integral \(I^j_+ \tau\) of the Taylorlet \(\tau\) is a Taylorlet as well.

First, we note that \(f = (I^j_+ \delta)(x_1 - q(x_2))\) is a tempered distribution for all \(j \in \mathbb{N}\). Let \(a > 0\), \(s \in \mathbb{R}^{n+1}\), \(t \in \mathbb{R}\). Then, the Taylorlet transform \(\mathcal{T}^{(n,\alpha)} f(a, s, t)\) is well-defined. By partial integration, we obtain for \(j \geq 1\)
\[
\mathcal{T}^{(n,\alpha)} f(a, s, t) = \left\{ \tau_{a,s,0}, f \right\} = \int_{\mathbb{R}} \left[ a \cdot I_{x_1, \pm} \tau_{a,s,0}(x) \cdot (I^j_\pm \delta)(x_1 - q(x_2)) \right]_{x_1 = -\infty}^{x_1 = +\infty} dx_2 + \int_{\mathbb{R}^2} a \cdot I_{x_1, \mp} \tau_{a,s,0}(x) \cdot (I^j_\pm \delta)(x_1 - q(x_2)) dx.
\]
We now show that the first term disappears. For this, we note that \(I^j_\pm \delta(x)\) exhibits only polynomial growth as \(|x| \to \infty\). Furthermore, with \(\tau = g \otimes h\), only \(g\) is altered...
by the operator $I_{x_1,\pm}$ while $h$ remains the same. Hence, we show that $I^j_\pm g \in S(\mathbb{R})$ for all $j < r$. By applying a Fourier transform to the function, we obtain

$$\left( I^j_\pm g \right)^\wedge (\omega) = \frac{\hat{g}(\omega)}{(\pm i \omega)^j}.$$  

Since $g$ has $r$ vanishing moments and $g \in S(\mathbb{R})$, we have $\hat{g}(\omega) = O(\omega^r)$ for $\omega \to 0$. Consequently, $\frac{\hat{g}(\omega)}{(\pm i \omega)^j} \in S(\mathbb{R})$ and hence also $I^j_\pm g \in S(\mathbb{R})$. We thus obtain

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = a \cdot \int_{\mathbb{R}^2} I_{x_1,\mp} \tau_{a,s,0}(x) \cdot (I^j_\pm \delta)(x_1 - q(x_2)) dx.$$ 

By induction, we get

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = a^j \cdot \left( I^j_{x_1,\mp} \tau_{a,s,0} \cdot \delta(x_1 - q(x_2)) \right).$$ 

This delivers an additional factor $a^j$ to the Taylorlet transform. Now, we examine the vanishing moments. By applying partial integration and utilizing $I^j_\pm g \in S(\mathbb{R})$, we obtain

$$\left| \int_{\mathbb{R}} (I^j_\pm g)(\pm t^k) t^m dt \right| = \left| \int_{\mathbb{R}} g(\pm t^k) t^{k+j+m} dt \right|.$$ 

Hence, $I^j_\pm g$ has $r - j$ vanishing moments of order $n$ and in case 2 with a highest approximation order of $k$ we obtain the decay rate

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0) = O\left(a^{j-1+(r-j)[1-(k+1)\alpha]}\right) \text{ for } a \to 0.$$ 

It remains to show that the restrictiveness condition of $g$ guarantees $I^j_\pm g(0) \neq 0$. For this, we apply the formula for iterated integrals stating that

$$I^j_\pm g(u) = \int_{-\infty}^u (u - v)^{j-1} g(v) dv.$$  

Hence, we obtain

$$I^j_\pm g(0) = (-1)^{j-1} \int_{-\infty}^0 g(v) v^{j-1} dv = \int_{\mathbb{R}} g(v) v^{j-1} dv + (-1)^j \int_0^\infty g(v) v^{j-1} dv \neq 0.$$ 

The statement $I^j_\pm g(0) \neq 0$ can be proved similarly. Hence, for all $j < r$, the function $I^j_{x_1,\pm}$ is a restrictive analyzing Taylorlet of order $n$ with $r - j$ vanishing moments, i.e.,

$$\int_{\mathbb{R}} I^j_{x_1,\pm} \left( \frac{0}{u} \right) du \neq 0.$$ 

Consequently, with the additional factor $a^j$, we get the decay rate

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0) = O\left(a^{j-1}\right) \text{ for } a \to 0.$$ 

\[\Box\]
6 Numerical examples

In this section, we illustrate the main result numerically. To this end, we present a procedure for the detection of the location, the orientation, and the curvature of an edge, based on the Taylorlet transform. As an example, we consider the sharp edge of a function

$$f(x) = 1_{\mathbb{R}_+}(x_1 - q(x_2)), \quad x \in \mathbb{R}^2,$$

with \(q \in C^\infty(\mathbb{R})\) as singularity function.

6.1 Construction of the Taylorlet

For the implementation of the Taylorlet transform in Matlab, we used Taylorlets of order 2 with 3 vanishing moments. They were constructed via the general setup in Section 3 starting from the function \(\phi(t) = e^{-t^2}\). Through this procedure, we obtain the Taylorlet

$$\tau(x) = g(x_1) \cdot h(x_2), \quad (20)$$

where

$$g(x_1) = \frac{64}{8!} \cdot (1 + x_1) \cdot (315 - 51660 x_1^2 + 286020 x_1^4 - 349440 x_1^6 + 142464 x_1^8 \left.- 21504 x_1^{10} + 1024 x_1^{12}\right) \cdot e^{-x_1^2},$$

$$h(x_2) = e^{-x_2^2},$$

which is shown in Fig. 1. To speed up computation time, we employed the one-dimensional adaptive Gauss-Kronrod quadrature \texttt{quadgk} in Matlab for the evaluation of the integrals. In order to reduce the Taylorlet transform to a one-dimensional integral, we utilize partial integration, i.e.,

$$\{\tau_{ast}(x), 1_{\mathbb{R}_+}(x_1 - q(x_2))\} = \{I_{x_1, +\tau_{ast}}(x_1 - q(x_2))\} = \int_{-\infty}^{\infty} I_{x_1, +\tau_{ast}}(q(t))dt.$$

Hereby, the antiderivative of \(\tau\) w.r.t. \(x_1\) can be determined analytically by computing the antiderivative of \(g\), i.e.,

$$\int_{-\infty}^{t} g(x_1)dx_1 = -\frac{32}{8!} \cdot e^{-t^2} \left(-9 - 630 t - 324 t^2 + 34020 t^3 + 25668 t^4 \right. \left.- 100800 t^5 - 86784 t^6 + 71040 t^7 + 65664 t^8 - 15872 t^9 \right. \left.- 15360 t^{10} + 1024 t^{11} + 1024 t^{12}\right).$$

6.2 Detection procedure

Due to Theorem 2, the decay rate of the Taylorlet transform changes depending on the highest approximation order of the shearing variable. We can exploit this pattern in a step-by-step search for consecutive Taylor coefficients of the singularity function \(q\). To this end, we first compute the Taylorlet transform of a function with \(\alpha > 1\) and
with varying shearing variable $s_0$ while $s_k = 0$ for all $k \in \{1, \ldots, n\}$. The choice of $\alpha$ and the restrictiveness of the Taylorlet ensure a decay rate of

$$\mathcal{T}^{(0, \alpha)} f(a, s_0, t) \sim 1 \quad \text{for } a \to 0$$

for $s_0 = q(t)$ due to Theorem 2.

We then consider the propagation of the local maxima w.r.t. $s_0$ through the scales. Since the choice $s_0 = q(t)$ leads to the lowest decay rate, we can expect the local maxima near $s_0 = q(t)$ to converge towards this value for decreasing scales in a similar fashion as in the method of wavelet maximum modulus by Mallat and Hwang [9]. Subsequently, we fix $s_0$ to the value $q(t)$, change $\alpha$ such that $\alpha \in \left(\frac{1}{2}, 1\right)$, and search for the matching value of $s_1$ in the same way as in the preceding step for $s_0$. Because of the restrictiveness, the Taylorlet transform has a decay rate of $\mathcal{T}^{(1, \alpha)} f(a, s, t) \sim 1$ for $a \to 0$, if additionally $s_1 = q'(t)$. Due to the vanishing moment condition, the decay rate of the Taylorlet transform is considerably higher for $s_1 \neq q'(t)$. Hence, the method of maximum modulus is still applicable. With the same argumentation, we can repeat this procedure for all shearing variables $s_k$ with a choice $\alpha \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$ up to the order of the Taylorlet.

### 6.3 Images

In order to better visualize the local maxima, we normalized the absolute value of the Taylorlet transform in the presented plots such that the maximum value in each scale is 1. Due to this normalization w.r.t. the local maxima on a compact interval regarding the respective shearing variable, discontinuities w.r.t. the dilation parameter can appear (e.g., around $-\log_2 a = 1$ in the bottom right image of Table 2).

Tables 1 and 2 contain plots of the Taylorlet transform $\mathcal{T} f(a, s, t)$ of the function $f(x) = 1_{\mathbb{R}_+} (x_1 - \sin x_2)$ for $t \in \{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\}$. The vertical axis shows the dilation parameter in a binary logarithmic scale while the horizontal axis shows location, slope, and parabolic shear. The respective true values can be found in the following table and are indicated by a vertical red line in the plots.

| $t$    | $q(t)$ | $q'(t)$ | $q''(t)$ |
|--------|--------|---------|----------|
| 0      | 0      | 1       | 0        |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{1}{2} \sqrt{3}$ | $-\frac{1}{2}$ |
| $\frac{\pi}{4}$ | $\frac{1}{2} \sqrt{3}$ | $\frac{1}{2}$ | $-\frac{1}{2} \sqrt{3}$ |
| $\frac{\pi}{2}$ | 1      | 0       | -1       |

As stated in the previous subsection, the values of $\alpha$ change with $s_i$. For the detection of the location, we use $\alpha = 1.01$; for the slope, we have $\alpha = 0.51$; and during the search for the parabolic shear, we set $\alpha = 0.34$. We can observe the paths of the local maxima w.r.t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, $\mathcal{T} f(a, s, t)$ decays fast for $a \to 0$, if $s_k \neq q^{(k)}(t)$, and slow
Table 1  Plots of the Taylorlet transform $\mathcal{T} f(a, s, t)$ for $f(x) = 1_{\mathbb{R}_+}(x_1 - \sin x_2)$, where $t \in \{0, \frac{\pi}{6}\}$

| $t$  | 0       | $\frac{\pi}{6}$ |
|------|---------|-----------------|
| $s_0$ | ![Plot](image1) | ![Plot](image2) |
| $s_1$ | ![Plot](image3) | ![Plot](image4) |
| $s_2$ | ![Plot](image5) | ![Plot](image6) |

The vertical axis shows the dilation parameter in a logarithmic scale $-\log_2 a$. The horizontal axis shows the location $s_0$ (top), the slope $s_1$ (center), and the parabolic shear $s_2$ (bottom). The respective true value is indicated by the vertical red line. The values of $\alpha$ change with $s_i$: for $s_0$, we use $\alpha = 1.01$; for $s_1$, we have $\alpha = 0.51$; and during the search for $s_2$, we set $\alpha = 0.34$. The Taylorlet transform was computed for points $(a, s_i)$ on a $300 \times 300$ grid. We can observe the paths of the local maxima w.r.t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, the local maxima display a fast convergence to the correct value $q^{(k)}(t)$.

In Tables 1 and 2, we can notice an influence of the order of the respective shearing variable on the convergence speed, as the local maxima appear to have a slower convergence to the correct value in the search for a higher Taylor coefficient. The reason for this occurrence is the dependence of the decay rate of the Taylorlet transform on the order $k$. In the search for the constant Taylor coefficient $q(t)$, the Taylorlet
The vertical axis shows the dilation parameter in a logarithmic scale $-\log_2 a$. The horizontal axis shows the location $s_0$ (top), the slope $s_1$ (center), and the parabolic shear $s_2$ (bottom). The respective true value is indicated by the vertical red line. The values of $\alpha$ change with $s_i$: for $s_0$, we use $\alpha = 1.01$; for $s_1$, we have $\alpha = 0.51$; and during the search for $s_2$, we set $\alpha = 0.34$. The Taylorlet transform was computed for points $(a, s_i)$ on a $300 \times 300$ grid. We can observe the paths of the local maxima w.r.t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, the local maxima display a fast convergence to the correct value.

Table 2  Plots of the Taylorlet transform $\mathcal{T} f(a, s, t)$ for $f(x) = \mathbb{1}_{\mathbb{R}^+}(x_1 - \sin x_2)$, where $t \in \{\frac{\pi}{3}, \frac{\pi}{2}\}$

| $t$   | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|-------|-----------------|-----------------|
| $s_0$ | ![Image]         | ![Image]         |
| $s_1$ | ![Image]         | ![Image]         |
| $s_2$ | ![Image]         | ![Image]         |

The transform decays faster than any polynomial for $a \to 0$, if $s_0 \neq q(t)$. When the algorithm looks for the $k$th Taylor coefficient of $q$ in $t$, the shearing variable $s \in \mathbb{R}^{k+1}$ has a highest approximation order of $k - 1$ everywhere except for the point where $s_k = q^{(k)}(t)$. In this context, for $s_k \neq q^{(k)}(t)$, the Taylorlet transform has a decay rate of

$$\mathcal{T}^{(n,a)} f(a, s, t) = \mathcal{O}(a^{1 - 1 + (r - j)[1 - k\alpha]})$$

for $a \to 0$ due to Theorem 2. When we insert the parameters $j = 1$ (since $f$ is 1-feasible) and $r = 3$ (as $\tau$ has 3 vanishing moments of order 2) and the respective values of $\alpha$ which
we use for the search for $q^{(k)}(t)$ for $k \in \{1; 2\}$, we obtain the following decay rates for $s_k \neq q^{(k)}(t)$:

| $k$ | $\alpha$ | Decay rate for $a \to 0$ |
|-----|-----------|--------------------------|
| 1   | 0.51      | $O \left( a^{0.98} \right)$ |
| 2   | 0.34      | $O \left( a^{0.64} \right)$ |

### 7 Conclusion

In order to detect higher order geometric information, we introduced the Taylorlet transform which is based on the continuous shearlet transform, and in addition to dilation, translation and classical shears utilize shears of higher order. The transform allows for an extraction of the position, the orientation, the curvature, and other higher order geometric information of distributed singularities (Theorem 2). A fast detection of these features can be guaranteed using the concept of vanishing moments of higher order. Additionally, we presented a constructive algorithm to build functions with the needed properties. First numerical studies showed its potential for future applications.

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