Stationary problems for equation of the KdV type and dynamical $r$-matrices

P.P. Kulish

Department of Theoretical Physics and IFIC, Valencia University,
E-46100, Brjassot (Valencia), Spain

S. Rauch-Wojciechowski

Department of Mathematics, Linkoping University, S-58183, Linkoping, Sweden

A.V. Tsiganov

Department of Mathematical and Computational Physics, Institute of Physics,
St.Petersburg University, 198 904, St.Petersburg, Russia

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Abstract

We study a quite general family of dynamical $r$-matrices for an auxiliary loop algebra $\mathcal{L}(su(2))$ related to restricted flows for equations of the KdV type. This underlying $r$-matrix structure allows to reconstruct Lax representations and to find variables of separation for a wide set of the integrable natural Hamiltonian systems. As an example, we discuss the Henon-Heiles system and a quartic system of two degrees of freedom in detail.

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1 Introduction

The main aim of this paper is to study the $r$-matrix formulation for mechanical systems embedded as restricted flows into KdV type equations and to investigate its connection with Lax representations and with the method of separation of variables. Lax representations are essential for solving these mechanical systems either by linearization on the Jacobian of determinant curve of the Lax matrix or by determining variables of separation through the functional Bethe ansatz.

Restricted flows are easiest understood as stationary flows of soliton hierarchies with sources. Many well known integrable mechanical systems, such as Henon-Heiles, Garnier, Neumann (and many other) can be embedded into the KdV and other soliton hierarchies as restricted flows.

The main advantage of such embedding is that important structure elements for restricted flows, such as Lax representation, bi-Hamiltonian formulation, Newton parametrization can be systematically derived from the underlying soliton hierarchies. But it is not the case, as yet, for the $r$-matrix formulation of these mechanical systems. There is no direct way of transporting the well known $r$-matrix formulation for soliton hierarchies to their stationary and restricted flows. The underlying $r$-matrix structure for finite-dimensional systems appears to be more complicated then for integrable PDE’s. A new type of, so called, dynamical $r$-matrices depending both on the spectral parameter and on dynamical variables has to be considered. The first example of such $r$-matrix has been found for the Calogero-Moser system and then for the parabolic and elliptic separable potentials.

It appears that the parabolic and elliptic $r$-matrices belong to a quite general family of dynamical $r$-matrices which we introduce in this paper through a convenient ansatz which generalizes our $r$-matrix from. We study the general algebraic properties of these $r$-matrices and derive Lax representations through proper specializations. These specializations are guided by the known integrable cases of natural Hamiltonian systems related to restricted flows for equations of the KdV type. Further we use these Lax representations for finding variables of separation explain how starting with variables of separation these Lax representations be reconstructed by using the Sklyanin approach.

The paper is organized as follows.

In section 2 we consider additive algebraic deformations of linear $r$-matrix algebras on an auxiliary loop algebra $g = L(su(2))$. Kernels of these $r$-matrices have either rational, trigonometric and elliptic dependence on the spectral parameter. We consider deformations $L(\lambda) = L_0(\lambda) + F(\lambda) \in L(su(2))$ of a certain Lax matrix $L_0(\lambda) \in L(su(2))$, which satisfies the standard $r$-matrix brackets. These deformed Lax matrices obey a dynamical $rs$-matrix algebra. From dynamical point of view the deformed Lax matrices correspond to systems with additively shifted integrals of motion

$$I_k^{\text{new}} = I_k^{\text{old}} + f_k.$$  

Further for $r$-matrices of the $XXZ$ and $XXX$ type we consider projection of deformations $F(\lambda)$ on finite-dimensional Poisson (ad$^*$-invariant) subspaces $L_{M,N}(su(2))$. Their Lax matrices $L_{M,N}(\lambda)$ obey the $rs$-matrix brackets as well. At the end of section 2 we construct Lax pairs from the $rs$-matrix algebra and prove that these Lax pairs are related to stationary flows for equations of the KdV type.

In section 3 we express matrix elements of the matrices $L_0(\lambda)$ and $L_{M,N}(\lambda)$, defined on a direct sum of loop algebras $\oplus L(su(2))$, through canonically conjugated variables of separation and prove that these matrices obey the $rs$-matrix brackets. Such matrices describe a geodesic motion and a potential motion on a Riemannian manifold with complex diagonal meromorphic Riemannian metrics of special type. Corresponding potentials are the finite gap potentials on these manifolds. The problem of quantization is discussed briefly.

In section 4 we consider finite-dimensional integrable systems of natural type and investigate three types of canonical transformations of variables prescribed by the linear $r$-matrix structure. Two of these are pure coordinate transformations to generalized elliptic coordinates and to Jacobi coordinates, while the third one provides an example of a momentum dependent change of variables. We exemplify of our approach with the Henon-Heiles system and a quartic potential of two degrees of freedom.

In the concluding remarks we discuss shortly an application of our constructions to other nonlinear integrable PDE’s such as the AKNS and the sine-Gordon hierarchies.
2 Deformations of linear $r$-matrix algebra

2.1 Additive deformations

Let us recall basic algebraic constructions in the $r$-matrix scheme \[25, 26\] for loop algebras, which we shall use in this paper. A loop algebra $g = \mathcal{L}(a)$ is defined as follows. Let $a$ be a Lie algebra, then $\mathcal{L}(a)$ consists of Laurent polynomials $X(\lambda) = \sum x_j \lambda^j$ with coefficients in $a$ for which commutator is defined as $[x \lambda^i, y \lambda^j] = [x, y] \lambda^{i+j}$. There is an obvious decomposition of $g = \mathcal{L}(a)$ into a linear sum of two subalgebras
\[ g = g_+ \oplus g_- , \quad g_+ = \oplus_{j \geq 0} a \lambda^j , \quad g_- = \oplus_{j < 0} a \lambda^j . \] (2.1)

Let $P_\pm$ be projection operator onto $g_\pm$ parallel to $g_\mp$. A standard $R$-matrix associated with $\mathcal{L}(a)$ is defined as
\[ R = P_+ - P_- . \] (2.2)

We shall use tensor form of $R$-brackets which is more common in the inverse scattering method \[12, 13\]. If the algebra $a$ is identified with its dual then the standard $R$-matrix (2.2) is the Hilbert transform in $\mathcal{L}(a)$ with the kernel
\[ r(\lambda, \mu) = \frac{t}{\lambda - \mu} , \] (2.3)
where $t \in a \otimes a$ is a kernel of the identity operator in $a$. The adjoint kernel is $r^t(\lambda, \mu) = -r(\lambda, \mu)$. For twisted subalgebras $G(a) \subset \mathcal{L}(a)$ the corresponding decomposition (2.1) yields kernels with trigonometric and elliptic dependence on the spectral parameter \[12, 13, 25\]. In the quantum inverse scattering method $r$-matrices with rational, trigonometric and elliptic dependence on the spectral parameter are called $r$-matrices of $XXX$, $XXZ$ and $XYZ$ types respectively \[12, 13, 17\].

In this paper we consider the Lie algebra $a = su(2)$ only. We fix also representation of the algebra $su(2)$ and shall work in the matrix notation, although the underlying constructions are independent of particular matrix representation. In the following we shall consider spin one half representations of $su(2)$ or shall use the direct sum of these representations. This means that we restrict ourselves to two-dimensional auxiliary space in which $su(2)$ matrices act.

There are two important Lie-Poisson brackets ($R$-brackets) \[24, 25, 31\] which can be considered as linear classical limit of the fundamental commutator relations in the quantum inverse scattering method \[28, 13\]. They are the $r$-bracket
\[ \{ L(\lambda), L(\mu) \} = [r(\lambda, \mu), \hat{L}(\lambda) + \hat{L}(\mu)] , \] (2.4)
and the $rs$-bracket
\[ \{ \hat{L}(\lambda), \hat{L}(\mu) \} = [r(\lambda, \mu), \frac{1}{2} \hat{L}(\lambda) + \frac{1}{2} \hat{L}(\mu)] + [s(\lambda, \mu), \frac{1}{2} \hat{L}(\lambda) - \frac{1}{2} \hat{L}(\mu)] , \] (2.5)
which is related to the reflection equations \[28, 32\]. Here we use the standard notations $\hat{L}(\lambda) = L(\lambda) \otimes I$, $\hat{L}(\mu) = I \otimes L(\mu)$. The brackets (2.4–2.5) are Lie-Poisson brackets, if the matrices $r(\lambda, \mu)$ and $s(\lambda, \mu)$ satisfy the modified classical Yang-Baxter equations \[24, 25, 31\].

For a traceless $2 \times 2$ matrix $L_0(\lambda)$ (which we shall call Lax matrix)
\[ L_0(\lambda) = \sum_{k=1}^3 s_k(\lambda) \cdot \sigma_k \] (2.6)
which obeys the linear $r$-matrix algebra \[24\] the $r$-matrix is an antisymmetric function of one argument $r(\lambda, \mu) \equiv r(\lambda - \mu)$ and $r(\lambda) = -r(-\lambda)$. Namely, put
\[ r(\lambda) = \sum_{k=1}^3 w_k(\lambda) \cdot \sigma_k \otimes \sigma_k . \] (2.7)
The $\sigma_k$ are Pauli matrices and coefficients $w_k(\lambda)$ are functions of spectral parameter only. They are determined by construction of $R$-matrix on loop algebra $\mathcal{L}(a)$ \[24, 25, 26\] or by the requirement that the $r$-matrix obeys classical Yang-Baxter equations \[12, 13, 17\].
It follows from the algebra (2.4) that the determinant \( d_0(\lambda) \equiv \det L_0(\lambda) \) can be taken as a generating function of commuting integrals of motion \( I_k \) since

\[
\{ d_0(\lambda), d_0(\mu) \} = 0, \quad d_0(\lambda) = \sum_k I_k \lambda^k, \quad \lambda, \mu \in \mathbb{C}
\]  

implies

\[
\{ I_k, I_j \} = 0.
\]

A similar property is valid for the \( rs \)-algebra (2.3) too.

Here we consider deformations \( L(\lambda) \in \mathcal{L}(\mathfrak{a}) \) of the Lax matrix \( L_0(\lambda) \in \mathcal{L}(\mathfrak{a}) \) with \( a = su(2) \). The main stimulus for considering such deformations has been inspired by the study of Lax representation and \( r \)-matrix formulation for restricted flows of the KdV and coupled KdV hierarchies [8, 18], which describe many physically interesting integrable natural Hamiltonian systems such as: Neumann, Garnier, Henon-Heiles systems and an infinite family of integrable polynomial potentials [24]. The relevant deformations of \( L_0(\lambda) \) had the form

\[
L(\lambda) = \left( \begin{array}{cc} a & b \\ G(b, \lambda) + c & -a \end{array} \right) (\lambda) \equiv L_0(\lambda) + F(b, \lambda)
\]  

where matrix \( L_0(\lambda) \in \mathcal{L}(\mathfrak{a}) \) and hence the entries \( a(\lambda), b(\lambda) \) and \( c(\lambda) \) are Laurent series satisfying Poisson brackets (2.4) with the \( r \)-matrix of the XXX type. The deformation \( G(b, \lambda) \) had a special form connected to KdV equation [13, 15].

Here we find a natural generalization of this result where the deformed matrix \( L(\lambda) \) belongs the same phase space. The additive deformation of \( L_0(\lambda) \) has the form

\[
L(\lambda) = \sum_{k=1}^{3} \left( s_k(\lambda) + \alpha_k(\lambda) \cdot \frac{f(\lambda)}{2b(\lambda)} \right) \cdot \sigma_k = L_0(\lambda) + F(\lambda),
\]

where

\[
\bar{b}(\lambda) = \sum_{i=1}^{3} \alpha_i(\lambda) s_i(\lambda),
\]

and

\[
\alpha_1^2(\lambda) + \alpha_2^2(\lambda) + \alpha_3^2(\lambda) = 0.
\]

Here functions \( \alpha_j(\lambda) \) and the arbitrary function \( f(\lambda) \) are complex-valued functions of the spectral parameter \( \lambda \) only. The condition (2.13) guarantees that determinant of \( L(\lambda) \) has the additive property

\[
d(\lambda) = d_0(\lambda) + f(\lambda),
\]

which means that integrals of the motion \( d(\lambda) = \sum I_k \lambda^k \) of the deformed integrable systems differ by some constants

\[
I_k^{\text{new}} = I_k^{\text{old}} + f_k, \quad f_k \in \mathbb{C}.
\]

The requirement that the deformed Lax matrix \( L(\lambda) \) satisfy the \( rs \)-bracket (2.3) with the same \( r \)-matrix \( r(\lambda - \mu) \) yields four algebraic equations for the unknown functions \( \alpha_k(\lambda), \ k = 1, 2, 3 \) and a certain function \( g(\lambda, \mu) \).

**Theorem 1** The deformed Lax matrix (2.11) satisfies the linear \( rs \)-algebra (2.3), if

\[
w_j(\lambda, \mu) \alpha_j(\mu) \alpha_i(\lambda) - w_i(\lambda, \mu) \alpha_i(\mu) \alpha_j(\lambda) = g(\lambda, \mu) \cdot \alpha_k(\lambda),
\]

where \( (j, i, k) \) are cyclic permutations of indices \( (1, 2, 3) \) and the scale function \( g(\lambda, \mu) \) depends only on the spectral parameters \( \lambda \) and \( \mu \). The corresponding matrix \( s(\lambda, \mu) \) is given by

\[
s(\lambda, \mu) = \sum_{i,j=1}^{3} \alpha_{ij}(\lambda, \mu) \sigma_i \otimes \sigma_j,
\]

with

\[
\alpha_{ij}(\lambda, \mu) = w_i(\lambda, \mu) \cdot \frac{\alpha_i(\mu) \alpha_j(\mu) f(\mu)}{\bar{b}^2(\mu)} - w_j(\lambda, \mu) \cdot g(\lambda, \mu) \cdot \frac{\alpha_i(\lambda) \alpha_j(\lambda) f(\lambda)}{\bar{b}^2(\lambda)}
\]  

(2.15)
Proof: The proof is a direct but lengthy computation.

Notice that the function \( f(\lambda) \) is quite arbitrary here. A compact form of the equation (2.13) follows from the ideas of Sklyanin in [27].

**Lemma 1** The condition (2.13) of Theorem 4 for the coefficients \( \alpha_k(\lambda) \) is equivalent to equation

\[
\{ \tilde{b}(\lambda), \tilde{b}(\mu) \} = g(\lambda, \mu) \cdot \tilde{b}(\lambda) - g(\mu, \lambda) \cdot \tilde{b}(\mu).
\]  

(2.16)

**Proof:** This lemma is proved by direct substitution of the equality (2.13) into the equation (2.16) and vice versa.

Let us present some of the best known [12, 13, 17] \( r \)-matrices and the related solutions for \( \alpha_k(\lambda) \) and \( g(\lambda, \mu) \). The \( r \)-matrix of the \( XX \) and \( XXZ \) types are

\[
w_1 = w_2 \equiv w(\lambda) = \frac{\eta}{\varphi(\lambda)} \quad \text{and} \quad w_3(\lambda) = \frac{\eta \varphi'(\lambda)}{\varphi(\lambda)},
\]

(2.17)

with \( \varphi(\lambda) = \lambda \) in the \( XX \) case,

and \( \varphi(\lambda) = \sinh \lambda \) in the \( XXZ \) case,

(2.18)

here \( \eta \) is constant and \( \varphi'(\lambda) \) denotes derivative with respect to \( \lambda \).

An interesting particular class of solutions to equations (2.12, 2.13) is distinguished by the conditions

\[
\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 = 0.
\]

(2.19)

They represent two points on the circle in complex plane with the phase difference equal to \( \pi/2 \), for instance \( \alpha_1 = 1 \), \( \alpha_2 = i \). For these solutions the function \( g(\lambda, \mu) \) is equal to zero \( (g(\lambda, \mu) = 0) \). A different example of solutions for \( r \)-matrix of \( XXZ \) type is

\[
\alpha_1 = \cosh \lambda, \quad \alpha_2 = i \sinh \lambda, \quad \alpha_3 = i.
\]

(2.20)

Usually one uses the following rational parametrization for \( r \)-matrix of \( XXZ \) type:

\[
w_1(u - v) = w_2(u - v) = \frac{\eta 2uv}{u^2 - v^2}, \quad w_3(u - v) = \frac{\eta u^2 + v^2}{u^2 - v^2},
\]

(2.21)

\( \varphi(u) = u - \frac{1}{u} \), \quad \text{with} \quad u = e^\lambda, \quad v = e^\mu,

as, for an example, in the description of the sine-Gordon equation [12].

An elliptic \( r \)-matrix of the \( YXZ \) type has

\[
w_1(\lambda) = \frac{\Theta_{11}'}{\Theta_{10}} \frac{\Theta_{10}(\lambda, k)}{\Theta_{11}(\lambda, k)}, \quad w_2(\lambda) = \frac{\Theta_{11}'}{\Theta_{00}} \frac{\Theta_{00}(\lambda, k)}{\Theta_{11}(\lambda, k)},
\]

(2.22)

\[
w_3(\lambda) = \frac{\Theta_{11}'}{\Theta_{00}} \frac{\Theta_{01}(\lambda, k)}{\Theta_{11}(\lambda, k)},
\]

with \( \alpha_1 = -1; \quad \alpha_2 = i \frac{\text{dn}(\lambda, k)}{k \text{cn}(\lambda, k)}; \quad \alpha_3 = \frac{k'}{k} \frac{1}{\text{cn}(\lambda, k)} \).

(2.23)

Here \( k \) and \( k' \) denote modulus of the Jacobi elliptic functions and \( \Theta_{ij}(\lambda, k) \) are the elliptic theta function in the notation of [34]. We have to emphasize that the functions \( \tilde{b}(\lambda) \) (2.11) with the coefficients \( \alpha_k(\lambda) \) (2.21) and the function \( g(\lambda, \mu) \) were introduced in [27] for the quadratic \( r \)-matrix algebra as solution of the equation (2.16). The function \( g(\lambda, \mu) \) is independent from second parameter \( \mu \) function and it is equal to

\[
g(\lambda, \mu) = \frac{\Theta_{11}'}{\Theta_{10}} \frac{\Theta_{10}(\lambda - K, k)}{\Theta_{11}(\lambda - K, k)}.
\]
We recall from [27] that, for the purpose of separation variables, Sklyanin considered a matrix $L_0(\lambda)$ which is similar to the Lax matrix $L_0(\lambda)$:

$$\hat{L}_0(\lambda) = \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right) (\lambda) = U^{-1}(\lambda)L_0(\lambda)U(\lambda),$$

(2.22)

with

$$U(\lambda) = \left( \begin{array}{cc} \Theta_{01} & \Theta_{00} \\ -\Theta_{00} & -\Theta_{01} \end{array} \right) (\xi, \tilde{k}), \quad \lambda = \frac{2\xi}{1+\tilde{k}}, \quad \tilde{k} = 2\sqrt{k}.$$

Here matrix $L_0(\lambda)$ obeys the linear $r$-matrix algebra (2.4) with the $r$-matrix (2.21). The similar matrix $\hat{L}_0(\lambda)$ satisfies the r-bracket (2.4) with $r$-matrix of the form

$$\rho(\lambda, \mu) = \frac{1}{2} U^{-1}(\lambda)U^{-1}(\lambda) r(\lambda, \mu) U(\lambda) U(\lambda)$$

$$= \sum_{j=1}^{3} w_j(\lambda - \mu, k) \cdot \sigma_j \otimes \sigma_j +$$

$$+ [w_3(\mu - K, k) - w_3(\lambda - K, k)] \cdot \sigma_3 \otimes \sigma_3 +$$

$$+ iw(\mu - K, k) \cdot \sigma_3 \otimes \sigma_2 - iw(\lambda - K, k) \cdot \sigma_2 \otimes \sigma_3;$$

(2.23)

$$w(\lambda - \mu, k) \equiv w_1(\lambda - \mu, k) = w_2(\lambda - \mu, k) = \eta \frac{\Theta_{11}^{(1)}(\lambda - \mu, k)}{\Theta_{11}^{(0)}(\lambda - \mu, k)},$$

$$w_3(\lambda - \mu) = \eta \frac{\Theta_{11}^{(1)}(\lambda - \mu, k)}{\Theta_{11}^{(0)}(\lambda - \mu, k)},$$

here $K$ is a standard elliptic integral of the module $k$. Notice that now two first coefficients of the matrix $\rho(\lambda)$ are equal $w_1(\lambda) = w_2(\lambda)$ in same way as it was in the $XXX$ and $XXZ$ cases.

It is well known that $r$-brackets (2.4) is the Lie-Poisson brackets if $r$-matrix obeys the classical Yang-Baxter equations [12, 25, 26]. The Yang-Baxter equation is an equation on matrices in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We use the familiar tensor notations $L_j$ for the matrix acting as matrix $L$ in the $j$'s factor of the product and trivially acting in the remaining factors, for example $L_1 = L \otimes I \otimes I$. In similar way a matrix $r_{ij}$ is acting trivially in the third $k$’s factor and it works as an matrix $r$ in the product of the other $ij$’s factors. Classical Yang-Baxter equation for pure numerical $r$-matrices has the form

$$[d_{12}^+(\lambda, \mu), d_{13}^+(\lambda, \nu)] + [d_{23}^+(\lambda, \nu), d_{32}^+(\nu, \mu)] + [d_{13}^+(\nu, \mu), d_{12}^+(\lambda, \nu)] = 0.$$

(2.24)

Following [31, 11] we introduce dynamical Yang-Baxter equations for the dynamical $rs$-matrix. Let us consider matrices $d^{\pm} \equiv r \pm s$ corresponding to particular class of solutions $\alpha_k$ (2.19).

**Theorem 2** For the $rs$-algebra (2.5) with matrices (2.17, 2.18) the following equations are valid

$$[d_{12}^+(\lambda, \mu), d_{13}^+(\lambda, \nu)] + [d_{23}^+(\lambda, \nu), d_{32}^+(\nu, \mu)] + [d_{13}^+(\nu, \mu), d_{12}^+(\lambda, \nu)] +$$

$$+[L_2(\mu), d_{13}^+(\lambda, \nu)] - [L_3(\nu), d_{12}^+(\lambda, \mu)] +$$

$$+[X(\lambda, \mu, \nu), L_2(\mu) - L_3(\nu)] = 0.$$

(2.25)

The matrix $X(\lambda, \mu, \nu)$ is

$$X(\lambda, \mu, \nu) = \beta(\lambda, \mu, \nu) \cdot \sigma_- \otimes \sigma_- \otimes \sigma_-,$$

(2.26)

$$\beta(\lambda, \mu, \nu) = \frac{f(\lambda)b^{-3}(\lambda) \cdot (\mu - \nu) + f(\mu)b^{-3}(\mu) \cdot (\nu - \lambda) + f(\nu)b^{-3}(\nu) \cdot (\lambda - \mu)}{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)}.$$

The other two equations are obtained from (2.22) by cyclic permutation.

**Proof:** This Theorem also can be proved by a straightforward computation.

Tensor $X(\lambda, \mu, \nu)$ is completely symmetric tensor with respect to any permutations of auxiliary spaces. For the quite arbitrary solutions $\alpha_k$ to the equations (2.13, 2.14) we can introduce an asymmetric tensors $X^{(i,j,k)}(\lambda, \mu, \nu)$. Then dynamical Yang-Baxter equations take the general form introduced in [31].
2.2 Special deformations of the XXX and XXZ cases and Lax representation

This section is based on list of formulae collected in [11, 18, 32] and on meaning its in general r-matrix approach to loop algebras.

Our purpose is to apply the rs-matrix formulation for description of natural Hamiltonian systems in order to solve them through separation of variables. For separating variables we shall apply the functional Bethe ansatz [27, 29]. The Bethe ansatz has an asymmetric matrix form [13, 29] (variables of separation are introduced as zeroes of the entry $b(\lambda)$). Therefore we restrict ourselves to a particular class of deformations obtained by certain projections of the matrix

$$L(\lambda) = \begin{pmatrix} a & b \\ f(\lambda)b^{-1} + c & -a \end{pmatrix} (\lambda) = L_0(\lambda) + \begin{pmatrix} 0 & 0 \\ f(\lambda)b^{-1} & 0 \end{pmatrix},$$  \hspace{1cm} (2.27)

where the entries $a(\lambda), b(\lambda)$ and $c(\lambda)$ are absolutely convergent Laurent series in the XXX case or absolutely convergent Fourier series in the XXZ case

$$a(\lambda) = \sum_k a_k \lambda^k, \hspace{1cm} \text{for XXX case},$$

$$a(\lambda) = \sum_k a_k \exp(k \cdot \lambda), \hspace{1cm} \text{for XXZ case},$$

and similarly for $b(\lambda)$ and $c(\lambda)$.

It is well known [25] that the standard R-bracket on $L(\lambda)$ associated with (2.1–2.2) has a large collection of finite-dimensional Poisson ($ad^*_R$-invariant) subspaces

$$L_{M,N} = \bigoplus_{j=-M}^N a^* \lambda^j, \hspace{1cm} \text{provided } M \geq 0; \ N \geq -1. \hspace{1cm} (2.28)$$

In other words the subspaces $L_{M,0}$ and $L_{1,N}$ are invariant under the coadjoint action of the subalgebras $g_+$ and $g_-$ (2.1).

Let us introduce the following projection operators $[\cdot]_{MN}$ on these Poisson subspaces

$$[z]_{MN} = \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_{MN} \equiv \sum_{k=-M}^{N} z_k \lambda^k, \hspace{1cm} (2.29)$$

$$[z]_{MN} = \left[ \sum_{k=-\infty}^{+\infty} z_k \exp(k \cdot \lambda) \right]_{MN} \equiv \sum_{k=-M}^{N} z_k \exp(k \cdot \lambda).$$

for the Laurent and for the Fourier series. In particular in the XXX case

$$[z]_+ = \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_+ \equiv \sum_{k=0}^{+\infty} z_k \lambda^k, \hspace{1cm} (2.30)$$

denotes the Taylor projection.

We shall define a new additive deformation of $L_0(\lambda)$ as

$$L_{MN}(\lambda) = L_0(\lambda) + F_{MN}(b, \lambda) \equiv \begin{pmatrix} a & b \\ G_{MN}(b, \lambda) + c & -a \end{pmatrix} (\lambda),$$  \hspace{1cm} (2.31)

where $G_{MN}(b, \lambda)$ is a function of the spectral parameter $\lambda$ and of the entry $b(\lambda)$. It reads

$$G_{MN}(b, \lambda) = [f(\lambda) \cdot b^{-1}(\lambda)]_{MN},$$  \hspace{1cm} (2.32)

$$f(\lambda) = \sum_{k=-\infty}^{+\infty} f_k \lambda^k, \hspace{1cm} \text{or} \hspace{1cm} f(\lambda) = \sum_{k=-\infty}^{+\infty} f_k \exp(k \lambda). \hspace{1cm} (2.33)$$

For the Taylor projection this deformation takes the form

$$G_N(b, \lambda) = [f_N(\lambda) \cdot b^{-1}(\lambda)]_+, \hspace{1cm} \text{where } f_N(\lambda) = \sum_{k=0}^{N} f_k \lambda^k. \hspace{1cm} (2.34)$$
The essential feature of this deformation is, in comparison with (2.10), that the determinant of the modified matrix $L_{MN}(\lambda)$ is different from $d_0(\lambda) + f(\lambda)$

$$d(\lambda) = d_0(\lambda) + b \cdot G_{MN} \neq d_0(\lambda) + f(\lambda)$$

and therefore integrals of motion $I_k^{\text{new}}$ of the deformed system are functionally different from the undeformed integrals $I_k^{\text{id}}$, and yet the $L_{MN}(\lambda)$ matrix belongs to a certain rs-algebra.

**Theorem 3** The matrix $L_{MN}(\lambda)$ (2.34) satisfies the linear rs-matrix algebra (2.5), with the matrix $s_{MN}(\lambda, \mu)$ given by

$$s(\lambda, \mu) = \alpha_{MN}(\lambda, \mu) \sigma_+ \otimes \sigma_-, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. $$

The function $\alpha_{MN}(\lambda, \mu)$ is defined by

$$\alpha_{MN}(\lambda, \mu) = -w(\lambda - \mu) \left( [f(\lambda)b^{-2}(\lambda)]_{MN} - [f(\mu)b^{-2}(\mu)]_{MN} \right), \quad (2.35)$$

where $w(\lambda) \equiv w_1(\lambda) = w_2(\lambda)$.

*Proof:* The proof is a straightforward calculation.

The Theorem 3 has a transparent meaning in the framework of the general r-matrix approach to loop algebras. The Theorem 3 asserts that the matrix $s(\lambda, \mu)$ can be restricted to subspaces $L_{M,N}$ (2.28) and subspaces $L_{M,N}$ is a common Poisson subspace for the rs-brackets (2.5) with matrix $s_{MN} = [s(\lambda, \mu)]_{MN}$. Since the deformations (2.6) are directly connected with the KdV equation (18), Theorem 3 describes the relationship between compatible Poisson structures of the KdV equation and the Virasoro algebra [2]. Matrices $d^\pm = r \pm s_{MN}$ obey dynamical Yang-Baxter equations, which are obtained by restriction of the equation (2.27) (2.26) to the corresponding subspace $L_{M,N}$. For certain matrices $L_{MN}(\lambda)$ and $s_{MN}$ this proposition has been proved in [1].

In order to describe Lax representations for integrable Hamiltonian systems related to matrices $L_0(\lambda)$ and $L_{MN}(\lambda)$ we introduce matrices [25, 26]

$$d_{12}(\lambda, \mu) = r(\lambda, \mu) + s(\lambda, \mu), \quad \text{and} \quad d_{21}(\lambda, \mu) = r(\lambda, \mu) - s(\lambda, \mu).$$

In terms of these matrices the Lie-Poisson rs-brackets reads:

$$\{L(\lambda), L(\mu)\} = [d_{12}(\lambda, \mu), \frac{1}{2} L(\lambda)] + [d_{21}(\lambda, \mu), \frac{1}{2} L(\mu)], \quad (2.36)$$

For constructing Lax representations we shall make use of the following lemma [3].

**Lemma 2** If matrix $L(\lambda)$ obeys the Lie-Poisson algebra (2.34) then the spectral invariants

$$H_n(\lambda) = \text{tr} (L^n), \quad n = 1, 2, \ldots, \quad (2.37)$$

are integrals of motion in involution $\{H_n, H_m\} = 0$ and the Lax equation are given by

$$\dot{L}(\mu) = \{H_n(\lambda), L(\mu)\} = [M_n(\lambda, \mu), L(\mu)], \quad (2.38)$$

with

$$M_n(\lambda, \mu) = n \text{tr}_1 \left( L^{-1}(\lambda) d_{21}(\lambda, \mu) \right), \quad n = 1, 2, \ldots. \quad (2.39)$$

where $\text{tr}_1$ means trace in the first auxiliary space and dot over $L$ means derivative with respect to time corresponding to the Hamiltonian $H_n(\lambda)$.

In two dimensional auxiliary space

$$d(\lambda) \equiv \text{det} L(\lambda) = -\frac{1}{2} (\text{tr} (L^2)) = -\frac{1}{2} H_2(\lambda).$$

In order to get a Hamiltonian $H$, which does not depend on the spectral parameter, one needs a linear functional (a projection) $\Phi_\lambda$ which selects, for example (in XXX case), a coefficient at certain power of $\lambda$

$$H = \frac{1}{2} \Phi_\lambda [H_2(\lambda)] = \Phi_\lambda [d(\lambda)]. \quad (2.40)$$
For instance it can be defined as a residue of order \( m \) at \( \lambda_0 \)

\[
\Phi_{\lambda}[z] = \text{Res}_{\lambda_0}^m z(\lambda) = \frac{1}{(m-1)!} \frac{d^{m-1}}{d\lambda^{m-1}} \left((\lambda - \lambda_0)^m z(\lambda)\right) \bigg|_{\lambda=\lambda_0}
\]

(2.41)

Then the Lax representation for the Lax matrices (2.6 or 2.27) with the Hamiltonian (2.40) has the form

\[
\dot{L}(\mu) = \{H, L(\mu)\} = \{\frac{1}{2} \Phi_{\lambda}[\text{tr}_1 L^2(\lambda)], L(\mu)\} = [M(\mu), L(\mu)],
\]

(2.42)

\[
M(\mu) = \Phi_{\lambda}\left[\text{tr}_1 \left(L(\lambda) \cdot d_{21}(\lambda, \mu)\right)\right] = \Phi_{\lambda}[M(\lambda, \mu)].
\]

In particular for the initial Lax matrix \( L_0 \) (2.4) one gets \( s = 0; \ d_{21}(\lambda, \mu) = r(\lambda - \mu) \) and then

\[
\dot{L}_0(\mu) = [M_0(\mu), L_0(\mu)],
\]

(2.43)

\[
M_0(\mu) = \Phi_{\lambda}\left[2 \sum_{k=1}^{3} s_k(\lambda) \cdot w_k(\lambda - \mu) \cdot \sigma_k\right]
\]

(2.44)

In this paper we start with the requirement that

\[
M_0 = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

(2.45)

and choose \( L_0(\lambda) \) and \( \Phi_{\lambda} \) appropriately. It is a rather strong restriction for the matrices \( L_0 \) (2.6) and for the corresponding Hamiltonians. This requirement comes from the study of \( rs \)-representations for natural Hamiltonian systems related to stationary flows for equations of KdV type [18, 32].

As an immediate consequence of the Lax representation (2.42) with the matrix \( M_0 \) (2.45) we can rewrite the Lax matrix \( L_0 \) in the form

\[
L_0(\mu) = \begin{pmatrix} \frac{-b_x}{2} & b \\ -b_{xx} & b_x \end{pmatrix}, \quad b_x = \{H_0, b\},
\]

(2.46)

where \( H_0 \) is a Hamiltonian corresponding to \( L_0 \). Since the determinant of \( L_0(\lambda) \) is a generating function of integrals of motion we see that it obeys the equation

\[
-\{H_0, d_0(\lambda)\} = -d_{0,xx}(\lambda) = \partial_x^3 \cdot b = b_{xxx} = 0.
\]

(2.47)

Now for a fixed triad \( (L_0, M_0, \Phi_{\lambda}) \) we introduce a modified matrix \( L(\lambda) \) (2.27). In this case the matrix \( M(\mu) \) is constructed according to (2.42) with \( d_{21}(\lambda, \mu) = r - s \) and with the use of the same projector \( \Phi_{\lambda} \). It reads

\[
M(\mu) = \begin{pmatrix} 0 & 1 \\ -u(\mu) & 0 \end{pmatrix}, \quad u(\mu) = f(\mu)b^{-2}(\mu).
\]

(2.48)

due to the linearity of \( \Phi_{\lambda} \)

\[
\Phi_{\lambda}\left[u_1 \cdot \left(-f(\lambda)b^{-1}(\lambda) + c(\lambda) + (f(\lambda)b^{-2}(\lambda) - f(\mu)b^{-2}(\mu)) \cdot b(\lambda)\right)\right] = -f(\mu)b^{-2}(\mu) = -u(\mu).
\]

(2.49)

Then from the Lax representation (2.42) it follows that the deformed matrix \( L(\lambda) \) has to be equal to

\[
L(\lambda) = \begin{pmatrix} \frac{-b_x}{2} & b \\ -b(\lambda)u(\lambda) - \frac{b_{xx}}{2} & b_x \end{pmatrix}, \quad b_x = \{H, b\}.
\]

(2.50)
where the Hamiltonian \( H = \Phi_\lambda [d(\lambda)] \) is obtained by applying the old projector \( \Phi_\lambda \) to the modified determinant. The determinant \( d(\lambda) \) of the modified matrix \( L(\lambda) \) obeys the equation

\[
- d_x(\lambda) = \left( \frac{1}{4} \partial_x^3 + u \partial_x + \frac{1}{2} u_x \right) \cdot b = B_1[u] \cdot b = 0. \tag{2.51}
\]

Here \( B_1[u] \) is the Hamiltonian pencil operator for the coupled KdV equation \([7, 8]\).

For the deformed (according to \([2.31]\)) matrix \( L_{MN}(\lambda) \) the Lax equation is constructed analogously \([2.48, 2.50]\) and \([2.51]\). One has to replace \( u(\mu) = f(\mu)b^{-2}(\mu) \) with the potential \( u_{MN}(\mu) = [f(\mu)b^{-2}(\mu)]_{MN} \).

Equation \([2.51]\) shows that our deformation is connected to the stationary problems for equation of the KdV type. The Hamiltonian and some properties of motion for the finite-dimensional integrable systems connected with stationary flows of the KdV equations were considered in \([7, 8, 11]\). Lax representations, bi-Hamiltonian structures, Newton representations and some other properties for these systems connected with stationary flows of the KdV equations were studied in \([7, 8, 11]\). In the next section we tie these different methods to the Lie algebraic method of constructing matrices \( L(\lambda) \) on the loop algebras.

In the next section we shall assume that the Lax matrix \( M_0 \) has the form

\[
M_0 = \begin{pmatrix} v_0 & u_0 \\ 0 & -v_0 \end{pmatrix}. \tag{2.52}
\]

It introduces certain technical complications without influence on the conceptual structure.

3 Construction of Lax matrices \( L(\lambda) \) for separable systems

3.1 Separation of variables

The main problem for a given integrable Hamiltonian system with a complete set of functionally independent and commuting integrals of motion \( H_1, \ldots, H_n \) is to determine its solutions. The construction in the Liouville theorem has a local character and there is no general way to describe solutions globally unless some additional information about the system is known, such as a complete, spectral parameter dependent Lax representation or variables of separation.

Here we study the Lax matrices satisfying the linear \( rs \)-algebra \([2.4, 2.5]\) and method of separation of variables. By separation of variables we mean here construction of \( n \) pairs of canonically conjugate variables

\[
\{ u_j, u_k \} = \{ v_j, v_k \} = 0, \quad \{ v_j, u_k \} = \delta_{jk}, \quad j, k = 1, \ldots, n \tag{3.1}
\]

and functions \( \Psi_k \) such that

\[
\Psi_k(u_k, v_k, H_1, H_2, \ldots, H_n) = 0, \quad k = 1, \ldots, n \tag{3.2}
\]

where \( H_k \) are integrals of motion in involution \([22]\). Note that the canonical transformation from the original variables \( (x_k, p_k), k = 1, \ldots, n \) to variables of separation \( (u_k, v_k), k = 1, \ldots, n \) may not necessarily be a pure coordinate change. It can involve both coordinates and momenta.

For the Lax matrices \( L(\lambda) \) satisfying \( r \)-matrix algebra with the \( r \)-matrix of the XXX type the problem of determining variables of separation has been essentially solved by Sklyanin (see Theorem 4 below).

Here we provide a solution of the converse problem how to construct Lax matrices \( L(\lambda) \) satisfying the \( rs \)-matrix algebra for an integrable system given in terms of variables of separation. We express first \( L(\lambda) \) in terms of variables of separation \( (u_k, v_k), k = 1, \ldots, n \) and then we shall use the canonical transformations naturally determining from the \( r \)-matrix structure to the "physical" variables \( (x_k, p_k), k = 1, \ldots, n \).

**Theorem 4 (Sklyanin [27, 29])** Coordinates \( u_k, v_k, k = 1, \ldots, n \) defined as

\[
b(\lambda = u_k) = 0 \quad v_k = a(\lambda = u_k) \tag{3.3}
\]
are canonically conjugate variables of separation, if the matrix $L(\lambda)$ obeys the $r$-matrix algebra with $r$-matrices \((2.17)\) or \((2.18)\). Separation equations are
\[
v_k^2 = -d(u_k),
\]
where $d(\lambda) \equiv \det L(\lambda)$.

Proof: We repeat the Sklyanin proof for the XXX and XXZ cases, because this technique will be used later.

Let us list commutation relations for $b(\mu)$ and $a(\lambda)$,
\[
\{b(\lambda), b(\mu)\} = \{a(\lambda), a(\mu)\} = 0, \quad \{a(\lambda), b(\mu)\} = w_3(\lambda - \mu)b(\mu) - w(\lambda - \mu)b(\lambda),
\]
where we denote that $w(\lambda - \mu) \equiv w_1(\lambda - \mu) \equiv w_2(\lambda - \mu)$. The equalities $\{u_i, v_j\} = 0$ follow from \((3.5)\).

To derive the equality $\{v_j, b(\mu)\} = w_3(u_j - \mu)b(\mu)$.

It together with the equation
\[
0 = \{v_j, b(u_i)\} = \{v_j, b(\mu)\} |_{\mu = u_i} + b'(u_i)\{v_j, u_i\},
\]
gives
\[
\{v_j, u_i\} = -\frac{1}{b'(u_i)}\{v_j, b(\mu)\} |_{\mu = u_i} = \eta\delta_{ij}.
\]
Equalities $\{v_i, v_j\} = 0$ can be verified in a similar way:
\[
- \{v_i, v_j\} = \{a(u_i), a(u_j)\}
= \{a(u_j), a(\mu)\} |_{\mu = u_i} + a'(u_i)\{u_i, a(u_j)\}
= a'(u_j)\{a(u_j), u_i\} + a'(u_i)\{u_i, a(u_j)\} = 0.
\]
After the substituting $\lambda = u_k$ into the identity
\[
a^2(\lambda) + b(\lambda)c(\lambda) = -d(\lambda)
\]
we obtain separation equations
\[
v_k^2 = -d(u_k), \quad k = 1, \ldots, n,
\]
where $d(\lambda)$ is the determinant of $L(\lambda)$.

In the XYZ case we have to use entries $\tilde{a}(\lambda)$ and $\tilde{b}(\lambda)$ of the similar matrix $\tilde{L}(\lambda)$ \((2.22)\). By the Lemma the zeroes of entry $\tilde{b}(\lambda)$ are commuting variables $\{u_i, u_j\} = 0$. For the rest of the proof we refer to \([27]\).

### 3.2 Construction of Lax matrices $L(\lambda)$

A general construction of matrices $L_0(\lambda)$ with multiple poles on a direct sum of loop algebras $\oplus L(su(2))$ has been presented in detail in \([23]\). The Lax matrix is assumed in the form
\[
L_0(\lambda) = \sum_{\alpha=1}^3 \sum_{j}^N s^\alpha_j w_\alpha(\lambda - \nu_j) \sigma_\alpha,
\]
where $\sigma_\alpha$ are Pauli matrices, $w_\alpha(\lambda)$ are coefficients of the corresponding $r$-matrix and $\nu_j$ are simple poles of the fixed divisor $D = \{\nu_1, \ldots, \nu_N\}$. Residues $s^\alpha_j$ are the standard linear coordinates on $su(2)^*$ with Lie-Poisson brackets
\[
\{s^\alpha_j, s^\beta_k\} = \delta_{jk} \varepsilon_{\alpha\beta\gamma} \cdot s^\gamma_j
\]
The corresponding phase space is the direct sum of $N$ copies $su(2)^*$. These matrices describe systems of $N$ interacting Euler tops. The case of higher order of poles can be treated as a degeneration of \([3.7]\). It describes different tops \([25]\) as well.

We shall express matrix elements of $L_0(\lambda)$ through canonically conjugate variables. Let us consider meromorphic matrix-function $L_0(\lambda)$ and the coefficients $w_1(\lambda - \mu) = w_2(\lambda - \mu)$ of the $r$-matrices \((2.17-2.18)\) with one distinguished point at infinity (it is zero by both arguments and their residues at infinity are equal to $-1$). Let us choose the linear functional $\Phi_\lambda$ as the higher residue fixed order $K$ at infinity

$$\Phi_\lambda[z] = -\text{Res}_\infty^K z(\lambda).$$

(3.8)

Then \((2.42)\) and \((2.53)\) gives

$$-\text{Res}_\infty^K \left( \frac{b(\lambda)}{\varphi(\lambda - \mu)} \right) = u_0,$$

(3.9)

$$-\text{Res}_\infty^K \left( \frac{c(\lambda)}{\varphi(\lambda - \mu)} \right) = 0,$$

(3.10)

$$-\text{Res}_\infty^K \left( \frac{a(\lambda)\varphi'(\lambda - \mu)}{\varphi(\lambda - \mu)} \right) = v_0,$$

(3.11)

In particular $u_0 = 1$, $v_0 = 0$ for the matrix $M_0$ \((2.45)\). In the $XXX$ case it means that entries $b(\lambda)$ and $a(\lambda)$ have the highest in $\lambda$ terms in power $K + 1$ equal to $u_0$ and $v_0$, respectively. The entry $c(\lambda)$ has the highest term of order $K$ only. Note that the conditions for the entry $a(\lambda)$ and $c(\lambda)$ \((3.10)\) in the Lax pair $(L_0, M_0)$ \((2.43)\) follow from the first condition \((2.13)\) for entry $b(\lambda)$ and from the definitions $a(\lambda) = -b_x/2 = -\partial_x b/2$ and $c(\lambda) = -b_{xx}/2$, since the derivative $\partial_x$ commutes with the functional $\Phi_\lambda$.

According to the construction in \([23]\) we can fix one divisor $D$ consisting of poles in the matrix-function $L_0(\lambda)$ and a second divisor $U$ consisting of zeros of the entry $b(\lambda)$. Let they have the form

$$D = \{(e_j, m_j) \mid j = 1, \ldots, \tilde{m}; \},$$

$$U = \{(u_k, n_k) \mid k = 1, \ldots, \tilde{n}; \},$$

(3.12)

$$u_k \neq \infty, \quad e_j \neq u_k, \quad \forall j, k.$$

where $D, U \subset \mathbb{CP}_1 = \mathbb{C} \cup \{\infty\}$ for the $XXX$ model and $D, U \subset \mathbb{CP}_1/2\pi \mathbb{Z}$ for the $XXZ$ model. We consider the simple zeroes $u_k$, $n_k \equiv 1$; $k = 1, \ldots, n$, only. This restriction is related to the $r$-matrix structure. The poles at $e_j$ can be multiple poles and their orders $m_j$ are fixed. In the formulæ below each pole is counted according to its multiplicity and we don’t use special indices when it is clear from a context.

Now we introduce the following ansatz for the entry $b(\lambda)$ according to \([23, 27]\)

$$b(\lambda) = u_0 \cdot \prod_{k=1}^n \frac{\varphi(\lambda - u_k)}{\prod_{j=1}^n \varphi(\lambda - e_j)},$$

(3.13)

$$\varphi(\lambda) \equiv w^{-1}(\lambda) = \lambda \quad \text{in the $XXX$ case},$$

$$\varphi(\lambda) = \sinh \lambda \quad \text{in the $XXZ$ case},$$

where $u_0$ is either a constant or a dynamical variables. The function $\varphi(\lambda)$ depends from the type of the $r$-matrix and the functional $\Phi_\lambda$ \((3.8)\) is residue with $K = 1$ for $n < m$ and $K = n - m + 1$ for $n \geq m$.

The entry $a(\lambda)$ has $m$ poles at points $e_j$ of the divisor $D$ and its values at $n$ points $u_k$ are given $(a(\lambda = u_k) = v_k)$ \((3.3)\). The asymptotic behavior at infinity is fixed by Lax representation \((3.9)\). By Lagrange interpolation formula the entry $a(\lambda)$ can be represented as

$$a(\lambda) = b(\lambda) \cdot \left( v_0 + \sum_{i=1}^n v_i \cdot w_3(\lambda - u_i) \cdot g_{ii} \right),$$

(3.14)

where

$$g_{ii} = \left[ \frac{\partial b}{\partial \lambda}_{\lambda = u_i} \right]^{-1} \text{Res}_{u_i} \bigg|_{\lambda = u_i} b^{-1}(\lambda) = \frac{\prod_{j=1}^n \varphi(u_i - e_j)}{\prod_{k \neq i} \varphi(u_i - u_k)},$$

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The $w_3(\lambda) = \varphi'(\lambda)/\varphi(\lambda)$ is the coefficient in the definition of the corresponding $r$-matrix. Variables $u_0$, $v_0$ are connected to the Lax matrix $M_0$ (2.52).

Further, from the Lax representation (2.46) for the initial matrix $L_0(\lambda)$, we can construct the entry $a(\lambda)$. It reads as

$$a(\lambda) = -\frac{b_x}{2} = \frac{b(\lambda)}{2} \cdot \left( -\frac{\dot{u}_0}{u_0} + \sum_{i=1}^{n} \dot{u}_i \cdot w_3(\lambda - u_i) \right),$$

(3.15)

where $\dot{u}_k \equiv u_{k,x} = \partial_x u_k$, and $w_3(\lambda) = \frac{\varphi'(\lambda)}{\varphi(\lambda)}.$

We set now, for the simplicity, $u_0 = 1$ and $v_0 = 0$ as in the Lax matrix (2.43). Definition of entry $c(\lambda)$ follows from (2.46) as well

$c(\lambda) = -\frac{b_x}{2} = \frac{a^2(\lambda)}{b(\lambda)} + \frac{b(\lambda)}{2} \cdot \sum_{i=1}^{n} \left( \dot{u}_i \cdot w_3(\lambda - u_i) - \dot{u}_i^2 \frac{\partial w_3(\lambda - u_i)}{\partial u_i} \right),$

(3.17)

where $\dot{u}_i$ and $\ddot{u}_i$ are defined below.

From comparison of two forms of the entry $a(\lambda)$ (3.14) and (3.15) we can introduce the following Hamiltonian

$$H_0 = \sum_{i=1}^{n} v_i^2 \cdot g_{ii} = \sum_{i=1}^{n} v_i^2 \cdot \left[ \frac{\partial b(\lambda)}{\partial \lambda} \right]^{-1}_{\lambda = u_i},$$

(3.18)

using the Hamilton-Jacobi equations.

The remaining symbols in the entry $c(\lambda)$ are defined as

$$\dot{u}_i \equiv \frac{\partial H_0}{\partial v_i} = 2 \cdot v_i \cdot g_{ii},$$

$$\ddot{u}_i = 2 \sum_{j=1}^{m} v_j^2 \cdot g_{ij} \cdot \varphi'(u_i - e_j) \varphi(u_i - e_j) +$$

$$+ 2 \sum_{k=1}^{n} [(v_k + v_i)^2 g_{ik}g_{kk} - v_i^2 g_{kk}g_{ii}] \varphi(u_i - u_k) \varphi(u_i - u_k),$$

where $\varphi(\lambda) = w_1^{-1}(\lambda) = \lambda$ or $\varphi(\lambda) = \sinh(\lambda)$. Then a direct calculation shows that the Hamiltonian (3.18) is equal to the Hamiltonian (2.41) $H_0 = \Phi_{\lambda} [d_0(\lambda)]$. The matrix $L_0(\lambda)$ with entries (3.13), (3.14) and (3.17) obeys the $r$-matrix structure (2.4) as we can check it directly. Let us calculate

$$\{ a(\lambda), b(\mu) \} = w_3(\lambda - \mu)b(\mu) - w(\lambda - \mu)b(\lambda) =$$

$$= -\sum_{k=1}^{n} \frac{g_{kk} \cdot \varphi'(\mu - u_k) \cdot \varphi(\lambda - u_k)}{\varphi(\lambda - u_k) \varphi(\mu - u_k)} \cdot b(\lambda)b(\mu) = b(\lambda)b(\mu) \sum_{k=1}^{n} g_{kk} \left( \frac{w_3(\lambda - \mu)}{\varphi(\lambda - u_k)} - \frac{w(\lambda - \mu)}{\varphi(\mu - u_k)} \right)$$

$$- \sum_{k=1}^{n} \frac{g_{kk} \cdot \varphi'(\mu - u_k)}{\varphi(\lambda - u_k) \varphi(\mu - u_k)} = \sum_{k=1}^{n} \frac{\varphi'(\lambda - \mu) \varphi(\mu - u_k) - \varphi(\lambda - u_k)}{\varphi(\lambda - \mu) \varphi(\lambda - u_k) \varphi(\mu - u_k)}$$

The last equality is true due to the additive theorem for $\varphi(\lambda)$.

So, we have constructed the initial matrix $L_0(\lambda)$ in the term of variables of separation. From the initial matrices $L_0(\lambda)$ we can construct modified matrices $L_{MN}(\lambda)$ (2.31). These matrices describe integrable Hamiltonian system with the Hamiltonian $H = \Phi_{\lambda} [d(\lambda)]$ and a complete set of the integrals of motion can be obtained from determinants of these matrices as well. The determinant $d(\lambda)$ is a meromorphic function with the divisors of poles determined by divisors of the entries of $L_0(\lambda)$ or of
$L_{MN}(\lambda)$. Therefore the rational algebraic functions $d(\lambda)$ can be written as a quotient of two product or it can be decomposed into simple fraction according to the Mittag-Leffler theorem

$$d(\lambda) = \frac{P^N_d(\lambda)}{Q_d^N(\lambda)} = \prod_{k=1}^N \varphi(\lambda - h_k) \prod_{j=1}^m \varphi(\lambda - e_j) =$$

$$= P_d^{N-m}(\lambda) + \sum_{j,i} H_{ji} \cdot \varphi^{-i}(\lambda - e_j) = \prod_{k=1}^{N-m} \varphi(\lambda - H_k) + \sum_{j,i} H_{ji} \cdot \varphi^{-i}(\lambda - e_j),$$

where $N = n$ for the initial matrix $L_0(\lambda)$. Both decomposition provide us with sets of integrals of motion.

**Lemma 3** Zeros $h_k$ of the function $P_d^N(\lambda)$ can be chosen as a first set of integrals of motion in involution for the Hamiltonian systems related to the Lax matrices $L_0(\lambda)$ and $L_{MN}(\lambda)$. Zeros $H_k$ of the function $P_d^{N-m}(\lambda)$ and the residues $H_{ji}$ at points $e_j$ constitute a second set of the integrals of motion in involution.

**Proof:** The proof of involutivity is based on the equality (2.8) and follows the same pattern as the proof of Theorem 4. Completeness and functional independence of these integrals have been proved by Mishchenko and Fomenko (see review [25]).

We summarize these considerations as

**Proposition 1** The Lax matrix $L_0(\lambda)$ describes geodesic motions on the Riemannian manifold with the complex diagonal meromorphic Riemannian metrics

$$g^{kj} = \delta^j_k \left[ \frac{\partial b(\lambda)}{\partial \lambda} \right]^{-1} = \delta^j_k \cdot \prod_{i \neq j} \varphi(u_i - e_j) \prod_{k \neq i} \varphi(u_i - u_k).$$

defined on moduli of $n$-dimensional Jacobi varieties in terms of complex elliptic (root) coordinates $u_k$. The Hamiltonian system (3.18)

$$H_0 = \sum_i v_i^2 g_{ii}$$

has a complete system of first integrals $v_k$ and it defines a Lagrangian submanifold of the phase space $C^{2m}$ which has the form of symmetric product ($(\Gamma \times \ldots \times \Gamma)/\sigma_n$) of $n$ copies of Riemannian surfaces.

A modified Lax matrix $L_{MN}(\lambda)$ describes potential motion on the same Riemannian manifold with a Hamiltonian of natural type

$$H = H_0 + V_{MN}(u_1, u_2, \ldots, u_n; f_{-M}, f_{-M+1}, \ldots, f_N),$$

where $f_j$ are coefficients of the function $f(\lambda) = \sum f_k \lambda^k$ or of $f(\lambda) = \sum f_k e^{k\lambda}$. These matrices obey the $rs$-matrix structure (2.2) with the $s_{MN}$-matrix given by the Theorem 4.

In the potential case we reconstruct, for instance, motions under permutational symmetric potential classically (via point transformation) separable in generalized elliptic coordinates [25]. For $m \leq n + 1$ we have here the well known Jacobi problem of geodesic motions on quadrics [3, 7], corresponding quadrics are defined by condition (3.9)

$$-\text{Res}_{\lambda \rightarrow \mu} \left[ \frac{b(\lambda)}{\varphi(\lambda - \mu)} \right] = 1, \quad \iff \quad Q^n \left\{ \sum \frac{x_j^2}{e_j} = 1 \right\}$$

with variables $x_j$ will be introduced in section 4.1.
For these systems we can define a Riemann matrix of periods of holomorphic differentials and by using the Abel-Jacobi map we can find the action-angle variables. The potentials $V_{MN}$ are the finite gap potentials on these manifolds [33]. For details about geodesic and potential motion with metric (3.20) in the XXX case see [4, 5].

Thus we have constructed initial matrices $L_0(\lambda)$ and modified matrices $L_{MN}(\lambda)$ (2.27–2.31) which correspond to a geodesic motion and a potential motion on Riemannian manifolds with metric (3.20), respectively. For the XXX case the initial matrix $L_0(\lambda)$ (3.13, 3.14, 3.17) was considered in [16] and the Hamiltonian $H_0$ (3.13) describes free motion on Riemannian spaces of constant curvature.

The above results describe motions on Jacobi variety $((\Gamma \times \ldots \times \Gamma)/\sigma_n)$ which is a symmetric product of uniform Riemannian surfaces

$$\Gamma : \psi_k^2 = -d(\lambda)|_{\lambda=u_k} .$$

This scheme can be generalized to the case of the more general Jacobi variety $((\Gamma_1 \times \Gamma_2 \ldots \times \Gamma_n))$ with different Riemannian surfaces

$$\Gamma^{(j)} : \psi_k^{(j)} = -d^{(j)}(\lambda)|_{\lambda=u_k} , \quad k = 1, \ldots, n_j, \ j = 1, \ldots, n .$$

where $k$ and $j$ are two numbers indexing curves. Motion on such manifolds can be described with the use of the block matrices

$$L(\lambda) = \oplus_j^n L^{(j)}(\lambda, u^{(j)}_1, u^{(j)}_2, \ldots; v^{(j)}_1, v^{(j)}_2, \ldots),$$

acting in the auxiliary space

$$V_{\text{aux}} = \oplus_j^n V^{(j)}_{\text{aux}} .$$

Here $j$ numbers blocks and $k$ denotes number of degree of freedom related to each block. Each block

$L^{(j)}(\lambda, u^{(j)}_1, u^{(j)}_2, \ldots; v^{(j)}_1, v^{(j)}_2, \ldots)$

is a matrix of type $L_0(\lambda)$ or $L_{MN}(\lambda)$ (2.27), respectively. These manifolds have been investigated in [33] and such block matrices were considered in [15, 32].

In the XYZ case parameterization of the entry $b(\lambda)$ is more complicated. For instance, it has to comply with the equality (2.16)

$$\{b(\lambda), \tilde{b}(\mu)\} = g(\lambda, \mu) \cdot \tilde{b}(\lambda) - g(\mu, \lambda) \cdot \tilde{b}(\mu) .$$

In order to bypass this difficulty we can use two-dimensional lattice averaging [12] for the constructing matrices $L_0(\lambda)$ from the corresponding matrices of the rational model.

In quantum mechanics the metric (3.20) defines standard operator of the Laplace-Beltrami type and stationary Schrödinger equations

$$\nabla_j^\dagger \nabla_j \Psi + (E - V_{MN}(u_1, u_2, \ldots, u_n; f_{-M}, \ldots, f_{-N})) \Psi = 0$$

on the $n$-dimensional Riemannian manifolds [37]. Quantum separation equation for geodesic motion and potential motion and their properties have been considered in [33] but not in the $r$-matrix approach. In the framework of the linear $r$-matrix formalism separation of variables has a direct quantum counterpart which is described in [27, 28, 29]. In the quantum case we replace variables $u_i, v_i$ by operators and Poisson brackets (2.4) by the commutators

$$\left[ L(\lambda), L(\mu) \right] = \left[ r(\lambda, \mu), L(\lambda) + L(\mu) \right] .$$

Note that dynamical $s_{MN}(\lambda, \mu)$ matrices depend on coordinates only. Separated equations have the following form

$$\left( \frac{d^2}{du_k^2} + U(u_k; e_1, \ldots, e_m; f_{-M}, \ldots, f_{-N}; h_1, \ldots, h_n) \right) \psi_k(u_k, h_1, \ldots, h_n) = 0$$
where potential $U(u_j)$ is a rational algebraic function of the variables $u_j$ parametrized by $e_{ij}$, $i = 1 \ldots m$; and $f_j$, $j = -M \ldots N$ and $h_j$ are the eigenvalues integrals of motion. According to [28], common multiparticle eigenfunction of quantum integrals of motion is equal to

$$
\Psi(u_1, \ldots, u_n) = \prod_{k=1}^{n} \psi_k(u_k).
$$

A quantum operator $L_0(\lambda)$ for free motion in the XXX case was investigated in [28, 19, 16] while, the quantum modified operator $L(\lambda)$ for potential motion was considered in [11, 18].

4 Canonical transformations of variables

We can represent meromorphic matrix-functions $L(\lambda)$ by means of generators of various Lie groups, which one can consider as the various phase spaces for the integrable systems. This leads to links among different integrable systems: the Neumann problem, Gaudin magnets and Euler tops [24, 1]. In this paper we shall consider finite-dimensional integrable systems of natural type and shall investigate three types of canonical transformations of variables prescribed by linear $r$-matrix structure (2.4,2.5).

4.1 Generalized elliptic coordinates

If the limit orders of poles $m_j$ at the points $e_j$ are fixed then the Poisson brackets at different poles are independent and the corresponding orbits are a direct product of orbits of algebra $su(2)$. This factorization is independent of the points $e_j$ [25]. By the Mittag-Leffler theorem we can decompose rational functions $b(\lambda)$ (3.13) and $a(\lambda)$ (3.14) into partial fractions

$$
b(\lambda) = P_{b}^{n-m}(\lambda) + \sum_{j,i} b_{ji} \cdot \varphi^{-i}(\lambda - e_j) = \prod_{k=1}^{n-m} \varphi(b_k - \lambda) + \sum_{j,i} b_{ji} \cdot \varphi^{-i}(\lambda - e_j), \tag{4.1}
$$

$$
a(\lambda) = P_{a}^{n-m-1}(\lambda) + \sum_{j,i} a_{ji} \cdot \varphi^{-i}(\lambda - e_j) = \prod_{k=1}^{n-m-1} \varphi(a_k - \lambda) + \sum_{j,i} a_{ji} \cdot \varphi^{-i}(\lambda - e_j),
$$

where $b_k$ and $a_k$ are zeroes of the polynomials $P_b^{n-m}(\lambda)$ and $P_a^{n-m-1}(\lambda)$, respectively. The residues $b_{ji}$ and $a_{ji}$ at the points $e_j$ are linear coordinates on the algebra $su(2)^*$ (3.7). We use later two representations for generators of this algebra in terms of canonical variables

$$
b_j = x_j^2, \quad a_j = x_j p_j, \quad c_j = p_j^2, \quad b_j = 2x_j X_j, \quad a_j = x_j p_j + X_j p_j, \quad c_j = 2p_j p_j. \tag{4.2, 4.3}
$$

We omit here the double indexes $ji$ which are necessary for higher order poles and we used notation $c_j$ for residues of entry $c(\lambda)$. Variables $x_j$, $p_j$ and $X_j$, $P_j$ are canonically conjugate coordinates and momenta. Generalized elliptic coordinates correspond to the first representation (4.2) and in our scheme they are a simple consequence of the Mittag-Leffler theorem for meromorphic functions.

The other $(n-m)$ variables can be chosen as zeroes of the polynomial $P_a^{n-m}(\lambda - x_k) = 0$ and as the values of function $P_a^{n-m-1}(\lambda)$ at these zeroes $x_k$

$$
P_b^{n-m}(\lambda = x_k) = 0 = P_a^{n-m-1}(\lambda = x_k) \tag{4.4}
$$

Variables $x_k$, $k = m + 1, \ldots, n$ are calculated from the coefficients of the polynomial $P_b^{n-m}(\lambda) = \sum B_k \lambda^k$. For instance in the XXX case with simple poles these coefficients are equal to

$$
B_k = \left. \frac{1}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} b(\lambda) \right|_{\lambda = 0} + (-1)^k \sum_{j=1}^{m} \frac{x_j^2}{e_j^{k+1}}, \tag{4.5}
$$

then the conjugate (to the $x_k$) momenta $p_k$, $k = m + 1, \ldots, n$ are

$$
p_k = a(\lambda)|_{\lambda = x_k} - \sum_{j=1}^{m} \frac{x_j p_j}{x_k - e_j}. \tag{4.6}
$$

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Theorem 4 Variables \( x_j, p_j \) \( j = 1, \ldots, m \) and \( x_k, p_k \) \( k = m + 1, \ldots, n \) are a system of canonically conjugate variables.

Proof The Mittag-Leffler theorem defines a unique, invertible map between old \( n \) pairs of variables \((u_k, v_k)\), \( k = 1, \ldots, n \) and the new \( n \) pairs of variables \((x_k, p_k)\), \( k = 1, \ldots, n \). The proof of canonical conjugation of these variables is based on the \( r \)-matrix structure. The technique developed by Sklyanin (see \cite{27} and proof of the Theorem 4) has to be employed.

In the case of higher order poles we have an additional freedom in representing residues through canonical variables. There are two approaches to this problem. The first one is connected with standard degenerations of simple divisors, for example \( e_j \rightarrow e_{j+1} \) or \( e_j \rightarrow \infty \). This approach is developed in \cite{10} \cite{13}. Our approach is different.

We shall show how the variables \((q_j, p_j)\) are connected to Newton type variables for stationary flows of KdV type equations introduced in \cite{24}. We shall consider the \( XXX \) case but generalization for the \( XXZ \) case is transparent. The \( r \)-matrix algebra is associative; a linear combination of any two matrices \( L_1 \) and \( L_2 \) satisfying \cite{(2.4)} satisfies \cite{(2.4)} too. Let us consider the polynomial part of \( L_0(\lambda) \) corresponding to poles at infinity. The determinants of \( L_0(\lambda) \) and \( L(\lambda) \) from \cite{(2.47)} \cite{(2.51)} are

\[
d_0(\lambda) = \frac{b \cdot b_{xx}}{2} - \frac{b_x^2}{4},
\]

\[
d(\lambda) = \frac{b \cdot b_{xx}}{2} - \frac{b_x^2}{4} + b^2 \cdot u_{MN}(\lambda).
\]

They are integrated forms of the KdV recursion relations \cite{24}.

A simple substitution for the entries of matrix \( L_0(\lambda) \)

\[
b(\lambda) = B^2, \quad a(\lambda) = -b_x/2 = -BB_x, \quad c(\lambda) = -b_{xx}/2 = -B_x^2 - BB_{xx},
\]

turns determinants \cite{(4.7)} \cite{(4.8)} into the form

\[
d_0(\lambda) = B^3B_{xx}, \quad d(\lambda) = B^3B_{xx} + B^4 \left[ \frac{f(\lambda)}{B^4} \right]_{MN},
\]

if we use an explicit formula for the potential \( u_{MN}(\lambda) \). These equations have the form of Newton equations for \( B \)

\[
B_{xx} = d_0(\lambda)B^{-3},
\]

\[
B_{xx} = d(\lambda)B^{-3} - B \left[ \frac{f(\lambda)}{B^4} \right]_{MN},
\]

If we assume that \( B = \sum_{j=0}^{N} q_{N-j} \lambda^j \) is a polynomial then its coefficients \( q_j \) obey the Newton equation of motion \cite{(4.11)} with \( d(\lambda) = \sum I_k \lambda^k \), where \( I_k \) are integrals of motion. It becomes clear in terms of variables \( q_j \) that deformation of integrals of motion \( I_k \) affects only the potential (\( q \)-dependent) part. The kinetic (momentum dependent) part of \( I_k \) remains unchanged.

Coefficients of all entries \( a(\lambda), b(\lambda) \) and \( c(\lambda) \) are easily expressed in terms of coefficients of \( B \). Let us fix the highest order of \( \lambda \) in entry as \( K \), then

\[
b(\lambda) = \sum_{j=0}^{K} b_j \lambda^j, \quad a(\lambda) = \sum_{j=0}^{K} a_j \lambda^j, \quad c(\lambda) = \sum_{j=0}^{K} c_j \lambda^j,
\]

\[
\text{with} \quad b_K = 1, \quad a_K = 0, \quad c_K = 0
\]

due to the definition of \( M_0 \) and \( \Phi(\lambda) \). Let

\[
B(\lambda) = \sum_{j=0}^{K} q_{K-j} \lambda^j, \quad \text{with} \quad q_0 = 1,
\]
where the set $\mathcal{B}$ we consider as a formal set. After substitution of these definitions into (4.9) we obtain

$$ b_j = \sum_{i=0}^{K-j} q_i q_{K-j-i}, \quad a_j = -\sum_{i=0}^{K-j} q_{i,x} q_{K-j-i}, \quad (4.14) $$

$$ c_j = -\sum_{i=0}^{K-j} q_{i,x} q_{K-j-i,x} - \sum_{i=0}^{K-j} q_{i,xx} q_{K-j-i}, $$

where $q_x$ is a derivative with respect to $x$ and we used the Newton formulae for a product of sets. From $q_0 = 1$ and the definitions (4.14) the restrictions (4.13) for the higher coefficients of entries of matrix $L_0$ follow automatically. Canonically conjugate (to the coordinates $q_j$) momenta $p_j$ can be derived from the $r$-matrix algebra (2.4) and from the definitions (4.12–4.14)

$$ p_j = q_{K+1-j,x}, \quad \{p_j, q_k\} = \delta_{jk}. $$

As an example, we present first polynomials

$$ K = 1, \quad b(\lambda) = \lambda + 2q_1, \quad -a(\lambda) = p_1, $$

$$ K = 2, \quad b(\lambda) = \lambda^2 + 2\lambda x_1 + (2x_2 + x_1^2), $$

$$ -a(\lambda) = \lambda p_2 + (p_1 + p_2 x_1) , $$

$$ K = 3, \quad b(\lambda) = \lambda^3 + 2\lambda^2 x_1 + \lambda(2x_2 + x_1^2) + 2(x_3 + x_1 x_2), $$

$$ -a(\lambda) = \lambda^2 p_3 + \lambda(p_2 + p_3 x_1) + (p_1 + p_2 x_1 + p_3 x_2), $$

All integrals of motion can be expressed in terms of new variables directly from the equations (4.10) by taking residues. Notice that the kinetic part of the Hamiltonian $H = \Phi_\lambda [d(\lambda)]$ has the nondiagonal form

$$ T = \sum_{j=1}^{K+1} p_j p_{K-j}. \quad (4.16) $$

Definition of variables of separation is not changed by deforming the Lax matrix $L_0(\lambda)$ (2.27). For instance, the free motion Hamiltonian $H = T + V_{MN}$ separate in the same coordinate systems. Later we shall show some examples illustrating the above scheme.

### 4.2 Jacobi coordinates

Let us take a simplest matrix $L_0^{(k)}(\lambda)$ with the entry $b(\lambda)$ of (3.13) which has only one zero

$$ L_0^{(k)}(\lambda, v_k, u_k) = \left( \begin{array}{cc} -v_k & \varphi(\lambda - u_k) \\ 0 & v_k \end{array} \right). \quad (4.17) $$

It obeys the linear $r$-matrix algebra (2.4) with the $r$-matrix (2.17) or (2.18). Let

$$ L_0(\lambda) = \oplus_k L_0^{(k)}(\lambda, v_k, u_k) \quad (4.18) $$

what means that we associate to our system a matrix $L$, which has blocks structure and each block $L_0^{(k)}$ obeys the $r$-algebra with the same $r$-matrix.

A modified matrix $L(\lambda)$ (2.27) is equal to

$$ L(\lambda) = \oplus_k L^{(k)}_{MN}(\lambda, v_k, u_k), \quad (4.19) $$

where blocks $L^{(k)}_{MN}(\lambda, v_k, u_k)$ are

$$ L^{(k)}_{MN}(\lambda, v_k, u_k) = \left( \begin{array}{cc} -v_k & \varphi(\lambda - u_k) \\ \varphi(\lambda - u_k) & v_k \end{array} \right)_{MN} $$

$$ = \left( \begin{array}{cc} -v_k & \varphi(\lambda - u_k) \\ c_{MN}(\lambda, u_k) & v_k \end{array} \right). \quad (4.20) $$
These matrices are deformations of the initial matrices \(L_0(\lambda, v_k, u_k)\) \(\text{(4.17)}\) and each block \(L_{MN}^{(k)}\) obeys the rs-algebra with a common r-matrix but different (since they depend on \(u_k\)) matrices \(\alpha_{MN}^{(k)}\) constructed according to \(\text{(2.35)}\).

The matrix \(L_0\) \(\text{(4.17)}\) and the modified matrix \(L_{MN}\) \(\text{(4.20)}\) have an internal structure \(\text{[1]}\)

\[
L_0(\lambda, v_k, u_k) = 
\begin{pmatrix}
-v_k & \varphi(\lambda - u_k) \\
0 & v_k
\end{pmatrix} = 
\begin{pmatrix}
-\sum \alpha_j p_j^{(k)} & \varphi(\lambda - \sum \alpha_j^{-1} q_j^{(k)}) \\
0 & \sum \alpha_j p_j^{(k)}
\end{pmatrix}
\]

\(\text{(4.21)}\)

if variables \(v_k\) and \(u_k\) are considered as linear combinations of certain canonical variables \(p_j^{(k)}, q_j^{(k)}\) \(j = 1, \ldots, K\). If we consider a system with of \(n\) degree of freedom and require that Hamiltonian has a natural canonical form with the kinetic energy form

\[
T^{(n)} = \sum_{k=1}^{n} v_k^2 = \sum q_j^2
\]

then the internal structure of \(\text{(4.21)}\) leads to the Jacobi transformations for an \(n\)-degrees of freedom systems. It is then a pure coordinate change of variables. As before free motion equations corresponding to the undeformed matrix and motion with potential corresponding to modified matrix separate in the same system of coordinate.

### 4.3 Momentum dependent change of variables

Let us consider functions \(B_j(\lambda)\) and \(A_j(\lambda)\), which are algebraic functions of entries of the matrix \(L(\lambda)\) and which satisfy suitable Poisson brackets, for example

\[
\{A_j(\lambda), A_k(\mu)\} = \{B_j(\lambda), B_k(\mu)\} = 0, \\
\{A_j(\lambda), B_k(\mu)\} = \delta_{jk} \cdot g(\lambda, \mu) \cdot (B_j(\lambda) - B_k(\mu)).
\]

We shall require that zeroes of the functions \(B_j(\lambda)\) and values of functions \(A_j(\lambda)\) at these zeroes define a new system of coordinates. Because functions \(B_j(\lambda)\) and \(A_j(\lambda)\) are meromorphic functions it will also be interesting to consider the corresponding residues. As an example, we present such transformation related to Henon-Heiles system \(\text{[32]}\).

Let us restrict to the XXXX case. In the same way as for the Jacobi transformations we consider the block matrices \(L(\lambda)\) \(\text{(4.17)}\) with the simplest blocks \(\text{(4.21)}\) \(\text{[32]}\).

**Lemma 5** The map \((u, v) \rightarrow (x, p)\) given by

\[
x_k^2 = \alpha \frac{d_k(\lambda) - d_{k+1}(\lambda)}{b_k(\lambda) - b_{k+1}(\lambda)} \bigg|_{\lambda=0} = \alpha \in \mathbb{R}, \quad k = 1, \ldots, n - 1,
\]

\[
p_k = \frac{a_k(\lambda) - a_{k+1}(\lambda)}{b_k(\lambda) - b_{k+1}(\lambda)} \bigg|_{\lambda=0} \cdot x_k;
\]

\(\text{(4.22)}\)

\[
x_n = \beta \sum_{k=1}^{n} b_k(\lambda) \bigg|_{\lambda=0} - \sum_{k=1}^{n-1} \frac{\alpha \beta}{2} \left( \frac{p_k}{x_k} \right)^2, \quad \beta \in \mathbb{R},
\]

\[
p_n = \frac{1}{\beta} \sum_{k=1}^{n} a_k(\lambda) \bigg|_{\lambda=0} + \alpha \beta \sum_{k=1}^{n-1} \frac{p_k}{x_k} \cdot \left( 1 + \frac{p_k^2}{x_k^2} \right);
\]

where \(a_k(\lambda)\) and \(b_k(\lambda)\) are the entries of matrices \(L_{MN}(\lambda, v_k, u_k)\); \(k = 1, \ldots, n\) \(\text{(4.20)}\) and \(d_k(\lambda)\) are the corresponding determinants, is a canonical transformation.
Proof. We fix the variable $x_k^2$ and the Hamiltonian $H = \sum_{k=1}^{n} d_k(\lambda = 0)$. Then a corresponding momentum $p_k = \{H, x_k\}$ follows from the $rs$-algebra (2.7)

$$
2 xp_k = \{H, x_k^2\}
= \alpha \left\{ \dfrac{d_k(\lambda) + d_{k+1}(\lambda), d_k(\mu) - d_{k+1}(\mu)}{b_k(\mu) - b_{k+1}(\mu)} \right\}_{\mu=0, \lambda=0}
= -\alpha \dfrac{d_k(\mu) - d_{k+1}(\mu)}{(b_k(\mu) - b_{k+1}(\mu))^2} \cdot \{d_k(\lambda) + d_{k+1}(\lambda), b_k(\mu) - b_{k+1}(\mu)\}_{\mu=0, \lambda=0}
= 2\alpha \dfrac{d_k(\mu) - d_{k+1}(\mu)}{b_k(\mu) - b_{k+1}(\mu)} \cdot \{d_k(\mu) - d_{k+1}(\mu), b_k(\mu) - b_{k+1}(\mu)\}_{\mu=0}
= 2 \dfrac{a_k(\mu) - a_{k+1}(\mu)}{b_k(\mu) - b_{k+1}(\mu)} \cdot x_k^2
$$

where we have used the explicit form of the $L_{MN}$ matrices (4.20) and the technique developed by Sklyanin [28]. The variable $x_n$ is fixed by the condition $\{x_k, x_n\} = 0$ and the corresponding momentum $p_n = \{H, x_n\}$ is calculated from the $rs$-algebra and from the Hamilton-Jacobi equations.

By the proof we have used the following relations

$$
\{b_k(\lambda), a_j(\mu)\} = \dfrac{\delta_{kj}}{\mu - \lambda} (b_k(\mu) - b_k(\lambda)),
\{b_k(\lambda), c_j(\mu)\} = 2\dfrac{\delta_{kj}}{\mu - \lambda} (a_k(\mu) - a_k(\lambda)) = 0,
\{b_k(\lambda), b_j(\mu)\} = 0,
$$

which are determined by the $r$-matrix only and the one relation $\{d_k(\lambda), d_j(\mu)\} = 0$, which is given the $rs$-algebra (2.7) with $s_{MN}^{(k)}$-matrices (2.35).

We emphasize that it is a non-pure coordinate change of variables and that a free motion on manifold and a potential motion on it separate in different systems of coordinates. The canonical transformation (4.22) is a technical result as yet, but it is known that the nonlinear equations behind the particular systems separated in these variables and in the Jacobi variables are related by a Miura map [14].

4.4 Examples

We restrict ourselves to the $r$-matrix algebra of the $XXX$ type and shall consider in detail only the Taylor projection (2.30).

As an example, we discuss here the Henon-Heiles system and a quartic system of two degrees of freedom. Their potentials are polynomials of degree three and four, respectively. The Henon-Heiles potential reads $(A, B, C$ are constant):

$$
H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(Ax^2 + By^2) + x^2y + Cy^3.
$$

It has been extensively studied both in nonintegrable and integrable regimes. It is known [14, 20] to be integrable for the following three sets of parameters

$$
(i) \quad A = B, \quad C = 1/3
$$
$$
(ii) \quad A, B \text{ arbitrary} \quad \quad C = 2
$$
$$
(iii) \quad 16A = B, \quad C = 16/3
$$

(4.23)

The quartic potential is described by

$$
H = \frac{1}{2}(p_x^2 + p_y^2) + Ax^4 + Bx^2y^2 + Cy^4.
$$
It is known \[20\] to be integrable for the following three sets of parameters

\[
\begin{align*}
(i) & \quad B = 6A, \quad C = A \\
(ii) & \quad B = 12A, \quad C = 16A \\
(iii) & \quad B = 6A, \quad C = 8A
\end{align*}
\]

(4.24)

Separation of variables of these Hamiltonians takes place in the coordinate systems defined in sections 4.1–4.3.

The type (i) systems are related to the Jacobi coordinates and to the block matrices \(L(\lambda)\) \((4.17, 4.19)\). Let us consider entries \(c_N(\lambda, u_k)\) in matrices \(L_N(\lambda, u_k, v_k)\) \((4.20)\) which are the following polynomials of two variables \(\lambda\) and \(u_k\):

\[
c_N(\lambda, u_k) = \left[ \sum_{i=0}^{N} f^{(k)}_i \lambda^i \cdot \sum_{j=0}^{+\infty} u_k^j \lambda^{j+1} \right] = \sum_{i=0}^{N-1} \lambda^i \sum_{j=0}^{N-1-i} u_k^j f^{(k)}_{j+i+1}.
\]

(4.25)

We remind that brackets \([\cdot, \cdot]_+\) denote a Tailor projection \((2.30)\). Determinants \(d^{(k)}_N\) of matrices \(L_N^{(k)}(\lambda, u_k, v_k)\) \((4.20)\) are equal to

\[
d^{(k)}_N = \sum_{i=1}^{N} f^{(k)}_j (u_k^i - \lambda^j) - v_k^2.
\]

(4.26)

A generating function of integrals of motion can be taken as a determinant of the matrix \((4.19)\) \(d(\lambda) = \prod_{k=1}^{n} d^{(k)}_N\). Hamiltonians for these systems can be defined by

\[
H^N_N = \frac{d^{(n-1)N}}{d\lambda^{(n-1)N}} d(\lambda) \bigg|_{\lambda=0},
\]

(4.27)

and their explicit form up to a constant factor reads

\[
H^N_N = \sum_{k=1}^{n} \left[ f^{(k)}_N \right]^{-1} v_k^2 - \sum_{k=1}^{n} \left[ f^{(k)}_N \right]^{-1} \sum_{j=1}^{N} f^{(k)}_j \cdot u_k^j
\]

\[
= T^{(n)} + \beta V^{(n)}_N, \quad \beta \in \mathbb{R},
\]

(4.28)

where \(T^{(n)}\) denotes kinetic energy and \(V^{(n)}_N\) is a potential. If higher coefficients \(f^{(k)}_N\) of the polynomials \(f^{(k)}(\lambda)\) are the same for all degrees of freedom \(f^{(k)} = f^{(j)}_N\) for \(\forall k, j\) then the hamiltonians \((4.27)\) can be rewritten as

\[
H^N_N = \sum_{k=1}^{n} d^{(k)}_N \bigg|_{\lambda=0}.
\]

(4.29)

In the case of the Laurent projection \([\cdot, \cdot]_{MN}\) \((2.29)\) Hamiltonians \((4.28)\) are equal to

\[
H^N_N = \sum_{k=1}^{n} \left[ f^{(k)}_N \right]^{-1} (v_k^2 - \sum_{j=-(M-1)}^{N} f^{(k)}_j \cdot u_k^j),
\]

(4.30)

Now by exploiting the internal structure of the matrix \(L(\lambda)\) and by using the respective Jacobi transformations \((4.21)\) we shall illustrate the general scheme through the cases of two and three degrees of freedom. For two degrees of freedom after the transformation \(u_1 = q_1 + q_2, u_2 = q_1 - q_2\) the homogeneous potentials \(V^{(2)}_N\) of degrees \(j = 1, 2, \ldots, N\) read:

\[
\begin{align*}
V_1^{(2)} &= e_1^+ q_1 + e_2^- q_2; \\
V_2^{(2)} &= V_1^{(2)} + e_2^+ (q_1^2 + q_2^2) + e_2^- q_1 q_2; \\
V_3^{(2)} &= V_2^{(2)} + e_3^+ (q_1^3 + 3q_1 q_2^2) + e_3^- (q_3^3 + 3q_1 q_2^2); \\
V_N^{(2)} &= \sum_{j=1}^{N} \sum_{i=0}^{j} C^{(j)}_{i} f^{(1)}_{j} (-1)^i \cdot f^{(2)}_{i} q_1^{j-i} \cdot q_2; 
\end{align*}
\]

(4.31)
where \( c_j^N = \left( f_N^{(1)} + f_N^{(3)} \right) \) and \( C_j^i \) are the binomial coefficients.

For three degrees of freedom and \( T = \sum R_k^2 \) we can choose the coefficients \( f_N^{(k)} \) in such way that the Jacobi transformation has the simplest possible form

\[
\begin{align*}
    u_1 &= (q_1 - 2q_2 + q_3), \quad u_2 = -3(q_1 - q_3), \quad u_3 = (q_1 + q_2 + q_3)
\end{align*}
\]

with \( f_N^{(1)} = 6, f_N^{(2)} = 18, f_N^{(3)} = 3 \). The first homogeneous potentials \( V_N^{(3)} \) of degree \( j = 1, 2, \ldots N \) are

\[
\begin{align*}
    V_1^{(3)} &= (f_1^{(1)} - 3f_1^{(2)} + f_1^{(3)})q_1 + (f_1^{(3)} - 2f_1^{(1)})q_2 + (f_1^{(1)} + 3f_1^{(2)} - f_1^{(3)})q_3; \\
    V_2^{(3)} &= V_1^{(2)} + (f_2^{(2)} - 2f_2^{(3)})q_1q_2 + (4f_2^{(1)} + f_2^{(3)})q_2^2 \\
    &\quad + (2f_2^{(3)} - 4f_2^{(1)})(q_1q_2 + q_1q_3) + 2(f_2^{(1)} - 9f_2^{(2)} + f_2^{(3)})q_1q_3; \\
    V_N^{(3)} &= V_N^{(3)} + \sum_{j+k+l=N} f_{jkl} q_1^j q_2^k q_3^l;
\end{align*}
\]

where the \( f_{jkl} \) coefficients are expressed through \( f_j^{(k)} \) and the binomial coefficients \( C_j^i \). For two degrees of freedom the Hamiltonians \( H_3^{(2)} \) and \( H_3^{(3)} \) coincides with the Hamiltonians of the Henon-Heiles system and of the quartic system of type (i).

Systems of type (iii) are described by the same block matrices \( L(\lambda) \) and by the same function \( f_{MN}(\lambda) \) as type (i) systems. Systems of type (i) are separated in Jacobi systems of coordinates while systems of type (iii) separate after non-pure coordinate canonical transformation of variables described in Lemma 3.

The type (ii) systems are separated in generalized elliptic coordinates. The corresponding matrices \( L(\lambda) \) were introduced in [13, 11] and have been investigated in detail there.

Hamiltonians (4.23) and (4.24) correspond to a special choice of the function \( f_N(\lambda) \); for the Henon-Heiles system

\[
    f_3(\lambda) = -\frac{1}{6}(\lambda^3 - \frac{1}{2}A\lambda^2 - \frac{3}{2}A^2\lambda).
\]

The corresponding particular Lax equations and separation of variables were studied in [15] for the special choice of the function \( f_N(\lambda) \), variables of separation for particular quartics potentials were discussed in [22]. Some extensions of these system are known [21]. They contain more general functions \( f_{MN}(\lambda) \) and can be investigated within our scheme which describes the infinite family of rational potentials.

5 Conclusions

We have developed here the \( rs \)-matrix scheme for natural finite-dimensional integrable systems connected to the KdV and the coupled KdV hierarchies. But it applies to other hierarchies of integrable nonlinear evolution equations as well. For the hierarchy of the KdV type equations the Lax matrix \( M_0 \) is independent of the spectral parameter. The corresponding matrices for the AKNS hierarchy and for the SG hierarchy have one pole at infinity or two poles at infinity and at zero correspondingly. For applying our scheme in these cases one has to redefine the projector \( \Phi_\lambda \):

\[
    \Phi_\lambda = \varphi(\mu) \cdot \text{Res}_{\lambda=\infty}^K + \text{Res}_{\lambda=\infty}^{K-1}, \quad \varphi(\mu) = \mu \quad \text{or} \quad \varphi(\mu) = \mu + \frac{1}{\mu}.
\]

These two choices of the function \( \varphi(\mu) \) correspond to the AKNS and to the SG hierarchy if we use the second rational parametrization for the \( r \)-matrix of XXZ type [22] for the SG equation. The Hamiltonians in terms of the root variables \( u_k \) of finite-dimensional systems connected with stationary flows for this hierarchies were introduced in [3, 4].

In order to apply our approach to restricted flows of the AKNS hierarchy of equations [21] we have to fix the form of the matrix \( M_0 \) as:

\[
    M_0(\mu) = \begin{pmatrix} \mu & w \\ u & -\mu \end{pmatrix},
\]
and to choose the projector \( \Phi_{\lambda \mu} \) in the form (6.1).

In the next step one has to consider the undeformed matrix \( L_0(\lambda) \) corresponding to this choice of \( M_0 \) and \( \Phi_{\lambda \mu} \) and to prove that it obeys the \( r \)-matrix algebra. Let the matrix \( L_0(\lambda) \) has the form

\[
L_0(\lambda) = L_0^\infty(\lambda) + L_0^D(\lambda)
\]

where we split \( L_0(\lambda) \) into the polynomial (pole at infinity) part and the \( D \)-divisor part corresponding to finite poles. One finds the following sequence of \( L_0^\infty \) matrices yielding the prescribed \( M_0 \) and \( \Phi_{\lambda \mu} \) and satisfying the \( r \)-matrix algebra (2.4):

\[
L_0^\infty(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
L_0^\infty(\lambda) = \begin{pmatrix} \lambda & q_1 \\ p_1 & -\lambda \end{pmatrix}, \quad \{p_1, q_1\} = 1,
\]

\[
L_0^\infty(\lambda) = \begin{pmatrix} \lambda^2 - \frac{q_1 q_2}{2} & \lambda q_1 - 2p_2 \\ \lambda q_2 - 2p_1 & -\lambda^2 + \frac{q_1 q_2}{2} \end{pmatrix}, \quad \{p_2, q_2\} = 1.
\]

The analog of the parametrization related to the Newton representation \([4.15] [24]\) for the recurrent construction of these matrices is not known as yet. In this case the Hamiltonians are equal to \( H = \Phi_{\lambda \mu} [d(\lambda)] \), but they do not depend on the spectral parameter \( \mu \) since the corresponding residues are equal to zero.

The matrix \( M_0 \) is more symmetric by its entries in this case and respective integrable systems are of non-natural type \([27]\) one can use various solutions of the equations (2.19) for the deformation coefficients \( \alpha_j \). Systems with such deformations will be investigated in details in the forthcoming publications.

Finally we remark that the \( rs \)-matrix scheme presented in this paper applies to the discrete time analogs of soliton hierarchies. It yields an \( rs \)-matrix description of integrable symplectic maps.

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