Field Theory On The World Sheet: Crystal Formation

Korkut Bardakci

Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

In a previous work, a world sheet field theory which sums planar $\phi^3$ graphs was investigated. In particular, a solitonic solution of this model was constructed, and quantum fluctuations around this solution led to a string picture. However, there were two problems which were not treated satisfactorily: An ultraviolet divergence and a spurious infrared divergence. Here we present an improved treatment, which eliminates the ultraviolet divergence in the normal fashion by mass and coupling constant renormalization. The infrared problem is taken care of by choosing a classical background which forms a one dimensional
crystal. The resulting picture is a hybrid model with both string and underlying field theory excitations. Only in the dense graph limit on the world sheet a full string picture emerges.
1. Introduction

The present work is the continuation of a series of earlier papers [1, 2] on the same subject. It has a lot in common with especially reference [1], but, it also has two important new features which we think are significant advances over [1]. For the convenience of the reader, we will first present a brief discussion of the problem at hand and the results of the earlier work, and then focus on what is new in the present article.

The idea is to sum the planar graphs of the $\phi^3$ field theory in both 3 + 1 and 5 + 1 dimensions, starting with the world sheet picture developed in [3], which in turn was based on the pioneering work of 't Hooft [4]. This picture, which we briefly review in section 2, makes use of the mixed light cone parametrization of planar graphs, similar to the one employed in string theory. In the next section, we describe the field theory on the world sheet, developed in [5], which reproduces these graphs. This theory is formulated in terms of a complex scalar field and a two component fermionic field; a central role is played by the field $\rho$ (eq.(4)), a composite of the fermions, which roughly measures the density of graphs on the world sheet. Just as in [1,2], here also we are mainly interested in high density graphs, to be defined more precisely later in terms of $\rho$. The basic idea, which motivated some of the very early work [6, 7], is that a densely covered world sheet would naturally have a string description. To find such a string picture has been a goal of the present, as well as of the earlier work.

The world sheet field theory discussed above suffers from two kinds of divergences: One of them is the standard field theoretic ultraviolet divergence which we will address later on. The second one is a (spurious) infrared divergence due to the choice of the light cone coordinates. We find our previous treatment of these problems unsatisfactory, and we readdress them in the present work. As before, we start by temporarily discretizing the $\sigma$ coordinate of the world sheet in steps of length $a$. This sort of cutoff has been extensively used both in field theory [8] and in string theory [9]. Our major goal in this paper is, in addition to eliminating the ultraviolet divergence by renormalization, to take the limit of zero grid spacing without encountering any singularity.

Sections 2 and 3 are mostly a review, but in section 4, we start diverging from the earlier work. An important feature of the world sheet field theory is that the existence of solitonic classical solutions. In [1] and [2], the classical solution was constructed with the help of the mean field approximation. An
unusual feature of the classical solution is that the corresponding classical energy is ultraviolet divergent. The natural renormalization prescription is to eliminate this divergence by means a bare mass counter term. Unfortunately, in [1], the structure of divergent term did not allow such a cancellation, and the interaction vertex had to be modified in somewhat ad hoc fashion in order to achieve such a cancellation. This is an unsatisfactory feature of the earlier work which we avoid in the present paper. This problem can be traced back to a somewhat premature application of the mean field approximation: \( \rho \) is originally a kind of spin variable which only takes on the values 0 and 1, but in the mean field approximation, it becomes a classical continuous variable in the range \( 0 \leq \rho \leq 1 \). It is this approximate treatment of \( \rho \) that is the source of the trouble. Instead, in section 4, we solve the equations of motion, treating the scalar field \( \phi \) classically, but keeping the fermions and \( \rho \) fully quantum mechanical. We find that mass renormalization can be carried out without any ad hoc modifications. This one of the new features of the present work that is an improvement over reference [1].

The main results of section 4 are the two ultraviolet finite expressions for the classical energy: Eq.(22) in \( 3 + 1 \) dimensions and eq.(25) in \( 5 + 1 \) dimensions. In the latter case, in addition to mass renormalization, the coupling constant also has to be renormalized. The next step is to find the field configuration corresponding to the ground state that minimizes this energy. Since we cannot do this exactly, it is at this point that we introduce the mean field approximation in section 5, which was already used extensively in the previous work. Here, apart from a different starting point due to renormalization, we also choose a different mean field background compared to the one chosen in [1] and [2]. The crucial point about this background, defined by eqs.(27) and (28), is that it vanishes except on lines equally spaced by a distance \( L \) (Fig.4). By letting the grid space \( a \) go to zero while keeping \( L \) finite and independent of \( a \), we are able to define a sensible continuum limit. In this limit, both the classical energy and the quantum corrections about it computed in section 7, all stay finite. This is to be contrasted with the \( 1/a^2 \) divergence in the classical energy and in some of the spectrum found in [1] and [2]. We also note that this new background breaks translation invariance in \( \sigma \); it is therefore natural to identify it with a one dimensional crystal.

In section 6, we search for the ground state by minimizing the energy with respect to both \( L \) and \( \rho_0 \), the ground state expectation value of \( \rho \). Starting with \( 3 + 1 \) dimensions, we first vary with respect to \( L \) at fixed \( \rho_0 \) and find a minimum. Next, varying with respect to \( \rho_0 \), we find that the energy goes
to minus infinity as $\rho_0 \to 1$. Since as $\rho_0 \to 1$, $L \to 0$ (eq.(36)), this is the limit of densely covered world sheet, which is expected to lead to string formation. So by lowering its energy, the model is dynamically driven towards $\rho_0 \to 1$. To investigate this limit without encountering singular expressions, we introduce a cutoff on the density of graphs (eq.(40)), which corresponds to an upper bound on $\rho_0$ less than one. It is important to notice that this is simply a restriction on the choice of graphs; there is no change of the world sheet dynamics. Later on, in section 8, we discuss the limit $\rho_0 \to 1$. In $5+1$ dimensions, things work out differently. The classical energy vanishes at its minimum, $L$ is determined as a function of $\rho_0$ by eq.(43) but $\rho_0$ itself, at least in this approximation, is undetermined.

In the next section, section 7, we compute second order quantum fluctuations around the classical background. This section is technically very similar to corresponding material in [1] and [2]. However, there is one important difference: In the previous work, in limit $a \to 0$, the energies of many excited states went to infinity. Here, this limit is smooth, and the excited states all remain at finite energy. Instead of analyzing the spectrum in full generality, we focus on the excitations defined by eq.(49), and determine their contribution to the action. These states, originally studied in [1] and [2], are candidates for string excitations.

We investigate the action for the candidate string states in section 8. This action for $3+1$ dimensions is not yet a string action; we show that, only in the dense graph limit of $\rho_0 \to 1$, it tends as a limit to the light cone string action. A plausible picture of the model for $\rho_0 < 1$ is the following: The spectrum is a combination of a heavy sector, consisting of the states of the original field theory, and lower lying states consisting of string excitations. In the limit $\rho_0 \to 1$, the masses of the heavy states go to infinity, whereas the string states stay finite. We tentatively identify a parameter $\gamma$ (eq.(60)), proportional to $1 - \rho_0$, which could serve as an expansion parameter in the dense graph limit. In section 9, we summarize our conclusions and discuss directions for future research.

2. The World Sheet Picture

The planar graphs of $\phi^3$ can be represented [4] on a world sheet parameterized by the light cone coordinates $\tau = x^+$ and $\sigma = p^+$ as a collection of horizontal solid lines (Fig.1), where the n’th line carries a D dimensional transverse momentum $q_n$. Two adjacent solid lines labeled by n and n+1
correspond to the light cone propagator

\[ \Delta(p_n) = \frac{\theta(\tau)}{2p^+} \exp \left( -i\tau \frac{p_n^2 + m^2}{2p^+} \right), \quad (1) \]

where \( p_n = q_n - q_{n+1} \) is the momentum flowing through the propagator. A factor of the coupling constant \( g \) is inserted at the beginning and at the end of each line, where the interaction takes place. Ultimately, one has to integrate over all possible locations and lengths of the solid lines, as well as over the momenta they carry.

The propagator (1) is singular at \( p^+ = 0 \). It is well known that this is a spurious singularity peculiar to the light cone picture. To avoid this singularity, and as well as other technical reasons, it is convenient to temporarily discretize the \( \sigma \) coordinate in steps of length \( a \). A useful way of visualizing the discretized world sheet is pictured in Fig.2. The boundaries of the propagators are marked by solid lines as before, and the bulk is filled by dotted lines spaced at a distance \( a \). For the time being, we will keep \( a \) finite, and later, we will show how one can safely take the limit \( a \to 0 \). For convenience, the \( \sigma \) is compactified by imposing periodic boundary conditions at \( \sigma = 0 \) and \( \sigma = p^+ \). In contrast, the boundary conditions at \( \tau = \pm\infty \) are left arbitrary.

3. The World Sheet Field Theory

It was shown in [5] that the light cone graphs described above are reproduced by a world sheet field theory, which we now briefly review. We
introduce the complex scalar field $\phi(\sigma, \tau, q)$ and its conjugate $\phi^\dagger$, which at time $\tau$ annihilate (create) a solid line with coordinate $\sigma$ carrying momentum $q$. They satisfy the usual commutation relations

$$[\phi(\sigma, \tau, q), \phi^\dagger(\sigma', \tau, q')] = \delta_{\sigma,\sigma'} \delta(q-q').$$

The vacuum, annihilated by the $\phi$'s, represents the empty world sheet.

In addition, we introduce a two component fermion field $\psi_i(\sigma, \tau)$, $i = 1, 2$, and its adjoint $\bar{\psi}_i$, which satisfy the standard anticommutation relations. The fermion with $i = 1$ is associated with the dotted lines and $i = 2$ with the solid lines. The fermions are needed to avoid unwanted configurations on the world sheet. For example, multiple solid lines generated by the repeated application of $\phi^\dagger$ at the same $\sigma$ would lead to overcounting of the graphs. These redundant states can be eliminated by imposing the constraint

$$\int dq \phi^\dagger(\sigma, \tau, q) \phi(\sigma, \tau, q) = \rho(\sigma, \tau),$$

where

$$\rho = \bar{\psi}_2 \psi_2,$$

which is equal to one on solid lines and zero on dotted lines. This constraint ensures that there is at most one solid line at each site.

Fermions are also needed to avoid another set of unwanted configurations. Propagators are assigned only to adjacent solid lines and not to non-adjacent
ones. To enforce this condition, it is convenient to define,

$$E(\sigma_i, \sigma_j) = \prod_{k=i+1}^{j-1} (1 - \rho(\sigma_k)),$$

(5)

for $\sigma_j > \sigma_i$, and zero for $\sigma_j < \sigma_i$. The crucial property of this function is that it acts as a projection: It is equal to one when the two lines at $\sigma_i$ and $\sigma_j$ are separated only by the dotted lines; otherwise, it is zero. With the help of $E$, the free Hamiltonian can be written as

$$H_0 = \frac{1}{2} \sum_{\sigma, \sigma'} \int dq \int dq' \frac{E(\sigma, \sigma')}{\sigma' - \sigma} ((q - q')^2 + m^2)$$

$$\times \phi^\dagger(\sigma, q)\phi(\sigma, q)\phi^\dagger(\sigma', q')\phi(\sigma', q')$$

$$+ \sum_\sigma \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q)\phi(\sigma, q) - \rho(\sigma) \right),$$

(6)

where $\lambda$ is a Lagrange multiplier enforcing the constraint (3). The evolution operator $\exp(-i\tau H_0)$, applied to states, generates a collection of free propagators, without, however, the prefactor $1/(2p^+)$.

Using the constraint (3), the free Hamiltonian can be written in a form more convenient for later application:

$$H_0 = \frac{1}{2} \sum_{\sigma, \sigma'} G(\sigma, \sigma')(\frac{1}{2} m^2 \rho(\sigma)\rho(\sigma') + \rho(\sigma') \int dq q^2 \phi^\dagger(\sigma, q)\phi(\sigma, q)$$

$$- \int dq \int dq' (q \cdot q') \phi^\dagger(\sigma, q)\phi(\sigma, q)\phi^\dagger(\sigma', q')\phi(\sigma', q')$$

$$+ \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q)\phi(\sigma, q) - \rho(\sigma) \right),$$

(7)

where we have defined

$$G(\sigma, \sigma') = \frac{E(\sigma, \sigma') + E(\sigma', \sigma)}{|\sigma - \sigma'|}.$$  

(8)

Next, we introduce the interaction term. Two kinds of interaction vertices, corresponding to $\phi^\dagger$ creating a solid line or $\phi$ destroying a solid line, are pictured in Fig.3. We also have to take care of the prefactor $1/(2p^+)$ in (1) by attaching it to the vertices. Here, as in [5], we choose a symmetric distribution of this factor, by attaching a factor of

$$V = \frac{1}{\sqrt{8 p_{12}^+ p_{23}^+ p_{31}^+}} = \frac{1}{\sqrt{8 (\sigma_2 - \sigma_1)(\sigma_3 - \sigma_2)(\sigma_3 - \sigma_1)}}$$

(9)
to each vertex. Different ways of splitting the prefactor $1/(2p^+)$ result in non-symmetric vertices; here we choose the standard symmetric form. The interaction term in the Hamiltonian can now be written as

$$H_I = g \sum_\sigma \int dq \left( \mathcal{V}(\sigma) \rho_+(\sigma) \phi(\sigma, q) + \rho_-(\sigma) \mathcal{V}(\sigma) \phi^\dagger(\sigma, q) \right),$$

(10)

where $g$ is the coupling constant. $\rho_\pm$ are given by

$$\rho_+ = \bar{\psi}_1 \psi_2, \quad \rho_- = \bar{\psi}_2 \psi_1,$$

and

$$\mathcal{V}(\sigma) = \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 > \sigma} \frac{W(\sigma_1, \sigma, \sigma_2)}{\sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)}},$$

(11)

where,

$$W(\sigma_1, \sigma_2) = \rho(\sigma_1) E(\sigma_1, \sigma_2) \rho(\sigma_2).$$

(12)

Here is a brief explanation of the origin of various terms in $H_I$: The factors of $\rho_\pm$ are there to pair a solid line with an $i = 2$ fermion and a dotted line with an $i = 1$ fermion. The factor of $\mathcal{V}$ ensures that the pair of solid lines 12 and 23 in Fig.3 are separated by only dotted lines, without any intervening solid lines. Apart from an overall factor, the vertex defined above is very similar to the bosonic string interaction vertex in the light cone picture. Taking advantage of the properties of $E$ discussed following eq.(5), we have written an explicit representation of this overlap vertex.

Finally, the total Hamiltonian is given by

$$H = H_0 + H_I$$

7
and the corresponding action by

\[ S = \int d\tau \left( \sum_{\sigma} \left( i\bar{\psi}\partial_{\tau}\psi + i \int dq \phi^\dagger \partial_{\tau}\phi \right) - H(\tau) \right). \tag{13} \]

4. The Semi Classical Solution And Renormalization

In this section, our goal is to search for classical solutions to the equations of motion that follow from the action (7). The idea is to eliminate terms linear in \( \phi \) and \( \phi^\dagger \) in \( H_I \) by setting

\[ \phi = \phi_0 + \phi_1, \quad \phi^\dagger = \phi_0^\dagger + \phi_1^\dagger. \tag{14} \]

Here, \( \phi_0 \) is the field fixed by the equations of motion, and \( \phi_1 \) represents the fluctuations around the classical background. There remains the question of what to do about the fermions. In the previous work [1,2], fermions were bosonized, and the fermionic bilinears \( \rho, \rho_{\pm} \) were treated as fixed classical background fields. In contrast, here, for the time being, the fermions, as well as the field \( \lambda \) will be treated exactly; in particular, we wish to preserve the relations

\[ \rho^2(\sigma) = \rho(\sigma), \quad \rho_+(\sigma)\rho_-(\sigma) = 1 - \rho(\sigma), \quad \rho_-(\sigma)\rho_+(\sigma) = \rho(\sigma), \tag{15} \]

which follow from \( \rho \) being a discrete variable, taking on only the values zero and one. As we shall shortly see, they are needed to show that the self mass divergence can be absorbed into the mass term already present in the action, without need for ad hoc counter terms. The field \( \phi_0 \) is a kind of hybrid: The equations of motion for \( \phi \) are used, but the fermions are still fully quantum mechanical. This is why we use the term semi classical.

We choose \( \phi_0 \) so that it depends only on \( q^2 \) (rotation invariance). As a result, the term that has the factor \( q \cdot q' \) on the right hand side of eq.(7) does not contribute, and the equation of motion for \( \phi_0 \) reduces to

\[ \left( -\frac{1}{2}G(\sigma,\sigma') \rho(\sigma') q^2 + \lambda(\sigma) \right) \phi_0(\sigma,q) + g \rho_-(\sigma) V(\sigma) = 0, \tag{16} \]

with a conjugate equation for \( \phi_0^\dagger \). The solution can be written as

\[ \phi_0(\sigma,q) = -g \frac{\rho_-(\sigma) V(\sigma)}{\lambda(\sigma) + \frac{1}{2}G(\sigma,\sigma') \rho(\sigma') q^2} \]

\[ = -g \sum_{\sigma_1<\sigma} \sum_{\sigma<\sigma_2} \frac{\rho_-(\sigma) W(\sigma_1,\sigma_2)}{\lambda(\sigma) + \frac{1}{2}q^2 \left( \frac{\sigma_2-\sigma_1}{(\sigma_2-\sigma_1)(\sigma_2-\sigma_1)} \right) \sqrt{(\sigma-\sigma_1)(\sigma_2-\sigma_1)(\sigma_2-\sigma)}}. \tag{17} \]
To derive the second line in this equation, we expand the denominator in powers of \( \frac{1}{2}G(\sigma, \sigma') \rho(\sigma') q^2 \) and repeatedly use the identity
\[
G(\sigma, \sigma') \rho(\sigma') W(\sigma_1, \sigma_2) = \left( \delta_{\sigma', \sigma_2} \frac{1}{\sigma_2 - \sigma} + \delta_{\sigma', \sigma_1} \frac{1}{\sigma - \sigma_1} \right) \rho_- W(\sigma_1, \sigma_2),
\]
which can easily be derived by expressing \( G \) and \( W \) in terms of the \( \rho \)'s (eqs. (8,12)), and making use of the relations (15). We should also mention that since \( \rho_{\pm} \) do not commute with \( \rho \), the ordering of these factors matters, and the factors \( \rho_- \) and \( W \) in eq.(17) and (18) are ordered correctly.

We can now compute the classical hamiltonian \( H_c \) by letting \( \phi \to \phi_0 \) in \( H \) (eqs.(7,10)). There is, however, an ultraviolet divergence, resulting from the integration over \( q \), which has to be addressed. So far, the transverse dimension \( D \) has been arbitrary, but now we have to make a choice. We specialize to the case \( D = 2 \), where the only ultraviolet divergence is a logarithmic divergence in the self mass. The result is
\[
H_c = -2\pi g^2 \sum_{\sigma} \sum_{\sigma_1 < \sigma_2} \sum_{\sigma_1, \sigma_2} W(\sigma_1, \sigma_2) \ln \left( \frac{\Lambda^2}{\lambda(\sigma) (\sigma_2 - \sigma_1)^2} \right) \\
- \sum_{\sigma} \lambda(\sigma) \rho(\sigma),
\]
where \( \Lambda \) is an ultraviolet cutoff needed because of the mass divergence. In deriving this result, one needs the identity
\[
W(\sigma_1, \sigma_2) \rho(\sigma) W(\sigma'_1, \sigma'_2) = \delta_{\sigma_1, \sigma'_1} \delta_{\sigma_2, \sigma'_2} W(\sigma_1, \sigma_2),
\]
which follows from the definition of \( W(\sigma_1, \sigma_2) \) (eq.(12)) and the identities (15). One can also understand it geometrically from the overlap properties of the vertices in Fig.3. Apart from an overall factor, these are structurally the same as the corresponding string vertices, and in particular, they satisfy the same overlap relations.

The classical hamiltonian can be renormalized by replacing the cutoff \( \Lambda \) by a an arbitrary finite mass \( \mu \). This amounts to introducing a counter term
\[
2\pi g^2 \sum_{\sigma} \sum_{\sigma_1 < \sigma_2} \sum_{\sigma_1, \sigma_2} W(\sigma_1, \sigma_2) \ln \left( \frac{\Lambda^2}{\mu^2} \right) = \frac{2\pi g^2}{a} \sum_{\sigma_1 < \sigma} \sum_{\sigma_2} W(\sigma_1, \sigma_2) \ln \left( \frac{\Lambda^2}{\mu^2} \right).
\]
(21)
Noticing that the term to be summed over is $\sigma$ independent, the sum over this variable was done explicitly. Now, since, from their definition, 

$$\rho(\sigma) G(\sigma, \sigma') \rho(\sigma') = \frac{W(\sigma, \sigma') + W(\sigma', \sigma)}{|\sigma - \sigma'|},$$

the above counter term is simply proportional to the $m^2$ term on the right hand side of (7). It can therefore be identified with a cutoff dependent part of the mass term. In fact, after eliminating the cutoff dependent part, the remaining finite portion of the mass term can be completely absorbed into the definition of $\mu^2$, and from now on, we shall assume that this has been done. The renormalized classical hamiltonian is then given by

$$H^r_c = -2\pi g^2 \sum_\sigma \sum_{\sigma_1 < \sigma < \sigma_2} W(\sigma_1, \sigma_2) \frac{\lambda(\sigma)}{(\sigma_2 - \sigma_1)^2} \ln \left( \frac{\mu^2}{\lambda(\sigma)(\sigma_2 - \sigma)(\sigma - \sigma_1)} \right)$$

$$- \sum_\sigma \lambda(\sigma) \rho(\sigma),$$

(22)

We would like to emphasize that so far no approximation has been made, and therefore the above equation for the classical part of the hamiltonian is exact. This why the mass renormalization can be carried out without introducing ad hoc terms not present in the original action. Of course, $H_c$ is not the whole story; terms that depend on $\phi_3$, as well as terms involving derivatives with respect to $\tau$ are not present in the classical hamiltonian. In the following sections, we will carry out an expansion to second in the fluctuations around the classical solutions, and show that the terms we compute are all ultraviolet finite.

Next we consider $D = 4$, corresponding to $\phi^3$ in six dimensions. The self mass is now quadratically divergent, but this divergence can be eliminated by a mass counter term exactly as in the case $D = 2$. There remains, however, a residual logarithmic divergence:

$$H_c = -4\pi^2 g_0^2 \sum_\sigma \sum_{\sigma_1 < \sigma < \sigma_2} \lambda(\sigma) W(\sigma_1, \sigma_2) \frac{(\sigma_2 - \sigma)(\sigma - \sigma_1)}{(\sigma_2 - \sigma_1)^3}$$

$$\times \ln \left( \frac{\lambda(\sigma)}{\Lambda^2 (\sigma_2 - \sigma)(\sigma - \sigma_1)} \right) - \sum_\sigma \lambda(\sigma) \rho(\sigma).$$

(23)

This divergence can be eliminated by renormalizing the bare coupling constant $g_0$ by setting

$$g_0^2 = \frac{g_r^2}{\ln (\Lambda^2/\mu^2)},$$

(24)
where $g_r$ is the renormalized coupling constant and $\mu$ an arbitrary mass parameter. We recall that $\phi^3$ is asymptotically free in 6 space-time dimensions, and the above relation between the bare and renormalized couplings is the well known lowest order result. In the limit $\Lambda \to \infty$, the renormalized $H_c$ is given by

$$H^r_c = 4\pi^2 g^2_r \sum_\sigma \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma_1} \lambda(\sigma) \left( W(\sigma_1, \sigma_2) \frac{(\sigma_2 - \sigma)(\sigma - \sigma_1)}{(\sigma_2 - \sigma_1)^3} - \rho(\sigma) \right). \tag{25}$$

5. The Meanfield Approximation

As we have already pointed out, the expressions for $H^r_c$ for $D = 2$ and $D = 4$ (eqs.(22, 25)) are exact. The fermionic fields $\psi$ and $\bar{\psi}$ and the lagrange multiplier $\lambda$ are still fully quantum mechanical, to be integrated over in the functional integral. Clearly, it is not possible to do the functional integrals indicated above exactly. It is at this point that we finally have to introduce some sort of approximation. The scheme we choose is the mean field method, already extensively used in the previous work on this subject [1,2]. The reason for postponing the introduction of this approximation is connected with renormalization: We were able to absorb the self mass divergence into the mass term already present in the action by making use of the overlap relations (20). These relations in turn followed from the identities (15), which are violated in the mean field approximation. By avoiding any approximation before deriving the finite renormalized expressions for $H^r_c$, we are able to bypass this problem.

The mean field approximation amounts to expanding the fields $\rho$, $\lambda$ and $\phi$ about their classical expectation values $\rho_c$ and $\lambda_c$:

$$\rho(\sigma) = \rho_c(\sigma) + \rho_1(\sigma), \quad \lambda(\sigma) = \lambda_c(\sigma) + \lambda_1(\sigma), \quad \phi(\sigma) = \phi_c(\sigma) + \phi_1(\sigma), \tag{26}$$

where $0 \leq \rho_c \leq 1$. In the earlier work, $\rho_0(\sigma)$ and $\lambda_0(\sigma)$ were taken to be constants independent of $\sigma$, and also, of course, of $\tau$. Here, we make a different choice for these classical background fields. We set

$$\rho_c(\sigma) = \rho_0, \quad \lambda_c(\sigma) = \lambda_0, \tag{27}$$

at $\sigma = \sigma_0 + nL$, and

$$\rho_c(\sigma) = 0, \quad \lambda_c(\sigma) = 0, \tag{28}$$
for $\sigma \neq \sigma_0 + nL$. Here $n$ runs over all integers, and $\sigma_0$ is arbitrary. We have also effectively let $p^+ \to \infty$, and we will discuss this later on. This structure is pictured in Fig.4: $\rho_c$ and $\lambda_c$ are constants $\rho_0$ and $\lambda_0$ on what we call hybrid lines, separated by intervals of distance $L$, and they vanish elsewhere. Hybrid lines, to be defined more precisely in section 5, are superpositions of solid and dotted lines. Since $\sigma$ is discretized in units of $a$, $L$ is an integer multiple of $a$. The important point is that as we eventually let $a \to 0$, $L$ will be kept fixed and finite.

It remains to specify $\phi_c$. We take

$$\phi_c = 0$$

(29)

at $\sigma \neq \sigma_0 + nL$, and at $\sigma = \sigma_0 + nL$, we simply set $\rho = \rho_0$ and $\lambda = \lambda_0$ in eq.(17) for $\phi_0$. The only remaining question is what to do about $\rho_\pm$. Actually, since $H_c$ depends only on $\rho$, we do not really need to know $\rho_\pm$ individually. However, for the sake of completeness, we note that, in the classical approximation,

$$\rho_+ = \rho_- = \sqrt{\rho - \rho^2}.$$ 

(30)

This follows both from the bosonization of the fermions [1,2], and also from the mean field approximation we will discuss shortly. We finally note that the classical fields are non-trivial only on the hybrid lines; they are zero elsewhere. This will be important for having a finite $a \to 0$ limit.
The mean field approximation $\rho(\sigma) \to \rho_c(\sigma)$ amounts to replacing $\rho$ by its expectation value

$$\rho_c(\sigma) = \langle \sigma | \rho(\sigma) | \sigma \rangle,$$

where the fermionic state $|\sigma\rangle$ is defined by

$$|\sigma\rangle = \left( \sqrt{\rho_0} \bar{\psi}_2(\sigma) + \sqrt{1 - \rho_0} \bar{\psi}_1(\sigma) \right) |0\rangle,$$

for $\sigma = \sigma_0 + nL$, and,

$$|\sigma\rangle = |0\rangle,$$

for $\sigma \neq \sigma_0 + nL$. What was represented by a hybrid line in Fig.4 is concretely the state $|\sigma\rangle$, a linear superposition of solid and dotted lines. This state can also be thought of as a variational ansatz [5] for minimizing the classical energy with respect to the parameter $\rho_0$. The novelty of this approach compared to the earlier work is that in addition to $\rho_0$, we will also treat $L$ as a variational parameter. Whereas in [1,2], $L$ is in effect set equal to the lattice spacing $a$, here it is freed from that restriction and allowed to vary in order to minimize the classical energy.

It is clearly advantageous to have as many free parameters as possible in a variational calculation, so the freeing of $L$ from the restriction $L = a$ is a welcome development. This new flexibility has another big advantage: So long as $L$ is kept fixed, the limit $a \to 0$ is smooth (non-singular). This point will be discussed further in the following sections, but it can already be gleaned from eq.(6). If one sets $L = a$, two hybrid lines are allowed to be at a distance $a$ apart. Since the hybrid lines are part of the time solid lines, this means that the denominator $\sigma' - \sigma$ of the first term in eq.(6) for $H_0$ can be equal to $a$, becoming singular as $a \to 0$. On the other hand, if the hybrid lines are a distance $L$ apart, the minimum value of this denominator is $L$, which remains finite as $a \to 0$. In fact, in the next section, we will show that the value of $L$ that minimizes $H_c$ depends only on $\rho_0$ and it is independent of $a$.

How do we know that the classical background we have chosen is the right one? We have chosen it because it was the simplest regular configuration we could think of which was non-singular in the limit $a \to 0$. It then clearly has lower energy the the background chosen in reference [1], where the classical energy becomes infinite in the limit $a \to 0$. Of course, one could not exclude the possibility of more complicated backgrounds with even lower classical energy.
6. The Ground State Of The Model

We start with the model at $D = 2$ (3+1 space-time dimensions). To find the ground state, $H^r_c$ of eq.(22) has to be minimized with respect to the chosen classical background. We note that in the triple sum, the $\sigma$’s are restricted to the hybrid lines at $\sigma = \sigma_0 + nL$, and at these locations, $\rho$ and $\lambda$ are given by eqs.(27) and (28). This means that, for example, we evaluate $W(\sigma_1, \sigma_2)$ by setting $\rho(\sigma_1) = \rho(\sigma_2) = \rho_0$ and by assigning a factor of $1 - \rho_0$ to each hybrid line in between $\sigma_1$ and $\sigma_2$ in eq.(12). As a result

$$H^r_c = -\frac{2\pi g^2 \rho_0^2 p^+}{L^3} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{(1 - \rho_0)^{n_1 + n_2 - 1}}{(n_1 + n_2)^2} \ln \left( \frac{\mu^2}{\lambda_0 L} \frac{n_1 + n_2}{n_1 n_2} \right) - \frac{\lambda_0 \rho_0 p^+}{L}. \quad (34)$$

Strictly speaking, the above sums over the $n$’s, instead of going all the way to infinity, should have an upper cutoff of the order of $p^+/L$, since $\sigma$ is restricted by $0 \leq \sigma \leq p^+$. However, this only makes a difference for small values of $\rho_0$, and in what follows, we will be interested only the values of $\rho_0$ near one, for which it is a good approximation to let the upper limit go to infinity. This equivalent to decompactifying the world sheet by letting $p^+ \to \infty$, as was done earlier.

Next, we minimize $H^r_c$ with respect to $\lambda_0$ and $L$ at fixed $\rho_0$ by setting

$$\frac{\partial H^r_c}{\partial \lambda_0} = 0, \quad \frac{\partial H^r_c}{\partial L} = 0. \quad (35)$$

The solution to these equations is

$$\lambda_0 = \frac{2\pi g^2 \rho_0 (1 - \rho_0)}{L^2} f_1(\rho_0),$$
$$L = \frac{g^2 \rho_0 (1 - \rho_0) f_1(\rho_0)}{\mu^2} \exp \left( \frac{1}{3} - \frac{f_2(\rho_0)}{f_1(\rho_0)} \right), \quad (36)$$

where,

$$f_1(\rho_0) = \sum_{n=0}^{\infty} \frac{n + 1}{(n + 2)^2} (1 - \rho_0)^n,$$
$$f_2(\rho_0) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{(1 - \rho_0)^{n_1 + n_2 - 2}}{(n_1 + n_2)^2} \ln \left( \frac{n_1 + n_2}{n_1 n_2} \right). \quad (37)$$
and the classical hamiltonian (energy) at these values of $L$ and $\lambda_0$ and at some fixed value of $\rho_0$ reduces to

$$H^r_c = -\frac{2\pi\mu^6 p^+}{3g^4\rho_0(1-\rho_0)^2f_1^2(\rho_0)} \exp\left(3\frac{f_2(\rho_0)}{f_1(\rho_0)} - 1\right). \quad (38)$$

We note that

a) The dependence on $a$, the spacing of the grid in $\sigma$, has completely disappeared. Therefore, the limit $a \to 0$ is trivial.

b) A new discrete structure has emerged: These are the hybrid lines spaced at intervals of length $L$. Notice, however, that the world sheet remains smooth, and the hybrid lines are effectively zero branes placed at regular intervals in the bulk. Their spacing $L$ is fixed, but the location $\sigma_0$ (eq.(28)) is arbitrary. The energy is independent of $\sigma_0$ because of translation invariance in $\sigma$. By fixing $\sigma_0$, this invariance is (spontaneously) broken. We are therefore witnessing the formation of a one dimensional crystal. One can then identify the fluctuations about the classical solution with the vibrations of this crystal, among them will be a zero mode corresponding to the broken translation invariance. The inclusion of this mode will presumably restore translation invariance. We will have more to say about this in the next section.

c) The classical energy (38) is negative. Since $H_0$ of eq.(6) is positive semi-definite, it is the interaction that is responsible for changing the sign of the energy. Both the sign of the energy and the existence of an optimal spacing $L$ can be understood as follows: In the expression for $H_0$, the first term acts as a repulsive potential between adjacent hybrid lines and pushes them apart. But there is an entropic attractive force which balances this repulsion and leads to a stable configuration. The origin of the entropic force has to do with the counting of configurations. $H^r_c$ calculated above is really the free energy

$$F = E - TS,$$

which takes into account the entropy arising from the counting of configurations. A hybrid line involves transitions between solid and dotted lines (Fig.(5)), and as such represents the superposition of a multitude of configurations, giving rise to increased entropy. The entropic force and the negative sign of the free energy comes from the contribution of the hybrid lines to the entropy. The entropic term in the free energy favors the increase in the number of hybrid lines, hence leading to their close spacing. Balancing this is the repulsive term in $H_0$, which is trying to keep the hybrid lines apart.
So far, $\rho_0$ has been fixed in the interval $0 \leq \rho_0 \leq 1$. We should now consider minimizing $H^r_\rho$ with respect to $\rho_0$. It turns out that $H^r_\rho$ is a steadily decreasing function which goes from zero at $\rho_0 = 0$ to $-\infty$ as $\rho_0 \to 1$:

$$H^r_\rho \to -\frac{16\pi p^+ \mu^6}{3g^4 e(1 - \rho_0)^2}. \quad (39)$$

This clearly a singular limit which we will investigate in more detail later on, meanwhile, we will introduce a cutoff on the average number of solid lines on the world sheet by setting

$$\sum_\sigma \rho(\sigma) \to \frac{\rho_0 p^+}{L} \leq \kappa, \quad (40)$$

where $\kappa$ is a fixed constant. Since $\rho_0 \to 1$ corresponds to $\kappa \to \infty$, a finite $\kappa$ corresponds to an upper bound on $\rho_0$ less than unity, and it is at this value of $\rho_0$ that the the minimum of $H^r_\rho$ is reached. Instead of this sharp cutoff, it is possible to introduce a smooth cutoff by adding to the action an external source proportional to, for example,

$$\int d\tau \sum_\sigma (1 - \rho(\sigma, \tau))^2.$$

However, in what follows, for the sake of simplicity, we will simply fix the value of $\rho_0$ at some value less than one.

It seems that what we have done is to exchange one cutoff for another: We have let the grid spacing $a$ go to zero, but instead, we have imposed an upper limit on the world sheet density of graphs measured by $\rho$. There is, however, a big difference between the two cutoffs. The grid in $\sigma$ distorts the world sheet, and by eliminating it, the original continuum world sheet picture is recovered. In contrast, the cutoff imposed by eq.(40) corresponds to a selection of the graphs; we are putting an upper bound of $\kappa$ on the average number of solid lines and hence on the number of propagators on the world sheet. The dynamics is still represented by the same action (13); we have simply chosen to study the set of of graphs subject to the restriction (40). Later, we will discuss the delicate limit $\kappa \to \infty, \rho_0 \to 1$.

We end this section by a brief discussion of the ground state of the model for $D = 4$ (6 space-time dimensions). To find ground state for $D = 4$, we set

$$\frac{\partial H^r_\rho}{\partial \lambda(\sigma)} = 0.$$
in eq.(25), which gives,

$$\rho(\sigma) = \sum_{\sigma_1<\sigma} \sum_{\sigma_2<\sigma} W(\sigma_1,\sigma_2) \frac{(\sigma_2-\sigma)(\sigma-\sigma_1)}{(\sigma_2-\sigma_1)^3}. \quad (41)$$

Notice that in contrast to the case $D = 2$, this equation does not fix $\lambda_0$. Here $\lambda$ acts as a lagrange multiplier and sets

$$H^r_c = 0. \quad (42)$$

We now evaluate the right hand side of the above equation in the mean field approximation: As before, the double sum is over only the hybrid lines, where we set $\rho = \rho_c = \rho_0$. Solving for $L$, we have,

$$L = 4\pi g^2_c \rho_0 (1-\rho_0) \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{n_1 n_2}{(n_1+n_2)^3} (1-\rho_0)^{n_1+n_2-2}. \quad (43)$$

In contrast to the case $D = 2$, here $\lambda_0$ is arbitrary, but $L$ is determined in terms of $\rho_0$ as before.

7. Quadratic Fluctuations Around The Classical Background

In this section, we will study the quantum fluctuations around the classical background by expanding in powers of the field fluctuations $\phi_1$ (eq.(14)). This expansion will be carried out only to second order. Later, we will discuss how this expansion may be fitted into a systematic perturbation series. In the interests of keeping this paper to a reasonable length, we will not try to do a complete second order calculation; we will only study some chosen terms of interest. To start with, $\rho$ and $\lambda$ will be frozen at their classical given by eqs.(27) and (28), only the field $\phi$ will be expanded to second order in $\phi_1$. We should stress, however, there is no obstacle to carrying out a complete second order calculation, except for lack of interest.

It is convenient to set

$$\phi_1 = \phi_{1,r} + i \phi_{1,i},$$

where $\phi_{1,r,i}$ are hermitian fields. The contribution to the action second order in $\phi_1$ is given by

$$S^{(2)} = S_{k,e} - \int d\tau H^{(2)}(\tau) = S_{k,e} + S_{p,e}, \quad (44)$$
where,

\[ S_{k.e} = 2 \sum_{\sigma} \int d\tau \int dq \, \partial_r \phi_{1,r} \phi_{1,r}, \]  

(45)

Since the action is quadratic in both \( \phi_{1,i} \) or \( \phi_{1,r} \), one can carry out the functional integral over one of these fields before writing down \( H^{(2)} \). We choose to integrate over \( \phi_{1,i} \), with the result,

\[ S_{k.e} \rightarrow \sum_{\sigma} \int d\tau \int dq \left( \frac{\partial_r \phi_{1,r}(\sigma, \tau, q)}{\lambda_c(\sigma)} \right)^2 + \frac{1}{2} \sum_{\sigma, \sigma'} G(\sigma, \sigma') \rho_c(\sigma') q^2; \]  

(46)

and, somewhat schematically,

\[ H^{(2)} \rightarrow \sum_{\sigma} \lambda_c(\sigma) \int dq \, \phi_{1,r}^2(\sigma, q) + \sum_{\sigma, \sigma'} G(\sigma, \sigma') \left( \frac{1}{2} \rho_c(\sigma') \int dq \, q^2 \phi_{1,r}^2(\sigma, q) \right) - 2 \int dq \int dq' \, (q \cdot q') (\phi_0 \phi_{1,r})_{\sigma,q} (\phi_0 \phi_{1,i})_{\sigma',q'} \]  

(47)

As stated earlier, in this equation \( \rho \) and \( \lambda \) are fixed at their classical values given by eqs.(27) and (28). We would like to emphasize that the \( \sigma \) sums in the above expressions are over the hybrid lines which are spaced by \( L \); the grid spacing \( a \) has completely disappeared.

The above action, being quadratic in \( \phi_{1,r} \), can be diagonalized. Again, in the interests of brevity, we will not carry out a full analysis, but instead, describe the general features of the spectrum, and work out in detail the sector of the model of particular interest. One important feature is that limit \( a \rightarrow 0 \) can be taken without causing any blowup or singularity in the spectrum. This is in contrast to the earlier work [1,2], where, in the limit \( a \rightarrow 0 \), part of the spectrum went to infinity as \( 1/a^2 \). This is due to different classical background we have here: As we have pointed out earlier, the factor that is the source of possible singularity is \( 1/|\sigma - \sigma'| \) in \( G(\sigma, \sigma') \). But with the classical background we have, this factor never becomes singular since, located on the hybrid lines, \( \sigma \) and \( \sigma' \) are separated by at least a distance \( L \), and \( L \) stays fixed and finite as \( a \rightarrow 0 \).

So far, we have kept \( \rho_0 < 1 \). In the case \( D = 2 \), this was done with the help of a cutoff (eq.(40)). It is of considerable interest to see what happens in the limit \( \rho_0 \rightarrow 1 \). In this limit, which we will call the high density limit, \( L \sim 1 - \rho_0 \rightarrow 0 \) (eq.36), and from a cursory examination of eq.(6), one expects the spectrum to blow up as \( 1/L^2 \). This in parallel with the \( 1/a^2 \)
behaviour found in [1,2]. Just as in that case, part of the spectrum stays finite in this limit. This because the world sheet dynamics is invariant under the translation
\[ \mathbf{q} \to \mathbf{q} + \mathbf{r}, \]  
(48)
where \( \mathbf{r} \) is a constant vector. This invariance is broken spontaneously by the classical solution, which is not translation invariant. This situation is familiar from soliton and instanton physics; as a consequence of Goldstone’s theorem, a massless zero mode develops. Protected by Goldstone’s theorem, this mode remains massless also in the limit \( L \to 0 \), and therefore there is no blowup in the spectrum. We will call this sector of the model the light sector, and identify and quantize it by the collective coordinate method for the case \( D = 2 \). The modes whose energies go to infinity as \( L \to 0 \) will be called the heavy modes.

Consider the field configuration
\[ \phi_{1,r} \to \phi_0 (\sigma, \mathbf{q} + \mathbf{v}(\sigma, \tau)) - \phi_0 (\sigma, \mathbf{q}), \]  
(49)
where \( \phi_0 \) is the classical solution (17). What we have done is to promote the constant vector \( \mathbf{r} \) into the collective coordinate \( \mathbf{v}(\sigma, \tau) \). We now replace \( \phi_{1,r} \) in \( S_{k,e} \) in eq.(46) by the above expression, expand to second order in \( \mathbf{v} \), and do the finite integral over \( \mathbf{q} \) explicitly. There are couple of helpful simplifications which we note below:

a) Before applying the mean field approximation, it is best to use the identities (18) and (20) in order to get rid of \( G \) and reduce quadratic terms in \( W \) to linear ones. This simplifies the final expression considerably.

b) Since the classical solution in the mean field approximation is \( \tau \) independent, the only \( \tau \) dependence is in \( \mathbf{v} \). This is why the final expression for \( S_{k,e} \) is quadratic in \( \partial_\tau \mathbf{v} \):

\[
S_{k,e} \to \frac{\pi g^2}{6} \int d\tau \sum_\sigma \sum_{\sigma_1 < \sigma} \sum_{\sigma_2 < \sigma} \frac{W(\sigma_1, \sigma_2)}{(\sigma - \sigma_1)(\sigma_2 - \sigma_1)(\sigma_2 - \sigma)} \lambda_0^3 (\partial_\tau \mathbf{v}(\sigma, \tau))^2 \\
= \frac{\pi g^2 \rho_0^3 (1 - \rho_0)}{6 \lambda_0^3 L^3} \int d\tau \sum_\sigma (\partial_\tau \mathbf{v})^2.
\]  
(50)
Here \( \lambda_0 \) and \( L \) are given by (36) and,
\[
f_3(\rho_0) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{(1 - \rho_0)^{n_1+n_2-2}}{n_1 n_2 (n_1 + n_2)}.
\]
Next, we make the replacement (49) in $H^{(2)}$ (eq.(47)). This is easily evaluated by shifting $q$ integration by

$$q \rightarrow q - v$$

and then using

$$\int dq \phi_0\phi_0 = \rho.$$  

We have,

$$H^{(2)} \rightarrow \frac{1}{2} \sum_{\sigma,\sigma'} G(\sigma, \sigma') \left( \int dq \rho(\sigma') q^2 (\phi_0^\dagger \phi_0)_{\sigma, q+v(\sigma)} - \int dq \int dq' (q \cdot q') (\phi_0^\dagger \phi_0)_{\sigma, q+v(\sigma)} (\phi_0^\dagger \phi_0)_{\sigma', q'+v(\sigma')} \right) \rightarrow \frac{1}{2} \sum_{\sigma,\sigma'} G(\sigma, \sigma') \rho(\sigma) \rho(\sigma') \left( v^2(\sigma) - v(\sigma) \cdot v(\sigma') \right).$$  

(51)

Finally, applying the mean field approximation to the last line gives

$$H^{(2)} \rightarrow \frac{1}{2} \rho_0^2 \sum_{n \neq n'} \frac{(1 - \rho_0)^{|n-n'-1|}}{|n-n'| L} \left( v^2(\sigma = nL) - v(\sigma = nL) \cdot v(\sigma' = n'L) \right).$$  

(52)

The above expression can be diagonalized by defining

$$v(nL) = \frac{L}{2\pi} \int_{-\pi/L}^{\pi/L} dk e^{-iknL} \tilde{v}(k),$$

and rewriting it in terms of $\tilde{v}$:

$$H^{(2)} \rightarrow \rho_0^2 \int_{-\pi/L}^{\pi/L} dk \tilde{v}(k) \cdot \tilde{v}(-k) \ln \left( 1 + \frac{2 (1 - \rho_0)}{\rho_0^2} (1 - \cos(kL)) \right).$$  

(53)

Up to an overall constant, the spectrum as a function of $k$ is given by the function

$$\ln \left( 1 + \frac{2 (1 - \rho_0)}{\rho_0^2} (1 - \cos(kL)) \right).$$

This is a periodic function with the period $2\pi/L$. This kind of spectrum is to be expected in view of the formation of a periodic structure (crystal) on the world sheet. We should point out that although we have treated $k$ as a
continuous variable, being conjugate to the compact variable \(0 \leq \sigma \leq p^+\), it is really a discrete variable quantized in units of \(2\pi/p^+\). So the integral over \(k\) should really be replaced by a discrete sum. Since, however, we have tacitly assumed the ratio \(p^+/L\) to be large, the integral is a good approximation to the sum.

We now briefly discuss quadratic fluctuations in \(v\) for the case \(D = 4\). Again, starting with the field configuration (49), we repeat the steps leading to eqs.(46) and (47). The result for \(S_{p.e}\) is unchanged, again given by eq.(53). \(S_{k.e}\) can be computed again by applying eq.(46), leading to

\[
S_{k.e} \rightarrow \frac{\pi^2 g_0^2 \rho_0 (1 - \rho_0) f_1(\rho_0)}{6 \lambda_0^2 L^2} \int d\tau \sum_\sigma (\partial_\tau v(\sigma, \tau))^2, \tag{54}
\]

where \(f_1\) is given by (37).

So far, \(\lambda_0\) was arbitrary, but now, for the model to be renormalizable, we must require the ratio

\[
g_0^2/\lambda_0^2
\]

to be cutoff independent, so that

\[
\lambda_0^2 \rightarrow \frac{1}{\ln (\Lambda^2/\mu^2)} \tag{55}
\]

as \(\Lambda \rightarrow \infty\).

8. Dense Graphs On The World Sheet

We start with \(D = 2\), and take the limit of dense graphs: \(\rho_0 \rightarrow 1, L \rightarrow 0\) in both \(S_{k.e}\) (eq.(50)) and in

\[
S_{p.e} = \int d\tau H^{(2)}(\tau).
\]

In this limit, two simplifications occur:

a) As \(\rho_0 \rightarrow 1\), the action becomes local. By a local action we mean an action which correlates two \(v\)'s separated by at most by a distance \(L\). Notice that \(S_{k.e}\) is already local, but \(H^{(2)}\) contains terms of the form

\[
v(nL) \cdot v((n + n')L)
\]

with \(n' > 1\), which are non-local. However, in the limit \(\rho_0 \rightarrow 1\), all of these non-local terms are suppressed by factors of \(1 - \rho_0\).
b) As $L \to 0$, $\sigma$ becomes continuous. The sums over $\sigma$ turn into integrals, and one can expand in powers of $L$:

$$v((n+1)L) - v(nL) \to L \partial_\tau v(\sigma).$$

We start with $S_{k.e}$. We let $\rho_0 \to 1$, or, equivalently, $L \to 0$, use eq.(36) for $\lambda_0$, and convert the sum over $\sigma$ into an integral:

$$S_{k.e} \to \frac{C}{2\mu^4} \int d\tau \int d\sigma (\partial_\tau v)^2,$$

where $C$ is a numerical constant:

$$C = \frac{\exp(2/3)}{48\pi^2}.$$

The important point is that, $S_{k.e}$ is independent of $L$, so that a finite limit is reached as $L \to 0$. We also note that the result is independent of $g$.

Next, we consider eq.(53). Expanding to leading order in $L$ by

$$1 - \cos(kL) = \frac{1}{2} k^2 L^2,$$

and letting $\rho_0 \to 1$ gives

$$S_{p.e} \to -\frac{L^2}{4\pi} \int d\tau \int dk k^2 \tilde{v}(\tau, k) \cdot \tilde{v}(\tau, -k) = -\frac{1}{2} \int d\tau \int d\sigma (\partial_\tau v(\tau, \sigma))^2.$$

The limit is again finite and $g$ independent. Finally, the limit of the full second order action is given by

$$S^{(2)} \to \int d\tau \int_0^{p^+} d\sigma \left( \frac{C}{2\mu^4} (\partial_\tau v)^2 - \frac{1}{2} (\partial_\sigma v)^2 \right).$$

This the action for a transverse string in the lightcone coordinates. The slope is given by

$$\alpha' = \left( 4\pi^2/C \right)^{1/2} \mu^2,$$

and depends only on $\mu$. It is important to notice that the string picture is an approximate one. In reality, $1 - \rho_0$ is small but not zero, and there is a heavy sector with masses proportional to

$$1/L^2 \simeq 1/(1 - \rho_0)^2.$$
The higher string excitations with masses comparable to the masses of the states in the heavy sector mix with these states, and the string picture breaks down. It is plausible to identify the heavy sector with the original field theory spectrum: Since the model at $D = 2$ is trivially asymptotically free, a weakly coupled field $\phi^3$ theory is expected to be valid at high energies. On the other hand, at low energies, the picture developed here suggests that low lying bound states of the model form a string. So we have a hybrid picture combining field theory and string theory: At low energies, the string picture is the relevant one, and at high energies, field theory takes over.

It is of interest to notice that in the dense graph limit, the combination

$$\gamma = g^2 (1 - \rho_0) \quad (60)$$

acts as an effective coupling constant. For example, the classical action (38) is proportional to $1/\gamma^2$, the lowest order quantum corrections (58) are independent of $\gamma$. This is the behaviour expected from a weak coupling expansion in $\gamma$ and it can be traced back to the structure of the interaction term in eq. (1). From this perspective, the heavy sector, with masses inversely proportional to $\gamma$, can be thought of as a sector of solitons. In the dense graph limit, it may be possible to do a systematic expansion in $\gamma$, without any appeal to the mean field approximation. We hope to further develop this idea in the future.

Finally, we comment briefly on the dense graph limit at $D = 4$. This limit is more problematic in this case, since $\lambda_0$ is undetermined. We note that, as $\rho_0 \to 1$, $L \sim 1 - \rho_0$ (eq. (36)), and in order to have a sensible string picture with a finite slope, we have to require $S_{k,e}$ (eq. (54)) to remain finite. Therefore, in this limit, in addition to eq. (55), the following condition has to be imposed on $\lambda_0$:

$$\lambda_0 \sim 1/(1 - \rho_0).$$

From the string perspective, this is a natural requirement: We are demanding that the string slope remain finite in the dense graph limit. However, it would be very desirable to confirm this by means of an alternative treatment based on the original field theory model.

9. Conclusions

The main contribution of the present article is the correct handling of the two divergences that plagued the previous work [1,2] on the world sheet field theory [5] for the $\phi^3$ interaction. The ultraviolet divergence is eliminated by
the mass and coupling constant terms already present in the model, without any ad hoc modifications, which was an unsatisfactory feature of [1]. The other problem is a spurious infrared divergence, which requires the discretization of the world sheet coordinate $\sigma$. Here we are able to take the limit grid spacing $a \to 0$ smoothly, without encountering any blow up in the spectrum. This is achieved by choosing a classical background different from the one chosen in [1,2]. The new background consists of equally spaced parallel lines that form a one dimensional crystal. Possible divergences are avoided by keeping the spacing of lines fixed as $a \to 0$.

We feel that, except for manifest Lorentz invariance\(^3\) all the major technical problems associated with the $\phi^3$ model have been resolved, at least within the context of the mean field approximation. It is time to apply the techniques developed for $\phi^3$ to more physical models, such as gauge theories\(^4\). An intermediate step would be to introduce, in addition to $\phi^3$, a $\phi^4$ interaction. This is a more physical model, and to some extent, mimics a gauge theory. We hope to investigate this possibility in the near future.

### Acknowledgement

This work was supported in part by the director, Office of Science, Office of High Energy Physics of the U.S. Department of Energy under Contract DE-AC02–05CH11231.

### References

1. K.Bardakci, JHEP **1003** (2010) 107, [arXiv:0912.1304](https://arxiv.org/abs/0912.1304).
2. K.Bardakci, JHEP **0903** (2009) 088, [arXiv:0901.0949](https://arxiv.org/abs/0901.0949).
3. K.Bardakci and C.B.Thorn, Nucl.Phys. **B 626** (2002) 287, [hep-th/0110301](https://arxiv.org/abs/hep-th/0110301).
4. G.'t Hooft, Nucl.Phys. **B 72** (1974) 461.
5. K.Bardakci, JHEP **0810** (2008) 056, [arXiv:0808.2959](https://arxiv.org/abs/0808.2959).
6. H.P.Nielsen and P.Olesen, Phys.Lett. **B 32** (1970) 203.
7. B.Sakita and M.A.Virasoro, Phys.Rev.Lett. **24** (1970) 1146.

---

\(^3\)See [10] for an investigation of renormalization and Lorentz invariance in the light cone formulation.

\(^4\)For some initial steps taken towards more physical theories, see [11, 12].
8. A.Casher, Phys.Rev. D 14 (1976) 452.

9. R.Giles and C.B.Thorn, Phys.Rev. D 16 (1977) 366.

10. C.B.Thorn, Nucl.Phys. B 699 427, hep-th/0405018. D.Chakrabarti, J.Qiu and C.B.Thorn, Phys.Rev. D 74 (2006) 045018, hep-th/0602026.

11. C.B.Thorn, Nucl.Phys. B 637 (2002) 272, hep-th/0203167. S.Gudmundsson, C.B.Thorn and T.A.Tran, Nucl.Phys. B 649 (2003) 3-38, hep-th/0209102.

12. C.B.Thorn and T.A.Tran, Nucl.Phys. B677 (2004) 289, hep-th/0307203.