MINIMAL GENERATING SETS FOR THE FIRST SYZYGINES OF A MONOMIAL IDEAL

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The object of this note is to produce two minimal generating sets of the first syzygies of a monomial ideal, given the minimal generating set of the ideal. In the first section we set up notation and give a preliminary result. In the second section we describe one small generating set that is not quite minimal. In the third section we describe a different but analogous small generating set. In the final section we describe how to reduce both of these to minimal generating sets.

1. Notation

Let $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $k$. Let $I$ be a monomial ideal in $R$. Let $G(I)$ be the minimal generating set of $I$. For any monomial $m$ in the lcm-lattice of $G(I)$, (i.e. the least common multiple lattice of $G(I)$ ordered by divisibility), define

$$\Gamma_m = \{ \tau : \tau \text{ is a square free monomial, } \tau|m, \text{ and } \frac{m}{\tau} \text{ is in } I \}$$

where

$$\frac{m}{\tau}$$

means the monomial obtained by dividing $m$ by $\tau$.

Also define

$$L_{<m} = \{ S : S \subset G(I), n_S|m, \text{ but } n_S \neq m \}$$

where $n_S$ is the least common multiple of the monomials in $S$. Clearly, $n_S$ is in the lcm-lattice of $G(I)$. The vertices of $L_{<m}$ may be identified with the elements of the minimal generating set, $G(I)$.

$\Gamma_m$ is an abstract finite simplicial complex. (Think of the square free monomials as sets, where divisibility is replaced by containment.) The facets (maximal faces) of $\Gamma_m$ are among the faces of the form

$$\sqrt{m}$$

where $\gamma$ is in $G(I)$ and $\gamma|m$. Here the root sign

$$\sqrt{n}$$

stands for the largest square free monomial dividing the monomial $n$. This is sometimes called the support of $n$. 

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$L_{<m}$ is also an abstract finite simplicial complex. (Use the usual containment relation.)

For each $\Gamma_m$ which is disconnected, let $C_{1,m}, C_{2,m}, \ldots, C_{k_m,m}$ be the connected components (as finite simplicial complexes).

Select in $C_{i,m}$, for each $i$, one facet of the form $\sqrt{m}/\gamma$. From all the $\gamma$’s such that $\sqrt{m}/\gamma$ equals this face select one and label it $\gamma_{i,m}$. Clearly, we can make this choice because all facets are of this form.

(Later we will make similar, but somewhat more natural choices for $L_{<m}$.)

For each disconnected $\Gamma_m$ let

$$Y_m = \left\{ \frac{m}{\gamma_{i,m}} \otimes \gamma_{i,m} - \frac{m}{\gamma_{j,m}} \otimes \gamma_{j,m} \right\} \subset R \otimes_k R$$

for all pairs $i, j$ such that $1 \leq i < j \leq k_m$. (If $\Gamma_m$ is connected let $Y_m = \emptyset$, the empty set.)

**Proposition 1.** For each $m$ such that $\Gamma_m$ is disconnected and for each pair $\gamma_{i,m}, \gamma_{j,m}$ where $i \neq j$,

$$m = \text{lcm}(\gamma_{i,m}, \gamma_{j,m})$$

where the right hand side means the least common multiple of the two $\gamma$’s.

**Proof.** If not, there exists a variable, $x$, such that $x \mid \frac{m}{\gamma_{i,m}}$ and $x \mid \frac{m}{\gamma_{j,m}}$. But then the faces $\sqrt{m}/\gamma_{i,m}$ and $\sqrt{m}/\gamma_{j,m}$ overlap and cannot be in distinct components.

Now, for all $n$ and $m$ in the lcm-lattice, let

$$Y_{\leq m} = \bigcup_{n \mid m} Y_n$$

$$Y = \bigcup_n Y_n$$

For all $\gamma \neq \gamma'$ in $G(I)$ let $n_{\gamma, \gamma'}$ be the $\text{lcm}(\gamma, \gamma')$.

Consider the set

$$\left\{ \frac{n_{\gamma, \gamma'}}{\gamma} \otimes \gamma - \frac{n_{\gamma, \gamma'}}{\gamma'} \otimes \gamma' \right\}$$

From Diana Taylor’s resolution we conclude that this set generates the first syzygies of the ideal $I$.

For a given $m$ in the lcm-lattice consider all pairs $\gamma, \gamma'$ such that $n_{\gamma, \gamma'} = m$.

Let

$$T_m = \left\{ \frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma'} \otimes \gamma' \right\}$$

quantified over all such pairs.

Let

$$T = \bigcup_m T_m$$

Clearly $T$ is the above set that generates the first syzygies.
If every element of a subset $Q$ of an $R$ module is a linear combination over $R$ of elements of another subset $P$ we shall say that “$P$ spans $Q$”. We shall also say that elements of $Q$ are “in the span of $P$”.

2. First Generating Set

**Theorem 1.** $Y$ spans $T$.

*Proof.* We will prove by induction on the lcm-lattice that

$$Y_{\leq m} \text{ spans } T_m$$

Clearly this will suffice to prove the theorem.

Let $\gamma$ and $\gamma'$ be any pair of elements in $G(I)$. Let $m = n_{\gamma, \gamma'}$ and consider the element

$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma'} \otimes \gamma'$$

It suffices to prove that this is in the span of $Y_{\leq m}$.

**Case 1:**

$\sqrt{\frac{m}{\gamma}}$ and $\sqrt{\frac{m}{\gamma'}}$ are faces of the same connected component of $\Gamma_m$.

Then there exists a sequence of elements of $G(I)$,

$$\gamma = \gamma_1, \gamma_2, \ldots, \gamma_l = \gamma'$$

such that $\sqrt{\frac{m}{\gamma_i}}$ and $\sqrt{\frac{m}{\gamma_{i+1}}}$ overlap for $i = 1, 2, \ldots, l - 1$.

So, for each pair $\gamma_i, \gamma_{i+1}$ there exists a variable $x_i$, such that

$$x_i \frac{m}{\gamma_i} \text{ and } x_i \frac{m}{\gamma_{i+1}}$$

for $i = 1, 2, \ldots, l - 1$. Hence $n_{\gamma_i, \gamma_{i+1}} \neq m$

Let $n_{\gamma_i, \gamma_{i+1}} = n_i$, for $i = 1, 2, \ldots, l - 1$. Then $n_i|m$, and $n_i \neq m$. By induction we assume that $Y_{\leq n_i} \text{ spans } T_{n_i}$.

Now,

$$\frac{n_i}{\gamma_i} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in $T_{n_i}$ by the definition of $T_{n_i}$. We have that

$$Y_{\leq n_i} \subset Y_{\leq m}$$

since $n_i|m$.

So,

$$\frac{n_i}{\gamma_i} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in the span of $Y_{\leq m}$ for $i = 1, 2, \ldots, l - 1$.

Thus

$$\frac{m}{n_i} \left( \frac{n_i}{\gamma_i} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1} \right)$$

which equals

$$\frac{m}{\gamma_i} \otimes \gamma_i - \frac{m}{\gamma_{i+1}} \otimes \gamma_{i+1}$$
is in the span of $Y_{\leq m}$.
But
$$\sum_{i=1}^{l-1} \left( \frac{m}{\gamma_i} \otimes \gamma_i - \frac{m}{\gamma_{i+1}} \otimes \gamma_{i+1} \right)$$
telescopes to
$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma'} \otimes \gamma'$$
. So this is also in the span of $Y_{\leq m}$. Thus Case 1 is proved.

**Case 2:**
$\sqrt{\frac{m}{\gamma}}$ and $\sqrt{\frac{m}{\gamma'}}$ are faces of different connected components of $\Gamma_m$. Say
$$\sqrt{\frac{m}{\gamma}}$$ is in $C_{j,m}$
and
$$\sqrt{\frac{m}{\gamma'}}$$ is in $C_{k,m}$
Thus $\sqrt{\frac{m}{\gamma}}$ and $\sqrt{\frac{m}{\gamma'}}$ are in the same connected component $C_{j,m}$. Hence by Case 1,
$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma_{j,m}} \otimes \gamma_{j,m}$$
is in the span of $Y_{\leq m}$.
Similarly,
$$\frac{m}{\gamma_{k,m}} \otimes \gamma_{k,m} - \frac{m}{\gamma'} \otimes \gamma'$$
is in the span of $Y_{\leq m}$.
But
$$\frac{m}{\gamma_{j,m}} \otimes \gamma_{j,m} - \frac{m}{\gamma_{k,m}} \otimes \gamma_{k,m}$$
is in $Y_m$ itself.
These three add up to
$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma'} \otimes \gamma'$$
\[\square\]

3. **Second Generating Set**

We now give the analogous results for the simplicial complexes $L_{<m}$.
For each $L_{<m}$ which is disconnected let the $C_{i,m}$'s be the connected components as in the previous case. Let $\gamma_{i,m}$ be any vertex in $C_{i,m}$. Define the set $Y_{m,L}$ by
$$Y_{m,L} = \left\{ \frac{m}{\gamma_{i,m}} \otimes \gamma_{i,m} - \frac{m}{\gamma_{j,m}} \otimes \gamma_{j,m} \right\} \subset R \otimes_k R$$
using these new $\gamma_{i,m}$'s
Proposition 2. For each $m$ such that $L_{<m}$ is disconnected and for each pair $\gamma_{i,m}, \gamma_{j,m}$ where $i \neq j$,

$$m = \text{lcm}(\gamma_{i,m}, \gamma_{j,m})$$

Proof. This is even easier to prove than Proposition 1, since if

$$m \neq \text{lcm}(\gamma_{i,m}, \gamma_{j,m})$$

then there is actually an edge of $L_{<m}$ between the two $\gamma$'s. \hfill \square

Define $Y_{\leq m,L}$ and $Y_L$ analogously to $Y_{\leq m}$ and $Y$. $T_m$ and $T$ are defined as before.

Theorem 2. $Y_L$ spans $T$. \hfill 1

Proof. Exactly as before we prove by induction on the lcm-lattice that

$$Y_{\leq m,L} \text{ spans } T_m$$

Let $\gamma$ and $\gamma'$ be any element of $G(I)$. Let $m = n_{\gamma,\gamma'}$ and consider the element

$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma'} \otimes \gamma'$$

It suffices to prove that this is in the span of $Y_{\leq m,L}$.

Case 1: $\gamma$ and $\gamma'$ are vertices of the same connected component of $L_{<m}$.

Then there exists a sequence of elements of $G(I)$,

$$\gamma = \gamma_1, \gamma_2, \ldots, \gamma_l = \gamma'$$

such that the edge $\{\gamma_i, \gamma_{i+1}\}$ is in $L_{\leq m}$ for $i = 1, 2, \ldots, l - 1$.

Let $n_i = \text{lcm}(\gamma_i, \gamma_{i+1})$. For $i = 1, 2, \ldots, l - 1$.

Then $n_i | m$, but $n_i \neq m$. Thus

$$\frac{n_i}{\gamma} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in $T_{n_i}$.

By induction we assume that

$$\frac{n_i}{\gamma} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in the span of $Y_{\leq n_i,L}$ and hence in the span of $Y_{\leq m,L}$.

The argument now goes exactly as before to complete Case 1.

Case 2: $\gamma$ and $\gamma'$ are vertices of different connected components of $L_{\leq m}$.

\hfill \footnote{The author needed the first generating set. He slightly modified the proof given here, which is by Victor Reiner, to get the proof in the previous section.}
The argument again goes as before, and I leave it as an exercise to complete the proof of the theorem.

4. Minimal Generating Sets for the First Syzygies

By Theorem 2.1, page 523 of [GPW], and Proposition 1.1 of [BH] we have that the minimal number of 1st syzygies of $I$ (which is the same as the minimal number of 2nd syzygies of $R/I$) is given by the formula

$$b_2(R/I) = \sum_m \dim \tilde{H}_0(\Gamma_m) = \sum_m \dim \tilde{H}_0(L_{<m})$$

where $m$ is quantified over all monomials in the lcm-lattice.

For each $m$ in the lcm-lattice, $\dim \tilde{H}_0(\Gamma_m) = \dim \tilde{H}_0(L_{<m})$ is one less than $k_m$, the number of connected components of $\Gamma_m$, respectively, $L_{<m}$.

For each $m$ such that $\Gamma_m$, respectively, $L_{<m}$ is disconnected we consider the set

$$Z_m, \text{ respectively, } Z_{m,L} = \left\{ \frac{m}{\gamma_{i,m}} \otimes \gamma_{i,m} - \frac{m}{\gamma_{1,m}} \otimes \gamma_{1,m} \right\}$$

for $i = 2, 3, \ldots, k_m$.

The cardinality of this set is also one less than $k_m$. On the other hand, by taking differences, we see that this set spans $Y_m$, respectively, $Y_{m,L}$. Quantifying over all $m$ in the lcm-lattice, we see that the set

$$Z, \text{ respectively, } Z_L = \bigcup_m Z_m, \text{ respectively, } Z_{m,L}$$

spans $Y$, respectively, $Y_L$ and hence $T$ by Theorem 1, respectively Theorem 2.

By the formula cited above from [GPW] we see that the cardinality of $Z$, respectively, $Z_L$ is the number of minimal generators of the first syzygies.

Since $Z$, respectively, $Z_L$ has the correct cardinality and spans $T$, each must be a minimal generating set for the first syzygies.

References

[BH] W. Bruns and J. Herzog, Semigroup rings and simplicial complexes, J. Pure. Appl. Algebra 122 (1997), 185-208.

[GPW] V. Gasharov, I. Peeva and V. Welker, The lcm-lattice in monomial resolutions, Math. Res. Lett. 6, (1999), 521-532.