Szekeres models: a covariant approach

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Abstract
We exploit the $1+1+2$ formalism to covariantly describe the inhomogeneous and anisotropic Szekeres models. It is shown that an *average scale length* can be defined *covariantly* which satisfies a 2d equation of motion driven from the *effective gravitational mass* (EGM) contained in the dust cloud. The contributions to the EGM are encoded to the energy density of the dust fluid and the free gravitational field $E_{ab}$. We show that the quasi-symmetric property of the Szekeres models is justified through the existence of 3 independent *intrinsic Killing vector fields* (IKVFs). In addition the notions of the apparent and absolute apparent horizons are briefly discussed and we give an alternative gauge-invariant form to define them in terms of the kinematical variables of the spacelike congruences. We argue that the proposed program can be used in order to express Sachs’ optical equations in a covariant form and analyze the confrontation of a spatially inhomogeneous irrotational overdense fluid model with the observational data.

Keywords: inhomogeneous models, spacelike congruences, Szekeres solutions

1. Introduction

The discovery of the accelerated phase of the Universe [1] and the detailed chartography of the temperature anisotropies in the cosmic microwave background (CMB) [2–4] make evident the fact that additional degrees of freedom are necessary in order to fit the observational data to the underlying geometry. The current observational ‘strategy’ states that the Universe is ‘almost’ homogeneous and isotropic at very large scales (∼100 Mpc), which imposes the highly symmetric Friedmann–Lemaître–Robertson–Walker (FLRW) geometry as the standard cosmological model. Within the FLRW spacetime, the accelerated expansion is usually interpreted by introducing a Λ-term dark sector that affects the cosmological evolution, whereas
the small temperature fluctuations occur due to the presence of local inhomogeneities and are described as first order perturbations of the background model.

However one can then argue that the presence, the form and the evolution of local density and expansion inhomogeneities and anisotropies seeded the formation of the Universe at small or medium scales (<10 Mpc) and influence the travel of light rays, causing various effects like focusing and lensing and thus giving an alternative explanation of the SN data out of the Λ CDM scheme. Therefore one must take into account exact inhomogeneous models which can be seen as ‘perturbations’ in an FLRW background. In this respect there is an increased interest of using inhomogeneous configurations within the standard cosmological paradigm like the Lemaître–Tolman (LT) model or Szekeres models, which represent an immediate generalisation of the former (see [5] and papers cited therein). Both can be matched to the FLRW at large scales, and it has been shown that they confront with the observational data without the need of a cosmological constant due to the rich variety of the matter density profiles accommodating these models.

At first glance, Szekeres’ solution [6, 7] (or its generalization to include a non-zero cosmological constant [8]) seems to be general enough within the class of inhomogeneous models due to the non-existence of isometries, i.e. Killing vector fields (KVFs) [9, 10] or, as far as we know, any other kind of symmetry. Nevertheless these models exhibit various special features (like the conformal flatness of the hypersurfaces $t = \text{const}$ [11]) which make them more tractable than expected [12]. In this paper we provide a covariant way to reproduce the key ingredients and describe, in a gauge-invariant form, the family of Szekeres models by using the well-established theory of spacelike congruences. In fact the $1 + 1 + 2$ splitting technique is the most appropriate tool for analyzing Szekeres models due to the decoupling of the spatial divergence and curl equations from the evolution equations of the kinematical and dynamical variables and the existence of 2d hypersurfaces of constant curvature at $t = \text{const}$ and $r = \text{const}$. As a result the formalism presented here can be used, in principle, to analyze light propagation and structure formation within an FLRW background.

The paper is organized as follows: in section 2 we give the evolution and constraint (spatial derivatives) equations of the associated kinematical and dynamical quantities. A further $1 + 2$ splitting is applied in section 3, revealing the role of the corresponding kinematical variables of the spacelike congruences in the dynamics. Using these results, in section 4 we define the rate of change or the expansion rate of the 2d surface area and the effective gravitational mass of the shells of dust, which is constituted of the total energy density $\rho$ of the matter fluid and the contribution from the free gravitational field (encoded in the electric part of the Weyl tensor). For the quasi-spherical case, due to the constant curvature feature of the hypersurface $t, r = \text{constant}$, we show that there exists a $SO(2)$ group of intrinsic Killing vector fields (IKVFs). We conclude in section 5 and discuss briefly the notions of apparent horizons (AH) and absolute apparent horizons (AAH) of Szekeres models [14] by proposing a covariant definition of them. We also discuss a possible application of the suggested program regarding the influence of the gravitational field on the light beams that come from a specific direction of an isotropic source (S) and pass through certain fluid configurations.

Throughout this paper the following conventions have been used: the pair $(\mathcal{M}, g)$ denotes the spacetime manifold endowed with a Lorentzian metric of signature $(−, +, +, +)$, spacetime indices are denoted by lower case Latin letters $a, b, \ldots = 0, 1, 2, 3$, spacelike eigenvalue indices are denoted by lower case Greek letters $\alpha, \beta, \ldots = 1, 2, 3$ and we have used geometrised units such that $8\pi G = 1 = c$. 


2. Description of the Szekeres models

The average velocity of the matter in the Universe is identified with a unit timelike vector field \( u^a \) \((u^a u_a = -1)\) tangent to the congruence of worldlines of the fundamental (preferred) observers. The Einstein field equations (EFEs) for a pressureless perfect fluid can be expressed in terms of the Ricci tensor as

\[
R_{ab} = \frac{\rho}{2}(u_a u_b + h_{ab}) + \Lambda g_{ab}
\]

(2.1)

where \( \rho \) is the energy density of the matter fluid as measured by the comoving observers \( u^a \), \( \Lambda \) is the cosmological constant and \( h_{ab} \equiv g_{ab} + u_a u_b \) is the projection tensor normally to \( u^a \).

The matter velocity \( u^a \) is geodesic (because of the vanishing of the pressure) and we also assume that it is irrotational (for an interesting treatment of Szekeres models with respect of tilted observers see [17]). In the case of a vanishing cosmological constant, the EFEs can be written in terms of the kinematical quantities of the geodesic and irrotational timelike congruence by using the Ricci identities

\[
2u_{a[i]} = R_{dab} u^d.
\]

(2.2)

Projecting equation (2.2) along the \( u^a \) we obtain the evolution equations [18]

\[
H = -H^2 - \frac{1}{3} \sigma_{ab} \sigma^{ab} - \frac{1}{6} \rho
\]

(2.3)

\[
\dot{\sigma}_{ab} = -2H \sigma_{ab} - \sigma^c_{[ab]} \sigma_{cb} + E_{ab} + \frac{1}{3} (\sigma_c d \sigma^{cd}) h_{ab}
\]

(2.4)

for the overall volume expansion (or Hubble parameter) rate \( 3H = \theta = u_c h^{ab} \) and the shear tensor \( \sigma_{ab} = h_{ij} h_{kl} u_{(i,j)} - \frac{\theta}{3} h_{ij} \) which describes the rate of distortion of the rest space of \( u^a \) in different directions (i.e. the change of its shape). On the other hand projecting normally to \( u^a \), equation (2.2) leads to the constraint equations (i.e. the spatial divergence and curl equations or equivalently the fully projected derivatives normal to the timelike congruence \( u^a \))

\[
h_{ab} \sigma_{kc} = 2h^k_j H_{jk}
\]

(2.5)

\[
H^a b^{ab} e_{ab} = 0
\]

(2.6)

where \( e^{abcd} \) is the 4-dimensional volume element and a dot denotes covariant differentiation along the direction of the vector field \( u^a \).

In order to obtain a closed set of equations we use the Bianchi identities

\[
R_{ab[cd,e]} = 0.
\]

(2.7)

The associated \( u^a \) and \( h_{ab} \) — projections of (2.7) give the energy and momentum conservation equations plus two evolution and two constraint equations for the electric and magnetic part of the Weyl tensor, which in the case of Szekeres models takes the form (we recall that the magnetic part of the Weyl tensor vanishes [19])

\[
E_{ab} = \frac{1}{2} \rho \sigma_{ab} - 3H E_{ab} + 3E_{cd} \sigma_{bc} - (E_c d \sigma^{cd}) h_{ab}
\]

(2.8)

\[
\dot{\rho} = -3 \rho H
\]

(2.9)
\[ h_{\alpha \beta} E_{\alpha \beta} = \frac{1}{3} h_{\alpha \beta}^{ab} \]  
\[ h_{\alpha \beta}^{ab} R_{\alpha \beta} = 0 \]  
\[ \epsilon^{abcd} F_{abcd} = 0. \] (2.10)  
(2.11)  
(2.12)

In the above system of equations (2.3), (2.4) and (2.8)–(2.12) we observe that the spatial divergence and curl equations (2.5), (2.6), (2.10) and (2.11) have been decoupled from the evolution equations of the kinematical and dynamical variables. As a result the evolution of Szekeres models is completely independent from the spatial variations of the physical variables and is described by a set of first-order ordinary differential equations (ODEs).

It is convenient to employ a set of three mutually orthogonal and unit spacelike vector fields \([x^a, y^a, z^a]\) which can be uniquely chosen as eigenvectors of the shear tensor. Equation (2.12) implies that the shear \(\sigma_{ab}\) and the electric part \(E_{ab}\) tensors commute, hence they share the eigenframe \([x^a, y^a, z^a]\). We write

\[ E_{ab} = E_{a} x_{a} x_{b} + E_{2} y_{a} y_{b} + E_{3} z_{a} z_{b} \]  
\[ \sigma_{ab} = \sigma_{a} x_{a} x_{b} + \sigma_{2} y_{a} y_{b} + \sigma_{3} z_{a} z_{b} \]  
where \(E_{a}, \sigma_{a}\) are the associated eigenvalues satisfying the trace-free conditions

\[ \sum_{a} E_{a} = \sum_{a} \tau_{a} = 0. \] (2.13)  
(2.14)  
(2.15)

Furthermore it has been shown that each of \([x^a, y^a, z^a]\) is hypersurface orthogonal [19], i.e.

\[ x_{a} x_{bc} = y_{a} y_{bc} = z_{a} z_{bc} = 0. \] (2.16)

Due to the hypersurface property of the timelike and spacelike vector fields each pair \([u^a, x^a]\), \([u^a, y^a]\), \([u^a, z^a]\) generates a 2d integrable submanifold of \(\mathcal{M}\) represented by the 2-form, e.g. \(F_2 = u \wedge x\). It turns out that at each point the spacetime manifold admits orthogonal 2-surfaces spanned by the vectors \([y^a, z^a]\) with a corresponding surface element the dual 2-form \(\tilde{F}_2\) (or equivalently the simple bivector \(C_{xy} = y \wedge z\)).

It should be emphasized that the above considerations hold, in general, for homogeneous and inhomogeneous Petrov type I silent models. However, in the present work we are interested in spatially inhomogeneous setups in which case only the Petrov type D subclass of models exists [12] and is described by the Szekeres family of solutions satisfying

\[ E_2 = E_3 \Leftrightarrow \sigma_2 = \sigma_3. \] (2.17)

In this case, local orthogonal coordinates can be found such that the metric can be written [6]

\[ ds^2 = g_{ab} dx^a dx^b = -dt^2 + e^{2A} dx^2 + e^{2B} (dy^2 + dz^2). \] (2.18)

The general solution of the resulting set of EFEs for a dust fluid implies that the smooth functions \(A(t, r, y, z), B(t, r, y, z)\) have the form

\[ e^A = \frac{S_r - S(\ln E)_r}{\sqrt{f + \epsilon}}, \quad e^B = \frac{S}{E} \]  
\[ \epsilon \] (2.19)

We shall see in section 4 that the value of \(\epsilon = 1, -1, 0\) describes the topology of the 2d hypersurface \(\mathcal{X}\) (i.e. the distribution \(\mathbf{F}_0 = 0\), and the function \(S(t; r)\) is isotropic representing a
generalized scale factor that corresponds locally to the length scale of the dust cloud satisfying the equation of motion

\[ (S^2)^{\frac{1}{2}} = \frac{2M(r)}{S} + f \]  

(2.20)

where \( M(r), f(r) \) are functions of the radial coordinate and \( E(r, y, z) \) controls the 2d anisotropy of \( \mathcal{X} \)

\[ E = A(r)(y^2 + z^2) + B(r)y + C(r)z + D(r). \]  

(2.21)

The functions \( A(r), B(r), C(r) \) and \( D(r) \) are subjected to the algebraic constraint

\[ 4AD - B^2 - C^2 = \epsilon. \]  

(2.22)

The relation (2.22) implies that the ‘anisotropy’ function \( E(r, y, z) \) can be written [13–16]

\[ E(r, y, z) = \frac{V}{2} \left\{ \left( \frac{y - Y(r)}{V} \right)^2 + \left( \frac{z - Z(r)}{V} \right)^2 + \epsilon \right\} \]  

(2.23)

where \( Y(r) = -B \cdot V, Z(r) = -C \cdot V \) and \( V(r) \) are functions of \( r \). With these identifications the Szekeres spacetime takes the form

\[ ds^2 = -dt^2 + S^2 \left\{ \left( \ln S/E \right)^2 \right\} dt^2 + \frac{4(dy^2 + dz^2)}{V^2 \left[ \left( \frac{y - Y}{V} \right)^2 + \left( \frac{z - Z}{V} \right)^2 + \epsilon \right]^2} \]  

(2.24)

The local form of the metric (2.24) manifests the constancy of the curvature of the distribution \( \mathcal{X} \) which in turn restricts the coordinate dependence of several crucial kinematical/dynamical quantities. Furthermore equation (2.24) has the additional advantage of exhibiting in a transparent and natural way important subfamilies with higher degree symmetries. For example, the spherically/hyperbolic/plane (LRS) symmetric model follows from the choice \( B = C = 0 \) and \( A = \epsilon D = 1/2 (\epsilon = 0), D = 0 (\epsilon = 0) \).

Szekeres models fall into two classes (I and II) depending on the radial dependence of the metric functions. Apart from (2.24), several other forms of the metric have been used in the literature, e.g. describing quasi-spherical collapsing clouds of dust, generalizations of the Kantowski–Sachs LRS cosmological models or to emphasize similarities of the dynamics of the Szekeres models with the linear perturbations of the Friedmann–Robertson–Walker model [9, 20, 21]. Because we are interested in using Szekeres models in order to analyze the effect of local inhomogeneities in the cosmological observations, it is also convenient to choose (pseudo-)spherical coordinates to express the local form of the metric [22]. In this case we get

\[ ds^2 = -dt^2 + \frac{E^2[(S(E))^{-1}]}{f + \epsilon} d\phi^2 + S^2(f^2 + \epsilon^2 \Sigma^2) d\phi^2. \]  

(2.25)

The function \( E(r, \theta, \phi) \) reads

\[ E = (A - \epsilon D) \Lambda(\epsilon, \theta) + B \Sigma(\epsilon, \theta) \cos \phi + C \Sigma(\epsilon, \theta) \sin \phi + \epsilon^2 (A + \epsilon D) \]

where \( \Sigma(\epsilon, \theta) = (\sin \theta, \sinh \theta, \theta) \), \( \Lambda(\epsilon, \theta) = (\cos \theta, \cosh \theta, \theta^2) \) for \( \epsilon = 1, -1, 0 \), respectively.
3. 1 + 1 + 2 decomposition

The constraint equations (2.5), (2.6), (2.10) and (2.11) can be conveniently reformulated in terms of the corresponding kinematical quantities of the spacelike congruences generated from each of the spacelike vector fields \( \{x^a, y^a, z^a\} \). In particular it has been shown that the first derivatives of the spacelike eigenvector fields are decomposed as [12]

\[
\begin{align*}
    x_{a;b} &= T_{ab}(x) + \frac{\epsilon^c}{2} p_{ab}(x) + (x^a)y_b \\
    y_{c;b} &= T_{ab}(y) + \frac{\epsilon^c}{2} p_{ab}(y) + y^d y_b \\
    z_{a;b} &= T_{ab}(z) + \frac{\epsilon^c}{2} p_{ab}(z) + z^d z_b
\end{align*}
\]

(3.1)

where\(^1\) [23]

\[
\begin{align*}
    E_x &= x_{a;b} p^{ab}(x), \\
    E_y &= y_{a;b} p^{ab}(y), \\
    E_z &= z_{a;b} p^{ab}(z)
\end{align*}
\]

(3.4)

\[
T_{ab}(x) = \alpha(y^a y_b - z^a z_b), T_{ab}(y) = \beta(z^a z_b - x^a x_b),
\]

\[
T_{ab}(z) = \gamma(x^a y_b - y^a x_b)
\]

(3.5)

are the rates of the 2d surface area spatial expansion and of the (traceless) shear tensor of the spacelike congruences respectively and we have used the notation

\[
(K_{a})^\gamma \equiv K_{a...\gamma} x^k, K'_{a} \equiv K_{a...\gamma}^\gamma, \\
(K_{a})^\gamma \equiv K_{a...\gamma}^\gamma.
\]

(3.6)

Using the Ricci identity (2.2) it is straightforward to show the following commutation relations for every scalar quantity \( S \)

\[
\begin{align*}
    (\ast S) &= (\ast S) - \ast(S_1 + H) \\
    (S') &= (\ast S) - S'(\gamma_2 + H) \\
    (\tilde{S}) &= (\ast S) - \tilde{S}(\gamma_3 + H).
\end{align*}
\]

(3.7) (3.8) (3.9)

The projection tensors \( p_{ab}(e) \) associated with \( e = \{x, y, z\} \) are given by

\[
p_{ab}(e) \equiv g_{ab} + u_a u_b - e_a e_b = h_{ab} - e_a e_b
\]

(3.10)

and are identified with the corresponding metrics of the 2d spaces \( \mathcal{X} \) (screen spaces) orthogonal to each vector field of the pairs \( \{u^a, x^a\}, \{u^a, y^a\}, \{u^a, z^a\} \) at any spacetime event. Each of the screen spaces is an assembly of 2-surfaces which are different from each other. However, due to the vanishing of the spatial twist \( \mathcal{R}_{ab}(e) = p^k_a p^l_b e_{k[l]} \) and Greenberg vector \( N_a = p^k_a \mathcal{L}_{wk} \) the screen space \( \mathcal{X} \) is a genuine 2d surface which is a submanifold of the observers’ instantaneous rest space [12].

We shall restrict our considerations to Szekeres models (i.e. spatially inhomogeneous irrotational silent (SIIS) models of Petrov type D) for which \( T_{ab}(x) = 0 = \alpha \) [12]. The evolution

\(^1\) In [24] the 1 + 1 + 2 covariant formalism has been exploited in studying covariant perturbations of a Schwarzschild black hole using a completely different notation of the various quantities.
equations (2.4), (2.8) and (2.9) and the divergence equations (2.5) and (2.10) are written in the form

\[ H = -H^2 - 2\sigma_3^2 - \frac{1}{6}\rho \]  
\[ \dot{\sigma}_3 = -2H\sigma_3 + (\sigma_3)^2 - E_3 \]  
\[ \dot{E}_3 = -\frac{1}{2}\rho\sigma_3 - 3(H + \sigma_3)E_3 \]  
\[ \dot{\rho} = -\rho^0 \]  
\[ (\rho)^\prime = -3[2(E_3)^\prime + 3E_3E_3] \]  
\[ (H)^\prime = -\frac{1}{2}[2(\sigma_3)^\prime + 3\sigma_3E_3] \]

\[ \rho' = 6E_3\left( \frac{E_3}{2} - \beta \right), \quad \dot{\rho} = 6E_3\left( \frac{E_3}{2} + \gamma \right) \]

\[ H' = \left( \frac{E_3}{2} - \beta \right)\sigma_3, \quad \dot{H} = \left( \frac{E_3}{2} + \gamma \right)\sigma_3. \]

Projecting (2.6) and (2.11) along \( y^a z^b \), \( z^a x^b \) and \( x^a y^b \), the shear and electric part constraints can be expressed as spatial variations of the corresponding eigenvalues along the individual spacelike curve

\[ \sigma_3' = -\left( \frac{E_3}{2} - \beta \right)\sigma_3, (\sigma_3)' = -\left( \frac{E_3}{2} + \gamma \right)\sigma_3 \]

\[ E_3' = -\left( \frac{E_3}{2} - \beta \right)E_3, (E_3)' = -\left( \frac{E_3}{2} + \gamma \right)E_3 \]

The above set of equations must be augmented with the Ricci identity and the Jacobi identities for the orthonormal triad \( \{x^a, y^a, z^a\} \). In particular, the \( u^a \) – projected Ricci identity gives evolution equations of the kinematical quantities of the spacelike congruences [12]

\[ (E_3) = -E_3(\sigma_3 + H) \]

\[ \left( \frac{E_3}{2} + \beta \right) = -\left( \frac{E_3}{2} + \beta \right)(\sigma_3 + H) \]

\[ \left( \frac{E_3}{2} - \beta \right) = -\left( \frac{E_3}{2} - \beta \right)(\sigma_3 + H) \]

\[ \left( \frac{E_3}{2} + \gamma \right) = -\left( \frac{E_3}{2} + \gamma \right)(\sigma_3 + H) \]

\[ \left( \frac{E_3}{2} - \gamma \right) = -\left( \frac{E_3}{2} - \gamma \right)(\sigma_3 + H). \]
The constraint equations arise from the spatial projections of the trace part of the Ricci identity of each spacelike vector field. We note that an appropriate linear combination of the constraint equation leads to the corresponding Jacobi identities or, equivalently, to the twist-free property $\mathcal{R} = 0$ of the spacelike congruences

$$ (\mathcal{E}_x)_y = 0, (\mathcal{E}_x)_y = 0 $$

(3.25)

$$ \left( \frac{\mathcal{E}_x}{2} + \beta \right)^{x}_y = -\beta \mathcal{E}_x \left( \frac{\mathcal{E}_x}{2} - \beta \right)^{x}_y - 2\beta \left( \frac{\mathcal{E}_x}{2} + \gamma \right) $$

(3.26)

$$ \left( \frac{\mathcal{E}_x}{2} - \gamma \right)^{x}_y = \gamma \mathcal{E}_x \left( \frac{\mathcal{E}_x}{2} + \gamma \right)^{x}_y - 2\gamma \left( \frac{\mathcal{E}_x}{2} - \beta \right). $$

(3.27)

Furthermore, from the $h^k_
u$–projected Ricci identity for the vector fields $\{x^a, y^a, z^a\}$ we obtain spatial propagation equations of the spacelike expansion and shear rates [12]. This set of equations completely characterizes the dynamics of Szekeres models in terms of the kinematical quantities of the timelike and spacelike congruences. Nevertheless, as we shall see in the next section, a more detailed analysis of the above set of equations reveals a number of ‘hidden’ properties of Szekeres models with sound geometrical and physical usefulness.

4. Geometrical and dynamical covariant analysis

The fact that the dust fluid velocity is geodesic and irrotational implies that the hypersurfaces $t = \text{const.}$ form an integrable submanifold $\mathcal{S}$ of $\mathcal{M}$ with metric $h_{ab}$ and extrinsic curvature $\Theta_{ab} = u_{(a;b)}$. Then Gauss equation implies

$$ 3R_{abcd} = h^k_ch^l_dR_{kab} + 2\Theta_{a(c}H_{k)b} $$

(4.1)

i.e. the 3d curvature tensor $3R_{abcd}$ of $\mathcal{S}$ is expressed in terms of the projected curvature tensor of the spacetime plus extrinsic curvature corrections. The 3d curvature tensor is completely determined by the Ricci tensor and the scalar curvature of the 3-spaces $\mathcal{S}$, which by virtue of (4.1) are given by

$$ 3R_{ab} = E_{ab} - H\sigma_{ab} + \sigma^k_{a;b} + \frac{2}{3}(\rho - 3H^2)h_{ab} $$

(4.2)

$$ 3R = 2(\rho + 3(\sigma^2 - H^2)). $$

(4.3)

Essentially the last relation represents the generalized Friedmann equation in the Szekeres models and, as we shall see below, it coincides with the equation of motion of the sheets constituting the clouds of dust.

It is interesting to remark that when the following condition holds

$$ (H + \sigma^2)^2 = 2\left(\frac{\rho}{6} + E_3\right) $$

(4.4)

then, using equations (4.2)–(4.3), the 3d Ricci tensor can be written

$$ 3R_{ab} = \frac{3R}{4}\left(\lambda_{ab} + h_{ab}\right). $$

(4.5)

Taking the spatial divergence of equation (4.5) and using the $h^i_jh^{ik} 3R_{ij;k} = 1/2h^i_j 3R_{ij}$ we obtain


Equation (4.6) means that when equation (4.4) holds then either the spatial slices are flat, i.e. 
\[ 3R_{ab} = 0 \] (\( \Leftrightarrow E_3 = (\sigma_3 + H)\sigma_3 \)) or there exists a spacelike KVF parallel to the eigenvector \( x^a \).

It can be seen that for the models \( \mathcal{E}_a = 0 \) the condition (4.4) is necessary and sufficient for the flatness of the 3d space \( \mathcal{S} \).

Because Szekeres models represent a collapsing/expanding dust matter configuration, it will be helpful to investigate the influence of the matter content into the curvature of the 2d screen space \( \mathcal{X} \). This can be achieved following a similar approach as before. The hypersurface \( t = \text{const.} \) and \( r = \text{const.} \) is an integrable submanifold of \( \mathcal{S} \) since the associated spacelike vorticity tensor and the Greenberg vector vanish [12]. In particular, the corresponding Gauss equation for the submanifolds \( \mathcal{S} \) reads

\[
2R_{abcd} = p_a^l p_b^m p_c^k p_d^l 3R_{ijkl} + 2K_{a[l}K_{db]} \tag{4.7}
\]

where \( K_{ab} = p_a^l p_b^m x_l(x_m) \equiv \nabla_{(a} x_{b)} \) is the extrinsic curvature, \( \nabla_{(a} \) denotes the proper covariant derivative and \( 2R_{abcd} \) is the curvature tensor of the screen space \( \mathcal{X}(x) \) with metric \( \gamma_{ab} = h_{ab} - x_a x_b \). Contracting twice equation (4.7) and using (4.3) we obtain

\[
2R = 4\left( \frac{\rho}{6} + E_3 \right) - 2(H + \sigma_3)^2 + \frac{\mathcal{E}_a^2}{2} \tag{4.8}
\]

The last equation can be regarded as the equation of motion of the dust shells. In particular equation (4.8) shows how the scalar curvature of the 2-space \( \mathcal{X}(x) \) is affected by the kinematics/dynamics and vice versa. For example when the condition (4.4) holds, the curvature of the 2d space \( \mathcal{X}(x) \) is completely determined by the spacelike expansion \( \mathcal{E}_a \). It follows that the flat 3d space \( \mathcal{S} \) (for the models with \( \mathcal{E}_a = 0 \)) is foliated by 2d sheets of positive curvature. Furthermore for models admitting a spacelike KVF parallel to the eigenvector \( x^a \) \( (\mathcal{E}_a = 0) \), equation (4.4) is the necessary and sufficient condition for the flatness \( (2R = 0) \) of the 2d screen space \( \mathcal{X}(x) \).

Equation (4.8) suggests that one must analyze the role of the quantities

\[
w_1 \equiv H + \sigma_3, w_2 \equiv \frac{\rho}{6} + E_3 \tag{4.9}
\]

in manufacturing the Szekeres models.

Let us consider first the quantity \( w_1 \) and observe that it can be written as

\[
w_1 = H + \sigma_3 = \frac{1}{2} u_{a,b} p^{ab} \tag{4.10}
\]

i.e. \( w_1 \) is the trace of the first derivatives of the four-velocity projected in the screen space \( \mathcal{X}(x) \).

This implies that, in complete analogy with the meaning of \( H \), the quantity \( w_1 \) represents the rate of change or the expansion rate of the surface area of the 2-space \( \mathcal{X}(x) \). We shall refer to it as area expansion. We point out that, although Szekeres models are in general quasisymmetric (i.e. no isotropic 2d sections exist), the surface area expands isotropically (due to equations (3.17) and (3.18))

\[
(w_1)^+ = (w_1)^- = 0. \tag{4.11}
\]

It would be convenient to define an average length scale \( \ell \) according to

\[
w_1 \equiv \frac{\dot{\ell}}{\ell}. \tag{4.12}
\]
We note that the intuitive definition (4.12) is dictated directly from (3.15).

Taking into account the above definitions, the length \( \ell \) completely determines the surface area of \( X \times x \), which scales \( \ell^2 \) as the 2-spaces \( \mathcal{X} \) evolve. The conditions of isotropy (4.11) allow us to define covariantly the scalar \( S \) according

\[
\frac{\dot{S}}{S} = \frac{\dot{\ell}}{\ell}
\]

(4.13)

where \( S' = \dot{S} = 0 \) such that the average length can be written as \( \ell = S \cdot N \) with \( \dot{N} = 0 \). The scalar \( S \) can be seen as auxiliary quantity without obvious geometrical and physical meaning. Nevertheless we will show covariantly and computationally in subsequent paragraphs that \( S \) satisfies the same dynamical equation (2.20) as the standard areal ‘radius’ of the Szekeres family. Therefore we can argue that the physical interpretation of \( S \) is that it represents a generalized scale factor and corresponds, for each \( t = \text{const.} \) and \( r = \text{const.} \), to the length scale of the dust cloud (or the radius shell in the case \( 2\mathcal{R}(t = \text{const.}, r = \text{const.}) > 0 \)). Furthermore the covariant (static) scalar \( N \) exhibits the anisotropy of the 2d screen space and the quasi-symmetrical feature of the Szekeres models (formally, in the coordinates adapted in (2.24), it corresponds to \( E^{-1} \)).

Similarly, the spatial (\( x^- \))-change of \( \ell \) is controlled by the expansion rate of the spacelike congruence generated by the vector field \( x^a \)

\[
\mathcal{E}_x = 2(\ell')^a \ell. \quad (4.14)
\]

Consequently for the Szekeres models that do not admit a spacelike KVF parallel to \( x^a \), the condition \( \mathcal{E}_x = 0 \iff (\ell')^a = 0 \) implies that the curves of the spacelike congruence converge to a single curve. Physically this means that different dust regions collide and stick together in a single sheet allowing the formation of a crossing singularity [14]. This situation is similar to the generation of a shell-crossing singularity in the case of locally rotationally symmetric (LRS) spatially inhomogeneous irrotational silent models (LT solution corresponds to the positive 2d curvature subclass).

On the other hand it can easily be seen that the quantity \( w_2 \) is defined as the sum of the total energy density \( \rho \) of the dust fluid plus the contribution from the free gravitational field (encoded in the eigenvalue \( E_3 \) of the electric part of the Weyl tensor) and shares the same ‘isotropic’ property with \( w_1 \)

\[
(w_2)' = (w_2)^- = 0. \quad (4.15)
\]

Using the first of (3.15) and the relation (4.14) we get

\[
\left( \frac{\rho}{6} \right)' + (E_3)^a = 0 \Rightarrow (w_2)^a = \left[ \left( \frac{\rho}{6} + E_3 \right) \ell^a \right]' = \frac{\rho \ell^2 (\ell')^a}{2}. \quad (4.16)
\]

Therefore we can rewrite \( w_2 \) in the following useful form

\[
w_2 \equiv \frac{\rho}{6} + E_3 = \frac{\mathcal{M}_{\text{eff}}}{\ell^3} \quad (4.17)
\]

where

\[
(\mathcal{M}_{\text{eff}})^a = \frac{\rho \ell^2 (\ell')^a}{2} \quad (4.18)
\]

is interpreted as the effective gravitational mass (EGM) contained in the dust cloud with length scale \( \ell \).
This interpretation is strictly correct only in the quasi-spherical case \( 2R(t = \text{const.}, r = \text{const.}) > 0 \) however, for the sake of simplicity, we shall continue to refer it as EGM. It should be emphasized that equations (4.11), (4.15) and (4.8) show that \( 2R \) is constant along the \( y^a, z^a \)—directions therefore the screen space is of constant curvature and we refer Szekeres models as quasi-symmetrical.

We note that in a series of interesting papers [25–29] the LT and Szekeres models are studied in terms of a similar set of quantities, namely the ‘\( q - s \)calars’ which are coordinate independent functionals or ‘quasi-local’ variables (in the spirit of the integral definition of the quasi local Misner-Sharp mass-energy function). However, our gauge-invariant approach ‘assigns’ a unique geometric or dynamical identity to each covariant quantity and clarifies how the geometry of the 2d screen space \( \mathcal{X}(x) \) is affected from the dynamics (and vice-versa). In particular the conservation of the EGM holds during the evolution irrespective the curvature of the 2-space \( \mathcal{X}(x) \). This can be shown by noticing that equations (3.13) and (3.14) imply \( \dot{w}_2 = -3w_1w_2 \), therefore propagation of (4.17) and use of (4.12) gives

\[
\mathcal{M}_{\text{eff}} = 0. 
\]  

(4.19)

With these identifications, equation (4.8) is written

\[
(\ell')^2 = \frac{2M_{\text{eff}}}{\ell} + \frac{\epsilon_2^2 - 2\epsilon_2^2}{4\ell^2} 
\]  

(4.20)

and represents the equation of motion (the Hamiltonian) of the sheets of the dust matter configuration.

In terms of the generalized scale factor \( S \) and the arbitrary scalar \( M_t = M_{\text{eff}}N^{-3/2} \) \((M_t)^0 = (M_t)' = 0 \) the last equation is expressed as

\[
(S')^2 = \frac{2M_t}{S} + \frac{\epsilon_2^2 - 2\epsilon_2^2}{4S^2}. 
\]  

(4.21)

We observe that the evolution of the average length scale of the dust cloud \( S \) depends on the function \( M_t \), the effective curvature term and an arbitrary function \( S_0 \) \((S_0 = S'_0 = S_0 = 0) \) that corresponds to the local time of the big bang \((S = 0) \). On the other hand any density profile within the Szekeres models is attributed to the arbitrary function \( M_t \) and the contribution of the free gravitational field \( E_3 \) satisfying equation (3.19).

At this point some comments regarding the coordinate representation of the above considerations are in order. We are interested only in the case of the quasi-spherical case, i.e. \( 2R > 0 \). The local form of the eigenvectors of the shear and electric part tensors in the spherical coordinate chart \( \{t, r, \theta, \phi\} \) are \( x^\alpha = b^{-1}e^\alpha_r, y^\alpha = S^{-1}E\delta^\alpha_\theta, z^\alpha = S^{-1}E(\sin \theta)^{-1}\delta^\alpha_\phi \) and we have set \( b = E(S/E)_r(f + \epsilon)^{-1/2} \). A trivial computation gives

\[
\sigma_1 + H = -2\sigma_3 + H = \frac{b}{b'}, \quad \sigma_3 + H = \frac{S}{S} 
\]  

(4.22)

\[
\mathcal{E}_r = \frac{2b^{3/2}}{S}, \quad \mathcal{E}_\theta = \frac{b}{2S} + \beta = \frac{E}{S} \left( \ln \frac{\sin \theta}{E} \right)_\beta = \frac{b'}{b}, 
\]  

\[
\frac{\mathcal{E}_\phi}{2} = \beta = \frac{E}{S} \frac{b_\beta}{b} = \frac{b'}{b}. 
\]  

(4.23)
\[
\frac{\mathcal{E}_z}{2} + \gamma = \frac{E}{S \sin \theta} \frac{b}{b},
\]
with \(\ell = S \cdot E^{-1} \sin \theta\) (note that \(N = E^{-1} \sin \theta\)). Interestingly the linear combination of the \(y, z\) kinematical variables has also a similar geometrical interpretation, representing the spatial variation of the average length scale \(\ell\) of the screen space and the scale \(\ell' = b\) along the \(y^a, z^b\) curves.

The energy density of the dust fluid follows from equation (4.18) (note that \(M_0(r)\) is now a function of the radial coordinate)

\[
\rho = \frac{2}{S^3} \left( \frac{M_0}{r} + 3 M_0 E_r \right).
\]

The generalized Friedmann equation (4.21) takes the familiar coordinate form [14]

\[
(\dot{S})^2 = \frac{2 M_0}{S} + f
\]

for all topologies of the 2d screen space. As a result the function \(f(r)\) represents an effective curvature term and can be also interpreted as twice the energy per unit mass of the particles in the shells of matter at constant \(r\).

We conclude this section by noticing that although the Szekeres family of models does not have, in general, Killing vector fields (KVF), it does however admit a 3d set of intrinsic symmetries (in the spirit e.g. of [30]). As we have seen in the previous section the projection tensor \(p_{ab}\) corresponds to the metric of \(X\) with associated covariant derivative \(\bar{\nabla}^a\). For the quasi-spherical class, the screen space has constant and positive curvature hence there exists an \(SO(2)\) group of intrinsic KVF (IKVF). Using the coordinate form (2.24) of the spacetime metric, we can verify that the vector fields

\[
X_1 = [z - Z(r)] \partial_y - [y - Y(r)] \partial_z
\]

\[
X_2 = \left\{ 1 + V^{-1} [y - Y(r)]^2 - V^{-2} [z - Z(r)]^2 \right\} \partial_y + 2 V^2 \left[ y - Y(r) \right] [z - Z(r)] \partial_z
\]

\[
X_3 = 2 V^2 \left[ y - Y(r) \right] [z - Z(r)] \partial_y + \left\{ 1 + V^{-2} [z - Z(r)]^2 - V^{-2} [y - Y(r)]^2 \right\} \partial_y
\]

satisfy \(\nabla_\alpha X_{(\alpha\mu)} = 0\) where \(\alpha = 1, 2, 3\). We note that the ‘isotropy’ of the quantities \(w_1, w_2\) and \(\mathcal{E}_z\) is the direct consequence of the existence of the IKVF.

5. Discussion

Inhomogeneous models can be seen as exact ‘perturbations’, to first or higher order, of the standard cosmological model and therefore the analysis of their geometry and dynamics should shed light on many questions regarding the effect of density fluctuations in the evolution of the universe. With respect to FLRW models and their richer variety of matter density profiles that we can accommodate, they could also impose further restrictions leading to a better fine-tuning of the observational parameters [31].
To this end a covariant description of the significant class of inhomogeneous Szekeres solutions has been presented using the $1 + 1 + 2$ split of spacetime. We have shown how the geometry and the dynamics of Szekeres models are described in terms of the kinematical quantities of the spacelike congruences. It was possible to identify an effective gravitational mass constituted of the matter density of the dust fluid and the contribution of the free gravitational field represented by the electric part of the Weyl tensor.

We note that several issues regarding the topology of Szekeres models can also be successively dealt with using the results of the present paper. For example, the existence of apparent horizons (AH) [32] and absolute apparent horizons (AAH) [33] has been considered within Szekeres models and their differences illustrated. In the case of the former and for the quasi-spherical subclass, an AH is located at $S = 2M$. Indeed let us consider the non-geodesic null radial vector field $n^a = u^a - x^a$ that represents the direction of outgoing null curves. Expressing their distortion in terms of the $u^a, x^a$ variables for an observer in the screen space we get

$$p_{ab} n^a = p_{ab} - H n^b = \frac{\mathcal{E}}{2} n^b(x).$$

(5.1)

The null rays converge to a single curve when the induced expansion rate of their surface area vanishes, which implies that

$$p_{ab} n^a = 0 \implies \sigma_3 + H = \frac{\mathcal{E}}{2} = 0 \implies (\sigma_3 + H)^2 = \frac{\mathcal{E}^2}{4}.$$  

(5.2)

From equation (4.21) or equivalently (4.8) we deduce that the surface $2M_1 = S$ corresponds to the AH of the Szekeres models. On the other hand for an AAH the null vector $n^a = \sqrt{3} u^a + \chi^a + z^a$ is ‘almost’ radial in the sense of [33] and in our approach must satisfy $p_{ab} n^a = 0$.

With that said in order to explore the possibility of having accelerating expansion in the SIIS models, we cannot use the standard definition for the expansion $H$ because the cosmological expansion in inhomogeneous cosmologies depends on the direction of observation and the Hubble parameter $H$ corresponds to an average of the expansion rates and the variation in different directions is hidden. Therefore it seems more natural to study and analyze the influence of the gravitational field on the light beams that come from a specific direction and pass through certain fluid configurations. In this spirit let us briefly discuss how the suggested program of the $1 + 1 + 2$ split of the Szekeres models can be conveniently applied for the determination of the luminosity distance $d_L$ of an isotropic source (S)

$$d_L = (1 + z)^2 \frac{\sqrt{\Lambda_0}}{\sqrt{\Omega_b}}$$

(5.3)

where $\Lambda_0$ is the physical cross-section of the light beam and $\Omega_b$ is the solid angle formed from the source to the position of the observer (O). The above expression allows the determination of the luminosity distance $d_L$ as a function of the redshift $z$ through the knowledge of $\Lambda_0$. The latter can be determined by using the Sachs optical equations [34]. These describe the evolution of the expansion and shear of the beam along its null trajectory in a similar way as in the case of the timelike and spacelike congruences. Then we can use them in order to study light propagation in the SIIS models and qualitatively estimate the induced increase of the luminosity distance relative to the FLRW background, thus generalizing the corresponding results for the LT models.

2 We easily verify using the covariant formalism of the present paper that the most general Szekeres model does not permit radial null geodesics [22], which is a reminder of their non-symmetric structure.
In general geometries the observed cosmological redshift $z$ is covariantly defined by the differential relation [35]

$$d \ln (1 + z) = -d \ln (k^a u_a)$$

(5.4)

where $k^a(v)$ is a null and geodesic vector tangent to the congruence of null curves $\xi^a(\nu)$ with affine parameter $\nu$, representing the paths of the light rays originating from S. Here $J_a$ is a spacelike vector field pointing in the direction of the observed light beam

$$k^a \equiv \frac{d \xi^a}{d \nu} = [k^0 u^a + J^a], \quad k^0 \equiv -k^a u_a$$

(5.5)

$$k^a u_a = 0, \quad k_{a b} k^b = 0$$

(5.6)

where the vector field $J^a$ is expressed as linear combination of the eigen-basis $\{x^a, y^a, z^a\}$

$$J^a = \lambda_1 x^a + \lambda_2 y^a + \lambda_3 z^a$$

(5.7)

and the dimensionless parameters $\lambda_\alpha$ satisfy the orthonormality condition

$$\sum_\alpha (\lambda_\alpha)^2 = (k^0)^2.$$  

(5.8)

Essentially the parameters $\lambda_\alpha$ correspond to the directional cosines of the null geodesics [36] relative to the orthonormal triad $\{x^a, y^a, z^a\}$ which is parallel-propagated [12] along the worldline of the four-velocity $u^a$ of the dust fluid. The distortion of the light beam is then encoded in the shear, rotation rates and the expansion of the null geodesic congruence, which give rise to the average cross section $\xi$ and their evolution along $\xi(\nu)$ can be conveniently formulated in terms of the kinematical quantities of the timelike and spacelike congruences.

We conclude by noticing that in many situations with clear geometrical or physical importance, apart from the existence of a preferred timelike congruence there may also exist a preferred spacelike direction representing an intrinsic geometrical/dynamical feature of a model or a physical situation. Consequently a further $1 + 2$ splitting of the 3d space naturally arises, leading to the concept of the $1 + 1 + 2$ decomposition of the spacetime manifold. As a result the $1 + 1 + 2$ covariant analysis reported in this paper revealed a number of ‘hidden’ properties of Szekeres models with sound geometrical and physical usefulness and provides an appropriate framework to study the effect of small or large inhomogeneities in the cosmological expansion. This will be the subject of a forthcoming work.

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