Mitosis algorithm for Grothendieck polynomials

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Abstract

In this article we will introduce the way to extend the mitosis algorithm for Schubert polynomials, developed by Ezra Miller, to the case of Grothendieck polynomials.

Introduction

The main subject of the present note is the extension of the results obtained by Ezra Miller in [3] to the case of Grothendieck polynomials. The mitosis algorithm was firstly introduced in [5] as a combinatorial rule that allows to compute the coefficients of Schubert polynomials inductively in terms of special combinatorial objects called pipe dreams, developed by Fomin and Kirillov.

Originally, Schubert polynomials are defined by downward induction on weak Bruhat order where the induction step is represented by applying of the corresponding divided difference operator. At the same time, according to the formula of Billey, Jockusch, and Stanley, the coefficients of each Schubert polynomial might be obtained from the set of diagrams in an $n \times n$ grid called reduced pipe dreams (or rc-graphs). Thus, mitosis might be considered as an analogue of the applying of the divided difference operator: namely, if $w$ and $v$ are permutations (and $v < w$ in the weak Bruhat order) then by using this algorithm we can obtain the set of reduced pipe dreams corresponding to the Schubert polynomial $S_v(x)$ (denoted by $\mathcal{RP}(v)$) from the set $\mathcal{RP}(w)$. The note [3] provides a short proof of this fact, based on elementary combinatorial properties of reduced pipe dreams ([3], Theorem 15).

In turn, the object of our study — Grothendieck polynomials (denoted by $G_w(x)$, where $w$ is an arbitrary permutation) — might be considered as a generalization of Schubert polynomials. As well as Schubert polynomials they also can be defined inductively by using isobaric divided difference operators and in terms of pipe dreams (only now not necessarily reduced ones). In particular, if $G_w(x)$ and $\mathfrak{S}_w(x)$ are the Schubert and Grothendieck polynomials of a permutation $w$ then $\mathfrak{S}_w(x)$ is the sum over all monomials of $G_w(x)$ of the minimal degree. Thus, it is reasonable to suggest that mitosis is somehow applied to the case of Grothendieck polynomials (i.e. to pipe dreams in general). In the first part of our note we will describe the way mitosis acts on pipe dreams in general. The second part will be devoted to the theorem, analogous to the Theorem 15 of [5]. For that we will slightly modify the original mitosis algorithm.

The plan of the paper is as follows: in the first section we will give the inductive definitions of Schubert and Grothendieck polynomials, introduce to the reader the concept...
of pipe dreams and describe the way Schubert and Grothendieck polynomials can be defined combinatorially. The second section is devoted to the original mitosis algorithm: we give a brief description of this algorithm and prove theorem 2 describing the way mitosis acts on the set of pipe dreams $P(w)$ corresponding to a permutation $w$. Section 3, analogously to [5], provides a special involution on $P(w)$ that is crucial for the proof of the main theorem. The final section is devoted to the main theorem of the paper (Theorem 4.1), which gives us an inductive way to construct Grothendieck polynomials in terms of the pipe dreams.

1 Pipe dreams

In this section we will give the inductive definitions of Schubert and Grothendieck polynomials, and introduce to the reader the concept of pipe dreams. Denote by $s_i = (i, i + 1)$ the corresponding elementary transposition of $S_n$. It is well known that the set $\{s_i| i = 1, \ldots, n-1\}$ generates the group $S_n$; with the following relations:

$$s_i^2 = 1$$
$$s_is_j = s_js_i \text{ if } |i - j| > 1$$
$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}.$$

Thus, we can say that the set $\{s_i| i \in \mathbb{N}\}$ generates the group $S_\infty$ with the relations described above. Let $w$ be an arbitrary element of $S_\infty$. Then we can define a sequence $a_1, \ldots, a_k$ of minimal length such that $w = s_{a_1} \cdots s_{a_k}$. The number $k$ is called the length of $w$ and denoted by $l(w)$ (remark that the sequence itself can be defined in more than one way).

Let $w$ be an arbitrary element of $S_\infty$ and $s_i$ be an elementary transposition. Then we say that $ws_i > w$ if $l(ws_i) = l(w) + 1$. Otherwise we have $l(ws_i) = l(w) - 1$ which means that $l(w) = l((ws_i)s_i) = l(ws_i) + 1$ and, accordingly, $w > ws_i$. Now, by using the property of transivity we can introduce on $S_\infty$ a partial order. It is well known that for each $n \in \mathbb{N}$ the greatest element of $S_n$ is the order reversing permutation, denoted by $w_n^{(n)} = (n \ldots 1)$.

Now we can define Schubert and Grothendieck polynomials inductively. For that we will also need to introduce the sets of divided difference operators, denoted by $\partial_i$, and isobaric divided difference operators, denoted by $\pi_i$. These linear operators act on the ring $\mathbb{Z}[x_1, x_2, \ldots]$ as follows:

$$\forall f \in \mathbb{Z}[x_1, x_2, \ldots], \quad \partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}, \quad \pi_i(f) = \partial_i(f - x_{i+1}f). \quad (1.1)$$

Here $s_i(f)$ is a polynomial, obtained from $f$ by interchanging the variables $x_i$ and $x_{i+1}$. Obviously, the result of applying of each operator is also a polynomial with integer coefficients.

**Definition 1.1:** For an arbitrary element $w$ of $S_\infty$ the corresponding Schubert polynomial can be defined inductively in compliance with the following rules:
1) For the order reversing permutation \( w_0^{(n)} \) the following equality holds:

\[
\mathcal{S}_{w_0^{(n)}}(x) = x_1^{n-1}x_2^{n-2} \cdots x_{n-1};
\]

2) \( \partial_i(\mathcal{S}_w(x)) = \mathcal{S}_{ws_i}(x) \), if \( l(ws_i) = l(w) - 1 \).

The corresponding Grothendieck polynomial is defined practically in the same way. Namely:

1) For \( w_0^{(n)} \) the Grothendieck polynomial \( G_{w_0^{(n)}}(x) \) equals the corresponding Schubert polynomial:

\[
G_{w_0^{(n)}}(x) = x_1^{n-1}x_2^{n-2} \cdots x_{n-1};
\]

2) \( \pi_i(G_w(x)) = G_{ws_i}(x) \), if \( l(ws_i) = l(w) - 1 \).

Now we will give the definition of a pipe dream. Consider the direct product \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \), represented in the form of a table extending infinitely south and east (the box, located in \( i \)-th row and \( j \)-th column indexed by pair \((i, j)\)). Then a pipe dream is a finite subset of \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \), with its elements marked with the + symbol (see the example below).

**Example 1.1:** The following pipe dream represents the set \( \{(1, 1); (1, 2); (1, 5); (2, 2); (3, 2); (5, 1)\} \).

```
+ + +
+   +
+   +
+  +
```

Now in every box with + we put the symbol \( \downarrow \), and in every empty box — the symbol \( \uparrow \). Thus, we obtain the network of strands, crossing each other at the positions belonging to the pipe dream and avoiding each other at other positions. The pipe dream is reduced if each pair of strands crosses at most once. Now for every \( w \in S_\infty \) we define the set \( \mathcal{RP}(w) \) of reduced pipe dreams, such that for each element \( D \) of this set the strand entering the \( i \)-th row, exits from the \( w(i) \)-th column.

**Example 1.2:** The depicted pipe dream is reduced and corresponds to the permutation \( w = (261354) \).
Note that for simplicity we usually do not draw the "sea" of wavy strands, whose entries and exits are equal.

Now Schubert polynomials can be defined combinatorially. Namely, for an arbitrary pipe dream $D$ we will introduce the following notation

$$x_D = \prod_{(i,j) \in D} x_i.$$ 

The following proposition is a nontrivial theorem, which proof will be not be given here (See [4], Theorem 1).

**Theorem 1.1:** For an arbitrary element $w$ of $S_{\infty}$ the following equality holds:

$$S_w(x) = \sum_{D \in R\mathcal{P}(w)} x_D. \quad (1.2)$$

As a consequence, the coefficients of the polynomial $S_w(x)$ are positive.

Consider now an arbitrary pipe dream $B$, whose strands can cross each other more than once. For each element $(i, j)$ of $D$ we will define its *antidiagonal index* by number $i + j - 1$. Then, by moving across the table from right to left and from top to bottom and associating to each element of $B$ its antidiagonal index, we will obtain a sequence $(i_1, \ldots, i_k)$, where $k$ is the number of crosses of $B$. The corresponding permutation $w$ is produced by multiplying the elementary transpositions $s_{i_1} \cdots s_{i_k}$ in compliance with the following rules:

$$s_i^2 = s_i$$
$$s_is_j = s_js_i \text{ if } |i - j| > 1$$
$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}.$$ 

In other words, we omit transpositions that decrease length (the corresponding operation is called the *Demazure product*). The set of all pipe dreams, whose Demazure products are equal to $w$, is denoted by $P(w)$. The set $R\mathcal{P}(w)$ is a subset of $P(w)$. Also in case of a reduced pipe dream the Demazure product is equivalent to the standard group operation.

**Example 1.3:** $B$ is a nonreduced pipe dream, belonging to $P(1423)$.

$$B = \begin{array}{|c|c|c|}
\hline
+ & + & + \\
\hline
+ & + & + \\
\hline
+ & + & + \\
\hline
\end{array}$$

The following theorem is a generalization of Theorem 1.1 for the case of Grothendeick polynomials (See [5], Theorem A).
Theorem 1.2: For an arbitrary element $w$ of $S_\infty$ the following equality holds:

$$
\mathfrak{G}_w(x) = \sum_{B \in P(w)} (-1)^{|B| - l(w)} x_B.
$$

(1.3)

Here by $|B|$ we denote the number of crosses of $B$.

2 Mitosis algorithm

The algorithm of mitosis recursion was developed by Ezra Miller and represents combination of inductive and combinatorial definitions of Schubert polynomials. More precisely, it allows us to obtain the set $\mathcal{RP}(ws_i)$ of reduced pipe-dreams of $ws_i$ from the set $\mathcal{RP}(w)$ if $l(ws_i) = l(w) - 1$. Here we will briefly recall the algorithm.

Consider an arbitrary pipe dream $D$. Introduce the following notations:

$$
\text{start}_i(D) = \min (\{j | (i, j) \notin D\} \cup \{n + 1\}).
$$

(2.1)

In other words, $\text{start}_i(D)$ is the maximal column index, such that each box in the $i$-th row, located strictly to the left from $(i, \text{start}_i(D))$ is marked with cross. Also by $\mathcal{J}_i(D)$ we denote the subset of the numbers of columns, located strictly to the left from $(i, \text{start}_i(D))$, such that for every $j \in \mathcal{J}_i(D)$ the box $(i + 1, j)$ is empty.

Then each element $p$ of $\mathcal{J}_i(D)$ can be associated with a new pipe dream $D_p$, constructed in the following way: first the cross in the box $(i, p)$ is deleted from $D$, then every cross in $i$-th row, located to the left from $(i, p)$ with in’s column index, belonging to $\mathcal{J}_i(D)$ is moved down to the empty box below it.

Now we can define the mitosis operator.

**Definition 2.1:** The $i$-th mitosis operator (denoted by $\text{mitosis}_i(D)$) sends $D$ to the set $\{D_p | p \in \mathcal{J}_i(D)\}$.

**Example 2.1:**

![Pipe Dreams and Mitosis Example](image)

Here $i = 3$, and $\mathcal{J}_i(D)$ contains 1, 2 and 4.

If $\mathcal{C}$ is a set of pipe dreams, then by $\text{mitosis}_i(\mathcal{C})$ we mean the union $\bigcup_{D \in \mathcal{C}} \text{mitosis}_i(D)$ over all elements of $\mathcal{C}$. If $\mathcal{J}_i(D) = \emptyset$, then $\text{mitosis}_i(D) = \emptyset$.

Now consider an element $w$ of $S_\infty$ and elemental transposition $s_i$, such that $l(ws_i) = l(w) - 1$. The main result of the corresponding article by E. Miller ([2]) is the following statement:
Theorem 2.1: Disjoint union \( \bigsqcup_{D \in \mathcal{RP}(w)} \) mitosis\(_i\)(\(D\)) coincides with the set \( \mathcal{RP}(w_s) \) of reduced pipe dreams of the permutation \(w_s\).

Together with the formula (1.3) this statement gives as the capability to obtain Schubert polynomials inductively, in terms of pipe dreams. Here we will not give the proof of this theorem (see [2]). Nevertheless, further we will use the same methods and terms, slightly modifying them.

Let’s start from the fact that theorem 2.1 states, that mitosis operator sends reduced pipe dreams from \( \mathcal{RP}(w) \) to \( \mathcal{RP}(w_s) \), so it is reasonable to assume, that the same is right for all elements of \( \mathcal{P}(w) \). Nevertheless, it is not quite true. For our further reasonings we will have to introduce the family of operations of the pipe dreams, which, in accordance with the original article ([2]) will be called chute moves.

Henceforth, we will distinguish between three types of chute moves. Under the action of a chute move some elements of a pipe dream change places and some are deleted in a certain way. These elements are located in two neighbouring rows of the pipe dream.

Definition 2.2: All three types of chute moves (which henceforth will be indexed by numbers 1,2 and 3) can be defined graphically in the following way:

1)  
\[
\begin{array}{cccc}
  + & + & + & + \\
  + & + & + & + \\
\end{array}
\rightarrow
\begin{array}{cccc}
  + & + & + & + \\
  + & + & + & + \\
\end{array}
\]

2)  
\[
\begin{array}{cccc}
  + & + & + & + \\
  + & + & + & + \\
\end{array}
\rightarrow
\begin{array}{cccc}
  + & + & + & + \\
  + & + & + & + \\
\end{array}
\]

3)  
\[
\begin{array}{cccc}
  + & + & + & + \\
  + & + & + & + \\
\end{array}
\rightarrow
\begin{array}{cccc}
  + & + & + & + \\
  + & + & + & + \\
\end{array}
\]

As we can see, for any chute move we can uniquely describe an inverse one. The following statement is right for both chute moves and inverse chute moves:

Theorem 2.2: If a pipe dream \(D\) belongs to the set \(\mathcal{P}(w)\), then the result of applying of any of the chute moves 1-3 also belongs to \(\mathcal{P}(w)\).

Proof: We will prove the statement only for the cases of chute move -1 and chute move-2, since chute move-3 can be introduced as a result of consistent applying of chute moves 2 and 1.

1) Let us suppose that the \(m\)-th and \((m + 1)\)-th rows of \(D\) look like:

\[
\begin{array}{cccc}
  B' & + & + & + \\
  A' & + & + & + \\
\end{array}
\]

so that we can apply chute move-1 (here numbers, located under the boxes of the \(i\)-th row and below the boxes of the \(i + 1\)-th row are the indexes of the corresponding antidiagonals
and the letters $A$, $B$, $A'$ and $B'$ are the corresponding subwords, obtained by "reading" of the pipe dream $D$ in the way, described above). Then the subword obtained by reading the $m$-th and $(m + 1)$-th rows is the following:

$$A s_{i+k} s_{i+k-1} \cdots s_{i+1}s_i A' B s_{i+k} s_{i+k-1} \cdots s_{i+2}s_{i+1} B'.$$

From the properties of the Demazure product it follows that this subword is equivalent to the following:

$$A A' s_{i+k} s_{i+k-1} \cdots s_{i+1}s_i s_{i+k} s_{i+k-1} \cdots s_{i+2}s_{i+1} B B'.$$

This means that for the case of chute move-1 we are reduced to proving the equivalence of the words

$$(s_{i+k} s_{i+k-1} \cdots s_{i+1}s_i s_{i+k} s_{i+k-1} \cdots s_{i+2}s_{i+1})$$

and

$$(s_{i+k-1} s_{i+k-2} \cdots s_{i+1}s_i s_{i+k-1} \cdots s_{i+2}s_{i+1})$$

with respect to the Demazure product. We will carry out the proof inductively (the parameter of induction is the *move length* denoted by $L$, by which we mean the length of the corresponding $2 \times L$ chuteable rectangle). The inductive basis is the simple equivalence of the words $(s_i s_{i+1} s_i) \cong (s_{i+1} s_i s_{i+1})$.

Let us suppose that our statement is proven for the case $L \leq k + 1$. Then for our initial word $(s_{i+k} s_{i+k-1} \cdots s_{i+1}s_i s_{i+k} s_{i+k-1} \cdots s_{i+2}s_{i+1})$ the following equivalence relations take place:

$$(s_{i+k} s_{i+k-1} \cdots s_{i+1}s_i s_{i+k} s_{i+k-1} \cdots s_{i+2}s_{i+1}) \cong$$

(obtained by shifting $s_{i+k}$ to the left until "collision" with $s_{i+k}$)

$$\cong (s_{i+k} s_{i+k-1}s_{i+k}) (s_{i+k-2} \cdots s_i s_{i+k-1} \cdots s_{i+1}) \cong$$

(obtained by applying the equivalence relation $s_{i+k} s_{i+k-1}s_{i+k} \rightarrow s_{i+k-1}s_{i+k} s_{i+k-1}$)

$$\cong (s_{i+k-1}s_{i+k}) (s_{i+k-1} \cdots s_i s_{i+k-1} \cdots s_{i+1}) \cong$$

(obtained by applying the induction hypothesis)

$$\cong (s_{i+k-1} s_{i+k}) (s_{i+k-2} \cdots s_i s_{i+k-1} \cdots s_i) \cong$$

(obtained by shifting $s_{i+k}$ to the right until "collision" with $s_{i+k-1}$)

$$\cong (s_{i+k-1} s_{i+k-2} \cdots s_{i+1}s_i s_{i+k} \cdots s_{i+1}s_i).$$

So the statement is proven for the case $L = k + 2$ and the induction step is fulfilled.
2) Let us now suppose that the $m$-th and $(m+1)$-th rows of $D$ look like the figure above, so that we can apply chute move-2. By carrying out the arguments analogously to 1), we reduce our statement to proving the equivalence of the words

$$(s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i)$$

and

$$(s_{i+k-1} s_{i+k-2} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i).$$

Again we apply the induction on the parameter $L$. The induction basis is now the equivalence relation $(s_i s_i) \equiv s_i$. Then, analogously to 1), for our initial word $$(s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i)$$

the following equivalence relations take place:

$$(s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} \hat{s}_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i) \equiv$$

( obtained by shifting $s_i$ to the right until "collision" with $s_{i+1}$)

$$(s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i) \equiv$$

( obtained by applying the equivalence relation $s_i s_{i+1} s_i \rightarrow s_{i+1} s_i s_{i+1}$)

$$(s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_i) \equiv$$

( obtained by applying the induction hypothesis)

$$(s_{i+k-1} s_{i+k-2} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_i) \equiv$$

( obtained by applying the equivalence relation $s_{i+1} s_i s_{i+1} \rightarrow s_{i+1} s_i$)

$$(s_{i+k-1} s_{i+k-2} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots \hat{s}_i) \equiv$$

( obtained by shifting $s_i$ to the left until "collision" until $s_{i+1}$)

$$(s_{i+k-1} s_{i+k-2} \ldots s_{i+1} s_i s_{i+k} s_{i+k-1} \ldots s_{i+1} s_i) \equiv$$

Q.E.D.

**Remark 2.1:** Note that in case of reduced pipe dreams chute moves 2 and 3 cannot be used, because their chuteable rectangles are given by double crossing of the pair of strands:
Also in case of applying chute move-1 to a reduced pipe dream preservation of the permutation can easily be proven graphically. Indeed, all stands, except the pair involved in the conversion, remain “untouched”. Also the crossing of two involved strands in the upper right corner of the chutable rectangle is replaced with the crossing in the lower left corner and the exits of these two strands stay the same (see the figure below).

Now we can redefine the mitosis algorithm in terms of chute moves:

**Proposition 2.3:** Let \( D \) be a pipe dream and \( j_{\text{min}} \) be a minimal column index, such that \((i+1, j) \notin D\) and for any \( p \leq j \) \((i, p)\) belongs to \( D \). Then each \( D_p \in \text{mitosis}_i(D) \) obtained from \( D \) by

1) deleting \((i, j_{\text{min}})\), and then,

2) moving to the right, one by one applying chute moves-1, so that \((i, p)\) is the last cross, moved from the \( i \)-th row to \( i + 1 \)-th row.

From this proposition we can see that all elements of \( \text{mitosis}_i(D) \) correspond to the same permutation, because all of them are the results of chute moves, applied to \( D \setminus (i, j_{\text{min}}) \). Now we are ready to prove the main statement of this section:

**Theorem 2.3:** Let \( w \) be an element of \( S_\infty \), \( s_i \) — an elementary transposition and let \( D \) belong to \( \mathcal{P}(w) \), with the condition that \( l(ws_i) = l(w) - 1 \). Then, if the set \( \text{mitosis}_i(D) \) is not empty, then it lies entirely in either \( \mathcal{P}(ws_i) \) or \( \mathcal{P}(w) \).

**Proof:** From our reasoning above we can see that if \( D \setminus (i, j_{\text{min}}) \) belongs to \( \mathcal{P}(ws_i) \) (accordingly, to \( \mathcal{P}(w) \)), then the same is true of all the elements of \( \text{mitosis}_i(D) \).

Let \( D \) look like

![Diagram](image-url)
For every pipe dream $B$ the corresponding word will be denoted by $\text{word}(B)$. For every word $v$ and pipe dream $B$ the corresponding Demazure product will be denoted by $\text{Demaz}(v)$ ($\text{Demaz}(B)$ accordingly). Then the following equalities hold:

$$w = \text{Demaz}(D) = \text{Demaz}(AA's_{i+j_{\min}-1} \ldots s_{i}B's_{i+j_{\min}-1} \ldots s_{i+1}B).$$

By applying to the left word a transform, analogous to chute move-1, and given that $s_i$ commutes with the subword $B'$, obtain

$$w = \text{Demaz}(D) = \text{Demaz}(AA's_{i+j_{\min}-2} \ldots s_{i}B's_{i+j_{\min}-1} \ldots s_{i+1}Bs_i) = \text{Demaz}(\text{word}(D')s_i).$$

Here $D'$ is the result of removing $(i, j_{\min})$ from $D$. Denote by $\tilde{w}$ the permutation $\text{Demaz}(\text{word}(D'))$. Then, according to the definition of the Demazure product, two cases are possible:

1) $l(\tilde{w}s_i) = l(\tilde{w}) - 1$. Then we have $w = \text{Demaz}(\text{word}(D')s_i) = \tilde{w}$ and $D'$ belongs to $\mathcal{P}(\text{word}(D')s_i)$.

2) $l(\tilde{w}s_i) = l(\tilde{w}) + 1$. Then we have $w = \text{Demaz}(\text{word}(D')s_i) = \tilde{w}s_i$, which means that $\tilde{w} = w s_i$ and $D'$ belongs to $\mathcal{P}(w s_i)$ (note, that in both cases $l(ws_i) = l(w) - 1$). Q.E.D.

In compliance with the proven theorem we divide $\mathcal{P}(w)$ into three disjoint sets: $\mathcal{P}_s(w)$ (the set of all pipe dreams, which are sent to $\mathcal{P}(ws_i)$), $\mathcal{P}_t(w)$ (the set of all pipe dreams, which are sent to $\mathcal{P}(w)$), and $\mathcal{P}_\emptyset(w)$ (the set of all pipe dreams, which are sent to the empty set). Here and further by mitosis we mean $\text{mitosis}_s$ with condition that $l(ws_i) = l(w) - 1$. This partition will be used later.

3 Intron mutations

Let $D$ be an arbitrary pipe dream with a fixed row index $i$. Index the boxes in in $i$-th and $(i+1)$-th rows as shown in the following figure:

\[
\begin{array}{cccccc}
(i) & 1 & 2 & 3 & 4 & \ldots \\
(i+1) & 1 & 3 & 5 & 7 & \ldots \\
\end{array}
\]

Then henceforth by an intron we mean a $2 \times m$ rectangle, located in these two adjacent rows such that:

1) the first and the last boxes of this rectangle are empty,

2) no column can be located to the right from a $\square$ column or a $\blacksquare$ column and no column can be located to the right from a $\blacksquare$ column. (The first (last) box of a rectangle is the box with the maximal (minimal) index according to the ordering described above.)

An intron $C$ is maximal, if the empty box with largest index before $C$ (if there is one) is located in the $i+1$-th row and the empty box with smallest index after $C$, is located in
Lemma 3.1: Let \( D \) be a pipe dream and \( C \subseteq D \) be an intron. Then by applying a sequence of chute moves and inverse chute moves we can transform \( C \) to a new intron \( \tau(C) \) with the following properties:

1) the set of \( \square \) columns in \( C \) coincide with the set of \( \square \) columns in \( \tau(C) \), and

2) the number \( c_i \) of crosses in the \( i \)-row of \( C \) coincide with the number \( \tilde{c}_{i+1} \) of crosses in the \( i+1 \)-th row of \( \tau(C) \) and vice versa.

Proof: Suppose that \( c_i \geq \tilde{c}_{i+1} \). Then we will carry out the proof inductively with the parameter \( c = c_i - \tilde{c}_{i+1} \). In case \( c = 0 \) we obviously have \( C = \tau(C) \). Now if \( c > 0 \) we consider the leftmost \( \square \) column (denote its index by \( p \)). Moving to the left from this column we will, sooner or later, find a column of the type \( \square \) or \( \square \). If it is a \( \square \) column, then we can apply chute move-1 and thereby chute the cross from the \( i \)-th row to the \( i+1 \)-th one. Since the result of this conversion will also be an intron, the proof is reduced to the induction hypothesis. If it looks like a \( \square \), then, owing to the fact that \( c > 0 \) and, thereafter, there is more than one \( \square \) column in \( C \), the corresponding fragment of \( C \) will look like

\[
\begin{array}{ccccc}
(i) & + & + & + & + \\
(i+1) & + & + & + & + \\
\end{array}
\]

Then by applying chute move-2 and reverse chute move-3 the way it shown on the following figure, we will bring the corresponding fragment of \( C \) to a form

\[
\begin{array}{ccccc}
(i) & + & + & + & + \\
(i+1) & + & + & + & + \\
\end{array}
\]

and thereby again chute the cross from the \( i \)-th row to the \( i+1 \)-th one. Thus the proof is again reduced to the induction hypothesis.

In case \( c_i < \tilde{c}_{i+1} \) we just flip the argument \( 180^\circ \).

The transformation \( \tau \) is called \textit{intron mutation}. Note, that the intron \( \tau(C) \) is defined uniquely and, by construction, \( \tau(\tau(C)) = C \), i.e. \( \tau \) is an involution.

Now we are ready to the main statement of this section:

Theorem 3.1: Let \( w \) be an element of \( S_\infty \). Then for each \( i \in \mathbb{N} \) there is an involution \( \tau_i : \mathcal{P}(w) \rightarrow \mathcal{P}(w) \), such that for any \( D \in \mathcal{P}(w) \) the following conditions take place:

1) \( \tau_i(D) \) coincides with \( D \) in all rows with indexes, different from \( i \) and \( i+1 \).
2) $\text{start}_i(D) = \text{start}_i(\tau_i(D))$ and $\tau_i(D)$ agrees with $D$ in all columns with indices strictly less than $\text{start}_i(D)$.

3) $l_i(\tau_i(D)) = l_{i+1}(D)$ (here $l_i(-)$ - is the number of crosses in the $r$-th row, located to the right or in column with index $\text{start}_i(-)$)

**Proof:** Let $D$ belong to $P(w)$. Consider all crosses of the union of the $i$-th and $(i + 1)$-th rows, located to the right or in column with index $\text{start}_i(D)$. Then, according to the definition of $\text{start}_i(D)$, we can find a minimal rectangle with empty last box, starting at the column $\text{start}_i(D)$ and containing all these crosses. Since the first box of this rectangle is also empty, it can be uniquely represented in form of a disjoint union of maximal introns and rectangles, completely filled with crosses. Apply to an every maximal intron the intron mutation, described above. Since every mutation is a sequence of chute moves and reverse chute moves, the obtained pipe dream obviously belongs to $P(w)$. Owing the fact that each mutation is an involution and that the result of applying a mutation to a maximal intron is also a maximal intron, the obtained transformation is also an involution. Properties 1)-3) are obvious from the construction scheme. ■

**Remark 3.1:** Note that the partition $P(w) = P_\emptyset(w) \sqcup P_s(w) \sqcup P_I(w)$ is invariant under the constructed involution. This fact will be used henceforth.

### 4 Mitosis theorem

In the second section we have defined the way mitosis acts on the set $P(w)$. Nevertheless, in order to prove the main theorem of this article we will have to slightly modify it’s initial definition:

**Definition 4.1:** Let $l(ws_i) = l(w) - 1$, $D$ belong to $P(w)$ and $J_i(D) = j_1, \ldots, j_k$. Then operator $\text{mitosis}'_i$ sends $D$ to the set

$$\{ D_{j_1}; D_{j_1} + D_{j_2}; D_{j_2}; \ldots; D_{j_{k-1}}; D_{j_{k-1}} + D_{j_k}; D_{j_k} \}.$$ 

Here by ”sum” we mean the union of the elements of the corresponding pipe dreams.

(As we can see, the elements of the set $\text{mitosis}'_i(D)$ form some kind of a chain, where links are elements of $D_{j_m}$ and the result of the two links’ cohesion is a sum of the corresponding pipe dreams).

Let us show that pipe dreams of $\text{mitosis}'_i(D)$ represent the same permutation. Indeed, let $D_{j_m}$ and $D_{j_{m+1}}$ look like

\[
\begin{array}{ccccc}
D_{j_m} & \cdots & + & + & + \\
& & + & + & + \\
& & + & + & + \\
\end{array}
\]

\[
\begin{array}{ccccc}
D_{j_{m+1}} & \cdots & + & + & + \\
& & + & + & + \\
& & + & + & + \\
\end{array}
\]

Then the sum $D_{j_m} + D_{j_{m+1}}$ looks like
\[ D_{jm} + D_{jm+1} \ldots \begin{array}{ccc} + & + & + \\
+ & + & + \\
\end{array} \ldots \]

It’s easy to see that \( D_{jm} + D_{jm+1} \) is obtained from \( D_{jm} \) by applying the inverse chute move-2. Consequently, \( D_{jm} + D_{jm+1} \) represent the same permutation. Thus, the partition \( \mathcal{P}(w) = \mathcal{P}_s(w) \cup \mathcal{P}_t(w) \cup \mathcal{P}_d(w) \), constructed for mitosis, is also preserved by \( \text{mitosis}'_i \).

Now we are ready to formulate the main theorem of this paper:

**Theorem 4.1:** If the condition \( l(ws_i) = l(w) - 1 \) is satisfied, then the following equalities take place:

1) \[ \bigcup_{D \in \mathcal{P}_s(w)} \text{mitosis}'_i(D) = \mathcal{P}(ws_i) \]

2) \[ \bigcup_{D \in \mathcal{P}_t(w)} \text{mitosis}'_i(D) = \mathcal{P}(\varnothing) \].

**Proof:** The disjunctivity of the unions on the left sides of the equalities 1) and 2) is obvious: indeed, each element of the image of \( \text{mitosis}'_i \) agrees with its preimage everywhere, except \( i \)-th and \((i + 1)\)-th rows. In these two adjacent rows they also coincide to the right from the leftmost column of the type \[ ] or \[ \). The rest of the diagram is restored uniquely, according to the corresponding algorithm.

Remind that if condition \( l(ws_i) = l(w) - 1 \) is satisfied, then \( \mathcal{G}_{ws_i}(x) \) is obtained from \( \mathcal{G}_w(x) \) by applying on the operator \( \pi_i \).

Consider now arbitrary element \( D \) of the \( \mathcal{P}_s(w) \), such that \( J = |\mathcal{J}_i(D)| \). Then \( x^D = x^J x^{D'} \), where \( D' \) is a pipe dream, obtained from \( D \) by removing all crosses in the columns with indices, belonging to \( \mathcal{J}_i(D) \). We also have the following equality:

\[
\sum_{E \in \text{mitosis}'_i(D)} (-1)^{|E| - l(ws_i)} x_E = (-1)^{|D| - l(w)} \left( \sum_{d=1}^{J} x^{J-d} d_{i+1} - \sum_{d=1}^{J-1} x^{J-d+1} d_{i+1} \right) x^{D'} =
\]

\[
= (-1)^{|D| - l(w)} \pi_i(x^{J}_i) x^{D'}. \]

Now, if \( \tau_i(D) = D \), then \( x^{D'} \) is symmetric in the variables \( x_i \) and \( x_{i+1} \). Consequently,

\[
(-1)^{|D| - l(w)} \pi_i(x^{J}_i) x^{D'} = (-1)^{|D| - l(w)} \pi_i(x^{J}_i x^{D'}) = \pi_i((-1)^{|D| - l(w)} x^D). \]

If \( \tau_i(D) \neq D \), then \( x^{D'} + s_i(x^{D'}) \) is symmetric in the variables \( x_i \) and \( x_{i+1} \) and for this sum we have

\[
(-1)^{|D| - l(w)} \pi_i(x^{J}_i)(x^{D'} + s_i(x^{D'}))1 = (-1)^{|D| - l(w)} \pi_i(x^{J}_i(x^{D'} + s_i(x^{D'}))) = \pi_i((-1)^{|D| - l(w)} (x^D + x_{\tau_i(D)})). \]

(Recall, that \( \tau_i(D) \) also lies in \( \mathcal{P}_s(w) \) and \(|D| = |\tau_i(D)|\))

Thus, by grouping the elements of \( D \in \mathcal{P}_s(w) \) in accordance with the involution \( \tau_i \), we obtain the following equality:

\[
\sum_{E \in \text{mitosis}'_i(\mathcal{P}_s(w))} (-1)^{|E| - l(ws_i)} x_E = \pi_i( \sum_{D \in \mathcal{P}_s(w)} (-1)^{|D| - l(w)} x_D ). \tag{4.1} \]
The analogous equality can also be obtained in case of $P_I(w)$. Now if $D \in P_I(w)$, i.e. $D' = D$ and $\tau_i(D) = D$, then $D$ is symmetric in the variables $x_i$ and $x_{i+1}$ and, consequently,

$$(1) = (-1)^{|D| - l(w)}x_D.$$ 

Also if $\tau_i(D) \neq D$ then the sum $x_D + x_{\tau_i(D)}$ is symmetric in the variables $x_i$ and $x_{i+1}$ and, consequently,

$$(1) = (-1)^{|D| - l(w)}(x_D + x_{\tau_i(D)}).$$

By grouping the elements of $P_I(w)$ with accordance with the involution $\tau_i$, we obtain the following equality

$$\pi_i \left( \sum_{D \in P_I(w)} (-1)^{|D| - l(w)}x_D \right) = \sum_{D \in P_I(w)} (-1)^{|D| - l(w)}x_D. \tag{4.2}$$

Using these three equalities, we obtain the following result:

$$\sum_{E \in mitosis'(P_s(w))} (-1)^{|E| - l(ws_i)}x_E + \sum_{E \in mitosis'(P_I(w))} (-1)^{|E| - l(ws_i)}x_E + \sum_{D \in P_E} (-1)^{|D| - l(w)}x_D =$$

$$= \pi_i(\mathbf{G}_{ws_i}(x)) = \mathbf{G}_{ws_i}(x) = \sum_{B \in P(ws_i)} (-1)^{|B| - l(ws_i)}x_B. \tag{4.3}$$

Now in order to prove 2) we only need to construct for an arbitrary $D$ from $P_I(w)$ a preimage in $P_I(w)$. Consider the first column in the corresponding adjacent pair of rows, which is different from $\square$ and $\square$. If its a $\square$ column, then, by applying the transformation, inverse to the one on the page 11, we will transform each $\square$ column on the left to the type $\square$ and thereby obtain the required element of $P_I(w)$. And if it’s $\square$, then, by applying the inverse chute move-1, we will transform $D$ to $B \setminus (i,j_{min})$, where $B$ is a pipe dream with nonempty set $mitosis'_i(B)$. It follows from the theorem 2.1 that $B$ belongs to $P_I(w)$. Obviously, $B$ is a preimage of $D$. Thus the action of $mitosis'_i$ on $P_I(w)$ is surjective so 2) is proved. This means, that the second and third sums on the left side of the equality (4.3) are canceling out. By inputting the value $(-1)$ to the equality we will obtain the proof of 1. 

Thus, the constructed algorithm allows us to obtain all elements of $P(ws_i)$ from the set $P(w)$ in case $l(ws_i) = l(w) - 1$, giving us the inductive method of constructing Grothendieck polynomials, but now in terms of pipe dreams.

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References

[1] Bergeron, N., & Billey, S. (1993). RC-graphs and Schubert polynomials. *Experimental Math*, 2 (4): 257-269.

[2] Miller, E. (2003). *Alternating formulas for K-theoretic Quiver polynomials*. Available from: http://arxiv.org/pdf/math/0312250.pdf

[3] Miller, E. (2002). *Mitosis recursion for coefficients of Schubert polynomials*. Available from: http://arxiv.org/pdf/math/0212131v1.pdf

[4] Sara C. Billey, William Jockusch, and Richard P. Stanley (1993), *Some combinatorial properties of Schubert polynomials*. J. Algebraic Combin. 2 (1993), no. 4, 345–374.

[5] Allen Knutson and Ezra Miller (2001), *Grobner geometry of Schubert polynomials*. Annals of Mathematics, 161 (2005), 1245–1318.